Energy and Momentum Eigenspectrum of the Hulthén-screened Cosine Kratzer Potential Using Proper Quantization Rule and SUSYQM Method

Kaushal R Purohit (kaushalsep1996@gmail.com)
Sardar Vallabhbhai National Institute of Technology

Rajendrasinh H PARMAR
Sir P T Science College, Modasa

Ajay Kumar Rai
Sardar Vallabhbhai National Institute of Technology

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Energy and Momentum eigenspectrum of the Hulthén-screened cosine Kratzer potential using proper quantization rule and SUSYQM method

Kaushal R Purohit 1, Rajendrasinh H Parmar 2 and Ajay Kumar Rai

1 Department of Physics, Sardar Vallabhbhai National Institute of Technology, Surat, Gujarat-395 007, India,
2 Sir P T Science College, Modasa, Gujarat-383 315, India,

Abstract

Using the Qiang-Dong proper quantization rule (PQR) and the supersymmetric quantum mechanics approach, we obtained the eigenspectrum of the energy and momentum for time independent and time dependent Hulthén-screened cosine Kratzer potentials. For the suggested time independent Hulthén-screened cosine Kratzer potential, we solved the Schrödinger equation in D dimensions (HSCKP). The Feinberg-Horodecki equation for time-dependent Hulthén-screened cosine Kratzer potential was also solved (tHSCKP). To address the inverse square term in the time independent and time dependent equations, we employed the Greene-Aldrich approximation approach. We were able to extract time independent and time dependent potentials, as well as their accompanying energy and momentum spectra. In three-dimensional space, we estimated the rotational vibrational (RV) energy spectrum for many homodimers (\(H_2, I_2, O_2\)) and heterodimers (\(MnH, ScN, LiH, HCl\)). We also used the recently introduced formula approach to obtain the relevant eigen function. We also calculated momentum spectra for the dimers \(MnH\) and \(ScN\). The method is compared to prior methodologies for accuracy and validity using numerical data for heterodimer \(LiH, HCl\) and homodimer \(I_2, O_2, H_2\). The calculated energy and momentum spectra are tabulated and analysed.

Email address: kaushalsep1996@gmail.com (Kaushal R Purohit 1, Rajendrasinh H Parmar 2 and Ajay Kumar Rai)

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1. Introduction

In the literature, there are exact or approximate solvable potential models. These hypothetical models have sparked a lot of curiosity since they provide us a glimpse into the quantum mechanical system. To solve the various potentials, a variety of strategies are available. The asymptotic iteration method (AIM) \cite{1, 2}, the ansatz method \cite{3}, the Nikiforov-Uvarov (NU) technique \cite{4, 5}, the factorization technique \cite{6, 7}, the formula method \cite{8}, and the supersymmetric quantum mechanics (SUSYQM) \cite{9, 10}.

Furthermore, many quantization approaches exist to address potential issues. The Wentzel-Kramers-Brillouin (WKB) quantization method \cite{11}, the supersymmetric WKB (SWKB) quantization technique \cite{12}, and the exact/improved quantization rule (EQR/IQR) \cite{13, 14, 15} are these methods. \cite{16, 17}, etc. Qiang–Dong appropriate quantization rule To produce energy eigenvalues, the Qiang–Dong proper quantization procedure can be applied to any absolutely solvable potential. \cite{16}

The Hulthén potential \cite{18, 19} is a very important short-range potential that is widely used to describe the bound and continuum states of interaction physical systems in physics and chemistry, and it has been applied to a variety of fields including chemical physics, condensed matter physics, atomic-molecular physics, particle and nuclear physics \cite{20}. Hulthén potential is solved analytically using several approaches such as SUSYQM, NU, AIM, and so on.

The Kratzer potential (KP) \cite{21} is applied in quantum chemistry and atomic-molecular physics. In quantum physics, it is applied to describe the interactions of molecular structure. The KP has been widely used both in relativistic and non-relativistic regime in three or higher dimensions using various methods \cite{1, 5, 9, 10, 6, 7, 11, 8, 2, 11, 3, 12, 13, 14, 15, 16, 17, 22}.
Combination of two or more potentials in both relativistic and non-relativistic realm has been great interest. The essence of mixing more than one physical potential model is to have many applications \[23\]. Kratzer potential and various type mixed Kratzer potential have been studied by different/same researcher via same/various methods such as Kratzer potential \[24, 25\] by NU method same potential studies by Bayrak et al. \[1\] within the framework of AIM. Modified Kratzer plus ring shape potential studies by S.M. Ikhdair \[6\] via NU method. Hellmann–Kratzer potential model investigated by C.O. Edet et al. \[26, 27\] via WKB formalism. Screened Kratzer potential analyzed by A N Ikot et al. \[28\] using NU method. Recently, we studies screened cosine Kratzer potential and obtained various molecular properties using NU method \[29\]. Screened Coulomb potential (SCP) plus Kratzer potential analyzed by Edet C.O et al. \[30\] using NU method. Manning-Rosen plus Hellmann potential studies by I.B. Ita et al. \[31\] via WKB approximation scheme and same potential investigated by H. Louis et al. \[31\] via proper quantization rule method in July 2019. Solution of the energy dependent molecular Kratzer potential have been obtained by A. N Ikot et al. \[32\] via asymptotic iteration method. Hulthén plus Kratzer potential in D space investigated by Obu, J.A. et al. \[33\] under relativistic and non-relativistic treatment using NU method. Bound state solution of the Schrödinger equation for inversely quadric Yukawa plus Kratzer-Fues potential have been obtained using the WKB method \[34\]. Approximate analytical solutions of Klein-Gordon equation obtained for Kratzer potential using ansatz method \[3\]. In march 2020, Onyenegecha et al. \[35\] have been studied the modified Mobius square plus Kratzer potential via NU method.

In this paper, we used combination of two potential, the Hulthén potential and the screened cosine Kratzer potential (SCKP) \[29\], known as Hulthén-screened cosine Kratzer potential (HSCKP) is investigated via the Qiang–Dong proper quantization rule. Numerous researchers applied this method to obtain energy eigenvalues for various potentials \[8, 12, 13, 16, 36, 37, 38, 39, 40, 41, 42, 43, 44\].
The time independent Hulthén potential define as \([18, 19]\)

\[
V(r) = -\frac{V_0 e^{-\alpha r}}{1 - e^{-\alpha r}} \tag{1}
\]

whereas recently investigated SCKP written as \([20]\)

\[
-2D_e \left( \frac{a}{r} - \frac{b}{2r^2} \right) e^{-\delta \alpha r} \cosh \delta \lambda \alpha r \tag{2}
\]

Newly proposed HSCKP given by

\[
V(r) = -\frac{V_0 e^{-\alpha r}}{1 - e^{-\alpha r}} - 2D_e \left( \frac{a}{r} - \frac{b}{2r^2} \right) e^{-\delta \alpha r} \cosh \delta \lambda \alpha r \tag{3}
\]

where \(D_e\) is the dissociation energy, \(a \equiv r_e, b \equiv r_e^2, \delta, \lambda\) and \(\alpha\) are the screening parameters, \(r_e\) is the equilibrium bond length and the interatomic distance \(r\).

We obtain energy eigenvalues for the HSCKP in D dimensional space using non relativistic Schrödinger equation via PQR method in this paper. Using formula method given by Falaye et al. \([8]\), we obtain wave function because the proper quantization rule is not applicable to compute the eigen functions of the given system.

In this paper, we also solved the time dependent HSCKP (tHSCKP) written as

\[
V(t) = -\frac{V_0 e^{-\alpha t}}{1 - e^{-\alpha t}} - 2D_e \left( \frac{a}{t} - \frac{b}{2t^2} \right) e^{-\delta \alpha t} \cosh \delta \lambda \alpha t \tag{4}
\]

The Feinberg-Horodecki (FH) equation is a space-like counterpart of the Schrödinger equation. From the relativistic Feinberg equation \([45]\), FH equation was derived by Horodecki \([46]\). Feinberg-Horodecki equation has been demonstrated in the possibility of describing biological systems \([47, 48]\) in terms of the time-like supersymmetric quantum mechanics \([49]\). The space-like solutions of the FH equation can be applied to test its relevance in various branches of science including medicine, biology and physics. Since last decade, numerous researchers solved the FH equation with different time dependent potentials.

For FH equation the exact momentum state solution with the rotating time-dependent Deng-Fan oscillator are relevant within the pattern of generalized
parametric NU (pNU) method. Bera and Sil \cite{50} found exact solutions of the Feinberg-Horodecki equation for the time-dependent Wei-Hua oscillator and Manning-Rosen potentials by the NU method.

Recently, FH equation studied by Altug and Sever \cite{51} for time-dependent pöschl-Teller potential and found its space like coherent state. Celia et al. \cite{52} has studied time-dependent mass and frequency and a perturbative potential to construct coherent states for two systems. Recently in 2020, Farout et al. \cite{53} obtained the quantized momentum solution of the FH equation with general potential model using NU method. In July 2020, Farout and Ikhdair, solved the FH equation with the time-dependent screened Kratzer-Hellmann potential model \cite{54} and obtained the approximated eigensolutions of momentum states. In August 2020, Farout et al. \cite{55} obtained the quantized momentum eigenvalues with space-like coherent eigenstates using FH equation with the Kratzer potential plus screened coulomb potential. In 2015, Ojonubah and Onate \cite{56} obtained exact solutions of Feinberg-Horodecki equation for time-dependent Tietz-Wei diatomic molecular potential. The quantized momentum eigenvalues and corresponding wave functions are found in the framework of supersymmetric quantum mechanics.

We also obtained momentum eigensolution of the tHSCKP via supersymmetric quantum mechanics.

Under some constraints on potentials parameters and screening parameters (for HSCKP and tHSCKP), we reduces HSCKP and tHSCKP into various time independent and time dependent potentials such as screened cosine Kratzer potential \cite{29}, screened Kratzer potential \cite{28}, Kratzer potential, the generalized cosine Yukawa potential (GCYP), inversely quadratic Yukawa potential (IQYP) \cite{57}, Hellmann potential and the Hulthén potential.

We calculate numerically eigen spectrum for the $MnH, ScN, HCl, LiH$, and $H_2, I_2, O_2$ dimer in three dimensions for various values of the $n$ and $\ell$ quantum numbers. At the end, we compare the numerical results of the energy spectrum for the HSCKP, screened Kratzer potential (SKP), and KP of the $LiH, HCl, H_2, I_2, O_2$ dimer. We obtained momentum spectra for selected dimer.
The work is divided in six sections: Section 2 Reviews of methodology. Solution of the HSCKP and tHSCKP are presented with deduced potentials in section 3. In section 4, Numerical results are discussed. The brief concluding remarks given section 5.

2. Reviews of the methodology

We employed Qiang-Dong proper quantization rule (PQR) method to solve the Schrödinger equation in D dimensions for time independent HSCKP and SUSYQM method have been employed to solve the Feinberg-Horodecki equation for time dependent HSCKP.

2.1. Proper quantization rule and formula method

One dimensional Schrödinger equation is expressed as

$$\frac{d^2 \psi(z)}{dz^2} + \frac{2\mu}{\hbar^2} (E - V(z)) \Psi(z) = 0$$  \hspace{1cm} (5)

Equation (5) can be written as

$$\frac{d^2 \Psi(z)}{dz^2} + k(z)^2 \Psi(z) = 0, \hspace{1cm} k(z) = \sqrt{\frac{2\mu}{\hbar^2} (E - V(z))}.$$ \hspace{1cm} (6)

where $\mu$ and $k(z)$ denotes the reduced mass of the interacting particles and momentum respectively. The Schrödinger equation can be written in the form of the Riccati equation as

$$\frac{d\chi(z)}{dz} + \chi(z)^2 + k(z)^2 = 0$$ \hspace{1cm} (7)

Here, $\chi(z) = \frac{1}{\Psi(z)} \frac{d\Psi(z)}{dz}$. Several researchers applied PQR to physical potential problem [25-35] both in relativistic and non-relativistic regime in the three and higher dimensions. Ma and Xu [12, 13] by carefully studying the one dimensional Schrödinger equation generalized to the three dimensions radial Schrödinger equation with spherically symmetric potential simplify making the replacement $z \to r$. The proper quantization rule written as

$$\int_{r_1}^{r_2} k(r)dr - \int_{r_{01}}^{r_{02}} k_0(r)dr - n\pi = 0$$ \hspace{1cm} (8)
Here, \( r_1 \) and \( r_2 \) are two turning points determined by \( E = V_{eff}(r) \). \( n \) is number of nodes of \( \chi(r) \) in the region \( E \geq V(r) \) and it is larger than the number of nodes of the wave function \( \Psi(r) \) by one.

Accordingly, it is required to first calculate the integral on the LHS of equation (8) and then replace spectrum of energy \( E_n \) by the lowest energy level \( E_0 \) in the result to obtain the RHS integral of equation (8). PQR method has been applied in numerous physical systems to obtain the exact energy spectrum of many exactly solvable quantum mechanical systems [12, 13, 38, 39, 40, 42, 58].

In the formula method [8], the eigen equations with any physical potential of interest are convert into the second order differential equation as [59]

\[
\frac{d^2 \Psi(g)}{dg^2} + \frac{(a_1 - a_2 g) \, d\Psi(g)}{g(1 - a_3 g)} + \frac{(ag^2 + bg + c)}{g^2(1 - a_3 g)^2} \Psi(g) = 0
\] (10)

We obtain energy spectrum and corresponding eigen function respectively using the coordinate transformation variable of \( g = g(r) \) as

\[
a_5^2 = \left[ a_4^2 - a_2^2 - \left( \frac{1 - 2n}{2a_3} - \frac{1}{2a_3} \left[ a_2 - \sqrt{(a_3 - a_2)^2 - 4a} \right] \right) \right]^2 = 0, \quad a_3 \neq 0. \tag{11}
\]

and

\[
\psi(g) = N_{n\ell} g^{a_4} (1 - a_3 g)^{a_5} \, _2F_1(-n, n + 2(a_4 + a_5) + \frac{a_2}{a_3} - 1; 2a_4 + a_1, a_3 g). \tag{12}
\]

where

\[
a_4 = \frac{(1 - a_1) + \sqrt{(1-a_1)^2 - 4c}}{2} \tag{13}
\]

\[
a_5 = \frac{1}{2} + \frac{a_1}{2} - \frac{a_2}{2a_3} + \sqrt{\left( \frac{1}{2} + \frac{a_1}{2} - \frac{a_2}{2a_3} \right)^2 - \left( \frac{a}{a_3^2} + \frac{b}{a_3} + c \right)} \tag{14}
\]

\( N_{n\ell} \) is constant of normalization and \( _2F_1 \) is the hypergeometric function.
2.2. SUSYQM method

It is possible to define two nilpotent operators $\Phi$ and $\Phi^\dagger$ for $N = 2$ in SUSUQM. They satisfy the following commutation and anti-commutation relations [9, 10, 60] as

$$[H, \Phi] = [H, \Phi^\dagger] = 0 \quad (15)$$

$$\{\Phi, \Phi\} = 0, \quad \{\Phi^\dagger, \Phi^\dagger\} = 0, \{\Phi, \Phi^\dagger\} = H \quad (16)$$

$H$ is the supersymmetric Hamiltonian operator and $\Phi$ and $\Phi^\dagger$ are known as the supercharges operators define as [60, 61],

$$\Phi = \begin{pmatrix} 0 & 0 \\ S^- & 0 \end{pmatrix} \quad (17)$$

and

$$\Phi^\dagger = \begin{pmatrix} 0 & S^+ \\ 0 & 0 \end{pmatrix} \quad (18)$$

where $S^-$ is bosonic operator and $S^+$ is its adjoint. The Hamiltonian $H$ in terms of $S^-$ and $S^+$ can be defined

$$H = \begin{pmatrix} S^+ S^- & 0 \\ 0 & S^- S^+ \end{pmatrix} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \quad (19)$$

where the $H_{\pm}$ are named as the Hamiltonian of supersymmetric-partner. If we have zero ground state energy for $H$ (i.e $E_0 = 0$), we can always represent the Hamiltonian as a product of a linear differential operators pairs in a factorable form. Therefore, the ground state $\Psi_0(z)$ obeys the schrodinger equation as follows:
\[ H \Psi_0(z) = -\frac{\hbar^2}{2m} \frac{d^2 \Psi_0(z)}{dz^2} + V(z) \Psi_0(z) = 0 \]  

hence

\[ V(z) = \frac{\hbar^2}{2m} \frac{1}{\Psi_0(z)} \frac{d^2 \Psi_0(z)}{dz^2} \]  

This result makes us possible to globally reconstruct the above potential from the information of its ground state wave functions that contain zero nodes. Therefore, factorizing of \( H \) quite easy by using the following ansatz

\[ H_- = -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) = S^+ S^- \]  

where

\[ S^- = \frac{\hbar}{\sqrt{2m}} \frac{d}{dz} + U(z) \]  

\[ S^+ = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dz} + U(z) \]

Now, the riccati equation for \( U(z) \),

\[ V_-(z) = U^2(z) - \frac{\hbar}{\sqrt{2m}} U'(z) \]

Here, \( U(z) \) can express it in terms of \( \Psi_0(z) \) by

\[ U(z) = -\frac{\hbar}{\sqrt{2m}} \frac{\Psi'_0(z)}{\Psi_0(z)} \]

We obtain this solution by noticing that when \( S^- \Psi_0(z) = 0 \) is satisfied, we have \( H_- \Psi_0(z) = S^+ S^- \Psi_0(z) = 0 \). We then introduce the operator \( H_+ = S^- S^+ \) which is written by reversing the order of the \( H_- \) components. After a bit
simplification, we find that $H_+$ is nothing but the Hamiltonian for new potential $V_+(z)$ as,

\[
H_+ = \frac{\hbar^2}{2m}\frac{d^2}{dz^2} + V_+(z), \quad \text{where} \quad V_+(z) = U^2(z) + \frac{\hbar}{\sqrt{2m}}U'(z) \tag{27}
\]

Here, $V_+(z)$ and $V_-(z)$ are supersymmetric partner potentials. For example, when the ground state energy of $H_1$ is $E_0$ with eigenfunction $\Psi_0$ form Eq. (22), we can always write \[60, 61\],

\[
H_1 = \frac{\hbar^2}{2m}\frac{d^2}{dz^2} + V_1(z) = S^+S^- + E_0 \tag{28}
\]

with

\[
S^+_1 = \frac{\hbar}{\sqrt{2m}}\frac{d}{dz} + U_1(z), \quad S^-_1 = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dz} + U_1(z) \tag{29}
\]

and

\[
V_1(z) = U^2_1(z) - \frac{\hbar}{\sqrt{2m}}U'_1 + E_0, \quad \text{where} \quad U_1(z) = -\frac{\hbar}{\sqrt{2m}}\frac{d\ln\Psi_0}{dz} \tag{30}
\]

The SUSY partner Hamiltonian $H_2$ is defined by

\[
H_2 = S^-_1S^+_1 + E_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + V_2(z) \tag{31}
\]

with

\[
V_2(z) = U^2_1(z) + \frac{\hbar}{\sqrt{2m}}U'_1 + E_0 = V_1(z) + \frac{2\hbar}{\sqrt{2m}}U'_1(z) = V_1(z) - \frac{\hbar^2}{m}\frac{d^2}{dz^2}(\ln\Psi_0) \tag{32}
\]

$V_1(z)$ and $V_2(z)$ are potentials corresponds to Hamiltonian $H_1$ and $H_2$ respectively. From equation (31), the energy eigenvalues and eigen functions corresponds to $H_1$ and $H_2$, are obtained as \[60, 61\],

\[
E^2_n = E^1_{n+1}, \tag{33}
\]
\[ \Psi_n^2 = (E_{n+1}^1 - E_0^1)^{-\frac{1}{2}} P_1^+ \Psi_{n+1}^1 \quad \text{and} \quad \Psi_{n+1}^1 = (E_n^2 - E_0^1)^{-\frac{1}{2}} P_1^+ \Psi_n^2 \] (34)

where \( E_n^m \) represent the energy eigenvalue, with \( n \) and \( m \) denote the energy level and \( mth \) Hamiltonian \( H_m \), respectively. Hence, it is clear that if \( H_1 \) has \( k \geq 1 \) bound state with corresponding eigenvalues \( E_n^1 \), as well as eigen function \( \psi_n^1 \) defined in \( 0 < n < k \), then we can always generate a hierarchy of \( (k-1) \) Hamiltonians, i.e. \( H_2, H_3, ..., H_k \) such that the \( (H_m) \) has the same spectrum of eigenvalue as \( H_1 \), apart from the fact that the first \( (m-1) \) eigenvalue of the Hamiltonian \( H \) are absent in \( H \) [60, 61].

\[ H_m = P_m^+ P_m - E_{m-1}^1 = -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V_m(z) \] (35)

where

\[ S_m^- = \frac{\hbar}{\sqrt{2m}} \frac{d}{dz} + U_m(z) \quad \text{and} \quad S_m^+ = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dz} + U_m(z) \] (36)

\[ U_m(z) = -\frac{\hbar}{\sqrt{2m}} \frac{d \ln \Psi_0^m}{dz}, \quad m = 2, 3, 4, ...k. \] (37)

We also have

\[ E_n^m = E_{n+1}^{m-1} = ... = E_{n+m-1}^1, \] (38)

\[ \Psi_n^m = (E_{n+m-1}^1 - E_{m-2}^1)^{-\frac{1}{2}} ... (E_{n+m-1}^1 - E_0^1)^{-\frac{1}{2}} S_{m-1}^- ... S_1^- \Psi_{n+m-1}^1. \] (39)

\[ V_m(z) = V_1(z) - \frac{\hbar^2}{m} \frac{d^2}{dz^2} \ln(\Psi_0^1 ... \psi_0^{(m-1)}). \] (40)

By knowing all the eigen functions and eigenvalues of \( H_1 \), we also obtain the corresponding eigenfunction \( \Psi_n^1 \) and energy eigenvalues \( E_n^1 \) of the \( (k-1) \) Hamiltonian \( (H_2, H_3, ..., H_k) \).
3. Solution of the HSCKP and tHSCKP

In this section, we obtained solution of the HSCKP and tHSCKP using PQR method and SUSYQM method respectively.

3.1. Solution of the HSCKP in D dimensional space via PQR method

For spherically symmetric potential, D dimensional Schrödinger equation expresses as \[62\]

\[
\frac{\hbar^2}{2\mu} \left( \nabla^2_D + V(r) \right) \Psi_{n\ell m}(r, L_D) = E_{n\ell} \Psi_{n\ell m}(r, L_D) \tag{41}
\]

\(\nabla^2_D\) - the Laplacian operator defined as

\[
\nabla^2_D = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) - \frac{\Lambda^2_D(L_D)}{r^2} \tag{42}
\]

where \(\mu, V(r), \hbar, L_D\), are reduced mass, potential, reduced Planck constant and angular coordinate respectively where as \(E_{n\ell}\) is energy eigenvalues. The total wave function \(\Psi_{n\ell m}(r, L_m)\) written as

\[
\Psi_{n\ell m}(r, L_D) = R_{n\ell}(r) Y^m_{\ell}(L_D) \tag{43}
\]

where \(Y^m_{\ell}(L_D)\) is hyperspherical harmonic where as \(R_{n\ell}(r)\) hyper radial wave function. For the operator \(\Lambda^2_D(L_D)\), the eigen spectrum of hyperspherical harmonic functions in D dimensional space is given by

\[
\Lambda^2_D(L_D) Y^m_{\ell}(L_D) = \ell(\ell + D - 2) Y^m_{\ell}(L_D) \tag{44}
\]

Using Eqs. (42), (43) and Eq. (44), Eq. (41) reduces as

\[
\frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial R_{n\ell}(r)}{\partial r} \right) + \frac{2\mu}{\hbar^2} [E_{n\ell} - V_{\text{eff}}(r)] R_{n\ell}(r) = 0 \tag{45}
\]

where

\[
V_{\text{eff}} = V(r) + \frac{\hbar^2}{2\mu} \frac{S(S+1)}{r^2}, \quad S = \ell + (D - 3)/2 \tag{46}
\]
Now, choosing the radial wave function as
\[ R_{n\ell}(r) = \frac{1}{r^{(D-1)/2}} G_{n\ell}(r) \] (47)
Substituting Eqs. (3), (46) and (47) in Eq. (45), we obtain simplified Schrödinger equation as \[ 2\mu \left[ E_{n\ell} + \frac{V_0 e^{-\alpha r}}{1 - e^{-\alpha r}} + 2D_e \left( \frac{a}{r} - \frac{b}{2r^2} \right) e^{-\delta\alpha r} \cosh \delta \alpha r - \frac{\hbar^2}{2\mu} \frac{S(S+1)}{r^2} \right] G_{n\ell}(r) = 0 \] (49)
From Eq. (49)
\[ V_{\text{eff}}(r) = -\frac{V_0 e^{-\alpha r}}{1 - e^{-\alpha r}} - 2D_e \left( \frac{a}{r} - \frac{b}{2r^2} \right) e^{-\delta\alpha r} \cosh \delta \alpha r + \frac{\hbar^2}{2\mu} \frac{S(S+1)}{r^2} \] (50)
for \( \delta = 1/2 \) and \( \lambda = 1 \) we obtain
\[ V_{\text{eff}}(r) = -\frac{V_0 e^{-\alpha r}}{1 - e^{-\alpha r}} - \frac{D_e a e^{-\alpha r}}{r} + \frac{D_e b e^{-\alpha r}}{2r^2} + \frac{D_e b e^{-\alpha r}}{2r^2} + \frac{\hbar^2}{2\mu} \frac{S(S+1)}{r^2} \] (51)
Now, employ a Greene-Aldrich approximation for the centrifugal term \[ \frac{1}{r^2} \approx \frac{\alpha^2}{(1 - e^{-\alpha r})^2} \Rightarrow \frac{1}{r} \approx \frac{\alpha}{1 - e^{-\alpha r}} \] (52)
\[ V_{\text{eff}}(r) = -\frac{(V_0 + D_e a a)e^{-\alpha r}}{1 - e^{-\alpha r}} - \frac{D_e a a}{(1 - e^{-\alpha r})} + \frac{D_e b a^2 e^{-\alpha r}}{2(1 - e^{-\alpha r})^2} + \frac{\hbar^2}{2\mu} \frac{S(S+1)}{r^2} \] (53)
where
\[ S'(S'+1) = \frac{2\mu}{\hbar^2} \frac{D_e b}{2} + S(S+1), \] (54)
To obtain solution for the proposed potential via the proper quantization rule, we applying \( p \) as
\[ p = \frac{e^{-\alpha r}}{1 - e^{-\alpha r}}, \quad \frac{dp}{dr} = -\alpha(p+1) \] (55)
Using this transformation, Eq. (53) can be written as
\[ V_{\text{eff}}(p) = -(V_0 + D_e a a)p - D_e a a(p+1) + \frac{D_e b a^2}{2} (p^2 + p) + \frac{\hbar^2}{2\mu} \frac{S'(S'+1)}{2}(p+1)^2 \] (56)
\[ V_{\text{eff}}(p) = (\beta + \frac{D_e a_0^2}{2}) p^2 - \left(2D_e a + V_0 - \frac{D_e a_0^2}{2} - 2\beta\right) p + (\beta - D_e a_0) \]

\[ V_{\text{eff}}(p) = \gamma_1 p^2 - \gamma_2 p + \gamma_3, \]  \hspace{1cm} (57)

where

\[ \beta = \alpha \frac{\hbar^2}{\mu} (S' + 1), \]
\[ \gamma_1 = \beta + \frac{D_e a_0^2}{2}, \]
\[ \gamma_2 = 2D_e a + V_0 - \frac{D_e a_0^2}{2}, \]
\[ \gamma_3 = \beta - D_e a_0 \]  \hspace{1cm} (58)

Now, we can determine two turning points \( p_1 \) and \( p_2 \) as

\[ E_{\text{nt}} - V_{\text{eff}}(p) = 0 \]

using Eq. (58)

\[ -\gamma_1 p^2 + \gamma_2 p - \gamma_3' = 0 \]  \hspace{1cm} (59)

\[ p_1 = \frac{-\gamma_2 + \sqrt{\gamma_2^2 - 4\gamma_1 \gamma_3'}}{2\gamma_1}, \quad p_2 = \frac{\gamma_2 + \sqrt{\gamma_2^2 - 4\gamma_1 \gamma_3'}}{2\gamma_1} \]  \hspace{1cm} (60)

where \( \gamma_3' = (\gamma_3 - E_{\text{nt}}) \)

\[ p_1 + p_2 = \frac{\gamma_2}{\gamma_1}, \quad p_1 p_2 = \frac{\gamma_3'}{\gamma_1} \]  \hspace{1cm} (61)

The momentum \( k(p) \) between the two turning point \( p_1 \) and \( p_2 \) is given as

\[ k(p) = \frac{\sqrt{2\mu}}{\hbar^2} (-\gamma_1 p^2 + \gamma_2 p - \gamma_3') \]  \hspace{1cm} (62)

Equation (7) becomes (31)

\[ -\alpha p (1 + p) \frac{d\chi(p)}{dp} + \chi(p)^2 = -\frac{2\mu}{\hbar^2} (-\gamma_1 p^2 + \gamma_2 p - \gamma_3') \]  \hspace{1cm} (63)

Using Sturm-Liouville theorem and taking \( \chi(p) = A_1 p + A_2 \) in Eq. (63), we obtain
\[ A_2^2 + A_1(2A_2 - \alpha)p + A_1(A_1 - \alpha)p^2 = -\frac{2\mu h^2}{\bar{h}^2}(E_0 - \gamma_3) - \frac{2\mu h^2}{h^2} \gamma_2 p + \frac{2\mu h^2}{h^2} \gamma_1 p^2 \] (64)

Equating coefficient in above equation and after simplifying, we obtain

\[ A_1 = \alpha \left( \frac{1}{2} + \frac{1}{2} \left[ \sqrt{1 + 4S'(S' + 1) + \frac{8\mu h^2}{h^2} \alpha^2 \gamma_1} \right] \right) \] (65)

\[ A_2 = \frac{\alpha}{2} - \frac{\mu^2 \gamma_2}{A_1 h^2}, \quad E_0 = -\frac{h^2 \alpha^2}{2 \mu} \left( \frac{1}{2} + \frac{1}{2} \left[ \sqrt{1 + 4S'(S' + 1) + \frac{8\mu h^2}{h^2} \alpha^2 \gamma_1} \right] - \frac{\mu \gamma_2}{h^2 \alpha^2} \right)^2 - \gamma_3 \] (66)

Now

\[ \int_{r_1}^{r_2} k_n(r)dr = -\int_{p_1}^{p_2} \sqrt{\frac{2\mu}{h^2}(-\gamma_1 p^2 + \gamma_2 p - \gamma_3)} \frac{dp}{\alpha p(1 + p)} \] (67)

Let

\[ 2p + 1 = q; \quad dp = \frac{dq}{2} \] (68)

Using above Eq. (68) in Eq. (67), we obtain

\[ -\int_{p_1}^{p_2} \sqrt{\frac{2\mu}{h^2}(-\gamma_1 p^2 + \gamma_2 p - \gamma_3)} \frac{dp}{\alpha p(1 + p)} = -\frac{2\mu \gamma_1}{\alpha^2 h^2} \int_{q_1}^{q_2} \frac{1}{q^2 - 1} \sqrt{-q^2 + \frac{\gamma_2 q - \gamma_3}{\gamma_1}} dq \] (69)

\[ = -\frac{\pi}{2} \frac{2\mu \gamma_1}{\alpha^2 h^2} \left( \sqrt{(p_1 + 1)(p_2 + 1)} - \sqrt{(p_1 - 1)(p_2 - 1) + 2} \right) \] (70)

From Eq. (60),

\[ \int_{r_1}^{r_2} k_n(r)dr = -\pi \sqrt{\frac{2\mu \gamma_1}{\alpha^2 h^2}} \left( \sqrt{\frac{\gamma_2}{\gamma_1} + \frac{\gamma_3}{\gamma_1} + 1 - \sqrt{-\frac{\gamma_3}{\gamma_1} - 1} \right) \] (71)

Using Eq. (68) in Eq. (71),

\[ \int_{r_1}^{r_2} k_n(r)dr = -\pi \sqrt{\frac{2\mu(\beta + D_e \alpha^2)}{\alpha^2 h^2}} \times \left( \sqrt{\frac{2D_e \alpha + V_0 - D_e \alpha^2 - 2\beta}{\beta + D_e \alpha^2} + \beta - D_e \alpha - E_{nt} \over \beta + D_e \alpha^2}} + 1 - \sqrt{\beta - D_e \alpha - E_{nt} \over \beta + D_e \alpha^2} - 1 \right) \] (72)
Using Eq. (66) into Eq. (72) and solving Eq. (8), we obtain 

\[ E_n\ell = -D_ea\alpha + \frac{\hbar^2\alpha^2}{2\mu}S(S' + 1) - \frac{\hbar^2\alpha^2}{2\mu} \times \]

\[ \left[ \frac{2S'(S' + 1)}{(n + \frac{1}{2}) + 2\sqrt{S' + \frac{1}{2}} + 2} \right] \]

\[ E_n\ell = -D_ea\alpha + \frac{D_eb\alpha^2}{2} + \frac{\hbar^2\alpha^2}{2\mu} S(S + 1) - \frac{\hbar^2\alpha^2}{2\mu} \left[ S(S + 1) + \frac{2\mu}{\hbar^2\alpha^2} \left( \frac{D_eb\alpha^2}{2} - V_0 - 2D_ea\alpha \right) \right] + \frac{(n + \rho)}{2} \]

(73)

where

\[ \rho = \frac{1}{2} + \sqrt{\left( S + \frac{1}{2} \right)^2 + \frac{2\mu D_e b}{\hbar^2}} \]  

(75)

To obtain eigen function of HSCKP via formula method [8], we uses an appropriate coordinate transformation \( p = e^{-\alpha r} \) in Eq. (10)

\[ \frac{d^2\Psi(p)}{dp^2} + \left( \frac{1 - p}{p(1 - p)} \right) \frac{d\Psi(p)}{dp} + \frac{BP_1 p^2 + BP_2 p + B_3}{p^2(1 - p)^2} \Psi(p) = 0 \]

(76)

where

\[ B_1 = \frac{2\mu}{\hbar^2\alpha^2} (E_n\ell - V_0 - D_ea\alpha), \quad B_2 = \frac{2\mu}{\hbar^2\alpha^2} \left( V_0 - 2E_n\ell - \frac{D_e b \alpha^2}{2} \right), \]

\[ B_3 = \frac{2\mu}{\hbar^2\alpha^2} (E_n\ell + D_e a \alpha) - S'(S' + 1) \]

(77)

Equating Eq. (10) with Eq. (76), we obtain

\[ a_1 = a_2 = a_3 = 1 \]

\[ \psi(p) = N_{n\ell} p^{a_4} (1 - p)^{a_5} \left( \frac{1}{2} \right)_2 F_1 (-n, n + 2(a_4 + a_5); 2a_4, p), \]

(79)

where

\[ a_4 = \sqrt{S(S + 1) - \frac{2\mu}{\hbar^2\alpha^2} \left( E_n\ell + D_e a \alpha - \frac{D_e b \alpha^2}{2} \right)}, \quad a_5 = \frac{1}{2} + \sqrt{S + \frac{1}{2}^2 + \frac{2\mu D_e b}{\hbar^2}} \]

(80)
3.2. Solution of the FH equation for tHSCKP via SUSYQM method

The space-like counterpart of the Schrödinger equation presented by

\[ \frac{\hbar^2}{2\mu c^2} \frac{d^2 F_n(t)}{dt^2} + V(t) F_n(t) = cP_n F_n(t) \]  

(81)

where \( \mu \) is mass of the particle, \( t \) is the space-like parameter \( x \), \( c \)-light velocity and \( P \) is quantized momentum with quantum number \( n = 0, 1, 2, 3,... \). Equation (81) was derived by Horodecki [46] from the relativistic Feinberg equation [45].

For \( \delta = 1/2 \) and \( \lambda = 1 \) we obtain The time dependent effective HSCKP

\[ V(t) = -\frac{V_0 e^{-\alpha t}}{1 - e^{-\alpha t}} - \frac{D_e a}{t} + \frac{D_e b}{2t^2} - \frac{D_e ae^{-\alpha t}}{t} + \frac{D_e be^{-\alpha t}}{2t^2} \]  

(82)

Now, employ a Greene-Aldrich approximation for the centrifugal term [63]

\[ \frac{1}{t^2} \approx \frac{\alpha^2}{(1 - e^{-\alpha t})^2} \Rightarrow \frac{1}{t} \approx \frac{\alpha}{(1 - e^{-\alpha t})} \]  

(83)

\[ V_{\text{eff}}(t) = -\mu'(V_0 + D_e a\alpha)e^{-\alpha t} - \frac{\mu' D_e a\alpha}{1 - e^{-\alpha t}} + \frac{\mu' D_e b\alpha^2 e^{-\alpha t}}{2(1 - e^{-\alpha t})^2} + \frac{\mu' D_b}{(1 - e^{-\alpha t})^2} \]  

(84)

Now, Eq. (81) can be written as

\[ -\frac{d^2 F_n(t)}{dt^2} + V_{\text{eff}} F_n(t) = P'_n F_n(t) \]  

(85)

Here

\[ V_{\text{eff}}(t) = \frac{\zeta_1 e^{-2\alpha t} + \zeta_2 e^{-\alpha t} + \zeta_3}{(1 - e^{-\alpha t})^2} \]  

(86)

and

\[ P'_n = \mu' cP_n \]  

(87)

where \( \mu' = \frac{2\mu c^2}{\hbar^2} \), and \( \zeta_1 = \mu' (V_0 + D_e a\alpha), \zeta_2 = \mu' \left( \frac{D_e b\alpha^2}{2} - V_0 \right), \zeta_3 = \mu' \left( \frac{D_b}{2} - D_e a\alpha \right) \)

In SUSY quantum mechanics, the ground state wave function \( F_0(r) \) in term of superpotential \( U(t) \) define as [9] [10] [60],

\[ F_0(t) = \exp \left( -\int U(t) dr \right) \]  

(88)
From Eq. (25) and Eq. (27) the partner potential is obtained as

\[ V_{\pm} = U^2(t) \pm \frac{dU}{dt} \]  

(89)

Now, the associated Riccati equation is of the form

\[ U^2(t) \pm \frac{dU}{dt} = V_{\text{eff}} - P_0 \]  

(90)

where \( V_{\text{eff}} \) presents in Eq. (87). Now, we proposed superpotential \( U(t) \) as

\[ U(t) = T_1 e^{-\alpha t} + T_2 \]  

(91)

Substituting Eqs. (87) and (91) into Eq. (90), and considering \( T_2 > 0 \) we obtain

relation between parameters [64, 49, 65]

\[ T_1 = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + (\zeta_1 + \zeta_2 + \zeta_3)} \]  

(92)

\[ = -\alpha \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\zeta_1 + \zeta_2 + \zeta_3}{\alpha^2}} \right) \]  

(93)

\[ T_2 = \frac{T_1}{2} + \frac{\zeta_3 - \zeta_1}{2T_1} \]  

(94)

\[ P_0 = \zeta_3 - T_2^2 \]  

(95)

Using Eq. (91) into Eq. (89), we obtained \( V_-(t) \) and \( V_+(t) \) as [64, 49, 65]

\[ V_-(t) = U^2(t) - \frac{dU}{dt} = \frac{T_1(T_1 - \alpha)e^{-2\alpha t}}{(1 - e^{-\alpha t})^2} + \frac{(2T_1T_2 + \alpha T_1)e^{-\alpha t}}{(1 - e^{-\alpha t})} + T_2^2 \]  

(96)

\[ V_+(t) = U^2(t) + \frac{dU}{dt} = \frac{T_1(T_1 + \alpha)e^{-2\alpha t}}{(1 - e^{-\alpha t})^2} + \frac{(2T_1T_2 - \alpha B_1)e^{-\alpha t}}{(1 - e^{-\alpha t})} + T_2^2 \]  

(97)

Above two partner potential Eqs. (96) and (97) satisfy the condition of the shape invariant potential

\[ V_+(t, b_0) = V_-(t, b_0) + K(b_1) \]  

(98)
Where, \( b_0 = T_1 \) and \( b_j \) is function of \( b_0 \), therefor \( b_1 = f(b_0) = b_0 - \alpha \). In general, \( b_n = f(b_0) = b_0 - n\alpha \). so, we observe that using mapping of the form \( T_1 \rightarrow T_1 - \alpha \) holds the shape invariance. From Eq. (98), we can write \[64, 65,\]

\[ K(b_1) = \left( \frac{b_0^2 + \zeta_3 - \zeta_1}{2b_0} \right)^2 - \left( \frac{b_1^2 + \zeta_3 - \zeta_1}{2b_1} \right)^2 \quad (99) \]

\[ K(b_2) = \left( \frac{b_0^2 + \zeta_3 - \zeta_1}{2b_1} \right)^2 - \left( \frac{b_2^2 + \zeta_3 - \zeta_1}{2b_2} \right)^2 \quad (100) \]

\[ \vdots \]

\[ K(b_n) = \left( \frac{b_{n-1}^2 + \zeta_3 - \zeta_1}{2b_{n-1}} \right)^2 - \left( \frac{b_n^2 + \zeta_3 - \zeta_1}{2b_n} \right)^2 \quad (101) \]

The energy eigenvalues corresponds to potential \( V_1(t, b_0) \) potential obtain as \[64, 65,\]

\[ cP_n^{(-)} = \sum_{i=1}^{n} (b_i) = \left( \frac{b_0^2 + \zeta_3 - \zeta_1}{2b_0} \right)^2 - \left( \frac{b_n^2 + \zeta_3 - \zeta_1}{2b_n} \right)^2 \quad (102) \]

Now, total energy eigenvalues in SUSY quantum mechanics can be written as \[9, 10, 65,\]

\[ P'_n = P_n^{(-)} + P_0 = P_n^{(-)} + \zeta_3 - T_2^2 \quad (103) \]

Using Eqs. (87) and (99), we get

\[ \mu' cP'_n = \zeta_3 - \left( \frac{\zeta_3 - \zeta_1}{2(T_1 - n\alpha)} + \frac{T_1 - n\alpha}{2} \right)^2 \quad (104) \]

Finally, using Eq. (87) and (93) into Eq. (104), we obtain momentum eigenvalues of the tHSCKP as

\[ P_n = \frac{1}{c} \left( -D_e a\alpha + \frac{D_e a\alpha^2}{2} - \frac{\hbar^2 \alpha^2}{2\mu c^2} \left[ \frac{2\mu c^2}{n^2 a^2} \left( \frac{D_e b_0^2}{2} - V_0 - 2D_e a\alpha \right) \right] + \frac{(n + \rho')}{2} \right)^2 \quad (105) \]
\[ \rho' = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu c^2 D_e b}{\hbar^2}} \]  

(106)

Corresponding momentum states from Eq. (107) written as,

\[ F_n(t) = N_n p_1(1) a_4' (1 - p_1(1))^a_5' \ 2F_1(-n, n + 2(a_4' + a_5'); 2a_4', p_1(1)), \]  

(107)

where

\[ p_1 = e^{-\alpha t}, \quad a_4' = \sqrt{\frac{-2\mu c^2}{\hbar^2 \alpha^2}} \left( E_n \alpha - D_e a \alpha - \frac{D_e \alpha^2}{2} \right), \quad a_5' = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu c^2 D_e b}{\hbar^2}} \]  

(108)

3.3. Time independent and time dependent recovered potentials

In this subsection, we recovered time independent and time dependent various potentials and corresponding energy and momentum spectrum.

3.3.1. SCKP

Taking \( \lambda = \delta = 1 \), and \( V_0 = 0 \) in Eq. \( \text{[3]} \), HSCKP convert into SCKP as \( \text{[29]} \)

\[ V_{SCKP}(r) = -2D_e \left( \frac{a}{r} - \frac{b}{2r^2} \right) e^{-\alpha r} \cosh \alpha r \]

Energy eigenvalues corresponds to SCKP from Eq. \( \text{[74]} \)

\[ E_{n\ell} = 2D_e \alpha (b \alpha - a) + \frac{2\hbar^2 \alpha^2}{\mu} S(S + 1) - \frac{2\hbar^2 \alpha^2}{\mu} \left[ \frac{S(S + 1) + \frac{n D_e}{\alpha} \left( \frac{b - \frac{2a}{\alpha}}{2} \right) + \frac{(n + \rho)}{2}}{2(n + \rho)} \right]^2 \]  

(109)

Time dependent SCKP and corresponding momentum eigenvalues from Eq. \( \text{[105]} \)

\[ V_{tSCKP}(t) = -2D_e \left( \frac{a}{t} - \frac{b}{2t^2} \right) e^{-\alpha t} \cosh \alpha t \]

\[ P_n = \frac{1}{c} \left( 2D_e \alpha (b \alpha - a) - \frac{2\hbar^2 \alpha^2}{\mu c^2} \left[ \frac{\mu c^2 D_e}{\hbar^2} \left( \frac{b - \frac{2a}{\alpha}}{2} \right) + \frac{(n + \rho)}{2} \right]^2 \right) \]  

(110)
3.3.2. Screened Kratzer potential

For $\delta = 1$, $\lambda = 0$ and $V_0 = 0$, Eq. (3) convert into SKP as

$$V_{SKP}(r) = -2De \left( \frac{a}{r} - \frac{b}{2r^2} \right) e^{-\alpha r}$$

corresponding energy spectrum from Eq. (74)

$$E_{nl} = \frac{\hbar^2 \alpha^2}{2\mu} S(S+1) - \frac{\hbar^2 \alpha^2}{2\mu} \left[ \frac{S(S+1) - 4\mu Da}{\hbar^2 \alpha} + \frac{(n+\rho)}{2} \right]^2$$ (111)

Time dependent SKP and corresponding momentum eigenvalues from Eq. (105)

$$V_{tSKP}(t) = -2De \left( \frac{a}{t} - \frac{b}{2t^2} \right) e^{-\alpha t}$$

$$P_n = \frac{1}{c} \left( -\frac{\hbar^2 \alpha^2}{2\mu c^2} \left[ \frac{-4\mu Da}{\hbar^2 \alpha} + \frac{(n+\rho)}{2} \right]^2 \right)$$ (112)

3.3.3. Kratzer potential

Taking $\delta = \alpha = V_0 = 0$, Eq. (3) convert into KP potential as

$$V_{KP}(r) = -2De \left( \frac{a}{r} - \frac{b}{2r^2} \right)$$ (113)

From Eq. (74), we obtained energy eigenvalues in D dimensions as

$$E_{nl} = -\frac{2\mu D_e^2 a^2}{\hbar^2 \left( n + \sqrt{\frac{2\mu D_e \alpha \hbar^2}{\hbar^2 \alpha}} + \frac{L + \frac{1}{2}}{ \alpha} \right)^2}$$ (114)

Time dependent KP and corresponding momentum eigenvalues from Eq. (105)

$$V_{tKP}(t) = -2De \left( \frac{a}{t} - \frac{b}{2t^2} \right)$$ (115)

$$P_n = \frac{1}{c} \left( -\frac{2\mu c^2 D_e^2 a^2}{\hbar^2 \left( n + \frac{1}{2} + \sqrt{\frac{2\mu c^2 D_e \alpha}{\hbar^2}} + \frac{1}{4} \right) \right)$$ (116)
3.3.4. GCYP

Setting $\delta = \lambda = 1$ and $b = V_0 = 0$ in Eq. (3), we obtain GCYP as

$$V_{GCYP}(r) = -\frac{2D_e a}{r} e^{-ar} \text{cosh} r$$

(117)

eigenvalues spectrum for GCYP from Eq. (74)

$$E_{n\ell} = \frac{\hbar^2 \alpha^2}{\mu} S(S+1) - 2D_e a a - \frac{2\hbar^2 \alpha^2}{\mu} \left[ \frac{\frac{\mu c^2 D_e a}{\hbar^2 \alpha}}{2(n+S+1)} + \frac{(n+S+1)}{2} \right]^2$$

(118)

Time dependent GCYP and corresponding momentum eigenvalues Eq. (105)

$$V_{tGCYP}(t) = -\frac{2D_e a}{t} e^{-at} \text{cosh} t$$

(119)

$$P_n = \frac{1}{c} \left( -2D_e a a - \frac{2\hbar^2 \alpha^2}{\mu c^2} \left[ \frac{2\mu c^2 D_e a}{\hbar^2 \alpha} + \frac{(n+1)}{2} \right]^2 \right)$$

(120)

3.3.5. inverse quadric Yukawa potential (IQYP)

For $a = 0, \lambda = V_0 = 0, \delta = 1$, and $2D_e b = -V_1$ in Eq. (3), HSCKP covert into the IQY potential as

$$V_{IQYP}(r) = -V_1 \frac{1}{r^2} e^{-ar}$$

(121)

Energy spectrum for IQYP potential from Eq. (74)

$$E_{n\ell} = \frac{\hbar^2 \alpha^2}{2\mu} S(S+1) - \frac{\hbar^2 \alpha^2}{2\mu} \left[ \frac{S(S+1)}{2(n+\frac{1}{2} + \sqrt{-\frac{\mu V_1}{\hbar^2} + (S+\frac{1}{2})^2})} + \frac{(n+1)}{2} \right] \left( -\frac{\mu V_1}{\hbar^2} + (S+\frac{1}{2})^2 \right)^2$$

(122)

Time dependent IQYP and corresponding momentum eigenvalues from Eq. (105)

$$V_{tIQYP}(t) = -\frac{V_1}{2t^2} e^{-at}$$

(123)

$$P_n = \frac{1}{c} \left( -\frac{\hbar^2 \alpha^2}{2\mu c^2} \left[ \frac{(n+\frac{1}{2} + \sqrt{-\frac{\mu V_1}{\hbar^2} + (S+\frac{1}{2})^2})}{2} \right]^2 \right)$$

(124)
3.3.6. Yukawa potential (YP)

The HSCKP reduces to the YP for $b = \delta = V_0 = 0$ in Eq. (3)

$$V_{YP}(r) = -\frac{2Dea}{r}e^{-ar}$$

(125)

Corresponding energy spectrum for YP from Eq. (74)

$$E_{nt} = \frac{\hbar^2 \alpha^2}{2\mu} S(S + 1) - \frac{\hbar^2 \alpha^2}{2\mu} \left[ \frac{S(S + 1) - \frac{4\mu Da}{\hbar^2 \alpha}}{2(n + S + 1)} + \frac{(n + S + 1)}{2} \right]^2$$

(126)

Time dependent YP and corresponding momentum eigenvalues from Eq. (105)

$$V_{tYP}(t) = -\frac{2Dea}{t}e^{-at}$$

(127)

$$P_n = \frac{1}{c} \left( -\frac{\hbar^2 \alpha^2}{2\mu c^2} \left[ -\frac{4\mu c^2 D_e a}{\hbar^2 \alpha} \frac{1}{2(n + 1)} + \frac{(n + 1)}{2} \right]^2 \right)$$

(128)

3.3.7. Hulthén potential (HP)

For $D_e = 0$ in Eq. (3), HSCKP covert into the Hulthén potential as [18, 19]

$$V_{HP}(r) = -\frac{V_0 e^{-ar}}{1 - e^{-ar}}$$

(129)

In D dimensions, energy eigenvalues for Hulthén potential, from Eq. (74)

$$E_{nt} = \frac{\hbar^2 \alpha^2}{2\mu} S(S + 1) - \frac{\hbar^2 \alpha^2}{2\mu} \left[ \frac{S(S + 1) - \frac{2\mu V_0}{\hbar^2 \alpha}}{2(n + S + 1)} + \frac{(n + S + 1)}{2} \right]^2$$

(130)

Time dependent HP and corresponding momentum eigenvalues from Eq. (105)

$$V_{tHP}(t) = -\frac{V_0 e^{-at}}{1 - e^{-at}}$$

(131)

$$P_n = \frac{1}{c} \left( -\frac{\hbar^2 \alpha^2}{2\mu c^2} \left[ -\frac{2\mu c^2 V_0}{\hbar^2 \alpha} \frac{1}{2(n + 1)} + \frac{(n + 1)}{2} \right]^2 \right)$$

(132)

3.3.8. Hellmann potential

For $V_0 = b = 0$ and $\lambda = 1/2, \delta = 1$ in Eq. (3), HSCKP covert into the Hellmann potential as []

$$V(r) = -\frac{V_1}{r} - \frac{V_1 e^{-ar}}{r}$$

(133)
Table 1: spectroscopic parameters for some dimer, where $D_e$ in eV and $\mu$ in amu. 

| Dimer | $D_e$  | $\mu$  | $r_e$ (Å) | $r_e$ (Å⁻¹) |
|-------|--------|--------|-----------|-----------|
| MnH   | 1.67   | 0.9989984 | 1.753     | 1.59737   |
| ScN   | 4.56   | 10.682771 | 1.768     | 1.50680   |
| LiH   | 2.515283695 | 0.8801221 | 1.5956    | 1.1280    |
| HCl   | 4.611906117 | 0.9801045 | 1.2746    | 1.8677    |
| $H_2$ | 4.744600000 | 0.5039100 | 0.7416    | 1.9426    |
| $I_2$ | 1.581791863 | 63.452235 | 2.6620    |           |
| $O_2$ | 5.156658828 | 7.9974575 | 1.2080    |           |

Tacking $V_1 = B$ for first part of above equation and $V_1 = -C$ in exponential part of potential. We can write Hellman potential as

$$V(r) = -\frac{B}{r} + C e^{-\alpha r}$$

(134)

In D dimensions, energy eigenvalues for Hellmann potential from Eq. (74)

$$E_{n\ell}^D = \frac{\hbar^2 \alpha^2}{2\mu} S(S+1) - \alpha B - \frac{\hbar^2 \alpha}{2\mu} \left[ S(S+1) + \frac{2\mu}{\hbar^2 \alpha} (C-B) \right] \frac{2(n+S+1)}{2(n+S+1)} + \frac{(n+S+1)^2}{2}$$

(135)

Time dependent Hellmann potential and corresponding momentum eigenvalues from Eq. (105)

$$V(t) = -\frac{B}{t} + C e^{-\alpha t}$$

(136)

$$P_n = \frac{1}{c} \left( -\alpha B - \frac{\hbar^2 \alpha}{2\mu c^2} \left[ \frac{2\mu^2}{\hbar^2 \alpha} (C-B) \right] \frac{2(n+S+1)}{2(n+S+1)} + \frac{(n+S+1)^2}{2} \right)$$

(137)
Table 2: spectroscopic parameters for various diatomic molecules

| Molecule | $D_e$(eV) | $t_e$(timeunit) | $\mu$(a.m.u) |
|----------|-----------|-----------------|--------------|
| TiH      | 2.05      | 1.781           | 0.987371     |
| ScN      | 4.56      | 1.768           | 10.682771    |
| $H_2$    | 4.7446    | 0.7416          | 0.50391      |
| CuLi     | 1.74      | 2.310           | 6.259494     |
| $I_2$    | 1.58179   | 2.6620          | 63.452235    |

Figure 1: Different time dependent potentials (in eV) model tHSCKP (v1), tSCKP (v2) and tYP(v5) against time $t$ (in sec) with $V_0 = 0.1$ and $\alpha = 0.001$ for MnH dimer.

Figure 2: Different time dependent potentials (in eV) model tHSCKP (v1), tSCKP (v2) and tYP(v5) against dissociation energy $D_e$ (in eV) with $V_0 = 0.1$, $t = 0.5$ and $\alpha = 0.001$ for MnH dimer.

Figure 3: Different time dependent potentials (in eV) model tHSCKP (v1), tSCKP (v2) and tYP(v5) against time $t$ (in sec) with $V_0 = 0.1$ and $\alpha = 0.1$ for ScN dimer.

Figure 4: Different time dependent potentials (in eV) model tHSCKP (v1), tSCKP (v2) and tYP(v5) against dissociation energy $D_e$ (in sec) with $V_0 = 0.1$, $t = 0.5$ and $\alpha = 0.1$ for ScN dimer.
Table 3: RV energy spectrum of the HSCKP for MnH and ScN dimer with different quantum numbers \((n, \ell)\) in three dimensions

| n | \(\ell\) | MnH       | ScN        |
|---|-------|-----------|------------|
| 0 | 0     | -0.1749955901 | -0.4993956221 |
| 1 | 0     | -0.3043388977  | -0.5557478349 |
| 1 | 0     | -0.3004835839  | -0.5554013780 |
| 2 | 0     | -0.4367287638  | 0.61236415060 |
| 1 | 0     | -0.4329778500  | -0.6120194882 |
| 1 | 0     | -0.4254934723  | -0.6113302194 |
| 3 | 0     | -0.5721387025  | -0.6692439560 |
| 1 | 0     | -0.5684892640  | -0.6689010768 |
| 1 | 0     | -0.5612070010  | -0.6682153741 |
| 2 | 0     | -0.5503249456  | -0.6671869591 |
| 4 | 0     | -0.7105446661  | -0.7263866490 |
| 1 | 0     | -0.7069940034  | -0.7260455420 |
| 2 | 0     | -0.6999085029  | -0.7253633833 |
| 3 | 0     | -0.6893196330  | -0.7243402830 |
| 4 | 0     | -0.6752741480  | -0.7229764064 |
| 5 | 0     | -0.8519247807  | -0.7837916384 |
| 1 | 0     | -0.8484703966  | -0.7834522929 |
| 2 | 0     | -0.8415767097  | -0.7827736564 |
| 3 | 0     | -0.8312737124  | -0.7817558381 |
| 4 | 0     | -0.8176059721  | -0.7803990019 |
| 5 | 0     | -0.8006321355  | -0.7787033660 |
Table 4: Momentum spectrum of the tHSCKP for MnH and ScN dimer with different quantum number n and screening parameter α.

| n  | MnH          |          | ScN          |          |
|----|--------------|----------|--------------|----------|
|    | α = 0.1      | α = 0.01 | α = 0.005    | α = 0.1  | α = 0.01 | α = 0.005 |
| 0  | -2.028674094 | -11.911286721 | -31.829369114 | -4.656617461 | -11.810787313 | -22.708304443 |
| 1  | -1.948191140 | -11.445055495 | -30.586750820 | -4.62264840  | -11.723550236 | -22.540910783 |
| 2  | -1.872364810 | -11.005624491 | -29.415395212 | -4.588293732 | -11.637272154 | -22.375357210 |
| 3  | -1.800846490 | -10.590801466 | -28.309955562 | -4.554698581 | -11.551939061 | -22.211616855 |
| 4  | -1.733319694 | -10.198951105 | -27.265577341 | -4.521473930 | -11.467537210 | -22.04963335  |
| 5  | -1.669496533 | -9.828354569  | -26.277844832 | -4.48614422  | -11.384053098 | -21.889470745 |
| 6  | -1.609114680 | -9.477502163  | -25.342734379 | -4.456114798 | -11.301473471 | -21.73103649  |
| 7  | -1.551934694 | -9.145015896  | -24.456733434 | -4.423969891 | -11.219785309 | -21.574267069 |
| 8  | -1.497376660 | -8.829635932  | -23.616003987 | -4.392174629 | -11.138975828 | -21.419206472 |
| 9  | -1.446323131 | -8.530208632  | -22.817951625 | -4.360724028 | -11.059032472 | -21.265807766 |
| 10 | -1.397507257 | -8.245675997  | -22.059596460 | -4.329613195 | -10.979942907 | -21.114047287 |

Figure 5: Different potentials (in eV) model HSCPK (v1), SCKP (v2), SKP (v3) and KP(v4)) against r (in Å) a interatomic distance for a = 2, b = 4, V0 = 1, De = 6 and α = 0.09.

Figure 6: Different potentials (in eV) model HSCPK (v1), SCKP (v2), SKP (v3) and KP(v4)) vs D, a dissociation energy (in eV) for r = 2, a = 2, b = 4, V0 = 0.2 and α = 0.09.
Table 5: RV energy eigenvalues of the SCKP for LiH and H\textsubscript{2} dimer with various values of the \( n \) and \( \ell \) quantum numbers in three dimensions and its comparison.

| \( n \) | \( \ell \) | LiH (Present) | LiH [29] | \( H\textsubscript{2} \) (Present) | \( H\textsubscript{2} \) [29] |
|---|---|---|---|---|---|
| 0 | 0 | -9.064652347 | -9.064592907 | -0.757132418 | -0.757132418 |
| 1 | 0 | -9.054274081 | -9.054214756 | -1.547462035 | -1.547462035 |
| 1 | 1 | -9.054275052 | -9.054215727 | -1.506342444 | -1.506342444 |
| 2 | 0 | -9.067973823 | -9.067914606 | -2.371907148 | -2.371907148 |
| 1 | 1 | -9.067986897 | -9.067927680 | -2.332898306 | -2.332898306 |
| 2 | 2 | -9.068013021 | -9.067953805 | -2.255541560 | -2.255541560 |
| 3 | 0 | -9.104461961 | -9.104402843 | -3.230081482 | -3.230081482 |
| 1 | 1 | -9.104486967 | -9.104427848 | -3.193091132 | -3.193091132 |
| 2 | 2 | -9.104536916 | -9.104477799 | -3.119712784 | -3.119712784 |
| 3 | 3 | -9.104611689 | -9.104552571 | -3.011257311 | -3.011257311 |
| 4 | 0 | -9.162562128 | -9.162503095 | -4.121662917 | -4.121662917 |
| 1 | 1 | -9.162599303 | -9.162540271 | -4.08661483 | -4.08661483 |
| 2 | 2 | -9.162673555 | -9.162614523 | -4.017058228 | -4.017058228 |
| 3 | 3 | -9.162784688 | -9.162725657 | -3.914079826 | -3.914079826 |
| 4 | 4 | -9.162932411 | -9.162873381 | -3.779236677 | -3.779236677 |
| 5 | 0 | -9.241199473 | -9.241140511 | -5.046381114 | -5.046381114 |
| 1 | 1 | -9.241249396 | -9.241190434 | -5.013199577 | -5.013199577 |
| 2 | 2 | -9.241349107 | -9.241290145 | -4.94738422 | -4.94738422 |
| 3 | 3 | -9.241498334 | -9.241439374 | -4.849781412 | -4.849781412 |
| 4 | 4 | -9.241696676 | -9.241637716 | -4.721955520 | -4.721955520 |
| 5 | 5 | -9.241943599 | -9.241884641 | -4.565676629 | -4.565676629 |
Table 6: RV energy eigenvalues of the KP for $I_2$ and $O_2$ dimer with different $n$, $\ell$ quantum numbers in three dimensions and its comparison.

| $n$ | $\ell$ | $I_2$ (Our work) | $I_2$ [29] | $I_2$ [70] | $O_2$ (Our work) | $O_2$ [29] | $O_2$ [70] |
|-----|--------|-----------------|------------|------------|-----------------|------------|------------|
| 0   | 0      | -1.5790825649  | -1.5790825649 | -1.5790825765 | -5.1263584907 | -5.1263584907 | -5.1263586200 |
| 1   | 0      | -1.5736871157  | -1.5736871157 | -1.5736871503 | -5.0666407663 | -5.0666407663 | -5.0666411467 |
|     | 1      | -1.5736778902  | -1.5736778902 | -1.5736779249 | -5.0662919381 | -5.0662919381 | -5.0662923214 |
| 2   | 0      | -1.5683192724  | -1.5683192724 | -1.5683193296 | -5.0079604870 | -5.0079604870 | -5.0079611102 |
|     | 1      | -1.5683100940  | -1.5683100940 | -1.5683101514 | -5.0076177016 | -5.0076177016 | -5.0076183271 |
|     | 2      | -1.5682917377  | -1.5682917377 | -1.5682917952 | -5.0069322711 | -5.0069322711 | -5.0069329023 |
| 3   | 0      | -1.5629788468  | -1.5629788468 | -1.5629789266 | -4.9502937621 | -4.9502937621 | -4.9502946186 |
|     | 1      | -1.5629697153  | -1.5629697153 | -1.5629697795 | -4.9499568798 | -4.9499568798 | -4.9499577391 |
|     | 2      | -1.5629514526  | -1.5629514526 | -1.5629515326 | -4.9492832534 | -4.9492832534 | -4.9492841183 |
|     | 3      | -1.562940594   | -1.562940594  | -1.562941397  | -4.9482731590 | -4.9482731590 | -4.9482740326 |
| 4   | 0      | -1.5576656526  | -1.5576656526 | -1.5576657546 | -4.8936173815 | -4.8936173815 | -4.8936184638 |
|     | 1      | -1.5576565676  | -1.5576565676 | -1.5576566697 | -4.8932862680 | -4.8932862680 | -4.8932873530 |
|     | 2      | -1.5576383980  | -1.5576383980 | -1.5576385003 | -4.8926241763 | -4.8926241763 | -4.8926252668 |
|     | 3      | -1.5576111444  | -1.5576111444 | -1.5576112469 | -4.8916313768 | -4.8916313768 | -4.8916324755 |
|     | 4      | -1.5575748077  | -1.5575748077 | -1.5575749105 | -4.8903082749 | -4.8903082749 | -4.8903093844 |
| 5   | 0      | -1.5523795049  | -1.5523795049 | -1.5523796290 | -4.8379087980 | -4.8379087980 | -4.8379100982 |
|     | 1      | -1.5523704662  | -1.5523704662 | -1.5523705903 | -4.8375832323 | -4.8375846252 | -4.8375846252 |
|     | 2      | -1.5523523890  | -1.5523523890 | -1.5523525133 | -4.8369325034 | -4.8369325034 | -4.8369381116 |
|     | 3      | -1.5523252739  | -1.5523252739 | -1.5523253985 | -4.8359560600 | -4.8359560600 | -4.8359579221 |
|     | 4      | -1.5522891221  | -1.5522891221 | -1.5522892469 | -4.8346560268 | -4.8346560268 | -4.8346573535 |
|     | 5      | -1.5522439346  | -1.5522439346 | -1.5522440598 | -4.8330312942 | -4.8330312942 | -4.8330326341 |
Table 7: RV energy spectrum of the KP for LiH and HCl dimer with different \( n, \ell \) quantum numbers in three dimensions and its comparison.

| \( n \) | \( \ell \) | LiH (Our work) | LiH [28] | LiH [70] | HCl (Our work) | HCl [28] | HCl [70] |
|-------|-------|----------------|----------|----------|----------------|----------|----------|
| 0     | 0     | -2.467310100   | -2.467293778 | -2.467310304 | -4.534753155 | -4.541847882 | -4.541848211 |
| 1     | 0     | -2.375818641   | -2.375802636 | -2.375819921 | -4.386752765 | -4.393727024 | -4.393727956 |
|       | 1     | -2.374107386   | -2.374091378 | -2.374107972 | -4.384318781 | -4.391292904 | -4.391293850 |
| 2     | 0     | -2.289323364   | -2.289307674 | -2.289324266 | -4.25881301  | -4.252735636 | -4.252737112 |
|       | 1     | -2.287704688   | -2.287688996 | -2.287705602 | -4.243563600 | -4.250417718 | -4.250419208 |
|       | 2     | -2.284474275   | -2.284458584 | -2.284475515 | -4.238935859 | -4.245789526 | -4.245791052 |
| 3     | 0     | -2.207467005   | -2.207451626 | -2.207468200 | -4.111688159 | -4.118423404 | -4.118425371 |
|       | 1     | -2.205934349   | -2.205918968 | -2.205935555 | -4.109479448 | -4.116214408 | -4.116216389 |
|       | 2     | -2.202875519   | -2.202860140 | -2.202876749 | -4.105069242 | -4.111803616 | -4.111805631 |
|       | 3     | -2.198303413   | -2.198288040 | -2.198304679 | -4.098471914 | -4.105205380 | -4.105207449 |
| 4     | 0     | -2.129923673   | -2.129908602 | -2.129925128 | -3.983757778 | -3.990375014 | -3.990377425 |
|       | 1     | -2.128471048   | -2.128455976 | -2.128472514 | -3.981651330 | -3.988268222 | -3.998270645 |
|       | 2     | -2.125571862   | -2.125556792 | -2.125733505 | -3.977445237 | -3.984061424 | -3.984063879 |
|       | 3     | -2.121238181   | -2.121223116 | -2.121239701 | -3.971153046 | -3.977765152 | -3.977770657 |
|       | 4     | -2.115487942   | -2.115472884 | -2.115489505 | -3.962794945 | -3.969408570 | -3.969411138 |
| 5     | 0     | -2.056395599   | -2.056380834 | -2.056397256 | -3.861706429 | -3.868206938 | -3.868209749 |
|       | 1     | -2.055017529   | -2.055002762 | -2.055019226 | -3.859696027 | -3.866196140 | -3.866198963 |
|       | 2     | -2.052267068   | -2.052252304 | -2.052268786 | -3.85681641  | -3.862180950 | -3.862183802 |
|       | 3     | -2.048155517   | -2.048140758 | -2.048157264 | -3.849676057 | -3.856174134 | -3.856177032 |
|       | 4     | -2.042699680   | -2.042684928 | -2.042701466 | -3.841698323 | -3.848194720 | -3.848197679 |
|       | 5     | -2.035921689   | -2.035906942 | -2.035923525 | -3.831773601 | -3.838267840 | -3.838270872 |
Table 8: Energy eigenvalues of the Hellmann potential for different quantum numbers $n, \ell$ in three dimensions with $2\mu = \hbar = 1$ corresponds to potential parameters $B = 2, C = -1$ with various values of screening parameter $\alpha$ and comparison with numerical results with pNU \cite{71} Amplitude phase method\cite{72} and SUSY \cite{73}.

| States | $n$ | $\ell$ | $\alpha$ | [PQR] | [pNU] | [AP] | [SUSY] |
|--------|----|------|--------|-------|-------|------|-------|
| $1s$   | 0  | 0    | 0.001  | -2.250500 | -2.250500 | -2.248981 | -2.250500 |
|        |    |      | 0.005  | -2.252506 | -2.252506 | -2.244993 | -2.244993 |
|        |    |      | 0.010  | -2.255025 | -2.255025 | -2.240030 | -2.255025 |
| $2s$   | 1  | 0    | 0.001  | -0.563001 | -0.563001 | -0.561502 | -0.563750 |
|        |    |      | 0.005  | -0.565025 | -0.565025 | -0.557549 | -0.568750 |
|        |    |      | 0.010  | -0.567000 | -0.567000 | -0.552697 | -0.575025 |
| $2p$   | 0  | 1    | 0.001  | -0.562250 | -0.562250 | -0.561502 | -0.562999 |
|        |    |      | 0.005  | -0.561256 | -0.561256 | -0.557541 | -0.564975 |
|        |    |      | 0.010  | -0.560025 | -0.560025 | -0.552664 | -0.567400 |
| $3s$   | 2  | 0    | 0.001  | -0.250502 | -0.250502 | -0.249004 | -0.251500 |
|        |    |      | 0.005  | -0.252556 | -0.252556 | -0.245110 | -0.257506 |
|        |    |      | 0.010  | -0.255225 | -0.255225 | -0.240435 | -0.265025 |
| $3p$   | 1  | 1    | 0.001  | -0.250168 | -0.250168 | -0.249004 | -0.251165 |
|        |    |      | 0.005  | -0.250867 | -0.250867 | -0.245102 | -0.255801 |
|        |    |      | 0.010  | -0.251802 | -0.251802 | -0.240404 | -0.261536 |
| $3d$   | 0  | 2    | 0.001  | -0.249500 | -0.249500 | -0.249003 | -0.250496 |
|        |    |      | 0.005  | -0.247506 | -0.247506 | -0.245086 | -0.252406 |
|        |    |      | 0.010  | -0.245025 | -0.245025 | -0.240341 | -0.254625 |
| $4s$   | 3  | 0    | 0.001  | -0.141129 | -0.141129 | -0.139633 | -0.142250 |
|        |    |      | 0.005  | -0.143225 | -0.143225 | -0.135819 | -0.148756 |
|        |    |      | 0.010  | -0.146025 | -0.146025 | -0.131380 | -0.156900 |
| $4p$   | 2  | 1    | 0.001  | -0.140940 | -0.140940 | -0.139632 | -0.142061 |
|        |    |      | 0.005  | -0.142264 | -0.142264 | -0.135811 | -0.147777 |
|        |    |      | 0.010  | -0.144056 | -0.144056 | -0.131350 | -0.154856 |
| $4d$   | 1  | 2    | 0.001  | -0.140564 | -0.140564 | -0.139632 | -0.141683 |
|        |    |      | 0.005  | -0.140351 | -0.140351 | -0.135795 | -0.145827 |
|        |    |      | 0.010  | -0.140156 | -0.140156 | -0.131290 | -0.150806 |
| $4f$   | 0  | 3    | 0.001  | -0.140000 | -0.140000 | -0.139631 | -0.141117 |
Table 9: Momentum eigenvalues of the Hellmann potential for different quantum numbers

$n$ corresponds to potential parameters $B = 5\text{eV}, C = 3\text{eV}$ with various values of screening parameter $\alpha$.

| HP    | $\alpha$ | $n = 0$ | $n = 0$ | $n = 1$ | $n = 1$ | $n = 2$ | $n = 2$ |
|-------|----------|---------|---------|---------|---------|---------|---------|
| $I_2$ | 0.001    | 0.00421072 | 0.00421072 | 0.00421071 | 0.00421071 | 0.00421071 | 0.00421071 |
|       | 0.01     | 0.04210710 | 0.04210710 | 0.04210651 | 0.04210651 | 0.04210552 | 0.04210552 |
|       | 0.1      | 0.42105329 | 0.42105329 | 0.42099419 | 0.42099419 | 0.42089569 | 0.42089569 |
|       | 1.0      | 4.20875995 | 4.20875995 | 4.20284999 | 4.20284999 | 4.19300006 | 4.19300006 |
| $TiH$ | 0.001    | 0.00365092 | 0.00365092 | 0.00365054 | 0.00365054 | 0.00364991 | 0.00364991 |
|       | 0.01     | 0.03649784 | 0.03649784 | 0.03645986 | 0.03645986 | 0.03639656 | 0.03639656 |
|       | 0.1      | 0.36383901 | 0.36383901 | 0.36004104 | 0.36004104 | 0.35371110 | 0.35371110 |
|       | 1.0      | 3.52445118 | 3.52445118 | 3.14465473 | 3.14465473 | 2.51166065 | 2.51166065 |
| $ScN$ | 0.001    | 0.00806206 | 0.00806206 | 0.00806203 | 0.00806203 | 0.00806197 | 0.00806197 |
|       | 0.01     | 0.08061963 | 0.08061963 | 0.08061612 | 0.08061612 | 0.08061026 | 0.08061026 |
|       | 0.1      | 0.80609098 | 0.80609098 | 0.80573995 | 0.80573995 | 0.80515490 | 0.80515490 |
|       | 1.0      | 8.05037891 | 8.05037891 | 8.01527566 | 8.01527566 | 7.95677024 | 7.95677024 |
| $H_2$ | 0.001    | 0.00351834 | 0.00351834 | 0.00351760 | 0.00351760 | 0.00351636 | 0.00351636 |
|       | 0.01     | 0.03516114 | 0.03516114 | 0.03508673 | 0.03508673 | 0.03496269 | 0.03496269 |
|       | 0.1      | 0.34937893 | 0.34937893 | 0.34193712 | 0.34193712 | 0.32953412 | 0.32953412 |
|       | 1.0      | 3.27053519 | 3.27053519 | 2.52635468 | 2.52635468 | 1.28605383 | 1.28605383 |
| $CuLi$| 0.001    | 0.00401938 | 0.00401938 | 0.00401932 | 0.00401932 | 0.00401922 | 0.00401922 |
|       | 0.01     | 0.04019200 | 0.04019200 | 0.04018601 | 0.04018601 | 0.04017602 | 0.04017602 |
|       | 0.1      | 0.40174030 | 0.40174030 | 0.40114121 | 0.40114121 | 0.40014273 | 0.40014273 |
|       | 1.0      | 3.99943033 | 3.99943033 | 3.93952133 | 3.93952133 | 3.83967301 | 3.83967301 |
Figure 7: Different potentials (in eV) model HSCKP (v1), SCKP (v2), SKP (v3) and KP(v4)) against a potential parameter for $r = 2$, $De = 6$, $V_0 = 0.2$, $b = 4$ and $\alpha = 0.09$.

Figure 8: Various potentials (in eV) model HSCKP (v1), SCKP (v2), SKP (v3) and KP(v4)) vs. a potential parameter for $r = 2$, $a = 2$, $V_0 = 0.2$, $De = 6$ and $\alpha = 0.09$.

Figure 9: Various potentials (in eV) model (HSCKP (v1), SCKP (v2) and SKP (v3)) vs. $\alpha$ a screening parameter (in Å$^{-1}$) for $r = 2$, $De = 6$, $a = 2$, $V_0 = 0.2$ and $b = 4$.

Figure 10: The energy eigenvalues $E_{nl}$ (in eV) of the HSCKP vs. $\alpha$ a screening parameter (in Å$^{-1}$) with $n = n_0 = 0$, $n = n_1 = 1$, $n = n_2 = 2$, $n = n_3 = 3$ a different vibrational quantum number for $h = V_0 = \mu = 1$, $De = 3$, $a = 2$, $b = 4$ for constant rotational quantum number $\ell = 0$. 

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Figure 11: The energy eigenvalues $E_{n\ell}$ (in eV) of the HSCKP vs. $\alpha$ a screening parameter (in Å$^{-1}$) with $n = n_0 = 0, n = n_1 = 1, n = n_2 = 2, n = n_3 = 3$ a various vibrational quantum number for $\hbar = V_0 = \mu = 1, \, D_e = 3, a = 2, \, b = 4$ for constant rotational quantum number $\ell = 1$.

Figure 12: The energy eigenvalues $E_{n\ell}$ (in eV) of the HSCKP vs. $\alpha$ a screening parameter (in Å$^{-1}$) with $n = n_0 = 0, n = n_1 = 1, n = n_2 = 2, n = n_3 = 3$ a different vibrational quantum number for $\hbar = V_0 = \mu = 1, \, D_e = 3, a = 2, \, b = 4$ for constant rotational quantum number $\ell = 2$.

Figure 13: The energy eigenvalues $E_{n\ell}$ (in eV) of the HSCKP vs. $\alpha$ a screening parameter (in Å$^{-1}$) with $n = n_0 = 0, n = n_1 = 1, n = n_2 = 2, n = n_3 = 3$ a different vibrational quantum number for $\hbar = V_0 = \mu = 1, \, D_e = 3, a = 2, \, b = 4$ for constant rotational quantum number $\ell = 3$.

Figure 14: The energy eigenvalues $E_{n\ell}$ (in eV) of the HSCKP vs. $\alpha$ a dissociation energy (in eV) with $n = n_0 = 0, n = n_1 = 1, n = n_2 = 2, n = n_3 = 3$ a various vibrational quantum number for $\hbar = V_0 = \mu = 1, \, D_e = 3, a = 2, \, b = 4$ for constant rotational quantum number $\ell = 0$.

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Figure 15: The energy eigenvalues $E_{n\ell}$ (in eV) of the HSCKP vs. $D_\text{e}$ a dissociation energy (in eV) with $n = n_0 = 0$, $n = n_1 = 1$, $n = n_2 = 2$, $n = n_3 = 3$ a various vibrational quantum number for $\hbar = V_0 = \mu = 1$, $D = 3$, $a = 2$, $b = 4$ for constant rotational quantum number $\ell = 1$.

Figure 16: The energy eigenvalues $E_{n\ell}$ (in eV) of the HSCKP vs. $D_\text{e}$ a dissociation energy (in eV) with $n = n_0 = 0$, $n = n_1 = 1$, $n = n_2 = 2$, $n = n_3 = 3$ a various vibrational quantum number for $\hbar = V_0 = \mu = 1$, $D = 3$, $a = 2$, $b = 4$ for constant rotational quantum number $\ell = 2$.

Figure 17: The energy eigenvalues $E_{n\ell}$ (in eV) of the HSCKP vs. $D_\text{e}$ a dissociation energy (in eV) with $n = n_0 = 0$, $n = n_1 = 1$, $n = n_2 = 2$, $n = n_3 = 3$ a various vibrational quantum number for $\hbar = V_0 = \mu = 1$, $D = 3$, $a = 2$, $b = 4$ for constant rotational quantum number $\ell = 3$.

Figure 18: The energy eigenvalues $E_{n\ell}$ (in eV) of the HSCKP vs $n$ a vibrational quantum number for $\ell = \ell_0 = 0$, $\ell = \ell_1 = 1$, $\ell = \ell_2 = 2$, $\ell = \ell_3 = 3$ a various rotational quantum numbers for $\hbar = V_0 = \mu = 1$, $\alpha = 0.09$, $a = 2$ and $b = 4$ in three dimensions.
4. Results and discussions

The behavior of the various potentials in terms of tHSCKP as a function of time \( t \) and dissociation energy \( D_e \) is graphically depicted. The Hulthén-screened cosine Kratzer potential is provided as a function of several potential parameters such as \( r, D_e, a, b \) and \( \alpha \). The HSCKP’s RV energy behavior is plotted using various parameters \( \alpha, D_e, \) and quantum number \( n \).

We use Eq. (74) and Eq. (105) to determine the RV energy eigenvalues for the \( MnH \) and \( ScH \) dimers of the HSCKP and tHSCKP, which are presented in table 3 and table 4, respectively. We also use Eq. (109) to determine the RV energy eigenvalues for the \( LiH \) and \( H2 \) dimers for SCKP. Table 5 shows that the numerical findings obtained in ref. [29] and the computed numerical results for SCKP coincide. We calculate the RV energy eigenvalues for \( HCl, I2, O2, \) and \( LiH, \) dimer for the KP using eq. (114). Tables 6 and 7 show the numerical results for these dimers, together with a comparison of the values reported in refs. [29, 70]. In tables 8 and 9, we calculate the energy and momentum spectra for the time independent and time dependent Hellmann potentials, respectively. With varied quantum numbers \( n, \ell \), we employed spectroscopic parameters from table 1 and table 2. Through calculate the numerical results in table 3 to table 9, we utilised \( \hbar c = 1973.29eV\text{Åand}1amu = 931.494028 \times 10^6eV/c^2. \)

Figure 1 shows plots of the various potentials vs time \( t \). It indicates that the tHSCKP and SCKP behave the same way, whereas for the \( MnH \) dimer, YP increases quickly at first and then grows slowly after \( t = 0.2 \). The behaviour of the various potentials with regard to the dissociation energy \( D_e \) for the \( MnH \) dimer is shown in Fig. 2. \( V_1 \) and \( V_2 \) progressively increase, and \( V_5 \) gradually drops. In Fig. 3, we show the behaviour of various potentials vs time \( t \) for the \( ScN \) dimer. The plots demonstrate that \( V_1 \) and \( V_2 \) fall rapidly at first and then steadily grow after \( t = 1.5 \), whereas \( V_5 \) increases exponentially. Figure 4 depicts the behaviour of various potentials in terms of dissociation energy. The plots show a progressive decline in \( V_1, V_2, V_3, \) and \( V_5 \). The variation in potential \( V_1 = HSCKP, V_2 = SCKP, V_3 = SKP \) and \( V_4 = KP \) corresponds
to intermolecular distance \( r \) for \( a = 2, b = 4, V_0 = 1, D_e = 6, \) and \( \alpha = 0.09, \) as shown in Fig. (5). In Fig. (6), the potential \( V_1, V_2, V_3, \) and \( V_4 \) are shown against the dissociation energy \( D_e \) for the parameters \( V_0 = 0.2, \alpha = 0.09, a = 2, b = 4, \) and \( r = 2. \) Potential \( V_1, V_2, V_3, \) and \( V_4 \) diminishes as \( D_e \) rises. We plot the potential \( V_1, V_2, V_3, \) and \( V_4 \) versus potential parameter \( a \) and \( b \) for \( V_0 = 0.2, \alpha = 0.09, b = 4, D_e = 6, r = 2, \) respectively, in Figs. (7) and (8). In Fig. (9), we plot the potential \( V_1, V_2, V_3, \) and \( V_4 \) vs. the screening parameter \( \alpha \) for the parameters \( V_0 = 0.2, a = 2, b = 4, D_e = 6, \) and \( r = 2. \)

RV energy of the HSCKP is plotted against \( \alpha \) a screening parameter with \( n = n_0 = 0, n = n_1 = 1, n = n_2 = 2, n = n_3 = 3 \) a different vibrational quantum number for \( h = V_0 = \mu = 1, D_e = 3, a = 2, b = 4 \) for constant value \( \ell = 0. \) In Fig. (10). RV energy of the HSCKP is plotted against \( \alpha \) with \( n = n_0 = 0, n = n_1 = 1, n = n_2 = 2, n = n_3 = 3 \) a different vibrational quantum number for \( h = V_0 = \mu = 1, D_e = 3, a = 2, b = 4 \) for constant value \( \ell = 1, 2, 3 \) shown in Figs. (11), (12), (13) respectively.

The energy spectrum of the HSCKP is displayed against \( D_e, \) a dissociation energy with \( n = n_0 = 0, n = n_1 = 1, n_2 = 2, n_3 = 3 \) in Fig. (14). for constant value \( \ell = 0 \) a distinct vibrational quantum number, \( h = V_0 = \mu = 1, a = 2, b = 4. \) With \( n = n_0 = 0, n = n_1 = 1, n_2 = 2, n_3 = 3, \) the RV energy of the HSCKP is shown against the dissociation energy \( D_e. \) \( h = V_0 = \mu = 1, a = 2, b = 4; h = V_0 = \mu = 1, a = 2, b = 4; h = V_0 = \mu \) for the constant values \( \ell = 1, 2, 3 \) illustrated in Figures (15), (16), (17). The energy spectrum of the HSCKP vs \( n \) for various rotating quantum numbers \( \ell = \ell_0 = 0, \ell = \ell_1 = 1, \ell = \ell_2 = 2, \ell = \ell_3 = 3 \) is shown in Fig. (18). in three dimensions for constant values \( h = V_0 = \mu = 1, D_e = 3, a = 2, \alpha = 0.09, D_e = 2b = 4. \)
5. Conclusions

In this paper, we used the correct quantization rule and the SUSYQM approach to solve the non-relativistic Schrödinger equation and the F-H equation for the HSCKP $tHSCKP$ and obtain the energy and momentum spectrum. We were able to recover a variety of time independent and time dependent potentials, as well as their related energy and momentum spectra. SCKP, tSCKP, SKP, tSKP, KP, tKP, GCYP, tGCYP, YP, tYP, IQYP, tIQYP, HP, tHP, Hellmann potential, and time dependent Hellmann potential are the derived potentials by establishing the parameters.

Table 3 and tables 5 to 8 show the computed RV energy eigenvalues for the $MnH, ScN, H_2, I_2, O_2, LiH, HCl$ dimer, which corresponds to the HSCKP and other potentials. The HSCKP vs a radius $r$, $D_e$, parameters $a$, $b$, and the screening parameter $\alpha$ are all discussed. The RV energy’s behaviour is displayed against the screening parameter $\alpha$, dissociation energy $D_e$, and vibrational quantum number $n$. For $MnH$ and $ScN$ dimers, we also plotted the behaviour of various potentials against time $t$ and dissociation energy $D_e$. Tables 4 and 9 provide the numerical findings of the momentum spectra for various dimers that correlate to the $tHSCKP$ and Hellmann potentials, respectively. Our findings potentially have applications in condensed matter, atomic-molecular physics, particle and nuclear physics, and quantum chemistry, among other fields of physical and chemical science.

6. Declaration

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**Code availability:** NA

**Authors Contribution:**
Kaushal R Purohit: Numerical calculation and graphical representation.
Rajendrasinh H Parmar: Methodology and obtained solution of the HSCKP
Ajay Kumar Rai: Results -discussion and conclusions.

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