Distinction of the Steinberg representation and a conjecture of Prasad and Takloo-Bighash

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Abstract

We prove a conjecture of Prasad and Takloo-Bighash for discrete series of inner forms of the general linear group over a non archimedean local field, in the case of Steinberg representations.

Introduction

Let $F$ be a non archimedean local field of characteristic not 2 and $E$ a quadratic extension of $F$. Let $D$ be a central division $F$-algebra of dimension $d^2$. If $n$ is a positive integer, then $M_n(D)$ is a simple central $F$-algebra of dimension $n^2d^2$. As $E/F$ is quadratic, $E$ is embedded in $M_n(D)$ as an $F$-subalgebra if and only if $nd$ is even. Then $C_{M_n(D)}(E)$ (the centralizer of $E$ in $M_n(D)$) is an $E$-algebra. We denote by $N_{rd,F}$ (resp. $N_{rd,E}$) the reduced norm of $GL(n, D)$ (resp. $(C_{M_n(D)}(E))^\times$) as well as its restriction to any subgroup.

We consider the following conjecture of Prasad and Takloo-Bighash:

Conjecture ([PTB11], Conjecture 1). Let $A = M_n(D)$. Let $\pi$ be an irreducible admissible representation of $A^\times$ such that the corresponding representation (via Jacquet-Langlands) of $GL(nd, F)$ is generic with central character $\omega_\pi$. Let $\mu$ be a character of $E^\times$ such that $\mu|_{E^\times} = \omega_\pi$. If the character $\mu \circ N_{rd,E}$ of $(C_{M_n(D)}(E))^\times$ appears as a quotient in $\pi$ restricted to $(C_{M_n(D)}(E))^\times$, then :

1. the Langlands parameter of $\pi$ takes values in $GSp_{nd}(\mathbb{C})$ with similitude factor $\mu|_{F^\times}$.
2. the epsilon factor $\varepsilon([\frac{1}{2}, \pi \otimes Ind_{E}^{F}(\mu^{-1})]) = (-1)^{n}\omega_{E/F}(-1)^{\frac{d}{2}}\mu(-1)^{\frac{d}{2}}(\omega_{E/F} \text{ is the quadratic character of } F^\times \text{ with kernel the norms of } E^\times)$.

If $\pi$ is a discrete series representation of $A^\times$, then these two conditions are necessary and sufficient for the character $\mu \circ N_{rd,E}$ of $(C_{M_n(D)}(E))^\times$ to appear as a quotient in $\pi$ restricted to $(C_{M_n(D)}(E))^\times$.

Let us set $G = GL(n, D)$ and $H = (C_{M_n(D)}(E))^\times$. We consider this conjecture for the Steinberg representation and we prove it by first establishing some $H$-distinction results. The main results of this paper are :

Theorem. Let $n$ be a positive integer and let $\mu$ be a character of $E^\times$. $E$ is embedded in $M_n(D)$ if and only if $nd$ is even. We set $St(1) = St(n, 1)$ the Steinberg representation of $G$ and $\tilde{\mu} := \mu \circ N_{rd,E}$.

• If $d$ is even, $H = (C_{M_n(D)}(E))^\times = GL(n, C_D(E))$ and $St(n, 1)$ is $\tilde{\mu}$-distinguished under $H$ if and only if
  \[- \mu|_{F^\times} = 1 \text{ and } \mu \neq 1 \text{ if } n \text{ is even.}\]
\[ \mu = 1 \text{ if } n \text{ is odd.} \]

- If \( d \) is odd and \( n \) is even, \( H = (C_{n-1}(D)}(E) = GL(n/2, D \otimes_F E) \) and \( St(n,1) \) is \( \mu \)-distinguished under \( H \) if and only if \( \mu_{F^*} = 1 \) and \( \mu 
eq 1 \).

**Theorem.** (Prasad and Takloo-Bighash conjecture, Steinberg case) Let \( A = A_{n}(D) \) and \( \pi = St(n,1) \) which is an irreducible admissible representation of \( A^\times = GL(n,D) = G \). Recall that \( \pi \) corresponds via Jacquet-Langlands correspondence to \( St(nd,1) \) (the Steinberg representation of \( GL(nd,F) \)) with central character \( \omega_{\pi} = 1 \). Let \( \mu \) be a character of \( E^\times \) such that \( \mu_{F^*}|_{F^*} = \omega_{\pi} = 1 \).

Then, the character \( \mu \circ N_{r.d,E} \) of \( H = (C_{n}(D)}(E) \) appears as a quotient in \( \pi \) restricted to \( H \) if and only if:

1. the Langlands parameter of \( \pi \) takes values in \( GL_{nd}(\mathbb{C}) \) with similitude factor \( \mu_{F^*} \).
2. the epsilon factor satisfies \( \epsilon(\frac{1}{2}, \pi \otimes Ind_E^{F}(\mu^{-1})) = (-1)^n \omega_{E/F}(-1)^{\mu_{F^*}} \) (where \( \omega_{E/F} \) is the quadratic character of \( F^\times \) with kernel the norms of \( E^\times \)).

Notice that when the character \( \mu \circ N_{r.d,E} \) of \( H \) extends to \( G \), the conjecture can be reformulated in the following more appealing way, and this is the version of the conjecture considered in [FMW17].

**Theorem.** Let \( St(n,\chi) \) be the Steinberg representation of \( G = GL(n,D) \). \( St(n,\chi) \) is \( H \)-distinguished if and only if it is symplectic and \( \epsilon(\frac{1}{2}, BC_E(St(n,\chi))) = (-1)^n \) (where \( BC_E \) denotes the base change to \( E \)).

In this latter reference, the authors partially prove the conjecture for supercuspidal representations of \( GL(n,\mathbb{H}) \) where \( \mathbb{H} \) is the quaternion division algebra over a \( p \)-adic field.

We recall that, for \( \mu \) a character of \( H \), a representation \((\pi,V)\) of \( G \) is said to be \( \mu \)-distinguished under \( H \) (\( H \)-distinguished if \( \mu = 1 \) and just distinguished if there is no possible confusion) if the space of \( H \)-homomorphisms \( Hom_H(\pi,\mu) \) is non-zero i.e. if there exists a non-zero linear form \( L \) on \( V \) such that \( L(\pi(h).v) = \mu(h)L(v) \) for all \( h \in H \), for all \( v \in V \).

Now, any character of \( GL(n,D) \) can be written as \( \chi \circ N_{d,F} \) where \( \chi \) is a character of \( F^* \). For \( \chi \) a character of \( F^* \), the Steinberg representation \( St(n,\chi) \) of \( GL(n,D) \) (denoted \( St(\chi) \) if the context is clear) is given by \( ind_P^{G}(\chi \circ N_{d,F})/\sum_P ind_P^{G}(\chi \circ N_{d,F}) \) where \( P_0 \) denotes the minimal standard parabolic subgroup of \( G \) and where the standard parabolic subgroups \( P \) in the sum correspond to a partition of \( n \) with all elements equal to 1 except one of them which is equal to 2.

In the first part of this paper, we recall some useful elementary results to study the distinction, mainly Frobenius reciprocity and Mackey theory.

In Sections 2 and 3, we study the distinction of the Steinberg representation according to the parity of \( d \). We follow the method used by Matringe in [Mat16] which we recall now. We start by determining a set of representatives of double cosets \( P \backslash G/H \) for \( P \) a standard parabolic subgroup of \( G \). These representatives allow us to apply Mackey theory, which is, with Frobenius reciprocity and modulus characters computations, an essential tool to establish a necessary condition for the distinction of the Steinberg representation.

First, we show that \( Hom_H(ind_{P_0}^G(1),\tilde{\mu}) \) is at most one dimensional, hence \( Hom_H(St(1),\tilde{\mu}) \) as well. It moreover implies that \( \dim(Hom_H(St(1),\tilde{\mu})) = 1 \) if and only if there is a nonzero \( \tilde{\mu} \)-equivariant linear form on \( ind_{P_0}^G(1) \) which vanishes at each term of \( \sum_{P \text{ of type (1,1,2,2,1,1,...)}} ind_P^G(1) \).

In particular the necessary condition on \( \mu \) for \( St(1) \) to be \( \tilde{\mu} \)-distinguished will come from the fact that \( ind_{P_0}^G(1) \) must also be \( \tilde{\mu} \)-distinguished in this case. On the other hand, if \( ind_{P_0}^G(1) \) is \( \tilde{\mu} \)-distinguished, we have an explicit \( \tilde{\mu} \)-equivariant linear form on its space given by an integral.

To get our sufficient condition, we will show that this linear form does not vanish on \( ind_{P_0}^G(1) \)
for a well chosen standard parabolic subgroup $P$ of type $(1, \ldots, 1, 2, 1, \ldots, 1)$ when $\mu$ not of the correct form.

In the last section, we explain how to reformulate the Prasad and Takloo-Bighash conjecture for the Steinberg representation and we do the epsilon factor calculation in order to prove it.

**Notation** We let $P_0$ denote the minimal standard parabolic subgroup (of upper triangular matrices), $M_0$ its standard Levi subgroup and $N_0$ its unipotent radical. We set $P_0^-$ the subgroup of $G$ of lower triangular matrices. We denote by $\Phi$ the roots of the center $Z(M_0)$ of $M_0$ acting on the Lie algebra $\text{Lie}(G)$, by $\Phi^+$ those corresponding to the restriction of this action on $\text{Lie}(N_0)$, and by $\Phi^-$ those corresponding to the restriction of this action on $\text{Lie}(N_0^-)$. In particular $\text{Lie}(N_0) = \oplus_{\alpha \in \Phi^+} \text{Lie}(N_0)$ and $\text{Lie}(N_0^-) = \oplus_{\alpha \in \Phi^-} \text{Lie}(N_0)$, with obvious notation. If $P$ is a parabolic subgroup of $G$ containing $P_0$, with standard Levi factor $M$, we denote by $\Phi_M$ the roots of $Z(M_0)$ on $\text{Lie}(M)$. We define $\Phi_M^+$ and $\Phi_M^-$ in a similar fashion as above.

For $\chi$ a character of a parabolic subgroup $P$, $\text{ind}^G_P(\chi)$ is the un-normalized parabolic induction. We let $\Delta_X$ denote the modular character of a locally compact totally disconnected topological group $X$ i.e. such that $\lambda(xg) = \Delta_X(g)\lambda(x)$ for all $x, g \in X$, for $\lambda$ a left Haar measure on $X$. We set $\delta_X := \Delta_X^{-1}$.

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## 1 General results

We gather together some useful results (written as in [Mat16]).

We only consider smooth representations on complex vector spaces. Let $X$ be a locally compact totally disconnected space, and $\mathcal{L}$ a locally compact totally disconnected group acting continuously and properly on $X$. If $\chi$ is a character of $\mathcal{L}$, we denote by $C^\infty_c(\mathcal{L}\backslash X, \chi)$ the space of smooth functions on $X$, with compact support mod $\mathcal{L}$, and which transform by $\chi$ under left translation by elements of $\mathcal{L}$. If $X$ is a group $Q$ which contains $\mathcal{L}$, then we write $\text{ind}^Q_\mathcal{L}(\chi)$ for $C^\infty_c(\mathcal{L}\backslash Q, \chi)$, which is a representation of $Q$ by right translation. We shall often use the following two theorems, which are respectively Frobenius reciprocity and Mackey theory for compactly induced representations. The first one is a consequence of Proposition 2.29 of [BZ76].

**Proposition 1.1.** Let $\chi$ be a character of $L$, then the vector space $\text{Hom}_Q(\text{ind}^G_L(\chi) \mu)$ is isomorphic to $\text{Hom}_L(\Delta_X, \mu)$, where $\Delta$ is the quotient of the modulus character of $L$ by that of $Q$.

The next one is a consequence of Theorem 5.2 of [BZ77].

**Proposition 1.2.** Let $P$ be a parabolic subgroup of $G$, and $\mu$ be a character of $P$. Take a set of representatives $(u_1, \ldots, u_r)$ of $P\backslash G/H$, ordered such that $X_i = \bigsqcup_{k=1}^r Pu_k H$ is open in $G$ for each $i$. Then $\text{ind}^G_P(\mu)$ is filtered by the $H$-submodules $C^\infty_c(\mathcal{P}\backslash X_i, \mu)$, and

$$C^\infty_c(\mathcal{P}\backslash X_i, \mu)/C^\infty_c(\mathcal{P}\backslash X_{i-1}, \mu) \simeq C^\infty_c(\mathcal{P}\backslash P u_i H, \mu).$$

Finally, we recall the following result from [HW93], which is Proposition 3.4 in there. It in particular implies that if $P$ contains a minimal $\tau$-split parabolic subgroup (see below), then $C^\infty_c(\mathcal{P}\backslash PH, \mu)$ is a subspace of $\text{ind}^G_P(\mu)$.

**Proposition 1.3.** Let $P$ be a parabolic subgroup of $G$, and $\tau$ be an $F$-rational involution of $G$. The class $PH$ is open if and only if $P$ contains a minimal parabolic $\tau$-split subgroup $P'$ (which means that $\tau(P')$ and $P'$ are opposite parabolic subgroups).
2 Case d even

2.1 Preliminaries

We fix $F$ a non archimedean local field of characteristic not 2 and $E$ a quadratic extension. We let $| \cdot |_F$ denote the normalized absolute value on $F$ and $N_{E/F}$ the norm map from $E^*$ to $F^*$. Let $D$ be a central division $F$-algebra of dimension $d^2$ with $d = 2d'$ an even positive integer. As 2 divides $d^2$, $E$ can be seen as a subfield of $D$. Moreover, as $\text{car}(F) \neq 2$, there exists $\delta$ in $E$ with $\delta^2 \in F \setminus F^2$ such that $E = F[\delta]$. As $E$ is a subfield of $D$, we can consider the centralizer of $E$ in $D : C_D(E)$. It is easy to see that $C_D(E)$ is a division $E$-algebra. The double centralizer property says that $[E : F][C_D(E) : F] = [D : F] = d^2$ so $[C_D(E) : F] = d^2/2$ and that $E = C_D(C_D(E))$ so the center of $C_D(E)$ is $E$. In summary, $D' := C_D(E)$ is a central division $E$-algebra of dimension $d^2/4 = d^2/2$. We recall that $N_{d,F}$ denotes the reduced norm on $GL(n, D)$ and $N_{r,d,E}$ denotes the reduced norm of $GL(n, D')$ as well as its restriction to any subgroup.

It is easy to see that $C_D(E)$ is the fixed points set $D'$ of $D$ under the involution $\sigma = \text{int}(\delta)$. As $\sigma$ is an involution different of identity, $-1$ is an eigenvalue of $\sigma$ so we can chose $i \in D$ (which is thus not in $D'$) such that $\sigma(i) = -i$ and $D = D' \oplus iD'$. We can easily see that $C_{M_n(D)}(E) = M_n(C_D(E)) = M_n(D')$.

If we denote again by $\sigma$ the involution of $GL(n, D)$ which is given by applying $\text{int}(\delta)$ to each entry of a matrix of $GL(n, D)$, then $H = GL(n, D')$ is the fixed points of $G = GL(n, D)$ under $\sigma : H = G^\sigma$.

2.2 Representatives of $P \backslash G / H$

Let $P$ be a standard parabolic subgroup of $G = GL(n, D)$, corresponding to a partition $\vec{n} = (n_1, \ldots, n_r)$ of $n$. We denote $I(\vec{n})$ the set of symmetric matrices with natural number entries such that the sum of the $i$-th row equals $n_i$ for all $i$ in $\{1, \ldots, r\}$. Let $B = (e_1, \ldots, e_n)$ be the canonical basis of $D^n$. For $s = (n_{i,j})_{1 \leq i, j \leq r} \in I(\vec{n})$, we set

$$B_{i,j} = (e_{(n_1+\cdots+n_{i-1}+n_{i,j-1})}, \ldots, e_{(n_1+\cdots+n_{i-1}+n_{i,j-1}+1)}, \ldots, e_{(n_1+\cdots+n_{i-1}+n_{i,j})})$$

We set $u_{s} \in GL(n, D)$ the matrix in the basis $B$ of the endomorphism of $D^n$ mapping $\text{Vect}(B_{i,j})_D$ to itself and $\text{Vect}(B_{i,j} \cup B_{j,i})_D$ to itself for $i \neq j$ such that $u_{s}$ restricted to $\text{Vect}(B_{i,i})_D$ is $I_{n_i,i}$ and $u_{s}$ restricted to $\text{Vect}(B_{i,j} \cup B_{j,i})_D$ is

$$\begin{pmatrix}
I_{n_{i,j}} & -I_{n_{i,j}} \\
I_{n_{i,j}} & I_{n_{i,j}}
\end{pmatrix}.$$

Proposition 2.1. The set of elements $u_{s}$ for $s$ in $I(\vec{n})$ as described above is a set of representatives of the double cosets $P \backslash G / H$.

Proof. The proof is similar to the odd case of $\text{Mat}_{16}$ and with more precisions in $\text{Mat}_{11}$. $\square$

Remark 2.1. If we denote $\mathbb{H}$ the division algebra of quaternions over $F$ (i.e. of dimension 4 over $F$), then $C_{\mathbb{H}}(E) = E$. Moreover, if we denote $\nu'$ an element of $\mathbb{H} \setminus E$ such that $\mathbb{H} = E \oplus \nu'E$ and $\sigma(\nu') = -\nu'$ and $u_{s}'$ the same matrix as $u_{s}$ where $\nu$ has been replaced by $\nu'$, then the map $u_{s} \mapsto u_{s}'$ for $s \in I(\vec{n})$ induces a bijection between $P(D) \backslash GL(n, D) / GL(n, D')$ and $P(\mathbb{H}) \backslash GL(n, \mathbb{H}) / GL(n, E)$.

Now, for $s$ in $I(\vec{n})$, we set $w_{s} = u_{s} \sigma(u_{s}^{-1})$. We can write $I = [1, n]$ as the ordered disjoint union $I = I_{1,1} \cup I_{1,2} \cup \cdots \cup I_{1,r} \cup I_{2,1} \cup \cdots \cup I_{r,1} \cup \cdots \cup I_{r,r-1} \cup I_{r,r}$, with $I_{i,j}$ of length $n_{i,j}$. We can check that $w_{s}$ is the matrix of the permutation of $n$ sending $I_{i,i}$ to itself identically and $I_{i,j}$ to $I_{j,i}$ when $i \neq j$ such that $w_{s} (n_{1,1}+\cdots+n_{i,j-1}+k) = n_{1,1}+\cdots+n_{j,i-1}+k$.
for \( k \in \{1, \ldots, n_{i,j}\} \).

We also set

\[
\sigma_s : G = GL(n, D) \to GL(n, D) \text{ by } x \mapsto w_s \sigma(x) w_s^{-1}
\]

Then, \( u_s H u_s^{-1} \) is the group of fixed points of \( G \) under the involution \( \sigma_s : u_s H u_s^{-1} =: G^{\sigma_s} \).

Now, for \( s \in I(\tilde{n}) \), we can consider the standard parabolic subgroup of \( G \) attached to \( s \) and its standard decomposition denoted by : \( P_s = M_s N_s \). We notice that as \( s \) can be seen as a subpartition of \( (n_1, \ldots, n_r) \) (corresponding to the standard parabolic subgroup \( P \)), \( P_s \) is included in \( P \).

We need now to study \( P_s \cap u_s H u_s^{-1} \) and especially its decomposition.

The same proof as in Lemma 3.2 and Proposition 3.2 of [Mat16] shows the following proposition:

**Proposition 2.2.** For \( s \in I(\tilde{n}) \), we have \( w_s(\Phi^-_M) \subset \Phi^- \), \( w_s(\Phi^+_M) \subset \Phi^+ \). Moreover, \( P \cap u_s H u_s^{-1} = P_s \cap u_s H u_s^{-1} \) and \( P_s \cap u_s H u_s^{-1} \) is the semidirect product of \( M_s \cap u_s H u_s^{-1} \) and \( N_s \cap u_s H u_s^{-1} \).

We can make the subgroup \( M^{\sigma_s}_s = M_s \cap u_s H u_s^{-1} \) explicit:

\[
M^{\sigma_s}_s = \{ diag(a_{1,1}, a_{1,2}, \ldots, a_{1,r}, a_{2,1}, \ldots, a_{r,r}) : a_{j,i} = \sigma(a_{i,j}) \in GL(n_{i,j}, D) \}.
\]

Finally, we have the following equality of characters:

**Proposition 2.3.** \((\delta_{P_s^\sigma}(t))_{M^{\sigma_s}_s} = (\delta_{P_s^\sigma})_{M^{\sigma_s}_s}\)

**Proof.** As said in Proposition 4.4 of [Mat11], thanks to Lemma 1.10 of [KT08], as \( \sigma_s \) is defined over \( E \), it is enough to check the equality on the \( E \)-split component \( Z^{\sigma_s}_s \) (the maximal \( E \)-split torus in the center of \( M_s \)) of \( M^{\sigma_s}_s \). Here,

\[
Z^{\sigma_s}_s = \left\{ \begin{pmatrix}
\lambda_{1,1} I_{n_{1,1}} & & & 0 \\
& \lambda_{1,2} I_{n_{1,2}} & & \\
& & \ddots & \\
0 & & & \lambda_{r,r} I_{n_{r,r}}
\end{pmatrix} : \lambda_{i,j} = \lambda_{j,i} \in E^* \right\}
\]

For \( t \in Z^{\sigma_s}_s \), \( \delta_{P_s^\sigma}(t) = [N_{rd,E}(Ad(t)|_{Lie(N_s)})]|_F = \prod_{\alpha \in \Phi^+, \Phi^-} |N_{rd,E}(\alpha(t))|_F \).

Now, for \( \alpha \in \Phi \), let \( N_{\alpha,w_{\alpha}(\alpha)} = \{ x \in Lie(N_s) + Lie(N_{w_{\alpha}(\alpha)}) : \sigma_s(x) = x \} \); it is a right \( D^\prime \)-vector space of dimension \( |\alpha|, w_{\alpha}(\alpha) \rangle \). Then, for \( t \in Z^{\sigma_s}_s \), we have:

\[
\delta_{P_s^\sigma}(t) = \prod_{\{\alpha \in \Phi^+, \Phi^- \in \Phi^+, \Phi^- \}} |N_{rd,E}(Ad(t)|_{N_{\alpha,w_{\alpha}(\alpha)})]|_E = \prod_{\{\alpha \in \Phi^+, \Phi^- ; w_{\alpha}(\alpha) \in \Phi^+, \Phi^- \}} |N_{rd,E}(\alpha(t))|_E.
\]

Moreover, we have (see Proposition 4.4 of [Mat11]):

\[
\prod_{\{\alpha \in \Phi^+, \Phi^- \in \Phi^+, \Phi^- \}} |N_{rd,E}(\alpha(t))|_E = \prod_{\{\alpha \in \Phi^+, \Phi^- ; w_{\alpha}(\alpha) \in \Phi^+, \Phi^- \}} |N_{rd,E}(\alpha(t))|_E = 1 \tag{1}
\]

Thus, \( \delta_{P_s^\sigma}(t) = \prod_{\{\alpha \in \Phi^+, \Phi^- \}} |N_{rd,E}(\alpha(t))|_E \).

Finally, as \( \alpha(t) \in D^\prime \), we have

\[
|N_{rd,F}(\alpha(t))|_F = |N_{E/F} \circ N_{rd,E}(\alpha(t))|_F = |N_{E/F} \circ N_{rd,E}(\alpha(t))|_E^{1/2} = |N_{rd,E}(\alpha(t))|_E
\]

which gives the equality of characters. \( \square \)
2.3 Distinguished Steinberg representations

For \( \mu \) a character of \( E^* \), we set \( \tilde{\mu} = \mu \circ N_{r,d,E} \). In this section, we will study whether \( St(1) \) is \( \tilde{\mu} \)-distinguished under \( H \) or not, according to the character \( \mu \) of \( E^* \). We denote by \( St(1) \) the Steinberg representation \( ind_{P_{\emptyset}}^{G}(1)/\sum_{\mu} ind_{P_{\emptyset}}^{G}(1) \) where \( P \) describes the standard parabolic subgroups of \( G \) corresponding to a partition of \( n \) of type \((1, \ldots, 1, 2, 1, \ldots, 1)\).

First, we suppose that \( St(1) \) is \( \tilde{\mu} \)-distinguished under \( H \) and we find a necessary condition on \( \mu \) in the following proposition:

**Proposition 2.4.** Suppose that \( St(1) \) is \( \tilde{\mu} \)-distinguished under \( H \). Then, \( \mu_{|F^*} = 1 \) if \( n \) is even and \( \mu = 1 \) if \( n \) is odd. Moreover, only the open orbit \( P_{\emptyset}u_sH \) where \( s = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \) supports a \( \tilde{\mu} \)-equivariant linear form and \( \dim(Hom_H(St(1), \tilde{\mu})) = 1 \).

**Proof.** The idea of the proof is the same as those of Propositions 3.4 and 3.5 of [Mat16] so we do not give all details. Suppose that \( St(1) \) is \( \tilde{\mu} \)-distinguished, then \( ind_{P_{\emptyset}}^{G}(1) \) is also \( \tilde{\mu} \)-distinguished so

\[
\exists s \in I(\tilde{n}) \text{ such that } Hom_{P_{\emptyset}^s}^G(\frac{\Delta_{F^*}}{\Delta_{G^*}}\tilde{\mu}, 1) \neq \{0\}
\]

where \( \tilde{\mu}_s(x) = \tilde{\mu}(u_s^{-1}xu_s) \) for \( x \in u_sHu_s^{-1} \).

Now, suppose that \( w_s \) has at least one fixed point, then if we consider \( M = \text{diag}(1, \ldots, 1, a, 1, \ldots, 1) \) with \( a \in F^* \) in the \( i \)-th row, the previous equality of characters and the fact that \( \tilde{\mu}_{F^*} \) is unitary (considering the central character of \( ind_{P_{\emptyset}}^{G}(1) \) and its \( \tilde{\mu} \)-distinction) imply that \( i = \frac{d+1}{2} \) so \( n \) is odd and \( w_s \) has only one fixed point.

Thus, if \( n \) is even, \( w_s \) has no fixed point. Then we get \( s = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \) and \( \tilde{\mu}(u_s^{-1}M_{\emptyset}^{s*}u_s) = 1 \).

Let us prove that \( N_{r,d,E}(u_s^{-1}M_{\emptyset}^{s*}u_s) = F^* \). First, we show how \( D \) can be embedded in \( M_2(D') \). \( M_2(D') \) identifies to \( \text{End}(D)_{D'} \) via the basis \((1, i)\) of the right \( D' \)-vector space \( D \). Moreover, we have

\[
D \otimes_F E \xrightarrow{a \otimes e} \text{End}(D)_{D'} \text{ with } (d \mapsto ade)
\]

so \( D \) can be embedded in \( M_2(D') \) via

\[
f : D \xrightarrow{a} M_2(D') \text{ with } a \mapsto a \otimes 1 \in D \otimes_F E \simeq M_2(D')
\]

Thus, \( N_{r,d,E}(f(D^*)) = N_{r,d,F}(D^*) = F^* \) (a splitting field \( L \) for \( D \) is also a splitting field for \( D \otimes_F E \) and \((D \otimes_F E) \otimes_E L = D \otimes_F L \)). We finish by noticing that \( u_s^{-1}M_{\emptyset}^{s*}u_s = f(D) \). Thus, \( N_{r,d,E}(u_s^{-1}M_{\emptyset}^{s*}u_s) = F^* \) so we obtain \( \mu_{|F^*} = 1 \).

If \( n \) is odd, we get \( s = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \), \( \mu_{F^*} = 1 \) and, as \( w_s \) has one fixed point, we have the extra condition \( \tilde{\mu}(D^*) = 1 \). As \( N_{r,d,E}(D^*) = E^* \), we obtain \( \mu = 1 \).
We notice that, as $\sigma(u_n^{-1}P_0u_n) = u_n^{-1}P_0^{-1}u_n$, then $u_n^{-1}P_0u_nH$ is open in $G$ (thanks to Proposition \textbf{1.3}) so $P_0u_nH$ is open in $G$ too.

Now, we will exhibit a non-zero $\tilde{\mu}$-equivariant linear form on $\text{ind}^G_{P_0}(1)$. To do that, we will follow the strategy of Sections 4.2 and 4.3 of \cite{Mat14}.

For $z \in \mathbb{C}$, we denote by $\delta_z$ the character $(\delta_{P_0})^z$. For $f \in \text{ind}^G_{P_0}(1)$, we denote by $f_z$ the only element in $\text{ind}^G_{P_0}(\delta_z)$ such that $f_z|_K = f|_K$ with $K = \text{GL}(n, \mathcal{O}_D)$. The map

$$\phi : \text{ind}^G_{P_0}(1) \to \text{ind}^G_{P_0}(\delta_z)$$

is a $K$-isomorphism. We also set $s_0 = \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) \in I(\mathcal{H}), u_0 = u_{s_0}$ and $\sigma_0 = \sigma_{s_0}.$

\textbf{Proposition 2.5.} Suppose that $\mu|_{F^*} = 1$ if $n$ is even and $\mu = 1$ if $n$ is odd. For $f$ in $\text{ind}^G_{P_0}(1)$, the integral $I_{n,z}(f_z) = \int_{u_0^{-1}P_0u_0 \cap H} \hat{P}^{-1}(h)f_z(u_0h)dh$ converges for $Re(z)$ large enough and there exists $Q \in \mathbb{C}[X]$ such that $Q(q^{-z})I_{n,z}(f_z)$ belongs to $\mathbb{C}[q^{\pm 1}]$ for all $f$ in $\text{ind}^G_{P_0}(1)$ (with $q = |k_F|$). Moreover, for $z' \in \mathbb{C}$, there exists $I_{n,z'}$ in $\mathbb{N}$ such that

$$L_{z'} = \lim_{z \to z'} (1 - q^{z'-z})^{-1}I_{n,z}$$

belongs to $\text{Hom}_H(\text{ind}^G_{P_0}(\delta_{z'}), \hat{\mu}) \setminus \{0\}$.

\textbf{Proof.} This is due to the Bernstein principle for meromorphic continuation of equivariant linear forms (see Corollary 2.12 in \cite{Mat14} which is stated for $\text{GL}(n, F)$ but is also true for $\text{GL}(n, D)$). First, notice that

$$P_0^{s_0} = P_0 \cap u_0 H u_0^{-1} = \{ \text{diag}(a_1, \ldots, a_{n/2}, \sigma(a_{n/2}), \ldots, \sigma(a_1)) ; a_i \in D^* \} \text{ if } n \text{ is even and}$$

$$P_0^{s_0} = \{ \text{diag}(a_1, \ldots, a_{n-1}, b, \sigma(a_{n-1}), \ldots, \sigma(a_1)) ; a_i \in D^*, b \in D^* \} \text{ if } n \text{ is odd.}$$

Thus, as $N_{r,d,F}(a) = N_{r,d,F}(\sigma(a))$ for $a \in D^*$, then $\delta_z(u_0^{-1}P_0u_0 \cap H) = 1$ and $\delta_z(u_0^{-1}P_0u_0 \cap H) = 1$ so $I_{n,z}(\mathbb{C}=(P_0 \setminus P_0 u_0 H, \delta_z))$ is non-zero and well defined for all $z \in \mathbb{C}$. As $\mathbb{C}=(P_0 \setminus P_0 u_0 H, \delta_z) \subset \text{ind}^G_{P_0}(\delta_z)$ (because $P_0 u_0 H$ is open in $G$), it means that for all $z \in \mathbb{C}$, there exists $f_z$ in $\text{ind}^G_{P_0}(\delta_z)$ such that $I_{n,z}(f_z) \neq 0$.

To see that $I_{n,z}(f_z)$ is absolutely convergent for $Re(z)$ large enough, first we notice that $|\mu^{-1}| = |\mu^{-1}| \circ N_{r,d,E}$ can be written $|\mu^{-1}| = |\cdot|_E \circ N_{r,d,E}$ for $\alpha$ a complex number. As $|\cdot|_E = |\cdot|_{E} \circ N_{r,d,E}/H$ and as $N_{r,d,E}/H = E/F \circ N_{r,d,E}/H$, then $|\mu^{-1}| = |\cdot|_{E} \circ N_{r,d,E}/H$ so $|\mu^{-1}|$ can be extended to a character of $G$ that we denote by $\chi$. Now, we have to prove that $\int_{\mathbb{C}=(P_0 \setminus P_0 u_0 \cap H)} \chi(h)f_z(u_0h)dh$ is absolutely convergent. We can see $f_z \mapsto \int_{u_0^{-1}P_0 u_0 \cap H} \chi(h)f_z(u_0h)dh$ as a function of $\chi f_z$ which belongs to $\chi \otimes \text{ind}^G_{P_0}(\delta_z)$. Then, the absolute convergence comes from Theorems 2.8 and 2.16 of \cite{BDO87}.

Finally, we can use the Bernstein principle for meromorphic continuation of equivariant linear forms, because the space $\text{Hom}_H(\text{ind}^G_{P_0}(\delta_z), \hat{\mu})$ is of dimension $\leq 1$ for all $q \cdot s$ and $s$. Indeed, we prove it as in the beginning of the proof of Proposition \textbf{2.4}. We suppose that $\text{ind}^G_{P_0}(\delta_z)$ is $\bar{\mu}$-distinguished under $H$. It implies that $\bar{\mu}|_{F^*}$ is unitary (considering the central character). Then, we apply Mackey theory and Frobenius reciprocity and we get that there exists a unique $s$ (it is $s_0$) such that $\text{Hom}_{P_0^{s_0}}(\Delta_{P_0^{s_0}} \delta_z, \bar{\mu}_s)$ is non-trivial. Thus, we obtain that $\text{Hom}_{P_0^{s_0}}(\Delta_{P_0^{s_0}} \delta_z, \bar{\mu}_s)$ is of dimension $1$ so $\text{Hom}_H(\text{ind}^G_{P_0}(\delta_z), \bar{\mu})$ is $1$-dimensional too. \hfill $\square$
Now, we can give two results of distinction, according to the parity of $n$ :

**Proposition 2.6.** If $n$ is odd and if $\mu = 1$, then $\text{St}(1)$ is $\mu$-distinguished.

*Proof.* Taking $z' = 0$, we get that $L_0$ is a non-zero $\mu$-equivariant linear form on $\text{ind}(\mathcal{G}_\theta^F)$. Its characteristic function is thus $\mu$-distinguished. We end this proof as in Proposition 3.6 of [Mat16].

**Proposition 2.7.** If $n$ is even and if $\mu_{F^*} = 1$ and $\mu \neq 1$, then $\text{St}(1)$ is $\mu$-distinguished.

*Proof.* The proof is the same as [Proposition 2.6].

To finish this case, we have the following proposition:

**Proposition 2.8.** If $n$ is even and if $\mu = 1$, then $\text{St}(1)$ is not $\mu$-distinguished.

To prove this theorem, we need first two lemmas about the integral $I_{n,z}$. We denote by $\Phi$ the constant function equal to 1 in $\text{ind}(\mathcal{G}_\theta^F)$. Then, for any $z \in \mathbb{C}$, $f_z = f\Phi_z$. If $n = 2$, we denote $\Phi$ by $\Phi_2$.

**Lemma 2.1.** Suppose that $\mu = 1$ and that $n = 2$, then up to a unit in $\mathbb{C}[q^{\pm}]$, we have:

$$I_{2,z}(\Phi_2) = L(dz - \frac{d}{2}, 1_{E^*})L(0, 1_{E^*})/L(2dz, 1_{F^*}),$$

where $L$ is the usual Tate $L$-factor. In particular, $I_{2,0}(\Phi_2) \neq 0$.

*Proof.* The proof is the same as Proposition 4.5 of [Mat16] with $u_0 = \begin{pmatrix} 1 & -t \\ 1 & t \end{pmatrix}$. We only give the beginning and the end. We set $\nu_F := |N_{r_d,F}()|_F$. For $\text{Re}(z)$ large enough, we have:

$$\Phi_2(g) = \nu_F(g)dz^2 \int \Phi((0, t)g)\nu_F(t)dz^2/L(2dz, 1_{F^*}).$$

Finally, if we set $\epsilon = (\nu_F(h^{-1})\nu_F(u_0))dz$ and denote by $\Phi_0$ the characteristic function of $\mathcal{M}(2, \mathcal{O})$ (and recalling that $d' = \frac{d}{2}$ is the index of $D'$ over $E$), we obtain:

$$I_{2,z}(\Phi_2) = \epsilon \int H \Phi_0(h)\nu_F(h)dz^2 dh/L(2dz, 1_{F^*})$$

$$= \epsilon \int H \Phi_0(h)\nu_F(h)dz^2 dh/L(2dz, 1_{F^*})$$

$$= \epsilon L(2dz - \frac{1}{2}(2d' - 1), 1_H)/L(2dz, 1_{F^*})$$

by definition of $Z$ and $L$-functions

$$= \epsilon L(2dz - \frac{1}{2}(3d' - 1), 1_{F^*})L(2dz - \frac{1}{2}(d' - 1), 1_{D^*})/L(2dz, 1_{F^*})$$

by inductivity relation of the Godement-Jacquet $L$-factor $L(z, 1_H)$

$$= \epsilon L(2dz - d', 1_E)L(2dz, 1_E)/L(2dz, 1_{F^*})$$

If $z = 0$, $L(2dz, 1_{F^*})$ and $L(2dz, 1_{F^*})$ have one simple pole and $L(2dz - d', 1_{E^*})$ has no pole so $I_{2,0}(\Phi_2) \neq 0$.

We recall Proposition 4.6 of [Mat16] :

**Lemma 2.2.** Suppose that $\mu = 1$. For $n = 2m$, let $P$ be the standard parabolic subgroup of $G$ corresponding to the partition $\bar{n} = (1, \ldots, 1, 2, 1, \ldots, 1)$ with $n_m = 2$. Then, there is $f$ in $\text{ind}(\mathcal{G}_\theta^F)$ such that $I_{n,z}(f\Phi_2) = I_{2,z}(\Phi_2)$. In particular, taking $z = 0$, one has $I_{u_{0}}(f) = I_{2,0}(\Phi_2) \neq 0$.

*Proof.* The proof is exactly the same as in Proposition 4.6 of [Mat16].
Finally, we come back to the proof of the proposition:

**Proof of the proposition.** Suppose that \( \mu = 1 \) and \( S(1) \) is \( \mu \)-distinguished (i.e. \( H \)-distinguished). Then, \( ind_H^G(1) \) is \( H \)-distinguished so there exists \( L \) a non-zero \( H \)-invariant linear form on \( ind_H^G(1) \). As \( \dim(\text{Hom}_H(ind_H^G(1),1)) = 1 \) (thanks to Proposition 2.4), then \( L \) equals to \( L_0 \) up to a non-zero scalar. As \( S(1) \) is distinguished, \( L_{\text{ind}_H^G(1)} \) must be equal to zero for all standard parabolic subgroups \( P \) of type \( (1, 1, 2, 1, \ldots, 1) \). Moreover, as said in Proposition 2.4, \( L_0 \) restricts non trivially to \( C^\infty(P_0 \setminus \mathcal{P}_{\Phi_0} H, 1) \subset \text{ind}_H^G(1) \).

As \( I_{n, 0}(C^\infty(P_0 \setminus \mathcal{P}_{\Phi_0} H, 1)) \) is non zero (and is well defined), this implies that \( L_0 = I_{n, 0} \). Now, we take \( f \) and \( \mathcal{P} \) as in Lemma 2.2. Then we have \( L_0(f) = I_{n, 0}(f) \neq 0 \) which contradicts the distinction of \( S(1) \).

\[\square\]

### 3 Case \( d \) odd and \( n \) even

#### 3.1 Preliminaries

We set \( n = 2m \) and we suppose that \( d \) (the index of \( D \) over its center \( F \)) is odd. Let us consider \( D \otimes_F E \) which is a central division \( E \)-algebra of dimension \( d^2 \) (thanks to Wedderburn structure theorem and Hasse’s invariant). We can choose \( \delta \in (D \otimes_F E) \) such that \( \Delta := \delta^2 \in F \) (for example, \( \delta = 1 \otimes x \) with \( x \in E \setminus F \) such that \( x^2 \) is in \( F \)). Then, \( D \otimes_F E \) is of dimension 2 over \( D \) so we can write \( D \otimes_F E \) as \( D \otimes \delta D \) and \( D \otimes_F E \) identifies as a right \( D \)-vector space of dimension 2.

Now, if we let \( (e_1, \ldots, e_m) \) denote the canonical basis of \( (D \otimes_F E)^m \), then the right \( D \)-vector space \( (D \otimes_F E)^m \) identifies with the \( D \)-vector space \( D^{2m} \) via the basis \( B = (e_1, \ldots, e_m, \delta e_m, \ldots, \delta e_1) \) of \( D^{2m} \) so \( \text{End}((D \otimes_F E)^m) \cong \text{End}(D^{2m}) \). Now, if \( u \in \text{End}((D \otimes_F E)^m) \), it is easy to see that \( u \in \text{End}(D \otimes_F E)^m \otimes_{D \otimes_F E} \) if and only if \( u \) commutes with the multiplication by \( \delta \) (denoted \( \mu_3 \)). In the basis \( B \), the matrix of the endomorphism \( \mu_3 \) is given by

\[
\begin{pmatrix}
\Delta & \cdots & \cdots \\
\vdots & \ddots & \cdots \\
1 & \cdots & \cdots
\end{pmatrix}
=: U_{\Delta_{2m}}
\]

so \( M_m(D \otimes_F E) \) can be viewed as the fixed points of \( M_{2m}(D) \) under the involution \( \text{int}_{U_{\Delta_{2m}}} \) of \( M_{2m}(D) \) (where \( \text{int}_{U_{\Delta_{2m}}}(x) = U_{\Delta_{2m}} x U_{\Delta_{2m}}^{-1} \)).

Now, \( E \) can be embedded in \( M_{2m}(D) \) via

\[
x \in E \mapsto \begin{pmatrix} 1 \otimes x & & \\ & \ddots & \\ & & 1 \otimes x \end{pmatrix} \in M_m(D \otimes_F E) \subset M_{2m}(D)
\]

and it is easy to check that \( g \in C_{M_{2m}(D)}(E) \) if and only if \( g \) commutes with \( U_{\Delta_{2m}} \), i.e. if and only if \( g \in M_m(D \otimes_F E) \). Thus \( C_{M_{2m}(D)}(E) = M_m(D \otimes_F E) \) and we set \( G = GL(2m, D) \), \( H = GL(m, D \otimes F) \) with \( \sigma = \text{int}_{U_{\Delta_{2m}}} \). We recall that \( N_{rd,F} \) denotes the reduced norm on \( GL(n, D) \) and \( N_{rd,E} \) denotes the reduced norm on \( GL(m, D \otimes F) \) (as well as its restriction to any subgroup).

#### 3.2 Representatives of \( P \setminus G/H \)

Let \( P \) be a standard parabolic subgroup of \( G = GL(2m, D) \) corresponding to a partition \( \tilde{n} = (n_1, \ldots, n_r) \) of \( n = 2m \). We define \( I(\tilde{n}) \) to be the set of symmetric matrices \( s = (n_{i,j}) \in M_r(\mathbb{N}) \)
with positive integral entries, even on the diagonal, and such that the sum of the \(i\)-th row is equal to \(n_i\) for all \(i\) in \(\{1, \ldots, r\}\).

As each \(n_{i,j}\) is even for \(i \in [1, r]\), we can write \(n_{i,j} = 2m_{i,j}\) and we can write \(n\) as an ordered sum of integers in two different ways:

\[
\begin{align*}
n &= m_{1,1} + n_{1,2} + \cdots + n_{1,r} + m_{2,2} + n_{2,3} + \cdots + n_{2,r} + m_{3,3} + \cdots + m_{r-1,r-1} + n_{r-1,r} + m_{r,r} \\
&\quad + m_{r,r} + n_{r,r-1} + m_{r-1,r-1} + \cdots + m_{3,3} + n_{r,2} + \cdots + n_{3,2} + m_{2,2} + n_{r,1} + \cdots + n_{2,1} + m_{1,1} \\
&= n_1 + \cdots + n_{1,r} + n_{2,1} + \cdots + n_{2,r} + \cdots + n_{r,1} + \cdots + n_{r,r}
\end{align*}
\]

\((1^\text{st} \text{ ordering})\)

\[(2^\text{nd} \text{ ordering corresponding to the lexicographical ordering})\]

We denote by \(w_s\) the matrix of the permutation (still denoted \(w_s\)) defined as follows:

If \(i \in [1, r]\), then for \(k \in [1, n_i]\), we set

\[
w_s(m_{1,1} + \cdots + m_{i-1,i-1} + n_{i-1,i} + \cdots + n_{i-1,r} + k) = n_{1,1} + \cdots + n_{i-1,1} + \cdots + n_{i,i-1} + k,
\]

and

\[
w_s(m_{1,1} + \cdots + m_{i+1,i+1} + n_{i+1,i} + \cdots + n_{i+1,r} + k) = n_{1,1} + \cdots + n_{i-1,1} + \cdots + n_{i,r-1} + k + m_{i,i}.
\]

If \(i < j\), for \(k \in [1, n_{i,j}]\), we set

\[
w_s(m_{1,1} + \cdots + m_{i,i} + n_{i,i+1} + \cdots + n_{j-1,j-1} + k) = n_{1,1} + \cdots + n_{i,r} + \cdots + n_{j,1} + \cdots + n_{i,j-1} + k
\]

and

\[
w_s(m_{1,1} + \cdots + m_{i+1,i+1} + n_{i,i+1} + \cdots + n_{j+1,j} + k) = n_{1,1} + \cdots + n_{i,r} + \cdots + n_{j,1} + \cdots + n_{i,j-1} + k.
\]

In other words, \(w_s\) sends an integer of rank \(k\) according to the \(1^\text{st}\) ordering to the integer of rank \(k\) corresponding to the \(2^\text{nd}\) ordering.

A proof similar to Proposition 3.1 of \cite{Mat16} shows the following result:

**Proposition 3.1.** Let \(\bar{n}\) be a partition of \(n\) and \(P\) be a standard parabolic subgroup of \(G\) corresponding to this partition, then \(G = \bigcup_{x \in I(\bar{n})} P w_s H\).

**Remark 3.1.** There is a bijection between \(P_n(D) \setminus GL(n, D) / GL(m, D \otimes F) \) and \(P_n(F) \setminus GL(n, F) / GL(m, E)\) via the identity map of \(\{w_s | s \in I(n)\}\).

Now, for \(s \in I(\bar{n})\), we set \(t_s = w_s U_{2n}^{-1}.\) It is a monomial matrix (so it is in \(N_G(M_0)\)) and if we let \(\tau_s\) denote the image of \(t_s\) in \(\sigma_n = N_G(M_0)/M_0\), then \(\tau_s\) is a permutation matrix of order 2, given by the formula \(\tau_s = w_s w^{-1}\) (where \(w = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\)).

Then, we can see that \(w_s H w_s^{-1}\) is the group of the fixed points of \(G\) under the involution:

\[
\sigma_s : x \mapsto t_s x t_s^{-1}
\]

We need to know exactly how acts the permutation \(\tau_s\). One checks that \(\tau_s\) is the involution of \([1, n_2] = I = I_{1,1} \cup I_{1,2} \cup \cdots \cup I_{1,r} \cup \cdots \cup I_{r-1,r-1} \cup I_{r,r} \cup I_{r+1,r} \cup I_{r+1,r+1} \cdots \cup I_{r,r}\) (with \(I_{i,j}\) of length \(n_{i,j}\)), which stablises each \(I_{i,j}\), acting on it as the symmetry with respect to its midpoint, and which stablises \(I_{i,j} \cup I_{j,i}\) (for \(i < j\)) and acts on this union of intervals as the symmetry with center the midpoint of the interval joining the left end of \(I_{i,j}\) and the right end of \(I_{j,i}\).
For \( s \in I(\tilde{n}) \), we denote by \( P_s \) the standard parabolic subgroup of \( G \) corresponding to the sub-
partition \( s \) of \( \tilde{n} \). As usual, we denote \( P = MN \) and \( P_s = M_sN_s \) the standard Levi decomposition of \( P \) and \( P_s \). Then, again as in the even case, we have the following proposition :

**Lemma 3.1.** For \( s \in I(\tilde{n}) \), one has \( \tau_s(\Phi_M^-) \subset \Phi^- \), \( \tau_s(\Phi_M^+) \subset \Phi^+ \).

**Proposition 3.2.** For any \( s \in I(\tilde{n}) \), one has \( P \cap w_sHw_s^{-1} = P_s \cap w_sHw_s^{-1} \), and \( P_s \cap w_sHw_s^{-1} \) is the semidirect product of \( M_s \cap w_sHw_s^{-1} \) and \( N_s \cap w_sHw_s^{-1} \).

We will now let \( P_{s*}^r \) denote \( P_s \cap w_sHw_s^{-1} \) and \( M_{s*}^r \) denote \( M_s \cap w_sHw_s^{-1} \). We can explicitly describe the group \( M_{s*}^r \) : an element \( a \in M_{s*}^r \) is of the form :

\[
a = \text{diag}(a_{1,1}, a_{1,2}, \ldots, a_{1,r}, a_{2,1}, \ldots, a_{r,r})
\]

where \( a_{i,i} \in GL(n_{i,i}, D) \) satisfies \( a_{i,i} = U_{\Delta} a_{i,i} U_{\Delta}^{-1} (U_{\Delta} := U_{\Delta_{n_{i,i}}}) \) i.e. \( a_{i,i} \in GL(n_{i,i}, D \otimes F) \) and \( a_{i,j} \in GL(n_{i,j}, D) \) satisfies \( a_{i,j} = wa_{j,i}w^{-1} \) if \( i \neq j \) (\( w = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \)) for all \( i, j \) in \( \{1, \ldots, r\} \).

**Proposition 3.3.** We have the following equality of characters :

\[
(\delta_{P_{s*}^r})_{|M_{s*}^r} = (\delta_{P_s})_{|M_{s*}^r}^{1/2}.
\]

Proof. Again, the method is the same as in Proposition 4.4 of [Mat11]. Thanks to Lemma 1.10 of [KT08], it is enough to check the equality on the \( F \)-split component \( Z_{s*}^r \), the maximal \( F \)-split torus in the center of \( M_s \) of \( M_{s*}^r \) because \( s \) is defined over \( F \). Thus, we will consider each character on \( Z_{s*}^r \), that is to say, the subgroup of the matrices of the form

\[
\begin{pmatrix}
\lambda_{1,1}I_{n_{1,1}} & & \\
& \lambda_{1,2}I_{n_{1,2}} & \\
& & \ddots \\
& & & \lambda_{r,r}I_{n_{r,r}}
\end{pmatrix}
\]

with \( \lambda_{i,j} = \lambda_{j,i} \in F^* \) and \( n_{i,i} \) even.

For \( \alpha \in \Phi \), we set \( N_{\alpha, \tau_s(\alpha)} = \{ x \in \text{Lie}(N_s) + \text{Lie}(N_{\tau_s(\alpha)}) ; \sigma_s(x) = x \} \). It’s a \( D \)-vector space of dimension 1.

For \( t \in Z_{s*}^r \), we have :

\[
\delta_{P_{s*}^r}(t) = \prod_{\{\alpha, \tau_s(\alpha)\} \subset \Phi^+ - \Phi^+_s} |N_{rd,F}(Ad(t)|_{N_{\alpha, \tau_s(\alpha)}})|_F = \prod_{\{\alpha \in \Phi^+ - \Phi^+_s ; \tau_s(\alpha) \in \Phi^+ - \Phi^+_s\}} |N_{rd,F}(\alpha(t))|_F^{1/2}
\]

The second equality of (4) comes from the fact that if \( \{\alpha_0, \tau_s(\alpha_0)\} \subset \Phi^+ - \Phi^+_s \), then \( |N_{rd,F}(Ad(t)|_{N_{\alpha_0, \tau_s(\alpha_0)}})|_F = |N_{rd,F}(\alpha_0(t))|_F \) and the power \( 1/2 \) comes from the fact that \( \tau_s \) has no fixed point whereas \( N_{\alpha_2, \tau_s(\alpha)} \) is of dimension 1.

As in the even case, \( \prod_{\{\alpha \in \Phi^+ - \Phi^+_s ; \tau_s(\alpha) \in \Phi^+ - \Phi^+_s\}} |N_{rd,F}(\alpha(t))|_F^{1/2} \) the second equality of (4) comes from the fact that \( \tau_s \) has no fixed point whereas \( N_{\alpha_2, \tau_s(\alpha)} \) is of dimension 1.

Finally, by definition we have :

\[
\delta_{P_s}(t) = |N_{rd,F}(Ad(t)|_{\text{Lie}(N_s)})|_F = \prod_{\{\alpha \in \Phi^+ - \Phi^+_s\}} |N_{rd,F}(\alpha(t))|_F.
\]

and we have the characters equality.

\( \square \)
3.3 Distinguished Steinberg representations

In this part, we will study whether the Steinberg representation is \( \mu \circ N_{rd,E} \)-distinguished under \( H \) or not according to the character of \( E^* \) considered \( \mu \). For \( \mu \) a character of \( E^* \), we set \( \tilde{\mu} := \mu \circ N_{rd,E} \).

We recall that \( St(1) \) is the Steinberg representation \( \text{ind}^E_P(1)/S \) with \( S = \sum_P \text{ind}^E_P(1) \) where the standard parabolic subgroups \( P \) in the sum correspond to a partition \( n \) of \( n \) with all \( n_i \)'s equal to 1 except one which is 2.

First, we give a necessary condition on \( \mu \) to allow \( St(1) \) to be \( \tilde{\mu} \)-distinguished.

**Proposition 3.4.** If \( St(1) \) is \( \tilde{\mu} \)-distinguished under \( H \), then \( \mu_{|P^*} = 1 \). Moreover, only the open orbit \( P \backslash H \) supports a \( \tilde{\mu} \)-equivariant linear form and \( \dim(\text{Hom}_H(St(1), \tilde{\mu})) = 1 \).

**Proof.** As the method is the same as in the proof of Proposition 2.4, we will only underline the most important points.

First, we recall that \( \tilde{\mu}_{|P^*} \) is unitary. Then, \( St(1) \) being \( \tilde{\mu} \)-distinguished implies, by Frobenius reciprocity, Mackey theory and Proposition 2.3, that there exists \( s \) in \( \text{I}(\tilde{\mu}) \) such that \( (\sigma \mu_{\tilde{\mu}})^{1/2} | \text{M}^* = \tilde{\mu}_{|\text{M}^*} \) where \( \tilde{\mu}_s(x) = \tilde{\mu}(w_s^{-1}xw_s) \) for \( x \) in \( w_sHw_s^{-1} \).

Thus, we get \( s = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \) and \( \mu \circ N_{rd,E} \begin{pmatrix} (a_1) \\ \vdots \\ (a_m) \end{pmatrix} = 1 \) for all \( (a_1, \ldots, a_m) \) in \( (D^*)^m \). As \( \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \) is the embedding of \( \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \) of \( \text{M}_m(D \otimes E) \) in \( \text{M}_{2m}(D) \), it implies that \( \mu \circ N_{rd,E}(D^*) = 1 \).

Let us show that \( N_{rd,E}(D^*) = 1 \). There exists \( L/F \) an extension of dimension \( d \) such that \( D \otimes F L \cong \text{M}_d(L) \). As \( d \) is odd, \( L \otimes F E \) is a field and it is a \( d \)-dimensional extension of \( E \). Thus, we have the following natural commutative diagram:

\[
\begin{array}{ccc}
D \otimes_F E & \rightarrow & \text{M}_d(L \otimes_F E) \\
\uparrow & & \uparrow \\
D & \rightarrow & \text{M}_d(L)
\end{array}
\]

which implies \( N_{rd,E}(D^*) = N_{rd,E}(D^*) = F^* \). We deduce that \( \mu_{|P^*} = 1 \).

We end this proof by noticing that as \( \sigma(P) = P_{0}^{-} \), then \( P_{0}H \) is open in \( G \) thanks to Proposition 1.3.

Now, we show that \( St(1) \) is \( \tilde{\mu} \)-distinguished under \( H \) if and only if \( \mu_{|P^*} = 1 \) and \( \mu \neq 1 \) in the following two propositions:

**Proposition 3.5.** If \( \mu_{|P^*} = 1 \) and \( \mu \neq 1 \), then \( St(1) \) is \( \tilde{\mu} \)-distinguished under \( H \).
Proof. Suppose \( \mu_{F^*} = 1 \) and \( \mu \neq 1 \). We have \( \sigma(P_0) = P_0^- \) and if \( a \) is in \( M_0^{\sigma} \), then \( a \) is of the form

\[
\begin{pmatrix}
    a_1 \\
    \vdots \\
    a_m \\
    \vdots \\
    a_1
\end{pmatrix}
\]

with \( a_1, \ldots, a_m \in D^* \) and \( \delta_{F_0}^{-1/2} \mu^{-1}(a) = 1 \). Thus, thanks to Theorem 2.8 of [BD08], \( \text{ind}^G_{F_0}(1) \) is \( \tilde{\mu} \)-distinguished. We end the proof as in Proposition 3.6 of [Mat16].

Finally, we get the non-distinguished case:

**Proposition 3.6.** If \( \mu = 1 \), then \( St(1) \) is not \( \tilde{\mu} \)-distinguished under \( H \).

**Proof.** We do not give the proof because it is similar to the one of Theorem 3.1 in [Mat16].

## 4 Prasad and Takloo-Bighash conjecture

Let us summarize our results:

**Theorem 4.1.** Let \( n \) be a positive integer and let \( \mu \) be a character of \( E^{\times} \). \( E \) is embedded in \( \mathcal{M}_n(D) \) if and only if \( nd \) is even. We set \( G = GL(n,D) \) and \( St(1) = St(n,1) \) the Steinberg representation of \( G \). We recall that \( \tilde{\mu} \) denotes \( \mu \circ N_{rd,E} \).

- If \( d \) is even, \( H = (C_{\mathcal{M}_n(D)}(E))^{\times} = GL(n,C_D(E)) \) and \( St(n,1) \) is \( \tilde{\mu} \)-distinguished under \( H \) if and only if
  - \( \mu_{F^*} = 1 \) and \( \mu \neq 1 \) if \( n \) is even.
  - \( \mu = 1 \) if \( n \) is odd.

- If \( d \) is odd and \( n \) is even, \( H = (C_{\mathcal{M}_n(D)}(E))^{\times} = GL(n/2,D \otimes_F E) \) and \( St(n,1) \) is \( \tilde{\mu} \)-distinguished under \( H \) if and only if \( \mu_{F^*} = 1 \) and \( \mu \neq 1 \).

Now, let us write and prove the Conjecture 1 of [PTB11] for the Steinberg representation \( St(n,1) \) (which is a discrete series representation):

**Theorem 4.2.** (Prasad and Takloo-Bighash conjecture, Steinberg case) Let \( A = \mathcal{M}_n(D) \) and \( \pi = St(n,1) \) which is an irreducible admissible representation of \( A^{\times} = GL(n,D) = G \). Recall that \( \pi \) corresponds via Jacquet-Langlands correspondence to \( St(nd,1) \) (a representation of \( GL(nd,F) \)) with central character \( \omega_\pi = 1 \). Let \( \mu \) be a character of \( E^{\times} \) such that \( \mu^{\frac{2d}{F^*}}|_{F^*} = \omega_\pi = 1 \). Then, the character \( \mu \circ N_{rd,E} \) of \( H = (C_{\mathcal{M}_n(D)}(E))^{\times} \) appears as a quotient in \( \pi \) restricted to \( H \) if and only if:

1. the Langlands parameter of \( \pi \) takes values in \( GSp_{nd}(\mathbb{C}) \) with similitude factor \( \mu_{F^*} \).
2. the epsilon factor satisfies \( \epsilon\left(\frac{1}{2}, \pi \otimes Ind_E^F(\mu^{-1})\right) = (-1)^n \omega_E/F(-1)^{\frac{2d}{F^*}} \) (where \( \omega_E/F \) is the quadratic character of \( F^{\times} \) with kernel the norms of \( E^{\times} \)).

As \( N_{rd,F} = N_{E,F} \circ N_{rd,E} \), then if \( \mu \) is a character of \( E^{\times} \), \( \mu \circ N_{rd,E} \) can be extended to a character of \( G \) if and only if there exists \( \chi \) a character of \( F^{\times} \) such that \( \mu = \chi \circ N_{E,F} \).
Let us rephrase Theorem 4.2 in this case. Let $\pi$ and $\mu$ be as in the conjecture and suppose that there exists $\chi$ a character of $F^*$ such that $\mu = \chi \circ N_{E/F}$. We denote by $W_F$ the Weil group of $F$ and $BC_E$ denotes the base change to $E$.

- As $\mu = \chi \circ N_{E/F}$, the statement "$\mu \circ N_{r_d,E}$ appears as a quotient in $\pi$ restricted to $H$" is equivalent to saying that $St(n,1)$ is $\chi \circ N_{r_d,E}$-distinguished under $H$. This is again equivalent to $(\chi \circ N_{r_d,F})^{-1}(\pi(n,1)) = St(n,\chi^{-1})$ is $H$-distinguished.

- Now, let us consider the 1st point of the conjecture. The Langlands parameter of $\pi = St(n,1)$ is $Sp(nd) =: \Phi$ (where $Sp(nd)$ denotes the unique irreducible representation of $SL(2, \mathbb{C})$ of dimension $nd$). The 1st assertion in the conjecture means:

There exists $\langle \cdot, \cdot \rangle$ a nondegenerate alternating bilinear form on $\mathbb{C}^{nd} := V$ such that

$$\forall w \in W_F, \forall v, v' \in V, \quad \langle \Phi(w).v, \Phi(w).v' \rangle = \mu_{|F^*}(w) \langle v, v' \rangle = \chi^2 < v, v' >. \quad (5)$$

As the Langlands parameter of $St(n,\chi^{-1})$ is $Sp(nd) \otimes \chi^{-1} =: \Psi$, statement (5) is equivalent to:

There exists $\langle \cdot, \cdot \rangle$ a nondegenerate alternating bilinear form on $V$ such that

$$\forall w \in W_F, \forall v, v' \in V, \quad \langle \Psi(w).v, \Psi(w).v' \rangle = \langle v, v' \rangle$$

which is exactly the definition of $St(n,\chi^{-1})$ being symplectic.

- Finally, we consider the 2nd point of the conjecture and we formulate the epsilon factor in another way:

$$\epsilon\left(\frac{1}{2}, \pi \otimes Ind_E^F(\mu^{-1})\right) = \epsilon\left(\frac{1}{2}, Sp(nd) \otimes Ind_E^F((\chi \circ N_{E/F})^{-1})\right)$$

$$= \epsilon\left(\frac{1}{2}, Ind_E^F(Sp(nd) \otimes \chi_E^{-1})\right)$$

where $\chi_E = \chi \circ N_{E/F} = \mu$

$$= \omega_{E/F}(-1)^{\frac{nd}{2}} \epsilon\left(\frac{1}{2}, Sp(nd) \otimes \chi_E^{-1}\right)$$

$$= \omega_{E/F}(-1)^{\frac{nd}{2}} \epsilon\left(\frac{1}{2}, BC_E(St(n,\chi^{-1}))\right)$$

so the 2nd point of the conjecture is equivalent to: $\epsilon\left(\frac{1}{2}, BC_E(St(n,\chi^{-1}))\right) = (-1)^n$.

To sum up, under the additional hypothesis that the character $\mu \circ N_{r_d,E}$ of $H$ can be extended to a character of $G$, Theorem 4.2 is equivalent to the following, which is a reformulation similar to Conjecture 1.4 of [FMW17]:

**Theorem 4.3.** Let $St(n,\chi)$ be the Steinberg representation of $G = GL(n,D)$. $St(n,\chi)$ is $H$-distinguished if and only if it is symplectic and $\epsilon\left(\frac{1}{2}, BC_E(St(n,\chi))\right) = (-1)^n$.

**Remark 4.1.** In [FMW17], Conjecture 1.4 (a reformulation of Conjecture 1 of Prasad and Takloo-Bighash in [PTBH11]) is stated for general representations but only in the quaternionic case. It is checked for supercuspidal representations with extra conditions.

**Proof of Theorem 4.2.** We use the usual $\epsilon, \gamma$ and $L$-factors as defined in Godement-Jacquet (see [GJ72]); we omit the third parameter in $\epsilon$ and $\gamma$ which is a non-trivial additive character of $E$ but trivial on $F$.

According to the preceding reformulation (the three points above), we have to prove that $St(n,1)$ is $\mu$-distinguished if and only if $\mu_{|F^*} = 1$ and $\epsilon\left(\frac{1}{2}, Sp(nd) \otimes \mu^{-1}\right) = (-1)^n$ that is to say $\epsilon\left(\frac{1}{2}, St(nd,\mu^{-1})\right) = (-1)^n$ where $St(nd,\mu^{-1})$ is the Steinberg representation of $GL(nd,E)$ with parameter $\mu^{-1}$. 

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For $s \in \mathbb{C}$,
\[
\gamma(s, St(nd, \mu^{-1})) = (\mu^{-nd}(-1))^n d^{-1} \epsilon(s, St(nd, \mu^{-1})) \frac{L(1-s, St(nd, \mu^{-1}))}{L(s, St(nd, \mu^{-1}))}.
\]
On the other hand,
\[
\gamma(s, St(nd, \mu^{-1})) = \gamma(s + \frac{1 - nd}{2}, \mu^{-1}) \times \gamma(s + \frac{1 - nd}{2} + 1, \mu^{-1}) \times \cdots \times \gamma(s + \frac{nd - 1}{2}, \mu^{-1}).
\]

If $\mu_{|F^*} = 1$, then $\tilde{\mu} = \bar{\mu}$ (where $\bar{\mu}$ denotes the Galois twist of $\mu$) so
\[
L(s, \mu^{-1}) = L(s, \bar{\mu}^{-1}) \quad \text{and} \quad \epsilon(s, \mu^{-1}) \epsilon(1-s, \mu^{-1}) = 1 \quad \forall s \in \mathbb{C}.
\]
Hence, if $\mu_{|F^*} = 1$, $\gamma(s, \mu^{-1}) = \epsilon(s, \mu^{-1}) \frac{L(1-s, \mu^{-1})}{L(s, \mu^{-1})}$, $\forall s \in \mathbb{C}$, so $\gamma(s, St(nd, \mu^{-1}))$ equals
\[
\epsilon(s + \frac{1 - nd}{2}, \mu^{-1}) \times \cdots \times \epsilon(s + \frac{nd - 1}{2}, \mu^{-1}) \times \frac{L(1-s, \mu^{-1})}{L(s, \mu^{-1})} \cdots L(1-s, \mu^{-1}) = \frac{L(1-s, \mu^{-1})}{L(s, \mu^{-1})} \times \cdots \times L(1-s, \mu^{-1}).
\]
Now for $\mu$ a character of $E^*$ such that $\mu_{|F^*} = 1$ and $s$ a real number, we need to know when $L(s, \mu^{-1})$ has a pole. Let $\varpi_E$ be a uniformizer of $E$ and $q_E$ denote the cardinality of the residue field of $E$.

If $\mu$ is non-ramified, $L(s, \mu^{-1}) = \frac{1}{1 - \mu^{-1}(\varpi_E)} q_E$ and $L(s, \bar{\mu}^{-1}) = L(s, \mu^{-1}) = \frac{1}{1 - \mu(\varpi_E)} q_E$ so $L(s, \mu^{-1})$ has a pole equivalent to:
\[
\mu^{-1}(\varpi_E) = q_E = \mu(\varpi_E) \Leftrightarrow s = 0 \quad \text{and} \quad \mu(\varpi_E) = 1 \quad \text{i.e.} \quad \mu = 1
\]
because $\mu$ is non-ramified.

If $\mu$ is ramified, $L(s, \mu^{-1}) = 1$ so $L(s, \mu^{-1})$ has no pole.
To conclude, $L(s, \mu^{-1})$ has a pole if and only if $s = 0$ and $\mu = 1$.

Finally, if $\mu_{|F^*} = 1$, $\gamma(\frac{1}{2}, St(nd, \mu^{-1})) = \epsilon(\frac{1}{2}, St(nd, \mu^{-1}))$ and also $\gamma(\frac{1}{2}, St(nd, \mu^{-1})) = \frac{L(\frac{nd}{2}, \mu^{-1})}{L(1-s, \mu^{-1})} \times \cdots \times \frac{L(1-s, \mu^{-1})}{L(s, \mu^{-1})}$ so:

- $(\mu_{|F^*} = 1$ and $\mu \neq 1) \Leftrightarrow (\epsilon(\frac{1}{2}, St(nd, \mu^{-1})) = 1$ and $\mu_{|F^*} = 1$).
- $(\mu = 1) \Leftrightarrow (\epsilon(\frac{1}{2}, St(nd, \mu^{-1})) = \lim_{s \to \frac{1}{2}} \frac{L(s, \mu^{-1})}{L(1-s, \mu^{-1})} = \lim_{s \to 0} \frac{1-q_E}{s} = 1$ and $\mu_{|F^*} = 1$).

This is equivalent to $St(n, 1)$ is $\tilde{\mu}$-distinguished if and only if $\mu_{|F^*} = 1$ and $\epsilon(\frac{1}{2}, St(nd, \mu^{-1})) = (-1)^n$.

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