CLASSIFICATION OF POSITIVE SOLUTIONS FOR FULLY NONLINEAR ELLIPTIC EQUATIONS IN UNBOUNDED CYLINDERS

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ABSTRACT. In this paper, we consider the positive viscosity solutions for certain fully nonlinear uniformly elliptic equations in unbounded cylinder with zero boundary condition. After establishing an Aleksandrov-Bakelman-Pucci maximum principle, we classify all positive solutions as three categories in unbounded cylinder. Two special solution spaces (exponential growth at one end and exponential decay at the another) are one dimensional, independently, while solutions in the third solution space can be controlled by the solutions in the other two special solution spaces under some conditions, respectively.

1. Introduction. The theory of viscosity solutions gives a solid framework to study fully nonlinear elliptic equations, and we refer the readers interested in it to [5]. With the help of the theory, we would like to study certain fully nonlinear uniformly elliptic equations in unbounded cylinders. As we know, the structure of positive solutions to linear elliptic equations has been studied extensively. Martin [14] gave a method for uniquely representing any positive harmonic function in an arbitrary domain in $\mathbb{R}^3$ by an integral on the minimal Martin boundary. His results have been extended to second order elliptic operators with a zero potential by Shur [15]. In 1985, Landis and Nadirashvili [13] considered the Dirichlet problem for linear uniformly elliptic equations

$$a_{ij}(x)D_{ij}u(x) = 0$$

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or

\[ D_i(a_{ij}(x)D_ju(x)) = 0 \]

in cones. They proved that the space of the positive solutions in a cone with zero boundary value is one-dimensional. These results have been extended by Jun Bao and some of the authors [4] in unbounded cylinders. They showed that there are two special solutions with exponential growth at one end while exponential decay at the other and all the positive solutions are linear combinations of these two. Related work, based on sectors, cones, cylinders or second order elliptic equation with lower order terms and with inhomogeneous term, can be found in [1, 10, 11, 18].

In this paper, we would like to consider the structure of the homogeneous fully nonlinear equations. The main difficulty lies in the nonlinearity of elliptic equations, which prevents us from using the arguments which are valid for linear elliptic equations directly. To overcome this defect, some preparations need to be done. For example, the homogeneous fully nonlinear equations should satisfy positive homogeneity, and the maximum principle in unbounded cylinders should be established under some conditions. As we all know that, in 1995, Caffarelli and Cabrè in [5] proved the classical Alexandrov-Bakelman-Pucci (ABP) maximum principle for viscosity solutions in bounded domains. We also know that Capuzzo-Dolcetta together with Leoni and Vitolo [9] established the validity of the weak maximum principle for solutions in unbounded domains under some geometric conditions. However, these ABP maximum principles in bounded domains or in unbounded domains are far from satisfying our expectations since what we considered is on unbounded cylinders. Therefore, we would like to extend firstly the above ABP maximum principle to unbounded cylinders. We use a new method to obtain ABP maximum principle in unbounded cylinders just only for solutions which are bounded from above. The new method is mainly based on the classical ABP maximum principle, weak Harnack inequality and iterations. From our perspective, the ABP maximum principle in unbounded cylinders is the keystone for our theory as well as a great contribution of this paper.

If we say ABP maximum principle in unbounded cylinders is a basic tool in our paper, so are the boundary Harnack inequalities. The boundary Harnack inequalities are very important and useful when studying the boundary behavior or free boundary problems for linear elliptic equations, or for homogeneous fully nonlinear equations. And they have been proved in various settings, see for instance [3, 7]. In our paper, the boundary Harnack inequalities are essential for proving the asymptotic behaviors of positive solutions or comparing positive solutions. The proofs can be seen in [3]. Besides, we show the boundary Hölder estimate through the extension up to the boundary of the weak Harnack inequality in [5], which differs from the approach in [5].

Following the original strategy of Jun Bao and some of the authors [4] as well as the two fundamental tools, this paper deals with the following fully nonlinear uniformly elliptic equations

\[
\begin{aligned}
F(D^2u(x)) &= 0, \quad x \in \mathcal{C}, \\
 u(x) &= 0, \quad x \in \partial \mathcal{C}, \\
 u(x) &> 0, \quad x \in \mathcal{C},
\end{aligned}
\]  

(1.1)

in unbounded cylinders \( \mathcal{C} = \mathcal{D} \times \mathbb{R} \subseteq \mathbb{R}^n \), where \( \mathcal{D} \) is a bounded Lipschitz domain in \( \mathbb{R}^{n-1} \) with \( n \geq 2 \) and is fixed throughout this paper. \( F(M) \) is a real valued function defined on \( S^{n \times n} \), where \( S^{n \times n} \) is the space of real \( n \times n \) symmetric matrices.
We always assume that $F$ is uniformly elliptic in $S_{\lambda}^{n \times n}$ if there exist constants $\lambda$ and $\Lambda$ such that $0 < \lambda \leq \Lambda$ and

$$\lambda \|N\| \leq F(M + N) - F(M) \leq \Lambda \|N\|$$

(1.2)

for any $N \geq 0$, which means $N$ is a semi-positive definite matrix.

In order to classify the solutions of (1.1) and analyze the solutions' structure, we also assume that

$F$ is smooth except at the origin, concave, $F(0) = 0$. (1.3)

In addition, we assume that $F$ satisfies positive homogeneity

$F(\eta M) = \eta F(M)$, for all $\eta \geq 0$. (1.4)

To state our main results, we give some notations for the reader's convenience. Let $x = (x_1, \ldots, x_{n-1}, y) = (x', y)$ be a typical point of $\mathbb{R}^{n-1} \times \mathbb{R}$. For $E \subset \mathbb{R}$, we denote $C_E := D \times E = \{(x', y) \in \mathbb{R}^n \mid x' \in D, y \in E\}$ and $\partial C_E := \partial D \times E = \{(x', y) \in \mathbb{R}^n \mid x' \in \partial D, y \in E\}$. For any $y \in \mathbb{R}$, we write $C_y := C_{(y)}$, $C_y^+ := C_{(y, +\infty)}$ and $C_y^- := C_{(-\infty, y)}$. For simplicity, we denote that $C^+ := C_0^+$, $C^- := C_0^-$. In addition, we denote $U$ as the set of positive solutions to problem (1.1). For any $u \in U$, let $\bar{u}(y) := \sup_{x' \in D} u(x', y)$ and $m(u) := \inf_{y \in \mathbb{R}} \bar{u}(y)$. Write $U^+ := \{u \in U \mid \lim_{y \to +\infty} u(x', y) = 0\}$, $U^- := \{u \in U \mid \lim_{y \to -\infty} u(x', y) = 0\}$, and $U^\diamond := \{u \in U \mid$ there exists a point $x^* = (x'^*, y^*) \in \mathcal{C}$ such that $u(x^*) = m(u) > 0\}$. We also assume, without loss of generality, $0' \in D$.

Finally, in this paper, we assume that $f \in C(\mathcal{C})$ and $\|f\|_{L^2(\mathcal{C})} = \sup_{y \in \mathbb{R}} \|f\|_{L^2(C_{(y, +\infty)})} < +\infty$. Then, one of our main results is ABP maximum principle in unbounded cylinders for solutions in $S(\lambda, \Lambda, f)$ (the definition of $S(\lambda, \Lambda, f)$ can be found in the section 2), which is one of the most important fundamental tools in our theory.

**Theorem 1.1.** [Aleksandrov-Bakelman-Pucci maximum principle] Assume $u(x) \in S(\lambda, \Lambda, f)$ for $x \in \mathcal{C}$, and $u(x)$ is bounded from above. Then we have

$$\sup_{x \in \mathcal{C}} u^+(x) \leq \sup_{x \in \partial \mathcal{C}} u^+(x) + C \|f\|_{L^2(\mathcal{C})},$$

where $C$ only depends on $n, \lambda, \Lambda$ and $\text{diam}(D)$.

**Corollary 1.** Assume $u(x) \in S(\lambda, \Lambda, f)$ for $x \in \mathcal{C}^+$, and $u(x)$ is bounded from above. Then we have

$$\sup_{x \in \mathcal{C}^+} u(x) \leq \sup_{x \in \partial \mathcal{C}^+} u^+(x) + C \|f\|_{L^2(\mathcal{C}^+)},$$

where $C$ only depends on $n, \lambda, \Lambda$ and $\text{diam}(D)$.

**Theorem 1.2.** Assume that $F$ satisfies (1.2), (1.3). Then there exists a unique bounded solution $u \in C(\mathcal{C}) \cap W^{2, q}_{\text{loc}}(\mathcal{C})$ for some $q_0 > 0$ and $q > q_0$ to the Dirichlet problem

$$\begin{cases}
F(D^2 u(x)) = f(x), & x \in \mathcal{C}, \\
u(x) = 0, & x \in \partial \mathcal{C}.
\end{cases}$$

The next main result is about the exponential decay properties of bounded solutions in $\mathcal{C}^+$ (or in $\mathcal{C}^-$).
Theorem 1.3. Let $F$ satisfy (1.2) and $F(0) = 0$. Assume $u(x)$ is bounded from above and satisfies
\[
\begin{cases}
F(D^2u(x)) \geq f(x), & x \in C^+,
\vspace{1em}
u(x) = 0, & x \in \partial D \times (0, +\infty).
\end{cases}
\]
Then there exist positive constants $\alpha, C_0$ and $C_1$ depending only on $n, \lambda, \Lambda$ and $\text{diam}(D)$, such that
\[u(x) \leq C_1\tilde{u}(0)e^{-\alpha y} + C_1\|f\|_{L^2_C(C^+)}, \quad x \in C^+.
\]

Theorem 1.4. Let $F$ satisfy (1.2),(1.3),(1.4). Then for the problem (1.1), there exist positive constants $\alpha, \beta, C$ and $C'$, such that for any $u \in U^+, v \in U^-$ and $w \in U^\vee$ we have
\[
\begin{align*}
-C - \alpha y &\leq \ln\left(\frac{\tilde{u}(y)}{\tilde{u}(0)}\right) \leq C' - \beta y, \quad y \in (-\infty, +\infty), \\
-C' + \beta y &\leq \ln\left(\frac{\tilde{v}(y)}{\tilde{v}(0)}\right) \leq C + \alpha y, \quad y \in (-\infty, +\infty), \\
-C + \alpha |y - y^*| &\leq \ln\left(\frac{\tilde{w}(y)}{\tilde{w}(y^*)}\right) \leq C' + \beta |y - y^*|, \quad y \in (-\infty, +\infty).
\end{align*}
\]
In the third conclusion, we use an assumption that $w(x^* = m(w))$ for $x^* = (x^*, y^*) \in C$.

Finally, we pursue further the structure of solutions to (1.1).

Theorem 1.5. Let $F$ satisfy (1.2),(1.3),(1.4). Assume that $U^+$ and $U^-$ are solutions spaces of the problem (1.1) which are defined as above. Then $U^+$ and $U^-$ are nonempty sets. Furthermore, $U^+$ and $U^-$ are one dimensional, i.e. we can take $u \in U^+$ and $v \in U^-$ such that
\[U^+ = \{au \mid a > 0\} \quad \text{and} \quad U^- = \{bv \mid b > 0\}.
\]

Theorem 1.6. Let $F$ satisfy (1.2),(1.3),(1.4). Assume that $U^\vee$ and $U^+$ are the defined solutions spaces for the problem (1.1) in $C_{(-\tilde{y}, \tilde{y})}$ with $\tilde{y} \leq 0$. Then, for any $u \in U^\vee$ and $v \in U^+$, we have
\[u(x) = (\alpha^* + o(1))u(x), \quad x \to -\infty,
\]
where $\alpha^*$ is a positive constant.

Similarly, we have the following Theorem.

Theorem 1.7. Let $F$ satisfy (1.2),(1.3),(1.4). Assume that $U^\vee$ and $U^-$ are the defined solutions spaces of the problem (1.1) in $C_{[\tilde{y}, +\infty)}$ with $\tilde{y} \geq 0$. Then, for any $u \in U^\vee$, $v \in U^-$, we have
\[u(x) = (\beta^* + o(1))u(x), \quad x \to +\infty,
\]
where $\beta^*$ is a positive constant.

Our paper is organized as follows. In Section 2, we recall some basic results for viscosity solutions of fully nonlinear uniformly elliptic equations. In Section 3, we extend weak Harnack inequality onto the boundary, and then get the boundary H"older estimates and the global H"older estimates. In Section 4, we mainly prove ABP maximum principle in unbounded cylinders. In section 5, we prove the existence and uniqueness of bounded solutions and the exponential decay of bounded solutions. The last section, we prove two structure theorems.
2. Preliminary results. In this section, we recall some basic properties of continuous viscosity solutions of fully nonlinear second order uniformly elliptic equations of the form

\[ F(D^2u(x), x) = f(x), \quad x \in \Omega, \quad (2.1) \]

where \( \Omega \) is an open domain in \( \mathbb{R}^n \). The usual references for these properties are [5, 8].

\( F(M, x) \) is a real valued function defined on \( S^{n \times n} \times \Omega \), where \( S^{n \times n} \) is the space of real \( n \times n \) symmetric matrices. What’s more, \( F \) and \( f \) are continuous in \( x \in \Omega \).

We say \( F \) is uniformly elliptic in \( S^{n \times n} \times \Omega \) if there exist constants \( \lambda \) and \( \Lambda \) such that \( 0 < \lambda \leq \Lambda \) and

\[ \lambda \|N\| \leq F(M + N, x) - F(M, x) \leq \Lambda \|N\|, \]

where \( N \) is semi-positive definite symmetric matrix, and \( \|N\| = \sup_{|x|=1} |Nx| \). Therefore, \( F(M, x) \) is monotone increasing and Lipschitz in \( M \in S^{n \times n} \), and \( \|N\| \) is equal to the maximum eigenvalue of \( N \) whenever \( N \geq 0 \). Next we give the definition of viscosity solution of (2.1).

**Definition 2.1.** A continuous function \( u \) in \( \Omega \) is a viscosity subsolution of (2.1) in \( \Omega \), if for any \( x_0 \in \Omega \) and any function \( \phi \in C^2(\Omega) \) such that \( u - \phi \) has a local maximum at \( x_0 \) there holds

\[ F(D^2\phi(x_0), x_0) \geq f(x_0). \]

Symmetrically, a continuous function \( u \) in \( \Omega \) is a viscosity supersolution of (2.1) in \( \Omega \), if for any \( x_0 \in \Omega \) and any function \( \phi \in C^2(\Omega) \) such that \( u - \phi \) has a local minimum at \( x_0 \) there holds

\[ F(D^2\phi(x_0), x_0) \leq f(x_0). \]

We say a continuous function \( u \) in \( \Omega \) is a viscosity solution of (2.1) in \( \Omega \) if it is a viscosity subsolution and a supersolution of (2.1) in \( \Omega \).

**Remark 1.** Noting that if \( u \) is a viscosity supersolution of \( F(D^2u(x), x) = f(x) \) in \( \Omega \), then \( v = -u \) is a viscosity subsolution of \( G(D^2v(x), x) = -f(x) \) in \( \Omega \), where

\[ G(M, x) = -F(-M, x), \quad M \in S^{n \times n}, \]

and \( G \) is again uniformly elliptic.

In the class of uniformly elliptic operators, there are two extremal ones, well known as Pucci maximum and minimal operators, respectively

\[ \mathcal{M}^+(M, \lambda, \Lambda) = \mathcal{M}^+(M) = \lambda \sum_{\varepsilon_i > 0} e_i + \lambda \sum_{\varepsilon_i < 0} e_i, \]

\[ \mathcal{M}^-(M, \lambda, \Lambda) = \mathcal{M}^-(M) = \lambda \sum_{\varepsilon_i > 0} e_i + \Lambda \sum_{\varepsilon_i < 0} e_i, \]

where \( e_i \) are eigenvalues of \( M \in S^{n \times n} \), and \( 0 < \lambda \leq \Lambda \) are constants. It is easy to see

\[ \mathcal{M}^- = -\mathcal{M}^+ \]

from the above definitions. We refer to [5] for other basic properties of the Pucci operators.

In the following, we give the definition of the class \( S \). The usefulness of the class \( S \) is in avoiding linearization.
Definition 2.2. Let \( f \in C(\Omega) \) and \( \lambda \leq \Lambda \) two positive constants.

We denote \( S(\lambda, \Lambda, f) \) the space of continuous functions \( u \) in \( \Omega \) such that
\[
\mathcal{M}^+(D^2 u, \lambda, \Lambda) \geq f(x), \quad x \in \Omega
\]
in the viscosity sense.

Similarly, we denote \( \overline{S}(\lambda, \Lambda, f) \) the space of continuous functions \( u \) in \( \Omega \) such that
\[
\mathcal{M}^-(D^2 u, \lambda, \Lambda) \leq f(x), \quad x \in \Omega
\]
in the viscosity sense.

We set
\[
S(\lambda, \Lambda, f) = S(\lambda, \Lambda, f) \cap \overline{S}(\lambda, \Lambda, f),
\]
and call the functions in \( S(\lambda, \Lambda, f), \overline{S}(\lambda, \Lambda, f), S(\lambda, \Lambda, f) \) subsolutions, supersolutions, solutions, respectively.

Here we state two propositions which will be used in the sequel.

Proposition 1. Let \( u \) satisfy \( F(D^2 u(x), x) \geq f(x) \) in the viscosity sense in \( \Omega \). Then we have
\[
u \in S\left(\frac{\lambda}{n}, \Lambda, f(x) - F(0, x)\right).
\]
More generally, for any \( \phi \in C^2(\Omega) \), we have
\[
u - \phi \in S\left(\frac{\lambda}{n}, \Lambda, f(x) - F(D^2 \phi(x), x)\right).
\]

Proposition 2. Let \( u \) be a viscosity subsolution of \( F(D^2 w) = 0 \) in \( \Omega \) and \( v \) be a viscosity supersolution of \( F(D^2 w) = 0 \) in \( \Omega \). Then
\[
u - v \in S\left(\frac{\lambda}{n}, \Lambda, 0\right), \quad x \in \Omega.
\]

3. The boundary Hölder estimates. Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded domain and satisfy a uniformly exterior cone condition. We will prove the boundary Hölder estimate for solutions \( u \in S(\lambda, \Lambda, f) \). When \( f = 0 \), the boundary Hölder estimate in [5] is proved by constructing a barrier function. In this paper, the main approach follows the lines of [3]. The basic tool is the so called weak Harnack inequality (Theorem 4.8) in [5]. To be specific, we will extend the weak Harnack inequality to the boundary, that is, to balls intersecting the boundary of the domain \( \Omega \subseteq \mathbb{R}^n \), where the solutions are defined. For this purpose, we will need suitable extensions of such solutions outside \( \Omega \). Precisely, take concentric balls \( B_R \subseteq B_{R/\theta} \), where \( \theta \in (0,1) \), such that
\[
B_R \cap \Omega \neq \emptyset, \quad B_{R/\theta} \setminus \Omega \neq \emptyset.
\]

For \( u \in C(\bar{\Omega}) \) and \( u \geq 0 \), we consider the following continuous extension \( u^- \) of function \( u \)
\[
u^-(x) = \begin{cases} \min\{u(x), m\}, & \text{if } x \in \Omega, \\ m, & \text{if } x \notin \Omega, \end{cases}
\]
where \( m = \inf_{\partial \Omega \cap B_{R/\theta}} u(x) \).

Lemma 3.1. (Boundary Weak Harnack Inequality) Let \( u(x) \in \bar{S}(\lambda, \Lambda, f) \) and \( u \geq 0 \) in \( \Omega \), where \( f \in C(\bar{\Omega}) \), then we have
\[
\left( \frac{1}{|B_R|} \int_{B_R} (u^-)^{p_0})^{1/p_0} \right) \leq C \left( \inf_{\partial \Omega \cap B_R} u + R ||f^+||_{L^n(\Omega \cap B_{R/\theta})} \right),
\]
where \( p_0 \) and \( C \) depend both on \( n, \lambda, \Lambda \) and \( \theta \).
Proof. For any $\varepsilon > 0$, set

$$m_\varepsilon = \inf_{L^2(\overline{\Omega})} u,$$

where $I_\varepsilon(\partial \Omega) = \{ x \in B_{R/\theta} \cap \overline{\Omega} : \text{dist}(x, \partial \Omega) \leq \varepsilon \}$. Moreover, for $x \in B_{R/\theta}$, define

$$u_{m_\varepsilon}^-(x) = \begin{cases} \min\{u(x), m_\varepsilon\}, & \text{if } x \in \Omega, \\ m_\varepsilon, & \text{if } x \notin \Omega, \end{cases}$$

and

$$f_\varepsilon(x) = f^+(x)\rho\left(\frac{\text{dist}(x, \mathbb{R}^n \setminus \Omega)}{\varepsilon}\right),$$

where

$$\rho(t) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ 1, & \text{if } t \geq 1. \end{cases}$$

It is easy to check that $u_{m_\varepsilon}^- \in C(B_{R/\theta})$, $u_{m_\varepsilon}^- \geq 0$, $f_\varepsilon \in C(\overline{B}_{R/\theta})$ and

$$M^-\left(\mathcal{D}^2u_{m_\varepsilon}\right) \leq f_\varepsilon(x), \quad x \in B_{R/\theta}. $$

Therefore $u_{m_\varepsilon}^- \in \overline{S}(\lambda, \Lambda, f_\varepsilon)$. By applying the scaled weak Harnack inequality (Theorem 4.8) in [5] with $u_{m_\varepsilon}$ and $f_\varepsilon$ to get

$$\left(\frac{1}{|B_R|} \int_{B_R} (u_{m_\varepsilon}^-)^{p_0}\right)^{1/p_0} \leq C\left(\inf_{B_R} u_{m_\varepsilon}^- + R\|f_\varepsilon\|_{L^\infty(\overline{B}_{R/\theta})}\right).$$

Noticing that $\inf_{B_R} u_{m_\varepsilon}^- \leq \inf_{\Omega \cap B_R} u$, $0 \leq f_\varepsilon \leq f^+$ in $\Omega$, and $f_\varepsilon = 0$ outside $\Omega$, we have $\lim_{\varepsilon \to 0} m_\varepsilon = m$. Then by Fatou’s lemma, we obtain

$$\int_{B_R} (u_{m_\varepsilon}^-)^{p_0} \leq \liminf_{\varepsilon \to 0} \int_{B_R} (u_{m_\varepsilon}^-)^{p_0}.$$

Thus we obtain the desired result. \qed

Lemma 3.2. Let $f(x) \in C(\overline{\Omega})$ and $u(x) \in S(\lambda, \Lambda, f)$ in $\Omega$. Then there exists $\rho_0 > 0$ such that for any ball $B_\rho$ centered at $\overline{\Omega}$ with $\rho \leq \rho_0$, we have

$$\text{osc}_{\Omega \cap B_\rho} u \leq C\left(\rho^\alpha(\rho_0^{-\alpha}) \sup_{\Omega \cap B_{\rho_0}} |u| + \|f\|_{L^\infty(\overline{B}_{\rho_0})} + \delta(\rho_0)\right),$$

where $\delta(\rho) = \text{osc}_{\partial B_{\rho} \cap \overline{\Omega}} u$, $C$ and $\alpha$ both depend on $n, \lambda, \Lambda$ and $\rho_0$.

Proof. Let $x_0 \in \partial \Omega$. By the uniform cone condition, for some $\rho_1 > 0$ and some $\xi > 0$, the balls $B$ with center at $x_0$ and radii $2\rho(\rho \leq \rho_1)$ satisfy $|B \setminus \Omega| \geq |\xi B|$. We may assume without loss of generality that $\rho \leq \inf\{\frac{\rho_0}{2}, \rho_1\}$. Write $M_0 = \sup_{\Omega \cap B_{\rho_0}(x_0)} |u|$

$$M_4 = \sup_{\Omega \cap B_{\rho_0}(x_0)} u, \quad M_4 = \inf_{\Omega \cap B_{\rho_0}(x_0)} u, \quad M_1 = \sup_{\Omega \cap B_{\rho_0}(x_0)} u, \quad M_1 = \inf_{\Omega \cap B_{\rho_0}(x_0)} u.$$

It is easy to check that $(u - M_4) \in S(\lambda, \Lambda, f)$ and $(M_4 - u) \in S(\lambda, \Lambda, f)$ in $B_{4\rho}(x_0)$. Moreover, $(u - M_4) \geq 0$ and $(M_4 - u) \geq 0$ in $B_{4\rho}(x_0)$.

We can apply the Lemma 3.1 to each of the functions $u - M_4$ and $M_4 - u$ in $B_{4\rho}(x_0)$. Hence we obtain

$$\left(\frac{m - M_4}{|B_{2\rho}(x_0) \setminus \Omega|}\right)^{1/p_0} \leq C\left(\frac{1}{|B_{2\rho}(x_0)|} \int_{B_{2\rho}(x_0)} ((u - M_4)^{m - M_4})^{p_0}\right)^{1/p_0} \leq C(m_1 - m_4 + \rho\|f\|_{L^\infty(\overline{B_{4\rho}(x_0)})})$$
cone condition, we have
prove the maximum principle in unbounded cylinders
Maximum principle in unbounded cylinders.
4. Assume that $k$, that the diameter of $C$
Lemma 4.1.
Then the desired result follows from Lemma 8.23 in [12].
Proof. For the existence of solutions for the Dirichlet problem (4.1), one can see Corollary 3.10 in [6].
On one hand, by the definition of \( v(x) \), we have

\[
\begin{align*}
(u - v)(x) & \in S(\lambda/n, \Lambda, 0), \\
(u - v)(x) & \leq 0,
\end{align*}
\]

where \( v \in C(1, n, 0) \) is a universal constant. Therefore, for \( x \in \mathcal{C}(k, k + 2) \), we obtain

\[
\|v\|_{L^2(C(1, n, 0))} \leq 1 + C_\varepsilon_0, \quad x \in \mathcal{C}(k, k + 2),
\]

where \( C \) depends only on \( n, \lambda, \Lambda \) and \( \text{diam}(\mathcal{D}) \).

Hence, we have

\[
u(x) \leq u(x) \leq 1 + C_\varepsilon_0, \quad x \in \mathcal{C}(k, k + 2).
\]

On the other hand, since \( v \in S(\lambda, \Lambda, f) \), by ABP maximum principle (Theorem 3.6) in [5], we have

\[
v(x) \leq 1 + C\|f\|_{L^2(C(1, k + 2))} \leq 1 + C_\varepsilon_0, \quad x \in \mathcal{C}(k, k + 2),
\]

where \( C \) depends only on \( n, \lambda, \Lambda \) and \( \text{diam}(\mathcal{D}) \).

Now take \( u(x) \leq v(x) \leq 1 + C_\varepsilon_0 \), and

\[
\mathcal{M}^- (D^2 (1 + C_\varepsilon_0 - v(x))) = \mathcal{M}^- (-D^2 v(x)) = -\mathcal{M}^+ (D^2 v(x)) = -f(x)
\]

in \( \mathcal{C}(k, k + 2) \). Hence \( 1 + C_\varepsilon_0 - v \in S(\lambda, \Lambda, -f) \) in \( \mathcal{C}(k, k + 2) \). We apply the weak Harnack inequality (Theorem 4.8) in [5] to \( (1 + C_\varepsilon_0 - v) \) in \( \mathcal{C}(k, k + 2) \), and we obtain for some \( \eta > 0 \)

\[
C_1 \leq \frac{1}{C''(k, k + 2)} \int_{C''(k, k + 2)} (1 + C_\varepsilon_0 - v)^\eta \frac{1}{\delta} \leq C \left\{ \inf_{C''(k, k + 2)} (1 + C_\varepsilon_0 - v) + \|f\|_{L^2(C(1, n, 0))} \right\} \leq C \left\{ \inf_{C''(k, k + 2)} (1 + C_\varepsilon_0 - v + \varepsilon_0) \right\},
\]

where \( C''(k, k + 2) \) is a compact subset of \( C'(k, k + 2) \) such that \( v \leq \frac{3}{4} \) in \( C''(k, k + 2) \), and \( C_1 \) is a universal constant. Therefore, for \( x \) with \( \text{dist}(x, \partial \mathcal{C}(k, k + 2)) \geq \sigma_0 \), we have

\[
1 - v(x', k + 1) \geq \frac{C_1}{C} - (1 + C)\varepsilon_0 \geq \frac{C_1}{2C} > 0
\]

by taking \( \varepsilon_0 \leq \frac{C_1}{2C(1 + C)} \). Noting that \( 1 - v(x', k + 1) \geq \frac{1}{2} \) for the rest of the points near the boundary, then the result follows by setting \( \delta = \min \left( \frac{C_1}{2C}, \frac{1}{2} \right) \).

Now we can prove the ABP maximum principle in unbounded cylinders \( \mathcal{C} \).

**Proof of Theorem 1.1:** We assume \( \sup_{x \in \partial \mathcal{C}} u^+(x) = 0 \). Otherwise, we can consider \( v(x) = u(x) - \sup_{x \in \partial \mathcal{C}} u^+(x) \). Therefore, we only need to prove

\[
u^+(x) \leq C\|f\|_{L^2(C)}, \quad x \in \mathcal{C}.
\]
Suppose $u(x) \leq M$, since $u$ is bounded from above. Here $M$ could be very large. Also let $\|f\|_{L^n(\mathcal{C})} = F_0$. Set $\mathcal{C}_{(k-1,k+1)} = \{x = (x', y) \in \mathbb{R}^n \mid x' \in D, k - 1 < y < k + 1\}$, $k \in \mathbb{Z}$. In order to apply Lemma 4.1, we consider the function

$$\hat{u}(x) = \frac{\varepsilon_0 u(x)}{\varepsilon_0 \max\{\hat{u}(k-1), \hat{u}(k+1)\} + \|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}}, \quad x \in \mathcal{C}.$$ 

Since $\varepsilon_0 \max\{\hat{u}(k-1), \hat{u}(k+1)\} + \|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}$ maybe equal to zero, we can assume $f$ is not identically zero. If not, we can choose an arbitrary $\varepsilon > 0$ such that $\varepsilon_0 \max\{\hat{u}(k-1), \hat{u}(k+1)\} + \|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})} + \varepsilon > 0$ and eventually let $\varepsilon$ tend to 0.

Then, for any $x \in \mathcal{C}_{(k-1,k+1)}$, $\hat{u}(x)$ satisfies

$$\begin{cases} 
\hat{u}(x) \in S(\lambda, \Lambda, \hat{f}), & x \in \mathcal{C}_{(k-1,k+1)}; \\
\hat{u}(x) \leq 0, & x \in \partial_b \mathcal{C}_{(k-1,k+1)}; \\
\hat{u}(x) \leq 1, & x \in \partial \mathcal{C}_{(k-1,k+1)},
\end{cases}$$

where

$$\hat{f}(x) = \frac{\varepsilon_0 f(x)}{\varepsilon_0 \max\{\hat{u}(k-1), \hat{u}(k+1)\} + \|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}}.$$ 

Now we can apply Lemma 4.1 to $\hat{u}(x)$ in $\mathcal{C}_{(k-1,k+1)}$. It follows that there exists a constant $\delta \in (0, 1)$ such that $\hat{u}(x', k) \leq (1 - \delta)$ for $x' \in D$. That is

$$u(x', k) \leq (1 - \delta)\frac{(1 - \delta)}{\varepsilon_0}\{\varepsilon_0 \max\{\hat{u}(k-1), \hat{u}(k+1)\} + \|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\}$$

$$= (1 - \delta)\{\max\{\hat{u}(k-1), \hat{u}(k+1)\} + \frac{(1 - \delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\}$$

$$\leq (1 - \delta)M + \frac{(1 - \delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}.$$

By the definition of $\hat{u}(y)$, we have

$$\hat{u}(k) \leq (1 - \delta)\{\max\{\hat{u}(k-1), \hat{u}(k+1)\} + \frac{(1 - \delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}\}$$

$$\leq (1 - \delta)M + \frac{(1 - \delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-1,k+1)})}.$$ 

Then, by induction, we have for any $k \in \mathbb{Z},$

$$\hat{u}(k) \leq (1 - \delta)\max\{(1 - \delta)\max\{\hat{u}(k-2), \hat{u}(k)\} + \frac{(1 - \delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k-2,k)})},$$

$$\quad (1 - \delta)\max\{\hat{u}(k), \hat{u}(k+2)\} + \frac{(1 - \delta)}{\varepsilon_0}\|f\|_{L^n(\mathcal{C}_{(k,k+2)})}\} + \frac{(1 - \delta)}{\varepsilon_0}F_0$$

$$\leq (1 - \delta)^2\max\{\max\{\hat{u}(k-2), \hat{u}(k)\}, \max\{\hat{u}(k), \hat{u}(k+2)\}\} + \frac{(1 - \delta)^2}{\varepsilon_0}F_0$$

$$+ \frac{(1 - \delta)}{\varepsilon_0}F_0$$

$$\vdots$$

$$\leq (1 - \delta)^m M + \frac{F_0}{\varepsilon_0} \sum_{i=1}^{m} (1 - \delta)^i.$$
where $u$ to know that there exists a unique viscosity solution

Firstly, we consider the following equations in the bounded cylinder

$\frac{\partial}{\partial t}u \leq \Delta u + f(x,t)$

where $C$ only depends on $n, \lambda, \Lambda$ and $\text{diam}(D)$. Letting $m \to \infty$, we have $\hat{u}(y) \leq (\frac{1-\delta}{\varepsilon_0\delta} + C)f_0$ for $y \in \mathbb{R}$. Therefore, we obtain

$$u^+(x) \leq (\frac{1-\delta}{\varepsilon_0\delta} + C)\|f\|_{L^p}(C), \quad x \in C,$$

where $(\frac{1-\delta}{\varepsilon_0\delta} + C)$ depends on $n, \lambda, \Lambda$ and $\text{diam}(D)$.

Similarly, we can obtain the ABP maximum principle in half cylinder $C^+$ (or $C^-$) by using the same method. We omit the proof of Corollary 1 here.

5. The existence and properties of solutions in unbounded cylinders. In this section, we assume $F$ does not depend on $x$. Firstly, we prove the existence and uniqueness of bounded solutions in unbounded cylinders. Then we show the exponential decay of bounded solutions.

**Proof of Theorem 1.2:** Firstly, we consider the following equations in the bounded cylinder

$$
\begin{cases}
F(D^2u(x)) = f(x), & x \in C_{(-N,N)}, \\
u(x) = 0, & x \in \partial C_{(-N,N)},
\end{cases}
$$

where $N \in \mathbb{Z}^+$. For this Dirichlet problem, one can refer to Theorem 3.1 in [16] to know that there exists a unique viscosity solution $u_N(x) \in C(\overline{C}_{(-N,N)}) \cap W^{2,q}(C_{(-N,N)})$ for some $q_0 > 0$ and $q > q_0$.

By ABP maximum principle (Theorem 3.6) in [5], we have

$$\|u_N\|_{L^\infty(C_{(-N,N)})} \leq C_N \|f\|_{L^p(C_{(-N,N)})},$$

where $C_N$ depends only on $n, \lambda, \Lambda$ and $N$.

We will prove there exists a constant $C_0 > 0$ not depending on $N$, such that

$$\|u_N\|_{L^\infty(C_{(-N,N)})} \leq C_0 \|f\|_{L^p(C_{(-N,N)})}.$$

For convenience, we denote $M = \|u_N\|_{L^\infty(C_{(-N,N)})}$.

For any $\xi$ satisfying $-N + 1 \leq \xi \leq N - 1$, it is clear that $C_{(-N+1,N-1)} \subset C_{(-N,N)}$. Arguing as in the proof of Theorem 1.1 in $C_{(-N+1,N-1)}$, we have

$$u_N(x', \xi) \leq (1-\delta)M + \frac{1-\delta}{\varepsilon_0} \|f\|_{L^p(C_{(-N,N)})}, \quad x' \in D.$$

Then we obtain

$$\sup_{\xi \in (-N+1,N-1)} \hat{u}_N(\xi) \leq (1-\delta)M + \frac{1-\delta}{\varepsilon_0} \|f\|_{L^p(C_{(-N,N)})},$$

that is,

$$\sup_{x \in C_{(-N+1,N-1)}} u_N(x) \leq (1-\delta)M + \frac{1-\delta}{\varepsilon_0} \|f\|_{L^p(C_{(-N,N)})},$$
We use the same arguments for \(-u\) to get further that
\[
\sup_{x \in C(-N+1, N-1)} |u_N(x)| \leq (1 - \delta)M + \frac{1 - \delta}{\varepsilon_0} \|f\|_{L^\infty(C(-N, N))}.
\]
For any \(x \in C(-N, -N+1)_\), by using ABP maximum principle (Theorem 3.6) in [5] again, we have
\[
\|u_N\|_{L^\infty(C(-N, -N+1))} \leq \|u_N\|_{C^n(\partial C(-N, -N+1))} + C_1 \|f\|_{L^n(C(-N, -N+1))}
\]
\[
\leq \sup_{x \in C(-N+1, N-1)} |u_N(x)| + C_1 \|f\|_{L^n(C(-N, N))}
\]
\[
\leq (1 - \delta)M + \frac{1 - \delta}{\varepsilon_0} \|f\|_{L^n(C(-N, N))} + C_1 \|f\|_{L^n(C(-N, N))}
\]
\[
=(1 - \delta)M + C_2 \|f\|_{L^n(C(-N, N))},
\]
where \(C_2 = \frac{1 - \delta}{\varepsilon_0} + C_1\), depending only on \(n, \lambda, \Lambda\) and \(\text{diam}(D)\).

For any \(x \in C(-N, N)_\), similarly, we have
\[
\|u_N\|_{L^\infty(C(-N, N))} \leq (1 - \delta)M + C'_2 \|f\|_{L^n(C(-N, N))},
\]
where \(C'_2 = \frac{1 - \delta}{\varepsilon_0} + C'_1\) depending only on \(n, \lambda, \Lambda\) and \(\text{diam}(D)\).

Therefore, for any \(x \in C(-N, N)_\), we have
\[
\|u_N\|_{L^\infty(C(-N, N))} \leq (1 - \delta)M + 2C_3 \|f\|_{L^n(C(-N, N))},
\]
where \(C_3 = \max\{C_2, C'_2\}\) which depends only on \(n, \lambda, \Lambda, \text{diam}(D)\). That is, from the definition of \(M\), we have \(M \leq (1 - \delta)M + 2C_3 \|f\|_{L^n(C(-N, N))}\). Thus we obtain
\[
\|u_N\|_{L^\infty(C(-N, N))} = M \leq \frac{2}{\delta} C_3 \|f\|_{L^n(C(-N, N))} = C_0 \|f\|_{L^n(C(-N, N))},
\]
where \(C_0 = \frac{2}{\delta} C_3\) which depends only on \(n, \lambda, \Lambda\) and \(\text{diam}(D)\).

Thus, by the boundary Hölder estimate (Lemma 3.2), there exists a constant \(C_* > 0\) depending only on \(n, \lambda, \Lambda\) and \(D\) such that
\[
[u_N(x)]_{C^{\alpha}(C(-N, N))} \leq C_*, \quad \alpha \in (0, 1).
\]

Thus, for any bounded domain \(C_{[-L, L]}\) with \(l \geq 0\), by Arzela-Ascoli theorem, there exists a subsequence of \(\{u_N(x)\}\) which uniformly converges in \(C_{[-L, L]}\). Without loss of generality, we can assume that there exists a function \(u(x)\) such that \(u_N(x)\) uniformly converges to \(u(x)\) in \(W^{2,q}_{L^q}(C) \cap C(\bar{C})\). Therefore, \(u(x)\) is bounded in \(C\) and satisfies (1.2). By ABP maximum principle (Theorem 1.1), we know \(u\) is the desired unique bounded solution.

Next, we give the proof of the exponential decay of bounded solutions with inhomogeneous term \(f\).

**Proof of Theorem 1.3:** Assume \(\|f\|_{L^2(C^+)} = F_0\). Since \(u\) is bounded from above, we can apply Corollary 1 to get
\[
\hat{u}(y) \leq \hat{u}(0) + C\|f\|_{L^2(C^+)} = \hat{u}(0) + CF_0, \quad y \in (0, +\infty).
\]
Applying Lemma 4.1, there exists a constant \(\delta \in (0, 1)\) such that
\[
\hat{u}(1) \leq \frac{1 - \delta}{\varepsilon_0} \{\varepsilon_0 \max\{\hat{u}(0), \hat{u}(2)\} + \|f\|_{L^n(C_{(0, 2)})}\}
\]
\[
\leq (1 - \delta) \max\{\hat{u}(0), \hat{u}(0) + C\|f\|_{L^2(C^+)}\} + \frac{1 - \delta}{\varepsilon_0} \|f\|_{L^n(C_{(0, 2)})}.
\]
Lemma 6.1. (Carleson estimate) Assume that \( \Omega \) is a bounded Lipschitz domain such that \( 0 \in \partial \Omega \). Assume further that \( u \in C(B_{4R} \cap \Omega) \) with \( R \in (0, \frac{R_0}{4}] \), satisfies

\[
\begin{cases}
F(D^2 u(x)) = 0, & x \in \Omega \cap B_{4R}, \\
u(x) > 0, & x \in \Omega \cap B_{4R}, \\
u(x) = 0, & x \in \partial \Omega \cap B_{4R}.
\end{cases}
\]

Similarly,

\[
\hat{u}(2) \leq \frac{1 - \delta}{\varepsilon_0} \{ \varepsilon_0 \max \{ \hat{u}(1), \hat{u}(3) \} + \| f \|_{L^\infty(C^{1,1}(\varepsilon_0, \varepsilon_0))} \}
\]

\[
\leq (1 - \delta) \max \{ (1 - \delta) \max \{ \hat{u}(0), \hat{u}(2) \} + \frac{(1 - \delta)}{\varepsilon_0} \| f \|_{L^\infty(C(\varepsilon_0, \varepsilon_0))},
(1 - \delta) \max \{ \hat{u}(2), \hat{u}(4) \} + \frac{(1 - \delta)}{\varepsilon_0} \| f \|_{L^\infty(C(\varepsilon_2, \varepsilon_1))} \}
\]

\[
\leq (1 - \delta)^2 \max \{ \hat{u}(0) + CF_0, \hat{u}(0) + CF_0 \} + \frac{(1 - \delta)^2}{\varepsilon_0} F_0 + \frac{1 - \delta}{\varepsilon_0} F_0
\]

\[
= (1 - \delta)^2 (\hat{u}(0) + CF_0) + \frac{(1 - \delta)^2}{\varepsilon_0} F_0 + \frac{1 - \delta}{\varepsilon_0} F_0.
\]

Doing the operation repeatedly, we obtain

\[
\hat{u}(k) \leq (1 - \delta)^k (\hat{u}(0) + CF_0) + \sum_{i=1}^{k} \frac{(1 - \delta)^i}{\varepsilon_0} F_0 \leq (1 - \delta)^k (\hat{u}(0) + CF_0) + \frac{F_0}{\varepsilon_0} \cdot \frac{1 - \delta}{\delta}.
\]

Therefore, we have the following estimate, for \( x = (x', y) \in \mathcal{C}^+ \),

\[
u(x) \leq \max \{ \hat{u}(\lceil y \rceil), \hat{u}(\lceil y \rceil + 1) \} + CF_0
\]

\[
\leq (1 - \delta)^{\lceil y \rceil} (\hat{u}(0) + CF_0) + \frac{F_0}{\varepsilon_0} \cdot \frac{1 - \delta}{\delta} + CF_0
\]

\[
\leq (1 - \delta)^{\lceil y \rceil - 1} (\hat{u}(0) + CF_0) + \frac{F_0}{\varepsilon_0} \cdot \frac{1 - \delta}{\delta} + CF_0
\]

\[
= \frac{(\hat{u}(0) + CF_0)}{(1 - \delta)} e^{\delta (1 - \delta)} + \frac{F_0}{\varepsilon_0} \cdot \frac{1 - \delta}{\delta} + CF_0
\]

\[
= \frac{(\hat{u}(0) + CF_0)}{(1 - \delta)} e^{-\alpha y} + \frac{F_0}{\varepsilon_0} \cdot \frac{1 - \delta}{\delta} + CF_0
\]

where \( \alpha = -\ln(1 - \delta) > 0 \). Since \( F_0 = \| f \|_{L^\infty(\mathcal{C}^+)} \), we have

\[
u(x) \leq C_0 (\hat{u}(0) + F_0) e^{-\alpha y} + \frac{1 - \delta}{\varepsilon_0 \delta} + CF_0
\]

\[
\leq C_0 \hat{u}(0) e^{-\alpha y} + \frac{1 - \delta}{\varepsilon_0 \delta} + C + CF_0 \| f \|_{L^\infty(\mathcal{C}^+)}, \quad x \in \mathcal{C}^+.
\]

6. The structure of solutions. In this section, we will show the structure of solutions to the uniformly elliptic equation. We start to boundary Harnack principles for fully nonlinear equations which were proved in [2].

Lemma 6.1. (Carleson estimate) Assume that \( \Omega \) is a bounded Lipschitz domain such that \( 0 \in \partial \Omega \). Assume further that \( u \in C(B_{4R} \cap \Omega) \), with \( R \in (0, \frac{R_0}{4}] \), satisfies

\[
\begin{cases}
F(D^2 u(x)) = 0, & x \in \Omega \cap B_{4R}, \\
u(x) > 0, & x \in \Omega \cap B_{4R}, \\
u(x) = 0, & x \in \partial \Omega \cap B_{4R}.
\end{cases}
\]
Let $A_R \in \Omega \cap B_{R/2L}$ be a point such that $d(A_R, \partial \Omega) > R/(4L^2)$. Then there exists a constant $C > 1$ which is independent of $u$ and of the radius $R$ such that

$$\sup_{\Omega \cap B_{R/4}} u \leq Cu(A_R),$$

where $d(A_R, \partial \Omega)$ is the distance from point $A_R$ to $\partial \Omega$, and $L = \max\{l, 2\}$, $l$ is a Lipschitz constant which is related to Lipschitz domain $\Omega$.

From the Lemma 6.1, we have the following lemma, which is the key to proving the asymptotic behaviors of positive solutions.

**Lemma 6.2.** Let $u \in U$. There exists a universal constant $C$, depending only on $n, \lambda, \Lambda$ and $D$, such that for any $y \in \mathbb{R}$ we have

$$u(x) \leq Cu(0', y), \quad x \in C_{(y-2,y+2)}.$$

**Lemma 6.3.** (Boundary Harnack Inequality) Assume that $\Omega$ is a bounded Lipschitz domain such that $0 \in \partial \Omega$, and $u_i \in C(B_{4R} \cap \Omega), i = 1, 2$, with $R \in (0, \frac{R_0}{4})$, satisfies

$$\begin{cases}
F(D^2 u_i(x)) = 0, & x \in \Omega \cap B_{4R}, \\
u_i(x) > 0, & x \in \Omega \cap B_{4R}, \\
u_i(x) = 0, & x \in \partial \Omega \cap B_{4R}.
\end{cases}$$

Let $A_R \in \Omega \cap B_{R/2L}$ be a point such that $d(A_R, \partial \Omega) > R/(4L^2)$ and $u_1(A_R) = u_2(A_R) > 0$. Then there exists a constant $C > 1$ which is independent of $u_i, i = 1, 2$, and of the radius $R$, such that

$$\sup_{\Omega \cap B_{R/4}} \frac{u_1}{u_2} \leq C,$$

where $d(A_R, \partial \Omega)$ is the distance from point $A_R$ to $\partial \Omega$, and $L = \max\{l, 2\}$, $l$ is a Lipschitz constant which is related to Lipschitz domain $\Omega$.

Similarly, we have the following lemma in some bounded cylinder.

**Lemma 6.4.** Let $u_1, u_2 \in U$, and $u_1(0', 0) = u_2(0', 0)$. Then there exists a universal constant $K$ such that

$$\frac{1}{K}u_2(x) \leq u_1(x) \leq Ku_2(x), \quad x \in C_{[-k,k]}, \ k \in \mathbb{N}^+.$$

**Remark 2.** If the condition $u_1(0', 0) = u_2(0', 0)$ in Lemma 6.4 is replaced by $u_1(0', 0) \leq (\geq)u_2(0', 0)$, then for a universal constant $K$ it holds that $u_1(x) \leq Ku_2(x)$ ($u_1(x) \geq \frac{1}{K}u_2(x)$) in $C_{[-k,k]}$. In fact, if letting us assume $u_1(0', 0) \leq u_2(0', 0)$ and setting $v(x) = \frac{u_1(0', 0)}{u_2(0', 0)}u_2(x)$, then we have $v(0', 0) = u_1(0', 0)$. Since $v(x) \in U$ by the homogeneity of $F$, we have

$$\frac{1}{K}u_1(x) \leq v(x), \quad x \in C_{[-k,k]}, \ k \in \mathbb{N}^+$$

for a universal constant $K$. This implies that $u_1(x) \leq Ku_2(x)$ for $x \in C_{[-k,k]}$ and $k \in \mathbb{N}^+$.

From the Lemma 6.4, we have the following lemma which can compare solutions.

**Lemma 6.5.** For any $u_1, u_2 \in U$, if there is a point $x_0 = (x_0', y_0) \in C$ such that $u_1(x_0) = u_2(x_0)$, then there exists a universal constant $\tau$ such that

$$\tau u_2(x', y_0) \leq u_1(x', y_0) \leq \frac{1}{\tau} u_2(x', y_0), \quad x' \in D.$$
Proof. Set \( \tilde{u}_1(\tilde{x}', y) = u_1(x'_0 - \tilde{x}', y_0 - y) \), \( \tilde{u}_2(\tilde{x}', y) = u_2(x'_0 - \tilde{x}', y_0 - y) \). Then we get \( \tilde{u}_1(0', 0) = u_2(0', 0) \). From Lemma 6.4, there exists a constant \( K > 0 \) such that \( \frac{1}{K} \tilde{u}_2(\tilde{x}) \leq \tilde{u}_1(\tilde{x}) \leq K \tilde{u}_2(\tilde{x}) \), \( \tilde{x} \in C_{[-k, k]} \). Taking \( y = 0 \), we have \( \frac{1}{K} \tilde{u}_2(\tilde{x}', 0) \leq \tilde{u}_1(\tilde{x}', 0) \leq K \tilde{u}_2(\tilde{x}', 0) \), i.e.

\[
\frac{1}{K} u_2(x'_0 - \tilde{x}', y_0) \leq u_1(x'_0 - \tilde{x}', y_0) \leq K u_2(x'_0 - \tilde{x}', y_0).
\]

Taking \( x' = x'_0 - \tilde{x}' \) \( \in D \) and \( \tau = \frac{1}{K} \), then we get our result

\[
\tau u_2(x', y_0) \leq u_1(x', y_0) \leq \tau u_2(x', y_0), \quad x' \in D.
\]

\[\square\]

**Remark 3.** If the condition \( u_1(x_0) = u_2(x_0) \) in Lemma 6.5 is replaced by \( u_1(x_0) \leq (\geq) u_2(x_0) \), then there exists a universal constant \( \tau \) such that \( u_1(x', y_0) \leq \frac{1}{\tau} u_2(x', y_0) \) \( (u_1(x', y_0) \geq \tau u_2(x', y_0)) \) for \( x' \in D \).

Now we show the solution set \( U \) can be divided into three parts under some conditions.

**Proposition 3.** Assume \( u \in U \). Then \( m(u) = \inf_{y \in \mathbb{R}} \hat{u}(y) \geq 0 \).

1. If \( m(u) = \inf_{y \in \mathbb{R}} \hat{u}(y) = 0 \), then either of the following alternatives holds:
   (i) There exists a sequence \( \{x_j = (x'_j, y_j)\} \subset C \) such that \( \lim_{j \to \infty} y_j = +\infty \),
   \( \lim_{j \to \infty} u(x'_j, y_j) = 0 \) and \( \hat{u}(y) \) is a strictly decreasing function in \((\infty, +\infty)\).
   (ii) There exists a sequence \( \{x_j = (x'_j, y_j)\} \subset C \) such that \( \lim_{j \to \infty} y_j = -\infty \),
   \( \lim_{j \to \infty} u(x'_j, y_j) = 0 \) and \( \hat{u}(y) \) is a strictly increasing function in \((-\infty, +\infty)\).

2. If \( m(u) > 0 \), then there exists \( x'^* = (x'^*, y'^*) \in C \) such that \( u(x'^*) = m(u) \) and \( \hat{u}(y'^*) \) is strictly increasing in \([y'^*, +\infty)\) and strictly decreasing in \((-\infty, y'^*)\).

**Proof.** By the definition of \( m(u) \) and \( \hat{u}(y) \), we can assume there exists a minimizing sequence \( \{x_j = (x'_j, y_j)\} \subset C \) such that \( u(x'_j, y_j) = \sup_{x' \in D} u(x', y_j) = \hat{u}(y_j) \) and \( \lim_{j \to \infty} u(x_j) = m(u) \).

(1) If \( m(u) = 0 \), then \( \lim_{j \to \infty} u(x_j) = m(u) = 0 \).

We will prove that there exists a subsequence of \( \{x_j = (x'_j, y_j)\} \subset C \) (still denoted \( \{x_j = (x'_j, y_j)\} \)) such that \( \lim_{j \to \infty} y_j = +\infty \) or \( \lim_{j \to \infty} y_j = -\infty \).

If we suppose the above statement is not true, then \( \{x_j = (x'_j, y_j)\} \) is bounded in \( C \). Then there exists a subsequence of \( \{x_j = (x'_j, y_j)\} \), still denoted \( \{x_j = (x'_j, y_j)\} \), and a point \( x_0 = (x'_0, y_0) \in C \) such that \( \lim_{j \to \infty} x_j = x_0 \). By the continuity of \( u(x) \), we get \( u(x_0) = 0 \). Thus we know \( x_0 \in \partial C \), that is, \( x_0 = (x'_0, y_0) \) with \( x'_0 \in \partial D \).

Due to \( \hat{u}(y_0) > 0 \) and the continuity of \( \hat{u}(y) \) in \( C \) (the detailed proof can be seen in Lemma 2.5 of [4]), there exist constants \( \epsilon_0, \delta_0 > 0 \) such that \( \hat{u}(y) > \epsilon_0 > 0 \) where \( \|y - y_0\| < \delta_0 \). Letting \( j \) be sufficiently large, we have \( \hat{u}(y_j) \geq \epsilon_0 > 0 \). It follows that \( u(x_j) \geq \epsilon_0 > 0 \) for sufficiently large \( j \). Therefore, \( u(x_0) = \lim_{j \to \infty} u(x_j) \geq \epsilon_0 > 0 \), which is a contradiction.

If we assume \( \lim_{j \to \infty} y_j = +\infty \) (in subsequence sense), we will show \( \hat{u}(y) \) is a strictly decreasing function in \((-\infty, +\infty)\).

Similarly, supposing it’s not true, then there exist \(-\infty < \tilde{y} < \bar{y} < +\infty \) such that \( 0 < \hat{u}(\tilde{y}) \leq \tilde{u}(\bar{y}) \). Due to \( \lim_{j \to \infty} y_j = +\infty \) and \( \lim_{j \to \infty} \tilde{u}(y_j) = 0 \), we can take \( j \) large enough such that \( \bar{y} < \tilde{y} < y_j \), \( u(x'_j, y_j) \leq \tilde{u}(\bar{y}) \) for \( x' \in D \). Considering \( \Omega := C_{(\bar{y}, y_j)} \), by the definition of \( \tilde{u} \), we can obtain a point \( \bar{x} \in \Omega = C_{(\tilde{y}, y_j)} \subset C \) such that \( \tilde{u}(\bar{x}) = \tilde{u}(\bar{y}) \geq \sup_{\Omega} u(x) \), which contradicts Maximum Principle (Corollary 3.7) in [5]. Therefore, we obtain our conclusion.
If we assume \( \lim_{j \to \infty} y_j = -\infty \) (in subsequence sense), we can use the same method to show \( \hat{u}(y) \) is a strictly increasing function in \((\infty, +\infty)\).

(2) If \( m(u) > 0 \), we will prove there exist a subsequence of \( \{x_j = (x'_j, y_j)\} \subset \mathcal{C} \) (still denoted \( \{x_j = (x'_j, y_j)\} \)) such that \( \lim_{j \to \infty} y_j = \pm \infty \) and \( \lim_{j \to \infty} \hat{u}(y_j) \) exists and \( m(u) > 0 \). If \( \lim_{j \to \infty} y_j = -\infty \), we can prove \( \hat{u}(y) \) is strictly increasing in \( \mathbb{R} \) by using the arguments as above. Therefore \( u(x) \) is bounded in \( \mathcal{C}(\infty, 0) \). Then, by applying Theorem 1.3, \( \lim_{y \to -\infty} \hat{u}(y) = 0 \), which contradicts (6.1). If \( \lim_{j \to \infty} y_j = +\infty \), we can get a contradiction by a similar method.

Therefore, we have proved that there exist a subsequence of \( \{x_j = (x'_j, y_j)\} \subset \mathcal{C} \) (still denoted \( \{x_j = (x'_j, y_j)\} \)) and \( x^* \in \mathcal{C} \) such that \( \lim_{j \to \infty} x_j = x^* \) and \( \lim_{j \to \infty} u(x_j) = u(x^*) = m(u) > 0 \).

Next, we prove \( \hat{u}(y) \) is strictly increasing in \([y^*, +\infty)\). If we suppose that it’s not true, then there exists \( y^* < \bar{y} < \tilde{y} < +\infty \) such that \( \hat{u}(\bar{y}) \geq \hat{u}(\tilde{y}) \geq \hat{u}(y^*) = m(u) > 0 \). Thus we get a local maximum point \( \tilde{x} = (x'_i, \bar{y}) \in \mathcal{C}(y^*, \bar{y}) \), which contradicts the Maximum principle (Corollary 3.7) in [5]. We can use the similar method to prove \( \hat{u}(y) \) is strictly decreasing in \((\infty, y^*)\).

\[ \square \]

**Remark 4.** From the above Proposition, we derive that the solution set \( U = U^+ \cup U^- \cup U^\nu \). From the proof, we know that \( U^+ \cap U^- = U^- \cap U^\nu = U^+ \cap U^\nu = \emptyset \).

**Proposition 4.** For any \( u \in U^+ \), \( v \in U^- \) and \( w \in U^\nu \), there exist constants \( p, q > 0 \) such that

\[ pu(x) \leq w(x), \quad qv(x) \leq w(x), \quad x \in \mathcal{C}. \]

**Proof.** From Proposition 3, for \( w \in U^\nu \), there exists \( x^* = (x'^*, y^*) \in \mathcal{C} \) such that \( \hat{u}(y) \) is strictly increasing in \([y^*, +\infty)\) and strictly decreasing in \((\infty, y^*)\).

For any \( u \in U^+ \), since \( \hat{u}(y) \) is strictly decreasing in \( \mathbb{R} \), we have \( u(x) \) is bounded in \( \mathcal{C}_{[0, +\infty)} \). From Remark 2, there exists a \( \kappa_0 > 0 \) such that \( \kappa_0 u(x) \leq w(x) \) for \( x \in \mathcal{C}_{[0]} \). By the positive homogeneity of \( F \), \( F(D^2(\kappa_0 u)) = 0 \). Hence, by Proposition 2, we have \( (\kappa_0 u - w) \in \mathcal{C}(\lambda/\kappa_0, 0) \in \mathcal{C}(0, +\infty) \). Since \( (\kappa_0 u - w) \) is bounded from above, by the Maximum Principle (Corollary 1), we have

\[ \kappa_0 u(x) \leq w(x), \quad x \in \mathcal{C}_{[0, +\infty)}. \]

(6.2)

We claim that there exists a constant \( p > 0 \) such that \( pu(x) \leq w(x) \) for \( x \in \mathcal{C} \). We suppose by contradiction that there exists a sequence \( \{x_j = (x'_j, y_j)\} \subset \mathcal{C} \) such that \( \frac{1}{j} u(x_j) > w(x_j) \).

Firstly, we prove there exists a subsequence of \( \{x_j = (x'_j, y_j)\} \) (still denoted \( \{x_j = (x'_j, y_j)\} \)) such that \( \lim_{j \to \infty} y_j = -\infty \). It is easy to observe that the sequence \( \{y_j\} \) is bounded from above from (6.2). In fact, if not, letting \( j \) sufficiently large, we can obtain \( x_j = (x'_j, y_j) \in \mathcal{C}_{[0, +\infty)} \). Then we get a contradiction since \( w(x_j) \geq \kappa_0 u(x_j) > \kappa_0 u w(x_j) \). If the sequence \( \{y_j\} \) is bounded in \( \mathbb{R} \), we can assume \( |y_j| \leq C, \quad j \geq 1 \). By Remark 2 and Maximum Principle (Corollary 1), there exists a constant \( \kappa_C > 0 \) such that \( \kappa_C u(x) \leq w(x) \) for \( x \in \mathcal{C}_{[-C, +\infty)} \). Taking \( j \) large enough such that \( \frac{1}{j} \leq \kappa_C \), then we get \( w(x_j) \leq \kappa_C u(x_j) \leq w(x_j), \quad x_j \in \mathcal{C}_{[-C, +\infty)} \), which is a contradiction.
Now we can assume the sequence \( \{y_j\} \) is decreasing. Suppose \( \tau \) is the constant in Remark 2. For any \( j \geq 1 \), we consider \( w(x) \) and \( \frac{1}{\tau_j} u(x) \) in \( C_{(y_j, 0)} \). Since there exists a point \( x_j \in C_{(y_j)} \) such that \( w(x_j) < \frac{1}{\tau_j} u(x_j) \), then by Remark 2 we have \( w(x) \leq \frac{1}{\tau_j} u(x) \) for \( x \in C_{(y_j)} \). That is \( w(x) - \frac{1}{\tau_j} u(x) \leq 0 \) for \( x \in C_{(y_j)} \). By using Maximum Principle (Corollary 3.7) in [5], we obtain

\[
\frac{w(x) - \frac{1}{\tau_j} u(x)}{w(0)} \leq 0, \quad x \in C_{(y_j, 0)}.
\]

Due to \( \lim_{j \to \infty} y_j = -\infty \), we have \( \lim_{j \to \infty} [w(x) - \frac{1}{\tau_j} u(x)] = w(x) \leq w(0) \) for \( x \in C_{(-\infty, 0)} \). Since \( w \in U^\nu \), we know from Proposition 3 that \( w(x) \) is not bounded in \( C_{(-\infty, 0)} \). Thus we get a contradiction. Therefore, there exists a constant \( p > 0 \) such that \( pu(x) \leq w(x) \) for \( x \in C \).

By using the same method, we can prove \( qw(x) \leq w(x) \) for \( x \in C \) and some constant \( q > 0 \).

**Proposition 5.** For any \( u, v \in U^+ \), there exists a constant \( d > 0 \) such that

\[
u(x) \leq dv(x), \quad x \in C.
\]

**Proof.** With Remark 2, for any \( y \in \mathbb{R} \), there exists a constant \( d_y > 0 \) such that \( u(x) \leq d_y v(x) \) for \( x \in C_{(y)} \). By applying maximum principle (Corollary 1), we have \( u(x) \leq d_y v(x) \) for \( x \in C_{[y, +\infty)} \). Taking \( y = 0 \), we get

\[
u(x) \leq d_0 v(x), \quad x \in C_{[0, +\infty)}.
\]

Next we prove (6.3) by contradiction. Suppose there exists \( \{x_j = (x'_j, y_j)\} \subset C \) such that \( u(x_j) > jv(x_j) \). With Remark 2, there exists a constant \( \tau \) such that \( u(x', y_j) \geq \tau jv(x', y_j) \) for \( x' \in D \). Taking \( j \) sufficiently large such that \( \tau j > d_0 \), then we have

\[
u(x) \geq \tau jv(x) > d_0 v(x) \geq u(x), \quad x \in C_{[y, +\infty)}.
\]

Thus we obtain a contradiction.

**Proof of Theorem 1.4:** From Lemma 6.2 and by using the similar arguments in the proof of Theorem 1.2 in [4], we can prove Theorem 1.4. Here we omit the details.

**Proof of Theorem 1.5:** We just analyze the structure of \( U^+ \). One can use the similar method to study the structure of \( U^- \).

Firstly, for any \( u, w \in U^+ \), we define

\[
E = \{ k > 0 \mid u(x) \leq kw(x), \ x \in C \}, \quad k_+ = \inf E.
\]

\[
F = \{ l > 0 \mid l \geq lw(x), \ x \in C \}, \quad k_- = \sup F.
\]

From Proposition 5, there exist \( k_0, l_0 > 0 \) such that \( k_0 \in E \) and \( l_0 \in F \). By continuity, we have \( k_+ \in E \) and \( k_- \in F \). It is clear that \( 0 < k_- \leq k_+ < +\infty \).

Then, if we are able to prove that \( k_+ = k_- \), we are done.

In fact, set \( w_1 = k_+ w - u \geq 0 \) and \( w_2 = u - k_- w \geq 0 \) in \( C \). It is clear that \( w_1 + w_2 = (k_+ - k_-)w \) in \( C \). Then fixing \( x_0 \in D \), we have

\[
(w_1 + w_2)(x_0, y) = (k_+ - k_-)w(x_0, y).
\]

Therefore, for some \( y_0 \in \mathbb{R} \), we have

\[
w_1(x_0, y_0) \geq \frac{1}{2}(k_+ - k_-)w(x_0, y_0), \quad (6.4)
\]
or
\[ w_2(x_0', y_0) \geq \frac{1}{2} (k^+ - k^-) w(x_0', y_0). \quad (6.5) \]

Suppose (6.4) holds, say
\[ (k^+ w - u)(x_0', y_0) \geq \frac{1}{2} (k^+ - k^-) w(x_0', y_0). \]

We claim that for any \( x' \in D \), there exists a universal constant \( C_0 > 0 \) such that
\[ (k^+ w - u)(x', y_0) \geq \frac{C_0}{2} (k^+ - k^-) w(x', y_0). \quad (6.6) \]

Then we have
\[ [k^+ - \frac{C_0}{2} (k^+ - k^-)] w \geq u, \quad x \in C_{\{y_0\}}. \]

By maximum principle (Corollary 1), we have
\[ [k^+ - \frac{C_0}{2} (k^+ - k^-)] w \geq u, \quad x \in C_{\{y_0, +\infty\}}. \]

Then from (6.8) and (6.9), we know \((6.10)\) and \((6.11)\), i.e.
\[ \alpha v(x) \leq u(x) \leq \beta v(x). \quad (6.10) \]

Proof of Theorem 1.6: We divide the proof into two steps.

Step 1. For any \( u \in U^-, \ v \in U^+ \) in \( C_{(-\infty, \tilde{y})} \) with \( \tilde{y} \leq 0 \), there exist universal constants \( 0 < \alpha < \beta < +\infty \) such that
\[ \alpha v(x) \leq u(x) \leq \beta v(x). \]
The left inequality of (6.10) obviously holds from Proposition 4. Therefore, we only need to prove the right inequality of (6.10). By Remark 2, for \( x_0 = (x_0', y_0) \in C \), we have
\[
\frac{1}{C} \frac{u(x', y_0)}{v(x', y_0)} \leq \frac{u(x_0', y_0)}{v(x_0', y_0)} \leq C \frac{u(x', y_0)}{v(x', y_0)}, \quad x' \in D,
\]
(6.11)
where \( C \) is a universal constant.

We suppose by contradiction that the right inequality of (6.10) is not true. Then, for any \( j \in \mathbb{N}^+ \), there exists a sequence \( \{x_j = (x_j', y_j)\} \subseteq C_{(- \infty, \tilde{y})} \) with \( \tilde{y} \leq 0 \) such that
\[
\frac{u(x_j', y_j)}{v(x_j', y_j)} \geq j.
\]
From (6.11), we obtain
\[
\frac{u(x_j', y_j)}{v(x_j', y_j)} \geq \frac{j}{C}, \quad x \in C_{(y_j)}.
\]
By the Maximum Principle (Theorem 1.1), we have
\[
u(x', y) \geq \frac{j}{C} v(x', y), \quad x \in C_{[y_j, +\infty)}.
\]
Since \( y_j \leq \tilde{y} \leq 0 \), if we take \( y = 0 \), then we have \( u(x', 0) \geq \frac{\tilde{j}}{C} v(x', 0) \). Letting \( j \) tends to \( +\infty \), we obtain a contradiction.

Step 2. There exists a positive constant \( \alpha^* \) such that
\[
u(x) = (\alpha^* + o(1))v(x), \quad x \to -\infty.
\]
(6.12)
In fact, for any \( j \in \mathbb{N}^+ \), let us define
\[
P_j = \inf\{p > 0 \mid u(x) \leq pv(x), \ x \in C_{(-\infty, -j)}\},
\]
\[
Q_j = \sup\{q > 0 \mid u(x) \geq qv(x), \ x \in C_{(-\infty, -j)}\}.
\]
From step 1, we know \( 0 < Q_j \leq P_j < +\infty \). We claim there exists a subsequence \( \{j_k\} \subseteq \{j\} \) satisfying \( j_1 \leq j_2 \leq \cdots \) and a constant \( \varepsilon \in (0, 1) \) depending only on \( n \) such that
\[
P_{j_{k+1}} - Q_{j_{k+1}} \leq \varepsilon (P_{j_k} - Q_{j_k}).
\]
(6.13)
Obviously, \( \{P_{j_k}\} \) is a decreasing sequence, and \( \{Q_{j_k}\} \) is an increasing sequence. From (6.13), there exists a positive constant \( \alpha^* \) such that \( \lim_{k \to \infty} P_{j_k} = \alpha^* = \lim_{k \to \infty} Q_{j_k} \). Therefore, we obtain the result (6.12).

Next we prove the claim (6.13). Noting that for any point \( x_0' \in D \), we have for \( l = 1, 2, \ldots \)
\[
P_{j_k} v(x_0', -j_k - l) \geq u(x_0', -j_k - l) \geq Q_{j_k} v(x_0', -j_k - l) > 0, \quad j_k \in \mathbb{N}^+.
\]
Therefore, we have
\[
u(x_0', -j_k - l) \geq Q_{j_k} v(x_0', -j_k - l) + \frac{1}{2} (P_{j_k} - Q_{j_k}) v(x_0', -j_k - l).
\]
(6.14)
or
\[
u(x_0', -j_k - l) \leq P_{j_k} v(x_0', -j_k - l) - \frac{1}{2} (P_{j_k} - Q_{j_k}) v(x_0', -j_k - l).
\]
(6.15)
If (6.14) holds, that is, for \( l = l_j \) and \( j \geq 1 \) we have
\[
u(x_0', -j_k - l_j) - Q_{j_k} v(x_0', -j_k - l_j) \geq \frac{1}{2} (P_{j_k} - Q_{j_k}) v(x_0', -j_k - l_j).
\]
Then we can use the same method in Theorem 1.5 to prove
\[
u(x', -j_k - l_j) - Q_{j_k} v(x', -j_k - l_j) \geq \frac{1}{2C} (P_{j_k} - Q_{j_k}) v(x', -j_k - l_j), \quad x \in C_{(-j_k - l_j)}.
\]
Here \( C \) depends only on \( n \).
For $l_j \in \mathbb{N}^+$, we let $j = 1, 2, \cdots$. Write $j_{k+1} = j_k + l$. By the ABP maximum principle (Corollary 3.7) in [5], we have

$$Q_{j_k} v(x) + \frac{1}{2C} (P_{j_k} - Q_{j_k}) v(x) \leq u(x), \quad x \in C_{[-j_{k+2}, -j_{k+1}]}.$$  

Then for any $y \in (-\infty, -j_{k+1}]$, we have

$$u(x) \geq [(1 - \frac{1}{2C}) Q_{j_k} + \frac{1}{2C} P_{j_k}] v(x), \quad x \in C_{(-\infty, -j_{k+1}]}.$$  

(6.16)

By the definition of $Q_j$ and (6.16), we obtain

$$Q_{j_{k+1}} \geq (1 - \frac{1}{2C}) Q_{j_k} + \frac{1}{2C} P_{j_k}.$$  

(6.17)

Thus combining with (6.17), we have

$$P_{j_{k+1}} - Q_{j_{k+1}} \leq P_{j_k} - (1 - \frac{1}{2C}) Q_{j_k} - \frac{1}{2C} P_{j_k} = (1 - \frac{1}{2C}) (P_{j_k} - Q_{j_k}).$$

Taking $\varepsilon = (1 - \frac{1}{2C}) \in (0, 1)$, we get (6.13).

If (6.15) holds, with the same method, we can also get (6.13).

The proof of Theorem 1.7 is similar to that of Theorem 1.6. Here we omit the proof.

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