A Surgery Theory for Manifolds of Bounded Geometry

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1 Introduction

In this paper we introduce a new application of geometric topology to the study of Riemannian metrics. The purpose of this application is to classify metrics of bounded geometry up to smooth quasi-isometry on an open manifold.

A manifold of bounded geometry is a non-compact manifold whose geometric complexity is bounded. Such manifolds can be described metrically as having sectional curvature bounded in absolute value and injectivity radius bounded below. Cheeger’s finiteness theorem is equivalent to saying that in the PL sense such a manifold has a triangulation with a uniform bound on the number of simplices in the link of each vertex. Universal coverings of compact manifolds and leaves of foliations lie within this class of open manifolds. In fact, Gromov has remarked that every manifold of bounded geometry is the leaf of a lamination of the infinite dimensional compact space of Riemannian metrics. It is still an open question, however, whether every manifold of bounded geometry is the leaf of a foliation of a compact manifold. (If the foliation is $C^1$ the answer is no, see [6]).

Two non-compact manifolds are said to be smoothly quasi-isometric when there exists a diffeomorphism $f$ between them so that their distance-metrics satisfy

$$\frac{1}{c}d(x, y) \leq d'(f(x), f(y)) \leq cd(x, y).$$

The classification problem for open manifolds, analogous to the diffeomorphism classification problem for compact manifolds is the following: Given $M$ and $N$ and a suitable map $f : M \to N$, when is $f$ boundedly homotopic to a smooth quasi-isometry?

The “answer” to the classification problem here is not an explicit one, but rather one in the form of a surgery exact sequence, none of whose terms can be computed in general, as is the case with compact manifolds. However, we will be able to present the computation for a large class of examples. We next describe the surgery theory (and correspondingly new notions of algebraic topology) which will be developed here.

Surgery theory for high-dimensional manifold classification was introduced by Milnor [43, 44, 41] in the late 1950s and early 1960s. The surgery exact sequence was first proven by Kervaire and Milnor for homotopy spheres in [11], and for high-dimensional manifolds by Browder [12, 13], Novikov [46, 48], Sullivan [50], Casson [56] and Wall [64]. Freedman [30, 31, 32] extended surgery classification to 4-manifolds with some restrictions on the fundamental group. Connell and Hollingsworth [22] introduced controlled algebraic topology in the 1960s. Anderson and Hsiang [1], Chapman [16, 17], Ferry [28], Quinn [52, 53, 54], Pedersen and Weibel [51] developed $\varepsilon$-controlled and boundedly controlled topology in the 1970s and 1980s. Ferry and Pedersen [29] introduced boundedly controlled surgery in the late 1980s. Our result on the uniqueness for $\mathbb{R}^n$ extends the result proven by Siebenmann in 1968 [61].

The surgery theory developed in this paper is an $L$-theoretic analogue of the index theorem of Roe in the sense that both Roe’s index and the surgery
obstruction lie in groups that, if a Baum-Connes [9] or Borel type conjecture were true (Baum-Connes for Roe’s coarse theory has been shown to be false by Higson, Lafforgue and Skandalis [38]), could be expressed as $L_\infty$ homology with coefficients in a spectrum. Weinberger observed in 1990 that boundedly controlled surgery should be analogous to Roe’s coarse index theory [58]. Roe’s coarse index theory [58, 39] has been shown to be related to boundedly controlled surgery for some spaces in the sense that both the index and the surgery obstruction lie in exotic homology with coefficients in a spectrum [25]. A character map (following Connes and Moscovici [23]) from the cyclic homology of Roe’s uniformly smoothing algebra to uniformly finite homology has been constructed by Block and Weinberger [11]. Roe’s coarse index theory has also been used to prove the homotopy invariance of rational Pontrjagin classes (originally due to Novikov [47]), see [50].

The main results of this classification theory for bg manifolds contrast sharply with other known results on the topology of non-compact manifolds. For example, Euclidean and hyperbolic space are homeomorphic to each other in the boundedly controlled category. However, the quasi-isometric classification of universal covers of manifolds with fundamental group a surface group exhibits the following phenomenon: there exist manifolds $X$ and $Y$ and a homotopy equivalence $f : X \to Y$ so that $f^*(p_i) - p_i \neq 0$, where $p_i$ denotes the $i$-th Pontrjagin class, and $f$ lifts to a map which is boundedly, but non-equivariantly, homotopic to a quasi-isometry on the universal covers. In distinction to this, every homotopy equivalence of manifolds with free abelian fundamental group which lifts to a map which is boundedly homotopic to a quasi-isometry must preserve Pontrjagin classes; see [5].

In [3] we introduced a new PL category, with objects simplicial complexes with bounded combinatorial complexity, and maps with bounded combinatorial complexity (see Definitions 2.1-2.14). In this category, hyperbolic $n$-space $H^n$ has a different “homotopy type” (see Definitions 2.15-2.17) than euclidean $n$-space $R^n$. There is an invariant of this “homotopy type”, the uniformly finite homology, denoted $H_{eff}^*(X; G)$ which is defined in Section 3, where $G$ is an abelian group equipped with a norm, so that

$$H_{0}^{eff}(H^n, \mathbb{Z}) = 0$$

and

$$H_{0}^{eff}(R^n, \mathbb{Z}) \neq 0.$$ 

This homology theory can be thought of as $L_\infty$-homology with coefficients in $G$. In addition to this notion of a “homotopy type” we introduced the notion of a “simple homotopy type”, discussed in section 4, with a smooth quasi-isometry being an example of this new type of “simple homotopy equivalence”. We also introduce the notion of a “structure set” (see Definition 5.10) in this new category, $S_{TOP}^{(q,s)}(X)$, which is the set of “homotopy equivalences” of manifolds of bounded geometry to $X$ in this category, modulo “homeomorphisms” (see Definition 2.11) in this category.

The main classification result proven here is:
Theorem 1.1 Let $M^k$ be a compact manifold. Then the bg simple structure set of $M \times \mathbb{R}^n$, $k + n \geq 5$ is

$$S_{\text{bg}}^{s,\text{TOP}}(M \times \mathbb{R}^n) = H^0_{\text{eff}}(\mathbb{R}^n; S_{\text{TOP}}(M \times D^n, \partial)) \oplus \ldots \oplus H^n_{\text{eff}}(\mathbb{R}^n; S_{\text{TOP}}^{2-n}(M))$$

where $S_{\text{TOP}}^i(M)$ denotes the fiber of the assembly map of Ranicki’s lower $L$-theory, and we take the convention $S_{\text{TOP}}^1(N, \partial N) = S_{\text{TOP}}^2(N, \partial N)$, where $N$ is a compact PL manifold with boundary $\partial N$.

From this we derive two results:

Theorem 1.2 Let $M^n$, $n \geq 5$ be a uniformly contractible smooth manifold of bounded geometry. Suppose further that there is a surjective map

$$f : M \to \mathbb{R}^n$$

which is EPL in the sense of [10] or a coarse map of bounded geometry in the sense of [3]. Then $M$ is smoothly quasi-isometric to $\mathbb{R}^n$.

We also have the following result:

Theorem 1.3 Let $M^k$ be a compact manifold and

$$f : M \times \mathbb{T}^n \to N^{k+n}$$

a homotopy equivalence, $k + n \geq 5$. Then the free abelian cover of $f$

$$\tilde{f} : M \times \mathbb{R}^n \to \tilde{N}$$

is bg homotopic to a quasi-isometry if and only if the lift of $f$ to a finite cover is homotopic to a diffeomorphism.

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2 Preliminaries

In this section we discuss the general results needed for the quasi-isometry classification of manifolds of bounded geometry. Bounded geometry was first studied by Cheeger and Gromov in [20]. We recall the definitions of simplicial complexes of bounded geometry, homotopy equivalences of bounded geometry and notions of controlled topology which will be used in the paper. These definitions are taken from [3].
Definition 2.1 A simplicial complex $X$ has bounded geometry if there is a uniform bound on the number of simplices in the link of each vertex of $X$.

Definition 2.2 A simplicial map $f : X \to Y$ of simplicial complexes of bounded geometry is said to have bounded geometry if the inverse image of each simplex $\Delta$ of $Y$ contains a uniformly bounded number of simplices of $X$. The uniform bound is called the complexity of the map.

For continuous maps, there is a notion of bounded geometry, which can be found in [10] where it is called EPL:

Definition 2.3 Let $X$ and $Y$ be metric spaces. A coarse map of bounded geometry is (not necessarily continuous) map $f : X \to Y$ satisfy the conditions:

i. A condition similar to a uniform Lipschitz condition. That is, given $r > 0$, there is a uniform $s > 0$ depending only on $r$ so that $f(B(x,r)) \subset B(f(x),s)$, where $B(x,r)$ denotes the metric ball of radius $r$ around $x$.

ii. It is effectively proper. That is, given $r > 0$ there exists a uniform $s > 0$ depending only on $r$ so that $f^{-1}(B(f(y),r)) \subset B(y,s)$.

We next recall the conditions for bounded geometry on the Riemannian metric of a smooth manifold.

Definition 2.4 A complete Riemannian manifold $M$ is said to have bounded geometry if its injectivity radius $\text{inj}_M > c > 0$ for some constant $c$ and its sectional curvature is bounded in absolute value. Recall that the injectivity radius of a complete Riemannian manifold is the infimum of the injectivity radii at each point of $M$. The injectivity radius at a point is the maximum radius for which the exponential map is injective.

Definition 2.5 A smooth map of bounded geometry is a smooth map which is effectively proper so that the $C^2$ norm of $f$ is uniformly bounded.

Definition 2.6 A subdivision of a simplicial complex of bounded geometry is said to be uniform if

i. Each simplex is subdivided a uniformly bounded number of times on its $n$-skeleton, where the $n$-skeleton is the union of $n$-dimensional sub-simplices of the simplex.

ii. The distortion $\sup(\text{length}(e),\text{length}(e)^{-1})$ of each edge $e$ of the of the subdivided complex is uniformly bounded in the metric given by the barycentric coordinates of the original complex.

Definition 2.7 A metric space $P$ is a bg polyhedron if:

i. It is topologically a subset $P \subset \mathbb{R}^n$.

ii. Each point $a \in P$ has a cone neighborhood $N = aL$ of $P$ in the given Euclidean space, where $L$ is compact and there is a uniform upper bound for all $a \in P$ for the number of simplices needed to triangulate $L$. 
Definition 2.8 A map \( f : P \to Q \) between bg polyhedra is bg PL if it is piecewise linear and has bounded distortion, i.e. the distortion of the image of a simplex is uniformly bounded. This is equivalent to saying that the graph of \( f \) is a bg polyhedron.

Definition 2.9 A PL manifold of bounded geometry is a bg polyhedron so that each point \( x \in M \) has a neighborhood in \( M \) which is PL homeomorphic to an open set of \( \mathbb{R}^n \), with a uniform bound on the distortion of the PL homeomorphism over \( M \).

Remark 2.1 A PL map of bounded geometry is an equivalence class of simplicial maps of bounded geometry under the equivalence of uniform subdivision. This follows by writing bg polyhedra as unions of simplices.

Definition 2.10 Let \( f : M \to N \) be a smooth map between Riemannian manifolds. Then \( f \) has bounded dilatation if there is a constant \( C \) so that

\[
|f_* v| \leq C |v|
\]

for all \( v \in TM \). \( f \) is called a smooth quasi-isometry if it is a diffeomorphism and both \( f \) and \( f^{-1} \) have bounded dilatation.

We shall also need to use the following:

Definition 2.11 Let \( f : M \to N \) be a continuous map between PL manifolds of bounded geometry, then \( f \) is a bg homeomorphism (or equivalently a continuous quasi-isometry) if \( f \) has a continuous inverse \( f^{-1} \) such that the distance metrics, \( d \) of \( M \) and \( d' \) of \( N \) satisfy

\[
\frac{1}{c} d(x, y) \leq d'(f(x), f(y)) \leq cd(x, y)
\]

We recall the following Theorem due essentially to Cheeger, Müller and Schrader \[19, 21\], from \[3\]:

Theorem 2.1 Let \( M \) be a smooth manifold with a Riemannian metric of bounded geometry. Then \( M \) admits a triangulation as a simplicial complex of bounded geometry whose metric given by barycentric coordinates is quasi-isometric to the metric on \( M \) induced by the Riemannian structure. This triangulation is unique up to uniform subdivision. Conversely, if \( M \) is a simplicial complex of bounded geometry which is a triangulation of a smooth manifold, then this smooth manifold admits a metric of bounded geometry with respect to which it is quasi-isometric to \( M \).

Corollary 2.1 A smooth map which can be simplicially approximated by a simplicial map of bounded geometry for appropriate triangulation of the source and target, can be approximated by a smooth map of bounded geometry. Conversely, any smooth map of bounded geometry can be simplicially approximated by a PL map of bounded geometry.
Definition 2.12 Let $M^n \subset N^{n+q}$, then $N$ is an abstract regular neighborhood of bounded geometry if $N$ collapses via a bg map to $M$.

Definition 2.13 A bounded geometry $q$-block bundle $\xi^q$ consists of a total space $E(\xi)$ and a bg simplicial complex $K$ so that $|K| \subset E(\xi)$ satisfying

i. For each $n$-cell $\sigma_i \in K$, there exists an $(n+q)$-ball $\beta_i \subset E(\xi)$ so that $(\beta_i, \sigma_i) \simeq (I^{n+q}, I^n)$

ii. $E(\xi)$ is the union of blocks $\beta_i$.

iii. The interiors of blocks are disjoint.

iv. They are compact polyhedra and fall into a finite number of types (as simplicial complexes).

v. Let $L = \sigma_i \cap \sigma_j$, then $\beta_i \cap \beta_j$ is the bounded union of blocks over cells of $L$.

$\xi^q, \eta^q/K$ are bg isomorphic if there is a bg homeomorphism $h : E(\xi) \to E(\eta)$, $h|K = 1, h(\beta_i(\xi)) = \beta_i(\eta), \sigma_i \in K, \xi \sim \eta$ or $\xi$ equivalent to $\eta$, if there exist uniform subdivisions $\xi', \eta'$ so that $\xi' \simeq \eta'$. Let $X = |K|$. Let $I_q(K)$ denote the set of bg isomorphism classes of $q$-block bundles over $K$, $I_q(X)$ the set of bg equivalence classes over $X$. Then amalgamation gives a bijection between the two sets.

The following Theorem is from [9].

Theorem 2.2 Let $N^{n+q}$ be a bg abstract regular neighborhood of $M^n$ and suppose $L \subset K$ so that $(M, \partial M) = (|K|, |L|)$. Then there is $\xi^q/K$ with $E(\xi) = N$.

Definition 2.14 Let $M, N \subset Q$ be bg submanifolds of the bg manifold $Q$, all in the bg PL category. Let $\xi$ a normal bg block bundle on $M$. Then $N$ is bg transverse to $M$ with respect $\xi$ if there is a uniform subdivision $\xi'$ of $\xi$ so that $N \cap E(\xi) = E(\xi'|N \cap M)$.

Theorem 2.3 (Transversality Theorem [9]) Let $M, N \subset Q$ be bg submanifolds of the bg manifold $Q$. There is an ambient isotopy of bounded geometry of $Q$ carrying $N$ by transverse to $M$ with respect $\xi$.

Proof. This is word for word as in [59], with bounded geometry in front of every term except subdivision, which must have the word uniform in front of it instead. We recall that the strategy of proof in [59] is to use the Zeeman unknotting theorem to induct on the skeleta of the dual triangulation of $M$. The analogous unknotting theorem for the bounded geometry case states that if one has an infinite collection of bg embedded spheres, each of which can be unknotted in the PL category, then one can unknot them by a bg PL isotopy. The rest also goes through verbatim as stated above.

Example: Consider the submanifold $y = e^{-|x|} \sin(x)$ of $\mathbb{R}^2$, which is a smooth submanifold of bounded geometry. This submanifold intersects the line $y = 0$ transversely in an infinite number of points, but is not bg transverse. The embedding of this submanifold into $\mathbb{R}^2$ cannot be simplicially as a bg PL embedding.
so that the intersection is still a countably infinite number of points, since this
requires \( \mathbb{R}^2 \) to be triangulated in such a way that the volume of a simplex is not
bounded below. However this embedding can still be \( C^0 \)-approximated by a \( \text{bg} \)
PL map which is \( y = 0 \) outside a compact set, which can be made arbitrarily
large. This is clearly not \( \text{bg} \) transverse since the dimension of the intersection
is positive outside the compact set. If we isotop this curve by adding a small
constant to \( y \) in the equation above, it becomes \( \text{bg} \) transverse.

**Definition 2.15** A homotopy of bounded geometry between two maps \( f_0 \) and \( f_1 \)
of bounded geometry between simplicial complexes \( X \) and \( Y \) of bounded geometry
is a map of bounded geometry \( F : X \times I \to Y \) so that \( F | X \times 0 = f_0 \) and
\( F | X \times 1 = f_1 \). We write this \( f_0 \sim_{\text{bg}} f_1 \).

**Definition 2.16** A homotopy equivalence of bounded geometry is a map \( f \) of
bounded geometry so that there is a map \( g \) of bounded geometry with \( f \circ g \) and
\( g \circ f \) \( \text{bg} \) homotopic to the identity.

**Definition 2.17** A CW-complex of bounded geometry is defined to be a CW-
complex with a uniformly bounded number of cells attached to each cell and
a finite number of homeomorphism types of attaching maps. A \( \text{bg} \) \( n \)-cell is a
discrete collection of \( n \)-cells \( \Sigma \times I^n \), equipped with an attaching map \( \psi : \Sigma \times I^n \to X \).
Two attaching maps \( \psi_1, \psi_2 : \Sigma \times I^n \to X \) are of the same homeomorphism
type if there is a cellular homeomorphism \( h : X \to X \) so that \( h\psi_1 h^{-1} = \psi_2 \).

**Definition 2.18** Let \( X_1 \) and \( X_2 \) be spaces equipped with continuous maps \( p_1, p_2 \)
to a metric space \( Z \). Then a map \( f : X_1 \to X_2 \) is boundedly controlled if
there exists an integer \( m \geq 0 \) so that for all \( z \in Z, r \geq 0, p_1^{-1}(B_r(z)) \subseteq
f(p_2^{-1}(B_{r+m}(z))), \) where \( B_r(z) \) denotes the metric ball in \( Z \) of radius \( r \) about \( z \).
Or, equivalently, there is a constant \( m \geq 0 \) so that
\[
\text{dist}_Z(p_2 \circ f(x), p_1(x)) < m
\]
for all \( x \in X \).

The next proposition is from \[3\].

**Proposition 2.1** Let \( X \) and \( Y \) be a simplicial complexes of bounded geometry
equipped with a map of bounded geometry to a simplicial complex \( Z \) of bounded
geometry. Then any map \( f : X \to Y \) is boundedly controlled only if it has
bounded geometry.

**Proof.** The property of having bounded geometry for a map is similar to injec-
tivity, except that the inverse image is uniformly bounded, rather than being a
point. So if \( q \circ f \) is of bounded geometry, \( f \) must be. Hence if \( f \) is boundedly
controlled, then \( | q \circ f(x) - p(x) | < M \) for all \( x \), so that by the bounded geometry
of the triangulation, \( q \circ f \) has bounded geometry, so that \( f \) does also.

We shall use this result also for smooth maps, but in this case one must take
care, as the following example shows. Consider the map \( f(x) = x^2 \) and the map
$g(x) = \sqrt{x}$ on $\mathbb{R}$. Neither of these maps is $bg$. However, $f \circ g$ is $bg$. This seems to contradict both the observation above and the correspondence between smooth and PL maps. However, this example fails because neither $f$ nor $g$ are simplicial with respect to a $bg$ triangulation of $\mathbb{R}$. Since smooth control maps will be used in the sequel, this phenomenon must be taken into account and smooth maps will be required to be simplicial with respect to a $bg$ triangulation.

We note also that the converse of this proposition is false if taken in its most literal sense: bounded geometry does not imply bounded control. For example, consider multiplication by 2 on $\mathbb{R}$, controlled over itself by the identity. However, every map of bounded geometry $f : X \to Y$ can be considered to be boundedly controlled over $Y$, taking the control maps to be $f$ and the identity, respectively. We will thus consider the two notions as equivalent from now on.

The following definitions are due to Anderson and Munkholm [2].

**Definition 2.19** Let $X$ be a space controlled over a metric space $Z$ by a control map $p$. Denote by $\mathcal{P} G_1(X)$ to be the category whose objects are pairs $(x, K)$ where $K \in \mathcal{P}$ is an object of $\mathcal{P}$ and a morphism $(x, K) \to (y, L)$ is a pair $(\omega, i)$ where $i \in \mathcal{P}(K, L)$ is a morphism in $\mathcal{P}$ from $K$ to $L$ and $\omega$ is a homotopy class of paths in $p^{-1}(L)$ from $y$ to $p^{-1}(i(x))$.

**Definition 2.20** The controlled homotopy groups $\pi_n^c(X, p)$ are defined to be the functor

$$\pi_n^c(X) : \mathcal{P} G_1(X) \to \mathcal{C}$$

where $\mathcal{C}$ is the category of pointed sets, groups or abelian groups defined by setting

$$\pi_n^c(X, p)(x, K) = \pi_n(p^{-1}(K), x)$$

and $\pi_n^c(\omega, i)$ is the composite of the change of basepoint isomorphism $\omega_*$ induced from $\omega$ and the homomorphism induced from the inclusion $i$.

**Definition 2.21** The controlled homology $H_n^c(X, p)$ (with integer coefficients) of a space controlled via $p : X \to Z$ is defined to be the pro-system $H_n(p^{-1}(B(r, z)))$ via the maps $B(r, z) \to B(r + 1, z)$.

**Definition 2.22** If $(X, p)$ and $(Y, q)$ are spaces controlled over $Z$ by control maps $p$ and $q$ respectively, then $X$ is coextensive with $Y$ if there exists an integer $m \geq 0$ so that if $p^{-1}(B_r(z)) \neq 0$ then $q^{-1}(B_{r+m}(z)) \neq 0$ and the same with the roles of $p$ and $q$ reversed.

We shall also use the category $C_M(R)$ for a metric space $M$ introduced by Pedersen and Weibel [39, 51]. It is shown by Anderson and Munkholm [2], pp.263-4, that in the case where the metric space $M$ is path connected, the metric is proper, and satisfies the condition that if $B(r, z) \subset B(s, z)$ then $B(r + 1, z) \subset B(s + 1, z)$, which is clearly satisfied for the case where $M$ is a manifold of bounded geometry, then $C_M(\mathbb{Z} \pi_1(X))$ is equivalent to finitely generated free modules over $\mathbb{Z} \mathcal{P} G_1(X)$, in the sense that there is an additive
functor between them which is an equivalence of categories, which is natural with respect to boundedly controlled maps. This additive functor induces an isomorphism on algebraic K-theory, which is natural with respect to boundedly controlled maps.

The next six definitions are from [29]. We define controlled chains and cochains as in [29] as follows:

**Definition 2.23** An object $A$ in $\mathcal{C}_M(R)$ is a collection of finitely generated free right $R$-modules $A_x$, one for each $x \in M$, such that for each ball $C \subset M$ of finite radius, only finitely many $A_x$, $x \in C$ are nonzero. A morphism $\phi : A \to B$ is a collection of automorphisms $\phi^*_y : A_x \to B_y$ such that there exists $k = k(\phi)$ such that $\phi^*_x = 0$ for $d(x,y) > k$.

The composition of $\phi : A \to B$ and $\psi : B \to C$ is given by $(\psi \circ \phi)^*_y = \sum_{x \in M} \psi^*_y \phi^*_x$. The composition $(\psi \circ \phi)$ satisfies the local finiteness and boundedness conditions whenever $\psi$ and $\phi$ do.

**Definition 2.24** The dual of an object $A$ in $\mathcal{C}_M(R)$ is the object $A^*$ with $(A^*)_x = A^*_x = \text{Hom}_R(A_x, R)$ for each $x \in M$. $A^*_x$ is naturally a left $R$-module, which we convert to a right $R$-module by means of the anti-involution. If $\phi : A \to B$ is a morphism, then $\phi^* : B^* \to A^*$ and $(\phi^*)^*_y(h) = h \circ \phi^*_y$, where $h : B_x \to R$ and $\phi^*_y : A_y \to B_x$. $\phi^*$ is bounded whenever $\phi$ is. Again, $\phi^*$ is naturally a left module homomorphism which induces a homomorphism of right modules $B^* \to A^*$ via the anti-involution.

**Definition 2.25** Consider a map $X \to M$

i. The map $p : X \to M$ is eventually continuous if there exist $k$ and a covering $\{U_\alpha\}$ of $X$, such that the diameter of $p(U_\alpha)$ is less than $k$.

ii. A bounded CW complex over $M$ is a pair $(X,p)$ consisting of a CW complex $X$ and an eventually continuous map $p : X \to M$ such that there exists $k$ such that $\text{diam}(p(C)) < k$ for each cell $C$ of $X$. $(X,p)$ is called proper if the closure of $p^{-1}(D)$ is compact for each compact $D \subset M$. We consider $(X,p_1)$ and $(X,p_2)$ to be the same, if there exists $k$ so that $d(p_1(x), p_2(x)) < k$ for all $x$.

**Definition 2.26** Consider a bounded CW complex $(X,p)$

i. The bounded CW complex $(X,p)$ is (-1)-connected if there is a $k \in \mathbb{R}_+$ so that for each point $m \in M$, there is a point $x \in X$ such that $d(p(x), m) < k$.

ii. $(X,p)$ is 0-connected if for every $d > 0$ there exist $k = k(d)$ so that if $x, y \in X$ and $d(p(x), p(y)) \leq d$, then $x$ and $y$ may be joined by a path in $X$ whose image in $M$ has diameter $< k(d)$. Note that 0-connected does not imply -1-connected.

**Definition 2.27** Let $p : X \to M$ be 0-connected, but not necessarily (-1)-connected.

i. $(X,p)$ has trivial bounded fundamental group if for each $d > 0$ there exist $k = k(d)$ so that for every loop $\alpha : S^1 \to X$ with $\text{diam}(p \circ \alpha(S^1)) < d$, there is
a map $\pi : D^2 \to X$ so that the diameter of $p \circ \pi : D^2 \to X$ so that the diameter of $p \circ \pi(D^2)$ is smaller than $k$.

ii. $(X, p)$ has bounded fundamental group $\pi$ if there is a $p$ cover $\tilde{X}$ so that $\tilde{X} \to M$ has trivial bounded fundamental group.

**Definition 2.28** If $X$ is a CW complex, we will denote the cellular chains of $\tilde{X}$ by $C_\#(X)$ considered as a chain complex of free right $\mathbb{Z}_\pi(X)$-modules. When $p : X \to M$ is a proper bounded CW complex with bounded fundamental group, we can consider $C_\#(X)$ to be a chain complex in $C_M(\mathbb{Z}_\pi(X))$ as follows: For each cell $C \in X$, choose a point $c \in C$ and let $D_\#(X)_y$ be the free submodule of $C_\#(X)$ generated by cells for which $p(c) = y$. The boundary map is bounded, since cells have a fixed maximal size. We will denote the cellular chains of $\tilde{X}$ by $D_\#(X)$ when we consider them as a chain complex in $C_M(\mathbb{Z}_\pi(X))$ and by $C_\#(X)$ when we consider them as an ordinary chain complex of $\mathbb{Z}_\pi(X)$ modules. We will denote $D_\#(X)^\ast$ by $D_\#(X)$. If $(X, \partial X)$ is a bounded CW pair, $D_\#(X, \partial X)$ denotes the relative cellular chain complex regarded as a chain complex in $C_M(\mathbb{Z}_\pi(X))$.

The following two theorems are modifications of the corresponding ones in [2] and can be found in [3].

**Theorem 2.4 (Whitehead Theorem)** If $f : (X, p) \to (Y, q)$ is a map of bounded geometry of CW complexes of bounded geometry, controlled by maps to $Z$ of bounded geometry, then $f$ is a bg homotopy equivalence if $(Y, q)$ is coextensive with $(X, p)$ and for all $n \geq 0$, $f_* : \pi_c^n(X, p) \to f_* \pi_c^n(Y, q)$ is an isomorphism.

**Theorem 2.5 (Hurewicz Theorem)** If $X$ is a simplicial complex of bounded geometry, then

i. $\pi_1(X)^{ab} = H_1(X)$

ii. If $\pi_i(X) = 0$ for $i \leq n - 1$ and $n \geq 2$ then, $H_i^c(X) = 0$ for $i \leq n - 1$, and $H_n^c(X) = \pi_n(X)$.

### 3 Uniformly Finite Homology

In this section we review $L^\infty$ cohomology and uniformly finite homology as defined by Gromov, Block-Weinberger [10], Roe [57] and Gersten [33, 34]. Whyte [67] has shown that the Poincaré dual of the $A$-class in $0$-dimensional uniformly finite homology is an obstruction to the existence of a metric of positive scalar curvature on an open manifold.

**Definition 3.1** A normed abelian group is an abelian group $G$ equipped with a norm function:

$$| \cdot | : G \to \mathbb{R}_+$$

which is not necessarily continuous, but so that the induced function on the Cayley graph of $G$ is non-decreasing as one moves away from the identity element in $G$. 

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Definition 3.2 Let $X$ be a bg simplicial complex. The $i$-dimensional fine uniformly finite homology groups of $X$ with coefficients in the normed group $(G, |\cdot|)$, denoted $H^{i\text{uff}}_*(X;G,|\cdot|)$ are defined to be the homology groups of the complex of infinite simplicial chains whose coefficients are in $l^\infty$ with respect to the norm $|\cdot|$ on $G$. Define the group of $q$-chains $C^{q\text{uff}}_*(X;G,|\cdot|)$ to be the group of formal sums of $q$-simplices in $X$, $c = \sum a_\sigma \sigma$ so that there exists $K > 0$ depending on $c$ so that $|a_\sigma| \leq K$ and the number of simplices $\sigma$ lying in a ball of given size is uniformly bounded. The boundary is defined to be the linear extension of the simplicial boundary.

We shall also need uniformly finite cohomology with coefficients in the normed group $(G, |\cdot|)$.

Definition 3.3 Let $X$ be a bg simplicial complex. The $i$-dimensional fine uniformly finite cohomology groups of $X$ with coefficients in the normed group $(G, |\cdot|)$, denoted $H^{i\text{uff}}_*(X;G,|\cdot|)$ are defined to be the cohomology groups of the complex of infinite simplicial cochains $c \in \text{Hom}(C^q_*(X;G),G)$ which satisfy $|c(\sigma)| \leq K$ for all simplices $\sigma \in X$ and fixed $K > 0$ depending on $c$. The coboundary is defined to be the simplicial coboundary.

Definition 3.4 Let $f : X \to Y$ be a simplicial map of bounded geometry, and let $(G,|\cdot|)$ be a normed group. Then the induced map

$$f_* : H^{i\text{uff}}_*(X;G,|\cdot|) \to H^{i\text{uff}}_*(Y;G,|\cdot|)$$

is defined by $f_*([c]) = [f \circ c]$, where $c : C \to X$, $C$ a simplicial complex and $c$ a bg simplicial map, represents a class in $H^{i\text{uff}}_*(X;G,|\cdot|)$. This is well-defined because $f$ commutes with the boundary.

We now set a convention for the coefficients which will be used throughout the rest of the paper: the group $\mathbb{R}$ will always be normed with the absolute value, as will $\mathbb{Z}$. The groups $\text{Wh}(\pi)$, $K_*(\mathbb{Z}\pi)$, $\text{S}^{PL}(M \times D^n, \partial)$, $\text{S}^{TOP}(M \times D^n, \partial)$ can be written for the groups $\pi$ considered in this paper, as the countable direct sum of finite abelian groups with a free abelian group. The terms in the direct sum will be given the absolute value norm as subsets of $C$. Assuming this convention, we will drop the symbol $|\cdot|$ in uniformly finite homology from now on.

Definition 3.5 We recall for completeness the definition of locally finite homology. Let $X$ be a simplicial complex which is also a metric space. Define $C^{\text{uff}}_*(X;G)$, $G$ an abelian group, to be the group of formal sums of simplices in $X$, $c = \sum a_\sigma \sigma$, $a_\sigma \in G$ so that the number of simplices $\sigma$ lying in a particular ball in $X$ is bounded. Not that $H^{\text{uff}}_*(X;G) = H^{\text{uff}}_*(X;G)$ for any finite abelian group $G$.

We now give some calculations of $H^{\text{uff}}_*(X;\mathbb{Z})$ which show:
i. It depends on the metric structure of $X$ and not just on its topology: e.g. $H^n$ and $\mathbb{R}^n$ have different uf homology, even though they are diffeomorphic.

ii. It can be very large: for $X = \mathbb{R}$, $H^0_{uf}(X; \mathbb{Z})$ is an uncountably generated $\mathbb{R}$-module.

iii. There are deep connections between properties of $H^*_uf(X; \mathbb{Z})$ and infinite group theory. For example amenability, Gromov hyperbolicity and weak forms of rigidity are determined by $H^*_uf(X; \mathbb{Z})$.

We will recall only the statements of the results in [3]. For sketches of proofs see [3].

**Proposition 3.1** The 0-dimensional uniformly finite homology group of $\mathbb{R}$ is

$$H^0_{uf}(\mathbb{R}; \mathbb{Z}) = H^0_{uf}(\mathbb{R}; \mathbb{R}) = \{ \phi : \mathbb{Z} \to \mathbb{Z} | \| \delta \phi \|_\infty < \infty \}$$

where $\delta \phi(n) = \phi(n) - \phi(n-1)$ and $\| \cdot \|_\infty$ is the $L^\infty$ norm.

**Theorem 3.1** Let $X$ be a simply connected symmetric space of non-positive curvature of rank $r$. Then

$$H^i_{uf}(X; \mathbb{Z}) = 0$$

for $i \leq n - r - 1$

$$H^i_{uf}(X; \mathbb{Z}) \neq 0$$

for $i \geq n - r$.

**Proposition 3.2** Let $X$ be a manifold with curvature pinched between two negative constants. Then

$$H^i_{uf}(X; \mathbb{Z}) = 0$$

for $i < n - 1$.

Gersten [34] has proven theorems relating Gromov hyperbolicity to the vanishing of $H^i_{uf}(X; \mathbb{Z})$. See [34] for details.

Whether or not an infinite group is amenable is determined by its uniformly finite homology.

In defining amenability, we will ultimately quote [10] where essentially the following theorem is proven: a bg simplicial complex $X$ is non-amenable if and only if $H^0_{uf}(X; \mathbb{Z}) = 0$.

However, before we present this definition, we will review the classical definitions of amenability.

**Definition 3.6** An infinite discrete group $\Gamma$ is said to be amenable if and only if it satisfies the two equivalent conditions:

i. There is a bounded linear functional $\mu : l^\infty(\Gamma) \to \mathbb{R}$ with $\inf_{g \in \Gamma} (f(g)) \leq \mu(f) \leq \sup_{g \in \Gamma} (f(g))$ and for all $g \in \Gamma$, $\mu(g \cdot f) = \mu(f)$, where $g \cdot f(x) = f(g^{-1}x)$. 

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ii. For every \( k \) in the interval \((0, 1)\) and arbitrary finite set of elements \( a_1, \ldots, a_n \) in \( \Gamma \) there is a finite subset \( E \) of \( \Gamma \), \( E \neq 0 \), so that

\[
\#(E \cap a_i \cdot E) \geq k\#(E)
\]

for \( i = 1, \ldots, n \).

Condition (i) is due to Von Neumann, and was the first criterion for amenability to be introduced. Condition (ii) is due to F"olner and is known as the F"olner condition.

The following criterion for amenability applies to finite dimensional simplicial complexes, and is due to Brooks and Gromov.

**Definition 3.7** An \( n \)-dimensional simplicial complex of bounded geometry is said to be amenable if there is a sequence of \( n \)-dimensional compact subcomplexes \( X_i \subset X \) so that the \( X_i \) exhaust \( X \) and \( \text{Vol} \partial X_i / \text{Vol} X_i \to 0 \), where \( \text{Vol} \partial X_i \) means the number of \( n-1 \)-simplices on the boundary of \( X_i \).

This criterion is related to amenability of groups in the following way:

**Proposition 3.3** An infinite discrete group \( \Gamma \) is amenable if and only if the universal cover of any compact manifold with fundamental group \( \Gamma \) is amenable.

We now come to the theorem of Gromov and Block and Weinberger \[10\]. Gromov found a criterion for the amenability of an infinite covering using differential forms. Block and Weinberger generalized this to arbitrary simplicial complexes of bounded geometry. Recall that \( H^p_\beta(X; R) \) is the bounded de Rham cohomology of \( X \) with real coefficients, where \( X \) is a smooth manifold of bounded geometry.

**Definition 3.8** Denote by \( \Omega^p_\beta(M) \) the Banach space of \( p \)-forms on a complete, oriented Riemannian manifold \( M \) which are bounded in the norm

\[
\| \alpha \| = \sup \{|\alpha(x)| + |d\alpha(x)| : x \in M\}
\]

This gives rise to a complex \( d_i : \Omega^i_\beta(M) \to \Omega^{i+1}_\beta(M) \). The bounded de Rham groups are defined by

\[
H^p_\beta(M) = \frac{\text{Ker} \ d_p}{\text{Im} \ d_{p-1}}
\]

Note that we are not taking the closure of \( \text{Im} \ d \) in this definition.

**Theorem 3.2** A non-compact manifold of smooth bounded geometry \( X \) is amenable if and only if

\[
H^0_\text{eff}(X; \mathbb{Z}) \neq 0
\]

or equivalently,

\[
H^0_\beta(X; \mathbb{R}) \neq 0
\]
We now adopt for the purpose of the next proposition, the following criterion for amenability, due to Block-Weinberger.

**Proposition 3.4** Let $X = B\Gamma$ be the classifying space of $\Gamma$ considered as a simplicial complex, $\tilde{X}$ an amenable covering of $X$. Then the map $H_\ast(X; \mathbb{R}) \to H_\ast^{uff}(\tilde{X}; \mathbb{R})$ given by taking each cycle to its lift to $\tilde{X}$ is injective.

**Proof.** The following proof was suggested to the author by J.Block and S.Weinberger. We use the invariant mean to construct a left inverse. The main step of the proof is to show that the complexes $C_\ast^{uff}(\tilde{X}; \mathbb{Z})$ and $C_\ast(\Gamma; l_\infty(\Gamma))$ are isomorphic. Here $\Gamma$ acts on $l_\infty(\Gamma)$ via $\gamma \cdot f(x) = f(\gamma \cdot x)$, for any $f : \Gamma \to \mathbb{R}$ in $l_\infty(\Gamma)$ and $C_\ast(\Gamma; l_\infty(\Gamma))$ is the standard bar resolution with respect to this action. The main step is just a matter of unraveling the various definitions. Once we have the desired isomorphism, one constructs the map $C_\ast(\Gamma; l_\infty(\Gamma)) \to C_\ast(X; \mathbb{R})$ induced by applying the invariant mean to the coefficients. This gives the desired left inverse and the proposition follows.

We have produced an injection with real coefficients. Use of the de Rham theorem proven below will result in an injection with rational coefficients.

We will prove next the de Rham and Poincaré duality theorems due to the author, J. Block and S. Weinberger, which also is proven in [4]. We first need a refinement of the notion of uniform subdivision.

**Definition 3.9** A regular uniform subdivision is defined in the following manner. Let $\sigma = [p_0, \ldots, p_m]$ be a simplex in $\mathbb{R}^k$, $k \geq m$. The vertices of the standard subdivision $S\sigma$ of $\sigma$ are the points

$$p_{ij} = \frac{1}{2}(p_i + p_j), i \geq j.$$

Define a partial ordering of the vertices of $S\sigma$ by setting

$$p_{ij} \leq p_{kl},$$

if $i \geq k, j \geq l$. The simplices of $S\sigma$ are the increasing sequences of vertices with respect to the above ordering.

We define a regular uniform subdivision to be any uniform subdivision which is a sequence of standard subdivisions.

Regular uniform subdivision of a bg simplicial complex $K$ produces a bg simplicial complex $K'$ and induces a map $s : C_\ast^{uff}(K; G) \to C_\ast^{uff}(K'; G)$ which has bounded norm.

**Theorem 3.3 (de Rham theorem)** Let $M$ be a smooth $n$-dimensional manifold of bounded geometry. Then there is an isomorphism

$$H_\ast^{uff}(M; \mathbb{R}) \simeq H_\ast(M).$$
Proof. The idea is to use Whitney and de Rham maps, with the triangulation constructed by Theorem 2.1 serving to show that these maps are well-defined. The proof here imitates the ones found in \[24\], \[66\]. Define the de Rham map

\[ \int : \Omega^i(M) \to C^i_{\text{aff}}(M; \mathbf{R}) \]

by

\[ \int (\omega) \cdot \sigma = \int_{\sigma} \omega. \]

The de Rham map commutes with regular uniform subdivision. To define the Whitney map, which is the chain homotopy inverse of the de Rham map, let \( c_{\sigma} \) be the cochain which assigns the value 1 to \( \sigma \) and the value 0 to every other simplex.

For each point \( q \) in \( M \) write \( q \) in barycentric coordinates as 

\[ q = \sum_{\nu} \alpha(q) q_{\alpha}, \]

where \( q_{\alpha} \) runs over the vertices of the triangulation, and \( \alpha \) is the corresponding index. For each \( \alpha \), let \( Q_{\alpha} \) be subsets of \( M \) so that \( Q_{\alpha} \) is the set of all \( p \) with \( \nu_\alpha(p) \geq \frac{1}{n+1} \); \( Q'_\alpha \) is the set of all \( q \) with \( \nu_\alpha(q) \leq \frac{1}{n+2} \). Then \( Q_{\alpha} \subset \text{star}(q_{\alpha}) \), \( \text{Int}(Q'_\alpha) \subset M - \text{star}(q_{\alpha}) \). Let \( \phi_\alpha(p) \) be a smooth non-negative real function in \( M \) which has all of its derivatives uniformly bounded over \( \alpha \) and so that each is positive in \( Q_{\alpha} \) and zero in \( Q'_\alpha \). Construct the partition of unity

\[ \phi_\alpha(p) = \frac{\phi'_\alpha(p)}{\sum_{\beta} \phi'_\beta(p)} \]

We will choose normalizations of the \( \phi'_\alpha \) below, so we will assume \( \phi'_\alpha \) normalized suitably for the moment. Take any \( p \in M \); since \( p \) has at most \( n+1 \) non-zero barycentric coordinates at least one of these, say \( \nu_\beta(p) \) is \( \geq \frac{1}{n+1} \). Hence \( p \in Q_\beta \), \( \phi'_\beta(p) > 0 \) and \( \phi_\alpha(p) \) is defined for all \( \alpha \). Define the Whitney map \( W \) on each \( c_{\sigma} \) by

\[ W(c_{\sigma}) = r! \sum_{i=0}^{r} (-1)^i \phi_{\alpha_i} d\phi_{\alpha_i} \land ... \land d\phi_{\alpha_i} \land ... \land d\phi_{\alpha_r}, \]

where \( \sigma = q_{\alpha_0}...q_{\alpha_r} \), and \( \phi_{\alpha} \) is defined above. We can then extend by linearity. Note that

\[ \text{supp} W(c_{\sigma}) \subseteq \sigma. \]

Furthermore, because the triangulation is uniform, the resulting form is bounded. In fact, the map \( W \) defines a bounded map on chains in the norm defined by taking the supremum of the coefficients of the chain.

We now choose normalizations so that the de Rham theorem will be true. Choose normalizations of the \( \phi'_\alpha \) so that

\[ \int W(c_{\sigma_\alpha})\sigma_\beta = \int_{\sigma_\beta} W(c_{\sigma_\alpha}) = \delta^\beta_\alpha, \]

since \( \text{supp} W(c_{\alpha}) \subset \sigma_\alpha \). Using this normalization, we have inductively for \( \sigma' \) a face of \( \sigma \),

\[ \int_{\sigma} W(c_{\sigma}) = \int_{\sigma} W(c_{\sigma'}) = \int_{\partial\sigma} W(c_{\sigma'}) = \int_{\sigma'} W(c_{\sigma'}) = 1. \]
The normalize all of the lower dimensional cochains. Define $f^*$ and $W^*$ to be the maps induced on cohomology by $f$ and $W$ respectively.

We claim that $f^*$ and $W^*$ are inverses of each other. In fact, by Stokes’ theorem we can observe that $f$ and $W$ are chain maps. That $fW = Id$ is proved by observing that the normalization conditions prove the result for each $c(\sigma)$ and then extending by linearity. That $W^* f^* = Id$ follows from a series of easy estimates. Let $\omega$ be a closed differential form on $\Omega^\beta(M; \mathbb{R})$. Suppose the cohomology class $[\int \omega] = 0$ in $H^{n_i}_{\text{aff}}(M; \mathbb{R})$. Since integration commutes with regular uniform subdivision, we have $[\int' \omega] = 0$, where $\int'$ denotes the integration map over a uniformly subdivided triangulation. Fix $\epsilon_1 > 0$. We claim that there exists a regular uniform subdivision so that

$$\| \omega - W \cdot \int \omega \| < \epsilon_1$$

Where $\| \cdot \|$ is the norm on $\Omega^\beta(M)$. Note that we have the estimate

$$| \omega(x) - W \cdot \int \omega | \leq C \cdot \text{diam}(\tau) \cdot \sup_{x \in \tau} | \frac{\partial \omega}{\partial x} |$$

over each simplex $\tau$ of the triangulation, where $C$ can be chosen independently of $\omega, \tau$. One the obtains an estimate in terms of the mesh of the triangulation $h$

$$\| \omega - W \cdot \int \omega \| \leq \sum_{\tau \in K} \int_{\tau} | \omega(x) - W \cdot \int \omega(x) | dV$$

$$\leq 4C \cdot h^N m \| \omega \|$$

where $m$ denotes the multiplicity of the intersections of the bg coordinate charts of $M$. Since $\int \omega$ is a boundary, we can find a cochain $f$ so that

$$\| \int \omega - \delta f \| \leq \epsilon_2$$

Then

$$\| \omega - dW f \| \leq \| \omega - W \cdot \int \omega \| + \| W \| \cdot \| \int \omega - \delta f \|$$

$$\leq \epsilon_1 + \| W \| \cdot \epsilon_2$$

Since $\epsilon_2$ is chosen independently of $\epsilon_1$ and both can be made arbitrarily small, we obtain that the cohomology class $[\omega] = 0$. This proves the de Rham theorem.

We shall also prove Poincaré duality, following the proof in \[35\].

**Theorem 3.4 (Poincaré duality)** If $M$ is a manifold of bounded geometry, there is an isomorphism

$$H^{n-i}_{\text{aff}}(M; Z) = H^{n_i}_{\text{aff}}(M; Z)$$

To prove the theorem we need the following definitions from \[55\]:

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Definition 3.10 The bg simplicial complex $K$ is ordered, so that for each simplex $\sigma \in K$ the set

$$K^*(\sigma) = \{ \tau \in K \mid \tau > \sigma, \mid \tau \mid = \mid \sigma \mid +1 \}$$

$$K_*(\sigma) = \{ \tau \in K \mid \tau < \sigma, \mid \tau \mid = \mid \sigma \mid -1 \}$$

Definition 3.11 The star and link of a simplex $\sigma \in K$ in a simplicial complex $K$ are the subcomplexes defined by

$$\text{star}_K(\sigma) = \{ \tau \in K \mid \sigma \tau \in K \}$$

$$\text{link}_K(\sigma) = \{ \tau \in K \mid \sigma \tau \in K, \sigma \cap \tau = \emptyset \}$$

where $\sigma \tau$ is the simplex spanned by $\sigma$ and $\tau$. The dual cell of $\sigma$ is the contractible subcomplex of the barycentric subdivision $K'$ defined by

$$D(\sigma, K) = \{ \hat{\sigma}_0 \hat{\sigma}_1 ... \hat{\sigma}_p \in K' \mid \sigma \leq \sigma_0 < \sigma_1 < ... < \sigma_p \}$$

where $\hat{\sigma}$ is the barycenter of $\sigma$. The barycentric subdivision of the link of $\sigma \in K$ is isomorphic to the boundary of the dual cell $D(\sigma, K)$

$$(\text{link}_K(\sigma))' \simeq \partial D(\sigma, K)$$

The star and link in $K'$ of the barycentre $\hat{\sigma} \in K'$ of $\sigma \in K$ are given by the joins

$$\text{star}_{K'}(\hat{\sigma}), \text{link}_{K'}(\hat{\sigma}) = \partial \sigma^* * (D(\sigma, K), \partial D(\sigma, K))$$

Proof of Poincaré duality. We will prove Poincaré duality following the proof for compact manifolds in [35]. Let $A$ and $B$ be two cycles in a PL manifold $M$, intersecting transversely at $p$. Let $\iota_p(A \cdot B)$ be the signed intersection of $A$ with $B$ at $p$ (which is either +1 or -1). Note that $D(\sigma)$ transversely intersects $\sigma$ at a point $p$. We choose orientations of $\sigma$ and $D(\sigma)$ so that $\iota_p(\sigma, D(\sigma)) = +1$ for any simplex $\sigma$ in $K$.

We now relate the boundary operator $\partial$ on the complex $\{\sigma_\alpha\}$ to the coboundary operator $\delta$ on $\{D(\sigma_\alpha)\}$. Note first that if $\sigma$ has vertices $\sigma^0, ..., \sigma^k$, then the dual cell is given by the $(k+1)$-fold intersection of the dual $n$-cells to the vertices. So the cells appearing in the coboundary $\delta D(\sigma)$ will just be the $k$-fold intersection of the dual cells of the faces of $\sigma$. We have the relation

$$\delta(D(\sigma)) = (-1)^{n-k+1}D(\partial \sigma)$$

(see [35] pp.54-55 for a proof). From this we see that the map $\sigma \to D(\sigma)$ induces an isomorphism between the complex $(C^*_\text{eff}(K, \mathbb{Z}), \partial)$ of uniformly finite chains of the original simplicial decomposition of $M$ and the complex $(C^*_\text{eff}(K', \mathbb{Z}), \delta)$ of cochains in the dual cell decomposition. This proves Poincaré duality.

We will also need the analogue of the Eilenberg-Zilber theorem for ordinary uniformly homology and cohomology theories. This theorem is due to the author and J.Block and also appears in [4].
Theorem 3.5 (Eilenberg-Zilber Theorem) Let $C_{n}^{uf}(X; \mathbb{Z})$ be the uniformly finite cellular $n$-chains with coefficients in $\mathbb{Z}$. Then

$$C_{n}^{uf}(X \times Y; \mathbb{Z}) = \bigoplus_{i+j=n} C_{i}^{uf}(X; C_{j}^{uf}(Y; \mathbb{Z}))$$

where $C_{i}^{uf}(X; C_{j}^{uf}(Y; \mathbb{Z}))$ means uniformly finite $i$-chains with values in the uniformly finite $j$-chains. More explicitly, we take uniformly finite chains with coefficients in the normed group $C_{j}^{uf}(Y; \mathbb{Z})$, where the norm of a chain is defined to be the supremum of its coefficients. The boundary map is given by the Leibnitz rule.

Proof Write $M = \{ \sigma_{n}^{\alpha} \}_{\alpha,k}$ and $N = \{ \sigma_{n}^{\alpha} \}_{\alpha,k}$. The products $\sigma_{n}^{\alpha} \times \sigma_{n}^{\beta}$ give a cell decomposition of the product $M \times N$ with the boundary operator

$$\partial(\sigma_{n}^{\alpha} \times \sigma_{n}^{\beta}) = \partial\sigma_{n}^{\alpha} \times \sigma_{n}^{\beta} + (-1)^{k}\sigma_{n}^{\alpha} \times \partial\sigma_{n}^{\beta}.$$  

We compute the chains as cellular chains by the triangulations of $X \times Y$, $X$ and $Y$. In general, we have:

$$(X \times Y)^{(n)} = \bigcup_{j} X^{(j)} \times Y^{(n-j)}$$

We note also that $l^{\infty}(S \times T) = l^{\infty}(S;l^{\infty}(T))$, where $S$ and $T$ are discrete sets.

We claim that the homology of this chain complex is $H_{n}^{uf}(X \times Y)$. We have a map

$$C_{n}^{uf}(M \times N; \mathbb{Z}) \to \bigoplus_{i+j=n} C_{i}^{uf}(M; C_{j}^{uf}(N; \mathbb{Z}))$$

which we have shown to be an isomorphism. By the construction of the boundary, this induces a map on homology. This clearly takes the boundary to the boundary, and is thus an isomorphism.

4 The Whitehead Group

We first recall some definitions regarding the bg Whitehead group of a simplicial complex of bounded geometry.

Definition 4.1 Let $X$ be a CW complex of bounded geometry. An expansion of bounded geometry is a bg CW complex $Y$ so that

i. $(Y,X)$ is a bg CW pair.

ii. $Y = X \cup_{f} (\Sigma \times I^{r}) \cup_{g} (\Sigma \times I^{r+1})$ for bg $(r+i)$-cells $\Sigma \times I^{r+i}$, $i = 0,1$ and attaching maps $f,g$.

iii. There is a characteristic map $\psi_{r+1} : \Sigma \times I^{r+1} \to Y$ for the bg $(r+1)$-cell so that $\psi_{r+1} \mid \Sigma \times I : \Sigma \times I^{r} \to Y$ is characteristic for the bg $r$-cell. If $Y$ is a bg expansion of $X$, $Y$ is said to bg collapse to $X$. 

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Definition 4.2 Let $DR^{bg}$ be the collection of all pairs $(Y, X)$ so that $X$ is a bg strong deformation retract of $Y$. The collection of equivalence classes of such a pairs under elementary bg expansions and collapses rel $X$ is denoted $Wh^{bg}(X)$.

We refer [2] for the definitions of controlled bases, controlled modules and $RPG_1(X)$-modules.

Definition 4.3 Let $X$ be a metric space, $R$ a ring. A controlled basis is a pro-object defined as follows. The basis is a pair $(S, \sigma)$ where $S$ is a set and $\sigma$ is a function from $S$ to the collection of open sets in a control space $X$. A morphism from $(S, \sigma)$ to $(T, \rho)$ is given by a map $\alpha : S \rightarrow T$ along with a natural transformation $\rho \alpha : C^n \sigma$, where $C^n$ is the operation on metric balls in the control space which increases the radius by $n$.

Definition 4.4 Let $(S, \sigma)$ be a controlled basis. The free $RPG_1(X)$-module $F(\sigma)$ with basis $(S, \sigma)$ is a functor from $\mathcal{P}G_1(X)$ to the category of $R$-modules defined as follows:

i. For any $b$ an object in $\mathcal{P}G_1(X)$, $F(\sigma)(b)$ is the free $R$-module on $\{(\beta, s) \mid s \in S, \beta \in \mathcal{P}G_1(X)(\sigma(s), b)\}$, where $\mathcal{P}G_1(X)(\sigma(s), b)$ is the set of morphisms from $\sigma(s)$ to $b$ in $\mathcal{P}G_1(X)$.

ii. For any $\gamma \in \mathcal{P}G_1(X)(b, c)$, $\gamma : F(\sigma)(b) \rightarrow F(\sigma)(c)$ has $\gamma(\beta, s) = (\gamma\beta, s)$.

Definition 4.5 The category of controlled free bg $\mathcal{Z}PG_1(X)$ modules is defined to be the category of controlled modules so that in any ball of fixed radius the modules fall into a finite number of types. Morphisms are defined to be morphisms of the modules so that if the control space is partitioned into neighborhoods of a fixed radius, the restrictions fall into a finite number of equivalence classes. By abuse of notation, we will denote this category by $\mathcal{Z}PG_1(X)^{bg}$.

Definition 4.6 We define $K_1(\mathcal{Z}PG_1(X)^{bg})$ to be the abelian group generated by $[F, \alpha]$ where $F$ is a controlled free bg module and $\alpha$ is an automorphism of $F$ so that

i. $[F, \alpha] = [F', \alpha']$ if there is an isomorphism $\phi : F \rightarrow F'$ so that $\phi\alpha = \alpha'\phi$.

ii. $[F \oplus F', \alpha \oplus \alpha'] = [F, \alpha] + [F, \alpha']$

iii. $[F, \alpha\beta] = [F, \alpha] + [F, \beta]$.

Definition 4.7 $Wh^{bg}(\mathcal{P}G_1(X))$ is defined to be the quotient of $K_1(\mathcal{P}G_1(X)^{bg})$ defined by taking the quotient by the subgroup of elements of the form

$[F(\sigma), u_{F(\sigma)}]$

and

$[F(\sigma), F(\alpha, \nu)]$

where $(S, \sigma)$ is any bg basis over $\mathcal{P}G_1(X)$, $u_{F(\sigma)}$ is multiplication by a unit, and $F(\alpha, \nu)$ is an automorphism of bases.
Definition 4.8 Let $\mathcal{A}$ be a small additive category. The idempotent completion $\hat{\mathcal{A}}$ is the category with objects morphisms $p: A \to A$ of $\mathcal{A}$ so that $p^2 = p$ and morphisms given by morphisms $\phi: A_1 \to A_2$ so that $\phi = p_1 \phi p_2$, where $p_i: A_i \to A_i$ are the source and target.

Definition 4.9 $K_{0}^{bg}(X)$ the $bg$ projective class group of $X$, where $X$ is a $bg$ simplicial complex, is defined to be $K_0$ of the idempotent completion of the category $\mathbf{ZPG}_1(X)^{bg}$. There is a homomorphism

$$\text{rank}: K_{0}^{bg}(X) \to H_{0}^{\text{uf}}(X; \mathbb{Z})$$

given as follows. Let $m \in K_{0}^{bg}(X)$ be a given element. Then one can find a representative for $m$ which has basis elements only at each vertex of the simplicial complex $X$. Thus to a given vertex one can naturally associate a free module. Define $\text{rank}(m)$ to be the uniformly finite 0-chain $\text{rank}(m) = \sum r_x x$, where $x$ is the rank of the free module constructed above. This clearly gives a map $K_{0}^{bg}(X) \to C_{0}^{\text{uf}}(X; \mathbb{Z})$, and we observe that by taking infinite process tricks into account which cancel the ranks, we can pass to homology. The kernel of this map is the reduced $bg$ projective class group of $X$ and is denoted $\tilde{K}_{0}^{bg}(X)$. It measures the obstruction to a $bg$ projective module being free.

Definition 4.10 Let $X$ be a $bg$ simplicial complex. Define the group $K_{-i}^{bg}(X)$, for $i > 0$, to be $K_{-i}(\mathbf{ZPG}_1(X)^{bg})$.

The following theorem is proven in [3]:

**Theorem 4.1** There is an isomorphism between the geometric Whitehead group, and the algebraic Whitehead group

$$\text{Wh}^{bg}(M) \simeq \text{Wh}^{bg}(\mathbf{PG}_1(M)).$$

Before stating the $bg$ s-cobordism theorem, we remark on the necessary notions of handlebody theory. It is a classical fact that one can construct a handlebody from a triangulation of a PL manifold, so that Theorem 2.1 yields a handlebody decomposition for a $bg$ PL manifold. To prove the $bg$ s-cobordism theorem, which will relate $bg$ h-cobordisms to the $bg$ controlled Whitehead group, we need to introduce a $bg$ controlled handlebody decomposition. This is identical to the controlled handlebody decomposition of [2], except that one uses the triangulation introduced in section 2 in place of the usual triangulation. Furthermore, all of the handle constructions and operations work identically as in [2] except that one must be carried out in a uniform manner. $bg$ transversality must also be used in place of ordinary transversality everywhere. We omit the details.

**Theorem 4.2 (s-cobordism theorem)** Let $M$ be a PL manifold of bounded geometry controlled by a $bg$ map over a uniformly contractible space $Z$. Then $Wh^{bg}(M)$ is in one-to-one correspondence with set of $h$-cobordisms over $M$, whenever $M$ is a PL manifold of bounded geometry of dimension $\geq 5$. In particular, if $W$ is a $bg$ h-cobordism which is $bg$ simple homotopy equivalent to one end. Then it is $bg$ PL homeomorphic to $M \times I$. 

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Theorem 4.3 (Smooth s-cobordism theorem) A smooth bg s-cobordism of dimension $\geq 6$ is bg diffeomorphic to a product.

5 Surgery Theory

The first four definitions are taken from [3]. They are all based on the controlled surgery theory of Ferry-Pedersen [29]. See also [37].

Definition 5.1 A Poincaré duality space $Y$ of bounded geometry over a simplicial complex $X$ of bounded geometry is defined to be a simplicial complex of bounded geometry over $X$ so that there is a fundamental class $[Y]$ in the top dimensional locally finite homology $[Y] \in H_{lf}^{n}(Y; \mathbb{Z})$ of $Y$ so that taking the cap product $[Y] \cap -$ : $D_{n}^+(Y) \to D_{n-1}^-(Y)$ induces a homotopy equivalence of controlled chain complexes. $Y$ is a simple bg Poincaré duality space if the torsion of $[Y] \cap -$ is trivial in $Wh_{bg}(Y)$.

Definition 5.2 Let $p : (Y, \partial Y) \to X$ be a proper bounded CW pair so that $X$ has bounded fundamental group $\pi$. The pair $(Y, \partial Y)$ is an $n$-dimensional bg Poincaré duality pair if $\partial Y$ is an $(n-1)$-dimensional bg Poincaré complex with orientation double covering the pullback of the orientation double covering on $X$ and if there is an element $[Y] \in H_{lf}^{n}(Y, \partial Y; \mathbb{Z})$ such that $[Y] \cap -$ : $D_{n}^+(X) \to D_{n-1}^-(X, \partial X)$ is a homotopy equivalence of chain complexes. $(Y, \partial Y)$ is a simple bg Poincaré duality space, if the torsion of $[Y] \cap -$ is trivial in $Wh_{bg}(Y)$.

Definition 5.3 A bg spherical fiber space $E$ over a bg simplicial complex $X$ is a bundle $p : E \to X$ with fiber a sphere $S^{k}$ so that $E$ is bg simplicial complex and $p$ is a bg map.

The following definition is from [29]:

Definition 5.4 Let $(X, p)$ be a bounded Poincaré duality space. Construct a proper embedding $X \subset \mathbb{R}^{n}$, $n - \dim X \geq 3$. Let $W$ be a regular neighborhood of $X$ and $r : W \to X$ a retraction. $W \to M$ has a bounded fundamental group, and we can triangulate sufficiently finely to get a bounded CW structure on $W$. Then the controlled Spivak normal fibration is the fibration $\partial W \to X$. Let $F$ be the homotopy fiber of this fibration.

Lemma 5.1 ([29]) The fibre $F$ is homotopy equivalent to sphere of dimension $n - \dim X - 1$.

Definition 5.5 Let $X$ be a bg Poincaré duality complex. The bg Spivak normal fibration is defined by giving the controlled Spivak fibration of $X$ a bg structure by observing that the projection map can be taken to be boundedly controlled and hence is bg.
Definition 5.6 Let $X^n$ be a bg Poincaré duality space over a bg simplicial complex $M$ and let $\nu$ be a bg PL block bundle over $X$. A bounded geometry surgery problem is a triple $(W^n, \phi, F)$ where $\phi : W \to X$ is a bg map from an $n$-manifold $W$ to $X$ such that $\phi_*([W]) = [X]$ and $F$ is a stable trivialization of $\tau_W \oplus \phi^*\nu$. Two problems $(W, \phi, F)$ and $(W, \phi, F)$ are equivalent if there exist an $(n+1)$-dimensional manifold $P$ with $\partial P = W \amalg W$, a bg map $\Phi : P \to X$ extending $\phi$ and $\phi$, and a stable trivialization of $\tau_P \oplus \Phi^*\nu$ extending $F$ and $\Phi$.

Definition 5.7 The bg surgery group of a simplicial complex of bounded geometry is defined as follows. An unrestricted object consists of:

1. A bg Poincaré pair $(Y, X)$ over a bg complex $M$.
2. A bg map $\phi : (W, \partial W) \to (Y, X)$ of pairs of degree $1$, where $W$ is a PL manifold of bounded geometry and $\phi \mid \partial W \to X$ is a bg simple homotopy equivalence.
3. A bg stable framing $F$ of $\tau_W \oplus \phi^*(\tau)$, where $\tau$ is the bg Spivak normal fibration of $(Y, X)$.
4. A map $\omega : Y \to K$, where $K$ is bg complex so that the pullback of the double cover of $K$ to $Y$ is orientation preserving.

The surgery group is then defined to be the cobordism group of such unrestricted objects. It is denoted $L^b_{m}(K)$.

Unrestricted objects, however, cannot be used for surgery. To take care of this one introduces restricted objects, for which the map $\omega$ induces an isomorphism of the fundamental groups.

The next definition is based on [2].

Definition 5.8 Let $T$ be a discrete set. A bg $r$-handle is a bg pair $(T \times (D^r, S^{r-1}) \times D^{n-r})$ with a bg control map over a bg complex $Z$. A uniform surgery is an exchange of $T_1 \times S^r \times D^{n-r}$ by $T_1 \times D^{r+1} \times S^{n-r-1}$ where $T_1$ is a discrete set, and the regluing PL homeomorphisms along each $S^r \times S^{n-r-1}$ in $T_1 \times S^r \times S^{n-r-1}$ lie in a finite set.

We recall the following result from [3]:

Lemma 5.2 Let $(X^n, \partial X)$ be a bg Poincaré duality space over $M$, $n \geq 6$. Consider a bg surgery problem $\phi : (W, \partial W) \to (X, \partial X)$. Then $\phi$ is equivalent to a bg surgery problem $\Phi : (W, \partial W) \to (X, \partial X)$ so that $\phi$ is bg [2] connected over $M$ and is $[\frac{n-1}{2}]$ connected when restricted to the boundary.

The following theorem is proven analogously to the corresponding $\pi - \pi$ theorem in [29].

Theorem 5.1 Let $\phi$ be a bg map of pairs $\phi : (X, Y) \to (N, M)$. Suppose the inclusion $Y \subset X$ induces an isomorphism on the controlled fundamental groups. Then one can uniformly surger $\phi$ to obtain a homotopy equivalence.

Proof. By Lemma 5.4 of [29] we may do surgery up to the middle dimension. This means that cancelling cells in the controlled algebraic mapping cone of the
corresponding controlled chain complexes

\[ D_\#(W', \partial W'; W', \partial W') \]

yields a complex which is 0 through dimension \( \left\lfloor \frac{n}{2} \right\rfloor \). Here

\[ W' \xrightarrow{\phi'} X \]

is the surgery problem obtained so that \( \phi' \) is an inclusion which is the identity below the middle dimension. The \( k + 1 \) dimensional generators are represented by \( k \)-dimensional discs \( D \) in \( \partial X \) whose boundaries lie in \( \partial W' \). Note that there is a parallel copy of \( D \) in a collar neighborhood of \( \partial W' \) which is contained in \( W' \). Now use cell trading to change \( D_\# \) to

\[ 0 \to D'_{k+3} \xrightarrow{\partial} D'_{k+2} \xrightarrow{\partial} D_{k+1} \to 0 \]

together with a homotopy \( s \) so that \( s\partial + \partial s = 1 \) except at degree \( k + 1 \). Corresponding to each generator of \( D_{k+2} \) we introduce a pair of cancelling \( (k-1) \)-and \( k \)-handles and excise the interior of the \( (k-1) \)-handle from \( (W, \partial W) \). The modified chain complex is:

\[ 0 \to D_{k+3} \to D_{k+2} \to D_{k+1} \oplus D_{k+2} \to 0 \]

All generators of \( D_{k+1} \oplus D_{k+2} \) are represented by discs. We may represent any linear combination of these discs by an embedded disc, and these embedded discs may be assumed disjoint by a piping argument which uses the \( \pi - \pi \) condition in the hypothesis. We do surgery on the following elements: For each generator \( x \) of \( D_{k+1} \), we do surgery on \( (x - s\partial x, sx) \) and for each generator \( y \) of \( D_{k+1} \), we do surgery on \( (0, \partial y) \). This results in a contractible chain complex and completes the even dimensional case. The odd dimensional case follows by crossing with \( S^1 \) and splitting back. This can be done by a codimension 1 splitting technique which is simpler than Theorem 5.5 of [3] and therefore will not be worked out in detail.

The following is a direct corollary of the above, as in [64].

**Theorem 5.2** Let \( \phi : (W, \partial W) \to (Y, X) \) be an unrestricted object which is \( bg \) \( 1 \)-connected and a \( bg \) homotopy equivalence on the boundary. Then there is a \( bg \) normal cobordism rel \( \partial W \) to a \( bg \) homotopy equivalence if and only if the equivalence class of \( \phi \) in \( L^{bg}(K) \) vanishes.

**Definition 5.9** Define \( NI^{bg,PL}(X) \) to be cobordism group of triples \( (M, \phi, F) \), where \( M \) is a \( PL \) manifold of bounded geometry, \( \phi \) a degree one \( bg \) normal map to \( X \) and \( F \) a stable \( bg \) trivialization of \( \tau_M \oplus \phi^*\nu \), where \( \nu \) is the \( bg \) normal bundle of \( X \).

**Definition 5.10** Define the simple \( bg \) \( PL \) structure set \( S^{bg,s}_{PL}(X) \), where \( X \) is a \( PL \) manifold of bounded geometry to be the set of \( bg \) simple homotopy equivalences \( \phi : N \to X \) modulo the equivalence relation \( \phi \sim \phi' : N' \to X \), if there is a \( PL \) quasi-isometry \( h : N \to N' \) so that \( \phi' \circ h = \phi \).
Define the simple bg TOP structure set \( S^{bg,s}_{TOP}(X) \), where \( X \) is a PL manifold of bounded geometry to be the set of bg simple homotopy equivalences \( \phi : N \rightarrow X \) modulo the equivalence relation \( \phi \sim \phi' : N' \rightarrow X \), if there is a continuous quasi-isometry \( h : N \rightarrow N' \) so that \( \phi' \circ h = \phi \).

**Proposition 5.1** There is an exact sequence

\[
S^{bg,s}_{PL}(X) \rightarrow NI^{bg,PL}(X) \rightarrow L^{bg}_{n}(X)
\]

We next introduce an algebraic version of this theory, following Ranicki. We recall that Ranicki has introduced the L-theory of an additive category. We refer to Ranicki’s book [55] for the relevant definitions. The following treatment of algebraic surgery is based directly and heavily on [55].

The following theorem relates the surgery groups defined above to the L-theory of an additive category.

**Definition 5.11** Let \( K \) be a simplicial complex of bounded geometry. Let \( A \) be an additive category, \( \pi \) a group. Then the category \( C^{bg}_{K}(A[\pi]) \) is defined to be the one whose objects are formal direct sums

\[
M = \sum_{x \in K} M(x)
\]

of objects \( M(x) \) in \( A[\pi] \), which fall into a fixed finite number of types inside of each ball of fixed radius in \( K \). Here \( A[\pi] \) is the category with the one object \( M[\pi] \) for each object \( M \) in \( A \), and with morphisms linear combinations of morphisms \( f_{g} : M \rightarrow N \) in \( A \) of the form

\[
f = \sum_{g \in \pi} n_{g} f_{g} : M[\pi] \rightarrow N[\pi],
\]

with \( \{ g \in \pi \mid f_{g} \neq 0 \} \) finite. We use the notation \( C^{bg}_{K}(A[\pi]) \) for \( C^{bg}_{R}(A[\pi]) \). For any commutative ring \( R \) there is an identification

\[
A^{b}(R)[\pi] = A^{b}(R[\pi])
\]

with \( A^{b}(R) \) the additive category of based finitely generated free \( R \)-modules. Write the category \( C^{bg}_{X}(A^{b}(R[\pi])) \) as \( C^{bg}_{X}(R[\pi]) \).

**Theorem 5.3** Let \( X \) be a manifold of bounded geometry with bounded fundamental group \( \pi_{1}(X) \). Then \( L^{bg}_{n}(X) = L^{bg}_{n}(C^{bg}_{X}(Z[\pi_{1}(X)]) \). Moreover every element of \( L^{bg}_{n}(C^{bg}_{X}(Z[\pi_{1}(X)]) \) is realized as the obstruction on a surgery problem with target \( N \times I \) and homotopy equivalence on the boundary for an arbitrary \( n-1 \) dimensional bg PL manifold \( N \) with bounded fundamental group \( \pi_{1}(X) \).

**Proof.** This is proven as in [29]. In the even dimensional case we first obtain a highly connected surgery problem. We obtain a chain complex homotopy equivalent to \( K^{\#}(M) \) which is concentrated in dimensions \( k+2, k+1 \) and \( k \), and
a contracting homotopy $s$ which is obtained from Poincaré duality. Introducing cancelling $k + 1$ and $k + 2$ handles, we may shorten this chain complex to a 2-term chain complex

$$0 \to K'_k \to K'_k \to 0$$

We can then do further surgery to get a chain complex concentrated in one degree. Denote the remaining module by $A$. Poincaré duality produces an isomorphism $\phi : A \to A^*$ which determines the intersections of different generators, i.e. $\phi(e_i)(e_j)$ determines the intersections of $e_i$ and $e_j$ when $e_i$ and $e_j$ are different. Now total order the basis and define a map $\nu : A \to A^*$ so that $\nu(e_i)(e_j)$ is 0 when $i > j$ and the intersection counted with sign in $\mathbb{Z}\pi_1(X)$ when $i \leq j$.

By symmetrization $\nu + \epsilon \nu^* = \phi$, hence an isomorphism. This represents the surgery obstruction. Using -1 and 0-connectedness, if the surgery obstruction is zero, we may find a Lagrangian so that doing uniform surgery on this Lagrangian produces a by homotopy equivalence.

In odd dimensions we do surgery below the middle dimension, and proceeding as above we may obtain a length 2 chain complex

$$0 \to K_{k+1} \to K_k \to 0$$

Now do surgeries on embedded $S^k \times D^{k+1}$ so that, denoting the trace of the surgery by $W$, the chain complexes $K_#(W,M), K_#(W)$ and $K_#(W,M')$ are homotopy equivalent to chain complexes which are zero except in dimension $k + 1$. One way to do this could be to do surgeries to all the generators of $K_k$. Denote the resulting manifold by $M'$. The surgery obstruction is now defined to be the following formation

$$(K_{k+1}(W,M) \oplus K_{k+1}(W,M'), K_{k+1}(W,M), K_{k+1}(W))$$

where the first Lagrangian is the inclusion on the first factor, and the second Lagrangian is induced by the pair of inclusions. Poincaré duality shows that these are indeed Lagrangians. This is a well-defined element in the odd L-group.

Realization also follows as in [29].

Let $\mathcal{B}(A)$ be the additive category of finite chain complexes in $A$ and chain maps.

**Definition 5.12** A subcategory $\mathcal{C} \subseteq \mathcal{B}(A)$ is closed if it is a full additive subcategory so that the algebraic mapping cone $C(f)$ of any chain map $f : C \to D$ in $\mathcal{C}$ is an object of $\mathcal{C}$. A chain complex $C$ in $\mathcal{A}$ is $\mathcal{C}$-contractible if it belongs to $\mathcal{C}$. A chain map $f : C \to D$ in $\mathcal{A}$ is a $\mathcal{C}$-equivalence if $C(f)$ is $\mathcal{C}$-contractible.

An n-dimensional quadratic complex $(C, \psi)$ in $\mathcal{A}$ is $\mathcal{C}$-contractible if $C$ and $C^{n-*}$ are $\mathcal{C}$-contractible. An n-dimensional quadratic complex $(C, \psi)$ in $\mathcal{A}$ is $\mathcal{C}$-Poincaré if the chain complex

$$\partial C = S^{-1}C((1 + T)\psi_0 : C^{n-*} \to C)$$

is $\mathcal{C}$-contractible.
Definition 5.13 Let $\Lambda = (A, B, C)$ be a triple of additive categories, where $A$ has a chain duality $T : A \to B(A)$ and a pair $(B, C \subseteq B)$ of closed subcategories of $B(A)$ so that for any object $B$ of $B$

i. The algebraic mapping cone $C(1 : B \to B)$ is an object of $C$

ii. The chain equivalence $\epsilon(B) : T^2(B) \to B$ is a $C$-equivalence.

Then $\Lambda$ is said to be an algebraic bordism category.

Definition 5.14 For any additive category with chain duality $A$ there is defined an algebraic bordism category $\Lambda(A) = (A, B(A), C(A))$

with $B(A)$ the category of finite chain complexes in $A$, and $C(A) \subseteq B(A)$ the subcategory of contractible complexes.

Definition 5.15 Let $K$ be a simplicial complex. Let $\Lambda = (A, B, C)$ be an algebraic bordism category. An $n$-dimensional quadratic complex $(C, \psi)$ in $\Lambda$ is an $n$-dimensional quadratic complex in $A$ which is $B$-contractible and $C$-Poincaré. The quadratic $L$-group $L_n(\Lambda)$ is the cobordism group of $n$-dimensional quadratic complexes in $\Lambda$.

Definition 5.16 Let $K$ be a simplicial complex. An object $M$ in an additive category $A$ is said to be $K$-based if it is expressed as a direct sum

$$M = \sum_{\sigma \in K} M(\sigma)$$

of objects $M(\sigma)$ in $A$ so that $\{\sigma \in K : M(\sigma) \neq 0\}$ is finite in ball of fixed radius in $K$. A morphism $f : K \to N$ of $K$-based objects is a collection of morphisms in $A$

$$f = \{f(\tau, \sigma) : M(\sigma) \to N(\tau) : \sigma, \tau \in K\}$$

of objects $M(\sigma)$ in $A$ so that $\{\sigma \in K : M(\sigma) \neq 0\}$ is finite in ball of fixed radius in $K$.

Definition 5.17 Let $K$ be a simplicial complex of bounded geometry. Let $A^{uf}(K)$ be the additive category of $K$-based objects in $A$ which fall into a finite number of types in each ball of fixed radius in $K$. Define a uniformly finite assembly map

$$A^{uf}(K) \to C_{B}^{bg}(A[\pi])$$

by associating to $M$ the $bg$ controlled module $\hat{M}$ which has the value $M(\sigma)$ at the barycenter of $\sigma$ and is 0 everywhere else.

Definition 5.18 Let $A(R)^{uf}(K)$ be the additive category of $K$-based objects in $A(R)$ which fall into a fixed number of types in a ball of fixed radius, with morphisms $f : M \to N$ such that $f(\tau, \sigma) = 0 : M(\sigma) \to N(\tau)$ unless $\tau \geq \sigma$ so that $f(M(\sigma)) \subseteq \sum_{\tau \geq \sigma} N(\tau)$. 

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Definition 5.19 Define three algebraic bordism categories:
i. local, uniformly finite, finitely generated free \((R,K)\)-modules
\[ \Lambda^{uf}(R,K) = (A^{uf}(R,K), B^{uf}(R,K), C^{uf}(R,K)) \]
where \(B^{uf}(R,K)\) is the category of finite chain complexes of f.g. free uniformly finite \((R,K)\)-modules. An object in \(C^{uf}(R,K)\) is finite f.g. free uniformly finite \((R,K)\)-module chain complex \(C\) such that each \([C][\sigma] (\sigma \in K)\) is a contractible f.g. free \(R\)-module chain complex.

ii. global, uniformly finite, finitely generated free \((R,K)\)-modules
\[ \Lambda^{uf}(R,K) = (A^{uf}(R,K), B^{uf}(R,K), C^{uf}(R,K)) \]
with \(C^{uf}(R,K) \subseteq B^{uf}(R,K)\)
the subcategory of finite finitely generated free uniformly finite \((R,K)\)-module chain complexes \(C\) which assemble to contractible finitely generated free \(C^{bg}(R[\pi])\)-module chain complexes.

iii. \(C^{bg}(A[\pi])\), along with the categories of chain complexes and contractible chain complexes over it.

Definition 5.20 The quadratic \(bg\) structure groups of \((R,K)\) are the cobordism groups
\[ S^{bg}_n(R,K) = L_{n-1}(A^{uf}(R,K), C^{uf}(R,K)) \]
Definition 5.21 Let \(M\) be a manifold of bounded geometry. We define an object which we will refer to as uniformly finite homology with coefficients in the \(L\)-spectrum,
\[ H^{uf}_{n}(M; L) = L_n(\Lambda(Z)^{uf}(M)) \]
For the moment this is a purely formal definition. At the end of this paper we will show that for \(M = N \times R^n\), \(N\) a compact manifold, this object is in fact the uniformly finite homology with coefficients in the \(L\)-groups.

Theorem 5.4 A normal map of \(n\)-dimensional PL manifolds of bounded geometry \((f,b) : N \rightarrow M\) determines an element, the normal invariant
\[ [f,b] \in H^{uf}_{n}(M; L) \]
which assembles to the surgery obstruction.

Proof. Let \(X\) be the polyhedron of an \(n\)-dimensional geometric \(bg\) Poincaré complex with a homotopy equivalence \(g : M \rightarrow X\) so that both \(g\) and \(gf : N \rightarrow X\) are \(bg\) transverse to the dual cell decomposition \(\{D(\tau,X) : \tau \in X\}\) of \(X\). The restrictions of \(f\) define a uf cycle of degree 1 \(bg\) normal maps of \((n-|\tau|)-dimensional manifolds with boundary
\[ \{(f(\tau), b(\tau)) : \{N(\tau)\} \rightarrow \{M(\tau)\} \}

with 
\[ M(\tau) = g^{-1}D(\tau, X), N(\tau) = (gf)^{-1}D(\tau, X), \tau \in X \]
so that \( M(\tau) = \{pt.\} \) for \( n \)-simplices \( \tau \in X^{(n)} \). The controlled kernel \( uf \) cycle
\[ \{(C(f(\tau)^1), \psi(b(\tau))) : \tau \in X\} \]
of \( (n-|\tau|) \)-dimensional quadratic Poincaré pairs \( A(Z) \) is a 1-connective, \( n \)-dimensional quadratic Poincaré complex in \( A(Z)^{uf}(M) \) allowing the definition
\[ [f, b] = \{(C(f(\tau)^1), \psi(b(\tau)))\} \in L_n(\Lambda(Z^{uf}(M))) \]

Finally, we need the uniformly finite version of the algebraic \( \pi - \pi \) theorem.

**Theorem 5.5** The global \( uf \) assembly maps define isomorphisms
\[ L_n(A^{uf}(R, K)) \simeq L_n(C^{bg}_K(A[\pi_1(K)]) \]

**Proof.** The proof is virtually the same as the one given by Ranicki in [55] in the compact case, so we will restrict ourselves to those points where the two proofs differ. In the compact case, the Hurewicz theorem is used to compare the \( (R, K) \)-module with its image under the assembly. In the \( bg \) case, this Hurewicz theorem is replaced by the controlled Hurewicz theorem of Anderson and Munkholm. The rest is verbatim the same as the one in Ranicki except that modules over the entire space are replaced by controlled modules.

**Theorem 5.6 (Algebraic Surgery Exact Sequence)** Let \( K \) be a \( bg \) simplicial complex. There is an exact sequence
\[ L_{n+1}(C^{bg}_K(Z\pi_1(K)) \to S^{bg}_{n+1}(Z, K) \to H_n^{uf}(K; L) \to L_n(C^{bg}_K(Z\pi_1(K))) \]

**Proof.** This is simply the exact sequence of Ranicki [55], Proposition 3.9.

**Theorem 5.7** The algebraic \( bg \) structure group \( S^{bg}_n(Z, K) \) of a uniform triangulation \( K \) of a \( bg \) PL manifold \( M \), is isomorphic to the topological structure set \( S^{bg}_{TOP}(M) \), with group structure given by characteristic variety addition.

**Proof.** As in [55] p.198, if \( f : N \to M \) is a homotopy equivalence the quadratic complex giving the normal invariant of \( f \) is globally contractible, allowing the definition
\[ s(f) = \{ C(f(\tau)^1, \psi(b(\tau))) \} \in S_{n+1}(M) \]
This proves the result.

We wish to express the PL \( bg \) surgery exact sequence as a version of Ranicki's exact sequence. In order to do this we need to incorporate the Casson-Sullivan obstruction to the uniqueness of triangulations. The triangulation of
the polyhedron of a $bg$ PL manifold is not unique. In other words, there are $bg$ homeomorphic $bg$ PL manifolds, which are not $bg$ PL homeomorphic. Any $bg$ homeomorphism of polyhedra can be approximated by a $bg$ PL map, however this map does not necessarily have a $bg$ PL inverse. This leads to the non-uniqueness of the $bg$ PL triangulation, which means that there are non-equivalent (via a $bg$ PL homeomorphism) triangulations (or $bg$ PL types) in the $bg$ homeomorphism type of a $bg$ PL manifold. One can show by obstruction theory that, if one disregards the $n-3$ skeleton, the triangulation on a $bg$ PL manifold is unique up to $bg$ PL homeomorphism, within that $bg$ homeomorphism type. However, there is an ambiguity when one gets to the $n-3$ skeleton caused by the fact that the signature of a topological 4-manifold is divisible by 8, whereas the signature of a PL or smooth manifold is divisible by 16. (See pp.182-183 of [55].) The ambiguity is resolved by the Casson-Sullivan obstruction. This can be written as with the Casson-Sullivan invariant above, we obtain the Kirby-Siebenmann homeomorphism type. However, there is an ambiguity when one gets to the $bg$ homeomorphism type of a $bg$ PL manifold. One can show by obstruction theory that, if one disregards the $n-3$ skeleton, the triangulation on a $bg$ PL manifold is unique up to $bg$ PL homeomorphism, within that $bg$ homeomorphism type. However, there is an ambiguity when one gets to the $n-3$ skeleton caused by the fact that the signature of a topological 4-manifold is divisible by 8, whereas the signature of a PL or smooth manifold is divisible by 16. (See pp.182-183 of [55].) The ambiguity is resolved by the Casson-Sullivan obstruction, which is a local obstruction defined following [55]. Let $f : M \to M'$ be a $bg$ homeomorphism, $(W^{n+1}; M, M')$ a $bg$ PL cobordism with $F|_M = \text{id}$, $F|_{M'} = f$ 

$$(F, B) : (W^{n+1}; M, M') \to M \times ([0, 1]; \{0\}, \{1\})$$

and let $\sigma_*(F, B) = (C, \psi)$ be the $bg$ surgery obstruction of $(F, B)$. Then we define an invariant

$$\kappa(f) = \sum_{\sigma \in \Omega(n-3)} (\text{signature}(C(\sigma), \psi(\sigma))/8) \sigma$$

which is an element of $H^3(M; \mathbb{Z}_2)$. That this classifies triangulations follows exactly as in the compact case. The Casson-Sullivan invariant clearly defines a map

$L^b_{n+1}\{M\} \oplus H^i_{n-3}(M; \mathbb{Z}_2) \to \mathcal{S}^{bg,s}_{PL}(M)$

by assigning the homotopy equivalence $f : M' \to M$ to the image of the surgery problem $(F, B) : W \to M \times [0, 1]$.

We also need to define the Kirby-Siebenmann invariant. This lies in $H^i_{n-4}(M; \mathbb{Z}_2)$. If we let $(g, c) : N \to M$ be an EPL degree one normal map, where $N$ is a topological manifold which is a $bg$ Poincaré duality complex controlled over $M$, we can define the surgery obstruction of $(g, c), \sigma_*(g, c)$ in terms of the cellular chains of $N$ as in [29] and note that this defines a $bg$ surgery obstruction $\sigma_*(g, c) \in L^b_{n}(M)$. If we take the image of this obstruction in $H^i_{n-4}(M; \mathbb{Z}_2)$, as with the Casson-Sullivan invariant above, we obtain the Kirby-Siebenmann obstruction. This can be written

$$g_*\kappa(N) = \sum_{\sigma \in \Omega(n-4)} (\text{signature}(C(\sigma), \psi(\sigma))/8) \sigma$$

where $\sigma_*(g, c) = (C, \psi)$.

We obtain from these theorems the following algebraic exact sequence which was suggested to the author by A. Ranicki:

**Theorem 5.8** Let $M$ be a manifold of bounded geometry. Then there is an exact sequence

$$L^b_{n+1}(M) \oplus H^i_{n-3}(M; \mathbb{Z}_2) \to \mathcal{S}^{bg,s}_{PL}(M) \to H^i_{n-4}(M; \mathbb{L})$$

$$\to L^b_{n}(M) \oplus H^i_{n-4}(M; \mathbb{Z}_2)$$
Proof. We now prove exactness at $S_{PL}^{bg,s}(M)$. A bg homotopy equivalence $f : M' \to M$ has zero normal invariant in $H_3^{bg}(M;L)$ if there is a bg PL normal bordism $(F',B') : (W;M',M') \to M \times ([0,1];\{0\},\{1\})$ to a bg homeomorphism $f' : F' \to M$. Hence $W,F',B',f''$ are the same as $W,F,B$ and $f$ in the definition of the Casson-Sullivan invariant above. Thus the kernel of the map $S_{PL}^{bg,s}(M) \to H_n^{bg}(M;L)$ is equal to the image of the map defined above in the definition of the Casson-Sullivan invariant. That is in this case, $(M,f)$ is in the image of $(\sigma_*(F',B'),\kappa(f'))$. This proves exactness at $S_{PL}^{bg,s}(M)$.

Next we prove exactness at $H_n^{bg}(M;L)$. The kernel of the map

$$H_n^{bg}(M;L) \to L_n(M) \oplus H^{lf}_{n-4}(M;\mathbb{Z})$$

is equal to the set of degree one normal maps $f : M' \to M$ with zero surgery obstruction and zero Kirby-Siebenmann invariant. Suppose that $f : M' \to M$ is a degree one normal map with zero surgery obstruction and zero Kirby-Siebenmann obstruction. Then by the definition, $f$ is a bg PL structure on $M$. This proves exactness at $H_n^{bg}(M;L)$.

Finally, we prove exactness at $H_{n+1}^{lf}(M;\mathbb{Z}_2) \oplus L_n^{bg}(M)$. The image of the map $H_n^{bg}(M;L) \to L_{n+1}(M) \oplus H^{lf}_{n-3}(M;\mathbb{Z}_2)$ is the assembly of the normal invariants of $(F,B) : N \to M \times I$ to surgery obstructions, which by the usual proof of the surgery exact sequence of Wall [64] is the kernel of the map $L_n^{bg}(M) \oplus H^{lf}_{n-3}(M;\mathbb{Z}_2) \to S_{PL}^{bg,s}(M)$.

Finally, we need to prove Siebenmann periodicity. This will follow immediately from the above exact sequence. Note first that if $\sigma^*(f)$ is the surgery obstruction of $f : M \to N$ then

$$\sigma^*(f) = \sigma^*(f \times \mathbb{C}P^2)$$

This defines an isomorphism of the bg $L$-groups $L_n^{bg}(M) \simeq L_{n+4}^{bg}(M)$. Next we define the bg “resolution obstruction”. Let $(C \to D,(d\psi,\psi))$ be an $n$-dimensional locally Poincaré globally contractible quadratic pair in $\mathcal{A}^{u}(Z,X)$ with $C$ 1-connective and $D$ 0-connective, then the image of the algebraic complex

$$x = \sum_{\tau \in X^{(n)}}((D/C)(\tau),(\delta \psi/\psi)(\tau)) \in H_n^{bg}(X;L_0(Z)) = Z$$

in $S^{bg,s}_{TOP}(X)$ is the “resolution obstruction”. Note that this obstruction is necessarily an element of $Z$. It is clear that we then have the following standard periodicity result (cf. [62], [42], [15], [45], [65], [63])

**Theorem 5.9** The topological structure set of a manifold $M$ of bounded geometry is almost 4-fold periodic:

$$S_{TOP}^{bg,s}(M) \simeq S_{TOP}^{bg,s}(M \times D^4,\partial) \oplus Z$$
6 Whitehead Group of $M \times \mathbb{R}^n$

The technique we use to calculate the Whitehead group is the same as used in [3], but more elaborate due to the fact that there are non-compact directions involved. We will introduce a filtration of $W^b_\text{bg}(M \times \mathbb{R}^n)$ which is obtained by easing the restrictions on bounded geometry in various perpendicular directions. This technical modification allows us to inductively prove the theorem using methods of [3].

The following definition was suggested by S. Cappell.

**Definition 6.1** Let $p : X \to \mathbb{R}^n$ be a control map. $p$ is said to be $\text{bg}(r)$, if, via the decompositions $\mathbb{R}^n - r \times \mathbb{R}^r$, $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n - r$, $\mathbb{R}^r \times \mathbb{R}^n - r$, ..., $\mathbb{R}^n - r$, the restriction of $p$ to a regular neighborhood $D^r \times \mathbb{R}^n - r$ of each codimension $n - r$ hyperplane is $\text{bg}$. Thus $\text{bg}(n)$ is the same as $\text{bg}$ and $\text{bg}(0)$ is the same as bounded control. The other notions can be considered intermediate between $\text{bg}$ and bounded control.

**Remark 6.1** If we consider $\text{bg}$ control to uniform control of the complexity with complexity bound $K$, then in $\mathbb{R}^2$ we can consider $\text{bg}(1)$ control to be uniform control of complexity with complexity bound growing along lines parallel to the $x$- or $y$-axes as either the $x$- or $y$-coordinate increases. Note that because these must be a uniform bound on each line, the constant is being allowed to increase along the diagonal. In general, $\text{bg}(r)$ control is equivalent to uniform control of complexity along hyperplanes $\mathbb{R}^n - r$ which is allowed to grow along an $r$-dimensional “diagonal” hyperplane.

**Definition 6.2** $H^\text{aff}_*(\mathbb{R}^n; Wh_*(\pi_1(M)))$ is the abelian group

$$H^\text{aff}_0(\mathbb{R}^n; Wh(\pi_1(M))) \oplus ... \oplus H^\text{aff}_n(\mathbb{R}^n; K_{1-n}(\mathbb{Z}\pi_1(M)))$$

**Theorem 6.1** $W^b_\text{bg}(M \times \mathbb{R}^n) = H^\text{aff}_*(\mathbb{R}^n; Wh_*(\pi_1(M)))$

**Remark 6.2** Observe that this is a “Bass-Heller-Swan” [8], [26], [27], [7] formula with no Nil terms. The reason for the absence of Nil terms in this formula is that the “splitting” performed in the proof below is performed with a sufficiently large separation between hyperplanes. Almost by definition, an element of one of the Nil terms vanishes over a large separation (i.e. larger than the nilpotency of the element). We note further that the splitting takes place at the level of individual elements, rather than over a set of representatives for whole Whitehead group.

Proof of 5.1. We proved this theorem for $n = 1$ in [3]. We recall the proof here. We have a map

$$W^h_\text{bg}(M \times \mathbb{R}) \to W^h_\text{bdd}(M \times \mathbb{R}) = K_0(\mathbb{Z}\pi_1(M))$$

given by considering a $\text{bg}$ controlled h-cobordism as a boundedly controlled one. We claim that if an h-cobordism is in the kernel of this map, then it can be
simultaneously split at the integer points of $\mathbb{R}$. This will be seen to follow by observing that $Wh^{bd}(M \times \mathbb{R})$ can be identified with $\tilde{K}_0(\mathbb{Z}\pi_1(M))$ which consists of splitting obstructions.

Following Pedersen, we give an explicit description of how this may be done. Consider the integer points of $\mathbb{R}$. Over each integer point is a module $A(j)$. Suppose now we are given a controlled automorphism $\alpha$ with bound $k$. Consider the strip $l-2k \leq j \leq l+2k$. Define $\phi([A,\alpha])$ by

$$\mathcal{I} = \bigoplus_{j=l-2k}^{l+2k} A(j)$$

$$\phi([A,\alpha]) = \sum_{l \in 4k\mathbb{Z}} (\mathcal{I}, \alpha p_\perp - \alpha^{-1} - \mathcal{I}, p_\perp) x_l.$$

Here $p_\perp$ is the projection onto the half-line below $j = l$ and $x_l$ is the $l$-th vertex of the triangulation of $\mathbb{R}$ by intervals of length $2k$. We claim first that this defines an element of $\tilde{K}_0(\mathbb{Z}\pi_1(M))$. This follows by inspection, since $\alpha p_\perp - \alpha^{-1}$ differs from $p_\perp$ only in a band around $j = l$ and these each, by inspection, can be seen to agree for each $l$. In fact, because of the bounded geometry, one can, after uniform stabilization, represent all of these elements by the same module. Next we claim that this element is the splitting obstruction. We see this for the case $l = 0$, the other cases being the same. If $[A,\alpha]$ is in the kernel of the map $Wh^{bd}(M \times \mathbb{R})$ then there are modules $A'$ and $A''$ so that $[\mathcal{I} \oplus A' \oplus A'', p_\perp \oplus 1 \oplus 0]$ is isomorphic to $[\mathcal{I} \oplus A' \oplus A', \alpha p_\perp - \alpha^{-1} \oplus 0]$. This implies that there is a bounded automorphism $\beta$ so that $\beta \alpha p_\perp = p_\perp \beta \alpha$. Thus $\beta \alpha$ has the property that it preserves the module below and above $l = 0$ and hence $\beta$ preserves the half spaces below and above $l = 2k + 1$. This then implies that the corresponding h-cobordism can be split by a sequence of expansions and collapses. This follows from the fact that since one can make the automorphism equal to the identity on each point, one can make the h-cobordism standard over that point by a sequence of expansions and collapses. Choosing these sufficiently far apart so that they do not interfere with each other, we can split at each integer point.

This shows that $Wh^{bg}(M \times \mathbb{R})$ can be written as a direct sum, with one summand being represented by splitting obstructions in $\tilde{K}_0(\mathbb{Z}\pi_1(M))$ and the kernel of the map $Wh^{bg}(M \times \mathbb{R}) \to \tilde{K}_0(\mathbb{Z}\pi_1(M))$ given by h-cobordisms split over $\mathbb{R}$ which is represented by elements of $C_{0f}^f(\mathbb{R}; Wh(\pi_1(M)))$. We claim that two such elements are equivalent in $Wh^{bg}$ if and only if they are homologous in $C_{0f}^f$. For if an element is null-homologous, then there exists a bounded infinite process trick cancelling its torsion. Conversely, if two elements are equivalent, one can split relatively on a representative to $M \times I$ and using the new splitting construct a homology between them. Also the map $H_{0f}^i(\mathbb{R}; Wh(\pi_1(M))) \to Wh^{bg}(M \times \mathbb{R})$ is given by gluing together representatives. This is clearly injective, for if one can glue together representative h-cobordisms so that their torsions is zero, the s-cobordism theorem shows that the representative had to be null-homologous.
This gives a short exact sequence:

$$0 \rightarrow H^0(\mathbb{R}, Wh(\pi_1(M))) \rightarrow Wh^{bd}(M \times \mathbb{R}) \rightarrow \tilde{K}_0(\mathbb{Z}\pi_1(M)) \rightarrow 0$$

which is split and surjective onto the last term by the following argument. By applying the infinite process trick of Swan, one sees that one can choose the representatives in $\tilde{K}_0(\mathbb{Z}\pi_1(M)) = Wh^{bd}(M \times \mathbb{R})$ which have bounded geometry. Given an element $(B, p) \in \tilde{K}_0(\mathbb{Z}\pi_1(M))$, one constructs an element of $Wh^{bd}(M \times \mathbb{R})$ by defining $A(j) = B$ and mapping $B$ to itself by $p$ and $1 - p$ successively. This has bounded geometry and gives rise to an automorphism whose image is $(B, p)$. Thus we have given an element of $Wh^{bg}(M \times \mathbb{R})$.

To do the general case, we first consider the case $n = 2$. Observe that there are forgetful maps

$$Wh^{bg}(M \times \mathbb{R}^2) \rightarrow Wh^{bg}(1)(M \times \mathbb{R}^2) \rightarrow Wh^{bd}(M \times \mathbb{R}^2)$$

Now note that the last group is known, due to Pedersen [49], to be

$$Wh^{bd}(M \times \mathbb{R}^2) \simeq K_{-1}(\mathbb{Z}\pi_1(M)).$$

We claim that if a geometrical representative of an element of $Wh^{bg}(1)(M \times \mathbb{R}^2)$ is the kernel of the forgetful map $K_{-1}(\mathbb{Z}\pi_1(M))$, then it is $bg$ split along parallel lines, and conversely. For if an element is $bg$ split along parallel lines, then by the infinite process trick of [49], one can move each module at each lattice point in $Wh^{bd}(M \times \mathbb{R}^2)$ out to infinity without violating the loosened boundedness condition. The proof is simply that, as Pedersen shows in [49], if a module is split along a single line in $Wh^{bd}(M \times \mathbb{R}^2)$, then it is equivalent to zero, by the usual infinite process trick, which requires no uniform bounds on the complexity of the module.

The converse goes by an analogue of [3] with $C_{1}^{bg}(\mathbb{Z}\pi_1(M))$ in place of $\mathbb{Z}\pi_1(M)$. As in the above argument, we define the splitting obstruction in one direction, $\phi([A, \alpha]) \in Wh^{bg}(1)(M \times \mathbb{R}^2)$ by

$$\phi([A, \alpha]) = \sum_{l \in 4k\mathbb{Z}} ([\mathcal{A}, \alpha p^{l-\alpha^{-1}}] - [\mathcal{A}, p^{l}])x_{l}$$

Elements of the kernel of this map are split by the argument of [49]. Moreover the map is the forgetful map to $K_{-1}(\mathbb{Z}\pi_1(M))$. Furthermore, applying Swan’s infinite process trick yields a surjection onto $K_{-1}(\mathbb{Z}\pi_1(M))$ and shows in fact that any element of $K_{-1}(\mathbb{Z}\pi_1(M))$ can be represented by a $bg(1)$-controlled element over $\mathbb{R}^2$. Given an element

$$(B, p) \in \tilde{K}_0^{bd}(M \times \mathbb{R}) \simeq K_{-1}(\mathbb{Z}\pi_1(M))$$

one constructs an element of $Wh^{bd}(M \times \mathbb{R}^2)$ by defining

$$A(j_1, j_2) = B(j_1)$$
and mapping $B$ to itself by $p$ and $1 - p$ successively, where instead of being considered as controlled $\mathbb{Z}\pi_1(M)$-modules, we consider them as modules over $C_1(\mathbb{Z}\pi_1(M))$. If we apply \[\mathcal{H}\] to the category $C_1(C^{0}_{bg}(\mathbb{Z}\pi_1(M)))$, we obtain for the Whitehead group of this category $\tilde{K}^{bg}_{0}(M \times \mathbb{R})$. Applying the method of calculation above to this Whitehead group, we obtain

$$\tilde{K}^{bg}_{0}(M \times \mathbb{R}) = H^{a\text{ff}}_{0}(\mathbb{R}; \tilde{K}_0(\mathbb{Z}\pi_1(M)) \oplus K_{-1}(\mathbb{Z}\pi_1(M))).$$

From this we can see that $(B, p)$ can be represented by a $bg$-controlled element over $\mathbb{R}$. This has bounded geometry in the $x$- and $y$-directions and gives rise to an automorphism whose image is $(B, p)$, hence it can be thought of as an element of $Wh^{bg(1)}(M \times \mathbb{R}^2)$. Thus we have given a splitting back to $Wh^{bg(1)}(M \times \mathbb{R}^2)$.

We next analyze the map $Wh^{bg}(M \times \mathbb{R}^2) \rightarrow Wh^{bg(1)}(M \times \mathbb{R}^2)$. We claim that the kernel of this map consists of elements $bg$ split simultaneously in both directions. To see this, note that if an element is $bg(1)$ and simultaneously split, one can move the modules along the lattice points until one reaches the diagonal and then out along the diagonals to infinity. This is because the complexity is no longer uniformly bounded, but is allowed to increase, remaining uniform only along parallel lines, as one goes away from the origin. Thus the kernel of the map is represented by elements of $C^{a\text{ff}}_{0}(\mathbb{R}^2; Wh(\pi_1(M)))$, modulo infinite process tricks which identify different elements of $Wh^{bg}(M \times \mathbb{R}^2)$, which yields $H^{a\text{ff}}_{0}(\mathbb{R}^2; Wh(\pi_1(M)))$, as in [3].

We will be finished with the calculation as soon as we have analyzed the kernel of the map $Wh^{bg(1)}(M \times \mathbb{R}^2) \rightarrow K_{-1}(\mathbb{Z}\pi_1(M))$ and show that both maps considered are split surjective. We claim that the obstruction to splitting along parallel lines (separately) is given by

$$C^{a\text{ff}}_{0}(\mathbb{R}; C^{a\text{ff}}_{1}(\mathbb{R}; \tilde{K}_0(\mathbb{Z}\pi_1(M))) \oplus C^{a\text{ff}}_{0}(\mathbb{R}; C^{a\text{ff}}_{0}(\mathbb{R}; K_{-1}(\mathbb{Z}\pi_1(M))))).$$

We clearly only have to compute this in one direction, since both summands are isomorphic, and represent the same splitting problem. But by the computation for the case of $\mathbb{R}$, the splitting obstruction can be represented by an element of $C^{a\text{ff}}_{0}(\mathbb{R}; K_{0}(\mathbb{Z}\pi_1(M)) \oplus K_{-1}(\mathbb{Z}\pi_1(M)))$. The projection of this element to $K_{-1}(\mathbb{Z}\pi_1(M))$ vanishes, by the standard infinite process trick, since it is split in both directions. We now observe that since $C^{a\text{ff}}_{1}(\mathbb{R}; \mathbb{Z}) = \mathbb{Z}$, we have the isomorphism

$$C^{a\text{ff}}_{0}(\mathbb{R}; G) \simeq C^{a\text{ff}}_{1}(\mathbb{R}; C^{a\text{ff}}_{0}(\mathbb{R}; G))$$

which proves the statement.

To prove the theorem for in general for any $n$ we need to show that the kernel of the map

$$Wh^{bg(r)}(M \times \mathbb{R}^n) \rightarrow Wh^{bg(r-1)}(M \times \mathbb{R}^n)$$

is $H^{a\text{ff}}_{0}(\mathbb{R}^n; K_{-n+r}(\mathbb{Z}\pi_1(M)))$. We prove this inductively by splitting off an $\mathbb{R}$ factor, and using the Eilenberg-Zilber theorem. We note that the kernel of this map consists of elements that are split in $r$ directions, since these can be taken
to infinity by an infinite process trick, along a diagonal hyperplane, using the loosened boundedness. The splitting obstruction then lies in $K^r_0(M \times R^{n-1})$, which is represented by elements of $C^u_{0}(R; K^r_0(M \times R^{n-2}))$ by the inductive hypothesis, since the elements of the other summands are zero, being split in more than $r$ directions. Using the Eilenberg-Zilber theorem completes the argument.

7 Proof of the Main Theorem

We apply the splitting theorem [3]. We review the statement of this theorem:

**Theorem 7.1** Let $h : M \to N$ be a by homotopy equivalence, where $M$ is the Cartesian product of a compact manifold with $R^n$ and $N$ is by over $R^n$. If $X \subset N$ is a by codimension 1 submanifold which is the transverse inverse image of $R^n-1 \times Z$, then $h$ can be split along $X$ if and only if an obstruction in a summand $\tilde{K}^r_0(N)$ of $Wh^b_0(N)$ vanishes, and the components of $X$ are sufficiently separated from each other.

**Definition 7.1** We denote by $\mathcal{S}_{PL}^{bg,i}(M)$ the PL by structure set with simple homotopy equivalences replaced by maps with torsion in $K^i_{bg}(M)$ equal to zero. Similarly, let $\mathcal{S}_{TOP}^{bg,i}(M)$ be the topological by structure set with simple homotopy equivalences replaced by maps with torsion in $K^i_{bg}(M)$.

**Definition 7.2** Let $M$ be a compact PL manifold. Then we define the group $H^u_{s}(R^n; \mathcal{S}_{TOP}^{s}(M))$ to be the group

$$H^u_{s}(R^n; \mathcal{S}_{TOP}^{s}(M)) = H^u_{0}(R^n; \mathcal{S}_{TOP}^{s}(M \times D^n, \partial)) \oplus ...$$

$$... \oplus H^u_{n}(R^n; \mathcal{S}_{TOP}^{2-n}(M \times D^{n-k}, \partial)) \oplus ... \oplus H^u_{n}(R^n; \mathcal{S}_{TOP}^{2-n}(M))$$

Where we set the convention $\mathcal{S}_{TOP}^{1}(N, \partial N) = \mathcal{S}_{TOP}^{h}(N, \partial N)$, for a compact PL manifold $N$ with boundary $\partial N$.

**Definition 7.3** Let $M$ be a compact PL manifold. We define the group

$$H_{s}(T^n; Wh_{s}(G)) = H_0(T^n; Wh(G)) \oplus ... \oplus H_n(T^n; K_{1-n}(ZG))$$

and

$$H_{s}(T^n; \mathcal{S}_{TOP}^{s}(M)) = H_0(T^n; \mathcal{S}_{TOP}^{s}(M \times D^n, \partial)) \oplus ... \oplus H_n(T^n; \mathcal{S}_{TOP}^{2-n}(M))$$

where we set the convention $\mathcal{S}_{TOP}^{1}(N, \partial N) = \mathcal{S}_{TOP}^{h}(N, \partial N)$, where $N$ is a compact PL manifold with boundary $\partial N$.

**Proof of 1.1** We will work in the PL category and apply the PL by surgery exact sequence to get the result in the topological category. We apply Theorem 7.1 in the following manner.
Let $N$ be simple homotopy equivalent to $M \times \mathbb{R}^n$. We can apply the splitting theorem, since the splitting obstruction vanishes, to obtain a splitting of $N$ along parallel hyperplanes. Each split pieces is $bg$ simple homotopy equivalent to $M \times \mathbb{R}^{n-1} \times I$. The boundary of each split piece is $bg$ simple homotopy equivalent to $M \times \mathbb{R}^{n-1}$, and each boundary piece is $bg$ PL homeomorphic to next one. We thus have a well defined map from $S^{bg,s}_{PL}(M \times \mathbb{R}^n) \to S^{bg,h}_{PL}(M \times \mathbb{R}^{n-1})$. A given $N$ in the kernel of this map gives rise to a chain in $C^{bg}_{0}(\mathbb{R}; S^{bg,s}_{PL}(M \times \mathbb{R}^{n-1} \times I, \partial))$. By applying the $bg$ splitting theorem, we see that two such yield $bg$ PL homeomorphic representatives if and only if they are homologous. In addition, one must establish the structure on the boundary of $M \times \mathbb{R}^{n-1} \times I$.

We next perform a second splitting transverse to the first. For this we apply the splitting theorem to each piece and then observe that the separation of the splitting depends only on the complexity of the given homotopy equivalence, which is uniformly bounded by hypothesis. Thus the splittings on each piece can be aligned with each other and we obtain a transverse splitting.

We continue the analysis of the splitting as before, splitting $M \times \mathbb{R}^{n-1} \times I$ along parallel hyperplanes, obtaining a manifold simple homotopy equivalent to $M \times \mathbb{R}^{n-2} \times D^2$. A component of the boundary of the split piece is $bg$ homotopy equivalent to $M \times \mathbb{R}^{n-2} \times I$, and the boundary of this manifold is $bg$ projective homotopy equivalent to $M \times \mathbb{R}^{n-2}$. The boundary of the first splitting has been split once more, giving rise to elements of $S^{h,bg}(M \times \mathbb{R}^{n-2})$ and so on.

We obtain a series of exact sequences of sets which we claim to be split:

$$0 \to C_{0}^{aff}(\mathbb{R}; S^{1-i,bg}_{PL}(M \times \mathbb{R}^{n-1} \times I, \partial)) \to S^{i-1,bg}_{PL}(M \times \mathbb{R}^{n-i}) \to S^{i,bg}_{PL}(M \times \mathbb{R}^{n-i-1})$$

the splitting being given by Cartesian product with $\mathbb{R}$. We can prove the theorem inductively by observing that

$$S^{1-i,bg}_{PL}(M \times \mathbb{R}^{n-i}) = C_{1}^{aff}(\mathbb{R}; S^{i-1,bg}_{PL}(M \times \mathbb{R}^{n-i}))$$

and applying the PL $bg$ surgery exact sequence, the algebraic $bg$ surgery exact sequence, the five-lemma and the Eilenberg-Zilber theorem.

We also claim that the splitting preserves the group structures on the various structure sets involved. This can be seen, by either using the “characteristic variety addition” on the structure sets, or the algebraic definition of the TOP structure set.

**Proof of Theorem 1.2.** This follows from the first main theorem except for some low dimensional difficulties, which are care of by Siebenmann periodicity, since

$$S^{bg}_{TOP}(\mathbb{R}^n) = H_*(\mathbb{R}^n; S^{TOP}_*(pt)) = 0$$

where we use Siebenmann periodicity to set the convention $S^{TOP}(D^i, \partial) = 0$, $i \leq 4$ although the PL Poincaré conjecture is not known in the cases $i = 3, 4$. (See [60] for a treatment of this in the compact case). We carry this argument out following [60]. We then use the $bg$ PL exact sequence and the finiteness of the group of homotopy spheres to derive the result in the smooth category.
Let $\phi : M \to N$ be a bg degree 1 normal map, $F$ a framing of $\tau(M) \oplus \phi^* v$ where $\tau(M)$ is the tangent PL block bundle of $M$, $v$ the stable normal bundle of $N$. Let $u$ be the stable normal bundle of $CP^2$, and let $G$ be a stable framing of $\tau(CP^2) \oplus u$. Then $F \times G$ is a framing of

$$\tau(M \times CP^2) \oplus (\phi \times id)^*(v \times u) = (\tau M \oplus \phi^* v) \times (\tau(CP^2) \oplus u).$$

The bg surgery obstruction of $M \times CP^2$, $\phi \times id$, $F \times G$ is then the same as that of $M, \phi, F$. This allows us to carry out the splitting along parallel 1-, 2- and 3-dimensional hyperplanes in $R^n$, by replacing $R^n$ with $R^n \times CP^2$ and then peeling off the $CP^2$.

**Proof of Theorem 1.3.** We first observe that a map $f : M \times T^n \to N$ lifts to a map which is boundedly homotopic to a bg simple homotopy equivalence on the free abelian cover if and only if it lifts to a map which is simple on a finite cover. This follows from the theorem of Bass-Heller-Swan:

$$Wh(Z^n \times G) = H_*(T^n; Wh_*(G)) \oplus Nils$$

along with the calculation of the Whitehead group in section 5, proposition 3.4 and the observation that any given element of the Nils vanishes on a finite cover. In fact, since $H^{\text{eff}}_*(R^n; G)$ is torsion-free for any abelian group $G$ the kernel of the map

$$H_*(T^n; Wh_*(G)) \to H^{\text{eff}}_*(R^n; Wh(G))$$

is torsion. This can also be seen by a transfer argument.

We then apply the calculation of the structure set, Theorem 1.1 above. By theorem 5.1 of [60], the structure set of $M \times T^n$ is given by

$$S_{\text{TOP}}^s(M \times T^n) = H_*(T^n; S_{\text{TOP}}^s(M))$$

Applying proposition 3.4, we obtain a rational injection

$$H_*(T^n; S_{\text{TOP}}^s(M)) \otimes Q \to H^{\text{eff}}_*(R^n; S_{\text{TOP}}^s(M)) \otimes Q.$$

We therefore need to check that an element of the kernel of the map, which is a torsion element in $H_i(T^n; S^{\text{TOP}}_{\text{TOP}}(M \times D^{n-1}, \partial))$, becomes trivial on a finite cover. But to see this, simply observe that taking an $n$-fold cover of a split structure on $M \times T^n$ has the effect of adding the structure on $(M \times D^{n-1}, \partial)$ to itself $n$ times. We can then apply the bg PL surgery exact sequence. This proves the result in the PL category. For the result in the smooth category, observe that by [60] if $N$ is PL homeomorphic to $M \times T^n$ then a finite cover of $N$ is diffeomorphic to $M \times T^n$.

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