Abstract. We establish a connection between two areas of independent interest in representation theory, namely Koszul duality and higher homological algebra. This is done through a generalization of the notion of $T$-Koszul algebras, for which we obtain a higher version of classical Koszul duality. Our approach is motivated by and has applications for $n$-hereditary algebras. In particular, we characterize an important class of $n$-$T$-Koszul algebras of highest degree $a$ in terms of $(na - 1)$-representation infinite algebras. As a consequence, we see that an algebra is $n$-representation infinite if and only if its trivial extension is $(n + 1)$-Koszul with respect to its degree 0 part. Furthermore, we show that when an $n$-representation infinite algebra is $n$-representation tame, then the bounded derived categories of graded modules over the trivial extension and over the associated $(n + 1)$-preprojective algebra are equivalent. In the $n$-representation finite case, we introduce the notion of almost $n$-$T$-Koszul algebras and obtain similar results.

Contents

1. Introduction
1.1. Conventions and notation
2. Preliminaries
2.1. Graded algebras, modules and extensions
2.2. Graded algebras as dg-categories
2.3. Graded Frobenius algebras
3. Higher Koszul duality
4. Tilting objects, equivalences and Serre functors
5. On $n$-hereditary algebras
6. Higher Koszul duality and $n$-representation infinite algebras
7. Higher almost Koszulity and $n$-representation finite algebras
References
1. INTRODUCTION

Global dimension is a useful measure for the objects one studies in representation theory of finite dimensional algebras. However, while algebras of global dimension 0 and 1 are exceptionally well understood, it seems quite difficult to develop a general theory for algebras of higher global dimension. This is a background for studying the class of $n$-hereditary algebras $[7,14,15,17,18,21,23]$. These algebras play an important role in higher Auslander–Reiten theory $[19,20,26]$, which has been shown to have connections to commutative algebra, both commutative and non-commutative algebraic geometry, combinatorics, conformal field theory, and homological mirror symmetry $[1,8,9,16,24,37]$. An $n$-hereditary algebra has global dimension less than or equal to $n$ and is either $n$-representation finite or $n$-representation infinite. As one might expect, these notions coincide with the classical definitions of representation finite and infinite hereditary algebras in the case $n = 1$.

Like in the classical theory, $n$-hereditary algebras have a notion of (higher) preprojective algebras. If $A$ is $n$-representation infinite and the $(n + 1)$-preprojective $\Pi_{n+1}A$ is graded coherent, there is an equivalence $\mathcal{D}^b(\text{mod } A) \simeq \mathcal{D}^b(\text{qgr } \Pi_{n+1}A)$, where $\text{qgr } \Pi_{n+1}A$ denotes the category of finitely presented graded modules modulo finite dimensional modules $[34,35]$. On the other hand, the bounded derived category of a finite dimensional algebra of finite global dimension is always equivalent to the stable category of finitely generated graded modules over its trivial extension $[13]$. Combining these two equivalences, and using the notation $\Delta A$ for the trivial extension of $A$, one obtains

$$\text{gr}(\Delta A) \simeq \mathcal{D}^b(\text{qgr } \Pi_{n+1}A).$$

The equivalence above brings to mind the acclaimed Bernštejn–Gel’fand–Gel’fand-correspondence, which can be formulated as $\text{gr } \Lambda \simeq \mathcal{D}^b(\text{qgr } \Lambda')$ for a finite dimensional Frobenius Koszul algebra $\Lambda$ and its graded coherent Artin–Schelter regular Koszul dual $\Lambda'$ $[4]$. The BGG-correspondence is known to descend from the Koszul duality equivalence between bounded derived categories of graded modules over the two algebras, as indicated in the following diagram

$$\mathcal{D}^b(\text{gr } \Lambda) \xrightarrow{\simeq} \mathcal{D}^b(\text{gr } \Lambda') \xrightarrow{\simeq} \text{gr } \Lambda \xrightarrow{\simeq} \mathcal{D}^b(\text{qgr } \Lambda').$$

It is natural to ask whether something similar is true in the $n$-representation infinite case, i.e. if the equivalence (1.1) is a consequence of some higher Koszul duality pattern. This is a motivating question for this paper.

**Motivating question.** Is the equivalence (1.1) a consequence of some higher Koszul duality pattern?
One reasonable approach to this question is to study generalizations of the notion of Koszulity. A positively graded algebra $\Lambda$ generated in degrees 0 and 1 with semisimple degree 0 part is known as a Koszul algebra if $\Lambda_0$ is a graded self-orthogonal module over $\Lambda$. This means that $\text{Ext}^i_{\text{gr}}(\Lambda_0, \Lambda_0(\langle j \rangle)) = 0$ whenever $i \neq j$, where $\langle - \rangle$ denotes the graded shift. Using basic facts about Serre functors and triangulated equivalences, one can show that a similar statement holds for $\Delta A$ with respect to its degree 0 part $(\Delta A)_0 = A$ in the case where $A$ is $n$-representation infinite. Here, the algebra $A$ is clearly not necessarily semisimple, but it is of finite global dimension.

In [12] Green, Reiten and Solberg present a notion of Koszulity for more general graded algebras, where the degree 0 part is allowed to be an arbitrary finite dimensional algebra. Their work provides a unified approach to Koszul duality and tilting equivalence. Koszulity in this framework is defined with respect to a module $T$, and thus the algebras are called $T$-Koszul. Madsen [33] gives a simplified definition of $T$-Koszul algebras, which he shows to be a generalization of the original one whenever the degree 0 part is of finite global dimension.

We generalize Madsen’s definition to obtain the notion of $n$-$T$-Koszul algebras, where $n$ is a positive integer and $n = 1$ returns Madsen’s theory. In Theorem 3.9 we prove that an analogue of classical Koszul duality holds in this generality, and we recover a version of the BGG-correspondence in Proposition 3.11. Moreover, Theorem 6.4 provides a characterization of an important class of $n$-$T$-Koszul algebras of highest degree $a$ in terms of $(na - 1)$-representation infinite algebras. More precisely, we show that a finite dimensional graded Frobenius algebra of highest degree $a \geq 1$ is $n$-$T$-Koszul if and only if $\tilde{T} = \bigoplus_{i=0}^{a-1} \Omega^{-ni}T(i)$ is a tilting object in the associated stable category and the endomorphism algebra of this object is $(na - 1)$-representation infinite. As a consequence, we see in Corollary 6.6 that an algebra is $n$-representation infinite if and only if its trivial extension is $(n + 1)$-Koszul with respect to its degree 0 part. Furthermore, we show in Corollary 6.9 that when $A$ is $n$-representation infinite, then the higher Koszul dual of its trivial extension is given by the associated $(n + 1)$-preprojective algebra. Combining this with our version of the BGG-correspondence, Corollary 6.10 gives an affirmative answer to our motivating question. In particular, we see that when an $n$-representation infinite algebra $A$ is $n$-representation tame, then the bounded derived categories of graded modules over $\Delta A$ and over $\Pi_{n+1}A$ are equivalent, and that this descends to give an equivalence $\text{gr}(\Delta A) \simeq D^b(qgr \Pi_{n+1}A)$.

Having developed our theory for one part of the higher hereditary dichotomy, we ask and provide an answer to whether something similar holds in the higher representation finite case. Inspired by and seeking to generalize the notion of almost Koszul algebras as developed by Brenner, Butler and King [6], we arrive at the definition of almost $n$-$T$-Koszul algebras. This enables us to show a similar characterization result as in the $n$-$T$-Koszul case, namely Theorem 7.17.
Altogether, we establish a connection between two areas of independent interest in representation theory, namely Koszul duality and higher homological algebra. Notice that a relationship between Koszulity and $n$-hereditary algebras is also studied in [5], and more recently in [11]. In some sense, parts of the theory we develop is a generalized Koszul dual version of results in [11, 35]. Note that many of our results are novel already in the case $n = 1$. This demonstrates that questions arising from higher homological algebra can lead to new insight also in the classical case.

This paper is organized as follows. In Section 2, we highlight relevant facts about graded algebras, before the definition and general theory of $n$-$T$-Koszul algebras is presented in Section 3. In Section 4, we give an overview of the notions of tilting objects and Serre functors, and construct an equivalence which will be heavily used later on. As a foundation for the rest of the paper, Section 5 is devoted to recalling definitions and known facts about $n$-hereditary algebras. Note that this section does not contain new results. In Section 6, we state and prove our results on the connections between $n$-$T$-Koszul algebras and higher representation infinite algebras. Finally, almost $n$-$T$-Koszul algebras are introduced in Section 7, and we develop their theory along the same lines as was done in Section 6.

1.1. Conventions and notation. Throughout this paper, let $k$ be an algebraically closed field and $n$ a positive integer. All algebras are algebras over $k$. We denote by $D$ the duality $D(−) = \text{Hom}_k(−, k)$.

Notice that $A$ and $B$ always denote ungraded algebras, while the notation $\Lambda$ and $\Gamma$ is used for graded algebras. We work with right modules, homomorphisms act on the left of elements, and we write the composition of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ as $g \circ f$. We denote by $\text{Mod} A$ the category of $A$-modules and by $\text{mod} A$ the category of finitely presented $A$-modules.

We write the composition of arrows $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ in a quiver as $\alpha \beta$. In our examples, we use diagrams to represent indecomposable modules. This convention is explained in more detail in Example 6.5.

Given a set of objects $U$ in an additive category $\mathcal{A}$, we denote by $\text{add} U$ the full subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of objects in $U$. If $\mathcal{A}$ is triangulated, we use the notation $\text{Thick} \mathcal{A}(U)$ for the smallest thick subcategory of $\mathcal{A}$ which contains $U$. When it is clear in which category our thick subcategory is generated, we often omit the subscript $\mathcal{A}$.

Note that we have certain standing assumptions given at the beginning of Section 3 and Section 6.

2. Preliminaries

In this section we recall some facts about graded algebras which will be used later in the paper. In particular, we observe how a graded algebra can be considered as
a dg-category concentrated in degree $0$. This plays an important role in our proofs in Section 3. We also provide an introduction to a class of algebras which will be studied in Section 6 and Section 7, namely the graded Frobenius algebras.

2.1. Graded algebras, modules and extensions. Consider a graded $k$-algebra $\Lambda = \bigoplus_{i \in \mathbb{Z}} \Lambda_i$. The category of graded $\Lambda$-modules and degree $0$ morphisms is denoted by $\text{Gr}\Lambda$ and the subcategory of finitely presented graded $\Lambda$-modules by $\text{gr}\Lambda$. Recall that $\text{gr}\Lambda$ is abelian if and only if $\Lambda$ is graded right coherent, i.e. if every finitely generated homogeneous right ideal is finitely presented.

Given a graded module $M = \bigoplus_{i \in \mathbb{Z}} M_i$, we define the $j$-th graded shift of $M$ to be the graded module $M\langle j \rangle$ with $M\langle j \rangle_i = M_{i-j}$. The following basic result relates ungraded extensions to graded ones.

**Lemma 2.1** (See [36, Corollary 2.4.7]). Let $M$ and $N$ be graded $\Lambda$-modules. If $M$ is finitely generated and there is a projective resolution of $M$ such that all syzygies are finitely generated, then

$$\text{Ext}^i_{\Lambda}(M, N) \simeq \bigoplus_{j \in \mathbb{Z}} \text{Ext}^i_{\text{Gr}\Lambda}(M, N\langle j \rangle)$$

for all $i \geq 0$.

A non-zero graded module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is said to be concentrated in degree $m$ if $M_i = 0$ for $i \neq m$. When $\Lambda$ is finite dimensional and $M$ finitely generated, there is an integer $h$ such that $M_h \neq 0$ and $M_i = 0$ for every $i > h$. We call $h$ the highest degree of $M$. In the same way, the lowest degree of $M$ is the integer $l$ such that $M_l \neq 0$ and $M_i = 0$ for every $i < l$.

2.2. Graded algebras as dg-categories. Recall that a dg-category is a $k$-linear category in which the morphism spaces are complexes over $k$ and the composition is given by chain maps. We refer to [27] for general background on dg-categories.

In [32, Section 4] it is explained how one can encode the information of a graded algebra as a dg-category concentrated in degree $0$. This is useful, as it enables us to apply known techniques developed for dg-categories to get information about the derived category of graded modules. Let us briefly recall this construction, emphasizing the part which is used in Section 3.

Given a graded algebra $\Lambda = \bigoplus_{i \in \mathbb{Z}} \Lambda_i$, we associate the category $\mathcal{A}$, in which $\text{Ob}(\mathcal{A}) = \mathbb{Z}$ and the morphisms are given by $\text{Hom}_\mathcal{A}(i, j) = \Lambda_{i-j}$. Multiplication in $\Lambda$ yields composition in $\mathcal{A}$ in the natural way. Observe that the Hom-sets of $\mathcal{A}$ behaves well with respect to addition in $\mathbb{Z}$, namely that for any integers $i$ and $j$, we have

\begin{equation}
\text{Hom}_\mathcal{A}(i, 0) \simeq \text{Hom}_\mathcal{A}(i + j, j).
\end{equation}

The category of right modules over $\mathcal{A}$, meaning $k$-linear functors from $\mathcal{A}^{\text{op}}$ into $\text{Mod} k$, is equivalent to $\text{Gr}\Lambda$. Similarly, as $\mathcal{A}$ is a dg-category concentrated in
degree 0, dg-modules over $\mathcal{A}$ correspond to complexes of graded $\Lambda$-modules. Consequently, one obtains $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\text{Gr} \, \Lambda)$, i.e. that the derived category of the dg-category $\mathcal{A}$ is equivalent to the usual derived category of $\text{Gr} \, \Lambda$.

Instead of starting with a graded algebra, one can use this construction the other way around. Given a dg-category $\mathcal{A}$ concentrated in degree 0, for which the objects are in bijection with the integers and the condition (2.1) is satisfied, we can identify the category with the graded algebra

$$\Lambda = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_\mathcal{A}(i, 0),$$

in the sense that $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\text{Gr} \, \Lambda)$. Notice that the fact that certain Hom-sets coincide is necessary in order to be able to use composition in our category to define multiplication in $\Lambda$.

2.3. Graded Frobenius algebras. Recall that twisting by a graded algebra automorphism $\phi$ of a graded algebra $\Lambda$ yields an autoequivalence $(-)_\phi$ on $\text{gr} \, \Lambda$. Given $M$ in $\text{gr} \, \Lambda$, the module $M_\phi$ is defined to be equal to $M$ as a vector space with right $\Lambda$-action $m \cdot \lambda = m\phi(\lambda)$, while $(-)_\phi$ acts trivially on morphisms.

A finite dimensional positively graded algebra $\Lambda$ is called graded Frobenius if $D\Lambda \simeq \Lambda(-a)$ as both graded left and graded right $\Lambda$-modules for some integer $a$. Notice that if $\Lambda$ is concentrated in degree 0, we recover the usual notion of a Frobenius algebra. Observe also that the integer $a$ in our definition must be equal to the highest degree of $\Lambda$, as $(D\Lambda)_i = D(\Lambda_{-i})$. We will usually assume $a \geq 1$.

Being graded Frobenius is equivalent to being Frobenius as an ungraded algebra and having a grading such that the socle is contained in the highest degree.

Lemma 2.2. Let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a finite dimensional algebra of highest degree $a$. The following are equivalent:

(1) $\Lambda$ is graded Frobenius.

(2) There exists a graded automorphism $\mu$ of $\Lambda$ such that $1\Lambda_\mu(-a) \simeq D\Lambda$ as graded $\Lambda$-bimodules.

(3) $\Lambda$ is Frobenius as ungraded algebra and has a grading satisfying $\text{Soc} \, \Lambda \subseteq \Lambda_a$.

Proof. If $\Lambda$ is graded Frobenius, [35, Lemma 2.9] implies that there exists a graded automorphism $\mu$ of $\Lambda$ such that

$$D\Lambda \simeq 1\Lambda_\mu(-a) \simeq \mu^{-1}1\Lambda(-a)$$

as graded $\Lambda$-bimodules. It is hence clear that (1) is equivalent to (2).

To see that (1) is equivalent to (3), use that graded lifts of finite dimensional modules are unique up to isomorphism and graded shift [33, Lemma 2.5.3] together with the fact that $\text{Soc} \, D\Lambda \subseteq (D\Lambda)_0$. □

The automorphism $\mu$ of a graded Frobenius algebra $\Lambda$ as in the lemma above, is unique up to composition with an inner automorphism and is known as the graded
Nakayama automorphism of $\Lambda$. We call $\Lambda$ graded symmetric if $\mu$ can be chosen to be trivial, and note that this notion also descends to the ungraded case.

One class of examples which will be important for us, is that of trivial extension algebras. Recall that given a finite dimensional algebra $A$, the trivial extension of $A$ is $\Delta A := A \oplus DA$ as a vector space. The trivial extension is an algebra with multiplication $(a, f) \cdot (b, g) = (ab, ag + fb)$ for $a, b \in A$ and $f, g \in DA$. We consider $\Delta A$ as a graded algebra by letting $A$ be in degree 0 and $DA$ be in degree 1. Observe that $\Delta A$ is graded symmetric as it is symmetric as an ungraded algebra and satisfies $\text{Soc} \Delta A \subseteq (\Delta A)_1$.

The stable category of finitely presented graded modules over a graded algebra $\Lambda$ is denoted by $\text{gr} \Lambda$. If $\Lambda$ is self-injective, the category $\text{gr} \Lambda$ is a Frobenius category, and $\text{gr} \Lambda$ is triangulated with shift functor $\Omega^{-1}(-)$. Notice that every Frobenius algebra is self-injective. Observe that twisting by a graded automorphism $\phi$ of $\Lambda$ descends to an autoequivalence $(-)_\phi$ on $\text{gr} \Lambda$. This functor commutes with taking syzygies and cosyzygies, as well as with graded shift.

We will often consider syzygies and cosyzygies of modules over self-injective algebras even when we do not work in a stable category. Whenever we do so, we assume having chosen a minimal projective or injective resolution, so that our syzygies and cosyzygies do not have any non-zero projective summands. Because of our convention with respect to (representatives of) syzygies and cosyzygies, the notions of highest and lowest degree make sense for these too.

Throughout the paper, we often need to consider basic degree arguments, as summarized in the following lemma. We include a short proof for the convenience of the reader.

**Lemma 2.3.** Let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a finite dimensional self-injective graded algebra of highest degree $a$ and $\text{Soc} \Lambda \subseteq \Lambda_a$. The following statements hold:

1. Given any non-zero element $x \in \Lambda$, there exists $\lambda \in \Lambda$ such that $x\lambda \in \Lambda_a$ is non-zero.
2. Let $P$ be an indecomposable projective graded $\Lambda$-module of highest degree $h$. Then, given any non-zero element $x \in P$, there exists $\lambda \in \Lambda$ such that $x\lambda \in P_h$ is non-zero.
3. Let $M$ and $P$ be finitely generated graded $\Lambda$-modules with $P$ indecomposable projective. Denote the highest degree of $P$ by $h$. Then, for every non-zero morphism $f \in \text{Hom}_{\text{gr} \Lambda}(M, P)$, there exists an element $x \in M$ such that $f(x) \in P_h$ is non-zero.
4. Let $M$ be an non-projective finitely generated graded $\Lambda$-module of highest degree $h$ and lowest degree $l$. Then the highest degree of $\Omega^i M$ is less than or equal to $h$ in the case $i \leq 0$ and greater than or equal to $l + a$ in the case $i > 0$.
5. Assume $a \geq 1$, and let $M$ and $N$ be modules concentrated in degree 0. Then $\text{Hom}_{\text{gr} \Lambda}(M, N) \simeq \text{Hom}_{\text{gr} \Lambda}(M, N)$.
(6) Let $M$ be a module concentrated in degree 0. Then
$$\text{Hom}_{\text{gr} \Lambda}(M, \Omega^i M\langle j \rangle) = 0$$
for $i, j < 0$.

(7) Let $M$ be a module concentrated in degree 0. Then
$$\text{Hom}_{\text{gr} \Lambda}(M, \Omega^i M\langle j \rangle) = 0$$
for $i > 0$ and $j \geq 1 - a$.

**Proof.** Combining the assumption $\text{Soc} \Lambda \subseteq \Lambda_0$ with the facts that $\text{Rad} \Lambda$ is nilpotent and $\text{Soc} \Lambda = \{ y \in \Lambda \mid y \text{Rad} \Lambda = 0 \}$, one obtains (1).

Part (2) follows from (1), as projectives are direct summands of free modules. For (3), let $y \in M$ such that $f(y) \neq 0$. By (2), there exists an element $\lambda \in \Lambda$ such that $f(y)\lambda \in P_h$ is non-zero. Consequently, the element $x = y\lambda$ yields our desired conclusion.

In order to prove (4), let us first consider the case $i \leq 0$. The statement clearly holds if $i = 0$. Observe next that $\text{Soc} M$ has highest degree $h$. Hence, the injective envelope of $M$ also has highest degree $h$. Since $M$ is non-projective, the cosyzygy $\Omega^{-1} M$ is a non-zero quotient of this injective envelope, and consequently has highest degree at most $h$. We are thus done by induction.

For the case $i > 0$, note that each summand in the projective cover of $M$ has highest degree greater than or equal to $l + a$. As $\Omega M$ is a submodule of this projective cover, it follows from (3) that $\Omega M$ also has highest degree greater than or equal to $l + a$. Moreover, the syzygy is itself non-projective of lowest degree greater than or equal to $l$, so the claim follows by induction.

To verify (5), notice that there can be no non-zero homomorphism $M \to N$ factoring through a $\Lambda$-projective. Otherwise, one would have non-zero homomorphisms $M \to \Lambda\langle i \rangle$ and $\Lambda\langle i \rangle \to N$ for some integer $i$. The former is possible only if $i = -a$ by (3). However, if $i = -a$, the latter is impossible as $\Lambda\langle -a \rangle$ is generated in degree $-a$.

Observe that (6) is immediate in the case where $M$ is projective. Otherwise, note that the highest degree of $\Omega^i M$ is at most 0 by (4). Hence, the highest degree of $\Omega^i M\langle j \rangle$ is less than or equal to $j$. As $j < 0$, this yields our desired conclusion.

For (7), it again suffices to consider the case where $M$ is non-projective. Applying (4), our assumptions yield that the highest degree of $\Omega^i M\langle j \rangle$ is greater than or equal to 1. By (3), this gives $\text{Hom}_{\text{gr} \Lambda}(M, \Omega^i M\langle j \rangle) = 0$, as syzygies are submodules of projectives.

3. **Higher Koszul duality**

Throughout the rest of this paper, let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a positively graded algebra, where $\Lambda_0$ is a finite dimensional algebra augmented over $k^r$ for some $r > 0$. We assume that $\Lambda$ is locally finite dimensional, i.e. that $\Lambda_i$ is finite dimensional as a vector space over $k$ for each $i \geq 0$. 
In this section we define more flexible notions of what it means for a module $T$ to be graded self-orthogonal and an algebra to be $T$-Koszul than the ones introduced by Madsen [33, Definition 3.1.1 and 4.1.1]. This enables us to talk about $T$-Koszul duality for a more general class of algebras. In particular, we obtain a higher Koszul duality equivalence in Theorem 3.9 and we recover a higher version of the BGG-correspondence in Proposition 3.11. Note that the ideas in this section are similar to the ones in [33]. For the convenience of the reader, we nevertheless give concise proofs of this section’s main results, to show that the arguments work also in our generality.

It should be noted that it is also possible to derive Theorem 3.9 by using [33, Theorem 4.3.4]. This strategy involves regrading the algebras so that they satisfy Madsen’s definition of graded self-orthogonality and tracking our original (derived) categories of graded modules through his equivalence. We spell this out in greater detail after our proof of Theorem 3.9. Proceeding in this way, one can recover generalized analogues of many of the results in [33]. We make no essential use of these results, but this approach could be relevant for future related work.

We remark that we believe it to be undesirable to work with the regraded algebras throughout, since – as will become clear – the resulting graded module categories are in some sense too big. Moreover, we consider endomorphism algebras of tilting objects, and it is less convenient to study regraded versions of these. In particular, as we want to relate our results to existing ones involving graded modules over trivial extensions or preprojective algebras, we cannot always work directly with the regraded algebras.

In order to state our main definitions, let us first recall the notion of a tilting module.

**Definition 3.1.** Let $A$ be a finite dimensional algebra. A finitely generated $A$-module $T$ is called a **tilting module** if the following conditions hold:

1. $\text{proj. dim}_A T < \infty$;
2. $\text{Ext}^i_A(T, T) = 0$ for $i > 0$;
3. There is an exact sequence
   $$0 \to A \to T^0 \to T^1 \to \cdots \to T^l \to 0$$
   with $T^i \in \text{add } T$ for $i = 0, \ldots, l$.

We now define what it means for a module to be graded $n$-self-orthogonal.

**Definition 3.2.** Let $T$ be a finitely generated basic graded $A$-module concentrated in degree 0. We say that $T$ is **graded $n$-self-orthogonal** if

$$\text{Ext}^i_A(T, T(j)) = 0$$

for $i \neq nj$.

Usually, it will be clear from context what the parameter $n$ is, so we often simply say that a module satisfying the description above is graded self-orthogonal.
Notice that this definition of graded self-orthogonality is more general than the one given in [33]. More precisely, the two definitions coincide exactly when $n$ is equal to 1. In this case, examples of graded self-orthogonal modules are given by $\Lambda_0$ in the classical Koszul situation or tilting modules if $\Lambda = \Lambda_0$. Moreover, we see in Section 6 that $n$-representation infinite algebras provide examples of modules which are graded $n$-self-orthogonal for any choice of $n$.

In general, a graded self-orthogonal module $T$ might have syzygies which are not finitely generated, so Lemma 2.1 does not apply. However, the following proposition gives a similar result for graded self-orthogonal modules. This is an analogue of [33, Proposition 3.1.2]. The proof is exactly the same, except that we use our more general version of what it means for $T$ to be graded self-orthogonal. 

**Proposition 3.3.** Let $T$ be a graded $n$-self-orthogonal $\Lambda$-module. Then 
\[
\text{Ext}_{\Lambda}^n(T, T) \simeq \text{Ext}_{\text{gr } \Lambda}^n(T, T^i)
\]
for all $i \geq 0$.

Using our definition of a graded self-orthogonal module $T$, we also get a more general notion of what it means for an algebra to be Koszul with respect to $T$.

**Definition 3.4.** Assume $\text{gl.dim} \ A_0 < \infty$ and let $T$ be a graded $\Lambda$-module concentrated in degree 0. We say that $\Lambda$ is $n$-$T$-Koszul or $n$-Koszul with respect to $T$ if the following conditions hold:

1. $T$ is a tilting $\Lambda_0$-module.
2. $T$ is graded $n$-self-orthogonal as a $\Lambda$-module.

**Remark 3.5.** In Definition 3.2 we require a graded $n$-self-orthogonal module to be basic for consistency with [33]. As a consequence of this choice, we later assume that certain algebras are basic, for instance in Corollary 6.6. Note that this assumption does not usually play an important role in our proofs, and could be omitted if one is willing to consider $n$-Koszul algebras with respect to a possibly non-basic module $T$.

Like in the classical theory, we want a notion of a Koszul dual of a given $n$-$T$-Koszul algebra.

**Definition 3.6.** Let $\Lambda$ be an $n$-$T$-Koszul algebra. The $n$-$T$-Koszul dual of $\Lambda$ is given by $\Lambda! = \bigoplus_{i \geq 0} \text{Ext}_{\text{gr } \Lambda}^n(T, T^i)$.

Note that while the notation for the $n$-$T$-Koszul dual is potentially ambiguous, it will in this paper always be clear from context which $n$-$T$-Koszul structure the dual is computed with respect to.

By Proposition 3.3 we get the following equivalent description of the $n$-$T$-Koszul dual.

**Corollary 3.7.** Let $\Lambda$ be an $n$-$T$-Koszul algebra. Then there is an isomorphism of graded algebras $\Lambda! \simeq \bigoplus_{i \geq 0} \text{Ext}_{\Lambda}^n(T, T)$.
Higher Koszul Duality and Connections with n-Hereditary Algebras

Given a set of objects \( \mathcal{U} \subseteq \mathcal{D}^b(\text{gr} \Lambda) \), let \( \text{Thick}^{(-)}(\mathcal{U}) \) denote the smallest thick subcategory of \( \mathcal{D}^b(\text{gr} \Lambda) \) which contains \( \mathcal{U} \) and is closed under graded shift. Using that \( \Lambda_0 \) has finite global dimension and that \( T \) is a tilting \( \Lambda_0 \)-module, one can see that \( T \) generates the entire bounded derived category of \( \text{gr} \Lambda \) whenever \( \Lambda \) is a finite dimensional \( n \)-\( T \)-Koszul algebra.

**Lemma 3.8.** Let \( \Lambda \) be a finite dimensional \( n \)-\( T \)-Koszul algebra. We then have \( \text{Thick}^{(-)}(T) = \mathcal{D}^b(\text{gr} \Lambda) \).

The proof of Theorem 3.9 uses notions and techniques of dg-homological algebra. Since this is the only section where these are used, we refer the reader to [27] for an introduction. Notice that we have more or less adopted the notation of that source for the reader’s convenience. In particular, recall from [27] that given a dg-category \( B \), we define the category \( H^0 B \) to have the same objects as \( B \) and morphisms given by taking the 0-th cohomology of the morphism spaces in \( B \). Similarly, also the category \( \tau \leq 0 B \) has the same objects as \( B \), and morphisms given by taking subtle truncation.

We are now ready to state and prove the main result of this section, namely to show that we obtain a higher Koszul duality equivalence. This recovers [33, Theorem 4.3.4] in the case where \( n = 1 \) and is a version of [3, Theorem 2.12.6] in the classical Koszul case.

**Theorem 3.9.** Let \( \Lambda \) be a finite dimensional \( n \)-\( T \)-Koszul algebra and assume that \( \Lambda^! \) is graded right coherent. There is then an equivalence \( \mathcal{D}^b(\text{gr} \Lambda) \simeq \mathcal{D}^b(\text{gr} \Lambda^!) \) of triangulated categories.

**Proof.** Consider the full subcategory \( \mathcal{U} = \{ T^{\langle i \rangle}[ni] \mid i \in \mathbb{Z} \} \) of \( \mathcal{D}^b(\text{gr} \Lambda) \). Using a standard lift [27, Section 7.3], we replace \( \mathcal{U} \) by a dg-category \( B \) which has objects \( \{ P^{\langle i \rangle}[ni] \} \), where \( P \) is some graded projective resolution of \( T \) and

\[
\text{Hom}_B(P^{\langle i \rangle}[ni], P^{\langle j \rangle}[nj])^k = \prod_{m \in \mathbb{Z}} \text{Hom}_{\text{gr} \Lambda}(P^{m+ni^{\langle i \rangle}}, P^{m+nj+k^{\langle j \rangle}}).
\]

In other words, morphism spaces are given by all homogeneous maps of complexes that are also homogeneous of degree 0 with respect to the grading of \( \Lambda \). The morphism spaces are complexes with the standard super commutator differential defined by

\[
d(f) = d_{P^{\langle j \rangle}[nj]} \circ f - (-1)^k f \circ d_{P^{\langle i \rangle}[ni]},
\]

for \( f \) in \( \text{Hom}_B(P^{\langle i \rangle}[ni], P^{\langle j \rangle}[nj])^k \).

By Lemma 3.8, we have \( \text{Thick}(\mathcal{U}) = \text{Thick}^{(-)}(T) = \mathcal{D}^b(\text{gr} \Lambda) \). Since we have used a standard lift and idempotents split in \( \mathcal{D}^b(\text{gr} \Lambda) \), we get that \( \text{Thick}(\mathcal{U}) = \mathcal{D}^b(\text{gr} \Lambda) \) is equivalent to \( \mathcal{D}^\text{perf}(B) \), i.e. the subcategory of perfect objects.

As \( T \) is graded \( n \)-self-orthogonal, the cohomology of each morphism space in \( B \) is concentrated in cohomological degree 0. Hence, we get a zigzag of dg-categories

\[
H^0 B \leftrightarrow \tau_{\leq 0} B \hookrightarrow B
\]
in which the dg-functors induce quasi-equivalences. Thus, we also get an equivalence $\mathcal{D}(\mathcal{H}^0 \mathcal{B}) \simeq \mathcal{D}(\mathcal{B})$ \cite{27} Sec. 7.1-7.2 and 9.1. This equivalence descends to one on the compact or perfect objects, and so we get $\mathcal{D}^{\text{perf}}(\mathcal{H}^0 \mathcal{B}) \simeq \mathcal{D}^{\text{perf}}(\mathcal{B})$.

The dg-category $\mathcal{H}^0 \mathcal{B}$ is concentrated in degree 0, its objects are in natural bijection with the integers and we can identify it with a graded algebra as described in Section 2.2. As we wish this algebra to be positively graded, we let the object $P\langle i \rangle [ni]$ in $\mathcal{H}^0 \mathcal{B}$ correspond to the integer $-i$. This yields the algebra
\[
\bigoplus_{i \geq 0} \text{Hom}_{\mathcal{H}^0 \mathcal{B}}(P, P\langle i \rangle [ni]) \simeq \bigoplus_{i \geq 0} \text{Ext}^{ni}_{\text{gr} \Lambda}(T, T\langle i \rangle) = \Lambda^1.
\]

It now follows that $\mathcal{D}(\mathcal{H}^0 \mathcal{B}) \simeq \mathcal{D}(\text{Gr} \Lambda^1)$, which again yields an equivalence $\mathcal{D}^{\text{perf}}(\mathcal{H}^0 \mathcal{B}) \simeq \mathcal{D}^{\text{perf}}(\text{Gr} \Lambda^1)$. As in the ungraded case, compact objects of $\mathcal{D}(\text{Gr} \Lambda^1)$ coincides with perfect complexes, i.e. bounded complexes of finitely generated graded projective modules \cite{27} Theorem 5.3]. Using that $\Lambda^1$ is graded right coherent, it remains to show that $\Lambda^1$ has finite global dimension as a graded algebra, since this hence yields $\mathcal{D}^{\text{perf}}(\text{Gr} \Lambda^1) \simeq \mathcal{D}^b(\text{gr} \Lambda^1)$.

Let $S$ denote the direct sum of all simple unshifted graded $\Lambda$-modules, i.e. the summands of $\Lambda/\text{Rad} \Lambda$, and consider the full subcategory $U' = \{ S\langle i \rangle \mid i \in \mathbb{Z} \}$ of $\mathcal{D}^b(\text{gr} \Lambda)$. Notice that $\text{Thick}(U') = \text{Thick}^-\langle\cdot\rangle(S) = \text{Thick}^-\langle\cdot\rangle(T)$. Similarly as earlier in the proof, we use a standard lift to replace $U'$ by a dg-category $\mathcal{B}'$ and deduce that $\text{Thick}(U') \simeq \mathcal{D}^{\text{perf}}(\mathcal{B}')$. It hence follows that $\mathcal{D}^{\text{perf}}(\text{Gr} \Lambda^1) \simeq \mathcal{D}^{\text{perf}}(\mathcal{B}')$. As part (3) of \cite{27} Corollary 9.2] is satisfied in this case, our equivalence on the perfect derived categories implies existence of a dg-Morita equivalence on the ambient derived categories. Using a version of \cite{30} Lemma 5.7] for augmented dg-categories, see also \cite{29} Proposition 2.2.4.1], one can show that $\mathcal{B}'$ is smooth as a dg-category. As smoothness is invariant under dg-Morita equivalence \cite{31} Theorem 3.17], this yields that $\mathcal{B}$ is smooth as a dg-category. Consequently, we can conclude that $\Lambda^1$ has finite global dimension as a graded algebra \cite{30} Lemma 3.6], which finishes our proof. \(\square\)

Let us now provide more details on how to obtain the above theorem and generalized analogues of other results in \cite{33} using the equivalence constructed there. Observe first that given $\Lambda = \oplus_{i \geq 0} \Lambda_i$, satisfying the assumptions in Theorem 3.9 one can rescale the grading so that the regraded algebra $\Lambda^e$ is $T$-Koszul in the sense of \cite{33} Definition 4.1.1]. To be precise, let $\Lambda^e_i = \Lambda_j$ if $i = nj$ for some integer $j$ and $\Lambda^e_i = 0$ otherwise. The category $\text{gr} \Lambda$ embeds into $\text{gr} \Lambda^e$ as the full subcategory consisting of modules which are non-zero only in degrees multiples of $n$. As the embedding is exact, it induces a triangulated functor between the corresponding derived categories. By \cite{32} Lemma 13.17.4], this functor yields an equivalence $\mathcal{D}^b(\text{gr} \Lambda) \cong \mathcal{D}^b_{\text{gr} \Lambda}(\text{gr} \Lambda^e)$, where $\mathcal{D}^b_{\text{gr} \Lambda}(\text{gr} \Lambda^e)$ denotes the full subcategory of $\mathcal{D}^b(\text{gr} \Lambda^e)$ consisting of objects with cohomology in $\text{gr} \Lambda$. 

Using that $\Lambda^\rho$ is $T$-Koszul and noticing that $(\Lambda^!\rho)^{\mathsf{e}} \simeq (\Lambda^\rho)^{\mathsf{!}}$, we get by [33, Theorem 4.3.4] the equivalence in the upper row of the diagram

\[
\begin{array}{ccc}
\mathcal{D}^b(\text{gr } \Lambda^\rho) & \cong & \mathcal{D}^b(\text{gr } \Lambda) \\
\uparrow & & \uparrow \\
\mathcal{D}^b(\text{gr } \Lambda^\rho) & \cong & \mathcal{D}^b(\text{gr } \Lambda^!\rho)
\end{array}
\]

In order to deduce Theorem [33, Theorem 3.9] from this, we need to show that the equivalence restricts as indicated by the dashed arrow. It is sufficient to show that objects which are non-zero only in degrees multiples of $n$ are sent to objects satisfying the same property. Examining the construction of the equivalence, we see that it is essentially the same as the one given in the proof of Theorem [33, Theorem 3.9] in the case $n = 1$. Consequently, we are done if the equivalences in the zig-zag and the equivalence from $\text{Thick}(\mathcal{U})$ to $\mathcal{D}^{\text{perf}}(\mathcal{B})$ satisfy the desired condition.

For the former equivalences, this is easily verified and is left to the reader, whereas for the latter, we begin by first recalling some necessary notions. Let $\mathcal{A}$ be the dg-category obtained by regarding the graded algebra $\Lambda^\rho$ as a category as outlined in Section [27, Section 1.2] for the definition of the dg-category $\text{Dif } \mathcal{A}$, and [27, Section 6.2] for the definition of the triangulated functor $\mathcal{R}H_X$ for $X$ an $\mathcal{A}$-$\mathcal{B}$-dg-bimodule. If $\mathcal{A}$ is an ordinary algebra concentrated in cohomological degree $0$, the objects of the category $\text{Dif } \mathcal{A}$ are complexes of modules over $\mathcal{A}$ and the morphisms are given by homogeneous maps which do not necessarily respect the differentials. In this case, the functor $\mathcal{R}H_X$ would be quasi-isomorphic to regular $\mathcal{R}\text{Hom}_s$. The theory of standard lifts [27, Section 7.3] implies that the equivalence $\text{Thick}(\mathcal{U}) \to \mathcal{D}^{\text{perf}}(\mathcal{B})$ is the restriction of the functor $\mathcal{R}H_X : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$, where $X$ is the $\mathcal{A}$-$\mathcal{B}$-dg-bimodule given by $X(j, k)^l = P^l_{j+k}$, which has property (P) as defined in [27, Section 3.1]. Hence, we get

\[
\mathcal{R}H_X(M)^l_k = \text{Hom}_{\text{Dif } \mathcal{A}}(X(?^l, k), M)^l_k = \prod_{m \in \mathbb{Z}} \text{Hom}_{\text{Gr } \Lambda^\rho}(P^{m-k}(-k), M^{m+l}) \cong \mathcal{R}\text{Hom}_{\text{Gr } \Lambda^\rho}(P^l(-k)[k], M)^l.
\]

If $n$ does not divide $k$, this is zero whenever $M$ is non-zero only in degrees that are multiples of $n$. Hence, one obtains that Madsen’s equivalence between $\mathcal{D}^b(\text{gr } \Lambda^\rho)$ and $\mathcal{D}^b(\text{gr } \Lambda^\rho)^{\mathsf{e}}$ restricts to yield an equivalence between $\mathcal{D}^b(\text{gr } \Lambda)$ and $\mathcal{D}^b(\text{gr } \Lambda^!)$ as claimed.

In our following two propositions, we denote by $K : \mathcal{D}^b(\text{gr } \Lambda) \to \mathcal{D}^b(\text{gr } \Lambda^!)$ the equivalence from Theorem [33, Theorem 3.9]. Since shifting by 1 in $\text{gr } \Lambda$ corresponds to shifting by $n$ in $\text{gr } \Lambda^\rho$, the argument above together with [33, Proposition 3.2.1] yield the following.
Proposition 3.10. Let \( \Lambda \) be a finite dimensional \( n\)-T-Koszul algebra and assume that \( \Lambda^! \) is graded right coherent and has finite global dimension. We then have \( K(M(i)) = K(M)(-i)[-ni] \) for \( M \in \mathcal{D}^b(\text{gr } \Lambda) \).

We finish this section by showing that an analogue of the BGG-correspondence holds in our generality. Recall that \( qgr \Lambda^! \) is defined as the localization of \( \text{gr } \Lambda^! \) at the full subcategory of finite dimensional graded \( \Lambda^! \)-modules. We hence have a natural functor \( \mathcal{D}^b(\text{gr } \Lambda^!) \to \mathcal{D}^b(qgr \Lambda^!) \). In the case where \( \Lambda \) is graded Frobenius, there is a well-known equivalence \( \mathcal{D}^b(\text{gr } \Lambda)/\mathcal{D}^\text{perf}(\text{gr } \Lambda) \approx \text{gr } \Lambda \) \cite[Theorem 2.1]{[11]}. Note that we recall this result as Theorem 4.2 in our next section. One consequently obtains a functor

\[
\mathcal{D}^b(\text{gr } \Lambda) \to \mathcal{D}^b(\text{gr } \Lambda)/\mathcal{D}^\text{perf}(\text{gr } \Lambda) \approx \text{gr } \Lambda.
\]

These two functors give the vertical arrows in the diagram in our proposition below.

Proposition 3.11. Let \( \Lambda \) be a finite dimensional \( n\)-T-Koszul algebra and assume that \( \Lambda^! \) is graded right coherent and has finite global dimension. If \( \Lambda \) is graded Frobenius, then the equivalence \( K \) descends to yield \( \text{gr } \Lambda \approx \mathcal{D}^b(qgr \Lambda^!) \), as indicated in the following diagram

\[
\begin{array}{ccc}
\mathcal{D}^b(\text{gr } \Lambda) & \xrightarrow{K} & \mathcal{D}^b(\text{gr } \Lambda^!)
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \downarrow
\end{array}
\]

\[
\text{gr } \Lambda \xrightarrow{\approx} \mathcal{D}^b(qgr \Lambda^!).
\]

Proof. Since \( DA \) is injective, we get that the \( k \)-th cohomology of \( R \mathcal{H}^X(\text{DA}(i))_j \) is zero unless \( k = ni = -nj \), in which case it is isomorphic to

\[
\text{Hom}_{\mathcal{D}^b(\text{gr } \Lambda)}(T, \text{DA}) \approx \text{Hom}_{\text{gr } \Lambda}(T, \text{DA})
\]

\[
\approx \text{Hom}_{\text{gr } \Lambda^{op}}(\Lambda, DT)
\]

\[
\approx DT.
\]

Chasing this through the equivalences in the zig-zag in the proof of Theorem 3.9, we notice that this stalk complex has the \( \Lambda^! \)-action one expects, i.e. the action induced by letting \( \Lambda^!_0 \approx \text{End}_{\text{gr } \Lambda}(T) \approx \text{End}_{\Lambda^!_0}(T) \) act on \( T \) on the left by endomorphisms. Our argument above hence yields that \( K \) restricts to an equivalence \( \text{Thick}^{(\cdot)}(\text{DA}) \approx \text{Thick}^{(\cdot)}(DT) \).

Since tilting theory implies that \( DT \) is a tilting module over \( \text{End}_{\Lambda^!_0}(T) \), one deduces that \( \text{Thick}^{(\cdot)}(DT) \) is the full subcategory of \( \mathcal{D}^b(\text{gr } \Lambda^!) \) of all objects with finite dimensional cohomology. As \( qgr \Lambda^! \) is the localization of \( \text{gr } \Lambda^! \) at the Serre subcategory of finite dimensional \( \Lambda^! \)-modules and the quotient functor in this case is known to have a left adjoint, we get that

\[
\mathcal{D}^b(\text{gr } \Lambda^!)/\text{Thick}^{(\cdot)}(DT) \approx \mathcal{D}^b(qgr \Lambda^!)
\]
is an equivalence by [42, Lemma 13.17.2-3].

The triangulated quotient functor $Q: D^b(\text{gr}\ \Lambda^!)/\text{Thick}^{(-1)}(DT) \to D^b(\text{gr}\ \Lambda^!)$ has kernel $\text{Thick}^{(-1)}(DT) \simeq K \text{Thick}^{(-1)}(D\Lambda)$, and hence composing it with $K$ induces a triangulated functor

$$\overline{K}: D^b(\text{gr}\ \Lambda)/\text{Thick}^{(-1)}(D\Lambda) \to D^b(\text{qgr}\ \Lambda^!)$$

satisfying $\overline{K} \circ P = Q \circ K$ by the universal property of quotient categories, in which $P$ is the quotient functor

$$P: D^b(\text{gr}\ \Lambda) \to D^b(\text{gr}\ \Lambda)/\text{Thick}^{(-1)}(D\Lambda).$$

As $\text{gr}\ \Lambda \simeq D^b(\text{gr}\ \Lambda)/\text{Thick}^{(-1)}(D\Lambda)$ by [41, Theorem 2.1] and it is straightforward to check that $\overline{K}$ is an equivalence, we are hence done. □

4. Tilting objects, equivalences and Serre functors

Tilting objects and the equivalences they provide play a crucial role throughout the rest of this paper. In this section we recall relevant notions and apply one of Yamamura’s ideas to give an explicit construction of an equivalence which will be heavily used in Section 6 and Section 7. We also describe the correspondence of Serre functors induced by this equivalence.

**Definition 4.1.** Let $\mathcal{T}$ be a triangulated category. An object $T$ in $\mathcal{T}$ is a tilting object if the following conditions hold:

1. $\text{Hom}_\mathcal{T}(T, T[i]) = 0$ for $i \neq 0$;
2. $\text{Thick}_\mathcal{T}(T) = \mathcal{T}$.

The first condition in the definition above is often referred to as rigidity.

A triangulated category is called algebraic if it is triangle equivalent to the stable category of a Frobenius category. Recall that when $\Lambda$ is a self-injective graded algebra, the category $\text{gr}\ \Lambda$ is Frobenius, and consequently the stable category $\text{gr}\ \Lambda$ is an algebraic triangulated category. By Keller’s tilting theorem [27, Theorem 4.3], we hence know that if $T$ is a tilting object in $\text{gr}\ \Lambda$ and $B = \text{End}_{\text{gr}\ \Lambda}(T)$ has finite global dimension, then there is a triangle equivalence $\text{gr}\ \Lambda \simeq D^b(\text{mod } B)$.

While Keller’s result is proved by applying general techniques from dg-homological algebra, we need a more explicit description of this equivalence. Recall first that $\text{gr}\ \Lambda$ can be realized as the quotient category $D^b(\text{gr}\ \Lambda)/D^\text{perf}(\text{gr}\ \Lambda)$.

**Theorem 4.2** (See [41, Theorem 2.1]). Let $\Lambda$ be finite dimensional and self-injective. Then the canonical embedding $\text{gr}\ \Lambda \to D^b(\text{gr}\ \Lambda)$ induces an equivalence $\text{gr}\ \Lambda \simeq D^b(\text{gr}\ \Lambda)/D^\text{perf}(\text{gr}\ \Lambda)$ of triangulated categories.

Denote by $G$ the quasi-inverse to the equivalence described in Theorem 4.2 and by $P$ the projection functor $D^b(\text{gr}\ \Lambda) \to D^b(\text{gr}\ \Lambda)/D^\text{perf}(\text{gr}\ \Lambda)$. As $T$ has a natural structure as a left $B$-module, we can consider the left derived tensor functor

$$D^b(\text{mod } B) \xrightarrow{\otimes_B T} D^b(\text{gr}\ \Lambda).$$
Note that when we think of the tilting object $T$ in $\text{gr} \Lambda$ as a graded $\Lambda$-module, we choose a representative without projective summands.

We now give an explicit description of the equivalence $\text{gr} \Lambda \simeq \mathcal{D}^b(\text{mod} B)$. This construction and proof is essentially the same as [13, Proposition 3.14], but we show that it also works in our more general setup.

**Proposition 4.3.** Let $\Lambda$ be finite dimensional and self-injective and assume that $\text{gl.dim} \Lambda_0 < \infty$. Consider a tilting object $T$ in $\text{gr} \Lambda$ and denote its endomorphism algebra by $B = \text{End}_{\text{gr} \Lambda}(T)$. Then the composition

$$F : \mathcal{D}^b(\text{mod} B) \xrightarrow{-\otimes^L_B T} \mathcal{D}^b(\text{gr} \Lambda) \xrightarrow{P} \mathcal{D}^b(\text{gr} \Lambda)/\mathcal{D}^\text{perf}(\text{gr} \Lambda) \xrightarrow{G} \text{gr} \Lambda$$

is an equivalence of triangulated categories.

**Proof.** Observe first that rigidity of $T$ yields

$$\text{Hom}_{\mathcal{D}^b(\text{mod} B)}(B, B[i]) \simeq \text{Hom}_{\text{gr} \Lambda}(T, \Omega^{-i}T)$$

for every $i \in \mathbb{Z}$. As $F(B)$ is isomorphic to $T$ in $\text{gr} \Lambda$, this means that the restriction of $F$ to the subcategory $\mathcal{X} = \{B[i] \mid i \in \mathbb{Z}\}$ is fully faithful. As $\Lambda_0$ has finite global dimension, so has $B$ by [13, Corollary 3.12]. Consequently, one obtains

$$\text{Thick}(B) = \text{Thick}(\mathcal{X}) = \mathcal{D}^b(\text{mod} B).$$

Using that $\mathcal{X}$ is closed under translation, this implies that $F$ is fully faithful. Since $\text{Thick}(T) = \text{gr} \Lambda$ and idempotents split in $\mathcal{D}^b(\text{mod} B)$, the functor $F$ is also essentially surjective, and hence an equivalence. \[\square\]

In the same way as $B$ is the preimage of $T$ under our equivalence above, we can also describe projective $B$-modules in terms of summands of $T$. Given a decomposition $T \simeq \bigoplus_{i=0}^r T^i$ of $T$, let $e_i : T \rightarrow T^i \hookrightarrow T$ denote the $i$-th projection followed by the $i$-th inclusion. This yields a decomposition $B \simeq \bigoplus_{i=0}^r P^i$ of $B$ into projectives $P^i = e_iB$. Notice that the projective $P^i$ is the preimage of the summand $T^i$ under the equivalence $F$, as $e_iB \otimes^L_B T \simeq e_iT = T^i$.

From Section 6 and on, the following notion will be crucial.

**Definition 4.4.** Let $\mathcal{T}$ be a $k$-linear Hom-finite triangulated category. An additive autoequivalence $\mathcal{S}$ on $\mathcal{T}$ is called a Serre functor provided there exists a bifunctorial isomorphism

$$\text{Hom}_{\mathcal{T}}(X, Y) \simeq D \text{Hom}_{\mathcal{T}}(Y, \mathcal{S}X)$$

for all objects $X$ and $Y$ in $\mathcal{T}$.

We want to compare the Serre functor on $\mathcal{D}^b(\text{mod} B)$ to that of $\text{gr} \Lambda$ when $\Lambda$ is a graded Frobenius algebra of highest degree $a$ with Nakayama automorphism $\mu$. In this case, it follows from Auslander–Reiten duality, see [2] and [40, Proposition I.2.3], combined with the characterization in Lemma 2.2 that $\Omega(-)\mu(-a)$ is a Serre functor on $\text{gr} \Lambda$. As $B$ is a finite dimensional algebra of finite global dimension, the derived Nakayama functor $\nu(-) = -\otimes_B^L DB$ is a Serre functor on $\mathcal{D}^b(\text{mod} B)$. 
By uniqueness of the Serre functor, the equivalence $F$ from Proposition 4.3 yields a commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}^b(\text{mod } B) & \xrightarrow{F} & \text{gr } \Lambda \\
\downarrow \nu & & \downarrow \Omega(-) \nu(-a) \\
\mathcal{D}^b(\text{mod } B) & \xrightarrow{F} & \text{gr } \Lambda.
\end{array}
$$

Note that throughout the rest of this paper, we often use the equivalence from Proposition 4.3 and the correspondence of the Serre functors described in the diagram above without making the reference explicitly.

5. On $n$-hereditary algebras

The class of $n$-hereditary algebras was introduced in [17] and consists of the disjoint union of $n$-representation finite and $n$-representation infinite algebras. In this section we recall some definitions and basic results from [17, 22, 23]. This forms a necessary background for exploring connections between the notion of $n$-T-Koszulity and higher hereditary algebras, which is the topic our next two sections. Note that Section 5 does not contain any new results.

Throughout this section, let $A$ be a finite dimensional algebra. Recall that if $A$ has finite global dimension, then the derived Nakayama functor $\nu(-) = - \otimes^L_A D A$ is a Serre functor on $\mathcal{D}^b(\text{mod } A)$. We use the notation $\nu_n = \nu(-)[{-n}]$. The algebra $A$ is called $n$-representation finite if $\text{gl.dim } A \leq n$ and $\text{mod } A$ contains an $n$-cluster tilting object. We have the following criterion for $n$-representation finiteness in terms of the subcategory

$$
\mathcal{U} = \text{add}\{ \nu^i_n A \mid i \in \mathbb{Z} \} \subseteq \mathcal{D}^b(\text{mod } A).
$$

**Theorem 5.1** (See [23, Theorem 3.1]). Assume $\text{gl.dim } A \leq n$. The following are equivalent:

1. $A$ is $n$-representation finite;
2. $DA \in \mathcal{U}$;
3. $\nu \mathcal{U} = \mathcal{U}$.

In particular, an algebra $A$ with $\text{gl.dim } A \leq n$ is $n$-representation finite if and only if for any indecomposable projective $A$-module $P_i$ is an integer $m_i \geq 0$ such that $\nu_n^{-m_i}(P_i)$ is indecomposable injective. We will need the following well-known property of $n$-representation finite algebras.

**Lemma 5.2** (See [17, Proposition 2.3]). Let $A$ be $n$-representation finite. For each indecomposable projective $A$-module $P_i$, we then have $\text{H}^l(\nu_n^{-m_i}(P_i)) = 0$ for $l \neq 0$ and $0 \leq m \leq m_i$, where $m_i$ is given as above.

Moving on to the second part of the $n$-hereditary dichotomy, recall that $A$ is called $n$-representation infinite if $\text{gl.dim } A \leq n$ and $\text{H}^i(\nu_n^{-j}(A)) = 0$ for $i \neq 0$ and $j \geq 0$. 
The following basic lemma will be needed in our next two sections. This fact should be well-known, but we include a proof as we lack an explicit reference. In the proof we abuse notation by letting $\nu$ denote both the derived Nakayama functor and the ordinary Nakayama functor, as context allows one to determine which one is intended.

**Lemma 5.3.** Let $\text{gl.dim} \ A < \infty$ and assume that for each indecomposable projective $A$-module $P$, we have $H^i(\nu_n^{-1}(P)) = 0$ for $i \notin \{0, -n\}$. Then $\text{gl.dim} \ A \leq n$. If there is at least one non-injective projective $A$-module, then $\text{gl.dim} \ A = n$.

**Proof.** To show $\text{gl.dim} \ A \leq n$, it is sufficient to check that $\text{inj.dim} \ A \leq n$, as $A$ has finite global dimension.

Let $P$ be an indecomposable projective $A$-module. Assume that in computing $\nu_n^{-1}(P)$ we use a minimal injective resolution $I^\bullet$ of $P$. As $\text{gl.dim} \ A < \infty$, this resolution is finite. If $\text{inj.dim} \ P = m \notin \{0, n\}$, our assumption yields $H^m(\nu^{-1}(P)) \simeq H^{m-n}(\nu_n^{-1}(P)) = 0$.

However, if there is no cohomology in degree $m$, this implies that the morphism $\nu^{-1}(I^{m-1} \to I^m)$ is an epimorphism. As $\nu^{-1}(I^m)$ is projective, this morphism must split. Since $\nu^{-1}$ is an equivalence when restricted to add $DA$, this contradicts the minimality of the resolution $I^\bullet$, and we can conclude that $\text{inj.dim} \ P = 0$ or $n$. In particular, one obtains $\text{inj.dim} \ A \leq n$. If there exists $P$ non-injective, we clearly get the second claim. \hfill $\square$

Like in the classical theory of hereditary algebras, the class of $n$-hereditary algebras also has an appropriate version of (higher) preprojective algebras which is nicely behaved. Given an $n$-hereditary algebra $A$, we denote the $(n+1)$-preprojective algebra of $A$ by $\Pi_{n+1}A$. Recall from [23, Lemma 2.13] that

$$\Pi_{n+1}A \simeq \bigoplus_{i \geq 0} \text{Hom}_{D^b(A)}(A, \nu_n^{-1}(A)).$$

If $A$ is $n$-representation finite, the associated $(n+1)$-preprojective is finite dimensional and self-injective, whereas in the $n$-representation infinite case, the $(n+1)$-preprojective is infinite dimensional graded bimodule $(n+1)$-Calabi–Yau of Gorenstein parameter $1$.

**Remark 5.4.** Note that terminology related to the classes of algebras discussed in this section varies in the literature. For instance, an $n$-representation finite algebra is called ‘$n$-representation-finite $n$-hereditary’ in [25]. This terminology is very reasonable, but as we need to mention $n$-representation finite algebras frequently, we stick to the notion from [22] for brevity.
6. Higher Koszul duality and \( n \)-representation infinite algebras

In this section we investigate connections between \( n \)-representation infinite algebras and the notion of higher Koszulity. Let us first present our standing assumptions.

Setup. Throughout the rest of this section, let \( \Lambda = \bigoplus_{i \geq 0} \Lambda_i \) be a finite dimensional graded Frobenius algebra of highest degree \( a \geq 1 \) with \( \text{gl.dim} \Lambda_0 < \infty \). Let \( T \) denote a basic graded \( \Lambda \)-module which is concentrated in degree 0 and a tilting module over \( \Lambda_0 \). Consider a decomposition \( T \cong \bigoplus_{i=0}^t T_i \) into indecomposable summands and assume that twisting by the Nakayama automorphism \( \mu \) of \( \Lambda \) only permutes these summands. This means that we have a permutation, for simplicity also denoted by \( \mu \), on the set \( \{1, \ldots, t\} \) such that \( T_i \mu \cong T_{\mu(i)} \). For our fixed positive integer \( n \), we consider the module \( \tilde{T} = \bigoplus_{i=0}^{a-1} \Omega^{-ni} T(i) \).

We denote the endomorphism algebra \( \text{End}_{\text{gr} \Lambda}(\tilde{T}) \) by \( B \).

One should note that in the classical case, the Nakayama automorphism induces a permutation of the simples, i.e. the module corresponding to our \( T \). This justifies the assumption that twisting by the Nakayama automorphism of \( \Lambda \) only permutes the indecomposable summands of \( T \). Note that using this, we immediately obtain \( T_i \mu \cong T_i \), and hence \( \Omega T_i(-a) \cong \Omega T(-a) \).

Our first aim in this section is to describe the endomorphism algebra \( B \) as an upper triangular matrix algebra of finite global dimension. We start by recalling the following lemma.

Lemma 6.1 (See [10, Corollary 4.21 (4)]). Let \( A \) and \( A' \) be finite dimensional algebras and \( M \) an \( A' \otimes_k A \)-module. Then the algebra

\[
\begin{bmatrix}
A & M \\
0 & A'
\end{bmatrix}
\]

has finite global dimension if and only if both \( A \) and \( A' \) have finite global dimension.

In Lemma 6.2 we describe \( B \) as an upper triangular matrix algebra associated to the graded algebra \( \Gamma = \bigoplus_{i \geq 0} \text{Ext}^n_{\text{gr} \Lambda}(T, T(i)) \). Notice that in the case where \( \Lambda \) is \( n \)-\( T \)-Koszul, the algebra \( \Gamma \) coincides with the \( n \)-\( T \)-Koszul dual \( \Lambda' \).

Lemma 6.2. The algebra \( B = \text{End}_{\text{gr} \Lambda}(\tilde{T}) \) is isomorphic to the upper triangular matrix algebra

\[
B \cong \begin{pmatrix}
\Gamma_0 & \Gamma_1 & \cdots & \Gamma_{a-1} \\
0 & \Gamma_0 & \cdots & \Gamma_{a-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Gamma_0
\end{pmatrix},
\]
where $\Gamma = \bigoplus_{i \geq 0} \text{Ext}^n_{\mathfrak{gr} \Lambda}(T, T(i))$. In particular, the global dimension of $B$ is finite.

**Proof.** For $0 \leq i, j \leq a - 1$, we consider

$$\text{Hom}_{\mathfrak{gr} \Lambda}(\Omega^{-n}T(j), \Omega^{-n}T(i)) \simeq \text{Hom}_{\mathfrak{gr} \Lambda}(T, \Omega^{-n(i-j)}T(i-j)).$$

In the case $i < j$, we note that $|i - j| \leq a - 1$ and so Lemma 2.3 (7) applies. Consequently,

$$\text{Hom}_{\mathfrak{gr} \Lambda}(T, \Omega^{-n(i-j)}T(i-j)) \simeq \text{Hom}_{\mathfrak{gr} \Lambda}(T, \Omega^{-n(i-j)}T(i-j)) = 0.$$

If $i = j$, one obtains $\text{End}_{\mathfrak{gr} \Lambda}(T)$, which is isomorphic to $\text{End}_{\mathfrak{gr} \Lambda}(T) = \Gamma_0$ by Lemma 2.3 (5). For $i > j$, we get

$$\text{Hom}_{\mathfrak{gr} \Lambda}(T, \Omega^{-n(i-j)}T(i-j)) \simeq \text{Ext}^n_{\mathfrak{gr} \Lambda}(T, T(i-j)) = \Gamma_{i-j}.$$

Computing our matrix with respect to the decomposition

$$\tilde{T} = \Omega^{-a}T(a-1) \oplus \cdots \oplus \Omega^{-n}T(1) \oplus T,$$

this yields our desired description.

To see that $B$ is of finite global dimension, notice that $\Gamma_0 \simeq \text{End}_{\Lambda_0}(T)$. As $\text{End}_{\Lambda_0}(T)$ is derived equivalent to $\Lambda_0$, which is of finite global dimension, Lemma 6.1 applies and the claim follows. $\square$

Note that we could also have deduced that $B$ is of finite global dimension from [43, Corollary 3.12]. In the main result of this section, Theorem 6.4, we characterize when our algebra $\Lambda$ is $n$-$T$-Koszul in terms of $B$ being $(na-1)$-representation infinite. Our next lemma provides an important step in the proof of this result.

Recall that given a graded $\Lambda$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$, each graded part $M_i$ is also a module over $\Lambda_0$. On the other hand, every $\Lambda_0$-module is trivially a graded $\Lambda$-module concentrated in degree 0. In the proof of Lemma 6.3, we repeatedly vary between thinking of graded $\Lambda$-modules concentrated in one degree and modules over the degree 0 part.

We use the notation $M_{\geq i}$ for the submodule of $M$ with

$$(M_{\geq i})_j = \begin{cases} M_j & j \geq i \\ 0 & j < i, \end{cases}$$

while the quotient module $M/M_{\geq i+1}$ is denoted by $M_{\leq i}$. Note that $M_i$ is isomorphic to $M_{\geq i}/M_{\geq i+1}$.

**Lemma 6.3.** The module $\tilde{T}$ generates $\mathfrak{gr} \Lambda$ as a thick subcategory, i.e. we have $\text{Thick}_{\mathfrak{gr} \Lambda}(\tilde{T}) = \mathfrak{gr} \Lambda$.

**Proof.** We divide the proof into two steps. In the first part, we show that the set of objects $\{A_0(i)\}_{i \in \mathbb{Z}}$ generates $\mathfrak{gr} \Lambda$ as a thick subcategory. In the second part, we prove that this set is contained in $\text{Thick}_{\mathfrak{gr} \Lambda}(\tilde{T})$, which yields our desired conclusion.
Part 1:

Notice first that every graded $\Lambda$-module which is concentrated in degree $i$ is necessarily contained in the thick subcategory generated by $\Lambda_0(i)$. To see this, apply $(i)$ to a finite $\Lambda_0$-projective resolution of the module, split up into short exact sequences and use that thick subcategories have the 2/3-property on distinguished triangles.

Let $M$ be an object in $\text{gr} \Lambda$. Denote the highest and lowest degree of $M$ by $h$ and $l$, respectively. Observe that $M_{\geq h} = M_h$. By the argument above, we know that $M_j$ is in $\text{Thick}_{\text{gr} \Lambda} (\{ \Lambda_0(i) \}_{i \in \mathbb{Z}})$ for every $j$. Considering the short exact sequences (6.1)

$$0 \longrightarrow M_{j+1} \longrightarrow M_j \longrightarrow M_j \longrightarrow 0$$

for $j = l, \ldots , h-1$, we can hence conclude that also $M_{\geq l} = M$ is in our subcategory. This proves that $\text{Thick}_{\text{gr} \Lambda} (\{ \Lambda_0(i) \}_{i \in \mathbb{Z}}) = \text{gr} \Lambda$.

Part 2:

As thick subcategories are closed under direct summands and translation, we immediately observe that $T(i)$ is in $\text{Thick}_{\text{gr} \Lambda} (\tilde{T})$ for $i = 0, \ldots , a-1$. Since $T$ is a tilting module over $\Lambda_0$, and $\Lambda_0(i)$ thus has a finite coresolution in $\text{add} \ T(i)$, this implies that $\Lambda_0(i)$ is in $\text{Thick}_{\text{gr} \Lambda} (\tilde{T})$ for $i = 0, \ldots , a-1$. Note that by our argument in Part 1, we hence know that every module which is concentrated in degree $i$ for some $i = 0, \ldots , a-1$, is contained in our subcategory.

Consider the short exact sequences (6.1) for $M = \Lambda$, and recall that the module $\Lambda_{\geq 0} = \Lambda$ is projective and hence zero in $\text{gr} \Lambda$. By a similar argument as before, this yields that $\Lambda_a$ is contained in $\text{Thick}_{\text{gr} \Lambda} (\tilde{T})$. We next explain why this entails that also $\Lambda_0(a)$ is in our subcategory.

Since $\Lambda$ is graded Frobenius, we have $\Lambda(-a) \simeq D \Lambda$ as graded right $\Lambda$-modules, and thus $D \Lambda_0 \simeq \Lambda_a$ as $\Lambda_0$-modules. As $\Lambda_0$ has finite global dimension, this implies that $\Lambda_0$ is contained in $\text{Thick}_{D^b(\text{gr} \Lambda)} (\Lambda_a(-a))$. Composing the equivalence from Theorem 4.2 with the associated quotient functor, one obtains a triangulated functor $Q : D^b(\text{gr} \Lambda) \to \text{gr} \Lambda$. From the chain of subcategories

$$\text{Thick}_{D^b(\Lambda_0)} \Lambda_a(-a) \subseteq \text{Thick}_{D^b(\text{gr} \Lambda)} \Lambda_a(-a) \subseteq Q^{-1}(\text{Thick}_{\text{gr} \Lambda} \Lambda_a(-a)),$$

we see that $\Lambda_0(a)$ is in $\text{Thick}_{\text{gr} \Lambda} (\Lambda_a)$, which again is contained in $\text{Thick}_{\text{gr} \Lambda} (\tilde{T})$.

Shifting the short exact sequences involved by positive integers and using the same argument as above, one obtains that $\Lambda_0(i)$ is in $\text{Thick}_{\text{gr} \Lambda} (\tilde{T})$ for all $i \geq 0$. That $\Lambda_0(i)$ is in $\text{Thick}_{\text{gr} \Lambda} (\tilde{T})$ for all $i < 0$ is shown similarly using the short exact sequences

$$0 \longrightarrow \Lambda_j \longrightarrow \Lambda_{\leq j} \longrightarrow \Lambda_{\leq j-1} \longrightarrow 0$$

for $j = 1, \ldots , a$. We can hence conclude that $\Lambda_0(i)$ is in $\text{Thick}_{\text{gr} \Lambda} (\tilde{T})$ for every integer $i$, which finishes our proof. \qed
We are now ready to state and prove the main result of this section.

**Theorem 6.4.** The following statements are equivalent:

1. $\Lambda$ is $n$-$T$-Koszul.
2. $\tilde{T}$ is a tilting object in $\text{gr} \Lambda$ and $B = \text{End}_{\text{gr} \Lambda}(\tilde{T})$ is $(na-1)$-representation infinite.

**Proof.** We begin by proving (1) implies (2). To see that $\tilde{T}$ is a tilting object, notice first that it generates $\text{gr} \Lambda$ by Lemma 6.3. Thus, we need only check rigidity, i.e. that $\text{Hom}_{\text{gr} \Lambda}(\tilde{T}, \Omega^{-l}T) = 0$ whenever $l \neq 0$. Splitting up on summands of $\tilde{T} = \oplus_{i=0}^{a-1} \Omega^{-ni}T(i)$ and reindexing appropriately, we see that it is enough to show

$$(6.2) \quad \text{Hom}_{\text{gr} \Lambda}(T, \Omega^{-(nk+l)}T(k)) = 0 \text{ for } l \neq 0$$

for any integer $k$ with $|k| \leq a - 1$.

Assume $nk + l = 0$. Now $l \neq 0$ implies $k \neq 0$, so the condition above is satisfied as our morphisms are homogeneous of degree 0.

Let $nk + l > 0$. Now,

$$\text{Hom}_{\text{gr} \Lambda}(T, \Omega^{-(nk+l)}T(k)) \simeq \text{Ext}^{nk+l}_{\text{gr} \Lambda}(T, T(k)),$$

which is zero for $l \neq 0$ as $\Lambda$ is $n$-$T$-Koszul.

It remains to verify (6.2) in the case where $nk + l < 0$. As $|k| \leq a - 1$, part (7) of Lemma 6.2 applies. We hence see that (6.2) is satisfied also in this case, which means that $T$ is a tilting object in $\text{gr} \Lambda$.

Recall from Lemma 6.2 that $B$ has finite global dimension. To see that $B$ is $(na-1)$-representation infinite, we use that $T$ is a tilting object in $\text{gr} \Lambda$. Hence, the equivalence and correspondence of Serre functors described in Section 4 yields

$$(6.3) \quad \text{Hom}_{\text{gr} \Lambda}(\tilde{T}, \Omega^{-(na+i)}\tilde{T}(ai)) \simeq \text{Hom}_{\text{D}^b(B)}(B, \nu^{-i}(B)[nai - i + l])$$

$$\simeq \text{Hom}_{\text{D}^b(B)}(B, \nu^{-i}_{na-1}(B)[l])$$

$$\simeq H^i(\nu^{-i}_{na-1}(B)),$$

where we have implicitly used that $T_\mu \simeq T$ and that the functors $\Omega^{\pm l}(-)$, $\langle \pm 1 \rangle$ and $(-)_\mu$ commute.

Splitting up on summands of $\tilde{T}$ and reindexing appropriately, we notice that $\text{Hom}_{\text{gr} \Lambda}(\tilde{T}, \Omega^{-(na+i)}\tilde{T}(ai)) = 0$ for $l \neq 0$ and $i > 0$ if and only if (6.2) is satisfied for $k > 0$. The latter follows by the same argument as in our proof of rigidity above, so we can conclude that $H^i(\nu^{-i}_{na-1}(B)) = 0$ for $i > 0$ and $l \neq 0$. Note that when $i = 0$ and $l \neq 0$, we have $H^i(\nu^{-i}_{na-1}(B)) = H^i(B) = 0$. Consequently, our algebra $B$ is $(na-1)$-representation infinite by Lemma 5.3.

To show that (2) implies (1), we verify that given any integer $k$, one obtains $\text{Ext}^{nk+l}_{\text{gr} \Lambda}(T, T(k)) = 0$ for $l \neq 0$. If $nk + l \leq 0$, this is immediately satisfied, so
assume \( nk + l > 0 \). As before, we now have

\[
\text{Ext}_{\text{gr}\Lambda}^{nk+l}(T, T(k)) \simeq \text{Hom}_{\text{gr}\Lambda}(T, \Omega^{-(nk+l)}T(k)).
\]

If \( k < 0 \), this is zero by Lemma 2.3 (6), so it remains to check the case where \( k \) is non-negative.

Observe that the isomorphism

\[
\text{Hom}_{\text{gr}\Lambda}(\tilde{T}, \Omega^{-(nai+l)}\tilde{T}(ai)) \simeq H^l(\nu_{na-1}^{-i}(B))
\]

from (6.3) still holds, as \( \tilde{T} \) is assumed to be a tilting object in \( \text{gr}\Lambda \). As \( B \) is \((na - 1)\)-representation infinite, we know that \( H^l(\nu_{na-1}^{-i}(B)) = 0 \) for \( i \geq 0 \) and \( l \neq 0 \). The isomorphism above hence yields that (6.2) is satisfied for \( k \geq 0 \).

This allows us to conclude that \( T \) is graded \( n \)-self-orthogonal. As \( T \) is a tilting module over \( \Lambda_0 \) by our standing assumptions, we have hence shown that \( \Lambda \) is \( n\)-\( T \)-Koszul. \( \square \)

To illustrate our characterization result, we consider an example. As can be seen below, we use diagrams to represent indecomposable modules. The reader should note that in general one cannot expect modules to be represented uniquely by such diagrams, but in the cases we look at, they determine indecomposable modules up to isomorphism.

**Example 6.5.** Let \( A \) denote the path algebra of the quiver

![Quiver Diagram](image)

modulo the ideal generated by paths of length two. The trivial extension \( \Delta A \) is given by the quiver

![Trivial Extension Quiver](image)

with the trivial extension relations, i.e. all length two zero relations with the exception of \( \alpha_i\alpha'_i \) and \( \alpha'_i\alpha_i \). Instead, these latter paths satisfy all length two commutativity relations, i.e. \( \alpha_1\alpha'_1 - \alpha_2\alpha'_2, \alpha_3\alpha'_3 - \alpha'_1\alpha_1, \alpha'_1\alpha_4 - \alpha'_3\alpha_3, \alpha'_2\alpha_2 - \alpha_4\alpha'_4 \). Moreover, we let \( \Delta A \) be graded with the trivial extension grading.

The indecomposable projective injectives for \( \Delta A \) can be given as the diagrams
where the (non-subscript) numbers represent elements of a basis for the module, each of which is annihilated by all the idempotents except for $e_i$ with $i$ equal to the number. The subscript numbers represent the degree of the basis element.

Let $T$ be the tilting $A$-module given by the direct sum of the following modules

\[
\begin{array}{cccc}
3_0 & 1_0 & 2_0 \\
1_1 & 2_1 & 3_0 & 2_0 & 3_0.
\end{array}
\]

The initial two terms of the minimal injective $\Delta A$-resolution of the first summand of $T$ as well as the first two cosyzygies can be given as

\[
\begin{array}{cccc}
3_{-1} & 4_{-1} & 3_0 & 1_0 \\
1_0 & 1_0 & 0 & 2_0 & 4_{-1} & 3_{-1} & 2_{-1} & 4_{-1}.
\end{array}
\]

Looking at this part of the resolution, it is not so obvious that $T$ is graded 2-self-orthogonal as a $\Delta A$ module, whereas by using the equivalence $D^b(\text{mod } A) \simeq \text{gr } \Delta A$ or by degree arguments as we have done before, it is immediate that $\tilde{T} \simeq T$ is a tilting object in $\text{gr } \Delta A$. It is also easy to check that $\text{End}_{\text{gr } \Delta A}(T)$ is isomorphic to the hereditary algebra given by the path algebra of the quiver of $A$, which is representation infinite. Using Theorem 6.4, we can hence conclude that the algebra $\Delta A$ is 2-$T$-Koszul.

Note that this example also illustrates that, as has been remarked on in the literature before, one cannot always expect nice minimal resolutions of $T$ for (generalized) $T$-Koszul algebras.

As a consequence of Theorem 6.4, our next corollary shows that an algebra is $n$-representation infinite if and only if its trivial extension is $(n + 1)$-Koszul with respect to its degree 0 part. This result is inspired by connections between $n$-representation infinite algebras and graded bimodule $(n + 1)$-Calabi–Yau algebras of Gorenstein parameter 1, as studied in [1, 17, 28, 35]. In some sense, the corollary below is a $T$-Koszul dual version of [17, Theorem 4.36].

Note that in the first part of Corollary 6.6, we set $T = \Lambda_0$ and hence assume that the Nakayama automorphism of $\Lambda$ only permutes the summands of $\Lambda_0$. This is trivially satisfied whenever our algebra is graded symmetric.

**Corollary 6.6.** If $a = 1$, our algebra $\Lambda$ is $(n + 1)$-Koszul with respect to $T = \Lambda_0$ if and only if $\Lambda_0$ is $n$-representation infinite. In particular, we obtain a bijective
correspondence
\[
\begin{align*}
\{ \text{isomorphism classes} \} & \iff \{ \text{isomorphism classes of graded symmetric finite} \\
& \quad \{ \text{of basic } n\text{-representation} \}
\quad \{ \text{dimensional algebras of highest degree } 1 \text{ which are} \\
& \quad \{ \text{infinite algebras} \} \} \\
& \quad \{ (n+1)\text{-Koszul with respect to their degree } 0 \text{ part} \}
\end{align*}
\]
where the maps are given by \( A \mapsto \Delta A \) and \( \Lambda_0 \mapsto \Lambda \).

**Proof.** Notice that \( \text{End}_{\text{gr}A}(\Lambda_0) \simeq \text{End}_{\text{gr}A}(\Omega^nA_0) \simeq \Lambda_0 \) by Lemma 2.3 (5). Observe that \( \text{Hom}_{\text{gr}A}(\Lambda_0, \Omega^{-i}A_0) \simeq \text{Hom}_{\text{gr}A}(\Omega^nA_0, \Lambda_0) = 0 \) for all \( i \neq 0 \). This follows by degree considerations similar to those used in the proof of Lemma 2.3 and using the fact that syzygies of \( \Lambda_0 \) are generated in degrees greater or equal to 1. Combining this with Lemma 6.3 one obtains that \( \Lambda_0 \) is a tilting object in \( \text{gr}A \), and consequently our first statement follows from Theorem 6.4.

We get the bijection as a special case of this, as \( \Delta A \) is a graded symmetric finite dimensional algebra of highest degree 1 and \( \Lambda \simeq \Delta \Lambda_0 \) as graded algebras in the case where \( \Lambda \) is symmetric. \( \square \)

Our aim for the rest of this section is to use the theory we have developed to provide an affirmative answer to our motivating question from the introduction. As in the case of the generalized AS-regular algebras studied by Minamoto and Mori in [35], the notion of quasi-Veronese algebras is relevant.

**Definition 6.7.** Let \( \Gamma = \bigoplus_{i \in \mathbb{Z}} \Gamma_i \) be a \( \mathbb{Z} \)-graded algebra and \( r \) a positive integer. The \( r \)-th quasi-Veronese algebra of \( \Gamma \) is a \( \mathbb{Z} \)-graded algebra defined by
\[
\Gamma^{[r]} = \bigoplus_{i \in \mathbb{Z}} \left( \begin{array}{cccc}
\Gamma_{ri} & \Gamma_{ri+1} & \cdots & \Gamma_{ri+r-1} \\
\Gamma_{ri-1} & \Gamma_{ri} & \cdots & \Gamma_{ri+r-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma_{ri-r+1} & \Gamma_{ri-r+2} & \cdots & \Gamma_{ri} 
\end{array} \right).
\]

In Proposition 6.8 we show that if \( \Lambda \) is \( n \)-T-Koszul, then the \( na \)-th preprojective algebra of \( B = \text{End}_{\text{gr}A}(\Gamma) \) is isomorphic to a twist of the \( a \)-th quasi-Veronese of \( \Lambda' \). In order to make this precise, notice first that a graded algebra automorphism \( \phi \) of a graded algebra \( \Gamma \) induces a graded algebra automorphism \( \phi^{[r]} \) of \( \Gamma^{[r]} \) by letting \( \phi^{[r]}(\gamma_{j,k}) = (\phi(\gamma_{j,k})) \). Here we use the notation \( (\gamma_{j,k}) \) for the matrix with \( \gamma_{j,k} \) in position \( (j,k) \). Recall also that we can define a possibly different graded algebra \( (\phi) \Gamma \) with the same underlying vector space structure as \( \Gamma \), but with multiplication \( \gamma \cdot \gamma' = \phi(\gamma)\gamma' \) for \( \gamma' \) in \( \Gamma_i \).

Recall that \( \mu \) is the Nakayama automorphism of \( \Lambda \), and denote our chosen isomorphism \( T_\mu \simeq T \) from before by \( \tau \). Note that twisting by \( \mu \) might non-trivially permute the summands of \( T \). In the case where \( \Lambda \) is \( n \)-T-Koszul, let \( \overline{\Gamma} \) be the graded algebra automorphism of \( \Lambda' \) defined on the \( i \)-th component
\[
\Lambda'^i = \text{Ext}_{\text{gr}A}^{ni}(T, T^{[i]}) \simeq \text{Hom}_{\text{gr}A}(T, \Omega^{-ni}T^{[i]})
\]
by the composition
\[ \text{Hom}_{\text{gr} \Lambda}(T, \Omega^{-ni}T\langle i \rangle) \xrightarrow{(-)_{\mu}} \text{Hom}_{\text{gr} \Lambda}(T_{\mu}, \Omega^{-ni}T_{\mu}\langle i \rangle) \xrightarrow{(-)^{\tau}} \text{Hom}_{\text{gr} \Lambda}(T, \Omega^{-ni}T\langle i \rangle), \]
where
\[ (\gamma)^{\tau} = \Omega^{-ni} (\tau) \circ \gamma \circ \tau^{-1} \]
for \( \gamma \) in \( \text{Hom}_{\text{gr} \Lambda}(T_{\mu}, \Omega^{-ni}T_{\mu}\langle i \rangle) \).

Before showing Proposition 6.8, recall that a decomposition of \( T \) yields a decomposition of \( B = \text{End}_{\text{gr} \Lambda}(T) \). In the proof below, we denote the summands of \( T \) by \( X^i = \Omega^{-ni}T\langle i \rangle \), while \( P^i \) is the projective \( B \)-module which is the preimage of \( X^i \) under the equivalence \( \mathcal{D}^b(\text{mod } B) \overset{\sim}{\to} \text{gr} \Lambda \) from Proposition 4.3.

**Proposition 6.8.** Let \( \Lambda \) be \( n-T \)-Koszul. Then \( \Pi_{na}B \simeq \langle (\Omega^{-1})^a \rangle (\Lambda^i)^{[a]} \) as graded algebras. In particular, we have \( \Pi_{na}B \simeq (\Lambda^i)^{[a]} \) in the case where \( \Lambda \) is graded symmetric.

**Proof.** As \( \Lambda \) is \( n-T \)-Koszul, we know from Theorem 6.4 that \( T \) is a tilting object in \( \text{gr} \Lambda \) and that \( B \) is \((na - 1)\)-representation infinite. The \( i \)-th component of the \( na \)-th preprojective algebra of \( B \) is given by \( (\Pi_{na}B)_i = \text{Hom}_{\mathcal{D}^b(B)}(B, \nu_{na,i}^{-1}B) \). For \( 0 \leq j, k \leq a - 1 \), we hence consider
\[
\text{Hom}_{\mathcal{D}^b(B)}(P^k, \nu_{na,i}^{-1}P^j) \simeq \text{Hom}_{\text{gr} \Lambda}(X^k, \Omega^{-(na-1)i-j}X^{j+1}_{\mu^{-1}}, \langle ai \rangle)
\simeq \text{Hom}_{\text{gr} \Lambda}(T, \Omega^{-n(ai+j-k)}T_{\mu^{-1}}, \langle ai + j - k \rangle)
\overset{(*)}{\simeq} \text{Ext}^{n(ai+j-k)}_{\text{gr} \Lambda}(T, T_{\mu^{-1}}, \langle ai + j - k \rangle) \simeq \Lambda^{i}_{ai+j-k}.
\]
Notice that the first isomorphism is a consequence of the equivalence and correspondence of Serre functors described in Section 4 while \((*)\) is obtained by applying Lemma 2.3 (5) and (7). The last isomorphism follows from the assumption \( T_{\mu} \simeq T \).

Computing our matrix with respect to the decomposition
\[ B \simeq P^{n-1} \oplus \cdots \oplus P^1 \oplus P^0, \]
this yields
\[
(\Pi_{na}B)_i \simeq \begin{pmatrix}
\Lambda^i_{ai} & \Lambda^1_{ai+1} & \cdots & \Lambda^1_{ai+a-1} \\
\Lambda^1_{ai-1} & \Lambda^1_{ai} & \cdots & \Lambda^1_{ai+a-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda^1_{ai-a+1} & \Lambda^1_{ai-a+2} & \cdots & \Lambda^1_{ai}
\end{pmatrix},
\]
which shows that our two algebras are isomorphic as graded vector spaces.
In order to see that the multiplications agree, consider the diagram

\[
\begin{array}{ccc}
(P^j, \nu^{-i'}_{na-1}P^j) \otimes (P^k, \nu^{-i}_{na-1}P^j) & \rightarrow & (P^k, \nu^{-i+i'}_{na-1}P^j) \\
\downarrow & & \downarrow \\
(\nu^{-i}_{na-1}P^j, \nu^{-i+i'}_{na-1}P^j) \otimes (P^k, \nu^{-i}_{na-1}P^j) & \rightarrow & (P^k, \nu^{-i+i'}_{na-1}P^j) \\
\downarrow & & \downarrow \\
(X^j_{\mu^{-1}}(ai), X^{i'}_{\mu^{-1}}(a(i+i'))) \otimes (X^k, X^{j'}_{\mu^{-1}}(ai)) & \rightarrow & (X^k, X^{j'}_{\mu^{-1}}(a(i+i'))) \\
\downarrow & & \downarrow \\
\Lambda^1_{a(i+1)} \otimes \Lambda^1_{a(i)+j-k} & \rightarrow & \Lambda^1_{a(i)+j-k}.
\end{array}
\]

For simplicity, we have here suppressed the Hom-notation and denoted \(\Omega^{-m_i}(-i)\) by \((-i)\). The horizontal maps are given by multiplication or composition, and the vertical maps give our isomorphism of graded algebras. In particular, the middle two horizontal maps are merely composition, whereas the top and bottom horizontal maps are the multiplication of \(\Pi_{na}B\) and \((\Omega^{-1}[a])(\Lambda^{1}[a])\), respectively. Moreover, the bottom vertical maps are given by

\[
f \otimes g \mapsto \prod_{i=0}^{i'-1} \tau^{-1}_{\mu^{-1}}(a'i+j'-j) \circ f_{\mu^{-1}}(-ai-j) \otimes \prod_{i=0}^{i+1} \tau^{-1}_{\mu^{-1}}(ai+j-k) \circ g(-k)
\]

and

\[
f \circ g \mapsto \prod_{i=0}^{i'+1} \tau^{-1}_{\mu^{-1}}(ai+i') + j' - k) \circ (f \circ g)(-k).
\]

As the diagram commutes, we can conclude that \(\Pi_{na}B \simeq (\Omega^{-1}[a])(\Lambda^{1}[a])\) as graded algebras. If \(\Lambda\) is assumed to be graded symmetric, the Nakayama automorphism \(\mu\) can be chosen to be trivial, so one obtains \(\Pi_{na}B \simeq (\Lambda^{1}[a])\). \(\square\)

In the corollary below, we show that the \((n+1)\)-th preprojective of an \(n\)-representation infinite algebra is isomorphic to the \(n-T\)-Koszul dual of its trivial extension. This is a \(T\)-Koszul dual version of [35 Proposition 4.20].

**Corollary 6.9.** If \(A\) is basic \(n\)-representation infinite, then \(\Pi_{n+1}A \simeq (\Delta A)^1\) as graded algebras.

**Proof.** Let \(A\) be a basic \(n\)-representation infinite algebra. It then follows from Corollary 6.6 that \(\Delta A\) is \((n+1)\)-Koszul with respect to \(A\). By Lemma 2.3 part (5), one obtains \(\text{End}_{gr}\Delta A(A) \simeq \text{End}_{gr}\Delta A(A) \simeq A\). Recall that \(\Delta A\) is graded symmetric of highest degree 1. Applying Proposition 6.8 to \(\Delta A\) hence yields our desired conclusion. \(\square\)
We are now ready to give an answer to our motivating question from the introduction, namely to see that we obtain an equivalence
\[ \text{gr}(\Delta A) \simeq D^b(\text{qgr} \Pi_{n+1} A) \]
which descends from higher Koszul duality in the case where \( A \) is \( n \)-representation infinite and \( \Pi_{n+1} A \) is graded right coherent.

Recall that an \( n \)-representation infinite algebra \( A \) is called \( n \)-representation tame if the associated \((n+1)\)-preprojective \( \Pi_{n+1} A \) is a noetherian algebra over its center [17, Definition 6.10]. Notice that a noetherian algebra is graded right coherent, so our result holds in this case.

**Corollary 6.10.** Let \( A \) be a basic \( n \)-representation infinite algebra with \( \Pi_{n+1} A \) graded right coherent. There is then an equivalence \( D^b(\text{gr} \Delta A) \simeq D^b(\text{gr} \Pi_{n+1} A) \) of triangulated categories which descends to an equivalence \( \text{gr}(\Delta A) \simeq D^b(\text{qgr} \Pi_{n+1} A) \).

In particular, this holds if \( A \) is \( n \)-representation tame.

**Proof.** We get the equivalence \( D^b(\text{gr} \Delta A) \simeq D^b(\text{gr} \Pi_{n+1} A) \) by Theorem 3.9 combined with Corollary 6.6 and Corollary 6.9. By Proposition 3.11, this equivalence descends to yield \( \text{gr}(\Delta A) \simeq D^b(\text{qgr} \Pi_{n+1} A) \). \( \square \)

7. Higher almost Koszulity and \( n \)-representation finite algebras

In our previous section, we gave connections between higher Koszul duality and \( n \)-representation infinite algebras. Having developed our theory for one part of the higher hereditary dichotomy, it is natural to ask whether something similar holds in the \( n \)-representation finite case. To provide an answer to this question, we introduce the notion of higher almost Koszulity. As before, this should be formulated relative to a tilting module over the degree 0 part of the algebra, which is itself assumed to be finite dimensional and of finite global dimension. Notice that after having presented the definitions and basic examples, we prove our results given the same standing assumptions as in Section 6.

Our definition of what it means for an algebra to be almost \( n \)-T-Koszul is inspired by and generalizes the notion of almost Koszulity, as introduced in [6]. Let us hence first recall the definition of an almost Koszul algebra.

**Definition 7.1 (See [6, Definition 3.1]).** Assume that \( \Lambda_0 \) is semisimple. We say that \( \Lambda \) is (right) almost Koszul if there exist integers \( p, q \geq 1 \) such that

1. \( \Lambda_i = 0 \) for all \( i > p \);
2. There is a graded complex
   \[ 0 \to P^{-q} \to \cdots \to P^{-1} \to P^0 \to 0 \]
   of projective right \( \Lambda \)-modules such that each \( P^{-i} \) is generated by its component \( P^{-i}_i \) and the only non-zero cohomology is \( \Lambda_0 \) in internal degree 0 and \( P^{-q}_i \otimes_{\Lambda_0} \Lambda_p \) in internal degree \( p + q \).

If \( \Lambda \) is almost Koszul for integers \( p \) and \( q \), one also says that \( \Lambda \) is \((p, q)\)-Koszul.
Roughly speaking, by iteratively taking tensor products over the degree 0 part, we see that if $\Lambda$ is almost Koszul, then $\Lambda_0$ has a somewhat periodic projective resolution which is properly piecewise linear for $p > 1$. This may remind one of the behaviour of the inverse Serre functor of an $n$-representation finite algebra on indecomposable projectives. However, note that for the latter the periods may be different for different indecomposable projectives. This highlights one additional area in which we must generalize the notion of almost Koszulity, namely that the length of the period of graded $n$-self-orthogonality can vary for different summands of our tilting module.

Motivated by our observations above, let us now define what it means for a module to be almost graded $n$-self-orthogonal. Recall that we consider a fixed decomposition $T \cong \bigoplus_{i=1}^{t} T^i$ into indecomposable summands.

**Definition 7.2.** Let $T \cong \bigoplus_{i=1}^{t} T^i$ be a finitely generated basic graded $\Lambda$-module concentrated in degree 0. We say that $T$ is almost graded $n$-self-orthogonal if for each $i \in \{1, \ldots, t\}$, there exists an object $T' \in \text{add } T$ and positive integers $l_i$ and $g_i$ such that the following conditions hold:

1. $\Omega^{-l_i} T^i \cong T'(-g_i)$;
2. $\text{Ext}^j_{\text{gr } \Lambda}(T, T^i(k)) = 0$ for $j \neq nk$ and $j < l_i$.

This leads to our definition of what it means for an algebra to be almost $n$-T-Koszul.

**Definition 7.3.** Assume $\text{gl.dim } \Lambda_0 < \infty$ and let $T$ be a graded $\Lambda$-module concentrated in degree 0. We say that $\Lambda$ is almost $n$-T-Koszul or almost $n$-Koszul with respect to $T$ if the following conditions hold:

1. $T$ is a tilting $\Lambda_0$-module.
2. $T$ is almost graded $n$-self-orthogonal as a $\Lambda$-module.

Whenever we work with an almost $n$-T-Koszul algebra, we use the notation $l_i$ and $g_i$ for integers given as in Definition 7.2.

As a first class of examples, we verify that Definition 7.3 is indeed a generalization of Definition 7.2.

**Example 7.4.** Let $\Lambda$ be a $(p,q)$-Koszul algebra. We show that $\Lambda$ is almost 1-Koszul with respect to $\Lambda_0$. It is immediate that $\text{gl.dim } \Lambda_0 < \infty$ and that $\Lambda_0$ is a tilting module over itself. To see that $\Lambda$ is almost 1-Koszul with respect to $\Lambda_0$, we must hence check that $\Lambda_0$ is almost graded 1-self-orthogonal as a $\Lambda$-module. Note that by letting $l_i = q + 1$ and $g_i = p + q$ for every $i \in \{1, \ldots, t\}$, we get that condition (2) of Definition 7.2 implies conditions (1) and (2) of Definition 7.2. To see this, we use the fact that an algebra is left $(p,q)$-Koszul if and only if it is right $(p,q)$-Koszul [3, Proposition 3.4]. Hence, we get a left projective resolution of $\Lambda_0$, which can be dualized to yield a right injective resolution of $\Lambda_0$.

Trivial extensions of $n$-representation finite algebras provide another important class of examples of algebras satisfying Definition 7.3 as can be seen through the
theory we develop in the rest of this section. Our main result is Theorem 7.17 which is an almost \( n \)-\( T \)-Koszul analogue of the characterization result in Section 6, i.e. Theorem 6.4. We divide the proof of Theorem 7.17 into a series of smaller steps. In order to state our precise result, we need information about the relation between the integers \( l_i \) and \( g_i \) of an almost \( n \)-\( T \)-Koszul algebra. As will become clear from the proof of our characterization result, the notion given in the definition below is sufficient. Recall that we consider a fixed decomposition \( T \cong \bigoplus_{i=1}^{t} T^i \) into indecomposable summands.

**Definition 7.5.** An almost \( n \)-\( T \)-Koszul algebra \( \Lambda \) of highest degree \( a \) is called \((n, m, \sigma)\)-\( T \)-Koszul or \((n, m, \sigma)\)-Koszul with respect to \( T \) if for each \( i \in \{1, \ldots, t\} \), there exists non-negative integers \( m_i \) and \( \sigma_i \) with \( \sigma_i \leq a - 1 \) such that

1. \( l_i = n a m_i - n \sigma_i + 1 \);
2. \( g_i = a(n m_i + 1) - \sigma_i \);
3. There is no integer \( k \) satisfying \( 0 < nk < l_i \) and \( \Omega^{-nk} T^i \cong T'(-k) \) with \( T' \in \text{add} T \).

We say that an algebra is \((n, m, \sigma)\)-\( T \)-Koszul if it is \((n, m_i, \sigma_i)\)-\( T \)-Koszul with \( m_i = m \) and \( \sigma_i = \sigma \) for all \( i \).

One can think of part (3) in the definition above as a minimality condition for each \( l_i \), as explained in the following remark.

**Remark 7.6.** When \( T \) is almost graded \( n \)-self-orthogonal, part (3) in the definition above is equivalent to that there exist no integers \( l'_i \) and \( g'_i \) with \( l'_i < l_i \) satisfying Definition 7.2. Note in particular that given such integers, one must have \( l'_i = ng'_i \), as \( T \) is almost graded \( n \)-self-orthogonal. This contradicts the third requirement in Definition 7.5.

Similarly as in Example 7.4, we see that almost Koszul algebras give rise to natural examples of algebras which are \((n, m_i, \sigma_i)\)-\( T \)-Koszul.

**Example 7.7.** Let \( \Lambda \) be a \((p, q)\)-Koszul algebra in the sense of Definition 7.1 and assume that \( \Lambda \) is graded Frobenius of highest degree \( a \geq 2 \). Then \( \Lambda \) is \((1, m, \sigma)\)-Koszul with respect to \( \Lambda_0 \), where \( m \) and \( \sigma \) are the unique integers such that \( q = am - \sigma \) with \( 0 \leq \sigma \leq a - 1 \). Note that as \( p = a \), it follows from Example 7.4 that part (1) and (2) of Definition 7.5 are satisfied. As the integers \( l_i \) and \( g_i \) do not depend on the parameter \( i \), we simply denote them by \( l \) and \( g \).

It remains to check minimality, i.e. that part (3) of Definition 7.5 holds. Assume to the contrary that there exist integers \( l' \) and \( g' \) as described in Remark 7.6. As \( n = 1 \), this in particular means that

\[
(7.1) \quad \Omega^{-l'} \Lambda_0 \cong \Lambda_0(-l').
\]

By the existence of the almost Koszul resolution from Definition 7.3, we have an epimorphism \( I^{l'-1} (1 - l') \to \Omega^{-l'} \Lambda_0 \), where \( I^{l'-1} \) is a summand of \( D \Lambda \) as
graded modules. Since $\Lambda$ is graded Frobenius and hence $DA \simeq \Lambda(-a)$, the module $I^{l'-1}(1-l')$ is a direct summand of $\Lambda(1-l'-a)$. Consequently, the top of $I^{l'-1}(1-l')$ is concentrated in degree $1-l'-a$. However, by the isomorphism (7.1), the projective module $I^{l'-1}(1-l')$ projects onto a semisimple module concentrated in degree $-l'$. This yields that $\text{Top} I^{l'-1}(1-l')$ is concentrated in degree $-l'$, which is a contradiction as $a \geq 2$.

Recall that a Dynkin quiver is said to have bipartite orientation if every vertex is either a sink or a source. Just as in the study of almost Koszul algebras in [6], trivial extensions of bipartite Dynkin quivers provide an important class of algebras which are $(n, m_i, \sigma_i)$-T-Koszul.

**Example 7.8.** Let $\Lambda$ be given by the quiver

$$
1 \xrightarrow{\alpha_0} 2 \xleftarrow{\alpha_0'} 3
$$

with relations $\alpha_0\alpha_1', \alpha_1\alpha_0'$, and $\alpha_0'\alpha_0 - \alpha_1'\alpha_1$. This algebra is graded symmetric of highest degree 2 with grading induced by letting the arrows be in degree 1. The indecomposable projective injectives can be represented by the diagrams

$$
\begin{array}{ccc}
1_0 & 2_0 & 3_0 \\
2_1 & 1_1 & 3_1 \\
2_2 & 1_2 & 3_2 \\
\end{array}
$$

where the subscripts indicate the degrees of the basis elements. Computing injective resolutions of the simples, one can check directly that $\Lambda$ is $(1, 1, 0)$-Koszul with respect to $\Lambda_0 = \Lambda/\text{Rad} \Lambda$, i.e. it is $(1, m_i, \sigma_i)$-$\Lambda_0$-Koszul with $(m_i)_{i=1}^3 = (1, 1, 1)$ and $(\sigma_i)_{i=1}^3 = (0, 0, 0)$. Moreover, one can verify that $\Lambda_0$ is a tilting object in $\text{gr} \Lambda$ with 1-representation finite endomorphism algebra. Note that this is a specific case of what we prove more generally in our characterization result for $(n, m_i, \sigma_i)$-T-Koszul algebras given in Theorem 7.17. In this example, we see that the endomorphism algebra of $\Lambda_0$ in $\text{gr} \Lambda$ decomposes as the direct sum of the endomorphism algebras of

$$
\begin{array}{ccc}
1_0 & 2_0 & 3_0 \\
1_0 & 2_0 & 3_0 \\
\end{array}
$$

and

$$
\begin{array}{ccc}
1_0 & 2_0 & 3_0 \\
1_0 & 2_0 & 3_0 \\
\end{array}
$$

which are respectively isomorphic to the path algebras of the quivers

$$1 \longleftarrow 2 \longrightarrow 3$$

and

$$1 \longrightarrow 2 \longleftarrow 3.$$
Note that $\Lambda$ in the example above is the trivial extension of a bipartite Dynkin quiver of type $A_3$ endowed with the grading given by putting arrows in degree 1. The behaviour exhibited in the example is typical of the general case, and we summarize this in the following proposition. See for instance [13, Section 3.1] for an overview of the Coxeter numbers of different Dynkin quivers.

**Proposition 7.9.** Let $Q$ be a bipartite Dynkin quiver with Coxeter number $h \geq 4$. Consider $\Lambda = \Delta kQ$ with grading given by putting arrows in degree 1. Then $\Lambda$ is $(1, \frac{h-2}{2}, 0)$-$\Delta_0$-Koszul if $h$ is even and $(1, \frac{h-1}{2}, 1)$-$\Delta_0$-Koszul otherwise.

**Proof.** As $Q$ is a bipartite Dynkin quiver and $h \geq 4$, it follows from [6, Proposition 3.11, Corollary 4.3] that $\Lambda$ is $(2, h-2)$-Koszul in the sense of Definition 7.1. Our conclusion now follows by the argument in Example 7.7. □

From now on, we make the same standing assumptions as we did in order to develop our theory in Section 6.

**Setup.** Throughout the rest of this paper, we use the standing assumptions described at the beginning of Section 6.

Given these assumptions, let us first show that the data of an $(n, m_i, \sigma_i)$-$T$-Koszul algebra determines a permutation on the set $\{1, \ldots, t\}$ in a natural way.

**Lemma 7.10.** Let $\Lambda$ be $(n, m_i, \sigma_i)$-$T$-Koszul. There is then a permutation $\pi$ on the set $\{1, \ldots, t\}$ such that $\Omega^{-l_i}T^i \simeq T^\pi(i)(-g_i)$ for each $i \in \{1, \ldots, t\}$.

**Proof.** Let $i \in \{1, \ldots, t\}$. As $T$ is almost graded $n$-self-orthogonal, there exists an object $T' \in \text{add} \, T$ such that $\Omega^{-l_i}T^i \simeq T'(-g_i)$.

Recall that $T$ is concentrated in degree 0 and that $a \geq 1$. Since it follows from Lemma 2.2 that $\text{Soc} \, \Lambda \subseteq \Lambda_a$, this implies that $T'$ is not projective as a $\Lambda$-module by Lemma 2.3 (3). As $\Omega^{-l}(-)$ is an equivalence on the stable category, the object $T'$ is indecomposable, and consequently $T' \simeq T''$ for some $i' \in \{1, \ldots, t\}$. This allows us to define the map

$$\pi: \{1, \ldots, t\} \to \{1, \ldots, t\}$$

by setting $\pi(i) = i'$.

We next show that $\pi$ is injective and hence a permutation. Let $\pi(i) = \pi(j)$ and assume $l_i \neq l_j$. Without loss of generality, we consider the case $l_i > l_j$. Our assumption yields $\Omega^{-(l_i-l_j)}T^i \simeq T'^{(g_i-g_j)}$. Observe that the integers $l'_i = l_i - l_j$ and $g'_i = g_i - g_j$ hence satisfy Definition 7.2. Note in particular that $0 < l'_i < l_i$ and that positivity of $l'_i$ combined with $T$ being...
almost graded $n$-self-orthogonal implies positivity of $g_i$. This contradicts part (3) of Definition 7.5 by Remark 7.6, so we must have $l_i = l_j$, which implies $T^i \simeq T^j$. As $T$ is basic, this means that $i = j$, which finishes our proof.

Using our fixed decomposition $T \simeq \oplus_{i=1}^t T^i$ together with the definition of $\tilde{T}$, we see that the algebra $B = \text{End}_{\text{gr}\Lambda}(\tilde{T})$ decomposes as

$$B \simeq \bigoplus_{i=1}^t \bigoplus_{j=0}^{a-1} \text{Hom}_{\text{gr}\Lambda}(\tilde{T}, X^{i,j}),$$

where $X^{i,j} = \Omega^{-nj} T^i(j)$. Hence, the indecomposable projective $B$-modules $P^{i,j} = \text{Hom}_{\text{gr}\Lambda}(\tilde{T}, X^{i,j})$ are indexed by the set

$$J = \{(i, j) \mid 1 \leq i \leq t \text{ and } 0 \leq j \leq a - 1\}.$$ 

Notice that if $\tilde{T}$ is a tilting object in $\text{gr}\Lambda$, then $X^{i,j}$ is the image of $P^{i,j}$ under the equivalence $D^b(\text{mod } B) \simeq \text{gr}\Lambda$, which was explicitly constructed in Proposition 4.3.

Given a permutation $\sigma$ on the index set $J$, we let $\sigma^L_i$ and $\sigma^R_i$ be defined by

$$\sigma(i, j) = (\sigma^L_j(i), \sigma^R_i(j)).$$

We are now ready to state and prove the first part of our characterization result. Note that this direction in the proof of Theorem 7.17 explains and justifies the somewhat technical definition of an $(n, m_i, \sigma_i)$-$T$-Koszul algebra.

**Theorem 7.11.** If $\tilde{T}$ is a tilting object in $\text{gr}\Lambda$ and $B = \text{End}_{\text{gr}\Lambda}(\tilde{T})$ is $(na - 1)$-representation finite, then there exist integers $m_i$ and $\sigma_i$ such that $\Lambda$ is $(n, m_i, \sigma_i)$-$T$-Koszul.

**Proof.** By [14, Proposition 0.2], there is a permutation $\sigma$ on $J$ such that for every pair $(i, j)$ in $J$ there is an integer $m_{i,j} \geq 0$ with

$$\nu_{na-1}^{-m_{i,j}} P^{i,j} \simeq P^{\sigma(i,j)},$$

as $B$ is $(na - 1)$-representation finite. Applying $\nu_{na-1}^{-1}$ on both sides, we get

$$\nu_{na-1}^{-m_{i,j}-1} P^{i,j} \simeq P^{\sigma(i,j)}[na - 1].$$

Since $\tilde{T}$ is a tilting object in $\text{gr}\Lambda$, we have an equivalence $D^b(\text{mod } B) \simeq \text{gr}\Lambda$ as described in Proposition 4.3. Using that $X^{i,j} = \Omega^{-nj} T^i(j)$ is the image of $P^{i,j}$ under this equivalence, combined with the correspondence of Serre functors, one obtains

$$\Omega^{-(na-1)(m_{i,j}+1)-(m_{i,j}+1)} X^{i,j}_{\mu^{-m_{i,j}-1}} (a(m_{i,j} + 1)) \simeq \Omega^{-(na-1)} X^{\sigma(i,j)}.$$

This again yields

$$\Omega^{-(na-1)} X^{\mu^{-m_{i,j}-1}(i,j)} \simeq X^{\sigma(i,j)}[-a(m_{i,j} + 1)],$$

(7.2)
as \((-\mu\)_µ) commutes with cosyzygies and graded shifts and permutes the summands of \(T\). It follows that for each pair \((i, j)\) in \(J\), we get
\[
(7.3) \quad \Omega^{-nam_i,j-1-n(j-\sigma_i^R(j))}T^{\mu^{-m_i,j-1}(i)} \simeq T^{\sigma_i^R(i)}(-a(m_i,j + 1) + \sigma_i^R(j) - j).
\]
Twisting by \(\mu^{m_i,j+1}\) and setting \(j = 0\), one obtains
\[
(7.4) \quad \Omega^{-(nam_i,0-n\sigma_i^R(0)+1)}T^i \simeq T^{\mu^{m_i,1+1}(\sigma_i^R(i))}(-a(m_i,0 + 1) + \sigma_i^R(0)).
\]

Letting \(m_i := m_{i,0}\) and \(\sigma_i := \sigma_i^R(0)\), we hence see that \(l_i\) and \(g_i\) can be chosen so that part (1) in the definition of being almost graded \(n\)-self-orthogonal is satisfied in this case. Note that \(g_i\) of this form is always positive, so is \(l_i\), as can be seen by applying Lemma 2.3 (6).

In order to show part (3) of Definition 7.3, consider an integer \(k\) satisfying \(0 < nk < l_i\). Note that we can write \(k = qa - r\) with \(q \geq 1\) and \(0 \leq r \leq a - 1\). Aiming for a contradiction, assume that there is an integer \(j \in \{1, \ldots, t\}\) with
\[
\Omega^{-n(qa-r)}T^i \simeq T^{j(r)\ldots(qa-r)}.
\]
Twisting by \((-\mu\)_\(-\eta\)) and using the equivalence \(\mathcal{D}^b(\text{mod } B) \simeq \text{gr } \Lambda\) in a similar way as in the beginning of this proof, we obtain
\[
\nu^{-q}_{na-1}P^{\mu,0}_i \simeq P^{-q-j,\eta}_i.
\]
Applying \(\nu^{-q}_{na-1}\) on both sides yields
\[
(7.5) \quad \nu^{-q-1}_{na-1}P^{\mu,0}_i \simeq I^{-q-j,\eta}_i[-na + 1].
\]
From the assumption \(nk < l_i\) along with the description of \(l_i\), we deduce that \(0 \leq q - 1 \leq m_i\). As long as \(na > 1\), the expression (7.5) hence contradicts Lemma 5.2, so we can conclude that the third condition of Definition 7.5 is satisfied. If \(na = 1\), the algebra \(B\) is semisimple. In particular, this implies that \(l_i = 1\), so the condition is trivially satisfied in this case.

It remains to prove that \(T\) satisfies part (2) of Definition 7.2 i.e. that for each \(i \in \{1, \ldots, t\}\), we have \(\text{Ext}^{nk_i+1}_{\text{gr } \Lambda}(T, T^i(k)) = 0\) for \(l \neq 0\) and \(nk_i + l < l_i\). If \(nk_i + l \leq 0\), this is immediately clear, so we can assume \(nk_i + l > 0\). This yields
\[
\text{Ext}^{nk_i+1}_{\text{gr } \Lambda}(T, T^i(k)) \simeq \text{Hom}_{\text{gr } \Lambda}(T, \Omega^{-(nk_i+l)}T^i(k)).
\]
In the case \(k < 0\), this is zero by Lemma 2.3 (6), and we can thus assume \(k \geq 0\).

As \(\tilde{T}\) is a tilting object in \(\text{gr } \Lambda\), a similar argument as in the proof of Theorem 6.4 yields an isomorphism
\[
(7.6) \quad \text{Hom}_{\text{gr } \Lambda}(\tilde{T}, \Omega^{-(nam_i+1)}X^{\mu^{-m}(i),\eta}(am)) \simeq H^l(\nu^{-m}_{na-1}(P^{i,j}))
\]
for every pair \((i, j)\) in \(J\). By Lemma 5.2, we know that \(H^l(\nu^{-m}_{na-1}(P^{i,j})) = 0\) for \(l \neq 0\) and \(0 \leq m \leq m_{i,j}\) as \(B\) is \((na - 1)\)-representation finite. Using that
(−)µ is an equivalence on gr Λ, that ˜Tµ ≃ ˜T and splitting up on summands of 
T = ⊕s0Ω−nsT ⟨s⟩, this yields 
(7.7) Homgr Λ(T, Ω−(na−s+j)+l)Ti ⟨am − s + j⟩ = 0 
for l ≠ 0 and 0 ≤ m ≤ mi,j. We simplify this by letting j = 0. Hence, we have 
mi,j = mi. In the case k ≤ am, we can write k = am − s for appropriate values 
of m and s, so (7.7) implies our desired conclusion in this case. If k > am, we use 
the isomorphism Ti = ViTkTπ(i) ⟨−gi⟩ to rewrite 
Homgr Λ(T, Ω−(nk+l)Ti ⟨l⟩) ≃ Homgr Λ(T, Ω−(nk+l)Tπ(i) ⟨k − gi⟩).
When nk + l < li, this is 0 by Lemma 2.3 (7). To see this, notice that the 
assumption k > am combined with the definition of g, yields k − g ≥ 1 − a. This 
finishes our proof.

□

Before giving a result which explains why our choices of mi and σ are reasonable, 
we need the following lemma.

Lemma 7.12. If ˜T is a tilting object in gr Λ, then the algebra B = Endgr Λ(˜T) is 
basic.

Proof. As ˜T is a tilting object in gr Λ, it suffices to show that ˜T is basic. Note 
that the indecomposable summands of ˜T are of the form T ⟨i⟩ with 0 ≤ i ≤ t 
and 0 ≤ j ≤ a − 1. Assume that we have isomorphic summands 
Ω−njTi ⟨j⟩ ≃ Ω−nlTk ⟨l⟩.
If j = l, it follows that i = k as T is basic. Without loss of generality, we hence 
assume j > l. Consider now 
Homgr Λ(Ti, Ti) ≃ Homgr Λ(Ti, Ω−(l−j)Tk ⟨l − j⟩),
which is non-zero as Ti ≠ 0. This contradicts Lemma 2.3 (7), as l − j ≥ 1 − a and 
−n(l − j) > 0, so we can conclude that (i, j) = (k, l).

□

Recall from [14] Proposition 0.2 and the proof of Theorem 7.11 that when B is 
(na − 1)-representation finite, there is a permutation σ on J such that for every 
pair (i, j) in J there is an integer mt,i,j ≥ 0 with 
ν−mi,j na−1 Pk,j ≃ Iσ(i,j).
As before, we use the notation 
σ(i, j) = (σj(i), σi(j)).
The proposition below provides more information about how the permutation σ 
and the integers mt,i,j associated to B being (na−1)-representation finite are related 
to the parameters mi and σi.
Proposition 7.13. If $\tilde{T}$ is a tilting object in $\text{gr} \Lambda$ and $B = \text{End}_{\text{gr} \Lambda}(\tilde{T})$ is $(na - 1)$-representation finite, then $\Lambda$ is $(n, m_i, \sigma_i)$-$T$-Koszul with $m_i = m_{i,0}$ and $\sigma_i = \sigma_i^R(0)$ and we have

$$\sigma_i^R(j) = \begin{cases} 
\sigma_i + j & \text{if } \sigma_i + j \leq a - 1 \\
\sigma_i + j - a & \text{if } \sigma_i + j > a - 1
\end{cases}$$

and

$$m_{i,j} = \begin{cases} 
m_i & \text{if } j \leq \sigma_i^R(j) \\
m_i - 1 & \text{if } j > \sigma_i^R(j).
\end{cases}$$

Additionally, if $\pi$ is the permutation on $\{1, \ldots, t\}$ induced by $\Lambda$ being $(n, m_i, \sigma_i)$-$T$-Koszul, we have

$$\sigma_j^L(i) = \mu^{-m_{i,j} - 1}(\pi(i)).$$

Proof. Recall first that $\Lambda$ is $(n, m_i, \sigma_i)$-$T$-Koszul with $m_i = m_{i,0}$ and $\sigma_i = \sigma_i^R(0)$ by Theorem 7.11 and its proof. From now, consider a fixed integer $i \in \{1, \ldots, t\}$ and let $0 \leq j \leq a - 1$.

Our next aim is to verify the first two equations in the formulation of the proposition. Note that to get the desired expression for $\sigma_i^R(j)$, it is enough to show that

$$\sigma_i^R(j) = \begin{cases} 
\sigma_i^R(0) + j & \text{if } j \leq \sigma_i^R(j) \\
\sigma_i^R(0) + j - a & \text{if } j > \sigma_i^R(j).
\end{cases}$$

To see that this is sufficient, observe that given the expression above, one has $j \leq \sigma_i^R(j)$ if and only if $\sigma_i + j \leq a - 1$. Indeed, if $j \leq \sigma_i^R(j)$, our formula gives

$$\sigma_i^R(j) = \sigma_i^R(0) + j = \sigma_i + j,$$

so $\sigma_i + j \leq a - 1$. On the other hand, the assumption $j > \sigma_i^R(j)$ yields

$$\sigma_i^R(j) = \sigma_i^R(0) + j - a = \sigma_i + j - a,$$

which implies $\sigma_i + j > a - 1$.

Assume $j \leq \sigma_i^R(j)$. Observe that one obtains

$$\Omega^{-nam_{i,j}} X^{\mu^{-m_{i,j} - 1}(i), 0} \simeq X^{\sigma(i,j)(-0,j)},$$

by applying $\Omega(-j)\langle -j \rangle$ to (7.2). Our assumption yields $0 \leq \sigma_i^R(j) - j \leq a - 1$, so we can run the argument at the beginning of the proof of Theorem 7.11 in reverse to get

$$\nu_{na-1}^{-m_{i,j}} \sim \nu_{na-1}^{-m_{i,j}} \sim I^{\sigma(i,j)(-0,j)}.$$
If \( na > 1 \), we deduce that \( m_{i,j} = m_{i,0} \) and \( I^{\sigma(i,j)-(0,j)} \simeq I^{\sigma(i,0)} \). If \( na = 1 \), then \( B \) is semisimple. This implies \( m_{i,j} = m_{i,0} = 0 \), and the same conclusion thus follows. In particular, this yields
\[
\sigma(i,j) - (0,j) = \sigma(i,0)
\]
as \( B \) is basic. Consequently, we obtain our desired expressions for \( \sigma_i^R(j) \) and \( m_{i,j} \) once we have made the substitutions \( m_i = m_{i,0} \) and \( \sigma_i = \sigma_i^R(0) \).

For the second case, assume \( j > \sigma_i^R(j) \). Note that we now necessarily have \( na > 1 \) as \( m_i = 0 \) implies \( \sigma_i = 0 \). Apply \( \Omega^{-n(a-j)}(-)(a-j) \to (7.2) \) to get
\[
\Omega^{-na(m_{i,j}+1)-1}X^{\mu_i(m_{i,j}+1)(i),0} \simeq X^{\sigma(i,j)+(0,a-j)}(-a((m_{i,j}+1)+1)).
\]
Our assumption yields \( 0 < \sigma_i^R(j) + a - j \leq a - 1 \). Twisting by \( (-)^{\mu-1} \) and again reversing the argument at the beginning of the proof of Theorem 7.11, we hence obtain
\[
\nu_{na-1}^{-m_{i,j}+1})P_{i,0} \simeq I^{\mu-1(\sigma_i^R(i)),\sigma_i^R(j)+a-j}.
\]
Similarly as above, this leads to our desired expressions for \( \sigma_i^R(j) \) and \( m_{i,j} \).

It remains to check that \( \sigma_i^R(i) = \mu_i^{-m_{i,j}-1}(\pi(i)) \). This follows by applying what we have shown so far to (7.3).

Our next aim is to prove the other direction of this section’s main result. Let us first give an overview of some useful observations.

**Lemma 7.14.** Let \( \Lambda \) be \((n, m_i, \sigma_i)\)-T-Koszul. The following statements hold for \( 1 \leq i \leq t \):

1. We have \( \pi \circ \mu = \mu \circ \pi \), where \( \pi \) is the permutation on \( \{1, \ldots, t\} \) induced by \( \Lambda \) being \((n, m_i, \sigma_i)\)-T-Koszul.
2. The constants \( l_i \) and \( g_i \) satisfy \( l_i = l_{\mu(i)} \) and \( g_i = g_{\mu(i)} \).
3. The constants \( m_i \) and \( \sigma_i \) satisfy \( m_i = m_{\mu(i)} \) and \( \sigma_i = \sigma_{\mu(i)} \).
4. We have \( g_i \geq a \). Moreover, if \( m_i = 0 \), then \( \sigma_i = 0 \).

**Proof.** For part (1) and (2), recall that \( \Omega^\pm(-) \) and \( \langle \pm 1 \rangle \) both commute with \( (-)^{\mu} \). This implies that \( \Omega^{-l_{\mu}(i)}g_i \simeq T^{\mu(i)}\pi(i) \) and \( \Omega^{-l_{\mu}(i)}T^{\mu(i)}g_{\pi(i)} \simeq T^{\pi(i)}\mu(i) \), and hence arguments similar to those in Remark 7.6 and Lemma 7.10 are sufficient.

Comparing the expressions for \( g_i \) and \( g_{\mu(i)} \), we see that part (3) follows from (2) by a number theoretical argument.

Part (4) is a consequence of the definition of \( l_i \) and \( g_i \). To be precise, it is clear that \( m_i = 0 \) implies \( \sigma_i = 0 \) as \( l_i \) is positive. Using this, the assumption \( \sigma_i = a - 1 \) yields our first statement.

Compared to the case for \( n\)-T-Koszul algebras, it is somewhat more involved to show that \( \overline{T} \) is a tilting object in \( \text{gr} \Lambda \) whenever \( \Lambda \) is \((n, m_i, \sigma_i)\)-T-Koszul. We hence prove this as a separate result.

**Proposition 7.15.** If \( \Lambda \) is \((n, m_i, \sigma_i)\)-T-Koszul, then \( \overline{T} \) is a tilting object in \( \text{gr} \Lambda \).
Proof. Since Lemma 6.3 yields $\text{Thick}_{\mathfrak{gr} \Lambda}(\tilde{T}) = \mathfrak{gr} \Lambda$, we only need to check rigidity. As in the proof of Theorem 6.4, it is enough to verify that
\[ \text{Hom}_{\mathfrak{gr} \Lambda}(T, \Omega^{-(nk+l)}T^i(k)) = 0 \text{ for } l \neq 0 \]
for any integer $k$ with $|k| \leq a - 1$. In the cases $nk + l = 0$ and $nk + l < 0$, the argument is exactly the same as in the proof of Theorem 6.4, so assume $nk + l > 0$. For each summand $T^i$ of $T$, one now obtains
\[ \text{Hom}_{\mathfrak{gr} \Lambda}(T, \Omega^{-(nk+l)}T^i(k)) \simeq \text{Ext}_{\mathfrak{gr} \Lambda}^{nk+l}(T, T^i(k)). \]
In the case $nk + l < l_i$, this is zero for $l \neq 0$ as $T$ is almost graded $n$-self-orthogonal. Otherwise, we use the isomorphism $T^i \simeq \Omega^k \Omega^{-\pi(i)}(-g_i)$ to rewrite the expression above. In the case $nk + l = l_i$, we get
\[ \text{Hom}_{\mathfrak{gr} \Lambda}(T, \Omega^{-(nk+l-l_i)}T^\pi(i)(k-g_i)) = \text{Hom}_{\mathfrak{gr} \Lambda}(T, T^\pi(i)(k-g_i)). \]
This is zero as $|k| \leq a - 1$ together with Lemma 7.14 (4) yields $k - g_i < 0$. If $nk + l > l_i$, one obtains
\[ \text{Hom}_{\mathfrak{gr} \Lambda}(T, \Omega^{-(nk+l-l_i)}T^\pi(i)(k-g_i)) \simeq \text{Ext}_{\mathfrak{gr} \Lambda}^{nk+l-l_i}(T, T^\pi(i)(k-g_i)). \]
As $nk + l - l_i > 0$ and $k - g_i < 0$, the first expression can not be written as an $n$-multiple of the second. If $nk + l - l_i < l_\pi(i)$, we are hence done. Otherwise, we iterate the argument until we reach our desired conclusion. \hfill \Box

We are now ready to show the other direction of Theorem 7.17.

**Theorem 7.16.** If $\Lambda$ is $(n, m_i, \sigma_i)T$-Koszul, then $\tilde{T}$ is a tilting object in $\mathfrak{gr} \Lambda$ and $B = \text{End}_{\mathfrak{gr} \Lambda}(\tilde{T})$ is $(na - 1)$-representation finite.

**Proof.** Since $\tilde{T}$ is a tilting object in $\mathfrak{gr} \Lambda$ by Proposition 7.15, we only need to show that $B = \text{End}_{\mathfrak{gr} \Lambda}(\tilde{T})$ is $(na - 1)$-representation finite. Let us first use the integers $m_i$ and $\sigma_i$ to define $\sigma_i^R(j)$, $m_{i,j}$ and $\sigma_i^L(i)$ for $(i,j) \in J$ by the formulas in the formulation of Proposition 7.15. Note that this yields $0 \leq \sigma_i^R(j) \leq a - 1$, as well as $1 \leq \sigma_i^L(i) \leq t$ and $m_{i,j} \geq 0$. The latter is a consequence of Lemma 7.13 (4).

Using that $\Lambda$ is assumed to be $(n, m_i, \sigma_i)T$-Koszul, we see that (7.4) is satisfied. Furthermore, we can run the argument at the beginning of the proof of Theorem 7.11 in reverse, using that $T$ is a tilting object in $\mathfrak{gr} \Lambda$. Consequently, one obtains
\[ \nu_{na-1}^{-m_{i,j}} P^{i,j} \simeq I^{\sigma(i,j)} \]
for every indecomposable projective $B$-module $P^{i,j}$, where
\[ \sigma(i,j) := (\sigma_j^L(i), \sigma_i^R(j)). \]

Our next aim is to show that $\sigma$ is a permutation on $J$. As $J$ is a finite set, it is enough to check injectivity. Recall that $\mu$ and $\pi$ are permutations, and hence injective. Combining this with Lemma 7.13 (1) and (3), notice that also $\sigma_0^L$ is injective.
Assume that $\sigma(i, j) = \sigma(k, l)$ for $(i, j)$ and $(k, l)$ in $J$. If $j \leq \sigma^R_l(j)$ and $l \leq \sigma^R_k(l)$, we see that
\[ \sigma^L_0(i) = \sigma^L_j(i) = \sigma^L_l(k) = \sigma^L_0(k), \]
so $i = k$ by injectivity of $\sigma^L_0$. As we in this case also have
\[ \sigma^L_0(0) + j = \sigma^R_0(j) = \sigma^R_k(l) = \sigma^L_0(0) + l, \]
it follows that $j = l$, so $\sigma$ is injective. The argument in the case $j > \sigma^R_l(j)$ and $l > \sigma^R_k(l)$ is similar.

By symmetry, it remains to consider the case where $j \leq \sigma^R_l(j)$ and $l > \sigma^R_k(l)$. Here, the assumption $\sigma(i, j)$ yields
\[ \sigma^L_0(i) = \sigma^L_j(i) = \sigma^L_l(k) = \mu(\sigma^L_0(k)). \]
Consequently, Lemma 7.14 (1) and (3) imply that $i = \mu(k)$ and $\sigma^R_0(0) = \sigma^R_k(0)$. As we in this case also have
\[ \sigma^R_0(0) + j = \sigma^R_l(j) = \sigma^R_k(l) = \sigma^R_0(0) + l - a, \]
this means that $j = l - a$, contradicting the assumption $0 \leq j, l \leq a - 1$. Hence, this case is impossible, and we can conclude that $\sigma$ is a permutation.

It now follows that every indecomposable injective, and hence also $DB$, is contained in the subcategory
\[ \mathcal{U} = \text{add}\{\nu_{na-1}B \mid l \in \mathbb{Z}\} \subseteq \mathcal{D}^b_{\text{mod}}(B). \]
By Theorem 5.1, it thus remains to prove that $\text{gl.dim} \ B \leq na - 1$. To show this, observe first that $B$ has finite global dimension by Lemma 6.2. As $\widetilde{T}$ is a tilting object in $\text{gr} \ \Lambda$, it follows from (7.3) in the proof of Theorem 7.11 that we have
\[ H^i(\nu_{na-1}(P^{i,j})) \simeq \text{Hom}_{\text{gr} \ \Lambda}(\widetilde{T}, \Omega^{-(na+l)}X^{\mu^{-1}(i),j}a)) \]
\[ \simeq \bigoplus_{s=0}^{a-1} \text{Hom}_{\text{gr} \ \Lambda}(T, \Omega^{-(na+j-s+l)}T^i(a + j - s)) \]
for every pair $(i, j)$ in $J$. We want to show that this is zero whenever $l \notin \{1 - na, 0\}$. Note that the argument for this is similar to the proof of Proposition 7.15. In particular, it is enough to consider the case $n(a + j - s) + l \geq l_i$ for each $i$, since the remaining cases are covered by our previous proof. Using that $\Omega^{-i}T^i \simeq T^{\pi(i)}(-g_i)$, the summands in our expression above can be rewritten as
\[ \text{Hom}_{\text{gr} \ \Lambda}(T, \Omega^{-n(\sigma_i + j - s - am_i) - (na - 1 + l)}T^{\pi(i)}(\sigma_i + j - s - am_i)). \]
If $n(\sigma_i + j - s - am_i) + na - 1 + l < l_{\pi(i)}$, this is non-zero only when $l$ is as claimed. Otherwise, Lemma 7.14 (4) implies that we get a negative graded shift in the next step of the iteration, and we are done by the same argument as in the proof of Proposition 7.15. From this, one can see that the assumptions in Lemma 5.3 are satisfied, and hence $\text{gl.dim} \ B \leq na - 1$. Applying Theorem 5.1, we conclude that $B$ is $(na - 1)$-representation finite, which finishes our proof. \[ \square \]
Altogether, combining Theorem 7.11 and Theorem 7.16, we have now proved this section’s main result. Recall that we use the standing assumptions described at the beginning of Section 6.

**Theorem 7.17.** The following statements are equivalent:

1. There exist integers $m_i$ and $\sigma_i$ such that $\Lambda$ is $(n, m_i, \sigma_i)$-$T$-Koszul.
2. $\tilde{T}$ is a tilting object in $\text{gr} \Lambda$ and $B = \text{End}_{\text{gr} \Lambda}(\tilde{T})$ is $(na - 1)$-representation finite.

Moreover, the parameters $m_i$, $\sigma_i$ and the permutation $\pi$ obtained from $\Lambda$ being $(n, m_i, \sigma_i)$-$T$-Koszul correspond to the parameter $m_{i,j}$ and the permutation $\sigma$ obtained from $B$ being $(na - 1)$-representation finite as described in Proposition 7.13.

We now present some consequences of our characterization theorem similar to the ones in Section 6. Notice that unlike the corresponding result for $n$-representation infinite algebras, the following corollary is not – as far as we know – an analogue of anything existing in the literature. Mutatis mutandis, the proof is the same as that of Corollary 6.6 and is hence omitted. The parameters of $\Lambda$ and $\Lambda_0$ in the statement correspond as described in Theorem 7.17.

**Corollary 7.18.** If $a = 1$, our algebra $\Lambda$ is $(n + 1, m_i, \sigma_i)$-Koszul with respect to $T = \Lambda_0$ if and only if $\Lambda_0$ is $n$-representation finite. In particular, we obtain a bijective correspondence

$$\begin{align*}
\left\{ \text{isomorphism classes of basic } n \text{-representation finite algebras} \right\} & \quad \leftrightarrow \quad \left\{ \text{isomorphism classes of graded symmetric finite dimensional algebras of highest degree 1 which are } (n + 1, m_i, \sigma_i) \text{-Koszul with respect to their degree 0 parts} \right\},
\end{align*}$$

where the maps are given by $A \mapsto \Delta A$ and $\Lambda_0 \hookrightarrow \Lambda$.

Just like in Section 6, it is natural to consider the notion of an almost $n$-$T$-Koszul dual of a given almost $n$-$T$-Koszul algebra.

**Definition 7.19.** Let $\Lambda$ be an almost $n$-$T$-Koszul algebra. The almost $n$-$T$-Koszul dual of $\Lambda$ is given by $\Lambda^! = \oplus_{i \geq 0} \text{Ext}^n_{\text{gr} \Lambda}(T, T(i))$.

As before, note that while the notation $\Lambda^!$ is potentially ambiguous, it is for us always clear from context which structure the dual is computed with respect to.

Our next proposition shows that if $\Lambda$ is $(n, m_i, \sigma_i)$-$T$-Koszul, then the $na$-th preprojective algebra of $B = \text{End}_{\text{gr} \Lambda}(\tilde{T})$ is isomorphic to a twist of the $a$-th quasi-Veronese of $\Lambda^!$. The proof is exactly the same as that of the corresponding result in Section 6, namely Proposition 6.8.
Proposition 7.20. Let $\Lambda$ be $(n, m_i, \sigma_i)$-$T$-Koszul. Then $\Pi_{na}B \simeq (\omega^{-1}\omega)(\Lambda^1)^{[a]}$ as graded algebras. In particular, we have $\Pi_{na}B \simeq (\Lambda^1)^{[a]}$ in the case where $\Lambda$ is graded symmetric.

The proof of our final corollary is similar to that of Corollary 6.9 and is hence omitted.

Corollary 7.21. If $A$ is basic $n$-representation finite, then $\Pi_{n+1}A \simeq (\Delta A)^1$ as graded algebras.

Acknowledgements. The authors would like to thank Steffen Oppermann for helpful discussions and Louis-Philippe Thibault and Øyvind Solberg for careful reading and helpful suggestions on a previous version of this paper. They also thank Bernhard Keller for pointing out that $\Lambda!$ has finite global dimension as a graded algebra provided $\Lambda$ is a finite dimensional $n$-$T$-Koszul algebra, simplifying the statement in Theorem 3.39. The authors profited from use of the software QPA to compute examples which motivated parts of the paper.

Parts of this work was carried out while the first author participated in the Junior Trimester Program “New Trends in Representation Theory” at the Hausdorff Research Institute for Mathematics in Bonn. She would like to thank the Institute for excellent working conditions.

REFERENCES

[1] Claire Amiot, Osamu Iyama, and Idun Reiten, Stable categories of Cohen-Macaulay modules and cluster categories, Amer. J. Math. 137 (2015), no. 3, 813–857.
[2] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, Representation theory of Artin algebras, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995.
[3] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996), no. 2, 473–527.
[4] Joseph N. Bernstein, Izrail’ M. Gel’fand, and Sergei I. Gel’fand, Algebraic vector bundles on $\mathbb{P}^n$ and problems of linear algebra, Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 66–67.
[5] Alexey I. Bondal and Alexander E. Polishchuk, Homological properties of associative algebras: the method of helices, Izv. Ross. Akad. Nauk Ser. Mat. 57 (1993), no. 2, 3–50; English transl., Russian Acad. Sci. Izv. Math. 42 (1994), no. 2, 219–260.
[6] Sheila Brenner, Michael C. R. Butler, and Alastair D. King, Periodic algebras which are almost Koszul, Algebr. Represent. Theory 5 (2002), no. 4, 331–367.
[7] Erik Darpő and Osamu Iyama, $d$-representation-finite self-injective algebras, Adv. Math. 362 (2020), 106932, 50.
[8] Tobias Dyckerhoff, Gustavo Jasso, and Yankı Lekili, The symplectic geometry of higher Auslander algebras: symmetric products of disks, Forum Math. Sigma 9 (2021), Paper No. e10, 49.
[9] David E. Evans and Mathew Pugh, The Nakayama automorphism of the almost Calabi-Yau algebras associated to $SU(3)$ modular invariants, Comm. Math. Phys. 312 (2012), no. 1, 179–222.
[10] Robert M. Fossum, Phillip A. Griffith, and Idun Reiten, Trivial extensions of abelian categories, Lecture Notes in Mathematics, Vol. 456, Springer-Verlag, Berlin-New York, 1975.
[11] Joseph Grant and Osamu Iyama, *Higher preprojective algebras, Koszul algebras, and super-potentials*, Compos. Math. **156** (2020), no. 12, 2588–2627.

[12] Edward L. Green, Idun Reiten, and Oyvind Solberg, *Dualities on generalized Koszul algebras*, Mem. Amer. Math. Soc. **159** (2002), no. 754, xvi+67.

[13] Dieter Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. **62** (1987), no. 3, 339–389.

[14] Martin Herschend and Osamu Iyama, *n*-representation-finite algebras and twisted fractionally Calabi–Yau algebras, Bull. Lond. Math. Soc. **43** (2011), no. 3, 449–466.

[15] Martin Herschend, Osamu Iyama, Hiroyuki Minamoto, and Steffen Oppermann, *Representation theory of Geigle-Lenzing complete intersections*, to appear in Mem. Amer. Math. Soc.

[16] Martin Herschend, Osamu Iyama, and Steffen Oppermann, *n*-representation infinite algebras, Adv. Math. **252** (2014), 292–342.

[17] Osamu Iyama, *Auslander correspondence*, Adv. Math. **210** (2007), no. 1, 51–82.

[18] Osamu Iyama, *Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories*, Adv. Math. **210** (2007), no. 1, 22–50.

[19] Osamu Iyama and Steffen Oppermann, *n*-representation-finite algebras and n-APR tilting, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6575–6614.

[20] Gustavo Jasso and Sondre Kvamme, *An introduction to higher Auslander–Reiten theory*, Bull. Lond. Math. Soc. **51** (2019), no. 1, 1–24.

[21] Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1.

[22] Kenji Lefèvre-Hasegawa, *Sur les A∞-catégories*. PhD thesis, Université Denis Diderot - Paris 7, November 2003.

[23] Valery A. Lunts, *Categorical resolution of singularities*, J. Algebra **323** (2010), no. 10, 2977–3003.

[24] Dag Oskar Madsen, *Ext-algebras and derived equivalences*, Colloq. Math. **104** (2006), no. 1, 113–140.

[25] Hiroki Masai and Izuru Mori, *The structure of AS-Gorenstein algebras*, Adv. Math. **226** (2011), no. 5, 4061–4095.
[36] Constantin Năstăsescu and Freddy Van Oystaeyen, *Methods of graded rings*, Lecture Notes in Mathematics, vol. 1836, Springer-Verlag, Berlin, 2004.

[37] Steffen Oppermann and Hugh Thomas, *Higher-dimensional cluster combinatorics and representation theory*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 6, 1679–1737.

[38] Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.

[39] The QPA-team, *QPA - Quivers, path algebras and representations* (Version 1.31; 2018), https://folk.ntnu.no/oyvinso/QPA/.

[40] Idun Reiten and Michel Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366.

[41] Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.

[42] The Stacks Project Authors, *Stacks Project* (2018), https://stacks.math.columbia.edu

[43] Kota Yamamura, *Realizing stable categories as derived categories*, Adv. Math. **248** (2013), 784–819.

Department of mathematical sciences, NTNU, NO-7491 Trondheim, Norway

Email address: johanne.haugland@ntnu.no

Email address: mads.sandoy@ntnu.no