BERNSTEIN SPACES, SAMPLING, AND RIESZ-BOAS INTERPOLATION FORMULAS IN MELLIN ANALYSIS

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ABSTRACT. The goal of the paper is to consider Bernstein-Mellin subspaces in the Lebesgue-Mellin spaces and establishing for functions in these subspaces new sampling theorems and Riesz-Boas high-order interpolation formulas.

1. INTRODUCTION

In a series of interesting papers by C. Bardaro at al [4]-[12], and also by P. Butzer and S. Jansche [17]-[19] authors developed in the framework of Mellin analysis analogs of such important topics as Sobolev spaces, Bernstein spaces, Bernstein inequality, Paley-Wiener theorem, Riesz-Boas interpolation formulas, different sampling results. Many of their results were obtained by using the notion of polar-analytic functions developed in [4]-[8].

The objective of the present paper is to present a very different approach to the same topics based solely on the fact that the family of Mellin translations defined as

\[(1.1) \quad U_c(t)f(x) = e^{ct}f(e^t x), \quad U_c(t+\tau) = U_c(t)U_c(\tau), \quad c \in \mathbb{R},\]

forms a one-parameter \(C_0\)-group of isometries in appropriate function spaces (see below). As one can see, its infinitesimal generator is the operator

\[(1.2) \quad \frac{d}{dt}U_c(t)f(x)|_{t=0} = x\frac{d}{dx}f(x) + cf(x) = \Theta_c f(x).\]

In this paper we are guided by our abstract theory of sampling and interpolation in Banach spaces which was developed in [23], [24]. At the same time, our approach is very specific and direct and we are not using the language of one-parameter groups. The fact that we consider a very concrete situation allows us to obtain results which we did not have in our general development. All our results hold true for a general group of translations \(U_c\) with any \(c \in \mathbb{R}\). However, for the sake of simplicity we consider only the case \(c = 0\) and adapting notations \(U_0 = U, \Theta_0 = \Theta\).

In section 2 we define analog of Bernstein spaces using Bernstein-type inequality for the operator \(\Theta\). Our analog of the Paley-Wiener Theorem is Theorem 2.2. In section 3 we prove four sampling theorems: Theorem 3.2 Theorem 3.7. Our formula (3.3), which is a generalization of the Valiron-Tschakaloff sampling theorem, looks exactly like one proved in [1], Th. 6. Other three theorems in this section seems to be new. They all deal with the regularly spaced sampling points. In contrast, section 4 contains two sampling theorems which are using irregularly spaced sampling points. One of these theorems is a generalization of a sampling theorem which belongs to J.R. Higgins [20] and another one to C. Seip [28].

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In section 2 we discuss what we call the Riesz-Boas interpolation formulas. The famous Riesz interpolation formula [25], [26], [22] gives expression of the derivative of a trigonometric polynomial as a linear combination of its translates:

\[ (\frac{d}{dt}) P(t) = \frac{1}{4\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{\sin^2 \frac{2k-1}{4n}\pi} P\left(\frac{2k-1}{2n}\pi + t\right), \quad t \in \mathbb{T}. \]

This formula was extended by Boas [2], [3], (see also [1], [22], [27]) to functions in the Bernstein class \( B^\infty_\sigma(\mathbb{R}) \) in the following form

\[ (\frac{d}{dt}) f(t) = \frac{n}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} f\left(\frac{\pi}{n}(k-1/2) + t\right), \quad t \in \mathbb{R}. \]

In turn, the formula (1.4) was extended in [10] to higher powers \((d/dt)^m\). Our objective in section 3 is to obtain similar formulas for \( m \in \mathbb{N} \) where the operator \( d/dt \) is replaced by the operator \( \Theta = x \frac{d}{dx} \). When \( m = 1 \) such formula for the operator \( \Theta \) was established in [3].

Obviously, the goals of the present article are quite close to some of the objectives of the papers [11]-[12]. It would be very interesting and instructive to do a rigorous comparison of our approaches and outcomes. However, the fact that our papers are based on a rather different ideas makes a such comparison not easy. A serious juxtaposition of our treatments would require substantial increase of the length of the present article. We are planning to do such analysis in a separate paper.

2. Bernstein spaces

2.1. Mellin translations. For \( p \in [1, \infty[ \), denote by \( \| \cdot \|_p \) the norm of the Lebesgue space \( L^p(\mathbb{R}_+) \). In Mellin analysis, the analogue of \( L^p(\mathbb{R}_+) \) are the spaces \( X^p(\mathbb{R}_+) \) comprising all functions \( f : \mathbb{R}_+ \rightarrow \mathbb{C} \) such that \( f(\cdot)(\cdot)^{-1/p} \in L^p(\mathbb{R}_+) \) with the norm \( \| f \|_{X^p(\mathbb{R}_+)} := \| f(\cdot)(\cdot)^{-1/p} \|_p \). Furthermore, for \( p = \infty \), we define \( X^\infty \) as the space of all measurable functions \( f : \mathbb{R}_+ \rightarrow \mathbb{C} \) such that \( \| f \|_{X^\infty} := \sup_{x>0} |f(x)| < \infty \).

In spaces \( X^p(\mathbb{R}_+) \) we consider the one-parameter \( C_0 \)-group of operators \( U(t), t \in \mathbb{R} \), where

\[ U(t)f(x) = f(e^t x), \quad U(t + \tau) = U(t)U(\tau), \]

whose infinitesimal generator is

\[ \frac{d}{dt} U(t)f(x)|_{t=0} = x \frac{d}{dx} f(x) = \Theta f(x). \]

The domain of its power \( k \in \mathbb{N} \) is denoted by \( \mathcal{D}^k(\Theta) \) and defined as the set of all functions \( f \in X^p(\mathbb{R}_+), 1 \leq p \leq \infty, \) such that \( \Theta^k f \in X^p(\mathbb{R}_+) \). The domains \( \mathcal{D}^k(\Theta), \ k \in \mathbb{N}, \) can be treated as analogs of the Sobolev spaces. The general theory of one-parameter semi-groups of class \( C_0 \) (see [13], [21]) implies that the operator \( \Theta \) is closed in \( X^p(\mathbb{R}_+) \) and the set \( \mathcal{D}^\infty(\Theta) = \cap_k \mathcal{D}^k(\Theta) \) is dense in \( X^p(\mathbb{R}_+) \). By using the following formula (see [17], p. 355)

\[ \Theta^k f(x) = \sum_{r=0}^{k} S(k, r)x^r f^{(r)}(x), \]

\( S(k, r) \) being Stirling numbers of the second kind, one can give more explicit description of the Mellin-Sobolev spaces (see [17], p. 357) in the spirit of the Bochner’s definition of the classical one-dimensional Sobolev spaces.
2.2. Bernstein spaces. Let’s remind that in the classical analysis a Bernstein class \([1], [22]\), which is denoted as \(B^p_\sigma(\mathbb{R})\), \(\sigma > 0\), \(1 \leq p \leq \infty\), is a linear space of all functions \(f : \mathbb{R} \to \mathbb{C}\) which belong to \(L^p(\mathbb{R})\) and admit extension to \(\mathbb{C}\) as entire functions of exponential type \(\sigma\). A function \(f\) belongs to \(B^p_\sigma(\mathbb{R})\) if and only if the following Bernstein inequality holds

\[
\|f^{(k)}\|_{L^p(\mathbb{R})} \leq \sigma^k \|f\|_{L^p(\mathbb{R})},
\]

for all natural \(k\). Using the distributional Fourier transform

\[
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\xi x}dx, \quad f \in L^p(\mathbb{R}), \quad 1 \leq p \leq \infty,
\]

one can show (Paley-Wiener theorem) that \(f \in B^p_\sigma(\mathbb{R})\), \(1 \leq p \leq \infty\), if and only if \(f \in L^p(\mathbb{R})\), \(1 \leq p \leq \infty\), and the support of \(\hat{f}\) (in sense of distributions) is in \([-\sigma, \sigma]\).

**Definition 1.** The Bernstein space subspace \(B^p_\sigma(\Theta)\), \(\sigma > 0\), \(1 \leq p \leq \infty\), is defined as a set of all functions \(f\) in \(X^p(\mathbb{R}_+)\) which belong to \(D^\infty(\Theta)\) and for which

\[
\|\Theta^k f\|_{X^p(\mathbb{R}_+)} \leq \sigma^k \|f\|_{X^p(\mathbb{R}_+)}, \quad k \in \mathbb{N}.
\]

**Theorem 2.1.** A function \(f \in D^\infty(\Theta)\) belongs to \(B^p_\sigma(\Theta)\), \(\sigma > 0\), \(1 \leq p \leq \infty\), if and only if the quantity

\[
\sup_{k \in \mathbb{N}} \sigma^{-k} \|\Theta^k f\|_{X^p(\mathbb{R}_+)} = R(f, \sigma)
\]

is finite.

**Proof.** It is evident that if \(f \in B^p_\sigma(\Theta)\) then \(2.5\) holds. Next, for an \(h \in X^q(\mathbb{R}_+), 1/p + 1/q = 1\), consider a scalar-valued function

\[
\Phi(t) = \int_{\mathbb{R}_+} f(e^t x)h(x) \frac{dx}{x}.
\]

We note that

\[
\left( x \frac{d}{dx} \right)^k f(x) = \left( \frac{d}{dt} \right)^k f(e^t x)|_{t=0},
\]

and since \(f \in D^\infty(\Theta)\) we conclude that \(f(e^t x)\) is infinitely differentiable at \(t = 0\). Now we have that

\[
\Phi(t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \left( \frac{d}{dt} \right)^k \Phi(0) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \int_{\mathbb{R}_+} \left( \frac{d}{dt} \right)^k f(e^t x)|_{t=0} h(x) \frac{dx}{x} =
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} t^k \int_{\mathbb{R}_+} \left( x \frac{d}{dx} \right)^k f(x)h(x) \frac{dx}{x},
\]

This series is absolutely convergent since by the assumption

\[
|\Phi(t)| \leq \sum_{k=0}^{\infty} \frac{1}{k!} t^k \left| \int_{\mathbb{R}_+} \Theta^k f(x)h(x) \frac{dx}{x} \right| \leq
\]

\[
R(f, \sigma) \sum_{k=0}^{\infty} \frac{1}{k!} t^k \sigma^k \|f\|_{X^p(\mathbb{R}_+)} \|h\|_{X^q(\mathbb{R}_+)} = R(f, \sigma) \|f\|_{X^p(\mathbb{R}_+)} \|h\|_{X^q(\mathbb{R}_+)} e^{\sigma t}.
\]
It implies that $\Phi$ can be extended to the complex plane $\mathbb{C}$ by using its Taylor series (2.2). Moreover, as the estimate (2.2) shows the inequality
\[
|\Phi(z)| \leq R(f, \sigma) \|f\|_{X^p(\mathbb{R})} \|h\|_{X^q(\mathbb{R})} e^{\sigma |z|}, \quad z \in \mathbb{C},
\]
will hold. In addition, $\Phi$ is bounded on the real line by the constant $\|f\|_{X^p(\mathbb{R})} \|h\|_{X^q(\mathbb{R})}$. In other words, we proved that if $f \in B^p_\sigma(\Theta)$, $h \in X^q(\mathbb{R})$, $1/p + 1/q = 1$, then $\Phi$ belongs to the regular Bernstein space $B^{\infty}_\sigma(\mathbb{R})$. This fact allows to apply to $\Phi$ the classical Bernstein inequality in the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ with the uniform norm:
\[
\left|\left(\frac{d}{dt}\right)^k \Phi(0)\right| \leq \sup_t \left|\left(\frac{d}{dt}\right)^k \Phi(t)\right| \leq \sigma^k \sup |\Phi(t)|.
\]
Since
\[
\left(\frac{d}{dt}\right)^k \Phi(0) = \int_{\mathbb{R}_+} \left(\frac{d}{dt}\right)^k f(e^t x)|_{t=0} h(x) \frac{dx}{x} = \int_{\mathbb{R}_+} \Theta^k f(x) h(x) \frac{dx}{x}
\]
we obtain
\[
\left|\int_{\mathbb{R}_+} \Theta^k f(x) h(x) \frac{dx}{x}\right| \leq \sigma^k \|f\|_{X^p(\mathbb{R})} \|h\|_{X^q(\mathbb{R})}
\]
Choosing $h$ such that $\|h\|_{X^p(\mathbb{R})} = 1$ and
\[
\int_{\mathbb{R}_+} \Theta^k f(x) h(x) \frac{dx}{x} = \|\Theta^k f\|_{X^p(\mathbb{R})}
\]
we obtain the inequality
\[
\|\Theta^k f\|_{X^p(\mathbb{R})} \leq \sigma^k \|f\|_{X^p(\mathbb{R})}, \quad k \in \mathbb{N}.
\]
Theorem is proved. $\square$

The following analog of the Paley-Wiener Theorem follows from the proof of the previous theorem.

**Theorem 2.2.** The following conditions are equivalent:

1. $f$ belongs to $B^p_\sigma(\Theta)$, $1 \leq p \leq \infty$;
2. for every $g \in X^q(\mathbb{R})$, $1/p + 1/q = 1$, the function
\[
\Phi(z) = \int_{\mathbb{R}_+} f(e^t x) g(x) \frac{dx}{x}, \quad z \in \mathbb{C},
\]

belongs to the regular space $B^{\infty}_\sigma(\mathbb{R})$, i.e. it is an entire function of exponential type $\sigma$ which is bounded on the real line.

### 3. Sampling theorems in Mellin analysis

#### 3.1. A week Shannon type sampling theorem

Below we are going to use the following known fact (see [13], p. 46).

**Theorem 3.1.** If $h \in B^{\infty}_\sigma(\mathbb{R})$, then for any $0 < \gamma < 1$ the following formula holds

\[
\Phi(z) = \sum_{k \in \mathbb{Z}} h \left(\frac{k \pi}{\sigma}\right) \text{sinc} \left(\gamma^{-1} \frac{\sigma}{\pi} z - k\right), \quad z \in \mathbb{C},
\]

where the series converges uniformly on compact subsets of $\mathbb{C}$.
By using Theorem 3.1 we obtain our First "Weak" Sampling Theorem.

**Theorem 3.2.** If \( f \in B^p_0(\Theta), \ 1 \leq p \leq \infty \) then for all \( g \in L^q(\Theta), \ 1/p + 1/q = 1 \), and all \( 0 < \gamma < 1 \) the following formula holds

\[
\int_{\mathbb{R}_+} f(\tau x) g(x) \frac{dx}{x} =
\]

\[
\sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}_+} f(e^{\gamma k \pi / \sigma} x) g(x) \frac{dx}{x} \right) \text{sinc} \left( \gamma^{-1} \frac{\sigma}{\pi} \ln \tau - k \right), \ \tau \in \mathbb{R}_+
\]

where the series converges uniformly on compact subsets of \( \mathbb{R}_+ \).

**Proof.** According to Theorem 2.2 for any \( f \in B^p_0(\Theta), \ 1 \leq p \leq \infty \) and any \( g \in X^q(\mathbb{R}_+), \ 1/p + 1/q = 1 \), the function

\[
(3.3) \quad \Phi(t) = \int_{\mathbb{R}_+} f(e^t x) g(x) \frac{dx}{x}, \ t \in \mathbb{R}.
\]

belongs to \( B^\infty_0(\mathbb{R}) \). Applying Theorem 3.1 we obtain

\[
\int_{\mathbb{R}_+} f(e^t x) g(x) \frac{dx}{x} =
\]

\[
\sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}_+} f(e^{\gamma k \pi / \sigma} x) g(x) \frac{dx}{x} \right) \text{sinc} \left( \gamma^{-1} \frac{\sigma}{\pi} \ln \tau - k \right), \ t \in \mathbb{R},
\]

where the series converges uniformly on compact subsets of \( \mathbb{R} \). Setting \( t = e^t \) or \( t = \ln \tau \) gives for we obtain: for any \( f \in B^p_0(\Theta), g \in X^q(\mathbb{R}_+), \ 1/p + 1/q = 1 \),

\[
\int_{\mathbb{R}_+} f(\tau x) g(x) \frac{dx}{x} =
\]

\[
\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}_+} f(e^{\gamma k \pi / \sigma} x) g(x) \frac{dx}{x} \text{sinc} \left( \gamma^{-1} \frac{\sigma}{\pi} \ln \tau - k \right), \ \tau \in \mathbb{R}_+
\]

where the series converges uniformly on compact subsets of \( \mathbb{R}_+ \).

Theorem is proved. \( \square \)

### 3.2. A sampling formula for Mellin convolution.

Note, that according to [17] the Mellin convolution is defined as

\[
F \ast_M G(z) = \int_{\mathbb{R}_+} F \left( \frac{z}{u} \right) G(u) \frac{du}{u}.
\]

**Theorem 3.3.** For any \( f \in B^p_0(\Theta) \) and \( h \in X^q(\mathbb{R}_+), \ 1/p + 1/q = 1 \), \( 1 < p < \infty \) the following formula holds

\[
f \ast_M h(\tau) = \sum_{k \in \mathbb{Z}} f \ast_M h \left( \frac{\gamma k \pi}{\sigma} \right) \text{sinc} \left( \gamma^{-1} \frac{\sigma}{\pi} \ln \tau - k \right), \ \tau \in \mathbb{R}_+
\]

where the series converges uniformly on compact subsets of \( \mathbb{R}_+ \).
Proof. In the formula (3.1) we replace \( g(x), x > 0, h(1/x), x > 0, \) and then perform the substitution \( x = 1/y \). After all the formula (3.1) takes the form

\[
f \ast_{M} h(\tau) = \int_{\mathbb{R}^+} f\left(\frac{\tau}{y}\right) h(y) dy = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^+} f\left(\frac{e^{\gamma k \pi / \sigma}}{y}\right) h(y) dy \operatorname{sinc}\left(\gamma^{-1} \sigma \ln \tau - k\right) =
\]

(3.7)

where the series converges uniformly on compact subsets of \( \mathbb{R}^+ \).

Theorem is proven. \( \square \)

Remark 3.4. Note that (3.6) is an analog of the formula

\[
f \ast g(t) = \sum_{k \in \mathbb{Z}} f(\gamma k \pi / \sigma) \operatorname{sinc}\left(\gamma^{-1} \sigma t / \pi - k\right),
\]

where \( f \in B_{p}^{\sigma}(\mathbb{R}), 1 \leq p < \infty, g \in L^{q}(\mathbb{R}), 1/p + 1/q = 1, 0 < \gamma < 1, \) and

\[
f \ast g(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) g(t - x) dx,
\]

is the classical convolution.

### 3.3. Valiron-Tschakaloff-type sampling formulas.

The next theorem contains an analog of the Valiron-Tschakaloff sampling/interpolation formula [15] in Mellin analysis.

**Theorem 3.5.** If \( f \in B_{p}^{\sigma}(\Theta), 1 < p < \infty, x \in \mathbb{R}^+, t \in \mathbb{R}, \) then

\[
f(\tau) = \operatorname{sinc}\left(\frac{\sigma}{\pi} \ln \tau\right) f(1) + \ln \tau \operatorname{sinc}\left(\frac{\sigma}{\pi} \ln \tau\right) (\partial_x f)(1) + \sum_{k \in \mathbb{Z} \setminus \{0\}} f\left(\frac{e^{k \pi / \sigma}}{\sigma} \ln \tau\right) k \operatorname{sinc}\left(\frac{\sigma}{\pi} \ln \tau - k\right), \quad \tau \in \mathbb{R}^+.
\]

The series converges absolutely and uniformly on compact subsets of \( \mathbb{R}^+ \).

**Proof.** As we know (see Theorem 2.2), if \( f \in B_{p}^{\sigma}(\Theta) \), then the function

\[
\Phi(t) = \int_{\mathbb{R}^+} f(e^t x) \frac{dx}{x}, \quad t \in \mathbb{R},
\]

belongs to \( B_{\sigma}^{\infty}(\mathbb{R}) \). By applying to it the Valiron-Tschakaloff sampling/interpolation formula which holds for functions in \( B_{\sigma}^{\infty}(\mathbb{R}) \) (see [15])

\[
h(t) = t \operatorname{sinc}\left(\frac{\sigma t}{\pi}\right) h'(0) + \operatorname{sinc}\left(\frac{\sigma t}{\pi}\right) f(0) + \sum_{k \neq 0} \frac{\sigma t}{k \operatorname{\pi}} \operatorname{sinc}\left(\frac{\sigma t}{\pi} - k\right) h\left(\frac{k \pi}{\sigma}\right), \quad h \in B_{\sigma}^{\infty}(\mathbb{R}),
\]

where convergence is absolute and uniform on compact subsets of \( \mathbb{R} \), we obtain

\[
\Phi(t) = t \operatorname{sinc}\left(\frac{\sigma t}{\pi}\right) \Phi'(0) +
\]
or for \( f \in B_p^q(\mathbb{R}) \), \( 1 < p < \infty \),

\[
\int_{\mathbb{R}^+} f(e^t x)g(x) \frac{dx}{x} = \int_{\mathbb{R}^+} \left[ \sin \left( \frac{\sigma t}{\pi} \right) f(x) + t \sin \left( \frac{\sigma t}{\pi} \right) (x \partial_x f)(x) \right] g(x) \frac{dx}{x} +
\]

\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\sigma t}{k \pi} \sin \left( \frac{\sigma t}{\pi} - k \right) \int_{\mathbb{R}^+} f(e^{k\pi/\sigma} x)g(x) \frac{dx}{x},
\]

where convergence is absolute and uniform on compact subsets of \( \mathbb{R} \). Since the following series converges in the norm of \( X^p(\mathbb{R}^+) \):

\[
\left\| \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\sigma t}{k \pi} \sin \left( \frac{\sigma t}{\pi} - k \right) f(e^{k\pi/\sigma} x) \right\|_{X^p(\mathbb{R}^+)} \leq \|f\|_{X^p(\mathbb{R}^+)} \sum_{k \neq 0, \pi \in \mathbb{N}} \left( \frac{1}{|\sigma t/\pi - k|^p} \right) \left( \frac{1}{|k|^q} \right) < \infty,
\]

we can rewrite (3.3) as

\[
\int_{\mathbb{R}^+} f(e^t x)g(x) \frac{dx}{x} = \int_{\mathbb{R}^+} \left[ \sin \left( \frac{\sigma t}{\pi} \right) f(x) + t \sin \left( \frac{\sigma t}{\pi} \right) (x \partial_x f)(x) \right] g(x) \frac{dx}{x} +
\]

\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\sigma t}{k \pi} \sin \left( \frac{\sigma t}{\pi} - k \right) \int_{\mathbb{R}^+} f(e^{k\pi/\sigma} x)g(x) \frac{dx}{x}.
\]

The last equality holds for any \( g \in X^q(\mathbb{R}^+) \), \( 1/p + 1/q = 1 \), and since for \( 1 < p < \infty \) the space \( X^q(\mathbb{R}^+) \) contains all functionals for \( X^p(\mathbb{R}^+) \), we can conclude that the following equality holds

\[
f(e^t x) = \sin \left( \frac{\sigma t}{\pi} \right) f(x) + t \sin \left( \frac{\sigma t}{\pi} \right) (x \partial_x f)(x) +
\]

\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\sigma t}{k \pi} \sin \left( \frac{\sigma t}{\pi} - k \right) f(e^{k\pi/\sigma} x).
\]

Substituting \( x = 1 \) into (3.3) we obtain

\[
f(e^t) = \sin \left( \frac{\sigma t}{\pi} \right) f(1) + t \sin \left( \frac{\sigma t}{\pi} \right) (\partial_x f)(1) +
\]

\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\sigma t}{k \pi} \sin \left( \frac{\sigma t}{\pi} - k \right) f(e^{k\pi/\sigma}), \quad t \in \mathbb{R}.
\]

For \( \tau = e^t, \quad t = \ln \tau, \ t \in \mathbb{R} \), one has

\[
f(\tau) = \sin \left( \frac{\sigma}{\pi} \ln \tau \right) f(1) + \ln \tau \sin \left( \frac{\sigma}{\pi} \ln \tau \right) (\partial_x f)(1) +
\]

\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} f(e^{k\pi/\sigma}) \frac{\sigma}{k \pi} (\ln \tau) \sin \left( \frac{\sigma}{\pi} \ln \tau - k \right), \quad \tau \in \mathbb{R}^+.
\]
Theorem is proven. □

For every \( f \in B^p_\sigma(\Theta) \), \( g \in X^q(\mathbb{R}_+) \), \( 1/p + 1/q = 1 \), let’s introduce the function \( \Psi \) defined as follows:

\[
\Psi(t) = \frac{1}{t} (\Phi(t) - \Phi(0)) = \int_{\mathbb{R}_+} \frac{f(e^t x) - f(x)}{t} g(x) \frac{dx}{x},
\]

if \( t \neq 0 \) and

\[
\Psi(0) = d \frac{d}{dt} \Phi(t)_{|t=0} = \int_{\mathbb{R}_+} \Theta f(x) g(x) \frac{dx}{x},
\]

if \( t = 0 \).

Lemma 3.6. If \( f \in B^p_\sigma(\Theta) \), \( g \in X^q(\mathbb{R}_+) \), \( 1/p + 1/q = 1 \), then \( \Psi(t) \) defined in (3.16), (3.17) belongs to \( B^r_\sigma(\Theta) \) for any \( r > 1 \).

Proof. We remind [22] that a function \( h(t) = \sum_{k} a_k t^k \) is an entire function of the exponential type \( \sigma \) if and only if the following condition holds

\[
\lim_{k \to \infty} k \sqrt{k} \cdot |a_k| \leq \sigma.
\]

Since \( \Phi \) belongs to classical Bernstein class \( B^\infty_\sigma(\mathbb{R}) \) (see Theorem [22]), it is an entire function of the exponential type \( \sigma \). Thus \( \Phi(t) = \sum_k c_k t^k \) with \( \lim_{k \to \infty} k \sqrt{k} \cdot |c_k| \leq \sigma \).

Now we have that \( \Psi(t) = \sum_k c_k t^{k-1} \), where one clearly has \( \lim_{k \to \infty} k \sqrt{k} \cdot |c_{k+1}| \leq \sigma \), which means that \( \Psi \) is an entire function of the exponential type \( \sigma \). Moreover, for any \( r > 1 \) the function \( \Psi \) is in \( L^r(\mathbb{R}) \) since

\[
|\Psi(t)| = \left| \int_{\mathbb{R}_+} \frac{f(e^t x) - f(x)}{t} g(x) \frac{dx}{x} \right| \leq 2 \frac{\|f\|_{X^p(\mathbb{R}_+)} \cdot \|g\|_{X^q(\mathbb{R}_+)}}{|t|}.
\]

Thus \( \Psi \) belongs to \( B^r_\sigma(\Theta) \) for any \( r > 1 \).

Lemma is proven. □

Theorem 3.7. If \( f \in B^p_\sigma(\Theta) \), \( 1 < p < \infty \), then the following sampling formulas hold for \( \tau \in \mathbb{R}_+ \)

\[
f(\tau) = f(1) + \ln \tau \left( \partial_x f(1) \right) \text{sinc} \left( \frac{\sigma}{\pi} \ln \tau \right) +
\]

\[
\ln \tau \sum_{k \neq 0} \frac{f \left( e^{\frac{k\pi}{\tau}} \right) - f(1)}{\frac{k\pi}{\tau}} \text{sinc} \left( \frac{\sigma}{\pi} \ln \tau - k \right),
\]

(3.18)

where the series converges uniformly on compact subsets of \( \mathbb{R}_+ \).

Proof. Consider the same \( \Psi \) as before. Since \( \Psi \in B^r_\sigma(\Theta) \) it implies (see [14], p.46) the following formula

\[
\Psi(t) = \sum_{k \in \mathbb{Z}} \Psi \left( \frac{k}{\tau} \right) \text{sinc} \left( \frac{\sigma}{\pi} t - k \right),
\]

the series being uniformly convergent on each compact subset of \( \mathbb{R} \).
Thus
\[
\int_{\mathbb{R}^+} \frac{f(x') - f(x)}{t} g(x) \frac{dx}{x} = \int_{\mathbb{R}^+} \Theta f(x) g(x) \frac{dx}{x} \sin\left(\frac{\sigma t}{\pi}\right) + \\
\sum_{k \in \mathbb{Z}\setminus\{0\}} \left( \int_{\mathbb{R}^+} \frac{f(e^{k\pi} x) - f(x)}{\frac{k}{\sigma} \pi} g(x) \frac{dx}{x} \right) \sin\left(\frac{\sigma t}{\pi} - k \right).
\]

The last formula can be rewritten as
\[
\int_{\mathbb{R}^+} \frac{f(x') g(x)}{x} \frac{dx}{x} = \int_{\mathbb{R}^+} f(x) g(x) \frac{dx}{x} + \left( \int_{\mathbb{R}^+} \Theta f(x) g(x) \frac{dx}{x} \right) \sin\left(\frac{\sigma t}{\pi}\right) + \\
t \sum_{k \in \mathbb{Z}\setminus\{0\}} \left( \int_{\mathbb{R}^+} \frac{f(e^{k\pi} x) - f(x)}{\frac{k}{\sigma} \pi} g(x) \frac{dx}{x} \right) \sin\left(\frac{\sigma t}{\pi} - k \right).
\]

Since the series
\[
\sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{f(e^{k\pi} x) - f(x)}{\frac{k}{\sigma} \pi} \sin\left(\frac{\sigma t}{\pi} - k \right),
\]
converges in \(X^p(\mathbb{R}^+)\) we can write the equality
\[
\int_{\mathbb{R}^+} f(x') g(x) \frac{dx}{x} = \\
\int_{\mathbb{R}^+} \left( f(x) + t \Theta f(x) \sin\left(\frac{\sigma t}{\pi}\right) + t \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{f(e^{k\pi} x) - f(x)}{\frac{k}{\sigma} \pi} \sin\left(\frac{\sigma t}{\pi} - k \right) \right) g(x) \frac{dx}{x}.
\]

Because this equality holds true for every \(g \in X^q(\mathbb{R}^+), 1p + 1/q = 1, 1 < p < \infty\), we finally coming to the formula
\[
f(x') = f(x) + \\
(3.19) \quad t(x \partial_x f)(x) \sin\left(\frac{\sigma t}{\pi}\right) + t \sum_{k \neq 0} \frac{f(e^{k\pi} x) - f(x)}{\frac{k}{\sigma} \pi} \sin\left(\frac{\sigma t}{\pi} - k \right).
\]

By setting \(x = 1\) we obtain
\[
f(e^t) = f(1) + \\
(3.20) \quad t(\partial_x f)(1) \sin\left(\frac{\sigma t}{\pi}\right) + t \sum_{k \neq 0} \frac{f(e^{k\pi}) - f(1)}{\frac{k}{\sigma} \pi} \sin\left(\frac{\sigma t}{\pi} - k \right),
\]
and for \(\tau = e^t, t = \ln \tau\), one has (3.7). Theorem is proven.
\[\square\]
4. Two theorems which involve irregular sampling

The following fact was proved by J.R. Higgins in [20].

**Theorem 4.1.** Let \( \{t_k\}_{k \in \mathbb{Z}} \) be a sequence of real numbers such that
\[
\sup_{k \in \mathbb{Z}} |t_k - k| < 1/4.
\]

Define the entire function
\[
G(z) = (z - t_0) \prod_{k \in \mathbb{Z}} \left( 1 - \frac{z}{t_k} \right) \left( 1 - \frac{z}{t_{-k}} \right).
\]

Then for all \( f \in B^2_\pi(\mathbb{R}) \) we have
\[
f(t) = \sum_{k \in \mathbb{Z}} f(t_k) \frac{G(t)}{G'(t_k)(t - t_k)},
\]
uniformly on every compact subset of \( \mathbb{R} \).

As we know (Lemma 3.6) for every \( f \in B^p_\pi(\Theta) \), \( g \in X^q(\mathbb{R}_+), 1/p + 1/q = 1 \), the function \( \Psi \) defined as
\[
\Psi(t) = \int_{\mathbb{R}_+} \frac{f(e^t x) - f(x)}{t} g(x) \frac{dx}{x},
\]
if \( t \neq 0 \) and
\[
\Psi(0) = \int_{\mathbb{R}_+} \Theta f(x) g(x) \frac{dx}{x},
\]
if \( t = 0 \), belongs to \( B^2_\pi(\mathbb{R}) \). Applying to it Theorem 4.1 we obtain the following.

**Theorem 4.2.** Under assumptions and notations of Theorem 4.1, for every \( f \in B^p_\pi(\Theta) \), \( g \in X^q(\mathbb{R}_+), 1/p + 1/q = 1 \), \( 1 < p < \infty \), the following formula holds
\[
\Psi(t) = \sum_{k \in \mathbb{Z}} \Psi(t_k) \frac{G(t)}{G'(t_k)(t - t_k)},
\]
uniformly on every compact subset of \( \mathbb{R} \).

Note, that if the sequence \( \{t_k\} \) does not contain zero then the formula (4.5) takes the form
\[
\int_{\mathbb{R}_+} \frac{f(e^t x) - f(x)}{t} g(x) \frac{dx}{x} = \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}_+} \frac{f(e^t x) - f(x)}{t_k} g(x) \frac{dx}{x} \right) \frac{G(t)}{G'(t_k)(t - t_k)}.
\]
But in the case \( t_0 = 0 \) the formula (4.5) has the form
\[
\int_{\mathbb{R}_+} \frac{f(e^t x) - f(x)}{t} g(x) \frac{dx}{x} = \left( \int_{\mathbb{R}_+} \Theta f(x) g(x) \frac{dx}{x} \right) \frac{G(t)}{G'(0)t} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \int_{\mathbb{R}_+} \frac{f(e^t x) - f(x)}{t_k} g(x) \frac{dx}{x} \right) \frac{G(t)}{G'(t_k)(t - t_k)}.
\]

In the paper by C. Seip [28] the following result can be found.
Theorem 4.3. Under assumptions and notations of Theorem 4.1, for any $0 < \delta < \pi$ and all $f \in B^\infty_{\pi - \delta}(\mathbb{R})$ the following holds true

$$f(t) = \sum_{k \in \mathbb{Z}} f(t_k) \frac{G(k)}{G'(t_k)(t - t_k)},$$

uniformly on all compact subsets of $\mathbb{R}$.

This Theorem together with Theorem 2.2 imply the following theorem.

Theorem 4.4. If $f \in B^p_{\pi - \delta}(\Theta), 0 < \delta < \pi, 1 \leq p \leq \infty$, then under assumptions and notations of Theorem 4.1 the following formula holds uniformly on compact subsets of $\mathbb{C}$

$$\Phi(t) = \sum_{k \in \mathbb{Z}} \Phi(t_k) \frac{G(t)}{G'(t_k)(t - t_k)},$$

or

$$\int_{\mathbb{R}^+} f(e^tx)g(x) \frac{dx}{x} = \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}^+} f(e^{tk}x)g(x) \frac{dx}{x} \right) \frac{G(t)}{G'(tk)(t - tk)},$$

where $g \in X^q(\mathbb{R}^+), 1/p + 1/q = 1, 1 \leq p \leq \infty$.

5. Riesz-Boas interpolation formulas

We introduce the following bounded operators in the spaces $X^p(\mathbb{R}^+), 1 \leq p \leq \infty$.

$$\mathcal{R}^{(2m-1)}(\sigma)f(x) =$$

(5.1) \[
\left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} f \left( e^{\pi(i (k - 1/2)x)} \right), \quad f \in X^p(\mathbb{R}^+), \; \sigma > 0, \; m \in \mathbb{N},
\]

and

$$\mathcal{R}^{(2m)}(\sigma)f(x) =$$

(5.2) \[
\left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} f \left( e^{\pi x} \right), \quad f \in X^p(\mathbb{R}^+), \; \sigma > 0, \; m \in \mathbb{N},
\]

where $A_{m,k}$ and $B_{m,k}$ are defined as

$$A_{m,k} = (-1)^{k+1} \text{sinc}^{(2m-1)} \left( \frac{1}{2} - k \right) =$$

(5.3) \[
\frac{(2m - 1)!}{\pi (k - \frac{1}{2})^{2m}} \sum_{j=0}^{m-1} \frac{(-1)^j}{(2j)!} \left( \pi(k - \frac{1}{2}) \right)^{2j}, \quad m \in \mathbb{N},
\]

for $k \in \mathbb{Z},$

(5.4) \[
B_{m,k} = (-1)^{k+1} \text{sinc}^{(2m)}(-k) = \frac{(2m)!}{\pi k^{2m+1}} \sum_{j=0}^{m-1} \frac{(-1)^j(\pi k)^{2j+1}}{(2j + 1)!}, \quad m \in \mathbb{N},
\]

for $k \in \mathbb{Z} \setminus \{0\},$ and

(5.5) \[
B_{m,0} = (-1)^{m+1} \frac{\pi^{2m}}{2m + 1}, \quad m \in \mathbb{N}.
\]
Both series converge in $X^p(\mathbb{R}_+)$, $1 \leq p \leq \infty$, and their sums are (see [16])
\begin{equation}
(5.6) \quad \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} |A_{m,k}| = \sigma^{2m-1}, \quad \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} |B_{m,k}| = \sigma^{2m}.
\end{equation}

Since $\|f(e^\cdot)\|_{X^p(\mathbb{R}_+)} = \|f\|_{X^p(\mathbb{R}_+)}$ it implies that
\begin{equation}
(5.7) \quad \|R^{(2m-1)}(\sigma)f\|_{X^p(\mathbb{R}_+)} \leq \sigma^{2m-1} \|f\|_{X^p(\mathbb{R}_+)} , \quad f \in X^p(\mathbb{R}_+).
\end{equation}

**Theorem 5.1.** For $f \in X^p(\mathbb{R}_+)$, $1 \leq p \leq \infty$, the next two conditions are equivalent:
\begin{enumerate}
  \item $f$ belongs to $B^p_\sigma(\Theta)$, $\sigma > 0$, $1 \leq p \leq \infty$,
  \item the following Riesz-Boas-type interpolation formulas hold true for $r \in \mathbb{N}$
\end{enumerate}
\begin{equation}
(5.8) \quad \Theta^r f = R^{(r)}(\sigma)f, \quad \text{or explicitly}
\end{equation}
\begin{equation*}
\left(\frac{d}{dx}\right)^{2m-1} f(x) = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} f \left( e^{\frac{\pi k}{2}} x \right),
\end{equation*}
\begin{equation*}
\text{and}
\end{equation*}
\begin{equation*}
\left(\frac{d}{dx}\right)^{2m} f(x) = \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} f \left( e^{\frac{\pi k}{2}} x \right),
\end{equation*}
where each of the serious converges absolutely and uniformly on $\mathbb{R}_+$.

**Proof.** We are proving that (1) $\rightarrow$ (2). According to Theorem [22] if $f \in B^p_\sigma(\Theta)$, $\sigma > 0$, $1 < p < \infty$, then the function $\Phi(t) = \int_{\mathbb{R}_+} f(e^t x)g(x)\frac{dx}{x}$ for any $g \in X^p(\mathbb{R}_+)$, $1/p + 1/q = 1$, belongs to $B^p_\sigma(\mathbb{R})$. Thus by [16] we have
\begin{equation}
\Phi^{(2m-1)}(t) = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} \Phi \left( t + \frac{\pi}{\sigma}(k - 1/2) \right), \quad m \in \mathbb{N},
\end{equation}
\begin{equation}
\Phi^{(2m)}(t) = \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} \Phi \left( t + \frac{\pi k}{\sigma} \right), \quad m \in \mathbb{N}.
\end{equation}
Together with
\begin{equation}
\left(\frac{d}{dt}\right)^k \Phi(t) = \int_{\mathbb{R}_+} \Theta f(e^t x)g(x)\frac{dx}{x},
\end{equation}
it shows
\begin{equation}
\int_{\mathbb{R}_+} \Theta^{2m-1} f(e^t x)g(x)\frac{dx}{x} = \left(\frac{\sigma}{\pi}\right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} \int_{\mathbb{R}_+} f \left( e^{(t+\pi)(k-1/2)} x \right) g(x)\frac{dx}{x}, \quad m \in \mathbb{N},
\end{equation}
and also
\begin{equation}
\int_{\mathbb{R}_+} \Theta^{2m} f(e^t x)g(x)\frac{dx}{x} = \left(\frac{\sigma}{\pi}\right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} \int_{\mathbb{R}_+} f \left( e^{(t+\pi k)} x \right) g(x)\frac{dx}{x}, \quad m \in \mathbb{N}.
Since both series (5) and (5) converge in $X^p(\mathbb{R}_+)$ and the last two equalities hold for any $g \in X^q(\mathbb{R}_+)$, 1/p + 1/q = 1, we obtain the next two formulas

$$\Theta^{2m-1} f(x) = \left( \frac{\sigma}{\pi} \right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} f \left( e^{(t+\pi k-1/2) x} \right), \quad m \in \mathbb{N},$$

$$\Theta^{2m} f(x) = \left( \frac{\sigma}{\pi} \right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} f \left( e^{(t+\pi k) x} \right), \quad m \in \mathbb{N}.$$ 

In turn, when $t = 0$ these formulas become formulas (5.8). The fact that (2) → (1) easily follows from the formulas (5.8) and (5).

Theorem is proved. □

**Corollary 5.1.** If $f$ belongs to $B_p^\sigma(\Theta), 1 < p < \infty$, then for any $\sigma_1 \geq \sigma, \sigma_2 \geq \sigma$ one has

$$R^{(r)}(\sigma_1) f = R^{(r)}(\sigma_2) f, \quad r \in \mathbb{N}.$$ 

Let us introduce the notation

$$R^1(\sigma) = R^{(1)}(\sigma).$$

One has the following "power" formula which follows from the fact that operators $R(\sigma)$ and $\Theta$ commute on any $B_p^\sigma(\Theta)$.

**Corollary 5.2.** For any $r \in \mathbb{N}$ and any $f \in B_p^\sigma(\Theta), 1 < p < \infty$,

$$\Theta^r f = R^{(r)}(\sigma) f = R^r(\sigma) f,$$

where $R^r(\sigma) f = R(\sigma) (\ldots (R(\sigma)) f)$.

Let us introduce the following notations

$$R^{(2m-1)}(\sigma, N) f(x) = \left( \frac{\sigma}{\pi} \right)^{2m-1} \sum_{|k| \leq N} (-1)^{k+1} A_{m,k} f \left( e^{\pi(k-1/2) x} \right),$$

$$R^{(2m)}(\sigma, N) f(x) = \left( \frac{\sigma}{\pi} \right)^{2m} \sum_{|k| \leq N} (-1)^{k+1} B_{m,k} f \left( e^{\pi k x} \right).$$

One obviously has the following set of approximate Riesz-Boas-type formulas.

**Theorem 5.2.** If $f \in B_p^\sigma(\Theta), 1 < p < \infty$, and $r \in \mathbb{N}$ then

$$\Theta^r f \approx R^{(r)}(\sigma, N) f + O(N^{-2}),$$

or explicitly

$$\left( x \frac{d}{dx} \right)^{2m-1} f(x) \approx \left( \frac{\sigma}{\pi} \right)^{2m-1} \sum_{|k| \leq N} (-1)^{k+1} A_{m,k} f \left( e^{\pi(k-1/2) x} \right) + O(N^{-2}),$$

and

$$\left( x \frac{d}{dx} \right)^{2m} f(x) \approx \left( \frac{\sigma}{\pi} \right)^{2m} \sum_{|k| \leq N} (-1)^{k+1} B_{m,k} f \left( e^{\pi k x} \right) + O(N^{-2}).$$
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