Central limit theorem, deformed exponentials and superstatistics

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Abstract

We show that there exists a very natural, superstatistics-linked extension of the central limit theorem (CLT) to deformed exponentials (also called q-Gaussians): This generalization favorably compares with the one provided by S. Umarov and C. Tsallis [arXiv:cond-mat/0703533], since the latter requires a special "q-independence" condition on the data. On the contrary, our CLT proposal applies exactly in the usual conditions in which the classical CLT is used. Moreover, we show that, asymptotically, the q-independence condition is naturally induced by our version of the CLT.

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I. INTRODUCTION

The central limit theorems (CLT) can be ranked among the most important theorems in probability theory and statistics and plays an essential role in several basic and applied disciplines, notably in statistical mechanics. Pioneers like A. de Moivre, P.S. de Laplace, S.D. Poisson, and C.F. Gauss have shown that the Gaussian function is the attractor of independent additive contributions with a finite second variance. Distinguished authors like Chebyshev, Markov, Liapounov, Feller, Lindeberg and Lévy have also made essential contributions to the CLT-theory.

The random variables to which the classical CLT refers are required to be independent. Subsequent efforts along CLT lines have established corresponding theorems for weakly dependent random variables as well (see some pertinent references in [1, 2, 3]). However, the CLT does not hold if correlations between far-ranging random variables are not negligible (see [4]).

Recent developments in statistical mechanics that have attracted the attention of many researchers deal with strongly correlated random variables ([5] and references therein). These correlations do not rapidly decrease with any increasing distance between random variables and are often referred to as global correlations (see [6] for a definition). Is there an attractor that would replace the Gaussians in such a case?

The answer is in the affirmative, as shown in [1, 2, 3], with the deformed or q-Gaussian playing the starring role. It is asserted in [2] that such a theorem cannot be obtained if we rely on classic algebra: it needs a construction based on a special algebra, which is called q-algebra [15]. The goal of this communication is to show that a q-generalization of the central limit theorem becomes indeed possible and in a very simple way without recourse to q-algebra.

A. Systems that are q-distributed

Consider a system $S$ described by a random vector $X$ with $d$—components whose covariance matrix reads

$$K = \langle XX^t \rangle \equiv EXX^t,$$

the superscript $t$ indicating transposition. We say that $X$ is $q$—Gaussian (or deformed Gaussian—) distributed if its probability distribution function writes as described by Eqs. (2)—(3) below.
• in the case $1 < q < \frac{d+4}{d+2}$

$$f_{X,q}(X) = \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1} - \frac{d}{2}\right) |\pi\Lambda|^{1/2}} (1 + X^t \Lambda^{-1} X)^{\frac{1}{2-q}}$$ \hspace{1cm} (2)$$

with matrix $\Lambda$ being related to $K$ in the fashion

$$\Lambda = (m - 2) K.$$ \hspace{1cm} (3)

The number of degrees of freedom $m$ is defined in terms of the dimension $d$ of $X$ as

$$m = \frac{2}{q - 1} - d.$$ \hspace{1cm} (4)

• in the case $q < 1$

$$f_{X,q}(X) = \frac{\Gamma\left(\frac{2-q}{q-1} + \frac{d}{2}\right)}{\Gamma\left(\frac{2-q}{1-q}\right) |\pi\Sigma|^{1/2}} (1 - X^t \Sigma^{-1} X)^{\frac{1}{2-q}}$$ \hspace{1cm} (5)$$

where the matrix $\Sigma$ is related to the covariance matrix via $\Sigma = pK$. We introduce here a parameter $p$ defined as

$$p = \frac{2 - q}{1 - q} + d.$$ \hspace{1cm} (6)

II. THE ROAD TOWARDS A NEW CLT

As stated above, several attempts to generalize the central limit theorem (CLT) have been published recently [1, 2, 3], the aim being to have the Gaussian attractor replaced by the q-Gaussian attractor. We recall here the standard multivariate version of the CLT.

**Theorem 1.** Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random vectors in $\mathbb{R}^d$ with expectation $E[X_i] = 0$ and covariance matrix $E[X_iX_i^t] = K$ and let

$$W_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i.$$ \hspace{1cm} (7)$$

Then $W_n$ converges weakly to a Gaussian vector $W$ with covariance matrix $K$, or equivalently stated [16]

$$\forall t \in \mathbb{R}^d, \lim_{n \to +\infty} \Pr\{W_n \leq t\} = \Phi_1(t) = \frac{1}{|2\pi K|^{1/2}} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_d} e^{-\frac{X^t K^{-1} X}{2}} dX.$$ \hspace{1cm} (8)
The basic idea leading towards non-conventional CLTs is to find conditions under which convergence to the usual normal cumulative density function (cdf) $\Phi_1$ with covariance matrix $K$ can be replaced by convergence to a $q-$Gaussian cdf

$$\Phi_q(t) = \int_{-\infty}^{t_1} \ldots \int_{-\infty}^{t_d} f_{X,q}(x) \, dx_1 \ldots dx_d$$

with $q > 1$, $f_{X,q}$ as defined in (2) and parameter $m$ defined by (4) or, for $q < 1$,

$$\Phi_q(t) = \int_{-\infty}^{t_1} \ldots \int_{-\infty}^{t_d} f_{X,q}(x) \, dx_1 \ldots dx_d$$

with $f_{X,q}$ as defined in (5) and parameter $p$ defined by (6). We note that both cases $m \to +\infty$ and $p \to +\infty$ correspond to convergence $q \to 1$ to the Gaussian case.

In two recent contributions, S. Umarov and C. Tsallis highlight the existence of such a central limit theorem, in the univariate [2] and multivariate [1] case, provided there exists a certain kind of dependence, called $q-$independence, between random vectors $X_i$. This $q-$independence condition is expressed in terms of the notions of $q-$Fourier transform $F_q$ and of $q-$product $\otimes_q$[1, 2] as

$$F_q[X_1 + X_2] = F_q[X_1] \otimes_q F_q[X_2]$$

which reduces to conventional independence for $q = 1$.

We recall that the $q-$product of $x \in \mathbb{C}$ and $y \in \mathbb{C}$ is

$$x \otimes_q y = (x^{1-q} + y^{1-q} - 1)^{\frac{1}{1-q}}$$

and the $q-$Fourier transform of a function $f(x), x \in \mathbb{R}^d$, is

$$F_q[f](\xi) = \int_{\mathbb{R}^d} (f^{1-q}(x) + (1 - q) \, ix^t \xi)^{\frac{1}{1-q}} \, dx.$$ 

However, this approach suffers from the lack of physical interpretation for such special dependence; moreover, the $q-$Fourier transform is a nonlinear transform (unless $q = 1$) which makes its use rather difficult.

Another approach, as described in [8], consists in keeping the independence assumption between vectors $X_i$ while replacing the $n$ terms in (7) by a random number $N(n)$ of terms. That is, if the random variable $N(n)$ follows a negative binomial distribution so as to diverge in a specified way, then convergence to a $q-$Gaussian distribution occurs whenever convergence occurs in the usual sense.
In the present contribution we show that there exists a much more natural way to extend the CLT, based on the Beck-Cohen notion of superstatistics \[9\] (see the discussion in \[10\]). Our starting point is the same as that in Umarov’s approach (i.e., assuming some kind of dependence between the summed terms). However, the manner in which we introduce this dependence among data is a natural one that can be interpreted in the physical framework of the Cohen-Beck physics (see \[14\] for an interesting overview).

III. PRESENT RESULTS

Our present results can be conveniently condensed by stating two theorems, according to the value of parameter \(q\). The essential idea is that of suitably introducing a chi-distributed random variable \(a\) that is independent (case \(q > 1\)) or dependent (case \(q < 1\)) of the data \(X_i\), and then constructing the following scale mixture (typical of superstatistics \[10\])

\[
Z_n = \frac{1}{a\sqrt{n}} \sum_{i=1}^{n} X_i. \tag{11}
\]

A. The case \(q > 1\)

**Theorem 2.** If \(X_1, X_2, \ldots\) are i.i.d. random vectors in \(\mathbb{R}^d\) with zero mean and covariance matrix \(K\), and if \(a\) denotes a random variable chi-distributed with \(m\) degrees of freedom, scale parameter \((m - 2)^{-1/2}\), and chosen independent of the \(X_i\), then random vectors

\[
Z_n = \frac{1}{a\sqrt{n}} \sum_{i=1}^{n} X_i \tag{12}
\]

converge weakly to a multivariate \(q\)-Gaussian vector \(Z\) with covariance matrix \(K\). Equivalently stated:

\[
\forall t \in \mathbb{R}^d, \lim_{n \to +\infty} \Pr \{Z_n \leq t\} = \Phi_q (t); \tag{13}
\]

with cdf \(\Phi_q (t)\) defined as in \[9\]. Moreover,

\[
q = \frac{m + d + 2}{m + d} > 1. \tag{14}
\]

**Proof.** First we note that the \(\chi\)-density with \(m\) degrees of freedom and scale parameter \(\frac{1}{\sqrt{m-2}}\) is

\[
f_a (a) = \frac{2^{1-\frac{m}{2}} (m - 2)^{\frac{m}{2}}}{\Gamma (\frac{m}{2})} a^{m-1} e^{-\frac{a^2 (m-2)}{2}}. \]
Now, by the multivariate central limit theorem above \[17\]
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \Rightarrow N
\]
where \( N \) is a normal vector in \( \mathbb{R}^d \) with covariance matrix \( K \). Applying from \[11\] its result [Th. 2.8] we deduce that
\[
Z_n \Rightarrow \frac{N}{a}
\]
where \( \frac{N}{a} \) follows a q-Gaussian distribution with covariance matrix \( K \) and parameter \( q \) defined by \[4\].

\[\]

B. The case \( q < 1 \)

The extension of theorem \[2\] to the case \( q < 1 \) proceeds as follows.

**Theorem 3.** If \( X_1, X_2, \ldots \) are i.i.d. random vectors in \( \mathbb{R}^d \) with zero mean and covariance matrix \( K \), and if \( a \) is a random variable independent of the \( X_i \) that is chi-distributed with \( m \) degrees of freedom and scale parameter \( \sqrt{m - 2} \), then the random vectors
\[
Y_n = \frac{1}{b \sqrt{n}} \sum_{i=1}^{n} X_i
\]
with
\[
b = \sqrt{a^2 + \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \right)^t \Lambda^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \right)}
\]
converge weakly to a multivariate q-Gaussian vector \( Y \) with covariance matrix \( K \) and distribution function given by \[9\]. Moreover,
\[
q = \frac{m - 4}{m - 2} < 1.
\]

**Proof.** If \( Z \) has a characteristic distribution function (cdf) given by \[9\], then \[8\]
\[
Y = \phi(Z) = \frac{Z}{\sqrt{1 + Z^t \Lambda^{-1} Z}}
\]
has cdf given by \[10\]. Since the function \( \phi = \mathbb{R}^d \to \{ Y \in \mathbb{R}^d | Y^t \Lambda^{-1} Y \leq 1 \} \) is continuous, the desired result is deduced by application of the continuous mapping theorem (see from \[12\] its Theorem 2.3, p.7).
Remark 1. We note that $Y_n$ in (15) is a normalized version of $Z_n$ in (12); however, the fluctuation term $a$ is replaced by a fluctuation term

$$b = \sqrt{a^2 + \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \right)^t \Lambda^{-1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \right)}$$

that involves the value of the sum itself - and thus is not independent of this sum anymore. Thus the case $q < 1$ can be considered as a fluctuating version of the usual CLT for which the fluctuation depends of the state of the system. Moreover, it is clear that as $n$ increases, the distribution of the fluctuation $b$ gets closer to a chi distribution with $m + d$ degrees of freedom.

C. Link with $q$–independence

Although the extension of the CLT proposed above differs from the ones developed in [1], a link can be established between both approaches for large values of $n$ and for $q > 1$ as follows. Note that we assume $q > 1$ in the rest of the paper.

Theorem 4. (linking theorem) Assume $1 < q < 1 + \frac{2}{d}$. Consider $n = n_0 + n_1$ together with the division of sum $Z_n$ in (12) into two parts as

$$Z_n = \frac{1}{a\sqrt{n}} \left( \sum_{i=1}^{n_0} X_i + \sum_{i=n_0+1}^{n} X_i \right) = Z_n^{(1)} + Z_n^{(2)}.$$ (18)

Assume that the characteristic function $\phi$ of $X_i$ is such that $\int_{\mathbb{R}^d} |\phi|^\nu \, dt < \infty$ for some $\nu \geq 1$, and that data $X_i$ are symmetric ($X_i$ and $-X_i$ have the same distribution). Then random vectors $Z_n^{(1)}$ and $Z_n^{(2)}$ are asymptotically $q$–independent in the sense that

$$\forall \epsilon > 0, \exists N \text{ such that } n_0 > N, n_1 > N \Rightarrow ||F_q[Z_n^{(1)} + Z_n^{(2)}] - F_q[Z_n^{(1)}] \otimes F_q[Z_n^{(2)}]||_\infty < \epsilon$$

with $q_1 = z(q) = \frac{2q+d(1-q)}{2+d(1-q)}$.

For didactic reasons we postpone the proof of this result until next Section. We deduce from it that, asymptotically, the CLT theorem (2) exactly generates the $q$–independence condition required for application of the particular CLT version proposed in [1, 2].
IV. PROOF OF THE LINKING THEOREM

A. Introduction

In order to simplify the proof we will assume that vectors $X_i$ verify a stronger version of the CLT than the one stated in theorem 1, namely the CLT in total variation. Now, the total variation divergence between two probability densities $f$ and $g$ is

$$d_{TV}(f,g) = \frac{1}{2} \int_{\mathbb{R}^d} |f - g|.$$  (19)

If $U$ and $V$ are random vectors distributed according to $f$ and $g$ respectively, we will denote

$$d_{TV}(U,V) = d_{TV}(f,g).$$

The total variation version of the CLT writes as follows (see [12] Th. 2.31.)

**Theorem 5.** (CLT in total variation) Assume that $X_1, X_2, \ldots$ are i.i.d random vectors of $\mathbb{R}^d$ with zero expectation, covariance matrix $K$ and characteristic function $\phi$ such that $\int |\phi|^{\nu} dt < \infty$ for some $\nu \geq 1$. If $W_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ and $W$ is a normal vector in $\mathbb{R}^d$ with covariance matrix $K$ then

$$\lim_{n \to +\infty} d_{TV}(W_n, W) = 0.$$  (20)

Let us introduce the following notations: $\tilde{Z}_n$ denotes a version of sum (12) where all $X_i$ are replaced by i.i.d. Gaussian vectors $N_i \in \mathbb{R}^d$ with covariance matrix $K$, i.e.,

$$\tilde{Z}_n = \frac{1}{a \sqrt{n}} \left( \sum_{i=1}^{n_q} N_i + \sum_{i=n_q+1}^{n} N_i \right) = \tilde{Z}_n^{(1)} + \tilde{Z}_n^{(2)}$$

The proof of theorem is based on the fact that vectors $\tilde{Z}_n^{(1)}$ and $\tilde{Z}_n^{(2)}$ are exactly $q$–independent (as seen in subsection IV B below). Since $n$ is large, according to the above total variations theorem, $Z_n^{(1)}$ and $Z_n^{(2)}$ are close to their $q$–Gaussian counterparts $\tilde{Z}_n^{(1)}$ and $\tilde{Z}_n^{(2)}$, respectively (see Lemma IV B below). It remains to check that closeness between these vectors can be stated in terms of their $q$–transforms. We proceed in five steps, that invoke technical lemmas that are the subject of Subsection C below. These steps are:

- step 1: components $\tilde{Z}_n^{(1)}$ and $\tilde{Z}_n^{(2)}$ are exactly $q$–independent, as is proved in Thm. of subsection IV B below.
• step 2: let us fix $\epsilon > 0$, and write
\[
\begin{align*}
\| F_q[Z_n^{(1)} + Z_n^{(2)}] - F_q[Z_n^{(1)}] \otimes_{q_1} F_q[Z_n^{(2)}] \|_{\infty} \\
\leq \| F_q[Z_n^{(1)} + Z_n^{(2)}] - F_q[Z_n^{(1)} + \tilde{Z}_n^{(2)}] \|_{\infty} \\
+ \| F_q[\tilde{Z}_n^{(1)}] \otimes_{q_1} F_q[\tilde{Z}_n^{(2)}] - F_q[Z_n^{(1)}] \otimes_{q_1} F_q[Z_n^{(2)}] \|_{\infty}
\end{align*}
\]

• step 3: the first term $\| F_q[Z_n^{(1)} + Z_n^{(2)}] - F_q[\tilde{Z}_n^{(1)} + \tilde{Z}_n^{(2)}] \|_{\infty} = \| F_q[Z_n] - F_q[\tilde{Z}_n] \|_{\infty}$ can be bounded as follows
\[
\| F_q[Z_n] - F_q[\tilde{Z}_n] \|_{\infty} \leq 2d_{TV}(Z_n, \tilde{Z}_n) \leq 2d_{TV}(X_n, \tilde{X}_n)
\]
where the first inequality follows from Lemma 3 and the second one from Lemma 1 below. Thus a value $N_1$ can be chosen so that $n_0 > N_1$ and $n_1 > N_1$ ensure that this term is smaller than $\frac{\epsilon}{2}$.

• step 4: the second term $\| F_q[\tilde{Z}_n^{(1)}] \otimes_{q_1} F_q[\tilde{Z}_n^{(2)}] - F_q[Z_n^{(1)}] \otimes_{q_1} F_q[Z_n^{(2)}] \|_{\infty}$ can be bounded by applying Lemma 4 for a large enough value of $n = n_0 + n_1$, say $n > N_2$, we have
\[
\begin{align*}
\| F_q[\tilde{Z}_n^{(1)}] \otimes_{q_1} F_q[\tilde{Z}_n^{(2)}] - F_q[Z_n^{(1)}] \otimes_{q_1} F_q[Z_n^{(2)}] \|_{\infty} &\leq 2d_{TV}(Z_n^{(1)}, \tilde{Z}_n^{(1)}) + 2d_{TV}(Z_n^{(2)}, \tilde{Z}_n^{(2)}) \\
\end{align*}
\]
Finally, from the total variation CLT, there exists a value $N_3$ such that $n_0 > N_3$ and $n_1 > N_3$ implies that each of both total variation divergences is smaller than $\frac{\epsilon}{4}$.

• step 5: The consideration of $N = \max(N_1, N_2, N_3)$ is then seen to prove the linking theorem.

We turn now our attention to those results that we have used in this proof.

B. Components of $q$–Gaussian vectors are $q$–independent

We first begin to check that “sub-vectors” extracted from $q$–Gaussian vectors are exactly $q$–independent; this results is obvious from the fact that, by the CLT given in [1] (Thm. 4.1), these sub-vectors can be considered as limit cases of sequences of $q$–independent sequences. However, the mathematical verification of this property is of an instructive nature and we proceed to give it. For readability, we will say that $X \sim (q, d)$ if $X$ is a $q$–Gaussian vector of dimension $d$ and nonextensivity parameter $q$. 

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Theorem 6. If $1 < q_0 < 1 + \frac{2}{a}$ and vector $X = [X_1^t, X_2^t]^t \sim (q_0, 2d)$ with parameter $q_0 > 1$ then vectors $X_1 \sim (q, d)$ and $X_2 \sim (q, d)$ and they are $q$–independent:

$$F_q[X_1 + X_2] = F_q[X_1] \otimes_{q_1} F_q[X_2]$$  

with $q = z(q_0) = \frac{2q_0 + d(1-q_0)}{2 + d(1-q_0)} > 1$ and $q_1 = z(q) > 1$.

Proof. Since $X_1 \sim (q, d)$, we know from the Corollary 2.3 of [1] that $F_q[X_1] \sim (q_1, d)$. Moreover, since $X_1$ and $X_2$ are components of the same $q$–Gaussian vector, from [8] we deduce that $X_1 + X_2 \sim (q, d)$ so that $F_q[X_1 + X_2] \sim (q_1, d)$. Finally, it is easy to check that since $F_q[X_1] \sim (q_1, d)$ and $F_q[X_2] \sim (q_1, d)$ then $F_q[X_1] \otimes_{q_1} F_q[X_2] \sim (q_1, d)$. The fact that both terms have same covariance matrices is straightforward, what proves the result.

We note that $q$–correlation (21) corresponds to $q$–independence of the third kind as listed in Table 1 of [1]. We pass now to the consideration of the four Lemmas invoked in the proof of the linking theorem.

C. Technical lemmas

As we are concerned with scale mixtures of Gaussian vectors, we need the following lemma.

Lemma 1. If $U$ and $V$ are random vectors in $\mathbb{R}^d$ and $a$ is a random variable independent of $U$ and $V$ then

$$d_{TV} \left( \frac{U}{a}, \frac{V}{a} \right) \leq d_{TV} (U, V).$$  

(22)

Proof. The distributions of scale mixtures $U/a$ and $V/a$ write, in terms of the distributions of $U$ and of $V$, in the fashion

$$f_{U/a} (x) = \int_{\mathbb{R}^+} \frac{1}{a^d} f_a (a) f_U \left( \frac{x}{a} \right) da, \quad g_{V/a} (x) = \int_{\mathbb{R}^+} \frac{1}{a^d} f_a (a) f_V \left( \frac{x}{a} \right) da.$$  

(23)
It thus follows that
\[
\begin{align*}
d_{TV} \left( \frac{U}{a}, \frac{V}{a} \right) &= \frac{1}{2} \left( \int_{\mathbb{R}^d} |f_{U/a}(x) - f_{V/a}(x)| \, dx \right) \\
&= \frac{1}{2} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^+} \frac{1}{a^d} f_a(a) \left( f_U \left( \frac{x}{a} \right) - f_V \left( \frac{x}{a} \right) \right) \, da \, dx \right) \\
&\leq \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^+} \frac{1}{a^d} f_a(a) |f_U \left( \frac{x}{a} \right) - f_V \left( \frac{x}{a} \right)| \, da \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^+} f_a(a) \, da \int_{\mathbb{R}^d} |f_U(z) - f_V(z)| \, dz \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \left| f_U - f_V \right| = d_{TV}(U, V) .
\end{align*}
\]

We also needed above the following

**Lemma 2.** For \( q > 1 \) and \( \Re(z) \geq 0 \), the function

\[
\psi_{q,z} : \mathbb{R}^+ \rightarrow \mathbb{C} \\
\left( x \right) \mapsto (x^{1-q} + z)^{\frac{1}{1-q}}
\]

is a Lipschitz function with unit constant:

\[
|\psi_{q,z}(x_1) - \psi_{q,z}(x_0)| \leq |x_1 - x_0| ,
\]

**Proof.** We have

\[
|\psi_{q,z}(x_1) - \psi_{q,z}(x_0)| \leq \sup_{x_0 \leq x \leq x_1} |\psi'_{q,z}(x)||x_1 - x_0| ,
\]

where

\[
\psi'_{q,z}(x) = \frac{1}{\left(1 + z|x^{q-1}\right)^{\frac{1}{q-1}}},
\]

with \( \frac{q}{q-1} > 0 \), so that, since \( x > 0 \) and \( \Re(z) \geq 0 \),

\[
|\psi'_{q,z}(x)| = \frac{1}{|1 + z|x^{q-1}|^{\frac{1}{q-1}}} \leq 1 .
\]

Two straightforward consequences of such inequality are the following lemmas, that we have also used above.
Lemma 3. For any random vectors $U$ and $V$, if $q \geq 1$, the following inequality holds

$$\|F_q[U] - F_q[V]\|_\infty \leq 2d_{TV}(U, V).$$  \hspace{1cm} (28)

Proof. This result is a straightforward consequence of inequality (34) of reference [1]. However, an elementary proof writes as follows: denote $f_U$ and $f_V$ the respective probability densities of $U$ and $V$. Then, $\forall \xi \in \mathbb{R}^d$,

$$|F_q[U](\xi) - F_q[V](\xi)| \leq \int_{\mathbb{R}^d} \left| (f_U^{1-q}(x) + (1-q)ix^t\xi)\frac{1}{1-q} - (f_V^{1-q}(x) + (1-q)ix^t\xi)\frac{1}{1-q} \right| dx$$

As $\Re((1-q)ix^t\xi) = 0$ and $f_U \geq 0$, by lemma 2 the integrand is bounded by $|f_U(x) - f_V(x)|$. Since this holds $\forall \xi \in \mathbb{R}^d$, the desired result follows.

We remark here that inequality (28) is a simple generalization of the well-known $q = 1$ case, in which $F_{q=1}$ corresponds to the classical Fourier transform. Thus a well-known result of the Fourier theory is reproduced, namely

$$\|F_1[U] - F_1[V]\|_\infty \leq 2d_{TV}(U, V).$$

As another consequence of lemma 2 we have

Lemma 4. For notational simplicity, let us denote as $Z_1 = Z_1^{(1)}$, $Z_2 = Z_2^{(2)}$, $\bar{Z}_1 = \bar{Z}_1^{(1)}$ and $\bar{Z}_2 = \bar{Z}_2^{(2)}$ those random vectors defined in part IV.A. Then, for $n$ large enough,

$$\|F_q[Z_1](\xi) \otimes_{q_1} F_q[Z_2](\xi) - F_q[\bar{Z}_1](\xi) \otimes_{q_1} F_q[\bar{Z}_2](\xi)\|_\infty \leq 2d_{TV}(Z_1, \bar{Z}_1) + 2d_{TV}(Z_2, \bar{Z}_2).$$

Proof. For any $\xi \in \mathbb{R}^d$,

$$|F_q[Z_1](\xi) \otimes_{q_1} F_q[Z_2](\xi) - F_q[\bar{Z}_1](\xi) \otimes_{q_1} F_q[\bar{Z}_2](\xi)|$$

$$\leq |F_q[Z_1](\xi) \otimes_{q_1} F_q[Z_2](\xi) - F_q[\bar{Z}_1](\xi) \otimes_{q_1} F_q[Z_2](\xi)|$$

$$+ |F_q[\bar{Z}_1](\xi) \otimes_{q_1} F_q[Z_2](\xi) - F_q[\bar{Z}_1](\xi) \otimes_{q_1} F_q[\bar{Z}_2](\xi)|$$

$$= |\psi_{q_1,F_q^{1-q_1}[Z_2](\xi)-1}(F_q[Z_1](\xi)) - \psi_{q_1,F_q^{1-q_1}[Z_2](\xi)-1}(F_q[\bar{Z}_1](\xi))|$$

$$+ |\psi_{q_1,F_q^{1-q_1}[\bar{Z}_1](\xi)-1}(F_q[Z_2](\xi)) - \psi_{q_1,F_q^{1-q_1}[\bar{Z}_1](\xi)-1}(F_q[\bar{Z}_2](\xi))|$$

Since $\bar{Z}_2$ is $q$-Gaussian, and since $1 < q < 1 + \frac{2}{n}$, there exists an $\alpha_2 \geq 0$ (as given in equation (15) of reference [1]) such that $F_q^{1-q_1}[\bar{Z}_2](\xi) - 1 = \alpha_2(q_1 - 1)|\xi|^2$ so that, since $q_1 >}$
0, it follows that \( F_{q}^{-q_{1}}[\tilde{Z}_{2}](\xi) \geq 1 \). From the CLT in total variation, we can choose \( n \) large enough so that \( d_{TV}(F_{q}[Z_{2}], F_{q}[\tilde{Z}_{2}]) \) is arbitrarily small, which in turns implies, by Lemma 3 that \( |F_{q}[Z_{2}](\xi) - F_{q}[\tilde{Z}_{2}](\xi)| \) is arbitrarily small as well. By continuity of the function \( x \mapsto x^{1-q_{1}} - 1 \), and since \( F_{q}[Z_{2}] \) is real-valued by the symmetry of the data, this ensures that \( F_{q}^{-q_{1}}[Z_{2}](\xi) - 1 \geq 0 \). Thus, the first term can be bounded using lemma 2 in the fashion

\[
|\psi_{q_{1},F_{q}^{-q_{1}}[Z_{2}](\xi)}^{-1}(F_{q}[Z_{1}](\xi)) - \psi_{q_{1},F_{q}^{-q_{1}}[Z_{2}](\xi)}^{-1}(F_{q}[\tilde{Z}_{1}](\xi))| \leq |F_{q}[\tilde{Z}_{1}](\xi) - F_{q}[Z_{1}](\xi)|.
\]

Accordingly, since \( \tilde{Z}_{1} \) is \( q \)-Gaussian, there exists \( \alpha_{1} \geq 0 \) such that \( F_{q}^{-q_{1}}[\tilde{Z}_{1}](\xi) - 1 = \alpha_{1}(q_{1} - 1)\xi^{2} \), hence \( F_{q}^{-q_{1}}[\tilde{Z}_{1}](\xi) \geq 1 \). Recourse again to lemma 2 yields

\[
|\psi_{q_{1},F_{q}^{-q_{1}}[\tilde{Z}_{1}](\xi)}^{-1}(F_{q}[Z_{1}](\xi)) - \psi_{q_{1},F_{q}^{-q_{1}}[\tilde{Z}_{1}](\xi)}^{-1}(F_{q}[\tilde{Z}_{2}](\xi))| \leq |F_{q}[\tilde{Z}_{2}](\xi) - F_{q}[Z_{2}](\xi)|.
\]

Applying now lemma 3 to each of both terms above yields

\[
|F_{q}[Z_{1}](\xi) \otimes_{q_{1}} F_{q}[Z_{2}](\xi) - F_{q}[\tilde{Z}_{1}](\xi) \otimes_{q_{1}} F_{q}[\tilde{Z}_{2}](\xi)| \leq 2d_{TV}(Z_{1}, \tilde{Z}_{1}) + 2d_{TV}(Z_{2}, \tilde{Z}_{2}).
\]

As this holds for any value of \( \xi \in \mathbb{C} \), the result follows.

\[ \square \]

V. CONCLUSIONS

We have here dealt with non-conventional central limit theorems, whose attractor is a deformed or \( q \)-Gaussian. Based on the Beck-Cohen notion of superstatistics [9], with scale mixtures relating random variables à la Eq. (11), it has been shown that there exists a very natural extension of the central limit theorem to these deformed exponentials that quite favorably compares with the one provided by S. Umarov and C. Tsallis [arXiv:cond-mat/0703533]. This is so because the latter requires a special “\( q \)-independence condition on the data”. On the contrary, our CLT proposal applies exactly in the usual conditions in which the classical CLT is used. However, links between ours and the Umarov-Tsallis treatment have also been established, which makes the here reported CLT a hopefully convenient tool for understanding the intricacies of the physical processes described by power-laws probability distributions, as exemplified, for instance, by the examples reported in [5] (and references therein).

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[5] Among literally hundreds of references see, for instance, M. Gell-Mann and C. Tsallis, Eds. Nonextensive Entropy: Interdisciplinary applications (Oxford University Press, Oxford, 2004); A. Plastino and A. R. Plastino, Braz. J. of Phys., 29 (1999) 50; C. Vignat, A. Plastino, Phys. Lett. A 365 (2007) 370.

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[15] This should not be confused with quantum algebra as defined in [13]

[16] Note that inequality between vectors $W_n \leq t$ denotes the set of $d$ component-wise inequalities
\[ \{ W_n(k) \leq t_k; 1 \leq k \leq d \}. \]

[17] note that, below, symbol $\Rightarrow$ denotes weak convergence