Is nonsymmetric gravity related to string theory?

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Abstract

In this work we raise the question whether nonsymmetric gravity and string theory are related. We start making the observation, that the gravitational field $g_{\mu \nu}$ and the nonsymmetric gauge field $A_{\mu \nu}$ arising in the low energy limit in the string theory are exactly the same two basic fields used in four dimensions in nonsymmetric gravity. We argue, that this connection between nonsymmetric gravity and string theory at the level of the gauge fields $g_{\mu \nu}$ and $A_{\mu \nu}$ is not, however, reflected at the level of the corresponding associated actions. In an effort to find a connection between such an actions we discover a new gravitational action, which suggests an alternative version of the bosonic string in which the target and the world-volume metrics are unified.

PACS numbers: 04.20.Fy, 04.50.+h, 11.25.-w

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In this paper we make, first, a number of observations, which may suggest a relation between nonsymmetric gravitational theory [NGT] [1] and string theory (ST) [2]. Motivated by such an observations we compare the action of NGT with the low energy action of the ST. We find, that such two actions are, in fact, unrelated. In this process we learn, that it becomes very important to unify the target space-time metric $g_{\mu\nu}$ and the antisymmetric gauge field $A_{\mu\nu}$ with the world-volume metric $\sqrt{-g} g^{ab}$ in just one object $g^{ab}_{\mu\nu}$: the unify metric $g^{ab}_{\mu\nu}$, that should be considered as a very general object, and that only as a particular case $g^{ab}_{\mu\nu}$ should be written as $g^{ab}_{\mu\nu} = \sqrt{-g} g^{ab}_{\mu\nu} g^{\mu\nu} + \epsilon^{ab}_{\mu\nu} A_{\mu\nu}$, where $\epsilon^{ab}$ is the Levi-Civita tensor in (1+1) dimensions. This particular form of the metric $g^{ab}_{\mu\nu}$ is in agreement with the suggestion arised in the analysis of the present work.

Another interesting aspect of the present work is that the Euclidean analysis of the problem at hand leads us to discover a different way to understand the complex numbers [3] (see appendix). Moreover, in this paper, assuming the metric $g^{ab}_{\mu\nu}$ as a basic fundamental object in the ST we conjecture an action, which presumably should be obtained in the low energy limit of a ST based on the metric $g^{ab}_{\mu\nu}$. Such an action is not of the nonsymmetric gravity type and reduces to the low energy limit to an action of the ordinary ST.

Let us start making the following observations. In ST two of the massless modes, besides the dilaton, are the symmetric second-rank tensor $g_{\mu\nu} = g_{\nu\mu}$, representing the gravitational field potential and the antisymmetric second-rank tensor $A_{\mu\nu} = -A_{\nu\mu}$. It turns out, that exactly the same kind of tensor are two basic fields, used as starting point in the theory called “nonsymmetric gravitational theory”. This observation raises the question whether the low energy limit in ST is related to NGT. An answer to this question may be of physical interest, because if these two theories are not, in fact, related, then nonsymmetric gravity may suggest an interesting alternative theory to ST, which low energy behavior leads precisely to nonsymmetric gravity. Otherwise, it may be useful to understand how nonsymmetric gravity arises from ordinary ST.
Another suggestion, that a relation may exist between NGT and ST comes from the work due to Sabbata and Gasperini \[4\], in which a formal coincidence is shown in the linear approximation between the Hermitian theory of gravity (an alternative formulation of NGT) and N=1 supergravity \[5\], which at the same time arises in the low energy limit of superstrings.

Before we proceed further let us make some general remarks about the NGT. Its origin may be traced back to the work of Einstein and Straus \[6\]. These authors related the gauge field $A_{\mu\nu}$ with the electromagnetic field tensor. This relation, however, was not very successful. Since the work of Einstein-Straus other alternative formulation have been proposed \[7\]. The central idea in these alternatives is to consider the gauge field $A_{\mu\nu}$ not as associated to the electromagnetic field, but rather to the gravitational field itself. Here we will consider the most recent version of the NGT \[1\], which presumably in the linear approximation yields a theory free of ghost poles and tachyons and avoids the formation of black holes \[8\] (see ref. \[9\] for some controversy about this point). So, NGT has a number of important features, which make it a very interesting theory of gravity by itself.

The starting point in the NGT is the decomposition of the fundamental tensor $g_{\mu\nu}$:

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]},$$  \hspace{1cm} (1)

where

$$g_{(\mu\nu)} = \frac{1}{2} (g_{\mu\nu} + g_{\nu\mu}), \quad g_{[\mu\nu]} = \frac{1}{2} (g_{\mu\nu} - g_{\nu\mu}).$$

The action (for pure gravity) is assumed to be \[1\]

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu},$$  \hspace{1cm} (2)

with

$$R_{\mu\nu} = W_{\mu\nu,\beta} - \frac{1}{2} (W_{\mu\nu,\beta} + W_{\nu\beta,\mu}) - W_{\alpha\nu} W_{\mu\beta} + W_{\alpha\beta} W_{\mu\nu},$$  \hspace{1cm} (3)

where
\[ W_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \frac{2}{3} \delta_{\mu}^\lambda W_{\nu}. \] (4)

In (4) \( \Gamma_{\mu\nu}^\lambda \) is the connection with decomposition

\[ \Gamma_{\mu\nu}^\lambda = \Gamma_{(\mu\nu)}^\lambda + \Gamma_{[\mu\nu]}^\lambda. \] (5)

On the other hand, the low energy limit of the ST leads to the action

\[ S = \int d^{26} x \sqrt{-G} \left( \mathcal{R} + \frac{1}{12} F_{\mu\nu\alpha} F^{\mu\nu\alpha} \right), \] (6)

where \( G_{\mu\nu} = G_{\nu\mu} \) is the metric in 26 dimensions; \( \mathcal{R} \) is the scalar curvature Riemann tensor defined in terms of \( G_{\mu\nu} \), and

\[ F_{\mu\nu\alpha} = \partial_{[\mu} A_{\nu\alpha]}, \] (7)

is the completely antisymmetric field strength defined in terms of the gauge field \( A_{\mu\nu} = -A_{\nu\mu} \). In (6) we should also consider the dilaton contribution \( (\nabla \phi)^2 \), but for the purpose of this work we dropped from (6). It is important to note, that in order to derive (6) in the low energy limit of ST it is assumed, that \( G_{\mu\nu} \) and \( A_{\mu\nu} \) are slow varying background fields.

The action (6) may be obtained as the low energy limit of the quantum theory associated to the string action

\[ S = \frac{1}{2} \int d^2 \xi \left( \sqrt{-g_{ab}} G_{\mu\nu} \partial_a x^\mu \partial_b x^\nu + \epsilon^{ab} A_{\mu\nu} \partial_a x^\mu \partial_b x^\nu \right), \] (8)

where \( g_{ab} \) is the metric on the world-surface swept out by the string in its evolution.

If we want to find a relation between NGT and ST we need to compare the two actions (4) and (6). We first notice, that in both cases the basic fields are the symmetric gauge field \( g_{(\mu\nu)} \) in the case of NGT and \( G_{\mu\nu} \) in the case of ST, and antisymmetric gauge field \( g_{[\mu\nu]} \) in the case of NGT and \( A_{\mu\nu} \) in the case of ST. In fact, we can make the comparison more directly, if we use the following definitions:

\[ g_{(\mu\nu)} = G_{\mu\nu}, \] (9)

\[ g_{[\mu\nu]} = A_{\mu\nu}. \] (10)
So, at the level of the gauge fields NGT and ST differ only in the dimensionality of the space-time: (3+1)-dimensions in the case of NGT and (25+1)-dimensions in the case of the bosonic ST. This observation do not provides and essential difference since in principle one can attempt to generalize the action (2) to higher dimensions. Assuming that the basic gauge fields \( G_{\hat{\mu} \hat{\nu}} \) and \( A_{\hat{\mu} \hat{\nu}} \) in NGT and ST are the same, we need to concentrate in the integrands of the two actions (2) and (6). At first sight it seems hopeless to find a connection between such integrands, since the integrand in (2) looks more geometrical, than the integrand in (6). However, if we success in writing (6) in a more geometrical way, we could find real differences or real similarities between NGT and ST. Of course, we can also use the fact, that

\[
\mathbb{g}_{\hat{\mu} \hat{\nu}} = G_{\hat{\mu} \hat{\nu}} + A_{\hat{\mu} \hat{\nu}},
\]

in order to develop (2). In what follows we will follow the former strategies.

In order to achieve our goal let us first assume, that in (11) \( G_{\hat{\mu} \hat{\nu}} \) is a real symmetric tensor and \( A_{\hat{\mu} \hat{\nu}} \) is a pure imaginary antisymmetric tensor. With this assumption it is not difficult to show, that the metric \( g_{\hat{\mu} \hat{\nu}} \) is an Hermitian matrix, that satisfies

\[
g_{\hat{\mu} \hat{\nu}}^{\dagger} = g_{\hat{\nu} \hat{\mu}},
\]

where the symbol ”\( \dagger \)” denotes conjugate transpose.

Now, let us introduce the following definitions:

\[
g_{\hat{\mu} \hat{\nu}0} \equiv G_{\hat{\mu} \hat{\nu}},
\]

and

\[
g_{\hat{\mu} \hat{\nu}1} \equiv A_{\hat{\mu} \hat{\nu}}.
\]

So, with this notation the metric \( g_{\hat{\mu} \hat{\nu}a} \) with \( a = 0, 1 \) becomes our basic object. In addition we need to introduce the metric

\[
g_{abc} = \begin{cases} 
g_{ab0} = \frac{g_{ab}}{\sqrt{-g}} \\
g_{ab1} = -\epsilon_{ab}. \end{cases}
\]
Here $g_{ab}$ is a $(1+1)$ dimensional metric, $g$ is the determinant of $g_{ab}$ and $\epsilon_{ab}$ is a two dimensional Levi-Civita tensor with $\epsilon_{01} = -1$.

The inverse of $g_{abc}$ can be taked as

$$g^{abc} = \begin{cases} 
  g^{ab0} = \sqrt{-g} g^{ab} \\
  g^{ab1} = \epsilon^{ab}.
\end{cases} \tag{16}$$

It is interesting to observe, that the metric $g_{abc}$ can be obtained by using a vielbien field in two dimensions $e_a^i$ and the flat metric

$$\eta_{ijc} = \begin{cases} 
  \eta_{ij0} = \eta_{ij} \\
  \eta_{ij1} = \epsilon_{ij}.
\end{cases} \tag{17}$$

In fact, we have

$$g_{abc} = e^{-1} e_a^i e_b^j \eta_{ijc}, \tag{18}$$

where $e$ is the determinant of $e_a^i$.

The important point here is that using (19) we can define the unify metric

$$g^{ab} \equiv g^{abc} g_{c\hat{\mu}\hat{\nu}}. \tag{19}$$

Summing over the index $c$ in (19) we discover, that

$$g^{ab} = \sqrt{-g} g^{ab} G_{\hat{\mu}\hat{\nu}} + \epsilon^{ab} A_{\hat{\mu}\hat{\nu}}, \tag{20}$$

where we used the definitions (13) and (14). It is not difficult to see, that using (20) the bosonic string action (8) can be written in terms of the unify metric $g^{ab}$, as

$$S = \frac{1}{2} \int d^2 \xi \ g^{ab} \partial \xi^\mu \partial \xi^\nu. \tag{21}$$

This action was proposed in a previous work [12]. The central idea was to consider the metric $g^{ab}_{\hat{\mu}\hat{\nu}}$ as a fundamental basic object such that, as a particular case adopt the form (20).

From this point of view either using $\beta$-function [10] or Fradkin and Tseytlin [11] procedure one should expect to obtain in a low energy limit field equation derived from an action with the metric $g^{ab}_{\hat{\mu}\hat{\nu}}$ as a fundamental field.
What could be the form of such an action? Of course, in the particular case, in which $g_{\hat{\mu}\hat{\nu}}$ has the form (20), such an action should reduce to the action (6). Just like in ordinary ST $G_{\hat{\mu}\hat{\nu}}$ leads to the geometrical action $\int d^{26}x \sqrt{-G} \mathcal{R}$, one should expect that the action, that we are looking for has a geometrical form with $g_{\hat{\mu}\hat{\nu}}$ as a basic object. Looking things from this point of view we can first attempt to construct the geometry behind the metric $g_{\hat{\mu}\hat{\nu}}$ or its associated metric $g_{\hat{\mu}\hat{\nu}} \epsilon_{\hat{\alpha}} = g_{\hat{\mu}\hat{\nu}} g_{\epsilon\delta}$. For this purpose let us define the analog of Christoffel symbols for $g_{\hat{\mu}\hat{\nu}} \epsilon_{\hat{\alpha}}$,

$$\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} \equiv \frac{1}{2} \left( g_{\hat{\alpha}\hat{\mu},\hat{\nu}} + g_{\hat{\nu}\hat{\alpha},\hat{\mu}} - g_{\hat{\delta}\hat{\mu},\hat{\alpha}} \right), \tag{22}$$

Notice, that the position of the indices in the last term is important. It is not difficult to see, that

$$\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} \equiv \Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}0} = \frac{1}{2} \left( G_{\hat{\alpha}\hat{\mu},\hat{\nu}} + G_{\hat{\nu}\hat{\alpha},\hat{\mu}} - G_{\hat{\mu}\hat{\nu},\hat{\alpha}} \right), \tag{23}$$

is the ordinary definition of the Christoffel symbol, and that

$$F_{\hat{\mu}\hat{\nu}\hat{\alpha}} \equiv F_{\hat{\mu}\hat{\nu}\hat{\alpha}1} = \frac{1}{2} \left( A_{\hat{\alpha}\hat{\mu},\hat{\nu}} + A_{\hat{\nu}\hat{\alpha},\hat{\mu}} + A_{\hat{\mu}\hat{\nu},\hat{\alpha}} \right), \tag{24}$$

is the completely antisymmetric field strength considered in (6). In (23) and (24) we used the Eqs. (13) and (14).

The curvature may be defined as

$$R_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}} \equiv \partial_{\hat{\alpha}} \Gamma_{\hat{\nu}\hat{\beta}\hat{\mu}} - \partial_{\hat{\beta}} \Gamma_{\hat{\nu}\hat{\alpha}\hat{\mu}} + \Gamma_{\hat{\nu}\hat{\alpha}\hat{\lambda}} \Gamma_{\hat{\beta}\hat{\lambda}\hat{\tau}} g^{\hat{\lambda}\hat{\tau}} a e f b g_{\epsilon\delta} g_{\epsilon\delta} g_{\epsilon\delta} - \Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} \epsilon_{\hat{\beta}} g^{\hat{\lambda}\hat{\tau}} a e f b g_{\epsilon\delta} g_{\epsilon\delta} g_{\epsilon\delta}. \tag{25}$$

General covariance implies, that the geometrical action, that we are looking for should have the form

$$S^a = \int d^{26}x \sqrt{-g} g^{\hat{\mu}\hat{\alpha}} g^{\hat{\beta}\hat{\nu}} g_{\epsilon\delta} g_{\epsilon\delta} g_{\epsilon\delta} R_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}} g^{\epsilon\delta}, \tag{26}$$

where g is the determinant associated to $g_{\hat{\mu}\hat{\nu}}$. Of course, except that $S^a$ is a complex action (see appendix), this action is in completely analogy to the Einstein-Hilbert action. Furthermore, we conjecture, that it should be possible to obtain the action (26), by quantization.
procedure of the action (21) and taking the low energy limit just in the same way that the
action (2) is obtained from (8). Since the metric (20) can be understood as a particular
solution of the field equation derived from (26) we should expect that (26) reduces to (8).

In fact, it is possible to show, that by choosing $g_{abc}$ as $\eta_{abc}$ and dropping cubic terms of
the antisymmetric gauge field $A_{\hat{\mu}\hat{\nu}}$ and higher in (26), the action $S^0$ reduces to the action $S$, while $S^1$ turns out to be a total derivative. The important point here, however, is that
we can now compare (26) and (2) since both have a geometrical structure. We find, in fact,
that both actions turn out to be different and represent distinct theories.

It is possible, that the action (2) may correspond to a different ST. It will be curious
to find the associated ST. However, one should expect that such a new ST must be based
on an action with different structure, than the action (21). Anyhow, in view that NGT has
unacceptable Global Asymptotics [13] one should expect, that such a new ST presents some
difficulties.

Note added. After the present work has been prepared for publication we became aware,
that a relation between NGT and ST has also been considered by Moffat [14]. It should be
stressed, however, that the analysis developed here based in fundamental strings essentially
diffs, when is compared with the analysis of the ref. [14] based on cosmic strings.

APPENDIX A:

In [3] the complex numbers were discussed from another point of view. The central idea
in such a reference is to avoid to use the imaginary complex number $i$, the one, that satisfies
$i^2 = -1$, and to see complex numbers as a vector $A^a$ (with a=1,2) in the plane with the
property that two vectors $A^a$ and $B^b$ are multiplied according to the rule

$$A^aB^b\omega_{abc} = C^c,$$  \hspace{1cm} (A1)

where

$$\omega_{ij1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$  \hspace{1cm} (A2)
and

$$\omega_{ij2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (A3)$$

Since the matrices (A2) and (A3) can be understood as reflexions in the plane with respect to the x axis and with respect to the line y = x, respectively, C1 in (A1) may be understood as the component of Cc obtained by making a reflexion along the x axis of the vector Bb and the reflected vector been projected into the vector Aa. Similarly, C2 may be understood as the second component of Cc obtained by making first one reflexion along the line y = x of the vector Bb, and then making the scalar product between the reflected vector and Aa. An interesting observation is that (A2) and (A3) satisfy a Clifford algebra.

Acknowledgments

This work was supported in part by CONACyT Grant 4862-E9406 and by the Coordinación de Investigación Científica de la UMSNH.
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