Entropy of black holes in $\mathcal{N} = 2$ supergravity

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Abstract: Using the formalism of isolated horizons, we construct space of solutions of asymptotically flat extremal black holes in $\mathcal{N} = 2$ pure supergravity in 4 dimensions. We prove that the laws of black hole mechanics hold for these black holes. Further, restricting to constant area phase space, we show that the spherical horizons admit a Chern–Simons theory. Standard way of quantizing this topological theory and counting states confirms that entropy is indeed proportional to the area of horizon.

Keywords: Black hole mechanics; Black hole entropy; Supergravity

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1. Introduction

Black holes are designated as “the simplest macroscopic objects” made out of spacetime [1]. Though classically they are the simplest, quantum mechanically, they encode all the complexity of a quantum system. In fact, it is expected that black holes will turn out to provide crucial clues for a quantum theory of gravity just as the hydrogen atom helped us unravel the secrets of the atomic system. The most important developments in the last few decades established the thermal aspects of black holes. In particular, it is now understood that black holes behave as macroscopic states in thermal equilibrium and their dynamical laws (laws of black hole mechanics) are exactly similar to laws of thermodynamics [2]. This observation prompted Bekenstein and Hawking to argue that black holes indeed have temperature related to the surface gravity and their entropy is related to the area of the black hole horizon [3, 4]. However, one needs statistical interpretation for such thermodynamic arguments. It is expected that any quantum theory of gravity should be able to specify the microstates of the black hole spacetime and the leading term in Boltzmann definition of entropy would be proportional to the area of the horizon.

Supergravity and string theories are leading candidates of quantum theory. Black holes occurring in these theories are subjects of intense study to understand the classical and quantum nature of these objects. These black hole solutions can be interpreted as self-gravitating solitons interpolating between different vacua of the theory [5–7]. The extremal Reissner–Nordstrom solution arising in pure $\mathcal{N} = 2$ supergravity in 4-dimensions is the simplest example [8–13]. The solitons in string theory (and low energy effective actions) play a central role in understanding string dualities. Alternatively, string theory has been used to investigate quantum properties of extremal black holes [14–18]. Extremal black holes in $\mathcal{N} = 2$ supergravity are BPS solitons (see [19] for other interpretations of the term BPS). One computes entropy of these black holes in perturbative regime and use the BPS property to extrapolate to non-perturbative region. Since these BPS solutions have high degree of supersymmetry as isometries, the number of states quantum over large variation of modular parameters, remains identical in either limits (note that in extended supergravity theories, the extremal solutions may be BPS as well as non-BPS solutions). More solutions of the BPS type exist in higher dimensions with various degree of supersymmetry and have proved to be of great interest for establishing various subtleties related to the entropy calculation (see [20–26]).

A useful method of finding the ‘macroscopic’ entropy of black holes (in different theories of gravity) is to utilize the Killing horizon formalism and the Wald formula, which classically defines the entropy of a black holes as the Noether charge corresponding to the Killing vector.

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generating the horizon [27–29]. For general relativity, the Wald entropy formula correctly reproduces the area of horizons cross-section as the entropy of black holes. However, for theories of gravity with higher curvature corrections, the entropy is not merely proportional to the area but receives corrections from the higher order terms [30]. If the higher curvature interactions are included along with the Einstein Lagrangian, the area increase theorem (the second law) is not generally valid, but interestingly, the Wald entropy increases and it is this function which satisfies the second law [31, 32]. This result has important consequences, particularly for black holes in supergravity, where such higher curvature corrections are arise naturally, as well as from string corrections. In these theories, the entropy as determined by the Wald formula is not the area of the black hole but interestingly, the statistical entropy as computed from the string theory matches with the Wald formula. Thus, the Wald formula has become an important tool in evaluating the black hole entropy and in validating the the microscopic computation of entropy [33]. The microscopic computation of entropy (in string theory) for three-charge supersymmetric black holes in five dimensions was first carried out by Strominger and Vafa and it matches with the area of the horizon [14, 34] in the supergravity approximation. Furthermore, the above result has been extended for nearly extremal black holes in which the nonextremality is treated as a perturbation [35, 36]. Similar to the case of five dimensions, the entropy has also been computed for black holes in four dimensions and the microscopic entropy computed exactly matches with the area of the horizon [37, 38]. It has also been observed that the higher derivative and higher curvature terms in string theory induce corrections to the Bekenstein–Hawking formula for entropy of black holes. These correction terms to the black hole entropy matches exactly with the finite-size corrections to microscopic black hole entropy [39–41]. It may also be noted that the formalism of entropy functional developed in [42–44], is a modification of the Wald formalism and has been successfully applied to computation of entropy of extremal rotating as well as non-rotating black holes. This formalism uses the near horizon geometry of the black holes to compute the entropy.

The Wald formula has led us to a better understanding of the black hole entropy. However, it is well known that the application of the Wald Noether charge approach to extremal black holes is subtle. The reasons are as follows: the derivation of Wald formula depends crucially on the existence of a bifurcate surface. On this surface, the Noether charge depends only on the metric, the matter fields, and their derivatives and all the dependence on the Killing vector field vanishes. Furthermore, it has also been shown in [29] that the entropy function evaluated at any section of the Killing horizon is identical to the one evaluated on the bifurcation two-sphere, provided the surface gravity is non-vanishing. Thus, the existence of a bifurcation surface is important for the construction of the entropy function. On the other hand, it has been proved in [45, 46], that the surface gravity is a non-zero constant on the Killing horizon if and only if the Killing horizon can be analytically extended globally to include a bifurcating two-sphere. For extremal Killing horizons, for which the surface gravity is vanishing (but constant), such a bifurcation surface does not exist and neither can it be analytically extended to contain such a point. Then, it may seem that entropy computation, as in the Noether method may not go through. However, as advocated in [47, 48], entropy of extremal black holes are to be defined through limits of their non-extremal counterparts. To implement this point of view, one notes that the bifurcate surface may not be part of the physical spacetime and by taking extremal limits, explicit calculations have shown that indeed the entropy remains well defined and finite [33, 41]. These computations in fact supports the conclusions reached in [47, 48], where it has been argued that, at least in general relativity, the entropy of both the extremal and non-extremal black holes, if they belong to the same space of solutions (related through appropriate limiting procedure), is proportional to area of their horizons. However, taking these limits is subtle, since entropy of black holes in a space of solutions which contains only extremal solutions may not be proportional to the area [49–52]. In other words, the computation of entropy for extremal black holes is subtle and in this paper, we propose to use the formalism of isolated horizons to clarify some aspects of this problem. As we shall show below, the isolated horizon formalism supports both the extremal as well as non-extremal horizons in its phase space. Our aim in this paper is to use the techniques of isolated horizons to calculate the entropy of spherical black holes in pure \( N = 2 \) supergravity. Expectedly, if extremal horizons are in the class as the non-extremal black holes, the entropy of extremal black holes may be clearly defined and a sequence of solutions can be constructed in this space of solution which will have their surface gravity limiting to zero. So in this phase space of isolated horizons, taking extremal limit for non-extremal black holes should be well defined.

Isolated horizon (IH) [53–60] is a local definition of black hole horizon. Unlike event horizons or the Killing horizon, this definition does not require the knowledge of spacetime external to the horizon. The knowledge of the entire spacetime is redundant. The IH boundary conditions are the minimal set of conditions that any generic black hole horizon (extremal/non-extremal) is expected to satisfy. The most important characterization of IH is that they are expansion-free. This condition separates an arbitrary null surface from a black hole horizon. For example, the Minkowski light cone expands in the future (and also in the past) and hence is not an IH. This is expected because the
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Minkowski null-cone behaves as a horizon only for the Rindler observers. The condition of expansion-freeness implies that no matter field falls inside the horizon. Moreover, the boundary conditions imply that there exists Killing vectors fields on the black hole horizon only. Thus there might be radiation arbitrarily close to the horizon but are not allowed to cross it. This allows a large class of solutions to satisfy the IH boundary conditions. Indeed, the space of solutions of any theory of gravity admitting an IH as an inner boundary is larger than that of the Killing or the event horizon (these definitions require Killing vector fields outside the horizon too). The original formulation of IH however, had extremal and non-extremal horizons in distinct phase spaces. The Weak Isolated Horizon (WIH) uses a set of milder conditions than IH and puts these two classes of solutions in equal footing [51, 60]. The IH (or the WIH) formulation does not require bifurcation spheres to establish the laws of black holes mechanics or to determine the entropy. In other words, the IH (or the WIH) formalism provides the ideal set-up to study mechanics of extremal black hole horizons arising in string theory or supergravity. The IH formulation is also useful to compute the entropy of horizons in loop quantum gravity (LQG) approach. The point of view of this approach is that the horizon supports the effective degrees of freedom that arise out of a well defined interaction between the bulk and the boundary configurations. These microstates reside on the null inner boundary of the spacetime but capture all the essential features of the entire spacetime. In other words, the quantum states for the black hole entropy counting are localized on the horizon only. These quantum states must arise from some effective quantum field theory residing on the horizon. It not very difficult to guess the nature of the theory on horizon. Since the surface is null, it does not support a metric theory. It is only natural that the effective theory on this null surface be a topological theory [61–63]. The theory turns out to be a $U(1)$ Chern–Simons theory [64]. Entropy of the horizon can then be obtained by directly geometrically quantizing this Chern–Simons theory [65, 66]. Alternatively, the authors of [67, 68] used techniques of conformal field theory to obtain the entropy and corrections to all orders in Planck length. The counting of states gives the entropy directly proportional to the area [69–72]

We shall use the IH formalism approach to deal with the black hole solutions in $\mathcal{N}=2$ supergravity. In [60], it was proposed that the IH formalism introduces an ideal set-up to study black hole solutions in string theory and supergravity. In [73–75], the authors formulated the precise notions required to make IH amiable to supergravity solutions. However, the detail compatibility study between the IH boundary conditions and the conditions arising from the supergravity theory is still to be carried out. Moreover, the derivation of the laws of black hole mechanics and the calculation of entropy (as is done for IH in GR) are needed for detailed understanding of the non-perturbative aspects of spacetime. The aim of the present paper is to fill this gap.

We will be interested in classical spacetimes which are purely bosonic solutions of supergravity theories. The solutions which retain some supersymmetry of the theory are labelled BPS saturated. The BPS condition bounds the mass (measured at infinity) from below by a function of the asymptotic charges of the fields (for $\mathcal{N}=2$, by charges of the Maxwell field) (however, see [19]). When the bound is attained, the classical spacetime admits Killing spinor fields. The supersymmetry transformations generated by these Killing spinors are such that the bosonic fields are left invariant while the supersymmetry transformations of the fermionic fields vanish. These conditions on the fields can be turned into a set of first order differential equation called the Killing spinor equations (KSE). Alternatively, given the KSEs for a supergravity theory, their solution leads to classical configurations with unbroken supersymmetry. However, only a few of these configurations actually solve the corresponding supergravity equations of motion. Solutions of KSEs for different supergravity theories are of great interest.

Our main interest in this paper is to study the black holes in $\mathcal{N}=2$ supergravity using the IH formalism. As a concrete example, we shall consider the extremal Reissner–Nordstrom black hole solution in $\mathcal{N}=2$ supergravity. This is also a well solution in the Einstein–Maxwell theory which in fact, is a consistent truncation of the $\mathcal{N}=2$ supergravity. For these global solutions, the mass (and charge) measured by an asymptotic observer equals the mass (and charge) defined on the horizon. Thus, this particular solution is supersymmetric or BPS for any observer, at asymptopia or at the horizon. These kind of solutions will be called globally supersymmetric. In the context of the IH formalism, the most general cases are those where one has access only to mass and charge defined locally on the horizon and does not have any knowledge of the nature of the exterior spacetime. In that case, it is natural to consider solutions (or configurations) which saturate the BPS bound solely on the horizon, i.e. local horizon mass equal the charge of the field equated on the horizon. Indeed, IH formalism can incorporate solutions which are supersymmetric on the horizon only while the bulk spacetime may have no residual supersymmetry of the theory. We repeat that, while it is enough for the IH formalism to require KSEs to hold just on the horizon, the

Footnote: The KSEs are different for different theories but since we are interested in classical configuration, it remains true for any other set of fermion field.
configurations which solve the KSEs globally (like the extremal RN solution) will also naturally be part of the IH phase space.

As mentioned before, we shall investigate the applicability of the IH formalism for black hole solutions arising in supergravity. We will use the global and supersymmetric Reissner–Nordstrom solution arising in pure $\mathcal{N} = 2$ supergravity for consistency study. First, we need to check that in the region outside the horizon (when the Killing vector is timelike), the solution of KSEs (equations arising because of BPS condition) imply a Reissner–Nordstrom like configuration i.e. static with asymptotically flat geometry, invariant under half of the the $\mathcal{N} = 2$ supersymmetries. Second, when the Killing vector is null, for e.g. on the horizon, the KSEs give rise to configurations whose geometric structures are consistent with the ones derived from IH boundary conditions. We shall call a horizon supersymmetric weak isolated horizon (SWIH) if the conditions for existence of Killing spinors on the horizon are compatible with the IH boundary conditions. In other words, on a SWIH, the KSEs arising because of the BPS nature of the horizon will be consistent with the IH boundary conditions. Once the SWIH is defined, the next task is to construct the phase-space of the theory of gravity in hand with appropriate boundary conditions. Our phase space will consist of all solutions of $\mathcal{N} = 2$ supergravity which satisfy the SWIH boundary condition at the horizon and are asymptotically flat at infinity. In this paper, we shall use the generalization of the Holst action [76] as the action for $\mathcal{N} = 2$ pure supergravity [77]. Using well known techniques of covariant phase space, one can construct the symplectic structure on this SWIH phase space [79, 80].

The first law for the SWIH will then follow from this symplectic structure. It will also follow from this symplectic structure that the topological theory on fixed area phase space of SWIH is a $U(1)$ Chern–Simons theory.

The plan of this paper is as follows: First, we spell out the isolated boundary conditions to be imposed on a generic null surface. Second, we study the constraints arising from Killing spinor equations and show that they are consistent with IH boundary conditions. Thirdly, we construct the space of solution of $\mathcal{N} = 2$ pure supergravity which satisfy the SWIH boundary conditions at horizon and are asymptotically flat at infinity. We construct the symplectic structure and prove the laws of black hole mechanics. Next, we shall go to fixed area phase space and and identify the $U(1)$ Chern–Simons theory as the boundary theory.

2. Isolated horizon boundary conditions

We consider a 4-dimensional spacetime manifold $\mathcal{M}$ equipped with a metric $g_{ab}$ having Lorentzian signature $(-,+,+,+)$ and a null hypersurface $\Delta$. Let $\ell^a$ be a future directed null normal on $\Delta$. However, if $\ell^a$ is a future directed null normal, then so is $\xi \ell^a$, where $\xi$ is any positive function on $\Delta$. Two null normals $\ell^a$ and $\ell^{'a}$ on $\Delta$ will be called equivalent if $\ell^{'a} = \xi \ell^a$. Thus, $\Delta$ naturally admits an equivalence class of null normals. We shall denote this equivalence class by $[\xi \ell^a]$. Let us denote by $q_{ab} \triangleq g_{ab}$ the degenerate intrinsic metric on $\Delta$ which is induced by the spacetime metric $g_{ab}$. Thus $q_{ab}$ has a signature $(0,+,+)$. For details, please see the Appendix 1 which contains the details of this geometrical construction along with an example of the Schwarzschild horizon. The A tensor $q^{ab}$ will be called an inverse of $q_{ab}$ if it obeys the condition $q^{ab} q_{ac} q_{bd} \triangleq q_{cd}$. The inverse metric $q^{ab}$, however, is not unique as one can redefine it as $q^{ab} \rightarrow q^{ab} + k(a \cdot b)$, where $k^a$ is any vector field tangential with $\Delta$. The expansion $\theta(\ell)_{(a)}$ of the null normal $\ell^a$ is then defined by $\theta(\ell)_{(a)} = q^{ab} \nabla_a \ell_b$, where $\nabla_a$ is the spacetime covariant derivative compatible with $g_{ab}$. Note that the expansion $\theta(\ell)_{(a)}$ is insensitive to the ambiguity in the inverse metric but it varies under the scaling of the null normal $\ell^a$ in the equivalence class $[\xi \ell^a]$ by $\theta(\ell)_{(a)} \rightarrow \xi \theta(\ell)_{(a)}$.

In what follows, we shall work with the Newmann-Penrose (NP) null tetrad basis $(e^a, m^a, \bar{m}^a, \bar{e}^a)$, $n_a$ being normal to the foliation of $\Delta$ by $S^2$ and $m^a, \bar{m}^a$ are tangential to 2-spheres. The basis vector obey the orthonormality conditions $\ell, n = -1 = m, \bar{m}$, others being zero. This is specially suited for the present study because one of the
null-normals in the equivalence class $[\xi l]$ coincide with the basis vector $l$. Moreover, in this basis, many components of the connection vanish making the calculations much simpler than that in the coordinate basis. For future calculations with supergravity, it will be convenient to use the spinor basis alongside \([54, 64, 86]\). We can find a spin dyad \((l^A, \sigma^A)\) with normalization condition \(l^A l_A = 1\). The null vectors \((l, n, m, \bar{m})\) are related to these dyad basis by the following relations:

\[
\ell^a = i\sigma^a_{\cdot AA} \sigma^A \delta^A, \quad n_a = i\sigma_a^{\cdot AA} l_A \bar{t}_A, \\
m^a = i\sigma^a_{\cdot AA} \sigma^A \bar{\eta}^A, \quad \bar{m}^a = i\sigma_a^{\cdot AA} \delta^A l_A,
\]

where \(\sigma^a_{\cdot AA}\) is called the soldering form.

The isolated horizon boundary conditions for spherically symmetric cases can be stated in terms of the spin dyads as follows \([54, 64]\). The surface \(\Lambda\) will be called a non-expanding horizon (NEH) if

1. \(\Lambda\) is topologically \(S^2 \times \mathbb{R}\).
2. The spin dyads \((\sigma_a, l_A)\) are constrained to satisfy

\(\sigma^A \nabla_a \sigma_A = 0\) and \(l^A \nabla_a l_A = \mu \bar{m}_a\)

where, \(\mu\) is a real, nowhere-vanishing, spherically symmetric function, and \(\nabla_a\) denotes the unique torsion-free connection compatible with \(\sigma^a_{\cdot AA}\).
3. All equations of motion hold on \(\Lambda\) and the forms of the fields are such that \(-T^a_{\cdot ba}\) is causal and \(e \equiv T_{ab} l^a \bar{\eta}^b\) is spherically symmetric.

We will study the consequences of these conditions after we have pointed out the restrictions from \(N = 2\) pure Supergravity.

### 3. Conditions from \(N = 2\) pure supergravity

The purpose of this section is to establish the compatibility of the isolated horizon boundary conditions to the conditions obtained from the KSEs of \(N = 2\) supergravity. In other words, we intend to show that on the horizon, the KSEs can be put in a form which are precisely the same as the IH boundary conditions.\(^5\) This approach was also addressed in \([73–75]\). However, for our purpose, which includes construction of the symplectic structure, deriving the first law of black hole mechanics and to understand the origin of entropy of the black holes arising in \(N = 2\) supergravity, further details about spacetime connection and its curvature will be required. This needs a study of all the constraints available from the KSEs.

Before diving into formal calculations, let us try to comprehend the method. As mentioned previously, we are interested in BPS configurations. In other words, we look for classical configurations which have some residual supersymmetry and hence satisfy the KSEs for \(N = 2\) supergravity. All such configurations (not necessarily solutions of equation of motion) for \(N = 2\) have already been determined and classified in \([8, 11, 12]\) by explicitly solving the KSEs. It then remains to identify, in the above classification, weather there exists BPS configuration(s) (i.e. member(s) of solution of KSEs) which also satisfy the isolated horizon boundary conditions. In an elaborate way, we can pick each configuration and check weather it actually satisfies the IH boundary conditions. Alternatively and equivalently, we might show that (in case when the Killing vector is null, see the paragraph just above Eq. (11) below), the conditions arising out of the KSEs can be put in a form which will be exactly identical to the isolated horizon boundary conditions. This will easily establish that under these circumstances, there exists classical configurations of \(N = 2\) pure supergravity which will satisfy the isolated horizon boundary conditions. All such configurations might not be solutions of equation of motion, only a few will be. We shall argue that there also exists solutions of equations of motion, for the theory under consideration, in this space of configurations. As an example, shall explicitly show that the Reissner–Nordstrom black hole, which solution of \(N = 2\) pure supergravity equations of motion, solves the KSEs and its horizon satisfies the isolated horizon boundary conditions. This will be done in two steps: first, we shall show that when there is a timelike static Killing vector, the configuration obtained by solving the KSEs is identical to the external region (of the horizon) of the Reissner–Nordstrom spacetime. Secondly, when the Killing vector is null, the KSEs can be put in a form identical to the IH boundary conditions which are also the ones satisfied by the horizon of Reissner–Nordstrom spacetime. Thus, the Reissner–Nordstrom spacetime will become consistent with IH formalism. This will also demonstrate that other black hole solutions in supergravity theories can be understood using the isolated horizon formulation.

Let us now look into the KSEs of the \(N = 2\) pure supergravity. This theory has graviton and Maxwell field as the bosonic fields and two gravitini as their fermionic counterparts \([10]\). We are interested in the bosonic sector, with the gravitini fields set to zero since this sector will give us the classical spacetime solutions. Supersymmetry transformations are generated by gauge-spinor fields \(\varepsilon^A = (\varepsilon_A, \beta_A)\) where \(A\) is the internal \(O(2)\) index and \(A\) is the...
spinor index. We start with the standard Killing spinor equations \([11, 12, 73, 74]\).

\[
\begin{align*}
\nabla_{AA'} z_{B} &= -\sqrt{2} \phi_{AB} \beta_{A'} \\
\nabla_{AA'} \beta_{B} &= \sqrt{2} \phi_{AB} \bar{\beta}_{A'} z_{A},
\end{align*}
\tag{4}
\]

where \(\phi_{AB}\) is the anti self-dual part of \(F_{ab}\), i.e.,

\[
F_{ab} = \sigma_{a}^{A'A'} \sigma_{b}^{B'B'} F_{AA'B'B} = (\phi_{AB} \epsilon_{A'B'} + \epsilon_{AB} \bar{\phi}_{A'B'}). \tag{5}
\]

For further calculations, it is useful to define the function \(V = z_{A} \beta_{A'}\). Also define the vector fields:\(^{6}\)

\[
L_{a} \equiv L_{AA'} = z_{A} \bar{a}_{A'}, \quad N_{a} \equiv N_{AA'} = \bar{\beta}_{A} \beta_{A'},
\]

and \(M_{a} \equiv M_{AA'} = z_{A} \beta_{A'}\). \(\tag{6}\)

We first concentrate on the case \(V \neq 0\). It follows from the equations in \((4)\) that \(J_{a} = (L_{a} - N_{a})\) and \(M_{a}\) are local gradients. A combination of the two vector fields \(M_{a}\) and \(M_{a}\) can be used to define the \((\theta, \phi)\) plane \([11, 12]\). From \((4)\), we also get that the vector field \(K^{a} = (L^{a} + N^{a})\) is timelike and Killing:

\[
\nabla_{a} K_{b} + \nabla_{b} K_{a} = 0 \tag{7}
\]

All the static configurations admitting solution to the Eq. \((4)\) are in the Majumdar–Papapetrou class \([8, 11, 12]\). If \(r_{H}\) defines the horizon, the space of solutions of Eq. \((4)\) include the \(r > r_{H}\) part of the extremal Reissner–Nordstrom spacetime. This is the only static solution (in the Majumdar–Papapetrou class) with a single horizon. It follows that when the Killing vector is timelike, a solution of the Killing spinor equation is indeed the spacetime exterior to the Reissner–Nordstrom horizon.

The degenerate sector, \(V = 0\) is somewhat subtle and needs care. Before going into that, let us study the \(V \neq 0\) case in greater detail. From Eq. \((7)\), we get:

\[
K^{a} \nabla_{a} K_{a} = 0 \tag{8}
\]

Since the vector field \(J_{a}\) is orthogonal to \(K^{a}\) i.e. \(K^{a} J_{a} = 0\), we get from \((8)\) that:

\[
K^{a} \nabla_{a} K_{a} = \chi J_{a}, \tag{9}
\]

where \(\chi\) is some function. \(^{7}\) For \(V \neq 0\), the vector fields \(K^{a}\) and \(J^{a}\) can be used to define two orthogonal directions. Indeed the these can describe the \(\left(^{0}(r-t)^{0}\right)\) plane. A combination of the other two vector fields \(M_{a}\) and \(\bar{M}_{a}\) define the \((\theta, \phi)\) plane.

On the horizon, \(K^{a}\) become null and hence, is automatically geodetic:

\[
K^{a} \nabla_{a} K_{a} = \bar{\chi} \bar{K}_{a}, \tag{10}
\]

where \(\bar{\chi}\) is some function on the horizon. Equations \((10)\) and \((9)\) together imply that on the horizon, \(J^{a} \to \bar{K}^{a}\). Thus the entire \(\left(^{0}(r-t)^{0}\right)\) plane degenerates (to a line) on the horizon. We will then identify the horizon to be the surface where, \(J^{a} = \bar{K}^{a}\) (modulo rescaling by functions). We shall take this as our criterion for defining the horizon \(\Delta\). We will see below that in this precise sense, \(V = 0\) defines a horizon.

When \(V\) is vanishing, \(\bar{\beta}^{A} \equiv \bar{K}^{A}\), where the function \(K\) on \(\Delta\) is such that (see \((4)\))

\[
z_{B} \nabla_{AA'} K \equiv \sqrt{2} \left(1 + K \bar{K}\right) \bar{a}_{A'} \phi_{AB}. \tag{11}
\]

From Eq. \((6)\), we also obtain that \(V = 0\) implies \(J^{a} = \bar{K}^{a}\) (modulo rescaling by functions). The configurations which solve the Killing spinor Eq. \((4)\) for \(V = 0\) actually describe null surfaces. The standard coordinate system used in the exterior collapses on such surfaces. For example, in the Reissner–Nordstrom solution, the \(\left(^{0}(r-t)^{0}\right)\) plane degenerates on the horizon. The trick to lift such degeneracy of coordinate systems is to introduce by hand an auxiliary vector field. \(^{8}\) We shall use this option. We introduce two normalised spinors \(o_{A}\) and \(l_{A}\) as such that \(r_{H} o_{A} \equiv 1\). The spinor \(o_{A}\) is such that \(o_{A} = \epsilon^{ig} z_{A}\), where, \(g\) is a real function on \(\Delta\). These spinors can be used to construct a null tetrad basis \((\ell, n, m, \bar{m})\) [compare with Eq. \((2)\)].

For further calculations, we need the form of the Maxwell field \((\phi_{AB})\). Observe that the Eq. \((11)\) can be rewritten as:

\[
\nabla_{AA'} K \equiv \sqrt{2} e^{2g} \left(1 + K \bar{K}\right) \bar{a}_{A'} \phi_{AB}. \tag{12}
\]

Since, \(K\) is a function on \(\Delta\), the right hand side of \((12)\) can depend on \(n_{a}, m_{a}, \bar{m}_{a}\). Then, \(\phi_{AB}\) can be of the form [see Eq. \((2)\)]

\[
\phi_{AB} \equiv \phi_{0} t_{A} t_{B} + \phi_{1} \left(t_{A} o_{B} + o_{A} t_{B}\right) + \phi_{2} o_{A} o_{B}. \tag{13}
\]

However, we shall see that \(\phi_{0} = 0\) and there cannot be any term proportional to \(t_{A} t_{B}\). This is because, the surface \(\Delta\) is null and the energy-momentum tensor \((T_{ab})\) must be such that the vector field \(-T_{a} e^{ig}\) is causal or null. For the Maxwell fields, the energy-momentum tensor is given by:

\[
T_{ab} = \frac{1}{4\pi} \left[ F_{ac} F^{c}_{b} - \frac{1}{4} g_{ab} F^{2} \right] \tag{14}
\]

\(^{6}\) From now on, we shall omit the soldering form \(\sigma_{a}^{A'A'}\). Double spinor indices of the same type will indicate one spacetime index, for e.g. \(\sigma_{0} = \sigma_{a}^{a}\).

\(^{7}\) More generally, since \(M_{a}\) and \(\bar{M}_{a}\) are also orthogonal to \(K^{a}\), \(K^{a} M_{a} = 0\) etc. the Eq. \((9)\) should be \(K^{a} \nabla_{a} K_{a} = \chi J_{a} + \bar{\chi} \bar{M}_{a}\). However, we are working with static solutions having timelike Killing vector field \(K^{a}\). This implies that \(K^{a} \nabla_{a} K_{a}\) will not include the \((\theta, \phi)\) components. In other words, we can concentrate only on the deformations of the \(\left(^{0}(r-t)^{0}\right)\) plane, keeping aside the ‘sphere’ \((\Delta, \bar{\Delta})\) part.

\(^{8}\) If \(e^{g}\) generates the null surface, one can introduce the auxiliary null vector field \(n^{a}\) such that \(\ell n = -1\).
Using the eqns. (5), (14) and form of $\ell^a$ in terms of spin-dyads [see Eq. (2)], the above-mentioned restriction on $T_{ab}$ implies that $\phi_b = \phi_{ab} a^b \overset{\Delta}{=} 0$. For future convenience, we shall call $1/2\pi |\phi_b|^2 \overset{\Delta}{=} e$. The other component of the Maxwell field $\phi_2$ will be kept unrestricted.

Let us now determine the constraints on the null-normals on $\Delta$. Using Eqs. (4) and (13), we get

$$\nabla_a o_B \overset{\Delta}{=} (\sqrt{2}i R_{1B} - i\nabla_B g) o_B : -\tilde{a}_a o_B \tag{15}$$

From the normalization condition $r^4 a_1 \overset{\Delta}{=} 1$ and Eq. (15), we can obtain the action of the gradient operator on $i_A$. We restrict the form to be

$$\nabla_a t_B \overset{\Delta}{=} \tilde{a}_a t_B + \mu m o_B, \tag{16}$$

where $\mu$ is a function on $\Delta$. With the Eqs. (15) and (16) in hand, we can proceed to study their consequences. Note that these equations are precisely the same form as the IH boundary conditions. Moreover, the constraints on the Maxwell field derived from Eq. (13) are such that they satisfy the conditions matter field must satisfy on an IH (see Sect. 2).

4. Consequences of boundary conditions

In this section, we shall study the kinematical consequences of the boundary conditions. In what follows, we shall always restrict to horizons which are spherical, i.e. the null surface $\Delta$ will be foliated by spheres. Using Eq. (2), we find that the Eqs. (15) and (16) imply

$$\nabla_A \ell^b \overset{\Delta}{=} o_a^{(1)} \ell^b \tag{17}$$

$$\nabla_a n^b \overset{\Delta}{=} -o_a^{(1)} n^b + \mu m n^b + \tilde{a}_a n^b \tag{18}$$

$$\nabla_m n^b \overset{\Delta}{=} V_a^{(m)} n^b + \mu m n^b \tag{19}$$

where, $o_a^{(1)} = -2\text{Re}(\tilde{a}_a)$ and $V_a^{(m)} = -2\text{Im}(\tilde{a}_a)$ are one-forms on $\Delta$. The superscripts $\ell$ and $m$ indicate that $o_a^{(1)}$ and $V_a^{(m)}$ depend on the transformation of these vector fields. Also, note that $V_a^{(m)}$ is purely imaginary.

Several consequences follow from these equations [51, 54]. Firstly, since $\ell^a$ is null (and generates $\Delta$), it is automatically geodetic and shear-free. Moreover from Eq. (17), the expansion $\Theta^{(1)}$ vanishes on $\Delta$. All these restrictions, the Raychaudhuri equation for $\ell^a$ and the energy condition further imply that $\ell^a$ is also shear-free. From Eq. (17), we get that the null-vector field $\ell^a$ is a Killing vector on $\Delta$.

$$\mathcal{L}_\ell g_{ab} \overset{\Delta}{=} 0. \tag{20}$$

i.e. the IH boundary conditions imply that it is enough to have a Killing vector only on $\Delta$. Further, the volume form of the 2-spheres foliating $\Delta$ given by $2\epsilon = \mu m \wedge \tilde{m}$, is also lie dragged: $\mathcal{L}_\ell 2\epsilon \overset{\Delta}{=} 0$. To see this, use the Cartan formula $\mathcal{L}_\ell 2\epsilon = d(\ell^2 \epsilon) + \ell^a d^2 \epsilon$, and the Eq. (19). The surface gravity of $\ell^a$ is denoted by $k_{(1)}$:

$$\ell^b \nabla_b \ell^a \overset{\Delta}{=} k_{(1)} \ell^a. \tag{21}$$

The Eqs. (17) and (21) together imply that $k_{(1)} \overset{\Delta}{=} o_a^{(1)} \ell^a$.

The properties of the vector field $n^a$ follow similarly. It is twist-free, shear-free, has spherically symmetric expansion $\Theta(n) \overset{\Delta}{=} 2\mu$ and vanishing $\pi(= \ell^a m^b \nabla_a n_b)$ on $\Delta$. This now shows that the function $\mu$ is actually related to the expansion of the vector field $n^a$.

Before proceeding further, let us discuss the issues related to the available gauge freedom for the spin-dyads on $\Delta$. The most general transformation that preserves the normalization of the dyad $(i_A, o_A)$ is [54]

$$(r^1, o^A) \rightarrow (e^{\Theta - \theta} r^1, e^{-\Theta + \theta} o^A), \tag{22}$$

where $\Theta$ and $\theta$ are real functions on $\Delta$. Under (2), transformations, the null vectors are

$$\ell^a \rightarrow \tilde{\epsilon} \ell^a \quad n^a \rightarrow \frac{1}{\tilde{\epsilon}} n^a \quad m^a \rightarrow e^{i\theta} m^a \quad \tilde{m}^a \rightarrow e^{-i\theta} \tilde{m}^a, \tag{23}$$

where, $\tilde{\epsilon} = e^{-2\Theta}$ and $\tilde{\theta} = 2\theta$ are functions on $\Delta$. The one-forms $o_a^{(1)}$ and $V_a^{(m)}$ transform like gauge fields under rescaling of $\ell^a$ and $m^a$ respectively:

$$o_a^{(1)} \overset{\Delta}{=} o_a^{(1)} + \nabla a \ln \tilde{\epsilon} \tag{24}$$

$$V_a^{(m)} \overset{\Delta}{=} V_a^{(m)} + i\nabla a \tilde{\theta} \tag{25}$$

Consequently the surface gravity also depends on the rescaling of $\ell^a$, $k_{(1)} \overset{\Delta}{=} k_{(1)} + \mathcal{L}_\ell \tilde{\epsilon}$. In this paper, we are interested in asymptotically flat global solutions, i.e where the observer one has access to the infinity. In these special cases, the vector field can always be normalized with respect to infinity. In other words, we can set $\Theta = 0$ so that there is no scaling ambiguity in the evaluation of the surface gravity $k_{(1)}$. The function $\theta$ however remains unrestricted.

For further calculations, we shall need the curvature of the one-form fields $o_a^{(1)}$ and $V^{(m)}$. They are given by [51, 64]:

$$d o_a^{(1)} \overset{\Delta}{=} 2\text{Im}(\Psi_2)^2 \epsilon, \tag{26}$$

$$d V^{(m)} \overset{\Delta}{=} \frac{2}{i} \left( \text{Re}(\Psi_2 - \Phi_{11} - \frac{R}{24}) \right)^2 \epsilon \tag{27}$$

where, $\Psi_2$ and $\Phi_{11}$ are components of the Weyl and the Ricci tensor respectively (see [1, 86]). The spherical symmetry of the horizon implies that $d o_a^{(1)} \overset{\Delta}{=} 0$ or in other words, $o_a^{(1)}$ is a pure gauge and can be made to vanish by a choice of gauge. We are interested in extremal black holes.
These have vanishing surface gravity. For future calculations, we shall always set \( \omega^{(i)} = 0 \). Equation (15) shows that this can be done for some special choice of the function \( g \).

5. Laws of black hole mechanics and entropy

In this section, we shall derive the zeroth and the first law of black hole mechanics. We will construct the symplectic structure for \( \mathcal{N} = 2 \) supergravity with appropriate boundary conditions and derive the first law. Thereafter, we will restrict to fixed area part of the phase space and derive the effective theory residing on the horizon. This will give us clues to calculate the entropy.

5.1. The zeroth law

We call \( \Delta \) a Weakly Isolated Horizon (WyIH) if \( \mathcal{L}_{(i)}^{\Delta} \neq 0 \). The requirement can be justified as follows. The quantity \( \omega^{(i)} \) is analogue of the extrinsic curvature on the null hypersurface [56]. Since \( \ell^a \) is a Killing vector on \( \Delta \), the above condition implies that the entire data on the phase space is lie dragged by \( \ell^a \). Using the Cartan equation for lie derivative, it follows that the surface gravity is constant throughout the horizon. For extremal and spherical black holes, \( \omega^{(i)} = 0 \). The restriction of WyIH implies that the surface gravity \( \kappa^{(i)} \) is zero and remains constant on \( \Delta \).

5.2. The first law

For the first law, we need to study the dynamics. For this, we need an action which will specify the dynamics and interactions of the geometric and matter degrees of freedom. We will use the Holst action modified for \( \mathcal{N} = 2 \) supergravity theory [77].

5.2.1. Holst type modification of \( \mathcal{N} = 2 \) supergravity action

The \( \mathcal{N} = 2 \) supergravity has an Abelian gauge field \( A \), the tetrad fields \( e^i_\alpha \) and two superpartner gravitinos. These gravitinos have chiral projections. One of them, denoted by \( \psi^a_\alpha \), has positive chirality and the other, denoted by \( \bar{\psi}_{\dot{a}} \), has negative chirality. For writing the first order action, we also introduce \( SO(3,1) \) Lie algebra-valued connection one form \( A_{IJ} \). The modified action for \( \mathcal{N} = 2 \) supergravity is given by [77]:

\[
S_{SG2} = S_{SSG2} + S_{MSG2}
\]

where, the term \( S_{SSG2} \) is the standard supergravity action for \( \mathcal{N} = 2 \):

\[
S_{SSG2} = \int_M d^4x \left[ \frac{1}{2} \sum_{a\beta} \bar{F}_{a\beta} \bar{F}^{a\beta} \right] + \frac{1}{2} \sum_{a\beta} \bar{F}_{a\beta} \bar{F}^{a\beta} + \frac{1}{4} \sum_{a\beta} \bar{F}_{a\beta} \bar{F}^{a\beta} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{2} \sum_{a\beta} \bar{F}_{a\beta} \bar{F}^{a\beta} + \frac{1}{4} \sum_{a\beta} \bar{F}_{a\beta} \bar{F}^{a\beta}
\]

where, \( F = da, \bar{\Sigma}_{a\beta} = (e^a_\alpha \wedge e^\alpha_\beta) \) and \( F_{IJ} \) is curvature of \( A_{IJ}, i.e. F_{IJ} = dA_{IJ} + A_{IK} \wedge A^K_J \). The supercovariant field strength is given by

\[
\bar{F}_{ab} = \bar{\partial}_a A_b - \frac{1}{\sqrt{2}} (\bar{\psi}^a \gamma^a \gamma^b \bar{\psi} + \bar{\psi} \gamma^a \gamma^b \bar{\psi})
\]

and the + and − signs denote the self dual and anti self dual fields, \( F^+_ab = \frac{1}{2} (F_{ab} + \ast F_{ab}) \) and \( \ast F_{ab} = \frac{1}{2} e_{abcd} F^{cd} \). The other part of the action \( S_{MSG2} \) is given by:

\[
S_{MSG2} = \int_M d^4x \left[ \frac{1}{2} \sum_{a\beta} \bar{F}_{a\beta} \bar{F}^{a\beta} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{2} \sum_{a\beta} \bar{F}_{a\beta} \bar{F}^{a\beta} + \frac{1}{4} \sum_{a\beta} \bar{F}_{a\beta} \bar{F}^{a\beta}
\]

where \( \gamma \) is the Barbero-Immirzi parameter. Equations of motion and other ramifications can be found in [77].

We are interested in black hole solutions. Moreover, we are restricting to Reissner–Nordstrom type of configurations. These are global supersymmetric solutions. From the point of view of classical solutions, the degrees of freedom of this theory are equivalent to the Holst action of the Einstein–Maxwell system, which is a consistent truncation of the modified Holst action for \( \mathcal{N} = 2 \) supergravity given in Eq. (29). In other words, for studying global classical solutions, we can consistently put the fermion fields to be zero. The KSEs for (29) are identical to that of the Einstein–Maxwell system. In what follows, we shall use the following action:

\[
S_H = \frac{1}{16\pi G} \int_M \Sigma_{IJ} \wedge F^{IJ} - \frac{1}{16\pi G} \int_M e^I \wedge e^J \wedge F^{IJ} - \frac{1}{8\pi} \int_M F \wedge \ast F
\]

(33)

where, \( \Sigma_{IJ} = \frac{1}{2} \epsilon_{IKL} e^K \wedge e^L \) is a 2-form and \( \epsilon_{IKL} \) is the completely antisymmetric tensor in internal space. Variation of the action with respect to the connection \( A_{IJ} \) leads to:

\[
D \Sigma_{IJ} = 0
\]

(34)

It then follows from (34) that \( A_{IJ} \) is the spin connection. Then the variation of the action (33) with respect to the tetrads \( e^I_\alpha \) give Einstein–Maxwell equations [85].
boundary terms that arise from the variation of the action get contributions from the inner and outer boundaries. However, IH (or SWIH) boundary conditions and asymptotic flatness ensure that these boundary terms vanish, making the action principle well-defined [51].

We shall also need the expression of the product of tetrads

\[ \delta_\alpha(\ell^a, n^a, m^a, \bar{m}^a) = 0. \]

Given these internal null vectors and the tetrad \( e^I_a \), we can construct the null vectors \( (\ell^a, n^a, m^a, \bar{m}^a) \) through \( e_a = e^I_a \ell_I \). We can use these information to find the connection. To do this, we first note that since the internal null vectors are fixed, \( \delta_\alpha(\ell^a, n^a, m^a, \bar{m}^a) = 0 \), for the internal vector \( \ell^I \) we get

\[ \nabla_\alpha e^I_\ell = A^I_\alpha J e^I_\ell. \] (35)

Similar expressions can be obtained for the other internal vectors. Using the Eq. (17), the full connection turns out to be:

\[ A^I_\alpha = \frac{2\mu}{m} \ell^I \bar{m} J + 2 \mu \ell^I m J + 2 V^{(m)} m J \] (36)

We define the following Lie algebra valued connection \( A^{(H)}_{\mu} \) for ease of further computation [51, 85]

\[ A^{(H)}_{\mu} = \frac{1}{2} \left( A_{\mu} - \frac{\gamma}{2} \epsilon_{\mu}^{KL} A_{KL} \right) \]

\[ \equiv V^{(m)} (\gamma \ell^I m J + m J \bar{m} J) + \mu m \ell^I m J (1 - i_\mu) \]

\[ + \mu \ell^I m J (1 - i_\mu), \] (37)

We shall also need the expression of the product of tetrads on the horizon \( \Delta \). It is easily determined to be:

\[ e^I \wedge e^I = -2 n \wedge m \bar{m}^I j - 2 n \wedge m \ell^I \bar{m}^I + 2 i m J \bar{m} J \epsilon \] (38)

5.2.2. Symplectic structure

Given the Lagrangian 4-form, there exists specific prescription for constructing the symplectic structure on the space of solutions [27, 28, 79, 80]. One obtains on-shell, the symplectic one-form \( \Theta \) (a spacetime 3-form) from the variation of the Lagrangian, \( \delta L = d \Theta(\delta) \) where \( \delta \) is an arbitrary vector field in the phase space. A brief introduction to the covariant phase-space approach and construction of symplectic structure is given in Appendix 1. For the case in hand, we get

\[ \Theta(\delta) = -\frac{1}{8\pi G_7} \delta_\epsilon (e^I \wedge e^I) \wedge A^{(H)}_{\mu} + \frac{1}{4\pi} \delta A \wedge F \] (39)

From \( \Theta(\delta) \), one then constructs the symplectic current

\[ J(\delta_1, \delta_2) = \delta_1 \Theta(\delta_2) - \delta_2 \Theta(\delta_1) \].

This is closed on-shell and integrating over the entire spacetime, we get (see Fig. 1):

\[ \left( \int_{M_+} - \int_{M_-} \right) J(\delta_1, \delta_2) = \frac{1}{8\pi G_7} \int_{\Delta} [\delta_1^2 \epsilon \wedge \delta_2 iV^{(m)}] - (1 \leftrightarrow 2) \]

\[ + \frac{1}{4\pi} \int_{\Delta} \lbrack \delta_1 A \wedge \delta_2^* F - (1 \leftrightarrow 2) \] (40)

To construct the symplectic structure, we must be careful that no data flows out of the phase space because of our choice of foliation. To ensure this, we will check that the symplectic structure is independent of the choice of foliation. We introduce potentials \( \mu(m) \) and \( \phi(\ell) \):

\[ L_{\mu} \mu(m) \equiv \ell^2 V^{(m)} \] and \[ L_{\ell} \phi(\ell) = \Phi(\ell) \equiv -\ell^2 A_{a} \] (41)

A straightforward calculation shows that (see [51, 60]):

\[ \left( \int_{M_+} - \int_{M_-} \right) J(\delta_1, \delta_2) = \frac{1}{8\pi G_7} \left( \left( \int_{S_+} - \int_{S_-} \right) [\delta_1 \ell \wedge \delta_2 \mu(m)] - (1 \leftrightarrow 2) \right) \]

\[ - \frac{1}{4\pi} \left( \left( \int_{S_+} - \int_{S_-} \right) [\delta_1 F \wedge \delta_2 \ell \phi(\ell)] - (1 \leftrightarrow 2) \right) \] (42)

So only a special combination of the bulk and boundary symplectic current is independent of the choice of foliation. The symplectic structure is that of a Einstein–Maxwell system (see [51] for detailed derivation):

\[ \Omega(\delta_1, \delta_2) = \frac{1}{8\pi G_7} \int_{M} [\delta_1 (e^I \wedge e^I) \wedge A^{(H)}_{\mu} - (1 \leftrightarrow 2)] \]

\[ - \frac{1}{8\pi G_7} \int_{S_+} [\delta_1^2 \epsilon \wedge \delta_2 \mu(m)] - (1 \leftrightarrow 2) \]

\[ + \frac{1}{4\pi} \int_{S_+} [\delta_1 F \wedge \delta_2 \ell \phi(\ell)] - (1 \leftrightarrow 2) \]

\[ - \frac{1}{4\pi} \int_{S_-} [\delta_1 F \wedge \delta_2 \ell \phi(\ell)] - (1 \leftrightarrow 2) \] (43)

The first law can now be derived using this symplectic structure (43). Let us understand the conceptual basis of
this proof. IH is a local definition of a horizon and the first law is expected to relate variations of local quantities that are defined only at the horizon without any reference to the rest of the spacetime. For example, the surface gravity \( \kappa(\delta) \) is defined locally at the horizon. For the first law, the IH formalism enables us to define local energy (for horizons carrying other charges, such as angular momentum, electric potential etc., we must also provide local definitions for them). In spacetime, energy is associated with a timelike Killing vector field. Given any vector field \( W \) in spacetime, it naturally induces a vector field \( \delta_w \) in the phase space. The phase space vector field \( \delta_w \) is the generator of time translation in the phase space. If time translation is a canonical transformation in the phase space then \( \delta_w \) defines a Hamiltonian function \( H_w \). So to find out the Hamiltonian function associated with energy, we must look for phase space transformations that keep the symplectic structure invariant (canonical transformations). The vector fields tangent to these canonical flows are the Hamiltonian vector fields. To check whether a vector field \( \delta_w \) is globally Hamiltonian, one constructs a one-form \( X_W \) where \( X_W(\delta) := \Omega(\delta, \delta_w) \), where \( \delta_w \) is the lie flow \( \mathcal{L}_W \) generated by the spacetime vector field \( W^a \) when tensor fields are varied. The necessary and sufficient condition for the vector field \( \delta_w \) to be a globally Hamiltonian vector field is that the one-form \( X_W \) is to be exact, \( X_W = \text{d}H_W \) where \( \text{d} \) is the exterior derivative in phase space and \( H_W \) is the corresponding Hamiltonian function. In other words, the vector field \( \delta_w \) is globally Hamiltonian if and only if \( X_W(\delta) = \delta H_W \) for any vector field \( \delta \) in the phase space. The vector fields \( W^a \) are also restricted by the condition that it should be tangential on \( \Delta \). Now being a null surface, the WIH has only three tangential directions, one null and the two other spacelike. The closest analog of ‘time’ translation on WIH is therefore translation along the null direction. It is generated by the vector field \( \partial_{\ell} \). For global solutions this null normal vector field becomes timelike outside the horizon and is expected to match with the asymptotic time-translation for asymptotically flat spacetimes.

Using the above considerations, the first law of supersymmetric horizons turns out to be:

\[
X_{(\ell)}(\delta) \triangleq \Phi_{(\ell)} \delta Q_{\Lambda} + \delta E_{(\ell)}, \tag{44}
\]

where \( E_{(\ell)} \) is the ADM energy obtained when \( \ell^a \) matches with the time translation at infinity and \( Q \triangleq (1/4\pi) \int_{\Sigma} \mathcal{F} \) is the charge of the electromagnetic field on the horizon. The right hand side of (44) is an exact variation if and only if \( \Phi_{(\ell)} \) is a function of \( Q_{\Lambda} \) alone. The phase space is characterized by charge and so \( \Phi_{(\ell)} \) is a function of \( Q_{\Lambda} \). Define a quantity \( E_{\Lambda} \)

\[
\delta E_{\Lambda} \triangleq \Phi_{(\ell)} \delta Q_{\Lambda} \tag{45}
\]

such that \( H_{\ell} = E_{(\ell)} - E_{\Lambda} \) where \( H_{\ell} \) is the associated Hamiltonian function \( \Phi_{(\ell)} = \delta H_{\ell} \). It is natural to interpret \( E_{\Lambda} \) as the locally defined energy of the WIH and (45) as the first law of the WIH. The quantity \( H_{\ell} \) receives contributions both from the bulk as well as the boundary symplectic structures and stands for the energy of the region between the WIH and the spatial infinity. The ADM energy \( E_{(\ell)} \) is the sum total of these two energies. For the global solutions we are interested in, it is well known that the first law is equivalent to:

\[
\delta E_{\text{ADM}} \triangleq \Phi_{(\ell)} \delta Q_{\Lambda} \tag{46}
\]

i.e., \( E_{\Lambda} = E_{\text{ADM}} \). To see this, observe that for all global solutions, \( H_{\ell} = 0 \). This is because when there is a global Killing vector field, \( \delta_{\ell} \) induces infinitesimal gauge transform and is thus a gauge direction,

\[
\Omega(\delta, \delta_{\ell}) = \delta H_{\ell} = 0, \tag{47}
\]

for all \( \delta \) on the phase space. So, for any connected component of the phase space consisting of the spacetimes with global Killing vector field, \( H_{\ell} \) is a constant. This constant can only be some spacetime quantity and can be the cosmological constant. For the present case, the cosmological constant vanishes and hence \( H_{\ell} \) vanishes too and hence the energy measured at \( \Delta \) is same as that measured by any ADM observer. This means that this first law is exactly equivalent to that for the event horizons, with ADM replaced by \( \Delta \) in (44). This is a consistency check for the IH formulation.

5.3. Chern–Simons theory and entropy

In the introduction, we have said that the effective theory residing on the horizon can only be a topological theory. In this section, we shall outline the derivation of the Chern–Simons theory on \( \Delta \), details are similar to the ones in [51].

Let us now restrict to fixed area and fixed charge phase space. Define the connection component \( V^{(H)} = iV^{(m)}/2 \). In this case of spherical symmetry, it can be shown that the Gauss-Bonnet theorem implies that the Eq. (27) reduce to [51, 64]

\[
dV^{(H)} \triangleq -\epsilon [2\pi / A_{s}^{2}] \tag{48}
\]

This condition is also called the quantum horizon condition. The subscript \( s \) indicates that we are in spherically symmetric phase space. Putting Eq. (48) in (40) and integrating by parts, we see that:
\[ \Omega(\delta_1, \delta_2) = \frac{1}{16\pi G_I} \int_M \left[ \delta_1 (e' \wedge e') \wedge \delta_2 \mathcal{A}_I^{(H)} - \delta_2 (e' \wedge e') \wedge \delta_1 \mathcal{A}_I^{(H)} \right] + \frac{1}{8\pi G_I} \frac{\mathcal{A}_I^{(H)}}{\pi} \int_S \left\{ \delta_1 V^{(H)g} \wedge \delta_2 V^{(H)g} \right\} - \frac{1}{4\pi} \int_M \left[ \delta_1 \mathcal{F} \wedge \delta_2 \mathcal{A} - (1 - 2) \right]. \]

where \( V^{(H)g} = V^{(H)} + d\mu_{(m)}/2. \) Note that the Maxwell field does not give any contribution to the entropy (see [51]). The boundary symplectic structure turns out to be that of \( U(1) \) Chern–Simons theory.\(^{10}\) The level of the theory \( k = \mathcal{A}_I^{(H)}/4\pi G_I \) takes integer values on quantization.

The entropy of the horizon can be obtained by quantization of \( U(1) \) Chern–Simons theory and hence counting states. The details of the quantization technique and various ramifications have been calculated in details [65]. The counting of states and entropy computation was first done in [65]. Better state counting methods have since been proposed [81–83] and the one put forward in [84] has carefully reconsidered some intricacies in the counting. The essential idea is the following: Consider a horizon of area \( \mathcal{A}_I^{(H)} \). To compute the entropy, these states are relevant which satisfy the quantum horizon condition and have the fixed area of value \( \mathcal{A}_I^{(H)} \). Entropy is obtained by taking logarithm of this value. The detailed counting of the microscopic quantum states of black hole is based on loop quantum gravity. It is proposed that the states are characterized by means of spin network basis [85]. If an edge with label \( j_i \) ends at the horizon \( S_\Lambda \), it creates a puncture with label \( j_i \). The area of the horizon will be given by the value \( 8\pi L_p^2 \sum_i \sqrt{j_i(j_i + 1)} \), \( L_p \) being the Planck length. The punctures are also labeled by the half-integers \( m_i \) where \(-j_i \leq m_i \leq j_i \). The quantum horizon condition relates this eigenstates to that of Chern–Simons theory. The requirement that the horizon is a sphere imposes the constraint \( \sum_i m_i = 0 \). Thus the quantum state associated with cross-section of horizon are characterized by punctures and spin quantum numbers \( j, m \) associated to each punctures label the states. Counting of states establishes that the entropy is indeed proportional to the area of the horizon.

6. Conclusions

The objective of this paper was to introduce a new way of calculating the entropy of extremal black holes in supergravity theories. This involves subtleties of taking extremal limits on the space of solutions. Instead of modifying the Wald formulation, we argued that the isolated horizon (IH) formulation of black hole may be better suited to address the problems of extremal limits. We matched the boundary conditions precisely and showed that it is possible to include the black holes arising in pure \( \mathcal{N} = 2 \) supergravity in the space of solutions this theory with IH as an inner boundary. Moreover, we proved the laws of black hole mechanics for these black holes and then went on to show that the entropy of these black holes can be easily determined by quantizing the effective Chern–Simons theory that resides on the inner boundary of these black holes.

Let us now spell out the the advantages of this framework. First, note that the supersymmetric generalisation of the isolated horizon formalism does not require the entire spacetime to be supersymmetric. It is very much a possibility that the spacetime just outside the horizon (and may be the entire spacetime) is non-supersymmetric (in the sense that there are no Killing spinors that generate supersymmetry as isometries) because of presence of time dependent fields like electromagnetic and gravitational, while only the horizon itself is supersymmetric. This implies that the horizon supports some Killing spinors and is in equilibrium. This is a general situation: only the horizon is required to be supersymmetric without any restriction the the nature of the spacetime outside the horizon. So, this formulation admits a larger class of spacetime in its phase-space than those admitted by the Killing Horizon framework or the event horizon framework which require a part or the entire spacetime to be supersymmetric respectively. For example, the formalism of entropy functional developed in [42, 44], depends on the near horizon structure. This is understandable since the entropy functional formalism is a reformulation of the Wald entropy formula, based on the Killing horizon framework. In fact, the appearance of the AdS space in the near horizon, which is a crucial ingredient in the entropy functional formula, implies that the near horizon geometry is maximally supersymmetric. On the other hand, the formalism developed here does not depend on the supersymmetry of the near horizon structure and only the supersymmetry of the horizon is enough to develop and calculate the entropy. Secondly, the method of determining the entropy is direct. It does not depend on the geometry or asymptotic structures (as is done for example in Kerr–CFT approach [87]) but is based on the quantization of the horizon topological theory induced as a result of the bulk-boundary gravitational interaction.

Let us take this opportunity to discuss the issues related to the gauge group of the Chern–Simons theory residing on the horizon. In [64–66], the authors argued that the theory on the horizon is a \( U(1) \) Chern–Simons theory. However, in [67, 68], the authors determined the dimensionality of the boundary Hilbert space by arguing that the gauge group must be SU(2) and that a relation between the horizon Chern–Simons theory and the conformal blocks of the the Wess–Zumino model on a two-sphere. This issue was
Par59 This metric may also be constructed using the one forms \( \ell^a \) and \( n^a \), through \( q_{ab} = g_{ab} + 2 \ell_a n_b \), introduced below in this section.

If the spacetime signature is Lorentzian, any null surface will have one null vector and two spacelike vectors. For the case in hand, these will be \((\partial/\partial v)^a\), \((\partial/\partial \theta)^a\) and \((\partial/\partial \phi)^a\). The vector field \((\partial/\partial v)^a\) is null and a Killing vector field of \((50)\). It is also clear that topologically, \( \Delta = S^2 \times \mathbb{R} \), where \( \mathbb{R} \) is the null direction parametrised by \( v \).

It is useful to introduce a linear combination of the coordinate basis vectors. We call the basis the Newman-Penrose basis:

\[
\ell^a = \left( \frac{\partial}{\partial v} \right)^a - (1 - 2M/r) \left( \frac{\partial}{\partial r} \right)^a
\]

\[
n^a = - \left( \frac{\partial}{\partial r} \right)^a
\]

\[
m^a = \frac{1}{\sqrt{2r_\Delta}} \left[ \left( \frac{\partial}{\partial \theta} \right)^a + i \frac{\sin \theta}{\sin \theta} \left( \frac{\partial}{\partial \phi} \right)^a \right]
\]

(54)

It is not difficult to see that \( \ell . n = -1, m . \bar{m} = 1 \) an the rest are equal to zero. On \( \Delta \), we have:

\[
\ell^a \triangleq \left( \frac{\partial}{\partial v} \right)^a
\]

(55)

\[
m^a \triangleq \frac{1}{\sqrt{2r_\Delta}} \left[ \left( \frac{\partial}{\partial \theta} \right)^a + i \frac{\sin \theta}{\sin \theta} \left( \frac{\partial}{\partial \phi} \right)^a \right]
\]

(56)

Interestingly, the vector field \( n^a \) does not exist on \( \Delta \) because it is not tangential. Thus, the tangent space \( T_p \Delta \) consists of \((\ell^a, m^a, \bar{m}^a)\). The relevant one forms are:

\[
\ell_a \triangleq 0
\]

(57)

\[
n_a \triangleq -(dv)_a
\]

(58)

\[
m_a \triangleq \frac{r_\Delta}{\sqrt{2}} [(d\theta)_a + i \sin \theta (d\phi)_a]
\]

(59)

From the expression of the null normal \( \ell_a \) in \( \mathcal{M} \) [see Eq. \((52)\)], we get \( \nabla_a \ell_b = -\frac{M}{r} \delta_a^s \delta_{b'} \). This implies that \( \nabla_a \ell_b \triangleq 0 \).

(60)

Thus, for the horizon of the Schwarzschild solution, it follows naturally that \( \nabla_a \ell_b \triangleq 0 \). Thus, all the conditions of isolated horizon are satisfied for the horizon of the Schwarzschild spacetime.

Appendix 2: The formalism of covariant phase-space

The covariant phase-space formalism has been explained in great detail in \([78–80]\). In the following, we give a short introduction to this method with the help of an example from particle mechanics. In one dimensions, the Lagrangian, written as a 1-form is given by
where \( m \) is the mass of the particle, \( q \) is the coordinate and \( V(q) \) is the potential. For any arbitrary variation \( \delta \) of the configuration variable \( q \), the variation of the Lagrangian is:

\[
\delta L = [m \dot{q} \delta q - V(q) \delta \dot{q}] dt
\]

(61)

where the \( \Theta(\delta) = m \dot{q} \delta q \). This quantity \( \Theta(\delta) \) is called the symplectic potential. The symplectic current which is defined as the second variation of the Lagrangian, is given by \( J(\delta_1, \delta_2) = \delta_1 \Theta(\delta_2) - \delta_2 \Theta(\delta_1) \), where \( \delta_1 \) and \( \delta_2 \) are independent variations. In this particular case of one-dimensional potential problem, this current is also called the symplectic structure, denoted by \( \Omega(\delta_1, \delta_2) \).

\[
\Omega(\delta_1, \delta_2) = (\delta_1 \rho \delta_2 q - \delta_2 \rho \delta_1 q),
\]

(64)

where we have used \( m \dot{q} = \rho \). This quantity \( \Omega(\delta) \) is a two form on the phase-space with \( \delta_1 \) and \( \delta_2 \) as two arbitrary vector fields in the phase-space.

Now, given a vector field \( X^a \) on the spacetime, it induces a vector field \( \delta X = \delta X^a \partial_a \) on the phase space. The vector field \( X^a \) is called a Hamiltonian vector field if \( \delta \partial X = \delta h X \). The quantity \( h X \) is called the Hamiltonian function corresponding to the vector field \( X^a \) on the spacetime. For example, let us take \( X^a = \tilde{t}^a = (\tilde{t}/\partial \tilde{t})^{a} \), which is the generator of time-translation. Then, we get that

\[
\Omega(\delta, \dot{\delta}) = \delta \dot{\rho} L - \tilde{t} \dot{\rho} \delta q
\]

\[
= \delta \left[ \frac{\tilde{t}^2}{2m} + V(q) \right] \equiv \delta H,
\]

(65)

where, we have used the on-shell condition. Note that \( H = \tilde{t}^2/2m + V(q) \), is the usual Hamiltonian function.

In the Sect. 5, we have used the first order gravity Lagrangian to find the symplectic structure and the first law arises as the Hamiltonian corresponding to the vector field generating the horizon.

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