Entanglement for multipartite systems of indistinguishable particles

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Abstract

We analyze the concept of entanglement for multipartite system with bosonic and fermionic constituents and its generalization to systems with arbitrary parastatistics. We use the representation theory of symmetry groups to formulate a unified approach to this problem in terms of simple tensors with appropriate symmetry. For an arbitrary parastatistics, we define the S-rank generalizing the notion of the Schmidt rank. The S-rank, defined for all types of tensors, serves for distinguishing entanglement of pure states. In addition, for Bose and Fermi statistics, we construct an analog of the Jamiołkowski isomorphism.

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1 Introduction

Confronted with entangled states of a bipartite system, Schrödinger said that entanglement was not ...one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical line of thought... [1].

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Note that Schrödinger’s interests for this aspect of Quantum Mechanics originated from the EPR paper. Indeed, the entanglement for states of composite systems created quite an embarrassment for theoretical physicists not sharing the point of view of the “Copenhagen interpretation” of quantum mechanics. As a matter of fact, the entanglement played a rather important role in the development of quantum mechanics, as it obliged the physics community to face the nonlocal nature of the description of natural processes. Nowadays, however, entanglement is considered as an important resource for quantum computation, quantum information and quantum teleportation.

The usual definition of entangled states is given by defining first separable pure states as decomposable tensor products and then declaring a state to be entangled if it is not separable.

In the case of a system containing identical constituents, the problem of the definition of entanglement has to be reconsidered, since the indistinguishability produces some amount of entanglement by its own. For instance, a non-zero skew-symmetric 2-tensor is never decomposable. We cannot therefore simply transpose to the Hilbert space of the composite system the requirement that non-entanglement is guaranteed by the factorization of the total state into a simple tensor product of vectors corresponding to the two subsystems. We have to analyze better the meaning of entanglement per se and refine the concept of separability.

Several nonequivalent ways of identification and quantification of the phenomenon were proposed. In [2] the authors explicitly stated the problem of distinguishing non-local correlations from those caused by the Pauli principle measured in fundamental experiments, aiming at checking the validity of quantum mechanics involving two identical fermions. To quantify the amount of extra correlations, the authors used the number of terms in the decomposition of the wave function in terms of elementary $2 \times 2$ ‘Slater determinants’ (see Section 6 below). Similar ideas, based on the lengths of the canonical decomposition of the wave function, were used in [3] for description of correlations in double ionization of atoms in strong electromagnetic fields. A scheme of extracting the correlations caused by the Pauli principle in fermionic systems in order to characterize a genuine entanglement, became important in analysis of elementary quantum gates based on quantum dots [4]. Subsystems here are no longer separated by macroscopic distances; on the contrary, they occupy the same spatial regions and their indistinguishability becomes relevant. The problem was analyzed in [5] and further in [6] in terms of modern theory of entanglement. Measures of correlations were constructed in analogy with the distinguishable particles case, again by employing algebraic properties of the coefficient matrix in the expansion of a state in terms of basis states.

Similar ideas can be applied, and in fact were applied, to bosons [6, 7]. In the bosonic case non-entangled states are again identified with simple symmetric tensors (see Section 4 below). Thus, in the case of two bosons a state is non-entangled if and only if it is a tensor product of two identical one-particle states. This definition should be contrasted with another one advocated already in [8], where bosonic and fermionic systems were treated in parallel. To identify two partite systems with minimal identical-particle correlations, i.e., without entanglement, one considers a situation when two particles are confined to two non-overlapping spatial regions in which they are subjected to spatially confined independent measurements. The correlations are minimal if, for every independent choice of measurements, the probability of outcomes factorizes into the product of single particles probabilities of outcomes. For two fermions, the result reduces to the demand that the wave function of the whole system is a simple antisymmetric tensor (a single Slater determinant), $|\Psi\rangle \sim (|\psi\rangle \otimes |\phi\rangle - |\phi\rangle \otimes |\psi\rangle )$, i.e., to the very definition adopted by other authors cited above. For bosons, however, the definition leads to two different classes of minimally correlated state. In addition to tensor products of identical one-particle
states, also wave functions in the form of Slater permanents i.e., symmetrized tensor products of orthogonal one-particle states, \(|\Psi\rangle \sim (|\psi\rangle \otimes |\phi\rangle + |\phi\rangle \otimes |\psi\rangle)\), \(\langle \psi|\phi \rangle = 0\), are classified as minimally entangled.\(^4\) The definition can be easily extended to many particle systems.\(^{[9]}\)

In a series of papers by Ghirardi et. al.\(^{[10, 11, 12]}\), a definition of non-entangled state was based on the concept of a complete set of properties possessed by a constituent of a composite system. For a bi-partite system, in a pure state \(\rho\), we say that a one of the constituents has a complete set of properties if and only if there exists a rank-one projection operator \(P\) acting in a single particle space \(\mathcal{H}\) such that \(\text{Tr}(E\rho) = 1\), where \(E = P \otimes I + I \otimes P - P \otimes P\) is a projection operator acting in \(\mathcal{H} \wedge \mathcal{H}\) or \(\mathcal{H} \vee \mathcal{H}\) for, respectively, fermions and bosons. This idea resembles the one advocated in the above cited papers of Herbut, but rather than invoking directly ‘local’ measurements it stresses properties of subsystem states. The result is the same. For fermions, non-entangled states are antisymmetrized tensor products, whereas for bosons they split into two classes: products of two identical states, or symmetrized products of two orthogonal states. Whereas for fermions the criterion of non-entanglement, based on the number of coefficients in expansion in terms of elementary \(2 \times 2\) Slater determinants, remains functional, it is not the case for bosons. Here, states expressible as linear combinations of two Slater permanents are non-entangled if the both coefficients of expansion are the same – such states can be expressed as symmetrized products of two orthogonal vectors.\(^{[9, 10]}\) A concept of entanglement for indistinguishable particles based on a measurability of correlations was recently considered also in\(^{[13]}\).

In this paper we attempt to approach this problem from a mathematical viewpoint, where structures and available mathematical constructions are used as a guide. A fundamental character of our work depends also on reviewing and generalizing basic concepts of mathematical foundations in understanding entanglement. We shall provide few physical considerations in the conclusions, where we also briefly compare our results with other outlined above. Here, let us mention only that our natural and unifying mathematical model strongly suggests non-entanglement for bosons to be associated with tensor products of identical states.

The paper is organized in the following way. In Section 2 we recall definitions and basic facts from tensor algebra underlying our analysis of entanglement for multipartite bosonic and fermionic systems. In Section 3 we introduce the concept of duality and analyze contractions between dual spaces of tensors, which allows us to define the S-rank of a tensor, generalizing the Schmidt rank, and then the simplicity of a tensor in Section 4. Section 5 contains various characterizations of simplicity of a tensor in general, and in the bosonic or fermionic class. In Section 6 we define entanglement for bosonic and fermionic multipartite states and provide a simple characterization of entanglement for pure states in terms of bilinear functions in coefficients of representing tensors. In analysis of entanglement of distinguishable particles, the so called Jamiołkowski isomorphism played the role of a very useful tool. We give an extension of it for boson and fermions in Section 7. The mathematically rigorous generalization of entanglement to multipartite system with arbitrary parastatistics, another novel invention in this paper, is given in Section 8.

Note that such an unifying approach allowed also for description of pure non-entangled states as the images of generalized Segre maps, in full analogy with the case of distinguishable particles.\(^{[14, 15]}\) These questions, however, we decided to postpone to a separate paper.

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2 Tensor algebras

To describe some properties of systems composed of indistinguishable particles and to fix the notation, let us start with introducing corresponding tensor algebras associated with a Hilbert space $H$. For simplicity, we assume that $H$ is finite-, say, $n$-dimensional, but a major part of our work remains valid also for Hilbert spaces of infinite dimensions. Note only that in the infinite dimensions the corresponding tensor product $H_1 \otimes H_2$ is the tensor product in the category of Hilbert spaces, i.e., corresponding to Hilbert-Schmidt norm.

In the tensor power $H^{\otimes k} = H \otimes \cdots \otimes H$, we distinguish the subspaces: $H^{\vee k} = H \vee \cdots \vee H$ of totally symmetric tensors, and $H^{\wedge k} = H \wedge \cdots \wedge H$ of totally antisymmetric ones, together with the symmetrization, $\pi^\vee_k : H^{\otimes k} \to H^{\vee k}$, and antisymmetrization, $\pi^\wedge_k : H^{\otimes k} \to H^{\wedge k}$, projectors:

$$\pi^\vee_k (f_1 \otimes \cdots \otimes f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)},$$

$$\pi^\wedge_k (f_1 \otimes \cdots \otimes f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)}.$$

Here, $S_k$ is the group of all permutations $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$, and $(-1)^\sigma$ denotes the sign of the permutation $\sigma$. Note that with every permutation $\sigma \in S_k$ there is a canonically associated unitary operator $U_\sigma$ on $H^{\otimes k}$ defined by

$$U_\sigma (f_1 \otimes \cdots \otimes f_k) = f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)},$$

so that the map $\sigma \mapsto U_\sigma$ is an injective unitary representation of $S_k \to U(H^{\otimes k})$. We will write simply $\sigma$ instead of $U_\sigma$, if no misunderstanding is possible. Symmetric and skew-symmetric tensors are characterized in terms of this unitary action by $\sigma(v) = v$ and $\sigma(v) = (-1)^\sigma v$, respectively, for all $\sigma \in S_k$.

We put, by convention, $H^{\otimes 0} = H^{\vee 0} = H^{\wedge 0} = \mathbb{C}$. It is well known that the obvious structure of a unital graded associative algebra on the graded space $H^\otimes = \bigoplus_{k=0}^\infty H^{\otimes k}$ (the tensor algebra) induces canonical unital graded associative algebra structures on the spaces $H^\vee = \bigoplus_{k=0}^\infty H^{\vee k}$ (called the bosonic Fock space) and $H^\wedge = \bigoplus_{k=0}^\infty H^{\wedge k}$ (called the fermionic Fock space) of symmetric and antisymmetric tensors, with the multiplications

$$v_1 \vee v_2 = \pi^\vee(v_1 \otimes v_2),$$

$$w_1 \wedge w_2 = \pi^\wedge(w_1 \otimes w_2).$$

Here, of course,

$$\pi^\vee = \bigoplus_{k=0}^\infty \pi^\vee_k : H^\otimes \to H^\vee,$$

and

$$\pi^\wedge = \bigoplus_{k=0}^\infty \pi^\wedge_k : H^\otimes \to H^\wedge,$$

are the symmetrization and antisymmetrization projections. Note that the multiplication in $H^\vee$ is commutative, $v_1 \vee v_2 = v_2 \vee v_1$, and the multiplication in $H^\wedge$ is graded commutative, $w_1 \wedge w_2 = (-1)^{k_1 k_2} w_2 \wedge w_1$, for $w_i \in H^{\wedge k_i}$. 


Denote with $H^*$ the complex dual space of $H$. The Hermitian product $\langle \cdot | \cdot \rangle$ on $H$ induces a canonical bijection between $H$ and $H^*$ which, in the Dirac’s notation, reads

$$H \ni |x\rangle \mapsto \langle x| \in H^*.$$  

We must stress that this is not an isomorphism of complex linear spaces, since the above map is anti-linear. Note, however, that the symmetric tensor algebra $H^\vee$ can be canonically identified with the algebra $Pol(H^*)$ of polynomial functions on the complex dual $H^*$ of $H$. Indeed, any $f \in H$ can be identified with a linear function $x_f$ on $H^*$ by means of the canonical pairing $\langle \cdot , \cdot \rangle : H \times H^* \to \mathbb{C}$ between the dual spaces, as $x_f(y) = \langle f, y \rangle$ (we must distinguish this pairing from the Hermitian product on $H$). This can be extended to an isomorphism of commutative algebras in which $f_1 \vee \cdots \vee f_k$ corresponds to the homogenous polynomial $x_{f_1} \cdots x_{f_k}$. Similarly, one identifies $H^\wedge$ with the Grassmann algebra $Grass(H^*)$ of polynomial (super)functions on $H^*$. Here, however, with $f \in H$ we associate a linear function $\xi_f$ on $H^*$ regarded as and odd function: $\xi_f \xi_f' = -\xi_{f'} \xi_f$. In the language of supergeometry one speaks about the purely odd manifold $\Pi H^*$ obtained from the standard (purely even) linear manifold $H^*$ by changing the parity of linear functions. In this sense, $H^\wedge$ is the algebra of holomorphic (super)functions on the complex supermanifold $H^*$.

If we fix a basis $e_1, \ldots, e_n$ in $H$ and associate with its elements even linear functions $x_1, \ldots, x_n$ on $H^*$, and odd linear functions $\xi_1, \ldots, \xi_n$ on $\Pi H^*$, then $H^\vee \simeq \mathbb{C}[x_1, \ldots, x_n]$, i.e. $H^\vee$ becomes isomorphic with the algebra of complex polynomials in $n$ commuting variables. Similarly, $H^\wedge \simeq \mathbb{C}[\xi_1, \ldots, \xi_n]$, i.e., $H^\wedge$ is isomorphic with the algebra of complex Grassmann polynomials in $n$ anticommuting variables. The subspaces $H^\vee_k$ and $H^\wedge_k$ correspond to homogeneous polynomials of degree $k$. It is straightforward that homogeneous polynomials $x_{i_1} \cdots x_{i_m}$, with $k_1 + \cdots + k_n = k$, form a basis of $H^\vee_k$, while homogeneous Grassmann polynomials $\xi_{i_1} \wedge \cdots \wedge \xi_{i_k}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, form a basis of $H^\wedge_k$. In consequence, $\dim H^\vee_k = \binom{n+k-1}{k}$ and $\dim H^\wedge_k = \binom{n}{k}$, so the gradation in the fermionic Fock space is finite (for finite-dimensional $H$).

Of course, we can put together both algebras and consider the tensor product $H^\vee_k \otimes H^\wedge_l$ with $\dim H_1 = n$ and $\dim H_2 = m$, which is a graded associative algebra with a bi-gradation $\mathbb{N} \times \mathbb{N}$ concentrated in $\mathbb{N} \times \{0, 1, \ldots, m\}$,

$$H^\vee \otimes H^\wedge = \bigoplus_{(k,l) \in \mathbb{N} \times \{0,1,\ldots,m\}} H^\vee_k \otimes H^\wedge_l. \quad (7)$$

Here, $H^\vee_k \otimes H^\wedge_l$ represent systems composed of $k$ bosons, described by a Hilbert space of dimension $n$, and $l$ fermions in the Hilbert space of dimension $m$. The whole algebra $H^\vee \otimes H^\wedge$ is the algebra of polynomial functions on the linear supermanifold $H^\vee_1 \times \Pi H^\wedge_2$ of dimension $(n,m)$. Such functions are written as finite complex linear combinations

$$\sum_{k_1, \ldots, k_n} a_{k_1, \ldots, k_n} x_{i_1}^{k_1} \cdots x_{i_n}^{k_n} \xi_{i_1} \cdots \xi_{i_l}. \quad (8)$$

Note that any basis $\{e_1, \ldots, e_n\}$ in $H$ induces a basis $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} | i_1, \ldots, i_k \in \{1, \ldots, n\}\}$ in $H^\otimes_k$. Therefore, any $u \in H^\otimes_k$ can be uniquely written as a linear combination

$$u = \sum_{i_1, \ldots, i_k = 1}^n u^{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}. \quad (9)$$
If \( u \in \mathcal{H}^\vee k \), then the tensor coefficients \( u^{i_1 \ldots i_k} \) are totally symmetric and, after applying the symmetrization projection to (9), we get

\[
    u = \sum_{i_1, \ldots, i_k = 1}^{n} u^{i_1 \ldots i_k} e_{i_1} \vee \ldots \vee e_{i_k}.
\]

(10)

Similarly, if \( u \in \mathcal{H}^\wedge k \), the tensor coefficients \( u^{i_1 \ldots i_k} \) are totally antisymmetric and

\[
    u = \sum_{i_1, \ldots, i_k = 1}^{n} u^{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}.
\]

(11)

We will refer to the coefficients \( u^{i_1 \ldots i_k} \) as to the coefficients of \( u \) in the basis \( \{e_1, \ldots, e_n\} \).

### 3 Tensor duality and contractions

Starting with the canonical duality (pairing) \( \langle , \rangle \) between vectors from \( \mathcal{H} \) and covectors from \( \mathcal{H}^* \), with an obvious prolongation to a paring between \( \mathcal{H}^\otimes k \) and \( (\mathcal{H}^*)^\otimes k \),

\[
    \langle f_1 \otimes \cdots \otimes f_k, g_1 \otimes \cdots \otimes g_k \rangle = \prod_{i=1}^{k} \langle f_i, g_i \rangle,
\]

(12)

and viewing symmetric and antisymmetric tensors as canonically embedded in the tensor algebra, we find canonical pairings between \( \mathcal{H}^\vee k \) and \( (\mathcal{H}^*)^\vee k \), as well as between \( \mathcal{H}^\wedge k \) and \( (\mathcal{H}^*)^\wedge k \).

For \( f_1, \ldots, f_k \in \mathcal{H} \) and \( g_1, \ldots, g_k \in \mathcal{H}^* \), we get

\[
    \langle f_1 \vee \cdots \vee f_k, g_1 \vee \cdots \vee g_k \rangle = \frac{1}{(k!)^2} \sum_{\sigma, \tau \in S_k} \prod_{i=1}^{k} \langle f_{\sigma(i)}, g_{\tau(i)} \rangle = \frac{1}{k!} \text{per}(\langle f_i, g_j \rangle).
\]

(13)

Here, \( \sum_{\tau \in S_k} \prod_{i=1}^{k} a_{i\tau(i)} = \text{per}(a_{ij}) \) is the permanent of the matrix \( A = (a_{ij}) \). Similarly,

\[
    \langle f_1 \wedge \cdots \wedge f_k, g_1 \wedge \cdots \wedge g_k \rangle = \frac{1}{k!} \text{det}(\langle f_i, g_j \rangle).
\]

(14)

Quite similarly one can prove that the natural Hermitian product on \( \mathcal{H}^\otimes \), for which the grading is the decomposition into orthogonal subspaces and

\[
    \langle f_1 \otimes \cdots \otimes f_k | f_1' \otimes \cdots \otimes f_k' \rangle = \prod_{i=1}^{k} \langle f_i | f_i' \rangle,
\]

(15)

induces Hermitian products on the subspaces \( \mathcal{H}^\vee k \) and \( \mathcal{H}^\wedge k \) which read, respectively,

\[
    \langle f_1 \vee \cdots \vee f_k | f_1' \vee \cdots \vee f_k' \rangle = \frac{1}{k!} \text{per}(\langle f_i | f'_j \rangle),
\]

(16)

\[
    \langle f_1 \wedge \cdots \wedge f_k | f_1' \wedge \cdots \wedge f_k' \rangle = \frac{1}{k!} \text{det}(\langle f_i | f'_j \rangle).
\]

(17)

Given a basis \( f_1, \ldots, f_n \) of \( \mathcal{H} \) and the dual basis \( f_1^*, \ldots, f_n^* \) of \( \mathcal{H}^* \), we have the induced bases:

\[
    f_1^{k_1} \vee \cdots \vee f_n^{k_n}, \quad k_1 + \cdots + k_n = k
\]

(18)
Similarly, 
\[ v = (f_1^*)^{k_1} \lor \cdots \lor (f_n^*)^{k_n}, \quad k_1 + \cdots + k_n = k, \]  
and
\[ f_1^* f_1 \land \cdots \land f_k^* f_k, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n. \]

Analogously, any orthonormal basis \( e_1, \ldots, e_n \) of \( H \) induces canonical orthonormal bases
\[ \sqrt{k!} e_1 \land \cdots \land e_k, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \]

of \( H^{\land k} \), and
\[ \sqrt{k!} e_1 \lor \cdots \lor e_k, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \]

of \( H^{\lor k} \).

The canonical pairings between \( H^{\lor k} \) and \( (H^*)^{\lor k} \) on one hand, and \( H^{\land k} \) and \( (H^*)^{\land k} \) on the other, can be generalized to certain 'pairings' (contractions or inner products) between \( H^{\lor k} \) and \( (H^*)^{\lor k} \) on one hand, and \( H^{\land k} \) and \( (H^*)^{\land k} \) on the other. For the standard simple tensors \( f = f_1 \otimes \cdots \otimes f_k \in H^{\otimes k} \) and \( g = g_1 \otimes \cdots \otimes g_l \in (H^*)^{\otimes l} \), we just put
\[ \iota_g f = (f_1 \otimes \cdots \otimes f_{l+1} \otimes \cdots \otimes f_k) \]
if \( l \leq k \) and \( \iota_g f = 0 \) if \( l > k \), and extend it by linearity to all tensors. It is easy to see now that, if \( v = f_1 \lor \cdots \lor f_k \in H^{\lor k} \subset H^{\otimes k} \) and \( v = g_1 \lor \cdots \lor g_l \in (H^*)^{\lor l} \subset (H^*)^{\otimes l} \) then \( \iota_g v \in H^{\lor (k-l)} \).

Similarly, \( \iota_w \omega \in H^{\land (k-l)} \), if \( w \in H^{\land k} \subset H^{\otimes k} \) and \( \omega \in (H^*)^{\land l} \subset (H^*)^{\otimes l} \). Explicitly,
\[ \iota_{g_1 \lor \cdots \lor g_l} f_{l+1} \lor \cdots \lor f_k = \frac{1}{k! l!} \sum_{S \in S(l,k-l), \tau \in S_l} \prod_{j=1}^l (f_{\sigma(j)}, g_{\tau(j)}) f_{\sigma(l+1)} \otimes \cdots \otimes f_{\sigma(k)} \]
\[ = \frac{(k-l)!}{k!} \sum_{S \in S(l,k-l), \tau \in S_l} \prod_{j=1}^l (f_{S(j)}, g_{\tau(j)}) f_{S(l+1)} \lor \cdots \lor f_{S(k)}, \]
where \( S(l,k-l) \) denotes the set of all \((l,k-l)\) shuffles.

For skew-symmetric tensors,
\[ \iota_{g_1 \land \cdots \land g_k} f_1 \land \cdots \land f_k = \frac{1}{k! l!} \sum_{S \in S_k, \tau \in S_l} (-1)^{\sigma} (-1)^{\tau} \prod_{j=1}^l (f_{\sigma(j)}, g_{\tau(j)}) f_{\sigma(l+1)} \land \cdots \land f_{\sigma(k)} \]
\[ = \frac{(k-l)!}{k!} \sum_{S \in S(l,k-l), \tau \in S_l} (-1)^{\sigma} (-1)^{\tau} \prod_{j=1}^l (f_{S(j)}, g_{\tau(j)}) f_{S(l+1)} \land \cdots \land f_{S(k)}, \]

In particular,
\[ \iota_{g_1 \lor \cdots \lor g_k} f_{l+1} \lor \cdots \lor f_k = (f_1 \lor \cdots \lor f_k, g_1 \lor \cdots \lor g_k) = \frac{1}{k!} \per((f_i|g_j)), \]
and
\[ \iota_{g_1 \wedge \cdots \wedge g_k} f_1 \wedge \cdots \wedge f_k = \langle f_1 \wedge \cdots \wedge f_k, g_1 \wedge \cdots \wedge g_k \rangle = \frac{1}{k!} \det(\langle f_i|g_j \rangle). \] (25)

Moreover,
\[ \iota_{g_1 \vee \cdots \vee g_{k-1}} f_1 \vee \cdots \vee f_k = \frac{1}{k} \sum_{j=1}^{k} \langle f_1 \vee \cdots \vee f_k, g_1 \vee \cdots \vee g_{k-1} \rangle f_j \] (26)
\[ = \frac{1}{k} \sum_{j=1}^{k} \text{per}(\langle f_i|g_s \rangle_{i \neq j}) f_j, \]
and
\[ \iota_{g_1 \wedge \cdots \wedge g_{k-1}} f_1 \wedge \cdots \wedge f_k = \frac{1}{k} \sum_{j=1}^{k} (-1)^{k-j} \langle f_1 \wedge \cdots \wedge f_k, g_1 \wedge \cdots \wedge g_{k-1} \rangle f_j \] (27)
\[ = \frac{1}{k} \sum_{j=1}^{k} (-1)^{k-j} \det(\langle f_i|g_s \rangle_{i \neq j}) f_j. \]

4 The S-rank

There are many concepts of a rank of a tensor used to describe its complexity. One of the simplest and most natural is the one based on the inner product operators defined in the previous section. We will call it the S-rank, since it turns out to be a natural generalization of the Schmidt rank of 2-tensors.

**Definition 4.1.** Let \( u \in \mathcal{H}^\otimes k \). By the S-rank of \( u \), we understand the maximum of dimensions of the linear spaces \( \iota_{\mathcal{H}}^{k-1} \sigma(u) \), for \( \sigma \in S_k \), which are the images of the contraction maps
\[ (\mathcal{H}^*)^\otimes (k-1) \ni \mu \mapsto \iota_{\mu} \sigma(u) \in \mathcal{H}. \] (28)

**Theorem 4.1.** If \( u \in \mathcal{H}^\vee k \) (resp., \( u \in \mathcal{H}^\wedge k \)), then the S-rank of \( u \) equals the dimension of the linear space being the image of the contraction map
\[ (\mathcal{H}^*)^\vee (k-1) \ni \mu \mapsto \iota_{\mu} u \in \mathcal{H}, \] (29)
(resp.,
\[ (\mathcal{H}^*)^\wedge (k-1) \ni \mu \mapsto \iota_{\mu} u \in \mathcal{H}. \] (30)

**Proof:** It follows immediately from the observation that a contraction of a symmetric (res., antisymmetric) tensor \( u \) with a tensor \( \mu \in (\mathcal{H}^*)^\otimes i \) is the same as its contraction with the symmetrization (resp., antisymmetrization) of \( \mu \), \( \iota_{\mu} u = \iota_{\mu} \sigma(u) \) (resp. \( \iota_{\mu} u = \iota_{\mu} \gamma(u) \)) and that, for \( \sigma \in S_k \), \( \sigma(u) = \pm u \).

\( \square \)

**Example 4.1.** Let \( e_1, e_2 \) be orthogonal vectors in \( \mathcal{H} \). The vector
\[ u = e_1 \vee e_2 = \frac{1}{2} (e_1 \otimes e_2 + e_2 \otimes e_1) \] (31)
has S-rank 2. Indeed, for each \( x \in \mathcal{H} \),
\[ \iota_{x} u = \frac{1}{2} \iota_{x} (e_1 \otimes e_2 + e_2 \otimes e_1) = \frac{1}{2} (\langle x|e_1 \rangle e_2 + \langle x|e_2 \rangle e_1), \]
so that \( \iota_{\mathcal{H}} u \) is spanned by \( \{e_1, e_2\} \).
Example 4.2. Let us take $\mathcal{H}$ of dimension 3 and an orthonormal basis $e_i$, $i = 1, 2, 3$. Let $u \in \mathcal{H}^\otimes 4$ in the polynomial notation reads
\[ u = c_1^4 + c_2^4 + 16c_1^2 c_2 + 4c_1^3 e_1 + 8c_2^3 e_2 + 4c_1^2 e_1^2 + 8c_2 e_2^3 + 32c_1 e_3^3 + 32c_2 e_3^3 \]
\[ + 6c_1^2 e_2^2 + 24c_1^2 e_3^2 + 24c_2^2 e_3^2 + 24c_1 e_2 e_3 + 24c_1 e_2 e_3 + 48e_1 e_2 e_3. \]
The vectors
\[ e_1^3, e_2^3, e_3^3, e_1^2 e_2, e_1 e_3^2, e_1 e_2 e_3, e_2 e_3, e_1 e_2 e_3 \]
form a basis of $\mathcal{H}^\otimes 3$ and it can be directly checked with the use of (26) that any of these vectors, say $\mu$, contracted with $u$ gives a vector proportional to $e_1 + e_2 + 2e_3$. For instance, the only non-zero parts of $\iota_\mu u$ for $\mu = e_2^3$ are
\[ \iota_{e_2^3}(u) = \iota_{e_2^3} (e_2^4 + 4e_1 e_2^3 + 8e_3 e_3) = e_2 + e_1 + 2e_3 \]
and
\[ \iota_{e_3^3}(u) = \iota_{e_3^3} (8e_1^3 e_3 + 24e_1^2 e_3^2 + 24e_1 e_2 e_3) = \|e_1^2 e_3\|^2 (6e_1 + 12e_2 + 6e_2) = 2(e_1 + 2e_2 + e_2) . \]
It follows that $\iota_{\mathcal{H}^\otimes 3}(u)$ is 1-dimensional, so the S-rank of $u$ is 1.

Example 4.3. For $\mathcal{H}$ with the basis as above, consider $w \in \mathcal{H}^\wedge 2$ of the form
\[ w = e_1 \wedge e_2 + e_2 \wedge e_3 . \]
Let us see that the S-rank of $w$ is 2. Indeed,
\[ \iota_{e_1} w = \frac{1}{2} e_2, \quad \iota_{e_2} w = \frac{1}{2} (e_3 - e_1), \quad \iota_{e_3} w = -\frac{1}{2} e_2 , \]
so that $\iota_{\mathcal{H}} w$ is spanned by $e_2$ and $e_3 - e_1$.

Example 4.4. For $\mathcal{H}$ with an orthonormal basis $e_1, e_2, e_3, e_4$ the tensor
\[ w = e_1 \wedge e_2 + e_3 \wedge e_4 \]
has the S-rank 4. Indeed,
\[ \iota_{e_1} w = \frac{1}{2} e_2, \quad \iota_{e_2} w = -\frac{1}{2} e_1, \quad \iota_{e_3} w = \frac{1}{2} e_4, \quad \iota_{e_4} w = -\frac{1}{2} e_3 . \]

Theorem 4.2. (a) The minimal possible S-rank of a non-zero tensor $u \in \mathcal{H}^\otimes k$ equals 1. A tensor $u \in \mathcal{H}^\otimes k$ is of S-rank 1 if and only if $u$ is decomposable, i.e., $u$ can be written in the form
\[ u = f_1 \otimes \cdots \otimes f_k, \quad f_i \in \mathcal{H}, \quad f_i \neq 0. \]
Such tensors span $\mathcal{H}^\otimes k$.

(b) The minimal possible S-rank of a non-zero tensor $v \in \mathcal{H}^\wedge k$ equals 1. A tensor $v \in \mathcal{H}^\wedge k$ is of S-rank 1 if and only if $v$ can be written in the form
\[ v = f \wedge \cdots \wedge f, \quad f \in \mathcal{H}, \quad f \neq 0. \]
Such tensors span $\mathcal{H}^\wedge k$. 

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(c) The minimal possible S-rank of a non-zero tensor \( w \in \mathcal{H}^\wedge k \) equals \( k \). A tensor \( w \in \mathcal{H}^\wedge k \) is of S-rank \( k \) if and only if \( w \) can be written in the form

\[
 w = f_1 \wedge \cdots \wedge f_k, \tag{37}
\]

where \( f_1, \ldots, f_k \in \mathcal{H} \) are linearly independent. Such tensors span \( \mathcal{H}^\wedge k \).

**Proof:** The ‘if’ parts of the above statements are obvious, so we shall prove ‘only if’. If the dimension of \( \mathcal{I}^{\wedge k-1} u \) is 1, thus the space is spanned by some \( f = f_k \in \mathcal{H} \), then clearly \( u = u' \otimes f \) for some \( u' \in \mathcal{H}^{\otimes (k-1)} \). If, in turn, the tensor is symmetric, then clearly \( u \) is proportional to \( f \). In the general case we get a similar fact for \( \sigma(u) \) with \( \sigma \in S_k \), so \( u = f_1 \otimes \cdots \otimes f_k \). If \( u \) is skew-symmetric and \( f_1, \ldots, f_k \) span \( \mathcal{I}^{\wedge k-1} \mathcal{H} \), then clearly \( u \) is proportional to \( f_1 \wedge \cdots \wedge f_k \). That simple tensors span the corresponding spaces is pretty well known for general and antisymmetric tensors. For symmetric tensors it follows from the fact that powers of linear functions span the spaces of polynomials with coefficients in a field of characteristic 0. \( \Box \)

**Definition 4.2.** Tensors of minimal S-rank in \( \mathcal{H}^{\otimes k} \) (resp., \( \mathcal{H}^{\vee k}, \mathcal{H}^\wedge k \)) we will call simple (resp., simple symmetric, simple antisymmetric).

Theorem 4.2 immediately implies the following.

**Corollary 4.1.** The S-rank is 1 for simple and simple symmetric tensors, and it is \( k \) for simple antisymmetric tensors from \( \mathcal{H}^\wedge k \). Simple tensor have the form (35), simple symmetric tensors have the form (36), and simple antisymmetric tensor have the form (37).

**Example 4.5.** The symmetric tensor \( u \) defined in (32), Example 4.2, has the S-rank 1, so it is simple. As a matter of fact,

\[
 u = (e_1 + e_2 + 2e_3)^1. 
\]

Also the antisymmetric tensor \( w \) defined in (33), Example 4.3, is simple:

\[
 w = e_2 \wedge (e_3 - e_1). 
\]

**Remark 4.1.** Of course, for distinguishable particles there is no need to use the same Hilbert space \( \mathcal{H} \) in the tensor products. We can use the tensor product \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k \) instead, with simple tensors being decomposable: \( u = f_1 \otimes \cdots \otimes f_k, f_i \in \mathcal{H}_i \).

5 Various characterizations of simple tensors

For \( \mathcal{H}_0^{\otimes 2} \), where \( \mathcal{H}_0 = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k \), we denote with \( \sigma_i : \mathcal{H}_0^{\otimes 2} \to \mathcal{H}_0^{\otimes 2} \), \( i = 1, \ldots, k \), the transposition with respect to the \( i \)-th and the \( (k+i) \)-th factor,

\[
 \sigma_i(a_1 \otimes \cdots \otimes a_{2k}) = a_1 \otimes \cdots \otimes a_{i-1} \otimes a_{k+i} \otimes a_{i+1} \otimes \cdots \otimes a_{2k+i-1} \otimes a_i \otimes a_{k+i+1} \otimes \cdots \otimes a_{2k}, \tag{38}
\]

and with \( \tau_k : \mathcal{H}_0^{\otimes 2} \to \mathcal{H}_0^{\otimes 2} \) – the cyclic permutation that moves the last factor into the first place:

\[
 \tau_k(a_1 \otimes \cdots \otimes a_{2k}) = a_{2k} \otimes a_1 \otimes \cdots \otimes a_{2k-1}. 
\]
Theorem 5.1. For \( u \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k \) the following are equivalent:

(a) \( u \) is simple;

(b) \( \forall \sigma \in S_k \ \forall \mu_1, \mu_2 \in \mathcal{H}^* \otimes \cdots \otimes \mathcal{H}^* \sigma_{(1)} \otimes \cdots \otimes \sigma_{(k-1)} \ u_{\mu_1} \sigma(u) \wedge u_{\mu_2} \sigma(u) = 0, \)

(c) \( \sigma_i(u \otimes u) = u \otimes u \) for \( i = 1, \ldots, k. \)

Proof: (a) \( \Rightarrow \) (b) is obvious in view of Theorem 4.2 (a). Also (b) \( \Rightarrow \) (c) is clear, so assume \( \sigma_k(u \otimes u) = u \otimes u. \) This implies that the dimension of the space \( \mathcal{H}_1^{k-1} u \) is 1. Indeed, if this space is spanned by linearly independent vectors \( g_1, \ldots, g_r, \) then \( u = \sum_{j=1}^r u_j \otimes g_j \) for some linearly independent \( u_j \in \mathcal{H}^{\otimes (k-1)}. \) Since \( \sigma_k(u \otimes u) = u \otimes u \) means that

\[
\sum_{j,s=1}^r u_j \otimes g_j \otimes u_s \otimes g_s = \sum_{j,s=1}^r u_j \otimes g_s \otimes u_s \otimes g_j,
\]

we conclude that \( r = 1, \) so \( u = u' \otimes f_k \) for some \( f_k \in \mathcal{H} \) and \( u' \in \mathcal{H}^{\otimes (k-1)}. \) A similar reasoning, applied to the identities \( \sigma_i(u \otimes u) = u \otimes u \) with \( i = 1, \ldots, k-1, \) implies that \( u = f_1 \otimes \cdots \otimes f_k. \)

Theorem 5.2. For a symmetric tensor \( v \in \mathcal{H}^{\wedge k} \) the following are equivalent

(a) \( v \) is simple symmetric,

(b) \( \forall \nu_1, \nu_2 \in (\mathcal{H}^*)^{\wedge (k-1)} \ \nu_1 v \wedge \nu_2 v = 0, \)

(c) \( v \otimes v = \sigma_k(v \otimes v). \)

Proof: (a) \( \Rightarrow \) (b) is obvious in view of Theorem 4.2 (b). As (b) implies \( v = v' \otimes f_k \) for some \( f_k \in \mathcal{H}, \) also (b) \( \Rightarrow \) (c) is clear. The condition (c), in turn, for symmetric tensor yields \( v \otimes v = \sigma_i(v \otimes v) \) for all \( i = 1, \ldots, k, \) so \( v = f_1 \otimes \cdots \otimes f_k \) as above, thus \( v = f \otimes \cdots \otimes f \) by symmetry.

Theorem 5.3. For an antisymmetric tensor \( w \in \mathcal{H}^{\wedge k} \) the following are equivalent

(a) \( w \) is simple antisymmetric,

(b) \( \forall \omega \in (\mathcal{H}^*)^{\wedge (k-1)} w \wedge \iota_\omega w = 0, \)

(c) \( (\tau_{k+1} \otimes id_{\mathcal{H}^{\otimes (k-1)}})(\tau_k(w \otimes w)) = 0. \)

Proof: (a) \( \Rightarrow \) (b) is obvious in view of Theorem 4.2 (c). Also (b) \( \Rightarrow \) (a) is clear and well known. We shall show that (b) and (c) are equivalent.

Let us write \( w \) as a sum of simple antisymmetric tensors, \( w = \sum_{j=1}^m w_j, \) with minimal \( m. \) Write \( w_j = f_j^1 \wedge \cdots \wedge f_j^k \) and denote \( w_s^j = f_j^1 \wedge \cdots \wedge f_j^s \wedge \cdots \wedge f_j^k, \) \( s = 1, \ldots, k. \) Here, ”\( \wedge \)" stands for the omission. As the number of simple tensors is minimal, the tensors \( w_s^j \) are linearly independent in \( \mathcal{H}^{\wedge (k-1)}. \)

Observe now that

\[
w_j = \sum_s \frac{(-1)^{k-s}}{k} w_s^j \otimes f_j^s,
\]
so that
\[(\pi_{k+1} \otimes id_{\mathcal{H}^{(n-1)}})(\tau_k(w \otimes w)) = \sum_{j,s} \frac{(-1)^{k-s}}{k} f^s_j \wedge w \otimes w^s_j.\]

Since \(w^s_j\) are linearly independent, the latter vanishes if and only if all \(f^s_j \wedge w\) vanish, that is clearly equivalent to (b).

\[\square\]

**Remark 5.1.** The conditions (c) in Theorems 5.1, 5.2, and 5.3 have formally this advantage over the corresponding conditions (b) that they are directly verifiable, as they do not contain general quantifiers referring to infinite sets.

Note that all we have said remains valid for an arbitrary vector space over a field of characteristics 0. The Hermitian structure played no role yet.

Let us note first that the S-rank is associated with a point in a projective space rather than with a tensor itself. Hence, we can restrict considerations to tensors of length 1 (as vectors in the Hilbert space \(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k\)). According to Theorem 5.1, a tensor \(u \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k\) is simple if and only if \(u \otimes u = \sigma_i(u \otimes u)\) for all \(i = 1, \ldots, k\), where \(\sigma_i\) interweaves the two copies of \(\mathcal{H}_i\) in \(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k\). Note that \(\sigma_i\) acts as a unitary operator in \((\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k)^\otimes 2\).

This means, in turn, that
\[\sum_{i=1}^k \|u \otimes u - \sigma_i(u \otimes u)\|^2 = 0.\] (39)

The latter we can write as
\[\sum_{i=1}^k \langle u \otimes u - \sigma_i(u \otimes u) | u \otimes u - \sigma_i(u \otimes u) \rangle = 0,\] (40)

which, for tensors of length 1 is equivalent to \(\sum_{i=1}^k \text{Re}\langle u \otimes u | \sigma_i(u \otimes u) \rangle = k\), or, finally, to
\[\text{Re}\left( u \otimes u \left| \sum_{i=1}^k \sigma_i \right. \right) (u \otimes u) = k.\]

The Schwarz inequality yields now \(u \otimes u = \bar{\sigma}(u \otimes u)\), where \(\bar{\sigma} = \frac{1}{k} \sum_{i=1}^n \sigma_i\), i.e.,
\[\bar{\sigma}(x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_k) = \frac{1}{k} \sum_{i=1}^n x_1 \otimes \cdots \otimes y_i \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes y_k.\]

In this way we have proven the following.

**Theorem 5.4.** Let \(u \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k\), \(||u|| = 1\). Then, \(u\) is simple if and only if
\[\langle u \otimes u, \bar{\sigma}(u \otimes u) \rangle = 1.\]

For symmetric tensors, a similar fact can be proven analogously.

**Theorem 5.5.** Let \(u \in \mathcal{H}^{\otimes k}\), \(||u|| = 1\). Then, \(u\) is simple symmetric if and only if
\[\langle u \otimes u, \sigma_k(u \otimes u) \rangle = 1.\]
The structure of a Hilbert space $\mathcal{H}$ has been used in \cite{4, 5, 6} to define simple symmetric and simple antisymmetric tensors for tensors of order 2 by means of the Slater decomposition. Both decompositions are direct consequences of a possibility to write a complex symmetric (antisymmetric) matrix in a diagonal (block-diagonal) form by a unitary change of basis (Takagi theorem \cite{16}). The existence of the Slater decompositions means that any symmetric tensor $v \in \mathcal{H} \vee \mathcal{H}$ and any antisymmetric tensor $w \in \mathcal{H} \wedge \mathcal{H}$ can be written as

$$v = \sum_{i=1}^{n} \lambda_i e_i \vee e_i, \quad \lambda_i > 0, \quad (41)$$

$$w = \sum_{i=1}^{n} \lambda_i f_i \wedge f_{n+i}, \quad \lambda_i > 0, \quad (42)$$

for some orthonormal systems $(e_i)$ and $(f_i)$ of $\mathcal{H}$. The Slater rank is the number $n$ of terms in these decompositions. The above are clearly symmetric and antisymmetric analogues of the Schmidt decomposition: any tensor $u \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can be written in the form

$$u = \sum_{i=1}^{n} \lambda_i e_i \otimes f_i, \quad \lambda_i > 0, \quad (43)$$

for some orthonormal systems: $(e_i)$ in $\mathcal{H}_1$, and $(f_i)$ in $\mathcal{H}_2$.

**Theorem 5.6.** For any 2-tensor $u \in \mathcal{H}_1 \otimes \mathcal{H}_2$ its Schmidt rank equals its S-rank.

**Proof:** It is clear that the Schmidt decomposition \cite{13} implies that the S-rank of $u$ is $n$. Conversely, if the S-rank of $u$ is $n$, then $u$ can be written in the form \cite{13} with the only difference that the systems $(e_i)$ and $(f_i)$ are merely linearly independent. But this is a standard procedure, used in the proof of existence of the Schmidt decomposition, that we can choose $(e_i)$ and $(f_i)$ orthonormal.

\qed

Hence, the 2-tensors are simple (resp., simple symmetric, simple antisymmetric), if there exists a Schmidt (resp. Slater) decomposition with a single $\lambda_i > 0$, i.e. they have the Schmidt (Slater) rank 1. Unfortunately, there are no direct analogues of these decompositions for tensors of higher orders.

On the other hand, as we have already seen, such type of a decomposition is not necessary to define (and check) which tensors are simple symmetric (resp., simple antisymmetric), as the S-rank can serve in these cases.

## 6 Entanglement for multipartite Bose and Fermi systems

Using the concept of simple tensors we can define simple (non-entangled or separable) and entangled pure states for multipartite systems of bosons and fermions.

**Definition 6.1.**

(a) A pure state $\rho_v$ on $\mathcal{H}^{\vee k}$ (resp., on $\mathcal{H}^{\wedge k}$), $\rho_v = \frac{|v\rangle \langle v|}{||v||^2}$, with $v \in \mathcal{H}^{\vee k}$ (resp., $v \in \mathcal{H}^{\wedge k}$), $v \neq 0$, is called a bosonic (resp., fermionic) simple (or non-entangled) pure state, if $v$ is a simple symmetric (resp., antisymmetric) tensor. If $v$ is not simple symmetric (resp., antisymmetric), we call $\rho_v$ a bosonic (resp., fermionic) entangled state.
(b) A mixed state $\rho$ on $\mathcal{H}^{\otimes k}$ (resp., on $\mathcal{H}^{\wedge k}$) we call bosonic (resp., fermionic) simple (or non-entangled) mixed state, if it can be written as a convex combination of bosonic (resp., fermionic) simple pure states. In the other case $\rho$ is called bosonic (resp., fermionic) entangled mixed state.

According to Theorem 6.2, bosonic simple pure $k$-states are of the form

$$|e\lor\cdots\lor e\rangle\langle e\lor\cdots\lor e|$$

for some unit vector $e \in \mathcal{H}$, and fermionic simple pure $k$-states are of the form

$$k!|e_1\land\cdots\land e_k\rangle\langle e_1\land\cdots\land e_k|$$

for some orthonormal system $e_1,\ldots,e_k$ in $\mathcal{H}$.

Fixing a base in $\mathcal{H}$ results in defining coefficients $[u^{i_1\ldots i_k}]$ of $u \in \mathcal{H}^{\otimes k}$. Formulae characterizing simple tensors, thus simple pure states, can be written in forms of quadratic equations with respect to these coefficients as follows. The corresponding characterization of entangled pure states are obtained by negation of the latter.

**Theorem 6.1.** The pure state $\rho_u$, associated with a tensor $u = [u^{i_1\ldots i_k}] \in \mathcal{H}^{\otimes k}$, is entangled if and only if there exist $i_1,\ldots,i_k,j_1,\ldots,j_k$, and $s = 1,\ldots,k$ such that

$$u^{i_1\ldots i_s\ldots i_k}u^{j_1\ldots j_s\ldots j_k} \neq u^{i_1\ldots i_s\ldots i_k}u^{j_1\ldots j_s\ldots j_k}. \quad (44)$$

**Proof:** The tensor $u \otimes u$ has coefficients $u^{i_1\ldots i_s\ldots i_k}u^{j_1\ldots j_s\ldots j_k}$, so Eq. (44) expresses the fact that $u \otimes u \neq \sigma_s(u \otimes u)$, and thus Theorem 6.1 is a direct consequence of Theorem 5.1.

**Theorem 6.2.** The bosonic pure state $\rho_v$, associated with a symmetric tensor $v = [v^{i_1\ldots i_k}] \in \mathcal{H}^{\wedge k}$, is bosonic entangled if and only if there exist $i_1,\ldots,i_k,j_1,\ldots,j_k$, such that

$$v^{i_1\ldots i_k\ldots i_k}v^{j_1\ldots j_k\ldots j_k} \neq v^{i_1\ldots i_k\ldots i_k}v^{j_1\ldots j_k\ldots j_k}. \quad (45)$$

**Proof:** A direct consequence of Theorem 5.2.

**Example 6.1.** The bosonic pure state associated with the symmetric tensor $u$ defined in (31) is entangled, as we know it has the S-rank 2. To apply our characterization (15), note that the only non-zero coefficients are $u^{12} = u^{21} = \frac{1}{4}$. Hence,

$$0 = u^{11}u^{22} \neq u^{12}u^{21} = \frac{1}{4}. $$

**Example 6.2.** The bosonic pure state associated with the symmetric tensor $u$ defined in (32) is non-entangled, as we know it has the S-rank 1. To apply our characterization (15), note that, as easily checked, $u^{i_1i_2i_3i_4} = 2^{i(3)}$, where $i(3)$ is the number of 3’s in the sequence $i = (i_1, i_2, i_3, i_4)$. Now it is clear that

$$u^{i_1i_2i_3i_4}u^{j_1j_2j_3j_4} = 2^{i(3)+j(3)},$$

where $i(3) + j(3)$ is the number of 3’s in the sequence $(i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4)$ which remains unchanged under permutations. In particular,

$$u^{i_1i_2i_3i_4}u^{j_1j_2j_3j_4} = u^{i_1i_2i_3i_4}u^{j_1j_2j_3j_4}. $$
Theorem 6.3. The fermionic pure state $\rho_w$, associated with an antisymmetric tensor $w = [w^{i_1\ldots i_k}] \in H^\wedge k$, is fermionic entangled if and only if there exist $i_1, \ldots, i_{k+1}, j_1, \ldots, j_{k-1}$ such that

$$w^{[i_1\ldots i_k j_{k+1}]j_1\ldots j_{k-1}} \neq 0,$$

where the left-hand side is the antisymmetrization of $w^{i_1\ldots i_k}w^{i_{k+1}j_1\ldots j_{k-1}}$ with respect to indices $i_1, \ldots, i_{k+1}$.

Proof: A direct consequence of Theorem 5.3.

In view of the above characterizations, it is obvious that the sets of entangled (entangled bosonic, entangled fermionic) pure states are open: pure states sufficiently close to entangled ones are entangled.

Example 6.3. The fermionic pure state associated with the antisymmetric tensor $w$ defined in (33) is non-entangled, as we know its S-rank is 2. To apply our characterization (46), let us note that the only non-zero coefficients are $w^{12} = -w^{21} = w^{34} = -w^{43} = \frac{1}{2}$. The antisymmetrization $w^{[i_1 i_2 i_3]}$ can be non-zero only if $(i_1, i_2, i_3)$ is a permutation of $(1, 2, 3)$. But, as easily seen,

$$w^{[12}w^{3]} = \frac{1}{3} (w^{12}w^{3} - w^{32}w^{1}) = 0$$

for $j = 1, 2, 3$.

Example 6.4. The fermionic pure state associated with the antisymmetric tensor $w$ defined in (34) is entangled, as we know its S-rank is 4. To apply our characterization (46), let us note that the only non-zero coefficients are $w^{12} = -w^{21} = w^{34} = -w^{43} = \frac{1}{2}$. As easily seen,

$$w^{[12}w^{34]} = \frac{1}{3} w^{12}w^{34} = \frac{1}{12} \neq 0.$$

7 Jamiołkowski isomorphisms for bi-partite systems of bosons and fermions

In the theory of entanglement there exists a useful tool for investigating, on one hand, entanglement properties of states and, on the other, structure of positivity-preserving and hermiticity-preserving maps between matrix algebras called the Jamiołkowski isomorphism [17]. Originally proposed in [18] as an instrument for checking the property of preserving positive semi-definiteness for a linear map between two matrix algebras in finite dimensional spaces, it was later used to prove the so-called operator form representation for linear maps on quantum states (i.e. positive semi-definite operators) [19]. The construction can be extended to the infinite-dimensional setting if the restriction to Hilbert-Schmidt operators is imposed [20] which significantly heightens the usefulness of the Jamiołkowski isomorphism in cases when finite-dimensional description of q quantum system is not suitable. An in-depth description of the applicability the Jamiołkowski isomorphism is given in [21] where various properties of entanglement and separability of states are parallelized with features of the corresponding linear maps.

One of the most straightforward applications of the Jamiołkowski isomorphism is a characterization of separability of a state in terms of the rank of the corresponding map. In the following we show how this description can be extended to states of indistinguishable particles.
Let us recall (see e.g. [20]) that for two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we have the following diagram consisting of Jamiołkowski isomorphisms:

$$\mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2), \mathcal{L}_2(\mathcal{H}_1)) \xrightarrow{\mathcal{J}} \mathcal{L}_2(\mathcal{L}_2(\mathcal{H}_2, \mathcal{H}_1))$$

Here, with $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_2)$ we denote the Hilbert space of Hilbert-Schmidt maps from $\mathcal{H}_1$ into $\mathcal{H}_2$. Of course, this space reduces to all complex linear maps, if $\mathcal{H}_1$ or $\mathcal{H}_2$ is finite-dimensional. Note that we write shorter $\mathcal{L}_2(\mathcal{H})$ for $\mathcal{L}_2(\mathcal{H}, \mathcal{H})$. Note also that according to the obvious identification

$$\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)^* \simeq \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2),$$

induced by the natural pairing

$$\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1) \times \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \ni (A, B) \mapsto \text{Tr}(A \circ B) \in \mathbb{C},$$

we can rewrite the above diagram in the form

$$\mathcal{H}_1 \otimes \mathcal{H}_1^* \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^* \xrightarrow{\mathcal{J}} \mathcal{H}_1 \otimes \mathcal{H}_2^* \otimes \mathcal{H}_2 \otimes \mathcal{H}_1^*$$

in which the Jamiołkowski isomorphisms reduce to appropriate permutations of tensors. In the case when $\mathcal{H}_1 = \mathcal{H}_2$ the whole picture reduces to

$$\mathcal{L}_2(\mathcal{H}) \xrightarrow{\mathcal{J}} \mathcal{L}_2(\mathcal{H})$$

We can now decompose $\mathcal{H} \otimes \mathcal{H}$ into $(\mathcal{H} \vee \mathcal{H}) \oplus (\mathcal{H} \wedge \mathcal{H})$ and reduce our diagram to maps on each of these components. In this way we get

$$\mathcal{L}_2^{sa}(\mathcal{H} \otimes \mathcal{H}^*) \xrightarrow{\mathcal{J}} \mathcal{L}_2^{sa}(\mathcal{H} \otimes \mathcal{H}^*)$$
Here, $L_2^sa(\mathcal{H} \otimes \mathcal{H}^*)$ is the space of self-adjoint Hilbert-Schmidt operators $\Phi$, $\Phi = \Phi^*$, on $\mathcal{H} \otimes \mathcal{H}^*$ = $L_2(\mathcal{H})$ according to the identification $L_2(\mathcal{H}) \simeq (L_2(\mathcal{H}))^*$ (see [48]) related with the pairing [49]. Note that this is different from hermicity, $\Phi^* \neq \Phi^\dagger$, since $\Phi^\dagger$ depends on $\Phi$ anti-linearly.

Indeed, $\Phi$ written in the Dirac notation as $\Phi = \lambda_{ijkl}|e_i \rangle \otimes \langle e_j| \otimes |e_k \rangle \otimes \langle e_l|$ is self-adjoint if and only if $\text{Tr}(A \circ \Phi(B)) = \text{Tr}(\Phi(A) \circ B)$ for all $A, B \in L_2(\mathcal{H})$, that applied to $A = |e_j \rangle \otimes \langle e_i|$ and $B = |e_l \rangle \otimes \langle e_k|$ yields $\lambda_{ijkl} = \lambda_{klij}$, so that we deal with maps coming from symmetric or antisymmetric tensors. What is more, since

$$J(\{e_i \otimes \{e_j\otimes |e_k \rangle \otimes \langle e_l\}) = |e_i \rangle \otimes \{e_j\otimes |e_k \rangle \otimes \langle e_l\}$$

we have a further splitting

$$L_2^sa(\mathcal{H} \otimes \mathcal{H}^*) = L_2^sa^+(\mathcal{H} \otimes \mathcal{H}^*) \oplus L_2^sa^-(\mathcal{H} \otimes \mathcal{H}^*)$$

where $L_2^sa^\pm(\mathcal{H} \otimes \mathcal{H}^*)$ consists of these $\Phi$ for which $J(\Phi) = \pm \Phi$. Finally, we end up with bosonic and fermionic Jamiołkowski maps:

$$\begin{array}{ccc}
L_2^sa^+(\mathcal{H} \otimes \mathcal{H}^*) & \xrightarrow{J^+} & L_2^sa^+(\mathcal{H} \otimes \mathcal{H}^*) \\
\mathcal{J}_i^+ & \xrightarrow{J^+} & \mathcal{J}_2^+
\end{array} \quad \begin{array}{ccc}
L_2^sa^+(\mathcal{H} \otimes \mathcal{H}^*) & \xrightarrow{J^-} & L_2^sa^+(\mathcal{H} \otimes \mathcal{H}^*) \\
\mathcal{J}_i^- & \xrightarrow{J^-} & \mathcal{J}_2^-
\end{array}$$

and

$$\begin{array}{ccc}
L_2^sa^-(\mathcal{H} \otimes \mathcal{H}^*) & \xrightarrow{J^+} & L_2^sa^-(\mathcal{H} \otimes \mathcal{H}^*) \\
\mathcal{J}_i^+ & \xrightarrow{J^+} & \mathcal{J}_2^+
\end{array} \quad \begin{array}{ccc}
L_2^sa^-(\mathcal{H} \otimes \mathcal{H}^*) & \xrightarrow{J^-} & L_2^sa^-(\mathcal{H} \otimes \mathcal{H}^*) \\
\mathcal{J}_i^- & \xrightarrow{J^-} & \mathcal{J}_2^-
\end{array}$$

If now $\rho = |v\rangle \langle v|$ is a pure state in $\mathcal{H}^{\otimes 2}$ corresponding to a vector $v \in \mathcal{H}^\otimes 2$ with a Slater decomposition $v = \sum_{i=1}^n \lambda_i \epsilon_i \vee \epsilon_i$, $\lambda_i > 0$, so that the Slater rank is $r$, then $\rho = J_1^+(\Phi)$, with

$$\Phi = \sum_{i,j} \lambda_i \lambda_j |e_i \rangle \otimes \{e_j\otimes |e_i \rangle \otimes \langle e_j\}$$

being a map from $L_2^sa^+(\mathcal{H} \otimes \mathcal{H}^*)$ of rank $r^2$.

Similarly, if $\rho = |w\rangle \langle w|$ is a pure state in $\mathcal{H}^\otimes 2$ corresponding to a vector $w \in \mathcal{H}^\otimes 2$ with a Slater decomposition $w = \sum_{i=1}^n \mu_i \epsilon_i \wedge \epsilon_i$, $\mu_i > 0$, so that the Slater rank is $r$, then $\rho = J_1^-(\Phi)$, with

$$\Phi = \sum_{i,j} \mu_i \mu_j (|f_i \rangle \otimes \{f_j\otimes |f_{n+i} \rangle \otimes \{f_{n+j}\}|f_j\rangle \otimes |f_i \rangle \otimes \langle f_{n+j}|)$$

being a map from $L_2^sa^-(\mathcal{H} \otimes \mathcal{H}^*)$ of rank $4r^2$. In this way we get a characterization of bosonic and fermionic simple pure states in terms of the corresponding Jamiołkowski isomorphisms.

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Theorem 7.1. A pure state $\rho$ in $\mathcal{H}^{\otimes 2}$ (resp., $\mathcal{H}^{\wedge 2}$) is bosonic (resp., fermionic) simple if and only if $\rho = J_1^+(\Phi)$ (resp., $\rho = J_1^-(\Phi)$) for $\Phi \in \mathcal{L}_{2}^{a+}(\mathcal{H} \otimes \mathcal{H}^*)$ (resp., $\Phi \in \mathcal{L}_{2}^{a-}(\mathcal{H} \otimes \mathcal{H}^*)$) of rank 1 (resp., 4).

Remark 7.1. Of course, choosing a basis in $\mathcal{H}$ we can represent the above maps by matrices and gets the Jamiołkowski isomorphism in the form of permutation of indices, more familiar to physicists. The above form has the advantage that it does not depend on the basis, i.e. is canonical and covariant.

8 Entangled states of composite systems with generalized parastatistics

Our approach to the entanglement of composite systems for identical particles is so general and natural that it allows for an immediate implications also for generalized parastatistics.

Observe first that simple tensors of length 1 in $\tilde{\mathcal{H}} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ form an orbit of the group $U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_k)$ acting on $\tilde{\mathcal{H}}$ in the obvious way. In fact, each such tensor can be written as $e_1^i \otimes \cdots \otimes e_k^i$ for certain choice of orthonormal bases $e_1^j, \ldots, e_k^{j_1}$ in $\mathcal{H}_j$, $j = 1, \ldots, k$. This means that simple tensors are just vectors of highest (or lowest – depending on the convention) weight of the compact Lie group $U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_k)$ relative to some choice of a maximal torus and Borel subgroups. If indistinguishable particles are concerned, the symmetric and antisymmetric tensors in $\mathcal{H}^k$ form particular irreducible parts of the ‘diagonal’ representation of the compact group $U(\mathcal{H})$ in the Hilbert space $\mathcal{H}^{\otimes k}$, defined by

$$U(x_1 \otimes \cdots \otimes x_k) = U(x_1) \otimes \cdots \otimes U(x_k). \tag{55}$$

Recall that we identify the symmetry group $S_k$ with the group of certain unitary operators on the Hilbert space $\mathcal{H}^k$ in the obvious way,

$$\sigma(x_1 \otimes \cdots \otimes x_k) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}. \quad \text{Note that the operators of } S_k \text{ intertwine the unitary action of } U(\mathcal{H}).$$

In the cases of the symmetric and antisymmetric tensors, we speak about Bose and Fermi statistics, respectively. But, for $k > 2$, there are other irreducible parts of the representation (55), associated with invariant subspaces of the $S_k$-action, that we shall call (generalized) parastatistics. Any of these $k$-parastatistics (i.e. any irreducible subspace of the tensor product $\mathcal{H}^{\otimes k}$) is associated with a Young tableau $\alpha$ with $k$-boxes (chambers) as follows (see e.g. [22, 23]).

Consider partitions of $k$: $k = \lambda_1 + \cdots + \lambda_r$, where $\lambda_1 \geq \cdots \geq \lambda_r \geq 1$. To a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is associated a Young diagram (sometimes called a Young frame or a Ferrers diagram) with $\lambda_i$ boxes in the $i$th row, the rows of boxes lined up on the left. Define a tableau on a given Young diagram to be a numbering of the boxes by the integers $1, \ldots, k$, and denote by $Y$ the set of all such Young tableaux. Finally, put $Y(k)$ to be the set of all Young tableaux with $k$ boxes. Given a tableau $\alpha \in Y(k)$ define two subgroups in the symmetry group $S_k$:

$$P = P_\alpha = \{ \sigma \in S_k : \sigma \text{ preserves each row of } \alpha \}$$

and

$$Q = Q_\alpha = \{ \sigma \in S_k : \sigma \text{ preserves each column of } \alpha \}. \quad \text{In the space of linear operators on } \mathcal{H}^{\otimes k} \text{ we introduce two operators associated with these subgroups:}$$

$$a_\alpha = \sum_{\sigma \in P} \sigma, \quad b_\alpha = \sum_{\sigma \in Q} (-1)^{\sigma} \sigma. \tag{56}$$
Finally, we define the Young symmetrizer

\[ c_\alpha = a_\alpha \circ b_\alpha = \sum_{\tau \in P, \sigma \in Q} (-1)^{\sigma} \circ \sigma. \tag{57} \]

It is well known that \( \pi_\alpha = \frac{1}{\mu(\alpha)} c_\alpha \), for some non-zero rational number \( \mu(\alpha) \), is an orthogonal projector and that the image \( \mathcal{H}_\alpha \) of \( c_\alpha \) is an irreducible subrepresentation of \( \mathcal{U}(\mathcal{H}) \), i.e. the parastatistics associated with \( \alpha \). As a matter of fact, these representations for Young tableaux on the same Young diagram are equivalent, so that the constant \( \mu(\alpha) \) depends only on the Young diagram \( \lambda \) of \( \alpha \) (does not depend on the enumeration of boxes), \( \mu(\alpha) = \mu(\lambda) \), and is related to the multiplicity \( m(\lambda) \) of this irreducible representation in \( \mathcal{H}^\otimes k \) by \( \mu(\lambda) \cdot m(\lambda) = k! \).

For a given Young diagram \( \lambda \), the map

\[ \epsilon_\lambda = \frac{1}{\mu(\lambda)^2} \sum_{\alpha \in Y_\lambda} c_\alpha \tag{58} \]

is an orthogonal projection, called the central Young symmetrizer, onto the invariant subspace being the sum of all copies of the irreducible representations equivalent to that with a parastatistics from \( Y_\lambda \).

The symmetrization \( \pi^\vee \) (antisymmetrization \( \pi^\wedge \)) projection corresponds to a Young tableau with just one row (one column) and arbitrary enumeration. It is well known that any irreducible representation \( \mathcal{H}_\alpha \) of \( \mathcal{U}(\mathcal{H}) \) contains cyclic vectors which are of highest weight relative to some choice of a maximal torus and Borel subgroups in \( \mathcal{U}(\mathcal{H}) \). We will call them \( \alpha \)-simple vectors or simple vectors in \( \mathcal{H}_\alpha \). Note that such vectors can be viewed as generalized coherent states \[24\]. They were also regarded as the ‘most classical’ states by several authors \[25\]. These are exactly the tensors associated with simple (non-entangled) pure states for composite systems of particles with (generalized) parastatistics. This is because \( \alpha \)-simple tensors represent the minimal amount of quantum correlations for tensors in \( \mathcal{H}_\alpha \): the quantum correlations forced directly by the particular parastatistics.

**Example 8.1.** (a) For \( k = 2 \) we have just the obvious splitting of \( \mathcal{H}^\otimes 2 \) into symmetric and antisymmetric tensors: \( \mathcal{H}^\wedge 2 \oplus \mathcal{H}^\vee 2 \).

(b) For \( k = 3 \), besides symmetric and antisymmetric tensors, we have two additional irreducible parts associated with the Young tableaux

\[ \alpha_1 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \text{and} \quad \alpha_2 = \begin{array}{c} 1 \\ 3 \\ 2 \end{array}, \tag{59} \]

respectively. Hence,

\[ \mathcal{H}^\otimes 3 = \mathcal{H}^\wedge 3 \oplus \mathcal{H}^{\alpha_1} \oplus \mathcal{H}^{\alpha_2} \oplus \mathcal{H}^\vee 3, \tag{60} \]

with

\[ \pi^{\alpha_1} : \mathcal{H}^\otimes 3 \to \mathcal{H}^{\alpha_1}, \tag{61} \]

\[ \pi^{\alpha_1}(x_1 \otimes x_2 \otimes x_3) = \frac{1}{3} (x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_1 \otimes x_3 - x_3 \otimes x_2 \otimes x_1 - x_1 \otimes x_3 \otimes x_2), \tag{62} \]

and

\[ \pi^{\alpha_2} : \mathcal{H}^\otimes 3 \to \mathcal{H}^{\alpha_2}, \tag{63} \]

\[ \pi^{\alpha_2}(x_1 \otimes x_2 \otimes x_3) = \frac{1}{3} (x_1 \otimes x_2 \otimes x_3 + x_3 \otimes x_2 \otimes x_1 - x_2 \otimes x_1 \otimes x_3 - x_2 \otimes x_3 \otimes x_1). \tag{64} \]
The simple tensors (the highest weight vectors) in $H^{\alpha_1}$ can be written as
\[ v^{\alpha_1}_\lambda = \lambda(e_1 \otimes e_1 e_2 - e_2 \otimes e_1 e_1), \] (65)
for certain choice of an orthonormal basis $e_i$ in $H$ and $\lambda \neq 0$. Analogously, the simple tensors in $H^{\alpha_2}$, in turn, take the form
\[ v^{\alpha_2}_\lambda = \lambda(e_1 \otimes e_2 e_1 - e_2 \otimes e_1 e_1). \] (66)

For $\dim(H) = 3$, the simple tensors of length 1 form an orbit of the unitary group $U(H)$ of the (real) dimension 7 in $H^{\alpha_1}$ and $H^{\alpha_2}$. The simple symmetric tensors of length 1 form an orbit of the dimension 5, and the simple antisymmetric ones (of length 1) – an orbit of the dimension 1. The dimensions of the irreducible representations are: $\dim(H^{\wedge 3}) = 1$, $\dim(H^{\vee 3}) = 10$, $\dim(H^{\alpha_1}) = \dim(H^{\alpha_2}) = 8$.

Let $H^\alpha \subset H^{\otimes k}$ denotes the irreducible component of the tensor representation of the unitary group $U(H)$ in $H^{\otimes k}$ associated with a Young diagram $\alpha \in Y(k)$.

**Definition 8.1.**

(a) We say that a pure state $\rho \in H^{\otimes k}$ obeys a parastatistics $\alpha \in Y(k)$ (is a pure $\alpha$-state in short) if $\rho$ is represented by a nonzero tensor $v \in H^\alpha$, i.e.
\[ \rho = \rho_v = \frac{|v \langle v|}{||v||^2}. \] (67)

In other words, $\rho$ is a pure state on the Hilbert space $H^\alpha$.

(b) A pure state $\rho \in H^{\otimes k}$, obeying a parastatistics $\alpha$ is called a simple pure state for the parastatistics $\alpha$ (simple pure $\alpha$-state, in short), if $\rho$ is represented by an $\alpha$-simple tensor in $H^\alpha$. If $\rho$ is not simple $\alpha$-state, we call it entangled pure $\alpha$-state.

(c) A mixed state $\rho$ on $H^\alpha$ we call a simple (mixed) state for the parastatistics $\alpha$ (simple $\alpha$-state in short), if it can be written as a convex combination of simple pure $\alpha$-states. In the other case, $\rho$ is called entangled mixed $\alpha$-state.

For an arbitrary parastatistics (Young tableau) $\alpha \in Y(k)$ with the partition (Young diagram) $\lambda = (\lambda_1, \ldots, \lambda_r)$, consider the map
\[ i_\alpha : H^{x(r)} \rightarrow H^{\otimes k}, \quad (x_1, \ldots, x_r) \rightarrow x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)}, \]
where $\alpha(i)$ is the number of the raw in which the box with the number $i$ appears in the tableaux $\alpha$. In other words, we make a tensor product of $k$ vectors from $\{x_1, \ldots, x_r\}$ by putting $x_j$ in the places indicated by the number of the boxes in the $j$th row. For instance, the Young tableaux from Example 8.1 give $i_{\alpha_1}(x_1, x_2) = x_1 \otimes x_1 \otimes x_2$ and $i_{\alpha_2}(x_1, x_2) = x_1 \otimes x_2 \otimes x_1$. It is clear that $i_\alpha(x_1, \ldots, x_r)$ is an eigenvector of $a_\alpha$.

It is easy to see now that the S-rank of the tensor $\pi^\alpha(x_{\alpha(1)} \otimes \cdots \otimes x_{\alpha(k)})$, for $x_1 \wedge \cdots \wedge x_r \neq 0$, is $r$ and that this is the minimal S-rank for tensors from $H^\alpha$. Hence, the minimality of the S-rank is a good characteristic also for simple $\alpha$-tensors.

**Theorem 8.1.** A tensor $u \in H^\alpha$ is $\alpha$-simple if and only if $u$ has the minimal S-rank among all non-zero tensors from $H^\alpha$. This minimal S-rank equals $r$ – the number of rows in the corresponding Young diagram.
9 Conclusions

States of identical particles exhibit \textit{a priori} correlations caused merely by (anti)symmetry of the wave function in the case of fermions or bosons. It is thus reasonable to treat as an analogue of the entanglement encountered in systems of distinguishable particles only an additional amount of correlation going beyond that stemming from symmetry requirements.

We proposed a way of treating all non-classical correlation, i.e., those which can be identified with the ‘genuine entanglement’ and not caused merely by symmetries. This unifies all cases: of distinguishable particles, fermions, and bosons, and can be easily extended to hypothetical multipartite systems consisting of particles subjected to arbitrary parastatistics.

We defined simple (non-entangled) pure states as one-dimensional selfadjoint projectors associated with simple tensors obeying appropriate symmetries identifying particles as bosons, fermions, etc. Consequently, simple (non-entangled) mixed states were defined as convex combinations of simple (non-entangled) pure ones. Such an unifying approach allowed also for description of such tools, known from the entanglement theory, as the Jamiołkowski isomorphism and Schmidt rank, to systems with other symmetries. The introduced concept of \textit{S-rank} not only provides us with a tool for distinguishing entanglement of pure states with a given parastatistics, but is interesting also \textit{per se}, as it offers the simplest characterization of highest weight vectors we know.

In the case of two fermionic subsystems our approach identifies non-entangled pure states to be the same as in all other approaches mentioned in Introduction, i.e., we identify them with simple antisymmetric tensors in the meaning explained in Section 4 above. In the bosonic case, from the geometric point of view, we clearly have \textit{a priori} two inequivalent types of non-entanglement: tensor products of identical states and states associated with symmetrizations of products of orthogonal vectors. Non-entangled states of two different types are not connected by local unitary transformations which is in contrast to the familiar situation of distinguishable particles and intuitions build upon the fact that all separable states of distinguishable particles can be obtained from a single one by local transformations. Although this is obviously acceptable, it poses an open fundamental problem what is a physical meaning of two geometrically inequivalent types of non-entanglement.

In our approach we adopted the view that non-entangled pure bosonic states are simple symmetric tensors - tensor products of identical vectors. We find at least two arguments justifying this choice. In \textsuperscript{7} it was pointed that all states which are symmetrizations of products of distinct vectors can be used to perform such clearly ‘non-classical’ tasks like teleportation. This definitely remains in conflict with the basic intuition connecting non-entanglement with the purely classical world. Second, from purely mathematical point of view, only tensor product of identical vectors provide highest weight vectors of the corresponding representation of the unitary group, like in the other cases.

Another achievements of our paper are explicit characterizations of simplicity (non-entanglement) for pure states in terms of directly verifiable, quadratic in coefficients, conditions which are computationally much easier than those proposed in the literature.

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