YOUNG TABLEAUX AND CRYSTAL $B(\infty)$
FOR FINITE SIMPLE LIE ALGEBRAS

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ABSTRACT. We study the crystal base of the negative part of a quantum group. An explicit realization of the crystal is given in terms of Young tableaux for types $A_n$, $B_n$, $C_n$, $D_n$, and $G_2$. Connection between our realization and a previous realization of Cliff is also given.

1. Introduction

Quantum group $U_q(g)$ is a $q$-deformation of the universal enveloping algebra over a Lie algebra $g$, and crystal bases reveal the structure of $U_q(g)$-modules in a very simplified form. As these $U_q(g)$-modules are known to be $q$-deformations of modules over the original Lie algebras, knowledge of these structures also affects the study of Lie algebras.

The crystal $B(\infty)$, which is the crystal base of the negative part $U_q^-(g)$ of a quantum group, has received attention since the very birth of crystal base theory [7, 8]. This is not only because it is an essential part of the grand loop argument proving the existence of crystal bases, but because it gives insight into the structure of quantum group itself.

Much effort has been made [1, 3, 4, 13, 14] to give explicit description of the crystals $B(\infty)$ over various Kac-Moody algebras. A related well known result is that it is possible to describe the highest weight crystal $B(\lambda)$ over finite simple Lie algebras, in terms of Young tableaux [6, 11]. This has lead to the belief that it should be possible to give a similar description for $B(\infty)$ also. So far, only the $A_n$ type has been dealt with in this direction.

In the current work, we restrict ourselves to finite simple Lie algebras of types $A_n$, $B_n$, $C_n$, $D_n$, and $G_2$. These are the cases for which Young tableau realization of crystal $B(\lambda)$ is known. For these cases, we explicitly describe $B(\infty)$ in terms of Young tableaux. Our result for $A_n$ type will be equivalent to that of [3], but it will be obtained through a completely different approach. As most of these cases were dealt with in Cliff’s [1] realization given in terms of a completely different object, we give an isomorphism between our realization and that of Cliff.

The previous work [3] gives a Young tableaux realization of $B(\infty)$ for the $A_n$ case, and then uses this to create yet another realization based on Nakajima monomials, an object which was originally introduced to characterize highest weight crystals [10, 12]. We expect results of this paper also to lead to similar realizations of $B(\infty)$ in terms of Nakajima monomials for the remaining finite types $B_n$, $C_n$, $D_n$, and $G_2$.

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The paper is organized as follows. We start by introducing the notion of large semi-standard tableaux. Then, in the following section, these are collected together into a set, an equivalence relation is given to the set, and a crystal structure is given to the resulting set of equivalence classes. In Section 4, this new crystal is shown to be isomorphic to $\mathcal{B}(\infty)$. Our main result is given in Section 5, where a set of representatives for our new crystal is explicitly presented. This gives a new realization of $\mathcal{B}(\infty)$. The last section connects our result with another realization of $\mathcal{B}(\infty)$ given by Cliff [1].

2. LARGE SEMI-STANDARD TABLEAUX

Throughout this paper, we shall be dealing with finite Lie algebras of types $A_n$, $B_n$, $C_n$, $D_{n+1}$, and $G_2$. Unless explicitly stated otherwise, all our discussions will hold true for each of these types. Notice that the subscript for $D$-type is different from the others. This is to simplify our later writings, and does not imply any restriction on the range of $D$-types we are considering. For the $G_2$ case, $n = 2$ should be assumed.

We shall assume knowledge of the basic theory of crystal bases, and related standard notation, for example, as given in the books [2, 5], will be used.

The crystal base of $U_q^-(\mathfrak{g})$, first introduced in [8], will be denoted by $\mathcal{B}(\infty)$. For each of the finite classical types, we shall use the definitions of semi-standard tableaux as given by Kashiwara and Nakashima [11]. For the $G_2$ type, we shall take the Young tableau realization of highest weight crystal $\mathcal{B}(\lambda)$ given in [6] as the definition of semi-standard tableaux. Since the first of these two works is a rather well known result, and since the second is very similar in spirit to the first, we refer readers to the original papers and shall not repeat the complicated definitions here.

The alphabet to be used inside the boxes constituting the Young tableaux for each type will be denoted commonly by $J$, and it will be equipped with an ordering $\prec$, as given in [6, 11]. For example, in the $C_n$ case, it would be

$$J = \{1 \prec 2 \prec \cdots \prec n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}\}.$$ 

Also, based on results of the same papers, we shall identify elements of the highest weight crystal $\mathcal{B}(\lambda)$ with semi-standard tableaux.

For later use, we recall the Kashiwara operator action on these tableaux. We first read the boxes in the tableau through the far eastern reading and write down the boxes in tensor product form. That is, we read through each column from top to bottom starting from the rightmost column, continuing to the left. The following diagram gives an example.

$$\begin{array}{cccc} 1 & 1 & 3 & 1 \\ 2 & 3 & 0 & 0 \\ 1 & 0 & & \end{array} = \begin{array}{cccccccccccc} 1 \otimes 3 \otimes 0 \otimes 1 \otimes 3 \otimes 1 \otimes 0 \otimes 1 \otimes 2 \otimes 3 \end{array}$$

Then, we apply the tensor product rule to decide on which box to apply $\tilde{f}_i$ or $\tilde{e}_i$ to. After application of Kashiwara operator to one of the boxes, these are gathered back into the original form.

In practice, the tensor product rule on multiple tensors can be applied through calculation of the $i$-signature. This is done as follows.

(1) First, under each tensor component $x$, write down $\varepsilon_i(x)$-many 1s followed by $\varphi_i(x)$-many 0s.
(2) Then, from the long sequence of mixed 0s and 1s, successively cancel out every occurrence of (0,1) pair until we arrive at a sequence of 1s followed by 0s, reading from left to right. This is called the $i$-signature of the whole tensor product form.

(3) To apply $\tilde{f}_i$ to the whole product, apply it to the single tensor component corresponding to the leftmost 0 remaining in the $i$-signature. If no 0 remains, the result of $\tilde{f}_i$ action is set to zero.

(4) Similarly, for $\tilde{e}_i$, apply it to the component corresponding to the rightmost 1, or set it to zero when no 1 remains.

We wish to restrict the set of dominant integral weights $P^+$ slightly for some of the classical types.

- **$A_n$ case:** $\hat{P}^+ := P^+$.
- **$B_n$ case:** $\hat{P}^+ := \{\lambda \in P^+ | \lambda(h_n) \text{ is even}\}$.
- **$C_n$ case:** $\hat{P}^+ := P^+$.
- **$D_{n+1}$ case:** $\hat{P}^+ := \{\lambda \in P^+ | \lambda(h_n) = \lambda(h_{n+1})\}$.
- **$G_2$ case:** $\hat{P}^+ := P^+$.

Notice that for $\lambda \in \hat{P}^+$, elements of $B(\lambda)$ become the most generic tableaux, in the sense that they do not involve any half-size boxes or other complications. It is also clear that given any $\lambda \in P^+$, we may always find a larger $\mu \in \hat{P}^+$, that is, one such that $\mu - \lambda \in P^+$.

We borrow the notion of large semi-standard tableaux from [1]. For the remainder of this paper, the top row of a tableau shall always be counted as the first row.

**Definition 2.1.** A semi-standard tableau $T$ of shape $\lambda \in \hat{P}^+$ is large if it consists of $n$ non-empty rows, and if for each $1 \leq i \leq n$, the number of $i$-boxes in the $i$-th row is strictly greater than the number of all boxes in the $(i+1)$-th row. In particular the $n$-th row of $T$ contains at least one $n$-box.

For each finite type, denote by $T(\lambda)^L$, the set of all large semi-standard tableaux of shape $\lambda$.

Once again, we remind readers that we are giving this definition for each of the types $A_n$, $B_n$, $C_n$, $D_{n+1}$, and $G_2$. The $n$ appearing in the definition is meant to be the same $n$ used as subscripts for the algebra types, with $n = 2$ for the $G_2$ case.

In Figure 1, for some of the finite types, we give examples of semi-standard tableaux. The ones on the left column are large, and the ones on the right are not large.

3. **THE NEW CRYSTAL $\mathcal{T}(\infty)$**

Let us collect all large tableaux into one set (separately for each finite type).

(1) $\mathcal{T}^L = \cup_{\lambda \in P^+} T(\lambda)^L$.

We shall define an equivalence relation on this set.

**Definition 3.1.** Two tableaux $T_1, T_2 \in \mathcal{T}^L$ are related if for each $1 \leq i \leq n$ and $j \in J$ such that $j > i$, the number of $j$-boxes appearing in the $i$-th rows of $T_1$ and $T_2$ are equal.
\begin{figure}

| \text{case} | T_1 | T_2 |
|-----------|-----|-----|
| $A_2$     | 1 1 1 1 1 1 2 2 3 | 1 1 1 2 3 3 |
| $B_3$     | 1 1 1 1 1 1 3 2 2 3 | 1 1 1 3 3 2 |
| $C_4$     | 2 2 2 2 2 2 3 1 1 1 1 4 | 2 2 3 |
| $D_4$     | 1 1 1 1 1 1 1 1 4 2 2 3 | 1 1 2 3 4 |
| $G_2$     | 1 1 1 1 2 2 0 3 3 2 1 | 1 2 0 3 3 2 |

\end{figure}

It is trivial to verify that the above gives an equivalence relation. We fix a notation

\[(2) \quad \mathcal{T}(\infty) := \mathcal{T}^L / \sim\]

for the set of equivalence classes. This section is devoted to providing $\mathcal{T}(\infty)$ with a crystal structure.

Let us start with the Kashiwara operators.

**Lemma 3.2.** Fix an $i \in I$.

1. If tableau $T$ is large, then $\tilde{f}_i T$ is never zero.
2. Given any element of $\mathcal{T}(\infty)$, it is always possible to choose its representative $T \in \mathcal{T}^L$ in such a way that $\tilde{f}_i T$ is large.
3. If $T_1, T_2 \in \mathcal{T}^L$ belong to the same equivalence class and $\tilde{f}_i T_1$ and $\tilde{f}_i T_2$ are both large, then $\tilde{f}_i T_1$ and $\tilde{f}_i T_2$ belong to the same equivalence class.
4. If tableau $T$ is large, then $\tilde{e}_i T$ is either zero or large.
5. If $T_1, T_2 \in \mathcal{T}^L$ belong to the same equivalence class, then either $\tilde{e}_i T_1$ and $\tilde{e}_i T_2$ are both zero, or $\tilde{e}_i T_1$ and $\tilde{e}_i T_2$ belong to the same equivalence class.

**Proof.**

1. It suffices to show that, after all cancelling out, at least one 0 remains in the $i$-signature for $T$. Consider the rightmost $i$-block on the $i$-th row of $T$. The signature to be written under it in the tensor form of $T$ is 0. Notice that the condition large guarantees it to be the lowest block in its column. Such careful consideration of both the conditions large and semi-standard for each of the finite types will show that the signature 0 under that block will not be cancelled out by signatures from blocks contained in any of the columns sitting to its left.

2. Given any $T \in \mathcal{T}^L$ such that $\tilde{T} = b$, let us create a larger representative of $b$. First, construct a column consisting of $i$ boxes, with $k$-box sitting on the $k$-th row ($1 \leq k \leq i$). Consider the rightmost $i$-box sitting on the $i$-th row of $T$ and insert the constructed column to its left. It is clear that this new tableau $T'$ is a (large) representative for $b$. 


Now, during the proof of item 1 of this lemma, we saw that if we apply \( \tilde{f}_i \) to \( T' \), it will act on either the rightmost \( i \)-block on the \( i \)-th row of \( T' \), or on one of the boxes sitting in columns to its right. Due to the column we have inserted, neither case will affect the largeness of \( T' \), and hence the result is obtained.

3) The tableaux \( T_1 \) and \( T_2 \) (or any other large tableaux) will take the following form.

\[
\begin{array}{cccccc}
1 & \cdots & 2 & \cdots & \cdots & i \\
\end{array}
\]

We already know from item 1 of this lemma that \( \tilde{f}_i \) will not act on any of the boxes contained in the unshaded part. It will act on either the light shaded \( i \)-box or on the dark shaded part. Notice that, for all finite types, \( j \)-boxes with \( j < i \) do not contribute to \( i \)-signatures, hence except for the \( i \)-box, none of the light shaded part affects the \( i \)-signature for the two tableaux. Also, by definition of the equivalence relation, the dark shaded parts will be identical for the two. Thus all ingredients that become involved in the \( i \)-signatures for the two tableaux are identical and hence \( \tilde{f}_i \) will act on two corresponding boxes contained in the two tableaux. This will result in the two tableaux being related even after \( \tilde{f}_i \) action.

4) and 5) These should be trivial to prove once proofs for previous items are understood. \( \square \)

It is now clear that, given \( b \in \mathcal{T}(\infty) \) and \( i \in I \), we may define

\[
\begin{align*}
\tilde{f}_i b &= \tilde{f}_i T \in \mathcal{T}(\infty), \\
\tilde{e}_i b &= \tilde{e}_i T \in \mathcal{T}(\infty) \cup \{ 0 \}
\end{align*}
\]

by choosing an appropriate representative \( T \) for \( b \).

Lemma 3.3. If \( T_1 \in \mathcal{T}(\lambda_1)^L \) and \( T_2 \in \mathcal{T}(\lambda_2)^L \) are related to each other with \( \text{wt}(T_1) = \lambda_1 - \xi_1 \) and \( \text{wt}(T_2) = \lambda_2 - \xi_2 \), then \( \xi_1 = \xi_2 \).

Based on this trivial lemma, for \( \bar{T} \in \mathcal{T}(\infty) \) with \( T \in \mathcal{T}(\lambda)^L \), we can define

\[
\text{wt}(\bar{T}) = \text{wt}(T) - \lambda.
\]

To complete the description of the crystal structure, it only remains to define

\[
\begin{align*}
\varepsilon_i(\bar{T}) &= \varepsilon_i(T), \\
\varphi_i(\bar{T}) &= \varepsilon_i(\bar{T}) + \text{wt}(\bar{T})(h_i).
\end{align*}
\]

As before, these may be shown to be well-defined.

Theorem 3.4. The operator given by equations (3) to (7), define a crystal structure on \( \mathcal{T}(\infty) \).

This theorem may be proved by a step by step checking of the definition for an abstract crystal.
4. Crystal isomorphism

In this section, an isomorphism between crystal $B(\infty)$ and the crystal $T(\infty)$, constructed in the previous section, will be given. We start by recalling the following theorem from [8].

**Theorem 4.1.** Let $\pi_\lambda : U_q^-(g) \to V(\lambda)$ be the $U_q^-(g)$-linear homomorphism sending $1$ to $u_\lambda$. Then

1. $\pi_\lambda(L(\infty)) = L(\lambda)$. Hence $\pi_\lambda$ induces the surjective homomorphism $\bar{\pi}_\lambda : L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda)$.
2. By $\bar{\pi}_\lambda$, $\{b \in B(\infty); \bar{\pi}_\lambda(b) \neq 0\}$ is isomorphic to $B(\lambda)$.
3. $\tilde{f}_i \circ \bar{\pi}_\lambda = \bar{\pi}_\lambda \circ \tilde{f}_i$.
4. If $b \in B(\infty)$ satisfies $\bar{\pi}_\lambda(b) \neq 0$, then $\tilde{e}_i \bar{\pi}_\lambda(b) = \bar{\pi}_\lambda(\tilde{e}_i b)$.

We shall adopt the notation $\bar{\pi}_\lambda$ introduced in this theorem. Let us prepare for the definition of an explicit mapping from $B(\infty)$ to $T(\infty)$.

**Lemma 4.2.**

1. Given any $b \in B(\infty)$, there exists a $\lambda \in \hat{P}^+$, for which $\bar{\pi}_\lambda(b)$ is large.
2. Given any $b \in B(\infty)$, if both $\bar{\pi}_\lambda(b)$ and $\bar{\pi}_{\lambda'}(b)$ are large, then the two belong to the same equivalence class. In particular, any large highest weight elements $u_\lambda$ and $u_{\lambda'}$ are related.

**Proof.**

1) Simply put, depending on the distance of $b$ (in terms of $\tilde{f}_i$) from the highest element $u_\infty$, we can always choose $\lambda$ large enough so that $\bar{\pi}_\lambda(b)$ is still large. A more careful proof can be written by modifying the proof of [1, Lemma 3.2].

2) That any two large $u_\lambda$ and $u_{\lambda'}$ are related follows from the definition for the equivalence relation. Starting from this point, we may use induction with the help of Lemma 3.2 (3) to obtain the result. □

We are now ready to define the mapping

$$\psi : B(\infty) \to T(\infty).$$

Given any $b \in B(\infty)$, choose $\lambda \in \hat{P}^+$ for which $\bar{\pi}_\lambda(b)$ is large, and set

$$\psi(b) = \bar{\pi}_\lambda(b).$$

The above lemma shows that this is well-defined. We thus arrive at one of our main results.

**Theorem 4.3.** The mapping (8) is an isomorphism between $B(\infty)$ and $T(\infty)$.

**Proof.** It is easy to check that $\psi$ preserves $\varepsilon_i$, $\varphi_i$, and wt. Notice that in the definition of $\psi$, the case $\bar{\pi}_\lambda(b) = 0$ is never encountered. Hence items 3 and 4 of Theorem 4.1 show that $\psi$ is a strict crystal morphism. Also, since we know $|B(\infty)_{-\varepsilon}| = |T(\lambda)_{\lambda-\varepsilon}|$ for all sufficiently large $\lambda$ (Cor. 4.4.5 of [8]), the mapping is bijective. □
5. An explicit description of $\mathcal{B}(\infty)$

To achieve our final goal of giving an explicit description of $\mathcal{B}(\infty)$ in terms of tableaux, it suffices to describe an explicit set of representatives for $\mathcal{T}(\infty) = \mathcal{T}^L / \sim$ and translate the various operators on $\mathcal{T}(\infty)$ to that on the representative set.

**Definition 5.1.** A tableau $T \in \mathcal{T}^L$ is *marginally* large, if for $1 \leq i \leq n$, the number of $i$-boxes in the $i$-th row of $T$ is greater than the number of all boxes in the $(i+1)$-th row by exactly one. In particular, the $n$-th row of $T$ should contain one $n$-box.

It is clear that the set of marginally large tableaux forms a set of representatives for $\mathcal{T}(\infty) \cong \mathcal{B}(\infty)$. In passing, we remark that the difference of numbers considered above does not have to be *one* to obtain a representative set. It suffices to fix it to some non-negative number for each row.

We shall now describe this representative set more explicitly for each finite type. For each case, we shall present a set of alphabets to be used inside the boxes forming the tableaux, together with an ordering on the set. Next, a set of conditions that should be satisfied by the tableaux is presented. The set of all tableaux subject to the given conditions will be the set of representatives for $\mathcal{T}(\infty) \cong \mathcal{B}(\infty)$.

These descriptions were obtained by considering all conditions defining semi-standard tableaux together with the condition *marginally large*. Trickiest part of the notion semi-standard involves something called configuration, but the condition large ensures that no such configuration can occur, and we obtain a vast simplification. The final description we give below are thus much simpler than the definition for semi-standard tableaux.

After giving out the explicit representative sets, we shall describe action of Kashiwara operators on these sets, in a manner which is applicable commonly to all cases. We leave translation of other operators $\psi_i$, $\epsilon_i$, and $\varphi_i$ to the readers.

Subsections 5.1 to 5.6 can be seen as the main contribution of this paper.

### 5.1. $A_n$ case. Alphabet:

$$J = \{1 \prec 2 \prec \cdots \prec n \prec n + 1\}.$$

Conditions:

1. Tableau consists of $n$ rows.
2. For $1 \leq i \leq n$, the $i$-th row of the leftmost column is an $i$-box.
3. Box indices weakly increase (w.r.t. $\prec$) as we go to the right.
4. For $1 \leq i \leq n$, the number of $i$-boxes in the $i$-th row is larger than the total number of boxes appearing in the $(i+1)$-th row by exactly one.

**Example 5.2.** The set of representatives for $\mathcal{T}(\infty)$, in the $A_2$ case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

$$T = \begin{array}{cccc}
1 & \cdots & 1 & 2 \\
2 & 3 & 3 & \end{array}$$

The element corresponding to highest weight element $u_\infty$ is

$$T_\infty = \begin{array}{c}
1 \\
2
\end{array}.$$
5.2. $B_n$ case. Alphabet:

\[ J = \{ 1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1} \} . \]

Conditions:

1. Tableau consists of $n$ rows.
2. For $1 \leq i \leq n$, the $i$-th row of the leftmost column is an $i$-box.
3. Box indices weakly increase (w.r.t. $\prec$) as we go to the right.
4. For $1 \leq i \leq n$, the number of $i$-boxes in the $i$-th row is larger than the total number of boxes appearing in the $(i+1)$-th row by exactly one.
5. All entries on the $i$-th row are less than or equal to $\bar{i}$ (w.r.t. $\prec$).
6. Index 0 appears at most once in each row.

**Example 5.3.** The set of representatives for $T(\infty)$, in the $B_3$ case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

\[
T = \begin{array}{cccccccccccc}
1 & \cdots & \cdots & \cdots & \cdots & 1 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\
2 & \cdots & 2 & 3 & 3 & 6 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

The element corresponding to highest weight element $u_\infty$ is

\[
T_\infty = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

5.3. $C_n$ case. Alphabet:

\[ J = \{ 1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1} \} . \]

Conditions:

1. Tableau consists of $n$ rows.
2. For $1 \leq i \leq n$, the $i$-th row of the leftmost column is an $i$-box.
3. Box indices weakly increase (w.r.t. $\prec$) as we go to the right.
4. For $1 \leq i \leq n$, the number of $i$-boxes in the $i$-th row is larger than the total number of boxes appearing in the $(i+1)$-th row by exactly one.
5. All entries on the $i$-th row are less than or equal to $\bar{i}$ (w.r.t. $\prec$).

**Example 5.4.** The set of representatives for $T(\infty)$, in the $C_3$ case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

\[
T = \begin{array}{cccccccccccc}
1 & \cdots & \cdots & \cdots & \cdots & 1 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\
2 & \cdots & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

The element corresponding to highest weight element $u_\infty$ is

\[
T_\infty = \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

5.4. $D_{n+1}$ case. Alphabet:

\[ J = \{ 1 < 2 < \cdots < n < \frac{n+1}{n+1} < \bar{n} < \cdots < \bar{2} < \bar{1} \} . \]

Conditions:

1. Tableau consists of $n$ rows.
For $1 \leq i \leq n$, the $i$-th row of the leftmost column is an $i$-box.

(3) Box indices weakly increase (w.r.t. $\prec$) as we go to the right.

(4) For $1 \leq i \leq n$, the number of $i$-boxes in the $i$-th row is larger than the total number of boxes appearing in the $(i + 1)$-th row by exactly one.

(5) All entries on the $i$-th row are less than or equal to $\bar{i}$ (w.r.t. $\prec$).

(6) $n+1$ and $\bar{n}+1$ do not appear on the same row.

Example 5.5. The set of representatives for $T(\infty)$, in the $D_4$ case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size. Either one of 4 or $\bar{4}$ may take the place of each of the letters $x$, $y$, and $z$.

$$T = \begin{array}{cccccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\
2 & \cdot & \cdot & \cdot & 2 & 1 & 3 & 3 & 3 \\
3 & x & x & x & x & y & y & 3 & 3 \\
\end{array}.$$ 

The element corresponding to highest weight element $u_\infty$ is

$$T_\infty = \begin{array}{c}
1 1 1 \\
2 2 \\
3 \\
\end{array}.$$ 

5.5. $G_2$ case. Alphabet:

$$J = \{1 \preceq 2 \preceq 3 \preceq 0 \preceq \bar{3} \preceq \bar{2} \preceq \bar{1}\}.$$ 

Conditions:

(1) Tableau consists of 2 rows.
(2) For $1 \leq i \leq 2$, the $i$-th row of the leftmost column is an $i$-box.
(3) Box indices weakly increase (w.r.t. $\prec$) as we go to the right.
(4) For $1 \leq i \leq 2$, the number of $i$-boxes in the $i$-th row is larger than the total number of boxes appearing in the $(i + 1)$-th row by exactly one.
(5) Only 2 and 3 appear as indices on the second row.
(6) Index 0 appears at most once on the first row.

Example 5.6. The set of representatives for $T(\infty)$, in the $G_2$ case, consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

$$T = \begin{array}{cccccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 3 & 3 & 0 & 3 & 3 \\
\end{array}.$$ 

The element corresponding to highest weight element $u_\infty$ is

$$T_\infty = \begin{array}{c}
1 1 \\
2 \\
\end{array}.$$ 

5.6. Kashiwara operators. To apply $\tilde{f}_i$ to one of the representatives, we go through the following procedure.

(1) Apply $\tilde{f}_i$ to the tableau as usual. That is, write it in tensor product form, apply tensor product rule, and assemble back into original tableau form.
(2) If the result is a large tableau, we are done. It is automatically marginally large.
(3) If the result is not large, the $\tilde{f}_i$ was applied to the rightmost $i$-box on the $i$-th row. Insert one column consisting of $i$ rows to the left of the box $\tilde{f}_i$ acted upon. The added column should have a $k$-box at the $k$-th row for $1 \leq k \leq i$. 


To apply $\tilde{e}_i$ to one of the representatives, we go through the following procedure.

1. Apply $\tilde{e}_i$ to the tableau as usual.
2. If the result is zero or a marginally large tableau, we are done.
3. Otherwise, the result is large but not marginally large. The $\tilde{e}_i$ operator has acted on the box sitting to the right of the rightmost $i$-box on the $i$-th row. Remove the column containing the changed box. It will be of $i$ rows and have a $k$-box at the $k$-th row for $1 \leq k \leq i$.

*Example 5.7.* In the Figures 2 and 3, we illustrate the top part of crystal $\mathcal{T}(\infty)$ for finite types $B_3$ and $G_2$. The dark shaded blocks are the ones $\tilde{f}_i$ has acted upon, and the light shadings show columns inserted to preserve largeness.
5.7. **Comparison with a previous $A_n$ result.** Our result on $A_n$-type can be found in an earlier work [3]. There, the approach was very different, relying on a work of Cliff [1], and the final result was written in a slightly different form.

The only difference with the current result is that, there, infinitely many copies of our leftmost column was added to the left of each representative. This has the advantage of having the Kashiwara operators look slightly more natural. We do not insert or remove columns to remain marginally large, but push or pull infinite rows instead.

In the current work, we chose not to add these infinitely many columns, so as to keep our representatives within the frames of Young tableaux. The choice between these two presentations seems to be a matter of taste.

6. **Relationship with another work**

In this section, we recall Cliff’s [1] combinatorial description of $B(\infty)$ and give an isomorphism between this and our own realization. Only the finite classical types will be dealt with in this section, as Cliff did not deal with $G_2$ case.

6.1. **Another realization of $B(\infty)$**. Let us first recall the crystals

$$B_i = \{b_i(k)|k \in \mathbb{Z}\}$$

defined for each $i \in I$ and introduced in [9]. The crystal $B_i$ reacts to the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$ by decrementing or incrementing the inner index $k$, but maps everything to zero under other $\tilde{f}_j$ and $\tilde{e}_j$. We shall not be concerned with its exact crystal structure.

Kashiwara has shown [9] the existence of an injective strict crystal morphism

$$\Psi : B(\infty) \rightarrow B(\infty) \otimes B_{i_k} \otimes B_{i_{k-1}} \otimes \cdots \otimes B_{i_1}$$

for any sequence $S = i_1, i_2, \cdots, i_k$ of numbers in $I$ and Cliff [1] uses this to give a combinatorial description of $B(\infty)$ for all finite classical types. To explain this result, we first define some crystals $B(i)$ and fix a notation for $\beta_i \in B(i)$.

- **$A_n$ type** ($1 \leq i \leq n$)
  $$B(i) = B_n \otimes B_{n-1} \otimes \cdots \otimes B_i$$

  and

  $$\beta_i = b_n(-k_{i,n}) \otimes b_{n-1}(-k_{i,n-1}) \otimes \cdots \otimes b_i(-k_{i,i}) \in B(i).$$

- **$B_n$ type** ($1 \leq i \leq n$)
  $$B(i) = B_i \otimes B_{i_{n-1}} \otimes B_n \otimes B_{i_{n-2}} \otimes \cdots \otimes B_i$$

  and

  $$\beta_i = b_i(-k_{i,i+1}) \otimes b_{i+1}(-k_{i,i+2}) \otimes \cdots \otimes b_{n-1}(-k_{i,n}) \otimes b_n(-k_{i,n}) \otimes b_{n-1}(-k_{i,n-1}) \otimes \cdots \otimes b_i(-k_{i,i}) \in B(i).$$

- **$C_n$ type** ($1 \leq i \leq n$)
  $$B(i) = B_i \otimes B_{i_{n-1}} \otimes B_n \otimes B_{i_{n-2}} \otimes \cdots \otimes B_i$$

  and

  $$\beta_i = b_i(-k_{i,i+1}) \otimes b_{i+1}(-k_{i,i+2}) \otimes \cdots \otimes b_{n-1}(-k_{i,n}) \otimes b_n(-k_{i,n}) \otimes b_{n-1}(-k_{i,n-1}) \otimes \cdots \otimes b_i(-k_{i,i}) \in B(i).$$
\[ B(i) = B_1 \otimes \cdots \otimes B_{n-1} \otimes B_{n+1} \otimes B_n \otimes \cdots \otimes B_l, \]
\[ B(n) = B_{n+1} \otimes B_n, \]

and
\[ B_n = B_{n+1} \otimes B_n, \]
\[ \beta_i = b_i(-k_{i,i+1}) \otimes b_{i+1}(-k_{i,i+2}) \otimes \cdots \otimes b_{n-1}(-k_{i,\bar{n}}) \in B(i), \]
\[ \beta_n = b_{n+1}(-k_{n,n+1}) \otimes b_n(-k_{n,n}) \in B(n). \]

The following result appears in [1].

**Proposition 6.1.** Image of the injective crystal morphism
\[ \Psi : B(\infty) \to B(\infty) \otimes B(1) \otimes B(2) \otimes \cdots \otimes B(n), \]
is given by
\[ \Psi(B(\infty)) = \{ u_\infty \otimes \beta_1 \otimes \beta_2 \otimes \cdots \otimes \beta_n \}. \]
The indices for the components of \( \beta_i \) in this expression are subject to the restrictions given below for each type.

- **A\( n \)** type:
  \[ 0 \leq k_{i,n} \leq k_{i,n-1} \leq \cdots \leq k_{i,i} \]
  for all \( 1 \leq i \leq n \).

- **B\( n \)** type:
  \[ 0 \leq k_{i,\bar{i}+1} \leq k_{i,\bar{i}+2} \leq \cdots \leq k_{i,i} \leq k_{i,n}/2 \leq k_{i,n-1} \leq \cdots \leq k_{i,i} \]
  for all \( 1 \leq i \leq n \). (note: \( k_{i,n}/2 \) need not be an integer)

- **C\( n \)** type:
  \[ 0 \leq k_{i,\bar{i}+1} \leq k_{i,\bar{i}+2} \leq \cdots \leq k_{i,i} \leq k_{i,n} \leq k_{i,n-1} \leq \cdots \leq k_{i,i} \]
  for all \( 1 \leq i \leq n \).

- **D\( n+1 \)** type:
  \[ 0 \leq k_{i,\bar{i}+1} \leq k_{i,\bar{i}+2} \leq \cdots \leq k_{i,i} \leq \min(k_{i,n}, k_{i,n+1}) \leq \max(k_{i,n}, k_{i,n+1}) \leq k_{i,n-1} \leq \cdots \leq k_{i,i} \]
  for all \( 1 \leq i \leq n \).

This proposition gives a combinatorial description of \( B(\infty) \).

6.2. **Isomorphism between two descriptions of** \( B(\infty) \). We now have two explicit descriptions of the same crystal \( B(\infty) \), namely, \( \Psi(B(\infty)) \) and representatives of \( \mathcal{T}(\infty) \). In this section, we provide maps between the two in both directions that are crystal isomorphisms.

With the help of tensor product rules, it is easy to check the compatibility of these maps with Kashiwara operators. Hence we shall only write out the maps and give no proofs.
6.2.1. $A_n$ case. Element $u_\infty \otimes \beta_1 \otimes \cdots \otimes \beta_n \in \Psi(B(\infty))$ is sent to the tableau whose $i$-th row $(1 \leq i \leq n)$ consists of

\begin{align*}
(k_{i,n})\text{-many } (n + 1) \text{s}, \\
(k_{i,j} - k_{i,j})\text{-many } j \text{s, for each } i < j \leq n, \text{ and} \\
((n - i + 1) + \sum_{r=1}^{n} k_{r,r})\text{-many } i \text{s.}
\end{align*}

Conversely, for each tableau with the $i$-th row consisting of $b^j_i$-many $j$'s $(i < j \leq n + 1)$ and some number of $i$'s, we may map it to the element $u_\infty \otimes \beta_1 \otimes \cdots \otimes \beta_n \in \Psi(B(\infty))$, where

$$k_{i,r} = \sum_{j=r+1}^{n+1} b^j_i$$

for $1 \leq i \leq n$ and $i \leq r \leq n$.

6.2.2. $B_n$ case. Element $u_\infty \otimes \beta_1 \otimes \cdots \otimes \beta_n \in \Psi(B(\infty))$ is sent to the tableau whose shape we describe below row-by-row.

- For $1 \leq i \leq n - 1$, the $i$-th row consists of

\begin{align*}
(k_{i,i+1})\text{-many } \bar{i} \text{s,} \\
(k_{i,j} - k_{i,j})\text{-many } \bar{j} \text{s, for each } i < j \leq n - 1, \\
((k_{i,n}/2 - k_{i,n})])\text{-many } \bar{n} \text{s,} \\
((A + B) - (A' + B'))\text{-many } 0 \text{s,} \\
((k_{i,n-1} - k_{i,n}/2])\text{-many } n \text{s,} \\
(k_{i,j} - k_{i,j})\text{-many } \bar{j} \text{s, for each } i < j \leq n - 1, \text{ and} \\
((n - i + 1) + \left[\frac{k_{n,n+1}}{2}\right] + \sum_{r=i+1}^{n-1} k_{r,r})\text{-many } i \text{s.}
\end{align*}

- The $n$-th row consists of

\begin{align*}
((k_{n,n}/2 - k_{n,n}])\text{-many } \bar{n} \text{s,}

(2(B - B'))\text{-many } 0 \text{s, and}

\text{one } n.
\end{align*}

Here, $A = k_{i,n-1} - k_{i,n}/2$, $B = k_{i,n}/2 - k_{i,n}$, $A' = [k_{i,n-1} - k_{i,n}/2]$, and $B' = [k_{i,n}/2 - k_{i,n}]$, for each $i \in I$.

Conversely, for each tableau with the $i$-th row consisting of $b^j_i$-many $j$'s $(i < j \leq n)$ and some number of $i$'s, we may map it to the element $u_\infty \otimes \beta_1 \otimes \cdots \otimes \beta_n \in \Psi(B(\infty))$, where, for $1 \leq i \leq n - 1$,

\begin{align*}
k_{i,r} &= \sum_{j=r+1}^{n} b^j_i + b^0_i + \sum_{j=i}^{n} b^j_i \quad \text{for } i \leq r \leq n - 1, \\
k_{i,n} &= 2(\sum_{j=i}^{n} b^j_i) + b^0_i, \\
k_{i,\bar{r}} &= \sum_{j=i}^{r-1} b^j_i \quad \text{for } i + 1 \leq r \leq n,
\end{align*}

and

$$k_{n,n} = 2b^n_n + b^n_0.$$
6.2.3. $C_n$ case. Element $u_\infty \otimes \beta_1 \otimes \cdots \otimes \beta_n \in \Psi(B(\infty))$ is sent to the tableau whose shape we describe below row-by-row.

- For $1 \leq i \leq n - 1$, the $i$-th row consists of
  \[ k_{i,i+1} \text{-many } \bar{i} \text{s, } (k_{i,j} - k_{i,j+1}) \text{-many } \bar{j} \text{s, for each } i < j \leq n - 1, \]
  \[ (k_{i,n} - k_{i,n+1}) \text{-many } \bar{n} \text{s, } (k_{i,j+1} - k_{i,j}) \text{-many } j \text{s, for each } i < j \leq n, \]
  \[ (n - i + 1) + \sum_{r=i+1}^{n} k_{r,r} \text{-many } i \text{s.} \]

- The $n$-th row consists of
  \[ k_{n,n} \text{-many } \bar{n} \text{s and one } n. \]

Conversely, for each tableau with the $i$-th row consisting of $b_i^i$-many $j$'s ($i \prec j \preceq i$) and some number of $i$'s, we may map it to the element $u_\infty \otimes \beta_1 \otimes \cdots \otimes \beta_n \in \Psi(B(\infty))$, where, for $1 \leq i \leq n - 1$,

\[
k_{i,r} = \sum_{j=r+1}^{n} b_j^i + \sum_{j=1}^{n} b_j^i \quad \text{for } i \leq r \leq n - 1,
\]

\[
k_{i,n} = \sum_{j=i}^{n} b_j^i,
\]

\[
k_{i,\bar{r}} = \sum_{j=i}^{r-1} b_j^i \quad \text{for } i + 1 \leq r \leq n,
\]

and

\[
k_{n,n} = b_n^n.
\]

6.2.4. $D_{n+1}$ case. Element $u_\infty \otimes \beta_1 \otimes \cdots \otimes \beta_n \in \Psi(B(\infty))$ is sent to the tableau whose shape we describe below row-by-row.

- For $1 \leq i \leq n - 1$, the $i$-th row consists of
  \[ k_{i,i+1} \text{-many } \bar{i} \text{s, } (k_{i,j} - k_{i,j+1}) \text{-many } \bar{j} \text{s, for each } i < j \leq n - 1, \]
  \[ (k_{i,n} - k_{i,n+1} - \max\{0, k_{i,n} - k_{i,n+1}\}) \text{-many } \bar{n} \text{s, } (\max\{0, k_{i,n+1} - k_{i,n}\}) \text{-many } (n + 1) \text{s, } \]
  \[ (k_{i,n+1} - k_{i,n} - \max\{0, k_{i,n+1} - k_{i,n}\}) \text{-many } n \text{s, } (k_{i,j} - k_{i,j+1}) \text{-many } j \text{s, for each } i < j \leq n - 1, \]
  \[ (C + (n - i + 1) + \sum_{r=i+1}^{n} k_{r,r}) \text{-many } i \text{s.} \]
• The $n$-th row consists of
  \[
  (k_{n,n} - \max\{0, k_{n,n} - k_{n,n+1}\})-\text{many } \pi \text{s},
  (\max\{0, k_{n,n+1} - k_{n,n}\})-\text{many } (n+1) \text{s},
  \]
  \[
  (\max\{0, k_{n,n} - k_{n,n+1}\})-\text{many } (n+1) \text{s}, \text{ and one } n.
  \]
  Here, $C = k_{n,n} + \max\{0, k_{n,n+1} - k_{n,n}\} = k_{n,n+1} + \max\{0, k_{n,n} - k_{n,n+1}\}$.

Conversely, for each tableau with the $i$-th row consisting of $b_i$-many $j$'s ($i \prec j \preceq i$) and some number of $i$'s, we may map it to the element $u_\infty \otimes \beta_1 \otimes \cdots \otimes \beta_n \in \Psi(B(\infty))$, where, for $1 \leq i \leq n - 1$,

\[
  k_{i,r} = \sum_{j=r+1}^{n+1} b_j^i + \sum_{j=1}^{n+1} b_j^i \quad \text{for } i \leq r \leq n - 1,
  \]

\[
  k_{i,n} = b_{n+1}^i + \sum_{j=1}^{n} b_j^i,
  \]

\[
  k_{i,n+1} = \sum_{j=i}^{n+1} b_j^i,
  \]

\[
  k_{i,\bar{r}} = \sum_{j=1}^{n} b_j^i \quad \text{for } i + 1 \leq r \leq n,
  \]

and

\[
  k_{n,n} = b_{n+1}^n + \sum_{j=n}^{n} b_j^n,
  \]

\[
  k_{n,n+1} = \sum_{j=n}^{n+1} b_j^n.
  \]

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