On the Maximum ABC Spectral Radius of Connected Graphs and Trees

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Abstract
Let $G = (V, E)$ be a connected graph, where $V = \{v_1, v_2, \cdots, v_n\}$ and $m = |E|$. $d_i$ will denote the degree of vertex $v_i$ of $G$, and $\Delta = \max_{1 \leq i \leq n} d_i$. The ABC matrix of $G$ is defined as $M(G) = (m_{ij})_{n \times n}$, where $m_{ij} = \sqrt{(d_i + d_j - 2)/(d_i d_j)}$ if $v_i v_j \in E$, and 0 otherwise. The largest eigenvalue of $M(G)$ is called the ABC spectral radius of $G$, denoted by $\rho_{ABC}(G)$. Recently, this graph invariant has attracted some attentions. We prove that $\rho_{ABC}(G) \leq \sqrt{\Delta + (2m - n + 1)/\Delta - 2}$. As an application, the unique tree with $n \geq 4$ vertices having second largest ABC spectral radius is determined.

Keywords: ABC matrix, Eigenvalues, ABC spectral radius, Upper bounds, Trees.

1 Introduction

Let $G = (V, E)$ be a simple connected graph. Suppose $V = \{v_1, v_2, \cdots, v_n\}$ and $m = |E|$. If $c = m - n + 1 \geq 0$, then $G$ is called a $c$-cyclic graph. In particular, $G$ is called a tree and a unicyclic graph if $c = 1, 2$, respectively. As usual, $S_n$, $P_n$, $C_n$, and $K_n$ will denote the star, path, cycle, and complete graph with $n$ vertices, respectively.

Let $d_i$ denote the degree of vertex $v_i$, and $\Delta = \max_{1 \leq i \leq n} d_i$. The atom-bond connectivity index (ABC index in short) of $G$ is defined [1] as $ABC(G) = \sum_{v_i v_j \in E} f(d_i, d_j)$, where

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\( f(x, y) = \sqrt{(x^2 + y^2)} \). Since this index can predict well the heat of formation of alkanes (see [2,3]), it became a hot topic in the past few years (see [4-30]).

In 2017, Estrada [31] defined the ABC matrix of \( G \) as \( M = M(G) = (m_{ij})_{n \times n} \), where \( m_{ij} = f(d_i, d_j) \) if \( v_i v_j \in E \), and 0 otherwise. The chemical background of this matrix was explicated in [31]. The eigenvalues of \( M \) are called the ABC eigenvalues of \( G \). Because \( M \) is non-negative, symmetric, and irreducible, any ABC eigenvalue of \( G \) is real. In particular, the largest ABC eigenvalue of \( G \) is called its ABC spectral radius, and denoted by \( \rho_{ABC}(G) \). Obviously, \( \rho_{ABC}(G) \) is positive and simple. Moreover, there exists a unique vector \( x > 0 \) such that \( \rho_{ABC}(G) = \max_{\|y\|=1} y^T M y = x^T M x \), which is known as the Perron vector of \( M \).

Estrada [31] proved that \( \frac{2}{n} \rho_{ABC}(G) \leq \rho_{ABC}(G) \leq \max_{1 \leq i \leq n} M_i \), with both equalities iff \( M_1 = M_2 = \cdots = M_n \), where \( M_i = \sum_{1 \leq j \leq n} m_{ij} \). Recently, Chen [32] presented another lower bound of \( \rho_{ABC}(G) \) in terms of \( R_{-1}(G) \), which is the sum of \( \frac{1}{d_i d_j} \) over all edges \( v_i v_j \in E \). Chen [32] further proposed the problem of characterizing graphs with extremal ABC spectral radius for a given graph class. Soon, this problem for trees, connected graphs, and unicyclic graphs were solved by Chen [33], Ghorbani et al. [34], and Li et al. [35], respectively.

**Lemma 1.1** [33]. Let \( T \) be a tree with \( n \geq 3 \) vertices. Then

\[
\sqrt{2} \cos \frac{\pi}{n + 1} \leq \rho_{ABC}(G) \leq \sqrt{2n - 4},
\]

with left (right) equality iff \( G \cong P_n \) (resp. \( G \cong S_n \)).

**Lemma 1.2** [34]. Let \( G \) be a connected graph with \( n \geq 3 \) vertices. Then

\[
\sqrt{2} \cos \frac{\pi}{n + 1} \leq \rho_{ABC}(G) \leq \sqrt{2n - 4},
\]

with the left (right) equality iff \( G \cong P_n \) (resp. \( G \cong K_n \)).

**Lemma 1.3** [35]. Let \( G \) be a unicyclic graph with \( n \geq 4 \) vertices. Then

\[
\sqrt{2} = \rho_{ABC}(C_n) \leq \rho_{ABC}(G) \leq \rho_{ABC}(S_n + e),
\]

with the left (right) equality iff \( G \cong C_n \) (resp. \( G \cong S_n + e \)).

For convenience, let \( \mathcal{C}(n) \) be the set of connected graphs with \( n \) vertices, and \( \mathcal{G}(m, n) \) the set of connected graphs with \( n \) vertices and \( m \) edges. In the present paper, we consider upper bounds of \( \rho_{ABC} \) for connected graphs. In Section 2, it is shown that, if \( G \in \mathcal{G}(m, n) \) and \( \Delta(G) = \Delta \), then \( \rho_{ABC}(G) \leq \sqrt{\Delta + (2m - n + 1)/\Delta - 2} \). As an
application, in Section 3, we characterize the unique tree with $n \geq 4$ vertices having the second largest ABC spectral radius. Finally, some problems are proposed in Section 4.

2 Some upper bounds of the ABC spectral radius

In this section, we present two upper bounds of $\rho_{ABC}$ of connected graphs.

**Theorem 2.1.** If $G \in \mathcal{G}(m, n)$ and $\Delta(G) = \Delta$, then

$$\rho_{ABC}(G) \leq \sqrt{\Delta + (2m - n + 1)/\Delta - 2}.$$ 

Moreover, the bound is attainable.

**Proof.** Let $M = M(G)$, $D = 2m - n + 1$, and $x = (\sqrt{d_1}, \sqrt{d_2}, \cdots, \sqrt{d_n})^T$. From the Perron-Frobenius theory, it suffices to confirm the claim: $(Mx)_i \leq \sqrt{d_i \Delta + D}/\Delta - 2$ or $[(Mx)_i/\sqrt{d_i}]^2 \leq \Delta + D/\Delta - 2$ holds for $1 \leq i \leq n$.

If $d_i < D/\Delta$, then $(Mx)_i = \sum_{v_i v_j \in E} f(d_i, d_j) \sqrt{d_j} \leq d_i \sqrt{d_i + D/\Delta - 2} < \sqrt{d_i \Delta + D/\Delta - 2}$.

Hence assume $d_i \geq D/\Delta$. By using the Cauchy-Schwarz Inequality we have

$$(Mx)_i = \sum_{v_i v_j \in E} \sqrt{(d_i + d_j - 2)/d_i} \leq \sqrt{d_i^2 - 2d_i + \sum_{v_i v_j \in E} d_j} \leq \sqrt{d_i^2 - 2d_i + [2m - d_i - (n - d_i - 1)]} = \sqrt{d_i^2 - 2d_i + D}.$$

Thus we have $[(Mx)_i/\sqrt{d_i}]^2 \leq (d_i^2 - 2d_i + D)/d_i = d_i + D/d_i - 2$. Since $\eta(x) = x + D/x - 2$ is a Nike function and $D/\Delta \leq d_i \leq \Delta$, it follows that

$$[(Mx)_i/\sqrt{d_i}]^2 \leq \max\{\eta(D/\Delta), \eta(\Delta)\} = \Delta + D/\Delta - 2.$$ 

Finally, to see the bound is attainable, one can take $S_n$ and $K_n$ as examples. The proof is thus completed. $\blacksquare$

Let $\theta(m, \Delta) = \sqrt{\Delta + (2m - n + 1)/\Delta - 2}$. For fixed $m$, the monotonicity of $\theta$ with respect to $\Delta$ is clear. Hence Theorem 2.1 can easily produce a upper bound of $\rho_{ABC}$ for subsets of $\mathcal{G}(m, n)$. For example, an upper bound for $c$-cyclic graphs is obtained as follows.

**Corollary 2.2.** Let $G$ be a $c$-cyclic graph with $n \geq 3$ vertices, and $c \leq (n - 1)/2$. Then

$$\rho_{ABC}(G) \leq \sqrt{n - 2 + 2c/(n - 1)}.$$
Proof. Since \( m = n - 1 + c \) and \( c \leq (n - 1)/2 \), by direct calculations we have

\[
\theta(m, 2) = \sqrt{(n - 1)/2 + c} \leq \theta(m, n - 1) = \sqrt{n - 2 + 2c/(n - 1)},
\]

and the conclusion follows from Theorem 2.1. ■

It is easily seen that, Theorem 2.1 can reproduce the upper bound part of Lemma 1.1. However, if we consider upper bounds of \( \rho_{ABC} \) for a subset of \( C_n \), whose elements have various sizes (numbers of edges), Theorem 2.1 may be not so convenient to applied directly. Hence we deduce the following result.

**Corollary 2.3.** If \( G \in G(m, n) \) and \( \Delta(G) = \Delta \), then \( \rho_{ABC}(G) \leq \sqrt{\Delta + k - 2} \), where \( k = \lceil (2m - n + 1)/\Delta \rceil \).

**Proof.** From \( k = \lceil (2m - n + 1)/\Delta \rceil \geq (2m - n + 1)/\Delta \), the conclusion holds immediately from Theorem 2.1. ■

Though Corollary 2.3 is weaker than Theorem 2.1, the upper bound \( \theta'(m, \Delta) = \sqrt{\Delta + k - 2} \) has better property than \( \theta(m, \Delta) \). In fact, for fixed \( m \), \( \theta'(m, \Delta) \) almost strictly decreases with \( \Delta \). To see this, let \( \Delta_1 > \Delta_2 \) and \( D = 2m - n + 1 \). We illustrate the fact with the following two cases.

**Case 1.** \( D/k \leq \Delta_2 < \Delta_1 < D/(k - 1) \). Then \( \theta'(m, \Delta_2) < \theta'(m, \Delta_1) \) obviously.

**Case 2.** \( D/k \leq \Delta_2 < D/(k - 1) \leq \Delta_1 \). Then

\[
\theta'(m, \Delta_2) = \sqrt{\Delta_2 + k - 2} \\
\leq \sqrt{\Delta_1 + k - 3} \\
= \theta'(m, \Delta_1),
\]

with equality iff \( \Delta_2 = D/(k - 1) - 1 = \Delta_1 - 1 \).

By the monotonicity of \( \theta'(m, \Delta) \) with respect to \( \Delta \), we are able to reproduce the upper bound part of Lemma 1.2.

**Corollary 2.4** [34]. Let \( G \) be a connected graph with \( n \geq 3 \) vertices. Then

\[
\rho_{ABC}(G) \leq \sqrt{2n - 4},
\]

with equality iff \( G \cong K_n \).

**Proof.** By the monotonicity of \( \theta' \) we have \( \theta'(m, \Delta) \leq \theta'(m, n - 1) \leq \theta'(n(n - 1)/2, n - 1) \), with all the equalities iff \( m = n(n - 1)/2 \), that is, \( G \cong K_n \). The conclusion thus follows from Corollary 2.3. ■
In order to further illustrate the application of Theorem 2.1 and Corollary 2.3, in this section we determine the tree with \( n \geq 4 \) vertices, whose ABC spectral radius is the second largest. For convenience, let \( \mathcal{T}_n \) be the set of trees with \( n \) vertices, and \( \mathcal{T}^{(\Delta)}_n = \{ T \in \mathcal{T}_n | \Delta(T) = \Delta \} \). Let \( T_i, i = 1, 2, 3, 4 \), be the trees shown in Figure 1. \( T_1 \) is just the double star \( S_{n-3}^{(1)} \). If \( T \in \mathcal{T}^{(\Delta)}_n \), then \( T \) contains \( S_{\Delta+1} \) as its (induced) subgraph, hence it is easily seen that \( \mathcal{T}^{(n-1)}_n = \{ S_n \} \), \( \mathcal{T}^{(n-2)}_n = \{ S_{n-3,1} \} \), and \( \mathcal{T}^{(n-3)}_n = \{ T_2, T_3, T_4 \} \).

Our aim in this section is to prove the following conclusion.

**Theorem 3.1.** If \( n \geq 4 \) and \( T \in \mathcal{T}_n - \{ S_n, S_{n-3,1} \} \), then

\[
\rho_{ABC}(T) < \rho_{ABC}(S_{n-3,1}) < \rho_{ABC}(S_n).
\]

We need some more preliminaries before presenting the proof of Theorem 3.1.

For two vertices \( u \) and \( v \) of a graph \( G \), they are said to be equivalent, denoted by \( u \sim v \), if there is an automorphism of \( G \) sending \( u \) to \( v \). By symmetry, the following result is immediate.

**Lemma 3.2.** Let \( x \) be the Perron vector of the ABC matrix \( M(G) \) of a connected graph \( G \). If \( u \sim v \), then \( x_i = x_j \).

**Lemma 3.3.** \( \rho_{ABC}(S_{n-3,1}) > \sqrt{n - 3.5} \) if \( n \geq 4 \).

**Proof.** Let \( \rho = \rho_{ABC}(S_{n-3,1}) \), and label the vertices of \( S_{n-3,1} \) as in Figure 1. Based on Lemma 3.2, let \( x = (x_1, x_2, x_3, x_4, \ldots, x_4)^T \) be the Perron vector of \( M = M(S_{n-3,1}) \). From \( \rho x = Mx \) we have

\[
\begin{align*}
\rho x_1 &= (n - 3) \sqrt{\frac{n - 3}{n - 2}} x_4 + \sqrt{\frac{1}{2}} x_2 \\
\rho x_2 &= \sqrt{\frac{1}{2}} x_1 + \sqrt{\frac{1}{2}} x_3 > \sqrt{\frac{1}{2}} x_1 \\
\rho x_4 &= \frac{n - 3}{n - 2} x_1
\end{align*}
\]
Hence \( \rho^2 x_1 = (n - 3)\sqrt{\frac{n-3}{n-2}} \rho x_2 + \sqrt{\frac{1}{2}} \rho x_2 > \frac{(n-3)^2}{n-2} x_1 + \frac{1}{2} x_1 \), and we arrive at
\[
\rho^2 > \frac{(n-3)^2}{n-2} + \frac{1}{2} > n - 4 + 0.5 = n - 3.5,
\]
which completes the proof. ■

**Lemma 3.4.** If \( n \geq 6 \) and \( T \in \{T_2, T_3, T_4\} \), then \( \rho_{ABC}(T) < \sqrt{n - 3.5} \).

**Proof.** If \( n = 6 \), the conclusion can be verified easily. Hence assume \( n \geq 7 \). Label the vertices of \( T \) as in Figure 1. Let \( M = M(T) \) and \( x = (\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})^T \). Based on Lemma 2.1, we prove the result by confirming \((Mx)_i/\sqrt{d_i} \leq \sqrt{n - 3.5}\) for \( 1 \leq i \leq n \).

We have \((Mx)_i/\sqrt{d_i} \leq \sqrt{d_i - 2 + \sum_{v_i, v_j \in E} d_j/d_i}\) from the proof of Theorem 2.1. Hence
\[
(Mx)_1/\sqrt{d_1} \leq \sqrt{n - 5 + (n - 1)/(n - 3)} = \sqrt{n - 4 + 2/(n - 3)} \leq n - 3.5.
\]
For \( i \geq 2 \), because \( d_i \leq 3 \) and \( n \geq 7 \), it follows
\[
(Mx)_i/\sqrt{d_i} \leq \max\{\sqrt{n - 4}, (n - 1)/2, (n + 2)/3\} < \sqrt{n - 3.5},
\]
and the proof is completed. ■

Now we present the proof of Theorem 3.1.

**Proof of Theorem 3.1.** It is easily seen that \( \mathcal{T}_4 = \{S_4, S_{1,1} \cong P_4\} \), \( \mathcal{T}_5 = \{S_5, S_{2,1}, P_5\} \), and \( \mathcal{T}_6 = \{S_6, S_{3,1}, T_2, T_3, T_4, P_6\} \), so the conclusion holds if \( n \leq 6 \) from Lemmas 1.1, 3.3, and 3.4. Hence assume \( n \geq 7 \).

If \( \Delta \geq n - 3 \), the conclusion follows from Lemmas 3.3 and 3.4. Otherwise, if \( \Delta \leq n - 4 \), then \( \rho_{ABC}(T) \leq \theta'(n - 1, n - 4) = \sqrt{n - 6 + \left(\frac{n - 1}{n - 4}\right)} \leq \sqrt{n - 4} \) from Corollary 2.3.

The proof is thus completed. ■

### 4 Further discussions

In this paper, it is shown that \( \rho_{ABC}(G) \leq \sqrt{\Delta + (2m - n + 1)/\Delta - 2} \) if \( G \in \mathcal{G}(m, n) \) and \( \Delta(G) = \Delta \). The bound is attained by \( S_n \) and \( K_n \). Firstly, the following problem may be worth consideration.

**Problem 4.1.** Characterize the graphs \( G \in \mathcal{G}(m, n) \) such that
\[
\rho_{ABC}(G) = \sqrt{\Delta(G) + (2m - n + 1)/\Delta(G) - 2}.
\]

As well, the double star \( S_{n-3,1} \) is shown to be the unique tree having second largest ABC spectral radius in \( \mathcal{T}_n \), \( n \geq 4 \). Recall that, Lin et al. [36] ordered trees by their
(adjacent) spectral radius $\lambda_1$, and showed that, if $T_1$ and $T_2$ are two trees with $n \geq 4$ vertice and $\Delta(T_1) > \Delta(T_2) \geq (2n)/3 - 1$, then $\lambda_1(T_1) > \lambda_1(T_2)$. Naturally, the following question is interesting.

**Question 4.2.** Let $G_1$ and $G_2$ be two graphs in a subset of $\mathcal{G}(m, n)$. Is there some integer $l(m, n)$ (depending on $n$ and/or $m$), such that if $\Delta(G_1) > \Delta(G_2) \geq l(m, n)$, then $\rho_{ABC}(G_1) > \rho_{ABC}(G_2)$?

This question may be difficult to answer at the present, even for trees, and the following two problems are worth investigation in advance.

**Problem 4.3** Order graphs in some classes of connected graphs by their ABC spectral radii.

**Problem 4.4.** Establish non-trivial lower bounds of $\rho_{ABC}(G)$ for a graph $G \in \mathcal{G}(m, n)$ (in terms of $m$, $n$, and $\Delta(G)$).

**References**

[1] E. Estrada, L. Torres, L. Rodríguez, I. Gutman, An atom-bond connectivity index: Modelling the enthalpy of formation of alkanes, *Indian J. Chem.* 37A (1998) 849-855.

[2] E. Estrada, Atom-bond connectivity and the energetic of branched alkanes, *Chem. Phys. Lett.* 463 (2008) 422-425.

[3] I. Gutman, J. Tošović, S. Radenković, S. Marković, On atom-bond connectivity index and its chemical applicability, *Indian J. Chem.* 51A (2012) 690-694.

[4] B. Furtula, A. Graovac, D. Vukičević, Atom-bond connectivity index of trees, *Discr. Appl. Math.* 157 (2009) 2828-2835.

[5] K. C. Das, Atom-bond connectivity index of graphs, *Discr. Appl. Math.* 158 (2010) 1181-1188.

[6] R. Xing, B. Zhou, Z. Du, Further results on atom-bond connectivity index of trees, *Discr. Appl. Math.* 157 (2010) 1536-1545.

[7] R. Xing, B. Zhou, F. Dong, On atom-bond connectivity index of connected graphs, *Discr. Appl. Math.* 159 (2011) 1617-1630.
[8] J. Chen, X. Guo, Extreme atom-bond connectivity index of graphs, *MATCH Commun. Math Comput. Chem.* 65 (2011) 713-722.

[9] K. C. Das, I. Gutman, B. Furtula, On atom-bond connectivity index, *Chem. Phys. Lett.* 511 (2011) 452-454.

[10] J. Chen, J. Liu, X. Guo, Some upper bounds for the atom-bond connectivity index of graphs, *Appl. Math. Lett.* 25 (2012) 1077-1081.

[11] I. Gutman, B. Furtula, M. Ivanović, Notes on trees with minimal atom-bond connectivity index, *MATCH Commun. Math Comput. Chem.* 67 (2012) 467-482.

[12] W. Lin, T. Gao, Q. Chen, X. Lin, On the atom-bond connectivity index of connected graphs with a given degree sequence, *MATCH Commun. Math. Comput. Chem.* 69 (2013) 571-578.

[13] W. Lin, X. Lin, T. Gao, X. Wu, Proving a conjecture of Gutman concerning trees with minimal ABC index, *MATCH Commun. Math. Comput. Chem.* 69 (2013) 549-557.

[14] I. Gutman, B. Furtula, M. B. Ahmadi, S. A. Hosseini, P. Salehi Nowbandegani, M. Zarrinderakht, The ABC index conundrum, *Filomat* 27 (2013) 1075-1083.

[15] D. Dimitrov, On structural properties of trees with minimal atom-bond connectivity index, *Discr. Appl. Math.* 172 (2014) 28-44.

[16] D. Dimitrov, On structural properties of trees with minimal atom-bond connectivity index II: Bounds on $B_1$- and $B_2$-branches, *Discr. Appl. Math.* 204 (2016) 90-116.

[17] Z. Du, C. M. da Fonseca, On a family of trees with minimal atom-bond connectivity index, *Discr. Appl. Math.* 202 (2016) 37-49.

[18] D. Dimitrov, Z. Du, C. M. da Fonseca, On structural properties of trees with minimal atom-bond connectivity index III: Trees with pendent paths of length three, *Appl. Math. Comput.* 282 (2016) 276-290.

[19] D. Dimitrov, On structural properties of trees with minimal atom-bond connectivity index IV: Solving a conjecture about the pendent paths of length three, *Appl. Math. Comput.* 313 (2017) 418-430.
[20] D. Dimitrov, Z. Du, C. M. da Fonseca, Some forbidden combinations of branches in minimal-ABC trees, *Discr. Appl. Math.* **236** (2018) 165-182.

[21] K. C. Das, S. Elumalai, I. Gutman, On ABC index of graphs, *MATCH Commun. Math. Comput. Chem.* **78** (2017) 459-468.

[22] D. Dimitrov, Efficient computation of trees with minimal atom-bond connectivity index, *Appl. Math. Comput.* **224** (2013) 663-670.

[23] W. Lin, J. Chen, Q. Chen, T. Gao, X. Lin, B. Cai, Fast computer search for trees with minimal ABC index based on tree degree sequences, *MATCH Commun. Math. Comput. Chem.* **72** (2014) 699-708.

[24] W. Lin, C. Ma, Q. Chen, J. Chen, T. Gao, B. Cai, Parallel search trees with minimal ABC index with MPI + OpenMP, *MATCH Commun. Math. Comput. Chem.* **73** (2015) 337-343.

[25] D. Dimitrov, N. Milosavljević, Efficient computation of trees with minimal atom-bond connectivity index revisited, *MATCH Commun. Math. Comput. Chem.* **79** (2018) 431-450.

[26] W. Lin, J. Chen, Z. Wu, D. Dimitrov, L. Huang, Computer search for large trees with minimal ABC index, *Appl. Math. Comput.* **338** (2018) 221-230.

[27] D. Dimitrov, Z. Du, C. M. da Fonseca, The minimal-ABC trees with $B_1$-branches, *PLoS ONE* **13**(4): e0195153. https://doi.org/10.1371/journal.pone.0195153.

[28] Z. Du, D. Dimitrov, The minimal-ABC trees with $B_1$-branches II, *IEEE Access* **6** (2018) 66350-66366.

[29] Y. Zheng, W. Lin, Q. Chen, L. Huang, Z. Wu, Characterizing trees with minimal ABC index with computer search: A short survey, *J. Discret. Appl. Math.* **1** (2018) 1-9.

[30] Z. Du, D. Dimitrov, The minimal-ABC trees with $B_2$-branches, *Comp. Appl. Math.* **39**, 85 (2020). https://doi.org/10.1007/s40314-020-1119-7.

[31] E. Estrada, The ABC matrix, *J. Math. Chem.* **55** (2017) 1021C1033.
[32] X. Chen, On ABC eigenvalues and ABC energy, *Linear Algebra Appl.* **544** (2018) 141-157.

[33] X. Chen, On extremality of ABC spectral radius of a tree, *Linear Algebra Appl.* **564** (2019) 159-169.

[34] M. Ghorbania, X. Li, M. Hakimi-Nezhaada, J. Wang, Bounds on the ABC spectral radius and ABC energy of graphs, *Linear Algebra Appl.* **598** (2020) 145-164.

[35] X. Li, J. Wang, On the ABC spectra radius of unicyclic graphs, *Linear Algebra Appl.* **596** (2020) 71-81.

[36] W. Lin, X. Guo, Ordering trees by their largest eigenvalues, *Linear Algebra Appl.* **418** (2006) 450-456.