A Remark on Leray’s Problem on Stationary Navier–Stokes Flows with Large Fluxes in Infinite Cylindrical Domains

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Abstract We consider Leray’s problem on stationary Navier–Stokes flows with arbitrary large fluxes in an unbounded cylinder with several exits to infinity. For a stationary Navier–Stokes flow with large fluxes in the unbounded cylinder, we prove that, if the difference between the pressure of the main flow and the pressure of the Poiseuille flow with the same flux in a branch of the cylinder remains bounded as \(|x| \to \infty\), then the flow behaves at infinity of the branch like the Poiseuille flow.

Keywords Leray’s problem, Navier–Stokes equations, large flux, unbounded cylindrical domains

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1 Introduction and Main Result

Let

\[ \Omega = \bigcup_{i=0}^{m} \Omega^i \]

be a cylindrical domain of \(C^2\)-class of \(\mathbb{R}^3\), where \(\Omega^0\) is a bounded domain and \(\Omega^i, i = 1, \ldots, m\), are disjoint semi-infinite straight cylinders, that is, in possibly different coordinates,

\[ \Omega^i = \{ x' = (x^i_1, x^i_2, x^i_3) \in \mathbb{R}^3 : x^i_3 > 0, x'' = (x^i_1, x^i_2) \in \Sigma^i \}, \]

where \(\Sigma^i \subset \mathbb{R}^2, i = 1, \ldots, m\), is a bounded domain and \(\Omega^i \cap \Omega^j = \emptyset\) for \(i \neq j\). Without loss of generality, we assume for each \(i = 1, \ldots, m\) that the coordinate system which is fixed in \(\Omega^i\) is such that \(x'^i, x'^i_3\) denote the variables with respect to the cross section \(\Sigma^i\) and the axial direction of \(\Omega^i\), respectively.

Let us consider the stationary Navier–Stokes system

\[ -\Delta U + (U \cdot \nabla)U + \nabla P = 0 \quad \text{in} \ \Omega, \]
\[ \text{div} \ U = 0 \quad \text{in} \ \Omega, \]
\[ U = 0 \quad \text{on} \ \partial \Omega. \]  

We impose an additional condition for the behavior of the velocity field at infinity as

\[ \lim_{|x| \to \infty} U(x) = u_\infty, \]
where \( u_\infty \) coincides at infinity of each exit \( \Omega^i \), \( i = 1, \ldots, m \), with the Poiseuille flow \( \mathbf{v}_i \) in \( \Omega^i \) corresponding to the prescribed flux \( \Phi_i \).

Poiseuille flows in an infinite straight cylinder \( \Sigma \times \mathbb{R} \) are often referred to as flows parallel to the axial direction. In the stationary case, the Poiseuille flow \( \mathbf{v} = (0,0,v(x')) \) and the corresponding Poiseuille flow pressure \( \Pi(x_3) = -kx_3 + b \) are simply given by the Poisson equation

\[
- \Delta' v = k, \quad v|_{\partial \Sigma} = 0,
\]

where, if \( \Sigma \) is a Lipschitz domain, then \( k = c(\Sigma) \Phi \), where \( c(\Sigma) = \frac{1}{\Sigma} \int_{\Sigma} |\nabla' g|^2 \, dx' \), \( - \Delta' g = 1 \), \( g|_{\partial \Sigma} = 0 \).

Note, due to the solenoidal condition for the fluid, that if \( U \) satisfies (1.3), then

\[
\int_{\Sigma} \mathbf{U} \cdot \mathbf{n}^i \, dx^i = \Phi_i \quad \text{(1.4)}
\]

should necessarily hold true, where \( \mathbf{n}^i \) is the unit vector along the positive axial direction of \( \Omega^i \). Moreover, the flux \( \Phi_i \) should be independent of \( x^i_3 \) over \( \Omega^i \) for \( i = 1, \ldots, m \) and

\[
\sum_{i=1}^{m} \Phi_i = 0 \quad \text{(1.5)}
\]

should be naturally assumed.

Classical Leray’s problem is to show whether or not the problem (1.2)–(1.4) will admit a solution. Leray’s problem seems to have been proposed, see [1], by Leray himself to Ladyzhenskaya, who in [11] attempted an existence proof under no restrictions on the viscosity.

There is a number of papers dealing with stationary Leray’s problem. Fundamental contribution to Leray’s problem was made by Amick in [1], where the existence of unique weak solution to (1.2)–(1.4) was proved under a smallness assumption on the total flux \( \sum_{i=1}^{m} |\Phi_i| \), see also [2, 5, 6, 9, 10] and [13–17]. However, it has been shown, up to now, that Leray’s problem in infinite cylindrical domains is solved positively only under smallness assumptions on the total flux, and the problem for arbitrary large total flux is known as one of the most challenging problems in the theoretical fluid dynamics; for Leray’s and related problems we refer, in particular, to [7, Chapter VI, Sections 1 and 2] and [8, Chapter XI, Sections 1–4], cf. also [3], Introduction and references cited therein for more details.

In this paper, we aim at considering Leray’s problem in \( \Omega \) for large total flux; we present a condition on the flow pressure to allow a weak solution (1.2), (1.4) to behave like Poiseuille flows at \( |x| \to \infty \).

In order to explain the main result of the paper, let us give the definition of the weak solution to the system (1.2), (1.4). Let

\[
C^\infty_{0,\sigma}(\Omega) := \{ \varphi \in C^\infty_0(\Omega)^n : \text{div} \varphi = 0 \}
\]

and domains \( \Omega_N^i \) and \( \Omega_{N,N+1} \) be respectively given as

\[
\Omega_N^i = \{ x \in \Omega^i : x^i_3 \leq N \}, \quad \Omega_{N,N+1}^i = \{ x \in \Omega^i : N \leq |x| \leq N + 1 \}, \quad \Omega_{N,N+1} = \bigcup_{i=1}^{m} \Omega_{N,N+1}^i.
\]

**Definition 1.1** A vector field \( \mathbf{U} : \Omega \to \mathbb{R}^3 \) is called a weak solution to (1.2), (1.4), if it satisfies the conditions (i)–(v):