SIMPLEX TRIANGULATION INDUCED SCALE-FREE NETWORKS

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Abstract. We propose a simple rule that generates scale-free networks with very large clustering coefficient and very small average distance. These networks are called simplex triangulation networks (STNs) as they can be considered as a kind of network representation of simplex triangulation. We obtain the analytic results of power-law exponent $\gamma = 2 + \frac{1}{d-1}$ for $d$-dimensional STNs, and clustering coefficient $C$. We prove that the increasing tendency of average distance of STNs is a little slower than the logarithm of the number of nodes in STNs. In addition, the STNs possess hierarchical structure as $C(k) \sim k^{-1}$ when $k \gg d$ that in accord with the observations of many real-life networks.

Keywords. complex networks, simplex triangulation, scale-free networks, small-world networks, clustering coefficient, average distance

AMS (MOS) subject classification: 05C75, 05C80

1 Introduction

Recently, empirical studies indicate that the networks in various fields have some common characteristics, which inspires scientists to construct a general model [1-3]. The most important characteristics are scale-free property and small-world effect. The former means that the degree distribution obeys power law as $p(k) \propto k^{-\gamma}$, where $k$ is the degree and $p(k)$ is the probability density function for the degree distribution. $\gamma$ is called the power-law exponent, and is usually between 2 and 3 in real-life networks. The latter involves two factors: small average distance as $L \sim \ln N$ or even smaller and great clustering coefficient $C$, where $L$ is the average distance, $N$ is the number of nodes in the network, and $C$ is the probability that a randomly selected node’s two randomly picked neighbors are neighbor. One of the most well-known models is Watts and Strogatz’s small-world network (WS network), which can be constructed by starting with a regular network and randomly moving one endpoint of each edge with probability $p$[4]. Another significant one is Barabási and Albert’s scale-free network model (BA network)[5]. The BA model suggests that two main ingredients of self-organization of a network in a scale-free structure are growth and preferential attachment. However, both WS and BA networks fail to mimic the reality in some aspects. Therefore,
a significant problem is how to generate networks displaying both scale-free property and small-world effect [6-8].

In this paper, we propose a simple rule that generates scale-free networks with very large clustering coefficient and very small average distance. These networks are called simplex triangulation networks (STNs) as they can be considered as the one-dimensional framework of simplex triangulation in the view of algebraic topology [9]. Strictly speaking, if $\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_d$ are the linear independent points in $\mathbb{R}^d$, the set $\{\sum_{i=0}^{d} \lambda_i \vec{a}_i | \lambda_i \geq 0, \sum_{i=0}^{d} \lambda_i = 1\}$ is the $d$-simplex [9]. For instance, 0-simplex is a single vertex, 1-simplex is a line segment, 2-simplex is a triangle, 3-simplex is a tetrahedron, and so on. Informal speaking, simplex triangulation is the process to triangulate original simplex into several sub-simplices. For example, choose an arbitrary point inside a $d$-simplex, $\vec{a}$, and link this point to all the vertices of this simplex, then this simplex will be triangulated into $d + 1$ sub-simplices.

Our model starts with a $d$-simplex, where $d \geq 2$ is an integer. The network representation of this simplex contains $d + 1$ nodes and $\frac{d(d+1)}{2}$ edges. Then, at each time step, a simplex is randomly selected, and a new node is added inside this simplex and linked to all its $d + 1$ vertices (nodes). During each time step, the number of simplices, nodes and edges increases by $d$, one, $d + 1$, respectively. Repeat this simple rule, one can get the $d$-STN of arbitrary order that he likes.

## 2 The degree distribution

As we have mentioned above, the degree distribution is one of the most important statistical characteristics of networks. Since many real-life networks are scale-free networks, whether the networks are of power-law degree distribution is a criterion to judge the validity of the model.

Since after a new node is added to the network, the number of simplices increases by $d$, we can immediately get that when the network is of order $N$, the number of simplices is:

$$N_s = d(N - d - 1) + 1 = dN - d^2 - d + 1 \quad (1)$$

Let $N_s^i$ be the number of simplices containing the node $i$, the probability that a newly added node will link to the $i$th node is $N_s^i/N_s$. Apparently, $N_s^i$ will increase by $d - 1$ while $k_i$ increases by one, and when the node is newly added, it is of degree $d + 1$ and contained by $d + 1$ different simplices, thus

$$N_s(k) = k(d - 1) - (d + 1)(d - 2), \quad (2)$$

where $N_s(k)$ denotes the number of simplices containing a node of degree $k$. Let $n(N, k)$ be the number of nodes with degree $k$ when $N$ nodes are present, now we add a new node to the network, $n(N, k)$ evolves according
to the following rate equation[10]:

\[ n(N + 1, k + 1) = n(N, k) \frac{N_s(k)}{N_s} + n(N, k + 1)(1 - \frac{N_s(k + 1)}{N_s}) \]  

(3)

When \( N \) is sufficient large, \( n(N, k) \) can be approximated as \( Np(k) \), where \( p(k) \) is the probability density function for the degree distribution. In terms of \( p(k) \), the above equation can be rewritten as:

\[ p(k + 1) = \frac{N}{N_s}[p(k)N_s(k) - p(k + 1)N_s(k + 1)] \]  

(4)

Using Eq.(1) & (2) and the expression \( p(k + 1) - p(k) = \frac{dp}{dk} \), we can get the continuous form of Eq.(4):

\[ (2d - 1 - \frac{d^2 + d - 1}{N})p(k + 1) + [k(d - 1) - (d + 1)(d - 2)] \frac{dp}{dk} = 0 \]  

(5)

For large \( k \) (\( p(k + 1) \approx p(k) \), \( k \gg d \)) and \( N \gg d \), we have:

\[ (2d - 1)p(k) + (d - 1)k \frac{dp}{dk} = 0 \]  

(6)

This lead to \( p(k) \propto k^{-\gamma} \) with \( \gamma = 2 + \frac{1}{d-1} \in (2, 3] \), in accord with many real-life networks with power exponent between 2 and 3.

### 3 The clustering coefficient

The clustering coefficient of a node is the ratio of the total number of existing edges between all its neighbors and the number of all possible edges between them. The clustering coefficient \( C \) of the whole network is defined as the average of all nodes’ clustering coefficient. At first, let us derive the analytical expression of \( C(k) \) denoting the clustering coefficient of a node with degree \( k \). When a node is newly added into the network, its degree and clustering coefficient are \( d + 1 \) and 1. After that, if its degree increases by one, then its new neighbor must link to its \( d \) existing neighbors. Hence we have:

\[ C(k) = \frac{d(d+1)/2 + d(k - d - 1)}{k(k-1)/2} = \frac{d(2k - d - 1)}{k(k - 1)} \]  

(7)

The clustering coefficient of the whole network can be obtained as the mean value of \( C(k) \) with respect to the degree distribution \( p(k) \):

\[ C = \int_{k_{\text{min}}}^{k_{\text{max}}} C(k)p(k)dk, \]  

(8)

where \( k_{\text{min}} = d + 1 \) is the minimal degree and \( k_{\text{max}} \gg k_{\text{min}} \) is the maximal degree. Combine Eq.(7) and Eq.(8), note that \( p(k) = Ak^{\frac{d-1}{d-1}} \) and
\[ \int_{k_{\text{min}}}^{k_{\text{max}}} A p(k) \, dk = 1, \]

we can get the analytical result of \( C \) by approximately treating \( k_{\text{max}} \) as \( +\infty \). For example, when \( d = 2 \), the clustering coefficient is

\[ C = \frac{46}{3} - 36 \ln 2 \approx 0.7366, \quad (9) \]

and when \( d = 3 \), it is

\[ C = 18 + 36 \sqrt{2} \arctan \sqrt{2} + \frac{9}{2} \pi - 18 \sqrt{2} \pi \approx 0.8021 \quad (10) \]

For larger \( d \), the expression is too long thus it will not be shown here. The integral values for \( d = 4, 5, 6, 7, 8, 9, 10 \) are 0.8406, 0.8660, 0.8842, 0.8978, 0.9085, 0.9171, and 0.9241, respectively.

It is remarkable that, the clustering coefficient of BA networks is very small and decreases with the increasing of network order, following approximately \( C \sim \frac{\ln N}{N} \) \cite{11}. Since the data-flow patterns show a large amount of clustering in interconnection networks, the STNs may perform better than BA networks. In addition, the demonstration exhibits that most real-life networks have large clustering coefficient no matter how many nodes they have.

That agrees with the case of STNs but conflicts with that of BA networks. Furthermore, many real-life networks are characterized by the existence of hierarchical structure\cite{12}, which can usually be detected by the negative correlation between the clustering coefficient and the degree. The BA network, which does not possess hierarchical structure, is known to have the clustering coefficient \( C(x) \) of node \( x \) independent of its degree \( k(x) \), while the STN has been shown to have \( C(k) \sim k^{-1} \), in accord with the observations of many real networks\cite{12}.

### 4 The average distance

Using symbol \( d(i, j) \) to represent the distance between \( i \) and \( j \), the average distance of STN with order \( N \), denoted by \( L(N) \), is defined as: \( L(N) = \frac{\sigma(N)}{N(N-1)} \), where the total distance is: \( \sigma(N) = \sum_{1 \leq i < j \leq N} d(i, j) \). Since newly added node will not affect the distance between existing nodes, we have:

\[ \sigma(N + 1) = \sigma(N) + \sum_{i=1}^{N} d(i, N + 1) \quad (11) \]

Assume that the node \( N + 1 \) is added into the simplex \( y_1 y_2 \cdots y_{d+1} \), then the Eq.(11) can be rewritten as:

\[ \sigma(N + 1) = \sigma(N) + \sum_{i=1}^{N} (D(i, y) + 1) \quad (12) \]
where $D(i, y) = \min \{d(i, y_j), j = 1, 2, \cdots, d + 1 \}$. Constrict this simplex continuously into a single node $y$, then we have $D(i, y) = d(i, y)$. Since $d(y_j, y) = 0$, Eq.(12) can be rewritten as:

$$\sigma(N + 1) = \sigma(N) + N + \sum_{i \in \Gamma} d(i, y) \quad (13)$$

where $\Gamma = \{1, 2, \cdots, N\} - \{y_1, y_2, \cdots, y_{d+1}\}$ is a node set with cardinality $N - d - 1$. The sum $\sum_{i \in \Gamma} d(i, y)$ can be considered as the total distance from one node $y$ to all the other nodes in STN with order $N - d$. In a rough version, the sum $\sum_{i \in \Gamma} d(i, y)$ is approximated in terms of $L(N - d)$ as:

$$\sum_{i \in \Gamma} d(i, y) \approx (N - d - 1) L(N - d) \quad (14)$$

Apparently,

$$(N - d - 1) L(N - d) = \frac{2\sigma(N - d)}{N - d} < \frac{2\sigma(N)}{N} \quad (15)$$

Combining Eqs.(13), (14) and (15), one can obtain the inequality:

$$\sigma(N + 1) < \sigma(N) + N + \frac{2\sigma(N)}{N} \quad (16)$$

Consider (15) as an equation, then we have:

$$\frac{d\sigma(N)}{dN} = N + \frac{2\sigma(N)}{N} \quad (17)$$

This equation leads to

$$\sigma(N) = N^2 \ln N + H \quad (18)$$

where $H$ is a constant. As $\sigma(N) \sim N^2 L(N)$, we have $L(N) \sim \ln N$. Which should be pay attention to, since Eq. (16) is an inequality indeed, the precise increasing tendency of $L$ may be a little slower than $\ln N$.

5 Conclusion remarks

In conclusion, in respect that the simplex triangulation networks are of very large clustering coefficient and very small average distance, they are not only the scale-free networks, but also small-world networks. Since many real-life networks are both scale-free and small-world, STNs may perform better in mimicking reality than BA and WS networks. In addition, STNs possess hierarchical structure in accord with the observations of many real networks.

Further more, we propose an analytic approach to calculate clustering coefficient, and an interesting technique to estimate the growing trend of
average distance. Since in the earlier studies, only few analytic results about these two quantities of networks with randomicity are reported, we believe that our work may enlightened readers on this subject.

We also have done the corresponding numerical simulations, the simulation results agree with the analytical ones very well.

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