On the Yang-Mills two-loop effective action with wordline methods

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We derive the two-loop effective action for covariantly constant field strength of pure Yang-Mills theory in the presence of an infrared scale. The computation is done in the framework of the worldline formalism, based on a generalization procedure of constructing multiloop effective actions in terms of the bosonic worldline path integral. The two-loop \(\beta\)-function is correctly reproduced. This is the first derivation in the worldline formulation, and serves as a nontrivial check on the consistency of the multiloop generalization procedure in the worldline formalism.

I. INTRODUCTION

The physics of strong but slowly varying chromomagnetic and electric fields may provide some insight to the non-trivial vacuum structure of QCD. For example, the Euler-Heisenberg Lagrangian in QED exhibits truly non-perturbative effects; it allows the investigation of the non-linear regime of QED, and has also been studied beyond one loop, see e.g. \([1, 2, 3]\). In QCD non-linearities already present on the classical level due to its non-Abelian nature. It is yet unknown how the full effective action changes beyond one loop. The computation of multiloop terms in the effective action is cumbersome, in particular for a non-Abelian gauge group. These computations simplify if background field methods are employed and the multiloop terms are evaluated for specifically chosen background configurations such as covariantly constant or selfdual configurations. This gives access to beta functions and parts of the effective action beyond one loop, see e.g. \([4, 5, 6]\). Wordline methods \([7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]\) have been shown to lead to a striking simplification of certain computations. A summation of Feynman diagrams is already implemented without loop momentum integrals and Dirac traces \([10, 11]\).

In the present work we provide for the first time a numerically accessible expression for the two loop effective action of Yang-Mills theory for covariantly constant fields in the presence of a physical infrared cut-off (at finite correlation length). This computation serves many purposes: firstly, it completes the construction of multiloop worldline methods for Yang-Mills theories as initiated in \([13, 14]\). The two-loop beta function serves as a nontrivial consistency check. Secondly, it is the necessary input for an RG-improved non-perturbative computation of the effective action within a Wilsonian framework. Finally one can study the stabilisation of the Savvidy vacuum beyond one loop.

II. WORDLINE REPRESENTATION BEYOND ONE LOOP

We briefly recapitulate the analysis of \([13, 14]\). The starting point is the generating functional of pure Yang-Mills theory in the presence of a background field configuration \(A\),

\[
Z[A] = \int Da \exp -S[a, A],
\]

where the gauge-fixed Yang-Mills action is given by

\[
S[a, A] = \frac{1}{2} \int_x \text{tr} F_{\mu\nu}(a + A)^2 - \frac{1}{2x} \int_x (\text{tr} (Da)^2 - \text{Tr} \ln(-D(A)D(a + A))),
\]

with \(\text{tr} \epsilon^{ab} = -\delta^{ab}/2\) in the fundamental representation, \(D(A) = \partial - igA\), and \(gF_{\mu\nu} = i[D_\mu, D_\nu]\). Within a perturbative expansion \(Z[A]\) reads

\[
Z[A] = \sum_n Z_n[A],
\]

where \(Z_n\) comprises the \(n\)-loop contribution to the generating functional. The generating functional \([3]\) is gauge invariant under the gauge transformation \(A \rightarrow A + D\omega\), and its logarithm is the gauge invariant Wilsonian effective action of pure Yang-Mills. The one loop contribution \(Z_1[A]\) for a subset of field configurations, e.g. covariantly constant fieldstrength \(F\), has been computed in \([12]\), for related results within standard methods see \([21]\). So far, a full computation of the two loop contribution \(Z_2[A]\) is lacking. After some algebra, \([11]\) can be turned into a more convenient representation for \(Z_2[A]\), see \([13]\). However, in all representations the formal expression \([11]\) suffers both from UV and IR divergences. In the present work we regularise and renormalise these divergences separately: for the UV divergences we employ dimensional regularisation for analytic purposes and construct a gauge-invariant proper-time cutoff interesting for numerical work. The divergences are then cured by appropriate counter terms. Additionally we introduce a physical IR cut-off. IR divergences are absent, if putting the theory into a box of size \(L\). Effectively this can be implemented by introducing gauge invariant masses \(m \sim 1/L\) to the propagating degrees of freedom of the theory. The latter also has the advantage of implementing a physical mass gap on the level of the Green functions. This offers a path towards a self-consistent
investigation of QCD in the confining regime in an effective field theory approach that is quite close to the fundamental theory. We emphasise that the approach leads to a fully gauge invariant effective action. However, the choice of the gauge-fixing parameter is directly related to the choice of different physical boundary conditions on the surface of the box.

Within this framework the renormalised two loop contribution \( Z_2[A] \) is provided by \[13\]

\[
Z_2[A] = \exp\left\{ \frac{i}{2} \int_{y_1,y_2} \left( \frac{i}{2} \frac{\delta}{\delta \alpha_{\mu
u}^{(0)}(y_1)} \delta_{\mu
u}^{(2)}(y_1) + i \delta \right) \left( \frac{i}{2} \frac{\delta}{\delta \beta_{\nu}(y_1)} \delta_{\nu}^{(2)}(y_1) + i \frac{\delta}{\delta \beta_{\nu}(y_2)} \right) \right\}
\times \exp\left\{ \frac{i}{16} \frac{\delta}{\delta \alpha_{\mu
u}^{(1)}} \frac{\delta}{\delta \alpha_{\mu
u}^{(1)}} \exp\left\{ -\frac{1}{2} \text{Tr} \ln \Delta^{-1} \right\} \exp\left\{ \text{Tr} \ln(D^2 + m^2 - D/\beta f) \right\} \bigg|_{\alpha=\beta=0} + \text{c.t.} \right., \tag{4}
\]

with antisymmetric tensor field \( \alpha_{\mu
u} = -\alpha_{\nu\mu} \),

\[
(\Delta^{-1})_{\mu\nu} = (\Delta_{\mu\nu})^{-1} + 2f_{abc}\alpha_{\mu\nu}^c, \tag{5}
\]

and

\[
\Delta_{\mu\nu}^{ab} = [-g_{\mu\nu}D^2 + 2F_{\mu\nu}f^{abc} + m^2 + (1 - \frac{1}{\xi})D_{\mu}D_{\nu}]^{-1}. \tag{6}
\]

We emphasise again that \( m^2 \) serves a twofold though related purpose. First it accounts for a possible non-perturbative mass-gap, secondly it mimics the implementation of a finite volume. The space-time integration over \( y_1, y_2 \) can be used for regularising the generating functional \[8\] by means of the dimensional regularisation

\[
\int_y \int d^D y, \quad \text{with} \quad D = 4 - 2\epsilon.
\]

In Feynman gauge \( \xi = 1 \), the expression simplifies as the last term in \[6\] drops out, and the non-trivial tensor structure disappears. In \[4\] we have not specified the counter terms indicated by c.t., that shall be discussed later. Within the representation introduced above the effective action derived from the generating functional \[4\] reads

\[
\Gamma = \Gamma_{\text{gluon}} + \Gamma_{\text{ghost}}, \tag{7}
\]

with

\[
\Gamma_{\text{ghost}} = -\frac{1}{2} \int_{y_1,y_2} \delta \int \Delta \delta \beta \text{Tr} \ln(D^2 + m^2 - D/\beta f) \bigg|_{\beta=0} + \text{c.t.}, \tag{8}
\]

the contribution of purely gluonic loops, \( \Gamma_{\text{gluon}} \), has been computed in \[14\], and the result is summarised in Appendix \[E\]. We complete the analysis of \[14\] by computing the ghost contribution \( \Gamma_{\text{ghost}} \) as well as the renormalisation insertions. To that end we turn \[8\] into Euclidean wordline integrals, and arrive at \[13\]

\[
\Gamma_{\text{ghost}} = -\frac{1}{2} \int_{y_1,y_2} \int_{\tau}^{\infty} \int_{\tau}^{\infty} \int_{\tau}^{\infty} \left[ D x | T_3 \right] \left[ D x | T_1 \right] \left[ D x | T_2 \right] \left[ D w | T_3 \right] \left[ D w | T_2 \right] \left[ D w | T_1 \right] \\
\times \left( Pe^{\sum T_3 M(w)} a c_i e^{\sum T_1 N(x)} c_j \frac{\delta}{\delta \beta (y_2)} (\lambda) f_g (P e^{\sum T_2 N(x)} y^i \frac{\delta}{\delta \beta (y_1)} (\lambda)) \right) \bigg|_{y_1=y_2} + \text{c.t.}, \tag{9}
\]

with the abbreviations

\[
\int [D x] T F[x] = \int [D x] e^{-\int_{\tau}^{\tau} (F_{\mu\nu} - \delta_{\mu\nu} \frac{1}{2} A_{\mu}(x)^2)} F[x],
\]

and

\[
M_{\mu\nu}^{ab}(x) = 2i (F_{\mu\nu} - \delta_{\mu\nu} \frac{1}{2} A_{\mu}(x)^2) (\lambda)^{ab},
N(x)^{ab} = -i A_{\mu}^c (\lambda)^{abc}, \tag{10}
\]

Eq. \[9\] stands for the fully renormalised ghost contribution to the two loop effective action. We have employed an additional ultraviolet regularisation in the proper-time integrations which entails a gauge invariant momentum cut-off. Such a cut-off scheme is amiable to numerical computation, whereas the dimensional regularisation facilitates analytic computations. In the following we shall conveniently project onto either regularisation by simply switching off either the regularisation parameter
\[ I = \text{ghost proper-time integral is given by} \]

\[ \Gamma = \frac{1}{2} \tau \partial \tau \ln \left( \frac{\rho}{\tau} \right) + \frac{1}{2} \partial \tau \partial \tau \ln \left( \frac{\rho}{\tau} \right) + \partial \tau \ln \left( \frac{\rho}{\tau} \right) \]

where \( n^a \) is a constant unit vector in color space with \( n^a n^a = 1 \). The gauge fields \[ \mathbf{A}_\mu \] satisfy the Fock-Schwinger gauge \[ \mathbf{A}_\mu \mathbf{x}_\mu = 0 \]. With \[ \mathbf{A}_\mu \] we rewrite the Lorentz matrices \( M \) and \( N \) as

\[ M(x) = 2i(\mathbf{F}_{\mu\nu} - \delta_{\mu\nu} \frac{1}{4} x^\rho \mathbf{F}_{\rho\nu} x^\gamma) \otimes \mathbf{T} \]
\[ N(x) = -\frac{i}{2} x^\mu \mathbf{F}_{\mu\nu} x^\nu \otimes \mathbf{T} \],

with \( \mathbf{T} = n^a \lambda^a \). The computation of \( \Gamma_{\text{ghost}} \) is straightforward but tedious [22]. It results in

\[ \Gamma_{\text{ghost}} = \frac{1}{2(4\pi)^D} \int_0^\infty dT_1 dT_2 dT_3 \]
\[ \times e^{-m^2 \tau} I_{\text{ghost}}[T_1, T_2, T_3; \mathcal{F}] + \text{c.t.} \tag{13} \]

with \( T = T_1 + T_2 + T_3 \). The integrand \( I_{\text{ghost}} \) of the proper-time integral is given by

\[ I_{\text{ghost}} = \int_0^\infty \frac{2}{T_3} \left( \frac{2}{T_5} \right) \left( \mathcal{F} \cot \mathcal{F}_T G^{-1} \cos 2\mathcal{F}_T \right) \]
\[ + \left( G^{-1} \mathcal{F}_T^2 \cot \mathcal{F}_T \right) \left( \cot \mathcal{F}_T \right) \]
\[ - \frac{2}{T_3} \left( \mathcal{F} G^{-1} \sin 2\mathcal{F}_T + \left( \mathcal{F}^2 G^{-1} \right) \right) \]
\[ \partial \tau \det \frac{1}{2} \mathbf{R} \],

where

\[ G = \mathcal{F} \cot \mathcal{F}_T + \mathcal{F} \cot \mathcal{F}_T + \frac{1}{T_3} \]
\[ R = \frac{\mathcal{F}^2}{\sin \mathcal{F}_T \sin \mathcal{F}_T (1 + \mathcal{F}_T (\cot \mathcal{F}_T + \cot \mathcal{F}_T))} \].

The expression [13] with [14] is numerically accessible, after the counter terms in [13] are specified.

### III. RENORMALISATION

Now we discuss the UV subtraction terms hidden in the counter terms that render \( Z_2, Z_2 \) finite, and in particular, [13] finite. Apart from applying a standard dimensional regularisation convenient for analytic considerations, we have introduced a gauge invariant UV regularisation by cutting off the proper-time integrals in [13] at a finite proper time \( \tau \), \( T_i \geq \tau \). This translates via a Laplace transform into a gauge invariant momentum cut-off if the effective action is formulated in terms of momentum loops. Such a scheme makes numerical computations accessible where the dimensional regularisation only can be employed in exceptional cases. Indeed, a fully non-perturbative wordline formulation of quantum field theories would provide a tool for devising gauge-invariant momentum cut-off schemes on the non-perturbative level which would be highly interesting. The ghost action [9] is then written as \( \Gamma = \Gamma_{\text{ghost}} + c \cdot \text{c.t.} \)

\[ \Gamma_{\text{ghost}} = \frac{1}{2(4\pi)^D} \int_0^\infty dT_1 dT_2 dT_3 \]
\[ \times e^{-m^2 \tau} I_{\text{ghost}}[T_1, T_2, T_3; \mathcal{F}] \tag{16} \]

The regularised expression \( \Gamma_{\text{ghost}} \) diverges if the regularisation parameters \( \tau, \epsilon \) are removed. Here we first concentrate on the proper-time regularisation with \( \epsilon = 0 \). Then the regularised expression \( \Gamma_{\text{ghost,reg}} \) in [16] diverges with powers of \( 1/\tau \), more precisely with \( 1/\tau^n (\ln \tau)^m \). Moreover, since we are dealing with the two-loop effective action, the divergent terms are not necessarily polynomial, and the counter terms cannot be determined in a polynomial expansion. The non-polynomial terms can be attributed to the divergence of one loop sub-diagrams which can be used to construct the related counter terms. However, here we want to set-up a procedure with which these counter terms can be derived systematically from \( \Gamma_{\text{ghost}} \) by means of derivatives. Such a procedure mimics the standard BPHZ-type subtraction schemes in momentum space. The divergences in \( \tau \) are extracted by appropriate \( \tau \)-derivatives with the help of the identity

\[ \tau \partial \tau \tau^{-n} (\ln \tau)^m = -\tau^{-n} (\ln \tau)^m \left( n - \frac{m}{\ln \tau} \right) \tag{17} \]

Applying the above \( \tau \)-derivative to \( \Gamma_{\text{ghost}} \), the physically finite term drops out. Hence appropriate subtractions

\[ \Gamma = \Gamma_{\text{reg}} - \sum_n c_n (\tau \partial \tau)^n \Gamma_{\text{reg}} \tag{18} \]

render the action \( \Gamma \) finite. Note that such a procedure in principle only properly provides the renormalised effective action by a careful discussion of the finite renormalisation that originates in the subtractions in [13]. In the present two loop case it suffices to only take one \( \tau \)-derivative, e.g.

\[ \tau \partial \tau \Gamma_{\text{ghost,reg}} = \frac{\tau}{2(4\pi)^D} \int_0^\infty dT_1 dT_2 dT_3 \]
\[ \times e^{-m^2 \tau} I_{\text{ghost}}[T_1, T_2, T_3; \mathcal{F}] \tag{19} \]

This reduces the number of proper-time integrations and makes the divergence structure analytically accessible. This procedure deserves further studies.

We still have to compute the traces over the field strength. Following [23], we work in the Lorentz frame in which the electric and magnetic fields are parallel, and thereby the field strength takes on a simple form with
only two non-zero symplectic block elements, i.e., it can be written as

\[ F = a\sigma_1 + b\sigma_2 \tag{20} \]

with

\[ \sigma_1 = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}, \tag{21} \]

and

\[ a = \epsilon, \quad b = -i\eta, \tag{22} \]

where \( \epsilon \) and \( \eta \) are the magnitudes of the magnetic and electric fields, respectively.

We close with a remark on the explicit computation of the proper-time integrals. In particular for numerical purposes it is advantageous to convert the integrals into less divergent expressions. Indeed, \( I_{\text{ghost}} \) as well as the corresponding integrand \( \tilde{I}_{\text{ghost}} \) can be integrated analytically over \( T_3 \) and hence can be written as a total \( T_3 \)-derivative. The computations are deferred to Appendix A and Appendix B respectively and the results read

\[ \Gamma_{\text{ghost}} = \frac{1}{2(4\pi)^D} \int_\tau^\infty \prod_{i=1}^3 dT_i e^{-m^2 T} \tag{23} \]
\[ \times \partial T_3 \tilde{I}_{\text{ghost}}[T_1, T_2, T_3; F] + \text{c.t.}, \]

where \( \tilde{I}_{\text{ghost}} \) is given in (A.4), and

\[ \Gamma_{\text{gluon}} = -\frac{1}{2(4\pi)^D} \int_\tau^\infty \prod_{i=1}^3 dT_i e^{-m^2 T} \tag{24} \]
\[ \times \partial T_3 \tilde{I}_{\text{gluon}}[T_1, T_2, T_3; F] + \text{c.t.}, \]

where \( \tilde{I}_{\text{gluon}} \) is given in (B.11). The counter terms in (23), (24) are \( \tau \)-dependent and can be computed from the derivative procedure outlined in (17), (18).

**IV. TWO LOOP \( \beta \) FUNCTION**

In the remainder of this work we concentrate on the question of full two-loop consistency of the wordline formalism suggested in [13, 14], the construction of which we have completed here. To that end we discuss the running of the coupling at two loop which is universal in mass-independent renormalisation schemes. As this concerns an analytic computation we employ a dimensional regularisation with \( \tau = 0 \). Moreover, we use a minimal subtraction scheme that renders the renormalisation constants mass-independent, and hence projects onto the the universal result for the two loop \( \beta \)-function. How such a mass-independent scheme is fixed in the presence of general IR cutoffs has been discussed in detail in [22], and we can straightforwardly use the related arguments for the mass terms for gluon and ghosts employed in the present work. The \( \beta \)-function can be read-off from the running of the wave function renormalisation \( Z_A \) of the gauge field. Expanding the two loop contribution to the effective action \( \Gamma_{\text{2,reg}} \) in powers of \( F \) we are led to

\[ \Gamma_{\text{2}}[A] = \frac{Z_A}{4} \int_x \text{tr} F^2 + O(F^3). \tag{25} \]

The \( F^2 \)-coefficient of the ghost effective action \( \Gamma_{\text{ghost,reg}} \) reads

\[ \frac{4g^4}{(4\pi)^4} \left\{ -\frac{10}{3} C_1' - 2C_2' - \frac{8}{3} C_3' + \frac{3}{2} C_4' \right\} \quad \frac{1}{4} \int_x \text{tr} F^2, \tag{26} \]

where the coefficients \( C_i' \) are given by [14]

\[ C_1' = (4\pi)^2 \int_0^\infty \prod_{i=1}^3 dT_i T_1^2 T_2^2 \Omega - e^{-m^2 T}, \]
\[ C_2' = (4\pi)^2 \int_0^\infty \prod_{i=1}^3 dT_i T_1^2 T_2 T_3^2 \Omega - e^{-m^2 T}, \]
\[ C_3' = (4\pi)^2 \int_0^\infty \prod_{i=1}^3 dT_i T_1^3 T_2 T_3 T_3 \Omega - e^{-m^2 T}, \]
\[ C_4' = (4\pi)^2 \int_0^\infty \prod_{i=1}^3 dT_i T_1^2 T_2 T_3 T_3 \Omega - e^{-m^2 T}, \tag{27} \]

with \( \Omega = T_1 T_2 + T_2 T_3 + T_3 T_1 \). The divergent parts of the coefficients \( C_i' \) read

\[ C_1' = \frac{1}{6\epsilon^2} + \left( -\frac{5}{3} + \frac{\rho_m \eta}{3 \epsilon} \right) + O(\epsilon^0), \]
\[ C_2' = \frac{1}{6\epsilon^2} + \left( \frac{1}{18} - \frac{\rho_m \eta}{3 \epsilon} \right) + O(\epsilon^0), \]
\[ C_3' = \frac{1}{12\epsilon^2} + \left( \frac{1}{72} + \frac{\rho_m \eta}{6 \epsilon} \right) + O(\epsilon^0), \]
\[ C_4' = \frac{1}{2\epsilon} + O(\epsilon^0), \tag{28} \]

with

\[ \rho_m = \gamma_E + \ln \frac{m^2}{4\pi^2}, \quad \gamma_E = \text{Euler const}. \tag{29} \]

Now we are in the position to perform the UV renormalisation for the ghost term. Note that by power counting, the UV divergence can appear at most in the quadratic term in the expansion, we write

\[ \Gamma_{\text{ghost,ren}} = \Gamma_{\text{ghost}} - \frac{1}{4} C'_{\text{ghost}} \int_x \text{tr} F^2 \]
\[ + \frac{1}{4} C'_{\text{finite}} \int_x \text{tr} F^2, \tag{30} \]

where

\[ C'_{\text{ghost}} = \frac{4g^4}{(4\pi)^4} \left\{ -\frac{10}{3} C_1' - 2C_2' - \frac{8}{3} C_3' + \frac{3}{2} C_4' \right\} \]
\[ - \left( \frac{1}{3} C_1' + \frac{1}{3} C_2' + \frac{5}{6} C_3' + C_4' \right) \epsilon, \tag{31} \]
and \( C'_{\text{finite}} \) is its finite part. This introduces the minimal subtraction scheme in the ghost part. The integral in \( C'_{\text{ghost}} \) is changed by the additional integrands proportional to \( C'_{\text{ghost}} F^2 \) rendering a finite expression. We remark that it can be explicitly checked that the renormalisation constants are mass-independent at two loop. This constitutes a mass-independent RG-scheme and hence \( \beta_2 \) is universal.

The two loop \( \beta \)-function is provided by

\[
\beta = -g \mu \partial_\mu \ln Z_g
= -g \left( \beta_1 C_A \left( \frac{g}{4\pi} \right)^2 + \beta_2 C_A^2 \left( \frac{g}{4\pi} \right)^4 + O(g^6) \right).
\]

The background field formalism allows us to directly extract the two loop \( \beta \)-function from \( Z_A \): the effective action \( \Gamma[A] \) is gauge invariant and consequently the combination \( gA \) is RG-invariant, leading to \( Z_g = Z_A^{-1/2} \), and hence

\[
\beta = \frac{1}{2} g \mu \partial_\mu \ln Z_A.
\]

With (32) and (33) we conclude that

\[
Z_A = 1 + \frac{\beta_1}{\epsilon} C_A \frac{g^2}{(4\pi)^2} + \frac{\beta_2}{\epsilon} C_A^2 \frac{g^4}{(4\pi)^4} + O(g^6),
\]

and we directly read off the two loop \( \beta \)-function from the subtraction terms computed in the last section. Using (28) in (26) we arrive at the ghost contribution \( \beta_{2,\text{ghost}} \) to the two loop coefficient \( \beta_2 \),

\[
\beta_{2,\text{ghost}} = \frac{11}{6}.
\]

The gluon loop contribution has been computed in \[14\] as \( \beta_{2,\text{gluon}} = -11/2 \). It is left to compute the contributions of the one loop counter terms. They arise from the insertion of the one loop RG constants of coupling and propagating field at one loop. Note that the propagating field is the fluctuation field \( a \) with one loop wave function renormalisation

\[
\delta Z_a = \frac{1}{\epsilon} \frac{g^2}{(4\pi)^2} \left( \frac{5}{3} + \frac{1}{2}(1 - \xi) \right).
\]

The worldline counter terms reduce to the standard one loop graphs. The corresponding diagrams shown in Fig.1 and 2 result from the one-loop renormalisation of the fluctuation field \( a \) and its vertices, while those in Fig.3 and 4 arise from the one-loop renormalisation of ghost field and its vertices.

The computation of the counter term in Fig.1 requires a gluon mass renormalisation with \( m^2 \rightarrow Z_a Z_m m^2 \). In Feynman gauge it is given by

\[
Z_a Z_m = 1 - \frac{1}{\epsilon} \frac{g^2 C_A}{(4\pi)^2}.
\]

This counter term has been considered in \[14\], which gives a contribution of \( 10/3\epsilon \) to the two loop coefficient

\[
\beta_2 = -\frac{11}{2} + \frac{11}{6} + \frac{35}{6} + \frac{10}{6} + \frac{1}{6} = \frac{17}{3},
\]

which agrees with the well-known result, e.g. \[4,5\]. As we are only interested in diagrams with external background fields we could have rescaled the fluctuation field \( a \) and

\[
\begin{align*}
\text{Fig. 1 + Fig. 2} : & \quad \frac{35}{6\epsilon}, \\
\text{Fig. 3 + Fig. 4} : & \quad \frac{1}{6\epsilon},
\end{align*}
\]

and we are led to

\[
\beta_2 = -\frac{11}{2} + \frac{11}{6} + \frac{35}{6} + \frac{10}{6} + \frac{1}{6} = \frac{17}{3},
\]

which agrees with the well-known result, e.g. \[4,5\]. As we are only interested in diagrams with external background fields we could have rescaled the fluctuation field \( a \) and
the ghost field with the renormalisation factors. With these rescaled fields, the diagrams above reduce to terms proportional to the renormalisation of the gauge-fixing term introduced by this rescaling and the renormalisation of the mass terms which also changes by this rescaling (e.g. the ghost mass renormalises with these rescaled fields). This has been used in \[3\]. Of course, this does not change the result. For comparison we list the different contributions

| \(\beta_2\)-contributions | \(m^2 \neq 0\) | \(m^2 = 0\) |
|--------------------------|---------------|-------------|
| two-loop diagrams        | \(-\frac{11}{3\epsilon}\) | \(\frac{7}{3\epsilon}\) |
| Fig.1+2                  | \(-\frac{5}{18\epsilon}\) or \(\frac{80}{9\epsilon}\) | \(\frac{35}{6\epsilon}\) or \(\frac{10}{3\epsilon}\) |
| Fig.3+4                  | \(0 + \frac{1}{3\epsilon}\) or \(\frac{1}{3\epsilon}\) | \(0\) |
| total                    | \(\frac{17}{3\epsilon}\) | \(\frac{17}{3\epsilon}\) |

In the middle column the different contributions from a direct computation (right), and from one with rescaled fluctuation fields (left) are listed.

We can use the above results on the consistent renormalisation in the presence of an infrared mass-scale to define the renormalised two loop contribution \(\Gamma_2[A]\) by means of appropriate subtractions instead of the dimensional regularisation used above. This allows us to numerically compute the full two loop effective action \(\Gamma = \Gamma_1 + \Gamma_2\) as a function of \(\mathcal{F}\).

V. OUTLOOK

In the present work we have completed the wordline construction of the two loop effective action initiated in \[12, 13\]. In particular we have provided a crucial consistency check of the construction by computing the universal two loop \(\beta\)-function within the wordline formalism.

We also have set-up a practical ultraviolet BPHZ-type renormalisation scheme in the proper-time which makes numerical computations accessible. For example, this can be used to numerically compute the two loop effective action for covariantly constant fields.

The inclusion of fermions in the present approach is straightforward, and is, in our opinion, the physically most interesting extension of the present work.

Acknowledgements We thank G. Dunne, H. Gies and C. Schubert for useful discussions.

A. GHOST CONTRIBUTION

One of the \(T\)-integrations in \[13\] can be done analytically. This is achieved by integrating \(\mathcal{I}_{\text{ghost}}\) over \(T_3\). Performing all the traces in the integrand of \[14\] and summing over them gives

\[
A = \frac{2a}{1 + a a_1 T_3} \left( -a T_3 + 2 \cosh(2a T_2) \coth(a T_1) \\
+ a T_3 \coth(a T_1) \coth(a T_2) + 2 \sinh(2a T_2) \right) + (a \leftrightarrow b),
\]

with \(a_1 = \coth(a T_1) + \coth(a T_2)\). The square-rooted term reads

\[
B = \frac{C_1}{(1 + a a_1 T_3)(1 + b b_1 T_3)},
\]

where \(C_1 = a^2 b^2 \csch(a T_1) \csch(b T_1) \csch(a T_2) \csch(b T_2)\). We define

\[
\mathcal{I}_{\text{ghost}}[T_1, T_2, T_3; \mathcal{F}] = \int_0^{T_3} d T_3' \mathcal{I}_{\text{ghost}}[T_1, T_2, T_3'; \mathcal{F}],
\]

after analytically performing the integration over \(T_3\) one finds (\(b_1 = a_1 (a \leftrightarrow b)\))

\[
\mathcal{I}_{\text{ghost}} = \int_y \left\{ \left[ \frac{A C_1 (1 + a a_1 T_3) - (1 + b b_1 T_3)}{a a_1 - b b_1} \right] \\
- \frac{C_1 C_2}{a a_1 (a a_1 - b b_1)^2 (1 + a a_1 T_3)} (-a a_1 + b b_1) \\
- a a_1 (1 + b b_1 T_3) \ln(1 + a a_1 T_3) \\
+ a a_1 (1 + b b_1 T_3) \ln(1 + b b_1 T_3) \\
+ (a \leftrightarrow b) \right\} - [T_3 \rightarrow 0],
\]

with

\[
C_2 = 2 a^2 \left( -1 - 2 a \cosh(2a T_2) \coth(a T_1) \\
+ \coth(a T_1) \coth(a T_2) - 2 a \sinh(2a T_2) \right).
\]

B. GLUONIC CONTRIBUTION

The gluon loop contribution to the two loop effective action reads \[14\]

\[
\Gamma_{\text{gluon}} = -\frac{1}{2} \int d T_1 d T_2 d T_3 e^{-m^2 T} \mathcal{I}_{\text{gluon}} + \text{c.t.}
\]

\[
= -\frac{1}{2} (4\pi)^{-D} \int_0^{\infty} d T_1 d T_2 d T_3 e^{-m^2 T} \int_y \frac{\mathcal{F}^2}{\det \left( \frac{x^2 T_3}{\Delta \mathcal{F} \sin \mathcal{F} T_2} \right)} (2 \sin \mathcal{F} T_1 \cos 2 \mathcal{F} (T_1 + 2 T_2))
\]

\[
\mathcal{F}^2 T_3 \Delta \mathcal{F} \sin \mathcal{F} T_2
\]

\[
\sin \mathcal{F} T_2
\]

\[
\cos 2 \mathcal{F} (T_1 + 2 T_2)
\]
\[-2 \sin \mathcal{F}(T_1 + T_2) \cos \mathcal{F}(2T_1 + 3T_2) + \left(1 - 2 \cos 2\mathcal{F}(T_1 + T_2) \right) \sin(3\mathcal{F}T_2) \cos \mathcal{F}(T_1 - T_2) \right) \\
\left[ 1 + \frac{\mathcal{F}}{\Delta_{\mathcal{F}}} \left[ 4 \sin \mathcal{F}T_1 \sin \mathcal{F}T_2 \sin 2\mathcal{F}(T_1 + T_2) \right] - 2 \sin \mathcal{F}T_1 \cos \mathcal{F}(2T_1 + 3T_2) \\
- 2 \sin \mathcal{F}T_2 \cos \mathcal{F}(T_1 - T_2) \\
- \sin \mathcal{F}(T_1 + T_2) \cos 2\mathcal{F}(T_1 - T_2) \right) \right] \\
+ \text{tr} \cos 2\mathcal{F}T_2 \left[ \frac{\mathcal{F}^2 T_3}{\Delta_{\mathcal{F}} \sin \mathcal{F}T_2} \left( \sin \mathcal{F}(T_1 + T_2) \right) \times \cos \mathcal{F}(2T_1 + T_2) - \sin \mathcal{F}T_1 \cos 2\mathcal{F}(T_1 + T_2) \right) \\
+ \frac{\mathcal{F}}{\Delta_{\mathcal{F}}} \left( 3 \sin \mathcal{F}T_1 \cos \mathcal{F}(2T_1 + T_2) \right) \\
+ 2 \sin \mathcal{F}T_1 \sin \mathcal{F}(T_1 + T_2) \right) \right] \\
+ \text{tr} \left( \frac{\mathcal{F}^2 T_3}{\Delta_{\mathcal{F}} \sin \mathcal{F}T_2} \left( \sin \mathcal{F}(T_1 - T_2) \right) \times \cos \mathcal{F}T_2 \sin \mathcal{F}(T_1 - T_2) \\
- \sin \mathcal{F}T_1 \cos 2\mathcal{F}T_2 \right) \\
- \sin \mathcal{F}(T_1 + T_2) \right) \right) \right] \\
+ \delta(T_3) \text{tr} \left( \cos 2\mathcal{F}(T_1 - T_2) \right) \\
- \delta(T_3) \text{tr} \left( \cos 2\mathcal{F}T_2 \right) \right] \right] \\
+ \frac{1}{2} (4\pi)^{-D} \int_0^\infty dT_1 dT_2 \int_y \frac{\sin^2 \mathcal{F}T_1 \sin \mathcal{F}T_2}{\det \frac{\mathcal{F}^2}{\sin \mathcal{F}T_1 \sin \mathcal{F}T_2}} \left( \text{tr} \cos 2\mathcal{F}(T_1 + T_2) - \text{tr} \cos 2\mathcal{F}(T_1 - T_2) \right) + \text{c.t.} \right)

\begin{align*}
\Delta_{\mathcal{F}} &= \sin(\mathcal{F}T_1) \sin(\mathcal{F}T_2) + \mathcal{F}T_3 \sin[\mathcal{F}(T_1 + T_2)]. \quad (B.7)
\end{align*}

Following the same procedure as we extracted Eq. (26) from [13], we obtain the renormalisation part of the effective action above at the second order of \(\mathcal{F}\)

\[ \Gamma_{\text{gluon}} = \frac{4g_0^4}{(4\pi)^4} \left( - \frac{11}{2\epsilon} \right) \int_y \left( - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right) + \mathcal{O}(\epsilon^0), \quad (B.8) \]

and hence the contribution of the gluon loops to the two loop \(\beta\)-function coefficient is \(-11/2\epsilon\).

For a numerical evaluation of \(\Gamma_{\text{gluon}}\) a less singular representation is advantageous. To that end we write \(\Gamma_{\text{gluon}}\) as

\[ \Gamma_{\text{gluon}} = - \left( \frac{1}{2} (4\pi)^{-D} \int_0^\infty \frac{3}{D} \sum_{i=1}^3 e^{-m^2 T} dT_i \right) \times \partial_{T_3} \hat{\mathcal{I}}_{\text{gluon}}[T_1, T_2, T_3; \mathcal{F}] + \text{c.t.}, \quad (B.9) \]

with \(\partial_{T_3} \hat{\mathcal{I}}_{\text{gluon}} = \mathcal{I}_{\text{gluon}}\), and hence similarly to the ghost-contribution it reads

\[ \hat{\mathcal{I}}_{\text{gluon}}[T_1, T_2, T_3; \mathcal{F}] = \int_0^3 dT_3 \mathcal{I}_{\text{gluon}}[T_1, T_2, T_3; \mathcal{F}], \quad (B.10) \]

The \(T_3\)-integral in the definition of \(\hat{\mathcal{I}}_{\text{gluon}}\) can be performed analytically and yields

\[ \hat{\mathcal{I}}_{\text{gluon}} = \int_y \left\{ \left[ \frac{A' C_1 (\ln(1 + aa_1 T_3) - \ln(1 + bb_1 T_3))}{aa_1 - bb_1} \right] + \left( C_1 C_3 \theta(T_3) + \frac{2aC_1(a_1 C_5 - C_4)}{a_1 (aa_1 - bb_1)^2 (1 + aa_1 T_3)} \right) \times \left( - aa_1 + bb_1 - aa_1 (1 + bb_1 T_3) \ln(1 + aa_1 T_3) + aa_1 (1 + bb_1 T_3) \ln(1 + bb_1 T_3) + (a \leftrightarrow b) \right) \right\} \right] - \left[ T_3 \to 0 \right], \quad (B.11) \]

with \(\theta(T_3)\) the step function and

\[ A' = \frac{2a(C_5 + C_4 a_3 T_3)}{1 + aa_1 T_3} - 4(-1 + D) \cosh(2aT_1) \delta(T_2) + 2 \delta(T_3) \left( \cosh(2a(T_1 - T_2)) - 2 \cosh(2aT_1) \right) + \cosh(2bT_1) \right) \cosh(2aT_2) - 2 \sinh(2aT_1) \sinh(2aT_2) \right) + \left( a \leftrightarrow b \right). \quad (B.12) \]

The abbreviations \(C_3, C_4\) and \(C_5\) read

\[ C_3 = 2 \cosh(2a(T_1 - T_2)) - 4 \sinh(2aT_1) \sinh(2aT_2) - 4 \left( \cosh(2aT_1) + \cosh(2bT_1) \right) \cosh(2aT_2), \]
\[ C_4 = 4 \cosh (2b(T_1 + T_2)) \left(-1 + \coth(aT_1) \coth(aT_2)\right) + \left(2 \cosh(2bT_2) \cosh(a(T_1 + T_2)) \right) \]
\[ + 2 \cosh (a(T_1 - T_2)) + \cosh (a(3T_1 + T_2)) \right) \csch(aT_1) \csch(aT_2), \]

and
\[ C_5 = 2 \cosh (2a(T_1 + T_2)) \coth(aT_2) + 2 \cosh (2b(T_1 + T_2)) \coth(aT_2) - 2 \cosh (a(T_1 - 2T_2)) \csch(aT_1) \]
\[ + 6 \cosh (a(2T_1 + 2T_2)) \csch(aT_1) \left( \cosh(2aT_2) + \cosh(2bT_2) \right) - 2 \cosh (a(2T_1 + 3T_2)) \csch(aT_2) \]
\[ + \csch(aT_1) \csch(aT_2) \sinh (a(T_1 + T_2)) \left(2 \cosh(2aT_1) \cosh(2aT_2) + 2 \cosh(2aT_1) \cosh(2bT_2) \right) \]
\[ - \cosh (2a(T_1 - T_2)) \right) \right) - 4 \sinh (2a(T_1 + T_2)). \]