Application of regularization maps to quantum mechanical systems in 2 and 3 dimensions

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Abstract

We extend the Levi-Civita (L-C) and Kustaanheimo-Stiefel (K-S) regularization methods that maps the classical system where a particle moves under the combined influence of $\frac{1}{r}$ and $r^2$ potentials to a harmonic oscillator with inverted sextic potential and interactions to corresponding quantum mechanical counterparts, both in 2 and 3 dimensions. Using the perturbative solutions of the Schrödinger equation of the later systems, we derive the eigen spectrum of the Hydrogen atom in presence of an additional harmonic potential. We have also obtained the mapping of a particle moving in the shifted harmonic potential to H-atom using Bohlin-Sundman transformation, for quantum regime. Exploiting this equivalence, the solution to the Schrödinger equation of the former is obtained from the solutions of the later.

1 Introduction

The Levi-Civita (L-C) [1] and Kustaanheimo-Stiefel (K-S) [2] regularization schemes convert singular differential equations into regular ones. This is achieved by a change of variable, after re-expressing the equations in complex co-ordinates, followed by a reparametrisation of time. In these schemes, existence of a constant of motion (normally, conserved energy) crucial in implementing the regularization map. Under these maps, a singular system on constant energy surface is mapped to a regular system.

These methods have been applied successfully to classical particle moving under the influence of $\frac{1}{r}$ potential in 2 and 3 dimensions, mapping these systems to harmonic oscillator in 2 and 4 dimensions, respectively [1][2]. These mappings have been extended to quantum mechanical systems also, showing the equivalence between 2 and 3 dimensional H-atom and harmonic oscillator in 2 and 4 dimensions [3]. In both classical as well as quantum mechanical cases, these regularization schemes map the singular, linear differential equation to a regular linear equation. In 2 dimensions, the equations of motion of the H-atom is first expressed in terms of complex coordinates where as in 3-dimensions, it is expressed in terms of

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more involved. Let recently \[7\]. Starting from the Schrödinger equation of a particle in presence of to be related to harmonic oscillator with inverse sextic potential and interactions in the classical cases, Note that K-S regularized Hamiltonian has an additional Schrödinger equations are shown to be mapped to each other by a non-inertial transformation. Further, the corresponding Schrödinger is mapped to that of the H-atom. oscillator to H-atom correspondence \[6\], here too, we see that only parity even states of the shifted H-atom mapping between the eigenfunctions and eigenvalues of these two systems. As in the case of usual obtain the eigenfunctions and eigenvalues. The regularization maps, viz: Levi-Civita and K-S maps, allows us to by treating the sextic potential and interactions as perturbations to harmonic oscillator and derive the strength of sextic potentials and interactions. The three dimensional Schrödinger equation describing H-atom with additional harmonic potential is shown also shown to be mapped to Schrödinger equation in 4-dimensional dimensions, we map it to the Schrödinger equation describing a particle in presence of harmonic, inverted dimension. The studies on the map between 3-dimensional H-atom and 4-dimensional harmonic oscillator brought out the role of hidden symmetries of the H-atom \[4\]. There also exist a map between the harmonic oscillator to Kepler problem in 2-dimensions known as Bohlin-Sundman transformation \[5\] which also uses a similar procedure of first expressing the equations of motion in terms of complex coordinates, followed by a reparametrization of time variable. Here too the conserved energy of the former decides the strength of the potential of the latter system. This map has been generalized to the realm of quantum mechanics \[6\].

It is of intrinsic interest to investigate the application of these regularization schemes to other quantum mechanical systems and we take up this study in this paper. It has been shown recently that the Kepler problem with an additional oscillator potential can be mapped to harmonic oscillator in presence of inverted sextic potential and interactions \[7\]. This map was derived in 2-dimensions using Levi-Civita(L-C) regularization \[1\] and in 3-dimensions using Kustaanheimo-Stiefel(K-S) regularization \[2\] schemes, respectively. The Levi-Civita map is a covering map of degree 2, \( L.C.: \mathbb{C}^2/\{0\} \to T^*S^2 \setminus S^2 \). We write \((q,p)\) for coordinates of \( T^*\mathbb{C} = \mathbb{C} \times \mathbb{C}, \) then in practice the Levi-Civita regularization of planar two body collisions are carried out by the variable substitution \((q,p) \mapsto (q^3, \frac{q}{p})\). Kustaanheimo-Stiefel(K.S.) map is technically more involved. Let \( \mathbb{H} \) denotes the space of quaternions. The K.S. map is defined as

\[
K.S.: T^*(\mathbb{H} \setminus \{0\}) \to \mathbb{H} \times \mathbb{H}, \quad \text{s.t.} \ (z,w) \mapsto (\bar{z} \bar{i}w, \frac{\bar{z}iw}{2|z|^2}).
\]

Note that K-S regularized Hamiltonian has an additional \(S^1\) symmetry resulting from a Hamiltonian \(S^1\)-action on the symplectic manifold \( T^*(\mathbb{H} \setminus \{0\}) \).

Here, we generalize these two regularization schemes to the case of quantum mechanical particle moving under the influence of the combined effect of \(\frac{1}{r}\) and \(r^2\) potentials in 2 and 3 dimensions, which were shown to be related to harmonic oscillator with inverse sextic potential and interactions in the classical cases, recently \[7\]. Starting from the Schrödinger equation of a particle in presence of \(\frac{1}{r}\) and \(r^2\) potentials in 2-dimensions, we map it to the Schrödinger equation describing a particle in presence of harmonic, inverted sextic potentials and interactions. The three dimensional Schrödinger equation describing H-atom with additional harmonic potential is shown also to be mapped to Schrödinger equation in 4-dimensions describing a particle under the influence of harmonic, inverted sextic potentials and interactions. In both cases, we find that the strength of sextic potential and interactions are controlled by the coefficient of the harmonic potential present in the former systems. The strength of harmonic potential in the later is governed by the conserved energy of the former systems. We then solve these Schrödinger equations by treating the sextic potential and interactions as perturbations to harmonic oscillator and derive the eigenfunctions and eigen values. The regularization maps, viz: Levi-Civita and K-S maps, allows us to obtain the eigenfunctions and eigen values of the H-atom with additional harmonic potential.

Using the regularization map between the shifted harmonic oscillator and H-atom, we obtain the mapping between the eigenfunctions and eigen values of these two systems. As in the case of usual oscillator to H-atom correspondence \[6\], here too, we see that only parity even states of the shifted H-atom is mapped to that of the H-atom.

We also explicitly construct the canonical transformation that relates a generalization of Kepler Hamiltonian having a time dependence to a time independent Hamiltonian. Further, the corresponding Schrödinger equations are shown to be mapped to each other by a non-inertial transformation.
This paper is organized as follows. In the next section, we give a brief summary of the essential results concerning the mapping between Kepler problem augmented with a harmonic potential to harmonic oscillator with inverted sextic potential and interactions, both in 2 and 3 dimensions [7]. Here the former are singular systems where as the later are regular systems. We also recall the results of generalization of he Bohlin-Sundman map connecting shifted harmonic oscillator to Kepler problem in 2 dimensions. In Sec.3, we derive the mapping between the Schrödinger equation of 2-dimensional Hydrogen atom in presence of harmonic potential and that of the harmonic oscillator coupled with inverted sextic potential and interactions in 2-dimensions. We then, by treating the inverted sextic potential and interactions as perturbations, solve the Schrödinger equation for harmonic oscillator and obtain eigenfunctions and eigen values. Using the mapping between the Schrödinger equations, we then obtain the eigenfunctions and eigen values of the Hydrogen atom augmented with a harmonic potential. We generalize this study to 3-dimensional Hydrogen atom in presence of an additional oscillator potential and obtain the mapping of the corresponding Schrödinger equation to that of 4-dimensional harmonic oscillator coupled with inverted sextic potential and interaction in Sec.4. Further, we derive the eigenfunctions and eigen values from the Schrödinger equation of 4-dimensional harmonic oscillator with inverted sextic potential and interactions by treating these as perturbations. Using this wave function and eigen values, we derive the wave function and eigen values of Hydrogen atom in presence of harmonic potential in 3-dimensions. In Sec.5, the mapping between the Schrödinger equation describing shifted harmonic oscillator to H-atom is used to obtain the spectrum of the former from that of the later. Our concluding remarks are given in the last section. In the appendix A, we show that the system described in Eqn.(2.1) below is related to a generalization of the Kepler motion whose Hamiltonian has a time dependence, by a canonical transformation and then we generalize this to quantum mechanical level. In appendix B, we show that these two systems are related by a non-inertial co-ordinate transformation by mapping their Schrödinger equations.

2 Review of regularization of Kepler problem in presence an additional confining potential

In this section, we briefly recall essential results of the generalization of the Levi-Civita(L-C) regularization [1] and Kustaanheimo-Stiefel(K-S) regularization [2] to Kepler problem augmented with an additional oscillator potential, both in 2-dimensions and 3-dimensions, respectively [7]. We also summarize the Bohlin-Sundman map applied to shifted harmonic oscillator in 2-dimensions. We will use these results in the next sections.

The regularization methods [1][2] map linear, singular, Kepler equations in 2 and 3 dimensions to regular and linear equations describing harmonic oscillators in 2 and 4 dimensions, respectively. In [7], it has been shown that the linear, singular equation describing the motion of a particle under the combined influence of Kepler and oscillator potential in 2 and 3 dimensions are mapped to regular, but still non-linear equations describing harmonic oscillator with inverted sextic potential and interactions. These methods of regularization crucially depend on the existence of constant of motion(energy) and consist of re-writing equations of motion using re-parameterization of time followed by a re-definition of co-ordinates. Finally, the conserved energy expression is used to re-express certain terms appearing in the equations of motion.

We will generalize these mappings to the corresponding quantum mechanical systems in this paper.

2.1 Regularization of motion under the combined potential of $\frac{1}{r}$ and $r^2$ in two dimensions

In this subsection, we summarize the mapping of equations of motion of a particle moving in $\frac{1}{r}$ potential with an additional oscillator potential to that of a particle moving under the influence of a harmonic oscillator potential, but coupled with inverted sextic potential and interactions.

The equation of motion of a particle moving under the combined influence of Kepler and oscillator
potentials is
\[ m \ddot{X}_i - \frac{m\lambda^2}{4} X_i + \frac{k}{(X_1^2 + X_2^2)^{\frac{3}{2}}} = 0, \quad i = 1, 2. \]  

(2.1)

These equations are shown to follow as Euler-Lagrange equations from the Lagrangian
\[ L = \frac{m}{2}(\dot{X}_1^2 + \dot{X}_2^2) + \frac{m\lambda^2}{8}(X_1^2 + X_2^2) - \frac{m\lambda^2}{2}(X_1 \dot{X}_1 + X_2 \dot{X}_2) + \frac{k}{r}. \]  

(2.2)

Here an over dot represents differentiation with respect to time \( t \). Note that the third term on RHS is a total derivative term and will not contribute to equations of motion. The corresponding Hamiltonian
\[ H = \frac{p_{X_1}^2 + p_{X_2}^2}{2m} + \frac{\lambda^2}{2}(X_1 p_{X_1} + X_2 p_{X_2}) - \frac{k}{r}, \]  

(2.3)

where \( r = \sqrt{X_1^2 + X_2^2} \) is a constant of motion. Since we consider the bounded motion, we express the negative energy in terms of velocities and co-ordinates, viz;
\[ -E = \frac{m}{2}(\dot{X}_1^2 + \dot{X}_2^2) - \frac{\lambda^2}{8}m(X_1^2 + X_2^2) - \frac{k}{r}. \]  

(2.4)

We re-express the above conserved energy and the equation of motion in Eqn.(2.1) in terms of complex variable \( Z = X_1 + iX_2 \) and then we applied Levi-Civita regularization obtaining
\[ \frac{d^2 U}{d\tau^2} + \frac{E}{2mc^2}U - \frac{3\lambda^2}{16c^2}|U|^4U = 0, \]  

(2.5)

where \( Z = U^2 \). Also note that we have used reparametrization of time variable and introduced a new time \( \tau \) through \( dt = \frac{c}{r}d\tau \), where \( c \) is a proportionality constant. It is easy to see that these equations follow from the Lagrangian
\[ L = \frac{m}{2}(U_1'^2 + U_2'^2) - \frac{E}{4c^2}(U_1^2 + U_2^2) + \frac{1}{32} \frac{m\lambda^2}{c^2} (U_1^2 + U_2^2)^3, \]  

(2.6)

where ‘\( \tau \)’ denotes differentiation with respect to \( \tau \). This Lagrangian describes an oscillator in 2-dimensions with inverted sextic potential and couplings. Note that, for small \( \lambda \), this is a system of uncoupled harmonic oscillators with perturbations involving couplings and inverted sextic potential. The Hamiltonian following from the above Lagrangian is given by
\[ H = \frac{P_{U_1}^2}{2m} + \frac{P_{U_2}^2}{2m} + \frac{E}{4c^2}(U_1^2 + U_2^2) - \frac{1}{32} \frac{m\lambda^2}{c^2} (U_1^2 + U_2^2)^3. \]  

(2.7)

Note here that the strength oscillator potential in the above is given by the conserved energy \( E \) of the former system while the strength of the inverted sextic potential and interactions are controlled by the coefficient \( \lambda \) of the additional oscillator potential in the former system (see Eqn. (2.2)).

### 2.2 Regularization of 3-dimensional Kepler problem in presence an additional confining potential

Applying the Kustaanheimo-Stiefel(K-S) transformation [2], equivalence between the motion under the combined action of Kepler potential and an oscillator potential in three dimension and perturbed 4-dimensional Harmonic Oscillator was established [7].

The equations of motion of a particle moving in 3-dimensions, under the action of \( \frac{1}{r} \) and \( r^2 \) potentials is
\[ m \ddot{X}_i - \frac{m\lambda^2}{4} X_i + \frac{k}{r^3} X_i = 0, \quad i = 0, 1, 2. \]  

(2.8)
The Hamiltonian for this system is

\[ H = \sum_{i=0}^{2} \frac{p_i^2}{2m} + \sum_{i=0}^{2} \frac{\lambda}{2} X_i P_i - \frac{k}{r}. \]  

(2.9)

This constant of motion, in terms of velocities and positions is

\[ -E = -\frac{E}{m} = \sum_{i=0}^{2} \\dot{X}_i^2 - \sum_{i=0}^{2} \frac{\lambda^2}{8} \dot{X}_i^2 - \frac{\mu}{r}. \]  

(2.10)

Applying re-parameterization of time and co-ordinate transformation (K-S transformation)

\[ \frac{d}{dt} = \frac{1}{4r} \frac{d}{d\tau} \quad \text{and} \quad X = U U^*, \]  

(2.11)

where \( U \) is quaternions (see [7] for details), the singular Eqn. (2.8) are mapped to

\[ U''_i + (8 \mathcal{E} - 3\lambda^2 |U|^4) U_i = 0, \quad i = 0, 1, 2, 3. \]  

(2.12)

Note that the \( \lambda^2 \) dependent term carries the signature of additional oscillator potential present in the initial system. These equations are the E-L equations following from the Lagrangian

\[ \mathcal{L}_{SO} = \frac{1}{2} m \left( \sum_{i=0}^{3} (U_i')^2 - 4m\mathcal{E} \sum_{i=0}^{3} U_i^2 + \frac{m\lambda^2}{2} (\sum_{i=0}^{3} U_i^2)^3 \right), \quad i = 0, 1, 2, 3. \]  

(2.13)

and the corresponding Hamiltonian is

\[ H = \sum_{i=0}^{3} \frac{\hat{p}_i^2}{2m} + 4m\mathcal{E} \sum_{i=0}^{3} U_i^2 - \frac{m\lambda^2}{2} (\sum_{i=0}^{3} U_i^2)^3. \]  

(2.14)

Note that the coefficient of the oscillator potential in the above is nothing but the conserved energy of the former system. This Lagrangian/Hamiltonian, describes a harmonic oscillator with an inverted sextic potential and interactions in 4-dimensions.

### 2.3 Mapping of shifted Harmonic oscillator : Bohlin-Sundman Map

Here we summarize the generalization of Bohlin-Sundman transformation applied to equations of a particle in harmonic oscillator potential whose coefficient is the shifted frequency of the oscillator. This maps the equation of motion to that of a particle moving in \( \frac{1}{r} \) potential.

A time-dependent co-ordinate transformation \( x_i = q_i e^{\frac{\lambda}{r}} \) allows one to re-express the below equations of motion in two-dimensions,

\[ q_i'' + \lambda q_i' + \Omega^2 q_i, \quad i = 1, 2 \]  

(2.15)

as

\[ x_i'' + \tilde{\Omega}^2 x_i = 0, \quad i = 1, 2 \]  

(2.16)

where \( \tilde{\Omega}^2 = \Omega^2 - \frac{\lambda^2}{r} \). These shifted harmonic oscillator equations follow from the Lagrangian

\[ L = \frac{m}{2} (x_1'^2 + x_2'^2) - \frac{m}{2} \tilde{\Omega}^2 (x_1^2 + x_2^2) - \frac{m\lambda}{2} x_1 x_2^2, \]  

(2.17)

where \( x_i' = \frac{dx_i}{d\tau} \) and corresponding Hamiltonian is

\[ H = \frac{1}{2m} (p_1^2 + p_2^2) + \frac{m\Omega^2}{2} (x_1^2 + x_2^2) + \frac{\lambda}{4} (x_1 p_1 + p_1 x_1 + x_2 p_2 + p_2 x_2). \]  

(2.18)
This conserved quantity in terms of velocities and coordinates becomes

\[ H = \frac{m}{2} (x_1'^2 + x_2'^2) + \frac{m\tilde{\Omega}^2}{2} (x_1^2 + x_2^2). \quad (2.19) \]

In terms of complex coordinate \( \omega = x_1 + ix_2 \) Eqn. (2.16) becomes

\[ \omega'' + \tilde{\Omega}^2 \omega = 0. \quad (2.20) \]

and Eqn. (2.19) becomes

\[ H = \frac{m}{2} [\bar{\omega}'\omega + \tilde{\Omega}^2 \bar{\omega} \omega] \equiv E. \quad (2.21) \]

Applying the Bohlin-Sundman transformation and re-parameterization of time, i.e.,

\[ \omega \rightarrow Z = \omega^2 \quad \text{and} \quad \bar{Z} \frac{dZ}{dt} = \frac{\bar{\omega} d\omega}{2 \, d\tau} \quad (2.22) \]

Eqn. (2.20) (using Eqn. (2.21)) becomes

\[ \frac{d^2 Z}{dt^2} = - \frac{E}{4m |Z|^3}. \quad (2.23) \]

Here \( E \), the conserved energy is related to strength of Kepler potential as \( k = \frac{E}{4} \). This equation is the Kepler’s equation in 2-dim, written in the complex co-ordinate \( Z = X_1 + iX_2 \). The above equation can be shown to come from the Hamiltonian,

\[ \mathcal{H} = \frac{P_{X_1}^2}{2m} + \frac{P_{X_2}^2}{2m} - \frac{k}{\sqrt{X_1^2 + X_2^2}}, \quad (2.24) \]

where \( k = \frac{E}{4} \).

3 Mapping of the Schrödinger Equation with \( \frac{1}{r} \) and \( r^2 \) potentials in two dimensions

In this section, we obtain the spectrum of the Schrödinger equation corresponding to the system described by the Hamiltonian in Eqn. (2.24). For this, we first map the Schrödinger equation for the Hamiltonian in Eqn. (2.23) to that of harmonic oscillator with inverted sextic potential and couplings, whose Hamiltonian is given in Eqn. (2.7). After solving the second of these using the perturbative methods, and using the map, we calculate the eigenvalues and eigenfunctions of the former system.

For this, we start with the Hamiltonian

\[ \hat{\mathcal{H}} = \frac{(P_{X_1}^2 + m \lambda X_1^2)}{2m} + \frac{\lambda}{4} (X_1 P_{X_1} + P_{X_1} X_1) + \frac{\lambda}{4} (X_2 P_{X_2} + P_{X_2} X_2) - \frac{k}{r}, \quad (3.1) \]

where we have symmetrized the \( X_i P_{X_i} \) terms. After re-expressing above Hamiltonian as

\[ \hat{\mathcal{H}} = \frac{(P_{X_1} + \frac{m \lambda X_1}{2})^2}{2m} + \frac{(P_{X_2} + \frac{m \lambda X_2}{2})^2}{2m} - \frac{\lambda^2 m}{8} (X_1^2 + X_2^2) - \frac{k}{\sqrt{X_1^2 + X_2^2}}, \quad (3.2) \]

we set up the Schrödinger equation

\[ \hat{\mathcal{H}} \Phi = E \Phi. \quad (3.3) \]

We now apply a transformation \( \eta \hat{\mathcal{H}} \eta^{-1} \Phi = E \eta \Phi \) to above equation and redefine \( \eta \hat{\mathcal{H}} \eta^{-1} = H \) and \( \eta \Phi = \psi \). Taking

\[ \eta = \exp \left( \frac{im \lambda (X_1^2 + X_2^2)}{4 \hbar} \right) \]

(3.4)
and noting \( \eta(P_X + \frac{m\lambda X}{2}) = P_X^n \), we find the transformed Hamiltonian to be

\[
H = \frac{(P_X^2 + P_Y^2)}{2m} - \frac{\lambda^2 m}{8} (X_1^2 + X_2^2) - \frac{k}{r}.
\]

(3.5)

This Hamiltonian describes a 2-dimensional H-atom with an additional inverted harmonic potential. In the polar coordinates, this Hamiltonian becomes

\[
H = \frac{1}{2m} (P_r^2 + P_\theta^2) - \frac{\lambda^2 m}{8} mr^2 - \frac{k}{r}
\]

(3.6)

and we will now find solution to the Schrödinger equation, in polar coordinates, i.e.,

\[
\left[-\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) - \frac{\lambda^2 m}{8} r^2 - \frac{k}{r} \right] \psi(r, \theta) = E\psi(r, \theta).
\]

(3.7)

where we have used the realization

\[
P_r = -i\hbar \left( \frac{\partial}{\partial r} + \frac{N-1}{2r} \right)
\]

(3.8)

\[
P_\theta = -i\hbar \frac{\partial}{\partial \theta}
\]

(3.9)

Note that, in our case, \( N = 2 \), in the above.

To find the solution to above equation, Eqn. (3.7), we first generalize the equivalence of above system to that described by the Schrödinger equation corresponding to the Hamiltonian in Eqn. (2.7) and use the solution of the latter, obtained using perturbative approach. For this we re-express the Hamiltonian in Eqn. (2.7) in polar coordinates, viz:

\[
\hat{H} = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2m} + \frac{m\Omega^2 \rho^2}{2} - \frac{\lambda^2 m}{32\pi^2} \rho^6
\]

(3.10)

where we have used \( \frac{m\Omega^2}{2} = \frac{E}{\hbar^2} \). The corresponding Schrödinger equation is given by

\[
\left[-\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right) + \frac{m\Omega^2 \rho^2}{2} - \frac{\lambda^2 m}{32\pi^2} \rho^6 \right] \psi(\rho, \phi) = \tilde{E}\psi(\rho, \phi).
\]

(3.11)

To see the equivalence of the Schrödinger equations given in Eqn. (3.11) and Eqn. (3.7), we have to apply the coordinate transformations corresponding to the Levi-Civita map (\( Z = U^2 \)) used in relating these two systems, classically. That is, we apply the relations:

\[
r = \rho^2, \theta = 2\phi, \text{ and choose } c = \frac{1}{4}
\]

(3.12)

and by a change of variables, map above Hamiltonian operators (and thus the Schrödinger equations), where we make further identifications

\[
E = -\frac{1}{8} m\Omega^2, \quad \text{and } 4k = \tilde{E}.
\]

(3.13)

1Here the Hamiltonian in Eqn. (3.1) in polar co-ordinate is \( \hat{H} = \frac{1}{2m}(P_r^2 + P_\theta^2) - \frac{\lambda^2 m}{8} (P_r^2 + rP_r) - \frac{k}{r} \). We note that with \( \eta = \exp \left( \frac{m\lambda X}{4\hbar} \right) \), we get \( H = \eta \hat{H} \eta^{-1} \), where \( H \) is same as the one given in the Eqn. (3.5). Thus, the transformation from \( \hat{H} \) in Eqn. (3.2) to \( \hat{H} \) in Eqn. (3.5) can be implemented in polar co-ordinates, directly giving the Hamiltonian in Eqn. (3.6).

2One also need to implement the re-parameterization of time variables, but this will not affect the relations \( -i\hbar \frac{\partial}{\partial \rho} = E\psi(r, \theta) \) and \( -i\hbar \frac{\partial}{\partial \phi} = \tilde{E}\psi(\rho, \phi) \).

3Under the map, \( Z = U^2 \) the conserved angular momentum \( J_H \) of the modified H-atom and that of perturbed oscillator \( J_O \) are related as \( 2J_H = J_O \) and this fixes \( C = \frac{1}{4} \).
We now solve the Schrödinger equation in Eqn. (3.11), by treating the $\lambda^2$ dependent term as a perturbation. After writing $\tilde{E} = \tilde{E}^0 + \lambda^2 \tilde{g}$ as the unperturbed energy, we readily find the energy eigenvalue and eigenfunction as [8]

$$\tilde{E}^0_{n_r,l} = \hbar \Omega_0 (2n_r + |l| + 1),$$  \hspace{1cm} (3.14)

$$\psi^0_{n_r,l}(\rho, \phi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{3}{2} \rho^2} \rho^{|l|} F_1 (-n_r; |l| + 1; \alpha \rho^2 e^{i\phi}),$$  \hspace{1cm} (3.15)

respectively. In the above, we have defined $\alpha = m \Omega_0 / \hbar$. Using the perturbative scheme, we find the first order correction to the eigen value to be

$$\tilde{E}^r_{n_r,l} = -\frac{\lambda^2 m}{64 c^2} \frac{1}{\alpha^2} \left( \frac{n_r + |l|}{n_r} \right) \left( \frac{n_r - 4}{n_r} \right) \Gamma(|l| + 1) \tilde{F}_2 (-n_r; |l| + 4, 4; |l| + 1, 4 - n_r; 1) A_{n_r,l},$$  \hspace{1cm} (3.16)

where $A_{n_r,l} = \left( \frac{n_r! |l|!}{(n_r + |l|)!} \right)^2 \frac{1}{\alpha}$ and the first order correction to the eigenfunction is

$$\tilde{\psi}^r_{n_r,l}(\rho, \phi) = \sum_{n_r \neq n_r'} \frac{AB}{\tilde{E}_{n_r} - \tilde{E}_{n_r'}} \psi^0_{n_r',l'}(\rho, \phi),$$  \hspace{1cm} (3.17)

where

$$A = \frac{\lambda^2 m}{64 c^2} \frac{1}{\alpha^2} \frac{n_r! |l|!}{(n_r + |l|)! (n_r' + |l|)!} \left( \frac{n_r' + |l|}{n_r} \right) \left( \frac{n_r - 4}{n_r} \right) \Gamma(|l| + 1) \tilde{F}_2 (-n_r; |l| + 4, 4; |l| + 1, 4 - n_r; 1).$$  \hspace{1cm} (3.18)

$$B = \left( \frac{n_r + |l|}{n_r} \right) \left( \frac{n_r - 4}{n_r} \right) \Gamma(|l| + 1) \tilde{F}_2 (-n_r; |l| + 4, 4; |l| + 1, 4 - n_r; 1).$$  \hspace{1cm} (3.19)

In calculating these corrections we have used certain identities given in [9]. Under the map, $Z = U^2$, (or equivalently, $r = \rho^2, \theta = 2\phi$), conserved angular momentum $J_H$ of the modified H-atom and that of perturbed oscillator $J_O$ are related as $2J_H = J_O$ and this, apart from fixing $C = \frac{l}{2}$ (see Eqn. (3.12)), relates the angular momentum quantum numbers of modified H-atom, $\tilde{l}$ and that of the perturbed oscillator $l$ as

$$\tilde{l} = \frac{l}{2}.$$  \hspace{1cm} (3.20)

This implies that $n_{\rho} = \frac{n_r}{2}$ i.e., when $n_{\rho} \in \mathbb{Z}^+$ (i.e., even integer), $n_r \in \mathbb{Z}$ and when $n_{\rho} \in \mathbb{Z}^-$ (i.e., odd integer), $n_r \in \mathbb{Z}_\frac{Z}{2} \Rightarrow n_r \sim$ half integer, which is not allowed. Thus, we see that the eigenfunctions and eigen values corresponding to even values of $n_{\rho}$ are mapped to the eigenfunctions and eigen values of the H-atom with additional oscillator potential. We will comment on the possible mapping of eigenfunctions and eigen values corresponding to odd integer values of $n_{\rho}$ in the concluding remarks.

Now, using $r = \rho^2, \theta = 2\phi, l = 2\tilde{l}$ and $\alpha = 2\beta$ in Eqn. (3.14, 3.15, 3.16, 3.17), we find the eigen value and eigenfunction corresponding to the Schrödinger equation in Eqn. (3.17)

$$E = E_{n_r,l}^0 + E_{n_r,l}^r,$$  \hspace{1cm} (3.21)

$$\psi_{n_r,l} = \psi_{n_r,l}^0 + \psi_{n_r,l}^r,$$  \hspace{1cm} (3.22)

where

$$E_{n_r,l}^0 = -\frac{mk^2}{2\hbar^2 (n_r + |l| + \frac{1}{2})^2},$$  \hspace{1cm} (3.23)

$$E_{n_r,l}^r = \frac{\lambda^2 m}{8(2\beta)^4} B_{n_r,l} \left( \frac{n_r + 2\tilde{l}}{n_r} \right) \left( \frac{n_r - 4}{n_r} \right) \Gamma(2|\tilde{l}| + 1) F,$$  \hspace{1cm} (3.24)

where $F = 3F_2 (-n_r, 2|\tilde{l}| + 4, 2|\tilde{l}| + 1, 4 - n_r; 1)$

and $B_{n_r,l} = (2\beta)^{-2|\tilde{l}|} \left( \frac{n_r (2|\tilde{l}| + 1)}{n_r + 2|\tilde{l}| + 1} \right)^2.$  \hspace{1cm} (3.25)
and similarly, we find

$$\psi^0_{n_r,l} = \frac{1}{\sqrt{2\pi}} e^{i\theta} l^{|\lambda|} e^{-\beta r} F_1(-n_r; 2|l|+1; 2\beta r)$$  \hfill (3.27)

$$\psi'_{n_r,l} = \sum_{n_r \neq n_r'} \frac{\tilde{B} \left( \frac{n_r+2|l|}{n_r} \right) \left( \frac{n_r-4}{n_r} \right) \Gamma(2|l|+4) \tilde{F} \psi^0_{n_r,l}}{E^0_{n_r,l} - E^0_{n_r',l}}$$  \hfill (3.28)

where

$$\tilde{B} = -\frac{\lambda^2}{8(2\beta)^{4+2|l|}} \frac{n_r!(2|l|)!}{n_r + 2|l|} \frac{n_r'!(2|l|)!}{n_r' + 2|l|}$$  \hfill (3.29)

$$\tilde{F} = \frac{3}{2} F_2(-n_r, 2|l|+4; 2|l|+1, 4-n_r; 1)$$  \hfill (3.30)

Note here that the Hamiltonian in Eqn.(3.1) and one in Eqn.(3.5) are gauge equivalent. Thus, measured quantities such as energy eigen values will be same for both Hamiltonian but energy eigenfunction will modified. From the relation $\eta \Phi = \psi$ we get energy eigenfunction of the Hamiltonian in Eqn.(3.1) as

$$\Phi = \exp \left( -\frac{im\lambda^2 r^2}{4h} \right) (\psi^0_{n_r,l} + \psi'_{n_r,l})$$  \hfill (3.31)

where $\psi^0_{n_r,l}$ and $\psi'_{n_r,l}$ are given in Eqn.(3.27) and Eqn.(3.28), respectively.

4 Mapping of the Schrödinger Equation with potentials $\frac{1}{r}$ and $r^2$ in three dimensions

Here, in this section, we obtain the eigen values and eigenfunctions of the Schrödinger equation corresponding to the Hamiltonian in Eqn.(2.9). This is derived from the perturbative solution of Schrödinger equation corresponding to Eqn.(2.13), by first establishing the equivalence between these two Schrödinger equations.

For this, we start with the Hamiltonian

$$\mathcal{H} = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{\lambda}{4} (X_0 P_{X_0} + P_{X_0} X_0) + \frac{\lambda}{4} (X_1 P_{X_1} + P_{X_1} X_1) + \frac{\lambda}{4} (X_2 P_{X_2} + P_{X_2} X_2) - \frac{k}{r}$$  \hfill (4.1)

where we have symmetrized the $X_i P_{X_i}$ terms. After re-expressing above Hamiltonian as

$$\mathcal{H} = \frac{(P_{X_0} + \frac{m \lambda X_0}{2})^2}{2m} + \frac{(P_{X_1} + \frac{m \lambda X_1}{2})^2}{2m} + \frac{(P_{X_2} + \frac{m \lambda X_2}{2})^2}{2m} - \frac{\lambda^2 m}{8} (X_0^2 + X_1^2 + X_2^2) - \frac{k}{\sqrt{X_0^2 + X_1^2 + X_2^2}}$$  \hfill (4.2)

we set up the Schrödinger equation

$$\hat{\mathcal{H}} \Phi = E \Phi.$$  \hfill (4.3)

We now apply a transformation $\eta \hat{\mathcal{H}} \eta^{-1} \eta \Phi = E \eta \Phi$ to above equation and redefine $\eta \hat{\mathcal{H}} \eta^{-1} = H$ and $\eta \Phi = \psi$. Taking

$$\eta = \exp \left( \frac{im\lambda(X_0^2 + X_1^2 + X_2^2)}{4h} \right)$$  \hfill (4.4)

and noting that $\eta (P_{X_i} + \frac{m \lambda X_i}{2}) \eta^{-1} = P_{X_i}^m$, we find the transformed Hamiltonian to be

$$H = \frac{(P_{X_0}^2 + P_{X_1}^2 + P_{X_2}^2)}{2m} - \frac{\lambda^2 m}{8} (X_0^2 + X_1^2 + X_2^2) - \frac{k}{r}.$$  \hfill (4.5)
This Hamiltonian describes a 3-dimensional H-atom with an additional inverted harmonic potential. In the spherical polar coordinates, this Hamiltonian becomes:

\[
H = \frac{1}{2m}(P_r^2 + P_\theta^2 + P_\phi^2) - \frac{\lambda^2}{8}mr^2 - \frac{k}{r}
\]  

and we will now find solution to the Schrödinger equation, in spherical-polar coordinates, i.e.,

\[
\left[ -\frac{\hbar^2}{2m} \left( \nabla_3^2 \right) - \frac{\lambda^2m}{8}r^2 - \frac{k}{r} \right] \Psi(r, \theta, \phi) = E\Psi(r, \theta, \phi).
\]

where \( \nabla_3^2 \) is Laplacian in 3-dimensional spherical coordinates, i.e,

\[
\nabla_3^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.
\]

To find the solution to above equation, Eqn.\((4.7)\), we first generalize the equivalence of above system to that described by the Schrödinger equation corresponding to the Hamiltonian in Eqn.\((2.14)\) and use the transformation of co-ordinates in Eqn.\((2.11)\) in relating these two systems. That is, we apply the relations

\[
\dot{\psi} = \frac{1}{4\nu} \nabla_4^2 \Psi
\]

and by a change of variables, map above Hamiltonian operators (and thus the Schrödinger equations), where we make further identifications

\[
E = -\frac{1}{8}m\omega_0^2, \quad \text{and} \quad 4k = \tilde{E}
\]

and a constraint \( \dot{\psi} = 0 \), where \( \dot{\psi} = U_3\dot{\theta}_0 - U_2\dot{\theta}_1 + U_1\dot{\theta}_2 - U_0\dot{\theta}_3 \).

We now solve the Schrödinger equation in Eqn.\((4.9)\) perturbatively. First we find energy eigen value and eigenfunction for 4-dimensional harmonic oscillator as (see \([11]\))

\[
\tilde{E}_{N,K}^0 = (N + 2K + 1)\hbar \omega_0,
\]

\[
\Psi_{NLMK}(\rho, \theta, \psi, \phi) = A_{NLMK} \rho^{N-1}Y_{NLM}(\theta, \psi, \phi)e^{-\frac{\rho^2}{\lambda^2}}L_{N+K}^N(\alpha \rho^2)
\]

respectively. Here,

\[
A_{NLMK} = 2\frac{\alpha^{\frac{N-K}{2}}}{\lambda^{\frac{1}{2}}}(K! \frac{N+K}{2}!)^{-\frac{1}{2}},
\]

Here the Hamiltonian in Eqn.\((4.1)\), in spherical-polar co-ordinate is \( \tilde{H} = \frac{1}{m}(P_r^2 + P_\theta^2 + P_\phi^2) - \frac{\lambda^2}{8}(P_r + rP_r) - \frac{k}{r} \). We note that with \( \eta = \exp \left( \frac{im\omega_0^2}{\hbar^2} \right) \), we get \( \tilde{H} = \eta \tilde{H} \eta^{-1} \), where \( \tilde{H} \) is same as the one given in the Eqn.\((4.1)\). Thus, the transformation from \( \tilde{H} \) in Eqn.\((4.2)\) to \( \tilde{H} \) in Eqn.\((4.5)\) can be implemented in spherical-polar co-ordinates, directly giving the Hamiltonian in Eqn.\((4.6)\).

One also need to implement the re-parameterization of time variables, but this will not affect the relations \( -i\hbar \frac{\partial \phi(r, \theta, \phi)}{\partial \phi} = \tilde{E}\psi(r, \theta, \phi) \) and \( -i\hbar \frac{\partial \phi(r, \theta, \phi)}{\partial \phi} = \tilde{E}\psi(r, \theta, \phi, \phi) \).
where $N \in N - 0, K \in N, L = 0, 1, ..., N - 1,$ and $M = -L, -L + 1, ..., L.$

In the above, we have defined $\alpha = m\omega_0/\hbar$. Using the perturbative method, we find the first order correction to the eigen value to be

$$\tilde{E}_{N,K} = C \left( \frac{2N + K}{N + K} \right) \left( \frac{N + K - 4}{N + K} \right) \Gamma(N + 4) \beta F_2(-(N + K), N + 4; N + 1, 4 - N - K; 1),$$

(4.15)

where $C = -\frac{\lambda^2 m}{4 \alpha^4 N} A^2_{NLMK}$ and the first order correction to the eigen function is

$$\Psi'_{NLMK} = \sum_{K \neq K'} \frac{AB}{(E_{N,K}^0 - E_{N,K'}^0)} \Psi_{NLMK'},$$

(4.16)

where

$$A = -\frac{\lambda^2 m}{4} \frac{1}{\alpha^4 N} A_{NLMK'} A_{NLMK} \left( \frac{2N + K'}{N + K'} \right) \left( \frac{N + K - 4}{N + K} \right) \Gamma(N + 4),$$

(4.17)

$$B = \beta F_2(-(N + K'), N + 4; N + 1, 4 - (N + K); 1).$$

(4.18)

In calculating these corrections we have used certain identities given in [9].

Now, applying K-S transformation in Eqs. (4.12, 4.13, 4.15, 4.16), we obtain the eigenfunction for the Schrödinger equation in Eqn.(4.7)

$$\Psi_{nlm}(Hydrogen) = \sum_{N,L,M,K} I(nlmNLMK)(\Psi_{NLMK} + \Psi'_{NLMK}),$$

(4.19)

where wave function for perturbed hydrogen atom problem in 3-dim, $\Psi_{nlm} = \Psi_{nlm}^0 + \Psi_{nlm}'$ can be calculated using above equation by following the method discussed in [11]. Also using Eqn.(4.11), Eqn.(4.12) and Eqn.(4.15) we obtained eigenvalues for perturbed hydrogen-atom problem (see details [11], [12]),

$$E_n(Hydrogen) = -\frac{k^2 m}{2\hbar^2 n^2} + D^2 \tilde{E}_{N,K}, \ n = 1, 2...$$

(4.20)

with identification $N + 2K + 1 = 2n$ and here $D = \sum_{N,L,M,K} I(nlmNLMK)$, which can be explicitly calculated by using method discussed in [11].

Here the Hamiltonian in Eqn.(4.1) and in Eqn.(4.3) are gauge equivalent. Thus, all measured quantities like energy eigen values will remain same for both Hamiltonian. From the relation $\eta \Phi = \Psi$ we get energy eigenfunction of the Hamiltonian in Eqn.(4.11) as

$$\Phi = exp \left( -\frac{i m \lambda r^2}{4\hbar} \right) (\Psi_{nlm}^0 + \Psi_{nlm}')$$

(4.21)

5 Mapping of the Schrödinger Equation of shifted harmonic oscillator

In the subsection.(2.3), we have summarized the study of mapping the equations of motion of a shifted harmonic oscillator to that of Kepler problem via the equations of motion of a shifted harmonic oscillator. Now, we will derive the relation between the Schrödinger equations corresponding to these systems. Mapping of Schrödinger equations harmonic oscillator and H-atom has been derived in [6] and we adapt this results for the shifted oscillator we have studied in [7].

The Hamiltonian corresponding to the shifted oscillator is

$$H = \frac{1}{2m} (p_1^2 + p_2^2) + m\Omega^2/2 (x_1^2 + x_2^2) + \lambda/4 (x_1 p_1 + p_1 x_1 + x_2 p_2 + p_2 x_2).$$
As earlier, we transform this Hamiltonian to an equivalent one, by the transformation \( \hat{H} = \eta H \eta^{-1} \) where \( \eta = e^{i \frac{\hbar}{\alpha} \lambda (x_1^2 + x_2^2)} \) and obtain

\[
\hat{H} = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{\hbar \tilde{\Omega}}{2}(x_1^2 + x_2^2),
\]

(5.1)

which is the Hamiltonian for harmonic oscillator with frequency \( \tilde{\Omega} = (\Omega^2 - \frac{\lambda^2}{4}) \). The eigenfunction satisfying the corresponding Schrödinger equation, in polar co-ordinates

\[
\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \right) + \frac{\hbar \tilde{\Omega}^2 \rho^2}{2} \right] \psi = E\psi
\]

(5.2)

is

\[
\psi = \frac{1}{\sqrt{2\pi}} e^{i \phi} \psi (\rho)
\]

(5.3)

where

\[
\psi (\rho) = e^{-\frac{i}{2} \alpha \rho^2} \varphi (\rho (|l|+1, \alpha \rho^2)).
\]

(5.4)

Here \( F(a, b, c) \) is Confluent Hypergeometric Function and we have used \( \alpha = m \tilde{\Omega}/\hbar \). The eigen value is given by

\[
E = \hbar \tilde{\Omega}(2n_\rho + |l|+1)
\]

(5.5)

Next, we apply the Bohlin-Sundman map (see Eqn.(5.2)) to above equation, i.e., in polar co-ordinate, we apply

\[
\rho^2 \rightarrow r; \phi \rightarrow \frac{\theta}{2}; \alpha \rightarrow 2\beta
\]

(5.6)

and after simple algebra, get the eigen value equation to be

\[
\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) + \frac{\hbar \tilde{\Omega}^2 r^2}{2} \right] \psi (r, \theta) = \frac{E}{4r} \psi (r, \theta)
\]

(5.7)

Substituting \( \psi (r, \theta) = \frac{1}{\sqrt{2\pi}} e^{i \phi} \psi (r) \), we re-write the above equation as

\[
\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) - \frac{\hbar^2}{2mr^2} - \frac{E}{4r} \right] \psi (r) = -\frac{m \tilde{\Omega}^2}{8} \psi (r)
\]

(5.8)

Now using Eqn.(5.2) and renaming \( E_k \) as \( E_H \) with the identification \( E/4 = k \), we re-express above equation as

\[
\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) - \frac{\hbar^2}{2mr^2} - \frac{k}{r} \right] \psi (r) = E_H \psi (r).
\]

(5.9)

We now easily identify this as the Schrödinger equation describing H-atom. Since we have \( 2\phi = \theta \) (see Eqn.(5.6)), we find that the orbital quantum numbers are related as \( l = 2 \vec{l} \). Using this and Eqn.(5.6) in Eqn.(5.4) we find the eigenfunction \( \psi (r, \theta) \) of Eqn.(5.9) and thus obtain

\[
\psi (r, \theta) = \frac{1}{\sqrt{2\pi}} e^{i \phi} e^{-\beta r \vec{l}|\vec{l}|} F(-n_r, 2|\vec{l}|+1, 2\beta r)
\]

(5.10)

where \( \beta = m \tilde{\Omega}/2\hbar \). From Eqn.(5.5), we also find the eigen value to be

\[
E_H = -\frac{mk^2}{2\hbar^2} \frac{1}{(n_r + |\vec{l}|+\frac{1}{2})^2}
\]

(5.11)

Since \( \vec{l} = \frac{l}{2} \), we note that of the eigenfunctions in Eqn.(5.10) only those corresponding to even values of \( n_r \) are mapped to the eigenfunctions \( \psi (r, \theta) \) given in Eqn.(5.10), as in [6]. Note here that states with even parity of shifted harmonic oscillator get to mapped to wave function of Hydrogen atom.
6 Conclusion

We have showed the equivalence between the Schrödinger equation corresponding to $\frac{1}{r}$ potential augmented with a harmonic oscillator potential in 2 and 3 dimensions to the Schrödinger equation corresponding to a particle moving under the influence of harmonic potential with an additional inverted, sextic potential and interactions in 2 and 4 dimensions, respectively. In both cases, the coefficient of the additional oscillator potential controls the strength of inverted sextic potential and interactions. For both case, using perturbative solution of the later, we obtain the eigenfunctions and eigen values of the former. Our results reduce to the standard results of the mapping between H-atom and harmonic oscillator in the vanishing limit of the co-efficient of additional oscillator potential, i.e., $\lambda \to 0$.

We then extend the mapping of motion of shifted harmonic oscillator to Kepler problem to the mapping between the corresponding Schrödinger equations. Using this, we map eigenfunctions and eigen values of these two systems. This map shows that the eigenfunctions of (shifted) harmonic oscillator corresponding to even eigen values are mapped to that of H-atom, as in [6]. It has been shown in [6] that the eigenfunctions corresponding to the odd integer eigen values are mapped to that of charged vortex system. Here too, we note that the eigen value equation in Eqn. (5.2) in terms of the complex co-ordinate $\omega = x_1 + ix_2 = \rho e^{i\phi}$ is

\[ (4 \frac{\partial^2}{\partial \omega \partial \bar{\omega}} + \frac{m \tilde{\Omega}^2}{2} \omega) \psi(\omega, \bar{\omega}) = E \psi(\omega, \bar{\omega}). \]

Under the parity transformation, $\omega \to \pm \omega$ and we have $z \to z$. Thus we see that under $\omega \to \pm \omega$, $arg(z)$ changes by $2\pi$. Thus parity eigenstates $\psi_\sigma(\omega, \bar{\omega}) = \psi_\sigma(\omega, \bar{\omega}) e^{i\sigma arg(\omega)}$, where $\sigma = 0$ for even and $\sigma = \frac{1}{2}$ for odd states, respectively, are mapped to $\psi_\sigma(\bar{z}, z) e^{i\sigma arg(z)}$. The parity even states, having zero phase factor will get mapped to the eigenfunctions of H-atom. The non-zero phase factor for the parity odd states will lead to the mapping of these states to that of charged magnetic vortex system as in [6]. This discussion is relevant for the mapping of eigenfunctions and eigen values obtained in section 3 and 4. Here too, we have seen that only the eigenfunctions corresponding to the even eigen values of harmonic oscillator with inverted sextic potential are mapped to that of the system with combination of $\frac{1}{r}$ potential and harmonic potential. It is thus natural to expect that the eigenfunctions corresponding to odd integer eigen values will be related to a H-atom with an additional oscillator potential in presence of a magnetic vortex.

Note that both the L-C and K-S regularization schemes map the solution of Kepler problem on a constant energy surface to that of harmonic oscillator. In the systems studied here, we have augmented the $\frac{1}{r}$ potential with an additional $r^2$ potential and here also this constant energy surface plays the same role. We have seen that the regular system we obtained in 2 and 4 dimensions had the coefficient of harmonic oscillator potential given by the conserved energy of the initial systems. This fact was crucial in connecting the eigen values of the corresponding Schrödinger equations. For the classical Kepler problem, this restriction of constant energy surface was relaxed in the Moser-regularization [13] and the Ligon-Schaaf regularization [14]. It will be of interest to see their generalization to the quantum systems studied here.

In the appendices we show the connection between the damped systems to the system described by Eqn. (2.1) using canonical transformation as well as using a non-inertial transformation of co-ordinates.

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Appendix : A

In this appendix, we show that the system described by the Eqn. (2.1) is related by a canonical transformation to a dammed system [15].

The Hamiltonian given in Eqn. (2.3)

\[ \mathcal{H} = \frac{(P_{X_1}^2 + P_{X_2}^2)}{2m} + \frac{\lambda}{2} (X_1 P_{X_1} + X_2 P_{X_2}) - \frac{k}{\sqrt{X_1^2 + X_2^2}} \]
is mapped to the Hamiltonian\(^6\)

\[
H = \frac{e^{-\lambda t}(p_{x_1}^2 + p_{x_2}^2)}{2m} - \frac{ke^{-\lambda t}}{\sqrt{x_1^2 + x_2^2}},
\]

(A.1)

by the canonical transformation generated by \(F_2(x_i, P_{X_i}) = x_i e^{\frac{\lambda t}{2}} P_{X_i}\), as \(\mathcal{H}(X_i, P_{X_i}) = H(x_i, p_{x_i}) + \frac{\partial F_2}{\partial t}\).

Here we have used the relations obtained from this generating function,

\[
p_{x_i} = e^{\frac{\lambda t}{2}} P_{X_i}, \quad X_i = x_i e^{\frac{\lambda t}{2}}.
\]

(A.2)

The corresponding Hamiltonian operators are related by [17]

\[
\hat{\mathcal{H}}(\hat{X}_i, \hat{P}_{X_i}) = \hat{H}(\hat{x}_i, \hat{p}_{x_i}) + \frac{\partial \hat{F}_2}{\partial t},
\]

(A.3)

where we use \(\hat{\cdot}\) indicate the operator nature explicitly. This can be expressed using an unitary operator as (see [17])

\[
\hat{\mathcal{H}}(\hat{X}_i, \hat{P}_{X_i}) = \hat{H}(\hat{x}_i, \hat{p}_{x_i}) + i\hbar \frac{\partial \hat{U}^\dagger}{\partial t},
\]

(A.4)

where \(\hat{U} = e^{\frac{i}{\hbar} [\hat{x}_i \hat{P}_{X_i} + \hat{P}_{X_i} \hat{x}_i] e^{-\frac{\lambda t}{2}}}\).

Appendix : B

In this appendix, we show that a non-inertial transformation relates the model given in Eqn.(2.1) to a damped system.

The time dependent Schrödinger equation corresponding to the Hamiltonian in Eqn.(2.3) is

\[
\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial X_i^2} - i\hbar \frac{\lambda}{4} X_i \frac{\partial}{\partial X_i} + \left( \frac{\partial}{\partial X_i} \right) X_i - \frac{k}{\sqrt{X_1^2 + X_2^2}} \right] \psi(X_i, t) = i\hbar \frac{\partial \psi(X_i, t)}{\partial t} 
\]

(B.1)

Now we will apply non-inertial transformation

\[
X_i \rightarrow x_i = X_i e^{-\frac{\lambda \tilde{t}}{2}}, \quad t = \tilde{t}
\]

(B.2)

the above Schrödinger equation, mapping it to

\[
\left[ -\frac{\hbar^2}{2m} (e^{-\lambda \tilde{t}}) \frac{\partial^2}{\partial x_i^2} - \frac{ke^{-\lambda t}}{\sqrt{x_1^2 + x_2^2}} \right] \tilde{\psi}(x_i, \tilde{t}) = i\hbar \frac{\partial \tilde{\psi}(x_i, \tilde{t})}{\partial \tilde{t}}
\]

(B.3)

where \(\tilde{\psi}(x_i, \tilde{t}) = \psi(x_i e^{-\lambda \tilde{t}}, \tilde{t})\). In obtaining above mapping, we have used [18]

\[
\frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t} - \frac{\lambda}{2} \frac{x_i}{\partial X_i}; \quad \frac{\partial}{\partial \hat{X}_i} = e^{-\frac{\lambda \tilde{t}}{2}} \frac{\partial}{\partial x_i}
\]

(B.4)

Note that this is the Schrödinger corresponding to the Hamiltonian given in Eqn.(A.1), showing the mapping between the quantum systems under the above non-inertial transformation.

\(^6\)which is derived from the Bateman-Caldirola-Kanai [16] type Lagrangian given by \(L = e^{\lambda t} \left[ \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) + \frac{ke^{-\frac{\lambda t}{2}}}{r} \right] \) where \(r = \sqrt{x_1^2 + x_2^2}, m = \frac{m_1 + m_2}{m_1 m_2}\) and \(k = Gm_1 m_2\)
References

[1] T. Levi-Civita, Acta Math. 42, 99 (1920); E. L. Stiefel and G. Scheifele, Linear and Regular Celestial Mechanics (Springer-Verlag, 1971).

[2] P. Kustaanheimo and E. Stiefel, J. Reine Angew. Math. 218 (1965) 204.

[3] H. Duro and H. Kleinert, Phys. Lett. B 84, 185 (1979); H. Grinberg, J. Maration and H. Vucetich, J. Math. Phys. 25 (1984) 2648.

[4] N. Mukunda, Phys. Rev. 155 (1967) 1383; D. M. Fradkin, Prog. Theor. Phys. 37 (1967) 798; A. Barut, C. K. E. Schneider and R. Wilson, J. Math. Phys. 20, 91979) 2244.

[5] K. Bohlin, Bull. Astro. 28, 144 (1911).

[6] A. Nersessian, V. Ter-Antonyan and M. Tsulaia, Mod. Phys. Lett. A11, 1605 (1996).

[7] E. Harikumar, S. K. Panja and P. Guha, Eur. Phys. J. Plus. 136 904 (2021).

[8] X. Fu, IOP Conf. Ser.: Earth Environ. Sci 295, 032042 (2019).

[9] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, series and products, 7th ed. 2007 (Elsevier Academic Press, 2007), pp.1001; L. Poh-aun, S-h Ong and H. M. Srivastava, Int. J. Computer Math. 78, 303 (2001)

[10] T. Das and A. Arda, Adv. High Energy Phys. 2015, 137038 (2015).

[11] M. Kibler, A. Ronveaux and T. Negadi, J. Math. Phys. 27, 1541 (1986).

[12] D. Bergmann and Y. Frishman, J. Math. Phys. 6, 1855 (1965).

[13] J. Moser, Commun. Pure Appl. Math. 23, 609 (1970).

[14] T. Ligon and M. Schaaf, Rep. Math. Phys. 9, 281 (1976).

[15] H. Dekker, Phys. Rep. 80 (1981) 1; R. W. Hasse, J. Math. Phys. 16 (1975) 2005; Quantum Dissipative Systems, U. Weiss, 4th Edition (World Scientific, 2008).

[16] H. Bateman, Phys. Rev. 38, 815 (1931); P. Caldirola, Nuovo Cimento 18, 393 (1941); E. Kanai, Prog. Theor. Phys. 3, 440 (1950).

[17] J. H. Kim and H. W. Lee, Can. J. Phys. 77, 411 (1999).

[18] S. Takagi Prog. Theor. Phys. 85, 463 (1991); ibid, Prog. Theor. Phys. 85, 723 (1991); Prog. Theor. Phys. 86, 783 (1991).