Nonparametric tests in linear model with autoregressive errors

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Abstract
In the linear regression model with possibly autoregressive errors, we construct a family of nonparametric tests for significance of regression, under a nuisance autoregression of model errors. The tests avoid an estimation of nuisance parameters, in contrast to the tests proposed in the literature. A simulation study illustrate their good performance.

Keywords Autoregression · Autoregression rank scores · Linear regression · Rank test · Regression rank scores
1 Introduction

The standard assumption of the serial independence and identical distribution of model errors in the linear regression model is often violated, and we should admit a possible autoregressive structure of model errors. We refer to McKnight et al. (2000), who applied a double bootstrap method to analyze linear models with autoregressive errors. Some authors consider the linear model with the autoregressive errors, but with their known innovation distribution. Alpuim and El-Shaarawi (2008) studied the ordinary least squares (OLS) estimator under the $p$-order autoregressive (AR($p$)) error term and the normal innovations. Tuaç et al. (2018) considered linear regression model with AR($p$) errors with Student’s t-distribution, and used the conditional maximum likelihood estimation of model parameters. In (2020), Tuaç et al. proposed an autoregressive regression procedure based on the skew-normal and skew-$t$ distributions. Güney et al. (2020a) considered the conditional maximum Lq-likelihood (CML$q$) estimation of regression parameters in model with autoregressive error terms with their normal innovations.

In the real life, the distribution of the data set is unknown, while it can be heavy-tailed and may contain outliers. Then we should take recourse to nonparametric models without a specific distribution assumptions. The most powerful tools for estimation and other inference in this area are the regression and autoregression quantiles. For testing the hypotheses the most convenient are the regression and autoregression rank scores, which are duals to the quantiles.

The quantile regression was introduced by Koenker and Bassett (1978) and by their followers. The regression rank scores were introduced by Gutenbrunner and Jurečková (1992) and an extension of this concept to the autoregressive model is due to Koul and Saleh (1995). The broad class of regression rank scores tests was developed by Gutenbrunner and Jurečková (1992) and Gutenbrunner et al. (1993). The optimal autoregression rank scores tests in the AR model were constructed by Hallin and Jurečková (1999). El Bantli and Hallin (2001) constructed the Kolmogorov-Smirnov type test in AR model based on the autoregression rank scores, following the KS test in the linear model based on regression rank scores, proposed by Jurečková (1991). The useful concept of the averaged autoregression quantile, and its asymptotics were studied by Güney et al. (2020b).

2 Statement of the model

In the present paper, we assume that our data follow a linear regression model whose model errors can be possibly autoregressive. The corresponding probability distributions are generally unknown, only satisfy some assumptions. In this setup, we shall verify the hypothesis of no linear regression under possible nuisance autoregression of model errors.

We consider the linear regression model of order $s$, whose model errors follow a stationary autoregressive process of order $p$:

$$ y_t = \beta_0 + x_t^T \beta^* + \varepsilon_t = \beta_0 + x_{t1} \beta_1 + \cdots + x_{ts} \beta_s + \varepsilon_t, \quad (2.1) $$
Here $y_t$ is the response variable, $x_t = (x_{t0}, \cdots, x_{ts})^T$ are the regressors and $\beta_j, \ j = 0, \cdots, s$ are unknown regression parameters. We assume that $x_{t0} = 1$ for all $t$, hence that $\beta_0$ is an intercept.

Moreover, $\varphi_0, \varphi_1, \cdots, \varphi_p$ are unknown autoregression parameters, where the intercept $\varphi_0$ is added for mathematical convenience and can be 0. The innovations $u_t$ are assumed being independently and identically distributed (i.i.d.) with a continuous distribution function $F$ and density $f$, generally unknown but satisfying

$$E(u_t) = 0, \quad \text{Var}(u_t) = \sigma^2 < \infty. \quad (2.3)$$

A stationary solution $\varepsilon_t$ of Eq. (2.2) exists (and is also the unique stationary solution) if and only if $\varphi(z) = 1 - \sum_{j=1}^{p} \varphi_j z^j \neq 0$ for all $|z| = 1$ (Brockwell and Davis (1991), p.75, eq. 3.1.4). Additionally, we assume that the error model given in Eq. (2.2) is a strictly stationary process, and so the $\varepsilon_t$ share a common marginal distribution and thus share the same quantiles. The distribution function $F$ of $u_t$ is unknown, but we assume that it is increasing on the set $\{u : 0 < F(u) < 1\}$. Because of the identifiability, we assume that the starting observations $(y_{-p+1}, \cdots, y_0)$ are known.

For the convenience, we also write (2.2) in the form

$$u_t = \Phi(B) \varepsilon_t \quad (2.4)$$

where $B$ is called the backshift operator. Then the linear regression model with AR(p) error term given in Eq. (2.1) can be expressed as

$$\Phi(B) y_t = y_t - \varphi_0 - \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p},$$
$$\Phi(B) x_t = x_t - \varphi_0 - \varphi_1 x_{t-1} + \cdots + \varphi_p x_{t-p}. \quad (2.5)$$

hence

$$\Phi(B) y_t = (\Phi(B) x_t)^T \beta + u_t. \quad (2.6)$$

In the model (2.1), we construct the tests of the hypothesis:

$$H_0 : \beta^* = 0, \quad \text{with} \ \beta_0, (\varphi_0, \varphi_1, \cdots, \varphi_p)^T \neq 0 \ \text{unspecified}.$$

Our tests are nonparametric; the test of $H_0$ is based on the autoregression rank scores and on the linear autoregression rank statistics for the model (2.1) and (2.2), calculated under hypothesis $H_0$, with the weights corresponding to the alternative.

### 3 Rank tests for $H_0$

We shall be testing the absence of regression

$$H_0 : \beta^* = 0, \quad \beta_0, \varphi_0, \varphi_1, \cdots, \varphi_p \ \text{unspecified}.$$
against the general alternative

\[ K : \beta^* \neq 0, \quad \beta_0, \varphi_0, \varphi_1, \ldots, \varphi_p \text{ unspecified.} \]

Our tests, based on autoregression rank scores, are asymptotically efficient, under some conditions on distribution tails and on the regression matrix, against the local (Pitman) regression alternative

\[ K_n : \beta^* = n^{-1/2} \beta_n^*, \quad \text{with } \beta_n^* \in IR^s \text{ fixed,} \]

(3.1)
similarly as the rank tests of regression without the nuisance autoregression.

Under \( H_0 \), the observations follow the model

\[ y_t = \beta_0 + \varepsilon_t, \quad t = 1, \ldots, n. \]

This is in fact the hypothetical autoregressive model

\[ H_0 : y_t = \beta_0 + \varphi_0 y_{t-1} + \cdots + \varphi_p y_{t-p} + \varepsilon_t, \quad t = 1, \ldots, n \]

(3.2)

which we like to test against the alternative \( K_n \). We shall use the notation

\[ \begin{align*}
\mathbf{y}_t^* & = (y_{t-1}, \ldots, y_{t-p})^\top \\
\mathbf{y}_t & = (1, y_{t-1}, \ldots, y_{t-p})^\top, \quad t = 0, \ldots, n - 1
\end{align*} \]

(3.3)

and consider the random matrices of the respective orders \( n \times p \) and \( n \times (p + 1) \)

\[ \begin{align*}
\mathbf{Y}_n^* & = \begin{bmatrix}
\mathbf{y}_1^* \\
\vdots \\
\mathbf{y}_n^* 
\end{bmatrix}, \quad \mathbf{Y}_n = \begin{bmatrix}
\mathbf{y}_1^\top \\
\mathbf{y}_2^\top \\
\vdots \\
\mathbf{y}_n^\top 
\end{bmatrix}.
\end{align*} \]

(3.4)

For convenience, denote also

\[ \begin{align*}
\mathbf{x}_t^* & = (x_{t1}, \ldots, x_{ts})^\top, \quad \mathbf{x}_t = (1, x_{t1}, \ldots, x_{ts})^\top = (1, \mathbf{x}_t^*^\top)^\top \\
\mathbf{X}_n^* & = \begin{bmatrix}
\mathbf{x}_1^*^\top \\
\vdots \\
\mathbf{x}_n^*^\top 
\end{bmatrix}, \quad \mathbf{X}_n = \begin{bmatrix}
\mathbf{x}_1^\top \\
\mathbf{x}_2^\top \\
\vdots \\
\mathbf{x}_n^\top 
\end{bmatrix}.
\end{align*} \]

The autoregression rank scores \( \mathbf{\hat{a}}_n(\alpha) = (\mathbf{\hat{a}}_{n1}(\alpha), \ldots, \mathbf{\hat{a}}_{nn}(\alpha))^\top \) under hypothesis \( H_0 \) are defined as the solution vector of the linear programming problem

\[ \begin{align*}
\sum_{t=1}^{n} y_t \mathbf{\hat{a}}_{nt}(\alpha) : \quad & = \max \\
\sum_{t=1}^{n} \left( \mathbf{\hat{a}}_{nt}(\alpha) - (1 - \alpha) \right) : \quad & = 0 \\
\mathbf{Y}_n^\top \left( \mathbf{\hat{a}}_n(\alpha) - (1 - \alpha) \mathbf{1}_n \right) : \quad & = 0 \\
\mathbf{\hat{a}}_n(\alpha) \in [0, 1]^n, \quad & 0 \leq \alpha \leq 1.
\end{align*} \]

(3.5)
The autoregression rank scores are \textit{autoregression-invariant}. More precisely, (3.5) implies that \( \hat{a}_n(\alpha) \) can be also formally written as a solution of the linear program

\[
\begin{align*}
\sum_{t=1}^{n} u_t \hat{a}_{nt}(\alpha) : \quad & \max \\
\sum_{t=1}^{n} (\hat{a}_{nt}(\alpha) - (1 - \alpha)) : \quad & 0 \\
Y_n^\top (\hat{a}_n(\alpha) - (1 - \alpha)1_n) : \quad & 0 \\
\hat{a}_n(\alpha) \in [0, 1]^n, \quad 0 \leq \alpha \leq 1.
\end{align*}
\]

where \( u_n = (u_1, \cdots, u_n)^\top \) is the unobservable white noise process.

We shall construct a new family of tests of hypothesis \( H_0 \) for the model (2.1), based on autoregression rank scores, and analyze the asymptotic distribution of the test criterion under the null hypothesis as well as under contiguous alternatives. Surprisingly, no preliminary estimation of \( \varphi \) is needed in order to compute autoregression rank score statistics, in contrast with the aligned rank methods (Puri and Sen (1985) and others).

The regression rank scores tests were originally introduced in Gutenbrunner and Jurečková (1992) and Gutenbrunner et al. (1993). They were then extended to the autoregression model in Koul and Saleh (1995) and in Hallin and Jurečková (1999). Let us also mention the tests of independence of two autoregressive time series based on AR scores, which were constructed in Hallin et al. (1999). The new tests developed in the present paper combine the linear regression model with autoregression rank scores.

The tests are developed under the assumption that the (unknown) density \( f \) of \( u_t \) belongs to the family \( \mathcal{F} \) of exponentially tailed densities, satisfying (2.3) and the following conditions on the tails:

(F1) \( f \) is positive and absolutely continuous, with finite Fisher information \( \mathcal{I}(f) = \int \left( \frac{f'(x)}{f(x)} \right)^2 f(x) dx < \infty \); moreover, there exists \( K_f \geq 0 \) such that \( f \) has two bounded derivatives \( f' \) and \( f'' \) for all \( |x| > K_f \);

(F2) \( f \) is monotonically decreasing to 0 as \( x \to \pm \infty \) and its distribution function \( F \) satisfies

\[
\lim_{x \to -\infty} \frac{-\log F(x)}{b|x|^r} = \lim_{x \to \infty} \frac{-\log(1 - F(x))}{b|x|^r} = 1
\]

for some \( b > 0 \) and \( r \geq 1 \).

Other properties of densities in \( \mathcal{F} \), consequences of (F1) and (F2), are summarized in Hallin and Jurečková (1999).

Furthermore, the following standard conditions are imposed on the regression matrix \( X_n^* \):

(X1) The matrix \( A_n = n^{-1} \sum_{t=1}^{n} X_n^* X_n^\top \) of order \( s \) is positive definite for \( n \geq n_0 \).

(X2) \( n^{-1} \sum_{t=1}^{n} \|x_{nt}\|^4 = \mathcal{O}(1) \) as \( n \to \infty \).

(X3) \( \lim_{n \to \infty} \max_{1 \leq t \leq n} \left\{ n^{-1} x_{nt}^\top A_n^{-1} x_{nt} \right\} = 0 \).
We introduce the projection matrix $H_n$ and the projection $\hat{X}_n^*$ of $X_n^*$ on the space spanned by the columns of $Y_n$:

$$H_n = Y_n^T(Y_n^TY_n)^{-1}Y_n, \quad \hat{X}_n^* = H_nX_n^*. \quad (3.6)$$

Define two random matrices $D_n$ and $Q_n$ of orders $(p+1) \times (p+1)$ and $s \times s$:

$$D_n = n^{-1}Y_n^TY_n, \quad Q_n = n^{-1}(X_n^* - \hat{X}_n^*)(X_n^* - \hat{X}_n^*). \quad (3.7)$$

We assume that

$$\lim_{n \to \infty} IE(D_n) = D, \quad \lim_{n \to \infty} IE(Q_n) = Q \quad (3.8)$$

where $D$ and $Q$ are positive definite matrices.

For constructing the rank tests of $H_0$ we choose a nondecreasing, square integrable score generating function $J : (0, 1) \to \mathbb{R}$, such that $J(1-u) = -J(u)$, $0 < u < 1$ and that $J'(u)$ exists for $u \in (0, \alpha_0) \cup (1 - \alpha_0, 1)$ and, in this domain, $J$ satisfies the Chernoff-Savage condition

$$|J'(u)| \leq c(u(1-u))^{-1+\delta}, \quad 0 < \delta < \frac{1}{4}. \quad (3.9)$$

The typical choices of $J$ are:

(i) Wilcoxon scores (optimal for $f$ logistic): $J(u) = u - \frac{1}{2}$, $0 < u < 1$. The scores are $\hat{b}_{n:t} = -\int_0^1 J(u) - \frac{1}{2} d\hat{a}_{n:t}(u) = \int_0^1 J^2(u) du - \frac{1}{2}$ while $A^2(J) = \frac{1}{12}$ and $\gamma(J, F) = \int f^2(x) dx$.

(ii) Normal (van der Waerden) scores (asymptotically optimal for $f$ normal): $J(t) = \Phi^{-1}(u)$, $0 < u < 1$, $\Phi$ being the d.f. of standard normal distribution. Here $A^2(J) = 1$ and $\gamma(J, F) = \int f(F^{-1}(J(x))) dx$.

(iii) Median (sign) scores: $J(u) = \frac{1}{2} \text{sign}(u - \frac{1}{2})$, $0 < u < 1$.

The rank scores generated by $J$ are defined as $\hat{b}_n = (\hat{b}_{n1}, \ldots, \hat{b}_{nn})^T$ with

$$\hat{b}_{nt} = -\int_0^1 J(u) d\hat{a}_{nt}(u), \quad t = 1, \ldots, n. \quad (3.10)$$

The proposed tests of $H_0$ are based on the linear autoregression rank statistics $S_n$:

$$S_n = n^{-\frac{1}{2}}(X_n^* - \hat{X}_n^*)^T\hat{b}_n. \quad (3.11)$$

As the test criterion for testing $H_0$ against $K_n$ we propose the quadratic form in $S_n$:

$$T_n = S_n^TQ_n^{-1}S_n/A^2(J) \quad (3.12)$$
where

\[ A^2(J) = \int_0^1 (J(t) - \bar{J})^2 dt, \quad \bar{J} = \int_0^1 J(t) dt. \quad (3.13) \]

Notice that the test statistic \( T_n \) requires no estimation of nuisance parameters, since the functional \( A(J) \) depends only on the score function and not on (the unknown) \( F \). We shall show that the asymptotic distribution of \( T_n \) under \( H_0 \) is central \( \chi^2 \) with \( s \) degrees of freedom, hence it is asymptotically distribution free. Under \( K_n \) it is noncentral \( \chi^2 \) with \( s \) degrees of freedom and noncentrality parameter dependent on \( J \) and \( F \), but not on the nuisance parameters. In this way, it is asymptotically equivalent to the rank test of \( H_0 \) in the situation without nuisance autoregression.

4 Asymptotic behavior of the test of \( H_0 \)

Let us return to the model (2.1). Assume that the matrices \( X_n^* \) and \( Y_n^* \) satisfy conditions (3.6)–(3.8). We want to test the hypothesis

\[ H_0 : \beta^* = 0 \quad (\beta_0, \varphi \text{ unspecified}) \]

against the alternative

\[ K_n : \beta^* = n^{-1/2} \beta_x^* \quad (\beta_x^* \in IR_s \text{ fixed}). \]

Let \( \hat{a}_n(\alpha) = (\hat{a}_1(\alpha), \ldots, \hat{a}_n(\alpha)) \) be the autoregression rank scores corresponding to the submodel under \( H_0 \), i.e.

\[ y_t = \beta_0 + \varphi_0 + \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p} + u_t, \quad t = 1, \ldots, n. \]

Let \( J : (0, 1) \mapsto IR \) be a nondecreasing and square integrable score-generating function such that \( J(1 - u) = -J(u), \ 0 < u < 1 \), satisfying (3.9). Define the rank scores \( \hat{b}_n = (\hat{b}_1, \cdots, \hat{b}_n)^\top \) by the relation (3.10). Consider the test statistics

\[ T_n = S_n^\top Q_n^{-1} S_n / A^2(J) \]

defined in (3.11)–(3.13). The critical region of the test is based on the asymptotic distribution of \( T_n \) under \( H_0 \). This is described in the following theorem.

**Theorem 1** Assume that the distribution \( F \) of the innovations \( u_t \) satisfies (F1)–(F2) and the regression matrix \( X_n \) satisfies (X1)–(X3). Let \( T_n \) be generated by the function \( J \) satisfying (3.9), nondecreasing and square integrable on \((0, 1)\).

(i) Then, under \( H_0 \), the asymptotic distribution of \( T_n \) is central \( \chi^2 \) with \( s \) degrees of freedom.

(ii) Under \( K_n \), the asymptotic distribution of \( T_n \) is noncentral \( \chi^2 \) with \( s \) degrees of freedom and the noncentrality parameter

\[ \eta^2 = \beta_x^\top Q\beta_x \cdot \gamma^2(J, F) / A^2(J) \]

where
\[ \gamma(J, F) = - \int_0^1 J(v) d f(F^{-1}(v)). \]  

(4.1)

Hence, the test rejects \( H_0 \) on the significance level \( \tau \in (0, 1) \) if \( T_n > \chi^2_s(1 - \tau) \) where \( \chi^2_s(1 - \tau) \) is the 100(1 - \( \tau \))\%-quantile of the \( \chi^2 \) distribution with \( s \) degrees of freedom. The asymptotic distribution under \( K_n \) also shows that the Pitman efficiency of the test coincides with that of the classical rank test for regression in the situation without the autoregressive error terms.

**Proof** The structure of the test follows the ideas of Gutenbrunner et al. (1993) and Hallin and Jurečková (1999) and in principle it follows the structure of the ordinary rank tests for regression. Notice that we combine the autoregression rank scores with the nonrandom covariates. The asymptotic \( \chi^2 \) distribution of the test criterion follows from the asymptotic linearity of the linear AR rank statistics, and this in turn follows from the asymptotic quadracity of the check function for the quantiles. The nonrandom covariates \( X^*_n \) play the role under alternatives. Remind that the autoregression rank scores \( \hat{a}_n(\alpha) \) for model (3.2) are dual in the sense of linear programming to the \( \alpha \)-autoregression quantile \( \hat{\rho}_n(\alpha) \), which is defined as a solution of

\[
\min \left\{ \sum_{t=1}^n h_\alpha(y_t - y_{t-1}^\top r) : r \in \mathbb{R}_{p+1} \right\},
\]

where \( h_\alpha(u) = |u|(1 - \alpha)I[u < 0] + \alpha I[u > 0] \), \( u \in \mathbb{R}_1 \). First we need the local approximation of the convex function \( \sum_{t=1}^n h_\alpha(y_t - y_{t-1}^\top r) \) by a function quadratic in \( r \), after a suitable standardization, uniform for \( \|r\| \leq C \) and for \( \alpha \in \alpha_n^*, (1 - \alpha_n^*) \). This approach follows the ideas of Hallin and Jurečková (1999), Theorem 3.3 and Gutenbrunner et al. (1993), Theorem 4.1, based on elaborated chaining argument. In turn it leads to the uniform asymptotic linearity of the linear autoregression rank scores statistics, and to an approximation of the test criterion.

More precisely, we obtain that under \( H_0 \), the linear autoregression rank scores statistic admits the representation with the sum of independent summands

\[
Q_n^{-1/2} S_n = n^{-1/2} Q_n^{-1/2} (X^*_n - \hat{X}^*_n)^\top \tilde{b}_n + o_p(1)
\]

as \( n \to \infty \), where \( \tilde{b}_n = (\tilde{b}_{n1}, \ldots, \tilde{b}_{nn})^\top \) and \( \tilde{b}_{nt} = J(F(u_{nt})) \), \( t = 1, \ldots, n \). The representation enables to apply the central limit theorem for independent summands, and it leads to proposition (i).

This same representation holds under the sequence of alternatives \( K_n \), which is contiguous with respect to the sequence of null distributions (with the densities \( \prod_{t=1}^n f(u_t) \)). The noncentral asymptotic \( \chi^2 \) distribution then follows from the LeCam and Hájek theory. \( \Box \)
Table 1  Powers of test ($T_n$ with Wilcoxon scores) for various sample size ($n$), autoregressive parameter ($\varphi_1$), regression parameter ($\beta_1$) and $\alpha = 0.05$.

| $\beta_1$ | n   | $\varphi_1 = -0.9$ | $\varphi_1 = -0.5$ | $\varphi_1 = 0$ | $\varphi_1 = 0.5$ | $\varphi_1 = 0.9$ |
|-----------|-----|-------------------|-------------------|----------------|----------------|----------------|
| -1        | 25  | 0.2541            | 0.2755            | 0.2892         | 0.2995         | 0.3204         |
|           | 50  | 0.3580            | 0.4093            | 0.4737         | 0.5369         | 0.5707         |
|           | 100 | 0.8886            | 0.8660            | 0.8423         | 0.7814         | 0.7329         |
|           | 200 | 0.9736            | 0.9798            | 0.9835         | 0.9826         | 0.9846         |
|           | 2000| 1.0000            | 1.0000            | 1.0000         | 1.0000         | 1.0000         |
| 0         | 25  | 0.0581            | 0.0577            | 0.0559         | 0.0565         | 0.0578         |
|           | 50  | 0.0587            | 0.0521            | 0.0513         | 0.0509         | 0.0475         |
|           | 100 | 0.0494            | 0.0491            | 0.0493         | 0.0544         | 0.0502         |
|           | 200 | 0.0492            | 0.0529            | 0.0498         | 0.0512         | 0.0511         |
|           | 2000| 0.0508            | 0.0510            | 0.0527         | 0.0499         | 0.0499         |
| 0.5       | 25  | 0.1151            | 0.1082            | 0.1109         | 0.1113         | 0.1177         |
|           | 50  | 0.1338            | 0.1435            | 0.1593         | 0.1784         | 0.1942         |
|           | 100 | 0.3760            | 0.3456            | 0.3163         | 0.2813         | 0.2539         |
|           | 200 | 0.5043            | 0.5146            | 0.5306         | 0.5461         | 0.5515         |
|           | 2000| 1.0000            | 1.0000            | 1.0000         | 1.0000         | 1.0000         |
| 1         | 25  | 0.2552            | 0.2673            | 0.2897         | 0.2958         | 0.3208         |
|           | 50  | 0.3634            | 0.4049            | 0.4751         | 0.5329         | 0.5836         |
|           | 100 | 0.8904            | 0.8728            | 0.8424         | 0.7912         | 0.7315         |
|           | 200 | 0.9732            | 0.9825            | 0.9830         | 0.9843         | 0.9865         |
|           | 2000| 1.0000            | 1.0000            | 1.0000         | 1.0000         | 1.0000         |
| 2         | 25  | 0.6527            | 0.7294            | 0.7797         | 0.7800         | 0.7623         |
|           | 50  | 0.8503            | 0.9213            | 0.9661         | 0.9780         | 0.9861         |
|           | 100 | 1.0000            | 1.0000            | 1.0000         | 0.9999         | 0.9987         |
|           | 200 | 1.0000            | 1.0000            | 1.0000         | 1.0000         | 1.0000         |
|           | 2000| 1.0000            | 1.0000            | 1.0000         | 1.0000         | 1.0000         |
| 4         | 25  | 0.9814            | 0.9976            | 0.9998         | 0.9989         | 0.9912         |
|           | 50  | 0.9999            | 1.0000            | 1.0000         | 1.0000         | 1.0000         |
|           | 100 | 1.0000            | 1.0000            | 1.0000         | 1.0000         | 1.0000         |
|           | 200 | 1.0000            | 1.0000            | 1.0000         | 1.0000         | 1.0000         |
|           | 2000| 1.0000            | 1.0000            | 1.0000         | 1.0000         | 1.0000         |

5 Computation and simulation study

In this section the behavior of the proposed test is illustrated and its computation is described. In the following simulation study the power of the test is estimated under various setting of the parameters of the model.
Both the number of regressors \( s \) and the order of autoregressive process \( p \) are set to 1. So the response variable \( y_t \) is generated according to the linear regression model

\[
y_t = \beta_0 + x_t \beta_1 + \varepsilon_t, \quad (5.1)
\]

\[
\varepsilon_t = \varphi_0 + \varphi_1 \varepsilon_{t-1} + u_t, \quad t = 1, 2, \ldots, n. \quad (5.2)
\]

The simulations showed that the proposed test works well even for larger \( s \) or \( p \), the choice \( s = p = 1 \) was done only for clearer description and easier interpretation of the results. \( \beta_0 \) is an intercept, thus \( x_{t0} = 1 \) for all \( t \). The second column of the regression matrix \( (x_{11}, \cdots, x_{n1})^\top \) is generated as a random sample from the uniform distribution on interval \((0, 1)\). The intercept \( \beta_0 \) is set to \(-1\), \( \varphi_0 \) to 0. The regression parameter \( \beta_1 \) is set to various values from \(-4\) to \(4\), the autoregression parameter \( \varphi_1 \) to values from \(-0.9\) to \(0.9\) and the sample size \( n \) from 25 to 2000. The innovations \( u_t \) are generated as \( (i.i.d.) \) from standard normal distribution.

We used the Wilcoxon scores to compute the test statistic \( T_n \) from (3.12) which is now central \( \chi^2 \) with 1 degree of freedom under \( H_0 : \beta_1 = 0 \). The function \( rq() \) from the R package quantreg (Koenker 2019) is used for the computation of the process of autoregression rank scores (3.5). All the calculations were done by the statistical software R (R Core Team 2020). Table 1 shows the empirical powers of the test for each combination of the model parameters. The results are based on 10,000 replications.

Table 1 shows that the power of the test is not much affected by the value of the autoregressive parameter \( \varphi_1 \) under this setting. Clearly the power increases with increasing absolute value of regression parameter \( \beta_1 \) and sample size \( n \) as expected.

However, if the second column of the regression matrix \( (x_{11}, \cdots, x_{n1})^\top \) is taken as an increasing equidistant sequence from 0 to 1 and the autoregressive parameter is positive and large, the ability of the test to detect the linear trend is much lower. The power of the test seems to be negatively influenced by interactions between the linear trend and positive autocorrelation.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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