Exploring the fractional quantum Hall effect with electron tunneling

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In this talk I present a summary of recent work on tunnel junctions of a fractional quantum Hall fluid and an electron reservoir, a Fermi liquid. I consider first the case of a single point contact. This is an exactly solvable problem from which much can be learned. I also discuss in some detail how these solvable junction problems can be used to understand many aspects of the recent electron tunneling experiments into edge states. I also give a detailed picture of the unusual behavior of these junctions in their strong coupling regime. A pedagogical introduction to the theories of edge states is also included.

1 Introduction

The fractional quantum Hall effect (FQHE) has uncovered a number of unique and unexpected quantum behaviors of two-dimensional electrons. Experiments of electron tunneling from normal Fermi liquids into FQH states are unique tools to explore the properties of the excitations of FQH states and to understand directly the behavior of these fluids. Because in the bulk these fluids are incompressible, i.e. all excitations have a finite energy gap, it is very difficult to tunnel into the bulk of these fluids. However, it is possible to tunnel electrons into the edge states of droplets of these fluids.

In this talk I will discuss a theoretical description of tunneling of electrons into FQH edge states. In the first part of the talk I give a quick review of the current understanding of the physics of the bulk FQH state and of the chiral Luttinger picture of the edge states. This part of the talk is pedagogical and I included it at the request of the organizers. Much of this material is quite standard, and here I follow closely the hydrodynamic theory of X. G. Wen[1]. In the second part of the talk I will discuss work that I have done recently, in collaboration with Claudio Chamon and Nancy Sandler, on junctions of FQH edges with normal Fermi liquids. Most of the results that I presented in this part of the talk were originally published in papers we coauthored (references [2] and [3]).

Let us begin by considering the standard setup of the FQHE. That is we will consider a two-dimensional electron gas (2DEG) formed in an Al As-Ga As heterostructure in the presence of a high perpendicular magnetic field $B$. As usual we will measure the electron density in terms of the filling fraction $\nu$ of the Landau level defined as $\nu = N_e/N_\phi$, where $N_e$ is the number of electrons in the 2DEG and $N_\phi$ is the total number of flux quanta piercing the 2DEG. When the FQHE occurs, the 2DEG behaves as an incompressible quantum charged fluid. In the absence of impurities this can only happen when certain charge-flux commensurability conditions are met and the filling fraction takes a set of specific (fractional) values. (Impurities trap the excitations above these ground states in localized states, and that is how the FQHE becomes observable.)

The most prominent FQH states seen in experiment are arranged in the sequence

$$\nu = \frac{p}{2np \pm 1}$$  \hspace{1cm} (1)

where $p = 1, 2, \ldots, \infty$ and $n = 0, 1, 2, \ldots, \infty$, are two integers that label the states. The $\pm$ sign refers to the FQH of electrons and to the particle-hole reversed sequence. Clearly $0 < \nu < 1/(2n)$ for $+$, while $1/(2n) < \nu < 1/(2n - 1)$. For $p = 1$ we find the states in the Laughlin sequence $\nu = 1/(2n + 1)$. The sequence of states with the largest gaps has $n = 1$ and it is formed by the states $\nu = p/(2p + 1) = 1/3, 2/5, 3/7, \ldots$.

Since the bulk states are fully gapped the only gapless excitations of this fluids reside at the edge: they are deformations of the edge. Wen has emphasized that such incompressible
fluids are topological fluids in that, if placed on a surface with non-trivial topology, their Hilbert spaces will be sensitive only to the topology of the surface and not to the local properties of the material.

In 1982 B. Halperin introduced a very simple and appealing picture of the edge states of non-interacting electrons in Landau levels (see figure 2). In this picture, the bulk Landau levels, which have an energy spacing of $\hbar \omega_c$, are bent upwards adiabatically near the edge of the sample by the confining potential, with an electric field strength $E$. For a disk-shaped sample the single particle states are eigenstates of angular momentum (for a rotationally invariant confining potential). If $E_F$ denotes the Fermi energy of the electrons, for a macroscopically large sample there are many states with single particle energies in the vicinity of $E_F$. The level spacing of these states is small as the radius of the electron droplet gets big. In this limit the Hilbert space of these states consists on a set of evenly spaced energy levels with a finite density of states. These states represent electron excitations that move in a direction specified by the sign of the magnetic field $B$ (or by the sign of their charge). The edge states propagate at the drift velocity $v = cE/B$, where $c$ is the speed of light.

Hence, the electrons close to the edge behave like a chiral right moving Fermi system. In this picture, which is a good description of a $\nu = 1$ state, the only effect of the Coulomb interactions among the electrons is a simply a change of the speed at which the electrons at the edge propagate. If the total number of electrons is fixed, the excitations of the edge are charge neutral electron-hole pairs. States labelled by sets of electron-hole pairs can also be regarded as geometric deformations of the edge of the fluid. This picture led Wen to formulate a hydrodynamic theory of the edge states.

1.1 Bulk Incompressible States

The physics of the bulk incompressible states is by now quite well understood. The sequence of Laughlin states $\nu = 1/(2n + 1)$ (with $0 \leq n < \infty$) are (almost exactly) described by the celebrated Laughlin wave functions. The states in the Jain sequences can be described using...
Jain wave functions\(^7\), which are generalizations of the Laughlin wave functions. In the lowest Landau level (LLL) the single particle states (in the circular gauge) are

\[
\psi_r(z) \propto z^r e^{-|z|^2/(4\ell_0^2)}
\]  

where \(\ell_0\) is the magnetic length, \(\ell_0 = \sqrt{\frac{e\hbar}{cB}}\) and \(z\) is the (complex) coordinate of the electron, \(z = x + iy\). Let \(m\) denote the odd integer \(m = 2n + 1\). For the Laughlin states, with filling fraction \(\nu = 1/m\), the Laughlin wave function for \(N\) electrons in a magnetic field with \(N_\phi = mN\) flux quanta is

\[
\Psi_m(z_1, \ldots, z_N) \propto \prod_{i<j=1}^{N} (z_i - z_j)^m e^{-\frac{1}{4\ell_0^2} \sum_{i=1}^{N} |z_i|^2}
\]  

The elementary excitations in the bulk are the Laughlin quasiholes. The wave function for a Laughlin quasihole at \(z_0\) is

\[
\Psi_m(z_0; z_1, \ldots, z_N) = \prod_{i=1}^{N} (z_0 - z_i) \Psi_m(z_1, \ldots, z_N)
\]  

Laughlin argued\(^8\) that, since in this state each electron picks up an extra unit of angular momentum, this state is equivalent to the effect of introducing (adiabatically) an infinitesimally thin solenoid which threads one flux quantum at \(z_0\). He further showed that the charge defect localized at \(z_0\), the quasiholes, has positive charge equal to \(+e/m\). As a consequence of incompressibility there is an extra (negative) charge at the boundary equal to \(-e/m\). Hence,
quasiholes make the edge swell by the right amount to accommodate the extra charge. Furthermore, a semiclassical argument shows that these objects have fractional statistics

$$\Psi_m(z_0, z_1', z_1, \ldots, z_N) = e^{\pm i\pi/m} \Psi_m(z_0', z_0; z_1, \ldots, z_N)$$  \hspace{1cm} (5)$$

Thus, the spectrum of excitations of these bulk incompressible states are quasiholes with fractional charge and fractional statistics. These results can also be (and have been) derived by field-theoretic methods involving Chern-Simons gauge fields.

Alternatively we may also use a hydrodynamic picture of the bulk. The hydrodynamic description is a highly economical way to summarize the universal data of the bulk FQH states: the Hall conductance, the charge and the statistics of the bulk excitations. It can be derived explicitly from the Chern-Simons (mean-field) picture of the FQH states or by a set of phenomenological arguments based on conservation laws. For the sake of simplicity we will follow the latter approach, also introduced by Wen. The fundamental idea is quite simple. The 2DEG is a charged fluid and, as consequence of charge conservation, it can be described in terms of the locally conserved 3-current $j_\mu$

$$j_\mu = (j_0, \vec{j})$$  \hspace{1cm} (6)$$

where $j_0$ is the local charge density $\rho$, $\vec{j}$ is the local charge current, and $\mu = 0, x, y$ (or $\mu = 0, 1, 2$). Local charge conservation means that $j_\mu$ obeys the continuity equation

$$\partial_\mu j^\mu = 0$$  \hspace{1cm} (7)$$
which is to say that \(j_\mu\) is a locally conserved current. Since the 3-divergence of the current vanishes, it must be a curl of a vector field \(a_\mu\) (since space-time is three dimensional). Thus we write

\[
  j_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial^\nu a^\lambda
\]

(the factor of \(\frac{1}{2\pi}\) is introduced for later convenience). Observe that the 3-current is invariant if we make the local transformation (or redefinition) \(a_\mu \rightarrow a_\mu + \partial_\mu \Lambda\), where \(\Lambda\) is an arbitrary, smooth single-valued function. Hence there is a gauge symmetry that is natural in this description.

What do we know about the FQH state? First of all we know that the Hall conductance has the exact value \(\sigma_{xy} = \frac{\nu e^2}{\bar{h}}\). If we denote by \(A_\mu\) an external electromagnetic vector potential that we will use to probe the system (i.e. it is neither part of the uniform magnetic field nor of the confining potential), the effective action must have a local coupling of the form \(j_\mu A^\mu\). Also the effective action must be invariant under gauge transformations of the hydrodynamic gauge field \(a_\mu\). This effective action must be local, be odd under time reversal (and parity) and have as few derivatives as possible. In addition it should contain only the universal data of the FQH state which is a set of dimensionless numbers. Thus, the only terms that can be allowed must be dimension three gauge invariant operators. For \(\nu = 1/m\), the unique choice that satisfies all these constraints is

\[
  S_{\text{eff}} = \frac{m}{4\pi} \int d^3x \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu a^\lambda + \frac{e}{2\pi} \int d^3x \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda
\]

(9)

where we have used eq. 8 to write the \(j_\mu A^\mu\) coupling. It is straightforward to show that the current induced by \(A_\mu\) is

\[
  J^{\text{em}}_\mu = \sigma_{xy} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda
\]

(10)

Hence, we get the correct value of the Hall conductance for \(\nu = 1/m\).

In this picture, the elementary excitations (quasiholes) look like vortices. Since we are interested in the low energy regime, we can assume that the quasiholes are quasistatic (namely, that their gap is very large) and as such they can be pictured by a set of (prescribed) vortex currents \(j^{\mu}_{\text{vortex}}\), minimally coupled to the hydrodynamic field \(a_\mu\),

\[
  S_{\text{vortex}} = \int d^3x \ a^\mu j^{\mu}_{\text{vortex}}
\]

(11)

It is very easy to show that the excitations described by Eq. (11) have charge \(Q = e/m\) and fractional statistics \(\theta = \pi/m\). Hence, the vortices are the Laughlin quasiholes.

1.2 Hydrodynamic Picture of Bulk and Edge (after X. G. Wen)

There is a very appealing picture of the dynamics of the edge states, which was also introduced by Wen. The theory is particularly simple for the case of the Laughlin states (which have filling fraction \(\nu = 1/m\)). In this case the edge is described in terms of a single hydrodynamic field. The picture of the more general FQH state is more complex and depends on details such as possible edge reconstructions. A good general discussion can be found in Wen’s review and in my recent paper with Ana Lopez. Here I will consider only the edges of Laughlin states.

Consider a droplet of the incompressible FQH fluid. For simplicity, we will picture the edge of the fluid in its ground state as a straight line (the horizontal line of figure 4). In this picture, an excitation is a deformation of the edge and in figure 4 is shown as a wavy curve. Let \(h\) denote the local displacement (or height) of the fluid at a point \(x\) along the edge. Because the fluid is incompressible, the change of the local charge density \(\delta \rho\) is proportional to the displacement \(h\), \(\delta \rho \propto h\). The energy of this excitation is the work done against the confining potential to create this displacement is

\[
  \text{energy} \propto \int dx [\delta \rho(x)]^2
\]

(12)
Figure 4: An edge distortion.

which plays the role of a classical Hamiltonian. At the quantum level, the dynamics of the edge is specified by the energy \( \frac{1}{2} x^2 \), and by the commutation relation of the edge degrees of freedom. Since the edge degrees of freedom are chiral (namely, the move at the drift velocity), the local density operators \( \delta \rho(x) \) have to obey appropriate commutation relations to lead to chiral propagation along the edge. The (equal time) commutation relations are

\[
[\delta \rho(x), \delta \rho(x')] = i\pi \text{sign}(x - x')
\]

These commutation relations can be solved by introducing a field \( \phi(x) \), known as the chiral boson field, with the action

\[
S_{\text{edge}} = \frac{1}{4\pi} \int dx dt \partial_x \phi (\partial_t \phi - v \partial_x \phi)
\]

where \( v \) is the effective velocity of the edge waves. The chiral boson is related to the density fluctuations by

\[
\delta \rho(x) = \sqrt{\frac{\nu}{2\pi}} \partial_x \phi
\]

With these definitions it is straightforward to show that the density operators obey the commutation relations of Eq. (13) and that the energy of the excitations is given by Eq. (12). In order to have a completely well defined effective quantum theory we need to specify the Hilbert space on which these operators act. Thus we need to specify what type of excitations are allowed and what is their charge and statistics. Since the bulk FQH state can only support excitations with a charge which must be a multiple of the quasiparticle charge. From Eq. (15) we find that if we add (or remove) an amount of charge \( Q \) to the system, the chiral boson must obey the following boundary condition

\[
Q = \oint_{\text{edge}} dx \delta \rho(x) = \sqrt{\frac{\nu}{2\pi}} [\phi(L) - \phi(0)]
\]

where the integral runs over a (closed) edge of length \( L \). Here we have set the electric charge to unity. Hence, since in the ground state sector \( Q = 0 \), the chiral boson obeys periodic boundary conditions (PBCs),

\[
\phi(x) = \phi(x + L)
\]

Next we define an operator \( \psi_e(x) \) which creates an electron, namely a fermion with charge 1. In terms of the chiral boson, the electron operator is given by the vertex operator

\[
\psi_e(x) \propto e^{-\sqrt{\nu} \phi(x)}
\]
The electron propagator is (with an \( i\epsilon \) regulator implied)
\[
\langle T\psi_e^\dagger(x,t)\psi_e(x',t') \rangle \sim \frac{1}{(z-z')^m}
\] (19)
which has a pole of order \( m \). In Eq. (19) \( T \) stands for time ordering and \( z = x - vt \).

Likewise, the quasiparticle operator \( \psi_{qp}(x) \) is given by
\[
\psi_{qp}(x) \propto e^{i\sqrt{\nu}\phi(x)}
\] (20)
which has the propagator
\[
\langle T\psi_{qp}^\dagger(x,t)\psi_{qp}(x',t') \rangle \sim \frac{1}{(z-z')^{1/m}}
\] (21)
The power \( 1/m \) in this propagator implies that it has a branch cut (instead of a pole) which says that the statistics of the quasiparticle is \( \pi/m \).

We now notice that both the electron and the quasiparticle operator are invariant under the transformation \( \phi \to \phi + 2\pi nR \) where \( n \) is an arbitrary integer and the compactification radius (or Luttinger parameter) \( R = \sqrt{\nu} \). Hence, the addition of \( n \) quasiparticles of charge \( Q = n\nu = n/m \) implies that the chiral boson acquires the twisted boundary condition
\[
\phi(x+L) - \phi(x) = \frac{2\pi}{\sqrt{\nu}}Q = 2\pi nR
\] (22)
However, it is easy to see that the electron operator remains single valued
\[
\psi_e(x+L) = \psi_e(x)
\] (23)
as it is required by the single-valuedness of its wave functions, while the quasiparticle operator instead obeys twisted boundary conditions,
\[
\psi_{qp}(x+L) = e^{i2\pi nu}\psi_{qp}(x)
\] (24)
and are multi-valued, as required by the branch cut in Eq. (21).

The set of conditions that we have discussed in this section constitute a complete definition of the Hilbert space of the edge states. In the second part of this talk I will use this machinery to explore a number of interesting physical questions found in the problem of tunneling of electrons into FQH edge states.

2 Model of a Quantum Point Contact Junction

Let us consider first the problem of a quantum point contact junction between an electron reservoir and the edge of a Laughlin FQH state. Typically the electron reservoir is a three-dimensional electron gas (3DEG). In particular, we will have in mind the situation of the tunnel junctions of the experiments of A. Chang and coworkers\cite{14}, see figure 5. In this experiment, the 3DEG acts as a reservoir of electrons. A 3DEG is well described by Fermi liquid theory. It is believed that the edge that results from this device is “atomically sharp”. If so, presumably there is no edge reconstruction taking place and the electron density of the 2DEG falls off rapidly and monotonically close enough to the edge. However, it is unclear to what degree these assumption do hold and, in fact, in ref. \cite{15} it is argued that there is a substantial amount of charge redistribution close to the edge. In any event, even if this were the case, it is most likely that tunnel of electrons from the 3DEG reservoir to the edge of the FQH state in the 2DEG will take place at special places where the tunneling amplitude is largest. In other words, the
broad edge can be reagrded as a (large) set of weak tunneling centers. In ref. C. Chamon and I developed a theory of tunneling that applies to this regime. Thus, I will concentrate first on the description of a single tunneling center, a quantum point contact (QPC), depicted in figure 6.

For the case of a single QPC the tunneling problem is substantially simpler to describe. The theory of the QPC is highly reminiscent of the physics of quantum impurities in metals. In fact, for a QPC, the problem reduces to a quantum impurity (or rather a tunneling problem) in a one-dimensional strongly correlated (chiral) system. In particular, it is very simple to see that, just as in the case of impurities in metals, of the infinitely many degrees of freedom of the 3DEG, only one channel is able to tunnel. The difference is that while in the case of the quantum impurity in the bulk of a metal, only the S-wave channel scatters off the impurity. In contrast, in the case of the the semi-infinite geometry of the QPC only the electrons in the ℓ = 1 m = 0 channel are able to tunnel. Nevertheless, what matters is that there is one and only one channel needed to describe a QPC. Details of the properties of the states in the channel are absorbed in the definition of the tunnel matrix element Γ.
Thus, in this picture, the 3DEG reduces to a semi-infinite one-dimensional Fermi liquid. In the limit of zero tunneling matrix element, $\Gamma \to 0$, the total charge current at the “end point” at $x = 0$ vanishes exactly. The states of a Fermi liquid are essentially equivalent to those of a free Fermi system, up to a renormalization of the Fermi velocity (screening effects and other renormalizations of the coupling constants of the 3DEG in the bulk play almost no role in the present situation and can be ignored.)

![Diagram](image1.png)

Figure 7: Only one channel of the 3DEG couples at the quantum point contact.

The electron states on the half-line are incoming and outgoing waves, as shown in figure 7, satisfying zero-current boundary conditions at the endpoint. However, exactly as in the problem of magnetic impurities in metals, this system exactly equivalent to a system of free chiral fermions on a full line. Thus, the states of the electrons that participate in tunneling processes are described by a theory of free chiral fermions. This also happens to be the theory of the edge states of a 2DEG at $\nu = 1$. Therefore the QPC is equivalent to a junction between the edge of a $\nu = 1$ QH state and the edge of a $\nu = 1/m$ FQH state.

![Diagram](image2.png)

Figure 8: The QPC between a 3DEG and a $\nu = 1/m$ FQH edge is equivalent to a junction between the same FQH edge and a $\nu = 1$ edge.

We can now use the picture developed in the first part of the talk, section 1, to write down
an effective Lagrangian for the theory of a QPC between a $\nu = 1$ state (an electron reservoir: a 3DEG or any other Fermi liquid) and the edge of a $\nu = 1/m$ Laughlin FQH state. To this end let us introduce to chiral bosons, $\phi_1$ and $\phi_2$ which I will use to represent the Hilbert space of the edges of the FQH state and the Fermi liquid respectively. The total Lagrangian has the form

$$L = L_{\text{FQH}} + L_{\text{reservoir}} + L_{\text{tunnel}}$$

where $L_{\text{FQH}}$ is the Lagrangian of the edge states of a $\nu = 1/m$ FQH fluid,

$$L_{\text{FQH}} = \frac{1}{4\pi} \partial_x \phi_1 (\partial_t \phi_1 - \partial_x \phi_1)$$

and $L_{\text{reservoir}}$ is the Lagrangian for the electrons in the reservoir,

$$L_{\text{reservoir}} = \frac{1}{4\pi} \partial_x \phi_2 (\partial_t \phi_2 - \partial_x \phi_2)$$

where we have set the velocities of both systems to unity $v_1 = v_2 = 1$. This is justified for a single QPC.

Finally, $L_{\text{tunnel}}$ represents the tunneling of electrons,

$$L_{\text{tunnel}} = \Gamma \delta(x) e^{i\omega_0 t} : e^{\sqrt{\nu} \phi_1} :: e^{-i\phi_2} : + h. c. \)$$

Here, $\Gamma$ denotes the tunneling matrix element for electrons, $\omega_0 = eV/h$ is the “Josephson” frequency (for a voltage drop of $V$, which we will set to zero for the most part of our discussion), and $\exp \sqrt{\nu} \phi_1$ and $\exp -i\phi_2$ are the operators that create an electron at the FQH edge and destroy an electron at the reservoir respectively.

In reference [2] it was shown that this problem can be solved exactly. Here I will give a summary of that solution. The first step is to observe that by an unitary (actually orthogonal) transformation of the form

$$ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \)$$

the Lagrangian can be brought to the form

$$L = \frac{1}{4\pi} \partial_x \varphi_1 (\partial_t \varphi_1 - \partial_x \varphi_1) + \frac{1}{4\pi} \partial_x \varphi_2 (\partial_t \varphi_2 - \partial_x \varphi_2) + \Gamma \delta(x) e^{\sqrt{g'} (\varphi_1 - \varphi_2)} + h. c. \)$$

where we have chosen the angle $\theta$ to be the solution of

$$\cos 2\theta = \frac{2\sqrt{\nu}}{1 + \nu} \)$$

and the “effective filling factor” $g'$ is

$$g' = \frac{1}{2} (1 + \nu^{-1}) \)$$

In other terms, we have mapped the junction between a $\nu = 1$ edge and another edge of a FQH state at filling factor $\nu$ to a problem of tunneling of charge $1$ particles between two identical states at filling factor $\nu' = g'$. Hence, the new (rotated) chiral bosons have compactification radius $R = \sqrt{g'}$. For the special case of $\nu = 1/3$ we find $\nu' = g' = 1/2$. Notice that, the statistics of the effective charge $1$ particles that tunnel between the two equivalent edges is $\theta_{\text{eff}} = \frac{\pi}{g'}$, and for the particular case of $\nu = 1/3$ they are bosons.
Finally, we define the orthogonal fields \( \varphi_\pm \)

\[
\varphi_\pm = \frac{1}{\sqrt{2}} (\varphi_1 \pm \varphi_2)
\]

(33)
in terms of which the Lagrangian decouples into two pieces, \( L_+ \) and \( L_- \). In Eq. (33) \( L_+ \) is the Lagrangian of a free chiral boson

\[
L_+ = \frac{1}{4\pi} \partial_x \varphi_+ (\partial_t \varphi_+ - \partial_x \varphi_+)
\]

(34)

with compactification radius \( R_+ = \sqrt{g'/2} \). The field \( \varphi_+ \) represents the fluctuations of the conserved charge current of the combined system.

The Lagrangian \( L_- \) contains the physics of the tunneling processes. It has the form

\[
L_- = \frac{1}{4\pi} \partial_x \varphi_- (\partial_t \varphi_- - \partial_x \varphi_-) + \Gamma \delta(x) e^{i\sqrt{2g} \varphi_-} + \text{h. c.}
\]

(35)
The compactification radius \( R_- \) of the field \( \varphi_- \) is also \( R_- = \sqrt{g'/2} \).

It is useful to introduce now an alternative equivalent form of the Lagrangian of Eq. (35) in which instead of working with the chiral boson field \( \varphi_- \), defined on a full line of length \( L \), one defines a non-chiral boson field \( \varphi_- \) on a half-line, \( 0 \leq x < L/2 \). The chiral field is identified with the right and left moving components \( \varphi_R \) and \( \varphi_L \) as follows

\[
\varphi_-(x) = \begin{cases} 
\varphi_R(x), & \text{for } x > 0 \\
\varphi_L(-x), & \text{for } x < 0 \\
\varphi_R(x) + \varphi_L(x), & \text{for } x \geq 0
\end{cases}
\]

(36)

Hence, we have folded the line and we now have a non-chiral boson field on a half-line. This procedure is very common for solving quantum impurity problems.

The action for the field \( \varphi_- \) on the half-line is

\[
S[\varphi_-] = \frac{1}{8\pi} \int_0^{L/2} \int_{-\infty}^{+\infty} dt \left[ (\frac{\partial \varphi_-}{\partial t})^2 - (\frac{\partial \varphi_-}{\partial x})^2 \right] + \int_{-\infty}^{+\infty} dt \ 2\Gamma \cos \left( \frac{1}{\sqrt{2g}} \varphi_-(0,t) \right)
\]

(37)

where the field \( \varphi_- \) obeys Neumann boundary conditions both at \( x = 0 \) and at \( x \to \infty \). Notice that the compactification radius of \( \varphi_- \) is \( R = \sqrt{2g} \). The Lagrangian of Eq. (37) is known as the boundary sine-Gordon theory. It is an integrable quantum field theory and a lot of useful information is known about it.
Hence, upon folding, the problem that we need to understand is a boundary sine-Gordon field theory with Neumann (N) boundary conditions at both ends of the line, \( \partial_x \phi(x,t)|_{0,L/2} = 0 \). By inspection of the Lagrangian of Eq. (37) we can see that the effect of the tunneling operator is a change in the boundary condition (BC) at \( x = 0 \). Indeed, for \( \Gamma = 0 \) we have a Neumann BC at \( x = 0 \), whereas for \( \Gamma \to \infty \) we have a Dirichlet (D) BC at \( x = 0, \phi_-(0,t) = 0 \) (see top part of fig. 10). Finally we make a duality transformation on this theory. The duality transformation exchanges oscillators with kinks (or solitons). In the folded picture it is equivalent to a replacement of the field \( \phi_- \) by its Cauchy-Riemann dual \( \tilde{\phi}_- \),

\[
\begin{align*}
\partial_x \tilde{\phi}_- &= \partial_t \phi_- \\
\partial_t \tilde{\phi}_- &= -\partial_x \phi_-
\end{align*}
\] (38)

It is straightforward to see that if the field \( \phi_- \) obeys a Neumann (Dirichlet) BC the dual field \( \tilde{\phi}_- \) obeys a Dirichlet (Neumann) BC.

In addition to a change of boundary conditions, under duality the compactification radius \( R_- = \sqrt{2g} \) of the non-chiral field \( \phi_- \) becomes

\[
\tilde{R}_- = \frac{2}{R_-} = \sqrt{\frac{2}{g'}}
\] (39)

Hence, the dual value of \( g \) is \( \tilde{g} = 1/g' \). Likewise, the dual coupling constant becomes

\[
\tilde{\Gamma} \propto \Gamma^{-g'}
\] (40)

and duality maps strong coupling to weak coupling and vice versa. The work of Fendley, Salaur and Warner has confirmed the non-perturbative validity of these duality results.

We are now ready to discuss the properties of the QPC. Here I will follow closely the results of my work with Chamon which, in turn, relied heavily on earlier results of Fendley, Ludwig ans Saleur. In particular, it is possible to compute exactly the differential tunneling conductance \( G_t = \frac{dI}{dV} \) of the QPC, where \( V \) is the voltage drop between the FQH edge and the reservoir, \( V = V_R - V_D \) (see figure 11). The boundary sine-Gordon theory is a free scalar field coupled to the vertex operator \( \mathcal{O} = \exp(\frac{i}{\sqrt{2g}}\phi_-(0,t)) \) at the boundary. The (boundary) scaling dimension of this operator, at the weak coupling fixed point \( \Gamma \to 0 \), is \( d_{\mathcal{O}} = g'^{-1} = (m + 1)/2 \). Thus, for \( \nu = 1/3 \), this is operator has boundary dimension \( 2 > 1 \) and it is an irrelevant operator.
Therefore, the weak coupling fixed point is stable. The effective coupling vanishes as the energy scale $V$ (the voltage) is lowered, and all quantities, such as the conductance, that depend on $\Gamma$ scale to zero in this regime.

Conversely, the strong coupling fixed point is (infrared) unstable. In this regime, $\Gamma \to \infty$ and $\tilde{\Gamma} \to 0$. The (boundary) scaling dimension of this operator in the dual theory (at the fixed point $\tilde{\Gamma} \to 0$), $\tilde{O} = \exp(i\sqrt{2g'}\hat{\varphi}_{-}(0,t))$, is $d_{\tilde{O}} = g' = 2/(m+1) < 1$, and it is always relevant. Thus, for $\nu = 1/3$, the scaling dimension of this operator is $\frac{1}{2}$.

By using the results of Fendley, Ludwig ans Saleur, Chamon and I calculated the differential tunnel conductance $G_t$ and found it to be

$$G_t = \tilde{g} \frac{e^2}{h} \left\{ \frac{\sum_{n=1}^{\infty} c_n(1/\tilde{g}) \left( \frac{V}{2T_K} \right)^{2n(1/\tilde{g}-1)} - 1}{\sum_{n=1}^{\infty} c_n(\tilde{g}) \left( \frac{V}{2T_K} \right)^{2n(\tilde{g}-1)}} \right\} , \quad \frac{V}{2T_K} < e^\delta$$

where $T_K$ is an energy scale set by the tunnel amplitude (the “Kondo” scale)

$$T_K \propto |\Gamma|^{-1/(\tilde{g}^{-1}-1)}$$

and the coefficients $c_n$ are given by

$$c_n(\tilde{g}) = (-1)^{n-1} \frac{\Gamma(n\tilde{g} + 1)}{\Gamma(n+1)} \frac{\Gamma(1/2)}{\Gamma(n(\tilde{g} - 1) + 1/2)} .$$

where $\Gamma(z)$ is the gamma function. The domains of convergence of the dual series are restricted by $\delta = [\tilde{g}\ln \tilde{g} + (1 - \tilde{g})\ln(1 - \tilde{g})]/[2(\tilde{g} - 1)]$, where $\delta = \ln \gamma$.

It interesting to determine the asymptotic behavior of the differential tunnel conductance $G_t$ for $V \gg T_K$ and $V \ll T_K$. In the former (strong coupling) regime we fing that $G_t$ saturates to the value

$$\lim_{V \gg T_K} G_t = \tilde{g} \frac{e^2}{h}$$

(44)
Notice that \( \tilde{g} \) is not equal to the filling factor \( \nu \). Hence, the large voltage differential tunnel conductance of a QPC is not equal to the quantum Hall conductance of the bulk FQH state. In particular, for \( \nu = 1/3 \), \( \tilde{g} = \frac{1}{g'} = \frac{1}{2} \) and \( G_t \to \frac{e^2}{2T_K} \).

Conversely, in the weak coupling regime, we get instead the well known scaling behavior predicted by Wen \cite{22}, and Kane and Fisher \cite{23}

\[
\lim_{V \ll T_K} G_t = \tilde{g} \frac{e^2}{h} c_1 \left( \frac{1}{g'} \right) \left( \frac{V}{2T_K} \right)^{2(\frac{1}{g'} - 1)}
\]

\( 3 \) Tunneling Current, Conductance and Chang’s Experiments

Hence, \( G_t(V \gg T_K) \) is not the Hall conductance of the \( \nu = 1/m \) bulk FQH state the electrons are tunneling into. What is it? It is the bulk Hall conductance of the problem in which charge 1 particles are tunneling between equivalent FQH states with effective filling factor \( \nu_{\text{eff}} = g' \). (This is the problem that was solved by Fendley, Ludwig and Saleur using the Thermodynamic Bethe Ansatz \cite{21}.) These results for a quantum point contact strongly suggest that the large voltage conductance depends on the nature of the contact.

In view of this observation it is instructive to calculate the asymptotic large voltage tunnel current and conductance for several cases (see figure 12).

\( 3.1 \) Uniform Tunneling

Let us begin with case (a) of figure 12, in which the two reservoirs (source and sink) are equilibrated with different parts of the edge of the FQH at filling factor \( \nu = 1/m \). In this case, the tunneling current is equal to the quantized Hall current of the device and the conductance is the bulk quantum Hall conductance. We will show below how this limit is attained.

\( 3.2 \) One Point Contact

This is the same case we considered before. In the strong tunneling region the tunneling current \( I_t \) is found to be proportional to the voltage drop across the junction

\[
I_t = G_t(V_R - V_{in})
\]

In this regime, \( V \gg T_K \), Eq. 41 predicts the behavior

\[
G_t \to \frac{1}{g'} \frac{e^2}{h} \left( \frac{V}{2T_K} \right)^{2(\frac{1}{g'} - 1)}
\]

(47)

where \( g'^{-1} = \frac{1}{2}(1 + \nu^{-1}) \). The reservoir to the left of the \( \nu = 1/m \) FQH state is in equilibrium with the edge. Thus, \( V_{in} = V_L \). But, inside the FQH state the injected (tunnel) current becomes the Hall current (which circulates around the edge). Thus \( V_+ = V_{out} \) is

\[
V_+ - V_- = \frac{I_t}{\nu e^2/h}
\]

(48)

From what we conclude that

\[
V_+ - V_- = \frac{G_t}{\nu e^2/h} (V_R - V_-) = \frac{2}{1+\nu} (V_R - V_L)
\]

(49)

is the Hall voltage, and \( V_+ \) is given by

\[
V_+ = \frac{2}{1+\nu} V_R + \frac{\nu - 1}{\nu + 1} V_L
\]

(50)
These equations show that the single point contact is a simple realization of the dc voltage transformer proposed by Chklovsky and Halperin. These equations show that the single point contact is a simple realization of the dc voltage transformer proposed by Chklovsky and Halperin. 

Hence, the two-terminal conductance of case (b) is

\[ \text{Conductance} \equiv \frac{I_t}{V_R - V_L} = \frac{2\nu}{1 + \nu} \frac{e^2}{h} \]

(51)

For \( \nu = 1/3 \) we find the saturation (maximum) value \( \frac{1}{2} \frac{e^2}{h} \) that we discussed above.
3.3 Two Point Contacts

In case (c) we have a FQH droplet connected by two separate QPC's to two independent reservoirs at voltages $V_R$ and $V_L$ respectively. There is no equilibration going on in this case. The same line of argument used for case (b) now tells us that

$$V_+ - V_- = \frac{\nu}{g'}(V_R - V_L) = \frac{2}{1 + \nu}(V_R - V_-)$$

$$V_- - V_+ = \frac{1}{\nu g'}(V_L - V_+) = \frac{2}{1 + \nu}(V_L - V_-)$$

Hence,

$$V_+ = \frac{1 + \nu}{2\nu}V_R + \frac{\nu - 1}{2\nu}V_L$$

$$V_- = \frac{1 + \nu}{2\nu}V_L + \frac{\nu - 1}{2\nu}V_R$$

Therefore,

$$V_+ - V_- = \frac{1}{\nu}(V_R - V_L)$$

The tunneling current $I_t$ is found to be

$$I_t = \frac{e^2}{h}(V_+ - V_-)$$

which implies that the two-terminal conductance is

$$G = \frac{e^2}{h}$$

instead of the FQH conductance. This result is a simple consequence of charge conservation. It is analogous to the statement that in a wire the two-terminal conductance does not depend on the Luttinger parameter $\gamma$.

3.4 Multiple Contacts, Equilibration and Chang’s Experiments

Finally, let us consider case of figure [13]. In this case we have many QPC’s. However, we will assume that the QPC’s are sufficiently far apart from each other that the reflected amplitudes equilibrate with the source reservoir (the “battery”) and that as a result there is no interference between QPC’s. Notice, however, that the FQH edge always remains coherent. Furthermore we will consider an array of weak tunnel junctions so that each junction remains in the perturbative regime.

In this case, which presumably applies to the actual experimental setup of A. Chang and coworkers [4], the total differential conductance $G_t$ at temperature $T$ and the total voltage drop $V = V_0 - V_R$, is given by

$$G = \nu e^2 \frac{e}{h} \left\{ 1 - \frac{e^{-\frac{V}{\frac{2\pi T}{k}}}}{e^{-\frac{1}{\Gamma^2(g)} \left(1 - e^{-\frac{2\pi T}{k}}\right)^{2(g-1)}} \left(\frac{V}{2\pi T}\right)^{2(g-1)} + 1} \right\}.$$  

(57)
where $g = \tilde{g}^{-1}$. Here we have introduced the effective scale $T^{\text{eff}}_K$

$$
\left( \frac{1}{T^{\text{eff}}_K} \right)^2 = \frac{1}{3\nu} \sum_{n=1}^{N} \left( \frac{1}{T^{(n)}_K} \right)^2
$$

(58)

The expression for the differential tunnel conductance of Eq. 57 is used in reference 2 (see figure 14 below) to fit the experimental data of A. Chang and coworkers.\(^1\)

At large voltages $V \gg T^{\text{eff}}_K$, the differential tunnel conductance approaches the bulk FQH conductance,

$$
\lim_{V \gg T^{\text{eff}}_K} G_t = \nu \frac{e^2}{h}
$$

(59)

which shows that the edge and reservoir are in equilibrium. For $T \ll T_K$, the differential conductance of Eq. 57 exhibits scaling behavior, $G_t \propto V^\alpha$, with an exponent $\alpha = 2g - 1$. For $\nu = 1/3$, we find $\alpha = 3$.

It is also possible to calculate the asymptotic conductance of a set of $N$ junctions in the strong coupling limit provided they are sufficiently far apart that there is no interference. For a set of $N_L$ and $N_R$ junctions between the FQH fluid and a left (L) reservoir, and a right (R) reservoir, the result is\(^2\)

$$
G_{N_L,N_R} = \frac{1 - \left( \frac{\nu-1}{\nu+1} \right)^{N_L}}{1 - \left( \frac{\nu-1}{\nu+1} \right)^{N_L+N_R}} \nu \frac{e^2}{h}
$$

(60)

Notice that these values of $G_t$ depend on both $N_L$ and $N_R$.

4 Strong coupling regime, non-Fermi liquid behavior and Andreev processes

Finally, I want to discuss the physics of the strong coupling regime ($\Gamma \to \infty$) of a single quantum point contact between an Laughlin state and a Fermi liquid. This regime exhibits a number of
very interesting and fascinating behaviors. In the first part of this talk we saw that the behavior of a single point contact is governed by the non-trivial crossover energy scale $T_K$. Thus, we will be interested in the behavior of the junction at $T = 0$ and $V \gg T_K$. In what follows I will assume that for voltages in this range, but smaller than a natural cutoff energy scale $D < \hbar \omega_c$, where $\omega_c$ is the cyclotron frequency. Here I will make the implicit assumption that “more irrelevant” operators can be neglected in the description of the junction. In principle such operators exist and their effects becomes large at strong coupling. However, for any reasonable model of the junction their coupling constants are small. Thus there should be a reasonably wide voltage range over which the effects of these operators can be ignored. Much of the discussion of this section is based on the results of my work with Sandler and Chaamon.

Let us consider the strong coupling regime $\Gamma \to \infty$ of a single junction. Alternatively, using duality, we can think of this regime as $\tilde{\Gamma} \to 0$. we noted before that the strong coupling fixed point is infrared unstable since the tunneling operator (in the dual picture) $\cos(\sqrt{2g} \tilde{\phi}_-) \chi$ has (boundary) scaling dimension $g' = 2/(1 + m) < 1$. (Here I am using the “unfolded” picture and $\tilde{\phi}_- \chi$ is a chiral field.) In what follows I will use the filling factor $\nu$ (instead of the denominator $m$) to avoid notational confusions.

Let us consider the scattering process depicted in figure 15. In the incoming state we have $m$ electrons on the Fermi liquid side (with total charge $me$), and $n$ quasiparticles on the FQH side (with total charge $n e^*$. (Here $m$ is an arbitrary integer, unrelated to the denominator of $\nu$!). Likewise, the outgoing state has $q$ electrons (and charge $qe^*$) and $p$ quasiparticles (and charge $pe^*$). The S-matrix associated with this process is contained in the correlation function

$$\langle :: e^{i\sqrt{g} \phi_1^{\text{out}}} e^{iq\phi_2^{\text{out}}} :: e^{-i\sqrt{g} \phi_1^{\text{in}}} e^{-im\phi_2^{\text{in}}} :: \rangle \quad (61)$$

where I have used the original fields (unrotated and undualized!). In the weak coupling fixed point $\Gamma = 0$ this amplitude factorizes (since the coupling constant is zero). The elementary scattering processes at the weak coupling fixed point are elastic reflections of electrons and quasiparticles at the junction (which behaves here as a hard wall).
The situation is drastically different at the strong coupling fixed point. Not only there is no factorization but, instead of perfect reflection, there are non-trivial selection rules for the allowed scattering processes. An elementary calculation shows that processes such as the simple reflection of a single electron at the junction is forbidden: the amplitude vanishes exactly at $\tilde{\Gamma} = 0$. The same applies to the reflection of a single FQH quasiparticle at the junction. In contrast, as shown in figure 16, the elementary allowed processes are of two types: (a) $e - e^*$ scattering, that is, the scattering of an electron off a quasiparticle, without a charge exchanged across the junction, and (b) the Andreev process in which there are $k + 1$ quasiparticles in the initial state (for $\nu = 1/(2k + 1)$ and $k$ a positive integer) and the final state has one transmitted electron and $k$ quasiholes. This process is analogous to an Andreev reflection at a normal-superconducting (NS) interface. (In the NS problem, the initial state has one electron and the final state has a Cooper pair plus a reflected hole.)

The selection rules exactly at $\tilde{\Gamma} = 0$, for the case of a more general process of figure 15, can be summarized by the matrix equation

$$\begin{pmatrix} q \\ p \end{pmatrix} = M \begin{pmatrix} m \\ n \end{pmatrix}, \quad M = \begin{pmatrix} \frac{1 - \nu}{1 + \nu} & \frac{2\nu}{1 + \nu} \\ \frac{1 + \nu}{1 + \nu} & \frac{1 - \nu}{1 + \nu} \end{pmatrix}. \quad (62)$$

Since $m, n, p, q$ are all integers, then it is easy to see that not all combinations of integers are allowed since the entries of the matrix $M$ are fractional numbers. We can think of this equation as a map from the lattice $(m, n) \in \mathbb{Z}_2$ onto itself (the points $(p, q)$). The allowed processes at $\tilde{\Gamma} = 0$ form the sublattice of dark dots shown in figure 17 for the special case of $\nu = 1/7$.
Figure 17: Lattice of scattering processes at different orders in the dual coupling constant.

A sublattice of allowed processes is spanned by the vectors \( \vec{a}_1 = (1, 1) \) (which represents elastic electron-quasiparticle scattering) and \( \vec{a}_2 = (0, k + 1) \) (the Andreev process).

For small \( \tilde{\Gamma} \), the in and out quantum numbers are now related by

\[
\left( \begin{array}{c} q \\ p \end{array} \right) = M \left( \begin{array}{c} m \\ n \end{array} \right) + lt, \quad t = \frac{2\nu}{1+\nu} - \frac{1+\nu}{1+\nu} \tag{63}
\]

and \( |l| \) is the order of the expansion \( (\tilde{\Gamma}^{|l|}) \). These higher order processes are shown as the shaded dots on the lattice of figure [17]. For example, the process in which there is just one electron in the incoming state and also just one electron in the outgoing state (a reflection), is forbidden at \( \tilde{\Gamma} = 0 \) (which is to say, the one-body S-matrix for electrons is zero), but it is non-zero at order \( \tilde{\Gamma} \). However, an explicit calculation shows that this amplitude has a branch cut (in energy or momentum) instead of a pole as in weak coupling. Hence it is no longer possible to give a particle interpretation to this process since there is no single particle scattering but instead only multiparticle processes: the leading contribution has a broad spectrum and looks like “incoherent” scattering.

Moreover, it is possible to show that the junction has a non-zero finite entropy at the strong coupling fixed point. In fact, this phenomenon is well known to in the context of the effective theory of the junction, the boundary sine-Gordon problem. Boundary sine-Gordon is a (boundary) conformal field theory (CFT). As we noted above, the effect of the tunneling operator is to induce a flow of boundary conditions. This is a very general phenomenon in boundary CFT’s, studied in great detail and generality by Affleck and Ludwig[26]. They conjectured that in all boundary CFT’s there exists a boundary degeneracy (or entropy) which flows (under \( \tilde{\Gamma} \) in our case) much as the central (Virasoro) charge flows in the bulk CFT. In particular Fendley, Saleur and Warner[20] used the Thermodynamic Bethe Anstaz and found that, in the thermodynamic limit, the total entropy flows from the non-extensive but finite value

\[
S(0) = \lim_{T \to 0} \left( -\frac{F}{T} \right) = \frac{1}{2} \ln(k + 1) \tag{64}
\]
at \( \tilde{\Gamma} = 0 \), to the value \( S(0) = 0 \) at \( \Gamma = 0 \). Here \( F \) is the total Free energy of the boundary sine-Gordon theory. In fact, for the special case of a junction of a \( \nu = 1/3 \) FQH fluid and a Fermi liquid, \( \nu = 1 \), we find \( S(0) = -\frac{1}{\pi} \ln 2 \), which is the boundary entropy of the two-channel Kondo problem!

Behaviors of these sort, both in the \( S \)-matrix and in the entropy, have been found previously in the multi-channel Kondo problem, which has a “non-Fermi liquid” fixed point. Clearly the physics of the strongly coupled junction is very similar.

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