1 Introduction

The aim of this paper is to apply algebro-geometric Szegö kernels to the asymptotic study of a class of trace formulae in equivariant geometric quantization and algebraic geometry.

Let \((M, J)\) be a connected complex projective manifold, of complex dimension \(d\), and let \(A\) be an ample line bundle on it. Let, in addition, \(G\) be a compact and connected \(g\)-dimensional Lie group acting holomorphically on \(M\), in such a way that the action can be linearized to \(A\). For every \(k \in \mathbb{N}\), the spaces of global holomorphic sections \(H^0(M, A^\otimes k)\) are linear representations of \(G\), and therefore may be equivariantly decomposed over its irreducible representations.

More precisely, let \(g\) be the Lie algebra of \(G\), and let \(\Lambda \subseteq g^*\) be a set of weights parametrizing the family of all finite-dimensional irreducible representations of \(G\). For every \(\varpi \in \Lambda\), denote by \(V_{\varpi}\) the corresponding \(G\)-module. Given \(\varpi \in \Lambda\) and a linear representation of \(G\) on a finite dimensional vector space \(W\), we shall denote by \(W_{\varpi} \subseteq W\) the \(\varpi\)-isotype of \(W\), that is, the maximal invariant subspace of \(W\) equivariantly isomorphic to a direct sum of copies of \(V_{\varpi}\). For every \(k \in \mathbb{N}\), we then have equivariant direct sum decompositions

\[
H^0(M, A^\otimes k) = \bigoplus_{\varpi \in \Lambda} H^0(M, A^\otimes k)_{\varpi}.
\]

We can find a \(G\)-invariant Hermitian metric \(h\) on \(A\) such that the unique compatible connection has curvature \(\Theta = -2i\omega\), where \(\omega\) is a Kähler form on \(M\). The choice of \(\omega\) determines a volume form on \(M\), with respect to which \(\text{vol}(M) = \frac{\pi^d}{d!} \int_M c_1(A)^d\).

*Address: Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano Bicocca, Via R. Cozzi 53, 20125 Milano, Italy; e-mail: roberto.paoletti@unimib.it
By way of motivation, suppose that a reductive connected algebraic group \( \tilde{G} \) acts on a complex projective manifold \( M \); then for any line bundle \( A \) on \( M \) there exists a positive integer \( l \) such that the action linearizes to \( A^{\otimes l} \) [KFM]. [D]. Let \( G \subseteq \tilde{G} \) be a maximal compact subgroup. If \( A \) is ample and \( h' \) is an Hermitian metric on it whose compatible connection has Kähler normalized curvature \( \omega' \), we may replace \( h' \) and \( \omega' \) by their \( G \)-averages \( h \) and \( \omega \). The action of \( G \) on \( M \) is then holomorphic and Hamiltonian with respect to \( \omega \).

These choices induce natural \( G \)-invariant Hermitian structures on every space of global sections of \( A^{\otimes k} \), and (1) is a unitary equivariant isomorphism. For \( k \in \mathbb{N} \) and \( \varpi \in \Lambda \), we shall denote by \( P_{\varpi,k} : H^0(M, A^{\otimes k}) \to H^0(M, A^{\otimes k})_{\varpi} \) the orthogonal projector. With abuse of language, if \( C^\infty(M, A^{\otimes k}) \) is the space of all smooth global sections of \( A^{\otimes k} \), we shall also denote by \( P_{\varpi,k} \) the orthogonal projector \( C^\infty(M, A^{\otimes k}) \to H^0(M, A^{\otimes k})_{\varpi} \).

Let, furthermore, \( \gamma : M \to M \) be a biholomorphism, also admitting a linearization to a unitary automorphism \( \tilde{\gamma} \) of \((A, h)\); in particular, \( \gamma \) is a symplectomorphism of \((M, \omega)\). We shall also denote by \( \tilde{\gamma} \) the induced linearization on \( A^{\otimes k} \), for every \( k \in \mathbb{Z} \). For every \( k \in \mathbb{N} \), let \( \tilde{\gamma}_k : H^0(M, A^{\otimes k}) \to H^0(M, A^{\otimes k}), s \mapsto \tilde{\gamma} \circ s \circ \gamma^{-1} \) be the unitary automorphism induced by \( \tilde{\gamma} \).

We shall make the additional assumption that \( \tilde{\gamma} \) commutes with the action of \( G \) on \( A \); this is equivalent to the condition that \( \Phi \) be \( \gamma \)-invariant, and therefore \( \gamma(\Phi^{-1}(0)) = \Phi^{-1}(0) \). Under these circumstances, \( \tilde{\gamma}_k \) preserves the decomposition (1); in other words, for every \( \varpi \in \Lambda \) we have

\[
\tilde{\gamma}_k(H^0(M, A^{\otimes k})_{\varpi}) = H^0(M, A^{\otimes k})_{\varpi}.
\]

As a simple example, consider the unitary action of \( S^1 \) on \( \mathbb{C}^{d+1} \) given by \( t \cdot (z_0, \ldots, z_d) = (t z_0, t^{-1} z_1, \ldots, t^{-1} z_d) \), and let \( \Gamma \in U(d+1) \) be any unitary diagonal matrix. This induces an holomorphic Hamiltonian circle action on \( \mathbb{P}^d \), commuting with the holomorphic symplectomorphism \( \gamma \) of \( \mathbb{P}^d \) induced by \( \Gamma \); both have tautological linearizations to the hyperplane line bundle.

More generally, assume given a holomorphic Hamiltonian action of a Lie group \( H \) on \((M, J, \omega)\), linearizing to \( A \); for \( h \in H \), let \( \psi_h : M \to M \) and \( \tilde{\psi}_h : A \to A \) be the associated maps. If \( G \subseteq H \) is a compact and connected subgroup, and \( h \) centralizes \( G \), then the restriction of the action to \( G \) and \( \gamma =: \psi_h, \tilde{\gamma} =: \tilde{\psi}_h \) satisfy the previous hypothesis.

Let us now describe the main object of study of this paper:
Definition 1.1. For $f \in C^\infty(M)$, let $M_f : C^\infty(M, A^\otimes k) \to C^\infty(M, A^\otimes k)$ be the multiplication operator $s \mapsto f s$. Let $\gamma : M \to M$, $\tilde{\gamma} : A \to A$ be as above.

1. For every $\varpi \in \Lambda$ and $k \in \mathbb{N}$, we introduce the equivariant Toeplitz operator

$$T_f^{(\varpi,k)} =: P_{\varpi,k} \circ M_f \circ P_{\varpi,k} : H^0(M, A^\otimes k) \to H^0(M, A^\otimes k),$$

which we shall view as an endomorphism of $H^0(M, A^\otimes k)_\varpi$.

2. More generally, we may consider for every $\varpi \in \Lambda$ and $k \in \mathbb{N}$ the compositions

$$\Psi_{\varpi,k} = \Psi_{\varpi,k}(\gamma, f) =: \tilde{\gamma}_k \circ T_f^{(\varpi,k)} : H^0(M, A^\otimes k)_\varpi \to H^0(M, A^\otimes k)_\varpi.$$

We shall show that, under familiar assumptions in the theory of symplectic reductions, the trace of $\Psi_{\varpi,k}$ admits an asymptotic expansion as $k \to +\infty$, and explicitly describe its leading term.

Of course, in the action free case and with $f = 1$ the Lefschetz fixed point formula of [AS] gives an exact expression for $\text{trace}(\tilde{\gamma}_k)$, but even in this case it may be of some interest that Szegö kernels, and more precisely their scaling limits ([BdMS], [BSZ], [SZ]), provide a relatively elementary approach to the leading asymptotics. On the other hand, the action free case with $\tilde{\gamma} = \text{id}$ has been studied in [BdMG], and more recently, under wider hypothesis on the symplectic structure, in [B]. If $f = 1$ and $\tilde{\gamma}$ is the identity, then the trace of $\Psi_{\varpi,k}$ computes $\dim H^0(M, A^\otimes k)_\varpi$, a natural object of study in the setting of symplectic reduction and geometric quantization since the landmarks [GS1], [GS2]; in fact, exact formulae for these dimensions are provided, for each given $k$, by the principle $[Q,R] = 0$ [M]. A rather elementary approach to the asymptotics for $k \to +\infty$ has been given in [P1], based on microlocal techniques.

In the general case, where the linearization and the equivariant Toeplitz operator are considered on the same footing, the leading coefficient in the asymptotic expansion for $\text{trace}(\Psi_{\varpi,k})$ is the product of the leading coefficient of the Lefschetz fixed point formula on the symplectic reduction of $(M, \omega)$ and a certain $G$-average of $f$, with a weighting that depends on $\varpi$ and $\gamma$.

In order to state the result more precisely, we need to describe some invariants associated to $\gamma$, $\tilde{\gamma}$ and the linearization of the $G$-action.

Given the linearization, the action of $G$ on $(M, 2\omega)$ is Hamiltonian; let $\Phi : M \to g^*$ be the corresponding moment map. We shall assume that $0 \in g^*$ is a regular value of $\Phi$, and that $G$ acts freely on $\Phi^{-1}(0)$; with
minor complications, the arguments below apply however to the case where
the (necessarily finite) stabilisers of the points in \( \Phi^{-1}(0) \) all have the same
cardinality.

Let

\[ p : \Phi^{-1}(0) \longrightarrow M_0 =: \Phi^{-1}(0)/G \]  

be the projection onto the symplectic reduction \( M_0 \) of \( M \). Thus \( p \) is a principal \( G \)-bundle, and the Kähler structure \((\omega, J)\) of \( M \) descends in a natural
manner to the quotient Kähler structure \((\omega_0, J_0)\) of \( M_0 \).

Let us list the definitions that will build up the statement of Theorem 1.

**Definition 1.2.** Associated to \( \gamma \) and the \( G \)-action we have the following
objects.

1. Since it commutes with the \( G \)-action, \( \gamma \) descends to a holomorphic
symplectomorphism of \((M_0, \omega_0, J_0)\), that we shall denote by \( \gamma_0 : M_0 \rightarrow M_0 \).

2. Let \( F_1, \ldots, F_\ell \subseteq M_0 \) be the connected components of its fixed locus,
\( \text{Fix}(\gamma_0) \): for every \( l = 1, \ldots, \ell \), \( F_l \) is a complex submanifold of \( M_0 \) of,
say, complex dimension \( d_l \). Let \( c_l =: (d - g) - d_l \) denote its complex
codimension.

3. If \( l = 1, \ldots, \ell \) and \( r \in F_l \), let \( N_{l,r} \) be the normal space to \( F_l \subseteq M_0 \)
at \( r \), and let \( \gamma_r : N_{l,r} \rightarrow N_{l,r} \) be the unitary map induced by the holomorphic
differential of \( \gamma_0 \) at \( r \). Then
\[
\text{id}_{N_{l,r}} - \gamma_r^{-1} \text{ is non-singular, and}
\]
\[
c_l(\gamma) =: \det \left( \text{id}_{N_{l,r}} - \gamma_r^{-1} \right) \tag{3}
\]
is constant on \( F_l \).

4. If \( m \in p^{-1}(F_l) \) for some \( l \), since \( G \) acts freely on \( \Phi^{-1}(0) \) there exists a
unique \( g_m \in G \) such that \( \gamma(m) = \mu g_m(m) \), where \( \mu : G \times M \rightarrow M \)
is the given action on \( M \); the conjugacy class of \( g_m \) only depends on \( l \).

5. If \( l = 1, \ldots, \ell \), \( \chi_\omega(g_m) \) does not depend on the choice of \( m \in p^{-1}(F_l) \);
we shall set \( \chi_\omega(F_l) =: \chi_\omega(g_m), \ (m \in p^{-1}(F_l)) \).

To see that the conjugacy class of \( g_m \) only depends on \( l \), set \( \tilde{F}_l =: p^{-1}(F_l) \)
and suppose \( m, n \in \tilde{F}_l \). If \( \pi(m) = \pi(n) \), there exists \( h \in G \) such that
\( n = \mu h(m) \). Therefore, \( \mu_{g_m g_n^{-1}} (n) = \mu h(\gamma(m)) = \gamma(\mu h(m)) = \gamma(n) \), whence
\( g_{\mu h(m)} = h g_m h^{-1}, \forall h \in G, m \in p^{-1}(F_l) \). If \( \pi(m) \neq \pi(n) \), let \( \eta : [0, 1] \rightarrow F_l \)
be a smooth path such that \( \eta(0) = \pi(m), \eta(1) = \pi(n) \). The principal \( G \)-
bundle \( p \) has a natural connection; let \( \eta^\sharp : [0, 1] \rightarrow \tilde{F}_l \) be the unique horizontal
lift of \( \eta \) such that \( \eta^*(0) = m \). Then \( \mu_g \circ \eta^* \), \( \gamma \circ \eta^* : [0, 1] \to \tilde{F}_l \) are both horizontal lifts of \( \eta \), and satisfy \( \mu_g \circ \eta^*(0) = \gamma(m) = \gamma \circ \eta^*(0) \); hence they are equal. Therefore, \( \mu_g \circ \eta^*(1) = \gamma \circ \eta^*(1) \), and so \( g_{\eta^*(1)} = g_m \). Since \( \eta^*(1) \) is in the same orbit as \( n \), we conclude by the previous considerations that \( g_m \) is conjugate to \( g_n \).

**Definition 1.3.** Given \( f \in C^\infty(M) \), we define \( \bar{f} \in C^\infty(M_0) \) as the \( G \)-average of \( f \), viewed as a smooth function on \( M_0 \). In other words,

\[
\bar{f}(m_0) =: \int_G f\left( \mu_g(m) \right) d\nu(g) \quad (m_0 \in M_0),
\]

where \( m \in p^{-1}(m_0) \subseteq \Phi^{-1}(0) \).

**Definition 1.4.** The ample line bundle \( A \) descends to an ample line bundle \( A_0 \) on \( M_0 \), and the linearization \( \tilde{\gamma} \) descends to a linearization \( \tilde{\gamma}_0 \) on \( A_0 \). If \( l = 1, \ldots, \ell \) there exists a unique \( h_l \in S^1 \) such that \( \tilde{\gamma}_0(r) : A_0(r) \to A_0(r) \) is multiplication by \( h_l \) for every \( r \in F_l \), where \( A_0(r) \) is the fiber of \( A_0 \) at \( r \).

With the above notation, we then have:

**Theorem 1.** Suppose \( \omega \in \Lambda \), \( f \in C^\infty(M) \) and \( \gamma : M \to M \), with unitary linearization \( \tilde{\gamma} : A \to A \), are given as above, so that \( \tilde{\gamma} \) commutes with the action of \( G \) on \( A \). Let \( \Psi_{\omega,k} \) be as in Definition 1.1. Then:

i): If \( \Phi^{-1}(0) = \emptyset \), then \( \Psi_{\omega,k} = 0 \) for \( k \gg 0 \).

ii): If \( \Phi^{-1}(0) \neq \emptyset \), assume that \( 0 \in g^* \) is a regular value of \( \Phi \) and that \( G \) acts freely on \( \Phi^{-1}(0) \). Then as \( k \to +\infty \) there is an asymptotic expansion

\[
\text{trace}(\Psi_{\omega,k}) \sim \dim(V_{\omega}) \sum_{l=1}^\ell \left( \frac{k}{\pi} \right)^d_l \frac{h^k_l}{c_l(\gamma)} \chi_{\omega}(F_l) \int_{F_l} \bar{f} \cdot \left( 1 + \sum_{a \geq 1} k^{-a/2} c_{\omega(a)} \right).
\]

As a test case, suppose that \( f = 1 \) and \( \tilde{\gamma} \) is the identity map. Then the asymptotic expansion of the Theorem reduces to:

\[
\dim H^0(M, A^k)_{\omega} \sim \dim(V_{\omega})^2 \left( \frac{k}{\pi} \right)^{d-\gamma} \cdot \left( 1 + \sum_{a \geq 1} k^{-a/2} c_{\omega(a)} \right).
\]

Up to a different normalization convention for the volume form, this agrees with Theorem 2 of [PT] (where only powers of \( k^{-1} \) appear).
The proof is largely based on the microlocal theory of the Szegö kernel in [BdMS], and on its developments in [Z2], [BSZ], [SZ].

To motivate the role of asymptotic expansions for Szegö kernels, let us dwell again on the action free case, first with $\tilde{\gamma} = \text{id}$. Thus, given $f \in C^\infty(M)$, we are considering the asymptotics of the trace of the level-$k$ Toeplitz operator $T_f^{(k)} =: P_k \circ M_f \circ P_k : H^0 (M, A^{\otimes k}) \to H^0 (M, A^{\otimes k})$; here $P_k$ is the level-$k$ Szegö projector, that is, the full orthogonal projector onto $H^0 (M, A^{\otimes k})$. An asymptotic expansion in this case has been proved in [BdMG], §13. On the other hand, inserting the diagonal asymptotic expansion for the level-$k$ Szegö kernel of [Z2] in the Schwartz kernel of $T_f^{(k)}$ leads to

$$\text{trace} \left( T_f^{(k)} \right) \sim \left( \frac{k}{\pi} \right)^d \int_M f \cdot \text{vol}_M + \text{L.O.T.}..$$

(4)

Trying to adopt the same approach to the case of the Lefschetz fixed point formula of [AS], thus now with $f = 1$, one is led however to consider the asymptotics of the Szegö kernel over off-diagonal points in $M \times M$ of the form $(m, \gamma(m))$, and this motivates the appearance of scaling limits around the fixed locus of $\gamma$ into the picture. In the general equivariant case, the asymptotic concentration of the equivariant Szegö kernels determines a further localization around the zero locus of the moment map.

It is in order to conclude this introduction by emphasizing that there is a broader scope for the methods and techniques appearing in this paper. Firstly, although our present focus is on the simpler holomorphic context, the following analysis could be generalized to the symplectic almost complex category, in view of the microlocal description in [SZ] of the almost complex analogues of Szegö kernels. On the other hand, the study of compositions akin to those in Definition 1.1 is particularly relevant to the theory of Toeplitz quantization [Z1], where one studies quantum maps associated to contact transformations, whose underlying symplectic maps are generally not holomorphic. In fact, all the ingredients of this article are already in place in the unitarization process of [Z1], where the non-unitarity of quantum maps induced by a contactomorphism is corrected by composing with appropriate Toeplitz operators, and in some cases certain trace formulae of this type are given.

Acknowledgments. I am very indebted to the referee for several stimulating comments on possible developments and for suggesting various improvements to the exposition.
2 Proof of Theorem 1

The first statement of the Theorem is an immediate consequence of the theory of [GS1]. More precisely, since \( \Phi(M) \subseteq g^* \) is a compact subset, if \( \Phi^{-1}(0) = \emptyset \) then \( \varpi \not\in k \Phi(M) \), hence \( H^0(M, A^{\otimes k})_{\varpi} = 0 \), for \( k \gg 0 \).

In order to prove the second statement, we shall lift the problem to the CR structure of the associated circle bundle. Let \( X \subseteq A^* \) be the unit circle bundle, with \( S^1 \)-action \( r : S^1 \times X \to X \). The connection 1-form \( \alpha \in \Omega^1(X) \) defines a contact structure and a volume form on \( X \). In terms of the latter, we shall tacitly identify (generalized) densities, half-densities and functions on \( X \) and \( X \times X \).

With this in mind, there is a natural Hermitian structure on \( L^2(X) \), and for every \( k \in \mathbb{N} \) there are standard unitary isomorphisms \( C^\infty(M, A^{\otimes k}) \cong C^\infty(X)_k \); the latter is the space of all smooth functions on \( X \) such that \( f(rt(x)) = t^k f(x), \forall t \in S^1, x \in X \). By restriction, we obtain the unitary isomorphisms \( H^0(M, A^{\otimes k}) \cong H(X)_k \), where \( H(X)_k \subseteq C^\infty(X)_k \) is the \( k \)-th isotypical component of the Hardy space of \( X \). Given \( s \in C^\infty(M, A^{\otimes k}) \), we shall denote its image by \( \hat{s} \in C^\infty(X)_k \).

Furthermore, given our assumptions, the action \( \mu : G \times M \to M \) naturally lifts to an action of \( G \) on \( X \), \( \mu_X : G \times X \to X \), as a group of contactomorphisms; to lighten notation, we shall often write \( \mu \) for \( \mu_X \) where no misunderstanding seems likely.

Similarly, \( \tilde{\gamma} : A \to A \) induces a contactomorphism \( \gamma_X : X \to X \), and \( \gamma_X \circ \mu_g = \mu_g \circ \gamma_X, \forall g \in G \).

Thus \( G \) acts on \( H(X)_k \) by pull-back, \( g : f \mapsto f \circ \mu_g^{-1} \), and the isomorphisms \( H^0(M, A^{\otimes k}) \cong H(X)_k \) are equivariant for this action. In terms of the equivariant unitary isomorphism \( s \mapsto \hat{s} \), we may rewrite (1) as

\[
H(X)_k = \bigoplus_{\varpi \in \Lambda} H(X)_{\varpi,k}. \tag{5}
\]

Similarly, for all \( s \in C^\infty(M, A^{\otimes k}) \) we have \( \hat{\gamma_k(s)} = \hat{s} \circ \gamma_X^{-1} \in C^\infty(X)_k \).

If \( \left\{ s_j^{(\varpi,k)} \right\}_j \) is any orthonormal basis of \( H_{\varpi,k}(X) \), then

\[
\Pi_{\varpi,k}(x,y) = \sum_j s_j^{(\varpi,k)}(x) \overline{s_j^{(\varpi,k)}(y)} \quad (x,y \in X)
\]

is the \((\varpi,k)\)-equivariant Szegő kernel, that is, the distributional kernel of the orthogonal projector onto the subspace \( H(X)_{\varpi,k} \subseteq H(X) \); in particular, it does not depend on the choice of \( \left\{ s_j^{(\varpi,k)} \right\}_j \). Thus, since \( \left\{ s_j^{(\varpi,k)} \circ \gamma_X \right\}_j \) is
also an orthonormal basis of $H_{\omega,k}(X)$, we have
\[
\Pi_{\omega,k}(\gamma_X(x), \gamma_X(y)) = \Pi_{\omega,k}(x, y), \quad \forall x, y \in X.
\]

Let $\text{dens}_{X \times X}$ and $\text{dens}_X$ denote the volume densities of $X \times X$ and $X$, respectively. Then
\[
\begin{align*}
\text{trace}(\Psi_{\omega,k}) &= \int_{X \times X} \Pi_{\omega,k}(\gamma_X^{-1}(x), y) f(y) \Pi_{\omega,k}(y, x) \text{dens}_{X \times Y}(x, y), \\
&= \int_{X \times X} \Pi_{\omega,k}(x, \gamma_X(y)) f(y) \Pi_{\omega,k}(y, x) \text{dens}_{X \times X}(x, y) \\
&= \int_X \Pi_{\omega,k}(\gamma_X^{-1}(y), y) f(y) \text{dens}_X(y).
\end{align*}
\]

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\begin{align*}
\text{trace}(\Psi_{\omega,k}) &= \int_{X \times X} \Pi_{\omega,k}(\gamma_X^{-1}(x), y) f(y) \Pi_{\omega,k}(y, x) \text{dens}_{X \times Y}(x, y), \\
&= \int_{X \times X} \Pi_{\omega,k}(x, \gamma_X(y)) f(y) \Pi_{\omega,k}(y, x) \text{dens}_{X \times X}(x, y) \\
&= \int_X \Pi_{\omega,k}(\gamma_X^{-1}(y), y) f(y) \text{dens}_X(y).
\end{align*}
\]

The strategy to determine the asymptotics of $\text{trace}(\Psi_{\omega,k})$ is then to insert in (6) the asymptotic expansion for the scaling limits of $\Pi_{\omega,k}$ determined in [P2]. To this end, we shall apply a number of reductions, at each step disregarding a contribution to the integral in (6) which is $O(k^{-\infty})$.

Let us define $R(\Phi) : = \{(m, n) \in M \times M : \Phi(m) = 0, n \in G \cdot m\}$, $I(\Phi) : = (\pi \times \pi)^{-1}(R(\Phi))$. In other words,
\[
I(\Phi) = \{(x, y) \in X \times X : \Phi \circ \pi(x) = 0, y \in (G \times S^1) \cdot x\}.
\]

**Lemma 2.1.** Uniformly on compact subsets of $X \times X \setminus I(\Phi)$, as $k \to +\infty$ we have $\Pi_{\omega,k}(x, y) = O(k^{-\infty})$.

Lemma 2.1 whose proof will be postponed, implies the following: if we fix an arbitrarily small $G$-invariant tubular neighborhood $V \subseteq M$ of $\Phi^{-1}(0)$, perhaps after disregarding a rapidly decaying contribution we may replace the integration over $X$ in (6) by an integration over $\pi^{-1}(V)$. We shall express this by writing
\[
\text{trace}(\Psi_{\omega,k}) \sim \int_{\pi^{-1}(V)} \Pi_{\omega,k}(\gamma_X^{-1}(y), y) f(y) \text{dens}_X(y).
\]

In particular, we may assume without loss that $G$ acts freely on $\overline{V}$.

As a further reduction, let us define
\[
S_k := \{m \in V : \text{dist}_M(G \cdot m, G \cdot \gamma(m)) < 2k^{-2/5}\},
\]
\[
S_k' := \{m \in V : \text{dist}_M(G \cdot m, G \cdot \gamma(m)) < k^{-2/5}\}.
\]

Let $\{\sigma_k, \sigma_k'\}$ be a partition of unity on $V$ subordinate to the open cover $\{S_k, S_k'\}$. We may assume $\sigma_k(m) = \sigma(k^{2/5} \text{dist}_M(G \cdot m, G \cdot \gamma(m)))$, for a fixed smooth function $\sigma : \mathbb{R} \to \mathbb{R}$. We shall write $\sigma_k$ for $\sigma_k \circ \pi$, $\sigma_k'$ for $\sigma_k' \circ \pi$. Inserting the equality $\sigma_k + \sigma_k' = 1$ in (7), the integral splits in two summands. One of these gives a negligible contribution to the asymptotics as $k \to +\infty$:
Lemma 2.2. As $k \to +\infty$, we have

\[
\int_{\pi^{-1}(S_k')} \sigma'_k(y) \Pi_{\varpi,k} (\gamma^{-1}_X(y), y) \ f(y) \text{dens}_X(y) = O \left( k^{-\infty} \right).
\]

Proof. It suffices to show that $|\Pi_{\varpi,k} (\gamma^{-1}_X(y), y)| = O (k^{-\infty})$ uniformly for $y \in \pi^{-1}(S_k')$. To this end, recall that

\[
\Pi_{\varpi,k} (\gamma^{-1}_X(y), y) = \dim(V_{\varpi}) \int_G \chi_{\varpi}(g) \Pi_k (\mu_g \circ \gamma^{-1}_X(y), y) \ d\nu(g).
\]

To simplify notation, let us write $\text{dist}_M$ for the composition $\text{dist}_M \circ (\pi \times \pi) : X \times X \to \mathbb{R}$. If $y \in \pi^{-1}(S_k')$, then $\text{dist}_M (\mu_g \circ \gamma^{-1}_X(y), y) > k^{-2/5}$ for all $g \in G$. By the off-diagonal estimates on the Szegö kernel of $[C]$, we conclude that there exist constants $C, D > 0$ such that $|\Pi_k (\mu_g \circ \gamma^{-1}_X(y), y)| < C e^{-D k^{1/10}}$, for all $g \in G, k \in \mathbb{N}$ and $y \in \pi^{-1}(S_k')$. The statement follows in view of (8).

Q.E.D.

Given Lemma 2.2,

\[
\text{trace} (\Psi_{\varpi,k}) \sim \int_{\pi^{-1}(S_k')} \sigma_k(y) \Pi_{\varpi,k} (\gamma^{-1}_X(y), y) \ f(y) \text{dens}_X(y).
\]

We next concentrate the integral on progressively shrinking neighborhoods of $\Phi^{-1}(0)$. More precisely, we set

\[
T_k =: \left\{ m \in S_k : \text{dist}_M (m, \Phi^{-1}(0)) < 2k^{-1/3} \right\},
\]

\[
T'_k =: \left\{ m \in S_k : \text{dist}_M (m, \Phi^{-1}(0)) > k^{-1/3} \right\},
\]

and let $\{\tau_k, \tau'_k\}$ be a partition of unity on $S_k$ subordinate to the open cover $\{T_k, T'_k\}$. We may assume that $\tau_k(m) = \tau \left( \sqrt[k]{\text{dist}_M (m, \Phi^{-1}(0))} \right)$, for a fixed smooth function $\tau : \mathbb{R} \to \mathbb{R}$.

We shall write $\tau_k$ for $\tau_k \circ \pi$ and similarly for $\tau'_k$. Again, insertion of the equality $\tau_k + \tau'_k = 1$ in (9) splits the integral in the sum of two terms, one of which is rapidly decaying as $k \to +\infty$:

**Proposition 2.1.** $\Pi_{\varpi,k} (\gamma^{-1}_X(y), y) = O \left( k^{-\infty} \right)$ uniformly for $y \in \pi^{-1}(T'_k)$.

Before commencing the proof of Proposition 2.1, let us notice that it implies:

**Corollary 2.1.** As $k \to +\infty$, we have

\[
\int_{\pi^{-1}(T'_k)} (\sigma_k \cdot \tau'_k)(y) \Pi_{\varpi,k} (\gamma^{-1}_X(y), y) \ f(y) \text{dens}_X(y) = O \left( k^{-\infty} \right).
\]
Corollary 2.2. Let $C > 0$ be as in the statement of Lemma 2.3. Then
\[
\Pi_k \left( g^{-1}(y), y \right) = O \left( k^{-\infty} \right) ,
\]
uniformly for $y \in \pi^{-1}(S_k)$ and $g \in G$ satisfying $\dist_G(g, e) > C k^{-2/5}$.

Let us now set
\[
G_k =: \left\{ g \in G : \dist_G(g, e) < 2 C k^{-2/5} \right\} , \quad G'_k =: \left\{ g \in G : \dist_G(g, e) > C k^{-2/5} \right\} ,
\]
and let $\{ \gamma_k, \gamma'_k \}$ be a smooth partition of unity on $G$ subordinate to the open cover $\{ G_k, G'_k \}$.
Let \( \exp_G : \mathfrak{g} \to G \) be the exponential map, and let \( E \subseteq \mathfrak{g} \) be an open neighborhood of 0 which is mapped diffeomorphically under \( \exp_G \) to an open neighborhood \( U = \exp_G(E) \) of \( e \). Since \( G_k \subseteq U \) for \( k \gg 0 \), we may view \( \gamma_k \) as a real valued smooth map on \( \mathfrak{g} \) supported on \( E \). With this interpretation, we may assume that \( \gamma_k(\xi) = \gamma \left( k^{2/5} \xi \right) (\xi \in \mathfrak{g}) \), for a certain fixed smooth function \( \gamma \) on \( \mathfrak{g} \).

Inserting the relation \( \gamma_k + \gamma'_k = 1 \) in (8), integration over \( G \) splits as the sum of two terms. In the summand containing \( \gamma'_k \), integration is over \( G' \); therefore, by Corollary 2.2 if \( y \in \pi^{-1}(S_k) \) the integrand is uniformly \( O (k^{-\infty}) \). Hence we need only worry about the summand containing \( \gamma_k \).

To prove Proposition 2.1, we are thus reduced to proving that uniformly on \( y \in \pi^{-1}(T'_k) \) we have

\[
\int_{G_k} \gamma_k(g) \chi_\omega (gg(y)) \Pi_k \left( \mu_{gg(y)} \circ \gamma_\omega^{-1}(y), y \right) \, d\nu(g) = O \left( k^{-\infty} \right).
\]

(13)

To this end, we shall now invoke the parametrix of the Szegö kernel produced in [BdMS], and apply an integration by parts as in the proof of the stationary phase Lemma.

Let us recall that the Szegö kernel on \( X \times X \subseteq L^* \times L^* \) may be microlocally represented as a Fourier integral operator of the form

\[
\Pi(x,y) = \int_0^{+\infty} e^{it\psi(x,y)} \ s(x,y,t) \ dt,
\]

(14)

where the complex phase has prescribed Taylor expansion along the diagonal of \( L^* \times L^* \), and the amplitude is a semiclassical symbol admitting an asymptotic expansion \( s(x,y,t) \sim \sum_{j \geq 0} t^{d-j} s_j(x,y) \) [BdMS].

Following [Z2] and [SZ], let us take Fourier components, and perform the change of variable \( t \sim kt \), so that the left hand side of (13) may be rewritten

\[
\frac{k}{2\pi} \int_{G_k} \int_0^{+\infty} \int_{-\pi}^{\pi} \gamma_k(g) \chi_\omega (gg(y))
\]

\[
\cdot e^{ik \left[ t\psi(\mu_{gg(y)} \circ r_{ei\theta} \circ \gamma_\omega^{-1}(y), y) - \theta \right]} s \left( \mu_{gg(y)} \circ r_{ei\theta} \circ \gamma_\omega^{-1}(y), y, kt \right) \, d\nu(g) \, dt \, d\theta
\]

\[
= \frac{k}{2\pi} \int_0^{+\infty} \int_{-\pi}^{\pi} \gamma \left( k^{2/5} \xi \right) \chi_\omega (e^{\xi} g(y))
\]

\[
\cdot e^{ik \left[ t\psi(\mu_{e^{\xi}g(y)} \circ r_{ei\theta} \circ \gamma_\omega^{-1}(y), y) - \theta \right]} s \left( \mu_{e^{\xi}g(y)} \circ r_{ei\theta} \circ \gamma_\omega^{-1}(y), y, kt \right) \ H_G(\xi) \, d\xi \, dt \, d\theta.
\]

In the latter expression, integration over \( G_k \) has been written as an integral over the Lie algebra \( \mathfrak{g} \) by the exponential map \( \exp_G(\xi) = e^{\xi} \), and \( H_G(\xi) \, d\xi \)
is the pull-back to $g$ of the Haar measure on $G$ by $\exp_g$. Now (15) is an oscillatory integral, with phase

$$\Psi(\xi, t, \theta, y) =: t\psi\left(\mu_{e^{g(y)}} \circ r_e \circ \gamma_X^{-1}(y), y\right) - \theta,$$

depending parametrically on $y$.

If $\xi = 0$ and $\gamma(\pi(y)) = \pi(y)$, so that $g(y) = e$, then $\Psi = it\left(1 - e^{i(\theta + \theta_0)}\right) - (\theta + \theta_0)$. The latter phase was considered in [Z2], [SZ]; in this case, $\left|\frac{\partial \Psi}{\partial \theta}\right| > \frac{1}{2}$ when $t < \frac{1}{2}$.

If more generally $y \in \pi^{-1}(S_k)$ and $g \in G_k$, then

$$\text{dist}_\gamma(\mu_{g(y)} \circ r_e \circ \gamma_X^{-1}(y), y) \lesssim k^{-2/5},$$

therefore, by continuity for $k \gg 0$ and $t < \frac{1}{2}$ we have $\left|\frac{\partial \Psi}{\partial \theta}\right| > \frac{1}{3}$, say. Hence the contribution from the locus $t < \frac{1}{2}$ is $O(k^{-\infty})$.

Similarly, in view of the arguments in §3 of [SZ], one can see that in the same range the contribution coming from $t \geq 4$, say, is rapidly decreasing.

On the upshot, after disregarding a rapidly decaying contribution, we are left with the oscillatory integral:

$$\frac{k}{2\pi} \int_0^4 \int_{1/2}^\pi \int_{-\pi}^\pi e^{ik\Psi(\xi, t, \theta, y)} \cdot \gamma(k^{2/5}\xi) \chi_\infty\left(e^\xi g(y)\right) s\left(\mu_{e^{g(y)}} \circ r_e \circ \gamma_X^{-1}(y), y, kt\right) H_G(\xi) d\xi dt d\theta$$

$$= \frac{k}{2\pi} \int_0^4 \int_{1/2}^\pi \int_{-\pi}^\pi e^{ik\Psi(\xi, t, \theta, y)} \gamma(k^{2/5}\xi) S(\xi, kt, \theta, y) d\xi dt d\theta,$$

where $S$ is obviously defined; integration in $\xi$ is supported on a ball centered at $0 \in g$ and of radius $\sim k^{-2/5}$.

Let us now focus on the directional derivative of $\Psi$ with respect to $\xi \in g$. This is $\partial_\xi \Psi = t \partial_\xi \chi_\psi$, where $\xi_\chi \in \mathfrak{X}(X)$ is the vector field generated by $\xi$. Recall from [BdMS] that for any $x \in X$, the differential of $\psi \in C^\infty(X \times X)$ at $(x, x)$ is $d_{(x,x)}\psi = (\alpha_x, -\alpha_x)$; more generally, for any $x \in X$ and $e^{i\theta_0} \in S^1$ we have

$$d_{(e^{i\theta_0}x, x)}\psi = (e^{i\theta_0}\alpha_{e^{i\theta_0}x}, -e^{i\theta_0}\alpha_x).$$

Now if $y \in \pi^{-1}(S_k)$ there exists a unique $e^{i\theta(y)} \in S^1$ such that

$$\text{dist}_\gammaX(\mu_{g(y)} \circ r_e \circ \gamma_X^{-1}(y), y) = \text{dist}_\gamma\left(\mu_{g(y)} \circ \gamma^{-1} \circ \pi(y), \pi(y)\right) \leq 2k^{-2/5}.$$

Therefore, there exists $D' > 0$ such that for all $y \in \pi^{-1}(S_k)$ and $g \in G_k$

$$\text{dist}_\gammaX(\mu_{gg(y)} \circ r_e \circ \gamma_X^{-1}(y), y) \leq D' k^{-2/5}.$$
It follows from (17) and (18) that if \((x, y) \in S_k\) and \(g \in G_k\), then
\[
d_{(\mu_{g \phi}(y), \gamma^{-1}(y), y)} \psi = \left( e^{-i\theta(y)} \alpha_{\mu_{g \phi}(y), \gamma^{-1}(y)} - e^{-i\theta(y)} \alpha_y \right) + O \left( k^{-2/5} \right).
\]

Let us now make use of the assumption \(y \in \pi^{-1}(T_k')\). Since \(0 \in g^*\) is a regular value of \(\Phi\), perhaps after restricting \(V\) there exists a constant \(E' > 0\) such that \(\|\Phi(m)\| \geq E' \text{dist}_M(m, \Phi^{-1}(0))\), \(\forall m \in V\). Therefore, since \(\text{dist}_M(\cdot, \Phi^{-1}(0))\) is invariant under the \(G\)-action and \(\gamma\), for \(y \in \pi^{-1}(T_k')\) we have
\[
\left\| \Phi \left( \mu_{g(y)} \circ \gamma^{-1} \circ \pi(y) \right) \right\| \geq E' k^{-1/3}.
\]
Equivalently, \(\forall y \in \pi^{-1}(T_k')\) there exists \(\eta = \eta(y) \in g\) of unit length such that \(\Phi^\eta = \langle \Phi, \eta \rangle\) satisfies \(\Phi^\eta \left( \mu_{g(y)} \circ \gamma^{-1} \circ \pi(y) \right) \geq E' k^{-1/3}\). Recalling the definition of \(G_k\), setting \(E =: E'/2\), say, and letting \(k \gg 0\), we have:

**Lemma 2.4.** There exists \(E > 0\) such that for \(k \gg 0\) the following holds:
\(\forall y \in \pi^{-1}(T_k')\), there exists \(\eta = \eta(y) \in g\) of unit length such that \(\Phi^\eta =: \langle \Phi, \eta \rangle\) satisfies
\[
\left\| \Phi^\eta \left( \mu_{g(y)} \circ \gamma^{-1} \circ \pi(y) \right) \right\| \geq E k^{-1/3}.
\]
for all \(g \in G_k\).

Recall that \(\Phi^\eta = -\alpha(\eta_x)\), where \(\eta_x \in \mathfrak{X}(X)\) denotes the smooth vector field generated by \(\eta \in g\). Given this and (19), we conclude the following: If \(y \in \pi^{-1}(T_k')\), \(e^\xi \in G_k\), and \(\eta \in g\) is as in Lemma 2.4 then
\[
|\partial_{\eta} \Psi| = t |\partial_{\eta_x} \psi| = t \left| \alpha_{\mu_{e^\xi g}(y), \gamma^{-1}(y)} (\eta_x) \right| + O \left( k^{-2/5} \right)
\]
(20)
\[
= t \left| \Phi^\eta \left( \mu_{e^\xi g(y)} \circ \gamma^{-1} \circ \pi(y) \right) \right| + O \left( k^{-2/5} \right) \geq \frac{E}{2} k^{-1/3} + O \left( k^{-2/5} \right) \geq \frac{E}{3} k^{-1/3},
\]
for all \(k \gg 0\), since we are assuming \(t \geq \frac{1}{2}\).

Let \(\{\eta_j\}\) be an orthonormal basis of \(g\). By (20), for every \(y \in \pi^{-1}(T_k')\) there exists \(j\) such that \(\left| \frac{\partial \Psi}{\partial \eta_j} (\xi, t, \theta, y) \right| > \frac{E}{2g} k^{-1/3}\), whenever \(e^\xi \in G_k\). In other words, if for every \(j = 1, \ldots, g\) we set \(s_j := \frac{\partial \Psi}{\partial \eta_j}\) and
\[
V_j := \left\{ y \in \pi^{-1}(T_k') : |s_j(\xi, t, \theta, y)| > \frac{E}{2g} k^{-1/3}, \forall \xi \in \exp_{G_k}^{-1}(G_k), t \in [1/2, 4], \theta \in [0, 2\pi] \right\},
\]
then \(\{V_j\}\) is an open cover of \(\pi^{-1}(T_k')\). Let \(\{\varrho_j\}\) be a partition of unity subordinate to this cover; the differential operator on \(\pi^{-1}(T_k') \times g\)
\[
L := \sum_{j=1}^{g} \left( \frac{\varrho_j}{s_j} \right) \frac{\partial}{\partial \eta_j}
\]
satisfies $L(\Psi) = 1$, hence $L(e^{ik\Psi}) = ike^{ik\Psi}$. Recall that in (16) integration in $d\xi$ is compactly supported; let us iteratively integrate by parts, so as to obtain

$$
\int g e^{ik\Psi} \gamma(k^{2/5} \xi) S \, d\xi = \left( \frac{-i}{k} \right) \int g L(e^{ik\Psi}) \gamma(k^{2/5} \xi) S \, d\xi
$$

$$
= \left( \frac{i}{k} \right) \sum_{j=1}^{g} \int g e^{ik\Psi} \frac{\partial \gamma(k^{2/5} \xi)}{\partial \eta_j} \, d\xi
$$

$$
= \left( \frac{i}{k} \right)^2 \sum_{j_1, j_2=1}^{g} \int g e^{ik\Psi} \frac{\partial}{\partial \eta_{j_2}} \left( \frac{\partial \gamma(k^{2/5} \xi)}{\partial \eta_{j_1}} \left( \frac{\partial \gamma(k^{2/5} \xi)}{\partial \eta_{j_1}} \left( \cdots \right) \right) \right) \, d\xi
$$

$$
= \cdots
$$

$$
= \left( \frac{i}{k} \right)^r \sum_{j_1, \ldots, j_r=1}^{g} \int g e^{ik\Psi} Y_J(\gamma(k^{2/5} \xi) S) \, d\xi \quad (22)
$$

where for any multiindex $J = (j_1, \ldots, j_r)$ and any smooth function $v$ we have set

$$
Y_J(v) = \frac{\partial}{\partial \eta_{j_r}} \left( \frac{\partial}{\partial \eta_{j_{r-1}}} \left( \frac{\partial}{\partial \eta_{j_{r-2}}} \left( \cdots \frac{\partial}{\partial \eta_{j_1}} \left( \frac{\partial}{\partial \eta_{j_1}} v \right) \cdots \right) \right) \right).
$$

For any multindex $B = (b_1, \ldots, b_g)$ let us define $s^B = s_1^{b_1} \cdots s_g^{b_g}$.

The following may be proved by induction on $r$:

**Lemma 2.5.** For any $r \in \mathbb{N}$ and $J \in \{1, \ldots, g\}^r$, we have

$$
Y_J(\gamma(k^{2/5} \xi) S) = \sum_{q} k^{2a_q/5} f_q \cdot \frac{\varphi_q^{B_q}}{s^{B_q}}, \quad (23)
$$

where:

- $f_q = P_q(s) \cdot \gamma^{(e_q)}(k^{2/5} \xi) \cdot S(f_q)$, where $P$ is a differential operator with no zero order term, and $\gamma^{(e_q)}$, $S(f_q)$ are (possibly) higher order derivatives of $\gamma$ and $S$ with respect to $\xi$;

- $b'_j > 0$ if $b_j > 0$;

- $a_q + |B_q| \leq 2r$ for every $q$.

In view of the rescaling $t \sim kt$, the leading term in $S$ and its derivatives grows like $k^d$. Therefore, the $q$-th summand in Lemma 2.5 is bounded by

$$
C k^{d + \frac{2}{5} a_q + \frac{1}{3} |B_q|} \leq C k^{d + \frac{1}{5} (a_q + |B_q|)} \leq C k^{d + \frac{2}{5} r}.
$$
In view of (22) we get
\[
\left| \int \int e^{ik\Psi} \gamma (k^{2/5} \xi) S d\xi \right| \leq C_r k^{d-\tau/5}
\]
for any \( r \in \mathbb{N} \). This completes the proof of Proposition 2.1.

Q.E.D.

Given Corollary 2.1, we conclude
\[
\text{trace} (\Psi_{w,k}) \sim \int_{\pi^{-1}(T_k)} \varsigma_k(y) \Pi_{w,k} (\gamma_{X^{-1}}(y), y) f(y) \text{dens}_X(y),
\]
where we have set \( \varsigma_k =: \sigma_k \cdot \tau_k \).

Let now \( \text{Fix}(\gamma_0) \subseteq M_0 \) be the fixed locus of \( \gamma_0 : M_0 \to M_0 \), and set \( \text{Fix}(\gamma_0) =: p^{-1} (\text{Fix}(\gamma_0)) \subseteq \Phi^{-1}(0) \), where \( p \) is as in (2).

**Proposition 2.2.** There exists \( C > 0 \) such that
\[
\text{dist}_M \left( (G \cdot m, \text{Fix}(\gamma_0)) \right) \leq C k^{-1/3}, \quad \forall m \in T_k.
\]

**Proof.** If \( m \in T_k \subseteq S_k \), then \( \text{dist}_M \left( (G \cdot m, G \cdot \gamma(m)) \right) < 2 k^{-2/5} \) by (7), and \( \text{dist}_M \left( (m, \Phi^{-1}(0)) \right) < 2 k^{-1/3} \) by (11).

Let \( q \in \Phi^{-1}(0) \) be such that \( \text{dist}_M \left( (m, \Phi^{-1}(0)) \right) = \text{dist}_M \left( (m, q) \right) \). Since \( \Phi^{-1}(0) \) is \( G \)-invariant, \( \text{dist}_M \left( (m, q) \right) = \text{dist}_M \left( (G \cdot m, G \cdot q) \right) \). As \( \gamma \) commutes with the action and preserves the metric, we also have
\[
\text{dist}_M \left( (m, q) \right) = \text{dist}_M \left( (\gamma(m), \gamma(q)) \right) = \text{dist}_M \left( (G \cdot \gamma(m), G \cdot \gamma(q)) \right).
\]

Since \( G \) acts freely and isometrically on \( V \), there is a Riemannian metric on the manifold \( V_0 =: V/G \) such that the projection \( \hat{p} : V \to V_0 \) is a Riemannian submersion. Hence, \( \text{dist}_M \left( (G \cdot m, G \cdot n) \right) = \text{dist}_{V_0} \left( \hat{p}(m), \hat{p}(n) \right) \), \( \forall m, n \in V \). By the triangle inequality on \( V_0 \),
\[
\text{dist}_M \left( (G \cdot q, G \cdot \gamma(q)) \right)
\leq \text{dist}_M \left( (G \cdot q, G \cdot m) \right) + \text{dist}_M \left( (G \cdot m, G \cdot \gamma(m)) \right) + \text{dist}_M \left( (G \cdot \gamma(m), G \cdot \gamma(q)) \right)
= 2 \text{dist}_M \left( (m, G \cdot \Phi^{-1}(0)) \right) + \text{dist}_M \left( (G \cdot m, G \cdot \gamma(m)) \right)
\leq 4 k^{-1/3} + 2 k^{-2/5} < 5 k^{-1/3}
\]
if \( k \gg 0 \).
Set $q_0 =: p(q) \in M_0$. Since $\text{dist}_M \left( q, \tilde{\text{Fix}}(\gamma_0) \right) = \text{dist}_{M_0} \left( q_0, \text{Fix}(\gamma_0) \right)$,
\[
\text{dist}_M \left( m, \tilde{\text{Fix}}(\gamma_0) \right) \leq \text{dist}_M \left( m, q \right) + \text{dist}_M \left( q, \tilde{\text{Fix}}(\gamma_0) \right) \\
\leq 2k^{-1/3} + \text{dist}_{M_0} \left( q_0, \text{Fix}(\gamma_0) \right).
\tag{27}
\]

**Lemma 2.6.** There exists a constant $C > 0$ such that for all $k \gg 0$ we have
\[
\text{dist}_{M_0} \left( q_0, \text{Fix}(\gamma_0) \right) \leq Ck^{-1/3}.
\]

**Proof.** By (23), for all $k \gg 0$ we have
\[
\text{dist}_{M_0} \left( q_0, \gamma_0(q_0) \right) = \text{dist}_M \left( G \cdot q, G \cdot \gamma(q) \right) < 5k^{-1/3}.
\tag{28}
\]

Let now $F_1, \ldots, F_{\ell} \subseteq M_0$, with normal bundles $N_1, \ldots, N_{\ell}$, be as in Definition [12]. Let $\exp_l : N_l \rightarrow M_0$ be the exponential map, $(q'_0, n) \mapsto \exp_l(q'_0, n) =: \exp_{q'_0}(n)$. For $\epsilon > 0$, let $N_{l}^{(\epsilon)} =: \{(q'_0, n) \in N_l : ||n|| < \epsilon\}$. Choose $\epsilon > 0$ so small that $\exp_l$ induces a diffeomorphism between $N_l^{(\epsilon)}$ and an open neighborhood $F_l^{(\epsilon)} \subseteq M_0$ of $F_l$, and $F_{l_1}^{(\epsilon)} \cap F_{l_2}^{(\epsilon)} = \emptyset, \forall l_1 \neq l_2 \in \{1, \ldots, \ell\}$.

If $q'_0 \in F_l$, the normal exponential map $\exp_{q'_0} : N_{l(q'_0)} \rightarrow M_0$ is an isometric immersion at the origin. By compactness of $F_l$, perhaps after decreasing $\epsilon$ we may assume that if $q'_0 \in F_l$ and $(q'_0, n), (q'_0, n') \in N_{l(q'_0)} \cap N_{l,\epsilon}$ then
\[
2 \|n - n'\| \geq \text{dist}_{M'_0} \left( \exp_{N_l}(q'_0, n), \exp_{N_l}(q'_0, n') \right) \geq \frac{1}{2} \|n - n'\|.
\tag{29}
\]

There exists $\delta > 0$ such that
\[
\text{dist}_{M_0} \left( q_0, \text{Fix}(\gamma_0) \right) \geq \epsilon \Rightarrow \text{dist}_{M_0} \left( q_0, \gamma_0(q_0) \right) \geq \delta.
\]

Thus, if $k \gg 0$ and (28) holds, then $q_0 \in \bigcup_{l=1}^{\ell} F_l^{(\epsilon)}$; hence, $q_0 = \exp_{N_l}(q'_0, n)$ for some $(q'_0, n) \in N_l^{(\epsilon)}$. Given that $\gamma_0 : M_0 \rightarrow M_0$ is a Riemannian isometry, we have $\exp_{M_0} \circ d\gamma_0 = \gamma_0 \circ \exp_{M_0} : TM_0 \rightarrow M_0$. In view of (29), we deduce
\[
\text{dist}_{M_0} \left( q_0, \gamma_0(q_0) \right) = \text{dist}_{M_0} \left( \exp_{M_0}(q'_0, n), \gamma_0 \circ \exp_{M_0}(q'_0, n) \right) \\
= \text{dist}_{M_0} \left( \exp_{M_0}(q'_0, n), \exp_{M_0} \circ d\gamma_0 \gamma_0(n) \right) \geq \frac{1}{2} \|d\gamma_0 \gamma_0(n) \cdot n\| \\
\geq \frac{1}{2} \inf \{ |\lambda_i - 1| \} \|n\| \geq \frac{1}{4} \inf \{ |\lambda_i - 1| \} \text{dist}_{M_0} \left( q_0, \text{Fix}(\gamma_0) \right).
\tag{30}
\]

Since $\lambda_i \neq 1$ for all $i$, the statement follows from (28) and (30). Q.E.D.
Proposition 2.2 now follows from (27) and Lemma 2.6.

Q.E.D.

It is now in order to give the:

Proof of Lemma 2.1. This follows from a simplified version of the previous arguments. Recall that
\[ \Pi_{\omega,k}(x,y) = \dim(V_{\omega}) \int_G \chi_{\omega}(g) \Pi_k(\mu_g(x), y) d\nu(g), \] for all \((x,y) \in X \times X\).

If \(R \subseteq X \times X \setminus I(\Phi)\) is compact, there exists \(\delta > 0\) such that \(\forall (x,y) \in R\) we have
\[ \max \left\{ \text{dist}_M(G \cdot \pi(x), G \cdot \pi(y)), \text{dist}_M(\pi(x), \Phi^{-1}(0)) \right\} \geq \delta. \]

The case \(\text{dist}_M(G \cdot \pi(x), G \cdot \pi(y)) \geq \delta\) can be handled by the arguments in the proof of Lemma 2.2, replacing the lower bound \(k^{-2/5}\) used there with \(\delta\).

To deal with the case \(\text{dist}_M(\pi(x), \Phi^{-1}(0)) > \delta\), we may by the same argument restrict to the case where \(\text{dist}_M(G \cdot \pi(x), G \cdot \pi(y)) < \epsilon\), for some fixed but arbitrarily small \(\epsilon > 0\). Invoking the fact that the Szegö kernel is smoothing away from the diagonal, we conclude that we only miss a rapidly decaying contribution if we restrict the \(G\)-integration in (31) to the open subset \(G(x,y) = \{ g \in G : \text{dist}_M(\mu_g \circ \pi(x), \pi(y)) < 2\epsilon \}\), say (the introduction of an appropriate partition of unity on \(G\) is understood). Now the hypothesis implies that \(\| \Phi \circ \pi(x) \| > \delta'\) for some \(\delta' > \delta\). By the arguments leading to the proof of Lemma 2.4 and (20), it follows that if \(0 < \epsilon < 1\) then \(\left| \partial_n \Psi \right| > \delta'/4\) on the range of integration. The statement then follows by a simpler version of the argument following (20).

Q.E.D.

Summing up, \(T_k\) is a shrinking open neighborhood of \(\widetilde{\text{Fix}(\gamma_0)}\). To obtain an asymptotic expansion for \(\text{trace}(\Psi_{\omega,k})\), we shall insert in (25) the scaling limit asymptotics for \(\Pi_{\omega,k}\) proved in [P2]. Scaling asymptotics are most naturally stated in local Heisenberg coordinates; therefore, we shall cover \(\pi^{-1}(\widetilde{\text{Fix}(\gamma_0)})\) by invariant open sets with a ‘transverse Heisenberg structure’, providing convenient coordinates to perform the integration.

For \(r \in \mathbb{N}\) and \(\epsilon > 0\), let \(B_r(\epsilon)\) be the open ball centered at \(0 \in \mathbb{R}^r\) and of radius \(\epsilon\). Referring the reader to [SZ] for the precise definitions, we recall that a system of local Heisenberg coordinates for the circle bundle \(X\) centered at a given \(x \in X\) is determined by the following data: i) a preferred local chart \(\phi : B_{2d}(\epsilon) \to U \subseteq M\) for the (almost) Kähler manifold \((M, \omega, J)\) centered
at $m = \pi(x)$, and ii) a preferred local frame $e : U \to A$ at $m$ satisfying $e^*(m) = x$, where $\langle e^*, e \rangle = 1$. That $f$ is a preferred local chart at $m$ means that it trivializes the unitary structure of $T_m M$. In the present integrable setting, $f$ may be chosen holomorphic, but this is not necessary, and won’t be assumed in the following; $f$ is at any rate always holomorphic and symplectic at $0 \in B_{2d}(\epsilon) \subseteq \mathbb{C}^d$. In fact, upon choosing an orthonormal complex basis of $T_m M$, the exponential map $\exp_m : \mathbb{C}^d \cong T_m M \to M$ restricts to a preferred local chart on $B_{2d}(\epsilon)$, for some $\epsilon > 0$; at places it will simplify our arguments to make this choice. Explicitly, given $f$ and $e$ the associated Heisenberg local chart is then

$$\psi : B_{2d}(\epsilon) \times (-\pi, \pi) \to \pi^{-1}(U), \quad (z, \vartheta) \mapsto e^{i\vartheta \langle e^* (f(z)) \rangle \| e^* (f(z)) \|}.$$ 

Following [SZ], we set

$$x + w =: \psi(w, 0)$$

if $w \in T_m M \cong \mathbb{C}^d$, $\|w\| < \epsilon$. In this notation, $T_m M$ is implicitly identified with the horizontal subspace $\text{Hor}_x(X) \subseteq T_x X$ for the connection.

Given any $x \in X$, it is always possible to find local Heisenberg coordinates centered at $x$, and this construction may be deformed smoothly with $x$; that is, given any $x \in X$ there exist $x \in U \subseteq X$ open and a smooth map $\Psi : U \times B_{2d}(\epsilon) \times (-\pi, \pi) \to X$, such that for any $y \in U$ the partial map $\psi(y) =: \Psi(y, \cdot, \cdot) : B_{2d}(\epsilon) \times (-\pi, \pi) \to X$ is a Heisenberg local chart for $X$ centered at $y$. We may, and will, assume without loss that

$$\Psi(r_{e^* e_0}(y), z, \vartheta) = \Psi(y, z, \vartheta - \vartheta_0)$$

whenever the two sides are defined.

In the present equivariant setting, suppose $x \in X$ and let $\psi : B_{2d}(\epsilon) \times (-\pi, \pi) \to X$ be a system of local Heisenberg coordinates centered at $x$, associated to the preferred choices $f$ and $e$. Then for any $g \in G$ we obtain a system of Heisenberg coordinates centered at $\mu_g(x)$ by considering the composition $\psi_g =: \mu_g \circ \psi$. Clearly, $\psi_g$ is associated to the preferred choices $\mu_g \circ f$ and $\hat{\mu}_g(e)$; here $\hat{\mu}$ denotes the action on the collection of local sections of $A$.

Let us set, for ease of notation, $M' =: \Phi^{-1}(0) \subseteq M$, $X' =: \pi^{-1}(\Phi^{-1}(0)) \subseteq X$, and let us denote by $\pi : X' \to M'$ the projection. Then $G$ acts freely on
\( M' \) and \( X' \), and we have the commutative diagram:

\[
\begin{array}{ccc}
X' & \xrightarrow{\tilde{p}} & X_0 = X'/G \\
\pi' & \downarrow & \downarrow \pi_0 \\
M' & \xrightarrow{p} & M_0 = M'/G
\end{array}
\]

where the vertical arrows are principal \( S^1 \)-bundles and the horizontal arrows are principal \( G \)-bundles. \( \pi' \) is \( G \)-equivariant and \( \tilde{p} \) is \( S^1 \)-equivariant. Given subsets \( V \subseteq M_0 \) and \( U \subseteq M \), we shall set \( X_0(V) =: \pi_0^{-1}(V) \subseteq X_0 \), \( X(V) =: \pi^{-1}(V) \subseteq X \).

Suppose given:

- \( m_0 \in F_l \subseteq \text{Fix}(\gamma_0) \subseteq M_0; \)
- an open subset \( V \subseteq F_l \) with \( m_0 \in V; \)
- a smooth section \( \sigma : V \to \tilde{F}_l =: p^{-1}(F_l) \) of the principal \( G \)-bundle \( \tilde{F}_l \to F_l \).

Then there exists a unique smooth section \( \tilde{\sigma} : X_0(V) \to X' \) of \( \tilde{p} \) which is a lift of \( \sigma \), that is, such that \( \sigma \circ \pi_0 = \pi' \circ \tilde{\sigma} : X_0(V) \to M'. \) \( \tilde{\sigma} \) is necessarily \( S^1 \)-equivariant.

By the above, we may also suppose given a smooth map

\[ \Psi : X_0(V) \times B_{2\delta}(\epsilon) \times (-\pi, \pi) \to X, \]

such that for any \( x_0' \in X_0(V) \) the partial map \( \psi(x_0') := \Psi(x_0', \cdot, \cdot) \) is a Heisenberg local chart for \( X \) centered at \( \tilde{\sigma}(x_0') \), with image containing some fixed open neighborhood \( X(U) \supseteq \pi^{-1}(\sigma(m_0)) \). We shall write

\[
\left( \psi(x_0') \right)^{-1} = \left( z_1(x_0'), \ldots, z_d(x_0'), \vartheta(x_0') \right) : U \to B_{2\delta}(\epsilon) \times (-\pi, \pi)
\]

for the corresponding Heisenberg local chart. Here we identify \( \mathbb{R}^{2d} \cong \mathbb{C}^d \) in the standard manner, and \( z_j(x_0') : U \to \mathbb{C} \) is a smooth function. Let \( a_j(x_0') := \Re \left( z_j(x_0') \right), b_j(x_0') := \Im \left( z_j(x_0') \right) : X(U) \to \mathbb{R} \). Clearly, \( a_j(x_0') \) and \( b_j(x_0') \) descend to \( U \), and form the system \( f(x_0') \) of preferred local coordinates on \( M \) centered at \( \pi \circ \tilde{\sigma}(x_0') \) which underlies \( \psi(x_0') \).

After composing with a suitable local diffeomorphism of \( M \), smoothly varying with \( x_0' \), we may assume that for every \( x_0' \in X_0(V) \) the following conditions are satisfied by \( f(x_0') \):
1. \(\Phi^{-1}(0) \cap U = \left\{ b_{d-g+1}^{(x'_0)} = \cdots = b_d^{(x'_0)} = 0 \right\} \);

2. \(\tilde{F}_i \cap U = \left\{ z_{d+1}^{(x'_0)} = \cdots = z_{d-g}^{(x'_0)} = b_{d-g+1}^{(x'_0)} = \cdots = b_d^{(x'_0)} = 0 \right\} \).

By the previous discussion, composing with the \(G\)-action we then obtain a smooth map (we write \(\mu\) for \(\mu_x\))

\[
\tilde{\Psi} : G \times X_0(V) \times B_{2d}(\epsilon) \times (-\pi, \pi) \to X, \quad (g, x'_0, z, \vartheta) \mapsto \mu_g \left( \Psi(x'_0, z, \vartheta) \right),
\]

such that for any \((g, x'_0) \in G \times X_0(V)\) the partial map

\[
\psi^{(g,x'_0)} = \tilde{\Psi}(g, x'_0, \cdot, \cdot) : B_{2d}(\epsilon) \times (-\pi, \pi) \to X \tag{35}
\]

is a Heisenberg local chart for \(X\) centered at \(\mu_g \circ \tilde{\sigma}(x'_0)\), and whose image contains \(\mu_g(X(U))\). We shall denote by

\[
\left( \psi^{(g,x'_0)} \right)^{-1} = \left( \psi^{(x'_0)} \right)^{-1} \circ \left( \mu_g \right)^{-1} \tag{36}
\]

the corresponding Heisenberg local coordinates. By the \(G\)-invariance of \(\Phi^{-1}(0)\) and \(\tilde{F}_i\) we obtain that for every \((g, x'_0) \in G \times V\)

1. \(\Phi^{-1}(0) \cap \mu_g(U) = \left\{ b_{d-g+1}^{(g,x'_0)} = \cdots = b_d^{(g,x'_0)} = 0 \right\} \);

2. \(\tilde{F}_i \cap \mu_g(U) = \left\{ z_{d+1}^{(g,x'_0)} = \cdots = z_{d-g}^{(g,x'_0)} = b_{d-g+1}^{(g,x'_0)} = \cdots = b_d^{(g,x'_0)} = 0 \right\} \).

Having in mind the identifications

\[
\mathbb{R}^{2d} \cong \mathbb{R}^{2d_1} \times \mathbb{R}^{2c_1} \times \mathbb{R}^{2g} \cong \mathbb{C}^d_1 \times \mathbb{C}^c_1 \times \mathbb{C}^g,
\]

the following is a straightforward consequence of the previous discussion:

**Lemma 2.7.** Suppose \(\epsilon > 0\) is sufficiently small, and define

\[
\Upsilon : G \times X(V) \times B_{2c_1}(\epsilon) \times B_g(\epsilon) \to X
\]

by

\[
\Upsilon \left( g, x'_0, z, b \right) = \tilde{\Psi} \left( g, x'_0, (0, z, ib, 0) \right) = \psi^{(g,x'_0)} \left( 0, z, ib, 0 \right),
\]

where \(z = (z_{d+1}, \cdots, z_{d-g}) \in B_{2d_1}(\epsilon) \subseteq \mathbb{C}^{c_1}\), and \(0\) denotes the origin of \(\mathbb{C}^{d_1}\). Then:
1. $\Upsilon$ is an equivariant diffeomorphism onto a $(G \times S^1)$-invariant open neighborhood $B$ of $G \cdot \tilde{\sigma}(X_0(V)) \subseteq X$.

2. In terms of $\Upsilon$, $\text{Fix}(\gamma_0) \cap B$ is defined by the conditions $z = 0$, $b = 0$; in other words, $\Upsilon^{-1}\left(\text{Fix}(\gamma_0)\right) = G \times X(V) \times \{0\} \times \{0\}$.

Before we proceed, let us dwell on the local structure of $M$ along $\tilde{F}_l = p^{-1}(F_l) \subseteq M'$. For any $m \in M$, let $g_M(m) \subseteq T_mM$ be the vector subspace generated by the infinitesimal action of $g$. Thus $g_M$ is a rank-g vector sub-bundle of $TM$ on some invariant open neighborhood of $M'$. If $m \in M'$, we have the unitary direct sum decompositions

$$T_mM = J_m(g_M(m)) \oplus T_mM', \quad T_mM' = g_M(m) \oplus H_m; \quad (37)$$

here $H_m =: T_mM' \cap (g_M(m))^\perp$ is a complex subspace, that gets unitarily identified with $T_{p(m)}M_0$ under $m_0p$ (the superscript $\perp$ stands for ‘Euclidean orthocomplement’). Thus if $m \in \tilde{F}_l$ with this identification we also have

$$H_m \cong T_{p(m)}M_0 = T_{p(m)}F_l \oplus (N_l)_{p(m)}, \quad T_m\tilde{F}_l \cong T_{p(m)}F_l \oplus g_M(m), \quad (38)$$

where $(N_l)_{p(m)}$ denotes the fiber at $p(m)$ of the normal bundle $N_l$ of $F_l \subseteq M_0$.

If $m \in M'$, let us set

$$(T_mM)_t =: J_m(g_M(m)), \quad (T_mM)_v =: g_M(m), \quad (T_mM)_h =: H_m. \quad (39)$$

Here the suffix $t$ stands for ‘transverse to $M'$’, $v$ stands for ‘vertical for the principal $G$-bundle structure of $M' \to M_0$’, $h$ for ‘horizontal’.

If in addition $m \in \tilde{F}_l$, with a slight abuse of language, let us set

$$(T_mM)_{h,tg} =: T_{p(m)}F_l, \quad (T_mM)_{h,nor} =: (N_l)_{p(m)}. \quad (40)$$

Thus, $(T_mM)_h = (T_mM)_{h,tg} \oplus (T_mM)_{h,nor}$. Here the suffix $h,tg$ stands for ‘horizontal and tangent to $\tilde{F}_l$’, $h,nor$ for ‘horizontal and normal to $\tilde{F}_l$’. Accordingly, if $m \in \tilde{F}_l$ any $v \in T_mM$ may be decomposed as $v = v_t + v_v + v_{h,tg} + v_{h,nor}$.

Let us consider again the statement of Lemma 2.7. Suppose $(g, x'_0) \in G \times X(V)$. Since $\tilde{\Psi}^{(g,x'_0)}$ is a local Heisenberg chart for $X$ centered at $x =:\mu_g \circ \tilde{\sigma}(g,x'_0)$, and satisfying 1. and 2. above, any $z \in \mathbb{C}^\ell$ gets identified with an appropriate $v_{h,nor} \in (T_mM)_{h,nor}$ where $m = \pi(x) \in \tilde{F}_l$. Similarly, if $b \in \mathbb{R}^g$ then $ib$ gets identified with an appropriate $v_t \in (T_mM)_t$. Following (32), for sufficiently small $z \in \mathbb{C}^\ell$ and $b \in \mathbb{R}^g$ we then have

$$\Upsilon\left(g, x'_0, z, b\right) = \mu_g \circ \tilde{\sigma}(x'_0) + v_{h,nor} + v_t.$$
For $\epsilon > 0$ sufficiently small, we can then replace $B$ in Lemma 2.7 by

$$
B =: \left\{ \mu_g(\tilde{\sigma}(x'_0)) + v_{h,nor} + v_t : (g, x'_0) \in G \times X(V), \right. \\
\left. v_{h,nor} \in (T_{\mu_g \circ \tilde{\sigma}(x'_0)}M)_{h,nor}, \quad v_t \in (T_{\mu_g \circ \tilde{\sigma}(x'_0)}M)_t, \quad \|v_{h,nor}\|, \|v_t\| < \epsilon \right\},
$$

an invariant open neighborhood of $\pi^{-1}(p^{-1}(V)) \subseteq \pi^{-1}\left(\overline{\text{Fix}(\gamma_0)}\right)$.

Let $B_g(\epsilon) \subseteq \mathbb{R}^g$, $B_{2d_l}(\epsilon) \subseteq \mathbb{C}^{d_l}$, $B_{2c_i}(\epsilon) \subseteq \mathbb{C}^{c_i}$ the open balls of radius $\epsilon$ centered at the origin. The parameterization (41) defines a diffeomorphism

$$
T : G \times X(V) \times B_g(\epsilon) \times B_{2c_i}(\epsilon) \longrightarrow B,
$$

$$
T(g, x'_0, v_t, v_{h,nor}) =: \mu_g(\tilde{\sigma}(x'_0)) + v_{h,nor} + v_t.
$$

Now for every $l = 1, \ldots, \ell$ let $\{V_{ij}\}_{j}$ be a finite open cover of $F_l$, such that on every $V_{ij}$ there is defined a smooth section $\sigma_{ij} : V_{ij} \rightarrow M'$ of $p$; in particular, $\{p^{-1}(V_{ij})\}_{i,j}$ is an invariant open cover of $\overline{\text{Fix}(\gamma_0)}$.

For every $l, j$, let $B_{ij} \subseteq X$ be defined by (41) with $(V_{ij}, \sigma_{ij})$ in place of $(V, \sigma)$; hence $B_{ij}$ is an $S^1 \times G$-invariant open neighborhood of $\pi^{-1}(p^{-1}(V_{ij}))$.

Finally, set $E =: \bigcup_{ij} B_{ij}$. Then $E$ is an $S^1 \times G$-invariant open neighborhood of $\pi^{-1}\left(\overline{\text{Fix}(\gamma_0)}\right) \subseteq X$, and $\{B_{ij}\}_{i,j}$ is an open cover of $E$. Let $\{\tau_{ij}\}_{i,j}$ be a smooth partition of unity on $E$ subordinate to this cover. After averaging, we may assume that each $\tau_{ij}$ is $S^1 \times G$-invariant; hence each $\tau_{ij}$ descends in a natural manner to a smooth function $\tilde{\tau}_{ij}$ on $F_l$, and $\{\tilde{\tau}_{ij}\}_{i,j}$ is a smooth partition of unity on $\overline{\text{Fix}(\gamma_0)}$, subordinate to the open cover $\{V_{ij}\}_{i,j}$.

Let us now return to (25). Since $T_k$ is a shrinking open neighborhood of $\overline{\text{Fix}(\gamma_0)}$ as $k \rightarrow +\infty$, we have $\pi^{-1}(T_k) \subseteq E$ for all $k \gg 0$, hence

$$
\pi^{-1}(T_k) = \bigcup_{i,j} \pi^{-1}(T_k) \cap B_{ij}.
$$

Thus $\pi^{-1}(T_k) \cap B_{ij}$ is a shrinking open neighborhood of $\pi^{-1}(p^{-1}(V_{ij}))$. Inserting the relation $\sum_{i,j} \tau_{ij} = 1$ in (25) yields

$$
\text{trace}(\Psi_{\pi,k}) \sim \sum_{i,j} \text{trace}(\Psi_{\pi,k})_{i,j},
$$

where

$$
\text{trace}(\Psi_{\pi,k})_{i,j} =: \int_{\pi^{-1}(T_k) \cap B_{ij}} \tau_{ij}(y) \zeta_k(y) \Pi_{\pi,k} \left( \gamma_{X^{-1}}(y), y \right) f(y) \text{dens}_X(y).
$$

Let us now estimate asymptotically each summand (43). To simplify our notation, in the following formulae we shall temporarily fix a pair $(l, j)$, and occasionally write $\sigma, \tilde{\sigma}$ for $\sigma_{ij}, \tilde{\sigma}_{ij}$.
We can parametrize $\pi^{-1}(T_k) \cap B_{ij}$ by (41). Given (7), (10), and Proposition 2.2 $\|v_t\|, \|v_{h,\text{nor}}\| \leq k^{-1/3}$. Inserting (41) in (43), we shall rescale $\|v_t\|, \|v_{h,\text{nor}}\|$ by a factor $k^{-1/2}$ and integrate the rescaled variables over a ball of radius $\approx k^{1/6}$. In other words, we shall write

$$y = \mu_g \circ \tilde{\sigma}(x_0') + \frac{1}{\sqrt{k}} (v_t + v_{h,\text{nor}}),$$

where $v = v_t + v_{h,\text{nor}} \in T_{\mu_g \circ \tilde{\sigma}(x_0')} M$; the latter is unitarily identified with $\mathbb{C}^d$ by means of the given Heisenberg local coordinates. Taylor expanding, we obtain with $m'_0 = \pi_0(g_0) \in V_{ij} \subseteq F_i$:

$$f(y) = f \circ \pi(y) \sim f(\mu_g \circ \sigma(m'_0)) + \sum_{j \geq 1} k^{-j/2} f_j(v, w),$$

$$\tau_j(y) = \tau_j(\pi(y)) \sim \tau_j^0(m'_0) + \sum_{j \geq 1} k^{-j/2} \tau_j(v),$$

$$s_k(y) = s_k(\pi(y)) \equiv 1 \quad \text{if} \quad \|v_t + v_{h,\text{nor}}\| \leq k^{1/10}.$$
• the $a_{\omega j}$'s are polynomials in $v$, $w$ whose coefficients depend on $x$ and $\omega$.

• Let $R_N(x, w, v)$ be the remainder term following the first $N$ summands in (46); then for $\|w\|, \|v\| \lesssim k^{1/6}$ we have the ‘large ball estimate’:

$$ \left| R_N(x, w, v) \right| \leq C_N k^{d-(g+N+1)/2} e^{-\frac{1-\epsilon}{2}(\|w_n-v_n\|^2+2\|w_n\|^2+2\|v_n\|^2)}. $$

(47)

To obtain an asymptotic expansion for $\Pi_{\omega, k}(\gamma_X^{-1}(y), y)$ in (43), with $y$ as in (44), we need the the Heisenberg local coordinates of $\gamma_X^{-1}(y)$. As a first step, let us work out the underlying preferred coordinates of $\gamma^{-1}(\pi(y))$.

More precisely, recall that our construction involves a moving Heisenberg local chart $\Psi = \Psi_{\omega}$ as in (34), now with $(V_{ij}, \sigma_{ij})$ in place of $(V, \sigma)$. Underlying $\Psi$, there is a moving preferred local chart

$$ \mathcal{F} : V_{ij} \times B_{2d}(\epsilon) \to M, $$

(48)

such that for any $m_{ij}^0 \in V_{ij}$ the partial map $f^{(m_{ij}^0)} = \mathcal{F}(m_{ij}^0, \cdot, \cdot)$ is a preferred local chart for $M$ centered at $\sigma(m_{ij}^0)$, with image containing some fixed open subset $U \subseteq M$. It will simplify our exposition to assume, as we may, that for every $m_{ij}^0 \in V$ we have

$$ f^{(m_{ij}^0)} = \exp_{\sigma(m_{ij}^0)} \circ \varrho_{m_{ij}^0} : B_{2d}(\epsilon) \subseteq \mathbb{C}^d \cong T_{\sigma(m_{ij}^0)}M \to M, $$

for an appropriate $\epsilon > 0$; here $\varrho_{m_{ij}^0} : \mathbb{C}^d \cong T_{\sigma(m_{ij}^0)}M$ is a smoothly varying unitary isomorphism, induced by the choice of an orthonormal frame for $T^{(1,0)}M$ on a neighborhood of $\sigma_{ij}(V_{ij})$.

The same then holds for every pair $(g, m_{ij}^0) \in G \times V_{ij}$. More precisely, since $\mu_g : M \to M$ is a Riemannian isometry, for every $(g, m_{ij}^0) \in G \times V_{ij}$ the composition

$$ \varrho_{(g, m_{ij}^0)} := d_{\sigma(m_{ij}^0)} \mu_g \circ \varrho_{m_{ij}^0} : \mathbb{C}^d \to T_{\sigma(m_{ij}^0)}M $$

is unitary, where write $\sigma = \sigma_{ij}$, and $\mu_g \circ \exp_{\sigma^{(m_{ij}^0)}} = \exp_{\mu_g \sigma^{(m_{ij}^0)}} \circ d_{\sigma^{(m_{ij}^0)}} \mu_g$. By construction,

$$ f^{(g, m_{ij}^0)} = \mu_g \circ f^{(m_{ij}^0)} = \mu_g \circ \exp_{\sigma^{(m_{ij}^0)}} \circ \varrho_{m_{ij}^0} = \exp_{\mu_g \sigma^{(m_{ij}^0)}} \circ d_{\sigma^{(m_{ij}^0)}} \mu_g \circ \varrho_{m_{ij}^0} = \exp_{\mu_g \sigma^{(m_{ij}^0)}} \circ \varrho_{(g, m_{ij}^0)}. $$

Given $(g, m_{ij}^0) \in G \times V_{ij}$ and $v \in T_{\mu_g \sigma^{(m_{ij}^0)}}M$ with $\|v\| < \epsilon$, we shall set $\mu_g \circ \sigma_{ij}(m_{ij}^0) + v = f^{(g, m_{ij}^0)} \circ \varrho_{(g, m_{ij}^0)^{-1}}(v)$.

For every $m \in \text{Fix}(\gamma_0)$ there exists a unique $g_m \in G$ such that $\gamma(m) = \mu_{g_m}(m)$ (Definition 1.2). Hence $\forall (g, m_{ij}^0) \in G \times M$, with $\kappa = g g_{\sigma^{(m_{ij}^0)}}^{-1} g^{-1}$,

$$ \gamma^{-1}(\mu_g \circ \sigma(m_{ij}^0)) = \mu_{\kappa} \circ \mu_g \circ \sigma(m_{ij}^0) = \mu_{\kappa} \circ \sigma(m_{ij}^0). $$

(49)
Since $\gamma : M \to M$ is also a Riemannian isometry, by (49) we also have

$$\gamma^{-1} \circ \exp_{\mu_g \circ \sigma(m'_0)} = \exp_{\gamma^{-1} \circ \mu_g \circ \sigma(m'_0)} \circ d_{\mu_g \circ \sigma(m'_0)} \gamma^{-1} = \exp_{\mu_g \circ \sigma(m'_0)} \circ d_{\mu_g \circ \sigma(m'_0)} \gamma^{-1}. \tag{50}$$

With the previous convention, (50) implies that $\forall v \in T_{\mu_g \circ \sigma(m'_0)} M$

$$\gamma^{-1}(\mu_g \circ \sigma(m'_0) + v) = \mu_{\kappa_g} \circ \sigma(m'_0) + d_{\mu_g \circ \sigma(m'_0)} \gamma^{-1}(v). \tag{51}$$

Lifting this to $X$, we deduce that $\forall (g, x'_0) \in G \times X(V_{ij})$

$$\gamma^{-1}_X(\mu_g \circ \sigma(x'_0) + v) = r_{e^{i\beta(g, x'_0, v)}}(\mu_{\kappa_g} \circ \sigma(x'_0) + d_{\mu_g \circ \sigma(m'_0)} \gamma^{-1}(v)), \tag{52}$$

with $m'_0 =: \pi_0(x'_0) \in V_{ij} \subseteq F_l$, for an appropriate smooth real function $\beta : G \times X(V_{ij}) \times B_{2d}(\epsilon) \to \mathbb{R}$, uniquely determined up to an integer multiple of $2\pi$. To determine $\beta$, recall that $h_l = e^{i\theta_l} \in S^1$ is uniquely determined by the condition $\tilde{\gamma}_0((r, a)) = (r, h_l a)$, $\forall r \in F_l \subseteq M_0$, $(r, l) \in A_0(r)$; here $A_0(r)$ is the fiber of $A_0$ at $r$, and $\tilde{\gamma}_0 : A_0 \to A_0$ is the linearization of $\gamma_0$ (Definition 1.4).

**Lemma 2.8.** Perhaps after adding a suitable integer multiple of $2\pi$, we may assume that $\beta - \theta_l$ vanishes to third order at $v = 0$, that is,

$$\beta(g, x'_0, v) = \theta_l + \sum_{|I|+|J|=3} c_{I,J}(g, x'_0)v^I \overline{v}^J + R(g, x'_0, v),$$

where $R(g, x'_0, \cdot)$ vanishes to fourth order at $v = 0$.

**Proof.** Let $\gamma_{x_0} : X_0 \to X_0$ be the contactomorphism induced by $\gamma_X$ by passage to the quotient. In other words, $\gamma_{x_0}$ is the restriction to $X_0$ of the dual linearization $(\tilde{\gamma}_0^{-1})^* \circ A^*_0$. Let us momentarily write $x'_0 = (m'_0, \eta)$, where $m'_0 = \pi(x'_0) \in F_l \subseteq \text{Fix}(\gamma_0)$, and $\eta \in A^*_0(m'_0)$ has unit norm. We obtain

$$\gamma_{x_0}^{-1}(x'_0) = (\gamma_0^{-1}(m'_0), \eta \circ \tilde{\gamma}_0) = (m'_0, e^{i\theta_l} \eta) = r_{e^{i\theta_l}}(x'_0). \tag{53}$$

On the other hand, (52) with $v = 0$ descends on $X_0$ to the relation

$$\gamma_{x_0}(x'_0) = r_{e^{i\beta(g, x'_0, 0)}}(x'_0). \tag{54}$$

Now (53) and (54) imply that $\beta(g, x'_0, 0) - \theta_l = 2\pi c$ for some $c \in \mathbb{N}$; by continuity, $c$ is constant, and we may assume without loss that $c = 0$. 

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Next, we make use of the fact that $\gamma_X$ is a contactomorphism, that is, $\gamma_X(\alpha) = \alpha$.

Let the Heisenberg local chart $\psi(y,x'_0)$ be as in (35), with image an open neighborhood $X(y,x'_0) \subseteq X$ of $\mu_0 \circ \sigma(x'_0)$. Let $(z^{(g,x'_0)}, \varphi^{(g,x'_0)}): X(y,x'_0) \to \mathbb{C}^1 \times \mathbb{R}$ be the associated local coordinates, as in (36). We write $\varphi^{(g,x'_0)} = a^{(g,x'_0)} + ib^{(g,x'_0)}$, with $a^{(g,x'_0)}$, $b^{(g,x'_0)}$ real-valued. Then by [SZ], §1 the local expression of $\alpha$ on $X(y,x'_0)$ has the form

$$\alpha = d\varphi^{(g,x'_0)} + a^{(g,x'_0)} db^{(g,x'_0)} - b^{(g,x'_0)} da^{(g,x'_0)} + \beta^{(g,x'_0)} \left( z^{(g,x'_0)} \right),$$

where $\beta^{(g,x'_0)} (z^{(g,x'_0)}) = O \left( \| z^{(g,x'_0)} \| ^2 \right)$. A similar expression, with $g$ replaced by $\kappa g$, holds for $\alpha$ on $X^{(\kappa g,x'_0)}$. Since $\left( \gamma_X^{-1} \right)^* \left( \varphi^{(\kappa g,x'_0)} \right) = \varphi^{(g,x'_0)} + \beta$, and $d_{\mu_0 \circ \sigma(m'_0)} \gamma^{-1}$ is unitary, hence symplectic, (52) implies

$$\alpha = d\varphi^{(g,x'_0)} + a^{(g,x'_0)} db^{(g,x'_0)} - b^{(g,x'_0)} da^{(g,x'_0)} + \beta^{(g,x'_0)} \left( z^{(g,x'_0)} \right)$$

$$= \left( \gamma_X^{-1} \right)^* \left( d\varphi^{(\kappa g,x'_0)} + a^{(\kappa g,x'_0)} db^{(\kappa g,x'_0)} - b^{(\kappa g,x'_0)} da^{(\kappa g,x'_0)} + \beta^{(\kappa g,x'_0)} \left( z^{(\kappa g,x'_0)} \right) \right)$$

$$= d\varphi^{(g,x'_0)} + d\beta + a^{(g,x'_0)} db^{(g,x'_0)} - b^{(g,x'_0)} da^{(g,x'_0)} + \left( \gamma_X^{-1} \right)^* \left( \beta^{(\kappa g,x'_0)} \left( z^{(\kappa g,x'_0)} \right) \right).$$

We deduce

$$d\beta = \beta^{(g,x'_0)} \left( z^{(g,x'_0)} \right) - \left( \gamma_X^{-1} \right)^* \left( \beta^{(\kappa g,x'_0)} \left( z^{(\kappa g,x'_0)} \right) \right) = O \left( \| z^{(g,x'_0)} \| ^2 \right).$$

Q.E.D.

**Corollary 2.3.** Let $y = y_k$ ($k = 1, 2, \ldots$) be as in (44), and set $\kappa := g g_{\sigma(m'_0)}^{-1} \in G$, $v := v_{t} + v_{n, \text{nor}} \in T_{\mu_0 \circ \sigma(m'_0)}M$. In the Heisenberg local chart $\psi^{(\kappa g,x'_0)}$,

$$\gamma_X^{-1}(y) = r_{e^{i\beta_k(g,x'_0,v)}} \left( \mu_{\kappa g} \circ \bar{\sigma}(x'_0) + \frac{1}{2 \sqrt{k}} d_{\mu_0 \circ \sigma(m'_0)} \gamma^{-1}(v) \right),$$

where $\beta_k(g, x'_0, v) \sim \theta_t + \sum_{j \geq 0} k^{-(3+j)/2} b_j(g, x'_0, v)$ as $k \to +\infty$.

On the upshot, with $y$ as in (44), we obtain

$$\Pi_{\varpi, k} \left( \gamma_X^{-1}(y), y \right) = \Pi_{\varpi, k} \left( r_{e^{i\beta_k(g,x'_0,v)}} \left( \mu_{\kappa g} \circ \bar{\sigma}(x'_0) + \frac{d_{\mu_0 \circ \sigma(m'_0)} \gamma^{-1}(v)}{\sqrt{k}} \right), \mu_{g} \circ \bar{\sigma}(x'_0) + \frac{v}{\sqrt{k}} \right)$$

$$= e^{ik\beta_k(g,x'_0,v)} \Pi_{\varpi, k} \left( \mu_{\kappa g} \circ \bar{\sigma}(x'_0) + \frac{d_{\mu_0 \circ \sigma(m'_0)} \gamma^{-1}(v)}{\sqrt{k}}, \mu_{g} \circ \bar{\sigma}(x'_0) + \frac{v}{\sqrt{k}} \right),$$

(55)
where \( v = v_t + v_{h,nor} \in T_{\mu_0}M_0 \). Now
\[
\mu_0 \circ \bar{\sigma}(x'_0) + \frac{v}{\sqrt{k}} = \mu_{k-1} \left( \frac{1}{\sqrt{k}} d_{\mu_0} \circ \bar{\sigma}(x'_0) + \frac{1}{\sqrt{k}} d_{\mu_0} \circ \bar{\sigma}(x'_0) \right)
\]

Now we remark that \( v_v = v_{h,tg} = 0 \); furthermore, \( d_{\mu_0} \) and \( d_{\gamma^{-1}} \) preserve the decomposition of tangent vectors described in (37) - (40), and the norm of each component. Let \( v_0 \in T_{m'_0}M_0 \) correspond to \( v_h \in H_{\mu_0} \subseteq T_{\mu_0}M_0 \). Any \( r \in T_{m'_0}M_0 \) may be decomposed as \( r = r_{tg} + r_{nor} \), where \( r_{tg} \in T_{m'_0}F_1, r_{nor} \in \left( T_{m'_0}F_1 \right) \). In our case, \( v_0 = v_{0,nor} \), whence
\[
(d_{\mu_0} \circ \sigma(m'_0) \gamma^{-1}(v))_0 = d_{m'_0} \gamma^{-1}(v_{0,nor}) = d_{m'_0} \gamma^{-1}(v)_{nor}.
\]

Thus,
\[
\psi_2(d_{\mu_0} \circ \sigma(m'_0) \mu_0(v)_h, d_{\mu_0} \circ \sigma(m'_0) \gamma^{-1}(v)_h) = \psi_2(v_{0,nor}, d_{m'_0} \gamma^{-1}(v)_{nor}).
\]

Therefore, by (46) we deduce
\[
\Pi_{\omega,k} \left( \mu_{k-1} \left( \frac{1}{\sqrt{k}} d_{\mu_0} \circ \bar{\sigma}(x'_0) + \frac{1}{\sqrt{k}} d_{\mu_0} \circ \bar{\sigma}(x'_0) \right) \right) \sim \left( \frac{k}{\pi} \right)^{d-g/2} \frac{2^{g/2}}{V_{\text{eff}(x'_0)}} \chi_{\varpi}(\kappa) e^{-2\gamma\|v\|^2} e^{\psi_2(v_{0,nor}, d_{m'_0} \gamma^{-1}(v)_{nor})} \cdot \left( 1 + \sum_{j \geq 1} a_{\omega j} (x, v) k^{-j/2} \right).
\]

Taking conjugates, we obtain from (55) and (57) that for \( y \) given by (44) we have
\[
\Pi_{\omega,k} \left( \gamma^{-1}_X(y), y \right) \sim \left( \frac{k}{\pi} \right)^{d-g/2} \frac{2^{g/2}}{V_{\text{eff}(x'_0)}} e^{i k \theta_l} \chi_{\varpi} \left( g_{\sigma(m'_0)} \right) e^{-2 \|v_l\|^2} e^{\psi_2(d_{m'_0} \gamma^{-1}(v_{0,nor}), v_{0,nor})} \cdot \left( 1 + \sum_{j \geq 1} b_{\omega j} (x, v) k^{-j/2} \right).
\]

Given (45), (58), and Corollary 2.3, as \( k \to +\infty \) the integrand of (43) (omitting \( \gamma_k(x, y) \)) admits an asymptotic expansion
\[
\tau_j(y) \Pi_{\omega,k} \left( \gamma^{-1}_X(y), y \right) f(y) \quad (59)
\]
\[
\sim \tau_0^0(m'_0) f(m_\sigma(m'_0)) \left( \frac{k}{\pi} \right)^{d-g/2} \frac{2^{g/2}}{V_{\text{eff}(x'_0)}} e^{i k \theta_l} \chi_{\varpi}(F_l) e^{-2 \|v_l\|^2} e^{\psi_2(d_{m'_0} \gamma^{-1}(v_{0,nor}), v_{0,nor})} \cdot \left( 1 + \sum_{j \geq 1} c_{\omega j} (x, v) k^{-j/2} \right),
\]

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for certain polynomials $c_{\omega j}$ in $v$; here $\chi_{\omega}(F_l)$ is as in Definition 1.2.

(44) is obtained by composing the parametrization $T = T_{lj}$ in (42) with a rescaling by $k^{-1/2}$. Let us denote this composition by $T_{lj}(k)$. Accordingly, (43) may be viewed as an integral over $G \times X(V) \times \mathbb{R}^g \times \mathbb{C}^c l$, with respect to the density $(T_{lj}(k))^*(\text{dens}_X)$. In view of Lemma 3.9 of [DP], and by the construction of $T_{lj}(k)$ using Heisenberg coordinates and rescaling, this admits an asymptotic expansion of the form

$$(T_{lj}(k))^*(\text{dens}_X)(y) \sim k^{-c_l-g/2} V_{\text{eff}}(m'_0) |d\nu(g)| \text{dens}_{x_0}(x'_0) |dv| \cdot \left(1 + \sum_{a \geq 1} k^{-a/2} d_a(\theta, x'_0, v)\right),$$

where $|dv|$ is the Lebesgue measure on $\mathbb{R}^g \times \mathbb{C}^c l$, and each $d_a(\theta, x'_0, v)$ is a polynomial in $v$.

Before proceeding, we need to establish that the asymptotic expansion obtained by multiplying (59) and (60) can be integrated term by term; to this end, let us pause on the remainder term. By the considerations preceding (44), integration $v$ is over a ball of radius $\approx k^{1/6}$ in $\mathbb{R}^g \times \mathbb{C}^c l$. On the domain of integration, therefore, the remainder term in (58) satisfies (47), with $w_h$ replaced by $d\gamma^{-1}(v_h)$. When we multiply the asymptotic expansions, therefore, one of the typical contributions due to the remainder terms is bounded by

$$C \left(\frac{k}{\pi}\right)^{d-g/2-(N+1)/2} p_N(v) e^{-2(1-\epsilon)\|v_t\|^2 - \frac{1-\epsilon}{2} \|v_{h,\text{nor}} - d\gamma^{-1}(v_{h,\text{nor}})\|^2},$$

where $p_N$ is some polynomial in $v = v_h + v_t$, and $N$ is a positive integer, that may be assumed to grow to infinity with the length of our expansion. On the other hand, $D^{-1} > 0$ is the operator norm of $(I - d\gamma^{-1})^{-1}$ acting on any fiber of the normal bundle $N_l$. The other terms can be handled in a similar way.

We can thus integrate term by term, and this proves the existence of an asymptotic expansion for trace $(\Psi_{\omega,k})_{lj}$ in (43) as $k \to +\infty$, and therefore for trace $(\Psi_{\omega,k})$. Let us now explicitly compute the leading term.

Since the region where $\zeta_k \neq 1$ yields a contribution to the integral which is $O(k^{-\infty})$, in the following we shall set $\zeta_k = 1$. 
We have
\[ \int_{\mathbb{R}^q} e^{-2\|v_t\|^2} dv_t = \int_{\mathbb{R}^q} e^{-2\|x\|^2} dx = \left( \frac{\pi}{2} \right)^{q/2}. \] (62)

Next, let \( \Lambda_l \in U(c_l) \) denotes the unitary matrix representing the restriction of \( d_{m_0'} \gamma_0 \) to the normal space of \( F_l \) at \( m_0' \), in the induced coordinates. The conjugacy class of \( \Lambda_l \) only depends on \( l \). Let \( (v_1, \ldots, v_{c_l}) \) be an orthonormal basis of \( \mathbb{C}^{c_l} \) composed of eigenvectors of \( \Lambda_l \), with corresponding eigenvalues \( \lambda_1, \ldots, \lambda_{c_l} \in S^1 \setminus \{1\} \). If \( v_{0, \text{nor}} = \sum_{j=1}^{c_l} a_j v_j \) we have
\[ \psi_2(\Lambda_l^{-1} v_{0, \text{nor}}, v_{0, \text{nor}}) = \sum_{j=1}^{c_l} (\lambda_j - 1) |a_j|^2. \] (63)

Therefore, recalling that \( \int_{\mathbb{C}} e^{s|u|^2} du = -\pi/s \) if \( \Re(s) < 0 \), we get
\[ \int_{\mathbb{C}^{c_l}} e^{\psi_2(d_{m_0'} \gamma_0^{-1} (v_{0, \text{nor}}), v_{0, \text{nor}})} dv_{0, \text{nor}} = \prod_{j=1}^{c_l} \int_{\mathbb{C}} e^{(\lambda_j - 1) |u|^2} du = \pi^{c_l} \prod_{j=1}^{c_l} \frac{1}{1 - \lambda_j} \]
\[ = \pi^{c_l} \frac{\det (\text{id}_{(N_l)_{m_0'}} - d_{m_0'} \gamma_0^{-1}|_{N_l, m_0'})}{c_l(\gamma)} \]

where \( c_l(\gamma) \) is as in (3). On the upshot, the leading term of the asymptotic expansion for (43) is:
\[ \left( \frac{k}{\pi} \right)^{d_l} \frac{e^{ik\theta_l}}{c_l(\gamma)} \dim(V_{x_0}) \chi_{x_0}(F_l) \cdot \int_{\mathbb{R}^q} \tau_{lj}(m_0') \cdot \left( \int_{G} f(\mu_g \circ \sigma(m_0')) \, d\nu(g) \right) \, \text{dens}_{X_0}(x_0) \]

where \( m_0' = \pi_0(x_0') \in V_{lj} \). To complete the proof of Theorem 1, we need only sum over \( l, j \).

Q.E.D.

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