An elementary approach to solve recursions relative to the enumeration of S-Motzkin paths

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ABSTRACT
S-Motzkin paths (bijective to ternary trees) and partial version of them are calculated using only elementary methods from linear algebra.

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1. Introduction

Motzkin paths consist, as their cousins, the Dyck paths, of up steps and down steps, but additionally of horizontal steps. They start at the origin, end at the x-axis, and never go below the x-axis. The encyclopedia [1] contains more than 600 related items.

The subfamily of Motzkin paths called S-Motzkin paths has been introduced in [7] and analysed in [9]. Further papers on the subject are [3,4].

S-Motzkin paths are Motzkin paths that consist of \( n \) up steps, \( n \) horizontal steps, and \( n \) down steps. Ignoring down steps \((d)\), the sequence of horizontal \((h)\) and up \((u)\) steps must look like \textit{huhu . . . hu}. They are enumerated by

\[
\frac{1}{2n + 1} \binom{3n}{n}.
\]

In [5], partial S-Motzkin paths were introduced and analysed: If one stops the left-to-right traversal of an S-Motzkin path at some time, we might still be at level \( k \); since flat steps and up steps interchange, one needs to introduce two families to analyse this more general scenario of not necessarily ending on the x-axis, and also one can consider the S-Motzkin paths from right to left, ending at any prescribed level.

Let \( a_{n,k} \) denote the number of partial S-Motzkin paths of length \( n \) that end at height \( k \) and the last non-down step is an up step. Furthermore, let \( b_{n,k} \) denote the number of partial S-Motzkin path of length \( n \) that end at height \( k \) and the last non-down step is a level step. The natural choice for the initial values are \( a_{0,0} = 1 \) and \( b_{0,0} = 0 \).

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The analysis in [5] is appealing and instructive but uses heavy analytic machinery. The present paper aims at a (relatively) elementary treatment, using only systems of linear equations and Cramer’s rule to solve them. Some determinants appear, and they follow some third-order recursions that can be solved. The help of Maple and the package Gfun [8] is gratefully acknowledged.

As one reviewer suggested, we briefly describe the approach chosen in the earlier paper [5]. At the core of the analysis are the bivariate generating functions $A(z, u) = \sum_{n,k} a_{n,k} z^n u^k$ and $B(z, u) = \sum_{n,k} b_{n,k} z^n u^k$, which turn out to be rational functions with a cubic denominator in the variable $u$. The kernel method allows them to cancel one of the factors from the denominator and this makes the generating function quite explicit. Following the same approach with $C(z, u) = \sum_{n,k} c_{n,k} z^n u^k$ and $D(z, u) = \sum_{n,k} d_{n,k} z^n u^k$ (for related sequences $c_{n,k}$ and $d_{n,k}$, described briefly below and in more detail later), it comes as a surprise (at least in the beginning) that two of three factors need to be cancelled. This leads in either case to a set of two equations that can be solved.

It is easily seen that the following recurrence relations hold (details to follow):

$$a_{n,k} = b_{n-1,k-1} + a_{n-1,k+1},$$
$$b_{n,k} = a_{n-1,k} + b_{n-1,k+1}.$$

In the same spirit, we get the other pair of recursions (details in a later section)

$$c_{n,k} = c_{n-1,k-1} + d_{n-1,k},$$
$$d_{n,k} = d_{n-1,k-1} + c_{n-1,k+1}.$$

We may note that $a_{n,0} = c_{n,0}$, since both just enumerate S-Motzkin paths; we get

$$a_{3n,0} = c_{3n,0} = \frac{1}{2n+1} \binom{3n}{n}.$$

The following sections show how to solve these two systems of recursions by basically elementary methods.

By ‘solving’ we mean to find explicit expressions for the respective generating functions. One can then go one step further and read off coefficients, but that is not new, as it was already shown in [5].

The following shows an S-Motzkin path of length 21; the red part alone is a partial S-Motzkin path ending on level 3, and the blue (dashed) part alone, read from right to left is a (reversed) partial S-Motzkin path ending on level 3. Section 2 is on the enumeration of the red objects, and Section 3 is on the enumeration of the blue (dashed) objects.

While the analysis in this paper is focused on the elementary analysis of partial S-Motzkin paths, we strongly believe that the approach with determinants is of more general interest. Furthermore, in order to make this paper readable independently of [5], a few repetitions were unavoidable (Figure 1).

2. The first pair of recursions

The objects of this section are the numbers $a_{n,k}$, $b_{n,k}$ and their recursions.
Figure 1. An example of an S-Motzkin path of length 21, separated in a red part in the beginning and a blue (dashed) part at the end.

Definition 2.1: The numbers \(a_{n,k}\) \((b_{n,k})\) are defined as the number of partial S-Motzkin paths of length \(n\), ending at level \(k\) such the last non-down step is an up (level) step.

These sequences are coupled by the recursions:
\[
\begin{align*}
a_{n,k} &= b_{n-1,k-1} + a_{n-1,k+1}, \\
b_{n,k} &= a_{n-1,k} + b_{n-1,k+1}.
\end{align*}
\]

The recurrence relations are valid for \(n \geq 1\) and \(k \geq 0\); the quantity \(b_{n-1,-1}\) must be interpreted as 0. Furthermore, the initial conditions that make sense are \(a_{0,0} = 1\) and \(b_{0,0} = 0\). The recursions can be understood when distinguishing the cases of the last step leading to level \(k\). Note further that \(a_{n,0}\) is the number of S-Motzkin paths of length \(n\).

We use generating functions
\[
\begin{align*}
f_k &= f_k(z) = \sum_{n \geq 0} a_{n,k} z^n \quad \text{and} \quad g_k &= g_k(z) = \sum_{n \geq 0} b_{n,k} z^n.
\end{align*}
\]

Then the sets of recursions translate into
\[
\begin{align*}
f_k &= zg_{k-1} + zf_{k+1} + \lfloor k = 0 \rfloor, \\
g_k &= zf_k + zg_{k+1},
\end{align*}
\]

where we employed Iverson’s notation. Our goal is to eliminate one set of generating functions: We find
\[
\begin{align*}
f_k - zf_{k+1} &= -z(2g_k - g_{k-1}) + zf_{k+1} - z^2f_{k+2} + \lfloor k = 0 \rfloor, \\
\end{align*}
\]

and further
\[
\begin{align*}
f_k &= z^2f_k - 2zf_{k+1} - z^2f_{k+2} + \lfloor k = 0 \rfloor.
\end{align*}
\]

These recursions are best expressed in matrix form. It requires an infinite matrix, but we limit it by a parameter \(h\), so that traditional methods from linear algebra can be used. In the following example, \(h = 5\), since the functions \(f_0, \ldots, f_4\) are involved. At an appropriate time, we will push this parameter \(h\) to infinity.

\[
\begin{pmatrix}
1 & -z \\
-z^2 & 1 & -2z & z^2 \\
-z^2 & 1 & -2z & z^2 \\
-z^2 & 1 & -2z \\
-z^2 & 1 & -2z \\
\end{pmatrix}
\begin{pmatrix}
f_0 \\
f_1 \\
f_2 \\
f_3 \\
f_4 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

As we see, the first row of the matrix is different. In order to overcome this, we introduce the following determinants (they all depend on the variable \(z\), but we do not write this
explicitly most of the time):

\[ Dh = D_h(z) = \det \begin{pmatrix} 1 & -2z & z^2 \\ -z^2 & 1 & -2z & z^2 \\ -z^2 & 1 & -2z & z^2 \\ -z^2 & 1 & -2z & z^2 \end{pmatrix} \]

and

\[ D^*_h = D^*_h(z) = \det \begin{pmatrix} 1 & -z \\ -z^2 & 1 & -2z & z^2 \\ -z^2 & 1 & -2z & z^2 \\ -z^2 & 1 & -2z & z^2 \end{pmatrix}, \]

in both cases \( h \) refers to the numbers of rows/columns. Expanding \( D^*_h \) according to the first row, say, we find

\[ D^*_h = D_{h-1} - z^3 D_{h-2}, \]

and it is easier to work with the set of determinants \( D_k \), since the structure is more regular. Expanding this determinant \( D_k \) (again according to the first row, say) we get the recursion

\[ D_k - D_{k-1} + 2z^3 D_{k-2} - z^6 D_{k-3} = 0, \]

which has the characteristic equation \( X^3 - X^2 + 2z^3 X - z^6 = 0 \). In order to deal with the cubic equation successfully, it is a good idea to use an auxiliary variable: \( z^3 = t(1-t)^2 \). This strategy has been applied before in other publications, for instance in [6]. Then the 3 roots have a beautiful form:

\[ r_1 = (t-1)^2, \quad r_2 = \frac{t}{2}(2-t+W), \quad r_3 = \frac{t}{2}(2-t-W), \]

with \( W = \sqrt{4t-3t^2} \). Consequently

\[ D_h = Ar^h_1 + Br^h_2 + Cr^h_3, \]

and the constants will be worked out from the initial conditions \( D_0 = 1, D_1 = 1, D_2 = 1 - 2z^3 \). We find (of course using computer algebra!)

\[ A = \frac{t - 1}{3t - 1}, \quad B = \frac{3t^2 - 4t - W}{(3t - 1)(3t - 4)}, \quad C = \frac{3t^2 - 4t + W}{(3t - 1)(3t - 4)}. \]

Now we will use these determinants to solve for the unknown generating functions \( f_i \), using Cramer’s rule. Allowing a finite determinant with \( h \) rows and columns, means that we solve for the \( h \) functions \( f_0, f_1, \ldots, f_{h-1} \), and we get

\[ f_i = \frac{z^{2i} D_{h-i-1}}{D^*_h} = \frac{z^{2i} D_{h-i-1}}{D_{h-1} - z^3 D_{h-2}}. \]

Now it is time to consider the limit \( h \to \infty \). It is not too difficult to see that the
contributions from the roots $r_2$ and $r_3$ will disappear. This follows from the expansions
\[ r_{2,3} = -\frac{1}{2} t^2 \pm t^{3/2} + \cdots \]
for small $t$, and $t \approx x$. In this way, we find
\[ f_i = \lim_{h \to \infty} \frac{z^{2i} D_{h-i-1}}{D_{h-1} - z^3 D_{h-2}} = \lim_{h \to \infty} \frac{z^{2i} A_r^{h-i-1}}{A_r^{h-1} - z^3 A_r^{h-2}} = \frac{t^i}{r_1^{i-1} - z^3 r_1^{-2}} = \frac{t^i}{z^i(1 - t)}. \]
From the system of recursions, we further find
\[ zg_k = f_{k+1} - zf_{k+2} = \frac{t^{k+1}}{z^{k+1}(1 - t)} - \frac{t^{k+2}}{z^{k+1}(1 - t)} = \frac{t^{k+1}}{z^{k+1}} \]
and finally
\[ g_k = \frac{t^{k+1}}{z^{k+2}}. \]

3. The second pair of recursions

A (partial) reverse S-Motzkin path is a (partial) S-Motzkin path read from right to left.

Definition 3.1: The numbers $c_{n,k}$ ($d_{n,k}$) are defined as the number of partial S-Motzkin paths of length $n$, ending at level $k$ such the last non-up step is an level (down) step.

The system of recursions is
\[ c_{n,k} = c_{n-1,k-1} + d_{n-1,k}, \]
\[ d_{n,k} = d_{n-1,k-1} + c_{n-1,k+1}; \]
these recurrences hold for $n \geq 1$ and $k \geq 0$; $c_{n-1,-1}$ and $d_{n-1,-1}$ must be interpreted as zero, and the initial values are $c_{0,0} = 1$ and $d_{0,0} = 0$. The system is again obtained by distinguishing the two instances of the last step leading to level $k$. We set
\[ \varphi_k = \varphi_k(z) = \sum_{n \geq 0} c_{n,k} z^n \quad \text{and} \quad \psi_k = \psi_k(z) = \sum_{n \geq 0} d_{n,k} z^n. \]
The translation of the recurrences to generating functions is
\[ \varphi_k = z\varphi_{k-1} + z\psi_k + [k = 0], \]
\[ \psi_k = z\psi_{k-1} + z\varphi_{k+1}. \]
The matrix of interest is the transposed version of before:

\[
M = \begin{pmatrix}
1 & -z^2 & -2z & z^2 \\
-2z & 1 & -z^2 & z^2 \\
z^2 & -2z & 1 & -z^2 \\
z^2 & -2z & 1 & -z^2
\end{pmatrix}
\]

The equations of interest are

\[
M \begin{pmatrix}
\varphi_0 \\
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{pmatrix} = \begin{pmatrix}
1 \\
-2z \\
0 \\
0
\end{pmatrix}
\quad \text{and} \quad
M \begin{pmatrix}
\psi_0 \\
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix} = \varphi_0 \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

It is easier to work with the second version, and using that \( \varphi_0 = \frac{x^2}{1-t} = f_0 \), which follows from a combinatorial argument about reading the same combinatorial objects once from left to right resp. from right to left. If one wants to avoid such an argument, one can work with the first system, which is similar but a bit messier. We work with \((1, 0, 0, \ldots)^T\) and multiply the solution later by the factor \( \frac{x^2}{1-t} \).

We want to apply Cramer’s rule to solve the system. Each component will then be a quotient of two determinants. The determinant in the denominator is always the same and the same as before. Now let \( D_{n,i} = D_{n,i}(z) \) be the determinant with \( n \) rows, and index \( i \) is replaced by a single 1 in the first row and otherwise 0 in row 1 and column \( i \). These numbers satisfy a recursion, which can be obtained by hand, but it is easier to use a computer (gfund and Maple):

\[
D_{n,i} = z^{i-1} \tau_i D_{n-i} - z^{i+2} \tau_{i-1} D_{n-i-1},
\]

where

\[
\sum_{i \geq 0} \tau_i w^i = \frac{w}{1 - 2w + w^2 - z^3 w^3}.
\]

We have this explicit form

\[
\tau_i = \sum_{0 \leq l \leq (i-1)/3} \binom{i - 1}{2l + 1} z^{3l}
\]

which is beautiful but will not be used.

For the application of Cramer’s rule, we must consider

\[
\frac{D_{n,i}}{D_n} = z^{i-1} \tau_i \frac{D_{n-i}}{D_n} - z^{i+2} \tau_{i-1} \frac{D_{n-i-1}}{D_n}
\]

and its limit for \( n \to \infty \), which follows from

\[
\lim_{n \to \infty} \frac{D_{n-i}}{D_n} = (t - 1)^{-2i}.
\]
Consequently, we get
\[
\lim_{n \to \infty} \frac{D_{n,i}}{D_n} = z^{i-1} \tau_i (t - 1)^{-2i} - z^{i+2} \tau_{i-1} (t - 1)^{-2i-2} = \frac{t^i}{z^{2i+1}} (\tau_i - t \tau_{i-1}).
\]

But this expression is simpler than \(\tau_i\) alone:
\[
\sum_{i \geq 0} (\tau_i - t \tau_{i-1}) w^i = \frac{w(1 - tw)}{1 - 2w + w^2 - z^3 w^3} = \frac{1}{1 - 2w + w^2 + tw - 2tw^2 + t^2 w^2}.
\]

Since the denominator is only quadratic (in \(w\)), there is a Binet formula:
\[
\tau_i - t \tau_{i-1} = \frac{\mu_i^3 - \mu_i^2}{W}
\]
with
\[
\mu_2 = \frac{-t + 2 - W}{2}, \quad \text{and} \quad \mu_3 = \frac{-t + 2 + W}{2}, \quad \text{and} \quad W = \sqrt{4t - 3t^2} = \mu_3 - \mu_2.
\]

The final answer is then (notice the shift, since \(i\) corresponds to \(\psi_{i-1}\))
\[
\psi_i = \frac{z^2 t^{i+1}}{1 - t z^{2i+3}} \frac{\mu_3^{i+1} - \mu_2^{i+1}}{W} = \frac{t^{i+1}}{z^{2i+1}(1 - t)} \frac{\mu_3^{i+1} - \mu_2^{i+1}}{\mu_3 - \mu_2}.
\]

From
\[
\varphi_k = \frac{1}{z} \psi_{k-1} - \psi_{k-2},
\]
the values \(\varphi_1, \varphi_2, \ldots\) can be computed; \(\varphi_0\) is already known, and consistent with \(\varphi_0 = 1 + z \psi_0 = \frac{1}{1 - t}\).

4. Computing coefficients

For completeness, we discuss how to compute the coefficients of the generating functions that we obtained. This is more or less a repetition of what we did already in our earlier paper [5].

The first ingredient is the inversion of \(x = t(1 - t)^2\), i.e. \(t\) expressed as a series in \(x\):
\[
[x^n] t^k = \frac{k}{n} [w^{n-k}] \frac{1}{(1 - w)^{2n}} = \frac{k}{n} \binom{3n - k - 1}{n - k} \Rightarrow t^k = \sum_{n \geq k} \binom{3n - k - 1}{n - k} \frac{k}{n} x^n.
\]

This can be done using the Lagrange inversion formula, or, equivalently by contour integration:
\[
[x^n] t^k = \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} t^k = \frac{1}{2\pi i} \oint \frac{(1 - t)(1 - 3t)dt}{t^{n+1}(1 - t)^{2n+2}} t^k
\]
\[
= \frac{1 - 3t}{(1 - t)^{2n+1}} \binom{3n - k}{n - k} - 3 \binom{3n - k - 1}{n - k - 1} = \frac{k}{n} \binom{3n - k - 1}{n - k}.
\]

The contours are in all instances small circles around the origin.
And now
\[ a_{n,k} = [z^n] f_k(z) = [z^n] \left( \frac{t^k}{z^k(1-t)} \right) = [z^{n+k}] \left( \frac{t^k}{1-t} \right) = [x^{(n+k)/3}] \left( \frac{t^k}{1-t} \right). \]
\[ = \frac{1}{2\pi i} \oint \frac{dx}{x^{(n+k)/3+1}} \frac{t^k}{(1-t)} \]
\[ = \frac{1}{2\pi i} \oint \frac{dt}{t^{(n+k)/3+1}} \frac{(1-t)(1-3t)dt}{(1-t)^{2(n+k)/3+2}} t^k \]
\[ = [t^{(n-2k)/3}] \frac{1-3t}{(1-t)^{2(n+k)/3+2}} \]
\[ = \left( \frac{n+1}{n-2k/3-1} \right) - 3 \left( \frac{n}{n-2k/3-2} \right). \]

This works if \( n \equiv 2k \mod 3 \) and \( n \geq k \); other values are zero. Note that the forms given in [5] are different but equivalent. Similarly
\[ b_{n,k} = [z^n] g_k(z) = [z^n] \left( \frac{t^{k+1}}{z^{k+2}} \right) = [x^{(n+k+2)/3}] t^{k+1} \]
\[ = \frac{1}{2\pi i} \oint \frac{dx}{x^{(n+k+2)/3+1}} t^{k+1} \]
\[ = \frac{1}{2\pi i} \oint \frac{dt}{t^{(n+k+2)/3+1}} \frac{(1-t)(1-3t)dt}{(1-t)^{2(n+k+2)/3+2}} t^{k+1} \]
\[ = [t^{(n-2k-1)/3}] \frac{1-3t}{(1-t)^{2(n+k+2)/3+1}} \]
\[ = \left( \frac{n+1}{n-2k-1/3} \right) - 3 \left( \frac{n}{n-2k-1/3-1} \right). \]

This works if \( n \equiv 2k+1 \mod 3 \) and \( n \geq 2k+1 \); other values are zero.

Now we show how to extract
\[ d_{n,k} = [z^n] \psi_k(z) = [z^n] \frac{t^{k+1}}{z^{2k+1}(1-t)} \frac{\mu_3^{k+1} - \mu_2^{k+1}}{\mu_3 - \mu_2}. \]

The identity [2, equation (22)] will be useful for the calculation; it is classical and goes by the name of Girard-Waring
\[ \frac{\mu_3^{k+1} - \mu_2^{k+1}}{\mu_3 - \mu_2} = \sum_{i=0}^{\lceil k/2 \rceil} (-1)^i \binom{k-i}{i} (\mu_2 + \mu_3)^{k-2i} (\mu_2 \mu_3)^i \]
\[ = \sum_{i=0}^{\lceil k/2 \rceil} (-1)^{i+k} \binom{k-i}{i} (t-2)^{k-2i} (t-1)^{2i}. \]

Therefore
\[ d_{n,k} = [z^{n+2k+1}] t^{k+1} \sum_{i=0}^{\lceil k/2 \rceil} (-1)^{i+k+1} \binom{k-i}{i} (t-2)^{k-2i} (t-1)^{2i-1} \]
\[ \left[ z^{n+2k+1} \right] t^{k+1} (t-1)^M = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x^{(n+2k+1)/3+1}} t^{k+1} (t-1)^M = \frac{1}{2\pi i} \oint_{|t|=1} \frac{(1-t)(1-3t)dt}{t^{(n+2k+1)/3+1}(1-t)^2(n+2k+1)/3+2} t^{k+1} (t-1)^M = (-1)^M \left[ t^{(n-k+1)/3-1} \right] \frac{1-3t}{(1-t)^2(n+2k+1)/3-M+1} = (-1)^M \left[ \left( \frac{n+k-M}{(n-k+1)/3-1} \right) - 3 \left( \frac{n-k-M-1}{(n-k+1)/3-2} \right) \right]. \]

This works for \( n \equiv k - 1 \mod 3 \). The explicit formulæ for \( c_{n,k} \) and \( d_{n,k} \) are given in [5].

**Note**

1. This convenient notation is quite common these days: \([P] = 1\) if condition \( P \) is true otherwise \([P] = 0\).

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