Upper bounds on the rate in device-independent quantum key distribution

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Quantum key distribution (QKD) is a method to distribute secret key among sender and receiver by the transmission of quantum particles (e.g. photons). Device-independent quantum key distribution (DIQKD) is a version of quantum key distribution with a stronger security demand in that sender and receiver do not need to rely on the inner workings of their device, which consequently can come from an untrusted vendor. We study the rate at which DIQKD can be carried out for a given quantum channel connecting sender and receiver and provide a technique to upper bound the achievable rate. As a result, we show that the rate of device independent secure key can for some channels be much smaller (even arbitrarily so) than that of QKD. We do so, by first providing definition of device independent secure key rate, which is of independent interest. For bipartite states we formulate a sufficient condition for a gap between QKD and DIQKD key rates and examples of 8-qubit bipartite state exhibiting the gap between device dependent and device independent key.

Ever since the invention of the BB84 protocol [1], the strength point of quantum key distribution (QKD) has been its ability to provide information-theoretically secure key distribution [2]. However it is practically difficult to guarantee that the protocols are implemented correctly without introducing any side channel [3]. These side channels are usually present in the quantum part of the implementation, therefore there is a lot of interest in using quantum-device-independent QKD, or simply device-independent QKD (DIQKD), which requires the protocols to guarantee security based only on the statistics of the classical values generated by the protocol [4–9]. By construction then such protocols will be secure also against any side channel that could be present in the quantum devices. Recent results have finally been able to prove the security of a (class) of device-independent QKD protocols against general adversaries, while still achieving rates comparable to standard QKD [9], which we will call henceforth device-dependent QKD. Although the accuracy of devices that can allow for implementation of current protocols needs to be increased [10], as a building block, the proof of principle experiment exhibiting a loophole-free violation of the CHSH inequality has been already demonstrated [11–13].

Every DIQKD protocol $\mathcal{P}$, in the ideal case of no external influence, is a classical post-processing procedure run on some honest device consisting of a pair of a quantum state and set of measurements $(\mathcal{M}, \rho)$. The honest device is such that given no attack, namely if the measurements from $\mathcal{M}$ are indeed performed on (many copies of) $\rho$, the test designed as a part of the protocol $\mathcal{P}$ will be passed with high proba-
bility and subsequently, the processing done by \(\mathcal{P}\) on the obtained data will result in maximal key length. The length per device use is called the key rate of the protocol \(\mathcal{P}\). For the practical protocols known so far (see [14] and references therein), the honest state \(\rho\) is in general a maximally entangled state mixed with a maximally mixed state. The measurements from \(\mathcal{M}\) correspond to some Bell inequality [15–17], that is maximally violated on state \(\rho\) whenever the latter state is close to \(\rho\).

An important benchmark of the device \((\mathcal{M}, \rho)\) then becomes how much quantum device-independent key rate one can gain from it if we had the best protocol. The device may be honest, and then one asks for the maximal possible device-independent key rate that the provider of the device can deliver, maximizing over protocols. On the other hand \((\mathcal{M}, \rho)\) can be initially honest, but later changed due to some external factor. In fact this is what the honest parties are interested in: how much key will they gain under certain realistic circumstances, e.g. noise in the communication channel used by the device that can change the state \(\rho\) distributed between the honest parties into some other state \(\sigma\)? Similarly, the honest measurement \(\mathcal{M}\) can drift to some other measurement \(\mathcal{N}\) due to systematic error, hacking, etc.

In this manuscript we pose and study a stronger problem, namely how much quantum device-independent key rate \(K_{DI}^D(\rho)\) can the honest parties achieve from the state \(\rho\), optimized also over the choice of measurements \(\mathcal{M}\), on top of the choice of protocols \(\mathcal{P}\). This is a question that a provider of DIQKD devices can ask aiming at delivering the best device that can work under circumstances that would allow only for access to the state \(\rho\).

We present fundamental limitations in the form of non-trivial upper bounds on \(K_{DI}^D\). A trivial upper bound is the so called device-dependent distillable key \(K_D\), for which there exist different equivalent definitions [18–21]. This is the amount of key rate secure against quantum adversary that one can obtain from independent copies of a state \(\rho\) using trusted quantum local operations and classical communication (LOCC). Since such protocols are not limited to measure \(\rho\) by some POVM \(\mathcal{M}\), as it is the case for device-independent ones, the inequality \(K_{DI}^D \leq K_D\) is clear. Moreover due to quantum de Finetti techniques [20, 22], in the device-dependent scenario one can perform some tests in the protocol that uniquely identify the underlying state and guarantee that it is close to the expected form \(\rho^\otimes n\), so that even without trusting the source of states one can achieve the key rate \(K_D(\rho)\). Such a strong guarantee on the underlying measurement and state of the device is impossible to achieve in the device-independent setting, because there exist infinite different measurements and states that lead to the same device, thus opening a way to new upper bounds that are true in the device-dependent case.

An important result was the discovery that bound-entangled states states, and thus states positive under partial transposition (PPT states), for which the distillation of pure entanglement at non-zero rate is impossible [23], can produce key at non-zero rate [18, 19]. It was shown there, that some PPT states can approximate states with perfect key arbitrarily well, but also fall arbitrarily close to separable states after partial transposition. An analogous result for channels was shown in [24–26].

As the main result, we show that for PPT states \(\rho\)

\[
K_{DI}^D(\rho) \leq \min\{K_D(\rho^\Gamma), K_D(\rho)\}
\]

(1)

where \(\rho^\Gamma\) is the partial transpose of state \(\rho\). The PPT states that show a gap between distillable entanglement and distillable key thus also become examples of states for which \(K_D(\rho)\) is large while \(K_{DI}^D(\rho^\Gamma)\) is small. This implies, that there are PPT states \(\rho\) which in spite of having positive quantum device-dependent key, cannot serve to produce key for any quantum device-independent protocol with the same rate. This proves a fundamental gap between quantum device-independent and quantum device-dependent secure key distribution. More precisely, using more computable upper bounds on \(K_D\) such as the relative entropy of entanglement \(E_r\) [19, 27], we obtain the extreme examples of states \(\rho_d\) of increasing dimension \(d\) for
which

\[ 0 \leftarrow d \rightarrow \infty K_D^{DI}(\rho_d) \ll K_D(\rho_d) \rightarrow d \rightarrow \infty 1 \]  

(2)

We further provide a sufficient condition for a class of states to exhibit the gap between keys, and based on states given in [28] we explicitly construct an 8-qubit bipartite state (of 16 $\otimes$ 16 dimension) with $K_D \geq 0.925284$ and $K_D^{DI} \leq 0.00396825$.

Note that narrowing the question to PPT states is well posed, as some of them display a Bell violation [29]. The problem of to what extent one can violate a Bell inequality on PPT states or almost bound-entangled states with key has been first studied in [30]. There, using techniques from [31] based on the impossibility of swapping some states of key across a repeater, it was argued that if a state is hardly distinguishable by Local Operations and Classical Communication (LOCC) from a separable state (which admits a locally-realistic model), then it cannot violate to high extent any Bell inequality with sufficiently small number of inputs and/or outputs. In [30] the problem of the rate of violation on asymptotically many copies of the state was also studied, and again via similar techniques to [31], proved that for some state it is small. However it was left as an open problem, studied here, the relation between low rate of non-locality and the amount of distillable device-independent key.

As our second main contribution, we work out analogous results for quantum channels. In this case the device is a tuple $(M, \rho, \Lambda)$. The state $\rho$ is a bipartite state of Alice and the part that is an input to the channel $\Lambda$, and $M$ is a product measurement performed at Alice and Bob after the use of the channel $\Lambda$. We then define the device-independent private capacity of $\Lambda$ and provide an upper bound on it in terms of device independent measures of the channel $\Lambda$ composed with partial transposition map. There are known channels for which the device-dependent distillable key stays constant, has asymptotically device-independent distillable key. The key ingredient is a symmetry, the invariance under partial transposition, coming from the locality of the measurements and the device-independence. Our approach is general, so that in particular we obtain upper bounds in terms of both device-independent squashed entanglement and device-independent relative entropy of entanglement. This is important, because squashed entanglement and relative entropy of entanglement are incomparable entanglement measures. For example, for some states the squashed entanglement is a tighter upper bound on distillable key than the relative entropy of entanglement [31].

Our approach differs from [33], where key from statistical distributions is considered. Recently, upper bounds showing difference between one-way and two-way quantum device-independent key has been shown in [34] for the case of particular protocol, based on the CHSH inequality. In approach presented here, the bounds are valid for two-way key distillation protocol, any Bell inequality, given known state of the honest implementation that has positive partial transposition.

I. MAIN RESULTS

In order to provide with aforementioned examples, we define the device-independent relative entropy of entanglement as an upper bound on the device-independent distillable key, and use it to show that a known family of states, for which the device-dependent distillable key stays constant, has asymptotically zero device-independent distillable key. The key ingredient is a symmetry, the invariance under partial transposition, coming from the locality of the measurements and the device-independence. Our approach is general, so that in particular we obtain upper bounds in terms of both device-independent squashed entanglement and device-independent relative entropy of entanglement. This is important, because squashed entanglement and relative entropy of entanglement are incomparable entanglement measures. For example, for some states the squashed entanglement is a tighter upper bound on distillable key than the relative entropy of entanglement [31].

The bipartite local measurement $M$ of a device is a collection of POVMs for Alice and POVMs for Bob. Collectively they are indexed by a pair of finite inputs $(x,y) \in X \times Y$ (one for Alice and one for Bob). Let $(a,b) \in A \times B$ be their output sets. Without loss of generality we can assume that these sets are the same every time the device is used. We can then identify the device measurement just by its POVM elements as

\[ M = \{ M^x_a \otimes M^y_b \} \]  

(3)
Together with a quantum state $\rho$, $\mathcal{M}$ defines the device $(\mathcal{M}, \rho)$, namely the conditional probability distribution
\[ p(ab|x_y) = \text{tr} \left[ \rho \cdot M^a_x \otimes M^b_y \right]. \]

With the definition of device in place, we can define the device-independent distillable key $\tilde{\mathcal{K}}^{DI}_{\mathcal{D}}(\rho)$ of a quantum state $\rho$. Informally, $\tilde{\mathcal{K}}^{DI}_{\mathcal{D}}(\rho)$ it is a supremum over the finite key rates $\kappa$ achieved by all possible measurements $\mathcal{M}$ and all possible protocols $\mathcal{P}$ with $(\mathcal{M}, \rho)$ as the honest implementation of the device.

The finite key rate of a fixed protocol hides an infimum over all possible real implementations $(\mathcal{N}, \sigma)$ of the device $(\mathcal{M}, \rho)$, namely over all the real devices that pass the same test in the protocol and thus are compatible with the honest device. For this it is sufficient that the conditional probability distributions of the device are close for any input, a condition that we denote with
\[ (\mathcal{M}, \rho) \approx (\mathcal{N}, \sigma). \]

Similarly we use equality $(\mathcal{M}, \rho) \triangleq (\mathcal{N}, \sigma)$, when the conditional probability distributions are the same. Finally, as for any definition of a rate, the definition of $\tilde{\mathcal{K}}^{DI}_{\mathcal{D}}$ also contains the asymptotic limits of block length and error parameters. This process is sufficiently general to include the recently proposed protocols of [9] and realistic future protocols. However, for our purpose of upper bounding $\tilde{\mathcal{K}}^{DI}_{\mathcal{D}}$, such a definition is exceedingly cumbersome. To simplify the treatment, we define, as a relaxation, the device-independent distillable key $\bar{\mathcal{K}}^{DI}_{\mathcal{D}}$, where the knowledge of the adversary is restricted to i.i.d. (or simply iid) attacks, and prove upper bounds on this new measure without loss of generality. We emphasize that
\[ \mathcal{K}^{DI}_{\mathcal{D}} \geq \bar{\mathcal{K}}^{DI}_{\mathcal{D}}. \]

Although there is no correspondent of the quantum de Finetti theorem for device-independent QKD [14], and thus we cannot prove the equivalence the these two definitions as it has been done for the device-dependent case [35, 36], it is demonstrated in [9] that in the case of CHSH game general attacks including memory are as powerful as the i.i.d. attacks, corresponding to the so-called collective attack in the case of device-dependent quantum key distribution. There are thus indications that $\mathcal{K}^{DI}_{\mathcal{D}}$ and $\bar{\mathcal{K}}^{DI}_{\mathcal{D}}$ might not be so different.

Before we study these problems, we shall comment on certain intuitions that are immediate from present literature on DIQKD. It is obvious that for any $\mathcal{M}$ there exist states (separable ones) $\sigma$ for which $\bar{\mathcal{K}}^{DI}_{\mathcal{D}}(\mathcal{M}, \sigma)$ is zero. This is because also device dependent key $\mathcal{K}_D$ is zero for separable states [19, 37], while clearly the latter upper bounds the former. It is also plausible that conversely, for a given state $\rho$ different choices of measurement settings $\mathcal{M}$ can lead to different values of rates of certain protocols (See how the values of CHSH inequality [16] and Chained Bell inequality [38] differ for a protocol proposed in [39]). However, it is expected that for general pure state there is no gap between $\mathcal{K}_D$ and $\bar{\mathcal{K}}^{DI}_{\mathcal{D}}$, i.e. there exist protocols in DIQKD that perform as good as in the device-dependent case, with key rate achieving $\mathcal{K}_D$. This is because of the fact, that pure states are self-testable [40], i.e. identifiable up to irrelevant factors via statistics drawn from certain settings $\mathcal{M}$.

More precisely, we prove in Theorem 1 that
\[ \mathcal{K}_D^{DI}(\rho) \leq \bar{\mathcal{K}}^{DI}_{\mathcal{D}}(\rho) := \sup_{\mathcal{M}} \inf_{(\mathcal{N}, \sigma) = (\mathcal{M}, \rho)} \mathcal{K}_D(\sigma). \]

Thus by a similarly adapted version, we obtain upper bounds for any known entanglement measure that upper bounds on the device-dependent distillable key. Namely, we can generally define for any entanglement measure $E$ the following device-independent version
\[ E^d(\rho) := \sup_{\mathcal{M}} \inf_{(\mathcal{N}, \sigma) = (\mathcal{M}, \rho)} E(\sigma) \leq E(\rho), \]

and if $E$ is either the distillable key $\mathcal{K}_D$ or an upper bound on it, then it follows that
\[ \bar{\mathcal{K}}^{DI}_{\mathcal{D}} \leq \mathcal{K}^{DI}_{\mathcal{D}} \leq \mathcal{K}_D \leq E^d \leq E. \]

In particular for $E$ being the squashed entanglement $E_{sq}$ [41] and the relative entropy of entanglement $E_r$ [42] we obtain
\[ \bar{\mathcal{K}}^{DI}_{\mathcal{D}} \leq E^d_{sq}, E^d_r. \]
As already mentioned, all pure states are self testable, meaning that for any pure state there exist a measurement such that any device that has equal statistic produces a locally equivalent state. As such $E^i(\rho) = E(\rho)$ for all pure states and any entanglement measure. During the completion of this work we learned that $E^i$ has been defined independently in [43], where they only compute lower bounds on this quantity; no lower bound on the operational definition of device-independent key was given.

The key ingredient for the gap is then the observation that, if $\rho$ is PPT, namely if its partial transpose is positive $\rho^T = \text{id} \otimes \tilde{T}(\rho) \geq 0$ and thus a valid state, then we have

$$\left( M^\Gamma, \rho^\Gamma \right) = \left( M, \rho \right)$$

where $M^\Gamma$ is intended as the device measurement obtained by applying the partial transposition to each measurement operator, which returns a valid local measurements due to all operators being product. It thus follows immediately that

$$K^{DI}_D(\rho) \leq \sup_{M} \lim_{\epsilon \to 0} \lim_{n \to \infty} E(\rho^\Gamma) = E(\rho^\Gamma).$$

and thus

$$\tilde{K}^{DI}_D(\rho) \leq E_{\text{qq}}(\rho^\Gamma), E_r(\rho^\Gamma).$$

Our result then follows immediately for all those examples of PPT states that are close to private bits, but that after partial transposition become close to separable states [19, 28, 30, 31, 44]. We specifically choose relative entropy of entanglement $E_r$ as an upper bound of distillable key [42], and estimate it for the partial transpose of some of these examples to prove that there exist families of states of increasing dimension for which the upper bound of Equation (10) approaches zero, in spite of the fact that $K_D$ is lower bounded by a constant. Thus there exist quantum states with positive device-dependent key rate, for which there does not exist any Bell inequality with finite number of measurements and outputs, that can certify a non-negligible amount of device-independent key.

II. PROOF OVERVIEW

The idea of the proof of the bound from Equation (7) is the following. We first find the most simple definition for the device-independent key $K^{DI}_D$ of a bipartite state $\rho$, obtained by enlarging the power of the honest parties, while restricting the power of the adversary. This gives an upper bound on the realistic rates, and it is sufficient for our purposes, since any upper bound on the simplified rate $K^{DI}_D(\rho)$ will be an upper bound on any realistic rate. Here, we focus directly on the simplified rate $K^{DI}_D$ [9, 14]. Our definition lets Alice and Bob to optimize over the measurement $M$ used to get the statistics on the given state $\rho$. The resulting device-independent key is the worst device-dependent i.i.d. key rate of devices compatible with $(M, \rho)$.

In our approach we mimic the definition of device-dependent distillable key, which we thus recall here to be defined as [19, 31, 44]

$$K_D(\rho) := \inf_{\Pi} \lim_{\epsilon \to 0} \lim_{n \to \infty} \kappa^\epsilon_n(\Pi, \rho)$$

$$= \inf_{\epsilon \to 0} \lim_{n \to \infty} \kappa^\epsilon_n(\rho)$$

where $\lim = \lim_{\epsilon \to 0} \lim_{n \to \infty}$, $\kappa^\epsilon_n(\Pi, \rho)$ is the $\epsilon$-perfect key rate of the state obtained as the output of protocol $\Pi$ acting on $n$ identical copies of the state $\rho$, and where we use $\kappa^\epsilon_n(\sigma) := \sup_{\Pi} \kappa^\epsilon_n(\Pi, \sigma)$ to denote the supremum achieved by all protocols for a given security parameter $\epsilon$ and block-length $n$. Here and below, we leave the protocols $\Pi$ implicit because we can black-box our argument around it.

Our definition of device-independent distillable key follows the same lines:

$$K^{DI}_D(M, \rho) := \inf_{\epsilon \to 0} \lim_{n \to \infty} \kappa^{DI,\epsilon}_n(M, \rho)$$

where

$$\kappa^{DI,\epsilon}_n(M, \rho) := \sup_{\Pi'} \kappa^\epsilon_n(\Pi', (N, \sigma))$$

is the maximal keyrate achieved for any security parameter $\epsilon$, blocklength or number of copies $n$, and measurement $M$ chosen by Alice and Bob, and where $\kappa^\epsilon_n(\Pi, (N, \sigma))$ is the
FIG. 1: The space of classical protocols are all the (classical) local operations with public communication (LOPC). A protocol will use LOPC to generate the input to the device and process the output. A device-dependent distillation protocol in constrast uses quantum local operations and public communication. The composition of a device-independent classical LOPC protocol and the device measurement is an quantum LOPC protocol acting on the quantum state (here we ignored the blocklength parameter $n$). The assumption of the measurement device performing a product measurement must be physically imposed, usually by imposing that the input-output delay is shorter than light-speed travel between the devices.

By simple min-max inequality we can swap the order of the optimization to get an upper bound, and by then relaxing to all device-dependent protocols we have

\[
\kappa^{\text{DI},\epsilon}_n(M,\rho) := \sup_{\Pi'} \inf_{(\mathcal{N},\sigma) \approx_{\epsilon}(M,\rho)} \kappa^{\Pi'}_n(\mathcal{N},\sigma) \] \quad (16)
\[
\leq \inf_{\mathcal{N},\sigma} \sup_{(\mathcal{N},\sigma) \approx_{\epsilon}(M,\rho)} \kappa^{\Pi}_n(\mathcal{N},\sigma) \] \quad (17)
\[
= \inf_{\mathcal{N},\sigma} \kappa^{\epsilon}_n(\sigma) \] \quad (18)

where the rates in Equations (17) and (18) are the same rates introduced in Equations (12) and (13). The relaxation of the protocols in Equation (17) is clear and displayed in Figure 1; the measurement $\mathcal{N}$ acts like a fixed pre-processing on the state, and thus for any protocol $\Pi$ acting on the devices, the composition of $\Pi$ with the $n$ measurements $\mathcal{N}$ is just a particular protocol acting on $n$-copies of the state $\sigma$. Removing this constraint can only increase the rate. We thus have

\[
K^{\text{DI}}_D(M,\rho) \leq \inf_{\epsilon>0} \lim_{n\to\infty} \inf_{(\mathcal{N},\sigma) \approx_{\epsilon}(M,\rho)} \kappa^{\epsilon}_n(\sigma) \quad (19)
\]

Using Equation (19) as our simplest starting point we find.

**Theorem 1.**

\[
K^{\text{DI}}_D(M,\rho) \leq \inf_{(\mathcal{N},\sigma) \approx_{\epsilon}(M,\rho)} K_D(\sigma)
\]

**Proof.** First notice that the min-max inequality is valid also as a inf-limsup inequality. Namely for any sequence of functions $f_n(x)$, we have

\[
\lim_{n\to\infty} \inf_x f_n(x) \leq \inf_x \lim_{n\to\infty} f_n(x) \quad (20)
\]

since we can rewrite the limit superior using infimum and supremum, and then use max-min inequality followed by the commutation of two infima:

\[
\lim_{n\to\infty} \inf_x f_n(x) = \inf_{n\geq0} \sup_{m\geq n} \inf_x f_n(x)
\]
\[
\leq \inf_x \lim_{n\to\infty} \sup_{m\geq n} f_n(x)
\]
\[
= \inf_x \lim_{n\to\infty} f_n(x).
\]
We use Equation (19) as our starting point and use Equation (20)
\[ K_{DI}(\mathcal{M}, \rho) \leq \inf_{\varepsilon > 0} \lim_{n \to \infty} \inf_{(N, \sigma) \in (\mathcal{M}, \rho)} \kappa_n^\varepsilon(\sigma) \] (21)
\[ \leq \inf_{\varepsilon > 0} \lim_{n \to \infty} \inf_{(N, \sigma) \in (\mathcal{M}, \rho)} \kappa_n^\varepsilon(\sigma). \] (22)

Independently we can always restrict the infimum to devices that are exactly equal to the original box, this only reduces the set of devices and increases the infimum
\[ K_{DI}(\mathcal{M}, \rho) \leq \inf_{\varepsilon > 0} \lim_{n \to \infty} \inf_{(N, \sigma) \in (\mathcal{M}, \rho)} \kappa_n^\varepsilon(\sigma). \] (23)

Since the infimum over devices is now independent of the security parameter, we can now simply commute the two infima
\[ K_{DI}(\mathcal{M}, \rho) \leq \inf_{(N, \sigma) \in (\mathcal{M}, \rho)} \inf_{\varepsilon > 0} \lim_{n \to \infty} \kappa_n^\varepsilon(\sigma) \] (24)
\[ = \inf_{(N, \sigma) \in (\mathcal{M}, \rho)} K_D(\sigma) \] (25)
reaching the claim.

III. CHANNEL PRIVATE CAPACITY

The same idea also works for private, or secret, capacity \( \mathcal{P}(\Lambda) \) of a channel \( \Lambda \), and thus for the most general setting for QKD which includes modelling, for example, the optical fiber itself instead of the states produced across the fiber. A general protocol around \( n \) iid copies of \( \Lambda \) is displayed in Figure 2.

In the channel setting, given a channel \( \Lambda \) from Alice to Bob, we define an honest device for \( \Lambda \) as a tuple \((\mathcal{M}, \rho, \Lambda)\), where \( \rho \) is a bipartite state of Alice and the input to the channel, and \( \mathcal{M} \) is a device measurement of Alice and Bob (the output of the channel). The conditional probability distribution is then obtained via
\[ p(ab|xy) = \text{tr}\left[(\text{id} \otimes \Lambda)(\rho) \cdot M_a^x \otimes M_b^y\right], \]
and we have the same definitions of equality and distance for two devices.

Again, there are multiple choices for the definition of device-independent private capacity, depending on the allowed adversaries. The device-dependent private capacity is of the form
\[ \mathcal{P}(\Lambda):= \inf_{\varepsilon > 0} \lim_{n \to \infty} \pi_n^\varepsilon(\Lambda), \] (26)
where \( \pi_n^\varepsilon(\Lambda) \) is the largest \( \varepsilon \)-perfect key rate obtained by the best privacy protocol that uses \( n \) identical copies of \( \Lambda \). The private capacity itself has different version, namely the two-way (\( P_2 \)), one-way (\( P_1 \)), or direct (\( P_0 \)) private capacities depending on whether two-way \((LOCC_2)\) or one-way classical communication \((LOCC_1)\), or only local operations \((LOCC_0 = LO)\) is allowed in the privacy protocol (practical protocols might still need communication for practical purposes, like testing, outside/around the privacy protocol). With increasing power comes increasing rates and thus \( P_0 \leq P_1 \leq P_2 \).

To define the device-independent private capacity we want to change \( \pi_n^\varepsilon \), like for the state scenario, and define
\[ \mathcal{P}^{DI}(\mathcal{M}, \rho, \Lambda):= \inf_{\varepsilon > 0} \lim_{n \to \infty} \pi_n^{DI,\varepsilon}(\mathcal{M}, \rho, \Lambda), \] (27)
where \( \pi_n^{DI,\varepsilon} \) will be the largest key rate optimized over privacy protocols, this time also including a minimization over the possible devices that are compatible with the honest device. Just like in the state setting, on top of different powers of the honest parties generating different definitions for \( \pi_n^{DI,\varepsilon} \), additional diversity is introduced considering different powers of the adversary, namely different classes of devices to compare against the honest device.

Remark 1. We can envision a fully general class as in Figure 5, the class of devices where the
channel is iid but the state is generated adaptively with two way communication, the class of devices where the channel is iid and the input state is a global fixed state as in Figure 7, or the class of fully iid devices where the channel, state and measurement are iid as in Figure 3.

Again, the upper bounds that we are interested in are upper bounds for all these capacities, and thus, for simplicity, it will suffice to focus on just the iid devices of Figure 3 with iid channels, states and measurements.

We denote with $DI_0$, $DI_1$ and $DI_2$ the devices where the channel is iid, memory is allowed, and that can respectively use none, one-way or two-way classical communication between the input-output rounds. Notice, that this communication does not happen between the time Alice and Bob give their inputs and receive their outputs (which would not allow for device independence), but either before the input is given or after the output is received. The devices can still share memory locally at Alice and Bob across each round, and thus we can further restrict the adversary as mentioned above and define the (iid-device independent) variants $IDI_0$, $IDI_1$ and $IDI_2$, where the whole devices are iid and are not allowed memory or communication from one round to the next. Notice that the iid assumption in the case of channels is much stronger and unnatural than for the state case, because even if the channel itself in the device is iid, the device might be not-iid because of the input state; in general even Alice and Bob need to use entangled input states to achieve any capacity. Therefore our use of $IDI_0$ devices is purely technical. Each choice defines a private capacity $P_{i}^{IDI}$ or $P_{i}^{IDI}$ for $i, j = 0, 1, 2$, which is upper bounded by a device-dependent capacity as a consequence of defining an upper bounding the corresponding finite rates as follows. Each choice of $i$ and $j$ defines the key rates $\pi_{i,n}^{DI_j,\varepsilon}$ and $\pi_{i,n}^{DI_j,\varepsilon}$, both bounded by a device-dependent key rate by making protocol, together with the state and measurement of the device, a specific device-dependent protocol:

$$\pi_{i,n}^{DI_j,\varepsilon}(\mathcal{M}, \rho, \Lambda) \leq \pi_{i,n}^{IDI_j,\varepsilon}(\mathcal{M}, \rho, \Lambda) \leq \sup_{\Pi \in \text{LOCC}_1(\mathcal{N}, \sigma, \Lambda) \approx \text{SI}(\mathcal{M}, \rho, \Lambda)} \inf_{(\mathcal{N}, \sigma, \Lambda) \in IDI_j} \kappa_n^\varepsilon(\Pi, (\mathcal{N}, \sigma, \Lambda'))$$

(28)

$$\leq \inf_{(\mathcal{N}, \sigma, \Lambda) \approx \text{SI}(\mathcal{M}, \rho, \Lambda)} \pi_{\max(i,j),n}^\varepsilon(\Lambda')$$

(29)

$$\leq \inf_{(\mathcal{N}, \sigma, \Lambda) \approx \text{SI}(\mathcal{M}, \rho, \Lambda)} \pi_{2,n}^\varepsilon(\Lambda')$$

(30)

where $\kappa_n^\varepsilon$ is the rate of achieved $\varepsilon$-perfect key, while $\pi_{i,n}^\varepsilon$ is the rate already optimized over $t$-way protocols acting on $n$ copies of $\Lambda'$.

The largest of these capacities is $P_{2}^{IDI_0}$, since iid devices, larger $i$, and smaller $j$ make for larger rates. Private capacities with $j < i$ are arguably less meaningful, because it allows less classical communication between the devices than it is allowed for Alice and Bob. We can then define different variants of the DI version of an entanglement measure (for channels), namely for a measure $E(\Lambda)$ we can define different device-independent optimizations $E_{i0}$, $E_{i1}$ and $E_{i2}$ depending on the communication allowed in the device. Since, as an argument to the entanglement measure, we are concerned with only one copy of the channel, the devices in the optimization in $E_{i1}$ are iid devices.

Taking the rate limits and with the same ar-
\[
P_{1}^{\text{DI}}(M, \rho, \Lambda) \leq \inf_{(N, \sigma, \Lambda') \in \text{ID} \text{I}_{j}} P_{\text{max}\{i,j\}}(\Lambda') \quad (\text{32})
\]

and in particular, with \( \ell := \max\{i, j\} \),

\[
P_{i}^{\text{DI}_j}(\Lambda) := \sup_{M, \rho} P_{i}^{\text{DI}_j}(M, \rho, \Lambda) \quad (\text{33})
\]

\[
\leq P_{\ell}^{\text{DI}_j}(\Lambda) := \sup_{M, \rho} \inf_{(N, \sigma, \Lambda') \in \text{ID} \text{I}_{j}} P_{i}(\Lambda') \quad (\text{34})
\]

\[
\leq P_{2}^{\text{DI}_2}(\Lambda) \quad (\text{35})
\]

Remark. Notice how we did not prove that \( P_{1}^{\text{DI}_2} \leq P_{2}^{\text{DI}_2} \), the crucial step being Equation (30). Take the example of \( i = 1 \), Alice and Bob are allowed only one-way communication, and \( j = 2 \), the devices could use two-way communication. The possibility of \( P_{1}^{\text{DI}_2} > P_{2}^{\text{DI}_2} \) means that the added power of two-way communication, could allow the device to switch to a channel \( \Lambda' \) with very bad one-way private capacity, e.g. \( P_{1}(\Lambda') = 0 \), however to mimic the statistics of the original channel, some key needs to be extracted using the two-way communication, and this could still be a better attack than simply finding the worst replacement using only one-way communication.

We can finally make the same use of the partial transpose map, which we denote with \( \varnothing \) (\( \varnothing(\rho) = \rho^T \)). If a channel \( \Lambda \) is such that \( \Lambda \circ \varnothing \) is also a channel, then any device for \( \Lambda \) can be transformed into a device for \( \Lambda \circ \varnothing \) with the exact same statistics; the case of \( \text{IDI}_0 \) is shown in

![FIG. 4: Relationship between the device-independent channel capacities.](image)

![FIG. 5: The most general way an adversary could implement any device in device-independent QKD, whether the honest implementation is iid, state based, channel based, or neither. Single lines are quantum systems, double line are classical systems, and \( x_i, y_i \) are the inputs and \( a_i, b_i \) the outputs of the device. This is almost the most general case allowed by the entropy accumulation protocol; indeed, joining such a device with inputs generated by a markov chain and copying the inputs as additional outputs produces and entropy accumulation channel [9]. There thus exist tests that passing them restricts the adversary enough to generate device-independent key at non-zero asymptotic rate. Restricting the structure of the real devices can only increase the rates.](image)

![FIG. 6: The strongest restriction on the devices of the adversary from Figure 5 in the case of state-based device-independent QKD. Each round is an iid copy of the same measurement \( \mathcal{N} \) on the same state \( \sigma \). A protocol around this device is a particular LOCC protocol on \( \sigma^{\otimes n} \).](image)
FIG. 7: Restriction of the devices of the adversary in the case of channel-based device-independent QKD. No communication between the two sides of the device is allowed aside from an iid channel $\Lambda'$. Here, the input state is generated at each round from a quantum memory, while in the main text we also restrict to single-copy iid states. Both lead to the same bound, but this one is easier to display. A protocol around this device is still a particular LOCC protocol around $\Lambda^{\otimes n}$.

IV. DISCUSSION

We have derived a general bound on quantum device-independent key. To this end, we have provided both theoretic and realistic definition of the latter, as the supremum over the most general class of protocols [9, 10], which is of independent interest. Using the bound we have shown that there is a gap between the amount of device-dependent and device-independent key which can be obtained from some states with positive partial transposition. Such states can not then serve as a base for the honest implementation of any DIQKD protocol with non-negligible key rate.

Notice that the technique with the transposition map has led to upper bounds on Bell non-locality in terms of faithfull measures of entanglement [30], taking inspiration on how the same technique has led to upper bounds on device-dependent repeater key rates [31]. In [46] the bounds on the device-dependent repeater key rates where improved using arguments beyond the use of the partial transpose and leading to upper bounds in terms of distillable entanglement. It is our hope that a similar connection can be found for device-independent distillable key and private capacity, and that improved bounds on them can be found in terms of distillable entanglement and quantum capacity using the techniques of [46]. This can potentially lead to bound for non PPT states and channels. The 8-qubit example is (as for now) the smallest known state with positive partial transposition that exhibits the gap between $K_{DD}$ and $K_{DI}$. It is also the first example that exhibits at the same time the same gap between $K_{DD}$ and repeated key, and gives hope that the NPT states (private bits) with limited repeater key given in [46] shares the same gap of the keys as described here for the PPT case.

In general, the gap $\Delta K((\rho, M)) := K_D(\rho) - K_{DI}(\rho, M))$ is a measure of trust towards a device $(\rho, M)$ (analogously for quantum channels). For example, because singlet state is self-testable, $\Delta K$ is zero for the singlet state with CHSH testing, meaning that this device does not need to be trusted which. However, this is not the case for some bound entangled states for which our results prove that $\Delta K$ is non-zero.

Note

After concluding the research on this article, we became aware of independent but closely related work [47].

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V. EXPLICIT CONSTRUCTIONS

Form the above results, it follows, that a sufficient condition for a PPT state to exhibit a gap in device-dependent and device-independent key is that $K_D(\rho) < K_D(\rho^\Gamma)$. We show below a sufficient condition for a wide class of states.

Theorem 2. If $\rho_{ABA'B'} \in PPT$ and has a matrix form

$$
\begin{bmatrix}
A_1 & 0 & 0 & C \\
0 & B_1 & 0 & 0 \\
0 & 0 & B_2 & 0 \\
C^\dagger & 0 & 0 & A_2
\end{bmatrix},
$$

such that its normalized diagonal blocks are separable states, and $||A_1||_1 = ||A_2||_1 = a$ and $||B_1||_1 = ||B_2||_1$, then (denoting $||C|| = c$) condition $H(a - c, a + c, (1 - 2a)/2, (1 - 2a)/2) < 2a$ (38) implies a gap $K_{DI}(\rho) < K_D(\rho)$.

Proof. By [48], and from what is observed in [31] we have that $K_D(\rho) \geq 1 - H(a - c, a + c, b, b)$. To see this we employ the well known technique of privacy squeezing [18]. The privacy squeezed state of $\rho$ reads:

$$
\rho_{ps} = \begin{bmatrix}
||A_1||_1 & 0 & 0 & ||C||_1 \\
0 & ||B_1||_1 & 0 & 0 \\
0 & 0 & ||B_2||_1 & 0 \\
||C^\dagger||_1 & 0 & 0 & ||A_2||_1
\end{bmatrix}, \tag{39}
$$

and it has at least as much distillable key obtained via Devetak-Winter protocol applied to its $AB$ system, as $\rho$. To lower bound the latter, one purifies $\rho_{ps}$ to system of Eve: $|\psi_{ABE}\rangle$ and then measures systems $AB$ in computational basis to obtain a classical-classical-quantum state $\rho_{ccq}$. Then the lower bound on the key is

$$
K_D(\rho) \geq I(A : B)_{\rho_{ccq}} - I(A : E)_{\rho_{ccq}}. \tag{40}
$$

Since $\rho_{ccq} = (a + c)/2(P_{00} + P_{11}) \otimes e_1 + (a - c)/2(P_{00} + P_{11}) \otimes e_2 + bP_{01} \otimes e_3 + bP_{10} \otimes e_4$, where $P_{ij} = |ij\rangle\langle ij|$ and $\{e_i\}_{i=1}^d$ form an arbitrary orthonormal basis. there is:

$$
I(A : B)_{\rho_{ccq}} - I(A : E)_{\rho_{ccq}} = 2 - H\left(\frac{a + c}{2}, \frac{(a + c)}{2}, \frac{(a - c)}{2}, \frac{(a - c)}{2}, b, b\right)
$$

and the lower bound follows. It is enough to place an upper bound on the key of the partially transposed state. We observe first that the $\rho^\Gamma$ (where $\Gamma = I \otimes T$ is partial transposition) is a state, since $\rho \in PPT$. Next, $\rho^\Gamma$ can be expressed as a convex combination of two states $2a\sigma + 2b\rho'$ where $\sigma = 1/2|00\rangle\langle 00| \otimes A_1 + |11\rangle\langle 11| \otimes A_2 \in SEP$. Then $K_D(\rho^\Gamma) \leq E_R(\rho^\Gamma) \leq 2bE_R(\rho')$ where we used the fact that the relative entropy of entanglement upper bounds $K_D$ [18, 19], that it is convex, and that zero for separable states. It is then sufficient to note that $E_R(\rho') \leq 1$. This follows from the fact, that the relative entropy is non-lockable [49]. It is enough to apply (at random) one of two unitary transformations - $I$ and $\sigma$, to perform von-Neumann measurement on the $AB$ subsystem of $\rho_{ABA'B'}$. This operation turns the state into $\rho''$ of the form

$$
\begin{bmatrix}
A_1 & 0 & 0 & 0 \\
0 & B_1 & 0 & 0 \\
0 & 0 & B_2 & 0 \\
0 & 0 & 0 & A_2
\end{bmatrix}, \tag{42}
$$

which is by assumption separable. The non-lockability property of $E_r$ assures than (see theorem 3 of [18]), that this measure does not drop down under a von-Neumann measurement by more than the entropy of the random variable that samples unitary transformations in the process of this measurement. It was enough to sample uniformly at random from two unitary transformations to turn the state into a one that has $E_r(\rho'') = 0$, hence the relative entropy of $\rho$ could not be larger than $h(\frac{1}{2}) = 1$.

To see that $K_D(\rho) > K_D(\rho^\Gamma)$, one observes that form the normalization condition $2a + 2b =
1, the condition $1 - H(a - c, a + c, b, b) > 2b$ reads equivalently form given in Equation (38). The final argument follows then from the main theorem.

The sufficient condition given in Equation (38) can be further expressed with 2 parameters only, utilizing the normalization condition $2a + 2b = 1$. We therefore express $c = \alpha a$ with $\alpha \in [0, 1]$ and $b = (1 - 2a)/2$ obtaining equivalent condition as a function of $\alpha$ and $a$:

$$H((1 + \alpha)a, (1 - \alpha)a, (1 - 2a)/2, (1 - 2a)/2) < 2a$$  \hspace{1cm} (43)

The allowed region of parameters $(\alpha, a)$ that satisfy the above condition is presented in Figure 8

![Figure 8](image)

FIG. 8: The shaded region is the set of pairs $(\alpha, a) \in [0, 1] \times [0.415, 0.5]$ leading to the gap between the device-independent key $K_{DI}$ and the device-dependent one $K_D$, for states of the form of Equation (37), according to parametrization $c = \alpha a$ and $b = (1 - 2a)/2$.

**Examples.** We focus on PPT states considered in [31] (obtained in [28]). They are $2d_s \times 2d_s$ dimensional states of the form:

$$\rho_p = (1 - p)\gamma_1 + p\sigma$$  \hspace{1cm} (44)

where $\gamma_1$ is certain private state while

$$\sigma = \frac{1}{2}(|01\rangle\langle 01| \otimes \sqrt{YY^\dagger} + |10\rangle\langle 10| \otimes \sqrt{Y^\dagger Y})$$

$$\quad = \frac{1}{2}(|01\rangle\langle 01| \otimes Y_1 + |10\rangle\langle 10| \otimes Y_2)$$

is a separable state for certain $Y$ of trace norm 1, $Y_1 = \sqrt{YY^\dagger}$, $Y_2 = \sqrt{Y^\dagger Y}$, and $p = \frac{1}{\sqrt{d_s + 1}}$. It is easy to see that the state is still PPT, and a structure of a private state mixed with a separable state, when $p$ is replaced with $p_n = \frac{(p/2)^n}{2^n((1-p/2)^n+(p/2)^n)}$, and the corresponding blocks of $\gamma_1$ and that of $\sigma$ gets tensored $n$ times. After certain number of iterations $n$, that correspond to recurrence protocol applied to the key part of Equation (44) [19] the state becomes one-way key distillable having positive rate of the Devetak-Winter protocol [48] equal to: $1 - H(\{1 - 2p_n, p_n, p_n\})$, where $H$ is the Shannon entropy. Indeed the eigenvalues of the privacy squeezed state of the matrix given in Equation (45) are $\{a + c, a - c, b, b\}$ with $a = \frac{(1 - 2p_n)}{2} = c$ and $b = p_n$ with $n \geq 1$. More precisely, will then consider the following class of states $\rho_n$ of the form:

$$\begin{bmatrix}
(\frac{1-p}{2}X_1)^{\otimes n} & 0 & 0 & (\frac{1-p}{2}X)^{\otimes n} \\
0 & (\frac{p}{2}Y_1)^{\otimes n} & 0 & 0 \\
0 & 0 & (\frac{p}{2}Y_2)^{\otimes n} & 0 \\
(\frac{1-p}{2}X^\dagger)^{\otimes n} & 0 & 0 & (\frac{1-p}{2}X^\dagger)^{\otimes n}
\end{bmatrix}$$  \hspace{1cm} (45)

normalized by a multiplicative factor $1/N_n = 1/2((1-p)/2)^n + (p/2)^n$, where $X = 1/(d_s\sqrt{d_s})\sum_{i,j=0}^{d_s - 1} u_{ij}|i\rangle\langle j|$, $X_1 = \sqrt{XX^\dagger}$, $X_2 = \sqrt{X^\dagger X}$, with $u_{ij}$ being (in general complex) numbers of modulus $\frac{1}{\sqrt{d_s}}$ so that $U = \sum_{i,j} u_{ij}|i\rangle\langle j|$ is a unitary matrix. For specificity of this example one can take $U$ to be (tensor power of) a Hadamard matrix[50]. The above class of states where found by Lukas Pankowski via the genetic algorithms as a mid-step to the findings of [28], and were described in [44] and first used in [31].

Now, aiming at constructing as low-dimensional state exhibiting a gap, as it is possible by the above construction, we focus on $d_s = 2$. We can see now, that for $m = 3$ i.e. the
state $\rho_m$ having in total 8 qubits (2 for key part and 6 for the shield) there is $K_D(\rho_3) \geq 0.925284$, while $K_D(\rho_3^\Gamma) \leq 0.00396825$. Hence we have the desired gap. It is also easy to see that the separation is true for all $m \geq 3$ achieving $K_D(\rho_m) \approx 1$ and $K_D(\rho_m^\Gamma) \approx 0$ already for $m = 4$. We have thus explicitly shown an 8 qubit state, which proves the separation between the device-dependent and the device-independent key.