How do exponential size solutions arise in semidefinite programming?

Gábor Pataki, Aleksandr Touzov

July 28, 2023

Abstract

A striking pathology of semidefinite programs (SDPs) is illustrated by a classical example of Khachiyan: feasible solutions in SDPs may need exponential space even to write down. Such exponential size solutions are the main obstacle to solve a long standing, fundamental open problem: can we decide feasibility of SDPs in polynomial time?

The consensus seems that SDPs with large size solutions are rare. However, here we prove that they are actually quite common: a linear change of variables transforms every strictly feasible SDP into a Khachiyan type SDP, in which the leading variables are large. As to “how large”, that depends on the singularity degree of a dual problem. Further, we present some SDPs coming from sum-of-squares proofs, in which large solutions appear naturally, without any change of variables. We also partially answer the question: how do we represent such large solutions in polynomial space?

Key words: semidefinite programming; exponential size solutions; Khachiyan’s example; reformulations; singularity degree

MSC 2010 subject classification: Primary: 90C22, 49N15; secondary: 52A40

OR/MS subject classification: Primary: convexity; secondary: programming-nonlinear-theory

1 Introduction

Contents

1 Introduction .................................................. 1

1.1 Notation and preliminaries .................................. 6

2 Main results and proofs ...................................... 7

2.1 Reformulating (P) and statement of Theorem 1 .......... 7

2.2 Proof of Theorem 1 ......................................... 12

2.3 Computing the exponents by Fourier-Motzkin elimination .................................. 22

3 When we do not even need a change of variables .......... 23

*Department of Statistics and Operations Research, University of North Carolina at Chapel Hill
4 Conclusion

Linear programs and polynomial size solutions  The classical linear programming (LP) feasibility problem asks whether a system of linear inequalities

$$x_1a_1 + \cdots + x_ma_m + b \geq 0$$

has a solution, where the $a_i$ and $b$ are column vectors with integer entries. When the answer is “yes”, then by a classical argument there exists a feasible rational $x$ whose size is at most $2m^2 \log m$ times the size of the matrix $[a_1, \ldots, a_m, b]$. When the answer is “no”, there is a certificate of infeasibility whose size is similarly bounded.

Here and in the sequel we define size (or bit-length) as in [34, Section 2.1]. Precisely, the size of a rational number/vector/matrix is the sum of the sizes of its elements plus $k$; and the size of an $k \times \ell$ rational matrix is the sum of the sizes of its elements plus $k \cdot \ell$. The size of a rational number/vector/matrix is essentially the number of bits needed to describe it in binary representation.

Semidefinite programs and exponential size solutions  Semidefinite programs (SDPs) are a far reaching generalization of linear programs, and in recent decades they have attracted widespread attention. An SDP feasibility problem can be formulated as

$$\begin{align*}
x_1A_1 + \cdots + x_mA_m + B & \succeq 0, \quad (P)
\end{align*}$$

where the $A_i$ and $B$ are symmetric matrices with integer entries. We assume that the $A_i$ are linearly independent, and as usual, $S \succeq 0$ means that the symmetric matrix $S$ is positive semidefinite.

In striking contrast to a linear program, in some cases all feasible solutions of $(P)$ have exponential size as a function of the number of variables. This surprising fact is illustrated by a classical convex feasibility problem of Khachiyan:

$$x_1 \geq x_2^2, \ x_2 \geq x_3^2, \ \ldots, \ x_{m-1} \geq x_m^2, \ x_m \geq 2. \quad (Khachiyan)$$

We first show how fast the variables grow in $(Khachiyan)$, so suppose $x$ is feasible in it. Then by a straightforward calculation we get $x_1 \geq 2^{m-1}$, so $\log_2 x_1 \geq 2^{m-1}$. Hence the size of $x_1$, and of any feasible solution is at least $2^{m-1}$.

We next show how to cast $(Khachiyan)$ in the form of $(P)$, so we write its quadratic constraints as

$$\begin{pmatrix}
x_i & x_{i+1} \\
x_{i+1} & 1
\end{pmatrix} \succeq 0 \text{ for } i = 1, \ldots, m-1. \quad (1.1)$$

Then we define a symmetric matrix $A(x)$ with $2m-1$ rows (and columns) as follows. We set the first $m-1$ two by two principal blocks of $A(x)$ equal to the matrices in $(1.1)$, and the lower right corner of $A(x)$ equal to $x_m - 2$. Then $A(x) \succeq 0$ holds if and only if $x$ satisfies $(Khachiyan)$. Finally, it is straightforward to put the problem $A(x) \succeq 0$ into the form of $(P)$ by defining suitable $A_i$ and $B$ matrices.

We show the feasible set of $(Khachiyan)$ with $m = 3$ on the left in Figure 1. Our goal is to illustrate how fast $x_1$ and $x_2$ grow with respect to $x_3$. Hence, to better visualize this growth rate (and not run out of space), we replaced the constraint $x_3 \geq 2$ by $2 \geq x_3 \geq 0$ and made $x_3$ increase from right to left.

Observe that Khachiyan’s example shows much more than exponential size solutions (in the number of variables) in an SDP may exist. After all, even in a linear program with unbounded feasible set solutions of any size exist! However, in $(Khachiyan)$ all solutions must have exponential size; and for that, the key is the hierarchy among the variables.
To be precise, from now on an “SDP with exponential size solutions” will mean an SDP in which all feasible solutions have exponential size in the number of variables.

Why are we interested in SDPs with exponential size solutions? Mostly because they are the main obstacle to solving the following fundamental open problem:

Can we decide feasibility of $(P)$ in polynomial time?

Indeed, algorithms that decide feasibility of $(P)$ in polynomial time must assume that a polynomial size solution exists (if there is a solution to start with): see a detailed exposition in [10]. Algorithms that optimize a linear function over the set defined by $(P)$ in polynomial time also need similar assumptions [20, 32, 8]. In contrast, the algorithm in [28] that achieves the best known complexity bound to decide feasibility of SDPs uses a fundamental result from the first order theory of reals [31], and it runs in polynomial time only in fixed dimension.

We know of few papers that deal directly with the complexity of SDP. However, several works study the complexity of a related problem, optimizing a polynomial subject to polynomial inequality constraints. On the positive side, some polynomial optimization problems are polynomial time solvable when the dimension is fixed: see [31, 4, 5, 3, 36, 39]. Further, polynomial size solutions exist in special cases [38]. On the other hand, several fundamental problems in polynomial optimization are NP-hard, see for example, [5, 22, 1, 2].

Khachiyan’s example inevitably leads to the following questions:

Do SDPs with exponential size solutions occur frequently? (1.2)

In such SDPs, can we represent the feasible solutions in polynomial space? (1.3)

The answer to (1.2) seems to be a “no”, since the only such SDP we know of is (Khachiyan). However, to question (1.3) we have hope to get a “yes” answer. After all, to convince ourselves that $x_1 := 2^{2^{m-1}}$ (with a suitable $x_2, \ldots, x_m$) is feasible in (Khachiyan), we do not need to write down $x_1$ explicitly: instead, we can just do a symbolic computation. Still, question (1.3) seems to be open.
soon, but for now we only need to know two facts. First, \( k \leq 1 \) holds when \((P)\) is a linear program; and second, that \( k = m \) holds in the SDP representation of \((Khachiyan)\).

An informal version of our main result follows.

**Informal Theorem 1** Suppose \( k \geq 2 \). Then there is an invertible matrix \( M \) such that the linear change of variables \( x \leftarrow Mx \) transforms \((P)\) into a problem \((P')\) with the following properties:

If \( x \) is strictly feasible in \((P')\), and \( x_k \) is sufficiently large, then \( x_1, x_2, \ldots, x_k \) obey a Khachiyan type hierarchy. Precisely, the inequalities

\[
x_1 \geq d_2x_2^2, \ x_2 \geq d_3x_3^3, \ldots, x_{k-1} \geq d_kx_k^k
\]

hold, where

\[
2 \geq \alpha_j \geq 1 + \frac{1}{k-j+1} \quad \text{for } j = 2, \ldots, k.
\]

Here the \( d_j \) and \( \alpha_j \) are positive constants that depend on the \( A_i \) on \( B \), and the last \( m - k \) variables, that we consider fixed.

Loosely speaking, Theorem 1 can be interpreted as: “a linear transformation uncovers a Khachiyan type hierarchy in all strictly feasible SDPs.”

Suppose \( x \) is as stated in Informal Theorem 1, in particular \( x_k \) is large and positive. Then \( x_1 \) is larger than \( x_k \).

How much larger? To find out, we combine the inequalities (1.4) and (1.5). First we look at the worst case, when \( \alpha_j = 2 \) for all \( j \), just like in \((Khachiyan)\). Then \( x_1 \) is at least constant times \( x_k^{2^k} \), so the size of \( x_1 \) is exponentially larger than that of \( x_k \). On the other hand, suppose \( \alpha_j = 1 + \frac{1}{k-j+1} \) for all \( j \). Then by an elementary calculation we get that even in this best case \( x_1 \) is at least constant times \( x_k^k \).

We next discuss why our assumptions to prove Theorem 1 are minimal. First, we must assume that \((P)\) has a strictly feasible solution. Indeed, there are SDPs without strictly feasible solutions, with large dual singularity degree, but no Khachiyan type hierarchy among the variables, and no large solutions. We discuss such an SDP after Example 1.

Next we explain why in Theorem 1 we must allow a linear change of variables. Suppose we perform a linear change of variables, say \( x \leftarrow Gx \) in \((Khachiyan)\), where \( G \) is a random, dense matrix. After this change \((Khachiyan)\) will be quite messy, and will have no variables that are obviously larger than others.. Thus, if we performed this transformation, then we must perform its inverse \( x \leftarrow G^{-1}x \) to get back to \((Khachiyan)\). Finally, we argue that in Theorem 1 we must focus on just a subset of variables and restrict the last of these variables to be sufficiently large. Indeed, suppose we replace the constraint \( x_m \geq 2 \) by \( x_m \geq 2 + x_{m+1} \) in \((Khachiyan)\), where \( x_{m+1} \) is a new variable. After this change \( x_1, \ldots, x_m \) can all be zero, so there is no longer a hierarchy among them, and none of them is forced to be large. Thus, we must restrict \( x_m \) to be larger than 1 (even though \( x_m > 1 \) is now not implied by the constraints) to restore the hierarchy.

Besides proving Theorem 1, we show that in SDPs coming from minimizing a univariate polynomial a Khachiyan type hierarchy, and large variables appear naturally; that is, without a change of variables, and without assuming that \( x_k \) is large enough. The same is true of an SDP published in [21] that proves nonnegativity of a linear function over a set described by quadratic constraints.

We will also partially answer the representation question (1.3) as follows. Inequalities (1.4) and (1.5) imply that whenever \( x \) is strictly feasible in the transformed SDP \((P')\), variables \( x_1, \ldots, x_k \) can take on large values. However, we will see that to verify that a strictly feasible \( x \) in \((P')\) exists, we will never have to compute these large values numerically. Instead, we will just do a symbolic computation to convince ourselves that suitable values of \( x_1, \ldots, x_k \) exist. See the discussion after the proof of Lemma 2.
**Related work**  Linear programs can be solved in polynomial time, as it was first proved by Khachiyan [12]; see Grötschel, Lovász, and Schrijver [10] for an exposition that handles important details like the necessary accuracy. Other landmark polynomial time algorithms for linear programming were given by Karmarkar [11], Renegar [30], and Kojima et al [13].

On the other hand, to decide SDP feasibility in polynomial time, we must assume that there is a polynomial size solution (provided there is a solution). We refer to [10] for such an algorithm based on the ellipsoid method. The algorithm of Porkolab and Khachiyan [28] is the fastest known algorithm to decide SDP feasibility; however, it runs in polynomial time only for fixed \( n \) and \( m \). The algorithm of [28] uses a foundational result of Renegar [31], which decides in polynomial time the feasibility of a system of polynomial inequalities in fixed dimension. We further refer to Nesterov and Nemirovskii [20] for foundational interior point methods to solve SDPs with an objective function. We also refer to Renegar [32] for a very clean treatment of interior point methods for convex optimization; and to DeKlerk and Vallentin [8] for a very precise bit complexity analysis of interior point methods to solve SDPs.

The complexity of SDP is closely related to the complexity of optimizing a polynomial subject to polynomial inequality constraints. To explain how, first consider a system of convex quadratic inequalities

\[ x^\top Q_i x + b_i^\top x + c_i \leq 0 \quad (i = 1, \ldots, m) \tag{1.6} \]

where the \( Q_i \) are fixed symmetric psd matrices, and \( x \in \mathbb{R}^n \) is the vector of variables. The question whether we can decide feasibility of (1.6) in polynomial time is also fundamental, open, and, in a sense, easier than the question of deciding feasibility of (P) in polynomial time. The reason is that (1.6) can be represented as an instance of (P) by choosing suitable \( A_i \) and \( B \) matrices. On the other hand, we can formulate semidefiniteness of a symmetric matrix variable by requiring the principal minors (which are polynomials) to be nonnegative.

Among positive results in polynomial optimization, we already mentioned Renegar’s paper [31]. Bienstock [4] proved that such problems can be solved in polynomial time, if the number of constraints is fixed, the constraints and objective are quadratic, and at least one constraint is strictly convex. Further, Sakaue et al [33] designed a practical algorithm to solve such problems with two constraints. The work of [4] builds on Barvinok’s fundamental result [3] that proved we can test in polynomial time whether a system of a fixed number of quadratic equations is feasible. It also builds on early work of Vavasis [38] which proved that a system with linear constraints and one quadratic constraint has a solution of polynomial size. In other important early work, Vavasis and Zippel [39] proved we can solve indefinite quadratic optimization problems with a ball constraint, in polynomial time. Other related papers are e.g., by Stern and Wolkowicz [36], and Pong and Wolkowicz [27]. These show that the trust region subproblem with an indefinite objective function can be viewed as a convex problem, and hence solved efficiently.

On the flip side, there are many hardness results. For example, Bienstock, del Pia, and Hildebrand [5] proved it is NP-hard to test whether a system of quadratic inequalities has a polynomial size rational solution, even if we know that the system has a rational solution. Pardalos and Vavasis [22] proved the fundamental problem of minimizing a (nonconvex) quadratic function subject to linear constraints is also NP-hard. The following problem is also classical, and was proven to be NP-hard only in 2013, by Ahmadi, Olshevsky, Parrilo, and Tsitsiklis [1]: can we test convexity of a polynomial? It is also NP-hard to test whether a polynomial optimization problem attains its optimal value, see Ahmadi and Zhang [2].

One of the tools we use is an elementary facial reduction algorithm. These algorithms were originally designed to ensure strong duality in conic optimization problems. They originated in the paper of Borwein and Wolkowicz [6], then simpler variants were given, for example, by Waki and Muramatsu [40] and in [24, 25]. For a recent comprehensive survey of facial reduction and its applications, see Drusvyatskiy and Wolkowicz [9].

In other related work, O’Donnell [21] presented an SDP that certifies nonnegativity of a polynomial via the sum of squares (SOS) proof system, and is essentially equivalent to (Khachiyan). Previously it was thought that sum-of-squares proofs, a popular tool in theoretical computer science, can be found in
polynomial time. However, due to O’Donnell’s work, it is now clear that this is not obviously the case. Precisely, the complexity of finding SOS proofs is just as open as the complexity of deciding feasibility of SDPs.

The plan of the paper In Subsection 1.1 we review preliminaries. In Subsection 2.1 we formally state Theorem 1 and illustrate it via two extreme examples. In Subsection 2.2 we prove it in a sequence of lemmas. In particular, in Lemma 5 we give a recursive formula, akin to a continued fractions formula, to compute the \( \alpha_j \) exponents in (1.4). As an alternative, in Subsection 2.3 we show how to compute the \( \alpha_j \) using the classical Fourier-Motzkin elimination for linear inequalities; this is an interesting contrast, since SDPs are highly nonlinear. In Section 3 we cover the case of SDPs coming from polynomial optimization and also revisit the example from [21]. Section 4 concludes with a discussion.

Our proofs are fairly elementary. We use Proposition 1, a convex analysis argument about positive semidefinite matrices and linear subspaces. However, other than that, we only rely on basic linear algebra, and on manipulating quadratic polynomials.

1.1 Notation and preliminaries

Matrices Given a matrix \( M \in \mathbb{R}^{n \times n} \) and \( R,S \subseteq \{1,\ldots,n\} \) we denote the submatrix of \( M \) corresponding to rows in \( R \) and columns in \( S \) by \( M(R,S) \). We write \( M(R) \) to abbreviate \( M(R,R) \).

We let \( S^n \) be the set of \( n \times n \) symmetric matrices and \( S^n_+ \) be the set of \( n \times n \) symmetric positive semidefinite (psd) matrices. The notation \( S^n_+ \) means that the symmetric matrix \( S \) is positive definite.

The inner product of symmetric matrices \( S \) and \( T \) is defined as \( S \cdot T := \text{trace}(ST) \).

Definition 1. We say that \( (C_1,\ldots,C_\ell) \) is a regular facial reduction sequence for \( S^n_+ \) if each \( C_i \) is in \( S^n \) and of the form

\[
C_1 = \begin{pmatrix}
  r_1 & n-r_1 \\
  I & 0 \\
  0 & 0
\end{pmatrix}, \ldots, C_i = \begin{pmatrix}
  r_1 + \cdots + r_{i-1} & r_i & n - r_1 - \cdots - r_i \\
  \times & \times & \times \\
  \times & I & 0 \\
  \times & 0 & 0
\end{pmatrix}
\]

for \( i = 1,\ldots,\ell \), where the \( r_i \) are nonnegative integers, and the \( \times \) symbols correspond to blocks with arbitrary elements.

To provide background, we next explain the parlance “facial reduction sequence.” For that, suppose \( Y \) is a psd matrix, which has zero \( \cdot \) product with \( C_1,\ldots,C_\ell \). Since \( C_1 \cdot Y = 0 \), the sum of the first \( r_1 \) diagonal elements of \( Y \) is zero. Since these diagonal elements are nonnegative, they must be all zero. Thus, since \( Y \succeq 0 \), its first \( r_1 \) rows and columns are zero. Hence, \( C_2 \cdot Y \) is the sum of the diagonal elements of \( Y \) in rows \( r_1+1,\ldots,r_1+r_2 \) and we similarly deduce that these rows (and corresponding columns) of \( Y \) are all zero.

Continuing, we learn that \( Y \) is reduced to live in the set

\[
F = \{ Y \in S^n_+ : \text{the first } r_1 + \cdots + r_\ell \text{ rows and columns of } Y \text{ are zero} \},
\]

and we know that \( F \) is a face of \( S^n_+ \). \(^1\)

Next we formalize what we mean by “performing the linear change of variables \( x \leftarrow Mx \) in (\( P \))” for some invertible matrix \( M \).

\(^1\)A convex subset \( F \) of \( S^n_+ \) is a face, if for any \( X,Y \in S^n_+ \) if the open line segment \( \{ \lambda X + (1-\lambda)Y : 0 < \lambda < 1 \} \) intersects \( F \), then both \( X \) and \( Y \) must be in \( F \).
Definition 2. We say that we reformulate \((P)\) if we apply to it some of the following operations (in any order):

1. Exchange \(A_i\) and \(A_j\), where \(i\) and \(j\) are distinct indices in \(\{1, \ldots, m\}\).
2. Replace \(A_i\) by \(\lambda A_i + \mu A_j\), where \(i\) and \(j\) are distinct indices in \(\{1, \ldots, m\}\), \(\lambda\) and \(\mu\) are reals, and \(\lambda \neq 0\).
3. Replace all \(A_i\) by \(T^\top A_i T\) and \(B\) by \(T^\top BT\), where \(T\) is a suitably chosen invertible matrix.

We also say that by reformulating \((P)\) we obtain a reformulation.

Reformulations were originally introduced to study various pathologies in SDPs, for example, unattained optimal values and duality gaps \([26]\); and infeasibility \([15]\). In this work we show that they help understand another classical pathology, exponential size solutions.

We next clarify some technicalities about reformulations. First, the above definition of a reformulation slightly differs from the one in \([26]\), where we also permit replacing \(B\) by \(B + \lambda A_i\) for some \(i\) index and \(\lambda\) real number. Second, operations (1) and (2) can be viewed as elementary row operations on a dual type system, say, on \(A_i \cdot Y = 0\) for \(i = 1, \ldots, m\).

Third, operation (3) does not influence the values of the \(x_i\), since \(x \in \mathbb{R}^m\) is feasible in \((P)\) before we apply operation (3), if and only if it is feasible in it afterwards. Thus, we only use operation (3) to put \((P)\) into a more convenient looking form.

We will rely on the following statement about the connection of \(S^n_+\) and a linear subspace. It is a special case of a classical, more general statement about the intersection of a linear subspace and a convex cone: see e.g., \([18, \text{Theorem 2}]\).

Proposition 1. Suppose \(L\) is a linear subspace of \(S^n\). Then exactly one of the following two alternatives is true:

1. There is a nonzero positive semidefinite matrix in \(L\).
2. There is a positive definite matrix in \(L^\perp\).

\[ \square \]

2 Main results and proofs

2.1 Reformulating \((P)\) and statement of Theorem 1

In our first lemma we present an algorithm to reformulate \((P)\) into a more convenient looking form. The algorithm is a simplified version of the algorithm in \([15]\) \(^2\). Both algorithms are specialized facial reduction algorithms applied to the dual semidefinite system defined in \((2.7)\).

Lemma 1. The problem \((P)\) has a reformulation

\[ x_1 A_1^1 + \cdots + x_k A_k^1 + x_{k+1} A_{k+1}^\prime + \cdots + x_m A_m^\prime + B' \succeq 0 \]

\((P')\)

with the following properties:

\(^2\)The algorithm of Lemma 1 uses only Proposition 1, whereas the algorithm of \([15]\) relies on a more involved theorem of the alternative.
\* \* \* is a nonnegative integer, and \((A'_1, \ldots, A'_k)\) is a regular facial reduction sequence.

- If \(r_1, \ldots, r_k\) is the size of the identity block in \(A'_1, \ldots, A'_k\), respectively, then \(n - r_1 - \cdots - r_k\) is the maximum rank of a matrix in

\[
\{ Y \succeq 0 \mid A_i \bullet Y = 0 \text{ for } i = 1, \ldots, m \}.
\] (2.7)

**Proof** We will reformulate \((P')\) in several steps. To start, we let \(L\) be the linear span of \(A_1, \ldots, A_m\) and apply Proposition 1. If item (2) holds, we let \(k = 0, A'_i = A_i\) for all \(i\), \(B' = B\), and stop.

If item (1) holds, then we choose a nonzero psd matrix \(V = \sum_{i=1}^m \lambda_i A_i\) in \(L\) and assume \(\lambda_1 \neq 0\) without loss of generality. We then choose a \(T\) invertible matrix so that

\[
T^\top VT = \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix},
\] (2.8)

where \(r_1\) is the rank of \(V\). We let \(A'_1 := T^\top VT, A'_i := T^\top A_i T\) for \(i \geq 2\), and \(B' = T^\top BT\).

Let \(r\) be the maximum rank of a psd matrix in \(L^\perp\) (i.e., in the set defined in (2.7)). Also, let \(L'_{\text{new}}\) be the linear span of \(A'_1, \ldots, A'_m\). We claim that \(r\) is also the maximum rank of a psd matrix in \(L'_{\text{new}}\). For that, let us choose a rank \(r\) matrix, say \(Y\), in \(L^\perp \cap S^+_n\). Then for all \(i\) we have

\[
0 = A_i \bullet Y = T^\top A_i T \bullet T^{-1} Y T^{-\top},
\]

where the last equality is from the definition of the \(\bullet\) product and the properties of the trace. Thus \(T^{-1} Y T^{-\top}\) is in \(L'_{\text{new}}\) and has rank \(r\). Similarly, from any psd matrix in \(L'_{\text{new}}\) we can construct a psd matrix in \(L^\perp\) with the same rank. This proves our claim.

Suppose that \(Y \in L'_{\text{new}} \cap S^+_n\); then \(A'_i \bullet Y = 0\). Since \(A'_i\) is now the \(T^\top VT\) matrix given in (2.8), the sum of the first \(r_1\) diagonal elements of \(Y\) is zero. Since \(Y\) is psd, the first \(r_1\) rows and columns of \(Y\) are zero.

We next construct an SDP

\[
\sum_{i=2}^m x_i F_i + G \succeq 0,
\]

where \(F_i\) is obtained from \(A'_i\) by deleting the first \(r_1\) rows and columns for \(i = 2, \ldots, m\), and \(G\) is obtained from \(B'\) in the same manner. By the above argument the maximum rank of a matrix in \(\{ Z \succeq 0 : F_i \bullet Z = 0 (i = 2, \ldots, m) \}\) is also \(r\), so we can proceed in a similar manner with this smaller SDP. When our process stops, we have the required reformulation. \(\Box\)

The reader may wonder why we require \((A'_1, \ldots, A'_k)\) to be a regular facial reduction sequence in \((P')\). Will we use them to verify that any \(Y\) in the set

\[
\{ Y \succeq 0 \mid A'_i \bullet Y = 0 \text{ for } i = 1, \ldots, m \} \quad (2.9)
\]

has its first \(n - r_1 - \cdots - r_k\) rows and columns equal to zero? We could indeed use them for this purpose, by an argument similar to the one after Definition 1 (the \(A'_i\) would play the role of the \(C_i\)). However, interestingly, such a \(Y\) will never appear in our arguments in Lemmas 2, 3, and later. Instead, we will use the staircase structure of the \(A'_i\) to prove results about large size solutions in \((P')\).

From now on we assume that

\[
k \text{ is the smallest integer that satisfies the requirements of Lemma 1.}
\]
Using the terminology of facial reduction, \( k \) is the singularity degree of the dual system (2.7). This concept was originally introduced by Sturm in [37] and used to derive error bounds, namely, bounds on the distance of a point from the feasible set of an SDP. For a broad generalization of Sturm’s result to conic systems over so-called amenable cones, see a recent result by Lourenço [16].

Note that since \((P)\) is strictly feasible, so is \((P')\). Since we will focus on the leading \( k \) variables in \((P')\), for the rest of the paper we fix \((\bar{x}_{k+1}, \ldots, \bar{x}_m)\) such that \((x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m)\) is strictly feasible in \((P')\) for some \(x_1, \ldots, x_k\).

From now on we will say that a number is a constant, if it depends only on the \(\bar{x}_i\), the \(A_i\), and \(B\). Theorem 1 will rely on such constants.

We now formally state our main result.

**Theorem 1.** Let \((P')\) be the reformulation of \((P)\) obtained in Lemma 1, \(k\) the singularity degree of the dual system (2.7), and assume \(k \geq 2\). Then

1. There is \((x_1, \ldots, x_k)\) such that \((x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m)\) is strictly feasible in \((P')\) and \(x_k\) is arbitrarily large.
2. If \((x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m)\) is strictly feasible in \((P')\) and \(x_k\) is sufficiently large, then

\[
x_j \geq d_{j+1}x_k^{\alpha_{j+1}} \text{ for } j = 1, \ldots, k-1,
\]

where

\[
2 \geq \alpha_{j+1} \geq 1 + \frac{1}{k-j} \text{ for } j = 1, \ldots, k-1.
\]

Here the \(d_j\) and \(\alpha_j\) are positive constants.

Note that even \(k \geq 1\) easily implies that the feasible set of \((P')\) (or equivalently, that of \((P)\)) is unbounded. Indeed, since \(A'_1 \succeq 0\), we can add an arbitrarily large multiple of the first unit vector to any feasible solution of \((P')\), and stay feasible. Of course, Theorem 1 proves much more than the feasible set of \((P')\) is unbounded: it proves a hierarchy among the variables in \((P')\).

The proof of Theorem 1 has three main parts. First, in Lemma 2 we prove item (1), that in strictly feasible solutions of \((P')\) we can have arbitrarily large \(x_k\). Lemma 3 is a technical statement about a certain parameter, called the tail-index of the \(A'_i\); this parameter depends on where the nonzero blocks of the \(A'_i\) are.

In the second part, Lemma 4 deduces from \((P')\) a set of quadratic inequalities. These are typically “messy”, namely they look like

\[
(x_1 + x_2 + x_3)(x_4 + 10x_5) > (x_2 - 3x_4)^2.
\]

Third, in Lemma 5 from these messy inequalities we first derive “cleaned up” versions, such as

\[
x_1x_4 > \text{constant } x_2^2.
\]

Then from these cleaned up inequalities we deduce the inequalities (2.10) and a recursive formula to compute the \(\alpha_j\). Next, Lemma 6 proves that the \(\alpha_j\) exponents are a monotone function of the tail-indices

\footnote{We can also define the singularity degree of \((P)\). Since this problem is strictly feasible, its singularity degree is just zero.}
of the $A'_j$. Finally, Lemma 7 shows that minimal tail-indices of the $A'_j$ give the smallest possible $\alpha_j$ exponents. We then combine all lemmas and prove Theorem 1.

Before we get to the proof, we illustrate Theorem 1 via two extreme examples. Recall that Theorem 1 is about strictly feasible solutions of ($P'$), in which $x_k$ must be large enough. However, the following SDP examples are fairly simple, and in all of them we can derive interesting quadratic inequalities that hold for all feasible solutions.

**Example 1. (Khachiyan SDP)** Consider the SDP

$$\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \\ x_3 & x_4 \\ x_4 & 1 \end{pmatrix} \succeq 0,$$

($Kh$-SDP)

which can be written in the form of ($P'$) with the $A'_i$ matrices given below and $B'$ the matrix whose lower right corner is 1 and the remaining elements are zero:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The subdeterminants in ($Kh$-SDP) with three red, three blue, and three green corners, respectively, give the inequalities

$$x_1 \geq x_2^2, \quad x_2 \geq x_3^2, \quad x_3 \geq x_4^2$$

(2.12)

that appear in (Khachiyan). So the exponents in the inequalities (2.12) are the largest permitted by our bounds (2.11).

(For simplicity we constructed this SDP, so its feasible set does not imply the inequality $x_4 \geq 2$, which does appear in (Khachiyan).

What is $k$ in Example 1? By definition, it is the singularity degree of

$$\{Y \succeq 0 : A'_i \cdot Y = 0 \text{ for } i = 1, \ldots, 4\}$$

(2.13)

and we will next show that $k = 4$. For that, we first observe that the maximum rank of a matrix in this system is 1. Then we consider a reformulation of ($Kh$-SDP)

$$\sum_{i=1}^4 x_i A_i + B \succeq 0,$$

(2.14)

with two properties. First, for some $k \leq 4$ the sequence $(A_1, \ldots, A_k)$ is a regular facial reduction sequence. Second, the sizes of the identity blocks in the $A_i$ sum to $5 - 1 = 4$ (note that now $n = 5$). Since $k$ is the singularity degree of (2.14), it is minimal, so the identity blocks in $(A_1, \ldots, A_k)$ are nonempty. Thus $A_1$ is a positive multiple of $A'_1$, since $A'_1$ is the only nonzero psd matrix in the linear span of the $A'_i$. Similarly, it follows by induction for $i = 1, \ldots, k$ that each $A_i$ is a positive multiple of $A'_i$ plus a linear combination of $A'_{i-1}, \ldots, A'_1$. Thus $k = 4$ follows.

We note in passing that the feasible sets of ($Kh$-SDP) and of the derived quadratic inequalities (2.12) are not equal. For example $x = (256, 16, 4, 2)$ is not feasible in ($Kh$-SDP), but is feasible in (2.12).
However, we can construct an SDP that exactly represents \((\text{Khachiyan})\), as we described in Section 1. In that SDP a straightforward argument like the one we gave above proves that \(k = m\).

We next discuss whether we need to assume that a strictly feasible solution exists in \((P)\), in order to derive Theorem 1. On one hand, there are semidefinite programs which have no strictly feasible solutions, nor do they exhibit the hierarchy among the leading variables seen in Theorem 1. Indeed, we obtain such an SDP if in \((K\text{-SDP})\) we change \(x_1\) to \(x_1 + 1\) and the 1 entry in the bottom right corner to 0. This new SDP is represented by the same \(A'_i\) matrices, so the associated singularity degree of (2.13) is still four. Further, this new SDP is not strictly feasible, and \(x_2 = x_3 = x_4 = 0\) holds in any feasible solution, but \(x_1\) can be \(-1\). We can similarly create such an SDP with an arbitrary number of variables.

On the other hand, there are SDPs with no strictly feasible solution, which, however, do have a Khachiyan type hierarchy among the variables (and hence large size solutions). For that, we only need to take an SDP with a Khachiyan type hierarchy, and simply add all-zero rows and columns.

Example 2. (Mild SDP) As a counterpoint to \((K\text{-SDP})\) we next consider a mild SDP (we will see soon why we call it “mild”)

\[
\begin{pmatrix}
 x_1 & x_2 \\
 x_2 & x_3 & x_4 \\
 x_3 & x_4 & 1
\end{pmatrix} \succeq 0. \quad \text{(Mild-SDP)}
\]

We naturally write \((\text{Mild-SDP})\) in the form of \((P')\) with the \(A'_i\) matrices shown below and \(B'\) the matrix whose lower right corner is 1 and the remaining elements are zero:

\[
\begin{pmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
\end{pmatrix} \quad , \quad \begin{pmatrix}
 0 & 1 & 0 \\
 1 & 0 & 0 \\
 0 & 0 & 1
\end{pmatrix} \quad , \quad \begin{pmatrix}
 0 & 1 & 0 \\
 1 & 0 & 0 \\
 0 & 0 & 1
\end{pmatrix} \quad , \quad \begin{pmatrix}
 0 & 1 & 0 \\
 1 & 0 & 0 \\
 0 & 0 & 1
\end{pmatrix}.
\]

In \((\text{Mild-SDP})\) the subdeterminants with three red, three blue, and three green corners, respectively, yield the inequalities

\[x_1 x_3 \geq x_2^2, x_2 x_4 \geq x_3^2, x_3 \geq x_4^2. \quad (2.15)\]

Next from the inequalities in (2.15) we derive

\[x_1 \geq x_2^{4/3}, x_2 \geq x_3^{3/2}, x_3 \geq x_4^2. \quad (2.16)\]

as follows. We first copy the last inequality \(x_3 \geq x_4^2\) from (2.15) to (2.16). Next we plug \(x_3^{1/2} \geq x_4\) into the middle inequality in (2.15) to get \(x_2 \geq x_3^{3/2}\). We finally raise both sides of this last inequality to the power of \(2/3\) and plug it into the first inequality in (2.15) to deduce \(x_1 \geq x_2^{4/3}\).

To summarize, the exponents in the derived inequalities (2.16) are the smallest permitted by our bounds (2.11).

We invite the reader to verify that \(k = 4\) holds in Example 2; this can be done just like in Example 1.

To illustrate the difference between \((\text{Khachiyan})\) and the inequalities derived from \((\text{Mild-SDP})\), we show the set defined by the inequalities

\[x_1 x_3 \geq x_2^2, x_2 \geq x_3^2, 2 \geq x_3 \geq 0 \quad (2.17)\]

on the right in Figure 1. Note that the set defined by (2.17) is a three dimensional version of the set given in (2.15), which we normalized by adding upper and lower bounds on \(x_3\).
### 2.2 Proof of Theorem 1

In Lemmas 2–4 we will use the following notation:

\[
\begin{align*}
  r_j &= \text{size of the identity block in } A'_j \text{ for } j = 1, \ldots, k, \\
  \mathcal{I}_1 &= \{1, \ldots, r_1\}, \\
  \mathcal{I}_2 &= \{r_1 + 1, \ldots, r_1 + r_2\}, \\
  & \vdots \\
  \mathcal{I}_k &= \{r_1 + \cdots + r_{k-1} + 1, \ldots, r_1 + \cdots + r_k\}, \\
  \mathcal{I}_{k+1} &= \{r_1 + \cdots + r_k + 1, \ldots, n\}. 
\end{align*}
\] (2.18)

**Lemma 2.** There is \((x_1, \ldots, x_k)\) such that \((x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m)\) is strictly feasible in \((P')\) and \(x_k\) is arbitrarily large.

**Proof** Let

\[
Z := \sum_{i=k+1}^m \bar{x}_i A'_i + B'.
\]

Since there is \(x_1, \ldots, x_k\) such that \(\sum_{i=1}^k x_i A'_i + Z \succ 0\), and \(A'_i(\mathcal{I}_{k+1}) = 0\) for \(i = 1, \ldots, k\) we see that

\[
Z(\mathcal{I}_{k+1}) \succ 0.
\]

Recall that \(A'_k(\mathcal{I}_k) = I\) and the other elements of \(A'_k(\mathcal{I}_k \cup \mathcal{I}_{k+1})\) are zero. Hence by the definition of positive definiteness (a symmetric matrix \(G\) is positive definite if \(x^\top Gx > 0\) for all nonzero \(x\)) we see that the \(\mathcal{I}_k \cup \mathcal{I}_{k+1}\) diagonal block of \(x_k A'_k + Z\) is positive definite when \(x_k\) is large enough. Similarly, for any such \(x_k\) there is \(x_{k-1}\) so the \(\mathcal{I}_{k-1} \cup \mathcal{I}_k \cup \mathcal{I}_{k+1}\) diagonal block of \(x_{k-1}A'_{k-1} + x_k A'_k + Z\) is positive definite. We construct \(x_{k-2}, \ldots, x_1\) in a similar manner.

The proof of Lemma 2 is partly inspired by the paper of Lourenço et al [17], which used a similar process to construct a nearly feasible solution to a weakly infeasible semidefinite program.

The proof of Lemma 2 also partially answers the representation question (1.3). To explain how, for the moment let us ignore the requirement that we need to choose \(x_k\) to be large and just focus on completing \((\bar{x}_{k+1}, \ldots, \bar{x}_m)\) to a strictly feasible solution. The proof that the required \((x_1, \ldots, x_k)\) could be computed is fairly simple, and it is illustrated on Figure 2, where the red blocks stand for the larger and larger blocks that we make positive definite. So we can convince ourselves that \((x_1, \ldots, x_k)\) exist, even without computing their actual values.

![Figure 2: Verifying that \(x_1, \ldots, x_k\) exist, without computing them](image)

12
\[
A'_{j+1} = \begin{pmatrix}
\times & \times & \times & \times & \bullet \\
\times & \times & I & & \\
\times & x & & & \\
\times & & & & \\
\times & & & & \\
\end{pmatrix}.
\]

Figure 3: The tail-index of \(A'_{j+1}\)

From now on we will assume
\[
 r_1 + \cdots + r_k < n,
\] (2.19)
and we claim that we can do so without loss of generality. Indeed, suppose that the sum of all the \(r_j\) is \(n\). Then an argument like in the proof of Lemma 2 proves that \(A'_1, \ldots, A'_k\) have a positive definite linear combination. Hence the singularity degree of (2.7) is actually just 1; but we assumed \(k \geq 2\).

By (2.19) we see that \(I_{k+1} \neq \emptyset\).

To motivate our next definition we compare our two extreme examples from two viewpoints. From the first viewpoint we see that in \((Kh-SDP)\) the \(x_j\) variables in the upper offdiagonal positions are more to the right than in \((Mild-SDP)\). From the second viewpoint, in the inequalities (2.12) derived from \((Kh-SDP)\) the exponents are larger than in the inequalities (2.16) derived from \((Mild-SDP)\). We will see that these two facts are closely connected, so in the next definition we capture “how far to the right the \(x_j\) are in upper offdiagonal positions.”

**Definition 3.** We define the tail-index \(t_{j+1}\) of \(A'_{j+1}\) for \(j = 1, \ldots, k-1\) as
\[
t_{j+1} := \max \{ t : A'_{j+1}(I_j, I_t) \neq 0 \}.
\] (2.20)

In words, \(t_{j+1}\) is the index of the rightmost nonzero block of columns “directly above” the identity block in \(A'_{j+1}\). We illustrate the tail-index on Figure 3. Here and in later figures the \(\bullet\) blocks are nonzero, and we separate the columns indexed by \(I_{k+1}\) from the other columns by double vertical lines.

We further illustrate Definition 3 using Examples 1 and 2. In both of these, we have \(I_j = \{j\}\) for \(j = 1, \ldots, 5\). Thus, the tail-indices are \(t_2 = t_3 = t_4 = 5\) in \((Kh-SDP)\), whereas they are \(t_2 = 3, t_3 = 4, t_4 = 5\) in \((Mild-SDP)\).

**Lemma 3.**
\[t_{j+1} > j + 1 \text{ for } j = 1, \ldots, k - 1.\]

**Proof** We will use the following notation: for \(r, s \in \{1, \ldots, k+1\}\) such that \(r \leq s\) we let
\[
I_{r,s} := I_r \cup \cdots \cup I_s.
\] (2.21)

Let \(j \in \{1, \ldots, k - 1\}\) be arbitrary. To help with the proof, we picture \(A'_j\) and \(A'_{j+1}\) in equation (2.22). As always, the empty blocks are zero, and the \(\times\) blocks are arbitrary. The blocks marked by \(\otimes\) are \(A'_{j+1}(I_j, I_{(j+2),(k+1)})\) and its symmetric counterpart. We will prove that these blocks are nonzero and this will prove our lemma.
To get a contradiction, suppose the $\otimes$ blocks are zero. We first redefine $A'_j$ as $A'_j := \lambda A'_j + A'_{j+1}$ for some large $\lambda > 0$. Then by the definition of positive definiteness (a symmetric matrix $G$ is positive definite if $x^T G x > 0$ for all nonzero $x$) we find

$$A'_j (I_{j:(j+1)}) > 0.$$  

Let $Q$ be a matrix of suitable scaled eigenvectors of $A'_j (I_{j:(j+1)})$, define

$$T := \begin{pmatrix} I & Q \\ Q & I \end{pmatrix},$$

and let

$$A'_i := T^T A_i T$$

for $i = 1, \ldots, j, j+2, \ldots, k$.

After this transformation we have that $A'_j (I_{j:(j+1)}) = I$. Further, an elementary calculation shows that $(A'_1, \ldots, A'_j, A'_{j+2}, \ldots, A'_k)$ is a length $k-1$ regular facial reduction sequence that satisfies the requirements of Lemma 1. However, we assumed that the shortest such sequence has length $k$. This contradiction completes the proof.

In Lemma 4 we construct a sequence of polynomial inequalities that must be satisfied by any $(x_1, \ldots, x_k)$ that complete $(\bar{x}_{k+1}, \ldots, \bar{x}_m)$ to a strictly feasible solution. We need some more notation. Given a strictly feasible solution

$$(x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m)$$

we will write $\delta_j$ for an affine combination of the “$x$” and “$\bar{x}$” terms with indices larger than $j$. In other words,

$$\delta_j = \gamma_{j+1} x_{j+1} + \cdots + \gamma_k x_k + \gamma_{k+1} \bar{x}_{k+1} + \cdots + \gamma_m \bar{x}_m + \gamma_{m+1},$$

where the $\gamma_i$ are constants for $i = j+1, \ldots, m+1$.

We will actually slightly abuse this notation. We will write $\delta_j$ more than once, but we may mean a different affine combination each time. For example, suppose $k = 3$, and $m = 4$; then we may write $\delta_2 = 2x_3 + 3\bar{x}_4 + 5$ on one line, and $\delta_2 = x_3 - 2\bar{x}_4 - 3$ on another. Given that $\bar{x}_{k+1}, \ldots, \bar{x}_m$ are fixed, $\delta_k$ will always denote a constant.

**Lemma 4.** Suppose that $(x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m)$ is strictly feasible in $P'$. Then

$$p_j(x_1, \ldots, x_k) > 0 \text{ for } j = 1, \ldots, k-1,$$

for some $p_j$ polynomials defined as follows:

- if $t_{j+1} \le k$, then we choose $p_j$ as
  $$p_j(x_1, \ldots, x_k) = (x_j + \delta_j)(x_{t_{j+1}} + \delta_{t_{j+1}}) - (\beta_{j+1} x_{j+1} + \delta_{j+1})^2,$$

  where $\beta_{j+1}$ is a nonzero constant. In this case we call $p_j$ a type 1 polynomial.

- if $t_{j+1} = k + 1$, then we choose $p_j$ as
  $$p_j(x_1, \ldots, x_k) = (x_j + \delta_j) - (\beta_{j+1} x_{j+1} + \delta_{j+1})^2,$$

  where $\beta_{j+1}$ is a nonzero constant. In this case we call $p_j$ a type 2 polynomial.
Before we prove Lemma 4, we discuss it. First we note that $p_{k-1}$ will always be type 2, since by Lemma 3 (with $j = k - 1$) we have $t_k = k + 1$.

In Khachiyan’s example (Khachiyan) all inequalities come from type 2 polynomials, namely from $x_j - x_j^2$ for $j = 1, \ldots, k - 1$. In contrast, among the inequalities (2.15) derived from (Mild-SDP) the first two come from type 1 polynomials and the last one from a type 2 polynomial.

**Proof of Lemma 4** Fix $j \in \{1, \ldots, k - 1\}$. By the definition of $t_{j+1}$, there is a nonzero element in $A'_{j+1} (I_j, I_{j+1})$. Let us choose $\ell_1 \in I_j$ and $\ell_2 \in I_{j+1}$ such that the $(\ell_1, \ell_2)$ element of $A'_{j+1}$, which we denote by $(A'_{j+1})_{\ell_1, \ell_2}$, is nonzero.

As stated, suppose that $(x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m)$ is strictly feasible in $(P')$. For brevity, define

$$S := \sum_{i=1}^{k} x_i A'_i + \sum_{i=k+1}^{m} \bar{x}_i A'_i + B'. \quad (2.27)$$

We distinguish two cases.

**Case 1:** Suppose $t_{j+1} \leq k$. Below we show the matrices that will be important when we define $p_j$:

$$\begin{pmatrix}
    \times & \times & \times & \times & \times & \times \\
    \times & I & \times & \times & \times & \times \\
    \times & \times & \times & \times & \times & \times \\
    \times & \times & \times & \times & \times & \times \\
    \times & \times & \times & \times & \times & \times \\
    \times & \times & \times & \times & \times & \times \\
    \end{pmatrix} A'_j$$

$$\begin{pmatrix}
    \times & \times & \times & \times & \times & \times \\
    \times & \times & \times & \times & \times & \times \\
    \times & \times & \times & \times & \times & \times \\
    \times & \times & \times & \times & \times & \times \\
    \times & \times & \times & \times & \times & \times \\
    \times & \times & \times & \times & \times & \times \\
    \end{pmatrix} A'_{j+1}
$$

As usual, the empty blocks are zero and the $\times$ blocks may have arbitrary elements. (More precisely, $A'_{j+1} (I_{j+1}) = I$, but we do not indicate this in equation (2.28), since the other entries will suffice to derive the $p_j$ polynomial.) Also, the $\bullet$ blocks are nonzero.

Define $\beta_{j+1} := (A'_{j+1})_{\ell_1, \ell_2}$. Let $S'$ be the submatrix of $S$ that contains rows and columns indexed by $\ell_1$ and $\ell_2$. Then $S'$ looks like

$$S' = \begin{pmatrix} x_j + \delta_j & \beta_{j+1} x_{j+1} + \delta_{j+1} \\ \beta_{j+1} x_{j+1} + \delta_{j+1} & x_{j+1} + \delta_{j+1} \end{pmatrix}.$$  

We define $p_j(x_1, \ldots, x_k)$ as the determinant of $S'$, then $p_j$ is a type 1 polynomial as required in (2.25). Since $S' > 0$, we see that $p_j(x_1, \ldots, x_k) > 0$ and the proof in this case is complete.
Case 2: Suppose $t_{j+1} = k + 1$. Now $p_j$ will mainly depend on two matrices that we show below:

\[
\begin{pmatrix}
\times & \times & \times & \times \\
\times & I & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{pmatrix}
\quad \begin{pmatrix}
\times & \times & \times & \times \\
\times & I & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{pmatrix}
\]

Again, the \( \bullet \) blocks are nonzero.

Define $\lambda := (A'_{j+1})_{\ell_1,\ell_2}$ then from the definition of $\ell_1$ and $\ell_2$ we have $\lambda \neq 0$. Let $\mu := S_{\ell_2,\ell_2}$, then $S > 0$ implies $\mu > 0$. Also, since $\ell_2 \in I_{k+1}$, we see that $\mu$ depends only on $\bar{x}_{k+1}, \ldots, \bar{x}_m$, the $A'_i$ and $B'$, in other words it is a constant.

We again let $S'$ be the submatrix of $S$ that contains rows and columns indexed by $\ell_1$ and $\ell_2$. Then $S'$ looks like

\[
S' = \begin{pmatrix}
x_j + \delta_j & \lambda x_{j+1} + \delta_{j+1} \\ 
\lambda x_{j+1} + \delta_{j+1} & \mu 
\end{pmatrix}
\]

Define

\[
p_j(x_1, \ldots, x_k) := \frac{1}{\mu} \det S'.
\]

Since $S' > 0$, and $\mu > 0$, we have $p_j(x_1, \ldots, x_k) > 0$. Thus

\[
p_j(x_1, \ldots, x_k) = (x_j + \delta_j) - \left( \frac{\lambda}{\sqrt{\mu}} x_{j+1} + \frac{\delta_{j+1}}{\sqrt{\mu}} \right)^2.
\]

Hence $p_j(x_1, \ldots, x_k)$ is a type 2 polynomial in the form required in (2.26) with $\beta_{j+1} = \lambda/\sqrt{\mu}$. (Since $\mu$ is a constant, by our definition of $\delta_{j+1}$ in (2.23) we have that $\delta_{j+1}/\sqrt{\mu}$ is still $\delta_{j+1}$. ) The proof in this case is now complete.

Lemma 5. Suppose that $(x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m)$ is strictly feasible in $(P')$ and $x_k$ is sufficiently large. Then

\[
x_j \geq d_{j+1} x_{j+1}^{\alpha_{j+1}} \text{ for } j = 1, \ldots, k - 1,
\]

where the $d_{j+1}$ are positive constants and the $\alpha_{j+1}$ satisfy the recursion

\[
\alpha_{j+1} = \begin{cases} 
2 - \frac{1}{\alpha_j + 2 \cdots \alpha_{t_{j+1}}} & \text{if } t_{j+1} \leq k \\
2 & \text{if } t_{j+1} = k + 1
\end{cases}
\]

for $j = 1, \ldots, k - 1$.

Before we prove Lemma 5, we discuss it. We have $t_k = k + 1$ (by Lemma 3) hence Lemma 5 implies $\alpha_k = 2$. Hence, by induction the recursion (2.31) implies that $\alpha_j \in (1, 2]$ holds for all $j$ (naturally, we compute $\alpha_k, \alpha_{k-1}, \ldots, \alpha_2$ in this order). Thus, if $x_k$ is large enough, then $x_j > 0$ for $j = 1, \ldots, k$.

It is also interesting that formula (2.31) is reminiscent of a continued fractions formula.
To illustrate Lemma 5 we show how from \((\text{Mild-SDP})\) we can deduce the inequalities (2.16) much more quickly than we did before. Recall that in this example we have \(k = 4\). We compute the \(\alpha_{j+1}\) exponents by the recursion (2.31) as
\[
\begin{align*}
\alpha_4 &= 2 \quad \text{(since } t_4 = 5) \\
\alpha_3 &= 2 - 1/\alpha_4 = 3/2 \quad \text{(since } t_3 = 4) \\
\alpha_2 &= 2 - 1/\alpha_3 = 4/3 \quad \text{(since } t_2 = 3) \quad (2.32)
\end{align*}
\]

Next we sketch the proof of Lemma 5. We start with the inequalities \(p_j(x_1, \ldots, x_k) > 0\) derived in Lemma 4; these are satisfied by all strictly feasible solutions of \((P')\). Note that the \(p_j\) polynomials defined in (2.25) and (2.26) are quite messy. However, if \(p_j\) is a type 1 polynomial (defined in (2.25)), then we deduce a cleaned up inequality
\[
x_jx_{t_j+1} > \text{constant } x_{j+1}^2,
\]
assuming \(x_k\) is large enough. Similarly, if \(p_j\) is a type 2 polynomial (defined in (2.26)), then we derive a similarly cleaned up inequality
\[
x_j > \text{constant } x_{j+1}^2,
\]
assuming \(x_k\) is large enough. Then from the cleaned up inequalities we derive the required inequalities (2.30) and the recursion (2.31).

Next, since the proof of Lemma 5 is somewhat technical, we illustrate the cleaning up of the inequalities with an example.

**Example 3.** (Perturbed Khachiyan) As a warmup, first let us consider the SDP
\[
\begin{pmatrix}
x_1 & x_2 \\ x_2 & x_3 \\ x_2 & x_3 \\
x_2 & x_3 & 1
\end{pmatrix} \succeq 0, \quad (2.33)
\]
which is just a smaller version of \((\text{Kh-SDP})\). Thus the feasible solutions of (2.33) satisfy the inequalities
\[
x_1 \geq x_2^2, \quad x_2 \geq x_3^2. \quad (2.34)
\]
Next, let us consider a “perturbed” version of (2.33)

\[
\begin{pmatrix}
 x_1 - 2x_2 & x_2 - x_3 \\
 x_2 + x_3 & x_3 \\
 x_2 - x_3 & x_3 \\
\end{pmatrix} \succeq 0,
\]

which we obtain from (2.33) by replacing \(x_1\) by \(x_1 - 2x_2\) and \(x_2\) by \(x_2 \pm x_3\).

Suppose \((x_1, x_2, x_3)\) is feasible in (2.35) Then the quadratic inequalities (2.34) may not hold \(^4\). However, let us also assume \(x_3 \geq 10\). We then claim that the following slightly weaker inequalities do hold:

\[
\begin{align*}
 x_1 & \geq \frac{1}{2} x_2^2, \\
 x_2 & \geq \frac{1}{2} x_3^2.
\end{align*}
\]

To prove that, from the principal minors of (2.34) we first deduce the inequalities

\[
\begin{align*}
 x_1 - 2x_2 & \geq (x_2 - x_3)^2, \\
 x_2 + x_3 & \geq x_3^2.
\end{align*}
\]

Then (2.37) follows from (2.39) and \(x_3 \geq 10\) directly. Hence \(x_2 \geq 50\) also holds.

To prove (2.36), we lower bound the right hand side in (2.38) as

\[
\begin{align*}
 (x_2 - x_3)^2 & \geq (x_2 - \sqrt{2x_2})^2 \\
 & \geq \frac{1}{2} x_2^2,
\end{align*}
\]

where the first inequality is from \(x_2 \geq x_3\) and (2.37). The second inequality follows, since \(x_2 \geq 50\). Using this lower bound in (2.38) and \(x_2 \geq 0\), the desired inequality (2.36) follows.

We show the set described by the inequalities (2.34) which appear in (Khachiyan), and the feasible set described by the inequalities (2.38)-(2.39) on Figure 4. We normalized both sets by suitable bounds on \(x_3\). Note that \(x_3\) increases from right to left for better visibility.

**Proof of Lemma 5** We use an argument analogous to the one in Example 3. We use induction and show how to suppress the “\(\delta\)” terms in the type 1 and type 2 polynomials at the cost of making \(x_k\) large and choosing suitable \(d_j\) constants.

Suppose that \((x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m)\) is strictly feasible in \((P')\). Then by Lemma 4 the inequalities \(p_j(x_1, \ldots, x_k) > 0\) hold for \(j = 1, \ldots, k - 1\).

We first establish the base case for the induction. From the remark after the statement of Lemma 4 we recall that \(p_{k-1}\) is a type 2 polynomial. Hence

\[
p_{k-1}(x_1, \ldots, x_k) = (x_{k-1} + \delta_{k-1}) - (\beta_k x_k + \delta_k)^2 = (x_{k-1} + \gamma_k x_k + \delta_k) - (\beta_k x_k + \delta_k)^2 > 0,
\]

where the first equality is from the definition of \(p_{k-1}\) (see (2.26) with \(j = k - 1\)). Here \(\beta_k \neq 0\) is a constant. The second equality is from the definition of \(\delta_{k-1}\) (see (2.23)); where \(\gamma_k\) is a constant which may be zero.

\(^4\)For example, \(x = (5, 2, 2)\) is feasible in (2.35), but not in (2.34).
Since \( \delta_k \) is a constant, and \( \beta_k \neq 0 \), from (2.41) we deduce that \( x_{k-1} \geq d_k x_k^2 \) if \( x_k \) is sufficiently large, where \( d_k \) is a suitable positive constant. So the proof of the base case is complete.

For the inductive step we will adapt the \( O, \Theta \) and \( o \) notation from theoretical computer science. Given functions \( f, g : \mathbb{R}^k \rightarrow \mathbb{R}_+ \) we say that

(1) \( f = O(g) \) (in words, \( f \) is big-Oh of \( g \)) if there are positive constants \( C_1 \) and \( C_2 \) such that if \( (x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m) \) is strictly feasible in \( (P') \) and \( x_k \geq C_1 \), then

\[
f(x_1, \ldots, x_k) \leq C_2 g(x_1, \ldots, x_k).
\]

(2) \( f = \Theta(g) \) (in words, \( f \) is big-Theta of \( g \)) if \( f = O(g) \) and \( g = O(f) \).

(3) \( f = o(g) \) (in words, \( f \) is little-oh of \( g \)) if for all \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( (x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m) \) is strictly feasible in \( (P') \) and \( x_k \geq \delta \) then

\[
f(x_1, \ldots, x_k) \leq \epsilon g(x_1, \ldots, x_k).
\]

The usual calculus of \( O, \Theta \) and \( o \) carries over verbatim. We spell out one calculus rule that we will use repeatedly:

\[
f = o(h), g = \Theta(h) \Rightarrow f + g = \Theta(h).
\] (2.42)

In the implication (2.42) we say informally that we absorb the \( o(h) \) term into the \( \Theta(h) \) term.

For brevity, in the rest of this proof we will say that \( (x_1, \ldots, x_k) \) is good, if \( (x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_m) \) is strictly feasible in \( (P') \) and \( x_k \) is sufficiently large.

Suppose next that \( 1 < j + 1 \leq k - 1 \) and we have proved the following: for all good \( (x_1, \ldots, x_k) \) the inequalities

\[
x_{j+1} \geq d_{j+2} x_{j+2}^{\alpha_{j+2}} \\
x_{j+2} \geq d_{j+3} x_{j+3}^{\alpha_{j+3}} \\
\vdots \\
x_{k-1} \geq d_k x_k^{\alpha_k}
\]

(2.43)

hold, where \( d_{j+1}, \ldots, d_k \) are positive constants, and \( \alpha_{j+2}, \ldots, \alpha_k \) are positive constants derived from the recursion (2.31); further, \( \alpha_k = 2 \).

We will next show that for all good \( (x_1, \ldots, x_k) \) the inequality

\[
x_j \geq d_{j+1} x_{j+1}^{\alpha_{j+1}}
\]

(2.44)

holds, where \( d_{j+1} \) is another positive constant and \( \alpha_{j+1} \) is computed by the recursion (2.31).

For that, first note that the recursion (2.31) implies by straightforward induction that \( \alpha_k, \alpha_{k-1}, \ldots, \alpha_{j+2} \) are in the interval \([1, 2]\). So by the inequalities (2.43) we have

\[x_s = o(x_{\ell}) \text{ when } s > \ell \geq j + 1.\] (2.45)

We distinguish two cases.

**Case 1**: First suppose that \( t_{j+1} \leq k \), in other words, the quadratic polynomial \( p_j \) is type 1 (see (2.25)). Then we claim that for all good \( (x_1, \ldots, x_k) \) the following hold:

\[
0 < p_j(x_1, \ldots, x_k) = (x_j + \delta_j)(x_{t_{j+1}} + \delta_{t_{j+1}}) - (\beta_{j+1} x_{j+1} + \delta_{j+1})^2 \\
= (x_j + \gamma_{j+1} x_{j+1} + \delta_{j+1})(x_{t_{j+1}} + \delta_{t_{j+1}}) - (\beta_{j+1} x_{j+1} + \delta_{j+1})^2 \\
\leq (x_j + |\gamma_{j+1}| x_{j+1} + |\delta_{j+1}|)(x_{t_{j+1}} + |\delta_{t_{j+1}}|) - (\beta_{j+1} x_{j+1} + \delta_{j+1})^2 \\
= (x_j + \gamma_{j+1} x_{j+1} + o(x_{j+1}))(x_{t_{j+1}} + o(x_{t_{j+1}})) - (\beta_{j+1} x_{j+1} + o(x_{j+1}))^2.
\] (2.46)
Here $\beta_{j+1}$ is a nonzero constant and $\gamma_{j+1}$ is a constant which may be zero.

Indeed, in (2.46) the first inequality is by Lemma 4 and the first equality is from the definition of $p_j$ in (2.25). The second equality follows from the definition of $\delta_j$ in (2.23). The second inequality follows, since $j + 1 > 1$, hence by Lemma 3 we have $t_{j+1} > j + 1$, so for good $(x_1, \ldots, x_k)$ by the inequalities (2.43) we have $x_{t_{j+1}} > 0$ and $x_{t_{j+1}} + \delta_{t_{j+1}} > 0$. The third equality follows, since by (2.45) the term $\delta_{t_{j+1}}$ is a linear combination of $o(x_{j+1})$ terms; and by $t_{j+1} > j + 1$ and by (2.45) the term $\delta_{t_{j+1}}$ is a linear combination of $o(x_{t_{j+1}})$ terms.

We next claim that the last expression in (2.46) is upper bounded by

$$ (x_j + \Theta(x_j+1))\Theta(x_{t_{j+1}}) - \Theta(x_{j+1})^2. $$

For that, first assume $\gamma_{j+1} \neq 0$. Then by the rule (2.42) we absorb the $o(x_{j+1})$ terms into the terms with $x_{j+1}$; and the $o(x_{t_{j+1}})$ term into the term with $x_{t_{j+1}}$. Next, assume $\gamma_{j+1} = 0$. Then we use the bound $0 = \gamma_{j+1} x_{j+1} < \Theta(x_{j+1})$, and for the rest of the estimate we still use the absorbing rule (2.42).

We then continue (2.46) by using the upper bound (2.47):

$$ 0 < (x_j + \Theta(x_j+1))\Theta(x_{t_{j+1}}) - \Theta(x_{j+1})^2 $$

$$ = x_j \Theta(x_{t_{j+1}}) + \Theta(x_{j+1} x_{t_{j+1}}) - \Theta(x_{j+1})^2 $$

$$ \leq x_j \Theta(x_{t_{j+1}}) + o(x_{j+1}^2) - \Theta(x_{j+1})^2 $$

$$ \leq x_j \Theta(x_{t_{j+1}}) - \Theta(x_{j+1})^2, $$

where the second inequality follows, since $t_{j+1} > j + 1$ hence by (2.45) we have $x_{t_{j+1}} = o(x_{j+1})$. For the last inequality we absorb the $o(x_{j+1}^2)$ term into the $\Theta(x_{j+1}^2)$ term by the rule (2.42).

Next we use $t_{j+1} > j + 1$ and combine the inequalities (2.43) to learn that

$$ x_{t_{j+1}}^\alpha = O(x_{j+1}) $$

where $\alpha = \alpha_{j+2} \alpha_{j+3} \ldots \alpha_{t_{j+1}}$. Hence $x_{t_{j+1}} = O(x_{j+1}^{1/\alpha})$.

We next plug this last estimate into the last inequality in (2.48) and deduce

$$ 0 < x_j \Theta(x_{j+1}^{1/\alpha}) - \Theta(x_{j+1}^2). $$

We finally divide this last inequality by $x_{j+1}^{1/\alpha}$ and a constant, and deduce that

$$ x_j \geq d_{j+1} x_{j+1}^{2-1/\alpha} $$

for a suitable positive $d_{j+1}$ constant, if $x_k$ is large enough. Hence we can set $\alpha_{j+1} := 2 - 1/\alpha$ and the proof in this case is complete.

**Case 2** Suppose that $t_{j+1} = k + 1$, in other words, the quadratic polynomial $p_j$ is type 2. Then by Lemma 4 we get

$$ 0 < p_j(x_1, \ldots, x_k) = (x_j + \delta_j) - (\beta_{j+1} x_{j+1} + \delta_{j+1})^2 $$

$$ = (x_j + \gamma_{j+1} x_{j+1} + \delta_{j+1}) - (\beta_{j+1} x_{j+1} + \delta_{j+1})^2 $$

for some $\beta_{j+1} \neq 0$ and $\gamma_{j+1}$ constants, where $\gamma_{j+1}$ may be zero. Here the second equality is from the definition of $\delta_j$ in (2.23). By (2.45) we have $\delta_{j+1} = o(x_{j+1})$, hence

$$ x_j \geq d_{j+1} x_{j+1}^2 $$

for a suitable positive constant $d_{j+1}$ if $x_k$ is large enough. So we can set $\alpha_{j+1} = 2$, and the proof is complete.
As a prelude to Lemma 6, in Figure 5 we show three SDPs (for brevity we left out the $\geq$ symbols). The first is \((\text{Mild-SDP})\). The second and third arise from it by shifting \(x_2\) in the upper offdiagonal position to the right. Underneath we show the vector of the \((\alpha, \alpha, \alpha_3, \alpha_4)\) exponents in the inequalities derived by the recursion \((2.31)\).

We see that \(\alpha_2\) increases from left to right and Lemma 6 presents a general result of this kind.

**Lemma 6.** The \(\alpha_j\) exponents in \((2.11)\) are strictly increasing functions of the \(t_{j+1}\) tail-indices defined in Definition 3.

Precisely, suppose we derived the inequalities

\[ x_{\ell} \geq d_{\ell+1} x_{\ell+1}^{\alpha_{\ell+1}} \quad \text{for} \quad \ell = 1, \ldots, k-1 \]  \hspace{1cm} (2.49)

from \((P')\) using the recursion \((2.31)\). Here \(d_{\ell+1}\) is a positive constant for all \(\ell\).

Let \(j\) be an index in \(\{1, \ldots, k-1\}\) such that \(t_{j+1} \leq k\). Suppose we increase \(t_{j+1}\) by 1 (by changing \(A'_{j+1}\) in \((P')\)), then derive the inequalities

\[ x_{\ell} \geq f_{\ell+1} x_{\ell+1}^{\omega_{\ell+1}} \quad \text{for} \quad \ell = 1, \ldots, k-1, \]  \hspace{1cm} (2.50)

using the recursion \((2.31)\). Here \(f_{\ell+1}\) is a positive constant for all \(\ell\).

Then

\[ \omega_{\ell+1} = \begin{cases} \alpha_{\ell+1} & \text{if} \quad \ell > j \\ > \alpha_{\ell+1} & \text{if} \quad \ell = j \\ \geq \alpha_{\ell+1} & \text{if} \quad \ell < j. \end{cases} \]  \hspace{1cm} (2.51)

**Proof** Let us make all the assumptions and note that \(t_{j+1} \leq k\) implies that polynomial \(p_j\) is type 1.

We first prove \(\omega_{\ell+1} = \alpha_{\ell+1}\) for all \(\ell > j\). For that, we observe two facts. First, when we define the polynomials in Lemma 4, the only polynomial that refers to \(t_{j+1}\) is \(p_j\). Second, by the proof of Lemma 5, \(p_j\) is only used to derive the inequality \((2.44)\), and hence to determine the value of \(\alpha_{j+1}\). Of course, \(\alpha_{j+1}\) affects \(\alpha_j, \alpha_{j-1}, \ldots, \alpha_2\) via the recursion \((2.31)\). However, \(\alpha_{j+1}\) does not affect \(\alpha_{\ell+1}\) for \(\ell > j\). From these two facts our claim follows.

We next prove \(\omega_{j+1} > \alpha_{j+1}\). For brevity, let \(s := t_{j+1}\) and \(\alpha := \alpha_{j+2} \cdots \alpha_s\). We first observe that since \(s \leq k\), the recursion formula \((2.31)\) shows

\[ \alpha_{j+1} = 2 - \frac{1}{\alpha}. \]

We next examine how we compute \(\omega_{j+1}\) by formula \((2.31)\). We distinguish two cases. If \(s < k\), then in \((2.31)\) we use the top equation, so we get \(\omega_{j+1} = 2 - 1/(\alpha \cdot \alpha_{s+1})\). Since \(\alpha_{s+1} > 1\), we deduce \(\omega_{j+1} > \alpha_{j+1}\),
as wanted. If $s = k$, then $s + 1 = k + 1$, so in (2.31) we use the bottom equation. Thus $2 = \omega_{j+1}$ and $2 > \alpha_{j+1}$, so $\omega_{j+1} > \alpha_{j+1}$ again follows.

Finally, we prove $\omega_{\ell+1} \geq \alpha_{\ell+1}$ for $\ell > j$. Since we already proved this relation for all $\ell \leq j$, our claim follows by induction from the recursion formula (2.31).

Recall from Lemma 3 that the tail-index $t_{j+1}$ is at least $j + 2$ for all $j$. In our final lemma we examine the case when $t_{j+1}$ equals $j + 2$ for all $j$ and we derive a closed form solution for the $\alpha_{j+1}$ exponents.

**Lemma 7.** Suppose that $t_{j+1} = j + 2$ for $j = 1, \ldots, k - 1$. Then the recursion formula (2.31) yields

$$\alpha_{j+1} = 1 + \frac{1}{k - j} \text{ for } j = 1, \ldots, k - 1.$$  \hspace{1cm} (2.52)

**Proof** We use induction. First suppose $j = k - 1$. Since $p_{k-1}$ is of type 2, we see $\alpha_{j+1} = \alpha_k = 2$, as wanted. Next assume that $1 \leq j < k - 1$ and

$$\alpha_{j+2} = 1 + \frac{1}{k - j - 1}.$$  \hspace{1cm}

By the recursion (2.31) we get

$$\alpha_{j+1} = 2 - \frac{1}{\alpha_{j+2}} = 1 + \frac{1}{k - j},$$

as wanted.

**Proof of Theorem 1** The result follows from Lemmas 2 through 7. Precisely, by Lemma 2 variable $x_k$ can be arbitrarily large in a strictly feasible solution of $(P')$. By Lemma 4 we derive the polynomial inequalities (2.24). From these in Lemma 5 we derive the clean inequalities (2.30) via the recursion (2.31).

From the recursion (2.31) it directly follows that all $\alpha_{j+1}$ are at most 2. The lower bound on the $\alpha_{j+1}$ is proved as follows: by Lemma 6 the $\alpha_{j+1}$ are monotone functions of the tail-indices $t_{j+1}$. On the other hand, $t_{j+1} \geq j + 2$ for all $j$ by Lemma 3 and when $t_{j+1} = j + 2$ for all $j$, then by Lemma 7 we have $\alpha_{j+1} = 1 + 1/(k - j)$. The proof is now complete.

### 2.3 Computing the exponents by Fourier-Motzkin elimination

The recursion (2.31) gives a convenient way to compute the $\alpha_j$ exponents. Equivalently, we can compute the $\alpha_j$ via the well known Fourier-Motzkin elimination algorithm, designed for linear inequalities; this is an interesting contrast, since SDPs are highly nonlinear.

We do this as follows. If polynomial $p_j$ is of type 1, then we suppress the lower order terms to get

$$x_j x_{t_{j+1}} \geq \text{constant} x_{j+1}^2,$$  \hspace{1cm} (2.53)

see the last inequality in (2.48). If polynomial $p_j$ is of type 2, then we similarly suppress the lower order terms to deduce

$$x_j \geq \text{constant} x_{j+1}^2.$$  \hspace{1cm} (2.54)

After this, using that $x_1, \ldots, x_k$ are all positive, we rewrite the inequalities in terms of $y_j := \log_2 x_j$ for all $j$, then eliminate variables. For example, from the inequalities (2.15) we deduce

$$y_1 + y_3 \geq 2y_2$$

$$y_2 + y_4 \geq 2y_3$$

$$y_3 \geq 2y_4.$$  \hspace{1cm} (2.55)
We add $\frac{1}{2}$ times the last inequality in (2.55) to the middle one to get
\[ y_2 \geq \frac{3}{2} y_3. \]  
(2.56)

We then add $\frac{2}{3}$ times (2.56) to the first inequality in (2.55) to get
\[ y_1 \geq \frac{4}{3} y_2. \]  
(2.57)

Finally, (2.56), (2.57) and the last inequality in (2.55) translate back to the inequalities (2.16).

3 When we do not even need a change of variables

As we previously discussed, the linear change of variables $x \leftarrow Mx$ is necessary to obtain a Khachiyan type hierarchy among the variables. Nevertheless, in this section we show a natural SDP in which a Khachiyan type hierarchy occurs even without a change of variables; more precisely, the SDP is in the form of $(P')$. For completeness, we also revisit O’Donnell’s example from [21], and show that the SDP therein is also in the regular form of $(P')$.

Given a univariate polynomial of even degree $f(x) = \sum_{i=0}^{2n} a_i x^i$ with $a_{2n} > 0$, we consider the problem of minimizing $f$ over $\mathbb{R}$. We write this problem as
\[ \sup_{\lambda} -\lambda \quad s.t. \quad f - \lambda \geq 0. \]  
(3.58)

We will show that in the natural SDP formulation of (3.58) exponentially large variables appear naturally, although here by “exponentially large” we only mean in magnitude, not in size.

Since $f - \lambda$ is also a univariate polynomial, it is nonnegative if and only if it is a sum of squares (SOS), that is, iff $f - \lambda = \sum_{i=1}^t g_i^2$ for a positive integer $t$ and polynomials $g_i$. Define the vector of monomials $z = (1, x, x^2, \ldots, x^{n})^T$.

Then $f - \lambda$ is SOS if and only if (see [14, 19, 23, 35]) $f - \lambda = z z^T \cdot Q$ for some $Q \succeq 0$. We then match monomials in $f - \lambda$ and $z z^T \cdot Q$ and translate (3.58) into the SDP
\[ \max_{Q} -A_0 \cdot Q \quad s.t. \quad A_i \cdot Q = a_i \quad \text{for } i = 1, \ldots, 2n \]  
(3.59)

\[ Q \in S_{+}^{n+1}. \]

Here for all $i \in \{0, 1, \ldots, 2n\}$ the $(k, \ell)$ element of the matrix $A_i$ is 1 if $k + \ell = i + 2$ for some $k, \ell \in \{1, \ldots, n+1\}$, and all other entries of $A_i$ are zero.

For positive integers $k$ and $\ell$, let us define $E_{k\ell}$ as the $(k, \ell)$th unit matrix, whose $(k, \ell)$ and $(\ell, k)$ entries are 1, and the rest are zero.

**Lemma 8.** After permuting and renaming variables, the constraints of the dual problem of (3.59) can be written as
\[ \sum_{i=1}^{2n} x_i A'_i + E_{n+1,n+1} \succeq 0, \]  
(3.60)

with $A'_i = \sum_{k+l=2i} E_{k\ell}$ for $i = 1, \ldots, n$. Thus $(A'_1, A'_2, \ldots, A'_n)$ is a regular facial reduction sequence, and the constraint set (3.60) is in the form of $(P')$, with $k = n$. \(\square\)

\(23\) However, there are multivariate polynomials that are nonnegative, but not SOS.
Before we prove Lemma 8, we illustrate it. For that, suppose \( n = 3 \). Then by Lemma 8 the constraints (3.60) look like

\[
x_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 \end{pmatrix} + \sum_{i=4}^{6} x_i A'_i + E_{4,4} \succeq 0.
\]

Note that if we delete the term \( \sum_{i=4}^{6} x_i A'_i \) from this system, we obtain a smaller version of our previously discussed problem \( \text{(Mild-SDP)} \) (with three variables rather than four).

**Proof of Lemma 8:** The dual problem of (3.59) is

\[
\min_{s.t.} \quad \sum_{i=1}^{2n} a_i y_i \\
\sum_{i=1}^{2n} y_i A'_i + A_0 \succeq 0,
\]

(3.61)

whose constraints can be written as

\[
\begin{pmatrix}
1 & y_1 & y_2 & \cdots & y_n \\
y_1 & y_2 & \cdots & y_{n+1} \\
y_2 & \cdots & y_{n+2} & & \\
\vdots & & & & \\
y_n & y_{n+1} & y_{n+2} & \cdots & y_{2n}
\end{pmatrix} \succeq 0.
\]

Permuting rows and columns, this is equivalent to

\[
\begin{pmatrix}
y_{2n} & y_{2n-1} & y_{2n-2} & \cdots & y_n \\
y_{2n-1} & y_{2n-2} & \cdots & y_{n-1} \\
y_{2n-2} & \cdots & y_{n-2} & & \\
\vdots & & & & \\
y_n & y_{n-1} & y_{n-2} & \cdots & 1
\end{pmatrix} \succeq 0.
\]

(3.62)

Let us rename the variables so the even numbered ones come first, and the rest come afterwards, as

\[
x_1 := y_{2n}, \quad x_2 := y_{2n-2}, \quad \ldots \quad x_n := y_{2}; \\
x_{n+1} := y_{2n-1}, \quad x_{n+2} := y_{2n-3}, \quad \ldots \quad x_{2n} := y_{1}.
\]

(3.63)

Then the constraints (3.62) become as required in (3.60) with \( (A'_1, A'_2, \ldots, A'_n) \) being a regular facial reduction sequence. Finally, an argument just like the one after Example 1 shows that the singularity degree of \( \{ Y \succeq 0 : A'_i \bullet Y = 0 \text{ for } i = 1, \ldots, 2n \} \) is \( n \). For this last part, we leave the details to the reader.

We next claim that for all feasible solutions of (3.60) the inequalities

\[
x_j x_{j+2} \geq x_{j+1}^2 \text{ for } j = 1, \ldots, n - 2; \text{ and } x_{n-1} \geq x_n^2
\]

(3.64)

hold. Indeed, we can derive these by following the proof of Lemma 4, since the tail-indices (cf. Definition 3) are \( t_{j+1} = j + 2 \) for \( j = 1, \ldots, n - 1 \). Further, now the “\( \delta \)” terms that appear in Lemma 4 are all zero, so we do not have to worry about strict feasibility, nor about “making \( x_k \) large.” Hence by Lemma 7 we deduce that

\[
x_j \geq x_{j+1}^{\alpha_{j+1}} \text{ for } j = 1, \ldots, n - 1
\]

(3.65)
hold, where \( \alpha_{j+1} = 1 + 1/(n - j) \) for all \( j \). From these inequalities we then derive \(^6\)

\[
x_1 \geq x_n^n. \tag{3.66}
\]

We invite the reader to try a simpler alternative derivation of the inequalities above: first, from the order two subdeterminants in (3.60) directly deduce the inequalities (3.64); and second, from (3.64) eliminate variables to get the inequalities in (3.65).

We translate the inequalities in (3.65) and (3.66) back to the original \( y_j \) variables (using the correspondence (3.63)), and obtain the following result:

**Theorem 2.** Suppose that \( y \in \mathbb{R}^{2n} \) is feasible in (3.61). Then the following hold:

\[
y_{2(n-j+1)} \geq \frac{y_{2(n-j)}^{1+1/(n-j)}}{2(n-j)} \quad \text{for } j = 1, \ldots, n-1 \tag{3.67}
\]

\[
y_{2n} \geq y_2^n. \tag{3.68}
\]

We next connect Theorem 2 to other results in the literature.

First, Theorem 2 complements a result of Lasserre [14, Theorem 3.2], which states the following: if \( \bar{x} \) minimizes the polynomial \( f(x) \) then \( (y_1, y_2, \ldots, y_{2n}) = (\bar{x}, \bar{x}, \ldots, \bar{x}) \) is optimal in (3.61). On the one hand, Theorem 2 states bounds on all feasible solutions, on the other hand, it does not specify an optimal solution.

Second, the matrix in formula (3.62) is a Hankel matrix, i.e., its elements along the reverse diagonals are constant \(^7\). Theorem 2 compares the even numbered elements of this matrix, no matter what the odd numbered elements \( y_{2n-1}, y_{2n-3}, \ldots, y_3 \) are. Thus, it is related to recent results of Choi and Jafari [7] on partially defined matrices, which can be completed to be Hankel, and positive (semi)definite.

For completeness, we next revisit an example of O’Donnell in [21], and show how the SDP that appears in there is in the regular form of \( P' \).

**Example 4.** We are given the polynomial with \( 2n \) variables

\[
p(x, y) = p(x_1, \ldots, x_n, y_1, \ldots, y_n) = x_1 + \cdots + x_n - 2y_1
\]

and the set \( K \) defined by the equations

\[
2x_1y_1 = y_1, \quad 2x_2y_2 = y_2, \quad \ldots \quad 2x_{n-1}y_{n-1} = y_{n-1}, \quad 2x_ny_n = y_n, \\
x_1^2 = x_1, \quad x_2^2 = x_2, \quad \ldots \quad x_{n-1}^2 = x_{n-1}, \quad x_n^2 = x_n, \\
y_1^2 = y_2, \quad y_2^2 = y_3, \quad \ldots \quad y_{n-1}^2 = y_n, \quad y_n^2 = 0.
\]

Note that in the description of \( K \) the very last constraint \( y_n^2 = 0 \) breaks the pattern seen in the previous \( n-1 \) columns. We ask the following question:

- Is \( p(x, y) \geq 0 \) for all \( (x, y) \in K \)?

The answer is clearly yes, since for all \( (x, y) \in K \) we have \( x_1, \ldots, x_n \in \{0, 1\} \) and \( y_1 = \cdots = y_n = 0. \)

\(^6\)Precisely, by straightforward induction we get \( x_1 \geq x_n^{n/(n-j)} \) for \( j = 1, \ldots, n-1. \)

\(^7\)The matrix in (3.62) has a “1” in the lower right corner, but this can always be achieved by a normalization.
On the other hand, the sum of squares procedure verifies the “yes” answer as follows. Let 
\[ z = (1, x_1, \ldots, x_n, y_1, \ldots, y_n)^\top \]
be a vector of monomials. To certify that \( p(x, y) \) is nonnegative over \( K \), we seek \( \lambda, \mu, \nu \in \mathbb{R}^n \) and \( Q \succeq 0 \) such that
\[
p(x, y) = z^\top Q z + \lambda_1(x_1 y_1 - y_1) + \mu_1(x_1^2 - x_1) + \nu_1(y_1^2 - y_1) \\
+ \lambda_2(x_2 y_2 - y_2) + \mu_2(x_2^2 - x_2) + \nu_2(y_2^2 - y_2) \\
+ \ldots \\
+ \lambda_n(x_n y_n - y_n) + \mu_n(x_n^2 - x_n) + \nu_n(y_n^2 - 0).
\]
Indeed, if we succeed and find such \( \lambda, \mu, \nu, Q \), then \( p(x, y) = z^\top Q z \geq 0 \) for all \( (x, y) \in K \).

The polynomials (in the \( x_i \) and \( y_i \)) on the two sides of (3.69) are equal exactly when all their coefficients are equal. Thus, matching coefficients in (3.69) on the left and right hand sides, O’Donnell [21] showed that any \( Q \) feasible in (3.69) looks like
\[
Q = \begin{pmatrix}
  u_1 & 0 & \ldots & 0 & 0 & -u_2 & \ldots & 0 & 0 & 0 & 0 \\
  0 & u_2 & \ldots & 0 & 0 & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & \ldots & u_{n-1} & 0 & 0 & \ldots & 0 & -u_n & 0 & 0 \\
  0 & 0 & \ldots & 0 & u_n & 0 & \ldots & 0 & 0 & -2 & 0 \\
  -u_2 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & \ldots & -u_{n-1} & 0 & 0 & \ldots & 1 & 0 & 0 & 0 \\
  0 & 0 & \ldots & -u_n & 0 & 0 & \ldots & 0 & 1 & 0 & 0 \\
  0 & 0 & \ldots & 0 & -2 & 0 & \ldots & 0 & 0 & 1 & 0 \\
  0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
for suitable \( u_1, \ldots, u_n \). Looking at \( 2 \times 2 \) subdeterminants of \( Q \) we see that the \( u_i \) satisfy
\[
u_1 \geq u_2^2, u_2 \geq u_3^2, \ldots, u_{n-1} \geq u_n^2, u_n \geq 4.
\]
which is the same as (Khachiyan), except we replaced the constant 2 by 4.

Recall that \( E_{k\ell} \) is the unit matrix in which the \( (k, \ell) \) and \( (\ell, k) \) entries are 1 and the rest zero. Define
\[ A_i' = E_{1i}, A'_i = E_{ii} - E_{i-1,n+i-1} \text{ for } i = 2, \ldots, n. \]
Then any \( Q \) feasible in (3.69) is written as
\[
Q = u_1 A'_1 + u_2 A'_2 + \cdots + u_n A'_n + B' \succeq 0,
\]
for a suitable \( B' \) (precisely, \( B' = -2E_{n,2n} + \sum_{i=n+1}^{2n} E_{ii} \)).

We see that \( (A'_1, \ldots, A'_n) \) is a regular facial reduction sequence, thus the system (3.72) is in the regular form of \( (P') \).

Note that (3.70) arises by concatenating \( 2 \times 2 \) psd blocks of the form (1.1), then permuting rows and columns. In other words, (3.70) is the exact representation of (Khachiyan) (apart from the constant 2 being replaced by 4 and the \( u_i \) being negated in the offdiagonal positions), that we discussed after Example 1.

Among followup papers of O’Donnell [21] we should mention the work of Raghavendra and Weitz [29] which gave SDPs which also have a sum-of-squares origin, and exponentially large size solutions. It would be interesting to see whether those SDPs are also in the regular form of \( (P') \).
4 Conclusion

Khachiyan’s SDP is a classical pathological problem in which the size of any feasible solution is exponential in the number of variables. Here we showed that Khachiyan’s SDP is far from being an isolated example: in any strictly feasible SDP a linear transformation induces a Khachiyan type hierarchy among a subset of the variables, and large size solutions. The number of variables in the hierarchy and “how large” they get, depends on the singularity degree of a dual problem. Further, such a hierarchy and large solutions naturally appear in SDPs that come from sum-of-squares optimization, without any change of variables.

We also studied how to represent large solutions of SDPs in polynomial space. Our main tool was the regularized semidefinite program \((P')\). If \((P)\) and \((P')\) are strictly feasible, then in the latter we can verify that a strictly feasible solution exists, without computing the actual values of the “large” variables \(x_1, \ldots, x_k\): see Figure 2. Further, SDPs that arise from polynomial optimization (Section 3) and the SDP that represents (Khachiyan) are naturally in the form of \((P')\). Hence in these SDPs we can also certify large solutions without computing their actual values.

Several questions remain open. For example, what can we say about large solutions in semidefinite programs that are not strictly feasible? The discussion after Example 1 shows that we do not have a complete answer yet.

Also, recall that we transform \((P)\) into \((P')\) by a linear change of variables (equivalent to operations (1) and (2) in Definition 2) and a similarity transformation (operation (3) in Definition 2). The latter has no effect on how large the variables are. We are thus led to the following question: are all SDPs with exponentially large solutions in the form of \((P')\) (perhaps after a similarity transformation)? In other words, can we always certify large size solutions in SDPs using a regular facial reduction sequence? Answering this question would help us answer the greater question: can we decide feasibility of SDPs in polynomial time?

Acknowledgement We are very grateful for helpful discussions to Richard Rimányi, in particular, to suggestions that led to the proof of Lemma 6. We are also very grateful to the anonymous referees and to Andrew B. Nobel whose suggestions helped us to improve the paper. We thank Alex Townsend for pointing out reference [7]. We gratefully acknowledge the support of National Science Foundation, award DMS-1817272.

References

[1] Amir Ali Ahmadi, Alex Olshevsky, Pablo A Parrilo, and John N Tsitsiklis. NP-hardness of deciding convexity of quartic polynomials and related problems. Mathematical Programming, 137(1):453–476, 2013. 3, 5

[2] Amir Ali Ahmadi and Jeffrey Zhang. On the complexity of testing attainment of the optimal value in nonlinear optimization. Mathematical Programming, pages 1–21, 2019. 3, 5

[3] Alexander I Barvinok. Feasibility testing for systems of real quadratic equations. Discrete & Computational Geometry, 10(1):1–13, 1993. 3, 5

[4] Daniel Bienstock. A note on polynomial solvability of the CDT problem. SIAM Journal on Optimization, 26(1):488–498, 2016. 3, 5

[5] Daniel Bienstock, Alberto Del Pia, and Robert Hildebrand. Complexity, exactness, and rationality in polynomial optimization. arXiv preprint arXiv:2011.08347, 2020. 3, 5

[6] Jonathan M. Borwein and Henry Wolkowicz. Regularizing the abstract convex program. J. Math. Anal. App., 83:495–530, 1981. 3, 5
[7] Hayoung Choi and Farhad Jafari. Positive definite Hankel matrix completions and Hamburger moment completions. *Linear Algebra and its Applications*, 489:217–237, 2016. 25, 27

[8] Etienne de Klerk and Frank Vallentin. On the turing model complexity of interior point methods for semidefinite programming. *SIAM Journal on Optimization*, 26(3):1944–1961, 2016. 3, 5

[9] Dmitriy Drusvyatskiy and Henry Wolkowicz. The many faces of degeneracy in conic optimization. *Foundations and Trends® in Optimization*, 3(2):77–170, 2017. 3, 5

[10] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*, volume 2. Springer Science & Business Media, 2012. 3, 5

[11] N Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4(4):373–395, 1984. 5

[12] Leonid Genrikhovich Khachiyan. A polynomial algorithm in linear programming. In *Doklady Akademii Nauk*, volume 244, pages 1093–1096. Russian Academy of Sciences, 1979. 5

[13] Masakazu Kojima, Shinji Mizuno, and Akiko Yoshise. A primal-dual interior point algorithm for linear programming. In *Progress in mathematical programming*, pages 29–47. Springer, 1989. 5

[14] Jean B Lasserre. Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11(3):796–817, 2001. 23, 25

[15] Minghui Liu and Gábor Pataki. Exact duality in semidefinite programming based on elementary reformulations. *SIAM J. Opt.*, 25(3):1441–1454, 2015. 7

[16] Bruno F Lourenço. Amenable cones: error bounds without constraint qualifications. *to appear, Mathematical Programming, arXiv preprint arXiv:1712.06221*, 2017. 9

[17] Bruno F. Lourenço, Masakazu Muramatsu, and Takashi Tsuchiya. A structural geometrical analysis of weakly infeasible sdps. *Journal of the Operations Research Society of Japan*, 59(3):241–257, 2016. 12

[18] Zhi-Quan Luo, Jos Sturm, and Shuzhong Zhang. Duality results for conic convex programming. Technical Report Report 9719/A, Erasmus University Rotterdam, Econometric Institute, The Netherlands, 1997. 7

[19] Yurii Nesterov. Squared functional systems and optimization problems. In *High performance optimization*, pages 405–440. Springer, 2000. 23

[20] Yurii Nesterov and Arkadii Nemirovskii. *Interior-point polynomial algorithms in convex programming*. SIAM, 1994. 3, 5

[21] Ryan O’Donnell. SOS is not obviously automatizable, even approximately. In *8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017. 4, 5, 6, 23, 25, 26

[22] Panos M Pardalos and Stephen A Vavasis. Quadratic programming with one negative eigenvalue is NP-hard. *Journal of Global optimization*, 1(1):15–22, 1991. 3, 5

[23] Pablo A Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical programming*, 96(2):293–320, 2003. 23

[24] Gábor Pataki. A simple derivation of a facial reduction algorithm and extended dual systems. Technical report, Columbia University, 2000. 3, 5

[25] Gábor Pataki. Strong duality in conic linear programming: facial reduction and extended duals. In David Bailey, Heinz H. Bauschke, Frank Garvan, Michel Théra, Jon D. Vanderwerff, and Henry Wolkowicz, editors, *Proceedings of Jonfest: a conference in honour of the 60th birthday of Jon Borwein*. Springer, also available from http://arxiv.org/abs/1301.7717, 2013. 3, 5
[26] Gábor Pataki. Characterizing bad semidefinite programs: normal forms and short proofs. *SIAM Review*, 61(4):839–859, 2019. 7

[27] Ting Kei Pong and Henry Wolkowicz. The generalized trust region subproblem. *Computational optimization and applications*, 58(2):273–322, 2014. 5

[28] Lorant Porkolab and Leonid Khachiyan. On the complexity of semidefinite programs. *Journal of Global Optimization*, 10(4):351–365, 1997. 3, 5

[29] Prasad Raghavendra and Benjamin Weitz. On the bit complexity of sum-of-squares proofs. *arXiv preprint arXiv:1702.05139*, 2017. 26

[30] James Renegar. A polynomial-time algorithm, based on newton’s method, for linear programming. *Mathematical programming*, 40(1):59–93, 1988. 5

[31] James Renegar. On the computational complexity and geometry of the first-order theory of the reals. part i: Introduction, preliminaries. the geometry of semi-algebraic sets. the decision problem for the existential theory of the reals. *Journal of symbolic computation*, 13(3):255–299, 1992. 3, 5

[32] James Renegar. *A Mathematical View of Interior-Point Methods in Convex Optimization*. MPS-SIAM Series on Optimization. SIAM, Philadelphia, USA, 2001. 3, 5

[33] Shinsaku Sakaue, Yuji Nakatsukasa, Akiko Takeda, and Satoru Iwata. Solving generalized CDT problems via two-parameter eigenvalues. *SIAM Journal on Optimization*, 26(3):1669–1694, 2016. 5

[34] Alexander Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, 1998. 2

[35] Naum Z Shor. Class of global minimum bounds of polynomial functions. *Cybernetics*, 23(6):731–734, 1987. 23

[36] Ronald J Stern and Henry Wolkowicz. Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations. *SIAM Journal on Optimization*, 5(2):286–313, 1995. 5

[37] Jos Sturm. Error bounds for linear matrix inequalities. *SIAM J. Optim.*, 10:1228–1248, 2000. 9

[38] Stephen A Vavasis. Quadratic programming is in NP. *Information Processing Letters*, 36(2):73–77, 1990. 3, 5

[39] Stephen A Vavasis and Richard Zippel. Proving polynomial-time for sphere-constrained quadratic programming. Technical report, Cornell University, 1990. 3, 5

[40] Hayato Waki and Masakazu Muramatsu. Facial reduction algorithms for conic optimization problems. *J. Optim. Theory Appl.*, 158(1):188–215, 2013. 3, 5