THE LEGENDRIAN SATELLITE CONSTRUCTION

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ABSTRACT. We examine the Legendrian analogue of the topological satellite construction for knots, and deduce some results for specific Legendrian knots and links in standard contact three-space and the solid torus. In particular, we show that the Chekanov-Eliashberg contact homology invariants of Legendrian Whitehead doubles of stabilized knots contain no nonclassical information.

1. Introduction

The spaces $\mathbb{R}^3$ and $S^1 \times \mathbb{R}^2$ both have a standard contact structure given by the kernel of the 1-form $dz - y \, dx$, where we view the solid torus $S^1 \times \mathbb{R}^2$ as $\mathbb{R}^3$ modulo the relation $(x, y, z) \sim (x + 1, y, z)$. We will assume that the reader is familiar with some basic concepts in Legendrian knot theory, such as front projections, the Thurston-Bennequin number, and the rotation number; see, e.g., [8], which we will use extensively.

The problem of classifying Legendrian knots in standard contact $\mathbb{R}^3$ up to Legendrian isotopy has attracted much recent attention. In this note, we study one particular construction on Legendrian knots, the Legendrian satellite construction, which relates knots in $\mathbb{R}^3$ and in $S^1 \times \mathbb{R}^2$. This is the Legendrian analogue of the satellite construction in the smooth category, which glues a link in the solid torus $S^1 \times \mathbb{R}^2$ into a tubular neighborhood of a knot in $\mathbb{R}^3$ to produce a link in $\mathbb{R}^3$. We examine some consequences of Legendrian satellites for the Legendrian knot classification problem in both $\mathbb{R}^3$ and $S^1 \times \mathbb{R}^2$; in particular, we recover previously-known results for knots in both spaces, and prove a new result in $S^1 \times \mathbb{R}^2$ (Proposition 2.11).

The motivation for this work is that Legendrian satellites may provide nontrivial, nonclassical invariants of stabilized Legendrian knots in $\mathbb{R}^3$. Here we recall that there are two stabilization operators $S_\pm$ on Legendrian knots, decreasing $tb$ by 1 and changing $r$ by $\pm 1$, which replace a segment of the knot’s front projection by a zigzag, as shown in Figure 1. Understanding stabilized knots is an important open problem in Legendrian knot theory; it also has repercussions for the classification of transverse knots.

It seems possible that satellites of stabilized knots may contain interesting information through the Chekanov-Eliashberg differential graded algebra invariant $\mathcal{H}$, which is derived from contact homology $\mathcal{H}$. We will show that, unfortunately, the DGAs of the simplest Legendrian satellites of stabilized knots do not encode any useful information. The computation used in the proof may be of interest as the first involved computation manipulating the DGA invariant directly, rather than using easier invariants such as Poincaré polynomials $\mathcal{P}$ or the characteristic algebra $\mathcal{A}$. In any case, more complicated satellites may well give nonclassical invariants of stabilized knots, as has been suggested by Michatchev [7].

We define the construction in Section 2 and show how it immediately implies facts about solid-torus links, including some that could not be shown using any previously known techniques. In
Figure 1. Stabilization of a Legendrian link, in the front projection.

Section 3, we show the result mentioned above about DGAs of satellites of stabilized knots; the key step is Lemma 3.4, which is proven in Section 4.

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2. Construction

As in $\mathbb{R}^3$, Legendrian links in the solid torus $S^1 \times \mathbb{R}^2$ may be represented by their front projections to the $xz$ plane, with the understanding that the $x$ direction is now periodic. If we view $S^1 \times \mathbb{R}^2$ as $[0, 1] \times \mathbb{R}^2$ with $\{0\} \times \mathbb{R}^2$ identified with $\{1\} \times \mathbb{R}^2$, then we can draw the front projection of a solid-torus Legendrian link as a front in $[0, 1] \times \mathbb{R}^2$ with the two boundary components identified. We depict the boundary components by dashed lines; see Figure 2 for an illustration. For a Legendrian link $\tilde{L}$ in the solid torus, let the endpoints of $\tilde{L}$ be $\tilde{L} \cap (\{0\} \times \mathbb{R}^2)$, that is, the points where the front for $\tilde{L}$ intersects the dashed lines.

Remark 2.1. Invariants of solid-torus links. There are three classical invariants of links on the solid torus: the Thurston-Bennequin number $tb$ and rotation number $r$ can be calculated from the front of a solid-torus Legendrian link exactly as in $\mathbb{R}^3$; and the winding number $w$ is the number of times the link winds around the $S^1$ direction of $S^1 \times \mathbb{R}^2$. Clearly the $tb$, $r$, and $w$ associated to any subset of the components of a solid-torus link also give invariants of the link.

In [9], L. Traynor and the author show that the Chekanov-Eliashberg DGA can be defined for links on the solid torus, thus yielding a nonclassical invariant. For certain links with two components, [10] defines another nonclassical invariant based on generating functions. We will give examples in this section of solid-torus knots which are not Legendrian isotopic, but which cannot be distinguished using any of these invariants.

We now introduce the Legendrian satellite construction. Let $L$ be an oriented Legendrian link in $\mathbb{R}^3$ with one distinguished component $L_1$, and let $\tilde{L}$ be an oriented Legendrian link in $S^1 \times \mathbb{R}^2$. We give two definitions of the Legendrian satellite $S(L, \tilde{L}) \subset \mathbb{R}^3$, one abstract, one concrete.

A tubular neighborhood of $L_1$ is a solid torus; the characteristic foliation on the boundary of this torus wraps around the torus $tb(L_1)$ times. By cutting the tubular neighborhood at a cross-sectional disk, untwisting it $tb(L_1)$ times, and regluing, we obtain a solid torus contactomorphic to $S^1 \times \mathbb{R}^2$ with the standard contact structure. Thus we can embed $\tilde{L} \subset S^1 \times \mathbb{R}^2$ as a Legendrian link in a tubular neighborhood of $L_1$. Replacing the component $L_1$ in $L$ by this new link gives $S(L, \tilde{L})$.

We can redefine $S(L, \tilde{L})$ in terms of the fronts for $L$ and $\tilde{L}$. First, we recall the definition of an $n$-copy from [7].

Definition 2.2. Given a Legendrian knot $K$, its $n$-copy is the link consisting of $n$ copies of $K$ which differ from each other through small perturbations in the transversal direction. In the front...
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Figure 2. Gluing a solid-torus link $\tilde{L}$ into an $\mathbb{R}^3$ link $L$, to form the satellite link $S(L, \tilde{L})$.

projection, the $n$-copy consists of $n$ copies of $K$, differing from each other by small shifts in the $z$ direction. The 2-copy is also known as the double.

Now suppose that $\tilde{L}$ has $n$ endpoints. By cutting along the dotted lines (i.e., the endpoints of $\tilde{L}$), we can embed $\tilde{L}$ as a Legendrian tangle in $\mathbb{R}^3$ with $2n$ ends. Replace the front of the first component $L_1$ of $L$ by the $n$-copy of $L_1$. Then choose a small segment of $L_1$ which is oriented from left to right; excise the corresponding $n$ pieces of the $n$-copy of $L_1$, and replace them by the front for $\tilde{L}$, cut along its endpoints. See Figure 2 for an illustration.

Definition 2.3. The resulting link $S(L, \tilde{L}) \subset \mathbb{R}^3$ is the Legendrian satellite of $L \subset \mathbb{R}^3$ and $\tilde{L} \subset S^1 \times \mathbb{R}^2$. We give $S(L, \tilde{L})$ the orientation derived from the orientations on $\tilde{L}$ (for the glued $n$-copy of $L_1$) and on $L$ (for the components of $L$ besides $L_1$).

The Legendrian satellite construction is motivated by the special case of Whitehead doubles (see Section 3), which were introduced by Eliashberg and subsequently used by Fuchs.
Figure 3. Pushing $\tilde{L}$ through singularities in $L$: a crossing, a right cusp, and a left cusp.

Figure 4. Pushing singularities in $\tilde{L}$ through a right cusp.

Remark 2.4. Classical invariants of Legendrian satellites. Before we show that $S(L, \tilde{L})$ is well-defined up to Legendrian isotopy, we note that the classical invariants of $S(L, \tilde{L})$ are easily computable from those of $L$ and $\tilde{L}$. Indeed, a straightforward computation with front diagrams yields

$$tb(S(L, \tilde{L})) = (w(\tilde{L}))^2 tb(L) + tb(\tilde{L})$$
$$r(S(L, \tilde{L})) = w(\tilde{L})r(L) + r(\tilde{L})$$

when $L$ is a knot, with a similar but slightly more complicated formula when $L$ is a multi-component link.

Lemma 2.5. $S(L, \tilde{L})$ is well-defined up to Legendrian isotopy.

Proof. We need to show that, up to Legendrian isotopy, $S(L, \tilde{L})$ is independent of the piece of the $n$-copy of $L_1$ which we excise and replace by $\tilde{L}$, as long as this piece is oriented left to right. The singularities of $\tilde{L}$ consist of crossings, left cusps, and right cusps; we imagine pushing these singularities one by one from one section of the $n$-copy of $L_1$ to another.

We can clearly push these singularities through any piece of $S(L, \tilde{L})$ which crosses a neighborhood of $\tilde{L}$ transversely; see the top diagram in Figure 3. Figure 4 shows that we can also push singularities through a right cusp in $L_1$, and clearly this argument extends to left cusps as well. We conclude that we can push all of $\tilde{L}$ through a cusp, resulting in the left-to-right mirror reflection of $\tilde{L}$; see the bottom diagrams in Figure 3. The lemma follows.

Proposition 2.6. $S(L, \tilde{L})$ is a well-defined operation on Legendrian isotopy classes; that is, if we change $L, \tilde{L}$ by Legendrian isotopies, then $S(L, \tilde{L})$ changes by a Legendrian isotopy as well.

Proof. We first consider Legendrian-isotopy changes of $\tilde{L}$. These fall into two categories: isotopies where the endpoints of $\tilde{L}$ remain fixed, and horizontal translations of $\tilde{L}$ (i.e., moving the dashed lines). The first category clearly preserves the Legendrian isotopy class of $S(L, \tilde{L})$. The second category consists of pushing singularities in $\tilde{L}$ through the dashed lines. But Figure 4 shows that
we can push individual singularities from one side of $\tilde{L}$ to the other, by moving the singularity all the way around the $n$-copy of $L_1$. Hence Legendrian-isotopy changes of $\tilde{L}$ do not change $S(L, \tilde{L})$.

Next consider Legendrian-isotopy changes of $L$. It suffices to show that $S(L, \tilde{L})$ does not change under Legendrian Reidemeister moves on $L$. Consider such a move, and push $\tilde{L}$ away from a neighborhood of the move. Then the fact that the Legendrian-isotopy class of $S(L, \tilde{L})$ does not change follows from Figure 5.

**Remark 2.7.** Both Proposition 2.6 and Corollary 2.8 below have been known for some time. It is easy, and probably more natural, to establish Proposition 2.6 using the global, non-front definition of Legendrian satellites; we chose to present the front proof because of its concreteness.

**Corollary 2.8.** Legendrian-isotopic knots in $\mathbb{R}^3$ have Legendrian-isotopic $n$-copies.

**Proof.** The $n$-copy of a knot $K$ is simply $S(K, \tilde{L}^{(n)})$, where $\tilde{L}^{(n)}$ is the union of $n$ unlinked loops which wind once around $S^1 \times \mathbb{R}^2$; see Figure 6 for an illustration of $\tilde{L}^{(2)}$. The result follows from Proposition 2.6.

**Corollary 2.9** ([7]). Suppose that $K$ is a stabilization of a Legendrian knot. The $n$-copy of $K$ is Legendrian isotopic to the $n$-copy with components cyclically permuted. More precisely, if $L_1, \ldots, L_n$ are the components of the $n$-copy of $K$, with $L_i$ slightly higher than $L_{i+1}$ in $z$ coordinate, then $(L_1, L_2, \ldots, L_n)$ is Legendrian isotopic to $(L_{1+k}, L_{2+k}, \ldots, L_{n+k})$ for any $k$, where indices are taken modulo $n$. 

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**Figure 5.** Legendrian Reidemeister moves on $n$-copies; in this illustration, $n = 3$.

**Figure 6.** The solid-torus links $\tilde{L}^{(2)}$ and $S_3$ (with obvious generalization to a family of links $S_n$), and the solid-torus Whitehead knots $W_{2k}$, $k \geq 0$, with $W_0$ and $W_2$ shown as examples. The box indicates $2k$ half-twists.
Proof. Suppose, without loss of generality, that \(K = S_n(K')\) for a Legendrian knot \(K'\). Then the \(n\)-copy of \(K\) is the Legendrian satellite \(S(K', S_n)\), where \(S_n\) is the solid-torus “\(n\)-copy stabilization link” depicted in Figure 6. It is easy to see that \(S_n\) is Legendrian isotopic to itself with components cyclically permuted; now apply Proposition 2.6.

We now present some applications of Proposition 2.6 to knots and links on the solid torus. Consider the link \(\tilde{L}(2)\) shown in Figure 6. The following result, established in [10] using generating functions, is also proven in [9] using the DGA for solid-torus links. The proof we give is yet another one.

**Proposition 2.10.** Write \(\tilde{L}(2) = (\tilde{L}_1, \tilde{L}_2)\). Then \((\tilde{L}_1, \tilde{L}_2)\) is not Legendrian isotopic to \((\tilde{L}_2, \tilde{L}_1)\).

Proof. In [8, Proposition 4.11], the author proves that the double of the figure eight knot is not Legendrian isotopic to the double with components swapped. The result now follows from Proposition 2.6.

Now consider the Whitehead knots \(W_{2k}\) shown in Figure 6. Each \(W_{2k}\) has \(r = w = 0\) and is thus topologically isotopic to its inverse (the same knot with the opposite orientation). By contrast, we can now show the following result.

**Proposition 2.11.** \(W_{2k}\) is not Legendrian isotopic to its inverse.

Proof. As usual, write \(-W_{2k}\) for the inverse of \(W_{2k}\), and let \(L\) be the double of the usual “flying-saucer” unknot in \(\mathbb{R}^3\). For \(k = 1\), it is easy to check that \(S(L, W_2)\) is precisely the oriented Whitehead link from [8, §4.5], and that \(S(L, -W_2)\) is the same link with one component reversed. Proposition 2.6 and [8, Proposition 4.12] then imply that \(W_2\) and \(-W_2\) are not Legendrian isotopic.

A calculation similar to the one in the proof of [8, Proposition 4.12], omitted here, shows that \(S(L, W_{2k})\) and \(S(L, -W_{2k})\) are not Legendrian isotopic for arbitrary \(k \geq 0\). The result follows.

The solid-torus DGA from [9] fails to distinguish between \(W_{2k}\) and its inverse. Proposition 2.11 is thus a result about solid-torus knots whose only presently known proof uses the Legendrian satellite construction.

### 3. Doubles

The Chekanov-Eliashberg DGA invariant vanishes for links which are stabilizations. The Legendrian satellite construction, however, seems to yield nontrivial nonclassical invariants of all Legendrian links; see Remark 3.7 below. On the other hand, the main result of this section shows that some of the simplest Legendrian satellites of stabilizations do not contain any new information.

**Definition 3.1.** The **Legendrian Whitehead double** of a Legendrian knot \(K\) in \(\mathbb{R}^3\) is \(S(K, W_0)\), where \(W_0\) is the knot shown in Figure 6. More generally, if \(\tilde{L}\) has two endpoints, then we call \(S(K, \tilde{L})\) a satellite double of \(K\).

As mentioned in Section 3, the Legendrian Whitehead double was originally defined by Eliashberg, with further study by Fuchs [8], who uses the notation \(\Gamma_{db\ell}(0, 0)\) for our \(S(K, W_0)\).

**Remark 3.2.** **Legendrian satellites and maximal \(tb\).** By Remark 2.4, the Legendrian Whitehead double of any Legendrian knot has Thurston-Bennequin number 1. As noted by J. Sabloff and the author, it is easy to show that the Legendrian Whitehead double maximizes \(tb\) in its topological class. This follows from the fact that \(g(S(K, W_0)) = 1\), along with Bennequin’s inequality \(tb(K) \leq 2g(K) - 1\) [9], where \(g(K)\) is the (three-ball) genus of \(K\). A similar argument shows that the usual double of any Legendrian knot maximizes \(tb\).
It is not true, however, that all satellite doubles maximize $tb$, even when $\tilde{L}$ maximizes $tb$. In particular, if $\tilde{L}$ has a half-twist $\subset$ next to its endpoints, and $K$ is a stabilization, then $S(K, \tilde{L})$ will also be a stabilization.

**Proposition 3.3.** If $K_1$ and $K_2$ are stabilized Legendrian knots in the same topological class with the same $tb$ and $r$, then the DGAs of the Legendrian Whitehead doubles of $K_1$ and $K_2$ are equivalent.

The key to proving Proposition 3.3 is the following result, whose proof we delay until Section 4.

**Lemma 3.4.** For any Legendrian knot $K$ which is a stabilization, the DGAs of $S(K, W_0)$ and of $S(S_+S_-(K), W_0)$ are equivalent.

**Proof of Proposition 3.3.** By a result of [6], any two Legendrian knots which are topologically identical and have the same $tb$ and $r$ are Legendrian isotopic after some number of applications of the double stabilization operator $S_+S_-$. That is, there exists an $n \geq 0$ such that $(S_+S_-)^nK_1$ and $(S_+S_-)^nK_2$ are Legendrian isotopic. The proposition now follows directly from Lemma 3.4.

**Remark 3.5.** It can in fact be shown that the DGA of the Legendrian Whitehead double of a stabilized knot depends only on the $tb$ and $r$ of the knot, and not on its topological class. In particular, we can recover the result of [5] that the DGA of a Legendrian Whitehead double always possesses an augmentation.

A slightly modified version of the proof of Lemma 3.4, omitted here for simplicity, establishes the following more general result.

**Proposition 3.6.** If $K_1$ and $K_2$ are stabilized Legendrian knots in the same topological class with the same $tb$ and $r$, and $\tilde{L}$ is any Legendrian link in $S^1 \times \mathbb{R}$ with two endpoints and winding number zero, then the DGAs of $S(K_1, \tilde{L})$ and $S(K_2, \tilde{L})$ are equivalent.

We believe that Proposition 3.6 actually holds for any satellite doubles of stabilized knots $K_1$ and $K_2$ with the same $tb$ and $r$, regardless of the winding number of $\tilde{L}$. However, the analogue of Lemma 3.4 is false if $\tilde{L}$ has winding number $\pm 2$, since $S(K, \tilde{L})$ and $S(S_+S_-(K), \tilde{L})$ have different $tb$; see Remark 2.4. Nevertheless, the argument of the proof of Lemma 3.4 shows that the characteristic algebra, at least, can never distinguish between satellite doubles of stabilized knots.

**Remark 3.7.** Invariants of stabilized Legendrian knots. As mentioned in the Introduction, it remains a very interesting open problem to find nonclassical invariants of stabilized Legendrian knots. There are currently no methods to prove that two stabilized knots with the same topological type, $tb$, and $r$ are not Legendrian isotopic.

We are hopeful that satellites more complicated than doubles will encode interesting information for stabilized knots. In particular, it seems that the $n$-copy of any Legendrian link maximizes Thurston-Bennequin number when $n \geq 2$, and thus probably has a nontrivial DGA. By Corollary 2.8, the DGAs of Legendrian satellites of a Legendrian link, including the $n$-copy, are Legendrian-isotopy invariants, which likely contain interesting nonclassical information in general. The problem we face when dealing with complicated satellites, however, is extracting useful information from the Poincaré polynomials or the characteristic algebra. See [7].

There is another approach to finding invariants of stabilized knots, which is probably more natural than investigating satellites. Eliashberg, Givental, and Hofer [8] have recently developed symplectic field theory, which generalizes contact homology; [4, §2.8] describes how this method yields invariants of Legendrian links, which would likely not vanish for stabilized knots. Unfortunately, no explicit combinatorial description, à la Chekanov, is presently known for the symplectic field theory associated to Legendrian links.
Figure 7. Whitehead doubles of stabilizations. In the lower diagrams, vertices (crossings and right cusps) are labelled, with vertex $a_i$ labelled by $i$.

4. Proof of Lemma 3.4

We assume familiarity with the definition of the DGA invariant, as formulated in [8]. We may suppose, without loss of generality, that $K = S_+(K')$ for some Legendrian $K'$. By using, if necessary, Legendrian Reidemeister moves (more precisely, IIb and the mirror of I from [8, Figure 1]), we may further assume that the rightmost cusp in $K'$ is oriented downwards. If we shift the zigzag in $K = S_+(K')$ next to the rightmost cusp in $K'$, then $K$ and $S(K, W_0)$ look like the diagrams in Figure 7 near the rightmost cusp.

The corresponding parts of $S_+(K)$ and $S(S_+(K), W_0)$ are also shown in Figure 7, and $S(K, W_0)$ and $S(S_+(K), W_0)$ are identical outside the regions depicted. It is easy to check that the degrees of all vertices not depicted are equal for the two Legendrian Whitehead doubles, and that the degrees of the vertices depicted are 1 for $a_1, a_2, a_3, a_4, a_8, a_9, a_{10}, a_{11}$ and 0 for $a_5, a_6, a_7, a_{12}, a_{13}, a_{14}, a_{15}$, in either diagram. Since the regions drawn are the rightmost parts of each double, the DGA for $S(S_+(K), W_0)$ is simply obtained from the DGA for $S(K, W_0)$ by making the following replacements:

- Replace $S(K, W_0)$ with $S(S_+(K), W_0)$.
- Shift the zigzag in $S(K, W_0)$ next to the rightmost cusp in $S(K, W_0)$.
- Label the vertices (crossings and right cusps) with $a_i$.

Note: The exact replacements and shifts are not specified in the text, but should be understood based on the diagram and the explanation provided.
We may then drop a
Under $\Phi$
If we apply the elementary automorphism
By applying the elementary automorphisms
$\partial a_1 \mapsto \partial a_1$, respectively.
where in the DGAs except in the equations above; $a_5, a_7$ appear additionally in $\partial a_1, \partial a_4$, respectively.
Our goal is to apply elementary automorphisms and algebraic stabilizations (see [2]) to the DGA for $S(K, W_0)$; until we obtain the DGA for $S(K, W_0)$. Start with the DGA for $S(S_+(K), W_0)$; we begin by rewriting $\partial a_3, \partial a_8, \partial a_9, \partial a_{10}, \partial a_{11}$ in a more manageable form.
We first wish to rewrite $\partial a_3$ as $\partial a_3 = t^{-1} - a_6a_7$. (Intuitively, this follows from the fact that $a_5 = a_7$ in the characteristic algebra or in the homology of the DGA.) We define the words $\alpha_1, \alpha_2, \alpha_3$ in the DGA as follows, and then compute $\partial \alpha_1, \partial \alpha_2, \partial \alpha_3$:
\[
\begin{align*}
\alpha_1 &= t a_{15} a_9 - a_{10} a_{13} \\
\alpha_2 &= t a_{11} - a_1 a_7 \\
\alpha_3 &= a_8 a_7 - a_{12} a_2
\end{align*}
\]
If we apply the elementary automorphism $a_3 \mapsto a_3 + a_6 a_3$, then we obtain $\partial a_3 = t^{-1} - a_6 a_7$.
In a similar fashion, we can successively replace $\partial a_{11}, \partial a_{10}, \partial a_9, \partial a_8$ as follows:
$\partial a_{11} = t^{-1} - a_{15} a_5$; $\partial a_{10} = t^{-1} - a_{5} a_{13}$; $\partial a_9 = 1 - ta_{12} a_6$.
For convenience, we now define
\[
\bar{a}_2 = (1 - t a_6 a_5) a_3 + a_6 a_2 a_7 \quad \implies \quad \bar{a}_2 = t^{-1} - a_6 a_5.
\]
By applying the elementary automorphisms $a_{11} \mapsto a_{11} + \bar{a}_2$ and $a_{15} \mapsto a_{15} + a_6$, we obtain $\partial a_{11} = -a_{15} a_5$. Similarly, we may write $\partial a_{10} = - ta_6 a_{14}$, $\partial a_9 = - a_5 a_{13}$, $\partial a_8 = - ta_{12} a_6$.
At this point, the DGA has the following form:
\[
\begin{align*}
\partial a_2 &= 1 - t a_5 a_6 \\
\partial a_3 &= t^{-1} - a_6 a_7 \\
\partial a_8 &= - t a_{12} a_6 \\
\partial a_5 &= a_6 a_7 = \partial a_{12} = \partial a_{13} = \partial a_{14} = \partial a_{15} = 0
\end{align*}
\]
We next eliminate $a_8, a_{11}, a_{12}, a_{15}$ through algebraic stabilization and destabilization. Introduce $e_1$ and $e_2$ of degree 0 and $-1$, respectively, with $\partial e_1 = e_2$ (and $\partial e_2 = 0$). Let $\Phi_1$ be the composition of the following elementary automorphisms in succession:
\[
a_{12} \mapsto a_{12} - e_1 a_5; \quad a_8 \mapsto a_8 - e_1 a_2; \quad e_1 \mapsto e_1 + ta_{12} a_6 - e_2 a_2.
\]
Under $\Phi_1$, the DGA changes as follows:
\[
\begin{align*}
\begin{cases}
\partial a_8 = - t a_{12} a_6 \\
\partial e_1 = e_2 \\
\partial a_{12} = 0 \\
\partial e_2 = 0
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
\partial a_8 = e_1 \\
\partial e_1 = 0 \\
\partial a_{12} = e_2 a_5 \\
\partial e_2 = 0
\end{cases}
\end{align*}
\]
We may then drop $a_8$ and $e_1$; these simply correspond to an algebraic stabilization.
Let $\Phi_2$ be the composition of the following maps:

\[
a_{15} \mapsto a_{15} - ta_{12}a_6; \quad a_{11} \mapsto a_{11} - ta_{12}\tilde{a}_2; \quad a_{12} \mapsto a_{12} + a_{15}a_5 - te_2a_5\tilde{a}_2; \quad a_{15} \mapsto a_{15} + e_2a_2.
\]

Under $\Phi_2$, the DGA now changes as follows:

\[
\begin{align*}
\partial a_{11} &= a_{15}a_5 \\
\partial a_{12} &= e_2a_5 \\
\partial a_{15} &= 0 \\
\partial e_2 &= 0
\end{align*}
\]

\[
\Phi_2 \rightarrow \begin{align*}
\partial a_{11} &= a_{12} \\
\partial a_{12} &= 0 \\
\partial a_{15} &= e_2 \\
\partial e_2 &= 0
\end{align*}
\]

We can now drop $a_{11}, a_{12}, a_{15}, e_2$; these correspond to two algebraic stabilizations.

Hence, up to algebraic stabilizations, we have eliminated $a_8, a_{11}, a_{12}, a_{15}$. An entirely similar process allows us to eliminate $a_9, a_{10}, a_{13}, a_{14}$. The resulting DGA is precisely the DGA of $S(K,W_0)$, as desired.

\[\square\]

Remark 4.1. The only part of this proof which uses the structure of $W_0$ is the calculation of the degrees of the vertices in Figure 7. To prove the more general case given in Proposition 3.6, we have to take more care vis-à-vis degrees, but the idea is the same. The proof also extends to knots which are not satellite doubles, but whose rightmost parts look like the bottom diagrams in Figure 7.

Remark 4.2. The method of the proof can also be used to show that the invariant $HC_{132}$ introduced by Michatchev [7] does not encode any nonclassical information for stabilized knots. The computation for this case involves more generators, but less algebra, than the computation performed in this section.

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