DISCRETE TWO-GENERATOR SUBGROUPS OF PSL₂ OVER NON-ARCHIMEDEAN LOCAL FIELDS

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Abstract. Let \( K \) be a non-archimedean local field with residue field of characteristic \( p \). We give necessary and sufficient conditions for a two-generator subgroup \( G \) of \( \text{PSL}_2(K) \) to be discrete, where either \( K = \mathbb{Q}_p \) or \( G \) contains no elements of order \( p \). We give a practical algorithm to decide whether such a subgroup \( G \) is discrete. We also give practical algorithms to decide whether a two-generator subgroup of either \( \text{SL}_2(\mathbb{R}) \) or \( \text{SL}_2(K) \) (where \( K \) is a finite extension of \( \mathbb{Q}_p \)) is dense. A crucial ingredient for this work is a structure theorem for two-generator groups acting by isometries on a \( \Lambda \)-tree.

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1. Introduction

The problem of identifying discrete two-generator subgroups of \( \text{PSL}_2(\mathbb{R}) \) has been extensively studied in the literature. A complete classification of such groups, and practical algorithms to decide whether or not a two-generator subgroup of \( \text{PSL}_2(\mathbb{R}) \) is discrete, are given in [13, 17, 24].

Here we prove analogous results for certain two-generator subgroups of \( \text{PSL}_2(K) \) over a non-archimedean local field \( K \) by studying the action on the associated Bruhat-Tits tree [26]. Our main classification result is the following; see Theorem A’ in Section 4 for a more detailed statement.

Theorem A. Let \( G \) be a discrete two-generator subgroup of \( \text{PSL}_2(K) \), where \( K \) is a non-archimedean local field with residue field of characteristic \( p \). If \( K = \mathbb{Q}_p \), or \( G \) contains no elements of order \( p \), then one of the following holds (where \( n \) and \( m \) are positive integers, parameterised by \( p \) as detailed in Theorem A’):

(a) \( G \) is finite, and either cyclic, dihedral, or isomorphic to \( A_4, S_4 \) or \( A_5 \);
(b) \( G \) is discrete and free of rank two;
(c) \( G \cong C_n \times C_m \);
(d) \( G \cong C_n \times \mathbb{Z} \);
(e) \( G \cong C_n \times \mathbb{Z} \), or \( G \cong \mathbb{Z} \);
(f) \( G \) is an HNN extension of either \( D_{2n+1} \) or \( A_4 \);
(g) \( G \) is isomorphic to either \( D_{2n+1} \ast C_2 D_2 \), \( A_4 \ast C_3 D_3 \), \( S_4 \ast C_4 D_4 \), \( A_5 \ast C_3 D_3 \) or \( A_5 \ast C_5 D_5 \).

Moreover, each of these possibilities can occur.

Throughout the paper, \( K \) will be used to denote a non-archimedean local field with finite residue field \( \mathbb{F}_q \) of characteristic \( p \). That is, \( K \) is either a finite extension of the \( p \)-adic numbers \( \mathbb{Q}_p \), or the field of formal Laurent series \( \mathbb{F}_q((t)) \) [25]. We will denote the valuation ring of \( K \) by \( \mathcal{O}_K \), and the uniformiser of \( K \) by \( \pi \).

When \( K = \mathbb{F}_q((t)) \), every order \( p \) element in \( \text{PSL}_2(K) \) is unipotent [18, p. 964]. Since cocompact discrete subgroups of \( \text{PSL}_2(K) \) do not contain unipotent elements [12, p. 10], we immediately obtain the following.

Corollary 1.1. The conclusion of Theorem A holds for discrete and cocompact two-generator subgroups of \( \text{PSL}_2(\mathbb{F}_q((t))) \).
Remark 1.2. As detailed in Theorem \(A\) (see Section 3), cases (f) and (g) only occur for certain congruence classes of \(q\).

Remark 1.3. All groups we obtain in case (g) are of the form \(G_1 \ast G_3 G_2\), where \(G_3\) is a maximal cyclic subgroup of both \(G_1\) and \(G_2\). These are also described in case (2) of [28] Theorem 3.5], which gives a list of discrete finitely generated subgroups of \(\text{PGL}_2\) over a \(p\)-adic field.

Remark 1.4. Our proof of Theorem \(A\) relies heavily on the fact that the set of points of the Bruhat-Tits tree fixed by a finite order element of \(G\) is preserved under non-trivial powers; see Corollary \(3.5\). If \(K \neq \mathbb{Q}_p\), then \(G\) may contain elements of order \(p\) with fixed point sets which are not necessarily preserved under non-trivial powers [19] Theorem 4.2]. We expect that significant further analysis will be required to complete an analogous classification for discrete two-generator subgroups of \(\text{PSL}_2(K)\) which contain elements of order \(p\).

Theorem \(A\) is a consequence of the following geometric theorem with corresponding labelling. Here, and throughout the paper, we will use \(l(X)\) to denote the translation length of an isometry \(X\) of a tree \(T\). We will also use \(\text{Fix}(A)\) to denote the fixed point set of an elliptic isometry \(A\) of \(T\), and \(\text{Ax}(B)\) to denote the axis of a hyperbolic isometry \(B\) of \(T\). Given a finite subpath \(P = [x, y]\) of \(\text{Ax}(B)\), where \(B\) translates \(x\) towards \(y\), we also refer to \(x\) (respectively \(y\)) as the initial (respectively terminal) vertex of \(P\).

Theorem B. Let \(G\) be a two-generator subgroup of \(\text{PSL}_2(K)\). If \(K = \mathbb{Q}_p\), or \(G\) contains no elements of order \(p\), then \(G\) is discrete if and only if there exists a generating pair \((A, B)\) for \(G\) such that one of the following holds:

(a) \(A\) and \(B\) are elliptic with \(\text{Fix}(A) \cap \text{Fix}(B) \neq \emptyset\), and \(G = \langle A, B \rangle\) is finite;
(b) \(A\) and \(B\) are hyperbolic, and \(\text{Ax}(A) \cap \text{Ax}(B)\) is either empty or a path of length \(\Delta < \min\{l(A), l(B)\}\);
(c) \(A\) and \(B\) are elliptic of finite order and \(\text{Fix}(A) \cap \text{Fix}(B) = \emptyset\);
(d) \(A\) is elliptic of finite order, \(B\) is hyperbolic of minimal translation length among the elements \(\{A^iB : i \in \mathbb{Z}\}\), and \(\text{Ax}(A) \cap \text{Ax}(B)\) is either empty or a path of length \(\Delta < l(B)\);
(e) \(A\) is elliptic of finite order, \(B\) is hyperbolic, and \(A\) and \(B\) commute;
(f) \(A\) is elliptic of finite order, \(B\) is hyperbolic and \(\text{Ax}(A) \cap \text{Ax}(B)\) is a path \(P\) of length \(\Delta = l(B)\). The group \(G_0 = \langle A, B, AB^{-1} \rangle\) is finite and does not contain a reflection in \(\text{Ax}(B)\) about the terminal vertex of \(P\);
(g) \(A\) is elliptic of finite order, \(B\) is hyperbolic and \(\text{Fix}(A) \cap \text{Ax}(B)\) is a path \(P\) of length \(\Delta = l(B)\). The group \(G_0 = \langle A, B, AB^{-1} \rangle\) is finite and contains a reflection in \(\text{Ax}(B)\) about the terminal vertex of \(P\).

Using Theorems \(A\) and \(B\) we obtain a practical algorithm (Algorithm 6.1) to decide whether or not a two-generator subgroup \(G\) of \(\text{PSL}_2(K)\) (where either \(K = \mathbb{Q}_p\), or \(G\) contains no elements of order \(p\)) is discrete. If \(G\) is discrete, then the algorithm returns the isomorphism type of \(G\) according to Theorem \(A\). Following a suggestion by Pierre-Emmanuel Caprace, we also obtain practical algorithms (Algorithms 6.3 and 6.11) to decide whether or not a two-generator subgroup of either \(\text{SL}_2(\mathbb{R})\) or \(\text{SL}_2(K)\) (where \(K\) is a finite extension of \(\mathbb{Q}_p\)) is dense. All three of these algorithms have been implemented in \textsc{Magma} [2] over appropriate subfields \([9]\).

Outline of paper: In Section 2, we prove a structure theorem (Theorem \(C\)) for (not necessarily discrete) two-generator groups acting by isometries on \(\Lambda\)-trees, which hinges on Klein-Maskit combination theorems given in [20] VIII]. In Section 3, we consider subgroups of \(\text{PSL}_2(K)\) acting on the corresponding Bruhat-Tits tree \(T_q\). Crucial to our results is the classification of the fixed point sets in \(T_q\) of certain finite order elements of \(\text{PSL}_2(K)\); see Proposition 3.4. The rest of this section is devoted to a systematic investigation of Theorem \(C\) for two-generator subgroups of \(\text{PSL}_2(K)\). In Section 4, we prove Theorem \(B\) and use it to prove Theorem \(A\). The latter proof relies on examples provided in Section 5. Finally, in Section 6, we present the practical algorithms identified above and discuss their implementation.
The following theorem is fundamental to our proof of Theorem A. It is stated in the context of Λ-trees, a wide class of metric spaces which includes ℝ-trees and simplicial trees; see [6]. The theorem crucially relies on Klein-Maskit combination theorems given in [20], and it also generalises Lemmas 2.1 and 2.2 of [15], which deal with cases (3) and (4). By replacing a Λ-tree \( T \) by an appropriate subdivision if necessary, we may assume that every isometry of \( T \) acts without inversions and is hence either elliptic or hyperbolic; see Lemma 1.3 of [6, Chapter 3].

**Theorem C.** Let \( G = \langle A, B \rangle \) be a group acting by isometries on a Λ-tree \( T \). After interchanging the roles of \( A \) and \( B \) if necessary, precisely one of the following holds:

1. \( A \) and \( B \) are elliptic, \( \text{Fix}(A) \cap \text{Fix}(B) \neq \emptyset \), and \( G \) fixes a point of \( T \).
2. \( A \) and \( B \) are hyperbolic, \( \text{Ax}(A) \cap \text{Ax}(B) \) is either empty or a path of length \( \Delta < \min\{l(A), l(B)\} \), and \( G = \langle A \rangle \ast \langle B \rangle \). Moreover, \( \text{Stab}_G(y) = \{e\} \) for every \( y \in \text{Ax}(A) \cup \text{Ax}(B) \).
3. \( A \) and \( B \) are elliptic, \( \text{Fix}(A) \cap \text{Fix}(B) = \emptyset \), and one of the following holds:
   (i) \( G = \langle A \rangle \ast \langle B \rangle \), \( \text{Stab}_G(x) = \langle A \rangle \) for every \( x \in \text{Fix}(A) \) and \( \text{Stab}_G(y) = \langle B \rangle \) for every \( y \in \text{Fix}(B) \);
   (ii) \( \text{Fix}(A') \cap \text{Fix}(B') \neq \emptyset \) for some non-trivial powers \( A' \) and \( B' \).
4. \( A \) is elliptic, \( B \) is hyperbolic, \( \text{Fix}(A) \cap \text{Ax}(B) \) is empty or a path of length \( \Delta < l(B) \), and one of the following holds:
   (i) \( G = \langle A \rangle \ast \langle B \rangle \) and \( \text{Stab}_G(y) = \langle A \rangle \) for every \( y \in \text{Fix}(A) \);
   (ii) \( A'B \) is elliptic for some non-trivial power \( A' \);
   (iii) \( \bigcup_{i=1}^{l(B)} \text{Fix}(A^i) \) is a path of length \( \Delta' \geq l(B) \).
5. \( A \) is elliptic, \( B \) is hyperbolic, \( \text{Fix}(A) \cap \text{Ax}(B) \) contains a geodesic ray, and one of the following holds:
   (i) \( A \) commutes with a power of \( B \) and \( \text{Ax}(B) \subseteq \text{Fix}(A) \);
   (ii) \( \text{Stab}_G(y) \) is infinite for every \( y \in \text{Fix}(A) \cap \text{Ax}(B) \).
6. \( A \) is elliptic, \( B \) is hyperbolic, \( \text{Fix}(A) \cap \text{Ax}(B) \) is a finite path of length \( \Delta \geq l(B) \) (with initial vertex \( x \) and terminal vertex \( y \)), and one of the following holds, where \( G_0 = \langle A, B^kAB^{-k} \rangle \) and \( k = \left\lfloor \frac{\Delta}{l(B)} \right\rfloor \):
   (i) \( G \) is an HNN extension \( G_0 *_{B} \), and \( G_0 = \text{Stab}_G(y) \);
   (ii) \( G_0 \) contains a subgroup \( H \) which fixes the path \([y, B^{k+1}x]\) and properly contains the subgroup \( \langle BAB^{-1}, \ldots, B^kAB^{-k} \rangle \);
   (iii) \( \Delta = k \cdot l(B) \) and there exists \( g \in G_0 \) such that \( gBy = B^{-1}y \).
7. \( A \) and \( B \) are hyperbolic, \( \text{Ax}(A) \cap \text{Ax}(B) \) is a path of length \( \Delta \geq \min\{l(A), l(B)\} \), and \( \min\{l(A), l(B)\} + \min\{l(AB), l(A^{-1}B)\} < l(A) + l(B) \).

**Remark 2.1.** Suppose that there is a positive lower bound on the translation length of all hyperbolic elements in \( G \) (for instance, when \( T \) is a simplicial tree). In case (7) of Theorem C repeatedly replacing an element of \( \{A, B\} \) with maximal translation length by an element of \( \{AB, A^{-1}B\} \) with minimal translation length will strictly reduce the sum of the translations lengths of the generators. Hence, after a finite number of steps, this process will produce a generating pair for \( G \) such that one of cases (1) – (6) of Theorem C occurs. Note that a similar argument is used in [8, Algorithm 4.1].

**Proof.** Case (1): \( G \) fixes every point of \( \text{Fix}(A) \cap \text{Fix}(B) \).

Case (2): Let \( S \) be an open segment of \( \text{Ax}(B) \) of length \( l(B) \) which contains \( \pi_B(\text{Ax}(A)) \), where \( \pi_B : T \to \text{Ax}(B) \) is the geodesic projection map. Let \( G_1 = \langle A \rangle \) and \( G_2 = \langle B \rangle \) with common subgroup \( J = \{e\} \), and set \( X_2 = \pi_B^{-1}(S) \) and \( X_1 = T \setminus X_2 \). Observe that \( X_1 \) and \( X_2 \) are invariant under \( J \), \( g_1(X_1) \subseteq X_2 \) for every \( g_1 \in G_1 \setminus \{e\} \), and \( g_2(X_2) \subseteq X_1 \) for every \( g_2 \in G_2 \setminus \{e\} \); see Figure 1 for the case where \( \text{Ax}(A) \cap \text{Ax}(B) \neq \emptyset \). Also no element of \( \text{Ax}(A) \subseteq X_2 \) can be the image of a point in \( X_1 \) under \( G_1 \). In the terminology of [20], \( X_1, X_2 \) is therefore a proper interactive pair for the groups \( G_1, G_2 \) and \( J \). Hence \( G = G_1 \ast_{J} G_2 = \langle A \rangle \ast \langle B \rangle \) by
Theorem A.10 of [20, VII]. Moreover, no non-trivial word in \( (A, B) \) can stabilise a point in \( \text{Ax}(A) \), so \( \text{Stab}_G(x) = \{ e \} \) for every point \( x \in \text{Ax}(A) \). By switching the roles of \( A \) and \( B \), it also follows that \( \text{Stab}_G(y) = \{ e \} \) for every \( y \in \text{Ax}(B) \).

![Figure 1. Combination theorem sets for Theorem C (2)](attachment:image1.png)

**Figure 1. Combination theorem sets for Theorem C (2)**

**Case (3):** We may suppose that the subtrees \( F(A) = \bigcup_{i: x^i \not= e} \text{Fix}(A^i) \) and \( F(B) = \bigcup_{j: y^j \not= e} \text{Fix}(B^j) \) are disjoint, otherwise subcase \((ii)\) holds. We then argue as in case \((2)\), but with \( F(A) \) and \( F(B) \) playing the roles of \( \text{Ax}(A) \) and \( \text{Ax}(B) \). Indeed, let \( S = \langle u, v \rangle \) be the unique geodesic from \( F(A) \) to \( F(B) \) and set \( G_1 = \langle A \rangle, \ G_2 = \langle B \rangle \) and \( J = \{ e \} \). Let \( \pi_S : T \to S \) be the geodesic projection map. Observe that \( X_1 = \pi_S^{-1}(b) \) and \( X_2 = \pi_S^{-1}(a) \) are invariant under \( J \), \( g_1(X_1) \subseteq X_2 \) for every \( g_1 \in G_1 \backslash \{ e \} \), and \( g_2(X_2) \subseteq X_1 \) for every \( g_2 \in G_2 \backslash \{ e \} \); see Figure 2. Moreover, no element of \( \text{Fix} \{ A \} \subseteq X_2 \) can be the image of a point in \( X_1 \) under \( G_1 \). Hence \( G = \langle A \rangle * \langle B \rangle \) by Theorem A.10 of [20, VII]. The only words in \( (A, B) \) that stabilise points in \( \text{Fix}(A) \) are powers of \( A \), so \( \text{Stab}_G(x) = \langle A \rangle \) for every point \( x \in \text{Fix}(A) \) and, similarly, \( \text{Stab}_G(y) = \langle B \rangle \) for every point \( y \in \text{Fix}(B) \).

![Figure 2. Combination theorem sets for Theorem C (3)](attachment:image2.png)

**Figure 2. Combination theorem sets for Theorem C (3)**

**Case (4):** Let \( F(A) = \bigcup_{i: x^i \not= e} \text{Fix}(A^i) \). We may suppose that \( F(A) \cap \text{Ax}(B) \) is either empty or a path of length \( \Delta' < l(B) \), as otherwise subcase \((iii)\) occurs. Let \( \pi_B : T \to \text{Ax}(B) \) be the geodesic projection map and let \( S = \langle u, v \rangle \) be an open segment of \( \text{Ax}(B) \) of length \( l(B) \) which contains \( \pi_B(F(A)) \). Let \( G_1 = \langle A \rangle, \ G_2 = \langle B \rangle \) and \( J = \{ e \} \), and set \( X_2 = \pi_B^{-1}(S) \) and \( X_1 = T \backslash X_2 \). Observe that \( X_1 \) and \( X_2 \) are invariant under \( J \), \( g_2(X_2) \subseteq X_1 \) for every \( g_2 \in G_2 \backslash \{ e \} \), and no element of \( \text{Fix}(A) \subseteq X_2 \) can be the image of a point in \( X_1 \) under \( G_1 \); see Figure 3 for the case where \( F(A) \cap \text{Ax}(B) \not= \emptyset \). If additionally \( g_1(X_1) \subseteq X_2 \) for every \( g_1 \in G_1 \backslash \{ e \} \), then \( G = \langle A \rangle * \langle B \rangle \) and \( \text{Stab}_G(y) = \langle A \rangle \) for every \( y \in \text{Fix}(A) \) by the same argument as in the previous case. Without loss of generality, we may hence suppose that there exists \( g \in G_1 \backslash \{ e \} \) such that \( g \cdot v \in X_1 \). Hence \( \pi_B(g \cdot v) \not\in S \), thus \( \text{Fix}(g) \cap \text{Ax}(B) \) is a single point \( x \) and \( g \) maps \( v \) to the unique other point of \( \text{Ax}(B) \backslash S \) which is equidistant from \( x \). In particular, \( g \) maps one of the two points of \( \text{Ax}(B) \) at distance \( \frac{l(B)}{2} \) from \( x \) to the other or, equivalently, there is an integer \( i \) such that \( A^iB \) is elliptic.
Case (5): Let \( y \in \text{Fix}(A) \cap \text{Ax}(B) \). After replacing \( B \) by \( B^{-1} \) if necessary, we may assume that \( B^i y \in \text{Fix}(A) \cap \text{Ax}(B) \) for every positive integer \( i \). Hence \( B^{-i} A B^i \in \text{Stab}_G(y) \) for every positive integer \( i \), so either \( \text{Stab}_G(y) \) is infinite (which is case (5)(ii)), or there are positive integers \( i < j \) such that \( B^{-i} A B^i = B^{-j} A B^j \). Suppose that the latter case holds, so that \( A \) commutes with \( B^{j-i} \) and \( \text{Ax}(B) = \text{Ax}(B^{j-i}) = \text{Ax}(B^{j-i} A^{-1}) = A \cdot \text{Ax}(B) \). If \( x \in \text{Ax}(B) \setminus \text{Fix}(A) \), then \( \text{Ax} \subseteq \text{Ax}(B) \) and it follows that \( \text{Ax} = x \), which is a contradiction. Thus \( \text{Ax}(B) \subseteq \text{Fix}(A) \) and case (5)(i) holds.

Case (6): Let \( \pi_B : T \rightarrow \text{Ax}(B) \) be the geodesic projection map and let \( b_-, b_+ \) denote the ends of \( \text{Ax}(B) \) on the boundary of \( T \), where \( B \) translates in the direction of \( b_+ \). Let \( J_1 = \langle A, B A B^{-1}, \ldots, B^{k-1} A B^{-(k-1)} \rangle \) and \( J_2 = \langle B A B^{-1}, \ldots, B^k A B^{-k} \rangle \) be subgroups of \( G_0 \), and let \( Z = \pi_B^{-1}(B^k x, B^{k+1} x) \), \( X_1 = \pi_B^{-1}(b_-, B^k x) \) and \( X_2 = \pi_B^{-1}(B^{k+1} x, b_+). \) Observe that \( B(Z \cup X_2) \subseteq X_2 \) and \( B^{-1}(Z \cup X_1) \subseteq X_1 \); see Figure 4. Note that \( J_1 \) fixes the non-trivial path \( [B^{k-1} x, B^k x] \) and it follows that \( X_1 \) is precisely invariant under \( J_1 \), that is, \( \text{Stab}_{G_0}(X_1) = J_1 \) and \( g(X_1) \cap X_1 = \emptyset \) for each \( g \in G_0 \setminus J_1. \) Moreover, \( y \in Z \) is fixed by \( G_0 \) and thus not a \( G_0 \)-translate of a vertex in \( X_1 \) or \( X_2 \). If additionally \( X_2 \) is precisely invariant under \( J_2 \), \( g(X_1) \subseteq Z \cup X_1 \) and \( g(X_2) \subseteq Z \cup X_2 \) for each \( g \in G_0 \), then, in the terminology of [20, VII.D], \( (Z, X_1, X_2) \) is a proper interactive triple for \( G = \langle G_0, B \rangle. \) Thus \( G = G_0 * B \) by Theorem D.12 of [20, VII] and also \( G_0 = \text{Stab}_G(y) \), which can be seen by applying Lemma D.11 of [20, VII] to \( y \). Hence we may suppose that one of these additional three conditions fails.

If \( X_2 \) is not precisely invariant under \( J_2 \), then there exists \( g \in G_0 \setminus J_2 \) and \( v \in X_2 \) such that \( g v \in X_2 \). Since \( G_0 \) fixes \( y \), and \( B^{k+1} x \) is the unique closest point of \( X_2 \) to \( y \), it follows that \( g \) fixes \( B^{k+1} x \). Thus the subgroup \( H = \langle J_2, g \rangle \) of \( G_0 \) fixes \( [y, B^{k+1} x] \) and properly contains \( J_2 \), which is subcase (ii).

On the other hand, suppose that there exists \( g \in G_0 \) such that \( g(X_2) \not\subseteq Z \cup X_2 \) (respectively \( g(X_1) \not\subseteq Z \cup X_1 \)). Since \( G_0 \) fixes \( [B^k x, y] \), it follows that \( y = B^k x \), and \( g \cdot B^k x = B^{-1} y \) (respectively \( g^{-1} \cdot B y = B^{-1} y \)). This corresponds to subcase (iii).
We may assume that $l(A) \leq l(B)$. By Lemmas 3.4 and 3.5 of [16, Chapter 3], $\min\{l(AB), l(A^{-1}B)\} \leq l(B) - l(A)$, from which the conclusion follows immediately. □

3. Subgroups of $\text{PSL}_2(K)$ and their actions on the Bruhat-Tits tree

In this section, we consider subgroups of $\text{PSL}_2(K)$ acting on the corresponding Bruhat-Tits tree $T_q$, where $K$ is a non-archimedean local field with finite residue field $\mathbb{F}_q$ of characteristic $p$. Note that we view $\text{PSL}_2(K)$ as a subgroup of the isometry group of $T_q$, so that $\text{PSL}_2(K)$ inherits the topology of pointwise convergence from the product topology on $T_q^\infty$.

As we will frequently refer to it throughout the paper, we state below a lemma of Kato describing necessary and sufficient conditions for discreteness of subgroups of $\text{PSL}_2(K)$.

Lemma 3.1. [10] Lemma 4.4.1] Let $G$ be a subgroup of $\text{PSL}_2(K)$.

1. If $G$ is discrete, then $\text{Stab}_G(y)$ is finite for every vertex $y$ of $T_q$.
2. If $\text{Stab}_G(y)$ is finite for some vertex $y$ of $T_q$, then $G$ is discrete.

The translation length of each element of $\text{PSL}_2(K)$ on $T_q$ is given by the following formula, where $v$ is the discrete valuation associated to $K$.

Proposition 3.2. [21, Proposition II.3.15] The translation length of $A \in \text{PSL}_2(K)$ on $T_q$ is given by

$$l(A) = -2 \min\{0, v(\text{tr}(\bar{A}))\},$$

where $\bar{A}$ is either of the two representatives of $A$ in $\text{SL}_2(K)$.

We recall the following well-known classification of finite subgroups of $\text{PSL}_2(K)$; we include a proof for completeness.

Proposition 3.3. Let $G$ be a non-trivial finite subgroup of $\text{PSL}_2(K)$, where either $K = \mathbb{Q}_p$ or $G$ contains no elements of order $p$. Precisely one of the following holds:

- $G \cong C_n$, where either $q \equiv \pm 1 \mod 2n$, or $q$ is even and $q \equiv \pm 1 \mod n$, or $K = \mathbb{Q}_p$ and $n = p \in \{2, 3\}$;
- $G \cong D_n$, where either $q \equiv \pm 1 \mod 2n$, or $K = \mathbb{Q}_2$ and $n = 3$;
- $G \cong A_4$ and either $p > 3$ or $K = \mathbb{Q}_3$;
- $G \cong S_4$ and $q \equiv \pm 1 \mod 8$;
- $G \cong A_5$ and $q \equiv \pm 1 \mod 10$.

Proof. Let $\bar{G}$ be a pre-image of $G$ in $\text{SL}_2(K)$. Since $\bar{G}$ is finite, it is conjugate into $\text{SL}_2(\mathcal{O}_K)$, and hence the kernel of the reduction map $\bar{G} \to \text{SL}_2(\mathbb{F}_q)$ is a pro-$p$ group [15, p. 964]. If $K \not\in \{\mathbb{Q}_2, \mathbb{Q}_3\}$, then $\bar{G}$ does not contain elements of order $p$, unless $p = 2$ and $-I \in \bar{G}$ [18, p. 972]. Since $-I$ is the unique involution in any special linear group, it follows that $G$ is isomorphic to a finite subgroup of $\text{PSL}_2(\mathbb{F}_q)$ and the classification follows from [10, Section 260].

On the other hand, if $K = \mathbb{Q}_p$ for $p \in \{2, 3\}$, then the proof of [18, Theorem 3.6] shows that $G$ is isomorphic to a finite subgroup of $\text{PSL}_2(\mathbb{F}_p)$. The result follows since $\text{PSL}_2(\mathbb{F}_2) \cong D_3$ and $\text{PSL}_2(\mathbb{F}_3) \cong A_4$. □

We also classify the fixed point sets of certain finite order elements of $\text{PSL}_2(K)$. In characteristic zero, this also follows from [19, Lemma 4.1].

Proposition 3.4. Let $A \in \text{PSL}_2(K)$ be a non-trivial element of finite order $n$. If $p \nmid n$ or $K = \mathbb{Q}_p$, then precisely one of the following holds:

- $\text{Fix}(A)$ consists of two adjacent vertices, $K = \mathbb{Q}_p$ and $n = p \in \{2, 3\}$;
- $\text{Fix}(A)$ is a single vertex, and either $q \equiv -1 \mod 2n$, or $q$ is even and $q \equiv -1 \mod n$;
- $\text{Fix}(A)$ is a bi-infinite ray, and either $q \equiv 1 \mod 2n$, or $q$ is even and $q \equiv 1 \mod n$. 

Proof. Let $\bar{A}$ be a representative of $A$ in $\SL_2(K)$ with minimal order (hence $\bar{A}$ has order $n$ if $n$ is odd, and order $2n$ if $n$ is even). Replacing $\bar{A}$ by an appropriate conjugate, if necessary, we may assume that $\bar{A} \in \SL_2(\mathcal{O}_K)$, so $\bar{A}$ fixes the vertex $v$ of $T_q$ corresponding to the standard lattice $\mathcal{O}_K$. By \cite[Proposition 11]{1}, the number of vertices in $\Fix(A)$ at distance $k$ from $v$ is precisely the number of roots of the characteristic polynomial of $A$ over $\mathcal{O}_K/\mathcal{O}_K^\times \cong \mathbb{Z}/p^k\mathbb{Z}$.

First suppose that $p \mid n$. Thus $K = \mathbb{Q}_p$ and, by Proposition \ref{3.3}, $n = p \in \{2, 3\}$. If $p = 2$, then $\bar{A}$ has order four and its characteristic polynomial is $\lambda^2 + 1$. This polynomial has one repeated root in $\mathbb{Z}/2\mathbb{Z}$ and no roots in $\mathbb{Z}/2^k\mathbb{Z}$ for $k > 1$, hence $\Fix(A) = \{v, w\}$ for some vertex $w$ adjacent to $v$. If $p = 3$, then $\bar{A}$ has order three, and its characteristic polynomial is $\lambda^2 + \lambda + 1$. This polynomial has one repeated root in $\mathbb{Z}/3\mathbb{Z}$ and no roots in $\mathbb{Z}/3^k\mathbb{Z}$ for $k > 1$. Hence $\Fix(A)$ also consists of two adjacent vertices in this case.

Now suppose that $p \nmid n$. By Proposition \ref{3.3} either $2n \mid q + 1$, or $q$ is even and $n \mid q + 1$. Observe from Hensel’s Lemma that, for $m$ coprime to $p$, $K$ contains an $m$-th root of unity if and only if $m \mid q - 1$. If $2n \mid q - 1$ (respectively $q$ is even and $n \mid q - 1$), then it follows that $\bar{A}$ is diagonalisable over $K$, so for every positive integer $k$ the characteristic polynomial of $\bar{A}$ has exactly two roots in $\mathbb{Z}/q^k\mathbb{Z}$. Hence $\Fix(A)$ is a bi-infinite ray through $v$. On the other hand, if $2n \mid q + 1$ (respectively $q$ is even and $n \mid q + 1$), then for every positive integer $k$ the characteristic polynomial of $\bar{A}$ has no roots in $\mathbb{Z}/q^k\mathbb{Z}$ and $\Fix(A) = \{v\}$.

We deduce that finite order elements of $\PSL_2(K)$ with order coprime to $p$, and finite order elements of $\PSL_2(\mathbb{Q}_p)$, have the same fixed point set as their non-trivial powers.

**Corollary 3.5.** If $A \in \PSL_2(K)$ is a non-trivial element of finite order $n$, and either $p \nmid n$ or $K = \mathbb{Q}_p$, then $\Fix(A) = \Fix(A^i)$ for every positive integer $i \not\equiv 0 \mod n$.

**Proof.** Let $i$ be a positive integer not divisible by $n$. If $p \mid n$, then $K = \mathbb{Q}_p$ and $n = p \in \{2, 3\}$ by Proposition \ref{3.3}. The result follows immediately since $A$ and $A^{-1}$ fix precisely the same points.

Hence suppose that $p \nmid n$ and let $m$ denote the order of $A^i$. Since $m$ is a divisor of $n$, every odd $q$ lies in the same congruence class modulo $2m$ as it does modulo $2n$, and every even $q$ lies in the same congruence class modulo $m$ as it does modulo $n$. It follows from Proposition \ref{3.3} that $\Fix(A)$ and $\Fix(A^i)$ are either both a single vertex or both a bi-infinite ray. Since $\Fix(A) \subseteq \Fix(A^i)$, the two fixed point sets must coincide.

We will frequently use the following special case of Corollary 3.5.

**Corollary 3.6.** Let $A \in \PSL_2(K)$ be a non-trivial element of finite order $n$, where either $p \nmid n$ or $K = \mathbb{Q}_p$. If $\Fix(A) \neq \Fix(A^2)$, then $A$ is an involution.

We also make the following observations.

**Lemma 3.7.** Let $A, B \in \PSL_2(K)$ be such that $B$ is hyperbolic and $AB$ has finite order. Suppose that either $K = \mathbb{Q}_p$, or the order of $AB$ is coprime to $p$. If $A$ is an involution which fixes a vertex $y \in \Ax(B)$, then $A$ is a reflection in $\Ax(B)$ about $y$ if and only if $AB$ is an involution.

**Proof.** If $A$ is a reflection in $\Ax(B)$ about $y$, then $(AB)^2$ fixes $y$ and $AB$ does not, so $AB$ is an involution by Corollary \ref{3.6}. On the other hand, if $AB$ is an involution, then $A \cdot B y = B^{-1} A y = B^{-1} y$ and $A \cdot B^{-1} y = A(AB) y = B y$. By induction, $A \cdot B^k y = B^{-k} y$ for all $k \in \mathbb{Z}$.

**Lemma 3.8.** Let $A \in \PSL_2(K)$ be a non-trivial element of finite order such that either $K = \mathbb{Q}_p$, or the order of $A$ is coprime to $p$. If $B \in \PSL_2(K)$ is hyperbolic (respectively elliptic with $\Fix(B)$ a bi-infinite ray), then $A$ and $B$ commute if and only if $\Fix(A) = \Ax(B)$ (respectively $\Fix(A) = \Fix(B)$).

**Proof.** Without loss of generality, we may assume that $A$ and $B$ are respectively represented by the following matrices in $\SL_2(K)$:

$$
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  \lambda & 0 \\
  0 & \lambda^{-1}
\end{bmatrix}.
$$
Hence $A$ and $B$ commute if and only if $b(\lambda - \lambda^{-1}) = c(\lambda - \lambda^{-1}) = 0$. Since $B$ is non-trivial, this occurs if and only if $b = c = 0$, in which case $A$ and $B$ fix the same two boundary points of $T_q$. The result follows from Proposition 3.3. □

Proposition 3.9. Let $G$ be a finite subgroup of $\text{PSL}_2(K)$ which fixes at least two vertices of $T_q$. If $K = \mathbb{Q}_p$, or $G$ contains no elements of order $p$, then $G$ is either cyclic or isomorphic to the Klein four-group $D_2$.

Proof. By Proposition 3.4, each element of $G$ fixes a bi-infinite ray, unless $K = \mathbb{Q}_p$ for $p \in \{2, 3\}$ and $G$ contains an element of order $p$ which fixes two adjacent vertices (in which case, $G$ fixes precisely those two vertices).

We first show that $G$ does not contain a non-abelian dihedral group. Indeed, suppose for a contradiction that $G$ contains a dihedral group $H$ of order $2m$, where $m > 2$. Let $A$ and $B$ be generators of $H$, where $A$ has order 2, $B$ has order $m$ and $ABA^{-1} = B^{-1}$. Note that $A \cdot \text{Fix}(B) = \text{Fix}(ABA^{-1}) = \text{Fix}(B)$, and either $\text{Fix}(B)$ consists of two adjacent vertices, $K = \mathbb{Q}_3$ and $m = 3$, or $\text{Fix}(B)$ is a bi-infinite ray. In the former case, $\text{Fix}(A)$ is a single vertex by Proposition 3.4, which is a contradiction. In the latter case, if $x \in \text{Fix}(B)$ then $Ax \in \text{Fix}(B)$ and, since $A$ fixes at least two vertices of $\text{Fix}(B)$, we obtain $x \not\in \text{Fix}(A)$. Hence $\text{Fix}(B) \subseteq \text{Fix}(A)$ and thus $\text{Fix}(A) = \text{Fix}(B)$, so $A$ and $B$ commute by Lemma 3.8 which is also a contradiction.

Since $G$ is one of the groups listed in Proposition 3.3, it must therefore be cyclic, or isomorphic to either $D_2$ or $A_4$. Suppose that $G \cong A_4$ (in particular, $p \neq 2$). Let $A$ and $B$ be generators of $G$, where $A$ has order 3 and $B$ has order 2. Note that $ABA^{-1} = B'$ for some element $B' \in G$ of order 2 which commutes with $B$. Since $p \neq 2$, $\text{Fix}(B)$ is a bi-infinite ray and it follows from Lemma 3.8 that $A \cdot \text{Fix}(B) = \text{Fix}(B') = \text{Fix}(B)$. As before, we conclude that $\text{Fix}(A) = \text{Fix}(B)$, which is a contradiction since $A$ and $B$ do not commute. Hence $G$ is either cyclic or isomorphic to $D_2$. □

We now consider certain two-generator subgroups $G$ of $\text{PSL}_2(K)$ which satisfy case (6) of Theorem C.

Lemma 3.10. Let $G = \langle A, B \rangle$ be a subgroup of $\text{PSL}_2(K)$ such that $A$ is elliptic, $B$ is hyperbolic, and $\text{Fix}(A) \cap \text{Ax}(B)$ is a finite path $P$ of length $\Delta \geq l(B)$. Suppose that $G_0 = \langle A, BAB^{-1}, \ldots, B^kAB^{-k} \rangle$ is finite, where $k = \lfloor \frac{\Delta}{l(B)} \rfloor$. If either $K = \mathbb{Q}_p$, or $G_0$ contains no elements of order $p$, then $\Delta = l(B)$ and the following hold:

- $G_0$ is isomorphic to one of the following groups:
  - $D_{2n+1}$, where $q \equiv 1 \mod 4$ and $q \equiv \pm 1 \mod (4n + 2)$;
  - $A_4$, where $q \equiv 1 \mod 6$;
  - $S_4$, where $q \equiv 1 \mod 8$;
  - $A_5$, where $q \equiv 1, 19 \mod 30$ or $q \equiv 1 \mod 10$.

- If $G_0$ is isomorphic to $S_4$ or $A_5$, then there exists $g \in G_0$ such that $g, gB$ and $gBA$ are involutions.

- There is no proper subgroup of $G_0$ which fixes at least two vertices of $T_q$ and properly contains $\langle A \rangle$ or $\langle BAB^{-1} \rangle$.

Proof. Let $\text{Fix}(A) \cap \text{Ax}(B) = \langle x, y \rangle$, where $B$ translates $x$ towards $y$. By Proposition 3.4, $\text{Fix}(A)$ is a bi-infinite ray and $q - 1$ is divisible by either twice the order of $A$ (if $q$ is odd) or the order of $A$ (if $q$ is even).

If $\Delta > l(B)$, then the subgroup $\langle A, BAB^{-1} \rangle$ of $G_0$ fixes the path $[Bx, y]$ (which contains at least two vertices). By Proposition 3.4, this subgroup is abelian. Thus $A$ and $BAB^{-1}$ commute, so $\text{Fix}(A) = \text{Fix}(BAB^{-1})$ by Lemma 3.8, a contradiction since $A$ fixes $x$ but $BAB^{-1}$ does not. Hence $\Delta = l(B)$ and $G_0 = \langle A, BAB^{-1} \rangle$ must be one of the groups listed in Proposition 3.3. We consider possible generating pairs for each such group, where both generators have the same order.
If $G_0$ is dihedral, then $A$ has order 2 and $ABAB^{-1}$ has order $m$. Thus $q \equiv 1 \mod 4$ and, by Proposition 3.3, $q \equiv \pm 1 \mod 2m$. Moreover, if $m$ is even, then Fix($ABAB^{-1}$) is a bi-infinite ray by Proposition 3.4. Since $A$ commutes with $(ABAB^{-1})^2$, Corollary 3.5 and Lemma 3.8 show that Fix($A$) = Fix($((ABAB^{-1})^2)^2$) = Fix($ABAB^{-1}$). This is again a contradiction (since $A$ fixes $x$ but $ABAB^{-1}$ does not), so $m$ is odd and the result follows.

For the remainder of the proof, we will fix a choice of representatives for $A$ and $B$ in $SL_2(K)$, and (by slight abuse of notation) define the trace of elements of $G$ using these representatives. We will include appropriate bracketing to indicate the use of the well-known trace identity for elements $X, Y \in SL_2(K)$:

\[
\text{tr}(X)\text{tr}(Y) = \text{tr}(XY) + \text{tr}(XY^{-1}).
\]

In particular, we will frequently use the identity

\[
(3.1) \quad \text{tr}(ABAB^{-1}) = \text{tr}(A) - \text{tr}(ABAB^{-1}).
\]

$G_0 \cong A_4$: The only pairs of elements which have the same order and generate $A_4$ are elements of order 3. Hence $A$ has order 3, and $q \equiv 1 \mod 6$ since Proposition 3.3 shows that $q$ is odd. Moreover, it can also be verified that $ABAB^{-1}$ and $ABAB^{-1}B^{-1}$ both have order 3. Since $\text{tr}(A) = \pm \sqrt{2}$, it follows from Equation (3.1) that $\text{tr}(ABAB^{-1}) = \text{tr}(ABAB^{-1}B^{-1}) = 1$. If $g = ABAB^{-1}B^{-1}A \in G_0$, then

\[
\text{tr}(g) = \text{tr}(ABAB^{-1}B^{-1})\text{tr}(A) - \text{tr}(ABAB^{-1}B^{-1}A^{-1})
= \text{tr}(A) - \text{tr}(A) = 0,
\]

\[
\text{tr}(gB) = \text{tr}(ABAB^{-1}B^{-1})\text{tr}(AB) - \text{tr}(ABAB^{-1}B^{-1}A^{-2}A^{-1})
= \text{tr}(AB) - \text{tr}(A^{-1}B^{-1}) = 0, \quad \text{and}
\]

\[
\text{tr}(gBA) = \text{tr}(ABAB^{-1}B^{-1})\text{tr}(ABA) - \text{tr}(ABAB^{-1}B^{-1}A^{-1}B^{-1}A^{-1})
= \text{tr}(ABA) - \text{tr}(A^{-1}B^{-1}A^{-1}) = 0.
\]

Hence $g, gB$ and $gBA$ are involutions.

$G_0 \cong S_4$: It can be easily verified that $A$ must have order 3 or 5. If $A$ has order 3, then $q \equiv 1, 19 \mod 30$ by Proposition 3.3. It can also be verified that $ABAB^{-1}$ and $ABAB^{-1}B^{-1}$ both have order 5. Hence Equation (3.1) shows that $t = \text{tr}(ABAB^{-1}B^{-1}) = 1 - \text{tr}(ABAB^{-1}) = \frac{1 + \sqrt{5}}{2}$.

Similarly, if $A$ has order 5, then $q \equiv 1 \mod 10$ and Equation (3.1) shows that one of $ABAB^{-1}$ and $ABAB^{-1}B^{-1}$ has trace 1, and the other has trace $\frac{1 - \sqrt{5}}{2}$. Without loss of generality, we may assume that $t = \text{tr}(ABAB^{-1}B^{-1}) = \frac{1 + \sqrt{5}}{2}$.

In both cases, if $g = (ABAB^{-1}B^{-1})^2A \in G_0$, then

\[
\text{tr}(g) = \text{tr}((ABAB^{-1}B^{-1})^2)\text{tr}(A) - \text{tr}((ABAB^{-1}B^{-1})^2A^{-1})
= \text{tr}(A)(t^2 - 2) - \text{tr}(A^{-1}(B^{-1}ABA^{-1}))
= \text{tr}(A)(t^2 - t - 2) + \text{tr}(ABAB^{-1}A^{-1}B^{-1})
= \text{tr}(A)(t^2 - t - 1) = 0.
\]

Similarly, it can be shown that

\[
\text{tr}(gB) = \text{tr}(AB)(t^2 - t - 1) = 0, \quad \text{and}
\]

\[
\text{tr}(gBA) = \text{tr}(ABA)(t^2 - t - 1) = 0.
\]

Thus $g, gB$ and $gBA$ are involutions.
Finally, suppose that $H$ is a proper subgroup of $G_0$ which fixes at least two vertices of $T_q$ and properly contains $\langle A \rangle$ or $\langle BAB^{-1} \rangle$. By considering the isomorphism types of $G_0$ listed above, $H$ is either a non-abelian dihedral group or $A_4$, but this contradicts Proposition 3.9.

**Lemma 3.11.** Let $G = \langle A, B \rangle$ be a subgroup of $\text{PSL}_2(K)$ such that either $K = \mathbb{Q}_p$, or $G$ contains no elements of order $p$. Suppose that $A$ is elliptic, $B$ is hyperbolic, and $\text{Fix}(A) \cap \text{Ax}(B)$ is a path of length $\Delta = l(B)$ with initial vertex $x$ and terminal vertex $y$. If $G_0 = \langle A, BAB^{-1} \rangle$ is finite and contains an element which maps By to $x$, then there exists $g \in G_0$ such that $g, gB$ and $gBA$ are involutions.

**Proof.** Suppose that $g \in G_0$ is such that $g \cdot By = x$. Observe that any element of the form $gBA^i$ has order 2 by Corollary 3.6 since $(gBA^i)^2$ fixes $y$ but $(gBA^i)$ does not.

In particular, $gB$ and $gBA$ are involutions and hence

$$BAB^{-1} = g^{-1}(gBA)B^{-1} = g^{-1}A^{-1}B^{-1}(g^{-1}B^{-1}) = g^{-1}A^{-1}g.$$

It follows that $G_0 = \langle A, g \rangle$.

Since $G_0$ is isomorphic to one of the finite groups listed in Lemma 3.10, it is readily verified (using a computational algebra package such as Magma) that some element of the form $A^ig$ has order 2. Such an element also maps By to $x$, and hence we may assume that $g$ is also an involution.

**Lemma 3.12.** Let $G = \langle A, B \rangle$ be a subgroup of $\text{PSL}_2(K)$ such that $A$ is elliptic of order $m$, $B$ is hyperbolic, and $\text{Fix}(A) \cap \text{Ax}(B)$ is a path $P$ of length $\Delta = l(B)$ with initial vertex $x$ and terminal vertex $y$. Let $G_0 = \langle A, BAB^{-1} \rangle$ and suppose that the following two conditions hold:

- There exists $g \in G_0$ such that $g, gB$ and $gBA$ are involutions;
- There is no proper subgroup $H$ of $G_0$ which fixes at least two vertices and properly contains $\langle A \rangle$.

Then $G = G_0 \ast_{\langle A \rangle} \langle A, gB \rangle \cong G_0 \ast_{C_m} D_m$ and $G_0 = \text{Stab}_G(y)$.

**Proof.** Let $G_1 = G_0$, $G_2 = \langle A, gB \rangle$ and $J = \langle A \rangle$. Recall that $\pi_B : T_q \rightarrow \text{Ax}(B)$ denotes the geodesic projection map, and that $b_-$ and $b_+$ denote the ends of $\text{Ax}(B)$, with $B$ translating from $b_-$ towards $b_+$. Consider the sets $X_1 = \pi_B^{-1}([b_-, x])$ and $X_2 = \pi_B^{-1}([y, b_+])$, which are invariant under $J$. Observe that $G_2$ is dihedral of order $2m$ since $gB(A)(gB)^{-1} = (A^{-1}B^{-1}gB = A^{-1}$. In particular, each element of $G_2 \setminus J$ can be written in the form $A^igB$. It can be shown inductively that $gB$ is a reflection in $\text{Ax}(B)$ about the midpoint of $[x, y]$, so it follows that $g_2(X_2) \subseteq X_1$ for every $g_2 \in G_2 \setminus J$; see Figure 5. Moreover, $y \in X_2$ is fixed by $G_1$, so it is not the image of any point of $X_1$ under $G_1$.

![Figure 5. Combination theorem sets for Lemma 3.12](image)

Now suppose for a contradiction that there exists $g_1 \in G_1 \setminus J$ and a vertex $z \in X_1$ such that $g_1z \notin X_2$. It follows that $\pi_B(g_1z) \in (b_-, y)$. Since $y$ is fixed by $G_1$, the non-trivial path
\[ \pi_B(\langle g_1 z, y \rangle \cap [x, y]) \text{ is fixed by } g_1. \] Thus \( H = \langle A, g_1 \rangle \) is a proper subgroup of \( G_0 \) that fixes at least two vertices and properly contains \( \langle A \rangle \), which contradicts our assumption. This shows that \( g_1(X_1) \subseteq X_2 \) for every \( g_1 \in G_1 \bmod J \). In the terminology of \([20\text{ VII}]\), \((X_1, X_2)\) is therefore a proper interactive pair for the groups \( G_1, G_2 \) and \( J \), and \( G = G_1 *_{J} G_2 \cong G_0 *_{C_m} D_m \) by Theorem A.10 of \([20\text{ VII}]\). Applying Lemma D.11 of \([20\text{ VII}]\) to \( y \) also shows that \( G_0 = \text{Stab}_G(y). \) \[ \square \]

4. Proof of Theorems A and B

We now prove our main results, starting with the proof of Theorem B.

**Proof of Theorem B.** By Remark \([22]\), there exists a generating pair \((A, B)\) for \( G \) such that one of cases (1) – (6) of Theorem C holds.

**Case (1):** \( G = \text{Stab}_G(y) \) for some vertex \( y \) of \( T_q \). By Lemma \([31]\) \( G \) is discrete if and only if it is finite, so this corresponds to Theorem C(a).

**Case (2):** \( G \) is discrete by Lemma \([31]\) so this corresponds to Theorem C(b).

**Case (3):** If \( G \) is discrete, then \( A \) and \( B \) have finite order by Lemma \([31]\). Conversely, if \( A \) and \( B \) have finite order, then Corollary \([35]\) shows that subcase (ii) does not occur and hence \( G \) is discrete by Lemma \([31]\). This corresponds to Theorem C(c).

**Case (4):** We may first assume that no element of the form \( A^iB \) is elliptic, as otherwise replacing \( B \) by \( A^iB \) gives a generating pair for \( G \) which lies in cases (1) or (3). Since \( \text{Ax}(B) \) and \( \text{Ax}(A^iB^{-1}) = A^i \cdot \text{Ax}(B) \) do not intersect with opposite orientations for any integer \( i \), it follows from \([22]\) Proposition 1.7 that \( B \) is hyperbolic of minimal translation length among all elements of the form \( A^iB \).

If \( G \) is discrete, then \( A \) has finite order by Lemma \([31]\). Conversely, if \( A \) has finite order, then Corollary \([35]\) shows that subcase (iii) does not occur and hence \( G \) is discrete by Lemma \([31]\). This case corresponds to Theorem C(d).

**Case (5):** It follows from Lemma \([31]\) Proposition \([34]\) and Lemma \([35]\) that \( G \) is discrete if and only if \( A \) has finite order, and \( A \) and \( B \) commute. This corresponds to Theorem C(e).

**Case (6):** First observe that \( G \) cannot be discrete in subcase (ii). Indeed, Lemma \([31]\) implies that \( G_0 \) (which fixes \( y \)) is infinite, so \( G \) is not discrete by Lemma \([31]\). Thus we may assume that one of subcases (i) and (iii) occurs. It follows from Lemma \([31]\) and Lemmas \([310, 312]\) that \( G \) is discrete if and only if \( G_0 = \langle A, BAB^{-1} \rangle \) is finite and the path \( P = \text{Fix}(A) \cap \text{Ax}(B) \) has length \( \Delta = l(B) \). Moreover, by Lemma \([37]\) subcases (i) and (iii) respectively correspond to cases (f) and (g) of Theorem B. \[ \square \]

As a consequence, we obtain the following more detailed version of Theorem A.

**Theorem A’.** Let \( G \) be a discrete two-generator subgroup of \( \text{PSL}_2(K) \), where \( K \) is a non-archimedean local field with finite residue field \( F \) of characteristic \( p \). If \( K = \mathbb{Q}_p \), or \( G \) contains no elements of order \( p \), then one of the following holds:

(a) \( G \) is one of the following finite groups:
   - \( C_n \), where either \( q \equiv \pm 1 \mod 2n \), or \( q \) is even and \( q \equiv \pm 1 \mod n \), or \( K = \mathbb{Q}_p \) and \( n = p \in \{2, 3\} \);
   - \( D_n \), where either \( q \equiv \pm 1 \mod 2n \), or \( K = \mathbb{Q}_2 \) and \( n = 3 \);
   - \( A_4 \), where \( p > 3 \) or \( K = \mathbb{Q}_3 \);
   - \( S_4 \), where \( q \equiv \pm 1 \mod 8 \);
   - \( A_5 \), where \( q \equiv \pm 1 \mod 10 \).

(b) \( G \) is discrete and free of rank two;

(c) \( G \) is one of \( C_n * C_m \), where for each \( t \in \{n, m\} \) either \( q \equiv \pm 1 \mod 2t \), or \( q \) is even and \( q \equiv \pm 1 \mod t \), or \( K = \mathbb{Q}_p \) and \( t = p \in \{2, 3\} \);

(d) \( G \) is one of \( C_n * \mathbb{Z} \), where either \( q \equiv \pm 1 \mod 2n \), or \( q \) is even and \( q \equiv \pm 1 \mod n \), or \( K = \mathbb{Q}_p \) and \( n = p \in \{2, 3\} \).

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(e) \( G \cong \mathbb{Z} \) or \( G \cong C_n \times \mathbb{Z} \), where either \( q \equiv 1 \mod 2n \), or \( q \) is even and \( q \equiv 1 \mod n \);

(f) \( G \) is an HNN extension of one of the following groups:

- \( D_{2n+1}, \) where \( q \equiv 1 \mod 4 \) and \( q \equiv \pm 1 \mod (4n+2) \);
- \( A_4 \), where \( q \equiv 1 \mod 6 \).

(g) \( G \) is isomorphic to one of the following groups:

- \( D_{2n+1} \ast_2 C_2 D_2, \) where \( q \equiv 1 \mod 4 \) and \( q \equiv \pm 1 \mod (4n+2) \);
- \( A_4 \ast_3 C_3 D_3, \) where \( q \equiv 1 \mod 6 \);
- \( S_4 \ast_4 C_4 D_4, \) where \( q \equiv 1 \mod 8 \);
- \( A_5 \ast_5 C_5 D_3, \) where \( q \equiv 1, 19 \mod 30 \);
- \( A_5 \ast_5 C_5 D_5, \) where \( q \equiv 1 \mod 10 \).

Moreover, each of these possibilities can occur.

**Proof.** If \( G \) is discrete, then one of cases (a) – (g) of Theorem 3 holds. In cases (a) – (e), the correspondingly labelled case of Theorem \( A' \) immediately follows by Proposition \ref{prop:3.3} and Theorem \( C \). In cases (f) and (g), the corresponding cases of Theorem \( A' \) follow from Theorem \( C \) and Lemmas \ref{lem:3.10}–\ref{lem:3.12}. The fact that each possibility described in Theorem \( A' \) can occur follows from the examples we exhibit in the next section. \( \Box \)

5. **Examples for Theorem \( A' \)**

In this section, we give explicit examples of the groups listed in Theorem \( A' \). A pair of representatives in \( \text{SL}_2(\mathbb{Q}_p) \) which generate a discrete and free subgroup of \( \text{PSL}_2(\mathbb{Q}_p) \) can be found in \cite{8} and, by replacing the role of \( p \) by the uniformiser \( \pi \) of \( K \), this gives examples for case (b) of Theorem \( A' \). Moreover, in considering the other cases below, we will demonstrate pairs of elements generating finite subgroups of \( \text{PSL}_2(K) \), which is case (a) of Theorem \( A' \). Hence we only consider cases (c) – (g).

**Case (c):** Let \( A, B \in \text{PSL}_2(K) \) be respectively represented in \( \text{SL}_2(K) \) by

\[
\begin{bmatrix}
0 & -1 \\
1 & t
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & -\pi^{-1} \\
\pi & s
\end{bmatrix},
\]

where \( s, t \in K \) are chosen so that \( A \) has order \( n \) and \( B \) has order \( m \) (where \( n, m \) depend on \( q \) as in Theorem \( A' \)). By Proposition \ref{prop:3.2}, \( s, t \in \mathcal{O}_K \) and it follows that \( AB \) is hyperbolic. Hence \( \text{Fix}(A) \cap \text{Fix}(B) = \emptyset \) by \cite{22} Proposition 1.8. Theorem \( C \) (3) and Corollary \ref{cor:3.5} show that \( G = \langle A, B \rangle \cong C_n \ast C_m \) and \( \text{Stab}_G(x) \cong C_n \) for every \( x \in \text{Fix}(A) \). Hence \( G \) is discrete by Lemma \ref{lem:3.1}.

**Case (d):** Let \( A, B \in \text{PSL}_2(K) \) be respectively represented in \( \text{SL}_2(K) \) by

\[
\begin{bmatrix}
0 & \pi^2 \\
-\pi^{-2} & t
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\pi^2 & \pi - 1 \\
1 & \pi^{-1}
\end{bmatrix},
\]

where \( t \in \mathcal{O}_K \) is chosen so that \( A \) has order \( n \) (which again depends on \( q \) as in Theorem \( A' \)). By Proposition \ref{prop:3.2}, \( l(B) = 2 \) and \( l(AB) = 4 > l(B) \). It follows from \cite{22} Proposition 1.7 that \( \text{Fix}(A) \cap \text{Ax}(B) = \emptyset \). Theorem \( C \) (4) and Corollary \ref{cor:3.5} show that \( G = \langle A, B \rangle \cong C_n \ast \mathbb{Z} \) and \( \text{Stab}_G(x) \cong C_n \) for every \( x \in \text{Fix}(A) \). Hence \( G \) is discrete by Lemma \ref{lem:3.1}.

**Case (e):** Let \( A, B \in \text{PSL}_2(K) \) be respectively represented in \( \text{SL}_2(K) \) by

\[
\begin{bmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\pi & 0 \\
0 & \pi^{-1}
\end{bmatrix},
\]

where \( \lambda \in K \). By Proposition \ref{prop:3.2}, \( l(B) = 2 \). If \( \lambda = \pm 1 \), then \( A \) is trivial and \( G = \langle A, B \rangle \cong \mathbb{Z} \), which is discrete by Lemma \ref{lem:3.1}. On the other hand, if \( q \equiv 1 \mod 2n \) (respectively \( q \) is even and \( q \equiv 1 \mod n \)) then we may choose \( \lambda \in K \) to be a \( 2n \)-th (respectively \( n \)-th) root of unity. Thus \( G = \langle A, B \rangle \cong C_n \times \mathbb{Z} \), which is discrete as the direct product of two discrete groups.
Cases (f) and (g): For the remaining cases, let $A, B \in \text{PSL}_2(K)$ be respectively represented in $\text{SL}_2(K)$ by

$$
\begin{bmatrix}
  a & 1 \\
  a(t-a) - 1 & t - a
\end{bmatrix}
\text{ and }
\begin{bmatrix}
  \pi & 0 \\
  0 & \pi^{-1}
\end{bmatrix},
$$

where $a, t \in K$. By slight abuse of notation, we will use these representatives to define the traces of elements of $G$.

By Proposition 3.2 $B$ is hyperbolic of translation length 2. Also note that

$$
(5.1) \quad s = \text{tr}(ABAB^{-1}) = (a^2 - at + 1)(2 - \pi^2 - \pi^{-2}) + t^2 - 2.
$$

We will choose certain values of $s, t \in O_K$ so that $A$ and $ABAB^{-1}$ have specific finite orders, as listed in Table 1. We will then show that:

- Such an element $A \in \text{PSL}_2(K)$ exists;
- The path $P = \text{Fix}(A) \cap \text{Ax}(B)$ has length $\Delta = l(B)$;
- The group $G_0 = \langle A, B \rangle$ is one of the finite groups listed in Lemma 3.10;
- The group $G = \langle A, B \rangle$ is discrete and is one of the groups described by cases (f) and (g) of Theorem 3.

Since we require that $A$ has finite order and $\text{Fix}(A)$ is a bi-infinite ray, we may assume that $t = \zeta + \zeta^{-1}$ for an appropriate root of unity $\zeta \in K$. Multiplying both sides of Equation (5.1) by $\pi^2$ and then reducing modulo $\pi$ yields the equation $a^2 - at + 1 = 0$. This equation has discriminant $(\zeta - \zeta^{-1})^2$, which is a non-zero square in $K$, so there are two distinct solutions in the residue field $O_K/\pi O_K \cong \mathbb{F}_q$. By Hensel’s Lemma, there exists $a \in K$ satisfying Equation (5.1), and hence such an element $A \in \text{PSL}_2(K)$ exists.

Now observe that

$$
\text{tr}(AB^2A^{-2}) = (s - t^2 + 2)(\pi^2 + \pi^{-2} + 2) + t^2 - 2.
$$

In each of the cases specified in Table 1 $s - t^2 + 2$ is non-zero modulo $\pi$ (since $q$ is odd) and hence $v(s - t^2 + 2) = 0$. It follows from Proposition 3.2 that $l(AB^2A^{-2}) = 4$. Proposition 1.8 of [22] shows that the distance between $\text{Fix}(A)$ and $B^2$. $\text{Fix}(A)$ is 2 and hence $\Delta = 2 = l(B)$.

| $G_0$ | Case | $t$ | ord($A$) | $s$ | ord($ABAB^{-1}$) | $s - t^2 + 2$ |
|-------|------|-----|-----------|----|------------------|----------------|
| $D_m$ | (f)  | 0   | 2         | $\omega_{2m} + \omega_{2m}^{-1}$ | $m$            | $\omega_{2m} + \omega_{2m}^{-1} + 2$ |
| $D_m$ | (g)  | 0   | 2         | $\omega_{m} + \omega_{m}^{-1}$  | $m$            | $\omega_{m} + \omega_{m}^{-1} + 2$ |
| $A_4$ | (f)  | 1   | 3         | 1  | 3                | 2              |
| $A_4$ | (g)  | 1   | 3         | 0  | 2                | 1              |
| $S_4$ | (g)  | $\sqrt{2}$ | 4   | 1  | 3                | 1              |
| $A_5$ | (g)  | $1 + \sqrt{5}$ | 5   | 1  | 3                | $\frac{3 + \sqrt{5}}{2}$ |
| $A_5$ | (g)  | $1 + \sqrt{5}$ | 5   | 1  | 3                | $\frac{3 + \sqrt{5}}{2}$ |

Table 1. Values of $s, t \in K$ for each group $G_0$

For $G_0$ to be dihedral, we require that $A$ has order 2, $ABAB^{-1}$ has order $m$ (for some odd positive integer $m$), $q \equiv 1 \mod 4$ and $q \equiv \pm 1 \mod 2m$ (in particular, $G_0$ contains no elements of order $p$). Hence we set $t = 0$ and $s = \omega + \omega^{-1}$, where $\omega \in K$ is either a $m$-th root of unity $\omega_{m}$ or a $2m$-th root of unity $\omega_{2m}$ (corresponding respectively to whether $ABAB^{-1}$ is represented in $\text{SL}_2(K)$ by a matrix of order $m$ or $2m$).

By von Dyck’s Theorem [11], $G_0 = \langle A, B \rangle$ is a quotient of $\langle x, y \mid x^2, y^2, (xy)^m \rangle \cong D_m$, but since $G_0$ contains an element of order $m$ it must be isomorphic to $D_m$. 
Now every involution in $G_0 \cong D_m$ is of the form $g = (ABAB^{-1})^iA$ for some $i \in \{0, \ldots, m-1\}$. Note that $\text{tr}(AB) = -\text{tr}(AB^{-1})$ and hence
\[
\text{tr}(gB) = \text{tr}((ABAB^{-1})^i A) \\
= \text{tr}((ABAB^{-1})^i A \text{tr}(A) - \text{tr}(ABAB^{-1} B^{-1} A^{-1}) \\
= \left(\omega^i + \omega^{-i}\right) \text{tr}(A) - \text{tr}(ABAB^{-1})^i \\
= \text{tr}(AB)[\omega^i + \omega^{-i} + \omega^{i-1} + \omega^{(i-1)}] + \text{tr}(BA^{-1} (ABAB^{-1})^{i-1}).
\]
By induction, we obtain
\[
\text{tr}(gB) = \text{tr}(AB)[\omega^i + \omega^{-i} + \omega^{i-1} + \omega^{(i-1)}] + \cdots + \omega + \omega^{-1} + 1].
\]
Since $AB$ is hyperbolic, $gB$ is either hyperbolic or an involution. Moreover, $gB$ is an involution if and only if
\[
0 = (\omega^i + \omega^{i-1} + \cdots + \omega + 1) + \omega^{-i}(\omega^{i-1} + \cdots + \omega + 1)
= \frac{\omega^{i+1} - 1}{\omega - 1} + \frac{\omega^{-i} - 1}{\omega - 1}
= \frac{\omega^{i+1} - \omega^{-i}}{\omega - 1}.
\]
If $\omega = \omega_m$, then $gB$ is an involution when $i = \frac{m-1}{2}$. A similar argument then shows that $gBA$ is also an involution. Hence $G \cong D_m \ast C_2 \ast D_2$ by Lemmas 3.10 and 3.12. On the other hand, if $\omega = \omega_{2m}$, then (since every element of $G_0$ is of the form $(ABAB^{-1})^iA^j$ for some $i \in \{0, \ldots, m-1\}$ and $j \in \{0,1\}$) it can be shown in a similar way that $gB$ is hyperbolic for every $g \in G_0$. This implies that Theorem $C(6)(iii)$ does not hold: if there exists $g \in G_0$ such that $gBy = B^{-1}y$, then $(gB)^2y = y$ and hence $gB$ is elliptic. Lemma 3.10 shows that Theorem $C(6)(ii)$ also does not hold. Hence Theorem $C(6)(i)$ holds and $G$ is an HNN extension of $D_m$. In both cases, $\text{Stab}_G(y) \cong D_m$ for some vertex $y$ of $T_q$, so $G$ is discrete by Lemma 3.1.

For $G_0$ to be isomorphic to $A_4$, we require that $A$ has order 3 and $q \equiv 1 \mod 6$ (in particular, $G_0$ contains no elements of order $p$). Hence we may choose $t = 1$. Let $s \in \{0,1\}$, so that $ABAB^{-1}$ has order 2 or 3. By Equation 6.1, $\text{tr}(ABAB^{-1}B^{-1}) = 1 - s$ and it follows from von Dyck’s Theorem that $(A, BAB^{-1})$ is a quotient of either $\langle x, y \mid x^3, y^2, (xy)^2 \rangle$ or $\langle x, y \mid x^3, y^2, (xy^{-1})^2 \rangle$, both of which are isomorphic to $A_4$. The only non-trivial normal subgroup of $A_4$ is $D_2$, and hence $(A, BAB^{-1}) \cong A_4$ by Lemma 3.10.

If $s = 0$, then let $g = ABAB^{-1}B^{-1}A \in G_0$ and observe that
\[
\text{tr}(g) = \text{tr}(ABAB^{-1}B^{-1}) \text{tr}(A) - \text{tr}(ABAB^{-1}B^{-1}A^{-1}) \\
= \text{tr}(A) - \text{tr}(A^{-1}) = 0, \quad \text{and}
\]
\[
\text{tr}(gB) = \text{tr}(ABAB^{-1}B^{-1}) \text{tr}(AB) - \text{tr}(ABAB^{-1}B^{-2}A^{-1}) \\
= \text{tr}(AB) - \text{tr}(A^{-1}B^{-1}) = 0, \quad \text{and}
\]
\[
\text{tr}(gBA) = \text{tr}(ABAB^{-1}B^{-1}) \text{tr}(ABA) - \text{tr}(ABAB^{-1}A^{-1}B^{-1}A^{-1}) \\
= \text{tr}(ABA) - \text{tr}(A^{-1}B^{-1}A^{-1}) = 0.
\]

Lemmas 3.10 and 3.12 hence show that $G \cong A_4 \ast C_2 \ast D_3$. On the other hand, if $s = 1$, then (since every element of $G_0$ is of the form $A^iBA^jB^{-1}A^k$ for some $i, j, k \in \{0,1,2\}$) it can be verified by similar trace computations that $gB$ is hyperbolic for every $g \in G_0$. By the same argument as in the dihedral case, Theorem $C(6)(iii)$ does not hold. Lemma 3.10 shows that Theorem $C(6)(ii)$ also does not hold, so $G$ is an HNN extension of $A_4$ by Theorem $C(6)(i)$. In both cases, $\text{Stab}_G(y) \cong A_4$ for some vertex $y$ of $T_q$, so $G$ is discrete by Lemma 3.1.

For $G_0$ to be isomorphic to $S_4$, we require that $A$ has order 4 and $q \equiv 1 \mod 8$. Hence we may choose $t = \sqrt{2}$. Set $s = 1$, so that $ABAB^{-1}$ has order 3. Note that $A^2BAB^{-1}$ is an involution,
In the former case, we may choose $A \mod 10$, or $\mod 5$. Finally, we consider the case where $G$ contains no elements of order $4$, so we deduce that $G_0 = \langle A, B \rangle$. As in the proof of Lemma 3.10, the element $g = ABA^{-1}B^{-1}A \in G_0$ is such that $g, gB$ and $gBA$ are involutions. Moreover, the only proper subgroup $H$ of $S_4$ which properly contains $C_4$ is $D_8$. Indeed, suppose for a contradiction that $H = \langle A, h \rangle \cong D_8$ for some involution $h \in G_0$. Thus $hAh^{-1} = A^{-1}$ and $h \cdot \text{Fix}(A) = \text{Fix}(A)$. If this group $H$ fixes at least two vertices, then this implies that $\text{Fix}(A) \subseteq \text{Fix}(h)$. Proposition 3.8 shows that $\text{Fix}(A) = \text{Fix}(h)$, and this contradicts Lemma 3.8 since $q$ is odd and $A$ has order 4. Thus Lemma 3.12 shows that $G \cong S_4 \ast C_4$ and $\text{Stab}_G(y) \cong S_4$ for some vertex $y$ of $T_q$, so $G$ is discrete by Lemma 3.11.

Finally, we consider the case where $G_0 \cong A_5$. We require that either $A$ has order 5 and $q \equiv 1 \mod 10$, or $A$ has order 3 and $q \equiv 1, 19 \mod 30$.

In the former case, we may choose $t = \frac{1+\sqrt{5}}{2}$. Set $s = 1$, so that $ABAB^{-1}$ has order 3. As in the previous case, observe that $\text{tr}(A^2BAB^{-1}) = 0$ and so von Dyck’s Theorem shows that $\langle A, B \rangle$ is a quotient of $\langle x, y \mid x^4, y^4, (xy)^5, (x^2y)^2 \rangle \cong A_5$. Since $A_5$ is simple, this gives $\langle A, B \rangle \cong A_5$. As in the proof of Lemma 3.10, the element $g = (ABA^{-1}B^{-1})^2A \in G_0$ is such that $g, gB$ and $gBA$ are involutions. Moreover, the only proper subgroup of $A_5$ which properly contains $C_5$ is $D_{10}$, so a similar argument to the $S_4$ case shows that $G \cong A_5 \ast C_5 \ast D_3$.

In the latter case, we may choose $t = 1$, so that $A$ has order 3. Set $s = \frac{1+\sqrt{5}}{2}$, so that $ABAB^{-1}$ has order 5. Observe that
\[
\text{tr}(A^2BAB^{-1}ABAB^{-1}) = \text{tr}(A)\text{tr}((ABAB^{-1})^2) - \text{tr}(BAB^{-1}ABAB^{-1}) = (s^2 - 2) - \text{tr}(ABAB^{-1}ABAB^{-1}) = (s^2 - 2) - \text{tr}(A)\text{tr}(ABAB^{-1}) + \text{tr}(A^{-1}B^{-1}ABA) = s^2 - s - 1 = 0.
\]

Since $A_5$ is simple, von Dyck’s Theorem shows that $\langle A, B \rangle \cong (x, y \mid x^4, y^4, (xy)^5, (x^2y)^2) \cong A_5$. Since $q$ is not divisible by 2, 3 or 5, $G_0$ contains no elements of order $p$ and thus Lemmas 3.10 and 3.12 show that $G \cong A_5 \ast C_5 \ast D_3$. In both cases, $\text{Stab}_G(y) \cong A_5$ for some vertex $y$ of $T_q$, so $G$ is discrete by Lemma 3.11.

6. ALGORITHMS FOR DISCREteness AND DENSITY

We now present an algorithm which takes as input two elements $A, B$ of $\text{PSL}_2(K)$ (where either $K = \mathbb{Q}_p$, or $G = \langle A, B \rangle$ contains no elements of order $p$) and decides whether or not the subgroup $G = \langle A, B \rangle$ is discrete. If $G$ is discrete, then the algorithm returns the isomorphism type of $G$ according to Theorem A. The algorithm relies on computing translation lengths on $T_q$ using Proposition 3.2.

**Algorithm 6.1.** Input: Two elements $A, B \in \text{PSL}_2(K)$, where either $K = \mathbb{Q}_p$ or the subgroup $G = \langle A, B \rangle$ of $\text{PSL}_2(K)$ contains no elements of order $p$.

Output: true: case (a) if $G = \langle A, B \rangle$ is discrete and of the type described in case (a) of Theorem A, and false otherwise.

1. **If** $G = \langle A, B \rangle$ is finite, **then return** true: case (a).
2. **Set** $X = A$ and $Y = B$.
3. **If** $l(X) > l(Y)$, **then swap** $X$ and $Y$.
4. **If** $l(X) > 0$, **then compute** $m = \min\{l(XY), l(X^{-1}Y)\}$. 

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(i) If $m \leq l(Y) - l(X)$ then replace $Y$ by an element from $\{XY, X^{-1}Y\}$ which has translation length $m$ and return to (3).

(ii) If $m > l(Y) - l(X)$, then return true:case (b).

(5) If $X$ has infinite order, then return false.

(6) Let $n$ be the order of $X$. If $l(X^iY) < l(Y)$ for some $i \in \{1, \ldots, n-1\}$, then replace $Y$ by $X^iY$.

(7) If $l(Y) = 0$, then return true:case (c) if $Y$ has finite order and $l(XY) > 0$, and otherwise return false.

(8) If $l([X,Y]) > 0$, then return true:case (d).

(9) If $[X,Y]$ has infinite order, then return false.

(10) If $[X,Y]$ is trivial, then return true:case (e).

(11) If $l([X,Y^2]) = 0$, then return false.

(12) Set $G_0 = \langle X, YXY^{-1} \rangle$. If $G_0$ is infinite, then return false.

(13) If there is no element $g \in G_0$ such that $g$ and $gY$ both have order 2, then return true:case (f).

(14) Return true:case (g).

**Remark 6.2.** Algorithm 6.1 includes several steps which involve determining the order of a subgroup or an element of $\text{PSL}_2(K)$. By Proposition 3.3 there is a short list of finite subgroups of $\text{PSL}_2(K)$ and hence a short list of possibilities to check at each of these steps.

**Remark 6.3.** It is straightforward to determine the precise isomorphism class of a group described by cases (a) – (g) of Theorem A, but we omit this step from the algorithm for brevity.

**Theorem 6.4.** Algorithm 6.1 terminates and produces the correct output.

**Proof.** The algorithm will always terminate, since the only recursive step is (4)(i) and this strictly reduces the integer $l(X) + l(Y)$; see also [8, Theorem 4.2]. It remains to prove that the algorithm is correct. If the algorithm returns:

- true:case (a), then $G$ is finite and hence discrete.
- true:case (b), then $G$ is discrete and free by [8, Corollary 3.6].
- true:case (c), then $X$ and $Y$ are elliptic of finite order. Since $l(XY) > 0$, their fixed point sets are disjoint by [22, Proposition 1.8]. Thus $G$ is discrete by Theorem H (c) and Theorem A (c) holds.
- true:case (d), then $X$ is elliptic of finite order and $Y$ is hyperbolic. Since $l([X,Y]) > 0$, the path $\text{Fix}(X) \cap \text{Ax}(Y)$ is either empty or of length shorter than $l(Y)$ by [22, Proposition 1.8]. Hence $G$ is discrete by Theorem H (d) and Theorem A (d) holds.
- true:case (e), then $X$ is elliptic of finite order (possibly trivial), $Y$ is hyperbolic, and $X$ and $Y$ commute. Hence $G$ is discrete by Theorem E (e) and Theorem A (e) holds.
- true:case (f), then $X$ is elliptic of finite order, $Y$ is hyperbolic, and $P = \text{Fix}(X) \cap \text{Ax}(Y)$ has length $l(B) \leq \Delta < 2l(B)$ by [22, Proposition 1.8] since $[X,Y]$ is elliptic and $[X,Y^2]$ is not. Also $G_0 = \langle A, BAB^{-1} \rangle$ is finite and hence $\Delta = l(Y)$ by Lemma 3.7. By Lemma E.7 $G_0$ does not contain a reflection in $\text{Ax}(Y)$ about the terminal vertex of $P$, so $G$ is discrete by Theorem E (f) and Theorem A (f) holds.
- true:case (g), then, by the same argument as above, $X$ is elliptic of finite order, $Y$ is hyperbolic, $G_0 = \langle A, BAB^{-1} \rangle$ is finite and $P = \text{Fix}(X) \cap \text{Ax}(Y)$ has length $l(Y)$. Moreover, Lemma E.7 shows that $G_0$ does contain a reflection in $\text{Ax}(Y)$ about the terminal vertex of $P$, hence $G$ is discrete by Theorem E (g) and Theorem A (g) holds.

On the other hand, if the algorithm returns false (at steps (5), (7), (9), (11) or (12)), then $G_0 = \langle X, YXY^{-1} \rangle$ is infinite (this follows from Lemma 3.10 in the case of step (11)). Since $G_0$ fixes the terminal vertex of $\text{Fix}(X) \cap \text{Ax}(Y)$, $G$ is not discrete by Lemma 3.1. □

**Remark 6.5.** Algorithm 6.1 has been implemented in Magma where the input is a pair of representative matrices in $\text{SL}_2(\mathbb{Q}_p)$; see [9]. It runs very efficiently (< 0.03s) for each of the examples discussed in Section 2. For randomly generated pairs of elements of $\text{SL}_2(\mathbb{Q}) \leq \text{SL}_2(\mathbb{Q}_p)$.
dense. We will use the following results, where $F$ is either $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$, and $X$ is accordingly the hyperbolic plane, the Riemann sphere, or the corresponding Bruhat–Tits tree.

Lemma 6.6. Let $G$ be a subgroup of $\text{SL}_2(F)$.

(i) If $G$ is not Zariski dense, then either $G$ fixes a point of $X$, or $G$ stabilises a set of one or two points on the boundary $\partial X$.

(ii) If $G$ fixes a point of $X$, then $G$ is not dense.

(iii) If $G$ stabilises a set of one or two points of $\partial X$, then $G$ is not Zariski dense.

Proof. (i): A Zariski closed proper subgroup of $\text{SL}_2(F)$ does not contain a non-abelian free subgroup. If $G$ is not Zariski dense, then it follows from [14, Section 3.1] that $G$ either fixes a point of $X$ or stabilises a set of one or two points of $\partial X$.

(ii): If $G$ fixes a point $x \in X$, then its closure in $\text{SL}_2(F)$ also fixes $x$. Since $\text{SL}_2(F)$ has no fixed points in $X$, this implies that $G$ is not dense.

(iii): Observe that if $G$ stabilises one or two points of $\partial X$, then the Zariski closure of $G$ is at most two-dimensional. Since $\text{SL}_2$ is 3-dimensional, $G$ is not Zariski dense. □

Retaining the notation from the previous lemma, we also obtain:

Lemma 6.7. If a subgroup $G$ of $\text{SL}_2(F)$ is Zariski dense, then either $G$ is discrete, $G$ is dense, or $G$ fixes a point of $X$.

Proof. Consider $\text{SL}_2(F)$ as a finite-dimensional Lie group over either $\mathbb{R}$ or $\mathbb{Q}_p$, and let $H$ denote the closure of $G$ in $\text{SL}_2(F)$. By Cartan’s closed subgroup theorem [4, Chapter III, §8 Theorem 2], $H$ is a Lie subgroup of $\text{SL}_2(F)$. Since $G$ is Zariski dense, the Lie algebra $\mathfrak{h}$ of $H$ is invariant under the adjoint action of $\text{SL}_2(F)$. Hence $\mathfrak{h}$ is an ideal of the simple Lie algebra $\mathfrak{sl}_2$ corresponding to $\text{SL}_2(F)$, so either $\mathfrak{h} = 0$ or $\mathfrak{h} = \mathfrak{sl}_2$. By [4, Chapter III, §4 Theorem 3], either $H$ is discrete or $H$ is open. Suppose that $G$ is neither discrete nor dense, so that $H$ is a proper open subgroup of $\text{SL}_2(F)$. By a result of Tits [21, 3.6.2] $H$ is bounded and thus the Bruhat–Tits fixed point theorem [5, Proposition 3.2.4] shows that $G$ fixes a point of $X$. □

As suggested by Pierre-Emmanuel Caprace, we can apply Algorithm 6.1 to obtain an algorithm which decides if a two-generator subgroup of $\text{SL}_2(K)$ is dense, where $K$ is either real or a $p$-adic field. We start with the $p$-adic field case, in which every dense subgroup of $\text{SL}_2(K)$ has the structure of an amalgamated free product [20, Chapter II, Theorem 3].

Algorithm 6.8. Input: Two elements $A, B \in \text{SL}_2(K)$, where $K$ is a finite extension of $\mathbb{Q}_p$.

Output: true if $G = \langle A, B \rangle$ is dense in $\text{SL}_2(K)$, and false otherwise.

(1) If Algorithm 6.1 shows that the corresponding subgroup $\overline{G}$ of $\text{PSL}_2(K)$ is discrete, then return false. Otherwise let $X$ and $Y$ be generators of $G$ at the step where Algorithm 6.1 terminates.

(2) If $l(Y) = 0$ and $l(XY) = 0$, then return false.

(3) If $l(Y) = 0$ and $l(XY) > 0$, then return true.

(4) If $\text{tr}((X,Y)) = 2$, then return false.

(5) If $[XYX^{-1}, Y]$ is trivial, then return false.

(6) Return true.

Theorem 6.9. Algorithm 6.8 terminates and produces the correct output.
Proof. If Algorithm 6.8 returns $\text{false}$ at step (1), then $G$ is not dense because it is discrete. For the remainder of the proof, we may therefore assume that $G = \langle X, Y \rangle$ is not discrete and (since Algorithm 6.1 applied to $G$ returns $\text{false}$) that $X$ is elliptic.

Suppose first that $Y$ is elliptic. By [22] Proposition 1.8, $XY$ is elliptic if and only if $X$ fixes a vertex of the associated Bruhat-Tits tree $T_q$. Hence if Algorithm 6.8 returns $\text{false}$ at step (2), then $G$ is not dense by Lemma 6.6 (ii). If $XY$ is hyperbolic, then $X$ and $Y$ cannot have a common fixed point on the boundary $\partial T_q$, as otherwise $\text{Fix}(X) \cap \text{Fix}(Y) \neq \emptyset$. Similarly, $X$ cannot interchange any fixed points of $Y$ on $\partial T_q$ (or vice versa). By Lemmas 6.6 and 6.7 $G$ is dense if Algorithm 6.8 returns $\text{true}$ at step (3).

Hence we may assume that $Y$ is hyperbolic and therefore fixes two points of $\partial T_q$. In particular, $G$ does not fix a vertex of $T_q$. By identifying the boundary $\partial T_q$ with the projective line $\mathbb{P}^1(K)$ [26, p. 72], we may assume (after conjugation if necessary) that $Y$ is a diagonal matrix. Hence a standard trace computation shows that $G$ fixes a point of $\partial T_q$ if and only if $\text{tr}([X, Y]) = 2$; see [3, Theorem 4.3.5(i)]. Similarly, $X$ interchanges the two fixed points of $Y$ on $\partial T_q$ if and only if $\text{tr}([X, Y]) \neq 2$ and $[XYX^{-1}, Y]$ is trivial; see [3, Theorem 4.3.5(ii)]. Thus if Algorithm 6.8 returns $\text{false}$ at steps (4) or (5), then $G$ is not dense by Lemma 6.6 (iii).

Finally, if Algorithm 6.8 reaches step (6), then $G$ is not discrete, it does not fix a vertex of $T_q$ and it does not stabilise a set of one or two points of $\partial T_q$. It follows from Lemma 6.6 (i) that $G$ is Zariski dense, and hence Lemma 6.7 shows that $G$ is dense. \hfill $\square$

Remark 6.10. Algorithm 6.8 has been implemented in MAGMA [9], and it will terminate under the same conditions discussed above for Algorithm 6.7. For randomly generated pairs of elements of $\text{SL}_2(\mathbb{Q}) \leq \text{SL}_2(\mathbb{Q}_p)$ (where each entry is a randomly generated numerator and denominator in $[-10^{10}, 10^{10}]$) the algorithm has an average runtime (across 1000 trials) of less than 0.004s for all primes $p < 17$.

Using similar techniques, we also obtain the following algorithm to decide whether a two-generator subgroup of $\text{SL}_2(\mathbb{R})$ is dense.

Algorithm 6.11. Input: Two elements $A, B \in \text{SL}_2(\mathbb{R})$.

Output: $\text{true}$ if $G = \langle A, B \rangle$ is dense in $\text{SL}_2(\mathbb{R})$, and $\text{false}$ otherwise.

1. If $\text{tr}([A, B]) = 2$, then return $\text{false}$.
2. If $[A, BAB^{-1}]$ or $[ABA^{-1}, B]$ is trivial, then return $\text{false}$.
3. If $\text{tr}([A, B]) < 2$ and Algorithm 1 of [17] shows that $G$ is discrete, then return $\text{false}$.
4. If $\text{tr}([A, B]) > 2$ and Algorithm 2 of [17] shows that $G$ is discrete, then return $\text{false}$.
5. Return $\text{true}$.

Theorem 6.12. Algorithm 6.11 terminates and produces the correct output.

Proof. Note that $\text{tr}([A, B]) = 2$ if and only if $A$ and $B$ have a common fixed point in their action as Möbius transformations of the Riemann sphere. Since the two fixed points (counted with multiplicity) of any such Möbius transformation either both lie on $\partial \mathbb{H}^2$ or are complex conjugates, it follows that $\text{tr}([A, B]) = 2$ if and only if $G$ fixes a point of $\mathbb{H}^2$ or $\partial \mathbb{H}^2$. Similarly, since $\text{SL}_2(\mathbb{R})$ preserves $\mathbb{H}^2$, $\text{tr}([A, B]) \neq 2$ and $[A, BAB^{-1}]$ is trivial if and only if $A$ is hyperbolic and $B$ interchanges the fixed points of $A$ on $\partial \mathbb{H}^2$ (and a similar argument holds for the case where $\text{tr}([A, B]) \neq 2$ and $[ABA^{-1}, B]$ is trivial). It follows from Lemma 6.6 (ii) and (iii) that $G$ is not dense if Algorithm 6.11 returns $\text{false}$ at steps (1) or (2).

We may now assume that $G$ does not fix a point of $\mathbb{H}^2$ and does not stabilise a set of one or two points of $\partial \mathbb{H}^2$. In particular, $G$ is Zariski dense by Lemma 6.6 (i), and $G$ is a non-elementary subgroup of $\text{SL}_2(\mathbb{R})$. Moreover, Algorithms 1 or 2 of [17] can be used to determine whether or not $G$ is discrete. Hence $G$ is not dense if Algorithm 6.11 returns $\text{false}$ at steps (3) or (4), and Lemma 6.7 shows that $G$ is dense if Algorithm 6.11 reaches step (5). \hfill $\square$
Remark 6.13. We have also implemented Algorithm 6.7 in MAGMA; see [9]. As for Algorithms 1 and 2 of [17], this implementation assumes that $G = \langle A, B \rangle$ is a subgroup of $\text{SL}_2(F)$, where $F$ is a real algebraic number field. For randomly generated pairs of elements of $\text{SL}_2(Q) \leq \text{SL}_2(\mathbb{R})$ (where each entry is a randomly generated numerator and denominator in $[-10^{10}, 10^{10}]$) the algorithm has an average runtime (across 1000 trials) of less than 0.002s.

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