A Two-Frequency-Two-Coupling model of coupled oscillators

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We considered the phase coherence dynamics in a Two-Frequency and Two-Coupling (TFTC) model of coupled oscillators, where coupling strength and natural oscillator frequencies for individual oscillators may assume one of two values (positive/negative). The bimodal distributions for the coupling strengths and frequencies are either correlated or uncorrelated. To study how correlation affects phase coherence, we analyzed the TFTC model by means of numerical simulation and exact dimensional reduction methods allowing to study the collective dynamics in terms of local order parameters. The competition resulting from distributed coupling strengths and natural frequencies produces nontrivial dynamic states. For correlated disorder in frequencies and coupling strengths, we found that the entire oscillator population splits into two subpopulations, both phase-locked (Lock-Lock), or one phase-locked and the other drifting (Lock-Drift), where the mean-fields of the subpopulations maintain a constant non-zero phase difference. For uncorrelated disorder, we found that the oscillator population may split into four phase-locked subpopulations, forming phase-locked pairs, which are either mutually frequency-locked (Stable Lock-Lock-Lock-Lock) or drifting (Breathing Lock-Lock-Lock-Lock), thus resulting in a periodic motion of the global synchronization level. Finally, we found for both types of disorder that a state of Incoherence exists; however, for correlated coupling strengths and frequencies, Incoherence is always unstable, whereas it is only neutrally stable for the uncorrelated case. Numerical simulations performed on the model show good agreement with the analytic predictions. The simplicity of the model promises that real-world systems can be found which display the dynamics induced by correlated/uncorrelated disorder.

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The synchronization of oscillators is a ubiquitous phenomenon that manifests itself in a vast range of settings in nature and technology, such as the beating of the heart, circadian clocks in the brain, metabolic oscillations in yeast cells, and life cycles of phytoplankton, pedestrians on a bridge locking their gait, metronomes on a swing, arrays of Josephson junctions, chemical oscillators, electric power grid networks and others. Studies have addressed collective dynamics emerging in coupled oscillator networks with properties giving rise to the formation subpopulation structures induced by either heterogeneous frequencies or interactions such as coupling strengths and/or phase-lags, with bimodal character. Recently, Hong et al. considered the emergence of collective states including traveling waves in a system with heterogeneous natural frequencies and positive/negative coupling strengths that were correlated with the given natural frequency. Here, we simplify this model by considering the case where both natural frequencies and coupling strengths may assume either positive or negative value, which may be correlated or uncorrelated. Thereby, we compared side-by-side the impact of correlated and uncorrelated disorder and study the collective dynamics based on numerical simulations and dimensional reduction methods. The resulting collective dynamics reflects the subpopulation structure imprinted by the natural frequencies/coupling strengths in a nontrivial way, depending on the amount of asymmetry of the disorder. On the one hand, the correlated model exhibits dynamic states where one oscillators belonging to one subpopulation are locked, while oscillators in the other subpopulation are adrift, or both subpopulations are phase-locked with constant phase difference; both states display traveling wave motion. On the other hand, for uncorrelated disorder we found a state of incoherent oscillations, and a state where all subpopulations are phase-locked, either with a drifting or constant phase difference; traveling wave motion is however absent. Our findings corroborate that traveling wave motion results from asymmetry in natural frequencies and coupling strengths.
with correlated disorder, rather than from disorder without correlation or disorder with non-zero variance.

I. INTRODUCTION

Synchronization phenomena occur in a large variety of systems in nature and technology, and a wide range of studies used both mathematical and physical models to uncover and understand the dynamics of synchronization. To explore the mechanisms behind collective synchronization, the paradigmatic Kuramoto model is a useful tool, and its variants capture many features of biological and physical systems in the real world, including pedestrians walking on a bridge, Josephson junctions, neural systems, metronomes, lasers and opto-mechanical systems. For example, the collective synchronization in Kuramoto model has attracted physicists’ attention because its governing equations can be related to the XY model for the spin magnetics, i.e., the Kuramoto model corresponds to an overdamped version of the Hamiltonian dynamics of the XY model in physics. The model has also attracted great theoretical attention because of its analytical tractability via the exact low dimensional description of the microscopic dynamics in terms of collective mean-field variables, for a review see.

The natural frequencies of the oscillators in the original Kuramoto model are randomly drawn from a unimodal distribution function such as the Gaussian one, while the coupling strength between all oscillators is the same value, and consequently, frequencies and coupling strengths are uncorrelated. The natural frequencies of the oscillators play two roles. The first is that the frequencies constitute driving forces in the system. The second is that they play the role of “disorder” due to their randomly distributed nature. This disorder in the oscillator frequencies tends to break synchrony and forces the oscillator phases to run away from each other; conversely, (positive) coupling strength is an antagonist to this disorder and enables the oscillators to entrain their phases. In Ref., one of the authors considered a system where the natural oscillator frequencies and coupling strengths are drawn from random distributions with finite variance. In particular, the authors considered the case where the distributions of the two parameters are symmetrically/asymmetrically correlated with each other and found that the correlation may induce interesting states including traveling waves. In the present study, we consider a minimal model where both natural frequencies and coupling strengths are centered around two distinct values and investigated how correlated/uncorrelated disorder in natural frequencies and coupling strengths affected the collective dynamics of the system. Coupling strengths hereby take on both positive and negative values. The motivation for this article is to study the effects of correlated/uncorrelated disorder by simplifying a previous model such that these effects are analytically tractable. To achieve this, we introduce a “Two-Frequencies-Two-Coupling (TFTC) model” where frequencies and coupling strengths may assume either of two values. In particular, we would like to address the following questions: Previous studies reported intriguing dynamic states such as traveling waves induced by correlated disorder; can we observe traveling waves despite the simplifications in this model, and what other dynamic states may appear? This paper is structured as follows. Sec. II defines the TFTC model of coupled oscillators, Sec. III gives a dimensional reduction to this system in terms of macroscopic collective dynamics of introduced by and . In Sec. IV we carry out a stability and bifurcation analysis for the dynamics resulting from the “correlated model”, where the natural oscillator frequency and coupling strength are correlated with each other, using numerical simulation and the dimensionally reduced equations. In Sec. V we carry out a similar analysis for the “uncorrelated model”, where natural oscillator frequencies and coupling strengths are randomly chosen, using numerical simulation, a self-consistency argument and the dimensionally reduced equations. Finally, Sec. VI provides a summary and discussion of our results.

II. MODEL

We consider a minimal model of coupled oscillators in which oscillators may assume either of two values for their natural frequencies and their coupling strengths — hence we refer to it as the “Two-Frequency and Two-Coupling (TFTC) model”, where the natural oscillator frequency and coupling strength are drawn from a bi-modal distribution function, \( g(\omega) = p\delta(\omega + q\gamma) + q\delta(\omega - p\gamma) \), (2) where \( \delta \) denotes the Kronecker-delta distribution. Thus, oscillators have either a negative frequency, \( \omega = -q\gamma \), with probability \( p \), or a positive frequency, \( \omega = p\gamma \), with probability \( q := 1 - p \). The parameter \( \gamma = |p\gamma - (q\gamma)| > 0 \) defines the spacing between the two peaks and, since \( \langle \omega \rangle = \int \omega g(\omega)d\omega = p(-q\gamma) + q(\gamma) = 0 \), the distribution has always zero mean. The coupling strength, \( \xi_k \), defines the interaction strength between oscillator, \( k \), and all other oscillators, \( j = 1, \ldots, N \), and is assumed to be
either positive or negative. The coupling strengths are drawn from the bimodal distribution function,

$$\Gamma(\xi) = p\delta(\xi - 1) + q\delta(\xi + 1). \quad (3)$$

We may either rescale the coupling strength, $\xi_k$, or frequencies (time), $\omega_k$. Here, we chose to keep the distance between peaks of the coupling strength fixed, while the distance between peaks of the frequencies remains tunable via $\gamma$. To simplify the problem, we assume that the parameter $p$ is identical in the two distributions given by (2) and (3). Thus, oscillators either have positive coupling strength ($\xi = 1$) with probability $p$, or negative coupling strength ($\xi = -1$) with probability $q = 1-p$.

By choosing the coupling strength and natural frequency according to (2) and (3), we introduce a certain type of disorder in the system. We consider two model variants:

\textbf{a. Correlated model.} We consider the case where the two types of disorders, namely, in natural frequencies and in coupling strengths, are \textit{correlated} with one another. One may envision various ways to introduce correlation between the two disorders; however, we consider a very simple way of correlating the two distributions of coupling strengths, $\xi_j$, and frequencies, $\omega_j$. Specifically, we observe that coupling strengths with either $\xi = +1$ or $\xi = -1$ split the population into two subpopulations, $S_1$ and $S_2$, containing a number of elements corresponding to integer values near $pN$ and $qN$, respectively. This can be achieved by defining the subpopulations as follows: $S_1 := \{1, \ldots, \iota(p)\}$ and $S_2 := \{\iota(p) + 1, \ldots, N\}$ with $\iota(p) := \lfloor p(N - 1) \rfloor$ for $0 < p < 1$; $S_1 := \{\}$ and $S_2 := \{1, \ldots, N\}$ for $p = 0$; and $S_1 := \{1, \ldots, N\}$ and $S_2 := \{\}$ for $p = 1$. Correlation between frequencies and coupling strengths is then invoked by the following rule:

$$\omega_j = \begin{cases} -q\gamma \xi_j, & \text{for } j \in S_1, \\ -p\gamma \xi_j, & \text{for } j \in S_2. \end{cases} \quad (4)$$

This choice for the correlated disorder divides oscillators into two sub-populations, $\sigma = 1, 2$, with properties:

$$(\xi^{(1)}, \omega^{(1)}) = (+1, -q\gamma),$$

$$(\xi^{(2)}, \omega^{(2)}) = (-1, +p\gamma). \quad (5)$$

\textbf{b. Uncorrelated model.} The natural frequencies $\omega$, drawn from the distribution $g(\omega)$, and the coupling strengths $\xi$, drawn from the distribution $\Gamma(\xi)$, are independent from one another. Thus, the uncorrelated model divides oscillators into four sub-populations, $\sigma = 1, 2, 3, 4$, reflecting the two properties assigned to the oscillators:

$$(\xi^{(1)}, \omega^{(1)}) = (+1, -q\gamma),$$

$$(\xi^{(2)}, \omega^{(2)}) = (-1, +p\gamma),$$

$$(\xi^{(3)}, \omega^{(3)}) = (+1, +p\gamma),$$

$$(\xi^{(4)}, \omega^{(4)}) = (-1, -q\gamma). \quad (6)$$

The grouping of properties resulting from these two models imparts a subpopulation structure that allows us to rewrite Eq. (1) as follows:

$$\dot{\phi}^{(\sigma)}_j = \omega^{(\sigma)} + \frac{1}{N} \sum_{\tau=1}^{M} \xi^{(\tau)}_k \sin(\phi^{(\tau)}_k - \phi^{(\sigma)}_j), \quad (7)$$

where $\phi^{(\sigma)}_j$ is the phase of oscillator $j = 1, \ldots, [S_\sigma]$ belonging to subpopulation $S_\sigma$, and $M = 2$ or $M = 4$ for the correlated and uncorrelated model, respectively.

\textbf{c. Characterization of collective dynamics.} The collective dynamic behavior observed for the governing equations (1) and both models may be characterized by the complex order parameter,

$$Z = Re^{i\Psi} = \frac{1}{N} \sum_{k=1}^{N} e^{i\phi_k}, \quad (8)$$

or the weighted complex order parameter,

$$W = S e^{i\Delta} = \frac{1}{N} \sum_{k=1}^{N} \xi_k e^{i\phi_k}. \quad (9)$$

Both order parameters measure the synchronization level in the oscillator population: for incoherent oscillations, phases spread uniformly on the circle such that $R = S = 0$; synchronized phase-locked motion can be characterized by $R = 1$. Note that the value of the weighted order parameter, $S = |W|$, can be smaller than 1 even if phase-locked motion occurs with $R = 1$. Accordingly, a perfectly synchronized/coherent state is characterized by $R = 1$ and $0 < S < 1$. By contrast, a state of “partial synchronization” implies that some of the oscillators exhibit synchronized phase-locked motion while others are adrift, and thus the state is characterized by $R < 1$; or an all-frequency locked state with distributed phases. The state with $R = 0$, but $S > 0$ is not possible in the current system, as we show further below using the numerical simulations, see also (9) and (8), and the comments following thereafter.

\section{III. DIMENSIONAL REDUCTION}

We restrict our analysis to the case of large systems in the continuum limit, $N \to \infty$. The continuum limit prompts a statistical description in terms of a density function describing the phases of oscillators, $\rho = \rho(\phi, \omega, \xi, t)$, which evolves according to the continuity equation

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial \phi} (\rho v) = 0, \quad (10)$$

where the velocity is given by

$$v = \omega + \text{Im}(W(t)e^{-i\phi}), \quad (11)$$
where the weighted order parameter,
\[ W(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \xi \Gamma(\xi') \Gamma(\xi) g(\omega') \rho(\xi', \omega', \phi', t) e^{i\phi'} d\phi' d\xi' d\omega', \]  
(12)
acts as a mean-field forcing on each oscillator. The Ott-Antonsen method\(^2\) formulates a solution for the phase density via Fourier series ansatz,
\[ f = \frac{1}{2\pi} g(\omega) \Gamma(\xi)(1 + f^++ \bar{f}^+) \]  
(13)
with
\[ f^+ = \sum_{k=1}^{\infty} a(\xi, \omega, t) e^{ik\phi}. \]  
(14)
where we assume that \( f^+ \) has an analytic continuation into the lower complex plane. We may recognize this ansatz for the phase density as the Poisson kernel, parameterized by \( a = re^{i\phi} \). Geometrically, the Ott-Antonsen manifold defines a two dimensional submanifold in the infinite-dimensional space of density functions. Substitution of this ansatz into (10) results in an infinite set of identical ordinary differential equations, the amplitude equations for each mode \( e^{ik\phi} \):
\[ \dot{a} = -i\omega a + \frac{1}{2}(W-Wa^2). \]  
(15)
When these amplitude equations are satisfied, the phase density \( \rho \) is restricted to the invariant Poisson manifold\(^2\). The integro-o.d.e. system defined by (15) and (12) can be further simplified. Substituting the Ott-Antonsen ansatz and carrying out the integral over the phases, we have
\[ W(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \xi \bar{a}(\xi, \omega, t) \Gamma(\xi) g(\omega) d\omega d\xi, \]  
(16)
and cleverly choosing the distribution functions \( g \) and \( \Gamma \) allow to take the contour integral in \( W \) over the lower complex plane and express \( W \) in terms of expressions in \( a^2 \). For the current models, evaluating the integral in \( W \) is particularly simple due to nature of choices for \( g \) and \( \Gamma \). Here, the particular choices for \( g(\omega) \) and \( \Gamma(\xi) \) give rise to the subpopulation structure explained for the correlated model in (5) and for the uncorrelated model in (6) which also organizes the macroscopic dynamics in terms of local order parameters, \( z_\sigma \), as we show next. The correlated model implies that \( g(\omega) \Gamma(\xi) = p \delta(\omega + q \gamma) \delta(\xi - 1) \) and \( q \delta(\omega - q \gamma) \delta(\xi + 1) \) and we therefore have
\[ W(t) = pz_1 - qz_2, \]  
(17)
where we defined
\[ z_1(t) := \bar{a}(\xi = +1, \omega = -q \gamma, t), \]
\[ z_2(t) := \bar{a}(\xi = -1, \omega = +q \gamma, t). \]  
(18)
Using \( g(\omega) \) as defined in Eq. (2) for the uncorrelated model, we obtain
\[ W(t) = p \int_{-\infty}^{\infty} \xi \bar{a}(\xi, -q \gamma, t) \Gamma(\xi) d\xi \]
\[ + q \int_{-\infty}^{\infty} \xi \bar{a}(\xi, q \gamma, t) \Gamma(\xi) d\xi \]
\[ = p^2 z_1 - pqz_4 + pqz_3 - q^2 z_2. \]  
(19)
where we defined
\[ z_1(t) := \bar{a}(\xi = -1, \omega = -q \gamma, t), \]
\[ z_2(t) := \bar{a}(\xi = -1, \omega = +q \gamma, t), \]
\[ z_3(t) := \bar{a}(\xi = +1, \omega = +q \gamma, t), \]
\[ z_4(t) := \bar{a}(\xi = +1, \omega = -q \gamma, t). \]  
(20)
Thus, the dynamics of \( z_\sigma \) are given in closed form by
\[ \dot{z}_\sigma = i\omega(\sigma) z_\sigma + \frac{1}{2} (W - Wz^2 \sigma), \]  
(21)
for each subpopulation \( \sigma = 1, 2 \) (correlated model) or \( \sigma = 1, 2, 3, 4 \) (uncorrelated model). Later, we shall use the complex order parameter which in the \( N \rightarrow \infty \) limit is defined as
\[ Z(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \Gamma(\xi') g(\omega') \rho(\xi', \omega', \phi', t) e^{i\phi'} d\phi' d\xi' d\omega'. \]  
(22)
For the correlated model, we find
\[ Z(t) = pz_1 + qz_2, \]  
(23)
and for the uncorrelated model,
\[ Z(t) = p^2 z_1 + pqz_2 + pqz_3 + q^2 z_2. \]  
(24)
We note that (19) and (21) may also be obtained from considering the dynamics of oscillator (sub-)populations with identical natural frequencies. Watanabe and Strogatz\(^2\) showed that the phase space of an oscillator population is foliated by 3-dimensional leaves determined by \( N_\sigma - 3 \) constants of motion \( \theta_j^\sigma \in \mathbb{R}/2\pi \mathbb{Z}, j = 1, \ldots, N_\sigma - 3 \). Dynamics for each subpopulation \( \sigma \) in (7) are constrained to submanifolds of dimension at most three, governed by\(^1, 38\),
\[ \dot{\rho}_\sigma = \frac{1 - \rho^2_\sigma}{2} \text{Re}(W e^{-i\varphi_\sigma}), \]  
(25a)
\[ \dot{\Phi}_\sigma = \omega_\sigma + \frac{1 + \rho^2_\sigma}{2\rho_\sigma} \text{Im}(W e^{-i\varphi_\sigma}), \]  
(25b)
\[ \dot{\Theta}_\sigma = \frac{1 - \rho^2_\sigma}{2\rho_\sigma} \text{Im}(W e^{-i\varphi_\sigma}), \]  
(25c)
where \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) represent the real and imaginary parts of a complex number, respectively. Suppose now
that the level of synchronization inside each subpopulation $\sigma$ is characterized by the magnitude $0 \leq r_\sigma \leq 1$ of the local complex order parameter given by

$$z_\sigma := r_\sigma e^{i\varphi_\sigma} = \frac{1}{N_\sigma} \sum_{k \in S_\sigma} e^{i\theta_k}.$$

(26)

Assuming that the constants of motion are uniformly distributed, $\theta_j^2 = 2\pi j/N_\sigma$, $j = 1, \ldots, N_\sigma - 3$, and that the number of oscillators tends to infinity, $N_\sigma \to \infty$, one can show that the equalities $r_\sigma = \rho_\sigma$ and $\varphi_\sigma = \Phi_\sigma$ hold and the dynamics of $\rho_\sigma = r_\sigma$ and $\Phi_\sigma = \varphi_\sigma$ decouple from the dynamics of $\Theta_\sigma$. Using these relations in Eq. (25a), recalling that $z_\sigma = r_\sigma e^{i\varphi_\sigma}$ and using that $N_1/N \to p$ and $N_2/N \to q$, we obtain from Eqs. (25a)-(25b) equations identical to (21) describing the dynamics on the Poisson (or Ott-Antonsen) manifold\textsuperscript{30}. For a review on dimensional reduction methods developed by Ott/Antonsen and Watanabe/Strogatz and their details, see\textsuperscript{26} and references therein.

IV. ANALYSIS FOR CORRELATED DISORDER

A. Numerical Simulations

We obtained first insights into the possible dynamic behavior for the model with correlated disorder via numerical simulations of Eqs. (1), using a fourth-order Runge-Kutta (RK4) integration scheme with a time step of $\Delta t = 0.01$, over a simulation time of $M_t = 2 \times 10^5$. For any given value of $p$, initial phases $\{\phi_i(0)\}$ were randomly drawn from a uniform distribution on the interval $[0,2\pi)$. Snapshots of asymptotic states of phases at time $t = M_t$ are shown in Fig. 1 for several values of $p$. For all reported values, the oscillator population splits into two subpopulations, where the first, $S_1$ (red), is formed by oscillators with $\xi^{(1)} = +1$ and $\omega^{(1)} = -q\gamma$, and the second, $S_2$ (blue), is formed by oscillators with $\xi^{(2)} = -1$ and $\omega^{(2)} = +p\gamma$.

We found that the system may exhibit at least two states:

i) The Lock-Drift state (LD) where oscillators split into two subpopulations, one phase-locked with $r_1 = 1$ ($S_1$) and the other ($S_2$) drifting with $r_2 < 1$, as shown in panels (a)-(c). The subpopulations are frequency-locked so that their phase difference $\delta := \varphi_2 - \varphi_1$ remains constant, as shown in the analysis further below.

ii) The Lock-Lock (LL) state where all oscillators split into two phase-locked subpopulations with $r_1 = r_2 = 1$ rotating at a constant frequency with fixed phase distance, $\delta > 0$, as shown in panel (d).

Note that the Drift-Lock state, that is, the symmetric counterpart of the LD state where the role between the two subpopulations is reversed, is not observed. We can understand this as follows. Oscillators in subpopulation $S_1$ with positive coupling strength $\xi^{(1)} = +1$ tend to minimize their phase differences, thus leading to “phase-locking behavior”. Vice versa, oscillators in subpopulation $S_2$ with negative coupling strength $\xi^{(2)} = -1$ tend to maximize their phase difference, thus leading to “drifting” behavior. Therefore, we observe the LD state where subpopulation $S_1$ and $S_2$ assume locked and drifting dynamic behavior, respectively; conversely, the DL state, where the roles of the two subpopulations is reversed, does not emerge, as discussed in Sec. IV C 4.

Furthermore, to gain insight into the ranges of existence of these state, we obtained the phase diagram shown in Fig. 2, where we measured the time asymptotic behavior of several macroscopic variables while varying $p$ for $\gamma = 0.05$ fixed, averaged over the time interval $[M_t/2, M_t]$ to remove transient behavior. These macroscopic variables are the complex order parameter $R = |Z|$ given by (8) and the weighted order parameter $S = |W|$ given by (9). Inspecting the phase diagram there appears to be a transition between LD and LL states at a critical $p = p_c$. An incoherent state with $R = S = 0$ is not observed. In the following, we attempt to explain these observations by analyzing the dynamics described by (27a)-(27c).
Recalling Eq. (17) and (23), we immediately recognize that these conditions result from letting \( r_1 = r_2 = 0 \) or \( r_1 = r_2 = 0 \). Eqs. (21) are given in polar coordinates and are hence singular in this point; we therefore instead inspect Eqs. (21) for \( M = 2 \) in complex coordinates and note that this incoherent state exists for any parameter choice. The associated Jacobian

\[
J_{\text{INC}} = \begin{pmatrix}
p/2 & \gamma q & -q/2 & 0 \\
-q r & p/2 & 0 & -q/2 \\
p/2 & 0 & -q/2 & -\gamma q \\
0 & p/2 & \gamma q & -q/2
\end{pmatrix}
\]  

has two pairs of complex conjugated eigenvalues,

\[
\lambda_{1,2} = \frac{1}{4} \left( -1 + 2 p + 2 i \gamma |1 - 2 p| \pm \sqrt{(1 - 2 p)^2 - 4 \gamma^2 + \text{sgn}(1 - 2 p) 4 i \gamma} \right),
\]

\[
\lambda_{3,4} = \frac{1}{4} \left( -1 + 2 p - 2 i \gamma |1 - 2 p| \pm \sqrt{(1 - 2 p)^2 - 4 \gamma^2 - \text{sgn}(1 - 2 p) 4 i \gamma} \right).
\]

Inspecting these eigenvalues numerically reveals that INC is unstable for almost all parameter choices: the eigenvalues are complex-valued with \( \text{Re} (\lambda_k) > 0 \) for \( 0 < p < 1 \) and \( \gamma > 0 \); exceptional cases occur for two cases, namely, for \( p = 0 \), where INC is stable; or for \( p < \frac{1}{2} \) and \( \gamma = 0 \), where INC is neutrally stable.

2. Lock-Lock (LL) state

Next we examine the Lock-Lock (LL) state where oscillators in both subpopulations \( S_1 \) and \( S_2 \) are phase locked, i.e., \( r_1^* = r_2^* = 1 \). These conditions immediately satisfy the fixed point conditions for (27a) and (27b) by definition; Eq. (27c) yields the fixed point condition

\[
\sin \delta^* = \frac{\gamma}{p - q} = \frac{\gamma}{2p - 1},
\]

with the explicit solutions

\[
\delta^*_{+,-} = \begin{cases} 
\arcsin \left( \frac{\gamma}{2p - 1} \right) + 2\pi k \\
\arcsin \left( \frac{\gamma}{2p - 1} \right) + 2\pi k
\end{cases},
\]

where \( k \in \mathbb{Z} \) and subscripts ‘+’ and ‘-’ label the two solution branches for (27c). These two solution branches are born in saddle-node bifurcations (SN\(_1\) and SN\(_2\)) located at \( p_1 := |\gamma - 1|/2 \) and \( p_2 := |\gamma + 1|/2 \), respectively. Thus, the existence of these branches is limited to \( 0 \leq p < p_1 \) and \( p_2 < p \leq 1 \) (see Fig. 3), and consequentially to \( \gamma \leq 1 \) since \( p_2 \leq 1 \).

The Jacobian of (27a)-(27c) for the LL state can be expressed as

\[
J_{\text{LL}} = \begin{pmatrix}
-p + (1 - p) \cos \delta^* & 0 & 0 \\
0 & 1 - p - p \cos \delta^* & 0 \\
-p \sin \delta^* & (1 - p) \sin \delta^* & (1 - 2p) \cos \delta^*
\end{pmatrix}
\]  

where \( \delta^* \) is given by (31). The Jacobian is tri-diagonal and we readily obtain the eigenvalues for LL by substituting the two solution branches for \( \delta^* \) and eliminating...
Thus, the LD state may appear like the symmetry breaking “chimera state” known from previous studies\textsuperscript{18,22} in the sense that one subpopulation of the oscillators displays perfect synchronization, but the other does not; however, the LD state occurring in Eqs. (1) has a different origin, since it arises due to the correlation of the two disorders, $\omega_i$ and $\xi_i$; moreover, unlike the chimera state, the LD state does not have a symmetric counterpart corresponding to a DL state (see Sec. IV C 4).

Fixed point conditions for Eqs. (27a)-(27c) are satisfied for the LD state with $r_1 = 1$ if in addition we demand stationary $\delta = \delta^*$ and $r_2 \neq 1$, i.e.,

\[
\begin{align*}
    r_2^* &= \frac{p}{q} \cos \delta^*, \quad (34a) \\
    \gamma &= \sin \delta^* \left( \frac{1 + r_2^2}{2r_2^2} p - qr_2^* \right). \quad (34b)
\end{align*}
\]

While it is possible to eliminate $r_2$ such as to obtain an equation containing $\delta$ as the only variable, we instead eliminate $\delta^*$ by using $1 = \cos^2 \delta^* + \sin^2 \delta^*$ and solving the two conditions above for

\[
\begin{align*}
    \cos \delta^* &= \frac{q}{p} r_2^*, \quad (35a) \\
    \sin \delta^* &= \frac{2r_2^*}{(1 + (r_2^*)^2)p - 2q(r_2^*)^2}. \quad (35b)
\end{align*}
\]

resulting in

\[
1 = T \left( \frac{(p - 1)^2}{p^2} + \frac{4\gamma^2}{((3p - 2)T + p)^2} \right), \quad (36)
\]

where $T := (r_2^*)^2 \geq 0$. This cubic polynomial can be solved for $r_2^* = +\sqrt{T}$ using computer assisted algebra, resulting in one real and two complex conjugated roots — too unwieldy to display here. Finally, we obtain from (34a) the fixed point solution shown in Fig. 3,

\[
\delta^* = \arccos \left( \frac{q}{p} r_2^* \right), \quad (37)
\]

where only the positive branch in Eq. (34a) is a valid solution since $p, q, r_2 \geq 0$ must be non-negative. The LD state exists for $0 \leq p < p_c$, where $p = p_c$ defines the transition from LD to LL state computed in Sec. IV C 5 further below. Numerically plotting the eigenvalues of this branch reveals that they are real and negative for all $0 \leq \gamma \leq 1$ with $0 \leq p < p_c$. While fixed points do exist for $p > p_c$, they are not physically meaningful since they have $r_2 > 1$ (we therefore do not show this branch in Fig. 3). However, their eigenvalues have positive real parts, thus prompting a transcritical bifurcation, denoted TC, at $p = p_c$. Furthermore, we observe that $r_2$ is monotonically increasing for $p < p_c$, but monotonically decreasing for $p > p_c$; as a consequence, the peak value of the relative phase between the two subpopulations is reached at $p = p_c$ where $r_2 = 1$ so that $\delta = \arccos \left( q/p \right)$. These results are summarized in Fig. 4.

3. Lock-Drift (LD) state

We examine the Lock-Drift (LD) state, where oscillators in the first subpopulation ($S_1$) with $\omega^{(1)} = -q\gamma$ and $\xi^{(1)} = +1$ show perfect synchronization, $r_1 = 1$, while oscillators in the second subpopulation ($S_2$) with $\omega^{(2)} = p\gamma$ and $\xi^{(2)} = -1$ are drifting incoherently with $r_2 < 1$. Numerically plotting the eigenvalues of the LD state reveals that only the second branch is linearly stable for $p_c < p \leq 1$, where $p_c$ denotes the critical point $p_c$ where LL state loses stability and connects to the LD state.
4. Absence of Drift-Lock state

The “Drift-Lock (D-L)” state with $r_1 = 0$ and $r_2 > 0$ does not emerge in the system. This is easily seen as follows. The oscillators in the first subpopulation $S_1$ with positive coupling strength $\xi^{(1)} = +1$ tend to minimize their phase difference, thus resulting in phase-locking behavior. On the other hand, oscillators in the second subpopulation $S_2$ with negative coupling strength $\xi^{(2)} = -1$ tend to maximize the phase differences, thus displaying drifting behavior.

5. Stability diagram

We establish a stability diagram for the three states discussed above: incoherence (INC), lock-lock (LL), lock-drift (LD). We have already shown that INC can only be (neutrally) stable for $p = 0$ (or $\gamma = 0$ with $p < 1/2$); we are left to determining the transition point between lock-lock and lock-drift states, i.e., the critical value $p_c$ at which the transition between stable LD and LL states occurs. To do this, we consider the fixed point condition for the LL state, Eq. (30), to be considered in the limit from above where $p \to p_c^+$ and $\delta^* \to \delta_c^*$; and the fixed point condition for the lock-drift state, Eq. (35a), in the limit from below where $p \to p_c^-$ and $r_2^* \to 1^-$. At this point, we have

$$\cos \delta_c = \frac{q_c}{p_c} \quad \text{and} \quad \sin \delta_c = \frac{\gamma_c}{2p_c - 1},$$

for the LL and LD states, respectively; both fixed point conditions satisfy $1 = \cos (\delta_c)^2 + \sin (\delta_c)^2$ simultaneously, so that

$$1 = \left(1 - \frac{p}{p_c}\right)^2 + \left(\frac{\gamma}{2p_c - 1}\right)^2,$$

which is equivalent to

$$8p_c^3 - (12 + \gamma_c^2)p_c^2 + 6p_c - 1 = 0,$$

provided that $p_c \neq 0, p_c \neq \frac{1}{2}$. Since $\gamma > 0$, we may infer the relationship

$$\gamma_c = \frac{(2p_c - 1)^{3/2}}{p_c},$$

which produces the stability diagram in Fig. 4.

We find that $p_c$ monotonically increases as $\gamma = \gamma_c$ increases, which is reasonable in the sense that a higher value of $p$ is required to make the oscillators synchronized for a wider distribution with increasing value of $\gamma$. Since values $p > 1$ are not meaningful so that $\gamma = 1$ constitutes an absolute limit for the existence of the LD state.

6. Global order parameters and traveling waves

We investigate the behavior of order parameters $W$ and $Z$ for the two stable equilibria found, LD and LL. In the limit of infinite oscillators, where $N_1/N \to p$ and $N_2/N \to q$, the complex order parameter (17) becomes

$$Z = Re^{i\psi} = pz_1 + qz_2 = pr_1 e^{i\varphi_1} + qr_2 e^{i\varphi_2},$$

which has magnitude

$$R = \sqrt{p^2 r_1^2 + 2pqr_1r_2 \cos \delta + q^2 r_2^2};$$

similarly, the weighted order parameter (19) is

$$W = Se^{i\Delta} = pz_1 - qz_2 = pr_1 e^{i\varphi_1} - qr_2 e^{i\varphi_2},$$

with magnitude

$$S = \sqrt{p^2 r_1^2 - 2pqr_1r_2 \cos \delta + q^2 r_2^2}.$$

Furthermore, we may determine the mean-field frequency or “wave speed” of the collective state (see App. B for a derivation),

$$\Omega := \frac{d}{dt}\Delta = \frac{d}{dt}\arg(W) = \frac{1}{S} \sqrt{|W|^2 - S^2},$$

where

$$|W|^2 = p^2 (r_1^2 + r_2^2 \varphi_1^2) + q^2 (r_2^2 + r_2^2 \varphi_2^2) + 2pq [(r_1 r_2 \varphi_2 - r_1 \varphi_2 r_1) \sin \delta - (r_1 h_2 + r_1 h_2 \varphi_2 \varphi_1) \cos \delta],$$

and

$$\dot{S} = \frac{1}{S} \left(p^2 r_1 \dot{r}_1 + q^2 r_2 \dot{r}_2 - pq \left[(r_1 r_2 + r_1 \dot{r}_2) \cos \delta - r_1 r_2 \dot{\delta} \sin \delta\right]\right),$$

FIG. 4. Stability diagram for the correlated model with infinite oscillators. The stability boundary $(p_c, \gamma_c)$ between lock-lock (LL) and lock-drift states (LD) given by (40) corresponds to a transcritical bifurcation (see text and Fig. 3).
with $\delta = \varphi_2 - \varphi_1$. Evaluating $R$, $S$, and $\Omega$ at the equilibria corresponding to LD and LL states, we are able to plot the behavior of $R$ and $S$ as a function of $p$ for $\gamma = 0.05$ as shown in Fig. 2. It should be clear that the nature of the LD state occurring for $p < p_c$ implies that $R < 1$; however, note that the LL state occurring for $p > p_c$ does not necessarily imply perfect synchronization for the complete system in the sense that $R = 1$, since the locked oscillators of the two subpopulations may assume non-identical mean-field phases, $(\delta = \varphi_2 - \varphi_1 \neq 0)$, which results in $R < 1$. Inspecting Fig. 3 we recognize that $R = 1$ is only obtained for $p = 1$ where $\delta = 0$. Indeed, evaluating the order parameter for the LL state, the asymptotic behavior for $p$ close to 1 is $R \sim 1 - \left(1 - \sqrt{1 - \gamma^2}\right) (1-p)+O\left((1-p)^2\right)$.

Note that $R = |Z| = 0$ is only possible if $|z_1| = |z_2| = 0$ as long as $p > 0$, $q > 0$; however, we found that such an INC is (almost always) unstable. As a consequence, we can also rule out the case where $S = |W| = 0$ or $S > 0$ with $R = 0$. Furthermore we note that the nonzero wave speed, $\Omega \neq 0$, seen in Fig. 2 implies the presence of the traveling wave, rather than being induced by other type of heterogeneity. While the wave speed could be set to zero by an appropriate choice of reference frame, we note that the wave speed $\Omega$ differs from the system’s mean natural frequency.

V. ANALYSIS FOR UNCORRELATED DISORDER

A. Numerical simulations

For the uncorrelated model, we first performed numerical simulations of Eq. (1) using a fourth-order Runge-Kutta (RK4) integration scheme with identical parameters as listed in Sec. IV A for the correlated model. Snapshots of asymptotic states are shown in Fig. 5 for several values of $p$. We first observed that, for all reported values, oscillators residing in the subpopulations $S_1$ and $S_2$, and in the subpopulations $S_2$ and $S_3$, respectively, are phase-locked. We found that the system may exhibit at least three states:

i) The Incoherent state (INC) where all subpopulations are desynchronized, i.e. $r_1 \approx r_2 \approx r_3 \approx r_4 \approx 0^{\circ}$, see panels a), b) and c).

ii) The Breathing Lock-Lock-Lock-Lock state (Breathing LLLL) where oscillators in each subpopulations are phase-locked, $r_1 = r_2 = r_3 = r_4 = 1$, but where the two mutually phase-locked subpopulation pairs $(S_1, S_2)$ and $(S_2, S_3)$ drift apart, i.e., their phase difference $\delta(t) := \phi_2(t) - \phi_1(t)$ increases with time.

iii) The Stable Lock-Lock-Lock-Lock state (Stable LLLL) where oscillators in each subpopulations are phase-locked, $r_1 = r_2 = r_3 = r_4 = 1$, and the phase-locked subpopulation pairs $(S_1, S_4)$ are frequency locked, i.e. their phase difference remains constant in time, $\delta = 0$, see panel d).

We also measured the asymptotic behavior for the order parameters, $R$ and $S$, averaged over the time window $[M_t/2, M_t]$ to quantify the collective synchronization level of the system, while varying the probability $p$. Fig. 6 shows the resulting time asymptotic behavior of $R$ and $S$ while varying $p$ for $\gamma = 0.05$ fixed. The incoherent (INC) state ($R = S = 0$) appears to exist only for $0 < p < p_c$, while the coherent LLLL states exist for $p > p_c$. However, note that the Ott/Antonsen Eqs. (50a)-(50d) reveal neutral stability of the incoherent state, as we discuss further below (Sec. V C 3). Moreover, random initial conditions for the local order parameters $z_r(0)$ evolve to arbitrary asymptotic order parameter values with $r_s(t) > 0$. Thus, we expect that initial phases deviating more strongly from $R = S = 0$ in (1) also asymptotically evolve to values with $R > 0, S > 0$, incongruent with Incoherence.

The critical value for this transition, $p_c$, may be deduced from a simple argument: we expect that the coherent state with $R > 0$ only exists for coupling strengths with positive mean given by

$$\langle \xi \rangle = p \cdot 1 + q \cdot (-1) = 2p - 1 > 0. \quad (48)$$
Since we cannot expect that the coherent state emerges for repulsive coupling, \( \langle \xi \rangle < 0 \), we obtain the critical value \( p_c = 1/2 \). Note that in the present study we chose \( \xi_j = 1 \) for \( j \in S_1 \) and \( \xi_j = -1 \) for \( j \in S_2 \) without loss of generality. We may instead assign general asymmetrically balanced values \( \langle \xi_+ \rangle \neq \langle \xi_- \rangle \), \( \xi = \xi_+ > 0 \) with probability \( p \) and \( \xi = \xi_- < 0 \) with probability \( 1 - p \). Then we have

\[
\langle \xi \rangle = p \xi_+ + (1 - p) \xi_- = (p + (p - 1)Q) \xi_+ \quad (49)
\]

where we define \( Q := -\xi_- / \xi_+ > 0 \). Again, the coherent state exists for \( \langle \xi \rangle > 0 \) only, thus determining a critical value given by \( p_c = Q / (1 + Q) \). Applying this to the present case with \( \xi_j = \pm 1 \) results in \( Q = 1 \) which yields our previous critical value of \( p_c = 1/2 \), as expected. This value agrees well with our numerical simulations, see Fig. 6.

In the following we explain the observed behavior using the dimensionally reduced equations derived in Sec. III and a self-consistency argument.

![Fig. 6](image)

**FIG. 6.** (Color Online) Phase diagram for the uncorrelated model obtained via numerical simulation of Eqs. (1). Equilibrium values for the order parameters \( R \) (red circles) and \( S \) (blue squares) are shown as a function of \( \gamma = 0.05 \) after transient behavior has vanished (see text). The system size is \( N = 10^5 \) and the data represent values averaged over 10 sample simulations with different initial conditions \( \{ \phi_0(0) \} \).

The incoherent state INC with \( R = S = 0 \) is observed for \( p < p_c \), while the coherent state LLLL with \( R > 0 \) and \( S > 0 \) emerges for \( p > p_c \), where \( p_c = 1/2 \). Theoretical predictions for \( R \) (magenta) given by Eq. (A9) and for \( S \) (cyan) given by Eqs. (59) and (A10), valid for \( p > p_c \), match the results obtained from numerical simulations very well. The inset shows the periodic behavior of \( R(t) \) in time for \( p = 0.52 \) (Breathing LLLL).

### B. Reduced dynamical equations

We explain the observed behavior by studying Eqs. (21) describing the dynamics for the local order parameters in (20) valid for the continuum limit with the \( M = 4 \) populations present in the uncorrelated model, given by

\[
\begin{align*}
\dot{z}_1 &= +iq\gamma z_1 + \frac{1}{2} (W - W z_1^2), \\
\dot{z}_2 &= -iq\gamma z_2 + \frac{1}{2} (W - W z_2^2), \\
\dot{z}_3 &= -iq\gamma z_3 + \frac{1}{2} (W - W z_3^2), \\
\dot{z}_4 &= +iq\gamma z_4 + \frac{1}{2} (W - W z_4^2),
\end{align*}
\]

where the weighted order parameter is given by (19),

\[
W = p^2 z_1 - pq z_4 + pq z_3 - q^2 z_2.
\]

We note that Eqs. (50a) and (50d) for subpopulations \( S_1 \) and \( S_4 \) and Eqs. (50b) and (50c) for subpopulations \( S_2 \) and \( S_3 \), have identical structure. Furthermore, numerical simulations (Sec. V.A) revealed asymptotic behavior for the LLLL states, i.e., \( |z_1(t) - z_4(t)| \rightarrow 0 \) and \( |z_2(t) - z_3(t)| \rightarrow 0 \) as \( t \rightarrow \infty \). This observation suggests the existence of a stable symmetric invariant subspace \( SS \) defined by \( z_1(t) = z_4(t) \) and \( z_2(t) = z_3(t) \) for all \( t \). We therefore first examined the dynamics confined to that subspace. Eqs. (50a) and (50b) govern this dynamics. Introducing polar coordinates \( z_0 = r_0 e^{i\varphi_0} \) and defining \( \delta := \varphi_2 - \varphi_1 \), we have

\[
\begin{align*}
\dot{r}_1 &= \frac{1 - r_1^2}{2} (p - q) (p r_1 + q r_2 \cos \delta), \\
\dot{r}_2 &= \frac{1 - r_2^2}{2} (p - q) (p r_1 \cos \delta + q r_2), \\
\dot{\delta} &= \gamma - \frac{1}{2} (p - q) \left( \frac{1 + r_2^2}{r_2^2 pr_1 + 1 + r_1^2 r_2^2} \right) \sin \delta.
\end{align*}
\]

### C. Equilibrium states

#### 1. Incoherent (INC) state

The incoherent state is defined by \( z_1 = z_2 = z_3 = z_4 = 0 \). The Jacobian for Eqs. (50a) and (50b) describing the dynamics in \( z_1 \) and \( z_2 \) on the symmetric subspace \( SS \), defined by \( z_1(t) = z_4(t) \), \( z_2(t) = z_3(t) \), expressed in Cartesian coordinates is

\[
J_{\text{INC}} = \begin{pmatrix}
p/2 & \gamma q & -q/2 & 0 \\
-\gamma q & p/2 & 0 & -q/2 \\
p/2 & 0 & -q/2 & -\gamma p \\
0 & p/2 & \gamma p & -q/2
\end{pmatrix}
\]
has two pairs of complex conjugated eigenvalues,
\[ \lambda_{1,2} = \frac{1}{4} \left( 2p - 1 + 2i\gamma|1 - 2p| \pm \sqrt{4i\gamma(2p - 1)|1 - 2p| - 4\gamma^2 + (1 - 2p)^2} \right) \]  
\[ \lambda_{3,4} = \frac{1}{4} \left( 2p - 1 - 2i\gamma|1 - 2p| \pm \sqrt{4i\gamma(2p - 1)|1 - 2p| - 4\gamma^2 + (1 - 2p)^2} \right) \]

Inspecting Re(\(\lambda_k\)) numerically for \(k = 1, 2, 3, 4\) we immediately see that INC is stable on the symmetric subspace \(SS\) only when \(p < p_c = 1/2\); otherwise, it is unstable.

2. Stable and breathing \(LLLL\) states

Locked states (\(LLLL\)) satisfy \(r_1 = r_2 = r_3 = r_4 = 1\). This also defines an invariant subspace (on the symmetric subspace \(SS\)) since the \(LLLL\) state implies \(\dot{r}_1 = \dot{r}_2 = 0\). In the following, we consider the dynamics and stability of \(LLLL\) states on \(SS\) as given by Eqs. (51a). Stationarity of the \(LLLL\) state requires the additional condition \(\delta = 0\), which implies the stationary phase difference
\[ \sin \delta^* = \frac{\gamma}{2p - 1}. \]  

We denote an equilibrium with \((r_1, r_2, \delta) = (1, 1, \delta^*)\) as a Stable \(LLLL\) state. Eq. (54) informs us that stable \(LLLL\) states are born in saddle-node bifurcations \(SN_1\) and \(SN_2\) at \(p_1 = |\gamma - 1|/2\) and \(p_2 = |\gamma + 1|/2\) and are constrained to the intervals \(0 \leq p \leq p_1\) and \(p_2 \leq p \leq 1\) (see Fig. 7).

To examine stability, consider the eigenvalues of the Jacobian for \(LLLL\),
\[ \lambda_1 = (1 - 2p) \cos \delta^*, \]  
\[ \lambda_2 = (1 - 2p)(1 + p(1 - \cos \delta^*)), \]  
\[ \lambda_3 = (1 - 2p)(p + (1 - p) \cos \delta^*). \]

We first note that all eigenvalues flip sign at \(p = p_c = 1/2\). It therefore suffices to consider eigenvalues restricted to the interval \(1/2 \leq p \leq 1\) where they share the common factor \((1 - 2p) < 0\). For \(p \in [p_2, 1]\), the lower branch \(\delta^* \in [0, \pi/2]\) has Re(\(\lambda_1(\delta^*)\)) < 0, whereas the upper branch with \(\delta^* \in [\pi/2, \pi]\) has Re(\(\lambda_1(\delta^*)\)) > 0. Since \(0 \leq \delta^* \leq \pi\) for \(p \in [p_2, 1]\) we have \(0 \leq \cos \delta^* \leq 1\) and \(\lambda_2, \lambda_3\) are real-valued. Minimizing and maximizing these two eigenvalues, we find that \((1 - 2p)(1 + p) < \lambda_2 < (1 - 2p) < 0\) and \((1 - 2p)p < \lambda_3 < (1 - 2p)\). As already mentioned, the signs of all eigenvalues are reversed for \(p < 1/2\). Therefore, the \(LLLL\) state \((1, 1, \delta^*)\) is stable for \(p \in [p_2, 1]\), as shown in Fig. 7.

For \(p_1 \leq p \leq p_2\), the phase difference \(\delta(t)\) is unlocked and evolves according to
\[ \dot{\delta} = \gamma + (2p - 1) \sin \delta. \]

We denote the resulting limit cycle, confined to the invariant suspace \(r_1 = r_2 = 1\), as the Breathing \(LLLL\) state.

Furthermore, for \(p = 1/2\) we have \(\dot{r}_1 = \dot{r}_2 \equiv 0\) (with \(\delta = \gamma\)), thus implying the presence of a degeneracy where \(0 \leq r_1 \leq 1\) and \(0 \leq r_2 \leq 1\) may assume arbitrary values. This is indicated as a the vertical dashed line in Fig. 7 (bottom).

Considering the numerical simulation results shown in Fig. 6, the Breathing \(LLLL\) state exists inside a small region \(1/2 < p \leq 0.525 = p_2\). This parameter region grows in size as \(\gamma\) is increased, which is shown as the gray shaded region on the stability diagram in Fig. 8.

3. Transverse stability of symmetric subspace \(SS\)

We so far only discussed stability on the symmetric (invariant) subset \(SS\) with \(z_1(t) = z_4(t)\) and \(z_2(t) = z_3(t)\).

![FIG. 7. Bifurcation diagram for uncorrelated disorder (\(\gamma = 0.65\)). Stable and unstable branches of INC and LLLL states are indicated as solid and dashed curves, respectively (Stability relates to the symmetric subspace \(SS\) given by Eqs. (51a)). Breathing LLLL states corresponding to limit cycles on the subspace \(r_1 = r_2 = 1\) are annihilated in the saddle-node bifurcation \(SN_2\).](image)
It remains unclear whether or not the subset \( SS \) is stable with respect to perturbations in directions transverse to itself, and in particular in the proximity of the LLLL states. Unfortunately, deciding this question in general turns out to be cumbersome since the associated variational equations do not appear decouple in suitable directions. However, numerical solutions of the governing equations (1) (see Figs. 5 and Figs. 6) and the four complex Ott-Antonsen equations in \( z_1, z_2, z_3, z_4 \) (see (50a)-(50d) or Appendix A) have confirmed stability for both Stable and Breathing LLLL states in transverse direction of \( SS \), for all parameters we tested.

For INC the Ott-Antonsen equations (50a)-(50d) yield four zero eigenvalues, and four eigenvalues that are either negative for \( p < 1/2 \) and positive for \( p > 1/2 \); furthermore, for \( p < 1/2 \), direct integration of Eqs. (50a)-(50d) reveals a degeneracy with respect to random initial conditions, as it is seen that \( r_1, r_2, r_3, r_4 \) converge to seemingly arbitrary values on \([0,1]\) as \( t \to \infty \), rather than just 0, while \( \phi_1(t) - \phi_4(t) \to 0 \) and \( \phi_2(t) - \phi_3(t) \to 0 \) as \( t \to \infty \).

4. Stability diagram

The preceding stability analysis for INC and LLLL states is summarized in the stability diagram shown in Fig. 8. The dotted line delineates the stability boundary where INC and Breathing LLLL swap stability, see also Fig. 7.

5. Global order parameters

On the symmetric subspace \( SS \), the global order parameters simplify to

\[
R = \sqrt{p^2 r_1^2 + 2pq r_1 r_2 \cos \delta + q^2 r_2^2} \quad (57)
\]

\[
S = \sqrt{(p - q)^2 (p^2 r_1^2 + 2pq r_1 r_2 \cos \delta + q^2 r_2^2)} \quad (58)
\]

which for INC \( (0 < p < 1/2) \) become \( R = S = 0 \); and for the stable LLLL state with \( p \in [p_2, 1] \) they become and \( R = \sqrt{p^2 + 2pq \cos \delta + q^2} \) and \( S = \sqrt{2p(p^2 + 2pq \cos \delta + q^2)} \), where \( \delta = \arcsin \gamma/(2p - 1) \).

The breathing LLLL state is bounded with \( R_{\text{min}} := 2p - 1 \leq R \leq 1 \approx R_{\text{max}} \) and \( (2p - 1)^2 < S < (2p - 1) \) since then \( |\cos \delta(t)| < 1 \). This aligns with the phase diagram provided in Fig. 6, with the exception of two minor differences: (i) numerical simulations in the Stable INC regime show that \( R \) stays close to \( R \approx 0 \). Possible explanations for this behavior are manifold: finite size effects, critical slowing down near the bifurcation point \( p_c = 1/2 \), and/or the aforementioned degeneracy of the INC state; (ii) results in the Breathing LLLL regime show values at the end of the simulation within the ranges specified above.

It is possible to determine an explicit expression for \( S = S(p) \) in the stable LLLL regime by deriving a self-consistency equation in the weighted order parameter (12) for the coherent (phase-locked) state based on Kuramoto’s classical argument\(^{27,28} \), see App. C:

\[
S = \sqrt{8p^4 - 16p^3 + 14p^2 + 2\sqrt{A} - 6p + 1}, \quad (59)
\]

where \( A := p^2(2p^2 - 3p + 1)^2(-\gamma^2 + 4p^2 - 4p + 1) \). This result is numerically confirmed using numerical simulations, as shown in Fig. 6.

VI. DISCUSSION

a. Summary. We have studied the collective dynamics in a network of coupled phase oscillators with disorder in natural frequencies and coupling strengths, which were correlated or uncorrelated. Specifically, we have assumed that the coupling strength and the natural frequency of each oscillator may assume only one of two values (positive or negative), amounting to a “Two-Frequency-Two-Coupling model”. The character and stability of the nontrivial dynamic states in the models with correlated/uncorrelated disorder depend on the interplay of the disorder asymmetry parameter, \( p \), and the frequency spacing, \( \gamma \). To explore how the different types of disorder influence the emergent phase coherence in the system, we performed numerical simulations revealing several nontrivial dynamic states. For the model with correlated disorder, oscillators split into two subpopulations where either one or both subpopulations are perfectly phase-locked, amounting to Lock-Drift (LD) or Lock-Lock (LL) states, respectively. Both states maintain a constant phase difference, the size of which is controlled by the disorder asymmetry \( p \). LD is stable for \( p < p_c \) and swaps stability with LL when \( p > p_c \). This observation can be rationalized by observing that a majority of oscillators experience attractive (\( \zeta = 1 \)) rather than repulsive coupling strength (\( \zeta = -1 \)) for large \( p \). Equilibria for the global order parameters, i.e., the weighted \( W \) and the unweighted \( Z \), depend on \( p \) in a nontrivial way (see Fig. 2). Furthermore, both states can be characterized by a traveling wave motion, corresponding to a non-zero
mean-field frequency $\Omega \neq 0$. At first sight, the LD state may resemble a chimera state, which also is characterized by one locked and one drifting subpopulation; however, the LD state is distinct since its symmetric counterpart, the DL state, is unstable, thus reflecting that the asymmetry inherent to the system itself rather than the system dynamics gives rise to asymmetric states. For the model with uncorrelated disorder, numerical simulations indicated that oscillators split into four subpopulations all of which are phase-locked (Lock-Lock-Lock-Lock / LLLL state); however, the two subpopulations with identical natural frequencies and opposing coupling strengths form pairs. These pairs are either frequency-locked with constant phase difference (Stable LLLL), or a drifting phase difference (Breathing LLLL), see Fig. 6. For uncorrelated disorder, we also observed a state of Incoherence (INC) where both global order parameters stay close to $R = S = 0$.

Next, we carried out a detailed bifurcation analysis for the local order parameters $z_0(t)$ describing the collective dynamics and the synchronization level in subpopulations formed by oscillators with identical attributes (natural frequency / coupling strength). While uncorrelated disorder allows for the formation of $M = 4$ subpopulations $(S_1, S_2, S_3, S_4)$, correlated disorder naturally implies the presence of only $M = 2$ subpopulations, $S_1$ and $S_2$, one with attractive coupling and negative frequency, the other with repulsive coupling and positive frequency. The order parameters $z_0$ satisfy Eqs. (21) which can be derived using the Ott-Antonsen method or the Watanabe-Strogatz method valid in the limit of $N \to \infty$ oscillators with uniformly distributed constants of motion.

For correlated disorder, the LD $(p < p_c(\gamma))$ and the LL $(p > p_c(\gamma))$ states swap stability in a transcritical bifurcation at the critical value $p = p_c(\gamma)$ which we determined analytically (see Eq. (40)). We also found analytical expressions of the unweighted and weighted order parameter $R = |Z|$ and $S = |W|$ at equilibrium, as well as for the non-zero mean-field frequency, $\Omega \neq 0$, thus prompting a traveling wave motion (Fig. 2).

For uncorrelated disorder, the dynamics of locked states are confined to the (invariant) symmetric subspace $SS$ implying that subpopulations $S_1$ and $S_4$, and $S_2$ and $S_3$ are mutually phase-locked ($z_1(t) = z_4(t)$ and $z_2(t) = z_3(t)$). While a proof for stability transverse to the symmetric subspace $SS$ remained elusive, both numerical simulations of (1) and direct numerical integration of (50a)-(50d) confirmed that $SS$ is attractive for two types of locked states. The Stable LLLL appears for $p > p_2$ and loses stability in a saddle-node bifurcation at $p = p_2$ on the invariant synchronized subspace defined by $r_1 = r_2 = 1$; this gives rise to the Breathing LLLL state which is stable for $1/2 < p < p_2$. The Breathing LLLL state is characterized by a drifting phase-relationship between subpopulations $S_1$ and $S_2$, i.e., their phase difference $\phi(t)$ increases monotonically and results in a periodic motion in $R(t)$ and $S(t)$. The Breathing LLLL state is remarkable in the sense that there is no external periodic driving acting on the system; i.e., the periodic synchronization emerges “spontaneously” when the coupling strengths and the natural frequencies are uncorrelated. Unlike for the correlated model, we did not find signs of traveling wave behavior with non-zero mean-field frequency $\Omega$; however, for the Breathing LLLL state, two subpopulation pairs $(S_1, S_2)$ and $(S_3, S_4)$ drift apart while their average frequency stays close to 0 — which we may refer to as a “Standing Wave”, alike states observed for oscillator populations with bimodal frequency distributions. Both LLLL states swap stability with the INC state at $p = 1/2$.

The Incoherent (INC) state is always unstable in the correlated model, in contrast to the uncorrelated model, where INC is neutrally stable for $p < 1/2$ on the symmetric subspace $SS$: Eqs. (21) exhibit for the INC state four negative and four zero eigenvalues for the INC state. Numerical integration of Eqs. (21) reveals degeneracy in the magnitude of local order parameters, i.e., $r_2(t)$ may attain arbitrary stationary values between 0 and 1, which do not match Incoherence $(R = S = 0)$, while $|\phi_1 - \phi_4| \to 0$ and $|\phi_2 - \phi_3| \to 0$ as $t \to \infty$. Therefore, we also expect that numerical simulations of Eqs. (1) may reveal states for $p < 1/2$ that have nonvanishing order parameters. However, introducing distributed frequencies of width $\Delta$ around each mode $(\omega = -q \gamma, p \gamma)$ results in additional terms of the form $-\Delta z_\sigma$ in (21) for $\sigma = 1, 2, 3, 4$. This removes the degeneracy and renders INC into a (stable) hyperbolic equilibrium, and we can say that INC is a robust state. Establishing a complete bifurcation diagram for distributed frequencies is beyond the scope of this study and remains subject for future research.

b. Relationship with other studies. The present study is closely related with previous work by Hong et al. where oscillators’ natural frequencies were drawn from a distribution with finite nonzero variance (contrasting the zero-width distribution considered here), in order to explore the effects of symmetrically and asymmetrically correlated disorder. It was found that asymmetrically correlated disorder induces traveling wave motion, characterized by non-zero mean-field frequency $\Omega \neq 0$; here, we found that correlated disorder still induces traveling waves when natural frequencies are bimodally distributed with zero variance, whereas uncorrelated disorder does not promote traveling waves. Thus, together with the simplifications implied by the present model, we may conclude that the traveling wave motion results from heterogeneity in terms of asymmetry in natural frequencies and coupling strengths, rather than it is a consequence of distributions with nonzero variance.

The correlated model also relates to several other studies addressing the collective dynamics of two interacting populations, either characterized by non-uniform interactions, see Abrams et al. and Martens et al.; or the dynamics of two population mod-
els combining both properties, see Montbrió et al.\textsuperscript{19}, Laing\textsuperscript{42} and Pietras\textsuperscript{43}. Most of these studies assume positive coupling strengths (exceptions include variants of the Kuramoto-Sakaguchi model with two populations\textsuperscript{42} where heterogeneous phase-lags may result in negative coupling strength), whereas the correlated model has \( W = pz_1 - qz_2 \). One may interpret the prefactors \( p \) and \(-q\) in one of two ways: (i) in the generic way as the correlated model was posed, namely, natural frequencies are bimodally distributed with asymmetric peaks, populated by a fraction of oscillators \( p \) and \( q \) obeying attractive and repulsive coupled, respectively; (ii) in the sense of (asymmetric) coupling strengths, i.e., when writing Eqs.\textsuperscript{(21)} in matrix-vector notation with the (vector) mean-field \( W_{\alpha} = \left( \frac{p}{q} - \frac{q}{p} \right) \cdot \left( z_1 \right) \) promotes attractive (or excitatory) coupling with strength \( p \) among oscillators within the first population and with the adjacent second population; and repulsive (or inhibitory) coupling \( q \) among oscillators within the second population and the adjacent first population. The mean-field for the uncorrelated model may also be interpreted in terms of coupling strengths in a similar fashion. Rewriting the mean-field in matrix-vector notation, we have \( W_{\alpha} = (2p-1) \left( \frac{p}{q} - \frac{q}{p} \right) \cdot \left( z_2 \right) \). Comparing \( W_{\alpha} \) with \( W_{\alpha} \) makes the different characters of the uncorrelated and the correlated model especially evident, as well as it elucidates why \( p > \frac{1}{2} \) or \( p < \frac{1}{4} \) results in predominantly attractive or repulsive coupling, promoting or hindering synchrony, respectively.

The models considered by Maistrenko et al.\textsuperscript{44} and Teichmann and Rosenblum\textsuperscript{45} coincides with our model Eqs.\textsuperscript{(1)}, but there are important differences. Our study concerns the effects of correlated/uncorrelated disorder on the long-term collective behavior and on their phase transitions towards synchrony for the thermodynamic limit \( N \rightarrow \infty \); these authors studied solitary states in finite oscillator systems, where a single oscillator ‘escapes’ from the synchronized frequency cluster as repulsive interactions increase, however they disappear in the thermodynamic limit \( N \rightarrow \infty \). Both models\textsuperscript{44,45} are restricted to subpopulations with equal size, \( N_1 = N_2 = N/2 \), corresponding to \( p = 1/2 \) in our model for the thermodynamic limit; here, we studied the general case with \( 0 \leq p \leq 1 \). Maistrenko et al.\textsuperscript{44} considered identical natural frequencies \( (\gamma = 0) \), while Teichmann and Rosenblum\textsuperscript{45}, considered the case with different natural frequencies in the subpopulations with attractive and repulsive (self-)interaction and found that the transition from a two-cluster synchrony to partial synchrony occurs via the formation of a solitary state for small frequency mismatch.

c. Outlook. Our analytical results are constrained to the dynamics on the Poisson manifold discovered by Ott and Antonsen\textsuperscript{4}\textsuperscript{40}; it would be interesting to investigate the dynamics off this manifold, too. Furthermore, it would be desirable to better understand how robust the Incoherent state in the correlated model is with regard to perturbations of the system. The simplicity of the model suggests that real-world systems can be found that display the dynamic states induced by correlated/uncorrelated disorder that we reported here. In this context it could be fruitful to identify intuitive mechanisms for, e.g., the breathing of the LLLL state, generating periodic behavior of \( R(t) \). Candidates for experimental systems might for instance be found in Josephson junction arrays\textsuperscript{9}, coupled Belousov-Zhabotinsky oscillators\textsuperscript{1,46-48}, and electro-chemical oscillators\textsuperscript{10,49}.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Appendix A: Full Ott-Antonsen equations for uncorrelated model

For completeness, we list the Ott-Antonsen equations for the uncorrelated model in polar coordinates, describing the complete dynamics on the Poisson manifold (i.e., on and off the symmetric subspace \( SS \)). Rather than performing a bifurcation analysis for this system, we numerically solved the four ordinary differential equations above for the fixed point conditions \( (\dot{r}_1 = \dot{r}_2 = \dot{r}_3 = \dot{r}_4 = 0) \) for \( p > p_c (= 1/2) \). Introducing \( z_\sigma \equiv r_\sigma e^{-i\theta} \), \( \sigma = 1, 2, 3, 4 \), we have

\[
\begin{align*}
\dot{r}_1 &= \frac{1}{2} \left[ p^2 r_1 - pqr_4 \cos \delta_{41} + pqr_3 \cos \delta_{31} \\
&\quad - q^2 r_2 \cos \delta_{21} - p^2 r_3^2 + pqr_4 r_1^2 \cos \delta_{41} \\
&\quad - pqr_3 r_1^2 \cos \delta_{31} + q^2 r_2 r_1^2 \cos \delta_{21} \right], \tag{A1} \\
\dot{r}_2 &= -q\gamma - \frac{1}{2r_1} \left[ pqr_4 \sin \delta_{41} - pqr_3 \sin \delta_{31} \\
&\quad + q^2 r_2 \sin \delta_{21} + pqr_4 r_1^2 \sin \delta_{41} \\
&\quad - pqr_3 r_1^2 \sin \delta_{31} + q^2 r_2 r_1^2 \sin \delta_{21} \right], \tag{A2}
\end{align*}
\]
where we defined the phase difference $\delta_k(t) := \theta_k(t) - \theta_l(t)$ with $k$ and $l = 1, 2, 3, 4$. Similarly, we find
\begin{equation}
\dot{r}_2 = \frac{1}{2} \left[ p^2 r_1 \cos \delta_{12} - pqr_4 \cos \delta_{42} + pqr_3 \cos \delta_{32} - q^2 r_2 - p^2 r_1^2 \cos \delta_{21} + pqr_4 r_2^2 \cos \delta_{42} - pqr_3 r_2^2 \cos \delta_{32} + q^2 r_3^2 \right],
\end{equation}
\begin{equation}
\dot{\gamma} = p \gamma - \frac{1}{2r_2} \left[ p^2 r_1 \sin \delta_{21} + pqr_4 \sin \delta_{42} - pqr_3 \sin \delta_{32} + p^2 r_1 r_2^2 \sin \delta_{21} + pqr_4 r_2^2 \sin \delta_{42} + pqr_3 r_2^2 \sin \delta_{32} \right],
\end{equation}
\begin{equation}
\dot{r}_3 = \frac{1}{2} \left[ p^2 r_1 \cos \delta_{31} - pqr_4 \cos \delta_{43} + pqr_3 - q^2 r_2 \cos \delta_{32} - p^2 r_1 r_3^2 \cos \delta_{31} + pqr_4 r_3^2 \cos \delta_{43} - pqr_3 r_3^2 \cos \delta_{31} + q^2 r_2 r_3^2 \cos \delta_{32} \right],
\end{equation}
\begin{equation}
\dot{\gamma} = p \gamma - \frac{1}{2r_3} \left[ p^2 r_1 \sin \delta_{31} + pqr_4 \sin \delta_{43} - q^2 r_2 \sin \delta_{32} + p^2 r_1 r_3^2 \sin \delta_{31} + pqr_4 r_3^2 \sin \delta_{43} - pqr_3 r_3^2 \sin \delta_{31} + q^2 r_2 r_3^2 \sin \delta_{32} \right],
\end{equation}
\begin{equation}
\dot{r}_4 = \frac{1}{2} \left[ p^2 r_1 \cos \delta_{41} - pqr_4 + pqr_3 \cos \delta_{43} - q^2 r_2 \cos \delta_{42} - p^2 r_1 r_4^2 \cos \delta_{41} + pqr_3^2 - pqr_4 r_4^2 \cos \delta_{42} \cos \delta_{41} + q^2 r_2 r_4^2 \cos \delta_{42} \right],
\end{equation}
\begin{equation}
\dot{\gamma} = -q \gamma - \frac{1}{2r_4} \left[ p^2 r_1 \sin \delta_{41} + pqr_3 \sin \delta_{43} - q^2 r_2 \sin \delta_{42} + p^2 r_1 r_4^2 \sin \delta_{41} + pqr_3 r_4^2 \sin \delta_{43} - q^2 r_2 r_4^2 \sin \delta_{42} \right].
\end{equation}
With Eq. (A1)-(A8), the order parameters \( R = |Z| \) and \( S = |W| \) are then given by
\begin{equation}
R = \left[ p^4 r_1^2 + p^2 q^2 r_4^2 + p^2 q^2 r_3^2 + q^4 r_2^2 + 2p^3 qr_1 r_4 \cos \delta_{41} + 2pq r_3 r_2 \cos \delta_{32} + 2p^3 qr_1 r_3 \cos \delta_{31} + 2p^2 q^2 r_1 r_2 \cos \delta_{21} + 2p^2 q^2 r_4 r_3 \cos \delta_{43} + 2pq^3 r_4 r_2 \cos \delta_{42} \right]^{1/2},
\end{equation}
and
\begin{equation}
S = \left[ p^4 r_1^2 + p^2 q^2 r_4^2 + p^2 q^2 r_3^2 + q^4 r_2^2 - 2p^3 qr_1 r_4 \cos \delta_{41} - 2pq r_3 r_2 \cos \delta_{32} + 2p^3 qr_1 r_3 \cos \delta_{31} - 2p^2 q^2 r_1 r_2 \cos \delta_{21} - 2p^2 q^2 r_4 r_3 \cos \delta_{43} + 2pq^3 r_4 r_2 \cos \delta_{42} \right]^{1/2}.
\end{equation}
Note that the \( R \) and \( S \) in Eq. (A9) and (A10) are valid only for \( p > p_c \) since the fixed points solutions are available only for \( p > p_c \). Eqs. (A9) and (A10) show a good agreement with the numerical simulation data, as seen in Fig. 6.

Appendix B: Wave Speed

We derive the wave speed stated in Eq. (45). We have \( W(t) = S(t)e^{i\Delta(t)} \) where \( W, S, \Delta \in \mathbb{R} \), and
\begin{equation}
\dot{W} = (\dot{S} + iS\dot{\Delta})e^{i\Delta},
\end{equation}
which implies \( \dot{\Delta} = -\frac{i}{S}(\dot{W}e^{-i\Delta} - \dot{S}). \) However, since \( \Delta \in \mathbb{R} \) we require \( \dot{\Delta} \in \mathbb{R} \), and we therefore define
\begin{equation}
\Omega := |\dot{\Delta}| = \frac{1}{S}|\dot{W}e^{-i\Delta} - \dot{S}|.
\end{equation}
Observing the identity,
\begin{equation}
\dot{W}e^{i\Delta} + \dot{W}e^{-i\Delta} = 2\dot{S},
\end{equation}
obtained by substituting (B1), and \( |z|^2 = z\bar{z}, z, \bar{z} \in \mathbb{C} \), we obtain:
\begin{equation}
|\dot{W}e^{-i\Delta} - \dot{S}|^2 = |\dot{W}|^2 - \dot{S}(|\dot{W}e^{i\Delta} + \dot{W}e^{-i\Delta}) + \dot{S}^2
\end{equation}
\begin{equation}
= \sqrt{|\dot{W}|^2 - \dot{S}^2}.
\end{equation}
The wave speed is therefore
\begin{equation}
|\dot{\Delta}| = \frac{1}{S}\sqrt{|\dot{W}|^2 - \dot{S}^2}.
\end{equation}

Appendix C: Self-consistency argument for weighted order parameter

The weighted order parameter in (12) allows us to re-cast the continuous version of the governing equations (11) into the following form,
\begin{equation}
\dot{\phi} = \omega - S \sin(\phi - \Delta).
\end{equation}
We expect that a phase-locked solution with $\dot{\phi} = 0$ with constant order parameter $W$ may exist when the effective coupling $S$ is sufficiently large to overcome the spread of the natural frequencies (i.e., when $S > |\omega|$). Accordingly, the locked phases are given by

$$\phi = \Delta + \sin^{-1}(\omega/S).$$  \hspace{1cm} (C2)

In the continuum limit of $N \to \infty$, the weighted order parameter (16) is expressed as follows:

$$W = Se^\Delta = \langle \xi \rangle \int e^{i(\Delta + \sin^{-1}(\omega/S))} g(\omega) d\omega,$$

$$= e^{i\Delta \langle \xi \rangle} \left[ \int_{-S}^{S} \sqrt{1 - (\omega/S)^2} g(\omega) d\omega \right].$$  \hspace{1cm} (C3)

where the stationary probability distribution function given by $p_\omega(\phi, \omega, \xi) = \delta(\phi - (\Delta + \sin^{-1}(\omega/S)))$, and $\langle \xi \rangle = \int \xi \Gamma(\xi) d\xi = 2p - 1$ is the mean value of the distribution in $\xi$. Since oscillators assume either of two values for both coupling strengths and frequencies, we assume there is no contribution to the integral from drifting oscillators. Carrying out the integral, we find that $S$ is implicitly given by the self-consistency equation

$$S = (2p - 1)[p\sqrt{1 - (q\gamma/S)^2} + q\sqrt{1 - (p\gamma/S)^2}].$$  \hspace{1cm} (C4)

We note that $S$ in Eq. (C4) must satisfy the conditions $S \geq q\gamma$ and $S \geq p\gamma$ in order to be real-valued; in other words, the interval of $p$ is restricted for which the self-consistency equation produces real values for $S$.

Substituting $q = 1 - p$, Eq. (C4) can find an exact solution for $S$ using algebraic manipulation software given by

$$S = \sqrt{8p^4 - 16p^3 + 14p^2 + 2p(4p^2 - 4p + 1)},$$  \hspace{1cm} (C5)

where $A := p^2(2p^2 - 3p + 1)^2(-\gamma^2 + 4p^2 - 4p + 1)$. This result is numerically confirmed using numerical simulations (Fig. 6).

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