Continuous Linear Operators On Infinite Quasi-Sobolev Spaces $\ell^m_\infty$

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Abstract. In this study, the concept of infinite quasi-Sobolev spaces $\ell^m_\infty$, where $m \in \mathbb{R}$, is considered. These spaces have been proved as quasi-Banach spaces, as well as Banach spaces, while they neither Hilbert spaces nor quasi-Hilbert spaces. Some kinds of linear operators such as continuous, bounded, closed and completely continuous for operators which map $\ell^m_\infty$ or $\ell^m_1$ into $\ell^m_\infty$ are discussed.

Key Words: Quasi-Banach space, quasi-Hilbert space, quasi-Sobolev space, closed operator, completely continuous operator

1. Introduction and Preliminaries

Quasi-normed space $(U, \| \cdot \|)$ or simply $U$ is a real vector space with a quasi-norm $\| \cdot \|$, which is a positive definite, absolutely homogeneous functional such that there is a constant $K \in [1, \infty)$, $\| u + v \| \leq K (\| u \| + \| v \|)$ for all $u, v \in U$.

Also, a function $\| \cdot \|$ be a norm $\| \cdot \|$ if $K = 1$, thus it is a generalization of a norm function.

Definitely, a quasi-normed space $U$ is metrizable, thus the concept of completeness is correct, and it is called a quasi-Banach space [1, 2].

Related to a quasi-normed space $U$, is an inner product space if and only if the following equation is hold:

$$\| v + w \|^2 + \| v - w \|^2 = 2 \| v \|^2 + 2 \| w \|^2, \quad \forall v, w, \in V$$  (1)

If the equality:

$$\| v + w \|^2 - \| v - w \|^2 = 8 \| \tau (v, w) \|^2 + \| \tau (w, v) \|^2, \quad \forall v, w, \in U$$  (2)

is satisfied, then $U$ is said to be a quasi-inner product, where $\tau (v, w)$ and $\tau (w, v)$ are Gateaux derivatives, A Gateaux derivative of $\| \cdot \|$, $\tau (v, w)$ at $v \in U$ in the direction $w \in U$ is defined as:

$$\tau (v, w) = \frac{\| v \|^2}{2} \left( \lim_{h \to +0} \frac{\| v + hw \|^2 - \| v \|^2}{h} + \lim_{h \to -0} \frac{\| v + hw \|^2 - \| v \|^2}{h} \right)$$  (3)
where $h \in \mathbb{R}$. Similarly, $\tau(w, v)$ at $w \in U$ in the direction $v$ is defined. If $U$ is a quasi-Banach space, then it is called a quasi-Hilbert space [3].

In [4], a sequence $\{2^{k\sigma}\}, k \in \mathbb{N}$ used to define a space $\ell_p^\sigma(A)$, where $A$ is a set of sequences, $1 \leq p < \infty, \sigma$ is a real number.

In [5,6], we have been used a set of all monotonically increasing eigen values $\{\lambda_k\} \subset \mathbb{R}^+$ such that $\lim_{K \to \infty} \lambda_k = +\infty$, of an operator which was defined on Sobolev spaces to construct quasi-Sobolev spaces $\ell_p^m$, where $0 < p < \infty$ and $m \in \mathbb{R}$ which are defined as:

$$\ell_p^m = \{ v = \{v_k\} : \sum_{k=1}^{\infty} \lambda_k^m |v_k|^p < +\infty \}.$$  

Also, a sequence $\{\lambda_k\}$ was used to define some types of operators on these spaces.

In this study, we devote transference above ideology using $\{\lambda_k\}$ to construct sequence space $\ell_\infty^m$ and to define continuous operators on these spaces. For every $m \in \mathbb{R}$, $\ell_\infty^m$ is called an infinite quasi-Sobolev space and is defined as:

$$\ell_\infty^m = \{ v = \{v_k\} : \sup_k \lambda_k^m |v_k| < +\infty \}.$$  

When $m = 0$ then $\ell_\infty^0 = \ell_p^0$, $0 < p \leq \infty$.

**Theorem 1.1** [5, 6]. Sequence spaces $\ell_p^m$, $0 < p < \infty$, are quasi-Banach spaces, and they are Banach spaces only when $1 \leq p < \infty$.

**Remark 1.2** [3]. Not all spaces $\ell_p^m$, where $0 < p < \infty$, are quasi-Hilbert spaces, such as, $\ell_1^m$, $\ell_2^m$, while $\ell_4^m$ be a quasi-Hilbert space, where a functional $\tau(v, w)$ in $\ell_p^m$ defines as:

$$\tau(v, w) = \sum_{k=1}^{\infty} \lambda_k^{mp} |v_k|^p (\text{sng } v_k) w_k, \forall v \in \ell_p^m / \{0\},$$

$$\text{sng } v_k = \begin{cases} 1, & v_k > 0 \\ 0, & v_k = 0 \\ -1, & v_k < 0 \end{cases}. $$

A linear operator $T: U \to V$ where $U$ and $V$ are normed spaces. If domain of $T$, $D(T) = U$, then $T$ is continuous if $\lim_{K \to \infty} \lim_{K' \to \infty} \| Tu_k - Tu \| = 0$ whenever a sequence $\{u_k\}$ converges to $u$ in $U$; bounded if there exists a constant $M > 0$, $\| Tu \| \leq M \cdot \| u \|$; and completely continuous if for every bounded sequence $\{u_k\}$ in $U$, $\| Tu_k \|$ has a convergent subsequence in $V$. A linear operator $T$ is closed if for every sequence $\{u_k\}$ in $D(T)$ converges to $u$ and $\| Tu_k \|$ converges to $y$ it holds $u$ in $D(T)$ and $Tu = y$ [7, 8].

**Lemma 1.3**[7]. Let $U$ and $V$ are normed spaces.

1. Any subset of a Banach space is closed if and only if it is complete.

2. A linear operator $T: U \to V$ is continuous if and only if it is bounded.

**Theorem 1.4** [7]. Let $U$ and $V$ are Banach spaces. A linear operator defined on $U$ into $V$ is continuous if and only if it is closed.
In the second section of this work, a proof of an infinite quasi-Sobolev spaces $\ell_\infty^m$, for every $m \in \mathbb{R}$ as quasi-Banach space is confirmed, while it is not a quasi-Hilbert space and a relationship between $\ell_1^m$ and $\ell_\infty^m$ is presented, while in the third section, a linear operator which is defined on $\ell_\infty^m$ or $\ell_1^m$, is proved as continuous, imply it is closed. Also, continuity of an operator is insufficient to be completely continuous operator.

2. An Infinite Quasi- Sobolev Spaces

In this part, we review Banach, Hilbert, quasi-Banach and quasi-Hilbert space for a sequence space $\ell_\infty^m$, with given the relationship between $\ell_1^m$ and $\ell_\infty^m$.

**Theorem 2.1.** For every $m \in \mathbb{R}$, an infinite quasi-Sobolev space $\ell_\infty^m$ is a quasi-Banach space with a function $\|u\|_q = \sup_k \lambda_k^m |u_k|$.

**Proof.**
A positive definite property and an absolute homogeneous property are obvious. Since,

$$\|u + v\|_q \leq \sup_k \lambda_k^m |u_k + v_k| \leq \sup_k \lambda_k^m |u_k| + \sup_k \lambda_k^m |v_k|$$

where $K=1$, then the triangle inequality is satisfied.

To prove the completeness of $\ell_\infty^m$. Consider $\{u_r\}, r \in \mathbb{N}$ is a fundamental sequence in $\ell_\infty^m$. Then, its coordinate sequences $u_k^r, k \in \mathbb{N}$ are fundamental sequences, and converge to $u_k^e$ according to completeness of $\mathbb{R}$. Let $u = \{u_r\}$ we have $\|u - u\|_q \rightarrow 0$. This implies that $\{u_r\} \rightarrow u$, and this is the desired result.

**Theorem 2.2.** Let $U$ is a quasi-inner product space, then:

1. it is an inner product space if and only if the equation (1) holds.
2. it is an inner-product space if and only if the following equivalence holds:
   $$q\|v + w\| = q\|v - w\| \leftrightarrow \tau(v, w) = 0, \forall u, v \in U$$

**Proof.**
proof (1) is coming from Theorem 1.7 and Proposition 2.8 in [3], where a quasi-norm function : $q\|\cdot\|^2 = <v, v>, \forall v \in U$.

proof (2) is very technical and proceeds in a same way into version in a normed space[9].

**Remark 2.3.** Since $\ell_2^m$ is a Hilbert space, for every $m \in \mathbb{R}$ [3], then the equation (2) and the equivalence (4) are satisfied, where $\tau(v, w) = <v, w>$, $\forall v, w \in \ell_2^m$. But, $\ell_4^m$ is not a Hilbert space, indeed,

suppose $u, v\in\ell_4^1$, where $v = \{v_k\} = \{1,1,1,0, \ldots \}$, $w = \{w_k\} = \{1,1,-1,0, \ldots \}$, and $\{\lambda_k\} = \{k\}, k \in \mathbb{N}$. Since $\tau(v, w) = 0$, and $1.48 \equiv q\|v + w\| 
eq q\|v - w\| \equiv 1.36$, then the equivalence (4) is not satisfied.

**Remark 2.4.** An infinite quasi-Sobolev space $\ell_\infty^m$, $m \in \mathbb{R}$ neither Hilbert space nor quasi-Hilbert space, since equations (1) and (2) are not satisfied, as shown in the following example:
Example 2.5. Suppose \( \{ v_k \}, \{ w_k \} \in \ell^0_{\infty} \), where \( \{ v_k \} = \left\{ \frac{1}{2^k}, \frac{3}{2^k}, \ldots, 1 - \frac{1}{2^k}, \ldots \right\} \), \( \{ w_k \} = \left\{ 0, \frac{1}{2^k}, \frac{2}{2^k}, \ldots, 1 - \frac{1}{2^{k-1}}, \ldots \right\} \), then from the equation (3), it is easy to show that \( \tau(v, w) = \tau(w, v) = 1 \). Thus, the right hand of the equation (2) equals 16, while the left hand equals 15.9961, where \( q \| v + w \| = 2 \) and \( q \| v - w \| = 0.25 \). Therefore, \( \ell^\infty \) is not a quasi-inner product space. Also, it is not an inner-product space, since the left hand of equation (1) equals 4.0625, while the right hand of it equals 4.

Theorem 2.6. For every \( m \in \mathbb{R} \), \( \ell^m_1 \subset \ell^m_{\infty} \) and \( q \| u \| \leq q \| u \| \leq \| u \| \).

Proof. Let \( u = \{ u_k \} \in \ell^m_1 \), then \( \sum_{k=1}^{\infty} \lambda_k^m \| u_k \| < +\infty \). Since, \( \forall k \in \mathbb{N} \) we have: \( \lambda_k^m \| u_k \| < \sum_{k=1}^{\infty} \lambda_k^m \| u_k \| , m \in \mathbb{R} \). This implies that \( \sup_k \lambda_k^m \| u_k \| < +\infty \), hence \( \{ u_k \} \in \ell^m_{\infty} \), so \( \ell^m_1 \subset \ell^m_{\infty} \).

And, \( q \| u \| \leq \sup_k \lambda_k^m \| u_k \| = \sup_k \lambda_k^m \| u_k \| \leq \| u \| \).

3. Continuous Linear Operators

In this section, we use equivalence of boundedness and continuity for linear operators on Banach spaces. Let \( \{ \lambda_k \} \subset \mathbb{R}^+ \) is a monotonically increasing sequence such that \( \lim_{K \rightarrow \infty} \lambda_k = +\infty \).

Theorem 3.1. An operator \( T : \ell^m_{\infty} \rightarrow \ell^m_{\infty} \), \( m \in \mathbb{R} \) such that \( Tu = \lambda_k^{-1} u_k \), \( k \in \mathbb{N} \) is a continuous linear operator.

Proof. It is clear that \( T \) is well-defined, since to each element in domain of \( T \), we have an unique element in codomain of \( T \). Obviously, Linearity of \( T \), since \( T(\alpha u + \beta w) = \alpha \lambda_k^{-1} u_k + \beta \lambda_k^{-1} w_k = \alpha Tu + \beta Tw, \forall \alpha, \beta \in \mathbb{R} \) and \( \forall u, v \in \ell^m_{\infty} \).

Now, let a sequence \( u = \{ u_k \} \) converges to \( u^* = \{ u_k^* \} \) in \( \ell^m_{\infty} \), that is, \( \lim_{k \rightarrow \infty} \| u - u^* \| = \lim_{k \rightarrow \infty} \sup \lambda_k^m \| u_k - u_k^* \| = 0 \) then we have

\[ \| Tu - Tu^* \| = \sup_k \lambda_k^m \| \lambda_k^{-1} u_k - \lambda_k^{-1} u_k^* \| = \lambda_k^{-1} \| u_k - u_k^* \| \leq \sup_k \lambda_k^{-1} \| u - u^* \| , \forall k \in \mathbb{N}. \]

Putting \( \gamma = \sup_k \lambda_k^{-1} \) then \( \| Tu - Tu^* \| \leq \gamma \| u - u^* \| . \) Taking limit to the sides, we get \( \lim_{k \rightarrow \infty} \| Tu - Tu^* \| = 0 \). Since \( u \) is an arbitrary element in \( \ell^m_{\infty} \) then \( T \) is continuous.

Corollary 3.2. A linear operator \( T \) which maps \( \ell^m_{\infty} \) into \( \ell^m_{\infty} \) such that \( Tu = \lambda_k^{-1} u_k \) is closed.

Proof of this corollary comes from Theorem 1.4 and Theorem 3.1.
**Theorem 3.3.** A continuous linear operator $T: \ell_1^m \to \ell_\infty^m$ such that $Tu = \lambda_k^{-1}u_k$, is completely continuous.

**Proof.**

Let $G$ be a closed subset in $\ell_\infty^m$ and $\{u_k\}$ be any bounded sequence in $G$, then $\{u_k\}$ has $\{u_{k_s}\}$ as a convergent subsequence in $\ell_\infty^m$. Since $G$ is complete by Lemma 1.3, then $\{u_k\}$ converges to an element $u^* = \{u_k^*\}$ in $G$. Thus, $\{u_{k_s}\}$ converges to $u^*$ in $G$, that is, $\lim_{k_s \to \infty} \| u_{k_s} - u^* \| = 0$.

Now, since $T$ is continuous, so

$$\lim_{k_s \to \infty} \| Tu_{k_s} - Tu^* \| = \lim_{k \to \infty} \| \lambda_k^{-1}u_{k_s} - \lambda_k u_k^* \| = 0 \text{ as } k_s \to \infty,$$

that is, $Tu_{k_s} \to Tu^*$. Thus, $\{Tu_{k_s}\}$ contains a subsequence converges to $Tu^*$. Hence, $T$ is completely continuous.

**Theorem 3.4.** A linear operator $T: \ell_1^1 \to \ell_\infty^m$, $Tu = \lambda_k^{-1}u_k$, $\forall u = \{u_k\} \in \ell_1^1$, $m \in \mathbb{R}$ is completely continuous.

**Proof.**

Since, $\forall k \in \mathbb{N}, \forall u = \{u_k\} \in \ell_1^1$ we have $\lambda_k^{-1}|u_k| < \sum \lambda_k^{-1}|u_k|$, then that

$$\| Tu \|_{\ell_\infty^m} = \sup_k \lambda_k^{-1}|u_k| \leq \sup_k \lambda_k^{-1} \sum \lambda_k^{-1}|u_k| \leq M. \| u \|_{\ell_1^m}, \text{ where } M > 0.$$

$T$ is bounded, and it is continuous.

Similarly to Theorem 3.3, an operator $T$ is completely continuous, where $\{Tu_k\}$ contains a subsequence converges in $\ell_\infty^m \forall \{u_k\} \in \ell_1^m$, $m \in \mathbb{R}$.

**Remark 3.5.** Every completely continuous operator is continuous, since discontinuity of an operator $T$ which is defined on $\ell_1^1$ or $\ell_\infty^m$ into $\ell_\infty^m$ would imply existence a sequence $\{u_k\}$ such that $\|u_k\| \leq 1$ and $\| Tu_k \| \to \infty$ and this implies that $T$ is not completely continuous. Conversely, may be not true, as shown in the following example:

**Example 3.6.** Consider a linear operator $T: \ell_\infty^1 \to \ell_\infty^1$, $Tu = u$. Suppose $\{u_k\}$ such that $u_k = 2e_k$ is any bounded sequence in $\ell_\infty$ where $\{e_k\}$ is a sequence of all zeroes except in the $k$-th spot where there appears 1 and $\|e_k\| = 1$, then, for any $k \neq r$,

$$\| Tu_k - Tu_r \| = 2\| e_k - e_r \| = 2.$$

Hence, $T$ is continuous, and also it is closed. But this operator is not completely continuous, because, any subsequence of $\{Tu_k\}$ is not converge.

4. Acknowledgements

The author would like to express thanks to College of Science-Musyansiriyah University for supporting this work. Special thanks to unknown referees for their careful reading and helpful comments.
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