A generalization of the Gram-Schmidt Algorithm

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Abstract

In this article we discuss the special set of unit length vectors \{s_1, s_2, \ldots, s_n\} in any \(n\)-dimensional real vector space with the property that the inner product of any two distinct of them are identical. In other words these vectors are (mutually) equiangular. We will present a method that generates a set of equiangular vectors (EVs) with the prescribed acute angle between them (\(\theta\)) via the combinations of given linearly independent vectors \{v_1, \ldots, v_n\}. In general equiangular vectors \(s_i\) can establish a matrix columnwise which we define it equiangular matrix (EM). We derive some properties and results from these matrices. Moreover, there are some new types of matrix factorizations that can be described based on equiangular matrices.

Keywords: Equiangular matrix, Equiangular vectors, Gram matrix, Equiangular tight frame.

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1. Introduction

The Gram-Schmidt (GS) procedure takes a set of linearly independent vectors \{v_1, \ldots, v_n\} of any inner product space, and then generates a set of orthogonal vectors \{q_1, \ldots, q_n\} which are constructed successively one-by-one so that they span the same subspace [2, 15, 17]. GS Algorithm is used in several application areas, such as theory of least squares and solving eigenvalue problems [3, 9]. At the step \(k + 1\) of the Classical GS Algorithm, the normalized vector \(q_{k+1}\) is obtained as follows.

\[
q_{k+1} = \frac{v_{k+1} - \sum_{i=1}^{k} \text{proj}_{q_i} v_{k+1}}{\|v_{k+1} - \sum_{i=1}^{k} \text{proj}_{q_i} v_{k+1}\|_2}.
\]

where \(\text{proj}_{q_i} v_{k+1} = \langle v_{k+1}, q_i \rangle q_i\) denotes the projection of \(v_{k+1}\) onto the previously computed \(q_i\). So \(Q = [q_1, \ldots, q_n]\) is an orthogonal (or unitary) matrix. Now suppose \(S = [s_1, \ldots, s_n]\) is a real matrix with unit norm columns in such a way that the angle between any pair of column vectors is \(\theta\). From the definition of standard inner product, and the law of cosines, the radian measure of the angle between nonzero vectors \(v, w\) in any inner product space is denoted by the scalar \(\theta \in [0, \pi]\) so that [13]

\[
\cos \theta = \frac{\langle v, w \rangle}{\|v\|_2 \|w\|_2}.
\]

For simplicity, assume that \(\|s_i\|_2 = 1\) for \(i = 1, \ldots, n\). Hence for \(i \neq j\), \(s_i^T s_j = [S^T S]_{ij} = \cos \theta\), where \(\cos \theta \in [-1, 1]\). We shall propose a method of producing a set of equiangular vectors \{s_1, s_2, \ldots, s_m\} with the ‘acute’ angle \(\theta\) via a set of linearly independent vectors \{v_1, v_2, \ldots, v_m\} of any inner product space, successively one-by-one, so that they span the same subspace. Any EVs \(s_i\) are the natural geometric generalization of an orthonormal basis. In this paper we deal with subspaces of \(\mathbb{R}^n\) with the standard inner product and Euclidean norm.

This paper is organized as follows. In section 2, using the standard orthogonal basis \{e_1, e_2, \ldots, e_n\}, we construct a standard equiangular basis \{s_1, s_2, \ldots, s_m\} with the angle \(\theta\). Next we focus on constructing equiangular vectors of the 3-dimensional case. In section 3, we generalize the approach of making the equiangular vectors of the general \(n\)-dimensional case. Then we summarize this method in an algorithm. Finally, the rational outcome

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of this method which is the factorization of a matrix $A$ that has full-rank, as a multiplication of an equiangular matrix $S$ by an upper triangular $R$. In section 4, we obtain the inverse of equiangular matrices with the eigenvalues of an equiangular matrix. In section 5, we study and develop new types of matrix factorization such as $SRS^{-1}$ and $SDST^T$ which are related to square and symmetric matrices, respectively. Accordingly, some comparative and particular results will be attainable. Finally, in section 6, a set of $n + 1$ equiangular vectors in $\mathbb{R}^n$ can be regarded as a sequence of equiangular tight frame (ETF).

Throughout this paper, $\| X \|$ denotes the 2-norm of matrix $X$; $\| x \|$ denotes the Euclidean norm of vector $x$. Moreover, All matrices in this paper are real. For simplicity, we denote the transpose of the inverse of $A$ as $A^{-T}$. Without loss of orthogonality we can let

$$v_2 = q_2 + \cot \theta s_1,$$

where $q_2 = \|v\|^{-1}v$ is orthogonal to $s_1$. The next unit norm equiangular vector which belongs to span$\{v_1, v_2\}$ is $s_2 = \frac{v_2}{\|v_2\|}$. For the non-acute angles between $v_1$ and $v_2$ we can proceed the same manner with slightly difference.

Note that any set of equiangular vectors $\{s_1, \ldots, s_n\}$ with the acute angle $\theta$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is linearly independent. Suppose $x_1 s_1 + \ldots + x_n s_n = 0$, for arbitrary scalars $x_i$. Premultiplying by $s_1^T$ gives a system of linear equations $Ax = 0$, where $x = [x_1, x_2, \ldots, x_n]^T$ and $A$ has ones on the main diagonal and $\cos \theta$ on all off-diagonal. Simply we can check det$(A) \neq 0$, because $\cos \theta \neq 1$, hence $x = 0$. For this reason, we can present a well-ordering equiangular basis in $\mathbb{R}^n$ analogous to the standard orthogonal basis $e_1, e_2, \ldots, e_n$. Consider

$$s_i = \frac{x e + (1 - x)e_i}{\sqrt{1 + (n - 1)x^2}} \quad i = 1, \ldots, n, \quad x \in (0, 1),$$

where $e$ is the vector of ones. It is easy to see that the vectors $s_i$ are EVs. Let $s_j^T s_j = \frac{2x + (n - 2)x^2}{1 + (n - 1)x^2} = \alpha$, then $x = \frac{\sqrt{1 - \alpha}}{1 - \alpha}$, so each $s_i$ depends on $\alpha = \cos \theta$, where $\theta \in (0, \frac{\pi}{2})$ is the angle between any two distinct vectors of them Figure 2. The angle between the $s_i$’s in (2.2) depends on $x$, i.e., $s_i \rightarrow e_i$ as $x \rightarrow 0$ and correspondingly $s_i \rightarrow \frac{e_i}{\sqrt{n}}$ as $x \rightarrow 1$, so when $x$ varies in the interval $(0, 1)$, the angle between the equiangular vectors $s_i$ will vary in the interval $(0, \frac{\pi}{2})$. The next two lemmas are need to follow up the discussion on the third equiangular vector.

**Lemma 2.1.** Suppose that $s_1, s_2, \ldots, s_k$ are equiangular vectors with the angle $\theta \in (0, \frac{\pi}{2})$. Then $\| \sum_{i=1}^{k} s_i \| = \sqrt{k (1 + (k - 1)\alpha)}$, where $\alpha = \cos \theta$. 

![Figure 1: Vector $\hat{v}_2$ makes the angle $\theta$ with $s_1$.](image)
Proof. Without loss of generality assume that each $s_i$ has n-1 entries, say $t > 0$ and the last one $s = \sqrt{1 - (n - 1)t^2}$, $t \in (0, \frac{1}{n})$. So an alternative form of $k$ vectors $s_i$ in addition to (2.2) is demonstrated as follows.

Let $s_i = [t, \ldots, t, t \text{ times}^i, \ldots, t]^T, \quad i = 1, \ldots, k; \quad k = 2, \ldots, n.$

Therefore $\sum_{i=1}^{k} s_i = [\sqrt{1 - (n - 1)t^2} + (k - 1)t][1, \ldots, 1, 0, \ldots, 0]^T + k t [0, \ldots, 0, 1, \ldots, 1]^T$, so

\[
\| \sum_{i=1}^{k} s_i \| = k \| \sqrt{1 - (n - 1)t^2} + (k - 1)t \|^2 + (n - k)^2 t^2 \\
= 2k(k - 1)t \sqrt{1 - (n - 1)t^2} + k(k - 1)^2 t^2 - k(n - 1)t^2 + k + nk^2 t^2 - k^3 t^2 \\
= 2k(k - 1)t \sqrt{1 - (n - 1)t^2} + k(k - 1)(n - 2)t^2 + k .
\]

Since $\alpha = s_j^T s_j = 2t \sqrt{1 - (n - 1)t^2} + (n - 2)t^2$, then $\| \sum_{i=1}^{k} s_i \|^2 = k(k - 1)\alpha + k = k(1 + (k - 1)\alpha)$, and $\| \sum_{i=1}^{k} s_i \| = \sqrt{k(1 + (k - 1)\alpha)}$.

The next problem is about the relationship between two special angles $\varphi$ and $\eta$, where the first one is between the vectors $\sum_{i=1}^{k} s_i$ and $s_{k+1}$, and the next one is between the vectors $\sum_{i=1}^{k} s_i$ and $s_j$, $(1 \leq j \leq k < n)$. An interesting triangular relationship will be attainable as follows.

\[
\cos \varphi = \frac{\langle s_{k+1}, \sum_{i=1}^{k} s_i \rangle}{\| s_{k+1} \| \cdot \| \sum_{i=1}^{k} s_i \|} = \frac{k\alpha}{\sqrt{k(1 + (k - 1)\alpha)}} ,
\]

\[
\cos \eta = \frac{\langle s_j, \sum_{i=1}^{k} s_i \rangle}{\| s_j \| \cdot \| \sum_{i=1}^{k} s_i \|} = \frac{1 + (k - 1)\alpha}{\sqrt{k(1 + (k - 1)\alpha)}} .
\]

Multiplying the above equalities together implies that

\[
\cos \varphi \cdot \cos \eta = \cos \theta = \alpha . \tag{2.3}
\]

Indeed $\varphi$, $\eta$ and $\theta$ are the angles of a trihedral angle, in such a way that two faces corresponding to $\varphi$ and $\eta$ are perpendicular to each other. Actually it is the first law of cosine in the spherical pyramid which we prove it: [18].

**Lemma 2.2.** Let $a$, $b$ and $c$ be three independent vectors in $\mathbb{R}^n$ $(n \geq 3)$ which make a trihedral angle with the angles $\varphi$, $\eta$ and $\theta$ as illustrated in Figure 3 in such a way that the dihedral angle between two faces opposite to $\theta$ is $\pi/2$. Therefore

\[
\cos \theta = \cos \varphi \cdot \cos \eta .
\]
cos \theta = \frac{(a,c)}{\|a\| \cdot \|c\|} = \frac{(\text{proj}_a a + (a - \text{proj}_a a), \text{proj}_a c + (c - \text{proj}_a c))}{\|a\| \cdot \|c\|}
\quad = \frac{(\text{proj}_a a) \cdot (\text{proj}_a c)}{\|a\| \cdot \|c\|} = \cos \varphi \cdot \cos \eta.

Proof.

\begin{align*}
\cos \varphi &= \frac{(a, c)}{\|a\| \cdot \|c\|} = \frac{(\text{proj}_a a + (a - \text{proj}_a a), \text{proj}_a c + (c - \text{proj}_a c))}{\|a\| \cdot \|c\|} \\
&= \frac{(\text{proj}_a a) \cdot (\text{proj}_a c)}{\|a\| \cdot \|c\|} = \cos \varphi \cdot \cos \eta.
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Three vectors $a$, $b$ and $c$ form a pyramid, so that two faces which are relevant to $\varphi$ and $\eta$ are perpendicular.}
\end{figure}

The next step is on transforming the third vector $v_3$ to an equiangular vector with $s_1$ and $s_2$. This leads to define the special subspace $P_2 \subset \text{span}\{v_1, v_2, v_3\}$ so that any vector $p$ of $P_2$ has the same angle with $s_1$ and $s_2$, $(\theta_p)$. Indeed $P_2$ and the 2-dimensional subspace of $s_1$ and $s_2$ are perpendicular and the intersection of them is a line spanned by $s_1 + s_2$. Note that $(p, s_1) = (p, s_2)$ so $(p, s_1 - s_2) = 0$; therefore, $s_1 - s_2$ is the orthogonal complement of $P_2$, then dim($P_2$) = 2. Let $v_3$ be the projection of $v_2$ onto $P_2$, thus it makes the same angle with $s_1$ and $s_2$. We have

$$\hat{v}_3 = \text{proj}_{P_2} v_3 = v_3 - \text{proj}_{s_1 - s_2} v_3$$

(2.4)

The vector which makes the angle $\theta$ with both $s_1$ and $s_2$ is denoted by $\hat{v}_3$. Definitely $\hat{v}_3 \in P_2$. The angle between $\hat{v}_3$ and $s_1 + s_2$ is denoted by $\varphi$ (see lemma 2.2.), that is less than $\theta$, $(\varphi < \theta)$. According to the definition of $\eta$ clearly $\eta = \frac{\theta}{2}$. From (2.3), $\varphi$ is attainable with respect to $\theta$. The purpose is to evaluate $\hat{v}_3$ via $v_3$ without falling out of $P_2$. In contrast to the previous step, $\hat{v}_3$ and $s_1 + s_2$ can be regarded as $v_2$ and $s_1$ in (2.1), respectively, so they can be substituted as well. Moreover, cot $\theta$ is replaced by cot $\varphi$ as follows.

$$\frac{\cos \varphi}{\cos \eta} = \frac{\cos \varphi}{\cos \frac{\theta}{2}}, \quad \sin \varphi = \frac{\sqrt{\cos^2 \frac{\theta}{2} - \cos^2 \theta}}{\cos \frac{\theta}{2}}, \quad \text{then} \quad \cot \varphi = \frac{\sqrt{2} \cos \theta}{\sqrt{1 + \cos \theta - 2 \cos^2 \theta}}.$$

Let $v = \hat{v}_3 - (\hat{v}_3, s_1 + s_2) \frac{s_1 + s_2}{\|s_1 + s_2\|^2}$, from (2.4)

\begin{align*}
v &= v_3 - \text{proj}_{s_1 - s_2} v_3 - (v_3 - \text{proj}_{s_1 - s_2} v_3, s_1 + s_2) \frac{s_1 + s_2}{\|s_1 + s_2\|^2} \\
&= v_3 - \text{proj}_{s_1 - s_2} v_3 - (v_3, s_1 + s_2) \frac{s_1 + s_2}{\|s_1 + s_2\|^2} \\
&= v_3 - (\text{proj}_{s_1 - s_2} v_3 + \text{proj}_{s_1 + s_2} v_3).
\end{align*}

Note that $s_1 - s_2 \perp s_1 + s_2$ so $v$ is orthogonal to $s_1, s_2$, because span$\{s_1 - s_2, s_1 + s_2\} =$span$\{s_1, s_2\}$. We have

$$\hat{v}_3 = v + \cot \varphi \frac{\|s_1 + s_2\|}{\|s_1 + s_2\|^2},$$

Without loss of generality we can take

$$\hat{v}_3 = q_3 + \cot \varphi \frac{s_1 + s_2}{\|s_1 + s_2\|^2},$$

(2.5)

where $q_3 = \|v\|^{-1}v$ is orthogonal to $s_1, s_2$. By normalizing the next unit norm equiangular vector to $s_1, s_2$ which belongs to span$\{v_1, v_2, v_3\}$ is $s_3 = \frac{\hat{v}_3}{\|\hat{v}_3\|}$. 

\[4\]
3. The generation of more than three equiangular vectors

Providing a general procedure to make more than three equiangular vectors gradually, is analogous to the three-dimensional case. After \( k \) steps the vectors \( s_1, \ldots, s_k \) have been produced \((k \geq 3)\). Define subspace \( P_k = \bigcap P_{ij} \subset \text{span}\{v_1, \ldots, v_{k-1}\} \), where \( P_{ij} = \{p \mid \langle k \rangle = s_i - s_j\} \), for \( 1 \leq i, j \leq k \). The vector \( v_{k+1} \) must be projected onto the subspace \( P_k \) to get \( \bar{v}_{k+1} \), thereby the vector \( s_{k+1} \) which makes the angle \( \theta \) with the previous \( s_i \)'s will be obtained via \( \bar{v}_{k+1} \). For simplicity, we can project \( v_{k+1} \) onto the orthogonal subspace of \( P_k \) \((W_k)\) which includes \( \text{span}\{s_i - s_j\}_{i,j=1}^k \), then using (2.4) in case of \( k + 1 \)-dimensional, implies

\[
v_{k+1} = \text{proj}_{\{v_1, \ldots, v_{k+1}\}} v_{k+1} = \text{proj}_{W_k} v_{k+1} = \text{proj}_{W_k} \bar{v}_{k+1} = \text{proj}_{W_k} v_{k+1} + \bar{v}_{k+1},
\]

and then

\[
\bar{v}_{k+1} = v_{k+1} - \text{proj}_{W_k} v_{k+1} + v_{k+1}.
\]

We must find an orthogonal basis \( B_{W_k} = \{B_i\}_{i=1}^k \) of \( W_k \). Before finding it, we introduce another basis of \( W_k \) which has equiangular vectors: \( A_{W_k} = \{A_i\}_{i=1}^k = \{s_1 - s_k, s_2 - s_k, \ldots, s_{k-1} - s_k\} \). To check the linear independence take \( \sum_{i=1}^{k-1} x_i(s_i - s_k) = 0 \) so \( \sum_{i=1}^{k-1} x_i = \sum_{i=1}^{k-1} x_i(s_i - s_k) = 0 \). Since \( s_1, \ldots, s_k \) are linearly independent, then \( x = x_1 = \cdots = x_{k-1} = 0 \). It remains to prove \( W_k \subset \text{span}\{A_{W_k}\} \), (because the reverse is trivial).

Let \( w \in W_k \). By the definition of \( W_k \), \( w \) can be written as

\[
w = \sum_{i=1}^k x_i(s_i - s_j) = \sum_{i=1}^k x_i(s_i - s_k) = \sum_{i=1}^{k-1} y_i(s_i - s_k).
\]

So \( w \in \text{span}\{A_{W_k}\} \). It implies \( W_k = \text{span}\{A_{W_k}\} \) and \( \dim(W_k) = k - 1 \), therefore \( \dim(P) = k + 1 - (k - 1) = 2 \) holds for each step.

We now present a simple method that takes a set of EVs such as \( A_{W_k} \) and thereby generates an orthogonal basis \( B_{W_k} = \{B_1, \ldots, B_{k+1} - 1, k\} \) without using the GS procedure. If \( k = 2 \) then \( B_{W_k} = A_{W_k} = \{s_1 - s_2\} \). If \( k = 3 \) then \( A_{W_k} = \{A_{1,3}, A_{2,3}\} = \{s_1 - s_3, s_2 - s_3\} \). Since \( A_{1,3} \) and \( A_{2,3} \) are equiangular by the same norm, then we can take \( B_{1,3} = A_{1,3} - A_{2,3} = s_1 - s_2 \) and \( B_{2,3} = A_{1,3} + A_{2,3} = s_1 + s_2 - 2s_3 \). So \( B_{W_k} = \{s_1 - s_2, s_1 + s_2 - 2s_3\} \).

If \( k = 4 \) so in this case \( A_{W_k} = \{s_1 - s_4, s_2 - s_4, s_3 - s_4\} \). Let \( B_{3,4} = A_{1,4} + A_{2,4} + A_{3,4} = s_1 + s_2 + s_3 - 3s_4 \). Take \( C_{1,4} = A_{1,4} - A_{3,4} \) and \( C_{2,4} = A_{2,4} - A_{3,4} \). We see that both of \( C_{1,4}, C_{2,4} \) are orthogonal to \( B_{3,4} \), but \( C_{1,4} \notin C_{2,4} \). It is enough to choose two special linear combinations of them which are orthogonal, similar to the previous step. So let \( B_{1,4} = C_{1,4} - C_{2,4} = A_{1,4} - A_{2,4} = s_1 - s_2 \) and \( B_{2,4} = C_{1,4} + C_{2,4} = A_{1,4} + A_{2,4} - 2A_{3,4} = s_1 + s_2 - 2s_3 \). So we have \( B_{W_k} = \{s_1 - s_2, s_1 + s_2 - 2s_3, s_1 + s_2 + s_3 - 3s_4\} \).

For \( k > 4 \) \( W_k \) will be obtained recurrently, e.g., if \( k = 5 \) we have \( B_{1,5} = s_1 - s_2, B_{2,5} = s_1 + s_2 - 2s_3, B_{3,5} = s_1 + s_2 + 3s_4, B_{4,5} = s_1 + 2s_2 + 3s_4 - 4s_5 \). The following set is the general form of \( B_{W_k} \) for \( k = 2, \ldots, n - 1 \)

\[
B_{W_k} = \{s_1 - s_2, s_1 + s_2 - 2s_3, \ldots, s_2 - s_{k-1}, s_2 + s_{k-1} - (k - 1)s_{k-1}, s_2 + s_{k-2} - (k - 2)s_{k-2}, \ldots, s_2 - s_1, s_2 + s_1 - (k - 1)s_1\}.
\]

Since \( B_{i,j} = B_{i,k} = \cdots = B_{j,k} \) for \( i = 1, \ldots, k - 1 \), so we let \( B_i = B_{i,j} \), where \( B_i = s_1 + \cdots + s_i - s_{i+1} \). From \( 3.1 \) \( \bar{v}_{k+1} = v_{k+1} - \sum_{j=1}^{k-1} \text{proj}_{B_j} v_{k+1} \). It remains to substitute \( \sum_{j=1}^k s_j \), and \( \sum_{j=1}^{k-1} \text{proj}_{B_j} v_{k+1} \) instead of \( s_1 + s_2 \) and \( \text{proj}_{s_1 - s_2} v_3 \) in (2.5), respectively to evaluate \( \bar{v}_{k+1} \). Moreover by (2.3)

\[
\cos \varphi = \frac{\cos \theta}{\sqrt{1 + (k - 1)(\cos \theta)}} \quad \text{and} \quad \sin \varphi = \frac{\sqrt{1 + (k - 1)(\cos \theta)} - k \cos^2 \theta}{\sqrt{1 + (k - 1)(\cos \theta)}} \quad \text{so}
\]

\[
\cot \varphi = \frac{\sqrt{1 + (k - 1)(\cos \theta)} - k \cos^2 \theta}{\sqrt{1 + (k - 1)(\cos \theta)}}
\]

Let \( v = v_{k+1} - \left( \text{proj}_{\sum_{j=1}^k s_j} v_{k+1} + \sum_{j=1}^{k-1} \text{proj}_{B_j} v_{k+1} \right) \). Analogous to \( 3.2 \) we note that \( \{B_1, \ldots, B_{k-1}, \sum_{j=1}^k s_j\} \) is a set of orthogonal vectors, so \( v \) is orthogonal to \( s_1, \ldots, s_k \), because \( \text{span}\{B_1, \ldots, \sum_{j=1}^k s_j\} = \text{span}\{s_1, \ldots, s_k\} \).

We have \( \bar{v}_{k+1} = v + \cot \varphi ||v|| \sum_{j=1}^k s_j \), Again we can take

\[
\bar{v}_{k+1} = q_{k+1} + \cot \varphi ||s|| \sum_{j=1}^k s_j
\]
where \( q_{k+1} = \|v\|^{-1}v \) is orthogonal to the previous \( s_i \)'s. So the \((k+1)^{th}\) unit norm equiangular vector to \(s_1, \ldots, s_k\) which belongs to \(\text{span}\{v_1, \ldots, v_k\}\) is \(s_{k+1} = \frac{\hat{v}_{k+1}}{\|\hat{v}_{k+1}\|}\).

A simpler formula for \(\cot\varphi\) is \(\cot\varphi = \frac{\sqrt{\kappa \cos \theta}}{\sqrt{\kappa + (k-1) \cos \theta - k \cos \theta}} = \sqrt{\frac{\kappa}{\sec \theta + \kappa}} = \sqrt{\frac{\kappa}{\sec \theta - \kappa}}\). In what follows, the ultimate algorithm is obtained: The projection of \(v_k\) onto \(s\) and \(B_k\) in line 3 can be performed as the modified Gram-Schmidt approach (MGS) to reduce the loss of orthogonality which leads to decreasing the loss of being equiangular in floating point arithmetic.

**Algorithm 2**

1. Set \(s_1 = \frac{v_1}{\|v_1\|}; \hat{s} = 0; r = 0;\)
2. for \(k = 2: n\) do
3. \(\hat{s} = \hat{s} + s_{k-1}; s = \frac{\hat{s}}{\|\hat{s}\|}; q = \|v_k - ((v_k, s) \cdot s + r)\|^{-1} \cdot (v_k - ((v_k, s) \cdot s + r));\)
4. \(\hat{v}_k = q + \left(\frac{\sqrt{k - 1}}{\sqrt{(\sec \theta - 1)(\sec \theta + k - 1)}}\right) \cdot s; s_k = \frac{\hat{v}_k}{\|\hat{v}_k\|}; r = 0;\)
5. \(B_{k-1} = \left(\sum_{j=1}^{k-1} s_j\right) - (k-1)s_k;\)
6. for \(i = 1: k - 1\) do
7. \(r = r + \langle v_k, B_i \rangle \cdot \|B_i\|^2;\)
8. end for
9. end for

as the modified Gram-Schmidt approach (MGS) to reduce the loss of orthogonality which leads to decreasing the loss of being equiangular in floating point arithmetic.

**Loss of orthogonality.** The formula (3.2) reduces to the orthogonalization process if \(\alpha = 0\) (in contrast with GS formula). In this case at the step \(k+1\), we have

\[
\hat{v}_{k+1} = v_{k+1} - \left(\text{proj}_{B_k} v_{k+1} + \ldots + \text{proj}_{B_{k-1}} v_{k+1} + \text{proj}_{B_1} v_{k+1}\right); \quad k = 1, 2, \ldots, n-1,
\]

where \(B_k = \sum_{i=1}^{k} s_i\). The formula (3.3) shows that the unit vector \(s_{k+1}\) is constructed by means of the set of orthogonal vectors \(\{B_1, \ldots, B_{k-1}, B_k\}\) instead of the previous set of orthonormal vectors \(\{s_1, \ldots, s_k\}\), analogous to GS algorithm. By comparison, this formula evaluates the orthogonal vector \(\hat{v}_{k+1}\) via the previous \(s_i\) indirectly, unlike the GS procedure. The sequence \(B_1, \ldots, B_{k-1}, B_k\) is a particular basis for a \(k\)-dimensional space, because this set amends the loss of orthogonality of the vectors \(s_1, \ldots, s_k\) in the GS one. For instance if \(k = 2\) we have \(B_1 = s_1 - s_2, B_2 = s_1 + s_2\) which are orthogonal more accurate than \(s_1, s_2\) in presence of roundoff. It implies that the next vector \(s_3\) which is constructed via \(B_1, B_2\) by GS Algorithm, (hear by Algorithm 1) is not affected more by rounding errors. For \(k = 3\) we first make the new basis \(B_1 = s_1 - s_2, B_2 = s_1 + s_2 - 2s_3\) and \(B_3 = s_1 + s_2 + s_3\) to generate the next orthogonal vector \(s_4\) via them. We can check the loss of orthogonality of \(B_1, B_2, B_3\); after simplifying \(\langle B_1, B_2 \rangle = \langle B_1, B_3 \rangle = \langle B_2, B_3 \rangle\) are rounded errors than the inner products of \(s_1, s_2\) and \(s_3\) regardless of the propagated errors via the normalizing of computed vectors. We can proceed until \(s_1, \ldots, s_n\) will be constructed with less deviation from loss of orthogonality.

We simply define any matrix \(S = [s_1, \ldots, s_n]\) which is characterized by the property that its column vectors have the same angle to be an \(n\)-by-\(m\) *Equiangular matrix*. Also \(S_\alpha\) is defined to be the set of all \(n\)-by-\(m\) equiangular matrices with \(\cos \theta = \alpha\). For convenience, we call *Equiangular matrix* the square ones.

**Example 3.1.** Let \(A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \\ 1 & 1 & 1 \\ \end{bmatrix}\), applying Algorithm 1 to \(A\) by \(\theta_1 = \frac{\pi}{4}\) and \(\theta_2 = \frac{3\pi}{4}\) gives us

\[
S_1 = \begin{bmatrix} 0.5 & -0.1942 & 0.5052 & -0.1266 \\ 0.5 & -0.0532 & 0.0410 & 0.7514 \\ 0.5 & 0.2904 & -0.3488 & -0.2302 \\ 0.5 & 0.3064 & 0.7576 & 0.6053 \\ \end{bmatrix}
\]

and \(S_2 = \begin{bmatrix} 0.5 & -0.0991 & 0.5567 & 0.0417 \\ 0.5 & 0.1228 & 0.1675 & 0.7174 \\ 0.5 & 0.3865 & -0.1154 & -0.0392 \\ 0.5 & 0.9140 & 0.8055 & 0.6943 \end{bmatrix}\), respectively.

The process of making the equiangular vectors provides a matrix factorization by generating an upper triangular matrix within the algorithm.

**Theorem 3.2.** Suppose \(A \in \mathbb{R}^{m \times n}\) has full-rank \((n \leq m)\). For any given \(\alpha \in (0, 1)\) there is an equiangular matrix \(S \in S_\alpha \subseteq \mathbb{R}^{m \times n}\) and an upper triangular \(R \in \mathbb{R}^{n \times n}\) so that \(A = SR\).
Proof. Applying Algorithm 1 to the columns of $A = [a_1, \ldots, a_n]$ to derive $\tilde{v}_k = \|\tilde{v}_k\| \cdot s_k = a_k + \sum_{i=1}^{k-1} d_i s_i$ followed by (3.2), implies the representation of $a_k$ as a linear combination of $s_1$ through $s_k$ as

$$a_k = -\sum_{i=1}^{k-1} d_i s_i + \|\tilde{v}_k\| s_k = [s_1, \ldots, s_k] \begin{bmatrix} -d_1 \\ \vdots \\ -d_{k-1} \\ \|\tilde{v}_k\| \end{bmatrix} = S_k r_k .$$

Choose $\hat{S}_{n-k}$ so that $S = [S_k, \hat{S}_{n-k}]$ is an $m$-by-$n$ equiangular matrix (via Algorithm 1), and then

$$a_k = [S_k, \hat{S}_{n-k}] \begin{bmatrix} r_k \\ 0 \end{bmatrix} = S \begin{bmatrix} r_k \\ 0 \end{bmatrix} .$$

This is the $k$th column of the two sides of $A = SR$.

The $(i, i)$ entry of $R$ ($r_{ii}$) equals $\|\tilde{v}_i\|$. Against of the notion that this factorization might be unique with respect to a fixed angle, it won’t occur. Indeed, the number of the SR factorization of a matrix $A \in \mathbb{R}^{m \times n}$ of full rank, is $2^{n-1}$. Because for making equiangular the vector $v_2$ with $s_1$, there are two choices for $s_2$, either the left side of $s_1$ or the right side of it, but Algorithm 1 chooses one side. To become equiangular the vector $v_3$ with $s_1, s_2$, there are also two choices for considering $s_3$: either the left side of $s_1 + s_2$ or the right side of it. So we have two choices at each step until the $n$th column vector of $A$.

**Example 3.3.** Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$, applying algorithm 1 to $A$ by $\theta_1 = \frac{\pi}{6}$ results $A = S_1 R_1$ where

$$S_1 = \begin{bmatrix} 0.5 & 0.0000 & 0.2321 & 0.2321 \\ 0.5 & 0.5774 & 0.1384 & 0.3854 \\ 0.5 & 0.5774 & 0.6808 & 0.2003 \\ 0.5 & 0.5774 & 0.6808 & 0.8543 \end{bmatrix}$$

and

$$R_1 = \begin{bmatrix} 2.0000 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2.0000 \\ 1.7321 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1.1444 \\ 2.0312 \\ 1.8436 \\ 1.6834 \end{bmatrix}$$

and $A = S_2 R_2$ where

$$S_2 = \begin{bmatrix} 0.5 & -0.6088 & -0.0301 & -0.0301 \\ 0.5 & 0.4580 & -0.4597 & 0.1080 \\ 0.5 & 0.4580 & 0.6276 & -0.2691 \\ 0.5 & 0.4580 & 0.6276 & 0.9566 \end{bmatrix}$$

and

$$R_2 = \begin{bmatrix} 2.0000 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3.1413 \\ 0.9574 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3.6476 \\ 1.3078 \\ 0.9197 \\ 0.8159 \end{bmatrix} .$$

**Corollary 3.4.** Any symmetric positive definite (SPD) matrix $A$ can be factored as a cholesky factorization plus a rank-one matrix.

Proof. Suppose $A \in \mathbb{R}^{n \times n}$ is a SPD matrix, then $A$ can be written as $B^T B$, where $B$ is a nonsingular matrix. SR factorization followed by Algorithm 1 implies

$$A = B^T B = R^T S^T S R = R^T \begin{bmatrix} 1 & \cdots & \alpha \\ \vdots & \ddots & \vdots \\ \alpha & \cdots & 1 \end{bmatrix} R = R^T [(1 - \alpha) I_n + \alpha e e^T] R$$

$$= (1 - \alpha) R^T R + \alpha (R^T e)(R^T e)^T = \text{Cholesky} + \text{rank} - \text{one},$$

where $e$ is the vector of ones.

4. Inverse and eigenvalue problems

We need to introduce a matrix that is significantly important to identify Equiangular matrices, which is said to be the Gram matrix and is a natural generalization of the identity matrix $I$.

**Definition 4.1.** (Gram matrix of an equiangular matrix $S$) For given EM $S \in S_n$ define $G_\alpha = S^T S$, i.e., $[G_\alpha]_{ij} = s_i^T s_j$. Thus $G_\alpha$ has one on its main diagonal and $\alpha$ on all off-diagonal. $G_\alpha$ can be written as $I + \alpha S$, where $S$ has zero on its main diagonal and $\alpha$ on all off-diagonal which is called the Seidel matrix.

Assume that $S = [s_1, \ldots, s_n] \in S_n$, so

$$S^T S = \begin{bmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & \ddots & \cdots & \alpha \\ \vdots & \ddots & \ddots & \vdots \\ \alpha & \cdots & \alpha & 1 \end{bmatrix} = G_\alpha .$$

(4.1)
\[ S^{-1}S^{-T} = G_\alpha^{-1} \] and \[ S^{-1} = G_\alpha^{-1}S^T. \] It can be checked that \( G_\alpha^{-1} = kG_h \), where \( k = \frac{1+(n-2)\alpha}{1-\alpha(1+(n-1)\alpha)} \) and \( h = \frac{\alpha}{1+(n-2)\alpha}. \) Taking these facts together implies

\[ S^{-1} = kG_hS^T. \] (4.2)

This formula can be regarded as the generalization of inverse of orthonormal matrices, in the sense that if \( \theta = \pi/2 \) then \( k = 1, \) \( G_h = G_0 = I \) and \( S^{-1} = S^T. \) It seems the rows of \( S^{-1} \) are equiangular. Let \([S]_{ij} = s_{ij}, \) by (4.2), \([S^{-1}]_{ij} = k(s_{ji} + h \cdot \sum_{k \neq i} s_{jk}). \) Therefore, \( S^{-1} \) is evaluated by \( O(n^2) \) arithmetical operations. This implies that the matrix equation \( Sx = b \) can be solved by \( O(n^2) \) operations, where \( S \) is an equiangular matrix.

**Theorem 4.2.** Let \( S \in S_\alpha \subset \mathbb{R}^{n \times n} \) with the acute angle \( \theta, \) and \( h, k \) are the same as before, then \( S^{-1} \) has equiangular rows with the angle \( \theta', \) where \( \cos \theta' = h. \) Furthermore, the size of any row of \( S^{-1} \) is \( \sqrt{k}. \)

**Proof.** Suppose that \( s_i \) and \( s'_i \) are the columns of \( S \) and the rows of \( S^{-1} \), respectively for \( i = 1, \ldots, n. \) Let \( \tau_i \) be the angle between two vectors \( s_i \) and \( s'_i \). We have \( 1 = s'_i \cdot s_i = \|s'_i\|\|s_i\| \cos \tau_i = \|s'_i\| \cos \tau_i, \) so \( \cos \tau_i = \frac{1}{\|s'_i\|}. \)

It means that \( \tau_i \in (0, \frac{\pi}{2}) \). We can verify that the two vectors \( s_i \) and \( s'_i \) make the angles \( \theta \) and \( \frac{\pi}{2} \) with all of vectors \( \tilde{S}_i = \{s_j \}_j \neq i \), respectively. Therefore, these two vectors are in a subspace called \( P_{n,i} \) (analogous to \( P_k \) in §3), in which every vector makes the same angle with the vectors of \( \tilde{S}_i \). Of course, another third vector which has the above property is \( z_i = \sum_{j \neq i} \frac{1}{\pi} s_j \). As shown before, \( P_{n,i} \) is a plane, hence \( \{s_i, s'_i, z_i\} \in P_{n,i}. \)

Since the angle between \( s_i \) and \( z_i \) is \( \frac{\pi}{2} \) and the angle between \( s_i \) and \( s'_i \), \( \tau_i \) is acute, then \( s_i \) must be laid between two ones else. If the angle between \( s_i \) and \( z_i \) is denoted by \( \varphi_i \), refer to (4.2) \( \cos \varphi_i = \frac{\sqrt{n-\alpha}}{\sqrt{1+(n-2)\alpha}}. \) We can verify that \( \cos \varphi_i < 1 \) for all \( \alpha, \) therefore, \( \varphi_i \) is an acute angle and \( \varphi_i + \tau_i = \frac{\pi}{2}. \) From Pythagoras theorem \( 1 = \cos^2 \varphi_i + \cos^2 \tau_i = \frac{(n-1)\alpha^2}{(n-2)\alpha+1} + \frac{1}{\|s'_i\|}, \) thus

\[ \|s'_i\| = \sqrt{\frac{1+(n-2)\alpha}{(1-\alpha)(1+(n-1)\alpha)}}, \quad i = 1, \ldots, n \] (4.3)

Taking inner product of the two rows of \( S^{-1} \) and (4.2) give

\[ s'_i s'_j = k^2[h \cdots i^{th} \cdots h] G_\alpha [h \cdots j^{th} \cdots h]^T \]

\[ = k^2[(\alpha + h + (n-2)\alpha)h] [1 \cdots i^{th} \cdots 1 + (1 + (n-1)\alpha)h] [0 \cdots i^{th} \cdots 0] [h \cdots j^{th} \cdots h]^T \]

\[ = k^2[(\alpha + h + (n-2)\alpha)((n-2)h+1) + (1 + (n-1)\alpha)h] \]

\[ = -\alpha \]

\[ \frac{(1-\alpha)(1+(n-1)\alpha)}{(1-\alpha)(1+(n-1)\alpha)} = h. \]

Hence the cosine of the angle between any \( s'_i \) and \( s'_j \) is the same (\( \theta' \)) and equals \( \frac{-\alpha}{1+(n-2)\alpha} = h. \)

So \( \cos \theta' = h. \) Obviously \( h < 0, \) thus \( \theta' \) is an obtuse angle, i.e., \( \theta' \in (\frac{\pi}{2}, \pi). \)

From (4.3) the matrix \( k^{-\frac{1}{2}} S^{-1} \) is Equiangular but in the state of rowwise, so \( k^{-\frac{1}{2}} S^{-T} \in S_h. \) We call a matrix such as \( k^{-\frac{1}{2}} S^{-1}, \) row-equiangular. For example if \( n = 2 \) then \( \alpha = -\alpha \) and \( \theta = \pi - \theta. \) We illustrate \( s_i \) and \( s'_i \) in Figure 4 (i = 1, 2).

**Corollary 4.3.** If \( S \in S_\alpha, \) then \( kG_\alpha G_h = I. \)

**Proof.** Since \( S^{T} S = G_\alpha \) and \( S^{-1} \) is the row-equiangular, by the size of the rows \( \sqrt{k}, \) then \( S^{-1}. S^{-T} = k G_h. \) By (4.2) we end the proof.
From Definition 4.1 the eigenvalues of the Seidel matrix $S$ because $S$ is a symmetric matrix. In other words, $S$ is orthogonally diagonalizable. This choice has two advantages: The first one is that the vector sum of columns of $S$ exists an orthogonal matrix $Q$ as noted in § 3.9. The eigenvalues of $QG$ will be the unique principal square root of $K = \begin{bmatrix} 3/7 & -2/7 & 6/7 \\ 6/7 & 3/7 & -2/7 \\ -2/7 & 6/7 & 3/7 \end{bmatrix}$, and $R = \begin{bmatrix} 1 & -1 & -0.6436 & -0.4760 \\ 0 & 1 & 1.1547 & -0.4082 \\ 0 & 0 & 0 & 1.2247 \end{bmatrix}$. We see that $R$ is row-equiquangular. Because $RR^T = S^{-1}QQ^T S^{-T} = S^{-1}S^{-T} = (S^T S)^{-1} = G_\alpha^{-1} = k G_h$, where $\alpha = \cos \theta$.

Now we discuss the eigenvalues of the Equiangular and Gram matrices. Actually we present a lower and upper bound for the eigenvalues of an Equiangular matrix $S$ relative to the eigenvalues of its corresponding matrix $G_{\alpha}$.

**Lemma 4.7.** The eigenvalues of $G_{\alpha}$ are $1 - \alpha$ and $1 + (n - 1)\alpha$ for $\alpha \in (0, 1)$. So $G_{\alpha}$ is SPD.

**Proof.** We have $G_{\alpha} e = a e e^T e + (1 - \alpha)I_n e = n e + (1 - \alpha)e = (1 + (n - 1)\alpha)e$ so $(1 + (n - 1)\alpha, e^T)$ is an eigenpair of $G_{\alpha}$. Now take $x = [x_1, \ldots, x_n]^T$ with the property that $x_1 + \ldots + x_n = 0$, we have $G_{\alpha}x = a e e^T x + (1 - \alpha)I_n x = (1 - \alpha)x$. Thus $1 - \alpha$ is the second eigenvalue of $G_{\alpha}$ with the algebraic multiplicity $n - 1$. From Definition 4.1 the eigenvalues of the Seidel matrix $S$ are $-1$ and $n - 1$.

As noted in § 2 a set of EVs $s_1, \ldots, s_n$ can be located into the positive coordinate axes so that $n - 1$ entries of any of them are $t$ and the last one is $s = \sqrt{1 - (n - 1)t^2}$. So the matrix $S = [s_1, \ldots, s_n] = S_n$ if $\alpha = 2ts + (n - 2)t^2$. We can simply verify that

$$s = \sqrt{1 + (n - 1)(n - 2) + n + 2(n - 1)\sqrt{1 - \alpha}(1 + (n - 1)\alpha)} = \sqrt{\frac{\alpha n + (\alpha - 2) \sqrt{1 - \alpha} \sqrt{1 + (n - 1)\alpha}}{n}}.$$ (4.4)

Recall from § 2 that we set $t \in (0, \frac{1}{\sqrt{n}})$. Thus we must select the plus sign in the formula of $s$ and vice versa in $t$. This choice has two advantages: The first one is that $s \in (\sqrt{n}, 1)$, then $S$ will be SPD with the positive eigenvalues. In other words, $S$ can be rewritten as $sG_{\alpha}$, so from Lemma 4.7, $\sigma(sG_{\alpha}) = \{s - t, s + (n - 1)t\}$. We denote $sG_{\alpha}$ as $G(s, t)$ which has $s$ on the main diagonal and $t$ on all off-diagonal. The next one is that $G(s, t)$ will be the unique principal square root of $G_{\alpha}$, i.e., $G(s, t) = G_{\alpha}^{1/2}$. Since $G(s, t), S_n \subset S_n$, then there exists an orthogonal matrix $Q$ so that $Q = GQ(s, t)$. It can be interpreted as a transformation of the orthogonal matrices to the equiquangular matrices and vice versa ($Q = SG^{-1}(s, t)$). More precisely the column vectors of $G, S$ are nested so that the vector sum of columns of $S, (Se)$ and columns of $Q, (Qe)$ are in the same direction, because $Sc = QG(s, t)e = (s + (n - 1)t)Qe$.

We now assume $(\mu, x)$ be the eigenpair of $G(s, t)$, then $G(s, t)x = \mu x$ and $G(s, t)x = G(s, t)(s, t)x = s + (n - 1)t s + (n - 1)t)$. So $(\mu, x)$ is the eigenpair of $G_{\alpha}$ and $\lambda = \mu^2$, where $\lambda$ is the eigenvalue of $G_{\alpha}$. In other words $\mu$ is either $\sqrt{1 - \alpha}$ or $\sqrt{1 + (n - 1)\alpha}$, which are equal to $s - t, s + (n - 1)t$, respectively. Therefore we get the simple form for $s, t$ as follows.

$$s = \sqrt{\frac{1 + (n - 1)\alpha + (n - 1)\sqrt{1 - \alpha}}{n}}, \quad t = \frac{\sqrt{1 + (n - 1)\alpha - \sqrt{1 - \alpha}}}{n}.$$ (4.5)
Example 4.8. Suppose that we have a Gram matrix as \( G = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix} \). The structure of the matrix \( G(s,t) \) depends on the location of \( s, t \) as follows.

\[
G(s,t) = \begin{bmatrix}
0.9428 & 0.2357 & 0.2357 \\
0.2357 & 0.9428 & 0.2357 \\
0.2357 & 0.2357 & 0.9428
\end{bmatrix}, \quad 0 < t < \frac{1}{\sqrt{n}} , \quad G(s,t) = \begin{bmatrix}
0 & 0.7071 & 0.7071 \\
0.7071 & 0 & 0.7071 \\
0.7071 & 0.7071 & 0
\end{bmatrix}, \quad \frac{1}{\sqrt{n}} < t \leq \frac{1}{\sqrt{n-1}}.
\]

Since \( \sigma(G(s,t)) = \{ s - t, s + (n - 1)t \} \), then the eigenvalues of the first one are 0.7071, 0.7071, 1.4142 and those of the second one are -0.7071, -0.7071, 1.4142. As mentioned earlier we prefer to use the first one.

We are interested in an upper and lower bound for eigenvalues of an equiangular matrix \( S \).

Theorem 4.9. Let \( (\lambda, x) \) be an eigenpair of an equiangular matrix \( S \in S_n \) with \( ||x|| = 1 \) then the following boundaries hold

\[
\lambda_{\min} G(s,t) \leq |\lambda| \leq \lambda_{\max} G(s,t),
\]

where \( \alpha = 2ts + (n - 2)t^2 \). Furthermore if \( x = [x_1, \ldots, x_n]^T \), then \( |\sum_{i=1}^n x_i| \geq 1 \) if and only if \( |\lambda| \geq 1 \).

Proof. We have \( Sx = \lambda x \), then \( x^*S^T = \lambda^*x^* \) and \( x^*S^TSx = \lambda \lambda^*x^*x = |\lambda|^2||x||^2 = |\lambda|^2 \).

On the other hand

\[
x^*S^TSx = x^*G_\alpha x = x^*(\alpha e^Te + (1 - \alpha)I)x = \alpha\left( \sum_{i=1}^n x_i^2 \right) \left( \sum_{i=1}^n x_i^2 \right) + (1 - \alpha)||x||^2
\]

\[
= \alpha \sum_{i=1}^n x_i^2 + 1 - \alpha = \alpha |e^Tx|^2 + 1 - \alpha,
\]

where \( e \) is the vector of ones. The last term equals \( |\lambda|^2 \), thus

\[
|\lambda| = \sqrt{\alpha |e^Tx|^2 + 1 - \alpha}.
\]

Equation (4.6) describes the relationship between eigenvalues and eigenvectors of \( S \). \( \max |e^Tx| \) is attained for \( x = \pm \frac{\alpha e^T}{||e^T||} \) (\( x_i = \frac{\alpha}{\sqrt{n}} \) or \( x_i = -\frac{\alpha}{\sqrt{n}} \)), so \( |\lambda| \leq \sqrt{\alpha n + 1 - \alpha} = \sqrt{1 + (n - 1)\alpha} = \lambda_{\max} G(s,t) \). Likewise, \( \min |e^Tx| \) is attained for \( x \in \ker(e^Tx) \) which means \( e^Tx = 0 \), so \( |\lambda| \geq \sqrt{1 - \alpha} = \lambda_{\min} G(s,t) \). From (4.6), \( |e^Tx| \geq 1 \) if and only if \( |\lambda| \geq 1 \).

\[
\Box
\]

Proposition 4.10. The condition number of every matrix \( S \in S_n \) relative to 2-norm is equal to \( \sqrt{1 + \frac{\alpha}{1 - \alpha}} \).

Proof. Since \( \|S\|_2 = \sqrt{\lambda_{\max}(S^TS)} \) and \( \|S^{-1}\|_2 = \sqrt{\lambda_{\min}((S^TS)^{-1})} = \frac{1}{\sqrt{\lambda_{\max}(S^TS)}} = \frac{1}{\sqrt{\lambda_{\min}(S^TS)}} \), then we have the simple reasoning \( \kappa_2(S) = \sqrt{\frac{\lambda_{\max}(S^TS)}{\lambda_{\min}(S^TS)}} = \sqrt{\frac{1 + (n - 1)\alpha}{1 - \alpha}} = \sqrt{1 + \frac{\alpha}{1 - \alpha}} \).

\[
\Box
\]

Obviously \( S \) tends to the ill-conditioned matrix as \( \alpha \to 1 \).

5. Other applications

Proposition 5.1. Given \( A \in R^{n\times n} \), then for every \( r \in (0,1) \) there exists an equiangular matrix \( S \in S_n \) so that \( S^{-1}AS = T \) is a block upper triangular, with 1-by-1 and 2-by-2 blocks on the diagonal. The eigenvalues of \( A \) are the eigenvalues of diagonal blocks of \( T \). The 1-by-1 blocks correspond to real eigenvalues, and the 2-by-2 blocks to complex conjugate pairs of eigenvalues.

Proof. From the Real Schur form, \( A = QΛQT \), where \( Q \) is an orthogonal matrix and \( Λ \) is a block upper triangular. Based on Theorem 3.2 there is a factorization \( Q = SR \) for given \( r \in (0,1) \), where \( S \in S_n \) and \( R \) is upper triangular. Taking these together implies \( A = QΛQT = SRΛ^{-1}S^{-1} = STS^{-1} \), in which \( T = RΛ^{-1} \) is also a block upper triangular, where the blocks conformable with that of \( Λ \). Because the diagonal entries of \( R \) are the reciprocal of those of \( R^{-1} \).

\[
\Box
\]
For some matrix $A$, there is an $\alpha \in (0,1)$ so that $S^{-1}AS = T = \text{diag}(t_1, \ldots, t_n)$, where $S \in S_\alpha$. So in this case $A$ has $n$ equiangular eigenvectors.

We now want to know more about the matrix $SS^T$, where $S \in S(\alpha)$. The symmetric matrix $SS^T$ is similar to $G_\alpha$, because $S^{-1}SS^TS = G_\alpha$. So from Lemma 4.7, $\sigma(SS^T) = \{1 - \alpha, 1 + (n - 1)\alpha\}$ with the algebraic multiplicity $n - 1$ at $1 - \alpha$. Furthermore, by Theorem 4.2

$$(SS^T)^{-1} = S^{-T}S^{-1} = k(k^{-1/2}S^{-T} \cdot k^{-1/2}S^{-1}) = k(\hat{S}S^T),$$

where $\hat{S} = k^{-1/2}S^{-T} \in S_h$. Conversely, suppose that $A \in \mathbb{R}^{n \times n}$ is symmetric matrix with two eigenvalues $\beta$, $\gamma = \beta + n(1 - \beta)$ with algebraic multiplicity $n - 1$ at $\beta$, where $0 < \beta < 1 < \gamma$. Taking $\alpha = 1 - \beta$, implies that $\sigma(SS^T) = \{1 - \alpha, 1 + (n - 1)\alpha\}$. Hence from Schur form $A = Q(SS^T)Q^T$, where $Q$ and $S$ are orthogonal and equiangular matrix, respectively ($S \in S_\alpha$). So $A = (QS)(QS)^T = \hat{S}S^T$. Therefore $\hat{S} \in S_\alpha$. We can deduce the following Lemma.

**Lemma 5.2.** Suppose that a symmetric matrix $A \in \mathbb{R}^{n \times n}$ has two nonzero distinct eigenvalues $\beta$, $\gamma$ with the algebraic multiplicity $n - 1$ at $\beta$, then $A$ can be factorized uniquely as $rSS^T$, where $r \neq 0$ and $S$ is an equiangular matrix.

**Proof.** There are two cases:

(a) $\beta < \gamma$

We find two scalars $r, \alpha \in (0,1)$ that satisfy $r(1 - \alpha) = \beta$ and $r(1 + (n - 1)\alpha) = \gamma$. So $r = \frac{\gamma - \beta + \alpha \gamma}{\gamma - \beta + (1 + \alpha)\gamma}$ and $\alpha = \frac{\gamma - \beta}{\gamma - \beta + (1 + \alpha)\gamma}$. Then two eigenvalues $\beta, \gamma$ will be reformulated as $r(1 - \alpha), r(1 + (n - 1)\alpha)$, respectively. As noted already, $A$ can be factorized as $rSS^T$, where $S \in S_\alpha$.

(b) $\beta > \gamma$

In this case $A^{-1}$ has two eigenvalues $\beta^{-1}, \gamma^{-1}$. Based on case (a), $A^{-1} = r'\hat{S}S^T$, with the new definitions of $r', \alpha'$. Then from (5.1) $A = \frac{1}{r}(SS^T)^{-1} = \frac{1}{r}(SS^T) = rSS^T$, where $S \in S_h$, so that $h' = \frac{1 - \alpha}{1 + (n - 1)\alpha}$.

We now are interested to know that which matrices have the equiangular eigenvectors.

**Theorem 5.3.** Suppose $A \in \mathbb{R}^{n \times n}$ is similar to $A^T$ as $A = BA^TB^{-1}$, where $B$ is symmetric and has two nonzero distinct eigenvalues $\beta, \gamma$ so that $\gamma > \beta$ with multiplicity $n - 1$ at $\beta$. In the Schur form $A = QTQ^T$ we factorize $Q = SR$ by algorithm so that $S \in S_\alpha$, where $\alpha = \frac{\gamma - \beta + \alpha \gamma}{\gamma - \beta + (1 + \alpha)\gamma}$. Now if $B = rSS^T$, where $r = \frac{\gamma - \beta + \alpha \gamma}{\gamma - \beta + (1 + \alpha)\gamma}$, then $A$ has $n$ equiangular eigenvectors, which form the columns of $S$.

Note that if $A$ is symmetric, then $B = rQQ^T = rI$, where $Q$ is an orthogonal matrix.

**Proof.** We have $A = BA^TB^{-1} = rSS^TA^T \cdot (SS^T)^{-1} = S(S^T \cdot A^T \cdot S^{-1}) S^{-1} = S(S^{-1}AS)^T S^{-1}$, so $S^{-1}AS = (S^{-1}AS)^T$ that means $S^{-1}AS$ is symmetric. On the other hand $A = QTQ^T = SRTR^{-1}S^{-1}$ so $S^{-1}AS = RTR^{-1}$ which is upper triangular. So the symmetric and upper triangular matrix $A = S^{-1}AS$ is diagonal. Therefore $A = SAS^{-1}$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with eigenvalues of $A$ on its diagonal and the column vectors of $S$ are the equiangular eigenvectors of $A$.

For the case of $\gamma < \beta$ we can consider $B^{-1}$ instead of $B$ and consequently the result of the Theorem 6.3 holds. The next theorem provides a method to produce such factorization $SS^T$ for some symmetric matrices $A$. The question is: under what conditions a symmetric matrix $A$ can be factorized as $SS^T$? The next theorem gives the mentioned factorization so that restricts the symmetric matrices to the equiangular matrices.

**Theorem 5.4.** (Generalization of the symmetric schur factorization) Suppose a symmetric matrix $A$ has all distinct eigenvalues, then there are an equiangular matrix $S \in S_\alpha$ for some $\alpha \in (0,1)$ and a real diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ which is uniquely determined up to the $\alpha$ so that $A = SDS^T$. In addition, Consider the Schur form $A = QAQ^T$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, then $\text{trace}(D) = \text{trace}(\Lambda)$.

**Proof.** If we produce a diagonal matrix $D$ so that matrix $G(s,t)DG(s,t)$ is factorized as the Schur form $G(s,t)DG(s,t) = PAP^T$, where $\Lambda = Q^TAQ$, hence we will end the proof:

$A = QPTPAP^T \cdot PQ^T = QPTG(s,t)DG(s,t)PQ^T$. The symmetric equiangular matrix $G(s,t)$ is of $S_h$ or $G(s,t)G(s,t) = G_h$, therefore, $QPTG(s,t)$ will be another equiangular matrix say $S$. Finally $A = SDS^T$. 

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It remains to find $D$. To establish a relationship between $D$ and $\Lambda$, note that $G(s, t)DG(s, t) = G(s, t)DG^2(s, t)G^{-1}(s, t) = G(s, t)DG_\alpha G^{-1}(s, t)$. This expression specifies that $DG_\alpha$ must be similar to $\Lambda$, so the characteristic polynomial of them are the same. We now obtain the characteristic polynomial of $DG_\alpha$.

$$\det(xI - DG_\alpha) = \det\begin{bmatrix} x - d_1 & -\alpha d_1 & \cdots & -\alpha d_1 \\ -\alpha d_2 & x - d_2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -\alpha d_n & \cdots & x - d_n \end{bmatrix} = (-1)^n \alpha^n \prod_{i=1}^n d_i \det\begin{bmatrix} x - \frac{d_i}{\alpha d_i} & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & \frac{1}{\alpha d_i} \end{bmatrix}.$$

By take $A_n = \begin{bmatrix} x_1 & 1 & \cdots & 1 \\ 1 & x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & x_n \end{bmatrix}$, where $x_i = \frac{x - d_i}{\alpha d_i}$, we have the following recurrence

$$\det(A_n) = (x_n - 1) \det(A_{n-1}) + (-1)^{n-1} (1 - x_1) \det(\bar{A}_{n-1}),$$

where $\bar{A}_{n-1} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$.

$$\det(\bar{A}_{n-1})$$ will be computed recursively as

$$\det(\bar{A}_{n-1}) = -(x_{n-1} - 1) \det(\bar{A}_{n-2}) = \cdots = (-1)^{n-2} (x_2 - 1) \cdots (x_{n-1} - 1) = (-1)^{n-2} \prod_{i=2}^{n-1} (x_i - 1).$$

Therefore we have

$$\det(A_n) = (x_n - 1) \det(A_{n-1}) + (-1)^{n-1} (1 - x_1) \cdot (-1)^{n-2} \prod_{i=2}^{n-1} (x_i - 1)$$

$$= (x_n - 1) \cdot \det(A_{n-1}) + \prod_{i=1}^{n-1} (x_i - 1) = \prod_{i \neq n} (x_i - 1) + \cdots + \prod_{i \neq 1} (x_i - 1) + \prod_{i=1}^{n} (x_i - 1).$$

By replacing $\det(A_n)$ in the first equality by the last one we have

$$\det(xI - DG_\alpha) = (-1)^n \alpha^n \left( \prod_{i=1}^n d_i \left[ \prod_{i \neq n} \left( \frac{x - d_i}{\alpha d_i} - 1 \right) + \prod_{i \neq 1} \left( \frac{x - d_i}{\alpha d_i} - 1 \right) + \prod_{i=1}^n \left( \frac{x - d_i}{\alpha d_i} - 1 \right) \right] \right)$$

$$= (-1)^n \alpha^n \left( \prod_{i=1}^n d_i \right) \left[ (-\alpha d_n \prod_{i \neq n} (x - d_i(1 - \alpha)) - \cdots - \alpha d_i \prod_{i \neq 1} (x - d_i(1 - \alpha)) + \prod_{i=1}^n (x - d_i(1 - \alpha)) \right] = \cdots$$

$$= x^n + (\alpha + (\alpha - 1)) \left( \sum_{i=1}^n d_i \right) x^{n-1} + (-2\alpha(\alpha - 1) + (\alpha - 1)^2) \left( \sum_{1 \leq i < j \leq n} d_i d_j \right) x^{n-2} + \cdots$$

$$+ (-(n - 1)\alpha(\alpha - 1)^{n-2} + (\alpha - 1)^{n-1}) \left( \sum_{1 \leq i < j \leq n} d_i d_j \right) x + (-(n - 1)\alpha(\alpha - 1)^{n-1} + (\alpha - 1)^n) d_1 \cdots d_n$$

$$= x^n - (\sum_{i=1}^n d_i) x^{n-1} + (1 - \alpha^2) \left( \sum_{1 \leq i < j \leq n} d_i d_j \right) x^{n-2} - \cdots - (\alpha - 1)^{n-2}((\alpha - 2)\alpha + 1) \left( \sum_{1 \leq i < j \leq n} d_i d_j \right) x$$

$$- (\alpha - 1)^{n-1} (1 + (n - 1)\alpha) d_1 \cdots d_n.$$

On the other hand, the characteristic polynomial of $\Lambda$ is

$$p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n) = x^n - (\sum_{i=1}^n \lambda_i) x^{n-1} + (\sum_{1 \leq i < j \leq n} \lambda_i \lambda_j) x^{n-2} - \cdots$$

$$+ (-1)^{n-1}(\sum_{1 \leq i_1 < \cdots < i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}}) x + (-1)^n \lambda_1 \cdots \lambda_n.$$

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The necessary condition for being possible this factorization is that the roots of $p$ of the polynomial coefficients of $p$ does not have not all real roots. Thus $g$ is diagonal that is a contradiction. Thus we can say $A$ symmetric, so the results of Theorem 6.4., e.g., $\Lambda = \text{diag}(0, ..., 0)$, are the real roots of the following polynomial $x^2 - 2rx + r^2 = 0$. In other words there is a real $0 < \alpha < 1$ depends on $n$ possibly near to zero so that $g(x)$ has all real roots. When $A$ is not diagonalizable, there is a simple counter example for the results of Theorem 6.4., e.g., $\Lambda = \text{diag}(0, 1, 1)$ then $D = \text{diag}(0, 1 + \sqrt{\frac{\alpha^2}{1-\alpha}}, 1 - \sqrt{\frac{\alpha^2}{1-\alpha}})$ is not real and symmetric, so $A$ will never be factorized as $SDS^T$. Another counter example is when $A$ be a factor of identity matrix: Let $A = rI = SDS^T$, then $rS^{-1}S^{-T} = rG^{-1} = D$ so from Corollary 1, $rkG_{c} = D$, which means $G_{c}$ is diagonal that is a contradiction. Thus we can say $g(x)$ does not have all real roots or $g(x)$ has at least two complex roots.

In general, we have proven the following proposition in general.

**Proposition 5.5.** Assume that $r$ and $\alpha$ are two nonzero real scalars so that $\alpha \in (0, 1)$ then the following polynomial does not have all real roots.

$$g_n(x) = x^n - nrx^{n-1} + \binom{n}{2}r^2x^{n-2} - \binom{n}{3}r^3x^{n-3} + \cdots + (-1)^{n-1}r^{n-1}x - (-1)^n r^n. \quad (5.4)$$

**Proof.** From Theorem 6.4. it suffices to take $A = rI$. So $\Lambda = rI$ and since there is no real symmetric $D$ which satisfies $A = SDS^T$, then by (5.2) we can evaluate the coefficients of $g(x)$ as follows.

$$c_1 = \sum_{i=1}^{n} r = nr,$$
$$c_2 = \frac{1}{1-\alpha^2} \sum_{1 \leq i < j \leq n} r^2 = \frac{\binom{n}{2}r^2}{1-\alpha^2},$$
$$\vdots$$
$$c_n = \frac{r^n}{(1-\alpha)^n(1 + (n-1)\alpha)}.$$

**Example 5.6.** It is easy to check the accuracy of proposition 6.5., for case of $n = 2, 3$ with a nonzero real $r$: When $n = 2$, $g_2(x) = x^2 - 2rx + r^2$, then the roots of $g_2$ are complex: $x_{1,2} = r \pm \frac{r\alpha}{\sqrt{1-\alpha^2}}$, so $g_2 > 0$. When
n = 3, \(g_3(x) = x^3 - 3x^2 + \frac{3x^2}{1-\alpha} - \frac{x^3}{(1-\alpha)^3(1+2\alpha)}\), by calculating the derivative \(g'_3(x) = 3x^2 - 6x + \frac{3\alpha^2}{1-\alpha^2}\), we observe \(g'_3 = 3g_3 > 0\). Therefore \(g_3\) is an increasing function, so it has only one real root, and the others are complex. We can verify that \(g'_m = mg_{m-1}\), by induction \(g_{n-1}\) has at most \(n-3\) real roots. It implies by intermediate value theorem that \(g_n\) has at most \(n-2\) real roots which is less than \(n\).

\[\Box\]

**Example 5.7.** Let see if \(\Lambda = \text{diag}(1, 2, 3)\), for which scalar \(\alpha\) there exist two matrices \(S, D\) satisfies \(\Lambda = SDS\). If \(p(x) = x^3 - 6x^2 + 11x - 6\) and \(g(x) = x^3 - 6x^2 + \frac{11x}{(1-\alpha)^3(1+2\alpha)}\). Considering MATLAB roots function, the roots of \(g\) are all real when \(\alpha \leq 0.1843\) by rounding.

\[\Box\]

Lemma 5.2 shows that the results of Theorem 5.4 holds for a class of nondiagonalizable matrices which have two eigenvalues with the multiplicity \(n-1\) at one of them. In these cases the diagonal matrix \(D\) will be a factor of identity matrix. For example assume \(\Lambda = \text{diag}(1, 1, 1)\), then from Lemma 5.2, \(r = \frac{1}{2}\), \(\alpha = \frac{1}{2}\) and \(D\) will be equal to \(\frac{1}{2}I_3\). Also by (4.3), \(s = 0.9856\), \(t = 0.1196\), thus \(G(s, t) = 0.9856G_{0.1213}\). Then by hypothesis of the Theorem 6.4.

\[\Box\]

We survey two special case of Proposition 5.5 to give some realizable intuitions:

**Example 5.8.** Take \(r = 1-\alpha\), so the general term of \(g_n(x)\) is \(c_k = \frac{n_k}{1-\alpha} \left(1 + \frac{1}{1-\alpha}\right)^{\alpha - 1} \left(1 + \frac{k - 1}{1-\alpha}\right)^{\alpha - 1}\). By simplifying \(c_k = \frac{(n_k)^k}{1-\alpha} \left(1 + \frac{1}{1-\alpha}\right)^{\alpha - 1} \left(1 + \frac{k - 1}{1-\alpha}\right)^{\alpha - 1}\).

Example 5.9. Take \(r = 1 + (k - 1)\alpha\), so the general term of \(g_n(x)\) is \(c_k = \frac{n_k}{1-\alpha} \left(1 + \frac{1}{1-\alpha}\right)^{\alpha - 1} \left(1 + \frac{k - 1}{1-\alpha}\right)^{\alpha - 1}\). By simplifying \(c_k = \frac{n_k}{1-\alpha} \left(1 + \frac{1}{1-\alpha}\right)^{\alpha - 1} \left(1 + \frac{k - 1}{1-\alpha}\right)^{\alpha - 1}\).

6. Equiangular vectors as a sequence of Equiangular frames

In this section another aspect of EVs can be regarded, which is the possibility of assuming them as the equiangular tight frames (ETFs). The theory of frames plays a fundamental role in signal processing, image processing, data compression and more which defined by Duffin and Schaeffer [7].

**Definition 6.1.** A sequence \(f_n (n \in \mathbb{Z})\) of elements in a Hilbert space \(H\) is called a frame if there are constants \(A, B > 0\) so that

\[A \|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2 \leq B \|f\|^2, \text{ for all } f \in H. \quad (6.1)\]

The scalars \(A, B\) are called lower and upper frame bounds, respectively. The largest \(A > 0\) and the smallest \(B > 0\) satisfying the frame inequalities for all \(f \in H\) are called the optimal frame bounds. The frame is a tight frame if \(A = B\) and a normalized tight frame or Parseval frame if \(A = B = 1\). A frame is called overcomplete in the sense that at least one vector can be removed from the frame and the remaining set of vectors will still form a frame for \(H\) (but perhaps with different frame bounds). ETFs potentially have many more practical and theoretical applications [3] [12]. An equiangular tight frame is a set of vectors \(\{f_i\}_{i=1}^m\) in \(\mathbb{R}^n\) (or \(\mathbb{C}^n\)) that satisfies the following conditions [16].

1. \(\|f_i\|_2 = 1, \quad i = 1, \ldots, m.\)
2. \(\langle f_i, f_j \rangle = \alpha, \quad \forall i \neq j\) and a constant \(\alpha.\)
3. \(\frac{1}{m} \sum_{i=1}^m \langle f_i, f_i \rangle f_i = f, \quad \forall f \in \mathbb{R}^n\) (or \(\mathbb{C}^n\)).

Taking inner product of the condition 3 by \(f\) implies that the mentioned vectors form a tight frame. We set \(S_n = [f_1, \ldots, f_m]\). It is not hard to verify that \(\frac{m}{n} S_n S_n^T f = f\), so \(S_n S_n^T = \frac{m}{n} f f\), which is equivalent to the condition 3.
3'. The matrix $S_n$ forms an equiangular tight frame, i.e., $S_n S_n^T = \frac{m}{n} I_n$.

The constant $\alpha$ is prescribed in terms of $n, m$. Actually, condition 2 and 3 together imply that

$$\alpha = \sqrt{\frac{m-n}{n(m-1)}},$$

which is the smallest possible scalar for $\alpha$ for a set of equiangular unit norm vectors in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Due to the theoretical and numerous practical applications, equiangular tight frames are noticeably the most important class of finite-dimensional frames which have applications in signal processing, communications, coding theory, sparse approximation and more. \cite{10, 10}

Example 6.2. (Orthonormal Bases). When $m = n$, the ETFs are unitary (and orthogonal) matrices. apparently, the inner product between distinct vectors is zero, ($\alpha = 0$).

Gerzon (Theorem 3.5 in \cite{13}) proved that the number of equiangular lines in $\mathbb{R}^n$ cannot be more than $\binom{n+1}{2}$. Similar proof implies that this maximum number cannot be more than $n^2$ in the $n$-dimensional complex space $\mathbb{C}^n$. The following lemma specifies the maximum number of EVs of $\mathbb{R}^n$ \cite{4}.

Lemma 6.3. The maximum number of equiangular vectors in $\mathbb{R}^n$ is $n + 1$, so that the cosine of the angle between them is $-\frac{1}{n}$.

Proof. The proof is attainable by induction. The case of $n = 1$ is trivial. For $n = 2$ a largest set of EVs consists of three vectors which make the angle $\frac{2\pi}{3}$ ($\alpha = \cos \frac{2\pi}{3} = -\frac{1}{2}$). We can represent them as the normalized column vectors of a $2 \times 3$ matrix as

$$S_2 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}; \quad n = 2, \alpha = -\frac{1}{2}.$$ 

For $n = 3$ it is easy to see that a largest set of EVs consists of four vectors in such a way that they make a regular triangular pyramid or tetrahedron when they pass through the origin. So the cosine of the angle between distinct vectors is $-\frac{1}{3}$. We can represent them as the normalized column vectors of a $3 \times 4$ matrix as

$$S_3 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ 0 & 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{bmatrix}; \quad n = 3, \alpha = -\frac{1}{3}.$$ 

We observe that the $2 \times 3$ submatrix in the lower right corner of $S_3$ is a factor of $S_2$, so its column vectors are equiangular by the angle $\frac{2\pi}{3}$. Now imagine there is five EVs $s_1', s_2', \ldots, s_5'$ in $\mathbb{R}^3$ so that $s_i'^T \cdot s_j' = \alpha'$. These vectors can be transformed under a proper rotation in $\mathbb{R}^3$ to the new vectors for which $s_1'$ lies in the $x$-axis spanned by $e_1$ and $s_2'$ lies in the $xy$-plane spanned by $e_1, e_2$. So without loss of generality we assume that $s_1', \ldots, s_5'$ form a matrix columnwise as follows.

$$S_5' = \begin{bmatrix} 1 & \alpha' & \alpha' & \alpha' & \alpha' \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \end{bmatrix}; \quad n = 3, \alpha = \alpha'.$$

in which the column vectors are equiangular. So the $2 \times 4$ submatrix in the lower right corner of $S_3'$ has four equiangular columns in $\mathbb{R}^2$ which is a contradiction with $S_2$. By this argument we can construct $n + 1$ EVs in $\mathbb{R}^n$ via $n$ EVs in $\mathbb{R}^{n-1}$ which form a $(n - 1) \times n$ matrix.

$$S_{n-1} = \begin{bmatrix} 1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \end{bmatrix}; \quad \alpha = -\frac{1}{n-1}.$$ 

Actually $S_n$ which is constructed by $n + 1$ EVs columnwise, has a $(n - 1) \times n$ submatrix in the lower right corner, which is a factor of $S_{n-1}$. Therefore the $n \times (n + 1)$ block matrix $S_n$ can be written as follows.

$$S_n = \begin{bmatrix} 1 & \frac{-1}{n} e^T \\ 0 & \rho S_{n-1} \end{bmatrix}; \quad \alpha = -\frac{1}{n},$$ 

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where \( e \) is the vector of ones. We must find the variable \( \rho \) so that the column vectors of \( S_n \) have unit norm. Therefore regarding to the second column \([-\frac{1}{n}, \rho, 0, \ldots, 0]^T\), we have \( \rho = \sqrt{n-1} \). We observe that the column vectors of \( S_n \) construct a regular simplex which is a regular polytope. Finally it can be seen that the maximum number of EVs in \( \mathbb{R}^n \) never exceed \( n+1 \).

Notice that we can extend the matrix \( S_n \) to the factor of orthogonal matrix by adding a new row \( \sqrt{\frac{n}{n-1}} e \) in the bottom. The new matrix can be written as \( \sqrt{\frac{n}{n-1}} Q_n \). (see condition 3’). So \( S_n \) can be viewed as the normalized projection of the columns of \( Q_n \) onto the orthogonal complement of the vector \( e_n = [0, 0, \ldots, 1]^T \), which is spanned by of the sum of the column basis of \( Q_n \). It implies that in \( S_n \) the row sums all equal to zero, i.e., if \([S_n]_{ij} = s_{ij} \in \mathbb{R}^{n \times (n+1)}\) then \( s_{ij} + \sum_{j=1}^{n+1} s_{ij} = 0 \) (**).

**Theorem 6.4.** Let \( \{s_1, s_2, \ldots, s_{n+1}\} \) be the set of equiangular vectors in \( \mathbb{R}^n \) by the angle \( \theta \), where \( \cos \theta = -\frac{1}{n} \) for \( n \geq 1 \). Then the sequence \( \{s_i\}_{i=1}^{n+1} \) is a tight frame with \( A = \frac{n+1}{n} \).

**Proof.** From definition of ETF the result is obvious \((m = n+1)\). However we prove it by Definition \([6.1] \). We have to show
\[
\frac{n+1}{n} ||x||^2 = \sum_{i=1}^{n+1} (x, s_i)^2, \quad \text{for all } x \in \mathbb{R}^n. \tag{6.2}
\]

Without loss of generality let \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) is chosen to be a normalized vector \( (||x|| = 1) \). For convenience assume that \( s_1, \ldots, s_{n+1} \) are the column vectors of the matrix \( S_n \) as noted in the Lemma \([6.3] \). The prove is by induction. For \( n=1,2 \), \( (6.2) \) trivial: \( \sum_{i=1}^{3} (x, s_i)^2 = x_1^2 + (-\frac{1}{n} x_1 + \rho s_2 x_2)^2 + (-\frac{1}{n} x_1 + \rho s_3 x_3)^2 + \ldots \)

\[
= \left(1 + \frac{n}{n^2} \right) x_1^2 - \frac{2}{n} x_1 \left( s_2 x_2 + s_3 x_3 \right) + \rho^2 s_2 x_2^2 + \rho^2 s_3 x_3^2 + \ldots
\]

\[
= \left[ s_2 x_2 + s_3 x_3 + \ldots \right] + \rho^2 s_2 x_2^2 + \rho^2 s_3 x_3^2 + \ldots
\]

\[
= \frac{n+1}{n} x_1^2 \quad \text{for } n=1,2.
\]

As noted Algorithm \([\ref{alg:2}] \) is not designed for the angles greater than \( \frac{\pi}{2} \). To solve this problem, it is better to consider the figure \( \frac{\pi}{2} < \theta < \pi \) where \( \theta = \pi - \theta \), the formula \( (2.1) \) still works. Because \( \tilde{\nu}_2 = v + \cot \Theta ||v||, s_1 = v - \cot \Theta ||v||, s_1 \), so the new \( \tilde{\nu}_2 \) is the symmetry of the prior \( \tilde{\nu}_2 \) (related to \( \theta \)) with respect to the vector \( v \) as the axis of the symmetry. This new formula also can be written as \( \tilde{\nu}_2 = v - \cot \Theta ||v||, s_1 \).

This method can happen for \( \tilde{\nu}_k \ (k \geq 3) \). So it is enough to replace the plus sign in the line 4 of Algorithm \([\ref{alg:2}] \) to the minus sign as follows
\[
\tilde{\nu}_k = u - \left( \sqrt{k-1}/\sqrt{((\sec \Theta - 1)(\sec \Theta + k-1))} \right) \cdot s. \tag{6.3}
\]

Moreover, \( (6.3) \) determines the possible range of \( \Theta \) that can be used to make a set of equiangular vectors. Since \( \sec \Theta < -1 \), then we must have \( (\sec \Theta + k-1) < 0 \). It implies
\[
\cos \Theta > -\frac{1}{k-1}. \tag{6.4}
\]

The relation \( (6.4) \) shows that the supremum of \( \Theta \) in such a way that a set of \( n \) linearly independent vectors can be ordered as the equiangular vectors is \( \cos^{-1}(\frac{1}{n-1}) \). So in this case the Algorithm \([\ref{alg:2}] \) will fail and accordingly,
the equiangular vectors with the noted angle return to a set of \( n \) equiangular vectors with the rank of \( n - 1 \), which is equivalent to the Lemma 6.3.

**How to construct an ETF by Algorithm 1** It is considerable that a set of \( n + 1 \) vectors \( v_1, \ldots, v_{n+1} \) with the rank of \( n \) in \( \mathbb{R}^n \) can be transformed to an equiangular frame by the Algorithm 1. Assume that \( v_1, \ldots, v_n \) are linearly independent. We apply Algorithm 1 by the angle \( \cos^{-1}(-\frac{1}{n}) \) as mentioned earlier. Then we get the equiangular vectors \( s_1, \ldots, s_n \) as an equiangular non-tight frame. To specify the last vector \( s_{n+1} \) we utilize the matrix \( S_n \). Clearly there is an orthogonal matrix so that \( QS_n = [s_{n+1}, s_1, \ldots, s_n] \). Hence \( s_{n+1} = Qc_1 = q_1 \). We also can write \( Q[-\frac{1}{n}e, \rho S_{n-1}^T]^T = [s_1, \ldots, s_n] \), so \( Q = [s_1, \ldots, s_n][-\frac{1}{n}e, \rho S_{n-1}^T]^{-T} \). Now it is enough to find \( q_1 \). Note that if \( s_1, \ldots, s_n \) are linearly dependent, then the Algorithm 1 will fail. To obviate this failure in any step of it we must exchange the linearly dependent vector by the next one until \( n \) equiangular vectors are produced as desired. The art of Algorithm 1 causes to ask that is it possible to produce a step-by-step method to construct a set of equiangular lines with the maximum number?

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