The Born Rule Dies

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The Born rule may be stated mathematically as the rule that probabilities in quantum theory are expectation values of a complete orthogonal set of projection operators. This rule works for single laboratory settings in which the observer can distinguish all the different possible outcomes corresponding to the projection operators. However, theories of inflation suggest that the universe may be so large that any laboratory, no matter how precisely it is defined by its internal state, may exist in a large number of very distantly separated copies throughout the vast universe. In this case, no observer within the universe can distinguish all possible outcomes for all copies of the laboratory. Then normalized probabilities for the local outcomes that can be locally distinguished cannot be given by the expectation values of any projection operators. Thus the Born rule dies and must be replaced by another rule for observational probabilities in cosmology. The freedom of what this new rule is to be is the measure problem in cosmology. A particular volume-averaged form is proposed.

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I. INTRODUCTION

In quantum theory, all probabilities for a system are supposed to be encoded in its quantum state. This may be true, but there is the question of how to decode the quantum state to give these probabilities. In traditional quantum theory, using the statistical interpretation of the wavefunction given by the Nobel Prize-winning Born rule [1], the probabilities are given by the expectation values of projection operators. Once a possible observation is specified (including the corresponding projection operator), then its probability is given purely by the quantum state as the expectation value the state assigns to the projection operator.

This prescription seems to work well in ordinary single laboratory settings, where the observer can distinguish between a set of different outcomes. Then distinct observations are mutually exclusive, so that different ones cannot both be observed. If one assigns a projection operator to each possible distinct observation in a complete exhaustive set, then these projection operators will be orthonormal, and their expectation values will be non-negative and sum to unity, which are conditions necessary for them to be interpreted as the probabilities of the different possible observations.

However, in cosmology there is the possibility that the universe is so large that there are many copies of each laboratory and observer, no matter how precisely the laboratory and observer are defined. (Here what one interprets to be a ‘laboratory’ can be as small as a single particle or as large as the entire earth or solar system, or maybe even as large as an entire causal diamond [2].) Similarly, the observer can be as small or as large as one might consider an observer to be, perhaps as small as whatever part of the brain of a tiny animal leads to a single conscious perception, or as large as a whole community, such as a human scientific information gathering and utilizing system [3], or even as large as the set of all humans and other conscious beings within the solar system. For this discussion, however, I would like to restrict to cases in which the different parts of the ‘observer’ are in communication with each other, so that I shall not count as a single ‘observer’ collections of more than one entirely separate civilization, not in communication with each other, say very widely separated within the universe so that they actually or effectively have no causal contact with each other.)

This possibility of many identical copies of each laboratory and observer raises the problem [3,4,5,6,7] that two observations that are seen as distinct for an observer are not mutually exclusive in a global viewpoint of the entire universe; both can occur for different copies of the laboratory and observer (though neither copy may be aware of that). This would not be a problem for a putative superobserver who can observe all possible sets of observations by all observers over the entire universe, but it is a problem for the assignment of normalized probabilities for the possible observations that are distinct for each copy of the observer. The result [5,7] is that one cannot get such a set of normalized probabilities as the expectation values of projection operators in the full quantum state of the universe.

One can use Tegmark’s language [5,8,10] of a ‘bird’ for a superobserver that can ‘observe’ (not necessarily in the traditional quantum sense of making a measurement that disturbs the system) or know the entire history of the universe (or its entire quantum state), and a ‘frog’
for a localized observer that is entirely in communication or causal contact with itself within the universe. The problem is that traditional quantum theory with its Born rule applies to birds but not to frogs, if the universe is so large that there are many exact copies of the frogs. In other words, traditional quantum mechanics is for the birds.

What we need instead is a replacement of Born’s rule that applies within the universe to us frogs that might exist in many copies, identical except for our location relative to our distant surroundings that we are not aware of. (If different frogs are aware that their surroundings are different, that awareness makes the frogs themselves different and so not exact copies of each other. But if the differences in the surroundings are not reflected in the internal states of the frogs, say if the surroundings are so far away that they are not in causal contact with the frogs, then the frogs can be identical and yet be at different locations as seen by the birds.)

One can still postulate that there are rules for getting the probabilities of all possible frog observations from the quantum state, but then the question arises as to what these rules are. Below we shall give examples of several different possibilities for these rules, showing that they are not uniquely determined. Thus, they are logically independent of the question of what the quantum state is. Therefore, the quantum state just by itself is insufficient to determine the probabilities of observations for us frogs within the universe.

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II. THE SET OF ALL POSSIBLE OBSERVATIONS

One goal of science is to produce good theories $T_i$ that enable one to predict the probabilities of results of observations by localized observers within the universe (‘frogs’, rather than ‘birds’ that are hypothetical superobservers for the entire universe). For brevity, I shall refer to the results of observations simply as ‘observations.’ Here for simplicity I shall assume that there is a countable set of possible distinct observations $O_j$ out of some exhaustive set of all such observations.

If one imagines a continuum for the set of observations (which seems to be logically possible, though not required), in that case I shall assume that they are binned into a countable number of exclusive and exhaustive subsets that each may be considered to form one distinct observation $O_j$. Then the goal of a good complete theory $T_i$ is to calculate the probability $P_j(i) = P(O_j|T_i)$ for each possible observation $O_j$, given the theory $T_i$.

The set of possible observations might be, for example, all possible conscious perceptions [28, 61], all possible data sets for one person, all possible contents for an eprint arXiv, all possible data sets for a human scientific information gathering and utilizing system [3], or all possible data sets for any community of observers. However, as mentioned above, I shall assume that the observer, whether a single organism or a whole community, is in complete causal contact with itself and is not a collection of disconnected organisms or communities.

In order that the sum of the probabilities for all members of the set of possible observations be unity, I shall also assume that the set of observations is mutually exclusive, so that any particular observation is a unique and distinct member of the set. In particular, each observation is to be complete and not a subset of another observation, to avoid double counting and unnormalized probabilities. The simplest way I know to impose this is to take observations to be conscious perceptions, all that an organism is phenomenally aware of at once [28, 61]. I shall generally have in mind this example of what an observation is, but my argument applies to any definition of an observation so long as the possible observations are mutually exclusive.

For example, if one defined observations to be data obtained by or held by a community of observers rather than by a single organism, one needs to define what comprises the community so that subsets of the community are not separately counted. That is, suppose one has organisms $A$, $B$, and $C$ with observed data $\alpha$, $\beta$, and $\gamma$ respectively. One could define communal observers to be any subset of the set $\{A, B, C\}$ of organisms and observations to be the data of that subset. One allowed choice of what the observations are would be $\{\alpha, \beta, \gamma\}$, with these three distinct observations (analogous to my personal preference for taking the observations to be conscious perceptions, assuming that each conscious perception can be attributed to one or another of the organisms but not jointly to any combination of them). Another allowed choice would be to take the entire community of $A$, $B$, and $C$ as one observer, so that the observation would be the combined data of $\alpha$, $\beta$, and $\gamma$, making one single observation, the one-element set $\{\alpha \beta \gamma\}$. But in this paper I do not allow one to consider sets of observations such that one observation within the set is part of another observation within the set, so that the two cannot be considered mutually exclusive. That is, I do not allow the set of observers to be taken to be all possible nonzero subsets of the set of the three organisms, the set $\{A, B, C, AB, AC, BC, ABC\}$ with observations $\{\alpha, \beta, \gamma, \alpha \beta, \alpha \gamma, \beta \gamma, \alpha \beta \gamma\}$, since, for example, the observation $\alpha$ is part of the observation $\alpha \beta$, so that these two
observations are not mutually exclusive.

Thus for this paper, the set of possible (frog) observations must be distinct, so that one observation is not simply part of another observation also considered to be within the set. This requirement seems simple to meet if the observations are considered to be conscious perceptions, since if organism A has conscious perception $\alpha$ and organism B has conscious perception $\beta$, the combination $\alpha\beta$ is not a conscious perception but a combination of two. (Even for a single organism, its conscious perceptions at two different times are distinct conscious perceptions, and the combination of the two is not a conscious perception, an awareness that is perceived at once by any frog within the universe. When one is aware of both the past and present, this is not actually a combination of a past conscious perception and a present conscious perception, but a single present conscious perception that has elements within it both of awareness of the present and of awareness of memories of the past.) This paper allows observations to be more general than conscious perceptions (for those readers sceptical about the fundamental nature of conscious perceptions), but for the conclusions of this paper, one is not allowed to consider a set of possible observations such that one element of the set contains another element rather than being distinct from it.

To give another illustration of the restriction, consider the idea that observations are finite strings of binary digits within a toy universe made up of an infinite string of binary digits. It would be acceptable to take the set of possible observations to be the set of all binary strings of a certain fixed length, since all such strings would be distinct. However, if one tried to take the set of possible observations to be the set of all binary strings of all finite lengths, then the observations would not be distinct, since, for example, the string $\{0101\}$ would be contained within the string $\{01010\}$ and so would not be distinct: if one observation was of $\{01010\}$, then the observation $\{0101\}$ would also necessarily occur.

This example does not imply that one could not consider the set of possible observations to include strings of different length, so long as the length is part of the observation. For example, one could define the set of observations to be all finite substrings of the universe’s infinite string that start and end with the string 00 and have no 00 inside (between the two 00’s at the ends). Then the set of possible observations would be the strings $\{0000\}$, $\{00000\}$, $\{00100\}$, $\{000100\}$, $\{001000\}$, $\{001100\}$, etc. (with all other strings being longer than these ones explicitly listed and with $\{000000\}$ not being a possible observation since it has a 00 between the two 00’s at the ends). The idea is that if in this toy model possible observations are defined to be certain strings of digits or integers, then no allowed strings should be proper subsets of other allowed strings.

Note that in this paper I am not assuming that observations are eigenstates or eigenvalues of Hermitian operators, or that they correspond to subspaces of a Hilbert space. Locally they may have that form, but the fact that the location of the observation is indeterminate (cannot be known by the frogs making the observation) means that observations are not globally eigenstates or eigenvalues or subspaces of a Hilbert space. Therefore, theorems such as Gleason’s [42] that may be taken to imply the uniqueness of Born’s rule need not apply to the frog observations being considered in this paper.

**III. PROBABILITIES FOR OBSERVATIONS**

Once one has defined a mutually exclusive set of all possible observations $O_j$, a goal of science is to produce good theories $T_i$ that each give normalized nonnegative probabilities $P_j(i) \equiv P(O_j|T_i)$ for the observations $O_j$,

$$\sum_j P_j(i) = 1, \quad (3.1)$$

for each theory $T_i$.

One might think that once one has the quantum state, there would be a standard answer to the question of the probabilities for the various possible observations. For example [5, 7], one might take traditional quantum theory (what I there called standard quantum theory) to use Born’s rule and hence give the probability $P_j(i)$ of the observation as the expectation value, in the quantum state given by the theory $T_i$, of a projection operator $P_j$ onto the observational result $O_j$. That is, one might take

$$P_j(i) = \langle P_j \rangle_i, \quad (3.2)$$

where $\langle \rangle_i$ denotes the quantum expectation value of whatever is inside the angular brackets in the quantum state $i$ given by the theory $T_i$. This traditional approach (Born’s rule) works in the case of a single laboratory setting where the projection operators onto different observational results are orthogonal, $P_j P_k = \delta_{jk} P_j$ (no sum over repeated indices).

However [6, 7], in the case of a sufficiently large universe, one may have many laboratories that are locally identical (e.g., without consideration of the surroundings that are not at all reflected by the quantum state or data within the laboratory, data that are accessible only to a hypothetical superobserver or bird). Then, within different copies of these locally identical laboratories (different only from the bird perspective), one can have observation $O_j$ occurring ‘here’ and observation $O_k$ occurring ‘there’ in a compatible way, so that $P_j$ and $P_k$ are not orthogonal. Then the traditional quantum probabilities given by Born’s rule, Eq. (3.2), will not be normalized to obey Eq. (3.1). An explicit proof of this will now be given for a toy model.
IV. TOY MODEL PROOF THAT BORN’S RULE DOES NOT WORK

To illustrate the problem with Born’s rule and prove that we cannot obey Eq. (3.1) with Born-rule probabilities Eq. (3.2), let us consider a toy model for a universe at one moment of time in which each component of the quantum state has \(N\) regions that can each have either no observer, denoted by 0, or one observer (which, as discussed above, can be taken to be an entire communicating community, so long as it is defined so that proper subsets of the community are not also counted as additional observers) with observation \(O_j\), denoted by \(j\). (Different values of \(N\) model different sizes of universes produced by differing amounts of inflation in the cosmological measure problem.) Write the quantum state as a superposition, with complex coefficients \(a_N\), of component states \(|\psi_N\rangle\) that each have different numbers \(N\) of regions:

\[
|\psi\rangle = \sum_{N=1}^{\infty} a_N |\psi_N\rangle, \tag{4.1}
\]

where \(|\psi_M|\psi_N\rangle = \delta_{MN}. Let each of these component states \(|\psi_N\rangle\) of fixed size (number of regions being \(N\)) be called a size eigenstate, or an \(N\)-state if we want to denote the size or number of regions \(N\).

Furthermore, write each component state with a definite number \(N\) of regions as a superposition of orthonormal states in the tensor product of \(N\) regions that can each be labeled by either having no observation, 0, or by having the observation \(j\), in the region \(L\), \(1 \leq L \leq N\). That is, if we let \(\mu_j\) be either 0 if the region \(L\) has no observer or else \(j\) if the region \(L\) has the observation \(j\), then the state for a definite \(N\) can be written as

\[
|\psi_N\rangle = \sum_{\mu_1, \mu_2, \ldots, \mu_N} b_{\mu_1 \mu_2 \ldots \mu_N} |\mu_1 \mu_2 \ldots \mu_N\rangle, \tag{4.2}
\]

where the component state \(|\mu_1 \mu_2 \ldots \mu_N\rangle\) has \(\mu_1\) (either no observation, 0, or the observation \(O_j\) that is denoted by \(j\)) in the first region, \(\mu_2\) in the second region, and so on with all \(\mu_L\) for \(0 \leq L \leq N\) up through \(\mu_N\) in the \(N\)th region. Let each of these component states be called a bird’s-eye eigenstate of the observations, or a bird eigenstate. (Each frog can only know of its particular \(O_j\) and not what the bird eigenstate is or even what its own location \(L\) is within the bird eigenstate with its \(N\) regions.) The full quantum state \(|\psi\rangle\) is then a superposition of size eigenstates \(|\psi_N\rangle\) given by Eq. (4.1), with each size eigenstate being itself a superposition of bird eigenstates \(|\mu_1 \mu_2 \ldots \mu_L\rangle\) given by Eq. (4.2).

It is also convenient to define an \(NN_j\) eigenstate as the sum, for fixed \(N\), given by Eq. (4.2) restricted to the terms that have a fixed number \(N_j\) of occurrences of the observation \(j\) within the string of regions. One can further define an \(NN_j\) eigenstate as the sum, for fixed \(N\), given by Eq. (4.2) restricted not only to the terms that have a fixed number \(N_j\) of occurrences of the observation \(j\) but also to terms that have the fixed number \(N_k\) of the observation \(k\), \(j \neq k\). One could then include just these sums for each \(N\) in the sum over \(N\) given by Eq. (4.1) to get the corresponding \(N_j\) and \(N_j\) eigenstates that are eigenstates of the occurrence of precisely \(N_j\) observations \(j\) and of the mutual occurrence of precisely \(N_j\) observations \(j\) and precisely \(N_k\) observations \(k\). Of course, distinguishing all of these eigenstates is only possible for hypothetical superobservers or birds who can see the entire universe; it is not possible for the frogs that can see only what is in their individual regions.

Now let us assume these very minimal principles:

**No Extra Vision Principle** (NEVP):

The probabilities of observations that have zero amplitudes to occur anywhere in the quantum state are zero; one cannot see what is not there in the quantum state.

**Probability Symmetry Principle** (PSP):

If the quantum state is an eigenstate of equal number of observations of two different observations, then the probabilities of these two observations are equal.

The No Extra Vision Principle is the assumption that frogs can’t see what birds don’t see, that if for some \(j\) the bird quantum probability \(p_{N_Lj}\) is zero for all \(N\) and \(L\) in theory \(T_j\), then the frog probability \(P_j(i) = P(O_j|T_i)\) is also zero. In other words, I assume that if there is zero quantum probability for the bird to see the observation occurring anywhere, then the probability is also zero for the frogs to make that observation.

If one did not make some assumption rather like that, it is hard to see how the quantum state could determine the frog probabilities in a reasonable way. I suppose one might take a mystical attitude and postulate that frogs can see visions of what has no basis in the physical quantum state, but for this paper I shall assume that anything that can be observed within the universe has a corresponding amplitude in the quantum state.

(1) That is, no visions or even revelations from God that do not directly correspond to physical reality external to the observer, since they can be caused by the part of the quantum state of the observer himself/herself/itself. In principle that part of the quantum state could even be directly caused by God Himself without having to go through the usual mechanism of coming in from external stimuli, though of course external stimuli can also be directly caused by God and can be so even in the case in which they are correctly described by theories of physics; the two descriptions are not logically contradictory. Nothing in this paper assumes or denies the possibility of such visions and/or revelations, but I am assuming that whatever is observed does have a corresponding nonzero amplitude in the quantum state to mediate it. This might be taken to be an analogue in physics of the Biblical claim that the Word became flesh.)

The Probability Symmetry Principle is the assumption that if the full quantum state \(|\psi\rangle\) is an \(N_j\) eigenstate with \(N_j = N_k\) in theory \(T_i\), then \(P_j(i) = P_k(i)\) for \(j \neq k\).
(though this equality is also trivially true for \( j = k \)). The idea of this principle is that if the quantum state has equal definite numbers of two different observations, there seems to be no good reason to assign one of them greater probability than the other. The only thing that distinguishes them (other than what the observations are intrinsically) is their location, and it is hard to see why that should favor one over another. Furthermore, if one imagined some sort of diffeomorphism invariance that allowed one to translate any one location to any other, there would be no absolute distinction to the locations, so why should one assign different probabilities to observations at the different locations? (Of course, one could say that from the bird’s eye view, though not to the frogs, the relative locations can differ, so it might be that one of the observations of \( j \) is next to another observation of \( j \), whereas there is no observation of \( k \) next to another observation of \( k \), but it is hard to see why this nonlocal relative information that only the birds have should affect the probabilities of the local frog observations.)

To show that Born’s rule, Eq. (3.2), necessarily violates the normalization condition, Eq. (3.1), when the No Extra Vision Principle and the Probability Symmetry Principle are assumed, consider the size eigenstates with \( N = 2 \) that have the form

\[
|\psi\rangle = b_{11}\langle 11| + b_{12}\langle 12| + b_{21}\langle 21| + b_{22}\langle 22|,
\]

so that in each component, precisely two observations are made that are either \( j = 1 \) or \( j = 2 \), with the state normalized, \( |b_{11}|^2 + |b_{12}|^2 + |b_{21}|^2 + |b_{22}|^2 = 1 \).

Now if \( b_{11} = 1 \), so \( |\psi\rangle = |11\rangle \) (an \( N_jN_k \) eigenstate with \( N_1 = 2 \) and \( N_2 = 0 \), then only the observation 1 occurs as seen by the birds, so \( P_1 = 1 \) and \( P_2 = 0 \) by the No Extra Vision Principle and by the normalization requirement. Similarly, if \( b_{22} = 1 \), so \( |\psi\rangle = |22\rangle \) (an \( N_jN_k \) eigenstate with \( N_1 = 0 \) and \( N_2 = 2 \), then only the observation 2 occurs, so then \( P_1 = 0 \) and \( P_2 = 1 \). On the other hand, a state orthogonal to both of these \( N_jN_k \) eigenstates, \( |\psi\rangle = b_{12}\langle 12| + b_{21}\langle 21| \) with \( b_{11} = b_{22} = 0 \) is an \( N_jN_k \) eigenstate with \( N_1 = N_2 = 1 \) and so by the Probability Symmetry Principle has \( P_1 = P_2 = 2 \), both of which must be 1/2 to be normalized, obeying Eq. (3.1).

The proof that Born’s rule fails to give this proceeds by showing that there is no pair of orthogonal projection operators \( P_1 \) and \( P_2 \) in this 4-state system whose expectation values reproduce these probabilities by Born’s rule Eq. (3.2) for all such quantum states.

For Born’s rule to give, for \( |\psi\rangle = |11\rangle \), \( P_1 = (P_1) \equiv \langle \psi|P_1|\psi\rangle = \langle 11|P_1|11\rangle = 1 \), the projection operator \( P_1 \) must have the form \( |11\rangle\langle 11| + P_1' \) where \( |11\rangle\langle 11| \) = 0. Similarly, for Born’s rule to give, for \( |\psi\rangle = |22\rangle \), \( P_2 = (P_2) \equiv \langle \psi|P_2|\psi\rangle = \langle 22|P_2|22\rangle = 1 \), the projection operator \( P_2 \) must have the form \( |22\rangle\langle 22| + P_2' \) where \( |22\rangle\langle 22| \) = 0.

Furthermore, for Born’s rule to give, for \( |\psi\rangle = |11\rangle \), \( P_2 = (11|P_2|11\rangle = 0 \), we need \( |11\rangle\langle 11| \) = 0. Similarly, for Born’s rule to give, for \( |\psi\rangle = |22\rangle \), \( P_1 = (22|P_1|22\rangle = 0 \), we need \( |22\rangle\langle 22| \) = 0. Since \( P_1 \) and \( P_2 \) are to be orthogonal projection operators, \( P_1P_1 = P_1 \), \( P_2P_2 = P_2 \), and \( P_1P_2 = P_2P_1 = 0 \), one can easily show that \( P_1' \) and \( P_2' \) themselves must be orthogonal projection operators within the subspace of states of the form \( b_{12}\langle 12| + b_{21}\langle 21| \) that are orthogonal to \( |11\rangle \) and \( |22\rangle \).

That is, to give the right probabilities for the states \( |\psi\rangle = |11\rangle \) and \( |\psi\rangle = |22\rangle \) by Born’s rule, and to give nonzero probabilities when \( N_1 = N_2 = 1 \), the projection operators must have the form \( P_1 = |11\rangle\langle 11| + |\psi_{12}\rangle\langle \psi_{12}| \) and \( P_2 = |22\rangle\langle 22| + |\psi_{12}'\rangle\langle \psi_{12}'| \) where \( |\psi_{12}\rangle = \cos \theta |12\rangle + \sin \theta e^{-i\phi} |21\rangle \) and \( |\psi_{12}'\rangle = -\sin \theta e^{-i\phi} |12\rangle + \cos \theta |21\rangle \) are two orthogonal states within the subspace orthogonal to \( |11\rangle \) and \( |22\rangle \), with the two arbitrary real parameters \( \theta \) and \( \phi \).

So far we have not run into the problem, but now we do. For a generic \( N_jN_k \) eigenstate with \( N_1 = N_2 = 1 \), \( |\psi\rangle = b_{12}\langle 12| + b_{21}\langle 21| \), Born’s rule does not give \( P_1 = (P_1) = 1/2 \) or \( P_2 = (P_2) = 1/2 \). In particular, if the quantum state has \( |\psi\rangle = |12\rangle \), then \( (P_1) = 1 \) and \( (P_2) = 0 \) instead of \( P_1 = P_2 = 1/2 \) for this \( N_jN_k \) eigenstate as required by the Probability Symmetry Principle.

One might object that if one were allowed to choose the projection operators \( P_1 \) and \( P_2 \) after the state is known, one can avoid the problem altogether. For example, if one knows that the quantum state is the particular \( N_jN_k \) eigenstate with \( N_1 = N_2 = 1 \) that is \( |\psi\rangle = b_{12}\langle 12| + b_{21}\langle 21| \), then with any orthogonal state \( |\psi'\rangle = b_{21}\langle 12| - b_{12}\langle 21| \) one can choose \( |\psi'\rangle = (|\psi\rangle + i|\psi'\rangle)/\sqrt{2} \) and \( |\psi''\rangle = (|\psi\rangle - i|\psi'\rangle)/\sqrt{2} \) to get \( P_1 = |11\rangle\langle 11| + |\psi_{12}\rangle\langle \psi_{12}| \) and \( P_2 = |22\rangle\langle 22| + |\psi_{12}'\rangle\langle \psi_{12}'| \) that would then give \( (P_1) = (P_2) = 1/2 \), in accord with the Probability Symmetry Principle. However, it is not correctly to have to know the quantum state in order to choose the projection operators whose expectation values in that state are the probabilities. If one were allowed to do that, one could always choose the orthonormal projection operators to give any normalized set of nonnegative probabilities one wanted, no matter what the quantum state is. In order for the probabilities to depend on the quantum state in some reasonable way, the rule for extracting the probabilities from the state should not be allowed to depend on the quantum state in such an ad hoc manner.

One might also object to the Probability Symmetry Principle, but it seems highly unnatural to give that up. One could instead just pick a preferred location and then normalize the relative probabilities for the various observations to occur there. Even if one had diffeomorphism invariance so that intrinsically there is no preferred location, one might just choose the relative location that, say, has the smallest rms distance to all other observations, as seen by the bird. But such ad hoc choices to favor one location over another certainly seem unlikely to be the way that nature really works, though one might want to consider it further before rejecting it completely.

Thus we have seen that if we require that the projection operators be chosen independently of the quantum state, and if the observational probabilities are to obey
the Probability Symmetry Principle (equal probabilities
for different observations that definitely occur an equal
number of times), such probabilities cannot be given by
Born’s rule as the expectation value of the projection op-
erators, Eq. (3.2), if we allow states with more than one
observation actually occurring that are not just quantum
alternatives.

Thus one needs a formula different from Born’s rule for
normalized frog probabilities of a mutually exclusive and
exhaustive set of possible observations, when distinct ob-
servations within the complete set cannot be described by
orthogonal projection operators. Here, by an ‘exhaustive’
set of possible observations, I mean all that can be locally
distinguished by the frogs, without considering the dis-
tinctions that the birds may see by being able to identify
the different locations (different distant surroundings) of
the copies of the laboratories and observers.

V. COMPARISON WITH CLASSICAL THEORY

One might note that conceptually this problem is not
peculiar to quantum theory. If one has a classical the-
ory, one might say that the analogue of the expectation
values of projection operators is the set of 0’s and 1’s for
whether something does not occur or does occur in the
classical behavior. One might then say that the analogue
of Born’s rule in classical theory is that the probability
is 0 for something that does not occur and 1 for some-
thing that does occur. This is fine for the view of a
bird that can see the entire classical behavior and can see
whether or not something occurs. However, for the
view of a frog within the classical universe that cannot
see the whole thing, one wants normalized probabilities
for the frog observations. If more than one observation
can occur within the classical behavior, then they cannot
have normalized probabilities that are each 1. One might
take the number of times a particular observation occurs
(or the time interval during which it occurs, if it occurs
continuously over a finite range of time) and divide by
the total for all possible observations to get a normalized
probability for the particular observation, but if there are
more than one actual observations, this normalized prob-
ability for the observation is not 1 and so is not given by
the classical analogue of Born’s rule as the expectation
value of a projection operator.

Perhaps the problem is so obvious in classical theory
that few would propose that normalized probabilities for
a set of possible observations in classical theory would all
be 0’s or 1’s if they considered cases in which more than
one observation can actually occur. But in quantum the-
ory, the expectation values of projection operators can be
anywhere between 0 and 1 inclusive, so perhaps it was
not so obvious that there also one cannot use these expec-
tation values as normalized probabilities for observations
if more than one can actually occur.

Another reason why this problem was not widely recog-
nized is that what we can see of the universe is so small, in
some sense, that most macroscopic systems within it are
complex enough to be unique within the limited part of
the universe we can observe. Atoms and small molecules
of course are not complex enough to be unique within
what we can see; we believe there are billions of identical
atoms and molecules within each of us. However, even
objects as small as snowflakes have far, far more possible
configurations (presumably very roughly the exponential
of some not-too-small fractional power, perhaps 2/3, of
the number of molecules, say $10^{20}$, in a snowflake, I would
guess at least $10^{10}$ possible molecular configurations
if the fractional power is not smaller than one half) than
the number of particles in the observable part of the uni-
verse (which is much, much less than $10^{2.5}$, which is
roughly the exponential of the one-eighth power of the
number of molecules in a snowflake; surely the fractional
power is not so small as one eighth). Therefore, it is plau-
sible that no two snowflakes in our part of the universe
are identical.

(There may be some snowflake configurations, such as
some precisely regular crystals, that are so much more
probable than the average irregular configurations that
it is conceivable that more than one of them might occur
within the observable universe. I simply do not know
the probabilities for such perfect crystals with a defi-
nite symmetric arrangement of water molecules, which
by minimizing the energy would maximize the probabil-
ity for a microscopic configuration in a thermal state.
However, it is surely the case that most snowflakes are
sufficiently irregular that it is highly improbable that any
others of those have the same microscopic arrangements
of molecules, say using the quantum criterion that the
trace of the product of the density matrices for the rela-
tive locations of the hydrogen and oxygen nuclei within
the two snowflakes is greater than, say, one half, to avoid
the objections of those who might say no two snowflakes
at different locations would have exactly the same quan-
tum state in the sense of having exactly the same density
matrices, with exactly the same probabilities of all the
excited energy eigenstates.)

With much of what we can see being very likely unique
within our observable universe, when one ignores the pos-
sibility that the universe extends far beyond what we can
see, one can also ignore the possibility that our observa-
tion, at least if it is sufficiently detailed, occurs more than
once. However, when we consider the possibility that the
universe is exponentially larger than what we can see of
it (either from a model of an infinite universe, such as a
simply connected $k = -1$ Friedmann-Robertson-Walker
model, or from a finite model that has been expanded to
enormously large sizes by inflation), we must face the
possibility that a local observation by a frog, no mat-
ter how detailed, may not be unique from the bird per-
spective. Then from the bird perspective, the different
observations are not mutually exclusive, so the sum of
their probabilities within the bird perspective can exceed
unity. However, from the frog perspective, its observa-
tion is a unique realization out of the set of possible frog
observations, so we frogs want normalizable probabilities for our possible frog observations. This discrepancy between the exclusivity of the observations from the frog perspective and their mutual compatibility from the bird perspective is what prevents the frog probabilities from all being 0 or 1 in classical theory and from being expectation values of orthonormal projection operators in quantum theory. It may be that the Born rule works for the birds, but it does not work for us frogs.

VI. EXAMPLES OF DIFFERENT OBSERVATIONAL PROBABILITIES FOR THE SAME QUANTUM STATE

Let us demonstrate the logical freedom in the rules for the observational probabilities $P_j(i) \equiv P(O_j|T_i)$ by exhibiting various examples of what they might be. For simplicity, let us restrict attention to theories $T_i$ that all give the same pure quantum state $|\psi\rangle$, which in the toy model above with different regions at one moment of time can be written as the superposition given by Eqs. (4.1) and (4.2), i.e., as

$$|\psi\rangle = \sum_{N=0}^{\infty} a_N |\psi_N\rangle = \sum_{N=0}^{\infty} a_N \sum_{\mu_1,\mu_2,...,\mu_N} b_{\mu_1\mu_2...\mu_N} |\mu_1\mu_2...\mu_N\rangle$$

(6.1)

in terms of the bird eigenstates $|\mu_1\mu_2...\mu_N\rangle$ in which the first of the $N$ regions has $\mu_1 = 0$ if no observation occurs there or $\mu_1 = j$ if the observation $O_j$ occurs there, and similarly for all other regions $L$ up to $N$ for each $N$.

As an example of such a state that I shall use to compare the results of the various replacements of Born’s rule is the following superposition of two bird eigenstates of different sizes:

$$|\psi\rangle = \cos \theta |m_0; m_1; m_2\rangle + \sin \theta |n_0; n_1; n_2\rangle$$

(6.2)

where $|m_0; m_1; m_2\rangle$ means the bird eigenstate with $N_m = m_0 + m_1 + m_2$ regions such that $\mu_L = 0$ (no observation) for the first $m_0$ regions, $0 \leq L \leq m_0$, $\mu_L = 1$ (the observation $O_1$) in the next $m_1$ regions, $m_0 + 1 \leq L \leq m_0 + m_1$, and $\mu_L = 2$ (the observation $O_2$) in the last $m_2$ regions, $m_0 + m_1 + 1 \leq L \leq N = m_0 + m_1 + m_2$,

and where, similarly, $|n_0; n_1; n_2\rangle$ means the bird eigenstate with $N_n = n_0 + n_1 + n_2$ regions such that $\mu_L = 0$ for the first $n_0$ regions, $\mu_L = 1$ in the next $n_1$ regions, and $\mu_L = 2$ in the last $n_2$ regions. To illustrate some points I wish to make below, unless stated otherwise I shall assume the generic case in which none of the integers $m_0, n_1, m_2, n_0, n_1$, and $n_2$ are zero or are equal to each other.

Let us suppose that $P_{\mu}^L$ is a complete set of orthogonal projection operators for the region $L$, either for there to be no observation ($\mu = 0$) in that region or else for the observation $O_j$ to occur ($\mu = j$) there, so $P_{\mu}^L P_{\nu}^L = \delta_{\mu\nu} P_{\mu}^L$ (no sum over $\mu$) and $\sum_{\mu} P_{\mu}^L = I$, the identity operator. (In a more realistic model, I would not assume that any of these projection operators is rank one, having only pure state eigenstates, so they would not pick out a unique basis, but in my toy model I am assuming that the state in each region is given uniquely by the value of $\mu$ there for the bird eigenstates.) For simplicity, I am also assuming that all of the different regions are spacelike separated (e.g., are at the same time), so that the Hilbert space for each number of regions $N$ is the tensor product of the Hilbert spaces for each region. Thus I am assuming that each $P_{\mu}^L$ (which acts on the full quantum state in the tensor product) acts nontrivially only on its region $L$ and acts trivially (as the identity) on all the other regions. In particular, this means that not only do all of the $P_{\mu}^L$ with the same $L$ commute with each other by their relation as a complete set of orthonormal projection operators, but also all of the $P_{\mu}^L$ for different $L$ commute with each other as well.

If the quantum state were the size eigenstate $|\psi_N\rangle$, then the bird quantum probability that the observation $O_j$ occurs in the region $L$ (in the view of the bird that can tell what the region $L$ is) would be

$$p_{NLj} = \langle \psi_N | P_j^L | \psi_N \rangle.$$  

(6.3)

However, in reality, even just in the component state $|\psi_N\rangle$ for $N > 1$, there are other regions where the observation could occur, so the frog-view probability $P_j(i)$ for the observation $O_j$ in the theory $T_i$ can be some $i$-dependent function of all the $p_{NLj}$'s. The freedom of this function is part of the independence of the observational probabilities from the quantum state itself.

The frog probabilities logically need not even be functions of the bird probabilities. However, at least in my toy model of a universe with separate regions at one moment of time, so that all the projection operators defined above (which the birds can observe) commute, it seems plausible that they would be. (It is a further challenge to describe what happens when frog observations occur at different times, so that the corresponding projection operators in the bird view do not commute, but I shall not address this issue here.)

In particular, I shall assume the No Extra Vision Principle, that if for all $N$ and $L$, the bird quantum probability $p_{NLj}$ in theory $T_i$ is zero for some $j$, then the frog probability $P_j(i) \equiv P(O_j|T_i)$ is also zero for that $j$. I shall also continue to assume the Probability Symmetry Principle. For my simplified toy example state given by Eq. (6.2), the PSP would imply that if $\theta = 0$, so the state were a single bird eigenstate, and if this state had $m_1 = m_2$ (equal numbers of regions with the observations $O_1$ and $O_2$ that are each assumed to occur once and only once within each respective region), then the frog probabilities for these two different observations would be equal, $P_1 = P_2 = 1/2$, with the value of $1/2$ determined by the fact that in this simplified bird eigenstate there is zero bird probability for any other observation (and hence, by the NEVP assumption, zero frog proba-
bility $P_j$ also), so $P_1$ and $P_2$ must sum to unity to obey the normalization condition Eq. (3.1).

Although I am arguing that Born’s rule is not sufficient for determining the frog probabilities from the quantum state, it is surely the case that whatever the rule is, the resulting probabilities will depend in some way on the quantum state as well. The PSP and NEVP assumptions above seem to be rather natural minimal requirements for this dependence. Now I want to give several examples, $T_1$ to $T_9$, illustrating the freedom of the rule within these minimal restrictions, and with a subset of these examples, $T_3$ to $T_5$, satisfying one other natural restriction.

A useful procedure for getting normalized frog probabilities $P_j(i)$ for theory $T_1$ is first to define some method for getting unnormalized nonnegative relative frog probabilities $p_j(i)$ and then simply normalize them by $P_j(i) = p_j(i)/\sum_k p_k(i)$. So in the following examples, I shall give examples of rules for calculating relative frog probabilities $p_j(i)$ from the bird probabilities $p_{NLj}$ that require the bird’s knowledge of the size $N$ and region label $L$ that the frogs do not have. The different indices $i$ will denote different theories, in this case different rules for calculating the probabilities, since for simplicity we are assuming that all the theories have the same quantum state $|\psi\rangle$ and hence the same set of bird probabilities $p_{NLj}$.

Next, let us turn to different possible examples.

(1) For theory $T_1$, use the No Extra Vision assumption to take the relative frog probability to be zero, $p_j(1) = 0$, if all the $p_{NLj}$ are zero for all $N$ and $L$ for that $j$, so that the birds have zero quantum probability to see any frogs making the observation $O_j$, but set $p_j(1) = 1$ for each frog observation $O_j$ that has at least one positive $p_{NLj}$ for that $j$. One might interpret this $p_j(1)$ to be the ‘existence probability’ of the observation $O_j$ in the Everett many worlds interpretation of the quantum state. That is, theory $T_1$ is essentially taking the Everett many worlds interpretation to imply that if there is any nonzero amplitude for the observation to occur, it definitely exists somewhere in the many worlds (and hence has existence probability unity). If the total number of observations that have nonzero amplitudes (at least one positive bird probability $p_{NLj}$ for each such $j$) is $N_1$, then the normalized frog probabilities in this theory $T_1$ are $P_j(1) = p_j(1)/N_1$. This would be the theory that every observation that actually does exist, as seen by the birds who look at the entire many-worlds quantum state, is equally probable, and that the observations that have zero bird quantum probabilities in all components of the many-worlds quantum state do not exist for the frogs either and hence have zero frog probabilities $P_j(1)$.

The problem with this theory $T_1$ is that for almost all quantum states, almost all observations will exist in the Everett sense, making their number $N_1$ nearly as large as the number of all possible observations, which I am assuming is enormous. Then the normalized frog probabilities $P_j(1) = p_j(1)/N_1$ will all be very tiny, giving extremely low likelihood to the theory $T_1$. Thus, in a Bayesian analysis, unless one assigned this theory a prior probability very near unity in comparison with the priors one assigned to theories having much higher likelihoods, it seems that surely this theory would have extremely low posterior probability. Therefore, I strongly suspect that other theories can be constructed that would be much more probably true, given our ordered observations that do not appear as if they have been randomly chosen with equal probabilities from a set of nearly all possible observations.

(2) For theory $T_2$, one might try to hang onto Born’s rule as closely as possible by constructing the global projection operator

$$P_j = I - \prod_L (I - P_j^L)$$  \hspace{1cm} (6.4)

and using it in Eq. (3.2) to get, not the normalized frog probabilities $P_j(2)$ (since these expectation values will not be normalized), but rather to get the relative frog probabilities $p_j(2) = \langle \text{P}_j \rangle = \langle \psi|\text{P}_j|\psi\rangle$. This would not be the full many-worlds existence probability 0 or 1 described for theory $T_1$ (which would be 0 if $p_j(2) = 0$ and 1 if $p_j(2) > 0$), but it might be regarded as the quantum probability for a superobserver (a bird) to find that at least one instance of the observation $O_j$ occurs if one imagines the bird making a quantum observation of all the frog observers in the quantum state $|\psi\rangle$.

Indeed, this $p_j(2)$ is essentially in quantum language what Hartle and Srednicki [3] propose for the probability of an observation (without making the distinction between bird probabilities and frog probabilities), the quantum probability that the observation occurs at least somewhere. This is fine for bird probabilities for the existence of observations that for them are not mutually exclusive. However, because the different $P_j$’s defined this way are not orthogonal, the resulting quantum probabilities $p_j(2)$ given by Born’s rule will not be normalized to obey Eq. (3.1). This lack of normalization is a consequence of the fact that even though it is assumed that two different observations $O_j$ and $O_k$ (with $j \neq k$) cannot both occur within the same region $L$, one can have $O_j$ occurring within one region and $O_k$ occurring within another region. Therefore, the existence of the observation $O_j$ at least somewhere is not incompatible with the existence of the distinct observation $O_k$ somewhere else, so the sum of the bird quantum existence probabilities $p_j(2)$ is not constrained to be unity. Thus they cannot be used directly as the normalized frog probabilities.

However, it would be perfectly legal to interpret the unnormalized $p_j(2)$’s as relative probabilities for the frogs, and from them construct the corresponding normalized probabilities $P_j(2) = p_j(2)/\sum_k p_k(2)$ for the frogs, with $M_2 = \sum_k p_k(2)$. This is what I am defining theory $T_2$ to do.

This theory $T_2$ seems likely to have a higher likelihood than $T_1$, since presumably there will be many of the $M_1$ observations that have $p_j(2) > 0$ and hence $p_j(1) = 1$ but yet have $p_j(2) \ll 1$, so that $M_2 = \sum_k p_k(2)$ is signif-
...M than to M is, I would assume that generically M2 is relatively much smaller than M1, at least in absolute terms (that is, I would assume that generically \( M_1 - M_2 \gg 1 \) if the number of possible observations is very large), though it would depend on the quantum state and the set of projection operators \( P_j^f \) whether or not \( M_2 \) is relatively much smaller than \( M_1 \) and \( M_2/M_1 \ll 1 \). If indeed \( M_2/M_1 \ll 1 \), and if our observations are an example of an \( O_j \) with \( p_j(2) \) near unity, then \( P_j(2) \gg P_j(1) \), so our observations would assign a much higher likelihood to theory \( T_2 \) than to \( T_1 \). However, I still suspect that the universe may be so large that a huge number of observations may have a bird quantum probability near unity to occur at least somewhere, so that not only \( M_1 \) but also \( M_2 \) would be so large that the resulting likelihood \( P_j(2) = p_j(2)/M_2 \) for theory \( T_2 \) would be also be enormously lower than that for other theories of similar elegance (and hence presumably to be assigned similar prior probabilities) that one may be able to construct.

For the next sequence of rules from extracting frog observational probabilities from a given quantum state, I shall assume the

**Probability Fraction Principle (PFP):**

If the quantum state has a definite fraction for the ratio of each possible observation to the total number of all possible observations, then the probability of each observation is that fraction.

In a restriction of our toy model in which the quantum state \( |\psi\rangle \) is a single bird eigenstate \( |\mu_1\mu_2 \ldots \mu_N\rangle \) with either zero or one definite observation within each region, let

\[
N_j = \sum_{L=1}^{N} p_{NLj}
\]

\[
= \sum_{L=1}^{N} \langle \mu_1\mu_2 \ldots \mu_N | P_j^L | \mu_1\mu_2 \ldots \mu_N \rangle \quad (6.5)
\]

be the number of observations \( O_j \) (the number of regions containing this particular observation, each region assumed to contain at most one observation). Further, let

\[
N_O = \sum_{j} N_j = \sum_{L=1}^{N} \sum_j p_{NLj}
\]

\[
= \sum_{L=1}^{N} \sum_j \langle \mu_1\mu_2 \ldots \mu_N | P_j^L | \mu_1\mu_2 \ldots \mu_N \rangle 
\]

\[
= \sum_{L=1}^{N} \langle \mu_1\mu_2 \ldots \mu_N | P_O^L | \mu_1\mu_2 \ldots \mu_N \rangle \quad (6.6)
\]

be the total number of all observations (the total number of regions \( N \) minus the number of regions with \( \mu_j = 0 \) or no observation), where \( P_O^L = \sum_j P_j^L \) is the total projection operator onto having any observation at all (regardless of which particular \( O_j \) it is) in the region \( L \). Since each bird eigenstate has a definite value of each \( N_j \) and of \( N_O \), it has the definite fraction \( f_j = N_j/N_O \) for the observation \( O_j \). If the full quantum state were only this single bird eigenstate, the Probability Fraction Principle would imply that the probability of the observation \( O_j \) is that fraction, \( P_j = f_j \).

The Probability Fraction Principle also implies that if the full quantum state is a superposition of different bird eigenstates with the same fractions \( f_j = N_j/N_O \) for each (though the \( N_j \) and \( N_O \) need not be the same for all component bird eigenstates, only their ratio), then \( P_j = f_j \). That is, the quantum state need not be an eigenstate of all the numbers of observations \( N_j \), but only of their ratios. For example, the quantum state of Eq. (6.2) for general \( \theta \) (a superposition of two bird eigenstates, \( |m_0; m_1; m_2\rangle \) with \( N_1 = m_1 \) and \( N_2 = m_2 \) and \( |n_0; n_1; n_2\rangle \) with \( N_1 = n_1 \) and \( N_2 = n_2 \), with different values of \( N_1 \) and of \( N_2 \)) is not an eigenstate of \( N_1 \) and of \( N_2 \) but would be an eigenstate of the fractions \( f_1 = N_1/(N_1 + N_2) \) and of \( f_2 = N_2/(N_1 + N_2) \) if \( m_1/n_1 = m_2/n_2 \). For such an eigenstate of the fractions, the Probability Fraction Principle implies that \( P_1 = f_1 \) and \( P_2 = f_2 \).

Note that the Probability Fraction Principle implies the Probability Symmetry Principle but not conversely, so that the Probability Fraction Principle is a stronger principle. For example, theories \( T_1 \) and \( T_2 \) satisfy the PSP but not the PFP, as one can see for the superposition above that is a fraction eigenstate with \( f_1 \neq f_2 \): both \( T_1 \) and \( T_2 \) give \( P_1 = 1/2 \neq f_1 \) and \( P_2 = 1/2 \neq f_2 \). This fact might be taken as another possible reason for rejecting theories \( T_1 \) and \( T_2 \) (or for assigning them low prior probabilities, though I argued above that I suspect they would end up with low posterior probabilities anyway just from their low likelihoods if indeed there are a huge number of possible observations that \( T_1 \) and \( T_2 \) would assign nearly equal, and hence very low, probabilities \( P_j \), with the particular \( P_j \) that the theory assigns to our observation to be used as the likelihood of the theory in a Bayesian analysis).

After imposing the Probability Fraction Principle, the main remaining freedom in the rule for assigning observation probabilities from the quantum state is how to weight different components of the quantum state with different fractions \( f_j \) that have different numbers of regions \( N \) and/or different numbers of observations \( N_O \) (and hence also with possibly different numbers \( N - N_O \) of regions with no observations). For example, different weightings below can correspond to the difference between using volume weighting or not in inflationary cosmology [1].

For theory \( T_3 \), corresponding to volume weighting, let the unnormalized relative probabilities \( p_j(3) \) be the expectation values of the numbers \( N_j \) of times the observation \( O_j \) occurs, so

\[
p_j(3) = \sum_L \langle \psi | P_j^L | \psi \rangle = \sum_{N=1}^{\infty} \sum_{L=1}^{N} |a_N|^2 p_{NLj}. \quad (6.7)
\]

Then, as always, normalize these to get the normalized
observational probabilities

\[ P_j(3) = \frac{p_j(3)}{\sum_k pk(3)}. \tag{6.8} \]

(4) For theory \( T_3 \), corresponding to volume averaging, let the unnormalized relative probabilities \( p_j(4) \) be the expectation values of the fraction of regions \( N_j/N \) in which the observation \( O_j \) occurs (out of all \( N \) regions, not just out of the regions with observations), so

\[ p_j(4) = \frac{\sum_{N=1}^{\infty} \frac{1}{N} \sum_{L=1}^{N} |a_N|^2 \rho_{NLj}}{\sum_k pk(4)}. \tag{6.9} \]

then giving normalized observational probabilities

\[ P_j(4) = \frac{p_j(4)}{\sum_k pk(4)}. \tag{6.10} \]

Theory \( T_3 \) in its sum over \( L \) weights each component state \( |\psi_N\rangle \) by the number of observational regions in which the observation \( O_j \) occurs. On the other hand, theory \( T_4 \) has an average over \( L \) for each total number \( N \) of regions, so that component states \( |\psi_N\rangle \) with larger \( N \) do not tend to dominate the probabilities for observations just because of the greater number of observation regions within them. Theory \( T_3 \) is analogous to volume weighting in the cosmological measure, and theory \( T_4 \) is analogous to volume averaging.

(5) For theory \( T_5 \), which might be said to be observational averaging, let the unnormalized relative probabilities \( p_j(5) \) be the expectation values of the fraction of observations, \( f_j = N_j/N_O \), in which the observation \( O_j \) occurs (out of just the \( N_O \) regions with observations in each eigenstate of the total number \( N_O \) of all regions with one observation), so

\[ p_j(5) = \frac{\sum_{N=1}^{\infty} |a_N|^2 \sum_{\mu_1,\mu_2,\ldots,\mu_N} |b_{\mu_1\mu_2\ldots\mu_N}|^2 \frac{N_j}{N_O}}{\sum_k pk(5)}. \tag{6.11} \]

with

\[ \frac{N_j}{N_O} = \frac{\langle \mu_1\mu_2\ldots\mu_N | P_{Lj} \rho_{Lj} | \mu_1\mu_2\ldots\mu_N \rangle}{\langle \mu_1\mu_2\ldots\mu_N | P_{Oj} \rho_{Oj} | \mu_1\mu_2\ldots\mu_N \rangle}. \tag{6.12} \]

This then leads to the normalized observational probabilities

\[ P_j(5) = \frac{p_j(5)}{\sum_k pk(5)}. \tag{6.13} \]

Theory \( T_5 \) would correspond to the procedure of collapsing the full quantum state \( |\psi\rangle \) to one of its bird eigenstates \( |\mu_1\mu_2\ldots\mu_N\rangle \) with probability given by the absolute square of its amplitude, \( |a_N|^2 |b_{\mu_1\mu_2\ldots\mu_N}|^2 \), and then saying in that bird eigenstate the probability of the observation \( O_j \) is the corresponding fraction \( f_j = N_j/N_O \) of all observations that are type \( j \).

In [63] I implicitly assumed that theory \( T_3 \) is the probability rule in Everett many-worlds quantum theory and that theory \( T_5 \) is the probability rule in collapse versions of quantum theory. Then I pointed out that in principle from these two different probability distributions for observations, one could test between these two versions of quantum theory. I still think that \( T_3 \) would be a more natural rule than \( T_5 \) in Everett many-worlds quantum theory, and that \( T_5 \) is the most natural rule in collapse versions of quantum theory (assuming that the collapse is to a bird eigenstate with a definite observation, or none at all, in each region), but I no longer believe that \( T_3 \) is the only possible rule within Everett many-worlds quantum theory. Although \( T_5 \) would naturally arise within collapse versions of quantum theory, it is logically possible it could also arise in many-worlds versions. Therefore, whereas I would now say that finding that \( T_3 \) (or \( T_4 \)) gives much higher probability for our observation than \( T_5 \) would tend to support many-worlds over collapse, it is no longer obvious to me that finding that \( T_3 \) gives much higher probabilities for our observation than \( T_3 \) or \( T_5 \) would necessarily support collapse versions of quantum theory over many-worlds versions without collapse.

Except for theory \( T_1 \), all of the rules above may be viewed as modifications of Born’s rule, Eq. \( (6.2) \), by replacing the projection operators \( P_j \) with some other observation operators \( Q_j(i) \) normalized so that \( \sum_j Q_j(i) = 1 \), giving

\[ P_j(i) = \langle Q_j(i) \rangle. \tag{6.14} \]

Of course, one also wants \( P_j(i) \geq 0 \) for each \( i \) and \( j \), so one needs to impose the requirement that the expectation value of each observation operator \( Q_j(i) \) in each theory \( T_i \) be nonnegative. The simplest way to do this would be to require that each observation operator \( Q_j(i) \) be a positive operator. However, since a complete theory must both specify the quantum state (here denoted by \( \langle \ldots \rangle \), the linear map from operators, replacing the \( \ldots \) in this expression, to complex numbers that are the quantum expectation values of the operators in the quantum state) and specify the observation operators \( Q_j(i) \), it is logically possible that the observation operators \( Q_j(i) \) need not be positive but just have positive expectation values in the quantum state \( \langle \ldots \rangle \) for the same theory \( T_i \).

The main point of this paper is that in cases with more than one copy of the observer, such as in a large enough universe, one cannot simply use the expectation values of projection operators as the probabilities of observations. This means that if Eq. \( (6.14) \) is to apply, each theory must assign a set of observation operators \( Q_j(i) \), corresponding to the set of possible observations \( O_j \), that are not projection operators, whose expectation values are used instead as the probabilities of the observations. Since these observation operators are not given directly by the formalism of traditional quantum theory (e.g., as projection operators by Born’s rule), they must be added to the formalism by each particular complete theory.

In other words, a complete theory \( T_i \) cannot be given merely by the dynamical equations and initial conditions
(the quantum state), but it also requires the set of observation operators $Q_j(i)$ whose expectation values are the probabilities of the observations $O_j$ in the complete set of possible observations by the localized observers (frogs) within the universe. (Alternatively, they may be given by some other rule for the probabilities, if they are not to be expectation values of operators.) The probabilities are not given purely by the quantum state but have their own logical independence in a complete theory.

For the theories $T_2$–$T_5$, we can write the observation operators $Q_j(i)$ in the following forms, with $(Q_j)_i = \langle \psi | Q | \psi \rangle$ when the quantum state is the pure state $| \psi \rangle$ for all of the theories $T_i$ under consideration, as we have been taking it to be in this paper:

$$Q_j(2) = \frac{P_j}{\sum_k P_k}_i,$$

$$Q_j(3) = \sum_k P_L^j P_N^i,$$

$$Q_j(4) = \sum_k^{\infty} \sum_{L=1}^N \frac{1}{N} \sum_{k=1}^P N_P^j P_N^i,$$

$$Q_j(5) = \sum_k^{\infty} \sum_{N=1}^N \sum_{L=1}^N \sum_{k=1}^P \frac{1}{N} \sum_{L=1}^N \sum_{k=1}^P N_P^j P_N^i,$$

(6.15)

where $P_N = | \psi_N \rangle \langle \psi_N |$ is the projection operator onto the component state with $N$ total regions, and $P_{N_{NO}}$ is the projection operator onto the state with $N$ total regions and $N_0$ regions with observations. The expectation values of the numerators of these expressions are the relative probabilities $p_j(i)$, and the expectation values of the full expressions are the normalized probabilities $P_j(i) \equiv P(O_j | T_i)$ for the observations $O_j$ made by the localized observers (frogs) within the universe that do not have access to the bird’s eye view of where they are within the universe or of how many observations of the various types occur within it.

Besides $T_1$, we can have other theories in which the observational probabilities $P_j(i)$ are not given by the expectation value of any observation operators $Q_j(i)$ chosen independent of the quantum state (up to normalization). For example, the relative probabilities $p_j(i)$ may be given nonlinearly in terms of quantum expectation values.

(6) For theory $T_6$, let

$$p_j(6) = \langle P_j | \psi \rangle^c = \langle \psi | [I - \prod_k^L (I - P_L^j)] | \psi \rangle^c = p_j(2)^c \quad (6.16)$$

and then get the normalized observational probabilities

$$P_j(6) = \frac{p_j(6)}{\sum_k P_k(6)}, \quad (6.17)$$

where the exponent $c$ is an arbitrary positive constant here and below.

(7) For theory $T_7$, let

$$p_j(7) = \langle \sum_L P_L^j | \psi \rangle^c = \langle \psi | \sum_L P_L^j | \psi \rangle^c = p_j(3)^c, \quad (6.18)$$

and then

$$P_j(7) = \frac{p_j(7)}{\sum_k P_k(7)}. \quad (6.19)$$

(8) For theory $T_8$, let

$$p_j(8) = \langle \sum_k \sum_{N=1}^N \frac{1}{N} \sum_{L=1}^N \sum_{j=1}^P N_P^j P_N^i \rangle^c = p_j(4)^c, \quad (6.20)$$

and then

$$P_j(8) = \frac{p_j(8)}{\sum_k P_k(8)}. \quad (6.21)$$

(9) For theory $T_9$, let

$$p_j(9) = \langle \sum_k \sum_{N=1}^N \sum_{L=1}^N \sum_{j=1}^P \frac{1}{N} \sum_{L=1}^N \sum_{k=1}^P N_P^j P_N^i \sum_{L=1}^N \sum_{k=1}^P N_P^j P_N^i \rangle^c = p_j(5)^c, \quad (6.22)$$

and then

$$P_j(9) = \frac{p_j(9)}{\sum_k P_k(9)}. \quad (6.23)$$

The theories $T_6$–$T_9$ are the nonlinear generalizations of $T_2$–$T_5$, respectively, and reduce to those linear probability rules for $c = 1$.

Let us write what these nine rules would give for the quantum state of Eq. (6.12), $| \psi \rangle = \cos \theta | m_0; m_1; m_2 \rangle + \sin \theta | n_0; n_1; n_2 \rangle$. Let me also choose numbers for the parameters of this state to represent very crudely an equal-amplitude ($\cos \theta = \sin \theta = 1/\sqrt{2}$) superposition of the present observable part of the universe and of what that part may become after a time of $10^{36}$ times its present age, where the precise value of this time is not important but here is taken as the time by which it would be probable for our asymptotically de Sitter region to have decayed if one uses the first number in Eq. (89) of [50] as the decay rate per-four-volume.

A human brain has a volume of about $10^{101}$ Planck volumes, and the observable universe today has a volume of about $10^{185}$ Planck volumes, or about $10^{84}$ times the volume of a human brain, so if we divide up the present universe into brain-sized regions, we will get of the order of $10^{84}$ regions, of which of the order of $10^{10}$ are occupied by human brains. I shall let $O_1$ represent an ordinary human observation, and $O_2$ represent an observation by a Boltzmann brain [3] for concreteness, so this probability is negligible in the present observable part of the universe. Therefore, let us say that the number of regions of the first component of the quantum state (the present observable universe) with no observations is $m_0 = 10^{84}$; the number with an ordinary human observation $O_1$ is $m_1 = 10^{10}$, and the
number with a Boltzmann brain observation $O_2$ is $m_2 = 0$.

The second component of the quantum state, roughly $10^{56}$ Hubble times to the future, will have a volume (assuming an asymptotically exponential de Sitter growth at the rate given by the present value of the cosmological constant) very roughly $10^{10^{56}}$ times that of a human brain, so I shall take the number of regions without any observations then to be $n_0 = 10^{10^{56}}$. Stars and ordinary observers will have almost entirely died out by then, so I shall take the number of ordinary human observations then to be $n_1 = 0$. Boltzmann brains might occupy a fraction of the universe then that is very roughly $10^{-10^{52}}$, but by then the universe will have expanded so enormously large that the total expected number of Boltzmann brains will be huge, giving the number of Boltzmann brain observations to be, say, $n_2 = 10^{10^{56} - 10^{52}}$. For simplicity I am ignoring all other observations but the ordinary human and Boltzmann brain observations $O_1$ and $O_2$ respectively. Note that here I have abandoned the generic case by setting $m_2 = n_1 = 0$.

Now let us see what the nine rules, theories $T_1$–$T_9$, give for the probabilities $P_1(i)$ and $P_2(i)$ for observations $O_1$ and $O_2$ for each of the nine value of $i$.

(1) Theory $T_1$, that each observation has equal probability if it has a nonzero amplitude, gives unnormalized relative probabilities $p_1(1) = p_2(1) = 1$ and normalized observational probabilities $P_1(1) = P_2(1) = 1/2$, the same for ordinary human and Boltzmann brain observations. If one considered all possible observations rather than just these two, theory $T_1$ would assign a very low probability for all of them, so the likelihood that this theory is right would be very low, where ‘likelihood’ is used with the standard technical meaning of being the conditional probability $P_i(i) = P(O_i|T_i)$ of our observation $O_i$, conditional upon the theory $T_i$.

(2) Theory $T_2$, that each observation has a relative (frog) probability that is the same as the bird probability for that observation to exist at least somewhere, would give $p_1(2) = \cos \theta$ (since only the first component, with amplitude $\cos \theta$, gives rise to the human observation $O_1$ and then with certainty if the state were that component) and $p_2(2) = \sin \theta$ (since only the second component, with amplitude $\sin \theta$, gives rise to the Boltzmann brain observation $O_2$ and then with certainty if the state were that component). Since these are already normalized (because of the special case that $m_2 = n_1 = 0$), the normalized frog observational probabilities are the same. For our choice of equal amplitudes, $\cos \theta = \sin \theta = 1/\sqrt{2}$, we get $P_1(2) = P_2(2) = 1/2$, in this case the same as theory $T_1$. If one considered all possible observations and labeled our observation as $O_1$, then because there may be some observations that are unlikely to occur anywhere, one would have $P_1(2)$ somewhat larger than $P_1(1)$ from theory $T_1$, but still $P_1(2)$ is likely to be so small that theory $T_2$, as well as theory $T_1$, would be assigned very low likelihood as a result of our observation.

(3) Theory $T_3$, that the relative frog observation probabilities are the expected numbers of such observations (the analogue of volume weighting in the cosmological measure problem [6]) would give $p_1(3) = m_1 \cos^2 \theta + n_1 \sin^2 \theta = 0.5 \times 10^{10}$ and $p_2(3) = m_2 \cos^2 \theta + n_2 \sin^2 \theta = 0.5 \times 10^{10^{56} - 10^{52}}$. Then the normalized probabilities would be

$$P_1(3) = \frac{m_1 \cos^2 \theta + n_1 \sin^2 \theta}{(m_1 + m_2) \cos^2 \theta + (n_1 + n_2) \sin^2 \theta} \approx 10^{-10^{56} - 10^{52}},$$

$$P_2(3) = \frac{m_2 \cos^2 \theta + n_2 \sin^2 \theta}{(m_1 + m_2) \cos^2 \theta + (n_1 + n_2) \sin^2 \theta} \approx 1.$$ (6.24)

Theory $T_3$ has the huge numerical dominance of the expectation value for the number of Boltzmann brains result in their huge dominance for the observational probabilities, so that the probability for the human observation $O_1$ is minuscule. Our ordinary human observation would give extremely low likelihood to this theory, so unless one gave it very nearly all of the prior probability to be correct before considering our observation, it would be statistically ruled out by our observation at an enormously high confidence level. This is a manifestation of the Boltzmann brain problem [3, 4, 5, 6, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90].

(4) Theory $T_4$, that the relative frog observation probabilities are the expectation values of the fraction of regions in which in observation occurs (the analogue of volume averaging in the cosmological measure problem [6]) would give the relative probabilities

$$p_1(4) = \frac{m_1 \cos^2 \theta}{m_0 + m_1 + m_2} + \frac{n_1 \sin^2 \theta}{n_0 + n_1 + n_2} \approx 0.5 \times 10^{-74},$$

$$p_2(4) = \frac{m_2 \cos^2 \theta}{m_0 + m_1 + m_2} + \frac{n_2 \sin^2 \theta}{n_0 + n_1 + n_2} \approx 0.5 \times 10^{-10^{52}}.$$ (6.25)

This then gives the normalized probabilities

$$P_1(4) = \frac{m_1 \cos^2 \theta}{(m_1 + m_2) \cos^2 \theta + (n_1 + n_2) \sin^2 \theta} + \frac{n_1 \sin^2 \theta}{(n_0 + n_1 + n_2) \sin^2 \theta} \approx 1,$$

$$P_2(4) = \frac{m_2 \cos^2 \theta}{(m_1 + m_2) \cos^2 \theta + (n_1 + n_2) \sin^2 \theta} + \frac{n_2 \sin^2 \theta}{(n_0 + n_1 + n_2) \sin^2 \theta} \approx 10^{-10^{52} - 74}.$$ (6.26)

In theory $T_4$ the volume averaging greatly suppresses the Boltzmann brains, since they are so dilute in the far future component of the quantum state, much more dilute than human brains are in our present component of the quantum state. Therefore, almost all of the normalized probability is for the ordinary human observation.
$O_1$. Of the simple theories $T_1$–$T_5$ that do not have the arbitrary exponent $c$, it would seem that the volume-averaged $T_5$ gives the most hope for giving the highest probability for our observation (the likelihood of the theory, to be multiplied by the prior probability of the theory to get the relative posterior probability of the theory in a Bayesian analysis).

(5) Theory $T_5$, that the relative frog observation probabilities are the expectation values of the fractions of all observations that are of the particular type (what might be said to be observational averaging, or what would most naturally be given by collapse versions of quantum theory), would give the relative probabilities

$$p_1(5) = \frac{m_1 \cos^2 \theta + n_1 \sin^2 \theta}{m_1 + m_2} + \frac{n_1 \sin^2 \theta}{n_1 + n_2} = 0.5,$$

$$p_2(5) = \frac{m_2 \cos^2 \theta + n_2 \sin^2 \theta}{m_1 + m_2} + \frac{n_2 \sin^2 \theta}{n_1 + n_2} = 0.5. \tag{6.27}$$

This then gives the normalized probabilities

$$P_1(5) = P_2(5) = \frac{1}{2}. \tag{6.28}$$

This theory $T_5$ is ambivalent whether it makes ordinary human observations or Boltzmann brain observations more probable. For the particular example used here, it would give a likelihood very nearly half that of theory $T_4$, so both of these theories might seem to be good candidate theories, with likelihoods not too small. However, if one extended the example to an enormous superposition of many different quantum states, I would suspect that most of probability assigned by theory $T_5$ would go to the bulk of the quantum components in which it seems likely that life would be much more rare than it is in our component, simply because the conditions are much less conducive for life there. But in the components in which the conditions are much less conducive for life, I would suspect that life would be quite different, and nearly all of the observations would have a significantly different character than ours do. Then it would seem that our observations would have much lower probabilities in theory $T_5$ than in theory $T_4$.

In collapse versions of quantum theory, in which the rule $T_5$ would arise quite naturally, it would seem most probable that the quantum state would collapse to one of the presumably many more components in which life is very rare (on a per-volume basis), and in which most life that does occur is much very different from ours. Therefore, it seems likely that our particular observations would be extremely improbable in this scenario, much more improbable than they would be in a component of the quantum state like ours in which many effective coupling constants and other parameters of the local state seem finely tuned for life and in which life is presumably not nearly so rare as it would be if the parameters of our component were not so suitable for life.

Of course, life might be extremely rare, on a per-volume basis or even a per-planet basis, for all of the components of the quantum state, but that need not affect the normalized probabilities of our observations, since one is necessarily selecting for an observation rather than just randomly selecting a region of space which may or may not have an observation. How rare our type of observation is, out of the volume of space or out of the number of planets, rather than out of all types of observations that occur with similar probabilities, does not affect the likelihoods that our observations impute to the theories that predict these probabilities. Therefore, it is of no disadvantage to a theory to predict that life, even within the component of the quantum state that is most conducive to life, is extremely rare on a per-volume basis.

What does seem likely to make the probability of our observation small in the actual quantum state of the universe (whatever reasonable, i.e., simple, rule is used to deduce the probabilities from the quantum state) is the fact that the quantum state seems likely to support a large number of different observations with roughly equal number, spatial frequency, or frequency among the set of all observations. However, one might expect that this number, although no doubt large, is not nearly so large as the set of all possible observations, so that the probability of an observation within this dominant subset would be much larger than the probability of an observation chosen at random, with nearly equal probabilities, from the set of all possible observations, as theories $T_1$ and $T_2$ seem likely to give.

In theory $T_4$ one could partially explain the apparent fine tuning as the selection effect of having the probabilities weighted by the density of observations per volume (at least if the ultimate theory allowed this fine tuning to occur in some components of the quantum state, which itself might be a nontrivial requirement suggesting design). (The theory $T_2$ could also partially explain the apparent fine tuning if it did not have the Boltzmann brain problem.) Theory $T_5$ does not incorporate this selection effect, so it would seem that if we want to explain the apparent fine tuning within our part of the universe (though not yet explaining an apparent design of the complete theory of physics that we hope shall predict quantum components with the local fine tuning), we should reject theories $T_1$, $T_2$, and $T_5$. If we also reject theories $T_6$–$T_9$ because of the extra complication of their exponent $c$ and consider only theories $T_3$ and $T_4$, then it seems that the Boltzmann brain problem may lead us to prefer theory $T_4$, which is indeed what I am advocating. On the other hand, perhaps there is some other solution of the Boltzmann brain problem that is less complicated than revising the very simple $T_3$ to the slightly more complicated $T_3$, in which case one might be able to stick with $T_3$. However, so far I have not seen other solutions to the Boltzmann brain problem that seem simpler than changing the volume-weighted $T_3$ to the volume-averaged $T_4$.

Although the theories $T_6$–$T_9$, with their arbitrary exponent $c$, seem uglier than the theories $T_1$–$T_5$ (so that I
personally would assign them lower prior probabilities), they are further proof of the existence of other possible rules for extracting normalized frog observational probabilities from the quantum state, so let us continue the discussion to see what they assign for the two-component quantum state given by Eq. (6.2).

(6) Theory $T_6$, the nonlinear analogue of $T_2$, gives the unnormalized relative probabilities $p_1(6) = p_1(2)^c = \cos^2 \theta = 2^{-c}$ and $p_1(6) = p_1(2)^c = \cos^2 \theta = 2^{-c}$, so for our quantum state with both components having equal amplitude, $T_6$ gives $P_1(6) = P_2(6) = 1/2$, exactly the same as theory $T_2$. However, the probabilities would be different from those of $T_2$ if we had chosen a state with $\cos^2 \theta \neq \sin^2 \theta$.

(7) Theory $T_7$, the nonlinear analogue of the volume-weighted $T_3$, gives the relative probabilities $p_1(7) = (m_1 \cos^2 \theta + n_1 \sin^2 \theta)^c = (0.5 \times 10^{10})^c$ and $p_2(7) = (m_2 \cos^2 \theta + n_2 \sin^2 \theta)^c = (0.5 \times 10^{10^{10^{10}}})^c$. Then the normalized probabilities would be $P_1(7) \sim 10^{-10^{10^{10^{10}}}}$ and $P_2(7) \sim 1$, again enormously favoring Boltzmann brains over ordinary human brains unless $c$ were taken to be extraordinarily small, in which case the probabilities would revert to those of the highly unpredictable theory $T_1$ in the limit of $c$ becoming arbitrarily small.

(8) Theory $T_8$, the nonlinear analogue of the volume-averaged $T_4$, gives the relative probabilities

$$p_1(8) = \left( \frac{m_1 \cos^2 \theta}{m_0 + m_1 + m_2} + \frac{n_1 \sin^2 \theta}{n_0 + n_1 + n_2} \right)^c \sim (0.5 \times 10^{-74})^c,$$

$$p_2(8) = \left( \frac{m_2 \cos^2 \theta}{m_0 + m_1 + m_2} + \frac{n_2 \sin^2 \theta}{n_0 + n_1 + n_2} \right)^c \sim (0.5 \times 10^{-10^{10^2}})^c. \quad (6.29)$$

Normalizing these then gives the normalized probabilities

$$P_1(8) \approx 1,$$

$$P_2(8) \sim 10^{-10^{10^{10^{10^2}}}}. \quad (6.30)$$

Again, unless $c$ is extraordinarily tiny, this theory $T_8$ very strongly favors human observations over Boltzmann brain observations in this toy model. Presumably $T_8$ would give likelihoods close to what $T_4$ gives if $c$ is close to 1. However, other than as an illustration of the freedom that one has in choosing the rules for extracting the frog observational probabilities from the quantum state, I do not see much motivation for complicating the simple linear volume-averaged theory $T_4$ by going to its nonlinear generalization $T_8$, though it is conceivable that a further analysis might show that it gives significantly higher probabilities for our observations, for some suitable $c$, than the theory $T_4$ that effectively has $c = 1$.

(9) Finally, theory $T_9$, the nonlinear analogue of the observationally-averaged or quantum collapse theory $T_5$, gives the unnormalized relative probabilities of the frog observations as

$$p_1(9) = \left( \frac{m_1 \cos^2 \theta}{m_1 + m_2} + \frac{n_1 \sin^2 \theta}{n_1 + n_2} \right)^c = 2^{-c},$$

$$p_2(9) = \left( \frac{m_2 \cos^2 \theta}{m_1 + m_2} + \frac{n_2 \sin^2 \theta}{n_1 + n_2} \right)^c = 2^{-c}. \quad (6.31)$$

Just as the theory $T_5$ did, for our particular two-component quantum state $T_9$ gives equal normalized probabilities for the observations,

$$P_1(9) = P_2(9) = \frac{1}{2}, \quad (6.32)$$

which again is not very informative and plausibly gives a very low likelihood for this theory when one goes to a more realistic model with a huge number of possible observations.

**VII. CONCLUSIONS**

The examples show that there is not just one unique rule for getting observational probabilities from the quantum state. It remains to be seen what the correct rule is. Of the four examples given above with linear probability rules (relative probabilities given by the first power of the expectation values of certain operators), that is, theories $T_2$–$T_5$, I suspect that with a suitable quantum state, theory $T_4$ would have the highest likelihood $P_i(i)$, given our actual observations, since theory $T_2$ would have the normalized probabilities nearly evenly distributed over a huge number of possible observations, theory $T_3$ seems to be plagued by the Boltzmann brain problem $[4]$, and theory $T_5$ would seem to favor components of the quantum state much more hostile to life than ours and hence probably having the dominant form of observers quite different from us. Theories $T_6$–$T_9$ can be arbitrarily close to $T_2$–$T_5$ respectively if $c$ is arbitrarily close to 1, but with their arbitrary constant $c$, they seem more complex than $T_2$–$T_5$ and hence might naturally be assigned lower prior probabilities. One might conjecture that the fairly simple theory $T_4$ can be implemented in quantum cosmology to fit observations better than other alternatives $[4]$.

Thus we see that in a universe with the possibility of multiple copies of an observer, observational probabilities are not given purely by the quantum state, but also by a rule to get them from the state. There is logical freedom in what this rule is (or in what the observation operators $Q_i(i)$ are if the rule is that the probabilities are the expectation values of these operators). In cosmology, finding the correct rule is the measure problem. Preliminary evidence suggests that the volume-averaged rule $T_4$ is the best possibility considered so far.
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