THE QUINTIC COMPLEX MOMENT PROBLEM

H. EL-AZHAR, A. HARRAT, K. IDRISSI, AND E. H. ZEROUALI

Abstract. Let \( \gamma^{(m)} \equiv \{ \gamma_{ij} \}_{0 \leq i+j \leq m} \) be a given complex-valued sequence. The truncated complex moment problem (TCMP in short) involves determining necessary and sufficient conditions for the existence of a positive Borel measure \( \mu \) on \( \mathbb{C} \) (called a representing measure for \( \gamma^{(m)} \)) such that \( \gamma_{ij} = \int \mathbb{C} z^i d\mu \) for \( 0 \leq i + j \leq m \). The TCMP has been completely solved only when \( m = 1, 2, 3, 4 \).

We provide in this paper a concrete solution to the quintic TCMP (that is, when \( m = 5 \)). We also study the cardinality of the minimal representing measure. Based on the bivariate recurrences sequences’ properties with some Curto–Fialkow’s results, our method intended to be useful for all odd-degree moment problems.

1. Introduction

Given a doubly indexed finite sequence of complex numbers

\[ \gamma \equiv \gamma^{(m)} = \{ \gamma_{ij} \}_{0 \leq i+j \leq m} = \{ \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0m}, \ldots, \gamma_{m0} \} \]

with \( \gamma_{00} > 0 \) and \( \gamma_{ij} = \gamma_{ji} \) for \( 0 \leq i + j \leq m \). The truncated complex moment problem (in short, TCMP) associated with \( \gamma \) entails finding a positive Borel measure \( \mu \) supported in the complex plane \( \mathbb{C} \) such that

\[ \gamma_{ij} = \int \mathbb{C} z^i d\mu \quad (0 \leq i + j \leq m); \]

A sequence \( \{ \gamma_{ij} \}_{0 \leq i+j \leq m} \) satisfying (1.1) will be called a truncated moment sequence and the solution \( \mu \) is said to be a representing measure associated to the sequence \( \{ \gamma_{ij} \}_{0 \leq i+j \leq m} \).

In [34] J. Stochel has shown that solving TCMP solves the widely studied Full Moment Problem (see, for example, [1, 2, 3, 17, 29, 30, 33, 36]). More precisely, a full moment sequence \( \{ \gamma_{ij} \}_{i,j \in \mathbb{Z}^+} \) admits a representing measure if and only if each of its truncation \( \gamma^{(m)} \) admits a representing measure.

The truncated complex moment problem serves as a prototype for several other moment problems to which it is closely related. Its application can be found in subnormal operator theory [31, 24, 35], polynomial hyponormality [12] and joint hyponormality [4, 5]. It is also related to the optimization theory [26, 25, 27, 28, 29] and arise in pure and applied mathematics and in the sciences in general.

For the even case \( m = 2n \), Curto and Fialkow developed in a series of papers an approach for TCMP based on positivity and flat extensions of the moment matrix, see Section 2. This allowed them to find solutions for various particular cases of
truncated moment problems (see, for instance, [6, 8, 7, 10, 11, 21, 20]). However, only the cases $m = 2$ and $m = 4$ are completely solved (cf. [6, 9, 19, 14]).

In the odd case $m = 2n + 1$, a general solution to some partial cases of the TCMP can be found in [22] and [23] as well as a solution to the truncated matrix moment problem; a solution to the cubic complex moment problem (when $m = 3$) was given in [23], see also [16]. The solution is based on commutativity conditions of matrices determined by $\{\gamma_{ij}\}_{0 \leq i+j \leq 2n+1}$.

Therefore, only the cases $m = 1, 2, 3$ and 4 (the quadratic, the cubic and the quartic moment problem) have been completely achieved. All the other cases (quintic, sixthic, ...) are open and interest several authors; as indicated in many recent papers (see, for instance, [13, 15, 16, 37, 38]).

In this paper, we provide a concrete solution to the, almost all, quintic moment problem (i.e. $m = 5$) when one desires a minimal representing measure. To this aim, we investigate the structure of recursive complex-valued bi-indexed sequences and we combine the obtained observations with some results due to R. Curto and L. Fialkow, to provide a new technique for solving the odd-degree TCMP. We notice that our techniques furnish a short solution to the cubic moment problem (we omit the proof because the cubic moment problem is already solved, see [16, 23]) and expected to be useful for higher odd-degree truncated moment problems.

Let $\gamma^{(5)} = \{\gamma_{ij}\}_{0 \leq i+j \leq 5}$ be a given complex valued bi-sequence. We associate with $\gamma^{(5)}$ the next two matrices that will play a crucial role in our approach.

$$M(2) := \begin{pmatrix}
\gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\
\gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \\
\end{pmatrix},$$

$$B := \begin{pmatrix}
\gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\
\gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\
\gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{50} \\
\end{pmatrix}.$$  

Let us recall that thanks to Douglas factorization theorem, we have $\text{Rang } B \subseteq \text{Rang } M(2)$ if, and only if, there exists a matrix $W$ such that $B = M(2)W$. We will show, in Section 2, that the Hermitian matrix $W^{*}M(2)W$ is symmetric with respect to the second diagonal, then one can set

$$W^{*}M(2)W = \begin{pmatrix}
a & b & c & d \\
b & e & f & c \\
e & f & e & b \\
d & c & b & a \\
\end{pmatrix}$$

As we will see in the sequel, the entries $a, b, e$ and $f$ in the matrix $W^{*}M(2)W$ encodes the complete information on the cardinal of the support of the minimal representing measure.

**Theorem 1.1.** Let $\gamma^{(5)} = \{\gamma_{ij}\}_{1 \leq i \leq 5}$ be a given finite sequence, such that $M(2) \geq 0$, $\text{Rang } B \subseteq \text{Rang } M(2)$ and $a \neq e$ or $b = f$.

Then the quintic moment problem, associated with $\gamma^{(5)}$, admits a solution $\mu$. Moreover, The smallest cardinality of supp $\mu$ is

- $\text{card supp } \mu = r \iff a = e$ and $b = f$,
- $\text{card supp } \mu = r + 1 \iff a \neq e$ and $b - f < |b - f|$,
- $\text{card supp } \mu = r + 2 \iff a > e$ and $b - f \geq |b - f|$,

where $r := \text{card } M(2)$ and the numbers $a, b, e$ and $f$ are given by (1.3).

Since (as we will show in Section 2) $M(2) \geq 0$ and $\text{Rang } B \subseteq \text{Rang } M(2)$ are two necessary conditions for the quintic TCMP, associated with $\gamma^{(5)}$, then Theorem 1.1 provides
a concrete solution to the quintic complex moment problem, except for the case \( a \neq e \) or \( b = f \). The difficulty that we encountered in solving the remaining case ( \( a \neq e \) or \( b = f \)) is technical, not a failure in the method, see Section 5.

This paper is organized as follows. In Section 2, we will give useful tools and results usually used in the treatment of the truncated complex moment problems. We will investigate in Section 3 the complex-valued recursive bi-sequences and we will exhibit important results for quintic TCMP in Section 4. Finally, in Section 5, we solve the quintic complex moment problem together with the minimal support problem.

### 2. Preliminaries

First, we recall a fundamental necessary condition. To this end, let us assume that \( \gamma^{(2n)} \equiv \{ \gamma_{ij} \}_{i,j \leq 2n} \) is a given moment sequence and let \( \mu \) be the associated representing measure, then, for every \( p \equiv \sum_{h,k} a_{hk}z^h z^k \in \mathbb{C}[z, \bar{z}] \),

\[
0 \leq \int |p|^2 d\mu = \sum_{h,k,k',k''} a_{hk} a_{k'k''} \int z^{h+k'} \bar{z}^{k+k''} = \sum_{h,k,h',k'} a_{hk} a_{k'h'k''} \gamma_{h+k',k+h''},
\]

or, equivalently, the moment matrix \( M(n) \equiv M(n)(\gamma^{(2n)}) \), defined below, is semi-definite positive.

\[
M(n) := 
\begin{pmatrix}
M[0,0] & M[0,1] & \ldots & M[0,n] \\
M[1,0] & M[1,1] & \ldots & M[1,n] \\
\vdots & \vdots & \ddots & \vdots \\
M[n,0] & M[n,1] & \ldots & M[n,n]
\end{pmatrix},
\]

where

\[
M[i, j] = 
\begin{pmatrix}
\gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \gamma_{i+j,0} \\
\gamma_{i-1,j+1} & \gamma_{i,j} & \cdots & \gamma_{i+j-1,1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{0,i+j} & \gamma_{1,i+j-1} & \cdots & \gamma_{j,i}
\end{pmatrix}.
\]

Considering the lexicographic order,

\[
1, Z, \bar{Z}, Z^2, Z\bar{Z}, \bar{Z}^2, \ldots, Z^n, Z^{-1}, \bar{Z}, \ldots, Z^nZ^{-1}, Z^n\bar{Z},
\]

to denote rows and columns of the moment matrix \( M(n) \). For example, The \( M(3) \) matrix is

\[
\begin{bmatrix}
1 & Z & \bar{Z} & Z^2 & Z\bar{Z} & \bar{Z}^2 & Z^3 & Z^2\bar{Z} & Z\bar{Z}^2 & \bar{Z}^3 \\
\gamma_{00} & \gamma_{01} & \gamma_{01} & \gamma_{02} & \gamma_{01} & \gamma_{01} & \gamma_{03} & \gamma_{02} & \gamma_{01} & \gamma_{01} \\
\gamma_{10} & \gamma_{11} & \gamma_{11} & \gamma_{12} & \gamma_{11} & \gamma_{11} & \gamma_{13} & \gamma_{12} & \gamma_{11} & \gamma_{11} \\
\gamma_{01} & \gamma_{02} & \gamma_{02} & \gamma_{03} & \gamma_{02} & \gamma_{02} & \gamma_{04} & \gamma_{03} & \gamma_{02} & \gamma_{02} \\
\gamma_{11} & \gamma_{12} & \gamma_{12} & \gamma_{13} & \gamma_{12} & \gamma_{12} & \gamma_{14} & \gamma_{13} & \gamma_{12} & \gamma_{12} \\
\gamma_{02} & \gamma_{03} & \gamma_{03} & \gamma_{04} & \gamma_{03} & \gamma_{03} & \gamma_{05} & \gamma_{04} & \gamma_{03} & \gamma_{03} \\
\gamma_{12} & \gamma_{13} & \gamma_{13} & \gamma_{14} & \gamma_{13} & \gamma_{13} & \gamma_{15} & \gamma_{14} & \gamma_{13} & \gamma_{13} \\
\gamma_{03} & \gamma_{04} & \gamma_{04} & \gamma_{05} & \gamma_{04} & \gamma_{04} & \gamma_{06} & \gamma_{05} & \gamma_{04} & \gamma_{04} \\
\end{bmatrix}.
\]
Observe in passing that each block $M[i, j]$ has a Toeplitz form. That is each of its diagonals contains constant entries. On the other hand, it is easy to see that the matrix $M(n)$ detects the positivity of the Riesz functional given by
\[ \Lambda_{\gamma}(\mathbf{z}) : p(\mathbf{z}, \mathbf{z}) \equiv \sum_{0 \leq i+j \leq 2n} a_{ij} \mathbf{z}^i \rightarrow \sum_{0 \leq i+j \leq 2n} a_{ij} \gamma_{ij} \]
onumber
on the cone generated by the collection $\{p(\mathbf{z}) : p \in \mathbb{C}[\mathbf{z}, \mathbf{z}]\}$, where $\mathbb{C}[\mathbf{z}, \mathbf{z}]$ is the vector space of polynomials in two variables with complex coefficients and total degree less than or equal to $n$.

It is an immediate observation that the rows $\mathbf{z}^i \mathbf{z}^j$, columns $\mathbf{z}^i \mathbf{z}^j$ entry of the matrix $M(n)$ is equal to $\Lambda_{\gamma}(\mathbf{z}) (\mathbf{z}^{i+j} \mathbf{z}^{i+j+k}) = \gamma_{i+j,k}$. For reason of simplicity, we identify a polynomial $p(\mathbf{z}, \mathbf{z})$ with its coefficient vector $p = (a_{ij})$ with respect to the basis of monomials of $\mathbb{C}[\mathbf{z}, \mathbf{z}]$ in degree-lexicographic order. Clearly, $M(n)$ acts on these coefficient vectors as follows:
\[ (M(n)p, q) = \Lambda_{\gamma}(\mathbf{z})(p \mathbf{z}). \]

A theorem of Smul’jan [32] shows that a block matrix
\[ (2.5) \quad M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0, \]
if and only if
\[ (i) \quad A \geq 0, \]
\[ (ii) \quad \text{there exists a matrix } B \text{ such that } B = AW, \]
\[ (iii) \quad C \geq W^*AW. \]

Since $A = A^*$, we obtain $W^*AW$ is independent of $W$ provided that $B = AW$. Moreover, $\text{rank } M = \text{rank } A \Leftrightarrow C = W^*AW$ for some $W$ such that $B = AW$. Conversely, if $A \geq 0$, any extension $M$ satisfying $\text{rank } M = \text{rank } A$ (if this condition is satisfied, we will say that $M$ is a flat extension of $A$) is necessarily positive. Notice also that from the expression
\[ \begin{pmatrix} I & 0 \\ -W^* & I' \end{pmatrix} M \begin{pmatrix} I & -W \\ 0 & I' \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C - W^*AW \end{pmatrix}, \]
where $I$ and $I'$ denote the unit matrices, we deduce that
\[ \text{rank } M = \text{rank } A + \text{rank } (C - W^*AW). \]

By Smul’jan’s theorem, $M(n) \geq 0$ admits a (necessarily positive) flat extension
\[ (2.7) \quad M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \]
in the form of a moment matrix $M(n+1)$ if and only if
\[ (i) \quad B = M(n)W \text{ for some } W, \]
\[ (ii) \quad C = W^*M(n)W \text{ is a Toeplitz matrix}. \]

We have the next result due to Curto and Fialkow,

**Theorem 2.1.** [6, Theorem 5.13] The finite sequence $\gamma^{(2n)}$ has a rank $M(n)$-atomic representing measure if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$. That is, $M(n)$ can be extended to a positive moment matrix $M(n+1)$ satisfying $\text{rank } M(n+1) = \text{rank } M(n)$.

An important step in our approach is to show that the Hermitian matrix $W^*M(n)W$ is persymmetric, that is, it is symmetric across its lower-left to upper-right diagonal. For this purpose, we introduce first some additional notation.

We denote the successive columns of $W$ and $B$ (given as in Expression (2.7)) by $W_{[\mathbf{z}^{2n+1}, W_{[\mathbf{z}^{n+1}}, \ldots, W_{[\mathbf{z}^{n+1}}, B_{[\mathbf{z}^{2n+1}, B_{[\mathbf{z}^{2n+1}}, \ldots, B_{[\mathbf{z}^{2n+1}}, \text{ respectively.} \]
Let us consider the \( (n+1)(n+2) \) matrix built as follows,
\[
M_\varphi(n) := J_0 \oplus J_1 \oplus \cdots \oplus J_n;
\]
where \( J_p = (\delta_{i+j,p})_{0 \leq i,j \leq p} \) with \( \delta_{i,j} \) is the Kronecker symbol given by \( \delta_{i,k} = 1 \) for \( k = i \) and zero otherwise. For example
\[
J_0 = (1), \quad J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

**Lemma 2.2.** Let \( M_\varphi(n), M(n) \) and \( B_{\mathbb{Z}^{n-1},i} \) \((i = 0, \ldots, n)\) be as above, then

1. \((M_\varphi(n))^2 = I\).
2. \((M_\varphi(n))^* = M_\varphi(n)\).
3. \(M_\varphi(n)B_{\mathbb{Z}^{n-1},i} = B_{\mathbb{Z}^{n-1},i}^*\) \((i = 0, \ldots, n)\).
4. \(M_\varphi(n)M(n) = \overline{M(n)M_\varphi(n)}\).

**Proof.** The assertions (1), (2) and (3) are obvious. Only the third assertion requires a proof. To this aim, we recall that \( M(n) = [M(i,j)]_{0 \leq i,j \leq n}, \) see (2.1). Therefore
\[
[M_\varphi(n)]M(n) = \left[ \bigoplus_{i=0}^{n} J_i \right] [M(i,j)]_{i,j \leq n} = [J_iM(i,j)]_{i,j \leq n} = \overline{M(i,j)J_i}_{i,j \leq n} = \overline{[M(i,j)]}_{i,j \leq n} \left[ \bigoplus_{i=0}^{n} J_i \right] = \overline{M(n)M_\varphi(n)}.
\]

\(\Box\)

**Proposition 2.3.** Let \( n \) be a given integer and let \( M(n) \) and \( W \) be as above, then \( W^* M(n)W \) is a Hermitian Persymmetric matrix.

**Proof.** Setting \( W^* M(n)W = (c_{ij})_{0 \leq i,j \leq n} \), then we have
\[
c_{n-j,n-i} = W_{\mathbb{Z}^{n-1},j}^* M(n)W_{\mathbb{Z}^{n-1},i}.
\]
By multiplying left both sides of the fourth equation in Lemma 2.2 by \( M_\varphi(n) \) we obtain
\[
M_\varphi(n)M_\varphi(n)M(n) = \overline{M_\varphi(n)M(n)M_\varphi(n)}.
\]
and hence, by applying Lemma 2.2-(1), we have
\[
M(n) = \overline{M(n)M_\varphi(n)}.
\]
It follows, from (2.8) and (2.10), that
\[
c_{n-j,n-i} = W_{\mathbb{Z}^{n-1},j}^* M_\varphi(n) \overline{M(n)M_\varphi(n)W_{\mathbb{Z}^{n-1},i}}.
\]
The fact that \( M_\varphi(n) \) is self-adjoint allows to write
\[
c_{n-j,n-i} = (M_\varphi(n)W_{\mathbb{Z}^{n-1},j})^* \overline{M(n)} \left( M_\varphi(n)W_{\mathbb{Z}^{n-1},i} \right).
\]
By using the assertions (3) and (4), in Lemma 2.2, we deduce that:
\[
\overline{M(n)M_\varphi(n)W_{\mathbb{Z}^{n-1},i}} = M_\varphi(n)M(n)W_{\mathbb{Z}^{n-1},i} = M_\varphi(n)B_{\mathbb{Z}^{n-1},i} = B_{\mathbb{Z}^{n-1},i}.
\]
Therefore, (2.12) implies that
\[ c_{n-j,n-i} = (M_\varphi(n)W_{Z^i,n}^*)B_{Z^j,n-i} \]
\[ = W_{Z^i,n}^* M_\varphi(n) M(n) W_{Z^j,n-i} \]
\[ = ((M(n)M_\varphi(n))W_{Z^i,n})^* W_{Z^j,n-i} \]
\[ = M(n)W_{Z^i,n}^* W_{Z^j,n-i} \]
\[ = W_{Z^i,n}^* M(n) W_{Z^j,n-i} \]
\[ = c_{i,j}. \]

This concludes the proof of the Proposition 2.3. \( \square \)

3. Complex-valued Recursive Bi-Sequences

Let \( \gamma^{(n)} \equiv \{\gamma_{ij}\}_{i+j \leq n} \), with \( \gamma_{ij} = \gamma_{ji} \) and \( n \in \mathbb{N} \cup \{+\infty\} \), be a given complex-valued sequence and let \( P_{Z^d} = \sum_{\substack{l+k \leq d \\text{ and} \ (l,k) \neq (e,d-e)}} a_{ik} z^l \) be in \( \mathbb{C}[z] \), the vector space of polynomials in two variables with complex coefficients and total degree less than or equal to \( d \) (we assume that \( d \leq n \)). The sequence \( \gamma^{(n)} \) is said to be recursive, associated with a generating polynomial \( z^d - P_{Z^d} \), if
\[ \gamma_{i+1,d-1} = \Lambda_{\gamma(n)}(z^d) P_{Z^{d-1}}, \quad \text{for all } i + j \leq n - d, \]
or, equivalently, if
\[ \gamma_{i+1,d-1} = \sum_{\substack{l+k \leq d \\text{ and}(l,k) \neq (e,d-e)}} a_{ik} \gamma_{l+i,k+j} \quad (i + j \leq n - d). \]

We notice that, because of the equality \( \gamma_{ij} = \gamma_{ji} \), Equation (3.2) is equivalent to the following one:
\[ \gamma_{d-e+i,e+j} = \sum_{\substack{l+k \leq d \\text{ and}(l,k) \neq (e,d-e)}} a_{ik} \gamma_{l+i,k+j}, \]
for all integers \( i \) and \( j \), with \( i + j \leq n - d \).

Therefore, \( z^{d-e} - P_{Z^{d-e}} \) (where \( P_{Z^{d-e}} = \overline{P_{Z^d}} \)) is also a generating polynomial, associated with \( \gamma^{(n)} \); that is,
\[ \gamma_{d-e+i,e+j} = \Lambda_{\gamma(n)}(z^{d-e}) P_{Z^{d-e}}, \quad i + j \leq n - d. \]

The following proposition provides a connection, via \( \Lambda \), between the polynomials \( P_{Z^{d+f+1}} \) and \( P_{Z^{d+f+1}} \).

**Proposition 3.1.** Let \( \gamma^{(n)} \equiv \{\gamma_{ij}\}_{i+j \leq n} \) be a recursive bi-sequence and let \( z^{f+1} - P_{Z^{f+1}} \) be an associated generating polynomial, then
\[ \Lambda_{\gamma(n)}(z^{f+1} P_{Z^{f+1}}) = \Lambda_{\gamma(n)}(z^{f+1} P_{Z^{f+1}}), \quad l + k \leq n - 2f - 2. \]

**Proof.** For all integers \( l \) and \( k \), with \( l + k \leq n - 2f - 2 \), we have
\[ \Lambda_{\gamma(n)}(z^{f+1} P_{Z^{f+1}}) = \gamma_{l+k+1,f+k+1} \]
\[ = \gamma_{l+k+1,f+k+1} \]
\[ = \Lambda_{\gamma(n)}(z^{f+1} P_{Z^{f+1}}) \]
\[ = \Lambda_{\gamma(n)}(z^{k+1} P_{Z^{f+1}}). \] \( \square \)
It is well known that the (classical singly indexed recursive sequence can be defined by the initial data and the, associated recurrence relation (or, characteristic polynomial), see [18]. In a similar way, one can define recursive bi-sequences as observed below.

**Remark 3.2.**

i) A generating polynomial \( z^P - P_z \) (or, equivalently, \( \overline{z} - P_{\overline{z}} \)), with \( \deg P_z < e \), together with the initial data \( \{ \gamma_{ij} \}_{i,j<e} \) and the equality \( \overline{\gamma}_{ij} = \gamma_{ji} \), are said to define the sequence \( \gamma^{(n)} \).

ii) For a generating polynomial \( \overline{z} z^{e-1} - P_{\overline{z} z^{-1}} \), with \( \deg P_{\overline{z} z^{-1}} < e \), we need (all) the data \( \{ \gamma_{ij} \}_{i,j<e} \cup \{ \gamma_{0j} \}_{j=e,...,n} \) and the equality \( \overline{\gamma}_{ij} = \gamma_{ji} \) to define the recursive bi-sequence \( \gamma^{(n)} \).

In the next lemmas, we provide useful results for solving the quintic moment problem.

**Lemma 3.3.** Let \( \gamma^{(s)} \equiv \{ \gamma_{ij} \}_{i,j \leq s} \), with \( \overline{\gamma}_{ij} = \gamma_{ji} \), be a truncated bi-sequence and let \( z^P - P_z \) (where \( P_z = \beta z^3 + R_z \) and \( R_z \in \mathbb{C}_2[\overline{z}, z] \)) be an associated generating polynomial. Assume that \( \overline{z} z^P - P_{\overline{z}}z^{e-1} \) (where \( P_{\overline{z} z^{-1}} = \alpha z^3 + R_{\overline{z} z^{-1}} \), \( \alpha \neq 0 \) and \( R_{\overline{z} z^{-1}} \in \mathbb{C}_2[\overline{z}, z] \)) is a generating polynomial for \( \gamma^{(s)} \cup \{ \gamma_{34}, \gamma_{43} \} \), then \( \overline{z} z^P - P_{\overline{z}}z^{e-1} \) is a generating polynomial for \( \gamma^{(s)} \).

**Proof.** We have \( z^P - P_z \) is a generating polynomial for \( \gamma^{(s)} \), that is,

\[
\gamma_{i,j+4} = \Lambda_{i,s}(\overline{z} z^P) = \beta \gamma_{i,j+3} + \Lambda_{i,s}(\overline{z} z^P), \quad i + j \leq 4.
\]

As showing in (3.4), the last equality (3.7) is equivalent to

\[
\gamma_{i+4,j} = \Lambda_{i,s}(\overline{z} z^P) = \beta \gamma_{i+3,j} + \Lambda_{i,s}(\overline{z} z^P), \quad i + j \leq 4;
\]

where \( \overline{P}_z := P_z = \overline{z} z^3 + \overline{P}_{\overline{z}}z^{-1} \).

Also, the polynomial \( \overline{z} z^P - P_{\overline{z}}z^{e-1} \) is a generating one for \( \gamma^{(s)} \cup \{ \gamma_{34}, \gamma_{43} \} \); that is, for all \( i + j \leq 3 \) and \( (i, j) = (2, 2), (1, 3) \):

\[
\gamma_{i+1,j+2} = \Lambda_{i,s}(\overline{z} z^P) = \alpha \gamma_{i,j+3} + \Lambda_{i,s}(\overline{z} z^P),
\]

or, equivalently, for \( i + j \leq 3 \) and \( (i, j) = (2, 2), (1, 3) \):

\[
\gamma_{i+2,j+1} = \Lambda_{i,s}(\overline{z} z^P) = \alpha \gamma_{i,j+3} + \Lambda_{i,s}(\overline{z} z^P),
\]

where \( P_{\overline{z} z^{-1}} := \overline{P}_{\overline{z}}z^{-1} = \beta \overline{z} z^3 + \overline{R}_{\overline{z} z^{-1}}z^{-1} \).

We have to show that (3.7) remains valid for all integers \( i, j \), with \( i + j \leq 5 \). To this end we consider the recursive bi-sequence \( \hat{\gamma}^{(s)} \equiv \{ \hat{\gamma}_{ij} \}_{i+j \leq s} \) defined by

\[
\begin{align*}
\hat{\gamma}_{i+1,j+2} &= \Lambda_{s}(\overline{z} z^P), \quad (i + j \leq 5), \\
\hat{\gamma}_{i,j} &= \gamma_{i,j} \quad \text{otherwise};
\end{align*}
\]

and we will show that \( \hat{\gamma}^{(s)} = \gamma^{(s)} \). Notice that since \( \overline{z} z^P - P_{\overline{z}}z^{e-1} \) is a generating polynomial for \( \gamma^{(s)} \), then \( \overline{z} z^P - P_{\overline{z}}z^{e-1} \) is an other one. Thus

\[
\hat{\gamma}_{i+2,j+1} = \Lambda_{s}(\overline{z} z^P), \quad (i + j \leq 5).
\]

It follows from (3.7) and (3.9) that, for \( n + m \leq 6 \), \( n = 0 \) and \( (n, m) = (3, 4), (4, 3) \):

\[
\gamma_{nm} = \Lambda_{s}(\overline{z} z^m) = \Lambda_{s}(\overline{z} z^m) := \hat{\gamma}_{nm}.
\]

Remark that if \( \hat{\gamma}_{nm} = \gamma_{nm} \), then \( \hat{\gamma}_{nm} = \overline{\gamma}_{nm} = \overline{\gamma}_{nm} = \gamma_{nm} \).

Therefore, we need to show (3.11), only, for the integers \( n = m = (2, 5), (1, 6); (1, 7), (2, 6), (3, 5), (4, 4). \)
\[ \gamma_{25} = \Lambda_{\gamma}(z^{2}P_{z^{2}}), \]
\[ = \Lambda_{\gamma}(R_{\gamma}; P_{z^{2}}), \quad \text{utilizing } (3.5), \]
\[ = \overline{\alpha} \Lambda_{\gamma}(z^{2}P_{z^{2}}) + \Lambda_{\gamma}(z^{2}R_{\gamma}; P_{z^{2}}) \]
\[ = \overline{\alpha} \gamma_{34} + \Lambda_{\gamma}(z^{2}R_{\gamma}; P_{z^{2}}), \quad \text{applying } (3.5), \]
\[ (3.12) \quad = \overline{\alpha} \gamma_{34} + \Lambda_{\gamma}(z^{2}R_{\gamma}; P_{z^{2}}), \quad \text{use } \deg z^{4}R_{\gamma}P_{z^{2}} \leq 6 \text{ and } (3.11), \]
\[ = \Lambda_{\gamma}(z^{2}P_{z^{2}}); \quad \text{according to Proposition } 3.1, \]
\[ = \gamma_{25}. \quad \text{from } (3.10). \]

\[ \gamma_{16} = \Lambda_{\gamma}(z^{2}P_{z^{2}}), \quad \text{use} (3.5), \]
\[ = \Lambda_{\gamma}(R_{\gamma;} P_{z^{2}}), \quad \text{employ } (3.8) \text{ and } \deg P_{z^{2}} \leq 3, \]
\[ = \Lambda_{\gamma}(\alpha z^{2}P_{z^{2}} + R_{\gamma}P_{z^{2}}) \]
\[ = \alpha_{07} \Lambda_{\gamma}(z^{2}P_{z^{2}}), \quad \text{utilizing } (3.5), \]
\[ = \alpha_{07} \Lambda_{\gamma}(z^{2}R_{\gamma}; P_{z^{2}}), \quad \text{using } (3.11) \text{ and } \deg z^{4}R_{\gamma}P_{z^{2}} \leq 6, \]
\[ = \Lambda_{\gamma}(\alpha z^{7} + z^{4}R_{\gamma}P_{z^{2}}) \]
\[ = \Lambda_{\gamma}(z^{4}P_{z^{2}}) \]
\[ = \gamma_{16}, \quad \text{according to } (3.9). \]

Thus, the equality (3.11) is valid for every integer \( n \) and \( m \) with \( n + m \leq 7 \). In other words,
\[ (3.14) \quad \gamma_{n m} = \Lambda_{\gamma}(z^{n}z^{m}) = \Lambda_{\gamma}(z^{n}z^{m}) := \gamma_{n m} \quad (n + m \leq 7). \]

And thus one can generalize the relation (3.7) as follows
\[ (3.15) \quad \gamma_{i+1,j+2} = \Lambda_{\gamma}(z^{i}z^{j}P_{z^{2}}) = \alpha \gamma_{i,j+3} + \Lambda_{\gamma}(z^{i}z^{j}R_{\gamma}P_{z^{2}}) \quad (i + j \leq 4). \]

Now, let us show (3.11) in the remaining cases \( (n + m = 8) \).
\[ (3.16) \quad \gamma_{08} = \gamma_{08}, \quad \text{by the construction of } \gamma^{(8)}, \text{ see } (3.9). \]
\( \gamma_{26} = \Lambda_{\gamma}(8) (\mathcal{T}^2 z^2 P_{\tau z}), \)
\[ \text{according to (3.5)} \]
\[ = \Lambda_{\gamma}(8) (\mathcal{T} P_{\tau z} P_{\tau z}), \]
\[ = \Lambda_{\gamma}(8) (\mathcal{T} z^4 P_{\tau z}), \]
\[ \text{use deg} \mathcal{T} P_{\tau z} \leq 4 \text{ and (3.15)}, \]
\[ \text{utilizing (3.5)}, \]
\[ (3.18) \]
\[ = \alpha \Lambda_{\gamma}(8) (\mathcal{T} z^7) + \Lambda_{\gamma}(8) (\mathcal{T} z^4 R_{\tau z}) \]
\[ = \alpha \Lambda_{\gamma}(8) (\mathcal{T} z^7) + \Lambda_{\gamma}(8) (\mathcal{T} z^4 R_{\tau z}), \]
\[ \text{by using (3.17) and (3.14)}, \]
\[ = \Lambda_{\gamma}(8) (\mathcal{T} z^4 P_{\tau z}) \]
\[ = \hat{\gamma}_{26}, \]
\[ \text{according to (3.9)}. \]

Before continue the proof, of these lemma, let us remark that the Relation 3.14 implies that, for all \( i + j \leq 5, \)

\[ \Lambda_{\gamma}(8) (\mathcal{T}^{i+1} z^{j+2}) = \hat{\gamma}_{i+1,j+2} = \Lambda_{\gamma}(8) (\mathcal{T} z^i (\alpha z^3 + R_{\tau z})), \]

and thus
\[ \Lambda_{\gamma}(8) (\mathcal{T} z^{i+3}) = \frac{1}{\alpha} \Lambda_{\gamma}(8) (\mathcal{T} z^i (\mathcal{T} z^2 - R_{\tau z})) \quad (i + j \leq 5). \]

Now,

\[ \gamma_{35} = \Lambda_{\gamma}(8) (\mathcal{T}^2 z P_{\tau z}), \]
\[ = \Lambda_{\gamma}(8) (\mathcal{T} P_{\tau z} P_{\tau z}), \]
\[ = \Lambda_{\gamma}(8) (\mathcal{T} z^4 P_{\tau z}), \]
\[ \text{because deg} \mathcal{T} z P_{\tau z} \leq 7, \]
\[ = \Lambda_{\gamma}(8) (\mathcal{T} z^6) - \frac{1}{\alpha} \Lambda_{\gamma}(8) (z^5 R_{\tau z}), \]
\[ \text{applying (3.14)}, \]
\[ (3.20) \]
\[ = \frac{1}{\alpha} \gamma_{26} - \frac{1}{\alpha} \Lambda_{\gamma}(8) (z^5 R_{\tau z}), \]
\[ \text{remark that deg} z^5 R_{\tau z} \leq 7, \]
\[ = \frac{1}{\alpha} \gamma_{26} - \frac{1}{\alpha} \Lambda_{\gamma}(8) (z^5 R_{\tau z}), \]
\[ = \frac{1}{\alpha} \gamma_{26} - \frac{1}{\alpha} \Lambda_{\gamma}(8) (z^5 R_{\tau z}), \]
\[ = \Lambda_{\gamma}(8) (\mathcal{T} z^5) \]
\[ = \hat{\gamma}_{35}. \]
\[ \gamma_{44} = \Lambda_{\gamma(8)}(z^4 P_z^4) \]
\[ = \Lambda_{\gamma(8)}(z^4 P_z^4) \]
\[ = \Lambda_{\gamma(8)}(\frac{1}{\alpha}(P_{z^2} - R_{z^2}) z P_z^4), \]
\[ = \frac{1}{\alpha} \Lambda_{\gamma(8)}(z^4 P_z^4) - \frac{1}{\alpha} \Lambda_{\gamma(8)}(z P_z^4 R_{z^2}), \]
\[ = \frac{1}{\alpha} \Lambda_{\gamma(8)}(z^4 P_z^4) - \frac{1}{\alpha} \Lambda_{\gamma(8)}(z P_z^4 R_{z^2}), \]
\[ = \frac{1}{\alpha} \gamma_{35} - \frac{1}{\alpha} \Lambda_{\gamma(8)}(z P_z^4 R_{z^2}), \]
\[ = \frac{1}{\alpha} \gamma_{35} - \frac{1}{\alpha} \Lambda_{\gamma(8)}(z P_z^4 R_{z^2}), \]
\[ = \Lambda_{\gamma(8)}(\frac{1}{\alpha}(P_{z^2} - R_{z^2}) z P_z^4), \]
\[ = \Lambda_{\gamma(8)}(z^4 P_z^4) \]
\[ = \gamma_{44}. \]

This finishes the proof of Lemma 3.3.

\[ \square \]

4. Solving the quintic moment problem

Let \( \gamma^{(5)} \equiv \{ \gamma_{ij} \}_{i+j \leq 5} \) be a given complex-valued bi-sequence, with \( \gamma_{00} > 0 \) and \( \gamma_{ij} = \gamma_{ji} \) for \( i + j \leq 5 \). The quintic moment problem involves determining necessary and sufficient conditions for the existence of a positive Borel measure \( \mu \) on \( \mathbb{C} \) (called a representing measure for \( \gamma^{(5)} \)) such that

\[ \gamma_{ij} = \int z^i \overline{z}^j d\mu, \quad \text{for } i + j \leq 5. \]

In this section we will show that in almost all cases the classical necessary conditions \( M(2) \geq 0 \) and \( \text{Rang } B \subseteq \text{Rang } M(2) \), for some \( W \), (with \( M(2) \) and \( B \) are as in (2.7)) guarantee the existence of at most \( (r + 2) \)-atomic (here \( r := \text{rank } M(2) \)) representing measure for \( \gamma^{(5)} \).

According to Proposition 2.3, the Hermitian \( 4 \times 4 \)-matrix \( W^* M(2) W \) is symmetric with respect to the second diagonal, then one can set

\[ W^* M(2) W = \begin{pmatrix} a & b & c & d \\ b & e & f & c \\ c & f & e & b \\ d & e & b & a \end{pmatrix} \]

The next Theorem gives a concrete solution to the quintic complex moment problem, except for the case \( a = e \) and \( b \neq f \).

**Theorem 4.1.** Let \( \gamma^{(5)} \equiv \{ \gamma_{ij} \}_{i+j \leq 5} \) be a given sequence, we assume that \( M(2) \geq 0 \) and \( \text{Rang } B \subseteq \text{Rang } M(2) \), and \( a \neq e \) or \( b = f \). Then the quintic moment problem, associated with \( \gamma^{(5)} \), admits a solution \( \mu \). Moreover, The smallest cardinality of \( \text{supp } \mu \) is

- \( \text{card } \text{supp } \mu = r \iff a = e \text{ and } b = f \),
- \( \text{card } \text{supp } \mu = r + 1 \iff a \neq e \text{ and } \frac{a - e}{2} < |b - f| \),
- \( \text{card } \text{supp } \mu = r + 2 \iff a > e \text{ and } \frac{a - e}{2} \geq |b - f| \),

where \( a, b, e \) and \( f \) are as in (4.1).
Before we develop the proof of our theorem, let us introduce some notations. For $n \in \{3, 4\}$; let $\gamma^{(2n)} \equiv \{\gamma_{ij}\}_{i+j \leq 2n}$ be a truncated complex bi-sequence and let $M(n)$ be the associated moment matrix. As before, we denote by $B(n)$ and $C(n)$ the $(n - 1) \times n$-matrix and the $n \times n$-matrix, respectively, such that

\[
M(n) = \begin{pmatrix}
M(n-1) & B(n) \\
B^*(n) & C(n)
\end{pmatrix}
\]

Let $\mathfrak{B} \equiv \mathfrak{B}(n) \equiv \{Z \bar{Z}^\prime\}_{i,j} \in \mathfrak{B}$ (where $\mathfrak{R} \equiv \mathfrak{R}(n) \subseteq \{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$) be a basis for the column space of $M(n)$. Let us remark that the $r \times r$-matrix $M(n)_{[r]}$, where $r \equiv r(n) \equiv \text{card} \ \mathfrak{R}(n)$, the restriction of the moment matrix $M(n)$ to the basis $\mathfrak{B}$, is invertible.

**Proof of Theorem 4.1.** The main idea is to extend the initial data $\gamma^{(5)}$ to an even-degree $\gamma^{(6)}$ (by adding the sixtic moments $\gamma_{60} = \gamma_{06}, \gamma_{51} = \gamma_{15}, \gamma_{42} = \gamma_{24}$ and $\gamma_{33} \in \mathbb{R}$) such that the associated moment matrix $M(3)$, for an appropriate choice of the missing moments, is either a flat extension of $M(2)$ or admits admits a flat extension $M(4)$. Thus Theorem 2.1 yields that $M(3)$ has a representing measure; and as a consequence, $\gamma^{(5)}$ also admits a representing measure $\mu$. It is also proved that the smallest cardinality of $\text{supp} \ \mu$ will be $r := \text{rank} \ M(2)$ or $r + 1$ or $r + 2$.

By virtue of the Smul'jan’s Theorem, we need to find a Toeplitz square matrix $C(3)$, built with the new, sixtic, moments as entries and such that $C(3) - W^*M(2)W \geq 0$. Setting

\[
C(3) - W^*M(2)W = \begin{pmatrix}
\gamma_{42} - b & \gamma_{51} - c & \gamma_{60} - d \\
\gamma_{42} - b & \gamma_{33} - e & \gamma_{42} - f \\\n\gamma_{33} - e & \gamma_{15} - f & \gamma_{24} - b \\\n\gamma_{15} - f & \gamma_{24} - b & \gamma_{33} - a
\end{pmatrix},
\]

we will distinguish two cases:

**Case I:** $a = c$ and $b = f$. In this case the matrix $W^*M(2)W$ is a Toeplitz one, then it suffice to consider that $C(3) = W^*M(2)W$. According to (2.6), the matrix $M(3)$ is a flat extension of $M(2)$ and thus $\gamma^{(6)}$ (and in force $\gamma^{(5)}$) has a $r$-representing measure.

**Case II:** $a \neq c$. We proceed in two steps for this case. Obviously, the matrix $W^*M(2)W$ is not a Toeplitz one. Therefore, for every choice of a Toeplitz $4 \times 4$-matrix $C(3)$, we have $\text{rank} \ (C(3) - W^*M(2)W) \geq 1$. We will show, in first step, that the smallest possible $\text{rank}$ of $C(3) - W^*M(2)W$ will be either 1 or 2. In the second step, we will show that the moment matrix $M(3)$, obtained by extending $\gamma^{(5)}$ with the entries of some suitable $C(3)$, has a flat extension and thus admits a $\text{rank} M(3)$-atomic representing measure, see Theorem 2.1.

**Step 1:** *(construction of $C(3)$).* Firstly, let us observe that

\[
\text{rank}(C(3) - W^*M(2)W) = 1 \text{ and } C(3) - W^*M(2)W \geq 0
\]

if and only if we have

\[
(0) \ \gamma_{33} > \text{max}(a, c),
\]

\[
(i) \ | \gamma_{42} - b| = \sqrt{(\gamma_{33} - a)(\gamma_{33} - e)} \text{ and } |\gamma_{42} - f| = \gamma_{33} - e.
\]

\[
(ii)(\gamma_{15} - f)(\gamma_{24} - b) = (\gamma_{33} - a)(\gamma_{24} - f),
\]

\[
(iii)(\gamma_{15} - f)(\gamma_{24} - b)^2 = (\gamma_{33} - a)^2(\gamma_{24} - f) \text{ and } |\gamma_{15} - f|^2 = (\gamma_{33} - a)^2.
\]

Remark that the equalities $(i)$ provide the compatibility of the two equalities in $(iii)$ and vice versa.

The condition $(i)$ means that $\gamma_{42}$ is in the intersection of the two next circles $C = C(b, \sqrt{(\gamma_{33} - a)(\gamma_{33} - e)})$, of radius $\sqrt{(\gamma_{33} - a)(\gamma_{33} - e)}$ and centered at $b$, and $C' = C(f, \gamma_{33} - e)$. 


It is an easy geometrical observation to see that, the two circles $C$ and $C'$ have a nonempty intersection if, and only if, there exists $\gamma_{33} > \max(a, e)$, such that

$$\gamma_{33} - a - b < 0.$$  

Furthermore, from (0) and (4.10), we derive that

$$\gamma_{15} = \gamma_{33} - a - b < 0.$$  

Subcase II-1: $a < e$ or $a > e$ and $|b - f| > \frac{\gamma_{33}}{2}$. It suffices to choose $\gamma_{33}$ verifying (4.6), and thus $\gamma_{42}$ exists (as the point intersection of the two circles $C$ and $C'$). Furthermore, from (0) and (i) we derive that

$$\gamma_{42} - b \cdot \gamma_{42} - f \neq 0$$  

The equality (ii) gives the moment $\gamma_{15}$ and (iii) supplies $\gamma_{06}$, and this complete the construction of a Toeplitz matrix $C(3)$ for which $\text{rank}(C(3) - W^*M(2)W) = 1$. Note that, $\text{rank}(M(3)\cup(Z^1)) = \text{rank} M(2) + 1 = \text{rank} M(3)$. Hence, in $M(3)$, the columns $\overline{ZZ}^2, \overline{Z}'Z$ and $\overline{Z}^1$ are a linear combination of the columns $\mathcal{B}(2) \cup \{Z^2\}$. In particular, we can set

$$\overline{ZZ}^2 = \mathcal{R}_{\mathcal{B}(2)}(Z, \overline{Z}) = \alpha Z^3 + \mathcal{R}_{\mathcal{B}(2)}(Z, \overline{Z}),$$

with

$$\alpha = \frac{\det \begin{vmatrix} M(2)_{\mathcal{B}(2)} & \overline{Z} \overline{Z}^2_{\mathcal{B}(2)} \\ \overline{Z}^2_{\mathcal{B}(2)} & \gamma_{42} \end{vmatrix}}{\det \begin{vmatrix} M(2)_{\mathcal{B}(2)} & \overline{Z} \overline{Z}^2_{\mathcal{B}(2)} \\ \overline{Z}^2_{\mathcal{B}(2)} & \gamma_{33} \end{vmatrix}} = \frac{\det \begin{vmatrix} M(2)_{\mathcal{B}(2)} & \overline{Z} \overline{Z}^2_{\mathcal{B}(2)} \\ \overline{Z}^2_{\mathcal{B}(2)} & \gamma_{33} - a \end{vmatrix}}{\det \begin{vmatrix} M(2)_{\mathcal{B}(2)} & \overline{Z} \overline{Z}^2_{\mathcal{B}(2)} \\ \overline{Z}^2_{\mathcal{B}(2)} & \gamma_{33} - a \end{vmatrix}} = \frac{\gamma_{42} - b}{\gamma_{33} - a} > 0; \text{ by virtue of (4.7).}$$

Subcase II-2: $a > e$ and $\frac{\gamma_{33}}{2} > |b - f|$. Then $\text{rank}(C(3) - W^*M(2)W) \geq 2$ for every $4 \times 4$ Toeplitz matrix $C(3)$. Let us choose the sixth moments as follows

$$\begin{cases} \gamma_{33} > \max(a, e), \\ |\gamma_{42} - a| = \sqrt{\gamma_{33} - a} \gamma_{33} - e, \end{cases}$$

$$\gamma_{15} = \gamma_{33} - a - b \gamma_{42} = \gamma_{24} - \gamma_{15} \gamma_{24} - \gamma_{24} \gamma_{15}.$$  

Let us remark that as the first subcase II-1, we have

$$\gamma_{42} - b \gamma_{33} - a = \sqrt{\gamma_{33} - e \gamma_{33} - a} \neq 0.$$  

The moment defined in (4.10) construct a Toeplitz matrix $C(3)$ for which $\text{rank}(C(3) - W^*M(2)W) = 2$. Indeed, it suffices to observe that

- $(C(3) - W^*M(2)W)(Z^3) = \frac{\gamma_{33} - a}{\gamma_{42} - b}(C(3) - W^*M(2)W)(\overline{Z}Z^2)$,
- $(C(3) - W^*M(2)W)(\overline{Z}^2) = \frac{\gamma_{33} - a}{\gamma_{42} - b}(C(3) - W^*M(2)W)(\overline{Z}^2 Z)$. 

• the columns \((C(3) - W^*M(2)W)(Z^3)\) and \((C(3) - W^*M(2)W)(Z^3')\) are nonlinear (because \((i)\) cannot be verified).

Therefore, in \(M(3)\), the columns \(ZZ^2\) is a linear combination of the columns \(B(2) \cup \{Z^3\}\).

For reason of simplicity, we adopt the notation of the Relation (4.8), that is,
\[(4.12) \quad ZZ^2 = P_{zz}(Z, \overline{Z}) = \alpha Z^3 + R_{zz}(Z, \overline{Z}),\]
Where
\[\alpha = \frac{\gamma_{42} - b}{\gamma_{33} - a} \neq 0\]
by using (4.11).

We conclude that, in the both cases \(II-1\) and \(II-2\), we have extended the initial data \(\gamma^{(3)}\) to \(\gamma^{(6)}\) so that the associated moment matrix \(M(3)\) has the following columns relation
\[(4.13) \quad ZZ^2 = P_{zz}(Z, \overline{Z}) = \alpha Z^3 + R_{zz}(Z, \overline{Z}), \quad \text{with} \quad \alpha \neq 0.\]
We also note that since \(a \neq e\) we get,
\[(4.14) \quad |\alpha| = \frac{\gamma_{42} - b}{\gamma_{33} - a} = \frac{\sqrt{(\gamma_{33} - a)(\gamma_{33} - e)}}{\gamma_{33} - a} \neq 1\]

**Step 2:** \((M(3)\) has a flat extension, and thus a representing measure). We will build moments \(\{\gamma_{i,j}\}_{i+j=7,8}\) for which the moment matrix \(M(4)\) is a flat extension of \(M(3)\).

The relation (4.13) yields that
\[\langle M(3)ZZ^2, ZZ^2 \rangle = \langle M(3)P_{zz}(Z, \overline{Z}), ZZ^2 \rangle, \quad \text{for all} \quad i + j \leq 3.\]
By applying (2.4), one obtain
\[(4.15) \quad \Lambda_{i,j}(0)(\mathfrak{z}^{i+1}z^{j+2}) = \Lambda_{i,j}(0)(\mathfrak{z}^i z^j P_{zz}), \quad i + j \leq 3.\]

Since \(|\alpha| \neq 1\), we derive that there exists a complex number \(\gamma_{43} = \overline{\gamma_{43}}\) such that
\[(4.16) \quad \gamma_{43} = \Lambda(\mathfrak{z}^3 z P_{zz}),\]
that is,
\[\gamma_{43} = \alpha^{34} + \sum_{i+j \leq 2} \alpha_{i,j} \gamma_{i+3,j+1}.\]

It follows, from (4.16) and (4.15), that \(\mathfrak{z}z^2 = P_{zz}2\) is a generating polynomial of \(\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}\).

Since \(\left(\begin{array}{c} M(2) | B(2) \\ \mathfrak{z}_z^2 | B(2) \end{array}\right)^* \gamma_{33} \mathfrak{z}_z^2 | B(2) \rangle > 0\), then there exists a (unique) vector, say
\[P_{zz} = \beta z^3 + R_{zz} = \beta z^3 + \sum_{\mathfrak{z}z^2 \in B(2) \cup \{\mathfrak{z}z^2\} \cup \{\mathfrak{z}z^2\}^* \gamma_{33}} \beta_{i,j} \mathfrak{z}^i z^j \]
the associated polynomial, such that
\[\left(\begin{array}{c} M(2) | B(2) \\ \mathfrak{z}_z^2 | B(2) \end{array}\right)^* \gamma_{33} \mathfrak{z}_z^2 | B(2) \rangle = ((\gamma_{04}, \gamma_{14}, \gamma_{05}, \gamma_{24}, \gamma_{15}, \gamma_{06}) B(2), \gamma_{34})^T.\]

Therefore the sequence \(\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}\) verifies that
\[(4.17) \quad \gamma_{i,j+4} = \Lambda(\mathfrak{z}^i z^j P_{zz}), \quad \text{for all} \quad i + j \leq 2 \quad \text{and} \quad (i, j) = (3, 0);\]
\[(4.18) \quad \gamma_{i+4,j} = \Lambda(\mathfrak{z}^i z^j P_{zz}), \quad \text{for all} \quad i + j \leq 2 \quad \text{and} \quad (i, j) = (0, 3).\]
Thus \(z^4 = P_{zz}\) is a generating polynomial of \(\gamma^{(6)} \cup \{\gamma_{34}, \gamma_{43}\}\).
We will build a sequence $\gamma^{(0)} = \{\gamma_{ij}\}_{i+j \leq 8}$, the extension of $\gamma^{(0)} \cup \{\gamma_{34}, \gamma_{43}\}$, by using a generating polynomial $P_{\gamma}$ and the initial data $\{\gamma_{ij}\}_{i+j \leq 3}$, that is,

$$
\gamma_{i,j+4} = \Lambda(\mathcal{Z} z^j P_{\gamma}) = \beta \gamma_{i,j+3} + \sum_{i+k \in \mathbb{N}(2)} \beta_k \gamma_{i,j+1+k} \quad (i + j \leq 4)
$$

or, equivalently,

$$
\gamma_{i+4,j} = \Lambda(\mathcal{Z} z^j P_{\gamma}) = \overline{\beta} \gamma_{i+3,j} + \sum_{i+k \in \mathbb{N}(2)} \overline{\beta_k} \gamma_{i+k,j+1} \quad (i + j \leq 4).
$$

Hence, lemma 3.3 implies that $z^2 - P_{\gamma} z^4$ and $z^4 - P_{\gamma} z^2$ are two generating polynomials of $\gamma^{(0)}$. Therefore, in $M(4)$, the columns $Z^4, \mathcal{Z} Z^4, \mathcal{Z}^2 Z^4, \mathcal{Z}^3 Z, \mathcal{Z}^4$ are a linear combination of the columns $\langle Z^Z Z^4 \rangle_{i+j \leq 3}$ and thus $M(4)$ is a flat extension of $M(3)$. Indeed, it suffices to observe that $P_{\gamma} z, P_{\gamma} z^2, P_{\gamma} z^3, P_{\gamma} z^4 \in V \equiv \text{Vect}(Z^4, \mathcal{Z}^3, Z^2, \mathcal{Z} Z, \mathcal{Z}^2, Z, Z, 1)$ and thus $zP_{\gamma} z^2, \mathcal{Z} P_{\gamma} z^3, \mathcal{Z}^2 P_{\gamma} z^4 \in V$; also one have, for all $i + j \leq 4$,

\[
\begin{align*}
\langle M(4) Z^4, \mathcal{Z} Z^5 \rangle &= \langle M(4) P_{\gamma} z, \mathcal{Z} Z^5 \rangle; \\
\langle M(4) Z^4, \mathcal{Z} Z^5 \rangle &= \langle M(4) P_{\gamma} z, \mathcal{Z} Z^5 \rangle; \\
\langle M(4) Z^4, \mathcal{Z} Z^5 \rangle &= \langle M(4) z P_{\gamma} z^2, \mathcal{Z} Z^5 \rangle; \\
\langle M(4) Z^4, \mathcal{Z} Z^5 \rangle &= \langle M(4) \mathcal{Z} P_{\gamma} z^3, \mathcal{Z} Z^5 \rangle; \\
\langle M(4) Z^4, \mathcal{Z} Z^5 \rangle &= \langle M(4) \mathcal{Z}^2 P_{\gamma} z^4, \mathcal{Z} Z^5 \rangle.
\end{align*}
\]

This finishes the proof of the theorem.

5. Example

We consider the quintic sequence,

$\gamma_{00} = 6$
$\gamma_{01} = 1 + i$
$\gamma_{02} = 2i$
$\gamma_{03} = 2 + 2i$
$\gamma_{04} = 0$
$\gamma_{05} = 4 + 4i$

$\gamma_{10} = 1 - i$
$\gamma_{11} = 6$
$\gamma_{12} = 2 - 2i$
$\gamma_{13} = 4i$
$\gamma_{14} = -4 + 4i$

$\gamma_{20} = -2i$
$\gamma_{21} = 2 - 2i$
$\gamma_{22} = 8$
$\gamma_{23} = 4 - 4i$

$\gamma_{30} = -2 - 2i$
$\gamma_{31} = -4i$
$\gamma_{32} = 2 - 2i$
$\gamma_{33} = 4 - 4i$

$\gamma_{40} = 0$
$\gamma_{41} = -4 - 4i$
$\gamma_{42} = 2 - 2i$
$\gamma_{43} = 4 - 4i$

then our matrices are

$$
M(2) = \begin{pmatrix}
6 & 1 + i & 1 - i & 2i & 6 & -2i \\
1 - i & 6 & -2i & 2 + 2i & 2 - 2i & -2 - 2i \\
1 + i & 2i & 6 & -2 + 2i & 2 + 2i & -2 - 2i \\
-2i & 2 - 2i & -2 - 2i & 8 & -4i & 0 \\
6 & 2 + 2i & 2 - 2i & 4i & 8 & -4i \\
2i & -2 + 2i & 2 + 2i & 0 & 4i & 8
\end{pmatrix}
$$

and

$$
B = \begin{pmatrix}
-2 + 2i & 2 + 2i & 2 - 2i & -2 - 2i & 4i & 8 & -4i & 0 \\
4i & 8 & -4i & 0 & 4 + 4i & 4 - 4i & -4 - 4i & -4 + 4i \\
-4 + 4i & 4 - 4i & -4 - 4i & -4 + 4i & -4 - 4i & 4 + 4i & 4 + 4i & -4 - 4i
\end{pmatrix}.
$$
The fact that $M(2)$ is positive definite implies,

$$W = (M(2))^{-1} B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
\frac{3}{4} + \frac{3i}{4} & \frac{3}{4} - \frac{i}{4} & \frac{1}{4} - \frac{i}{4} & \frac{1}{4} + \frac{3i}{4} \\
0 & 0 & 1 & 0
\end{pmatrix}$$

and

$$W^* M(2) W = \begin{pmatrix}
12 & -8i & -4 & 8i \\
8i & 12 & -8i & -4 \\
-4 & 8i & 12 & -8i \\
-8i & -4 & 8i & 12
\end{pmatrix}$$

Since $a = e = 12$ and $b = f = -8i$, according to theorem 4.1, our sequence is a moment matrix for a 6 atomes measure. In fact, from $W$, we can see that $Z^3 + \frac{3i}{4}(Z^2 - Z^2) - Z$ and $Z^2 Z + \frac{3i}{4}(Z^2 - Z^2) - Z$ are two characteristic polynomials for the moment sequence. The comment roots of the two polynomials are

$$\{\pm 1, \pm i, 0, 1 + i\}$$

Finally get that $\mu = \delta_1 + \delta_{-1} + \delta_i + \delta_{-1} + \delta_0 + \delta_{1+i}$.

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Mohammed V University in Rabat., Rabat, Morocco
E-mail address: elazharhamza@gmail.com

Mohammed V University in Rabat., Rabat, Morocco
E-mail address: ayoub1993harrat@gmail.com

Mohammed V University in Rabat., Rabat, Morocco
E-mail address: kaissar.idrissi@gmail.com

Mohammed V University in Rabat., Rabat, Morocco
E-mail address: zerouali@fsr.ac.ma