Reformulating Scalar-Tensor Field Theories

as Scalar-Scalar Field Theories

Using Lorentzian Cofinsler Spaces

by

Gregory W. Horndeski
2814 Calle Dulcinea
Santa Fe, NM 87505-6425
email: horndeskimath@gmail.com

November 14, 2019
ABSTRACT

In this paper I shall show how the notions of Finsler geometry can be used to construct a similar geometry using a scalar field, f, on the cotangent bundle of spacetime. This will enable me to use the second vertical derivatives of f, along with the differential of the scalar field φ, to construct a Lorentzian metric on spacetime that depends upon φ. f will be chosen so that the resultant metric on spacetime has the form of a LFRW metric, with the t equal constant slices being flat. When the Horndeski Lagrangians are evaluated for this choice of geometry the quartic and quintic Lagrangians are third order in φ, but reduce to non-degenerate second-order Lagrangians plus a divergence. Upon varying φ in these “scalarized” Horndeski Lagrangians, equations will be obtained which admit self-inflating universe solutions, provided the coefficient functions in the Lagrangians are chosen suitably. This approach is also used to study solutions of the most general conformally invariant scalar-tensor field theory which is flat space compatible (i.e., such that the Lagrangians are well-defined when either the space is flat or the scalar field is constant). There too the coefficient functions can be chosen to give self-inflating universes. Arguments will be presented to show that it may be possible to construct model Universes that begin explosively, and then settle down to period of much quieter acceleration, which can be followed by a collapse to its original, pre-expansion, state.
Table of Contents

Section 1: Introduction ................................................ page 4

Section 2: Cofinsler Spaces and Scalar-Scalar Field Theories ........ page 11

Section 3: Lagrangians and Field Equations ........................ page 18

Section 4: Delineating Future Tasks and
          Constructing Universes........................................ page 42

Acknowledgements ................................................. page 56

Bibliography ....................................................... page 57

Figures ............................................................. page 60
Section 1: Introduction

I shall begin with a few remarks which are intended to help explain why I considered using ideas from Finsler geometry to study scalar-tensor field theories.

When a layman asks me what a scalar field is I usually answer with an example, by telling them that the temperature at each point in (say) this room, at each instant of time is a good example of a scalar field. However, that is not really true. After all temperature is a measure of energy, and such measurements are observer dependent—but I wish to spare them the confusing details. However, if I said the temperature at each point in the room at each instant of time measured by observers at rest with respect to me, then I would have a valid scalar field.

The important thing to be gleaned from the temperature example is, as is well-known, energy measurements are observer dependent. I shall now show how this can be used to construct an energy scalar field on the subbundle, TLUM, of time-like unit vectors on a Lorentzian spacetime $V_4 = (M, g)$, with signature $(-, +, +, +)$. This in turn will get us thinking about Finsler Spaces as suitable venues for studying gravitation.

Let $P$ be a point of $M$, and let $v_P$ be the unit TL ($:=\text{timelike}$) tangent vector to the world line of an observer, $O_P$, at $P$. If $\mu_p \in T_p M$ is the energy-momentum vector of a particle passing through $P$, then the energy of that particle with respect to $O_p$ is $E(\mu_p, v_P) := g(\mu_p, v_P)$. We can now define the energy with respect to $O_p$ of all particles
passing through $P$ by

$$E(v_p) := - \sum_{\text{all particles passing through } P} g(\mu_p, v_p).$$  \hspace{1cm} \text{Eq.1.1}$$

The function $E: TLUM \to \mathbb{R}$ is closely related to a scalar field $\mathcal{E}$ on $TLM$, the bundle of $TL$ vectors on $M$, which is an open subbundle of $TM$. If $v \in TLM$ then

$$\mathcal{E}(v) := E((-g(v,v))^{1/2}v).$$  \hspace{1cm} \text{Eq.1.2}$$

In general relativity it is more convenient to deal with the energy density of matter fields, which is obtained from the energy-momentum tensor, $T$, of those fields. $T$ is an $(0,2)$ tensor field, and gives rise to an energy density scalar function $\mathcal{E}_T$ on $TLM$, which is defined by

$$\mathcal{E}_T(v) := -\frac{T(v,v)}{g(v,v)}.$$  \hspace{1cm} \text{Eq.1.3}$$

Since energy resides as a scalar field in $TM$, it seems reasonable to try to reformulate gravity as a field theory on $TLM$ or $TLUM$. This is where Lorentzian Finsler Geometry can be useful. Such geometries are relatively new in comparison to Finsler Geometries which arose in the early 1900's, Finsler [1]. Excellent treatises on Finsler Geometry are provided by Rund [2], and Chern, et al [3]. I shall not bother giving the complete definition of Finsler Spaces, but a few cursory remarks are in order. I shall be a bit more precise when we get to pseudo-Finsler spaces.

If we have a curve $c = c(t)$, $t_i \leq t \leq t_f$, in a Riemannian Manifold $V_n = (M,g)$, we
define the length of $c$ by

$$L(c) := \int_{t_i}^{t_f} [g(c',c')]^{1/2} \, dt ,$$  \hspace{1cm} \text{Eq.1.4}$$

where $c'$ is the tangent vector to $c$, which can be viewed as a curve in $TM$. If $c = c(\bar{t}) := c(t(\bar{t}))$, $\bar{t}_i \leq \bar{t} \leq \bar{t}_f$ is a reparameterization of $c$, with $\frac{dt}{d\bar{t}} > 0$, then $L(c) = L(\bar{c})$, and hence the length of a curve is well-defined.

Finsler Geometry generalizes this notion of length using a function $f:TM\backslash\{0\} \to \mathbb{R}^+$, which is such that $\forall \lambda \in \mathbb{R}^+$ and $v \in TM\backslash\{0\}$, $f(\lambda v) = \lambda^2 f(v)$. We can then use $f$ to define the length of curves $c = c(t)$, $a \leq t \leq b$, in $M$ by

$$L_f(c) := \int_a^b |f(c')|^{1/2} \, dt .$$

(We do not really need the absolute value signs in the above integrand, but they are required when we consider pseudo-Finsler Spaces in which $f$ maps into $\mathbb{R}$.) Since $f$ is positively homogeneous of degree 2, $L_f(c)$ is well-defined; i.e., independent of parameterization.

If $f$ is to define a Finsler Space, then there exists a second very important condition that it must satisfy. But in order to not be too repetitive I shall hold off stating that condition until we get to Pseudo-Finsler Spaces in just a moment.

If $V_n = (M,g)$ is a Riemannian or pseudo-Riemannian space, then $g$ naturally gives rise to a function $f$ on $TM$ defined by

$$f(v) = |g(v,v)|.$$
We shall call Finsler spaces of this type, trivial Finsler Spaces. Examples of non-trivial Finsler spaces are provided in [2] and [3].

I would like to point out that the Finsler function is usually denoted by F in the literature on the subject. However, physicists usually use F for the electromagnetic field tensor. Thus to avoid confusion, and to also allow the introduction of electromagnetic fields into our discussion, I chose f to denote the Finsler function.

I shall now state the formal definition of pseudo-Finsler Spaces following the presentation given by Bejancu & Farran [4].

Let M be an n-dimensional smooth manifold with tangent bundle TM, and let $\pi: TM \to M$ denote the natural projection. So if $v \in TM$ is a tangent vector at $P \in M$, then $\pi(v) = P$. If $x$ is a chart of M with domain $U$, it naturally gives rise to a chart $(\chi, y)$ in TM, with domain $\pi^{-1}U$, which is defined by

$$v = v^i \frac{\partial}{\partial x^i} \in \pi^{-1}U \to (\chi(v), y(v)),$$

with $\chi = x \circ \pi$, $y(v) = [v^i]$, and Einstein’s summation convention is being used. If $x'$ is another chart of M with domain $U'$, and $U \cap U' \neq \emptyset$, then on $U \cap U'$ we can write $x^i = x'^i(\chi^j)$, and conversely $x^i = x'(\chi^j)$. In this case the standard charts $(\chi', y')$ and $(\chi, y)$ are overlapping charts of TM. On the overlap we have

$$\chi'^j = \chi'^i(\chi^i), \quad y'^j = y^h \frac{\partial x'^j}{\partial x^h} \circ \pi$$

and conversely

$$\chi^j = \chi^i(\chi^i), \quad y^j = y^h \frac{\partial \chi^j}{\partial \chi^h} \circ \pi$$

Eq.1.5
\[ \chi^i = \chi^i(\chi^n), \quad y^i = y^m \frac{\partial \chi^i}{\partial x^m} \circ \pi \quad \text{Eq.1.6} \]

Now for the definition of a pseudo-Finsler space. Consider a positive integer q<n, and a smooth function f: N→R, where N is an open submanifold of TM with π(N) = M, and 0:M→TM is the zero section. N is required to be invariant under dilation: i.e., \( \forall \lambda \in \mathbb{R}^+ \), and \( v \in N, \lambda v \in N \). Lastly f is required to satisfy the following two conditions:

(i) f is positively homogeneous of degree 2, i.e., \( \forall \lambda \in \mathbb{R}^+ \) and \( v \in N, f(\lambda v) = \lambda^2 f(v) \); and

(ii) if \( v \in N \) and \((\chi,y)\) is a standard chart of TM at v, then the matrix

\[ \left[ \frac{1}{2} \frac{\partial^2 f(v)}{\partial y^i \partial y^j} \right] \]

defines a quadratic form on \( \mathbb{R}^n \) with q negative eigenvalues and n−q positive eigenvalues, 0<q<n.

We say that the triple \( F^n := (M,N,f) \) is a pseudo-Finsler Manifold (or Space) of index q, and refer to f as the pseudo-Finsler function. When q=1 or n−1, \( F^n \) is said to be a Lorentzian Finsler Manifold.

One should note that due to Eqs.1.5 and 1.6 the signature of f is well-defined.

For suppose that \((\chi',y')\) is another standard chart at v. Then we have

\[ \frac{\partial f}{\partial y^i} \bigg|_v = \frac{\partial f}{\partial \chi^i} \bigg|_v \frac{\partial \chi^i}{\partial y^i} \bigg|_v + \frac{\partial f}{\partial y^i} \bigg|_v \frac{\partial y^i}{\partial y^i} \bigg|_v \quad \text{Eq.1.7} \]

Since \( \chi^i = \chi^i(\chi^n) \), the first term on the right-hand side of Eq.1.6 vanishes, and we can
use Eq.1.6 to rewrite Eq.1.7 as

\[ \frac{\partial f}{\partial y^i} = \frac{\partial f(\mathbf{v})}{\partial y^j} \frac{\partial x^j}{\partial x^i}. \]  

Eq.1.7

It should now be clear that

\[ \frac{\partial^2 f}{\partial y^i \partial y^j}(\mathbf{v}) = \frac{\partial^2 f}{\partial y^j} \frac{\partial x^j}{\partial x^i} (\pi(\mathbf{v})) \]  

Eq.1.8

Hence the index of \( \frac{\partial^2 f(\mathbf{v})}{\partial y^i \partial y^j} \) is independent of the standard chart of TM that we use. From Eq.1.8 it should be evident how the higher order “vertical derivatives” of \( f \) would transform under coordinate transformations, where by vertical derivatives I mean derivatives with respect to the \( y^i \)'s, which are the coordinates on the (vertical) fibres of TM.

Eq.1.8 also shows us that if we have a vector field \( \xi \) on M which is such that \( \xi_p \in N \forall P \in M \), then we can define a pseudo-Riemannian metric tensor \( g_{\xi} \) on M using \( f \) and \( \xi \). To do this let x be any chart of M, with corresponding standard chart \((\chi, y)\) of TM. Then we define the x components of \( g_{\xi} \) by

\[ g_{\xi,ij} := g_{\xi} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) := \frac{1}{2} \frac{\partial^2 f}{\partial y^i \partial y^j}(\xi). \]  

Eq.1.9

Due to Eq.1.8, \( g_{\xi} \) is a well-defined pseudo-Riemannian metric tensor on M. Similarly the \( k^{th} \) order vertical derivatives of \( f (k>0) \) could be used in conjunction with the vector field \( \xi \) to construct symmetric \((0,k)\) tensor fields on M.
In summary, if we have a pseudo-Finsler space \( F^n = (M, N, f) \), of index \( q \), and a vector field \( \xi \) on \( M \) whose range lies in \( N \), then we also have a pseudo-Riemannian space \( V_n = (M, g_\xi) \) of index \( q \).

Now let us confine our attention to \( n=4 \), with index 1, so we have a Lorentzian Finsler space. Earlier in this section I showed how we could introduce energy scalar fields \( \mathcal{E} \) and \( \mathcal{E}_T \) on an open submanifold of \( TM \) (see, Eqs.1.2 and 1.3). So the natural question to ask is: Can we use an energy scalar field and \( f \) as the basis of a gravitational field theory? We could then use \( f \) and \( \xi \) to give us a Lorentzian metric tensor on \( M \) in the manner described above. The scalar fields \( \mathcal{E} \) and \( f \) (or \( \mathcal{E}_T \) and \( f \)) in \( TM \) could be thought of as a generalization of the variables appearing in Poisson’s classical gravitational equation. To complete our task all we need are some field equations on \( TM \) governing a pair of scalar fields. Unfortunately no such equations exist due to the following

Proposition: If \( \mathcal{L} \) is a Lagrange scalar density on an \( n \)-dimensional manifold \( (n>1) \) which is a concomitant of two scalar fields, \( \psi_1 \) and \( \psi_2 \), along with their derivatives of arbitrary order, then
\[
\frac{\delta \mathcal{L}}{\delta \psi_1} = 0 \text{ and } \frac{\delta \mathcal{L}}{\delta \psi_2} = 0 .
\]

Proof: Since \( \mathcal{L} \) is a scalar density, its associated Euler-Lagrange tensors are interrelated by a Noether type mathematical identity. In [5] (see, the bottom of page
I derived that identity for the case in which the Lagrangian is a concomitant of a metric tensor field, a covariant vector field and a scalar field, along with their derivatives. That identity is easily generalized to the present case, and is given by

\[ \frac{\delta \mathcal{L}}{\delta \psi_1} d\psi_1 + \frac{\delta \mathcal{L}}{\delta \psi_2} d\psi_2 = 0. \]  

Eq. 1.10

Eq. 1.10 must hold \( \forall \) pair of scalar fields. Since \( n>1 \) this is only possible when

\[ \frac{\delta \mathcal{L}}{\delta \psi_1} = \frac{\delta \mathcal{L}}{\delta \psi_2} = 0. \]

This proposition destroys our aspiration to construct gravitational field equations governing an energy scalar field and \( f \) on \( TM \) using a Lorentzian Finsler Space and a vector field \( \xi \) on \( M \). However, my real desire is to construct some sort of generalized scalar-tensor theory using a Finsler-like structure. This task will be tackled in the following sections.

**Section 2: Cofinsler Spaces and Scalar-Scalar Field Theories**

Suppose that we have a 4-dimensional Lorentzian Finsler Space \( F^4 = (M,N,f) \) and a scalar field \( \varphi \) on \( M \). Is there a simple way for us to use \( f \) and \( \varphi \) to endow \( M \) with a Lorentzian metric tensor? At first you might think that if the gradient of \( \varphi \), \( \nabla \varphi \), lies in \( N \) then we could use it to build a metric on \( M \). But to build \( \nabla \varphi \) you need a metric tensor, \( g \), on \( M \), since locally \( \nabla^i \varphi = g^{ij} \varphi_j \). Unfortunately there is no easy way to use \( F^4 \) and \( \varphi \) to endow \( M \) with a Lorentzian structure. Thus we shall have to take a different
approach if we want to use the scalar field $\varphi$ in the construction of a Lorentzian metric tensor on $M$. To that end we note that $d\varphi$ naturally provides us with a map from $M$ into $T^*M$, the cotangent bundle of $M$. Hence what we need is a smooth map from $T^*M$ into $\mathbb{R}$, whose second derivatives in the vertical direction (i.e., in the direction tangent to the fibres of $T^*M$) constitute a Lorentzian quadratic form when evaluated at $d\varphi$. I shall now make these notions a bit more precise.

Let $M$ be an $n$-dimensional manifold with cotangent bundle $T^*M$. Recall that $T^*M := \bigcup_{p \in M} T_p^*M$, where $T_p^*M$ is the dual space of $T_pM$, the tangent space to $M$ at $P$. We let $\pi:T^*M \rightarrow M$ denote the canonical (or natural) projection, which maps a covector $\omega \in T_p^*M$ to $P$: $\pi(\omega) = P$. If $x$ is a chart of $M$ with domain $U$ then it naturally gives rise to a standard chart $(\chi, y)$ of $T^*M$ with domain $\pi^{-1}U$ defined by $\chi := x \circ \pi$ and $y(\omega) = y(\omega_i \left. dx^i \right|_P) := (\omega_1, \ldots, \omega_n) = (\omega_i))$. (There should be no confusion between the standard charts of $TM$ and $T^*M$, since it should always clear what space we are working in.)

So if we write $y: \pi^{-1}U \rightarrow \mathbb{R}^n$, as $y = (y_1, \ldots, y_n) = (y_i)$, then $y_i(\omega) = \omega_i$, where $\omega = \omega_i \left. dx^i \right|_P$. Now suppose $x$ and $x'$ are overlapping charts of $M$ with domains $U$ and $U'$. Then $(\chi, y)$ and $(\chi', y')$ are overlapping charts in $T^*M$. On the overlap we have

$$\chi^i = \chi^j (\chi'^j), \quad \chi'^i = \chi^i (\chi^j) \quad \text{Eq.2.1a}$$

and

$$y_i = y_j' \frac{\partial \chi'^j}{\partial \chi^i}, \quad y'_i = y_j \frac{\partial \chi^j}{\partial \chi'^i} \quad \text{Eq.2.1b}$$

where the definition of partial differentiation on a manifold was used to make
the substitutions

\[ \frac{\partial x^i}{\partial \chi^j} = \partial x^i \circ \pi \quad \text{and} \quad \frac{\partial x^j}{\partial \chi^i} = \partial x^j \circ \pi. \quad \text{Eq.2.2} \]

With these preliminaries disposed of I can now define an n-dimensional Pseudo-Cofinsler Space. Let \( f : T^*M \to \mathbb{R} \) be a smooth function defined on an open submanifold \( N \) of \( T^*M \) where \( \pi N = M \). Choose a positive integer \( q \) such that \( q \leq n \). If \( x \) is a chart of \( M \) with domain \( U \), and corresponding standard chart \( (\chi, y) \) of \( T^*M \) with domain \( \pi^{-1}U \), then we require that \( \forall \omega \in \pi^{-1}U \) the matrix

\[ \left[ \frac{1}{2} \frac{\partial^2 f}{\partial y_i \partial y_j} \right]_\omega \quad \text{Eq.2.3} \]

defines a quadratic form on \( \mathbb{R}^n \) of index \( q \). Due to Eqs.2.1a and 2.1b the index of the matrix presented in Eq.2.3 is independent of the standard chart chosen at \( \omega \in T^*M \). When these conditions are met the triple \( CF^n(M,N,f) \) is called an n-dimensional Pseudo-Cofinsler Space (or Manifold) of index \( q \), and \( f \) is called the Cofinsler function. When \( q = 1 \) or \( n - 1 \), \( CF^n \) is called a Lorentzian Cofinsler Space (or Manifold). In what follows our Lorentzian Cofinsler Spaces will have index 1.

Note that in the definition of Cofinsler manifolds I have not placed any homogeneity constraints upon \( f \), as is done in the Finsler case. This was done because there is no natural way to “lift” curves in \( M \) into \( T^*M \) to compute their lengths using \( f \). Consequently there is no reason to require \( f \) to be homogeneous.
Suppose we have an n-dimensional pseudo-Cofinsler Space $CF^n(M,N,f)$ and a scalar function $\varphi$ on $M$. If the range of $d\varphi$ lies in $N$; i.e., $d\varphi(M) \subset N$, then we can define the contravariant components of a pseudo-Riemannian metric tensor $g_\varphi$ on $M$ by

$$g_{\varphi}^{ij} := g_\varphi(dx^i,dx^j) := \frac{1}{2} \frac{\partial^2 f}{\partial y_i \partial y_j} (d\varphi),$$

where $x$ is any chart of $M$ with corresponding standard chart $(\chi, y)$ for $T^*M$. Due to Eqs.2.1 and 2.2 it is clear that the $g_{\varphi}^{ij}$'s are the local contravariant components of a pseudo-Riemannian metric tensor on $M$.

I shall now present an example of a 4-dimensional Lorentzian Cofinsler Space that will play a significant roll in what follows. To see where this example comes from recall that the LFRW $\coloneqq$ Lemaître-Friedmann-Roberston-Walker) metric is given by

$$ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),$$

where $k$ is a constant with units of $(\text{length})^{-2}$, $r$ has units of length, $a(t)$ is unitless and $c=1$. $k$ determines the curvature of the spaces $t = \text{constant}$, which are 3-surfaces of constant curvature. To make matters even simpler I shall take $k=0$ in Eq.2.5, and so our line element becomes:

$$ds^2 = -dt^2 + a(t)^2 (du^2 + dv^2 + dw^2),$$

where $x=(x^0, x^1, x^2, x^3) := (t, u, v, w)$, are the standard coordinates of $\mathbb{R}^4$, and the 3-spaces $t = \text{constant}$ are flat. What we would like is a function $f$ on $T^*\mathbb{R}^4$ which is such that
when we construct its associated metric $g$ using Eq.2.4, and a scalar field $\phi$ on $\mathbb{R}^4$, we get a line element similar to the one presented in Eq.2.6. Well, sure we could always choose the trivial function $f_T$ defined by

$$f_T := -y_t^2 + a(t)\pi^2(y_u^2 + y_v^2 + y_w^2),$$

where $((t,u,v,w)\pi, (y_t,y_u,y_v,y_w))$ is the standard chart of $T^*\mathbb{R}^4$ determined by the chart $(t,u,v,w)$ of $\mathbb{R}^4$. (When working with $T^*\mathbb{R}^4$ I shall adopt the custom of dropping the projection function $\pi$ from the standard chart, since $T^*\mathbb{R}^4$ is obviously diffeomorphic to $\mathbb{R}^4 \times \mathbb{R}^4$.) For this choice of Cofinsler function $d\phi$ would actually play no roll in the associated metric tensor. A more useful choice of Cofinsler function is

$$f := -y_t^2 + y_t^{-2}(y_u^2 + y_v^2 + y_w^2).$$

When $\phi=\phi(t)$, this Cofinsler function gives us

$$[g_\phi^{ij}] = \frac{1}{2} \frac{\partial^2 f}{\partial y_i \partial y_j} (d\phi) = \text{diag}(-1, \phi'^{-2}, \phi'^{-2}, \phi'^{-2})$$

where $\phi'$ denotes a derivative with respect to $t$. Evidently $g_{ij}dx^i dx^j$ (where here, and in what follows, I shall drop the subscript $\phi$ on the covariant and contravariant form of the metric tensor) gives us the line element presented in Eq.2.6 with $a(t) = \phi'(t)$.

At this point some remarks concerning units is in order. In what follows we shall use geometrized units in terms of which $c=G=1$, and all dimensioned quantities will have units of length, $L$, to some power. To convey the fact that a physical quantity $\Omega$ has units of $L^\lambda$ we shall write $\Omega \sim L^\lambda$. In terms of these units the coordinates $t, u, v,$
w ~ L^1, and g_{ij} ~ L^0. Consequently \varphi' ~ L^0, and hence for us \varphi ~ L^1. In terms of geometrized coordinates the y_i (as well as the y^i in TM) are unitless. This follows from the fact that

\[ y_i (d\varphi) = \frac{\partial \varphi}{\partial x^i} \sim L^0. \]

Since g^i^j ~ L^0, we can now use Eq.2.4 to deduce that f ~ L^0.

From Eq.2.6 we know that the metric on the t=constant slices is

\[ (\varphi')^2 (du^2 + dv^2 + dw^2). \]

Thus the unitless quantity |\varphi'| plays the roll of a scale factor in these slices. To get the physical distance between any two distinct points in a slice you just multiply their Euclidean distance by |\varphi'|. An interesting thing happens when \varphi' \to 0^+. When that occurs the metric is telling us that the physical distance between any two points in a slice is vanishing, while topologically the slice still remains \( \mathbb{R}^3 \). So when \varphi'=0 our physical spacetime is not crushed out of existence but remains intact. More will be said about this later when we deal with physical interpretation of solutions to the various field equations which I shall construct.

The largest subset N of T*\( \mathbb{R}^4 \) upon which the function f presented in Eq.2.7 is defined and smooth is

\[ N = \{ ((t,u,v,w),(y_t,y_u,y_v,y_w)) | y_t \neq 0 \}. \]

It should be noted that f is not homogeneous on N.

To determine if f really does define a Cofinsler Structure on N \( \subset T*\mathbb{R}^4 \), we need
to examine the matrix $g^{ij}$ on $N$ and not just for $d\phi$. It is easy to use Eq.2.7 to show that

$$
[g^{ij}] = \begin{bmatrix}
-1 & -2y_t^{-3}y_u & -2y_t^{-3}y_v & -2y_t^{-3}y_w \\
-2y_t^{-3}y_u & y_t^{-2} & 0 & 0 \\
-2y_t^{-3}y_v & 0 & y_t^{-2} & 0 \\
-2y_t^{-3}y_w & 0 & 0 & y_t^{-2}
\end{bmatrix}
$$

Eq.2.9

and, with some effort, its inverse $[g_{ij}]$ is found to be

$$
[g_{ij}] = D^{-1} \begin{bmatrix}
-1 & -2y_t^{-4}y_u & -2y_t^{-4}y_v & -2y_t^{-4}y_w \\
-2y_t^{-4}y_u & y_t^{-2}y_u & 4y_t^{-2}y_u y_w & 4y_t^{-2}y_u y_v \\
-2y_t^{-4}y_v & 4y_t^{-2}y_u y_u & y_t^{-2}y_v + 4y_t^{-2}y_u y_v + 4y_t^{-2}y_w \\
-2y_t^{-4}y_w & 4y_t^{-2}y_u y_u & -4y_t^{-2}y_v y_u & y_t^{-2}y_w + 4y_t^{-2}y_v y_w + 4y_t^{-2}y_v y_v
\end{bmatrix}
$$

Eq.2.10

where

$$
D := 1 + 4y_t^{-4}(y_u^2 + y_v^2 + y_w^2)
$$

Eq.2.11

and

$$
det [g_{ij}] = -y_t^6 D^{-1}.
$$

Eq.2.12

In view of Eq.2.12 it is clear that $f$ defines a Lorentzian Cofinsler structure on $N \subset T^*\mathbb{R}^4$.

In passing I would like to point out that a Finsler or Pseudo-Finsler Space of the form $(\mathbb{R}^n,N,f)$ which is such that the Finsler function $f$ is only a function of the $y^i$ coordinates is called a Minkowski Finsler or Minkowski Pseudo-Finsler Space. In keeping with that nomenclature we probably should refer to the Cofinsler Space just introduced as a Minkowskian Lorentzian Cofinsler Space. However, I shall not do that since, not only is it a mouthful of terminology, but Minkowski Spaces have a different meaning for Relativists, and I do not want to cause any undue confusion.
I shall call a Lorentzian Spacetime, $V_4 = (M,g)$ endowed with a scalar field $\varphi$, in which the metric tensor $g$ arises from a Lorentzian Cofinsler Structure $CF^4(M,N,f)$ in the manner presented in Eq.2.4, as a Scalar-Scalar Theory. It is tempting to think of $f$ as a generating function, or as a potential for the Lorentzian metric tensor on $M$. But that is not quite right, since both $f$ and $\varphi$ are required to generate $g$. Our next task is to develop field equations for scalar-scalar theories. This will be done forthwith.

**Section 3: Lagrangians and Field Equations**

Suppose that we have a Scalar-Scalar Field theory based on the Lorentzian Cofinsler Space $CF^4(M,N,f)$ and the scalar field $\varphi$. We would like to build a Lagrangian from $f$ and $\varphi$. The first problem we encounter is that $f$ and $\varphi$ are defined on different spaces. Well, this is easily circumvented by letting $\Phi := \varphi \circ \pi$, where $\pi : T^*M \to M$ is the natural projection. Now $f$ and $\Phi$ are two smooth scalar fields on $T^*M$. Unfortunately it is impossible to use these scalar fields on $T^*M$ to build a useful Lagrangian due to the Proposition presented at the end of Section 1.

So let us now consider the possibility of deriving field equations on $M$ using $f$ and $\varphi$. To that end let $\mathcal{L}$ be any second-order Lagrange scalar density on $M$ which is a concomitant of a metric tensor and scalar field $\varphi$. We replace the metric tensor by our cofinsler metric tensor built from the vertical derivatives of $f$ evaluated at $d\varphi$, to
obtain a Lagrangian of the form

\[ \mathcal{L} = \mathcal{L}(g^{ij}(d\phi); g^{ij}(d\phi)_{,hh}; g^{ij}(d\phi)_{,hh}, \varphi; \varphi_{,h}; \varphi_{,hh}) \]  

Eq.3.1

which is locally well-defined on the coordinate domains of M. However, we are now confronted with a problem which is unique to Cofinsler (and Finsler Theory for that matter); viz., how does one go about computing the variational derivatives of a Lagrangian of the form presented in Eq.3.1? Say we want to vary \( \varphi \) holding \( f \) in \( T^*M \) fixed. Since the \( g^{ij} \)'s are functions of \( d\phi \), any variation of \( \varphi \) will lead to a variation of the \( g^{ij} \)'s. Conversely, the general variation of

\[ g^{ij} = \frac{1}{2} \frac{\partial f}{\partial y_i \partial y_j} d\phi \]

would require a variation of both \( f \) and \( \varphi \). In my experience with classical field theories I have not encountered such a convoluted variational problem. Usually when one encounters a Lagrangian which is a concomitant of various fields (all defined on the same manifold), one computes the variational derivative by holding all fields fixed except for one which is varied. That is not possible with the Lagrangian given in Eq.3.1. I shall now describe a method that we can use to circumvent this impediment to computing field equations for Scalar-Scalar theories. To that end we need to re-examine some very familiar equations from General Relativity.

Let us consider the derivation of the Schwarschild solution. Recall that in this situation we are looking for a static, spherically symmetric, asymptotically flat solution
to Einstien’s Equations $G^{ij} = 0$. The symmetry demands imply that we can choose a coordinate system so that our ansatz metric has the form

$$ds^2 = -e^\nu dt^2 + e^{\lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$ \hspace{1cm} \text{Eq.3.2}

where $\nu$ and $\lambda$ are functions of $r$ with $\theta$ and $\phi$ being spherical polar coordinates. Conventionally one plugs this metric into $G^{ij} = 0$ to obtain the following equations (c.f. Adler, Bazin & Schiffer [6])

$$R_{00} = \frac{1}{2}e^{(\nu - \lambda)}[\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\nu'\lambda' + 2\nu'r^{-1}] = 0,$$

$$R_{11} = \frac{1}{2}[\nu'' + \frac{1}{2}(\nu')^2 - \frac{1}{2}\nu'\lambda' - 2\lambda'r^{-1}] = 0,$$

$$R_{22} = e^{-\lambda}[1 + \frac{1}{2}r(\nu' - \lambda')] - 1 = 0,$$

$$R_{33} = \sin^2 \theta R_{22},$$

where a prime denotes a derivative with respect to $r$, and the other $R_{ij}$ vanish. These equations are readily solved and yield $\lambda = -\nu$ with $e^\nu = 1 - 2mr^{-1}$, in terms of units with $c=G=1$.

However, there exist a second way to proceed to get the equations for $\lambda$ and $\nu$, which I shall describe, even though it is a bit impractical. We know that the Einstein field equations $G^{ij}=0$, can be derived by varying the metric tensor in $\mathcal{L}_E := (-g)^{1/2}R$. So let us build $\mathcal{L}_E$ from the line element given in Eq.3.2. This yields

$$\mathcal{L}_E = -2 \sin \theta e^{\frac{1}{2}(\nu - \lambda)}(r\lambda' - 1 + e^{\lambda}).$$

If we vary the scalar fields $\lambda$ and $\nu$ in this Lagrangian we obtain the following Euler-
Lagrange equations:
\[ \frac{\delta \mathcal{L}}{\delta \lambda} = \sin \theta \ e^{\frac{1}{2}(v^2 \lambda)} (1 + r \nu - \nu^2) = 0 \]

and
\[ \frac{\delta \mathcal{L}}{\delta \nu} = \sin \theta \ e^{\frac{1}{2}(v^2 \lambda)} (1 - r \lambda - \lambda^2) = 0 . \]

Using the assumption of asymptotic flatness, we can solve this pair of equations to obtain the usual Schwarzschild result.

This observation concerning the Schwarzschild equations suggests an approach to deriving field equations from a Lagrangian of the form presented in Eq.3.1. We should begin by guessing a form of \( f \) suitable to the problem at hand. Plug that ansatz \( f \) into \( \mathcal{L} \) along with \( \varphi \), and then vary the fields in \( \mathcal{L} \), just as we did in \( \mathcal{L}_E \) above.

To illustrate this approach to obtaining field equations for scalar-scalar field theories I shall derive some cosmological solutions for various choices of \( \mathcal{L} \). For these solutions it will be assumed that \( f \) has been chosen as in Eq.2.7, with \( \varphi = \varphi(t) \) and \( g^{ij} \) given by Eq.2.8. I have selected this form of \( f \) since it generates a LFRW line element.

Let us begin by examining the form of the Horndeski Lagrangians \( L_H \) (see, [5] or [7]) when
\[ [g_{ij}] = \text{diag}(-1, (\varphi')^2, (\varphi')^2, (\varphi')^2) . \]

In general the scalar-tensor Lagrangian \( L_H \) is given by
\[ L_H = L_2 + L_3 + L_4 + L_5 \]
Eq.3.4

where
\[ L_2 := g^{ij} K(\varphi, X) \quad \text{Eq.3.5} \]
\[ L_3 := g^{ij} G_3(\varphi, X) \Box \varphi \quad \text{Eq.3.6} \]
\[ L_4 := g^{ij} G_4(\varphi, X) R - 2g^{ij} G_{4,X}(\varphi, X)((\Box \varphi)^2 - \varphi_{ab} \varphi_{ab}) \quad \text{Eq.3.7} \]
and
\[ L_5 := g^{ij} G_5(\varphi, X) \varphi_{ab} G^{ab} + \frac{1}{2} g^{ij} G_{5,X}(\varphi, X)((\Box \varphi)^2 - 3\Box \varphi(\varphi_{ab} \varphi_{ab}) + 2\varphi_a \varphi_b \varphi_{bc} \varphi_{ac}), \text{Eq.3.8} \]
where \( g := |\det g_{ab}|, \ X := g^{ab} \varphi_a \varphi_b, \Box \) denotes a partial derivative with respect to \( X \), \( \varphi_a \) denotes a partial derivative with respect to the local coordinate \( x^a \), \( \varphi_{ab} \) denotes the components of the second covariant derivative of \( \varphi \), \( \Box \varphi := g^{ab} \varphi_{ab} \), \( R := g^{ij} R_{ij} \), \( R_{ij} := R^{h \ ij}_{h \ jh} \)
\( G^{ij} := \frac{1}{2} g^{ij} R \), and \( R_{ij}^{h \ hk} := \Gamma^h_{ij, hk} - \Gamma^h_{ik, jh} + \Gamma^m_{ij, km} - \Gamma^m_{mk, ij} \Gamma^h_{mj} \) with \( \Gamma^h_{ij} := \frac{1}{2} g^{hm}(g_{nm, i} + g_{mi, n} + g_{ij, mn}) \). Using the metric tensor given in Eq.3.3 we find that the nonzero components of \( \Gamma^h_{ij} \), \( R_{ij} \), \( R \), \( G_{ij} \) and \( \varphi_{ij} \) are given by
\[ \Gamma^t_{uu} = \Gamma^t_{vv} = \Gamma^t_{ww} = \varphi'\varphi'' \quad \text{Eq.3.9} \]
\[ \Gamma^u_{tu} = \Gamma^v_{tv} = \Gamma^w_{tw} = \varphi''/\varphi' \quad \text{Eq.3.10} \]
\[ R_{tt} = -3\varphi''/\varphi', \ R_{uu} = R_{vv} = R_{ww} = \varphi'\varphi'' + 2(\varphi'')^2, \ R = 6(\varphi''/\varphi' + (\varphi'')^2) \quad \text{Eq.3.11} \]
\[ G_{tt} = 3(\varphi''/\varphi')^2, \ G_{uu} = G_{vv} = G_{ww} = -(2\varphi''/\varphi' + (\varphi'')^2) \quad \text{Eq.3.12} \]
and
\[ [\varphi_{ab}] = \text{diag}(\varphi_{tt}, \varphi_{uu}, \varphi_{vv}, \varphi_{ww}) = \text{diag}(\varphi'', -\varphi''(\varphi')^2, -\varphi''(\varphi')^2, -\varphi''(\varphi')^2) \quad \text{Eq.3.13} \]
with
\[ \Box \varphi = -4\varphi'', \ X = -(\varphi')^2, \ g^{ij} = (\varphi')^3 \quad \text{Eq.3.14} \]
where a prime denotes a derivative with respect to \( t \), repeated indices \( t, u, v \) and \( w \) are not summed over, and it is assumed that \( \varphi' > 0 \). Employing Eqs. 3.10–3.14 we find that the Horndeski Lagrangians given in Eqs.3.5–3.8 become

22
\[ L_2 = (\phi')^3 K(\phi, (\phi')^2) \]  
\[ L_3 = -4G_3(\phi, -(\phi')^2)(\phi')^3 \phi'' \]  
\[ L_4 = 6G_4(\phi, -(\phi')^2)((\phi')^3\phi''' + \phi'(\phi'')^2) - 24G_{4,\chi}(\phi, -(\phi')^2)(\phi')^3(\phi'')^2 \]  
\[ L_5 = 6\phi G_5(\phi, -(\phi')^2)[(\phi'')^3 + \phi'(\phi'')^2] - 8G_{5,\chi}(\phi, -(\phi')^2)(\phi')^3(\phi'')^3. \]

At first glance the Lagrangians \( L_4 \) and \( L_5 \) seem to be of third order in \( \phi \), but the third order terms can be assimilated into divergences. Upon doing this and dropping the divergences, \( L_4 \) and \( L_5 \) become \( L_{4,SO} \) and \( L_{5,SO} \) ("SO" meaning "second-order"), and are given by

\[ L_{4,SO} = -6G_4(\phi, -(\phi')^2)(\phi')^2(\phi'')^2 - 6G_{4,\phi}(\phi, -(\phi')^2)(\phi')^3 \phi'' + -12G_{4,\chi}(\phi, -(\phi')^2)(\phi')^3(\phi'')^2 \]  
\[ \text{and} \]
\[ L_{5,SO} = -3G_5(\phi, -(\phi')^2)(\phi')^2(\phi'')^2 - 2G_{5,\chi}(\phi, -(\phi')^2)(\phi')^3(\phi'')^2. \]

At this point all savvy readers have probably leaped to their feet screaming and pointing at \( L_{4,SO} \) and \( L_{5,SO} \) as non-degenerate second-order Lagrangians, rotten with Ostrogradsky [8] type singularities (\textit{cf.}, Woodward [9]), and as such they should be thrown to the dogs! But I say, let us not be too hasty. After all, when you think about it, doesn’t the Universe seem to act like an unstable system? It seems to have exploded into existence, expanded incredibly fast, and even now it is still expanding at an accelerating rate. This behavior doesn’t seem like the hallmark of a stable system. So I say, let us give these Ostrogradsky violating Lagrangians a chance to see where they
lead us. You will be surprised.

In view of what we saw when we examined the Schwarzschild solution of Einstein’s equations, one might think that if we vary $\phi$ in the Lagrangians presented in Eqs. 3.15, 3.16, 3.19 and 3.20, the equations that we obtain will somehow just be what one gets from the usual scalar-tensor equations derived from the Lagrangians $L_2$, . . . , $L_5$ when the metric has the form presented in Eq. 3.4. However, that is not the case. For example let us consider the simple Lagrangian

$$L_S := g^{ij} K(\phi).$$ 

Eq. 3.21

The equations $\delta L_S = 0$, and $\delta L_S = 0$ are

$$\delta g_{ij} \frac{\partial}{\partial \phi}$$

$$g^{ij} K = 0 \text{ and } g^{ij} K_{,\phi} = 0,$$

where “$K$” denotes a derivative with respect to $\phi$. Consequently, $K$ must vanish. But if we evaluate $L_S$ for the metric given in Eq. 3.4 we obtain

$$L_S = K(\phi')^3.$$ 

Eq. 3.22

For this form of $L_S$ the equation $\delta L_S = 0$ is given by

$$K_{,\phi}(\phi')^3 + 3K\phi'\phi'' = 0,$$

which implies (since $\phi' \neq 0$) that

$$K(\phi')^3 = \beta^3,$$ 

Eq. 3.23

where $\beta$ is a constant. This equation can be integrated to yield
\[ \beta t + \kappa = \int K^{1/3} \, d\varphi, \quad \text{Eq.3.24} \]

with \( \kappa \) being an integration constant. We can now get numerous expressions for \( \varphi \) by using different choices of \( K \). E.g., if we want the Universe to grow exponentially as \( t \) increases from 0, we could choose \( K = \varphi^{-3} \) in Eq.3.24 to get

\[ \varphi = \alpha \, e^{\beta t}, \quad \text{and} \quad \varphi' = \alpha \beta \, e^{\beta t}, \quad \text{Eq.3.25} \]

where \( \alpha \approx L^1 \) is a positive constant, and \( \alpha \beta > 0 \).

If we choose \( K = \varphi^{3s} \), \( s \neq -1 \), then we can use Eq.3.24 to find that

\[ \varphi = [(s+1)\beta t + \kappa]^{1/(s+1)} \quad \text{and} \quad \varphi' = \beta [(s+1)\beta t + \kappa]^{-s/(s+1)} \]. \quad \text{Eq.3.26} \]

So if we take \( \kappa = 0 \), and choose \( s \) so that \( -1/2 < s < 0 \), then we obtain a Universe that expands explosively from \( t = 0 \), where \( \varphi' \) has a vertical tangent. If we take \( s = -2/5 \) then the metric tensor in this model is identical to the one appearing in a non-empty cosmological model of Einstein’s theory (see page 364 of Adler, et al., [6]).

Thus we see that the simple Lagrangian, \( L_s \), given in Eqs.3.21 and 3.22 is capable of providing us with interesting cosmological models. What is the Hamiltonian for this Lagrangian? Since \( L_s \) is of first order in \( \varphi \) we have

\[ H_s = P_s \varphi' - L_s, \]

where

\[ P_s := \frac{\partial L_s}{\partial \varphi'} = 3(\varphi')^3K. \]

Consequently we have \( H_s = 2(\varphi')^3K \), and hence due to Eq.3.23 we see that when the field equations are satisfied
\[ H_s = 2\beta^3, \]
a constant, which vanishes only for the trivial solution.

It is no surprise that \( H_s \) is a constant when the field equations hold, since that is to be expected. However, I would prefer that the Hamiltonian would vanish when evaluated for solutions to the field equations of scalar-scalar theories devoid of sources. My thinking is that for such source-free models the scalar fields, \( f \) and \( \varphi \), represent the geometry of the underlying Universe, and I do not think that the geometry should have energy (even though \( \varphi \) has units of length, which is also the unit of energy). Hence I would prefer that pure (source-free) scalar-scalar theories yield vanishing Hamiltonians when evaluated for solutions to the field equations. I shall now proceed to construct such solutions drawing upon other Lagrangians appearing in \( L_4 \).

For a Lagrangian, \( L \), which is third-order in \( \varphi = \varphi(t) \), the Ostrogradsky Hamiltonian is given by (see, Woodward [9])

\[
H := P_1 \dot{\varphi} + P_2 \varphi'' + P_3 \varphi''' - L, \quad \text{Eq.3.27}
\]

where

\[
P_1 := \frac{\partial L}{\partial \dot{\varphi}} - \frac{d}{dt} \frac{\partial L}{\partial \varphi''} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \varphi''} \quad \text{and} \quad P_3 := \frac{\partial L}{\partial \varphi''}. \quad \text{Eq.3.28}
\]

We do not really need the third-order formalism to determine the Hamiltonians of the Lagrangians given in Eqs.3.15-3.18. This is so because the third-order Lagrangians, \( L_4 \) and \( L_5 \) presented in Eqs.3.17 and 3.18 differ from the second-order Lagrangians
$L_{4SO}$ and $L_{5SO}$ by divergences, and as a result yield the same expressions for $H$, when $\varphi=\varphi(t)$.

One useful property of the Ostrogradsky Hamiltonian is that it satisfies the following identity

$$\frac{dH}{dt} = -\varphi' \frac{\delta L}{\delta \varphi}.$$  
Eq.3.29

Thus, as is well-known, when the field equations are satisfied, $H$ is a constant, and conversely, when $H$ is constant, the field equations are satisfied. Thus the equation $H=\text{constant}$, is a first integral of our field equations. In particular we shall be interested in the solutions to $H=0$, the so-called vacuum solutions to our empty space scalar-scalar theory. I have no interest in using the Legendre transformation (see, e.g., [9]) to re-express $H$ in terms of the canonical variables. $H$ will always be expressed in terms of $\varphi$ and its time derivatives.

In view of the above remarks, it is a straightforward matter to employ Eqs.3.15, 3.16, 3.19, 3.20, 3.27 and 3.28 to show that

$$H_2 = 2K(\varphi')^3 - 2K_{,\chi} (\varphi')^5$$  
Eq.3.30

$$H_3 = 4 G_{3,\varphi}(\varphi')^5$$  
Eq.3.31

$$H_4 = 12 G_{4}(\varphi')^2(\varphi''') + 24 G_{4,\chi}(\varphi')^4(\varphi''') + 12 G_{4,\chi}(\varphi')^3(\varphi'')^2 + 12 G_{4,\varphi}(\varphi')^3(\varphi'') +$$
$$+ 6 G_{4,\varphi}(\varphi')^5 + 24 G_{4,\chi\varphi}(\varphi')^5(\varphi'') - 24 G_{4,\chi\chi}(\varphi')^5(\varphi'')^2$$  
Eq.3.32

$$H_5 = 6 G_{5,\varphi}(\varphi')^4(\varphi''') + 6 G_{5,\varphi}(\varphi')^3(\varphi'')^2 + 12 G_{5,\chi}(\varphi')^4\varphi''(\varphi''') + 8 G_{5,\chi}(\varphi')^3(\varphi'')^3 +$$
For the present form of our metric and scalar field the Lagrangian $L_3$ and its corresponding Hamiltonian $H_3$ can be assimilated into $L_2$ and $H_2$. To see why let us choose $K$ to be

$$K := -2X^{-1} \int XG_{3,\phi}(\phi, X)dX.$$ 

Using this value of $K$ in Eq.3.30 we end up with the Hamiltonian $H_3$ given in Eq.3.31, which corroborates my claim that $L_3$ can be assimilated into $L_2$ for our present purposes.

I shall set $L_{2,4} := L_2 + L_{4SO}$ and $L_{2,5} := L_2 + L_{5SO}$, with the corresponding Hamiltonians $H_{2,4} = H_2 + H_4$ and $H_{2,5} = H_2 + H_5$. Recall that we are expressing our Hamiltonians in terms of $\phi$ and its time derivatives, and not in terms of canonical variables. This justifies my expressions for $H_{2,4}$ and $H_{2,5}$. I shall now show that the coefficient functions appearing in $H_{2,4}$ can be chosen so that the equation $H_{2,4} = 0$ admits solutions of the form $e^{\beta t}$ and $t^q (\beta$ and $q$ some real numbers), with the coefficient functions for these two different classes of solutions being unchanged. Note this differs from the situation with $L_5$ above, where different choices of $K$ were needed to get the solutions presented in Eqs.3.25 and 3.26. After showing how the equation $H_{2,4} = 0$, can be solved, I shall present solutions to $H_{2,5} = 0$.

We shall seek solutions to $H_{2,4} = 0$, when $K$ and $G_4$ have the form

$$+ 6G_{5,\phi}(\phi')\phi'' - 8G_{5,XX}(\phi')^2(\phi'')^3.$$ 

Eq.3.33
$K := A_2 \varphi^\mu [X]^\nu = A_2 \varphi^\mu (\varphi')^{2\nu}$ and $G_4 := A_4 \varphi^n |X|^\zeta = A_4 \varphi^n (\varphi')^{2\zeta}$, \hspace{1cm} \text{Eq.3.34}

where $A_2$, $A_4$, $\mu$, $\nu$, $\eta$ and $\zeta$ are numbers to be determined. To begin we place our ansatz expressions for $K$ and $G_4$ into the equation $H_{2,4} = 0$. Using Eqs.3.30, 3.32 and 3.34 we find

$$\frac{1}{2}H_{2,4} = A_2 (\nu+1)\varphi^\mu (\varphi')^{2\nu+3} + 6A_4 (1-2\zeta)\varphi^n (\varphi')^{2\zeta+2} \varphi'''' +$$

$$+ 6A_4 \zeta (1-2\zeta)\varphi^n (\varphi')^{2\zeta+1} (\varphi'')^2 + 6A_4 \eta (1-2\zeta)\varphi^{n-1} (\varphi')^{2\zeta+3} \varphi'' +$$

$$+ 3A_4 \eta (\eta-1)\varphi^{n-2} (\varphi')^{2\zeta+5}.$$ \hspace{1cm} \text{Eq.3.35}

We desire a solution to $H_{2,4} = 0$, of the form $\varphi = \alpha e^{\beta t}$, where $\alpha$ and $\beta$ are real constants with $\alpha \sim L^1$ and $\beta \sim L^{-1}$. Upon putting this form for $\varphi$ and its derivatives into Eq.3.35, we discover that in order for $\varphi = \alpha e^{\beta t}$ to be a solution to $H_{2,4} = 0$, we must have

$$\mu + 2\nu = \eta + 2\zeta$$ \hspace{1cm} \text{Eq.3.36}

and

$$(\nu+1)A_2 \beta^{(2\nu+3)} + A_4 \beta^{2\zeta+5} [3\eta + 3\eta^2 + 6 - 12\zeta \zeta - 6 \zeta - 12 \zeta^2] = 0$$ \hspace{1cm} \text{Eq.3.37}

with $\alpha$ being an arbitrary real number. It should be noted that Eqs.3.35 and 3.36 imply that the equation $H_{2,4} = 0$ is a homogeneous differential equation.

Next we seek a solution to $H_{2,4} = 0$ of the form

$$\varphi = \gamma (k_1 t + k_2)^\eta.$$ \hspace{1cm} \text{Eq.3.38}

where $\gamma \sim L^1$, $k_1 \sim L^{-1}$, $k_2 \sim L^0$ and $q \sim L^0$ are constants. (If we were working with unitless quantities we could pull $k_1$ out of the expression $k_1 t + k_2$ in Eq.3.38, and absorb it into $\gamma$ and $k_2$. However, since various constants in Eq.3.38 have units, doing this would
cause $\gamma$ to have very strange, and possibly irrational units. That is why I left $k_1$ where it is.) To determine this solution we insert the expression for $\varphi$ and its derivatives into Eq.3.35 and discover, after some effort, that in order for Eq.3.38 to satisfy the equation $H_{2,4} = 0$, we must have

$$q\mu + 2qv - 2v = q\eta + 2q\zeta - 2\zeta - 2 .$$

Eq.3.39

Eqs.3.36 and 3.39 combine to give us

$$v = \zeta + 1 ,$$

Eq.3.40

for our present choices of $K$ and $G_4$. This in turn implies that the $\beta$’s in Eq.3.37 cancel leaving us with

$$(\zeta+2)A_2 + A_4[3\eta + 3\eta^2 + 6 - 12\zeta - 6\zeta - 12\zeta^2 ] = 0 .$$

Eq.3.41

In addition Eqs.3.36 and 3.40 show that we must also have

$$\mu = \eta - 2 .$$

Eq.3.42

When these values for $\mu$ and $v$ are placed into Eq.3.35, with $\varphi$ given by Eq.3.38, the $\gamma$ and $k_1$ terms can be eliminated, as well as a lot of common factors of $q$, leaving us with

$$(\zeta+2)A_2q^2 + A_4[24\zeta q - 18\zeta - 6\zeta q^2 - 12\zeta q - 6\eta q - 12\zeta^2 q^2 + 24\zeta^2 q + - 12\zeta^2 + 6q^2 - 18q + 12 + 3\eta^2 q^2 + 3\eta q^2 ] = 0 .$$

Eq.3.43

If we multiply Eq.3.41 by $q^2$, and subtract that from Eq.3.43, we discover that all of the $q^2$ terms cancel, leaving us with
\[ A_4[24\zeta q - 18\zeta + 12\zeta \eta q - 6\eta q + 24\zeta^2 q - 12\zeta^2 - 18q + 12] = 0. \]  
Eq.3.44

Since we are assuming that \( A_4 \) is non-zero, Eq.3.44 implies that

\[
q = \frac{2\zeta^2 + 3\zeta - 2}{4\zeta^2 + 4\zeta + 2\zeta \eta - \eta - 3} = \frac{(2\zeta - 1)(\zeta + 2)}{(2\zeta - 1)(2\zeta + \eta + 3)}.
\]  
Eq.3.45

So if \( \zeta \neq \frac{1}{2} \), then Eq.3.45 tells us that

\[
q = \frac{\zeta + 2}{2\zeta + \eta + 3},
\]  
Eq.3.46

while if \( \zeta = \frac{1}{2} \) then Eq.3.44 holds for any choice of \( q \) and \( \eta \). If we place \( \zeta = \frac{1}{2} \) into Eqs.3.41 and 3.43 we obtain the same relationship between \( A_2 \) and \( A_4 \); viz.,

\[
A_2 = \frac{[6\eta(\eta - 1)A_4]}{5}.
\]  
Eq.3.47

In summary if we choose \( \zeta = \frac{1}{2} \), then we shall obtain solutions to \( H_{2,4} = 0 \) of the form \( \varphi = \alpha e^{\beta t} \) and \( \varphi = \gamma(k_1 t + k_2)^q \), where \( \alpha, \beta, \gamma, k_1, k_2 \) and \( q \) are arbitrary provided we choose

\[
K = A_2 \varphi^{1/2} |X|^{3/2} \quad \text{and} \quad G_4 = A_4 \varphi^q |X|^{1/2}
\]  
Eq.3.48

with \( A_2 \) and \( A_4 \) being related by Eq.3.47. This strikes one as ridiculous until you examine the form of \( H_{2,4} \) for this choice of \( \zeta \), \( K \) and \( G_4 \), which is given by the left-hand side of Eq.3.35. Upon doing so we find that \( H_{2,4} = 0 \), which clearly explains why that equation admits so many solution, it’s \( 0 = 0 \)!

Thus we can assume that \( \zeta \neq \frac{1}{2} \). Consequently Eq.3.41 provides us with the relationship between \( A_2 \) and \( A_4 \). If \( \zeta = -2 \) then we see that \( A_2 \) is arbitrary, while due
to Eq.3.46 q must vanish. This situation is uninteresting, so we shall now assume that \( \zeta \neq -2 \), and hence

\[
A_2 = 3A_4 \left[ \frac{4\eta\zeta + 4\zeta^2 + 2\zeta - \eta^2 - \eta - 2}{(\zeta + 2)} \right]. \tag{Eq.3.49}
\]

In summary, if we choose values of \( \eta, \zeta \) and \( A_4 \) (with \( \zeta \neq -\frac{1}{2} \) or \( -2 \), and \( 2\zeta + \eta + 3 \neq 0 \)), and use them to compute \( \nu, \mu, q \) and \( A_2 \) with Eqs.3.40, 3.42, 3.46 and 3.49, then \( \varphi = \alpha e^{\beta t} \) and \( \varphi = \gamma (k_1 t + k_2)^q \) will be solutions to \( H_{2,4} = 0 \) for all values of \( \alpha, \beta, \gamma, k_1 \) and \( k_2 \) for which \( \varphi' > 0 \). The selected values of \( \eta, \zeta \) and \( A_4 \) can be used in Eq.3.34 to provide us with expressions for \( K \) and \( G_4 \).

For Horndeski theory it has been shown (see, Baker, et al.,[10], Creminelli & Vernizzi [11], Sakstein & Jain [12] and Ezquiaga & Zumalacarregui [13]) that in order to guarantee that the speed of a gravitational wave, \( c_g \), is equal to the speed of light, \( c \), we need \( G_4 \) to be independent of \( X \) and \( G_5 \) approximately constant. If you look at the derivation of this result in the references cited it involves a perturbation analysis of the metric tensor. Whether a similar result applies to the class of scalar-scalar theories I have presented here is not clear. But in any case let us look at our \( H_{2,4} = 0 \) solutions when \( G_{4,X} = 0 \), which requires \( \zeta = 0 \). In this case we can use Eqs.3.40, 3.42, 3.46 and 3.49 to conclude that \( \mu = \eta - 2, \nu = 1, \)

\[
A_2 = -3A_4 [\eta^2 + \eta + 2]/2 \quad \text{and} \quad q = 2(\eta + 3)^{-1}, \tag{Eq.3.50}
\]

Hence our Hamiltonian, which appears on the left-hand side of Eq.3.35, is
\[ H_{2,4} = -12A_4(\eta+1)\phi^{n-2}(\phi')^5 + 12A_4\phi^n(\phi')^2\phi''' + 12\eta A_4\phi^{n-1}(\phi')^3\phi''. \quad \text{Eq.3.51} \]

I would like a solution for \( \phi \) of the form \( \phi = \gamma(k_1 t + k_2)^q \) which is such that \( \phi'(0) = 0 \), and \( \phi'(t) \) has a vertical tangent vector at \( t = 0 \) (more precisely we want \( \phi'(t) \to 0 \), and \( \phi''(t) \to \infty \) as \( t \to 0^+ \)). In view of the form of our metric given in Eq.3.3 this would correspond to a universe that expands infinitely fast at \( t=0 \). None of our exponential solutions can generate such a universe since they never have vertical tangent vectors. Nevertheless they turn out to be very useful as we shall see in the next section.

To obtain a \( \phi \) solution of the type I described in the above paragraph we need \( k_2 = 0 \), and \( q \) to lie in the range \( 1 < q < 2 \). In view of Eq.3.50 we see that this restriction on \( q \) implies that \( -2 < \eta < -1 \). The question now is: what should we choose as reasonable value of \( q \) between 1 and 2? Well, when people look for inflationary models of the universe they seek models in which the curvature tensor rapidly vanishes as the universe expands. Our present model has the \( t = \) constant slices being flat, and it is easily seen from Eq.3.11 that the scalar curvature, \( R \), for this model is given by

\[ R = 6(q-1)(2q-3)t^2, \quad \text{Eq.3.52} \]

which, serendipitously, vanishes for \( q = 3/2 \), which is in the middle of our admissible range for \( q \). For this value of \( q \), \( \eta = -5/3 \), and hence the coefficient functions \( K \) and \( G_4 \) appearing in \( L_{2,4} \) are

\[ K = (-14A_4/3)\phi^{-11/3}|X|, \quad G_4 = A_4\phi^{-5/3}. \]
One should also note that this solution is not flat for all time, since the Ricci tensor components are non-zero and go as $t^2$. In the next section I shall discuss how the explosive solution to $H_{2,4}=0$, can be combined with the exponential solutions to yield cosmological models which are strikingly similar to the usual inflationary model.

I shall now make a few remarks about the vacuum solutions to the scalar-scalar theory generated by the Lagrangian $L_2 + L_{5SO}$.

If we take $K = A_2 \phi^\mu |X|^{\nu}$ and $G_3 = A_3 \phi^\nu |X|^{\xi}$, then it is a straightforward matter to use Eqs.3.30 and 3.33 to show that

$$\frac{1}{2}H_{2,5} = (1+\nu)A_2 \phi^\mu (\phi')^{2\nu+3} + 3\chi A_3 \phi^{\chi-1} (\phi')^{2\xi+3} (\phi'')^2 - 4\xi^2 A_3 \phi^{\chi}(\phi')^{2\xi+1} (\phi'')^3 +$$

$$+ 3\chi (\chi-1)A_4 \phi^{\chi-2} (\phi')^{2\xi+5} \phi'' + 3\chi A_4 \phi^{\chi-1} (\phi')^{2\xi+4} \phi''' - 6\xi A_4 \phi^\chi (\phi')^{2\xi+2} \phi'' \phi'''. \quad \text{Eq.3.53}$$

For $\phi = \alpha e^{\beta t}$ to be a solution to $H_{2,5} = 0$, we must have

$$\mu + 2\nu = \chi + 2\xi + 1, \quad \text{Eq.3.54}$$

and

$$A_2 (\nu+1)\beta^{2\nu} + A_3 \beta^{2\xi+4} [3\chi + 3\chi^2 - 4\xi^2 - 6\xi] = 0. \quad \text{Eq.3.55}$$

As was the case for the equation $H_{2,4}=0$, the equation $H_{2,5}=0$, must be a homogeneous differential equation, due to Eq.3.54.

Next we seek a second solution to $H_{2,5} = 0$ of the form $\phi = \gamma (k_1 t + k_2)^\eta$. In order for a solution of this form to exist along with solutions of the form $\phi = \alpha e^{\beta t}$ we must have

$$q\mu + 2qv - 2v = q\chi + 2q\xi + q - 2\xi - 4.$$
Due to Eq.3.54 this equation simplifies to
\[ v = \xi + 2 , \quad \text{Eq.3.56} \]
which can be combined with Eq.3.54 to give us
\[ \mu = \chi - 3 . \quad \text{Eq.3.57} \]
Eq.3.56 implies that the \( \beta \) terms in Eq.3.55 cancel out leaving us with
\[ (v + 1)A_2 + A_5[3\chi^2 + 3\chi - 4\xi^2 - 6\xi] = 0 . \quad \text{Eq.3.58} \]
Eq.3.56 also permits us to cancel the \( \gamma \) and \( k_1 \) terms that arise when we evaluate
\[ H_{2,5} = 0 \text{ for } \varphi = \gamma(k_1 t + k_2)^q, \]
giving us
\[ (v+1)A_2q^3 + A_5[3\chi q^3 - 12\chi q^2 + 9\chi q - 4\xi^2 q^3 + 12\xi^2 q^2 - 12\xi^2 q + 4\xi^2 + 3\chi^2 q^3 + \\
- 3\chi^2 q^3 - 6\xi q^3 + 24\xi q^2 - 30\xi q + 12\xi] = 0 . \quad \text{Eq.3.59} \]
If we multiply Eq.3.58 by \( q^3 \) and subtract the resulting equation from Eq.3.59 we obtain
\[ A_5[-12\chi q^2 + 9\chi q + 12\xi^2 q^2 - 12\xi^2 q + 4\xi^2 - 3\chi^2 q^2 + 24\xi q^2 - 30\xi q + 12\xi] = 0 , \quad \text{Eq.3.60} \]
which is a quadratic equation relating \( q, \chi \) and \( \xi \).

Now recall what we are trying to do here. We want to choose the exponents \( \mu, v, \chi \) and \( \xi \) along with the coefficients \( A_2 \) and \( A_5 \) appearing in our expressions for \( K \) and \( G_5 \); viz.,
\[ K = A_2\varphi^v|X|^v \text{ and } G_5 = A_5\varphi^\gamma|X|^{\xi} \quad \text{Eq.3.61} \]
so that the resulting Hamiltonian vacuum equation \( H_{2,5} = 0 \), admits simultaneous
solutions of the form $\phi = \alpha e^{\beta t}$ and $\phi = \gamma (k_1 t + k_2)^q$. This can be achieved by choosing values of $\chi$ and $\xi$, which can then be used in Eqs 3.56 and 3.57 to give us values for $\mu$ and $v$. The values of $\chi$ and $\xi$ are then used in Eqs.3.58 and 3.60 to obtain values for $A_2$ and $q$. In the solutions so obtained $\alpha$, $\beta$, $\gamma$, $k_1$ and $k_2$ are only constrained by the condition that $\phi' > 0$.

To illustrate this approach to obtaining solutions to $H_{2,5} = 0$, let us choose $\xi = 0$, and leave the choice of $\chi$ temporarily unfixed. Under this assumption Eq.3.60 tells us that $q$ must satisfy

$$4\chi q^2 - 3\chi q + \chi^2 q^2 = 0.$$  

Eq.3.62

If $\chi = 0$ in Eq.3.62, then we can employ Eqs.3.53, 3.56 and 3.58 to deduce that $H_{2,5} = 0$, which is undesirable. Hence $\chi \neq 0$, and Eq.3.62 tells us that

$$q = 3(4 + \chi)^{-1}.$$  

Eq.3.63

To obtain solutions of the form $\phi = \gamma (k_1 t + k_2)^q$ which begin explosively at $t = 0$, we require $k_2 = 0$, with $1 < q < 2$, and hence Eq.3.63 implies that $-2.5 < \chi < -1$. If we would also wish $R = 0$ for our explosive solution, then we would need to choose $q = 1.5$ and $\chi = -2$. From Eq.3.52 we know that for fixed values of $t$, the minimum value of $R$ is attained when $q = 1.25$, which would require us to choose $\chi = -8/5$. The values for $A_2$ and $A_5$ for each of these solutions can be found from Eq.3.58. When $q = 1.5$ we have $A_2 = -2A_5$, and when $q = 1.25$ we have $A_2 = -24A_5/25$. 

36
When we dealt with the $H_{2,4} = 0$ equation above, we saw that for solutions of the form $\varphi = \gamma(k_1 t + k_2)^q$ to exist, $q$ must be given by Eq.3.46 in terms of $\eta$ and $\zeta$, with only one $q$ value corresponding to each choice of $\eta$ and $\zeta$. I shall now demonstrate that when dealing with the equation $H_{2,5} = 0$, we can choose $\chi$ and $\xi$ in $G_5 = A_5\varphi^x |X|^{\xi}$, so that solutions of the form $\varphi = \gamma(k_1 t + k_2)^q$ can exist for two different choices of $q$.

To that end let us choose $\chi = 0$. Then Eq.3.60 tells us that

$$\xi(12q^2 - 12q + 4) + \xi(24q^2 - 30q + 12) = 0 .$$

Eq.3.65

$\xi \neq 0$ in this equation, since if it did $H_{2,5}$ would reduce to $H_2$, due to Eq.3.53, which is not what we wish to investigate. Thus Eq.3.65 reduces to

$$\xi(6q^2 - 6q + 2) + 12q^2 - 15q + 6 = 0 ,$$

Eq.3.66

and hence

$$\xi = \frac{15q - 12q^2 - 6}{6q^2 - 6q + 2} .$$

Eq.3.67

Say we choose $q = \frac{3}{2}$. Then Eq.3.67 tells us that $\xi = \frac{-21}{13}$. If we put this value of $\xi$ into Eq.3.65 we find that $q = \frac{3}{2}$ or $\frac{4}{5}$. Using Eqs.3.56-3.58 we also find that $(\mu, v) = (-3, \frac{5}{13})$ and $A_2 = 7A_5/13$.

If we look for a solution with $q = \frac{5}{4}$, then following the above approach we obtain $\xi = \frac{-48}{31}$. This in turn gives rise to two values of $q$, viz., $q = \frac{5}{4}$, and $\frac{6}{7}$, with $(\mu, v) = (-3, \frac{14}{31})$, and $A_2 = 608A_5/155$.

I shall now summarized what we have just done with the Hamiltonian $H_{2,5}$. The
equation $H_{2,3} = 0$, with $K := A_2 \phi^a |X|^a$ and $G_5 := A_5 \phi^a |X|^a$, will admit simultaneous solutions of the form $\varphi = a e^{\beta t}$ and $\varphi = \gamma (k_1 t + k_2)^n$, with $\alpha$, $\beta$, $\gamma$, $k_1$ and $k_2$ arbitrary, except that $\varphi' > 0$, when:

(i) $(\mu, \nu) = (-5, 2), (\chi, \xi) = (-2, 0), A_2 = -2A_5, q = \frac{3}{2};$

(ii) $(\mu, \nu) = (-\frac{23}{5}, 2), (\chi, \xi) = (-\frac{8}{5}, 0), A_2 = -24A_5/25, q = \frac{5}{4};$

(iii) $(\mu, \nu) = (-3, \frac{5}{13}), (\chi, \xi) = (0, -\frac{21}{13}), A_2 = 7A_5/13, q = \frac{3}{2}$ or $\frac{4}{5};$ and

(iv) $(\mu, \nu) = (-3, \frac{14}{31}), (\chi, \xi) = (0, -\frac{48}{31}), A_2 = 608A_5/155, q = \frac{5}{4}$ or $\frac{6}{7}.$

For our last example we shall consider conformally invariant scalar-tensor field theories. If $L$ is the Lagrangian of a scalar-tensor field theory, then that theory will be said to be conformally invariant, if the field tensor densities

$$
\frac{g_{bc} \delta L}{\delta g_{ac}} \quad \text{and} \quad \frac{\delta L}{\delta \phi}
$$

are invariant under the conformal transformation which replaces $g_{ij}$ by $e^{2\sigma} g_{ij}$ throughout the field tensors, with $\sigma$ being an arbitrary scalar field. The scalar-tensor theory will also be said to be flat space compatible if $L$ is well-defined and differentiable when evaluated for either a flat metric tensor, or constant scalar field. In [14] (and also in [15]) I show that in a 4-dimensional space the Lagrangian of the most general conformally invariant scalar-tensor field theory which is flat-space compatible is given by

$$
L_c = L_{2c} + L_{3c} + L_{4c} + L_{UC}
$$

Eq.3.68
where
\[ L_{2C} := g^{\frac{1}{2}} K(\varphi) X^2, \quad \text{Eq.3.69a} \]
\[ L_{3C} := P(\varphi) \epsilon^{abcd} C_{\text{ef}} \text{C}_{\text{ef}} \quad \text{Eq.3.69b} \]
\[ L_{4C} := g^{\frac{1}{2}} B(\varphi) C_{\text{abcd}} C_{\text{abcd}} \quad \text{Eq.3.69c} \]
and
\[ L_{UC} := g^{\frac{1}{2}} U(\varphi) [-12 R^a_{\text{abc}} \varphi_b + 2 RX - 3(\Box \varphi)^2 - 6 \varphi^a \varphi_{ab} - 12 \varphi^a(\Box \varphi)_{a}] \quad \text{Eq.3.69d} \]
where
\[ C_{\text{abcd}} := R_{\text{abcd}} + \frac{1}{2}(g_{ad} R_{bc} + g_{bc} R_{ad} - g_{ac} R_{bd} - g_{bd} R_{ac}) + \frac{1}{6} R (g_{ac} g_{bd} - g_{ad} g_{bc}) \]
are the components of the Weyl tensor, \( K, P, B \) and \( U \) are differentiable functions of \( \varphi \), and \( \epsilon^{abcd} \) is the Levi-Civita tensor density.

We shall now seek solutions to the scalar-scalar theory based on the Lagrangian \( L_C \) with the metric tensor given by Eq.3.3 and corresponding line element
\[ ds^2 = -dt^2 + (\varphi')^2 [du^2 + dv^2 + dw^2] = (\varphi')^2 [-dt^2 + du^2 + dv^2 + dw^2]. \quad \text{Eq.3.70} \]
We see if we define a new coordinate \( \bar{t} \) by \( d\bar{t} := \varphi^{-1} dt \), then it is obvious that our metric is conformally flat. Consequently when we evaluate \( L_{3C} \) and \( L_{4C} \) for our ansatz metric they vanish, which certainly makes life a lot easier. So we can now employ Eqs.3.9-3.14 to deduce that
\[ L_C = K(\varphi')^7 - 12 U[2(\varphi')^4 \varphi'' + 7(\varphi')^3 (\varphi'')^2], \]
which can be rewritten as
\[ L_C = K(\varphi')^7 + 24 U \varphi (\varphi')^5 \varphi'' + 12 U(\varphi')^3 (\varphi'')^2 - \frac{d}{dt} (24 U(\varphi')^4 \varphi''). \quad \text{Eq.3.71} \]
Using Eq.3.27 and 3.71 we find that the Ostrogradsky Hamiltonian associated with \( L_C \)
is given by

\[ H_c = -24(\varphi')^3 \left[ \frac{d(U\varphi'^\prime\varphi'')}{dt} - \left[ \frac{1}{4}K - U_{\varphi\varphi} \right](\varphi')^4 \right] \]  

Eq.3.72

The vacuum solutions for the scalar-scalar theory based on the Lagrangian \( L_c \) are the solutions to \( H_c = 0 \). Since we require that \( \varphi' > 0 \), we can use Eq.3.72 to deduce that the vacuum solutions for \( \varphi \) must satisfy

\[ \frac{d}{dt} (U\varphi'^\prime\varphi'') = \left[ \frac{1}{4}K - U_{\varphi\varphi} \right](\varphi')^4. \]  

Eq.3.73

This is a fascinating equation since it bears a striking resemblance to an equation that we are all more familiar with from classical mechanics: \textit{viz.}, the equation relating angular momentum and torque for a point particle,

\[ \frac{d}{dt} l = \tau, \]

where \( l \) is the angular momentum of the particle under consideration and \( \tau \) is the torque on it. \( l = r \times p \), and \( \tau = r \times F \), where \( r \) is a radial vector to the particle, \( p \) is the linear momentum of the particle in question, and \( F \) is the force acting on it. In our metric \( g_{ij} \), \( \varphi' \) plays the roll of a scale factor, but for heuristic purposes let us interpret it as a “radius,” which would be the case if our coordinates were unitless. In that case \( U\varphi'^\prime\varphi'' \) appearing on the left-hand side of Eq.3.73 is the product of a radius and a velocity, with \( U \) playing the roll of mass, if we are going to interpret this as something like “angular momentum.” On the right-hand side of Eq.3.73 we have a “radius” \( \varphi' \) times
a “volume” \((\varphi')^3\), multiplied by \(\frac{1}{4}K - U_{\varphi\varphi}\), which, in keeping with our analogy, can be interpreted as force per unit volume. Thus the right-hand side of Eq.3.73 is like the magnitude of \(\mathbf{r} \times \mathbf{F}\). Now I realize that these equations are not intended to describe a rotating universe, but it is uncanny that they have a form analogous to the equations of a rotating rigid body. That is certainly something that one would not have expected.

Let us proceed to construct solutions to \(H_c = 0\), which is equivalent to solving Eq.3.73. We shall seek solutions similar to those that we found for \(H_{2,4} = 0\), and \(H_{2,5} = 0\). To that end I shall assume that \(K := A\varphi^a\) and \(U := B\varphi^b\), where \(A, B, a\) and \(b\) are real numbers to be determined so that Eq.3.73 admits simultaneous solutions of the form \(\varphi = \alpha e^{\beta t}\) and \(\varphi = \gamma (k_1 t + k_2)^q\). Upon setting \(\varphi = \alpha e^{\beta t}\) in Eq.3.73 it is easily shown that this expression for \(\varphi\) is a solution to Eq.3.73 if and only if

\[
a = b - 2, \text{ and } A = 4(b^2 + 2)B, \quad \text{Eq.3.74}
\]

and hence

\[
K = 4(b^2 + 2)B\varphi^{a-2}, \text{ and } U = B\varphi^b. \quad \text{Eq.3.75}
\]

Once again Eqs.3.74 and Eq.3.75 force the equation \(H_c = 0\), as formulated in Eq.3.73, to be a homogeneous differential equation.

Next we shall seek a solution to Eq.3.73 of the form \(\varphi = \gamma (k_1 t + k_2)^q\), under the assumption that Eq.3.74 holds. It is easily seen that under these assumptions this expression for \(\varphi\) will be a solution for arbitrary choices of \(\gamma, k_1, k_2\) provided
\[ q = \frac{3}{5 + b} . \]  

Eq.3.76

In order to obtain solutions with \( 1 < q < 2 \), we need \( -\frac{7}{2} < b < -2 \). So, for example, if we choose \( q = \frac{3}{2} \), in our expression for \( \varphi = \gamma(k_1 t + k_2)^9 \), so that the scalar curvature, \( R \), vanishes for this solution, then Eqs.3.75 and 3.76 tell us that

\[ a = -5, \ b = -3 \text{ and } A = 44B . \]  

Eq.3.76

Consequently to guarantee that \( \varphi = \alpha e^{\beta t} \) and \( \varphi = \gamma(k_1 t + k_2)^{3/2} \) are solutions to \( H_c = 0 \), with \( \alpha, \beta, \gamma, k_1 \text{ and } k_2 \) arbitrary (except for \( \varphi' > 0 \)), we require

\[ K = 44B\varphi^{-5} \text{ and } U = B\varphi^{-3} . \]  

Eq.3.77

In the next section I shall explain how the various solutions to the scalar-scalar field theories that we constructed here can be used to generate cosmological models.

Section 4: Delineating Future Tasks and Constructing Universes

As I mentioned earlier, in [10], [11], [12] and [13] it was pointed out that, in the context of Horndeski Theory, to guarantee that \( c = c_g \) we need \( G_{4,X} = 0 \) and \( G_z \) to be approximately a constant. This fact was arrived at using perturbation analysis of the metric tensor in those scalar-tensor theories. We now need a similar analysis to be performed on those scalar-scalar theories based upon the Lagrangians \( L_{2,4} \), \( L_{2,5} \) and \( L_C \), with corresponding Hamiltonians given in Eqs.3.35, 3.53 and 3.72. Completing this task in general might be difficult, in which case it would be nice to know what the relationship is between \( c \) and \( c_g \) in spaces which are given by some of the particular
solutions to $H_{2,4} = 0$, $H_{2,5} = 0$ and $H_c = 0$ constructed in Section 3. In attempting to complete this task the first thing that needs to be determined is how to go about doing perturbation analysis in scalar-scalar theories. My suggestion is that since we fixed the choice of the Cofinsler function $f$ in all the spaces constructed in the previous section, we should do the same thing when doing a perturbation analysis, and only vary $\varphi$.

In the previous section we assumed that $\varphi = \varphi(t)$. If we now assume that $\varphi$ has the more general form $\varphi = \varphi(x^i)$ we require a more complicated Hamiltonian. For a third-order Lagrangian the Ostrogradsky Hamiltonian takes the form of a matrix $H_{i,j}$ defined by

$$H_{i,j} := \left( \frac{\partial L}{\partial \varphi_{,ij}} - \frac{d}{dx^h} \frac{\partial L}{\partial \varphi_{,hj}} + \frac{d^2}{dx^h dx^k} \frac{\partial L}{\partial \varphi_{,hkj}} \right) \varphi_{,i} + \left( \frac{\partial L}{\partial \varphi_{,hj}} - \frac{d}{dx^k} \frac{\partial L}{\partial \varphi_{,khj}} \right) \varphi_{,hj} + \frac{\partial L}{\partial \varphi_{,hki}} \delta^i_j L \quad .$$

Eq.4.1

This Hamiltonian has the property that

$$\frac{dH_{i,j}}{dx^l} = -\varphi_{,i} \frac{\delta L}{\delta \varphi} .$$

Eq.4.2

Consequently if we have a solution to the Euler-Lagrange equation $\frac{\delta L}{\delta \varphi} = 0$, then $H_{i,j}$ is divergence-free. However, due to Eq.4.2 we see that, unlike the case when $\varphi = \varphi(t)$, solutions to $H_{i,j} = \text{constant}$, are solutions to the Euler-Lagrange equation, but there can be other solutions to the Euler-Lagrange equation that are not solutions to $H_{i,j} = \ldots$
constant. So $H_{ij} = \text{constant}$ is not a first integral of the Euler-Lagrange equation. This makes the quest for a solution when $\varphi = \varphi(x^i)$ more arduous.

Another problem with the Hamiltonian approach when $\varphi = \varphi(x^i)$, arises if $L_1$ and $L_2$ are two third-order Lagrangians which differ by a divergence, which is also at most third-order. In this case the Hamiltonians $H_{1i}^j$ and $H_{2i}^j$ that $L_1$ and $L_2$ generate via Eq.4.1, are not equal, although due to Eq.4.2, $H_{1i}^j - H_{2i}^j$ is identically divergence-free. Hence one cannot drop divergences from $L$ when computing $H_{ij}$. This is another reason why the Hamiltonian approach to pursuing solutions may not be that helpful when $\varphi = \varphi(x^i)$.

The solutions to scalar-scalar field theories that we obtained in Section 3 described empty space. How does one go about describing a more realistic situation involving matter? The easiest way to do this would be to adjoin to the Lagrangians $L_{2,4}, L_{2,5}$ and $L_C$ a matter Lagrangian $\mathcal{L}_M = \mathcal{L}_M (g_{ij}(d\varphi); g_{ij}(d\varphi)_n; \Psi_A; \Psi_{A,n})$, where the $\Psi_A$'s represent the matter fields. Now just compute the Euler-Lagrange equations as we did in Section 3, and search for solutions.

A second approach to introducing matter fields into scalar-scalar field theories would use the energy-momentum tensor $T^{ij}$ of the matter fields. If $\nabla \varphi$ is timelike (as is was in the last section), then the energy density for observers with worldlines which are integral curves of $|X|^{-\frac{1}{2}} \nabla \varphi$ would be (see Eq.1.3) $\mathcal{E}_1(\nabla \varphi) := -X^{-1}T^{ij} \varphi_i \varphi_j$. In this
case we could take $\mathcal{L}_M = g^{\mathcal{E}_1}(\nabla \phi)$, and then proceed as above.

In Section 3 we confined our attention to LFRW spaces which were such that the $t = \text{constant}$ slices were flat. It would be of interest to redo all of the calculations of Section 3 for the cases in which the $t = \text{constant}$ slices were either of constant positive or constant negative curvature.

When doing vector-tensor theory, the vector field is usually taken to be a covariant vector, $A = A_i dx^i$. E.g., this is the case in the Einstein-Maxwell field theory or my generalization of it [16]. Thus given a Lorentzian Cofinsler Space $CF^d = (M, N, f)$, one can use $A$ to construct a Lorentzian metric $g$ on $M$, provided that the range of $A$ lies in $N$. When $A$ has this property, and $x$ is any chart of $M$ with corresponding standard chart $(\chi, y)$ in $T^*M$, then the $x$ components of this metric tensor are given by

$$g^{ij} := g(dx^i, dx^j) = \frac{1}{2} \frac{\partial^2 f(A)}{\partial y_i \partial y_j}.$$

One can evaluate Lagrangians of vector-tensor field theories using this $g^{ij}$ and $A_i$ to obtain vector-scalar Lagrangians, which can be used to generate field equations in a manner analogous to what we did in the previous section. I am not sure whether the equations obtained in this manner will actually lead to anything new, since $A_i$ does not appear differentiated in $g^{ij}$, unlike what occurred in the scalar-scalar field theories investigated in Section 3.

In the standard description of inflationary models of the Universe (see, Guth
[17]), the Universe expands a little after \( t = 0 \) and stops, to achieve thermodynamic equilibrium, and then expands rapidly in an exponential manner for a bit, after which there is a moderate, albeit somewhat accelerated, expansion for eternity. I shall demonstrate that we can obtain something like that using the solutions generated in the previous section. I shall also show that we can get the solutions to stop expanding, and then deaccelerate to \( \varphi'(t) = 0 \), at sometime in the future.

To begin with, the equations \( H_{2,4} = 0, H_{2,5} = 0, \) and \( H_c = 0 \) all admit the spurious solution \( \varphi' = 0 \). We dispensed with this solution because in order for the metric given in Eq.3.3 to be non-degenerate we need \( \varphi' \neq 0 \). However, what if we took the \( \varphi'=0 \) solution seriously. It simply means that in the slices \( t = \) constant the physical distance between any two distinct points is zero, even though their Euclidean distance in these slices is non-zero. I maintain that it is the physical distance between the constituents of the \( t = \) constant slices that enters into thermodynamic considerations to determine whether we have thermodynamic equilibrium in these slices. So when \( \varphi' = 0 \) all of the particles in the \( t = \) constant slices are contiguous, and hence they must be in thermodynamic equilibrium. Consequently for our scalar-scalar theories based on the Lagrangians investigated in Section 3, we do not require the universe to expand for a while and stop to achieve thermodynamic equilibrium. We have thermodynamic equilibrium before the universe even begins to expand, because we can assume that \( \varphi' \)
= 0, until the universe’s expansion commences.

In Section 3 we sought solutions to our field equations of the form \( \varphi = \alpha e^{\beta t} \) and \( \varphi = \gamma(k_1 t + k_2)^3 \), for which \( \varphi' > 0 \). It is natural to at first restrict one’s attention to the case where \( \alpha > 0, \beta > 0, \gamma > 0 \) and \( k_1 > 0 \) (when \( q > 0 \)), since these give rise to expanding universes. However, the cases where \( \alpha < 0, \beta < 0, \gamma < 0 \) and \( k_1 < 0 \) are also interesting, since they represent collapsing universe solutions.

To illustrate how expanding and collapsing universes can be combined, let us consider some of the solutions to \( H_{2,4} = 0 \) obtained in the preceding section. If we want the space with \( \varphi = \gamma(k_1 t + k_2)^3 \) to have scalar curvature \( R = 0 \), we need to have \( q = \frac{3}{2} \). Due to Eq.3.50, and the remarks following Eq.3.52, we see that if we choose \( K \) and \( G_4 \) in \( L_{2,4} \) by

\[
K = -14A_4 \varphi^{-11/3} |X|/3 \quad \text{and} \quad G_4 = A_4 \varphi^{-5/3}, \quad \text{Eq.4.3}
\]

where \( A_4 \) is an arbitrary real number, then \( \varphi = \alpha e^{\beta t} \) and \( \varphi = \gamma(k_1 t + k_2)^{3/2} \) are solutions to \( H_{2,4} = 0 \), (\( \forall \) choice of \( \alpha, \beta, \gamma, k_1 \) and \( k_2 \) which are such that \( \varphi' > 0 \)), and hence solutions to the Euler-Lagrange equation associated with \( L_{2,4} \). So let us assume that our universe begins at \( t = 0 \) as \( \varphi = \gamma(k_1 t + k_2)^{3/2} \) with \( \varphi'(0) = 0 \). Thus we need \( k_2 = 0 \), and so \( \varphi' = (3\gamma k_1/2)(k_1 t)^{1/2} \), with \( \varphi'' = (3\gamma k_1^2/4)(k_1 t)^{-1/2} \). Unsurprisingly this solution begins explosively at \( t = 0 \), with a vertical tangent vector. As time increases this solution passes through a myriad of solutions of the form \( \varphi = \alpha e^{\beta t} \), where \( \alpha > 0 \), and \( \beta > 0 \). From Eq.3.11
we see that for these exponential solutions $R = 12\beta^2$, and hence is never 0. However, when $\beta > 0$ is very close to 0, the solution $\varphi = \alpha e^{\beta t}$, gives rise to $\varphi' = \alpha \beta e^{\beta t}$, which stays “fairly flat,” for a range of $t > 0$, although it continues an upward climb at an accelerating rate. My contention is that when our original universe, with $\varphi' = (3\gamma k_1/2)(k_1t)^{1/2}$, crosses a universe with $\varphi' = \alpha \beta e^{\beta t}$ at some time $t_1$, it can “jump” from the original $\varphi$ state, to the exponential $\varphi$ state, when $(3\gamma k_1/2)(k_1t_1)^{1/2} = \alpha \beta \exp(\beta t_1)$, which, due to Eq.3.3, guarantees that the metric of the universe is continuous across the jump. However, the first and second derivatives of the metric, in particular the scalar curvature $R$, are not continuous across the jump at time $t = t_1$, since $R$ jumps from $R = 0$, to $R = 12\beta^2$.

Now the natural question to ask is how does the initial exploding state pick which exponential state it wants to jump into when the $\varphi$’s are equal? Well, one way to avoid this problem of making choices is to assume that everything that can happen does; i.e., we are dealing with a multiverse. I shall make this thought a bit more precise. Suppose that $\forall$ choice of non-negative $\alpha, \beta, \gamma, k_1$ and $k_2 \in \mathbb{R}$, we define $\varphi_{\alpha,\beta} := \alpha e^{\beta t}$, and $\varphi_{\gamma,k_1,k_2} := \gamma(k_1t + k_2)^{3/2}$, where $\varphi'_{\alpha,\beta} > 0$, and $\varphi'_{\gamma,k_1,k_2} > 0$. Let $Q^{(3/2)}$ denote the subset of $(\mathbb{R}^+)^4 \times \mathbb{R}$ consisting of all $(\alpha, \beta, \gamma, k_1,k_2)$ which is such that the equation $\varphi'_{\alpha,\beta}(t) = \varphi'_{\gamma,k_1,k_2}(t)$ admits two solutions, and let $t_1$ be the smallest solution, and $t_2$ the second solution. Due to the nature of the exponential function we know that $Q^{(3/2)}$ is non-empty. Then the
multiverse would be an 9-dimensional space, \( \mathbf{MV} := \mathbb{Q}^{(3/2)} \times \mathbb{R}^4 \), with coordinates: \(((\alpha, \beta, \gamma, k_1,k_2), (t,u,v,w))\). For the model described above we choose a point \((\alpha, \beta, \gamma, k_1, 0)\) in \( \mathbb{Q}^{(3/2)} \), and attach to that point a universe whose metric is:

(i) degenerate for \(t<0\);  
(ii) determined by \(\varphi'_{\gamma,k_1,0}\) for \(0 \leq t < t_1\); and  
(iii) determined by \(\varphi'_{a,\beta}\) for \(t_1 < t\).

It should be noted that in general at the times \(t_1\) and \(t_2\) when \(\varphi'_{a,\beta} = \varphi'_{\gamma,k_1,0}\), we have \(\varphi_{a,\beta} \neq \varphi_{\gamma,k_1,0}\). So that if we try to construct one scalar field \(\varphi\) defined by

\[
\varphi := \begin{cases} 
0, & \text{for } t < 0, \\
\varphi_{\gamma,k_1,0}, & \text{for } 0 < t < t_1, \\
\varphi_{a,\beta}, & \text{for } t_1 < t,
\end{cases}
\]

then this \(\varphi\) does not admit a continuous “extension” through \(t=t_1\). However, \(\varphi'\) does admit a continuous extension through \(t=t_1\), if we define

\[
\varphi'(t_1) := \lim_{t \to t_1^+} \varphi_{a,\beta}(t) = \lim_{t \to t_1^-} \varphi_{\gamma,k_1,0}(t).
\]

\(\varphi\), as just defined, is an element of a class of functions that I call \(C^{k,1}\). More generally I define this class of functions in the following way. Let \(\psi: \mathbb{R} \to \mathbb{R}\) be defined on an interval of the form \((a,b) \subseteq \mathbb{R}\) (where \(a\) could be \(-\infty\) and \(b\) could be \(+\infty\), except at a set of distinct points \(\{P_i\}_{i \in I} \subseteq (a,b)\), where \(I \subseteq \mathbb{Z}\) (the set of integers) is an indexing set, which could be countably infinite. I shall assume that the set \(\{P_i\}_{i \in I}\) does not have a limit point, and thus \(D := (a,b) \setminus \{P_i\}_{i \in I}\) is an open subset of \(\mathbb{R}\). \(\psi\) is said to be of class \(C^{k,1}\)
on (a,b) (k ∈ N, the set of natural numbers) if:

(i) \( \psi \) is of class \( C^k \) on \( D \), and

(ii) \( \forall \; P_i, \; \lim_{t \to P_i^-} \psi'(t) = \lim_{t \to P_i^+} \psi'(t) =: \psi'(P_i) \).

Thus when a function is of class \( C^{k,1} \) on (a,b) it has a well-defined “first derivative” on (a,b), even though it is discontinuous at points in (a,b).

When building cosmological spaces from the solutions obtained in Section 3 the scalar field \( \phi \) we obtain by piecing solutions together when they cross will be of class \( C^{\infty,1} \), which is adequate for our needs since the metric tensor \( g_{ij} \) depends upon \( \phi' \) and not \( \phi \). Thus \( g_{ij} \) will be continuous, but its derivatives can experience discontinuities at those points where \( \phi \) jumps between solutions. In what follows whenever I talk about a physical quantity, such as the scalar curvature \( R \), being continuous across a point where \( \phi \) experiences discontinuity, it will be in the sense that its left and right limits are equal at those points.

I shall now describe two ways in which we can get a universe in the multiverse to end. Suppose that \( (\alpha, \beta, \gamma, k_1, 0) \in O(3/2) \), and that \( t = t_1 \) and \( t = t_2 \), with \( 0 < t_1 < t_2 \), are the times when \( \phi'_{\alpha, \beta} = \phi'_{\gamma, k_1, 0} \). We can choose a negative number \( \dot{\alpha} \) so that \( \phi_{\alpha, \beta}(t) := \dot{\alpha} e^{-\beta t} \) is a solution to \( H_{2,4} = 0 \), with \( \phi'_{\alpha, \beta} > 0 \), and \( \phi'_{\alpha, \beta}(t_2) = \phi'_{\alpha, \beta}(t_2) \). (It is easily seen that \( \dot{\alpha} = -\alpha \exp(2\beta t_2) \).) So as time, \( t \), moves through \( t_2 \), \( \phi \) can jump from \( \phi_{\alpha, \beta}(t) \) to \( \phi_{\dot{\alpha}, \beta}(t) \) in such a way that the metric tensor is continuous, but its first derivatives experience a
discontinuity, with \( R \) being continuous as we pass through \( t_2 \), since \( R = 12\beta^2 \) (see Figure 1 on page 60, where the solid blue curve represents the model universe’s history). In this model, the originally expanding universe, gradually collapses down to the \( \varphi' = 0 \) state, over an infinitely long period. Note that in this model the three branches of the universe meet when \( t = t_2 \) (although the first branch was left behind as \( t \) increased past \( t_1 \)). Of course we could have chosen other exponentially decreasing solutions of \( H_{2,4} = 0 \), to end our model universe, but for them \( R \) would also experience a discontinuity as \( t \) passes through \( t_2 \). I leave it to astrophysicists to choose values of the constants \( \alpha, \beta, \gamma \) and \( k_1 \), which enable this model can have properties similar to those observed so far in our universe.

Next, let us construct a model that ends more abruptly in a finite amount of time.

Let’s consider the three solutions to \( H_{2,4} = 0 \), given by

\[
\varphi_{1,1;\gamma,k} := \gamma_1 (k_{11} t)^{3/2}, \quad \varphi_{1;\alpha,\beta} := \alpha_1 \exp(\beta_1 t), \quad \varphi_{1,2;\gamma,k} := -\gamma_1 (2k_{11} t_2 - k_{11} t)^{3/2},
\]

Eq.4.4

where \((\alpha_1, \beta_1, \gamma_1, k_{11}, 0) \in \mathbb{Q}^{(3/2)}\) and \( t_2 \) is defined as above. (The reason why I have put all of these identifying symbols on \( \varphi \) will be clear in a moment.) The first two solutions in Eq.4.4 are defined for \( t \geq 0 \), while the last solution is defined for \( t \leq 2t_2 \). In addition we have \( \varphi'_{1,1;\gamma,k}(t_2) = \varphi'_{1;\alpha,\beta}(t_2) = \varphi'_{1,2;\gamma,k}(t_2) \). For this triple of solutions our universe begins explosively with \( \varphi \) in the \( 1,1;\gamma,k \) state, then jumps at \( t = t_1 \) into the \( 1;\alpha,\beta \) state, and then jumps at \( t = t_2 \) into the \( 1,2;\gamma,k \) state, where it continues until \( t = t_2 \), and the universe
comes to an abrupt end (see Figure 2). It should be noted that in this model the graph of the third leg of the universe is actually the reflection of the graph of the first leg of the universe in the line \( t=t_2 \), if that first leg is continued to time \( t=t_2 \). Evidently this universe need not end at time \( t = 2t_2 \), because we can find other solutions of the form \( \varphi = \gamma_2 (k_{21}t + k_{22})^{3/2} \) which begin explosively at a time after \( 2t_2 \). In a moment I shall construct a series of such models, but before I do, I would like to make one remark.

In the model just constructed, \( R \) jumps from \( R=0 \) to \( R=12\beta^2 \) as we pass through \( t=t_1 \), and then jumps from \( R=12\beta^2 \) back to \( R=0 \), as we pass through \( t=t_2 \). It makes one think that the universe had to absorb some curvature to jump from one \( \varphi \) state to the next, and then had to emit that curvature to return to the “ground state.” This is reminiscent of the electron in the hydrogen atom absorbing a photon to move from the ground state to a higher energy state, and then emitting that photon to return to the ground state. But in this analogy, where did the universe borrow the curvature from, and then return it? The obvious place is from the \( R=0 \) vacuum states. Perhaps the first \( R=0 \) vacuum state solution, \( \varphi_{1.1;\gamma,k} \), “creates” an amount of positive scalar curvature \( R \) which enables the second vacuum state, \( \varphi_{1;\alpha;\beta} \), to come into existence with \( \beta = (R/12)^{1/2} \). After a period of time this second vacuum state must return this scalar curvature, which it then “gives” to the third, \( R=0 \), vacuum state \( \varphi_{1.2;\gamma,k} \). Of course, this is just speculation on my part. What someone needs to do at this juncture is quantize the scalar field \( \varphi \),
and in the process the gravitational field $g_{ij}$, to see if that lends any credence to the above thoughts. Evidently the canonical quantization approach is not available, since the Lagrangians $L_4$, $L_5$ and $L_C$ that we have employed, have Ostrogradsky instability issues. Hence we shall require a new approach to quantize $\varphi$. Hopefully this new approach will give rise to an inequality relating the lifetime, $T$, and $\beta$ in the exponential solution. Perhaps, something like $T\beta < q = 3/2$, or $T\beta < q - 1 = 1/2$.

Now for more models.

$\forall$ integer $n \in \mathbb{N}$ (the set of natural numbers), let $t_{nb} > 0$, denote the time that our $n^{th}$ universe begins, with $t_{1b} = 0$. The $n^{th}$ universe will be built from three solutions of $H_{2,4} = 0$ defined by

\begin{align*}
\varphi_{n,1;\gamma,k} &= \gamma_n(k_{n1}(t - t_{nb}))^{3/2}, \quad t_{nb} \leq t \leq t_{n1}, \\
\varphi_{n;\alpha,\beta} &= \alpha_n \exp(\beta_n t), \quad t_{n1} \leq t \leq t_{n2}, \text{ and} \\
\varphi_{n,2;\gamma,k} &= -\gamma_n(k_{n1}(2t_{n2} - t_{nb} - t))^{3/2}.
\end{align*}

where $(\alpha_n, \beta_n, \gamma_n, k_{n1}, k_{n2} := -k_{n1} t_{nb}) \in Q^{(3/2)}$, with $t_{n1} < t_{n2}$ being the times when $\varphi_{n,1;\gamma,k}$ and $\varphi_{n;\alpha,\beta}$ meet. The time, $t_{ne}$, when this universe ends is $t_{ne} = 2t_{n2} - t_{nb}$, and the length of its duration is $t_{ne} - t_{nb} = 2(t_{n2} - t_{nb})$. It will be assumed that $\forall$ $n \in \mathbb{N}$, the constants $\alpha_n$, $\beta_n$, $\gamma_n$, $k_{n1}$ and $k_{n2} = -k_{n1} t_{nb}$ will have been chosen so that $t_{ne} \leq t_{(n+1)b}$.

It should now be clear that by stringing the above countably infinite collection of universes together, we have constructed a Universe comprised of an endless chain of
“mini-universes,” each of which begins explosively (with the first beginning at $\varphi'=0$, when $t=0$), then ends abruptly after a decline, with a period of exponential growth between these two extreme states, which can be made as flat as you wish (see Figure 3). One should note that the mini-universes given by Eq.4.5 spend as much time expanding as they do collapsing. In addition, if $\forall \ n \in \mathbb{N}$, $t_{ne}=t_{(n+1)b}$, then our mini-universes bounce from one universe into the next without any hesitation.

The models presented above, which were based upon solutions of $H_2,4=0$, could be made to end more abruptly if we wished. We could also extend the infinite chain of models presented in Figure 3 so that they extend back into the infinite past to obtain a Universe without beginning or end. This Universe would consist of a countably infinite series of mini-universes, each of which exist for only a finite amount of time from their beginning to their end.

In passing I would like to point out that we could also build a solution to $H_2,4=0$, just using exponential solutions $\varphi_{a,\beta} = a e^{\beta t}$, which is even more in keeping with the usual inflationary model. For $\varphi_{a,\beta} = a e^{\beta t}$, we have $\varphi'_{a,\beta} = a \beta e^{\beta t}$, and hence $\varphi'_{a,\beta}(0)=a \beta$. Thus for the non-constant exponential solutions we always have $\varphi'_{a,\beta}$ beginning at a non-zero value when time begins at $t=0$. But we can suppose that for $t<0$; i.e., prior to the expansion of the universe, the universe did not satisfy any particular equations and just sat there with the metric on the $t=$constant slices being $(a\beta)^2(du^2+dv^2+dw^2)$. 

54
Consequently when t<0 the universe had plenty of time to achieve thermodynamic equilibrium, in fact an infinite amount of time, if that was necessary, prior to the “awakening” of \( \varphi_{a,\beta} \) at t=0. As time advances through t=0 the universe described by \( \varphi_{a,\beta} \) could jump to another “very flat” exponential solution, \( \varphi'_{a',\beta'} \) with \( \beta' \) very close to zero. As time continues, this exponential solution could jump to another, say, decreasing exponential solution \( \varphi_{a',\beta'} \) (see, Figure 4). We could continue stringing exponential solutions together, but any such solution would describe a universe which continues forever due to the nature of our exponential solutions.

Everything we did with the solutions to \( H_{2,4} = 0 \), with \( q = \frac{3}{2} \), could be done with our solutions to \( H_{2,5} = 0 \) and \( H_c = 0 \), for \( q = \frac{3}{2} \) as well as other values of \( q \). However, when \( q \neq \frac{3}{2} \), Eq.3.11 tells us that the value of R for the solutions of the form \( \gamma(k_1t+k_2)^q \), is not constant, but is positive if \( \frac{3}{2} < q < 2 \), just as R is a positive constant for each solution of the form \( \alpha e^{\beta t} \). So perhaps when \( \frac{3}{2} < q < 2 \) a solution of the form \( \gamma(k_1t+k_2)^q \) can jump to a solution of the form \( \alpha e^{\beta t} \) when their R values are equal.

All of the examples of scalar-scalar theories I have presented in this paper are based upon the function f given in Eq.2.7 which is intended to be used for cosmological considerations. Other choices can be made for f to deal with different situations in the universe. E.g., to deal with a few astronomical bodies we could choose the trivial f; \( \text{viz.} \),
\[ f := g^{ij} \pi y_i y_j \]  

Eq.4.6

In this case when we build the metric tensor using Eq.2.4, with \( \phi \) chosen equal to a constant, we obtain a Lorentzian metric. When this metric is used in the Horndeski Lagrangian presented in Eq.3.4, with \( \phi \) constant, we obtain the Einstein Lagrangian with cosmological term. This shows how we can recover Einstein’s theory from a scalar-scalar field theory using the Horndeski Lagrangian. Being able to do this is fairly important due to the success of Einstein’s theory in describing situations with only a handful of astronomical bodies.

The next biggest region of astronomical interest would be galaxies and galactic clusters. I am not certain what choice of \( f \) would be suitable for this situation. If we simply chose the trivial \( f \) given in Eq. 4.6 above, with \( \phi \) not a constant, then the theory we would arrive at would be the usual Horndeski theory.

This completes my introduction to scalar-scalar field theories. I hope that you have found it interesting.

Acknowledgements

I would like to thank Dr. M. Zumalaćarregui and Dr. J.M. Ezquiaguia for discussions on some of the topics dealt with in this paper. I owe a debt of gratitude to Dr. A. Guarnizo Trilleras for allowing me to read his Ph.D. thesis, which introduced me
to the subject of beyond Horndeski theory, and got me back into working on scalar-tensor theories. Professor A. Silvestri, and the Lorentz Institute of Physics at Leiden University in the Netherlands, deserves my thanks for their financial support through the acquisition of one of my paintings dealing with Horndeski Theory for their Institution. Lastly, I need to thank Dr. Moon Nahm, M.D., for not only encouraging me to pursue the research I have presented here, but also for financial assistance through the purchase of a painting of mine for the Physics Department at Washington University in St. Louis, where I did my undergraduate studies.

**Bibliography**

[1] P. Finsler, “über Kurven und Flächen in Allgemeinen Räumen,” Ph.D. thesis, University of Göttingen, Göttingen, Germany, 1918.

[2] H. Rund, “The Differential Geometry of Finsler Spaces,” Springer-Verlag, 1959.

[3] S.-S. Chern, D. Bao & Z. Shen, “An Introduction to Riemannian-Finsler Geometry,” Springer-Verlag, 2000.

[4] A. Bejancu & H. R. Farran, “Geometry of Pseudo-Finsler Submanifolds,” Springer Science & Business Media B.V., 2000.

[5] G. W. Horndeski, “Invariant Variational Principles and Field Theories,” PhD. thesis, University of Waterloo, Waterloo, Ontario, 1973.
[6] R. Adler, M. Bazin & M. Schiffer, “Introduction to General Relativity,” McGraw-Hill Book Company, 1965.

[7] G. W. Horndeski, “Second-Order Scalar-Tensor Field Theories in a Four-Dimensional Space,” Inter. J. of Theo. Phys. 10 (1974), 363-384.

[8] M. Ostrogradsky, “Memories sur les equations differentielles relatives au problems des isoperimetrics,” Mem. Ac. St. Petersburg 4, (1850), 385.

[9] R. P. Woodard, “The Theorem of Ostrogradsky,” arXiv.org/abs/1506.02210, August, 2015.

[10] T. Baker, E. Bellini, P. G. Ferreira, M. Lagos, J. Noller & I. Sawicki, “Strong constraints on cosmological gravity from GW170817 and GRB170817A,” Phys. Rev. Lett. 119, 251301 (2017); arXiv.org/abs/1710.06494, October, 2017.

[11] P. Creminelli & F. Vernizzi, “Dark Energy after GW170817 and GRB170817A,” Phys. Rev. Lett. 119, 251302 (2017); arXiv.org/abs/1710.05877, December, 2017.

[12] J. Sakstein & B. Jain, “Implications of the Neutron Star Merger GW170817 for Cosmological Scalar-Tensor Theories,” Phys. Rev. Lett. 119, 251303 (2017); arXiv.org/abs/1710.05893, December, 2017.

[13] J. M. Ezquiaga & M. Zumalaácarregui, “Dark Energy after GW170817: dead ends and the road ahead,” Phys. Rev. Lett 119, 251304 (2017);
arXiv.org/abs/1710.05901, November, 2017.

[14] G. W. Horndeski, “Conformally Invariant Scalar-Tensor Field Theories in a 4-Dimensional Space,” arXiv.org/abs/1706.04827, June, 2017.

[15] G. W. Horndeski, “Conformally Invariant Scalar-Vector-Tensor Field Theories Consistent with Conservation of Charge in a Four-Dimensional Space,” Fund. J. of Modern Phys. 11, 93-133 (2018); arXiv.org/abs/1801.02490, January, 2019.

[16] G. W. Horndeski, “Conservation of Charge and the Einstein-Maxwell Field Equations,” J. Math. Phys. 17, 1980 (1976).

[17] A. H. Guth, “The Inflationary Universe: The Quest for a New Theory of Cosmic Origins,” Basic Books, 1998.
Figure 1

Figure 2
