Bounded automorphism groups of compact complex surfaces

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Abstract. We classify compact complex surfaces whose groups of bimeromorphic selfmaps have bounded finite subgroups. We also prove that the stabilizer of a point in the automorphism group of a compact complex surface of zero Kodaira dimension, as well as the stabilizer of a point in the automorphism group of an arbitrary compact Kähler manifold of nonnegative Kodaira dimension, always has bounded finite subgroups.

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§1. Introduction

One says that a group $\Gamma$ has bounded finite subgroups if there exists a constant $B = B(\Gamma)$ such that every finite subgroup of $\Gamma$ has order at most $B$. Otherwise one says that $\Gamma$ has unbounded finite subgroups. In many interesting cases automorphism groups or groups of birational selfmaps of algebraic varieties have bounded finite subgroups. For instance, this is the case for nonuniruled varieties with vanishing irregularity over fields of characteristic zero (see [12], Theorem 1.8, (i)), for varieties over number fields (see [12], Theorem 1.4, and [3], Theorem 1.1) and for nontrivial Severi-Brauer surfaces over fields of characteristic zero that contain all roots of unity (see [18], Corollary 1.5).

The main purpose of this paper is to prove the following assertion that classifies compact complex surfaces whose groups of bimeromorphic selfmaps have unbounded finite subgroups.

Theorem 1.1 (cf. [14], Lemma 3.5). Let $S$ be a compact complex surface of non-negative Kodaira dimension. Suppose that the group $\text{Bim}(S)$ of bimeromorphic selfmaps of $S$ has unbounded finite subgroups. Then $S$ is bimeromorphic to a surface of one of the following types:

- a complex torus;
- a bielliptic surface;
- a Kodaira surface;
- a surface of Kodaira dimension 1.

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Moreover, in the first three cases the group $\text{Bim}(\mathcal{S})$ always has unbounded finite subgroups. In the fourth case the group $\text{Bim}(\mathcal{S})$ has bounded finite subgroups if and only if this holds for its subgroup that consists of all selfmaps preserving the fibres of the pluricanonical fibration of $\mathcal{S}$.

One of the main steps in the proof of Theorem 1.1 is the following assertion which, as we think, is interesting on its own.

**Proposition 1.2.** Let $\mathcal{S}$ be a minimal compact complex surface of Kodaira dimension 1. Consider the pluricanonical fibration $\varphi: \mathcal{S} \to \mathbb{C}$. Then the image of the group $\text{Aut}(\mathcal{S})$ in $\text{Aut}(\mathcal{C})$ is finite.

Also, in this paper we prove the following result. For a complex manifold $\mathcal{X}$ and a point $P \in X$, by $\text{Aut}(\mathcal{X}; P)$ we denote the stabilizer of $P$ in the group $\text{Aut}(\mathcal{X})$.

**Proposition 1.3.** There exists a constant $B$ such that for every compact complex surface $\mathcal{S}$ of Kodaira dimension 0, every point $P \in \mathcal{S}$, and every finite subgroup $G \subset \text{Aut}(\mathcal{S}; P)$ the order of $G$ is at most $B$.

**Remark 1.4.** Let $\mathcal{S}$ be a compact complex surface of Kodaira dimension 1, and let $P$ be a point on $\mathcal{S}$. We do not know whether the group $\text{Aut}(\mathcal{S}; P)$ always has bounded finite subgroups. It is easy to show that this holds at least when the fibre of the pluricanonical fibration of $\mathcal{S}$ passing through $P$ is nonsingular and reduced at $P$.

Under the additional assumption that a manifold is Kähler we can prove boundedness of finite subgroups for the stabilizers of points in the automorphism groups in arbitrary dimension.

**Theorem 1.5.** Let $\mathcal{X}$ be a compact Kähler manifold of nonnegative Kodaira dimension, and let $P$ be a point on $\mathcal{X}$. Then the group $\text{Aut}(\mathcal{X}; P)$ has bounded finite subgroups.

We will prove one more assertion whose structure is similar to that of Proposition 1.3; it completes some results in [13].

**Proposition 1.6** (cf. [13], Corollary 8.10). There exists a constant $J$ such that for every compact complex surface $\mathcal{S}$ of Kodaira dimension 0 and every finite subgroup $G \subset \text{Bim}(\mathcal{S})$ there is an abelian subgroup of index at most $J$ in $G$.

In §2 we prove several auxiliary assertions. In §3 we study automorphism groups of elliptic fibrations and prove Proposition 1.2. In §4 we prove Theorem 1.1. In §5 we prove Proposition 1.3. In §6 we prove Theorem 1.5. In §7 we prove Proposition 1.6.

We use the following notation and conventions. A **complex manifold** is a smooth irreducible complex space. A **morphism** is a holomorphic map of complex spaces; a **fibration** is a morphism with connected fibres of positive dimension. A **typical fibre** of a fibration $\varphi: \mathcal{X} \to \mathcal{Y}$ is a fibre over a point of some nonempty subset of the form $\mathcal{Y} \setminus \Sigma$, where $\Sigma$ is a closed (analytic) subset in $\mathcal{Y}$. By $\mathcal{X}$ we denote the canonical line bundle on a (compact) complex manifold $\mathcal{X}$, and by $\kappa(\mathcal{X})$ we denote the Kodaira dimension of $\mathcal{X}$. By $T_P(\mathcal{X})$ we denote the tangent space to the complex manifold $\mathcal{X}$ at the point $P \in \mathcal{X}$. Given a complex manifold $\mathcal{X}$ and a morphism
φ: X → Y which is equivariant with respect to the group Aut(X), we denote by Aut(⟨X⟩φ the group that consists of all automorphisms mapping every fibre of φ to itself.

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§2. Preliminaries

In this section we prove some auxiliary results on compact complex surfaces. The following assertion is well known.

Lemma 2.1. Let S be a compact complex surface. Suppose that there are two divisors C_1 and C_2 on S such that C_1^2 ≥ 0 and C_1 · C_2 > 0. Then S is projective.

Proof. For n ≫ 0 one has

\[(nC_1 + C_2)^2 = n^2C_1^2 + 2nC_1 · C_2 + C_2^2 > 0,\]

so that S is projective by Theorem 6.2 in [4], Ch. IV.

Recall that a compact complex surface S is called minimal if it does not contain smooth rational curves with self-intersection −1. For every compact complex surface S there exists a minimal surface S' bimeromorphic to S (see [4], Ch. III, Theorem 4.5, for instance). Furthermore, if κ(S) ≥ 0, then the minimal surface S' is unique (see [4], Ch. III, Proposition 4.6); in this case there is a bimeromorphic morphism π: S → S'. Recall also that there exists a Kodaira-Enriques classification of minimal compact complex surfaces (see [4], Ch. VI).

Lemma 2.2 (see [13], Proposition 3.5, for instance). Let S be a nonruled minimal compact complex surface. Then Bim(S) = Aut(S).

Lemma 2.3. Let S be a complex manifold, and let φ: S → C be a morphism that is equivariant with respect to the group Aut(S). Let F = φ*(c), c ∈ C, be a fibre of φ. Suppose that there is an irreducible component F_1 ⊂ Supp(F) of multiplicity 1 in F. Then every finite subgroup of Aut(S) acts faithfully on F.

Proof. Suppose that the induced action on F of some nontrivial finite subgroup G ⊂ Aut(S) is trivial; in particular, G preserves the irreducible component F_1 and acts trivially on it. Choose a point P ∈ F_1 such that the morphism φ is smooth at P. The group G acts nontrivially in the tangent space T_P(S) (see [1], §2.2, or [13], Corollary 4.2, for instance). On the other hand, G acts trivially on the subspace T_P(F_1) ⊂ T_P(S). Since the morphism φ is smooth at P, the differential

\[dφ: T_P(S) → T_{φ(P)}(C)\]

is a surjective linear map whose kernel is identified with T_P(F_1). Moreover, this map is G-equivariant, where the action of G on C is taken to be trivial. Since G acts trivially on the tangent space T_{φ(P)}(C), we conclude that the action of G on the tangent space T_P(S) is trivial as well. The contradiction obtained shows that the action of G on F cannot be trivial.

The lemma is proved.
Recall that for a primary Kodaira surface the algebraic reduction gives a morphism onto an elliptic curve, and all its fibres are elliptic curves. This morphism is equivariant with respect to the automorphism group of the surface. For a secondary Kodaira surface the algebraic reduction is a morphism to a rational curve whose typical fibre is an elliptic curve.

**Lemma 2.4.** Let $S$ be a primary (respectively, secondary) Kodaira surface, and let $\varphi: S \to C$ be its algebraic reduction. Then the image $\Gamma$ of the group $\text{Aut}(S)$ in $\text{Aut}(C)$ has order at most 6 (respectively, 24).

**Proof.** If $S$ is a primary Kodaira surface, then the group $\Gamma$ does not contain elements that act on the elliptic curve $C$ by translations (see [17], Corollary 3.3). Since the automorphism group of an elliptic curve is a semi-direct product of the subgroup of translations with a cyclic group of order at most 6, we conclude that the order of $\Gamma$ is at most 6.

Let $S$ be a secondary Kodaira surface, and let $\varphi: S \to \mathbb{P}^1$ be its algebraic reduction. Then $\varphi$ has either three or four multiple fibres (see, for instance, the proof of Lemma 7.1 in [13]). Thus $\Gamma$ is isomorphic to a subgroup of the symmetric group on four elements. In particular, the order of $\Gamma$ is at most 24.

The lemma is proved.

**Lemma 2.5.** Let $S$ be a primary Kodaira surface and $\varphi: S \to C$ be its algebraic reduction. Let $G \subset \text{Aut}(S; \varphi)$ be a nontrivial subgroup of finite order. Then $G$ acts on $S$ without fixed points.

**Proof.** Without loss of generality, we may assume that the group $G$ is cyclic. Let $\gamma$ be its generator. Suppose that $G$ has a fixed point $P \in S$. Let $F$ be the fibre of $\varphi$ passing through $P$. Recall that $F$ is irreducible and nonmultiple (see [4], Ch. V, §5). By Lemma 2.3 the automorphism $\gamma$ acts nontrivially on the elliptic curve $F$. Suppose that $\gamma$ fixes some point $P$ on $F$. Note that the morphism $\varphi$ is smooth at $P$ (as well as at any other point on $S$). Arguing as in the proof of Lemma 2.3, we see that the differential 

$$d\varphi: T_P(S) \longrightarrow T_{\varphi(P)}(C)$$

allows us to identify the two-dimensional representation $T_P(S)$ of the cyclic group $G$ with the direct sum $T_P(F) \oplus T$, where the action of $G$ on the subspace $T \cong T_{\varphi(P)}(C)$ is trivial. By Corollary 4.7 in [13] there exists a (compact) curve $D \subset S$ consisting of fixed points of $\gamma$, such that $D$ passes through $P$ and the tangent space to $D$ at $P$ coincides with the subspace $T \subset T_P(S)$. Therefore, we have $\varphi(D) = C$. Since the surface $S$ is not projective, this gives a contradiction with Lemma 2.1.

The lemma is proved.

We will use the following classical theorem proved by Minkowski.

**Theorem 2.6** (see [16], Theorem 1, for instance). For every positive integer $n$ the group $\text{GL}_n(\mathbb{Q})$ has bounded finite subgroups.

**Corollary 2.7.** For every positive integer $n$ there exists a constant $B_T(n)$ such that for every $n$-dimensional complex torus $S$, every point $P$ on $S$, and every finite subgroup $G \subset \text{Aut}(S; P)$, the order of $G$ is at most $B_T(n)$.

It is well known that $\text{Aut}(S; P)$ is isomorphic to a subgroup of $\text{GL}_{2n}(\mathbb{Z})$ (see [13], Theorem 8.4, for instance). Therefore, the assertion follows from Theorem 2.6.
Corollary 2.8. There exists a constant $B_{K3}$ such that for every compact complex surface $S$ which is either a $K3$ surface or an Enriques surface, and for every finite subgroup $G \subset \text{Aut}(S)$, the order of $G$ is at most $B_{K3}$.

Proof. Let $S$ be either a $K3$ surface or an Enriques surface. Consider the representation of $\text{Aut}(S)$ in the cohomology group

$$\rho: \text{Aut}(S) \to \text{GL}(H^*(S,\mathbb{Q})).$$

In both cases we have to consider, the dimension of the vector space $H^*(S,\mathbb{Q})$ does not exceed 24 (see [4], Table 10, for instance). By Theorem 2.6 the orders of finite subgroups in the image of the representation $\rho$ are bounded by some constant that does not depend on $S$. On the other hand, if $S$ is a $K3$ surface, then the representation $\rho$ is faithful (see [6], Proposition 15.2.1, for instance). If $S$ is an Enriques surface, then the kernel of $\rho$ contains at most 4 elements (see [11]).

The corollary is proved.

Finite groups acting on $K3$ surfaces have been investigated rather closely; see [20] and the references there. This classification can be used for an adequate bound for the constant $B_{K3}$ in Corollary 2.8.

The following theorem is well known.

Theorem 2.9 (see [19], Corollary 14.3, for instance). Let $S$ be a compact complex surface of Kodaira dimension 2. Then the group $\text{Aut}(S)$ is finite.

§3. Elliptic surfaces

In this section we prove Proposition 1.2.

To describe fibres of elliptic fibrations we use the standard notation for types of degenerate fibres in the Kodaira classification (see [4], Ch. V, §7, or [10], Lecture I, §4). Recall that a fibration $\varphi: S \to C$ with compact fibres, where $S$ is a smooth (but possibly noncompact) complex surface and $C$ is a smooth (but possibly noncompact) curve, is called relatively minimal if the fibres of $\varphi$ do not contain smooth rational curves with self-intersection $-1$.

Given an elliptic fibration $\varphi: S \to C$, one can consider the function $J$ that associates to a point $c \in C$ the value of the $j$-invariant of the fibre $F = \varphi^*c$, provided that the fibre $F$ is smooth. It is straightforward to see that $J$ is a meromorphic function on $C$.

Lemma 3.1. Let $\Delta \subset \mathbb{C}$ be the unit disc. Let $S$ be a smooth (noncompact) complex surface, and let $\varphi: S \to \Delta$ be a fibration with compact fibres whose typical fibre is an elliptic curve. Suppose that $\varphi$ is relatively minimal. Consider the fibre $F = \varphi^*0$ over the point $0 \in \Delta$. Suppose that the fibre $F$ is nonmultiple, and has type $I_b$ or $I_b^*$ for some $b \geq 1$. Then the function $J$ has a pole at $0$.

Proof. Since the fibre $F$ is nonmultiple, at least one of its irreducible components is reduced (see [4], Ch. V, §7). Hence, in a small neighbourhood of the point $0$ the fibration $\varphi$ admits an analytic section. Thus we can assume that $\varphi$ is taken to the Weierstrass form (see [10], Lecture II). Now the assertion follows from [10], Table IV.3.1 (cf. [4], Ch. V, §10, Table 6, and [10], Proposition VI.1.1).
Recall that a fibration is called isotrivial if its fibres over the points in some dense open subset are isomorphic one to another.

Lemma 3.2 (cf. [15], §2.1). Let $\Delta \subset \mathbb{C}$ be the unit disc. Let $S$ be a smooth (noncompact) complex surface, and let $\varphi: S \to \Delta$ be a fibration with compact fibres whose typical fibre is an elliptic curve. Suppose that $\varphi$ is relatively minimal and isotrivial. Consider the fibre $F = \varphi^*0$ over the point $0 \in \Delta$. If $F$ is a nonmultiple fibre, then it cannot be of type $I_b$ or $I^*_b$ for $b \geq 1$; if $F$ is a multiple fibre, then it can only have type $mI_0$ for some $m \geq 2$.

Proof. Since the fibration $\varphi$ is isotrivial, the $j$-invariant of its fibres is a constant function on the set $\Delta \setminus \Lambda$, where $\Lambda$ is the set of images of all multiple fibres of $\varphi$. In particular, this function cannot have a pole at a point of $\Delta \setminus \Lambda$. Therefore, the assertion about nonmultiple fibres follows from Lemma 3.1.

Now suppose that the fibre $F$ is multiple. According to [4], Ch. V, §7, it can only have type $mI_r$, where $r \geq 0$ and $m \geq 2$. We need to exclude the case $r \geq 1$. To do this, we apply the standard construction from [4], Ch. III, §10, or [8], pp. 571, 572: it gives an isotrivial relatively minimal elliptic fibration over $\Delta$ with nonmultiple fibre of type $I_{r'}$ for some $r' > 0$. The latter gives a contradiction with what we have already proved above.

The lemma is proved.

We will need detailed information on automorphism groups of compact complex surfaces of Kodaira dimension 1 (similar computations were used in the proofs of Lemma 8.2 in [13] and Lemma 3.3 in [14]).

Lemma 3.3. Let $S$ be a minimal compact complex surface of Kodaira dimension 1. Consider the pluricanonical fibration $\varphi: S \to C$ and suppose that $C \cong \mathbb{P}^1$. Then the image $\Gamma$ of the group $\text{Aut}(S)$ in $\text{Aut}(C)$ is finite.

Proof. The group $\Gamma$ acts faithfully on $\mathbb{P}^1$. Assume that $\Gamma$ is infinite. Then there exists a point $P \in \mathbb{P}^1$ with an infinite $\Gamma$-orbit $\Xi_P \subset \mathbb{P}^1$ (indeed, if the orbit of every point is finite, $\Gamma$ contains a subgroup of finite index that fixes three points on $\mathbb{P}^1$). The fibres of $\varphi$ over all points of $\Xi_P$ are isomorphic to another. Since the $j$-invariant of the fibres of $\varphi$ is a meromorphic function on $\mathbb{P}^1$, we conclude that this function is constant, so that the elliptic fibration $\varphi$ is isotrivial. In particular, by Lemma 3.2 all multiple fibres of $\varphi$ have type $mI_0$.

Suppose that $\varphi$ has at least one singular nonmultiple fibre. Since the group $\Gamma$ is infinite, $\varphi$ has at most two singular nonmultiple fibres; indeed, otherwise $\Gamma$ would have a finite invariant subset in $\mathbb{P}^1$ of cardinality at least 3, which is impossible for an infinite subgroup of $\text{Aut}(\mathbb{P}^1)$. By Lemma 3.2 and the Kodaira classification the topological Euler characteristic of any singular nonmultiple fibre of $\varphi$ is positive and does not exceed 10. Thus,

$$0 < \chi_{\text{top}}(S) \leq 20.$$ 

By Noether’s formula we obtain

$$\chi(\mathcal{O}_S) = \frac{1}{12}(c_1(S)^2 + \chi_{\text{top}}(S)) = \frac{\chi_{\text{top}}(S)}{12},$$

which implies that $\chi(\mathcal{O}_S) = 1$. 

Let $F_i$ be all multiple fibres of $\varphi$ (considered with reduced structure), and let $m_i$ be the multiplicity of $F_i$. By the canonical bundle formula (see [4], Ch. V, Theorem 12.1) we have

$$K_S \sim \varphi^*(K_{\mathbb{P}^1} \otimes \mathcal{L}) \otimes \mathcal{O}_S \left( \sum (m_i - 1) F_i \right),$$

where $\mathcal{L}$ is some line bundle of degree $\chi(\mathcal{O}_S) = 1$ on $\mathbb{P}^1$. Keeping in mind that $\kappa(S) = 1$, we obtain

$$-1 + \sum \left( 1 - \frac{1}{m_i} \right) = \deg(\mathcal{K}_{\mathbb{P}^1} \otimes \mathcal{L}) + \sum \left( 1 - \frac{1}{m_i} \right) > 0.$$

Hence $\varphi$ has at least two multiple fibres. Together with a nonmultiple singular fibre that exists by assumption, this gives a finite $\Gamma$-invariant subset of $\mathbb{P}^1$ containing at least three points. The latter means that the group $\Gamma$ is finite, which contradicts our assumption.

Therefore, $\varphi$ has no nonmultiple singular fibres. Hence $\chi_{\text{top}}(S) = 0$, so that Noether’s formula gives $\chi(\mathcal{O}_S) = 0$. By the canonical bundle formula we have

$$-2 + \sum \left( 1 - \frac{1}{m_i} \right) > 0,$$

where the $m_i$ are the multiplicities of the multiple fibres of $\varphi$. This implies that $\varphi$ has at least three multiple fibres. Thus we again obtain a finite $\Gamma$-invariant subset of $\mathbb{P}^1$ containing at least three points, which gives a contradiction.

The lemma is proved.

Now we are ready to consider the general case.

**Proof of Proposition 1.2.** Let $\Gamma$ be the image of the group $\text{Aut}(S)$ in $\text{Aut}(C)$. The group $\Gamma$ acts faithfully on the curve $C$. In particular, if the genus $g(C)$ is at least 2, then $\Gamma$ is finite, because the whole group $\text{Aut}(C)$ is finite. On the other hand, by Lemma 3.3 we can assume that $g(C) \neq 0$. Therefore, it remains to consider the case $g(C) = 1$.

If $\varphi$ is not a smooth morphism, then the group $\Gamma$ has a nonempty finite invariant subset in $C$. Since $g(C) = 1$, this implies that $\Gamma$ is finite.

Therefore, we can assume that the morphism $\varphi$ is smooth, and thus all of its fibres are elliptic curves. In this case we have $\chi_{\text{top}}(S) = 0$. By Noether’s formula $\chi(\mathcal{O}_S) = 0$. By the canonical bundle formula we obtain

$$K_S \sim \varphi^*(K_C \otimes \mathcal{L}),$$

where $\mathcal{L}$ is some line bundle of degree $\chi(\mathcal{O}_S) = 0$ on $C$. Hence the Kodaira dimension of $S$ is nonpositive, which contradicts our assumption.

The proposition is proved.
§ 4. Proof of the main result

Proof of Theorem 1.1. We may assume that the surface $S$ is minimal. In this case $\text{Bim}(S) = \text{Aut}(S)$ by Lemma 2.2.

Suppose that $\kappa(S) = 0$. Let us use the classification of minimal compact complex surfaces of Kodaira dimension 0 (see [4], Ch. VI): the surface $S$ is either a $K3$ surface, or an Enriques surface, or a complex torus, or a bielliptic surface, or a Kodaira surface. If $S$ is either a $K3$ surface or an Enriques surface, then the group $\text{Aut}(S)$ has bounded finite subgroups (see Corollary 2.8 or [13], Lemma 8.8). On the other hand, if $S$ is either a complex torus, or a bielliptic surface, or a Kodaira surface, then the group $\text{Aut}(S)$ has unbounded finite subgroups (see [17], Theorem 1.1, (i)).

Suppose that $\kappa(S) = 1$. Consider the pluricanonical map $\varphi: S \to C$. Since $\varphi$ is equivariant with respect to the group $\text{Aut}(S)$, there is an exact sequence of groups

$$1 \to \text{Aut}(S)_\varphi \to \text{Aut}(S) \to \Gamma,$$

where $\Gamma$ is a subgroup of $\text{Aut}(C)$. According to Proposition 1.2, the group $\Gamma$ is finite. This implies that the group $\text{Aut}(S)$ has bounded finite subgroups if and only if this holds for $\text{Aut}(S)_\varphi$.

Finally, if $\kappa(S) = 2$, then the assertion follows from Theorem 2.9.

The theorem is proved.

In the case of compact Kähler surfaces one can make Theorem 1.1 a bit more precise. Recall that all Kodaira surfaces are non-Kähler. This follows from the observation that the first Betti number of a (primary or secondary) Kodaira surface is always odd (see [4], Ch. V, §5, case B, Ib)). On the other hand, the first Betti number of a compact Kähler surface must be even (see [4], Ch. IV, Theorem 3.1). Thus, Theorem 1.1 implies the following.

Corollary 4.1. Let $S$ be a compact Kähler surface with $\kappa(S) \geq 0$. Suppose that the group $\text{Bim}(S)$ has unbounded finite subgroups. Then $S$ is bimeromorphic to a surface of one of the following types:

- a complex torus;
- a bielliptic surface;
- a surface of Kodaira dimension 1.

Moreover, in the first two cases the group $\text{Bim}(S)$ always has unbounded finite subgroups.

Remark 4.2. While Theorem 1.1 classifies compact complex surfaces of nonnegative Kodaira dimension such that their group of bimeromorphic selfmaps has unbounded finite subgroups, we can ask a similar question about the automorphism groups of minimal surfaces of negative Kodaira dimension. In this case the finiteness of the whole automorphism group can be of interest. The automorphism groups of ruled surfaces were the subject of [21]; in particular, necessary and sufficient conditions are known for these groups to be finite. As regards higher dimensions, there are finiteness results for the automorphism groups of certain smooth Fano 3-folds [22] and certain rational affine varieties (see [23], for instance).
§ 5. Stabilizers of points

In this section we prove Proposition 1.3. We start by deriving two corollaries from the results in § 2.

**Corollary 5.1.** Let \( S \) be a bielliptic surface, and let \( P \) be a point on \( S \). Then the stabilizer \( \text{Aut}(S; P) \) has order at most 36.

**Proof.** Consider the Albanese map \( \varphi: S \to C \). It is equivariant with respect to the group \( \text{Aut}(S) \). Recall that \( \chi_{\text{top}}(S) = 0 \), so that by Noether’s formula we have \( \chi(\mathcal{O}_S) = 0 \). Therefore, it follows from the canonical bundle formula that \( \varphi \) has no multiple fibres.

Let \( F \) be the fibre of the morphism \( \varphi \) passing through \( P \). Consider the natural homomorphism

\[
\sigma: \text{Aut}(S; P) \to \text{Aut}(C) \times \text{Aut}(F).
\]

It follows from Lemma 2.3 that the homomorphism \( \sigma \) is injective. Furthermore, its image is contained in the subgroup \( \text{Aut}(C; \varphi(P)) \times \text{Aut}(F; P) \subset \text{Aut}(C) \times \text{Aut}(F) \).

Since both \( C \) and \( F \) are elliptic curves, we see that

\[
|\text{Aut}(S; P)| \leq |\text{Aut}(C; P) \times \text{Aut}(F; P)| \leq 36.
\]

The corollary is proved.

**Corollary 5.2.** Let \( S \) be a Kodaira surface, and let \( P \) be a point on \( S \). Let \( G \) be a finite subgroup in the stabilizer \( \text{Aut}(S; P) \). Then \( G \) has order at most 6 if \( S \) is a primary Kodaira surface, and at most 36 if \( S \) is a secondary Kodaira surface.

**Proof.** First suppose that \( S \) is a primary Kodaira surface. Consider the algebraic reduction \( \varphi: S \to C \), and set \( G' = G \cap \text{Aut}(S) \varphi \). By Lemma 2.4 the subgroup \( G' \subset G \) has index at most 6. Let \( F \) be a fibre of the morphism \( \varphi \) passing through the point \( P \). By Lemma 2.5 every nontrivial element of the group \( G' \) acts on \( F \) without fixed points. Therefore, the group \( G' \) is trivial, and \( |G| \leq 6 \).

Now suppose that \( S \) is a secondary Kodaira surface. Then there exists a canonical finite cover

\[
\theta: \tilde{S} \to S,
\]

where \( \tilde{S} \) is a primary Kodaira surface, and the degree of \( \theta \) is at most 6. In fact, in this case the class \( [\mathcal{K}_S] \in H^2(S, \mathbb{Z}) \) of the canonical line bundle \( \mathcal{K}_S \) is an \( n \)-torsion element with \( n = 2, 3, 4 \) or 6 (see [4], Ch. VI, §1). By the universal coefficient theorem it defines an \( n \)-torsion element in \( H_1(S, \mathbb{Z}) \) and so there exists a canonically defined subgroup of index \( n \) in the fundamental group \( \pi_1(S) \), which in turn defines our cover \( \theta \). In other words, the surface \( \tilde{S} \) is the analytic spectrum \( \text{Spec}_{\text{an}}(\mathcal{R}) \) of the canonical \( \mathcal{O}_S \)-algebra

\[
\mathcal{R} = \bigoplus_{i=0}^{n-1} \mathcal{K}_S^\otimes i.
\]

Since \( \theta \) is canonically defined, there is a surjective homomorphism \( \text{Aut}(\tilde{S}) \to \text{Aut}(S) \). In particular, there exists a finite subgroup \( \tilde{G} \subset \text{Aut}(\tilde{S}) \) with a surjective
homomorphism onto $G$. Let $\tilde{P}_0$ be one of the preimages of the point $P$ on $\tilde{S}$. Then $\tilde{G}$ contains a subgroup $\tilde{G}_0$ of index at most 6 such that $\tilde{G}$ fixes the point $\tilde{P}_0$. As we already proved above, one has $|\tilde{G}_0| \leq 6$, and hence

$$|G| \leq |\tilde{G}| \leq 6|\tilde{G}_0| \leq 36.$$ 

The corollary is proved.

**Remark 5.3.** We do not know whether the bounds obtained in Corollaries 5.1 and 5.2 are sharp.

Now we will complete the case of Kodaira dimension 0.

**Proof of Proposition 1.3.** Let $S$ be a compact complex surface of Kodaira dimension 0. Consider the minimal model $S'$ of the surface $S$. Then $\text{Aut}(S) \subset \text{Bim}(S')$; hence by Lemma 2.2 there is an embedding $\text{Aut}(S) \subset \text{Aut}(S')$. Consider the bimeromorphic morphism $\pi: S \to S'$. Since the minimal model $S'$ is unique, the morphism $\pi$ is equivariant with respect to the group $\text{Aut}(S)$. Therefore, the image $\pi(P)$ of the point $P$ is invariant under the group $\text{Aut}(S; P)$. Thus we can assume from the very beginning that the surface $S$ is minimal.

We use the classification of minimal compact complex surfaces of Kodaira dimension 0. If $S$ is either a $K3$ surface or an Enriques surface, the assertion follows from Corollary 2.8. If $S$ is a complex torus, the assertion follows from Corollary 2.7. If $S$ is a bielliptic surface, the assertion follows from Corollary 5.1. Finally, if $S$ is a Kodaira surface, the assertion follows from Corollary 5.2.

The proposition is proved.

### §6. Kähler manifolds

In this section we prove Theorem 1.5. Given a complex variety $X$, by $\text{Aut}^0(X)$ we denote the connected component of identity in the complex Lie group $\text{Aut}(X)$.

**Theorem 6.1** (see [5], Corollary 5.11). Let $X$ be a compact Kähler manifold of nonnegative Kodaira dimension. Then the group $\text{Aut}^0(X)$ is either trivial or is a complex torus.

**Lemma 6.2.** Let $X$ be a compact complex manifold. Suppose that $X$ is nontrivially acted on by a complex torus $T$. Then $T$ has no fixed points on $X$.

**Proof.** The action of $T$ on $X$ is given by a morphism

$$\Psi: T \times X \to X.$$ 

Suppose that some point $P \in X$ is fixed by $T$. Then the image $\Psi(T \times \{P\})$ is a point. On the other hand, since the action of $T$ on $X$ is nontrivial, for a typical point $Q \in X$ the image $\Psi(T \times \{Q\})$ is not a point. This is impossible by the rigidity theorem (see [9], Theorem 5.23).

The lemma is proved.
Corollary 6.3. Let $X$ be a compact Kähler manifold of nonnegative Kodaira dimension, and let $P$ be a point on $X$. Then the group

$$\text{Aut}^0(X; P) = \text{Aut}(X; P) \cap \text{Aut}^0(X)$$

is finite. Moreover, there exists a constant $B = B(X)$ independent of $P$ such that $|\text{Aut}^0(X; P)| \leq B$.

Proof. If the group $\text{Aut}^0(X)$ is trivial, there is nothing to prove. Thus we assume that $\text{Aut}^0(X)$ is nontrivial, and hence it is a complex torus by Theorem 6.1.

Suppose that the group $\text{Aut}^0(X; P)$ is infinite. Since it is a closed subgroup of $\text{Aut}^0(X)$, it contains some complex torus $T$. Thus $T$ acts on $X$ with the fixed point $P$, which contradicts Lemma 6.2. Therefore, the group $\text{Aut}^0(X; P)$ is finite.

Now consider the incidence relation

$$\Xi = \{ (\sigma, Q) \mid \sigma(Q) = Q \} \subset \text{Aut}^0(X) \times X,$$

and let $\pi: \Xi \to X$ denote the projection onto the second factor. Then $\Xi$ is a (possibly reducible) compact complex space, and a fibre of $\pi$ over a point $P$ is exactly the subgroup $\text{Aut}^0(X; P)$. The projection $\pi$ is a proper map. Since the fibres of $\pi$ are finite, we conclude that the map $\pi$ is finite. We claim that the number of points in the fibres of $\pi$ is bounded by some constant $B = B(X)$. Indeed, since the number of irreducible components of $\Xi$ is finite, we may replace $\Xi$ by its irreducible component. Now the number of points in a fibre is bounded by the degree of the map $\pi: \Xi \to \pi(\Xi)$.

The corollary is proved.

Now we are ready to prove the main result of this section.

Proof of Theorem 1.5. According to Corollary 6.3, the intersection $\text{Aut}(X; P) \cap \text{Aut}^0(X)$ has bounded order. Hence it is enough to check that the image of $\text{Aut}(X; P)$ in the quotient group

$$\Upsilon = \text{Aut}(X)/\text{Aut}^0(X)$$

has bounded finite subgroups. On the other hand, by Lemma 3.1 in [7] the whole group $\Upsilon$ has bounded finite subgroups.

The theorem is proved.

Corollary 6.4. Let $S$ be a minimal compact Kähler surface of Kodaira dimension 1. Suppose that the pluricanonical fibration $\varphi: S \to C$ has a singular fibre. Then the group $\text{Bim}(S)$ has bounded finite subgroups.

Proof. One has $\text{Bim}(S) = \text{Aut}(S)$ by Lemma 2.2. Consider the set $\Sigma \subset S$ of singular points of all singular fibres of $\varphi$, and choose a point $P \in \Sigma$. Since $\varphi$ is equivariant with respect to the group $\text{Aut}(S)$, this group acts on $\Sigma$. Hence the stabilizer $\text{Aut}(S; P)$ of the point $P$ has index at most $|\Sigma|$ in $\text{Aut}(S)$. Now it remains to apply Theorem 1.5.

The corollary is proved.
§ 7. Abelian subgroups

In this section we prove Proposition 1.6. We start with one of its particular cases, which we treat using the results of § 2.

**Corollary 7.1.** Let \( S \) be a Kodaira surface, and let \( G \) be a finite subgroup in \( \text{Aut}(S) \). Then \( G \) contains an abelian subgroup of index at most 6.

**Proof.** First suppose that \( S \) is a primary Kodaira surface. Consider the algebraic reduction \( \phi : S \to C \), and set \( G' = G \cap \text{Aut}(S)_\phi \). By Lemma 2.4 the subgroup \( G' \subset G \) has index at most 6. By Corollary 5.2 the group \( G' \) acts by translations on an elliptic curve that is a fibre of the morphism \( \phi \). In particular, this group is abelian.

Now suppose that \( S \) is a secondary Kodaira surface. As in the proof of Corollary 5.2, there exists a primary Kodaira surface \( \widetilde{S} \) and a finite subgroup \( \widetilde{G} \subset \text{Aut}(\widetilde{S}) \) with a surjective homomorphism onto \( G \). According to what we proved above, the group \( \widetilde{G} \) contains an abelian subgroup \( \widetilde{G}' \) of index at most 6. Its image in \( G \) will also be an abelian subgroup of index at most 6.

The corollary is proved.

Now we prove the general assertion.

**Proof of Proposition 1.6.** Let \( S \) be a compact complex surface of Kodaira dimension 0. We replace \( S \) by its minimal model. Then \( \text{Bim}(S) = \text{Aut}(S) \) by Lemma 2.2.

Let us use the classification of minimal compact complex surfaces of Kodaira dimension 0. If \( S \) is either a \( K3 \) surface or an Enriques surface, then the assertion follows from Corollary 2.8. If \( S \) is a complex torus, then the assertion follows from Corollary 2.7, since in this case the quotient of the group \( \text{Aut}(S) \) by the abelian subgroup acting by translations on \( S \) is isomorphic to the stabilizer of (any) point of \( S \). If \( S \) is a Kodaira surface, then the assertion follows from Corollary 7.1. Finally, if \( S \) is a bielliptic surface, then the assertion follows from the explicit description of the group \( \text{Aut}(S) \) (see [2], Table 3.2): according to this description \( \text{Aut}(S) \) contains a subgroup of index at most 24 isomorphic to the group of points of an elliptic curve.

The proposition is proved.

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