D-brane Scattering in
IIB String Theory and IIB Matrix Model

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Abstract

We consider two Dirichlet p-branes with lower dimensional brane charges and their scattering. We first calculate the cylinder amplitude of open string with suitable boundary conditions. We compare this result with that in the IIB matrix model. We find the agreement between them in the long distance, low velocity, or large field limit. We also find a way to investigate more general boundary conditions for open string.
1 Introduction

String theory is the best candidate for the unified theory of interactions. Dirichlet branes on which fundamental open string boundaries are attached are key objects to understand non-perturbative effects of string theory. That is, in the type IIB string theory, the D-string is a strong-weak coupling dual object of the fundamental string. Furthermore a matrix model for IIB string theory was presented by IKKT. They propose that the model is a non-perturbative definition of string theory. However, clear correspondences between string theory and its matrix model are not found yet. Of course in Ref [2], some correspondences between them were presented. Firstly, Green Schwarz action in Schild gauge can be related to the action of the matrix model in the case of the infinite matrix size. Next, the potential between two static D-strings in the matrix model was calculated. In Ref [3] the authors calculated the scattering amplitude of two D-strings based on the matrix model and they compared it with the result in string theory. They set the non-zero values of all commutators of the coordinate matrices to $\alpha \times \text{constant}$. However, in Ref [4], the authors insist that the value of the commutators in the matrix model $F^{MN}$ is the inverse of the field strength $F$ on the D-brane world volume in string theory in the large $F$ approximation. In this context, the authors in Ref [6] compared the potentials between two static D-branes in string theory and the IIB matrix model.

We consider $D_p$-branes with non-zero field strength in IIB string theory and in its matrix model (IKKT model). We calculate the amplitude of the scattering of two $D_p$-branes and compare it with the one of the IIB matrix model. Since the relative velocity $(2v)$ is non-zero, the configuration is not BPS. When $v \to 0$, the amplitude is expected to vanish. We find that the amplitudes in the two models are equal in three independent limits if we identify the respective field strengths suitably. Namely, we find how the "D-brane" configurations in the matrix model are translated to those in string theory.

In section 2, we calculate the amplitude of two $D_p$-branes with the field strength on their world volumes in bosonic sector. First, we explain the configuration of two D-branes in the flat space-time. Two $D_p$-branes are extended infinitely in the world volume direction. They have a constant relative velocity to each other and flat world volume. They are located in spatially parallel to each other. In this situation we consider the fundamental open string whose ends are attached on the two D-branes. Then we calculate the one loop amplitude (i.e. cylinder amplitude) of open string. When the equations of motion are chosen to be free field type, the boundary condition of open string is complex because of the non-zero field strength $F$ and the non-zero relative velocity $2v$. Previously, $v = 0$ case was calculated in [4], $F = 0$ case was also done in [1] and related case in IIA string theory was presented in [8]. We formulate a general mode expansion of the string coordinates in the flat world volume case, quantize them canonically, and construct the Virasoro algebra.

In section 3, we consider superstring based on the NSR formalism. In this
paper, the boundary condition for the world sheet fermion is defined by the super transformation of the bosonic one. Then we perform a calculation using the proper time representation of the amplitude. To compare the result with that of the matrix model, we choose the leading term in three independent expansions in terms of parameters, in the long distance \((b \gg \alpha')\) case where \(b\) is the distance between two D-branes, in the low velocity case, or in the large gauge field strength case.

In section 4, we consider the amplitude in the IKKT matrix model with a brief review in our notations.

In section 5, we compare the amplitudes in the two models and identify the field strength \(F\) in string theory with the value \(F^M\) of the commutator in the matrix model, in such a way that \(\det(\eta + F)F^M = -1\) where \(\eta\) is the target space metric. Then we find that the amplitudes in the two models are precisely the same when \(b\) is large for arbitrary \(v, F, v\) is small for arbitrary \(b, F, v\) or \(F\) is large for arbitrary \(b, v\) when the distance between the two Dp-branes is large for arbitrary velocity \(v\).

In section 6, there are conclusions and discussions. In Appendices, we explain a general method to study the open string boundary condition which is any linear combinations of Neumann and Dirichlet boundary conditions. Some detailed calculations are also explained there.

1.1 physical situation

First, we consider two flat identical D-branes with the same world volume dimensions in the \(D = 10\) dimensional Minkowski space-time whose coordinates are written as \(X^0, \cdots, X^{D-1}\). We restrict the world volume dimension: \(p + 1\) to be even because we consider the IIB string theory. We locate them spatially parallel to each other in the \(D\) dimensional Minkowski space-time. We call the common directions of the two Dp-branes \(1, \cdots, p\), where the world volumes of the Dp-branes are infinitely extended. We set two Dp-branes to have relative velocity, \(2v\), in the \((p + 1)\)-th direction. We set the coordinates’ value of the
$p + 2, \cdots, (D - 2)$-th directions of the two D$p$-branes to be 0 and locate them separately in the $(D - 1)$-th direction by the distance $b$. We restrict $v$ and $b$ to be positive constant in this paper.

### 1.2 correspondence between string theory and matrix model

Assuming that two D-branes interact by exchanging closed fundamental strings and any effect of two D-branes can be treated as open string boundary conditions, we calculate the amplitude of the fundamental string diagram with two boundaries, especially the cylinder amplitude. Since what is calculated in the matrix model is one loop diagram which is planar, we expect that the cylinder amplitude agrees with the sum of planar diagrams in the IIB matrix model. The cylinder amplitude can be thought of the one loop amplitude of an open string whose boundaries are fixed in the D-branes, which we are considering now.
2 Dp-brane-Dp-brane scattering for bosonic part

We would like to calculate the scattering amplitude, that is the effective action. To calculate the amplitude, we have to obtain the propagators in space-time, in another word, the normal ordered Hamiltonian with the correct zero point energy. Therefore, in the following part in this section we define a normal ordering, construct Virasoro operators, and find a physical state condition. Let us start with the following world sheet action of open string.

\[ S_{\text{bosonic}} = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \partial_\sigma X^\mu \partial^\sigma X_\mu + \frac{1}{2\pi\alpha'} \int d\tau A_\mu^0 \partial_\tau X^\mu \bigg|_{\sigma=0} - \frac{1}{2\pi\alpha'} \int d\tau A_\mu^\pi \partial_\tau X^\mu \bigg|_{\sigma=\pi}, \quad (2.1) \]

where \( A_\mu^0 \) and \( A_\mu^\pi \) are the gauge potentials on the D-brane world volume to which the \( \sigma = 0 \) and \( \sigma = \pi \) end of open string are attached, respectively. We can rewrite the action as follows.

\[ S_{\text{bosonic}} = \{ \text{terms whose deformation vanishes with free equations of motion} \} + S_{\text{boundary}}, \]

where

\[ S_{\text{boundary}} = -\frac{1}{4\pi\alpha'} X^\mu \left( \partial_\sigma X_\mu - 2 \partial_\tau A_\mu^0 \right) \bigg|_{\sigma=0} + \frac{1}{4\pi\alpha'} X^\mu \left( \partial_\sigma X_\mu - 2 \partial_\tau A_\mu^\pi \right) \bigg|_{\sigma=\pi}. \quad (2.2) \]

To get free equations of motion, we require

\[ \delta S_{\text{boundary}} = 0 . \quad (2.3) \]

3In this paper, the indices run as follows:

- \( \mu, \nu, \rho, \sigma = 0 \ldots D - 1 \) \quad in the D=10 dimensional space-time
- \( \alpha, \beta, \gamma = 0 \ldots p \) \quad in the p-brane world volume
- \( \lambda, \delta = 2 \ldots D - 1 \) \quad in the pure Neumann and pure Dirichlet directions
- \( a, b = 0, 1, \ldots \text{the meaning depends on the situations} \)
- \( i, j = 2 \ldots p + 1 \) \quad in the 0 mode directions
- \( k = p + 2 \ldots D - 1 \) \quad in the pure Dirichlet direction
- \( l, l' = 1, \ldots, \frac{p+1}{2} \) \quad for the field strength
- \( m, n \in \mathbb{Z} \) \quad for bosonic modes and \( b, c \) ghosts
- \( r, s \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2} \) \quad for fermionic modes and \( \beta, \gamma \) ghosts
Namely,
\[
\delta X^\mu \left( \partial_\sigma X_\mu - 2 \partial_\tau A^{0}_\mu \right)|_{\sigma = 0} = 0 , \\
\delta X^\mu \left( \partial_\sigma X_\mu - 2 \partial_\tau A^{\pi}_\mu \right)|_{\sigma = \pi} = 0 .
\] (2.4)

2.1 $p = 1$ case

In this subsection, we consider $p = 1$ case to understand how to treat the non-trivial boundary conditions.

First, imagine a pure (with no gauge field) D-string moving in the 2-nd direction with the constant velocity $v$. Its coordinates are restricted as
\[
X^2 = vX^0 ,
\] (2.5)
it is equivalent to
\[
\partial_\tau \left( X^2 - vX^0 \right) = 0 .
\] (2.6)

This is a Dirichlet boundary condition.

Second, in order to introduce gauge field on the world sheet of the D-string, look at it from the proper time coordinate: $\tilde{X}$ in the following way.
\[
\partial_a X^\mu \rightarrow \partial_a \tilde{X}^\mu = \Lambda^\mu_{\nu, a} \partial_a X^\nu ,
\] (2.7)
so that
\[
\partial_\tau \tilde{X}^2 = 0 ,
\] (2.8)
where
\[
\Lambda^\mu_{\nu, a} = \begin{pmatrix}
\frac{1}{\sqrt{1-v^2}} & 0 & \frac{-v}{\sqrt{1-v^2}} \\
0 & 1 & 0 \\
\frac{-v}{\sqrt{1-v^2}} & 0 & \frac{1}{\sqrt{1-v^2}} \\
0 & 0 & 1
\end{pmatrix}^\mu_{\nu} .
\] (2.9)

The boundary condition in this coordinate is, because of Lorentz invariance,
\[
\delta \tilde{X}^\mu \left( \partial_\sigma \tilde{X}_\mu - 2 \partial_\tau \tilde{A}^{0, \pi}_\mu \right) = 0 ,
\] (2.10)
on each end. To consider D-string, we impose the following conditions.
\[
\delta \tilde{X}^0 , \delta \tilde{X}^1 \neq 0 , \delta \tilde{X}^2 = \cdots = \delta \tilde{X}^9 = 0 .
\] (2.11)

Therefore
\[
\begin{align*}
\partial_\sigma \tilde{X}_0 - 2 \partial_\tau \tilde{A}^{0, \pi}_0 &= 0 , \\
\partial_\sigma \tilde{X}_1 - 2 \partial_\tau \tilde{A}^{0, \pi}_1 &= 0 , \\
\partial_\tau \tilde{X}^2 &= \cdots = \partial_\tau \tilde{X}^9 = 0 .
\end{align*}
\] (2.12)
We consider the following constant field strength of the gauge field.

\[ F_{\alpha\beta} = \begin{pmatrix} 0 & f_1 \\ -f_1 & 0 \end{pmatrix}, \]

\[ \tilde{A}_0^0 = -\frac{1}{2} F_{\alpha\beta} \tilde{X}^\beta, \quad \tilde{A}_0^\alpha = 0, \quad \alpha \neq 0, \]

(2.13)

where \( F_{\alpha\beta} \) is the same in each brane since we are considering identical branes.

Now, the boundary condition is

\[
\begin{align*}
\partial_\sigma \tilde{X}_0 + f_1 \partial_\tau \tilde{X}^1 &= 0 \\
\partial_\sigma \tilde{X}_1 - f_1 \partial_\tau \tilde{X}_0 &= 0 \\
\partial_\tau \tilde{X}^2 &= \cdots = \partial_\tau \tilde{X}^9 = 0
\end{align*}
\]

(2.14)

on each end.

We can rewrite the boundary condition as

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_\sigma \begin{pmatrix} \tilde{X}_0 \\ \tilde{X}_1 \\ \vdots \end{pmatrix} + \begin{pmatrix} 0 & f_1 & 0 \\ -f_1 & 0 & 0 \\ 0 & 1 & \vdots \end{pmatrix} \partial_\tau \begin{pmatrix} \tilde{X}^0 \\ \tilde{X}^1 \\ \vdots \end{pmatrix} = 0. \quad (2.15)
\]

By using the fact that

\[ \tilde{X}^1 = \tilde{X}_1, \quad \tilde{X}^0 = -\tilde{X}_0, \]

(2.16)

because of Minkowski space-time, we can rewrite eq. (2.15) again as

\[
\partial_\sigma \tilde{X}_L = M_0 \partial_\tau \tilde{X}_R, \quad M_0 := \begin{pmatrix} \frac{1 + f_1^2}{1 - f_1^2} & \frac{2 f_1}{1 - f_1^2} & 0 \\ \frac{2 f_1}{1 - f_1^2} & \frac{1 + f_1^2}{1 - f_1^2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (2.17)
\]

where we decomposed the coordinate functions by using the equations of motion

\[ \tilde{X}(\tau, \sigma) = \tilde{X}_L(\sigma^+) + \tilde{X}_R(\sigma^-), \quad \sigma^\pm := \tau \pm \sigma. \]

(2.18)

Then finally, we can get the following formula for the boundary condition in the original coordinate frame:

\[
\begin{align*}
\partial_\sigma X_L &= M_0 \partial_\tau X_R, \quad \sigma = 0 \\
\partial_\sigma X_L &= M_\pi \partial_\tau X_R, \quad \sigma = \pi, \quad M_\pi := \Lambda_\pi^{-1} M_0 \Lambda_\pi.
\end{align*}
\]

(2.19)

\[ ^4\text{We sometimes omit indices. Eq. (2.17) means}
\]

\[ \partial_\sigma \tilde{X}_L^{m}(\sigma^+) \bigg|_{\sigma=0, \pi} = M_0^{m} \partial_\sigma \tilde{X}_R^{m}(\sigma^-) \bigg|_{\sigma=0, \pi}. \]
2.2 mode expansions for general boundary conditions

We can treat general boundary condition with two Dp-branes as follows.

\[ \begin{align*}
\partial X_L(\tau) &= M \partial X_R(\tau) \quad \cdots \sigma = 0 \\
\partial X_L(\tau + \pi) &= \bar{M} \partial X_R(\tau - \pi) \quad \cdots \sigma = \pi
\end{align*} \tag{2.20} \]

where \( M \) and \( \bar{M} \) are any \( SO(D-1,1) \) matrices. In this section, we proceed in this general case for a while, and sometimes come back to our special case: two identical parallel Dp-branes’ scattering. Next, we have to perform mode expansions of the coordinate functions which satisfy the above boundary condition. The result is as follows.

\[ X(\tau, \sigma) = x + \sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \left[ M \int_{0}^{\sigma^+} + \int_{0}^{\sigma^-} d\rho e^{-i(m+iE)\rho} \alpha_m \right], \tag{2.21} \]

where

\[ E := \frac{1}{2\pi} \ln M^{-1}\bar{M}, \tag{2.22} \]

is restricted to be diagonalizable. \( x \) is the integration constant for \( \tau \) and \( \sigma \), which may depend on \( \alpha_m \)’s.

2.3 quantization

We perform the canonical quantization by applying the canonical commutation relations to the coordinates and their conjugate momenta. The only non-zero commutation relations are,

\[ [\partial_\tau X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] = -2\pi i \alpha' \delta(\sigma - \sigma') \eta_{\mu\nu}, \quad (0 < \sigma, \sigma' \leq \pi), \tag{2.23} \]

where

\[ \eta = \text{diag}(-1,1,\cdots,1). \tag{2.24} \]

We can get the commutation relations between modes from eq. (2.23). The results are

\[ [\alpha_m, \alpha_n] = (m + iE) \eta \delta_{m+n}, \tag{2.25} \]

\[ [x, \alpha_m] = i \sqrt{\alpha'/2} (1 + M) \eta, \tag{2.26} \]

\[ [x, x] = \frac{\pi i \alpha'}{2} (M - M^{-1}) \eta. \tag{2.27} \]

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5See Appendix A.

6For any Lorentz vectors \( a^\mu \) and \( b^\nu \), matrix \([a, b]\) is defined by its elements to be \([a^\mu, b^\nu]\).
Note that the component of the matrix $M$ in the pure Neumann or pure Dirichlet direction are 1 or $-1$, respectively. We can rewrite these relations as a more convenient form

$$\begin{align*}
\tilde{\alpha}_m, \tilde{\alpha}_n &= \begin{pmatrix} m + iE_1 & 0 \\ 0 & m \end{pmatrix} \eta_T \delta_{m+n} , \\
\tilde{x}, \tilde{\alpha}_m &= i\sqrt{\frac{\alpha'}{2}} \begin{pmatrix} 0 \\ N_{22} \end{pmatrix} \eta_T \delta_m , \\
\tilde{x}, \tilde{x} &= \frac{\pi i \alpha'}{2} \begin{pmatrix} -N_{11} \cot \eta_{T_{11}} & (ST)_{11}^1 - (ST)_{11} N_{11} \end{pmatrix} \eta_T \delta_{m+n} ,
\end{align*}$$

where

$$\begin{align*}
\tilde{\alpha}_m &= T^{-1} \alpha_m , \\
\tilde{x} &= Sx - \sqrt{\frac{\alpha'}{2}} \sum_{m \in \mathbb{Z}} C_m \tilde{\alpha}_m , \\
\eta_T &= T^{-1} \eta T^{-t} .
\end{align*}$$

In equations (2.28)-(2.33), matrices $T, S, N$ and $C_m$ are defined by $M$ and $E$ in the appendix B. In the case of two identical parallel Dp-brane scattering, the commutation relations are as follows

$$\begin{align*}
\tilde{\alpha}_m, \tilde{\alpha}_n &= \begin{pmatrix} 0 & m + ie \\ m - ie & 0 \end{pmatrix} \delta_{m+n} , \\
\tilde{x}, \tilde{p}^i &= i \delta^{ij} , \quad [\tilde{x}, \tilde{\alpha}_{m \neq 0}] = [\tilde{x}, \tilde{\alpha}_{m = 0}] = 0 , \\
\tilde{\alpha}_0^i &= \sqrt{2} \alpha \operatorname{diag}(\frac{1}{g}, 1 + m_2, \cdots, 1 + m_{p+1})^{ij} p^j , \\
g &= \sqrt{1 - f_i^2 (1 - v^2)} , \quad i, j = 2, \cdots, p + 1 ,
\end{align*}$$

where $v$ is the velocity of the D-branes in the $(p+1)$-th direction and $m_2, \cdots, m_{p+2}$ are the $2 \times 2$ matrices in the following form.

$$\begin{align*}
m_1 &= \begin{pmatrix} 1 + f_i^2 & 2f_i \\ 2f_i & 1 - f_i^2 \end{pmatrix} , \\
m_l &= \begin{pmatrix} 1 - f_i^2 & -2f_i \\ 2f_i & 1 + f_i^2 \end{pmatrix} , \quad l = 2, \cdots, p + 1
\end{align*}$$

See Appendix B. We would like to have complete pairs of canonical variables and their conjugate momenta with one to one correspondence, by transforming $(x, \alpha_m) \rightarrow (\tilde{x}, \tilde{\alpha}_m)$.

In this section and appendix B, we write any matrices $A$ as:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} ,$$

where the dimension of the square matrix $A_{11}$ is defined to be equal to the rank of $E$.

$t$ means transpose.
In eq. (2.35) $f_1, \cdots, f_{p+1}$ are the components of the U(1) gauge field strength:

$$F_{\alpha\beta} := \begin{pmatrix}
0 & f_1 & \cdots & 0 \\
-f_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & f_{p+1} & \cdots & 0
\end{pmatrix}_{\alpha\beta}, \quad (2.36)$$

defined in the proper flame of branes, as in the subsection 2.1.

### 2.4 Definition of the vacuum

We consider two identical parallel D$p$-branes scattering here. To define normal ordered operators, for instance, Virasoro operators, we identify the creation and the annihilation operators with respect to the vacuum. Since $\epsilon \neq 0$, the commutation relations for the modes include the following unusual relation as compared with the case of pure Neumann or pure Dirichlet.

$$[\hat{a}_0^\mu, \hat{a}_1^\mu] = i\epsilon. \quad (2.37)$$

Therefore we classify the modes as

$$\begin{cases}
\hat{a}_m^\mu_{>0} , \quad \hat{a}_0^1 & \cdots \text{annihilation operators} \\
\hat{a}_m^\mu_{<0} , \quad \hat{a}_0^0 & \cdots \text{creation operators}
\end{cases} \quad (2.38)$$

That is, the vacuum $|0\rangle$ is defined as

$$\hat{a}_0^1|0\rangle = \langle 0|\hat{a}_0^1 = \hat{a}_m^\mu_{>0}|0\rangle = \langle 0|\hat{a}_m^\mu_{<0} = 0. \quad (2.39)$$

Since exchange the role of $\hat{a}_0^0$ and $\hat{a}_0^1$ correspond to exchange the sign of the imaginary part of the Hamiltonian, we choose the role as above in order to the imaginary part to be negative. (Recall that $v$ is assumed to be positive.)

### 2.5 Virasoro algebra

We define the energy momentum tensor as

$$T_{ab} := -4\pi\alpha' \frac{\delta S_{\text{bosonic}}}{\delta h^{ab}}, \quad (2.40)$$

where $S_{\text{bosonic}}$ is the action in eq. (2.1) before the conformal gauge is chosen.

We introduce the holomorphic energy momentum tensor,

$$T(z) := \frac{1}{2\alpha' z^2} (T_{00} + T_{01}), \quad (2.41)$$

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where
\[ z := e^{i \sigma^\pm}. \] (2.42)

If we define
\[ \tilde{T}(z) := T(z) - \frac{i e(1 + i e)}{2z^2}, \] (2.43)
then
\[ \tilde{T}(z) \tilde{T}(w) = D \frac{1}{(z - w)^4} + 2 \tilde{T}(w) \frac{\partial \tilde{T}(w)}{z - w} + \text{regular} \] (2.44)
This is the same form with the simpler boundary conditions.

Virasoro operators are defined by
\[ \tilde{L}_m := \int_{|z|=1} \frac{dz}{2\pi i} z^{m+1} \tilde{T}(z). \] (2.45)

Now we can find the Virasoro algebra:
\[ [\tilde{L}_m, \tilde{L}_n] = \int \frac{dw}{2\pi i} w^{m+n+1} \text{Res} \frac{z^{m+1} \tilde{T}(z) \tilde{T}(w)}{z = w} = (m - n) \tilde{L}_{m+n} + \frac{D}{12} m(m^2 - 1) \delta_{m+n}. \] (2.46)

Note that the physical state condition by \( L_0 \) is different from the usual one, namely
\[ \left( \tilde{L}_0 - c \right) |\text{phys}\rangle = \left( \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_{-n}^\dagger \eta \alpha_n : - \frac{i e(1 + i e)}{2} - c \right) |\text{phys}\rangle = 0, \] (2.47)
where \( : \) means the normal ordering defined in the previous subsection and \( c \) is the intercept, which is 1 in the bosonic open string theory. Now we can calculate the one loop scattering amplitude for bosonic string. The formula is \( [10] \)
\[ A_{\text{bosonic}} = - \ln \det^{-\frac{1}{2}} \left( \tilde{L}_0 - c \right) = \frac{1}{2} \text{Tr} \ln \left( \tilde{L}_0 - c \right) = - \frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Tr} e^{-\pi t (\tilde{L}_0 - c)}. \] (2.48)

However we do not calculate this and we consider the superstring case in the next section.

\[ ^{10}\text{See Appendix C.} \]
3 Dp-brane-Dp-brane scattering in superstring theory

In this section we consider supersymmetric case. Let us introduce super partners for dynamical variables $X$ in the previous section. We choose RNS formalism in this paper and introduce fermion variable $\psi$ in the world sheet. In the super conformal gauge, the total action is

$$S_{\text{string}} = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left[ \partial_\mu X_\mu \partial^\alpha X_\mu - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right] + \frac{1}{4\pi\alpha'} \int d\tau F_{\mu\nu}^\psi \left[ X_\nu \partial_\tau X_\mu - \frac{i}{2} \bar{\psi}_\nu \rho_0^0 \psi_\mu \right] \bigg|_{\sigma=0}^{\sigma=\pi} + (\text{ghost}),$$

where

$$F_{\mu\nu}^\psi = \Lambda_{\pm \nu}^\rho \Lambda_{\pm \nu}^\sigma \left( \begin{array}{cc} F & 0 \\ 0 & 0 \end{array} \right)_{\rho\sigma},$$

where $\pm$ correspond to $\sigma = 0$ and $\sigma = \pi$, respectively and

$$\Lambda_{\nu}^\rho = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & 0 \\ \frac{v}{\sqrt{1-v^2}} & 1 \end{pmatrix}.$$

In eq. (3.2), $F$ defined in eq. (2.36) is a field strength on D-branes world volume and is set to be constant. Since we are considering identical branes, $F$ is the same in each brane. $\Lambda_{\nu}$ is a Lorentz boost matrix with the velocity $v$ in the $(p+1)$-th direction.

Since the spinor is written as follows

$$\psi = \left( \begin{array}{c} \psi_R \\ \psi_L \end{array} \right),$$

the world sheet global supersymmetry is expressed as

$$\delta_{\text{susy}} X_\mu = i \epsilon_{\text{susy}} (\psi_L^\sigma + \psi_R^\sigma)^\mu,$$

$$\delta_{\text{susy}} \psi_{L,R}^{\sigma \mu} = -2 \epsilon_{\text{susy}} e^{i(1-\alpha)\sigma \pm} \partial_{\pm} X_\mu,$$

Our convention of two dimensional Dirac matrix is

$$\rho_0 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \rho_1 = \left( \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right).$$
where $\epsilon_{\text{susy}}$ is independent of $\tau, \sigma$ and
\[
\psi_a^\alpha := e^{\frac{1 - a}{2} \sigma} \cdot \psi_L \quad \{ \begin{array}{l} a = 0 \quad \text{NS sector} \\ a = 1 \quad \text{R sector} \end{array} . \tag{3.6}
\]
Due to the existence of the boundary (i.e. D-brane), there is only one parameter for supersymmetry.

### 3.1 boundary condition

As in bosonic case, in order to get free equations of motion, we require
\[
\delta X \left( \eta \partial^1 X - F^v \partial_0 X \right) + \frac{i}{2} \delta \bar{\psi} \left( \eta \rho^1 \psi + F^v \rho^0 \psi \right) = 0 , \tag{3.7}
\]
or represent equivalently as
\[
\delta X \left[ (\eta + F^v) \partial X_L - (\eta - F^v) \partial X_R \right] = 0 , \tag{3.8}
\]
\[
\delta \bar{\psi}_L (\eta + F^v) \psi_L - \delta \bar{\psi}_R (\eta - F^v) \psi_R = 0 . \tag{3.9}
\]
In the proper time coordinate frame of D-branes, we impose the following boundary conditions for bosons and fermions at the both ends.

\[
\begin{cases}
(\eta + F^v) \partial X_L = (\eta - F^v) \partial X_R & \text{Neumann direction} \\
(\eta + F^v) \bar{\psi}_L = (\eta - F^v) \bar{\psi}_R & \\
\partial \tilde{X}_L = - \partial \tilde{X}_R & \text{Dirichlet direction} \tag{3.10}
\end{cases}
\]

In the original frame, that is
\[
\begin{cases}
\partial X_L = M_v \partial X_R \\
\psi_L = M_v \psi_R \\
\partial X_L = M_{-v} \partial X_R \\
\psi_L = (\sigma) = (\sigma = \pi) \tag{3.11}
\end{cases}
\]

---

12We determine boundary conditions for fermions as follows. If the boundary condition for bosons is
\[
(\partial_+ M \partial_-) X = 0 ,
\]
from eq. (3.8) as in the section 2, then we transform this by the supersymmetry and get the following equation.
\[
(\partial_+ M \partial_-) (\psi_L^\alpha + \psi_R^\alpha) = 0 .
\]
By using the equations of motion for fermions: $\partial_\pm \psi_L^\alpha = 0$, we get $\partial_\tau \left( \psi_L^\alpha - M \psi_R^\alpha \right) = 0$.
Then we impose the following boundary condition for fermions.
\[
\psi_L^\alpha = M \psi_R^\alpha .
\]
Eq. (3.9) determines the relation between $\delta \psi_L$ and $\delta \psi_R$.  

---
In eq. (3.11),

\[ M_v := \Lambda_v^{-1} M_0 \Lambda_v , \quad (3.12) \]

\[
M_0 := \begin{pmatrix}
  m_1 & 0 \\
  \vdots & \ddots \\
  0 & m_{\frac{p+1}{2}} & -1
\end{pmatrix}
\]

(3.13)

where \( m_1, \ldots, m_{\frac{p+1}{2}} \) are 2 \( \times \) 2 matrices which are defined in the previous section.

### 3.2 mode expansion

We can find the mode expansions for \( X \) and \( \psi \) to satisfy the above boundary condition as follows.

\[
X(\tau, \sigma) = x + \sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \left[ M_v \int_0^{\sigma^+} + \int_0^{\sigma^-} \right] \rho e^{-i(m+iE)\rho} \alpha_m , \quad (3.14)
\]

\[
\psi_L(\sigma^+) = \sqrt{\alpha'} M_v \sum_{r \in \mathbb{Z} - \frac{1-a}{2}} e^{-i(r+iE)\sigma^+} d_r , \quad (3.15)
\]

\[
\psi_R(\sigma^+) = \sqrt{\alpha'} \sum_{r \in \mathbb{Z} - \frac{1-a}{2}} e^{-i(r+iE)\sigma^-} d_r , \quad (3.16)
\]

where

\[
E := \frac{1}{2\pi} \ln M_v^{-1} M_{-v} , \quad (3.17)
\]

which is diagonalizable.

### 3.3 quantization

We can find that the commutation relations for fermionic modes \( d \) are the usual type:

\[
\{ d_r, d_s \} = \eta \delta_{r+s} , \quad (3.18)
\]

and bosonic commutators is as in the previous section:

\[
[\alpha_m, \alpha_n] = (m + iE) \eta \delta_{m+n} \\
[x, \alpha_m] = i \sqrt{\frac{2\alpha'}{\pi}} (1 + M_v) \eta \\
[x, x] = \frac{\alpha'}{2} (M_v - M_{-v}^{-1}) \eta
\]

(3.19)

Eq. (3.19) is rewritten as eq. (2.34). The ghost part is the same as usual because the boundary condition does not change it.
3.4 Virasoro algebra

The energy momentum tensor is

\[ T(z) := \frac{1}{\alpha'z^2} \left[ \partial_- X^\mu \partial_- X_\mu + \frac{i}{2} \psi_R^\mu \partial_- \psi_R^\mu \right] + \text{(ghost)} \]

\[ = T_{\text{boson}}(z) + T_{\text{fermion}}(z) + T_{\text{ghost}}(z) \]

\[ := \frac{1}{2} \sum_{m,n \in \mathbb{Z}} z^{-m-n-2} : \alpha_m^T \eta^T_n : \]

\[ + \frac{1}{2} \sum_{r,s \in \mathbb{Z}} z^{-r-s-2} : d_T^r \eta(s + iE)d_s : + \text{ (ghost)} . \]  

(3.20)

If we define a shifted energy momentum tensor as

\[ \tilde{T}(z) := T(z) - \frac{i\epsilon (1 + i\epsilon)}{2z^2} + \frac{aD - 8\epsilon^2}{16z^2} \]

\[ = T(z) + \frac{1}{z^2} \left( \frac{aD}{16} - \frac{i\epsilon}{2} \right) , \]  

(3.21)

then we get

\[ \tilde{T}(z) \tilde{T}(w) = \frac{\frac{3}{4}D}{(z-w)^{4}} + \frac{2T(w)}{(z-w)^{2}} + \frac{\partial_w T(w)}{z-w} + \text{ (regular + ghost contribution)} . \]  

(3.22)

This form is of usual type.

3.4.1 physical state condition

Virasoro operators are defined by

\[ L_m := \int_{|z|=1} \frac{dz}{2\pi i} z^{m+1} T(z) . \]  

(3.23)

Note that the physical state condition by \( L_0 \) is different from the usual one because energy momentum tensor \( \tilde{T} \) which satisfies the correct operator product expansions is shifted from \( T \) as eq. (3.21). That is,

\[ \left( L_0 - \frac{i\epsilon}{2} - \frac{1-a}{2} \right) |\text{phys} \rangle = 0 . \]  

(3.24)

\[^{13}\text{See appendix C.}\]
We further define
\[ L^a_0 := L_0 - \frac{i\epsilon}{2} - \frac{1-a}{2} \]
\[ = L^\alpha_0 + L^d_0 + L^{bc}_0 + L^{\beta\gamma}_0 - \frac{i\epsilon}{2} + \frac{1-a}{2}, \tag{3.25} \]
\[ L^\alpha_0 := \frac{1}{2} \sum_{n\in\mathbb{Z}} :\alpha - n\eta\alpha_n :, \tag{3.26} \]
\[ L^d_0 := \sum_{r=1}^{\infty} d^r_\eta(r + iE)d_r + \frac{a}{2} d_0^r \eta E d_0, \tag{3.27} \]
\[ L^{bc}_0 := \sum_{n=1}^{\infty} n (b_{-n}c_n + c_{-n}b_n), \tag{3.28} \]
\[ L^{\beta\gamma}_0 := \sum_{r=1}^{\infty} r (\beta_{-r}\gamma_r - \gamma_{-r}\beta_r), \tag{3.29} \]

where \( b, c, \beta, \) and \( \gamma \) are the standard ghosts. The normal ordering procedure in eq. (3.26), is found by eq. (2.38).

### 3.5 scattering amplitude

We calculate the scattering amplitude for two Dp-branes, namely, the one loop amplitude of open string whose ends are attached on the Dp-branes. The one loop amplitude can be expressed by the following proper time \( (t) \) integral of the exponentiated Hamiltonian, which annihilate the physical state. It is gotten in the previous subsection, that is \( L_0 \) with the correct constant shift. The shift can be found by imposing the physical state condition in the previous subsection. Therefore we use \( L_a^0 \) as Hamiltonian. The integral region is from 0 to \( \infty \) because of cylinder \( [9] \). Since we are choosing the NSR formalism, we have to sum over the spin structures which have \( 2 \times 2 = 4 \) combinations because of the cylinder amplitude. The formula of the amplitude is \( [10] \)

\[ A_{\text{string}} = -\ln \det e^{-tL^a_0} = \frac{1}{2} \text{Tr} \ln L^a_0 = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Tr} e^{-\pi tL^a_0} \]
\[ = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \sum_{a,b=0,1} C^a_b \text{Tr}_{\text{NSR}} (-)^b F e^{-\pi tL^a_0}, \tag{3.30} \]

where \( F = F^a + F^{\beta\gamma} \) is the fermion number operator, the summation is done over the spin structures which consist of GSO projection \( (b=0,1) \) and the summation of NS \( (a=0) \) and R \( (a=1) \) sectors. \( C^a_b \) are their weights:

\[ C^0_0 = -C^0_1 = -C^1_0 = \frac{1}{2}. \tag{3.31} \]
"Tr" means
\[ \text{Tr}_{\text{NSR}} = 2 \times \text{Tr} \times \text{Tr} \times \text{Tr} \times \text{Tr} \],
(3.32)
where the factor 2 is caused by the sum of the configurations of the exchanged ends of open string. Therefore
\[
\text{Tr}_{\text{NSR}} (-)^{bF} q^{L_0^a} = 2 \times q^{-\frac{bF}{2}} \times \text{Tr}_{\alpha} q^{L_0^a} \times \text{Tr}_{b,c} q^{L_0^b} \times \text{Tr}_{d}(-)^{bF} q^{L_0^d} \times \text{Tr}_{\beta,\gamma} (-)^{bF} q^{L_0^{\beta,\gamma}},
\]
(3.33)
where
\[ q := e^{-\pi t}. \]
(3.34)

We calculate these traces as follows.
\[
\text{Tr}_{\alpha} q^{L_0^a} = q^{\sum_{i=p+2}^{D-1} a_i^2} \times \text{Tr}_{p^i} q^{\sum_{i=p+2}^{D-1} a_i^2} \times \text{Tr}_{q^{L_0^a}} \mid \text{frequency term}
\]
\[ = q^{\frac{bF}{2}} \times V_p \int_{-\infty}^{\infty} \prod_{i=2}^{p+1} \frac{dp_i}{2\pi} q^{\frac{1}{2} a_i^2} \times (1 - q^{-i})^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-(D-2)} \mid 1 - q^{-i} \mid^{-2}, \]
(3.35)
\[
\text{Tr}_{b,c} q^{L_0^b} = \prod_{n=1}^{\infty} (1 - q^n)^2, \]
(3.36)
\[
\text{Tr}_{d}(-)^{bF} q^{L_0^d} = (1 - ab) \left( 2^4 \cos \frac{\pi t \epsilon}{2} \right)^a \times \prod_{n=1}^{\infty} \left( 1 + (-)^b q^{n-\frac{1-a}{2}} \right)^{-(D-2)} \mid 1 + (-)^b q^{n-\frac{1-a}{2}} \mid^{-2}, \]
(3.37)
\[
\text{Tr}_{\beta,\gamma} (-)^{bF} q^{L_0^{\beta,\gamma}} = \prod_{n=1}^{\infty} \left( 1 + (-)^b q^{n-\frac{1-a}{2}} \right)^{-2}. \]
(3.38)

In the above equations, the zero mode integral can be done by using the relation between \( p^i \) and \( \tilde{a}_i \) obtained in the previous section. The result can be expressed as follows by using Jacobi’s theta functions: \( \theta_{(a)} \).

\[ \text{Tr}_{\alpha,b,c} q^{(L_0^a + L_0^b + L_0^c)} = V_p g \prod_{l=2}^{\frac{bF}{2}} (1 + f_l^2) (4\pi^2 \alpha' t)^{-\frac{bF}{2}} e^{-\frac{bF}{4\pi^2 \alpha' t}} \]

\(^{14}V_p \) is the space volume of a Dp-brane. See appendix B and see also Ref. 9.

\(^{15}\)The definition and some properties are in an appendix in Ref. 11.

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\[
\text{Tr}_{d,\beta,\gamma} (-)^F q^{(L_0^d + L_0^\gamma)} = \frac{\theta \left( \frac{1}{2} \right) \theta \left( \frac{1}{4} \right)^3 \theta \left( \frac{1}{8} \right)^3}{q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n)^4}.
\]

Therefore we can find the scattering amplitude eq. (3.30) as

\[
A_{\text{string}} = -i(2\pi)^3 V_p g \prod_{l=2}^{p+1} (1 + f_l^2) \cdot \int_0^\infty \frac{dt}{t} (4\pi^2 \alpha' t)^{-\frac{1}{2}} e^{-\frac{\pi^2 \alpha' t}{4 \pi \omega}} \times \sum_{a,b=0,1} C \left( a \right) \theta \left( \frac{a}{b} \right) \left( 0 \mid \frac{1}{2} \right)^3 \theta \left( \frac{a}{b} \right) \left( \frac{t}{2} \mid \frac{it}{2} \right) = \theta \left( \frac{1}{4} \right) \left( \frac{t}{4} \mid \frac{it}{2} \right)^4.
\]

Using Jacobi identity:

\[
\sum_{a,b=0,1} C \left( a \right) \theta \left( \frac{a}{b} \right) \left( 0 \mid \frac{1}{2} \right)^3 \theta \left( \frac{a}{b} \right) \left( \frac{t}{2} \mid \frac{it}{2} \right) = \theta \left( \frac{1}{4} \right) \left( \frac{t}{4} \mid \frac{it}{2} \right)^4,
\]

and the property of the theta function under the modular transformation:

\[
\theta \left( \frac{b}{a} \right) \left( \frac{2\pi i}{2} \right) = \sqrt{2} e^{\frac{\pi i a b + \pi i a}{2}} \theta \left( \frac{1}{2} \mid \frac{2\pi i}{2} \right),
\]

\[
\theta \left( \frac{1}{4} \right) \left( 0 \mid \frac{2\pi i}{2} \right) = \frac{1}{\sqrt{2}} \theta \left( \frac{1}{4} \right) \left( 0 \mid \frac{2\pi i}{2} \right)
\]

we can further simplify the amplitude eq. (3.41) as

\[
A_{\text{string}} = V_p g \prod_{l=2}^{p+1} (1 + f_l^2) \cdot \int_0^\infty \frac{dt}{t} (4\pi^2 \alpha' t)^{-\frac{1}{2}} e^{-\frac{\pi^2 \alpha' t}{4 \pi \omega}} \times \frac{(\pi t)^3}{\theta \left( \frac{1}{4} \right) \left( 0 \mid \frac{2\pi i}{2} \right)^3 \theta \left( \frac{1}{4} \right) \left( \frac{t}{4} \mid \frac{it}{2} \right)^4}.
\]
4 Dp-brane-Dp-brane scattering in IIB matrix model

In this section, with our notation, we briefly review the calculation of the brane scattering amplitude in the IIB matrix model \[3\]. In this model, dynamical variables are \(^{\text{coordinates}}\) represented by \(N \times N\) matrices and \(N\) is assumed to be infinitely large.

The action is

\[ S_{\text{matrix}} = \alpha \Tr \left( -\frac{1}{4} [X_\mu, X_\nu]^2 - \frac{1}{2} \bar{\psi} \Gamma^\mu [X_\mu, \psi] \right) + \beta N. \tag{4.1} \]

The classical equations of motions for \(X^\mu\) from this action are

\[ [X^\mu, [X_\mu, X_\nu]] = 0, \tag{4.2} \]

which has the solutions that may be representing D-branes. In our situation: two identical parallel \(p\)-branes moving with relative velocity \(2v\), the following classical configuration is suitable \[3\].

\[ X^\mu = \begin{pmatrix} X^\mu_{(1)} & 0 \\ 0 & X^\mu_{(2)} \end{pmatrix}, \quad \psi = 0, \tag{4.3} \]

where \(X_{(I)}\) are \( \frac{N}{2} \times \frac{N}{2} \) matrices and their indices \(I = 1, 2\) mean the first and second branes.

First, in the proper time frame (written as \(\tilde{X}\)) for each brane, we set

\[ \left[ \tilde{X}^\alpha_{(I)}, \tilde{X}^\beta_{(I)} \right] = \frac{2\pi i \alpha'}{F^M_{\alpha \beta}} 1_{\frac{N}{2} \times \frac{N}{2}}, \quad \text{all other commutator} = 0, \tag{4.4} \]

\[ F^M = \begin{pmatrix} 0 & f^M_{l l'} & \cdots & f^M_{l_{p+1} l_{p+1}} \\ -f^M_{l l'} & 0 & \cdots & \vdots \\ \vdots & \cdots & 0 & f^M_{l_{p+1} l_{p+1}} \\ -f^M_{l_{p+1} l_{p+1}} & -f^M_{l_{p+1} l_{p+1}} & \cdots & 0 \end{pmatrix}, \quad \det F^M \neq 0, \tag{4.5} \]

in order to describe \(p\)-branes.

Since we are considering identical branes, \(F^M\) is independent of \(I = 1, 2\). The right hand side of eq. (4.4) is proportional to the identical matrix. To satisfy the eq. (4.4), we use \(\frac{p+1}{2}\) pairs of matrices \((q_l, p_l)\) with large enough dimensions.

\[
\begin{align*}
\tilde{X}^{2l-2}_{(I)} &= \sqrt{2\pi \alpha' f^M_{l l}} q_l \\
\tilde{X}^{2l-1}_{(I)} &= \sqrt{2\pi \alpha' f^M_{l_{p+1} l_{p+1}}} p_l \\
\tilde{X}^{p+1}_{(I)} &= \cdots = \tilde{X}^{D-2}_{(I)} = 0 \\
\tilde{X}^{D-1}_{(I)} &= \pm \frac{b}{2},
\end{align*}
\tag{4.6}
\]
Next, for the first and second branes, we perform the boost with velocity $v$ and $-v$ along the $(p+1)$-th axis, respectively as
\[
\begin{pmatrix}
X_0^{(l)} \\
X_{p+1}^{(l)}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{1-v^2}} & \frac{0}{\sqrt{1-v^2}} \\
\frac{0}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}}
\end{pmatrix}
\begin{pmatrix}
\tilde{X}_0^{(l)} \\
\tilde{X}_{p+1}^{(l)}
\end{pmatrix},
\]
where $\pm$ correspond to $I = 1, 2$. We can get the following solution:
\[
\begin{cases}
X^0 = \sqrt{2\pi\alpha'} f_1^M \left( q_0 \frac{1}{\sqrt{1-v^2}} 0 \right) \\
X^{2l-2} = \sqrt{2\pi\alpha'} f_1^M \left( q_l 0 \right) \quad l = 2, \ldots, \frac{p+1}{2} \\
X^{2l-1} = \sqrt{2\pi\alpha'} f_1^M \left( p_l 0 \right) \quad l = 1, \ldots, \frac{p+1}{2} \\
X^{p+1} = \sqrt{2\pi\alpha'} f_1^M \left( q_0 \frac{v}{\sqrt{1-v^2}} 0 \right) \\
X^{D-1} = \left( \frac{b}{2} 0 0 \right)
\end{cases}
\]
where $2v$ is the relative velocity between the branes.

The result of the calculation for the effective action (scattering amplitude) in the one loop level is [3]
\[
A_{\text{matrix}} = -i8V_1 \prod_{l=2}^{p+1} f_1^{M-2} \cdot \int_0^\infty \frac{dt}{t} (4\pi^2\alpha')^2 e^{-\frac{t^2}{4\pi^2 t}}
\times \frac{1}{f_1^M \cosh \epsilon^M \sin (f_1^M \sinh \epsilon^M t)}
\]

where
\[
\tanh \epsilon^M := v,
\]
and the fact that the matrix size $N$ is large enough is used.
5 Comparing string theory and matrix model

We compare the two scattering amplitudes in the IIB string theory and the IIB matrix model, respectively. They can be rewritten as follows:

\[ A_{\text{string}} = -\frac{i}{2} V_p \cdot \prod_{l=2}^{\frac{2n+4}{2}} \left( f_l^2 + 1 \right) \cdot \frac{v^3}{1 - v^2} \cdot \int_0^\infty \frac{dt}{t} \left( \frac{4\pi^2\alpha'}{t} \right)^{\frac{5}{2}} e^{-\frac{\sinh^2 \epsilon}{4\pi\alpha'}} \]
\[ \times \prod_{n=1}^{\infty} \left[ 1 - \left( \frac{\sinh \frac{\pi}{4} n^2 \epsilon}{\sinh \frac{\pi}{4} \epsilon} \right)^2 \right]^4, \quad \sinh \pi \epsilon = \frac{2\sqrt{1 - f^2(1 - v^2)}}{1 - v^2} (1 - f^2_l). \quad (5.1) \]

\[ A_{\text{matrix}} = -\frac{i}{2} V_p \cdot \prod_{l=2}^{\frac{2n+4}{2}} f_l^{M^2} \cdot \frac{v^3}{1 - v^2} \cdot \int_0^\infty \frac{dt}{t} \left( \frac{4\pi^2\alpha'}{t} \right)^{\frac{5}{2}} e^{-\frac{\sinh^2 \epsilon}{4\pi\alpha'}} \]
\[ \times \prod_{n=1}^{\infty} \left[ 1 - \left( \frac{f^M_l \sinh \epsilon^M}{\frac{\epsilon}{4}} \right)^2 \right]^4, \quad \sinh \epsilon^M = \frac{v}{\sqrt{1 - v^2}}. \quad (5.2) \]

Here we compare the imaginary parts of the phase shifts; \( \text{Im}(-iA) \) as:

\[ \text{Im}(-iA_{\text{string}}) = 8V_p \sum_{k=1,3,5,\ldots} \frac{1}{k} \left( \frac{\epsilon}{4\pi^2\alpha' k} \right)^{\frac{5}{2}} e^{-\frac{\epsilon^2}{4\pi\alpha'} \frac{2k}{4}} \]
\[ \times \prod_{l=2}^{\frac{2n+4}{2}} (1 + f_l^2) \cdot g \prod_{n=1}^{\infty} \coth \left( \frac{\pi kn}{\epsilon} \right). \quad (5.3) \]

\[ \text{Im}(-iA_{\text{matrix}}) = 8V_p \sum_{k=1,3,5,\ldots} \frac{1}{k} \left( \frac{f^M_l \sinh \epsilon^M}{\frac{\epsilon}{4}} \right)^{\frac{5}{2}} e^{-\frac{\epsilon^2}{4\pi\alpha'} \frac{2k}{4}} \]
\[ \times \prod_{l=2}^{\frac{2n+4}{2}} f_l^{M^2} \cdot \frac{\sqrt{1 - v^2}}{f_l^M}. \quad (5.4) \]
5.1 three special limits

Now, if we assume a relation between the $U(1)$ field strength $F$ in string theory and the value of commutator $F^M$ among coordinates in IIB matrix model as follows:

$$\prod_{l=2}^{p+1} \frac{(f^2_l + 1)}{1 - f^2_l} \prod_{l=2}^{p+1} \frac{f^M_l}{f^M_1} = 1 ,$$ \hspace{1cm} (5.5)

considering three different regions of parameters which are $b$, $v$ and $f_1$ as:

$$\begin{cases} b \gg \alpha' \quad \text{(long distance)} \\ v \ll 1 \quad \text{(low velocity)} \\ |f_1| \gg 1 \quad \text{(large field strength)} \end{cases} \hspace{1cm} (5.6)$$

then, in each region, the both amplitudes approach the same value as follows:

$$A_{\text{string matrix}} \rightarrow -\frac{i}{2} \Gamma \left(3 - \frac{p}{2}\right) \cdot \frac{V_p}{(4\pi^2\alpha')^{\frac{p}{2}}} \cdot \prod_{l=2}^{p+1} \frac{(f^2_l + 1)}{1 - f^2_l} \cdot \frac{v^3}{1 - v^2} \cdot \left(\frac{4\pi\alpha'}{b^2}\right)^{\frac{3-p}{2}}.$$ \hspace{1cm} (5.7)

In addition, in the case of $v \ll 1$ or $|f_1| \gg 1$ the imaginary parts of the phase shifts; eq. (5.3) and eq. (5.4), are also identical.

The main conclusion of this paper is that eq. (5.1) and eq. (5.2) are identical in three independent limit of parameters after identifying the field strengths in the models as eq (5.5). This condition is satisfied when we require

$$\det (\eta + F^M) = -1 .$$ \hspace{1cm} (5.8)

\footnote{We put $f_1$ pure imaginary which has the large absolute value, that is small real $f^M_1$}
6 Conclusions and Discussions

In this paper, we have considered two identical p-branes and compared their scattering amplitudes in the IIB string theory and the IIB matrix model. Here the p-brane has the lower brane charges in the various ways. That is, in the context of the string theory, there are arbitrary constant electromagnetic field strength $F$ on the p-brane world volume. On the other hand, in the context of the matrix model, there are various values ($F^M$) of the commutators among brane coordinates. Based on the matrix model, we should calculate the amplitude around the classical back-ground of the two body system of p-branes. However we have only the amplitude in the one loop level. The loop expansion is assumed to be the low energy expansion. Therefore the corresponding value based on the string theory is a cylinder diagram in the low energy limit, which is attached on the nontrivial background. It means the one loop calculation of open string with the corresponding boundary condition. After we have performed the low energy expansion that is the long distance expansion or small velocity expansion in each model, or small $f_1^M$ expansion in IIB matrix model and large $|f_1|$ expansion in IIB string theory, the leading terms of the amplitudes are found to be the same precisely if we identify $\det (\eta + F)F^M = -1$. When all $f_1^M$s are the same and $f_2 = 0$ in the all world volume directions, our result reproduces that in Ref. [3]. When $f_1^M \ll 1$, our result reproduces that in Ref. [5, 12]. Note that we have performed these expansions of three parameters independently. In this paper, we have obtained a clear correspondence between the IIB string theory and its IIB matrix model in the near BPS configuration: D-brane scattering. Since we also find a way to study general boundary conditions we can calculate various configurations of two body systems of branes. For example, D$p$-brane and D$q$-brane case for $p \neq q$ with different lower dimensional brane charges and with arbitrary angles. We can also consider the relation between IIA string theory and matrix model for M-theory [13]. Our next task is to clarify the correspondence for objects which are far from BPS. If there is no agreement between string theory and matrix model for such cases, we may need to look for a better (matrix) model.
Appendix A

In this appendix, we get commutation relations between modes’ \( \alpha_m \)'s and \( x \), namely, equations (2.25)-(2.27). The mode expansion for the coordinate shown in eq. (2.21) is

\[
X(\tau, \sigma) = x + M \tilde{X}^+ (\sigma^+) \quad , \\
\tilde{X}(\rho) := \sqrt{\frac{\alpha^\prime}{2}} \sum_{m \in \mathbb{Z}} \int_0^\rho d\rho e^{-i(m+iE)\rho} \alpha_m . \tag{A.1}
\]

To use eq. (A.1) not only in the region \((0 \leq \sigma \leq \pi)\) but also in the region \((-\pi \leq \sigma \leq \pi)\), we define extended variables in the following.

\[
\left\{ \begin{array}{l}
\partial X_L(\rho) := M \partial X_R(\rho) \quad \tau - \pi \leq \rho \leq \tau \\
\partial X_R(\rho) := M^{-1} \partial X_L(\rho) \quad \tau \leq \rho \leq \tau + \pi
\end{array} \right. \tag{A.2}
\]

Then we get useful commutation relations as follows.

\[
\left[ \partial X(\rho), \partial \tilde{X}(\rho') \right] = \pi i \alpha \delta^\prime (\rho - \rho') \eta , \quad \tau - \pi \leq \rho, \rho' \leq \tau + \pi . \tag{A.3}
\]

A.1 \( [\alpha_m, \alpha_n] \)

Since we can write

\[
\alpha_m = \frac{1}{2\pi} \sqrt{\frac{2}{\alpha'}} \int_{\tau-\pi}^{\tau+\pi} d\rho e^{i(m+iE)\rho} \partial \tilde{X}(\rho) , \tag{A.4}
\]

we get the commutation relation between \( \alpha_m \)'s as follows.

\[
[\alpha_m, \alpha_n] = \frac{1}{(2\pi)^2} \frac{2}{\alpha'} \int_{\tau-\pi}^{\tau+\pi} d\rho d\rho' \left[ e^{i(m+iE)\rho} \partial \tilde{X}(\rho), e^{i(n+iE)\rho} \partial \tilde{X}(\rho') \right] = \frac{1}{(2\pi)^2} \frac{2}{\alpha'} \int_{\tau-\pi}^{\tau+\pi} d\rho d\rho' e^{i(m+iE)\rho} \left[ \partial \tilde{X}(\rho), \partial \tilde{X}(\rho') \right] \left( e^{i(n+iE)\rho'} \right)^t \\
= \frac{i}{2\pi} \int_{\tau-\pi}^{\tau+\pi} d\rho d\rho' \delta^\prime (\rho - \rho') \left( e^{i(m+iE)\rho} \right)^t \left( e^{i(n+iE)\rho'} \right)^t = -\frac{i}{2\pi} \int_{\tau-\pi}^{\tau+\pi} d\rho d\rho' \delta (\rho - \rho') \left( \partial_\rho e^{i(m+iE)\rho} \right) \eta \left( e^{i(n+iE)\rho'} \right)^t + \frac{i}{2\pi} \int_{\tau-\pi}^{\tau+\pi} d\rho d\rho' \delta (\rho - \rho') e^{i(m+iE)\rho} \eta \left( e^{i(n+iE)\rho'} \right)^t \bigg|_{\rho = \tau + \pi}^{\rho = \tau - \pi} . \tag{A.5}
\]

Since we find that the last term in the eq. (A.5) is zero, the commutator is

\[
-\frac{i}{2\pi} \int_{\tau-\pi}^{\tau+\pi} d\rho \left( \partial_\rho e^{i(m+iE)\rho} \right) \eta \left( e^{i(n+iE)\rho'} \right)^t = \frac{m + iE}{2\pi} \int_{\tau-\pi}^{\tau+\pi} d\rho e^{i(m+iE)\rho} \eta \left( e^{i(n+iE)\rho} \right)^t . \tag{A.6}
\]
The integrand in eq. (A.6) is
\[ e^{i(m+n)\eta}, \] (A.7)
because of \( M \) and \( \bar{M} \) being \( SO(D-1,1) \) matrices. Therefore we get
\[ [\alpha_m, \alpha_n] = \delta_{m+n} (m + iE) \eta. \] (A.8)

**A.2 \([x, \alpha_n] \)**

We consider the following commutator
\[ [\partial_{\tau} X(\tau, \sigma), X(\tau, \sigma')] = [\partial_{\tau} X(\sigma^-), x] \]
\[ + M \left[ \partial_{\tau} X(\sigma^+), \bar{X}(\sigma^-) \right] M^t + \left[ \partial_{\bar{\sigma}} X(\sigma^-), \bar{X}(\sigma^-) \right] \]
\[ + M \left[ \partial_{\tau} X(\sigma^+), \bar{X}(\sigma^-) \right] + \left[ \partial_{\bar{\sigma}} X(\sigma^-), \bar{X}(\sigma^-) \right] M^t. \] (A.9)

Since
\[ [\partial_{\rho} \bar{X}(\rho), \bar{X}(\rho')] = \frac{\alpha'}{2} \sum_{m,n} e^{-i(m+n+1)\rho} [\alpha_m, \alpha_n] \int_0^{\rho'} d\tilde{\rho} \left( e^{-i(n+1)\tilde{\rho}} \right)^t \]
\[ = -\pi i \alpha' (\delta(\rho - \rho') - \delta(\rho)) \eta, \] (A.10)
we rewrite the commutator as
\[ [\partial_{\tau} X(\tau, \sigma), X(\tau, \sigma')] = [\partial_{\tau} X(\tau, \sigma), x] \]
\[ - 2\pi i \alpha' \delta(\sigma - \sigma') \eta - \pi i \alpha' (M - M^{-1}) \delta(\sigma + \sigma') \eta \]
\[ + \pi i \alpha' \left[ \delta(\sigma^+) (1 + M) + \delta(\sigma^-) (1 + M^{-1}) \right] \eta. \] (A.11)

Therefore
\[ [\partial_{\tau} X(\tau, \sigma), X(\tau, \sigma')] = -2\pi i \alpha' \delta(\sigma - \sigma') \eta, \quad 0 < \sigma, \sigma' \leq \pi, \] (A.12)
\[ [\partial_{\tau} X(\tau, \sigma), x] = -\pi i \alpha' \left[ \delta(\sigma^+) (1 + M) + \delta(\sigma^-) (1 + M^{-1}) \right] \eta, \] (A.13)
After integrating \{ \( e^{i(m+n+1)\rho} \times \text{eq. (A.13)} \) \} by \( \rho \) from \( \tau - \pi \) to \( \tau + \pi \), we get
\[ [x, \alpha_m] = i \sqrt{\frac{\alpha'}{2}} (1 + M) \eta. \] (A.14)
A.3 $[x, x]$

We consider the following commutator

\[
[X(\tau, \sigma), X(\tau, \sigma')] = [x, x] + 2iM \left[ \tilde{X}(\sigma^-), X(\sigma^-) \right] \eta M^t + 2iM \left[ \tilde{X}(\sigma^-), \tilde{X}(\sigma^-') \right] M^t + 2iM \left[ \tilde{X}(\sigma^-), \tilde{X}(\sigma^-) \right] M^t.
\]

(A.15)

We find

\[
[\bar{X}(\rho), \bar{X}(\rho')] = \frac{\alpha'}{2} \sum_{m,n} \int_0^\rho \int_0^{\rho'} d\tilde{\rho} e^{-i(m+iE)\tilde{\rho}} \left( e^{-i(n+iE)\tilde{\rho}} \right)^t
\]

\[
= \pi i \alpha' \left( \varepsilon(\rho' - \rho) + \varepsilon(\rho) - \varepsilon(\rho') \right) \eta,
\]

(A.16)

and

\[
[x, \tilde{X}(\rho)] = \pi i \alpha' (1 + M) \varepsilon(\rho) \eta,
\]

(A.17)

where

\[
\varepsilon(\rho) := \begin{cases} 
\frac{1}{2} & \rho > 0 \\
0 & \rho = 0 \\
-\frac{1}{2} & \rho < 0
\end{cases}.
\]

(A.18)

Therefore we get

\[
[X(\tau, \sigma), X(\tau, \sigma')] = [x, x] - i\pi \alpha' (M - M^{-1}) \varepsilon(\sigma + \sigma') \eta = 0, \quad 0 < \sigma, \sigma' \leq \pi,
\]

\[
[x, x] = \frac{\pi i \alpha'}{2} (M - M^{-1}) \eta.
\]

(A.19)

Appendix B

In this appendix, we rewrite the commutation relations between the modes in a more convenient form and apply the formula to our case. Namely, we derive equations (2.28)-(2.30). We recall equations (2.25)-(2.27):

\[
[\alpha_m, \alpha_{-m}] = \langle m + iE \rangle \eta
\]

\[
[x, \alpha_m] = i \sqrt{\frac{\alpha'}{2}} (1 + M) \eta
\]

\[
[x, x] = \frac{\pi i \alpha'}{2} (M - M^{-1}) \eta
\]

(B.1)
We would like to find the conjugate pairs for the modes. Our strategy is to transform the each right-hand side of equations (B.1) to become the following form, respectively.

\[
\begin{pmatrix}
* & 0 \\
0 & m \\
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 \\
0 & * \\
\end{pmatrix} \delta_m, \quad \begin{pmatrix}
* & 0 \\
0 & 0 \\
\end{pmatrix}.
\]  

(B.2)

**B.1 T**

First, we would like to diagonalize the matrix \( E \) by transforming \( \alpha_m \) by a matrix \( T \)

\[
\bar{\alpha}_m := T^{-1} \alpha_m,
\]

\[
E_D := T^{-1} E T := \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
\eta_T := T^{-1} \eta T^{-t}, \quad \text{Det} E_{11} \neq 0.
\]  

(B.3)

(B.4)

(B.5)

Then the commutation relations are

\[
[\bar{\alpha}_m, \bar{\alpha}_n] = (m + i E_D) \eta_T \delta_{m+n},
\]

\[
[x, \bar{\alpha}_m] = i \sqrt{\frac{\alpha'}{2}} (1 + M) T \eta_T.
\]  

(B.6)

(B.7)

**B.2 S, N**

Second, we would like to write the each side of the equation (B.7) as the following form by transforming \( x \)

\[
\begin{pmatrix}
* & 0 \\
* & * \\
\end{pmatrix},
\]

\[
y := S x.
\]  

(B.8)

How to determine the matrix \( S \) and \( N \) is as the following.

\[
S := K J, \quad K := \begin{pmatrix} 1 & -L_{12} L_{22}^{-1} \\ 0 & 1 \end{pmatrix},
\]

\[
L := J \tilde{L}, \quad \tilde{L} := (1 + M) T.
\]  

(B.9)

(B.10)

\( K \) is a matrix which transforms \( L_{12} \) to 0, if \( L_{22}^{-1} \) exists. \( J \) is a matrix which exchanges the rows of a matrix \( (\tilde{L}) \) so that \( L_{22}^{-1} \) becomes existing. That is,\footnote{The possibility is assumed here. If not so, we replace \( L_{22} \) by its part whose minor is not zero. In our case, we avoid pure Dirichlet directions: \( p + 2 \cdots D = 1 \).}
Det$L_{22} \neq 0$.

Then

$$N := S (1 + M) T = KL = \begin{pmatrix} L_{11} - L_{12} L_{22}^{-1} L_{21} & 0 \\ L_{21} & L_{22} \end{pmatrix} =: \begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}.$$  \tag{B.11}$$

Then the commutation relations are

$$[y, \tilde{\alpha}_m] = i \sqrt{\alpha'} N \tilde{\eta}, \quad \tag{B.12}$$

$$[y, y] = \frac{\pi i \alpha'}{2} \left( N \eta T (ST)^t - (ST) \eta T N^t \right). \tag{B.13}$$

**B.3 $C_m \cdots$ step 1**

Third, we would like to write the each side of the equation \textsuperscript{B.12} as the following form by transforming $y$.

$$\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \delta_m \cdot \quad \tag{B.14}$$

We write $y$ as follows:

$$\hat{x} := y - \sqrt{\alpha'} \sum_{m \in \mathbb{Z}} C_m \tilde{\alpha}_m. \tag{B.15}$$

Then

$$[\hat{x}, \tilde{\alpha}_m] = \sqrt{\alpha'} \left[ i \begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix} - C_m \begin{pmatrix} -m + i E_{11} & 0 \\ 0 & -m \end{pmatrix} \eta T \right]. \tag{B.16}$$

Therefore we chose matrix $C_m$ as

$$C_{m \neq 0} := i N (m + i E_D)^{-1}, \quad C_0 := \begin{pmatrix} N_{11} E_{11}^{-1} & c_{12} \\ N_{21} E_{11}^{-1} & c_{22} \end{pmatrix}, \tag{B.17}$$

where $c_{12}$ and $c_{22}$ will be defined in the next subsection.

Then we get

$$[\hat{x}, \tilde{\alpha}_m] = i \sqrt{\alpha'} \begin{pmatrix} 0 & 0 \\ 0 & N_{22} \end{pmatrix} \delta_m . \tag{B.18}$$
B.4 \( C_m \cdots \text{step 2} \)

Finally, we would like to write the each side of the equation (B.13) as the following form by choosing \( c_{12} \) and \( c_{22} \).

\[
\begin{pmatrix}
* & 0 \\
0 & 0
\end{pmatrix}.
\] (B.19)

We can find commutation relation for \( \hat{x} \)'s as follows:

\[ [\hat{x}, \hat{x}] = [y, y] + \pi i \alpha' \left( \begin{array}{c}
N_{11} \coth(\pi E_{11}) \eta_{T_{11}11} N_{11}^t \\
N_{21} \coth(\pi E_{11}) \eta_{T_{11}11} N_{11}^t
\end{array} \right)
\]

\[ + \left( \begin{array}{c}
\frac{i \alpha'}{2} \left( -N_{22} c_{12} N_{22}^t - N_{22} c_{22} N_{22}^t \right)
\end{array} \right), \] (B.20)

where we have used the formula

\[
\sum_{m \in \mathbb{Z}} \frac{i}{m + i E_{11}} = \pi \coth \pi E_{11}.
\] (B.21)

Therefore we choose \( c_{12} \) and \( c_{22} \) to satisfy following equations

\[
c_{12} N_{22}^t = \pi N_{11} \coth(\pi E_{11}) \eta_{T_{11}11} N_{21}^t + \pi \left( N \eta_{T} (ST)^t - (ST) \eta_{T} N^t \right)_{12},
\]

\[
c_{22} N_{22}^t - N_{22} c_{22}^t = \pi N_{21} \coth(\pi E_{11}) \eta_{T_{11}21} N_{21}^t + \pi \left( N \eta_{T} (ST)^t - (ST) \eta_{T} N^t \right)_{22}.
\] (B.22)

Then we get the following formula:

\[
[\hat{x}, \hat{x}] = \frac{\pi i \alpha'}{2} \left( N_{11} \coth \pi E_{11} \eta_{T_{11}11} N_{11}^t + N_{11} \eta_{T_{11}11} (ST)^t_{11} - (ST) \eta_{T_{11}11} N_{11}^t \right).
\] (B.23)

B.5 conjugate momentum for \( \hat{x} \)

We can define the conjugate momentum for the center of coordinate.

\[
[\hat{x}, \hat{a}_0] = i \sqrt{\frac{\alpha'}{2}} \begin{pmatrix} 0 & 0 \\
0 & N_{22}
\end{pmatrix}.
\] (B.24)

We assume that the determinant of the matrix \( N_{22} \) is not zero. If not the case, we replace \( N_{22} \) by its sub-matrix whose determinant is not zero.

We define the momenta \( p^i \):

\[
p^i := \sqrt{\frac{2}{\alpha'}} (N_{22}^{-1} \hat{a}_0)^i,
\] (B.25)

where the number of the independent momenta is determined by the rank of \( N_{22} \). \( i \) is the index in the momentum space. Then

\[
[\hat{x}^i, p^j] = i \delta^{ij} \mathbb{1}.
\] (B.26)

\[\text{If matrix } S \text{ is not } \eta\text{-orthogonal matrix, we must find the momentum for } x \text{ not } \hat{x}.\]
Now the trace of the zero mode in the phase space is defined as
\[
\text{Tr}_{p^i} := \int \prod_i \frac{L_i}{2\pi} dp^i, \quad (B.27)
\]
where \(L_i\) is a length of the area where \(X^i\) covers \([9]\).

**B.6 two identical parallel D-branes case**

We can find as follows after straight forward calculation
\[
\begin{align*}
[\hat{\alpha}_m, \hat{\alpha}_n] & = \left( \begin{array}{ccc} 0 & m + i\epsilon & 0 \\ m - i\epsilon & 0 & m1 \end{array} \right) \delta_{m+n}, \quad \epsilon := \frac{1}{\pi} \ln \left( \frac{g + v}{g - v} \right), \\
[\hat{\xi}^i, p^j] & = i\delta^{ij}, \quad [\hat{\xi}, \hat{\alpha}_{m\neq0}] = [\hat{\xi}, \hat{\alpha}_{m=0}] = [\hat{\xi}, \hat{\xi}] = 0, \\
\hat{\alpha}_0^i & := \sqrt{2\alpha^j \text{diag} \left( \frac{1}{g}, 1 + m_2^i, \ldots, 1 + m_{p+1}^i \right)} p^i, \\
g & := \sqrt{1 - f_1^2(1 - v^2)}, \quad i, j = 2 \cdots p + 1, \\
\end{align*}
\]
(B.28)

where \(m_2, \ldots, m_{p+1}\) are the \(2 \times 2\) matrices which are defined in eq. (2.35) and \(f_1\) is a component of the field strength \(F\) in eq. (2.36).

0, 1, \(p + 1\) components of matrices \(T, S,\) and \(N\) are
\[
\begin{align*}
T & \sim \left( \begin{array}{ccc} \frac{1}{\sqrt{2g}} & -\frac{1}{\sqrt{2g}} & 0 \\ \sqrt{1 - g^2} & \sqrt{1 - g^2} & \frac{1}{g} \end{array} \right), \quad S \sim \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right), \\
N & \sim \left( \begin{array}{ccc} \frac{1}{\sqrt{2g}} & -\frac{1}{\sqrt{2g}} & 0 \\ \sqrt{1 - g^2} & \sqrt{1 - g^2} & \frac{1}{g} \end{array} \right) \left( \begin{array}{ccc} \frac{1}{\sqrt{2g}} & -\frac{1}{\sqrt{2g}} & 0 \\ \sqrt{1 - g^2} & \sqrt{1 - g^2} & \frac{1}{g} \end{array} \right). \\
\end{align*}
\]
(B.29)

Note that \([\hat{\xi}, \hat{\xi}]\) is zero as in Ref. [1].

**Appendix C**

In this appendix we find energy momentum tensor which satisfies the ordinary operator product expansion (OPE). Namely, we would like to explain eq. (3.21) and eq. (3.22). For this purpose, we start with the calculation of OPE between naive energy momentum tensor of the world sheet. We normal order it...
with respect to the vacuum defined in the section 2. We define a naive energy momentum tensor as follows.

\[ T_{ab} := -4\pi\alpha' \frac{1}{\sqrt{h}} \frac{\delta S_{\text{string}}}{\delta h^{ab}}, \]  

where \( S_{\text{string}} \) in (C.1) is that of eq. (3.1). With

\[ z := e^{i\rho}, \]  

we define the holomorphic energy momentum tensor:

\[ T(z) := \frac{1}{2\alpha'z^2} (T_{00} + T_{01}) \]

\[ = T_{\text{boson}}(z) + T_{\text{fermion}}(z) + T_{\text{ghost}}(z) \]

\[ := \frac{1}{2} \sum_{m,n \in \mathbb{Z}} z^{-m-n-2} :\alpha_m^\dagger \alpha_n : \]

\[ + \frac{1}{2} \sum_{r,s \in \mathbb{Z}} z^{-r-s-2} :d_r^T \eta(s+iE)d_s : \]

\[ + \text{(ghost)}. \]  

(C.3)

We extract the singular parts of the OPE in the following subsections.

**C.1 OPE for bosonic part**

In this subsection, we calculate OPE between the bosonic part of the naive energy momentum tensor as follows:

\[ T_{\text{boson}}(z)T_{\text{boson}}(w) = \frac{1}{2} \mathrm{Tr} \left( \partial Z(z) \partial Z(w) \right)^T \eta_T \left( \partial Z(z) \partial Z(w) \right) \eta_T \]

\[ + \mathrm{Tr} \left( \partial Z(z) \partial Z(w) \right)^T \eta_T : \partial Z(z) \partial Z(w) : \eta_T \]

\[ + \text{regular( finite term with } w \to z) , \]  

(C.4)

where

\[ T_{\text{boson}}(z) = -\frac{1}{2} : \partial Z(z) T \eta_T \partial Z(z) : \]

\[ = \frac{1}{2} \sum_{m,n} : \bar{\alpha}_m^\dagger \eta_T \alpha_n : z^{-m-n-2} , \]  

(C.5)

\[ Z(z) := \sqrt{\frac{2}{\alpha'}} \tilde{X}(\rho = -i \ln z) \]
\[ \sum_{n \in \mathbb{Z}} z^{-n-iE} \alpha_n . \]  

(C.6)

First, we calculate the two point function for \( \partial Z \):

\[ \langle \partial Z(z)^\mu \partial Z(w)^\nu \rangle . \]  

(C.7)

We calculate this separately as follows.

\[ \langle \partial Z(z)^\lambda \partial Z(w)^\delta \rangle = -\sum_{m=1}^{\infty} \left( \frac{w}{z} \right)^m \frac{1}{z^w} \langle \bar{\alpha}_m^\lambda \alpha_{-m}^\delta \rangle \]  

= \frac{-1}{(z-w)^2} \eta_{T}^{\lambda \delta} , \]  

(C.8)

\[ \langle \partial Z(z)^a \partial Z(w)^b \rangle = -\frac{1}{zw} \sum_{m=1}^{\infty} \left( \frac{w}{z} \right)^m \left( z^{-i\epsilon} w^{-i\epsilon} \right) \langle \bar{\alpha}_m^a \alpha_{-m}^b \rangle \]  

= \frac{-1}{zw} \sum_{m=1}^{\infty} \left( \frac{w}{z} \right)^m \left( z^{-i\epsilon} w^{-i\epsilon} \right) \eta_{T}^{a b} . \]  

(C.9)

where \( \lambda, \delta = 2, \cdots, D - 1; a, b = 0, 1 \) and \( \epsilon^a := (-)^{a} \epsilon \).

In the case of \((a, b) = (0, 1)\), eq. (C.9) is

\[ -\frac{1}{zw} \sum_{m=0}^{\infty} (m \pm i \epsilon) \left( \frac{w}{z} \right)^m \eta_{T}^{0 1} \]  

= \frac{-1}{zw} \left( \frac{\partial w}{z} \right) \sum_{m=0}^{\infty} \left( \frac{w}{z} \right)^m \eta_{T}^{0 1} \]  

= \frac{-1}{w^2} \left( \frac{\partial w}{z} \right) \eta_{T}^{0 1} . \]  

(C.10)

In the case of \((a, b) = (1, 0)\), eq. (C.9) is

\[ -\frac{1}{zw} \sum_{m=0}^{\infty} (m \pm i \epsilon) \left( \frac{w}{z} \right)^m \eta_{T}^{1 0} \]  

= \frac{-1}{zw} \left( \frac{\partial w}{z} \right) \sum_{m=0}^{\infty} \left( \frac{w}{z} \right)^m \eta_{T}^{1 0} \]  

= \frac{-1}{w^2} \left( \frac{\partial w}{z} \right) \eta_{T}^{1 0} . \]  

(C.11)

That is,

\[ \langle \partial Z(z)^\mu \partial Z(w)^\nu \rangle = \begin{pmatrix} 0 & -\frac{1}{w^2} \left( \frac{\partial w}{z} \right) \eta_{T} \epsilon \left( \frac{z-w}{w} \right) & 0 \\ -\frac{1}{w^2} \left( \frac{\partial w}{z} \right) \eta_{T} \epsilon \left( \frac{z-w}{w} \right) & 0 & -\frac{1}{(z-w)^2} 1 \end{pmatrix} \]  

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Therefore we get
\[
T_{\text{boson}}(z)T_{\text{boson}}(w) = \frac{\partial}{(z-w)^4} + \frac{2}{(z-w)^2} \left( T_{\text{boson}}(w) - \frac{i\epsilon(1+i\epsilon)}{2w^2} \right) + \frac{1}{z-w} \partial_w \left( T_{\text{boson}}(w) - \frac{i\epsilon(1+i\epsilon)}{2w^2} \right) + \text{regular},
\]
\[
(C.13)
\]
where \( D = 10 \) is the space-time dimension.

### C.2 OPE for fermionic part

We calculate OPE between the fermionic part of the energy momentum tensor:
\[
T_{\text{fermion}} = -\frac{1}{2\alpha'z} : \psi_R \partial_z \psi_R :. \quad (C.14)
\]

We can calculate the two point function for \( \psi_R \) as follows.
\[
\langle \psi_R(z)^\mu \psi_R(w)^\nu \rangle = \alpha' \sum_{r,s} \left( \left( z^{-(r+iE)} d_r \right)^\mu \left( w^{-(s+iE)} d_s \right)^\nu \right)
\]
\[
= \alpha' \sum_{r,s} \left( z^{-(r+iE)} \right)^\mu \left( w^{-(s+iE)} \right)^\nu \delta_r \delta_s \left( \eta \right)^{\mu\nu} (1 - \frac{1}{2} \delta_r)
\]
\[
= \alpha' \sum_{0 \leq r \leq \frac{1}{2} - n} \left( z^{-(r+iE)} w^{-(r-iE)} \right)^\mu \left( \eta \right)^{\mu\nu} (1 - \frac{1}{2} \delta_r)
\]
\[
= \alpha' \sum_{0 \leq r \leq \frac{1}{2} - n} \left( \frac{w}{z} \right)^{r+iE} \eta \left( 1 - \frac{1}{2} \delta_r \right)
\]
\[
= \alpha' \sum_{0 \leq r \leq \frac{1}{2} - n} \left( \frac{w}{z} \right)^{r+iE} \eta \left( 1 - \frac{1}{2} \delta_n a \right)
\]
\[
= \alpha' \left( \frac{1}{1 - \frac{z}{w}} - a \right) \left( \frac{w}{z} \right)^{r+iE} \eta \left( 1 - \frac{1}{2} \delta_n a \right)^{\mu\nu}. \quad (C.15)
\]

Therefore, after straightforward calculation, we get
\[
T_{\text{fermion}}(z)T_{\text{fermion}}(w) = \frac{\partial}{(z-w)^4}
\]
\[ + \frac{2}{(z-w)^2} \left[ T_{\text{fermion}} + \frac{aD - 4 \text{Tr } E^2}{16w^2} \right] \]
\[ + \frac{1}{z-w} \partial_w \left[ T_{\text{fermion}} + \frac{aD - 4 \text{Tr } E^2}{16w^2} \right] \]
\[ + \text{(regular)} . \quad (C.16) \]

Note that \( \text{Tr } E = 0 \) is used in this calculation.

### C.3 shifted energy momentum tensor

Summarizing previous two subsections, using \( \text{Tr } E^2 = 2\epsilon^2 \), we define the following shifted energy momentum tensor:

\[
\tilde{T}(z) := T(z) - \frac{i\epsilon(1 + i\epsilon)}{2z^2} + \frac{aD - 8\epsilon^2}{16z^2} \\
= T(z) + \frac{1}{z^2} \left( \frac{aD}{16} - \frac{i\epsilon}{2} \right). \quad (C.17)
\]

We can get the following OPE.

\[
\tilde{T}(z)\tilde{T}(w) = \frac{3D}{(z-w)^4} + \frac{2\tilde{T}(w)}{(z-w)^2} + \frac{\partial_w \tilde{T}(w)}{z-w} + \text{regular + ghost contribution} . \quad (C.18)
\]

This form is the usual type and is invariant under the exchange of the \( z \) and \( w \) except for the regular part.

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