Exact solutions of boundary integral equation arising in vortex methods for incompressible flow simulation around elliptical and Zhukovsky airfoils

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Abstract. The problem of 2D incompressible flow simulation around airfoils using vortex methods is considered. An exact solution for the boundary integral equation with respect to a free vortex sheet intensity at the airfoil surface line that arises in such problems is obtained. The exact solution is constructed for flows around elliptical and Zhukovsky airfoils using the theory of complex potentials and conformal mappings technique. It is possible to take into account the influence of singularities in the flow domain — point vortices which simulate vortex wake. The obtained exact solutions can be used to verify and estimate the accuracy of numerical schemes for the boundary integral equation solution: such procedure is also described in details.

1. Introduction
The main idea of Lagrangian vortex methods [1, 2, 3, 4] for viscous incompressible flow simulation is the considering the vorticity as a primary calculated value. The influence of the airfoil on the flow according to the principle discovered by prof. N. E. Zhukovsky (and later generalized by S. A. Chaplygin, M. W. Kutta, W. Prager et al.) can be replaced with the influences of the attached and free vortex sheets and the attached source sheet placed on the airfoil surface line [5]. If the law of the airfoil motion is known, only free vortex sheet intensity is unknown: attached vortex and source sheet intensities can be calculated as the tangent and normal components of the airfoil surface line velocity, respectively. The intensity of the free vortex sheet can be found from the boundary condition satisfaction.

Taking into account, that vorticity generation takes place only on the airfoil surface line, it becomes clear, that the accuracy of the free vortex sheet intensity computation influences directly the quality of solution of the whole problem, because after being generated on the airfoil surface line free vortex sheet sheds to the flow and it forms vorticity distribution in the flow domain. There are number of researches [6, 7, 8, 9] about numerical schemes development for vortex sheet intensity computation, but high accuracy achieving is a nontrivial problem, and this question is still not fully investigated. However, for the schemes described in the above mentioned papers exact formulae are derived in [10] for the quadratures which appear in the coefficients of the corresponding matrices.
The intensity of the free vortex sheet can be found from the no-slip boundary condition at the airfoil surface line

\[ V(r, t) = V_K(r, t), \quad r \in K, \]

where \( V(r, t) \) is the flow velocity field; \( V_K(r, t) \) is the airfoil surface line velocity; \( K \) is the airfoil surface line. It is proved in [11] that when \( V(r, t) \) is expressed as generalized Helmholtz decomposition (that corresponds exactly to the influence of the above mentioned vortex and source sheets and vorticity distribution in the flow domain), the equation (1) can be satisfied in two different ways: through equality between normal or tangent components of the velocities. Both these approaches lead to boundary integral equation, but of different types.

When developing numerical schemes for these equations solution, it is important to provide the correct verification: it can be done using test problems with known exact analytical solution. The aim of this paper is to describe in details the approach to accuracy estimation of the approximate numerical solution of boundary integral equation arising in vortex methods by its comparison with exact analytical solutions for test problems.

2. Boundary integral equations

The flow velocity \( V(r, t) \) according to the Biot — Savart law and generalized Helmholtz decomposition [11] can be expressed through the vorticity distribution in the flow domain and the vortex and source sheet intensities on the surface line of the airfoil:

\[ V(r, t) = V_\infty + \int_S \frac{k \times (r - \xi)}{2\pi|r - \xi|^2} \Omega(\xi, t)d\xi + \int_K \frac{k \times (r - \xi)}{2\pi|r - \xi|^2} \gamma(\xi, t)d\xi + \int_K \frac{k \times (r - \xi)}{2\pi|r - \xi|^2} \gamma^{att}(\xi, t)dl_\xi + \int_K \frac{(r - \xi)}{2\pi|r - \xi|^2} q^{att}(\xi, t)dl_\xi, \]

where \( \Omega(\xi, t) = \Omega(\xi, t) \cdot k \) is the vorticity field inside the flow domain \( S; V_\infty \) is the incident flow velocity; \( \gamma(\xi, t) \) is the free vortex sheet intensity, distributed on the airfoil surface line \( K; \gamma^{att}(\xi, t) \) and \( q^{att}(\xi, t) \) are the attached vortex and source sheets intensities, respectively; \( k \) is unit vector orthogonal to the flow plane.

The vortex wake is normally simulated by \( N_v \) separate vortex elements, which positions \( r_s \) and circulations \( \Gamma_s, s = 1, \ldots, N_v, \) are known, so the vorticity distribution \( \Omega(r) \) in the flow domain is replaced with a linear combination

\[ \Omega(r) = \sum_{s=1}^{N_v} \Gamma_s \delta(r - r_s), \]

where \( \delta(r) \) is 2D Dirac delta-function, or with some ‘regularized’ distribution

\[ \Omega(r) = \sum_{s=1}^{N_v} \Gamma_s \delta_\epsilon(r - r_s), \]

where \( \delta_\epsilon \) is some smoothing kernel, which replaces the delta-function.

In this paper, we assume the airfoil to be rigid and immovable, so \( \gamma^{att}(\xi, t) = 0, q^{att}(\xi, t) = 0; \) time dependence hereafter is omitted.

As it was mentioned in the Introduction, there are two possible ways to satisfy equation (1): it is enough to write it down either for normal or for tangent components of the velocities. Taking into account (2) and (3) and denoting the known right-hand side as \( f_n(r) \) and \( f_\tau(r), \) respectively, we obtain the following boundary integral equations.
• Singular integral equation of the first kind following from the equality between normal
components,
\[ -\oint_{K} \frac{(r - \xi) \cdot \tau(\xi)}{2\pi |r - \xi|^2} \gamma(\xi) d\xi = f_n(r), \quad r \in K, \]
with Hilbert-type kernel \( Q_n \), where \( \tau(\xi) = k \times n(\xi) \) is unit tangent vector at the
corresponding point of the airfoil surface line; \( n(\xi) \) is outer normal unit vector. In order to
compute the principal value of the integral in Cauchy sense the specific quadrature formulae
should be used. In most cases, Discrete Vortices Method-type quadrature formulae are
used [3]; their accuracy is not very high and they require special surface line discretization
that can be not easy to provide in case of airfoils of complex shapes and deformable airfoils.

• Fredholm-type integral equation of the second kind which follows from the equality between
tangent velocity components [11],
\[ \oint_{K} \frac{(r - \xi) \cdot n(\xi)}{2\pi |r - \xi|^2} \gamma(\xi) d\xi = \frac{\gamma(r)}{2} = f_\tau(r), \quad r \in K, \]
which kernel \( Q_\tau \) is bounded by one half of maximal curvature of the airfoil (it means that
in case of smooth airfoil the kernel is uniformly bounded function) [9].

This approach seems to be much more accurate in comparison with the previous one; the
hierarchy of numerical schemes of the first and second order of accuracy has been developed
for it and adapted for flow simulation around smooth and non-smooth airfoils [7, 8].

The right-hand sides of both equations are normal and tangent components of flow velocity
at the corresponding point of the airfoil surface line, respectively,
\[ f_n(r) = -V_\omega(r) \cdot n(r), \quad f_\tau(r) = -V_\omega(r) \cdot \tau(r), \]
where
\[ V_\omega(r) = V_\infty + \sum_{s=1}^{N_v} \frac{k \times (r - r_s)}{2\pi |r - r_s|^2} \Gamma_s \]
is the influence of the incident flow and vortex elements which simulate vortex wake. Note, that
it is especially important to take into account correctly the influence of vortices which are placed
in proximity to the airfoil (in boundary layer).

3. Exact analytical solution

Exact analytical solution can be derived by using conformal mappings technique [12, 13] for the
airfoils of simple shapes: elliptical airfoils and Zhukovsky wing airfoils. These solutions are often
used as reference ones for accuracy estimation of numerical solutions.

Firstly, we need to write down the expression for the complex potential of the flow around
circular cylinder of radius \( R \) in the presence of point vortices in the flow domain. Let \( \xi \) to be the
point on the complex plane, and the center of the circle \( C \) of radius \( R \) is located at the origin
of the complex plane, \( W_\infty \) is complex value of the incident flow velocity. Let the vortices have
circulations \( \Gamma_s \) and they are placed at the points \( \xi_{s,1} = \rho_s e^{i\theta_s} \).

To construct the contribution of the \( s \)-th vortex to the complex potential of the flow, the
image vortex system should be introduced (figure 1): a vortex with circulation \((-\Gamma_s)\) placed at
the inverse (with respect to a circle) point
\[ \xi_{s,2} = \rho_s e^{i\theta_s} = \frac{R^2}{\rho_s} e^{i\theta_s} \]
and a vortex with circulation \( \Gamma_s \) placed at the center of circle.
Figure 1. The vortex in the flow region and the image system

The complex potential now can be written down as superposition of complex potentials of the vortex-free flow, the circulatory flow with arbitrary circulation $\Gamma$ and the flow induced by system of ‘real’ and ‘image’ separate vortices:

$$F(\xi) = W_\infty^* \xi + \frac{R^2 W_\infty}{\xi} + \frac{\Gamma}{2\pi i} \ln \frac{\xi}{R} + \sum_{s=1}^{N_v} \frac{\Gamma_s}{2\pi i} \left( \ln \frac{\xi - \xi_{s,1}}{R} - \ln \frac{\xi - \xi_{s,2}}{R} + \ln \frac{\xi}{R} \right),$$

where ‘$*$’ means complex conjugation; $\Gamma$ is circulation of the velocity field along a closed contour enclosing the circle $C$ and not embracing the vortices.

For an arbitrary airfoil $C'$, complex potential can be found if direct and inverse conformal mappings $z = \zeta(\xi)$ and $\xi = \hat{\zeta}(z)$ are known, which map the region outside the circle $C$ to the region outside the airfoil $C'$ and vice versa (figure 2), respectively, and satisfy the condition

$$\zeta'(\xi) \to \text{const}, \quad \xi \to \infty.$$

Figure 2. Complex planes $\xi$ and $z$ with circular and arbitrary airfoils, respectively, and conformal mappings between them

In order to construct a complex potential for flow around circular airfoil $C$ in $\xi$-plane which corresponds to the complex potential of the flow around the initial arbitrary airfoil in $z$-plane, it is necessary to map the vortices points from $z$-plane to $\xi$-plane and then construct image vortex system in $\xi$-plane.
Therefore, using these mappings we can write down the expression for the complex potential of the flow in the region outside the contour $C'$:

$$
F(\xi) = F(\zeta(z)) = W_\infty^* \zeta(z) + \frac{R^2 W_\infty}{\zeta(z)} + \frac{\Gamma}{2\pi i} \ln \frac{\zeta(z)}{R} +
+ \sum_{s=1}^{N_s} \frac{\Gamma_s}{2\pi i} \left( \ln \frac{\zeta(z) - \zeta(z_s)}{R} - \ln \frac{\zeta(z) - R^2/\zeta(z_s)}{R} + \ln \frac{\zeta(z)}{R} \right) = f(z).
$$

Here, $z_s$ are the positions on the complex plane $z$ of the vortices in the flow domain; complex velocity $W_\infty$ is expressed as $W_\infty = V_\infty \cdot \zeta'(\infty)^*$, where $V_\infty$ is the complex incident flow velocity.

For elliptical and Zhukovsky airfoils these mappings are known and they are based on Zhukovsky function $[12, 13]$:

$$
\zeta(\xi) = \frac{1}{2} \left( \xi + H + \frac{a^2}{\xi + H} \right), \quad \zeta(z) = z - H + \sqrt{z^2 - a^2}.
$$

where parameters $a$, $H$ and radius $R$ of the circle $C$ are determined by the shape of the airfoil.

For elliptical airfoil

$$
a = \sqrt{a_1^2 - b_1^2}, \quad R = a_1 + b_1, \quad H = 0,
$$

where $a_1$ and $b_1$ are the semiaxes of the ellipse.

For Zhukovsky airfoil

$$
R = d + \sqrt{a^2 + h^2}, \quad H = \frac{-ad + iRh}{\sqrt{a^2 + h^2}},
$$

where parameters $a$, $d$ and $h$ determine length, width and curvature of the airfoil, respectively.

Note that $\zeta'(\xi) \to \frac{1}{2}$ when $\xi \to \infty$, thus $W_\infty = \frac{1}{2} V_\infty$.

Introducing parametrization $\xi = Re^{i(t-\varphi)}$ with parameter $t$ on the surface line of the circle $C$ on $\xi$-plane, we obtain the expression for the complex potential of the flow on the airfoil surface line:

$$
\Phi(t) = F(Re^{i(t-\varphi)}) = W_\infty^* Re^{i(t-\varphi)} + W_\infty Re^{-i(t-\varphi)} +
+ \frac{t - \varphi}{2\pi i} \left( \Gamma + \sum_{s=1}^{N_s} \Gamma_s \right) + \sum_{s=1}^{N_s} \frac{\Gamma_s}{2\pi i} \ln \frac{Re^{i(t-\varphi)} - \rho_s e^{i\theta_s}}{Re^{i(t-\varphi)} - \rho_{s,2} e^{i\theta_s}}, \quad t \in [0, 2\pi),
$$

where $\rho_s e^{i\theta_s} = \zeta(z_s)$ is position of the $s$-th vortex being mapped onto $\xi$-plane; $\rho_{s,2} = R^2/\rho_{s,1}$. Value of the parameter $\varphi$ is chosen in such a way that $t = 0$ corresponds to the point on the major semiaxis in case of elliptical airfoil ($\varphi = 0$) and to the sharp edge in case of Zhukovsky airfoil ($\varphi = \arctan \frac{h}{a}$).

The point on the original airfoil $C'$ which corresponds to the parameter $t$ is the following:

$$
z(t) = \zeta(e^{i(t-\varphi)}) = \frac{1}{2} \left( \chi(t) + \frac{a^2}{\chi(t)} \right), \quad \chi(t) = Re^{i(t-\varphi)} + H.
$$

The vortex sheet intensity distribution can be found as the tangent component of the flow velocity at the corresponding point of the airfoil surface line, which in turn is equal to conjugate
complex velocity multiplied by the ‘complex tangent vector’, since the flow velocity on the airfoil surface line has only tangent component:

$$\gamma_{ex}(t) = V^*(t) \cdot \tau (t) = \frac{\Phi'(t)}{z'(t)} \cdot \frac{z'(t)}{|z'(t)|} = \Phi' \frac{z' \cdot \bar{z}' |z'|}{|z'|}.$$ 

As a result, we finally obtain the following formula for exact solution for vortex sheet intensity:

$$\gamma_{ex}(t) = 2|V_\infty| \sin(\varphi + \beta - t) + \left(\Gamma + W(t)\right) / (\pi R),$$

where term $W(t)$ expresses the influence of point vortex elements in the flow:

$$W(t) = \sum_{s=1}^{N_v} \frac{R \cos(t - \theta_s - \varphi) - \rho_{s,2}}{R \cos(t - \theta_s - \varphi) - (\rho_{s,1} + \rho_{s,2})/2} \Gamma_s.$$ 

Values $\rho_{s,1}$ and $\theta_s$ are computed in the following way

$$\rho_{s,1} = |\hat{\zeta}(z_s)|, \quad \theta_s = \text{Arg} \hat{\zeta}(z_s),$$

where that branch of square root in function $\hat{\zeta}(z)$ should be chosen, that provides condition $\rho_{s,1} > R$ satisfaction. The value $\rho_{s,2}$ can be found in accordance with (4).

Note, that the denominator of formulae (6) can be expressed through the parameters of the airfoil: for elliptical airfoil

$$\left| 1 - \frac{a^2}{(Re^{i(t-r)} + H)^2} \right| = \frac{2\sqrt{a_1^2 \sin^2 t + b_1^2 \cos^2 t}}{a_1 + b_1},$$

and for Zhukovsky wing airfoil

$$\left| 1 - \frac{a^2}{(Re^{i(t-r)} + H)^2} \right| = \frac{2}{(d \sin \varphi + h + R \sin(t - \varphi))^2 + (R \cos(t - \varphi) - d \cos \varphi)^2} \times$$

$$\times \left\{ \left( (d \sin \varphi + h + R \sin(t - \varphi))(d \sin \varphi + h + R \sin(t - \varphi)) + dR(\cos t - 1) \right)^2 + \right.$$

$$\left. + \left( R(\sin(t - \varphi) + \sin \varphi)(R \cos(t - \varphi) - d \cos \varphi) \right)^2 \right\}^{\frac{1}{2}}.$$ 

The value of total vorticity $\Gamma$ for elliptical airfoil can be chosen arbitrary, for our test problem we assume $\Gamma = 0$ for any angle of incidence; for Zhukovsky airfoil $\Gamma$ can be chosen from the Chaplygin — Zhukovsky — Kutta condition in such a way that velocity at the sharp edge is bounded: its value is proportional to incident flow velocity and depends on the shape of the airfoil and angle of incidence, as well as on vortex element positions in the flow:

$$\Gamma = -2\pi |V_\infty| \sin(\beta + \varphi) R - \sum_{s=1}^{N_v} \frac{R \cos(\theta_s + \varphi) - \rho_{s,2}}{R \cos(\theta_s + \varphi) - (\rho_{s,1} + \rho_{s,2})/2} \Gamma_s.$$
4. Model problems

The pictures given below show exact solutions for the model problems of flow simulation around the airfoils in case of presence of one vortex element in the flow domain near the airfoil. Of course, in real problems there are much more vortices near the airfoil, but in the framework of this paper, the single vortex more clearly demonstrates its influence on the solution.

In figure 3, a, the streamlines of the flow around elliptical airfoil with semiaxes $a_1 = 4$ and $b_1 = 1$ (centered at the origin) in the presence of one vortex element in the flow domain at point $r_s = (2, 1.5)$ with circulation $\Gamma_s = 10$ is shown. The incident flow at infinity has a unit velocity; the angle of incidence is equal to $\beta = \pi/6$. Figure 3, b shows the dependence of the exact solution for the vortex sheet intensity on parameter $t$ which parameterizes the airfoil surface line as described in the previous section.

![Figure 3](image)

**Figure 3.** The streamlines of flow around elliptical airfoil ($a_1/b_1 = 4$) in presence of vortex element (a) and exact solution for vortex sheet intensity (b)

Figures 4, a, and 4, b show the results for a similar problem for the Zhukovsky airfoil. Considered Zhukovsky airfoil has the following parameters: $a = 3$, $d = 0.5$, $h = 0.3$. The velocity of the incident flow as in previous case is unit and the angle of incidence is $\beta = \pi/6$. Vortex element is placed at the point $r_s = (1.1, 1)$ and has a circulation equal to $\Gamma_s = 10$.

![Figure 4](image)

**Figure 4.** The streamlines of flow around Zhukovsky airfoil ($a = 3$, $d = 0.5$, $h = 0.3$) in presence of vortex element (a) and exact solution for vortex sheet intensity (b)

5. The method of errors computation

Usually, in vortex methods for the boundary integral equation solving the airfoil surface line is approximated by a polygon consists of $N$ rectilinear segments (panels) in such way that angle
points or sharp edges of the initial airfoil coincide with panels endings. The numerical solution is normally assumed to be piecewise-constant or piecewise-linear function (discontinuities are at the panels endings). A hierarchy of such numerical schemes is described in [7].

There are three ways of the absolute errors calculation in different norms.

- Uniform norm defined in the the space of piecewise-continuous bounded functions (we assume that airfoil has a finite number of angle points or sharp edges):

\[ \| \Delta \gamma \|_C = \max_{t \in [0, 2\pi)} |\gamma(t) - \gamma_{ex}(t)|, \]

which characterizes the error in vortex sheet intensity distribution.

- The \( L_1 \)-norm

\[ \| \Delta \gamma \|_{L_1} = \int_0^{2\pi} |\gamma(t) - \gamma_{ex}(t)| dt, \]

which gives the integral of the error in the vortex sheet intensity distribution.

- The norm defined by the formula

\[ \| \Delta \gamma \|_{C_h} = \max_{1 \leq i \leq N} |\bar{\gamma}_i - \bar{\gamma}_{ex,i}|, \]

where \( \bar{\gamma}_i \) and \( \bar{\gamma}_{ex,i} \) are the average values of the numerical and exact solutions on the \( i \)-th panel.

The main problem is that the shape of the approximation polygon and the distribution of the numerical solution along the panels don’t have direct dependence on the parameter \( t \) parameterising the initial surface line (5) and the analytical solution for the initial airfoil given by equations (6). In order to define the relationship between analytical and numerical solutions, we have to specify the mapping \( T \rightarrow P \), where \( T \) is set of parameter \( t \) values equal to \([0; 2\pi)\), \( P \) is set of natural parameter \( p \) values for the approximation polygon: \( P = [0; L) \), where \( L \) is sum of panels lengthes.

To calculate the error according to the formula (7) and (8) the following approach seems to be reasonable. We denote the beginning and the ending of the \( i \)-th panel as \( r_i \) and \( r_{i+1} \), respectively. Let the values of the parameter \( t \) from the equations (5) and (6) specifying the initial surface line and exact solution for the initial airfoil at the points \( r_i \) and \( r_{i+1} \) are equal to \( t_i \) and \( t_{i+1} \), respectively. And finally, we denote by \( p_i \) and \( p_{i+1} \) the values of parameter \( p \), which corresponds to the beginning and the ending of the \( i \)-th panel.

In [7] numerical solution at the every panel is assumed to be a constant or linear function with respect to parameter \( p \). For any value \( \tilde{t} \) of the parameter \( t \) from the interval \([t_i, t_{i+1}]\), we can find the radius vector \( r(\tilde{t}) \) of the point on the initial curve and the corresponding value of vortex sheet intensity from the analytical formula. In order to find the value of the numerical solution corresponding to the point \( r(\tilde{t}) \), it is proposed to drop a perpendicular from this point to the rectilinear panel (Fig. 5) and find the value of the parameter \( \tilde{p} \) by the following way:

\[ \tilde{p} = p_i + \frac{(r(\tilde{t}) - r_i) \cdot (r_{i+1} - r_i)}{|r_{i+1} - r_i|}. \]

Using the described approach, one can find the value of numerical solution \( \gamma(\tilde{p}) \) and compare it with the exact solution \( \gamma_{ex}(\tilde{t}) \).
6. Conclusions
The problem of 2D incompressible flow simulation around airfoils using vortex methods is considered. An exact solution for the boundary integral equation with respect to a free vortex sheet intensity at the airfoil surface line is obtained. The exact solution is constructed for flows around elliptical and Zhukovsky airfoils using the theory of complex potentials and conformal mappings technique. It is possible to take into account the influence of singularities in the flow domain — point vortices which simulate vortex wake. The obtained exact solution can be used to verification and accuracy estimation of the numerical schemes for the boundary integral equation solution: such procedure is also described in details.

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