INDECOMPOSABLE DECOMPOSITION OF TENSOR PRODUCTS OF MODULES
OVER THE RESTRICTED QUANTUM UNIVERSAL ENVELOPING ALGEBRA
ASSOCIATED TO $\mathfrak{sl}_2$

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Abstract. In this paper we study the tensor category structure of the module category of the restricted quantum enveloping algebra associated to $\mathfrak{sl}_2$. Indecomposable decomposition of all tensor products of modules over this algebra is completely determined in explicit formulas. As a by-product, we show that the module category of the restricted quantum enveloping algebra associated to $\mathfrak{sl}_2$ is not a braided tensor category.

1. Introduction

In the study of quantum groups at roots of unity, the case of $p$-th roots of unity with a prime number $p$ has been considered more than other cases. However, the representation theory of restricted quantum groups at $n$-th roots of unity with an even integer $n$ has pointed out to be concerned with knot invariants ([MN]) and logarithmic conformal field theories ([FGST1], [FGST2]), and this type of algebra seems to be becoming a more interesting object. We study on the module category of the restricted quantum enveloping algebra associated to $\mathfrak{sl}_2$ at $2p$-th roots of unity with an arbitrary integer $p \geq 2$.

Let $q$ be a primitive $2p$-th root of unity in a fixed algebraically closed field of characteristic zero. The restricted quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is a Hopf algebra defined by generators and relations. Since by definition $\mathcal{U}_q(\mathfrak{sl}_2)$ is finite-dimensional, its module category can be examined by technique of Auslander-Reiten theory. Indecomposable left $\mathcal{U}_q(\mathfrak{sl}_2)$-modules of finite dimension are classified up to isomorphism by Suter ([Sut]) and Xiao ([X3]). There exist $2p$ simple modules (two of them are projective), $2p-2$ nonsimple projective modules, and several infinite sequences of other indecomposable modules of semisimple length 2. This result implies that all left $\mathcal{U}_q(\mathfrak{sl}_2)$-modules of finite dimension are classified, for such modules are expressed as direct sum of indecomposable modules uniquely by the general property of Artin algebras (Krull-Schmidt Theorem). The Auslander-Reiten quiver of this algebra is also known, especially $\mathcal{U}_q(\mathfrak{sl}_2)$ has a tame representation type.

Since $\mathcal{U}_q(\mathfrak{sl}_2)$ is a Hopf algebra, its module category has a structure of a tensor category naturally. In this paper we study this tensor category structure, which seems to be important in the connection with the topics mentioned above.

Section 2 is devoted to a review on $\mathcal{U}_q(\mathfrak{sl}_2)$ and its module category. In Section 3 we give formulas for indecomposable decomposition of tensor products of arbitrary indecomposable $\mathcal{U}_q(\mathfrak{sl}_2)$-modules. In [Sut] one can find formulas for tensor products of simple modules with simple or projective modules. For computing tensor products of other types of modules, we use the following facts which come from general properties of finite-dimensional Hopf algebras (See Appendix A):

(i) If $\mathcal{P}$ is a projective $\mathcal{U}_q(\mathfrak{sl}_2)$-module, $\mathcal{Z} \otimes_k \mathcal{P}$ and $\mathcal{P} \otimes_k \mathcal{Z}$ are also projective for any $\mathcal{U}_q(\mathfrak{sl}_2)$-module $\mathcal{Z}$.

(ii) Projective modules and injective modules coincide.

(iii) The category of finite-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-modules has a structure of a rigid tensor category. From the rigidity we have $\text{Ext}^2_{\mathcal{U}_q(\mathfrak{sl}_2)}(\mathcal{Z}_1 \otimes_k \mathcal{Z}_2, \mathcal{Z}_3) \cong \text{Ext}^2_{\mathcal{U}_q(\mathfrak{sl}_2)}(\mathcal{Z}_1, \mathcal{Z}_2 \otimes_k D(\mathcal{Z}_3))$ for arbitrary $\mathcal{U}_q(\mathfrak{sl}_2)$-modules $\mathcal{Z}_1$, $\mathcal{Z}_2$, and $\mathcal{Z}_3$, where $D(\mathcal{Z})$ is the standard dual of $\mathcal{Z}$.

As a result we know that the module category of $\mathcal{U}_q(\mathfrak{sl}_2)$ is not a braided tensor category if $p \geq 3$. As a corollary, we prove that $\mathcal{U}_q(\mathfrak{sl}_2)$ has no universal $R$-matrices for $p \geq 3$. An observation about this fact is given in the last section.

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2. Indecomposable modules over $\bar{U}_q(\mathfrak{sl}_2)$

Throughout the paper, we work on a fixed algebraically closed field $k$ with characteristic zero. All modules considered are left modules and finite-dimensional over $k$.

Let $p \geq 2$ be an integer and $q$ be a primitive $2p$-th root of unity. For any integer $n$, we set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Note that $[n] = [p - n]$ for any $n$.

In this section we summarize facts about the restricted quantum $\mathfrak{sl}_2$, which one can find in [Sut], [X3], [FGST2] and [Ari].

2.1. The restricted quantum group $\bar{U}_q(\mathfrak{sl}_2)$. The restricted quantum group $\bar{U} = \bar{U}_q(\mathfrak{sl}_2)$ is defined as an unital associative $k$-algebra with generators $E, F, K, K^{-1}$ and relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad K^{2p} = 1, \quad E^p = 0, \quad F^p = 0.$$  

This is a finite-dimensional algebra and has a Hopf algebra structure, where coproduct $\Delta$, counit $\epsilon$, and antipode $S$ are defined by

$$\Delta: E \mapsto E \otimes K + 1 \otimes E, \quad F \mapsto F \otimes 1 + K^{-1} \otimes F,$$

$$K \mapsto K \otimes K, \quad K^{-1} \mapsto K^{-1} \otimes K^{-1},$$

$$\epsilon: E \mapsto 0, \quad F \mapsto 0, \quad K \mapsto 1, \quad K^{-1} \mapsto 1,$$

$$S: E \mapsto -EK^{-1}, \quad F \mapsto -KF, \quad K \mapsto K^{-1}, \quad K^{-1} \mapsto K.$$

The category $\bar{U}$-mod of finite-dimensional left $\bar{U}$-modules has a structure of a tensor category associated with this Hopf algebra structure on $\bar{U}$.

2.2. Basic algebra. Let $A$ be an unital associative $k$-algebra of finite dimension. The basic algebra of $A$ is defined as follows: Let $A = \bigoplus_{i=1}^n \mathcal{P}_i^{m_i}$ be a decomposition of $A$ into indecomposable left ideals, where $\mathcal{P}_i \neq \mathcal{P}_j$ if $i \neq j$. For each $i$ take an idempotent $e_i \in A$ such that $Ae_i \cong \mathcal{P}_i$, and set $e = \sum_{i=1}^n e_i$. Then the subspace $B_A = eAe$ of $A$ has a $k$-algebra structure naturally and is called the basic algebra of $A$.

It is known (see [ASS], for example) that the categories of finite-dimensional modules over $A$ and $B_A$ are equivalent each other by $B_A$-mod $\rightarrow A$-mod; $Z \mapsto Ae \otimes_{B_A} Z$.

The basic algebra $B_p$ of $\bar{U}$ can be decomposed as a direct product $B_p \cong \prod_{s=0}^p B_s$ and one can describe each $B_s$ as follows:

- $B_0 \cong B_p \cong k$.
- For each $s = 1, \ldots, p - 1$, $B_s$ is isomorphic to the 8-dimensional algebra $B$ defined by the following quiver

$$V^+ \xrightarrow{\tau_1^+} V^- \xleftarrow{\tau_1^-} V^+ \xleftarrow{\tau_2^-} V^- \xrightarrow{\tau_2^+} V^+$$

with relations $\tau_i^+ \tau_i^- = 0$ for $i = 1, 2$, and $\tau_1^+ \tau_2^- = \tau_2^- \tau_1^+$.

The algebra $B$ is studied in [Sut] and [X3] and is known to have a tame module category. We shall review on the classification theorem of isomorphism classes of indecomposable $B$-modules. Note that one can identify a $B$-module with data $Z = (V_0^+, V_0^-, \tau_1^+ \tau_1^-, \tau_2^+ \tau_2^-, \tau_1^-, \tau_2^-)$, where $V_0^\pm$ is a vector space over $k$ and $\tau_i^+ \tau_i^-: V_0^\pm \rightarrow V_0^\pm$ for $i = 1, 2$ are $k$-linear maps satisfying $\tau_i^+ \tau_i^- = 0$, $\tau_1^+ \tau_2^- = \tau_2^- \tau_1^+$.

**Proposition 2.2.1.** Any indecomposable $B$-module is isomorphic to exactly one of modules in the following list:

- Simple modules

$$X^+ = (k, 0, 0, 0, 0), \quad X^- = (0, k, 0, 0, 0).$$


- **Projective-injective modules**

\[ \mathcal{P}^+ = (k^2, k^2, e_{1,1}, e_{2,1}, e_{2,2}), \quad \mathcal{P}^- = (k^2, k^2, e_{2,1}, e_{1,1}, e_{2,2}), \]

where for positive integers \( m, n \) and \( i = 1, \ldots, m \), \( j = 1, \ldots, n \) we denote the composition of \( j \)-th projection and \( i \)-th embedding \( k^m \rightarrow k \rightarrow k^n \) by \( e_{i,j} \).

- \( \mathcal{M}^+(n) = (k^{n-1}, k^n, \sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i}, 0, 0), \mathcal{M}^-(n) = (k^n, k^{n-1}, 0, 0, \sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i}) \) for each integer \( n \geq 2 \).
- \( \mathcal{W}^+(n) = (k^n, k^{n-1}, \sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i}, 0, 0), \mathcal{W}^-(n) = (k^{n-1}, k^n, 0, 0, \sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i}) \) for each integer \( n \geq 2 \).
- \( \mathcal{E}^+(n; \lambda) = (k^n, k^n, \varphi_1(n; \lambda), \varphi_2(n; \lambda), 0, 0), \mathcal{E}^-(n; \lambda) = (k^n, k^n, 0, 0, \varphi_1(n; \lambda), \varphi_2(n; \lambda)) \) for each integer \( n \geq 1 \) and \( \lambda \in \mathbb{P}^1(k) \), where

\[
(\varphi_1(n; \lambda), \varphi_2(n; \lambda)) = \begin{cases} 
(\beta \cdot \text{id} + \sum_{i=1}^{n-1} e_{i,i+1}, \text{id}) & (\lambda = [\beta : 1]) \\
(\text{id}, \sum_{i=1}^{n-1} e_{i,i+1}) & (\lambda = [1 : 1]) 
\end{cases}
\]

**Remark 2.2.2.** \( \{\mathcal{X}^+, \mathcal{X}^-, \mathcal{M}^+(n), \mathcal{W}^+(n), \mathcal{E}^+(n; \lambda)\} \) and \( \{\mathcal{X}^-, \mathcal{X}^+, \mathcal{M}^-(n), \mathcal{W}^-(n), \mathcal{E}^-(n; \lambda)\} \) both correspond to indecomposable modules over the algebra of the Kronecker quiver (two vertices with two arrows in the same direction). Using dimension vectors \( \mathcal{Z} \longrightarrow (\dim V^2, \dim V_Z) \). Those are parameterized by the positive root system of \( A_1^{(1)} \)-type (modules of types \( \mathcal{X}, \mathcal{M}, \mathcal{W} \) correspond to real roots, while those of type \( \mathcal{E} \) correspond to imaginary roots).

2.3. **Indecomposable modules.** Denote by \( \mathcal{C}(s) \) the full subcategory of \( \mathcal{U}\text{-mod} \) corresponding to \( B_s \)-modules (considered as \( B_{\mathcal{U}} \)-modules) for \( s = 0, \ldots, p \). Each indecomposable \( \mathcal{U} \)-module belongs to exactly one of \( \mathcal{C}(s) \) (\( s = 0, \ldots, p \)). Each of \( \mathcal{C}(0) \) and \( \mathcal{C}(p) \) has precisely one indecomposable module (denote \( \mathcal{X}^+_p \), \( \mathcal{X}^-_p \), respectively).

For \( s = 1, \ldots, p-1 \), indecomposable modules from \( \mathcal{C}(s) \) are classified by Proposition 2.2.1 as the following proposition.

**Definition 2.3.1.** For \( s = 1, \ldots, p-1 \), let \( \Phi_s \) be the composition of functors \( B_{\mathcal{U}} \longrightarrow B_{\mathcal{U}} \longrightarrow \mathcal{U}\text{-mod} \), where the first one is induced from \( B_{\mathcal{U}} \cong \prod_{s=0}^p B_s \rightarrow B_s \cong B \) and the second one is expressed in the previous subsection.

We denote by \( \mathcal{X}^+_s, \mathcal{X}^-_{p-s}, \mathcal{P}^+_s, \mathcal{P}^-_{p-s}, \mathcal{M}^+_s(n), \mathcal{M}^-_{p-s}(n), \mathcal{W}^+_s(n), \mathcal{W}^-_{p-s}(n), \mathcal{E}^+_s(n; \lambda), \mathcal{E}^-_{p-s}(n; \lambda) \) the images of \( \mathcal{X}^+, \mathcal{X}^-, \mathcal{P}^+, \mathcal{P}^-, \mathcal{M}^+(n), \mathcal{M}^-(n), \mathcal{W}^+(n), \mathcal{W}^-(n), \mathcal{E}^+(n; \lambda), \mathcal{E}^-(n; \lambda) \) by \( \Phi_s \).

**Proposition 2.3.2.** Each subcategory \( \mathcal{C}(s) \) (\( s = 1, \ldots, p-1 \)) has two simple modules \( \mathcal{X}^+_s \) and \( \mathcal{X}^-_{p-s} \), two indecomposable projective-injective modules \( \mathcal{P}^+_s \) and \( \mathcal{P}^-_{p-s} \), and three series of indecomposable modules:

- \( \mathcal{M}^+_s(n) \) and \( \mathcal{M}^-_{p-s}(n) \) for each integer \( n \geq 2 \),
- \( \mathcal{W}^+_s(n) \) and \( \mathcal{W}^-_{p-s}(n) \) for each integer \( n \geq 2 \),
- \( \mathcal{E}^+_s(n; \lambda) \) and \( \mathcal{E}^-_{p-s}(n; \lambda) \) for each integer \( n \geq 1 \) and \( \lambda \in \mathbb{P}^1(k) \).

Any indecomposable module in \( \mathcal{C}(s) \) is isomorphic to one of the modules listed above.

Properties of the module category of \( \mathcal{U} \), without taking the tensor category structure into account, can be proved by passing to \( B_{\mathcal{U}} \).

**Proposition 2.3.3.** (i) There are no \( \mathcal{U} \)-modules with semisimple length 3, and the only indecomposable modules with semisimple length 3 are projective modules \( \mathcal{P}^+_s \) with \( s = 1, \ldots, p-1 \).
(ii) \( \text{top} \mathcal{P}^+_s \cong \mathcal{X}^+_s, \text{top} \mathcal{M}^+_s(n) \cong (\mathcal{X}^+_s)^{n-1}, \text{top} \mathcal{W}^+_s(n) \cong (\mathcal{X}^+_s)^n, \text{top} \mathcal{E}^+_s(n; \lambda) \cong (\mathcal{X}^+_s)^n. \)
(iii) \( \text{soc} \mathcal{P}^+_s \cong \mathcal{X}^+_s, \text{soc} \mathcal{M}^+_s(n) \cong (\mathcal{X}^+_s)^n, \text{soc} \mathcal{W}^+_s(n) \cong (\mathcal{X}^+_s)^{n-1}, \text{soc} \mathcal{E}^+_s(n; \lambda) \cong (\mathcal{X}^+_s)^n. \)

**Proposition 2.3.4.** \( \dim_k \mathcal{X}^+_s = s, \dim_k \mathcal{P}^+_s = 2p, \dim_k \mathcal{M}^+_s(n) = pn-s, \dim_k \mathcal{W}^+_s(n) = pn-p+s, \dim_k \mathcal{E}^+_s(n; \lambda) = pn. \)

We describe some indecomposable modules explicitly by bases and action of \( \mathcal{U} \) on those.

**Proposition 2.3.5.** (i) \( \mathcal{X}^+_s \) (\( s = 1, \ldots, p \)) is isomorphic to the \( s \)-dimensional module defined by basis \( \{a_n\}_{n=0,\ldots,s-1} \) and \( \mathcal{U} \)-action given by

\[
K a_n = \pm q^{s-1-2n} a_n, \quad E a_n = \begin{cases} 
\pm [n] [s-n] a_{n-1} & (n \neq 0) \\
0 & (n = 0) 
\end{cases}, \quad F a_n = \begin{cases} 
a_{n+1} & (n \neq s-1) \\
0 & (n = s-1) 
\end{cases}.
\]
(ii) $E_s^{\pm}(1; \lambda) \ (s = 1, \ldots, p - 1, \lambda = \lambda_1 : \lambda_2)$ is isomorphic to the $p$-dimensional module defined by basis

\[ \{ b_n \}_{n=0, \ldots, s-1} \text{ and } U\text{-action given by} \]

\[ K b_n = \pm q^{s-1-2n} b_n, \quad K x_m = \mp q^{p-s-1-2m} x_m, \]

\[ E b_n = \begin{cases} \pm |n| (s-n) b_{n-1} & (n \neq 0) \\ \lambda_2 x_{p-s-1} & (n = 0) \end{cases}, \quad E x_m = \begin{cases} \mp |m| (p-s-m) x_{m-1} & (m \neq 0) \\ 0 & (m = 0) \end{cases}, \]

\[ F b_n = \begin{cases} b_{n+1} & (n \neq s-1) \\ \lambda_1 x_0 & (n = s-1) \end{cases}, \quad F x_m = \begin{cases} x_{m+1} & (m \neq p-s-1) \\ 0 & (m = p-s-1) \end{cases}. \]

2.4. Extensions. We describe the projective covers and the injective envelopes of indecomposable $\overline{U}$-modules which we use in the sequel.

**Proposition 2.4.1.** There exist following exact sequences

\[ 0 \to \mathcal{M}_{p-s}^\pm(n) \to (\mathcal{P}_s^\pm)^n \to \mathcal{M}_s^\pm(n+1) \to 0, \]

\[ 0 \to \mathcal{W}_{p-s}^\mp(n+1) \to (\mathcal{P}_s^\pm)^n \to \mathcal{W}_s^\mp(n) \to 0, \]

\[ 0 \to \mathcal{E}_{p-s}(n; -\lambda) \to (\mathcal{P}_s^\pm)^n \to \mathcal{E}_s^\pm(n; \lambda) \to 0 \]

for each $s=1, \ldots, p-1$, $n \geq 1$ and $\lambda \in \mathbb{P}^1(k)$, where we set $\mathcal{M}_{p-s}^\pm(1) = \mathcal{W}_{p-s}^\mp(1) = \mathcal{X}_s^\pm$. Moreover, each sequence gives the projective cover of the right term and the injective envelope of the left term.

The first extensions between indecomposable $\overline{U}$-modules can be calculated by passing to $B\text{-mod}$ and using the Auslander-Reiten formulas ($[ASS]$).

**Proposition 2.4.2.** (i) $\text{Ext}^1_U(\mathcal{E}_s^\pm(n; \lambda), \mathcal{X}_s^\pm) = 0$, \quad $\dim_k \text{Ext}^1_U(\mathcal{E}_s^\pm(n; \lambda), \mathcal{X}_{p-s}^\pm) = n$.

(ii) $\dim_k \text{Ext}^1_U(\mathcal{X}_s^\pm, \mathcal{E}_s^\pm(n; \lambda)) = n$, \quad $\text{Ext}^1_U(\mathcal{X}_{p-s}^\pm, \mathcal{E}_s^\pm(n; \lambda)) = 0$.

(iii) $\dim_k \text{Ext}^1_U(\mathcal{X}_s^\pm(n; \lambda), \mathcal{E}_s^\pm(n; \mu)) = \delta_{\lambda \mu} \min\{m, n\}$, \quad $\dim_k \text{Ext}^1_U(\mathcal{E}_s^\pm(n; \lambda), \mathcal{E}_{p-s}^\pm(n; -\mu)) = \delta_{\lambda \mu} \min\{m, n\}$.

**Proposition 2.4.3.** Let $s=1, \ldots, p-1$, $n \geq 2$ and $\lambda \in \mathbb{P}^1(k)$. Then there exist exact sequences

\[ 0 \to \mathcal{E}_s^\pm(n-1; \lambda) \to \mathcal{E}_s^\pm(n; \lambda) \to \mathcal{E}_s^\pm(1; \lambda) \to 0 \]

3. Calculation of tensor products

3.1. Tensor products of simple modules. Tensor products of simple $\overline{U}$-modules $\mathcal{X}_s^\pm \otimes \mathcal{X}_s' \ (\otimes -$ means $- \otimes -$, here and further) have been studied in $[S1]$. Here we present the results with some different notation.

**Definition 3.1.1.** For $s, s' = 1, \ldots, p$ with $s \leq s'$, define $I_{s,s'}$ and $J_{s,s'}$ by

\[ I_{s,s'} = \{ t = s' - s + 2i - 1 \mid i = 1, \ldots, s, \ t \leq 2p - s - s' \}, \]

\[ J_{s,s'} = \{ t = 2p - 2i + s + 1 \mid i = 1, \ldots, s, \ t \leq p \}, \]

and set $I_{s,s'} = I_{s',s}, \ J_{s,s'} = J_{s',s}$ for $s', s = 1, \ldots, p$ with $s > s'$.

Let us give an example. When $p = 5$, $I_{s,s'}$ and $J_{s,s'}$ are as the following table.

|    | 1 | 2 | 3 | 4 | 5 |
|----|---|---|---|---|---|
| 1  | {1} | {2} | {3} | {4} | 0 |
| 2  | {2} | {1,3} | {2,4} | {3} | 0 |
| 3  | {3} | {2,4} | {1,3} | {2} | 0 |
| 4  | {4} | {3} | {2} | {1} | 0 |
| 5  | 0  | 0  | 0  | 0  | 0  |

|    | 1 | 2 | 3 | 4 | 5 |
|----|---|---|---|---|---|
| 1  | 1 | 0 | 0 | 0 | 0 |
| 2  | 2 | 0 | 0 | 0 | 5 |
| 3  | 3 | 0 | 0 | 5 | 4 |
| 4  | 4 | 0 | 5 | 4 | 3,5 |
| 5  | 5 | 4 | 3,5 | 2,4 | 1,3,5 |

We collect some properties of $I_{s,s'}$ and $J_{s,s'}$ for later use, a proof of which is straightforward.

**Proposition 3.1.2.** Let $s, s', t, t' = 1, \ldots, p$.

(i) $I_{s,s'} \subseteq \{ 1, \ldots, p-1 \}$, $J_{s,s'} \subseteq \{ 1, \ldots, p \}$.

(ii) $I_{s,s'} \cap J_{s,s'} = \emptyset$.

(iii) If $s = 1, \ldots, p-1$, $I_{p-s,s'} = \{ p-t \mid t \in I_{s,s'} \}$. If $s = p$, $I_{p,s'} = \emptyset$.

(iv) $t \in I_{s,s'}$ implies $s' \in I_{s,t}$.

(v) $J_{s,s'} = J_{t,t'}$ if $s + s' = t + t'$. If $s + s' \leq p$, $J_{s,s'} = \emptyset$.

**Remark 3.1.3.** Since $J_{s,s'}$ depends only on $s + s'$ by (v), we denote it by $J_{s+s'}$ in the following.
Theorem 3.3.1. For $s, s' = 1, \ldots, p$ we have
\[
\mathcal{X}_s^+ \otimes \mathcal{X}_{s'}^+ \cong \bigoplus_{t \in I_{s,s'}} \mathcal{X}_t^+ \oplus \bigoplus_{t \in J_{s,s'}} \mathcal{P}_t^+, \\
\mathcal{X}_s^\pm \otimes \mathcal{X}_{s'}^\pm \cong \mathcal{X}_t^\pm \otimes \mathcal{X}_{s'}^\pm \cong \mathcal{X}_s^\pm, \\
\mathcal{P}_s^\pm \otimes \mathcal{X}_t^\pm \cong \mathcal{X}_t^\pm \otimes \mathcal{P}_s^\pm \cong \mathcal{P}_s^\pm,
\]
where we set $\mathcal{P}_p^\pm = \mathcal{X}_p^\pm$.

Remark 3.3.5. The second and third formulas of the theorem enable us to compute the tensor products $\mathcal{X}_s^- \otimes \mathcal{X}_{s'}^+$, $\mathcal{X}_s^+ \otimes \mathcal{X}_{s'}^-$, and $\mathcal{X}_s^- \otimes \mathcal{X}_{s'}^-$. For example, $\mathcal{X}_s^- \otimes \mathcal{X}_{s'}^+ \cong \mathcal{X}_t^- \otimes \mathcal{X}_{s'}^+ \cong \mathcal{X}_s^- \otimes \bigoplus_{t \in I_{s,s'}} \mathcal{X}_t^+ \oplus \bigoplus_{t \in J_{s,s'}} \mathcal{P}_t^-$. In the following this kind of procedure will be omitted.

3.2. Tensor products with projective modules. The tensor products of projective modules with simple modules are also computed in [Sut].

Theorem 3.2.1 (Sut). For $s = 1, \ldots, p - 1$ and $s' = 1, \ldots, p$ we have
\[
\mathcal{P}_s^\pm \otimes \mathcal{X}_{s'}^\pm \cong \mathcal{X}_{s'}^\pm \otimes \mathcal{P}_s^\pm \cong \bigoplus_{t \in I_{s,s'}} \mathcal{P}_t^+ \oplus \bigoplus_{t \in J_{s,s'}} (\mathcal{P}_t^+)^2 \oplus \bigoplus_{t \in J_{p-s,s'}} (\mathcal{P}_t^-)^2,
\]
Since the right side of the formula above is projective, we can calculate the tensor products of projective modules with arbitrary modules.

Corollary 3.2.2. Let $s = 1, \ldots, p - 1$ and $Z$ be an arbitrary $\mathcal{U}$-module. Then $\mathcal{P}_s^\pm \otimes Z$ and $Z \otimes \mathcal{P}_s^\pm$ remain the same up to isomorphism if one replaces $Z$ by the direct sum of its composition factors.

Example 3.2.3. For $s, s' = 1, \ldots, p - 1$ and $n \geq 2$ we have
\[
\mathcal{P}_s^\pm \otimes \mathcal{M}_s^+(n) \cong \mathcal{M}_s^+(n) \otimes \mathcal{P}_s^\pm \\
\cong \mathcal{P}_s^\pm \otimes ((\mathcal{X}_{p-s})^n) \otimes (\mathcal{X}_{p-s})^{n-1}) \\
\cong \bigoplus_{t \in I_{s,s'}} (\mathcal{P}_t^+)^n \otimes (\mathcal{P}_t^-)^n \oplus \bigoplus_{t \in J_{s,s'}} (\mathcal{P}_t^+)^{2n-2} \oplus \bigoplus_{t \in J_{p-s,s'}} (\mathcal{P}_t^-)^{2n-2},
\]
where in the last isomorphism we use Proposition 3.1.2 (iii).

3.3. Tensor products with $\mathcal{M}_s^\pm(n)$ and $\mathcal{W}_s^\pm(n)$. The tensor products of $\mathcal{M}_s^\pm(n)$ and $\mathcal{W}_s^\pm(n)$ can be calculated inductively by using their projective covers and injective envelopes stated in Proposition 2.4.1 and that both $- \otimes Z$ and $Z \otimes -$ are exact functors from $\mathcal{A}$-\text{mod} to itself (since $k$ is a field).

The conclusion of this subsection is as follows:

Theorem 3.3.1. For $s, s' = 1, \ldots, p - 1$ and $m, n \geq 2$ we have
\[
\mathcal{M}_s^+(m) \otimes \mathcal{M}_s^+(n) \cong \bigoplus_{t \in I_{s,s'}} \mathcal{M}_t^+(m) \oplus \bigoplus_{t \in J_{s,s'}} (\mathcal{P}_t^+)^{m-1} \oplus \bigoplus_{t \in J_{p-s,s'}} (\mathcal{P}_t^-)^n, \\
\mathcal{M}_s^+(m) \otimes \mathcal{W}_s^+(n) \cong \bigoplus_{t \in I_{s,s'}} \mathcal{W}_t^+(m) \oplus \bigoplus_{t \in J_{s,s'}} (\mathcal{P}_t^+)^{m-1} \oplus \bigoplus_{t \in J_{p-s,s'}} (\mathcal{P}_t^-)^n, \\
\mathcal{M}_s^+(m) \otimes \mathcal{W}_s^+(n) \cong \bigoplus_{t \in I_{s,s'}} (\mathcal{M}_{p-t}^-(m-n+1) \oplus (\mathcal{P}_t^+)^{(m-1)(n-1)} \oplus (\mathcal{P}_t^-)^{mn} \oplus (\mathcal{P}_t^-)^{m-1} \oplus (\mathcal{P}_t^-)^{m(n-1)}), \\
\mathcal{W}_s^+(m) \otimes \mathcal{W}_s^+(n) \cong \bigoplus_{t \in I_{s,s'}} (\mathcal{W}_t^+(m+n-1) \oplus (\mathcal{P}_t^+)^{(m-1)(n-1)} \oplus (\mathcal{P}_t^+)^{mn} \oplus (\mathcal{P}_t^-)^{mn} \oplus (\mathcal{P}_t^-)^{(m-1)(n-1)} \oplus (\mathcal{P}_t^-)^{(m-1)n} \oplus (\mathcal{P}_t^-)^{(m-1)n} \oplus (\mathcal{P}_t^-)^{(m-1)n}) \oplus (\mathcal{P}_t^-)^{(m-1)n} \oplus (\mathcal{P}_t^-)^{(m-1)n}).
\( M^+_s(m) \otimes W^+_s(n) \cong W^+_s(m) \otimes M^+_s(n) \)

\[
\cong \bigoplus_{t \in I_{s,s'}} \mathcal{Y}_t(m, n) \oplus \bigoplus_{t \in J_{p+-s,s'}} (P^+_t)^{(m-1)n} \oplus \bigoplus_{t \in J_{2p+-s,s'}} (P^-_t)^{(m-1)n-1} \\
\oplus \bigoplus_{t \in J_{p+-s,s'}} (P^-_t)^{(m-1)(n-1)} \oplus \bigoplus_{t \in J_{p+-s,s'}} (P^-_t)^{mn},
\]

where \( \mathcal{Y}_t(m, n) \) is defined by

\[
\mathcal{Y}_t(m, n) = \begin{cases} 
M^+_t(m - n + 1) \oplus (P^+_t)^{(m-1)n} & \text{if } m > n \\
X^+_p \oplus (P^+_t)^{(n-1)n} & \text{if } m = n, \\
W^-_{p-t}(n - m + 1) \oplus (P^+_t)^{(m-1)n} & \text{if } m < n 
\end{cases}
\]

and

\[
M^+_s(n) \otimes X^-_1 \cong X^-_1 \otimes M^+_s(n) \cong M^+_s(n), \\
W^+_s(n) \otimes X^-_1 \cong X^-_1 \otimes W^+_s(n) \cong W^+_s(n).
\]

**Proof.** We only prove the third formula in detail, for the proof of others is similar.

Suppose the formulas about \( M^+_s(n) \otimes X^-_1 \) and \( M^+_s(m) \otimes M^+_s(n-1) \) (here we set \( M^+_s(1) = X^-_{p-s} \) as before) have already shown. Proposition 2.4.1 and the previous lemma give an exact sequence

\[
0 \longrightarrow M^+_s(m) \otimes M^+_{p-s}(n-1) \longrightarrow M^+_s(m) \otimes (P^+_s)^{n-1} \longrightarrow M^+_s(m) \otimes M^+_s(n) \longrightarrow 0.
\]

By hypothesis the left term is isomorphic to

\[
\bigoplus_{t \in I_{s,p-s'}} (M^+_{p-t}(m + n - 2) \oplus (P^-_t)^{(m-1)(n-2)}) \oplus \bigoplus_{t \in J_{p+-s,s'}} (P^-_t)^{(m-1)n-1} \oplus \bigoplus_{t \in J_{2p+-s,s'}} (P^-_t)^{mn}.
\]

Note that we have by Proposition 3.1.2 (iii)

\[
\bigoplus_{t \in I_{s,p-s'}} (M^+_{p-t}(m + n - 2) \oplus (P^-_t)^{(m-1)(n-2)}) = \bigoplus_{t \in I_{s,s'}} (M^+_t(m + n - 2) \oplus (P^-_{p-t})^{(m-1)(n-2)}).
\]

On the other hand the central term is calculated as in Section 5.2 and is isomorphic to

\[
\bigoplus_{t \in I_{s,s'}} ((P^+_t)^{(m-1)n-1} \oplus (P^-_{p-t})^{m(n-1)}) \oplus \bigoplus_{t \in J_{p+-s,s'}} (P^+_t)^{(2m-2)(n-1)} \oplus \bigoplus_{t \in J_{2p+-s,s'}} (P^-_t)^{2m(n-1)} \\
\oplus \bigoplus_{t \in J_{p+-s,s'}} (P^-_t)^{2m(n-1)} \oplus \bigoplus_{t \in J_{2p+-s,s'}} (P^-_t)^{(2m-2)(n-1)}.
\]

Projective summands in the left term also appear in the central term since projective modules are injective, and we notice that an injective homomorphism from \( M^+_t(m + n - 2) \) to an injective module must factor through the injective envelope in Proposition 2.4.1. Consequently we know that the right term is isomorphic to

\[
\bigoplus_{t \in I_{s,s'}} (M^+_{p-t}(m + n - 1) \oplus (P^+_t)^{(m-1)(n-1)} \oplus (P^-_{p-t})^{m(n-1)-(m-1)(n-2)-(m+n-2)}) \\
\oplus \bigoplus_{t \in J_{p+-s,s'}} (P^+_t)^{(2m-2)(n-1)-(m-1)(n-1)} \oplus \bigoplus_{t \in J_{2p+-s,s'}} (P^-_t)^{2m(n-1)} \oplus \bigoplus_{t \in J_{p+-s,s'}} (P^-_t)^{(2m-2)(n-1)}.
\]

and the result follows.
3.4. Tensor products of $E_s^{±}(1; \lambda)$ with simple modules. In this subsection we shall compute $E_s^{±}(1; \lambda) \otimes X_r^{±}$ and $X_r^{±} \otimes E_s^{±}(1; \lambda)$, which is the only case we need explicit calculation in.

**Proposition 3.4.1.** For $s = 1, \ldots, p - 1$ and $\lambda \in \mathbb{P}(k)$ we have

$$E_s^{±}(1; \lambda) \otimes X_1^{−} \cong E_s^{±}(1; −\lambda),$$

$$X_1^{−} \otimes E_s^{±}(1; \lambda) \cong E_s^{±}(1; −(1)^{p−1}\lambda),$$

where for $c \in k$ and $\lambda = [\lambda_1 : \lambda_2] \in \mathbb{P}(k)$ we set $c\lambda = [c\lambda_1 : c\lambda_2]$.

**Proof.** Since $Z \otimes X_1^{−} \otimes X_1^{−} \cong Z \otimes X_1^{+} \cong Z$ for any $U$-module $Z$, it is enough to consider the case about $E_s^{±}(1; \lambda)$. By Proposition 2.3.3 we can assume $X_1^{+} = k\alpha$, $E_s^{±}(1; \lambda) = \bigoplus_{n=0}^{p−1} kb_n \oplus \bigoplus_{m=0}^{p−1} kx_m$ with $U$-action given as that proposition. Then $E_s^{±}(1; \lambda) \otimes X_1^{−}$ has basis $\{b_n \otimes a_0\}_{n=0,\ldots,s−1}$ $\Pi \{x_m \otimes a_0\}_{m=0,\ldots,p−s−1}$ and $U$-action on these vectors is as follows:

$$K(b_n \otimes a_0) = −q^{s−1−2n} b_n \otimes a_0, \quad K(x_m \otimes a_0) = q^{p−s−1−2m} x_m \otimes a_0,$$

$$E(b_n \otimes a_0) = \begin{cases} −[n][s−n]b_{n−1} \otimes a_0 & (n \neq 0) \\ −2\lambda_2 x_{p−s−1} \otimes a_0 & (n = 0) \end{cases}, \quad E(x_m \otimes a_0) = \begin{cases} [m][p−s−m]x_{m−1} \otimes a_0 & (m \neq 0) \\ 0 & (m = 0) \end{cases},$$

$$F(b_n \otimes a_0) = \begin{cases} b_{n+1} \otimes a_0 & (n \neq s−1) \\ λ_1 x_0 \otimes a_0 & (n = s−1) \end{cases}, \quad F(x_m \otimes a_0) = \begin{cases} x_{m+1} \otimes a_0 & (m \neq p−s−1) \\ 0 & (m = p−s−1) \end{cases}. $$

This shows immediately $E_s^{±}(1; \lambda) \otimes X_1^{−} \cong E_s^{±}(1; −\lambda)$.

Similarly, $X_1^{−} \otimes E_s^{±}(1; \lambda)$ has basis $\{a_0 \otimes b_n\}_{n=0,\ldots,s−1}$ $\Pi \{a_0 \otimes x_m\}_{m=0,\ldots,p−s−1}$ and $U$-action on these vectors is as follows:

$$K(a_0 \otimes b_n) = −q^{s−1−2n} a_0 \otimes b_n, \quad K(a_0 \otimes x_m) = q^{p−s−1−2m} a_0 \otimes x_m,$$

$$E(a_0 \otimes b_n) = \begin{cases} [n][s−n]a_0 \otimes b_{n−1} & (n \neq 0) \\ 2\lambda_2 a_0 \otimes x_{p−s−1} & (n = 0) \end{cases}, \quad E(a_0 \otimes x_m) = \begin{cases} −[m][p−s−m]a_0 \otimes x_{m−1} & (m \neq 0) \\ 0 & (m = 0) \end{cases},$$

$$F(a_0 \otimes b_n) = \begin{cases} −a_0 \otimes b_{n+1} & (n \neq s−1) \\ −λ_1 a_0 \otimes x_0 & (n = s−1) \end{cases}, \quad F(a_0 \otimes x_m) = \begin{cases} −a_0 \otimes x_{m+1} & (m \neq p−s−1) \\ 0 & (m = p−s−1) \end{cases}. $$

We take basis $\{(−1)^{n}a_0 \otimes b_n\}_{n=0,\ldots,s−1}\Pi \{(−1)^{m}a_0 \otimes x_m\}_{m=0,\ldots,p−s−1}$ and note that

$$E(a_0 \otimes b_0) = (−1)^{p−s−1}(−1)^{p−s−1}\lambda_2 a_0 \otimes x_{p−s−1}, \quad F((−1)^{p−s−1}a_0 \otimes b_{s−1}) = (−1)^{s}\lambda_1 a_0 \otimes x_0,$$

then we can see that $X_1^{−} \otimes E_s^{±}(1; \lambda) \cong E_s^{±}(1; µ)$ with $µ = [(−1)^sλ_1 : (−1)^{p−s−1}\lambda_2] = (−1)^{p−1}\lambda$. □

We shall next compute $E_s^{±}(1; \lambda) \otimes X_2^{±}$ and $X_2^{±} \otimes E_s^{±}(n; \lambda)$.

**Lemma 3.4.2.** Let $Z$ be a $U$-module and $s = 1, \ldots, p$.

(i) If $v \in Z$ satisfies

$$Kv = \pm q^{s−1}v, \quad Fv = \alpha Fv$$

for some $s = 1, \ldots, p − 1$ and $\alpha \in k$, then $\bigoplus_{n=1}^{p−1} kF^n v$ is a submodule of $Z$ isomorphic to $E_s^{±}(1; [1 : α])$.

(ii) If $v \in Z$ satisfies

$$Kv = \pm q^{−s−1}v, \quad Fv = 0$$

for some $s = 1, \ldots, p − 1$, then $\bigoplus_{n=1}^{p−1} kE^n v$ is a submodule of $Z$ isomorphic to $E_s^{±}(1; [0 : 1])$.

(iii) If $s = p$ and $v \in Z$ satisfies the conditions in (i) or (ii), then $\bigoplus_{n=1}^{p−1} kF^n v$ or $\bigoplus_{n=1}^{p−1} kE^n v$, respectively, is a submodule of $Z$ isomorphic to $X_2^{±}$.

**Proof.** The assertions follow by comparing the standard equations

$$EF^n = F^n E + [n]F^{n−1}q^{n+1}K − q^{n−1}K^{−1},$$

$$FE^n = E^n F − [n]E^{n−1}q^{n−1}K − q^{n−1}K^{−1}$$

with Proposition 2.3.3. □
**Proposition 3.4.3.** For \( s = 1, \ldots, p - 1 \) and \( \lambda = [\lambda_1 : \lambda_2] \in \mathbb{P}^1(k) \) we have

\[
\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \cong \mathcal{E}_{s-1}^+(1; \frac{s}{s-1} \lambda) \oplus \mathcal{E}_{s+1}^+(1; \frac{s}{s+1} \lambda),
\]

\[
\mathcal{X}_2^+ \otimes \mathcal{E}_s^+(1; \lambda) \cong \mathcal{E}_{s-1}^+(1; \frac{s}{s-1} \lambda) \oplus \mathcal{E}_{s+1}^+(1; \frac{s}{s+1} \lambda),
\]

where we put \( \mathcal{E}_{s-1}^+(1; \frac{s}{s-1} \lambda) = \mathcal{X}_p^- \) if \( s = 1 \), and \( \mathcal{E}_{s+1}^+(1; \frac{s}{s+1} \lambda) = \mathcal{X}_p^- \) if \( s = p - 1 \).

**Proof.** It is enough to show that the modules on the left-hand sides have submodules isomorphic to direct summands on the right-hand sides, because any nonzero \( \mathcal{U} \)-module cannot be isomorphic to a submodule of \( \mathcal{E}_{s-1}^+(1; \frac{s}{s-1} \lambda) \) and \( \mathcal{E}_{s+1}^+(1; \frac{s}{s+1} \lambda) \) simultaneously.

As in the proof of Proposition 3.4.1 we can take basis \( \{ b_n \otimes a_l \} \) \( \{ x_m \otimes a_l \} \) \( (n = 0, \ldots, s - 1, m = 0, \ldots, p - s - 1, l = 0, 1) \) of \( \mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \) in which \( \mathcal{U} \)-action on \( b_n, x_m, a_l \) is as Proposition 2.5.5. Let \( v = [s]q^n b_0 \otimes a_0 + \lambda_2 x_{p-s-1} \otimes a_1 \). Then \( K v = q^n v \) and, using the standard equality

\[
\Delta(F^n) = \sum_{k=0}^{n} q^k(n-k) \left[ n \atop k \right] F^n-k K^{-k} \otimes F^k
\]

(where \( \left[ n \atop k \right] = \frac{n!}{k!(n-k)!} \) with \( [k!] = \prod_{i=1}^{k} [i] \)), we have

\[
F^{p-1}v = [s]q^n (F^{p-1}b_0 \otimes a_0 + q^{p-2}[p-1]F^{p-2}K^{-1}b_0 \otimes Fa_0)
\]

\[
= [s] \lambda_1(q^n x_{p-s-1} \otimes a_0 - q^{-1}x_{p-s-2} \otimes a_1),
\]

\[
E v = [s]q^n E b_0 \otimes K a_0 + \lambda_2(E x_{p-s-1} \otimes K a_1 + x_{p-s-1} \otimes E a_1)
\]

\[
= ([s]q^{p-1} \lambda_2 + \lambda_2) x_{p-s-1} \otimes a_0 - [p-s-1]q^{-1} \lambda_2 x_{p-s-2} \otimes a_1
\]

\[
= [s] \lambda_1 \lambda_2(q^n x_{p-s-1} \otimes a_0 - q^{-1}x_{p-s-2} \otimes a_1).
\]

Hence if \( \lambda \neq [0 : 1] \), \( v \) satisfies the condition of (i) (or (iii) when \( s = p - 1 \)) of the previous lemma and thus \( \mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \) has a submodule isomorphic to \( \mathcal{E}_{s+1}^+(1; \frac{s}{s+1} \lambda) \). If \( \lambda = [0 : 1] \), one can verify that \( v = q^n x_{p-s-1} \otimes a_0 - q^{-1} x_{p-s-2} \otimes a_1 \) satisfies the condition of (ii) (or (iii) when \( s = p - 1 \)) of the previous lemma by using the equality

\[
\Delta(E^n) = \sum_{k=0}^{n} q^k(n-k) \left[ n \atop k \right] E^n-k K^k
\]

and in this case also \( \mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \) has a submodule isomorphic to \( \mathcal{E}_{s+1}^+(1; \frac{s}{s+1} \lambda) \). Similarly, if we set \( w = b_1 \otimes a_0 - q[s-1]b_0 \otimes a_1 \) then \( K w = q^{-2}w \) and

\[
F^{p-1}w = -[s] \lambda_1 x_{p-s-1} \otimes a_1, \quad E w = -[s-1] \lambda_2 x_{p-s-1} \otimes a_1
\]

hold, hence the previous lemma shows that \( \mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \) has a submodule isomorphic to \( \mathcal{E}_{s-1}^+(1; \frac{s}{s-1} \lambda) \) as desired (if \( \lambda = [0 : 1] \) set \( w = x_{p-s-1} \otimes a_1 \) then the same argument applies).

In the case of \( \mathcal{X}_2^+ \otimes \mathcal{E}_s^+(1; \lambda) \) one can prove the assertion by the same process: if we set \( v = [s]a_0 \otimes b_0 + \lambda_2 a_1 \otimes x_{p-s-1} \) and \( w = [s-1]a_1 \otimes b_0 - q^{-s-1}a_0 \otimes b_1 \), we have \( K v = q^n v, K w = q^{s-2}w \) and

\[
F^{p-1}v = -[s] \lambda_1(qa_0 \otimes x_{p-s-1} - a_1 \otimes x_{p-s-2}), \quad E v = [s+1] \lambda_2(qa_0 \otimes x_{p-s-1} - a_1 \otimes x_{p-s-2})
\]

\[
F^{p-1}v = -[s] \lambda_1 a_1 \otimes x_{p-s-1}, \quad E v = [s-1] \lambda_2 a_1 \otimes x_{p-s-1},
\]

which leads us to the desired results.

Using Proposition 3.4.1 and 3.4.3 we can calculate tensor products \( \mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \) and \( \mathcal{X}_2^+ \otimes \mathcal{E}_s^+(1; \lambda) \) inductively on \( s' \) as follows: if \( \mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \) has known \( t \leq s' - 1 \), the isomorphism

\[
(\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+_{s-1}) \cong \mathcal{E}_s^+(1; \lambda) \otimes (\mathcal{X}_2^+_{s-1} \otimes \mathcal{X}_2^+) \cong \mathcal{E}_s^+(1; \lambda) \otimes (\mathcal{X}_2^+_{s-2} \otimes \mathcal{X}_2^+)
\]

\[
\cong (\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+_{s-2}) \otimes (\mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+)
\]

determines the indecomposable decomposition of \( \mathcal{E}_s^+(1; \lambda) \otimes \mathcal{X}_2^+ \).

The explicit formulas are as follows:
Proposition 3.4.4. For $s, s' = 1, \ldots, p - 1$ and $\lambda \in \mathbb{P}^1(k)$ we have

$$\mathcal{E}^+_s(1; \lambda) \otimes \mathcal{X}^+_s \cong \bigoplus_{t \in I_{s,s'}} \mathcal{E}^+_t \left(1; \frac{s}{t}\right) \otimes \bigoplus_{t \in J_{s,s'}} \mathcal{P}^+_t \otimes \bigoplus_{t \in J_{p-s,s'}} \mathcal{P}^-_t,$$

$$\mathcal{X}^+_s \otimes \mathcal{E}^+_s(1; \lambda) \cong \bigoplus_{t \in I_{s,s'}} \mathcal{E}^+_t \left(1; (-1)^{s-1-1}\right) \otimes \bigoplus_{t \in J_{s,s'}} \mathcal{P}^+_t \otimes \bigoplus_{t \in J_{p-s,s'}} \mathcal{P}^-_t.$$

Proof. From the exact sequence

$$0 \rightarrow \mathcal{X}^-_p \otimes \mathcal{X}^+_s \rightarrow \mathcal{E}^+_s(1; \lambda) \otimes \mathcal{X}^+_s \rightarrow \mathcal{X}^+_s \otimes \mathcal{X}^+_s \rightarrow 0$$

and $\mathcal{X}^+_p \otimes \mathcal{X}^+_s \cong \bigoplus_{t \in I_{s,s'}} \mathcal{X}^-_t \otimes \bigoplus_{t \in J_{p-s,s'}} \mathcal{P}^-_t$, $\mathcal{X}^+_s \otimes \mathcal{X}^+_s \cong \bigoplus_{t \in I_{s,s'}} \mathcal{X}^+_t \otimes \bigoplus_{t \in J_{p-s,s'}} \mathcal{P}^+_t$ by Theorem 3.1.4 and Proposition 3.3.3 we have

$$\mathcal{E}^+_s(1; \lambda) \otimes \mathcal{X}^+_s \cong \bigoplus_{t \in I_{s,s'}} \mathcal{Z}_t \otimes \bigoplus_{t \in J_{p-s,s'}} \mathcal{P}^+_t \otimes \bigoplus_{t \in J_{p-s,s'}} \mathcal{P}^+_t$$

with an exact sequence $0 \rightarrow \mathcal{Z}^-_p \rightarrow \mathcal{Z}_t \rightarrow \mathcal{X}^+_s \rightarrow 0$ for each $t \in I_{s,s'}$.

On the other hand, Proposition 3.3.3 and the calculation shown before the proposition, we see that a nonprojective indecomposable summand of $\mathcal{E}^+_s(1; \lambda) \otimes \mathcal{X}^+_s$ must be of the form $\mathcal{E}^+_t(1; \frac{s}{t}\lambda)$ with $t = 1, \ldots, p - 1$. Then we have $\mathcal{Z}_t \cong \mathcal{E}^+_t(1; \frac{s}{t}\lambda)$ since $\mathcal{Z}_t$ cannot be projective. Thus we have the first formula.

The proof of the second formula is similar. \[ \square \]

3.5. Rigidity. For computing tensor products for the rest combination, we use a fact on finite-dimensional Hopf algebras.

Let $A$ be a finite-dimensional Hopf algebra over $k$. Then it is known that $A$-mod is a rigid tensor category (cf. Appendix A).

Definition 3.5.1. Let $Z$ be a $A$-module. We define an $A$-module structure on the standard dual $D(Z) = \text{Hom}_k(Z,k)$ by $(a \varphi)(v) = \varphi(S(a)v)$ for $a \in A$, $\varphi \in D(Z)$ and $v \in Z$.

As a consequence of the rigidity, we have the following proposition which is a central tool for computing tensor products (cf. Appendix A).

Proposition 3.5.2. For $A$-modules $Z_1$, $Z_2$, $Z_3$ and $n \geq 0$ we have

$$\text{Ext}^n_A(Z_1 \otimes Z_2, Z_3) \cong \text{Ext}^n_A(Z_1, Z_2 \otimes D(Z_3)),$$

$$\text{Ext}^n_A(Z_1, Z_2 \otimes Z_3) \cong \text{Ext}^n_A(D(Z_2) \otimes Z_1, Z_3).$$

Let us compute $D(\cdot)$ for our case $A = U$.

Proposition 3.5.3. For $s = 1, \ldots, p - 1$ and $\lambda \in \mathbb{P}^1(k)$ we have

$$D(\mathcal{X}^+_s) \cong \mathcal{X}^+_s, \quad D(\mathcal{E}^+_s(1; \lambda)) \cong \mathcal{E}^-_{p-s}(1; (-1)^s\lambda), \quad D(\mathcal{E}^-_{s}(1; \lambda)) \cong \mathcal{E}^+_p(1; (-1)^{p-s}\lambda).$$

Proof. We only prove for $\mathcal{E}^+_s(1; \lambda)$. The other parts are similar.

Take basis $\{b^1_n\}_{n=0,\ldots,p-s-1} \sqcup \{x^m\}_{m=0,\ldots,p-s-1}$ of $\mathcal{E}^+_s(1; \lambda)$ as Proposition 2.3.3. Let $\{b^1_n\}_{n=0,\ldots,s-1} \sqcup \{x^m\}_{m=0,\ldots,p-s-1}$ be the dual basis of this basis. Using the standard equalities

$$S(F^n) = (-1)^n F^n K^n q^{-n(n+1)}, \quad S(E^n) = (-1)^n E^n K^{-n} q^{-n(n-1)},$$

if we put $v = x^1_{p-s-1}$ then $Kv = -q^{p-s-1}v$ and

$$F^{p-s} = q^{p-s-1}(p-1)\lambda_1 a_0^+ \quad \text{and} \quad E v = q^{p-s+1} \lambda_2 a_0^+.$$

From Lemma 3.3.2 this yields $D(\mathcal{E}^+_s(1; \lambda)) \cong \mathcal{E}^-_{p-s}(1; \mu)$, where $\mu = \left[q^{(p-s-1)(p-1)}\lambda_1 : -q^{p-s+1}\lambda_2\right] = (-1)^s\lambda$ as desired (in the case $\lambda = 0 : 1$ the same argument as Proposition 3.4.3 is necessary). \[ \square \]

Proposition 3.5.4. For $s = 1, \ldots, p - 1$, $n \geq 0$ and $\lambda \in \mathbb{P}^1(k)$ we have

$$D(\mathcal{E}^+_s(n; \lambda)) \cong \mathcal{E}^-_{p-s}(n; (-1)^s\lambda), \quad D(\mathcal{E}^-_{s}(n; \lambda)) \cong \mathcal{E}^+_p(n; (-1)^{p-s}\lambda).$$

Proof. We prove the first formula, for the second one is proved similarly. Since $D$ preserves direct sum and dimension over $k$, we know that $D(\mathcal{E}^+_s(n; \lambda))$ is an indecomposable module of dimension $pn$, therefore this is of the form $\mathcal{E}^+_p(n; \mu)$ or is projective (the latter case could occur only if $n \leq 2$).

On the other hand by Proposition 3.5.2 we have

$$\dim_k \text{Ext}^1_D(\mathcal{E}^+_s(n; \lambda), \mathcal{X}^+_s).$$
Comparing these equalities with Proposition 2.4.2 we have $D(\mathcal{E}_s^+(n; \lambda)) \cong \mathcal{E}_{p-s}^-(n; (1)^s \lambda)$ as desired. □

### 3.6. Tensor products of $\mathcal{E}_s^+(n; \lambda)$ with simple modules.

Now we can calculate $\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_s^\pm$ and $\mathcal{X}_{s'}^\pm \otimes \mathcal{E}_s^+(n; \lambda)$ for general $n$.

**Theorem 3.6.1.** For $s, s' = 1, \ldots, p - 1$, $n \geq 1$ and $\lambda \in \mathbb{P}^1(k)$ we have

$$\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm \cong \bigoplus_{t \in I_{s, s'}} \mathcal{E}_t^+(n; \lambda) \otimes \bigoplus_{t \in J_{s, s'}} (P_t^+)^n \otimes \bigoplus_{t \in \lambda_n, J_{p-s, s}} (P_t^-)^n,$$

$$\mathcal{X}_{s'}^\pm \otimes \mathcal{E}_s^+(n; \lambda) \cong \bigoplus_{t \in I_{s, s'}} \mathcal{E}_t^+(n; \lambda) \otimes \bigoplus_{t \in J_{s, s'}} (P_t^+)^n \otimes \bigoplus_{t \in \lambda_n, J_{p-s, s}} (P_t^-)^n$$

and

$$\mathcal{E}_s^\pm(n; \lambda) \otimes \mathcal{X}_1^\pm \cong \mathcal{E}_{p-s}^\mp(n; (1)^s \lambda),$$

$$\mathcal{X}_1^\pm \otimes \mathcal{E}_s^\pm(n; \lambda) \cong \mathcal{E}_s^\mp(n; (1)^{p-1-s} \lambda).$$

**Proof.** We prove the first formula, for others are proved similarly. The same argument as Proposition 3.4.4 shows that there exists an isomorphism

$$\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm \cong \bigoplus_{t \in I_{s, s'}} \mathcal{E}_t^+(n; \lambda) \otimes \bigoplus_{t \in J_{s, s'}} (P_t^+)^n \otimes \bigoplus_{t \in \lambda_n, J_{p-s, s}} (P_t^-)^n$$

and an exact sequence $0 \rightarrow (\mathcal{X}_{s'}^\pm)^n \rightarrow Z_t \rightarrow (\mathcal{X}_s^\pm)^n \rightarrow 0$ for each $t \in I_{s, s'}$. Moreover, by the exact sequence in Proposition 2.4.3 and induction on $n$ we can assume that there exists an exact sequence

$$0 \rightarrow \mathcal{E}_t^+(n - 1; \lambda) \rightarrow Z_t \rightarrow \mathcal{E}_t^+(n; \lambda) \rightarrow 0$$

for each $t \in I_{s, s'}$.

Let $t \in I_{s, s'}$ then from Proposition 3.5.2 Proposition 3.5.3 and Proposition 2.4.2 we have

$$\dim_k \text{Ext}_D^1(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm, \mathcal{X}_s^\pm)$$

$$= \dim_k \text{Ext}_D^1(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm, \mathcal{X}_s^\pm) = \dim_k \text{Ext}_D^1(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm, \mathcal{X}_s^\pm) = 0,$$

$$\dim_k \text{Ext}_D^1(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm, \mathcal{X}_{p-s}^\pm)$$

$$= \dim_k \text{Ext}_D^1(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm, \mathcal{X}_{p-s}^\pm) = \dim_k \text{Ext}_D^1(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm, \mathcal{X}_{p-s}^\pm) = n,$$

$$\dim_k \text{Ext}_D^1(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm, \mathcal{X}_s^\pm(1; \mu))$$

$$= \dim_k \text{Ext}_D^1(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm, \mathcal{X}_s^\pm(1; \mu)) = \dim_k \text{Ext}_D^1(\mathcal{E}_s^+(n; \lambda) \otimes \mathcal{X}_{s'}^\pm, \mathcal{X}_s^\pm(1; \mu))$$

where we note that $\mathcal{E}_s^+(n; \lambda)$ has no nontrivial first extension with modules from $\mathcal{C}(u)$ with $u \neq s$, and that $s \in I_{s, s'}$ by Proposition 3.1.2 (iv). This yields $Z_t \cong \mathcal{E}_t^+(n; \lambda)$ as desired. □
Now we can calculate tensor products of $E^\pm_s(m; \lambda)$ with $M^+_s(n)$ or $W^+_s(n)$ by using projective covers and injective envelopes of $M^+_s(n)$, $W^+_s(n)$. The proof is analogous to that of Theorem 3.3.1 and is omitted.

**Theorem 3.6.2.** For $s, s' = 1, \ldots, p - 1$, $m \geq 1$, $n \geq 2$ and $\lambda \in \mathbb{P}^1(k)$ we have

$$E^\pm_s(m; \lambda) \otimes M^+_s(n) \cong \bigoplus_{t \in I_{s,s'}} \left( E^-_{p-t}(m; -\frac{[s]}{[t]} \lambda) \oplus (p^+_t)^m(n-1) \right) \oplus \bigoplus_{t \in J_{s,s'}} \left( (p^+_t)^{-m} \oplus (p^-_t)^{m(n-1)} \right),$$

$$M^+_s(n) \otimes E^\pm_s(n; \lambda) \cong \bigoplus_{t \in I_{s,s'}} \left( E^+_t(m; -1)^{s-1} \frac{[s]}{[t]} \lambda \oplus (p^+_t)^m(n-1) \right) \oplus \bigoplus_{t \in J_{s,s'}} \left( (p^+_t)^{-m} \oplus (p^-_t)^{m(n-1)} \right),$$

$$E^\pm_s(m; \lambda) \otimes W^+_s(n) \cong \bigoplus_{t \in I_{s,s'}} \left( E^+_t(m; -1)^{s'} \frac{[s]}{[t]} \lambda \oplus (p^+_t)^m(n-1) \right) \oplus \bigoplus_{t \in J_{s,s'}} \left( (p^+_t)^{-m} \oplus (p^-_t)^{m(n-1)} \right),$$

$$W^+_s(n) \otimes E^\pm_s(n; \mu) \cong \bigoplus_{t \in I_{s,s'}} \left( E^+_t(m; -1)^{s'} \frac{[s]}{[t]} \lambda \oplus (p^+_t)^m(n-1) \right) \oplus \bigoplus_{t \in J_{s,s'}} \left( (p^+_t)^{-m} \oplus (p^-_t)^{m(n-1)} \right).$$

### 3.7. Tensor products of $E^\pm_s(m; \lambda)$ and $E^\pm_s(n; \mu)$

We calculate $E^\pm_s(m; \lambda) \otimes E^\pm_s(n; \mu)$ by using rigidity as the previous subsection.

We note that there exist following exact sequences:

$$0 \rightarrow E^\pm_s(m; \lambda) \otimes (X^-_{p-s})^m \rightarrow E^\pm_s(m; \lambda) \otimes E^\pm_s(n; \mu) \rightarrow E^\pm_s(m; \lambda) \otimes (X^+_s)^n \rightarrow 0,$$

$$0 \rightarrow (X^-_{p-s})^m \otimes E^\pm_s(n; \mu) \rightarrow E^\pm_s(m; \lambda) \otimes E^\pm_s(n; \mu) \rightarrow (X^+_s)^m \otimes E^\pm_s(n; \mu) \rightarrow 0.$$

The left and right terms of these sequences are computed by using Theorem 3.6.1 which proves the next result:

**Proposition 3.7.1.** For $s, s' = 1, \ldots, p - 1$, $m, n \geq 1$ and $\lambda, \mu \in \mathbb{P}^1(k)$ we have

$$E^\pm_s(m; \lambda) \otimes E^\pm_s(n; \mu) \cong \bigoplus_{t \in I_{s,s'}} V_t(s, s'; m, n; \lambda, \mu) \oplus \bigoplus_{t \in J_{s,s'}} \left( (p^+_t)^m \oplus (p^-_t)^{m(n-1)} \right) \oplus \bigoplus_{t \in J_{2p-s-s'}} \left( (p^+_t)^{-m} \oplus (p^-_t)^{m(n-1)} \right),$$

where $V_t(s, s'; m, n; \lambda, \mu)$ is a module in $C(t)$. Moreover, there exist exact sequences

$$0 \rightarrow E^-_{p-t}(m; -\frac{[s]}{[t]} \lambda)^n \rightarrow V_t(s, s'; m, n; \lambda, \mu) \rightarrow E^+_t(m; \frac{[s]}{[t]} \lambda)^n \rightarrow 0,$$

$$0 \rightarrow E^-_{p-t}(n; (-1)^{s'} \frac{[s']}{[t']} \mu)^m \rightarrow V_t(s, s'; m, n; \lambda, \mu) \rightarrow E^+_t(n; (-1)^{s-1} \frac{[s]}{[t]} \lambda)^m \rightarrow 0.$$
\begin{theorem}
For \( s, s' = 1, \ldots, p - 1, t \in I, m, n \geq 1, \) and \( \lambda, \mu \in \mathbb{P}(k) \) we have
\[
\mathcal{V}_t(s, s'; m, n; \lambda, \mu) \cong \begin{cases}
\mathcal{E}_t^+(l, \nu_t) \oplus \mathcal{E}_{p-t}^-(l, -\nu_t) \oplus (\mathcal{P}_t^+)^{mn-l} & \left( \frac{|s|}{|t|} \lambda = (1)^{s-1} \frac{|s'|}{|t|} \mu = \nu_t \right) \\
(\mathcal{P}_t^+)^{mn} & \left( \frac{|s|}{|t|} \lambda \neq (1)^{s-1} \frac{|s'|}{|t|} \mu \right)
\end{cases},
\]
where we put \( l = \min\{m, n\} \).
\end{theorem}

\begin{proof}
We have
\[
\dim_k \text{Ext}_t^1(\mathcal{E}_s^+(m; \lambda) \otimes \mathcal{E}_{s'}^+(n; \mu), \mathcal{X}_t^+)
= \dim_k \text{Ext}_t^1(\mathcal{E}_s^+(m; \lambda), \mathcal{X}_t^+ \otimes \mathcal{E}_{p-s'}^-(n; (1)^{s'} \mu))
= \dim_k \text{Ext}_t^1(\mathcal{E}_s^+(m; \lambda), \mathcal{E}_{p-s}^-(n; (1)^{s'-t-1} \frac{|s|}{|t|} \mu))
= \begin{cases}
\min\{m, n\} & (1)^{s-1} \frac{|s|}{|t|} \lambda = \frac{|s'|}{|t|} \mu \\
0 & (1)^{s-1} \frac{|s|}{|t|} \lambda \neq \frac{|s'|}{|t|} \mu
\end{cases}, \quad (t \equiv s - s' + 1 \mod 2 \text{ for } t \in I, s')
\]
\[
\dim_k \text{Ext}_t^1(\mathcal{E}_s^+(m; \lambda) \otimes \mathcal{E}_{s'}^+(n; \mu), \mathcal{X}_{p-t}^+)
= \dim_k \text{Ext}_t^1(\mathcal{E}_s^+(m; \lambda), \mathcal{E}_{p-t}^+ \otimes \mathcal{E}_{p-s'}^-(n; (1)^{s'} \mu))
= \dim_k \text{Ext}_t^1(\mathcal{E}_s^+(m; \lambda), \mathcal{E}_s^+(n; (1)^{s'+t-1} \frac{|s|}{|t|} \mu))
= \begin{cases}
\min\{m, n\} & (1)^{s-1} \frac{|s|}{|t|} \lambda = \frac{|s'|}{|t|} \mu \\
0 & (1)^{s-1} \frac{|s|}{|t|} \lambda \neq \frac{|s'|}{|t|} \mu
\end{cases}.
\]
These equalities and the exact sequences in the previous proposition show that, if \( (1)^{s-1} \frac{|s|}{|t|} \lambda \neq \frac{|s'|}{|t|} \mu \), \( \mathcal{V}_t(s, s'; m, n; \lambda, \mu) \) is a projective module, hence it is isomorphic to \( (\mathcal{P}_t^+)^{mn} \).

From now on we assume \((1)^{s-1} \frac{|s|}{|t|} \lambda = \frac{|s'|}{|t|} \mu \) and denote this same value \( \frac{|s|}{|t|} \lambda = (1)^{s-1} \frac{|s'|}{|t|} \mu \) by \( \nu_t \). Then again from the equalities above, the nonprojective direct summand of \( \mathcal{V}_t(s, s'; m, 1; \lambda, \mu) \) is isomorphic to \( \mathcal{E}_t^+(1, \nu_t) \oplus \mathcal{E}_{p-t}^-(1, -\nu_t) \).

For general \( n \), using the result for \( n = 1 \) we have
\[
\dim_k \text{Ext}_t^1(\mathcal{E}_s^+(m; \lambda) \otimes \mathcal{E}_{s'}^+(n; \mu), \mathcal{E}_t^+(1; \nu_t))
= \dim_k \text{Ext}_t^1(\mathcal{E}_s^+(m; \lambda), \mathcal{E}_t^+(1; \nu_t) \otimes \mathcal{E}_{p-s'}^-(n; (1)^{s'} \mu))
= \dim_k \text{Ext}_t^1(\mathcal{E}_s^+(m; \lambda), \mathcal{E}_t^+(1; \lambda) \oplus \mathcal{E}_{p-s}^-(1; -\lambda))
= 2.
\]
This equality and the previous equalities show that the nonprojective direct summand of \( \mathcal{V}_t(s, s'; m, n; \lambda, \mu) \) is isomorphic to \( \mathcal{E}_t^+(\min\{m, n\}, \nu_t) \oplus \mathcal{E}_{p-t}^-(\min\{m, n\}, -\nu_t) \). The assertion follows.
\end{proof}

Theorem 3.3.4, Theorem 3.2.1, Corollary 3.2.2, Theorem 3.3.1, Theorem 3.6.1, Theorem 3.6.2, Proposition 3.7.1, Theorem 3.7.2 and obvious combination of them give indecomposable decomposition of tensor products of arbitrary \( U\)-modules.

From the results in this section we have

\begin{proposition}
(i) Let \( Z_1, Z_2 \) be \( \mathcal{U}_q(sl_2)\)-modules. If \( Z_1 \) nor \( Z_2 \) do not have any indecomposable summand of type \( \mathcal{E} \), we have \( Z_1 \otimes Z_2 \cong Z_2 \otimes Z_1 \).
(ii) If \( p = 2 \), for arbitrary \( \mathcal{U}_q(sl_2)\)-modules \( Z_1, Z_2 \) we have \( Z_1 \otimes Z_2 \cong Z_2 \otimes Z_1 \).
(iii) If \( p \geq 3 \), there exist \( \mathcal{U}_q(sl_2)\)-modules \( Z_1, Z_2 \) such that \( Z_1 \otimes Z_2 \neq Z_2 \otimes Z_1 \). In particular, \( \mathcal{U}_q(sl_2)\)-mod is not a braided tensor category.
\end{proposition}

\begin{proof}
(i) and (ii) are clear. For (iii), set \( Z_1 = \mathcal{E}_1^+(1; [1 : 1]) \) and \( Z_2 = \mathcal{X}_2^+ \).
\end{proof}

As a by-product we have

\begin{corollary}
If \( q \) is a primitive \( 2p \)-th root of unity, \( \mathcal{U}_q(sl_2) \) has no universal \( R \)-matrices for \( p \geq 3 \). That is, it is not a quasi-triangular Hopf algebra.
\end{corollary}
Remark 3.7.5. Let $U_q^{>0}$ be the $k$-subalgebra of $U_q(sl_2)$ generated by $E, K, K^{-1}$. It is a 2$p^2$-dimensional Hopf subalgebra of $U_q(sl_2)$. By the quantum double construction, $D(U_q^{>0}) = D(U_q^{>0}) \otimes U_q^{>0}$ has a structure of a quasi-triangular Hopf algebra. One can show that there is no surjective Hopf algebra homomorphism $D(U_q^{>0}) \to U_q(sl_2)$. This fact tells us $U_q(sl_2)$ can not be obtained from the usual quantum double construction, but it does not give a proof of non-existence of universal $R$-matrices.

4. COMPLEMENTS

4.1. A quasi-triangular Hopf algebra $\mathcal{D}$. The phenomenon which we showed in Proposition 3.7.3 can be explained partly by considering a finite dimensional Hopf $k$-algebra $\mathcal{D}$ which has a Hopf subalgebra isomorphic to $U$. $\mathcal{D}$ is defined by generators $e, f, t, t^{-1}$ and relations

$$tt^{-1} = t^{-1}t = 1, \quad te^{-1} = qe, \quad tf^{-1} = q^{-1}f,$$

$$ef - fe = \frac{t^2 - t^{-2}}{q - q^{-1}}, \quad t^4 = 1, \quad e^p = 0, \quad f^p = 0.$$

The Hopf algebra structure on $\mathcal{D}$ is given by

$$\Delta: e \mapsto e \otimes t^2 + 1 \otimes e, \quad F \mapsto f \otimes 1 + t^{-2} \otimes f,$$

$$t \mapsto t \otimes t, \quad t^{-1} \mapsto t^{-1} \otimes t^{-1},$$

$$\varepsilon: e \mapsto 0, \quad f \mapsto 0, \quad t \mapsto 1, \quad t^{-1} \mapsto 1,$$

$$S: e \mapsto -et^{-2}, \quad f \mapsto -t^2f, \quad t \mapsto t^{-1}, \quad t^{-1} \mapsto t.$$

$U$ can be embedded into $\mathcal{D}$ as a Hopf subalgebra by

$$v: E \mapsto e, \quad F \mapsto f, \quad K \mapsto t^2.$$

We remark that finite-dimensional indecomposable $\mathcal{D}$-modules are classified by Xiao ([X3], see also [X1], [X2]). Those are parametrized by the positive root system of type $A_1^{(1)}$ and some additional data.

As in [FGST1], $\mathcal{D}$ is a quasi-triangular Hopf algebra and has an universal $R$-matrix

$$\mathcal{R} = \frac{1}{4p} \sum_{m=0}^{p-1} \sum_{n,j=0}^{p-1} \frac{(q - q^{-1})^m}{m!} q^{\frac{m(m-1) + m(n-j) - n}{2}} e^m t^n \otimes f^n t^j \in \mathcal{D} \otimes \mathcal{D}.$$  

This shows that $\mathcal{D}$-mod is a braided tensor category.

Definition 4.1.1. Let $Z$ be a finite dimensional $U$-module. The $U$-action on $Z$ is defined by a $k$-algebra homomorphism $\rho: U \to \text{End}_k(Z)$. We call $Z$ liftable if there exists a $k$-algebra homomorphism $\rho': \mathcal{D} \to \text{End}_k(Z)$ such that $\rho = \rho' \circ v$. The map $\rho'$ is called a lifting of $\rho$.

The following lemma is easy to verify.

Lemma 4.1.2. Each indecomposable $U$-module except $E^\pm_s(n; \lambda)$ ($\lambda \neq [1 : 0], [0 : 1]$) is liftable. On the other hand, $E^\pm_s(n; \lambda)$ ($\lambda \neq [1 : 0], [0 : 1]$) is not liftable. As a by-product, a universal $R$-matrix $\mathcal{R}$ can act on $Z \otimes Z$ for liftable modules $Z_1, Z_2$, and if either $Z_1$ or $Z_2$ is $E^\pm_s(n; \lambda)$ ($\lambda \neq [1 : 0], [0 : 1]$), $\mathcal{R}$ can not act on $Z_1 \otimes Z_2$.

As we already mentioned, Xiao [X3] classify all finite-dimensional indecomposable $\mathcal{D}$-modules. In his list, there is the indecomposable $\mathcal{D}$-module $T^s(\alpha, \kappa, n)$ where $1 \leq s \leq p - 1$, $\alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$, $\kappa = (\kappa_1, \kappa_2) \in (k^\times)^2$ and $n$ is a positive integer. In Appendix B, we will give the explicit construction of $T^s(\alpha, \kappa, n)$.

Assume $\alpha \in \{\pm 1\}$. As a $U$-module, $T^s(\alpha, \kappa, n)$ decomposes into two indecomposable modules (for details, see Appendix B):

$$T^s(\alpha, \kappa, n) \cong E^+_{s}(n; \sqrt{\kappa_1 \kappa_2}) \oplus E^+_{s}(n; -\sqrt{\kappa_1 \kappa_2}).$$

Here we set $E^+_{s}(n; \beta) = E^+_{s}(n; [1 : \beta])$ for $\beta \in k$.

Let $Z$ be a liftable $U$-module and, by a fixed lifting $\rho': \mathcal{D} \to \text{End}_k(Z)$, we regard $Z$ as a $\mathcal{D}$-module. Since $\mathcal{D}$ has an universal $R$-matrix $\mathcal{R}$, there is an isomorphism of $\mathcal{D}$-modules:

$$\sigma \mathcal{R}: T^s(\alpha, \kappa, n) \otimes Z \sim Z \otimes T^s(\alpha, \kappa, n).$$
Here we denote by \( (ii) \) regarding \( D \) of \( \mu \) of \( A \). From now on we assume \( A \) is a Hopf algebra with coproduct \( \Delta \), counit \( \varepsilon \) and antipode \( S \). A right integral \( \mu \) of \( A \) is an element of \( D(A) \) satisfying
\[
(\mu \otimes \text{id})\Delta(a) = \mu(a)1_A
\]
for all \( a \in A \). Here \( 1_A \) is the unit of \( A \). The following theorem is due to Sweedler [Sw] (See also [R]).

**Theorem A.1.1 ([Sw]).** Assume \( A \) is a finite-dimensional Hopf algebra over \( \mathbb{K} \).

(i) Up to a scalar multiple, there uniquely exists a right integral \( \mu \).

(ii) Regarding \( A \) as a right \( A \)-module, \( D(A) \) has a left \( A \)-module structure. For a right integral \( \mu \), the map \( A \to D(A) \) defined by
\[
a \mapsto (a \to \mu)
\]
is an isomorphism of left \( A \)-modules.

(iii) \( S \) is bijective.

**Remark A.1.2.** The right integral of \( \mathcal{U}_q(\mathfrak{sl}_2) \) is given by
\[
\mu(F^i E^m K^n) = c\delta_{i,p-1}\delta_{m,p-1}\delta_{n,p+1} \quad (c \in k^\times).
\]
The following corollary follows from the second statement of the theorem:

**Corollary A.1.3.** If \( A \) is a finite-dimensional Hopf algebra, \( A \) is a Frobenius algebra. As a by-product, the following are equivalent:

(a) \( M \) is a projective \( A \)-module.

(b) \( M \) is an injective \( A \)-module.

**A.2. Rigid tensor categories.** In this subsection, we introduce a notion of rigid tensor categories following Bakalov and Kirillov, Jr. [BK].

Let \( \mathcal{C} \) be a tensor category with the bifunctor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) and the unit object \( 1 \in \text{Ob} \mathcal{C} \). For \( V \in \text{Ob} \mathcal{C} \), a right dual to \( V \) is an object \( D^R(V) \) with two morphisms
\[
e^V_l : D^R(V) \otimes V \to 1,
\]
\[
i^V_l : 1 \to V \otimes D^R(V),
\]
such that the two compositions
\[
V \cong 1 \otimes V \xrightarrow{i^V_l \otimes \text{id}_V} V \otimes D^R(V) \otimes V \xrightarrow{\text{id}_V \otimes e^V_l} V \cong V
\]
and
\[
D^R(V) \cong D^R(V) \otimes 1 \xrightarrow{id_{D^R(V)} \otimes i^V_l} D^R(V) \otimes V \otimes D^R(V) \xrightarrow{e^V_l \otimes id_{D^R(V)}} 1 \otimes D^R(V)\cong D^R(V)
\]
are equal to \( \text{id}_V \) and \( \text{id}_{D^R(V)} \), respectively.

Similarly to the above, we define a left dual of \( V \) to be an object \( D^L(V) \) with morphisms
\[
e^V_l : V \otimes D^L(V) \to 1,
\]
\[
i^V_l : 1 \to D^L(V) \otimes V
\]
and similar axioms.

**Definition A.2.1.** A tensor category $\mathcal{C}$ is called rigid if every object in $\mathcal{C}$ has right and left duals.

**Proposition A.2.2.** Let $\mathcal{C}$ be a rigid tensor category and $V_1, V_2, V_3$ objects in $\mathcal{C}$.

(i) $\text{Hom}_C(V_1, V_2 \otimes V_3) \cong \text{Hom}_C(D^R(V_2) \otimes V_1, V_3) \cong \text{Hom}_C(V_1 \otimes D^L(V_3), V_2)$.

(ii) $\text{Hom}_C(V_1 \otimes V_2, V_3) \cong \text{Hom}_C(V_1, V_3 \otimes D^R(V_2)) \cong \text{Hom}_C(V_2, D^L(V_1) \otimes V_3)$.

**Proof.** We only prove the first isomorphism of (ii). The others are proved by the similar way.

Define a map $\Phi: \text{Hom}_C(V_1 \otimes V_2, V_3) \to \text{Hom}_C(V_1, V_3 \otimes D^R(V_2))$ by

$$\Phi(f): V_1 \otimes V_2 \otimes K \xrightarrow{id \otimes i^L_{V_2}} V_1 \otimes V_2 \otimes D^R(V_2) \xrightarrow{f \otimes id_{D^R(V_2)}} V_3 \otimes D^R(V_2)$$

for $f \in \text{Hom}_C(V_1 \otimes V_2, V_3)$. We remark that, by the rigidity axioms, $\Phi(f)$ gives an element of $\text{Hom}_C(V_1, V_3 \otimes D^R(V_2))$. Similarly we define a well-defined map $\Psi: \text{Hom}_C(V_1, V_3 \otimes D^R(V_2)) \to \text{Hom}_C(V_1 \otimes V_2, V_3)$ by

$$\Psi(g): V_1 \otimes V_2 \xrightarrow{g \otimes id_{V_2}} V_3 \otimes D^R(V_2) \otimes V_2 \xrightarrow{id_{V_3} \otimes e^R_{V_2}} V_3 \otimes K \cong V_3$$

for $g \in \text{Hom}_C(V_1, V_3 \otimes D^R(V_2))$.

It is easy to see that $\Phi$ and $\Psi$ are inverse each other. Thus, we have the statement. \qed

A.3. The module category over a finite-dimensional Hopf algebra. Recall that $A$ is a finite-dimensional Hopf algebra over a field $K$. Let $A\text{-}\text{mod}$ be the category of finite-dimensional left $A$-modules. It has a structure of a tensor category associated with the Hopf algebra structure of $A$.

For a finite-dimensional left $A$-module $V$, we define two left module structure on $D(V) = \text{Hom}_K(V, K)$: for $a \in A$, $\lambda \in D(V)$ and $v \in V$,

$$(a \otimes \lambda)(v) = \lambda(S(a)v),$$

$$(a \otimes \lambda)(v) = \lambda(S^{-1}(a)v).$$

We denote by $D^R(V)$ the first left $A$-module structure on $D(V)$ and by $D^L(V)$ the second one.

**Remark A.3.1.** (i) Since $A$ is finite-dimensional, the antipode $S$ is bijective (Theorem A.1.1 (iii)). Thus, $S^{-1}$ is a well-defined anti-isomorphism of $A$. However, $(A, \Delta, \varepsilon, S^{-1})$ is not a Hopf algebra in general. More precisely $S^{-1}$ does not satisfy the axiom of an antipode.

(ii) If $S^2 \neq id_A$, $D^L(V)$ is not isomorphic to $D^R(V)$, in general. We remark that $S^2 \neq id_A$ for $A = \overline{U}_q(\mathfrak{sl}_2)$.

By the construction, it is easy to see that $D^R(D^L(V)) \cong V$ and $D^L(D^R(V)) \cong V$.

The following proposition is easy to verify.

**Proposition A.3.2.** Let $V$ be an object in $A\text{-}\text{mod}$, $\{v_i\}$ a basis of $V$ and $\{v_i^*\}$ the dual basis of $D(V)$.

(i) The $K$-linear maps $e^R_V: D^R(V) \otimes V \to K$ and $i^L_V: K \to V \otimes D^R(V)$ defined by

$$e^R_V(\lambda \otimes v) = \lambda(v) \quad \text{and} \quad i^L_V(\alpha) = \alpha \left( \sum_i v_i \otimes v_i^* \right)$$

are homomorphisms of left $A$-modules, where we regard $K$ as a left $A$-module via the counit $\varepsilon$. Therefore $D^R(V)$ is the right dual to $V$.

(ii) Similarly, the $K$-linear maps $e^L_V: V \otimes D^L(V) \to K$ and $i^R_V: K \to D^L(V) \otimes V$ defined by

$$e^L_V(v \otimes \lambda) = \lambda(v) \quad \text{and} \quad i^R_V(\alpha) = \alpha \left( \sum_i v_i^* \otimes v_i \right)$$

are homomorphisms of left $A$-modules. Therefore $D^L(V)$ is the left dual to $V$.

(iii) $A\text{-}\text{mod}$ is a rigid tensor category.

As a consequence of the rigidity of $A\text{-}\text{mod}$ and Proposition A.2.2, we have

**Corollary A.3.3.** Let $V_1, V_2, V_3$ be objects in $A\text{-}\text{mod}$.

(i) $\text{Hom}_A(V_1, V_2 \otimes V_3) \cong \text{Hom}_A(D^R(V_2) \otimes V_1, V_3) \cong \text{Hom}_A(V_1 \otimes D^L(V_3), V_2)$.

(ii) $\text{Hom}_A(V_1 \otimes V_2, V_3) \cong \text{Hom}_A(V_1, V_3 \otimes D^R(V_2)) \cong \text{Hom}_A(V_2, D^L(V_1) \otimes V_3)$.
Corollary A.3.4. Let \( P \) be a projective module. Then \( P \otimes V \) and \( V \otimes P \) are also projective for any object \( V \) in \( A\text{-mod} \).

Proof. We only show the projectivity of \( P \otimes V \). Let \( W_1 \) and \( W_2 \) be objects in \( A\text{-mod} \), and \( g: W_1 \to W_2 \) a surjective \( A \)-homomorphism. It is enough to show that

\[
g_*: \text{Hom}_A(P \otimes V, W_1) \to \text{Hom}_A(P \otimes V, W_2)
\]
is surjective. Let us consider the following diagram:

\[
\begin{array}{ccc}
\text{Hom}_A(P \otimes V, W_1) & \xrightarrow{g_*} & \text{Hom}_A(P \otimes V, W_2) \\
\text{Hom}_A(P, W_1 \otimes D^R(V)) & \xrightarrow{(g \otimes \text{id})_*} & \text{Hom}_A(P, W_2 \otimes D^R(V)) \\
\end{array}
\]
where the vertical arrows are the isomorphisms constructed in the proof of Proposition A.2.2. By the construction this diagram is commutative. Since \( P \) is projective, \( (g \otimes \text{id})_* \) is surjective. Thus \( g_* \) is also surjective.

Corollary A.3.5 (cf. Proposition 3.5.2). Let \( V_1, V_2, V_3 \) be objects in \( A\text{-mod} \). For any \( n \geq 0 \), we have the following.

(i) \( \text{Ext}^n_A(V_1, V_2 \otimes V_3) \cong \text{Ext}^n_A(D^R(V_2) \otimes V_1, V_3) \cong \text{Ext}^n_A(V_1 \otimes D^L(V_3), V_2) \).

(ii) \( \text{Ext}^n_A(V_1 \otimes V_2, V_3) \cong \text{Ext}^n_A(V_1, V_3 \otimes D^R(V_2)) \cong \text{Ext}^n_A(V_2, D^L(V_1) \otimes V_3) \).

Proof. We only prove the first isomorphism in (ii). Take a projective resolution of \( V_1 \):

\[
\cdots \xrightarrow{d_2} P_1(V_1) \xrightarrow{d_1} P_0(V_1) \xrightarrow{d_0} V_1 \to 0.
\]

Then

\[
\text{Ext}^n_A(V_1, V_3 \otimes D^R(V_2)) = \frac{\ker(d_{n+1}^*: \text{Hom}_A(P_n(V_1), V_3 \otimes D^R(V_2)) \to \text{Hom}_A(P_{n+1}(V_1), V_3 \otimes D^R(V_2)))}{\text{im}(d_n^*: \text{Hom}_A(P_n-1(V_1), V_3 \otimes D^R(V_2)) \to \text{Hom}_A(P_n(V_1), V_3 \otimes D^R(V_2)))}.
\]

Since \(- \otimes V_2\) is an exact functor, the sequence

\[
\cdots \xrightarrow{d_2 \otimes \text{id}_{V_2}} P_1(V_1) \otimes V_2 \xrightarrow{d_1 \otimes \text{id}_{V_2}} P_0(V_1) \otimes V_2 \xrightarrow{d_0 \otimes \text{id}_{V_2}} V_1 \otimes V_2 \to 0
\]
is exact. Moreover, since \( P_n(V_1) \otimes V_2 \) is projective for any \( n \geq 0 \), this sequence gives a projective resolution of \( V_1 \otimes V_2 \). Therefore we have

\[
\text{Ext}^n_A(V_1 \otimes V_2, V_3) = \frac{\ker((d_{n+1} \otimes \text{id}_{V_2})^*: \text{Hom}_A(P_n(V_1) \otimes V_2, V_3) \to \text{Hom}_A(P_{n+1}(V_1) \otimes V_2, V_3))}{\text{im}((d_n \otimes \text{id}_{V_2})^*: \text{Hom}_A(P_{n-1}(V_1) \otimes V_2, V_3) \to \text{Hom}_A(P_n(V_1) \otimes V_2, V_3))}.
\]

By the construction, there exists a commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_A(P_n(V_1), V_3 \otimes D^R(V_2)) & \xrightarrow{d_{n+1}^*} & \text{Hom}_A(P_{n+1}(V_1), V_3 \otimes D^R(V_2)) \\
\text{Hom}_A(P_n(V_1) \otimes V_2, V_3) & \xrightarrow{(d_{n+1} \otimes \text{id}_{V_2})^*} & \text{Hom}_A(P_{n+1}(V_1) \otimes V_2, V_3) \\
\end{array}
\]

This diagram induces an isomorphism \( \text{Ext}^n_A(V_1, V_3 \otimes D^R(V_2)) \cong \text{Ext}^n_A(V_1 \otimes V_2, V_3) \). \( \square \)
B.2. The module $T^*(\alpha, \kappa, n)$ and its decomposition as $\mathcal{U}$-module. Let $\alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$, \( \kappa = (\kappa_1, \kappa_2) \in (k^\times)^2 \) and $n$ be a positive integer. The indecomposable $\mathcal{D}$-module $T^*(\alpha, \kappa, n)$ is defined as follows:

The basis of $T^*(\alpha, \kappa, n)$ is \( \{ e^*_u(\alpha, m), \tilde{e}^*_u(\alpha, m) \mid 0 \leq u \leq p - 1, 1 \leq m \leq n \} \) and the action of $e, f, t^\pm$ is given as:

\[
  t^\pm e^*_u(\alpha, m) = \alpha^\pm q^{(s-1-2u)/2} e^*_u(\alpha, m), \quad t^\pm \tilde{e}^*_u(\alpha, m) = -\alpha^\pm q^{-{(s-1-2u)/2}} \tilde{e}^*_u(\alpha, m),
\]

\[
  e e^*_u(\alpha, m) = \begin{cases} 
    \alpha^2[u][s-u]e^*_{u-1}(\alpha, m) + \tilde{e}^*_{p-1}(\alpha, m-1) & (u \neq 0), \\
    \kappa_1 e^*_u(\alpha, m) + \tilde{e}^*_{p-1}(\alpha, m-1) & (u = 0),
  \end{cases}
\]

\[
  e \tilde{e}^*_u(\alpha, m) = \begin{cases} 
    \alpha^2[u][s-u]\tilde{e}^*_{u-1}(\alpha, m) & (u \neq 0), \\
    \kappa_2 \tilde{e}^*_u(\alpha, m) & (u = 0),
  \end{cases}
\]

\[
  f e^*_u(\alpha, m) = e^*_{u+1}(\alpha, m), \quad f \tilde{e}^*_u(\alpha, m) = \tilde{e}^*_{u+1}(\alpha, m),
\]

where $e^*_u(\alpha, 0) = e^*_u(\alpha, 0) = \tilde{e}^*_u(\alpha, 0) = 0$ and $e^*_u(\alpha, m) = \tilde{e}^*_u(\alpha, m) = 0$.

Assume $\alpha^2 = 1$. Consider an invertible $(2n \times 2n)$ matrix $Q$ which satisfies

\[
  Q^{-1} \begin{pmatrix} O & J(n; \sqrt{\kappa_1 \kappa_2}) \\ J(n; \kappa_1 \kappa_2) & O \end{pmatrix} Q = \begin{pmatrix} J(n; \sqrt{\kappa_1 \kappa_2}) & O \\ O & J(n; -\sqrt{\kappa_1 \kappa_2}) \end{pmatrix}
\]

where $J(n; \beta)$ is the $(n \times n)$-Jordan cell with the eigenvalue $\beta$. Define $b^*_u(\alpha, m), b^*_u(\alpha, m) (0 \leq u \leq p - 1, 1 \leq m \leq n)$ by

\[
  (b^*_u(\alpha, 1), \ldots, b^*_u(\alpha, n), b^*_u(\alpha, 1), \ldots, b^*_u(\alpha, n)) := (e^*_u(\alpha, 1), \ldots, e^*_u(\alpha, n), \tilde{e}^*_u(\alpha, 1), \ldots, \tilde{e}^*_u(\alpha, n)) Q
\]

and a $k$-linear isomorphism $\Psi : T^*(\alpha, \kappa, n) \longrightarrow \mathcal{E}^+_n(\kappa_1 \kappa_2) \oplus \mathcal{E}^+_n(-\sqrt{\kappa_1 \kappa_2})$ by

\[
  b^*_u(\alpha, m) \mapsto \begin{cases} 
    b^*_{u+s}(m) & (0 \leq u \leq s - 1), \\
    x^*_{u+s}(m) & (s \leq u \leq p - 1),
  \end{cases}
\]

where we denote by \( \{ b^*_{u+s}(m), x^*_{u+s}(m) \} \) the basis of $\mathcal{E}^+_n(\pm \sqrt{\kappa_1 \kappa_2})$ which is introduced in the previous subsection. By the construction, it is easy to see that $\Psi$ is an isomorphism of $\mathcal{U}$-modules.

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