A MIRROR THEOREM FOR Sym\(^d\)\(\mathbb{P}^r\)

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Abstract. We prove a mirror theorem for the symmetric product stack Sym\(^d\)\(\mathbb{P}^r\) := \([(\mathbb{P}^r)^d/S_d]\).

The theorem states that a generating function of Gromov-Witten invariants of Sym\(^d\)\(\mathbb{P}^r\) is equal to an explicit power series \(I_{\text{Sym}^d\mathbb{P}^r}\). This is part of a project to prove Ruan’s Crepant Resolution Conjecture for the resolution Hilb\(^{(d)}\)(\(\mathbb{P}^2\)) of the coarse moduli space of Sym\(^d\)\(\mathbb{P}^2\).

1. Introduction

Over the last 20 years, following predictions from string theory, mathematicians have proven a series of results known as mirror theorems; an incomplete list is [Giv98b, LLY97, Giv98a, BCFKvS00, Zin09, Li11, JK02, CCIT15, CCFK15, FLZ15, CCIT14]. These theorems reveal elegant patterns and structures embedded in the collection of (usually genus-zero) Gromov-Witten invariants of a fixed target manifold or orbifold \(X\). They also allow for computation of these invariants in many cases where direct computation is combinatorially difficult. However, the scope of these results is essentially limited to the world of toric geometry; specifically, \(X\) must either be a complete intersection in a toric stack or stack admitting a toric degeneration.

Our goal in this paper is to develop tools for mirror symmetry outside of toric geometry. The main theorem (Theorem 7.2) is a genus-zero mirror theorem for the orbifold Sym\(^d\)\(\mathbb{P}^r\), which is not possible using existing techniques. This is also the only known mirror theorem for a nonabelian orbifold, besides single points \([\bullet/G]\).

The most modern formulation of mirror theorems involves a geometric formalism introduced by Givental to encode the genus-zero Gromov-Witten invariants of \(X\). Generating functions of genus-zero Gromov-Witten invariants are interpreted as schematic points of a certain (germ of a) manifold \(L_X\), called the Givental cone of \(X\). The theorem is therefore stated as follows:

**Theorem 7.2** We introduce formal variables \(Q, z, \{t_i\}_{0 \leq i \leq r}\), and \(\{x_{\Pi}\}_{\Pi \in \text{Part}(d)}\), where Part\((d)\) is the set of partitions of \(d\). Then \(I_{\text{Sym}^d\mathbb{P}^r}(Q, t, x, z)\) lies on the Givental cone \(L_{\text{Sym}^d\mathbb{P}^r}\), where

\[
I_{\text{Sym}^d\mathbb{P}^r}(Q, t, x, z) := z \sum_{\sigma \in \text{Part}(d)} \sum_{\beta \geq 0} \exp \left( \sum_{i=0}^{r} t_i \left( [H_{\sigma,i}] / z + \beta \right) \right) Q^\beta \sum_{Z \geq 0\text{-labels } L = (L_\eta)} \sum_{\text{the parts of } \sigma \text{ with sum } \beta} \left( \sum_{k=(k_\Pi)_{\Pi \in \text{Part}(d)}, k_\Pi \geq 0} H(\sigma, x^k) \prod_{\Pi \in \text{Part}(d)} \frac{x^{k_\Pi}}{z^{k_\Pi} k!} \right) \left( \frac{|S_{\sigma}|}{|S_{\sigma,L}|} \right) \left( \prod_{\eta \in \sigma} \prod_{i=0}^{L_\eta} \frac{1}{\prod_{\gamma=1}^{L_\eta} \left( H_{\sigma,\eta,i} + \frac{\gamma}{\eta} z \right)} \right),
\]

where:
- \(1_{\sigma} \in H^*_{CR,T}(\text{Sym}^d\mathbb{P}^r)\) is the Chen-Ruan cohomology class of the twisted sector corresponding to the partition \(\sigma\).
- \([H_{\sigma,i}]\) and \([H_{\sigma,\eta,i}]\) are hyperplane classes defined in Section 3.2.
- \(H(\sigma, x^k)\) is the number of ways of factoring \(1 \in S_d\) as a product \(a_1 \cdots a_1 + \sum k_\Pi\), where the conjugacy classes (i.e. partitions) of the permutations \(a_j\) are given by the list \((\sigma, x^k)\), and
• $S_\sigma$ and $S_{\sigma,L}$ are the automorphism groups of the partition $S_\sigma$ and the labeled partition $S_{\sigma,L}$.

**Corollary 7.5** There is an equality $I_{\Sym^d \mathbb{P}^r}(Q, t, x, z) = J_{\Sym^d \mathbb{P}^r}(Q, \theta, z)$, where

$$J_{\Sym^d \mathbb{P}^r}(Q, \theta, z) = 1z + \theta + \sum_{\beta, n} \frac{Q^\beta}{n!} \left\langle \theta, \ldots, \theta, \frac{\gamma \phi}{z - \psi} \right\rangle_{0,n+1,\beta} \gamma^0$$

and $\theta = \sum_{i=0}^r \tau_{H_\sigma,i} + \sum_{\Pi \in \Part(d)} x_\Pi 1_\Pi$.

We have two motivations for working with $\Sym^d \mathbb{P}^r$, besides the fact that it is a relatively concrete nontoric and nonabelian orbifold.

1. **The crepant resolution conjecture.** Following physical predictions, Ruan [Ruan00], Bryan-Graber [BG09], and Coates-Iritani-Tseng [CIT09] made a conjecture relating the Gromov-Witten invariants of an orbifold $X$ to those of a crepant resolution of its coarse moduli space. This conjecture has been proven in the context of toric geometry [CIJ14]. However, the crepant resolution Hilb$(d)$(P$^2$) of the coarse moduli space of $\Sym^d \mathbb{P}^r$ was one of Ruan’s motivating examples; this case has now been open for a decade. Theorem 7.2 is a first step towards this case.

2. **Higher genus invariants of projective space.** Costello’s thesis expressed the genus $g$ Gromov-Witten invariants of a smooth projective variety $X$ in terms of the genus-zero Gromov-Witten invariants of $\Sym^{g+1} X$. Theorem 7.2 provides an efficient way of computing the latter for $X = \mathbb{P}^r$.

There are two main difficulties encountered in the nontoric case. The first is to correctly guess the explicit series $I_X$ appearing in the mirror theorem. There is a systematic approach that works in the context of toric orbifolds and (nonorbifold) GIT quotients, using moduli spaces of stable quasimaps ([CFK14]). Specifically, one calculates $I_X$ as an integral over a graph moduli space $\mathcal{Q}G(X)$, which parametrizes very degenerate maps $\mathbb{P}^1 \to X$ with basepoints. There is an obstruction to defining $\mathcal{Q}G(X)$ in the orbifold nontoric setting: orbifold points on $\mathbb{P}^1$ must be allowed to collide in families. Some of these collisions have been studied by Ekedahl [Eke95], but very little is known in general.

We were not able to rigorously define $\mathcal{Q}G(\Sym^d \mathbb{P}^r)$, but assuming its existence we guessed the function $I_{\Sym^d \mathbb{P}^r}(Q, t, x, z)$. We will study these spaces further in the future, but they play no role in our proof and are not mentioned in the rest of the paper.

After determining $I_{\Sym^d \mathbb{P}^r}(Q, t, x, z)$, we prove Theorem 7.2 using torus localization techniques, with respect to the natural $(\mathbb{C}^*)^r + 1$-action on $\Sym^d \mathbb{P}^r$. Torus localization in Gromov-Witten theory was developed by Kontsevich [Kon95] and Givental [Giv98b], and streamlined by Brown [Bro14]. In structure, our proof follows the mirror theorem for toric stacks [CCIT15], which uses orbifold calculations of Johnson [Joh14] and Liu [Liu13] in conjunction with Brown’s technique. We use localization to show that $I_{\Sym^d \mathbb{P}^r}(Q, t, x, z)$ satisfies a certain recursion, and that this recursion exactly characterizes points of $\mathcal{L}_{\Sym^d \mathbb{P}^r}$ (Theorem 6.4). The localization argument involves classifying torus-fixed stable maps combinatorially, or equivalently classifying 1-dimensional torus orbits.

The second difficulty arises here: while 1-dimensional torus orbits of a toric variety are isolated, in $\Sym^d \mathbb{P}^r$ they come in positive-dimensional families. In order to carry out the argument, we describe the base of each family explicitly (Section 4). The description applies in more generality, for example to symmetric products of toric varieties. This reduces the computation of Gromov-Witten invariants to the computation of integrals over the bases of these families, which we compute in Section 6.

Finally, and relatedly, we draw the reader’s attention to a technical aspect of the recursion in Theorem 6.4: condition (II). The recursion expresses Laurent coefficients with negative exponents of a certain generating function in terms of those with positive exponents. We have not seen this
The paper is organized as follows. Section 2 sets up combinatorial conventions, and reviews Atiyah-Bott torus localization, orbifold Gromov-Witten theory, and moduli spaces of curves called Losev-Manin spaces, which play a role in classifying the edge moduli spaces \( \mathcal{M}_e \). In Section 3, we review relevant facts about symmetric product orbifolds. Section 4 describes the torus action on \( \text{Sym}^d \mathbb{P}^r \), and Sections 5 and 6 contain the various calculations necessary to apply torus localization. Section 6 proves the characterization (Theorem 6.4) of the Givental cone \( \mathcal{L}_{\text{Sym}^d \mathbb{P}^r} \). Finally, Section 7 defines the explicit series \( I_{\text{Sym}^d \mathbb{P}^r} \), and proves that it lies on \( \mathcal{L}_{\text{Sym}^d \mathbb{P}^r} \).

2. Notation, conventions, and background

We work over \( \mathbb{C} \). We write \( H^*(X) := H^*(X, \mathbb{Q}) \). For a point \( x \) of an orbifold \( X \), we write \( G_x \) for the isotropy group of \( x \).

2.1. Multipartitions and graphs. A finite multiset \( \Pi \) is an unordered finite collection of elements \( a \) (we write \( a \in \Pi \)), possibly appearing more than once. Multisets are denoted with parentheses, e.g., \( (a, a, b) \). We write \( \text{MultiPart}(\Pi, a) \) for the number of times that \( a \) appears in \( \Pi \). We will sometime index multisets by (unordered) sets, e.g., \((a_i)_{i \in I}\). A submultiset \( \Pi' \subseteq \Pi \) is a multiset such that for \( a \in \Pi' \) with multiplicity \( m \), we have \( a \in \Pi \) with multiplicity at least \( m \). Unions are defined in the obvious way, e.g., \((a, a, b) \cup (a, c) = (a, a, a, b, c)\). The cardinality or length of a multiset is the number of elements, including multiplicities.

For \( d \in \mathbb{Z}_{>0} \), a partition of \( d \) is a multiset of positive integers whose sum (with multiplicities) is \( d \). The (finite) set of partitions of \( d \) is denoted \( \text{Part}(d) \). The ones partition of \( d \) is the multiset \( \{1, \ldots, 1\} \) of size \( d \). A nonnegative ordered partition of \( d \) is an ordered tuple of nonnegative integers whose sum is \( d \). The (finite) set of nonnegative ordered partitions of \( d \) of length \( r \) is denoted \( \mathbb{Z}\text{Part}(d, r) \).

If \( D \) is a multiset of positive integers, a multipartition of \( D \) is a multiset \( \Pi_d \) of \( d \) a partition of \( d \). The (finite) set of multipartitions of \( D \) is denoted \( \text{MultiPart}(D) \). The ones multipartition of \( D \) is the multipartition of \( D \) each of whose elements \( \Pi_d \) is the ones partition of \( d \). There is a forgetful map \( \text{MultiPart}(D) \to \text{Part}(\sum_{d \in D} d) \) sending \( \Pi_d \) to \( \bigcup_{d \in D} \Pi_d \).

If \( \Pi \) is a partition, we write \( S_{\Pi} \) for the group of automorphisms of the partition; e.g., for the partition \( \Pi = \{1, 1, 1, 2, 2\} \) of 7, we have \( S_{\Pi} \cong S_3 \times S_2 \). For \( \sigma = (\Pi_d)_{d \in D} \) a multipartition of \( D \), we define \( S_{\sigma} := \prod_{d \in D} S_{\Pi_d} \).

Let \( A \) be a set, and let \( \sigma = (\Pi_d)_{d \in D} \) be a multipartition of \( D \). An \( A \)-labeling \( L \) of \( \sigma \) is an assignment \( (L_p)_{p \in A} \) of an element of \( A \) to each part \( p \) of each \( \Pi_d \). Precisely, it is the data of a multiset \( \tilde{\sigma} = (\tilde{\Pi}_d)_{d \in D} \), where \( \tilde{\Pi}_d \) is a multiset of pairs \((p, a)\) with \( p \in \mathbb{Z}_{>0} \) and \( a \in A \), such that the multiset \( \sigma = (\Pi_d)_{d \in D} \) obtained by forgetting the second entry of each pair in each \( \tilde{\Pi}_d \) is equal to \( \sigma \). We define \( S_{\sigma, L} \) to be the subgroup of \( S_{\sigma} \) of permutations that preserve labels.

Let \( \Gamma = (V(\Gamma), E(\Gamma)) \) be a finite graph. We denote by \( E(\Gamma, v) \) the set of edges incident to \( v \). The valence \( \text{val}(v) \) of \( v \in V(\Gamma) \) is \( |E(\Gamma, v)| \). (This is different from some Gromov-Witten theory literature, which defines \( \text{val}(v) \) to include contributions from certain decorations on \( \Gamma \), described in Section 4.2.) A flag of \( \Gamma \) is a pair \((v, e) \in V(\Gamma) \times E(\Gamma) \) with \( e \in E(\Gamma, v) \). The set of flags of \( \Gamma \) is denoted \( F(\Gamma) \).

2.2. Notation for projective space. We denote a point of \( \mathbb{P}^r = \mathbb{P}(\mathbb{C}^{r+1}) \) by \([x_0 : x_1 : \cdots : x_r]\) with \( x_i \in \mathbb{C} \). We denote the coordinate points of \( \mathbb{P}^r \) by \( P_0, P_1, \ldots, P_r \), where \( P_i \) is the point where all coordinates by \( x_i \) vanish. We denote by \( L_{(i_1, i_2)} = L_{(i_2, i_1)} \) the coordinate line passing through \( P_{i_1} \) and \( P_{i_2} \). We write \( P_{(i_1, i_2)} \) for the midpoint of this line, i.e., the point where \( x_{i_1} = x_{i_2} \) and \( x_i = 0 \) for \( i \neq i_1, i_2 \).
2.3. **Equivariant cohomology.** We will consider actions of the torus \( T := (\mathbb{C}^*)^{r+1} \) on various spaces, e.g. \( \mathbb{P}^r \), \( \text{Sym}^d \mathbb{P}^r \), and \( \overline{M}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta) \). If \( T \) acts on a Deligne-Mumford stack \( X \), the equivariant cohomology \( H^*_T(X) \) is a module over \( H^*_T(\text{Spec} \mathbb{C}) \cong \mathbb{Q}[\alpha_0, \ldots, \alpha_r] \), where \( -\alpha_i \) is the weight of the character \( T \to \mathbb{C}^* \) defined by \( (\lambda_0, \ldots, \lambda_r) \mapsto \lambda_i \). We write \( H^*_T,\text{loc}(\text{Spec} \mathbb{C}) \) for the localization \( \mathbb{Q}(\alpha_0, \ldots, \alpha_r) \), and more generally \( H^*_T,\text{loc}(X) := H^*_T(X) \otimes_{H^*_T(\text{Spec} \mathbb{C})} H^*_T,\text{loc}(\text{Spec} \mathbb{C}) \). We will often use the Atiyah-Bott localization theorem, as well as Graber-Pandharipande’s generalization, the virtual localization theorem.

**Theorem 2.2** ([AB84], see [EG98] for statement in the Chow ring). Let \( T \) be a torus acting on a smooth compact manifold \( X \), with fixed point set \( F \). Then the map \( (\iota_F)_*: H^*_T,\text{loc}(F) \to H^*_T,\text{loc}(X) \) is an isomorphism, where \( (\iota_F)_* \) is the Gysin map associated to the inclusion \( F \hookrightarrow X \). The inverse map is \( \iota_F^*/e_T(N_F|_X) \), where \( e_T(N_F) \) is the equivariant Euler class of the normal bundle to \( F \). In particular, for \( \alpha \in H^*_T,\text{loc}(X, \text{Spec} \mathbb{C}) \), we have

\[
\int_X \alpha = \int_X (\iota_F)_* \left( \frac{\iota_F^* \alpha}{e_T(N_F)} \right) = \int_F \frac{\iota_F^* \alpha}{e_T(N_F)}.
\]

**Theorem 2.2** ([GP99]). Let \( X \) be a Deligne-Mumford stack with a \( T \)-action and a \( T \)-equivariant perfect obstruction theory \( E^* \). Again, let \( \iota_F: F \hookrightarrow X \) denote the inclusion of the fixed locus. Let \( [X]^\text{vir} \) denote the virtual fundamental class associated to \( E^* \). The \( T \)-fixed part of \( E^* \) defines a perfect obstruction theory on \( F \), with virtual fundamental class \( [F]^\text{vir} \). The virtual normal bundle \( N_F^\text{vir} \) to \( F \) is the \( T \)-moving part of \( E^* \). Then

\[
\int_{[X]^\text{vir}} \alpha = \int_{[F]^\text{vir}} \frac{\iota_F^* \alpha}{e_T(N_F^\text{vir})}.
\]

**Remark 2.3.** The proof in [GP99] requires that \( X \) have an equivariant embedding into a smooth Deligne-Mumford stack, but this condition was removed in [CKL15]. Also, it is usually convenient to write \( F \) as a union of connected components (or a union of unions of connected components) \( F_j \), in which case (1) becomes

\[
\int_{[X]^\text{vir}} \alpha = \sum_j \int_{[F_j]^\text{vir}} \frac{\iota_{F_j}^* \alpha}{e_T(N_{F_j}^\text{vir})}.
\]

2.4. **(Orbifold) Gromov-Witten theory.** Our objects of study are the moduli spaces \( \overline{M}_{0,n}(X, \beta) \) of \( n \)-marked genus zero stable maps to a smooth proper Deligne-Mumford stack \( X \) of degree \( \beta \), introduced in [CR02] and [AV02]. See [Liu13], Section 7 for an introduction to the subject (in all genus). Following [Liu13], we use the convention that all gerbes come with the data of a section.

In this paper we will have either \( X = \text{Sym}^d \mathbb{P}^r \) or \( X = BG \) for some finite group \( G \). We write \( (f: C \to X) \) for a \( \mathbb{C} \)-point of \( \overline{M}_{0,n}(X, \beta) \), and

\[
\begin{array}{ccc}
C & \xrightarrow{f} & X \\
\pi \downarrow & & \downarrow \pi \\
\overline{M}_{0,n}(X, \beta)
\end{array}
\]

for the universal curve and universal map.

A **Gromov-Witten invariant** is an integral of the form

\[
(\overline{\psi}_1^{a_1} \gamma_1, \ldots, \overline{\psi}_n^{a_n} \gamma_n)_{0,n,\beta} := \int_{[\overline{M}_{0,n}(X, \beta)]^\text{vir}} \prod_{j=1}^n \overline{\psi}_j^{a_j} \text{ev}_j * \gamma_j \in \mathbb{Q},
\]

where

\[
\begin{array}{c}
\mathbb{P}^r \\
\text{Sym}^d \mathbb{P}^r \end{array}
\]
We may allow $t^1$ to be a certain cohomology class, coming from the cotangent space to the coarse moduli space of $C^1$.\footnote{Note that $\psi_j = r_j \psi_j$, where $r_j$ is the size of the isotropy group at the mark $b_j$, and $\psi_j$ is the “stacky” cotangent class.}

- the “insertions” $\gamma_j$ are in the Chen-Ruan cohomology (see \cite{CR04}) $H^*_{CR}(X)$, and
- $\ev_j : \overline{M}_{0,n}(X,\beta) \to IX$ is the $j$th evaluation map.

If $X$ has an action of a torus $T$, it induces a natural $T$-action on $IX$ and $\overline{M}_{0,n}(X,\beta)$, and $\overline{M}_{0,n}(X,\beta)^{vir}$, $\psi_j$, and $\ev_j^* \gamma_j$ are naturally equivariant classes (where $\gamma_j \in H^*_{CR,T}(X)$). In this case (2) defines an equivariant Gromov-Witten invariant (an element of $H^*_T(\Spec \mathbb{C})$, denoted by $\langle \cdots \rangle_{0,n,\beta}$) via $T$-equivariant integration.

Following \cite{CCIT15}, the $T$-equivariant Novikov ring of $\Sym^d \mathbb{P}^r$ is

$$\Lambda^T_{\text{Nov}} := H^*_{T,\text{loc}}(\Spec \mathbb{C})[[Q]],$$

and Givental’s symplectic vector space is

$$\mathcal{H} := H^*_{CR,T,\text{loc}}(\Sym^d \mathbb{P}^r)[[Q]]((z^{-1})) = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where $\mathcal{H}^+ = H^*_{CR,T,\text{loc}}(\Sym^d \mathbb{P}^r)[[Q]][z]$ and $\mathcal{H}^- = z^{-1} H^*_{CR,T,\text{loc}}(\Sym^d \mathbb{P}^r)[[Q]][z^{-1}]$. Inside $\mathcal{H}$, there is a special subscheme $\mathcal{L}_{\Sym^d \mathbb{P}^r}$ called the Givental cone of $\Sym^d \mathbb{P}^r$, which encodes the genus-zero Gromov-Witten invariants of $\Sym^d \mathbb{P}^r$. (Precisely, $\mathcal{L}_{\Sym^d \mathbb{P}^r}$ is a formal germ of a subscheme, defined at $-1 \cdot z$, where $1 \in H^*_{CR,T,\text{loc}}(\Sym^d \mathbb{P}^r)$ is the fundamental class of the nontwisted sector.) Fix a basis $\gamma_\phi$ of $H^*_{CR,T,\text{loc}}(\Sym^d \mathbb{P}^r)$, with Poincaré dual basis $\gamma_\phi$. By definition, a $\Lambda^T_{\text{Nov}}[[x,\gamma]]$-valued point of $\mathcal{L}_{\Sym^d \mathbb{P}^r}$ is defined to be a power series

$$-1z + t(z) + \sum_{n=0}^\infty \sum_{\beta=0}^\infty \sum_{\phi} \frac{Q^\beta}{n!} \left\langle t(\psi), \ldots, t(\psi), \frac{\gamma_\phi}{-z - \psi} \right\rangle_{0,n+1,\beta} \gamma_\phi \in \mathcal{H}[[x]],$$

where $t(z) \in \langle Q, x \rangle \subseteq \mathcal{H}_+[[x]]$. $\mathcal{L}_{\Sym^d \mathbb{P}^r}$ has several important geometric properties that follow from relations between Gromov-Witten invariants: see Appendix B of \cite{CCIT09}, which also defines $\mathcal{L}_{\Sym^d \mathbb{P}^r}$ rigorously as a non-Noetherian formal scheme. For example, it is a cone in a certain sense, hence the name (Proposition B.2 of \cite{CCIT09}).

There is also a notion of a twisted Givental cone $\mathcal{L}^T_{\mathbb{X}}$. We need this notion only when $X = BG$, and even then in a special case. Let $E$ be a $T \times G$ representation. Then $R\pi_* f^* E \in K^0_T(\overline{M}_{0,n}(BG,\beta))$. A $\Lambda^T_{\text{Nov}}[[x]]$-valued point of $\mathcal{L}^T_{BG}$ is defined to be

$$-1z + t(z) + \sum_{n=0}^\infty \sum_{\beta=0}^\infty \sum_{\phi} \frac{Q^\beta}{n!} \left\langle t(\psi), \ldots, t(\psi), \frac{\gamma_\phi}{-z - \psi} \right\rangle_{0,n+1,\beta} \frac{BG,T_{\text{tw}}}{\gamma_\phi},$$

where $t(z) \in \langle Q, x \rangle \subseteq \mathcal{H}_+[[x]]$.

\begin{align}
\langle \psi_1^{\alpha_1}, \ldots, \psi_n^{\alpha_n} \rangle_{BG,T_{\text{tw}}} := \int_{\overline{M}_{0,n}(BG,\beta)^{vir}} \prod_{j=1}^n \psi_j^{\alpha_j} \ev_j^* \gamma_j \cup e_T^{-1}(R\pi_* f^* E).
\end{align}

We may allow $t(z)$ to be a power series in $z$, because $\psi$ is nilpotent in this case. Also, here $\gamma_\phi$ and $\gamma_\phi$ are dual bases of $H^*_T(BG)$ under the twisted Poincaré pairing, see \cite{CCIT15}.
2.5. Losev-Manin spaces. We recall certain moduli spaces of marked curves, studied originally by Losev and Manin \([LM00]\).

**Definition 2.4.** Let \(k \geq 1\), and fix a 2-element set \(\{0, \infty\}\). An \(0|k|\infty\)-marked Losev-Manin curve is a connected genus zero \(k + 2\)-marked nodal curve \((C, b_0, b_1, \ldots, b_k, b_\infty)\), satisfying:

- The irreducible components of \(C\) form a chain, with two leaves \(C_0\) and \(C_\infty\).
- The points \(b_0, b_1, \ldots, b_k, b_\infty\) are smooth points of \(C\), with \(b_0 \in C_0\) and \(b_\infty \in C_\infty\).
- \(b_i \neq 0\) and \(b_i \neq \infty\) for \(i = 1, \ldots, k\) (though it is possible that \(b_i = b_j\) for \(i \neq j\)), and
- Each irreducible component of \(C\) contains at least one point of \(b_1, \ldots, b_k\).

**Theorem 2.5** ([LM00], Theorems 2.2 and 2.6.3). The moduli space of \(0|k|\infty\)-marked Losev-Manin curves \(\overline{M}_{0|k}|\infty\) is a smooth, proper variety, and the natural morphism \(\varphi : \overline{M}_{0,k+2} \to \overline{M}_{0|k}|\infty\) is birational.

**Remark 2.6.** The spaces \(\overline{M}_{0|k}|\infty\) are special cases of moduli spaces of weighted stable curves, developed by Hassett [Has03], and Theorem 2.5 is a special case of Theorems 2.1 and 4.1 of [Has03]. Specifically, there is a natural isomorphism \(\overline{M}_{0|k}|\infty\to \overline{M}_{0,A}\), where \(A\) is the weight datum \((1, \epsilon, \epsilon, \ldots, \epsilon, 1)\) of length \(k + 2\), for \(\epsilon \leq 1/k\).

**Definition 2.7.** Let \(s \geq 1\) be an integer. An \(order s\) orbifold \(0|k|\infty\)-marked Losev-Manin curve is a \(k + 2\)-marked twisted curve \((C, b_0, b_1, \ldots, b_k, b_\infty)\) (in the sense of [Ols07]) whose coarse moduli space is a \(k\)-marked Losev-Manin curve, such that \(C\) has orbifold structure only at \(b_0, b_\infty\), and the nodes of \(C\), all of which have order \(s\).

By standard arguments about twisted curves, the moduli space \(\overline{M}_{0|k}|\infty\) of order \(s\) orbifold \(k\)-marked Losev-Manin curves has a natural isomorphism \(\overline{M}_{0|k}|\infty\to \overline{M}_{0,s}|\infty\) that comes from taking coarse moduli spaces. Our calculations in Section 6 will use the following fact, which is immediate from Lemma 2.3 of [Moo11].

**Lemma 2.8.** Let \(\psi_{0,LM}\) and \(\psi_{\infty,LM}\) denote the tautological cotangent classes at \(b_0\) and \(b_\infty\) on \(\overline{M}_{0|k}|\infty\). The pullbacks \(\varphi^*\psi_{0,LM}\) and \(\varphi^*\psi_{\infty,LM}\) along the reduction morphism \(\overline{M}_{0,k+2} \to \overline{M}_{0|k}|\infty\) are the cotangent classes \(\psi_0\) and \(\psi_\infty\), respectively.

**Remark 2.9.** Lemma 2.8 holds for order \(s\) orbifold Losev-Manin spaces, either using the cotangent classes \(\psi\) on the coarse moduli space, or replacing \(\overline{M}_{0,k+2}\) with a suitable (isomorphic) orbifold replacement \(\overline{M}_{0,k+2}^s\). (Precisely, \(\overline{M}_{0,k+2}^s\) parametrizes curves where \(b_0\) and \(b_\infty\) have order \(s\) orbifold structure, as do any nodes that separate \(b_0\) from \(b_\infty\).)

3. Symmetric product stacks

3.1. Two definitions of \(\text{Sym}^d X\). Let \(X\) be an algebraic stack. The \(d\)th power \(X^d\) has a natural action of \(S_d\). The \(d\)th symmetric product \(\text{Sym}^d X\) is the stack \([X^d/S_d]\). That is, for \(S\) a scheme, an object \(f : S \to \text{Sym}^d X\) of \(\text{Sym}^d X\) over \(S\) is a principal \(S_d\)-bundle \(\tilde{S}\) over \(S\), together with an \(S_d\)-equivariant map \(\tilde{f} : \tilde{S} \to X^d\). A morphism \((f : S \to \text{Sym}^d X) \to (g : T \to \text{Sym}^d X)\) over \(S \to T\) is a diagram

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\tilde{f}} & \tilde{T} \\
\downarrow \quad & & \quad \downarrow \\
S & \xrightarrow{f} & T
\end{array}
\]

6
such that the square is cartesian and the triangle commutes. If $X$ is smooth, then so is $\text{Sym}^d X$, since it has an étale cover by the smooth stack $X^d$. Similarly, if $X$ is Deligne-Mumford, then so is $\text{Sym}^d X$.

There is an equivalent characterization of $\text{Sym}^d X$, which will allow us to sidestep some of the complications of working with stacks. We define a stack $\tilde{\text{Sym}}^d$ that is naturally isomorphic to $\text{Sym}^d X$. Roughly, rather than parametrizing “$d$ ordered points of $X$ up to reordering”, $\tilde{\text{Sym}}^d X$ will parametrize “Maps $P : d(\bullet) \to X$,” where $d(\bullet) = \bigcup_{j=1}^d \text{Spec} \mathbb{C}$. Precisely, an object $f : S \to \tilde{\text{Sym}}^d X$ over $S$ is an étale map $\rho : S' \to S$ of degree $d$ (i.e. a bundle with fiber $d(\bullet)$), together with a map $f' : S' \to X$. A morphism $(f : S \to \tilde{\text{Sym}}^d X) \to (g : T \to \tilde{\text{Sym}}^d X)$ over $S \to T$ is a diagram

\[
\begin{array}{ccc}
S' & \longrightarrow & T' \\
\downarrow & & \downarrow \\
S & \longrightarrow & T
\end{array}
\]

There is a map $\text{Sym}^d X \to \tilde{\text{Sym}}^d X$ that sends:

\[
\begin{pmatrix}
\tilde{S} & \tilde{f} \to X^d \\
\downarrow & \\
S
\end{pmatrix}
\mapsto
\begin{pmatrix}
\tilde{S} \times S_d \{1, \ldots, d\} & \longrightarrow & X \\
\downarrow & \\
S
\end{pmatrix},
\]

where the $S_d$-action on $\{1, \ldots, d\}$ is the obvious one, and the map $\tilde{S} \times S_d \{1, \ldots, d\} \to X$ sends $(\tilde{s}, i) \mapsto \text{pr}_i \circ \tilde{f}(s)$, where $\text{pr}_i$ denotes the $i$th projection $X^d \to X$. It is easy to check that this is $S_d$-equivariant, and defines a morphism of stacks.

In the other direction, we may send:

\[
\begin{pmatrix}
S' & \rightarrow & X \\
\downarrow & & \\
S
\end{pmatrix}
\mapsto
\begin{pmatrix}
\text{Isoms}_S(S', \{1, \ldots, d\}) & \longrightarrow & X^d \\
\downarrow & \\
S
\end{pmatrix},
\]

where $\text{Isoms}_S(S', \{1, \ldots, d\})$ is the principal $S_d$-bundle given on small (étale) open sets $U \to S$ by the set of isomorphisms $S' \times_S U \to \{1, \ldots, d\} \times U$. Given such an isomorphism, $f'$ determines a $U$-valued point of $X^d$. It is again straightforward to check that this defines a map of stacks $\tilde{\text{Sym}}^d X \to \text{Sym}^d X$, and that it is an inverse to the previous map. For the rest of the paper we will use the descriptions interchangeably and denote them both by $\text{Sym}^d X$. It is useful to keep in mind the following diagram, where the cube is Cartesian and the left and right faces consist of étale maps:

(The composition $S' \to X^d \times S_d \{1, \ldots, d\} \xrightarrow{P} X$ is $f'$.)

### 3.2. The inertia stack of $\text{Sym}^d \mathbb{P}^r$. We describe the cyclotomic inertia stack $I \text{Sym}^d \mathbb{P}^r$, see Section 3 of [AGV08].

To a map $d(\bullet) \xrightarrow{P} \mathbb{P}^r$, we may assign a partition $\sigma \in \text{Part}(d)$, where parts correspond to points of $\text{Im}(P)$. This gives a stratification of $\text{Sym}^d \mathbb{P}^r$ into strata $(\text{Sym}^d \mathbb{P}^r)_\sigma$ indexed by $\sigma \in \text{Part}(d)$. We have $(\text{Sym}^d \mathbb{P}^r)_\sigma \cong \prod_{i \geq 1} \text{Sym}^{\text{Mult}(\sigma,i)} \mathbb{P}^r$, and each point of $(\text{Sym}^d \mathbb{P}^r)_\sigma$ has isotropy group isomorphic to $\prod_{\eta \in \sigma} S_{\sigma}$. 7
It follows that the components of $I\operatorname{Sym}^d\mathbb{P}^r$ are indexed by partitions of $d$. We denote the component associated to $\sigma \in \operatorname{Part}(d)$ by $(I\operatorname{Sym}^d\mathbb{P}^r)_\sigma$. This is isomorphic to (a gerbe over) $\prod_{i \geq 1} \operatorname{Sym}^{\operatorname{Mult}(\sigma,i)}\mathbb{P}^r$.

The (equivariant, nonorbifold) cohomology with rational coefficients may be computed explicitly, as the $S_d$-invariant cohomology of $H^*_{\text{Sym}}(\mathbb{P}^r)^d = \bigotimes_{j=1}^d H^*_T(\mathbb{P}^r)$. In particular, we have identifications $H^2_{\text{Sym}}(\mathbb{P}^r) \cong H^2_T(\mathbb{P}^r)^S_d \cong H^2_T(\mathbb{P}^r)$. We will later write $[H_i]$ for the element of $H^2_T(\mathbb{P}^r)$ that pulls back to $\sum_{j=1}^d \text{pr}_j^*[H_i] \in H^2_T(\mathbb{P}^r)^d$, where $\text{pr}_j$ is the $j$th coordinate map and $[H_i]$ is the equivariant fundamental class of the $i$th coordinate hyperplane.

Fix a component $(I\operatorname{Sym}^d\mathbb{P}^r)_\sigma$ of $I\operatorname{Sym}^d\mathbb{P}^r$. For $\eta \in \sigma$, we denote by $[H_{\sigma,\eta,i}]$ the class $[H_i]$ pulled back from the factor of $(I\operatorname{Sym}^d\mathbb{P}^r)_\sigma \cong \prod_{i \geq 1} \operatorname{Sym}^{\operatorname{Mult}(\sigma,i)}\mathbb{P}^r$ corresponding to $\eta$. We write $[H_{\sigma,i}]$ for $\sum_{\eta} [H_{\sigma,\eta,i}]$.

3.3. The tangent bundle to $\operatorname{Sym}^d X$. Now we assume $X$ is a smooth Deligne-Mumford stack. The two definitions of $\operatorname{Sym}^d X$ in Section 3.1 give two descriptions of the tangent bundle $T\operatorname{Sym}^d X$. First, since $\operatorname{Sym}^d X$ is isomorphic to $[X^d/S_d]$, $T\operatorname{Sym}^d X$ is the vector bundle on $\operatorname{Sym}^d X$ corresponding to the $S_d$-equivariant vector bundle $T(X^d)$ on $X^d$, where $S_d$ acts by the derivative. Consider the portion of Figure 3.1 with more maps named:

We claim that there is a natural isomorphism $T\operatorname{Sym}^d X \cong \rho_*(P^*TX)$. Since the square is cartesian and the maps are étale, we have

$$\text{pr}^*(\rho_*(P^*TX)) \cong \tilde{\rho}_*((\text{pr}')^*(P^*TX)) = \tilde{\rho}_*((\text{pr}' \circ P^*TX)).$$
Recall that $pr' \circ P$ is simply the "universal coordinate map," so since $\tilde{\rho}$ is a trivial étale cover, there is a canonical isomorphism

$$\tilde{\rho}_*(((pr' \circ P^*TX)) \cong \bigoplus_{t=1}^{d} P_t^*TX \cong T(X^d).$$

Since $\tilde{\rho}$ is $S_d$-equivariant, there is an induced $S_d$-action on $T(X^d)$ which agrees with the usual one. Thus the isomorphism descends to give $\rho_*(P^*TX) \cong T \ Sym^d X$.

4. The action of $(\mathbb{C}^*)^{r+1}$ on $\ Sym^d \mathbb{P}^r$

There is a natural action of $T := (\mathbb{C}^*)^{r+1}$ on $\mathbb{P}^r$. This induces a diagonal action of $(\mathbb{C}^*)^{r+1}$ on $(\mathbb{P}^r)^d$, which commutes with the action of $S_d$, hence acts on $\ Sym^d \mathbb{P}^r$. It is easy to check that this action agrees with that on $\overline{\ Sym}^d \mathbb{P}^r$ defined by postcomposition of $f' : S \to \mathbb{P}^r$ with the action on $\mathbb{P}^r$. The $T$-action on $\ Sym^d \mathbb{P}^r$ induces an action on $\overline{\ Sym}_{0,n}(\ Sym^d \mathbb{P}^r, \beta)$ for all $n$, and $\beta$.

The goal of this section is Theorem 4.18, which explicitly characterizes the $T$-fixed locus in $\overline{\ Sym}_{0,n}(\ Sym^d \mathbb{P}^r, \beta)$ in terms of combinatorial data called decorated trees, and the Losev-Manin spaces of Section 2.5.

4.1. T-fixed points and 1-dimensional orbits of $\ Sym^d \mathbb{P}^r$.

Proposition 4.1. (1) A point $(d(\bullet) \overset{P}{\to} \mathbb{P}^r) \in \ Sym^d \mathbb{P}^r$ is $T$-fixed if and only if $\operatorname{Im}(P) \subseteq \{P_0, \ldots, P_r\}$.

(2) $(d(\bullet) \overset{P}{\to} \mathbb{P}^r)$ is in a 1-dimensional $T$-orbit if and only if (it is not $T$-fixed and) $\operatorname{Im}(P) \subseteq \{P_0, \ldots, P_r\} \cup L_{i_1,i_2}$ for some $0 \leq i_1, i_2 \leq r$.

Proof. (1) is clear from the description of the $T$-action on $\overline{\ Sym}^d \mathbb{P}^r$, and the fact that $\{P_0, \ldots, P_r\}$ is the $T$-fixed locus of $\mathbb{P}^r$.

The $r$-dimensional subtorus defined by $t_{i_1} = t_{i_2}$ acts trivially on $\{P_0, \ldots, P_r\} \cup L_{i_1,i_2}$, proving the backwards direction of (2). If $\operatorname{Im}(P) \not\subseteq \{P_0, \ldots, P_r\} \cup L_{i_1,i_2}$, it is easy to show the $T$-orbit is at least 2-dimensional.

Remark 4.2. The $T$-fixed points of $\ Sym^d \mathbb{P}^r$ are in natural bijection with $Z\operatorname{Part}(d, r + 1)$, where the $i$th part is the number of points of $d(\bullet)$ mapping to $P_i$. We will use this identification from now on.

By the second part of 4.1, for each 1-dimensional $T$-orbit there are two associated indices $i_1$ and $i_2$. There is also associated (1) an element of $Z\operatorname{Part}(d')$, where $d' < d$ is the number of points of $P$ mapping to $\{P_0, \ldots, P_r\}$, and (2) a $(d - d')$-tuple of points of $L_{i_1,i_2}$, up to scaling.

4.2. T-fixed stable maps to $\ Sym^d \mathbb{P}^r$. It is well-known that if $X$ is a Deligne-Mumford stack with an action of a torus $T$, then a stable map $f : C \to X$ is $T$-fixed if and only if each component $C_\nu$ of $C$ each maps into the fixed locus $X^T$, or maps to the closure $\overline{U}$ of a 1-dimensional $T$-orbit $U$, with special points (nodes and marks) and ramification points mapping to $\overline{U} \setminus U$. (In the latter case we may regard $f|_{C_\nu}$ as a point of $\overline{\ Sym}_{0,2}(\overline{U}, \beta)$ for some $\beta$.) If $T$ has isolated fixed points, we refer to the two types of components as contracted and noncontracted, since those of the first type map to a single point of $X$.

Lemma 4.3. Let $(f : C \to \ Sym^d \mathbb{P}^r) \in \overline{\ Sym}_{0,2}(\ Sym^d \mathbb{P}^r, \beta)$ be a $T$-fixed stable map of degree $\beta > 0$ with irreducible source curve. (By the above, this is a ramified cover of a 1-dimensional orbit closure $\overline{U}$.) Denote by $b_1$ and $b_2$ the two marked points of $C$. Then:

- The associated étale cover $\rho : C' \to C$ from Section 3.1 is a disjoint union of rational connected components,
• Under the associated map $f' : C' \to \mathbb{P}^r$, each component of $C'$ is either contracted to a $T$-fixed point of $\mathbb{P}^r$, or maps to the coordinate line $L_{(i_1,i_2)}$, where $i_1$ and $i_2$ are the two indices associated to $\mathcal{U}$ from Section 4.1, and

• On each component $C'_{\eta}$ of the latter type, $\rho^{-1}(b_1)$ and $\rho^{-1}(b_2)$ are each a (single) fully ramified point.

• If $c_{\eta}$ is the degree of $\rho|_{C'_{\eta}} : C'_{\eta} \to C$ and $\beta_{\eta}$ is the (coarse) degree of $f'|_{C'_{\eta}} : C'_{\eta} \to L_{(i_1,i_2)}$, then the ratio $q := \beta_{\eta}/c_{\eta}$ is independent of $\eta$ (where $\eta$ runs over noncontracted components of $C'$).

Proof. The first three statements follow from the fact that $C$ has exactly two orbifold points, and from Proposition 4.1. It is straightforward to check that the last statement is equivalent to the fact that the $T$-action is compatible with the map $\rho$, i.e. that the action of $\lambda \in T$ corresponds to changing coordinates on $C$.

Remark 4.4. The same statement and proof apply to $\overline{M}_{0,1}(\text{Sym}^d \mathbb{P}^r, \beta)$ and $\overline{M}_{0,0}(\text{Sym}^d \mathbb{P}^r, \beta)$ and in these cases we have a slightly stronger statement: since $C$ has at most one orbifold point, it has no nontrivial étale cover, i.e. $C' \cong C \times \{1, \ldots, d\}$.

Following [Liu13], we now introduce combinatorial objects called decorated trees, which capture the combinatorial data of elements of $(\overline{M}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta))^T$. We will use them to write $(\overline{M}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta))^T$ as a disjoint union of explicit substacks. See 2.1 for notation in this section.

Definition 4.5. An $n$-marked genus-zero $\text{Sym}^d \mathbb{P}^r$-decorated tree $\tilde{\Gamma} = (\Gamma, \text{Mark}, \text{VEval}, q, \overline{\text{Mon}})$ is

- A tree $\Gamma$,
- A “marking map” $\text{Mark} : \{b_1, \ldots, b_n\} \to V(\Gamma)$,
- A “vertex evaluation map” $\text{VEval} = (\text{VEval}_0, \ldots, \text{VEval}_r) : V(\Gamma) \to \text{ZPart}(d, r + 1)$,
- A “degree ratio map” $q : E(\Gamma) \to \mathbb{Q}_{>0}$,
- A “marking monodromy” $\text{Mon} = (\text{Mon}_0, \ldots, \text{Mon}_r)$ that assigns to each $b_i \in \{b_1, \ldots, b_n\}$ an element of $\text{MultiPart}(\text{VEval}(\text{Mark}(b_i)))$, and
- A “flag monodromy map” also denoted $\text{Mon}$, that assigns to each flag $(v, e) \in F(\Gamma)$ an element of $\text{MultiPart}(\text{VEval}(v))$, subject to the conditions:

1. If $e$ is an edge connecting vertices $v$ and $v'$, then $\text{VEval}_k(v) = \text{VEval}_k(v')$ for all but exactly two $0 \leq k \leq r$, denoted $i_{\text{mov}}(v, e)$ and $i_{\text{mov}}(v', e)$,
2. If $e$ is an edge connecting vertices $v$ and $v'$, then for $i \neq i_{\text{mov}}(v, e)$, $i_{\text{mov}}(v', e)$, $\text{Mon}_i(v, e)$ is equal (as a partition of $\text{VEval}_i(v) = \text{VEval}_i(v')$) to $\text{Mon}_i(v', e)$. Furthermore, we have $\text{Mon}_{i_{\text{mov}}(v, e)}(v', e) \subseteq \text{Mon}_{i_{\text{mov}}(v', e)}(v, e)$, $\text{Mon}_{i_{\text{mov}}(v', e)}(v, e) \subseteq \text{Mon}_{i_{\text{mov}}(v', e)}(v', e)$, and the relation between complements holds: $\text{Mon}_{i_{\text{mov}}(v, e)}(v, e) \setminus \text{Mon}_{i_{\text{mov}}(v, e)}(v', e) = \text{Mon}_{i_{\text{mov}}(v', e)}(v', e) \setminus \text{Mon}_{i_{\text{mov}}(v', e)}(v, e)$,
3. If $v \in V(\Gamma)$ with $E(\Gamma, v) = \{e_v\}$ and $\text{Mark}^{-1}(v) = \emptyset$, then $\text{Mon}(v, e_v)$ is the ones multipartition of $\text{MultiPart}(\text{VEval}(v))$,
4. If $v \in V(\Gamma)$ with $E(\Gamma, v) = \{e_v\}$ and $\text{Mark}^{-1}(v) = \{b_v\}$, then $\text{Mon}(v, e_v) = \text{Mon}(b_v)$.
5. If $v \in V(\Gamma)$ with $E(\Gamma, v) = \{e_1^v, e_2^v\}$ and $\text{Mark}^{-1}(v) = \emptyset$, then $\text{Mon}(v, e_1^v) = \text{Mon}(v, e_2^v)$,

We introduce some notation:

• $\text{Mon}(v, e)$ (resp. $\text{Mon}(b)$) is naturally a $\{0, \ldots, r\}$-labeled multipartition, since it is a multipartition of the $\{0, \ldots, r\}$-indexed multiset $\text{VEval}(v)$ (resp. $\text{VEval}(\text{Mark}(b))$). For a part $i(\eta)$ of the underlying partition, we write $i(\eta)$ for its label.
• Let \( \text{Mov}(e) \) be the difference multiset \( \text{Mon}_{\text{mov}}(v, e) \setminus \text{Mon}_{\text{mov}}(v', e) \), which by condition 2 depends on \( e \) rather than \( (v, e) \). \( \text{Mov}(e) \) is the submultiset of “moving parts” of \( \text{Mon}(v, e) \) (or \( \text{Mon}(v', e) \)). We write \( \text{mov}(e) := |\text{Mov}(e)| \).

• \( \text{Stat}(e) := \text{Mon}(v, e) \setminus \text{Mov}(e) \) is an \( \{0, \ldots, r\} \)-labeled multipartition of \( \text{VEval}(v) \), where \( \text{VEval}(v) \) is obtained from \( \text{VEval}(v) \) by decreasing \( \text{VEval}_{\text{mov}}(v, e) \) by \( \text{mov}(e) \). \( \text{Stat}(v, e) \) is the submultiset of “stationary parts” of \( \text{Mon}(v, e) \), and by condition 2 depends only on \( e \).

• Let \( \text{Mon}(e) \) be the partition \( \bigcup_k \text{Mon}_k(v, e) \) of \( d \), which again by condition 2 depends only on \( e \). Note that unlike \( \text{Mon}(v, e) \) and \( \text{Mon}(b_i) \), \( \text{Mon}(e) \) is only a partition of \( d \), rather than a multipartition, and does not have a \( \{0, \ldots, r\} \)-labeling.

• For \( v \) satisfying any one of conditions 3, 4, or 5 we write \( \text{Mon}(v) \) for \( \text{Mon}(v, e_v) \) or \( \text{Mon}(v, e_v^1) = \text{Mon}(v, e_v^2) \).

• For an edge \( e \), let \( \beta(e) = \sum_{\eta \in \text{Mov}(e)} \beta_\eta(e) := \sum_{\eta \in \text{Mov}(e)} q(e) \eta \). Let \( \beta(\tilde{\Gamma}) = \sum_e \beta(e) \).

• Denote by \( \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta) \) the finite set of \( n \)-marked genus-zero \( \text{Sym}^d \mathbb{P}^r \)-decorated trees \( \tilde{\Gamma} \) with \( \beta(\tilde{\Gamma}) = \beta \). From now on we will call these simply “decorated trees” when no confusion is possible.

**Lemma 4.6.** There is a natural map

\[
\Psi : (\mathcal{M}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta))^T \to \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta).
\]

**Proof.** Let \( (f: (C, b_1, \ldots, b_n) \to \text{Sym}^d \mathbb{P}^r) \in (\mathcal{M}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta))^T \). Define sets \( V(\tilde{\Gamma}) \) equal to the set of connected components of \( f^{-1}(\text{Sym}^d \mathbb{P}^r)^T \), and \( E(\tilde{\Gamma}) \) the set of noncontracted irreducible components of \( C \). By Lemma 4.3, associated to each noncontracted irreducible component of \( C \) are two \( T \)-fixed points \( P_{i_1} \) and \( P_{i_2} \), so these define a graph \( \tilde{\Gamma} \). It is a tree because \( C \) has genus zero.

We now define the various decorations of \( \tilde{\Gamma} \). Let \( \text{Mark}(b_i) \) be the connected component of \( f^{-1}(\text{Sym}^d \mathbb{P}^r)^T \) containing \( b_i \). Let \( \text{VEval}(v) \) be the \( (r+1) \)-tuples representing the \( T \)-fixed point \( f(v) \), from Section 4.1. Let \( q(e) \) be the rational number \( \beta_\eta/e_\eta \) determined by Lemma 4.3. Let \( \text{Mon}(b_i) \) be the monodromy of \( f \) at \( b_i \). This is naturally a conjugacy class in the isotropy group \( G_{f(\text{Mark}(b_i))} \), and these are in natural bijection with MultiPart(\( \text{VEval}(\text{Mark}(b_i)) \)). Finally, let \( \text{Mon}(v) \), \( v \) be the monodromy of \( f \) at point \( \xi(v, e) \) where the connected component \( v \) meets the irreducible component \( e \), which is again naturally an element of MultiPart(\( \text{VEval}(v) \)).

Conditions 1 and 2 for decorated trees follow from the description in Lemma 4.3. Condition 3 follows from Remark 4.4. Condition 4 holds because for such \( v \), \( \xi(v, e_v) \) and \( b_v \) are the same point of \( C \). Condition 5 is true for the same reason.

**Notation 4.7.** Let \( (f : C \to \text{Sym}^d \mathbb{P}^r) \in \Psi^{-1}(\tilde{\Gamma}) \). If \( v \in V(\Gamma) \), then from Lemma 4.6 \( v \) corresponds to a subcurve of \( C \). We denote this by \( C_v \). Similarly, for \( e \in E(\Gamma) \), we write \( C_e \) for the corresponding irreducible component of \( C \). For \( (v, e) \in \Gamma \), we write \( \xi(v, e) \) for the point \( v \cap e \in C \), again using the notation of the proof of Lemma 4.6. We say \( (v, e) \) is a special flag if \( \xi(v, e) \) is a special point, equivalently if \( \text{val}(v) > 1 \) or \( \text{Mark}^{-1}(v) \neq \emptyset \). Note that the isotropy group at \( \xi(v, e) \) (resp. \( b_v \)) has order \( \text{lcm}(\text{Mon}(v, e_v)) \) (resp. \( \text{lcm}(\text{Mon}(b_v)) \)). For brevity we denote this by \( r(v, e) \) (resp. \( r_i \))

We adopt the following notation from [Liu13], corresponding to conditions 3 and 5 of Definition 4.5.

\[
\begin{align*}
V^1(\tilde{\Gamma}) &= \{ v \in V(\Gamma) \mid \text{val}(v) = 1, \text{Mark}^{-1}(v) = 0 \} \\
V^{1,1}(\tilde{\Gamma}) &= \{ v \in V(\Gamma) \mid \text{val}(v) = 1, \text{Mark}^{-1}(v) = 1 \} \\
V^2(\tilde{\Gamma}) &= \{ v \in V(\Gamma) \mid \text{val}(v) = 2, \text{Mark}^{-1}(v) = 0 \} \\
V^S(\tilde{\Gamma}) &= V(\Gamma) \setminus (V^1(\tilde{\Gamma}) \cup V^{1,1}(\tilde{\Gamma}) \cup V^2(\tilde{\Gamma})).
\end{align*}
\]

We call vertices in \( V^S(\tilde{\Gamma}) \) stable. A vertex \( v \) is stable if and only if \( C_v \) is 1-dimensional (rather than a single point).
Remark 4.8. Conditions (3) and (4) in Definition 4.5 are always satisfied for $\hat{\Gamma} \in \text{Im}(\Psi)$.

Definition 4.9. Let $\hat{\Gamma} \in \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta)$, and let $e_1, e_2 \in E(\Gamma)$. We say $e_1$ and $e_2$ are combinable, and write $e_1 \parallel e_2$, if there exists $v \in V^2(\hat{\Gamma})$ with $\{e_1, e_2\} = \{e_v^1, e_v^2\}$ and the following hold:

- $q(e_1) = q(e_2)$,
- $i^{\text{mov}}(v, e_1) = i^{\text{mov}}(v, e_2)$ and $i^{\text{mov}}(v, e_1) = i^{\text{mov}}(v, e_2)$.

Denote by $\mathcal{P} \subseteq (E(\Gamma)^3)$ the set of pairs $\{\{e_1, e_2\} : e_1 \parallel e_2\}$.

Definition 4.10. Let $(v, e) \in F(\Gamma)$. We say $(v, e)$ is a steady flag if either of the following holds:

1. $v \not\in V^2(\Gamma)$, or
2. $v \in V^2(\Gamma)$ and $\{e_v^1, e_v^2\} \not\in \mathcal{P}$.

Definition 4.11. Let $\hat{\Gamma} \in \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta)$ and let $e_1 \parallel e_2$ be a pair of combinable edges. We may define a new decorated tree $\text{Comb}(\hat{\Gamma}, e_1 \parallel e_2) \in \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta)$ by combining $e_1$ and $e_2$. In other words, we delete the vertex $v$ and the edges $e_1$ and $e_2$, and add an edge $e_{12} = (v_1, v_2)$ with $q(e_{12}) = q(e_1) = q(e_2)$, $\text{Mon}(v_1, e_{12}) = \text{Mon}(v_1, e_1)$, and $\text{Mon}(v_2, e_{12}) = \text{Mon}(v_2, e_2)$. (See Figure 2.) It is easy to check that $\text{Comb}(\hat{\Gamma}, e_1 \parallel e_2)$ satisfies the two conditions of a decorated tree, and that $\text{Mov}(e_{12}) = \text{Mov}(e_1) \cup \text{Mov}(e_2)$, and $\text{Mon}(e_{12}) = \text{Mon}(e_1) = \text{Mon}(e_2)$. There is a natural map $\phi_{e_1, e_2} : E(\Gamma) \to E(\text{Comb}(\hat{\Gamma}, e_1 \parallel e_2))$ with $\phi_{e_1, e_2}(e_1) = \phi_{e_1, e_2}(e_2) = e_{12}$, and $\phi_{e_1, e_2}(e) = e$ for $e \in E(\Gamma) \setminus \{e_1, e_2\}$.

Proposition 4.12. Let $\hat{\Gamma} \in \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta)$, and let $e_1 \parallel e_2$ and $e_1' \parallel e_2'$ be two distinct pairs of combinable edges of $\Gamma$. Then $\phi_{e_1, e_2}(e_1') \parallel \phi_{e_1, e_2}(e_2')$ as edges of $\text{Comb}(\hat{\Gamma}, e_1 \parallel e_2)$ and $\phi_{e_1', e_2'}(e_1) \parallel \phi_{e_1', e_2'}(e_2)$ as edges of $\text{Comb}(\hat{\Gamma}, e_1' \parallel e_2')$. Also, combining pairs commutes, i.e.

$$\text{Comb}(\text{Comb}(\hat{\Gamma}, e_1 \parallel e_2), e_1' \parallel e_2') \cong \text{Comb}(\text{Comb}(\hat{\Gamma}, e_1' \parallel e_2'), e_1 \parallel e_2),$$

and this isomorphism identifies the maps $\phi_{e_1', e_2} \circ \phi_{e_1, e_2}'$ and $\phi_{e_1', e_2'} \circ \phi_{e_1, e_2}$.

Proof. There are two cases, pictured in the left side of Figure 4.2 either the pairs $e_1 \parallel e_2$ and $e_1' \parallel e_2'$ share an edge, or they do not. Suppose we are in the first case, i.e. the top line of Figure 4.2. By definition of $\phi_{e_1, e_2}$, the edges $\phi_{e_1, e_2}(e_1')$ and $\phi_{e_1, e_2}(e_2')$ meet at $v'$ (precisely, at the corresponding vertex in $\text{Comb}(\hat{\Gamma}, e_1 \parallel e_2)$), and satisfy the three conditions of Definition 4.9. Thus $\phi_{e_1, e_2}(e_1') \parallel \phi_{e_1, e_2}(e_2')$. Similarly $\phi_{e_1', e_2'}(e_1) \parallel \phi_{e_1', e_2'}(e_2)$. To see that $\text{Comb}(\text{Comb}(\hat{\Gamma}, e_1 \parallel e_2), e_1' \parallel e_2') \cong \text{Comb}(\text{Comb}(\hat{\Gamma}, e_1' \parallel e_2'), e_1 \parallel e_2)$, we note that both are obtained from the tree in Figure 4.2 by replacing the three edges shown with a single edge $e$ connecting $v_1$ to $v_1'$. The decorations on this edge are:

- $q(e) := q(e_1) = q(e_2) = q(e_2')$,
- $\text{Mon}(e) := \text{Mon}(e_1) = \text{Mon}(e_2) = \text{Mon}(e_2')$,
Let $\Gamma_1, \Gamma_2, \Gamma_3$ be the associated maps. Write $\Psi$ where $\leq$ the set of $\{v_0, v_1, v_2\}$. Then states that for $\tilde{\Gamma}$ Corollary 4.13 may be restated as follows. Definition 4.11 determines a partial order $\leq$ if and only if $(e_1, e_2) \in \tilde{\Gamma}$, then $\phi_{\leq}(e_1) = \phi_{\leq}(e_2)$ if and only if $(e_1, e_2) \in \tilde{\Gamma}$. This follows from factoring $\phi_{\leq}$ as a sequence of edge combination maps as in Definition 4.11.

Corollary 4.13 may be restated as follows. Definition 4.11 determines a partial order $\leq$ on $\text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}, \beta)$, where $\tilde{\Gamma'} \leq \tilde{\Gamma}$ if $\tilde{\Gamma'}$ can be obtained from $\tilde{\Gamma}$ by combining edges. The Corollary then states that for $\tilde{\Gamma} \in \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}, \beta)$, there is a natural order-reversing bijection between $\{\tilde{\Gamma'} : \tilde{\Gamma'} \leq \tilde{\Gamma}\}$ and $\{$subsets of $\mathcal{P}(\tilde{\Gamma})\}$, where the latter is partially ordered by inclusion. In particular, associated to $\tilde{\Gamma}$ is a unique minimal decorated tree $\text{Comb}(\tilde{\Gamma}, \mathcal{P}(\tilde{\Gamma}))$. Denote by $\text{Trees}_{0,n}^{\min}(\text{Sym}^d \mathbb{P}, \beta)$ the set of $\leq$-minimal elements of $\text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}, \beta)$.

Theorem 4.14. Let $\tilde{\Gamma}_0 \in \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}, \beta)$. The closure of $\Psi^{-1}(\tilde{\Gamma}_0)$ is

$$\bigcup_{\tilde{\Gamma} \in \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}, \beta)} \Psi^{-1}(\tilde{\Gamma}_0),$$

where $\Psi$ is the map from Lemma 4.10.

Lemma 4.15. Let $\tilde{\Gamma}_0 = v_1 \bullet e \bullet v_2$, where each of $v_1$ and $v_2$ contains a single marked point, $b_1$ and $b_2$. Let $f : C \to \text{Sym}^d \mathbb{P}$ be in the closure of $\Psi^{-1}(\tilde{\Gamma}_0)$, and let $\rho : C' \to C$ and $f' : C' \to \mathbb{P}$ be the associated maps. Write $C'_{\eta}$ for a noncontracted irreducible component of $C'$, corresponding to $\eta \in \text{Mov}(e) \subseteq \text{Mon}(e)$, as described in Lemma 4.3. Denote by $L_e := L_{\text{mov}(v_1, e), \text{mov}(v_2, e)}$ the line in $\mathbb{P}$ connecting $P_{\text{mov}(v_1, e)}$ and $P_{\text{mov}(v_2, e)}$. Then:

1. $C$ and $C'_{\eta}$ are nodal chains of rational curves,
2. $f'|_{C'_{\eta}}$ maps one irreducible component of $C'_{\eta}$ to $L_e$ with degree $\beta_{\eta}(e) = q(e) \cdot \eta$ (on coarse moduli spaces), and is fully ramified at the two special points of this component, and
3. $f'|_{C'_{\eta}}$ contracts all other irreducible components of $C'_{\eta}$ to one of the endpoints of $L_e$. 

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Remark 4.15. Thus \( C \) is a chain. This proves claims (2) and (3).

Proof of Lemma. Let \( f : C \to \mathbb{P}^r \) be a family over \( S \) of stable maps whose generic fiber is in \( \Psi^{-1}(\tilde{\Gamma}_0) \), and let \( s \in S \) such that the fiber over \( s \) is the stable map \( f : C \to \text{Sym}^d \mathbb{P}^r \). After an \( \acute{e}tale \) base change \( \tilde{S} \to S \), \( C' \) is a union of connected components \( C'_\eta \) indexed by \( \text{Mon}(e) \), and the maps \( C'_\eta \to C \) have degrees determined by \( \text{Mon}(e) \).

Consider the Stein factorization relative to \( S \):

\[
C' \to C'_{\eta} \to \eta \to C.
\]

The pullbacks along \( \overline{f} \) of the divisors \( P_{\text{mov}(v_1,e)} \) and \( P_{\text{mov}(v_2,e)} \) on \( L_e \) are divisors on \( C'_{\eta} \), that by the definition of the Stein factorization do not contain a component of any fiber. On a generic fiber, these divisors are each supported on a single point, i.e. \( \rho^{-1}(b_1) \) and \( \rho^{-1}(b_2) \). Thus on the fiber \( C'_{\eta} \) over \( s \), the divisors are still supported on single points, and \( \rho^{-1}(b_1) \) and \( \rho^{-1}(b_2) \) each lie above one of these points. (Also, the points are distinct since \( \overline{f} \) is well-defined.)

As any component of \( C'_{\eta} \) maps surjectively to \( L_e \), the above implies that \( C'_{\eta} \) is irreducible. This proves claims (2) and (3).

Since \( f' \) is \( T \)-fixed, any irreducible components of \( C'_{\eta} \) that are contracted by \( f' \) map to either \( P_{\text{mov}(v_1,e)} \) or \( P_{\text{mov}(v_2,e)} \), i.e. they lie over either \( (\overline{f})^{-1}(P_{\text{mov}(v_1,e)}) \) or \( (\overline{f})^{-1}(P_{\text{mov}(v_2,e)}) \). Also, all nodes of \( C'_{\eta} \) lie over one of these two points. Since \( \eta \) was arbitrary, this shows that any irreducible component \( D \) of \( C' \) that is not contracted by \( f' \) has at most two special points, where a special point here means either a node or one of the points \( \rho^{-1}(b_1) \) and \( \rho^{-1}(b_2) \). Since \( \rho^{-1}(b_1) \) and \( \rho^{-1}(b_2) \) lie above distinct points of \( D \), \( D \) has exactly two special points.

If \( C \) is not a chain, some component has only one special point. By stability there is a component \( D \) of \( \rho^{-1}(D) \) that is not contracted by \( f' \), which contradicts the fact that \( D \) has two special points. Thus \( C \) is a chain, and it follows that each connected component \( C'_\eta \) is a chain. This proves claim (1). \( \square \)

Remark 4.16. In summary, the restriction to \( C'_\eta \) of a point in \( \Psi^{-1}(\tilde{\Gamma}_0) \) may be represented as in Figure 4.2 (where despite appearances we mean for the map to \( L_e \) to have a single preimage point over each of \( P_{\text{mov}(v_1,e)} \) and \( P_{\text{mov}(v_2,e)} \)).

Proof of Theorem 4.14. It is sufficient to consider the situation of Lemma 4.15. To see this, note that any \( \tilde{\Gamma}_0 \in \text{Trees}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta) \) may be decomposed into subtrees of the form in the Lemma, together with single-vertex trees, glued at marked points. There is a corresponding decomposition of \( \Psi^{-1}(\tilde{\Gamma}_0) \) as a product (up to a finite morphism), and this decomposition extends to the closure...
(see [AGV08], Section 5.2, or [Liu13], Section 9.2). Thus we may treat each factor of the product separately.

First, we show

\[ \Psi^{-1}(\tilde{\Gamma}_0) \subseteq \bigcup_{\tilde{\Gamma}, \tilde{\Gamma}_0 \leq \tilde{\Gamma}} \Psi^{-1}(\tilde{\Gamma}). \]

Let \((f : C \to \mathbb{P}^r) \in \Psi^{-1}(\tilde{\Gamma}_0)\). It follows from Lemma 4.15 that \(f^{-1}((\text{Sym}^d \mathbb{P}^r)^T)\) is exactly the set of nodes of \(C\), together with the two marked points. By stability, all irreducible components of \(C\) are noncontracted. Thus the tree \(\Psi(f : C \to \text{Sym}^d \mathbb{P}^r)\) is a chain with a vertex for each node and marked point, and an edge for each irreducible component.

Denote by \(v_1\) and \(v_2\) the leaves of \(C\), such that \(v_1 = \{b_1\}\) and \(v_2 = \{b_2\}\). For \(v \neq v_1, v_2\), we have \(\text{Mark}^{-1}(v) = \emptyset\). By claim 2 of Lemma 4.15, the degree ratios \(q(e)\) are equal for all edges \(e\). By the description of the connected components of \(C'\), the partitions \(\text{Mon}(e)\) are equal for all \(e\). Finally, deleting an edge \(e\) breaks \(C\) into two connected components, one containing \(v_1\) and one containing \(v_2\). Let \(v\) be on the component with \(v_1\), and \(v'\) on the component with \(v_2\), such that \(e = (v, v')\). Then from the proof of Lemma 4.15, we have \(i^{\text{mov}}(v, e) = i^{\text{mov}}(v_1, e_{12})\) and \(i^{\text{mov}}(v', e) = i^{\text{mov}}(v_2, e_{12})\). Thus any pair of adjacent edges in \(\Psi(f : C \to \text{Sym}^d \mathbb{P}^r)\) is combinable. Combining them all yields \(\tilde{\Gamma}_0\), i.e. \(\tilde{\Gamma}_0 \cong \Psi(f : C \to \text{Sym}^d \mathbb{P}^r)\).

For the converse, by induction on \(|E(\tilde{\Gamma})| - |\tilde{\Gamma}_0| = |E(\tilde{\Gamma})| - 1\), it is sufficient to show that

\[ \Psi^{-1}(\tilde{\Gamma}_0) \supseteq \bigcup_{\tilde{\Gamma}, \tilde{\Gamma}_0 \leq \tilde{\Gamma}} \Psi^{-1}(\tilde{\Gamma}). \]

Fix such a tree \(\tilde{\Gamma} = v_1 \cdot e_1 \cdot v_2 \cdot e_2 \cdot v_2\), and fix \((f : C \to \text{Sym}^d \mathbb{P}^r) \in \Psi^{-1}(\tilde{\Gamma})\). We will construct a family \(f : C \to \text{Sym}^d \mathbb{P}^r\) over \(\mathbb{C}\) whose restriction to \(0 \in \mathbb{C}\) is the map \(f : C \to \text{Sym}^d \mathbb{P}^r\).

By Lemma 4.3 and by representability of \(f : C \to \text{Sym}^d \mathbb{P}^r\), the orbifold points and nodes of \(C\) have order \(\text{lcm}(\text{Mon}(e_1)) = \text{lcm}(\text{Mon}(e_2))\). Thus \(C\) is isomorphic to \(V(xy) \subseteq [\mathbb{P}^2 / \mu_{\text{lcm}(\text{Mon}(e_1))}]\), where \(\mathbb{P}^2\) has coordinates \(x, y, z\), and \(\text{lc}(\text{Mon}(e_1))\) acts by multiplication by inverse roots of unity on the first two coordinates. Define \(C\) so that \(C_t = V(xy - tz^2)\) for \(t \in \mathbb{C}\). Precisely, \(C\) is an open subset of \([\mathbb{B}^f_{[1:0:0],[0:1:0]} \mathbb{P}^2 / \mu_{\text{lcm}(\text{Mon}(e_1))}]\).

For \(\eta \in \text{Mon}(e_1)\), there is an étale quotient map \(\tilde{\rho} : [\mathbb{P}^2 / \mu_\eta] \to [\mathbb{P}^2 / \text{lcm}(\text{Mon}(e_1))]\). As above, define \((C')_\eta = V(xy - tz^2) \subseteq [\mathbb{P}^2 / \mu_\eta]\).

We must now define a map \(f' : (C')_\eta \to \mathbb{P}^r\) for each \(\eta \in \text{Mon}(e_1)\). As \(\mathbb{P}^r\) is a variety, it is enough to define this on coarse moduli spaces. We choose isomorphisms of the fibers \((C'_{\eta})_0\) and \(C_0\) with \((C'_\eta)_0\) and \(C_0\) respectively, such that the maps \(\tilde{\rho}\) and \(\rho\) are identified. Then \(f'\) defines a map \(f'_{\eta} : (C'_{\eta})_0 \to L_{e_1} = L_{e_2}\) (The case where \(C'_\eta\) is contracted is trivial, so we assume it is not contracted.) By Lemma 4.15, after equivariantly identifying \(L_{e_1} \cong \mathbb{P}^1\), \(f'_{\eta}\) is given (without loss of generality, on coarse moduli spaces) by

\[
 [x : 0 : z] \mapsto [0 : 1] \\
 [0 : y : z] \mapsto [y^{\beta_\eta(e_1)} : z^{\beta_\eta(e_1)}].
\]

It remains to extend this to a map \(\tilde{f}' : C'_\eta \to L_{e_1}\) that is fixed with respect to the \(T\)-action, i.e. fully ramified over the endpoints of \(L_{e_1}\). We observe that the rational map

\[
 [x : y : z] \mapsto [y^{\beta_\eta(e_1)} : z^{\beta_\eta(e_1)}]
\]

is equivariant under the \(T\)-action.
is regular after blowing up the point \([1:0:0]\). This defines a map \(\tilde{f}'\) as desired. Doing this for all \(\eta\) simultaneously shows that \(f: C \rightarrow \text{Sym}^d \mathbb{P}^r\) is in \(Ψ^{-1}(\overline{Γ})_0\) as desired. \(\Box\)

Because \((\overline{M}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta))^T = \bigcup \overline{Ψ}^{-1}(\overline{Γ})\), we have:

**Corollary 4.17.** Let \(\tilde{Γ} \in \text{Trees}^{\text{min}}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta)\). Then \(\overline{Ψ}^{-1}(\overline{Γ})\) is an open and closed substack of \((\overline{M}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta))^T\). We denote it \(\overline{M}_F\).

The rest of this section proves the following:

**Theorem 4.18.** For a stable vertex \(v\) or edge \(e = (v_1, v_2)\) of a minimal decorated tree \(\tilde{Γ} = (Γ, \text{Mark}, \text{VEval}, q, \text{Mon}) \in \text{Trees}^{\text{min}}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta)\), we define

\[\overline{M}_v := \overline{M}_{0,\text{Mon}(v)}(BS_{\text{VEval}(v)}, 0)\]

\[\overline{M}_e := \left[\overline{M}^{\text{lc}(\text{Mon}(e))}_{v_1|\text{mov}(v)|v_2}/\left(\prod_{\eta \in \text{mov}(e)} \mu_{\beta_{\eta}(e)} \text{wr } S_\ell\right)\right],\]

where:

- \(\overline{\text{Mon}}(v)\) is the list of multipartitions \(\{\text{Mon}(b_i)\}_{i \in \text{Mark}^{-1}(v)} \cup \{\text{Mon}(v, e)\}_{(v, e) \in F(Γ)}\),
- \(\overline{M}^{\text{lc}(\text{Mon}(e))}_{v_1|\text{mov}(v)|v_2}\) is the order \(\text{lc}(\text{Mon}(e))\) orbifold Losev-Manin space with \(\text{mov}(e)\) marked points \(b_1, \ldots, b_{\text{mov}(e)}\) and labeling set \(\{v_1, v_2\}\), from Section 2.5,
- \(S_\ell\) is the group \(C_{\text{Stat}(e)} \times S_{\text{Mov}(e)}\), where \(C_{\text{Stat}(e)}\) is the centralizer of any element of the conjugacy class \(\text{Stat}(e)\) in \(\prod_{i=0}^{\ell} S_{\text{Stat}(e)_i}\), and acts trivially on the Losev-Manin space,
- A generator of \(\mu_{\beta_{\eta}(e)}\) acts by translating the marked point \(b_\eta\) by \(e^{2\pi i / q(e)}\), and
- \(\text{wr}\) denotes the wreath product.

Then the substack \(\overline{M}_F\) associated to \(\tilde{Γ}\) is isomorphic to

\[(4) \quad \left[\left(\prod_{v \in V^\delta(Γ)} \overline{M}_v \times \prod_{e \in E(Γ)} \overline{M}_e\right)/\text{Aut}(Γ)\right].\]

**Remark 4.19.** More precisely, \(\overline{M}_F\) has extra automorphisms coming from gluing at nodes, and is thus a gerbe over \(Γ\). Gluing of components is fibered over the rigidified inertia stack \(\tilde{Γ}\text{Sym}^d \mathbb{P}^r\) (see [AGV08] or [Liu13]). In particular, for each steady flag \((v, e)\) of \(Γ\), we get an extra factor of \(\left[ C_{\text{VEval}(v)}(\text{Mon}(v, e)) \right] / \tau(v, e) \) in the fundamental class of \(\overline{M}_F\), where \(C_{\text{VEval}(\text{Mon}(v, e))}\) is the centralizer of any element of the conjugacy class \(\text{Mon}(v, e)\) of \(G_{\text{VEval}(v)}\). (We make the usual correction for double counting when \(v \in V^2(Γ)\).)

**Proof of 4.18.** Using the gluing morphisms, we may write

\[\overline{M}_F \cong \left[\left(\prod_{v \in V(Γ)} \overline{M}_{0,\text{Mon}(v)}(BS_{\text{VEval}(v)}, 0) \times \prod_{e \in E(Γ)} \overline{M}_{0,\{\text{Mon}(e), \text{Mon}(e)\}}(L_e, \beta(e))^T\right)/\text{Aut}(Γ)\right],\]

We need to show that, for all \(e = (v_1, v_2) \in E(Γ)\), we have

\[\overline{M}_{0,\{\text{Mon}(e), \text{Mon}(e)\}}(L_e, \beta(e))^T \cong \left[\overline{M}^{\text{lc}(\text{Mon}(e))}_{v_1|\text{mov}(e)|v_2}/\left(\prod_{\eta \in \text{mov}(e)} \mu_{\beta_{\eta}(e)} \text{wr } S_\ell\right)\right].\]
Note that the left hand side is isomorphic to $\overline{\mathcal{M}}_{\tilde{\Gamma}_e}$ for $\tilde{\Gamma}_e = v_1 \cdots e \cdots v_2$, where the decorations on $\tilde{\Gamma}_e$ are induced from $\tilde{\Gamma}$. (Here the two vertices are labeled, i.e. $\text{Aut}(\tilde{\Gamma}_e) = 1$.) Write $P_e := P_{(\mu_{\text{mov}(v_1,e)}, \mu_{\text{mov}(v_2,e)})}$ for the midpoint of $L_e$. For $(f : C \to \mathbb{P}^r) \in \overline{\mathcal{M}}_{\tilde{\Gamma}_e}$, consider the preimage of $P_e$ under the associated map $f' : C' \to \mathbb{P}^r$. By Lemma 4.15, $C'$ is a union of connected components $C''_\eta$ for $\eta \in \text{Mon}(e)$, and if $\eta \in \text{Mov}(e)$ then the preimage of $P_e$ on $C''_\eta$ consists of $\beta_\eta(e)$ points on the single noncontracted component of $C''_\eta$. These points are $\mu_{\beta_\eta(e)}$-translates of each other, under the natural action that fixes the two special points.

After a principal $(\prod_{\eta \in \text{Mov}(e)} \mu_{\beta_\eta(e)} \wr S_e)$-cover $\overline{\mathcal{M}}_{\tilde{\Gamma}_e} \to \overline{\mathcal{M}}_{\tilde{\Gamma}_e}$, we may fix a labeling of the connected components $C''_\eta$, and label a distinguished preimage of $P_e$ on $C''_\eta$ for $\eta \in \text{Mov}(e)$. (The $S_e$-cover removes all automorphisms of stable maps induced by automorphisms of the image curve that commute with the monodromy at $v_1$ and $v_2$.) Remembering the images of these distinguished points under $\rho$ yields a nodal chain of rational curves with $\text{mov}(v)$ labeled marked points, none of which coincides with $v_1$ or $v_2$. The stability condition for $\overline{\mathcal{M}}_{0, \{\text{Mon}(e), \text{Mov}(e)\}}(L_e, \beta(e))$ implies that this is a Losev-Manin curve, with orbifold points of order $\text{lcm}((\text{Mon}(e)))$ at marked points and nodes.

This construction works in families, so it defines a map $\overline{\mathcal{M}}_{\tilde{\Gamma}_e} \to \overline{\mathcal{M}}_{\text{mov}(v_1,e) \wr S_e}$, which is equivariant by definition with respect to the action of $\prod_{\eta \in \text{Mov}(e)} \mu_{\beta_\eta(e)} \wr S_e$. This gives a map

$$\Phi : \overline{\mathcal{M}}_{\tilde{\Gamma}_e} \to \left[ \begin{array}{c} \overline{\mathcal{M}}_{\text{mov}(v_1,e) \wr S_e} \\ \left( \prod_{\eta \in \text{Mov}(e)} \mu_{\beta_\eta(e)} \wr S_e \right) \end{array} \right].$$

We now construct an inverse to this map. Let $(C, b_{v_1}, b_1, \ldots, b_{\text{mov}(v_1,e)}, b_{v_2}) \in \overline{\mathcal{M}}_{\text{mov}(v_1,e) \wr S_e}$ be a Losev-Manin curve whose points are indexed by the multiset $\text{Mov}(e)$. Fix a curve $C' = \bigcup_{\eta \in \text{Mon}(e)} C''_\eta$ with étale maps $\rho_\eta : C''_\eta \to C$ of degree $\eta$. This may be done uniquely up to isomorphism. Also, uniquely up to isomorphism (of $C'$ commuting with $\rho : C' \to C$), for each $\eta \in \text{Mov}(e)$ we may choose a preimage point $b'_\eta \in C''_\eta$ of the corresponding marked point $b_\eta \in C$. Finally, there is a unique map $f' : C' \to \mathbb{P}^r$ that sends:

- $C''_\eta$ to a $T$-fixed point, for $\eta \not\in \text{Mov}(e)$,
- $C''_\eta$ to $L_e$ with degree $\beta_\eta(e)$, with $b'_\eta$ mapping to $P_e$, $\rho^{-1}(b_{v_1})$ mapping to $P_{(\mu_{\text{mov}(v_1,e)})}$ and $\rho^{-1}(b_{v_2})$ mapping to $P_{(\mu_{\text{mov}(v_2,e)})}$, for $\eta \in \text{Mov}(e)$.

Again, this works in families, and defines a map $\tilde{\Theta} : \overline{\mathcal{M}}_{\text{mov}(v_1,e) \wr S_e} \to \overline{\mathcal{M}}_{\tilde{\Gamma}_e}$, which we claim is invariant under the action of $\prod_{\eta \in \text{Mov}(e)} \mu_{\beta_\eta(e)} \wr S_e$. Indeed, acting by $e^{2\pi i / \eta(e)}$ on $b'_\eta$ translates the preimage $b'_\eta$ by some power of $e^{2\pi i / \beta_\eta(e)}$, and commutes with $f'$. Thus $\tilde{\Theta}$ descends to a map

$$\Theta : \left[ \begin{array}{c} \overline{\mathcal{M}}_{\text{mov}(v_1,e) \wr S_e} \\ \left( \prod_{\eta \in \text{Mov}(e)} \mu_{\beta_\eta(e)} \wr S_e \right) \end{array} \right] \to \overline{\mathcal{M}}_{\tilde{\Gamma}_e},$$

which is by construction an inverse to $\Phi$. \qed

**Corollary 4.20.** The $(\prod_{\eta \in \text{Mov}(e)} \mu_{\beta_\eta(e)} \wr S_e)$-action on $\overline{\mathcal{M}}_{\text{mov}(v_1,e) \wr S_e}$ extends to the universal curve, so we have a universal curve on $\overline{\mathcal{M}}_{\tilde{\Gamma}_e}$, and by gluing, a universal curve on the left side of (4).

The isomorphism of 4.18 naturally identifies this with the universal curve on $\overline{\mathcal{M}}_{\tilde{\Gamma}_e}$.

**Proof.** The first statement is by definition of the action, and the second is immediate from the proof of Theorem 4.18. \qed
Remark 4.21. Theorem 4.18 shows in particular that $\overline{\mathcal{M}}_{F_{\xi}}$ is irreducible, so connected components of $(\overline{\mathcal{M}}_{0,n}(\text{Sym}^d \mathbb{P}^r, \beta))^T$ are indexed by minimal decorated trees with the additional data of a connected component of $\overline{\mathcal{M}}_{\text{Mon}(v)}(BS_{\text{Eval}(v)}, 0)$ for each $v$.

Notation 4.22. For a special flag $(v, e) \in F(\Gamma)$, we denote by $\psi_v^{\overline{\mathcal{M}}_e}$ the $\psi$-class on $\overline{\mathcal{M}}_e$ at the point labeled by $v$. If $v \in V^S(\tilde{\Gamma})$, we denote by $\psi_v^{\overline{\mathcal{M}}_v}$ the $\psi$-class on $\overline{\mathcal{M}}_v$ at the marked point $\xi(v, e)$. We use the same notation for the $\overline{\psi}$-classes.

5. THE VIRTUAL NORMAL BUNDLE AND VIRTUAL FUNDAMENTAL CLASS OF $\overline{\mathcal{M}}_F$

In this section we compute the Euler class of the virtual normal bundle to $\overline{\mathcal{M}}_F$, and show that the virtual fundamental class of $\overline{\mathcal{M}}_F$ is equal to its fundamental class. Many of the arguments are “classical,” and we refer the reader to [Liu13] for these.

In this section we fix $\tilde{\Gamma} \in \text{Trees}_{0,n}^\text{min}(\text{Sym}^d \mathbb{P}^r, \beta)$. Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_F$ and $\rho : \mathcal{C}' \to \mathcal{C}$ denote the universal curve and universal étale cover, respectively:

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{f'} & \mathbb{P}^r \\
\downarrow \rho & & \downarrow \pi \\
\mathcal{C} & \xrightarrow{f} & \text{Sym}^d \mathbb{P}^r \\
\overline{\mathcal{M}}_F
\end{array}
$$

By a standard argument (see [Liu13]), we have an exact sequence of $T$-equivariant sheaves on $\overline{\mathcal{M}}_{0,n+1}(\text{Sym}^d \mathbb{P}^r, \beta)$ giving the perfect obstruction theory\footnote{We will always use the notation in [G] for higher direct image sheaves, writing e.g. $R^i\pi_*(C, f^*T \text{Sym}^d \mathbb{P}^r)$ instead of $R^i\pi_*f^*T \text{Sym}^d \mathbb{P}^r$. This is because we will restrict $\pi$ to various substacks of $\mathcal{C}$, and wish to avoid renaming maps.}

\begin{equation}
0 \to \text{Aut}(\mathcal{C}) \to R^0\pi_*(\mathcal{C}, f^*T \text{Sym}^d \mathbb{P}^r) \to \text{Def}(\mathcal{C}, f) \to \\
\to \text{Def}(\mathcal{C}) \to R^1\pi_*(\mathcal{C}, f^*T \text{Sym}^d \mathbb{P}^r) \to \text{Obs}(\mathcal{C}, f) \to 0,
\end{equation}

where $\text{Aut}(\mathcal{C})$ (resp. $\text{Def}(\mathcal{C})$) is the sheaf on $\overline{\mathcal{M}}_{0,n+1}(\text{Sym}^d \mathbb{P}^r)$ of infinitesimal automorphisms (resp. deformations) of the marked source curve $\mathcal{C}$. (See [Liu13] for rigorous definitions.) For $(f : \mathcal{C} \to \text{Sym}^d \mathbb{P}^r) \in \overline{\mathcal{M}}_F$, we also have a normalization exact sequence computing the fibers of the middle terms:

\begin{equation}
0 \to H^0(C, f^*T \text{Sym}^d \mathbb{P}^r) \to \bigoplus_{\nu} H^0(C_{\nu}, f^*T \text{Sym}^d \mathbb{P}^r) \to \bigoplus_{\xi} H^0(\xi, f^*T \text{Sym}^d \mathbb{P}^r) \to \\
\to H^1(C, f^*T \text{Sym}^d \mathbb{P}^r) \to \bigoplus_{\nu} H^1(C_{\nu}, f^*T \text{Sym}^d \mathbb{P}^r) \to 0,
\end{equation}

where $\nu$ runs over the set of irreducible components of $C$, and $\xi$ runs over nodes of $C$. The sequences \cite{5} and \cite{6} each split as direct sums of two exact sequences: the $T$-fixed part and the $T$-moving part. We use the notations $\text{Aut}(\mathcal{C})^{\text{fix}}$ and $\text{Aut}(\mathcal{C})^{\text{mov}}$ (and similar) to denote the $T$-fixed subsheaf or subspace and its $T$-invariant complement. By definition (see [GP99]), the Euler class of the virtual normal bundle $e_T(N_F^{\text{vir}})$ is

\begin{equation}
e_T(\text{Def}(\mathcal{C}, f)^{\text{mov}}) = e_T(\text{Def}(\mathcal{C})^{\text{mov}})e_T(R^0\pi_*(\mathcal{C}, f^*T \text{Sym}^d \mathbb{P}^r)^{\text{mov}}) \\
e_T(\text{Aut}(\mathcal{C})^{\text{mov}})e_T(R^1\pi_*(\mathcal{C}, f^*T \text{Sym}^d \mathbb{P}^r)^{\text{mov}}) \in H^*_T(\overline{\mathcal{M}}_F),
\end{equation}
and the virtual fundamental class $[\mathcal{M}_\Gamma]^{vir}$ of $\mathcal{M}_\Gamma$ is $e_T(\text{Obs}(\mathcal{C}, f)^{\text{fix}})$. We compute the various terms of (5) and (6) one by one. It is convenient to compute by pulling back to the canonical $\text{Aut}(\Gamma)$-cover $\mathcal{M}_\Gamma^{\text{rig}}$ of $\mathcal{M}_\Gamma$, so that the correspondence between $C$ and $\hat{\Gamma}$ is more concrete.

**The sheaves $\text{Aut}(\mathcal{C})$ and $\text{Def}(\mathcal{C})$.** In the toric case, from [Liu13] we have

\[ e_T(\text{Aut}(\mathcal{C})^{\text{mov}}) = \prod_{v \in V^1(\hat{\Gamma})} e_T(T_{\xi(v,e_v)} \mathcal{C}) = \prod_{v \in V^1(\hat{\Gamma})} \psi_v \mathcal{M}_v. \]

The same argument and answer apply here, using (Theorem 4.14 and) the observation that combining edges gives a natural identification of $V^1(\hat{\Gamma})$. Briefly, moving automorphisms come from noncontracted components with only one special point, and correspond to vector fields on such a component that are nonvanishing at the nonspecial $T$-fixed point.

Similarly, in the toric case [Liu13] gives

\[ e_T(\text{Def}(\mathcal{C})) = \left( \prod_{(v,e) \in F(\Gamma)} (-\psi_v \mathcal{M}_v - \psi_e \mathcal{M}_e) \right) \left( \prod_{\xi \in V^S(\hat{\Gamma})} (\prod_{(v,e) \in F(\Gamma)} (-\psi_v \mathcal{M}_v - \psi_e \mathcal{M}_e)) \right). \]

This is again correct in our case. The factors in (9) come from smoothing nodes. (Classically, the deformation space of a node is the tensor product of the tangents spaces to the two branches.) Therefore the observation we need is that the nodes that do not appear in (9) have $T$-fixed deformation space. We will use the following notation.

**Definition 5.1.** A node $\xi$ is called steady\(^3\) if $T_\xi C_1 \otimes T_\xi C_2$ has a nontrivial torus action, where $C_1$ and $C_2$ are the branches $\xi$.

**Remark 5.2.** Steady nodes are exactly those of the form $\xi(v,e)$ for $(v,e)$ a steady flag. By Theorem 4.14 if $\Psi(f : C \to \text{Sym}^d \mathbb{P}^r) = \hat{\Gamma}$ (i.e. it is minimal), then all nodes of $C$ are steady nodes. Furthermore, the set of steady nodes is canonically identified for any two points of $\mathcal{M}_\Gamma^{\text{rig}}$.

The factors in (9) are in correspondence with steady nodes.

**The bundles $R^0\pi_*(\mathcal{C}, f^*T \text{Sym}^d \mathbb{P}^r)$ and $R^1\pi_*(\mathcal{C}, f^*T \text{Sym}^d \mathbb{P}^r)$.** We use the sequence (6). The computation is very similar to the original one by Kontsevich [Kon95] (and the orbifold computations of Johnson [Joh13] and Liu [Liu13]), but requires some care due to the edge moduli spaces.

Because normalization does not commute with base change, (6) only computes fibers of $R^i\pi_*(\mathcal{C}, f^*T \text{Sym}^d \mathbb{P}^r)$. However, normalization of steady nodes does commute with base change on $\mathcal{M}_\Gamma^{\text{rig}}$, by the canonical identification of nodes above. Thus we have the sequence

\[ 0 \to R^0\pi_*(\mathcal{C}, f^*T \text{Sym}^d \mathbb{P}^r) \to \bigoplus_{\xi} R^0\pi_*(\mathcal{C}_{\xi}, f^*T \text{Sym}^d \mathbb{P}^r) \to \bigoplus_{\xi} R^0\pi_*(\xi, f^*T \text{Sym}^d \mathbb{P}^r) \to \]

\[ \to R^1\pi_*(\mathcal{C}, f^*T \text{Sym}^d \mathbb{P}^r) \to \bigoplus_{\xi} R^1\pi_*(\mathcal{C}_{\xi}, f^*T \text{Sym}^d \mathbb{P}^r) \to 0, \]

where $\xi$ runs over maximal subcurves of $\mathcal{C}$ containing only non-steady nodes, and $\xi$ runs over steady nodes. ($\mathcal{C}_{\xi}$ may contain a single branch of a steady node, but not both branches.) Observe that either $\mathcal{C}_{\xi}$ is contracted by $f$, or each fiber $\mathcal{C}_{\xi}$ of $\mathcal{C}_{\xi}$ contains only noncontracted components.

By Section 3.3 we have

\[ R^i\pi_*(\mathcal{C}_{\xi}, f^*T \text{Sym}^d \mathbb{P}^r) = R^i\pi_*(\mathcal{C}_{\xi}, f^*T \text{Sym}^d \mathbb{P}^r) = R^i(\pi \circ \rho)_*(\mathcal{C}_{\xi}^f, (f^*T \mathbb{P}^r). \]

\(^3\)This is similar to the definition of a breaking node from [OP10].
the set indexing irreducible components is not contracted. The components (equivalently, irreducible (see [Liu13], Section 7.5). However, note that
where we also denote by \( C \) to apply the orbifold quantum Riemann-Roch theorem.

(15)

(13)

If \( C \) is contracted, then \((f')^*T\mathbb{P}^r\) is trivial on \( C' \). Thus we have

\[
R^i(\pi \circ \rho)_*(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \cong R^i(\pi \circ \rho)_*(C'_{\nu,\eta}, \mathcal{O}_{C'_{\nu,\eta}}) \otimes T_{\pi(\eta)}\mathbb{P}^r,
\]

where as usual we write \( P_{\pi(\eta)} \) for \( f'(C'_{\nu,\eta}) \). In particular,

(13)

\[
R^0\pi_*(C_\nu, f^*T\text{Sym}^d\mathbb{P}^r)^{\text{mov}} = R^0\pi_*(C_\nu, f^*T\text{Sym}^d\mathbb{P}^r)^{\text{fix}} = 0.
\]

The bundle \( R^1\pi_*(C_\nu, f^*T\text{Sym}^d\mathbb{P}^r)^{\text{mov}} \) is nontrivial, and is isomorphic to a Hurwitz-Hodge bundle (see [Liu13], Section 7.5). However, note that \( c_T(R^1\pi_*(C_\nu, f^*T\text{Sym}^d\mathbb{P}^r)) \) is the inverse of the twisting class from [4]. We will use this fact in Section 6 in our characterization of \( \mathcal{L}_{\text{Sym}^d\mathbb{P}^r} \), and in Section 7 to apply the orbifold quantum Riemann-Roch theorem.

Similarly for a steady node \( \xi(v, e) \), we have

(14)  \[ R^0\pi_*(\xi(v, e), f^*T\text{Sym}^d\mathbb{P}^r)^{\text{fix}} = 0 \]

(15)  \[ R^0\pi_*(\xi(v, e), f^*T\text{Sym}^d\mathbb{P}^r)^{\text{mov}} = T_{\text{V Eval}(v), \text{Mon}(v, e)} I \text{Sym}^d\mathbb{P}^r = \bigoplus_{\eta \in \text{Mon}(v, e)} T_{\pi(\eta)}\mathbb{P}^r. \]

Suppose \( C \) is not contracted. The components \( C'_{\nu,\eta} \) are in bijection with \( \text{Mon}(e) \), where \( e \) is the edge of \( \tilde{\Gamma} \) corresponding to \( C \). First, we argue that \( R^i(\pi \circ \rho)_*(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \) vanishes for all \( \eta \).

The normalization exact sequence for a fiber \( C'_{\nu,\eta} \) reads:

\[
0 \to H^0(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \to \bigoplus_{\nu \in \mathcal{U}} H^0(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \to \bigoplus_{\xi} H^0(\xi, (f')^*T\mathbb{P}^r) \to
\]

\[
\to H^1(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \to \bigoplus_{\nu \in \mathcal{U}} H^1(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \to 0,
\]

where we also denote by \( \nu \) the set indexing irreducible components \( C_\nu \) of \( C \) (equivalently, irreducible components \( C'_{\nu,\eta} \) of \( C'_{\nu,\eta} \)). For each \( \nu \in \mathcal{U} \), we have

(16)  \[ H^1(C_\nu, (f')^*T\mathbb{P}^r) = 0 \]

by convexity of \( \mathbb{P}^r \). We claim that the map

\[
\bigoplus_{\nu \in \mathcal{U}} H^0(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) \to \bigoplus_{\xi} H^0(\xi, (f')^*T\mathbb{P}^r)
\]

is surjective, so that \( H^1(C'_{\nu,\eta}, (f')^*T\mathbb{P}^r) = 0 \). (The map takes the difference of the sections on the two branches of a node.) If \( C'_{\nu,\eta} \) has a component \( C'_{\nu_0,\eta} \) not contracted by \( f' \), there is at most one, by Lemma 4.15. On any other component \( C'_{\nu,\eta} \), we have \( (f')^*T\mathbb{P}^r \cong \mathcal{O}_{C'_{\nu,\eta}} \otimes T\mathbb{P}^r \), i.e. \( H^0(C'_{\nu,\eta}, \mathcal{O}_{C'_{\nu,\eta}} \otimes T\mathbb{P}^r) \cong T\mathbb{P}^r \). Fix an arbitrary section \( s \in H^0(C'_{\nu_0,\eta}, (f')^*T\mathbb{P}^r) \). Then “working outward” from \( C'_{\nu_0,\eta} \) shows that the map is surjective. The case where \( f' \) contracts \( C'_{\nu,\eta} \) is similar and simpler.
Next, we compute \( R^0(\pi \circ \rho)_*(C_{\Gamma, \eta}^r, (f')^*T\mathbb{P}^r) \). If \( C_{\Gamma, \eta}^r \) is contracted, \((f')^*T\mathbb{P}^r\) is trivial and we have
\[
R^0(\pi \circ \rho)_*(C_{\Gamma, \eta}^r, (f')^*T\mathbb{P}^r) \cong T\mathbb{P}^r \otimes \mathcal{O}_{\overline{\mathcal{M}_{\Gamma}^{\rig}}} \]
by properness of \( \pi \circ \rho \). Suppose \( C_{\Gamma, \eta}^r \) is not contracted. Consider the Stein factorization of \( f'|_{C_{\Gamma, \eta}^r} \) relative to \( \pi \circ \rho \):

\[
\begin{array}{ccc}
C_{\Gamma, \eta}^r & \xrightarrow{sf} & C_{\Gamma, \eta}^r' \\
\pi \circ \rho & \xrightarrow{f'} & \mathbb{P}^r
\end{array}
\]

If \((f : C \to \text{Sym}^d \mathbb{P}^r)\) is in the dense open substack \( \Psi^{-1}(\tilde{\Gamma}) \subseteq \overline{\mathcal{M}_{\Gamma}^{\rig}} \), then \( C_{\Gamma, \eta}^r \) is irreducible, hence so is \( C_{\Gamma, \eta}^r' \). This, with the fact that \( C_{\Gamma, \eta}^r \) is not contracted, implies that \( sf \) is birational. By the projection formula for coherent sheaves,
\[
(\pi \circ \rho)_*(f')^*T\mathbb{P}^r = (\pi \circ \rho)_* sf^*(f''')^*T\mathbb{P}^r
= (\pi \circ \rho)_* sf sf^*(f''')^*T\mathbb{P}^r
= (\pi \circ \rho)_*(sf^*(f''')^*T\mathbb{P}^r \otimes sf_*(\mathcal{O}_{C_{\Gamma, \eta}^r}))
= (\pi \circ \rho)_*(f''')^*T\mathbb{P}^r.
\]

After an étale base change on \( \overline{\mathcal{M}_{\Gamma}^{\rig}} \), the map \((f')^*\) trivializes the family \( C_{\Gamma, \eta}^r \). Thus \( R^0(\pi \circ \rho)_*(C_{\Gamma, \eta}^r, (f''')^*T\mathbb{P}^r) \) is a trivial vector bundle.

Calculation of the \( T \)-weights of this vector bundle is identical to Kontsevich’s calculation in Section 3.3.4 of [Kon95], which uses the Euler sequence on \( \mathbb{P}^r \). The weights are
\[
\frac{A}{\beta_{\eta}(e)} \alpha_{\text{mov}(v_1, e)} + \frac{B}{\beta_{\eta}(e)} \alpha_{\text{mov}(v_2, e)} - \alpha_i,
\]
where \( 0 \leq A, B \leq \beta_{\eta}(e) \), \( A + B = \beta_{\eta}(e) \), and \( i \in \{0, \ldots, r\} \). Note that this is zero exactly when \( A = 0 \) and \( i = i^{\text{mov}}(v_2, e) \), or \( B = 0 \) and \( i = i^{\text{mov}}(v_1, e) \). (These factors contribute to \( e_T(R^0(\pi \circ \rho)_*(C_{\Gamma, \eta}^r, (f''')^*T\mathbb{P}^r)_{\text{rig}}) \). The Euler class \( e_T(R^0(\pi \circ \rho)_*(C_{\Gamma, \eta}^r, (f''')^*T\mathbb{P}^r)_{\text{mov}}) \) for \( \nu \) noncontracted is thus

\[
(\prod_{\eta \in \text{Stat}(e), i \neq i(\eta)} \prod_{\eta \in \text{Mov}(e)} (\alpha_{i(\eta)} - \alpha_i)) \left( \prod_{\eta \in \text{Mov}(e)} \prod_{0 \leq i \leq r \atop (A, i) \neq (0, i^{\text{mov}}(v_2, e)) \atop (B, i) \neq (0, i^{\text{mov}}(v_1, e))} \left( \frac{A}{\beta_{\eta}(e)} \alpha_{\text{mov}(v_1, e)} + \frac{B}{\beta_{\eta}(e)} \alpha_{\text{mov}(v_2, e)} - \alpha_i \right) \right)
\]

\[(17)\]

**Summary.** We collect the arguments of this section in the following two statements.

**Proposition 5.3.** For any minimal decorated tree \( \tilde{\Gamma} \), \( \overline{\mathcal{M}_{\Gamma}} \) is smooth, and the virtual fundamental class is equal to the fundamental class.
Proposition 5.4. The equivariant Euler class $e_T(N_{\mathcal{M}_\Gamma}^{vir})$ of the virtual normal bundle to $\mathcal{M}_\Gamma$ is

$$
\left( \prod_{v \in V^2(\Gamma)} \frac{(-\psi_v \mathcal{M}_v - \psi_\nu \mathcal{M}_\nu)}{(\psi_v \mathcal{M}_v - \psi_\nu \mathcal{M}_\nu)} \right) \prod_{v \in V^1(\Gamma)} \frac{\psi_\nu \mathcal{M}_\nu}{\psi_v \mathcal{M}_v} \\
\cdot \prod_{\eta \in Stat(e)} \prod_{i \neq i(\eta)} (\alpha_i(\eta) - \alpha_\iota) \prod_{\eta \in Mov(e)} \prod_{0 \leq i \leq r} \left( \frac{A}{\beta_\eta(\iota_1, e)} + \frac{B}{\beta_\eta(\iota_2, e)} \right)
$$

Proof of Proposition 5.3. Recall from Theorem 2.2 that the virtual fundamental class of $\mathcal{M}_\Gamma$ is obtained from the fixed part of the perfect obstruction theory on $\mathcal{M}_0, (\text{Sym}^d \mathbb{P}^r, \beta)$. By (14), the fixed part of $\bigoplus_\xi R^0 \pi_* (\xi, f^* T \text{Sym}^d \mathbb{P}^r)$ is zero. Thus by (10), we have

$$
R^1 \pi_* (\mathcal{C}, f^* T \text{Sym}^d \mathbb{P}^r) \cong \bigoplus_{\xi} R^1 \pi_* (\mathcal{C}_\xi, f^* T \text{Sym}^d \mathbb{P}^r).
$$

But we showed, in (13) and (16), that $\bigoplus_\mu R^1 \pi_* (\mathcal{C}_\mu, f^* T \text{Sym}^d \mathbb{P}^r)$ has no fixed part. Thus $R^1 \pi_* (\mathcal{C}, f^* T \text{Sym}^d \mathbb{P}^r)$ has no fixed part. By Proposition 5.5 of [BF97], the Proposition follows. (Smoothness already followed easily from Theorem 4.18) \qed

Proof of Proposition 5.4. The first line is the contribution from Def($\mathcal{C}$) and Aut($\mathcal{C}$), from (8) and (9). The second line is the contribution of noncontracted components $\mathcal{C}_\mu$ to $R \pi_* (\mathcal{C}, f^* T \text{Sym}^d \mathbb{P}^r)$, from (17) and (16). The third line is the contribution of steady nodes to $R \pi_* (\mathcal{C}, f^* T \text{Sym}^d \mathbb{P}^r)$, from (14). (The numerator corrects for the fact that $F(\Gamma)$ overcounts the steady nodes.) The last line is the contribution of contracted components $\mathcal{C}_\mu$ to $R \pi_* (\mathcal{C}, f^* T \text{Sym}^d \mathbb{P}^r)$, by definition. \qed

6. Characterization of the Givental cone $\mathcal{L}_{\text{Sym}^d \mathbb{P}^r}$

In this section, we apply the results of Sections 4.2 and 5 to give a criterion (Theorem 6.4) that exactly determines whether a given power series lies on the Givental cone $\mathcal{L}_{\text{Sym}^d \mathbb{P}^r}$.

Definition 6.1. Fix $(\mu, \sigma) \in (I \text{Sym}^d \mathbb{P}^r)^T$. Let $\Upsilon(\mu, \sigma) \subseteq \text{Trees}_{0,1} (\text{Sym}^d \mathbb{P}^r, \beta)$ be the set of 1-edge decorated trees $\tilde{\kappa} = v_1 \arrow{e} v_2$, with marking set $\{b_{n+1}\}$, with Mark$(b_{n+1}) = v_1$, such that $\mu = \text{Eval}(v_1)$ and $\sigma = \text{Mon}(v_1, e)$.

Notation 6.2. For $\tilde{\kappa} \in \Upsilon(\mu, \sigma)$ as in Definition 6.1, we write:

- $q(\tilde{\kappa}) := q(e)$,
- $\text{Mov}(\tilde{\kappa}) := \text{Mov}(e)$,
- $\text{mov}(\tilde{\kappa}) := \text{mov}(e)$,
Let \( \tilde{\kappa} \in \Upsilon(\mu, \sigma) \) and let \( a \in \mathbb{Z}_{>0} \) We define the recursion coefficient as
\[
RC(\tilde{\kappa}, a) = \frac{(-1)^{\mu_{\text{mov}}(\tilde{\kappa}) - a}}{q(\tilde{\kappa})^{\mu_{\text{mov}}(\tilde{\kappa})}} \left( \frac{\sigma_{\mu_{\text{mov}}(\tilde{\kappa})}}{\text{Mov}(\tilde{\kappa})} \right) \left( \text{mov}(\tilde{\kappa}) - 1 \right).
\]
\[
\prod_{\eta \in \text{Mov}(\tilde{\kappa})} \prod_{1 \leq B \leq \beta_{\sigma}(\eta)} \frac{1}{(B, \tilde{\kappa}) \neq (\beta_{\sigma}(\eta), \mu_{\text{mov}}(\tilde{\kappa}))} \left( \frac{\beta_{\sigma}(\eta) - B}{\beta_{\sigma}(\eta)} \alpha_{\text{mov}}(\tilde{\kappa}) + \frac{B}{\beta_{\sigma}(\eta)} \alpha_{\text{mov}}(\tilde{\kappa}) - \alpha_{\eta} \right),
\]
where \( \left( \frac{\sigma_{\mu_{\text{mov}}(\tilde{\kappa})}}{\text{Mov}(\tilde{\kappa})} \right) \) is the number of ways of choosing \( \text{Mov}(\tilde{\kappa}) \) as a subpartition of \( \sigma_{\mu_{\text{mov}}(\tilde{\kappa})} \) with specified parts.

The following theorem and its proof are adapted from Theorem 41 of [CCIT15], which in turn is adapted from Theorem 2 of [Bro14].

**Theorem 6.4.** Let \( f \) be a element of \( \mathcal{H}[[x]] \) such that \( f|_{Q=x=0} = -1z \), where \( 1 \) denotes the fundamental class of \( \text{Sym}^d \mathbb{P}^r \subseteq \text{I Sym}^d \mathbb{P}^r \). Then \( f \) is a \( \Lambda^T_{\text{mov}}[[x]] \)-valued point of \( L_{\text{Sym}^d \mathbb{P}^r} \) if and only if for each \( T \)-fixed point \( (\mu, \sigma) \in \text{I Sym}^d \mathbb{P}^r \), the following three conditions hold:

1. The restriction \( f_{(\mu, \sigma)} \) along \( \iota_{(\mu, \sigma)} : (\mu, \sigma) \mapsto \text{I Sym}^d \mathbb{P}^r \) is a power series in \( Q \) and \( x \), such that each coefficient of this power series is an element of \( H^T_{\text{I loc}}(\bullet)(z) \). Each coefficient is regular in \( z \) except for possible poles at \( z = 0, z = \infty \), and
   \[
   z \in \{ \overline{\varphi}(\tilde{\kappa}) : \tilde{\kappa} \in \Upsilon(\mu, \sigma) \}.
   \]

2. The Laurent coefficients of \( f_{(\mu, \sigma)} \) at the poles (other than \( z = 0 \) and \( z = \infty \)) satisfy the recursion relation:
   \[
   \text{Laur}(f_{(\mu, \sigma)}, (\overline{w} - z)^{-a}) = \sum_{\tilde{\kappa} \in \Upsilon(\mu, \sigma)} Q^\beta(\overline{w}) \text{RC}(\tilde{\kappa}, a) \text{Laur}(f_{(\mu' \tilde{\kappa}, \sigma')(\tilde{\kappa})}, (\overline{w} - z)^{\text{mov}(\tilde{\kappa}) - a})
   \]
   for \( a > 0 \), and

3. The restriction \( f_{\mu} \) along \( \iota_{\mu} : I \mu \mapsto \text{I Sym}^d \mathbb{P}^r \) is a \( \Lambda^T_{\text{mov}}[[x]] \)-valued point of \( L^T_{\sigma} \).

**Remark 6.5.** In \( \Lambda^T_{\text{mov}} \) is the equivariant Novikov ring associated to \( \text{Sym}^d \mathbb{P}^r \), not \( \mu \). In other words, \( \Lambda^T_{\text{mov}}[[x]] = H_{\text{CR,T loc}}(\mu)[[Q, x]] \).

**Remark 6.6.** The major difference between Theorem 6.4 and the corresponding theorems in [CCIT15] and [Bro14] is that condition (II) gives a recursive relation for all negative-exponent Laurent coefficients at \( z = w(\tilde{\kappa}) \), in terms of positive-exponent ones. In [CCIT15] and [Bro14], only stacks with isolated 1-dimensional \( T \)-orbits are considered. Thus the poles at \( z = w(\tilde{\kappa}) \) are simple, and a recursive relation is given for their residues.
Proof. Let \( f \) be a \( \Lambda^T_{\text{nov}}[[x]] \)-valued point of \( \mathcal{L}_{\text{Sym}^d \mathbb{P}^r} \). By definition, we can write

\[
\begin{align*}
f &= -1z + t(z) + \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{Q^\beta}{n!} \sum_{\phi} \gamma_{\phi} \left\langle \frac{t(z)}{z-\overline{w}}, \frac{1}{\overline{w}} \right\rangle_{\text{Sym}^d \mathbb{P}^r, T, 0,n+1,\beta} \\
&= -1z + t(z) + \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{Q^\beta n!}{n!} (ev_{n+1})_* \left( \prod_{j=1}^{n} ev_j^* t(\overline{w}) \cup \frac{1}{z-\overline{w}} \right) \cap \overline{\mathcal{M}}_{0,n+1}(\text{Sym}^d \mathbb{P}^r, \beta)^{\text{vir}} \right).
\end{align*}
\]

for \( t(z) \in \mathcal{H}_+[x] \) with \( t|_{Q=0} = 0 \). The restriction \( f_{(\mu,\sigma)} \) is then

\[
-\delta_{\sigma=(1,\ldots,1)} z + t_{(\mu,\sigma)} (z) + \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{Q^\beta n!}{n!} t_{(\mu,\sigma)}^* (ev_{n+1})_* \left( \prod_{j=1}^{n} ev_j^* t(\overline{w}) \cup \frac{1}{z-\overline{w}} \right) \cap \overline{\mathcal{M}}_{0,n+1}(\text{Sym}^d \mathbb{P}^r, \beta)^{\text{vir}} \right).
\]

Using the projection formula, we write

\[
\begin{align*}
l_{(\mu,\sigma)}^* &\left( (ev_{n+1})_* \left( \prod_{j=1}^{n} ev_j^* t(\overline{w}) \cup \frac{1}{z-\overline{w}} \right) \cap \overline{\mathcal{M}}_{0,n+1}(\text{Sym}^d \mathbb{P}^r, \beta)^{\text{vir}} \right) \\
&= |C_\mu(\sigma)| \left\langle (t_{(\mu,\sigma)}^*), (\mu,\sigma) \right\rangle_{\text{Sym}^d \mathbb{P}^r, T, 0,n+1,\beta}.
\end{align*}
\]

The first equality uses the identification of \( \mathcal{I}_{\text{Sym}^d \mathbb{P}^r, \text{ct}(\mu,\sigma)} \) with the identity map \( \text{Spec} \mathbb{C} \to \text{Spec} \mathbb{C} \) on coarse moduli spaces, and the factor \( |C_\mu(\sigma)| \) corrects for the isotropy at \( (\mu,\sigma) \in I \text{Sym}^d \mathbb{P}^r \). (Recall that \( C_\mu(\sigma) \) denotes the centralizer of any element of \( \sigma \) in \( G_\mu \).) In summary,

\[
f_{(\mu,\sigma)} = -\delta_{\sigma=(1,\ldots,1)} z + t_{(\mu,\sigma)} (z) + \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} |C_\mu(\sigma)| \frac{Q^\beta}{n!} \left\langle t(\overline{w}), \ldots, t(\overline{w}), \left(\frac{(\mu,\sigma)}{z-\overline{w}}\right) \right\rangle_{\text{Sym}^d \mathbb{P}^r, T, 0,n+1,\beta},
\]

where \( t_{(\mu,\sigma)} (z) := l_{(\mu,\sigma)}^* (t_{(\mu,\sigma)} (z)) \). Now we calculate \( \left\langle (\mu,\sigma) \right\rangle \) by virtual torus localization (see Theorem 2.2). Namely, we may write

\[
|C_\mu(\sigma)| \left\langle t(\overline{w}), \ldots, t(\overline{w}), \left(\frac{(\mu,\sigma)}{z-\overline{w}}\right) \right\rangle_{\text{Sym}^d \mathbb{P}^r, T, 0,n+1,\beta} = \sum_{\tilde{\Gamma} \in \text{Trees}_{0,n+1}(\text{Sym}^d \mathbb{P}^r, \beta)} \text{Constr}_{(\mu,\sigma)}(\tilde{\Gamma}).
\]

We can partition \( \text{Trees}_{0,n+1}(\text{Sym}^d \mathbb{P}^r, \beta) \) into three subsets:

(i) \( \tilde{\Gamma} \) such that \( (\text{VEval}(\text{Mark}(b_{n+1})), \text{Mon}(b_{n+1})) \neq (\mu,\sigma) \),
(ii) \( \tilde{\Gamma} \) such that \( (\text{VEval}(\text{Mark}(b_{n+1})), \text{Mon}(b_{n+1})) = (\mu,\sigma) \) and \( \overline{\mathcal{M}}(b_{n+1}) \in V^1(\Gamma) \), and
(iii) \( \tilde{\Gamma} \) such that \( (\text{VEval}(\text{Mark}(b_{n+1})), \text{Mon}(b_{n+1})) = (\mu,\sigma) \) and \( \text{Mark}(b_{n+1}) \in V^\mathcal{S}(\Gamma) \).
In some literature, e.g. [CFK14], decorated trees of type (i) are called recursion type and those of type (iii) are called initial type. Let \( v_1 := \text{Mark}(b_{n+1}) \) be the vertex containing the point \( b_{n+1} \). (We will see below, however, that in our setup both types are used recursively.)

For a tree \( \hat{\Gamma} \) of type (i) the restriction \( \text{ev}^*_{n+1}([[(\mu, \sigma)]] \) vanishes, hence \( \text{Contr}_{(\mu, \sigma)}(\hat{\Gamma}) = 0 \). For this reason, we may simplify our notation, and write \( \text{Contr}(\hat{\Gamma}) := \text{Contr}_{(\mu, \sigma)}(\hat{\Gamma}) \), where \( \mu = V\text{Eval}(\text{Mark}(b_{n+1})) \) and \( \sigma = \text{Mon}(\text{Mark}(b_{n+1})) \).

If \( \hat{\Gamma} \) is a tree of type (iii) then by Theorem 4.18 and Corollary 4.20 \( \psi_{n+1} \) is pulled back from \( \mathcal{M}_{0,\text{Mon}(v_1)}(BG_\mu, 0) \), where \( G_\mu \) is the isotropy group of \( \mu \). Since this stack parametrizes maps that factor through the fixed point \( \mu \), the action of \( T \) is trivial, hence

\[
H^*_{T,\text{loc}}(\mathcal{M}_{0,\text{Mon}(v_1)}(BG_\mu, 0)) \cong H^*(\mathcal{M}_{0,\text{Mon}(v_1)}(BG_\mu, 0)) \otimes H^*_{T,\text{loc}}(\bullet).
\]

In particular, \( \overline{\psi}_{n+1} \) is nilpotent. It follows that \( \text{Contr}(\hat{\Gamma}) \) is a polynomial in \( z^{-1} \), hence has a pole only at \( z = 0 \).

Finally, let \( \hat{\Gamma} \) be a tree of type (ii). By (1), we have

\[
\text{Contr}(\hat{\Gamma}) = |C_\mu(\sigma)| \int_{[\mathcal{M}_\Gamma]} \frac{1}{e_T(N_{\text{vir}}^{\Gamma})} \left( \prod_{j=1}^n \text{ev}^*_j t(\psi) \cup \frac{\text{ev}^*_{n+1}([[(\mu, \sigma)]])}{-z - \overline{\psi}_{n+1}} \right),
\]

where \( \iota_T \) is the inclusion \( \mathcal{M}_\Gamma \hookrightarrow \mathcal{M}_{0,n+1}(\text{Sym}^d \mathbb{P}^r, \beta) \), and \( [\mathcal{M}_\Gamma]' \) denotes the fundamental class, weighted by factors from Remark 4.19. Note that \( \iota_T^* \text{ev}^*_{n+1} \) factors through \( (\mu, \sigma) \), hence \( \iota_T^* \text{ev}^*_{n+1}([[(\mu, \sigma)]]) \) is the weight \( e_T(\mu, \sigma) \text{Sym}^d \mathbb{P}^r) \).

Then \( \hat{\Gamma} \) has a decorated subtree \( \hat{\kappa} \in \Upsilon(\mu, \sigma) \), obtained by removing all edges except for \( e := e_{v_1} \) (and necessary vertices), and all marked points except \( b_{n+1} \). Let \( \hat{\Gamma} \setminus \hat{\kappa} \) denote the tree obtained by pruning \( \hat{\kappa} \). That is, \( \hat{\Gamma} \setminus \hat{\kappa} \in \text{Trees}_{0,n+1}(\text{Sym}^d \mathbb{P}^r, \beta - \beta(\hat{\kappa})) \) is defined by \( V(\Gamma \setminus \kappa) = V(\Gamma) \setminus \{v_1\} \), \( E(\Gamma \setminus \kappa) = E(\Gamma) \setminus e \), and decorations Mark, VEval, \( q \), and Mon are unchanged, except \( \text{Mark}(b_{n+1}) := v_2 \), where \( v_2 \) is the common vertex of \( \hat{\kappa} \) and \( \hat{\Gamma} \setminus \hat{\kappa} \). Observe that an automorphism of \( \Gamma \) fixes \( b_{n+1} \), and therefore fixes \( e \), so we have \( \text{Aut}(\hat{\Gamma}) = \text{Aut}(\hat{\Gamma} \setminus \hat{\kappa}) \) and may write

\[
\mathcal{M}_\Gamma \cong \mathcal{M}_e \times \mathcal{M}_{\hat{\Gamma} \setminus \hat{\kappa}}
\]

by Theorem 4.18 We factor the \( T \)-equivariant map \( \mathcal{M}_\Gamma \to \text{Spec} \mathbb{C} \) through the second projection, i.e. we integrate over \( \mathcal{M}_e \), again using Remark 4.19

\[
\text{Contr}(\hat{\Gamma}) = |C_\mu(\sigma)| \int_{[\mathcal{M}_e \times \mathcal{M}_{\hat{\Gamma} \setminus \hat{\kappa}}]'} \left( \int_{[\mathcal{M}_e]} \frac{e_T(T(\mu, \sigma)I \text{Sym}^d \mathbb{P}^r)}{e_T(N_{\text{vir}}^{\Gamma})} \iota_T^* \left( \prod_{j=1}^n \text{ev}^*_j t(\psi) \cup \frac{1}{-z - \overline{\psi}_{n+1}} \right) \right).
\]

From Proposition 5.4 we may write

\[
\frac{e_T(T(\mu, \sigma)I \text{Sym}^d \mathbb{P}^r)}{e_T(N_{\text{vir}}^{\Gamma})} = \frac{1}{W} \frac{e_T(T(\mu(\hat{\kappa}), \sigma(\hat{\kappa}))I \text{Sym}^d \mathbb{P}^r)}{(-\overline{\psi}_e \mathcal{M}_{v_2} - \overline{\psi}_{v_2} \mathcal{M}_e)} e(N_{\text{vir}}^{\Gamma \setminus \hat{\kappa}}),
\]
where

\[
W := \frac{\prod_{\eta \in \Stat(\hat{\kappa})} \prod_{i \neq i(\eta)} (\alpha_i(\eta) - \alpha_i)}{e_T(T_{(\mu, a)} I \Sym^d \mathbb{P}^r)} \prod_{\eta \in \Mov(\hat{\kappa})} \prod_{0 \leq i \leq r (A, i) \neq (0, i_{\mov}(v_2, e))} A + B = \beta_\eta(\hat{\kappa}) \prod_{1 \leq B \leq \beta_\eta(\hat{\kappa})} \prod_{0 \leq i \leq r (B, i) \neq (0, i_{\mov}(v_2, e))} \left( \frac{A}{\beta_\eta(\hat{\kappa})} \alpha_{i_{\mov}(v_1, e)} + \frac{B}{\beta_\eta(\hat{\kappa})} \alpha_{i_{\mov}(v_2, e)} - \alpha_i \right) = \prod_{\eta \in \Mov(\hat{\kappa})} \prod_{0 \leq i \leq r (B, i) \neq (0, i_{\mov}(v_2, e))} \left( \frac{\beta_\eta(\hat{\kappa}) - B}{\beta_\eta(\hat{\kappa})} \alpha_{i_{\mov}(v_1, e)} + \frac{B}{\beta_\eta(\hat{\kappa})} \alpha_{i_{\mov}(v_2, e)} - \alpha_i \right) \in H^*_T,loc(Spec \mathbb{C})
\]

Note that the cancellation in the last step removes the terms with $B \neq 0$ in the product, and that $1/W$ is the product appearing $\mathbf{RC}(\hat{\kappa}, a)$.

To avoid confusion, we write $\overline{\psi}_{n+1}$ (resp. $\overline{\psi}_{n1}$) for the $\overline{\psi}$-class at the $(n + 1)$st marked point on $\overline{\mathcal{M}}_{\Gamma}$ (resp. $\overline{\mathcal{M}}_{\Gamma \setminus \hat{\kappa}}$), recalling that on $\Gamma \setminus \hat{\kappa}$ we defined Mark($b_{n+1}$) = $v_2$. We also have $t_{\Gamma}^k \overline{\psi}_{n+1} = \overline{\psi}_{v_1}$. The $T$-weight on $\overline{\psi}_{v_1}$ is $-\overline{w}(\hat{\kappa})$ (see Notation 6.2), so we have

\[
\overline{\psi}_{v_1} = \overline{\psi}_{v_1}^{\text{ne}} - \overline{w}(\hat{\kappa}) \in H^*_T(\overline{\mathcal{M}}_{\Gamma}) \cong H^*(\overline{\mathcal{M}}_{\Gamma}) \otimes H^*_T(Spec \mathbb{C}),
\]

where $\overline{\psi}_{v_1}^{\text{ne}}$ denotes the nonequivariant $\overline{\psi}$-class. Similarly $\overline{\psi}_{v_2} = \overline{\psi}_{v_2}^{\text{ne}} + \overline{w}(\hat{\kappa})$. Then since $t_{\Gamma}^k \psi_{v_1}$ is pulled back from $\overline{\mathcal{M}}_{\Gamma \setminus \hat{\kappa}}$,

\[
\text{Contr}(\Gamma) = \frac{|C_{\mu(\hat{\kappa})}| |C_{\mu'(\hat{\kappa})}(\sigma'(\hat{\kappa}))| e_T(T_{(\mu', \sigma')}(\hat{\kappa})) I \Sym^d \mathbb{P}^r)}{W} W \left( \overline{\psi}_{v_1} - \overline{w}(\hat{\kappa}) \right) \int_{\overline{\mathcal{M}}_{\Gamma \setminus \hat{\kappa}}} \left( t_{\Gamma}^k \left( \prod_{j=1}^n \ev_j^* \psi_{v_1} \right) \right) \int_{\overline{\mathcal{M}}_{\Gamma \setminus \hat{\kappa}}} \frac{1}{(-\psi_{n+1} - \psi_{v_2} - \overline{w}(\hat{\kappa})) \left(-z - \overline{w}_{\psi_{v_1}}^{\text{ne}} + \overline{w}(\hat{\kappa})\right)}.
\]

The factor $|C_{\mu'(\hat{\kappa})}(\sigma'(\hat{\kappa}))|/r(\hat{\kappa})$ comes from Remark 4.19. We compute the last integral using the fact that $w(\hat{\kappa})$ is invertible, and Lemma 2.8 which says we may integrate on $\overline{\mathcal{M}}_{k+2}$ instead of $\overline{\mathcal{M}}_e$. We use $r(\hat{\kappa})(-\psi_{n+1} - \psi_{v_2} - w(\hat{\kappa})) = \psi_{n+1} - \psi_{v_2}^{\text{ne}} - \overline{w}(\hat{\kappa})$. It is well-known (see e.g. [Koc01], Lemma 1.5.1) that

\[
\int_{\overline{\mathcal{M}}_{0,k}} \psi_{n+1}^m \psi_{v_2}^{k-3-m} = \binom{k-3}{m}.
\]
By Lemma 2.8, this identity holds on $\mathcal{M}_{0,k|\infty}$ also. Thus:

$$\int_{\mathcal{M}_{k}} \frac{1}{(-\psi_{n+1} - \psi_{v_2} - \overline{w(k)})} \left(- z - \psi_{v_1} + \overline{w(k)}\right)$$

$$= \frac{1}{|S_e|} \prod_{\eta \in \text{Mov}(\overline{\kappa})} \beta_{\eta}(\overline{\kappa}) \int_{\mathcal{M}_{v_1|\text{mov}(\overline{\kappa})}} \left(\sum_{m_1=0}^{\infty} \frac{(-\psi_{n+1} - \overline{w(k)})^{m_1}}{(\psi_{v_2})^{m_1}}\right) \left(\sum_{m_2=0}^{\infty} \frac{(-\psi_{v_1})^{m_2}}{(-z + \overline{w(k)})^{m_2+1}}\right)$$

$$= \frac{1}{|S_e|} \prod_{\eta \in \text{Mov}(\overline{\kappa})} \beta_{\eta}(\overline{\kappa}) \sum_{m_1+m_2=\text{mov}(\overline{\kappa})-1}^{\infty} \frac{(-\psi_{n+1} - \overline{w(k)})^{m_1}}{(\psi_{v_2})^{m_1}} \frac{(-z + \overline{w(k)})^{m_2+1}}{(-\psi_{v_1})^{m_2}}$$

(22)

$$= \frac{1}{|S_e|} \prod_{\eta \in \text{Mov}(\overline{\kappa})} \beta_{\eta}(\overline{\kappa}) \left(-z - \psi_{v_1} + \overline{w(k)}\right)^{\text{mov}(\overline{\kappa})-1} \left(-\psi_{n+1} - \overline{w(k)}\right)^{\text{mov}(\overline{\kappa})-1}$$

The last inequality is gotten in the backwards direction by writing the numerator as $(-z + \overline{w(k)}) + (-\psi_{n+1} - \overline{w(k)})^{\text{mov}(\overline{\kappa})-1}$ and expanding. We have

$$\text{Contr}(\bar{\Gamma}) = \frac{|C_{\mu}(\sigma)\mid C_{\mu'}(\sigma')(\overline{\kappa})|}{|S_e|} \prod_{\eta \in \text{Mov}(\overline{\kappa})} \beta_{\eta}(\overline{\kappa}) \cdot W(-z + \overline{w(k)})^{\text{mov}(\overline{\kappa})}$$

$$\cdot \left(\int_{\overline{\mathcal{M}_{\Gamma\backslash\kappa}}} \left(\prod_{j=1}^{\infty} \frac{1}{\text{ev}_{j}^{*}} \frac{t(\overline{\psi})}{e^{T}N_{\Gamma\backslash\kappa}^{|\overline{\psi}|}}\right) \frac{(-z + \psi_{v_1})^{\text{mov}(\overline{\kappa})-1}}{(-\psi_{n+1} - \overline{w(k)})^{\text{mov}(\overline{\kappa})}} \right).$$

(23)

For fixed $\beta_0$, and $n_0$, from (20), the coefficient of $Q^{\beta_0}x^{n_0}$ in $f_{(\mu,\sigma)}$ only has contributions from $\bar{\Gamma} \in \text{Trees}_{0,n}(\text{Sym}^{d}P_{r},\beta)$ for $\beta + n \leq \beta_0 + n_0$. This is because $t(z) \in (Q, x)$, so if $H[[x]]$ is graded by giving $Q$ and $x$ degree 1, then the $(n, \beta)$ term in (19) has degree at least $n + \beta$. In particular, $\bigcup_{\beta+n \leq \beta_0+n_0} \text{Trees}_{0,n}(\text{Sym}^{d}P_{r},\beta)$ is a finite set. Thus (20) and (23) realize the contribution to such a coefficient from trees of type (ii) as a finite sum of rational functions with poles at the weights $\kappa$. Together with the analysis above for types (i) and (iii) this proves that $f_{(\mu,\sigma)}$ satisfies condition (I) of the Theorem.

We consider the Laurent coefficient $\text{Laur}(\text{Contr}(\bar{\Gamma})), (w-z)^{-a})$. By (23), $\text{Laur}(\text{Contr}(\bar{\Gamma})), (w-z)^{-a})$ is zero if $w \neq \overline{w}(\overline{\kappa})$, or if $\text{mov}(\overline{\kappa}) < a$. Otherwise,

$$\text{Laur}(\text{Contr}(\bar{\Gamma})), (w-z)^{-a})$$

$$= \frac{1}{\text{mov}(\overline{\kappa}) - a} \left(\frac{d^{\text{mov}(\overline{\kappa})-a}}{(d(w(k) - z)^{\text{mov}(\overline{\kappa})-a}(w(k) - z)^{\text{mov}(\overline{\kappa})})} \right) \text{Contr}(\bar{\Gamma}) \right|_{z \to \overline{w}(\overline{\kappa})}$$

$$= \left(-1\right)^{\text{mov}(\overline{\kappa})-a} \left(\frac{C_{\mu}(\sigma)\mid C_{\mu'}(\sigma')(\overline{\kappa})|}{|S_e|} \prod_{\eta \in \text{Mov}(\overline{\kappa})} \beta_{\eta}(\overline{\kappa})\right)\left(\frac{\text{mov}(\overline{\kappa})}{a-1}\right)$$

$$\int_{\mathcal{M}_{\Gamma\backslash\kappa}} \left(\prod_{j=0}^{\infty} \text{ev}_{j}^{*} \frac{t(\overline{\psi})}{e^{T}N_{\Gamma\backslash\kappa}^{|\overline{\psi}|}}\right) \frac{(-z + \psi_{v_1})^{\text{mov}(\overline{\kappa})-1}}{(-\psi_{n+1} - \overline{w(k)})^{\text{mov}(\overline{\kappa})-a+1}}.$$

Now, summing over all $\bar{\Gamma}$ of type (ii) with associated subtree $\kappa$ yields

$$\left(-1\right)^{\text{mov}(\overline{\kappa})-a} \left(\frac{C_{\mu}(\sigma)\mid C_{\mu'}(\sigma')(\overline{\kappa})|}{|S_e|} \prod_{\eta \in \text{Mov}(\overline{\kappa})} \beta_{\eta}(\overline{\kappa})\right)\left(\frac{\text{mov}(\overline{\kappa})}{a-1}\right)$$

$$\left(\left.\text{Laur}(\text{Contr}(\bar{\Gamma})), (w-z)^{-a})\right|_{\text{Contr}(\bar{\Gamma})}\right)$$

$$= \left(-1\right)^{\text{mov}(\overline{\kappa})-a} \left(\frac{C_{\mu}(\sigma)\mid C_{\mu'}(\sigma')(\overline{\kappa})|}{|S_e|} \prod_{\eta \in \text{Mov}(\overline{\kappa})} \beta_{\eta}(\overline{\kappa})\right)\left(\frac{\text{mov}(\overline{\kappa})}{a-1}\right)$$

$$\left(\left.\text{Laur}(\text{Contr}(\bar{\Gamma})), (w-z)^{-a})\right|_{\text{Contr}(\bar{\Gamma})}\right).$$

(24)
On the other hand, the coefficient \( \text{Laur}(f_{(\mu',(\kappa'),\sigma'(\kappa'))}, (\bar{w}(\kappa) - z)^{\text{mov}(\kappa) - a}) \) is

\[
\sum_{\beta \geq 0, n \geq 0} \frac{|C_\mu(\sigma)|}{n!} Q^\beta \left\langle \frac{t(\bar{w}), \ldots, t(\bar{w})}{(-\psi_{n+1} - \bar{w}(\kappa)^{\text{mov}(\kappa) - a + 1})} \right\rangle_{\text{Sym}^d \mathbb{P}^r, T},_{0, n, + \beta}
\]

We compute \( |C_\mu(\sigma)| / |S_\sigma| \prod_{\eta \in \sigma} \eta \) explicitly:

\[
|C_\mu(\sigma)| = |S_\sigma| \prod_{\eta \in \sigma} \eta
\]

\[
\frac{|C_\mu(\sigma)|}{|S_\epsilon| \prod_{\eta \in \text{Mov}(\kappa)} \eta} = \frac{|S_\sigma| \prod_{\eta \in \text{Mov}(\kappa)} \eta}{|S_{\text{Stat}(\kappa)}| \prod_{\eta \in \text{Mov}(\kappa)} \eta} = \frac{1}{\text{Stat}(\kappa)^{\text{mov}(\kappa)}} \left( \frac{\sigma}{\text{Mov}(\kappa)} \right)
\]

With (24) and (25), this proves (II). Note that the contribution from all graphs of type (iii) (and the term \( t_{(\mu,\sigma)}(z) \)) is

\[
\tau_{(\mu,\sigma)}(z) := t_{(\mu,\sigma)}(z) + \sum_{\tilde{e} \in E(\tilde{\Gamma}, v), a \leq \text{mov}(\kappa)} Q^\beta \frac{\text{RC}(\kappa,a)}{(\bar{w}(\kappa) - z)^a} \text{Laur}(f_{(\mu',(\kappa'),\sigma'(\kappa'))}, (\bar{w}(\kappa) - z)^{\text{mov}(\kappa) - a}).
\]

The proof of condition (III) is identical to that of condition (C) in [CCIT15], and we reproduce the argument here for convenience.

Consider a decorated tree \( \tilde{\Gamma} \) of type (iii). We write \( v := \text{Mark}(b_{n+1}) \in V^S(\Gamma) \). The marked points of \( \tilde{\mathcal{M}}_v \) correspond to (1) elements of \( \text{Mark}^{-1}(v) \), and (2) edges \( e \in E(\Gamma, v) \). To \( e \) is associated a maximal subtree \( \Gamma_e \) containing \( v \), with \( E(\Gamma_e, v) = e \). We decorate \( \Gamma_e \) so that \( \text{Mark}^{-1}(v) = b \), and the rest of the decorations inherited from \( \tilde{\Gamma} \). We will then write \( \text{Contr}(\tilde{\Gamma}) \) in terms of \( \text{Contr}(\tilde{\Gamma}_e) \) for \( e \in E(\Gamma, v) \), and integrals over the vertex moduli space \( \tilde{\mathcal{M}}_v \).

We apply (21) again. After an étale base change \( \tilde{\mathcal{M}}_{\tilde{\Gamma}} \rightarrow \tilde{\mathcal{M}}_v \), we may label the subtrees \( \tilde{\Gamma}_e \).

(Write \( M \) for the degree of this base change.) We then write \( \tilde{\mathcal{M}}_{\tilde{\Gamma}} \cong \tilde{\mathcal{M}}_v \times \prod_{e \in E(\Gamma, v)} \tilde{\mathcal{M}}_{\Gamma_e} \). Now we again apply Proposition 5.4 to see that

\[
\frac{1}{e_T(N_{\text{vir}})} = e_T^{-1} (R \pi_*(C_v, f^*T \text{Sym}^d \mathbb{P}^r)) \prod_{e \in E(\Gamma, v)} e_T(T_{(\mu, \text{Mon}(v,e))} I \text{Sym}^d \mathbb{P}^r) \left( -\psi_e - \psi_e \right) e_T(N_{\text{vir}})_{\tilde{\Gamma}_e}
\]

Observe that \( e_T(T_{(\mu, \text{Mon}(v,e))} I \text{Sym}^d \mathbb{P}^r) \left( -\psi_e - \psi_e \right) \) is the insertion at \( b \) in \( \text{Contr}(\tilde{\Gamma}_e) \). Thus

\[
\text{Contr}(\tilde{\Gamma}) = \frac{1}{M} \int_{\tilde{\mathcal{M}}_v} \left( \prod_{e \in E(\Gamma, v)} |C_\mu(\sigma)| Q^\beta | \text{Contr}(\tilde{\Gamma}_e) \right)_{\bar{w}(\kappa) \mapsto \bar{w}_e} \cup \left( \prod_{b \in \text{Mark}^{-1}(v)} t_{(b)} \right)
\]

\[
\cup e_T(T_{(\mu, \sigma)} I \text{Sym}^d \mathbb{P}^r) \left( -\bar{w} - \bar{w}_{n+1} \right) e_T^{-1} (R \pi_*(C_v, f^*T \text{Sym}^d \mathbb{P}^r)).
\]

This is almost a twisted Gromov-Witten invariant of \( \text{VEval}(v) \), but not quite, since there are restrictions on the monodromies at the marked points. Summing over \( \tilde{\Gamma}_e \) for a single \( e \), with
Adding in the contributions from type (ii) graphs, summing (19) over \( \sigma \), where

\[ \tau = T \]

conditions of Theorem 6.4. That is, we show that it is a \( \Lambda \) point determined by \( t \) in fact

\[ f = d \]

terms of \( H \) these are determined since they are of the form:

\[ \beta \]

\[ x \]

show that it determines \( f \) determined by terms of \( \mu, \sigma \)

\[ \Pi \]

In this section we introduce the function \( I \) and \( III \) we may uniquely write

\[ f = \mu \]

where \( \tau (\mu, \sigma) (z) \) is the expression in (26), for some \( t_{(\mu, \sigma)} (z) \in t_{(\mu)} (H_+) [[x]] \). We claim that the set \( \{ t_{(\mu, \sigma)} (z) \} \) for all fixed points \( (\mu, \sigma) \) determines \( f \). By the localization isomorphism, if suffices to show that it determines \( f_{(\mu, \sigma)} \) for all \( (\mu, \sigma) \). We induct on the degree \( \beta + k \), where \( k \) is the exponent of \( x \). The base case \( \beta = k = 0 \) is taken care of by the assumption \( f_{(Q=x=0)} = -1z \). Assume the coefficients of \( f_{(\mu, \sigma)} \) up to degree \( \beta + k \) are determined by \( \{ t_{(\mu, \sigma)} \} \). Consider the coefficients of degree \( \beta + k + 1 \). Some of these appear in \( t(z) \), but these are given. Some of them appear in \( \tau (\mu, \sigma)(z) \), but these are determined since they are of the form: \( Q^{\beta(z)} \) multiplied by a factor determined by the inductive hypothesis. The sum of all of these terms is in \( H^{\ast}_{\ast, \ast} [[Q, x]] [[z]] \).

Finally, some of them appear in \( O(z^{-1}) \). However, condition (III) and (3) show that these are determined by terms of \( -1z + \tau (\mu, \sigma)(z) \) of degree at most \( \beta + k + 1 \). Since all such terms are determined by \( t_{(\mu, \sigma)} \) and induction, the degree \( \beta + k + 1 \) coefficients of \( f_{(\mu, \sigma)} \) are determined. Thus in fact \( f \) is determined by \( \{ t_{(\mu, \sigma)} (z) \} \).

Again by the localization isomorphism, the set \( \{ t_{(\mu, \sigma)} (z) \} \) corresponds uniquely to an element \( t(z) \in H_+[[x]] \) that restricts to each \( t_{(\mu, \sigma)} (z) \). This in turn corresponds uniquely to a \( \Lambda^{\ast}_{\ast, \ast, \ast} [[t, x]] \) valued point \( f_{\ast, \ast, \ast} \) of \( L_\ast \). By the uniqueness argument above we have \( f = f_{\ast, \ast, \ast} \).

\( f \)

Remark 6.7. No modifications are required to replace \( x \) with a tuple \( (x_1, \ldots, x_m) \).

7. The I-Function and Mirror Theorem

In this section we introduce the function \( I_{\ast, \ast, \ast} (Q, t, x, -z) \), and show that it satisfies the conditions of Theorem 6.4. That is, we show that it is a \( \Lambda^{\ast}_{\ast, \ast, \ast} [[t, x]] \) valued point of \( L_{\ast, \ast, \ast, \ast} \), where \( x = \{ x_\Pi \}_{\Pi \in \text{Part}(\ell)} \) are formal variables.
\section*{Definition 7.1.} The (extended) $I$-function is

\[ I_{\text{Sym}^d \mathbb{P}^r}(Q, t, x, z) = z \sum_{\sigma \in \text{Part}(d)} \sum_{\beta \geq 0} \exp \left( \sum_{i=0}^{r} t_i [H_{\sigma, i}] / (z + \beta) \right) Q^{\beta} \sum_{\text{labels } L = (L_\eta) \text{ of } \sigma \text{ with sum } \beta} \]

(27)

where:

- $1_\sigma \in H^*_{CR,T}(\text{Sym}^d \mathbb{P}^r)$ is the fundamental class of the twisted sector corresponding to $\sigma$,
- $[H_{\sigma, i}]$ and $[H_{\sigma, \eta, i}]$ are defined in Section 3.2,
- $x^k := \prod \sigma x_{\sigma,i}^{k_\sigma}$,
- $k! := \prod \sigma k_{\sigma}!$,
- $z^k := \prod \sigma z^{k_{\sigma}}$, and
- $H(\sigma, x^k)$ is the number of ways of factoring $1 \in S_d$ as a product $a_1 \cdots a_m$, where the conjugacy classes (i.e. partitions) of the permutations $a_j$ are given by the list $(\sigma, x^k)$.

Note that (27) uses the normal cup product on $H^*(\text{Sym}^d \mathbb{P}^r)$, not the Chen-Ruan product. As mentioned, we prove:

\begin{theorem}
$I_{\text{Sym}^d \mathbb{P}^r}(Q, t, x, -z)$ is a $\Lambda^T_{\text{nov}}[[t, x]]$-valued point of $\mathcal{L}_{\text{Sym}^d \mathbb{P}^r}$.
\end{theorem}

\begin{remark}
We instead show that $I_{\text{Sym}^d \mathbb{P}^r}(Q, t, x, -z)$ is a $\Lambda_{\text{nov}}[[t, x]]$-valued point of $\mathcal{L}_{\text{Sym}^d \mathbb{P}^r}$, where $I_{\text{Sym}^d \mathbb{P}^r}(Q, t, x, -z)$ is obtained from $I_{\text{Sym}^d \mathbb{P}^r}(Q, t, x, -z)$ by removing the exponential factor. The divisor equation in Gromov-Witten theory then implies Theorem 7.2.
\end{remark}

\begin{proof}
It is immediate that $I_{\text{Sym}^d \mathbb{P}^r}(0, 0, 0, -z) = -1z$. Per Theorem 6.4 it now suffices to prove conditions (I) (II) and (III). We write $L(\mu, \sigma)$ for the restriction of $I_{\text{Sym}^d \mathbb{P}^r}(Q, t, x, -z)$ to a $T$-fixed point $(\mu, \sigma) \in I_{\text{Sym}^d \mathbb{P}^r}$. We write $r_\sigma := \text{lcm}(\sigma)$. Then from (27),

(28)

\[ L(\mu, \sigma) = -z \sum_{\beta \geq 0} \exp \left( \sum_{i=0}^{r} t_i \left( \beta + \sum_{\ell \in \mu} r_\sigma (\alpha_\ell - \alpha_i) - z \right) \right) Q^{\beta} \sum_{\text{labels } L = (L_\eta) \text{ of } \sigma \text{ with sum } \beta} \]

(29)

\[ \left( \sum_{k=(k_\eta)\eta \in \text{Part}(d)} \frac{x^k H(\sigma, x^k)}{k!(z^k)} \right) \left( \frac{|S_\sigma|}{|S_{\sigma, L}|} \right) \left( \frac{1}{\prod_{\eta \in \sigma} \prod_{\gamma=1}^{L_\eta} \prod_{i=0}^{r} \left( r_\sigma (\alpha_\gamma(\eta) - \alpha_i) - \frac{z}{q} \right) \right) \]

It is clear that the coefficient of a single power in $t$, $x$, and $Q$ is a rational function in $z$. The poles of such a coefficient are (at worst) $z = 0$, $z = \infty$, and $z = \frac{r_\sigma (\alpha_i - \alpha_1)}{q}$, where $i_1 = i(\eta)$ for some $\eta \in \sigma$, and $q \in \frac{1}{\eta} \mathbb{Z}$. This is exactly the set of values arising as $\overline{w}(\kappa)$ for $\kappa \in \Upsilon(\mu, \sigma)$. This proves (I).
For [II] we work with the left and right sides of (18). Fix $a \equiv \frac{r_{\sigma}(\alpha_i - \alpha_j)}{q}$ and applied to $f = \mathcal{I}_{\text{Sym}^d}(Q, t, x, z)$ the factor

$$
\Omega := -z \left( \sum_{k=(k_1, \ldots, k_n) \in \text{Part}(d)} \frac{x^k H(\sigma, x^k)}{k!(-z)^k} \right) \left( \prod_{\eta \in \sigma} \prod_{\gamma \leq \eta \text{ or } L_{\text{q}a} < q_{\eta}} \prod_{\gamma = 1}^{L_{\eta}} \prod_{i = 0}^{r} \left( r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} z \right) \right)
$$

appears identically on both sides, so we may prove (18) instead for $L(\mu, \sigma)/\Omega$. We break up $L(\mu, \sigma)/\Omega$, the left-hand side of (18), into terms by the label $L$ (with sum $\beta$):

$$
T_{[\mu, \sigma], L}(z) := \frac{|S_{\sigma}|}{|S_{\sigma, L}|} \left( \prod_{\eta \in \sigma} \prod_{\gamma \leq \eta \text{ or } L_{\text{q}a} < q_{\eta}} \prod_{\gamma = 1}^{L_{\eta}} \prod_{i = 0}^{r} \left( r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} z \right) \right).
$$

Write $\sigma_T$ for the multiset $\{ \eta \in \sigma | i(\eta) = t, L_{\eta} \geq \eta q \}$. This consists of parts of $\sigma$ that are in $\text{Mov}(\tilde{\kappa})$ for some $\tilde{\kappa}$ with weight $\overline{w}$ based at $(\mu, \sigma)$. We compute

$$
\text{Laur}(T_{[\mu, \sigma], L}(z), (\overline{w} - z)^{-a}) = \frac{1}{(|\sigma_T| - a)!} \left( d^{[\sigma_T] - a} / d(\overline{w} - z)^{[\sigma_T] - a} \right) (\overline{w} - z)^{[\sigma_T]} T_{[\mu, \sigma], L}(z) \bigg|_{z \to \overline{w}}
$$

(31)

$$
|S_{\sigma}| / |S_{\sigma, L}| \left( \sum_{A} \prod_{(\eta, \gamma, i) \in A} \frac{\gamma/\eta}{r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} \overline{w}} \right) \prod_{\eta \in \sigma} \prod_{\gamma \leq \eta \text{ or } L_{\text{q}a} < q_{\eta}} \prod_{\gamma = 1}^{L_{\eta}} \prod_{i = 0}^{r} \left( r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} z \right)
$$

where $A$ ranges over $\{ |\sigma_T| - a \}$-tuples of factors in the denominator of $T_{[\mu, \sigma], L}(z)$, i.e. over unordered tuples of triples $(\eta, \gamma, i)$, with $\eta \in \sigma_T$, $1 \leq \gamma \leq L_{\eta}$, $0 \leq i \leq r$, and $(\gamma, i) \neq (\eta q_{\eta}, i_2)$. Observe that

$$
r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} \overline{w} = r_{\sigma} \left( \frac{\eta q_{\eta} - \gamma}{\eta q_{\eta}} \alpha_i + \frac{2}{\eta q_{\eta}} \alpha_{i_2} - \alpha_i \right).
$$

For the right-hand side of (18), let $\tilde{\kappa} \in \Upsilon(\mu, \sigma)$ with $\overline{w}(\tilde{\kappa}) = \overline{w}$. Such an edge corresponds to a subset $\text{Mov}(\tilde{\kappa}) \subseteq \sigma_T$. Let $L'(\tilde{\kappa})$ be the label of $\sigma'(\tilde{\kappa})$ obtained by decreasing $L_{\eta}$ by $\eta q_{\eta}$ for $\eta \in \text{Mov}(\tilde{\kappa})$, using an identification of $\sigma$ and $\sigma'(\tilde{\kappa})$ as partitions. (There is a factor of $|S_{\sigma, L'}(\tilde{\kappa})| / |S_{\sigma, L}|$ from different choices of identification.) As above we write $T_{[\mu'(\tilde{\kappa}), \sigma'(\tilde{\kappa})], L'(\tilde{\kappa})}(z)$ for the factors of $L(\mu'(\tilde{\kappa}), \sigma'(\tilde{\kappa}))$ coming from $\sigma_T$. (The meaning of the multiset $\sigma_T \subseteq \sigma$ has not changed.) Then as before (32)

$$
\text{Laur}(T_{[\mu'(\tilde{\kappa}), \sigma'(\tilde{\kappa})], L'(\tilde{\kappa})}(z), (\overline{w} - z)^{\text{mov}(\tilde{\kappa}) - a}) = \frac{1}{(|\sigma_T| - a)!} \left( d^{[\sigma_T] - a} / d(\overline{w} - z)^{[\sigma_T] - a} \right) (\overline{w} - z)^{[\sigma_T]} T_{[\mu'(\tilde{\kappa}), \sigma'(\tilde{\kappa})], L'(\tilde{\kappa})} \bigg|_{z \to \overline{w}}
$$

(33)

$$
= \frac{|S_{\sigma}| / |S_{\sigma, L}|}{(|\sigma_T| - a)!} \left( \sum_{A} \prod_{(\eta, \gamma, i) \in A} \frac{\gamma/\eta}{r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} \overline{w}} \right) \prod_{\eta \in \text{Mov}(\tilde{\kappa})} \prod_{\gamma = 1}^{L_{\eta}} \prod_{i = 0}^{r} \left( r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} \overline{w} \right)
$$

$$
= \frac{1}{|S_{\sigma}| / |S_{\sigma, L}|} \left( \frac{d^{[\sigma_T] - a} / d(\overline{w} - z)^{[\sigma_T] - a} (\overline{w} - z)^{[\sigma_T]} T_{[\mu'(\tilde{\kappa}), \sigma'(\tilde{\kappa})], L'(\tilde{\kappa})}}{\prod_{\eta \in \text{Mov}(\tilde{\kappa})} \prod_{\gamma = 1}^{L_{\eta}} \prod_{i = 0}^{r} \left( r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} \overline{w} \right)} \right)
$$

$$
= \frac{1}{|S_{\sigma}| / |S_{\sigma, L}|} \left( \frac{d^{[\sigma_T] - a} / d(\overline{w} - z)^{[\sigma_T] - a} (\overline{w} - z)^{[\sigma_T]} T_{[\mu'(\tilde{\kappa}), \sigma'(\tilde{\kappa})], L'(\tilde{\kappa})}}{\prod_{\eta \in \text{Mov}(\tilde{\kappa})} \prod_{\gamma = 1}^{L_{\eta}} \prod_{i = 0}^{r} \left( r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} \overline{w} \right)} \right)
$$

(34)

$$
= \frac{1}{|S_{\sigma}| / |S_{\sigma, L}|} \left( \frac{d^{[\sigma_T] - a} / d(\overline{w} - z)^{[\sigma_T] - a} (\overline{w} - z)^{[\sigma_T]} T_{[\mu'(\tilde{\kappa}), \sigma'(\tilde{\kappa})], L'(\tilde{\kappa})}}{\prod_{\eta \in \text{Mov}(\tilde{\kappa})} \prod_{\gamma = 1}^{L_{\eta}} \prod_{i = 0}^{r} \left( r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} \overline{w} \right)} \right)
$$

(35)

$$
q^{[\sigma_T] - \text{mov}(\tilde{\kappa})} \prod_{\eta \in \text{Mov}(\tilde{\kappa})} \prod_{\gamma = 1}^{L_{\eta}} \prod_{i = 0}^{r} \left( r_{\sigma}(\alpha_i - \alpha_i) - \frac{2}{\eta} \overline{w} \right)
$$

(36)
where \( B_\tilde{\kappa} \) runs over \((|\sigma_T| - a)\)-tuples of factors in the denominator. The product in the denominator over \( \eta \in \sigma_T \setminus Mov(\tilde{\kappa}) \) appears identically in \((\ref{eq:31})\), and the product over \( \eta \in Mov(\tilde{\kappa}) \) of the factors
\[
\left( r_\sigma(\alpha_{i_2} - \alpha_{i_1}) - \frac{\gamma}{\eta} \right) = \left( r_\sigma(\alpha_{i_1} - \alpha_{i_2}) - \left( q + \frac{\gamma}{\eta} \right) \right)
\]

appears in \((\ref{eq:31})\) via the substitution \( \gamma \mapsto \gamma - q\eta \). Together with the denominator of \( RC(\tilde{\kappa}, a) \), this makes up entire denominator of \((\ref{eq:31})\), excluding the sum over \( A \). The factor \( Q^\beta \) also appears on both sides, so it remains to prove:

\[
(33) \quad \sum_A \prod_{(\eta, \gamma, i) \in A} \frac{\gamma / \eta}{\eta, \gamma, i} = \sum_{\kappa} \sum_{B_\kappa} (-1)^{\text{move}(\kappa)} \left( \frac{\sigma_{i_1}^{\text{move}(\kappa)}}{\text{Mov}(\kappa)} \right) \left( \frac{\text{Mov}(\kappa) - 1}{a - 1} \right) \prod_{(\eta, \gamma, i) \in B_\kappa} \frac{\gamma / \eta}{\eta, \gamma, i}.
\]

We switch the order of summation on the right-hand side, and identify each tuple \( B_\kappa \) with one of the tuples \( A \) via the substitution \( \gamma / \eta \mapsto \gamma / \eta - q \) for \( \eta \in Mov(\kappa) \). We now want to prove:

\[
\sum_A \prod_{(\eta, \gamma, i) \in A} \gamma / \eta = \sum_A \sum_{\gamma / \eta \in \text{Move}_{\gamma / \eta}^{\gamma / \eta} \setminus \sigma_T, |\text{Move}| \geq a} (-1)^{\text{Move}| - a} \left( \frac{\sigma_{i_1}^{\text{Move}}}{\text{Mov}} \right) \left( \frac{\text{Move}| - 1}{a - 1} \right) \prod_{(\eta, \gamma, i) \in A} \gamma / \eta - q \prod_{\eta \notin \text{Mov} \setminus (\eta, \gamma, i) \in A, \eta \notin \text{Mov}} \gamma / \eta.
\]

We break up the right side further by fixing the set \( A' := \{(\eta, \gamma, i) \in A | \eta \in \text{Mov}\} \):

\[
(34) \quad \sum_{A} \sum_{A' \subseteq A} \prod_{(\eta, \gamma, i) \in A'} \gamma / \eta - q \prod_{(\eta, \gamma, i) \in A \setminus A'} \gamma / \eta \sum_{\eta \in \text{Mov} \setminus (\eta, \gamma, i) \in A', \eta \notin \text{Mov}} (-1)^{\text{Move}| - a} \left( \frac{\sigma_{i_1}^{\text{Moving}}}{\text{Mov}} \right) \left( \frac{\text{Move}| - 1}{a - 1} \right).
\]

The factor \( \left( \frac{\sigma_{i_1}^{\text{Mov}}}{\text{Mov}} \right) \) turns the second summation on the right into a sum over labeled submultisets \( \text{Mov} \subseteq \sigma_T \). We then use the straightforward combinatorial identity:

\[
\sum_{\text{multisets} \ Move \subseteq \sigma_T, |\text{Move}| \geq a} (-1)^{\text{Move}| - a} \left( \frac{\text{Move}| - 1}{a - 1} \right) = \begin{cases} 0 & A' \neq \emptyset \\ 1 & A' = \emptyset. \end{cases}
\]

Thus \((34)\) is equal to \( \sum_{A} \prod_{(\eta, \gamma, i) \in A} \gamma / \eta \), proving \((33)\) and \((\ref{eq:II})\).

Finally, we prove \((\ref{eq:III})\) using Tseng’s orbifold quantum Riemann-Roch operator. From \(\text{JK02, Proposition 3.4,}\) we compute that \( J_\mu(x, -z) \) is a \( \Lambda^T_{\text{Mov}}[[x]] \)-valued point of the nontwisted Givental cone \( L_\mu \), where

\[
J_\mu(x, z) := z \sum_{\sigma} \sum_{k, \beta} \frac{1_\sigma Q^\beta x^k H(\sigma, k)}{\beta! z^\beta k! z^k}.
\]

\(4\)Note that the coefficients of \( J_\mu(x, z) \) differ from the Gromov-Witten invariants \( \mu \) by a factor of \(|G|\); this convention is chosen so that \( J_\mu(x, z)|_{Q=x=0} = 1z \).
Here $\sigma$ runs over conjugacy classes in $G_\mu$. We introduce variables $Q_{\sigma,\eta}$ indexed by a multipartition $\sigma$ and part $\eta$, and define
\[
J^Q_\mu(x, -z) := -z \sum_{\sigma} \sum_{k, \beta, L} \frac{1_\sigma x^k H(\sigma, k)}{k!(-z)^k} \left( \frac{|S_\sigma|}{|S_{\sigma, L}|} \prod_{\eta \in \sigma} Q^L_{\sigma, \eta} \right)
\]
\[
L^Q_\mu := -z \sum_{\sigma, k, L} \left( \frac{1_\sigma x^k H(\sigma, x^k)}{k!(-z)^k} \right) \left( \frac{|S_\sigma|}{|S_{\sigma, L}|} \prod_{\eta \in \sigma} \frac{Q^L_{\sigma, \eta} \eta^L_{\eta}}{(-z)^{L_{\eta}} L_{\eta}!} \prod_{\gamma = 1}^{L_{\eta}} \prod_{i = 0}^L (\alpha_i - \eta - \frac{2}{z}) \right).
\]

Using combinatorics we may check that specializing $Q_{\sigma, \eta} = Q$ for all $\sigma, \eta$ recovers $J_\mu(x, z)$ and $L_\mu$.

From here, (III) essentially follows from the proof in [CCIT15], despite the fact that $T_\mu \Sym^d \mathbb{P}^r$ is not a direct sum of line bundles. We give an outline.

As in [CCIT15], we work with a general multiplicative characteristic class $c_s$. Denote by $\Delta_s$ the orbifold quantum Riemann-Roch operator, which by [Tse10] maps $L_\mu$ to $L^\tw_\mu$. Explicitly,
\[
\Delta_s := \bigoplus_{\sigma} \exp \left( \sum_{\eta \in \sigma} \sum_{0 \leq \ell \leq \eta - 1} \sum_{m \geq 0} s_{m-1} R(\sigma, \ell) \frac{B_m(\ell/\eta)}{m!} z^{m-1} \right),
\]
where $B_m$ is the $m$th Bernoulli polynomial, and $R(\sigma, \ell)$ is the rank of the eigenbundle of $T_\mu \Sym^d \mathbb{P}^r$ on which elements of $\sigma \subseteq G_\mu$ act with eigenvalue $e^{2\pi i k/\eta}$. Note that the values
\[
s_k = \begin{cases} 
- \log(\alpha_i(\eta)) - \alpha_i & k = 0 \\
(-1)^k \frac{(k-1)!}{(\alpha_i(\eta) - \alpha_i)^k} & k > 0 
\end{cases}
\]
recover the $T$-equivariant Euler class. Using the functional equation of the Bernoulli polynomials, we may check that $L^Q_{\mu, \sigma} = \Delta_s(J^\text{mod}_\mu(x, -z))$, where
\[
J^\text{mod}_\mu(x, -z) := -z \sum_{\sigma} \sum_{k, \beta, L} \frac{1_\sigma x^k H(\sigma, k)}{k!(-z)^k} \left( \frac{|S_\sigma|}{|S_{\sigma, L}|} \prod_{\eta \in \sigma} Q^L_{\sigma, \eta} \right)
\]
\[
\exp \left( \sum_{\eta \in \sigma} \sum_{\ell = 1}^{\eta - 1} \sum_{m \geq 0} s_{n+m-1} B_m(0) R(\sigma, \ell) \frac{z^{L_{\eta} + \ell}}{m!} \frac{(\ell/\eta)^n}{n!} z^{m-1} \right).
\]

Analyzing the floor function, we have
\[
\sum_{\ell = 1}^{\eta - 1} \frac{z^{\ell/\eta} - z^{L_{\eta} + \ell}}{n!} = \sum_{\ell = 1}^{\eta - 1} \frac{z^{L_{\eta} + \ell}}{n!} = \frac{(z^{L_{\eta}})^n}{n!}.
\]

Now (35) is equal to $P(Q_{\sigma, \eta} \frac{\partial}{\partial Q_{\sigma, \eta}})(J^Q_\mu(x, z))$, for
\[
P(a) = \exp \left( \sum_{\eta \in \sigma} \sum_{\ell = 1}^{\eta - 1} \sum_{m \geq 0} s_{n+m-1} B_m(0) R(\sigma, \ell) \frac{(z^{L_{\eta} + \ell}/\eta)^n}{n!} z^{m-1} \right).
\]

The inductive argument at the end of the proof of Theorem 4.6 of [CCIT09] shows that $J^\text{mod}_\mu(x, -z)$ is a $H^*_{T_\mu \text{loc}}(\Spec \mathbb{C})[[Q_{\sigma, \eta}, x]]$-valued point of $L_\mu$, and orbifold quantum Riemann-Roch then shows that $L^Q_\mu$ is a $H^*_{T_\mu \text{loc}}(\Spec \mathbb{C})[[Q_{\sigma, \eta}, x]]$-valued point of $L^\tw_\mu$. Specializing to $Q_{\sigma, \eta} = Q$ proves (III). \qed
Remark 7.4. The beginning terms of $I_{\text{Sym}^d \mathcal{P}^r}(Q, t, x, z)$ are

$$I_{\text{Sym}^d \mathcal{P}^r}(Q, t, x, z) = 1 + \sum_{\sigma} \sum_{i=0}^r t_i [H_{\sigma,i}] + \sum_{\Pi \in \text{Part}(d)} x_{\Pi} 1_{\Pi} + O(z^{-1}).$$

On the other hand, by definition, there is a unique $\Lambda_{\text{nov}}^T[t, x]$-valued point of $\mathcal{L}_{\text{Sym}^d \mathcal{P}^r}$ of this form, namely the $J$-function

$$J_{\text{Sym}^d \mathcal{P}^r}(Q, \theta, z) = 1 + \theta + \sum_{\beta, n} Q^n \left\langle \theta, \ldots, \theta, \frac{\gamma \phi}{z - \psi} \right\rangle_{\text{Sym}^d \mathcal{P}^r, T} \gamma^\phi,$$

where $\theta = \sum_{\sigma} \sum_{i=0}^r t_i [H_{\sigma,i}] + \sum_{\Pi \in \text{Part}(d)} x_{\Pi} 1_{\Pi}$.

Thus:

**Corollary 7.5.** $I_{\text{Sym}^d \mathcal{P}^r}(Q, t, x, z) = J_{\text{Sym}^d \mathcal{P}^r}(Q, \theta, z).$

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