Turing pattern of a reaction-diffusion predator-prey model with weak Allee effect and delay

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Abstract. In this paper, we investigate a reaction-diffusion model with gestation delay and weak Allee effect. The conditions of delay-induced instability are obtained after stability analysis. By using numerical simulation, we discuss the effects of gestation delay and weak Allee effect on the Turing pattern. The results show that weak Allee effect and gestation delay can change the pattern formations. This means Allee effect and delay play significant roles in spatial invasion of populations.

1. Introduction

Turing pattern of predator-prey model is an important issue in the field of biomathematics and population dynamics [1,2]. We consider a predator-prey model with hyperbolic mortality established by Zhang et al [3]. The model is as follows:

\[
\begin{align*}
\frac{\partial U}{\partial T} - d_1 \nabla^2 U &= au \left( 1 - \frac{u}{k} \right) - \frac{bUV}{c+U}, \\
\frac{\partial V}{\partial T} - d_2 \nabla^2 V &= \frac{mUV}{c+U} - h(V), \\
\frac{\partial U}{\partial n} &= \frac{\partial V}{\partial n} = 0, \\
U(\partial \Omega,0) > 0, V(\partial \Omega,0) > 0, X \in \Omega,
\end{align*}
\]

where $U$ and $V$ represent the population density of prey and predator, $a$ is the birth rate of prey, $k$ is the carrying capacity, $c$ is the prey density at which the predator has the maximum kill rate, $b$ is the maximum uptake rate of the prey, $m$ is the birth rate, and $h(V)$ reflects the predator death rate. $\nabla^2$ is the Laplacian operator $n$ is the outward unit normal vector of the boundary \( \partial \Omega \). $d_j$ are the diffusion coefficients of $U$ and $V$, respectively, and the third equation is the zero flux boundary condition. After nondimensionalization,

\[
U \rightarrow ku, V \rightarrow \frac{ac}{b} v, \frac{k}{c} \rightarrow \beta, \frac{a}{m} \rightarrow \alpha, T \rightarrow t.
\]
In order to make the model closer to reality, we introduce gestation delay and weak Allee effect [4], as follows:

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \nabla^2 u &= au\left(1-u(t-\tau)\right) - \frac{auv}{u + A}, \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} - d_2 \nabla^2 v &= \frac{\beta uv}{1 + \beta u} - h(v), \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0, \quad x \in \Omega, t > 0, \\
u(x, \theta) = \varphi(x, \theta) \geq 0, v(x, \theta) = \psi(x, \theta) \geq 0, \quad (x, \theta) \in \Omega \times (-\tau, 0),
\end{align*}
\]

where \( \frac{u}{u + A} (A > 0) \) is weak Allee effect. \( h(v) = \frac{\gamma v^2}{e + \eta v} \). For hyperbolic mortality, \( e \) and \( \eta \) are coefficients of light attenuation by water and self-shading in the context of plankton mortality, \( \gamma \) is the death rate of the predator, and \( \tau \) is the gestation delay.

When \( \tau = 0 \), the model is as follows:

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \nabla^2 u &= au(1-u) - \frac{auv}{u + A}, \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} - d_2 \nabla^2 v &= \frac{\beta uv}{1 + \beta u} - \frac{\gamma v^2}{e + \eta v}, \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0, \quad x \in \Omega, t > 0, \\
u(x, \theta) = \varphi(x, \theta) \geq 0, v(x, \theta) = \psi(x, \theta) \geq 0, \quad (x, \theta) \in \Omega \times (-\tau, 0).
\end{align*}
\]

In [5], following results have obtained by Liu et al for model (3).

(A1) The positive equilibrium is \( E_*(u_*, v_*) \), where

\[
u_* = \sqrt{\left(\frac{\beta \gamma - \gamma - \beta}{2 \beta} \right)^2 + \left(\frac{\gamma - \beta A}{\beta \gamma} \right)^2}, \quad v_* = \frac{\beta \gamma - \gamma - \beta}{2 \beta \gamma}. 
\]

(A2) The Jacobian matrix of model (3) at \( E_* \) is \( J_{E_*} = \begin{pmatrix} a_{10} & a_{01} \\ b_{01} & b_{01} \end{pmatrix} \), where

\[
\begin{align*}
a_{10} &= -2\alpha u_*^2 - 3\alpha A u_*^2 + au_*^2 + 2\alpha A u_*, \\
a_{01} &= -au_*/(1 + \beta u_*), \\
b_{01} &= \frac{\beta^2 u_*}{\gamma(1 + \beta u_*)^2}, \\
b_{01} &= -\frac{\beta u_*}{(1 + \beta u_*)^2}.
\end{align*}
\]

(A3) \( T = \frac{2\alpha u_*^2 - 3\alpha A u_*^2 + au_*^2 + 2\alpha A u_*}{(u_* + A)^2} - \frac{\alpha \beta u_*}{\gamma(1 + \beta u_*)^2} - \beta u_*/(1 + \beta u_*)^2. \)
\[ D = \left\{ \frac{-2\alpha u^2 - 3\alpha Au^2 + \alpha\beta u}{(u + A)^2} - \frac{\alpha\beta u}{(1 + \beta u)^2}, -\beta u + \frac{\alpha\beta^2 u^2}{1 + \beta u,} \right\}. \]

(A4) If \( T > 0 \), the positive equilibrium is unstable. If \( T < 0 \) and \( D > 0 \), the positive equilibrium is locally asymptotically stable.

2. Delay-induced instability
If \( \tau \) is small enough, we have following change [6].

\[ u(x, y, t - \tau) = u(x, y, t) - \tau \frac{\partial u}{\partial t}. \]

Next, we substitute this equation into (2), and get the following model

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \nabla^2 u &= \alpha u \left( 1 - \left( u(x, y, t - \tau) \frac{\partial u}{\partial t} \right) \right) \frac{u}{u + A} - \frac{\alpha u v}{1 + \beta u}, \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} - d_2 \nabla^2 v &= \frac{\beta \alpha u v}{1 + \beta u} - \frac{\gamma v^2}{1 + \beta u}, \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial n} &= 0, \quad x \in \Omega, t > 0, \\
u(x, \theta) &= \varphi(x, \theta) \geq 0, v(x, \theta) = \psi(x, \theta) \geq 0, \quad (x, \theta) \in \Omega \times (-\tau, 0).
\end{align*}
\]

Expanding Taylor’s series of the above model and neglecting the higher-order nonlinearities, we get

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \nabla^2 u &= \alpha u (1 - u) \frac{u}{u + A} - \frac{\alpha u v}{1 + \beta u} \\
&\quad - \tau f_{u(t-\tau)}(u(t-\tau), v) \frac{\partial u}{\partial t}, \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} - d_2 \nabla^2 v &= \frac{\beta \alpha u v}{1 + \beta u} - \frac{\gamma v^2}{1 + \beta u}, \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial n} &= 0, \quad x \in \Omega, t > 0, \\
u(x, \theta) &= \varphi(x, \theta) \geq 0, v(x, \theta) = \psi(x, \theta) \geq 0, \quad (x, \theta) \in \Omega \times (-\tau, 0),
\end{align*}
\]

where \( f_{u(t-\tau)}(u(t-\tau), v) = \alpha u (1 - u(t-\tau)) \frac{u}{u + A} - \frac{\alpha u v}{1 + \beta u} \).

It is easy to see that if \( f(u, v) = 0 \) and \( g(u, v) = 0 \) are satisfied at \( E_* = (u_*, v_*) \), we can get the model:
\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \nabla^2 u &= f_u(u,v)(u-u_\ast) + f_v(u,v)(v-v_\ast), \\
-\tau f_{u(t-\tau)}(u(t-\tau),v)\frac{\partial u}{\partial t}, &\quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} - d_2 \nabla^2 v &= g_u(u,v)(u-u_\ast) + g_v(u,v)(v-v_\ast), \\
&\quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0, \\
u(x, \theta) = \varphi(x, \theta) \geq 0, v(x, \theta) = \psi(x, \theta) \geq 0, \\
&\quad (x, \theta) \in \Omega \times (-\tau, 0), 
\end{align*}
\]

where \( f(u,v) = au(1-u) - \frac{auv}{u + A} \) and \( g(u,v) = \frac{\beta uv}{1 + \beta u} - \frac{\gamma v^2}{1 + \beta u + \eta v} \).

We put a small perturbation into the stable positive equilibrium point \( E_\ast = (u_\ast, v_\ast) \). Let \( u = u_\ast + \bar{u} \) and \( v = v_\ast + \bar{v} \), we get

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} - d_1 \nabla^2 \bar{u} &= a_{10}\bar{u} + a_{01}\bar{v} + \tau \frac{au^2}{u + A} \frac{\partial \bar{u}}{\partial t}, &\quad x \in \Omega, t > 0, \\
\frac{\partial \bar{v}}{\partial t} - d_2 \nabla^2 \bar{v} &= b_{10}\bar{u} + b_{01}\bar{v}, &\quad x \in \Omega, t > 0, \\
\frac{\partial \bar{u}}{\partial n} = \frac{\partial \bar{v}}{\partial n} &= 0, &\quad x \in \Omega, t > 0, \\
u(x, \theta) = \varphi(x, \theta) \geq 0, v(x, \theta) = \psi(x, \theta) \geq 0, &\quad (x, \theta) \in \Omega \times (-\tau, 0).
\end{align*}
\]

Assuming that the model has the following solution,

\[
\bar{u}(x,t) = \bar{u}_\ast e^{\tau t} \cos k \cdot x, \quad \bar{v}(x,t) = \bar{v}_\ast e^{\tau t} \cos k \cdot x,
\]

so,

\[
J^\tau_t = \begin{pmatrix}
a_{00} - k^2 d_1 & a_{00} \\
1 - \tau M & 1 - \tau M \\
b_{10} & b_{01} - k^2 d_2
\end{pmatrix},
\]

where

\[
M = \frac{au^2}{u + A}.
\]

It is easy to get

\[
T^\tau_t = \frac{a_{00} - k^2 d_1}{1 - \tau M} + b_{01} - k^2 d_2, \quad D^\tau_t = \frac{(a_{00} - k^2 d_1)(b_{01} - k^2 d_2) - a_{01}b_{00}}{1 - \tau M}.
\]

The model (2) undergoes a Hopf bifurcation at \( T^\tau_t = 0 \) when \( k = 0 \). So the threshold of the bifurcation can be obtained: \( \tau_H = \frac{b_{01} + a_{01}}{b_{01}M} \).
From (A4), we know that if $E_*$ is stable, $a_{10}b_{01} - a_{01}b_{10} > 0$,
\[
D_k^* = \left( \frac{a_{10} - k^2 d_2}{1 - \tau M} \right) (b_{01} - k^2 d_2) - \frac{a_{01}b_{10}}{1 - \tau M} > 0,
\]
so, we need to judge the following inequality
\[
T_k^* = \frac{a_{10} - k^2 d_2}{1 - \tau M} + b_{01} - k^2 d_2 > 0,
\]
when
\[
\tau > \frac{k^2 d_2 + k^2 d_1 - a_{10} - b_{01}}{Mk^2 d_2 - Mb_{01}}, \quad T_k^r = \frac{a_{10} - k^2 d_2}{1 - \tau M} + b_{01} - k^2 d_2 > 0.
\]
The instability condition are as follows:
\[
\begin{align*}
D &= \frac{-2\alpha u_*^3 - 3\alpha A u_*^2 + au_*^2 + 2aAu_*}{(u_* + A)^2} - \frac{\alpha \beta u_*}{\gamma(1 + \beta u_*)^2} - \frac{\beta u_*}{(1 + \beta u_*)^2} < 0, \\
T &= \frac{k^2 d_2 + k^2 d_1 - a_{10} - b_{01}}{Mk^2 d_2 - Mb_{01}}.
\end{align*}
\]

3. Numerical simulations
In this section, we use numerical simulation to study the effect of delay and weak Allee effect on Turing pattern formation.

First, we discuss the case without delay and weak Allee effect. In addition to the weak Allee effect and delay parameters, other parameters are fixed as $d_1 = 0.001$, $d_2 = 0.1$, $\alpha = 0.65$, $\beta = 6$, $\gamma = 0.5$, $e = 1$, $\eta = 0.5$. The pattern is shown in Fig.2. It is made up of spots and stripes. We call it the original pattern.

![Pattern formation](image)

**Fig. 1.** Pattern formations are spots and stripes for $A=0$ and $\tau = 0$. Time: $t=100$. 


Fig. 2. Pattern formations are spots and stripes for $A=0$ and $\tau = 0$. Time: $t=5000$.

Fig. 3. Pattern formations are spots and stripes for $A=0.02$ and $\tau = 0$. Time: $t=100$.

Fig. 4. Pattern formations are spots and shorter stripes for $A=0.02$ and $\tau = 0$. Time: $t=5000$. 
Second, we try to increase the weak Allee effect parameter to observe the changes in the pattern. As the weak Allee effect parameter increases, the stripes of the pattern become shorter than original pattern [5].

Liu et al proved that searching delay makes the original pattern become stripes [5]. In this paper, we try to introduce gestation delay, and find that there is no obvious changes than the original pattern.

Fig. 5. Pattern formations are spots and stripes for $A=0.01$ and $\tau = 0.5$. Time: $t=100$.

Finally, we try to consider the combined effect of weak Allee effect and delay. The change of the pattern is shown in Fig.7 and Fig.8. The type of pattern is obviously different from that in original pattern. We get the pattern formations are spots.
Fig. 7. Pattern formations are spots for $A=0.2$ and $\tau = 0.5$. Time: $t=100$.

Fig. 8. Pattern formations are spots for $A=0.2$ and $\tau = 0.5$. Time: $t=5000$.

4. Result and Conclusion
Our numerical simulation are performed for model (2). First, we control Allee effect and delay parameters are zero. Mixture patterns obtained with $A=0$, $\tau = 0$ and call it the original pattern. Next, we increase the value of Allee effect, the results show that Allee effect will reduce the length of the strips. Same method, we increase the value of delay and find that there is no obvious changes than the original pattern. Finally, we consider the weak Allee effect and delay to observe the changes in Turing pattern. Mixture patterns turn into spots patterns. So, this study believes Allee effect and delay play significant roles in spatial invasion of populations.

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