The theory of figures of Clairaut with focus on the gravitational modulus: inequalities and an improvement in the Darwin-Radau equation.

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Abstract

This paper contains a review of Clairaut’s theory with focus on the determination of a gravitational modulus $\gamma$ defined as $\left(\frac{C-I}{I}\right) \gamma = \frac{2}{3} \Omega^2$, where $C$ and $I$ are the polar and mean moment of inertia of the body and $\Omega$ is the body spin. The constant $\gamma$ is related to the static fluid Love number $k_2 = \frac{3 I - G \frac{1}{5}}{R}$, where $R$ is the body radius and $G$ is the gravitational constant. The new results are: a variational principle for $\gamma$, upper and lower bounds on the ellipticity that improve previous bounds by Chandrasekhar [6], and a semi-empirical procedure for estimating $\gamma$ from the knowledge of $m$, $I$, and $R$, where $m$ is the mass of the body. The main conclusion is that for $0.2 \leq I / (m R^2) \leq 0.4$ the approximation $\gamma \approx G \sqrt{\frac{2}{3} \frac{m^5}{I^3}} \overset{\text{def}}{=} \gamma_1$ is a better estimate for $\gamma$ than that obtained from the Darwin-Radau equation, denoted as $\gamma_{DR}$. Moreover, an inequality in the paper implies that the Darwin-Radau approximation may be valid only for $0.3 \leq I / (m R^2) \leq 0.4$ and within this range $|\gamma_{DR}/\gamma_1 - 1| < 0.14\%$.

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1 Introduction

Many celestial bodies are large extended objects, almost spherical in shape, almost rigid, and usually with spin. In most cases the deformation is caused by the motion itself and it is difficult to determine the laws that relate motion to deformation. This paper is about one of these laws: the static gravitational rigidity that relates the spin of a body to its flatness. The static gravitational rigidity is characterized by a single parameter \[ \gamma \] \[1\] : the gravitational modulus \[ \gamma \], which is closely related to the static fluid Love number \[ k_2 \]. The goal of this paper is to review the classical theory used in the determination of this parameter. The novelties are: the unusual approach to the subject in considering the mean moment of inertia of the body as a given quantity (usually this is the main unknown to be determined), a new variational principle for the static gravitational rigidity, and a new differential equation for \[ \gamma \] that allows for obtaining sharp upper and lower bounds for this quantity. These three new aspects are detailed in the following paragraphs.

A celestial body is considered large when self-gravitation dominates over any elastic stress. In this case the self-gravitational force is balanced by the hydrostatic pressure and at rest the resulting equilibrium shape is spherically symmetric with moment of inertia \[ I_0 \] along any axis passing through the center of mass. We are interested in small deformations of this spherical shape under slow rotational motion. *Deformations are always assumed to be small and incompressible* \[1\] Under these conditions it was showed by G. Darwin \[28\] that the trace of the moment of inertia tensor remains invariant, namely, if \( A \leq B \leq C \) denote the time-dependent principal moments of inertia of a body then

\[
I_0 = \frac{A + B + C}{3} \tag{1.1}
\]

does not depend on time.

To a given body of mass \( m \) and moment of inertia \( I_0 \) we can associate the *radius of inertia* that is a length scale given by

\[
R_I = \sqrt{\frac{5I_0}{2m}} \tag{1.2}
\]

The radius of inertia is the radius of a homogeneous ball of mass \( m \) and moment of inertia \( I_0 \). The constants \( m, I_0 \) and \( G \) (the gravitational constant) also define a time scale by means of

\[
\gamma_I = \omega_I^2 = \frac{4Gm}{5R_I^3} = \frac{2I_0G}{R_I^5} \tag{1.3}
\]

\[1\] For small deformations the incompressibility hypothesis is quite reasonable because the modulus of compressibility of solids and fluids is much larger than the modulus of shear.
The quantity \( \omega_1 \), which we call the *inertial frequency*, admits three interpretations: it is the smallest angular frequency of oscillations of a homogeneous spherical mass of liquid with mass \( m \) and moment of inertia \( I_0 \) (\cite{16} paragraph 262 Eq. (10)), it is \( 4/5 \) times the square of the angular velocity of a particle moving with a circular orbit of radius \( R_1 \) around a point mass \( m \), and it is the gravitational modulus, to be defined below, of a homogeneous spherical mass of liquid with mass \( m \) and radius \( R_1 \). The three quantities \( m, R_1, \) and \( \gamma_1 \) do not depend on the deformations of the body and may be considered as invariant properties of the given deformable body.

This entire paper is about the determination of the gravitational modulus \( \gamma \) that is defined in the following way. The effect of the centrifugal force upon an isolated body under uniform steady rotation is to flatten the body along the axis of rotation. Let \( \Omega > 0 \) be the constant angular spin rate\(^2\). The amount of flatness can be measured by the increase in the moment of inertia \( C \) along the axis of rotation. In \cite{24} and \cite{25}, under the hypothesis that \( C - I_0 \) is small, \(\gamma = \frac{2}{3} \Omega^2 \) (\cite{24} Eq. (1.5)). The factor \( C - I_0 \) in equation (1.5) represents the moment of inertia strain (nondimensional) while \( \frac{2}{3} \Omega^2 \) (time\(^2 \)) represents the centrifugal stress, therefore the gravitational modulus \( \gamma \) has the unusual dimension time\(^{-2} \). Notice that the definition of \( \gamma \) makes sense even when \( C - I_0 \) and \( \Omega^2 \) are large but, in this case, \( \gamma \) may be a nonlinear function of \( \Omega^2 \) \cite{36}. So, in order to keep the relation between \( C - I_0 \) and \( \Omega^2 \) within the linear range we impose the smallness of \( \frac{C - I_0}{I_0} \), namely hypothesis \(1.4\). Equivalently, we may impose the smallness of the nondimensional quantity

\[ \zeta_1 = \frac{\Omega^2 R_1^3}{Gm} \]

which is the ratio of the centrifugal to the gravitational acceleration at \( R_1 \).

\(^2\)A main issue in the dynamics of deformable bodies is to define a “body frame” or, equivalently, a notion of body rotation. In principle, each point of the body may rotate differently about a given inertial frame. The definition adopted in \cite{24} and \cite{25} is that of Tisserand: if \( L \) is the instantaneous angular momentum (vector) and \( I \) is the instantaneous moment of inertia (matrix) then the instantaneous angular velocity (vector) \( \Omega \) is defined by \( L = I \Omega \). The Sun, for instance, requires the use of this definition.
The gravitational modulus $\gamma$, which is constant, fails to satisfy equation (1.5) as $\zeta_1$ becomes large.

Notice that the geometric radius of the body $R$ (in this paper $R$ always refer to the volumetric mean radius) does not appear in the definition of $\gamma$.

Using

$$J_{2v} = \frac{C - A}{mR^2},$$

where $A$ is the moment of inertia along an axis passing through the equator and $J_{2v}$ is the dynamic form factor normalized by the volumetric mean radius. $\gamma$ can be written as

$$\gamma = \frac{I_0}{mR^2 J_{2v}}$$

(1.7)

The constant $\gamma$ is also related to the static fluid Love number $k_2$ (equation (14))

$$k_2 = 3I_0G \frac{1}{R^5} \gamma = \frac{3}{2} \left( \frac{R_1}{R} \right)^5 \frac{\gamma_1}{\gamma}$$

(1.8)

The radius $R$, which neither appears in the definition of $\gamma$ nor in the dynamic model presented in [25], is an important quantity in this paper. If in the non-rotating state, the radially symmetric density $\rho$ of the body is a non-increasing function of the radius, a hypothesis assumed throughout the paper, then the radius and the inertial radius satisfy

$$\left( \frac{R_1}{R} \right)^2 = \frac{5}{2} \frac{I_0}{mR^2} \leq 1$$

(1.9)

with $R_1/R = 1$ if, and only if, the body is homogeneous (this well known inequality is proved in Proposition 3.1 in Appendix A).

The paper is organized as follows. Sect. 2 contains a review of the theory of Clairaut which describes the flattening of a rotating body with radial stratification of density. This Section is subdivided into several subsections where we discuss different types of radial density distributions: piecewise constant, with a point-mass at the center, and induced by a polytropic gas. At the end of Section 2 we present the approximate theory of Darwin-Radau.

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3Our parameter $\gamma$ is closely related to the inverse of the parameter $\Lambda_2$ in equation (37.6) of [38], the difference is that their definition uses the radius of the body.

4The usual dynamic form factor $J_2 = (C - A)/mR^2$ is normalized by the equatorial radius $R_e$. The difference $(R_e - R)/R = \epsilon/3$ is of the order of the ellipticity $\epsilon$ of the body and $J_{2v} = J_2 R_e^2/R^2$.

5If the density non-increasing hypothesis is not assumed, then $R_1/R \leq 5/3$, where the equality holds for a spherical shell of radius $R$. 

3
This Section does not contain any new result but another way to look at old formulas which highlight some of their interesting features. For instance, the Darwin-Radau approximation gives rise to an estimate of $\gamma$ (equation (2.75)), denoted as $\gamma_{DR}$, that depends only on the ratio $R_1/R \leq 1$ and such that

$$
\frac{\gamma_{DR}}{\gamma_1} = 1 + \text{Term of order} \left(1 - \frac{R_1}{R}\right)^3
$$

(1.10)

Since the Darwin-Radau approximation is exact for $R_1/R = 1$ we conclude that $\gamma \approx \gamma_1$ whenever $R_1/R \approx 1$. A plot (Figure 3 (a)) shows that the approximation $\gamma \approx \gamma_1$ almost coincide with that by the Darwin-Radau equation for $0.89 < R_1/R \leq 1$. For a polytrope, $\gamma \approx \gamma_1$ is correct up to 3% within the range $0.7 < R_1/R \leq 1$ (Figure 3 (b)).

In Section 3 we present a variational principle for $\gamma$. There are at least two different variational characterizations of Clairaut’s equation: one due to Macke and Voss (see [19] Section 3.3) and another due to Rau [26]. The variational principle of Rau uses the adjoint equation to Clairaut’s equation and is very different from the variational principle of Macke and Voss. We also present a variational characterization of Clairaut’s equation, which is different but related to that of Macke and Voss, with the additional property that the value of the functional to be minimized has $\gamma$ as its minimal value. Although this variational principle can be used to estimate the value of $\gamma$, in this paper, its relevance is mostly conceptual.

In Section 4 we derive a new first order differential equation that allows for the determination of $\gamma$ without having to solve Clairaut’s equation. Using this equation we were able to show that for any non increasing density function

$$
\frac{3}{2} \left(\frac{R_1}{R}\right)^5 \left[\frac{5}{3} \left(\frac{R}{R_1}\right)^2 - 1\right] \leq \frac{\gamma}{\gamma_1} \leq \sqrt{\frac{35 \ 8575}{39 \ 8112}} \approx 1.001401
$$

(1.11)

This is the main result in the paper. Inequality (1.11) implies that the Darwin-Radau theory cannot be valid if $\frac{R_1}{R} < 0.86534\ldots$ (see the discussion close to equation 4.116). If

$$
\zeta = \frac{\Omega^2 R^3}{Gm}
$$

(1.12)

denotes the ratio of the centrifugal acceleration at the equator to the gravitational acceleration on the body surface and $\epsilon(R)$ denotes the ellipticity of the rotating figure of equilibrium, then inequalities (1.11), equation (1.17), and the relation ([36 equation (32.19)])

$$
3J_{2r} = 2\epsilon(R) - \zeta
$$

4
imply
\[ 2 \left( 1 - \frac{3}{5} \left( \frac{R_1}{R} \right)^2 \right) \leq \frac{\zeta}{\epsilon(R)} \leq 2 \left( 1 + 1.4979 \left( \frac{R_1}{R} \right)^5 \right)^{-1} \] (1.13)

or, equivalently,
\[ \frac{1}{3} \left( 2 - \frac{\zeta}{\epsilon(R)} \right) \leq \frac{I_0}{mR^2} \leq 0.400225 \left( \frac{4}{3} \frac{\epsilon(R)}{\zeta} - \frac{2}{3} \right)^{2/5}, \] (1.14)

or
\[ 2 \left( \frac{\zeta}{J_{2v}} + 3 \right)^{-1} \leq \frac{I_0}{mR^2} \leq 0.400225 \left( \frac{\zeta}{2J_{2v}} \right)^{-2/5}. \] (1.15)

For a homogeneous body \( \zeta/\epsilon(R) = 4/5 \) and \( \zeta/J_{2v} = 2 \) and the above inequalities are almost sharp, namely \( 0.4 \leq \frac{I_0}{mR^2} \leq 0.400225 \).

Inequalities 1.13 must be compared to those of Chandrasekhar [7], [6]
\[ \frac{4}{5} \leq \frac{\zeta}{\epsilon(R)} < 2 \]

obtained without any constraint on the ratio \( R_1/R \) (according to [7], equation (100), the first lower bound was known to Laplace). The lower bound in inequality (1.13) coincides with the lower bound by Chandrasekhar if \( R_1/R = 1 \) and the upper bound in inequality (1.13) coincides with the upper bound by Chandrasekhar if \( R_1/R = 0 \). For other values of \( R_1/R \) our inequalities, which essentially cannot be improved without further constraining the density, represent a great advance with respect to those of Chandrasekhar.

In Section 5 we present an application of the results in the previous sections to large bodies in the Solar system (Sun, Earth, Mars, Jupiter, Saturn, Uranus and Neptune). The value of \( \gamma \) is estimated by different means: integration of Clairaut’s equation using density functions available in the literature, the Darwin-Radau approximation, and equation (1.7) with values of \( \Omega \) and \( J_{2v} \) obtained from observations. The results are summarized in tables and figures given in Section 5.

Finally, Section 6 is a conclusion where we propose a way to estimate \( \gamma \) in terms of \( m, I_0, \) and \( R \). Polytropes are taken as archetypal models and it is verified that the values of \( \gamma \) obtained from this model are close to those estimated from observations.

The results in this paper require careful interpretation. Unless a body is homogeneous, for which \( R_1/R = 1 \) and \( \gamma/\gamma_1 = 1 \), there is no one-to-one relation between the normalized gravitational modulus \( \gamma/\gamma_1 \) and the ratio
Nevertheless, the remarkable Darwin-Radau theory and the results in this paper show that, for most large bodies in the Solar system, \( \gamma \) is essentially determined by \( m \), \( I_0 \), and \( R \), and for a body with \( 0.7 \leq R_1/R < 1 \) the value of \( \gamma \) is close to that of an equivalent uniform body with radius \( R_1 \). By no means the internal density \( \rho \) of the body of radius \( R_0 \), which must satisfy \( \rho(r) > 0 \) for \( R_1 < r < R \), has to be close to that of the uniform body of radius \( R_1 \). Moreover, the empirical conclusion of the paper is that, for a large body, if \( m \), \( I_0 \), and \( R \) are known then its gravitational modulus \( \gamma \) is approximately that of a polytrope with the same values of the known quantities (for \( 0.7 \leq R_1/R < 1 \) this approximation is even better than \( \gamma \approx \gamma_1 \)). Again, by no means the density of the given body must be close to that of a polytrope.

This work has been done under the premise that \( I_0 \) is given but “this is exactly the opposite to the actual practice followed by astronomers” (referee’s comment) who from observed values of \( \Omega \) and several gravitational moments \( J_2, J_4, J_6, \ldots \) obtain information about the internal mass distribution of the body including the value of \( I_0 \) (see [36] chapter 3). The next paragraph shows that in situations where astronomical observations are difficult the results in this paper can be used in the usual astronomer way.

The spin rate of an exoplanet can be estimated from near-infrared spectroscopic observations [30]. As argued in [15], “the shape of the transiting light curve might, in principle, reveal the shape of the planet, and in particular, its deviation from spherical symmetry”, namely \( \epsilon(R) \). So, in theory, it may be possible to estimate \( \xi \) and \( \epsilon(R) \) for an exoplanet and, using \( 3J_{2v} = 2\epsilon(R) - \xi \), also \( J_{2v} \). If this is the case, then equation (1.14) establishes upper and lower bounds to \( I_0/(mR^2) \). If the approximation \( \gamma \approx \gamma_1 \) is further adopted then equations (1.3), (1.7), (1.9), and (1.12) imply

\[
\frac{I_0}{mR^2} \approx 0.4 \left( \frac{2J_{2v}}{\xi} \right)^{2/5},
\]

which is a as Darwin-Radau type formula. The approximation may fail: for bodies with unusual density distributions (as for instance with a high density core surrounded by a homogeneous layer, see the “Thick shell Roche model” in Section 2.1.2), for fast spinning bodies (corrections of order \( \xi^2 \) are nonnegligible), and for bodies that are not in hydrostatic equilibrium.

\footnote{For given values of \( m \), \( I_0 \), and \( R \) such that \( R_1/R < 1 \) there is an internal density distribution that realizes the lower bound in inequality (1.11), the “Thick shell Roche model”, and another that realizes \( \gamma/\gamma_1 = 1 \), the “Homogeneous core Roche model” (see Section 2.1.2 for details). There is always an internal density that realizes an intermediate value of \( \gamma/\gamma_1 \).}
In order to test the validity of equation (1.16), the values of $m$, $R$, $\epsilon(R)$, and $\Omega$ for the Earth, Mars, Jupiter, Saturn, Uranus, and Neptune, taken from the “NASA fact sheets”, were used to estimate $J_2 = (2\epsilon(R) - \zeta)/3$ and, by means of equation (1.16), $\frac{1}{mR^2}$. The result was compared to the values of $\frac{1}{mR^2}$ given in the same reference. The relative difference between both values varies from 0.7% (for the Earth) to 20% (for Uranus) being in the average 10%. Notice that within Clairaut’s theory the approximation $\gamma \approx \gamma_1$ for the gravitational modulus $\gamma$ is much better than that for $\frac{1}{mR^2}$ given in equation (1.16). This may be explained by the variational principle: $\gamma$ is the minimum value of a functional that has as its minimum point the the ellipticity (times an arbitrary nonnull constant) as a function of the radius. An error of order $\delta$ in the minimum point becomes an error of order $\delta^2$ in the minimum value $\gamma$.

2 Clairaut’s Equation

The main goal in this section is to review some results about Clairaut’s equation and to solve it in some special cases. The theory of figures of Clairaut is explained in detail in [23], [19], and [36].

The Clairaut’s equation (first obtained in 1743) describes the equilibrium configuration of a spherically symmetric self-gravitating celestial body made of an incompressible fluid. The body is supposed to be rotating steadily with uniform angular speed $\Omega$ about a fixed axis $e_3$ passing through its center of mass. In the rotating frame the configuration must satisfy the stationary Euler’s equation given by

$$\frac{1}{\tilde{\rho}} \nabla p = -\nabla [\Phi + \Phi_c] \tag{2.17}$$

where $x$ is the position vector, $\tilde{\rho}(x)$ is the density, $p(x)$ is the pressure, $\Phi_c = -\frac{\Omega^2}{2}(x_1^2 + x_2^2)$ is the potential of the centrifugal force and $\Phi$ is the gravitational potential given by

$$\Phi(x) = -G \int_{\mathbb{R}^3} \frac{\tilde{\rho}(\tilde{x})}{|x - \tilde{x}|} d\tilde{x} \tag{2.18}$$

Equation (2.17) shows that $\nabla \tilde{\rho} \times \nabla [\Phi + \Phi_c] = 0$, so the level sets of all three functions $\tilde{\rho}$, $\Phi + \Phi_c$, and $p$ coincide.

For $\Omega = 0$ equation (2.17) has a solution with spherical symmetry. For $\Omega > 0$ sufficiently small we may expect the existence of solutions with level sets which are approximately ellipsoids of revolution. More precisely, if for $\Omega = 0$ the radius of a spherical shell of constant density is $r > 0$, then for
Ω > 0 this shell becomes ellipsoidal and is given by

\[
\frac{x_1^2 + x_2^2}{r^2(1 + \epsilon/3)^2} + \frac{x_3^2}{r^2(1 - 2\epsilon/3)^2} = 1 \tag{2.19}
\]

where \(\epsilon(r) > 0\) is the small flattening of the ellipsoid, defined as the ratio (equatorial radius - polar radius)/(equatorial radius).

In spherical coordinates \((r, \theta, \phi)\), with polar axis given by \(e_3\), it can be shown that (see equation (B.128) in Appendix B)

\[
\Phi(r, \theta) = \Phi_0(r) + \Phi_2(r) P_2(\cos \theta) + \mathcal{O}(\epsilon^2)
\]

where:

\[
P_2(\cos \theta) = \frac{1}{2} (3 \cos^2 \theta - 1),
\]

\[
\Phi_0(r) = -4\pi G \left\{ \frac{1}{r} \int_0^r a^2 \rho(a) da + \int_r^\infty a \rho(a) da \right\},
\]

\[
\Phi_2(r) = -\frac{8\pi}{15} G \left\{ \frac{1}{r^3} \int_0^r a^5 \epsilon(a) \rho'(a) da + r^2 \int_r^\infty \epsilon(a) \rho'(a) da \right\},
\]

\(\rho\) is the spherically symmetric density function of the body at rest with \(\rho(r) = 0\) if \(r > R\), and a prime denotes a derivative with respect to \(r\). The gravitational potential external to the body can also be written as

\[
\Phi(r, \theta) = -\frac{Gm}{r} + J_{2v} \frac{GmR^2}{r^3} P_2(\cos \theta) + \mathcal{O}(r^{-4}). \tag{2.21}
\]

where

\[
J_{2v} = -\frac{8\pi}{15} \frac{1}{mR^2} \int_0^\infty a^5 \epsilon(a) \rho'(a) da. \tag{2.22}
\]

Suppose that

\[
\Omega^2 = \mathcal{O}(\epsilon) \quad \text{and} \quad \zeta = \frac{\Omega^2 R^3}{Gm} \quad \text{are small}. \tag{2.23}
\]

Under these hypotheses a nontrivial argument, where terms of order \(\epsilon^2\) are neglected (see, for instance, [8], [19], or [36]), shows that

\[
-\frac{2}{3} r \epsilon(r) \Phi_0'(r) + \Phi_2(r) + \frac{\Omega^2 r^2}{3} = 0
\]

This is Clairaut’s integral equation that can be written as

\[
4\pi G \left\{ \epsilon(r) r^2 \int_0^r a^2 \rho(a) da + \frac{1}{5} \int_0^r a^5 \epsilon(a) \rho'(a) da + \frac{r^5}{5} \int_r^\infty \epsilon(a) \rho'(a) da \right\} = \frac{\Omega^2 r^5}{2}. \tag{2.24}
\]
Clairaut’s equation can be presented in different forms, which are interesting for different reasons. Let
\[ \bar{\rho}(r) = \frac{3}{r^3} \int_0^r a^2 \rho(a) da \] (2.25)

be the mean density inside the spheroidal shell of radius \( r \) and
\[ K(a, r) = \rho'(a)\rho'(r)F(a, r), \quad \text{where} \]
\[ F(a, r) = \begin{cases} \frac{a^5}{5} & \text{if } a \geq r \\ \frac{r^5}{5} & \text{if } r \geq a, \end{cases} \] (2.26)

be a positive symmetric function. Note that
\[ \rho'(r) = \frac{3}{r} \left[ \rho(r) - \bar{\rho}(r) \right]. \] (2.27)

Equation (2.24) multiplied by \( \rho' \) can be written as
\[ 4\pi G \left\{ \frac{r^5\bar{\rho}(r)\rho'(r)}{3} \epsilon(r) + \int_0^\infty K(a, r) \epsilon(a) da \right\} = \Omega^2 \frac{r^5\rho'(r)}{2}, \] (2.28)

where we used
\[ \int_0^\infty K(a, r) \epsilon(a) da = \rho'(r) \left\{ \frac{1}{5} \int_0^r a^5 \epsilon(a) \rho'(a) da + \frac{r^5}{5} \int_r^\infty \epsilon(a) \rho'(a) da \right\} \] (2.29)

Integration by parts and differentiation of equation (2.24) with respect to \( r \) gives
\[ (r \epsilon' + 2 \epsilon) \bar{\rho}(r) - 3 \int_r^\infty \epsilon'(a) \rho(a) da = \frac{15}{8} \Omega^2 G. \] (2.30)

Further differentiation with respect to \( r \) gives the Clairaut’s differential equation
\[ r^2 \epsilon'' - 6 \epsilon + 6 \frac{\rho}{\bar{\rho}} (r \epsilon' + \epsilon) = 0 \] (2.31)

that can also be written as
\[ r \epsilon'' + 6 \epsilon' + 2 \frac{\rho}{\bar{\rho}} (r \epsilon' + \epsilon) = 0, \] (2.32)

where we used equation (2.27).

Many realistic models for \( \rho \) have points of discontinuity as, for instance, the “Preliminary Reference Earth Model” (PREM) [10]. In the following we
rewrite Clairaut’s equation (2.24) in a way which is convenient for working with densities that satisfy the following hypothesis:

\( \rho \) is: non-increasing; piecewise \( C^2 \) with finitely many points of discontinuities \( 0 < r_1 < r_2, \ldots, < r_n \leq R \); and \( \rho(r) = 0 \) for \( r > R \). \hspace{1cm} (2.33)

Let

\[
q(r) = \frac{1}{5} \int_r^\infty \rho'(a)\epsilon(a)da \\
\sigma(r) = \frac{1}{5r^5} \int_0^r a^5 \rho'(a)\epsilon(a)da + q(r),
\]

which together with equation (2.20) imply

\[
\Phi_2(r) = -\frac{8\pi}{3} Gr^2 \sigma(r). \hspace{1cm} (2.35)
\]

If \( \rho \) is \( C^2 \) then Clairaut’s equation (2.24) can be written as the following boundary value problem

\[
q' = -\frac{1}{5} \rho' \epsilon \\
\sigma' = -\frac{5}{r}(\sigma - q) \\
\epsilon(r) = \frac{3}{\rho(r)} \left( \frac{\Omega^2}{8\pi G} - \sigma(r) \right) \\
q(0) = \sigma(0) = q_0, \text{ such that } q(R) = 0,
\]

where \( q(0) = \sigma(0) = q_0 \) must be understood as \( \lim_{r \to 0} [q(r) - \sigma(r)] = 0 \) with \( r > 0 \). If \( \rho \) has points of discontinuity as in hypothesis (2.33), then let \( \chi_j \) be the density jump at \( r_j \),

\[
\chi_j = \rho(r_j) - \lim_{r \to (r_j)^-} \rho(r) < 0 \quad \text{for} \quad j = 1, \ldots, n, \hspace{1cm} (2.37)
\]

where \( r \to (r_j)^- \) denotes the limit as \( r \) tends to \( r_j \) with \( r < r_j \). The derivative of \( \rho \) at \( r_j \), in distribution sense, is \( \rho'(r) = \chi_j \delta(r - r_j) \) where \( \delta \) is the Dirac \( \delta \)-measure. Using this fact and integrating equations (2.36) in a neighborhood of \( r_j \) we obtain the following jump conditions at \( r_j \):

\[
\Delta q(r_j) = q(r_j) - \lim_{r \to (r_j)^-} q(r) = -\frac{\chi_j}{5} \epsilon(r_j) \\
\Delta \sigma(r_j) = \sigma(r_j) - \lim_{r \to (r_j)^-} \sigma(r) = 0 \hspace{1cm} (2.38)
\]
Equations (2.36) plus the jump conditions (2.38) entirely determine the solution to Clairaut’s equation (2.24). We remark that $\sigma$ and $\epsilon$ are continuous functions even when $\rho$ is not and that if $r_n = R$ is a point of discontinuity, then $q(R) = 0$ in equation (2.36) must be understood as

$$0 = q(R) = \Delta q(r_n) + \lim_{r \to (r_j)^-} q(r) = -\frac{\chi_n}{3} \epsilon(r_n) + \lim_{r \to (r_j)^-} q(r)$$

Equations (2.35), (2.36), and (2.22) imply

$$J_{2v} = -2 \frac{\sigma(R)}{\rho(R)} = -\frac{8\pi R^3}{3m} \sigma(R),$$

and, the well known relation,

$$J_{2v} = \frac{1}{3} \left( 2\epsilon(R) - \frac{\Omega^2 R^3}{Gm} \right),$$

In order to solve the boundary value problem in equation (2.36) it is convenient to further change of variables as

$$w = q - \frac{\Omega^2}{8\pi G}, \quad y = \sigma - \frac{\Omega^2}{8\pi G}$$

Then equation (2.36) becomes

$$w' = 3 \frac{\rho'}{5 \rho} y$$

$$y' = -\frac{5}{r} (y - w)$$

$$\epsilon(r) = -\frac{3}{\rho(r)} y,$$

the boundary conditions become

$$w(0) = y(0) = w_0, \quad w(R) = -\frac{\Omega^2}{8\pi G},$$

and the jump conditions become

$$\Delta w(r_j) = 3 \frac{\chi_j}{5 \rho(r_j)} y(r_j)$$

$$\Delta y(r_j) = 0.$$
Proposition 2.1. Suppose that $\rho$ satisfies hypothesis (2.33). Then, for $\Omega > 0$ there exists a unique bounded solution to equation (2.24) (and therefore to problems (2.36) and (2.43)). This solution is strictly positive, non-decreasing, and $C^1$. For $\Omega = 0$ the only solution to equation (2.24) is $\epsilon(r) = 0$, $r \geq 0$.

The following algorithm allows for the solution to equation (2.24). Let $(\hat{w}, \hat{y})$ be the solution to the differential equation (2.43) with the initial condition $\hat{w}(0) = \hat{y}(0) = 1$ and the jump conditions (2.45). We claim that $\hat{w}(R) \neq 0$. Indeed, if $\hat{w}(R) = 0$ then $(\hat{y}, \hat{w})$ satisfies the boundary conditions (2.44) with $\Omega = 0$ and the corresponding $\epsilon$ solves equation (2.24) with $\Omega = 0$, which is impossible due to Proposition 2.1. The desired solution $(w, y)$ to the boundary value problem in equations (2.43), (2.44), and (2.45) satisfies the initial condition

$$w(0) = y(0) = -\frac{\Omega^2}{8\pi G \hat{w}(R)}$$

(2.46)

2.1 N-layer models

2.1.1 Piecewise constant density functions

In this paragraph we consider functions $\rho$ of the following form

$$\rho(r) = \begin{cases} 
\rho_0 = \text{constant} > 0 & \text{for } 0 = r_0 \leq r < r_1 \\
\rho_1 = \text{constant} > 0 & \text{for } r_1 \leq r < r_2 \\
\vdots \\
\rho_{n-1} = \text{constant} > 0 & \text{for } r_{n-1} \leq r < r_n = R \\
\rho_n = 0 & \text{for } R \leq r
\end{cases}$$

(2.47)

In this case the solution to equation (2.36) is continuously differentiable in each interval $r_{j-1} \leq r < r_j$, $j = 1, \ldots, n$. Using the definitions

$$q_j = q(r_j), \quad \sigma_j = \sigma(r_j), \quad \epsilon_j = \epsilon(r_j), \quad j = 0, \ldots, n,$$

and the initial conditions $q(0) = \sigma(0) = q_0$, we can write the solution to equation (2.36) as

$$q(r) = q_0, \quad \sigma(r) = q_0, \quad \epsilon(r) = \frac{3}{\rho_0} \left( \frac{\Omega^2}{8\pi G} - q_0 \right) \text{ for } 0 \leq r < r_1$$

(2.48)

The jump conditions at equation (2.38) imply

$$q_1 = q_0 - \frac{\rho_1 - \rho_0}{5} \epsilon_1$$

$$\sigma_1 = q_0$$

$$\epsilon_1 = \frac{3}{\rho_0} \left( \frac{\Omega^2}{8\pi G} - q_0 \right)$$

(2.49)
For \( r_j \leq r < r_{j+1} \), \( j = 1, \ldots n - 1 \), the solution to equation (2.36) can be written as

\[
q(r) = q_j \\
\sigma(r) = \sigma_j \frac{r^5_j}{r^5} + q_j \left(1 - \frac{r^5_j}{r^5}\right) \\
\epsilon(r) = \frac{3}{\rho(r)} \left(\frac{\Omega^2}{8 \pi G} - \sigma(r)\right)
\] (2.50)

Using the jump conditions in equation (2.38) we obtain that, for \( j = 2, \ldots n \),

\[
q_j = q_{j-1} - \frac{\rho_j - \rho_{j-1}}{5} \epsilon_j \\
\sigma_j = \sigma_{j-1} \frac{r^5_{j-1}}{r^5_j} + q_{j-1} \left(1 - \frac{r^5_{j-1}}{r^5_j}\right) \\
\epsilon_j = \frac{3}{\rho(r_j)} \left(\frac{\Omega^2}{8 \pi G} - \sigma_j\right)
\] (2.51)

For numerical computations it is convenient to add the following recursion relation for \( \rho(r_j) \) defined as

\[
\overline{\rho}_j = \rho_{j-1} + \frac{r^3_{j-1}}{r^3_j} (\overline{\rho}_{j-1} - \rho_{j-1}), \quad j = 1, \ldots, n, \quad \text{with} \quad \overline{\rho}_0 = \rho_0
\] (2.52)

Finally, the boundary condition \( q(R) = 0 \) implies

\[
q_n = 0
\] (2.53)

and equation (2.40) implies

\[
J_{2v} = -2 \frac{\sigma_n}{\rho_n} = -\frac{8 \pi R^3}{3 m} \sigma_n
\] (2.54)

The solution to this boundary value problem can be easily obtained using the transformation (2.42) and the initial condition (2.46). The method just presented is well known in geophysics as the “propagation matrix method”.

### 2.1.2 Two-layer models and generalized “Roche models”

Consider a density function with just two different values with \( x = r_1/R \) denoting the internal point of discontinuity. Then, after using a computer algebra system, we obtain

\[
J_{2v} = -2 \frac{\sigma_2}{\rho(R)} = \frac{\Omega^2 R^3}{G m} \left(\frac{\rho_1 (2 \rho_0 + 3 \rho_1) + (\rho_0 - \rho_1)(5 \overline{\rho}(R) + 3 \rho_1)x^5}{5 \overline{\rho}(R) - 3 \rho_1 (2 \rho_0 + 3 \rho_1) - 9 \rho_1 (\rho_0 - \rho_1)x^5}\right)
\] (2.55)
This expression, equation (1.7) and the definition \( \gamma_1 = (4/5)(Gm/R_1^3) \) imply
\[
\frac{\gamma}{\gamma_1} = \frac{1}{2} \left( \frac{R_1}{R} \right)^5 \frac{(5\overline{\rho}(R) - 3\rho_1)(2\rho_0 + 3\rho_1) - 9\rho_1(\rho_0 - \rho_1)x}{\rho_1(2\rho_0 + 3\rho_1) + (\rho_0 - \rho_1)(5\overline{\rho}(R) + 3\rho_1)x} \tag{2.56}
\]
This expression can be further simplified using the relations
\[\alpha = \frac{\rho_1}{\rho_0} \in (0, 1)\quad \text{and} \quad \overline{\rho}(R) = \rho_1 + x^3(\rho_0 - \rho_1),\]
the result is
\[
\frac{\gamma}{\gamma_1} = \frac{1}{2} \left( \frac{R_1}{R} \right)^5 \frac{3\alpha^2 (3x^5 - 5x^3 + 2) + \alpha (-9x^5 + 5x^3 + 4) + 10x^3}{\alpha^2 (5x^8 - 8x^5 + 3) + \alpha (-10x^8 + 8x^5 + 2) + 5x^8} \tag{2.57}
\]
where
\[
\frac{R_1}{R} = \sqrt{\frac{x^5 + x^3(1 - x)}{x^3 + x(1 - x^3)}} \tag{2.58}
\]
or
\[
\alpha = \frac{\left( \frac{R_1}{R} \right)^2 x^3 - x^5}{\left( \frac{R_1}{R} \right)^2 x^3 - x^5 + 1 - \left( \frac{R_1}{R} \right)^2} \tag{2.59}
\]
Notice that the condition \( 0 < \alpha \leq 1 \) and equation (2.59) imply that
\[
0 < x \leq \frac{R_1}{R} \tag{2.60}
\]
Two interesting limits of two layer bodies are discussed in the following. These limits are generalizations of the usual Roche model, which consists of a point mass surrounded by a medium so rarefied that its mass can be neglected.

**Homogeneous core Roche model:** is the body obtained as \( \alpha \to 0 \) while \( x \) remains fixed. So the limit body is just a homogeneous body of density \( \rho_0 \) and radius \( R_1 \) surrounded by a rarefied layer of thickness \( R - R_1 \) of negligible density. The family of homogeneous core Roche models is parameterized by \( x = R_1/R \in (0, 1) \) and, for all \( x \), \( \gamma/\gamma_1 = 1 \) in agreement with equations (2.58) and (2.57). The fact that for all \( x \), \( \gamma/\gamma_1 = 1 \), follows from our definition of rigidity that is associated with the dynamical flattening \((C - I_0)/I_0\) and not to the geometrical one \((a - c)/a\), where \((a, b, c)\) are the semi-axes of the ellipsoidal shape of the body. Therefore, the massless atmosphere has no dynamical effect.

**Thick shell Roche model:** The second limit is more interesting and occurs as \( x \to 0 \) while \( R_1/R \) remains fixed. In this case equation (2.59) implies
\[
\alpha = \frac{\left( \frac{R_1}{R} \right)^2 x^3 + O(x^5)}{1 - \left( \frac{R_1}{R} \right)^2 x^3} \tag{2.61}
\]
and equation (2.57) implies
\[
\lim_{x \to 0} \frac{\gamma}{\gamma_1} = \frac{1}{2} \left( \frac{R_1}{R} \right)^3 \left[ 5 - 3 \left( \frac{R_1}{R} \right)^2 \right]
\] (2.62)

The equation for the moment of inertia, \( I_0 = \frac{8\pi}{3} \int_0^R r^4 \rho(r) ds \), implies
\[
\overline{\rho}(R) \left( \frac{R_1}{R} \right)^2 = \rho_1 + x^5(\rho_0 - \rho_1),
\]
and from this equation and equation (2.61) follow
\[
\rho_1 \to \overline{\rho}(R) \left( \frac{R_1}{R} \right)^2 \quad \text{as} \quad x \to 0
\]
and
\[
\rho_0 = \frac{\rho_1}{\alpha} = \frac{\overline{\rho}(R)}{x^3} \left[ 1 - \left( \frac{R_1}{R} \right)^2 \right] + \mathcal{O}(x^{-1})
\]

So, if \( R_1/R < 1 \), the limit body obtained as \( x \to 0 \) represents a family of “Roche models” that consists of a point at the origin with mass \( m_0 = [1 - (R_1/R)^2]m \) and a surrounding homogeneous layer with mass \( m_1 = (R_1/R)^2m \). The sum \( m_0 + m_1 = m \) gives the total mass of the body and the moment of inertia is only due to the homogeneous layer with \( I_0 = 0.4 m_1 R^2 = 0.4 mR_1^2 \). Notice that the limit as \( R_1/R \to 0 \) represents the actual Roche model where the surrounding medium is so rarefied that the moment of inertia of the body can be neglected.

### 2.2 Polytropic models

The interior density distribution of stars and fluid planets can be easily determined under the hypothesis that they are made of a polytropic fluid such that pressure depends upon density as
\[
p = K \rho^{1+1/n}
\] (2.63)

where \( K > 0 \) is a constant and \( n \) is the polytropic index. For \( \Omega = 0 \), equation (2.17) for the hydrostatic equilibrium becomes
\[
\frac{p'}{\rho} = -\Phi' = -\frac{4\pi G}{r^2} \int_0^r a^2 \rho(a) da
\] (2.64)

where we used the expression for \( \Phi_0 \) given in equation (2.20). If we write
\[
\rho(r) = \lambda \theta^n(r),
\] (2.65)
where $\lambda > 0$, then equations (2.63) and (2.64) imply that $\theta$ must satisfy

$$\frac{K(n+1)\lambda^{1/n-1}}{4\pi G} \left( \frac{r^2 \theta'}{r^2} \right)' = -\theta^n$$

If a new spatial variable $\xi$ is defined as

$$r = \left( \frac{K(n+1)\lambda^{1/n-1}}{4\pi G} \right)^{1/2} \xi$$

then we obtain the so called Lane-Emden equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = \frac{(\xi^2 \theta')'}{\xi^2} = -\theta^n$$

The initial conditions are: $\theta(0) = 1$, which is just a normalization such that $\lambda = \rho(0)$, and $\theta'(0) = 0$, which is due to the regularity of the density at $r = 0$. For $0 < n < 5$ the Lane-Emden equation has a solution $\theta$ that reaches zero at a finite value of $\xi$ denoted as $\xi_1$. So, for each value of $n \in (0, 5)$ the solution to the Lane-Emden equation, which is referred as a polytrope, defines a body of radius

$$R = \left( \frac{K(n+1)\lambda^{1/n-1}}{4\pi G} \right)^{1/2} \xi_1 \implies r = \frac{R}{\xi}$$

Notice that, given $\lambda$ and $n$, $R$ is determined by $K$. The mass of the body is given by

$$m = 4\pi \int_0^R \rho(r)r^2dr = 4\pi \lambda \frac{R^3}{\xi_1^3} \int_0^{\xi_1} \theta^n(\xi)\xi^2d\xi$$

and the moment of inertia of the body is given by

$$I_o = \frac{8\pi}{3} \int_0^R \rho(r)r^4dr = \frac{8\pi}{3} \lambda \frac{R^5}{\xi_1^5} \int_0^{\xi_1} \theta^n(\xi)\xi^4d\xi$$

Equations (2.69) and (2.70) and the relation $I_o/(mR^2) = 0.4(R_1/R)^2$ imply

$$\left( \frac{R_1}{R} \right)^2 = \frac{5}{3} \frac{1}{\xi_1^2} \frac{\int_0^{\xi_1} \theta^n(\xi)\xi^4d\xi}{\int_0^{\xi_1} \theta^n(\xi)\xi^2d\xi}$$

This equation shows that $R_1/R$ is determined entirely by the polytropic index, it neither depends on $\lambda = \rho(0)$ nor on $K$. Figure which was obtained from the numerical integration of the Lane-Emden equation, shows that the relation of $R_1/R$ and $n$ is one-to-one and almost affine.
Figure 1: Graph of the polytropic index $n$ of a polytrope as a function of $R_I/R$ (left vertical axis) and graph of the ratio $\gamma/\gamma_1$ of a polytrope as a function of $R_I/R$ (right vertical axis).

Given a density distribution $\rho$ determined by a polytropic fluid it is possible to numerically integrate Clairaut’s equation to obtain the ratio $\gamma/\gamma_1$. As said in the Introduction, $\gamma$ is invariant under homothetic transformations that preserve the density, so $\gamma$ can be computed using a body of radius $R = \xi_1$. Moreover, $\gamma/\gamma_1$ is additionally invariant under the multiplication of $\rho$ by a constant (see, for instance, equation (3.78)), so $\gamma/\gamma_1$ does not depend on $\lambda$ either, and we obtain that $\gamma/\gamma_1$ is determined entirely by the polytropic index $n$. Since there is a one-to-one correspondence between $n$ and $R_I/R$, the value of $\gamma/\gamma_1$ for bodies modeled by polytropes is fully determined by the ratio $R_I/R$. The graph of $\gamma/\gamma_1$ as a function of $R_I/R$ is shown in Figure 1.
2.3 The Darwin-Radau approximation

The Darwin-Radau approximation consists of the substitution $\eta = \frac{r}{\epsilon} \epsilon'$ into equation (2.32) that leads to the equation

$$\frac{d}{dr} \left[ \bar{p} r^{5} (1 + \eta)^{1/2} \right] = 5 \bar{p} r^{4} \psi(\eta),$$

(2.72)

where

$$\psi(\eta) = \frac{1 + \eta/2 - \eta^2/10}{(1 + \eta)^{1/2}} \approx 1.$$  

The last approximation is based on the empirical fact that for any planet in the solar system the maximum difference $|\psi(\eta) - 1|$ is 0.026 ([8] pg 81). Equation (2.72) with $\psi(\eta) = 1$ can be explicitly integrated

$$(1 + \eta)^{1/2} = \frac{5}{\bar{p} r^{5}} \int_{0}^{r} \bar{p}(a) a^{4} da.$$  

(2.73)

From equation (2.27) we obtain

$$I_0 = \frac{8\pi}{3} \int_{0}^{\infty} r^{4} \rho(r) dr = \frac{8\pi}{9} \left( \bar{p}(R) R^{5} - 2 \int_{0}^{R} r^{4} \bar{p}(r) dr \right)$$

and from equation (2.30)

$$R \epsilon'(R) + 2 \epsilon(R) = \frac{15 \Omega^2}{8\pi G \bar{p}(R)}.$$  

From the last three equations we get

$$\epsilon(R) = \frac{5}{2} \frac{\Omega^2 R^3}{G m} \left( 1 + \left[ \frac{5}{2} \left( 1 - \frac{3 I_0}{2 m R^2} \right) \right]^{2} \right)^{-1},$$

and, using equation (2.41), we obtain

$$J_{2v} = \frac{5}{3} \frac{\Omega^2 R^3}{G m} \left( \left\{ 1 + \left[ \frac{5}{2} \left( 1 - \frac{3 I_0}{2 m R^2} \right) \right]^{2} \right\}^{-1} - \frac{1}{5} \right).$$  

(2.74)

Finally, using equations (1.7), (1.2), and (1.3) we obtain the Darwin-Radau approximation for $\gamma$:

$$\frac{\gamma}{\gamma_1} = \frac{3}{2} \left( \frac{R_1}{R} \right)^{5} \frac{1 + \left[ \frac{5}{2} - \frac{3}{2} \left( \frac{R_1}{R} \right)^{2} \right]^{2}}{4 - \left[ \frac{5}{2} - \frac{3}{2} \left( \frac{R_1}{R} \right)^{2} \right]^{2}}.$$  

(2.75)

Notice that if $R_1/R = 1 - \delta$ then the above formula gives $\gamma/\gamma_1 = 1 + \mathcal{O}(\delta^3)$, which is equation (1.10) in the Introduction.
3 A variational principle for the gravitational modulus $\gamma$.

In this section, until otherwise stated, we suppose that

$$\rho \in C^2[0, \infty), \ \rho'(r) < 0 \text{ for } 0 < r < R, \text{ and } \rho(r) = 0 \text{ for } R \leq r. \quad (3.76)$$

The gravitational modulus $\gamma$ has the following integral characterization. Multiply equation (2.28) by $\epsilon(r)$, integrate over $[0, R]$, and use equations (1.7) and (2.22) to obtain:

$$\gamma_I = -15G \int_0^\infty a^5 \rho(a) \rho'(a) \epsilon(a)^2 / 3 da + \int_0^\infty \int_0^\infty K(a, s) \epsilon(s) \epsilon(a) dsda \quad (3.77)$$

Moreover, using that $\gamma_I = 2I_G R^5 I_0$ and rescaling the space variables in the above integrals by $R$, we obtain:

$$\gamma = -\frac{15}{2} \frac{R^5}{R_1^5} \int_0^1 a^5 \rho'(a) \epsilon(a)^2 / 3 da + \int_0^1 \int_0^1 K(a, s) \epsilon(s) \epsilon(a) dsda \quad (3.78)$$

This expression shows that the nondimensional ratio $\gamma/\gamma_1$ depends on the three variables $m, I_0$, and $R$ exclusively by means of the ratio $R_1/R$. In particular, the multiplication of the density function of the body by a constant does not change the value of $\gamma/\gamma_1$.

Let $L^2_\rho = L^2([0, R], \rho)$ be the weighted space of Lebesgue square integrable functions with inner product given by

$$\langle f, g \rangle_\rho = -\int_0^R f(r) g(r) r^5 \rho'(r) dr \quad (3.79)$$

Let $N$ and $M$ be the operators on $L^2_\rho$ defined by

$$N[\epsilon](r) = -\left\{ \frac{1}{5r^5} \int_0^r a^5 \epsilon(a) \rho'(a) da + \frac{1}{5} \int_r^R \epsilon(a) \rho'(a) da \right\}$$

$$M[\epsilon](r) = -\frac{1}{r^5} \rho'(r) \int_0^r K(a, r) \epsilon(a) da$$

$$\quad + \frac{7}{3} \epsilon(r) \quad (3.80)$$

Using these operators, Clairaut’s equation (2.28) can be written as

$$M[\epsilon](r) - N[\epsilon](r) = \frac{\Omega^2}{8\pi G} \quad (3.81)$$
Consider the functional $V : L^2_\rho \to \mathbb{R}$ given by

$$V(u) = -\int_0^R \frac{a^5 \rho'(a) \rho'(a)}{3} u^2(a) da - \int_0^R \int_0^R K(a,s) u(s) u(a) ds da$$

$$= \langle u, Mu \rangle_\rho - \langle u, Nu \rangle_\rho$$

(3.82)

$$= \frac{3}{8\pi G} \int_0^R \left[ \Phi_2(s) - \Phi_0(s) \frac{2}{3} s u(s) \right] u(s) s^4 \rho'(s) ds$$

where $\Phi_0$ and $\Phi_2$ are the functions defined in equation (2.20) with $u$ replacing $\epsilon$ in the definition of $\Phi_2$.

**Lemma 3.1.** Suppose that $\rho$ satisfies hypothesis (3.76). Then, $V(u) > 0$ for all $u \in L^2_\rho$, $u \neq 0$.

**Proof.** The proof of the lemma requires the following.

**Proposition 3.1.** There exists a strictly positive function $u_1$ in $L^2_\rho$ and a positive number $\lambda_1$, they are solutions to the eigenvalue problem $N u_1 = \lambda_1 M u_1$, such that for any $u \in L^2_\rho$, with $u \neq u_1$, the following inequality holds

$$\frac{\langle u, Nu \rangle_\rho}{\langle u, Mu \rangle_\rho} < \frac{\langle u_1, Nu_1 \rangle_\rho}{\langle u_1, Mu_1 \rangle_\rho} = \lambda_1$$

**Proof.** Consider a new inner product defined by

$$\langle f, g \rangle_M = \langle f, Mg \rangle_\rho = -\int_0^R f(r) g(r) \frac{r^5 \rho'(r) \bar{\rho}(r)}{3} dr$$

Since $M[\epsilon](r) = \frac{\rho(r)}{3} \epsilon(r)$ and $\bar{\rho}(r) \geq \bar{\rho}(R) > 0$ the space $L^2_\rho$ with this new inner product is also a Hilbert space that will be denoted as $L^2_M$.

Note that $M$, defined on $L^2_M$, is invertible with inverse $M^{-1} u = 3u/\bar{\rho}$. We define a new operator on $L^2_M$ as $A = M^{-1} N$. In order to write $A$ more explicitly, it is convenient to rewrite the function $K$ given in equation (2.26) as

$$K(a,r) = a^5 \rho'(a) r^5 \rho'(r) P(a,r)$$

where

$$P(a,r) = \frac{F(a,r)}{a^5 r^5}$$

Notice that $P(a,r) = P(r,a)$ and

$$P(a,r) = \begin{cases} \frac{1}{5r^5} & \text{if } r \geq a, \end{cases}$$
which implies that \( P \) is continuous for \((a,r) \neq (0,0)\). Notice that

\[
A[u](r) = M^{-1}Nu(r) = -\frac{3}{p(r)} \int_0^R P(a,r)u(a)a^5\rho'(a)da \\
= -\int_0^R Q(a,r)u(a)\frac{\rho(a)}{3}a^5\rho'(a)da
\]

where \( Q(a,r) \) is the symmetric positive function

\[
Q(a,r) = \frac{3}{p(r)}\frac{3}{p(a)}P(a,r).
\]

This implies that \( A \) is symmetric, \( \langle u, Av \rangle_M = \langle v, Au \rangle_M \), and

\[
\int_0^R \int_0^R Q^2(a,r)\frac{\rho(a)}{3}a^5\rho'(a)\frac{\rho(r)}{3}r^5\rho'(r)dadr \\
= \int_0^R \int_0^R P^2(a,r)a^5\rho'(a)r^5\rho'(r)dadr \\
= 2\int_0^R \int_0^r P^2(a,r)a^5\rho'(a)da r^5\rho'(r)dr \\
= 2\int_0^R \int_0^r a^5\rho'(a)da \frac{1}{25r^5}\rho'(r)dr \leq \frac{R^2}{150}\|\rho'\|_\infty^2
\]

Therefore, by theorem VI-23 in [27], \( A \) is a Hilbert-Schmidt operator in \( L^2_M \) which implies that it is bounded and compact. Since \( Q(a,R) > 0 \), the operator \( A \) is a strongly positive operator in the sense that it maps a non-negative continuous function that is not identically equal to zero to a strictly positive function. All these properties imply that the Krein-Rutman theorem (Theorem 7.C in [34]) can be used to show the existence of a unique maximal eigenvalue \( \lambda_1 > 0 \) associated to a unique positive eigenfunction \( u_1 \) such that \( Au_1 = \lambda_1 u_1 \). The maximal eigenvalue of a Hilbert-Schmidt operator satisfies the inequalities:

\[
\frac{\langle u, Au \rangle_M}{\langle u, u \rangle_M} < \frac{\langle u_1, Au_1 \rangle_M}{\langle u_1, u_1 \rangle_M} = \lambda_1, \quad u \neq u_1
\]

Then the proposition follows from: \( A = M^{-1}N \), \( \langle f, g \rangle_M = \langle f, Mg \rangle_\rho \), and that a function is in \( L^2_M \) if and only if it is in \( L^2_\rho \). \( \square \)

We return to the proof of lemma 3.1. Let \( \epsilon \) be the solution to equation (2.28) given in proposition 2.1. Multiplying both sides of equation (2.28) by
and integrating over \([0, R]\) we obtain

\[-\frac{\Omega^2}{8\pi G} \int_0^\infty u_1(a)a^5 \rho'(a) da = \{\langle u_1, M\epsilon \rangle_\rho - \langle u_1, N\epsilon \rangle_\rho \}
= \{\langle u_1, M\epsilon \rangle_\rho - \langle Nu_1, \epsilon \rangle_\rho \} = -(1 - \lambda_1) \int_0^\infty u_1(a) \frac{a^5 \rho'(a) \rho_\epsilon}{3} \epsilon(a) da \]

So, using that \(\epsilon\) and \(u_1\) are positive, we obtain

\[1 - \lambda_1 = \frac{3\Omega^2}{8\pi G} \int_0^\infty u_1(a)a^5 \rho'(a) da > 0. \tag{3.83}\]

So, for any \(u \in L^2_\rho\),

\[V(u) = \langle u, Mu \rangle_\rho - \langle u, Nu \rangle_\rho = \lambda_1 \langle u, Mu \rangle_\rho - \langle u, Nu \rangle_\rho + (1 - \lambda_1) \langle u, Mu \rangle_\rho .\]

Using that \(\lambda_1 \langle u, Mu \rangle - \langle u, Nu \rangle_\rho \geq 0\) we get, for any \(u \in L^2_\rho[0, R]\),

\[V(u) \geq (1 - \lambda_1) \int_0^\infty \frac{\epsilon^5 \rho(r) \rho'(r)}{3} u^2(r) dr > 0, \tag{3.84}\]

where \(1 - \lambda_1 > 0\) is given in equation (3.83). This finishes the proof of the lemma.

The positivity of the functional \(V\), given in lemma 3.1, implies that the solution to Clairaut’s equation (3.81) has the following variational characterization. The operator

\[P[u] = (M - N)[u], \quad u \in L^2_\rho \tag{3.85}\]

is positive because \(V(u) = \langle u, Pu \rangle_\rho > 0\) if \(u \neq 0\). Clairaut’s equation (3.81) can be written as

\[P[\epsilon] = \frac{\Omega^2}{8\pi G} \tag{3.86}\]

and, up to a multiplicative constant, the solution to this equation is given by the variational principle (see [31] equation 3.82):

\[\max_{u \in L^2_\rho} \frac{\langle 1, u \rangle_\rho^2}{\langle u, Pu \rangle_\rho} = \max_{u \in L^2_\rho} \frac{\langle 1, u \rangle_\rho^2}{V(u)} = \frac{\langle 1, \epsilon \rangle_\rho^2}{V(\epsilon)}, \tag{3.87}\]

where the 1 inside brackets means the constant function equal to one. This variational characterization and equation (3.77) imply

\[\gamma = 15GI, \min_{u \in L^2_\rho} \frac{V(u)}{\langle 1, u \rangle_\rho^2} = 15GI, \frac{V(\epsilon)}{\langle 1, \epsilon \rangle_\rho^2} \tag{3.88}\]
and, using \( \gamma_1 = (4/5)(Gm/R_1^3) \),

\[
\frac{\gamma}{\gamma_1} = \frac{15}{2} R_1^5 \min_{u \in L^2_\rho} \langle 1, u \rangle_\rho^2 = \frac{15}{2} R_1^5 \frac{V(\epsilon)}{\langle 1, \epsilon \rangle_\rho^2}
\]

(3.89)

In Appendix B it is shown that

\[
V(u) = \frac{45}{32\pi^2 G} U(u) \quad \text{and} \quad \langle 1, u \rangle_\rho = -\frac{45}{8\pi^2 \Omega^2} U_c(u)
\]

(3.90)

where

\[
U(u) = \frac{1}{2} \int_{\mathbb{R}^3} \tilde{\rho}(x) \Phi(x) dx \quad \text{and} \quad U_c(u) = \int_{\mathbb{R}^3} \tilde{\rho}(x) \Phi_c(x) dx
\]

are the gravitational energy (at second order in the deformation \( u \)) and the centrifugal energy (at first order in \( u \)), respectively. So, the variational principle in (3.88) can be rephrased in terms of the gravitational energy and the centrifugal energy.

The variational principle (3.88) determines the solution to Clairaut’s equation up to a multiplicative constant. There is another variational principle (see [31] equation 3.80), closely related to (3.88), that fully determines the solution to Clairaut’s integral equation: \( \epsilon \) is a solution to equation (3.86) if and only if it minimizes

\[
E[u] = U[u] + U_c[u]
\]

(3.92)

3.1 Discontinuous mass distributions with \( \rho'(r) \leq 0 \)

The mass distribution of the “Preliminary Reference Earth Model” (PREM) [10] does not satisfy hypotheses (3.76). From a physical perspective this may not be relevant because the density distribution of the PREM can be arbitrarily well approximated (pointwise and in the \( L^2 \) sense) by densities which are smooth and strictly decreasing in the interval \((0, R)\). In this section

\[\text{The considerations in the following paragraphs and in Appendix B are due to one of the referees of the paper.}\]
we exhibit an approximation scheme that allows for the use of the variational principle in equation (3.88) when \( \rho : \mathbb{R} \to [0, \infty) \) satisfies the hypotheses in equation (2.33). It is convenient to extend \( \rho \) to \( r < 0 \) as an even function. We will regularize \( \rho \) using a mollifier. Let \( f : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) positive even function, with support in the interval \([-1, 1]\), with \( \int_{\mathbb{R}} f(r)dr = 1 \), and such that \( f'(r) < 0 \) for \( 0 < r < 1 \). For a given small \( \tau > 0 \), let \( g_\tau : \mathbb{R} \to \mathbb{R} \) be the function \( g_\tau(r) = \tau(R - |r|) \) for \( |r| \leq R \) and \( g_\tau(r) = 0 \) for \( |r| > R \). A regularized density function \( \rho_\tau \) is defined as

\[
\rho_\tau(r) = \frac{1}{\tau} \int_{-\infty}^{\infty} f\left(\frac{(r - a)/\tau}{\rho(a) + g_\tau(a)}\right) da
\]

For any \( \tau > 0 \), the function \( \rho_\tau \) is: \( C^\infty \), positive, and even. We claim that \( \rho'_\tau < 0 \) if \( 0 < r < R + \tau \). We split the proof of the claim into two parts: \( 0 < \tau \leq r \) and \( 0 < r < \tau \). For \( 0 < \tau \leq r \) the derivative of \( \rho_\tau \) is

\[
\rho'_\tau(r) = \frac{1}{\tau^2} \int_{-\infty}^{\infty} f'(\frac{(r - a)/\tau}{\rho(a) + g_\tau(a)}) [\rho(a) + g_\tau(a)] da
\]

\[
= \frac{1}{\tau^2} \int_{r-\tau}^{r} f'(\frac{(r - a)/\tau}{\rho(a) + g_\tau(a)}) [\rho(a) + g_\tau(a)] da
\]

\[
+ \frac{1}{\tau^2} \int_{r}^{r+\tau} f'(\frac{(r - a)/\tau}{\rho(a) + g_\tau(a)}) [\rho(a) + g_\tau(a)] da.
\]

Since: \( f' \) is odd, \( f'(\frac{(r - a)/\tau}{\rho(a) + g_\tau(a)}) > 0 \) \( (0 < \tau < R + \tau) \) \( (r - \tau < a < r) \), and \( \rho(a) + g_\tau(a) \) is positive and strictly decreasing for \( 0 < a < R \), the first term in the right hand side of the equation above is negative and it is larger in absolute value than the second term, which is positive. Therefore \( \rho'_\tau(r) < 0 \) for \( 0 < \tau \leq r < R + \tau \). For \( 0 < r < \tau \) the derivative of \( \rho_\tau \) is

\[
\rho'_\tau(r) = \frac{1}{\tau^2} \int_{0}^{2r} f'(\frac{(r - a)/\tau}{\rho(a) + g_\tau(a)}) da
\]

\[
+ \frac{1}{\tau^2} \int_{2r}^{r+\tau} f'(\frac{(r - a)/\tau}{\rho(a) + g_\tau(a)}) da
\]

\[
+ \frac{1}{\tau^2} \int_{r-\tau}^{0} f'(\frac{(r - a)/\tau}{\rho(a) + g_\tau(a)}) da.
\]

The integral in the first line is negative by the same argument given in the previous case. If we change the variables of the third integral as \( a \to 2r - a \), then the sum of the second and third integrals can be written as

\[
\frac{1}{\tau^2} \int_{2r}^{r+\tau} f'(\frac{(r - a)/\tau}{\rho(a) + g_\tau(a)}) \left\{ [\rho(a) + g_\tau(a)] - [\rho(a - 2r) + g_\tau(a - 2r)] \right\} da,
\]
where we used that $f'$ is odd and $\rho + g_\tau$ is even. This integral is negative because $f'((r-a)/\tau) > 0$ for $2r < a < r + \tau$ and $\rho(a) + g_\tau(a)$ is strictly decreasing for $0 < a < R$. As $\tau \to 0$ the mollifier $f(r/\tau)/\tau$ tends to the Dirac-$\delta$ distribution and therefore $\rho_\tau(r) \to \rho(r)$ and $\rho'_\tau(r) \to \rho'(r)$ whenever $r$ is a point of continuity of $\rho$. At a point of discontinuity $r_j$, $\rho_\tau(r) \to \chi_j \delta(r - r_j)$ where $\delta$ is the Dirac $\delta$-measure and $\chi_j$ is the density jump defined in equation (2.37).

For a given $\rho$ satisfying hypotheses (2.33), consider the functional $V$ defined on equation (3.82) restricted to the space of continuous functions on $[0, R]$. At a point of discontinuity of $\rho$, $\rho'$ must be understood as a $\delta$-distribution (what explains the necessity of restricting $V$ to the space of continuous functions). Let $V_\tau$ be the same functional defined using the regularized density $\rho_\tau$. For a given $u$, standard arguments in the theory of distributions show that $\lim_{\tau \to 0} V_\tau(u) \to V(u)$. By lemma 3.1 $V_\tau(u) > 0$ that implies the following.

**Theorem 3.1.** Suppose that the density distribution $\rho$ satisfies hypotheses (2.33) and $u$ is a continuous function on $[0, R]$. Then $V(u) \geq 0$. If $\rho$ has no points of discontinuity then the same result is valid for $u \in L^2[0, R]$.

Notice that if $\rho'(r) = 0$ for $r$ in some nonempty open interval in $[0, R]$, then $V(u) = 0$ for all functions $u$ with support in this interval.

The same argument shows that equation (3.88) is valid under hypotheses (2.33):

**Theorem 3.2.** Suppose that the density distribution $\rho$ satisfies hypotheses (2.33) and $u$ is a continuous function on $[0, R]$. Then

$$\gamma \leq 15 G \mathbf{I}_\rho \frac{V(u)}{\langle 1, u \rangle_\rho^2},$$

(3.93)

where $V$ is given in equation (3.82) and

$$\langle 1, u \rangle_\rho = -\int_0^R u(r)r^5 \rho'(r)dr.$$

If $\epsilon$ is the solution to the Clairaut’s equation then

$$\gamma = 15 G \mathbf{I}_\rho \frac{V(\epsilon)}{\langle 1, \epsilon \rangle_\rho^2}$$

(3.94)

We recall that the solution $\epsilon$ to the Clairaut’s equation is continuous even when $\rho$ has points of discontinuity.
3.2 Variational principle for piecewise constant mass distributions

At first consider the case of a homogeneous body of constant density $\rho_0$. In this case $-\rho'(r) = \rho_0 \delta(r-R)$ and from equations (3.79), (3.82), and (2.26)

$$\langle 1, u \rangle_{\rho} = -\int_0^R u(r)r^5\rho'(r)dr = \rho_0 \int_0^R u(r)r^5\delta(r-R)dr = \rho_0 R^5 u(R)$$

and

$$V(u) = -\int_0^R \frac{a^5\rho(a)\rho'(a)}{3}u^2(a)da - \int_0^R \int_0^R \rho'(a)\rho'(r)F(a,r)u(r)u(a)drda$$

$$= \rho_0 \int_0^R \frac{a^5\rho(a)}{3}\delta(a-R)u^2(a)da$$

$$- \rho_0^2 \int_0^R \delta(r-R)u(r) \int_0^R \delta(a-R)F(a,r)u(a)da \, dr$$

$$= \rho_0^2 \frac{R^5}{3}u^2(R) - \frac{\rho_0^2}{5}R^5u(R) \int_0^R \delta(r-R)u(r)dr = \frac{2}{15}\rho_0^2 R^5 u^2(R)$$

So, inequality (3.93) implies

$$\gamma \leq 15GI \frac{V(u)}{\langle 1, u \rangle_{\rho}^2} = \frac{2IG}{R^5} = \frac{4Gm}{5R^5}$$

Notice that the right hand side does not depend on $u$ and therefore the equality holds. This result agrees with that obtained directly from equations (2.24), (1.7), and (2.22).

The same computation can be done for a piecewise constant density distribution as that in equation (2.47). Using $\rho'(r) = -\sum \chi_j \delta(r-r_j)$, where from equation (2.37)

$$\chi_j = \rho_j - \rho_{j-1} \quad \text{for} \quad j = 1, \ldots n \quad (3.95)$$

the result is

$$\langle 1, u \rangle_{\rho} = -\int_0^R u(r)r^5\rho'(r)dr = -\sum_{j=1}^n \chi_j r_j^5 u(r_j)$$

and

$$V(u) = -\int_0^R \frac{a^5\rho(a)\rho'(a)}{3}u^2(a)da - \int_0^R \int_0^R \rho'(a)\rho'(r)F(a,r)u(r)u(a)drda$$

$$= -\frac{1}{3} \sum_{j=1}^n \chi_j \rho(r_j)r_j^5 u^2(r_j) - \sum_{j=1}^n \sum_{k=1}^n \chi_j \chi_k F(r_j, r_k)u(r_j)u(r_k)$$
where, from equation (2.26),

\[ F(r_j, r_k) = \begin{cases} 
  \frac{r_j}{r_k} & \text{if } r_j \geq r_k \\
  \frac{r_k}{r_j} & \text{if } r_k > r_j
\end{cases} \]

So, for any set of values \( \{u(r_1), \ldots, u(r_n)\} \in \mathbb{R}^n \) the following inequality must hold

\[
\gamma \leq 15GI_0 \frac{V(u)}{(1, u)^2} = 15GI_0 \frac{-\frac{1}{3} \sum_{j=1}^{n} \chi_j p(r_j)r_j^5u^2(r_j) - \sum_{j=1}^{n} \sum_{k=1}^{n} \chi_j \chi_k F(r_j, r_k)u(r_j)u(r_k)}{\left(- \sum_{j=1}^{n} \chi_j r_j^5u(r_j)\right)^2}
\]

(3.96)

with equality exact at \( \{\epsilon(r_1), \ldots, \epsilon(r_n)\} \) that is the solution to the discrete Clairaut’s equation discussed in Section 2.1.1.

4 A Riccati equation associated to \( \gamma \) and inequalities

The Darwin-Radau approximation is obtained from a first order differential, equation (2.72), associated to Clairaut’s equation. In this section, from Clairaut’s equation, we derive another first order differential equation and from this equation we obtain a sharp lower bound for \( \gamma/\gamma_1 \).

Equations (1.3), (1.7), and (2.23) can be combined to give

\[
\frac{\gamma}{\gamma_1} = 1 - 2 \left(\frac{R_1}{R}\right)^5 \frac{\zeta}{J_{2v}}, \text{ where } \zeta = \frac{\Omega^2 R^3}{Gm}.
\]

(4.97)

The substitution of \( \epsilon(R) = -\frac{3}{R^3} y(R) \) and \( w(R) = -\frac{\Omega^2}{8\pi G} \), equations (2.43) and (2.44), into \( J_{2v} = \frac{1}{3}(2\epsilon(R) - \zeta) \), equation (2.41), gives

\[
\frac{J_{2v}}{\zeta} = \frac{1}{3} \left( \frac{y(R)}{w(R)} - 1 \right)
\]

(4.98)

The last two equations give

\[
\frac{\gamma}{\gamma_1} = 3 \left(\frac{R_1}{R}\right)^5 \frac{v(R)}{1 - v(R)} \text{ where } v(r) = \frac{w(r)}{y(r)}
\]

(4.99)
This equation suggests to look for a differential equation for \( v \), which is readily obtained from equations (2.43) and (2.44):
\[
v' = \frac{5}{r} v (1 - v) + \frac{3 \rho'}{5 \rho} \quad \text{with} \quad v(0) = 1
\]  
(4.101)

Notice that \( v \) satisfies a Riccati equation (or a non homogeneous logistic equation).

If \( \rho \) has a point of discontinuity at \( r_j \) then the jump condition is
\[
\Delta v(r_j) = \frac{3}{5} \chi_j \rho(r_j) - \lim_{r \to (r_j)^-} \rho(r) < 0
\]  
(4.102)

Proposition 2.1 states that \( \varepsilon(r) \) does not change sign and so \( y(r) = -\frac{\varepsilon(r)}{3} \varepsilon(r) \). In the proof of Proposition (2.1) we showed that \( w(r) \) does not change sign either (see equation (A.120)) and, since \( v(0) = 1 \), \( v = w/y \) is always positive. Since \( \rho'/\rho \leq 0 \) and \( v(1 - v) \) is positive for \( 0 < v < 1 \) and negative for \( v > 1 \), the solution \( v(r) \) of equation (4.101) satisfies
\[
0 < v(r) \leq 1, \quad 0 \leq r \leq R.
\]  
(4.103)

**Theorem 4.1.** If \( \rho \) satisfies the hypothesis in equation (2.33) then, for given values of \( m, I_o, \) and \( R \),
\[
\frac{\gamma}{\gamma_1} \geq \frac{3}{2} \left( \frac{R_1}{R} \right)^5 \left[ \frac{5}{3} \left( \frac{R}{R_1} \right)^2 - 1 \right],
\]  
(4.104)

where the equality is verified for a “thick shell Roche model” described in Section 2.1.2 equation (2.62), which do not satisfy hypothesis (2.33). So, the value of \( \gamma/\gamma_1 \) is minimum for the Roche model that consists of a point at the origin with mass \( m_0 = [1 - (R_1/R)^2]m \) and a surrounding homogeneous layer with mass \( m_1 = (R_1/R)^2m \).

**Proof.** In the following we will assume that \( \rho \) is \( C^2 \). The same regularization argument presented in Section 3.2 implies that the theorem holds for discontinuous densities as those in hypothesis (2.33).

Equation (4.99) implies that inequality (4.104) holds if, and only if,
\[
v(R) \geq 1 - \frac{3}{5} \left( \frac{R_1}{R} \right)^2
\]  
(4.105)

\(^8\)Equations (1.8) and (4.99) imply that \( v(R) \) is entirely determined by the static fluid Love number \( k_2 \)
\[
k_2 = \frac{1 - v(R)}{v(R)}
\]  
(4.100)
In order to estimate \( v(R) \), we multiply equation (4.101) by \( r^5 \overline{\rho}(r) \) and integrate its left hand side by parts

\[
\int_0^R \overline{\rho}(r)r^5 v'(r)dr = \overline{\rho}(R)R^5 v(R) - \int_0^R \left[ (\overline{\rho}(r)r^5)' + (\overline{\rho}(r)r^5)'(v(r) - 1) \right]dr
\]

\[
= \overline{\rho}(R)R^5 v(R) - \overline{\rho}(R)R^5 - \int_0^R (\overline{\rho}(r)r^5)'(v(r) - 1)dr
\]

Then, using that

\[
\frac{3}{8\pi}I_0 = \int_0^R r^4 \rho(r)dr = -\frac{1}{5} \int_0^R r^5 \rho'(r)dr \quad \text{and} \quad \frac{9}{8\pi} \frac{I_0}{\overline{\rho}(R)R^5} = \frac{3}{5} \left( \frac{R_1}{R} \right)^2
\]

we obtain

\[
v(R) = 1 - \frac{3}{5} \left( \frac{R_1}{R} \right)^2 + \frac{1}{R^3 \overline{\rho}(R)} \int_0^R (1 - v) \left[ 5r^4 \overline{\rho}v - (r^5 \overline{\rho})' \right]dr
\]

So inequality (4.105) holds if

\[
5r^4 \overline{\rho}v - (r^5 \overline{\rho})' = r^4(5\overline{\rho}v - 3\rho - 2\overline{\rho}) = r^4 H(r) \geq 0,
\]

where we used \( \overline{\rho}(r) = \frac{3}{5} \left[ \rho(r) - \overline{\rho}(r) \right] \), equation (2.27). We will show that \( H(r) \geq 0 \) for \( 0 < r \leq R \).

Notice that

\[
H(0) = 5\overline{\rho}(0)v(0) - 3\rho(0) - 2\overline{\rho}(0) = 0,
\]

and, after some computation using equation (4.101), the definition of \( H \), and equation (2.27)

\[
rH'(r) = -5v(r)H(r) + 6[\overline{\rho}(r) - \rho(r)]. \tag{4.106}
\]

Let \( \hat{r} = \sup_{r \geq 0} \{ \overline{\rho}(r) - \rho(r) \} = 0 \}. Notice that \( \overline{\rho}(r) - \rho(r) > 0 \) for \( r > \hat{r} \).

The differential equation for \( H \), its differentiability at \( r = 0 \), and the initial condition \( H(0) = 0 \) implies that \( H(r) = 0 \) for \( 0 < r \leq \hat{r} \). For the same reasons \( v(r) = 1 \) for \( 0 \leq r \leq \hat{r} \), see equation (4.101).

We will show that there exists \( \overline{r} > \hat{r} \) sufficiently close to \( \hat{r} \) such that \( H(\overline{r}) > 0 \) and \( H(r) \geq 0 \) for \( \hat{r} < r \leq \overline{r} \). Since \( v(\hat{r}) = 1 \), there exist \( \hat{r} > \hat{r} \) sufficiently close to \( \hat{r} \) such that, for \( \hat{r} < r < \overline{r} \), \( 1 - 5v(r) < 0 \). Now, suppose that there exists a value of \( r_* \in (\hat{r}, \overline{r}) \) such that \( H(r_*) < 0 \). Then there exist \( r_* \in [\hat{r}, r_*] \) (possibly \( r_* = \overline{r} \)) such that \( H(r) < 0 \) for \( r_* < r \leq r_* \) and \( H(r_*') = 0 \). The integration of equation (4.106) gives

\[
r_*H(r_*) = \int_{r_*}^{r_*'} (1 - 5v(a))H(a)da + 6 \int_{r_*}^{r_*'} [\overline{\rho}(a) - \rho(a)]da,
\]

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which is impossible because the left hand side of this equation is strictly negative and the right hand side is strictly positive. So \( H(r) \geq 0 \) for \( r \in (\tilde{r}, \hat{r}) \) and again integration of equation (4.106) gives that \( H(\tilde{r}) > 0 \) for some \( \tilde{r} > \hat{r} \) sufficiently close to \( \hat{r} \). Now we claim that \( H(r) > 0 \) for \( r \geq \hat{r} \). Indeed, if there exists \( \tau > \hat{r} \) such that \( H(\tau) = 0 \), then equation (4.106) implies \( \tau H'(\tau) = 6(\gamma(\tau) - \rho(\tau)) > 0 \) which is impossible. Therefore \( H(r) \geq 0 \) for \( 0 \leq r \leq R \).

The next theorem establishes upper bounds for \( \gamma/\gamma_1 \). Let \( \Gamma : (0, 1] \to \mathbb{R} \) be the function

\[
\Gamma(R_1/R) = \sup_{\rho} \{\gamma/\gamma_1 : R_1/R \text{ is fixed}\},
\]

where the supremum is taken over all \( \rho \)'s that satisfies hypothesis (2.33). This function is non increasing due to the ill defined concept of geometric radius for bodies with low density external shells (see [2] for a discussion on the definition of the radius of a star). The proof that \( \Gamma \) is non increasing is based on the following argument: a density distribution that is positive only for \( r < \hat{R} \) can be extended to a larger radius \( R \) adding a negligible layer of mass that does not change the value of \( \gamma/\gamma_1 \) but decreases the value of \( R_1/R \). A detailed proof is the following. Suppose that for a certain value of \( \hat{R} \) there exists a density \( \hat{\rho} \) for which \( \hat{\gamma}/\hat{\gamma}_1 = \Gamma(\hat{R}_1/\hat{R}) \) (if the supremum of \( \gamma/\gamma_1 \) is not realized by any density \( \hat{\rho} \) then \( \hat{\rho} \) must be substituted for a maximizing sequence). For a small value of \( \tau > 0 \) let \( R_\tau \) be the largest value of \( r \) such that \( \hat{\rho}(r) > \tau \) for \( r < R_\tau \). For \( R > \hat{R} \) consider the new density function given by: \( \rho_\tau(r) = \hat{\rho}(r) \) for \( r < R_\tau \), \( \rho_\tau(r) = \tau \) for \( R_\tau \leq r < R \), and \( \rho_\tau(r) = 0 \) for \( r \geq R \). Notice that \( \rho_\tau \to \hat{\rho} \) as \( \tau \to 0 \) uniformly in the interval \([0, R]\). Therefore all the quantities \( m_\tau , I_{0\tau} , R_{1\tau} , \gamma_{1\tau} \) and \( \gamma_\tau \) tends to those respective quantities of \( \hat{\rho} \) as \( \tau \to 0 \) (note that for \( r > \hat{R} \) the solution \( \hat{v} \) to equation (4.101) satisfies \( \frac{1}{r^5 \hat{v}(r)} = \text{constant such that} \frac{1}{\hat{g}_1} \) in equation (4.99) remains constant). Therefore \( \Gamma(\hat{R}_1/\hat{R}) \geq \Gamma(\hat{R}_1/\hat{R}) \) for \( R \geq \hat{R} \) which implies that \( \Gamma \) is non increasing. Since for a homogeneous body \( R = \hat{R} \) and \( \gamma/\gamma_1 = 1 = \Gamma(1) \), we obtain that \( \Gamma(\hat{R}_1/\hat{R}) \geq 1 \) for \( R_1/R \leq 1 \). A body that realizes \( \gamma/\gamma_1 = 1 \) for any \( R_1/R \in (0, 1] \) is the homogeneous core Roche model of Section 2.1.2. The next theorem shows that the upper bound \( \gamma/\gamma_1 \leq 1 \) is almost correct.

**Theorem 4.2.** If \( \rho \) satisfies the hypothesis in equation (2.33) then, for given values of \( m \) and \( I_0 \),

\[
\frac{\gamma}{\gamma_1} \leq \sqrt{\frac{8575}{39}} \approx 1.001401 \quad \text{for all} \quad \frac{R_1}{R} \in (0, 1]
\]

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Proof. For a given \( \rho \), consider a family of auxiliary density functions \( \rho_s \), \( s > 0 \), obtained from \( \rho \) by means of a cutoff at radius \( s \), namely \( \rho_s(r) = \rho(r) \) for \( r < s \) and \( \rho_s(r) = 0 \) for \( r \geq s \). The idea of the proof relies upon the study of the function \( \Gamma(s) = \frac{\gamma}{\gamma_1}(s) \) for which a differential equation will be written.

The solution \( v_s(r) \) to equation (4.101), for the density \( \rho_s \), coincides with the solution \( v(r) \) to the same equation, for the density \( \rho \), as far as \( r < s \). At \( r = s \) a jump, possibly null, must be added to \( v_s \) and according to equation (4.102)

\[
v_s(s) = v(s_-) - \frac{3}{5} \frac{\rho(s_-)}{\bar\rho(s)}
\]

where \( s_- = \lim_{r \to s, r < s} r \). For \( r > s \), \( \rho'_s = 0 \) and equation (4.101) can be explicitly solved

\[
v_s(r) = \frac{v_s(s) r^5}{s^5 [1 - v_s(s)] + v_s(s) r^5}
\]

It is a remarkable fact that \( \frac{\rho}{\bar\rho} \) satisfies a differential equation which is similar to equation (4.101). Indeed, equation (2.27) implies

\[
\left( \frac{\rho}{\bar\rho} \right)' = \frac{3 \rho}{r \bar\rho} \left[ 1 - \frac{\rho}{\bar\rho} \right] + \frac{\rho'}{\bar\rho}
\]

and from this follows a very symmetric form of equation (4.101):

\[
v' = \frac{5}{r} v(1 - v) + \frac{3}{5} \left( f' - \frac{3}{r} f(1 - f) \right) \quad \text{where} \quad f = \frac{\rho}{\bar\rho}
\]

or

\[
\frac{d}{dr} \left( v - \frac{3 \rho}{5 \bar\rho} \right) = \frac{5}{r} v(1 - v) - \frac{9}{5r} f(1 - f)
\]

Notice that the left hand side of this last equation is the derivative of the function \( v_s(r) \) at \( r = s \) as given in equation (4.107). So, we define a new variable

\[
z(r) = v(r) - \frac{3}{5} \frac{\rho(r)}{\bar\rho(r)} = v(r) - \frac{3}{5} f(r)
\]

and from equation (4.110) and the initial values \( v(0) = 1 \) and \( f(0) = \rho(0)/\bar\rho(0) = 1 \) we obtain

\[
z' = \frac{5}{r} z(1 - z) + \frac{6}{5r} f(1 - 5z), \quad z(0) = \frac{2}{5}.
\]

We remark that \( v_s(s) = z(s) \). Let \( \beta(s) \) be the the inertial radius of \( \rho_s \) divided by \( s \), namely

\[
\beta^2(s) = \frac{5}{3} \frac{\int_0^s a^4 \rho da}{s^2 \int_0^s a^2 \rho da}
\]
A computation using the definitions \( f = \rho/\overline{p} \) and \( \overline{p} = \frac{3}{\pi} \int_{0}^{r} \alpha^{2} \rho da \) gives

\[
\frac{d}{ds} \beta^2 = \frac{1}{s} \left( - (3f + 2) \beta^2 + 5f \right) \tag{4.112}
\]

All these definitions and equation (4.99), namely \( \frac{\gamma}{\gamma_{1}} = \frac{3}{2} \left( \frac{R_{1}}{R} \right)^{5} \left[ \frac{H}{1 - \rho(0)} \right]^{\frac{1}{2}} \), imply that \( \Gamma(s) \), which is the value of \( \frac{\gamma}{\gamma_{1}} \) for the density function \( \rho_{s} \), is given by

\[
\Gamma(s) = \frac{3}{2} \beta^{5}(s) \frac{z(s)}{1 - z(s)}. \tag{4.113}
\]

Notice that \( \beta(0) = 1 \) and \( z(0) = 2/5 \) imply \( \Gamma(0) = 1 \). Differentiating both sides of equation (4.113) with respect to \( s \) and using equations (4.111) and (4.112) we obtain

\[
\frac{d}{ds} \Gamma = \frac{f}{10s^{5}} \left( 18\beta^{10} + (125\beta^{3} - 111\beta^{5})\Gamma - 32\Gamma^2 \right), \quad \Gamma(0) = 1, \tag{4.114}
\]

which is the desired equation. We recall that for any value of \( s \geq 0 \) the following inequalities are verified:

\[
0 \leq f(s) \leq 1 \quad \text{and} \quad 0 < \beta(s) \leq 1 \tag{4.115}
\]

with \( f(s) = 0 \), if and only, if \( \rho(s) = 0 \).

The right hand side of equation (4.114) has two factors: \( \frac{f}{10s^{5}} \), which is greater or equal to zero for \( s > 0 \), and \( P(\beta, \Gamma) = 18\beta^{10} + (125\beta^{3} - 111\beta^{5})\Gamma - 32\Gamma^2 \), which is a quadratic polynomial in \( \Gamma \). For a given value of \( \beta \in (0, 1] \), \( \Gamma \rightarrow P(\beta, \Gamma) \) has a positive root \( \Gamma = \Gamma_{1}(\beta) \) given by

\[
\Gamma_{1}(\beta) = \beta \frac{125 - 111\beta^{2} + 5\sqrt{5} \sqrt{117\beta^{4} - 222\beta^{2} + 125}}{64},
\]

and another negative root such that \( P(\beta, \Gamma) < 0 \) for \( \Gamma > \Gamma_{1}(\beta) \) and \( P(\beta, \Gamma) > 0 \) for \( 0 < \Gamma < \Gamma_{1}(\beta) \). The graph of \( \Gamma_{1}(\beta) \) for \( \beta \in (0.9, 1) \) is given in Figure 2. The point of maximum \( \beta \) of \( \Gamma_{1}(\beta) \) can be calculated in the following way. Equations \( \frac{d}{d\beta} P(\beta, \Gamma_{1}(\beta)) = 0 \) and \( \frac{d}{d\beta} \Gamma_{1}(\beta) = 0 \) give

\[
\Gamma_{1}(\beta) = \frac{180\beta^{9}}{-375\beta^{2} + 555\beta^{4}}
\]

which can be substituted into \( P(\beta, \Gamma_{1}(\beta)) = 0 \) to give an equation for \( \beta \)

\[
P(\beta, \Gamma_{1}(\beta)) = -168750(\beta - 1)\beta^{14}(\beta + 1)(39\beta^{2} - 35) = 0
\]

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The only root to this equation in the interval \((0,1)\) is \(\tilde{\beta} = \sqrt{35/39}\). The value of \(\Gamma_1(\beta)\) at \(\beta = \tilde{\beta}\) is \(\frac{8575}{8112} \sqrt{\frac{35}{39}}\).

The above argument shows that for: \(\Gamma = \frac{8575}{8112} \sqrt{\frac{35}{39}}\), \(0 < s, 0 < \beta < 1, 0 \leq f \leq 1\), the right hand side of equation (4.114), and therefore \(\frac{d}{ds} \Gamma\), is less than or equal to zero. So no solution to equation (4.114) that starts at \(\Gamma(0) = 1\), for any admissible \(f\) and \(\beta\) that satisfy inequalities (4.115), can cross above the line \(\Gamma = \frac{8575}{8112} \sqrt{\frac{35}{39}}\), which ends the proof of the theorem.

Remark: There are density functions for which \(\gamma/\gamma_1\) is larger than one: a polytrope of index \(n = 0.4604\) has \(R_I/R = 0.9102\) and \(\gamma/\gamma_1 \approx 1.0003 > 1\) (numerically estimated); and the parabolic density function \(\rho(r) = 1 - r^2\), for \(0 \leq r \leq \sqrt{28/48}\), and \(\rho(r) = 0\), for \(r \geq \sqrt{28/48}\), has \(R_I/R = 0.947331\) and \(\gamma/\gamma_1 \approx 1.0008\) (numerically estimated). I believe that the upper bound \(\frac{\gamma}{\gamma_1} \approx 1.001401\) is sharp, namely there are density functions for which the value of \(\gamma/\gamma_1\) gets arbitrarily close to 1.001401. If this is true, then it is an interesting mathematical problem to determine the limit density profile that maximizes \(\gamma/\gamma_1\). Both, the fact that the upper bound of \(\gamma/\gamma_1\) as a function of \(R_I/R\) is non increasing and that \(\gamma/\gamma_1 = 1\) for \(R_I/R = 1\), shows that our upper bound as a function of \(R_I/R\) can be sharp only for \(R_I/R \leq \sqrt{35/39}\).

Theorem 4.2 establishes a limit for the validity of the Darwin-Radau theory. Indeed, as illustrated in Figure 3 (a), the value of \(\gamma/\gamma_1\) in equation (2.75) given by the Darwin-Radau approximation is smaller than that in the upper bound given in theorem 4.2 if, and only if,

\[
\frac{R_I}{R} > 0.86534\ldots, \quad \text{(Validity of the Darwin-Radau approximation)}.
\]

From Figure 3 (b) it is possible to see that for a polytrope, within the range \(0.7 \leq R_I/R < 1\), the approximation \(\gamma = \gamma_1\) has a maximum relative error of the order of 3% while the Darwin-Radau approximation has a relative error of the order of 20%. Figure 4 shows \(\gamma/\gamma_1\) as a function of \(R_I/R \in (0,1)\) for: the Clairaut’s approximation, for the polytropes, for the thick shell Roche models presented in Section 2.1.2, which are the lower bounds for \(\gamma/\gamma_1\), and for the upper bound given in Theorem 4.2.

5 Computation of \(\gamma\) for some celestial bodies

In this section we compute the values of \(\gamma\) for some bodies in the solar system solving numerically Clairaut’s equation according to the algorithm described in the paragraph above equation (2.46). In the literature there is
more than one proposal of mass-distribution for the same body, in most cases we just chose one. Our goal is to compare the values obtained with: the direct integration of the Clairaut’s equation, the Darwin-Radau approximation, the upper and lower bounds in Theorems 4.1 and 4.2, and the value $\gamma_{ob} = \frac{1}{mR^2} \frac{\Omega^2}{J_{2v}}$ (equation (1.7)), where $\Omega$ and $J_{2v}$ are values found in the literature, which were estimated from observations.

In principle, the values of $\gamma_{ob}$ do not have to match the value computed using Clairaut’s equation for three reasons. The first is that in Clairaut’s theory only gravitational forces are taking into account while $\gamma_{ob}$ is due to gravity plus solid and fluid elastic forces. The dominance of the gravitational forces over the elastic forces tends to increase as the body increase. The second reason is that tide-dissipation is slowing down the spin of celestial bodies, so viscous forces, both in the fluid and in the solid part, may offset the system from equilibrium (this was the explanation found in [18] for the difference between the Earth flattening predicted under the hydrostatic hypothesis and the observed one). The third reason is that Clairaut’s theory is of first order in the small parameter $\frac{\Omega^2 R^3}{Gm}$, so, as this parameter increases, higher order corrections become more important. In spite of all these remarks, the values of $\gamma_{ob}$ are reasonably well approximated by the values found using Clairaut’s theory.

The results are summarized in: Table 1, which contains the data used
Remarks and notation:

a) The gravitational constant is $G = 6.67408 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$

b) $\gamma_C$ is the value computed integrating numerically Clairaut’s equation.

c) $\gamma_1 = (4/5)(Gm/R_1^3)$ is the value of $\gamma$ for a homogeneous body with mass $m$ and radius equal to the radius of inertia $R_1 = \sqrt{\frac{5I}{2m}}$.

d) $\gamma_{DR}$ is the value of $\gamma$ computed using the equation of Darwin-Radau, which requires $R_1/R > 1/\sqrt{3}$ (at $R_1/R = 1/\sqrt{3}$ the denominator of the right-hand side of equation (2.75) becomes zero), so it cannot be computed for the Sun.
e) $\gamma_P$ is the value of $\gamma$ under the hypothesis that the body is made of a polytropic fluid with an index determined by the ratio $R_1/R$ (see Section 2.2).

f) $\gamma_{ob} = \frac{I_o}{mR^2} \frac{\Omega^2}{J_2} = \frac{I_0}{mR^2} \frac{\Omega^2}{J_2}$ is the value of $\gamma$ where $\Omega$ and $J_2$ are numbers found in the literature, which were estimated from observations. All the equatorial radii $R_e$ used in this paper were taken from [33]. As remarked above $\gamma_{ob}$ may not represent the same physical quantity as $\gamma_C$, $\gamma_P$, or $\gamma_{DR}$.

g) The value $C/(mR^2)$, where $C$ is the polar moment of inertia, is more frequently found in the literature than $I_o/(mR^2)$. The two quantities are related by

$$\frac{I_o}{mR^2} = \frac{C}{mR^2} - \frac{2}{3} J_2 \Omega = \frac{C}{mR^2} - \frac{2}{3} R_e^2 J_2$$

h) For Mars, the value of $\gamma_{ob}/\gamma_1 = 0.9258$ is considerably smaller than that obtained from the Darwin-Radau approximation $\gamma_{DR}/\gamma_1 = 0.9999$. This difference is discussed in [35] (see p. 368) and it may be caused by non-gravitational internal tensions.

i) The Sun I. It was more difficult to obtain the several values of $\gamma$ for the Sun than for the other bodies. The Sun has a ratio $R_1/R \approx 0.4$ and therefore it has a higher concentration of mass at its core than the other bodies. It is well-known that the internal angular velocity of the Sun varies with the radius and this requires a modification of the Clairaut’s theory [36]. Nevertheless, this radial variation of angular velocity seems to be concentrated near the core (see [4] Figure 1) and we decided to apply the usual Clairaut’s theory with an averaged angular velocity $\Omega$ in the sense of Tisserand, which is defined by $L = I_o \Omega$ where $L$ is the Sun angular momentum. We found several proposals of internal density distributions for the Sun (Solar Standard Models) and we did computations with two of them.

j) The Sun II. The first density distribution we used is that in [5]. In this reference the authors provided all constants we needed except for $J_{2e}$. Their values are: $m = 1.9889 \times 10^{30}$kg, $R = 696000$km, $I_o = 7.60 \times 10^{46}$kg m$^2$ ($I_o/mR^2 = 0.0785$ and $R_1/R = 0.444$), $\Omega = 2.87 \times 10^{-6}$rad/s. The graph of the density function used in [5] is shown in Figure 5. We used $J_{2e} = 0.2295 \times 10^{-6}$ [32]. For this set of data we obtained: $\gamma_1 = 3.5965 \times 10^{-6}$s$^{-2}$, $\gamma_C/\gamma_1 = 0.5326$ (obtained from the numerical integration of Clairaut’s equation), and $\gamma_{ob}/\gamma_1 = 0.7872$. According to
Section 2.2 the ratio $R_I/R = 0.444$ corresponds to a polytrope of index $n = 2.948$ and a $\gamma_P/\gamma_1 = 0.811$. We observe that the values $\gamma_C/\gamma_1 = 0.5326$ and $\gamma_{ob}/\gamma_1 = 0.7872$ are very different, indeed $\gamma_{ob}$ is closer to the value $\gamma_P$ of the polytrope than to $\gamma_C$. Since $\frac{\Omega^2 R^3}{G m} = 0.021 \times 10^{-3}$ is very small, if this density model would be a good representative for the real density of the Sun then Clairaut’s theory should have given a better result. This model has a larger value of $I_0/mR^2 = 0.0785$ than others found in the literature for which $I_0/mR^2 \approx 0.07$. The data for this model are not presented in Tables 1 and 2.

k) *The Sun III.* Since with the density model in [5] we did not get a reasonable result we tried a second one that is given in [1]. All the results in [1] are normalized by the Solar radius that we chose as $R = 695700\text{km}$. Explicit values for $m$ and $I_0$ are not provided in [1], we obtained them in the following way. Integrating the density distribution given in [1], and shown in Figure 5, we obtained a value for the total mass of $1.985549 \times 10^{30}\text{kg}$. In order to calibrate the total mass to the standard value $m = 1.9885 \times 10^{30}\text{kg}$ we multiplied the densities provided in [1] by the small factor 1.9885/1.9855 $\approx 1.0015$. With this normalized density we computed $I_0 = 6.877 \times 10^{46}\text{kg m}^2$ that implies $I_0/mR^2 = 0.0715$ and $R_I/R = 0.423$. In order to obtain $\Omega$ we use the results in [4] in the following way. In this reference there is a graph of the variation of the angular rotation within the Sun as a function of the radius (see [4] Figure 1, the model which takes into account magnetic effects). This distribution supposes an average surface velocity of $2.9 \times 10^{-6}\text{rad/s}$, we multiplied it by the factor $2.87/2.9 \approx 0.99$ to obtain the most accepted value $\Omega = 2.87 \times 10^{-6}\text{rad/s}$ average angular velocity at the surface [3]. This changes the estimate $2.02 \times 10^{41}\text{kg m}^2/\text{s}$ for the solar total angular momentum $L$ in reference [4] to $2.00 \times 10^{41}\text{kg m}^2/\text{s}$ (see [14] for several other estimates of $L$). Then we defined the average angular velocity $\Omega = L/I_0 = 2.91 \times 10^{-6}\text{rad/s}$. In order to check the consistency of the models used in [1] and [4] we computed the total angular momentum using the density distribution in [1] and the varying angular velocity given in Figure 1 of [4], the result is $2.03 \times 10^{41}\text{kg m}^2/\text{s}$ which is close to the total angular momentum above. Integrating numerically Clairaut’s equation we obtained $\gamma_C/\gamma_1 = 0.6271$. The value $\gamma_{ob}/\gamma_1 = 0.6313$ was computed using $\Omega = 2.87 \times 10^{-6}\text{rad/s}$ as above and $J_{2v} = 0.2295 \times 10^{-6}$ [22]. Notice that $\gamma_C/\gamma_1$ and $\gamma_{ob}/\gamma_1$ are close. The polytrope that corresponds to the ratio $R_I/R = 0.423$ has index $n = 3.060$ and $\gamma_P/\gamma_1 = 0.7863$, which is 25% larger than the observed value $\gamma_{ob}/\gamma_1$. The density function of this polytropic approximation
normalized to have the same $m = 1.9885 \times 10^{30}\text{kg}$ is shown in Figure 5.

1) *The Sun IV.* There are different estimates of $J_{2e}$ for the Sun [29]. The quantity $\gamma_{ob}/\gamma_1$ in Table 2 is very sensitive to variations of $J_{2e}$ (and also of $\Omega$) while $\gamma_1$, $\gamma_C$, $\gamma_P$, and $\gamma_{DR}$ do not depend neither on $J_{2e}$ nor on $\Omega$. If we fix the quantities $m = 1.9885 \times 10^{30}\text{kg}$, $I_o = 6.877 \times 10^{46}\text{kg m}^2$ ($I_o/mR^2 = 0.0715$ and $R_1/R = 0.423$), and $\Omega = 2.87 \times 10^{-6}\text{rad/s}$ as in remark (k) and vary $J_{2e}$ from $1.65 \times 10^{-7}$ to $7.43 \times 10^{-7}$, which are the values in the last three lines of Table 1 of the historical surve y [29], then we obtain $0.1950 < \gamma_{ob}/\gamma_1 < 0.8781$ for the variation of $\gamma_{ob}/\gamma_1$. The lowest value 0.195 is close to the lower bound of Theorem 4.1, which is 0.169. If we restrict the variation of $J_{2e}$ to the values from INPOP2008 $J_2 = 0.182 \times 10^{-6}$ [11] to INPOP2017 $J_2 = 0.2295 \times 10^{-6}$ [32], which was the value adopted in this paper, then we obtain $0.6313 \leq \gamma_{ob}/\gamma_1 < 0.7961$. Notice that the value $\gamma_{ob}/\gamma_1 = 0.7961$, for $J_2 = 0.182 \times 10^{-6}$, is close to the value $\gamma_P/\gamma_1 = 0.7863$ for the polytrope with the same $R_1/R = 0.423$. The sensitivity of $\gamma_{ob}$ to variations of $J_2$, and other parameters as $\Omega$ and $I_o$, and the empirical difficulty in obtaining a sharp estimate of this value explains the variation in our previous determinations of $\gamma$ in [24] and [9] and also shows that in the future we may be enforced to change our estimate of $\gamma_{ob}$ for the Sun again. So, the simple estimate obtained with the polytropic approximation that does not match by 25% the value $\gamma_{ob}/\gamma_1 = 0.6313$, which we believe is the best at the moment, seems not bad.

6 Conclusion

Theorems 4.1 and 4.2 establish sharp inequalities for the gravitational modulus $\gamma$ as a function of the ratio $R_1/R$. These inequalities, which can be useful in the determination of physical properties of exoplanets, may be improved, or from a practical perspective substituted, in the following way.

The upper bound, $\gamma/\gamma_1 \geq 1.001401$, in Theorem 4.2 does not depend on $R_1/R$ because the geometric radius $R$ can be artificially large due to the presence of a thick layer of negligible mass. While this is not a drawback for bodies with $R_1/R$ close to one, which is the case for most planets in the solar system, it would be useful to have a more realistic upper (and also a lower) bound for $\gamma/\gamma_1$ when $R_1/R < 0.7$. One way to improve the inequalities in Theorems 4.1 and 4.2 would be to impose additional restrictions on the density function $\rho$, besides it being non increasing, or on the definition of $R$. 
Table 1: \( m \) =mass, \( R \) =volumetric mean radius, \( I_o \)=mean moment of inertia \((A + B + C)/3\), \( R_1 \)=inertial radius defined by \( I_o = 0.4mR_1^2 \) and related to \( I_o/(mR^2) \) by \( (R_1/R)^2 = \frac{5}{2} \frac{I_o}{mR^2} \), \( \Omega \)=spin angular velocity, \( J_2 = (C - A)/(mR^2) \)=dynamic form factor, where \( C \) is the polar moment of inertia and \( A \) is the equatorial moment of inertia of the rotating body. (a) The constants for the Sun were obtained according to remark (k) in the text. The value of \( \Omega \) for Saturn is from [17].

| Body   | \( m \) \((\times 10^{24} \text{ kg})\) | \( R \) \((\text{km})\) | \( \frac{1}{mR^2} \) | \( R_1/R \) | \( \Omega \) \((\times 10^{-6} \text{ s}^{-1})\) | \( J_2 \) \((\times 10^{-6})\) |
|--------|-----------------------------------|-----------------|-----------------|-------------|-----------------|-----------------|
| Sun\(^{(a)}\) | 1988500                             | 695700          | 0.0715          | 0.423       | 0.291           | 0.2295          |
| Earth\(^{[10]}\) | 5.974                               | 6371           | 0.331           | 0.909       | 7.2921          | 1082.6          |
| Mars\(^{[35]}\) | 0.6419                              | 3390           | 0.365           | 0.955       | 7.0882          | 1957.0          |
| Jupiter\(^{[13]}\) | 1899                                | 69911          | 0.264           | 0.816       | 17.585          | 14696           |
| Saturn\(^{[21]}\) | 568.3                               | 58232          | 0.228           | 0.754       | 16.53           | 16290           |
| Uramus\(^{[20]}\) | 86.81                               | 25388          | 0.227           | 0.754       | 10.121          | 3510.7          |
| Neptune\(^{[20]}\) | 102.4                               | 24622          | 0.238           | 0.772       | 10.833          | 3533.0          |

Table 2: \( \gamma_1 = (4/5)(Gm/R_1^3) \), \( \gamma_{DR} \) =value of \( \gamma \) obtained from the Darwin-Radau approximation (remark (d)), \( \gamma_P \) =value of \( \gamma \) for a body made of a polytropic fluid with an index determined by the ratio \( R_1/R \) (remark (e)), \( \gamma_C \) is the value of \( \gamma \) obtained from the numerical integration of Clairaut’s equation (the density functions are shown in Figure 5), \( \gamma_{ob} = \frac{1}{mR^2} \frac{\Omega^2 R^3}{Gm} \) (see remark (f)), \( \Omega^2 R^3/(Gm) \)=the centrifugal acceleration at the equator over the average gravitational acceleration on its surface (small quantity in Clairaut’s theory). In reference [20] there are two density models for Uranus and Neptune, (U1) and (N1) indicate the model we used (only the data file for the density of Neptune is provided in the supplementary data of [20], so the value of \( \gamma_C/\gamma_{ob} \) was computed only for Neptune). For Saturn the difference between \( \gamma_C/\gamma_1 \) and \( \gamma_{ob}/\gamma_1 \) may be attributed to corrections of higher order in the parameter \( \Omega^2 R^3/(Gm) = 0.139 \), which in this case is not so small.

| Body   | \( \gamma_1 \) \((\times 10^{-6} \text{ s}^{-2})\) | \( \gamma_{DR}/\gamma_1 \) | \( \gamma_P/\gamma_1 \) | \( \gamma_C/\gamma_1 \) | \( \gamma_{ob}/\gamma_1 \) | \( \frac{\Omega^2 R^3}{Gm} \) \((\times 10^{-3})\) |
|--------|-----------------------------------|-----------------|-----------------|-------------|-----------------|-----------------|
| Sun\(^{[1]}\) | 4.1761                              | 1.000           | 0.9998          | 0.9885      | 3.449           | 4.569           |
| Earth\(^{[10]}\) | 1.640                               | 0.9999          | 0.9999          | 1.0012      | 0.9258          | 83.39           |
| Mars\(^{[35]}\) | 1.009                               | 1.011           | 0.9963          | 0.9954      | 0.9802          | 139.0           |
| Jupiter\(^{[13]}\) | 0.5466                              | 1.055           | 0.9877          | 0.9650      | 0.9924          | 28.93           |
| Saturn\(^{[21]}\) | 0.3582                              | 1.037           | 0.9908          | 0.9868      | 0.9805          | 25.63           |
| Uramus\(^{[20]}\) (U1) | 0.6614                              | 1.055           | 0.9877          | 0.9650      | 0.9924          | 139.0           |
| Neptune\(^{[20]}\) (N1) | 0.7974                              | 1.037           | 0.9908          | 0.9868      | 0.9805          | 25.63           |
Figure 5: Mass density distribution $\rho$ and flatness $\varepsilon$, which was obtained from the numerical integration of Clairaut’s equation, for six of the bodies in Table 2. Three different density models were used for the Sun: the index “a” refers to the density distribution in [5] (remark (j)), the index “b” to the density distribution in [1] (remark (k)), and the index “p” to the density of a polytrope of index $n = 3.060$, which corresponds to $R_1/R = 0.423$, and mass $m = 1.9885 \times 10^{30}$kg (remark (k)). The density distribution for the Earth, Mars, Jupiter, Saturn, and Neptune were taken, respectively, from [10], [35] (model M13), [13], [12], and [20] (model N1).
Figure 6: The four graphs in this figure are: the lower and upper bounds for $\gamma/\gamma_1$, the Darwin-Radau approximation, and $\gamma_P/\gamma_1$ for polytropes. The points indicated with “obs” represent the values of $(R_1/R, \gamma_{ob}/\gamma_1)$, where $R_1/R$ and $\gamma_{ob}/\gamma_1$ are given in Tables 1 and 2 respectively, and those points indicated with “Clairaut” represent $(R_1/R, \gamma_C/\gamma_1)$, where $\gamma_C/\gamma_1$ is given in Table 2. The vertical line represents the possible values of $\gamma_{ob}/\gamma_1$ for the Sun as the value of $J_2$ varies from $J_2 = 0.182 \times 10^{-6}$ \textsuperscript{11} to $J_2 = 0.2295 \times 10^{-6}$ \textsuperscript{32}, this last value being that used to obtain $\gamma_{ob}/\gamma_1 = 0.6314$ (see remark (l)). For an enlargement of the region $0.7 \leq R_1/R \leq 1$ see Figure 7.
Figure 7: Magnification of the region $0.7 \leq R_t/R \leq 1$ of Figure 6.
Then the analysis of the solutions to the differential equations (4.10) and (4.11) under the new constraints would give the desired inequalities. As far as I know there is no well-accepted suggestion of further restrictions on $\rho$ or on the definition of $R$ in the physical literature (see [2] for a discussion in this direction). A way to avoid these general restrictions is to assume an archetypal model that we choose as the polytropes.

It has been a practice among researchers, as for instance Chandrasekhar, to use polytropes as a first approximation to more realistic stellar models. As discussed in Section 2.2, a polytrope is characterized by the polytropic index $n$ and by two more parameters that can be the mass and the radius or the density at the center and the constant $K$. It is remarkable that $n$ and $R/I/R$ are in one-to-one correspondence and, as shown in Figure 1, $n \approx (1 - R/I/R)^5$. The gravitational modulus of a polytrope, denoted as $\gamma_P$, depends only on the index $n$ and therefore it is determined by the value of $R/I/R$. For all planets in Figure 7, except for Saturn and Uranus, the value of $\gamma_P/\gamma_1$ is a good approximation for $\gamma_C/\gamma_1$, where $\gamma_C$ is the value of $\gamma$ obtained from the integration of Clairaut’s equation. For Saturn the relative difference $|\gamma_P - \gamma_C|/\gamma_P$ is 2.8% and for Uranus $\gamma_C$ was not computed. Figure 7 also shows that $\gamma_P/\gamma_1$ is a very good approximation even for the observed values $\gamma_{ob}/\gamma_1$ with the small deviations being possibly explained by the existence of non-gravitational stresses, transient behavior, or higher order corrections in the small parameter $\Omega^2 R^3/(Gm)$, as argued in the second paragraph of Section 5. For the Sun, if $J_2$ is chosen as $0.2295 \times 10^{-6}$ [32], then $|\gamma_P/\gamma_{ob} - 1| = 25%$; and if $J_2$ is chosen as $0.182 \times 10^{-6}$ [11], then $|\gamma_P/\gamma_{ob} - 1| = 1%$ (see the vertical line in Figure 6 and the remark (l) in Section 5). So within the range of different values of $J_2$ in the recent literature [29] the value of $\gamma_P/\gamma_1$ is acceptable even for the Sun. These considerations lead me to the following:

**Practical rule for the estimation of $\gamma$:** The mass $m$ and the moment of inertia $I_0$ of a large celestial body determine its inertial radius $R_I = \sqrt{\frac{3I}{2m}}$ and its square inertial frequency $\gamma_1 = (4/5)(Gm/R_I^3)$. If in addition the volumetric radius $R$ of the body is given, then the ratio $R_1/R$ and the graph in Figure 1 determine the value of $\gamma_P/\gamma_1$ for a polytrope. The gravitational modulus $\gamma$ of the body is approximately given by $\gamma/\gamma_1 \approx \gamma_P/\gamma_1$. If $R_1/R > 0.7$, what happens for the planets in the solar system, then $\gamma/\gamma_1 \approx \gamma_P/\gamma_1 \approx 1$.

9For the planets listed in Tables 1 and 2 the largest error of $|\gamma_P/\gamma_{ob} - 1|$ is for Mars, 8%, because Mars may not be in hydrostatic equilibrium. For the remaining planets the error is within 1%.
A Appendix: Proofs of some Propositions

The following simple result is widely stated in the literature with no proof or reference.

**Proposition A.1.** For any spherically symmetric integrable mass density distribution \( \rho \) with support in \([0, R]\):

\[
\left( \frac{R_1}{R} \right)^2 \leq \frac{5}{3},
\]

the value \( \left( \frac{R_1}{R} \right) = \frac{5}{3} \) being achieved when all the mass is uniformly distributed over a spherical shell of radius \( R \). If in addition \( \rho \) is non-increasing then \( \frac{R_1}{R} \leq 1 \).

In this case, \( R_1/R = 1 \) if and only if \( \rho \) is constant.

**Proof.** The definition of \( R_1 \) implies

\[
\left( \frac{R_1}{R} \right)^2 = \frac{5}{3} \frac{\int_0^R a^2 \rho(a) da}{\int_0^R a^2 \rho(a) da} \leq \frac{5}{3}
\]

If all the mass is concentrated on a spherical shell of radius \( R \), \( \rho(r) = (m/4\pi R^2)\delta(r - R) \), then integration gives \( (R_1/R)^2 = 5/3 \).

Now, suppose \( \rho \) is non-increasing and let

\[
\hat{\rho} = \frac{3}{R^3} \int_0^R \rho(r)r^2dr \quad \text{and} \quad \rho(r) = \hat{\rho} + f \implies \int_0^R fr^2dr = 0
\]

The signed density \( f \) is not null if, and only if, \( \rho \) is not constant. If \( f \) is not null then there exists a value \( \tau \in (0, R) \) such that \( f(r) \geq 0 \) for \( r < \tau \) and \( f(r) \leq 0 \) for \( r > \tau \) with \( \int_0^\tau fr^2dr > 0 \) and \( \int_\tau^R fr^2dr < 0 \). These considerations imply that if \( f \) is not null:

\[
\frac{3}{5} R_1^2 = \frac{\int_0^R a^4 \rho(a) da}{\int_0^R a^2 \rho(a) da} - \frac{\int_0^R a^4 \hat{\rho}da + \int_0^R a^4 f(a) da}{\int_0^R a^2 \hat{\rho} da}
\]

\[
= \frac{3R^2}{5} + \frac{\int_\tau^R a^4 f(a) da + \int_\tau^R a^4 f(a) da}{\hat{\rho}R^3/3}
\]

\[
< \frac{3R^2}{5} + \frac{\tau^2 \int_0^\tau a^2 f(a) da + \tau^2 \int_\tau^R a^2 f(a) da}{\hat{\rho}R^3/3} = \frac{3R^2}{5}
\]

so \( (R_1/R)^2 < 1 \). For a body with constant density \( \frac{R_1}{R} = 1 \).

We recall the statement of Proposition 2.1. \( \square \)
Proposition A.2. Suppose that $\rho$ satisfies hypothesis (2.33). Then, for $\Omega > 0$ there exists a unique bounded solution to equation (2.24) (and therefore to problems (2.36) and (2.43)). This solution is strictly positive, non-decreasing, and $C^1$. For $\Omega = 0$ the only solution to equation (2.24) is $\epsilon(r) = 0$, $r \geq 0$.

In the case $\rho$ is $C^2$ the proof of this result, and more, can be found in [23] chapter IV.

In order to solve equation (2.24) we will solve the boundary value problem in equations (2.43), (2.44), and (2.45). At first we show that any solution to equations (2.43) with the jump conditions (2.45) imply that $\epsilon$ is $C^1$. Within the intervals $(r_j, r_{j+1})$, $\epsilon(r) = \frac{3}{\rho(r)} y$ implies

\[
\epsilon' = -\frac{3}{\rho^2} (y \bar{\rho} - y \rho) = \frac{3}{r \rho^2} \left(5(y - w) \bar{\rho} + 3y(\rho - \bar{\rho})\right),
\]

where we used equations (2.43) and (2.27). Following the notation in equation (2.38)

\[
\Delta \epsilon'(r_j) = \frac{3}{r_j \rho'(r_j)} \left(-5[\Delta w(r_j)] \bar{\rho}(r_j) + 3y(r_j)[\Delta \rho(r_j)]\right) = 0,
\]

where we used $\Delta w(r_j) = \frac{3}{5} \frac{\chi_j}{\rho(r_j)} y(r_j)$ from equation (2.45) and $\Delta \rho(r_j) = \chi_j$ from equation (2.37). This shows that $\epsilon$ is $C^1$.

Inside the intervals $[r_j, r_{j+1})$ the solution to equation (2.43) also satisfies equation (2.32), namely

\[
re'' + 6 \epsilon' + 2 \frac{\bar{\rho}}{\rho} (r \epsilon' + \epsilon) = (r^6 \epsilon')' + 2 \frac{r^5 \bar{\rho}}{\rho} (r \epsilon' + \epsilon) = 0.
\]

At first consider the interval $[0, r_1)$ (if $\rho$ is $C^2$ everywhere, then $r_1 = \infty$). The regularity of $\rho$ at $r = 0$ implies that $\rho(r) = \rho(0) + r^2 \rho''(0)/2 + \ldots$ and $\bar{\rho}(r) = \rho(0) + r^2 \rho''(0)/3/10 + \ldots$. So, near the origin equation (2.32) can be written as

\[
\epsilon'' + 6 \frac{\epsilon'}{r} + \left(\frac{6\rho''(0)}{5\rho(0)} + O(r)\right) (r \epsilon' + \epsilon) = 0. \quad (A.117)
\]

If we impose that $\epsilon$ is bounded (twice continuously differentiable) at $r = 0$, then taking the limit as $r \to 0$ into this equation we obtain that $\epsilon'(0) = 0$, which implies that near the origin

\[
\epsilon(r) = \epsilon(0) + r^2 \epsilon''(0)/2 + \ldots \quad \text{where} \quad \epsilon''(0) = -\frac{6 \rho''(0)}{35 \rho(0)} \epsilon(0). \quad (A.118)
\]

In the following we assume that $\epsilon(0) \neq 0$. Since $\rho''(0) \leq 0$, for $r > 0$ sufficiently small $\epsilon(r) \epsilon'(r) \geq 0$. If $\epsilon(r) \epsilon'(r) \geq 0$ near $r = 0$, then let
\( \dot{a} = \sup_{0 < r < r_1} \{ \rho(r) = 0 \} \). If \( \dot{a} = r_1 \), then \( \epsilon'(r) = 0 \) for \( 0 \leq r \leq r_1 \). If \( \dot{a} < r_1 \), then \( \rho' < 0 \) in some interval \( (\hat{a}, \dot{a} + \delta) \subset (0, r_1) \) and equation (2.32) implies that \( \epsilon(r) \cdot \epsilon'(r) > 0 \) in a possibly smaller interval. Now, let \( \overline{a} = \sup_{a < r < r_1} \{ \epsilon(a) \neq 0 \} \) and suppose that \( \overline{a} < r_1 \). Then equation (2.32) implies \( \overline{a} \epsilon''(\overline{a}) = -2 \frac{\rho(\overline{a})}{\rho} \epsilon(\overline{a}) \) and, since \( \rho'(\overline{a}) < 0 \) and \( \epsilon^2(\overline{a}) > 0 \), we obtain \( \epsilon(\overline{a}) \epsilon''(\overline{a}) > 0 \). But this is impossible because the function \( F(a) = \epsilon(a) \epsilon'(a) \) would be positive for \( \overline{a} < a < \overline{a} \) and would satisfy \( F(\overline{a}) = \epsilon(\overline{a}) \epsilon'(\overline{a}) = 0 \) and \( F'(\overline{a}) = \epsilon(\overline{a}) \epsilon''(\overline{a}) > 0 \). So, \( \overline{a} = r_1 \) and \( \epsilon(r) \epsilon'(r) \geq 0 \) for \( 0 \leq r \leq r_1 \). If \( \epsilon(0) > 0 \) (\( \epsilon(0) < 0 \)), then \( \epsilon'(r) \geq 0 \) (\( \epsilon'(r) \leq 0 \)) for \( 0 \leq r \leq r_1 \) and \( \epsilon(r_1) = \epsilon(0) \) (\( \epsilon(r_1) \geq \epsilon(0) \)). If \( r_1 \leq R \) is a point of discontinuity of \( \rho \), then the same argument applied to the interval \( [0, r_1] \) can be used in the interval \( [r_1, r_2] \) to show that \( \epsilon(r) \epsilon'(r) \geq 0 \) for \( r_1 \leq r \leq r_2 \). The argument can be repeated up to the interval \( [r_n, \infty) \) to conclude that for any \( w_0 \neq 0 \) and

\[ w(0) = y(0) = w_0 \neq 0, \quad \epsilon(0) = -\frac{3}{\overline{\rho}(0)}w_0, \quad (A.119) \]

equation (2.43) with the jump conditions in equation (2.45) has a solution such that \( \epsilon(r) \neq 0 \) and \( \epsilon(r) \epsilon'(r) \geq 0 \) for \( 0 \leq r < \infty \). In the following we show that the \( w \)-component of this solution is always different from zero.

The second and third equations in (2.43) imply

\[ (r^5 y)' = 5r^4 w, \quad r^5 \frac{7 \rho}{3} = -r^5 y, \quad \text{and} \quad \frac{d}{dr} \left( r^5 \frac{7 \rho}{3} \right) = -5r^4 w. \]

This last equation and \( \overline{\rho}(r) = \frac{2}{3} \left[ \rho(r) - \rho(r) \right] \), equation (2.27), imply

\[ \frac{2}{15} \overline{\rho}(r)\epsilon(r) + \frac{1}{5} \rho(0)\epsilon(0) + \frac{r}{15} \overline{\rho}(r)\epsilon'(r) = -w(r). \quad (A.120) \]

Since \( \epsilon(r) \neq 0 \) and \( \epsilon(r) \epsilon'(r) \geq 0 \) for \( 0 \leq r < \infty \), the left hand side of this equation is either strictly positive or strictly negative.

The boundary value problem in equations (2.43), (2.44), and (2.45) can be solved with the following algorithm. Let \( (\hat{w}, \hat{y}) \) be the solution to the differential equation (2.43) with the initial condition \( \hat{w}(0) = \hat{y}(0) = 1 \) and the jump conditions (2.45). Since \( \hat{w}(R) \neq 0 \), the desired solution \( (w, y) \) to the boundary value problem is the solution to equation (2.43) with the jump conditions (2.45) that satisfies the initial condition

\[ w(0) = y(0) = -\frac{\Omega^2}{8 \pi G} \frac{1}{\hat{w}(R)}. \]

Since \( w(R) \neq 0 \) if \( w_0 \neq 0 \), for \( \Omega = 0 \) the only solution to equation (2.43) with the jump conditions (2.45) that satisfies the boundary condition
\( w(R) = 0 \) in equation (2.44) is the trivial solution. This implies that the solution to the boundary value problem is unique (the difference between two different solutions would be a nontrivial solution to the problem with \( \Omega = 0 \)).

B Appendix: The gravitational potential, the gravitational energy, and the centrifugal energy.

The main goal in this appendix is to show that the gravitational energy of a deformed body

\[
U(\epsilon) = \frac{1}{2} \int_{\mathbb{R}^3} \tilde{\rho}(x)\Phi(x)dx,
\]

(B.121)
satisfies

\[
U(\epsilon) = \frac{32\pi^2 G}{45} V(\epsilon),
\]

(B.122)
where: \( V \) is the functional defined in equation (3.82), \( \epsilon \) is an arbitrary small deformation as those in the theory of Clairaut, and the equality holds up to terms of order \( \epsilon^2 \). We assume that \( \rho \) and \( \epsilon \) are \( C^2 \). Approximation arguments, as those in section 3.1, may be used to prove equation (B.122) under the same hypotheses of Theorem 3.1.

The interesting identity (B.122) was pointed out by one of the referees of the paper and it is non trivial since it requires computations up to the order of \( \epsilon^2 \), instead of only those of order \( \epsilon \) used to derive Clairaut’s equation. The notation and the theory used in this appendix are those in [36].

Let \( s \) be the mean radius of the density contour \( \tilde{\rho}(r, \theta) = \rho(s) \), where \( \rho \) is the radial density function of the body at rest and \( \tilde{\rho} \) is density function of a deformed state. For a given \( \theta \) the equation \( \tilde{\rho}(r, \theta) = \rho(s) \) can be solved for \( r(\theta, s) \). If \( r(\theta, s) \) is expanded into a sum of Legendre polynomials \( P_l(\cos \theta) \) and only the terms corresponding to \( l = 0 \) and \( l = 2 \) are retained, then (36 equation 27.8)

\[
r(\theta, s) = s \left[ 1 + s_0(s) + s_2(s) P_2(\cos \theta) \right].
\]

(B.123)
Due to the definition of \( s \), \( r(\theta, s) \) must satisfy (36 equation 27.6)

\[
\frac{4\pi}{3} s^3 = \int_0^{\pi} \int_0^{\theta(\theta, s)} \int_0^{2\pi} a^2 \sin \theta \, d\theta \, d\phi = \frac{2\pi}{3} \int_0^{\pi} r^3(\theta, s) \sin \theta d\theta.
\]

(B.124)
The identity \( \tilde{\rho}(r, \theta) = \rho(s) \) implies that equation (B.124) is a necessary condition for the conservation of mass under a deformation.
In Clairaut’s theory, for each given \( s \in (0, R] \), the curve \( \theta \to r(\theta, s) \) is an ellipsoid with small flattening \( \epsilon(s) \), see equation (2.19). This implies that up to first order in \( \epsilon(s) \): \( s_0(s) = 0 \) and \( s_2(s) = -\frac{2}{3}\epsilon(s) \). For this choice of \( s_0 \) and \( s_2 \), equation \( \text{(B.124)} \) is verified up to order \( \epsilon \) but not at order \( \epsilon^2 \). So, under deformation, mass is conserved up to order \( \epsilon \) but not at order \( \epsilon^2 \). While this is not a problem for obtaining Clairaut’s equation, which is correct up to order \( \epsilon \), it is a problem in obtaining the correct expression for the gravitational energy, which is a quantity of order \( \epsilon^2 \). This difficulty is overcome with the choice \( s_0 = -\frac{4}{45}\epsilon^2 \), which ensures that equation \( \text{(B.124)} \) is verified up to order \( \epsilon^2 \) but does not change the flattening of the deformed ellipsoids up to order \( \epsilon \). Therefore equation \( \text{(B.123)} \) becomes

\[
    r(\theta, s) = s \left[ 1 - \frac{2}{3}\epsilon(s)P_2(\cos \theta) - \frac{4}{45}\epsilon^2(s) \right]. \tag{B.125}
\]

This equation can be inverted up to order \( \epsilon^2 \) to give

\[
    s(\theta, r) = r \left[ 1 + \frac{2}{3}\epsilon(r)P_2(\cos \theta) + \frac{4}{45}\epsilon^2(r) \right] + \frac{2}{9} \frac{d}{dr} \left[ r^2 \epsilon^2(r) \right] \left[ P_2(\cos \theta) \right]^2. \tag{B.126}
\]

This equation and \( \tilde{\rho}(r, \theta) = \rho(s) \) implies that, up to order \( \epsilon^2 \), the density of the deformed body is given by

\[
    \tilde{\rho}(r, \theta) = \rho(r) + \rho'(r) \left\{ \frac{2}{3}r\epsilon(r)P_2(\cos \theta) + \frac{4}{45}r^2\epsilon^2(r) + \frac{2}{9} \frac{d}{dr} \left[ r^2 \epsilon^2(r) \right] \left[ P_2(\cos \theta) \right]^2 \right\} + \frac{1}{2}\rho''(r) \frac{4}{9} r^2 \epsilon^2(r) [P_2(\cos \theta)]^2. \tag{B.127}
\]

In order to compute the gravitational potential of the deformed body

\[
    \Phi(x) = -G \int_{\mathbb{R}^3} \rho(\tilde{x}) \frac{d\tilde{x}}{|x - \tilde{x}|}
\]

we use the expansions

\[
    \int_{0}^{2\pi} \frac{d\phi}{|x - \tilde{x}|} = \frac{2\pi}{r} \sum_{n=0}^{\infty} \frac{(\hat{r})^n}{r^n} P_n(\cos \hat{\theta})P_n(\cos \tilde{\theta}) \quad \text{for} \quad r > \hat{r}
\]

and

\[
    \int_{0}^{2\pi} \frac{d\phi}{|x - \tilde{x}|} = \frac{2\pi}{\hat{r}} \sum_{n=0}^{\infty} \frac{(\hat{r})^n}{r^n} P_n(\cos \hat{\theta})P_n(\cos \tilde{\theta}) \quad \text{for} \quad \hat{r} > r.
\]
Then, from the orthogonality relations of the Legendre polynomials and from equation (B.127), we obtain

\[
\Phi(r, \theta) = \Phi_0(r) + \tilde{\Phi}_0(r) + [\Phi_2(r) + \mathcal{O}(\epsilon^2)] P_2(\cos \theta) + \mathcal{O}(\epsilon^3)
\]

where

\[
\Phi_0(r) = -4\pi G \left\{ \frac{1}{r} \int_0^r a^2 \rho(a) da + \int_r^\infty a \rho(a) da \right\}
\]

\[
\tilde{\Phi}_0(r) = -\frac{8\pi}{45} G \int_r^\infty a^2 \rho'(a) e^2(a) da
\]

\[
\Phi_2(r) = -\frac{8\pi}{15} G \left\{ \frac{1}{r^3} \int_0^r a^5 e(a) \rho'(a) da + \int_r^\infty e(a) \rho'(a) da \right\}
\]

(B.128)

where we used \( \int_0^\pi [P_2(\cos \theta)]^3 \sin \theta d\theta = 4/35 \). Notice that \( \Phi_0 = \mathcal{O}(\epsilon^0) \) and \( \Phi_2 = \mathcal{O}(\epsilon) \) are the functions in equation (2.20) that are used in the derivation of Clairaut’s equation.

The gravitational energy in equation (B.121) can be written as

\[
U = \pi \int_0^\pi \int_0^\infty \tilde{\rho}(r, \theta) \left[ \Phi_0(r) + \tilde{\Phi}_0(r) + [\Phi_2(r) + \mathcal{O}(\epsilon^2)] P_2(\cos \theta) \right] \sin \theta d\theta \frac{dr^3}{3}
\]

(B.129)

where terms of order \( \mathcal{O}(\epsilon^3) \) were neglected. In order to compute this integral it is convenient to perform the change of variables \( r = r(\theta, s) \) in equation (B.125) that implies

\[
\tilde{\rho}(r, \theta) = \rho(s)
\]

\[
dr^3 d\theta = \left\{ ds^3 - 2P_2(\cos \theta) d[s^3 \epsilon(s)] - \frac{4}{15} d[s^3 \epsilon^2(s)] 
\right. 
\]

\[
+ \left. \frac{4}{3} [P_2(\cos \theta)]^2 d[s^3 \epsilon^2(s)] + \mathcal{O}(\epsilon^3) \right\} d\theta
\]

\[
\Phi_0(r) = \Phi_0(s) - \Phi'_0(s) \frac{2}{3} P_2(\cos \theta) \epsilon(s)
\]

\[
+ \left\{ \Phi''_0(s) \frac{2}{9} [P_2(\cos \theta)]^2 s^2 - \Phi'_0(s) \frac{4}{45} s \right\} \epsilon^2(s) + \mathcal{O}(\epsilon^3)
\]

\[
\tilde{\Phi}_0(r) = \tilde{\Phi}_0(s) + \mathcal{O}(\epsilon^3)
\]

\[
\Phi_2(r) = \Phi_2(s) - \Phi'_2(s) \frac{2}{3} P_2(\cos \theta) \epsilon(s) + \mathcal{O}(\epsilon^2)
\]

Using all these relations the integral in equation (B.129) can be computed up to order \( \epsilon^2 \) and, after a long computation, it gives

\[
U = \frac{4\pi}{15} \int_0^R \rho'(s) \epsilon(s) s^3 \left[ \Phi_2(s) - \Phi'_0(s) \frac{2}{3} s \epsilon(s) \right] ds
\]

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This equation and equation (3.82) imply equation (B.122).

The centrifugal potential is given by

$$\Phi_c = -\frac{\Omega^2}{2}(x_1^2 + x_2^2)$$

and the centrifugal energy by

$$U_c(\epsilon) = \int_{\mathbb{R}^3} \rho(x) \Phi_c(x) dx = -\Omega^2 \pi \int_0^\pi \int_0^\infty \tilde{\rho}(r, \theta) r^2 \sin^2 \theta \, r^2 \sin \theta dr \quad (B.130)$$

Again it is convenient to perform the change of variables $r = r(\theta, s)$ given in equation (B.125). Using that $\sin^2 \theta = \frac{2(1 - P_2(\cos \theta))}{3}$ and $I_0 = \frac{8\pi}{3} \int_0^R \rho(a^4 da$, a computation keeping terms up to order $\epsilon$ gives

$$U_c(\epsilon) = -\frac{\Omega^2}{2} I_0 - \Omega^2 \frac{8\pi}{45} \int_0^R \rho(s)[s^5\epsilon(s)]$$

The first term, which does not depend on the deformation, will be neglected. So

$$U_c(\epsilon) = -\Omega^2 \frac{8\pi}{45} \int_0^R \rho(s)[s^5\epsilon(s)] = -\Omega^2 \frac{8\pi}{45} \langle 1, \epsilon \rangle_{\rho} \quad (B.131)$$

where the inner product $\langle \cdot, \cdot \rangle_{\rho}$ is defined in equation (3.79).

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