A regular equilibrium solves the extended HJB system

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Abstract

Stochastic control problems that do not satisfy the dynamic programming principle are known as time-inconsistent. The game-theoretic approach is to interpret such problems as games and look for equilibrium controls instead of optimal controls. The main result of the game-theoretic approach to time-inconsistent stochastic control is a verification theorem based on the extended HJB system. It says that solving the extended HJB system is a sufficient condition for equilibrium. In the present paper we show that solving the extended HJB system is a necessary condition for equilibrium, under regularity conditions. The controlled process is a general Itô diffusion.

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1 Introduction

Consider a controlled process $X^u$ with initial data $(t, x)$ and the problem of choosing a control $u$ that maximizes

$$J(t, x, u) := \mathbb{E}_{t,x}[F(x, X^u_T)] + G(x, \mathbb{E}_{t,x}[X^u_T]),$$

(1)

where the functions $F$ and $G$ are deterministic. This problem is inconsistent in the sense that if a control $u$ is optimal for the initial data $(t, x)$ then $u$ is generally not optimal for other initial data $(s, y)$. This means that the dynamic programming principle does hold. Control problems of this kind are known as time-inconsistent.

Time-inconsistent problems are typically studied using either a game-theoretic approach, a pre-commitment approach or using the notion of dynamic optimality. For a comparison see [5, 15, 16].

The game-theoretic approach is to interpret the problem as an optimization problem for a person whose preferences change when $(t, x)$ changes. The person is interpreted as comprising versions of herself, one version for each $(t, x)$. These versions of the person are interpreted as agents playing a sequential game.
regarding how to control the process $X^u$. The game-theoretic approach is formalized by the definition of equilibrium.

Time-inconsistent problems were first studied in finance and economics. Here the time-inconsistency arises mainly due to non-exponential discounting, endogenous habit formation and mean-variance utility; for a description see [3,5]. Each of these types of problems can be formulated and studied in the framework of the present paper.

The game-theoretic approach to time-inconsistency was first used in [18] where utility maximization problems were studied. Other influential papers on time-inconsistency in economics and finance include [12,14,17,19]. Early papers of a more mathematical kind include [9,13,11] where non-exponential discounting was studied. The first general results on the game-theoretic approach to time-inconsistent stochastic control are due to the authors of [2,3] who defined the extended HJB system, which is a system of simultaneously determined PDEs, and proved the aforementioned verification theorem. Similar PDE approaches for particular problems were studied in [1,11]. The game-theoretic approach was also studied in [20,21,22], although there a slightly different equilibrium definition compared to that of [2] and the present paper was used. In [8], the equilibrium of a time-inconsistent control problem was characterized by a stochastic maximum principle. Mean-variance optimization is likely the most studied time-inconsistent problem. Different versions of this problem have recently been studied in [4,6,7,16]. A class of related problems are the time-inconsistent stopping problems, see e.g. [5] and the references therein. For more comprehensive surveys of the literature on time-inconsistent problems we refer to [3,5,15,16].

The rest of the paper is organized as follows. In Section 2 we formulate the time-inconsistent stochastic control problem in more detail and give the definition of equilibrium. In Section 3 we define the extended HJB system and prove the main result, Theorem 3.9. To illustrate the main result we study a simple example in Section 3.1. Section 3.2 contains a more general version of the main result.

2 Problem formulation

Consider a stochastic basis $(\Omega, F, P, \mathcal{F})$ where $\mathcal{F}$ is the augmented filtration generated by a $d$-dimensional Wiener process $W$. Consider a constant time horizon $T < \infty$ and an $n$-dimensional controlled SDE

$$dX_s = \mu(s, X_s, u(s, X_s))ds + \sigma(s, X_s, u(s, X_s))dW_s, \quad X_t = x, \quad t \leq s \leq T, \quad (2)$$

where

- $u: [0, T] \times \mathbb{R}^n \to \mathbb{R}^k$, and

- $\mu: [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $\sigma: [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to M(n, d)$ are continuous and satisfy standard global Lipschitz and linear growth and conditions, see e.g. [13] sec 5.2. $M(n, d)$ denotes the set of $n \times d$ matrices.

We also consider a control constraint mapping $U: [0, T] \times \mathbb{R}^n \to 2^{\mathbb{R}^k}$ that will be used to restrict the class of admissible controls, see Definition 2.2. It is
throughout this paper assumed that $U$ and the functions $F$ and $G$ in (1) satisfy Assumption 2.1.

**Assumption 2.1** $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $G \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$. The control constraint mapping $U : [0, T] \times \mathbb{R}^n \to 2^{\mathbb{R}^k}$ is continuous.

**Definition 2.2** The set of admissible controls is denoted by $U$. A control $u$ is said to be admissible if the following conditions hold:

- $u(t, x) \in U(t, x)$ for each $(t, x) \in [0, T] \times \mathbb{R}^n$.
- For each initial data point $(t, x) \in [0, T] \times \mathbb{R}^n$, the SDE (2) has a unique strong solution with the Markov property, denoted by $X^u$, satisfying $E_{t,x}[||F(x, X^u_T)||] < \infty$ and $E_{t,x}[||X^u_T||] < \infty$.

**Definition 2.3** For any $u \in U$ the auxiliary functions $f_u : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $g_u : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ are defined by

\[
 f_u(t, x, y) = E_{t,x}[F(y, X^u_T)] \text{ and } g_u(t, x) = E_{t,x}[X^u_T].
\]

We are now ready to define the equilibrium for the time-inconsistent stochastic control problem corresponding to (1) in line with [2]. For a motivation of this type of equilibrium see [2, 3].

**Definition 2.4 (Equilibrium)**

- Consider a point $(t, x) \in [0, T] \times \mathbb{R}^n$, two controls $u, \hat{u} \in U$ and a constant $h > 0$. Let

\[
 u_h(s, y) := \begin{cases} u(s, y), & \text{for } t \leq s < t + h, \ y \in \mathbb{R}^n \\ \hat{u}(s, y), & \text{for } t + h \leq s \leq T, \ y \in \mathbb{R}^n. \end{cases}
\]

- The control $\hat{u} \in U$ is said to be an equilibrium control if, for any fixed initial data point $(t, x) \in [0, T] \times \mathbb{R}^n$ and any $u \in U$, it satisfies the equilibrium condition

\[
 \liminf_{h \searrow 0} \frac{J(t, x, \hat{u}) - J(t, x, u_h)}{h} \geq 0. \tag{3}
\]

- If $\hat{u}$ is an equilibrium control then $V_{\hat{u}}$ defined by $V_{\hat{u}}(t, x) = J(t, x, \hat{u})$ is said to be the corresponding equilibrium value function and the quadruple $(\hat{u}, V_{\hat{u}}, f_{\hat{u}}, g_{\hat{u}})$ is said to be the corresponding equilibrium.

The following definition will be used throughout this paper.

**Definition 2.5**

- The differential operator $A^u$ corresponding to the controlled SDE (2) is defined by

\[
 A^u = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mu_i(t, x, u(t, x)) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij}^2(t, x, u(t, x)) \frac{\partial^2}{\partial x_i \partial x_j}.
\]
For a function \( f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) placing the third variable as a superscript, \( f^y(t, x) = f(t, x, y) \), means that \( y \) is supposed to be taken as a constant parameter. For example, \( f^y \in C^{1,2}([0, T] \times \mathbb{R}^n) \) means that \( f(t, x, y) \) is continuously differentiable with respect to \( t \) and \( x \) twice continuously differentiable with respect to \( x \) for a fixed \( y \) and \( A^y f^y(t, x) \) involves only derivatives with respect to \( t \) and \( x \). Moreover, \( A^y f(t, x, x) \) should be interpreted as \( A^y f(t, x) \) with \( f(t, x) := f(t, x, y) \).

For a function \( g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^m \) we write \( g(t, x) = (g_1(t, x), \ldots, g_m(t, x))^T \) and let \( A^u g(t, x) := (A^u g_1(t, x), \ldots, A^u g_m(t, x))^T \). Moreover, the operator \( H^u \) is defined by

\[
H^u g(t, x) = G_g(x, g(t, x)) A^u g(t, x), \quad \text{where } G_g(x, y) := \frac{\partial G}{\partial y}(x, y). \tag{4}
\]

We use the notation

\[
G \circ g(t, x) = G(x, g(t, x)). \tag{5}
\]

We remark that [1], Definition [2], Definition [3] and [5] directly imply that

\[
V_\tilde{u}(t, x) = J(t, x, \tilde{u}) = f_\tilde{u}(t, x, x) + G \circ g_\tilde{u}(t, x). \tag{6}
\]

## 3 The main result

Let us first define the extended HJB system in line with [2].

**Definition 3.1 (Extended HJB system)** For \((t, x, y) \in [0, T) \times \mathbb{R}^n \times \mathbb{R}^n\),

\[
\begin{align*}
A^uf^y(t, x) &= 0, \tag{7} \\
f^y(T, x) &= F(y, x), \\
A^ug(t, x) &= 0, \tag{8} \\
g(T, x) &= x,
\end{align*}
\]

\[
\sup_{u \in U(t, x)} \{A^uV(t, x) - A^uf(t, x, x) + A^uf^2(t, x) + A^uG(t, x) + H^u g(t, x)\} = 0, \tag{9}
\]

where

\[
\tilde{u}(t, x) \in \arg \sup_{u \in U(t, x)} \{A^uV(t, x) - A^uf(t, x, x) + A^uf^2(t, x) + A^uG(t, x) + H^u g(t, x)\} = 0. \tag{10}
\]

**Remark 3.2** For a fixed function \( \tilde{u} \) the equations (7) and (8) are Kolmogorov backward equations. For fixed functions \( f \) and \( g \) the equation (9) is an HJB equation. The non-standard attribute of (7)–(9) is that \( \tilde{u}, f \) and \( g \) are not fixed in this way. Instead, (7)–(9) is system that is simultaneously determined through (10). Let us describe what constitutes a solution: If four functions \( V : [0, T] \times \mathbb{R}^n \to \mathbb{R}, f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^m \) and \( \tilde{u} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^k \), where \( \tilde{u}(t, x) \in U(t, x) \) for each \((t, x) \in [0, T] \times \mathbb{R}^n\), satisfy the following conditions then \((\tilde{u}, V, f, g)\) is a solution to the extended HJB system:
• The functions \( f^y \) and \( \bar{u} \) satisfy (7), for each fixed \( y \in \mathbb{R}^n \).

• The functions \( g \) and \( \bar{u} \) satisfy (8).

• The boundary condition \( V(T, x) = F(x, x) + G(x, x) \) is satisfied.

• For each fixed \( (t, x) \in [0, T) \times \mathbb{R}^n \), the inequality \( A^u V(t, x) - A^u f(t, x) + A^u f^y (t, x) - A^u G \circ g(t, x) + H^u g(t, x) \leq 0 \) holds for each constant \( u \in U(t, x) \), and it holds with equality for the constant \( u := \bar{u}(t, x) \).

In order to prove the main result of the this paper, Theorem 3.9, we need Lemma 3.4, Lemma 3.5 and Proposition 3.6 below. We remark that Lemma 3.4 and Lemma 3.5 are versions of the Feynman-Kac formula. A proof is included for the sake of completeness. We will use the following definition.

**Definition 3.3** Consider a control \( u \in U \). A function \( h : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \) is said to belong the set \( L^2_T(X^u) \) if, for each initial data point \( (t, x) \in [0, T) \times \mathbb{R}^n \),

\[
\mathbb{E}_{t,x} \left[ \int_t^T \left( |A^u h(s, X^u_s)| + \left| \frac{\partial h}{\partial x} (s, X^u_s) \sigma(s, X^u_s, u(s, X^u_s)) \right|^2 \right] ds \right] < \infty.
\]

**Lemma 3.4** Consider a continuous control \( u \in U \). Suppose that the auxiliary function \( f_u \) satisfies \( f_u^y \in C^{1,2}([0, T) \times \mathbb{R}^n) \cap L^2_T(X^u) \), for any fixed \( y \in \mathbb{R}^n \). Then, \( f_u^y \) is a solution to the PDE

\[
A^u f^y(t, x) = 0,
\]

\[
f^y(T, x) = F(y, x), \quad (t, x) \in [0, T) \times \mathbb{R}^n,
\]

for any fixed \( y \in \mathbb{R}^n \).

**Proof.** By definition \( f_u^y(t, x) = \mathbb{E}_{t,x}[F(y, X^u_{t+h})] \) and the boundary condition is therefore immediately satisfied. Consider an arbitrary point \( (t, x, y) \in [0, T) \times \mathbb{R}^n \times \mathbb{R}^n \). Let \( X^u \) be the strong solution to the SDE (2) for the initial data \( (t, x) \). Consider an arbitrary constant \( h > 0 \), satisfying \( t + h < T \). The Markov property gives us

\[
\mathbb{E}_{t,x} \left[ f_u^y(t + h, X^u_{t+h}) \right] - f_u^y(t, x) = 0.
\]

From Itô’s formula we obtain,

\[
\mathbb{E}_{t,x} \left[ f_u^y(t + h, X^u_{t+h}) \right] = f_u^y(t, x)
\]

\[
= \mathbb{E}_{t,x} \left[ \int_t^{t+h} A^u f_u^y(s, X^u_s)ds + \int_t^{t+h} \frac{\partial f_u^y}{\partial x}(s, X^u_s) \sigma(s, X^u_s, u(s, X^u_s))dW_s \right]
\]

\[
= \mathbb{E}_{t,x} \left[ \int_t^{t+h} A^u f_u^y(s, X^u_s)ds \right],
\]

where the Itô integral vanished since \( f_u^y \in L^2_T(X^u) \). It follows that

\[
\mathbb{E}_{t,x} \left[ \frac{\int_t^{t+h} A^u f_u^y(s, X^u_s)ds}{h} \right] = 0. \tag{11}
\]
Then, consider two controls \( g \) and \( \tilde{v} \). Suppose that the auxiliary functions \( f \) and \( \tilde{v} \) satisfy \( f \in C^{1,2}([0,T] \times \mathbb{R}^n) \cap L^2_t(X^u) \), for any fixed \( y \in \mathbb{R}^n \) and \( i = 1, \ldots, n \). Consider the initial data \((t, x) \in [0,T] \times \mathbb{R}^n \). Let

\[
\tilde{v}_h(s, y) := \begin{cases} 
v(s, y), & \text{for } t \leq s < t + h, \ y \in \mathbb{R}^n \\
\tilde{v}(s, y), & \text{for } t + h \leq s \leq T, \ y \in \mathbb{R}^n.
\end{cases}
\]

Then,

\[
\lim_{h \searrow 0} \frac{f_h(t, x, x) - f_{\tilde{v}_h}(t, x, x)}{h} = -A^u f_h(t, x), \tag{13}
\]

and

\[
\lim_{h \searrow 0} \frac{G \circ g_h(t, x) - G \circ g_{\tilde{v}_h}(t, x)}{h} = -H^u g_h(t, x), \tag{14}
\]

where \( v := \tilde{v}(t, x) \).

**Proof.** Consider the initial data \((t, x) \in [0,T] \times \mathbb{R}^n \) and an arbitrary constant \( h > 0 \) satisfying \( t + h < T \). Using Itô’s formula we obtain

\[
\mathbb{E}_{t,x} \left[ f_h(t + h, X_{t+h}^y) - f_h(t, x) \right] = \mathbb{E}_{t,x} \left[ \int_t^{t+h} A^\gamma f_h(s, X_s^\gamma) ds \right],
\]

where the Itô integral vanished as in Lemma 3.4. By definition, \( v_h \) and \( \tilde{v} \) coincide on \([t, t+h]\), except at the point \( t + h \). By definition, \( v_h(s, y) \) and \( \tilde{v}(s, y) \) coincide on \([t+h, T]\). Thus,

\[
\mathbb{E}_{t,x} \left[ f_h(t + h, X_{t+h}^y) \right] = \mathbb{E}_{t,x} \left[ E_{t+h, X_{t+h}^y} [F(x, X_t^y)] \right] = \mathbb{E}_{t,x} \left[ E_{t+h, X_{t+h}^x} [F(x, X_T^y)] \right] = f_{\tilde{v}_h}(t, x, x).
\]

The result follows from (11) and (12). \( \square \)

**Lemma 3.5** Consider a continuous control \( u \in U \). Suppose that the elements of the auxiliary function \( g_u \) satisfy \( g_u, i \in C^{1,2}([0,T] \times \mathbb{R}^n) \cap L^2_t(X^u) \), \( i = 1, \ldots, n \). Then \( g_u \) is a solution to the PDE

\[
A^u g(t, x) = 0, \\
g(T, x) = x, \ \text{for } (t, x) \in [0,T] \times \mathbb{R}^n.
\]

**Proof.** The proof is analogous to that of Lemma 3.4 and is omitted. \( \square \)

**Proposition 3.6** Consider two controls \( v, \tilde{v} \in U \) where \( v \) is continuous. Suppose that the auxiliary functions \( f \) and \( g \tilde{v} \) satisfy \( f \in C^{1,2}([0,T] \times \mathbb{R}^n) \cap L^2_t(X^v) \) for any fixed \( y \in \mathbb{R}^n \) and \( i = 1, \ldots, n \). Consider the initial data \((t, x) \in [0,T] \times \mathbb{R}^n \). Let

\[
\tilde{v}_h(s, y) := \begin{cases} 
v(s, y), & \text{for } t \leq s < t + h, \ y \in \mathbb{R}^n \\
\tilde{v}(s, y), & \text{for } t + h \leq s \leq T, \ y \in \mathbb{R}^n.
\end{cases}
\]

Then,

\[
\lim_{h \searrow 0} \frac{f_h(t, x, x) - f_{\tilde{v}_h}(t, x, x)}{h} = -A^v f_h(t, x), \tag{13}
\]

and

\[
\lim_{h \searrow 0} \frac{G \circ g_h(t, x) - G \circ g_{\tilde{v}_h}(t, x)}{h} = -H^v g_h(t, x), \tag{14}
\]

where \( v := \tilde{v}(t, x) \).

**Proof.** Consider the initial data \((t, x) \in [0,T] \times \mathbb{R}^n \) and an arbitrary constant \( h > 0 \) satisfying \( t + h < T \). Using Itô’s formula we obtain

\[
\mathbb{E}_{t,x} \left[ f_h(t + h, X_{t+h}^y) - f_h(t, x) \right] = \mathbb{E}_{t,x} \left[ \int_t^{t+h} A^\gamma f_h(s, X_s^\gamma) ds \right],
\]

where the Itô integral vanished as in Lemma 3.4. By definition, \( v_h \) and \( v \) coincide on \([t, t+h]\), except at the point \( t + h \). By definition, \( v_h(s, y) \) and \( \tilde{v}(s, y) \) coincide on \([t+h, T]\). Thus,

\[
\mathbb{E}_{t,x} \left[ f_h(t + h, X_{t+h}^y) \right] = \mathbb{E}_{t,x} \left[ E_{t+h, X_{t+h}^y} [F(x, X_t^y)] \right] = \mathbb{E}_{t,x} \left[ E_{t+h, X_{t+h}^x} [F(x, X_T^y)] \right] = f_{\tilde{v}_h}(t, x, x).
\]
From the above follows that
\[ f_{v_h}(t, x, x) - f_{\tilde{v}}(t, x) = \mathbb{E}_{t,x} \left[ \int_t^{t+h} A^\nu f_{\tilde{v}}(s, X^\nu_s) ds \right]. \]

Using arguments analogous to those in the proof of Lemma 3.4 we thus obtain
\[ \lim_{h \downarrow 0} \frac{f_v(t, x, x) - f_{v_h}(t, x, x)}{h} = \lim_{h \downarrow 0} \frac{-\mathbb{E}_{t,x} \left[ \int_t^{t+h} A^\nu f_{\tilde{v}}(s, X^\nu_s) ds \right]}{h} = -A^\nu_x f_{\tilde{v}}(t, x), \]
which, since \( v := v(t, x) \), means that (13) holds. Using the same arguments as above we obtain
\[ g_{v_h}(t, x) = g_{\tilde{v}}(t, x) + \mathbb{E}_{t,x} \left[ \int_t^{t+h} A^\nu g_{\tilde{v}}(s, X^\nu_s) ds \right]. \]

Standard Taylor expansion gives us
\[ G \left( x, g_{\tilde{v}}(t, x) + \mathbb{E}_{t,x} \left[ \int_t^{t+h} A^\nu g_{\tilde{v}}(s, X^\nu_s) ds \right] \right) = G \left( x, g_{\tilde{v}}(t, x) \right) + G_y \left( x, g_{\tilde{v}}(t, x) \right) \mathbb{E}_{t,x} \left[ \int_t^{t+h} A^\nu g_{\tilde{v}}(s, X^\nu_s) ds \right] + o(h). \]

Hence,
\[ \lim_{h \downarrow 0} \frac{G(x, g_{\tilde{v}}(t, x)) - G(x, g_{v_h}(t, x))}{h} = \lim_{h \downarrow 0} \frac{-G_y \left( x, g_{\tilde{v}}(t, x) \right) \mathbb{E}_{t,x} \left[ \int_t^{t+h} A^\nu g_{\tilde{v}}(s, X^\nu_s) ds \right] + o(h)}{h} \]
\[ = -G_y \left( x, g_{\tilde{v}}(t, x) \right) A^\nu_x g_{\tilde{v}}(s, x) = -\mathbf{H}^\nu g_{\tilde{v}}(t, x), \]
which means that (14) holds. \( \square \)

Let us now define what we mean by a regular equilibrium and prove the main result.

**Definition 3.7** An equilibrium \((\hat{u}, V_\hat{u}, f_{\hat{u}}, g_{\hat{u}})\) is said to be regular if:

(i). The equilibrium control \( \hat{u} \) is continuous.

(ii). \( f_{\hat{u}}, g_{\hat{u}, i} \in L^2(T, X^\nu) \) and \( f_{\hat{u}}, g_{\hat{u}, i}, \hat{f} \in C^{1,2}([0, T) \times \mathbb{R}^n) \) for each fixed \( y \in \mathbb{R}^n \) and \( i = 1, \ldots, n \), where \( \hat{f}(t, x) := f_{\hat{u}}(t, x, x) \).

(iii). For each \((t, x) \in [0, T) \times \mathbb{R}^n\) and each \( u \in U(t, x) \), there exists a continuous control \( u \) with \( u(t, x) = u \) such that \( f_{\hat{u}^u, i}, g_{\hat{u}, i} \in L^2(T, X^\nu) \).

Regarding the technical condition (iii) we remark that Assumption 2.1 ensures that for each \((t, x)\) and each \( u \in \hat{U}(t, x) \), there exists a continuous control \( \hat{u} \) with \( \hat{u}(t, x) = u \).
Remark 3.8 A simple problem with a regular equilibrium is studied in Section 3.1.

Theorem 3.9 A regular equilibrium \((\hat{u}, V_\hat{u}, f_\hat{u}, g_\hat{u})\) solves the extended HJB system.

Proof. It follows from Lemma 3.4 that the auxiliary function \(f_\hat{u}(t, x)\) and the equilibrium control \(\hat{u}\) satisfy (7), for each \(y \in \mathbb{R}^n\). It follows from Lemma 3.5 that the auxiliary function \(g_\hat{u}(t, x)\) and \(\hat{u}\) satisfy (8). Sufficient regularity for the use of these lemmas is provided by (i) and (ii) in Definition 3.7.

The boundary condition \(V(T, x) = F(x, x) + G(x, x)\) in (9) is trivially verified using (5), (6) and Definition 2.3.

Consider an arbitrary point \((t, x) \in [0, T) \times \mathbb{R}^n\). In order to prove that the equilibrium \((\hat{u}, V_\hat{u}, f_\hat{u}, g_\hat{u})\) is a solution to the extended HJB system we only have left to show that the following inequality holds for any \(u \in U(t, x)\) and that it holds with equality for \(u := \hat{u}(t, x)\):

\[
\begin{align*}
\mathbf{A}^u V_\hat{u}(t, x) - \mathbf{A}^u f_\hat{u}(t, x, x) + A^u f_\hat{u}(t, x) \\
- \mathbf{A}^u G \circ g_\hat{u}(t, x) + H^u g_\hat{u}(t, x) \leq 0.
\end{align*}
\]  

(15)

Consider an arbitrary \(u \in U(t, x)\). It directly follows from (6) that

\[
\begin{align*}
\mathbf{A}^u V_\hat{u}(t, x) = \mathbf{A}^u f_\hat{u}(t, x, x) + A^u G \circ g_\hat{u}(t, x),
\end{align*}
\]  

(16)

where differentiability is provided by (iii) and Assumption 2.1. Consider a continuous control \(U\) satisfying \(u(t, x) = u\) such that the regularity condition \(f_\hat{u}, g_\hat{u}, i \in L^2_T(X^u)\) is fulfilled, cf. (iii). We may then use Proposition 3.6 to obtain,

\[
\begin{align*}
&\lim_{n \searrow 0} \frac{f_\hat{u}(t, x, x) + G \circ g_\hat{u}(t, x) - (f_{u_n}(t, x, x) + G \circ g_{u_n}(t, x))}{h} \\
&= -H^u g_\hat{u}(t, x) - A^u f_\hat{u}(t, x) \\
&= -(\mathbf{A}^u V_\hat{u}(t, x) - \mathbf{A}^u f_\hat{u}(t, x, x) + A^u f_\hat{u}(t, x) \\
&- \mathbf{A}^u G \circ g_\hat{u}(t, x) + H^u g_\hat{u}(t, x)).
\end{align*}
\]  

(17)

where we in the last step used (16). Now use the definition of \(J(t, x, u)\) to obtain

\[
\begin{align*}
&\lim_{n \searrow 0} \frac{f_\hat{u}(t, x, x) + G \circ g_\hat{u}(t, x) - (f_{u_n}(t, x, x) + G \circ g_{u_n}(t, x))}{h} \\
&= \lim_{h \searrow 0} \frac{J(t, x, \hat{u}) - J(t, x, u_n)}{h} \\
&\geq 0,
\end{align*}
\]  

(18)

where the inequality is due to the assumption that \(\hat{u}\) is an equilibrium control, cf. the equilibrium condition (3). Recall that \(u \in U(t, x)\) was arbitrarily chosen. Hence, (17) and (18) imply that (15) holds for any \(u \in U(t, x)\).

Since \(f_\hat{u}\) and \(\hat{u}\) satisfy (7) for any \(y\) we know that \(\mathbf{A}^\hat{u} f_\hat{u}(t, x, x) = 0\). Since \(g_\hat{u}\) and \(\hat{u}\) satisfy (8) we know that \(\mathbf{A}^\hat{u} g_\hat{u}(t, x, x) = 0\) which with (11) gives us \(H^\hat{u} g_\hat{u}(t, x) = 0\). From (10) we know that \(\mathbf{A}^\hat{u} V_\hat{u}(t, x) = \mathbf{A}^\hat{u} f_\hat{u}(t, x, x) + \mathbf{A}^\hat{u} G \circ g_\hat{u}(t, x)\). Hence,

\[
\begin{align*}
\mathbf{A}^\hat{u} V_\hat{u}(t, x) - \mathbf{A}^\hat{u} f_\hat{u}(t, x, x) + A^\hat{u} f_\hat{u}(t, x) - \mathbf{A}^\hat{u} G \circ g_\hat{u}(t, x) + H^\hat{u} g_\hat{u}(t, x) = 0.
\end{align*}
\]

This means that (15) holds with equality for \(u := \hat{u}(t, x)\). \(\square\)
3.1 An example

To illustrate Theorem 3.9 we here study a very simple time-inconsistent control problem described as follows. The controlled SDE is one-dimensional and given by
\[
dX_s = -u(s, X_s)^2 ds + \sigma dW_s,
\]
where \( \sigma > 0 \) is constant. Admissible controls are restricted to the interval \( U = [-a, a] \) for some constant \( a > 0 \). The functions \( F \) and \( G \) are defined by \( F(x, y) = (x - y)^2 \) and \( G(x, y) = 0 \). This implies that
\[
J(t, x, u) = \mathbb{E}_t, x \left[ (X^u_T - x)^2 \right].
\]

We make the ansatz that an equilibrium control is given by \( \hat{u} = 0 \). With simple calculations we obtain the corresponding auxiliary functions, \( g_{\hat{u}}(t, x) = x \), and \( f_{\hat{u}}(t, x, y) = (x - y)^2 + \sigma^2(T - t) \).

It follows that,
\[
\frac{\partial f_{\hat{u}}(t, x)}{\partial t} = -\sigma^2, \quad \frac{\partial f_{\hat{u}}(t, x)}{\partial x} = 2x - 2y, \quad \text{and} \quad \frac{\partial^2 f_{\hat{u}}(t, x)}{\partial x^2} = 2.
\]

Let us now show that the control \( \hat{u} = 0 \) does indeed satisfy the equilibrium condition (3). Consider an arbitrary control \( u \in U \) and an arbitrary point \((t, x) \in [0, T) \times \mathbb{R}\). Using Itô’s formula and the derivatives above we obtain
\[
\mathbb{E}_t, x \left[ f_{\hat{u}}(t + h, X^u_{t+h}) \right] - f_{\hat{u}}(t, x) = \mathbb{E}_t, x \left[ \int_t^{t+h} A^u(s, X^u_s) ds + \sigma \int_t^{t+h} \frac{\partial f_{\hat{u}}(s, X^u_s)}{\partial x} dW_s \right],
\]

Using arguments similar to those in the proof of Proposition 3.6 we also obtain \( \mathbb{E}_t, x \left[ f_{\hat{u}}(t + h, X^u_{t+h}) \right] = f_{u_h}(t, x, x) \). Hence,
\[
f_{u_h}(t, x, x) - f_{\hat{u}}(t, x, x) = \mathbb{E}_t, x \left[ \int_t^{t+h} u(s, X^u_s)^2(2x - 2X^u_s) ds \right].
\]

Recall that \( G(x, y) = 0 \) and \( U = [-a, a] \). Hence,
\[
J(t, x, \hat{u}) - J(t, x, u_h) = f_{\hat{u}}(t, x, x) - f_{u_h}(t, x, x)
\]
\[
= \mathbb{E}_t, x \left[ \int_t^{t+h} u(s, X^u_s)^2(2X^u_s - 2x) ds \right]
\]
\[
\geq -2a^2 \mathbb{E}_t, x \left[ \int_t^{t+h} |X^u_s - x| ds \right].
\]

Hence,
\[
\lim_{h \to 0} \frac{J(t, x, \hat{u}) - J(t, x, u_h)}{h} \geq 0.
\]
We have thus shown that the equilibrium condition (3) holds and \( \hat{u} = 0 \) is therefore an equilibrium control. The corresponding equilibrium is

\[
(\hat{u}, V_\hat{u}, f_\hat{u}, g_\hat{u}) = (0, \sigma^2(T-t), (x-y)^2 + \sigma^2(T-t), x).
\]

The equilibrium is easily verified as regular. From Theorem 3.9 it therefore follows that the equilibrium solves the corresponding extended HJB system.

We remark that it is easy, in this example, to verify that this is the case.

### 3.2 A more general problem

In this section we include a running time function \( H \) and we also allow \( F \) and \( G \) to depend on the initial time \( t \). Specifically, in this section we consider

\[
J(t, x, u) := E_t,x \left[ \int_t^T H(t, x, r, X^n_t, u(r, X^n_t)) dr + F(t, x, X^n_T) \right] + G(t, x, X^n_T)
\]

where \( F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( G : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R} \) satisfy conditions analogous to those in Assumption 2.1 and \( H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R} \) is continuous and bounded. The definition of an admissible control is analogous to Definition 2.2. Here we let

\[
G \circ g(t, x) := G(t, x, g(t, x)),
\]

\[
H^n g(t, x) := G_y(t, x, g(t, x)) A^n g(t, x), \quad \text{and}
\]

\[
f_u(t, x, s, y) := E_{t, x} \left[ \int_t^T H(s, y, r, X^n_r, u(r, X^n_r)) dr + F(s, y, X^n_T) \right].
\]

The equilibrium definition is analogous to Definition 2.4. Placing the third and fourth variables of a function \( f : [0, T] \times \mathbb{R}^n \times [0, T] \times \mathbb{R}^n \to \mathbb{R} \) as superscripts, \( f^{s,y}(t, x) = f(t, x, s, y) \), means \( s \) and \( y \) are supposed to be taken as constant.

**Definition 3.10 (Extended HJB system II)** For \((t, x, s, y) \in [0, T) \times \mathbb{R}^n \times [0, T) \times \mathbb{R}^n\),

\[
A^u f^{s,y}(t, x) + H(s, y, t, x, \hat{u}(t, x)) = 0,
\]

\[
f^{s,y}(T, x) = F(s, y, x),
\]

\[
A^u g(t, x) = 0,
\]

\[
g(T, x) = x,
\]

\[
\sup_{u \in U(t,x)} \{ A^u V(t, x) - A^u f(t, x, t, x) + A^u f^{t,x}(t, x) - A^u G \circ g(t, x) + H^n g(t, x) + H(t, x, t, x, u) \} = 0,
\]

\[
V(T, x) = F(T, x, x) + G(T, x, x),
\]

where

\[
\hat{u}(t, x) \in \arg \sup_{u \in U(t,x)} \{ A^u V(t, x) - A^u f(t, x, t, x) + A^u f^{t,x}(t, x) - A^u G \circ g(t, x) + H^n g(t, x) + H(t, x, t, x, u) \} = 0.
\]
The definition of a regular equilibrium is analogous to that of Definition 3.7. Theorem 3.11 generalizes the main result of this paper to the present setting.

**Theorem 3.11** A regular equilibrium \((\hat{u}, V_{\hat{u}}, f_{\hat{u}}, g_{\hat{u}})\) solves the extended HJB system II.

**Proof.** The proof is very similar to that of Theorem 3.9 and is omitted. \(\square\)

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