The Possibility of Factorizable Contextual
Hidden Variable Theories

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Abstract

Considering an extended type of Bohm’s version of EPR thought experiment, we derive Bell’s inequality for the case of factorizable contextual hidden variable theories which are consistent with the predictions of quantum theory. Usually factorizability is associated with non-contextuality. Here, we show that factorizability is consistent with contextuality, even for the ordinary Bohm’s version of the EPR thought experiment.

1 Introduction

In the derivation of almost all Bell inequalities the assumption is made that all dynamical variables involved are not compatible. It is also a common understanding that in thought experiments like Bohm’s version [1] of EPR’s [2] (hereafter called EPRB), the violation of all Bell inequalities are due to non-local effects between the two correlated particles. In the case of systems involving two correlated particles, it is generally believed that Bell’s locality condition (i.e. factorizability) is equivalent to non-contextuality and that non-contextual hidden variable theories are incompatible with the standard

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quantum mechanics [3,4]. This kind of incompatibility has been primarily discussed by Gleason [5], Bell [6] and Kochen-Specker [7]. Nevertheless, the meaning of contextuality has not been expanded very well. In the present paper, we extend the notion of contextuality as originating from one of the following sources or both of them:

1) State preparation of the system [8,9];
2) Measured value of a dynamical variable is dependent on what other commuting dynamical variables are being measured along with it [3].

The significance of this distinction is made clear in section 2, where we make use of a thought experiment for four compatible observables, resembling the EPRB thought experiment, in which non-contextuality (in the sense of factorizability) is not violated. Considering the first version of contextuality, we show that the possibility of having compatibility between factorizable contextual hidden variable theories and quantum mechanics exists. Then, in section 3 we argue that even for the EPRB thought experiment this possibility remains open.

2 An EPRB-type Thought experiment

We consider a source of spin zero systems which emits two spin-1/2 particles. On its way, each particle meets two Stern-Gerlach apparatus (S.G$_1$ and S.G$_2$) at times $t_1$ and $t_2$ ($t_2 > t_1$), respectively. The spin of particle 1 (2) is being measured at either $t_1$ or $t_2$ along $\hat{a}$ ($\hat{b}$) or $\hat{a}'$ ($\hat{b}'$) respectively. We represent the results of each measurement in the following way:

$$\sigma_{a,1}^{(t_1)} \rightarrow A_1; \quad \sigma_{a',1}^{(t_2)} \rightarrow A_2; \quad \sigma_{b,2}^{(t_1)} \rightarrow B_1; \quad \sigma_{b',2}^{(t_2)} \rightarrow B_2$$

where $\sigma_{a,1}^{(t_1)}$ refers to the spin component of particle 1 measured along $\hat{a}$ at time $t_1$, and similarly for others. Using the units of $\hbar/2$, the quantities $A_1, B_1, A_2$, and $B_2$ take the values $\pm 1$. 
In a non-contextual hidden variable theory, we define non-contextuality as a conjunction of the following two assumptions for four compatible dynamical variables \( \sigma_{a,1}^{(t_1)}, \sigma_{a',1}^{(t_1)}, \sigma_{b,2}^{(t_1)}, \sigma_{b',2}^{(t_2)} \):

\[ C_1 \] The value obtained for the spin of each particle at \( t_2 \) is independent of the preparation made for the spin state of the same particle at an earlier time \( t_1 \) (i.e., the orientation of the Stern-Gerlach apparatus through which the particle passed at \( t_1 \)).

Here, it has been taken for granted that the result obtained for a particle’s property at any time is independent of the presence of any apparatus in the path of the same particle at a later time.

\[ C_2 \] For the whole system, the result obtained for each particle should be independent of the measurement made on the other particle at that time or at any other time. This is equivalent to Bell’s locality condition.

We consider this experiment in the light of a hidden variable theory and assume that the separation of the two particles 1 and 2 are space-like. Then we make use of the following postulates:

\[ P_1 \] The spin state of the particles 1 and 2 is described by a function of a collection of hidden variables called \( \lambda \), which belongs to the space \( \Lambda \). The parameter \( \lambda \) contains all the information which is necessary to specify the spin state of the system.
One can define a grand joint probability \( p_{GJP} \) for the dynamical variables \( \sigma^{(t_1)}_{a,1}, \sigma^{(t_1)}_{a',1}, \sigma^{(t_2)}_{b,2}, \sigma^{(t_2)}_{b',2} \) on the space \( \Lambda \), representing the statistical informations which are possible to obtain from the spin state of the system.

Using \( P_1 \), it would be possible to obtain the marginal probabilities from \( p_{GJP} \). For example, we have

\[
p(A_2, B_2|\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) = \sum_{A_1, B_1} A_1 B_1 p_{GJP}(A_1, B_1, A_2, B_2|\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) \quad (1)
\]

\[
p(A_1|\hat{a}, \hat{b}, \lambda) = \sum_{A_2, B_1, B_2} A_2 B_1 B_2 p_{GJP}(A_1, B_1, A_2, B_2|\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) \quad (2)
\]

The relation (1) gives the probability that the joint values of the spin components of the particles 1 and 2 along \( \hat{a}' \) and \( \hat{b}' \) at \( t_2 \) being \( A_2 \) and \( B_2 \) respectively, assuming that \( \lambda \) and the directions \( \hat{a}, \hat{b}, \hat{a}', \text{and} \hat{b}' \) are completely specified. Similarly, (2) gives the probability that the value of particle 1’s spin component along \( \hat{a} \) at \( t_1 \) being \( A_1 \), assuming that \( \lambda \) and \( \hat{a} \) and \( \hat{b} \) are known. This probability is independent of the directions \( \hat{a}' \) and \( \hat{b}' \) related to time \( t_2 \).

\( P_2 \). Any information about the values of the spin components of particles 1 and 2, which are spatially separated, originates from \( \lambda \) (common causes) and is not related to any influence of one particle over the other.

Due to \( P_2 \), probability functions of the type (1) should be factorizable:

\[
p(A_2, B_2|\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) = p(A_2|\hat{a}, \hat{a}', \lambda) p(B_2|\hat{b}, \hat{b}', \lambda) \quad (3)
\]

and for probability functions of the type (2) we have:

\[
p(A_1|\hat{a}, \hat{b}, \lambda) = p(A_1|\hat{a}, \lambda) \quad (4)
\]

Our postulate \( P_2 \) is equivalent to Bell’s locality condition and is also equivalent to \( C_2 \) in a non-contextual theory.
Using $P_1$ and $P_2$, one can derive a Bell-type inequality. This inequality is not violated by quantum mechanical predictions. To obtain this inequality, we define

$$E_{11}(\hat{a}, \lambda) = \sum_{A_1} A_1 p(A_1|\hat{a}, \lambda)$$

$$E_{22}(\hat{b}, \hat{b}^\prime, \lambda) = \sum_{B_2} B_2 p(B_2|\hat{b}, \hat{b}^\prime, \lambda)$$

$$E_{12}(\hat{a}, \hat{b}, \lambda) = E_{11}(\hat{a}, \lambda) E_{22}(\hat{b}, \lambda)$$

Here, $E_{11}(\hat{a}, \lambda)$ and $E_{22}(\hat{b}, \hat{b}^\prime, \lambda)$ are, respectively, the average values of the spin components of particle 1 along $\hat{a}$ at $t_1$ and particle 2 along $\hat{b}$ at $t_2$, and $E_{12}(\hat{a}, \hat{b}, \lambda)$ represents the average value of the product of the spin components of particles 1 and 2 along $\hat{a}$ and $\hat{b}$, respectively.

Now, one can show [10] that if the variables $x$, $y$, $x'$, $y'$ are confined to the interval $[-1, 1]$, then there exists a function $S$ defined by

$$S = xy + x'y' + x'y - x'y'$$

which lies in the interval $[-2, 2]$ . Here, we take

$$x = E_{11}(\hat{a}, \lambda); \quad y = E_{22}(\hat{b}, \hat{b}^\prime, \lambda); \quad x' = E_{11}(\hat{a}, \hat{a}^\prime, \lambda); \quad y' = E_{22}(\hat{b}, \lambda)$$

where all of these variables lie in the interval $[-1, 1]$ . Thus, we have

$$-2 \leq E_{12}(\hat{a}, \hat{b}, \lambda) E_{12}(\hat{a}, \hat{b}, \hat{b}^\prime, \lambda) + E_{12}(\hat{a}, \hat{b}, \hat{b}^\prime, \lambda) - E_{12}(\hat{a}, \hat{b}, \hat{a}^\prime, \lambda) \leq 2 \quad (8)$$

Multiplying through the probability density $\rho(\lambda)$ and integrating over $\Lambda$ ($\int_{\Lambda} \rho(\lambda) d\lambda = 1$ ), we get the following inequality at the quantum level

$$-2 \leq \langle \sigma_{a,1}^{(t_1)} \sigma_{b,2}^{(t_2)} \rangle + \langle \sigma_{a,1}^{(t_1)} \sigma_{b',2}^{(t_2)} \rangle + \langle \sigma_{a',1}^{(t_2)} \sigma_{b',2}^{(t_2)} \rangle - \langle \sigma_{a',1}^{(t_2)} \sigma_{b,2}^{(t_1)} \rangle \leq 2 \quad (9)$$

where, e.g., we have set the quantum expectation values as

$$\langle \sigma_{a,1}^{(t_1)} \sigma_{b,2}^{(t_2)} \rangle = \int_{\Lambda} E_{12}(\hat{a}, \hat{b}, \lambda) \rho(\lambda) d\lambda$$

From quantum mechanics we have (see appendix)
\[ \langle \sigma_{a,1}^{(t_1)} \sigma_{b,2}^{(t_1)} \rangle = - \cos \theta_{ab}; \quad \langle \sigma_{a,1}^{(t_2)} \sigma_{b',2}^{(t_2)} \rangle = - \cos \theta_{ab} \cos \theta_{b'b'}; \]

\[ \langle \sigma_{a',1}^{(t_1)} \sigma_{b,2}^{(t_1)} \rangle = - \cos \theta_{ab} \cos \theta_{aa'}; \quad \langle \sigma_{a',1}^{(t_2)} \sigma_{b',2}^{(t_2)} \rangle = - \cos \theta_{ab} \cos \theta_{aa'} \cos \theta_{bb'}; \]

where \( \theta_{kl} \) is the angle between \( \hat{k} \) and \( \hat{l} \) \( (\hat{k}, \hat{l} = \hat{a}, \hat{b}, \hat{a}', \hat{b}') \). Inserting these relations into (9), one obtains

\[ -2 \leq - \cos \theta_{ab} - \cos \theta_{ab} \cos \theta_{bb'} - \cos \theta_{ab} \cos \theta_{aa'} \cos \theta_{bb'} + \cos \theta_{ab} \cos \theta_{aa'} \leq 2 \]

This is never violated for any choice of \( \theta_{ab}, \theta_{bb'}, \) and \( \theta_{aa'} \). Thus, once \( P_1 \) and \( P_2 \) are assumed, there is no incompatibility between a hidden variable theory and quantum mechanics as far as this version of EPRB thought experiment is concerned.

Here, we have not made any use of \( C_1 \), but \( C_2 \) is applied in the form of \( P_2 \). This means that a particle is influenced by its past, as can been seen from the relations (6) and (7). Now, we try to see what happens when we impose \( C_1 \). We impose \( C_1 \) in the form of the following postulate:

\( \text{P}_3 \). The statistical results obtained for a spin component of a particle is independent of the sort of preparation made for the spin state of the same particle at an earlier time.

Using this postulate, one can write the relations (6) and (7) in the following forms

\[ E_2^{(t_2)}(\hat{b}, \hat{b}', \lambda) = E_2^{(t_2)}(\hat{b}', \lambda) = \sum_{B_2} B_2 p(B_2 | \hat{b}', \lambda) \quad (11) \]

\[ E_{12}(\hat{a}, \hat{b}, \hat{b}', \lambda) = E_1^{(t_1)}(\hat{a}, \lambda) E_2^{(t_2)}(\hat{b}', \lambda) \quad (12) \]

As a consequence of the conjunction of \( P_1 \) and \( P_3 \) one gets (1) in the form

\[ p(A_2, B_2 | \hat{a}', \hat{b}', \lambda) = \sum_{A_1, B_1} A_1 B_1 p_{GJP}(A_1, B_1, A_2, B_2 | \hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) \quad (13) \]

Now, by combining \( P_1, P_2 \) and \( P_3 \), the inequality (8) takes the form

\( p(A_2, B_2 | \hat{a}', \hat{b}', \lambda) \)
\[ -2 \leq E_{12}(\hat{a}, \hat{b}, \lambda) + E_{12}(\hat{a}', \hat{b}', \lambda) + E_{12}(\hat{a}', \hat{b}', \lambda) - E_{12}(\hat{a}', \hat{b}, \lambda) \leq 2 \quad (14) \]

In the integrated form of this inequality, the quantum expectations are defined in the following form

\[ \langle \sigma_{a,1}^{(t_1)} \sigma_{b',2}^{(t_2)} \rangle = \int_{\Lambda} E_{12}(\hat{a}, \hat{b}', \lambda) \rho(\lambda) d\lambda \quad (15) \]

The quantum expectation values in (15) are found to be

\[ \langle \sigma_{a,1}^{(t_1)} \sigma_{b',2}^{(t_2)} \rangle = -\cos \theta_{ab'} \quad (16) \]

and similarly for other expectation values. Insertion of these expectation values in the integrated form of (14) leads to the violation of it at the quantum level. Thus, the postulate \( P_3 \) leads to inconsistency with quantum mechanics. But, the possibility of consistency between a contextual factorizable hidden variable theory with quantum mechanics remains open. Here, the contextuality is only used as the negation of \( C_1 \), which means that the preparation of the spin state of each particle cannot be excluded from our calculations.

### 3 EPRB Thought Experiment

The combination of \( P_1 \) and \( P_3 \) in the previous experiment, provides the possibility of conversion of our thought experiment to that of EPRB. We show this case by taking the limit \( \lim_{t_2 \to t_1} (P_1 + P_3) = P_4 \) which can be stated in the following terms:

\[ P_4 \text{. There exists a limit for } t_2 \to t_1, \text{ under which the conjunction of the two postulates } P_1 \text{ and } P_3 \text{ remains valid.} \]

This justifies relations like (13) for the EPRB case. Now, we have

\[ \lim_{t_2 \to t_1} p(\sigma_{a',1}^{(t_2)} = A_2, \sigma_{b',2}^{(t_2)} = B_2|\hat{a}', \hat{b}', \lambda) = \]

\[ \lim_{t_2 \to t_1} \sum_{A_1B_1} A_1B_1 p_{GJP}(\sigma_{a,1}^{(t_1)} = A_1, \sigma_{b,1}^{(t_1)} = B_1, \sigma_{a',1}^{(t_2)} = A_2, \sigma_{b',2}^{(t_2)} = B_2|\hat{a}, \hat{b}, \hat{a}', \hat{b}', \lambda) \]

7
\[ p(\sigma_1^{(t_1)} = A_2, \sigma_2^{(t_1)} = B_2 | \hat{a}, \hat{b}, \lambda) \]  

Similarly, combining \( P_4 \) and \( P_2 \) leads to Bell’s inequality in the form of (14). Here, \( C_1 \) makes no sense, but \( C_2 \) is used in the form of \( P_2 \). However, we use \( C'_1 \) in the following form:

\[ C'_1 \]  

For a system consisting of a pair of particles 1 and 2, the spin state of the system is independent of the parameters which might be involved as a consequence of the state preparation of the system, and the value of a spin component along an arbitrary direction is completely determined by \( \lambda \).

This assumption has a more comprehensive character than \( C_1 \), because \( C_1 \) can be concluded from \( C'_1 \), as a special case.

Now, having introduced non-contextuality in a broader sense, we logically conclude that since the violation of \( P_3 \) leads to the violation of \( P_4 \), the violation of the integrated form of (14) in the EPRB experiment should be a result of the violation of \( P_4 \). But, what does this mean?

This means that the conjunction of \( P_1 \) and \( P_3 \) is not valid under the limit of \( t_2 \rightarrow t_1 \), so that the relation (17) can not be correct. The conjunction of \( P_1 \) and \( P_3 \), which leads to relations like (13), indicates that the possible values of a pair of spin component related to two particles are completely determined by \( \lambda \), when the measuring directions of the corresponding spin components are specified, and are independent of the factors which can be introduced by the preparation of the spin state of each particle at an earlier time. This attitude which led ultimately to conflict with the quantum mechanical predictions at the statistical level, was caused by \( P_3 \). The postulate \( P_4 \), which is defined by \( \lim_{t_2 \rightarrow t_1} (P_1 + P_3) \), indicates that the foregoing argument remains valid, when the spin state of each particle can be in principle prepared simultaneously at two different directions. Thus, the violation of \( P_3 \) (in the previous section) and of \( P_4 \) (in this section) both mean that the statistical result of any measurement should be locally assessed only in the context of the preparation factors, since there is no reason for the violation of \( P_2 \) in both cases.

As a consequence of the violation of \( P_4 \), we can use contextuality as the negation of \( C'_1 \), which means that the spin state of each particle cannot be independent of the preparation conditions. Thus, (17) is not valid and we have
base vectors, the singlet state of the source and is defined as a linear combination of two

\[ p(\sigma_{a',1}^{(t_1)}) = A'_2, \sigma_{b',2}^{(t_1)} = B'_2|X_1(\hat{a}'), X_2(\hat{b}), \lambda) \]

\[ \neq p(\sigma_{a',1}^{(t_1)}) = A_2, \sigma_{b',2}^{(t_1)} = B_2|X_1(\hat{a}'), X_2(\hat{b}), \lambda) \]  

where

\[ p(\sigma_{a',1}^{(t_1)}) = A_2, \sigma_{b',2}^{(t_1)} = B_2|X_1(\hat{a}'), X_2(\hat{b}), \lambda) = \]

\[ \sum_{A_1B_1} A_1B_1 p_{GJP}(\sigma_{a,1}^{(t_1)} = A_1, \sigma_{b,1}^{(t_1)} = B_1, \sigma_{a',1}^{(t_1)} = A_2, \sigma_{b',2}^{(t_1)} = B_2|X_1(\hat{a}'), X_2(\hat{b}), \lambda) \]  

Here, \( X_1(\hat{a}, \hat{a}') \) (\( X_2(\hat{b}, \hat{b}') \)) defines the particle 1 (2) in the context of \( \hat{a} \) and \( \hat{a}' \) (\( \hat{b} \) and \( \hat{b}' \)), which shows the state dependence of particle 1 (2) on the preparation factors \( \hat{a} \) and \( \hat{a}' \) (\( \hat{b} \) and \( \hat{b}' \)). Similarly, \( X_1(\hat{a}') \) (\( X_2(\hat{b}') \)) defines the state of particle 1 (2) in the context of \( \hat{a}' \) (\( \hat{b}' \)), which includes the preparation effects on the same particle. The above state dependences hold locally.

The failure of (17) for the EPRB case has been sometimes interpreted as a failure of the definition of \( p_{GJP} \) for the case of incompatible observables (see for example refs. [11]-[13]). We think this point of view is not essential, since in principle, the definition of \( p_{GJP} \) at a hidden variable level is tenable.

Appendix

In order to calculate the quantum mechanical expectation values in our proposed experiment, it is essential to define an original joint probability function which describes the possible outcomes of the spin components of the particles 1 and 2 at \( t_1 \) and \( t_2 \):

\[ P(A_1, B_1, A_2, B_2|\hat{a}, \hat{b}, \hat{a}', \hat{b}', \psi_0) = | \langle \psi_0|u_{a,1}^{(t_1)}(\pm), u_{b,2}^{(t_1)}(\pm) \rangle |^2 \]

\[ \times | \langle u_{a,1}^{(t_1)}(\pm), u_{b,2}^{(t_1)}(\pm)|u_{a',1}^{(t_2)}(\pm), u_{b'}^{(t_2)}(\pm) \rangle |^2 \]  

(A.1)

where the outcomes \( A_1, B_1, A_2 \) and \( B_2 \) take the values ±1. In (A.1), | \( \psi_0 \rangle \) is the singlet state of the source and is defined as a linear combination of two base vectors, | \( z+ \rangle \) and | \( z+ \rangle \), which correspond to the two eigenstates of \( \sigma_z \):

9
The quantum states (spin states) for two particles along an arbitrary direction relative to the z-axis, at $t_1$ or $t_2$ is represented by

$$|\psi_0\rangle = [|z+1\rangle \otimes |z-2\rangle - |z-1\rangle \otimes |z+2\rangle] \quad (A.2)$$

where $j = 1, 2$; $m = a, a'$ and $n = b, b'$. The individual spin states for particle 1 are defined as

$$|u_{m,1}(\pm)\rangle = \cos \frac{\hat{m}}{2} |z+1\rangle + \sin \frac{\hat{m}}{2} |z-1\rangle,$$

$$|u_{m,1}^-(\pm)\rangle = -\sin \frac{\hat{m}}{2} |z+1\rangle + \cos \frac{\hat{m}}{2} |z-1\rangle, \quad (A.3)$$

and similarly for particle 2. Now, the quantum mechanical counterpart of relation (10) is obtained in the following way

$$\langle \sigma_{a_1}^{(t_1)} \sigma_{b'2}^{(t_2)} \rangle = \sum_{A_1B_2} A_1B_2 P(A_1, B_2|\hat{a}, \hat{b}, \hat{b}', \psi_0) \quad (A.4)$$

where

$$P(A_1, B_2|\hat{a}, \hat{b}, \hat{b}', \psi_0) = \sum_{A_1B_1B_2} P(A_1, B_1, A_2, B_2|\hat{a}, \hat{b}, \hat{a'}, \hat{b'}, \psi_0) \quad (A.5)$$

is the marginal joint probability of the results $A_1$ and $B_2$. Using the relations (A.1)-(A.3), the relation (A.5) yields

$$P(A_1, B_2|\hat{a}, \hat{b}, \hat{b}', \psi_0) = \frac{1}{4} \left[ 1 - A_1B_2 \cos \theta_{ab} \cos \theta_{bb'} \right] \quad (A.6)$$

Inserting (A.6) into (A.4), one gets

$$\langle \sigma_{a_1}^{(t_1)} \sigma_{b'2}^{(t_2)} \rangle = -\cos \theta_{ab} \cos \theta_{bb'}$$

The other quantum expectation values are obtained in a similar way.
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