Width of the Whitehead double of a nontrivial knot

Zhenkun Li, Qilong Guo

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Abstract

In this paper, we prove that \( w(K) = 4w(J) \), where \( w(.) \) is the width of a knot and \( K \) is the Whitehead double of a nontrivial knot \( J \).

1 Introduction

Width is a knot invariant introduced in Gabai [1]. It has been studied intensively since then. Zupan in [4] conjectured that

\[ w(K) \geq n^2w(J), \]

where \( K \) is a satellite knot with companion \( J \) and wrapping number \( n \). This conjecture is still open. The authors of this paper proved in [2] a weaker version of the conjecture with wrapping number being replaced by winding number. In this paper we prove a special case of \( K \) having wrapping number 2 and winding number 0, which is not covered by the discussion in [2].

**Theorem.** Suppose \( J \) is a nontrivial knot and \( K \) is a Whitehead double of \( J \). Then we have

\[ w(K) = 4 \cdot w(J). \]

The gap between wrapping number and winding number in the proof in author’s previous paper [2] lies in lemma 4.4, which says that any properly embedded surface representing a generator of \( H_2(V, \partial V) \) has the number of intersection points with the satellite knot \( K \) being no smaller than the
absolute value of the winding number of $K$. Here $V$ is the tubular neighborhood of $J$ containing $K$. This is in general not true for wrapping number but for the special case that $K$ is a Whitehead double, we can somehow overcome this difficulty.

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### 2 Width of a Whitehead double

We will include the definition of Whitehead double here but for the width of a knot, readers are referred to [2]. In what follows, we will use capital letters $K, J, L$ to denote the knot or link classes while use lower case letters $k, j, l$ to denote particular knots or links within the corresponding classes.

**Definition 2.1.** Suppose $l_w = \hat{k} \cup \hat{j}$ is a Whitehead link in $S^3$. Let $\hat{V} = S^3 \setminus N(\hat{j})$ be the exterior of $\hat{j}$ containing $\hat{k}$ in its interior. Since $\hat{j}$ is the unknot in $S^3$, $\hat{V}$ is a solid torus and $\hat{k} \in \hat{V}$ can be thought as in figure 1. Let $j \subset S^3$ be a non-trivial knot and let $V = N(j)$ be the closure of a tubular neighborhood of $j$ in $S^3$. Let $f : \hat{V} \to S^3$ be an embedding such that $f(\hat{V}) = V$, and let $k = f(\hat{k})$. Then $k$ is called a Whitehead double of $j$.

![Figure 1: Pattern for Whitehead double](image)

**Lemma 2.2.** Suppose $k, j, V$ are defined as in definition 2.1, then any meridian disk $D$ of $V$ would intersect $k$ at least two times.
Remark 2.3. The above lemma actually means that the wrapping number of $k$ is 2. On the other hand, $[k] = 0 \in H_1(V)$ so the winding number of $k$ is 0.

As discussed in the introduction, the key lemma is the following.

Lemma 2.4. (Key lemma.) Suppose $S$ is a connected properly embedded planar surface inside $V$ which represents a generator of $H_2(V, \partial V)$, then $S$ intersects $k$ at least two times.

In order to prove the key lemma, we need the following one.

Lemma 2.5. Suppose $S$ is a connected, properly embedded planar surface inside a three ball $B$. Let $B_1$ and $B_2$ be the two components of $B \setminus S$. Then there is no Hopf link $l_h = l_1 \cup l_2$ inside $B$ such that

$l_i \subset B_i$ for $i = 1, 2$.

Proof. Assume, on the contrary, that there is a Hopf link $l_h = l_1 \cup l_2$ such that

$l_i \subset B_i$.

We can find a disk $E$ (actually a Seifert surface for $l_1$) such that

1. $\partial E = l_1$,
2. $E$ is in the interior of $B$.

Now since $l_2 \subset B_2$, we know that $l_2 \cap E = l_2 \cap (E \cap B_2)$. Suppose $E_0$ is a component of $E \cap B_2$. Since $\partial E \subset B_1$, we have $\partial E_0 \subset S$. Then we can do a surgery using $E_0$ on $S$: on $S$ cut a neighborhood of $\partial E_0$ and glue back two copies of $E_0$, denoted by $E_{0,+}$ and $E_{0,-}$ respectively, along the boundary created by the cutting. Since $S$ is a planar surface and any circle on a planar surface is separating, the result of the surgery is a disjoint union of two connected surfaces $S_1$ and $S_2$ and each one contains one copy of $E_0$.

Let us focus on $S_1$, and assume without loss of generality that $E_{0,+} \subset S_1$. Since $l_2$ is a circle inside $B$, we know that $L_2$ has algebraic intersection number 0 with $S_1$. Since $l \cap S = \phi$, the intersection of $l_2$ with $S_1$ must be all in $E_{0,+}$. Then $l_2$ must have an even intersection number with $E_{0,+}$, and hence has an even intersection number with $E_0$. Since $E_0$ is an arbitrary component of $E \cap B_2$, we know that $L_2$ must have an even intersection points with $E$, which contradicts to the fact that $l_2$ has linking number 1 with $l_1$, since $l_h = l_1 \cup l_2$ is a Hopf link. 

\[\boxempty\]
In the proof of lemma 2.4, we will also need the auxiliary function defined as follows.

**Definition 2.6.** Suppose $M$ is a compact 3-manifold with boundary and $S$ is a properly embedded surface in $M$ such that any components of $S$ separates $M$. Then we can define a map

$$C_{M,S} : (M \setminus S) \times (M \setminus S) \to \{\pm 1\}$$

as

$$C_{M,S}(x,y) = (-1)^{|\gamma \cap S|},$$

where $x, y \in M$ are two points and $\gamma$ is an arc connecting two points $x$ and $y$ that is transverse to $S$. $|\gamma \cap S|$ means the number of intersection points.

When $S$ is connected, $C_{M,S}(x,y)$ is 1 if and only if $x$ and $y$ lie in the same component of $M \setminus S$. If in general $S$ is not necessarily connected, we know that $C_{M,S}(x,y) = -1$ would still imply that $x$ and $y$ are not in the same component of $M \setminus S$.

Another good property of this function is the following equality: for any 3 points $x, y, z \in M \setminus S$, we have

$$C_{M,S}(x,y) \cdot C_{M,S}(y,z) = C_{M,S}(x,z). \quad (1)$$

Now we are ready to prove the key lemma.

**Proof of lemma 2.4.** Suppose $k$ is a Whitehead double of $j$, $V$ is the tubular neighborhood of $j$ containing $k$ and $S$ is a connected, properly embedded planar surface in $V$, representing a generator of $H_2(V, \partial V)$. Assume that $S$ and $k$ have less than 2 intersections, then since $[k] = 0 \in H_1(V)$, we know that $S$ and $k$ must be disjoint.

It is easy to see that there is a meridian disk $D$ of $V$, such that in $B = V \setminus N(D)$, that $D$ intersects $S$ transversely, and that after adding two small arcs to $(k - N(D))$ near $\partial N(D)$, we will get a Hopf link $l$ out of $k$. Here $N(D)$ is a neighborhood of $D$ in $V$. See figure 2.

If $D \cap S = \emptyset$, then we can apply lemma 2.5 directly to conclude a contradiction, since when $[S]$ represents a generator of $H_2(V, \partial V)$ and $l \cap S = \emptyset$, the two components of the Hopf link $l$ are in two different components of $B \setminus S$.

If $D \cap S \neq \emptyset$, we can assume that

$$D \cap S = \beta_1 \cup \beta_2 \ldots \cup \beta_m,$$
where $\beta_1, \ldots, \beta_n$ are intersection circles of the two surfaces and are in the order such that $\beta_j$ bounds a disk $D_j \subset D$ disjoint from any $\beta_i$ for $i < j$.

If all $D_i$ are disjoint from $k$, then we can do a series surgeries on $S$ with respect to $D_n, D_{n-1}, \ldots, D_1$ one by one to get a surface $S''$ so that $S''$ is disjoint from $D$ and $k$. We can pick any connected component of $S''$, and it will also have such properties and applying the argument above we can get the same contradiction.

Now we are in the most complicated case where some $D_i$ intersects $k$. Suppose $j_0$ is the greatest index such that $D_j \cap k \neq \emptyset$, then we claim that $D_{j_0}$ cannot have a unique intersection point with $k$. Suppose the contrary, then a sequence of surgeries on $S$ with respect to $D_n, \ldots, D_{j_0+1}$ would generate a surface $S'$ such that $S'$ represent a generator of $H_2(V, \partial V)$, $S'$ is disjoint from $k$ and

$$D_{j_0} \cap S' = \partial D_{j_0} = \beta_{j_0}.$$

Suppose $S'_0$ is the component of $S'$ containing $\beta_{j_0}$, then $S'_0$ is still a planar surface. A surgery on $S'_0$ with respect to $D_{j_0}$ would result in two surfaces $S'_1$ and $S'_2$, each of which contains one copy of $D_{j_0}$ and hence has a unique intersection point with $k$. This is impossible since $[k] = 0 \in H_1(V)$.

Hence we conclude that $D_{j_0}$ has two intersection points with $k$, which are all the intersection points of $D$ with $k$. Now, as before, we can do a sequence of surgeries on $S$ with respect to $D_n, \ldots, D_{j_0+1}, D_{j_0}$ to get $S'$. When doing last surgery with respect to $D_{j_0}$, we shall modify $k$ at the same time: cut $k$ along $D$ and glue two small arcs to the newly born boundary points near $D$ to get a Hopf link $l$ disjoint from $S'$ and $D$. See figure 3.

![Figure 2: Cutting off a neighborhood of $D$](image)
Now we can apply the remaining sequence of surgeries on \( S' \), with respect to \( D_{j_0-1}, ..., D_1 \), to get a surface \( S'' \) which represents a generator of \( H_2(V, \partial V) \) and is disjoint from \( D \) and \( l \).

![Figure 3: Surgery on \( S \)](image)

Now we can cut off a neighborhood \( N(D) \) of \( D \) from \( V \), which is disjoint from \( l \) and \( S'' \), to get a 3-ball \( B = V \setminus N(D) \). In order to see that the two components of \( l \) are in different components of \( B \setminus S'' \), we can pick points \( x, y \) on two arcs of \( l \) which do not belong to \( k \), and pick two points \( u, v \) near the boundary \( \partial N(D) \) but on different sides of \( D \). See figure 4.

![Figure 4: Points \( u, v, x, y \)](image)

There is a symmetry between the two pairs \( (x, u) \) and \( (y, v) \) with respect
to $D$ so we have
\[ C_{B,S''}(x, u) = C_{B,S''}(v, y), \]
where $C_{B,S''}$ is defined as in definition 2.6. Since $S''$ represents a generator of $H_2(V, \partial V)$, we have
\[ C_{B,S''}(u, v) = -1. \]
Using equality (1), we have
\[ C_{B,S''}(x, y) = C_{B,S''}(x, u) \cdot C_{B,S''}(u, v) \cdot C_{B,S''}(v, y) = -1. \]
Thus the two components of $l$ are in different components of $B \setminus S''$. Pick a connected component $S_0''$ which separates the two components of $l$, we can apply lemma 2.5 to get a contradiction.

**Theorem 2.7.** Suppose $J$ is a nontrivial knot (class) and $K$ is a Whitehead double of $J$, then we have
\[ w(K) = 4w(J). \]

**Proof.** It is easy to construct examples to show that
\[ w(K) \leq 4w(J). \]
In order to prove the reverse inequality, we repeat the whole argument as in the authors’ previous paper [2], with lemma 4.4 in that paper replaced by lemma 2.4 in the current paper. Then we can conclude that
\[ w(K) \geq 4w(J). \]
References

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