The Metrological Power of Nonclassical Single-Mode States

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The subject of quantum metrology is concerned with how quantum systems can be used to measure a physical quantity with a better precision than that possible with classical probes [1]. The precision possible with a quantum system is determined by the state in which it is prepared; certain nonclassical states enable a better precision than any classical state with the same energy or number of particles. As an example, a single mode prepared in a squeezed vacuum state [2] has been shown to be useful in a variety of metrological tasks including gravitational wave detection in laser interferometers [3], measurements of mechanical motion [4, 5], and biological imaging and sensing [6]. A central endeavor in quantum metrology is, therefore, to determine the maximum advantage that a nonclassical state can provide for a given metrological task, and how to achieve it. Since nonclassical states become harder to produce as their energy increases, classical (coherent) states are always available at much higher energies than non-classical states. Thus for practical purposes what we wish to know is the metrological advantage of a non-classical state when it is combined with arbitrary classical states with much higher energy. This quantity has been termed the metrological power of the non-classical state.

Our work here was initially inspired by recent attempts to determine the relationship between the metrological powers for force sensing (also known as “displacement” sensing) and phase sensing [7, 8]. To-date the metrological power of quantum states is known only for force sensing [7]. Here we determine the metrological power for all physical quantities for which it is defined. We find that the metrological power for every quantity is proportional to the maximized quadrature variance of the non-classical state. This elucidates the relationship between the metrological powers for all tasks, and reveals that the maximal quadrature variance is a universal measure of the metrological utility of non-classical states.

To use a quantum system for the metrology of a physical quantity, the quantity in question must affect the system in some way. This effect is usually given by a unitary transformation of the form \( U(\xi) = e^{-i\xi \hat{G}} \) in which \( \hat{G} \) is a Hermitian operator and \( \xi \) is the quantity to be measured. For a single mode with annihilation operator \( \hat{a} \), the transformation for force/acceleration metrology is \( \hat{G} = \hat{X}_a \equiv \hat{a} e^{-i\phi} / \sqrt{2} + \text{H.c.} \), where the optimal choice of \( \phi \) is determined by the initial state of the system [9, 10]. For phase metrology, \( \hat{G} = \hat{a} \hat{a}^\dagger \). A commonly employed method for phase sensing is the Mach-Zehnder interferometer [11–18], in which a nonclassical state is combined with a classical state. However, the Mach-Zehnder interferometer is not the most general configuration for performing phase sensing with classical states. Such a general scheme can utilize any number of additional modes containing coherent states, and mix these modes with that containing the non-classical state using an arbitrary passive linear network (PLN) [19–32].

Here we consider not only metrology of a single value (local metrology) but also distributed metrology [31–34] which involves estimating a linear combination of a number of independent values of the same physical quantity. We first obtain the metrological power for phase measurements and then extend the analysis to arbitrary single-mode transformations. The procedure we use to evaluate the metrological power for phase measurements is the following. We consider a general scheme for distributed phase sensing with a single mode nonclassical state, which is depicted in Fig. 1. We begin the analysis with a pure state. The expression for the metrological power is unwieldy and the key to simplifying it is to employ normal ordering of the mode operators [35]. Normal ordering also allows us to neatly separate the classical and nonclassical contributions to the total measurement precision. We are then able to explicitly optimize the parameters describing the passive linear network to determine both the maximum precision and the network required to achieve it. This gives us the metrological power for pure states which can then be extended to mixed states using a result conjectured by Tóth and Petz [36] and proved by Yu [37].

Preliminaries:—Before we begin our analysis we need to review some facts and define some notation. When a mode is prepared in state \( \hat{\rho} \) and the transformation \( U(\xi) \) applied
FIG. 1. (Color online) A general interferometric phase sensing scheme that applies multiple independent phase shifts \( \theta_j \) to a quantum state, \( \hat{\rho} \), augmented with classical resources. The latter consist of a linear network, coherent states \(|\alpha_j\rangle\), and a displacement \( D(\alpha_0) \) of the state \( \hat{\rho} \). The dashed frame represents a general passive linear network.

to it, the accuracy with which \( \xi \) can be determined by measuring the mode is captured by the quantum Fisher information (QFI), denoted by \( \mathcal{T}(\hat{\rho}, \hat{G}) \) [38]. For a pure state \( \hat{\rho} = |\psi\rangle\langle\psi| \), the QFI reduces to four times the variance of \( \hat{G} \), namely \( \mathcal{T}(\hat{\rho}, \hat{G}) \equiv 4V_{\psi}(\hat{G}) \) with \( V_{\psi}(\hat{G}) = \langle \hat{G}^2 \rangle - \langle \hat{G} \rangle^2 \) and \( \langle \hat{G} \rangle \equiv \langle \hat{G}|\psi\rangle \) for any operator \( \hat{A} \). For mixed states, the QFI can be written in terms of that for pure states as [36, 37]

\[
\mathcal{T}(\hat{\rho}, \hat{G}) = \min_{\{p_n, |\psi_n\rangle\}} \sum_n p_n \mathcal{T}(\langle \psi_n | \psi_n \rangle, \hat{G}),
\]

where the minimization is over all ensembles that decompose \( \hat{\rho} \) (all ensembles \( \{p_n, |\psi_n\rangle\} \) for which \( \hat{\rho} = \sum_n p_n |\psi_n\rangle\langle\psi_n| \) and \( p_n > 0, \forall n \)). The expression on the RHS of Eq. (1) is referred to as the convex roof of \( \mathcal{T} \).

The metrological power is defined as the increase in the QFI provided by a non-classical state over the best classical state with the same energy. When a non-classical state \( \hat{\rho} \) with energy \( E \) is used for metrology along with coherent states with total energy \( |\alpha|^2 \), the advantage that \( \hat{\rho} \) provides over a classical state with the same energy depends on \( |\alpha|^2 \). Denoting the QFI in this case by \( \mathcal{T}(\hat{\rho}, \alpha, \hat{G}) \), the metrological power is [7]

\[
\mathcal{M}_p(\alpha, \hat{G}) = \mathcal{T}(\hat{\rho}, \alpha, \hat{G}) - \mathcal{T}(\hat{\rho}_c, \alpha, \hat{G}),
\]

and is a function of \( |\alpha|^2 \), where \( \hat{\rho}_c \) is the classical state that achieves the maximum QFI \( \mathcal{T}(\hat{\rho}_c, \alpha, \hat{G}) \) for fixed \( |\alpha|^2 \) and \( E \). We will find that the metrological power for a transformation that contains products of \( 2p \) mode operators (e.g. \( \hat{G} = \hat{a}^{\dagger p-k}\hat{a}^k + H.c. \)) contains terms proportional to \( |\alpha|^2 \) with \( j = 0, 1, \ldots, p - 1 \). Thus when \( |\alpha|^2 \gg E \) the term proportional to \( |\alpha|^{2(p-1)} \) is the only significant contribution and determines the metrological power in this regime. Since this is the regime of most practical interest, here we define the metrological power as

\[
\mathcal{M}_p(\hat{G}) = \max_{\mathcal{PLN}} \lim_{|\alpha|^2/E \to \infty} \mathcal{M}_p(\alpha, \hat{G}),
\]

where \( \max_{\mathcal{PLN}} \) represents the maximization over an arbitrary passive linear network with a fixed classical energy \( |\alpha|^2 \). For force metrology one must maximize the quadrature angle \( \phi \) in \( \hat{G} \) (as defined above), which can be effectively achieved via the maximization of a linear network to optimize the QFI [7]. We will denote the metrological power for force measurement by \( \mathcal{M}_p^F \). Since the variance of every quadrature for every coherent state is \( 1/2 \), the metrological power for force measurement for a pure state \( |\psi\rangle \) is simply \( \mathcal{M}_p^F |\psi\rangle = 4 \max_{M} \sum_{j=0}^{M} w_j (\xi_j - 1/2) \).

Distributed metrology, which involves the estimation of a linear combination of \( m \) parameters \( \{\xi_j\} \), or (the simultaneous estimation of these parameters) is a straightforward generalization of single-parameter metrology to multiple modes [31–34, 39]. Defining mode operators \( \hat{a}_j, j = 0, \ldots, m - 1 \) for each of \( m \) modes, and operators \( \hat{G}_j = f(\hat{a}_j, \hat{a}_j^\dagger) \) in which \( f \) is some function, the probe system is acted upon by the product \( \hat{U}_\xi = \prod_j \exp[-i\xi_j \hat{G}_j] \). For a pure state \( |\psi\rangle \) the multi-parameter QFI, which we again denote by \( \mathcal{T} \), is now a matrix whose elements are

\[
\mathcal{T}_{jk}/4 = \langle \psi|\hat{G}_j \hat{G}_k|\psi\rangle - \langle \psi|\hat{G}_j|\psi\rangle \langle \psi|\hat{G}_k|\psi\rangle.
\]

For simultaneous estimation of all the parameters \( \xi_j \) with a set of unbiased estimators \( \Xi_j \), a bound applies to the covariance matrix, \( \text{cov}(\Xi) \), whose matrix elements are \( \text{cov}(\Xi)_{jk} \equiv \langle \Xi_j - \langle \Xi_j \rangle \rangle \langle \Xi_k - \langle \Xi_k \rangle \rangle \), where \( \langle O \rangle \) is the expected value of the quantity \( O \). This bound is the multi-parameter quantum Cramér-Rao bound [40], \( \text{cov}(\Xi) \geq (\mathcal{M}^F)^{-1} \), where \( \mathcal{M} \) is the number of independent repetitions of the metrology process. For estimating the linear combination \( \xi = \sum_j w_j \xi_j \), with \( \sum_j |w_j|^2 = 1 \), referred to as a global estimate, the lower bound on the variance of the estimate is \( \Delta^2 \xi \geq \sum_j |w_j|^4 / (\mathcal{M}^F) \) [32] in which \( \mathcal{M}^F = \sum_j w_j^2 \mathcal{T}_{jk} \), \( \Delta^2 \xi = \sum_j w_j^2 \xi_j^2 \). For distributed metrology we use the same definition for the metrological power, except we replace the QFI by \( \mathcal{T}_{\text{eff}}/|w|^4 \) maximized over the weighting coefficients \( w \).

**Metrological power for phase-measurement:** The general scheme for metrology of \( m \) parameters (in our case \( m \) phase shifts, \( \theta_j \), \( j = 0, \ldots, m - 1 \)) with a single mode state \( \hat{\rho} \) and \( m \) modes each containing a coherent state with amplitude \( \alpha_j \) is shown in Fig. 1. (We find that there is no utility in using more auxiliary modes than unknown parameters.) The mode with annihilation operator \( \hat{a}_0 \) contains the state \( \hat{\rho} \) which we are free to displace by a coherent amplitude \( \alpha_0 \). The total classical energy supplied is \( |\alpha|^2 \equiv \sum_j |\alpha_j|^2 \). Phase shift \( \theta_k \) is applied to mode \( \hat{b}_k \) via \( \exp[-i\theta_k \hat{b}_k] \), where \( \hat{b}_k \equiv \hat{b}_k^\dagger \hat{b}_k \) and \( \hat{b}_0 \) is related to the input modes \( \hat{a}_j \) by the unitary \( U_k = \prod_{j=0}^{k} U^{(j)} \).
We begin the analysis with a single-mode nonclassical pure state $|\psi\rangle$ at the first input mode and then generalize the results to an arbitrary mixed state. The elements of the QFI matrix are given by $F_{jk} = 4(\langle \hat{h}_{jk} \rangle - \langle \hat{h}_{j} \rangle \langle \hat{h}_{k} \rangle)$. The complexity of this expression for $F_{jk}$ comes from the fact that the transformation $\hat{b}_k = \sum_j u^{(k)}_{j} \hat{a}_j$ depends on different unitary matrices for different $k$ where the complex numbers $u^{(k)}_{j}$ are the elements of the $k^{th}$ row of the unitary matrix $U_k$.

The key to evaluating $F_{jk}$ is to use normal ordering of $\hat{b}_k$. We will denote the normal ordering of a product of mode operators in the usual way by sandwiching the product between colons. Thus $:\langle \hat{b}_j \hat{b}_j \rangle^2 = \langle \hat{b}_j \hat{b}_j \rangle \langle \hat{b}_j \hat{b}_j \rangle^2$. Writing $F_{jk}$ in terms of normally ordered products gives

$$F_{jk} = \sum_{i,j=0}^{n-1} u^{(k)}_{ij} \hat{a}_j$$

where we have defined $F_{jk}$ as four times the normally ordered covariance. Since this covariance vanishes for all classical states, and since $\langle \hat{b}_j \rangle$ (the energy of mode $j$) is effectively independent of $|\psi\rangle$ (recall that $E \ll |\langle \psi | \psi \rangle|^{1/2}$), it is $F_{jk}$ that is the nonclassical contribution to $F_{jk}$. The nonclassical metrological advantage for measuring the linear combination of the displacement of mode $\hat{a}_0$, the resulting replacement is

$$\hat{b}_k \rightarrow u_{0k} \hat{a}_0 + \sum_{j=0}^{n-1} u^{(k)}_{j} \hat{a}_j = u_{0k} \hat{a}_0 + f_k$$

where we have defined the complex amplitudes $f_k \equiv \sum_{j=0}^{n-1} u^{(k)}_{j} \hat{a}_j$. Making this replacement in $F_{w}$ we note that since the coherent energies, $|\langle \psi | \psi \rangle|^{1/2}$, are much larger than the quantum state energy $E$, the dominant terms will be those with the most factors of the $f_j$. Since all terms vanish that contain only a single occurrence of $\hat{a}_0$, the terms with most factors of $f_k$ contain exactly two factors of the mode operator (and two factors of $f_j$). Of these terms, all vanish except those in which one factor of $\hat{a}_0$ comes from the first factor in the product $\langle \hat{b}_j \hat{b}_j \rangle \langle \hat{b}_k \hat{b}_k \rangle$ and the other from the second. The result is that for large coherent resources $(|\psi|^2 / E \rightarrow \infty)$

$$\lim_{|\psi|^2 / E \rightarrow \infty} M_{\psi}(\alpha, \hat{h}, w) = \sum_{j} w_{j} \theta_{j} f_{j}$$

we need to maximize $|z|^2$ in Eq.(7) over all sets of unitary transformations $\{U_j\}$ and classical inputs $\{|\psi\rangle\}$. Since $|\langle f | \langle f \rangle \rangle = 1 \sum_{k=0}^{m-1} \sum_{k=0}^{m-1} |\langle f | \langle f \rangle \rangle = 1 |\langle f | \langle f \rangle \rangle = 1 |\langle f | \langle f \rangle \rangle = 1$, we have

$$|z|^2 \leq |\langle f | \langle f \rangle \rangle = 1 |\langle f | \langle f \rangle \rangle = 1 |\langle f | \langle f \rangle \rangle = 1 |\langle f | \langle f \rangle \rangle = 1.$$

The last inequality is saturated when $u_{\phi}^{(j)} = c \hat{a}_j$ for every $j$ for some complex constant $c$. Since $|u_{\phi}^{(j)}| \leq 1$ we must have $|c| \leq 1/ \max_{j} \max_{j} |w_{j}|$. We thus have

$$|z|^2 \leq |\langle f | \langle f \rangle \rangle = 1 |\langle f | \langle f \rangle \rangle = 1 |\langle f | \langle f \rangle \rangle = 1 |\langle f | \langle f \rangle \rangle = 1.$$

We can saturate this bound while selecting any value for $\phi$ merely by choosing the phases of $\alpha_j$ and $u_{\phi}^{(j)}$.

Having found the maximum in Eq.(7) we can now write down the metrological power for a distributed phase measurement with a given weighting distribution $\{w_{i}\}$, which is

$$M_{\psi}(\alpha, \hat{h}, w) = \max_{\phi} : V_{\psi}(\hat{X}_\phi) : = \frac{2|\langle \phi | \phi \rangle |^{2} M_{\psi}^{c}}{w_{\max}}.$$  

To obtain the maximum precision for measuring $m$ phase shifts we must choose the minimal value for $\max_i w_i$ which is achieved by setting $w_i = 1/m$ for all $i$. The result is

$$M_{\psi}(\alpha, \hat{h}, w) = \frac{2m^{2} |\phi|^{2} M_{\psi}^{c}}{w_{\max}}.$$  

**Parallel distributed sensing:** The maximum precision for distributed sensing, $E_{\text{Parallel}}$, is obtained by applying all the phase shift $\theta_j$ in sequence to a single mode. Ideally a distributed sensing scheme would involve applying each phase shift to a different mode, since this allows all the modes to be sent simultaneously to different locations to undergo the individual phase shifts. We will refer to this as parallel distributed sensing. If we wish to place this additional restriction on phase sensing then the configuration in Fig.1 reduces to that in Fig.2. In this case $M_{\psi}(\alpha, \hat{h}, w) = \frac{2m^{2} |\phi|^{2} M_{\psi}^{c}}{w_{\max}}.$

\[\begin{array}{c}
\text{Linear} \\
\text{Optical} \\
\text{Network} \\
\text{Measurement}
\end{array}\]
Here the $u_{jk}$ are the elements of a single unitary transformation, and so the vector $g \equiv (f_0, \ldots, f_{m-1})/|a|$ has unit norm, as does $u = (u_0, \ldots, u_{m-1})$. It is also useful to define the vector $x = (x_0, \ldots, x_{m-1})$ with $x_j = u^*_j g_j = u^*_j f_j/|a|$. It follows from the Cauchy-Schwartz inequality that the $l^1$ norm of $|x|$ is less than unity: $|x|^2 = \sum_j |u^*_j g_j|^2 \leq |u|^2 |g|^2 = 1$. With these definitions we can now write
\[ |x|^2 = |\alpha|^2 \sum_{j} |w_j x_j|^2 \leq |\alpha|^2 |w|^2 |x|^2 \leq |\alpha|^2 |w|^4, \] (13)

Here the first inequality is the Cauchy-Schwartz inequality. The second inequality follows from the fact that the first inequality is saturated when $x = ||x||_1 w$ and $||x||_1 \leq 1$. Inserting this tight bound into Eq. (7) we have the metrological power for parallel distributed phase measurement:
\[ M_{PD}^{\theta}(\hat{n}) = 2|\alpha|^2 M_{\psi}^F. \] (14)

We notice that the metrological power for parallel distributed phase measurement is identical to that for the measurement of a single phase obtained by setting $m = 1$ in Eq. (12), giving
\[ M_{\psi}(\hat{n}) = 2|\alpha|^2 M_{\psi}^F. \] (15)

So far all the results we have obtained for the metrological power for phase measurement are only for pure quantum states. We now extend these results to all quantum states. We notice that Eq. (12) is achieved when all the phases are encoded in series, which can be described effectively by the single phase $\theta$. Because of this we are able to apply Yu’s theorem to obtain the metrological power for mixed states [36, 37]. Using Eq. (1), we show that the mixed-state versions of the distributed (multi-parameter) metrological power is given by the convex roof of that for pure states (details are given in the supplement). As a result Eqs. (12), (14), and (15) are also true for mixed states.

**Metrological power for arbitrary transformations:** Consider an arbitrary single-mode transformation $K$ expressed as a normally-ordered power series of the annihilation and creation operators: $K = \sum_{q=0}^p \sum_{k_0}^q \sum_{j=0}^1 \sum_{\alpha_k}^q \hat{a}^{\dagger q-k} \hat{a}^k$. In this power series the number of mode operators in each term (the order of each term) is given by $q$. We will find that every term of order $q$ contributes to the metrological power a term proportional to $|\alpha|^{2q-2}$. The metrological power (defined in the limit of large $|\alpha|$) is thus undefined if the power series is infinite, and so we restrict ourselves to transformations for which the maximum value of $q$ is $p < \infty$. Further, since it is only those terms of order $p$ that contribute to the metrological power, we need only retain those terms and can write $K$ as
\[ K = \sum_{j=0}^p \kappa_j \hat{C}_j \] (16)

where we have defined $\hat{C}_j = \hat{a}^{\dagger (p-j)} \hat{a}^j$, and $\kappa_j = \kappa_{p-j}$, for Hermitian operators. For any arbitrary observable, we prove that the metrological power is proportional to that of displacement measurements in the supplemental materials. Here we just present the results of a special type of observables, where $\kappa_j$ are all real and we normalize $K$ so that $\sum_j \kappa_j = 1$.

Calculating the metrological power for $\tilde{K}$ in the distributed sensing scheme proceeds in a similar fashion to that for phase measurement, except that we perform a partial maximization part way through to show that the contribution of each covariance $\langle \hat{C}_j \rangle - \langle \hat{C}_j \rangle \langle \hat{C}_k \rangle$ is proportional to a quadrature covariance. The details are given in the supplement, and the result is
\[ M_{\psi}(\tilde{K}, \hat{u}) = \frac{2p^2}{w_{max}} |\alpha|^{2(p-1)} \max_{\phi} V(\hat{X}_\phi): \] (17)

so that
\[ M_{\psi}(\tilde{K}) = \frac{1}{2} p^2 m^2 |\alpha|^{2(p-1)} M_{\psi}^F. \] (18)

and as before this relation extends to all mixed states. For a phase measurement we take $\tilde{K} = \hat{a}^\dagger \hat{a}$ so that $p = 2$, only one of the $\kappa$’s is non-zero, being $\kappa_1 = 1$, and we recover the result in Eq. (12). For the self Kerr nonlinearity $\tilde{K} = \hat{a}^{12} \hat{a}^2$ so we have $p = 4$ and $\kappa_j = \delta_{j,2}$ giving
\[ M_{\psi}(\tilde{K}) = 8m^2 |\alpha|^6 M_{\psi}^F. \] (19)

The analysis for parallel distributed metrology of an arbitrary transformation proceeds in the same way, so we have
\[ M_{\psi}(\tilde{K}) = \frac{1}{2} p^2 |\alpha|^{2(p-1)} M_{\psi}^F. \] (20)

**Summary:** We have determined the metrological power for all single mode states, which we define as the quantum advantage provide by the state when classical states are freely available. The explicit optimization also shows the linear network required to obtain the maximum precision. The method we have used here can also be used to obtain explicit expressions for the quantum advantage when the available energy of classical states is not large compared to the non-classical state. In that case, however, it may not be possible to perform the optimization analytically. The method used here might also be useful in exploring the metrological power of multi-mode states, and this is an interesting area for future work.

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Supplement to “The Metrological Power of Nonclassical States”

The distributed metrological power of a mixed state

Based on Eqs. (3) and (5), the metrological power of a mixed state \( \hat{\rho} \) for distributed quantum metrology (as shown in Fig. 1) is defined as

\[
M_{p}(\hat{\rho}, w) = \max_{\text{PLN}} \lim_{|\alpha|/E \to \infty} \frac{1}{|w|^2} \sum_{jk} w_j w_k \left( F_{jk} - 4 \delta_{jk} \langle \hat{\rho} \rangle \right),
\]
(A.21)

where \( \max_{\text{PLN}} \) means the maximization over all possible linear networks in Fig. 1 and \( F_{jk} \) is the element of the Quantum Fisher information matrix (QFIM) \( F(\hat{\rho}) \) for the mixed state \( \hat{\rho} \).

According to the convexity of the QFIM [32],

\[
F(\hat{\rho}) \leq \sum p_i F(\{ |\psi_i\rangle \langle \psi_i| \}),
\]
(A.22)

for any decomposition of the quantum state \( \hat{\rho} = \sum p_i |\psi_i\rangle \langle \psi_i| \). Using the definition of the metrological power for phase sensing of a pure state, it follows from Eqs. (A.22) that

\[
M_{p}(\hat{\rho}, w) \leq \sum p_i M_{p,i}(\hat{\rho}, w)
\]
(A.23)

for any decomposition of \( \hat{\rho} \). Therefore, \( M_{p}(\hat{\rho}, w) \leq \min_{p_i,|\psi_i\rangle} \sum_i p_i M_{p,i}(\hat{\rho}, w) \). Substituting Eq. (12), we obtain

\[
M_{p}(\hat{\rho}, w) \leq 2m^2 |a|^2 \min_{p_i,|\psi_i\rangle} \sum_i p_i M_{p,i}^{F} = 2m^2 |a|^2 M_{p}^{F},
\]
(A.24)

where the last equality comes from the convex roof proof of the quantum Fisher information of a mixed state for single-parameter estimation (displacement sensing) [36, 37]. Noting that Eq. (12) can be satisfied in the scheme of all phases encoded in series (which is equivalent to an effective single parameter \( \xi = 1/m \sum j \xi_j \) applied \( m \) times) and the convex roof of the QFI for a single parameter (phase sensing) [36, 37], we prove that

\[
M_{p}(\hat{\rho}, w) = 2m^2 |a|^2 M_{p}^{F}
\]
(A.25)

for any mixed state. We find the equality is achieved in the situation of all phases encoded in series. However, this does not preclude other possible phase sensing schemes to achieve this maximum metrological power using Fig. 1.

Measuring an arbitrary transformation with a quantum state and arbitrary linear resources

1. System setup

We want to use an arbitrary transformation \( \hat{K}(\hat{a}, \hat{a}' \rangle \) for metrology, where \( \hat{K} \) is given by a sum of products of \( \hat{a} \) and \( \hat{a}' \), up to and including a number of products that contain \( p \) of these operators (but no more than \( p \)). First we rearrange the operators in every product by using \( [\hat{a}, \hat{a}'] = 1 \) so that \( \hat{K} \) is written in the normally-ordered form as \( \hat{K} = \sum_{j=0}^{p} \sum_{j=0}^{q} \kappa_{j} \hat{a}^{j} \hat{a}'^{\dagger} \).

We will find that terms with \( q \) contribute a value proportional to \( |a|^2 q^{2q-2} \). Thus only leading contributing terms will be those of order \( p \) for large \( |a| \). Thus for the purpose of calculating the metrological power, we can write all Hermitian operators \( \hat{K} \) as

\[
\hat{K} = \sum_{j=0}^{p} \kappa_{j} \hat{C}_{j}
\]
(A.26)

and define

\[
\hat{C}_{j} = \hat{a}^{j} \hat{a}'^{\dagger}.
\]
(A.27)

Since \( \hat{K} \) is an observable, it is Hermitian by requiring \( \kappa_{j} = \kappa_{j}\)'. As per the main text on the general distributed metrology,

\[
M_{p}(\hat{K}, w) = \max_{\text{PLN}} \lim_{|\alpha|/E \to \infty} \frac{1}{|w|^2} \sum_{w} w_{w} F_{w,w}^{2},
\]
(A.28)
where
\[ \mathcal{F}_{uv} := 4 \left[ \langle \hat{K}_u \hat{K}_v \rangle - \langle \hat{K}_u \rangle \langle \hat{K}_v \rangle \right] = 4 \sum_{jk} k_j k_k \left[ \langle \hat{C}^j_0, \hat{C}^k_0 \rangle - \langle \hat{C}^j_0 \rangle \langle \hat{C}^k_0 \rangle \right] . \] (A.29)

and \( \hat{K}_u \) and \( \hat{C}^j_0 \) are defined by replacing \( \hat{a} (\hat{a}^\dagger) \) with \( \hat{b}_u (\hat{b}^\dagger_u) \) in \( \hat{K} \) and \( \hat{C} \), respectively.

2. Evaluation of \( \mathcal{F}_{uv} \):

We now replace
\[ \hat{b}_u \rightarrow u^{(a)}_{u0} \hat{a}_0 + f_u \] (A.30)
in \( \langle \hat{C}^j_0 \hat{C}^k_0 \rangle - \langle \hat{C}^j_0 \rangle \langle \hat{C}^k_0 \rangle \), where \( f_u = \sum_{i=0}^{m-1} d^{(a)}_{ul} \alpha_i \) and \( u^{(a)}_{ul} \) are the elements of the \( u^{th} \) row of the unitary matrix \( U_u \) in the general metrology scheme (Fig. 1). We obtain
\[ \langle (u^{(a)}_{u0} \hat{a}_0 + f_u)^s (u^{(a)}_{v0} \hat{a}_0 + f_v)^t (u^{(a)}_{u0} \hat{a}_0 + f_u)^k \rangle - \langle (u^{(a)}_{u0} \hat{a}_0 + f_u)^s (u^{(a)}_{v0} \hat{a}_0 + f_v)^t (u^{(a)}_{u0} \hat{a}_0 + f_u)^k \rangle \] (A.31)
where \( s = p - j \) and \( t = p - k \). We now recall that for large \( |s| \), the dominant terms that appear from expanding the above expression have exactly two factors of the operators (either \( \hat{a}_0 \) or \( \hat{a}_0^\dagger \)) where one factor must come from the first expectation value in the second term above and the other from the second expectation value. Counting the dominant terms, we have
\[ \lim_{|s|/|t| \rightarrow \infty} \langle \hat{C}^j_0 \hat{C}^k_0 \rangle - \langle \hat{C}^j \rangle \langle \hat{C}^k \rangle = jk f_u^{s+t} f_v^{s-t} f_u^{p-j} u^{(v)}_{u0} (\hat{a}_0^\dagger \hat{a}_0^\dagger (\hat{a}_0^\dagger)^2) + st f_u^{s-t} f_v^{s+t} f_u^{p-j} u^{(v)}_{u0} (\hat{a}_0^\dagger \hat{a}_0^\dagger (\hat{a}_0^\dagger)^2) \]
\[ + jk f_u^{s+t} f_v^{s-t} f_u^{p-j} u^{(v)}_{u0} (\hat{a}_0^\dagger \hat{a}_0^\dagger (\hat{a}_0^\dagger)^2) + st f_u^{s-t} f_v^{s+t} f_u^{p-j} u^{(v)}_{u0} (\hat{a}_0^\dagger \hat{a}_0^\dagger (\hat{a}_0^\dagger)^2) \]
\[ = f_u^{p-j} f_u^{s+t} f_u^{s-t} \left[ \langle s f_u u^{(v)}_{u0} \hat{a}_0^\dagger \rangle + f_u f_u^{s+t} u^{(v)}_{u0} (\hat{a}_0^\dagger \hat{a}_0^\dagger (\hat{a}_0^\dagger)^2) \right] - \langle s f_u u^{(v)}_{u0} \hat{a}_0^\dagger \rangle f_u f_u^{s+t} u^{(v)}_{u0} (\hat{a}_0^\dagger \hat{a}_0^\dagger (\hat{a}_0^\dagger)^2) \] (A.32)

Thus, we find
\[ \lim_{|s|/|t| \rightarrow \infty} \mathcal{F}_{uv} := 4 \left( \langle \hat{A}_u \hat{A}_v \rangle - \langle \hat{A}_u \rangle \langle \hat{A}_v \rangle \right) , \] (A.33)
where \( \hat{A}_u = \sum_{j=0}^{p} k_j f_u^{s+p-1} f_u^{s-j-1} (p - j) f_u u^{(v)}_{u0} \hat{a}_0^\dagger \hat{a}_0 + j f_v f_u^{s+t} u^{(v)}_{u0} \hat{a}_0 \).

3. Unification of metrological powers

By substituting \( \mathcal{F}_{uv} \) in Eq. (A.28), we obtain
\[ M_{\phi} (\hat{K}, \omega) = \max_{\phi, \text{PLN}} \frac{4}{|\omega|^2} \sum_{uv} w_u w_v \left( \langle \hat{A}_u \hat{A}_v \rangle - \langle \hat{A}_u \rangle \langle \hat{A}_v \rangle \right) = \max_{\phi, \text{PLN}} \frac{8|\omega|^2}{|\omega|^2} : V_{\phi}(\hat{X}_{\omega}) : , \] (A.34)

where \( \phi = \text{arg} \ z \) and
\[ z = \sum_{u} w_u u^{(a)}_{u0} \sum_{j=0}^{p} k_j f_u^{s+p-1} f_u^{s-j-1} (p - j) . \] (A.35)

Here we note that (1) the relation between the general metrological power and the quadrature variance is guaranteed by the fact that \( \hat{A}_u \) is a quadrature operator of \( \hat{a}_0 \) since \( \sum_{j=0}^{p} k_j f_u^{s+p-1} f_u^{s-j-1} (p - j) u^{(a)}_{u0} = \left( \sum_{j=0}^{p} k_j f_u^{s+p-1} f_u^{s-j-1} u^{(a)}_{u0} \right) \) using \( k_j = k_j^{*} \). (2) The expression of \( z \) reduces to that of distributed quantum metrology when we take \( p = 2 \) and \( k_j = \delta_j \), where \( \delta_j \) is the Kronecker delta.

To obtain the connection between the general metrological power and that of the displacement sensing, \( M_{\phi}^{F} = 4 \max_{\phi, \delta} : V_{\phi}(\hat{X}_{\omega}) : \) by maximizing \( : V_{\phi}(\hat{X}_{\omega}) : \) through the phase of \( \omega \) and the amplitude \( \frac{8|\omega|^2}{|\omega|^2} \) using an arbitrary linear network. We find the maximum amplitude depends on \( k_j \) of the observable \( \hat{K} \), thus for any pure state \( |\phi\rangle \), we have
\[ M_{\phi} (\hat{K}, \mu) = \left( \max_{\mu} \left\{ \frac{2|\hat{e}|^2}{|w|^4} \right\} \right) M_{\phi}^F, \]  
(3.36)

where \( \phi_m \) is the phase when \( V_{\phi} (\hat{X}_m) \) is maximized for the state \( |\phi \rangle \), and \( \arg z = \phi_m \) is the constraint to make sure \( M_{\phi} (\hat{K}, \mu) \propto M_{\phi}^F \).

To maximize the amplitude, we consider

\[ |\hat{e}|^2 \leq \left| \sum_u w_u u^{(u)}_m \right|^2 \left| \sum_{j=0}^n \kappa_j (p-j) f_j u^{p-j-1} f^*_u \right|^2 \leq |\alpha|^2 (p-1) \left| \sum_j \kappa_j (p-j) e^{i(2j+1-p)} \right|^2, \]

(3.37)

where the second inequality is obtained according to the results in the general phase sensing in the main text and the relation \( |f_u| \leq |\alpha| \). The phase \( \theta = \arg f_u \) is assumed to be the same for different \( u \) and \( |z| \) is maximized by choosing \( \theta \) that maximizes the coefficient \( \sum_j \kappa_j (p-j) e^{i(2j+1-p)} \). The optimal phase in the quadrature variance can be obtained by choosing the phase of \( u^{(u)}_m \).

Now we have

\[ M_{\phi} (\hat{K}, \mu) = \frac{2|\alpha|^2 (p-1)}{w_{\max}^2} \max_{\theta} \left| \sum_j \kappa_j (p-j) e^{i(2j+1-p)} \right|^2 \]

(3.38)

By choose the minimal value for \( w_{\max} \) for \( m \) modes, we obtain

\[ M_{\phi} (\hat{K}) = 2m^2 |\alpha|^2 (p-1) \max_{\theta} \left| \sum_j \kappa_j (p-j) e^{i(2j+1-p)} \right|^2 M_{\phi}^F. \]

(3.39)

We note that the above relation can be achieved when all the parameters are encoded in series in one mode, giving an effective single-parameter estimation of the mean of these parameters. Therefore, similar to the distributed quantum sensing scheme, the above relation also extends to mixed states according to the convex roof of quantum Fisher information. Thus we have proved that the metrological powers of a quantum state for any unitary transformations are all proportional to that of displacement sensing.

4. Special considerations

Here we consider a special class of observables where \( \kappa_j \) are all real. Using \( \kappa_j = \kappa_{p-j} \), we have

\[ \max_{\theta} \left| \sum_{j=0}^p \kappa_j (p-j) e^{i(2j+1-p)} \right| = \frac{p}{2} \sum_{j=0}^p \kappa_j \]

(3.40)

We consider that \( \sum_{j=0}^p \kappa_j = 1 \) as a normalization condition in accordance with the usual phase sensing observable, where \( \hat{K} = \frac{1}{2} (\hat{a}^\dagger \hat{a} + \text{H.c.}) \). So the general metrological power becomes

\[ M_{\phi} (\hat{K}) = \frac{p^2}{2} m^2 |\alpha|^2 (p-1) M_{\phi}^F \]

(3.41)

Now we consider some examples. For a phase measurement we have

\[ K = a^\dagger a = \frac{1}{2} \hat{a}^\dagger \hat{a} + \text{H.c.} \]

(3.42)

so that only one of the \( \kappa \)'s is non-zero, being \( \kappa_1 = 1 \), and \( p = 2 \). So we have

\[ M_{\phi} (\hat{K}) = 2m^2 |\alpha|^2 M_{\phi}^F \]

(3.43)

in agreement with our previous result. For a Kerr non-linearity we have

\[ K = a^\dagger a^2 = \frac{1}{2} a^\dagger a^2 + \text{H.c.} \]

(3.44)

with \( \kappa_2 = 1 \) and \( p = 4 \), so that

\[ M_{\phi} (\hat{K}) = 8m^2 |\alpha|^6 M_{\phi}^F \]

(3.45)