Full chains of twists for symplectic algebras

Abstract
The problem of constructing the explicit form for full twist deformations of simple Lie algebras $g$ with twist carriers containing the maximal nilpotent subalgebra $N^+(g)$ is studied. Our main tool is the sequence of regular subalgebras $g_i$ in $U(g)$ that become primitive under the action of extended jordanian twists $F_E : U(g) \rightarrow U_E(g)$. It is demonstrated that the structure of the sequence $\{g_i\}$ is defined by the extended Dynkin diagram of algebra $g$. To construct the injection $g_i \subset U_E(g)$ the special deformations of algebras $U_E(g)$ are performed. They are reduced to the (cohomologically trivial) twists $F_s$. Thus it is proved that full chains of twists can be written in the canonical form $F_B = \prod F_{E_i}$. The links $F_{E_i}$ in such chains must contain not only the extended twists $F_E$ but also the factors $F_s$ whose form depend on the type of the series of classical algebra $g$. The explicit forms of universal $R$-matrices (and the $R$-matrices in the fundamental representations) corresponding to full chains of twists for classical simple Lie algebras are found. The properties of the construction are illustrated by the example of full chain of extended twists for algebra $sp(3)$. 

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1 Introduction

Triangular Hopf algebras $\mathcal{A}(m, \Delta, S, \eta, \epsilon; \mathcal{R})$ play an essential role in quantum group theory and applications. Quantization of antisymmetric $r$-matrices, $r = -r_{21}$, (solutions of classical Yang-Baxter equation (CYBE)) form an important class of such algebras. The corresponding triangular solutions of quantum Yang-Baxter equation (QYBE) can be constructed from the classical ones by means of Campbell-Hausdorff series. However, these constructions are obviously inappropriate for an efficient usage of quantum $\mathcal{R}$-matrices. If one provides the initial Lie algebra $g$ with primitive costructure $\Delta_{\text{prim}}$ and consider the corresponding Hopf algebra $U(g)$, then the solution $F \in U(g)^{\otimes 2}$ of the twist equation

$$(F)_{12} (\Delta_{\text{prim}} \otimes \text{id}) F = (F)_{23} (\text{id} \otimes \Delta_{\text{prim}}) F,$$  

(1)

allows one to find the solution of QYBE, namely: $\mathcal{R}_F = F_{21} F^{-1}$.

For a long time very few types of twists $\mathcal{F}$ were known in a closed form (jordanian twist and its extensions and Reshetikhin twist) [4, 5, 6, 7]. In particular, the jordanian twist [5], defined on the Borel subalgebra $B(2) = \{H, E \mid [H, E] = E\}$, has the twisting element

$$\Phi_J = \exp\{H \otimes \sigma\},$$

where $\sigma = \ln(1 + E)$. This twist generates the solution of QYBE: $\mathcal{R} = (\Phi_J)_{21}\Phi_J^{-1}$. The classical $r$-matrix $r = H \wedge E$ can be associated with it. It was demonstrated in [7] that there exist different extensions of this twist. For the algebra $U(sl(N))$ the element $\mathcal{F}_E \in U(sl(N))^{\otimes 2}$ of the form

$$\mathcal{F}_E = \Phi_E \Phi_J = \left(\prod_{i=2}^{N-2} \Phi_{E_i}\right) \Phi_J = \exp\left\{2\xi \sum_{i=2}^{N-1} \mathcal{E}_{1,i} \otimes \mathcal{E}_{i,N} e^{-\tilde{\sigma}}\right\} \exp\{H_{1,N} \otimes \tilde{\sigma}\}$$

(2)

is a solution of (1). Here $H_{1,N} = \mathcal{E}_{1,1} - \mathcal{E}_{N,N}$, $\tilde{\sigma} = \frac{1}{2} \ln(1 + 2\xi E)$ with $E = \mathcal{E}_{1,N}$ and $\{\mathcal{E}_{i,j}\}_{i,j=1,...,N}$ is the standard matrix basis for the linear algebra. Algebras deformed by extended jordanian twists can be also obtained via the specific contraction of Drinfeld-Jimbo quantizations $U_q(sl(N))$.

The minimal subalgebra $g_c \subset g$, necessary for the twist $\mathcal{F}_E$ to be defined, is called the carrier (sub)algebra for $\mathcal{F}_E$. The carrier algebra for the extended twist $\mathcal{F}_E$ is the multidimensional Heisenberg subalgebra of $sl(N)$.
with generators \( \{ E_{i,j}, E_{j,N} \mid i = 2, \ldots, N-1, j = 1, \ldots, N-1 \} \) extended by the Cartan element \( H_{1,N} \).

In the general case the solutions of the twist equation do not form an algebra with respect to multiplication in \( \mathcal{A} \); the product of twists must not be a twist. However, under certain conditions the compositions of extended twists constitute the solutions of the twist equation, as it was demonstrated in [9]. For infinite series of simple Lie algebras the twists (called “chains”) were found. They were constructed as products, where each factor is itself a twisting element for the initial algebra (see Section 3). The structure of these chains is determined by the fundamental symmetry properties of the corresponding root systems and is common for all simple Lie algebras. Such chains were called canonical.

In the case of chains of extended twists one can assume without loss of generality that their carrier algebras belong to the Borel subalgebra \( B^+(g) = N^+ + H \). The reason of such a restriction is that the chain structure is based on the solvability property of the carrier subalgebra. The Cartan subalgebra \( H \) is not always entirely contained in \( g \). To find the subalgebra \( H \cap g_c \) in \( g \) it is sufficient to consider the extended Dynkin diagram for \( g \) (see section 4). Thus the dimension criterion for a chain is the ratio of the dimension \( \dim N^+(g) \) and that of the nil-radical for chain’s carrier algebra. The chain is said to be full if \( g_c \supset N^+(g) \). Among the maximal canonical chains for classical Lie algebras only the chains for \( U(sl(N)) \) appear to be full. In the case of \( U(sp(N)) \) the carrier subalgebra for canonical chain of extended twists belongs to the regular subalgebra \( sl(N) \subset sp(N) \) (see [4]). Such chains are called improper for they are totally defined by the properties of \( sl(N) \)-subalgebra. Maximal canonical chains for orthogonal algebras are proper but not full.

The purpose of this paper is to construct full proper chains for all classical Lie algebras.

Let \( g_{\lambda_0}^\perp \) be a subalgebra in \( g \) with the root system orthogonal to the initial root \( \lambda_0 \) of jordanian twist \( \Phi_{\mathcal{J}_0} \) (\( \lambda_0 \) is the root of the generator \( E_{\lambda_0} \) in the Borel carrier subalgebra of \( \Phi_{\mathcal{J}_0} \)). Let \( \mathcal{F}_{\mathcal{E}_0} \) be the maximal (for \( g \)) extended twist of the form (2) with the jordanian factor \( \Phi_{\mathcal{J}_0} \). Consider the Cartan decomposition for the subalgebra \( g_{\lambda_0}^\perp \) in the form:

\[
g_{\lambda_0}^\perp = N^-(g_{\lambda_0}^\perp) + H(g_{\lambda_0}^\perp) + N^+(g_{\lambda_0}^\perp) = N^-(g_{\lambda_0}^\perp) + B^+(g_{\lambda_0}^\perp).
\]

The existence of canonical chains of extended twists is based on the so-called
“matreshka” effect [9]. It amounts to the primitivization of the cost ructure of the subalgebra $g^\perp_{\lambda_0}$ twisted by $F_{E_0}$. In the canonical chains of twists for $B_n$ and $D_n$ series (constructed in [9]) the subalgebras $N^+(g^\perp_{\lambda_0}) \setminus (N^+(g^\perp_{\lambda_0}) \cap g_c)$ are nontrivial, thus the canonical chain can not be full. The reason for this peculiarity is that the coproducts in the space of the subalgebra $g^\perp_{\lambda_0}$ are nontrivially deformed by the preceding links of the extended twists. It was demonstrated in [10] that the deformed universal enveloping algeb ras $U(g^\perp_{\lambda_0})$ contain not only the subalgebras $g^\perp_{\lambda_0}$ with deformed costructure but also the primitive ones equivalent to $g^\perp_{\lambda_0}$. This property of orthogonal classical Lie algebras allows to construct full chains of twists for them, i.e. chains with the carrier algebras $g_c \supset N^+(g^\perp_{\lambda_0})$ [11].

In Section 4 the decomposition of the root system consistent with the structure of extended twists for simple Lie algebra will be constructed. There the system of positive roots $\Lambda^+$ will be presented as a union of the initial root $\lambda_0$, the constituent roots $\{\lambda', \lambda''| \lambda' + \lambda'' = \lambda_0\}$ for $\lambda_0$ and the subsystem $\Lambda^\perp_{\lambda_0}$ of positive roots orthogonal to $\lambda_0$

$$\Lambda^+ = \lambda_0 \cup \{\lambda'\} \cup \{\lambda''\} \cup \Lambda^\perp_{\lambda_0}. $$

The systematic construction of full chains is based on this decomposition. For symplectic algebras we investigate the existence of proper and full chains of twists (see Section 5). Our main tool is the deformed carrier space mentioned above. As soon as full chains of twists can not be based on the sequences of classical injections $sp(1) \subset sp(2) \subset ... \subset sp(N - 1) \subset sp(N)$ we shall look for the necessary injections in the universal enveloping algebras $U(sp(N))$. The problem can be reduced to the construction of a sequences of injections $U(sp(1)) \subset ... \subset U(sp(N - 1)) \subset U(sp(N))$ where to construct the images of a subalgebra $sp(M)$ means to construct it’s nonlinear (in terms of initial gen- erators) realization in $U(sp(M+1))$. First we consider the maximal extended jordanian twists $F_{E_k}$ for $U(sp(N - k))$. The full proper chain $F_{E_0 \prec \cdots \prec (N-1)}$ of such twists for $U(sp(N))$ will be realized as the product of factors $F_{E_k'}$. Such construction can be considered as a generalization of the deformed jordanian twists used to define chains for orthogonal Lie algebras [11]. The universality of the primitivization effect for subalgebras $g^\perp_{\lambda_0}$ will be established and the recursion formula for full chains of twists for $U(sp(N))$ will be obtained.

To find the explicit expressions for full chains of twists for all classical Lie algebras we propose to introduce the additional twisting factors. Despite their cohomological triviality they realize the transitions from the deformed
to primitive subalgebras \( g_i^{(k)} \) in \( U_{\mathcal{F},F} \) (see Section 6). This allows to include
the deformed carrier spaces (specific to the full chains construction) in the
general scheme so that the latter remains similar to the canonical one.

The expressions for the appropriate multiparametric universal \( R \)-matrices
and \( R \)-matrices in the fundamental representation are presented. As an ex-
ample the case of \( U(\text{sp}(1)) \subset U(\text{sp}(2)) \subset U(\text{sp}(3)) \) is considered in Section 8.

## 2 Basic definitions

In this section we introduce the necessary notations, remind the definitions
and properties of twists. The algebras are considered below over the field \( C \)
of complex numbers.

A Hopf algebra \( \mathcal{A}(m, \Delta, \eta, \epsilon, S) \) \([1, 2, 12]\) with multiplication \( m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \),
coproduct \( \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \), unit \( \eta : C \to \mathcal{A} \), counit \( \epsilon : \mathcal{A} \to C \) and
antipode \( S : \mathcal{A} \to \mathcal{A} \) can be transformed \([12]\) by an invertible (twisting)
element \( F \in \mathcal{A} \otimes \mathcal{A} \), \( F = \sum f_i^{(1)} \otimes f_i^{(2)} \), into a twisted one \( \mathcal{A}_F(m, \Delta_F, \eta, \epsilon, S_F) \).
Hopf algebra \( \mathcal{A}_F \) has the same multiplication, unit and counit but the twisted
coproduct and antipode given by:

\[
\Delta_F(a) = \mathcal{F} \Delta(a) \mathcal{F}^{-1}, \quad S_F(a) = v S(a) v^{-1},
\]

with

\[
v = \sum f_i^{(1)} S(f_i^{(2)}), \quad a \in \mathcal{A}.
\]

To provide the necessary properties (coassociativity) for the coproduct \( \Delta_F \)
it is sufficient that the twisting element \( \mathcal{F} \) is a solution of the twist equations \([3]\):

\[
(\mathcal{F})_{12} (\Delta \otimes \text{id}) \mathcal{F} = (\mathcal{F})_{23} (\text{id} \otimes \Delta) \mathcal{F},
\]

\[
(\epsilon \otimes \text{id}) \mathcal{F} = (\text{id} \otimes \epsilon) \mathcal{F} = 1.
\]

If the initial algebra \( \mathcal{A} \) is quasitriangular with the universal element \( \mathcal{R} \), then
such is the twisted one \( \mathcal{A}_F(m, \Delta_F, \eta, \epsilon, S_F, \mathcal{R}_F) \) with the \( \mathcal{R} \)-matrix

\[
\mathcal{R}_F = \mathcal{F}_{21} \mathcal{R} \mathcal{F}^{-1}.
\]

Any quantization of a Lie bialgebra with antisymmetric classical \( r \)-matrix
can be described by a twist \([3]\). Consider a skew solution of the classical
Yang-Baxter equation (CYBE) \( r \in \Lambda^2 \mathfrak{g} \). Let \( 1 \subset \mathfrak{g} \) be the minimal Lie
subalgebra such that the bilinear form $b$ on $l$ induced by $r$ is non-degenerate. This form $b$ is a 2-cocycle for the algebra $l$, $b \in Z^2(l, C)$, i.e.

$$b([x, y], z) + b([z, x], y) + b([y, z], x) = 0.$$  

(6)

Algebra $l$ is called **quasi-Frobenius subalgebra** (of the initial Lie algebra $g$), if there exists a non-degenerate 2-cocycle $b$ defined on it. Algebra $l$ is called **Frobenius**, if there exists a linear functional $c \in l^*$, such that the form $c([x, y]) = b(x, y)$ is non-degenerate. If $l$ is a Frobenius algebra, then $b(x, y) \in B^2(l, C)$, i.e. the form $b$ is 2-coboundary. The classification of quasi-Frobenius subalgebras for $sl(n)$ is given in [13].

In Section 7 we shall demonstrate how the cocycles of quasi-Frobenius algebras can be directly used in twist constructions.

### 3 Extended twists

Extended jordanian twists are associated with the parametric set $\mathcal{L} = \{L(\alpha, \beta)_{\alpha + \beta = 1}\}$ of Frobenius algebras,

$$[H, E] = \delta E, \quad [H, A] = \alpha A, \quad [H, B] = \beta B,$$

$$[A, B] = \gamma E, \quad [E, A] = [E, B] = 0, \quad \alpha + \beta = \delta.$$  

(7)

The equation (4) for the carrier algebra $g_c = \mathcal{L}$ has the solutions

$$\mathcal{F}_E^{(\alpha, \beta)} = \exp\{A \otimes Be^{-\beta \sigma}\} \exp\{H \otimes \sigma\}$$  

(8)

and

$$\mathcal{F}_E^{(\alpha, \beta)} = \exp\{-B \otimes Ae^{-\alpha \sigma}\} \exp\{H \otimes \sigma\}.$$  

(9)

The twists $\mathcal{F}_E^{(\alpha, \beta)}$ and $\mathcal{F}_E^{(\alpha, \beta)}$ correspond to the classical $r$-matrix, $r = H \wedge E + A \wedge B$. Algebras of type $L$ can be found in any simple Lie algebra $g$ with rank$(g) > 1$.

It was demonstrated in [9], that (for classical Lie algebras) the twists of type (8) or (9) can be systematically composed into sequences named **chains of twists**. This possibility is based on the existence of sequences of regular injections

$$g_p \subset g_{p-1} \ldots \subset g_1 \subset g_0 = g$$  

(10)

and the properties of the invariant symmetric forms on the carrier spaces for extended twists. To construct a chain it is necessary to choose the **initial**
root $\lambda^k_0$ in the root system $\Lambda(g_k)$. In the root space $V_{\Lambda(g_k)}$ consider the subspace $V_{\lambda^k_0}^\perp$ orthogonal to the initial root $\lambda^k_0$. The root system $\Lambda(g_{k+1}) = \Lambda(g_k) \cap V_{\lambda^k_0}^\perp$ defines the subalgebra $g_{k+1}$. Consider the set $\pi_k$ of constituent roots for the root $\lambda^k_0$:

$$\pi_k = \left\{ \lambda', \lambda'' | \lambda' + \lambda'' = \lambda^k_0; \; \lambda' + \lambda^k_0, \lambda'' + \lambda^k_0 \notin \Lambda(g_k) \right\}$$

$$\pi_k = \pi'_k \cup \pi''_k; \quad \pi'_k = \{ \lambda' \}, \pi''_k = \{ \lambda'' \}.$$  \hspace{1cm} (11)

For a classical Lie algebra $g$ in $g_{\lambda^k_0}$ one can always find a subalgebra $g_{k+1} \subseteq g_{\lambda^k_0} \subseteq g_k$, whose generators have the primitive costructure after the twist $F_{E_k}$ have been applied to $U(g_k)$ \cite{[9]}. The phenomenon of primitivization of a subalgebra $g_{k+1} \subseteq g_k$ in $U_{E_k}(g_k)$ was called the “matreshka” effect. It allows one to construct chains of extended jordanian twists,

$$F_{E_0 \cdot p} = \prod_{\lambda' \in \pi'_p} \exp \left\{ E_{\lambda'} \otimes E_{\lambda^0_0 - \lambda'} e^{-\frac{i}{2} \sigma_{\lambda^0_0}} \right\} \cdot \exp \left\{ H_{\lambda^0_0} \otimes \sigma_{\lambda^0_0} \right\}.$$  

$$\prod_{\lambda' \in \pi''_{p-1}} \exp \left\{ E_{\lambda'} \otimes E_{\lambda^0_{p-1} - \lambda'} e^{-\frac{i}{2} \sigma_{\lambda^0_{p-1}}} \right\} \cdot \exp \left\{ H_{\lambda^0_{p-1}} \otimes \sigma_{\lambda^0_{p-1}} \right\}.$$  

$$\ldots$$

$$\prod_{\lambda' \in \pi''_0} \exp \left\{ E_{\lambda'} \otimes E_{\lambda^0_0 - \lambda'} e^{-\frac{i}{2} \sigma_{\lambda^0_0}} \right\} \cdot \exp \left\{ H_{\lambda^0_0} \otimes \sigma_{\lambda^0_0} \right\}.$$  \hspace{1cm} (12)

Here $H_{\lambda^0_0}$ is the generator dual to the root $\lambda^k_0$, $\sigma_{\lambda^0_0} = \ln(1 + E_{\lambda^0_0})$.

Chains of twists give an opportunity to quantize explicitly the large number of $r$-matrices corresponding to Frobenius subalgebras in simple Lie algebras \cite{[13]}. Some peculiarities were found in the construction of chains for the orthogonal classical algebras. In this case the subalgebra $g_{\lambda^k_0}^\perp$ is isomorphic to the direct sum $sl(2) \oplus g_{k+1}$. After the application of the twist $F_{E_k}$ the summand $sl(2)$ obtains the nontrivial costructure, whereas the coproducts for $g_{k+1} = so(N - 4(k + 1))$ remains primitive. Such primitive summands compose the carrier space for the canonical chains of type (12) (they are described in \cite{[2]}). These chains are based on the sequences of injections $so(N - 4(k + 1)) \subset so(N - 4k)$. It was demonstrated in \cite{[10] [11]} that $U_{E_k}(g_{\lambda^k_0})$ contains not only the deformed subalgebra $U_{E_k}(sl(k)(2))$, but also the primitive Hopf subalgebra isomorphic to $U(sl(k)(2))$. The generators of this algebra can be composed using the scalar products in the space of vector subrepresentations $d^p(so(N - 4(k + 1)))$. Such subrepresentations appear.
in the reduction of the adjoint representation of $so(N)$ to the subalgebra $so(N - 4(k + 1))$. These subrepresentations have the weight diagram formed by the projection of constituent roots \( \Pi \) on the space \( V(g_{\lambda_0}^+) \). As a result the primitive subalgebra \( g_{\lambda_0}^+ \) can be found in \( U_{\varepsilon_k}(g_{\lambda_0}^+) \) providing the evidence that the primitivization of the subalgebra orthogonal to the initial root is a general property of chain twist deformations. In this sense the chains of extended twists for \( B_N \) series are similar to that for \( A_N \) series. The difference is that in the \( B_N \)-case the primitive subalgebra equivalent to \( g_{\lambda_0}^+ \) is realized on the “deformed carrier space” \([10]\).

For symplectic simple Lie algebras the situation appears to be more complicated. When the twist \( F_{\varepsilon_k-1} \) is applied to \( g_{\lambda_0}^+ \subset U(sp(N - k)) \) most of generators acquire the nonprimitive coproducts. The possibility to construct the proper chain of twists like \([12]\) fails. However, as it will be shown below, the primitive subalgebra isomorphic to \( U(sp(N - (k + 1))) \) does exist and its generators can be obtained by the nonlinear transformations in the space of the subalgebra \( U_{\varepsilon_{k-1}}(sp(N - (k + 1))) \).

4 Extended Dynkin diagrams and chains of twists

Extended Dynkin diagrams for infinite series of simple Lie algebras \( g \) allow one to establish the structure of maximal chains of twists for the sequence of regular injections and also define the form of such sequences. Let us remind that the extended Dynkin diagrams allow one to describe all regular subalgebras of \( g \) \([14]\).

Consider the algebra \( A_N \) and the corresponding extended Dynkin diagram:

\[
\begin{array}{c}
\bullet \\
\circ \quad \cdots \quad \circ \\
\circ \\
\end{array}
\]

Let \( \{\alpha_i\} \) be the set of simple roots for the algebra \( sl(N + 1) \), \( \theta \) be the grey root: \( \theta \equiv - \sum_{i=1}^{N} \alpha_i \). In the orthonormal basis \( \{e_i\}_{i=1,\ldots,(N+1)} \) of the space \( \mathbb{R}^{N+1} \)
we have $\alpha_i = e_i - e_{i+1}$ and $-\theta = e_1 - e_{N+1}$. If one choose the root $-\theta$ to be the initial one, then it follows from the Dynkin diagram that $g_{\theta}^+ \approx sl(N-1)$, i.e. the twist $F_{\varepsilon_1}$, based on the long root $-\theta$, will have two constituent basic roots $\alpha_1$ and $\alpha_N$ in the set $\pi_1$ (see the definition (II)). The chain of regular injections for $A_N$ series will have the following form:

$$sl(2) \subset sl(4) \subset ... \subset sl(N-1) \subset sl(N+1)$$

or

$$sl(3) \subset sl(5) \subset ... \subset sl(N-1) \subset sl(N+1).$$

For $N = 2k - 1$ ($N = 2k$) the factor $\Phi_\varepsilon = \exp \{H_{\lambda_0} \otimes \sigma_{\lambda_0}\}$ (correspondingly $F_{\varepsilon_k} = \exp \left\{ E_{\lambda_0} \otimes E_{\lambda_0}^{-e_1} e_{-\sigma_{\lambda_0}} \right\} \cdot \exp \{H_{\lambda_0} \otimes \sigma_{\lambda_0}\}$) will be the last in the maximal chain of twists.

Let us remind that a chain of twists for an algebra $g$ is said to be full, if it’s carrier subalgebra contains the nilpotent subalgebra $N^+(g)$. In the case of $A_N$-series this means that the maximal canonical chains are full.

Extended Dynkin diagrams for $B_N$ and $D_N$ series have respectively the following form:

![Extended Dynkin diagrams](image)

It follows immediately that the structure of the spaces orthogonal to the initial root and the corresponding subalgebras are $g_{\theta}^+(B_N) \approx B_{N-2} \oplus sl(2)$ and $g_{\theta}^+(D_N) \approx D_{N-2} \oplus sl(2)$. Thus, the chains of injections for $B_N$ and $D_N$ series have the form:

$$so(3) \subset so(7) \subset ... \subset so(2N - 3) \subset so(2N + 1)$$

or

$$so(5) \subset so(9) \subset ... \subset so(2N - 3) \subset so(2N + 1)$$

8
and

\[ \text{so}(4) \subset \text{so}(8) \subset ... \subset \text{so}(2(N - 2)) \subset \text{so}(2N) \]

or

\[ \text{so}(6) \subset \text{so}(10) \subset ... \subset \text{so}(2(N - 2)) \subset \text{so}(2N). \]

For \( B_N \) series \( \Phi_{J_k} \) will be the last factor in the maximal chain of twists (where \( N = 2k - 1 \) if \( N \) is odd, and \( N = 2(k - 1) \) if \( N \) is even).

The \( D_N \)-series is remarkable for the appearance of two possibilities (depending on \( N \)) in the last but one step of a maximal chain of twists:

![Diagram](image)

The first diagram corresponds to even-odd orthogonal algebra, and the second one to even-even algebra. In the first case the subalgebra \( g^\perp_\theta \) is isomorphic to \( \text{sl}(4) \oplus \text{sl}(2) \) and \( \Phi_{J_k} \) (\( N = 2k - 1 \)) is the last factor in the maximal chain.

In the second case \( g^\perp_\theta \) is isomorphic to \( \text{sl}(2) \oplus \text{sl}(2) \oplus \text{sl}(2) \) and \( \Phi^1_{J_k}, \Phi^2_{J_k}, \Phi^3_{J_k} \) (\( N = 2k \)) are the last three factors in the maximal chain of twists. It was shown in [11], that for odd \( N \) (\( D_{2k+1} \) algebras) in \( \text{so}(2N) \) there always exists the independent Cartan generator which does not belong to the carrier algebra for the maximal chain of twists. In other cases, for \( \text{so}(2N) \) with even \( N \) and algebras of series \( B \), the carrier subalgebra for the maximal chain of twists coincides with the Borel subalgebra \( B^+(g) \).

For \( C_N \) series the extended Dynkin diagram has the form

![Diagram](image)

Here the subalgebra orthogonal to the initial root is isomorphic to \( C_{N-1} \) and the jordanian twist \( \Phi_{J_N} \) is the last factor in the maximal chain of twists. The Borel subalgebra \( B^+(sp(N)) \) appears to be the carrier subalgebra for the maximal chain.

**Proposition.** The system of positive roots \( \Lambda^+ \) for classical Lie algebra \( g \) with Cartan decomposition

\[ g = N^-(g) + H(g) + N^+(g) = N^-(g) + B^+(g) \]
can be presented as a union of the initial root $\lambda_0$, its constituent roots $\{\lambda', \lambda''| \lambda' + \lambda'' = \lambda_0\}$ and the subsystem $\Lambda^\perp_{\lambda_0}$ of positive roots orthogonal to $\lambda_0$:

$$\Lambda^+ = \lambda_0 \cup \{\lambda'\} \cup \{\lambda''\} \cup \Lambda^\perp_{\lambda_0}. \quad (13)$$

\[\text{♠ Proof.}\] Any highest root $\lambda_0 = -\theta$ is long. If $\text{rank}(g) > 1$ then the basic root $\alpha_i$ cannot be collinear to the highest one. For any positive root $\alpha \in \Lambda^+$ we have

$$(\lambda_0, \alpha) \geq 0.$$ 

The case of zero projection corresponds to the subsystem $\Lambda^\perp_{\lambda_0}$. For any $\{\alpha| (\lambda_0, \alpha) > 0, \alpha \neq \lambda_0\}$ the difference $\lambda_0 - \alpha$ is a positive root. Indeed

$$(\lambda_0 - \alpha, \lambda_0) = (\lambda_0, \lambda_0)(1 - \frac{1}{2}m(\alpha, \lambda_0)), \quad (14)$$

where the entry of the Cartan matrix $m(\alpha, \lambda_0)$ in our situation has a single value equal to 1 [15]. Hence, any positive root $\alpha$ noncollinear and nonorthogonal to $\lambda_0$ is constituent for $\lambda_0$ and the appropriate pair is $\{\alpha, \lambda_0 - \alpha\}$. ♠

We have shown that the set $\Lambda^+ \setminus \Lambda^\perp_{\lambda_0}$ consists of the root $\lambda_0$ and its constituent roots. It is obvious (see (14)) that any constituent root has a projection on $\lambda_0$ equal to $1/2$.

The decomposition (13) can be considered as an algebraic foundation for the system of full chains of extended twists for simple algebras.

## 5 Chains of twists for $U(sp(N))$

Consider the sequence of injections

$$U(sp(1)) \subset ... \subset U(sp(N-1)) \subset U(sp(N)). \quad (15)$$

In the root system

$$\Lambda(sp(N)) = \{\pm e_i \pm e_j, \pm 2e_i\}; \quad i, j = 1, ..., N$$

choose the vector $\lambda_0 = 2e_1$ to be the initial root. Introduce the following notations:

1. $e_{i \pm j} \equiv e_i \pm e_j, \quad i \neq j$ for the short roots;
2. \( E_{i \pm j} \) and \( E_{i^+} \) (respectively \( F_{i \pm j} \) and \( F_{i^+} \)) — the generators corresponding to the roots \( e_{i \pm j} \) and \( 2e_i \) (respectively \( -e_{i \pm j} \) and \( -2e_i \)), here \( i < j \);

3. \( \pi_1 \) — the set of constituent roots for \( \lambda_0 \),

\[
\pi_1 = \{ \lambda', \lambda'' | \lambda' + \lambda'' = 2e_1; \lambda' + 2e_1, \lambda'' + 2e_1 \not\in \Lambda \}.
\]

Obviously the constituent roots are \( \pi_1 = \{ e_{1-i}, e_{1+i} | i = 2, \ldots, N \} \).

Let \( H_{ii} \) be the Cartan generators dual to the roots \( 2e_i \). The Borel subalgebra for \( sp(N) \) in the basis \( \{ E_{i+i}, E_{i \pm j}, H_{ii} | i, j = 1, \ldots, N; i < j \} \) is given by the commutation relations

\[
\begin{align*}
[H_{ii}, E_n] &= \delta_{i,n} E_{n+n}, \\
[H_{ii}, E_n] &= \frac{1}{2}(\delta_{i,n} + \delta_{i,m}) E_{n+m}, \\
[H_{ii}, E_n] &= \frac{1}{2}(\delta_{i,n} - \delta_{i,m}) E_{n-m}.
\end{align*}
\]

It is natural to start the construction of a chain by the Jordanian twist

\[
\Phi_{J_1} = \exp\{H_{11} \otimes \sigma_{1+1}\}, \quad \sigma_{1+1} = \ln(1 + E_{1+1}).
\]

Since the triple of roots \( \{2e_1, e_{1-i}, e_{1+i} | i = 2, \ldots, N \} \) defines the subalgebra \( L'(\alpha, \beta) \) with \( \alpha = \beta = \frac{1}{2} \), the constituent roots \( \{ e_{1-i}, e_{1+i} | i = 2, \ldots, N \} \) allow one to construct the full extension \( \Phi_{E_1} \) of the twist \( \Phi_{J_1} \):

\[
\Phi_{E_1} = \prod_{i=2}^N e^{E_{1-i} \otimes E_{1+i} e^{-\frac{1}{2} \sigma_{1+1}}}. \tag{16}
\]

The extended twist

\[
\mathcal{F}_{E_1} = \Phi_{E_1} \Phi_{J_1},
\]

performs the deformation \( U(sp(N)) \rightarrow U_{E_1}(sp(N)) \). The cosstructure of the subalgebra \( U_{E_1}(sp(N-1)) \subset U_{E_1}(sp(N)) \) is defined by the following relations:

\[
\begin{align*}
\Delta_{E_1}(E_{i+i}) &= E_{i+i} \otimes 1 + 1 \otimes E_{i+i} + 2E_{1+i} \otimes E_{1+i} e^{-\frac{1}{2} \sigma_{1+1}} + E_{1+i}^2 \otimes e^{-\sigma_{1+1}}, \\
\Delta_{E_1}(E_{i+j}) &= E_{i+j} \otimes 1 + 1 \otimes E_{i+j} + E_{1+i} \otimes E_{1+j} e^{-\frac{1}{2} \sigma_{1+1}} + E_{1+j}^2 \otimes E_{1+i} e^{-\sigma_{1+1}}, \\
\Delta_{E_1}(E_{i-j}) &= E_{i-j} \otimes 1 + 1 \otimes E_{i-j}, \\
\Delta_{E_1}(F_{i+i}) &= F_{i+i} \otimes 1 + 1 \otimes F_{i+i} + 2E_{1-i} \otimes E_{1-i} e^{-\frac{1}{2} \sigma_{1+1}} - E_{1-i}^2 \otimes E_{1+i} e^{-\sigma_{1+1}}, \\
\Delta_{E_1}(F_{i+j}) &= F_{i+j} \otimes 1 + 1 \otimes F_{i+j} + E_{1-i} \otimes E_{1-j} e^{-\frac{1}{2} \sigma_{1+1}} + E_{1-j}^2 \otimes E_{1-i} e^{-\sigma_{1+1}} - E_{1-i} E_{1-j} \otimes E_{1+i} e^{-\sigma_{1+1}}, \\
\Delta_{E_1}(F_{i-j}) &= F_{i-j} \otimes 1 + 1 \otimes F_{i-j},
\end{align*}
\]
where $i, j = 2, \ldots, N$; $j > i$. Other generators of the subalgebra $U_{E_1}(sp(N - 1))$ retain the former primitive coproducts.

In order to construct a chain of twists corresponding to the sequence (15), it is necessary to find the carrier algebra for the jordanian twist (and its extensions) containing the generator of the initial root $\lambda_0^1 = 2e_2$ for $sp(N - 1)$ subalgebra. It is generally assumed that the subalgebra to be twisted must possess the primitive costructure, since in this case one may profit by using the known solution of the twist equation. The expressions (16) indicate that in our case it is impossible to apply the standard procedure of chain construction. The problem will be solved, if one can find in $U_{E_1}(sp(N))$ a primitive subalgebra isomorphic to $U(sp(N - 1))$ and generated by the vectors of the space $sp(N - 1)$ and the element $\sigma_{1+1}$.

Such a subalgebra does exist in $U_{E_1}(sp(N))$. To make sure of this consider the following set of generators:

\[
\begin{align*}
E'_{i+i} &= E_{i+i} - E^2_{i+i}e^{-\sigma_{i+1}}, \\
F'_{i+i} &= F_{i+i} - E^2_{i-i}, \\
E'_{i+j} &= E_{i+j} - E_{i+1}e_{i+1}E_{1+j}e^{-\sigma_{i+1}}, \\
F'_{i+j} &= F_{i+j} - E_{1-i}E_{1-j}, \\
E'_{i-j} &= E_{i-j}, \\
F'_{i-j} &= F_{i-j}, \\
H'_{ii} &= H_{ii}.
\end{align*}
\] (17)

Using the formulas (16) it is not difficult to check that the costructure of the generators $X'$ is primitive. The space with the basis $E'_{i\pm j}, F'_{i\pm j}, H'_{i+i}$ ($i, j = 2, \ldots, N; j \geq i$) forms an algebra $sp'(N - 1)$ equivalent to $sp(N - 1)$.

The existence of the primitive algebra $sp'(N - 1) \subset U(sp(N - 1))$ verifies the universality of the “matreshka” effect. The main point of this effect is that in the universal enveloping algebra deformed by the full extended twist there exists the primitive subalgebra, its root system consists of all the roots belonging to the hyperplane orthogonal to $\lambda_0$. In order to construct explicitly the chain of twists for regular injections (14), it is necessary to find the transformation of the basis leading to the primitive subalgebra like $sp'(N - 1)$. Note, that for the construction of a chain it is not important how the generators of $sp(N) \setminus sp(N - 1)$ are transformed.

Given a cocommutative Hopf algebra $U(sp'(N - 1))$, generated by $E'_{i\pm j}, F'_{i\pm j}, H'_{i+i}$ ($i, j = 2, \ldots, N; j \geq i$), we are able to perform the twisting procedure with $2e_2$ being the initial root, and so on.
In the $k$-th step we shall have a cocommutative Hopf algebra $U(sp^{k-1}(N-(k-1)))$ and the possibility to apply the composition

$$F_{E_k} = \Phi_{E_k} \Phi_{\tau_k}$$

which consists of the jordanian twist

$$\Phi_{\tau_{k-1}} = \exp\{H_{kk} \otimes \sigma_{k+k}^{(k-1)}\}$$

and its full extension

$$\Phi_{E_k} = \prod_{i>k} \exp\left\{E_{i-1}^{(k-1)} \otimes E_{i+k}^{(k-1)} e^{-\frac{1}{2}\sigma_{k+k}^{(k-1)}}\right\}.$$  

Here $\sigma_{k+k}^{(k-1)} = \ln(1 + E_{k+k}^{(k-1)})$.

Applying the sequence of twists of the type (18),

$$F_{B_0 \langle (k-1) \rangle} = \prod_{j=1}^{k} F_{E_j},$$

we shall obtain the subalgebra $U_{E_k}(sp^{k-1}(N-k))$ whose costructure will be defined by the following equalities:

$$\Delta_{E_k}(E_{i+j}^{(k-1)}) = E_{i+j}^{(k-1)} \otimes 1 + 1 \otimes E_{i+j}^{(k-1)} + 2E_{k+i}^{(k-1)} \otimes E_{k+i}^{(k-1)} e^{-\frac{1}{2}\sigma_{k+i}^{(k-1)}} +$$

$$+ E_{k+i}^{(k-1)} \otimes E_{k+i}^{(k-1)} e^{-\sigma_{k+i}^{(k-1)}},$$

$$\Delta_{E_k}(E_{i+j}^{(k-1)}) = E_{i+j}^{(k-1)} \otimes 1 + 1 \otimes E_{i+j}^{(k-1)} + 2E_{i+k}^{(k-1)} \otimes E_{i+k}^{(k-1)} e^{-\frac{1}{2}\sigma_{k+k}^{(k-1)}} +$$

$$+ E_{i+k}^{(k-1)} \otimes E_{i+k}^{(k-1)} e^{-\sigma_{k+k}^{(k-1)}},$$

$$\Delta_{E_k}(E_{i-j}^{(k-1)}) = E_{i-j}^{(k-1)} \otimes 1 + 1 \otimes E_{i-j}^{(k-1)} + 2E_{k+i}^{(k-1)} \otimes E_{k+i}^{(k-1)} e^{-\frac{1}{2}\sigma_{k+i}^{(k-1)}} -$$

$$- (E_{k-i}^{(k-1)})^2 \otimes E_{k+i}^{(k-1)} e^{-\sigma_{k+i}^{(k-1)}},$$

$$\Delta_{E_k}(E_{i-j}^{(k-1)}) = E_{i-j}^{(k-1)} \otimes 1 + 1 \otimes E_{i-j}^{(k-1)} + 2E_{k-j}^{(k-1)} \otimes E_{k-j}^{(k-1)} e^{-\frac{1}{2}\sigma_{k+j}^{(k-1)}} +$$

$$+ E_{k-j}^{(k-1)} \otimes E_{k-j}^{(k-1)} e^{-\sigma_{k+j}^{(k-1)}},$$

$$\Delta_{E_k}(F_{i+j}^{(k-1)}) = F_{i+j}^{(k-1)} \otimes 1 + 1 \otimes F_{i+j}^{(k-1)} + E_{k+i}^{(k-1)} \otimes E_{k+i}^{(k-1)} e^{-\frac{1}{2}\sigma_{k+i}^{(k-1)}} +$$

$$+ E_{k+i}^{(k-1)} \otimes E_{k+i}^{(k-1)} e^{-\sigma_{k+i}^{(k-1)}},$$

$$\Delta_{E_k}(F_{i-j}^{(k-1)}) = F_{i-j}^{(k-1)} \otimes 1 + 1 \otimes F_{i-j}^{(k-1)}.$$  

Thus the result of the sequence of nonlinear transformations is the reproduction in the $k$-th step of the costructure (10). This allows to continue
the construction of the chain. In $U_{\mathcal{E}_k}(sp^{k-1}(N - (k - 1)))$ there exists the primitive subalgebra isomorphic to $U(sp(N - k))$. It can be obtained by the nonlinear basis transformation

$$
E^{(k)}_{i+i} = E^{(k-1)}_{i+i} - \left( E^{(k-1)}_{k+i} \right)^2 e^{-\sigma(k-1)},
$$

$$
E^{(k)}_{i+j} = E^{(k-1)}_{i+j} - E^{(k-1)}_{k+i} E^{(k-1)}_{k+j} e^{-\sigma(k-1)},
$$

$$
F^{(k)}_{i+i} = F^{(k-1)}_{i+i} - \left( E^{(k-1)}_{k-i} \right)^2 ,
$$

$$
F^{(k)}_{i+j} = F^{(k-1)}_{i+j} - E^{(k-1)}_{k-i} E^{(k-1)}_{k-j} ,
$$

$$
E^{(k)}_{i-j} = E^{(k-1)}_{i-j},
$$

$$
F^{(k)}_{i-j} = F^{(k-1)}_{i-j},
$$

$$
H^{(k)}_{ii} = H^{(k-1)}_{ii}, \quad i, j = k + 1, \ldots, N; j > i.
$$

Continuing the specified procedure recursively one can obtain the explicit expression for the maximal chain of extended twists corresponding to the sequence of injections (13),

$$
\mathcal{F}_{\mathcal{E}_k(N-1)} = e^{H_{NN} \otimes \sigma(N-1)} \times
$$

$$
e^{E^{(N-2)}_{(N-1)-N} \otimes E^{(N-2)}_{(N-1)+N} \cdot \frac{1}{2} \sigma(N-2)} e^{H_{(N-1)+1} \otimes \sigma(N-2)} \times
$$

$$
\vdots
$$

$$
e^{E^{(N-2)}_{2-2} \otimes E^{(N-2)}_{2+2} \otimes \frac{1}{2} \sigma(N-2)} \times e^{E^{(N-2)}_{1+1} \otimes E^{(N-2)}_{1+1} \otimes \frac{1}{2} \sigma(N-2)} e^{H_{(N-1)+1} \otimes \sigma(N-2)} \times
$$

$$
e^{E^{(N-2)}_{1-1} \otimes \sigma(N-2)} \times e^{H_{(N-1)+1} \otimes \sigma(N-2)} e^{H_{(N-1)+1} \otimes \sigma(N-2)} \times
$$

$$
(22)
$$

## 6 Basis transformation and trivial twists

Now we shall consider the inner automorphisms of the deformed algebras $U_{\mathcal{E}_k}(sp(N - k))$ and demonstrate that the primitive generators of $U_{\mathcal{E}_k}(sp(N - k - 1))$ constructed in the previous section by means of basis transformations can be also obtained as a result of cohomologically trivial twist.

**Proposition.** In the subalgebra $sp(N - k - 1) \subset U_{\mathcal{E}_k}(sp(N - k))$ deformed by the $k$-th step of the chain $\mathcal{F}_{\mathcal{E}_k(N-1)}$ the coboundary twist

$$
\mathcal{F}_{s_k} = \left( s_k^{-1} \otimes s_k^{-1} \right) \Delta_{\mathcal{E}_k} (s_k)
$$

produces the same primitivization of generators as that obtained by the basis transformation (21). The element $s_k$ that generates this trivial twist has the
following form:

\[ s_k = \exp \left\{ \left( \sum_{i=k+1}^{N} E_{k-i} E_{k+i} \right) f(E_{k+k}) \right\}, \quad f(E_{k+k}) = -\frac{\sigma_{k+k}}{2E_{k+k}}. \]  \hspace{1cm} (23)

\[ \star \]

The twist \( \mathcal{F}_{s_k} \) must be placed after the factor \( \mathcal{F}_{\mathcal{E}_k} \in \mathcal{F}_{\mathcal{B}_0 \prec (N-1)} \). The symbol \( \Delta_{\mathcal{E}_k} \) in \( \mathcal{F}_{s_k} \) denotes the (initially primitive) coproduct twisted by the \( k \)-th link of the chain of extended twists (in the case under consideration this is the maximal canonical extended twist for the subalgebra \( U(sp(N - k)) \)).

**Proof.** Let \( s \) be the element that induces the inner automorphism for the algebra \( U(sp(N)) \) that results in the basis transformation \((17)\). This means that the generators \( Q \) of the deformed algebra \( U_{\mathcal{E}_1}(sp(N-1)) \) are connected with the generators \( Q' \) of the primitive subalgebra \( U(sp(N-1)) \) by the adjoint transformation

\[ Q' = sQs^{-1}. \]  \hspace{1cm} (24)

It is easy to check that (up to a scalar factor) the element \( s \) have the form:

\[ s = \exp \left\{ \left( \sum_{i=2}^{N} E_{1-i} E_{1+i} \right) f(E_{1+1}) \right\}. \]

The function \( f(E_{1+1}) \) can be fixed using the relations \((24)\). For example, taking \( Q = E_{2+2} \) and \( Q' = E'_{2+2} \) one gets:

\[ sE_{2+2}s^{-1} = e^{ad\left( \sum_{i=2}^{N} E_{1-i} E_{1+i} f(E_{1+1}) \right)} E_{2+2} \]

\[ = E_{2+2} - E_{1+2}^2 e^{-\sigma_{1+1}}. \]  \hspace{1cm} (25)

Hence \( f(E_{1+1}) \) must satisfy the equation

\[ e^{2E_{1+1} f(E_{1+1})} - 1 = -E_{1+1} e^{-\sigma_{1+1}} = -\left( 1 - e^{-\sigma_{1+1}} \right). \]

It follows that

\[ f(E_{1+1}) = -\frac{\sigma_{1+1}}{2E_{1+1}}. \]

Consider the generators \( X \in U(sp(N - 1)) \). According to the properties of the type \((25)\) and \((21)\) we have the following relation:

\[ \Delta_{\mathcal{E}_1}(X') = \Delta_{\text{prim}}(X') = X' \otimes 1 + 1 \otimes X' = \]

\[ = (s \otimes s) \left( \Delta_{\text{prim}} X \right) (s \otimes s)^{-1}. \]
Thus
\[ \Delta_{\text{prim}} X = (s \otimes s)^{-1} \Delta_{E_1} (X) (s \otimes s) = (s \otimes s)^{-1} \Delta_{E_1} (s) \Delta_{E_1} (s) = \mathcal{F}_s \Delta_{E_1} (X) \mathcal{F}_s^{-1}. \]

The expression \( \mathcal{F}_s = (s \otimes s)^{-1} \Delta_{E_1} (s) \) is the twisting element of the cohomologically trivial twist. If we apply this twist deformation to the subalgebra \( U_{E_1} (sp(N-1)) \) its initial (nondashed) generators acquire the primitive coproducts. So when we start to compose the next link of the chain the necessary twists are functions of the generators of the undeformed universal enveloping subalgebra (here it is \( U(sp(N-1)) \)).

The arguments presented above are valid for any link of the chain. In particular one can avoid the basis transformation (21) in the \( k \)-th step of the chain \( \mathcal{F}_{B_0 \prec (N-1)} \) by inserting the factor \( \mathcal{F}_{s_k} = (s_k^{-1} \otimes s_k^{-1}) \Delta_{E_k} (s_k) \) after (to the left of) the factor \( \mathcal{F}_{E_k} \). Here \( s_k \) is given by the formula (23).

As a result the maximal chain of twists can be written in the form
\[ \mathcal{F}_{B_0 \prec (N-1)} = \prod_{i=1}^{N} \mathcal{F}_{s_i} \mathcal{F}_{E_i}. \]

Notice that here the extended twists \( \mathcal{F}_{s_k} \) have the canonical form (12) in terms of the initial generators of \( sp(N) \). This transcription for the maximal chain of twists vividly demonstrates the universality of the “matreshka” effect.

The coboundary twists can be used to compensate the “deformations” of the spaces \( V_{\perp}^{1} \) in the chains of extended twists for any classical Lie algebra. For the linear algebras ( \( A_N \) series) these twists correspond to the identical transformation, \( \mathcal{F}_{s_k} = 1 \otimes 1 \), and the expression (26) describes the canonical chain.

For orthogonal algebras the coboundary twists are defined by the elements
\[ s_k = e^{(E_{2k-1} E_{2k} + \sum_{i=2k+1}^{N} (E_{(2k-1)+(i)} E_{(2k)-(i)} + E_{(2k-1)-(i)} E_{(2k)+(i)})) f(E_{(2k-1)+(2k)})} \]
for \( g \in B_N \) and
\[ s_k = e^{\sum_{i=2k+1}^{N} (E_{(2k-1)+(i)} E_{(2k)-(i)} + E_{(2k-1)-(i)} E_{(2k)+(i)})) f(E_{(2k-1)+(2k)})} \]
for \( g \in D_N \). In both cases (\( g \in B_N \) or \( D_N \)) the twist \( \mathcal{F}_{s_k} \) deforms only the \( sl(2) \)-subalgebra in \( g^{1}_{N_0} \). The latter conserves its primitivity when two twists \( \mathcal{F}_{E_k} \) and \( \mathcal{F}_{s_k} \) are successively applied.
Thus the expression (26) (with the appropriate choice of $s_k$) describes the maximal chain of twists for an arbitrary simple Lie algebra of the infinite series.

7 \textit{R-matrices and forms}

The coboundary twists $F_{s_k}$ do not contribute to classical $r$-matrices. The expression (22) allows one to use the formula (5) while finding the explicit form of the quantum $R$-matrix for the deformed algebra $U_{B_0<(N-1)}(sp(N))$:

$$R_{B_0<(N-1)} = \left( F_{B_0<(N-1)} \right)_{21} \left( F_{B_0<(N-1)} \right)^{-1}. \tag{27}$$

To fix the classical limit it is necessary to introduce deformation parameters. This can be performed by scaling the generators of the twisted algebra $U_{B_0<(N-1)}(sp(N))$:

$$E_{i+k} \rightarrow \xi \eta_k E_{i+k}; \quad k \geq i.$$

Considering the expression (27) in the small neighborhood of $\xi = 0$ one can extract the classical $r$-matrix

$$r = \sum_{k=1}^{N} \eta_k (H_{kk} \wedge E_{k+i} + \kappa_k \sum_{i=k+1}^{N} E_{k-i} \wedge E_{k+i}). \tag{28}$$

Here the multipliers $\eta_k$ are arbitrary. They signify the independence of the chain links in the quasiclassical limit. The discrete parameter $\kappa_k$ (with values $\kappa_k = 0, 1$) indicates whether the extension of the $k$-th jordanian twist has been included in the chain. (The degenerate “matreshka” effect takes place also for the jordanian twist without extensions.) The number of discrete parameters $\kappa_k$ is equal to $N - 1$. For full chains of twists we have $\kappa_k = 1$ for all $k$’s.

The Frobenius forms (they are 2-coboundaries)

$$\omega_p = \sum_{k=1}^{p} \gamma_k E_{k+k}^*([\cdot, \cdot]), \quad p = 1, \ldots, N, \tag{29}$$

correspond to the solution (19) of the twist equation. These forms are (by the definition) nondegenerate on the carrier algebra of the twist. The matrix $r = \omega_p^{-1}$ is defined on the same subalgebra.
For $\eta_k = \frac{1}{\gamma_k}$ in (28) the solution of CYBE corresponding to the matrix $r = \omega_N^{-1}$ coincides with (28).

Algebra $B^+(sp(N))$ has a nontrivial second cohomology group $H^2(B^+(sp(N)))$. It is not difficult to be verify that $H^*_i \wedge H^*_k$ is a nontrivial 2-cocycle. Therefore the forms

$$\hat{\omega}_p = \omega_p + \xi_{ik} H^*_i \wedge H^*_k, \quad i, k \leq N,$$

are the nondegenerate 2-cocycles. We can associate twists to the forms $\hat{\omega}_p$. To find them notice that application of the chain $F_{B_0 < (p-1)}$ gives rise to the set of primitive commuting elements $\{\sigma^{(k-1)}_{k+k} | k = 1, \ldots, p\}$. It follows that the algebra $U_{B_0 < (p-1)}(sp(N))$ admits the twist

$$\Phi_R = \exp\{\xi_{ij} \sigma_i \otimes \sigma_j\}, \quad \sigma_i \in \{\sigma^{(k-1)}_{k+k} | k = 1, \ldots, p\}.$$

Thus the composition $\Phi_R F_{B_0 < (p-1)}$ satisfies the twist equation for $U(sp(N))$. The corresponding $R$-matrix looks like

$$R_{RRB_0 < (p-1)} = \left(\Phi_R F_{B_0 < (p-1)}\right)_{21} \left(F_{B_0 < (p-1)}\right)^{-1} (\Phi_R)^{-1}.$$

Tending the deformation parameter to zero one can obtain the classical $r$-matrix

$$\hat{r} = r + \sum_{i=1}^{p} \sum_{k=i+1}^{p} \xi_{ik} E_{i+i} \wedge E_{k+k}.$$

If we put $p = N$ in (30), the matrix

$$\hat{\omega}_N^{-1} = r + \sum_{i=1}^{N} \sum_{k=i+1}^{N} \frac{\xi_{ik}}{\gamma_i \gamma_k} E_{i+i} \wedge E_{k+k}$$

coincides (up to redifinition of parameters) with the $r$-matrix (31).

As it was demonstrated in Section 6 devoted to cohomologically trivial twists for any simple Lie algebra of four infinite series there exists the general expression for the twisting element of the full chain. To construct the appropriate quantum system one needs the $R$-matrix in a finite-dimensional representation. In particular, one can use the fundamental representations of lowest dimension — the first fundamental representations. Notice that in these representations the basis transformations that we need are degenerate, i.e. the dashed generators coincide with the initial ones. This is the direct
consequence of the generators nilpotency in the first fundamental representation.

Let

\[ d(E_\lambda) = L_\lambda, \quad d(H_\lambda) = M_\lambda \]

be the generators of the Borel subalgebra in the first fundamental representation \( d \). Then for any series of Lie algebras the \( R \)-matrix in this representation will look like:

\[
d^{\otimes 2} (\mathcal{R}) = \prod_{\lambda' \in \pi'_k} e^{L_{\lambda - \lambda'} \otimes L_{\lambda'}} e^{L_{\lambda} \otimes M_{\lambda}} \cdots \prod_{\lambda' \in \pi'_0} e^{L_{\lambda_0 - \lambda'} \otimes L_{\lambda'} e^{L_{\lambda_0} \otimes M_{\lambda_0}}} \prod_{\lambda' \in \pi'_0} e^{-M_{\lambda_0} \otimes L_{\lambda_0} e^{-L_{\lambda'} \otimes L_{\lambda_0^{-1}} \lambda''} \cdots \prod_{\lambda' \in \pi'_k} e^{-M_{\lambda_k} \otimes L_{\lambda_k} e^{-L_{\lambda'} \otimes L_{\lambda_k^{-1}} \lambda''}}. \tag{32}
\]

Here it was taken into account that in this representation we have \( d(\sigma_{\lambda_k}) = \ln(1 + L_{\lambda_k}) = L_{\lambda_k} \).

Let us introduce the deformation parameters by the substitution:

\[ L_{\lambda_0} \rightarrow \xi \eta_k L_{\lambda_0}, \quad L_{\lambda_0^{-1}} \rightarrow \xi \eta_k L_{\lambda_0^{-1}}. \]

(The other generators remain unchanged and it is easy to check that this is the algebra automorphism.) Due to the fact that the only nonzero terms in the jordanian twists and the extensions in the representation \( d^{\otimes 2} \) are linear in \( \xi \) we get the simple expression for \( d^{\otimes 2} (\mathcal{R}) \):

\[
d^{\otimes 2} (\mathcal{R}) = [(1 + \xi \eta_k \sum_{\lambda' \in \pi'_k} L_{\lambda_0^{-1} \lambda'} \otimes L_{\lambda'}) (1 + \xi \eta_k L_{\lambda_0} \otimes M_{\lambda_0})] \times \\
\vdots \\
[(1 + \xi \eta_0 \sum_{\lambda' \in \pi'_0} L_{\lambda_0^{-1} \lambda'} \otimes L_{\lambda'}) (1 + \xi \eta_0 L_{\lambda_0} \otimes M_{\lambda_0})] \times \\
[(1 - \xi \eta_k M_{\lambda_0} \otimes L_{\lambda_0}) (1 - \xi \eta_0 \sum_{\lambda' \in \pi'_0} L_{\lambda'} \otimes L_{\lambda_0^{-1} \lambda'})] \times \\
\vdots \\
[(1 - \xi \eta_0 M_{\lambda_0} \otimes L_{\lambda_0}) (1 - \xi \eta_k \sum_{\lambda' \in \pi'_k} L_{\lambda'} \otimes L_{\lambda_0^{-1} \lambda'})]. \tag{33}
\]
Lemma:

\[ d^{\otimes 2}(\mathcal{R}) = I \otimes I + \xi \sum_{k=0}^{N} \eta_k \left\{ \sum_{\lambda' \in \pi_k} (L_{\lambda'_0} \wedge L_{\lambda'_k}) + L_{\lambda'_0} \wedge M_{\lambda'_k} + M_{\lambda'_0} \wedge L_{\lambda'_k} \right\} + \]
\[ + \xi^2 \sum_{k=0}^{N} \eta_k^2 \left\{ \sum_{\lambda' \in \pi_k} (L_{\lambda'_0} \wedge L_{\lambda'_k} \otimes L_{\lambda'_0} \wedge L_{\lambda'_k}) + L_{\lambda'_0} M_{\lambda'_k} \otimes M_{\lambda'_0} L_{\lambda'_k} \right\} + \]
\[ + \xi^2 \sum_{i<j, \lambda' \in \pi_i, \lambda'' \in \pi_j} \eta_i \eta_j (L_{\lambda'_0} \wedge L_{\lambda'_i} \otimes L_{\lambda'_0} \wedge L_{\lambda'_j} + L_{\lambda'_0} - \lambda'_i L_{\lambda'_0} \wedge L_{\lambda'_j} - \lambda'_j). \]

(34)

\[ \blacktriangleleft \]

The proof can be carried out by induction through the length of the chain. Apply the Lemma to the full chains of extended twists for the algebras \(A_{N-1} \). Replacing all the generators by the appropriate \(N \times N\) basic matrices, one can see that only the terms originating from the product of jordanian twists give the nontrivial contribution of the order \(\xi^2\):

\[ d^{\otimes 2}(\mathcal{R}) = I \otimes I + \]
\[ + \xi \sum_{k=0}^{\left[ \frac{N}{2} \right]} \eta_k \left\{ \sum_{i=2}^{N-k} (\mathcal{E}_{i,N-k} \wedge \mathcal{E}_{k+1,N}) + \mathcal{E}_{k+1,N-k} \wedge (\mathcal{E}_{k+1,N-k} - \mathcal{E}_{N-k,N-k}) \right\} + \]
\[ + \xi^2 \sum_{k=0}^{\left[ \frac{N}{2} \right]} \eta_k^2 \mathcal{E}_{k+1,N-k} \otimes \mathcal{E}_{k+1,N-k}. \]

(35)

For symplectic series all the three groups of terms give the nontrivial contribution of the second order:

\[ d^{\otimes 2}(\mathcal{R}) = I \otimes I + \xi \left\{ \sum_{i=1}^{N} \eta_i (\mathcal{E}_{i,(N+i)} \wedge (\mathcal{E}_{i,i} - \mathcal{E}_{(N+i),(N+i)}) + \]
\[ + \sum_{j=2}^{N} \sum_{i<j} \eta_j (\mathcal{E}_{i,(j+N+i)} + \mathcal{E}_{j,(i+N+i)}) \wedge (\mathcal{E}_{i,j} - \mathcal{E}_{(i+N),(j+N+i)}) \right\} + \]
\[ + \xi^2 \sum_{k=1}^{N} (N - k + 1) \eta_k^2 \mathcal{E}_{k,N+k} \otimes \mathcal{E}_{k,N+k} + \]
\[ + \xi^2 \sum_{i<j} \eta_i \eta_j (\mathcal{E}_{i,N+j} \otimes \mathcal{E}_{i,N+j} + \mathcal{E}_{j,N+i} \otimes \mathcal{E}_{j,N+i}) + \]
\[ + \xi^2 \sum_{i<j} (N - j) \eta_i \eta_j (\mathcal{E}_{i,N+j} \otimes \mathcal{E}_{i,N+j} + \mathcal{E}_{j,N+i} \otimes \mathcal{E}_{j,N+i}). \]

(36)

Here, for convenience the indices of \(\eta\) are shifted by one, i.e. the substitution \(\eta_k \rightarrow \eta_{k+1}\) is performed.

20
8 Chains of twists for $sp(3)$

As an example consider the full chain of twists for the algebra $sp(3)$. This is the simplest case where the specific structure of symplectic series and it’s difference from linear and orthogonal algebras can be seen. Consider the root system

$$
\Lambda(sp(3)) = \{ \pm e_i \pm e_j, \pm 2e_i \} \quad (i, j = 1, 2, 3).
$$

For the initial root $\lambda_0 = 2e_1$ the constituent roots are $\lambda' = e_1 - e_i$ and $\lambda'' = e_1 + e_i$.

The full proper chain of twists for $U(sp(3))$ should be based on the sequence of regular injections

$$
U(sp(1)) \subset U(sp(2)) \subset U(sp(3)).
$$

The roots of $\Lambda(sp(2))$ are orthogonal to the long root $\lambda_0 \in \Lambda(sp(3)) \setminus \Lambda(sp(2))$. In other words for the appropriate indexation of basic vectors in $\Lambda$ we have $\Lambda(sp(2)) \perp 2e_1$ and $\Lambda(sp(1)) \perp 2e_1, 2e_2$.

We start the construction of the full chain of twists by performing the jordanian twist with the carrier subalgebra generated by $\{H_{11}, E_{1+1}\}$:

$$
\Phi_{J_1} = \exp\{H_{11} \otimes \sigma_{1+1}\}, \quad \sigma = \ln(1 + E_{1+1}).
$$

The sets $\{2e_1, e_{1-2}, e_{1+2}\}$ and $\{2e_1, e_{1-3}, e_{1+3}\}$ define two extensions $\mathcal{E}^I$ and $\mathcal{E}^{II}$ for $\Phi_{J_1}$. So the full jordanian extended twist has the twisting element

$$
\mathcal{F}_{\mathcal{E}_1} = e^{H_{11} \otimes \sigma_{1+1}} e^{E_{1-3} \otimes E_{1+3} e^{\frac{1}{2} \sigma_{1+1}}} e^{E_{1-2} \otimes E_{1+2} e^{\frac{1}{2} \sigma_{1+1}}} e^{H_{11} \otimes \sigma_{1+1}}. \quad (37)
$$

Being applied to $U(sp(3))$ this twist deforms the subalgebra $g_{\lambda_0}^I = g_{(2e_1)}^I = \ldots$
$U_{\varepsilon_1}(sp(2))$ which obtains the costruction defined by the relations:

\[
\begin{align*}
\Delta_{\varepsilon_1}(E_{2+2}) &= E_{2+2} \otimes 1 + 1 \otimes E_{2+2} + 2E_{1+2} \otimes E_{1+2}e^{-\frac{i}{2}\sigma_{1+1}} + E_{1+1} \otimes E_{1}^2 e^{-\sigma_{1+1}}, \\
\Delta_{\varepsilon_1}(E_{2+3}) &= E_{2+3} \otimes 1 + 1 \otimes E_{2+3} + E_{1+2} \otimes E_{1+3}e^{-\frac{i}{2}\sigma_{1+1}} + E_{1+3} \otimes E_{1+2}e^{-\frac{i}{2}\sigma_{1+1}} + E_{1+1} \otimes E_{1+2}E_{1+3}e^{-\sigma_{1+1}}, \\
\Delta_{\varepsilon_1}(E_{3+3}) &= E_{3+3} \otimes 1 + 1 \otimes E_{3+3} + 2E_{1+3} \otimes E_{1+3}e^{-\frac{i}{2}\sigma_{1+1}} + E_{1+1} \otimes E_{1+2}E_{1+3}e^{-\sigma_{1+1}}, \\
\Delta_{\varepsilon_1}(F_{2+2}) &= F_{2+2} \otimes 1 + 1 \otimes F_{2+2} + 2E_{1-2} \otimes E_{1-2}e^{-\frac{i}{2}\sigma_{1+1}} - E_{1-2} \otimes E_{1+1}e^{-\sigma_{1+1}}, \\
\Delta_{\varepsilon_1}(F_{2+3}) &= F_{2+3} \otimes 1 + 1 \otimes F_{2+3} + E_{1-2} \otimes E_{1-3}e^{-\frac{i}{2}\sigma_{1+1}} + E_{1-3} \otimes E_{1-2}e^{-\frac{i}{2}\sigma_{1+1}} - E_{1-2}E_{1-3} \otimes E_{1+1}e^{-\sigma_{1+1}}, \\
\Delta_{\varepsilon_1}(F_{3+3}) &= F_{3+3} \otimes 1 + 1 \otimes F_{3+3} + 2E_{1-3} \otimes E_{1-3}e^{-\frac{i}{2}\sigma_{1+1}} - E_{1-3} \otimes E_{1+1}e^{-\sigma_{1+1}}.
\end{align*}
\] (38)

Now we shall use formulas (17) to construct the deformed generators in the twisted algebra $U_{\varepsilon_1}(sp(2))$,

\[
\begin{align*}
E'_{i+i} &= E_{i+i} - E_{1+1}^2 e^{-\sigma_{1+1}}, \\
F'_{i+i} &= F_{i+i} - E_{1-i}^2, \\
E'_{2+3} &= E_{2+3} - E_{1+2}E_{1+3}e^{-\sigma_{1+1}}, \quad i = 2, 3 \quad (39) \\
F'_{2+3} &= F_{2+3} - E_{1-2}E_{1-3}, \\
H'_{ii} &= H_{ii}.
\end{align*}
\]

The dashed generators are primitive and form the subalgebra $sp'(2)$.

To continue the construction of the chain consider the algebra $U(sp'(2))$ on the deformed carrier space generated by $E'$, $F'$ and $H'$. In the standard form we introduce for $sp'(2)$ an independent root system $\Lambda(sp'(2))$.

For the algebra $U(sp'(2))$ there exists the jordanian twist based on the long root $\lambda_{2+2} = 2e_2 \in \Lambda(sp'(2))$,

\[
\Phi_{J_2'} = \exp\{H_{22} \otimes \sigma_{2+2}'\}, \quad \sigma_{2+2}' = \ln(1 + E_{2+2}')
\] (40)

and it’s extension $\Phi_{\varepsilon_1'}$ defined by the roots $\{e_{2-3}, e_{2+3}\}$. In other words the algebra $U(sp'(2))$ admits the extended jordanian twist with the element

\[
F_{\varepsilon_1'} = e^{E_{2-3}' \otimes E_{2+3}'e^{-\frac{i}{2}\sigma_{2+2}'}} e^{H_{22} \otimes \sigma_{2+2}'},
\]

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Being applied to \( U(sp'(2)) \) it nontrivially transforms the costructure of the subalgebra \( sp'(1) \subset sp'(2) \). Let us again pass to the deformed space \( sp''(1) \subset U_{E_2 E_1}(sp(3)) \) with primitive generators (compare with the corresponding expressions in [10]):

\[
\begin{align*}
E''_3+3 &= E_3+3 - (E_{2+3})^2 e^{-\sigma_{2+2}}, \\
F''_3+3 &= F_3+3 - (E'_{2-3})^2, \\
H''_{33} &= H_{33}.
\end{align*}
\]

The next link, the last one for the full chain in case of \( U(sp(3)) \), is degenerate. It does not contain extensions. The subalgebra \( U(sp''((1)) \) on the twice deformed space is generated by the elements \( E'', F'' \) and \( H'' \). It admits the jordanian twist

\[
\Phi_J''(3) = \exp\{H_{33} \otimes \sigma''_{3+3}\}, \quad \sigma''_{3+3} = \ln(1 + E''_{3+3}).
\]

Thus, the jordanian twist

\[
\mathcal{F}_{E''_3} = \Phi_J''(3).
\]

is the last factor in our chain.

The full chain of twists for \( U(sp(3)) \) will have the following form:

\[
\mathcal{F}_{B_{0<2}} = e^{H_{33} \otimes \sigma''_{3+3}} \times e^{E'_{2-3} \otimes E'_{2+3}} e^{-E'_{2+2}} \times e^{H_{22} \otimes \sigma'_{2+2}} \times e^{E_{1-3} \otimes E_{1+3}} e^{-E_{1+1} + E_{1-2} \otimes E_{1+3} e^{-E_{1+1}}} e^{H_{11} \otimes \sigma_{1+1}}.
\]

(41)

Here the generators of \( sp'(2) \) are defined by the formulas (39) and the generator \( E''_{3+3} \) has the form

\[
E''_{3+3} = E_{3+3} - E^2_{1+3} e^{-\sigma_{1+1}} - (E_{2+3} - E_{1+2} E_{1+3} e^{-\sigma_{1+1}})^2 e^{-\sigma_{2+2}}.
\]

Notice, that the initial sequence of regular injections has also been deformed:

\[
U(sp''(1)) \subset U(sp'(2)) \subset U(sp(3)).
\]

Nevertheless this new sequence still consists of symplectic regular subalgebras. Thus the chain (41) is proper for the algebra \( U(sp(3)) \) (the improper chains of twists were described in [3]). Algebra \( B^+(sp(3)) \) is the carrier for the twist (41). Consequently this chain of extended twists is full.
The quantum $R$-matrix, corresponding to the twist (41), and its classical counterpart are as follows:

\[
R = (F_{B_0 < 2})^{-1},
\]

\[
r = \sum_{k=1}^{3} \eta_k (H_{k,k} \wedge E_{k+k} + \kappa_k \sum_{i=k+1}^{3} E_{k-i} \wedge E_{k+i}).
\]

The approach based on the cohomologically trivial twists, developed in the section 6, allows one to rewrite the expression (41) in the form

\[
F_{B_0 < 2} = \prod_{i=1}^{3} F_{s_i} F_{E_i},
\]

here

\[
s_1 = \exp\{(E_{1-2}E_{1+2} + E_{1-3}E_{1+3}) f(E_{1+1})\},
\]

\[
s_2 = \exp\{E_{2-3}E_{2+3} f(E_{2+2})\},
\]

\[
s_3 = 1.
\]

Finally let us notice that the recursive procedure proposed in Section 5 allows one to construct full chains of twists for any classical Lie algebra.

9 Conclusions

Explicitly constructed maximal proper chain of twists for $U(sp(N))$ terminates the process of composition of full proper chains for classical Lie algebras. Thus it was demonstrated that the “matreshka” effect is universal. The following proposition can be formulated: the Hopf algebra $U_{\gamma}(g)$ with simple $g$, deformed by the maximal (canonical) extended twist $F_{\gamma}$, contains a primitive subalgebra equivalent to the universal enveloping algebra $U(g^\perp_{\lambda_0})$ where the root system of $g^\perp_{\lambda_0} \subset g$ is orthogonal to the initial root $\lambda_0$. For classical algebras this statement was proved in Section 6. It would be interesting to check if its validity for the five exceptional Lie algebras.

There exists a variety of applications where the explicit expressions for twisting elements received above are essential. The explicit form of the universal $R$-matrix allows one to get the $R$-matrix in an arbitrary representation. Thus the first fundamental representation $d$ was exploited in Section 7 and the vector representation $d_v$ in \[\text{III}\].
The twist deformation of a coalgebra induces the modification of the Clebsh-Gordan coefficients in the decomposition of the tensor product of irreducible representation $d_i \otimes d_j$. New Clebsh-Gordan coefficients are defined by the action of the matrix operator $F = d_i \otimes d_j(F)$ of the twisting element $F$ on the initial ones [17].

Due to the embedding of simple Lie algebras in the appropriate Yangians $U(g) \subset Y(g)$, there exists a possibility to deform the Yangians by the chains of twists for the algebras $U(g)$ [18].

If we consider the Yangian corresponding to the symplectic algebra $sp(N)$ the $R$-matrix in the defining representation $d \otimes d \subset \text{Mat}(2N, C) \otimes \text{Mat}(2N, C)$ looks as follows [16]

$$uI + \mathcal{P} - u(u + N + 1)^{-1} \tilde{K}.$$

Here $u$ is the spectral parameter, $\tilde{K}$ is the rank one projector, that in the basis $(C^2 \otimes C^N) \otimes (C^2 \otimes C^N)$ can be written as $P \cdot \mathcal{K}$ with $P$ being the antisymmetrizer in $C^2 \otimes C^2$ and the operator $\mathcal{K}$ can be obtained from the permutation $\mathcal{P}$ in $C^N \otimes C^N$ by transposing the first tensor multiplier. The twisted $R$-matrix will have the form

$$R_F(u) = uF_{21}F^{-1} + \mathcal{P} - u(u + N + 1)^{-1}F_{21}\tilde{K}F^{-1}.$$  

This leads to the deformation of the Hamiltonian density in the integrable $sp(N)$-spin system (see [17]),

$$h_F = F\mathcal{P}F^{-1} - (N + 1)^{-1}F\tilde{K}F^{-1} = FhF^{-1}.$$

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