A quantum criticality perspective on the charging of narrow quantum-dot levels

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Understanding the charging of exceptionally narrow levels in quantum dots in the presence of interactions remains a challenge within mesoscopic physics. We address this fundamental question in the generic model of a narrow level capacitively coupled to a broad one. Using bosonization we show that for arbitrary capacitive coupling charging can be described by an analogy to the magnetization in the anisotropic Kondo model, featuring a low-energy crossover scale that depends in a power-law fashion on the tunneling amplitude to the level. Explicit analytical expressions for the exponent are derived and confirmed by detailed numerical and functional renormalization-group calculations.

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Introduction. Confined nanostructures offer a unique arena for thoroughly interrogating the interplay between interference and interactions while holding the promise of future applications. Particularly appealing are semiconductor quantum dots (QDs), for which the manipulation of spin1,2 and charge3 has recently been demonstrated. The precise and rapid control of switchable gate voltages renders these devices attractive candidates for a solid-state qubit4,5. The accurate manipulation of QD setups requires, however, detailed understanding of how charging proceeds. Indeed, interactions can substantially modify the orthodox picture of charging, whether by renormalizing the tunneling rates or by introducing nonmonotonicities into the population of individual levels6,7,8. Even the simplest two-level device, where each level harbors only a single spinless electron, displays remarkably rich behavior9.

We consider a situation in which the width of a narrow level is much smaller than the width of the other broad one. A disparity in widths is generic for QDs in the intermediate regime between integrable and chaotic9. It was reported in several artificial structures10,11, and has been exploited for charge sensing12,13. As the energy $\epsilon_-$ of the narrow level is raised, its occupation varies from 1 to 0 over a characteristic width $\Omega$. This energy scale, or the corresponding charge-fluctuation time scale $\hbar/\Omega$, manifests itself in charge sensing and transmission-phase measurements14. The effect of inter-level repulsion $U$ on $\Omega$ has been explored only in the large-$U$ limit, revealing novel correlation effects14,16,17,18. The physical mechanism determining $\Omega$ for moderate $U$ remains unclear9.

In this Letter we solve the fundamental question of the charging of a narrow QD level from a quantum-critical perspective. Due to the capacitative coupling $U$, every switching of the narrow level initiates restructuring of the broad level and its attached Fermi sea, in direct analogy with the x-ray edge singularity. For nonzero tunneling to the narrow level, coherent superpositions of these charge re-arrangements lead to Kondo physics19 with the charge state (0 or 1) acting as a pseudo-spin, and the energy of the narrow level acting as a Zeeman field. Using Abelian bosonization we show that $\Omega$, being the Kondo scale in the pseudo-spin language, depends on the charge state (0 or 1) acting as a pseudo-spin, and the direct hopping amplitude $b$. This form follows from a generic model of spinless electrons with two dot levels interacting via hopping amplitudes $V_+ \geq V_- \geq 0$ and a tilted magnetic field, whose components are $\epsilon_+ - \epsilon_-$. Using Abelian bosonization we show that $\Omega$, being the Kondo scale in the pseudo-spin language, depends on the charge state (0 or 1) acting as a pseudo-spin, and the direct hopping amplitude $b$.

Model and objective. Our specific model for charging is depicted schematically in the inset of Fig. 1, and is defined by the Hamiltonian ($\sigma$ is the pseudo-spin index)

$$\mathcal{H} = \sum_{\sigma=\pm} \sum_k \epsilon_k e_k^\dagger e_k + V_\sigma \sum_k (e_k^\dagger c_\sigma^+ + c_\sigma d_\sigma^+ + d_\sigma e_k)$$

(1)

Here, $d_\pm^\dagger (c_{\pm}^\dagger)$ creates an electron on the dot (in the leads), and $\Delta n_\pm$ equals $d_+^\dagger d_- - 1/2$. Equation (1) is a generalized Anderson impurity model with pseudo-spin-dependent tunneling amplitudes $V_+ \geq V_- \geq 0$ and a tilted magnetic field, whose components are $\epsilon_+ - \epsilon_-$. This form follows from a generic model of spinless electrons with two dot levels and two leads by simultaneous unitary transformations in the dot and the lead space16,17,18. The Hamiltonian (1) has recently gained considerable attention in connection with phase lapses, population inversion, and many-body resonances4. The energies $\epsilon_{\pm}$ are tuned using gate voltages. Depending on the specific realization, their tuning may inflict a similar change in $b$. We focus
on realizations where $\epsilon_+\pm$ can be tuned independently of $b$.

The bare energy scales that characterize tunneling in Eq. (1) are the level broadenings $\Gamma_{\pm}=\pi\rho V_{\pm}^2$ and the direct hopping amplitude $b$. The density of states (DOS) $\rho$ is taken to be equal for both bands without loss of generality. Our interest is in the charging properties of the narrow level $d_-$ as a function of $\epsilon_-$ in the limit where $b$ and $V_-$ are both small: $\Gamma_{-}, b \ll \Gamma_{+}$. Strictly at $b=V_-=0$ ergodicity of the microcanonical ensemble is broken as a new conserved quantity arises: $\hat{n}_\pm=d_\pm^\dagger d_\pm$ is either equal to 0 or 1. Comparing the total energies of the competing ground states with $\langle \hat{n}_- \rangle = 0$ and $\langle \hat{n}_- \rangle = 1$ as a function of $\epsilon_-$ one finds a critical value $\epsilon_- = \epsilon^*(\epsilon_+, U, V_+)$ at which the two become degenerate. For $\epsilon_+ = 0$, $\epsilon^*$ is pinned to zero by particle-hole symmetry if symmetric bands are prone to strong valence fluctuations (for $\epsilon_+ = 0$).

The tunneling term in Eq. (3), proportional to $A$, depends on the case of interest; one takes $A = b/2$ and the upper sign ($A = \sqrt{\Gamma_{+}\Gamma_{-}}$, lower sign) for $\Gamma_{-} = 0 (b = 0)$.
The value of $\delta_U = \arctan(U/2\Gamma_\perp)$ is fixed by matching the $b = \Gamma_\perp = 0$ scattering phase shifts of the ‘+’ band in the fermionic and the bosonic representations, for each sector with fixed integer occupancy of the ‘+’ level.

Next, we manipulate Eq. (3) by (i) applying the canonical transformation $\mathcal{H}' = U\mathcal{H}U$ with

$$\hat{U} = \exp \left[-i(2\delta_U/\pi)\Phi_+(0)\Delta \hat{n}_-\right],$$

(4)

and (ii) converting to the “spin” and “charge” fields $\Phi_s(x)$ and $\Phi_c(x)$. The latter are defined as $\Phi_s(x) = \Phi_+(x)$ and $\Phi_c(x) = -\Phi_-(x)$ for $\Gamma_\perp = 0$, and

$$\Phi_{s,c}(x) = \left[1 + (2\delta_U/\pi)^2\right]^{-1/2}\left[\Phi_+(x) \mp 2\delta_U/\pi \Phi_{\pm}(x)\right]$$

(5)

for $b = 0$ (the upper signs correspond to $\Phi_s$). In this manner, the Hamiltonian acquires the unified form

$$\mathcal{H}' = \sum_{\mu = s,c} \frac{b_{\nu \mu}}{4\pi} \int_{-\infty}^{\infty} [\nabla \Phi_\mu(x)]^2 dx + \epsilon_d d_+^\dagger d_- + A \left\{ e^{i\gamma \Phi_\mu(0)} d_+ d_- e^{-i\gamma \Phi_\mu(0)} \right\},$$

(6)

where $\gamma = \sqrt{1 + (2\delta_U/\pi)^2}$ for $b = 0$ and $\gamma = 1 - 2\delta_U/\pi$ for $\Gamma_\perp = 0$.

The very same Hamiltonian with $0 < \gamma < \sqrt{\tau}$ also describes the anisotropic Kondo model with $0 < J_z$, where in standard notation $A = J_\perp/\sqrt{\tau}$ and $\gamma = \sqrt{2[1-(2/\pi)\arctan(\pi\rho J_z/4)]}$ represent the transverse and longitudinal spin-exchange couplings respectively, and $\epsilon_d = \mu_B g H$ corresponds to a local magnetic field. This representation of the Kondo model is obtained by [24] (i) bosonizing the Kondo Hamiltonian with two bosonic fields $\Phi_1(x)$ and $\Phi_\perp(x)$, (ii) converting to the spin and charge fields $\Phi_{s,c}(x) = \Phi_1(x) \mp \Phi_\perp(x)/\sqrt{2}$, (iii) employing $\mathcal{H}' = T^\dagger \mathcal{H} T$ with $T = \exp[-i\sqrt{2}(\delta_U/\pi)\Phi_\perp(0)\tau_z]$, $\tau_z$ being the $z$ spin component and $\delta_\perp = \arctan(\pi\rho J_z/4)$, and (iv) representing the spin $\vec{\tau}$ in terms of the fermion $d_-^\dagger$. This establishes a mapping between our problem with either $b = 0$ or $\Gamma_\perp = 0$ and the anisotropic Kondo model. In particular, charging of the $d_-^\dagger$ level is mapped onto the magnetization of the Kondo impurity, relating the width $\Omega$ to the Kondo temperature $T_K$.

We can now exploit known results for the Kondo problem. Specifically, RG equations perturbative in $J_\perp$ but nonperturbative in $J_z$ [19] give $T_K \sim D_+(A/D_\perp)^{2/(2-\gamma^2)}$, which yields for our problem

$$\frac{\Omega}{\Gamma_\perp} \sim \left\{ \frac{1}{(\Gamma_\perp/\Gamma_\perp)^\alpha} \right\}$$

(7)

if $b = 0$,

$$\left\{ (b/\Gamma_\perp)^{2\beta} \right\}$$

if $\Gamma_\perp = 0$,

$$\alpha = \frac{1}{1 - (2\delta_U/\pi)^2}; \quad \beta = \frac{1}{2 - [1 - (2\delta_U/\pi)]^2}.$$  

Thus, $\Omega$ is a power law of the relevant tunneling amplitude with an exponent that varies smoothly with $U$. In going from $U = 0$ to $U \gg \Gamma_\perp$, $\alpha$ grows monotonically from 1 to $\pi U/(8\Gamma_\perp)$ while $\beta$ decreases from 1 to 1/2. The asymptote $\alpha = \pi U/(8\Gamma_\perp)$ coincides with the result of Ref. 17 [Eq. (29) with $c_0 = -U/2$], obtained using very different techniques. For $U = 0$, the noninteracting integer exponents are reproduced. Hence Eqs. (8) are precise both at small and large $U$. As shown next, these expressions remain highly accurate also at intermediate $U$, suggesting that they might actually be exact.

**Numerical analysis.** To test Eqs. (8), we computed $\alpha$ and $\beta$ numerically using the NRG [20] and FRG [21], each approach having its own distinct advantage. The NRG is extremely accurate in all parameter regimes of interest, while the FRG is approximative in $U$ but offers a far more flexible framework for scanning parameters. The width $\Omega = 1/(\pi \chi_c)$ was obtained with either method from the inverse charge susceptibility $\chi_c = d(\hat{n}_-)/d\epsilon$, evaluated at $\epsilon_- = 0$ and $T \to 0$. The exponents $\alpha$ and $\beta$ were extracted from log-log fits (see the inset to Fig. 2). Our results, summarized in Figs. [1] and [2], reveal excellent agreement between Eqs. (8) and the NRG, to within numerical precision. The agreement extends to all interaction strengths from small to large $U$, confirming the accuracy of Eqs. (8) at all $U$. The FRG results for $\alpha$ coincide with those of the NRG up to $U/\Gamma_\perp \approx 2$, above which they acquire a linear slope that is reduced by a factor of $8/\pi^2$ as compared to the NRG [17]. The exponent $\beta$ is accurately reproduced up to larger values of $U/\Gamma_\perp$. In particular, the FRG data for $\alpha$ and $\beta$ exactly reproduce the leading behaviors of Eqs. (8) at small $U$.

**Combination of $\Gamma_\perp$ and $b$.** The case where both $\Gamma_\perp$ and $b$ are nonzero lies beyond the scope of our bosonization treatment, but allows the formulation of a scaling law. To this end, consider the dimensionless quantity $\tilde{\Omega} = \Omega/D_\perp$, which depends on the three dimensionless parameters in Eq. (2): $\tilde{\Omega} = f(\tilde{V}, b, \delta_\perp)$, with $\tilde{V} = \sqrt{\Gamma_\perp/\Gamma_\perp} + b = b/D_\perp$. Given the exact RG trajectories, $\tilde{\Omega}$ evolves according to $\tilde{\Omega}' = \tilde{\Omega}/\xi = f(\tilde{V}', \tilde{b}', \delta_\perp', \xi')$ upon reducing the bandwidth from $D_\perp$ to $\xi D_\perp (0 < \xi < 1)$.
Here, primes denote renormalized parameters and \{\lambda_i'\} are the new couplings generated. At sufficiently weak tunneling the RG equations can be linearized with respect to the relevant couplings \(V'\) and \(\delta'\), resulting in their power-law growth with the exponents determined previously: \(V' = V\xi^{-1/2b}\) and \(\delta' = \delta k^{-1/2b}\). Note that \(\delta_U\) is left unchanged in this approximation, nor are there any new couplings generated. Consequently, \(f(V', \delta_U) = \xi f(V\xi^{-1/2b}, \delta k^{-1/2b}, \delta_U) = \Omega\) is a homogeneous function of \(\xi\), taking the general form \(f(V, \delta_U) = V^{2\alpha} \Omega(b^{2\beta}/\Gamma^{\alpha}, \delta_U)\). Finally, defining the coefficients \(A\) and \(B\) from \(\Omega_{b=0} = A\Gamma^{\alpha}\) and \(\Omega_{\Gamma=-} = B\Gamma^{2\beta}\), we arrive at the scaling form \[\Omega = A\Gamma^{\alpha} \mathcal{F}(B\Gamma^{2\beta}/A\Gamma^{\alpha}; \delta_U),\] (9)

with \(\mathcal{F}(0; \delta_U) = 1\) and \(\mathcal{F}(x \gg 1; \delta_U) = x\). In Fig. 3 we confirm the scaling form of Eq. (9) using FRG data.

**Extension to arbitrary \(\epsilon_+\).** Our discussion has focused thus far on \(\epsilon_+ = 0\). A nonzero \(\epsilon_+\) introduces the potential-scattering term \(H_{ps} = \epsilon_+ a : \psi_+\dagger(0)\psi_+(0)\) into Eq. (2). Consequently, \(\delta_U\) in Eq. (3) is replaced with two distinct parameters \(\delta_{\pm} = \text{arctan}(U \pm 2\epsilon_+/2\Gamma_+),\) assigned to \(\delta_{\pm} = \pm 1/2\), respectively. An identical derivation, only with \(2\delta_U\delta_\pm \rightarrow (\delta_+ - \delta_-)\delta_\pm + (\delta_+ + \delta_-)/2\) in Eq. (4), leads then to the same Hamiltonian (3) with two modifications: (i) \(\epsilon_+\), and thus \(\epsilon^*\), acquires a shift proportional to \(\delta_{+}^{2} - \delta_{-}^{2}\), and (ii) \(\delta_U\) is replaced with \((\delta_+ + \delta_-)/2\) in the expressions for \(\gamma\). The end results for \(\alpha\) and \(\beta\) are just Eqs. (5) with \(\delta_U \rightarrow (\delta_+ + \delta_-)/2\), which properly reduce to the noninteracting limit \(\alpha = \beta = 1\) when \(|\epsilon_+| \gg U, \Gamma_+\). The effect of nonzero \(\epsilon_+\) is negligible for \(|\epsilon_+| \ll \max\{U, \Gamma_+\}\). It becomes significant only as \(|\epsilon_+|\) approaches \(\max\{U, \Gamma_+\}\).

**Summary.** We have resolved the fundamental question of the charging of a narrow QD level capacitively coupled to a broad one. The zero-tunneling fixed point is critical in the sense of being unstable. Finite tunneling is a relevant perturbation, driving the system to a strong-coupling Fermi-liquid fixed point. The inverse charge-fluctuation time \(\Omega\) varies as a power of the bare tunneling amplitude, with a nonuniversal exponent that depends on the nature of tunneling, the strength of the capacitive coupling, and the width and position of the broad level. We have proven this scenario by devising a two-stage mapping of the original model onto the anisotropic Kondo problem, yielding accurate analytic expressions for the exponents. Our analytic predictions were confirmed by extensive numerical calculations within the frameworks of the NRG and FRG.

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[25] Note that $A = D_+^{1-\alpha} \bar{A}(\delta U)$ and $B = D_+^{1-2\beta} \bar{B}(\delta U)$ depend on both $\Gamma_+$ and $\delta U$. 