Determinant of the Laplacian on Tori of Constant Positive Curvature with one Conical Point

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Abstract. We find an explicit expression for the zeta-regularized determinant of (the Friedrichs extensions of) the Laplacians on a compact Riemann surface of genus one with conformal metric of curvature 1 having a single conical singularity of angle 4π.

1 Introduction

Let X be a compact Riemann surface of genus one and let P ∈ X. According to [1, Cor. 3.5.1], there exists at most one conformal metric on X of constant curvature 1 with a (single) conical point of angle 4π at P. The following simple construction shows that such a metric, m(X, P), in fact always exists (and, due to [1], is unique).

Consider the spherical triangle T = {(x₁, x₂, x₃) ∈ S³ ⊂ R³ : x₁ ≥ 0, x₂ ≥ 0, x₃ ≥ 0} with all three angles equal to π/2. Gluing two copies of T along their boundaries, we get the Riemann sphere CP¹ with metric m of curvature 1 and three conical points P₁, P₂, P₃ of conical angle π. Consider the two-fold covering

μ : X(Q) → CP¹

ramified over P₁, P₂, P₃ and some point Q ∈ CP¹ \ {P₁, P₂, P₃}. Lifting the metric m from CP¹ to the compact Riemann surface X(Q) of genus one via μ, one gets the metric μ∗m on X(Q) that has curvature 1 and the unique conical point of angle 4π at the preimage μ⁻¹(Q) of Q. Clearly, any compact surface of genus one is (biholomorphically equivalent to) X(Q) for some Q ∈ CP¹ \ {P₁, P₂, P₃}. Now let X be an arbitrary compact Riemann surface of genus one and let P be any point of X. Take Q ∈ CP¹ such that X = X(Q) and consider the automorphism α : X → X (the translation) of X sending P to μ⁻¹(Q). Then

m(X, P) = α∗(μ∗(m)) = (μ ◦ α)* (m).

Introduce the scalar (Friedrichs) self-adjoint Laplacian Δ(X, P) := Δm(X, P) on X corresponding to the metric m(X, P). For any P and Q from X the operators Δ(X, P) and Δ(X, Q) are isospectral and, therefore, the ζ-regularized (modified, i.e., with zero modes excluded) determinant det Δ(X, P) is independent of P ∈ X and, therefore, is
a function on moduli space $M_1$ of Riemann surfaces of genus one. The main result of
the present work is the following explicit formula for this function:

$$\det \Delta(X, P) = C_1|\Im\eta(\sigma)|^4 F(t) = C_2 \det \Delta^{(0)}(X) F(t),$$

where $\sigma$ is the $b$-period of the Riemann surface $X$, $C_1$ and $C_2$ are absolute constants,
$\eta$ is the Dedekind eta-function, $\Delta^{(0)}$ is the Laplacian on $X$ corresponding to the flat
conformal metric of unit volume, the surface $X$ is represented as the two-fold covering of
the Riemann sphere $\mathbb{C}P^1$ ramified over the points $0, 1, \infty$ and $t \in \mathbb{C} \setminus \{0, 1\}$, and

$$F(t) = \frac{|t|^{\frac{1}{4}}|t - 1|^{\frac{1}{4}}}{(|\sqrt{t} - 1| + |\sqrt{t} + 1|)^{\frac{1}{2}}}.$$

As is well known, the moduli space $M_1$ coincides with the quotient space

$$\left( \mathbb{C} \setminus \{0, 1\} \right) / G,$$

where $G$ is a finite group of order 6, generated by transformations $t \to \frac{1}{t}$ and $t \to 1 - t$.
A direct check shows that $F(t) = F(\frac{1}{t})$ and $F(t) = F(1 - t)$, and, therefore, the right
hand side of (1.1) is in fact a function on $M_1$.

**Remark 1.1** Using the classical relation (see, e.g., [2, f-la (3.35)])

$$t = -\left( \frac{\Theta_{11}^{(1)}(0 | \sigma)}{\Theta_{11}^{(1)}(0 | \sigma)} \right)^4,$$

one can rewrite the right-hand side as a function of $\sigma$ only.

The well known Ray–Singer relation $\det \Delta^{(0)} = C|\Im\eta(\sigma)|^4$ (see [10–12]) used in
(1.1) implies that (1.1) can be considered as a version of Polyakov’s formula (relating
determinants of the Laplacians corresponding to two smooth metrics in the same
conformal class) for the case of two conformally equivalent metrics on a torus: one
of them is smooth and flat, another is of curvature one and has exactly one singular
point.

2 **Metrics on the Base and on the Covering**

Here we find an explicit expression for the metric $m$ on the Riemann sphere $\mathbb{C}P^1$ of
curvature 1 and with three conical singularities at $P_1 = 0$, $P_2 = 1$, and $P_3 = \infty$.

The stereographic projection (from the south pole) maps the spherical triangle $T$
onto quarter of the unit disk $\{ z \in \mathbb{C} : |z| \leq 1, 0 \leq \text{Arg} z \leq \pi/2 \}$. The conformal map

$$z \mapsto w = \left( \frac{1 + z^2}{1 - z^2} \right)^2$$

sends this quarter of the disk to the upper half-plane $H$; the corner points $i, 0, 1$ go to
the points $0, 1,$ and $\infty$ on the real line. The push forward of the standard round metric

$$\frac{4|dz|^2}{(1 + |z|^2)^2}$$
on the sphere by this map gives rise to the metric
\[(2.2)\]
\[m = \frac{|dw|^2}{|w| |w - 1|(|\sqrt{w} + 1| + |\sqrt{w} - 1|)^2}\]
on $H$; clearly, the latter metric can be extended (via the same formula) to $\mathbb{C}P^1$. The resulting curvature one metric on $\mathbb{C}P^1$ (also denoted by $m$) has three conical singularities of angle $\pi$: at $w = 0$, $w = 1$, and $w = \infty$.

Consider a two-fold covering of the Riemann sphere by a compact Riemann surface $X(t)$ of genus 1:
\[(2.3)\]
\[\mu : X(t) \to \mathbb{C}P^1\]
ramified over four points: 0, 1, $\infty$, and $t \in \mathbb{C} \setminus \{0, 1\}$. Clearly, the pull back metric $\mu^* m$ on $X(t)$ is a curvature one metric with exactly one conical singularity. The singularity is a conical point of angle $4\pi$ located at the point $\mu^{-1}(t)$.

3 Variation of Spectral Zeta-function with Respect to $t$

The analysis from [5] in particular implies that one can introduce the standard Ray–Singer $\zeta$-regularized determinant
\[(3.1)\]
\[\det \Delta^{\nu} m := \exp\{-\zeta'_{\Delta^\nu m}(0)\}\]
of the (Friedrichs) self-adjoint Laplacian $\Delta^{\nu} m$ in $L^2(X(t), \mu^* m)$, where $\zeta'_{\Delta^\nu m}$ is the spectral zeta-function. In this section we establish a formula for the variation of $\zeta'_{\Delta^\nu m}(0)$ with respect to the parameter $t$ (the fourth ramification point of the covering (2.3)). The derivation of this formula coincides almost verbatim with the proof of [5, Proposition 6.1]; therefore, we give only few details.

For the sake of brevity we identify the point $t$ of the base $\mathbb{C}P^1$ with its (unique) preimage $\mu^{-1}(t)$ on $X(t)$.

Let $Y(\lambda, \cdot)$ be the (unique) special solution of the Helmholtz equation (here $\lambda$ is the spectral parameter) $(\Delta^m - \lambda) Y = 0$ on $X \setminus \{t\}$ with asymptotic $Y(\lambda)(x) = \frac{1}{x} + O(x)$ as $x \to 0$, where $x(\nu) = \sqrt{-\nu} - t$ is the distinguished holomorphic local parameter in a vicinity of the ramification point $t \in X(t)$ of the covering (2.3). Introduce the complex-valued function $\lambda \mapsto b(\lambda)$ as the coefficient near $x$ in the asymptotic expansion
\[Y(x, \overline{x}; \lambda) = \frac{1}{x} + c(\lambda) + a(\lambda) \overline{x} + b(\lambda) x + O(|x|^{2-\epsilon}) \quad \text{as } x \to 0.\]

The following variational formula is proved in [5, Proposition 6.1]:
\[(3.2)\]
\[\partial_t \left( - \zeta'_{\Delta^\nu m}(0) \right) = \frac{1}{2} \left( b(0) - b(-\infty) \right).\]
The value $b(0)$ is found in [5, Lemma 4.2]: one has the relation
\[(3.3)\]
\[b(0) = -\frac{1}{2} S_{Sch}(x) \bigg|_{x=0},\]
where $S_{Sch}$ is the Schiffer projective connection on the Riemann surface $X(t)$.

Since $\lambda = -\infty$ is a local regime, in order to find $b(-\infty)$, the solution $Y$ can be replaced by a local solution with the same asymptotic as $x \to 0$. A local solution $\tilde{Y}$
with asymptotic
\[ \tilde{Y}(u, \overline{w}; \lambda) = \frac{1}{u} + c(\lambda) + \bar{a}(\lambda)\overline{w} + \bar{b}(\lambda)u + O(|u|^{-2+\varepsilon}) \quad \text{as } u \to 0 \]
in the local parameter \( u^2 = z - s \) was constructed in [5, Lemma 4.1] by separation of variables; here \( z \) and \( w = \mu(P) \) (resp. \( s \) and \( t \)) are related by (2.1) (resp. by (2.1)) with \( z = s \) and \( w = t \) and \( \tilde{b}(-\infty) = \frac{1}{2} \). One can easily find the coefficients \( A(t) \) and \( B(t) \) of the Taylor series \( u = A(t)x + B(t)x^3 + O(x^5) \). As a local solution replacing \( Y \), we can take \( A(t)\tilde{Y} \). This immediately implies that \( b(-\infty) = A^2(t)\tilde{b}(-\infty) - B(t)/A(t) \). A straightforward calculation verifies that
\[ b(-\infty) = \partial_t \log \left( |t| |t - 1|(|\sqrt{t} + 1| + |\sqrt{t} - 1|) \right)^{1/4}. \]

Observe that the right-hand side in (3.4) is actually the value of \( \partial_t \log \rho(w, \overline{w})^{-1/4} \) at \( w = t \), where \( \rho(w, \overline{w}) \) is the conformal factor of the metric (2.2); this is also a direct consequence of [4, Lemma 4].

Substituting (3.3) and (3.4) into (3.2), we obtain the desired formula for the variation of \( \zeta'_{\Delta^m}(0) \) with respect to the parameter \( t \).

## 4 Explicit Formula for the Determinant

Equations (3.2), (3.3), and (3.4) imply that the determinant (3.1) can be represented as a product
\[ (1.1) \quad \det \Delta^{n^m} = C(I\sigma|\tau(t)|)^2\left| \frac{1}{|t| |t - 1|(|\sqrt{t} + 1| + |\sqrt{t} - 1|) \right|^{1/8}, \]

where \( \tau(t) \) is the value of the Bergman tau-function (see [7–9]) on the Hurwitz space \( H_{1,2}(2) \) of two-fold genus one coverings of the Riemann sphere, having \( \infty \) as a ramification point at the covering, ramified over \( 1, 0, \infty, \) and \( t \). More specifically, \( \tau \) is a solution of the equation
\[ \partial_t \log \tau = -\frac{1}{4} S_B(x) |x = 0, \]
where \( S_B \) is the Bergman projective connection on the covering Riemann surface \( X(t) \) of genus one and \( x \) is the distinguished holomorphic parameter in a vicinity of the ramification point \( t \) of \( X(t) \). We remind the reader that the Bergman and the Schiffer projective connections are related via the equation
\[ S_{Sch}(x) = S_B(x) - 4\pi (I\sigma)^{-1}v^2(x) \]
where \( v \) is the normalized holomorphic differential on \( X(t) \) and that the Rauch variational formula (see, e.g., [7]) implies the relation
\[ \partial_t \log \sigma = \frac{2}{7} (I\sigma)^{-1}v^2(x)|x = 0. \]

The needed explicit expression for \( \tau \) can be found e.g., in [9, f-la (18)] (it is a very special case of the explicit formula for the Bergman tau-function on general coverings of arbitrary genus and degree found in [8] as well as of a much earlier formula of Kitaev and Korotkin for hyperelliptic coverings [6]). Namely, [9, f-la (18)] implies that
\[ (4.2) \quad \tau = \eta^2(\sigma)^3 \left( \frac{v(\infty)^3}{v(P_1)v(P_2)v(Q)} \right)^{1/7} \]

where $P_1$ and $P_2$ are the points of the $X(t)$ lying over 0 and 1, $Q$ is a point of $X(t)$ lying over $t$ and $\infty$ denotes the point of the covering curve $X(t)$ lying over the point at infinity of the base $\mathbb{C}P^1$; $v$ is an arbitrary nonzero holomorphic differential on $X(t)$; and, say, $v(P_1)$ is the value of this differential in the distinguished holomorphic parameter at $P_1$. (One has to take into account that $\tau = t^{-2}$, where $t^{-1}$ is from [9].) Taking

$$v = \frac{dw}{\sqrt{w(w-1)(w-t)}},$$

and using the following expressions for the distinguished local parameters at $P_1$, $P_2$, $Q$, and $\infty$

$$x = \sqrt{w}; \quad x = \sqrt{w-1}; \quad x = \sqrt{w-t}; \quad x = \frac{1}{\sqrt{w}}$$

one arrives at the relations (where $\sim$ means = up to insignificant constants like $\pm 2$, etc.)

$$v(P_1) \sim \frac{1}{\sqrt{t}}; \quad v(P_2) \sim \frac{1}{\sqrt{t-1}}; \quad v(Q) \sim \frac{1}{\sqrt{t(t-1)}}; \quad v(\infty) \sim 1.$$

These relations together with (4.2) and (4.1) imply (1.1).

**Remark 4.1** The result of this paper can be generalized to hyperelliptic surfaces of genus $g \geq 2$. Indeed, choose $2g-1$ distinct points $Q_1, Q_2, \ldots, Q_{2g-1}$ in $\mathbb{C}P^1 \setminus \{P_1, P_2, P_3\}$ and consider the two-fold covering

$$\mu_g: X(Q_1, Q_2, \ldots, Q_{2g-1}) \to \mathbb{C}P^1$$

ramified over $Q_1, \ldots, Q_{2g-1}$ and $P_1, P_2, P_3$. The pullback $\mu_g^* m$ of the metric $m$ in (2.2) by $\mu_g$ is a metric of constant curvature 1 with conical points of angle $4\pi$ at $2g-1$ Weierstrass points of the hyperelliptic curve $X(Q_1, Q_2, \ldots, Q_{2g-1})$ (three remaining Weierstrass points are nonsingular points of the metric). Using the same methods as in the genus 1 case, one can derive an explicit expression for the determinant of the Laplacian in the metric $\mu_g^* m$ as a function on moduli space of hyperelliptic curves of genus $g$. For instance, in genus two one gets the following explicit expression

$$\det \Delta_{\mu^2: m} = C \mathcal{F}^{2/5} \Phi(t_1, t_2, t_3),$$

where

$$\mathcal{F} = (\det \mathcal{B})^{5/2} \prod_{\beta} |\Theta[\beta](0|\mathcal{B})|$$

is the Petersson norm $\| \Delta_2 \|$ of the Siegel cusp form $\Delta_2 = \prod_{\beta} \Theta[\beta](0|\mathcal{B})$ ($\beta$ runs through the set of 10 even characteristics) and

$$\Phi(t_1, t_2, t_3) = \frac{|t_1 t_2 t_3 (t_1 - 1)(t_2 - 1)(t_3 - 1)|^{1/2} |t_1 - t_2|^{1/2} |t_1 - t_3|^{1/2} |t_2 - t_3|^{1/2}}{\prod_{k=1}^{10} (|\sqrt{t_k} + 1| + |\sqrt{t_k} - 1|)^{1/2}},$$

where the points $Q_1, Q_2, Q_3, P_1, P_2, P_3$ are identified with the points $t_1, t_2, t_3, 0, 1, \infty$ of $\mathbb{C}P^1$. It is straightforward to check that the right-hand side of (4.1) is a function on the moduli space $\mathcal{M}_2$ of compact Riemann surfaces of genus 2 (it suffices to check that $\Phi(t_1, t_2, t_3) = \Phi(t_1^{-1}, t_2^{-1}, t_3^{-1}) = \Phi(1 - t_1, 1 - t_2, 1 - t_3)$).
Remark 4.2  [In response to referee comments] The necessary and sufficient condition on a triple of positive numbers $\theta_1, \theta_2, \theta_3$ for the existence of a conformal curvature one metric on the Riemann sphere $\mathbb{C}P^1$, with three conical singularities of angles $2\pi \theta_1, 2\pi \theta_2, 2\pi \theta_3$ at the points $0, 1, \infty$, respectively, was obtained in [3, 13]. Let $m = \rho(w, \overline{w})|dw|^2$ stand for the corresponding metric on $\mathbb{C}P^1$. Then the pull back metric $\mu^* m$ on $X(t)$ (here $\mu$ is the same as in (2.3)) is a curvature one metric with conical singularity of angle $4\pi$ located at the point $\mu^{-1}(t)$ and three conical singularities of angles $4\pi \theta_1, 4\pi \theta_2, 4\pi \theta_3$ at the points $\mu^{-1}(0), \mu^{-1}(1)$, and $\mu^{-1}(\infty)$, respectively. It turns out that the formula (3.2) (for the spectral zeta function of the Friedrichs self-adjoint extension of Laplacian $\Delta^{\mu_0}$) is still valid, where $b(0)$ is the same as before and $b(-\infty) = \partial_\nu \log \rho(w, \overline{w})^{-1/4}|_{w=0}$. For details, we refer the reader to [4]. As a generalization of (1.1), we thus obtain

\[
\det \Delta^{\mu_0} = C_1 \delta \sigma|\sigma(t)|^4 \sqrt{|t|^2 - t} \sqrt{\rho(t, \overline{t})} = C_2 \det \Delta^{(0)}(X) \sqrt{|t|^2 - t} \sqrt{\rho(t, \overline{t})},
\]

where $C_1$ and $C_2$ are absolute constants and $t$ can be expressed as a function of $\sigma$; see Remark 1.1. Having at hand an explicit expression for the conformal factor $\rho(w, \overline{w})$ (in the case $\theta_1 = \theta_2 = \theta_3 = 1/2$ we use (2.2)), one immediately gets the corresponding explicit formula for $\det \Delta^{\mu_0}$. Let us also note that (4.3) remains valid if $m = \rho(w, \overline{w})|dw|^2$ is any conical metric on $\mathbb{C}P^1$ and $t$ stays outside of the conical singularities of $m$.

Acknowledgments  We thank the anonymous referee for valuable constructive comments. The research of the second author was supported by NSERC. The second author thanks Max Planck Institute for Mathematics in Bonn for hospitality and excellent working conditions.

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