Fairly Dividing a Cake after Some Parts Were Burnt in the Oven

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Abstract

There is a heterogeneous resource that contains both good parts and bad parts, for example, a cake with some parts burnt, a land-estate with some parts heavily taxed, or a chore with some parts fun to do. The resource has to be divided fairly among $n$ agents, each of whom has a personal value-density function on the resource. The value-density functions can accept any real value - positive or negative. Can standard cake-cutting procedures, developed for positive valuations, be adapted to this setting? This paper focuses on the question of envy-free cake-cutting with connected pieces. It is proved that such a division exists for 3 agents. The proof uses a generalization of Sperner’s lemma, which may be of independent interest.

1 Introduction

Most research works on fair division assume that the manna (the resource to divide) is good, e.g, tasty cakes, precious jewels or fertile land-estates. A substantial minority of the works assume that the manna is bad, e.g, house-chores or night-shifts. Recently, Bogomolnaia et al. (2017) introduced the more general setting of mixed manna — every resource can be good for some agents and bad for others. Here are some illustrative examples.

1. A cake with some parts burnt has to be divided among children. Some consider the burnt parts bad, while others think they are tasty.

2. A land-estate has to be divided among heirs, where landowners are subject to taxation. The value of a land-plot to an heir may be either positive or negative, depending on his/her valuation of the land and tax status.

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3. A house-chore such as washing the dishes has to be divided among family members. Most of them consider this bad, but some of them may view dish-washing, in some parts of the day, as a perfect relaxation after spending hours in solving mathematical problems.

At first glance, one could think that mixed-manna could be reduced to good-manna and bad-manna in the following way. For each part of the resource: (a) if there is one or more agents who think it is good, then divide it among them using any known procedure for dividing goods; (b) otherwise, all agents think it is bad — divide it among them using any known procedure for dividing bads. However, this simple reduction does not work when there are additional requirements besides fairness, such as economic efficiency or connectivity. When such natural requirements are added, the mixed setting is much more challenging than the all-good or all-bad setting.

Bogomolnaia et al. (2017) focused on the setting of homogeneous divisible resources, where the additional requirement is Pareto-efficiency. This paper focuses on the classic problem of cake-cutting (Steinhaus, 1948) — dividing a single heterogeneous resource — where the additional requirement is connectivity. The cake-cutting problem comes in many tastes: the cake can be one-dimensional or multi-dimensional; the fairness criterion can be proportionality (each agent receives a piece he values as at least 1/n of the total) or envy-freeness (each agent receives a piece he values at least as much as the piece of any other agent); the pieces can be connected or disconnected; and more. See Brânzei (2015); Procaccia (2015) for recent surveys. All variants have been studied in the good-cake setting (all agents consider every piece of cake good). Some variants have also been studied in the bad-cake setting (all agents consider every piece of cake bad). So far, no variants have been studied in the general mixed-cake setting.

While all variants of the cake-cutting problem are interesting, this paper focuses on a specific variant in which (a) the cake is one-dimensional, (b) the fairness criterion is envy-freeness, (c) the pieces must be connected (see Section 2 for the formal model). The main question of interest in this paper is:

Does there always exist a connected envy-free division of a mixed cake?

It is known that the answer is “yes” both for good cakes and for bad cakes. Moreover, there are procedures for finding such a division for three agents, and for approximating such a division for any number of agents. However, the proofs are based on a specific combinatorial structure, based on the well-known Sperner’s lemma; this structure breaks down in the mixed-cake setting, so the existing proofs are inapplicable (see Section 3).

Working with mixed cakes requires a new, more general combinatorial structure. This structure is based on a generalization of Sperner’s lemma. Based on this structure, it is possible to prove the main result (Section 4):

A connected envy-free division of a mixed cake always exists for three agents.

Most parts of this structure are applicable for any number of agents, however, there is one part which I do not know how to generalize to four or more agents.
Therefore, the existence of connected envy-free division for four or more agents remains an open question.

The existence of a connected envy-free division for three agents implies that an existing approximation algorithm can be adapted to find an approximately-envy-free division of a mixed cake. The adaptation is straightforward and it is briefly discussed in Section 5.

2 Model

A cake — modeled as the interval [0, 1] — has to be divided among n agents. The agents are called A1, ..., An or Alice, Bob, Carl, etc. The cake should be partitioned into n pairwise-disjoint intervals, X1, ..., Xn (some possibly empty), whose union equals the entire cake. Interval Xi should be given to agent Ai such that the division is envy-free — each agent weakly prefers his piece over any other piece. Two models for the agents’ preferences are considered.

(A) Additive agents: each agent Ai has an integrable value-density function vi. The value of a piece is the integral of the value-density on that piece: Vi(Xj) = ∫x∈Xj vi(x)dx. A division is envy-free if each agent believes that his piece’s value is weakly more than the value of any other piece: ∀i, j : Vi(Xi) ≥ Vi(Xj).

(B) Selective agents: each agent Ai has a function pi that accepts a set of pieces X and returns one or more pieces from X. The interpretation is that the agent “prefers” each of the pieces in pi(X) over all other pieces in X (this implies that the agent is indifferent between the pieces in pi(X)). A division X is envy-free if each agent receives one of his preferred pieces: ∀i : Xi ∈ pi(X). The preference functions should be continuous — any piece that is preferred for a convergent sequence of partitions is preferred at the limit partition (equivalently: for each j, the set of partitions X in which Xj ∈ pi(X) is a closed set. See Ku (1999)).

(A) is the more common model in the cake-cutting world, while (B) is much more general: every additive agent is also a selective agent (with pi(X) := arg maxj∈{1,...,n} (Vi(Xj))), but selective agents may have non-additive valuations. Moreover, selective agents can even handle some externalities: the preference of an agent may depend on the entire set of pieces in the partition rather than just on his own piece (however, the preference may not depend on which agent receives what piece; see Brânzei et al. (2013) for a discussion of such externalities).

In the good-cake and bad-cake settings, additional assumptions are made on the agents’ preferences. These assumptions are removed in our mixed-cake setting, as shown in Table 1.
Approximately-envy-free division. There are two ways to define an approximate envy-free division. (A) With additive agents, the approximation is measured in units of value: an $\epsilon$-envy-free division is a division in which each agent believes that his piece's value is at most $\epsilon$ less than the value of any other piece: $\forall i,j : V_i(X_i) \geq V_j(X_j) - \epsilon$. The valuations are usually normalized such that the value of the entire cake is 1 for all agents, so $\epsilon$ is a fraction (e.g, 1% of the cake value).

(B) With selective agents there are no numeric values, so the approximation is measured in units of length: a $\delta$-envy-free division is a division in which, for every agent $A_i$, movement of the borders by any amount at most equal to $\delta$ results in a division in which $A_i$ believes that his interval is as highly valued as any other interval. If $\delta$ is sufficiently small (e.g. 0.01 millimeter) then an $\delta$-envy-free division is envy-free for all practical purposes, since a movement of 0.01 millimeter is not noticeable by humans.

Unless stated otherwise, all results in this paper are valid for selective agents, therefore also for additive agents.

3 Why Existing Procedures Do Not Work

With $n = 2$ agents, the classic “I cut, you choose” protocol produces an envy-free division whether the cake is good, bad or mixed. The fun begins at $n = 3$.

### 3.1 Moving-knives procedures and approximations

Three procedures for connected envy-free division for three additive agents are known: [Stromquist] (1980), [Robertson and Webb] (1998) pages 77-78 and [Barbanel and Brams] (2004). All of them use one or more knives moving continuously. They were originally designed for good cakes and later adapted to bad cakes. All of them crucially rely on a monotonicity assumption: all agents weakly prefer a piece to all its subsets (in the case of a good cake), or all agents weakly prefer a piece to all its supersets (in the case of a bad cake). However, monotonicity does not hold with a mixed cake, so these procedures cannot be used. See Appendix A for details and specific negative examples.

Recently, [Branzei and Nisan] (2017) presented two algorithms for finding an $\epsilon$-envy-free allocation for additive agents. These algorithms, too, do not work with a mixed cake; see Appendix A for details.
3.2 Simplex of partitions

For four or more agents (or even for three selective agents), no moving-knives procedures are known. A different approach, which works for any number of selective agents, was suggested by Stromquist (1980) and further developed by Su (1999). Every partition of the cake can be represented by the lengths of the intervals, $l_1, \ldots, l_n$, with $l_1 + \cdots + l_n = 1$. Therefore, the set of all partitions is represented by the $(n-1)$-dimensional unit simplex $\Delta^{n-1}$. See Figure 1.

We denote the $n$ vertices of $\Delta^{n-1}$ by $1, \ldots, n$, and call them the main vertices. Denote $[n] := \{1, \ldots, n\}$ = the set of main vertices. For every subset $I \subseteq [n]$ of the main vertices, $F_I$ denotes the face of $\Delta^{n-1}$ which is the convex hull of the main vertices in $I$.

Agent labelings. Given a partition of the cake into $n$ intervals, each agent has one or more preferred pieces. The preferences of agent $A_i$ can be represented by a function $L_i : \Delta^{n-1} \to 2^{[n]}$. The function $L_i$ maps each cake-partition (= a point in the unit simplex) to the set of pieces that the agent prefers in this partition (= a set of labels from $[n]$). The set of preferred pieces always contains at least one label; it may contain more than one label if the agent is indifferent between two or more pieces. This is particularly relevant in case the agent prefers an empty piece, since there are partitions in which there is more than one empty piece. In such a partition, $L_i$ will be the set of all empty pieces. See Figure 2. An envy-free division corresponds to a point $x$ in the partition-simplex where it is possible to select, for each $i$, a single label from $L_i(x)$, such that the
Figure 2: Possible labeling $L_i$ of a single agent. **Left:** the value of the entire cake is positive. Hence, in each main vertex $i$, the agent prefers only piece $i$, since it is the only non-empty piece. In the edges between two main vertices $i, j$, the agent prefers either $i$ or $j$.

**Right:** the value of the entire cake is negative, but it contains some positive parts. In each main vertex $i$, the agent prefers the two empty pieces — the two pieces that are NOT $i$. In the edges between two main vertices, all three labels may appear.

$n$ labels are distinct.

**Definition 3.1 (Envy-free simplex).** Let $T$ be a triangulation of $\Delta^{n-1}$ — a partition of $\Delta^{n-1}$ to $(n-1)$-dimensional sub-simplices such that each two sub-simplices are either disjoint or intersect in a common face.

Let $Vert(T)$ the set of vertices of $T$, and $L_i : Vert(T) \rightarrow 2^{[n]}$ the agent labelings of the vertices of $T$ (for $i \in \{1, \ldots, n\}$). An envy-free simplex is a sub-simplex of $T$ with vertices $(t_1, \ldots, t_n)$, such that, for each agent $A_i$, it is possible to select a single label from $L_i(t_i)$ such that the $n$ labels are distinct.

If the diameter of each sub-simplex in $T$ is at most $\delta$, then each envy-free simplex corresponds to a $\delta$-envy-free division. If, for every $\delta$, there is an envy-free simplex with diameter at most $\delta$, then the continuity of the preference functions $p_i$ implies the existence of an envy-free division; see Su (1999).

**Good Cakes.** In a partition of a good cake, there always exists a non-empty piece with a weakly-positive value, so it is always possible to assume that each agent prefers a non-empty piece. Therefore, every labeling $L_i$ satisfies Sperner’s boundary condition: the label of every triangulation-vertex in the face $F_I$ must be from the set $I$ (see Figure 2/Left). Succinctly:

$$\forall i \in [n] : \forall I \subseteq [n] : \forall x \in F_I : L_i(x) \subseteq I$$

Sperner’s lemma implies that for every $i$ there is a fully-labeled simplex — a simplex whose $n$ vertices are labeled by $L_i$ with $n$ different labels.
Figure 3: **Left**: Assignment of vertices to agents such that, in each sub-triangle, each vertex is owned by a different agent. **Right**: the corresponding combined labeling, created by the trick of Su (1999). The sub-triangle marked in blue is fully-labeled and it corresponds to an approximately-envy-free allocation.

**Lemma 3.2** (Sperner’s lemma). Let $T$ be a triangulation of $\Delta^{n-1}$. Let $L$ be a labeling $\text{Vert}(T) \rightarrow 2^n$. If $L$ satisfies Sperner’s boundary condition, then it has an odd number of fully-labeled simplices.

However, we seek not a fully-labeled simplex but an *envy-free simplex* (see Definition 3.1). Su (1999) presents a way to combine $n$ agent-labelings $L_1, \ldots, L_n$ to a single labeling $L^W$ in the following way. Each triangulation-vertex is assigned to one of the $n$ agents, such that in each sub-simplex, each of its vertices is owned by a single agent. See Figure 3/Left. Now, each vertex is labeled with the corresponding label of its owner: if a vertex $x$ is owned by agent $A_i$, then $L^W(x) := L_i(x)$. See Figure 3/Right. If all the $L_i$ satisfy Sperner’s boundary condition, then the combined labeling $L^W$ also satisfies Sperner’s boundary condition. Therefore, by Sperner’s lemma, $L^W$ has a fully-labeled simplex. By definition of $L^W$, this is an envy-free simplex.

**Bad Cakes** In a partition of a bad cake, the values of all non-empty pieces are weakly negative, so it is always possible to assume that each agent prefers an empty piece. In the main vertices, there are $n - 1$ empty pieces; the agent is indifferent between them, so we may label each main vertex with an arbitrary empty piece. We can always do this in a way that ensures that the labeling satisfies Sperner’s boundary condition. For example, if we label each main vertex $F_i$ by $i + 1$ (modulo $n$) then the labeling satisfies Sperner’s condition, so again an envy-free simplex exists.

**Mixed Cakes** The challenging case is that in which the value of the entire cake is negative, but the cake may contain both positive and negative pieces. Then, in each point, the agent may prefer either an empty piece or a non-empty
piece. Hence, the agent labelings no longer satisfy Sperner’s boundary condition; see Figure 2/Right. Here, our work begins.

4 Cutting Mixed Cakes

4.1 The Permutation Condition

The first step in handling a mixed cake is to define boundary conditions that are satisfied for all agent labelings, regardless of whether the cake is good, bad or mixed. Our boundary condition is based on the observation that different points on the boundary of the partition-simplex may represent the same physical cake-partition. For example, consider the three diamond-shaped points in Figure 4. In each of these points, the set of pieces is the same: \{[0, .8], [8, 1], \emptyset\}. Therefore, a consistent agent will select the same piece in all three partition, even though this piece might have a different index in each point. This means that the agent’s label in each of these points uniquely determines the agent’s labels in the other two points. For example, if the agent labels the top-left diamond point by “3”, this means that he prefers the empty piece, so he must label the top-right diamond point by “2” and the bottom-left diamond point by “1” (as in the figure).

To formalize this boundary condition we need several definitions.

**Definition 4.1.** Two point \(x, y \in \Delta^{n-1}\) are called friends if they have the same ordered list of non-zero coordinates.

For example, on \(\Delta^{3-1}\), the points (0, .2, .8) and (.2, 0, .8) and (.2, .8, 0) are friends, since their ordered list of non-zero coordinates is (.2, .8). But the point (0, .8, .2) is not their friend since the order is different.
How many friends does a point \( x \) have? If \( x \) is in the interior of \( \Delta^{n-1} \), then all its coordinates are nonzero, so it has no friends except itself. Suppose \( x \in F_I \), where \( I \subseteq [n] \) is some subset of the main vertices and \( F_I \) is a face of \( \Delta^{n-1} \) which is the convex hull of the main vertices in \( I \). Then, \( x \) has a single friend in each face \( F_I \) where \( |I| = |I'| \). So if \( |I| = k \) then \( x \) has \( \binom{n}{k} \) friends (including itself).

For example, the point \((0, .2, .8)\) is on the face \( F_{\{2,3\}} \), so it has \( \binom{3}{2} = 3 \) friends including itself.

Since our boundary conditions have a bite only for friends, we will consider from now on only triangulations that are “friendly” in the following sense:

**Definition 4.2.** A triangulation \( T \) is called **friendly** if, for every vertex \( x \in \text{Vert}(T) \), all the friends of \( x \) are in \( \text{Vert}(T) \).

For any \( k \in [n] \), let \( F_{-k} := F_{[n] \setminus \{k\}} \) be the face on which the empty piece is piece number \( k \). Our boundary condition is that the label of a vertex in \( F_{-n} \) uniquely determines the labels of all its friends. Specifically, suppose that the label of a vertex \( x \in F_{-n} \) is \( j \). Consider its friend \( y \in F_{-k} \). The label on \( y \) is determined by the following function:

\[
\pi_{-k}(j) := \begin{cases} 
  j & j < k \\
  j + 1 & k \leq j < n \\
  k & j = n
\end{cases}
\] (1)

For every \( k \), the function \( \pi_{-k} \) is a permutation (a bijection from \([n]\) to \([n]\)). \( \pi_{-n} \) is the identity permutation. Table 2 shows the three permutations for \( n = 3 \): \( \pi_{-3}, \pi_{-2} \) and \( \pi_{-1} \).

**Definition 4.3.** A labeling \( L : \text{Vert}(T) \rightarrow 2^{[n]} \) satisfies the **permutation condition** if, for every \( k \in [n] \), every face \( F_{-k} \) and every vertex \( y \in F_{-k} \):

\[ L(y) = \pi_{-k}(L(x)) \]

where \( x \) is the friend of \( y \) on the face \( F_{-n} \), and \( \pi_{-k} \) is defined by (1).

It can be verified that all labelings in Figures 4 and 2 satisfy the permutation condition.

The following observations will be important in the following subsections. The first observation is related to the **parity** of permutations. Recall that a permutation is **even**(odd) if it can be implemented by an even(odd) number of swaps.

**Observation 4.4.** The permutation \( \pi_{-k} \) is even(odd) if \( n - k \) is even(odd).

For example, Table 2 clearly shows that \( \pi_{-3} \) is even (it is the identity permutation), \( \pi_{-2} \) is odd (maps 123 to 132) and \( \pi_{-1} \) is even (maps 123 to 231).

The second observation is related to labels on faces \( F_I \) where \( |I| \leq n - 2 \). Such faces are intersections of two or more \( n - 1 \)-faces. For example, in \( \Delta^{3-1} \), the main vertex \( F_1 \) is the intersection of \( F_{-2} \) and \( F_{-3} \) (it corresponds to a
| Preferred piece: | Left | Right | Empty | \{ER\}ELRE\{EL\} |
|------------------|------|-------|-------|--------------------|
| Label on $F_{-3}$: | 1    | 2     | 3     | \{32\}3123\{31\} |
| Label on $F_{-2}$: | 1    | 3     | 2     | \{23\}2132\{21\} |
| Label on $F_{-1}$: | 2    | 3     | 1     | \{13\}1231\{12\} |

Table 2: Label-permutations that satisfy Definition 4.3 for $n = 3$. The rightmost column is provided as an example. It corresponds to the labeling in each edge in Figure 4/Right. Note that the labeling always goes from the vertex with the lower index (the Left vertex) to the vertex with the higher index (the Right vertex). E means that the agent prefers the Empty piece, R means the Right piece and L means the Left piece. Braces imply that there are multiple labels on the same point.

partition in which both piece 2 and piece 3 are empty). Since it is a friend of itself, the permutation condition implies a restriction on the possible labelings on $F_1$. Specifically, $2 \in L(F_1)$ if-and-only-if $3 \in L(F_1)$ (since $\pi_{-2}(2) = 3$ and $\pi_{-2}(3) = 2$). This makes sense: since all empty pieces are the identical, the agent prefers an empty piece if-and-only-if he prefers all empty pieces. This is summarized in the following observation:

**Observation 4.5.** If a labeling $L$ satisfies the permutation condition, then for every vertex $x \in F_{[n]}\setminus I$, either $L(x) \cap I = I$ or $L(x) \cap I = \emptyset$.

We would like to use the Permutation Condition to prove the existence of an envy-free simplex. However, the permutation condition is valid for a single agent; we have to find a way to combine $n$ different labelings that satisfy the permutation condition to a single labeling that satisfies it. This is handled in the following subsection.

### 4.2 Combining $n$ labelings to a single labeling

**Definition 4.6.** An ownership-assignment of a triangulation $T$ is a function from the vertices of the triangulation to the set of $n$ agents, $W : Vert(T) \rightarrow \{A_1, \ldots, A_n\}$.

**Definition 4.7.** Given a triangulation $T$, $n$ labelings $L_1, \ldots, L_n$, and an ownership-assignment $W$, the combined labeling $L^W$ is the labeling that assigns to each vertex in $Vert(T)$ the label/s assigned to it by its owner. I.e, if $W(x) = A_i$, then $L^W(x) := L_i(x)$.

**Definition 4.8.** An ownership-assignment $W$ is called **diverse** if in each sub-simplex in $T$, each vertex of the sub-simplex has a different owner.

**Definition 4.9.** An ownership-assignment $W$ is called **friendly** if it assigns friends to the same owner. I.e, if the vertices $x, y$ are friends (see Definition 4.1), then $W(x) = W(y)$.

The diversity condition was introduced by Su (1999). As an example, the ownership-assignment of Figure 3 is diverse. However, it is not friendly. For
Figure 5: **Left:** Barycentric subdivision of a triangle. **Right:** Barycentric triangulation of a triangle, with a friendly and diverse ownership assignment (here $A, B, C$ are agents $1, 2, 3$).

example, the two vertices near $(1, 0, 0)$ are friends since their coordinates are $(0.8, 2, 0)$ and $(0.8, 0, 2)$, but they have different owners ($B, C$). This means that the permutation condition might not hold in the combined labeling.

It is easy to construct a friendly ownership-assignment: go from $F_1$ towards $F_2$, assign the vertices to arbitrary owners, then assign the vertices from $F_1$ towards $F_3$ and from $F_2$ towards $F_3$ to the same owners. However, in general it will not be easy to extend this to a diverse assignment.

Does there always exist an ownership-assignment which is both diverse and friendly? Yes.

**Lemma 4.10.** For any $n \geq 3$ and any $\delta > 0$, there exists a friendly triangulation $T$ of $\Delta^{n-1}$ where the diameter of each sub-simplex is at most $\delta$, and an ownership-assignment of $T$ that is both friendly and diverse.

**Proof.** The construction is based on the Barycentric subdivision. The Barycentric subdivision of a simplex with main vertices $F_1, \ldots, F_n$ is constructed as follows:

Pick a permutation $\pi$ of the main vertices. For every prefix of the permutation, $\pi_1, \ldots, \pi_m$ (for $m \in \{1, \ldots, n\}$), define $v_m$ as their barycenter (arithmetic mean): $v_m := (\pi_1 + \cdots + \pi_m)/m$. We call $v_m$ a level-$m$ vertex. The vertices $v_1, \ldots, v_n$ define a sub-simplex.

Each permutation yields a different sub-simplex, so all in all, the barycentric subdivision of an $(n-1)$-dimensional simplex contains $n!$ subsimplices (see Figure 5/Left for the case $n = 3$). Note that each sub-simplex has exactly one vertex of each level $m \in \{1, \ldots, n\}$.

By recursively applying the barycentric subdivision to each sub-simplex (as in Figure 5/Right), we get the Barycentric triangulation. It is known that, in each subdivision, the diameter of the subsimplices is at most $n/(n+1)$ the

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1 The following description follows the Wikipedia page “Barycentric subdivision”. 

diameter of the original simplex. Hence, to get a barycentric triangulation in which the diameter of each sub-simplex is at most \( \delta \), we have to perform \( k \) steps of barycentric division, where \( k \) satisfies the inequality:

\[
\left( \frac{n}{n+1} \right)^k \leq \delta
\]

\[
k \geq \frac{\ln \delta}{\ln \frac{n}{n+1}} = \frac{\ln(1/\delta)}{\ln (1 + 1/n)}
\]

\[
\approx \frac{\ln(1/\delta)}{(1/n)} = n \ln(1/\delta).
\]

The ownership assignment is determined by the levels of the vertices in the last subdivision step: all the vertices whose level in the last subdivision is \( i \) are assigned to agent \( i \) (see Figure 5/Right). This ownership assignment is:

- diverse — since each subsimplex has exactly one vertex of each level.
- friendly — since, by symmetry, every two friends have the same level. \( \square \)

**Lemma 4.11.** Let \( L_1, \ldots, L_n \) be labelings of a friendly triangulation \( T \), all of which satisfy the Permutation Condition.

If \( W \) is a friendly ownership-assignment, then the combined labeling \( L^W \) also satisfies the Permutation Condition.

**Proof.** The Permutation Condition restricts only the labels of friends. Since all friends are labeled by the same owner, and the labeling of each owner satisfies the permutation condition, the combined labeling also satisfies the same condition. \( \square \)

Lemmas 4.10 and 4.11 reduce the problem of finding an envy-free simplex with \( n \) labelings, to the problem of finding a fully-labeled simplex with a single labeling. Now it remains to prove that a labeling that satisfies the Permutation Condition always has a fully-labeled simplex. To do this, we need to generalize Sperner’s lemma.
4.3 The Degree Lemma

This subsection presents a generalization of a lemma by Musin (2014), which is itself a generalization of Sperner’s lemma. It is based in the concept of degree of a labeling, which is defined below.

4.3.1 Degrees of affine mappings

Let $P, Q$ be two $n - 1$-dimensional simplices in $\mathbb{R}^{n-1}$. Let $f : P \to Q$ be a mapping that maps each vertex of $P$ to a vertex of $Q$. One way to define such mapping is to name the vertices of $Q$ by $Q_1, \ldots, Q_n$ and label the vertices of $P$ by integers in $1, \ldots, n$; then, for every vertex $v \in P$ labeled by $i$, $f(v) = Q_i$. It is known that there is a unique way to extend $f$ to an affine transformation from $P$ to $Q$ (see Musin (2014)). Define $\text{deg}(f)$ as the sign of the determinant of this transformation. So there are three cases:

- $\text{deg}(f) = +1$ means that $f$ is onto $Q$ and it can be implemented by translations, rotations and scalings (but no reflections);
- $\text{deg}(f) = -1$ means that $f$ is onto $Q$ and it can be implemented by translations, rotations, scalings and a single reflection;
- $\text{deg}(f) = 0$ means that $f$ is not onto $Q$ (i.e, it maps the entire $P$ into a single face of $Q$ which has dimension $n - 2$ or less).

The picture below shows three mappings with different degrees from different source simplices in $\mathbb{R}^3$ to the same target simplex $Q$:

We make several observations that relate the labeling to the degree.

- If $P$ is fully-labeled (each vertex has a unique label), then $f$ is onto $Q$, so $\text{deg}(f)$ is either $+1$ or $-1$. If it is not fully-labeled (two or more vertices have the same label), then $f$ is not onto $Q$ so $\text{deg}(f) = 0$.
- An odd permutation of the labels on $P$ requires a reflection, so it inverses the sign of $\text{deg}(f)$. An even permutation keeps $\text{deg}(f)$ unchanged.
- The following multiplicative property of the degree operator follows directly from the properties of determinants (or affine mappings):

$$\text{deg}(f \circ g) = \text{deg}(f) \cdot \text{deg}(g)$$ (2)
Consider now an \( n - 2 \)-dimensional simplex in \( \mathbb{R}^{n-1} \). It is contained in a hyperplane and this hyperplane divides \( \mathbb{R}^{n-1} \) to two half-spaces. Define an oriented simplex as a pair of a simplex and one of its two half-spaces.

Let \( P', Q' \) be two oriented \( n - 2 \)-dimensional simplices in \( \mathbb{R}^{n-1} \). Let \( f \) be a mapping that maps each vertex of \( P' \) to a vertex of \( Q' \), and maps the half-space attached to \( P' \) to the half-space attached to \( Q' \). There are infinitely many ways to extend \( f \) to an affine transformation, but all of them have the same degree. An example is shown below, where the half-space attached to each simplex is denoted by an arrow:

Consider now an \( n - 1 \)-dimensional simplex in \( \mathbb{R}^{n-1} \). Each of its \( n - 2 \)-dimensional faces divides \( \mathbb{R}^{n-1} \) to two half-spaces, one of which contains the interior of the simplex. Orient each face by attaching to it its half-space containing the simplex interior (figuratively, attach to it an arrow pointing inwards).

Let \( P, Q \) be two \( n - 1 \)-dimensional simplices in \( \mathbb{R}^{n-1} \), \( L \) a labeling of the vertices of \( P \), and \( f_L \) be the (unique) affine transformation from \( P \) to \( Q \) defined by this labeling. Let \( Q' \) be a fixed \( n - 2 \)-dimensional face of \( Q \). Define \( f'_L \) as the restriction of \( f_L \) to the pre-image of \( Q' \). There are several cases. They are illustrated below; in all illustrations, \( Q' \) is the face \( Q_1 Q_3 Q_4 \), and the arrow/s represent the orientation/s of its pre-image/s when they are transformed into \( Q' \) without reflection (the images look better in color).

If \( \deg(f_L) = +1 \), then the pre-image of \( Q' \) is a unique face \( P' \subseteq P \) (in the illustration, the face labeled with 134). \( f'_L \) maps \( P' \) onto \( Q' \). When both faces are oriented inwards, \( \deg(f'_L) = +1 \) since \( P' \) is transformed to \( Q' \) with no reflection.

If \( \deg(f_L) = -1 \) then again the pre-image of \( Q' \) is a unique face \( P' \subseteq P \). \( f'_L \) maps \( P' \) onto \( Q' \). When both faces are oriented inwards, \( \deg(f'_L) = -1 \) since \( P' \) is transformed to \( Q' \) with a single reflection.

If \( \deg(f_L) = 0 \) then there are two cases:
(a) $f_L$ is not onto $Q'$ (in the illustration, no face is labeled with 134) — it maps the entire simplex $P$ into another face of $Q$ (of dimension $n - 2$ or less). In this case, by definition $\text{deg}(f_L') = 0$.

(b) $f_L$ is onto $Q'$ — it maps exactly two different faces of $P$ onto $Q'$. (in the illustration, exactly two faces are labeled with 134). Call these faces $P'_+$ and $P'_-$. For each $i \in \{+, -\}$, let $f'_i$ be the mapping from $P'_i$ onto $Q'$. Then $\text{deg}(f'_+) = -\text{deg}(f'_-)$. 

Define the boundary degree of $f$ on $Q'$ as the following sum:

$$\text{deg}(f, Q') := \sum_{P' \in f^{-1}(Q')} \text{deg}(f' : P' \to Q')$$

where $f^{-1}(Q')$ is the set of faces of $P$ that are mapped to $Q'$ (as explained above, this set contains either 0 or 1 or 2 faces). From the above discussion, the following lemma follows:

**Lemma 4.12.** Let $P, Q$ be $n - 1$-dimensional simplices in $\mathbb{R}^{n-1}$, and let $f$ by an affine mapping that maps each vertex of $P$ to a vertex of $Q$. Then, for every $n - 2$-dimensional face $Q' \subseteq Q$:

$$\text{deg}(f, Q') = \text{deg}(f)$$

**Proof.** As explained in the above table, the left-hand side is +1 when $\text{deg}(f) = 1$, −1 when $\text{deg}(f) = -1$, and either 0 or +1 when $\text{deg}(f) = 0$. □

### 4.3.2 Degrees of labelings of triangulations

Let $T$ be a triangulation of a simplex $P$, and let $L : \text{Vert}(T) \to [n]$ be a labeling of the vertices of $T$. In each sub-simplex $t$ of the triangulation $T$, $L$ defines an affine transformation $f_{L,t} : t \to Q$ as explained in the previous subsection. Define the degree of $L$ as the sum of the degrees of all these transformations:

$$\text{deg}(L) := \sum_{t \in T} \text{deg}(f_{L,t})$$

---

2 Why exactly two faces of $P$ are mapped onto $Q'$? Because the image of $P$ has $n-1$ vertices, so $P$ is labeled with exactly $n-1$ different labels, so there are exactly two vertices of $P$ with the same label while all other $n-2$ vertices have different labels, so there are exactly two faces of $P$ that have the $n-1$ labels of $Q'$.

3 To see this, suppose that, before mapping $P'_+$ and $P'_-$ onto $Q'$, we first map $P'_+$ onto $P'_-$, with no reflection. Let $g$ be this mapping. Then, the half-space attached to $g(P'_+)$ does not contain the interior of $P$ (figuratively, the arrow attached to $g(P'_+)$ points outwards). To align the orientations we must use reflection, so an orientation-preserving mapping $g' : P'_- \to P'_+$ must have $\text{deg}(g') = -1$. Since $f'_+ = f'_+ \circ g'$, the multiplicative property (2) implies that $\text{deg}(f'_+) = -\text{deg}(f'_-)$. 15
Each fully-labeled sub-simplex of $T$ contributes either 1 or $-1$ to this sum — depending on the arrangement of labels: if the labels are an even permutation of the labels on $Q$, then the contribution is +1; if they are an odd permutation, the contribution is $-1$.

Given an $n-2$-dimensional face $Q' \subseteq Q$, define:

$$\text{deg}(L, Q') := \sum_{t \in T} \text{deg}(f_{L,t}, Q')$$

Suppose w.l.o.g. that $Q'$ is the face $Q_1 \ldots Q_{n-1}$. Then, each $n-2$-dimensional face of $T$ that is labeled with $1, \ldots, n-1$ contributes either 1 or $-1$ to this sum — depending on the arrangement of labels: if the labels are an even permutation of the labels on $Q'$ then the contribution is +1; if they are an odd permutation, the contribution is $-1$.

Consider an internal face in $T$ — a face common to two sub-simplices of $T$. This face contributes to $\text{deg}(L, Q')$ twice — once for each sub-simplex it is a face of. The orientation of this face in its two sub-simplices is opposite, since the interiors of these sub-simplices are in opposite directions of the face:

Therefore, the two contributions of this face to $\text{deg}(L, Q')$ cancel out, and its net contribution is 0. Hence, to calculate $\text{deg}(L, Q')$, it is sufficient to consider the faces labeled $1, \ldots, n-1$ on the boundary of the simplex $P$. Define:

$$\text{deg}_\partial(L, Q') := \sum_{t \in \partial T} \text{deg}(f'_{L,t}, Q')$$

where $\partial T$ is the collection of $n-2$-dimensional faces of $T$ on the boundary of the simplex $P$. Thus we have proved the main result of this subsection:

**Lemma 4.13 (The Degree Lemma).** For every labeling $L$ and every face $Q' \subseteq Q$:

$$\text{deg}(L) = \text{deg}_{\partial}(L, Q')$$

An illustration for $n = 3$ is shown in Figure 6. There are six fully-labeled triangles: five with an even permutation (counterclockwise arrows) and one with an odd permutation (clockwise arrows). Therefore:

$$\text{deg}(L) = 5 - 1 = 4$$
Let $Q' = Q_1Q_2$. At the boundary of $P$ there are four edges labeled with 1, 2, all in the same direction (counterclockwise). Therefore:

$$\deg_\partial(L, Q') = 4 - 0 = 4$$

in accordance with the Degree Lemma.

Sperner’s lemma is a special case of the Degree Lemma, since every labeling that satisfies Sperner’s condition has a boundary degree of 1 (see Appendix C).

The Degree Lemma reduces the problem of proving existence of a fully-labeled simplex, to the problem of proving that the boundary-degree is non-zero. Therefore, our next goal is to prove that the Permutation Condition of Subsection 4.1 implies that the boundary degree is non-zero. However, there is a technical difficulty: the permutation condition is defined for multi-valued labelings, while the degree is only defined for single-valued labelings. A multi-valued labeling $L : \text{Vert}(T) \to 2^{[n]}$ can be converted to a single-valued labeling by simply selecting, for each vertex $x \in \text{Vert}(T)$, a single label from the set $L(x)$. For our purposes, it is sufficient to prove the following claim:

(*) If a multi-valued labeling $L$ satisfies the Permutation Condition, then there exists a selection of single labels such that the resulting single-valued labeling has a non-zero boundary degree.

Currently, I know to prove the claim (*) only for $n = 3$. This is done in the next subsection.

4.4 From Permutation Condition to Boundary Degree

In this subsection $n = 3$, so the simplex $P$ is the unit triangle $\Delta^{3-1} = \Delta^2$. The simplex $Q$ is any triangle whose vertices are labeled by $Q_1, Q_2, Q_3$. The boundary degree of a labeling, $\deg_\partial(L, Q')$, is determined by the cyclic sequence of labels around the boundary of $P$ (in the counterclockwise direction). For any
two labels $i, j \in \{1, 2, 3\}$, denote by $\#_{ij}(L)$ the number of adjacent $i, j$ pairs in the cyclic sequence of labels around the boundary of $P$. Then:

$$\deg_\partial(L, Q_iQ_j) = \#_{ij}(L) - \#_{ji}(L)$$

For any two points $x, y$ on the boundary of $\Delta^{3-1}$, denote by $L[xy]$ the sequence of labels assigned by $L$ to the vertices encountered when walking from $x$ counterclockwise to $y$. It is convenient to define its average degree:

$$\deg_\partial(L[xy], \bar{Q}) := \frac{1}{3} \left( \deg_\partial(L[xy], Q_1Q_2) + \deg_\partial(L[xy], Q_2Q_3) + \deg_\partial(L[xy], Q_3Q_1) \right)$$

When we consider the entire cyclic sequence of labels, it is easy to see that $\deg_\partial(L, Q_1Q_2) = \deg_\partial(L, Q_2Q_3) = \deg_\partial(L, Q_3Q_1)$\footnote{One way to see this equality is: suppose we are going around the boundary of the triangle $Q$. We start at some vertex $Q_i$, turn around the boundary several times in both directions, ending at the starting point. The net number of times we travel from $Q_1$ to $Q_2$ (forward minus backward) equals the net number of times we travel from $Q_2$ to $Q_3$ and from $Q_3$ to $Q_1$, since all of these equal the net number of times we surround the triangle (counterclockwise minus clockwise). Another way to prove this is by induction on the sequence length: start with a trivial sequence e.g. 11, then add labels between the first and the last 1 and show that the equality is preserved. See the figure in page 17 for an illustration.} so all of these are equal to $\deg_\partial(L, Q)$. Our goal now is to prove that $\deg_\partial(L, Q)$ is non-zero.

Lemma 4.14. Let $L : \text{Vert}(T) \to 2^{[3]}$ be a labeling of a friendly triangulation of $\Delta^{3-1}$ that satisfies the Permutation Condition (Definition 4.3). Then, it is possible to select a single label per vertex such that the resulting labeling satisfies:

$$\deg(L) \mod 3 \neq 0$$

Proof. The permutation condition (and particularly Observation 4.5) implies that there are only two cases regarding the labels on the main vertices.

Positive case (Figure 7/Left): for each main vertex $F_i$, $i \in L(F_i)$ (this corresponds to the case that the owner of the main vertices values the entire
We select, for each vertex $F_i$, the single label $i$. Then, the sequence of labels on the face $F_{12}$ starts with 1 (the left piece) and ends with 2 (the right piece), so its average degree is $k + 1/3$ for some integer $k$. The sequence of labels on $F_{23}$ also starts with the left piece (2) and ends with the right piece (3); by the Permutation Condition, it is exactly the same sequence as in $F_{12}$, up to an even permutation. Therefore its average degree is $k + 1/3$ too. The sequence of labels on $F_{13}$ also starts with the left piece (1) and ends with the right piece (3); by the Permutation Condition, it is exactly the same sequence as in $F_{12}$, up to an odd permutation (see Observation 4.4). Therefore its average degree is $- (k + 1/3)$. When traveling around the boundary in the positive direction, the face $F_{13}$ is traveled backwards, so $\deg_\partial (L, \bar{Q}) = 3(k + 1/3) = 3k + 1$.

**Negative case** (Figure 2/Right): for each main vertex $F_i$, $L(F_i) = [n]\{i\}$ (this corresponds to the case that the owner of the main vertices values the entire cake as strictly negative). So we have $2^3 = 8$ ways to select a single label per main vertex. Here we cannot just pick one of the eight arbitrarily, since for each single selection, the boundary degree might be zero in some cases; two such cases are illustrated below.

The problem is that, for any single selection, the sequences of labels on the edges are *not* the same up to permutation. For example, suppose we select $L(F_1) = 2$. Piece number 2 is the rightmost piece on the face $F_{12}$, but it is the empty piece on the face $F_{13}$.

The trick we will use to overcome this problem is to calculate the sum of the boundary degree over all $2^3$ selections. We will prove that this sum is non-zero (modulo 3). This will imply that it is non-zero for at least one of the eight selections.

We calculate this sum for each face separately. First, consider the face $F_{12}$. Only the labels on the vertices $F_1, F_2$ are relevant here. If these labels are 2, 1 then the average degree is $k_1 - 1/3$ for some integer $k_1$; if they are 2, 3 then it is $k_2 + 1/3$; if they are 3, 1 then it is $k_3 + 1/3$; if they are 3, 3 then it is $k_4$.

---

5 At first glance, one could think that in the positive case each face $F_{i,j}$ should be labeled only with $i$ and $j$, since there always exists a non-empty piece with a positive value. While this is true for a single agent-labeling, it is *not* true for a combined labeling: it is possible that the labels on the main vertices belong to Alice while the adjacent labels on the faces $F_{i,j}$ belong to Bob (as in Figure 6), and Bob might think that the entire cake is negative.

6 At first glance, one could think that, if the triangulation is sufficiently fine, we will not have such anomalous cases, since by continuity, the label in each vertex sufficiently close to $F_1$ should be one of the labels in $F_i$. Like in the previous footnote, this is true for a single agent-labeling but false for a combined labeling.
We have to sum all these numbers, then multiply by 2 to count for the two selections on $F_3$. The result is $k + 2/3$ for some integer $k$.

Next, consider the face $F_{23} \equiv F_{-1}$. By the Permutation Condition, for every possible selection of labels on $F_1, F_2$, the images of these labels under the permutation $\pi_{-1}$ are a possible selection of labels on $F_2, F_3$ (even if the label on $F_2$ is not the same). Therefore, the sum of degrees for all possible selections on $F_{23}$ is the same as on $F_{12}$ — it is $k + 2/3$. Similarly, the sum on the face $F_{13}$ is $-(k + 2/3)$ but it is traveled backwards. Therefore, the sum of the boundary-degree over all $2^3$ selections is $3(k + 2/3) = 3k + 2$. Therefore, for at least one of these eight selections, the boundary degree modulo 3 is nonzero.

In all cases $\deg_\partial(L, \overline{Q}) \mod 3 \neq 0$. By the Degree Lemma, the same is true for $\deg(L)$.

I could not extend Lemma 4.14 to any $n > 3$; it is left as a conjecture:

**Conjecture 4.15.** Let $L : \text{Vert}(T) \to 2^n$ be a labeling of a friendly triangulation $T$ of $\Delta^{n-1}$ that satisfies the Permutation Condition. Then, it is possible to select a single label per vertex such that the resulting labeling satisfies:

$$\deg_\partial(L, \overline{Q}) \mod n \neq 0$$

hence:

$$\deg(L) \mod n \neq 0$$

The final theorem of this section ties the knots.

**Theorem 4.16.** For $n = 3$ selective agents, there always exists a connected envy-free division of a mixed cake.

**Proof.** Let $T$ be a barycentric triangulation of the partition-simplex $\Delta^{n-1}$. Let $W$ be a friendly and diverse ownership-assignment on $T$, which exists by Lemma 4.10. Ask each agent to label its vertices by the indices of its preferred pieces.

All $n$ labelings satisfy the Permutation Condition (see Subsection 4.1). Since $W$ is friendly, by Lemma 4.11 $L^W$ satisfies these conditions too.

Therefore, by Lemma 4.14 $\deg(L^W) \neq 0$, so $L^W$ has at least one fully-labeled simplex. Since $W$ is diverse, every such fully-labeled simplex of $L^W$ is an envy-free simplex.

All the above procedure can be done for finer and finer barycentric triangulations. This yields an infinite sequence of envy-free simplexes. This sequence has a converging subsequence. By the continuity of the preferences, the limit of this subsequence is an envy-free partition.

The main question left open is whether Conjecture 4.15 is true. If it is true, then Theorem 4.16 immediately becomes true for any $n$. 

5 Finding an Envy-Free Division

While the main focus in this paper is on the existential question of whether an envy-free allocation exists, the present section briefly relates to the algorithmic question of how to find such an allocation. Stromquist (2008) proved that connected envy-free allocations cannot be found in a finite number of queries even when all valuations are positive, so the best we can hope for is an approximation algorithm.

The following simple binary-search algorithm can be used to find a fully-labeled sub-simplex in a labeled triangulation. It is adapted from Deng et al. (2012):

1. If the triangulation is trivial (contains one sub-simplex), stop.
2. Divide the large simplex to two halves, respecting the triangulation lines.
3. In each half, calculate the degree of the labeling on the boundary.
4. Select one half in which the boundary degree is non-zero; perform the search recursively in this half.

While Deng et al. (2012) present this algorithm for the positive case, it works whenever the boundary degree of the original simplex is non-zero. Then, in step 4, the boundary degree of at least one of the two halves is non-zero, so the algorithm goes on until it terminates with a fully-labeled simplex. This is the case when there are \( n = 3 \) agents with arbitrary mixed valuations (Lemma 4.14). If Conjecture 4.15 is true, then this is also the case for any \( n \).

To calculate the runtime of the binary search algorithm, suppose the triangulation is such that each side of the original simplex is divided to \( D \) intervals. Then, the runtime complexity of finding a fully-labeled simplex is \( O(D^{n-2}) \) (Deng et al., 2011).

To calculate the complexity of finding a \( \delta \)-approximate envy-free allocation, we have to relate \( D \) to \( \delta \). As estimated in the proof of Lemma 4.10, to guarantee that the diameter of each subsimplex is at most \( \delta \), it is sufficient to have \( \approx n \ln(1/\delta) \) steps of the barycentric division. In each step, the number of intervals in each side is doubled, so \( D = O(1/\delta^n) \). So the total runtime complexity of finding a \( \delta \)-approximate envy-free allocation using the barycentric triangulation is \( O(1/\delta^{n(n-2)}) \).

Deng et al. (2012) note the slow convergence of the barycentric triangulation, and propose to use the Kuhn triangulation instead. This triangulation looks similar to the equilateral triangulations shown in Figure 3. In this triangulation, \( D = 1/\delta \) so the runtime complexity of the binary search is \( O(1/\delta^{n-2}) \). However, this triangulation does not support a diverse and friendly ownership-assignment.

7 Note that \( \delta \) is an additive approximation factor to the location of the borderlines. Deng et al. (2012) also consider an additive approximation to the values (e.g., each agent values another agent’s piece at most \( \epsilon \) more than his own piece). To relate the \( \delta \)-factor to the \( \epsilon \)-factor, they add an assumption that the valuation functions are Lipschitz continuous with a constant factor.
For $n = 3$, I found a similar triangulation that does support a diverse and friendly ownership-assignment (see Figure 8). This implies that, for $n = 3$, a $\delta$-envy-free division can be found in time $O(1/\delta)$. However, I do not know how to generalize this “hack” to $n > 3$; this topic is left for future work.
A Known Algorithms for Connected Envy-Freeness Do Not Work with Mixed Valuations

When there are at least 3 agents, connected envy-free cake-cutting cannot be found by a finite discrete procedure ([Stromquist, 2008]. For 3 agents, several non-discrete procedures are known. These procedures use *moving knives*.

A.1 Rotating-knife procedure

This beautiful procedure of [Robertson and Webb (1998, pages 77-78)](Robertson Webb) can be used only when the cake has at least two dimensions — it cannot be used when the cake is a one-dimensional interval. However, this author has a special fondness for two-dimensional cakes ([Segal-Halevi et al., 2017, 2015](Segal Halevi et al)) so he does not consider this a disadvantage.

For simplicity assume that the cake is a convex 2-dimensional object, though the procedure can be extended to more general geometric settings. When all value-densities are positive, the procedure works as follows.

Initially each agent marks a line parallel to the *y* axis, such that the cake to the left of its line equals exactly 1/3 by this agent’s valuation. The leftmost mark is selected; suppose this mark belongs to Alice. Alice receives the piece to the left of her mark, and the remainder has to be divided among Bob and Carl.

Alice places a knife that divides the remainder to two pieces equal in her eyes. She rotates the knife slowly such that the two pieces at the two sides of the knife remain equal (this is possible to do for every angle). By the intermediate value theorem, there exists an angle such that Bob thinks that the two pieces at the two sides of the knife are equal too. At this point, Bob shouts “stop”, the cake is cut, Carl picks the piece he prefers and Bob receives the last remaining piece.

For Alice, all three pieces have the same value, so she does not envy anyone; this is true even with mixed valuations. For Bob and Carl, the division of the remainder is like cut-and-choose so they do not envy each other; this too is true even with mixed valuations. When the valuations are positive, both Bob and Carl do not envy Alice, since her piece is contained in their leftmost 1/3 pieces so it is worth for them less than 1/3. However, this claim is true only when their value-densities are positive.

The procedure can be adapted to the case in which all value-densities are weakly-negative: in the first step, the rightmost mark is selected instead of the leftmost one. However, with mixed valuations this adaptation does not work either. For example, suppose that the cake is piecewise-homogeneous with 4 homogeneous parts, and the agents’ values to these parts are:

|   | Alice | Bob  | Carl |
|---|------|------|------|
| 1 | -1   | 1    | 3    |
| 2 | 2    | -2   | -2   |
| 3 | 2    | -4   | -2   |

For all agents, the entire cake is worth $-3$, so in the first step, each agent marks a line such that the cake to its left is worth $-1$. Thus Alice’s mark is after the
first slice to the left, Bob’s mark is after the second slice and Carl’s mark is after the third slice. Then:

- If Alice receives the piece to the left of her mark, then Bob might envy her even if he gets a half of the remainder;

- If Bob receives the piece to the left of his mark, then Carl might envy him even if he gets a half of the remainder;

- If Carl receives the piece to the left of his mark, then Alice might envy him even if she gets a half of the remainder.

So the procedure cannot be adapted, at least not in a straightforward way.

A.2 Two-moving-knives procedure

This procedure of Barbanel and Brams (2004) works also for a cake of one or more dimensions (to guarantee that the pieces are connected, it should be assumed that the cake is convex; all knives and cuts are parallel).

The first step is the same as in the rotating-knife procedure: the agents mark their 1/3 line and the leftmost mark is selected; suppose this mark belongs to Alice. In the second step, Alice divides the remainder to two pieces equal in her eyes. Then there are three cases:

- If Bob prefers the middle piece and Carl the right piece or vice versa, then each of them gets his preferred piece and Alice gets the leftmost piece.

- If both Bob and Carl prefer the middle piece, then Alice holds two knives at the two ends of the middle piece and moves them inwards, keeping the two external pieces equal in her eyes. When either Bob or Carl believes that the middle piece is equal to one of the external pieces, he shouts “stop” and takes that external piece. The non-shouter takes the middle piece and Alice takes the other external piece.

- If both Bob and Carl prefer the rightmost piece, then Alice holds two knives at the two ends of the middle piece and moves them rightwards, keeping the two leftmost pieces equal in her eyes; then the procedure proceeds as in the previous case.

When all valuations are positive, these are the only possible cases, since both Bob and Carl believe that Alice’s piece is worth at most 1/3. The procedure can be adapted to the case of all-negative valuations, by putting the two knives in the hand of the rightmost cutter. However, this adaptation does not work with mixed valuations, as shown by the example in the previous subsection.

A.3 Four-moving-knives procedure

This procedure of Stromquist (1980) was the first procedure for connected envy-free division. It requires a “sword” moved by a referee, and three knives moved
simultaneously by the three agents. It works for a convex cake in one or more dimensions; again all knives and cuts are parallel.

The sword moves constantly from the left end of the cake to its right end. Each agent holds his knife in a point that divides the cake to the right of the sword to two pieces equal in his eyes. The first agent that thinks that the leftmost piece is sufficiently valuable (equal to the piece at the left/right of the middle knife) shouts “stop” and receives the leftmost piece. Then, the middle knife cuts the remainder and each of the non-shouters gets a piece that contains its knife.

The correctness of this procedure depends on the assumption that the non-shouters will not envy the shouter (since otherwise they should have shouted earlier). However, this is true only if the piece to the left of the sword grows monotonically as the sword moves rightwards. When the valuations are mixed, the monotonicity breaks, and with it, the no-envy guarantee.

A.4 Approximation algorithms

For additive agents, Brânzei and Nisan (2017) present a general procedure for finding an \( \epsilon \)-approximation for any condition described by linear constraints. Whenever there exists an allocation that satisfies such a condition, their algorithm finds an allocation in which the value of each agent is at most \( \epsilon \) less than its required value. In particular, whenever an envy-free allocation exists, their algorithm finds an allocation in which each agent values its piece as at most \( \epsilon \) less than the piece of any other agent (the agents’ valuations are normalized such that the entire cake-value is 1 for all agents, so \( \epsilon \) is a fraction, e.g., 0.01 of the entire cake value). Their algorithm works as follows:

1. Each agent makes several marks on the cake, such that its value for the piece between each two consecutive marks is at most \( \epsilon \).

2. The algorithm checks all combinations of \( n - 1 \) marks; each such combination defines a connected division. If an envy-free division exists, then necessarily one of the checked divisions represents an \( \epsilon \)-envy-free division.

This algorithm works well for mixed cakes. In particular, for \( n = 3 \), an envy-free allocation exists, so an \( \epsilon \)-envy-free allocation will be found by the above procedure.

However, there is a “catch”. When the cake is good, the number of queries required is \( O(n/\epsilon) \), since each agent has to make \( O(1/\epsilon) \) marks. This is also true when the cake is bad; in this case, the values between each two consecutive marks will be \( -\epsilon \). However, when the cake is mixed, the number of marks might be arbitrarily large: each agent might have an unbounded number of \(+\epsilon\) and \(-\epsilon\) pieces.

For the case \( n = 3 \), Brânzei and Nisan (2017) present an improved approximation algorithm that finds an \( \epsilon \)-envy-free allocation in \( O(\log(1/\epsilon)) \) queries. However, this algorithm approximates the Barbanel-Brams two-knives procedure, which does not work with mixed cakes (see above).
Therefore, the query complexity of finding an $\epsilon$-envy-free allocation in a mixed cake is still wide open.

## B Envy-freeness with disconnected pieces

Without the connectivity requirement, more options for envy-free division are available.

### B.1 Exactly-equal and nearly-exactly-equal divisions

It is possible to divide the cake to pieces each of which is worth exactly $V_i(C)/n$ for every agent $i$. Such a division is envy-free whether the valuations are positive, negative or mixed. The existence of such partitions was proved by Dubins and Spanier (1961); later, Alon (1987) proved it can be done with a bounded number of cuts. However, this number is still much larger than $n-1$, so the pieces will not necessarily be connected. In fact, Alon (1987) proves that their number of cuts is optimal, so it is impossible to have an exactly-equal division with connected pieces, even with positive valuations, let alone mixed valuations.

Even without connectivity, it is impossible to find an exactly-equal division with a finite number of queries. The algorithm of Robertson and Webb (1998) uses a finite number of queries to find a nearly-exactly-equal division, which is also envy-free. At first glance, it seems this algorithm should work for mixed valuations too, but the details require more work.

### B.2 Trimming and enlarging

The first algorithm for envy-free division for three agents was devised by Selfridge and Conway (Brams and Taylor, 1996, pages 116-120). It introduced the idea of trimming. Let Alice cut the cake to three pieces equal in her eyes. Then ask Bob and Carl which piece they prefer. If they prefer different pieces then we are done. If the prefer the same piece, then let Bob trim this best piece so that it’s equal to his second-best piece. Now, Carl takes any piece he wants, Bob takes one of his two best pieces (at least one of these remains on the table), and Alice takes one of her three original pieces (at least one of these remains on the table). We have an envy-free division of a part of the cake; the trimmings remain on the table and are divided by a second step, which we skip here for brevity.

The idea of trimming a best piece to make it equal to the second-best piece lies in the heart of more sophisticated algorithms for $n$ agents, such as Brams and Taylor (1995) and Aziz and Mackenzie (2016). This idea crucially relies on all valuations being positive, so that trimming a piece makes it weakly less valuable for all agents.

When all valuations are negative, the analogue of trimming is enlarging — the smallest piece should be enlarged to make it equal to the second-smallest; however, it is not immediately clear how this enlargement can be done — where
should the extra cake come from? The first solution was devised by Reza Oskui (Robertson and Webb, 1998, pages 73-75) for three agents. The idea of enlarging pieces was further developed by Peterson and Su (2009), who presented an algorithm for $n$ agents. Their algorithm is discrete and requires a finite, but unbounded, number of queries.

When valuations are mixed, trimming or enlarging a piece can make it better for some players and worse for some other players. Therefore, it is not clear how any of these procedures can be adapted.

B.3 Dividing positive and negative parts separately

There is another simple trick that can be used when there is no connectivity requirement. The idea is to divide the cake to sub-cakes of two types:

1. Sub-cakes whose value is positive for at least one agent;
2. Sub-cakes whose value is negative for all agents.

Sub-cakes of the first kind should be divided among the agents who value them positively, using any algorithm for envy-free division with positive valuations; sub-cakes of the second kind should be divided among all $n$ agents, using any algorithm for envy-free division with negative valuations.

This algorithm can be done in finite time if-and-only-if, for each agent, the cake can be divided to a finite number of pieces, each of which is entirely-positive or entirely-negative (in other words, the number of switches between positive and negative value-density is finite for every agent).

Even with this condition, the algorithm does not fit the standard Robertson-Webb query model. This model allows to ask an agent to mark a piece of cake having a certain value, but there is no query of the form “mark the cake at a point where your value switches between negative and positive”.

Still, in practice this algorithm seems like the most reasonable alternative: it does not make sense to give a cake to an agent who thinks it is bad, when other agents think it is good.

C The Degree Lemma implies Sperner’s Lemma

Lemma C.1. If a labeling $L$ satisfies Sperner’s boundary condition, then for every face $Q’ \subseteq Q$:

$$|\deg_\partial(L, Q’)| = 1$$

Together with the Degree Lemma (Sub. 4.3), this lemma implies that, when $L$ satisfies Sperner’s boundary condition, $|\deg(L)| = 1$. This implies that the number of fully-labeled simplices must be odd — which is exactly Sperner’s lemma.
Proof of Lemma C.1. The proof is by induction on $n$ — the number of vertices in the main simplices $P$ and $Q$. W.l.o.g, let $Q'$ be the face $Q_1 \ldots Q_{n-1}$.

**Base:** $n = 3$. Sperner’s condition implies that only one face of $P$ is labeled by both 1 and 2; all nonzero contributions to $\deg_\partial(L, Q')$ come from this face. One end of this face is labeled by 1 and the other end by 2, so the number of 12 edges is one plus the number of 21 edges. Therefore, $|\deg_\partial(L, Q')| = 1$.

**Step:** we assume the lemma is true for $n-1$ and prove for $n$. Again, Sperner’s condition implies that only one face of $P$, say $P'$, is labeled by all the labels $1, \ldots, n-1$. All nonzero contributions to $\deg_\partial(L, Q')$ come from $P'$. Let $L'$ be the restriction of $L$ to the face $P'$. Now, $P'$ is an $n-2$-dimensional simplex labeled by a labeling $L'$ that satisfies Sperner’s condition. By the induction assumption, the boundary degree of $L'$ is $\pm 1$. Therefore, by the Degree Lemma, $|\deg(L')| = 1$. But $\deg_\partial(L, Q')$ exactly equals $\deg(L')$, so:

$$|\deg_\partial(L, Q')| = 1$$

which verifies the induction assumption. \(\square\)

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