Second-order differential equations for bosons with spin $j \geq 1$ and in the bases of general tensor-spinors of rank $2j$

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Abstract. A boson of spin $j \geq 1$ can be described in one of the possibilities within the Bargmann-Wigner framework by means of one sole differential equation of order twice the spin, which however is known to be inconsistent as it allows for non-local, ghost and acausally propagating solutions, all problems which are difficult to tackle. The other possibility is provided by the Fierz-Pauli framework which is based on the more comfortable to deal with second-order Klein-Gordon equation, but it needs to be supplemented by an auxiliary condition. Although the latter formalism avoids some of the pathologies of the high-order equations, it still remains plagued by some inconsistencies such as the acausal propagation of the wave fronts of the (classical) solutions within an electromagnetic environment. We here suggest a method alternative to the above two that combines their advantages while avoiding the related difficulties. Namely, we suggest one sole strictly $D(j,0) \oplus (0,j)$ representation specific second-order differential equation, which is derivable from a Lagrangian and whose solutions do not violate causality. The equation under discussion presents itself as the product of the Klein-Gordon operator with a momentum-independent projector on Lorentz irreducible representation spaces constructed from one of the Casimir invariants of the spin-Lorentz group. The basis used is that of general tensor-spinors of rank $2j$.

1 Introduction

High-spin $j \geq 1$ fields, both massive and massless, have always been among the principal topics in field theories. In particle physics they are needed in the description of the reported hadron resonances whose spins vary from 1/2 to 17/2 for baryons, and from zero to six for mesons. In gravity, bosons of higher spins can couple to the metric tensor and cause its deformation [1], and are besides this in demand in the physics of rotating black holes [2]. Gravitational interactions among high-spin fermions are also under discussion [3]. Various approaches to high-spin fields in general and bosonic in particular have been developed over the years (see [4] for a recent review, and [5] for a standard textbook), the Fierz-Pauli (FP) [6] and the Bargmann-Wigner (BW) frameworks [7] counting among them. In the following we briefly highlight these two methods for the sake of self consistency of the presentation, we comment on their problems, and suggest an alternative approach to high-spin description based on one sole strictly representation specific second-order differential equation which is free from inconsistencies.

1.1 General tensor-spinors and their restrictions within the Bargmann-Wigner and Fierz-Pauli frameworks for spin $j \geq 1$

The concepts underlying the two methods under discussion are most transparently presented within the framework of spinor calculus [8], relevant for gravity in the spinor form. Specifically, both methods depart from general tensor-spinors, here denoted by \((S^r_s)_{\alpha_1 \ldots \alpha_r}^{\beta_1 \ldots \beta_s}\), and defined as products of $r$ spinor components, \((\xi_{\alpha_i})\), with $s$ co-spinors components, \((\eta_{\beta_k})\) according to

\[
(S^r_s)_{\alpha_1 \ldots \alpha_r}^{\beta_1 \ldots \beta_s} = \xi_{\alpha_1}^{(1)} \ldots \xi_{\alpha_r}^{(r)} \eta_{\beta_1}^{(1)} \ldots \eta_{\beta_s}^{(s)}.
\]
Here the marks in the parentheses on top of the spinor components enumerate the different $sl(2,C)$ spinors to which they belong. For the sake of not overloading the notations, these marks will be suppressed in the following. The Bargmann-Wigner (BW) method specifically singles out the tensor-spinors, $S^n_0$ and $S^n_r$, namely, 

$$\text{BW: } (S^n_0)^{\alpha_1 \ldots \alpha_n} = \xi^{\alpha_1} \ldots \xi^{\alpha_n}, \quad (S^n_r)_{\beta_1 \ldots \beta_n}^{\bullet \ldots \bullet} = \eta_{\beta_1}^{\bullet} \ldots \eta_{\beta_n}^{\bullet},$$

embedded within the general direct product,

$$S^r_s \simeq \otimes_{i=1}^{i=n} \left[ D^{(1/2,0) \oplus (0,1/2)} \right]_i, \quad D^{(1/2,0) \oplus (0,1/2)} = \left( \begin{array}{c} \xi^1 \\ \xi^2 \\ \eta_1^\bullet \\ \eta_2^\bullet \end{array} \right), \quad r + s = n,$$

of $n$ four-component spinors, $D^{(1/2,0) \oplus (0,1/2)}$. Instead, the Fierz-Pauli framework is based on rank-$2n$ tensor-spinors, $S^n_0$, (subsequently denoted by $\Phi$) which have an equal number of undotted (spinor) and dotted (co-spinor) indexes, namely,

$$\text{FP: } \Phi^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} = \xi^{\alpha_1} \eta_{\beta_1}^{\bullet} \ldots \xi^{\alpha_n} \eta_{\beta_n}^{\bullet}.$$  

They emerge as the direct product of $n$ four-vectors, $D^{(1/2,1/2)}$, as

$$\text{BW: } \Phi \simeq \otimes_{i=1}^{i=n} \left[ D^{(1/2,1/2)} \right]_i, \quad D^{(1/2,1/2)} = \left( \begin{array}{c} \xi^1 \eta_1^{\bullet} \\ \xi^1 \eta_2^{\bullet} \\ \xi^2 \eta_1^{\bullet} \\ \xi^2 \eta_2^{\bullet} \end{array} \right).$$

The $S^r_s$ and $\Phi$ spaces in the respective eqs. (3) and (5) decompose into irreducible Lorentz group representation spaces as

$$\text{BW: } \otimes_{i=1}^{i=n} \left[ D^{(1/2,0) \oplus (0,1/2)} \right]_i = D^{(0,0)} \oplus \ldots \oplus D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})} \oplus \ldots \oplus D^{(\frac{1}{2},-\frac{1}{2})} \oplus D^{(-\frac{1}{2},\frac{1}{2})} \oplus \ldots \oplus D^{(m\frac{1}{2},m\frac{1}{2})} \oplus \ldots, \quad m \leq \frac{n}{4},$$

for the BW method, and as

$$\text{FP: } \otimes_{i=1}^{i=n} \left[ D^{(1/2,1/2)} \right]_i = D^{(0,0)} \oplus \ldots \oplus D^{(0,\frac{1}{2})} \oplus D^{(\frac{1}{2},0)} \oplus \ldots \oplus D^{(\frac{1}{2},\frac{1}{2}),}$$

for the FP method. Equation (6) shows that there are several irreducible Lorentz group representation spaces of different dimensionality which contain the highest possible spin $j = n/2$, such as the multiple-spin $D^{(n/4,n/4)}$, the two-spin $D^{(1/2,(n-1)/2) \oplus ((n-1)/2,1/2)}$, and the pure spin $D^{(n/2,0) \oplus (0,n/2)}$-spaces, the latter being of main interest for the BW method. On the other side, the highest possible spin in (7) is twice as big, given the fact that the spinor-tensor is of rank $2n$. It is considered as the highest spin $j = n$ in the $D^{(\frac{1}{2},\frac{1}{2})}$ irreducible sector of $\Phi$, the former containing multiple spins $j$ which fall within the range of $j \in [0,n]$.

In order to identify $D^{(\frac{1}{2},\frac{1}{2}) \oplus (0,\frac{1}{2})}$ in (6), the Bargmann-Wigner approach follows the following strategy:

- It constructs the four-spinor of rank $n$, here denoted by $\psi^{(n)}$ as

$$\psi^{(n)} = \left( S^n_0 \right) \quad \left( S^n_0 \right)^* = \left( \begin{array}{c} \psi_{1\alpha_2 \ldots \alpha_n} \\ \psi_{2\alpha_2 \ldots \alpha_n} \\ \psi_{1\beta_2 \ldots \beta_n} \\ \psi_{2\beta_2 \ldots \beta_n} \end{array} \right),$$

- It confines to symmetric spinor indexes alone according to

$$\text{Sym } \psi^{\alpha_1 \ldots (\alpha_r \ldots \alpha_e) \ldots \alpha_n} = \text{Sym } \psi^{\alpha_1 \ldots (\alpha_r \ldots \alpha_e) \ldots \alpha_n},$$

$$\text{Sym } \psi_{\beta_1 \ldots (\beta_r \ldots \beta_e) \ldots \beta_n} = \text{Sym } \psi_{\beta_1 \ldots (\beta_r \ldots \beta_e) \ldots \beta_n}.$$
It constructs the dynamics by requiring the symmetrized rank-\(n\) spinor to be eigenstate \(\forall i\) to the covariant projector, \(\Pi^{BW}(\partial)\), defined as

\[
\Pi^{BW}(\partial^{2j}) = \otimes_{r=1}^{\infty} \left[ \pi(\partial) \right]_r ,
\]

\[
\pi(\partial) = \frac{iD + m}{2m} ,
\]

with \(D\) being defined as

\[
D = \begin{pmatrix}
0 & 0 & \partial_{11} \partial_{12} \\
0 & 0 & \partial_{11} \partial_{22} \\
\partial_{12} \partial_{12} & 0 & 0 \\
\partial_{21} \partial_{12} & 0 & 0
\end{pmatrix} ,
\]

where

\[
\begin{pmatrix}
\partial_{11} \\
\partial_{12} \\
\partial_{21} \\
\partial_{22}
\end{pmatrix} = \partial^0 \sigma_0 + \nabla \cdot \vec{\sigma},
\]

\[
\begin{pmatrix}
\partial_{11} \\
\partial_{12} \\
\partial_{21} \\
\partial_{22}
\end{pmatrix} = \partial^0 \sigma_0 - \nabla \cdot \vec{\sigma}^T ,
\]

\[
\sigma_0 = \begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix} ,
\]

which ensures that the spinor under discussion satisfies the Dirac equation \(\forall i\) as:

\[
D \begin{pmatrix}
\xi^1 \\
\xi^2 \\
\eta^1 \\
\eta^2
\end{pmatrix} = -im \begin{pmatrix}
\xi^1 \\
\xi^2 \\
\eta^1 \\
\eta^2
\end{pmatrix} .
\]

Stated differently, the Bargmann-Wigner framework [7] for the description of particles with any spin \(j\) (preceded by a work of de Broglie [9] for spin 1) is based on the general principle for constructing irreducible representation spaces of the spin-Lorentz group (acting on the internal spin degrees of freedom alone) as parts of multiple direct products of its fundamental four-component spinor, \(\xi \oplus \eta\), the direct sum of a spinor, \(\xi\), and co-spinor, \(\eta\). The BW projector imposes the Dirac equation (and therefore the on-mass shell condition) on each one of the spinor components in the product, a reason for which it becomes of the order \(\partial^0 \equiv \partial^{2j}\) in the derivatives, as indicated in the parentheses of its notation. With the aid of the explicit expressions of \(\partial^{a\dagger}\) and \(\partial^{a}\) in (12) one finds the Bargmann-Wigner framework expressed in terms of the following two coupled high-order differential equations:

\[
j = n/2 \in D^{(\frac{j}{2}, 0)}(0, \frac{j}{2}) ; \quad \left( \partial^{a_1 \beta_1} \otimes \ldots \otimes \partial^{a_n \beta_n} \right) \text{Sym} \psi^{\text{a}1\ldots\text{a}n}_{\beta1\ldots\betan} = (-im)^n \text{Sym} \psi^{\text{a}1\ldots\text{a}n}_{\beta1\ldots\betan} ,
\]

\[
\left( \partial^{a_1 \beta_1} \otimes \ldots \otimes \partial^{a_n \beta_n} \right) \text{Sym} \psi^{\text{a}1\ldots\text{a}n}_{\beta1\ldots\betan} = (-im)^n \text{Sym} \psi^{\text{a}1\ldots\text{a}n}_{\beta1\ldots\betan} .
\]

It can be shown that the BW scheme selects precisely \(2(2j+1)\) degrees of freedom, as required for the description of the two representation spaces, \(D^{(j, 0)}\) and \(D^{(0, j)}\). In this fashion, spin \(j = n/2\) is described within the BW framework by means of a totally symmetric rank-\(2j\) four-component spinor and in terms of the higher-order differential equations in (14) and (15). However, as remarked above, such equations present serious difficulties in so far as they allow for unphysical non-local and ghost solutions [10, 11] which need a special effort to be excluded. A partial remedy to these problems is provided by the Fierz-Pauli framework in [6], according to which the \(\Phi\) tensor in (7) is taken as traceless and symmetric with respect to the \((\xi^i, \beta^j)\) pairs of spinor indexes (each pair being equivalent to a Lorentz index)

\[
\text{FP} : \quad j = n \in D^{(\frac{j}{2}, \frac{j}{2})} ; \quad \text{Sym} \Phi^{\text{a}1\ldots\text{a}n}_{\beta1\ldots\betan} = \text{Sym} \Phi^{\text{a}1\ldots\text{a}n}_{\beta1\ldots\betan} ,
\]

\[
\text{tr Sym} \Phi^{\text{a}1\ldots\text{a}n}_{\beta1\ldots\betan} = 0 ,
\]

and conditioned by

\[
\partial^{a_i \beta_i} \text{Sym} \Phi^{\text{a}1\ldots\text{a}n}_{\beta1\ldots\betan} = 0 .
\]
The latter equation acts as an auxiliary condition to the dynamics introduced by setting the \( \Phi \) tensor-spinor on its mass shell,

\[
(\partial^2 + m^2) \text{Sym}\frac{\phi^{\alpha_1 \ldots \alpha_n}}{\beta_1 \ldots \beta_n} = 0.
\]

(19)

Though the FP approach [6] circumvents some of the inconsistencies typical of high-order theories, such as the Ostrogradskian instability [11], it is still not completely consistent as it does not exclude ghosts and the propagation of its solutions upon coupling to an electromagnetic background can violate causality, this basically because of the violation of the auxiliary condition (18) in the presence of interaction.

To recapitulate, the BW scheme has the advantage to amount to one single wave equation, however of a high order, while the FP method amounts to a second-order equation but invokes an auxiliary condition, difficult to tackle upon gauging. The goal of the present work is to combine the advantages of both approaches and avoid their difficulties. For this purpose, we shall be seeking to construct within the general tensor-spinor basis \( S_j^r \) in (3) with \( r + s = n \), a second-order differential equation invoking, as in the FP approach, the on-mass-shell condition, but in such a way that the equation is \( (j, 0) \oplus (0, j) \) representation specific, free from inconsistencies and, differently from the Fierz-Pauli framework, without auxiliary conditions.

Our case is that the Lorentz group, when acting exclusively on the internal spin degrees of freedom, does indeed provide the adequate tools for the realization of such a program. Below we shall see that in working in the whole space of the \( S_j^r \) tensor-spinors in (3), dropping the conditions on the on-mass-shellness of the “constituent” spinors and the symmetry of the spinor indexes in (9), the uncomfortable high \( \partial^2 \) order of the related differential wave equations of the BW scheme is replaced by a second-order equation but without any need for auxiliary conditions.

However, before turning to the next section we wish to note that the Bargmann-Wigner approach is supposed to be consistent with that developed by Joos and Weinberg in [12,13], respectively, and in which the single spin-\( j \) is described in terms of a set of the even number of \( 2(2j + 1) \) functions, constituting a so called “bi-vector”, as

\[
\psi^{(j)} = \begin{pmatrix}
\psi_1^{(j)} \\
\vdots \\
\psi_{2j+1}^{(j)} \\
\psi_{2(2j+1)+1}^{(j)} \\
\vdots
\end{pmatrix} \simeq (j, 0) \oplus (0, j).
\]

(20)

The \( 2(2j + 1) \)-component wave function \( \psi^{(j)} \) satisfies a differential matrix equation of the same high order as the Bargmann-Wigner equation, given by,

\[
(i^{2j} [\gamma_{\mu_1 \mu_2 \ldots \mu_{2j}}]_{AB} \partial^{\mu_1} \partial^{\mu_2} \ldots \partial^{\mu_{2j}} - m^{2j} \delta_{AB}) \psi_B^{(j)}(x) = 0, \quad B \in [1, 2(2j + 1)].
\]

(21)

Here, \( [\gamma_{\mu_1 \mu_2 \ldots \mu_{2j}}]_{AB} \) are the elements of the generalized Dirac Hermitian matrices of dimensionality \( [2(2j + 1)] \times [2(2j + 1)] \), which transform as Lorentz tensors of rank \( 2j \). The complete sets of such matrices have been extensively studied in the literature for the purpose of constructing all possible bi-linear forms of the fields needed in the definitions of the generalized currents, both transitional and diagonal [14]. Notice that the wave equation in (21) emerges as a similarity transformation of the parity operator within the bi-vector space by the corresponding operator of the boost, which is of order \( j \) in the momenta and encodes the frame dependence of the representation space. In consequence, the spin-\( j \) degrees of freedom within the Joos-Weinberg method transform according to strictly irreducible \( (j, 0) \oplus (0, j) \) representation spaces of the Lorentz group algebra. Below we shall bring an example which shows that in reality the Bargmann-Wigner method is not fully consistent with the Joos-Weinberg approach because it does not necessarily guarantee the irreducibility of its predicted \( 2(2j + 1) \) degrees of freedom under Lorentz transformations. We then suggest an upgrade of the BW equations by which the aforementioned inconsistency is removed.

### 1.2 The algebra of the spin-Lorentz group, its Casimir invariants, and momentum-independent Lorentz projectors on irreducible representation spaces

The Lorentz group transforming the internal spin degrees of freedom, henceforth termed to as spin-Lorentz group, and denoted by \( \mathcal{L} \), is a subgroup of the complete Lorentz group, which acts besides on the spin also on the external spacetime. The \( \mathcal{L} \) generators, denoted by \( S_{\mu \nu} \), are quadratic \( d \times d \) constant matrices, where \( d \) fixes the finite dimensionality of the internal representation space, and encodes the spin value. For the special case of a pure spin, dimensionality
and spin are related as \( d = 2(2j + 1) \), while for representations of multiple spins, relations like \( d = \sum_i (2j_i + 1) \), or \( d = 2 \sum_i (2j_i + 1) \), can hold valid. Now, the direct product of \( \mathcal{L} \) and \( T_4 \), the group of translations in the external space-time, whose generators, \( i\partial_\mu \), represent the quantum mechanical operators, \( P_\mu \), of the components of the relativistic four-momentum, i.e. \( P_\mu = i\partial_\mu \), is generated by the following sub-set of the Poincaré algebra:

\[
\mathcal{L} : \quad [S_{\mu\nu}, S_{\rho\sigma}] = i(g_{\mu\rho}S_{\nu\sigma} - g_{\mu\sigma}S_{\nu\rho} + g_{\nu\rho}S_{\mu\sigma} - g_{\nu\sigma}S_{\mu\rho}),
\]

(22)

\[
T_4 : \quad [P_\mu, P_\lambda] = 0,
\]

(23)

\[
[S_{\mu\nu}, P_\lambda] = 0,
\]

(24)
to be termed as the “inhomogeneous spin-Lorentz group”. In contrast, within the algebra of the full Poincaré group (also termed to as “inhomogeneous Lorentz group”) the commutators in (24) are non-vanishing because there the Lorentz group generators in the internal spin space are supplemented by the angular momentum and boost operators, \( L_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \), which transform the external space-time, and which do not commute with the operators of translations. The algebra of the spin-Lorentz group, given in (22), has two Casimir invariants \([15]\), here denoted in their turn by \( F \) and \( G \) and defined as

\[
F_{AB} = \frac{1}{4} [S_{\mu\nu}]_{AB} [S_{\mu\nu}]_{DB},
\]

(25)

\[
G_{AB} = \frac{1}{8} \epsilon_{\mu\nu\alpha\beta} [S^{\mu\nu}]_{AC} [S_{\alpha\beta}]_{CB}, \quad A, B, C, D, \ldots = 1, \ldots, d.
\]

(26)

Due to the constancy of the quadratic \( d \times d \) dimensional matrices \( S_{\mu\nu} \), the \( F \) and \( G \) operators commute both with the generators of translation, \( P_\mu \), and with \( P^2 \), one of the Casimir invariants of the algebra of the full Poincaré group. In fact, the Casimir invariants of the algebra of the spin-Lorentz group, behave as invariants of the algebra, \( T_4 \times \mathcal{L} \), which generates the “inhomogeneous spin-Lorentz group”. This is a remarkable property which we will put at work in what follows. The two operators in (25) and (26), have the property of unambiguously identifying any irreducible finite-dimensional \( \mathcal{L} \) group representation space, here generically denoted by \( D^{(j_1, j_2)\oplus(j_2, j_1)} \), through their eigenvalues according to

\[
F D^{(j_1, j_2)\oplus(j_2, j_1)} = c_{(j_1, j_2)} D^{(j_1, j_2)\oplus(j_2, j_1)},
\]

(27)

\[
c_{(j_1, j_2)} = \frac{1}{2} \left( K(K + 2) + M^2 \right), \quad K = j_1 + j_2, \quad M = |j_1 - j_2|,
\]

(28)

\[
G D^{(j_1, j_2)} = r_{(j_1, j_2)} D^{(j_1, j_2)}, \quad G D^{(j_2, j_1)} = r_{(j_2, j_1)} D^{(j_2, j_1)},
\]

(29)

\[
r_{(j_1, j_2)} = -r_{(j_2, j_1)} = i(K + 1)M.
\]

(30)

The idea of the present work, exposed in the next section, is to employ the Casimir invariant \( F \) in the construction of a momentum-independent (static) projector on the irreducible sectors of the \( n = 2j \)-rank spinor in (6) and to explore the consequences. The article closes with brief conclusions and has one appendix, devoted to the Lagrangian description and the coupling to an electromagnetic field.

## 2 Momentum-independent projector on the \( D^{(n/2, 0)\oplus(0, n/2)} \) irreducible sector of the rank-\( (2j) \) tensor-spinors. The spin-Lorentz group projector method

We here are specifically interested in projectors on the irreducible Lorentz representations appearing in the rhs of eq. (6) which contain spin \( j = n/2 \). The first projector we wish to consider, here denoted by \( \mathcal{P}^{(n/2, 0)} \), is the one that identifies the \( D^{(n/2, 0)\oplus(0, n/2)} \) irrep. We construct it from \( F \) in (27) as

\[
\mathcal{P}^{(n/2, 0)} = \Pi_{kl} \times \left( \frac{F - c_{(j_k, j_l)}}{c_{(n/2, 0)} - c_{(j_k, j_l)}} \right),
\]

(31)

where, \( \Pi_{kl} \times \) denotes the ordinary product of the operators in parenthesis, the pairs of indexes \((k, l)\) run over all the \( D^{(j_k, j_l)} \) labels characterizing the irreducible representation spaces in the rhs of (6), and the constants \( c_{(j_k, j_l)} \) are those defined in (28). Equation (31) shows that the operator \( \mathcal{P}^{(n/2, 0)} \) has the property to nullify any irreducible representation space for which \( (j_k, j_l) \neq (n/2, 0) \). Instead, for \( (j_k, j_l) = (n/2, 0) \), it acts as the identity operator, meaning that \( \mathcal{P}^{(n/2, 0)} \) is a projector on \( D^{(n/2, 0)\oplus(0, n/2)} \). It is obvious that \( \mathcal{P}^{(n/2, 0)} \) is of zeroth order in the momenta (the derivatives). That is possible due to the direct product character \( T_4 \times \mathcal{L} \) of the symmetry group. Indeed, the spin-Lorentz group \( \mathcal{L} \), in factorizing from the group of translations, and in exclusively acting on the internal space, does not
need external transformations for the identification of its irreducible degrees of freedom. Furthermore, it is sufficient to require the \( D^{(n/2,0)\oplus(0,n/2)} \) states to be on their mass shell, and drop the requirement of on-mass-shellness of the “constituent” Dirac spinors. The reward will be a \( D^{(n/2,0)\oplus(0,n/2)} \)-specific second-order differential equation within the general \( S^n \) spinor-tensor basis in (6). In the following we formulate the method of tracking down the \( D^{(j,0)\oplus(0,j)} \) sector in (6) in terms of \( \Psi^{(n/2,0)}_F \), and explore consequences. However, before moving to this issue, we shall switch from the spinor–co-spinor (chiral) to the commonly used Dirac’s parity representation according to

\[
\begin{pmatrix}
\xi^1 \\
\xi^2 \\
\eta^1_2 \\
\eta^2_1
\end{pmatrix} \rightarrow
\begin{pmatrix}
\xi^1 + \eta^1_2 \\
\xi^2 + \eta^2_1 \\
\eta^1_2 - \xi^1 \\
\eta^2_1 - \xi^2
\end{pmatrix} = \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{pmatrix},
\]

(32)

in which case the dotted and undotted indexes are all replaced by regular Dirac indexes, \( a_i = 1, 2, 3, 4 \), for which we shall use small Latin letters. In what follows we systematically use the conventional Dirac spinors in the composition of the spinor-tensors meaning that instead of \( (S^n_s)^{a_1\ldots a_r}_{\beta_1\ldots\beta_s} \) in (3), with \( r + s = n \), we shall be using Dirac-spinor–tensors of rank \( n \), here denoted by \( \Psi^{(n)}_{a_1 a_2 \ldots a_n} \), with \( a_i = 1, 2, 3, 4 \), i.e. the following notational change will be undertaken

\[
(S^n_s)^{a_1\ldots a_r}_{\beta_1\ldots\beta_s} \rightarrow \Psi^{(n)}_{a_1 a_2 \ldots a_n}, \quad a_i = 1, 2, 3, 4, \quad r + s = n.
\]

(33)

2.1 One sole second-order differential wave equation for spin \( j \geq 1 \) in the basis of the tensor-spinor of rank \( 2j \)

The Lorentz group generators, \( [S^{(1)}]_{\mu\nu} \), in the Dirac spinor space, \( D^{(1/2,0)\oplus(0,1/2)} \), are textbook knowledge [15] and read,

\[
[S^{(1)}]_{\mu\nu} = \frac{i}{4} [\gamma_{\mu\nu}, \gamma_0] = \frac{1}{2} \gamma_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3,
\]

(34)

where \( \gamma_{\mu\nu} \) and \( \gamma_\nu \) stand for the standard Dirac matrices. Then, the spin-Lorentz group generators, \( [S^{(n)}]_{\mu\nu} \), in the reducible \( \Psi^{(n)}_{a_1 a_2 \ldots a_n} = \otimes_{i=1}^{i=n} D^{(1/2,0)\oplus(0,1/2)} \) basis in (3), and in the new notations of (33), are calculated following the standard prescription regarding generator construction in product spaces as

\[
[S^{(n)}]_{\mu\nu} = [S^{(1)}]_{\mu\nu} \otimes [\Pi_{i=1}^{i=n-1} \otimes 1] + 1 \otimes [S^{(1)}]_{\mu\nu} \otimes [\Pi_{i=1}^{i=n-2} \otimes 1] + \ldots + [\Pi_{i=1}^{i=n-1} \otimes 1] \otimes [S^{(1)}]_{\mu\nu},
\]

(35)

with \( [S^{(1)}]_{\mu\nu} \) in (34), and \( 1 \) standing for the \( 4 \times 4 \) unit matrix in the Dirac spinor space. In substituting (34) into (35), and then (35) into (31), the \( P_F^{(n/2,0)} \) projector of interest is explicitly constructed. Once again, it is seen that this projector is a static one and does not provide a wave equation. In order to introduce the free kinematics, we impose on the states spanning the \( D^{(n/2,0)\oplus(0,n/2)} \) representation space the indispensable mass shell condition via the Klein-Gordon equation, and find the following master equation:

\[
\text{This work} \quad \left( \partial^\mu \partial_\mu - m^2 \right) \Psi^{(n)}_{a_1 a_2 \ldots a_n} + m^2 \Psi^{(n)}_{a_1 a_2 \ldots a_n} = 0, \quad n = 2j.
\]

(36)

This is the one sole second-order differential equation which we suggest in place of (i) the high-order Bargmann-Wigner coupled system of equations in (14)–(15), and (ii) the Fierz-Pauli system of equations (18)–(19). As long as within the Bargmann-Wigner framework in (14)–(15), the two \( D^{(n/2,0)} \) and \( D^{(0,n/2)} \) building blocks of the \( (2j + 1) \) spin-\( j = n/2 \) degrees of freedom are not searched at all by the wave operator but are put by hand through the symmetrization of the tensor indexes, they transform irreducibly under \( SU(2) \) but, as we shall see below, their sum is reducible under Lorentz transformations. Instead, the wave operator in our suggested master equation (36), in acting on the complete set of indexes characterizing the Lorentz reducible tensor-spinors, unambiguously identifies in it the Lorentz irreducible \( D^{(n/2,0)\oplus(0,n/2)} \) sector, while neatly cutting out the rest. The Fierz-Pauli method identifies the spin-\( j \) degrees of freedom implicitly and by the aid of the auxiliary conditions, which is not conserved by the gauging procedure, and needs special adjustments. Stated differently, the master equation (36) describes a manifestly genuine and strictly representation-specific eigenvalue problem that is free from auxiliary conditions. It identifies the spin-\( j \) degrees of freedom directly, at once, and unambiguously, a reason for which it can be derived from a Lagrangian, and coupled to the electromagnetic field in the regular way of a minimal gauging (see the appendix).

In this manner, we furnished one single second-order differential wave equation for a particle of spin \( j = n/2 \) which we described in terms of a general (Dirac) spinor-tensor of rank \( 2j \). In the following, we work out for illustrative purposes the case of spin 1 residing in the \( D^{(1,0)\oplus(0,1)} \) irreducible sector of second-rank (Dirac) spinor-tensor, \( \Psi^{(2)} \).
2.2 The wave equation of spin 1 in $\Psi^{(2)}$. An illustrative example

As an illustrative example for our suggested method we consider the simplest case of an integer spin, namely spin 1 as embedded by tensor-spinors of second rank, $\Psi^{(2)}_{a_1a_2} = \psi_{a_1} \tilde{\psi}_{a_2} \simeq D^{(1/2,0)\oplus(0,1/2)} \otimes D^{(1/2,0)\oplus(0,1/2)}$, with $\psi_{a_1}$ defined in (33). The latter tensor is reducible under Lorentz transformations according to

$$\Psi^{(2)}_{a_1a_2} \simeq D^{(1/2,0)\oplus(0,1/2)} \otimes D^{(1/2,0)\oplus(0,1/2)} = D^{(1,0)\oplus(0,1)} \oplus 2D^{(0,0)} \oplus 2D^{(1/2,1/2)},$$  \(37\)

where the integer numbers in front of the irreps stand for their multiplicities in the direct sum. The rhs in (37) contains three different irreducible Lorentz representation spaces, whose eigenvalues with respect to the Casimir invariant $F$ in (27) are calculated from the expressions given in (28) as

$$c_{(0,0)} = 0 \quad \text{for} \quad (0,0),$$ \(38\)

$$c_{(1,0)} = 2 \quad \text{for} \quad (1,0) \oplus (0,1),$$ \(39\)

$$c_{(1/2,1/2)} = \frac{3}{2} \quad \text{for} \quad (1/2,1/2).$$ \(40\)

Therefore, according to (31), the projector $P_F^{(1,0)}$ on $(1,0) \oplus (0,1)$ emerges as

$$P_F^{(1,0)} = \frac{1}{2} (2F^2 - 3F).$$ \(41\)

Next we evaluate the generators within $\Psi^{(2)}_{a_1a_2}$ by the aid of (35) setting $n = 2$, substitute them in (25), and then insert the result for $F$ in (31). In so doing we calculate the following explicit expression for the $F$-Casimir invariant of the spin-Lorentz group algebra:

$$\Psi^{(2)}_{a_1a_2} : \quad F = \frac{1}{16} (1 \otimes \sigma^{\mu\nu} + \sigma^{\mu\nu} \otimes 1) (1 \otimes \sigma^{\mu\nu} + \sigma^{\mu\nu} \otimes 1).$$ \(42\)

In index notation, $F$ reads

$$F^{a_1a_2b_1b_2} = \frac{1}{8} \left[ 12 \delta^{a_1}_{b_1} \delta^{a_2}_{b_2} + (\sigma^{\mu\nu})^{a_1}_{b_1} (\sigma^{\mu\nu})^{a_2}_{b_2} \right].$$ \(43\)

Substitution of (43) into (41) amounts to the following explicit expression for the searched spin 1 Lorentz projector:

$$\left[ P_F^{(1,0)} \right]^{a_1a_2}_{b_1b_2} = \frac{1}{32} (\sigma^{\mu\nu})^{a_1}_{b_1} (\sigma^{\mu\nu})^{a_2}_{b_2} \left[ 12 \delta^{b_1}_{b_1} \delta^{b_2}_{b_2} + (\sigma^{\mu\nu})^{b_1}_{b_1} (\sigma^{\mu\nu})^{b_2}_{b_2} \right]$$

$$= \frac{1}{4} (\sigma^{\mu\nu})^{a_1}_{b_1} (\sigma^{\mu\nu})^{a_2}_{b_2} F^{b_1b_2}_{b_1b_2}.$$ \(44\)

Therefore, according to (36), the second-order differential wave equation for a particle of spin 1 described in terms of a second-rank Dirac spinor takes the following form:

$$\left( \partial_{\mu} \partial^{\mu} \left[ P_F^{(1,0)} \right]^{a_1a_2}_{b_1b_2} + m^2 \right) \Psi^{(2)}_{a_1a_2} = 0.$$ \(45\)

Along this line, any arbitrary spin $j$ can be described by means of a rank-2$j$ (Dirac) spinor-tensor satisfying a second-order differential equation. Below, the solutions to eq. (45), when considered in the momentum space upon the replacement, $i\partial_{\mu} \rightarrow p_{\mu}$, are obtained, and compared to the corresponding solutions appearing in the Bargmann-Wigner framework.

2.3 Comparison of spin-1 predictions following from the respective spin-Lorentz group projection method, and the Bargmann-Wigner framework

A set of linearly independent solutions in momentum space for the spin-1 states satisfying the equation in (45) can be constructed upon applying the Lorentz projector $P_F^{(1,0)}$ in (44) to the sixteen dimensional rank-2 tensor-spinors,


\[ [u_{\pm}(\mathbf{p}, \lambda)]^a [u_{\pm}(\mathbf{p}, \lambda)]^b, \]

as composed by the momentum space Dirac spinors of positive (+) and negative (−) parities,

\[
u_+ (\mathbf{p}, 1/2) \equiv u(\mathbf{p}, 1/2) = \frac{1}{\sqrt{2m(m + p_0)}} \begin{pmatrix} m + p_0 \\ 0 \\ p_3 \\ p_1 + ip_2 \end{pmatrix},
\]

\[
u_+ (\mathbf{p}, -1/2) \equiv u(\mathbf{p}, -1/2) = \frac{1}{\sqrt{2m(m + p_0)}} \begin{pmatrix} m + p_0 \\ 0 \\ p_1 - ip_2 \\ -p_3 \end{pmatrix},
\]

\[
u_- (\mathbf{p}, 1/2) \equiv v(\mathbf{p}, 1/2) = \frac{1}{\sqrt{2m(m + p_0)}} \begin{pmatrix} p_3 \\ p_1 + ip_2 \\ m + p_0 \\ 0 \end{pmatrix},
\]

\[
u_- (\mathbf{p}, -1/2) \equiv v(\mathbf{p}, -1/2) = \frac{1}{\sqrt{2m(m + p_0)}} \begin{pmatrix} p_1 - ip_2 \\ -p_3 \\ 0 \\ m + p_0 \end{pmatrix}.
\] (46)

The sixteen dimensional rank-2 Dirac spinors span the reducible Lorentz representation space \( D^{(1/2,0) \oplus (0,1/2)} \otimes D^{(1/2,0) \oplus (0,1/2)}. \) Executing the proper calculation, we find precisely six linearly independent combinations, as it should be, and list them in table 1 together with the corresponding Bargmann-Wigner spin-1 states, as a comparison.

In order to make the comparison between the two schemes manifest, we write down in the subsequent two equations the explicit rank-2 Dirac spinors following from the present work, \([w_+^{(2)}(\mathbf{p}, \lambda)]^{ab}\), on the one side, and from the Bargmann-Wigner approach, \([W_+^{(2)}(\mathbf{p}, \lambda)]^{ab}\), \([W_{-}^{(2)}(\mathbf{p}, 1)]^{ab}\), on the other, focusing on the particular case of \([w_+^{(2)}(\mathbf{p}, 1)]^{ab}\) and \([W_{+}^{(2)}(\mathbf{p}, 1)]^{ab}\), for the sake of concreteness. In so doing, we find the following normalized states:

\[
\begin{align*}
[w_+^{(2)}(\mathbf{p}, 1)]^{ab} &= \frac{\sqrt{2}}{4m(m + p_0)} \begin{pmatrix}
(m + p_0)^2 + p_3^2 & (p_1 + ip_2)p_3 & 2p_3(m + p_0) & (p_1 + ip_2)(m + p_0) \\
(p_1 + ip_2)p_3 & (p_1 + ip_2)^2 & (p_1 + ip_2)(m + p_0) & 0 \\
2p_3(m + p_0) & (p_1 + ip_2)(m + p_0) & (m + p_0)^2 + p_3^2 & (p_1 + ip_2)p_3 \\
(p_1 + ip_2)(m + p_0) & 0 & (p_1 + ip_2)p_3 & (p_1 + ip_2)^2
\end{pmatrix}
\end{align*}
\] (47)

and

\[
\begin{align*}
[W_+^{(2)}(\mathbf{p}, 1)]^{ab} &= \frac{1}{2m(m + p_0)} \begin{pmatrix}
(m + p_0)^2 & p_3(m + p_0) & (m + p_0)(p_1 + ip_2) & 0 \\
0 & 0 & 0 & 0 \\
0 & p_3(m + p_0) & 0 & p_3(p_1 + ip_2) \\
(p_1 + ip_2)(m + p_0) & 0 & p_3(p_1 + ip_2) & (p_1 + ip_2)^2
\end{pmatrix}.
\end{align*}
\] (48)

An inspection of these equations shows that the tensor calculated in eq. (47) of the present work has the following six independent degrees of freedom:

\[ w_{11} = w_{33}, \quad w_{12} = w_{21} = w_{34} = w_{43}, \quad w_{13} = w_{31}, \]
\[ w_{22} = w_{44}, \quad w_{14} = w_{41} = w_{25} = w_{32}, \quad w_{24} = w_{42}, \] (49)

and so does the Bargmann-Wigner solution in (48), though theirs are distinct from ours. The six independent spin-1 degrees of freedom following from the Bargmann-Wigner framework are:

\[ W_{11}, \quad W_{13} = W_{31}, \quad W_{14} = W_{41}, \]
\[ W_{33}, \quad W_{34} = W_{43}, \quad W_{44}, \] (50)

while the rest of the components are identically vanishing. In addition, upon comparing the tensor-spinor in (47) with the one in (48), we observe differences (modulo normalization factors), between the following four components:

\[ w_{11} \neq W_{11}, \quad w_{22} \neq W_{22}, \quad w_{33} \neq W_{33}, \quad w_{12} \neq W_{12}, \] (51)
Spin-1 Bargmann-Wigner states

\begin{align*}
\left[w_+^{(2)}(p, 1)\right]^{ab} &= \frac{1}{\sqrt{2}} \left( [u_+(p, 1/2)]^a [u_+(p, 1/2)]^b \\
&\quad + [u_-(p, 1/2)]^a [u_-(p, 1/2)]^b \right) \\
\left[w_+^{(2)}(p, 0)\right]^{ab} &= \frac{1}{2} \left( [u_+(p, 1/2)]^a [u_+(p, -1/2)]^b \\
&\quad + [u_+(p, -1/2)]^a [u_+(p, 1/2)]^b \\
&\quad + [u_-(p, 1/2)]^a [u_-(p, -1/2)]^b \\
&\quad + [u_-(p, -1/2)]^a [u_-(p, 1/2)]^b \right) \\
\left[w_+^{(2)}(p, -1)\right]^{ab} &= \frac{1}{\sqrt{2}} \left( [u_+(p, -1/2)]^a [u_+(p, -1/2)]^b \\
&\quad + [u_-(p, -1/2)]^a [u_-(p, -1/2)]^b \right) \\
\left[w_-^{(2)}(p, 1)\right]^{ab} &= \frac{1}{\sqrt{2}} \left( [u_-(p, 1/2)]^a [u_-(p, 1/2)]^b \\
&\quad + [u_-(p, -1/2)]^a [u_-(p, 1/2)]^b \right) \\
\left[w_-^{(2)}(p, 0)\right]^{ab} &= \frac{1}{2} \left( [u_-(p, 1/2)]^a [u_-(p, -1/2)]^b \\
&\quad + [u_-(p, -1/2)]^a [u_-(p, 1/2)]^b \right) \\
\left[w_-^{(2)}(p, -1)\right]^{ab} &= \frac{1}{\sqrt{2}} \left( [u_-(p, -1/2)]^a [u_-(p, -1/2)]^b \\
&\quad + [u_-(p, -1/2)]^a [u_-(p, -1/2)]^b \right)
\end{align*}

while the remaining components are equal. Below these differences are attributed to the reducibility of the BW states under Lorentz transformations. In the last two equations we use “rationalized” notations in which we dropped subscripts and arguments in the \(\left[w_+^{(2)}(p, 1)\right]^{ab}, \left[W_+^{(2)}(p)\right]^{ab}\) spinor components under consideration for the sake of keeping the formulas possibly more transparent.

Before proceeding further we notice that the six degrees of freedom of the \(D^{(1,0)} \otimes (0,1)\) representation space can equivalently be described in terms of the (also six) independent components of the totally anti-symmetric Lorentz tensor of second rank, \(B^{\mu \nu}\), according to

\[D^{(1,0)} \otimes (0,1) \simeq \begin{pmatrix} 0 & B^{01} & B^{02} & B^{03} \\ -B^{01} & 0 & B^{12} & B^{13} \\ -B^{02} & -B^{12} & 0 & B^{23} \\ -B^{03} & -B^{13} & -B^{23} & 0 \end{pmatrix} .\]

In so doing, spin 1 is described in terms of degrees of freedom equipped with Lorentz indexes. The link to the spinor-index notation considered in the present work is established by the following relation:

\[
\begin{pmatrix} f^1_1 f^2_1 \\ f^1_2 f^2_2 \end{pmatrix} = \begin{pmatrix} (E^3 - iH^3) \\ (E^1 - iE^2) + i(H^1 - iH^2) \end{pmatrix} ,
\]

where \(f^2_\alpha\) is a spinor-tensor of second rank in the undotted indexes, while the components of the vector \(E\), and the axial-vector \(H\) fields relate to Lorentz indexes according to

\[E^i = B^{0i}, \quad H^1 = B^{23}, \quad -H^2 = B^{13}, \quad H^3 = B^{12}, \quad i = 1, 2, 3.\]
Obviously, a basis choice does not affect the calculated values of the physical observables. Therefore, embedding (1, 0) ⊕ (0, 1) in particular, and (j, 0) ⊕ (0, j) in general by bases distinct from the tensor-spinors, is a matter of mere convenience and does not affect the physical properties of the particles. In view of this, a spin-1 particle transforming in (1, 0) ⊕ (0, 1) has to be characterized by the same set of physical observables in all approaches to high spins. In the following we present the case where the Bargmann-Wigner spin-1 spinors transform reducibly under Lorentz transformations, while our spinors, in parallel of those of Joos-Weinberg, transform strictly irreducibly, as it should be, if the spin under consideration were to reside entirely in $D^{(1,0)⊕(0,1)}$. For this purpose we construct the Lorentz projector, $P^{(1/2,1/2)}_F$, on the $D^{(1/2,1/2)}$ irreducible sector in the rhs of (6) and let it act on the Bargmann-Wigner spin-1 spinors.

### 2.4 Lorentz reducibility of the spin-1 Bargmann-Wigner states

Without entering into technical details, we limit ourselves to report that the projector of our interest is obtained along the line of the reasoning presented above and its general form is found as

$$P^{(1/2,1/2)}_F = \frac{4}{3}(2F - F^2). \quad (55)$$

In spinor index notation we calculated it as

$$\left[ P^{(1/2,1/2)}_F \right]^{a_1 a_2}_{b_1 b_2} = \delta^{a_1}_{b_1} \delta^{a_2}_{b_2} - \frac{1}{12} (\sigma_{\mu_1 \nu_1})^{a_1}_{c} (\sigma_{\mu_2 \nu_2})^{a_2}_{c} (\sigma_{\mu_3 \nu_3})^{a_3}_{c} \frac{d}{d_b} (\sigma^{\mu_3 \nu_3})^{a_3}_{c} - \frac{2}{12} (\sigma^{\mu \nu})^{a_1}_{b_1} (\sigma^{\mu \nu})^{a_2}_{b_2}. \quad (56)$$

When applied to any of the Bargmann-Wigner $[W_+ (p, \lambda)]^{ab} - [\overline{W}_+ (p, \lambda)]^{ab}$ spinors, non-vanishing projections on $D^{(1/2,1/2)}$ states (not shown here) are obtained. This means that the spin-1 states following from the Bargmann-Wigner framework are linear combinations of spin-1 states residing in the two distinct Lorentz irreducible representation spaces, $D^{(1,0)⊕(0,1)}$ and $D^{(1/2,1/2)}$. In other words, differently from the supposed consistency with the Joos-Weinberg approach (discussed at the end of sect. 1.1), and with our spin-1 degrees of freedom, the junction of the two spin-1 triplets generated by the Bargmann-Wigner approach violates the Lorentz-irreducibility, a shortcoming of serious consequences. Indeed, in order to satisfy Wigner’s definition of an elementary particle at the classical level, its spin degrees of freedom have to transform according to non-unitary finite dimensional irreducible representations of the spin-Lorentz group so that upon quantization, the states of continuous four-momenta could transform according to the infinite dimensional unitary representations of the Poincaré algebra. Within this context, the mixture of irreducible spin-Lorentz group representations has to be considered as unphysical and removed. The reason behind the request on irreducibility is that particles of equal spins, transforming in distinct spin-Lorentz group representation spaces, can have different physical properties. For example, in ref. [16] the electromagnetic multipole moments of particles with spin-1 residing in the four-vector, $(1/2, 1/2)$, on the one side, and within the strictly irreducible Joos-Weinberg spin-1 bi-vector $(1, 0) ⊕ (0, 1)$ (equivalent to our approach according to the line of reasoning around (52)–(54)), on the other side, have been explicitly calculated and compared. There, is was found that the Breit-frame electric quadrupole $(E2)$ moments of the two spin-1 particles under discussion are of equal magnitudes, but of different orientations, amounting to opposite signs. Mixing the two representation spaces will therefore distort the spin-1 particles under discussion are of equal magnitudes, but of different orientations, amounting to opposite signs.

The $(1/2, 1/2)$-component of the spin-1 BW field is easily eliminated upon application of the projector operator in (31), (44) to eqs. (14), (15) according to

$$\left( \partial^{\alpha_1 \beta_1} \otimes \ldots \otimes \partial^{\alpha_n \beta_n} \right) \left[ P^{(j=0)}_F \right]^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} = (im)^n \left[ P^{(j=0)}_F \right]^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} \text{Sym} \psi^{\kappa_1 \ldots \kappa_n},$$

$$\left( \partial^{\alpha_1 \beta_1} \otimes \ldots \otimes \partial^{\alpha_n \beta_n} \right) \left[ P^{(j=0)}_F \right]^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} = (im)^n \left[ P^{(j=0)}_F \right]^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} \text{Sym} \psi^{\kappa_1 \ldots \kappa_n}, \quad (57)$$

and setting $j = 1$. Upon this upgrade, the equivalence between the tensor-spinors following from the Bargmann-Wigner and our spin-Lorentz group projector methods is achieved, and the consistency with the Joos-Weinberg approach is reached. With that, the congruity among the predictions resulting from the three approaches under discussion regarding the physical observables characterizing the spin-$j$ particles transforming in $(j, 0) ⊕ (0, j)$ is warranted.

In the appendix we minimally gauge the equation in (36) and prove its causality, besides presenting the corresponding Lagrangian.
3 Conclusions

In the present work we suggested an approach to the description of any spin \( j \) at the classical field theoretical level and by means of the one sole second-order differential equation (36) operating on the general rank-2\( j \) (Dirac) spinor-tensor, \( \psi^{(n)} \), defined in (6). The key ingredient in (36) is a momentum-independent projector, \([\mathcal{P}^{(n/2,0)}_{F}]\), which we built up in (31) and (35), and by the aid of (34) from the Casimir invariant, \( F \) in (27), of the spin-Lorentz group, according to the algebra in (22) and (24). This projector has the property to unambiguously identify at once and as a whole the irreducible \( D^{(j,0)} \otimes (0,j) \) sector of \( \psi^{(n)} \), while neatly sorting out the rest. The second-order equation (36) is the one which replaces the high-order coupled equations in (14) and (15), on the one side, and the Fierz-Pauli second-order framework in (18) and (19), on the other. In being free from auxiliary conditions, eq. (36) allows for a minimal gauging, is local and causal, thus avoiding the most serious problems of Bargmann-Wigner’s higher-order equations and of the second-order framework by Fierz and Pauli. In being based on the eigenvalue problem of a quadratic invariant form of the relativistic space-time algebra, the suggested method describes a pair of spin-\( j \) particles of opposite parities. Fixing the parity of fermions would require further studies of the conditions for the bi-linearizations of the gauged second-order wave equation. For example, it is well known that the bi-linearization upon gauging of the most general Klein-Gordon operator,

\[
\partial^\mu \partial_{\mu} - g \frac{i \hbar \gamma^\mu}{4} [\gamma_\mu, \gamma_\nu] + m^2, \quad (58)
\]
to the gauged Dirac equation (and its conjugate) is possible only for \( g = 2 \) [17]. However, for bosons, the fields of main interest here, where particles and anti-particles are of equal parities, spin-\( j \) pairs, in particular abundant among light-flavor mesons with masses around and above 2000 MeV, would behave as chiral fields and could be of interest, among others, in strong processes where the chiral symmetry has been restored from the spontaneously broken Goldstone, to the manifest Wigner-Weyl mode.

It is to be noticed that our method can be extended [18] toward the high spins carried by the two-spin–valued representation spaces of the type \( D^{(1/2,(n-1/2)) \otimes ((n-1/2)/2,1/2)} \) in (6), which are beyond the reach of the Bargmann-Wigner method, and in which case the second-order wave equation is not obtained from the mass shell condition alone but from another covariant projector, \( \mathcal{P}^{(j,m)}_{W2} = (-W^2/m^2 - j(j - 1)\mu^2/m^2)/(2j) \), which fixes besides the mass, also the highest of the two spin degrees of freedom. Here, \( W^2 \) is the squared Pauli-Lubanski operator, the second Casimir invariant of the algebra of inhomogeneous spin-Lorentz group. Such a projector technique, however in the bases of Lorentz tensors, has been originated by Aurilia and Umezawa in [19] at the free spin-3/2 particle level, and employed independently in [20] at the interacting level. In the latter work the second-order differential equation resulting from \( \mathcal{P}^{(3/2,m)}_{W2} \) has been extended to include the most general terms allowed by relativity and containing \([\partial_\mu, \partial_\nu] \) commutators. Such terms, identically vanishing at the free particle level, provide upon gauging essential contributions proportional to the electromagnetic field strength tensor, \( F_{\mu\nu} \) and guarantee that the resulting wave equation is free from the Velo-Zwanziger problem [21] for a \( g \) factor taking the value of \( g = 2 \). As long as the Lorentz-tensors can be equivalently re-written to tensor-spinors along the prescription encoded by the above equations (52)–(54), we conclude that the technique used in [19,20] can be transcribed to the tensor-spinor level meaning that the general tensor-spinors considered here can be employed in the description of spin \( j \) residing in \( D^{(1/2,j-1/2)} \otimes (j-1/2,1/2) \). Within the context of this discussion, we believe that the relevance of our approach goes beyond its main advantage over the Bargmann-Wigner framework to correctly identify the strictly Lorentz-irreducible \( D^{(j,0)} \otimes (0,j) \) degrees of freedom, while simultaneously avoiding the high order of the differential wave equations.

Appendix A.

Appendix A.1. Lagrangian formulation and minimal gauging

In order to obtain the master equation in (36) from a Lagrangian we first introduce the tensor-spinor, \( [\mathcal{P}^{(n/2,0)}_{F}]_{a_1 \ldots a_n b_1 \ldots b_n} \), as

\[
[\mathcal{P}^{(n/2,0)}_{F}]_{a_1 \ldots a_n}^{b_1 \ldots b_n} \partial^\mu \partial_\mu = [\mathcal{P}^{(n/2,0)}_{F}]_{a_1 \ldots a_n}^{b_1 \ldots b_n} \partial^2, \quad (A.1)
\]

and write down the master equation as

\[
[\mathcal{P}^{(n/2,0)}_{F}]_{a_1 \ldots a_n}^{b_1 \ldots b_n} \partial^\mu \partial_\mu \psi^{(n)}_{b_1 \ldots b_n} = -m^2 \psi^{(n)}_{a_1 \ldots a_n}, \quad n = 2j. \quad (A.2)
\]

With this definition of \( \mathcal{P}^{(n/2,0)}_{F} \), the free master equation (A.2) can now be derived from the following Lagrangian:

\[
\mathcal{L}^{(n/2,0)}_{free} = \left( \partial^\mu \left[ \bar{\psi}^{(n)} \right]_{a_1 \ldots a_n} \right) \left[ \mathcal{P}^{(n/2,0)}_{F}]_{a_1 \ldots a_n}^{b_1 \ldots b_n} \left( \partial_\nu \left[ \psi^{(n)} \right]_{b_1 \ldots b_n} \right) - m^2 \left[ \bar{\psi}^{(n)} \right]_{a_1 \ldots a_n} \left[ \psi^{(n)} \right]_{a_1 \ldots a_n}. \quad (A.3)
\]
Now the electromagnetic interaction in (36) and (A.3) can be introduced in the regular way through minimal gauging amounting to
\[
\left[ F^{(n/2,0)}_{\mu\nu} \right]_{a_1 \ldots a_n}^{b_1 \ldots b_n} D^\mu D^\nu \tilde{\psi}^{(n)}_{b_1 \ldots b_n} = -m^2 \tilde{\psi}^{(n)}_{a_1 \ldots a_n}, \quad D^\mu = \partial^\mu + i e A^\mu, \tag{A.4}
\]
and
\[
L^{(n/2,0)} = \left( D^\mu \tilde{\psi}^{(n)} \right)^{a_1 \ldots a_n} \left[ F^{(n/2,0)}_{\mu\nu} \right]_{a_1 \ldots a_n}^{b_1 \ldots b_n} \left( D_\nu \tilde{\psi}^{(n)} \right)_{b_1 \ldots b_n} - m^2 \tilde{\psi}^{(n)} \tilde{\psi}^{(n)}_{a_1 \ldots a_n}, \tag{A.5}
\]
respectively, where \( \tilde{\psi}^{(n)}_{b_1 \ldots b_n} \) stands for the gauged solutions. To guarantee that the gauged solutions continue being eigenstates of the Lorentz projector, i.e. that they continue transforming according to the \((j, 0) \oplus (0, j)\) representation space, the tensor-spinor \( \left[ F^{(n/2,0)}_{\mu\nu} \right]_{a_1 \ldots a_n}^{b_1 \ldots b_n} \) has to satisfy the following condition:
\[
\left[ F^{(n/2,0)}_{\mu\nu} \right]_{a_1 \ldots a_n}^{c_1 \ldots c_n} \left[ F^{(n/2,0)}_{\mu\nu} \right]_{c_1 \ldots c_n}^{b_1 \ldots b_n} = \left[ F^{(n/2,0)}_{\mu\nu} \right]_{a_1 \ldots a_n}^{b_1 \ldots b_n}. \tag{A.6}
\]
This condition, however, does not fix \( \left[ F^{(n/2,0)}_{\mu\nu} \right]_{a_1 \ldots a_n}^{b_1 \ldots b_n} \) in an unique way and one is still left with a considerable freedom in the choice of this tensor. In the following subsection we provide the causality and hyperbolicity proof of the gauged equation (A.4) on the grounds of eq. (A.1) alone and without making any particular choice for \( \left[ F^{(n/2,0)}_{\mu\nu} \right]_{a_1 \ldots a_n}^{b_1 \ldots b_n} \).

### Appendix A.2. Causal propagation of the classical wave fronts of the gauged equation

In requiring the gauged solutions of eq. (A.4) to remain in \( D^{(n/2,0) \oplus (0, n/2)} \), we seek to expand them (in any Lorentz frame) in the basis of any complete set spanning this representation space. Specifically for the second-rank tensor, \( \tilde{\psi}^{(2)}_{ab}(x) \), one can choose as such a set the six independent tensors listed in (49), taken in the rest frame, i.e. \([w_{\pm 0}^2(0, \lambda)]_{ab}\). In so doing, the corresponding gauged \( \tilde{\psi}^{(2)}_{ab}(x) \) can be represented as
\[
\tilde{\psi}^{(2)}_{ab}(x) = \sum_{\lambda, \tau} a_{\lambda}(x, \lambda) \left[ w^{(2)}_{\tau}(0, \lambda) \right]_{ab}, \quad \lambda = -1, 0, +1, \quad \tau = +, -, \tag{A.7}
\]
In the case of a rank-\(n\) tensor-spinor, the latter equation generalizes to
\[
\tilde{\psi}^{(n)}_{a_1 \ldots a_n}(x) = \sum_{\lambda, \tau} a_{\lambda}(x, \lambda) \left[ w^{(n)}_{\tau}(0, \lambda) \right]_{a_1 \ldots a_n}, \quad \lambda = \frac{n}{2}, \ldots, \frac{n}{2}, \quad \tau = +, -, \tag{A.8}
\]
where \([w^{(n)}_{\tau}(0, \lambda)]_{a_1 \ldots a_n}\) stands for the \(2(2\frac{n}{2} + 1)\) degrees of freedom spanning the \(D^{(n/2,0) \oplus (0, n/2)}\) representation space at rest. Substituting (A.8) in (A.4), and multiplying from the left by any one of the conjugate states, \([\bar{w}^{(n)}_{\perp}(0, \lambda)]\), we obtain the following system of second-order partial differential equations for the coefficients \(a_{\pm}(x, \lambda)\) in (A.8):
\[
\left[ \bar{w}^{(n)}_{\perp}(0, \lambda') \right]_{a_1 \ldots a_n}^{a_1 \ldots a_n} \left[ F^{(n/2,0)}_{\mu\nu} \right]_{a_1 \ldots a_n}^{b_1 \ldots b_n} D^\mu D^\nu \tilde{\psi}^{(n)}_{b_1 \ldots b_n} = -m^2 a_{\pm}(x, \lambda'). \tag{A.9}
\]
In order to prove the causality and hyperbolicity of (A.9), we employ the Courant-Hilbert criterion [21] which requires one to calculate the characteristic determinant. The latter is found by replacing the highest order derivative by the components of the vector \(n^\mu\), the normal to the characteristic surfaces, and characterizing the propagation of the wave fronts of the solutions to the gauged equation. If the characteristic determinant vanishes for real-valued time-like components, \(n^0\), then the equation is hyperbolic. If in addition, also \(n^\mu n_\mu = 0\) holds valid, then the equation is causal. In now applying the Courant-Hilbert criterion to (A.9), we replace there the partial derivatives \(\partial^\mu\) by the vector \(n^\mu\) and arrive at
\[
\left[ \bar{w}^{(n)}_{\perp}(0, \lambda') \right]_{a_1 \ldots a_n}^{a_1 \ldots a_n} \left[ F^{(n/2,0)}_{\mu\nu} \right]_{a_1 \ldots a_n}^{b_1 \ldots b_n} n^\mu n^\nu \left[ w^{(n)}_{\perp}(0, \lambda) \right]_{b_1 b_2 \ldots b_n} =
\left[ \bar{w}^{(n)}_{\perp}(0, \lambda') \right]_{a_1 \ldots a_n}^{a_1 \ldots a_n} \left[ \bar{w}^{(n/2,0)}_{\perp} \right]_{a_1 \ldots a_n}^{b_1 \ldots b_n} n^2 \left[ w^{(n)}_{\perp}(0, \lambda) \right]_{b_1 b_2 \ldots b_n} =
n^2 \left[ \bar{w}^{(n)}_{\perp}(0, \lambda') \right]_{a_1 \ldots a_n}^{a_1 \ldots a_n} \left[ w^{(n)}_{\perp}(0, \lambda) \right]_{a_1 \ldots a_n} = (\pm 1)n^2 \delta \lambda \delta_{\pm}.
\]
In this way, a diagonal characteristic determinant is obtained whose vanishing requires \(n^2 = 0\), meaning that our gauged equation is both causal and hyperbolic.
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