THE FRIEDRICHS OPERATOR AND CIRCULAR DOMAINS

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Abstract. The Friedrichs operator of a domain in $\mathbb{C}^n$ is closely related to its Bergman projection and encodes crucial information (geometric, quadrature, potential theoretic etc.) about the domain. We show that the Friedrichs operator of a domain has rank one if the domain can be covered by a circular domain via a proper holomorphic map of finite multiplicity whose Jacobian is a homogeneous polynomial. As an application, we show that the Friedrichs operator is of rank one on the tetrablock, pentablock, and the symmetrized polydisc – domains of significance in the study of $\mu$-synthesis in control theory.

1. Introduction

Let $D \subset \mathbb{C}^n$, $n \geq 1$, be a domain. The Bergman space of $D$, denoted by $A^2(D)$, is the set of holomorphic functions in $L^2(D)$. The Friedrichs operator of $D$ is

$$ F: A^2(D) \to A^2(D) \text{ defined by } F(g) = B(\bar{g}) \quad (1) $$

where the Bergman projection $B$ is the orthogonal projection from $L^2(D)$ onto $A^2(D)$. The Bergman projection and the Friedrichs operator encode crucial information (geometric, quadrature, potential theoretic etc.) about the domain. For instance, the Friedrichs operator having finite rank translates to the domain satisfying a quadrature identity which, in turn, has close connections to the boundary geometry of the domain. For more on this, see Friedrichs [10] and Shapiro [17].

A domain $D$ is said to be circular if $e^{i\theta}z \in D$ whenever $z \in D$ and $\theta \in \mathbb{R}$. We first show that the Friedrichs operator of a circular domain containing 0 has rank one – see Proposition 2.2. This is an easy consequence of a projective representation of circular domains due to Azukawa [5]. Our main result in this article is the following generalization which we prove in Section 3.

Theorem 3.7. Let $D_1$ and $D_2$ be bounded domains in $\mathbb{C}^n$, $n \geq 1$, and $\phi: D_1 \to D_2$ be a proper holomorphic map of finite multiplicity. If $D_1$ is circular, 0 $\in D_1$, and $J\phi$ is a homogeneous polynomial, then the Friedrichs operator of $D_2$ is of rank one.

The approach of using a covering domain to study the Bergman projection and related objects has its origins in the work of Misra, Roy, and Zhang [11] and has been refined further by Trybula [18]. We build on results of Azukawa [5], to show that the Friedrichs operator associated to a certain weighted Bergman space on circular domain has rank one. We then prove generalizations of some results of Trybula which allow us to conclude our main result.

Using the above result we show the Friedrichs operator has rank one on many domains that are of significance in the $\mu$-synthesis problem from control theory. For a brief description of $\mu$-synthesis see §4.1. In some cases, the $\mu$-synthesis problem reduces to an...
interpolation problem from $\mathbb{D}$ to certain special domains. Some examples of these special domains are the tetrablock, pentablock, and symmetrized polydisc. Chen, Krantz, and Yuan [7] show that the Friedrichs operator of the symmetrized bidisc $G_2$ is of rank one. Their proof relies crucially on covering $G_2$ by $\mathbb{D}^2$, a Reinhardt domain. By allowing for covering domains to be circular (possibly non-Reinhardt), we prove the following in Section 4.

**Theorem 4.2.** The Friedrichs operator of the Tetrablock $E \subset \mathbb{C}^3$ has rank one.

**Theorem 4.4.** The Friedrichs operator of the Pentablock $P \subset \mathbb{C}^3$ has rank one.

**Theorem 4.6.** The Friedrichs operator of the Symmetrized polydisc $G_n \subset \mathbb{C}^n$, $n \geq 2$, has rank one.

We now begin with a quick overview of Bergman spaces, circular domains, and Bergman spaces of circular domains.

2. **Bergman Spaces and Circular Domains**

Let $D \subset \mathbb{C}^n$ be a bounded domain, $dV$ denote the Lebesgue measure in $\mathbb{C}^n$, and $L^2(D)$ denote the Hilbert space of square-integrable functions with the inner product

$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} dV(z). \quad (2)$$

The subspace of $L^2(D)$ consisting of holomorphic functions is called as the Bergman space of $D$ and we denote it by $A^2(D)$.

The Bergman space $A^2(D)$ is a closed subspace of $L^2(D)$ and this induces an orthogonal projection from $L^2(D)$ onto $A^2(D)$ known as the Bergman projection of $D$, denoted by $B_D$. The Friedrichs operator of $D$ is defined as $F_D: A^2(D) \to A^2(D)$, $F_D(g) = B_D(g)$.

The Bergman projection is an integral operator given by

$$B_D(f)(z) = \int_D K_D(z, w) f(w) dV(w), \quad (3)$$

where the integral kernel $K_D(z, w)$ is known as the Bergman kernel of $D$. We will drop the subscripts when the domain under consideration is clear from context. The Bergman kernel $K: D \times D \to \mathbb{C}$ is a reproducing kernel for the Bergman space and satisfies the following properties:

1. $k_w := K(\cdot, w) \in A^2(D)$ for all $w \in D$,
2. $\langle f, k_w \rangle = f(w)$ for all $f \in A^2(D)$ and $w \in D$, and
3. if $\{e_n\}$ is an orthonormal basis for $A^2(D)$, then
   $$K(z, w) = \sum e_n(z) \overline{e_n(w)}. \quad (4)$$

A domain $D \subset \mathbb{C}^n$ is said to be circular if $e^{\theta}z \in D$ for every $z \in D$ and $\theta \in \mathbb{R}$. Circular domains admit a characterization in terms of projective coordinates that is useful. Let $D \subset \mathbb{C}^n$ be a circular domain. Define

$$V = \{ (\zeta, r) \in \mathbb{CP}^{n-1} \times \mathbb{R}_{\geq 0}: r \psi(\zeta) \in D \} \quad (5)$$

where $\psi: \mathbb{CP}^{n-1} \to S^{2n-1}$ is such that $\pi \circ \psi = \text{Id}_{\mathbb{CP}^{n-1}}$ and $\pi: \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ is the canonical projection. The set $V$ is independent of the choice of $\psi$ and is called the representative domain for $D$. The domain $D$ can be recovered from $V$ as follows:

$$D = \{ re^{i\theta} \psi(\zeta): (\zeta, r) \in V, \theta \in \mathbb{R} \}. \quad (6)$$

The Bergman space of a circular domain admits a decomposition in terms of homogeneous polynomials. A holomorphic function $f$ on $D$ is said to be $k$-homogeneous, for
k ∈ ℤ, if \( f(\lambda z) = \lambda^k f(z) \) for all \( z ∈ D \) and \( \lambda ∈ ℂ \) with \( |\lambda| ∈ I(z) \) where \( I(z) \) is the connected component of the set \( \{ r ∈ ℝ: r ≠ 0, rz ∈ D \} \) that contains 1. Let \( H_k(D) \) denote the set of holomorphic functions on \( D \) that are \( k \)-homogeneous. If \( f \) is holomorphic in \( D \), then \( f = \sum_{k∈ℤ} f_k \), with \( f_k ∈ H_k \), and the series converges uniformly on compact subsets of \( D \) (see [5, Lemma 1.3]).

The following lemma is an expanded version of [5, Lemma 1.2]. To state this lemma we need to express integrals on \( D \) as integrals on \( V \) using the Fubini-Study metric in \( ℂP^{n−1} \). Let \( U = \{ π(z_1, ..., z_n): z_n ≠ 0 \} \). Let \( u_j: U → ℂ \), for \( 1 ≤ j ≤ n − 1 \), be defined by \( u_j(π(z)) = z_j/z_n \), and \( u = (u_1, ..., u_{n−1}) \). Let \( v \) denote the volume element on \( ℂP^{n−1} \) associated to the Fubini-Study metric. Then,

\[
v|_U = (1 + |u|^2)^{−n} u^* dv_{n−1}
\]

where \( dv_{n−1} \) is the volume element in \( ℂ^{n−1} \). Let \( α: U → S^{2n−1} \) be given by

\[
α = (1 + |u|^2)^{−1/2} (u_1, ..., u_{n−1}, 1).
\]

**Lemma 2.1.** Let \( D ⊂ ℂ^n \) be a circular domain with representative domain \( V ⊂ ℂP^{n−1} × ℝ_{≥0} \).

(a) For \( f, g ∈ A^2(D) \),

\[
⟨f, g⟩ = \int_{0}^{2π} \int_{(ζ, r) ∈ V \times ℂ} f(rα(ζ)e^{iθ})\overline{g(rα(ζ)e^{iθ})} r^{2n−1}v(ζ) ∧ dr ∧ dθ.
\]

(b) For \( f ∈ H_k, g ∈ H_ℓ \), and \( k ≠ ℓ \), we have \( ⟨f, g⟩ = 0 \).

(c) For \( f ∈ A^2(D) \), let \( f = \sum_{k∈ℤ} f_k \), with \( f_k ∈ H_k \), be the homogeneous expansion. Then, \( f_k ∈ A^2(D) \) for each \( k \).

(d) \( A^2(D) = ⋃_{k∈ℤ} H_k \cap A^2(D) \).

As an easy consequence of the above lemma, we get that the Friedrichs operator of a circular domain containing the origin is of rank one.

**Proposition 2.2.** Let \( D ⊂ ℂ^n \) be a circular domain with 0 ∈ \( D \). Then, the Friedrichs operator of \( D \) is of rank one.

**Proof.** For \( f ∈ A^2(D) \), \( f = \sum_{k≥0} f_k \), where \( f_k ∈ A^2(D) \) is \( k \)-homogeneous. Note that \( k ≥ 0 \) in this expansion since \( 0 ∈ D \). Let \( F_D \) be the Friedrichs operator of \( D \). Then,

\[
F_D(f) = B_D(\tilde{f}) = B_D(\sum_k \tilde{f}_k) = \sum_k B_D(\tilde{f}_k).
\]

For \( k > 0 \), Lemma 2.1 gives us that \( ⟨\tilde{f}_k, g_ℓ⟩ = 0 \) for all \( g_ℓ ∈ H_ℓ \) and \( ℓ ≥ 0 \). So, \( ⟨\tilde{f}_k, g⟩ = 0 \) for all \( g ∈ A^2(D) \). Since \( f_0 \) is a constant, \( F_D(f) = B_D(f_0) \) and the conclusion follows. □

If the circular domain \( D \) does not contain the origin, the Friedrichs operator can have arbitrary finite rank or infinite rank. To realize the former possibility consider the fat Hartogs triangle \( \{ |z|^γ < |w| < 1 \} \), \( γ > 0 \), and for the latter consider a product of annuli centred at the origin. For more on these examples and related ideas see Ravisankar and Zeytuncu [15].

3. Proper holomorphic mappings and the Friedrichs operator

In this section we present the relationship between the Bergman projection of a domain and its image under a proper holomorphic map of finite multiplicity. This, in turn, leads to a relationship between their Friedrichs operators.
Let $D_1$ and $D_2$ be two bounded domains in $\mathbb{C}^n$, $n \geq 1$, and $\phi : D_1 \rightarrow D_2$ be a proper holomorphic mapping with multiplicity $m$. Trybula [18] has shown that there is a closed subspace of $A^2(D_1)$ which is unitarily isomorphic to $A^2(D_2)$ (see also [11]). Let $J\phi$ denote the complex Jacobian of $\phi$ and $\nu(z) = |J\phi(z)|^2$. Let the weighted Bergman space $A^2(D_1, \nu)$ be the set of holomorphic functions in $L^2(D_1, \nu)$ equipped with the inner product $\langle f, g \rangle_\nu = \int_{D_1} f(z)\overline{g(z)}\nu(z)dV(z)$. We adopt the same strategy as Trybula [18] to generalize their results to our setting. We show that there is a closed subspace of $A^2(D_1, \nu)$ which is unitarily isomorphic to $A^2(D_2)$. For $f \in A^2(D_2)$, $f \circ \phi$ is well defined and holomorphic on $D_1$ and, by change of variables,

$$m \int_{D_2} f dv = \int_{D_1} (f \circ \phi) |J\phi|^2 dv.$$  \hfill (11)

Thus $f \circ \phi \in A^2(D_1, \nu)$. Define $\Gamma_\nu : A^2(D_2) \rightarrow A^2(D_1, \nu)$ by $\Gamma_\nu(f) = \frac{1}{\sqrt{m}} (f \circ \phi)$. Clearly $\Gamma_\nu$ is an isometric embedding. Therefore $\Gamma_\nu A^2(D_2)$ is a closed subspace of $A^2(D_1, \nu)$ that is isometrically isomorphic to $A^2(D_2)$.

Note that $\Gamma_\nu$ is unitary when understood as an operator from $A^2(D_2)$ onto $\Gamma_\nu A^2(D_2)$. The adjoint operator $\Gamma^*_\nu$ can be described as follows. Let $g \in \Gamma_\nu A^2(D_2)$. Then, $g(z) = g(w)$ whenever $\phi(z) = \phi(w)$ and $z, w \in D_1$. So, we can define $\tilde{g}$ on $D_2$ by $\tilde{g}(\phi(z)) = g(z)$. Then, $\tilde{g}$ is well defined and holomorphic on $D_2$. It is easy to verify, using (11), that $\tilde{g} \in A^2(D_2)$. Hence,

$$\Gamma^*_\nu(g) = \sqrt{m} \tilde{g}, \quad \text{for } g \in \Gamma_\nu A^2(D_2).$$  \hfill (12)

**Remark 3.1.** For $g \in \Gamma_\nu A^2(D_2)$, $\tilde{g} \in A^2(D_2)$ and $\tilde{g} \circ \phi = g$. Therefore $\Gamma^*_\nu = \Gamma_\nu^{-1}$ (where $\Gamma_\nu$ is considered a map onto its range) and

$$\Gamma_\nu A^2(D_2) = \{g \in A^2(D_1, \nu) : g(z) = g(w) \text{ whenever } \phi(z) = \phi(w) \text{ and } z, w \in D_1\}. \hfill (13)$$

The following two lemmas express the Bergman kernel of $D_2$ in terms of the weighted Bergman kernel of $D_1$.

**Lemma 3.2.** The orthogonal projection $P_\nu$ of $A^2(D_1, \nu)$ onto $\Gamma_\nu A^2(D_2)$ is given by

$$P_\nu g = \frac{1}{m} \sum_{j=1}^{m} (g \circ \phi^j \circ \phi), \quad \text{for } g \in A^2(D_1, \nu),$$  \hfill (14)

where $\{\phi^j\}_{1}^{m}$ are the local inverses of $\phi$.

**Proof.** For $g \in A^2(D_1, \nu)$, let $Qg$ denote the right hand side of (14). Note that $Qg \in A^2(D_1, \nu)$ since $Qg$ is holomorphic and

$$\|Qg\|_{(D_1, \nu)} = \frac{1}{m^2} \int_{D_1} \left| \sum_{j=1}^{m} (g \circ \phi^j \circ \phi) \right|^2 |J\phi|^2 dv \hfill (15)$$

$$\leq \frac{1}{m} \int_{D_1} \sum_{j=1}^{m} |(g \circ \phi^j \circ \phi)|^2 |J\phi|^2 dv = \|g\|_{(D_1, \nu)}. \hfill (16)$$

Using $\phi \circ \phi^j \circ \phi = \phi$, it is easy to verify that $Q^2 = Q$ and $Q \Gamma_\nu = \Gamma_\nu$. Since $Qg(z) = Qg(w)$ whenever $\phi(z) = \phi(w)$ and $z, w \in D_1$, we have $\tilde{Q}g \in A^2(D_2)$ with $\tilde{Q}g(\phi(z)) = Qg(z)$. Therefore the range of $Q$ coincides with the range of $\Gamma_\nu$. Since $Q$ is a projection and $\|Q\| = 1$, we conclude that $Q$ is the orthogonal projection onto $\Gamma_\nu A^2(D_2)$. \hfill \Box
Lemma 3.3. Let $K_{D_1}^\nu$ and $K_{D_2}$ be the Bergman kernels associated to the Bergman projections onto $A^2(D_1, \nu)$ and $A^2(D_2)$ respectively. Then

$$K_{D_2}(\phi(z), \phi(w)) = \sum_{j=1}^{m} K_{D_1}^\nu(\phi^j \circ \phi(z), w)$$

(17)

where $\{\phi^j\}_{1}^{m}$ are the local inverses of $\phi$.

**Proof.** For $f \in A^2(D_2)$ and $w \in D_1$, $K_{D_1}^\nu(\cdot, w) \in A^2(D_1, \nu)$ and hence

$$\langle \Gamma_\nu f, (I - P_\nu)K_{D_1}^\nu(\cdot, w) \rangle_{(D_1, \nu)} = 0$$

(18)

where $P_\nu$ is the orthogonal projection of $A^2(D_1, \nu)$ onto $\Gamma_\nu A^2(D_2)$. By the reproducing property of $K_{D_2}$ and $\Gamma_\nu$ being an isometry, we have

$$\langle f, \Gamma_\nu P_\nu K_{D_1}^\nu(\cdot, w) \rangle_{D_2} = \langle \Gamma_\nu f, P_\nu K_{D_1}^\nu(\cdot, w) \rangle_{(D_1, \nu)} = \langle \Gamma_\nu f, K_{D_1}^\nu(\cdot, w) \rangle_{(D_1, \nu)}$$

(19)

$$= \Gamma_\nu f(w) = \frac{1}{\sqrt{m}} f(\phi(w)) = \frac{1}{\sqrt{m}} \langle f, K_{D_2}(\cdot, \phi(w)) \rangle_{D_2}.$$ 

(20)

So, $K_{D_2}(\cdot, \phi(w)) = \sqrt{m} \Gamma_\nu(P_\nu K_{D_1}^\nu(\cdot, w))$. Now, by (12) and Lemma 3.2,

$$K_{D_2}(\phi(z), \phi(w)) = m P_\nu K_{D_1}(z, w) = \sum_{j=1}^{m} K_{D_1}^\nu(\phi^j \circ \phi(z), w).$$

(21)

□

**Remark 3.4.** The relation between the Bergman kernels $K_{D_1}$ and $K_{D_1}^\nu$ is given by

$$K_{D_1}(z, w) = J\phi(z)K_{D_1}^\nu(z, w)J\phi(w).$$

(22)

In fact, if $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis for $A^2(D_1, \nu)$, then $\{\varphi_n J\phi\}_{n=1}^{\infty}$ is an orthonormal basis for $A^2(D_1)$. Since $\phi \circ \phi^j \circ \phi = \phi$, we have $J\phi(\phi^j \circ \phi(z)), J\phi^j(\phi(z)) J\phi(z) = J\phi(z)$.

An alternate approach to deducing (17) is to use (22) along with Corollary 1 of [18].

The following results are consequences of Lemma 3.3.

**Lemma 3.5.** Let $B_{D_1}^\nu$ and $B_{D_2}$ be the Bergman projections associated to $A^2(D_1, \nu)$ and $A^2(D_2)$ respectively. Then,

$$B_{D_2}(g)(\zeta) = \frac{1}{m} \sum_{j=1}^{m} B_{D_1}^\nu(g \circ \phi)(\phi^j(\zeta)), \text{ for } g \in L^2(D_2) \text{ and } \zeta \in D_2,$$

(23)

where $\{\phi^j\}_{1}^{m}$ are the local inverses of $\phi$.

**Proof.** Let $g \in L^2(D_2)$. Then, $g \circ \phi \in L^2(D_1, \nu),$

$$B_{D_1}^\nu(g \circ \phi) = \int_{D_1} K_{D_1}^\nu(\cdot, w)(g \circ \phi)(w)|J\phi(w)|^2dV(w), \text{ and}$$

(24)

$$B_{D_2}(g) = \int_{D_2} K_{D_2}(\cdot, \eta)g(\eta)dV(\eta)$$

(25)

$$= \frac{1}{m} \int_{D_1} K_{D_2}(\cdot, \phi(w))g(\phi(w))|J\phi(w)|^2dV(w).$$

(26)
For $\zeta \in D_2$, there exists $z \in D_1$ such that $\zeta = \phi(z)$. Now, by Lemma 3.3,
\[
B_{D_2}(g)(\zeta) = B_{D_2}(g)(\phi(z)) = \frac{1}{m} \int_{D_1} K_{D_2}(\phi(z), \phi(w))(g \circ \phi)(w)|J\phi(w)|^2dV(w) 
\]
\[
= \frac{1}{m} \int_{D_1} \sum_{j=1}^{m} K'_{D_1}(\phi^j \circ \phi(z), w)(g \circ \phi)(w)|J\phi(w)|^2dV(w) 
\]
\[
= \frac{1}{m} \sum_{j=1}^{m} B''_{D_1}(g \circ \phi)(\phi^j(\zeta)). 
\]
\[
(27)
\]
\[
(28)
\]
\[
(29)
\]

**Corollary 3.6.** Let $T: \Gamma_{\nu}A^2(D_2) \to A^2(D_1, \nu)$ be defined by $T(f) = B''_{D_1}(\bar{f})$. If $T$ is of rank one, then so is the Friedrichs operator on $D_2$.

**Proof.** Since $T(\lambda) = \bar{\lambda}$ for any $\lambda \in \mathbb{C}$, the range of $T$ is the set of (complex) constant functions.

For $g \in A^2(D_2)$, we have $g \circ \phi \in \Gamma_{\nu}A^2(D_2)$ and $T(g \circ \phi)$ is a constant, say $a_0$. Now, by Lemma 3.5,
\[
B_{D_2}(g)(w) = \frac{1}{m} \sum_{j=1}^{m} B''_{D_1}(g \circ \bar{\phi})(\phi^j(w)) = a_0,
\]
for $w \in D_2$. \qed

We now prove the main result of this article.

**Theorem 3.7.** Let $D_1$ and $D_2$ be bounded domains in $\mathbb{C}^n$, $n \geq 1$, and $\phi: D_1 \to D_2$ be a proper holomorphic map of finite multiplicity. If $D_1$ is circular, $0 \in D_1$, and $J\phi$ is a homogeneous polynomial, then the Friedrichs operator of $D_2$ is of rank one.

**Proof.** With the setup as in the beginning of this section, $\Gamma_{\nu}A^2(D_2)$ is a closed subspace of $A^2(D_1, \nu)$ that is isometrically isomorphic to $A^2(D_2)$.

Since $J\phi$ is a homogeneous polynomial and $\nu = |J\phi|^2$, a version of Lemma 2.1 holds for $A^2(D_1, \nu)$. Note that $0 \in D_1$ and hence any homogeneous polynomial in $D_1$ is $k$-homogeneous for some $k \geq 0$. We use Lemma 2.1 to get that
\[
\langle f_k, f_\ell \rangle_{D_1, \nu} = \int_{D_1} f_k(x, y, z)\overline{f_\ell(x, y, z)}\nu(z)dV(x, y, z) 
\]
\[
= \int_{D_1} ((J\phi)f_k(x, y, z))\overline{((J\phi)f_\ell(x, y, z))}dV(x, y, z) = 0,
\]
for $f_k \in H_k$, $f_\ell \in H_\ell$, and $k \neq \ell$. Hence,
\[
A^2(D_1, \nu) = \bigoplus_{k \geq 0} H_k \cap A^2(D_1, \nu).
\]

**Lemma 2.1(a)** also gives us that $\langle f_k, f_\ell \rangle_{D_1, \nu} = 0$ for $f_k \in H_k$, $f_\ell \in H_\ell$ unless $k = \ell = 0$.

Let $T: \Gamma_{\nu}A^2(D_2) \to A^2(D_1, \nu)$ by $T(f) = B''_{D_1}(\bar{f})$. Write $f \in \Gamma_{\nu}A^2(D_2)$ as $f = \sum_{k \geq 0} f_k$, where $f_k \in H_k$, to get
\[
T(f) = B''_{D_1}(\bar{f}) = B''_{D_1}\left(\sum_{k \geq 0} \bar{f}_k\right) = \sum_{k \geq 0} B''_{D_1}(\bar{f}_k) = B''_{D_1}(\bar{f}_0) \text{ (a constant)}. 
\]
Thus $T$ is of rank one. Now, by Corollary 3.6, the Friedrichs operator of $D_2$ is of rank one. \qed
4. Friedrichs Operator of Domains Related to $\mu$-synthesis

In this section we consider three domains related to $\mu$-synthesis – the tetrablock, the pentablock and the symmetrized polydisc. For each of these domains, we first recall some important characterizations that define them. We then proceed to show, using Theorem 3.7, that their Friedrichs operators are of rank one. We now begin with a brief description of the $\mu$-synthesis problem.

4.1. $\mu$-Synthesis. The $\mu$-synthesis problem plays an important role in modeling structured uncertainties in control engineering. Here, $\mu$ is a cost function on matrices that denotes the structured singular value of a matrix relative to a subspace of linear transformations (see [6,9]). Let $M$ be a linear subspace of the complex $n \times m$ matrices $\mathbb{C}^{n \times m}$. The structured singular value of an $B \in \mathbb{C}^{m \times n}$, denoted by $\mu_M(B)$, is defined as

$$\mu_M(B) = \frac{1}{\inf \{\|X\| : X \in M, I - BX \text{ is singular}\}},$$

where $\|X\|$ denotes the operator norm of the matrix $X$. We set $\mu_M(B) = 0$ whenever $I - BX$ is non-singular for all $X \in M$.

Let $\lambda_1, \ldots, \lambda_k$ be distinct points in the unit disc $\mathbb{D} \subset \mathbb{C}$, and $B_1, \ldots, B_k \in \mathbb{C}^{m \times n}$. The $\mu$-synthesis problem is to find an analytic function $f : \mathbb{D} \to \mathbb{C}^{m \times n}$ such that

$$f(\lambda_j) = B_j, \text{ for } 1 \leq j \leq k, \text{ and } \mu_M(f(\lambda)) < 1, \text{ for } \lambda \in \mathbb{D}. \quad (36)$$

When $M$ is the set of scalar square matrices ($m = n$), $\mu_M$ coincides with the spectral radius. In that case, the $\mu$-synthesis problem reduces to the spectral Nevanlinna-Pick interpolation problem. Agler and Young [3] showed that this problem, in $m = n = 2$, reduces to an interpolation problem from $\mathbb{D}$ to the symmetrized bidisc $\mathbb{D}_2$. A similar phenomenon holds for certain other subspaces of $\mathbb{C}^{2 \times 2}$ reducing the $\mu$-synthesis problem to an interpolation problem into the domains tetrablock $\mathcal{E}$ and pentablock $\mathcal{P}$.

4.2. Tetrablock. The tetrablock is defined as

$$\mathcal{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - zx_1 - wx_2 + zwx_3 \neq 0, z, w \in \mathbb{D}\} \quad (37)$$

and can be characterized as follows.

**Theorem 4.1** ([1, Theorem 2.4]). For $x \in \mathbb{C}^3$, the following are equivalent.

1. $x \in \mathcal{E}$.
2. $|x_1 - \bar{x}_2x_3| + |x_1x_2 - x_3| < 1 - |x_2|^2$.
3. $|x_2 - \bar{x}_1x_3| + |x_1x_2 - x_3| < 1 - |x_1|^2$.
4. $|x_1 - \bar{x}_2x_3| + |x_2 - \bar{x}_1x_3| < 1 - |x_3|^2$.
5. There exists a symmetric $2 \times 2$ matrix $A = [a_{ij}]$ such that $\|A\| < 1$ and $x = (a_{11}, a_{22}, \det A)$.
6. $|x_3| < 1$ and there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that $|\beta_1| + |\beta_2| < 1$ and

$$x_1 = \beta_1 + \bar{\beta}_2x_3, \quad x_2 = \beta_2 + \bar{\beta}_1x_3.$$
of the points \((i, 1, 1), (1, i, 1)\) and \((1, 1, i)\) are in \(\overline{E}\). So, we use a proper holomorphic map from a circular domain to \(E\) to study the Friedrichs operator on \(E\).

**Theorem 4.2.** The Friedrichs operator of \(E\) is of rank one.

**Proof.** Let \(S\) be the unit ball, in the operator norm, of symmetric complex \(2 \times 2\) matrices. We view \(S\) as a domain in \(\mathbb{C}^3\):

\[
S = \{(x, y, z) \in \mathbb{C}^3 : |x|^2 + |y|^2 + 2|z|^2 < 1 + |xy - z^2|\}. \tag{39}
\]

Note that \(S\) is a circular domain containing the origin. By Theorem 4.1, we can write

\[
E = \{(x, y, xy - z^2) : (x, y, z) \in S\}. \tag{40}
\]

Let \(\Phi : S \to E\) be defined by \(\Phi(x, y, z) = (x, y, xy - z^2)\). It is a proper holomorphic map of multiplicity two with Jacobian \(J\Phi(x, y, z) = -2z\). The conclusion follows from Theorem 3.7 since the Jacobian is a homogeneous polynomial. \(\square\)

### 4.3. Pentablock

The next domain we consider is the pentablock

\[
\mathcal{P} := \{(a_{21}, \text{tr}(A), \det A) \in \mathbb{C}^3 : A = [a_{ij}] \in \mathbb{C}^{2 \times 2}, \|A\| < 1\}. \tag{41}
\]

Here are few alternate characterizations of \(\mathcal{P}\).

**Theorem 4.3** ([2, Theorem 1.1]). Let \((s, p) \in \mathbb{G}_2\) and \(a \in \mathbb{C}\). The following are equivalent.

1. \((a, s, p) \in \mathcal{P}\).
2. \(|a| < 1 - \frac{\frac{1}{2}s\beta}{1 + \sqrt{1 - |\beta|^2}}\), where \(\beta = \frac{s - sp}{1 - |p|^2}\).
3. \(2|a| < |1 - \lambda_2\lambda_1| + \sqrt{(1 - |\lambda_1|^2)(1 - |\lambda_2|^2)}\), where \(\lambda_1, \lambda_2 \in \mathbb{D}\) and \((s, p) = (\lambda_1 + \lambda_2, \lambda_1\lambda_2)\).

Similar to the tetrablock, the pentablock is also related to the \(\mu\)-synthesis problem where the structure is given by the upper triangular matrices \(N\) in \(\mathbb{C}^{2 \times 2}\). Additionally, by [2, Theorem 5.2], the pentablock can also be characterized as follows.

\[
\mathcal{P} = \{(a_{21}, \text{tr}(A), \det A) : A = [a_{ij}] \in \mathbb{C}^{2 \times 2}, \mu_N(A) < 1\}. \tag{42}
\]

The pentablock is a Hartogs domain over the symmetrized bidisc \(\mathbb{G}_2\). But \(\mathbb{G}_2\) itself is not a Hartogs domain (see [7, Proposition 6.3]). It is easy to check that the pentablock is not a Reinhardt domain: \((0, 2, 1) \in \overline{\mathcal{P}}\), but \((0, 2i, 1) \notin \overline{\mathcal{P}}\).

**Theorem 4.4.** The Friedrichs operator of \(\mathcal{P}\) is of rank one.

**Proof.** Consider the circular domain

\[
\mathcal{L} = \{(x, y, z) \in \mathbb{C}^3 : 2|z| < |1 - xy + \sqrt{(1 - |x|^2)(1 - |y|^2)}|\}. \tag{43}
\]

Let \(\Psi : \mathcal{L} \to \mathcal{P}\) be defined by \(\Psi(x, y, z) = (z, x + y, xy)\). It is easy to see that \(\Psi\) is a proper holomorphic covering map with multiplicity two. The Jacobian of \(\Psi\), \(J\Psi = x - y\), is a homogeneous polynomial. Then, Theorem 3.7 gives us the result. \(\square\)
4.4. Symmetrized polydisc. The last domain we consider is the symmetrized polydisc $G_n \subset \mathbb{C}^n$, a generalization of $G_2$ to higher dimensions. $G_n$ is defined to be the image of the symmetrization map $\pi_n: \mathbb{D}^n \to \mathbb{C}^n$ defined by

$$\pi_n(z_1, \ldots, z_n) = \left( \sum_{1 \leq i \leq n} z_i, \sum_{1 \leq i < j \leq n} z_i z_j, \ldots, \prod_{i=1}^n z_i \right).$$  \hfill (44)

However, the complex geometry and operator theoretic properties of $G_n$, $n \geq 3$, is starkly different from those of $G_2$ (see [12]).

The symmetrized polydisc is associated with spectral interpolation and hence with the $\mu$-synthesis problem. For $A \in \mathbb{C}^{n \times n}$, the spectral radius $r(A) < 1$ if and only if $\pi_n(\lambda_1, \ldots, \lambda_n) \in G_n$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$ counted with multiplicities. In fact, the interpolation problem into the spectral unit ball in $\mathbb{C}^{n \times n}$ is equivalent to the interpolation problem into $G_n$ (see [8, Theorem 2.1]).

We collect a few important characterizations of $G_n$ in the following theorem. More characterizations of $G_n$ can be found in [13].

**Theorem 4.5 ([8]).** For $(s_1, \ldots, s_n) \in \mathbb{C}^n$, the following are equivalent.

1. $(s_1, \ldots, s_n) \in G_n$.
2. $\sup_{|z| \leq 1} |f_s(z)| < 1$, where
   $$f_s(z) = \frac{n(-1)^n s_n z^{n-1} + (n-1)(-1)^{n-1} s_{n-1} z^{n-2} + \cdots + (-1)^1 s_1}{n - (n-1)s_1 z + \cdots + (-1)^{n-1} s_{n-1} z^{n-1}}.$$
3. $|s_n| < 1$ and there exists $(\beta_1, \ldots, \beta_{n-1}) \in G_{n-1}$ such that $s_j = \beta_j + \bar{\beta}_{n-j} s_n$ for $j = 1, \ldots, n-1$.

Chen, Krantz, and Yuan [7] have shown that the Friedrichs operator on $G_2$ is of rank one (see [7]). We now show that the same holds for all $G_n$.

**Theorem 4.6.** The Friedrichs operator on $G_n$ is of rank one.

**Proof.** The symmetrization map $\pi_n$ is a proper holomorphic covering map with multiplicity $n!$ and Jacobian $J\pi_n = \prod_{1 \leq i < k \leq n}(z_i - z_k)$. Since the Jacobian is a homogeneous polynomial, we are done by Theorem 3.7. \hfill $\square$

A generalization of the symmetrized polydisc called the extended symmetrized polydisc was introduced by the second author and Pal [13]. These domains are useful in studying the Schwarz lemma for $G_n$ (see [14]) and are related to the $\mu$-synthesis problem as well (see [16]). The extended symmetrized polydisc $\tilde{G}_n$, $n \geq 2$, is defined as follows.

$$\tilde{G}_n := \left\{ (y_1, \ldots, y_{n-1}, q) \in \mathbb{C}^n : q \in \mathbb{D}, y_j = \beta_j + \bar{\beta}_{n-j} q, \beta_j \in \mathbb{C} \text{ and } |\beta_j| + |\bar{\beta}_{n-j}| < \binom{n}{j} \text{ for } j = 1, \ldots, n-1 \right\}.$$ \hfill (45)

Note that $\tilde{G}_2 = G_2$, and $G_n \subset \tilde{G}_n$ for $n \geq 3$. However, $\tilde{G}_3$ is linearly isomorphic to the tetrablock $E$. Consequently, the Friedrichs operators of $G_2$ and $\tilde{G}_3$ are of rank one. It would be interesting to see if the Friedrichs operator continues to have rank one on $\tilde{G}_n$ for $n \geq 4$. \hfill $\square$
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