IN Variant Differential Operators on
Siegel-Jacobi Space

JAE-HYUN YANG

Abstract. For two positive integers $m$ and $n$, we let $\mathbb{H}_n$ be the Siegel upper half plane of degree $n$ and let $C^{(m,n)}$ be the set of all $m \times n$ complex matrices. In this article, we study differential operators on the Siegel-Jacobi space $\mathbb{H}_n \times C^{(m,n)}$ that are invariant under the natural action of the Jacobi group $Sp(n, \mathbb{R}) \ltimes H_R^{(n,m)}$ on $\mathbb{H}_n \times C^{(m,n)}$, where $H_R^{(n,m)}$ denotes the Heisenberg group. We give some explicit invariant differential operators. We present important problems which are natural. We give some partial solutions for these natural problems.

1. Introduction

For a given fixed positive integer $n$, we let

$$\mathbb{H}_n = \{ \Omega \in C^{(n,n)} \mid \Omega = \Omega^t, \quad \text{Im} \Omega > 0 \}$$

be the Siegel upper half plane of degree $n$ and let

$$Sp(n, \mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid M J_n M = J_n \}$$

be the symplectic group of degree $n$, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$ for two positive integers $k$ and $l$, $M^t$ denotes the transpose matrix of a matrix $M$ and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$ 

$Sp(n, \mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers $m$ and $n$, we consider the Heisenberg group

$$H_R^{(n,m)} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu^t \lambda \text{ symmetric} \}$$

2000 Mathematics Subject Classification. Primary 13A50, 32Wxx, 15A72.

Keywords and phrases: invariants, invariant differential operators, Siegel-Jacobi space.

This work was supported by Basic Science Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (41493-01) and partially supported by the Max-Planck-Institut für Mathematik in Bonn.
endowed with the following multiplication law
\[(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda' \mu' - \mu' \lambda')\]
with \((\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}\). We define the semidirect product of \(Sp(n, \mathbb{R})\) and \(H_{\mathbb{R}}^{(n,m)}\)
\[G^J = Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}\]
endowed with the following multiplication law
\[(M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) = (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}' \mu' - \tilde{\mu}' \lambda'))\]
with \(M, M' \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}\) and \((\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'\). Then \(G^J\) acts on \(\mathbb{H}_n \times \mathbb{C}^{(m,n)}\) transitively by
\[(1.2) \quad (M, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left(M \cdot \Omega, (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1}\right),\]
where \(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}\) and \((\Omega, Z) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}\). We note that the Jacobi group \(G^J\) is not a reductive Lie group and that the homogeneous space \(\mathbb{H}_n \times \mathbb{C}^{(m,n)}\) is not a symmetric space. We refer to [1] [6] [22] [23] [24] [25] [27] [28] [29] [30] [31] about automorphic forms on \(G^J\) and topics related to the content of this paper. From now on, for brevity we write \(\mathbb{H}_{n,m} = \mathbb{H}_n \times \mathbb{C}^{(m,n)}\), called the Siegel-Jacobi space of degree \(n\) and index \(m\).

The aim of this paper is to study differential operators on \(\mathbb{H}_{n,m}\) which are invariant under the natural action (1.2) of \(G^J\). The study of these invariant differential operators on the Siegel-Jacobi space \(\mathbb{H}_{n,m}\) is interesting and important in the aspects of invariant theory, arithmetic and geometry. This article is organized as follows. In Section 2, we review differential operators on \(\mathbb{H}_n\) invariant under the action (1.1) of \(Sp(n, \mathbb{R})\). We let \(\mathbb{D}(\mathbb{H}_n)\) denote the algebra of all differential operators on \(\mathbb{H}_n\) that are invariant under the action (1.1). According to the work of Harish-Chandra [7][8], we see that \(\mathbb{D}(\mathbb{H}_n)\) is a commutative algebra which is isomorphic to the center of the universal enveloping algebra of the complexification of the Lie algebra of \(Sp(n, \mathbb{R})\). We briefly describe the work of Maass [14] about constructing explicit algebraically independent generators of \(\mathbb{D}(\mathbb{H}_n)\) and Shimura’s construction [18] of canonically defined algebraically independent generators of \(\mathbb{D}(\mathbb{H}_n)\). In Section 3, we study differential operators on \(\mathbb{H}_{n,m}\) invariant under the action (1.2) of \(G^J\). For two positive integers \(m\) and \(n\), we let
\[T_{n,m} = \{ (\omega, z) | \omega = t\omega \in \mathbb{C}^{(n,n)}, z \in \mathbb{C}^{(m,n)} \}\]
be the complex vector space of dimension \(\frac{n(n+1)}{2} + mn\). From the adjoint action of the Jacobi group \(G^J\), we have the natural action of the unitary group \(U(n)\) on \(T_{n,m}\) given by
\[(1.3) \quad u \cdot (\omega, z) = (u \omega \cdot u, z \cdot u), \quad u \in U(n), (\omega, z) \in T_{n,m}.\]
The action (1.3) of $U(n)$ induces canonically the representation $\tau$ of $U(n)$ on the polynomial algebra $\text{Pol}(T_{n,m})$ consisting of complex valued polynomial functions on $T_{n,m}$. Let $\text{Pol}(T_{n,m})^{U(n)}$ denote the subalgebra of $\text{Pol}(T_{n,m})$ consisting of all polynomials on $T_{n,m}$ invariant under the representation $\tau$ of $U(n)$, and $\mathbb{D}(\mathbb{H}_{n,m})$ denote the algebra of all differential operators on $\mathbb{H}_{n,m}$ invariant under the action (1.2) of $G$. We see that there is a canonically defined linear bijection of $\text{Pol}(T_{n,m})^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$ which is not multiplicative. We will see that $\mathbb{D}(\mathbb{H}_{n,m})$ is not commutative. The main important problem is to find explicit generators of $\text{Pol}(T_{n,m})^{U(n)}$ and explicit generators of $\mathbb{D}(\mathbb{H}_{n,m})$. We propose several natural problems. We want to mention that at this moment it is quite complicated and difficult to find the explicit generators of $\mathbb{D}(\mathbb{H}_{n,m})$ and to express invariant differential operators on $\mathbb{H}_{n,m}$ explicitly. In Section 4, we gives some examples of explicit $G$-invariant differential operators on $\mathbb{H}_{n,m}$ that are obtained by complicated calculations. In Section 5, we deal with the special case $n = m = 1$ in detail. We give complete solutions of the problems that are proposed in Section 3. In Section 6, we deal with the case that $n = 1$ and $m$ is arbitrary. We give some partial solutions for the problems proposed in Section 3. In the final section, using these invariant differential operators on the Siegel-Jacobi space, we discuss a notion of Maass-Jacobi forms.

Acknowledgements: This work was in part done during my stay at the Max-Planck-Institut für Mathematik in Bonn. I am very grateful for the hospitality and financial support. I also thank the National Research Foundation of Korea for its financial support. Finally I would like to give my hearty thanks to Don Zagier, Eberhard Freitag, Rainer Weissauer, Hiroyuki Ochiai and Minoru Itoh for their interests in this work and fruitful discussions.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. The symbol “$:=$” means that the expression on the right is the definition of that on the left. For two positive integers $k$ and $l$, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$. For a square matrix $A \in F^{(k,k)}$ of degree $k$, $\text{tr}(A)$ denotes the trace of $A$. For any $M \in F^{(k,l)}$, $^tM$ denotes the transpose matrix of $M$. $I_n$ denotes the identity matrix of degree $n$. For $A \in F^{(k,k)}$ and $B \in F^{(k,k)}$, we set $B[A] = \overline{tAB}$. For a complex matrix $A$, $\overline{A}$ denotes the complex conjugate of $A$. For $A \in \mathbb{C}^{(k,k)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = \overline{tAB}$. For a positive integer $n$, $I_n$ denotes the identity matrix of degree $n$. 


2. Invariant Differential Operators on the Siegel Space

For a coordinate $\Omega = (\omega_{ij}) \in \mathbb{H}_n$, we write $\Omega = X + iY$ with $X = (x_{ij})$, $Y = (y_{ij})$ real. We put $d\Omega = (d\omega_{ij})$ and $d\overline{\Omega} = (d\overline{\omega}_{ij})$. We also put
\[
\frac{\partial}{\partial \Omega} = \left( \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{\Omega}} = \left( \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \overline{\omega}_{ij}} \right).
\]

Then for a positive real number $A$,
\[\begin{equation}
(2.1) \quad ds_{2n;A}^2 = A \operatorname{tr} \left( Y^{-1}d\Omega Y^{-1}d\overline{\Omega} \right)
\end{equation}\]
is a $Sp(n, \mathbb{R})$-invariant Kähler metric on $\mathbb{H}_n$ (cf. [19, 20]), where $\operatorname{tr}(M)$ denotes the trace of a square matrix $M$. H. Maass [13] proved that the Laplacian of $ds_{2n;A}^2$ is given by
\[\begin{equation}
(2.2) \quad \Delta_{n;A} = 4 A \operatorname{tr} \left( Y^t \left( Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right).
\end{equation}\]

And
\[dv_n(\Omega) = (\det Y)^{-(n+1)} \prod_{1 \leq i \leq j \leq n} dx_{ij} \prod_{1 \leq i \leq j \leq n} dy_{ij}\]
is a $Sp(n, \mathbb{R})$-invariant volume element on $\mathbb{H}_n$ (cf. [20, p. 130]).

For brevity, we write $G = Sp(n, \mathbb{R})$. The isotropy subgroup $K$ at $iI_n$ for the action (1.1) is a maximal compact subgroup given by
\[K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right| A^t A + B^t B = I_n, \quad A^t B = B^t A, \quad A, B \in \mathbb{R}^{(n,n)} \right\}.
\]

Let $\mathfrak{k}$ be the Lie algebra of $K$. Then the Lie algebra $\mathfrak{g}$ of $G$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where
\[\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -i X_1 \end{pmatrix} \right| X_1, X_2, X_3 \in \mathbb{R}^{(n,n)}, \quad X_2 = X^t, \quad X_3 = X^t \right\},
\]
\[\mathfrak{k} = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \right| X = X^t, \quad Y = Y^t \right\},
\]
\[\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \right| X = X^t, \quad Y = Y^t, \quad X, Y \in \mathbb{R}^{(n,n)} \right\}.
\]

The subspace $\mathfrak{p}$ of $\mathfrak{g}$ may be regarded as the tangent space of $\mathbb{H}_n$ at $iI_n$. The adjoint representation of $G$ on $\mathfrak{g}$ induces the action of $K$ on $\mathfrak{p}$ given by
\[\begin{equation}
k \cdot Z = k Z k^t, \quad k \in K, \quad Z \in \mathfrak{p}.
\end{equation}\]

Let $T_n$ be the vector space of $n \times n$ symmetric complex matrices. We let $\Psi : \mathfrak{p} \rightarrow T_n$ be the map defined by
\[\begin{equation}
(2.4) \quad \Psi \left( \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \right) = X + iY, \quad \left( \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \right) \in \mathfrak{p}.
\end{equation}\]
We let $\delta : K \longrightarrow U(n)$ be the isomorphism defined by
\begin{equation}
\delta \left( \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right) = A + iB, \quad \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K,
\end{equation}
where $U(n)$ denotes the unitary group of degree $n$. We identify $\mathfrak{p}$ (resp. $K$) with $T_n$ (resp. $U(n)$) through the map $\Psi$ (resp. $\delta$). We consider the action of $U(n)$ on $T_n$ defined by
\begin{equation}
h \cdot \omega = h\omega^t h, \quad h \in U(n), \quad \omega \in T_n.
\end{equation}
Then the adjoint action (2.3) of $K$ on $\mathfrak{p}$ is compatible with the action (2.6) of $U(n)$ on $T_n$ through the map $\Psi$.

Precisely for any $k \in K$ and $Z \in \mathfrak{p}$, we get
\begin{equation}
\Psi(kZ^t k) = \delta(k) \Psi(Z)^t \delta(k).
\end{equation}
The action (2.6) induces the action of $U(n)$ on the polynomial algebra $\text{Pol}(T_n)$ and the symmetric algebra $S(T_n)$ respectively. We denote by $\text{Pol}(T_n)^{U(n)}$ (resp. $S(T_n)^{U(n)}$) the subalgebra of $\text{Pol}(T_n)$ (resp. $S(T_n)$) consisting of $U(n)$-invariants. The following inner product $( , )$ on $T_n$ defined by
\begin{equation}
(Z, W) = \text{tr}(ZW), \quad Z, W \in T_n
\end{equation}
gives an isomorphism as vector spaces
\begin{equation}
T_n \cong T_n^*, \quad Z \mapsto f_Z, \quad Z \in T_n,
\end{equation}
where $T_n^*$ denotes the dual space of $T_n$ and $f_Z$ is the linear functional on $T_n$ defined by
\begin{equation}
f_Z(W) = (W, Z), \quad W \in T_n.
\end{equation}
It is known that there is a canonical linear bijection of $S(T_n)^{U(n)}$ onto the algebra $\mathbb{D}(H_n)$ of differential operators on $H_n$ invariant under the action (1.1) of $G$. Identifying $T_n$ with $T_n^*$ by the above isomorphism (2.8), we get a canonical linear bijection
\begin{equation}
\Theta_n : \text{Pol}(T_n)^{U(n)} \longrightarrow \mathbb{D}(H_n)
\end{equation}
of $\text{Pol}(T_n)^{U(n)}$ onto $\mathbb{D}(H_n)$. The map $\Theta_n$ is described explicitly as follows. Similarly the action (2.3) induces the action of $K$ on the polynomial algebra $\text{Pol}(\mathfrak{p})$ and the symmetric algebra $S(\mathfrak{p})$ respectively. Through the map $\Psi$, the subalgebra $\text{Pol}(\mathfrak{p})^K$ of $\text{Pol}(\mathfrak{p})$ consisting of $K$-invariants is isomorphic to $\text{Pol}(T_n)^{U(n)}$. We put $N = n(n + 1)$.

Let $\{\xi_\alpha | 1 \leq \alpha \leq N\}$ be a basis of a real vector space $\mathfrak{p}$. If $P \in \text{Pol}(\mathfrak{p})^K$, then
\begin{equation}
(\Theta_n(P)f)(gK) = \left. P \left( \frac{\partial}{\partial t_\alpha} \right) f \left( g \exp \left( \sum_{\alpha=1}^N t_\alpha \xi_\alpha \right) K \right) \right|_{(t_\alpha)=0},
\end{equation}
where $f \in C^\infty(H_n)$. We refer to [9, 10] for more detail. In general, it is hard to express $\Phi(P)$ explicitly for a polynomial $P \in \text{Pol}(\mathfrak{p})^K$.

According to the work of Harish-Chandra [7 8], the algebra $\mathbb{D}(H_n)$ is generated by $n$ algebraically independent generators and is isomorphic to the commutative ring...
\( \mathbb{C}[x_1, \ldots, x_n] \) with \( n \) indeterminates. We note that \( n \) is the real rank of \( G \). Let \( g_\mathbb{C} \) be the complexification of \( g \). It is known that \( D(\mathbb{H}_n) \) is isomorphic to the center of the universal enveloping algebra of \( g_\mathbb{C} \).

Using a classical invariant theory (cf. [11, 21], we can show that \( \text{Pol}(T_n)^U(n) \) is generated by the following algebraically independent polynomials

\[
q_j(\omega) = \text{tr}\left( (\omega\bar{\omega})^j \right), \quad \omega \in T_n, \quad j = 1, 2, \ldots, n.
\]

For each \( j \) with \( 1 \leq j \leq n \), the image \( \Theta_n(q_j) \) of \( q_j \) is an invariant differential operator on \( \mathbb{H}_n \) of degree \( 2j \). The algebra \( D(\mathbb{H}_n) \) is generated by \( n \) algebraically independent generators \( \Theta_n(q_1), \Theta_n(q_2), \ldots, \Theta_n(q_n) \). In particular,

\[
\Theta_n(q_1) = c_1 \text{tr}\left( Y^t \left( Y \frac{\partial}{\partial \Omega} \right) \frac{\partial}{\partial \Omega} \right) \quad \text{for some constant} \quad c_1.
\]

We observe that if we take \( \omega = x + iy \in T_n \) with real \( x, y \), then \( q_1(\omega) = q_1(x, y) = \text{tr}(x^2 + y^2) \) and

\[
q_2(\omega) = q_2(x, y) = \text{tr}\left( (x^2 + y^2)^2 + 2x(xy - yx)y \right).
\]

It is a natural question to express the images \( \Theta_n(q_j) \) explicitly for \( j = 2, 3, \ldots, n \). We hope that the images \( \Theta_n(q_j) \) for \( j = 2, 3, \ldots, n \) are expressed in the form of the trace as \( \Phi(q_1) \).

H. Maass [14] found algebraically independent generators \( H_1, H_2, \ldots, H_n \) of \( D(\mathbb{H}_n) \). We will describe \( H_1, H_2, \ldots, H_n \) explicitly. For \( M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in Sp(n, \mathbb{R}) \) and \( \Omega = X + iY \in \mathbb{H}_n \) with real \( X, Y \), we set

\[
\Omega_* = M_\ast \cdot \Omega = X_\ast + iY_\ast \quad \text{with} \quad X_\ast, Y_\ast \ \text{real}.
\]

We set

\[
K = \left( \Omega - \bar{\Omega} \right) \frac{\partial}{\partial \Omega} = 2iY \frac{\partial}{\partial \Omega},
\]

\[
\Lambda = \left( \Omega - \bar{\Omega} \right) \frac{\partial}{\partial \bar{\Omega}} = 2iY \frac{\partial}{\partial \bar{\Omega}},
\]

\[
K_\ast = \left( \Omega_* - \bar{\Omega}_\ast \right) \frac{\partial}{\partial \Omega_\ast} = 2iY_\ast \frac{\partial}{\partial \Omega_\ast},
\]

\[
\Lambda_\ast = \left( \Omega_* - \bar{\Omega}_\ast \right) \frac{\partial}{\partial \bar{\Omega}_\ast} = 2iY_\ast \frac{\partial}{\partial \bar{\Omega}_\ast}.
\]

Then it is easily seen that

\[
K_\ast = {}^t\left( C\bar{\Omega} + D \right)^{-1} {}^t\left\{ (C\Omega + D)^t K \right\},
\]

\[
(2.14) \quad \Lambda_\ast = {}^t\left( C\bar{\Omega} + D \right)^{-1} {}^t\left\{ (C\Omega + D)^t \Lambda \right\}.
\]
and
\begin{equation}
(2.15) \quad \Lambda t \left\{ (C\Omega + D)' \right\} = \Lambda t (C\Omega + D) - \frac{n+1}{2} (\Omega - \Omega^*) 'C.
\end{equation}

Using Formulas (2.13), (2.14) and (2.15), we can show that
\begin{equation}
(2.16) \quad \Lambda^* K_* + \frac{n+1}{2} K_* = t (C\Omega + D)^{-1} t \left\{ (C\Omega + D) \left( \Lambda K + \frac{n+1}{2} K \right) \right\}.
\end{equation}
Therefore we get
\begin{equation}
(2.17) \quad \text{tr} \left( \Lambda^* K_* + \frac{n+1}{2} K_* \right) = \text{tr} \left( \Lambda K + \frac{n+1}{2} K \right).
\end{equation}

We set
\begin{equation}
(2.18) \quad A^{(1)} = \Lambda K + \frac{n+1}{2} K.
\end{equation}

We define $A^{(j)} (j = 2, 3, \cdots, n)$ recursively by
\begin{equation}
(2.19) \quad A^{(j)} = A^{(1)} A^{(j-1)} - \frac{n+1}{2} \Lambda A^{(j-1)} + \frac{1}{2} \Lambda \text{tr} (A^{(j-1)})
+ \frac{1}{2} (\Omega - \Omega^*) t \left\{ (\Omega - \Omega^*)^{-1} t (t^* A^{(j-1)}) \right\}.
\end{equation}

We set
\begin{equation}
(2.20) \quad H_j = \text{tr} (A^{(j)}), \quad j = 1, 2, \cdots, n.
\end{equation}

As mentioned before, Maass proved that $H_1, H_2, \cdots, H_n$ are algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$.

In fact, we see that
\begin{equation}
(2.21) \quad -H_1 = \Delta_{n:1} = 4 \text{tr} \left( Y t \left( Y \frac{\partial}{\partial \Omega} \right) \frac{\partial}{\partial \Omega} \right),
\end{equation}
is the Laplacian for the invariant metric $ds^2_{n,1}$ on $\mathbb{H}_n$.

**Conjecture.** For $j = 2, 3, \cdots, n$, $\Theta_n(q_j) = c_j H_j$ for a suitable constant $c_j$.

**Example 2.1.** We consider the case $n = 1$. The algebra $\text{Pol}(T_1)^U(1)$ is generated by the polynomial
\[ q(\omega) = \omega \bar{\omega}, \quad \omega = x + iy \in \mathbb{C} \text{ with } x, y \text{ real}. \]

Using Formula (2.10), we get
\[ \Theta_1(q) = 4 y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \]
Therefore $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}[\Theta_1(q)] = \mathbb{C}[H_1]$. 

Example 2.2. We consider the case $n = 2$. The algebra $\text{Pol}(T_2)^{(2)}$ is generated by the polynomial

$$q_1(\omega) = \text{tr}(\omega \bar{\omega}), \quad q_2(\omega) = \text{tr}\left((\omega \bar{\omega})^2\right), \quad \omega \in T_2.$$ 

Using Formula (2.10), we may express $\Theta_2(q_1)$ and $\Theta_2(q_2)$ explicitly. $\Theta_2(q_1)$ is expressed by Formula (2.12). The computation of $\Theta_2(q_2)$ might be quite tedious. We leave the detail to the reader. In this case, $\Theta_2(q_2)$ was essentially computed in [4], Proposition 6. Therefore

$$\mathbb{D}(\mathbb{H}_2) = \mathbb{C}[\Theta_2(q_1), \Theta_2(q_2)] = \mathbb{C}[H_1, H_2].$$

In fact, the center of the universal enveloping algebra $\mathbb{U}(\mathfrak{g}_C)$ was computed in [4].

G. Shimura [18] found canonically defined algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$. We will describe his way of constructing those generators roughly. Let $K_C, \mathfrak{g}_C, \mathfrak{t}_C, \mathfrak{p}_C, \cdots$ denote the complexification of $K, \mathfrak{g}, \mathfrak{t}, \mathfrak{p}, \cdots$ respectively. Then we have the Cartan decomposition

$$\mathfrak{g}_C = \mathfrak{t}_C + \mathfrak{p}_C, \quad \mathfrak{p}_C = \mathfrak{p}_C^+ + \mathfrak{p}_C^-$$

with the properties

$$[\mathfrak{t}_C, \mathfrak{p}_C^\pm] \subset \mathfrak{p}_C^\pm, \quad [\mathfrak{p}_C^+, \mathfrak{p}_C^-] = [\mathfrak{p}_C^-, \mathfrak{p}_C^+] = \{0\}, \quad [\mathfrak{p}_C^+, \mathfrak{p}_C^-] = \mathfrak{t}_C,$$

where

$$\mathfrak{g}_C = \left\{ \left( \begin{array}{cc} X_1 & X_2 \\ X_3 & -tX_1 \end{array} \right) \mid X_1, X_2, X_3 \in \mathbb{C}^{(n,n)}, \quad X_2 = tX_2, \quad X_3 = tX_3 \right\},$$

$$\mathfrak{t}_C = \left\{ \left( \begin{array}{cc} A & -B \\ B & A \end{array} \right) \in \mathbb{C}^{(2n,2n)} \mid tA + A = 0, \quad B = tB \right\},$$

$$\mathfrak{p}_C = \left\{ \left( \begin{array}{cc} X & Y \\ Y & -X \end{array} \right) \in \mathbb{C}^{(2n,2n)} \mid X = tX, \quad Y = tY \right\},$$

$$\mathfrak{p}_C^+ = \left\{ \left( \begin{array}{cc} Z & iZ \\ iZ & -Z \end{array} \right) \in \mathbb{C}^{(2n,2n)} \mid Z = tZ \in \mathbb{C}^{(n,n)} \right\},$$

$$\mathfrak{p}_C^- = \left\{ \left( \begin{array}{cc} Z & -iZ \\ -iZ & -Z \end{array} \right) \in \mathbb{C}^{(2n,2n)} \mid Z = tZ \in \mathbb{C}^{(n,n)} \right\}.$$

For a complex vector space $W$ and a nonnegative integer $r$, we denote by $\text{Pol}_r(W)$ the vector space of complex-valued homogeneous polynomial functions on $W$ of degree $r$. We put

$$\text{Pol}^r(W) := \sum_{s=0}^{r} \text{Pol}_s(W).$$

$\text{ML}_r(W)$ denotes the vector space of all $\mathbb{C}$-multilinear maps of $W \times \cdots \times W$ ($r$ copies) into $\mathbb{C}$. An element $Q$ of $\text{ML}_r(W)$ is called symmetric if

$$Q(x_1, \cdots, x_r) = Q(x_{\pi(1)}, \cdots, x_{\pi(r)}).$$
for each permutation \(\pi\) of \(\{1, 2, \cdots, r\}\). Given \(P \in \text{Pol}_r(W)\), there is a unique element symmetric element \(P_*\) of \(\text{ML}_r(W)\) such that

\[
P(x) = P_*(x, \cdots, x) \quad \text{for all } x \in W.
\]

Moreover the map \(P \mapsto P_*\) is a \(\mathbb{C}\)-linear bijection of \(\text{Pol}_r(W)\) onto the set of all symmetric elements of \(\text{ML}_r(W)\). We let \(S_r(W)\) denote the subspace consisting of all homogeneous elements of degree \(r\) in the symmetric algebra \(S(W)\). We note that \(\text{Pol}_r(W)\) and \(S_r(W)\) are dual to each other with respect to the pairing

\[
\langle \alpha, x_1 \cdots x_r \rangle = \alpha_*(x_1, \cdots, x_r) \quad (x_i \in W, \alpha \in \text{Pol}_r(W)).
\]

Let \(\mathfrak{p}_C^*\) be the dual space of \(\mathfrak{p}_C\), that is, \(\mathfrak{p}_C^* = \text{Pol}_1(\mathfrak{p}_C)\). Let \(\{X_1, \cdots, X_N\}\) be a basis of \(\mathfrak{p}_C\) and \(\{Y_1, \cdots, Y_N\}\) be the basis of \(\mathfrak{p}_C^*\) dual to \(\{X_\nu\}\), where \(N = n(n + 1)\). We note that \(\text{Pol}_r(\mathfrak{p}_C)\) and \(\text{Pol}_r(\mathfrak{p}_C^*)\) are dual to each other with respect to the pairing

\[
\langle \alpha, \beta \rangle = \sum \alpha_*(X_{i_1}, \cdots, X_{i_r}) \beta_*(Y_{i_1}, \cdots, Y_{i_r}),
\]

where \(\alpha \in \text{Pol}_r(\mathfrak{p}_C)\), \(\beta \in \text{Pol}_r(\mathfrak{p}_C^*)\) and \((i_1, \cdots, i_r)\) runs over \(\{1, \cdots, N\}^r\). Let \(\mathcal{U}(\mathfrak{g}_C)\) be the universal enveloping algebra of \(\mathfrak{g}_C\) and \(\mathcal{W}(\mathfrak{g}_C)\) its subspace spanned by the elements of the form \(V_1 \cdots V_s\) with \(V_i \in \mathfrak{g}_C\) and \(s \leq p\). We recall that there is a \(\mathbb{C}\)-linear bijection \(\psi\) of the symmetric algebra \(S(\mathfrak{g}_C)\) of \(\mathfrak{g}_C\) onto \(\mathcal{U}(\mathfrak{g}_C)\) which is characterized by the property that \(\psi(X^r) = X^r\) for all \(X \in \mathfrak{g}_C\). For each \(\alpha \in \text{Pol}_r(\mathfrak{p}_C^*)\) we define an element \(\omega(\alpha)\) of \(\mathcal{U}(\mathfrak{g}_C)\) by

\[
\omega(\alpha) := \sum \alpha_*(Y_{i_1}, \cdots, Y_{i_r}) X_{i_1} \cdots X_{i_r},
\]

where \((i_1, \cdots, i_r)\) runs over \(\{1, \cdots, N\}^r\). If \(Y \in \mathfrak{p}_C\), then \(Y^r\) as an element of \(\text{Pol}_r(\mathfrak{p}_C^*)\) is defined by

\[
Y^r(u) = Y(u)^r \quad \text{for all } u \in \mathfrak{p}_C^*.
\]

Hence \((Y^r)_*(u_1, \cdots, u_r) = Y(u_1) \cdots Y(u_r)\). According to (2.25), we see that if \(\alpha(\sum t_i Y_i) = P(t_1, \cdots, t_N)\) for \(t_i \in \mathbb{C}\) with a polynomial \(P\), then

\[
\omega(\alpha) = \psi(P(X_1, \cdots, X_N)).
\]

Thus \(\omega\) is a \(\mathbb{C}\)-linear injection of \(\text{Pol}_r(\mathfrak{p}_C^*)\) into \(\mathcal{U}(\mathfrak{g}_C)\) independent of the choice of a basis. We observe that \(\omega(\text{Pol}_r(\mathfrak{p}_C^*)) = \psi(S_r(\mathfrak{p}_C))\). It is a well-known fact that if \(\alpha_1, \cdots, \alpha_m \in \text{Pol}_r(\mathfrak{p}_C^*)\), then

\[
\omega(\alpha_1 \cdots \alpha_m) - \omega(\alpha_m) \cdots \omega(\alpha_1) \in \mathcal{W}^{r-1}(\mathfrak{g}_C).
\]

We have a canonical pairing

\[
\langle \ , \ \rangle : \text{Pol}_r(\mathfrak{p}_C^*) \times \text{Pol}_r(\mathfrak{p}_C) \longrightarrow \mathbb{C}
\]

defined by

\[
\langle f, g \rangle = \sum f_*(\tilde{X}_{i_1}, \cdots, \tilde{X}_{i_r}) g_*(\tilde{Y}_{i_1}, \cdots, \tilde{Y}_{i_r}),
\]

where \(f_*\) (resp. \(g_*\)) are the unique symmetric elements of \(\text{ML}_r(\mathfrak{p}_C^*)\) (resp. \(\text{ML}_r(\mathfrak{p}_C^-)\)), and \(\{\tilde{X}_1, \cdots, \tilde{X}_N\}\) and \(\{\tilde{Y}_1, \cdots, \tilde{Y}_N\}\) are dual bases of \(\mathfrak{p}_C^*\) and \(\mathfrak{p}_C^-\) with respect to
the Killing form \( B(X, Y) = 2(n + 1) \text{tr}(XY) \), \( \widetilde{N} = \frac{n(n+1)}{2} \), and \((i_1, \cdots, i_r)\) runs over \(\{1, \cdots, \widetilde{N}\}\).

The adjoint representation of \(K_C\) on \(\mathfrak{p}^\pm_C\) induces the representation of \(K_C\) on \(\text{Pol}_r(\mathfrak{p}^+_C)\). Given a \(K_C\)-irreducible subspace \(Z\) of \(\text{Pol}_r(\mathfrak{p}^+_C)\), we can find a unique \(K_C\)-irreducible subspace \(W\) of \(\text{Pol}_r(\mathfrak{p}^-_C)\) such that \(\text{Pol}_r(\mathfrak{p}^-_C)\) is the direct sum of \(W\) and the annihilator of \(Z\). Then \(Z\) and \(W\) are dual with respect to the pairing (2.28).

Take bases \(\{\zeta_1, \cdots, \zeta_k\}\) of \(Z\) and \(\{\xi_1, \cdots, \xi_r\}\) of \(W\) that are dual to each other. We set

\[
(2.30) \quad f_Z(x, y) = \sum_{\nu=1}^k \zeta_\nu(x) \xi_\nu(y) \quad (x \in \mathfrak{p}^+_C, \ y \in \mathfrak{p}^-_C).
\]

It is easily seen that \(f_Z\) belongs to \(\text{Pol}_{2r}(\mathfrak{p}_C)^K\) and is independent of the choice of dual bases \(\{\zeta_\nu\}\) and \(\{\xi_\nu\}\). Shimura [13] proved that there exists a canonically defined set \(\{Z_1, \cdots, Z_n\}\) with a \(K_C\)-irreducible subspace \(Z_r\) of \(\text{Pol}_r(\mathfrak{p}^+_C)\) \((1 \leq r \leq n)\) such that \(f_{Z_1}, \cdots, f_{Z_n}\) are algebraically independent generators of \(\text{Pol}(\mathfrak{p}_C)^K\). We can identify \(\mathfrak{p}^-_C\) with \(T_n\). We recall that \(T_n\) denotes the vector space of \(n \times n\) symmetric complex matrices. We can take \(Z_r\) as the subspace of \(\text{Pol}_r(T_n)\) spanned by the functions \(f_{a\nu}(Z) = \det_r(\nu a Za)\) for all \(a \in GL(n, \mathbb{C})\), where \(\det_r(x)\) denotes the determinant of the upper left \(r \times r\) submatrix of \(x\). For every \(f \in \text{Pol}(\mathfrak{p}_C)^K\), we let \(\Omega(f)\) denote the element of \(D(\mathbb{H}_n)\) represented by \(\omega(f)\). Then \(D(\mathbb{H}_n)\) is the polynomial ring \(\mathbb{C}[\omega(f_{Z_1}), \cdots, \omega(f_{Z_n})]\) generated by \(n\) algebraically independent elements \(\omega(f_{Z_1}), \cdots, \omega(f_{Z_n})\).

3. Invariant Differential Operators on Siegel-Jacobi Space

The stabilizer \(K^J\) of \(G^J\) at \((iI_n, 0)\) is given by

\[
K^J = \left\{(k, (0, 0; \kappa)) \mid k \in K, \ k = \iota \kappa \in \mathbb{R}^{(m,m)}\right\}.
\]

Therefore \(\mathbb{H}_{n,m} \cong G^J/K^J\) is a homogeneous space of non-reductive type. The Lie algebra \(\mathfrak{g}^J\) of \(G^J\) has a decomposition

\[
\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,
\]

where

\[
\mathfrak{g}^J = \left\{(Z, (P, Q, R)) \mid Z \in \mathfrak{g}, \ P, Q \in \mathbb{R}^{(m,n)}, \ R = \iota R \in \mathbb{R}^{(m,m)}\right\},
\]

\[
\mathfrak{k}^J = \left\{(X, (0, 0, R)) \mid X \in \mathfrak{k}, \ R = \iota R \in \mathbb{R}^{(m,m)}\right\},
\]

\[
\mathfrak{p}^J = \left\{(Y, (P, Q, 0)) \mid Y \in \mathfrak{p}, \ P, Q \in \mathbb{R}^{(m,n)}\right\}.
\]

Thus the tangent space of the homogeneous space \(\mathbb{H}_{n,m}\) at \((iI_n, 0)\) is identified with \(\mathfrak{p}^J\).
If \( \alpha = \left( \begin{pmatrix} X_1 \\ Z_1 \\ -X_1 \end{pmatrix}, (P_1, Q_1, R_1) \right) \) and \( \beta = \left( \begin{pmatrix} X_2 \\ Z_2 \\ -X_2 \end{pmatrix}, (P_2, Q_2, R_2) \right) \) are elements of \( \mathfrak{p}^J \), then the Lie bracket \([\alpha, \beta]\) of \( \alpha \) and \( \beta \) is given by

\[
[\alpha, \beta] = \left( \begin{pmatrix} X^* \\ Z^* \\ -X^* \end{pmatrix}, (P^*, Q^*, R^*) \right),
\]

where

\[
\begin{align*}
X^* &= X_1X_2 - X_2X_1 + Y_1Z_2 - Y_2Z_1, \\
Y^* &= X_1Y_2 - X_2Y_1 + Y_1^tX_1 - Y_1^tX_2, \\
Z^* &= Z_1X_2 - Z_2X_1 + X_2Z_1 - X_1Z_2, \\
P^* &= P_1X_2 - P_2X_1 + Q_1Z_2 - Q_2Z_1, \\
Q^* &= P_1Y_2 - P_2Y_1 + Q_2^tX_1 - Q_1^tX_2, \\
R^* &= P_1^tQ_2 - P_2^tQ_1 + Q_2^tP_1 - Q_1^tP_2.
\end{align*}
\]

Lemma 3.1.

\([\mathfrak{k}^J, \mathfrak{k}^J] \subset \mathfrak{k}^J, [\mathfrak{k}^J, \mathfrak{p}^J] \subset \mathfrak{p}^J\).

Proof. The proof follows immediately from Formula (3.1). \(\square\)

Lemma 3.2. Let

\[
k^J = \left( \begin{pmatrix} A \\ -B \\ A \end{pmatrix}, (0, 0, \kappa) \right) \in K^J
\]

with \( \begin{pmatrix} A \\ B \\ A \end{pmatrix} \in K, \ \kappa = \kappa^t \in \mathbb{R}^{(m,m)} \) and

\[
\alpha = \left( \begin{pmatrix} X \\ Y \\ -X \end{pmatrix}, (P, Q, 0) \right) \in \mathfrak{p}^J
\]

with \( X = X^t, \ Y = Y^t \in \mathbb{R}^{(n,n)}, \ P, Q \in \mathbb{R}^{(m,n)} \). Then the adjoint action of \( K^J \) on \( \mathfrak{p}^J \) is given by

\[
Ad(k^J)\alpha = \left( \begin{pmatrix} X^* \\ Y^* \\ -X^* \end{pmatrix}, (P^*, Q^*, 0) \right),
\]

where

\[
\begin{align*}
X^* &= AX^tA - (BX^tB + BY^tA + AY^tB), \\
Y^* &= (AX^tB + AY^tA + BX^tA) - BY^tB, \\
P^* &= P^tA - Q^tB, \\
Q^* &= P^tB + Q^tA.
\end{align*}
\]

Proof. We leave the proof to the reader. \(\square\)
We recall that $T_n$ denotes the vector space of all $n \times n$ symmetric complex matrices. For brevity, we put $T_{n,m} := T_n \times \mathbb{C}^{(m,n)}$. We define the real linear map $\Phi : p^J \rightarrow T_{n,m}$ by

\begin{equation}
\Phi \left( \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P, Q, 0) \right) = (X + iY, P + iQ),
\end{equation}

where $\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in p$ and $P, Q \in \mathbb{R}^{(m,n)}$.

Let $S(n, \mathbb{R})$ denote the additive group consisting of all $n \times n$ real symmetric matrices. Now we define the isomorphism $\theta : K^J \rightarrow U(n) \times S(n, \mathbb{R})$ by

\begin{equation}
\theta(h, (0, 0, \kappa)) = (\delta(h), \kappa), \quad h \in K, \quad \kappa \in S(n, \mathbb{R}),
\end{equation}

where $\delta : K \rightarrow U(n)$ is the map defined by (2.5). Identifying $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$, we can identify $p^J$ with $T_n \times \mathbb{C}^{(m,n)}$.

**Theorem 3.1.** The adjoint representation of $K^J$ on $p^J$ is compatible with the natural action of $U(n) \times S(n, \mathbb{R})$ on $T_{n,m}$ defined by

\begin{equation}
(h, \kappa) \cdot (\omega, z) := (h \omega^t h, z^t h), \quad h \in U(n), \quad \kappa \in S(n, \mathbb{R}), \quad (\omega, z) \in T_{n,m}
\end{equation}

through the maps $\Phi$ and $\theta$. Precisely, if $k^J \in K^J$ and $\alpha \in p^J$, then we have the following equality

\begin{equation}
\Phi(Ad(k^J) \alpha) = \theta(k^J) \cdot \Phi(\alpha).
\end{equation}

Here we regard the complex vector space $T_{n,m}$ as a real vector space.

**Proof.** Let

$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K, \quad \kappa = \kappa^t \in \mathbb{R}^{(m,m)}$ and

$\alpha = \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P, Q, 0) \in p^J$

with $X = X^t, \; Y = Y^t \in \mathbb{R}^{(n,n)}, \; P, Q \in \mathbb{R}^{(m,n)}$. Then we have

$\theta(k^J) \cdot \Phi(\alpha) = (A + iB, \kappa) \cdot (X + iY, P + iQ)$

$= ((A + iB)(X + iY)^t (A + iB), (P + iQ)^t (A + iB))$

$= (X_* + iY_*, P_* + iQ_*)$

$= \Phi \left( \begin{pmatrix} X_* & Y_* \\ Y_* & -X_* \end{pmatrix}, (P_*, Q_*, 0) \right)$

$= \Phi(Ad(k^J) \alpha) \quad (by \ \text{Lemma \ 3.2}),$
where \( X_*, Y_*, Z_* \) and \( Q_* \) are given by the formulas (3.3), (3.4), (3.5) and (3.6) respectively.

We now study the algebra \( \mathbb{D}(\mathbb{H}_{n,m}) \) of all differential operators on \( \mathbb{H}_{n,m} \) invariant under the \textit{natural action} (1.2) of \( G^J \). The action (3.9) induces the action of \( U(n) \) on the polynomial algebra \( \text{Pol}_{n,m} := \text{Pol}(T_{n,m}) \). We denote by \( \text{Pol}_{n,m}^{U(n)} \) the subalgebra of \( \text{Pol}_{n,m} \) consisting of all \( U(n) \)-invariants. Similarly the action (3.2) of \( K \) induces the action of \( K \) on the polynomial algebra \( \text{Pol}(p^J) \). We see that through the identification of \( p^J \) with \( T_{n,m} \), the algebra \( \text{Pol}(p^J) \) is isomorphic to \( \text{Pol}_{n,m} \). The following \( U(n) \)-invariant inner product \((,)_* \) of the complex vector space \( T_{n,m} \) defined by

\[
((\omega, z), (\omega', z'))_* = \text{tr}(\omega\overline{\omega'}) + \text{tr}(z\overline{z'}), \quad (\omega, z), (\omega', z') \in T_{n,m}
\]
gives a canonical isomorphism

\[
T_{n,m} \cong T_{n,m}^*, \quad (\omega, z) \mapsto f_{\omega, z}, \quad (\omega, z) \in T_{n,m},
\]

where \( f_{\omega, z} \) is the linear functional on \( T_{n,m} \) defined by

\[
f_{\omega, z}(\omega', z') = (\omega', z')(\omega, z)_*, \quad (\omega', z') \in T_{n,m}.
\]

According to Helgason ([II], p. 287), one gets a canonical linear bijection of \( S(T_{n,m})^{U(n)} \) onto \( \mathbb{D}(\mathbb{H}_{n,m}) \). Identifying \( T_{n,m} \) with \( T_{n,m}^* \) by the above isomorphism, one gets a natural linear bijection

\[
\Theta_{n,m} : \text{Pol}_{n,m}^{U(n)} \rightarrow \mathbb{D}(\mathbb{H}_{n,m})
\]
of \( \text{Pol}_{n,m}^{U(n)} \) onto \( \mathbb{D}(\mathbb{H}_{n,m}) \). The map \( \Theta_{n,m} \) is described explicitly as follows. We put \( N_* = n(n + 1) + 2mn \). Let \( \{ \eta_\alpha \mid 1 \leq \alpha \leq N_* \} \) be a basis of \( p^J \). If \( P \in \text{Pol}(p^J)^K = \text{Pol}_{n,m}^{U(n)} \), then

\[
(\Theta_{n,m}(P)f)(gK^J) = \left[ P \left( \frac{\partial}{\partial t_\alpha} \right) f \left( g \exp \left( \sum_{\alpha=1}^{N_*} t_\alpha \eta_\alpha \right) K^J \right) \right]_{(t_\alpha) = 0},
\]

where \( f \in C^\infty(\mathbb{H}_{n,m}) \). In general, it is hard to express \( \Theta_{n,m}(P) \) explicitly for a polynomial \( P \in \text{Pol}(p^J)^K \). We refer to [II], p. 287.

We present the following \textit{basic} \( U(n) \)-invariant polynomials in \( \text{Pol}_{n,m}^{U(n)} \).

\[
q_j(\omega, z) = \text{tr}(\omega \overline{\omega}^{j+1}), \quad 0 \leq j \leq n - 1,
\]

\[
\alpha_{kp}^{(j)}(\omega, z) = \text{Re} \left( z (\overline{\omega})^j \overline{\omega'}_{kp} \right), \quad 0 \leq j \leq n - 1, \quad 1 \leq k \leq p \leq m,
\]

\[
\beta_{lq}^{(j)}(\omega, z) = \text{Im} \left( z (\overline{\omega})^j \overline{\omega'}_{lq} \right), \quad 0 \leq j \leq n - 1, \quad 1 \leq l < q \leq m,
\]

\[
f_{kp}^{(j)}(\omega, z) = \text{Re} \left( z (\overline{\omega})^j \overline{\omega'}z_{kp} \right), \quad 0 \leq j \leq n - 1, \quad 1 \leq k \leq p \leq m,
\]

\[
g_{kp}^{(j)}(\omega, z) = \text{Im} \left( z (\overline{\omega})^j \overline{\omega'}z_{kp} \right), \quad 0 \leq j \leq n - 1, \quad 1 \leq k \leq p \leq m,
\]

where \( \omega \in T_n \) and \( z \in \mathbb{C}^{(m,n)} \).
We present some interesting $U(n)$-invariants. For an $m \times m$ matrix $S$, we define the following invariant polynomials in $\text{Pol}_{n,m}^U$:

\begin{align}
\text{(3.17)} & \quad m_{j,S}^{(1)}(\omega, z) = \text{Re}\left(\text{tr}(\omega \zbar + t_z S \zbar)^j\right), \quad 1 \leq j \leq n, \\
\text{(3.18)} & \quad m_{j,S}^{(2)}(\omega, z) = \text{Im}\left(\text{tr}(\omega \zbar + t_z S \zbar)^j\right), \quad 1 \leq j \leq n, \\
\text{(3.19)} & \quad q_{k,S}^{(1)}(\omega, z) = \text{Re}\left(\text{tr}\left((t_z S \zbar)^k\right)\right), \quad 1 \leq k \leq m, \\
\text{(3.20)} & \quad q_{k,S}^{(2)}(\omega, z) = \text{Im}\left(\text{tr}\left((t_z S \zbar)^k\right)\right), \quad 1 \leq k \leq m, \\
\text{(3.21)} & \quad \theta_{i,k,j,S}^{(1)}(\omega, z) = \text{Re}\left(\text{tr}\left((\omega \zbar)^i (t_z S \zbar)^k (\omega \zbar + t_z S \zbar)^j\right)\right), \\
\text{(3.22)} & \quad \theta_{i,k,j,S}^{(2)}(\omega, z) = \text{Im}\left(\text{tr}\left((\omega \zbar)^i (t_z S \zbar)^k (\omega \zbar + t_z S \zbar)^j\right)\right),
\end{align}

where $1 \leq i, j \leq n$ and $1 \leq k \leq m$.

We define the following $U(n)$-invariant polynomials in $\text{Pol}_{n,m}^U$:

\begin{align}
\text{(3.23)} & \quad r_{j,k}^{(1)}(\omega, z) = \text{Re}\left(\det\left((\omega \zbar)^j (t_z \zbar)^k\right)\right), \quad 1 \leq j \leq n, 1 \leq k \leq m, \\
\text{(3.24)} & \quad r_{j,k}^{(2)}(\omega, z) = \text{Im}\left(\det\left((\omega \zbar)^j (t_z \zbar)^k\right)\right), \quad 1 \leq j \leq n, 1 \leq k \leq m.
\end{align}

We propose the following natural problems.

**Problem 1.** Find a complete list of explicit generators of $\text{Pol}_{n,m}^U$.

**Problem 2.** Find all the relations among a set of generators of $\text{Pol}_{n,m}^U$.

**Problem 3.** Find an easy or effective way to express the images of the above invariant polynomials or generators of $\text{Pol}_{n,m}^U$ under the Helgason map $\Theta_{n,m}$ explicitly.

**Problem 4.** Decompose $\text{Pol}_{n,m}$ into $U(n)$-irreducibles.

**Problem 5.** Find a complete list of explicit generators of the algebra $\mathbb{D}(\mathbb{H}_{n,m})$. Or construct explicit $G^J$-invariant differential operators on $\mathbb{H}_{n,m}$.

**Problem 6.** Find all the relations among a set of generators of $\mathbb{D}(\mathbb{H}_{n,m})$.

**Problem 7.** Is $\text{Pol}_{n,m}^U$ finitely generated? Is $\mathbb{D}(\mathbb{H}_{n,m})$ finitely generated?

Quite recently Minoru Itoh [12] solved Problem 1 and Problem 7.

**Theorem 3.2.** $\text{Pol}_{n,m}^U$ is generated by

\[ q_j(\omega, z), \alpha_{kp}^{(j)}(\omega, z), \beta_{iq}^{(j)}(\omega, z), f_{kp}^{(j)}(\omega, z) \text{ and } g_{kp}^{(j)}(\omega, z), \]

where $0 \leq j \leq n - 1$, $1 \leq k \leq p \leq m$ and $1 \leq l < q \leq m$. 
4. **Examples of Explicit $G_J$-Invariant Differential Operators**

In this section we give examples of explicit $G_J$-invariant differential operators on the Siegel-Jacobi space and the Siegel-Jacobi disk.

For $g = (M, (\lambda, \mu; \kappa)) \in G_J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, we set

$$\Omega_* = M \cdot \Omega = X_* + i Y_*, \quad X_*, Y_* \text{ real},$$

$$Z_* = (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1} = U_* + i V_*, \quad U_*, V_* \text{ real}.$$  \hfill (4.1)

For a coordinate $(\Omega, Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu})$ and $Z = (z_{kl})$, we put $d\Omega$, $\overline{d\Omega}$, $\frac{\partial}{\partial \Omega}$, $\frac{\partial}{\partial \overline{\Omega}}$ as before and set

$$Z = U + i V, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real},$$

$$dZ = (dz_{kl}), \quad d\overline{Z} = (d\overline{z}_{kl}),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{mn}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix}.$$  \hfill (4.2)

Then we can show that

$$d\Omega_* = t(C \Omega + D)^{-1} d\Omega(C \Omega + D)^{-1},$$

$$dZ_* = dZ(C \Omega + D)^{-1} + \{ \lambda - (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1} C \} d\Omega(C \Omega + D)^{-1},$$

$$\frac{\partial}{\partial \Omega_*} = (C \Omega + D) \left\{ (C \Omega + D) \frac{\partial}{\partial \Omega} \right\} + (C \Omega + D)^t \left\{ (C^t Z + C^t \mu - D^t \lambda)^t \left( \frac{\partial}{\partial \overline{Z}} \right) \right\}$$

and

$$\frac{\partial}{\partial Z_*} = (C \Omega + D) \frac{\partial}{\partial \overline{Z}}.$$  \hfill (4.3)

From [14, p. 33] or [20, p. 128], we know that

$$Y_* = t(C \overline{\Omega} + D)^{-1} Y(C \Omega + D)^{-1} = t(C \Omega + D)^{-1} Y(C \overline{\Omega} + D)^{-1}.$$  \hfill (4.4)

$$Y_* = t(C \overline{\Omega} + D)^{-1} Y(C \Omega + D)^{-1} = t(C \Omega + D)^{-1} Y(C \overline{\Omega} + D)^{-1}.$$  \hfill (4.5)
Using Formulas (4.1), (4.2) and (4.5), the author [29] proved that for any two positive real numbers $A$ and $B$,

\[
d s^2_{n,m;A,B} = A \text{tr} \left( Y^{-1} d\Omega Y^{-1} d\Omega \right) \\
+ B \left\{ \text{tr} \left( Y^{-1} tV V Y^{-1} d\Omega Y^{-1} d\Omega \right) + \text{tr} \left( Y^{-1} t(dZ) d\Omega \right) \\
- \text{tr} \left( V Y^{-1} d\Omega Y^{-1} t(dZ) \right) - \text{tr} \left( V Y^{-1} d\Omega Y^{-1} t(dZ) \right) \right\}
\]

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) of $G^J$.

The following lemma is very useful for computing the invariant differential operators. H. Maass [13] observed the following useful fact.

**Lemma 4.1.** (a) Let $A$ be an $m \times n$ matrix and $B$ an $n \times l$ matrix. Assume that the entries of $A$ commute with the entries of $B$. Then $t(AB) = tB tA$.

(b) Let $A, B$ and $C$ be a $k \times l$, an $n \times m$ and an $m \times l$ matrix respectively. Assume that the entries of $A$ commute with the entries of $B$. Then

\[
t(A t(BC)) = B t(A tC).
\]

*Proof.* The proof follows immediately from the direct computation. $\square$

Using Formulas (4.3), (4.4), (4.5) and Lemma 4.1, the author [29] proved that the following differential operators $M_1$ and $M_2$ on $\mathbb{H}_{n,m}$ defined by

\[
M_1 = \text{tr} \left( Y \frac{\partial}{\partial Z} \left( \frac{\partial}{\partial Z} \right) \right)
\]

and

\[
M_2 = \text{tr} \left( Y t \left( Y \frac{\partial}{\partial \Omega} \frac{\partial}{\partial \Omega} \right) \frac{\partial}{\partial Z} \right) + \text{tr} \left( V Y^{-1} tV \frac{\partial}{\partial \Omega} \frac{\partial}{\partial \Omega} \right) \\
+ \text{tr} \left( V t \frac{\partial}{\partial \Omega} \frac{\partial}{\partial Z} \right) + \text{tr} \left( tV t \frac{\partial}{\partial \Omega} \frac{\partial}{\partial Z} \right)
\]

are invariant under the action (1.2) of $G^J$. The author [29] proved that for any two positive real numbers $A$ and $B$, the following differential operator

\[
\Delta_{n,m;A,B} = \frac{4}{A} M_2 + \frac{4}{B} M_1
\]

is the Laplacian of the $G^J$-invariant Riemannian metric $ds^2_{n,m;A,B}$. 
Proposition 4.1. The following differential operator $K$ on $\mathbb{H}_{n,m}$ of degree $2n$ defined by

\begin{equation}
K = \det(Y) \det\left(\frac{\partial}{\partial Z}\left(\frac{\partial}{\partial Z}\right)\right)
\end{equation}

is invariant under the action (1.2) of $G^J$.

Proof. Let $K_{M, (\lambda, \mu; \kappa)}$ denote the image of $K$ under the transformation

$$(\Omega, Z) \mapsto ((M \cdot \Omega, (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1})$$

with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(\lambda, \mu; \kappa) \in H^R_{\mathbb{R}}$. If $f$ is a $C^\infty$ function on $\mathbb{H}_{n,m}$, using (4.4), (4.5) and Lemma 4.1, we have

$$K_{M, (\lambda, \mu; \kappa)}f = \det(Y) |\det(C \Omega + D)|^{-2} \det\left[(C \Omega + D)\frac{\partial}{\partial Z}\left\{ (C \Omega + D)\frac{\partial f}{\partial Z} \right\}\right]$$

$$= \det(Y) |\det(C \Omega + D)|^{-2} \det\left[(C \Omega + D)^t\left\{ (C \Omega + D)^t\left(\frac{\partial}{\partial Z}\left(\frac{\partial f}{\partial Z}\right)\right)\right\}\right]$$

$$= \det(Y) |\det(C \Omega + D)|^{-2} \det\left[(C \Omega + D)^t\left(\frac{\partial}{\partial Z}\left(\frac{\partial f}{\partial Z}\right)\right)^t(C \Omega + D)\right]$$

$$= \det(Y) \det\left(\frac{\partial}{\partial Z}\left(\frac{\partial f}{\partial Z}\right)\right)$$

$$= Kf.$$}

Since $M \in Sp(n, \mathbb{R})$ and $(\lambda, \mu; \kappa) \in H^R_{\mathbb{R}}$ are arbitrary, $K$ is invariant under the action (1.2) of $G^J$. \qed

Proposition 4.2. The following matrix-valued differential operator $T$ on $\mathbb{H}_{n,m}$ defined by

\begin{equation}
T = \left(\frac{\partial}{\partial Z}\right)^t Y^t \frac{\partial}{\partial Z}
\end{equation}

is invariant under the action (1.2) of $G^J$.

Proof. Let $T_{M, (\lambda, \mu; \kappa)}$ denote the image of $K$ under the transformation

$$(\Omega, Z) \mapsto ((M \cdot \Omega, (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1})$$
with \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}) \) and \( (\lambda, \mu; \kappa) \in H^{(n,m)}_{\mathbb{R}} \). If \( f \) is a \( C^\infty \) function on \( \mathbb{H}_{n,m} \), according to (4.4), (4.5) and Lemma 4.1, we have

\[
T_{M,(\lambda,\mu;\kappa)} f = t \left( (C\Omega + D) \frac{\partial}{\partial Z} \right) t( (C\Omega + D)^{-1} Y (C\Omega + D)^{-1} (C\Omega + D) \frac{\partial f}{\partial Z} \\
= \left( \frac{\partial}{\partial Z} \right) Y \frac{\partial f}{\partial Z} \\
= T f.
\]

Since \( M \in Sp(n, \mathbb{R}) \) and \( (\lambda, \mu; \kappa) \in H^{(n,m)}_{\mathbb{R}} \) are arbitrary, \( T \) is invariant under the action (1.2) of \( G^J \).

**Corollary 4.1.** Each \((k,l)\)-entry \( T_{kl} \) of \( T \) given by

\[
(4.11) \quad T_{kl} = \sum_{i,j=1}^n y_{ij} \frac{\partial^2}{\partial z_{ki} \partial z_{lj}}, \quad 1 \leq k, l \leq m
\]

is an element of \( \mathbb{D}(\mathbb{H}_{n,m}) \).

**Proof.** It follows immediately from Proposition 4.2. \( \square \)

Now we consider invariant differential operators on the Siegel-Jacobi bi disk. Let

\[
\mathbb{D}_n = \{ W \in \mathbb{C}^{(n,n)} \mid W = ^t W, \; I_n - \overline{WW} > 0 \}
\]

be the generalized unit disk.

For brevity, we write \( \mathbb{D}_{n,m} := \mathbb{D}_n \times \mathbb{C}^{(m,n)} \). For a coordinate \((W, \eta) \in \mathbb{D}_{n,m}\) with \( W = (w_{\mu\nu}) \in \mathbb{D}_n \) and \( \eta = (\eta_{kl}) \in \mathbb{C}^{(m,n)} \), we put

\[
dW = (dw_{\mu\nu}), \quad d\overline{W} = (d\overline{w}_{\mu\nu}),
\]

\[
d\eta = (d\eta_{kl}), \quad d\overline{\eta} = (d\overline{\eta}_{kl})
\]

and

\[
\frac{\partial}{\partial W} = \left( \frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial w_{\mu\nu}} \right), \quad \frac{\partial}{\partial \overline{W}} = \left( \frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{w}_{\mu\nu}} \right),
\]

\[
\frac{\partial}{\partial \eta} = \left( \begin{array}{ccc} \frac{\partial}{\partial \eta_{11}} & \cdots & \frac{\partial}{\partial \eta_{1m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \eta_{n1}} & \cdots & \frac{\partial}{\partial \eta_{nm}} \end{array} \right), \quad \frac{\partial}{\partial \overline{\eta}} = \left( \begin{array}{ccc} \frac{\partial}{\partial \overline{\eta}_{11}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{1m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{\eta}_{n1}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{nm}} \end{array} \right).
\]
We can identify an element \( g = (M, (\lambda, \mu; \kappa)) \) of \( G^J \), \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}) \) with the element
\[
\begin{pmatrix}
A & 0 & B & A^t \mu - B^t \lambda \\
\lambda & I_m & \mu & \kappa \\
C & 0 & D & C^t \mu - D^t \lambda \\
0 & 0 & 0 & I_m
\end{pmatrix}
\]
of \( Sp(m + n, \mathbb{R}) \).

We set
\[
T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{m+n} & I_{m+n} \\ iI_{m+n} & -iI_{m+n} \end{pmatrix}.
\]

We now consider the group \( G_*^J \) defined by
\[
G_*^J := T_*^{-1} G^J T_*.
\]
If \( g = (M, (\lambda, \mu; \kappa)) \in G^J \) with \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}) \), then \( T_*^{-1} g T_* \) is given by
\[
(4.12) \quad T_*^{-1} g T_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q_*} & \overline{P_*} \end{pmatrix},
\]
where
\[
P_* = \left( \frac{P}{2} (\lambda + i\mu) \begin{pmatrix} Q^t (\lambda + i\mu) - P^t (\lambda - i\mu) \\ I_h + i\kappa \end{pmatrix} \right),
\]
\[
Q_* = \left( \frac{Q}{2} (\lambda - i\mu) \begin{pmatrix} P^t (\lambda - i\mu) - Q^t (\lambda + i\mu) \\ -i\kappa \end{pmatrix} \right),
\]
and \( P, Q \) are given by the formulas
\[
(4.13) \quad P = \frac{1}{2} \left\{ (A + D) + i (B - C) \right\}
\]
and
\[
(4.14) \quad Q = \frac{1}{2} \left\{ (A - D) - i (B + C) \right\}.
\]
From now on, we write
\[
\left( \begin{pmatrix} P \\ Q \\ \overline{Q} \\ \overline{P} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} (\lambda + i\mu) \\ \frac{1}{2} (\lambda - i\mu); -i\kappa \end{pmatrix} \right) := \begin{pmatrix} P_* \\ Q_* \\ \overline{Q_*} \\ \overline{P_*} \end{pmatrix}.
\]
In other words, we have the relation
\[
T_*^{-1} \left( \begin{pmatrix} A \\ C \end{pmatrix}, \lambda, \mu; \kappa \right) T_* = \left( \begin{pmatrix} P \\ Q \\ \overline{Q} \\ \overline{P} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} (\lambda + i\mu) \\ \frac{1}{2} (\lambda - i\mu); -i\kappa \end{pmatrix} \right).
\]

Let \( H^{(m,m)}_C := \{ (\xi, \eta; \zeta) | \xi, \eta \in \mathbb{C}^{(m,n)}, \zeta \in \mathbb{C}^{(m,m)}, \zeta + \eta^t \xi \text{ symmetric} \} \) be the complex Heisenberg group endowed with the following multiplication
\[
(\xi, \eta; \zeta) \circ (\xi', \eta'; \zeta') := (\xi + \xi', \eta + \eta'; \zeta + \zeta' + \xi^t \eta' - \eta^t \xi').
\]
We define the semidirect product
\[ SL(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)} \]
endowed with the following multiplication
\[
\left( \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\xi, \eta; \zeta) \right) \cdot \left( \begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\xi', \eta'; \zeta') \right) = \left( \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\bar{\xi} + \xi', \bar{\eta} + \eta'; \zeta + \zeta' + \bar{\xi}'\eta' - \bar{\eta}'\xi') \right),
\]
where \( \bar{\xi} = \xi P' + \eta R' \) and \( \bar{\eta} = \xi Q' + \eta S' \).

If we identify \( H_{\mathbb{R}}^{(n,m)} \) with the subgroup
\[ \left\{ (\xi, \bar{\xi}i\kappa) \mid \xi \in \mathbb{C}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)} \right\} \]
of \( H_{\mathbb{C}}^{(n,m)} \), we have the following inclusion
\[ G_{\mathbb{R}}^J \subset SU(n, n) \ltimes H_{\mathbb{R}}^{(n,m)} \subset SL(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}. \]
We define the mapping \( \Theta : G^J \rightarrow G_{\mathbb{R}}^J \) by
\[
(4.15) \quad \Theta \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) := \left( \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \left( \frac{1}{2}(\lambda + i\mu), \frac{1}{2}(\lambda - i\mu); -i\frac{\kappa}{2} \right) \right),
\]
where \( P \) and \( Q \) are given by (4.13) and (4.14). We can see that if \( g_1, g_2 \in G^J \), then \( \Theta(g_1g_2) = \Theta(g_1)\Theta(g_2) \).

According to \[26, p. 250\], \( G_{\mathbb{R}}^J \) is of the Harish-Chandra type (cf. \[17, p. 118\]). Let
\[ g_* = \left( \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\lambda, \mu; \kappa) \right) \]
be an element of \( G_{\mathbb{R}}^J \). Since the Harish-Chandra decomposition of an element \( \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \) in \( SU(n, n) \) is given by
\[
\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_n & QS^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} P - QS^{-1}R & 0 \\ 0 & S^{-1}R \end{pmatrix} \begin{pmatrix} I_n & 0 \\ S^{-1}R & I_n \end{pmatrix},
\]
the \( P_*^+ \)-component of the following element
\[ g_* \cdot \left( \begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right), \quad W \in \mathbb{D}_n \]
of \( SL(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)} \) is given by
\[
(4.16) \quad \left( \begin{pmatrix} I_n & (PW + Q)(\overline{QW + P})^{-1} \\ 0 & I_n \end{pmatrix}, (0, (\eta + \lambda W + \mu)(\overline{QW + P})^{-1}; 0) \right).
\]
We can identify \( D_{n,m} \) with the subset
\[
\left\{ \left( \begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right) \mid W \in D_n, \ \eta \in \mathbb{C}^{(m,n)} \right\}
\]
of the complexification of \( G^{J}_{n,m} \). Indeed, \( D_{n,m} \) is embedded into \( P^{+}_{n,m} \) given by
\[
P^{+}_{n,m} = \left\{ \left( \begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right) \mid W = tW \in \mathbb{C}^{(n,n)}, \ \eta \in \mathbb{C}^{(m,n)} \right\}.
\]
This is a generalization of the Harish-Chandra embedding (cf. [17, p. 119]). Then we get the natural transitive action of \( G^{J}_{n,m} \) on \( D_{n,m} \) defined by
\[
(P, Q, \xi, \bar{\xi}; \kappa) \cdot (W, \eta) = \left( (PW + Q)(\Omega W + \mathcal{P})^{-1}, (\eta + \xi W + \bar{\xi})(\Omega W + \mathcal{P})^{-1} \right),
\]
where \( (P, Q, \mathcal{P}) \in G^{J}_{+}, \ \xi \in \mathbb{C}^{(m,n)}, \ \kappa \in \mathbb{R}^{(m,m)} \) and \( (W, \eta) \in D_{n,m} \).

The author [30] proved that the action (1.2) of \( G^{J}_{n,m} \) on \( \mathbb{H}_{n,m} \) is compatible with the action (4.17) of \( G^{J}_{n,m} \) on \( D_{n,m} \) through a partial Cayley transform \( \Phi : D_{n,m} \rightarrow \mathbb{H}_{n,m} \) defined by
\[
\Phi(W, \eta) := \left( i(I_n + W)(I_n - W)^{-1}, 2i \eta (I_n - W)^{-1} \right).
\]
In other words, if \( g_0 \in G^{J}_{n,m} \) and \( (W, \eta) \in D_{n,m} \),
\[
g_0 \cdot \Phi(W, \eta) = \Phi(g_0 \cdot (W, \eta)),
\]
where \( g_0 = T^{-1}_0 g_0 T_0 \). \( \Phi \) is a biholomorphic mapping of \( D_{n,m} \) onto \( \mathbb{H}_{n,m} \) which gives the partially bounded realization of \( \mathbb{H}_{n,m} \) by \( D_{n,m} \). The inverse of \( \Phi \) is
\[
\Phi^{-1}(\Omega, Z) = \left( (\Omega - iI_n)(\Omega + iI_n)^{-1}, Z(\Omega + iI_n)^{-1} \right).
\]
For \( (W, \eta) \in D_{n,m} \), we write
\[
(\Omega, Z) := \Phi(W, \eta).
\]
Thus
\[
\Omega = i(I_n + W)(I_n - W)^{-1}, \quad Z = 2i \eta (I_n - W)^{-1}.
\]
Since
\[
d(I_n - W)^{-1} = (I_n - W)^{-1}dW (I_n - W)^{-1}
\]
and
\[
I_n + (I_n + W)(I_n - W)^{-1} = 2 (I_n - W)^{-1},
\]
we get the following formulas from (4.20)
\begin{align}
Y &= \frac{1}{2i}(\Omega - \overline{\Omega}) = (I_n - W)^{-1}(I_n - WW)(I_n - \overline{W})^{-1}, \\
V &= \frac{1}{2i}(Z - \overline{Z}) = \eta (I_n - W)^{-1} + \overline{\eta} (I_n - \overline{W})^{-1}, \\
d\Omega &= 2i (I_n - W)^{-1}dW(I_n - W)^{-1}, \\
dZ &= 2i \left\{ d\eta + \eta (I_n - W)^{-1}dW \right\}(I_n - W)^{-1}.
\end{align}

Using Formulas (4.18), (4.20)-(4.24), the author [31] proved that for any two positive real numbers \(A\) and \(B\), the following metric \(ds^2_{n,m;A,B}\) defined by

\[
\begin{align*}
\frac{ds^2_{D_{n,m;A,B}}}{d\psi} &= 4A \text{tr}\left((I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}\right) \\
&+ 4B \left\{ \text{tr}\left((I_n - W\overline{W})^{-1}(d\eta)\beta\right) \\
&+ \text{tr}\left((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}(d\overline{\eta})\right) \\
&+ \text{tr}\left((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1}d\overline{W}(I_n - W\overline{W})^{-1}(d\eta)\right) \\
&- \text{tr}\left((I_n - W\overline{W})^{-1}\eta \overline{\eta}(I_n - \overline{W}W)^{-1}d\overline{W}(I_n - W\overline{W})^{-1}d\overline{W}\right) \\
&- \text{tr}\left(W(I_n - \overline{W}W)^{-1}\overline{\eta}\eta(I_n - W\overline{W})^{-1}d\overline{W}(I_n - W\overline{W})^{-1}d\overline{W}\right) \\
&+ \text{tr}\left((I_n - W\overline{W})^{-1}\eta \overline{\eta}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}\right) \\
&+ \text{tr}\left((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1}dW(I_n - W\overline{W})^{-1}dW\right) \\
&+ \text{tr}\left((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}\right) \\
&\times (I_n - \overline{W})(I_n - W)^{-1}dW(I_n - W\overline{W})^{-1}dW\right) \\
&- \text{tr}\left((I_n - W\overline{W})^{-1}(I_n - W)(I_n - \overline{W})^{-1}\overline{\eta}\eta(I_n - W)^{-1}\right) \\
&\times dW(I_n - \overline{W})^{-1}d\overline{W}\right\}
\end{align*}
\]

is a Riemannian metric on \(D_{n,m}\) which is invariant under the action (4.17) of the Jacobi group \(G^J_\ast\).
We note that if $n = m = 1$ and $A = B = 1$, we get
\[
\frac{1}{4} ds_{\mathbb{D}_{1,1;1,1}}^2 = \frac{dW d\bar{W}}{(1 - |W|^2)^2} + \frac{1}{(1 - |W|^2)} d\eta d\bar{\eta} + \frac{(1 + |W|^2)|\eta|^2 - \bar{W}\eta^2 - W\bar{\eta}^2}{(1 - |W|^2)^3} dW d\bar{W} + \frac{\eta\bar{W} - \bar{\eta}}{(1 - |W|^2)^2} dW d\eta + \frac{\bar{\eta}W - \eta}{(1 - |W|^2)^2} d\eta dW.
\]

From the formulas (4.20), (4.23) and (4.24), we get
\[
\frac{\partial}{\partial \Omega} = \frac{1}{2i} (I_n - W) \left[ t \left\{ (I_n - W) \frac{\partial}{\partial W} \right\} - t \left\{ \eta \left( \frac{\partial}{\partial \eta} \right) \right\} \right]
\]
and
\[
\frac{\partial}{\partial Z} = \frac{1}{2i} (I_n - W) \frac{\partial}{\partial \eta}.
\]

Using Formulas (4.20)-(4.22), (4.25), (4.26) and Lemma 4.1, the author [31] proved that the following differential operators $S_1$ and $S_2$ on $\mathbb{D}_{n,m}$ defined by
\[
S_1 = \sigma \left( (I_n - \bar{W}W) \frac{\partial}{\partial \eta} \left( \frac{\partial}{\partial \bar{\eta}} \right) \right)
\]
and
\[
S_2 = \text{tr} \left( (I_n - W\bar{W})^t \left( (I_n - W\bar{W}) \frac{\partial}{\partial W} \right) \frac{\partial}{\partial \bar{W}} \right) + \text{tr} \left( \eta - \bar{\eta} \right) \left( \frac{\partial}{\partial \eta} \right) \left( (I_n - \bar{W}W) \frac{\partial}{\partial \eta} \right) + \text{tr} \left( \bar{\eta} - \bar{\eta} \right) \left( \frac{\partial}{\partial \bar{\eta}} \right) \left( (I_n - W\bar{W}) \frac{\partial}{\partial \bar{\eta}} \right) - \text{tr} \left( \eta \bar{W} (I_n - \bar{W}W)^{-1} \eta \left( \frac{\partial}{\partial \eta} \right) \left( I_n - \bar{W}W \right) \frac{\partial}{\partial \eta} \right) - \text{tr} \left( \bar{\eta}W (I_n - W\bar{W})^{-1} \bar{\eta} \left( \frac{\partial}{\partial \bar{\eta}} \right) \left( I_n - W\bar{W} \right) \frac{\partial}{\partial \bar{\eta}} \right) + \text{tr} \left( \eta \bar{W} \left( I_n - \bar{W}W \right)^{-1} \eta \left( \frac{\partial}{\partial \eta} \right) \left( I_n - \bar{W}W \right) \frac{\partial}{\partial \eta} \right) + \text{tr} \left( \eta \left( I_n - \bar{W}W \right)^{-1} \eta \left( \frac{\partial}{\partial \eta} \right) \left( I_n - \bar{W}W \right) \frac{\partial}{\partial \eta} \right)
are invariant under the action \( (4.17) \) of \( G^J_* \). The author also proved that
\[
\Delta_{D_{n,m};A,B} := \frac{1}{A} S_2 + \frac{1}{B} S_1
\]
is the Laplacian of the invariant metric \( ds^2_{D_{n,m};A,B} \) on \( D_{n,m} \) (cf. [31]).

**Proposition 4.3.** The following differential operator on \( D_{n,m} \) defined by
\[
(4.28) \quad K_D = \det(I_n - WW) \det \left( \frac{\partial^t}{\partial \eta^t} \left( \frac{\partial}{\partial \eta} \right) \right)
\]
is invariant under the action \( (4.17) \) of \( G^J_* \) on \( D_{n,m} \).

**Proof.** It follows from Proposition 4.1, Formulas (4.21), (4.26) and the fact that the action \( (1.2) \) of \( G^J \) on \( H_{n,m} \) is compatible with the action \( (4.17) \) of \( G^J_* \) on \( D_{n,m} \) via the partial Cayley transform. \( \square \)

**Proposition 4.4.** The following matrix-valued differential operator on \( D_{n,m} \) defined by
\[
(4.29) \quad T_D := \left( \frac{\partial}{\partial \eta} \right) \left( I_n - WW \right) \frac{\partial}{\partial \eta}
\]
is invariant under the action \( (4.17) \) of \( G^J_* \) on \( D_{n,m} \).

**Proof.** It follows from Proposition 4.2, Formulas (4.21), (4.26) and the fact that the action \( (1.2) \) of \( G^J \) on \( H_{n,m} \) is compatible with the action \( (4.17) \) of \( G^J_* \) on \( D_{n,m} \) via the partial Cayley transform. \( \square \)

**Corollary 4.2.** Each \((k,l)\)-entry \( T_{D}^{kl} \) of \( T_D \) given by
\[
(4.30) \quad T_D^{kl} = \sum_{i,j=1}^{n} \left( \delta_{ij} - \sum_{r=1}^{n} w_{ir} w_{jr} \right) \frac{\partial^2}{\partial \eta_{kl} \partial \eta_{ij}}, \quad 1 \leq k, l \leq m
\]
is a \( G^J_* \)-invariant differential operator on \( D_{n,m} \).

**Proof.** It follows immediately from Proposition 4.4. \( \square \)

For two differential operators \( D_1 \) and \( D_2 \) on \( H_{n,m} \) or \( D_{n,m} \), we write
\[
[D_1, D_2] := D_1 D_2 - D_2 D_1.
\]
Then
\[
(4.31) \quad M_3 = [M_1, M_2] = M_1 M_2 - M_2 M_1
\]
is an invariant differential operator of degree three on \( H_{n,m} \) and
\[
(4.32) \quad P_{kl} = [K, T_{kl}] = KT_{kl} - T_{kl} K, \quad 1 \leq k, l \leq m
\]
is an invariant differential operator of degree \( 2n + 1 \) on \( H_{n,m} \).
Similarly
\[ S_3 = [S_1, S_2] = S_1S_2 - S_2S_1 \]
is an invariant differential operator of degree three on \( D_{n,m} \) and
\[ Q_{kl} = [K_D, T_{kl}^D] = K_D T_{kl}^D - T_{kl}^D K_D, \quad 1 \leq k, l \leq m \]
is an invariant differential operator of degree \( 2n + 1 \) on \( D_{n,m} \).

Indeed it is very complicated and difficult at this moment to express the generators of the algebra of all \( G^*_\infty \)-invariant differential operators on \( D_{n,m} \) explicitly.

5. The Case \( n = m = 1 \)

We consider the case \( n = m = 1 \). For a coordinate \((\omega, z)\) in \( T_{1,1} \), we write \( \omega = x + iy, \quad z = u + iv, \quad x, y, u, v \) real. The author [27] proved that the algebra \( Pol_{1,1}^{U(1)} \) is generated by
\[
q(\omega, z) = \frac{1}{4} \omega \overline{\omega} = \frac{1}{4} (x^2 + y^2), \\
\xi(\omega, z) = z \overline{z} = u^2 + v^2, \\
\phi(\omega, z) = \frac{1}{2} \text{Re} (z^2 \overline{\omega}) = \frac{1}{2} (u^2 - v^2)x + uvy, \\
\psi(\omega, z) = \frac{1}{2} \text{Im} (z^2 \overline{\omega}) = \frac{1}{2} (v^2 - u^2)y + uvx.
\]
In [27], using Formula (3.11) the author calculated explicitly the images
\[ D_1 = \Theta_{1,1}(q), \quad D_2 = \Theta_{1,1}(\xi), \quad D_3 = \Theta_{1,1}(\phi) \quad \text{and} \quad D_4 = \Theta_{1,1}(\psi) \]
of \( q, \xi, \phi \) and \( \psi \) under the Halgason map \( \Theta_{1,1} \). We can show that the algebra \( \mathbb{D}(\mathbb{H}_{1,1}) \) is generated by the following differential operators
\[
D_1 = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\
+ 2yv \left( \frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right),
\]
\[ D_2 = y \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right), \]
\[ D_3 = y^2 \frac{\partial}{\partial y} \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} \\
- \left( \frac{v}{\partial v} + 1 \right) D_2. \]
and

\[ D_4 = y^2 \frac{\partial}{\partial x} \left( \frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v \frac{\partial}{\partial u} D_2, \]

where \( \tau = x + iy \) and \( z = u + iv \) with real variables \( x, y, u, v \). Moreover, we have

\[ D_1 D_2 - D_2 D_1 = 2y^2 \frac{\partial}{\partial y} \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 4y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2 \left( v \frac{\partial}{\partial v} D_2 + D_2 - 2 D_2 \right). \]

In particular, the algebra \( \mathbb{D}(\mathbb{H}_{1,1}) \) is not commutative. We refer to [1, 27] for more detail.

Recently Hiroyuki Ochiai [15] proved the following results.

**Theorem 5.1.** We have the following relation

\[(5.1) \quad \phi^2 + \psi^2 = q \xi^2. \]

This relation exhausts all the relations among the generators \( q, \xi, \phi \) and \( \psi \) of \( \text{Pol}_{1,1}^{U(1)} \).

**Theorem 5.2.** We have the following relations

(a) \( [D_1, D_2] = 2D_3 \)
(b) \( [D_1, D_3] = 2D_1 D_2 - 2D_3 \)
(c) \( [D_2, D_3] = -D_2^2 \)
(d) \( [D_4, D_1] = 0 \)
(e) \( [D_4, D_2] = 0 \)
(f) \( [D_4, D_3] = 0 \)
(g) \( D_3^2 + D_4^2 = D_2 D_1 D_2 \)

These seven relations exhaust all the relations among the generators \( D_1, D_2, D_3 \) and \( D_4 \) of \( \mathbb{D}(\mathbb{H}_{1,1}) \).

We can prove the following

**Theorem 5.3.** The action of \( U(1) \) on \( \text{Pol}_{1,1}^{U(1)} \) is not multiplicity-free.

Finally we see that for the case \( n = m = 1 \), the seven problems proposed in Section 3 are completely solved.
Remark 5.1. According to Theorem 5.2, we see that $D_4$ is a generator of the center of $\mathbb{D}(\mathbb{H}_{1,1})$. We observe that the Laplacian

$$\Delta_{1,1;A,B} = \frac{4}{A} D_1 + \frac{4}{B} D_2 \quad (\text{see } (4.8))$$

of $(\mathbb{H}_{1,1}, ds_{1,1;A,B}^2)$ does not belong to the center of $\mathbb{D}(\mathbb{H}_{1,1})$.

6. The Case $n = 1$ and $m$ is arbitrary

Conley and Raum [5] found the $2m^2 + m + 1$ explicit generators of $\mathbb{D}(\mathbb{H}_{1,m})$ and the explicit one generator of the center of the universal enveloping algebra of $\mathfrak{g}$ of the Jacobi Lie algebra $\mathfrak{g}$. The number of generators of the center of $\mathbb{D}(\mathfrak{g})$ is $1 + \frac{m(m+1)}{2}$.

According to Theorem 3.2, $\Pol^U_{1,m}$ is generated by

\begin{align}
q(\omega, z) &= \tr(\omega \overline{z}), \\
\alpha_{kp}(\omega, z) &= \Re (z^t \overline{z})_{kp} = \Re (z_k \overline{z}_p), \quad 1 \leq k \leq p \leq m, \\
\beta_{lq}(\omega, z) &= \Im (z^t \overline{z})_{lq} = \Im (z_l \overline{z}_q), \quad 1 \leq l < q \leq m, \\
f_{kp}(\omega, z) &= \Re (z^t \overline{z})_{kp} = \Re (\overline{z}_k z_p), \quad 1 \leq k \leq p \leq m, \\
g_{kp}(\omega, z) &= \Im (z^t \overline{z})_{kp} = \Im (\overline{z}_k z_p), \quad 1 \leq k \leq p \leq m,
\end{align}

where $\omega \in T_1$ and $z \in \mathbb{C}^m$.

We let

$$\omega = x + iy \in \mathbb{C} \quad \text{and} \quad z = (z_1, \cdots, z_m) \in \mathbb{C}^m$$

with $z_k = u_k + iv_k$, $1 \leq k \leq m$, where $x, y, u_1, v_1, \cdots, u_m, v_m$ are real. The invariants $q, \alpha_{kp}, \beta_{lq}, f_{kp}$ and $g_{kp}$ are expressed in terms of $x, y, u_k, v_l (1 \leq k, l \leq m)$ as follows:

\begin{align}
q(\omega, z) &= x^2 + y^2, \\
\alpha_{kp}(\omega, z) &= u_k u_p + v_k v_p, \quad 1 \leq k \leq p \leq m, \\
\beta_{lq}(\omega, z) &= u_q v_l - u_l v_q, \quad 1 \leq l < q \leq m, \\
f_{kp}(\omega, z) &= x(u_k u_p - v_k v_p) + y(u_k v_p + v_k u_p), \quad 1 \leq k \leq p \leq m, \\
g_{kp}(\omega, z) &= x(u_k u_p + v_k v_p) - y(u_k u_p - v_k v_p), \quad 1 \leq k \leq p \leq m.
\end{align}

Theorem 6.1. The $1 + \frac{m(m+1)}{2}$ relations

\begin{equation}
f_{kp}^2 + g_{kp}^2 = q \alpha_{kk} \alpha_{pp}, \quad 1 \leq k \leq p \leq m
\end{equation}

exhaust all the relations among a set of generators $q, \alpha_{kp}, \beta_{lq}, f_{kp}$ and $g_{kp}$ with $1 \leq k \leq p \leq m$ and $1 \leq l < q \leq m$. 
**Theorem 6.2.** The action of $U(1)$ on $Pol_{1,m}$ is not multiplicity-free. In fact, if 

$$Pol_{1,m} = \sum_{\sigma \in \hat{U}(1)} m_{\sigma} \sigma,$$

then $m_{\sigma} = \infty$.

Problem 1, Problem 2, Problem 4, Problem 5 and Problem 7 were solved. Problem 3 can be handled. Finally Problem 6 is unsolved in the case that $n = 1$ and $m$ is arbitrary.

7. Final Remarks

Using $G^J$-invariant differential operators on the Siegel-Jacobi space, we introduce a notion of Maass-Jacobi forms.

**Definition 7.1.** Let 

$$\Gamma_{n,m} := \text{Sp}(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of $G^J$, where 

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(n,m)} | \lambda, \mu, \kappa \text{ are integral} \right\}.$$ 

A smooth function $f : \mathbb{H}_{n,m} \rightarrow \mathbb{C}$ is called a **Maass-Jacobi form** on $\mathbb{H}_{n,m}$ if $f$ satisfies the following conditions (MJ1)-(MJ3):

1. **(MJ1)** $f$ is invariant under $\Gamma_{n,m}$.
2. **(MJ2)** $f$ is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$ (cf. Formula (4.8)).
3. **(MJ3)** $f$ has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer $N$ such that

$$|f(X + iY, Z)| \leq C |p(Y)|^N$$

as $\det Y \rightarrow \infty$, where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

**Remark 7.1.** Let $D_*$ be a commutative subalgebra of $D(\mathbb{H}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f : \mathbb{H}_{n,m} \rightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $D_*$ if $f$ satisfies the conditions (MJ1), (MJ2)$_*$ and (MJ3): the condition (MJ2)$_*$ is given by

$$\text{(MJ2)$_*$} \ f \text{ is an eigenfunction of any invariant differential operator in } D_*.$$ 

It is natural to propose the following problems.

**Problem A:** Find all the eigenfunctions of $\Delta_{n,m;A,B}$.

**Problem B:** Construct Maass-Jacobi forms.
If we find a nice eigenfunction $\phi$ of the Laplacian $\Delta_{n,m;A,B}$, we can construct a Maass-Jacobi form $f_\phi$ on $H_{n,m}$ in the usual way defined by

\begin{equation}
(f_\phi)(\Omega, Z) := \sum_{\gamma \in \Gamma_{n,m}\setminus \Gamma_{n,m}} \phi(\gamma \cdot (\Omega, Z)),
\end{equation}

where

\[\Gamma_{n,m}^\infty = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right), (\lambda, \mu; \kappa) \in \Gamma_{n,m} \mid C = 0 \right\}\]

is a subgroup of $\Gamma_{n,m}$.

We consider the simple case $n = m = 1$ and $A = B = 1$. A metric $ds^2_{1,1;1,1}$ on $H_{1,1}$ given by

\[
ds_{1,1;1,1}^2 = \frac{y + v^2}{y^3} (dx^2 + dy^2) + \frac{1}{y} (du^2 + dv^2) - \frac{2v}{y^2} (dx \, du + dy \, dv)
\]

is a $G^J$-invariant Kähler metric on $H_{1,1}$. Its Laplacian $\Delta_{1,1;1,1}$ is given by

\[
\Delta_{1,1;1,1} = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (y + v^2) \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2yv \left( \frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right).
\]

We provide some examples of eigenfunctions of $\Delta_{1,1;1,1}$.

(1) $h(x,y) = y^s K_{s-\frac{1}{2}}(2\pi |a| y) e^{2\pi i ax}$ ($s \in \mathbb{C}$, $a \neq 0$) with eigenvalue $s(s - 1)$. Here

\[K_s(z) := \frac{1}{2} \int_0^\infty \exp \left\{ -\frac{z}{2} (t + t^{-1}) \right\} t^{s-1} dt,
\]

where $\text{Re} \, z > 0$.

(2) $y^s, y^sx, y^su$ ($s \in \mathbb{C}$) with eigenvalue $s(s - 1)$.

(3) $y^sv, y^svx$ with eigenvalue $s(s + 1)$.

(4) $x, y, u, v, xv, uv$ with eigenvalue 0.

(5) All Maass wave forms.

Let $\rho$ be a rational representation of $GL(n, \mathbb{C})$ on a finite dimensional complex vector space $V_\rho$. Let $M \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree $m$. Let $C^\infty(H_{n,m}, V_\rho)$ be the algebra of all $C^\infty$ functions on $H_{n,m}$ with values in $V_\rho$. We define the $|_{\rho, M}$-slash action of $G^J$ on $C^\infty(H_{n,m}, V_\rho)$ as follows:
If $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$,

$$f|_{\rho, \mathcal{M}}[(M, (\lambda, \mu; \kappa))](\Omega, Z) = e^{-2\pi i \text{tr}(\mathcal{M}((Z + \Omega + \lambda \tau + \kappa + \mu \lambda)\mathcal{M} + D)^{-1} C)} \cdot e^{2\pi i \text{tr}(\mathcal{M}((C \Omega + D)\mathcal{M} + D)^{-1})},$$

(7.2)

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(\lambda, \mu; \kappa) \in H_{n,m}^{(n,m)}$. We recall the Siegel's notation $\alpha[\beta] = \tau \beta \alpha \tau$ for suitable matrices $\alpha$ and $\beta$. We define $\mathbb{D}_{\rho, \mathcal{M}}$ to be the algebra of all differential operators $D$ on $\mathbb{H}_{n,m}$ satisfying the following condition

$$\langle Df \rangle|_{\rho, \mathcal{M}}[g] = D(f|_{\rho, \mathcal{M}}[g])$$

for all $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ and for all $g \in G^d$. We denote by $\mathcal{Z}_{\rho, \mathcal{M}}$ the center of $\mathbb{D}_{\rho, \mathcal{M}}$.

We define another notion of Maass-Jacobi forms as follows.

**Definition 7.2.** A vector-valued smooth function $\phi : \mathbb{H}_{n,m} \rightarrow V_\rho$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ of type $\rho$ and index $\mathcal{M}$ if it satisfies the following conditions $(\text{MJ1})_{\rho, \mathcal{M}}$, $(\text{MJ2})_{\rho, \mathcal{M}}$ and $(\text{MJ3})_{\rho, \mathcal{M}}$:

$(\text{MJ1})_{\rho, \mathcal{M}}$ $\phi|_{\rho, \mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{n,m}$.

$(\text{MJ2})_{\rho, \mathcal{M}}$ $f$ is an eigenfunction of all differential operators in the center $\mathcal{Z}_{\rho, \mathcal{M}}$ of $\mathbb{D}_{\rho, \mathcal{M}}$.

$(\text{MJ3})_{\rho, \mathcal{M}}$ $f$ has a growth condition

$$\phi(\Omega, Z) = O\left(e^{a \det Y} \cdot e^{2\pi \text{tr}(\mathcal{M}[V]Y^{-1})}\right)$$

as $\det Y \rightarrow \infty$ for some $a > 0$.

The case $n = 1$, $m = 1$ and $\rho = \det^k(k = 0, 1, 2, \cdots)$ was studied by R. Schmidt and R. Schmidt [1], A. Pitale [16] and K. Bringmann and O. Richter [3]. The case $n = 1$, $m = \text{arbitrary and } \rho = \det^k(k = 1, 2, \cdots)$ was dealt with by C. Conley and M. Raum [5]. In [3] the authors proved that the center $\mathcal{Z}_{\det^k, \mathcal{M}}$ of $\mathbb{D}_{\det^k, \mathcal{M}}$ is the polynomial algebra with one generator $\mathcal{C}_{\det^k, \mathcal{M}}$, the so-called Casimir operator which is an $\mathcal{M}$-slash invariant differential operator of degree three. Bringmann and Richter [3] considered the Poincaré series $\mathcal{P}_{\det^k, \mathcal{M}}$ (the case $n = m = 1$) that is a harmonic Maass-Jacobi form in the sense of Definition 7.2 and investigated its Fourier expansion and its Fourier coefficients. Here the harmonicity of $\mathcal{P}_{\det^k, \mathcal{M}}$ means that $\mathcal{C}_{\det^k, \mathcal{M}}\mathcal{P}_{\det^k, \mathcal{M}} = 0$, i.e., $\mathcal{P}_{\det^k, \mathcal{M}}$ is an eigenfunction of $\mathcal{C}_{\det^k, \mathcal{M}}$ with zero eigenvalue. Conley and Raum [5] generalized the results in [16] and [3] to the case $n = 1$ and $m$ is arbitrary.

**Remark 7.2.** In [2], Bringmann, Conley and Richter proved that the center of the algebra of differential operators invariant under the action of the Jacobi group over a complex quadratic field is generated by two Casimir operators of degree three. They also introduce an analogue of Kohnen’s plus space for modular forms of half-integral weight over $K = \mathbb{Q}(i)$, and provide a lift from it to the space of Jacobi forms over $K$. 
References

[1] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Progress in Mathematics, 163, Birkhäuser, Basel, 1998.

[2] K. Bringmann, C. Conley and O. K. Richter, *Jacobi forms over complex quadratic fields via the cubic Casimier operators*, preprint.

[3] K. Bringmann and O. K. Richter, *Zagier-type dualities and lifting maps for harmonic Maass-Jacobi forms*, Advances in Math. 225 (2010), 2298-2315.

[4] D. Bump and Y. J. Choie, *Derivatives of modular forms of negative weight*, Pure Appl. Math. Q. 2 (2006), no. 1, 111-133.

[5] C. Conley and M. Raum, *Harmonic Maass-Jacobi forms of degree 1 with higher rank indices*, arXiv:1012.289/v1 [math.NT] 13 Dec 2010.

[6] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, 55, Birkhäuser, Boston, Basel and Stuttgart, 1985.

[7] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I.*, Trans. Amer. Math. Soc. 75 (1953), 185-243.

[8] Harish-Chandra, *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. 83 (1956), 98-163.

[9] S. Helgason, *Differential operators on homogeneous spaces*, Acta Math. 102 (1959), 239-299.

[10] S. Helgason, *Groups and geometric analysis*, Academic Press, New York (1984).

[11] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proceedings, vol. 8 (1995), 1-182.

[12] M. Itoh, *On the Yang Problem*, preprint (2011).

[13] H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. 126 (1953), 44-68.

[14] H. Maass, *Siegel modular forms and Dirichlet series*, Lecture Notes in Math., vol. 216, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[15] H. Ochiai, *A remark on the generators of invariant differential operators on Siegel-Jacobi space of the smallest size*, preprint (2011).

[16] A. Pitale, *Jacobi Maass forms*, Abh. Math. Sem. Univ. Hamburg 79 (2009), 87-111.

[17] I. Satake, *Algebraic Structures of Symmetric Domains*, Kano Memorial Lectures 4, Iwanami Shoten, Publishers and Princeton University Press (1980).

[18] G. Shimura, *Invariant differential operators on hermitian symmetric spaces*, Ann. Math. 132 (1990), 237-272.

[19] C. L. Siegel, *Symplectic Geometry*, Amer. J. Math. 65 (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.

[20] C. L. Siegel, *Topics in Complex Function Theory: Abelian Functions and Modular Functions of Several Variables*, vol. III, Wiley-Interscience, 1973.

[21] H. Weyl, *The classical groups: Their invariants and representations*, Princeton Univ. Press, Princeton, New Jersey, second edition (1946).

[22] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg 63 (1993), 135–146.

[23] J.-H. Yang, *Singular Jacobi Forms*, Trans. Amer. Math. Soc. 347 (6) (1995), 2041-2049.

[24] J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms*, Canad. J. Math. 47 (6) (1995), 1329-1339 or arXiv:math.NT/0612502.

[25] J.-H. Yang, *A geometrical theory of Jacobi forms of higher degree*, Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda.), Sendai, Japan (1996), 125-147 or Kyungpook Math. J. 40 (2) (2000), 209-237 or arXiv:math.NT/0602267.

[26] J.-H. Yang, *The Method of Orbits for Real Lie Groups*, Kyungpook Math. J. 42 (2) (2002), 199-272 or arXiv:math.RT/0602056.
[27] J.-H. Yang, A note on Maass-Jacobi forms, Kyungpook Math. J. 43 (4) (2003), 547–566 or arXiv:math.NT/0612387.
[28] J.-H. Yang, A note on a fundamental domain for Siegel-Jacobi space, Houston J. Math. 32 (3) (2006), 701–712 or arXiv:math.NT/0507218.
[29] J.-H. Yang, Invariant metrics and Laplacians on Siegel-Jacobi space, Journal of Number Theory 127 (2007), 83–102 or arXiv:math.NT/0507215.
[30] J.-H. Yang, A partial Cayley transform for Siegel-Jacobi disk, J. Korean Math. Soc. 45, No. 3 (2008), 781-794 or arXiv:math.NT/0507216.
[31] J.-H. Yang, Invariant metrics and Laplacians on Siegel-Jacobi disk, Chinese Annals of Mathematics, Vol. 31B(1), 2010, 85-100 or arXiv:math.NT/0507217.

DEPARTMENT OF MATHEMATICS, INHA UNIVERSITY, INCHEON 402-751, KOREA
E-mail address: jhyang@inha.ac.kr