The STIRAP-based unitary decelerating and accelerating processes of a single free atom

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Abstract

The STIRAP-based unitary decelerating and accelerating processes have been proposed to realize the time- and space-compressing processes in the quantum control process to simulate the reversible and unitary state-insensitive halting protocol (Arxiv: quant-ph/0607144). A standard three-state STIRAP pulse sequence may act as a basic unitary decelerating sequence or a basic unitary accelerating sequence. A STIRAP-based unitary decelerating (accelerating) process then consists of a train of these basic STIRAP unitary decelerating (accelerating) sequences. The present work is focused on investigating analytically and quantitatively how the momentum distribution of a momentum superposition state of a pure-state quantum system such as a momentum Gaussian wave-packet state of a single freely moving atom affects the STIRAP state transfer in these decelerating and accelerating processes. The complete STIRAP state transfer and the unitarity of these processes are stressed highly in the investigation. It has been shown that the momentum distribution has an important influence upon the STIRAP state-transfer efficiency. In the ideal adiabatic condition these unitary decelerating and accelerating processes for a freely moving atom are studied in detail, and it is shown that they can be used to manipulate and control in time and space the center-of-mass position and momentum of a Gaussian wave-packet motional state of a free atom. Two general adiabatic conditions for the basic STIRAP decelerating and accelerating processes are derived analytically. They are strict and accurate. They can be used to set up a conventional STIRAP state-transfer experiment and also the basic STIRAP decelerating and accelerating processes. With the help of the STIRAP theory and the unitary quantum dynamics it confirms theoretically that the time- and space-compressing processes of the quantum control process (Arxiv: quant-ph/0607144) can be realized almost perfectly by the STIRAP-based unitary decelerating and accelerating processes in the ideal or nearly ideal adiabatic condition.

1. Introduction

The stimulated Raman adiabatic passage (STIRAP) processes are very important coherent double-photon processes [1, 2]. The STIRAP method has been extensively applied to the complete population transfer in energy levels of atoms
and molecules in the laser spectroscopy [3, 4], the laser cooling in neutral atomic ensembles [5, 6, 7, 8, 9] (here may include the conventional Raman adiabatic processes), and the atomic quantum interference experiments [10, 11, 12, 13, 14] as well as other science research fields. A standard three-state STIRAP pulse sequence [3, 4, 15, 17, 18] consists of a pair of Raman laser light beams which are selectively applied to three chosen energy levels of an atomic or molecular system. The largest advantage of the STIRAP method is that the complete population transfer or state transfer in an atomic or molecular system may be achieved by the STIRAP pulse sequence with delayed and overlapping Raman laser light beams [15, 16, 17], and the STIRAP method is tolerant to the experimental imperfections. The basic theory of the STIRAP method has been well set up [16, 17, 18]. The experimental confirmation for the STIRAP method has been carried out first in the atomic and molecular laser spectroscopy [3, 15] and then in diverse other science research fields. The basic STIRAP theoretical and experimental methods have been first extended to study and design the quantum interference experiments in cold atomic ensembles [10, 11, 12, 13, 14], which involve the atomic motional momentum transfer. The theory also has been developed to study the Raman laser cooling processes in a neutral atom ensemble [20, 21] by combining the velocity-selective coherent population trapping [5, 19]. In these two types of experiments [5, 10, 11, 12, 13, 14, 19, 20] the Raman laser pulse sequence such as the STIRAP pulse sequence usually consists of a pair of counterpropagating Raman laser light beams.

It has been shown that the dynamical state-locking pulse field plays a key role in constructing the reversible and unitary state-insensitive halting protocol [22] and solving efficiently the quantum search problem [22, 36]. A general state-locking pulse field [22] consists of a sequence of the time- and space-dependent electromagnetic pulse fields and could also contain the time- and space-dependent potential fields which could be generated by the external electric and/or magnetic field. A unitary decelerating (or accelerating) laser light pulse sequence that is used to decelerate (or accelerate) a moving free atom could be thought of as the component of a dynamical state-locking pulse field.

The standard three-state STIRAP pulse sequence may act as either a basic unitary decelerating sequence or a basic unitary accelerating sequence, which is dependent upon the parameter settings of the two counterpropagating Raman adiabatic laser light beams of the STIRAP pulse sequence. The STIRAP-based decelerating (accelerating) process then consists of a train of the basic STIRAP unitary decelerating (accelerating) sequences which are applied to a moving atom consecutively. It has been proposed that the STIRAP-based unitary decelerating and/or accelerating processes may be used to coherently manipulate the halting-qubit atom in the quantum control process to simulate the reversible and unitary state-insensitive halting protocol [22]. Thus, the STIRAP-based unitary decelerating and accelerating processes are the important building blocks of the reversible and unitary state-insensitive halting protocol and the efficient quantum search process based on the unitary quantum dynamics in time and space. A unitary decelerating process is much like the conventional laser cooling process in an atomic ensemble. The essential difference between the two processes
is that the unitary decelerating process is reversible and unitary, while the laser cooling process usually is irreversible. A unitary accelerating process could be more like the momentum transfer process in the quantum interference experiments of an atomic ensemble. The atomic momentum transfer process in these quantum interference experiments is generally transverse with respect to the initial atomic moving direction, while here the unitary accelerating process is longitudinal. The unitary decelerating and accelerating processes stress their unitarity and complete state transfer in the quantum control process.

A quantum computation tends to avoid using a space-dependent unitary operation as its basic building block, since such a unitary operation usually is more complicated and inconvenient to manipulate and control in a quantum system with respect to the conventional quantum-gate operations. It is necessary to manipulate and control at will the internal motion of an atom (or atomic ion) in quantum computation when the specific internal states of the atom such as the hyperfine ground electronic states or the nuclear spin states are taken as a quantum bit, but at the same time the atomic center-of-mass motion tends to be kept unchanged simply (or to be constrained simply) so that it does not affect these quantum-gate operations and hence in quantum computational theory it is usually not considered explicitly. However, as pointed out in the previous paper [22], it is of crucial importance to unitarily manipulate and control at will in time and space the center-of-mass motion, the internal motion, and the mutual cooperation and coupling of the two motions of the halting-qubit atom in order to realize the quantum control process to simulate the reversible and unitary state-insensitive halting protocol. The unitary manipulation and control in time and space for the atom is also a key step toward the realization to solve efficiently the quantum search problem. An electromagnetic wave field such as a laser light field can manipulate and control in time and space not only the center-of-mass motion and the internal electronic (or spin) motion of an atom separately, but also it can create and control the coupling between the center-of-mass motion and the internal motion of an atomic system. This is the theoretical fundament for the laser cooling of a neutral atomic ensemble [21, 23, 24] and the unitary decelerating and accelerating of a free atom [22].

One large advantage to use a laser light field to manipulate an atom is that the space-selective and / or the internal-state-selective unitary operations of the atom can be realized easily. The STIRAP-based unitary decelerating and accelerating processes could be a very useful double-photon method to coherently manipulate and control in time and space the center-of-mass motion, the internal motion, and the coupling of both the motions of a moving free atom. It usually uses a pair of Raman adiabatic laser light beams to couple the atomic center-of-mass motional state and internal electronic states (or spin polarization states) to realize the coupling between the two motions. The STIRAP-based unitary decelerating and accelerating processes have been proposed to realize the unitary time- and space-compressing processes which are necessary components of the quantum control process to simulate efficiently the state-insensitive reversible and unitary halting protocol [22]. On the other hand, as far as the Gaussian wave-packet state of a free atom is concerned, the STIRAP-based
unitary decelerating and accelerating processes could generate a type of time-
and space-dependent unitary propagators which can manipulate and control
the center-of-mass position and momentum of the atomic Gaussian wave-packet
state. This important result will be shown in the paper.

The basic STIRAP theory generally does not consider explicitly the effect of
the center-of-mass motional momentum distribution of an atomic or molecular
system on the STIRAP population transfer in the laser spectroscopy [16, 17,
18], although the Doppler effect in these atomic and molecular systems has also
been considered suitably in the STIRAP experiments [3, 15]. This could be
due to that the center-of-mass motion of an atom or molecule is generally much
slower than the internal electronic motion of the atom or molecule and could be
neglected in the basic STIRAP theory, and the STIRAP experimental settings
[3, 15] are also favorable for the experiments to minimize the center-of-mass
motional effect, for example, the STIRAP experiments may use a pair of co-
propagating Raman laser light beams. However, a pure-state quantum system
such as a single freely moving atom may be in a superposition of the center-of-
mass motional momentum states which may has a broad momentum distribu-
tion. It is generally hard to realize the complete state transfer in a quantum
system with a broad momentum distribution by the standard STIRAP method.
One therefore must consider the effect of the momentum distribution of a su-
perposition state of the quantum system on the STIRAP population or state
transfer. In concept the momentum distribution of a momentum superposition
state of a pure-state quantum system is essentially different from the conven-
tional momentum distribution of a quantum ensemble, the latter is a statistical
distribution of momentum of the particles which form the quantum ensemble.
But the effect of the momentum distribution on the STIRAP state transfer is
similar for the two types of quantum systems according to the unitary quantum
dynamical principle that both a closed quantum system and its ensemble obey
the same unitary quantum dynamics [22] (it seems that a quantum system in
the presence of an external electromagnetic field is not a closed quantum system,
but in theory such a quantum system may be treated conditionally as if it is a
closed quantum system, as shown in the section 11 in Ref. [40]). It has been
investigated in theory how the momentum distribution of an atomic ensemble
affects the population or state transfer in the atomic laser cooling based on the
velocity-selective coherent population trapping [5, 19, 20] which also uses the
Raman laser light beams, the STIRAP-based momentum transfer in the cold
atomic interference experiments [10, 12], and the conventional Raman-laser-
light-based cold atomic interference experiments [11], but these investigations
are usually either qualitative or numerical. It is important to investigate ana-
lytically and quantitatively how a superposition of the momentum states affects
the STIRAP state transfer in a pure-state quantum system when the STIRAP
method is used to perform the state transfer or the unitary operation in quan-
tum computation. This paper is devoted to such a theoretic investigation: how
a superposition of the momentum states affects the STIRAP state transfer in
a single freely moving atom. According to quantum mechanics a freely moving
atom in the presence of the STIRAP pulse sequence may be described by the
complete set of the product states of both the center-of-mass motional states and the internal states (the electronic states or spin polarization states) of the atom. The atomic center-of-mass motional state may be a superposition of the atomic momentum eigenstates. When the atom is transferred from one internal state to another by a STIRAP pulse sequence, the transfer efficiency is generally dependent upon the atomic center-of-mass motional state. The purpose to investigate this dependence is to understand how the momentum distribution of the center-of-mass motional state affects the transfer efficiency and then design a better STIRAP pulse sequence so as to achieve a complete STIRAP state transfer over the whole effective momentum distribution of the atomic center-of-mass motional state.

In order to manipulate and control the halting-qubit atom in time and space in the quantum control process [22] it is necessary to investigate the time evolution process of the halting-qubit atom in the STIRAP-based unitary decelerating and accelerating processes. In order to investigate quantitatively how a superposition of the momentum states affects the STIRAP state transfer for a single freely moving atom in the STIRAP-based decelerating or accelerating process it is also necessary to calculate the time evolution process of the atom in the presence of the Raman laser light field. For example, it needs to solve the unitary dynamical equation to set up a general adiabatic condition for the basic STIRAP decelerating and accelerating processes for the atom. It seems that a single freely moving atom in the presence of an external electromagnetic field such as the Raman laser light field is a simple quantum system, but the time evolution process of the atomic system is not so simple, partly because the interaction between the atom and the external electromagnetic field is usually time-dependent, and on the other hand, because the atomic internal motion and center-of-mass motion as well as the coupling of the two motions induced by the external electromagnetic field need to be considered explicitly and simultaneously in a theoretical treatment. The time evolution process of a free atom in the presence of a laser light field becomes so complex that it is generally difficult to solve exactly the unitary dynamical equation of the atomic system. It is also quite inconvenient even to use an approximation method to solve the unitary dynamical equation with a high accuracy. The STIRAP-based unitary decelerating and accelerating processes for a free atom are relatively simple, because these decelerating and accelerating processes are adiabatic and the Raman laser light beams of the STIRAP decelerating and accelerating pulse sequences affect only the three chosen atomic internal states. These special points may simplify greatly the investigation of the time evolution process of the atom in the STIRAP decelerating and accelerating processes. Because the time evolution process of the atom involves only the three chosen internal states of the atom, it may be investigated conveniently in the atomic three-internal-state subspace. On the other hand, it is well known that a unitary process of a free atom absorbing or emitting a photon has to obey the energy, momentum, and angular momentum conservation laws. The energy, momentum, and angular momentum conservation laws put a restriction on the time evolution process for the free atom during the STIRAP-based unitary decelerating and accelerat-
ing processes. These laws lead to that the atomic motional momentum cannot be changed arbitrarily by these Raman laser light beams in the STIRAP unitary decelerating and accelerating processes, but rather it can be changed only within the Raman-laser-light-induced momentum state subspace \[5, 19\]. Then the time evolution process of a free atom may be investigated conveniently in the Raman-laser-light-induced momentum state subspace in the STIRAP unitary decelerating and accelerating processes. Here the unitarity of these processes is emphasized again. This greatly simplifies the evaluation for the time evolution process.

In this paper the basic STIRAP theory \[15, 16, 17, 18\] has been developed to study and construct the STIRAP-based unitary decelerating and accelerating processes for a free atom by combining the quantum superposition principle \[25\] and the energy, momentum, and angular momentum conservation laws for the atomic photon absorption and emission processes \[5, 19\]. This research is focused on investigating analytically and quantitatively the effect of the momentum distribution of a superposition of momentum states of the atom on the STIRAP state transfer in these processes. Both the ideal and the real adiabatic condition for the basic STIRAP decelerating and accelerating processes are derived analytically. One important result of the paper is to confirm theoretically the time- and space-compressing processes of the quantum control process \[22\], which are realized by the STIRAP unitary decelerating and accelerating processes, with the help of the STIRAP state-transfer theory and the unitary quantum dynamics.

2. The Hamiltonian for a single atom in the presence of the Raman laser light beams

The STIRAP-based laser cooling processes and the unitary decelerating and accelerating processes are generally involved in the center-of-mass motion, the internal electronic motion, and the interaction between the two motions in an atomic system. A complete theoretical description for these processes need consider the atomic center-of-mass motion, the internal electronic motion, and the coupling of both the motions of the atom. When an atom is irradiated by an externally applied electromagnetic field such as the Raman laser light beams in the STIRAP experiments, the total Hamiltonian for the physical system consisting of the atom and the electromagnetic field may be generally written as \[25\]

\[
H = H_a + H_{rad} + H_{int} \tag{1}
\]

where \(H_a\) is the atomic Hamiltonian in the absence of the externally applied electromagnetic field,

\[
H_a = \sum_k \frac{1}{2m_k} p_k^2 + V_a \tag{2}
\]

\(H_{rad}\) the Hamiltonian for the electromagnetic field which may be written as

\[
H_{rad} = \frac{1}{8\pi} \int d^3x (E.E^* + B.B^*)
\]
and $H_{\text{int}}$ the interaction between the atom and the externally applied electromagnetic field,

$$
H_{\text{int}} = -\sum_k \frac{e_k}{m_k c} p_k A(r_k, t) + \sum_k \frac{e_k^2}{2m_k c} A(r_k, t)^2.
$$

The atomic Hamiltonian $H_a$ describes both the atomic center-of-mass motion and the internal electronic (or spin) motion of the atom, while the interaction $H_{\text{int}}$ induced by the external electromagnetic field creates the coupling between the atomic center-of-mass motion and the internal electronic motion. The external electromagnetic field usually is weak in the conventional STIRAP experiments and the interaction $H_{\text{int}}$ between the atom and the electromagnetic field could be considered as a perturbation.

It is usually inconvenient to use directly the total Hamiltonian of Eq. (1) to describe the STIRAP-based decelerating and accelerating processes, although it is the most exact to use the total Hamiltonian to treat these processes theoretically. Without losing generality one may use a simpler form of the total Hamiltonian of Eq. (1) to describe clearly the unitary decelerating and accelerating processes. It is well known that the semiclassical theory of electromagnetic radiation has been extensively used to describe the laser spectroscopy in the atomic and molecular systems [1, 16, 17, 18], the atomic coherent laser-cooling processes [20, 21, 23], and the atomic quantum interference experiments [10, 11, 12]. If the semiclassical theory is also reasonable for the unitary decelerating and accelerating processes, then the Hamiltonian $H_{\text{rad}}$ of the electromagnetic field itself may be omitted from the total Hamiltonian of Eq. (1). Though the semiclassical theory of electromagnetic radiation generally can not describe exactly the spontaneous emission in an atomic system [1, 25] and especially the spontaneous emission in the long-time atomic laser cooling process [26], it may be suited to describe the STIRAP-based unitary decelerating and accelerating processes as these unitary STIRAP processes can avoid the atomic spontaneous emission by setting the suitable experimental parameters. Furthermore, if the electric dipole approximation for the atomic system is also reasonable, then one may use conveniently the electric dipole interaction $H_d$ to replace the interaction $H_{\text{int}}$ to describe the unitary decelerating and accelerating processes. The electric dipole interaction $H_d$ may be written as

$$
H_d = -D \cdot E(x, t)
$$

where $D$ is the atomic electric dipole moment and $E(x, t)$ is the electric field of the externally applied electromagnetic field, and the coordinate $x$ in the electric field $E(x, t)$ is the center-of-mass position of the atom. The electric dipole approximation is reasonable when the wave length of the external electromagnetic field is much larger than the atomic dimension under study so that the atom may be considered as a point particle — a point electric dipole — in the electromagnetic field [1]. It has been shown that in the electric dipole approximation the electric-dipole Hamiltonian $H_d$ is really equivalent to the interaction $H_{\text{int}}$ up to a gauge transformation which is also a unitary transformation [1, 27].
The electric dipole approximation is very popular in the theoretical description of a variety of laser light and matter interactions. For example, one generally uses the electric-dipole approximation to deal with theoretically the dynamical process of atomic laser cooling in an atomic ensemble [1, 4, 5, 10, 19, 20, 26]. Now in the semiclassical theory of electromagnetic radiation and in the electric dipole approximation the total Hamiltonian (1) of the atom in the external electromagnetic field is reduced to the form

\[ H = \frac{P^2}{2M} + V(x) + H(r) + H_d \]  (4)

where the sum of the first three terms is just the atomic Hamiltonian \( H_a \) of Eq. (2). The center-of-mass Hamiltonian \( H_{cm} = \frac{P^2}{2M} + V(x) \) describes the atomic center-of-mass motion, while the internal Hamiltonian \( H(r) \) describes the internal electronic (or spin) motion of the atom. In the center-of-mass Hamiltonian \( H_{cm} \), the term \( H_K = \frac{P^2}{2M} \) with the atomic mass \( M \) and momentum \( P \) is the atomic kinetic energy and the term \( V(x) \) the atomic potential energy in an external potential field. In the unitary decelerating and accelerating processes of a free atom the external potential energy \( V(x) \) of the atom is zero, i.e., \( V(x) = 0 \). If there is an external electric (or magnetic) field during these processes, then the external potential energy \( V(x) \) could not be zero. For example, the potential energy \( V(x) \neq 0 \) when an atom in a harmonic potential well is applied by an external electromagnetic field. Generally, both the center-of-mass kinetic energy operator \( \frac{P^2}{2M} \) and the internal Hamiltonian \( H(r) \) are not commutable with the electric dipole interaction \( H_d \). The Hamiltonian \( H \) of Eq. (4) still needs to be further simplified so that it can describe conveniently the standard three-state STIRAP experiment. This simplification is based on the facts that the externally applied electromagnetic field in the STIRAP experiments can only affect some specific internal electronic energy levels of the atom and the internal electronic states are discrete. In the following only one-dimensional center-of-mass motion of the atom is considered.

Because there are the center-of-mass motion, the internal electronic motion, and even the coupling between the two motions of an atom in the STIRAP processes, one must use simultaneously both the atomic center-of-mass motional states and the internal electronic states of the atom to describe exactly the STIRAP processes. Suppose that the wave functions \( |\psi_P(x)\rangle \) and \( |\psi_j(r)\rangle \) are the eigenstates of the center-of-mass Hamiltonian \( H_{cm} \) and the internal Hamiltonian \( H(r) \), respectively,

\[ H_{cm}|\psi_P(x)\rangle = E_P|\psi_P(x)\rangle, \quad H(r)|\psi_j(r)\rangle = E_j|\psi_j(r)\rangle, \]

where \( E_P \) and \( E_j \) are the eigen-energy of the center-of-mass motional state \( |\psi_P(x)\rangle \) and the internal state \( |\psi_j(r)\rangle \), respectively. For the STIRAP processes of a free atom the eigen-energy \( E_P \) is equal to the atomic center-of-mass kinetic energy \( \frac{P^2}{2M} \) since the potential energy \( V(x) = 0 \) and \( H_{cm}|\psi_P(x)\rangle = H_K|\psi_P(x)\rangle = \frac{P^2}{2M}|\psi_P(x)\rangle \). According to quantum mechanics [25] the complete set \( \{|\psi_P(x)\rangle|\psi_j(r)\rangle\} \) of the product states of the center-of-mass motional states and the internal states of the atom can be used to describe completely
both the center-of-mass motion and the internal electronic motion as well as the coupling of the two motions of the atom. Now the total wave function $|\Psi(x, r, t)\rangle$ of the atom may be generally expressed as a linear combination of these product states [25],

$$|\Psi(x, r, t)\rangle = \sum_{j,P} a(j, P, t)|\psi_j(r)\rangle$$

where $a(j, P, t)$ is an expansion coefficient. In the decelerating and accelerating processes an atom may be in a given internal state, while its center-of-mass motional state is a superposition state. Then in this case the total wave function of the atom may be expanded as

$$|\Psi(x, r, t)\rangle = \sum_{j,p} a(j, p, t)|\psi_p(x,t)\rangle|\psi_j(r)\rangle$$

(5)

where the wave function $|\psi_p(x, t)\rangle$ is a superposition of the atomic center-of-mass motional states. For example, the center-of-mass motional state $|\psi_j(r)\rangle$ may be a wave-packet state of the halting-qubit atom in the quantum control process, which was denoted as $|CM, R\rangle$ in the previous paper [22]. In the decelerating and accelerating processes of a free atom any center-of-mass motional state $|\psi_p(x, t)\rangle$ of the atom may be expanded in terms of the complete eigenstate set $\{|\psi_P(x)\rangle\}$ [25],

$$|\psi_p(x, t)\rangle = \sum_P a(p, P) \exp\left(-\frac{i P^2}{\hbar 2M} t\right)|\psi_P(x)\rangle.$$  

(6)

Note that the state $|\psi_P(x)\rangle$ is also an eigenstate of the atomic center-of-mass momentum operator $P$ and in one-dimensional case it may be written as

$$|\psi_P(x)\rangle \equiv |P\rangle = \frac{1}{\sqrt{2\pi}} \exp(iPx/\hbar).$$

(7)

The momentum wave function $|P\rangle$ represents that the atom moves along the direction $+x$ with the center-of-mass motional velocity $P/M$.

In what follows it is supposed that the halting-qubit atom has a three-level Λ configuration for the three-state STIRAP experiments. In the STIRAP experiment the external electromagnetic field can have a real effect only on the specific three-state subspace $\{|\psi_0(r)\rangle, |\psi_1(r)\rangle, |\psi_2(r)\rangle\}$ of the internal electronic states of the atom. This three-state subspace will further simplify the Hamiltonian of Eq. (4) as the time evolution process of the internal states of the atom in the STIRAP experiment is confined in the three-state subspace. Those internal states of the atom outside the three-state subspace will not be affected by the external electromagnetic field and are not considered in the STIRAP experiment. Thus, it is sufficient to use the internal Hamiltonian $(H(r))$ projection onto the three-state subspace to describe the three-state STIRAP experiment. Since the internal states $\{|\psi_j(r)\rangle\}$ are the eigenstates of the internal Hamiltonian $H(r)$, this projection Hamiltonian onto the three-state subspace may be
given by
\[ H(r) = E_0|\psi_0(r)\rangle\langle \psi_0(r)| + E_1|\psi_1(r)\rangle\langle \psi_1(r)| + E_2|\psi_2(r)\rangle\langle \psi_2(r)|. \] (8)

On the other hand, the electric dipole interaction \( H_d \) of Eq. (3) can also be simplified further in the three-state subspace. In the STIRAP-based decelerating and accelerating processes the total external electromagnetic field generally consists of a pair of counterpropagating electromagnetic fields, which may be amplitude- and phase-modulation Raman laser light beams. The total electric field for the two counterpropagating linearly polarized Raman laser light beams may be expressed as
\[
E(x,t) = \frac{1}{2}E_{L0}(t) \exp[i(-k_{L0}x - \omega_{L0}t)] \\
+ \frac{1}{2}E_{L1}(t) \exp[i(k_{L1}x - \omega_{L1}t)] + C.C.
\] (9)
where \( C.C. \) stands for the complex (or hermite) conjugate of the first two terms. Notice that the three internal states \( \{ |\psi_k(r)\rangle \} \) have quite different energy eigenvalues. The first Raman laser light beam with the electric field \( E_{L0}(t) \) couples only the two internal states \( |\psi_0(r)\rangle \) and \( |\psi_2(r)\rangle \) of the atom, while the second with the electric field \( E_{L1}(t) \) connects only the two internal states \( |\psi_1(r)\rangle \) and \( |\psi_2(r)\rangle \). The frequency difference \( |\omega_{L0} - \omega_{L1}| \) should be near the resonance frequency of the two atomic internal energy levels \( |\psi_0(r)\rangle \) and \( |\psi_1(r)\rangle \). The first Raman laser light beam usually is named the pumping laser pulse and the second the Stokes laser pulse in the laser spectroscopy [15]. If among the three internal states the two states \( |\psi_0(r)\rangle \) and \( |\psi_1(r)\rangle \) which are usually the ground states have the same energy eigenvalues, then one may use a pair of \( \sigma_+ \) and \( \sigma_- \) circularly polarized laser light beams [19] to replace the present two Raman laser light beams, one circularly polarized laser light beam coupling only the two internal states \( |\psi_0(r)\rangle \) and \( |\psi_2(r)\rangle \) and another connecting only the two internal states \( |\psi_1(r)\rangle \) and \( |\psi_2(r)\rangle \). Then in the three-state (internal) subspace the electric dipole interaction \( H_d \) of Eq. (3) may be written as, in the rotating wave approximation [1],
\[
H_d = \hbar \Omega_{02}(t) \exp\{i(-k_{L0}x - \omega_{L0}t)\}|\psi_2(r)\rangle\langle \psi_0(r)| \\
+ \hbar \Omega_{12}(t) \exp\{i(k_{L1}x - \omega_{L1}t)\}|\psi_2(r)\rangle\langle \psi_1(r)| + C.C.
\] (10)
where the Rabi frequencies for the two Raman laser light beams are defined as
\[
\Omega_{02}(t) = -\frac{1}{2}\langle \psi_2(r)|D,E_{L0}(t)|\psi_0(r)\rangle, \\
\Omega_{12}(t) = -\frac{1}{2}\langle \psi_2(r)|D,E_{L1}(t)|\psi_1(r)\rangle.
\]
The electromagnetic field of the Raman laser light beams is usually weak in the conventional STIRAP experiments, so that the effect of the electromagnetic
field on the atom could be considered as a perturbation. Hence the rotating
wave approximation is reasonable. On the other hand, if each Raman laser
light beams in the STIRAP experiment is replaced with a pair of the laser
light beams with the orthogonal electric field vectors and the suitable phases
[38] or one circularly polarized laser light beam [19], then the rotating-wave
approximation may be eliminated and hence the electric dipole interaction (10)
may be constructed exactly. The total Hamiltonian of Eq. (4) associated with
the electric dipole interaction \( H_d \) of Eq. (10) and the internal Hamiltonian \( H(r) \)
of Eq. (8) and the product basis set \( \{ |\psi_P(x)\rangle |\psi_j(r)\rangle \} \) may be used conveniently
to describe the STIRAP-based unitary decelerating and accelerating processes
of a free atom. The transition matrix elements of the electric dipole interaction
\( H_d \) can be calculated in the product basis set,

\[
W(j', P' ; j, P) = \langle \psi_{j'}(r) | \langle \psi_{P'}(x) | H_d | \psi_P(x) \rangle | \psi_j(r) \rangle.
\]

These matrix elements are not zero only when both the internal states \( |\psi_j(r)\rangle \)
and \( |\psi_{j'}(r)\rangle \) are in the three-state (internal) subspace. They are also subjected
to the constraint of the energy, momentum, and angular momentum conserva-
tion laws for the atomic photon absorption and emission process in the STIRAP
decelerating and accelerating processes. This is an instance of the velocity-
selective rules which have been used in the atomic laser cooling processes [5,
19]. Below it is shown how the energy and momentum conservative laws have a
constraint on the electric dipole transition matrix elements \( \{ W(j', P' ; j, P) \} \) in
the three-state STIRAP experiments of the STIRAP-based unitary decelerating
and accelerating processes.

In the STIRAP experiments the internal states \( |\psi_0(r)\rangle \) and \( |\psi_1(r)\rangle \) of the
three-state subspace usually are taken as the hyperfine ground electronic states
\( |g_0\rangle \) and \( |g_1\rangle \) of an atom, while the internal state \( |\psi_2(r)\rangle \) may be taken as some
excited state \( |e\rangle \) of the atom. For example, the two internal states \( |g_0\rangle \) and
\( |g_1\rangle \) may be the hyperfine ground electronic states \( 3S_{1/2} \) \( (F = 1) \) and \( 3S_{1/2} \)
\( (F = 2) \) of sodium atom \( (Na) \), respectively, while the excited state \( |e\rangle \) may be
the excited electronic state \( 3P_{3/2} \) \( (F = 2) \) of the sodium atom. Suppose that at
the initial time in the STIRAP experiment the atom is in the ground internal
state \( |g_0\rangle \) and the center-of-mass momentum state \( |P'\rangle = (\sqrt{2\pi})^{-1} \exp(iP'x/\hbar), \)
which also means that the atom is in the product state \( |P'\rangle |g_0\rangle \) and it travels
along the direction +\( x \) with the velocity \( v' = P'/M \). Now a laser light field
propagating along the direction −\( x \) is applied to the atom. Then the moving
atom may absorb a photon from the laser light field if the frequency \( (\omega = k_0c) \)
of the laser light field is just equal to the transition frequency of the atom in
motion between the ground internal state \( |g_0\rangle \) and the excited state \( |e\rangle \) after the
Doppler effect is taken into account. After the atom absorbs a photon from the
laser light field, the atomic motional momentum becomes \( P' - \hbar k_0 \) according to
the momentum conservation law and hence the atom is decelerated by \( \hbar k_0/M \).
Since the atomic internal energy levels are discrete, the momentum change of
the moving atom is also discrete after the atom absorbs a photon. This means
that when the atom is excited to the internal state \( |e\rangle \), its motional momentum
can not take an arbitrary value but it has to be \( P' - \hbar k_0 \) due to the fact
that the atomic optical absorption process obeys the energy and momentum conservation laws [19]. After the atom absorbs a photon it jumps to the excited state $|e\rangle$ from the ground state $|g_0\rangle$ and its initial motional state $|P'\rangle$ is changed to $|P' - \hbar k_0\rangle$ and hence the atom is in the product excited state $|P' - \hbar k_0\rangle|e\rangle$.

This is just the atomic decelerating process based on the optical absorption mechanism [23, 24]. The atom in the product excited state $|P' - \hbar k_0\rangle|e\rangle$ may be further decelerated by another laser light field. This laser light field travels along the same direction $+x$ as the atom and its frequency ($\omega = k_1c, k_1 \neq k_0$) is equal to the transition frequency of the moving atom between the ground internal state $|g_1\rangle$ and the excited state $|e\rangle$ after the Doppler effect is taken into account. Thus, this laser light field may stimulate the atom in the excited internal state $|e\rangle$ to jump to the ground internal state $|g_1\rangle$. Because the energy difference between the two internal states $|g_1\rangle$ and $|e\rangle$ is quite different from that one between the two internal states $|g_0\rangle$ and $|e\rangle$, this laser light field will not affect the transition between the two internal states $|g_0\rangle$ and $|e\rangle$. Likewise, the first laser light field does not affect the transition between the two internal states $|g_1\rangle$ and $|e\rangle$. Different from the first decelerating process this atomic decelerating process is based on the stimulated optical emission mechanism [23, 24]. An atom in an excited state may jump to the ground state when it is stimulated by an external laser light field [25]. When the atom in the excited state $|e\rangle$ jumps to the ground state $|g_1\rangle$, it may emit coherently a photon to the laser light field. Since the atomic motion direction is the same as the propagation direction of the laser light field, the atom really sends part of its motional momentum to the laser light field and hence is decelerated in the stimulated transition process from the excited state $|e\rangle$ to the ground state $|g_1\rangle$. This part of motional momentum is just $\hbar k_1$ according to the momentum conservation law and accordingly the atom is decelerated by $\hbar k_1/M$. Thus, after the stimulated optical emission process the atom is in the ground internal state $|g_1\rangle$ and has to be in the motional state $|P' - \hbar k_0 - \hbar k_1\rangle$, that is, the atom is in the product state $|P' - \hbar k_0 - \hbar k_1\rangle|g_1\rangle$. Both the atomic optical absorption and emission processes are required to be unitary here, as pointed out before [22]. The unitary decelerating process based on the three-state STIRAP process just consists of the reversible optical absorption and emission processes mentioned above, where the two Raman laser light beams are adiabatic and usually counterpropagating. This basic STIRAP-based unitary decelerating process may be expressed in an intuitive form

$$
|P + \hbar k_0\rangle|g_0\rangle \rightarrow |P\rangle|e\rangle \rightarrow |P - \hbar k_1\rangle|g_1\rangle
$$

(11)

Here for convenience in the later discussion the atomic motion momentum $P'$ is denoted as $P + \hbar k_0$. Thus, a conventional three-state STIRAP pulse sequence may be really used as a basic unitary decelerating sequence if the two Raman laser light beams of the STIRAP pulse sequence are arranged suitably such that the moving atom is decelerated consecutively by the two Raman laser light beams. Obviously, the inverse process of the STIRAP-based unitary decelerating process (11) may be used to accelerate the atom in motion. However, it is more convenient to use directly the reversible optical absorption and emission processes to accelerate an atom in motion and this may be achieved by
setting suitably the parameters of the two Raman laser light beams of the STIRAP pulse sequence. In the STIRAP-based unitary accelerating process the first laser light field that induces the optical absorption process travels along the motional direction (+x) of the atom, while the second laser light field that stimulates the atomic optical emission process propagates along the opposite direction (−x) to the moving atom. When the atom in the initial product state \( |P'| (g_0) \) is excited to the product state \( |P' + \hbar k'_0 \rangle (e) \) by the first laser light field, it absorbs a photon from the laser light field and is accelerated by \( \hbar k'_0 / M \).

When the atom in the excited state \( |P' + \hbar k'_0 \rangle (e) \) jumps to the ground state \( |P' + \hbar k'_0 + \hbar k'_1 \rangle (g_1) \) under the stimulation of the second laser light field, it emits a photon to the laser light field and is accelerated further by \( \hbar k'_1 / M \). Therefore, the basic STIRAP-based unitary accelerating process may be expressed in an intuitive form (11a)

\[
|P - \hbar k'_0 \rangle \rightarrow |P \rangle (e) \rightarrow |P + \hbar k'_1 \rangle (g_1).
\]

In what follows only the basic STIRAP-based unitary decelerating process (11) is treated explicitly. In an analogous way, one can also deal with the basic STIRAP-based unitary accelerating process (11a).

It follows from the basic decelerating sequence (11) that the time evolution process of the atom in the unitary decelerating process (11) is restricted within the three-state (product state) subspace \( \{ |P + \hbar k_0 \rangle (g_0), |P \rangle (e), |P - \hbar k_1 \rangle (g_1) \} \) for a given atomic motion momentum \( P \). Then during the STIRAP-based unitary decelerating process (11) the total wave function \( |\Psi(x, r, t)\rangle \) of the atom at any instant of time \( t \) can be expanded in the three-state (product state) subspace \([5, 10, 12, 19, 20]\), according to the superposition principle in quantum mechanics [25],

\[
|\Psi(x, r, t)\rangle = \sum_P \rho_P \{ A_0(P, t) |P + \hbar k_0 \rangle (g_0) \\
+ A_1(P, t) |P \rangle (e) + A_2(P, t) |P - \hbar k_1 \rangle (g_1) \}.
\]

(12)

where the sum over the momentum \( P \) is due to the fact that the atom may be in a superposition of momentum states, as can be seen in Eq. (6), \( \rho_P \) is the time-independent amplitude which has the physical meaning that \( |\rho_P|^2 \) is the probability in the superposition to find the atom in the three-state subspace \( \{ |P + \hbar k_0 \rangle (g_0), |P \rangle (e), |P - \hbar k_1 \rangle (g_1) \} \) labelled by the momentum \( P \), and the time-dependent amplitudes \( \{ A_k(P, t) \} \) satisfies the normalization condition:

\[
|A_0(P, t)|^2 + |A_1(P, t)|^2 + |A_2(P, t)|^2 = 1.
\]

(13)

The amplitude \( \rho_P \) is time-independent because the two Raman laser light beams induce a change only within the three-state subspace \( \{ |P + \hbar k_0 \rangle (g_0), |P \rangle (e), |P - \hbar k_1 \rangle (g_1) \} \) for each given momentum \( P \) during the unitary decelerating process [19]. If at the initial time \( t_0 \) the atom is in the ground internal state \( |g_0 \rangle \) and in a wave-packet motional state, then the initial wave packet state of the atom may be expanded as

\[
|\Psi(x, r, t_0)\rangle = \sum_{P'} \rho(P') |P' \rangle (g_0).
\]

(12a)
Obviously, here the coefficients $A_0(P, t_0) = 1$ and $A_1(P, t_0) = A_2(P, t_0) = 0$, which can be deduced from Eqs. (12) and (12a) with $P' = P + \hbar k_0$, while $|\rho(P')|^2$ is the probability to find the atom in the three-state subspace $\{|P + \hbar k_0\rangle g_0\}, |P\rangle |e\rangle, |P - \hbar k_1\rangle |g_1\rangle\}$. As an example, the initial state $|\Psi(x, r, t_0)\rangle$ may be taken as the Gaussian wave-packet motional state of the halting-qubit atom in the right-hand potential well of the double-well potential field in the quantum control process [22]. The three-state (product state) subspace and the basic decelerating process (11) show that only special dipole transition matrix elements $\left\{ W(j', P'; j, P) \right\}$ can take nonzero values, that is, for a given momentum $P$ there are only four matrix elements to take nonzero value: $W(e, P; g_0, P + \hbar k_0)$, $W(e, P; g_1, P - \hbar k_1)$, $W(g_0, P + \hbar k_0; e, P)$, and $W(g_1, P - \hbar k_1; e, P)$. For the basic STIRAP decelerating process (11) the total electric field of the two Raman laser light beams can be explicitly obtained from equation (9) by setting the parameter sets: $(E_{L0}(t), k_{L0}, \omega_{L0}) = (E_{01}(t), k_0, \omega_0)$ and $(E_{L1}(t), k_{L1}, \omega_{L1}) = (E_{12}(t), k_1, \omega_1)$, where the first Raman laser light beam $E_{01}(t), k_0, \omega_0)$ couples the two internal states $|g_0\rangle$ and $|e\rangle$ and its propagating direction is opposite to the motional direction of the atom, while the second beam $E_{12}(t), k_1, \omega_1)$ couples the two internal states $|g_1\rangle$ and $|e\rangle$ and it travels along the motional direction $+(x)$ of the atom. Here suppose that the energy difference (measured in frequency unit) between the two ground internal states $|g_0\rangle$ and $|g_1\rangle$ is much larger than the detunings of the two Raman laser light beams. Now these four nonzero electric-dipole-transition matrix elements for the basic decelerating process (11) can be obtained with the help of the total electric field of Eq. (9) with these parameter settings, the electric dipole interaction $H_d$ of Eq. (10), and the momentum eigenstates of Eq. (7) as well as their orthogonalizations,

\[
W_{02}(t) = W_{20}^*(t) = \langle e|P|H_d[P + \hbar k_0]\rangle g_0\rangle = \hbar \Omega_{02}(t) \exp(-i\omega_0 t),
\]

\[
W_{20}(t) = \langle g_0|P + \hbar k_0|H_d|e\rangle,
\]

\[
W_{12}(t) = W_{21}^*(t) = \langle e|P|H_d[P - \hbar k_1]\rangle g_1\rangle = \hbar \Omega_{12}(t) \exp(-i\omega_1 t),
\]

\[
W_{21}(t) = \langle g_1|P - \hbar k_1|H_d|e\rangle.
\]

Here the star * stands for the complex conjugate and the Rabi frequencies $\Omega_{02}(t)$ and $\Omega_{12}(t)$ are defined in Eq. (10) with the states: $|\psi_0(r)\rangle = |g_0\rangle, |\psi_1(r)\rangle = |g_1\rangle$, and $|\psi_2(r)\rangle = |e\rangle$.

Now the time evolution process of the atom in the presence of the Raman laser light beams in the basic decelerating process (11) is described by the time-dependent Schrödinger equation:

\[
i\hbar \frac{\partial}{\partial t} \Psi(x, r, t) = H(t)\Psi(x, r, t).
\]

Here the total Hamiltonian $H(t)$ is given by Eq. (4), in which $V(x) = 0$ and $H(r)$ and $H_d$ are given by Eq. (8) and (10), respectively, while the wave function $|\Psi(x, r, t)\rangle$ is given by Eq. (12). By using the four nonzero electric-dipole-transition matrix elements and the orthonormalization of the momentum eigenstates of Eq. (7) the Schrödinger equation (14) can be reduced to a three-state
Schrödinger equation for a given momentum $P$, which may be written in the matrix form

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} A_0(P,t) \\ A_1(P,t) \\ A_2(P,t) \end{pmatrix} = \hat{H}(P,t) \begin{pmatrix} A_0(P,t) \\ A_1(P,t) \\ A_2(P,t) \end{pmatrix}$$

(14a)

where the three-state vector $(A_0(P,t), A_1(P,t), A_2(P,t))^T$ (here $T$ stands for the vector transpose) satisfies the normalization of Eq. (13) and the reduced Hamiltonian $\hat{H}(P,t)$ is a $3 \times 3$-dimensional Hermitian matrix,

$$\hat{H}(P,t) = \begin{bmatrix} \frac{(P + \hbar k_0)^2}{2M} + E_0 & W_{02}^*(t) & 0 \\ W_{02}(t) & \frac{P^2}{2M} + E_2 & W_{12}(t) \\ 0 & W_{12}^*(t) & \frac{(P - \hbar k_1)^2}{2M} + E_1 \end{bmatrix}.$$

Here the three basis vectors $|1\rangle = (1, 0, 0)^T$, $|2\rangle = (0, 1, 0)^T$, and $|3\rangle = (0, 0, 1)^T$ of the three-state vector space $\{A_0(P,t), A_1(P,t), A_2(P,t)\}$ stand for the three basis product states $|P + \hbar k_0\rangle|g_0\rangle$, $|P\rangle|e\rangle$, and $|P - \hbar k_1\rangle|g_1\rangle$ of the original three-state subspace $\{|P + \hbar k_0\rangle|g_0\rangle, |P\rangle|e\rangle, |P - \hbar k_1\rangle|g_1\rangle\}$, respectively. This reduced three-state Schrödinger equation can describe completely the three-state STIRAP experiments just like the original Schrödinger equation (14).

By making a unitary transformation on the three-state vector in Eq. (14a) [20, 12, 18, 25]:

$$\tilde{A}_0(P,t) = \exp\left[i\frac{(P + \hbar k_0)^2}{2M} + E_0\right] A_0(P,t),$$

(15a)

$$\tilde{A}_1(P,t) = \exp\left[i\frac{P^2}{2M} + E_2\right] A_1(P,t),$$

(15b)

$$\tilde{A}_2(P,t) = \exp\left[i\frac{(P - \hbar k_1)^2}{2M} + E_1\right] A_2(P,t),$$

(15c)

the Schrödinger equation (14a) is further reduced to the form

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \tilde{A}_0(P,t) \\ \tilde{A}_1(P,t) \\ \tilde{A}_2(P,t) \end{pmatrix} = H(P,t) \begin{pmatrix} \tilde{A}_0(P,t) \\ \tilde{A}_1(P,t) \\ \tilde{A}_2(P,t) \end{pmatrix},$$

(16)

Now the Hamiltonian $H(P,t)$ is a traceless Hermitian matrix,

$$H(P,t) = \begin{bmatrix} 0 & \tilde{W}_{02}(P,t) & 0 \\ \tilde{W}_{02}^*(P,t) & 0 & \tilde{W}_{12}(P,t) \\ 0 & \tilde{W}_{12}^*(P,t) & 0 \end{bmatrix},$$

where the time- and momentum-dependent complex parameters $\tilde{W}_{02}(P,t)$ and $\tilde{W}_{12}(P,t)$ are given respectively by

$$\tilde{W}_{02}(P,t) = \hbar \Omega_{02}(t) \exp\left\{-i\frac{2P k_0 + \hbar k_0^2}{2M} - (\omega_{02} - \omega_0)t\right\},$$

$$\tilde{W}_{12}(P,t) = \hbar \Omega_{12}(t) \exp\left\{-i\frac{-2P k_1 + \hbar k_1^2}{2M} - (\omega_{12} - \omega_1)t\right\},$$

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and the transition frequencies for the atomic internal states are defined by $\hbar \omega_{02} = E_2 - E_0$ and $\hbar \omega_{12} = E_2 - E_1$. Suppose that the Raman laser light beams are amplitude- and phase-modulating. Then the Rabi frequencies for the two Raman laser light beams (the pumping pulse ($\Omega_p(t)$) and the Stokes pulse ($\Omega_s(t)$)) are written as

$$
\Omega_{02}(t) = \Omega_p(t) \exp[-i\phi_0(t)], \quad \Omega_{12}(t) = \Omega_s(t) \exp[-i\phi_1(t)],
$$

and the parameters in the Hamiltonian $H(P,t)$ therefore are given by

$$
\tilde{W}_{02}(P,t) = \hbar \Omega_p(t) \exp[-i\alpha_p(P,t)], \quad \tilde{W}_{12}(P,t) = \hbar \Omega_s(t) \exp[-i\alpha_s(P,t)],
$$

where the phases $\alpha_p(P,t)$ and $\alpha_s(P,t)$ are dependent upon the momentum $P$,

$$
\alpha_p(P,t) = \left[ \frac{2Pk_0 + \hbar k_0^2}{2M} - (\omega_{02} - \omega_0) \right] t + \phi_0(t),
$$

$$
\alpha_s(P,t) = \left[ \frac{-2Pk_1 + \hbar k_1^2}{2M} - (\omega_{12} - \omega_1) \right] t + \phi_1(t).
$$

Now the Hamiltonian $H(P,t)$ can be rewritten in the explicit form

$$
H(P,t) = \hbar \begin{bmatrix}
0 & \Omega_p(t)e^{-i\alpha_p(P,t)} & 0 \\
\Omega_p(t)e^{i\alpha_p(P,t)} & 0 & \Omega_s(t)e^{-i\alpha_s(P,t)} \\
0 & \Omega_s(t)e^{i\alpha_s(P,t)} & 0
\end{bmatrix}
$$

This type of Hamiltonians often have been met in the three-state STIRAP experiments in the laser spectroscopy [4, 15, 17, 18] and in the atomic interference experiments [12]. The three-state Schrödinger equation (16) and the traceless Hamiltonian (17) are the theoretical basis to design the Raman adiabatic pulses of the STIRAP-based decelerating and accelerating processes. There is a special point in the STIRAP-based unitary decelerating and accelerating processes that the Hamiltonian (17) is dependent upon the center-of-mass momentum of the atom besides the frequency offsets $\{\omega_{k2} - \omega_k\} \ (k = 0 \text{ and } 1)$. This is similar to the situations of the STIRAP-based atomic laser cooling [20] and quantum interference experiments [10, 12]. The effect of the frequency offsets on the STIRAP population transfer has been examined in detail in the laser spectroscopy [18b].

Though here considers only the pure-state quantum system of a single atom instead of an atomic ensemble, the atomic momentum distribution could have a great effect on the population transfer of the atom from an internal state to another in the STIRAP decelerating and accelerating processes. This is because the atom may be in a superposition of the momentum eigenstates and hence has a momentum distribution. For example, though a freely moving atom is in a Gaussian wave-packet state in coordinate space, it is also in a superposition of the momentum eigenstates of the atom in momentum space. Here the position of the momentum $P$ in the unitary decelerating and accelerating processes is similar to that one of the frequency offsets in the STIRAP experiments of the conventional laser spectroscopy [18]. The frequency offsets (with respect to the
transition frequencies of given atomic internal energy levels) could be set at will for a single atom, since they are the parameters of the Raman laser light beams, while the superposition of momentum eigenstates of the atom (i.e. the atomic momentum distribution) is an inherent property of the atomic motional state. Whether or not the Raman adiabatic pulses obtained from the Schrödinger equation (16) and the Hamiltonian (17) are suitable for the decelerating and accelerating processes are dependent upon the atomic momentum distribution. The excitation bandwidth for a good STIRAP pulse sequence must be much larger than the effective spreading of the atomic momentum distribution.

In the quantum control process [22] to simulate the reversible and unitary halting protocol and the quantum search process the halting-qubit atom needs to be decelerated and accelerated by the STIRAP-based unitary decelerating and accelerating processes, respectively. As required by the quantum control process, if the halting-qubit atom is completely in the ground internal state \(|g_0\rangle\) at the initial time in the decelerating or accelerating process, then it must be converted completely into another ground internal state \(|g_1\rangle\) at the end of the process. Because the conversion efficiency from the initial state to the end state in these processes has a great effect on the performance of the quantum control process, it is of crucial importance to achieve a high enough conversion efficiency in these processes. This is the first guidance to design the STIRAP pulse sequences for the unitary decelerating and accelerating processes. On the other hand, in order that a high conversion efficiency is achieved in the STIRAP experiments one must also consider the decoherence effect due to the atomic spontaneous emission when the atom is in the excited internal state. The atomic spontaneous emission becomes an important factor to cause the decoherence effect when the atom is in a short-lifetime excited internal state in the STIRAP experiments. Therefore, the atom should avoid being in the excited internal state during the STIRAP experiments. This may be realized by setting the favorable detunings for the two Raman laser light beams. Since the three product states \(|P = \hbar k_0\rangle |g_0\rangle\), \(|P\rangle |e\rangle\), and \(|P = \hbar k_1\rangle |g_1\rangle\) are the eigenstates of the total atomic Hamiltonian \(H_a\) of Eq. (2) and have eigenenergy: \((P + \hbar k_0)^2/(2M) + E_0\), \(P^2/(2M) + E_2\), and \((P - \hbar k_1)^2/(2M) + E_1\), respectively. Then the energy difference between the ground state \(|P + \hbar k_0\rangle |g_0\rangle\) and the excited state \(|P\rangle |e\rangle\) is \((E_2 - E_0) - \hbar P k_0/M - \hbar^2 k_0^2/(2M)\) and the energy difference between the ground state \(|P - \hbar k_1\rangle |g_1\rangle\) and the excited state \(|P\rangle |e\rangle\) is \((E_2 - E_1) + \hbar P k_1/M - \hbar^2 k_1^2/(2M)\). Since the Raman laser light beam with the carrier frequency \(\omega_0\) couples the ground state \(|P + \hbar k_0\rangle |g_0\rangle\) and the excited state \(|P\rangle |e\rangle\), the detuning \(\Delta_p\) of the beam is given by

\[
\Delta_p(P) = (\omega_0^2 - \omega_0) - \left(\frac{k_0}{M}\right)P - \frac{\hbar k_0^2}{2M},
\]

while the detuning \(\Delta_s\) of another Raman laser light beam with the carrier frequency \(\omega_1\) is

\[
\Delta_s(P) = (\omega_1^2 - \omega_1) + \left(\frac{k_1}{M}\right)P - \frac{\hbar k_1^2}{2M}.
\]

By using the two formulae one may set conveniently the detunings \(\Delta_p\) and \(\Delta_s\).
for the STIRAP-based decelerating and accelerating processes. Both the detunings are dependent upon the motional momentum of the atom and also their frequency offsets, respectively. If the atom is in a wave-packet state or generally in a superposition of momentum states, then there is a distribution of momentum and the detuning settings should consider the momentum distribution. In next sections the parameters of the Raman laser light beams will be determined so as to arrive at the main goal that the conversion efficiency from the initial state to the end state in the decelerating and accelerating processes must be as close to 100% as possible. There the atomic momentum distribution must be taken into account. Thus, this is involved in the adiabatic condition for the three-state STIRAP state transfer of the decelerating and accelerating processes.

3. The basic differential equations for the STIRAP-based decelerating and accelerating processes

The three-state STIRAP experiments have been studied extensively both in theory and experiment in the laser spectroscopy [15, 16, 17, 18]. These studies tell ones which conditions the complete population transfer can be achieved by the STIRAP method among the energy levels of atomic and molecular systems. From the point of view of theory an important feature of the three-state STIRAP experiments is that there is the special eigenstate of the Hamiltonian (17) that corresponds to the atomic trapping state [28]. This special eigenstate is independent of the intermediate state [17], which is usually taken as the excited internal state of the atom in the STIRAP experiments. Under the adiabatic condition the complete state or population transfer through the special eigenstate can occur from the initial state directly to the final state, while the intermediate state is bypassed in the transfer process. Such an adiabatic state-transfer process is particularly favorable for the STIRAP-based decelerating and accelerating processes. This is because the atomic spontaneous emission could be avoided in the decelerating and accelerating processes when the excited state of the atom is bypassed in the STIRAP state-transfer processes. This also shows that the semiclassical theory of electromagnetic radiation is suited to treat the STIRAP experiments. On the other hand, from the experimental point of view the three-state STIRAP experiment needs to set suitably the experimental parameters for the Raman laser light beams. The important thing in experiment is the settings for the Rabi frequencies of the two Raman laser light beams and for the pulse delay between the two Raman laser light beams [16]. The atomic system should first interact with the Stokes pulse and then with the pumping pulse, and an appropriate overlapping between the two Raman laser light beams is also required in experiment [3, 4, 15, 16, 17, 18]. These requirements are generally related to the adiabatic condition of the three-state STIRAP experiment.

The special point for the STIRAP decelerating and accelerating processes is that one must consider the atomic momentum distribution when a general adiabatic condition is set up for these processes. According to the adiabatic theorem and the adiabatic approximation method in quantum mechanics [29, 30] one should first calculate the three eigenvectors and eigenvalues of the in-
stantaneous Hamiltonian $H(P, t)$ of Eq. (17),
\[ H(P, t)|g(P, t)\rangle = E(P, t)|g(P, t)\rangle. \tag{18} \]

The three eigenvectors \{|g(P, t)\rangle\} and their eigenvalues \{E(P, t)\} of the Hamiltonian $H(P, t)$ are usually named the adiabatic eigenvectors and eigenvalues, respectively. Notice that the three basis vectors of the three-state vector space \{\{A_0(P, t), A_1(P, t), A_2(P, t)\}^T\} are \|1\rangle = (1, 0, 0)^T, \|2\rangle = (0, 1, 0)^T, and \|3\rangle = (0, 0, 1)^T, respectively. The three basis vectors really stand for the three basis product states \|P + \hbar k_0\rangle|g_0\rangle, \|P\rangle|e\rangle, and \|P - \hbar k_1\rangle|g_1\rangle of the original three-state subspace for each given momentum $P$, respectively. Any one of the three eigenvectors of the Hamiltonian $H(P, t)$ may be expanded in terms of the three basis vectors. By the explicit Hamiltonian of Eq. (17) one can obtain one of the three eigenvectors [15, 17, 18],
\[ |g^0(P, t)\rangle = \exp[-i\gamma(t)]\{\cos \theta(t)|1\rangle - \sin \theta(t) \exp[-i(\alpha_p(P, t) - \alpha_s(P, t))]|3\rangle\} \tag{19a} \]
where the phase $\gamma(t)$ is a global phase, the mixing angle $\theta(t)$ and the Rabi frequency $\Omega(t)$ are defined respectively by
\[ \cos \theta(t) = \frac{\Omega_s(t)}{\Omega(t)}, \quad \sin \theta(t) = \frac{\Omega_p(t)}{\Omega(t)}, \quad \text{and} \quad \Omega(t) = \sqrt{\Omega_p(t)^2 + \Omega_s(t)^2}. \]

The eigenvalue corresponding to the eigenvector \|g^0(P, t)\rangle\) is $E^0 = \hbar \omega^0 = 0$. The eigenvector \|g^0(P, t)\rangle\) is special in that it does not contain the intermediate eigenvector \|2\rangle which contains the excited internal state \|e\rangle of the atom. It is the so-called atomic trapping state [28]. The adiabatic population transfer is achieved through the eigenvector \|g^0(P, t)\rangle\) in all the three-state STIRAP experiments [15, 17]. Therefore, when the initial state \|1\rangle is transferred to the final state \|3\rangle through the eigenvector \|g^0(P, t)\rangle\) in the STIRAP state-transfer process, the intermediate state \|2\rangle is not involved and hence is bypassed. Since the phase difference $\alpha_p(P, t) - \alpha_s(P, t)$ in the eigenvector \|g^0(P, t)\rangle\) is dependent upon the atomic motional momentum $P$ the adiabatic state transfer in the STIRAP experiments is really affected by the atomic momentum distribution. In the laser spectroscopy the similar trapping state is obtained [4, 15, 17, 18] but that state is not dependent upon the momentum $P$. Thus, there is not any theoretical problem involved in the effect of the momentum distribution on the adiabatic state transfer in the laser spectroscopy. Of course, in the laser spectroscopy the Doppler effect usually is considered in the STIRAP experiments of those quantum systems such as an atomic or molecular beam [15]. However, in the atomic laser cooling [5, 19, 20, 21], quantum interference experiments [10, 11, 12, 13, 14], and the atomic decelerating and accelerating processes the atomic momentum distribution generally needs to be considered explicitly. The other two adiabatic eigenstates of the Hamiltonian of Eq. (17) are given by [15, 17, 18]
\[ |g^\pm(P, t)\rangle = \frac{1}{\sqrt{2}} \exp[-i\delta(t)]\{\sin \theta(t)|1\rangle \mp \exp[-i\alpha_p(P, t)]|2\rangle \]
and their corresponding eigenvalues are $E^\pm \equiv \hbar \omega^\pm = \mp \hbar \Omega(t)$. Here the phase $\delta(t)$ is also a global phase. The phase difference $\alpha_p(P,t) - \alpha_s(P,t)$ in the adiabatic eigenvectors is given by

$$\alpha_p(P,t) - \alpha_s(P,t) = \phi_0(t) - \phi_1(t)$$

$$+ \{ -[\omega_{02} - \omega_0] + [\omega_{12} - \omega_1] + \frac{P(k_0 + k_1)}{M} + \frac{\hbar k_0^2 - \hbar k_1^2}{2M} \},$$

and there is a relation between the phase difference and the detunings:

$$\alpha_p(P,t) - \alpha_s(P,t) = [\phi_0(t) - \phi_1(t)] + [\Delta_s(P) - \Delta_p(P)]t.$$ \hspace{1cm} (20)

The phase difference is dependent upon both the frequency offsets $(\omega_{02} - \omega_0)$ and $(\omega_{12} - \omega_1)$ and also the momentum $P$. It could be considered that the term $(k_0 + k_1)P/M$ in the phase difference is generated by the Doppler effect. A large Doppler effect is not favorable for the decelerating and accelerating processes. Assume that the atomic motional state is a wave-packet state with a finite wave-packet spreading. Then in order to minimize the effect of the Doppler-effect term $(k_0 + k_1)P/M$ on the STIRAP state transfer in the decelerating and accelerating processes one must choose suitably the frequency offsets $(\omega_{02} - \omega_0)$ and $(\omega_{12} - \omega_1)$ for the two Raman laser light beams. Now $P_0$ and $\Delta P_M$ are denoted as the central position and the effective momentum bandwidth of the atomic momentum wave-packet state, respectively, and $\Delta P = P - P_0$ as the deviation of the position $P$ of the momentum wave-packet state from the central position $P_0$. For simplicity, hereafter assume that the momentum $P_0$ is always much greater than $\Delta P_M/2$ in the unitary decelerating process, this means that the atom always moves along the same direction $+x$ in the decelerating process. Suppose that the absolute amplitude at the position $P$ of the momentum wave-packet state decays exponentially as the deviation $\Delta P$. A typical example of such wave-packet states is Gaussian wave-packet states \cite{25, 31}. Then the amplitude of the position $P$ outside the effective momentum region $[P] = [P_0 - \Delta P_M/2, P_0 + \Delta P_M/2]$ in the momentum wave-packet state is almost zero and the probability to find that the atom is not in the effective momentum region $[P]$ is so small that it can be negligible. This is just the definition of the effective momentum bandwidth $\Delta P_M$ of the atomic momentum wave-packet state. Then it is sufficient to consider only the effective momentum region $[P]$ of the momentum wave-packet state when the adiabatic condition is investigated for the three-state STIRAP experiments of the decelerating and accelerating processes. Here the carrier frequencies $\omega_0$ and $\omega_1$ of the two Raman laser light beams are not determined until the adiabatic condition for the STIRAP experiments is examined later.

The unitary decelerating and accelerating processes of the quantum control process \cite{22} require that the atom in the initial internal state $|g_0\rangle$ ($|g_1\rangle$) be completely transferred to another internal state $|g_1\rangle$ ($|g_0\rangle$) and at the same time the initial atomic wave-packet motional state be completely converted into another
wave-packet motional state. It is well known that in an ideal condition the STIRAP population transfer process can achieve 100% transfer efficiency from one atomic internal state to another [3, 4, 15, 17, 18]. The adiabatic state-transfer channel for the three-state STIRAP experiments is formed generally through the special adiabatic eigenstate $|g_0(P, t)\rangle$ of the Hamiltonian $H(P, t)$ [17]. Therefore, if the unitary decelerating and accelerating processes want to achieve their main goal, they had better make full use of this adiabatic state-transfer channel. If now the atom is prepared in the adiabatic eigenstate $|g_0(P, t_0)\rangle$ of the Hamiltonian $H(P, t_0)$ at the initial time $t_0$, then according as the adiabatic theorem [25, 30] the atom will be in the adiabatic eigenstate $|g_0(P, t)\rangle$ of the Hamiltonian $H(P, t)$ at any instant of time $t$ in the adiabatic process for $t_0 \leq t \leq t_0 + T$ if the adiabatic condition is met, that is, if the time period $T$ is infinitely large or more generally if the eigenvectors of the Hamiltonian $H(P, t)$ vary infinitely slowly [30]. However, in practice the time period $T$ can not be infinitely large and rotating of the eigenvectors of the Hamiltonian are not infinitely slow. Thus, the ideal adiabatic condition can not met perfectly in practice. In fact, the quantum control process does not allow the time period $T$ of the adiabatic process to take an infinitely large value. Obviously, if the time interval $T$ is taken as a finite value but large enough or the eigenvectors of the Hamiltonian $H(P, t)$ rotate sufficiently slowly, then the adiabatic theorem still holds approximately and the real adiabatic state at the end time $t_0 + T$ of the adiabatic process will be very close to the ideal adiabatic eigenstate $|g_0(P, t_0 + T)\rangle$. Since the real adiabatic state at the end of the adiabatic process is close to the ideal adiabatic eigenstate $|g_0(P, t_0 + T)\rangle$, one may calculate the real wave-packet state of the atom at the end of the STIRAP-based unitary decelerating or accelerating process with the help of the ideal adiabatic eigenstate $|g_0(P, t_0 + T)\rangle$. Then this simplifies greatly the calculation for the real wave-packet state of the atom at the end of the decelerating or accelerating process, although such a calculation could generate an error for the real wave-packet state of the atom. Similarly, if the real transfer efficiency for the STIRAP process is calculated with the help of the ideal adiabatic eigenstate $|g_0(P, t_0 + T)\rangle$ instead of the real adiabatic state at the final time of the decelerating or accelerating process, then there exists certainly an error for the real transfer efficiency. However, these errors may be estimated. In fact, if one finds the real adiabatic condition for the STIRAP process, one may estimate these errors, as shown below. Therefore, for the STIRAP process satisfying the real adiabatic condition there is a simple scheme to calculate the real transfer efficiency and the real wave-packet state of the atom at the end of the STIRAP process: one may use the ideal adiabatic eigenstate $|g_0(P, t_0 + T)\rangle$ of the Hamiltonian $H(P, t_0 + T)$ to simplify the calculation of the real transfer efficiency and the real wave-packet state of the atom, then evaluate the generated errors and control these errors to be within the desired upper bound by setting suitably the experimental parameters for the STIRAP process. This is really the procedure to design the STIRAP pulse sequences for the unitary decelerating and accelerating processes. Thus, the problem to be solved below is how to design the STIRAP pulse sequence such that in the real adiabatic condition the final adiabatic state for the real adiabatic process is still
very close to the ideal adiabatic eigenstate $|g^0(P, t_0 + T)\rangle$ even if the time period $T$ takes a finite value.

In the STIRAP-based decelerating and accelerating processes the adiabatic evolution process of the atom could occur simultaneously and in a parallel form in these three-state subspaces \( \{|P + h|g_0\rangle, \{P\} e, \{P - h|g_1\rangle\} \) for all the given momentum $P$ within the effective momentum region $|P|$ if the adiabatic condition is met in the effective momentum region $|P|$. It is sufficient to examine the adiabatic evolution process of the atom in a three-state subspace \( \{|P + h|g_0\rangle, \{P\} e, \{P - h|g_1\rangle\} \) with a given momentum $P$ of the effective momentum region $|P|$ to set up the adiabatic condition for the STIRAP experiments of the decelerating and accelerating processes. Obviously, any atomic state in the three-state subspace during the adiabatic evolution process can be expanded in terms of the three basis vectors of the subspace. If now the three basis vectors of the three-state subspace \( \{|P + h|g_0\rangle, \{P\} e, \{P - h|g_1\rangle\} \) are taken as the three orthonormal adiabatic eigenvectors \( \{g^0(P, t)\}, \{g^\pm(P, t)\} \) of the Hamiltonian $H(P, t)$, then the atomic three-state vector $|\Phi(P, t)\rangle = (\tilde{A}_0(P, t), \tilde{A}_1(P, t), \tilde{A}_2(P, t))^T$ in the Schrödinger equation (16) at any instant of time $t$ in the adiabatic evolution process may be expanded as $[4, 5, 10, 12, 15, 17, 18, 19, 20, 25]$

$$
|\Phi(P, t)\rangle = a_0(P, t)|g^0(P, t)\rangle + a_+(P, t)|g^+(P, t)\rangle \exp[i \int_{t_0}^{t} dt' \Omega(t')] + a_-(P, t)|g^-(P, t)\rangle \exp[-i \int_{t_0}^{t} dt' \Omega(t')].
$$

By substituting the adiabatic eigenstates of Eqs. (19a) and (19b) into the state $|\Phi(P, t)\rangle$ of Eq. (21) one can find that the coefficients $\{a_0(P, t), a_\pm(P, t)\}$ in Eq. (21) are related to those $\{\tilde{A}_l(P, t), l = 0, 1, 2\}$ in Eq. (16) by

$$
\tilde{A}_0(P, t) = a_0(P, t) \cos \theta(t) \exp[-i \gamma(t)]
$$

$$
+ \frac{1}{\sqrt{2}} a_+(P, t) \sin \theta(t) \exp[-i \delta(t)] \exp[i \int_{t_0}^{t} dt' \Omega(t')]$$

$$
+ \frac{1}{\sqrt{2}} a_-(P, t) \sin \theta(t) \exp[-i \delta(t)] \exp[-i \int_{t_0}^{t} dt' \Omega(t')],
$$

(22a)

$$
\tilde{A}_1(P, t) = \frac{1}{\sqrt{2}} \exp[-i \alpha_p(P, t)] \exp[-i \delta(t)] \{-a_0(P, t) \exp[i \int_{t_0}^{t} dt' \Omega(t')]$$

$$+ a_-(P, t) \exp[-i \int_{t_0}^{t} dt' \Omega(t')]\}
$$

$$
\tilde{A}_2(P, t) = \exp[-i(\alpha_p(P, t) - \alpha_s(P, t))] \{-a_0(P, t) \sin \theta(t) \exp[-i \gamma(t)]
$$

$$+ \frac{1}{\sqrt{2}} a_+(P, t) \cos \theta(t) \exp[-i \delta(t)] \exp[i \int_{t_0}^{t} dt' \Omega(t')]\}
$$

(22b)
Thus, the atomic three-state vector \( |\Phi(P, t)\rangle \) may be determined if one knows the coefficients \( a_0(P, t) \) and \( a_\pm(P, t) \). Assume that at the initial time \( t_0 \) the atom is in the adiabatic eigenstate \( |g^0(P, t_0)\rangle \). Then the adiabatic theorem shows that the atom is kept in the adiabatic eigenstate \( |g^0(P, t)\rangle \) of the Hamiltonian \( H(P, t) \) at any instant of time \( t \) during the adiabatic process if the ideal adiabatic condition is met. Then it follows from the atomic state \( |\Phi(P, t)\rangle \) of Eq. (21) that if the ideal adiabatic condition is met, the time-dependent coefficient \( |a_0(P, t)\rangle \) should be kept almost unchanged and is very close to unity over the whole adiabatic process, while two other coefficients \( \{ |a_\pm(P, t)\rangle \} \) should be very close to zero. Generally, these coefficients may be evaluated by solving the Schrödinger equation (16) that the state \( |\Phi(P, t)\rangle \) of Eq. (21) obeys. Inserting the state \( |\Phi(P, t)\rangle \) of Eq. (21) into the Schrödinger equation (16) one obtains a set of the three differential equations with the variables \( a_0(P, t) \) and \( a_\pm(P, t) \),

\[
\frac{\partial}{\partial t} a_0(P, t) + i \omega_0(P, t) a_0(P, t) + a_+(P, t) \omega_{0,+}(P, t) \exp[-i \int_{t_0}^{t} dt' \Omega(t')] \\
+ a_-(P, t) \omega_{0,-}(P, t) \exp[-i \int_{t_0}^{t} dt' \Omega(t')] = 0, \tag{23a}
\]

\[
\frac{\partial}{\partial t} a_\pm(P, t) + i \omega_\pm(P, t) a_\pm(P, t) + a_\mp(P, t) \omega_{\pm,\mp}(P, t) \exp[\mp 2i \int_{t_0}^{t} dt' \Omega(t')] \\
+ a_0(P, t) \omega_{\pm,0}(P, t) \exp[\mp i \int_{t_0}^{t} dt' \Omega(t')] = 0, \tag{23b}
\]

where the coefficients \( \{ \omega_k(P, t), \omega_{k,l}(P, t) \} \) are defined as

\[
i \omega_k(P, t) = \langle g^k(P, t)|\partial g^k(P, t)/\partial t \rangle \ (k = 0, \pm),
\]

\[
\omega_{k,l}(P, t) = \langle g^k(P, t)|\frac{\partial}{\partial t}|g^l(P, t)\rangle \ (k \neq l, \ k, l = 0, \pm).
\]

These coefficients can be obtained explicitly from the adiabatic eigenstates of Eqs. (19),

\[
\omega_0(P, t) = - \frac{\partial}{\partial t} \gamma(t) - \sin^2 \theta(t) \frac{\partial}{\partial t} [\alpha_p(P, t) - \alpha_s(P, t)], \tag{24a}
\]

\[
\omega_+(P, t) = \omega_-(P, t) = - \frac{\partial}{\partial t} \delta(t) - \frac{1}{2} \frac{\partial}{\partial t} \alpha_p(P, t)
\]

\[
- \frac{1}{2} \cos^2 \theta(t) \frac{\partial}{\partial t} [\alpha_p(P, t) - \alpha_s(P, t)] \tag{24b}
\]

and

\[
\omega_{0,\pm}(P, t) = - \omega_{\pm,0}(P, t)^* = \frac{1}{\sqrt{2}} \exp[i(\gamma(t) - \delta(t))]
\]

23
\[ \times \left\{ \frac{\partial}{\partial t} \theta(t) + i \sin \theta(t) \cos \theta(t) \frac{\partial}{\partial t} [\alpha_p(P, t) - \alpha_s(P, t)] \right\}, \quad (24c) \]

\[ \omega_{+,-}(P, t) = \omega_{-,+}(P, t) = i \frac{1}{2} \frac{\partial}{\partial t} \alpha_p(P, t) \]

\[ - i \frac{1}{2} \cos^2 \theta(t) \frac{\partial}{\partial t} [\alpha_p(P, t) - \alpha_s(P, t)]. \quad (24d) \]

Though these coefficients contain the global phases \( \gamma(t) \) and \( \delta(t) \), the final results of the adiabatic process obtained from these coefficients will not be affected by these global phase factors \([18b]\), as can be seen later. In order to solve conveniently the equations (23) it could be better to make variable transformations \([25, 30]\):

\[ a_k(P, t) = b_k(P, t) \exp\left[-i \int_{t_0}^t dt' \omega_k(P, t')\right] \quad (k = 0, \pm). \quad (25) \]

Then with the new variables \( \{b_k(P, t)\} \) these equations (23) are changed to

\[ \frac{\partial}{\partial t} b_0(P, t) = \frac{\exp[i(\gamma(t_0) - \delta(t_0))] - \sqrt{2}}{\Theta(P, t)*} \]

\[ \times \left\{ b_+(P, t) \exp\left[i \int_{t_0}^t dt' \Omega_+(P, t')\right] + b_-(P, t) \exp\left[-i \int_{t_0}^t dt' \Omega_-(P, t')\right] \right\}, \quad (26a) \]

\[ \frac{\partial}{\partial t} b_\pm(P, t) = -i \frac{1}{2} b_\mp(P, t) \Gamma(P, t) \exp[\mp i \int_{t_0}^t dt' 2\Omega(t')] \]

\[ - \frac{\exp[-i(\gamma(t_0) - \delta(t_0))]}{\sqrt{2}} b_0(P, t) \Theta(P, t) \exp[\mp i \int_{t_0}^t dt' \Omega_\pm(P, t')]. \quad (26b) \]

The coefficients in Eqs. (26) are obtained from those in Eq. (24):

\[ \Omega_\pm(P, t) = \Omega(t) \pm \{[\dot{\alpha}_s(P, t) - \frac{1}{2} \dot{\alpha}_p(P, t)] \sin^2 \theta(t) \]

\[ + [\dot{\alpha}_p(P, t) - \frac{1}{2} \dot{\alpha}_s(P, t)] \cos^2 \theta(t) \}\}; \quad (27a) \]

\[ \Theta(P, t) = -\dot{\theta}(t) + i \frac{1}{2} \sin 2 \theta(t) [\dot{\alpha}_p(P, t) - \dot{\alpha}_s(P, t)], \quad (27b) \]

\[ \Gamma(P, t) = \sin^2 \theta(t) \dot{\alpha}_p(P, t) + \cos^2 \theta(t) \dot{\alpha}_s(P, t), \quad (27c) \]

where \( \dot{\theta}(t) = \frac{d}{dt} \theta(t) \), \( \dot{\alpha}_s(P, t) = \frac{d}{dt} \alpha_s(P, t) \), and so on. The equations (26) are the basic differential equations to describe completely the STIRAP-based decelerating and accelerating processes. The set of basic equations (26) is a generalization of the basic equations to describe the conventional three-state STIRAP experiments \([17, 16, 18]\) when the effect of a momentum distribution is taken into account on the STIRAP experiments. The basic equations (26) may be used to set up a general adiabatic condition for the three-state STIRAP-based decelerating and accelerating processes.
The basic differential equations (26) may be rewritten in the matrix form
\[ i \frac{\partial}{\partial t} B(P,t) = M(P,t) B(P,t). \] (28)

Here the normalization three-state vector \( B(P,t) \) is defined as \((b_0(P,t), b_+(P,t), b_-(P,t))^T\) and the 3 × 3-dimensional hermitian Hamiltonian \( M(P,t) \) is given by
\[
M(P,t) = \begin{bmatrix}
0 & M_{12} & M_{13} \\
M_{12}^* & 0 & M_{23} \\
M_{13}^* & M_{23}^* & 0
\end{bmatrix},
\]
in which the matrix elements \( \{M_{ij}\} \) are defined by
\[
M_{12} = \frac{i}{\sqrt{2}} \exp[i(\gamma(t_0) - \delta(t_0))]\Theta(P,t) \exp[i \int_{t_0}^{t} dt' \Omega_+(P,t')],
\]
\[
M_{13} = \frac{i}{\sqrt{2}} \exp[i(\gamma(t_0) - \delta(t_0))]\Theta(P,t) \exp[-i \int_{t_0}^{t} dt' \Omega_-(P,t')],
\]
\[
M_{23} = \frac{1}{2} \Gamma(P,t) \exp[-i \int_{t_0}^{t} dt' 2\Omega(t')].
\]

Notice that the momentum \( P \) in the hermitian Hamiltonian \( M(P,t) \) is a parameter instead of an operator. The three-state Schrödinger equation (28) may have the formal solution:
\[
B(P,t) = T \exp\{-i \int_{t_0}^{t} dt'M(P,t')\} B(P,t_0)
= \{1 + \left(\frac{1}{i}\right) \int_{t_0}^{t} dt_1 M(P,t_1) + \left(\frac{1}{i}\right)^2 \int_{t_0}^{t} \int_{t_0}^{t_1} dt_1 dt_2 M(P,t_1)M(P,t_2) + \ldots\} B(P,t_0).
\] (29)

The Dyson series solution (29) may be useful to set up a general adiabatic condition for the STIRAP-based decelerating and accelerating processes. The detailed discussion will appear in the section seven of the paper.

4. The STIRAP state-transfer process in the ideal adiabatic condition

Consider first the special case: the ideal adiabatic condition. The ideal adiabatic condition usually means that the time interval \( T \) of the adiabatic process is infinitely large or the adiabatic eigenstates of the Hamiltonian (17) rotate infinitely slowly. Here the ideal adiabatic condition means that for any instant of time of the adiabatic process the integrations of the basic differential equations (26) approach zero for any given momentum \( P \) within the effective momentum region \([P]\) of the atomic wave-packet motional state. The ideal adiabatic condition may be expressed as
\[
\int_{t_0}^{t} dt' \left[ \frac{\partial}{\partial t'} b_l(P,t') \right] \to 0, \ l = 0, \pm; \ t_0 \leq t \leq t_0 + T; \ P \in [P]. \] (30)
The ideal adiabatic condition (30) shows that the basic equations (26) have the solution \( b_l(P, t) \to b_l(P, t_0) \) \((l = 0, \pm)\) for any instant of time \( t \) of the adiabatic process and any given momentum \( P \) within the effective momentum region \([P]\). If at the initial time \( t_0 \) the atom is prepared in the adiabatic eigenstate \(|g^0(P, t_0)\rangle\), which is the initial atomic state \(|\Phi(P, t_0)\rangle\) of Eq. (21) with \( a_0(P, t_0) = 1 \) and \( a_{\pm}(P, t_0) = 0 \), then according to the adiabatic theorem \([25, 30]\) one should find that at any instant of time \( t \) of the adiabatic process the atom is in the adiabatic eigenstate \(|g^0(P, t)\rangle\) of the Hamiltonian \(H(P, t)\). In fact, by the equations (25) and (26) one can obtain \(|a_l(P, t)| \to |a_l(P, t_0)| \((l = 0, \pm)\) in the ideal adiabatic condition (30). Hence \(|a_0(P, t)| \to |a_0(P, t_0)| = 1 \) and \( a_{\pm}(P, t) \to a_{\pm}(P, t_0) = 0 \) for the atomic state \(|\Phi(P, t)\rangle\) at the instant of time \( t \) of the ideal adiabatic process, while this atomic state \(|\Phi(P, t)\rangle\) is just the adiabatic eigenstate \(|g^0(P, t)\rangle\) of the Hamiltonian \(H(P, t)\), as can be seen from Eq. (21). The basic equations (26) show that the ideal adiabatic condition (30) could be achieved if the coefficients \(\Theta(t)\) and \(\Gamma(P, t)\) approach zero sufficiently and the Rabi frequencies \(\Omega(t)\) and \(\Omega_{\pm}(P, t)\) are sufficiently large. The Rabi frequencies \(\Omega(t)\) and \(\Omega_{\pm}(P, t)\) can have a great effect on the adiabatic condition of the STIRAP experiments. This point is very important for the quantum control process, since a sufficiently long time period \(T\) of the adiabatic process is not accepted for the quantum control process. Then one may likely make the adiabatic condition to be met by setting suitably the Rabi frequencies of the Raman laser light beams, although the adiabatic condition is met usually by making the time interval \(T\) sufficiently large. A general adiabatic condition will be discussed in detail later.

In order to optimize the adiabatic condition the carrier frequencies \(\omega_0\) and \(\omega_1\) of the Raman laser light beams could be chosen suitably such that

\[
\alpha_P(P, t) = \frac{2P_{k_0} + \hbar k_0^2}{2M} - (\omega_{\omega_2} - \omega_0)|t + \phi_0(t) = \frac{\Delta P}{M}k_0t + \varphi_0(t) \tag{31}
\]

and

\[
\alpha_s(P, t) = \frac{-2P_{k_1} + \hbar k_1^2}{2M} - (\omega_{\omega_2} - \omega_1)|t + \phi_1(t) = -\frac{\Delta P}{M}k_1t + \varphi_1(t) \tag{32}
\]

where the momentum difference \(\Delta P = P - P_0\) and the momentum \(P_0\) is just the central position of the effective momentum region \([P]\) of a general momentum wave-packet state. When the carrier frequencies are chosen according to Eq. (31) and (32), the maximum momentum difference value within the effective momentum region \([P]\) is minimum. These equations (31) and (32) can be satisfied when the carrier frequencies \(\{\omega_l\} \(l = 0, 1\) are determined from the two equations:

\[
\frac{\hbar k_0^2}{2M} - (\omega_{\omega_2} - \omega_0) + c_0 = -\frac{P_0}{M}k_0 \tag{33a}
\]

and

\[
\frac{\hbar k_1^2}{2M} - (\omega_{\omega_2} - \omega_1) + c_1 = \frac{P_0}{M}k_1 \tag{33b}
\]
where the wave number $k_t = \omega_t/c$, $c_0 = \Delta_p(P_0)$, $c_1 = \Delta_s(P_0)$, and $\frac{d}{dt}\phi_1(t) = c_1 + \frac{\omega}{c}\phi_1(t)$. On the other hand, if the carrier frequencies are given in advance, then the detunings $c_0 = \Delta_p(P_0)$ and $c_1 = \Delta_s(P_0)$ may also be determined from these two equations $(33)$, respectively. In the decelerating and accelerating processes the momentum value $P_0$ is generally different for each basic STIRAP-based decelerating or accelerating process. Then one may adjust the carrier frequencies $\{\omega_i\}$ or the detunings $\{c_i\}$ so that the equations $(33)$ can be satisfied.

Now using the phase $\alpha_p(P,t)$ of Eq. (31) and $\alpha_s(P,t)$ of Eq. (32) one can rewrite the phase difference of Eq. (20) in the simple form

$$\alpha_p(P,t) - \alpha_s(P,t) = \frac{\Delta P}{M}(k_0 + k_1)t + \phi_0(t) - \phi_1(t). \quad (34)$$

If in the STIRAP experiments at the initial time $t_0$ the Rabi frequency $\Omega_s(t_0)$ of the Stokes pulse is much greater than the one $\Omega_p(t_0)$ of the pumping pulse, then at the initial time the mixing angle $\theta(t)$ in the adiabatic eigenstates of Eqs. (19) satisfies $[15, 16, 17, 18]$

$$\cos\theta(t_0) = \frac{\Omega_s(t_0)}{\Omega_p(t_0)} \rightarrow 1 \text{ or } \theta(t_0) \rightarrow 0. \quad (35)$$

In practice both the Rabi frequencies $\Omega_s(t_0)$ and $\Omega_p(t_0)$ usually could be relatively small at the initial time, but the Rabi frequency $\Omega_p(t_0)$ of the pumping pulse is much less than $\Omega_s(t_0)$ of the Stokes pulse. The initial mixing angle $\theta(t_0) \rightarrow 0$ leads the initial adiabatic eigenstates of the Hamiltonian $H(P,t_0)$ of Eq. (17) to the asymptotic forms

$$|g^0(P,t_0)\rangle \rightarrow \exp[-i\gamma(t_0)]|1\rangle, \quad (36a)$$

$$|g^\pm(P,t_0)\rangle \rightarrow \frac{1}{\sqrt{2}}\exp[-i\delta(t_0)]\{\pm \exp[-i\alpha_p(P,t_0)]|2\rangle + \exp[-i(\alpha_p(P,t_0) - \alpha_s(P,t_0))]|3\rangle\}. \quad (36b)$$

Notice that the three-state basis vectors $|1\rangle$, $|2\rangle$, and $|3\rangle$ stand for the basis vectors $|P + \hbar k_0\rangle|g_0\rangle$, $|P\rangle|e\rangle$, and $|P - \hbar k_1\rangle|g_1\rangle$, respectively. The adiabatic eigenstate $|g^0(P,t_0)\rangle$ of the Hamiltonian $H(P,t_0)$ is much simpler than $|g^\pm(P,t_0)\rangle$, the latter is more complex in that their expansion coefficients are dependent on the momentum. In general, at the initial time an atomic system may be prepared more easily in the adiabatic eigenstate $|g^0(P,t_0)\rangle$ up to a global phase factor. An important example is that at the initial time the atom is completely in the ground internal state $|g_0\rangle$ and in the Gaussian wave-packet motional state or in a superposition of the momentum states. Now consider that the initial state of the atom is prepared in the superposition state $|\Psi(x, r, t_0)\rangle$ given by Eq. (12a). By comparing Eq. (12) with Eq. (12a) one sees that the coefficients of the state $|\Psi(x, r, t_0)\rangle$ of Eq. (12) are given by $A_0(P,t_0) = 1$, $A_1(P,t_0) = 0$, and $A_2(P,t_0) = 0$ at the initial time $t_0$. Then it follows from Eqs. (15) that the coefficients $\{\bar{A}_i(P,t_0)\}$ can be obtained from $\{A_i(P,t_0)\}$:

$$\bar{A}_0(P,t_0) = \exp\left[\frac{i}{\hbar}\left(\frac{(P + \hbar k_0)^2}{2M} + E_0\right)t_0\right], \quad \bar{A}_1(P,t_0) = \bar{A}_2(P,t_0) = 0. \quad (37)$$
These coefficients associated with the initial mixing angle \( \theta(t_0) = 0 \) (here \( \theta(t_0) \) is so small that it can be taken as zero without losing generality) are inserted into Eqs. (22) one obtains, by solving the equations (22),

\[
a_0(P, t_0) = \exp[i\gamma(t_0)] \exp \left[ \frac{i}{\hbar} \frac{(P + \hbar k_0)^2}{2M} + E_0 t_0 \right], \quad a_+(P, t_0) = a_-(P, t_0) = 0.
\]  

(38)

Indeed, at the initial time the atom is completely in the adiabatic eigenstate \( |g^0(P, t_0)\rangle \) of the Hamiltonian \( H(P, t_0) \) of (17) because both the coefficients \( a_+(P, t_0) \) and \( a_-(P, t_0) \) are zero in the initial atomic state \( |\Phi(P, t_0)\rangle \). It follows from Eq. (25) that at the initial time \( t_0 \) the coefficient \( a_l(P, t_0) = b_l(P, t_0) \) \((l = 0, \pm)\). Then at the initial time \( t_0 \) the variables \( \{b_l(P, t_0)\} \) of the basic equations (26) can be obtained from those coefficients of Eq. (38),

\[
b_0(P, t_0) = \exp[i\gamma(t_0)] \exp \left[ \frac{i}{\hbar} \frac{(P + \hbar k_0)^2}{2M} + E_0 t_0 \right], \quad b_+(P, t_0) = b_-(P, t_0) = 0.
\]  

(39)

These coefficients \( \{b_l(P, t_0)\} \) of Eq. (39) provide the basic equations (26) with the initial values.

Now the atom with the initial wave-packet state \( |\Psi(x, r, t_0)\rangle \) of Eq. (12a) undergoes the STIRAP-based decelerating process (11) in the ideal adiabatic condition (30). Then one can calculate the atomic state at the end time \( t_f = t_0 + T \) of the ideal adiabatic process (11). This atomic state \( |\Psi(x, r, t)\rangle \) is still given by Eq. (12), but the coefficients in Eq. (12) need to be calculated explicitly. By integrating the basic equations (26) and using the ideal adiabatic condition (30) the solution to the basic equations (26) is given by

\[
b_l(P, t) = b_l(P, t_0), \quad l = 0, \pm; \quad t_0 \leq t \leq t_0 + T; \quad P \in [P].
\]  

(40)

Here the initial values \( \{b_l(P, t_0)\} \) are given by Eq. (39). Once the coefficients \( \{b_l(P, t)\} \) are obtained from Eqs. (40) one may use equation (25) to calculate the coefficients \( \{a_l(P, t)\} \):

\[
a_0(P, t) = b_0(P, t_0) \exp[i(\gamma(t) - \gamma(t_0))]
\]

\[
\times \exp \left\{ i \int_{t_0}^{t} dt' \sin^2 \theta(t') \frac{\partial}{\partial t'} [\alpha_p(P, t') - \alpha_s(P, t')] \right\},
\]  

(41a)

\[
a_+(P, t) = a_-(P, t) = 0.
\]  

(41b)

Here the frequency parameter \( \omega_0(P, t) \) of Eq. (24a) has been used. The fact that the coefficients \( a_+(P, t) = a_-(P, t) = 0 \) shows that the atom in the adiabatic eigenstate \( |g^0(P, t_0)\rangle \) of the Hamiltonian \( H(P, t_0) \) at the initial time \( t_0 \) is transferred adiabatically to the adiabatic eigenstate \( |g^0(P, t)\rangle \) of the Hamiltonian \( H(P, t) \) and finally to the desired adiabatic eigenstate \( |g^0(P, t_f)\rangle \) of the Hamiltonian \( H(P, t_f) \) at the end of the adiabatic process. This is just consistent with the prediction of the adiabatic theorem in quantum mechanics [25, 30]. By substituting the coefficients \( \{a_l(P, t), l = 0, \pm\} \) of Eqs. (41) into Eqs. (22) one obtains

\[
A_0(P, t) = \exp[-i\gamma(t)]a_0(P, t) \cos \theta(t),
\]  

(42a)
The coefficients $\bar{A}_1(P, t) = 0$, $\bar{A}_2(P, t) = -a_0(P, t) \exp[-i\gamma(t)] \exp[-i(\alpha_p(P, t) - \alpha_s(P, t))] \sin \theta(t)$. (42c)

The coefficient $\bar{A}_1(P, t) = 0$ shows clearly that in the ideal adiabatic condition the atomic excited internal state $|e\rangle$ indeed does not appear during the STIRAP state-transfer process. Though one has the coefficients $\{a_l(P, t), l = 0, \pm\}$ of Eqs. (41) at hand, it is not sufficient from Eqs. (42) to determine uniquely the coefficients $\{\bar{A}_k(P, t)\}$ if one does not know the mixing angle $\theta(t)$ at the end time $t_f = t_0 + T$ of the ideal adiabatic process. Suppose that at the end of the ideal adiabatic process the Rabi frequency $\Omega_p(t)$ for the pumping pulse is much greater than the one $\Omega_s(t)$ of the Stokes pulse. Then the mixing angle $\theta(t)$ at the end of the ideal adiabatic process takes the asymptotic form [15, 17]

$$\sin \theta(t_f) = \frac{\Omega_p(t_f)}{\Omega(t_f)} \to 1 \text{ or } \theta(t_f) \to \pi/2.$$ 

Now inserting the mixing angle $\theta(t_f) = \pi/2$ into Eqs. (42) one obtains uniquely

$$\bar{A}_0(P, t_f) = \bar{A}_1(P, t_f) = 0,$$ (43a)

$$\bar{A}_2(P, t_f) = -a_0(P, t_f) \exp[-i\gamma(t_f)] \exp[-i(\alpha_p(P, t_f) - \alpha_s(P, t_f))] \sin \theta(t_f).$$ (43b)

Thus, with the aid of Eqs. (15) and (43) the atomic three-state vector $\{A_0(P, t), A_1(P, t), A_2(P, t)\}^T$ at the end time $t_f$ of the ideal adiabatic process is determined by

$$A_0(P, t_f) = A_1(P, t_f) = 0,$$ (44a)

$$A_2(P, t_f) = -\exp\left[i\frac{\hbar}{2M} \left(\frac{(P + \hbar k_0)^2}{2M} + E_0\right) t_0\right] \exp\left[-i\frac{\hbar}{2M} \left(\frac{(P - \hbar k_1)^2}{2M} + E_1\right) t_f\right] \exp[-i(\alpha_p(P, t_f) - \alpha_s(P, t_f))] \int_{t_0}^{t_f} dt' \sin^2 \theta(t') \frac{\partial}{\partial t'} [\alpha_p(P, t') - \alpha_s(P, t')] \}}. (44b)

The coefficients $A_0(P, t_f) = A_1(P, t_f) = 0$ show that at the end of the ideal adiabatic process the atom is transferred completely to the ground internal state $|g_1\rangle$ from the initial internal state $|g_0\rangle$ by the basic STIRAP decelerating process (11). This is just the desired result of the ideal adiabatic process (11). By using the phase difference in Eq. (34) one can further express the coefficient $A_2(P, t_f)$ as

$$A_2(P, t_f) = \exp[i\beta(t_f, t_0)] \exp\left[i\frac{\hbar}{2M} \left(\frac{(P + \hbar k_0)^2}{2M} + E_0\right) t_0\right]$$

$$\times \exp[-i\frac{\hbar}{2M} \left(\frac{(P - \hbar k_1)^2}{2M} + E_1\right) t_f]$$

$$\times \exp\left\{-i\frac{\Delta P}{M} (k_0 + k_1) t_0 + \int_{t_0}^{t_f} dt' \cos^2 \theta(t')\right\}.$$ (44c)
where the global phase factor is given by

\[
\exp[i\beta(t_f, t_0)] = -\exp[-i(\varphi_0(t_f) - \varphi_1(t_f))]
\times \exp\{i \int_{t_0}^{t_f} dt' \sin^2 \theta(t') [\varphi_0(t') - \varphi_1(t')] \}.
\] (45)

Once the atomic three-state vector \( \{A_0(P, t), A_1(P, t), A_2(P, t)\}^T \) is obtained at the end time \( t_f \) of the basic STIRAP decelerating process (11), through the equation (12) one can calculate the motional state of the atom at the end of the ideal adiabatic process. In order to calculate conveniently the wave-packet motional state of the atom one may use the momentum \( P' = P + \hbar k_0 \) as variable to express the three-state vector \( \{A_0(P', t), A_1(P', t), A_2(P', t)\}^T \), and then the vector at the end time \( t_f \) can be determined by

\[
A_0(P', t_f) = A_1(P', t_f) = 0,
\]

\[
A_2(P', t_f) = \exp[i\beta(t_f, t_0)] \exp\left\{i \int_{t_0}^{t_f} \frac{P'^2}{2M} + E_0 \right\} t_0 \times \exp\left\{-i \frac{\hbar}{2M} \left( \frac{(P' - \hbar k_0 - \hbar k_1)^2}{2M} + E_1 \right)t_f \right\} \times \exp\left\{-i \Delta P_0 M (k_0 + k_1) [t_0 + \int_{t_0}^{t_f} dt' \cos^2 \theta(t')] \right\}.
\] (46a)

(46b)

Note that here the momentum difference \( \Delta P' \) still represents the deviation of the momentum point \( P' \) from the central point of the effective momentum region \( [P] \) in the initial wave-packet motional state.

For the STIRAP-based accelerating process (11a) in the ideal adiabatic condition the three-state vector \( \{A_0(P', t), A_1(P', t), A_2(P', t)\}^T \) at the end of the ideal adiabatic process (11a) should be determined by

\[
A_0(P', t_f) = A_1(P', t_f) = 0,
\]

\[
A_2(P', t_f) = \exp[i\beta'(t_f, t_0)] \exp\left\{i \int_{t_0}^{t_f} \frac{P'^2}{2M} + E_0 \right\} t_0 \times \exp\left\{-i \frac{\hbar}{2M} \left( \frac{(P' + \hbar k_0 + \hbar k_1)^2}{2M} + E_1 \right)t_f \right\} \times \exp\left\{i \Delta P_0 M (k_0 + k_1) [t_0 + \int_{t_0}^{t_f} dt' \cos^2 \theta(t')] \right\}.
\] (47a)

(47b)

This is because the propagating directions of the Raman laser light beams in the accelerating process are just opposite to those in the decelerating process, respectively.

5. The decelerating and accelerating processes of a Gaussian wave-packet state in the ideal adiabatic condition
As a typical example, consider that an atom with a Gaussian wave-packet state is decelerated by the basic STIRAP-based decelerating sequence (11). For simplicity, here consider the simple situation that the STIRAP decelerating process (11) satisfies the ideal adiabatic condition (30). The theory developed in previous sections can determine the wave-packet motional state of the atom at the end of the basic decelerating process (11). Suppose that the initial motional state of the atom is a standard Gaussian wave-packet state [25, 31] in one-dimensional coordinate space:

\[ \Psi_0(x,t_0) = \exp(i\varphi_0)\sqrt{\frac{(\Delta x)^2}{2\pi}} \exp\left\{ -\frac{1}{2} \frac{(x-z_0)^2}{(\Delta x)^2} + \frac{i\hbar T_0}{2M} \right\} \exp\{ip_0x/\hbar\}. \]  

(48)

Here the complex linewidth of the Gaussian wave-packet state \( \Psi_0(x,t_0) \) is defined as

\[ W(T_0) = (\Delta x)^2 + \frac{i\hbar T_0}{2M}. \]

The probability density of the state \( \Psi_0(x,t_0) \) is given by

\[ |\Psi_0(x,t_0)|^2 = \frac{1}{\sqrt{\pi}} \sqrt{\frac{1}{2}} \frac{1}{(\Delta x)^2} \exp\left\{ -\frac{1}{2} \frac{(x-z_0)^2}{(\Delta x)^2} + \frac{i\hbar T_0}{2M} \right\}. \]

By comparing \( |\Psi_0(x,t_0)|^2 \) with the standard Gaussian function \( G(x) = \varepsilon \sqrt{\pi}^{-1} \times \exp\left\{ -(x-x_0)^2/\varepsilon^2 \right\} \) one sees that the center-of-mass position and the wave-packet spreading of the state \( \Psi_0(x,t_0) \) are \( z_0 \) and \( \varepsilon_0 = \sqrt{2[(\Delta x)^2 + \frac{\hbar T_0}{2M(\Delta x)^2}]} \), respectively. If the atom is a free particle, the Gaussian wave-packet state \( \Psi_0(x,t_0) \) has an explicit physical meaning that the atom with the Gaussian wave-packet state moves along the direction +x with the velocity \( p_0/M \). One may expand the coordinate-space Gaussian wave-packet state \( \Psi_0(x,t_0) \) in terms of the momentum eigenstates \( \{|p\}\) of Eq. (7),

\[ \Psi_0(x,t_0) = \sum_p \rho_0(p,t_0)|p\rangle. \]  

(49)

The expansion coefficient or the amplitude \( \rho_0(p,t_0) \) is determined by

\[ \rho_0(p,t_0) = \int \Psi_0(x,t_0)\psi_p(x)^* \]  

(49a)

where \( \psi_p(x) = \frac{1}{\sqrt{2\pi}} \exp(ipx/\hbar) \) is just the momentum eigenstate \( |\psi_p(x)\rangle \) or \( |p\rangle \) of Eq. (7). By substituting the wave-packet state \( \Psi_0(x,t_0) \) of Eq. (48) and the momentum eigenstate \( |\psi_p(x)\rangle \) of Eq. (7) into the amplitude \( \rho_0(p,t_0) \) of Eq. (49a) one obtains, by a complex calculation,

\[ \rho_0(p,t_0) = \exp(i\varphi_0)\sqrt{\frac{2(\Delta x)^2}{\pi}} \exp\left\{ -(\Delta x)^2 \frac{(p-p_0)^2}{\hbar^2} \right\} \]  

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where the momentum eigenstates |\Psi_0\rangle from the state \Psi_0 is the Fourier transform state of the coordinate-space Gaussian wave-packet state \Psi_0. It also has a Gaussian shape and this can be seen more clearly from its absolute square:

$$|\rho_0(p, t_0)|^2 = \frac{2(\Delta x)^2}{\pi} \exp\{-2(\Delta x)^2 \left(\frac{p - p_0}{\hbar}\right)^2\}. \quad (50a)$$

This is a standard Gaussian function with the propagation-vector variable \(p = \frac{p}{\hbar}\). Therefore, it satisfies the normalization,

$$\int_{-\infty}^{+\infty} dk |\rho_0(k, t_0)|^2 = \int_{-\infty}^{\infty} dx |\Psi_0(x, t_0)|^2 = 1. \quad (51)$$

The center-of-mass position of the Gaussian function is \(p_0\) for the momentum variable or \(p_0/\hbar\) for the propagation-vector variable. The wave-packet spreading of the Gaussian function for the momentum \(p\) is determined by \(\Delta p = \hbar/\sqrt{2(\Delta x)}\) and for the propagation-vector variable \(k\) is given by \(\Delta k = \sqrt{2(\Delta x)^2}\). The state \(\rho_0(p, t_0)\) is called the momentum-space Gaussian wave-packet state of the state. There is a Gaussian decay factor \(\exp\{-\frac{1}{2(\Delta x)^2} \left(\frac{p - p_0}{\hbar}\right)^2\}\) in the momentum-space wave-packet state \(\rho_0(p, t_0)\), which decides the effective momentum region \([p]\) of the momentum wave-packet state. Obviously, the probability density \(|\rho_0(p, t_0)|^2\) approaches zero rapidly (exponentially) when the momentum \(p\) takes a value such that the deviation \(|p - p_0|\) is greater than the wave-packet spreading \(\Delta p\). Equation (49) really shows that the coordinate-space Gaussian wave-packet state \(\Psi_0(x, t_0)\) can be expanded in terms of the momentum-space Gaussian wave-packet states \(\{\rho_0(p, t_0)\}\).

Consider the superposition of the momentum wave-packet states \(\{\rho_0(p, t_0)\}\)

$$\Phi_0(x, t_0) = \sum_{|p - p_0| \leq \Delta P_M/2} \rho_0(p, t_0)|p\rangle, \quad (52)$$

where \(\Delta P_M\) is just the bandwidth of the effective momentum region \([P]\) of the momentum wave-packet state \(\rho_0(p, t_0)\). When the bandwidth \(\Delta P_M \to \infty\), the superposition state \(\Phi_0(x, t_0)\) is just the Gaussian wave-packet state \(\Psi_0(x, t_0)\), as can be seen from Eqs. (49) and (52). The deviation of the state \(\Phi_0(x, t_0)\) from the state \(\Psi_0(x, t_0)\) may be measured by the probability

$$P(\Psi_0(x, t_0) - \Phi_0(x, t_0)) = \sum_{|p - p_0| > \Delta P_M/2} \rho_0(p, t_0)|p\rangle|^2$$

$$= 2 \int_{[\Delta P_M/2 + p_0]/\hbar}^{\infty} dk |\rho_0(k, t_0)|^2$$

where the momentum eigenstates \(|p\rangle\) of Eq. (7) and their orthonormal relations and the momentum wave-packet state \(\rho_0(p, t_0)\) of Eq. (50) have been used. Now using the probability density \(|\rho_0(p, t_0)|^2\) of Eq. (50a) one has

$$P(\Psi_0(x, t_0) - \Phi_0(x, t_0)) = \frac{2}{\sqrt{\pi}} \int_{g_M}^\infty dy \exp(-y^2) \quad (53)$$

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where the lower integral limit is

\[ y_M = \left( \frac{\Delta P_M}{\Delta x} \right) \frac{(\Delta x)}{2\hbar} \]

The probability \( P\{\Psi_0(x,t_0) - \Phi_0(x,t_0)\} \) is bounded by [32]

\[
\frac{2}{\sqrt{\pi} y_M + \sqrt{y_M^2 + 2}} \exp(-y_M^2) \leq P\{\Psi_0(x,t_0) - \Phi_0(x,t_0)\} \leq \frac{2}{\sqrt{\pi} y_M + \sqrt{y_M^2 + 4/\pi}} \exp(-y_M^2).
\]

The important thing is that the probability \( P\{\Psi_0(x,t_0) - \Phi_0(x,t_0)\} \) decays exponentially with the number \( y_M^2 \). If the bandwidth \( \Delta P_M \) of effective momentum region \( [P] \) of the momentum wave-packet state \( \rho_0(p,t_0) \) is chosen such that the number \( y_M >> 1 \), then the probability \( P\{\Psi_0(x,t_0) - \Phi_0(x,t_0)\} \) is almost zero. As a result, the superposition state \( \Phi_0(x,t_0) \) is almost equal to the Gaussian wave-packet state \( \Psi_0(x,t_0) \).

If an atomic system is in a momentum superposition state which spreads from \(-\infty\) to \(+\infty\) in momentum space, then it is generally hard to achieve a complete STIRAP state transfer in the atomic system by the basic decelerating process (11), since the adiabatic condition can not be met over the whole momentum space \((-\infty, +\infty)\). On the other hand, an almost complete STIRAP state transfer could be achieved for an atomic wave-packet motional state with a finite wave-packet spreading by the basic decelerating process (11). This can be illustrated through the momentum wave-packet state \( \Phi_0(x,t_0) \) of Eq. (52). The state \( \Phi_0(x,t_0) \) is also a superposition of the momentum eigenstates of Eq. (7). All the momentum components of the state \( \Phi_0(x,t_0) \) are within the effective momentum region \( [P] = [p_0 - \Delta P_M/2, p_0 + \Delta P_M/2] \), as can be seen in Eq. (52). When the bandwidth \( \Delta P_M \) of the effective momentum region \( [P] \) is finite and satisfies \( p_0 - \Delta P_M/2 > \hbar k_0 + \hbar k_1 \), a complete STIRAP state transfer could be achieved for the state \( \Phi_0(x,t_0) \) within the effective momentum region \( [P] \) by the STIRAP decelerating process (11) if the ideal adiabatic condition (30) is met within the effective momentum region \( [P] \) for the decelerating process (11). Now using the same STIRAP pulse sequence (11) one can make an almost complete STIRAP state transfer for the Gaussian wave-packet state \( \Psi_0(x,t_0) \) as the state \( \Psi_0(x,t_0) \) is almost equal to the state \( \Phi_0(x,t_0) \) when the number \( y_M >> 1 \). Hereafter suppose that the number \( y_M >> 1 \) so that the Gaussian wave-packet state \( \Psi_0(x,t_0) \) can be replaced with the state \( \Phi_0(x,t_0) \) and vice versa without generating a significant error in evaluating the unitary decelerating and accelerating processes.

Now suppose that at the initial time \( t_0 \) the atom is in the Gaussian wave-packet motional state \( \Psi_0(x,t_0) \) of (48) and in the internal state \( |g_0\rangle \). Then the total product state of the atom at the initial time is given by

\[
\Psi_0(x,r,t_0) = \Psi_0(x,t_0)|g_0\rangle = \sum_p \rho_0(p,t_0)|p\rangle|g_0\rangle.
\]

By comparing the product state \( \Psi_0(x,r,t_0) \) with that state of Eq. (12a) one can see that the amplitude \( \rho_0(p,t_0) \) in the state \( \Psi_0(x,r,t_0) \) just corresponds to
the amplitude \( \rho(P') \) in the state of Eq. (12a). This means that at the initial time \( t_0 \) the probability to find the atom in the three-state subspace \( \{ |p\rangle |g_0\rangle, |p - \hbar k_0\rangle |e\rangle, |p - \hbar k_0 - \hbar k_1\rangle |g_1\rangle \} \) is given by \( |\rho_0(p, t_0)|^2 \). Note that during the STIRAP decelerating process (11) this probability is not changed with time. Obviously, the coefficients \( A_0(p, t_0) = 1, A_1(p, t_0) = 0, \) and \( A_2(p, t_0) = 0 \) at the initial time, as can be seen in Eq. (54). Now the initial atomic product state \( \Psi_0(x, r, t_0) \) of (54) undergoes the basic STIRAP decelerating process (11). Then at the end time \( t_f = t_0 + T \) of the decelerating process (11) the total product state of the atom is given by Eq. (12),

\[
\Psi_0(x, r, t_f) = \sum_p \rho_0(p, t_0) \{ A_0(p, t_f)|p\rangle |g_0\rangle \\
+ A_1(p, t_f)|p - \hbar k_0\rangle |e\rangle + A_2(p, t_f)|p - \hbar k_0 - \hbar k_1\rangle |g_1\rangle \}
\]

(55)

where in the ideal adiabatic condition (30) the coefficients \( \{ A_2(p, t_f), l = 0, 1, 2 \} \) are given by Eqs. (46) with the parameter settings \( P' = p \) and \( \Delta P' = p - p_0 \). Though in Eq. (55) the sum for the momentum \( p \) runs over only the effective momentum region \( [P] \), it will not generate a significant error if the sum really runs over the whole momentum region \( (-\infty, +\infty) \), as pointed out before. Since in the ideal adiabatic condition (30) the coefficients \( A_0(p, t_f) = A_1(p, t_f) = 0 \), the total product state (55) is reduced to the simple form

\[
\Psi_0(x, r, t_f) = \sum_p \rho_0(p, t_0) A_2(p, t_f)|p - \hbar k_0 - \hbar k_1\rangle |g_1\rangle.
\]

(56)

The product state \( \Psi_0(x, r, t_f) \) shows that in the ideal adiabatic condition (30) at the end of the decelerating process (11) the atom is completely in the ground internal state \( |g_1\rangle \) and also in the wave-packet motional state:

\[
\Psi_0(x, t_f) = \sum_p \rho_0(p, t_0) A_2(p, t_f)|p - \hbar k_0 - \hbar k_1\rangle.
\]

(57)

The initial product state of Eq. (54) and the final product state of Eq. (56) show that indeed, the atom is transferred completely from the initial internal state \( |g_0\rangle \) and the Gaussian wave-packet motional state \( \Psi_0(x, t_0) \) of (48) to the final internal state \( |g_1\rangle \) and the motional state \( \Psi_0(x, t_f) \) of (57), respectively, by the STIRAP decelerating sequence (11) in the ideal adiabatic condition (30). It can turn out that the motional state \( \Psi_0(x, t_f) \) of Eq. (57) is still a Gaussian wave-packet state. By the new variable \( q = (p - p_0)/\hbar \) the coefficient \( A_2(p, t_f) \) of Eq. (46b) with \( P' = p \) and \( \Delta P' = p - p_0 \) and the amplitude \( \rho_0(p, t_0) \) of Eq. (50) are respectively rewritten as

\[
A_2(p, t_f) = \exp[i\beta(t_f, t_0)] \exp\left[ \frac{i p_0^2}{\hbar^2} + E_0 \right] t_0 \\
\times \exp\left[ -\frac{i}{\hbar} \frac{(p_0 - \hbar k_0 - \hbar k_1)^2}{2M} + E_1 \right] t_f \exp[-iq^2 \frac{\hbar(t_f - t_0)}{2M}]
\]

34
\[ \times \exp[-i\frac{(p_0 - \hbar k_0 - \hbar k_1)}{M}(t_f - t_0)] \exp[-i\frac{\hbar(k_0 + k_1)}{M} \int_{t_0}^{t_f} dt' \cos^2 \theta(t')] \]

and

\[ \rho_0(p, t_0) = \exp(i\varphi_0)[\frac{2(\Delta x)^2}{\pi}]^{1/4} \exp\{-q^2[(\Delta x)^2 + i\frac{\hbar T_0}{2M}]\} \exp(-iqz_0). \]

Inserting the two coefficients and the momentum eigenstate \(|p - \hbar k_0 - \hbar k_1\rangle\) of Eq. (7) into Eq. (57) one obtains

\[ \Psi_0(x, t_f) = \exp(i\varphi_0)[\frac{2(\Delta x)^2}{\pi}]^{1/4} \exp[i\beta(t_f, t_0)] \exp[i \frac{p_0^2}{2M} + E_0]t_0] \]

\[ \times \exp[-\frac{i}{\hbar}(\frac{(p_0 - \hbar k_0 - \hbar k_1)}{2M} + E_1)t_f] \exp[i(p_0 - \hbar k_0 - \hbar k_1)x/\hbar] \]

\[ \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq \exp(-aq^2 + bq) \]

where the sum \(\sum_p\) has been replaced with the integral as the momentum \(p\) is continuous for a free atom, and the coefficients \(a\) and \(b\) are given by

\[ a = (\Delta x)^2 + i\frac{\hbar(T_0 + t_f - t_0)}{2M}, \]

\[ b = i[x - z_0 - \frac{(p_0 - \hbar k_0 - \hbar k_1)}{M}(t_f - t_0) - \frac{\hbar(k_0 + k_1)}{M} \int_{t_0}^{t_f} dt' \cos^2 \theta(t')]. \]

The Gaussian integral in the state \(\Psi_0(x, t_f)\) can be calculated by

\[ \int_{-\infty}^{\infty} dq \exp(-aq^2 + bq) = \sqrt{\frac{\pi}{a}} \exp(b^2/(4a)). \]  

(58)

Now by a simple calculation one obtains the final state:

\[ \Psi_0(x, t_f) = \exp[i\varphi_1(t_f, t_0)][(\Delta x)^2]^{1/4} \sqrt{\frac{1}{(\Delta x)^2 + i\frac{\hbar(T_0 + t_f - t_0)}{2M}}} \]

\[ \times \exp\{-\frac{1}{4} \frac{(x - z_1)^2}{[(\Delta x)^2 + i\frac{\hbar(T_0 + t_f - t_0)}{2M}]}\} \exp[ipx/\hbar]. \]  

(59)

Indeed, the motional state \(\Psi_0(x, t_f)\) is also a Gaussian wave-packet state just like the initial motional state \(\Psi_0(x, t_0)\) of Eq. (48). Here the Gaussian wave-packet state \(\Psi_0(x, t_f)\) has the center-of-mass position

\[ z_1 = z_0 + \frac{(p_0 - \hbar k_0 - \hbar k_1)}{M}(t_f - t_0) + \frac{\hbar(k_0 + k_1)}{M} \int_{t_0}^{t_f} dt' \cos^2 \theta(t'), \]

(60)

the wave-packet spreading

\[ \varepsilon = \sqrt{2(\Delta x)^2 + 2\frac{\hbar(T_0 + t_f - t_0)}{2M(\Delta x)}}, \]

(61)
the mean motional momentum

\[ p_1 = p_0 - \hbar k_0 - \hbar k_1, \]

and the global phase factor

\[ \exp[i\varphi_1(t_f, t_0)] = \exp(i\varphi_0) \exp[i\beta(t_f, t_0)] \]

\[ \times \exp\left\{ \frac{i}{\hbar} \left( \frac{p_0^2}{2M} + E_0 t_0 \right) \right\} \exp\left\{ -\frac{i}{\hbar} \left( \frac{(p_0 - \hbar k_0 - \hbar k_1)^2}{2M} + E_1 t_f \right) \right\}. \]

It is interesting to compare the final motional state \( \Psi_0(x, t_f) \) with the initial state \( \Psi_0(x, t_0) \) of the decelerating process (11). It needs only three parameters to characterize completely a Gaussian wave-packet motional state of a free particle: the center-of-mass position, the mean motional momentum (or velocity), and the complex linewidth. Here the wave-packet spreading is determined completely by the complex linewidth. For the first point, the atom is decelerated by \( \hbar k_0/M + \hbar k_1/M \) by the STIRAP pulse sequence (11) as expected, because the average motional momentum \( p_1 = p_0 - \hbar k_0 - \hbar k_1 \) of the final state \( \Psi_0(x, t_f) \) is smaller than \( p_0 \) of the initial state \( \Psi_0(x, t_0) \) and their difference is \( (\hbar k_0 + \hbar k_1) \). Here suppose that the initial velocity for the moving atom \( p_0/M > \hbar (k_0 + k_1)/M \).

For the second point, the atom moves a distance \( (z_1 - z_0) \) along the direction +x during the decelerating process. If a free atom moved along the direction +x with the velocities \( p_0/M \) and \( p_1/M \), respectively, then in the time interval \( T = t_f - t_0 \) the atom would move distances equal to \( T \times p_0/M \) and \( T \times p_1/M \), respectively. Here the velocities \( p_0/M \) and \( p_1/M \) are the atomic moving velocities before and after the decelerating process, respectively. One can expect that the distance \( (z_1 - z_0) \) should lie in between the distances \( T \times p_1/M \) and \( T \times p_0/M \), that is, \( T \times p_1/M < z_1 - z_0 < T \times p_0/M \). Indeed, the equation (60) shows this point. For the third point, the wave-packet spreading of the atom at the end of the decelerating process is larger than the initial one. If one compares the wave-packet spreading of Eq. (61) with the free-atom wave-packet spreading (see section 6 below), one can see that though the atom is irradiated by the Raman laser light beams, the wave-packet spreading of the atom during the decelerating process is not really affected by the Raman laser light beams and is just the same as that one of the atom in the absence of the Raman laser light beams. This point is important as the wave-packet spreading of the atom after the decelerating (or accelerating) process can be calculated easily. Note that the wave-packet spreading of a free atom becomes larger and larger as time increases, as can be seen in section 6 below.

From the experimental viewpoint one does not expect that after the decelerating process (11) the wave-packet spreading of the atomic momentum wave-packet state could become larger, because this may make a trouble for the design of the STIRAP pulse sequence (11). Fortunately, it can turn out that in the ideal adiabatic condition (30) the wave-packet spreading of the momentum
wave-packet state is not really affected by the STIRAP pulse sequence (11). One can expand the wave-packet motional state of Eq. (59) in terms of the momentum eigenstates \( \{p\} \) of Eq. (7): \( \Psi_0(x,t_f) = \sum_p \rho_1(p,t_f) |p\rangle \) just like the expansion (49), and just like the momentum wave-packet state \( \rho_0(p,t_0) \) the momentum wave-packet state \( \rho_1(p,t_f) \) can be calculated from Eq. (49a),

\[
\rho_1(p,t_f) = \exp[i\varphi_1(t_f,t_0)] \frac{2i(\Delta x)^2}{\pi} \frac{1}{\hbar} \exp\{-\frac{(\Delta x)^2}{\hbar} \left( \frac{p-p_1}{\hbar} \right)^2 \}
\times \exp\left\{-i\frac{\hbar}{2M} \left( \frac{p-p_1}{\hbar} \right)^2 \right\} \exp[-i\frac{(p-p_1)}{\hbar} z_1].
\]

Indeed, the probability density \( |\rho_1(p,t_f)|^2 \) is also a Gaussian function and it is really equal to \( |\rho_0(p,t_0)|^2 \) of the initial state \( \Psi_0(x,t_0) \) if the initial momentum \( p_0 \) is replaced with \( p_1 \). Thus, the wave-packet spreading of the Gaussian function \( |\rho_1(p,t_f)|^2 \) is equal to that one of \( |\rho_0(p,t_f)|^2 \) and is also given by \( (\Delta p) = \hbar/\sqrt{2(\Delta x)} \). This shows that the effective momentum region \( [P] \) of the initial momentum wave-packet state \( \rho_0(p,t_0) \) is not changed after the atom is decelerated by the STIRAP pulse sequence (11), although the center-of-mass position of the momentum wave-packet state is changed to \( p_1 = p_0 - \hbar k_0 - \hbar k_1 \) from the initial one \( p_0 \) after the atom is decelerated.

In the quantum control process the halting-qubit atom usually needs to be decelerated continuously [22] because each STIRAP pulse sequence (11) usually can decelerate the atom only by a small velocity value. As shown above, after the STIRAP decelerating process (11) the decelerated atom is in the product state \( \Psi_0(x,r,t_f) \) of Eq. (56), that is, the atom is in the internal state \( |g_1\rangle \) and the Gaussian wave-packet state \( \Psi_0(x,t_f) \) of Eq. (59). Now the atom needs to be decelerated further by another STIRAP pulse sequence. This basic STIRAP-based decelerating process may be expressed as

\[
|P + \hbar k_0\rangle |g_1\rangle \rightarrow |P\rangle |e\rangle \rightarrow |P - \hbar k_1\rangle |g_0\rangle.
\]

In this decelerating process the atom is changed from the internal state \( |g_1\rangle \) to \( |g_0\rangle \). This is opposite to the previous decelerating process (11). Therefore, the experimental parameters for the STIRAP pulse sequence (63) needs to be set suitably. Now the two Raman laser light beams in Eq. (9) for the STIRAP pulse sequence (63) should have the parameter settings: \( (E_{L0}(t), k_{L0}, \omega_{L0}) = (E_{01}(t), l_0, \omega_0) \) and \( (E_{L1}(t), k_{L1}, \omega_{L1}) = (E_{12}(t), l_1, \omega_1) \), where the first Raman laser light beam \( (E_{01}(t), l_0, \omega_0) \) (the pumping pulse) couples the two internal states \( |g_1\rangle \) and \( |e\rangle \) and it propagates along the opposite motional direction to the atom, while the second beam \( (E_{12}(t), l_1, \omega_1) \) (the Stokes pulse) connects the two internal states \( |g_0\rangle \) and \( |e\rangle \) and it travels along the motional direction of the atom. The atomic three-state subspace \( \{|\psi_h(r)\rangle\} \) in the STIRAP decelerating process (63) should be set by \( |\psi_0(r)\rangle = |g_1\rangle \), \( |\psi_1(r)\rangle = |g_0\rangle \), and \( |\psi_2(r)\rangle = |e\rangle \), and the transition frequencies between the atomic internal energy levels in the decelerating process (63) should be defined by \( \omega_{02}^0 = \omega_{12} = (E_2 - E_1)/\hbar \) and \( \omega_{12} = \omega_{02} = (E_2 - E_0)/\hbar \), respectively. If one makes an exchange \( E_0 \leftrightarrow E_1 \) in
all those results obtained in the previous decelerating process (11), then these results can be adopted in the present decelerating process (63).

Now suppose that at the initial time $t_0$ the atom is in the product state $\Psi_0(x, r, t_0)$ of Eq. (54). The atom first undergoes the STIRAP decelerating process (11) and hence the product state $\Psi_0(x, r, t_0)$ is completely transferred to the product state $\Psi_0(x, r, t_f)$ of Eq. (56) at the end time $t_f = t_0 + T$ of the decelerating process (11). Then the atom undergoes the second STIRAP decelerating process (63). Now one wants to calculate the atomic wave-packet product state at the end of the second decelerating process (63). At the initial time $t_1 = t_f$ of the second decelerating process (63) the atomic product state is given by $\Psi_1(x, r, t_1) = \Psi_1(x, t_1) |g_1\rangle$. Obviously, this product state is just the product state of the atom at the end of the first decelerating process (11). Thus, the initial motional state $\Psi_1(x, t_1)$ is the Gaussian wave-packet state of Eq. (59): $\Psi_1(x, t_1) = \Psi_0(x, t_f)$. Then in the ideal adiabatic condition (30) at the end time $t'_f = t_1 + T$ of the second decelerating process (63) the atom is completely in the product state:

$$\Psi_1(x, r, t'_f) = \Psi_1(x, t'_f) |g_0\rangle$$

(64)

where the wave-packet motional state $\Psi_1(x, t'_f)$ can be calculated by

$$\Psi_1(x, t'_f) = \sum_p \rho_1(p, t_1) A_2(p, t'_f) |p - \hbar l_0 - \hbar l_1\rangle.$$  

(65)

Here the amplitude $\rho_1(p, t_1)$ is given by Eq. (62) with the time $t_f = t_1$, the momentum eigenstate $|p - \hbar l_0 - \hbar l_1\rangle$ is still given by Eq. (7), and the coefficient $A_2(p, t'_f)$ with the center-of-mass momentum $P_0 = p_1$ and $\Delta p = p - p_1$ is written as

$$A_2(p, t'_f) = \exp[i\beta(t'_f, t_1)] \exp[i(\frac{p^2}{2M} + E_1)t_1]$$

$$\times \exp[-i(\frac{(p - \hbar l_0 - \hbar l_1)^2}{2M} + E_0)t'_f]$$

$$\times \exp\{-i\frac{\Delta p}{M}(l_0 + l_1)[t_1 + \int_{t_1}^{t'_f} \cos^2 \theta(t')] dt\},$$

(66)

where the global phase $\beta(t'_f, t_1)$ is still calculated by Eq. (45) with the related parameter settings such as the mixing angle $\theta(t)$ and the phase modulation functions $\varphi_0(t)$ and $\varphi_1(t)$ of the present STIRAP pulse sequence (63). Then by a complex calculation one can obtain from Eq. (65) the Gaussian wave-packet motional state at the end of the decelerating process (63):

$$\Psi_1(x, t'_f) = \exp[i\varphi_2(t'_f, t_1)][(\frac{\Delta x}{2\pi})^{3/4}] \left(1 + \frac{1}{(\Delta x)^2 + i\hbar(T_0 + (t_1 - t_0) + (t'_f - t_1))}\right)$$

$$\times \exp\{-\frac{1}{4} \frac{(x - z_2)^2}{(\Delta x)^2 + i\hbar(T_0 + (t_1 - t_0) + (t'_f - t_1))}\} \exp[ip_2 x/\hbar]$$

(67)
where the center-of-mass position \( z_2 \) is given by
\[
z_2 = z_1 + \frac{(p_1 - \hbar l_0 - \hbar l_1)}{M}(t'_f - t_1) + \frac{\hbar(l_0 + l_1)}{M} \int_{t_1}^{t'_f} dt' \cos^2 \theta_i(t'),
\]
(68)
The wave-packet spreading by
\[
\varepsilon = \sqrt{2(\Delta x)^2 + 2\frac{\hbar(T_0 + (t_1 - t_0) + (t'_f - t_1))}{2M(\Delta x)}}^2,
\]
(69)
the mean momentum by
\[
p_2 = p_1 - \hbar l_0 - \hbar l_1,
\]
and the global phase factor by
\[
\exp[i \varphi_2(t'_f, t_1)] = \exp[i \varphi_1(t_1, t_0)] \exp[i \beta_1(t'_f, t_1)] 
\times \exp\left[\frac{i}{\hbar} \frac{p_1^2}{2M} + E_1 t_1\right] \exp\left[\frac{i}{\hbar} \frac{(p_1 - \hbar l_0 - \hbar l_1)^2}{2M} + E_0 t'_f\right].
\]
Here both the basic decelerating sequences (11) and (63) are studied in detail as they are the basic STIRAP-based decelerating processes. Any unitary decelerating process in the quantum control process [22] may be constructed with a train of these two basic decelerating sequences.

When the atom is in the Gaussian wave-packet motional state \( \Psi_0(x, t_0) \) of Eq. (48) at the initial time \( t_0 \), the complex linewidth of the motional state is \( W(T_0) = (\Delta x)^2 + i\hbar T_0/(2M) \). After the first basic STIRAP decelerating process (11) the atom is in the Gaussian wave-packet motional state \( \Psi_0(x, t_1) \) \((t_1 = t_0 + T)\) of Eq. (59) and the state has the complex linewidth \( W(t_1 - t_0 + T) = (\Delta x)^2 + i\hbar(T_0 + t_1 - t_0)/(2M) \). Then after the second basic STIRAP decelerating process (63) the atom is in the Gaussian wave-packet motional state \( \Psi_1(x, t'_f) \) \((t'_f = t_1 + T)\) of Eq. (67) and the complex linewidth of the state is \( W(t'_f - t_0 + T) = (\Delta x)^2 + i\hbar(T_0 + (t_1 - t_0) + (t'_f - t_1))/(2M) \). Thus, one can see that the real part of the complex linewidth of the Gaussian wave-packet motional state of the atom keeps unchanged during these decelerating processes (11) and (63), while the imaginary part increases linearly with the time periods of these decelerating processes. This result is found not only in the decelerating processes but also in the accelerating processes and the free-particle motional process.

In a general case a unitary decelerating process may consist of a train of the two basic STIRAP decelerating processes (11) and (63). For convenience, here each basic STIRAP decelerating process is set to have the same time period \( T = t'_d \) and suppose that at the initial time of the unitary decelerating process the atom is in the internal state \( |g_0\rangle \) and has a large motional momentum such that the atom still moves along the initial direction \(+x\) even after the unitary decelerating process. The basic decelerating sequences (11) and (63) may form a basic STIRAP decelerating cycle in such a way that first the decelerating
sequence (11) and then the sequence (63) is applied to the decelerated atom. The unitary decelerating process may consist of many of these basic STIRAP decelerating cycles. Denote \( U_d(11) \) and \( U_d(63) \) as the unitary propagators of the basic decelerating processes (11) and (63), respectively. Then a general unitary decelerating process may be expressed as

\[
U_D(2N) = U_{2N}^d(63)U_{2N-1}^d(11)...U_{2n}^d(63)U_{2n-1}^d(11)...U_2^d(63)U_1^d(11)
\]  

(70a)

or

\[
U_D(2N - 1) = U_{2N-1}^d(11)...U_{2n}^d(63)U_{2n-1}^d(11)...U_2^d(63)U_1^d(11)
\]

(70b)

where \( U_{2n-1}^d(11) \) is the propagator of the \((2n - 1)\)th basic decelerating unit for \( n = 1, 2, ..., N \) in the unitary decelerating process \( U_D(2N) \) or \( U_D(2N - 1) \), while \( U_{2n}^d(63) \) is the propagator of the \(2n\)th basic decelerating unit. Each basic decelerating unit with an even index \(2n\) in the unitary decelerating process is taken as the basic decelerating process (63), while that with an odd index \((2n-1)\) is taken as the basic decelerating process (11). Thus, \( U_{2n-1}^d(11) = U_d^d(11) \) and \( U_{2n}^d(63) = U_d^d(63) \) for \( n = 1, 2, ..., N \). In particular, \( U_D(0) = E \) (the unit operator), \( U_D(1) = U_d(11) \), and \( U_D(2) = U_d^d(63)U_d^d(11) \). The unitary decelerating processes \( U_D(1) \) and \( U_D(2) \) have been investigated in detail in the preceding paragraphs. Obviously, the unitary decelerating process \( U_D(2N) \) consists of \(2N\) basic decelerating processes (11) and (63) alternately or \( N \) basic decelerating cycles, while \( U_D(2N - 1) \) consists of \( N \) basic decelerating process (11) and \( N - 1 \) basic decelerating process (63) alternately.

The time evolution process of the atom in the presence of the unitary decelerating sequence \( U_D(2N) \) or \( U_D(2N - 1) \) can be calculated exactly in the ideal adiabatic condition (30). For the simplest cases \( N = 1 \) and \( 2 \) the time evolution processes of the atom have already calculated in the ideal adiabatic condition (30) in the previous paragraphs. In order to calculate the time evolution process of the atom in a general unitary decelerating process one may first set up the recursive relation between the two atomic product states at the end of the unitary decelerating processes \( U_D(2n-1) \) and \( U_D(2n) \) for \( n = 1, 2, ..., N \). Since the initial internal state of the atom is \( |g_0\rangle \) in both the unitary decelerating processes \( U_D(2N) \) and \( U_D(2N - 1) \), after the unitary decelerating process \( U_D(2n) \) (or \( U_D(2n-1) \)) \( (1 \leq n \leq N) \) the final internal state of the atom is clearly \( |g_0\rangle \) (or \( |g_1\rangle \)). Then the initial internal states of the atom in the basic decelerating processes \( U_{2n-1}^d(11) \) and \( U_{2n}^d(63) \) should be \( |g_0\rangle \) and \( |g_1\rangle \), respectively. It is known that at the initial time \( t_0 \) the atom is in the Gaussian wave-packet motional state \( \Psi_0(x, t_0) \) of Eq. (48) and the product state \( \Psi_0(x, r, t_0) \) of Eq. (54). It is also known that after the unitary decelerating sequences \( U_D(1) \) and \( U_D(2) \) act on the initial product state \( \Psi_0(x, r, t_0) \) of Eq. (54) the initial motional state \( \Psi_0(x, t_0) \) of Eq. (48) is converted into the Gaussian wave-packet motional states \( \Psi_0(x, t_f) \) of Eq. (59) and \( \Psi_1(x, t'_f) \) of Eq. (67), respectively. This means that the unitary decelerating sequences \( U_D(1) \) and \( U_D(2) \) do not change the Gaussian shape of the atomic motional state. Therefore, it is reasonable to deduce that after the unitary decelerating process \( U_D(2n) \) for \( n = 0, 1, 2, ..., N \) the atom
is in the Gaussian wave-packet motional state:

$$\Psi_{2n}^d(x, t_{2n-1}^e) = \exp(i\varphi_{2n}^d)\left[\frac{(\Delta x)^2}{2\pi}\right]^{1/4} \sqrt{\frac{1}{(\Delta x)^2 + i\hbar/(2M)}} \times \exp\left\{-\frac{1}{4}(\Delta x)^2 + i\hbar/(2M)\right\} \exp\{iP_{2n}^d x/\hbar\}$$

and also in the atomic wave-packet product state:

$$\Psi_{2n}^d(x, r, t_{2n-1}^e) = \Psi_{2n}^d(x, t_{2n-1}^e)\langle g_0 | = \sum_p \rho(p, t_{2n-1}^e) | p \rangle | g_0 \rangle. \quad (72)$$

where $t_{2n-1}^e$ is the end time of the unitary decelerating process $U_D(2n)$. In an analogous way to calculating the amplitude $\rho_0(p, t_0)$ via the equation (49a) one can calculate the amplitude $\rho_0(p, t_{2n-1}^e)$ of Eq. (72) from the motional state $\Psi_{2n}^d(x, t_{2n-1}^e)$ of Eq. (71). The result is

$$\rho(p, t_{2n-1}^e) = \exp(i\varphi_{2n}^d)\frac{2(\Delta x)^2}{\pi}^{1/4} \exp\{-i\hbar/(2M)P_{2n}^d\} \times \exp\{-i\hbar/(2M)P_{2n}^d\} \exp\{iP_{2n}^d x/\hbar\}$$

It will prove below that the states $\Psi_{2n}^d(x, t_{2n-1}^e)$ of (71) and $\Psi_{2n}^d(x, r, t_{2n-1}^e)$ of (72) are indeed the wave-packet motional state and product state of the atom at the end of the unitary decelerating process $U_D(2n)$, respectively.

First of all, the product state $\Psi_0(x, r, t_0)$ of Eq. (54) and the motional state $\Psi_0(x, t_0)$ of Eq. (48) are the initial product state and motional state of the atom in the presence of the unitary decelerating process $U_D(2n)$ (or $U_D(2n - 1)$), respectively. Of course, these two states may also be formally thought of as the final wave-packet product state and motional state of the atom after the unitary 'decelerating' process $U_D(0) = E$ (the unity operator), respectively. This means that the atomic wave-packet product state $\Psi_0^d(x, r, t_{-1}^e)$ of Eq. (72) and the motional state $\Psi_0^d(x, t_{-1}^e)$ of Eq. (71) with $n = 0$ should be equal to $\Psi_0(x, r, t_0)$ of Eq. (54) and $\Psi_0(x, t_0)$ of Eq. (48), respectively,

$$\Psi_0^d(x, r, t_{-1}^e) = \Psi_0(x, r, t_0), \quad \Psi_0^d(x, t_{-1}^e) = \Psi_0(x, t_0),$$

while the momentum wave-packet state $\rho(p, t_{-1}^e)$ of Eq. (73) thus is just $\rho_0(p, t_0)$ of Eq. (50). Indeed, these equations (71), (72), and (73) show this point, if in Eqs. (71), (72), and (73) one sets the parameters: $n = 0$, $t_{-1}^e = t_0^e = t_0$, $\varphi_0^d = \varphi_0$, $T_d = T_0$, $\varepsilon_0^d = \varepsilon_0$, $P_0^d = P_0$, and $t_d = T$, where $T$ is the time period of the basic decelerating sequence (11). Next, the product state $\Psi_1(x, r, t_f^e)$ of Eq. (64) and the motional state $\Psi_1(x, t_f^e)$ of Eq. (67) with the time $t_f^e = t_f^e$ are just the product state $\Psi_1^d(x, r, t_f^e)$ of Eq. (72) and the motional state $\Psi_1^d(x, t_f^e)$ of Eq. (71) with $n = 1$ at the end time $t_f^e$ of the unitary decelerating process $U_D(2)$, respectively,

$$\Psi_1^d(x, r, t_f^e) = \Psi_1(x, r, t_f^e), \quad \Psi_1^d(x, t_f^e) = \Psi_1(x, t_f^e).$$
This can be confirmed by setting the following parameters in Eqs. (71) and (72):
\[ t_1^d = t_f = t_0 + t_d, \quad t_1^f = t_0 + t_d, \quad \varphi_2^d = \varphi_2(t_1^f, t_1^f), \quad z_2^d = z_2, \]
and \( P_2^d = p_2 \). One therefore shows that the motional state \( \Psi_{2n}^d(x, t_{2n-1}^d) \) of Eq. (71) and the product state \( \Psi_{2n}^d(x, r, t_{2n-1}^d) \) of Eq. (72) are correct for both the unitary decelerating processes \( U_D(0) \) \((n = 0)\) and \( U_D(2) \) \((n = 1)\). It will prove below that both the motional state \( \Psi_{2n}^d(x, t_{2n-1}^d) \) of Eq. (71) and the product state \( \Psi_{2n}^d(x, r, t_{2n-1}^d) \) of Eq. (72) are also correct for the unitary decelerating process \( U_D(2n) \) with \( n = 0, 1, \ldots, N \).

Suppose that the states \( \Psi_{2n}^d(x, t_{2n-1}^d) \) of Eq. (71) and \( \Psi_{2n}^d(x, r, t_{2n-1}^d) \) of Eq. (72) are correct for the unitary decelerating process \( U_D(2n) \). It is known that the internal state of the atom is \( |g_0\rangle \) at the end of the unitary decelerating process \( U_D(2(n + 1)) \). Then one needs only to prove that the motional state of Eq. (71) is also correct for the unitary decelerating process \( U_D(2(n + 1)) \). The propagator of the unitary decelerating process \( U_D(2(n + 1)) \) can be written as

\[
U_D(2(n + 1)) = U_{2n+2}^d(63)U_D(2n + 1) = U_{2n+2}^d(63)U_{2n+1}^d(11)U_D(2n).
\]

According to the assumption the motional state \( \Psi_{2n}^d(x, t_{2n-1}^d) \) is just the final motional state of the atom when the atom is acted on by the unitary propagator \( U_D(2n) \). Obviously, the motional state \( \Psi_{2n}^d(x, t_{2n-1}^d) \) of the initial motional state of the \((2n + 1)\)th basic decelerating process (11) with the propagator \( U_{2n+1}^d(11) \) in the unitary decelerating process \( U_D(2(n + 1)) \), where the initial time is denoted as \( t_{2n}^d = t_{2n-1}^d \) and there are the recursive relations:
\[
t_0^d = t_0, \quad t_1^d = t_0 + t_d, \quad t_2^d = t_0 + 2t_d, \quad \text{and} \quad t_{2n}^d = t_{2n-1}^d + t_d \quad (n > 0).
\]
Then the initial product state of the basic decelerating process \( U_{2n+1}^d(11) \) is given by \( \Psi_{2n}^d(x, r, t_{2n}^d) = \Psi_{2n}^d(x, r, t_{2n-1}^d) \) of Eq. (72). Now the initial product state \( \Psi_{2n}^d(x, r, t_{2n}^d) \) is acted on by the unitary propagator \( U_{2n+1}^d(11) \). Then it can turn out that at the end time \( t_{2n}^d = t_{2n}^d + t_d \) of the basic decelerating process \( U_{2n+1}^d(11) \) the initial motional state \( \Psi_{2n}^d(x, t_{2n}^d) \) and product state \( \Psi_{2n}^d(x, r, t_{2n}^d) \) are respectively transferred into the motional state:

\[
\Psi_{2n+1}^d(x, t_{2n}^d) = \exp(i\varphi_{2n+1}^d)\left(\frac{(\Delta x)^2}{2\pi}\right)^{1/4}\sqrt{\frac{1}{(\Delta x)^2 + i\frac{\hbar(T_d + 2n + 1)\theta}{2M}}} \times \exp\left\{-\frac{1}{4}\frac{(x - z_{2n}^d)^2}{(\Delta x)^2 + i\frac{\hbar(T_d + 2n + 1)\theta}{2M}}\right\}\exp\{iP_{2n+1}^d x/\hbar\}
\]
and the product state:

\[
\Psi_{2n+1}^d(x, r, t_{2n}^d) = \Psi_{2n+1}^d(x, t_{2n}^d)|g_1\rangle
\]
where
\[
P_{2n+1}^d = P_{2n}^d - \hbar k_0 - \hbar k_1,
\]
\[
z_{2n+1}^d = z_{2n}^d + \frac{P_{2n+1}^d}{M}t_d + \frac{\hbar(k_0 + k_1)}{M}\int_{t_{2n}^d}^{t_{2n}^d} dt' \cos^2 \theta(t'),
\]
\[ \exp(i\varphi_{2n+1}^d) = \exp(i\varphi_{2n}^d) \exp(i/\beta(t_{2n}^d t_{2n}^d)) \exp\left\{ i/\hbar \left( \frac{(P_{2n}^d)^2}{2M} + E_0 t_{2n}^d \right) \right\} \times \exp\left\{ -i/\hbar \left( \frac{(P_{2n+1}^d)^2}{2M} + E_1 t_{2n}^d \right) \right\}. \] (78)

The computational process from the initial state \( \Psi_{2n}^d(x, t_{2n}^d) \) to the final state \( \Psi_{2n+1}^d(x, t_{2n+1}^d) \) is the same as the previous one from the initial state \( \Psi_0(x, t_0) \) of (48) to the final state \( \Psi_0(x, t_f) \) of (59). There are the relations:

\[ \Psi_{2n+1}^d(x, r, t_{2n}^d) = U_{2n+1}^d(11) \Psi_{2n}^d(x, r, t_{2n}^d) = U_D(2n + 1) \Psi_0(x, r, t_0). \]

These relations show that both the states \( \Psi_{2n+1}^d(x, r, t_{2n}^d) \) and \( \Psi_{2n+1}^d(x, t_{2n}^d) \) are also the product state and the motional state of the atom at the end time \( t_{2n}^d = t_{2n}^d + t_d \) of the unitary decelerating process \( U_D(2n + 1) \), respectively. The atomic product state \( \Psi_{2n+1}^d(x, r, t_{2n}^d) \) of Eq. (75) at the end of the unitary decelerating process \( U_D(2n + 1) \) will be used to calculate the atomic product state at the end of the unitary decelerating process \( U_D(2n + 2) \). This computational process is the same as the previous one from the initial state \( \Psi_1(x, t_1) = \Psi_0(x, t_f) \) of (59) to the final state \( \Psi_1(x, t_f) \) of (67). There are the relations:

\[ \Psi_{2n+2}^d(x, r, t_{2n+1}^d) = U_{2n+2}^d(63) \Psi_{2n+1}^d(x, r, t_{2n}^d) = U_D(2n + 2) \Psi_0(x, r, t_0). \]

These relations show that the atomic product state \( \Psi_{2n+2}^d(x, r, t_{2n+1}^d) \) at the end of the unitary decelerating process \( U_D(2n + 2) \) can be obtained by applying the propagator \( U_{2n+2}^d(63) \) to the atomic product state \( \Psi_{2n+1}^d(x, r, t_{2n}^d) \) of Eq. (75). For convenient calculation, the atomic motional state \( \Psi_{2n+1}^d(x, t_{2n}^d) \) of Eq. (74) is rewritten as \( (n' = n + 1) \)

\[ \Psi_{2n'-1}^d(x, t_{2n'-1}^d) = \exp(i\varphi_{2n'-1}^d) \frac{(\Delta x)^2}{2\pi} \sqrt{\frac{1}{(\Delta x)^2 + i\hbar(T_d + (2n'-1)t_d)/2M}} \times \exp\left\{ -\frac{1}{4} \frac{(x-z_{2n'-1}^d)^2}{(\Delta x)^2 + i\hbar(T_d + (2n'-1)t_d)/2M} \right\} \exp\{ i P_{2n'-1}^d x/\hbar \}. \] (79)

Then the atomic product state of Eq. (75) can be rewritten as

\[ \Psi_{2n'-1}^d(x, r, t_{2n'-1}^d) = \Psi_{2n'-1}^d(x, t_{2n'-1}^d)|g_1) = \sum_p \rho(p, t_{2n'-1}^d)|p)|g_1) \] (80)

where the amplitude \( \rho(p, t_{2n'-1}^d) \) can be calculated from the motional state of Eq. (79) and is given by

\[ \rho(p, t_{2n'-1}^d) = \exp(i\varphi_{2n'-1}^d) \frac{2(\Delta x)^2}{\pi} \exp\left\{ -(\Delta x)^2 \frac{(p - P_{2n'-1}^d)^2}{\hbar} \right\} \times \exp\left\{ -i\hbar(T_d + (2n'-1)t_d) \frac{(p - P_{2n'-1}^d)^2}{2M} \right\} \exp\left\{ -\frac{i\hbar(T_d + (2n'-1)t_d)}{2M} z_{2n'-1}^d \right\}. \] (81)
Now the atomic product state \( \Psi^d_{2n'}(x, r, t^d_{2n'} - 1) \) of Eq. (80) is applied by the unitary propagator \( U_{2n'}^n(t_d) \) (63) \( (n' = n + 1) \). Then it can turn out that at the end time \( t^d_{2n'} = t^d_{2n'-1} + t_d \) of the unitary decelerating process \( U_D(2n + 2) \) the atomic wave-packet motional state takes the form

\[
\Psi^d_{2n'}(x, t^d_{2n'-1}) = \exp(i \varphi^d_{2n'})[\frac{(\Delta x)^2}{2 \pi}]^{1/4} \sqrt{\frac{1}{(\Delta x)^2 + i \hbar (t_d + 2n(t_d))}} \times \exp\{-\frac{1}{4} (\Delta x)^2 \} \exp\{i P^d_{2n'} x / \hbar\}.
\]

and the atomic product state is

\[
\Psi^d_{2n'}(x, r, t^d_{2n'} - 1) = \Psi^d_{2n'}(x, t^d_{2n'} - 1)|g_0\rangle,
\]

where

\[
P^d_{2n'} = P^d_{2n'-1} - \hbar l_0 - \hbar l_1,
\]

\[
z^d_{2n'} = z^d_{2n'-1} + \frac{P^d_{2n'}}{M} t_d + \frac{\hbar (l_0 + l_1)}{M} \int_{t^d_{2n'-1}}^{t^d_{2n'} - 1} dt' \cos^2 \theta(t'),
\]

\[
\exp(i \varphi^d_{2n'}) = \exp(i \beta^d_{2n'}(t^d_{2n'} - 1, t^d_{2n'} - 1)])
\]

\[
\times \exp\left[\frac{i}{\hbar} \left(\frac{(P^d_{2n'-1})^2}{2M} + E_1\right) t^d_{2n'-1}\right] \exp\left[-\frac{i}{\hbar} \left(\frac{(P^d_{2n'})^2}{2M} + E_0\right) t^d_{2n'} - 1\right].
\]

Now by comparing the motional state \( \Psi^d_{2n'}(x, t^d_{2n'} - 1) \) of Eq. (82) with the motional state \( \Psi^d_{2n'}(x, t^d_{2n'} - 1) \) of Eq. (71) and the product state \( \Psi^d_{2n'}(x, r, t^d_{2n} - 1) \) of Eq. (83) with the product state \( \Psi^d_{2n}(x, r, t^d_{2n} - 1) \) of Eq. (72) one can conclude by the mathematical principle of induction that the motional state \( \Psi^d_{2n'}(x, t^d_{2n} - 1) \) of Eq. (71) and the product state \( \Psi^d_{2n}(x, r, t^d_{2n} - 1) \) of Eq. (72) are indeed the states of the atom at the end of the unitary decelerating process \( U_D(2n + 1) \) for \( n = 0, 1, 2, \ldots, N \). In an analogous way, one can prove that the product state \( \Psi^d_{2n+1}(x, r, t^d_{2n+1}) \) of Eq. (75) and the motional state \( \Psi^d_{2n+2}(x, t^d_{2n+1}) \) of Eq. (74) are the states of the atom at the end of the unitary decelerating process \( U_D(2n + 1) \) for \( n = 0, 1, 2, \ldots, N - 1 \).

Now one can prove that the atomic motional momentum \( P^d_{2n+1} \) of Eq. (76) and \( P^d_{2n'} \) of Eq. (84) are given by, respectively,

\[
P^d_{2n+1} = P^d_0 - (n + 1) (\hbar k_0 + \hbar k_1) - n (\hbar l_0 + \hbar l_1),
\]

\[
P^d_{2n'} = P^d_0 - n' (\hbar k_0 + \hbar k_1) - n' (\hbar l_0 + \hbar l_1).
\]

It is known that the recursive relations for the atomic motional momentum are given by \( P^d_{2n+1} = P^d_{2n} - \hbar k_0 - \hbar k_1 \) for \( n = 0, 1, \ldots, N - 1 \), and \( P^d_{2n'} = P^d_{2n'-1} - \hbar k_0 - \hbar k_1 \) for \( n' = 1, 2, \ldots, N \), which are obtained from Eqs. (76) and (84), respectively. The two recursive relations together can lead directly to the two formulae for the atomic motional momentum \( P^d_{2n+1} \) and \( P^d_{2n'} \).
One also can calculate exactly the time evolution process of the atom in the unitary STIRAP-based accelerating process in the ideal adiabatic condition (30). There are also two basic STIRAP-based accelerating sequences which correspond to the basic decelerating sequences (11) and (63), respectively. One of which is already expressed as (11a). The basic accelerating sequence (11a) corresponds to the basic decelerating sequence (11). Another may be expressed in an intuitive form

\[ |P - \hbar l_0||g_1\rangle \rightarrow |P||e\rangle \rightarrow |P + \hbar l_1||g_0\rangle. \]  

(63a)

This basic accelerating sequence corresponds to the basic decelerating sequence (63). In an analogous way to constructing the unitary decelerating processes \( U_D(2N) \) and \( U_D(2N - 1) \) one may build up the unitary accelerating processes \( U_A(2N) \) and \( U_A(2N - 1) \) out of the basic accelerating sequences (11a) and (63a),

\[ U_A(2N) = U_{2N}^a(63a)U_{2N-1}^a(11a)\ldots U_{2n}^a(63a)U_{2n-1}^a(11a)\ldots U_{2}^a(63a)U_{1}^a(11a) \] 

(87a)

or

\[ U_A(2N - 1) = U_{2N-1}^a(11a)\ldots U_{2n}^a(63a)U_{2n-1}^a(11a)\ldots U_{2}^a(63a)U_{1}^a(11a) \] 

(87b)

where \( U_{2n-1}^a(11a) \) and \( U_{2n}^a(63a) \) for \( n = 1, 2, \ldots, N \) are the unitary propagators of the \((2n-1)\)–th basic accelerating sequence (11a) and \((2n)\)–th basic accelerating sequence (63a), respectively. Here also suppose that the atom is in the internal state \(|g_0\rangle\) at the initial time in both the unitary accelerating processes \( U_A(2N) \) and \( U_A(2N - 1) \).

The time evolution process of the atom in the unitary accelerating process \( U_A(2N) \) (and \( U_A(2N - 1) \)) can be calculated exactly in the ideal adiabatic condition (30) in a similar way to that one in the unitary decelerating process \( U_D(2N) \) (and \( U_D(2N - 1) \)). Actually, the recursive relations (71)–(78) and (79)–(86) of the unitary decelerating process \( U_D(2N) \) or \( U_D(2N - 1) \) can be used as well for the unitary accelerating process \( U_A(2N) \) or \( U_A(2N - 1) \) if one makes transformations: \( k_0 \rightarrow -k_0^a, k_1 \rightarrow -k_1^a, \) and \( \theta(t) \rightarrow \theta_a(t) \) in those recursive equations (71)–(78) and \( l_0 \rightarrow -l_0^a, l_1 \rightarrow -l_1^a, \) and \( \theta_l(t) \rightarrow \theta_{la}(t) \) in those recursive equations (79)–(86). As an example, suppose that the initial wave-packet motional state for the atom in the unitary accelerating process \( U_A(2n) \) is given by

\[ \Psi_0^a(x, t_0^a) = \exp(i\varphi_0^a)\sqrt{\frac{1}{2\pi}} \frac{1}{\sqrt{[(\Delta x)^2 + \frac{i}{2\hbar}]}} \times \exp\left\{-\frac{1}{4} \frac{[x - z_0^a]^2}{[(\Delta x)^2 + \frac{i}{2\hbar}]}) \right\} \exp[iP_0^a x/\hbar] \] 

(88)

and the atomic wave-packet product state by

\[ \Psi_0^a(x, r, t_0^a) = \Psi_0^a(x, t_0^a)|g_0\rangle = \sum_p \rho(p, t_0^a)|p\rangle|g_0\rangle. \] 

(89)
Then it can turn out that the momentum wave-packet state \( \rho(p, t_0^a) \) of the motional state \( \Psi_0^a(x, t_0^a) \) can be written as

\[
\rho(p, t_0^a) = \exp(i\varphi_0^a) \left[ \frac{2(\Delta x)^2}{\pi} \right]^{1/4} \exp\left\{ -(\Delta x)^2 \left( \frac{p - P_0^a}{\hbar} \right)^2 \right\} \]
\[
\times \exp\left\{ -i \left( \frac{\hbar T_n}{2M} \right) \left( \frac{p - P_0^a}{\hbar} \right)^2 \right\} \exp\left\{ -i \left( \frac{\hbar}{2M} \right) z_0^a \right\}.
\]

(90)

Now the initial wave-packet product state \( \Psi_0^a(x, r, t_0^a) \) of the atom undergoes the unitary accelerating process \( U_A(2n) \). Then it can be proved that at the end of the unitary accelerating process \( U_A(2n) \) the atomic wave-packet motional state is given by

\[
\Psi_{2n}^a(x, t_{2n-1}^a) = \exp(i\varphi_{2n}^a) \left[ \frac{(\Delta x)^2}{2\pi} \right]^{1/4} \sqrt{\frac{1}{(\Delta x)^2 + i \left( \frac{\hbar}{2M} \right) t_{2n-1}^a}} \]
\[
\times \exp\left\{ -\frac{1}{4} \left( \frac{x - z_{2n}^a}{\Delta x} \right)^2 \right\} \exp\left\{ i P_{2n}^a x / \hbar \right\}
\]

(91)

and the atomic wave-packet product state by

\[
\Psi_{2n}^a(x, r, t_{2n-1}^a) = \Psi_{2n}^a(x, t_{2n-1}^a) |g_0\rangle
\]

(92)

where the end time of the unitary accelerating process \( U_A(2n) \) is \( t_{2n-1}^a = t_{2n}^a = t_0^a + 2nt_a \) for \( n = 0, 1, 2, ... N \), the atomic motional momentum \( P_{2n}^a \) is given by

\[
P_{2n}^a = P_0^a + n(hk_0^a + h\ell_1^a) + n(h\ell_0^a + h\ell_1^a),
\]

(93)

and the center-of-mass position \( z_{2n}^a \) is determined from these recursive relations:

\[
z_{2k-1}^a = z_{2k-2}^a - \frac{P_{2k-1}^a}{M} t_a - \frac{h(k_0^a + k_1^a)}{M} \int_{t_0^a + (2k-1)t_a}^{t_0^a + (2k-2)t_a} dt' \cos^2 \theta_a(t'),
\]

(94a)

\[
z_{2k}^a = z_{2k-1}^a + \frac{P_{2k}^a}{M} t_a - \frac{h(l_0^a + l_1^a)}{M} \int_{t_0^a + (2k-1)t_a}^{t_0^a + (2k)t_a} dt' \cos^2 \theta_a(t'),
\]

(94b)

\[
P_{2k-1}^a = P_{2k-2}^a + h(k_0^a + k_1^a), \quad P_{2k}^a = P_{2k-1}^a + h(l_0^a + l_1^a),
\]

where the initial center-of-mass position and momentum are \( z_0^a \) and \( P_0^a \), respectively, the index \( 1 \leq k \leq n \), both the basic accelerating sequence (11a) and (63a) have the same duration \( t_a \), and the global phase factor \( \exp(i\varphi_{2n}^a) \) can also be calculated through the recursive relations similar to Eq. (78) and (86). In an analogous way, one also can calculate exactly the time evolution process of the atom in the unitary accelerating process \( U_A(2n - 1) \) in the ideal adiabatic condition (30).

6. The space- and time-compressing processes based on the unitary decelerating and accelerating processes
Suppose that in the quantum control process [22] the first unitary decelerating process consists of $2n_d$ basic STIRAP decelerating sequences (11) and (63) alternately, which is given by $U_D(2n_d)$ of Eq. (70a), and the ideal adiabatic condition (30) is met for all these basic decelerating sequences. According to the quantum control process the unitary decelerating sequence is applied selectively in the given spatial region $[D_L, D_R]$ in the right-hand potential well of the double-well potential field, where $D_L$ and $D_R$ are the left- and right-boundary positions of the spatial region in the coordinate axis, respectively. The halting-qubit atom can be decelerated by the unitary decelerating sequence $U_D(2n_d)$ only when the atom enters into the spatial region $[D_L, D_R]$. Thus, the spatial region $[D_L, D_R]$ may be called the decelerating spatial region. The decelerating spatial region must cover sufficiently the whole wave-packet motional state of the atom during the whole unitary decelerating process when the atom is decelerated in the decelerating region. The decelerating region is so wide that for the wave-packet motional state of the atom the Raman laser light beams of the electromagnetic fields. Suppose that the halting-qubit atom is in the product state $\Psi_0(x, r, t_0)$ of Eq. (54) (or in the motional state $\Psi_0(x, t_0)$ of Eq. (48) and the internal state $|g_0\rangle$) and in the decelerating region $[D_L, D_R]$ when the unitary decelerating sequence $U_D(2n_d)$ is turned on at the initial time $t_0$. Here the spatial position of an atom is defined as the center-of-mass position of the atomic wave-packet motional state. Now the center-of-mass position and wave-packet spread of the initial motional state $\Psi_0(x, t_0)$ are $z_0$ and $\varepsilon_0$, respectively. Since the halting-qubit atom is in the decelerating region $[D_L, D_R]$ at the time $t_0$, that is, $z_0 \in [D_L, D_R]$, both the distances $(z_0 - D_L)$ and $(D_R - z_0)$ must be much greater than the wave-packet spreading $\varepsilon_0$, that is, $(D_R - z_0) > (z_0 - D_L) > > \varepsilon_0$, meaning that the decelerating region $[D_L, D_R]$ covers sufficiently the whole initial wave-packet motional state $\Psi_0(x, t_0)$. The halting-qubit atom starts to be decelerated by the unitary decelerating process $U_D(2n_d)$ at the initial time $t_0$ and in the decelerating region $[D_L, D_R]$. With the help of the recursive relations (71)–(78) and (79)–(86) one can prove that at the end time $t^d_{2n_d} = t_0 + 2n_d\omega_d$ of the unitary decelerating process $U_D(2n_d)$ the motional state of the halting-qubit atom is given by

$$
\Psi^d_{2n_d}(x, t^d_{2n_d}) = \exp(i\varphi^d_{2n_d})\left[\frac{(\Delta x)^2}{2\pi}\right]^{1/4}\sqrt{\frac{1}{(\Delta x)^2 + i\frac{h(2\omega_d + 2\omega_{d+1})}{2M}}} \\
\times \exp\left\{-\frac{1}{4}\frac{(x - z^d_{2n_d})^2}{(\Delta x)^2 + i\frac{h(2\omega_d + 2\omega_{d+1})}{2M}}\right\}\exp\{iP^d_{2n_d}x/h\} 
$$

(95)

and the atomic product state by

$$
\Psi^d_{2n_d}(x, r, t^d_{2n_d}) = \Psi^d_{2n_d}(x, t^d_{2n_d})(g_0), 
$$

(96)

where the atomic motional momentum $P^d_{2n_d}$ is given by

$$
P^d_{2n_d} = p_0 - n_d(\hbar k_0 + \hbar k_1) - n_d(\hbar l_0 + \hbar l_1), 
$$

(97)
and the center-of-mass position $z_{2n_d}$ can be calculated by the recursive relations 
(77) and (85),

$$z_{2n-1}^d = z_{2n-2}^d + \frac{P_{2n-1}^d}{M} t_d + \frac{\hbar(k_0 + k_1)}{M} \int_{t_{2n-2}^d}^{t_{2n-2}^d} dt' \cos^2 \theta(t'),$$

(98a)

$$z_{2n}^d = z_{2n-1}^d + \frac{P_{2n}^d}{M} t_d + \frac{\hbar(l_0 + l_1)}{M} \int_{t_{2n-1}^d}^{t_{2n-1}^d} dt' \cos^2 \theta(t'),$$

(98b)

$$P_{2n-1}^d = P_{2n-2}^d - (\hbar k_0 + \hbar l_1), \quad P_{2n}^d = P_{2n-1}^d - (\hbar l_0 + \hbar l_1),$$

where $1 \leq n \leq n_d$, $z_0^d = z_0$, $P_0^d = p_0$, $T_d = T_0$, $t_{0}^d = t_0$, $t_{k+1}^d = t_k^d + t_d$, and $t_k^d = t_{k+1}^d$ for $0 \leq k \leq 2n_d - 1$, and the global phase factor \( \exp(i\varphi_{2n_d}^d) \) can be calculated by Eq. (78) and (86) with the initial phase \( \varphi_0^d = \varphi_0 \). Both the initial atomic product state $\Psi_0(x, r, t_0)$ of Eq. (54) and the final product state $\Psi_{2n_d}^d(x, r, t_{2n_d}^d)$ of Eq. (96) show that before and after the unitary decelerating process $U_D(2n_d)$ the halting-qubit atom is in the same internal state \( |g_0\rangle \), while its initial motional state $\Psi_0(x, t_0)$ of Eq. (48) is changed to the motional state $\Psi_{2n_d}^d(x, t_{2n_d}^d)$ of Eq. (95) after the unitary decelerating process. The final motional state $\Psi_{2n_d}^d(x, t_{2n_d}^d)$ of Eq. (95) has the center-of-mass position $z_{2n_d}^d$ and the wave-packet spreading

$$\varepsilon = \sqrt{\frac{2(\Delta x)^2 + 2\hbar(T_d + 2n_dt_d)}{2M(\Delta x)}}.$$  

Though the atom moves a distance $(z_{2n_d}^d - z_0)$ during the unitary decelerating process, the center-of-mass position $z_{2n_d}^d \in [D_L, D_R]$ as the atom is still in the decelerating region $[D_L, D_R]$ at the end of the unitary decelerating process. Then both the distances $(z_{2n_d}^d - D_L)$ and $(D_R - z_{2n_d}^d)$ must be much greater than the wave-packet spreading $\varepsilon$, that is, $(z_{2n_d}^d - D_L) > (D_R - z_{2n_d}^d) >> \varepsilon$. This means that the decelerating region $[D_L, D_R]$ also covers sufficiently the whole final motional state $\Psi_{2n_d}^d(x, t_{2n_d}^d)$. There are two extra constraint conditions on the decelerating region $[D_L, D_R]$. If the atom has not yet entered into the decelerating region when the unitary decelerating sequence is switched on or it leaves the decelerating region after the unitary decelerating process is switched off, then it will not be affected by the unitary decelerating sequence or by next unitary decelerating sequences. The two constraint conditions are stated below.

In the quantum control process [22] the halting-qubit atom may enter into the right-hand potential well from the left-hand one in any $i$-th cycle of the quantum program for $i = 1, 2, ..., m_r$. The $i$-th possible wave-packet motional state of the halting-qubit atom is just defined as the atomic motional state when the atom enters into the right-hand potential well in the $i$-th cycle of the quantum program. Thus, there is a different time for any possible atomic motional state such as the $i$-th wave-packet motional state to enter into the right-hand potential well. If the time period of each cycle of the quantum program is $\Delta T$, then the time difference between the $i$-th and $j$-th $(i < j)$
wave-packet motional states to enter into the right-hand potential well is given by $\Delta T(i, j) = (j - i)\Delta T$ for $i < j$ and $i, j = 1, 2, ..., m_r$. This time difference results in a center-of-mass distance in space between the two wave-packet motional states. If the halting-qubit atom moves along the direction $+x$ with the velocity $p_0/M$, then the distance is given by $\Delta L(i, j) = (j - i)\Delta T(p_0/M)$. Now examine two consecutive possible wave-packet motional states: the $i$–th and $(i + 1)$–th wave-packet motional states. Here for convenience the $i$–th wave-packet motional state is set to the motional state $\Psi_0(x, t_0)$ of Eq. (48). Then at the time $t_0$ the $i$–th wave-packet motional state is in the decelerating region $[D_L, D_R]$ and its center-of-mass position is $z_0$, while the center-of-mass position of the $(i + 1)$–th wave-packet motional state is clearly $[z_0 - \Delta T(p_0/M)]$. It is known that the total duration for the unitary decelerating sequence $U_D(2n_d)$ is $2n_d\epsilon_d$. Then at the end time $t_0 + 2n_d\epsilon_d$ of the unitary decelerating process the center-of-mass position of the $(i + 1)$–th wave-packet motional state becomes $[z_0 - (\Delta T - 2n_d\epsilon_d)(p_0/M)]$. Obviously, the distance between this center-of-mass position and the left-end position of the decelerating region $[D_L, D_R]$ is $D_L - [z_0 - (\Delta T - 2n_d\epsilon_d)(p_0/M)]$. Denote $\epsilon_i(t_0 + 2n_d\epsilon_d)$ as the wave-packet spreading of the $(i + 1)$–th wave-packet motional state at the time $t_0 + 2n_d\epsilon_d$. The wave-packet spreading $\epsilon_i(t_0 + 2n_d\epsilon_d)$ may be calculated with the help of the $i$–th wave-packet state $\Psi_0(x, t_0)$ and the free-particle propagator. Then this distance must be much greater than $\epsilon_i(t_0 + 2n_d\epsilon_d)$, that is, $D_L - [z_0 - (\Delta T - 2n_d\epsilon_d)(p_0/M)] \gg \epsilon_i(t_0 + 2n_d\epsilon_d)$, so that the $(i + 1)$–th wave-packet motional state is not affected by the unitary decelerating sequence $U_D(2n_d)$ during the whole unitary decelerating process. This is a constraint condition on the decelerating region $[D_L, D_R]$.

It is known that at the end time $t_0 + 2n_d\epsilon_d$ of the unitary decelerating process $U_D(2n_d)$ the $i$–th wave-packet motional state is the state $\Psi_{2n_d}(x, t_{2n_d})$ of Eq. (95), which has the center-of-mass position $z_{2n_d}^i$ and the motional momentum $p_{2n_d}^i$. After the unitary decelerating process the $i$–th wave-packet motional state (i.e., the halting-qubit atom) moves along the direction $+x$ with the velocity $p_{2n_d}^i/M$. Since the atom is usually decelerated greatly by the decelerating sequence $U_D(2n_d)$ the atomic velocity $p_{2n_d}^i/M$ is much less than the original velocity $p_0/M$. Obviously, the $i$–th wave-packet motional state moves to the position $z_{2n_d}^i + (\Delta T - 2n_d\epsilon_d)p_{2n_d}^i/M$ when next unitary decelerating sequence starts to work at the time $t_0 + \Delta T$. Then the distance between this position and the right-end position of the decelerating region $[D_L, D_R]$ is given by $z_{2n_d}^i + (\Delta T - 2n_d\epsilon_d)p_{2n_d}^i/M - D_R$. Denote $\epsilon_i(t_0 + \Delta T)$ as the wave-packet spreading of the $i$–th wave-packet motional state at the time $t_0 + \Delta T$. The wave-packet spreading $\epsilon_i(t_0 + \Delta T)$ can be calculated with the help of the motional state $\Psi_{2n_d}(x, t_{2n_d})$ of Eq. (95) and the free-particle propagator. Then this distance must be much greater than $\epsilon_i(t_0 + \Delta T)$, that is, $z_{2n_d}^i + (\Delta T - 2n_d\epsilon_d)p_{2n_d}^i/M - D_R \gg \epsilon_i(t_0 + \Delta T)$, so that, from the time $t_0 + \Delta T$ on, the $i$–th wave-packet motional state is no longer affected by the unitary decelerating sequences. This is another constraint condition on the decelerating region $[D_L, D_R]$.
The $i$–th wave-packet motional state is decelerated from the initial time $t_0$ to the end time $t_0 + 2n_dt_d$ by the unitary decelerating process $U_D(2n_d)$ in the decelerating region $[D_L, D_R]$. It moves a distance $z^d_{2n_d} - z_0$ along the direction $+x$ and spends the time $2n_dt_d$ and it is decelerated down to $P^d_{2n_d}/M$ from the initial velocity $p_0/M$ during the unitary decelerating process. According to the quantum control process [22] the $(i + 1)$–th wave-packet motional state arrives at the position $z_0$ in the decelerating region $[D_L, D_R]$ at the time $t_0 + \Delta T$. Then the $(i + 1)$–th wave-packet motional state at the time $t_0 + \Delta T$ is really equal to the $i$–th wave-packet motional state at the time $t_0$ up to a global phase factor, indicating that the $(i + 1)$–th wave-packet motional state at the time $t_0 + \Delta T$ is also equal to the motional state $\Psi_0(x, t_0)$ of Eq. (48) up to a global phase factor. Generally, according to the quantum control process each of these $m_r$ possible wave packet motional states is really equal to the motional state $\Psi_0(x, t_0)$ of Eq. (48) up to a global phase factor when the wave-packet motional state arrives at the same position $z_0$ in the decelerating region $[D_L, D_R]$. Just like the $i$–th wave-packet motional state at the time $t_0$ the $(i + 1)$–th wave-packet motional state at the time $t_0 + \Delta T$ is decelerated by the unitary decelerating process $U_D(2n_d)$. It also moves the distance $z^d_{2n_d} - z_0$ along the direction $+x$ and spends the time $2n_dt_d$ and it is also decelerated down to $P^d_{2n_d}/M$ from the initial velocity $p_0/M$ during the unitary decelerating process. Generally, each of these $m_r$ possible wave-packet motional states moves the same distance $z^d_{2n_d} - z_0$ along the direction $+x$ and also spends the same time $2n_dt_d$ and it is also decelerated down to the same velocity $P^d_{2n_d}/M$ from the same initial velocity $p_0/M$ during the unitary decelerating process. The difference among these $m_r$ possible wave-packet motional states is that the starting time is different to decelerate each one of these wave-packet motional states by the unitary decelerating process $U_D(2n_d)$.

In the quantum control process [22] the unitary decelerating sequence is used to decelerate the halting-qubit atom so that the center-of-mass distances between these $m_r$ possible wave-packet motional states of the atom can be narrowed greatly. Thus, the unitary decelerating process $U_D(2n_d)$ is really a space-compressing process for these possible wave-packet motional states. Since each one of these $m_r$ possible wave-packet motional states spends the same time $2n_dt_d$ in the unitary decelerating process $U_D(2n_d)$, the time difference $\Delta T(i, j) = (j - i)\Delta T$ between the $i$–th and $j$–th wave-packet motional states ($i < j$; $i, j = 1, 2, \ldots, m_r$) does not change before and after the unitary decelerating process. It is known that each possible wave-packet motional state has the initial moving velocity $p_0/M$ before the unitary decelerating process and the final moving velocity $P^d_{2n_d}/M > 0$ after the unitary decelerating process. Here the atomic moving velocity $P^d_{2n_d}/M$ can be obtained from Eq. (97),

$$P^d_{2n_d}/M = [p_0 - n_d(\hbar k_0 + \hbar k_1) - n_d(\hbar l_0 + \hbar l_1)]/M.$$  \hspace{1cm} (99)

If the number $n_d$ of the unitary decelerating process $U_D(2n_d)$ is chosen suitably, then the velocity $P^d_{2n_d}/M$ can be much less than the initial one $p_0/M$. Before the unitary decelerating process the distance between the $i$–th and $j$–th wave-
packet motional states is $\Delta L_0(i, j) = (j - i)\Delta T(p_0/M)$, since the velocity is $p_0/M$ and the time difference is $\Delta T(i, j) = (j - i)\Delta T$ before the unitary decelerating process. After the unitary decelerating process the atomic moving velocity is $P_{d2n_d}/M$ and the time difference is still $\Delta T(i, j) = (j - i)\Delta T$. Then after the unitary decelerating process the distance between the $i$-th and $j$-th ($i < j$) wave-packet motional states is equal to

$$\Delta L(i, j) = (j - i)\Delta T(P_{d2n_d}/M),$$

(100)

where $i < j$; $i, j = 1, 2, ..., m_r$. Since the velocity $(P_{2n_d}/M) << p_0/M$, the distance $\Delta L(i, j) << \Delta L_0(i, j)$, indicating that the spatial region to cover all these $m_r$ possible wave-packet motional states is greatly compressed after the unitary decelerating process. The distance $\Delta L(i, j)$ of Eq. (100) has been obtained in the previous paper [22], where the atomic velocity $(P_{d2n_d}/M)$ is denoted as $v_0$ after the unitary decelerating process. Then the ratio of the two distances $\Delta L(i, j)$ and $\Delta L_0(i, j)$ is the space-compressing factor for these wave-packet motional states after and before the unitary decelerating process, which can be calculated by

$$R_s = \frac{\Delta L(i, j)}{\Delta L_0(i, j)} = \frac{p_0 - n_d(\hbar k_0 + \hbar k_1) - n_d(\hbar l_0 + \hbar l_1)}{p_0}.$$ 

(101)

The space-compressing factor is not dependent upon the indices $i$ and $j$, since the time difference $\Delta T(i, j)$ does not change before and after the unitary decelerating process and since all these possible wave-packet motional states have the same initial motional momentum $p_0$ and also the same final motional momentum $P_{d2n_d}$ after each of these possible wave-packet motional states undergoes the same unitary decelerating process $U_D(2n_d)$ in the same decelerating region $[D_L, D_R]$.

Before the unitary accelerating process comes to making a real action on the halting-qubit atom, the atom needs to stay in the right-hand potential well for a time period to wait for the quantum program running to the end according to the quantum control process [22]. The time period during which the halting-qubit atom stays in the right-hand potential well is different and dependent upon how early the halting-qubit atom enters into the right-hand potential well from the left-hand one. When the atom enters into the right-hand potential well at an earlier time, it will stay in the right-hand potential well for a longer time. Denote that $T_s(i) = T_s - (i - 1)\Delta T$ with the index $i = 1, 2, ..., m_r$ is the time period during which the atom moves freely along the direction $+x$ in the right-hand potential well after the atom is decelerated by the unitary decelerating sequence $U_D(2n_d)$ and before the atom starts to be accelerated at the end time of the quantum program. The index $i$ indicates that the halting-qubit atom enters into the right-hand potential well from the left-hand one in the $i$-th cycle of the quantum program. Here suppose that the last unitary decelerating process $U_D(2n_d)$ is turned off before the quantum program comes to the end. The calculation for the time evolution process of the halting-qubit atom moving freely during the time period $T_s(i)$ needs to use the free-particle
unitary propagator. Now the unitary propagator of a free particle is written as

\[ G(x', t'; x, t) = \sqrt{\frac{M}{2\pi\hbar(t' - t)}} \exp\left[\frac{iM(x' - x)^2}{2\hbar(t' - t)}\right]. \]  

(102)

Then the time evolution process of an atom in a free-particle motion with the time period \( T = t' - t \) can be calculated by

\[ \Psi(x', t') = \int dx G(x', t'; x, t)\Psi(x, t). \]  

(103)

It is known that the \( i \)-th wave-packet motional state of the atom is given by the motional state \( \Psi_{d_{2n_d}}^d(x, t_{d_{2n_d}}^d) \) of Eq. (95) at the end time \( t_{d_{2n_d}}^d = t_0 + 2n_d t_d \) of the unitary decelerating process. When the wave-packet motional state \( \Psi_{d_{2n_d}}^d(x, t_{d_{2n_d}}^d) \) moves freely along the direction \(+x\) for the time period \( T_s(i) \) from the time \( t_{d_{2n_d}}^d \) to the time \( t_{d_{2n_d}}^d + T_s(i) \), it will change to another Gaussian wave-packet motional state. Thus Gaussian wave-packet motional state can be calculated from the equation (103) by taking the initial state \( \Psi(x, t) \) as \( \Psi_{d_{2n_d}}^d(x, t_{d_{2n_d}}^d) \) of Eq. (95), using the free-particle propagator \( G(x', t'; x, t) \) of Eq. (102), and denoting \( t' = t_{d_{2n_d}}^d + T_s(i) \) and \( t = t_{d_{2n_d}}^d \). By a complex calculation, in which the Gaussian integral (58) has been used, the final Gaussian wave-packet state \( \Psi(x', t') \) can be obtained explicitly, which now is renamed \( \Psi_{d_{2n_d}}^d(x, t_{d_{2n_d}}^d + T_s(i)) \),

\[ \Psi_{d_{2n_d}}^d(x, t_{d_{2n_d}}^d + T_s(i)) = \exp(i\varphi_{d_{2n_d}}^d) \exp\left\{-\frac{i(P_{2n_d}^d)^2 T_s(i)}{2\hbar M}\right\} \]

\[ \times \left\{ \frac{1}{\sqrt{2\pi}} \right\} \left[ (\Delta x)^2 + \frac{\hbar}{2M} \{T_d + 2n_d t_d + T_s(i)\} \right]^{1/4} \exp\left\{-\frac{1}{4} \left[ \frac{|x - z_{2n_d}^d - (P_{2n_d}^d / M) T_s(i)|^2}{(\Delta x)^2 + \frac{\hbar}{2M} \{T_d + 2n_d t_d + T_s(i)\}} \right] \right\} \]  

(104)

On the other hand, the atomic internal state \( |g_0\rangle \) and the motional momentum \( P_{d_{2n_d}} \) keep unchanged during the free-particle motion of the atom. Therefore, before the unitary accelerating process starts at the end of the quantum program, these \( m_r \) possible wave-packet motional states are given by \( \Psi_{d_{2n_d}}^d(x, t_{d_{2n_d}}^d + T_s(i)) \) of Eq. (104) for \( i = 1, 2, ..., m_r \) and each of them has a different center-of-mass position: \( z_{d_{2n_d}}^d + (P_{2n_d}^d / M) T_s(i) \), a different global phase factor:

\[ \exp(i\varphi_{d_{2n_d}}^d) \exp\{-i(P_{2n_d}^d)^2 T_s(i)/(2\hbar M)\}, \]

and a different complex linewidth:

\[ W_s(T_d + 2n_d t_d + T_s(i)) = (\Delta x)^2 + i \frac{\hbar(T_d + 2n_d t_d + T_s(i))}{2M}. \]  

(105)

An important fact is that the imaginary part of the complex linewidth of the motional state \( \Psi_{d_{2n_d}}^d(x, t_{d_{2n_d}}^d + T_s(i)) \) increases linearly with the time period \( T_s(i) \),
while the real part keeps unchanged. Though each one of these \( m_r \) possible wave-packet motional states has the same wave-packet spreading and the same complex linewidth \( W = (\Delta x)^2 + \frac{i \hbar (T_d + 2n_d T_d)}{2M} \) before the free-particle motion, as can be seen from the state \( \Psi_{2n_d}^d(x, t_{2n_d}^d) \) of Eq. (95), each possible wave-packet motional state has a larger wave-packet spreading and a different complex linewidth wave-packet motional states has the same wave-packet spreading and the same \( W \) starts to act on the halting-qubit atom at the end of the quantum program. Obviously, the first wave-packet motional state \( \Psi_{F}^d \) while the real part keeps unchanged. Though each one of these \( m_r \) possible wave-packet motional states and makes these wave-packet motional states broader. This difference may have a significant impact on the quantum control process [22]. On the other hand, the free-particle motion of the halting-qubit atom does not change the time differences and the distances in space between these \( m_r \) possible wave-packet motional states. This is because the motional momentum \( P_{2n_d}^d \) is the same for all these \( m_r \) possible wave-packet motional states and keeps unchanged during the free-particle motion. Thus, the distance between the \( i-th \) and \( j-th \) (\( i < j \)) wave-packet motional states is still given by \( \Delta L(i, j) \) of Eq. (100) and their time difference by \( \Delta T(i, j) = (j - i)\Delta T \). Particularly, the distance between two nearest wave-packet motional states is given by \( \Delta T(P_{2n_d}^d/M) \). Obviously, the halting-qubit atom moves a distance equal to \( (P_{2n_d}^d/M)T_s(i) \) along the direction \( +x \) in the time period \( T_s(i) \) of the free-particle motion. This distance is dependent upon the index \( i \). The first wave-packet motional state \( \Psi_i^d(x, t_{2n_d}^d + T_s) \) moves the largest distance \( (P_{2n_d}^d/M)T_s \) which decides mainly the dimensional size of the right-hand potential well, while the last wave-packet motional state \( \Psi_{m_r}^d(x, t_{2n_d}^d + T_s(m_r)) \) moves the shortest distance \( (P_{2n_d}^d/M)[T_s - (m_r - 1)\Delta T] \).

The atomic wave-packet states \( \{\Psi_i^d(x, t_{2n_d}^d + T_s(i))\} \) of Eq. (104) show that just before the unitary accelerating sequence is switched on, all these \( m_r \) possible wave-packet states of Eq. (104) are in the spatial region \( [x_0(m_r) - \varepsilon_d(m_r), x_0(1) + \varepsilon_d(1)] \), where \( x_0(j) \) and \( \varepsilon_d(j) \) \( (j = 1, 2, ..., m_r) \) are the center-of-mass position and the wave-packet spreading of the \( j-th \) wave-packet state \( \Psi_j^d(x, t_{2n_d}^d + T_s(j)) \), respectively. Suppose that all these \( m_r \) possible wave-packet states are accelerated uniformly by the unitary accelerating sequence and each possible wave-packet state moves the same distance \( L_A \) during the unitary accelerating process. The distance \( L_A \) will be obtained later. Obviously, after the unitary accelerating process all these \( m_r \) possible wave-packet motional states are in the spatial region \( [x_0(m_r) + L_A - \varepsilon_a(m_r), x_0(1) + L_A + \varepsilon_a(1)] \), where \( \varepsilon_a(j) \) \( (j = 1, 2, ..., m_r) \) is the wave-packet spreading of the \( j-th \) wave-packet motional state of the atom after the unitary accelerating process. Therefore, during the unitary accelerating process any possible wave-packet motional state of the halting-qubit atom is within the effective spatial region:

\[
[A_L, A_R] = [x_0(m_r) - \varepsilon_d, x_0(1) + L_A + \varepsilon_a]
\]
where $\varepsilon_a >> \varepsilon_a(1)$ and $\varepsilon_d >> \varepsilon_d(m_r)$. The effective spatial region $[A_L, A_R]$ covers all these $m_r$ possible wave-packet motional states of the atom during the whole unitary accelerating process. Now the spatial region of the unitary accelerating sequence must encompass sufficiently the whole effective spatial region $[A_L, A_R]$, so that for all these $m_r$ possible wave-packet motional states the Raman laser light beams of the unitary accelerating sequence can be thought of as infinite plane-wave electromagnetic fields, and the most important is that the unitary accelerating sequence can act on all these $m_r$ possible wave-packet motional states simultaneously and uniformly during the whole unitary accelerating process. The spatial region $[A_L, A_R]$ may be called the accelerating spatial region.

According to the quantum control process [22] the halting-qubit atom is accelerated by a unitary accelerating sequence at the end time of the quantum program. Here the unitary accelerating sequence may be given by $U_A(2n_a)$ of Eq. (87a), which consists of $n_a$ pairs of the basic STIRAP accelerating sequences (11a) and (63a) in an alternate form and each basic accelerating sequence has the same time period $t_a$. The unitary accelerating process $U_A(2n_a)$ has a total time period $2n_at_a$. The ideal adiabatic condition (30) is also met in the unitary accelerating process. Now one may use the recursive relations (88)–(94) to obtain the final wave-packet motional state of the halting-qubit atom after the atom is accelerated by the unitary accelerating sequence $U_A(2n_a)$. Here the starting time of the unitary accelerating process is the end time $t_{m_r}$ of the quantum program. At the initial time $t_{m_r}$ each possible wave-packet motional state of the halting-qubit atom is given by $\Psi^A_f(x, t_{2n_a}^d + T_s(j))$ of Eq. (104) for $j = 1, 2, ..., m_r$. All these $m_r$ possible wave-packet motional states start to undergo the same unitary accelerating process $U_A(2n_a)$ at the initial time $t_{m_r}$ simultaneously. In order to use the recursive relations (88)–(94) the initial motional state $\Psi_0^A(x, t_0^d)$ of Eq. (88) needs first to be obtained from the state $\Psi^A_f(x, t_{2n_a}^d + T_s(j))$ of Eq. (104). By comparing the initial state $\Psi_0^A(x, t_0^d)$ of Eq. (88) with $\Psi^A_f(x, t_{2n_a}^d + T_s(j))$ of Eq. (104) one can see that at the initial time $t_0^d = t_{m_r}$ the center-of-mass position, momentum, and global phase factor of the initial state $\Psi_0^A(x, t_0^d)$ are given by $z_0^d \equiv z_2^d(j) = _{2n_a}^d + (P_{2n_a}^d/M)T_s(j)$, $P_0^d = P_{2n_a}^d$, and $\exp[i\varphi_0^A(j)] \equiv \exp[i\varphi_{2n_a}^d(j)] = \exp(i\varphi_{2n_a}^d) \exp[-i(P_{2n_a}^d/2(\hbar M))T_s(j)]$, respectively, and in the complex linewidth $W(T_s)$ of the initial state $\Psi_0^A(x, t_0^d)$ the time interval $T_0^d \equiv T_s(j) = T_d + 2n_at_a + T_s(j)$. It is known that the initial internal state is $\{g_0(n)\}$. The initial atomic wave-packet product state then is given by $\Psi_0^A(x, r, t_0^d) = \Psi_f^A(x, t_{2n_a}^d + T_s(j)\{g_0\})$. After the unitary accelerating process $U_A(2n_a)$ the wave-packet motional state of the halting-qubit atom will take the form, according to the recursive relations (88)–(94),

$$
\Psi_{2n_a,j}^A(x, t_{2n_a}^d) = \exp[i\varphi_{2n_a}^d(j)][(\Delta x)^2/2\pi]^{1/4} \sqrt{1/(\Delta x)^2 + i\hbar(T_s(j) + 2n_at_a)}/2M \times \exp\{-1/(4(\Delta x)^2 + i\hbar(T_s(j) + 2n_at_a))/2M\} \exp\{iP_{2n_a}^d(x/\hbar)\}
$$

(106)
and the atomic wave-packet product state is given by

$$\Psi_{2n_a,j}^a(x, r, t_{2n_a}) = \Psi_{2n_a,j}^a(x, t_{2n_a}) |\psi_0\rangle,$$

where the end time of the unitary accelerating process is \(t_{2n_a}^a = t_m + 2na_t_a\), and the atomic motion momentum is given by

$$P_{2n_a}^a = P_{2n_a}^d + n_a (\hbar k_0^a + \hbar k_1^a) + n_a (\hbar l_0^a + \hbar l_1^a),$$

and the center-of-mass position \(z_{2n_a}^a(j)\) can be determined from the recursive relations:

$$z_{2k-1}^a(j) = z_{2k-2}^a(j) + \frac{P_{2k}^a}{M} t_a - \frac{\hbar (l_0^a + l_1^a)}{M} \int_{t_{2k-2}^a}^{t_{2k-1}^a} dt' \cos^2 \theta_a(t'),$$

$$z_{2k}^a(j) = z_{2k-1}^a(j) + \frac{P_{2k}^a}{M} t_a - \frac{\hbar (l_0^a + l_1^a)}{M} \int_{t_{2k-1}^a}^{t_{2k}^a} dt' \cos^2 \theta_a(t'),$$

where \(1 \leq k \leq n_a\). The global phase factor \(\exp[i \varphi_{2n_a}^a(j)]\) in Eq. (106) can also be obtained from the recursive relations similar to Eq. (78) and (86). Now one can find from the final motional state \(\Psi_{2n_a,j}^a(x, t_{2n_a}^a)\) of Eq. (106) that the moving distance \(L_A\) of the halting-qubit atom is \(L_A = z_{2n_a}^a(j) - z_0^a(j)\) during the unitary accelerating process, which appears in the accelerating region \([A_L, A_R]\) above. Note that the distance \(L_A\) is the same for each one of these \(m_r\) possible wave-packet motional states.

The unitary accelerating process tells some facts. For the first point, the halting-qubit atom indeed is accelerated by \(n_a (\hbar k_0^a + \hbar k_1^a) + n_a (\hbar l_0^a + \hbar l_1^a)\) and this accelerating process is uniform, that is, the accelerating process is the same for each one of these \(m_r\) possible wave-packet motional states \(\{\Psi_{j}^a(x, t_{2n_a}^d + T_a(j))\}\). Thus, after the unitary accelerating process the atom is accelerated to the velocity \((P_{2n_a}^a/M)\). For the second point, it can be seen from the motional states \(\{\Psi_{2n_a,j}^a(x, t_{2n_a}^a)\}\) of Eq. (106) that in the complex linewidth the imaginary part increases linearly with the time period of the unitary accelerating process and is increased by \(\hbar(2na_t_a)/(2M)\), which is also independent of any index value \(j\), while the real part keeps unchanged in the unitary accelerating process. For the third point, the distances between these \(m_r\) possible wave-packet motional states keep unchanged during the unitary accelerating process. This fact can be deduced from the recursive relations (109a) and (109b) because the motional momentum \(P_l^a (l = 0, 1, 2,..., 2n_a)\), the mixing angles \(\theta_a(t)\) and \(\theta_{ta}(t)\), and the wave numbers \((k_0^a + k_1^a)\) and \((l_0^a + l_1^a)\) all are independent of the index value \(j\). This means that each one of these \(m_r\) possible wave-packet motional states moves the same spatial distance during the unitary accelerating process. Since the distance \(\Delta L(i, j)\) between the \(i\)-th and \(j\)-th \((i < j)\) wave-packet motional states is still given by Eq. (100) and the atomic moving velocity is \((P_{2n_a}^a/M)\)
after the unitary accelerating process, the time difference between the two wave-packet states $\Psi_{2n_a,i}^a(x,t_{2n_a})$ and $\Psi_{2n_a,j}^a(x,t_{2n_a})$ then is given by

$$\Delta T(i,j) = \Delta L(i,j)/(P_{2n_a}/M) = (j - i)\Delta T(P_{2n_d}/P_{2n_a}),$$

(110)

where $i < j; i, j = 1, 2, \ldots, m_r$. It is known that the time difference $\Delta T_0(i,j) = (j - i)\Delta T$ before the unitary accelerating process. Since the atomic velocity $(P_{2n_a}/M)$ after the accelerating process is much greater than the velocity $(P_{2n_d}/M)$ before the accelerating process, the time difference $\Delta T(i,j) << \Delta T_0(i,j)$, indicating that the time differences are compressed greatly for these $m_r$ possible wave-packet motional states after the unitary accelerating process. Then the time-compressing factor for these possible wave-packet motional states after and before the unitary accelerating process $U_A(2n_a)$ can be calculated by

$$R_t = \frac{\Delta T(i,j)}{\Delta T_0(i,j)} = \frac{P_{2n_d}}{P_{2n_d}^2 + n_a(\hbar k_{0\lambda}^2 + \hbar k_{1\lambda}^2) + n_a(\hbar l_{0\lambda}^2 + \hbar l_{1\lambda}^2)}.$$  

(111)

The time-compressing factor $R_t$ is independent of the indices $i$ and $j$. Thus, the time-compressing process is uniform. The time-compressing factor $R_t$ has been obtained in the previous paper [22], where $R_t = (v_0/v)$ and $v_0$ and $v$ are denoted as the atomic moving velocities $P_{2n_d}/M$ and $P_{2n_a}/M$ before and after the unitary accelerating process, respectively.

7. General adiabatic conditions and the error estimation for the decelerating and accelerating processes

The starting point to set up a general adiabatic condition for a basic STIRAP decelerating or accelerating process is to solve the basic equations (23) to find the coefficients $\{a_k(P,t)\}$ or to solve the basic equations (26) to obtain the coefficients $\{b_k(P,t)\}$. Then it is to seek under what experimental conditions a real adiabatic condition for the basic STIRAP decelerating or accelerating process can be sufficiently close to the ideal adiabatic condition (30). This is a routine procedure in quantum mechanics [25]. There are three basic parameters to affect the real adiabatic condition of a STIRAP experiment: the time period of the STIRAP experiment, the Rabi frequencies and the phase-modulation functions of the Raman laser light beams. From the point of view of quantum computation one usually does not expect the quantum control process to consume a long time. However, a long time period of the STIRAP experiment usually can lead to that the adiabatic condition for the STIRAP experiment is met better [30]. If the time period of each basic STIRAP pulse sequence in the STIRAP-based unitary decelerating and accelerating processes is not long enough, then the adiabatic condition could not be met well. Then in this situation one may use jointly the time period, the Rabi frequencies, and even the phase-modulation functions to achieve a better adiabatic condition for these decelerating and accelerating processes. Actually, the Rabi frequencies of the Raman laser light beams are very important to achieve a better adiabatic condition for the STIRAP experiment [15, 18b]. Without losing generality here take the basic STIRAP decelerating sequence (11) as an example to discuss a
The obtained results can be used as well for other basic STIRAP decelerating and accelerating processes. The STIRAP adiabatic conditions have been discussed in detail in many references [15, 16, 17, 18] in the conventional STIRAP experiments without considering explicitly the atomic or molecular momentum distribution. The conventional adiabatic conditions [4, 15, 17, 18] usually are based on the first-order approximation solution to the basic equations similar to the present basic differential equations (26). These adiabatic conditions are usually a qualitative and approximate description to the adiabatic theorem. In the following two strict and different general adiabatic conditions are derived analytically. They are a quantitative description to the adiabatic theorem. The first general adiabatic condition is based on the Dyson series solution (29) of the basic differential equations (26). The second is based on a new method to solve the basic differential equations (26). This new method uses the equivalent transformations to solve the basic differential equations (26). That is, by making repeatedly the equivalent transformations the three basic differential equations (26) are transformed to the three equivalent linear algebra equations. Though the final solution to the basic differential equations (26) obtained with the new method is approximate, the truncation error of the approximation solution can be controlled as desired. The two general adiabatic conditions may be used to set up the conventional STIRAP experiments. Thus, they may be used to design the STIRAP pulse sequence to realize the perfect state (or population) transfer for a quantum ensemble of the atoms or molecules. But their more important application is that they may be used to set up the basic STIRAP unitary decelerating and accelerating processes for a free atom and an atomic or molecular ensemble.

The basic differential equations (26) or their matrix form (28) can be integrated formally. The formal solution to the basic equations (28) may be expressed as the Dyson series (29). Here one needs to use the initial condition of the basic STIRAP decelerating sequence (11). At the initial time $t_0$ of the basic STIRAP decelerating sequence (11) the three-state vector $B(P,t_0) = (b_0(P,t_0), b_+(P,t_0), b_-(P,t_0))^T$ is given by Eq. (39). The initial condition (39) has been used to set up the ideal adiabatic condition (30). For a real adiabatic condition the initial condition may be generally given in (130) below. At first the formal solution (29) may be rewritten as

$$B(P,t) = B(P,t_0) + E_r(P,t)$$

(112)

where $t_0 \leq t \leq t_0 + T$ and $T$ is the time period of the basic STIRAP decelerating process, and the error term $E_r(P,t)$ measures the deviation of a real adiabatic condition from the ideal adiabatic condition and it may be expressed as

$$E_r(P,t) = \left\{ \left( \frac{1}{t} \right) \int_{t_0}^{t} dt_1 M(P,t_1) + \left( \frac{1}{t} \right)^2 \int_{t_0}^{t} \int_{t_0}^{t_1} dt_1 dt_2 M(P,t_1) M(P,t_2) ight. + \left. \left( \frac{1}{t} \right)^3 \int_{t_0}^{t} \int_{t_0}^{t_1} \int_{t_0}^{t_2} dt_1 dt_2 dt_3 M(P,t_1) M(P,t_2) M(P,t_3) + ... \right\} B(P,t_0).$$

(113)
The upper bound of the error term is evaluated accurately below. Denote the maximum norm of the hermitian matrix $M(P, t_n)$ which is given in (28) in the time region $[t_0, t_{n-1}]$ as

$$
\|M(P, t_n)\|_{\text{max}} = \max_{t_0 \leq t_n \leq t_{n-1}} \{\|M(P, t_n)\|\},
$$

where $t_0 < ... < t_n < t_{n-1} < ... < t_1 < t_0 + T$. Obviously, there are the following relations for the maximum norms $\{\|M(P, t_n)\|\}$ :

$$
\|M(P, t_0)\|_{\text{max}} \leq ... \leq \|M(P, t_n)\|_{\text{max}} \leq \|M(P, t_{n-1})\|_{\text{max}}
$$

$$
\leq \|M(P, t_2)\|_{\text{max}} \leq \|M(P, t_1)\|_{\text{max}} = \|M(P, t)\|_{\text{max}}.
$$

(114)

Here the maximum norm $\|M(P, t)\|_{\text{max}}$ is defined as

$$
\|M(P, t)\|_{\text{max}} = \max_{t_0 \leq t \leq t_0 + T} \{\|M(P, t)\|\}.
$$

(115)

Then with the help of (113) and (114) it can turn out that the upper bound of the deviation $E_r(P, t)$ may be determined from

$$
\|E_r(P, t)\| \leq \exp[\|M(P, t)\|_{\text{max}} T] \times \|B^{(1)}(P, t)\|_{\text{max}},
$$

(116)

where the first-order approximation solution $B^{(1)}(P, t)$ to the basic differential equations (26) is given by

$$
B^{(1)}(P, t) = \frac{1}{i} \int_{t_0}^{t} dt_1 M(P, t_1) B(P, t_0), \ (t_0 \leq t \leq t_0 + T),
$$

(117)

while $\|B^{(1)}(P, t)\|_{\text{max}}$ is the maximum norm of the solution $B^{(1)}(P, t)$ in the time region $t_0 \leq t \leq t_0 + T$. This norm $\|B^{(1)}(P, t)\|_{\text{max}}$ is written as

$$
\|B^{(1)}(P, t)\|_{\text{max}} = \sqrt{(\|b_0^{(1)}(P, t)\|^2 + \|b_+^{(1)}(P, t)\|^2 + \|b_-^{(1)}(P, t)\|^2)}_{\text{max}}.
$$

(118)

On the other hand, it follows from the matrix $M(P, t)$ in (28) that the maximum norm $\|M(P, t)\|_{\text{max}}$ is bounded by

$$
\|M(P, t)\|_{\text{max}} \leq \sqrt{\sum_{i,j} |M_{ij}(P, t)|^2} \leq \frac{1}{\sqrt{2}} \{2|\Theta(P, t)| + |\Gamma(P, t)|\}_{\text{max}}.
$$

(119)

The adiabatic condition (116) is strict because it is required that at any instant of time in the whole STIRAP decelerating or accelerating process the deviation from the ideal adiabatic condition (30) be limited within a given small value, that is, the upper bound of the error term $E_r(P, t)$ is less than a given small value at any instant of time. Notice that in theory at the initial time $t_0$ the atom is prepared to be in the trapped state $|g^0(P, t_0)\rangle$ of (19a) completely. If the error term $E_r(P, t)$ is large, then this will mean that during the STIRAP decelerating or accelerating process there is a large probability for the atom to
be excited to the two eigenstates $|g^\pm(P,t)\rangle$ of the instantaneous Hamiltonian $H(P,t)$ of (17). It is known from (19b) that any one of the two eigenstates $|g^\pm(P,t)\rangle$ contains the excited internal state of the atom. Then the atom could be easily affected due to the atomic spontaneous emission if it is in any one of the eigenstates $|g^\pm(P,t)\rangle$. On the other hand, a high probability for the atom to stay in the trapped state $|g^0(P,t)\rangle$ may lead to that the atom is not easily affected by environment and may avoid the spontaneous emission. The adiabatic condition (116) indicates that the probability for the atom to leave the trapped state $|g^0(P,t)\rangle$ may be limited to a small value as desired during the STIRAP decelerating or accelerating process. Therefore, it ensures that the atom is almost completely in the trapped state $|g^0(P,t)\rangle$ during the STIRAP decelerating or accelerating process. The adiabatic condition (116) is more severe than those in the conventional STIRAP experiments [15, 17, 18]. The latter usually require that the probability for the atoms or molecules under investigation in the two eigenstates $|g^\pm(P,t)\rangle$ be much smaller than one. This is a qualitative description for the adiabatic theorem. The present adiabatic condition (116) is closely related to the requirement that Gaussian shape of the Gaussian wave-packet motional state of the decelerated or accelerated atom keep unchanged before and after the basic STIRAP decelerating and accelerating processes. It measures the deviation of a real adiabatic condition from the ideal adiabatic condition (30), while the deviation may occur not only in the two eigenstates $|g^\pm(P,t)\rangle$ but also in the trapped state $|g^0(P,t)\rangle$. The present adiabatic condition (116) limits the upper bound of the deviation to a given small value. This is a quantitative description for the adiabatic theorem. This results in that the present adiabatic condition (116) is more severe.

When the adiabatic condition (116) is met, the error term $||E_r(P,t_0+T)||$ of the final state at the time $t = t_0 + T$ is clearly not more than the upper bound (116) and the real error term $||E_r(P,t_0+T)||$ could be much less than the upper bound (116). It may be required in theory that the real error term $||E_r(P,t_0+T)||$ of the final state be less than some given value which is much less than the upper bound (116). This requirement is not severe in theory with respect to the adiabatic condition (116). It may be met by setting the suitable experimental parameters at the final time $t = t_0+T$ for the STIRAP decelerating or accelerating process. However, in practice the lower bound of the error term $E_r(P,t_0+T)$ of the final state is generally affected by the adiabatic condition (116). If the upper bound (116) is large, then the lower bound of the error term $E_r(P,t_0+T)$ usually is large too.

According to the superposition principle in quantum mechanics in a real adiabatic condition the atomic product state at any instant of time $t$ ($t_0 \leq t \leq t_0 + T$) in the basic STIRAP decelerating process (11) may be calculated from Eq. (12) ($P = P' - \hbar k_0$),

$$|\Psi_r(x,r,t)\rangle = \sum_P \rho(P)\{[A_0^A(P,t) + \delta_0^A(P,t)]|P + \hbar k_0\rangle|g_0\rangle
+ [A_1^A(P,t) + \delta_1^A(P,t)]|P\rangle|e\rangle + [A_2^A(P,t) + \delta_2^A(P,t)]|P - \hbar k_1\rangle|g_1\rangle\}$$

(120)
where the coefficients \( \{ A_k^i(P, t) \} \) for \( k = 0, 1, 2 \) are obtained in the ideal adiabatic condition (30), while the coefficients \( \{ \delta_k^A(P, t) \} \) measure the deviation of the real adiabatic condition from the ideal one. The product state (120) may be rewritten as

\[
|\Psi_r(x, r, t)\rangle = |\Psi_i(x, r, t)\rangle + E_r(x, r, t)
\]

where the wave-packet state \( |\Psi_i(x, r, t)\rangle \) is the atomic state at the time \( t \) in the basic STIRAP decelerating process (11) in the ideal adiabatic condition and it may be written as

\[
|\Psi_i(x, r, t)\rangle = \sum_P \rho(P) \{ A_0^i(P, t)|P + \hbar k_0\rangle|g_0\rangle + A_1^i(P, t)|e\rangle + A_2^i(P, t)|P - \hbar k_1\rangle|g_1\rangle \}
\]

and the error term \( E_r(x, r, t) \) is given by

\[
E_r(x, r, t) = \sum_P \rho(P) \{ \delta_0^A(P, t)|P + \hbar k_0\rangle|g_0\rangle + \delta_1^A(P, t)|e\rangle + \delta_2^A(P, t)|P - \hbar k_1\rangle|g_1\rangle \}.
\]

It turns out in the preceding section 5 that the final state \( |\Psi_r(x, r, t)\rangle \) with \( t = t_0 + T \) in the ideal adiabatic condition is a perfect Gaussian wave-packet state if the initial state of the basic STIRAP decelerating process (11) is a Gaussian wave-packet state. Obviously, it follows from (122) that the probability for the error term \( E_r(x, r, t) \) at any time \( t \) may be calculated by

\[
||E_r(x, r, t)||^2 = \sum_P \rho(P)^2 \left( ||\delta_0^A(P, t)||^2 + ||\delta_1^A(P, t)||^2 + ||\delta_2^A(P, t)||^2 \right).
\]

(Notice that the error probability (121) in the previous versions of this paper which is denoted as \( E_r(P, t) \) is just equal to \( 2||E_r(x, r, t)||^2 \) of (123)). In order to use directly the solution to the basic differential equations (26) or their matrix form (28) to calculate the upper bound of the error term \( E_r(x, r, t) \) one may use the coefficients \( b_0(P, t) \) and \( b_\pm(P, t) \) to express the error probability \( ||E_r(x, r, t)||^2 \) of (123). Notice that there is the unitary transformation between the two three-state vectors \( \{ b_0(P, t), b_+(P, t), b_-(P, t) \}^T \) and \( \{ A_0(P, t), A_1(P, t), A_2(P, t) \}^T \). The three-state vector \( \{ A_0(P, t), A_1(P, t), A_2(P, t) \}^T \) is first converted into the three-state vector \( \{ A_0(P, t), A_1(P, t), A_2(P, t) \}^T \) by the unitary transformation of (15a)-(15c), then into the three-state vector \( \{ a_0(P, t), a_+(P, t), a_-(P, t) \}^T \) by the unitary transformation of (22a)-(22c), and finally into the three-state vector \( \{ b_0(P, t), b_+(P, t), b_-(P, t) \}^T \) by the unitary transformation (25). Thus, under these unitary transformations there is the relation:

\[
(\{ A_0(P, t), A_1(P, t), A_2(P, t) \})^T = U_{Ab}(\{ b_0(P, t), b_+(P, t), b_-(P, t) \})^T \]

(124) where \( U_{Ab} \) is the unitary transformation between the two three-state vectors \( \{ b_0(P, t), b_+(P, t), b_-(P, t) \}^T \) and \( \{ A_0(P, t), A_1(P, t), A_2(P, t) \}^T \). If now the solution to the basic equations (26) in the ideal adiabatic condition is given
by \((b_0^b(P,t), b_1^b(P,t), b_2^b(P,t))^T\), then after the unitary transformation \(U_{AB}\) one obtains the three-state vector \((A_0^b(P,t), A_1^b(P,t), A_2^b(P,t))^T\) of the ideal adiabatic condition and then the state \(|\Psi_t(x,r,t)\rangle\) can be calculated from Eq. (121) by using the three-state vector. If the solution to the basic equations (26) in a real adiabatic condition is given by \((b_0(P,t), b_1^b(P,t), b_2^b(P,t))^T\) with \(b_k(P,t) = b_k^b(P,t) + \delta_k^b(P,t)\) for \(k = 0, +, -\), then after the unitary transformation \(U_{AB}\) one obtains the three-state vector \((A_0^b(P,t), A_1^b(P,t), A_2^b(P,t))^T\) of the real adiabatic condition, where \(A_k(P,t) = A_k^0(P,t) + \delta_k^b(P,t)\). Thus, there is the unitary transformation between the two three-state deviation vectors:

\[
(\delta_0^A(P,t), \delta_1^A(P,t), \delta_2^A(P,t))^T = U_{AB}(\delta_0^b(P,t), \delta_1^b(P,t), \delta_2^b(P,t))^T.
\]

It is well known that the unitary transformation \(U_{AB}\) does not change the norm of the three-state deviation vector \((\delta_0^b(P,t), \delta_1^b(P,t), \delta_2^b(P,t))^T\). This indicates that there is the relation:

\[
|\delta_0^A(P,t)|^2 + |\delta_1^A(P,t)|^2 + |\delta_2^A(P,t)|^2 = |\delta_0^b(P,t)|^2 + |\delta_1^b(P,t)|^2 + |\delta_2^b(P,t)|^2.
\]

This relation leads to that the error probability \(||E_r(x,r,t)||^2\) of (123) may be expressed as

\[
||E_r(x,r,t)||^2 = \sum_{P} |\rho(P)|^2 \{|\delta_0^b(P,t)|^2 + |\delta_1^b(P,t)|^2 + |\delta_2^b(P,t)|^2\}.
\]

It is convenient to calculate the error upper bound \(||E_r(x,r,t)||\) by using the equation (126), since the deviation vector \((\delta_0^b(P,t), \delta_1^b(P,t), \delta_2^b(P,t))^T\) can be obtained conveniently by solving the basic differential equations (26). Thus, an accurate error upper bound \(||E_r(x,r,t)||\) could be obtained directly by computing the equation (126) by using the deviation vector \((\delta_0^b(P,t), \delta_1^b(P,t), \delta_2^b(P,t))^T\) for the basic STIRAP decelerating or accelerating process. Obviously, the three-state deviation vector \((\delta_0^b(P,t), \delta_1^b(P,t), \delta_2^b(P,t))^T\) has the maximum norm or length over the effective momentum distribution region \([P]\) and in the time period \(t_0 \leq t \leq t_0 + T\),

\[
\sqrt{|\delta_0^b(P,t)|^2 + |\delta_1^b(P,t)|^2 + |\delta_2^b(P,t)|^2} \\
\leq \sqrt{|\delta_0^b(P,t)|^2 + |\delta_1^b(P,t)|^2 + |\delta_2^b(P,t)|^2}_{\text{max}}, \text{ for } P \in [P] \text{ and } t_0 \leq t \leq t_0 + T.
\]

Then the upper bound of the error term \(E_r(x,r,t)\) may be determined from

\[
||E_r(x,r,t)|| \leq \sqrt{|\delta_0^b(P,t)|^2 + |\delta_1^b(P,t)|^2 + |\delta_2^b(P,t)|^2}_{\text{max}}, \tag{127}
\]

where the normalization relation \(\sum_P |\rho(P)|^2 = 1\) is used and the truncation error is neglected for any momentum components outside the effective momentum region \([P]\). The inequality (127) is a general adiabatic condition for the basic STIRAP decelerating or accelerating process of a free atom in a wave-packet motional state. There is also a simpler method to obtain the error.
upper bound $||E_r(x,r,t)||$, as stated below. It uses the general adiabatic condition (116). It is known that the formal solution to the basic differential equations (26) or their matrix form (28) may be expressed as (112), where the solution in the ideal adiabatic condition (30) is given by $B(P,t) = B(P,t_0)$, as shown in Eq. (40) in the previous section 4. Then the equation (112) shows that the three-state deviation vector is just $E_r(P,t)$ and hence one has $E_r(P,t) = (\delta_0^b(P,t), \delta_+^b(P,t), \delta_-^b(P,t))^T$. Furthermore, the adiabatic condition (116) and the equation (126) show that there are the relations:

$$||E_r(x,r,t)|| = \left\{ \sum_P |\rho(P)|^2 ||E_r(P,t)||^2 \right\}^{1/2}$$

$$\leq \left\{ \sum_P |\rho(P)|^2 \exp[2(||M(P,t)||_{\max})T] \times ||B^{(1)}(P,t)||_{\max}^2 \right\}^{1/2}$$

$$\leq \exp[(||\dot{M}(P,t)||_{\max})T] \times ||\dot{B}^{(1)}(P,t)||_{\max},$$

where the relation $\sum_P |\rho(P)|^2 = 1$ is used and the truncation error has been neglected for any momentum components outside the effective momentum region $[P]$, and the maximum norms $||M(P,t)||_{\max}$ and $||B^{(1)}(P,t)||_{\max}^2$ are respectively defined as

$$||\dot{M}(P,t)||_{\max} = \max_{P \in [P]} (||M(P,t)||_{\max}),$$

$$||\dot{B}^{(1)}(P,t)||_{\max} = \max_{P \in [P]} ||B^{(1)}(P,t)||_{\max}.$$

The last inequality in (128) is a real adiabatic condition of the basic STIRAP decelerating or accelerating process for a free atom in a wave-packet motional state. It could be useful to design the basic STIRAP decelerating or accelerating process.

As first the adiabatic condition (128) requires one to calculate the norm $(||M(P,t)||_{\max})$ and the first-order approximation solution $B^{(1)}(P,t)$. It is easy to calculate the first-order approximation solution to the basic differential equations (26). Actually, the first-order approximation solution may be obtained from the equation (117). Here for convenience setting the global phases $\gamma(t_0) = \delta(t_0) = 0$ in the basic equations (26) and the phase-modulation functions of the Raman laser light beams to be $\varphi_0(t) = 0$ in Eq. (31) and $\varphi_1(t) = 0$ in Eq. (32). It should be pointed out that the following methods are available as well for the phase-modulation Raman laser light beams. It follows from Eqs. (31) and (32) that

$$\frac{d}{dt} \alpha_p(P,t) = \frac{\Delta P}{M} k_0, \quad \frac{d}{dt} \alpha_s(P,t) = -\frac{\Delta P}{M} k_1.$$

By inserting these two equations into Eqs. (27a)–(27c) one obtains

$$\Omega_\pm(P,t) = \Omega(t) \pm K_0(t) \Delta P, \quad \Theta(P,t) = -\dot{\theta}(t) + i K_1(t) \Delta P, \quad \Gamma(P,t) = \frac{k_0}{M} \Delta P - K_2(t) \Delta P,$$

(129a)

(129b)
where
\[ K_0(t) = \frac{1}{4M}[(k_0 - k_1) + 3(k_0 + k_1)\cos 2\theta(t)] , \]
\[ K_1(t) = \frac{k_0 + k_1}{2M}\sin 2\theta(t) , \quad K_2(t) = \frac{(k_0 + k_1)}{M}\cos^2\theta(t) . \]

The initial condition for the basic equations (26) is given by (39). If the initial condition (39) may be changed to the general form
\[ b_0(P, t) = \exp\left[\frac{i}{\hbar}\left(\frac{(P + \hbar k_0)^2}{2M} + E_0)\right)\cos \theta(t_0)\right] , \quad (130a) \]
\[ b_+ (P, t_0) = b_- (P, t_0) = \frac{1}{\sqrt{2}}\exp\left[\frac{i}{\hbar}\left(\frac{(P + \hbar k_0)^2}{2M} + E_0)\right)\sin \theta(t_0)\right] . \quad (130b) \]

Then in the initial condition (130) the first-order approximation solution to the basic equations (26) for the coefficient \( b_0(P, t) \) may be written as, by integrating by parts the integral (117),
\[ b_0^{(1)}(P, t) = b_0(P, t) - b_0(P, t_0) = C_0^{(1)}(P, t) + C_{0T}^{(2)}(P, t) , \quad (131a) \]
where the main term \( C_0^{(1)}(P, t) \) that is proportional to \( \Theta(P, t)^*/\Omega(t) \) is written as
\[ C_0^{(1)}(P, t) = \sqrt{2}b_+(P, t_0)\frac{\Theta(P, t_0)^*}{\Omega(t)}\exp[i\Delta P \int_{t_0}^{t} dt' K_0(t')][\sin[i \int_{t_0}^{t} dt' \Omega(t')] , \quad (131b) \]
and the secondary term \( C_{0T}^{(2)}(P, t) \) is given by
\[ C_{0T}^{(2)}(P, t) = i\sqrt{2}b_+(P, t_0)\int_{t_0}^{t} dt_1\left[\frac{i}{\Omega(t_1)}\frac{\Theta(P, t_1)^*}{\Omega(t_1)} - \frac{K_0(t_1)\Theta(P, t_1)^*}{\Omega(t_1)}\Delta P\right] \]
\[ \times \exp[i\Delta P \int_{t_0}^{t_1} dt' K_0(t')][\sin[i \int_{t_0}^{t_1} dt' \Omega(t')]]. \quad (131c) \]

The first-order solution for the coefficients \( b_\pm (P, t) \) is given by
\[ b_\pm^{(1)}(P, t) = b_\pm(P, t) - b_\pm(P, t_0) = C_\pm^{(1)}(P, t) + C_{\pm T}^{(2)}(P, t) + F_\pm^{(1)}(P, t) + F_{\pm T}^{(2)}(P, t) , \quad (132a) \]
where the main terms \( C_\pm^{(1)}(P, t) \) are given by
\[ C_\pm^{(1)}(P, t) = \mp i\frac{1}{\sqrt{2}}b_0(P, t_0)\frac{\Theta(P, t_0)}{\Omega(t_0)}\exp[-i\Delta P \int_{t_0}^{t} dt' K_0(t')][\exp[\mp i \int_{t_0}^{t} dt' \Omega(t')]] \]
\[ \pm \frac{i}{\sqrt{2}}b_0(P, t_0)\frac{\Theta(P, t_0)}{\Omega(t_0)} , \quad (132b) \]
and the secondary terms are

\[
\begin{align*}
C_{\pm T}^{(2)}(P, t) &= \pm \frac{1}{\sqrt{2}} b_0(P, t_0) \int_{t_0}^{t} dt_1 \left\{ i \frac{\partial}{\partial t_1} \left( \frac{\Theta(P, t_1)}{\Omega(t_1)} \right) + \frac{K_0(t_1) \Theta(P, t_1)}{\Omega(t_1)} \right\} \Delta P \\
&\quad \times \exp[-i \Delta P \int_{t_0}^{t_1} dt' K_0(t')] \exp[\mp i \int_{t_0}^{t_1} dt' \Omega(t')] \\
F_{\pm}^{(1)}(P, t) &= \pm \frac{1}{4} b_\mp(P, t_0) \frac{\Gamma(P, t)}{\Omega(t)} \exp[\mp i \int_{t_0}^{t} dt' 2\Omega(t')] \mp \frac{1}{4} b_\mp(P, t_0) \frac{\Gamma(P, t_0)}{\Omega(t_0)} , \\
F_{\pm T}^{(2)}(P, t) &= \mp \frac{1}{4} b_\mp(P, t_0) \int_{t_0}^{t} dt_1 \left\{ i \frac{\partial}{\partial t_1} \left( \frac{\Gamma(P, t_1)}{\Omega(t_1)} \right) \right\} \exp[\mp i \int_{t_0}^{t_1} dt' 2\Omega(t')] .
\end{align*}
\]

(132c)

(132d)

(132e)

The dominating terms in the first-order approximation solution of (131a) and (132a) are \( C_{\pm}^{(1)}(P, t) \), which are proportional to \( \Theta(P, t)/\Omega(t) \) and \( b_0(P, t_0) \). It can turn out by integrating by parts the integral (132c) that the terms \( C_{\pm T}^{(2)}(P, t) \) are proportional to \( \Omega(t)^{-2} \). Thus, the terms \( C_{\pm T}^{(2)}(P, t) \) are secondary in the first-order solution for a large Rabi frequency \( \Omega(t) \). Similarly, it can turn out that \( C_{0T}^{(2)}(P, t) \propto b_\mp(P, t_0)\Omega(t)^{-2} \) and \( F_{\pm T}^{(2)}(P, t) \propto b_\mp(P, t_0)\Omega(t)^{-2} \), by integrating by parts the integrals (131c) and (132e), respectively. Thus, these terms \( C_{0T}^{(2)}(P, t) \) and \( F_{\pm T}^{(2)}(P, t) \) are secondary with respect to the terms \( C_0^{(1)}(P, t) \) and \( F_\pm^{(1)}(P, t) \), respectively. On the other hand, the imperfection for the initial conditions \( b_\mp(P, t_0) = b_\pm(P, t_0) \neq 0 \) could mainly affect \( C_0^{(1)}(P, t) \) and \( F_\pm^{(1)}(P, t) \). Its magnitude is approximately proportional to the factors \( |b_\pm(P, t_0)||\Theta(P, t)/\Omega(t) | \) or \( |b_\pm(P, t_0)||\Gamma(P, t)/\Omega(t) | \). Since \( |b_\pm(P, t_0)| \ll |b_0(P, t_0)| \), these terms \( C_0^{(1)}(P, t) \) and \( F_\pm^{(1)}(P, t) \) are secondary with respect to the main terms \( C_{\pm}^{(1)}(P, t) \). Thus, the error upper bound \( ||\hat{B}^{(1)}(P, t)||_{\text{max}} \) in (128) may be determined from the main terms \( c_{\pm}^{(1)}(P, t) \). Now by inserting \( b_0^{(1)}(P, t) \) of (131a) and \( b_\pm^{(1)}(P, t) \) of (132a) into (118) it can be found that the norm \( ||B^{(1)}(P, t)|| \) is bounded by

\[
||B^{(1)}(P, t)|| \leq \frac{|\Theta(P, t)|}{\Omega(t)} + \frac{|\Theta(P, t_0)|}{\Omega(t_0)}
\]

\[
= \frac{\sqrt{\dot{\theta}(t)^2 + K_1(t)^2|\Delta P|^2}}{\Omega(t)} + \frac{\sqrt{\dot{\theta}(t_0)^2 + K_1(t_0)^2|\Delta P|^2}}{\Omega(t_0)} ,
\]

(133)

where those secondary terms of the first-order approximation solution are neglected and only the main terms \( C_{\pm}^{(1)}(P, t) \) of (132b) are used and \( |b_0(P, t_0)| \leq 1 \) is also used. It is known that the initial mixing angle \( \theta(t_0) \to 0 \), as can be seen in (35). Note that \( K_1(t) = (k_0 + k_1) \sin \theta(t)/(2M) \) and the time derivative \( \dot{\theta}(t) \) of the mixing angle is given by

\[
\dot{\theta}(t) = \frac{\dot{\Omega}_p(t) \cos \theta(t) - \dot{\Omega}_a(t) \sin \theta(t)}{\Omega(t)} .
\]
If the Rabi frequencies $\Omega_p(t)$ and $\Omega_s(t)$ are chosen suitably in experiment such that at the initial and final times the mixing angle satisfies the relations:

$$\tan \theta(t_0) = \Omega_p(t_0)/\Omega_s(t_0) \approx \dot{\Omega}_p(t_0)/\dot{\Omega}_s(t_0) \rightarrow 0,$$

$$\dot{\Omega}_p(t_0 + T) \cos \theta(t_0 + T) - \dot{\Omega}_s(t_0 + T) \sin \theta(t_0 + T) \rightarrow 0,$$

then the time derivative $\dot{\theta}(t_0) \approx 0$ and $\dot{\theta}(t_0 + T) \approx 0$. On the other hand, the momentum distribution satisfies $|\Delta P| \leq \Delta P_M/2$ for a momentum wave-packet state with an effective momentum bandwidth $\Delta P_M$. Therefore, at the initial time $|K_1(t_0)|\Delta P \leq |K_1(t_0)|\Delta P_M/2 = \Delta P_M(k_0 + k_1)\sin 2\theta(t_0)/(4M) \approx 0$ if the momentum wave-packet state has a finite wave-packet spread. Then in these conditions the error upper bound $||\hat{B}^{(1)}(P,t)||_{\text{max}}$ may be determined from, by neglecting the second term on the right side of (133),

$$||\hat{B}^{(1)}(P,t)|| \leq \left\{ \frac{\sqrt{\theta(t)^2 + (\Delta P_M)^2(k_0 + k_1)^2 \sin^2 2\theta(t)/(16M^2)}}{\Omega(t)} \right\}_{\text{max}}. \quad (134)$$

Here the subscript ‘$\text{max}$’ means that the function on the right-hand side of (134) is taken as the maximum value in the time period $t_0 \leq t \leq t_0 + T$ of the STIRAP process. The first-order approximation adiabatic condition is that the maximum value of the function on the right-hand side of (134) is controlled to be smaller than some desired small value. (Notice that this first-order adiabatic condition (134) is slightly different from that one (128a) in the previous versions of this paper). As shown below, in the initial and final time periods of the STIRAP process the adiabatic condition (134) still may be met even if the Rabi frequency $\Omega(t)$ is small in these time periods. This is a global adiabatic condition, since it is involved in the whole time period of the STIRAP process. Here the global adiabatic condition has a different definition from the conventional one in Ref. [4]. In the conventional STIRAP experiments [4, 15, 17, 18] the (local) adiabatic condition is defined as that at any instant of time of the STIRAP process the population or probability in the two eigenstates $|g^{±}(P,t)\rangle$ that contain the excited internal state is much smaller unity. This is approximately equivalent to the first inequality of (133) for the first-order approximation, where the inequality symbol ‘$\leq$’ is replaced with ‘$<<$’ . On the other hand, according to (119) the maximum matrix norm $||M(P,t)||_{\text{max}}$ is determined from

$$||M(P,t)|| \leq \frac{1}{\sqrt{2}} \{2|\Theta(P,t)| + |\Gamma(P,t)| \}$$

$$\leq \frac{1}{\sqrt{2}} \left\{ 2\sqrt{|\theta(t)|^2_{\text{max}} + (k_0 + k_1)^2(\Delta P_M)^2/(16M^2)} + \frac{(\Delta P_M)}{2M} \max(k_0, k_1) \right\} \quad (135)$$

where $|\theta(t)|_{\text{max}}$ is the maximum value of $|\theta(t)|$ in the whole STIRAP decelerating or accelerating process. After the upper bound $||\hat{B}^{(1)}(P,t)||_{\text{max}}$ and maximum norm $||M(P,t)||_{\text{max}}$ are determined from (134) and (135), respectively, one
may determine the upper bound of the error term $E_r(x,r,t)$ from (128). It can be seen from (134), (135), and (128) that the adiabatic condition (128) may be better satisfied for a small time derivative $\dot{\theta}(t)$, a short time period $T$, a large Rabi frequency $\Omega(t)$, and a narrow momentum wave-packet state ($\Delta P_M$ is small). If the time derivative $\dot{\theta}(t)$ is smaller, then the time period $T$ usually is larger. Thus, there is a compromise between the settings of the time derivative $\dot{\theta}(t)$ and the time period $T$ in experiment. Obviously, one has the relation for any basic STIRAP decelerating or accelerating process:

$$\theta(t_0 + T) - \theta(t_0) = \int_{t_0}^{t_0 + T} dt' \dot{\theta}(t') = \pi/2.$$  

This relation may be used to determine the time period $T$ if one knows the time derivative $\dot{\theta}(t)$ of the mixing angle.

It seems that the adiabatic conditions (116) and (128) could not be better satisfied in the initial and final time periods, since the Rabi frequency $\Omega(t)$ takes a smaller value in these time periods. It is well known that any STIRAP experiment requires that at the initial and final time the mixing angle $\theta(t)$ satisfy the constraint conditions:

$$\theta(t_0) \rightarrow 0, \theta(t_0 + T) \rightarrow \pi/2.$$  

The two constraint conditions are compatible with the adiabatic conditions (116) and (128). This can be seen from (134). At the initial time period the Rabi frequency $\Omega(t)$ takes generally a smaller value, but at the same time the mixing angle $\theta(t_0) \rightarrow 0$ and its time derivative $\dot{\theta}(t_0)$ may be set to a value close to zero, leading to that the value $||B^{(1)}(P,t)||$ may be kept at a smaller value at the initial time period. At the final time the mixing angle $\theta(t_0 + T) \rightarrow \pi/2$ or $\sin^2 2\theta(t_0 + T) \rightarrow 0$ and the time derivative $\dot{\theta}(t_0 + T)$ also may be set to a value close to zero. Then the value $||B^{(1)}(P,t)||$ still may be kept at a smaller value, although the Rabi frequency $\Omega(t)$ takes a smaller value at the final time period. These results show that in theory the adiabatic conditions (116) and (128) still may be met in the initial and final time periods as long as the Raman laser light beams of the STIRAP experiment are suitably designed. The adiabatic conditions (116) and (128) could be better used for a conventional three-state STIRAP experiment that uses a pair of copropagating Raman laser light beams and those STIRAP-based decelerating and accelerating processes of the atomic or molecular systems with a narrow momentum distribution.

The deviation $E_r(P,t)$ of (113) and its upper bound (116) are obtained for a single basic STIRAP decelerating sequence (11). If a unitary decelerating process consists of $n_d$ pairs of the basic STIRAP decelerating sequences (11) and (63), then the total deviation generated in the unitary decelerating process is bounded by

$$||E_r(P,t)|| \leq n_d \exp[[||\hat{M}_k(P,t)||_{\text{max}}]T] \times ||\hat{B}^{(1)}_k(P,t)||_{\text{max}}$$

$$+ n_d \exp[[||\hat{M}_l(P,t)||_{\text{max}}]T] \times ||\hat{B}^{(1)}_l(P,t)||_{\text{max}}, \quad (136)$$
where the subscript \( k \) marks the basic STIRAP decelerating sequence (11) that uses a pair of the Raman laser light beams with the Rabi frequency \( \Omega(t) \), the mixing angle \( \theta(t) \), the wave numbers \( k_1 \) and \( k_2 \), and the carrier frequencies \( \omega_{01} \) and \( \omega_{02} \), while the subscript \( l \) denotes the basic STIRAP decelerating sequence (63) that may use another pair of the Raman laser light beams with the Rabi frequency \( \Omega_l(t) \), the mixing angle \( \theta_l(t) \), the wave numbers \( k_{l1} \) and \( k_{l2} \), and the carrier frequencies \( \omega_{01}^l \) and \( \omega_{02}^l \). Both the basic STIRAP decelerating processes have the same time period \( T \).

The upper bound of the total deviation \( E_r(P,t) \) on the right-hand side of the inequality (136) for the unitary decelerating or accelerating process may be controlled by setting suitably the experimental parameters of the Raman laser light beams, which include the Rabi frequencies \( \Omega(t) \) and \( \Omega_l(t) \), the mixing angles \( \theta(t) \) and \( \theta_l(t) \), and the time derivatives \( \dot{\theta}(t) \) and \( \dot{\theta}_l(t) \), and so on.

The basic STIRAP decelerating or accelerating process is a time-dependent unitary quantum dynamical problem from the viewpoint of quantum mechanics. The three-state basic STIRAP decelerating or accelerating process is a quite simple unitary dynamical process, but it can describe completely the complex STIRAP-based unitary decelerating and accelerating processes of a free atom. It is relatively simple to solve approximately such a unitary dynamical problem as the three-state basic STIRAP decelerating or accelerating process in quantum mechanics, although this problem is time-dependent and it is difficult to solve exactly the basic differential equations (26) except for some special cases [39]. For example, one may use the conventional methods of successive approximations [33] to solve these basic differential equations approximately when the Rabi frequencies \( \Omega(t) \) is large. As mentioned before, the Dyson series solution (29) to the basic differential equations (26) or (28) may be used to calculate the deviation of a real adiabatic condition from the ideal adiabatic condition. The problem to be answered is that one needs to calculate how many leading terms in the Dyson series (29) so that the result obtained is enough accurate. The leading term number is dependent upon the desired error value and the maximum norm of the matrix \( \hat{M}(P,t) \) in (28). If one uses the first \( n \) terms on the right-hand side of (113) to calculate the error term \( E_r(P,t) \), then the residual term may be given by

\[
R(P,t) = \left\{ \left( \frac{1}{i} \right)^{n+1} \int_{t_0}^{t} \int_{t_0}^{t_1} ... \int_{t_0}^{t_n} dt_1 dt_2 ... dt_{n+1} M(P,t_1)M(P,t_2)...M(P,t_{n+1}) \right. \\
+ \left( \frac{1}{i} \right)^{n+2} \int_{t_0}^{t} \int_{t_0}^{t_1} ... \int_{t_0}^{t_{n+2}} dt_1 dt_2 ... dt_{n+2} M(P,t_1)M(P,t_2)...M(P,t_{n+2}) \right. \\
+ \ldots \ldots \} B(P,t_0). \tag{137}
\]

Then the residual term \( R(P,t) \) is bounded by

\[
\| R(P,t) \| \leq \sum_{k=n+1}^{\infty} \frac{(t-t_0)^k}{k!} (\| \hat{M}(P,t) \|_{\text{max}})^k.
\]
\[
\leq \frac{(||\hat{M}(P,t)||_{\text{max}}(t-t_0))^{n+1}}{(n+1)!} \exp[||\hat{M}(P,t)||_{\text{max}}(t-t_0)]
\]
where \( ||B(P,t_0)|| = 1 \) is used. Now \((n+1)! \approx \sqrt{2\pi(n+1)}(n+1)/e \)\(^{n+1}\). Then the upper bound of the residual term is determined from
\[
||R(P,t)|| \leq \frac{(||\hat{M}(P,t)||_{\text{max}}(t-t_0))^{n+1}}{(n+1)!e} \exp[||\hat{M}(P,t)||_{\text{max}}(t-t_0)]/2\sqrt{\pi(n+1)}
\]
If \((||\hat{M}(P,t)||_{\text{max}}(t-t_0)) < (n+1)\), then \(\exp[||\hat{M}(P,t)||_{\text{max}}(t-t_0)] < e\). Thus, if \((n+1)/e > ||\hat{M}(P,t)||_{\text{max}}(t-t_0)\exp[||\hat{M}(P,t)||_{\text{max}}(t-t_0)]\), then the residual term \(R(P,t)\) is exponentially small. Suppose that the upper bound of the error term \(E_r(P,t)\) is set to a given value \(\varepsilon_r : ||E_r(P,t)|| \leq \varepsilon_r\). Then this requires the residual term to satisfy \(||R(P,t)|| < \varepsilon_r\). The condition \(||R(P,t)|| < \varepsilon_r\) can be easily met by setting a minimum integer \(n\) such that
\[
\frac{1}{\sqrt{2\pi(n+1)!}} \left\{ ||\hat{M}(P,t)||_{\text{max}}T \right\}^{n+1} \exp[||\hat{M}(P,t)||_{\text{max}}T] < \varepsilon_r.
\]
Once the minimum integer \(n\) is determined, one may use the first \(n\) terms on the right-hand side of \(113\) to calculate the error term \(E_r(P,t)\) and its upper bound, while the residual term \(R(P,t)\) does not affect significantly the final result.

It can be seen from \(133\) that the first-order approximation solution shows that the error upper bound \(||B^{(1)}(P,t)||\) is mainly dependent upon the parameter \(||\Theta(P,t)||/\Omega(t)\) and almost independent of the parameters \(\Gamma(P,t)\) and \(K_0(t)\). Actually, in the first-order approximation solution the parameter \(\Gamma(P,t)\) appears in the secondary terms \(F_{\pm}^{(1)}(P,t)\) and \(F_{\pm}^{(2)}(P,t)\) and \(K_0(t_1)\) in the secondary terms \(C_{\pm}^{(2)}(P,t)\) (these parameters may appear in the phase factors, and if so, they do not make a contribution to the error upper bound). It is known that the parameter \(\Theta(P,t) = \dot{\Theta}(t) + i\frac{1}{2}(\Delta P/M)(k_0 + k_1) \sin 2\Theta(t)\). If the two Raman laser light beams are copropagating, then the wave-number sum \((k_0 + k_1)\) will be changed to the wave-number difference \((k_0 - k_1)\) in the parameter \(\Theta(P,t)\). Then the effect of the momentum distribution on the error upper bound \(||B^{(1)}(P,t)||\) will be greatly weakened and the momentum distribution will become a higher-order effect on the STIRAP state transfer. Therefore, in this sense the copropagating Raman laser light beams used to construct the STIRAP pulse sequence may be better than the counterpropagating ones to realize the perfect STIRAP state transfer. Unlike the parameter \(\Theta(P,t)\) these two parameters \(\Gamma(P,t)\) and \(K_0(t)\) are dependent on both the wave-number sum and difference. That the momentum distribution affects the STIRAP state transfer is mainly through the parameters \(\Gamma(P,t)\) and \(K_0(t)\), which appear in the higher-order terms in the Dyson series \(29\), if the copropagating Raman laser light beams are used in the STIRAP state transfer. In fact, the maximum norm \(||\hat{M}(P,t)||_{\text{max}}\) determined from \(135\) shows that it is proportional to \(||\Gamma(P,t)||\).
These parameters make an important effect for the momentum distribution on the STIRAP state transfer. Thus, that the conventional three-state STIRAP experiments [4, 15] use the copropagating Raman laser light beams is favorable for the perfect state transfer and may minimize the effect of the momentum distribution of the atomic and molecular systems under investigation on the perfect state transfer. However, in the basic STIRAP decelerating or accelerating process the counterpropagating Raman laser light beams are generally used so that the fast moving atom can be decelerated or accelerated more efficiently. On the other hand, the first-order solution (117) could not exactly account for the momentum distribution in any case that either copropagating or counterpropagating laser light beams is used in the STIRAP experiments. The adiabatic conditions (116) and (128) are accurate, but due to that there is an exponential correction factor in (116) and (128) they could not be met for a broad momentum distribution. A broad momentum distribution is often met in an atomic or molecular quantum ensemble. It is necessary to consider the effect of the momentum distribution when these physical ensembles are decelerated (or accelerated) by the STIRAP decelerating (or accelerating) pulse sequence. Thus, it is necessary to find a more useful adiabatic condition that can account for the effect of a broad momentum distribution on the STIRAP state transfer.

It is still complex to use the Dyson series solution (29) to calculate the error term $E_r(P,t)$ of (113) and its upper bound. In the following an equivalent transformation method based on the integration by parts to solve the basic differential equations (26) is proposed so that the error term $E_r(P,t)$ of (113) and its upper bound can be obtained conveniently in a high accuracy. On the other hand, by solving the basic differential equations (26) to obtain an enough accurate solution one may further use the solution to calculate conveniently the time evolution process for the basic STIRAP decelerating and accelerating processes for a free atom. Generally, it is quite inconvenient to calculate the time evolution process of an atomic decelerating or accelerating process by directly solving the Schrödinger equation. The present scheme is convenient to calculate the time evolution process of the basic STIRAP decelerating or accelerating process because it does not solve directly the original Schrödinger equation but solves the three first-order differential equations (26) that are equivalent to and much simpler than the original Schrödinger equation. The equivalent transformation method to solve the basic equations (26) is based on the fact that the Rabi frequency $\Omega(t)$ may be set to a large value in experiment. Though the solution to the basic equations (26) obtained by this method is approximate, the truncation error of the solution can be controlled as expected. The procedure to solve the basic equations (26) with the equivalent transformation method may be described below. By integrating the basic differential equations (26) one obtains the equivalent integral equations:

$$b_0(P,t) - b_0(P,t_0) = \frac{1}{\sqrt{2}} \int_{t_0}^{t} dt_1 \{b_+(P,t_1)\Theta(P,t_1)^*$$
where the five amplitudes are given by
\[ \Gamma_{+1}(P, t) = \frac{1}{4} \frac{\Gamma(P, t)}{\Omega(t)}, \quad \Gamma_{-1}(P, t) = \frac{1}{\sqrt{2}} \frac{\Theta(P, t)}{\Omega(t)}, \]
\[ \Gamma_{1}(P, t) = \frac{1}{8} \frac{\Gamma(P, t)^2 + 4\Theta(P, t)^2}{\Omega(t)}. \]

Hereafter the initial condition (39) is used for convenience. The equivalent transformation method is that by the integration by parts and another transformation (see below) the integral equations (141) may be approximately reduced to the three linear algebra equations. These three linear algebra equations are equivalent to the original integral equations (141) if the initial condition (39) is taken into account and when the truncation error can be neglected. At the first step of the equivalent transformation method the integrals on the right-hand sides of (141) are calculated by the integration by parts. Then the initial condition (39) is used to simplify the calculated results. If there are the time derivatives of the variables \( b_0(P, t) \) and \( b_{\pm}(P, t) \) in the integrands after the integration by parts, then one may use the basic differential equations (26) to replace these time derivatives. As an example, by integrating by parts the equation (141b) for the variable \( b_+(P, t) \) and then using the basic differential equations (26) and the initial condition (39), one can obtain the following equation:

\[
\begin{aligned}
&b_+(P, t) - b_{\pm}(P, t) = -i \frac{1}{2} \int_{t_0}^{t} dt_1 \{ b_+(P, t_1) \Gamma(P, t_1) \exp[i \int_{t_0}^{t_1} dt' \Theta(P, t')] \exp[-i \int_{t_0}^{t_1} dt' \Omega(t')] \}, \\
&- \frac{1}{\sqrt{2}} \int_{t_0}^{t} dt_1 \{ b_0(P, t_1) \Theta(P, t_1) \exp[-i \int_{t_0}^{t_1} dt' \Omega(t')] \}.
\end{aligned}
\]
\[ \Theta^-_{+1}(P,t) = \frac{1}{2} \left\{ \left( 1 + \frac{1}{2} \frac{\partial}{\partial t} \right) \Gamma_{+1}(P,t) \right\} + \left( \frac{1}{2} \frac{\Gamma(P,t)}{\Omega(t)} \right)^2 \],

\[ \Theta^0_{+1}(P,t) = \frac{1}{\sqrt{2}} \left\{ \frac{\partial}{\partial t} \left( \frac{\Theta(P,t)}{\Omega(t)} \right) + \frac{1}{4} \frac{\Gamma(P,t)\Theta(P,t)}{\Omega(t)} + \frac{K_0(t)\Theta(P,t)}{\Omega(t)} \right\} \Delta P. \]

The equation (142) is almost completely equivalent to the original equation (141b) for the variable \( b_+(P,t) \). The unique difference between the two equations is that the equation (142) uses the initial condition (39), while the original equation does not. The first four terms on the right-hand side of (142) may be considered as the main terms, since these terms have a greater contribution to the solution \( b_+(P,t) \) of (142). On the other hand, each one of the last two integrals on the right-hand side of (142) contains the integrands with the largely oscillatory phase factor \( \exp[-i \int_{\mathbf{t}_0}^{\mathbf{t}_1} dt' \Omega(t')] \) or \( \exp[-i \int_{\mathbf{t}_0}^{\mathbf{t}_1} dt' 2\Omega(t')] \).

These largely oscillatory phase factors make the two integrals secondary in the solution (142). This can be seen by integrating by parts the two integrals once again. If these two integrals are neglected, then one obtains the first-order approximation solution to the original equation (141b) for the variable \( b_+(P,t) \), since all the five amplitudes including \( \Gamma_{+1}(P,t) \), \( \Gamma^0_{+1}(P,t) \), etc., appearing in the equation (142) are inversely proportional to the Rabi frequency \( \Omega(t) \).

One may further obtain a better approximation solution than the first-order one. This can be done by integrating by parts the last two integrals on the right-hand side of (142) again. However, the fourth term (or the first integral) contains the solution \( b_+(P,t) \) itself on the right-hand side of (142). While one may substitute the first-order approximation solution of \( b_+(P,t) \) into the integral to obtain a better approximation, the calculation process becomes so complex that one can only obtain a lower-order approximation solution. In order to avoid this complex one may make a transformation on the solution \( b_+(P,t) \) to cancel the integral before integrating by parts the last two integrals. This transformation is given by

\[ \hat{b}_+(P,t) = b_+(P,t) \exp[-i \int_{\mathbf{t}_0}^{\mathbf{t}} dt_1 \Gamma^+_{+1}(P,t_1)]. \] (143)

This transformation is the key point to the present equivalent transformation method to solve the basic differential equations (26). By this transformation and the initial condition (39) the transformed solution \( \hat{b}_+(P,t) \) may be written as

\[ \hat{b}_+(P,t) = b_-(P,t) \hat{\Gamma}_+^-(P,t) \exp[-i \int_{\mathbf{t}_0}^{\mathbf{t}} dt' 2\Omega(t')] \]

\[ + i b_0(P,t) \hat{\Gamma}_+^{01}(P,t) \exp[-i \Delta P \int_{\mathbf{t}_0}^{\mathbf{t}} dt' K_0(t')] \exp[-i \int_{\mathbf{t}_0}^{\mathbf{t}} dt' \Omega(t')] \]

\[ - i b_0(P,t_0) \hat{\Gamma}_+^{01}(P,t_0) + i \int_{\mathbf{t}_0}^{\mathbf{t}} dt_1 \{ b_-(P,t_1) \hat{\Theta}_+^-(P,t_1) \exp[-i \int_{\mathbf{t}_0}^{\mathbf{t}_1} dt' 2\Omega(t')] \} \]
where the amplitudes satisfy the recursive relations:

\[ \hat{\Gamma}^{-1}_{+1}(P, t) = \Gamma^{-1}_{+1}(P, t) \exp[-i \int_{t_0}^{t} dt_1 \Gamma^+_{1+1}(P, t_1)] \],

\[ \hat{\Gamma}^{0}_{+1}(P, t) = \Gamma^{0}_{+1}(P, t) \exp[-i \int_{t_0}^{t} dt_1 \Gamma^+_{1+1}(P, t_1)] \],

\[ \hat{\Theta}^{-1}_{+1}(P, t) = [\Theta^{-1}_{+1}(P, t) + \Gamma^{-1}_{+1}(P, t) \Gamma^+_{1+1}(P, t)] \exp[-i \int_{t_0}^{t} dt_1 \Gamma^+_{1+1}(P, t_1)] \],

\[ \hat{\Theta}^{0}_{+1}(P, t) = [\Theta^{0}_{+1}(P, t) - \Gamma^{0}_{+1}(P, t) \Gamma^+_{1+1}(P, t)] \exp[-i \int_{t_0}^{t} dt_1 \Gamma^+_{1+1}(P, t_1)] \].

The transformation (143) and the integration by parts may be called the equivalent transformations as they do not generate any error term in these transformation processes. The transformation (143) does not improve essentially the first-order approximation solution, but it does simplify greatly the calculation process to further obtain a higher-order approximation solution. Now by integrating by parts the last two integrals on the right-hand side of (144). The amplitudes of the solution \( \hat{b}^1_{+1}(P, t) \) are related to those amplitudes of the solution \( \hat{b}^1_{+1}(P, t) \) of (142) by the recursive relations:

\[ \hat{\Gamma}^{-1}_{+1}(P, t) = \Gamma^{-1}_{+1}(P, t) \exp[-i \int_{t_0}^{t} dt_1 \Gamma^+_{1+1}(P, t_1)], \]

\[ \hat{\Gamma}^{0}_{+1}(P, t) = \Gamma^{0}_{+1}(P, t) \exp[-i \int_{t_0}^{t} dt_1 \Gamma^+_{1+1}(P, t_1)], \]

\[ \hat{\Theta}^{-1}_{+1}(P, t) = [\Theta^{-1}_{+1}(P, t) + \Gamma^{-1}_{+1}(P, t) \Gamma^+_{1+1}(P, t)] \exp[-i \int_{t_0}^{t} dt_1 \Gamma^+_{1+1}(P, t_1)], \]

\[ \hat{\Theta}^{0}_{+1}(P, t) = [\Theta^{0}_{+1}(P, t) - \Gamma^{0}_{+1}(P, t) \Gamma^+_{1+1}(P, t)] \exp[-i \int_{t_0}^{t} dt_1 \Gamma^+_{1+1}(P, t_1)]. \]

Now there is not any integral containing the solution \( \hat{b}^1_{+1}(P, t) \) itself on the right-hand side of (144). The amplitudes of the solution \( \hat{b}^1_{+1}(P, t) \) are related to those amplitudes of the solution \( \hat{b}^1_{+1}(P, t) \) of (142) by the recursive relations:

\[ b^2_{+1}(P, t) \equiv \hat{b}^1_{+1}(P, t) = b_{-}(P, t) \Gamma_{+2}^{-1}(P, t) \exp[-i \int_{t_0}^{t} dt' 2\Omega(t') \]

\[ + i b_{0}(P, t) \Gamma_{+2}^{0}(P, t) \exp[-i \Delta P \int_{t_0}^{t} dt' K_0(t')] \exp[-i \int_{t_0}^{t} dt' \Omega(t')] \]

\[ - i b_{0}(P, t_0) \Gamma_{+2}^{0}(P, t_0) + i \int_{t_0}^{t} dt_1 \{ b^2_{+1}(P, t_1) \Gamma^+_{1+1}(P, t_1) \}

\[ + i \int_{t_0}^{t} dt_1 \{ b_{-}(P, t_1) \Theta_{+2}^{-1}(P, t_1) \exp[-i \int_{t_0}^{t_1} dt' \Omega(t')] \}

\[ + \int_{t_0}^{t} dt_1 \{ b_{0}(P, t_1) \Theta_{+2}^{0}(P, t_1) \exp[-i \Delta P \int_{t_0}^{t_1} dt' K_0(t')] \exp[-i \int_{t_0}^{t_1} dt' \Omega(t')] \} \]

where the amplitudes satisfy the recursive relations:

\[ \Gamma^{+2}_{+1}(P, t) = \hat{\Gamma}^{+2}_{+1}(P, t) - \frac{1}{2} \frac{\hat{\Theta}^{+1}_{+1}(P, t)}{\Omega(t)}, \]

\[ \Gamma^{0}_{+2}(P, t) = \hat{\Gamma}^{0}_{+1}(P, t) + \frac{\hat{\Theta}^{0}_{+1}(P, t)}{\Omega(t)}. \]
The solutions $\hat{b}_k^k(P,t)$ to the transformed solution $\hat{b}_k^k(P,t)$ may be generally given by

$$\hat{b}_k^k(P,t) = b_k^k(P,t) \exp[-i \int_{t_0}^{t} dt_1 \Gamma_k^+(P,t_1)], \quad k = 1, 2, \ldots$$

The solutions $\hat{b}_k^k(P,t)$ and $b_k^k(P,t)$ are called the $k-$order exact solutions to the equation (141b) for the variable $b_+(P,t)$.

From the solution $\hat{b}_k^k(P,t)$ to the transformed solution $\hat{b}_k^k(P,t)$ the amplitudes \{\Gamma_{\alpha k}^+(P,t), \Theta_{\alpha k}^+(P,t)\} (\alpha = 0, -)$ of the solution $b_k^k(P,t)$ are transformed to the amplitudes \{\Gamma_{\alpha k}^+(P,t), \Theta_{\alpha k}^+(P,t)\} of the solution $\hat{b}_k^k(P,t)$ according to the recursive equations (145) if in the recursive relations (145) one makes the following replacements: \(\hat{\Gamma}_{\alpha 1}^+(P,t) \leftrightarrow \Gamma_{\alpha 1}^+(P,t), \quad \hat{\Theta}_{\alpha 1}^+(P,t) \leftrightarrow \Theta_{\alpha 1}^+(P,t), \quad \hat{\Gamma}_{\alpha k}^+(P,t) \leftrightarrow \Gamma_{\alpha k}^+(P,t), \quad \hat{\Theta}_{\alpha k}^+(P,t) \leftrightarrow \Theta_{\alpha k}^+(P,t),\) for \(\alpha = 0, -\) and \(\Gamma_{\alpha 1}^+(P,t) \leftrightarrow \Gamma_{\alpha 1}^+(P,t).\) On the other hand, from the solution $\hat{b}_k^k(P,t)$ to the solution $b_k^{k+1}(P,t)$ the recursive relations for their amplitudes are still given by (147) except the relation (147b), in which one needs to make the following replacements: \(\hat{\Gamma}_{\alpha 1}^+(P,t) \leftrightarrow \Gamma_{\alpha 1}^+(P,t), \quad \hat{\Theta}_{\alpha 1}^+(P,t) \leftrightarrow \Theta_{\alpha 1}^+(P,t), \quad \Gamma_{\alpha 1}^+(P,t) \leftrightarrow \Gamma_{\alpha 1}^+(P,t), \quad \Theta_{\alpha 1}^+(P,t) \leftrightarrow \Theta_{\alpha 1}^+(P,t), \quad \Gamma_{\alpha 1}^+(P,t) \leftrightarrow \Gamma_{\alpha 1}^+(P,t), \quad \Theta_{\alpha 1}^+(P,t) \leftrightarrow \Theta_{\alpha 1}^+(P,t), \quad \Gamma_{\alpha k}^+(P,t) \leftrightarrow \Gamma_{\alpha k}^+(P,t), \quad \Theta_{\alpha k}^+(P,t) \leftrightarrow \Theta_{\alpha k}^+(P,t),\) for \(\alpha = 0, -\) and \(\Gamma_{\alpha 1}^+(P,t) \leftrightarrow \Gamma_{\alpha 1}^+(P,t).\)
and $\Gamma_{k+1}^{+}(P, t) \leftrightarrow \Gamma_{k+1}^{-}(P, t)$. The relation (147b) is modified to the form

$$
\Gamma_{k+1}^{+}(P, t) = -\frac{1}{4} \frac{\Gamma(P, t)\hat{\Theta}_{k}^{-}(P, t)}{\Omega(t)} - \frac{1}{4} \frac{\Gamma(P, t)\hat{\Theta}_{k}^{0}(P, t)}{\Omega(t)}
$$

$$
+ \frac{1}{\sqrt{2}} \frac{\Theta(P, t)\hat{\Theta}_{k+1}^{0}(P, t)}{\Omega(t)} \mu_{k} \exp\left[ i \int_{t_0}^{t} dt_{1} \Gamma_{k+1}^{+}(P, t_{1}) \right]
$$

$$
\times \exp\left[ i \int_{t_0}^{t} dt_{1} \Gamma_{k}^{-}(P, t_{1}) \right] \exp\left[ i \int_{t_0}^{t} dt_{1} \Gamma_{k-1}^{+}(P, t_{1}) \right] \exp\left[ i \int_{t_0}^{t} dt_{1} \Gamma_{k+1}^{+}(P, t_{1}) \right].
$$

It can be found that after integrating by parts the last two integrals of the solution $\hat{b}^{k}_{+}(P, t)$ many times, the two integrals become less and less important in the solution. Actually, it can turn out that the amplitudes $\hat{\Theta}_{k}^{-}(P, t)$ and $\hat{\Theta}_{k}^{0}(P, t)$ of the solution $\hat{b}^{k}_{+}(P, t)$ is inversely proportional to the $k$-th power of the Rabi frequency $\Omega(t)$, that is, $\hat{\Theta}_{k}^{-}(P, t) \propto \Omega(t)^{-k}$ and $\hat{\Theta}_{k}^{0}(P, t) \propto \Omega(t)^{-k}$. Thus, by making only a few equivalent transformations of the integration by parts one can obtain a highly accurate solution to the original equation (141b) for the variable $b_{+}(P, t)$ even if the last two integrals are neglected.

Now it is easy to obtain the first-order approximation solution to the original equation (141b) for the variable $b_{+}(P, t)$ from the equation (144) by neglecting the last two integrals on the right-hand side of (144). The first-order approximation solution is a linear algebra equation with the three variables $\{b_{0}(P, t) - b_{0}(P, t_0), b_{\pm}(P, t)\}$ and is given by

$$
b_{+}(P, t) = b_{+}^{L}(P, t) = b_{-}(P, t)\Gamma_{k+1}^{-}(P, t) \exp\left[ -i \int_{t_0}^{t} dt' 2\Omega(t') \right]
$$

$$
+ i(b_{0}(P, t) - b_{0}(P, t_0))\Gamma_{k+1}^{0}(P, t) \exp\left[ -i \Delta P \int_{t_0}^{t} dt' K_{0}(t') \right] \exp\left[ -i \int_{t_0}^{t} dt' \Omega(t') \right]
$$

$$
+ i\beta_{0}(P, t_0)\Gamma_{k+1}^{0}(P, t) \exp\left[ -i \Delta P \int_{t_0}^{t} dt' K_{0}(t') \right] \exp\left[ -i \int_{t_0}^{t} dt' \Omega(t') \right]
$$

$$
- i\beta_{0}(P, t_0)\Gamma_{k+1}^{0}(P, t_0) \exp\left[ i \int_{t_0}^{t} dt' \Gamma_{k+1}^{+}(P, t_{1}) \right],
$$

(149)

while the truncation error is just the last two integrals:

$$
E_{1}^{+}(P, t) = \exp\left[ i \int_{t_0}^{t} dt' \Gamma_{k+1}^{+}(P, t_{1}) \right]
$$

$$
\times \{ i \int_{t_0}^{t} dt_{1} \left\{ b_{-}(P, t_{1})\hat{\Theta}_{k+1}^{-}(P, t_{1}) \exp\left[ -i \int_{t_0}^{t_1} dt' 2\Omega(t'') \right] \right\}
$$

$$
+ \int_{t_0}^{t} dt_{1} \left\{ b_{0}(P, t_{1})\hat{\Theta}_{k+1}^{0}(P, t_{1}) \exp\left[ -i \Delta P \int_{t_0}^{t_1} dt' K_{0}(t') \right] \exp\left[ -i \int_{t_0}^{t_1} dt' \Omega(t') \right] \right\}. \}
$$

By the integration by parts it can turn out that the error term $E_{1}^{+}(P, t)$ is bounded by

$$
|E_{1}^{+}(P, t)| \leq \frac{1}{2} \frac{|\hat{\Theta}_{k+1}^{-}(P, t)|}{\Omega(t)} + \frac{|\hat{\Theta}_{k+1}^{0}(P, t)|}{\Omega(t)} + \frac{|\hat{\Theta}_{k+1}^{0}(P, t_0)|}{\Omega(t_0)}
$$
and the truncation error is bounded by
\[ |E_{r2}(t)| \leq \frac{1}{2} \frac{|\Theta^{-2}(t)|}{\Omega(t)} + \frac{\left| \hat{\Theta}_{-2}^0(P, t_0) \right|}{\Omega(t)} + \frac{\left| \hat{\Theta}_{+2}^0(P, t_0) \right|}{\Omega(t)} + \int_{t_0}^{t} dt_1 \{ |\Gamma_3^+(P, t_1)| + |\Theta_{-3}(P, t_1)| + |\Theta_{+3}^0(P, t_1)| \}. \] (152)

Obviously, this error upper bound is inversely proportional to \( \Omega(t)^3 \). The first- and second-order approximation solutions may be used to set up the adiabatic condition.

By using the similar equivalent transformations mentioned above one may obtain the \( k \)-order exact solution \( b_k^-(P, t) \) \((k = 1, 2, \ldots)\) from the equation (141b) for the variable \( b_-(P, t) \),

\[ b_k^-(P, t) = b_-(P, t)\Gamma_+^+(P, t) \exp[i \int_{t_0}^{t} dt' \Omega(t')] \]

\[ + i b_0(P, t) \Omega hãy khuẩn(P, t) \exp[-i \Delta P \int_{t_0}^{t} dt' K_0(t')] \exp[i \int_{t_0}^{t} dt' \Omega(t')] \]

\[ - i b_0(P, t_0) \Gamma_1^0(P, t_0) + i \int_{t_0}^{t} dt_1 \{ b_k^-(P, t_1) \Gamma_+^-(P, t_1) \} \]

\[ + i \int_{t_0}^{t} dt_1 \{ b_+(P, t_1) \Theta_+^-(P, t_1) \exp[i \int_{t_0}^{t} dt' \Omega(t')] \} \]

\[ + \int_{t_0}^{t} dt_1 \{ b_0(P, t_1) \Theta_0^-(P, t_1) \exp[-i \Delta P \int_{t_0}^{t_1} dt' K_0(t')] \exp[i \int_{t_0}^{t_1} dt' \Omega(t')] \}. \] (153)
The equation (153) is equivalent to the original equation (141b) for the variable $b_-(P, t)$ if the initial condition (39) is taken into account. In particular, the first-order exact solution $b^1_-(P, t) \equiv b_-(P, t)$ and its five amplitudes are given by

\[
\Gamma^+_{-1}(P, t) = -\frac{1}{4} \frac{\Gamma(P, t)}{\Omega(t)}, \quad \Gamma^0_{-1}(P, t) = \frac{1}{\sqrt{2}} \frac{\Theta(P, t)}{\Omega(t)}, \quad (154a)
\]

\[
\Gamma^-_1(P, t) = -\frac{1}{8} \frac{\Gamma(P, t)^2 + 4|\Theta(P, t)|^2}{\Omega(t)}, \quad (154b)
\]

\[
\Theta^+_1(P, t) = \frac{1}{2} \{i \frac{\partial}{\partial t} \left( \frac{\Gamma(P, t)}{\Omega(t)} \right) + \frac{|\Theta(P, t)|^2}{\Omega(t)}\}, \quad (154c)
\]

\[
\Theta^0_1(P, t) = \frac{1}{\sqrt{2}} \{i \frac{\partial}{\partial t} \left( \frac{\Theta(P, t)}{\Omega(t)} \right) + \frac{1}{4} \frac{\Gamma(P, t)\Theta(P, t)}{\Omega(t)} + \frac{K_0(t)\Theta(P, t)}{\Omega(t)} \Delta P\}. \quad (154d)
\]

The equivalent transformation from the solution $b^k_-(P, t)$ of (153) to the solution $\hat{b}^k_-(P, t)$ is given by

\[
\hat{b}^k_-(P, t) = b^k_-(P, t) \exp[-i \int_{t_0}^{t} dt_1 \Gamma^-_k(P, t_1)]. \quad (155)
\]

By the transformation (155) and the initial condition (39) the fourth term on the right-hand side of (153) is cancelled and the solution $b^k_-(P, t)$ is changed to the transformed solution $\hat{b}^k_-(P, t)$:

\[
\begin{align*}
\hat{b}^k_-(P, t) &= b_+(P, t) \hat{\Gamma}^+_k(P, t) \exp[i \int_{t_0}^{t} dt' 2\Omega(t')] \\
+& b_0(P, t) \hat{\Gamma}^0_k(P, t) \exp[-i \Delta P \int_{t_0}^{t} dt' K_0(t')] \exp[i \int_{t_0}^{t} dt' 2\Omega(t')] \\
- & b_0(P, t_0) \hat{\Gamma}^0_k(P, t_0) + i \int_{t_0}^{t} dt_1 \{b_+(P, t_1) \hat{\Theta}^+_k(P, t_1) \exp[i \int_{t_0}^{t_1} dt' 2\Omega(t')] \\
+ & \int_{t_0}^{t} dt_1 \{b_0(P, t_1) \hat{\Theta}^0_k(P, t_1) \exp[-i \Delta P \int_{t_0}^{t_1} dt' K_0(t')] \exp[i \int_{t_0}^{t_1} dt' 2\Omega(t')] \}.
\end{align*} \quad (156)
\]

The recursive relations for the amplitudes of both the solutions $\hat{b}^-_-(P, t)$ and $\hat{b}^k_-(P, t)$ are given by

\[
\hat{\Gamma}^+_k(P, t) = \Gamma^+_k(P, t) \exp[-i \int_{t_0}^{t} dt_1 \Gamma^-_k(P, t_1)], \quad (157a)
\]

\[
\hat{\Gamma}^0_k(P, t) = \Gamma^0_k(P, t) \exp[-i \int_{t_0}^{t} dt_1 \Gamma^-_k(P, t_1)], \quad (157b)
\]

\[
\hat{\Theta}^+_k(P, t) = \{\Theta^+_k(P, t) + \Gamma^+_k(P, t)\Gamma^-_k(P, t)\} \exp[-i \int_{t_0}^{t} dt_1 \Gamma^-_k(P, t_1)], \quad (157c)
\]
\[ \hat{\Theta}^0_{-k}(P, t) = [\Theta^0_{-k}(P, t) - \Gamma^0_{-k}(P, t)\Gamma^-_{k}(P, t)] \exp[-i \int_{t_0}^{t} dt_1 \Gamma^-_{k}(P, t_1)]. \] (157d)

After making the integration by parts on the last two integrals on the right-hand side of (156) the solution \( \hat{b}^k_{-}(P, t) \) is changed to the \( (k+1) \)-order exact solution \( \hat{b}^{k+1}_{-}(P, t) \) which is also given by (153). The recursive relations between the amplitudes of both the solutions \( \hat{b}^k_{-}(P, t) \) and \( \hat{b}^{k+1}_{-}(P, t) \) are generally given by

\[
\begin{align*}
\Gamma^+_{-(k+1)}(P, t) & = \hat{\Gamma}^+_{-k}(P, t) + \frac{1}{2} \frac{\hat{\Theta}^+_{-k}(P, t)}{\Omega(t)}, \\
\Gamma^0_{-(k+1)}(P, t) & = \hat{\Gamma}^0_{-k}(P, t) - \frac{\hat{\Theta}^0_{-k}(P, t)}{\Omega(t)}, \\
\Gamma^-_{k+1}(P, t) & = \left[ \frac{1}{4} \frac{\Gamma(P, t)\hat{\Theta}^+(P, t)}{\Omega(t)} + \frac{\Theta_0(P, t)\hat{\Theta}^0_{-k}(P, t)}{\sqrt{2}} \right] \\
& \times \exp[i \int_{t_0}^{t} dt_1 \Gamma^+_{k}(P, t_1)] \exp[i \int_{t_0}^{t} dt_1 \Gamma^-_{k-1}(P, t_1)] \ldots \exp[i \int_{t_0}^{t} dt_1 \Gamma^+_{1}(P, t_1)], \\
\Theta^+_{-(k+1)}(P, t) & = i \frac{1}{\Omega(t)} \frac{\partial}{\partial t} \left( \frac{\hat{\Theta}^+_{-k}(P, t)}{\Omega(t)} \right) + \frac{\Theta_0(P, t)\hat{\Theta}^0_{-k}(P, t)}{\sqrt{2} \Omega(t)} + \frac{K_0(t)\hat{\Theta}^0_{-k}(P, t)}{\Omega(t)} \Delta P, \\
\Theta^0_{-(k+1)}(P, t) & = \frac{1}{2\sqrt{2}} \frac{\Theta(P, t)\hat{\Theta}^+_{-k}(P, t)}{\Omega(t)} + i \frac{\partial}{\partial t} \left( \frac{\hat{\Theta}^0_{-k}(P, t)}{\Omega(t)} \right) + \frac{K_0(t)\hat{\Theta}^0_{-k}(P, t)}{\Omega(t)} \Delta P.
\end{align*}
\] (158c, 158d, 158e)

The recursive relations (157c), (157d), (158d), and (158e) show that in the last two integrals of the \( k \)-order exact solution \( \hat{b}^k_{-}(P, t) \) of (156) the amplitudes \( \hat{\Theta}^+_{-k}(P, t) \) and \( \hat{\Theta}^0_{-k}(P, t) \) are inversely proportional to the \( k \)-th power of the Rabi frequency \( \Omega(t) \), that is, \( \hat{\Theta}^+_{-k}(P, t) \propto \Omega(t)^{-k} \) and \( \hat{\Theta}^0_{-k}(P, t) \propto \Omega(t)^{-k} \). Thus, by making a few equivalent transformations one may obtain a high-order approximation solution from the exact solution \( \hat{b}^k_{-}(P, t) \) or \( \hat{b}^k_{+}(P, t) \). The first-order approximation solution may be obtained from the exact solution \( \hat{b}^1_{-}(P, t) \) by neglecting the last two integrals on the right-hand side of (156) \( (k = 1) \),

\[ b_-(P, t) \equiv b^1_{-}(P, t) = b_+(P, t)\Gamma^+_{-1}(P, t) \exp[i \int_{t_0}^{t} dt' 2\Omega(t')] \]

\[ +i(b_0(P, t) - b_0(P, t_0))\Gamma^0_{-1}(P, t) \exp[-i\Delta P \int_{t_0}^{t} dt' K_0(t')] \exp[i \int_{t_0}^{t} dt' \Omega(t')] \]

\[ +ib_0(P, t_0)\Gamma^0_{-1}(P, t) \exp[-i\Delta P \int_{t_0}^{t} dt' K_0(t')] \exp[i \int_{t_0}^{t} dt' \Omega(t')] \]

\[ -ib_0(P, t_0)\Gamma^0_{-1}(P, t_0) \exp[i \int_{t_0}^{t} dt_1 \Gamma^-_{1}(P, t_1)], \] (159)
and the truncation error is just given by these last two integrals neglected, and by the integration by parts it can turn out that the truncation error is bounded by
\[ |E_{r1}(P, t)| \leq \frac{1}{2} \left( \frac{|\hat{\Theta}^+_{-1}(P, t)|}{\Omega(t)} + \frac{|\hat{\Theta}^0_{-1}(P, t)|}{\Omega(t)} \right) \]
\[ + \int_{t_0}^t dt_1 \{ |\Gamma_2^- (P, t_1)| + |\Theta^+_{-2}(P, t_1)| + |\Theta^0_{-2}(P, t_1)| \}. \]  
(160)

The error upper bound is proportional to \( \Omega(t)^{-2} \). The second-order approximation solution is obtained from the exact solution \( \hat{b}_2^0 (P, t) \) by neglecting the last two integrals on the right-hand side of (156) \((k = 2)\),
\[ b_2^0 (P, t) = b_+ (P, t) \Gamma^+_{-2} (P, t) \exp[i \int_{t_0}^t dt' 2\Omega(t')] \]
\[ + i b_0 (P, t) \Gamma^0_{-2} (P, t) \exp[-i \Delta P \int_{t_0}^t dt' K_0(t')] \exp[i \int_{t_0}^t dt' \Omega(t')] \]
\[ - i b_0 (P, t_0) \Gamma^0_{-2} (P, t_0) \exp[i \int_{t_0}^t dt_1 \Gamma_2^- (P, t_1)] \]
(161)

and the truncation error is bounded by
\[ |E_{r2}(P, t)| \leq \frac{1}{2} \left( \frac{|\hat{\Theta}^+_{-2}(P, t)|}{\Omega(t)} + \frac{|\hat{\Theta}^0_{-2}(P, t)|}{\Omega(t)} \right) \]
\[ + \int_{t_0}^t dt_1 \{ |\Gamma_3^- (P, t_1)| + |\Theta^+_{-3}(P, t_1)| + |\Theta^0_{-3}(P, t_1)| \}. \]  
(162)

The error upper bound is proportional to \( \Omega(t)^{-3} \). The first- and second-order approximation solutions may be used to set up the adiabatic condition below.

With the help of the equivalent transformations similar to those used above one may obtain the \( k \)-order exact solution for the variable \( b_0(P, t) \) from the equation (141a). At first the first-order exact solution may be given by
\[ b_0(P, t) - b_0(P, t_0) = i b_+ (P, t) \Gamma^+_{01} (P, t) \exp[i \Delta P \int_{t_0}^t dt' K_0(t')] \exp[i \int_{t_0}^t dt' \Omega(t')] \]
\[ + i b_0 (P, t) \Gamma^0_{01} (P, t) \exp[i \Delta P \int_{t_0}^t dt' K_0(t')] \exp[-i \int_{t_0}^t dt' \Omega(t')] \]
\[ + \int_{t_0}^t dt_1 \{ b_+ (P, t_1) \Theta^+_{01} (P, t_1) \exp[i \Delta P \int_{t_0}^{t_1} dt' K_0(t')] \exp[i \int_{t_0}^{t_1} dt' \Omega(t')] \]
\[ + \int_{t_0}^t dt_1 \{ b_+ (P, t_1) \Theta^+_{01} (P, t_1) \exp[i \Delta P \int_{t_0}^{t_1} dt' K_0(t')] \exp[-i \int_{t_0}^{t_1} dt' \Omega(t')] \} \]
(163)
where the four amplitudes are given by

\[ \Gamma_{01}^+(P, t) = -\Gamma_{01}^-(P, t) = -\frac{1}{\sqrt{2}} \frac{\Theta(P, t)^*}{\Omega(t)}, \]

\[ \Theta_{01}^+(P, t) = -\Theta_{01}^-(P, t) = i \frac{\partial}{\partial t} \frac{\Theta(P, t)^*}{\Omega(t)} - \frac{1}{2} \frac{\Gamma(P, t) \Theta(P, t)^*}{\Omega(t)} - \frac{K_0(t) \Theta(P, t)^*}{\Omega(t)} \Delta P. \]

A special point in the first-order exact solution (163) is that there is not any term containing the solution \( b_0(P, t) \) itself on the right-hand side of (163). Thus, one may make directly the integration by parts on the last two integrals on the right-hand side of (163). Here still denote that \( \delta_0^+(P, t) = b_0(P, t) - b_0(P, t_0) \) and \( \delta_0^-(P, t) = b_\pm(P, t) - b_\pm(P, t_0) \). Then the second-order exact solution \( \delta_0^{(2)}(P, t) \) may be expressed as

\[ \delta_0^{(2)}(P, t) \equiv \delta_0^+(P, t) = ib_+(P, t) \Gamma_{02}^+(P, t) \exp[i \Delta P \int_{t_0}^{t} dt' K_0(t')] \exp[i \int_{t_0}^{t} dt' \Omega(t')] \]

\[ + ib_-(P, t) \Gamma_{02}^-(P, t) \exp[i \Delta P \int_{t_0}^{t} dt' K_0(t')] \exp[-i \int_{t_0}^{t} dt' \Omega(t')] \]

\[ + ib_0(P, t_0) \Gamma_{02}^{(0)}(P, t) + i \int_{t_0}^{t} dt_1 \{ b_+(P, t_1) \Theta_{02}^+(P, t_1) \} \]

\[ + \int_{t_0}^{t} dt_1 \{ b_-(P, t_1) \Theta_{02}^-(P, t_1) \} \exp[i \Delta P \int_{t_0}^{t_1} dt' K_0(t')] \exp[-i \int_{t_0}^{t_1} dt' \Omega(t')] \]

\[ + \int_{t_0}^{t} dt_1 \{ b_0(P, t_1) \} \]

(164)

where the amplitudes are given by

\[ \Gamma_{02}^+(P, t) = \Gamma_{01}^+(P, t) - \frac{\Theta_{01}^+(P, t)}{\Omega(t)}, \quad \Gamma_{02}^-(P, t) = \Gamma_{01}^-(P, t) + \frac{\Theta_{01}^-(P, t)}{\Omega(t)}, \]

\[ \Gamma_{2}^{(0)}(P, t) = \int_{t_0}^{t} dt_1 \Gamma_{2}^+(P, t_1), \quad \Gamma_{2}^{(0)}(P, t) = -\sqrt{2} \frac{\Theta(P, t) \Theta_{01}^+(P, t)}{\Omega(t)}, \]

\[ \Theta_{02}^+(P, t) = -\Theta_{02}^-(P, t) = i \frac{\partial}{\partial t} \frac{\Theta_{01}^+(P, t)}{\Omega(t)} \]

\[ - \frac{K_0(t) \Theta_{01}^+(P, t)}{\Omega(t)} \Delta P - \frac{1}{2} \frac{\Gamma(P, t) \Theta_{01}^+(P, t)}{\Omega(t)} \]

Now the first-order approximation solution may be obtained from the first-order exact solution (163) by neglecting the last two integrals on the right-hand side of (163). It is given by

\[ \delta_0^b(P, t) = ib_+(P, t) \Gamma_{01}^+(P, t) \exp[i \Delta P \int_{t_0}^{t} dt' K_0(t')] \exp[i \int_{t_0}^{t} dt' \Omega(t')] \]

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\[ + i b_-(P, t) \Gamma_{01}^-(P, t) \exp[i \Delta P \int_{t_0}^t dt' K_0(t')] \exp[-i \int_{t_0}^t dt' \Omega(t')] \]  
(165)

and the truncation error is just the last two integrals on the right-hand side of (163), and it can turn out by the integration by parts that the truncation error is bounded by

\[ |E_{r1}^0(P, t)| \leq \frac{2|\Theta_{01}^+(P, t)|}{\Omega(t)} + \int_{t_0}^t dt_1 \{|\Gamma_2^0(P, t_1)| + 2|\Theta_{02}^+(P, t_1)|\}. \]  
(166)

This error upper bound is clearly proportional to \( \Omega(t)^{-2} \). A higher-order exact solution than the second-order one (164) may be obtained by integrating by parts the last two integrals in (164), but it needs first to eliminate the fourth term that contains the solution \( \delta_{b2}^0(P, t) \) itself on the right-hand side of (164). This can be done by making an equivalent transformation on the solution \( \delta_{b2}^0(P, t) \). In general, for the \( k \)-order exact solution \( \delta_{b}^k(P, t) \) \( (k = 2, 3, \ldots) \) the equivalent transformation is written as

\[ \delta_{b}^k(P, t) = \delta_0^k(P, t) \exp[-i \int_{t_0}^t dt_1 \Gamma_0^0(P, t_1)]. \]  
(167)

Then the \( k \)-order transformed solution \( \delta_{b}^k(P, t) \) may be expressed as

\[ \delta_{b}^k(P, t) = i b_+(P, t) \hat{\Gamma}_{0k}^+(P, t) \exp[i \Delta P \int_{t_0}^t dt' K_0(t')] \exp[i \int_{t_0}^t dt' \Omega(t')] \]

\[ + \int_{t_0}^t dt_1 \{ b_+(P, t_1) \hat{\Theta}_{0k}^+(P, t_1) \exp[i \Delta P \int_{t_0}^t dt' K_0(t')] \exp[i \int_{t_0}^t dt' \Omega(t')] \}

\[ + \int_{t_0}^t dt_1 \{ b_-(P, t_1) \hat{\Theta}_{0k}^-(P, t_1) \exp[i \Delta P \int_{t_0}^t dt' K_0(t')] \exp[-i \int_{t_0}^t dt' \Omega(t')] \} \]  
(168)

where the amplitudes satisfy the recursive relations:

\[ \hat{\Gamma}_{0k}^+(P, t) = \Gamma_{0k}^+(P, t) \exp[-i \int_{t_0}^t dt_1 \Gamma_0^0(P, t_1)], \]  
(169a)

\[ \hat{\Gamma}_{0k}^-(P, t) = \Gamma_{0k}^-(P, t) \exp[-i \int_{t_0}^t dt_1 \Gamma_0^0(P, t_1)], \]  
(169b)

\[ \hat{\Gamma}_{k}^{00}(P, t) = \int_{t_0}^t dt_1 \{ \frac{\partial}{\partial t_1} \Gamma_{k}^{00}(P, t_1) \} \exp[-i \int_{t_0}^{t_1} dt' \Gamma_0^0(P, t')], \]  
(169c)

\[ \hat{\Theta}_{0k}^+(P, t) = [\Theta_{0k}^+(P, t) - \Gamma_{0k}^+(P, t) \hat{\Gamma}_{k}^0(P, t)] \exp[-i \int_{t_0}^t dt_1 \Gamma_0^0(P, t_1)], \]  
(169d)

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\[
\hat{\Theta}_{0k}(P, t) = [\Theta_{0k}^0(P, t) - \Gamma_{-0k}^0(P, t)\Gamma_k^0(P, t)] \exp[-i \int_{t_0}^{t} dt' \Gamma_k^0(P, t_1)]. \quad (169e)
\]

From the \(k\)–order transformed solution \(\delta_{0k}^{bk}(P, t)\) to the \((k + 1)\)–order solution \(\delta_{0k}^{bk+1}(P, t)\) there is an equivalent transformation of the integration by parts. The \((k + 1)\)–order exact solution \(\delta_{0k}^{bk+1}(P, t)\) is still given by (164) as long as one makes the replacement: \(\delta_{0k}^{b2}(P, t) \leftrightarrow \delta_{0k}^{bk+1}(P, t)\) and the following replacements for the amplitudes:

\[
\Gamma_{02}^\alpha(P, t) \leftrightarrow \Gamma_{0k+1}^\alpha(P, t), \quad \Theta_{02}^\alpha(P, t) \leftrightarrow \Theta_{0k+1}^\alpha(P, t), \quad \alpha = +, -; \quad \Gamma_2^0(P, t) \leftrightarrow \Gamma_{k+1}^0(P, t), \quad \Gamma_0^2(P, t) \leftrightarrow \Gamma_{k+1}^0(P, t).
\]

The recursive relations for the amplitudes between the two solutions \(\delta_{0k}^{bk}(P, t)\) and \(\delta_{0k}^{bk+1}(P, t)\) \((k \geq 2)\) are given by

\[
\Gamma_{0k+1}^+(P, t) = \hat{\Gamma}_{0k}^+(P, t) - \frac{\hat{\Theta}_{0k}^+(P, t)}{\Omega(t)}, \quad \Gamma_{0k+1}^-(P, t) = \hat{\Gamma}_{0k}^-(P, t) + \frac{\hat{\Theta}_{0k}^-(P, t)}{\Omega(t)}, \quad (170a)
\]

\[
\Gamma_{k+1}^0(P, t) = \hat{\Gamma}_k^0(P, t) + \frac{1}{\sqrt{2}} \int_{t_0}^{t} dt_1 \left\{ - \Theta(P, t_1) \hat{\Theta}_{0k}^+(P, t_1) \frac{\hat{\Theta}_{0k}^+(P, t_1)}{\Omega(t_1)} + \Theta(P, t_1) \hat{\Theta}_{0k}^-(P, t_1) \frac{\hat{\Theta}_{0k}^-(P, t_1)}{\Omega(t_1)} \right\}, \quad (170b)
\]

\[
\Gamma_{k+1}^0(P, t) = -\frac{1}{\sqrt{2}} \Theta(P, t) \hat{\Theta}_{0k}^+(P, t) \frac{\hat{\Theta}_{0k}^+(P, t)}{\Omega(t)} + \frac{1}{\sqrt{2}} \Theta(P, t) \hat{\Theta}_{0k}^-(P, t) \frac{\hat{\Theta}_{0k}^-(P, t)}{\Omega(t)} \bigg\} \times \exp\left[i \int_{t_0}^{t} dt_1 \Gamma_k^0(P, t_1) \right] \exp\left[i \int_{t_0}^{t} dt_1 \Gamma_{k-1}^0(P, t_1) \right] \ldots \exp\left[i \int_{t_0}^{t} dt_1 \Gamma_2^0(P, t_1) \right],
\]

\[
\Theta_{0k+1}^+(P, t) = \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\hat{\Theta}_{0k}^+(P, t)}{\Omega(t)} \right) - \frac{K_0(t) \hat{\Theta}_{0k}^+(P, t)}{\Omega(t)} \Delta P - \frac{1}{2} \frac{\Gamma(P, t) \hat{\Theta}_{0k}^+(P, t)}{\Omega(t)}, \quad (170c)
\]

\[
\Theta_{0k+1}^-(P, t) = -i \frac{\partial}{\partial t} \left( \frac{\hat{\Theta}_{0k}^-(P, t)}{\Omega(t)} \right) + \frac{K_0(t) \hat{\Theta}_{0k}^-(P, t)}{\Omega(t)} \Delta P + \frac{1}{2} \frac{\Gamma(P, t) \hat{\Theta}_{0k}^-(P, t)}{\Omega(t)}. \quad (170d)
\]

It can turn out that the amplitudes \(\Theta_{0k}^+(P, t)\) and \(\Theta_{0k}^-(P, t)\) of the \(k\)–order exact solution \(\delta_{0k}^{bk}(P, t)\) are inversely proportional to \(\Omega(t)^{-k}\). Thus, the last two integrals of the \(k\)–order exact solution \(\delta_{0k}^{bk}(P, t)\) have a negligible contribution to the solution if the Rabi frequency \(\Omega(t)\) is large. Now the second-order approximation solution may be obtained from the second-order exact solution \(\delta_{0k}^{b2}(P, t)\) of (168),

\[
\delta_{0k}^{b2}(P, t) = i b_+(P, t) \Gamma_{02}^+(P, t) \exp\left[i \Delta P \int_{t_0}^{t} dt' K_0(t') \right] \exp\left[i \int_{t_0}^{t} dt' \Omega(t') \right] + i b_-(P, t) \Gamma_{02}^-(P, t) \exp\left[i \Delta P \int_{t_0}^{t} dt' K_0(t') \right] \exp\left[-i \int_{t_0}^{t} dt' \Omega(t') \right]
\]

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\[ -b_0(P, t_0)\{1 - \exp[i \int_{t_0}^t dt_1 \Gamma_2^0(P, t_1)]\}, \quad (171) \]

while the truncation error is given by the last two integrals on the right-hand side of (168), and the truncation error upper bound is determined from

\[ |E^0_\pm(P, t)| \leq \left| \frac{\hat{\Theta}^\pm_{02}(P, t)}{\Omega(t)} \right| + \left| \frac{\hat{\Theta}_{02}(P, t)}{\Omega(t)} \right| \]

\[ + \int_{t_0}^t dt_1 \{ |\Theta^0_{03}(P, t)| + |\Theta^+_{03}(P, t)| + |\Theta^-_{03}(P, t)| \}. \quad (172) \]

This error upper bound is clearly proportional to \( \Omega(t)^{-3} \).

Now by solving the three first-order approximation solutions (149), (159), and (165), which are all linear algebra equations, one may obtain the first-order approximation solution to the basic equations (141). At first according to the first-order approximation solutions (149), (159), and (165) and their truncation errors the first-order exact solution to the equations (141) may be formally written as, (this is really the first-order approximation solutions plus their truncation errors),

\[ b^+_+(P, t) = \alpha^+ + \beta^+_0(P, t) + \delta^+_0(P, t) + E^+_1(P, t), \quad (173a) \]
\[ b^-_- (P, t) = \alpha^- - \beta^-_0(P, t) + \delta^-_0(P, t) + E^-_1(P, t), \quad (173b) \]
\[ \delta^0_0(P, t) = \alpha^0_+ b^+_+(P, t) + \alpha^0_- b^-_-(P, t) + E^0_1(P, t), \quad (173c) \]

where the truncation errors \( E^+_1(P, t) \) and \( E^-_1(P, t) \) may be considered as small parameters and any other parameters such as \( \alpha^+, \alpha^-, \alpha^0, \) etc., can be obtained directly from the first-order approximation solutions (149), (159), and (165). These three equations are linear algebra equations as \( E^+_1(P, t) \) and \( E^-_1(P, t) \) are considered as the small parameters. The exact solution to the three linear algebra equations (173) may be written as

\[ \delta^0_\pm(P, t) = \delta^\pm (P, t) + E^\pm_1(P, t), \quad \delta^0_0(P, t) = \delta_0 (P, t) + E^0_1(P, t), \quad (174) \]

where the main terms are written as

\[ \delta^\pm (P, t) = \mp i \frac{1}{\sqrt{2}} b_0(P, t_0) F(P, t) \]

\[ \times \left\{ \frac{\Theta(P, t)}{\Omega(t)} \exp[-i \Delta P \int_{t_0}^t dt' K_0(t')] \exp[\mp i \int_{t_0}^t dt' \Omega(t')] \right\} \]

\[ - \frac{\Theta(P, t_0)}{\Omega(t_0)} \exp[\pm i \frac{1}{8} \int_{t_0}^t dt_1 \frac{\Gamma(P, t_1)}{\Omega(t_1)}] \cdot \frac{1}{4|\Theta(P, t_1)|^2} \], \quad (175a) \]

\[ \delta_0 (P, t) = 0. \quad (175b) \]

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The dominating error terms are given by

\[
F(P, t) = \frac{1 + \frac{1}{2} \frac{\Gamma(P, t)^2}{\Omega(t)^2}}{1 + \frac{1}{2} \frac{\Gamma(P, t)^2}{\Omega(t)^2} + \frac{1}{2} \frac{\Theta(P, t)^2}{\Omega(t)^2}}, \quad G_\pm(P, t) = \frac{\pm \frac{1}{2} \frac{\Gamma(P, t)^2}{\Omega(t)^2} + \frac{1}{2} \frac{\Theta(P, t)^2}{\Omega(t)^2}}{1 + \frac{1}{2} \frac{\Gamma(P, t)^2}{\Omega(t)^2} + \frac{1}{2} \frac{\Theta(P, t)^2}{\Omega(t)^2}}.
\]

The factors \( G_\pm(P, t) \) appear in the error terms \( E_{rt}^\pm(P, t) \) and \( E_{rt}^0(P, t) \), as can be seen below. Then the upper bound of the error term that is generated by the main terms \( \delta_{\pm m}(P, t) \) and \( \delta_{0m}(P, t) \) is determined from

\[
||E_r^{(1)}(P, t)|| = \sqrt{||\delta_{+m}(P, t)||^2 + ||\delta_{-m}(P, t)||^2 + ||\delta_{0m}(P, t)||^2} \leq F(P, t) \left[ |\Theta(P, t)| \frac{\Omega(t)}{\Omega(t_0)} + |\Theta(P, t_0)| \right].
\]

(176)

It is clear that this upper bound is proportional to \( \Omega(t)^{-1} \). Notice that the factor \( F(P, t) \) is unity approximately if \( |\Theta(P, t)|^2/\Omega(t)^2 \ll 1 \) and \( \Gamma(P, t)^2/\Omega(t)^2 \ll 1 \). By comparing the inequality (176) with the inequality (133) one can find that if the factor \( F(P, t) \) is unity, then the inequality (176) is really the inequality (133) that leads to the first-order approximation adiabatic condition (134).

Now investigate the secondary error terms \( E_{rt}^{\pm 1}(P, t) \) and \( E_{rt}^0(P, t) \) in the exact solution (174). These error terms contain the first-order truncation errors \( E_{rt}^{\pm 1}(P, t) \) and \( E_{rt}^0(P, t) \) (See: (150), (160), and (166)). They may be written as

\[
E_{rt}^{\pm 1}(P, t) = E_{rt}^{\pm 2}(P, t) + E_{rt}^{\pm 3}(P, t) + E_{rt}^{\pm 4}(P, t),
\]

(177a)

\[
E_{rt}^0(P, t) = E_{rt}^{02}(P, t) + E_{rt}^{03}(P, t) + E_{rt}^{04}(P, t) + E_{rt}^{05}(P, t),
\]

(177b)

where \( E_{rt}^{\pm 2}(P, t) \), \( E_{rt}^{02}(P, t) \propto \Omega(t)^{-2} \); \( E_{rt}^{\pm 3}(P, t) \), \( E_{rt}^{03}(P, t) \propto \Omega(t)^{-3} \); \( E_{rt}^{\pm 4}(P, t) \), \( E_{rt}^{04}(P, t) \propto \Omega(t)^{-4} \); \( E_{rt}^{05}(P, t) \propto \Omega(t)^{-5} \). It is not difficult to obtain the upper bounds for all these error terms, but one needs only to consider the dominating terms \( E_{rt}^{\pm 2}(P, t) \) and \( E_{rt}^{02}(P, t) \) that are proportional to \( \Omega(t)^{-2} \), since the truncation errors \( E_{rt}^{\pm 1}(P, t) \) and \( E_{rt}^0(P, t) \) are proportional to \( \Omega(t)^{-2} \) and the other error terms \( E_{rt}^{\pm k}(P, t) \) and \( E_{rt}^{0k}(P, t) \) \((k = 3, 4; l = 3, 4, 5)\) are higher-order and can be neglected with respect to the dominating terms \( E_{rt}^{\pm 2}(P, t) \) and \( E_{rt}^{02}(P, t) \). The dominating error terms are given by

\[
E_{rt}^{\pm 2}(P, t) = E_{rt}^{\pm 2}(P, t) F(P, t) \pm \frac{1}{\sqrt{2}} G_\pm(P, t) \exp\left[ \mp i \int_{t_0}^t dt' \Omega(t') \right]
\]

\[
\times \left\{ i b_0(P, t_0) \frac{\Theta(P, t_0)}{\Omega(t)} \exp\left[ -i \Delta P \int_{t_0}^{t_0} dt' K_0(t') \right] \exp\left[ \pm i \int_{t_0}^t dt' \Omega(t') \right]
\]

\[
- i b_0(P, t_0) \frac{\Theta(P, t_0)}{\Omega(t_0)} \exp\left[ \mp i \frac{1}{8} \int_{t_0}^t dt_1 \Gamma(P, t_1) \frac{2 + 2 |\Theta(P, t_1)|^2}{\Omega(t_1)} \right] \right\},
\]

(178a)
be expressed as a function on the right side of (180). Then the global adiabatic condition may be ensured if the effective momentum region \([P]t_0\) is not to be more than a desired small value over the entire time period \(t_0\) of the basic STIRAP decelerating or accelerating process. Denote the region \([P]t_0\) to be so small that it can be neglected. Then the norm for the error vector \(\|E_r^0(P,t), E_r^\pm(P,t), E_{r1}^0(P,t)\|^T\) is bounded by

\[
\|E_r^0(P,t)\| = \sqrt{|E_r^0(P,t)|^2 + |E_r^\pm(P,t)|^2 + |E_{r1}^0(P,t)|^2}
\]

\[
\leq \sqrt{|E_r^0(P,t)|^2 + |F(P,t)|^2(\|E_r^\pm(P,t)|^2 + |E_{r1}^0(P,t)|^2)}
\]

\[+ \frac{\|\Theta(P,t)\|}{\Omega(t)} \sqrt{\frac{|F(P,t)|^2}{\Omega(t)^2} + \frac{1}{2} |G_+(P,t)|^2 + \frac{1}{2} |G_-(P,t)|^2}.\] (179)

This error upper bound is proportional to \(\Omega(t)^{-2}\). Now it follows from (174) and (177) and then the inequalities (176) and (179) that the total deviation of a real STIRAP adiabatic process from the ideal one at any instant of time is bounded by

\[
\|E_r(P,t)\| = \sqrt{|\delta_r^0(P,t)|^2 + |\delta_r^\pm(P,t)|^2 + |\delta_{r1}^0(P,t)|^2}
\]

\[
\leq \|E_r^{(1)}(P,t)\| + \|E_r^{(2)}(P,t)\|
\]

\[\leq \frac{\|\Theta(P,t)\|}{\Omega(t)} \sqrt{\frac{|F(P,t)|^2}{\Omega(t)^2} + \frac{1}{2} |G_+(P,t)|^2 + \frac{1}{2} |G_-(P,t)|^2} + \frac{1}{\Omega(t)} \sqrt{\frac{|\Theta(P,t)|^2}{\Omega(t)^2} + \frac{1}{4} \frac{|\Theta(P,t)|^4}{\Omega(t)^2}}\]

\[\frac{1}{\Omega(t)} \sqrt{\frac{|\Theta(P,t)|^2}{\Omega(t)^2}} + \frac{1}{\Omega(t)} \sqrt{\frac{|\Theta(P,t)|^4}{\Omega(t)^2}} + \frac{1}{\Omega(t)} \sqrt{\frac{|\Theta(P,t)|^6}{\Omega(t)^2}}.\] (180)

where the upper bounds for the truncation errors \(|E_r^0(P,t)|, \ |E_r^\pm(P,t)|, \ |E_{r1}^0(P,t)| \) are obtained from (150), (160), and (166), respectively. In order to obtain the global adiabatic condition one needs to limit the maximum value on the right side of (180) not to be more than a desired small value over the effective momentum region \([P]\) and in the whole time period \(t_0 \leq t \leq t_0 + T\) of the STIRAP decelerating or accelerating process. Denote \(A_d(P,t)\) as the function on the right side of (180). Then the global adiabatic condition may be expressed as

\[
\|E_r(P,t)\| \leq A_d(P,t) \leq \max_{P \in [P], t_0 \leq t \leq t_0 + T} \{A_d(P,t)\} \leq \varepsilon_r,\] (181)
where $\varepsilon_r$ is the desired value and $\varepsilon_r << 1$. Unlike the adiabatic condition (128) there is not an exponential correction factor in the adiabatic condition (180) and (181). The adiabatic condition (180) and (181) consists of the first-order term that is proportional to $\Omega(t)^{-1}$ and the second-order terms that are proportional to $\Omega(t)^{-2}$, while the adiabatic condition (128) is the first-order term with the factor $F(P,t) = 1$ times the exponential correction factor. Thus, the adiabatic condition (180) and (181) is much less severe than the adiabatic condition (128), the latter is most severe for a quantum ensemble with a broad momentum distribution. However, just like the adiabatic condition (128) the adiabatic condition (180) and (181) is also strict and accurate (It is not difficult to include the omitted secondary error terms of (177) in the adiabatic condition (180) and (181)). Therefore, if any one of the two general adiabatic conditions is met, then a better STIRAP pulse sequence is obtained and the perfect state (or population) transfer may be realized by the STIRAP pulse sequence. The adiabatic condition (180) and (181) may be more useful in practice to realize the perfect STIRAP state (or population) transfer in an atomic or molecular ensemble with a broad momentum distribution. It may also be used to realize the STIRAP decelerating and accelerating processes in the laser cooling and the quantum coherence interference experiments of a cold atomic or molecular ensemble.

The adiabatic condition (180) and (181) is still slightly severe. This is because it is based on the first-order approximation solution to the basic equations (141), leading to that the upper bounds for the truncation errors $|E^+_{r1}(P,t)|$, $|E^-_{r1}(P,t)|$, and $|E^0_{r1}(P,t)|$ are not the lowest ones. As known before, these truncation errors are proportional to $\Omega(t)^{-2}$. A better adiabatic condition may be set up on the basis of the second-order approximation solution to the basic equations (141), which are given by (151), (161), and (171), and the corresponding truncation error upper bounds $|E^+_{r2}(P,t)|$, $|E^-_{r2}(P,t)|$, and $|E^0_{r2}(P,t)|$ are given by (152), (162), and (172), respectively. These upper bounds are proportional to $\Omega(t)^{-3}$. Thus, they are the lower ones with respect to those truncation error upper bounds of the first-order solution, leading to that the adiabatic condition based on the second-order approximation solution is better one.

It is known that in the adiabatic condition (180) and (181) the truncation errors $|E^+_{r1}(P,t)|$, $|E^-_{r1}(P,t)|$, and $|E^0_{r1}(P,t)|$ are proportional to $\Omega(t)^{-2}$ and the factor $F(P,t)$ is a function of the ratio $\Gamma(P,t)^2/\Omega(t)^2$. Then the parameter $\Gamma(P,t)$ appears only in those terms that are proportional to $\Omega(t)^{-2}$ on the rightest side of (180). This is different from the case that the parameter $\Theta(P,t)$ may appear in the first term on the rightest side of (180) that is proportional to $\Omega(t)^{-1}$. Thus, the parameter $\Gamma(P,t)$ has a smaller contribution to the adiabatic condition (180) and (181) than the parameter $\Theta(P,t)$. As pointed out before, the effect of the momentum distribution is dependent upon whether the Raman laser light beams of a STIRAP experiment are copropagating or counterpropagating. Consider that the Raman laser light beams are copropagating. Then the momentum distribution could have a relatively small effect, because in this case the parameter $\Theta(P,t)$ has a smaller value. This is correct for the first-order approximation adiabatic condition (134). However, it could not be
so simple for the inequality (180) in a quantum ensemble with a broad momentum distribution. When the first term on the right side of (180) is smaller due to a small parameter $\Theta(P, t)$, the second and third terms could become more important and have a dominating contribution to the adiabatic condition (180) and (181). Then in this case the parameters $\Gamma(P, t)$ and $K_0(t)$ become more important in the adiabatic condition (180) and (181), resulting in that the momentum distribution has a large effect on the adiabatic condition. Therefore, the momentum distribution needs to be considered explicitly even for a conventional STIRAP state (or population) transfer experiment in a quantum ensemble with a broad momentum distribution, which uses the copropagating Raman laser light beams. Obviously, for the STIRAP decelerating and accelerating processes in a quantum ensemble with a broad momentum distribution, which use the counterpropagating Raman laser light beams, one needs to consider generally the effect of the momentum distribution on the STIRAP state (or population) transfer.

The present adiabatic theoretical methods including the equivalent transformation method to solve the basic differential equations (26) not only can be used to set up a general adiabatic condition for the basic STIRAP decelerating and accelerating processes of a single atom or molecule or a quantum ensemble of the atoms or molecules, but also they will have an extensive application in other research fields such as the NMR spectroscopy (See: for example, Ref. [41]) and the magnetic resonance image (MRI).

8. Discussion

In the paper the standard three-state STIRAP population transfer theory in the laser spectroscopy [4, 15, 16, 17, 18] has been developed to describe theoretically the STIRAP-based unitary decelerating and accelerating processes of a single freely moving atom by combining the superposition principle in quantum mechanics [25] and the energy, momentum, and angular momentum conservation laws for the atomic photon absorption and emission processes [5, 19]. There are similar theoretical works or developments to describe the atomic laser cooling process [5, 19, 20, 21] in a neutral atom ensemble and the atomic quantum interference experiments [10, 12] in a cold atomic ensemble. There are also a number of works to investigate the atomic decelerating and accelerating processes by the laser light techniques [21, 23, 24]. However, the present work is focused on the analytical and quantitative investigation how the momentum distribution of a superposition of the momentum states of a pure-state quantum system such as a single freely moving atom affects the state-transfer efficiency in these STIRAP unitary decelerating and accelerating processes. It emphasizes the complete STIRAP state transfer and the unitarity of these processes. This means that in the present work any decoherence effect of the atomic system under study is not considered. A main purpose to investigate the effect of the momentum distribution on the STIRAP state transfer is to build up better STIRAP unitary decelerating and accelerating sequences, so that the time- and space-compressing processes of the quantum control process to simulate
the reversible and unitary state-insensitive halting protocol [22] can be realized through these decelerating and accelerating processes. Thus, this is involved in setting up a general adiabatic condition for the STIRAP unitary decelerating and accelerating processes. In the paper two general adiabatic conditions have been obtained analytically, one based on the Dyson series solution to the basic differential equations to govern the STIRAP processes, another based on the equivalent transformation method to solve the basic differential equations. Both the general adiabatic conditions may be used to set up a conventional STIRAP experiment and also the STIRAP-based decelerating and accelerating processes. A complete STIRAP state transfer could be achieved only in the ideal adiabatic condition. Generally, it is hard to achieve a complete state transfer for the STIRAP processes when the atomic momentum superposition state has a broad momentum distribution. However, in the ideal or nearly ideal adiabatic condition an almost complete STIRAP state transfer may be realized if the superposition of the momentum states has a small effective wave-packet spreading or a narrow momentum distribution. When the initial motional state of a freely moving atom is a Gaussian wave-packet state, the final motional state of the atom is still a perfect or almost perfect Gaussian wave-packet state after the atom undergoes the STIRAP unitary decelerating (or accelerating) process in the ideal or nearly ideal adiabatic condition. Therefore, in the paper it is shown that the time- and space-compressing processes of the quantum control process [22] can be realized almost perfectly through the STIRAP decelerating and accelerating processes in the ideal or nearly ideal adiabatic condition. This is one of the important results in the paper.

The standard STIRAP population transfer theory is generally based on the semiclassical theory of electromagnetic radiation. In the semiclassical theory the externally applied electromagnetic fields such as the Raman laser light beams are considered as the classical electromagnetic fields, while the atomic system itself and the interaction between the atomic system and the external electromagnetic fields are treated quantum mechanically. It has been shown that the semiclassical theory can describe almost perfectly the three-state STIRAP population transfer experiments of the atomic and molecular beams in the laser spectroscopy [4, 15, 16, 17, 18]. On the other hand, the semiclassical theory is also successful to describe the STIRAP-based laser cooling processes in a neutral atomic ensemble [20, 21] and especially the atomic quantum interference experiments in a cold neutral atom ensemble [10, 12, 13, 14]. In the paper the semiclassical theory also is directly employed to describe the STIRAP-based unitary decelerating and accelerating processes of a single atom. The semiclassical theory of electromagnetic radiation generally can not explain reasonably the atomic spontaneous emission [1, 25, 26]. However, it is suited to describe these STIRAP unitary decelerating and accelerating processes due to that these STIRAP processes can avoid the atomic spontaneous emission by setting suitably the experimental parameters.

As far as the Hamiltonian of Eq. (4) to describe the three-state STIRAP experiment of an atom system is concerned, there are three requirements: (i) the three-state subspace for the atomic internal states is closed under the Hamil-
tonian; (iii) the electric-dipole approximation is satisfied; and (iii) the rotating wave approximation (RWA) is reasonable. The first requirement can be satisfied if one chooses suitably the atom and its three internal states and the experimental parameters of the Raman laser light beams. Since the size of an atom is generally much less than the wavelengths of the Raman laser light beams at the optical frequencies (\(\sim \omega_{01}\) and \(\omega_{02}\)), the second requirement may be met generally. If the Raman laser light beams are strong, it can turn out that in the first approximation the strong laser light field may generate a Bloch-Siegert shift to the transition frequency of the atomic internal energy levels [1].

When the Rabi frequencies (\(\Omega_p(t)\) and \(\Omega_q(t)\)) of the Raman laser light beams are much less than the resonance frequencies (\(\omega_{01}\) and \(\omega_{02}\)) of the atomic internal energy levels and the detunings for the Raman laser light beams are small, the magnitude of the Bloch-Siegert shifts generated by the Raman laser light beams at the optical frequencies is very small and may be negligible and hence the rotating wave approximation is reasonable [1]. However, it is also convenient to correct the Bloch-Siegert shifts in the STIRAP experiment because one needs only to add the Bloch-Siegert shifts to the resonance frequencies (\(\omega_{01}\) and \(\omega_{02}\)). A simple evaluation for the Bloch-Siegert shifts can be seen in Ref. [1] and a general treatment may use the average Hamiltonian theory [34]. On the other hand, it is also possible to apply an extra laser light field for each one of the Raman laser light beams in the STIRAP experiment to compensate the rotating-wave approximation. In fact, if each one of the two Raman laser light beams in the STIRAP experiment is replaced with a pair of the laser light beams with the orthogonal electric field vectors and the suitable phases [38], one can eliminate the rotating-wave approximation. Similarly, one may also use the circularly polarized lights to prepare the dipole interaction Hamiltonian (10) without the rotating wave approximation [19, 38].

There is also another condition to be met that the electromagnetic field of any Raman laser light beam is considered as an infinite plane-wave electromagnetic field when calculating the time evolution process of a freely moving atom under the STIRAP unitary decelerating and accelerating processes. Here the infinite plane-wave electromagnetic field has spatially uniform amplitude and phase. This condition can be met only when the electromagnetic field can encompass sufficiently the whole wave-packet motional state of the moving atom. Note that the electromagnetic field of the Raman laser light beam propagates along a direction parallel to the atomic moving direction in one-dimensional space. Since the atom is in a Gaussian wave-packet motional state which has a finite wave-packet spreading, then one can set suitably the experimental parameters for the Raman laser light beam such that the electromagnetic field in space is much wider than the effective wave-packet spreading of the atomic motional state during the whole decelerating or accelerating process. Then in this case the electromagnetic field in space can be reasonably considered as an infinite and uniform plane-wave electromagnetic field for the atomic wave-packet motional state. The condition may be satisfied more easily for a heavy atom as the wave-packet motional state for such atom has a more narrow wave-packet spreading. If the electromagnetic field in space has a finite bandwidth less than
or comparable to the wave-packet spreading of an atomic motional state, then it can not be considered as an infinite and uniform plane-wave electromagnetic field and the electric dipole interaction of Eq. (10) with space-independent Rabi frequencies is not suited to describe the STIRAP process, since in this case the Rabi frequencies are dependent upon the spatial coordinate [35].

As an important result in the paper, it is shown that if a free atom is in a Gaussian wave-packet motional state at the initial time, then it is still in a Gaussian wave-packet motional state after it is decelerated (or accelerated) by the STIRAP-based unitary decelerating (or accelerating) sequence in the ideal or nearly ideal adiabatic condition. As far as a Gaussian wave-packet state is concerned, there are two types of time- and space-dependent unitary evolution processes that do not change the Gaussian wave-packet shape of the atomic motional state. The first type is that the unitary evolution processes may manipulate and control the center-of-mass position and/or momentum of a Gaussian wave-packet state but can not manipulate at will the complex linewidth of a Gaussian wave-packet state. The second type is that the unitary evolution processes may manipulate and control the complex linewidth of a Gaussian wave-packet state. The STIRAP-based unitary decelerating and accelerating processes belong to the first type. This type of the unitary evolution processes tend to have the property that in the unitary evolution process the imaginary part of the complex linewidth of a Gaussian wave-packet state increases linearly with the time period of the unitary evolution process, while the real part usually keeps unchanged, i.e., the wave-packet spreading of the Gaussian wave-packet state becomes larger and larger early or late as the time period increases. These unitary evolution processes which have the property also include the free-particle motion and the atomic bouncing process off a hard potential wall in the special case. Thus, the free-particle motion and the atomic bouncing process may be assigned to the first type. The Hamiltonians of the quantum systems to create the first type of the time- and space-dependent unitary propagators usually can not be singly used to generate their inverse unitary propagators without any help of the interactions from outside the quantum systems. Therefore, this type of unitary evolution processes do not have their own inverse unitary propagators in these quantum systems separated from the outside or their environment. Obviously, these separate quantum systems also include the isolated quantum systems in the quantum statistical physics and they are described completely by the Hamiltonians of the quantum systems. These unitary evolution processes could be considered to be self-irreversible in the sense that there do not exist their own inverse unitary propagators in the same separate quantum systems, although these processes obey the unitary quantum dynamics and their inverse unitary propagators could be generated with the help of the specific interactions from outside the quantum systems. Take a free-particle motion as a typical example. A free-particle motion can be described completely by the unitary propagator $U(t) = \exp[-ip^2t/(2m\hbar)]$. Of course, one may also choose the unitary propagator $U(t)^+ = \exp[ip^2t/(2m\hbar)]$ to describe the free-particle motion. However, once one chooses one of the two unitary propagators to describe the free-particle motion, another is the inverse propagator of the chosen
unitary propagator. The free particle may move toward the left or the right in the coordinate axis and both the motions can be described by the same unitary propagator \( U(t) \). Now the inverse free-particle motion is described by the unitary propagator \( U(t) \). The Hamiltonian \( H = p^2/2m \) of the free particle can really generate only the unitary propagator \( U(t) \) but can not really generate both the unitary propagator \( U(t) \) and its inverse propagator \( U(t) \) simultaneously by the Schrödinger equation \( i\hbar \partial U(t)/\partial t = H(t)U(t) \). One may take a hermite conjugate on the Schrödinger equation and then could obtain \( U(t)^+ \), but taking a Hermite conjugate is not a real physical process and hence there does not exist the inverse propagator \( U(t)^+ \) in the free-particle quantum system. Then the inverse free-particle motion will never really take place if the free particle is not acted on by any external interaction. In other words, one may argue that the time reversal process (i.e., the inverse free-particle motion) may be a real physical process [25, 30], but this process can not really take place for the free particle without any help of the specific external interactions. Then in this sense the free-particle motion is really self-irreversible, although it is governed by the unitary propagator \( U(t) \). Of course, it is possible to construct the inverse unitary propagator of the free particle if the specific external interactions are applied to the free particle. How to construct these inverse propagators of the type of time- and space-dependent unitary propagators mentioned above will be reported in next paper. As pointed out in the previous papers [22, 36], such a situation that a quantum system that obeys the unitary quantum dynamics can not really have both the unitary propagator and its inverse propagator is often met in the quantum systems which are used to implement a quantum computation. The spontaneously irreversible processes of isolated quantum systems in the quantum statistical physics [40] could be related to the situation. It has been stressed in the previous papers [22, 36, 37] that these irreversible processes should be understood through the unitary quantum dynamics instead of the stochastic process and probability statistics [40], and they could be related to the difference between the unitary evolution process and its inverse process. On the other hand, there also exist other quantum systems that have both the unitary propagators and their inverse propagators, where both the types of the unitary propagators can be generated by the same Hamiltonians of the quantum systems. A conventional harmonic oscillator is typically one of such quantum systems. In the quantum statistical physics such quantum systems obey completely the Poincaré’s recurrence theorem.

There are the second type of time- and space-dependent unitary propagators that can manipulate and control the complex linewidth of a Gaussian wave-packet motional state. A general quadratic Hamiltonian can generate a time- and space-dependent unitary propagator that can keep the Gaussian shape unchanged for a Gaussian wave-packet motional state under the action of the unitary propagator. Obviously, the Hamiltonian is different from those of the STIRAP unitary decelerating and accelerating processes. One needs this type of unitary propagators to manipulate and control the complex linewidth of a Gaussian wave-packet motional state to build up the quantum circuit for the reversible and unitary state-insensitive halting protocol and also needs their in-
verse unitary propagators to realize the efficient quantum search process [22, 36]. How to construct these unitary propagators will be reported in next paper (Arxiv: quant-ph/0708.2129).

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