Commuting Difference Operators with Polynomial Eigenfunctions

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Abstract. We present explicit generators \( \hat{D}_1, \ldots, \hat{D}_n \) of an algebra of commuting difference operators in \( n \) variables with trigonometric coefficients. The algebra depends, apart from two scale factors, on five parameters. The operators are simultaneously diagonalized by Koornwinder’s multivariable generalization of the Askey-Wilson polynomials. For special values of the parameters and via limit transitions, one obtains difference operators for the Macdonald polynomials that are associated with (admissible pairs) of the classical root systems: \( A_{n-1}, B_n, C_n, D_n \) and \( BC_n \). By sending the step size of the differences to zero, the difference operators reduce to known hypergeometric differential operators. This limit corresponds to sending \( q \to 1 \); the eigenfunctions reduce to the multivariable Jacobi polynomials of Heckman and Opdam. Physically the algebra can be interpreted as an integrable quantum system that generalizes the (trigonometric) Calogero-Moser systems related to classical root systems.

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1 Introduction

Over the past few years, progress has been made with the study of orthogonal polynomials depending on more than one variable. It has turned out that such polynomials are not restricted to the two well-studied families of classical orthogonal polynomials depending on one variable; they are known as (generalized) Calogero-Moser systems \([C, Su1, Su2, OP2]\).

It has been shown by Heckman and Opdam that the hypergeometric differential operator of which the polynomials are eigenfunctions is but one member of an algebra of commuting PDO’s having the multivariable Jacobi polynomials related to root systems \([HO, H1]\). Recently, a more elementary account of some of these results and \([M3]\) for lectures devoted then Macdonald’s polynomials coincide with the continuous \(q\)-Jacobi polynomials. (For information on continuous \(q\)-Jacobi polynomials see e.g. \([A W]\)).

A crucial ingredient in the construction of the relevant operator of which the polynomials are eigenfunctions is the existence of an \( \mathbb{A}^n \)-type multivariable versions of the Askey-Wilson polynomials \([AW]\). Again, the results pertaining to the root system \(BC_n\) have been generalized by Koornwinder \([K4]\). He finds \(BC_n\)-type multivariable versions of the \( BC_n \) polynomials.

The motion of \(BC_n\) particles in \( \mathbb{R}^n \) is but one member of an algebra of commuting PDO’s having the multivariable Jacobi polynomials related to root systems \([HO, H1]\). Recently, a more elementary account of some of these results and \([M3]\) for lectures devoted then Macdonald’s polynomials coincide with the continuous \(q\)-Jacobi polynomials. (For information on continuous \(q\)-Jacobi polynomials see e.g. \([A W]\)).

The studies carried out in \([H1, H2, HO]\) and \([OP1, OP2]\) for the generalization of the classical systems have already been shown by Heckman and Opdam to be but one member of an algebra of commuting PDO’s.
polynomials as their joint eigenfunctions $\{\text{H}, \Phi\}$. This algebra is generated by $n$ independent PDO’s. This state of affairs can be expressed by saying that the corresponding (generalized) Calogero-Moser system is quantum integrable. For $R = A_{n-1}$, Liouville integrability of the classical system, i.e. the existence of $n$ independent integrals in involution, was already proved by Moser using a Lax pair formulation. For a partial generalization of this result to the other classical (i.e. non-exceptional) root systems, see OP2 and OP3.

Just as the hypergeometric PDO, Macdonald’s difference operator for $R = A_{n-1}$ is related to certain known quantum systems of $n$ particles. The systems of interest were originally introduced as a relativistic generalization of the Calogero-Moser systems (classical) and (quantum). (See [R2] for a survey and connections with certain soliton PDE’s and exactly solvable quantum field theories). The relativistic generalization of the Calogero-Moser system is (quantum) integrable too; explicit formulas representing PDE’s and exactly solvable quantum field theories). The relativistic generalization of the Calogero-Moser system is (quantum) integrable too; explicit formulas representing PDE’s and exactly solvable quantum field theories). The relativistic generalization of the Calogero-Moser system is (quantum) integrable too; explicit formulas representing PDE’s and exactly solvable quantum field theories). The relativistic generalization of the Calogero-Moser system is (quantum) integrable too; explicit formulas representing PDE’s and exactly solvable quantum field theories). The relativistic generalization of the Calogero-Moser system is (quantum) integrable too; explicit formulas representing PDE’s and exactly solvable quantum field theories).

By sending the step size of the differences to zero (this corresponds to the limit $\epsilon \to 0$) our A∆O’s go over in PDO’s. Thus, we recover the commuting hypergeometric PDO’s associated with the classical root systems as a limit case.

Our difference operators constitute a new integrable quantum system of $n$ particles in dimension one. In this paper, however, we will not pay much attention to this interpretation of the A∆O’s; instead we will emphasize the connection with orthogonal polynomials. In a forthcoming paper the author intends to return to the question of integrability of these and related $n$-particle systems, at the level of both quantum and classical mechanics [OP3]. In particular, possible generalization to integrable systems consisting of commuting difference operators with elliptic functions as coefficients will be discussed in [OP4].

Before outlining the contents of this paper in more detail, let us mention two more connections of interest. For special values of the parameters (namely those corresponding to root multiplicities), the system of hypergeometric PDO’s coincides with the radial reduction of the algebra of invariant differential operators on certain symmetric spaces $G/K$. It seems natural to ask oneself the question whether, for special values of the parameters, our system of difference operators can be seen in some way as radial reduction of certain A∆O’s connected with quantum homogeneous spaces. Recent results on the quantum group interpretation of Macdonald’s $A_{n-1}$-type polynomials [K3, N] indeed seem to point in this direction. However, apart from this special case no relations of this kind are known to the author.

Recently, Cherednik introduced commuting difference operators connected with Knizh-
2 Introducing the Difference Operators

In this section the operators \( \hat{D}_r \), \( r = 1, \ldots, n \), are introduced and their combinatorial structure is discussed.

2.1 The Operator \( \hat{D}_r \)

In order to write down our difference operators we first introduce some notation. Let \( v_a(z) \) and \( v_b(z) \) be the following trigonometric functions:

\[
\begin{align*}
v_a(z) & = \frac{\sin \alpha_0 + z}{\sin \alpha z}, \\
v_b(z) & = \frac{\sin \alpha_0 + z \cos \alpha_1 + z}{\sin \alpha z} \frac{\cos \alpha_0 + \gamma + z \cos \alpha_1 + \gamma + z}{\cos \alpha_1 + \gamma + z},
\end{align*}
\]

(2.1)

(2.2)

with \( \alpha, \gamma \) and \( \mu, \mu_0, \mu_1, \delta \), \( (\delta = 0, 1) \) complex parameters. For later purposes, it is convenient to parametrize \( \gamma \) according to:

\[
\gamma \equiv i \beta / 2.
\]

(2.3)

We form the following multivariable functions using \( v_a \) and \( v_b \) as elementary constituents:

\[
V_{e,J,K} \equiv \prod_{j \in J} v_a(\varepsilon_j z_j) \prod_{j' < j} v_a(\varepsilon_j z_j + \varepsilon_j' z_j') v_a(\varepsilon_j z_j + \varepsilon_j' z_j' + 2\gamma)
\]

\[
\times \prod_{j \in J, k \in K} v_a(\varepsilon_j z_j + \varepsilon_k z_k) v_a(\varepsilon_j z_j - \varepsilon_k z_k),
\]

(2.4)

with \( J, K \subset \{1, \ldots, n\} \), \( J \cap K = \emptyset \), \( \varepsilon_j \in \{+1, -1\} \).

(2.5)

The variables \( x_1, \ldots, x_n \) are assumed to be real. The function \( V_{e,J,K} \) depends on the index sets \( J, K \) and on a collection of prescribed signs \( \varepsilon_j, j \in J \); it serves as a building block from which the coefficients of \( \hat{D}_r \) are constructed. The \( \Lambda \Delta \Omega \)’s read explicitly

\[
\hat{D}_r \equiv \sum_{J \subset \{1, \ldots, n\}, |J| = r} \sum_{\varepsilon_j, j \in J} (-1)^{s-1} \prod_{1 \leq s' \leq s} V_{e(J_s \setminus J_{s-1}), J_{s-1}, J_{s-2}, \ldots, J_1} \left( e^{-\beta \hat{\theta}_1 z_1} - 1 \right),
\]

(2.6)

\[
r = 1, \ldots, n,
\]

with \( J_0 \equiv \emptyset \) and

\[
\hat{\theta}_e, J \equiv \sum_{j \in J} \varepsilon_j \hat{\theta}_j, \quad \hat{\theta}_j \equiv \frac{1}{i} \frac{\partial}{\partial x_j}.
\]

(2.7)

Remarks

i. \( |J| \) denotes the cardinality of \( J \) and \( J^c \) is the complement set \( \{1, \ldots, n\} \setminus J \).

ii. The first summation in Eq. (2.6) is over all index sets \( J \subset \{1, \ldots, n\} \) with cardinality \( r \) and over all flippings of the signs \( \varepsilon_j \in \{+1, -1\}, j \in J \); the second summation is over all strictly increasing sequences of subsets in \( J \):

\[
\emptyset \subset J_1 \subset J_2 \subset \cdots \subset J_{s-1} \subset J_s = J, \quad 1 \leq s \leq |J|.
\]

(2.8)

iii. The exponential \( \exp(-\beta \hat{\theta}_j) \) acts on a field of variables \( x_1, \ldots, x_n \) as a (complex) shift:

\[
\left( e^{-\beta \hat{\theta}_j} f \right)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n).
\]

Hence, \( \hat{D}_r \) is indeed an analytic difference operator.

iv. The ‘hats’ in Eqs. (2.6), (2.7) are used rather than ordinary (complex) functions of \( \varepsilon_j \).

v. In the simplest case, i.e. for \( r = 1 \), Eq. (2.6) reads

\[
\hat{D}_1 = \sum_{1 \leq j \leq n} v_a(\varepsilon_j z_j) \prod_{k \neq j} v_a(\varepsilon_j z_j)
\]

with \( J_{-1} \equiv \emptyset \) and

\[
\hat{D}_r = \sum_{0 \leq s \leq r} \sum_{\varepsilon_j, j \in J} (-1)^s \prod_{1 \leq s' \leq s} V_{e(J_s \setminus J_{s-1}), J_{s-1}, J_{s-2}, \ldots, J_1} \left( e^{-\beta \hat{\theta}_1 z_1} - 1 \right)
\]

(2.9)

(2.10)

(2.11)

(2.12)

(2.13)

(2.14)

\[
W_{J_0} \equiv \sum_{1 \leq s \leq r} (-1)^s \sum_{\varepsilon_j, j \in J} (-1)^s \prod_{1 \leq s' \leq s} V_{e(J_s \setminus J_{s-1}), J_{s-1}, J_{s-2}, \ldots, J_1} \left( e^{-\beta \hat{\theta}_1 z_1} - 1 \right)
\]

(2.15)

\[
W_{J_0} \equiv 1.
\]

(2.16)

In Eq. (2.12), \( \hat{D}_r \) is written in terms of the \( \Lambda \Delta \Omega \)’s. Accordingly, \( J_0 \) in Eq. (2.12) reflects the fact that the coefficients of the translation operator do not commute with the translation operator. Hence, \( \hat{D}_r \) is a \( \Lambda \Delta \Omega \), which does not commute with the translation operator.

Note. From a physical point of view, one can view the \( \hat{D}_r \) as a Hamiltonian for a particle quantum system in dimension one. The functions of the type

\[
\left( e^{-\beta \hat{\theta}_j} - 1 \right)
\]

which does not commute with the particle quantum system in dimension one. The functions of the type

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2.2 Combinatorial Structure and Parameters

The increment sets \( J_1, J_2 \setminus J_1, \ldots, J_s \setminus J_{s-1} \) of the increasing sequence (2.8) form the blocks of a partition of \( J \); the second summation in (2.9) amounts to a sum over all ordered blocks. By breaking up \( V_{\epsilon,J,K} \) into three parts (Eq. (2.4))

\[
V_{\epsilon,J,K} = V_{1,J}^1 V_{2,J}^2 V_{3,J,K}^3
\]

(2.15)

with

\[
V_{1,J}^1 = \prod_{j \in J} v_a(\varepsilon_j x_j),
\]

(2.16)

\[
V_{2,J}^2 = \prod_{j,j' \in J, j < j'} v_a(\varepsilon_j x_j + \varepsilon_j' x_j' + 2\gamma),
\]

(2.17)

\[
V_{3,J,K}^3 = \prod_{j \in J, k \in K} v_a(\varepsilon_j x_j + x_k) v_a(\varepsilon_j x_j - x_k),
\]

(2.18)

one can rewrite \( \hat{D}_r \) (Eq. (2.9)) as

\[
\hat{D}_r = \sum_{J \subset \{1, \ldots, n\}, |J| = r, \varepsilon_j = \pm 1, j \in J} \left\{ V_{1,J}^1 V_{2,J}^2 V_{3,J,K}^3 \right\}
\]

(2.19)

\[
\times \sum_{\begin{array}{c}
s \subseteq J \setminus \varepsilon_J \setminus J_{s-1} \setminus J_{s-2} \\
1 \leq s' \leq r
\end{array}} (-1)^{s'-1} \prod_{1 \leq s'' \leq s} V_{3,J_{s''}\setminus J_{s''-1}}^3 V_{1,J_{s''}\setminus J_{s''-1}, J_{s''} \setminus J_{s''-1}}^1 \left( e^{-\beta \delta s J_1} - 1 \right).
\]

(2.20)

Eqs. (2.10), (2.12) and (2.13) are more compact than (2.11), but the latter has the virtue that different parts of the coefficient can be controlled independently. The index set \( J \) in Eq. (2.11) will be referred to as the cell. The first block \( J_1 \) determines the translator; this part of the cell will be called the nucleus. Notice that: i. \( V_{1,J}^1 V_{2,J}^2 V_{3,J,K}^3 \) depends on the cell \( J \) but not on its subdivision in blocks; ii. the product over \( V_{2,J_{s''}\setminus J_{s''-1}}^2 \) depends on the partition of \( J \), but not on the order of the blocks; iii. the product over \( V_{3,J_{s''}\setminus J_{s''-1}, J_{s''} \setminus J_{s''-1}}^3 \) depends both on the blocks and on their order.

The parameters \( \alpha \) and \( \beta \) are scale factors; \( \alpha \) determines the period of the trigonometric functions and \( \beta \) the complex shift of the translation operators \( \exp(\pm i \theta) \). Both parameters will be taken positive. The parameters \( \mu, \mu_s, \mu_s' \) and \( \mu_s'' \) determine the relative ‘weight’ of \( v_a \) and \( v_b \) in the coefficients of the \( \Delta \)-AO. For instance, \( v_a \equiv 1 \) for \( \mu = 0 \); therefore, \( v_a \) may be omitted for \( \mu = 0 \). The parameters \( \mu, \mu_s, \mu_s' \) and \( \mu_s'' \) will be assumed to be non-negative imaginary:

\[
\mu \equiv i \beta g, \quad \mu_s \equiv i \beta' g_s, \quad \mu_s' \equiv i \beta g_s', \quad (\delta = 0, 1)
\]

\[
\alpha, \beta > 0; \quad g, g_s, g_s' \geq 0.
\]

Note that the above restrictions on the parameters guarantee that: i. \( \exp(\pm i \theta) \) yields a purely imaginary shift; ii. the commuting part of the coefficient, viz. \( W_{\epsilon,J,r-s} \) (cf. Eqs. 2.13, 2.14), is real because \( v_a(z) = v_a(-z) \) for \( z \) real.

If one picks \( \alpha = 1/2 \), then \( \hat{D}_1 \) (Eq. (2.21)) is the generating function of the \( \Delta \)-AO, with Koornwinder’s different constants that determine the strengths of the various interactions. In this interpretation, \( \mu = 0 \) (i.e. \( \mu = 0 \)) yields a system with an external field.

2.3 \( g = 0 \): Reduction to Rank

By setting \( g = 0 \), the combinatorial structure of the coefficients in Eq. (2.11) no longer depend on the cell \( J \) = \{1, \ldots, n\}.\( ^{[K4]} \) It will be shown next that in this case \( \hat{D}_1 \) (Eq. (2.10)) coincides, up to an irrelevant multiplicative constant, with Koornwinder’s difference operator (Eq. (2.10))

\[
\hat{D}_1(x_j) \equiv \sum_{\varepsilon_j = \pm 1} v_b(\varepsilon_j x_j) \left( e^{\beta \hat{s} J_1} - 1 \right)
\]

(For \( n = 1, \hat{D}_1 \) coincides with \( \hat{D}_1(x_1) \). By summation of all terms in \( \hat{D}_r \) which correspond to \( J_1 = \{1\} \), Eq. (2.11) reduces to

\[
\hat{D}_r = \sum_{J \subset \{1, \ldots, n\}, |J| = r} \sum_{\varepsilon_j = \pm 1, j \in J} \left\{ V_{1,J}^1 V_{2,J}^2 V_{3,J,K}^3 \right\}
\]

\[
\times \sum_{\begin{array}{c}
s \subseteq J \setminus \varepsilon_J \setminus J_{s-1} \setminus J_{s-2} \\
1 \leq s' \leq r
\end{array}} (-1)^{s'-1} \prod_{1 \leq s'' \leq s} V_{3,J_{s''}\setminus J_{s''-1}}^3 V_{1,J_{s''}\setminus J_{s''-1}, J_{s''} \setminus J_{s''-1}}^1 \left( e^{-\beta \delta s J_1} - 1 \right).
\]

(2.20)

where

\[
c_0 \equiv 1, \quad c_p \equiv \sum_{1 \leq p_1 + \ldots + p_s = p} \binom{p}{p_1, \ldots, p_s}
\]

(2.21)

with

\[
N_{p,s} \equiv \sum_{p_1 + \ldots + p_s = p, p_j \geq 1} \binom{p}{p_1, \ldots, p_s}
\]

To verify this, think of \( N_{p,s} \) as the number of ways of distributing \( s \) distinguishable objects into \( p \) distinct slots.

**Lemma 2.1**

One has

\[
c_p \equiv \sum_{1 \leq p_1 + \ldots + p_s = p} \binom{p}{p_1, \ldots, p_s}
\]

(2.22)
Proof
The above interpretation of \( N_{p,s} \) leads to the recursion relation
\[
N_{p,s} = \sum_{s-1 \leq q \leq p-1} \binom{p}{p-q} N_{q,s-1}, \quad (p \geq s > 1), \quad N_{p,1} = 1. \tag{2.28}
\]
Substituting (2.28) in (2.25) yields a recursion relation for \( c_p \)
\[
c_p = -\sum_{0 \leq q \leq p-1} \binom{p}{q} c_q, \tag{2.29}
\]
whose unique solution is (2.27).

Definition 2.2 Let \( t_1, \ldots, t_n \) belong to a commutative algebra. The \( r \)th elementary symmetric function \( S_r \) of \( t_1, \ldots, t_n \) is defined as
\[
S_r(t_1, \ldots, t_n) = \sum_{J \subseteq \{1, \ldots, n\}, |J| = r} \prod_{j \in J} t_j, \quad r = 1, \ldots, n. \tag{2.30}
\]

Proposition 2.3 If \( g = 0 \), then
\[
\hat{D}_r = S_r(\hat{D}_1(t_1), \ldots, \hat{D}_1(t_n)), \quad r = 1, \ldots, n \tag{2.31}
\]
(with \( \hat{D}_1(t_j) \) defined by Eq. (2.24)).

Proof
Substituting (2.28) in (2.24) yields
\[
\hat{D}_r = \sum_{J \subseteq \{1, \ldots, n\}, |J| = r} \sum_{\substack{J \subseteq \{1, \ldots, n\}\; e_j = \pm 1, \; j \not \in J}} (-1)^{r-|J|} \epsilon_{\theta_{\hat{D}_r}^J} \tag{2.32}
\]
Using Eq. (2.16), one completes the proof of the proposition:
\[
\hat{D}_r = \sum_{J \subseteq \{1, \ldots, n\}, |J| = r} \prod_{j \not \in J} v_\theta(\epsilon_j x_j) \left( e^{-\beta_{\hat{D}_r}^J} - 1 \right) \tag{2.33}
\]
\[
= \sum_{J \subseteq \{1, \ldots, n\}, |J| = r} \prod_{j \not \in J} \hat{D}_1(t_j) = S_r(\hat{D}_1(t_1), \ldots, \hat{D}_1(t_n)). \tag{2.34}
\]

Note. Proposition 2.3 is in accordance with the previously noted fact that for \( g = 0 \) the particles of the quantum system become independent.

3 Simultaneous Diagonalization

In this section it is shown that Koornwinder’s polynomials form a basis of joint eigenfunctions of \( \hat{D}_1, \ldots, \hat{D}_n \). We prove that the different eigenvalues. As a result, we obtain an explicit expression for the abelian algebra generated by \( \hat{D}_1, \ldots, \hat{D}_n \).

For convenience, we will put \( \alpha \) = 1/2 from now on.

3.1 Trigonometric Polynomials

Let \( A = \mathbb{C}[\exp(ix_1), \ldots, \exp(ix_n)] \) be the algebra of trigonometric polynomials on \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \). The subalgebra \( \mathbb{A} \) of \( \mathbb{T} \) is spanned by the Fourier basis \( \{e^\lambda \} \) with \( \lambda \in \mathbb{R}^n \).

Let \( W \) be the (Weyl) group of permutations and sign flips of the variables such that \( W \supseteq S_n \times (\mathbb{Z}_2)^n \). The subalgebra \( \mathbb{A} \) of \( \mathbb{T} \) is spanned by the Fourier basis \( \{e^\lambda \} \) with \( \lambda \in \mathbb{R}^n \) and \( \lambda \in W \lambda \).

The lattice \( \mathcal{P} \) can be partially ordered in the following way:

\[ (\forall \lambda, \lambda' \in \mathcal{P}) : \quad \lambda' \leq \lambda \iff \sum_{1 \leq j \leq n} \lambda_j' \leq \lambda_j \quad (\lambda_j' \geq \lambda_j) \]

The above ordering induces a partial ordering on each dominant weight \( \lambda \in \mathcal{P}^+ \) associated with the highest weight \( \lambda \):

\[ \mathcal{A}^\lambda_{\text{W}} \equiv \text{span}\{m_{\lambda'}(\cdot)\}_{\lambda' \in \mathcal{P}} \]

Occasionally we will also use the notation

\[ |\lambda| = \sum_{1 \leq j \leq n} \lambda_j \]
3.2 Triangularity

In this subsection it is shown that $\hat{D}_r$ maps the highest weight spaces $A^{W}_\lambda$ into itself.

Definition 3.2 A linear operator $\hat{D} : A^{W}_\lambda \rightarrow A^{W}_\lambda$ is called triangular if

$$\hat{D}(A^{W}_\lambda) \subset A^{W}_\lambda, \quad \forall \lambda \in \mathcal{P}^+.$$  \hspace{1cm} (3.8)

One can rewrite Eq. (3.8) in a more illuminating way:

$$(\forall \lambda \in \mathcal{P}^+) : \quad \hat{D} m_\lambda = \sum_{\lambda' \in \mathcal{P}^+, \lambda' \preceq \lambda} [\hat{D}]_{\lambda, \lambda'} m_{\lambda'}, \quad \text{with} \quad [\hat{D}]_{\lambda, \lambda'} \in \mathbb{C}  \hspace{1cm} (3.9)$$

(i.e. $[\hat{D}]_{\lambda, \lambda'} = 0$ if $\lambda' \not\preceq \lambda$). In order to prove that $\hat{D}_r$ is triangular, we first need to verify that the operator maps $A^{W}_\lambda$ into itself.

Proposition 3.3 (invariance of $A^{W}_\lambda$)

$$\hat{D}_r(A^{W}_\lambda) \subset A^{W}_\lambda, \quad r = 1, \ldots, n.  \hspace{1cm} (3.10)$$

Proof

Acting with $\hat{D}_r$ Eq. (2.11) on a monomial $m_\lambda$ (3.3) yields the following $W$-invariant trigonometric function on the torus $T$:

$$(\hat{D}_r m_\lambda)(x) = \sum_{J \subseteq \{1, \ldots, n\}, |J| = r, \epsilon_j = \pm 1} \sum_{s \leq \rho \in P} V^2_{\epsilon \lambda J, J, x_\epsilon} \prod_{1 \leq \rho \leq s} V^2_{e_{\rho J \setminus J_{\rho - 1}}} \prod_{J \subseteq \{1, \ldots, n\}, |J| = r} \left[ m_\lambda(x + 2\gamma e_{\rho J, 1}) - m_\lambda(x) \right],  \hspace{1cm} (3.11)$$

with

$$e_{\epsilon J} = \sum_{j \in J} \epsilon_j e_j  \hspace{1cm} (3.12)$$

$\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{R}^n$. The r.h.s. of (3.11) is rational in the exponentials $\exp(i x_j), j = 1, \ldots, n$. In order to prove the proposition, we need to show that $\hat{D}_r m_\lambda$ (3.11) is actually a polynomial in $\exp(i x_1), \ldots, \exp(i x_n)$. Since the r.h.s. of (3.11) is symmetric in $x_1, \ldots, x_n$ it suffices to verify that $\hat{D}_r m_\lambda$, viewed as a function of $x_1$, is free of poles.

As a function of $x_1$, the terms in (3.11) may have poles caused by zeros in the denominators of the coefficients of the $A \Delta O$. These poles are located at (cf. Eqs. (2.16), (2.18) and (2.1), (2.2)):

$$x_1 = 0 \mod \pi, = \pm \gamma \mod \pi, = \pm x_j \mod 2\pi, j = 2, \ldots, n, = \pm x_j \pm 2\gamma \mod 2\pi, j = 2, \ldots, n.  \hspace{1cm} (3.13)$$

From now on the parameters $\gamma, \mu, \mu_0, \delta$ ($\delta = 0, 1$) and the remaining variables $x_2, \ldots, x_n$ are fixed in general position. Specifically, we choose these parameters and variables such that the poles in the terms of (3.11) are simple.

The residue at $x_1 = 0$ vanishes because $x_j$ vanishes because (3.11) is invariant under translations $x_j \rightarrow x_j + \pi, j = 1, \ldots, n$. Furthermore, because $Dm_\lambda(x)$ is even in $x_j$ over half the period (cf. Remark vii, Section 2.1):

$$x_j \rightarrow x_j + \pi, j = 1, \ldots, n$$

(with $|\lambda| = \sum_{j=1}^n \lambda_j$), we need only show that

$$x_1 = -\gamma = -x_j - 2\gamma  \hspace{1cm} (3.14)$$

(type I: $x_1 = -\gamma$).

The only terms in the r.h.s. of (3.11) that contribute to the residue at $x_1 = -\gamma$ in the sum of all $\hat{D}_r m_\lambda$ corresponding to a fixed cell $J$, with the signs prescribed, is zero. One may see this is because the general situation can be obtained by an appropriate flipping of the signs of the variables $x_j, j \in J$, with $1$. Again the general case (corresponding to an arbitrary cell $J$) can be obtained by an appropriate flipping of the signs.

We conclude that $\hat{D}_r m_\lambda$ is a $W$-invariant polynomial in $x_1, \ldots, x_n$. Consequently, $\hat{D}_r m_\lambda$ must be a polynomial in $x_1$, as claimed.

Proposition 3.3 says that $\hat{D}_r m_\lambda$ is a $W$-invariant polynomial in $x_1, \ldots, x_n$.

$$(\forall \lambda \in \mathcal{P}^+) : \quad \hat{D}_r m_\lambda = \sum_{\lambda' \in \mathcal{P}^+} \sum_{\lambda' \in \mathcal{P}^+} [\hat{D}]_{\lambda, \lambda'} m_{\lambda'}, \quad \text{with} \quad [\hat{D}]_{\lambda, \lambda'} \in \mathbb{C}$$

In order for $\hat{D}_r$ to be triangular one must have

$$\lambda' \in \mathcal{P}^{+}_{\lambda, \lambda'}  \hspace{1cm} (3.15)$$
We shall prove this property by studying the asymptotics of \( \hat{D}_\tau m_\lambda(x) \) for \( \text{Im } x_j \to -\infty \). The following limits will be useful (cf. Eqs. (2.1), (2.2) and (2.20)):

\[
\begin{align*}
\lim_{R \to \infty} v_0(z + i\varepsilon R) &= e^{\varepsilon \beta y/2}, \\
\lim_{R \to \infty} v_0(z + iR) &= e^{\varepsilon(\beta y_0 + \beta g_0 + \beta^* y_0^*)/2}
\end{align*}
\]

with \( \varepsilon = \pm 1 \).

**Proposition 3.4 (triangularity)**

\[
(\forall \lambda \in \mathcal{P}^+): \quad \hat{D}_\tau(A_\lambda^W) \subset A_\tau^W, \quad \tau = 1, \ldots, n.
\]

**Proof**

Fix an \( r \in \{1, \ldots, n\} \) and \( \lambda \in \mathcal{P}^+ \). Let \( \omega_k \equiv \sum_{1 \leq j \leq k} e_j \) (the \( k \)th fundamental weight) and introduce (cf. (3.17))

\[
M_{\lambda, r, k} \equiv \max \{ (\lambda', \omega_k) \mid \lambda' \in \mathcal{P}^+_{\lambda, r} \}.
\]

To derive a contradiction, assume (3.18) does not hold; i.e. assume that there exists a \( k \in \{1, \ldots, n\} \) such that

\[
M_{\lambda, r, k} > (\lambda, \omega_k) \left( \sum_{1 \leq j \leq k} \lambda_j \right).
\]

Now it is easy to verify the asymptotics

\[
m_{\lambda}(x - iR\omega_k) \sim e^{R(\lambda' - \omega_k)} \sum_{\lambda'' \in W_{\lambda, r}(\lambda')} e^{\lambda''}(x), \quad R \to +\infty,
\]

with

\[
W_{\lambda', k} \equiv \{ w \in W \mid (w, \lambda', \omega_k) = (\lambda', \omega_k) \},
\]

so using (3.10) we obtain

\[
\lim_{R \to \infty} e^{-RM_{\lambda, r, k}}(\hat{D}_\tau m_\lambda)(x - iR\omega_k) = \sum_{\lambda' \in \mathcal{P}^+_{\lambda, r}} [\hat{D}_\tau]_{\lambda, \lambda'} \left( \sum_{\lambda'' \in W_{\lambda', r}(\lambda')} e^{\lambda''}(x) \right).
\]

On the other hand, Eq. (3.24) combined with the limits (3.19) and (3.20) entails the following asymptotics for (3.11):

\[
(\hat{D}_\tau m_\lambda)(x - iR\omega_k) = O(e^{R(\lambda, \omega_k)}), \quad R \to \infty.
\]

Consequently,

\[
\lim_{R \to \infty} e^{-RM_{\lambda, r, k}}(\hat{D}_\tau m_\lambda)(x - iR\omega_k) = 0
\]

(because of inequality (3.23)).

The matrix elements \([\hat{D}_\tau]_{\lambda, \lambda'}\) in (3.26) are non-zero (by definition), and the exponentials \( e^{\lambda''}(x) \) (3.3) corresponding to different weights \( \lambda'' \in \mathcal{P} \) are linearly independent. Hence, by comparing the r.h.s. of Eqs. (3.26) and (3.28) one arrives at the desired contradiction. \( \square \)

### 3.3 The Spectrum

By extending \( \leq \) (Definition 3.1) to a linear ordering, it is easy to see that the triangular matrix \([\hat{D}_\tau]_{\lambda, \lambda'}\) has as consequence that the elements on the diagonal of the matrix \([\hat{D}_\tau]_{\lambda, \lambda'}\) (it will become clear in Section 3.5 that in our case these eigenvalues are semisimple). The purpose of the present subsection is to compute \([\hat{D}_\tau]_{\lambda, \lambda'}\).

Let \( y \in \mathbb{R}^n \) be a fixed vector subject to the condition

\[
y_1 > y_2 > \ldots > y_n.
\]

By combining the asymptotics (cf. (3.24))

\[
m_\lambda(iRy) \sim e^{R} \sum_{\lambda' < \lambda} \lambda' \chi_{\lambda'}(y), \quad R \to \infty,
\]

with Eq. (3.23), one finds (use \( \lambda' < \lambda \Rightarrow \sum_{\lambda' < \lambda} \lambda' \chi_{\lambda'}(y) = \sum_{\lambda' < \lambda} \lambda' \chi_{\lambda'}(y) = \lim_{R \to \infty} e^{-R} \chi_{\lambda'}(y) \) )

\[
[\hat{D}_\tau]_{\lambda, \lambda'} = \sum_{R \to \infty} e^{-R} \chi_{\lambda'}(y).
\]

In the next proposition, we will evaluate the matrix\( [\hat{D}_\tau]_{\lambda, \lambda'} \).

**Proposition 3.5 (eigenvalues)**

One has

\[
[\hat{D}_\tau]_{\lambda, \lambda'} = 2^r E_{r, n}(ch\beta(\lambda_1 + \rho_1), \ldots, \chi_{\lambda_n}(y))
\]

with

\[
E_{r, n}(t_1, \ldots, t_n; p_1, \ldots, p_n) \equiv \sum_{0 \leq s \leq r} (-1)^{s+r} \left( \sum_{J \subseteq \{1, \ldots, n\}} \prod_{i \in J} \rho_i \right)
\]

and

\[
\rho_j \equiv (n - j)g + (g_0 + g_1 + g_2)
\]

**Proof**

Using Eq. (2.13) we obtain

\[
(\hat{D}_\tau m_\lambda)(x) = \sum_{0 \leq s \leq r} \sum_{J \subseteq \{1, \ldots, n\}} \prod_{i \in J} \rho_i
\]

\( \square \)
with $V_{x,J}$ and $W_{x,p}$ defined by Eqs. (2.4) and (2.14), respectively. In order to compute the limit (3.32), we first derive some preliminary asymptotics:

i. $m_\lambda$ (cf. Eq. (3.24)):

$$m_\lambda(x + i\beta \varepsilon_j, y_j) \sim e^{R \sum_{j=1}^\infty \lambda_j y_j} e^{\sum_{j \in J} \varepsilon_j \lambda_j}, \quad R \to \infty.$$  (3.37)

From (3.19), (3.20) one deduces

$$\lim_{R \to \infty} V_{x,J}(x + iy_j) = e^{\beta \gamma N_0 J + (g_0 + g_1 + g'_1) M_{x,J}/2},$$  (3.38)

with

$$M_{x,J} = \sum_{j \in J} \varepsilon_j.$$

Using (2.14) and (3.41) it is not hard to see that $\lim_{R \to \infty} R_{x,J}(x + i\beta \varepsilon_j)$ exists and depends only on the cardinality of $I$.

ii. $V_{x,J}(x)$:

$$\lim_{R \to \infty} V_{x,J}(x) = e^{\beta \gamma N_0 J + (g_0 + g_1 + g'_1) M_{x,J}/2},$$  (3.39)

and

$$N_{x,J} = 2|\{j, k \in J \mid j < k, \varepsilon_j = +1\}| - 2|\{j, k \in J \mid j < k, \varepsilon_j = -1\}| + 2|\{j \in J \mid j < k, \varepsilon_j = +1\}| - 2|\{j \in J \mid j < k, \varepsilon_j = -1\}| + 2|\{j \in J \mid k, j, n \mid j < k, \varepsilon_j = +1\}| - 2|\{j \in J \mid j, k, n \mid j < k, \varepsilon_j = -1\}| = 2\sum_{j \in J} (n - j) \varepsilon_j.$$  (3.40)

Consequently,

$$\lim_{R \to \infty} V_{x,J}(x) = e^{\beta \gamma N_0 J + (g_0 + g_1 + g'_1) M_{x,J}/2},$$

with $\rho_j$ defined by (3.35).

Using (2.14) and (3.41) it is not hard to see that $\lim_{R \to \infty} W_{x,p}(x + i\beta \varepsilon_j)$ exists and depends only on the cardinality of $I$ and on $p$ (but not on the number of variables $n$). We define

$$F_{l,p} \equiv 2^{-p} \lim_{R \to \infty} W_{l,p}(x + iy_j).$$  (3.42)

Notice that (cf. Eq. (2.14))

$$F_{0,0} = 1, \quad m = 0, \ldots, n.$$  (3.43)

After these preliminaries, we are now ready to compute limit (3.32). Substituting (3.36) in (3.32), and making use of (3.37), (3.41) and (3.42), we obtain

$$[\tilde{D}_x]_{\lambda,\lambda} = 2^r \sum_{J \subseteq \{1, \ldots, n\}} \prod_{j \in J} \gamma \beta (\lambda_j + \rho_j) F_{n - s, r - s}.$$  (3.44)

It remains to calculate $F_{m,p}$, $1 \leq p \leq m \leq n$. Consider $\lambda = 0$:

$$\sum_{\sum_{0 \leq s \leq r} J \subseteq \{1, \ldots, n\}} \prod_{j \in J} \gamma \beta (\lambda_j + \rho_j).$$

For a fixed number of variables, this yields the same coefficients $F_{m,p}$ occur, now with $0 \leq p \leq m$, for $n = 1, \ldots, n-1$, and making use of Eq. (3.44), one obtains, therefore, the following relations for $\rho_j$ (unlike $\rho_j$) does not depend on the number of variables $n$ +1/2 $F_{m,p} = (-1)^{p/2}$

$$\sum_{\sum_{0 \leq s \leq r} J \subseteq \{1, \ldots, n\}} \prod_{j \in J} \gamma \beta (\lambda_j + \rho_j).$$

with condition (3.43). In Lemma B.3 of Appendix B, the above system has a unique solution:

$$F_{m,p} = (-1)^{p/2} \sum_{\sum_{0 \leq s \leq r} J \subseteq \{1, \ldots, n\}} \prod_{j \in J} \gamma \beta (\lambda_j + \rho_j).$$

Substituting (3.47) in (3.44) and using $\tilde{D}_x$ (3.33).

For $g = 0$ all components of the vector $\rho_j$ are linear,

$$\rho_1, \rho_2, \ldots, \rho_n = (g_0 + g_1 + g'_1)$$

Then, one can rewrite the above expression for $\gamma \beta (\lambda_j + \rho_j)$ symmetric functions (see the remark following)

$$[\tilde{D}_x]_{\lambda,\lambda} = 2^r \sum_{J \subseteq \{1, \ldots, n\}} \prod_{j \in J} (\lambda_j + \rho_j).$$

This equation is in agreement with Proposition 2.4 in Ref. [K4].

### 3.4 Symmetry

In Ref. [K4] the following weight function on the torus was introduced:

$$\Delta(x) \equiv \prod_{1 \leq j < k \leq n} d_{n}(x_j + x_k).$$

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\[
\begin{align*}
  d_c(z) &= d_c^+(z)d_c^-(z), & c = a, b \\
  d_a^+(z) &= \frac{(e^{iz}; e^{-\beta})_{\infty}}{(e^{-\beta}g_1e^{iz}; e^{-\beta})_{\infty}}, \quad \text{Eq. (3.52)} \\
  d_b^+(z) &= \frac{(e^{-2iz}; e^{-\beta})_{\infty}}{(e^{\beta}_0e^{iz}, e^{-\beta}_{s_1}e^{iz}, e^{-\beta}(s_1^0+1/2)e^{iz}, e^{-\beta}(s_1^0+1/2)e^{iz}; e^{-\beta})_{\infty}}. \quad \text{Eq. (3.53)}
\end{align*}
\]

The so-called $q$-shifted factorials are defined in the usual way:

\[
(a; q)_{\infty} \equiv \prod_{m=1}^{\infty} (1 - aq^m), \quad (a_1, \ldots, a_r; q)_{\infty} \equiv \prod_{i=1}^{r} (a_i; q)_{\infty}. \quad \text{Eq. (3.54)}
\]

Notice that the conditions on our parameters, viz. \[(2.20),\] guarantee that the infinite products in Eqs. (3.52) and (3.53) converge. Recall that in order to compare our formulas with respect to the measure $\Delta$ dx, the space of $W$-invariant polynomials $A^W$ is dense in $L^2_v(\mathbb{T}, \Delta dx)$. The purpose of the present section is to show that the $\Lambda$DO's $\hat{\varepsilon}$'s are $W$-invariant with respect to $\Delta dx$. We define

\[
\langle f, g \rangle_{\Delta} \equiv \int_{\mathbb{T}} f(z) g(z) \Delta dx, \quad f, g \in L^2_v(\mathbb{T}, \Delta dx). \quad \text{Eq. (3.55)}
\]

The space of $W$-invariant polynomials $A^W$ is a dense subspace of $L^2_v(\mathbb{T}, \Delta dx)$. The purpose of the present section is to show that the $\Lambda$DO's $D_1, \ldots, D_n$ are symmetric with respect to $\langle \cdot, \cdot \rangle_{\Delta}$. We need the following lemma:

**Lemma 3.6** Let $\alpha = 1/2$ and $z \in \mathbb{R}$; furthermore, let the parameters be subject to condition \[(2.24).\] Then the functions $d_c^{(\pm)} (c = a, b)$ satisfy the following first order difference equations:

i. $d_c^+$:

\[
\begin{align*}
  d_a^+(z + \beta) &= e^{-\beta(g_1+1/2)} v_a(z) d_a^+(z), \quad \text{Eq. (3.56)} \\
  d_b^+(z + \beta) &= e^{-\beta(s_1^0+1/2)} v_b(z) d_b^+(z); \quad \text{Eq. (3.57)}
\end{align*}
\]

ii. $d_c$:

\[
 v_c(z - \beta) d_c(z - \beta) = v_c(z) d_c(z), \quad c = a, b. \quad \text{Eq. (3.58)}
\]

**Proof**

i. Eq. (3.56) is an immediate consequence of definition (3.52):

\[
\frac{d_a^+(z + \beta)}{d_a^+(z)} = \frac{1 - e^{-\beta g_1e^{iz}}}{1 - e^{iz}} = e^{-\beta g_1e^{iz}} v_a(z). \quad \text{Eq. (3.59)}
\]

Eq. (3.57) can be reduced to the former case by observing that $d_a^+(z)$ factorizes:

\[
\begin{align*}
  d_a^+(z) &= \frac{(e^{iz}; e^{-\beta})_{\infty}}{(e^{-\beta}g_1e^{iz}; e^{-\beta})_{\infty}} \frac{(-e^{iz}; e^{-\beta})_{\infty}}{(-e^{-\beta}g_1e^{iz}; e^{-\beta})_{\infty}} \\
  &\times \frac{(e^{-\beta/2}e^{iz}; e^{-\beta})_{\infty}}{(e^{-\beta}(g_1^0+1/2)e^{iz}; e^{-\beta})_{\infty}} \frac{(-e^{-\beta/2}e^{iz}; e^{-\beta})_{\infty}}{(-e^{-\beta}(g_1^0+1/2)e^{iz}; e^{-\beta})_{\infty}}. \quad \text{Eq. (3.60)}
\end{align*}
\]

ii. Using Eqs. (3.56) or (3.57), respectively,

\[
 v_c(z - \beta) d_c(z - \beta) = v_c(z) d_c(z). \quad \text{Eq. (3.61)}
\]

Part ii. of the above lemma leads to the following corollary:

**Corollary 3.7** One has

\[
e^{\beta \theta_{c, c}} \langle V_c, \varepsilon \rangle_{\Delta} = \langle V_c, \varepsilon \rangle_{\Delta}.
\]

We now arrive at the main result of this subsection: The proof hinges on relation (3.62).

**Proposition 3.8 (symmetry)**

\[
\langle \hat{D}_\varepsilon, m_\lambda, m_{\lambda'} \rangle_{\Delta} = \langle m_\lambda, \hat{D}_\varepsilon \rangle_{\Delta}. \quad \text{Proposition (3.64)}
\]

**Proof**

First consider the following contour integral:

\[
\oint_{C_j} W_{c, \varepsilon \lambda} \int_{\mathbb{C}} \varepsilon \, \text{d}z,
\]

with $V_{c, \varepsilon \lambda}$ and $W_{c, \varepsilon}$ as in (2.4) and (2.14), respectively.

Let all parameters and the variables $z, \varepsilon$ be subject to the condition \[(2.24),\] and $\Delta(x)$ has simple poles inside $C_j$ and $\Delta(x)$. However, one easily verifies that an additional zero in $\Delta(x)$; similarly, poles inside $C_j$ cancel each other because the integration path to $\Delta(x)$, one may deform the contour $C_j = [-\pi, \pi] \cup [\pi, \pi - i \varepsilon_j \beta] \cup [\pi - i \varepsilon_j \beta, -\pi] \cup [\pi, \pi - i \varepsilon_j \beta] \cup [\pi - i \varepsilon_j \beta, -\pi]$. With this information, we can write

\[
\hat{D}_\varepsilon m_\lambda, m_{\lambda'} \rangle_{\Delta} = \sum_{|J|=0}^{|\varepsilon|} \sum_{c=0}^{c=1} \int_{\mathbb{T}} V_c, \varepsilon\lambda \, \text{d}z.
\]

Armed with this conclusion and Eq. (3.64), the Proposition can be written
Deformation of the integration paths of $x_j$, $j \in J$, from $[-\pi, \pi]$ to $[-\pi - i\varepsilon_j \beta, \pi - i\varepsilon_j \beta]$, followed by a change of variables $x_j \rightarrow x_j - i\beta \varepsilon_j$, $j \in J$, yields

$$
\langle \hat{D}_r, m_{\lambda}, m_{\lambda'} \rangle_\Delta = \sum_{|J|=\alpha, n \leq \varepsilon} \int_{x_j \rightarrow x_j - i\beta \varepsilon_j} W_{j^r, r-s} m_{\lambda} (e^{-i\beta \varepsilon_j} m_{\lambda'} \Delta x). 
$$

(3.67)

Using Corollary 3.7 and the fact that $W_{j^r, r-s}$ is real (for parameters subject to (2.20)) entails

$$
\langle \hat{D}_r, m_{\lambda}, m_{\lambda'} \rangle_\Delta = \sum_{|J|=\alpha, n \leq \varepsilon} \int m_{\lambda} W_{j^r, r-s} V_{s^r, j^s \varepsilon} (e^{-i\beta \varepsilon_j} m_{\lambda'}) \Delta dx
= \langle m_{\lambda}, \hat{D}_r m_{\lambda'} \rangle_\Delta.
$$

(3.68)

3.5 Diagonalization and Commutativity

If $g = g_0 = g_1 = g_1' = 0$, then $d_a = d_b = 1$ (see Eqs. (3.54), (3.52) and (3.60)), and thus $\langle \cdot, \cdot \rangle_\Delta$ reduces to the inner product on $\mathbb{T}$ with respect to Lebesgue measure ($\Delta = 1$). The basis of monomials $\{m_{\lambda}\}_{\lambda \in \mathbb{P}^+}$ is an orthogonal basis of $L^2_\mathbb{W}(\mathbb{T}, dx)$. For arbitrary parameters however, the orthogonality of the monomials with respect to $\langle \cdot, \cdot \rangle_\Delta$ no longer holds. By subtracting from $m_{\lambda}$ the orthogonal projection of $m_{\lambda}$ onto $\text{span}\{m_{\lambda'}\}_{\lambda' \in \mathbb{P}^+, \lambda' < \lambda}$ one obtains an alternative basis $\{p_{\lambda}\}_{\lambda \in \mathbb{P}^+}$ of $A^W$. This is the basis of Koornwinder polynomials.

**Definition 3.9** [Koornwinder polynomials]

Koornwinder’s polynomial $p_{\lambda} \in A^W_\lambda$ is defined by the conditions

$$
p_{\lambda} = m_{\lambda} + \sum_{\lambda' \in \mathbb{P}^+, \lambda' < \lambda} c_{\lambda, \lambda'} m_{\lambda'}
$$

and

$$
\langle p_{\lambda}, m_{\lambda'} \rangle_\Delta = 0, \quad \forall \lambda' \in \mathbb{P}^+, \lambda' < \lambda.
$$

(3.70)

We now prove that $\{p_{\lambda}\}_{\lambda \in \mathbb{P}^+}$ is a basis of joint eigenfunctions of $\hat{D}_1, \ldots, \hat{D}_n$. For convenience, the notation for the eigenvalues Eq. (3.33) is sometimes abbreviated by putting

$$
E_{r,n}(ch\beta\theta_1, \ldots, ch\beta\theta_n; ch\beta\rho_1, \ldots, ch\beta\rho_n) \rightarrow E_{r,n}(\theta), \quad (\theta \in \mathbb{R}^n).
$$

(3.71)

**Theorem 3.10** (eigenfunctions)

$$
\hat{D}_r p_{\lambda} = E_{r,n}(\lambda + \rho) p_{\lambda}, \quad \forall \lambda \in \mathbb{P}^+
$$

(3.72)

(with $r = 1, \ldots, n$ and $\rho = (\rho_1, \ldots, \rho_n)$, see Eq. (3.33)).

**Proof**

It follows from (3.64) and Proposition 3.1 that

$$
\langle \hat{D}_r - [\hat{D}_r], p_{\lambda} \rangle_\Delta = 0.
$$

On the other hand, one has (use the propositions 3.8, 3.4 and Eq. (3.70))

$$
\langle \hat{D}_r - [\hat{D}_r], p_{\lambda} \rangle_\Delta \text{ symmetry} = \langle p_{\lambda}, [\hat{D}_r] \rangle_\Delta
$$

Combining (3.73) and (3.74) entails $\hat{D}_r p_{\lambda}$ to complete the proof.

In Appendix C it is shown that if a difference or differential operator vanishes on $\text{span}\{m_{\lambda'}\}_{\lambda' \in \mathbb{P}^+, \lambda' < \lambda}$, then all its coefficients must be zero. Combining this with the commutativity of the $A\Delta O$'s:

**Theorem 3.11** (commutativity)

The operators $\hat{D}_1, \ldots, \hat{D}_n$ mutually commute.

**Proof**

The polynomials $\{p_{\lambda}\}_{\lambda \in \mathbb{P}^+}$ form a basis of $A^W$. In other words, the commutator $[\hat{D}_r, \hat{D}_s]$ vanishes on $A^W$:

$$
[\hat{D}_r, \hat{D}_s](A^W) = 0.
$$

It now follows from Proposition 3.5 in Appendix C that $[\hat{D}_r, \hat{D}_s][A^W] = 0$.

Consider the real algebra of difference operators

$$
\mathbb{D} \equiv \mathbb{R}[\hat{D}_r].
$$

It is clear that $\mathbb{D}$ is an abelian algebra (Theorem 3.10) simultaneously diagonalized by the Koornwinder polynomials.

**Theorem 3.12** ($\mathbb{D} \cong \mathbb{R}[ch\beta\theta_1, \ldots, ch\beta\theta_n]$)

For each symmetric function $S(\theta) \in \mathbb{R}[ch\beta\theta_1, \ldots, ch\beta\theta_n]$ operator $\hat{D} \in \mathbb{D}$ such that

$$
\hat{D} p_{\lambda} = S(\lambda + \rho)
$$

where

$$
E_{r,n}(\theta) = S_r(ch\beta\theta_1, \ldots, ch\beta\theta_n).
$$

Proof

$E_{r,n}(\theta)$ (3.71) is a linear combination of elementary symmetric functions.

$E_{r,n}(\theta) = S_r(ch\beta\theta_1, \ldots, ch\beta\theta_n)$. 

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I.d. stands for terms of lower degree in $\text{ch}\beta\theta_j$, $j = 1, \ldots, n$. The elementary symmetric functions form a set of algebraically independent generators of the symmetric algebra (this fact is the ‘fundamental theorem on symmetric functions’, see e.g. [4]). Hence, Eq. (3.78) implies that the same is true for the functions $E_{r,n}(\theta)$: every element in $\mathbb{R}[\text{ch}\beta\theta_1, \ldots, \text{ch}\beta\theta_n]|_{S_{\beta\theta}}$ can be written uniquely as a polynomial in $E_{r,n}(\theta)$, $r = 1, \ldots, n$.

Now we use Theorem 3.11 to conclude that for every symmetric function $S(\theta)$ there exists a difference operator $\hat{D} \in \mathcal{D}$ such that Eq. (3.77) holds. That such a difference operator $\hat{D}$ is unique follows from Proposition C.1 (Appendix C).

The symmetric functions $S(\theta) \in \mathbb{R}[\text{ch}\beta\theta_1, \ldots, \text{ch}\beta\theta_n]|_{S_{\beta\theta}}$ separate the points of the wedge

$$\{ \theta \in \mathbb{R}^n \mid \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \geq 0 \}. \quad (3.79)$$

To see this, first notice that $(-1)^{n-r}S_r(\text{ch}\beta\theta_1, \ldots, \text{ch}\beta\theta_n)$ is the coefficient of $x^{\alpha+n}$ in the characteristic polynomial $\det(T - \alpha I)$ of the diagonal matrix $T = \text{diag}(\text{ch}\beta\theta_1, \ldots, \text{ch}\beta\theta_n)$. Consequently, the values of $S_r(\text{ch}\beta\theta_1, \ldots, \text{ch}\beta\theta_n)$, $r = 1, \ldots, n$ determine $\theta$ in the wedge (3.79) uniquely. This fact combined with Theorem 3.12 can be used to prove the orthogonality of the basis $\{p_\lambda\}$:

**Corollary 3.13 (orthogonality)**

$$\langle p_\lambda, p_{\lambda'} \rangle_\Delta = 0, \quad \forall \lambda, \lambda' \in \mathcal{P}^+, \lambda \neq \lambda'. \quad (3.80)$$

**Proof**

Let $\lambda, \lambda' \in \mathcal{P}^+$, and $\lambda \neq \lambda'$. Since the symmetric functions in $\mathbb{R}[\text{ch}\beta\theta_1, \ldots, \text{ch}\beta\theta_n]|_{S_{\beta\theta}}$ separate the points of the wedge (3.79), there exists an $S(\theta) \in \mathbb{R}[\text{ch}\beta\theta_1, \ldots, \text{ch}\beta\theta_n]|_{S_{\beta\theta}}$ such that

$$S(\lambda + \rho) \neq S(\lambda' + \rho). \quad (3.81)$$

Hence, by Theorem 3.12, there exists a $\hat{D} \in \mathcal{D}$ for which $p_\lambda$ and $p_{\lambda'}$ are eigenfunctions corresponding to different eigenvalues. But then the polynomials $p_\lambda$ and $p_{\lambda'}$ must be orthogonal with respect to $\langle \cdot, \cdot \rangle_\Delta$ because $\hat{D}$ is symmetric, cf. Proposition 3.8.

**Note.** The orthogonality of the basis $\{p_\lambda\}_{\lambda \in \mathcal{P}^+}$ was already shown by Koornwinder [K]. His proof exploits the continuity of $\langle p_\lambda, p_{\lambda'} \rangle_\Delta$ in the parameters.

**Corollary 3.14 (self-adjointness)**

Every difference operator $\hat{D} \in \mathcal{D}$ is essentially self-adjoint on $A^W \subset L^2_W(\mathbb{T}, \Delta dx)$.

**Proof**

This an immediate consequence of the fact that every $A\Delta O$ in $\mathcal{D}$ acts as a real multiplication operator on the orthogonal basis $\{p_\lambda\}_{\lambda \in \mathcal{P}^+}$ of $L^2_W(\mathbb{T}, \Delta dx)$.

**Remarks**

i. Theorem 3.12 states that the system

$$\hat{D}_r \to \mathcal{H}_{E_{r,n}}(\theta)$$

induce a Harish-Chandra-type algebra isomorphism.

ii. Recall that $p_\lambda$ is defined as $m_\lambda$ minus the orthogonal projection of $\{m_\lambda'\}_{\lambda' < \lambda}$. Using the orthogonality of the following recursion relation for $p_\lambda$:

$$p_\lambda = m_\lambda - \sum_{\lambda' \in \mathcal{P}^+, \lambda' < \lambda} \langle p_\lambda, p_{\lambda'} \rangle_\Delta \hat{D} p_{\lambda'}$$

iii. If the partial ordering (3.3) is extended to a $\beta$-ordering, one is possible to orthogonalize the basis $\{m_\lambda\}$.

Corollary 3.14: The result does not depend on the ordering: the resulting orthogonal basis is independent of the measure $\Delta dx$.

iv. According to Theorem 3.12, the Hamiltonians $\hat{D}_r$ acts as a real multipli-

$$\hat{D}_r \to \mathcal{H}_{E_{r,n}}(\theta)$$

- The transition to the Hamiltonian PDO’s associated with the rooted tree $\{p_\lambda\}$ converges to the $BC_n$-type Jacobi polynomials.

In this section we will make the dependence on all objects of interest, e.g.: $\hat{D}_r, \beta, \Delta \beta$ and $p_\lambda$.

**4 $\beta \to 0$: The Transition to the $BC_n$-type PDO’s**

By sending the step size $\beta$ of the difference operators $BC_n$-type PDO’s associated with the rooted tree $\{p_\lambda\}$ to zero one finds that $\{p_\lambda\}$ converges to the $BC_n$-type Jacobi polynomials of Refs. [HO] and [H2].

In this section we will make the dependence on all objects of interest, e.g.: $\hat{D}_r, \beta, \Delta \beta$ and $p_\lambda$.

**4.1 Eigenfunctions**

Consider the following weight function on the torus $\mathbb{T}$

$$\Delta_0(x) \equiv \prod_{1 \leq j < k \leq n} |\sin(\alpha_j + \alpha_k)| \left( \prod_{1 \leq j \leq n} |\sin(\alpha_j)| \right)^2 |\cos(\alpha_j)|$$

and let $\langle \cdot, \cdot \rangle_{\Delta_0}$ be the inner product on $L^2(\mathbb{T})$. It introduces $W$-invariant polynomials on $\mathbb{T}$.

**Definition 4.1** $[BC_n$-type Jacobi polynomials$]$ The Jacobi polynomial $p_{\lambda,0} \in A^W$ is defined by

$$p_{\lambda,0} = m_\lambda + \sum_{\lambda' \in \mathcal{P}^+, \lambda' < \lambda} \langle p_{\lambda}, p_{\lambda'} \rangle_\Delta \hat{D} p_{\lambda'}$$

End of proof.
with the root system BC. Usually ∆ and ˜ weight function.

**Proposition 4.3** One has

\[ \langle p_{\lambda,0}, p_{\lambda',0} \rangle_{\Delta_0} = 0, \quad \forall \lambda, \lambda' \in \mathcal{P}^+, \lambda \neq \lambda'. \]  

and

\[ \langle p_{\lambda,0}, m_{\lambda'} \rangle_{\Delta_0} = 0, \quad \forall \lambda' \in \mathcal{P}^+, \lambda' < \lambda. \]  

Note. The limit β → 0 corresponds to the orthogonal projection (with respect to the Jacobi polynomials the q → 1 limit to the Jacobi polynomials of Heckman and Opdam was proved in [H1] (again for arbitrary root systems).
4.3 Operators

Expansion of $\hat{D}_{r,\beta}$ in $\beta$ yields a formal power series of the form

$$\hat{D}_{r,\beta} = \sum_{m=0}^{\infty} \hat{D}^{(m)} \beta^m. \tag{4.16}$$

The coefficients $\hat{D}^{(m)}$ are polynomials in the partials $\hat{\theta}_i$, $j = 1, \ldots, n$; this means that these coefficients are PDO’s. We define the leading differential operator $\hat{D}_{r,0}$ of $\hat{D}_{r,\beta}$ as the first nonzero coefficient in expansion (4.16):

**Definition 4.5** [leading PDO]

Let

$$m_r \equiv \min\{m \in \mathbb{N} \mid \hat{D}^{(m)} \neq 0 \} \tag{4.17}$$

(with $\hat{D}^{(m)}$ defined by expansion (4.16)). Then,

$$\hat{D}_{r,0} \equiv \hat{D}^{(m_r)} \tag{4.18}$$

called the leading PDO of $\hat{D}_{r,\beta}$.

The $BC_n$-type Jacobi polynomials are joint eigenfunctions of $\hat{D}_{1,0}, \ldots, \hat{D}_{n,0}$:

**Theorem 4.6** One has $m_r = 2r$ and

$$\hat{D}_{r,0} p_{\lambda,0} = E_{r,n} \left( (\lambda_1 + \rho_1)^2, \ldots, (\lambda_n + \rho_n)^2; \rho_1^2, \ldots, \rho_n^2 \right) p_{\lambda,0}, \quad \forall \lambda \in \mathcal{P}^+, \tag{4.19}$$

with $r = 1, \ldots, n$ and $\rho$ as in (3.35).

**Proof**

Consider the eigenvalue equation (3.72):

$$\dot{D}_{r,\beta} (p_{\lambda,\beta}) = 2^r E_{r,n} (\text{ch}\beta(\lambda_1 + \rho_1), \ldots, \text{ch}\beta(\lambda_n + \rho_n); \text{ch}\beta p_1, \ldots, \text{ch}\beta p_n) p_{\lambda,\beta}. \tag{4.20}$$

First, apply Taylor’s theorem to the l.h.s. of Eq. (4.20) and make use of Definition 4.5 and Limit (4.10) to conclude that

$$\hat{D}_{r,\beta} p_{\lambda,\beta} = \hat{D}_{r,0} p_{\lambda,0} \beta^{m_r} + o(\beta^{m_r}). \tag{4.21}$$

Next, use Proposition 4.4 and Limit (4.10) to derive the asymptotic behavior for $\beta \to 0$ of the r.h.s. of (4.20):

$$2^r E_{r,n} (\text{ch}\beta(\lambda_1 + \rho_1), \ldots, \text{ch}\beta(\lambda_n + \rho_n); \text{ch}\beta p_1, \ldots, \text{ch}\beta p_n) p_{\lambda,\beta} = E_{r,n} \left( (\lambda_1 + \rho_1)^2, \ldots, (\lambda_n + \rho_n)^2; \rho_1^2, \ldots, \rho_n^2 \right) p_{\lambda,0} \beta^{2r} + o(\beta^{2r}). \tag{4.22}$$

By Definition 4.5 and Proposition 4.3 it is possible to pick a $\lambda \in \mathcal{P}^+$ such that $\hat{D}_{r,0} p_{\lambda,0} \neq 0$, so $m_r \geq 2r$. It is not difficult to see that there also exist $\lambda \in \mathcal{P}^+$ such that

$$E_{r,n} \left( (\lambda_1 + \rho_1)^2, \ldots, (\lambda_n + \rho_n)^2; \rho_1^2, \ldots, \rho_n^2 \right) \neq 0,$$

so $m_r \leq 2r$. This entails $m_r = 2r$ and (4.19).

**Corollary 4.7**

$$\hat{D}_{r,0} = \lim_{\beta \to 0} \hat{D}_{r,\beta} \tag{4.23}$$

Explicit computation of (4.23) for $r = 1$ yields

$$\hat{D}_{1,0} = \sum_{1 \leq j \leq n} \theta_j^2 - 2i\alpha g \sum_{1 \leq j < k \leq n} \{ \cot(\alpha \theta_j) \theta_j \theta_k - \cot(\alpha \theta_k) \theta_j \theta_k \} \tag{4.24}$$

(with $\alpha = 1/2$ and $\theta_j$, $\theta_k$ as in Eq. (4.14)).

An immediate consequence of Theorem 4.6 is

$$\hat{D}_{r,0}, r = 1, \ldots, n.$$

**Corollary 4.8** *The differential operators $\hat{D}_{r,0}$*

Let $D_0 \equiv \mathbb{R}[\hat{D}_{1,0}, \ldots, \hat{D}_{n,0}]$. The algebra simultaneously diagonalized by the $BC_n$-type orthogonal polynomials is

**Theorem 4.9** $(D_0 \cong \mathbb{R}[\theta_1^2, \ldots, \theta_n^2] S_n)$

For every symmetric polynomial $S(\theta) \in \mathbb{R}[\theta_1^2, \ldots, \theta_n^2]$ the operator $D_0 \in D_0$ such that

$$\hat{D}_0 p_{\lambda,0} = S(\lambda + \rho, \beta).$$

**Corollary 4.10** *Self-adjointness*

The PDO’s $\hat{D}_0 \in D_0$ are essentially self-adjoint.

The proofs of Theorem 4.9 and Corollary 4.10 are completely analogous to the corresponding results of Theorem 3.12 and Corollary 3.14, respectively.

**Remarks i.** The operator $\hat{D}_{1,0}$ (1.24) can be explicitly computed as a hypergeometric PDO associated with the root system $BC_n$ with the usual notation which emphasizes the role of the hypergeometric functions $\tan(\alpha \theta_j)$ from (1.24) by means of the relations (2.17) of [36], and the remark under Definition 4.1.

**Remarks ii.** The existence of an abelian algebra of PDO’s commuting with the usual notation which emphasizes the role of the hypergeometric functions $\tan(\alpha \theta_j)$ from (1.24) by means of the relations (2.17) of [36], and the remark under Definition 4.1.

**Remarks iii.** The operator $\hat{D}_{1,0}$ (1.24) can be explicitly computed as a hypergeometric PDO associated with the root system $BC_n$ with the usual notation which emphasizes the role of the hypergeometric functions $\tan(\alpha \theta_j)$ from (1.24) by means of the relations (2.17) of [36], and the remark under Definition 4.1.
with $E_0 = 4a^2 \| p, \rho \|$ and $a = (n - j)g + (\tilde{g}_0 + \tilde{g}_1)/2$, cf. (3.3). Corollary 4.8 can be interpreted as the quantum integrability of the Sutherland system. For arbitrary root systems integrability follows from [H1, O2] (cf. Remark ii).

5 Special Cases Related to Classical Root Systems

By limit transitions and/or specialization of the parameters, the operators $\hat{D}_1, \ldots, \hat{D}_n$ reduce to commuting $\Lambda \Delta$O's, which are simultaneously diagonalized by Macdonald’s polynomials. Such difference operators are obtained for all Macdonald families associated with (admissible pairs of) the classical root systems: $A_{n-1}$, $B_n$, $C_n$, $D_n$ and $BC_n$.

Note. Most results in this section have an obvious counterpart for $\beta = 0$ (which amounts to $q = 1$).

5.1 Preliminaries

First, we outline very briefly some of the main points of the construction presented by Macdonald [M2]. A more detailed summary of his results can be found in [M4] and [K4]. (For our purposes, especially the second summary is useful.) Here, we only want to introduce some terminology which facilitates clarifying the connection between the preceding sections and Ref. M2. For general information on root systems the reader is referred to e.g. [B3]. Although most of the remaining part of the paper should be accessible without a detailed knowledge of root systems, a glance at the ‘planchers’ in Bourbaki [B1] might be of some help.

Ref. M2 uses the concept of admissible pairs of root systems. The pair $(R, S)$ is admissible if $R$ and $S$ are root systems (assumed irreducible) such that $S \subset R$ is reduced and generates the same Weyl group as $R$. Let $V$ be the real vector space spanned by $R$ and consider the torus $T_R \equiv V/(2\pi \mathbb{Z} R^\vee)$. Let $P^+_R$ be the dominant cone of the weight lattice of $R$ (which equals the character lattice of $T_R$) and let $A^W_R$ denote the algebra of $W$-invariant (trigonometric) polynomials on $T_R$ (this algebra is isomorphic to the $W$-invariant part of the group algebra over the weight lattice). To every admissible pair $(R, S)$ Macdonald associates a weight function $\Delta_{(R, S)}$ on $T_R$ and finds a corresponding orthogonal basis $\{p_{\lambda,(R, S)}\}_{\lambda \in P^+_R}$ of $A^W_R$. Furthermore, he introduces difference operators $D_n$ associated with the so-called (quasi-)minuscule weights $\sigma$ of $S^\vee$. These operators are diagonalized by the basis of Macdonald polynomials $\{p_{\lambda,(R, S)}\}_{\lambda \in P^+_R}$.

We will show that additional $\Lambda \Delta$O’s for Macdonald’s polynomials arise as special cases of $\hat{D}_1, \ldots, \hat{D}_n$ (2.4). This leads to difference operators associated with the fundamental weights of $S^\vee$, for every admissible pair consisting of classical root systems. These $\Lambda \Delta$O’s generate an abelian algebra $\mathbb{D}_{(R, S)}$ of difference operators, of which $\Lambda$ is the $W$-invariant part of the real group algebra.

Remark. If all roots in $R$ have the same length $S = R$ and there exists only one admissible pair $R, S$, then there are several possibilities for the $\Lambda$-operators: there are six such possibilities; these correspond to

5.2 The Root System $A_{n-1}$

Let $\hat{D}_{r, lead}$ consist of those terms in $\hat{D}_r$ that correspond to the leading terms of $\Delta_r$ (up to a multiplicative constant) with Macdonald’s difference operator (5.1):

$$\hat{D}_{r, lead} = \sum_{J \subset \{1, \ldots, n\}} V_{j, J} \frac{d^r_{\lambda}}{d^r_{\lambda}}(x_j + x_k)$$

One picks up these leading terms via the following limit:

$$\hat{D}_{r, lead} = \lim_{R \to \infty} e^{-\beta_1 \hat{g}_1 - \cdots - \beta_n \hat{g}_n}$$

with

$$\Lambda_R \equiv e^{-\sum_{i=1}^{n-1} \beta_i \hat{g}_i}$$

Let

$$\hat{D}_r = \hat{D}_+ \hat{D}_{r, lead} \hat{D}_+^{-1}$$

Conjugation of $\hat{D}_{r, lead}$ with $\hat{D}_+$ results in

$$\hat{D}_r = \hat{D}_+ \hat{D}_{r, lead} \hat{D}_+^{-1}$$

It is clear from Theorem 3.11 and Eq. (5.1) that $\hat{D}_r$ commutes with $\hat{D}_- = e^{-\sum_{i=1}^{n-1} \beta_i \hat{g}_i} \hat{D}_-$.

The first part causes a translation $x \to x + p$ (up to a multiplicative constant) with Macdonald’s operators $E_{\omega_r} = e_x \hat{D}_{r, A_{n-1}}$, $e_x \equiv e^{\lambda^T x}$.
The operator $E_{\omega}$ is associated with the $r$th fundamental weight $\omega$ of the root system $A_{n-1}$. The parameters in $\mathbb{M}_2^r$ are related to ours via Eq. (2.21).

**Notes**

i. For $r = 1$, Eq. (5.8) reduces to

$$D_1' = \sum_{1 \leq j < n} \left( \prod_{k \neq j} v_a(x_j - x_k) \right) e^{-\beta \hat{\theta}_j}.$$  \hfill (5.9)

ii. After transformation to Lebesgue measure, the operator $\hat{D}_r$ goes over in the $r$-th quantum integral $\hat{S}_r$ (Eq. (2.3)) of the relativistic Calogero-Moser system with trigonometric coefficients. More precisely, let

$$\Delta_{A_{n-1}}(x) \equiv \prod_{1 \leq j < k \leq n} d_a(x_j - x_k),$$  \hfill (5.10)

then

$$\hat{S}_r = \Delta_{A_{n-1}}^{1/2} \hat{D}_r' \Delta_{A_{n-1}}^{-1/2} = \sum_{J \subseteq \{1, \ldots, n\}} \left( \prod_{j \in J} v_a(x_j - x_k) \right)^{1/2} e^{-\beta \hat{\theta}_j} \left( \prod_{j \in J^c} v_a(x_k - x_j) \right)^{1/2}.$$  \hfill (5.11)

This relation between the $n$-particle relativistic CM system introduced by Ruijsenaars and Macdonald's difference operators for the root system $A_{n-1}$ was first observed by Koornwinder [K1]. It generalizes the relation between the $n$-particle Calogero-Sutherland system and the hypergeometric PDO's associated with Koornwinder's polynomials by a certain limit transition.

Let $m_{\lambda, \text{lead}}$ be the sum of terms in $m_{\lambda}$ (3.3) which are of the highest degree in $\exp(ix_j)$, $j = 1, \ldots, n$:

$$m_{\lambda, \text{lead}} = \sum_{\lambda' \in S_n \lambda} e^{\lambda'}, \quad \lambda \in \mathcal{P}^+.$$  \hfill (5.12)

Recall that according to Definition 3.3, $p_\lambda$ is a linear combination of monomials of the form

$$p_\lambda = \sum_{\lambda' \in \mathcal{P}^+, \lambda' \leq \lambda} c_{\lambda, \lambda'} m_{\lambda'}, \quad \lambda \in \mathcal{P}^+,$$  \hfill (5.13)

with $c_{\lambda, \lambda'}$ certain complex coefficients (which depend only on the parameters) such that (3.70) holds and $c_{\lambda, \lambda} = 1$. We set

$$p_{\lambda, \text{lead}} \equiv \sum_{\lambda' \in \mathcal{P}^+, \lambda' \leq \lambda} c_{\lambda, \lambda'} m_{\lambda', \text{lead}}, \quad \lambda \in \mathcal{P}^+.$$  \hfill (5.14)

Let $\omega \equiv e_1$ from the asymptotics

$$m_\lambda(x - iR\omega) \sim m_{\lambda, \text{lead}}(x) \text{ as } R \to \infty,$$

and Eqs. (5.13) and (5.14), one derives

$$p_{\lambda, \text{lead}}(x) = \lim_{R \to \infty} p_{\lambda}(x + iR\omega).$$

The polynomial $p_{\lambda, \text{lead}}$ is homogeneous of degree $\nu$. Consequently, a translation causes an automorphic phase factor:

$$\exp(iR(\lambda, \text{lead})),$$

and $p_{\lambda, \text{lead}}$ ends up with a basis of translation-invariant polynomials $p_{\lambda, A_{n-1}}$ below).

**Lemma 5.1** Let

$$F_\beta(\theta) = F_{\beta}(\theta).$$

Then, for all $\lambda, \lambda' \in \mathcal{P}^+$:

$$\lambda > \lambda' \Rightarrow F_{\beta}(\lambda) > F_{\beta}(\lambda'),$$

with $\rho$ given by (2.33).

**Proof**

It is clear from Definition 5.3 that $\lambda > \lambda'$ if

$$\lambda = \lambda' + \sum_{1 \leq j \leq n-1} a_j (e_j - e_{j+1}),$$

with at least one of the $a_j$'s positive. Obviously, it suffices to verify (5.21) for a convex function:

$$\text{ch}_\beta(x + a) = \text{ch}_\beta(x) + a \text{ch}_\beta'(x),$$

if $x \geq y$ and $a > 0$. 

\vspace{15pt} 

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Proposition 5.2 Let $\hat{D}_{r,A_{n-1}}$ be determined by (3.6), (4.4) and let $p_{\lambda,A_{n-1}}$ be defined by (5.14). Then
\[ \hat{D}_{r,A_{n-1}} p_{\lambda,A_{n-1}} = E_{r,A_{n-1}}(\lambda + \rho') p_{\lambda,A_{n-1}}, \tag{5.24} \]
with
\[ E_{r,A_{n-1}}(\theta) = e^{\lambda \rho} \left( \prod_{i=1}^{n} e^{x_i} \right) \left( \prod_{i=1}^{n} e^{\rho_i} \right), \tag{5.25} \]
\[ \rho_j' = \frac{g(n+1-2j)}{2}, \quad 1 \leq j \leq n. \tag{5.26} \]
($S_r$ denotes the $r$th elementary symmetric function (Definition 2.2)).

Proof
The operator $\hat{D}_r$ (5.3) is invariant both under permutations of $x_j$ and under translations of the form $x \rightarrow x + R \omega$ (with $\omega$ as in (5.13)). We use this and the asymptotics of $(\hat{D}_r m_{\lambda,lead})(-iRg)$ for $R \rightarrow \infty$ (with $y$ such that (3.30) holds) to derive (cf. Proposition 3.3, Proposition 5.4 and their proofs)
\[ \hat{D}_r m_{\lambda,lead} = \sum_{\lambda' \in P_+^+, \lambda' \leq \lambda} [\hat{D}_r]_{\lambda,\lambda'} m_{\lambda',lead}, \tag{5.27} \]
with
\[ [\hat{D}_r]_{\lambda,\lambda'} = S_r(e^{-\beta(\lambda_1 + \rho_1')}, \ldots, e^{-\beta(\lambda_n + \rho_n')}). \tag{5.28} \]
(The poles at $x_j = x_k, i \neq k$, cancel because of the permutation symmetry; the condition $|\lambda'| = |\lambda|$ in sum (5.27) stems from the translational invariance of $\hat{D}_r$.)

We will now show that $p_{\lambda,lead}$ is an eigenfunction of $\hat{D}_r$ (with eigenvalue (5.28)). Consider the eigenvalue equation (5.72) for $r = 1$:
\[ (\hat{D}_1 p_{\lambda})(x) = 2 \left( \sum_{1 \leq j \leq n} \left[ \log(\lambda_j + \rho_j) - \log(\rho_j) \right] \right) p_{\lambda}(x), \quad \lambda \in P^+, \tag{5.29} \]
with $\hat{D}_1$ given by (2.10). Substitute
\[ x \rightarrow x - iR \omega \tag{5.30} \]
and divide both sides of the equation by $exp(R(1)), \omega$; Sending $R \rightarrow \infty$ entails (use (4.19), (4.20) and (5.17))
\[ (c_1 \hat{D}_1 + c_1^{-1} (\hat{D}_n)^{-1} - c_2) p_{\lambda,lead} = 2 \left( \sum_{1 \leq j \leq n} \left[ \log(\lambda_j + \rho_j) - \log(\rho_j) \right] \right) p_{\lambda,lead}, \tag{5.31} \]
with
\[ c_1 = \frac{e^{-\beta g(n-1)/2} e^{-\beta(g_0 + g)}}{c_1 \prod_{1 \leq j \leq n} v_0(x_j - x_i)}, \quad c_2 = 2 \left( \sum_{1 \leq j \leq n} \log(\rho_j) \right). \tag{5.32} \]

(To verify equality *, first check that both parts of (5.33) are regular and bounded in $x$; consequently, these parts are constants because of Liouville’s theorem. One obtains (5.27), (5.28).

One obtains the expressions (5.24)-(5.26) for $r = 1$.

Notice that $m_{\lambda',A_{n-1}} = m_{\lambda,A_{n-1}}$ if $\lambda' - \lambda \in \mathbb{Z}$ can be relabeled by the projection of $P_+^+$ to those associated with $A_{n-1}$.

The polynomials $p_{\lambda,lead}$ can also be relabeled by the projection of $P$ onto the hyperplane $x_{\lambda_1} = \cdots = x_{\lambda_n} = 0$. (This follows from (5.14) for $r = 1, \ldots, n - 1$.) Since the operator $\hat{D}_r$ commutes with the difference operators up to a constant, we end up with

Corollary 5.3 The function $p_{\lambda,A_{n-1}}$ coincides correspondingly with the weight vector $\lambda - |\lambda|(e_{\lambda_1} + \cdots + e_{\lambda_n})$, as in (5.24).

Notes i. The transition $p_{\lambda,lead} \rightarrow p_{\lambda,A_{n-1}}$ amounts to ignoring the linear motion of the center of mass.

ii. For $\beta \rightarrow 0$ (i.e. $q = \exp(-\beta) \rightarrow 1$), the polynomials associated with $A_{n-1}$ can be found [12]. It relates the Jacobi polynomials
\[ p_{\lambda,A_{n-1}} = \frac{e^{-i|\lambda|(x_1 + \cdots + x_n)/n}}{\lim_{R \rightarrow \infty} e^{-i\beta R(\lambda)}}, \tag{5.33} \]
with $q \neq 1$. 

iii. Recently, a completely different limit $BC_n$ to those associated with $A_{n-1}$ has been found [12].
5.3 The Root Systems $B_n, C_n$ and $BC_n$.

In order to compare our results with [K2], it is convenient to carry out a reparametrization:

\[
\begin{align*}
\mu_0 &\rightarrow \nu_1 + \nu_2, \\
\mu_1 &\rightarrow \nu_2, \\
\nu_1' &\rightarrow \nu_1' + \nu_2', \\
\mu_1' &\rightarrow \nu_2',
\end{align*}
\]

with (cf. (2.24))

\[
\nu_3 \equiv i\beta k_3, \quad \nu_4' \equiv i\beta k_4', \quad k_3, k_4' \geq 0, \quad \delta = 1, 2.
\]

With these new parameters we rewrite $v_0(z)$ (2.2) and $d_k^2(z)$ (3.53) (recall also (3.60)):

\[
v_0(z) = \frac{\sin \alpha(\nu_1 + \nu_2 + z)}{\sin \alpha(\nu_2 + z)} \cdot \frac{\sin \alpha(\nu_1' + \nu_2' + \gamma + z)}{\sin \alpha(\nu_2' + \gamma + z)},
\]

and

\[
d_k^2(z) = \frac{(e^{(i\nu_2 + z)}; e^{-\beta})_{\infty}}{(e^{(i\nu_1 + \nu_2 + z)}; e^{-\beta})_{\infty}} \cdot \frac{(e^{(i\nu_2' + \gamma + z)}; e^{-\beta})_{\infty}}{(e^{(i\nu_1' + \nu_2' + \gamma + z)}; e^{-\beta})_{\infty}}
\]

For the following parameters $\Delta(x)$ (4.50) reduces to Macdonald’s weight function $\Delta_{(R,S)}$ with $R = B_n, C_n$ or $BC_n$ and $S = B_n$ or $C_n$:

| $S/R$ | $B_n$ | $C_n$ | $BC_n$ |
|-------|-------|-------|-------|
| $B_n$ | $\nu_1' = \nu_2' = \nu_2 = 0$ | $\nu_1' = \nu_2' = \nu_1 = 0$ | $\nu_1' = \nu_2' = 0$ |
| $C_n$ | $\nu_1' = \nu_1$ | $\nu_1' = \nu_1 = 0$ | $\nu_1' = \nu_1$ |
|       | $\nu_2' = \nu_2$ | $\nu_2' = \nu_2$ | $\nu_2' = \nu_2$ |

The cone of dominant weight vectors can be obtained from $\Delta_{(R,S)}$.

The relation with the parameters employed in Ref. [K2] reads:

\[
t_{\pm e_j} = e^{i\mu}, \quad t_{\pm e_j} = e^{i\nu_j}, \quad t_{\pm 2e_j} = e^{2i\nu_2}
\]

and

\[
q = \begin{cases} 
  e^{-\beta} & \text{if } S = B_n \text{ or } S = R = C_n \\
  e^{-\beta}/2 & \text{if } S = C_n \text{ and } R = B(C)n.
\end{cases}
\]

In order to verify that for the above parameters, $\Delta$ (4.50) indeed coincides with the weight functions introduced by Macdonald, it may be helpful to compare our expressions with Eqs. (3.1)-(3.5) of [K2], since the latter are rather explicit.

Next, we consider the Macdonald polynomials associated with $\Delta_{(R,S)}$. We distinguish two cases:

i. $R = (B)C_n$ 

In this case the torus $T_R = (\mathbb{R}^n/(2\pi \mathbb{Z}R^*))$ of $W$-invariant polynomials on $T_R$ coincides with the cone $P^+$, and we read

\[
\omega_k = e_1 + \cdots + e_p
\]

(our convention regarding the choice of the basis of dominant weights $P^+_R$, which consists of $k = 1, \ldots, n$, coincides with the cone $P^+_R$ and 3.4 of the above table, $\{p_\lambda\}_{\lambda \in P^+_R}$ reduces to the Macdonald basis $\{p(R,S)\lambda\}_{\lambda \in P^+_R}$).

ii. $R = B_n$

This case is a bit more complicated because the Macdonald torus $T_{B_n} = (\mathbb{R}^n/(2\pi \mathbb{Z}B_n^*))$ of the roots $\omega_k$ of $B_n$ are the same as in (5.42):

\[
\omega_n = (e_1 - e_2)
\]

The cone of dominant weight vectors can be obtained from $\Delta_{(R,S)}$.

The algebra of $W$-invariant polynomials:

\[
A_{B_n}^W = \text{span} \{ p_{\lambda} \}_{\lambda \in \Lambda^+}, \quad \text{where} \quad \Lambda^+ = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \}_{\lambda_1 \geq \cdots \geq \lambda_n, \sum \lambda_i = M}
\]

and

\[
x_j \rightarrow x_j + 2\pi \implies m_{\lambda + \delta \omega_j}(x)
\]

Combining (5.46) with the fact that the function $m_{\lambda + \delta \omega_j}(x)$ is periodic of period $2\pi$ on the hyperplanes $x_j = \pi$ (mod $2\pi$).

\[
m_{\omega_n}(x) = 2^n
\]

Thus, we have the following decomposition

\[
A_{B_n}^W = A_{B_n}^W
\]

(with $A^W$ as before). This decomposition is important for later considerations.

The situation is now as follows: just as in the case of the $B_n$-type Macdonald polynomials that are in the subspace $P_{B_n}^+$ of $P^n_{B_n}^+$, the above table gives

\[
\{p_{\lambda} \}_{\lambda \in \Lambda^+} \text{ of } A_{B_n}^W
\]

and

\[
\{p_{\lambda} \}_{\lambda \in \Lambda^+} \text{ of } A_{B_n}^W
\]

respectively. By specializing to $B_n$-type Macdonald polynomials that are in the subspace $P_{B_n}^+$ of $P^n_{B_n}^+$, the above table gives

\[
\{p_{\lambda} \}_{\lambda \in \Lambda^+} \text{ of } A_{B_n}^W
\]

and

\[
\{p_{\lambda} \}_{\lambda \in \Lambda^+} \text{ of } A_{B_n}^W
\]
By multiplying $m_{\omega_n}$ and the Macdonald polynomials with parameters as in (5.49), we obtain an orthogonal basis of $m_{\omega_n}A^W_n$; the latter polynomials coincide with the anti-periodic Macdonald polynomials. To be more explicit, we have:

$$m_{\omega_n} \Delta(B_n, S) \equiv (3.50)-(3.53) \text{ with parameters (5.49). The precise form of these relations can be obtained as in (5.51).}$$

By comparing the spectrum of the operators on both sides of the equation (cf. eqs. (3.33)-(3.35)), for $n \geq 1$, and (5.57):
For \( R = D_n \), the torus \( \mathbb{T}_R = \mathbb{R}^n/(2\pi \mathbb{Z}^n) \) is the same as for \( R = B_n \). The Weyl group, however, is smaller: only an even number of sign flips of the variables \( x_j, j = 1, \ldots, n \), is allowed. For \( k = 1, \ldots, n-2 \), the fundamental weights \( \omega_k \) of \( D_n \) are the same as in (5.42), but \( \omega_{n-1} \) and \( \omega_n \) are now given by the half-spin weights

\[
\omega_{n-1} = (e_1 + \cdots + e_{n-1} - e_n)/2, \quad \omega_n = (e_1 + \cdots + e_{n-1} + e_n)/2.
\]

(5.57)

It is not hard to see that the cone of dominant weights \( \mathcal{P}^+_{D_n} \) generated by \( \omega_k, k = 1, \ldots, n \), consists of the vectors

\[
(\lambda + \delta \omega_n)_{<} \equiv (\lambda_1 + \delta/2, \ldots, \lambda_{n-1} + \delta/2, \varepsilon(\lambda_n + \delta/2))
\]

with

\[
\lambda \in \mathcal{P}^+, \quad \delta = 0, 1, \quad \varepsilon = \pm 1.
\]

(5.59)

The Macdonald polynomials \( p_{\lambda+\nu,D_n} \), \( \lambda' \in \mathcal{P}^+_{D_n} \), form an orthogonal basis of \( L^2_V(D_n, \Delta D_n, dx) \). By combining the polynomials associated with \((\lambda + \delta \omega_n), \lambda \in \mathcal{P}^+ \) and \((\lambda + \delta \omega_n), \lambda \in \mathcal{P}^+ \), one obtains polynomials that are even in \( \epsilon \). These are related to Koornwinder’s polynomials in the following way (cf. Eqs. (5.50), (5.51)):

\[
\hat{D}n_{\lambda+\nu,D_n} = \sum_{1 \leq s \leq n} \prod_{1 \leq k \leq \lambda_n} v_s(\epsilon x_k) e^{-\beta(\epsilon \theta_1 + \cdots + \epsilon \theta_n)/2},
\]

(5.61)

These operators are proportional to Macdonald’s operators \( E_{\omega_{n-1}} \) and \( E_{\omega_n} \), which are associated with the half-spin weights (5.55).

**Appendix A: Cancellation of Poles**

In this appendix we prove two results, which were needed to demonstrate that \( \hat{D}_r \) maps \( \mathcal{A}^W \) into itself. It was claimed in the proof of Proposition (3.3) (Section 3.4) that the following expression:

\[
V_{j}^{\lambda} \sum_{0 \leq s \leq J} (-1)^s \prod_{1 \leq s' \leq s} V_{j+\lambda,D_n}^{s'},
\]

(with \( \lambda \in \mathcal{P}^+ \) and \( J \in \{1, \ldots, n\} \), \( J_0 \equiv \emptyset \)) and at \( x_1 = -x_j - 2\gamma, j = 2, \ldots, n \) (poles of zeros to the above regularity claims, thereby completing the proof).

Before turning to the details, let us outline of a sum of terms of the type

\[
(-1)^2 V_{j+\lambda,D_n}^{s'} \prod_{1 \leq s' \leq s} V_{j+\lambda,D_n}^{s'},
\]

where the index sets \( B_s \subset J, s' = 1, \ldots, s \)

\[
B_s = J_s \setminus J_{s'}
\]

The terms (A.3) are associated with the sequence \( \emptyset \subset J_1 \subset J_2 \subset \ldots \subset J \), (with the cell \( J \) fixed). Each term in (A.3) construct an involutive operation \( \sigma \) of a way that the terms in the denominators of the coefficient that we assume that the parameters \( \gamma, \mu \) are chosen in such a way that these poles are zero.

**Lemma A.1 (pole of type I)**

Let \( \gamma, \mu, \delta, \delta' \) \( (\delta = 0, 1) \) and \( x_2, \ldots, x_n \) be poles. Then (A.1) is regular as a function of \( x_1 \).

**Proof**

First note that the lemma is trivial if \( 1 \notin J \) and at \( x_1 \). But if \( 1 \in J \), then \( V_{j+\lambda,D_n}^{s'} \) gives rise to a pole.

Assume \( 1 \notin J \) and let \( B_s \) denote the block of the cell \( J \). Then \( (A.1) \) is regular as a function of \( x_1 \).

1. If \( |B_s| > 1 \), then \( \sigma \) maps (A.1) to the sequence \( \emptyset \subset J_1 \subset J_2 \subset \ldots \subset J_{s-1} \).

2. If \( |B_s| = 1 \) and \( s_1 > 1 \), then \( \sigma \) maps (A.1) to the sequence \( \emptyset \subset J_1 \subset J_2 \subset \ldots \subset J_{s-1} \).

3. If \( |B_s| = 1 \) and \( s_1 = 1 \) (i.e. \( J_1 = \{1\} \))
Phrased in words: unless $B_{s_1} = \{1\}$, the map $\sigma$ pulls the index 1 out of $B_{s_1}$ and places it in a newly created block, which is sandwiched between $B_{s_1} \setminus \{1\}$ and $B_{s_1+1}$ (case 1A); when $B_{s_1}$ contains only the index 1, then $\sigma$ merges the blocks $B_{s_1} = \{1\}$ and $B_{s_1-1}$ if $s_1 > 1$ (case 1B) or, if $s_1 = 1$, then it leaves the sequence (A.4) unchanged (case 2).

Thus defined, $\sigma$ is indeed an involution on the collection of sequences (A.4): the cases 1A and 1B are inverse to each other (see Fig. 1. below).

![Fig. 1. A graphical representation of the map $\sigma$.](image)

We claim that in the first situation (i.e. 1A or 1B) the pole at $x_1 = -\gamma$ in the term (A.2) (which is associated with (A.4)) cancels against the pole in the term corresponding with the $\sigma$-image of the sequence (A.4). To see this, we may assume that we are in situation 1A. One obtains the term corresponding to the sequence (A.3) from (A.2) by making the substitutions

$$s \rightarrow s + 1,$$

$$V_{B_{s_1}}^2 \rightarrow V_{B_{s_1} \setminus \{1\}} V_{\{1\}} V_{B_{s_1} \setminus \{1\}},$$

$$V_{B_{s_1} \setminus \{1\},J_{s_1}}^3 \rightarrow V_{B_{s_1} \setminus \{1\},(\setminus J_{s_1})\cup\{1\}} V_{\{1\},J_{s_1}},$$

$$m_3(x + 2\gamma e_{B_1}) \rightarrow m_3(x + 2\gamma e_{B_1} - 2\gamma \delta_{1,s_1} e_1)$$

( $\delta_{j,k}$ denotes the Kronecker symbol). This substitution in (A.2) amounts to replacing the part

$$(V_{B_{s_1}}^2 V_{B_{s_1} \setminus \{1\}}^2) \left[ m_3(x + 2\gamma e_{B_1}) - m_3(x) \right] \prod_{j \in B_{s_1} \setminus \{1\}} v_a(x_j + x_1) v_a(x_j + x_1 + 2\gamma) [m_3(x + 2\gamma e_{B_1}) - m_3(x)]$$

by

$$-V_{B_{s_1} \setminus \{1\},1}^3 \left[ m_3(x + 2\gamma e_{B_1} - 2\gamma \delta_{1,s_1} e_1) - m_3(x) \right] \prod_{j \in B_{s_1} \setminus \{1\}} v_a(x_j + x_1) v_a(x_j - x_1) [m_3(x + 2\gamma e_{B_1} - 2\gamma \delta_{1,s_1} e_1) - m_3(x)].$$

At $x_1 = -\gamma$ the r.h.s. of (A.11) and (A.12) $x_1 = -\gamma$ cancel.

If we are in situation 2., i.e. $B_1 = \{1\}$, then the pole in $V_{\{1\}}^3$ is compensated by a zero in the difference

$$[m_3(x + 2\gamma e_1) - m_3(x)].$$

This shows that the total residue at $x_1 = -\gamma$ is zero, completing the proof of the lemma.

**Lemma A.2 (poles of type II)**

Let $\gamma$, $\mu$, $\mu_1$, $\mu_2$ ($\delta = 0,1$) and $x_2, \ldots, x_n$ be poles. Then (A.4) is regular as a function of $x_1$.

**Proof**

The proof is very similar to that of Lemma A.1. Assume that $J$ does not contain the pair $\{1,j\}$, because $x_1 = -x_j - 2\gamma$.

Assume for the remaining part of the proof that $\sigma$ maps the collection of sequences (A.4) to

$$\emptyset \subsetneq J_1 \subsetneq \cdots \subsetneq J_{s_j-1} \subsetneq J_{s_j}.$$

1.A If the pair $\{1,j\}$ is contained in one of the blocks of sequence $\{1,j\}$, then $\sigma_j$ leaves the sequence (A.4) to itself.

2. If $B_1 = \{1,j\}$, then $\sigma_j$ maps sequence (A.4) to

$$\emptyset \subsetneq J_1 \subsetneq \cdots \subsetneq J_{s_j-1} \subsetneq J_{s_j}.$$

3. If the pair $\{1,j\}$ is not contained in any of the blocks of sequence (A.4), then

It is clear that $\sigma_j$ is an involution, the case of 2. below).
These substitutions amount to the following change in the term (A.2): replace the part
\[ x \] by
\[ x \]
\[ (A.1) \]
is zero, thus completing the proof of the lemma.

Consider situation 1., assuming case 1.A. The application of \( \sigma_j \) boils down to making the following substitutions in the associated term (A.2):
\[ s \to s + 1 \]
\[ V_{B_{j}} \]
\[ V_{B_{j} \setminus \{1, j\}} \]
\[ V_{B_{j} \setminus \{1, j\}} \]
\[ V_{B_{j} \setminus \{1, j\}} \]
\[ V_{B_{j} \setminus \{1, j\}} \]
These substitutions amount to the following change in the term (A.2): replace the part
\[ \prod_{k \in B_{j} \setminus \{1, j\}, k \neq 1, j} v_a(x_k + x_1) v_a(x_k + x_j) v_a(x_k + x_j + 2\gamma) \]
by
\[ - \prod_{k \in B_{j} \setminus \{1, j\}, k \neq 1, j} v_a(x_k + x_1) v_a(x_k - x_1) v_a(x_k + x_j) v_a(x_k - x_j) \]
\[ (A.19) \]
\[ (A.20) \]
At \( x_1 = -x_j - 2\gamma \), (A.19) and (A.20) differ only by sign. Consequently, the residues at \( x_1 = -x_j - 2\gamma \) of the corresponding terms in (A.1) add up to zero.

In situation 2. the pole in the coefficient of (A.2), which is caused by \( V_{B_{j}, 1} \), cancels against the zero in the difference of the monomial symmetric functions:
\[ [m_\lambda(x + 2\gamma e_{\{1, j\}})]_{x_1 = -x_j - 2\gamma} = 0. \]
\[ (A.21) \]
In situation 3. the denominator of the coefficient of (A.2) has no zero at \( x_1 = -x_j - 2\gamma \), so the term is regular at \( x_1 = -x_j - 2\gamma \).

We conclude from the above analysis that the total residue at \( x_1 = -x_j - 2\gamma \) in the sum (A.1) is zero, thus completing the proof of the lemma.

Fig. 2. A graphical representation of the map \( \sigma_j \).

Appendix B: Two Combinatorial Lemmas

In this appendix we prove two technical results on the eigenvalues of our difference operators; its solution resulted in explicit formulas for the eigenvalues of \( \hat{D} \)

Lemma B.1 The functions
\[ F_{m,p} = (-1)^p \sum_{1 \leq i_1 \leq \ldots \leq i_p \leq m-p} \left( \prod_{j \in J} t_j \right) F_{s} \]
form the unique solution of the linear system
\[ \sum_{J \subseteq \{1, \ldots, n\}, |J| = s} \left( \prod_{j \in J} t_j \right) F_{s} = 1, \]
with the convention \( F_{m,0} = 1 \).

Proof

After splitting off the term in (B.1) corresponding to \( |J| = s \), one arrives at
\[ F_{n,r} = - \sum_{J \subseteq \{1, \ldots, n\}, |J| = s} \left( \prod_{j \in J} t_j \right) F_{s} \]
It is clear that (B.1) with condition (B.3) cancels identically (for \( 1 \leq r \leq n \)). Hence, the system (B.1), (B.3) has a unique solution.

In order to prove that this solution is indeed given by Eq. (B.1), we must show that the expression
\[ \sum_{J \subseteq \{1, \ldots, n\}, |J| = s} \left( -1 \right)^s \left( \prod_{j \in J} t_j \right) F_{s} \]
vanishes identically (for \( 1 \leq r \leq n \)). To this end we observe that (B.5) consists of a sum
\[ \sum_{J \subseteq \{1, \ldots, n\}, |J| = s} \left( \prod_{j \in J} t_j \right) F_{s} \]

\[ \left[ m_\lambda(x + 2\gamma e_{\{1, j\}}) \right]_{x_1 = -x_j - 2\gamma} = 0. \]
\[ (A.21) \]
subject to the condition
\begin{equation}
1 \leq j_1 < j_2 < \cdots < j_s \leq n, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{r-s} \leq n - r + 1.
\end{equation}
(B.8)

We shall now show that these monomials cancel in pairs.

Let \( \sigma \) be the following operation defined on the above collection of pairs (B.7) with condition (B.3):

A. If \( j_1 < j_1 \) or \( s = 0 \), then
\begin{equation}
\{(j_1, j_2, \ldots, j_s), (i_1, i_2, \ldots, i_{r-s})\} \xrightarrow{\sigma} \{(i_1, j_1, \ldots, j_s), (i_2, \ldots, i_{r-s})\},
\end{equation}
(B.9)

B. If \( j_1 \geq j_1 \) or \( s = r \), then
\begin{equation}
\{(j_1, j_2, \ldots, j_s), (i_1, i_2, \ldots, i_{r-s})\} \xrightarrow{\sigma} \{(j_2, \ldots, j_s), (j_1, i_1, \ldots, i_{r-s})\}.
\end{equation}
(B.10)

Roughly speaking, \( \sigma \) compares the first entries of the two elements constituting the pair \( \text{[B.7]} \), and moves the smallest of these two to the first entry of the other element. One easily verifies that: first, \( \sigma \) is well defined in the sense that the image of \( \text{[B.7]} \) is again a pair satisfying \( \text{[B.8]} \); second, \( \sigma \) is an involution (\( \sigma^2 = \text{id} \)), the cases A. and B. being inverse to each other.

For the associated monomial \( \text{[B.9]} \), acting with \( \sigma \) amounts to an increase (case A.) or a decrease (case B.) of the number \( s \) by one, i.e. it flips the sign of the corresponding monomial. Therefore, combining the term \( \text{[B.9]} \) with a pair \( \text{[B.7]} \) with the one associated with its image under \( \sigma \) entails the vanishing of the sum \( \text{[B.3]} \), which completes the proof.

\[ \Box \]

Remark. If one replaces the upper bound \( n - r + 1 \) of the second summation in (B.3) by \( n \), then, for \( r = n \), expression (B.3) also vanishes. (Indeed, the above proof again applies). In this case the vanishing of (B.3) amounts to a well-known relation between the elementary symmetric functions and the complete symmetric functions (see e.g. [M1]).

Lemma B.2 The function
\begin{equation}
E_{r,n}(t_1, \ldots, t_n; p_r, \ldots, p_n) = \sum_{0 \leq s \leq r} (-1)^{r+s} \left( \sum_{1 \leq j_1 < \cdots < j_s \leq n} t_{j_1} \cdots t_{j_s} \right) \left( \sum_{r \leq i_1 \leq \cdots \leq i_{r-s} \leq n} p_{i_1} \cdots p_{i_{r-s}} \right), \quad 1 \leq r \leq n,
\end{equation}
(B.11)
is the unique solution of the recursion relation
\begin{equation}
E_{r,n}(t_1, \ldots, t_n; p_r, \ldots, p_n) = (t_n - p_n)E_{r-1,n-1}(t_1, \ldots, t_{n-1}; p_r, \ldots, p_n) + E_{r,n-1}(t_1, \ldots, t_{n-1}; p_r, \ldots, p_{n-1}),
\end{equation}
(B.12)
with the convention
\begin{equation}
E_{0,n} \equiv 1, \quad E_{r,n} \equiv 0 \text{ if } n < r.
\end{equation}
(B.13)

Proof
It is clear that (B.12) with condition (B.13) on \( n \). After splitting up the sum in (B.11) into the three blocks denoted \( \text{[B.10]} \), \( \text{[B.11]} \) and \( \text{[B.12]} \) it is easily verified that (B.12) indeed solves Eq. (B.12):

i. terms with \( j_s = n \);

ii. terms with \( j_s < n \) and \( i_{r-s} = n \);

iii. terms with \( j_s < n \) and \( i_{r-s} < n \);

Remark. In some cases Lemma B.2 can be used to obtain alternative expressions for \( \hat{D} \).

For instance, one easily verifies with the aid of (B.11) that
\begin{equation}
E_{r,n} = \sum_{j \leq \{1, \ldots, n\}} \left( \prod_{j \in J} (t - j) \right) \left( \prod_{j \notin J} (t - j) \right) \left( \prod_{j \notin J} (t - j) \right),
\end{equation}
In particular, \( E_{n,n}(t_1, \ldots, t_n; p_r, p_{r+1}, \ldots, p_n) = (t_1 - p_r) \cdots (t_n - p_{n-1}) \).

Appendix C: \( \hat{D}(A^W) = 0 \)

In this appendix we present a result due to S. N. M. Ruijsenaars [R3]. It shows that if an \( A\Delta O \) or PDO is zero on all symmetric \( r \)-vectors, then it is zero on all its \( n \)-vectors.

In this section we present a result due to S. N. M. Ruijsenaars [R3]. It shows that if an \( A\Delta O \) or PDO is zero on all symmetric \( r \)-vectors, then it is zero on all its \( n \)-vectors.

Proposition C.1 \( (\hat{D}(A^W) = 0 \Rightarrow \hat{D} = 0) \)

Let \( \hat{D} \) be an \( A\Delta O/PDO \) of the form (C.4).
\begin{equation}
\hat{D} \equiv 0,
\end{equation}
then
\begin{equation}
V_1(x) = V_2(x) = \cdots = 0.
\end{equation}
The fact that $v$ is in the kernel of $\hat{D}$ translates itself geometrically in the orthogonality of $t_\lambda$ and $v$:
\[
\hat{D} m_\lambda = 0 \iff (t_\lambda, v) = 0.
\] (C.8)

We will assume $v \neq 0$ and derive a contradiction. Let $\lambda^{(1)}, \ldots, \lambda^{(M)}$ be vectors in $\mathcal{P}^+$. One has $v \perp t_{\lambda^{(1)}}, \ldots, t_{\lambda^{(M)}}$. Therefore, the vectors $t_{\lambda^{(1)}}(x), \ldots, t_{\lambda^{(M)}}(x)$ must be linearly dependent for all $x \in \mathcal{U}$ for which $v(x) \neq 0$. Since $v(x)$ is continuous in $x$, there exists an open ball $B \subset \mathcal{U}$ on which $v(x) \neq 0$. The fact that the vectors $t_{\lambda^{(s)}}(x)$, $1 \leq s \leq M$ are real-analytic in $x$ then entails
\[
\det (\text{Col}[t_{\lambda^{(1)}}(x), \ldots, t_{\lambda^{(M)}}(x)]) = 0, \quad \forall x \in \mathcal{U}.
\] (C.9)

We will now show that an appropriate choice of the vectors $\lambda^{(1)}, \ldots, \lambda^{(M)}$ contradicts the vanishing of the above determinant.

Let $\lambda \in \mathcal{P}^+$ and $y \in \mathbb{R}^n$ be such that
\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0, \quad y_1 > y_2 > \cdots > y_n > 0.
\] (C.10)

From the asymptotics (cf. Eq. (3.37))
\[
(\hat{T}_n m_\lambda)(x)|_{x = iy, n} \sim \tau_{n, \lambda} e^{(\lambda,y) R}, \quad R \to \infty,
\] (C.11)

with
\[
\tau_{n, \lambda} = \begin{cases} 
- \beta(n, \lambda) & \text{(A} \Delta \text{O)} \\
 (\lambda_1)^{s_1} \cdots (\lambda_n)^{s_n} & \text{(PDO)}
\end{cases}
\] (C.12)

one derives:
\[
\lim_{R \to \infty} e^{(\lambda, x)} t_\lambda(x)|_{x = iy} = (\tau_{n^{(1)}, \lambda}, \ldots, \tau_{n^{(M)}, \lambda}).
\] (C.13)

Pick the vector $\lambda$ (subject to condition (C.10)) in such a way that
\[
\tau_{n^{(r)}, \lambda} \neq \tau_{n^{(p)}, \lambda}, \quad 1 \leq r < p \leq M.
\] (C.14)

That such a $\lambda$ exists follows in the A$\Delta$O case from (C.12) and the fact that the vectors $k^{(1)}, \ldots, k^{(M)}$ are distinct; in the PDO case one can pick distinct prime numbers for the components of $\lambda$.

We use $\lambda = (\lambda_1, \ldots, \lambda_n)$ to form the vectors $\lambda^{(1)}, \ldots, \lambda^{(M)}$ in the following way:
\[
\lambda^{(s)} = \begin{cases} 
(s - 1) (\lambda_1, \ldots, \lambda_n) & \text{(A} \Delta \text{O)} \\
((\lambda_1)^{s_1}, \ldots, (\lambda_n)^{s_n}) & \text{(PDO)}
\end{cases}, \quad 1 \leq s \leq M.
\] (C.15)

On the one hand, Eqs. (C.7) and (C.13) imply
\[
\tau \equiv \begin{vmatrix} 
\tau_{n^{(1)}, \lambda^{(1)}} & \cdots & \tau_{n^{(1)}, \lambda^{(M)}} \\
\vdots & \ddots & \vdots \\
\tau_{n^{(M)}, \lambda^{(1)}} & \cdots & \tau_{n^{(M)}, \lambda^{(M)}}
\end{vmatrix} = 0.
\] (C.16)

On the other hand, for the above choice of determinant:
\[
\tau_{n^{(r)}, \lambda^{(r)}} \equiv (\tau_{n^{(r)}, \lambda^{(r)}})_{1 \leq r \leq M}.
\]

Therefore,
\[
\tau = \prod_{1 \leq r < p \leq M} \tau_{n^{(r)}, \lambda^{(p)}}.
\]

Because $\lambda$ is chosen such that $\tau_{n^{(r)}, \lambda^{(r)}} \neq \tau_{n^{(p)}, \lambda^{(p)}}$ the determinant $\tau \neq 0$, contradicting (C.10).

Hence, $v$ (C.7) must be zero.

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