Energy-mass spectrum of Yang–Mills bosons is infinite and discrete

Alexander Dynin

Department of Mathematics, Ohio State University

Columbus, OH 43210, USA, dynin@math.ohio-state.edu

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Abstract

Non-perturbative antinormal quantization of relativistic Yang–Mills fields with a compact semisimple gauge group entails an infinite discrete energy–mass spectrum of $Z'$ gauge bosons. In particular, a quantum Yang–Mills theory existence and mass gap problem is solved.

To I. M. Gelfand with gratitude

1 Introduction

1.1 Problem

Large Hadron Collider ambitious goals include a long overdue sighting of the mass generating Higgs boson and discovery of new particles such as massive neutral, colorless, self-adjoint, spin-1 intermediate $Z'$ gauge bosons (RIZZO [25]), in particular, hypothetical glueballs of the strong force.

This paper presents a mathematically rigorous Higgsless quantum Yang–Mills theory on Minkowski 4-space that demonstrates an infinite discrete energy-mass spectrum of $Z'$ gauge bosons. The theory also solves the 7th problem of Clay Mathematics Institute "Millennium Prize Problems":

Prove that for any compact (semi-)simple global gauge group, a nontrivial quantum Yang–Mills theory exists on $\mathbb{R}^{1+3}$ and has
a positive mass gap. Existence includes establishing axiomatic properties at least as strong as the Gårding-Whightman axioms of the axiomatic quantum field theory. (JAFFE-WITTEN [23])
(Slightly edited)

Thus the problem consists of two parts:

A. To develop a sufficiently strong mathematically rigorous nontrivial quantum Yang–Mills theory on Minkowski time-space

B. To deduce from that theory that there is a positive mass gap in the energy-mass spectrum of Yang–Mills bosons.

Certainly, physicists have a well developed quasi mathematical quantum Yang–Mills theory (see, e. g., FADDEEV-SLAVNOV [14], and a mass gap is an experimental fact for weak and strong forces. Yet the Clay Institute has proposed a pure mathematical existence problem.

1.2 Outline

A1. In the temporal gauge, classical Yang–Mills fields (i. e., solutions of relativistic Yang–Mills equations) are in one–one correspondence with their Cauchy data. This parameterization of classical Yang–Mills fields is advantageous in two ways:

- The Cauchy data carry a positive definite scalar product.
- The constraint equation for the Cauchy data is elliptic.

The elliptic equation is solved via a gauge version of Helmholtz decomposition. The solution allows to pull back the energy–mass functional to a vector space of non-constrained Cauchy data.

A2. Our approach follows I. Segal’s program of quantization on a space of non-constrained Cauchy data (see, e. g., SEGAL[27]) along with Schwinger’s prescription (in the formulation of BOGOLJUBOV-SHIRKOV [7] Chapter II]) to quantize Noether’s invariant functionals of Cauchy data as time-dependent fields on \( \mathbb{R}^3 \). (Characteristically, there is no singular Lagrangian in the Schwinger’s scheme.)

Since the Yang–Mills energy-mass functional has the inverse length dimension, it is not conformally invariant on \( \mathbb{R}^3 \), and therefore its quantization may have a mass gap.
The quantum energy-mass operator is the time component of the quantized relativistic energy–momentum vector but its mass gap is uniformly positive in all Lorentz coordinate systems.

**A3.** Following the holomorphic 2nd quantization scheme of [21], [22], the quantum energy-mass operator is a continuous linear operator in a Gelfand triple of non-constrained Cauchy data (rather than in a conventionally non-specified Hilbert space). However we use a more convenient white noise calculus holomorphic Gelfand triple.

**B1.** Since the quantum energy-mass operator is not a selfadjoint operator on a Hilbert space, its spectrum is defined via the variational mini–max principle (cp., e. g., BEREZIN-SHUBIN[4] Appendix 2) with the caveat that the dimensions of participating subspaces are taken relative to the abelian von Neumann algebra generated by the orthogonal projectors onto \( n \)-particle states.

**B2.** To facilitate the mini–max principle, the quantization of the conserved energy polynomial is chosen to be antinormal (aka anti-Wick or Berezin quantization). The use of the antinormal quantization is a specific renormalization (cp. GLIMM-JAFFE[16]). The gain is twofold:

- The corresponding normal symbol of the energy-mass operator contains a quadratic mass term which is absent in the energy–mass functional.
- The expectation of the energy-mass operator majorizes the expectation number operator with von Neumann simple discrete spectrum.

Then, by the mini–max principle, the energy-mass operator has an infinite von Neumann discrete spectrum. (The proof is the centerpiece of the paper.)

**B3.** Finally, the von Neumann mass gap coincides with the usual mini–max mass gap. (Thus the fact that the von Neumann spectrum is discrete is the backbone of the solution of the Yang–Mills problem.)

### 1.3 Contents

Section 2 reviews basics of classical Yang–Mills dynamics.
Section 3 states and proves the gauge Helmholtz decomposition of Cauchy data.

Section 4 describes holomorphic functional quantization.

Section 5 is the formulation and a proof that energy–mass spectrum of Yang–Mills bosons is infinite and discrete.

Section 6 discusses briefly more conventional mathematical approaches to solution of the Yang–Milss mass gap problem: constructive Glimm–Jaffe and perturbative quantum fields theories. Both are not rigorous yet.

Next we speculate how the dimensionality of Yang–Mills energy–mass functional produces a ”dimensional transmutation” of the coupling constant and eventually ”asymptotic freedom” of Yang–Mills bosons at higher energies.

Finally, we demonstrate properties of our quantum quantum Yang–Mills that are comparable with Gárding–Whightman axioms.

Appendix section 7 presents a solution of Schrödinger equation for quantum Yang–Mills transition amplitudes via a mathematically rigorous version of Feynman integral.

2 Classical dynamics of Yang–Mills fields

2.1 Global gauge group

The global gauge group of a Yang–Mills theory is a connected semi-simple compact Lie group $G$.

By Cartan-Weyl classification, $G$ is a compact matrix group whose Lie algebra $\text{Ad}(G)$ is a direct sum of Lie algebras of the special unitary groups $SU(N), N \geq 2$, special orthogonal groups $SO(N), N \geq 3$, symplectic groups $Sp(N), N \geq 3$ and the exceptional groups $G_2, F_4, E_6, E_8$.

All connected compact simple Lie groups are the listed groups (except for $SO(4)$) and spinor groups $\text{Spin}(N), N \geq 3, N \neq 4$.

The notation $\text{Ad}(G)$ indicates that the Lie algebra carries the adjoint representation $\text{Ad}(g)a = gag^{-1}, g \in G, a \in \text{Ad}(G)$, of the group $G$ and the corresponding representation $\text{ad}(a)b = [a, b], a, b \in \text{Ad}(G)$, of the algebra $\text{Ad}(G)$.

In this paper we need just the positive definiteness of the negative $\text{Ad}$-invariant Killing inner product on $\text{Ad}(G)$:

$$X \star Y \equiv -\text{Trace}[\text{ad}(X)\text{ad}(Y)].$$

(1)
2.2 Local gauge group

Let the Minkowski space $\mathbb{M}$ be oriented and time oriented with the Minkowski metric signature $-, +, +, +$. In a Minkowski coordinate system $x^\mu, \mu = 0, 1, 2, 3$, the metric tensor is diagonal. In the natural unit system, the time coordinate $x^0 = t$. Thus $(x^\mu) = (t, x^i), i = 1, 2, 3$.

The local gauge group $\mathcal{G}$ is the group of infinitely differentiable $G$-valued functions $g(x)$ on $\mathbb{M}$ with the pointwise group multiplication. The local gauge Lie algebra $\text{Ad}(\mathcal{G})$ consists of infinitely differentiable $\text{Ad}(G)$-valued functions on $\mathbb{M}$ with the pointwise Lie bracket.

$\mathcal{G}$ acts via the pointwise adjoint action on $\text{Ad}(\mathcal{G})$ and correspondingly on $\mathfrak{g}$, the real vector space of gauge fields $A = A_\mu(x) \in \text{Ad}(\mathcal{G})$.

Gauge fields $A$ define the covariant partial derivatives
\[
\partial_{A\mu} X \equiv \partial_\mu X - \text{ad}(A_\mu)X, \quad X \in \text{Ad}(\mathcal{G}).
\]
This definition shows that in the natural units gauge connections have the mass dimension $1/|L|$. We omit before $\text{ad}(A_\mu)$ the dimensionless coupling constant.

Any $g \in \mathcal{G}$ defines the affine gauge transformation
\[
A_\mu \mapsto A_\mu^g := \text{Ad}(g)A_\mu - (\partial_\mu g)g^{-1}, \quad A \in \mathfrak{g},
\]
so that $A_\mu^g A_\nu^g = A_{\mu\nu}^g$.

Yang–Mills curvature tensor $F(A)$ is the antisymmetric tensor
\[
F(A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu].
\]
The curvature is gauge invariant:
\[
\partial_{A\mu}\text{Ad}(g) = \text{Ad}(g)\partial_{A\mu}, \quad \text{Ad}(g)F(A) = F(A^g).
\]

2.3 Cauchy problem for Yang–Mills fields.

By Fock-Weyl gauge relativity principle, a Yang–Mills theory should be covariant under gauge transformations.

Utiyama theorem (see, e. g., BLEEKER [5]): Any Lagrangian which is invariant under gauge transformations, is a function of the curvature $F(A)$. Such is is the Yang–Mills Lagrangian
\[
L = -(1/4)F(A)^{\mu\nu} \ast F(A)_{\mu\nu}
\]
The corresponding Euler–Lagrange equation \( \partial_{\mu} F(A)^{\mu \nu} = 0 \) is the Yang–Mills equation, a 2nd order non-linear partial differential

\[
\partial_{\mu} F^{\mu \nu} + [A_{\mu}, F^{\mu \nu}] = 0. \tag{7}
\]

The solutions \( A \) are called Yang–Mills fields. They form the configuration space \( \mathcal{M} \) of the classical Yang–Mills theory. Physicists use the term "on-shell" for restriction to \( \mathcal{M} \). The term "off-shell" means no such restriction.

The expanded form of (7) is

\[
\Box A^{\nu} - \partial^{\nu}(\partial_{\mu} A^{\mu}) + [A_{\mu}, \partial^{\nu} A^{\mu}] - [A_{\mu}, \partial^{\nu} A^{\mu}] + [A_{\mu}, [A^{\mu}, A^{\nu}]] = 0 \tag{8}
\]

where \( \Box \equiv \partial_{\mu} \partial^{\mu} = -\partial^{2} + \Delta \) is the d’Alembertian.

In view of gauge invariance, a Yang–Mills field \( A(t, x) \) is not uniquely defined by the Cauchy data \( A(0, x), \partial_{t} A(0, x) \). However in the temporal gauge \( A_{0}(t, x) = 0 \) the equation (7) splits into the evolution hyperbolic system

\[
\Box A^{i} - \partial^{i}(\partial_{\mu} A^{i}) + [A_{i}, \partial^{i} A^{i}] - [A_{i}, \partial^{i} A^{i}] + [A_{i}, [A^{i}, A^{j}]] = 0, \tag{9}
\]

and the constraint equation

\[
\partial^{i}(\partial_{\mu} A_{i}) + [A^{i}, \partial_{\mu} A_{i}] = 0. \tag{10}
\]

Then, by GOGANOVI-KAPITANSKII [18], the Cauchy problem for the evolution system (9) is uniquely solvable on the whole \( \mathbb{R}^{1+3} \) with smooth Cauchy data, so that the constraint equation is satisfied if

\[
\partial_{t}(\partial^{i} A_{i}(0, x)) + [A^{i}(0, x), \partial_{\mu} A_{i}]|_{\mu}(0, x) = 0. \tag{11}
\]

From now on we assume that all space derivatives of gauge fields \( A = A(t, x^{k}) \) vanish faster than any power of \( x^{k} \) as \( x^{k} \to \infty \), uniformly with respect bounded \( t \). (This condition does not depend on a Lorentz coordinate system.) Let \( \mathcal{G}_{0} \) denote the local Lie algebra of such gauge fields and \( \mathcal{G}_{1} \) denote the corresponding infinite dimensional local Lie group.

Then 3-dimensional integration of the divergence-free Noether current vector fields leads to Noether relativistic and gauge invariant on shell conservation laws. The 15-dimensional conformal group of symmetries of Yang–Mills equation produces 15 independent non-trivial conservation laws (see, e. g., GLASSEY-STRAUSS [15]). Four of them are the conservation of the energy-momentum relativistic vector.
On the other hand, gauge invariance of Yang–Mills equation under infinite dimensional group \( G_1 \) produces no non-trivial conservation law. In particular, such Yang–Mills fields are colorless (see, e.g., Glassey-Strauss [15]).

Choose a Lorentz coordinate system. Then we get the following matrix-valued time-dependent fields on \( \mathbb{R}^3 \):

**Gauge electric vector field** \( E(A) \equiv (F_{01}, F_{02}, F_{02}) \),

**Gauge magnetic pseudo vector field** \( B(A) \equiv (F_{23}, F_{31}, F_{12}) \).

Now the (non-trivial) energy–mass conservation law is that the time component

\[
P^0(A) \equiv \int d^3x (1/2)(E^i \star E_i + B^i \star B_i)
\]

of the relativistic energy-momentum vector is constant on-shell. Appropriately, \( P^0(A) \) has the mass dimension.

At the same time, by Glassey-Strauss Theorem [15], the energy–mass density \((1/2)(E^i \star E_i + B^i \star B_i)\) scatters asymptotically along the light cone as \( t \to \infty \). This is a mathematical reformulation of the physicists assertion that Yang–Mills fields propagate with the light velocity.

### 2.4 1st order formalism

Rewrite the 2nd order Yang–Mills equations (9) in the temporal gauge \( A_0(t,x^k) = 0 \) as the 1st order systems of the evolution equations for the time-dependent \( A_j(t,x^k) \), \( E_j(t,x^k) \) on \( \mathbb{R}^3 \) as

\[
\partial_t A_k = E_k, \quad \partial_t E_k = \partial_j F^j_k - [A_j, F^j_k], \quad F^j_k = \partial_j A_k - \partial_k A^j - [A^j, A_k].
\]

and the constraint equations

\[
[A^k, E_k] = \partial^k E_k, \quad \text{i.e.,} \quad \partial_{kA}E_k = 0
\]

By Goganov-Kapitanski [18], the evolution system is a semilinear first order partial differential system with finite speed propagation of the initial data, and the Cauchy problem for it with initial data at \( t = 0 \)

\[
a(x_k) \equiv A(0,x_k), \quad e(x_k) \equiv E(0,x_k)
\]

is **uniquely solvable** on \( \mathbb{R}^{1+3} \) in \( \mathcal{G}_0 \).
If the constraint equations are satisfied at $t = 0$, then, in view of the evolution system, they are satisfied for all $t$ automatically. Thus the 1st order evolution system along with the constraint equations for Cauchy data is equivalent to the 2nd order Yang–Mills system. Moreover the constraint equations are invariant under time independent gauge transformations. As the bottom line, we have

**Proposition 2.1** In the temporal gauge Yang-Mills fields $A$ are in one–one correspondence with their Cauchy data $(a,e)$ that have the gauge transversality property $\partial_a e = 0$.

### 3 Gauge Helmholtz decomposition

#### 3.1 Gauge vector calculus

Let $\mathcal{V}^0 = \mathcal{V}^0(\mathbb{R}^3)$ denote the real $\mathcal{L}^2$-space of $\text{Ad}(G)$-valued vector fields $a$ on $\mathbb{R}^3$. The corresponding Sobolev-Hilbert spaces on $\mathbb{R}^3$ are denoted $\mathcal{W}^s$, $s \geq 0$; and $\mathcal{W}_0^s$ are closures in $\mathcal{W}^s$ of the subspaces of smooth $\text{Ad}(G)$ valued functions with compact supports in $\mathbb{R}^3$. By Sobolev, if $s \geq 3/2$, then $\mathcal{W}^s$ consists of matrix-valued functions with bounded continuous partial derivatives up to the order $< s - (3/2)$. Moreover, if $s' > s$ then the imbedding $\mathcal{W}^{s'} \subset \mathcal{W}^s$ is compact.

Accordingly, $\mathcal{V}^s_0$ denote the spaces of vector fields with components in $\mathcal{W}^s_0$. The intersections $\mathcal{V}^\infty_0 \equiv \bigcap_s \mathcal{V}^s_0$ and $\mathcal{V}^s_0 \equiv \bigcap_s \mathcal{V}^s$ form nuclear Frechet spaces of smooth $\text{Ad}(G)$-valued scalar and vector fields on $\mathbb{R}^3$.

We get the Gelfand triple of real topological vector spaces

$$\mathcal{V}^\infty_0 \subset \mathcal{V}_0 \subset \mathcal{V}^{-\infty}_0,$$  \hspace{1cm} (16)

where $\mathcal{V}^{-\infty}_0$ is the nuclear space of $\text{Ad}(G)$-valued vector distributions on $\mathbb{R}^3$, i.e., the dual space of $\mathcal{V}_0^\infty$.

Let $\mathcal{G}^s_1 \subset \mathcal{W}^s$, $s > 3/2$, denote the infinite dimensional Frechet Lie group of $G$–valued functions on $\mathbb{R}^3$ whose Lie algebra is $\mathcal{V}^s_0$.

Local gauge transformations

$$a_k^g = \text{Ad}(g)a_k - (\partial_k g)g^{-1}, \quad g \in \mathcal{G}^{s+3/2}_1, \ a \in \mathcal{V}^s_0,$$  \hspace{1cm} (17)

define continuous left action of $\mathcal{G}^s_1$ on $\mathcal{V}^s_0$.
The intersection \( G^\infty_1 \equiv \bigcap_s G^s_1 \) is an infinite dimensional Lie group with the nuclear Lie algebra \( \mathcal{V}_0^\infty \). The local gauge transformations \( v^g \) by \( g \in G^\infty_1 \) define continuous left action \( G^\infty_1 \times \mathcal{V}_0^s \to \mathcal{V}_0^s \).

Moreover, by DELL’ANTONIO - ZWANZIGER [9], the action is continuous even when \( s = 0 \) and \( G^\infty_1 \) is endowed with the natural topology induced by \( L^2 \) topology of \( \mathcal{V}^0 \). The gauge orbits of this action are closed; and on every orbit the Hilbert norm \( \| a^g \| \) attains the absolute minimum at some gauge equivalent connection \( \tilde{a} \). (The result was proved under condition that the connections are defined on a compact manifold which in our case may be taken as the spherical compactification of \( \mathbb{R}^3 \).

Even earlier SEMENOV-TYAN-SHANSKII – FRANKE [28] have proved the

**Proposition 3.1** The minimizing connections \( \tilde{a} \) are divergence free: \( \partial^k \tilde{a}_k = 0 \).

Define the following linear partial differential operators on smooth Ad\((G)\) valued functions \( u \) and vector fields \( v_k \) on \( \mathbb{R}^3 \):

- **Gauge gradient** \( \text{grad}_a u \equiv \text{grad} u - [a, u] \),
- **Gauge divergence** \( \text{div}_a v \equiv \partial_k a_k - [a_k, v^k] \),
- **Gauge Laplacian** \( \triangle_a u \equiv \text{div}_a \text{grad}_a u \).

The 1st order partial differential operators \( -\text{grad}_a \) and \( \text{div}_a \) are adjoint with respect to the \( L^2 \) scalar product:

\[
\langle -\text{grad}_a u \mid v \rangle = \langle u \mid \text{div}_a v \rangle.
\]  

(18)

If \( a \in \mathcal{V}^{s+3}_0 \), then \( \text{div}_a \) is a continuous linear operator from \( \mathcal{V}^{s+1}_0 \) to \( \mathcal{V}^s \) and \( \text{grad}_a \) is continuous linear operator from \( \mathcal{V}^{s+1}_0 \) to \( \mathcal{V}^s \).

The gauge Laplacian \( \triangle_a \) is a 2nd order partial differential operator. Since its principal part is the usual Laplacian \( \triangle \), the operator \( \triangle_a \) is elliptic. Moreover, it has a unique extension to the Dirichlet domain \( \mathcal{W}^1_0 \) as an unbounded symmetric operator in the real Hilbert space \( L^2 \). We keep the notation \( \triangle_a \) for this extension.

**Proposition 3.2** The gauge Laplacian \( \triangle_a \) is an invertible operator from \( \mathcal{W}^{s+2}_0 \) onto \( \mathcal{W}^s_0 \) for all \( s \geq 0 \).

**Lemma 3.1** \( \triangle_a u = 0 \), \( u \in \mathcal{W}^1_0 \), if and only if \( u = 0 \).

\[
\triangleright \quad u \star [a, u] = -\text{Trace}(uau - uua) = 0 \quad \text{so that} \quad u \star \text{grad}_a u = u \star \text{grad} u = (1/2)\text{grad}(u \star u) = 0.
\]  

(19)
This shows that for \( u \in \mathcal{W}^1_0 \) we have \( \text{grad}_a u = 0 \) if and only if \( u = 0 \). \( \triangleright \)

Next, by the equality (18),
\[
\langle \triangle_a u \mid u \rangle = \langle -\text{grad}_a u \mid \text{grad}_a u \rangle, \quad u \in \mathcal{W}^1_0.
\]  

(20)

Thus \( \triangle_a u = 0, \ u \in \mathcal{W}^1_0 \), if and only if \( u = 0 \). \( \triangleright \)

Both Laplacian \( \triangle \) and gauge Laplacian \( \triangle_a \) map \( \mathcal{W}^{s+2}_0 \) into \( \mathcal{W}^s_0 \).

The Laplace operator is invertible from \( \mathcal{W}^{s+2}_0 \) onto \( \mathcal{W}^s_0 \) whatever \( s \geq 0 \) is. Since \( \triangle - \triangle_a \) is a 1st order differential operator, the operator \( \triangle_a : \mathcal{W}^{s+2}_0 \to \mathcal{W}^s_0 \) is a Fredholm operator of zero index. Then, by Lemma 3.1, the inverse \( \triangle_a^{-1} : \mathcal{W}^s_0 \to \mathcal{W}^{s+2}_0 \)
exists for all \( s \geq 0 \). \( \triangleright \)

Now proposition 3.2 shows that the operator \( \text{div}_a : \mathcal{V}^s_0 \to \mathcal{V}^{s-1}_0 \) is surjective and the operator \( \text{grad}_a : \mathcal{W}^s_0 \to \mathcal{V}^{s-1}_0 \) is injective. We have arrived to

**Theorem 3.1** The gauge Helmholtz operator
\[
P_a \equiv \text{grad}_a \triangle_a^{-1} \text{div}_a
\]  

is an \( \mathcal{L}^2 \)-bounded projector of \( \mathcal{V}^s \) onto the space of gauge longitudinal vector fields, i. e., the range of the operator \( \text{grad}_a : \mathcal{W}^{s+1}_0 \to \mathcal{V}^s_0 \).

The operator \( \mathbf{1} - P_a \) is an \( \mathcal{L}^2 \) bounded projector of \( \mathcal{V}^s \) onto the space of gauge transversal vector fields, i. e., the null space of the operator \( \text{div}_a : \mathcal{V}^s_0 \to \mathcal{V}^{s-1}_0 \).

\( \triangleright \) Both \( P_a \) and \( \mathbf{1} - P_a \) are pseudodifferential operators of order 0, and, therefore are \( \mathcal{L}^2 \)-bounded.

By computation,
\[
P_a^2 = P_a, \quad P_a \text{grad}_a = \text{grad}_a, \quad \text{div}_a (\mathbf{1} - P_a) = 0. \quad \triangleright
\]

The space \( \mathcal{C}^s_0, \equiv \mathcal{V}^s_0 \times \mathcal{V}^s_0, \ s \geq 0, \) of Cauchy data \( c \equiv (e, a) \) is the direct product of the spaces of gauge electric components \( e \) and the gauge magnetic components \( a \).

The Cauchy data \( c \in \mathcal{C}^s, \ s \geq 1, \) are gauge transversal if their gauge electric components are solutions of the constraint equation \( \text{div}_a e = 0 \). Let \( \mathcal{T}^s \subset \mathcal{C}^s \) denote the set of gauge transversal Cauchy data
\[
(a, e_a) \equiv (a, e - P_a(e)), \quad (a, e) \in \mathcal{C}^s, \ s \geq 1.
\]  

(22)

In particular, pure gauge magnetic fields \( (a, 0) \) are transversal.

The intersection \( \mathcal{T} \equiv \bigcap_s \mathcal{T}^s \) is a nuclear smooth Frechet manifold with the diagonal smooth action \( \phi^s = (e^s, a^s) \) of the compact local gauge group \( G_1 \). This manifold is a smooth Frechet submanifold in \( \mathcal{C} \equiv \bigcap_s \mathcal{C}^s \).

**Proposition 3.3** Projectors \( \mathbf{1} - P_a \) are gauge covariant.
Propositions 3.2 and 3.1 show that, if $v$ is a smooth vector field with compact support in $\mathbb{R}^3$ then $e_a \equiv (1 - P_a)v$ is smooth with support in the closure of $\mathbb{R}^3$, so that the pair $(e_a, a)$ is a transversal Cauchy data. Thus $T^s$ is a vector subbundle in $C^s$ over (A). Its fiber over $a$ is the null space of the projector $1 - P_a$.

The constraint space $T^s$ is gauge invariant. The gauge transformations are affine and invertible. Thus $T^s$, as the vector bundle, is gauge covariant.

4 Holomorphic quantization

4.1 Holomorphic states

In the complexification of the Gelfand triple (16)

$$C^\infty_0 \subset C^0_0 \subset C^\infty_0$$  \hspace{1cm} (23)

the real inner product on $\mathcal{V}^0_0$ is extended to the sesquilinear Hermitian inner product $z^*w$ on $C^\infty_0$ (antilinear on the left and linear on the right). The complex conjugation $z^*$ is the Hermitian matrix conjugation. The Hermitian form is extended to the sesquilinear form on $C^\infty_0 \times C^{\infty}_0$ so that the spaces $C^\infty_0$ and $C^{\infty}_0$ are mutually antidual. By the Riesz theorem, the Hilbert space $C^\infty_0$ is identified with its anti-dual, so that if $z \in C^\infty_0$ then $z^* \in C^{\infty}_0$.

By Bochner-Minlos theorem, space $C^{\infty}_0$ carries the probability Gauss Radon measure $dz^*dz e^{-z^*z}$. This symbolic expression is meaningful as a cylindrical measure on $C^{\infty}_0$ which extends to the Gauss measure. We use the same notation for both measures because it allows integration by parts and Fubini theorem which hold for integrals of cylindrical functions followed by limit transition to wider class of integrable functions.

An important Fernique’s theorem (see, e.g., BOGACHEV[6]) implies that there exists a positive constant $k$ such that if a functional $\Psi(z^*)$ on $C^{\infty}_0$ is continuous and $\Psi \prec e^{kz^*z}$ then $\Psi(z^*)$ is integrable with respect to Gauss measure.

The Bargmann space (see, e. g., BEREZIN[2], Chapter I) is the (complete) complex Hilbert space of Gâteaux entire functionals $\Psi = \Psi(z^*)$ on $C^{\infty}_0$ with conjugation

$$\Psi^* = \Psi^r(z) \equiv \overline{\Psi(z^*)}$$  \hspace{1cm} (24)

and integrable Hermitian sesquilinear inner product

$$\Psi^* \Phi \equiv \int dz^*dz e^{-z^*z} \Psi^r(z) \Phi(z^*)$$.  \hspace{1cm} (25)

The integral is is denoted also as the Gaussian contraction $\Psi^r(z) \Phi(z^*)$. 

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The exponential functionals
\[ e^{z} \equiv e^{z^*}, \quad z \in \mathbb{C}_0^0, \quad (26) \]
belong to \( \mathcal{B}^0 \) since \( e^{z^*} e^z = e^{z^*z} < \infty \). Indeed
\[ e^{z^*} e^{\xi} = \int dz^* dz e^{-z^*z} e^{z^*z+\xi} = e^{z^*} \int dz^* dz e^{-(z^*-z^*)(z-\xi)} = e^{z^*} \xi \quad (27) \]
They form a continual orthogonal basis of exponential functionals in \( \mathcal{B}^0 \) (see, e. g., BEREZIN [2, Chapter I]): If \( \Psi = \Psi(z^*) \in \mathcal{B}^0 \) then the Borel transform
\[ \Psi(z^*) = e^{-z^*z} \int d\xi d\xi^* e^{-\xi^*\xi} \tilde{\Psi}(\xi) e^{z^*\xi}, \quad \tilde{\Psi}(\xi) \equiv \Psi^* e^\xi. \quad (28) \]
is a unitary operator in \( \mathcal{B}^0 \).
The continual orthogonal basis is overcomplete since
\[ e^z = \int d\xi d\xi^* e^{-\xi^*\xi} e^{z^*\xi}. \quad (29) \]

Bargmann-Hida space \( \mathcal{B}^\infty \) is the vector space of of Gâteaux entire test functionals \( \Psi(z^*) \) on \( \mathbb{C}_0^\infty \) of the (topological) second order and minimal type, i. e., for any \( s \geq 0 \) and \( \varepsilon > 0 \) there exists a constant \( C > 0 \) such that
\[ |\Psi(z^*)| \leq Ce^{K\|z\|^2}, \quad z^* \in \mathbb{C}_0^{-s}. \quad (30) \]
The Bargmann-Hida space \( \mathcal{B}^\infty \) is a nuclear space of type (F), dense in \( \mathcal{B}^0 \) (see, e. g., OBATA [24, Section 3.6]). Actually, the topology of \( \mathcal{B}^\infty \) is defined by the norms
\[ ||\Psi||_{s,\varepsilon} \equiv \sup_{z^*} |\Psi(z^*)| e^{-\|z\|^2}, \quad (31) \]
Again, by OBATA [24, Section 3.6]), Borel transform is a topological automorphism of \( \mathcal{B}^\infty \). Bargmann-Hida space \( \mathcal{B}^{-\infty} \) of generalized functionals \( \Psi^*(z) \) on \( \mathbb{C}_0^\infty \) is the strong anti-dual space of \( \mathcal{B}^\infty \) (and, therefore, of type (DF)). The Borel transform \( \Psi^*(z) \) of \( \mathcal{B}^{-\infty} \) is defined as the anti-dual of the Borel transform of \( \mathcal{B}^\infty \) of \( \mathcal{B}^{-\infty} \) (and, therefore, a topological automorphism).

By (e. g., OBATA, [24, Section 3.6], the generalized functionals are characterized as entire functionals of the (bornological) second order on \( \mathbb{C}_0^\infty \), i. e., there exist positive constants \( C, K \) and \( s \geq 0 \) such that
\[ |\Psi(z)| \leq Ce^{K\|z\|^2}, \quad z \in \mathbb{C}_0^s. \quad (32) \]
We get the Bargmann-Hida Gelfand triple of holomorphic states
\[ \mathcal{B}^\infty \subset \mathcal{B}^0 \subset \mathcal{B}^{-\infty}. \quad (33) \]
The vector spaces $\mathcal{B}^\infty$ and $\mathcal{B}^{-\infty}$ are locally convex commutative topological algebras with the point-wise multiplication. Moreover we have Taylor expansions

$$\Psi(z^* + w^*) = \sum_{n=0}^\infty \frac{\partial^n \Psi(z^*)}{n!} w^n \quad \text{for } \Psi \in \mathcal{B}^\infty,$$

$$\Psi(z + w) = \sum_{n=0}^\infty \frac{\partial^n \Psi(z)}{n!} w^n \quad \text{for } \Psi \in \mathcal{B}^{-\infty}.$$  

By conjugating $z$ to $z^*$, we convert $\mathcal{C}C_0^\infty \subset \mathcal{C}C_0^0 \subset \mathcal{C}C_0^{-\infty}$ into the anti-linear Gelfand triple $\overline{\mathcal{C}}\mathcal{W}_0^\infty \subset \overline{\mathcal{C}}\mathcal{W}_0^0 \subset \overline{\mathcal{C}}\mathcal{W}_0^{-\infty}$. Their direct product

$$\mathcal{C}C_0^\infty \times \overline{\mathcal{C}}\mathcal{W}_0^\infty \subset \mathcal{C}C_0^0 \times \overline{\mathcal{C}}\mathcal{W}_0^0 \subset \mathcal{C}C_0^{-\infty} \times \overline{\mathcal{C}}\mathcal{W}_0^{-\infty}$$

carries the complex conjugation $(z, w^*) \mapsto (w, z^*)$.

The Bargmann-Hida Gelfand triple associated with (36) is denoted as

$$\mathcal{B}^\infty \subset \mathcal{B}^0 \subset \mathcal{B}^{-\infty},$$

Sesqui-entire functionals $\Theta(z, w^*) \in \mathcal{B}^{-\infty}$ are uniquely defined by $\Theta(z, z^*)$, their restrictions to the real diagonal. If $\Theta(z, z^*) = \Theta(z, z)$, then $\Theta(z, z^*)$ is a classical observable on the phase space $\mathcal{C}C_0^\infty$.

**Example.** The energy-mass functional $H(z, z^*)$ pulled from complex gauge transversal Cauchy data to $\mathcal{C}C_0^\infty$ via the complexified Helmholtz operator.

### 4.2 Quantization of classical observables

For $z \in \mathcal{C}C_0^\infty$, $z^* \in \overline{\mathcal{C}}\mathcal{W}_0^{-\infty}$ define four continuous operators of multiplication and directional complex differentiation (operators of creation and annihilation):

$$\xi : \mathcal{B}^\infty \rightarrow \mathcal{B}^\infty, \quad \xi^* \Psi(z^*) \equiv z \Psi(z^*) = (\xi z) \Psi(z^*);$$  

$$\xi^\dagger : \mathcal{B}^{-\infty} \rightarrow \mathcal{B}^{-\infty}, \quad \xi^\dagger \Psi(z) \equiv \partial_z \Psi(z);$$  

$$\xi^- : \mathcal{B}^\infty \rightarrow \mathcal{B}^{-\infty}, \quad \xi^- \Psi(z) \equiv z^* \Psi(z) = (z^* \xi) \Psi(z);$$  

$$\xi^\dagger^- : \mathcal{B}^{-\infty} \rightarrow \mathcal{B}^\infty, \quad \xi^\dagger^- \Psi(z^*) \equiv \partial_{z^*} \Psi(z^*).$$

The continuity of multiplications is straightforward and of directional differentiations is by Cauchy integral formula for the derivative of a holomorphic function.  

These operators generate strongly continuous abelian operator groups in $\mathcal{B}^\infty$ and $\mathcal{B}^{-\infty}$:

$$e^\xi : \mathcal{B}^\infty \rightarrow \mathcal{B}^\infty, \quad e^\xi \Psi(z^*) = e^{\xi z} \Psi(z^*);$$  

$$e^{\xi^\dagger} : \mathcal{B}^{-\infty} \rightarrow \mathcal{B}^{-\infty}, \quad e^{\xi^\dagger} \Psi(z) = \Psi(z + z);$$  

$$e^{\xi^-} : \mathcal{B}^\infty \rightarrow \mathcal{B}^{-\infty}, \quad e^{\xi^-} \Psi(z) = e^{\xi^*} \Psi(z);$$  

$$e^{\xi^\dagger^-} : \mathcal{B}^{-\infty} \rightarrow \mathcal{B}^\infty, \quad e^{\xi^\dagger^-} \Psi(z^*) = \Psi(z^* + z).$$
The only non-trivial commutator relations for the groups

\[ [e^{\hat{z}^\dagger}, e^\hat{z}] = e^{zz}, \quad [e^{\hat{z}^\dagger}, e^\hat{z}^\dagger] = e^{\hat{z}z}. \] (46)

imply the only non-trivial commutator relations for their generators

\[ [\hat{z}^\dagger, \hat{z}] = z^* z, \quad [\hat{z}^\dagger, \hat{z}^\dagger] = zz^*. \] (47)

The *normal quantization* \( \Theta(\hat{z}, \hat{z}^\dagger) \) of \( \Theta(z, z^*) \in \mathcal{B}^{-\infty} \) is defined as the continuous linear operator

\[ \Theta(\hat{z}, \hat{z}^\dagger) : \mathcal{B}^\infty \mapsto \mathcal{B}^{-\infty} \] (48)

via the sesquilinear quadratic form (in Einstein-Dewitt contraction notation)

\[ \Psi^*(z)\Theta(\hat{z}, \hat{z}^\dagger)\Psi(z^*) \equiv \bar{\Theta}(\xi^*, \xi) e^\zeta e^{\hat{z}^\dagger} \bar{\Psi}^*(\xi)\bar{\Psi}(\xi^*). \] (49)

The holomorphic \( \bar{\Theta}(\xi^*, \eta) \) is the *normal symbol* of the operator \( \Theta(\hat{z}, \hat{z}^\dagger) \) uniquely defined by its restriction \( \bar{\Theta}(\xi^*, \xi) \) to the real diagonal \( \eta = z \).

Similarly, the *kernel* \( K(z, y^*) \) of the operator \( \Theta(\hat{z}, \hat{z}^\dagger) \) is uniquely defined by its diagonal restriction

\[ K(z, z^*) \equiv e^{\hat{z}^\dagger}(z)\Theta(\hat{z}, \hat{z}^\dagger)e^{\hat{z}}(z^*) \] (50)

\[ = \bar{\Theta}(\xi^* , \xi)e^\zeta e^{\hat{z}^\dagger}(\xi^*)e^\xi(\xi) = \Theta(z, z^*) e^{zz^*} \in \mathcal{B}^{-\infty} \] (51)

Thus the correspondence between *quantum observables* \( \Theta(\hat{z}, \hat{z}^\dagger) \) and classical observables \( \Theta(z, z^*) \) is one-one:

\[ K(z, z^*) = \Theta(z, z^*) e^{zz^*}. \] (52)

Since \( e^{zz^*} \) is the integral kernel of the projection of \( \mathcal{B}^{-\infty} \Pi : \mathcal{B}^{-\infty} \mapsto \mathcal{B}^\infty \), the classical variable \( \Theta(z^*, z) \) is the *Berezin-Toeplitz* (aka *antinormal, or diagonal*) symbol of the operator \( \Theta(\hat{z}, \hat{z}^\dagger) \), i.e., the compression of the multiplication with \( \Theta(z^*, z) \) to \( \mathcal{B}^\infty \)

\[ \Theta(\hat{z}, \hat{z}^\dagger)\Psi(z^*) = e^{\hat{z}z^*}\Theta(z^*, z)\Psi(z). \] (53)

The symbol is called antinormal because

\[ \Psi^*(z)e^{zz^*}\Theta(z^*, z)\Psi(z^*) = \bar{\Psi}^*(\xi^*)\bar{\Theta}(\xi^* , \xi)e^{\hat{z}^\dagger}\bar{\Psi}(\xi). \] (46)

\[ = \bar{\Psi}^*(\xi^*)\bar{\Theta}(\xi^* , \xi)e^{\hat{z}^\dagger}\bar{\Psi}(\xi). \] (54)

Compare with the (opposite) normal operator ordering in (49).
For $\Theta \in \hat{B}^\infty$ we have, by Taylor expansion and integration by parts,

\[
\Theta(z, \hat{z}^*) = e^{-\hat{z}^* \hat{z}} \hat{\Theta}(\hat{z}^*, \hat{z}) e^{\hat{z}^* \hat{z}} = \int d\xi^* d\zeta \hat{\Theta}(\xi^*, \xi) e^{-\xi^* \xi} d\xi = \int d\xi^* d\xi e^{-\xi^* \xi} \hat{\Theta}(\xi^* - \xi^*, \xi - \xi) = \sum_{k,m} \frac{(-1)^{k+m}}{k! m!} \int d\xi^* d\xi e^{-\xi^* \xi} \frac{\partial^k \xi^*}{\partial \xi^*} \frac{\partial^m \xi}{\partial \xi} \hat{\Theta}(\xi^*, \xi) = e^{(1/2)\partial_{\xi^*} \partial_\xi} \hat{\Theta}(\xi^*, \xi)
\]

since $\hat{\xi}^* = \partial_{\xi}$. (Note the contraction $\partial_{\xi^*} \partial_\xi$ is an infinite dimensional complex Laplacian.)

Since $\hat{B}^\infty$ is dense in $\hat{B}^{-\infty}$ we get the relationship between the normal and antinormal symbols for all $\Theta \in \hat{B}^{-\infty}$ as

\[
\Theta(z^*, \hat{z}) = e^{(1/2)\partial_{\xi^*} \partial_\xi} \hat{\Theta}(\xi^*, \hat{z}) \quad \text{and} \quad \hat{\Theta}(\xi^*, \hat{z}) = e^{-(1/2)\partial_{\xi^*} \partial_\xi} \Theta(z^*, \hat{z}). \quad (55)
\]

\section{Energy-mass spectrum}

\subsection{Elementary free bosons}

The \textit{number operator} $\hat{z}^* \hat{z}^- : \hat{B}^\infty \to \hat{B}^\infty$ and $\hat{B}^{-\infty} \to \hat{B}^{-\infty}$ has the normal symbol $\Theta(z, \hat{z}^*) = \hat{z}^* \hat{z}$ and the antinormal symbol $\hat{\Theta}(\hat{z}^*, \hat{z}) = \hat{z} \hat{z}^* + 1/2$. The eigenvectors of $\hat{z}^* \hat{z}^+$ are continuous homogeneous polynomials of degree $n = 0, 1, 2, \ldots$, (\textit{n-particle states}) with the corresponding eigenvalues $n$.

In particular, the constant \textit{vacuum state} $\Omega_0 \equiv 1$ corresponds to the eigenvalue $n = 0$. In general, homogeneous polynomials of degree $n$ on a complex vector space are functionals whose restrictions to finite dimensional complex vector subspaces are finite dimensional homogeneous polynomials of degree $n$.

Thus the Bargmann-Hida triple is the orthogonal sum of \textit{n-particle Gelfand triples}

\[
\hat{B}_n^\infty \subset \hat{B}_n^0 \subset \hat{B}_n^{-\infty}, \quad n = 0, 1, 2, \ldots \quad (56)
\]

The \textit{quantization} $\hat{L}$ of a continuous linear operator $L$ on $\mathbb{C}^\infty_0$ is the continuous linear operator $\hat{L} \Psi(z^*) \equiv \Psi(L^* z^*)$ on $\hat{B}^\infty$. Its normal symbol is $e^{\hat{z}^* \hat{z}^-} z e^{\hat{z}^* \hat{z}^-}$. The \textit{differential quantization} of $L$ is the continuous linear operator $\hat{L}^* \equiv (\hat{L}^*)^* \hat{z}^+ \hat{z}^+ \in \hat{B}^\infty$ with the normal symbol $z^* L z$.

Accordingly, the quantization and differential quantizations of $L^+$, defined as $\hat{L}^+$ and $\hat{L}^+$, are continuous linear operators on $\hat{B}^{-\infty}$. 

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E. g., the quantized identity operator $\hat{1}$ is the identity operator, and $\hat{1}'$ is the number operator.

An operator $L_\infty$ on the Gelfand triple $C^\infty_0 \subset C^0_0 \subset C^{-\infty}_0$ is a bounded linear operator on $C^\infty_0$ that transform $C^\infty$ into $C^\infty$ (and then $C^{-\infty}$ into $C^{-\infty}$).

Let $U_\infty$ denote the group of unitary operators $U_\infty$ on the Gelfand triple. Then their quantizations $\hat{U}_\infty$ are unitary operators on the Gelfand triple $B^\infty_0 \subset B^0_0 \subset B^{-\infty}_0$ form the quantized unitary group $\hat{U}_\infty$. They commute with the number operator so that the $n$-particle Gelfand triples (56) are eigenspaces triples of $\hat{U}_\infty$.

### 5.2 von Neumann spectrum

Let $L$ be a continuous linear operator from $B^\infty$ to $B^{-\infty}$. We use the Courant-Weyl (aka Rayleigh-Ritz) variational mini-max principle, to define the von Neumann energy-mass spectrum of $L$ relative to von Neumann operator algebra generated by the orthogonal projectors onto $n$-particle states of the number operator $\hat{1}'$ (see, e. g., DIXMIER [10]).

The expectation of $L$ on $B^\infty$ is the functional

$$\langle L \rangle \equiv \Psi^* L \Psi / \Psi^* \Psi, \quad \Psi \in B^\infty, \quad \Psi \neq 0. \quad (57)$$

Let $\dim_1 W$ denote the von Neumann dimension of a nontrivial subspace $W \subset B^\infty$ relative this von Neumann algebra. Then the relative von Neumann eigenvalues are defined as

$$\lambda_0(L) \equiv \inf \langle L \rangle_\Psi, \quad \lambda_n(L) \equiv \sup \left\{ \inf \{ \langle L \rangle_\Psi, \Psi \in W^\perp \} : \dim_1 W > n \right\}. \quad (58)$$

Note that the conventional and von Neumann lowest eigenvalues are equal, and the conventional $n$-th eigenvalue is not bigger than the $n$-th von Neumann eigenvalue.

Thus the lowest spectral gap $\lambda_1 - \lambda_0$ in the von Neumann spectrum is not less than the lowest spectral gap in the conventional spectrum.

von Neumann spectrum is called discrete if all generalized eigenvalues $\lambda_n(L)$ have finite multiplicity in it. E. g., the number operator has infinite discrete von Neumann spectrum and each eigenvalue $\lambda_n(\hat{1}') = n$ has multiplicity one.

Since the ball $\| \cdot \|_0 \leq 1$ has a compact closure in $B^{-\infty}$, finite mini-max values $\lambda_n$ are limits of the expectations at corresponding generalized eigenfunctions of $L$.

We have a straightforward analogue of the Courant-Weyl-Rayleigh-Ritz theorem:

**Proposition 5.1** Suppose operators of $L_1$ and $L_2$ are such that the expectation $\langle L_1 \rangle$ is bounded from below, $\langle L_1 \rangle \leq \langle L_2 \rangle$, and the von Neumann spectrum of $L_1$ is discrete. Then the von Neumann spectrum of $L_2$ is discrete, and $\lambda_n(L_2) \geq \lambda_n(L_1)$. 
5.3 Energy-mass spectrum of Yang–Mills bosons.

The energy functional on smooth transversal Cauchy data with minimizing \( \tilde{a} \)

\[
H(\tilde{a}, e_{\tilde{a}}) = \int_{\mathbb{R}^3} d^3x \left( (d\tilde{a} - [\tilde{a}, \tilde{a}]) \ast (d\tilde{a} - [\tilde{a}, \tilde{a}]) + e_{\tilde{a}} \ast e_{\tilde{a}} \right)
\]

(59)

\[
= \int_{\mathbb{R}^3} d^3x \left( -\Delta \tilde{a} \ast \tilde{a} + [\tilde{a}, \tilde{a}] \ast [\tilde{a}, \tilde{a}] + e_{\tilde{a}} \ast e_{\tilde{a}} \right),
\]

(60)

since, by Proposition 3.1, the minimizing connections are divergence free (here \( \Delta = \partial_k \partial_k \) is the usual Laplacian).

Henceforth we deal only with minimizing connections and skip the “breve” notation.

Now the energy functional is, obviously, non-negative, and positive when \( a \) is non-zero. We consider this classical observable as the real valued antinormal symbol \( H(z, z^\ast) \) of the quantum energy-mass operator \( \hat{H} \) (so that \( (a, e) = (z + z^\ast) / \sqrt{2} \)).

**Theorem 5.1** Let \( H_a \) denote the magnetic part \( H(a, 0) \) of the mass–energy functional, and let \( \hat{1}_a \) denote the magnetic number operator with the normal symbol \( aa^\ast \).

Then

\[
\langle \hat{H} \rangle \geq \langle \hat{1}_a \rangle + kI,
\]

(61)

where \( kI \) is a constant scalar matrix field.

\[ \triangleright \]

Since \( H(a, e_a) \geq H_a \), we have \( \langle \hat{H} \rangle \geq \langle \hat{H}_a \rangle \), by (53), for their quantum expectations.

The normal symbol of \( \hat{H}_a \) is

\[
H_a(z^\ast, z) = e^{(1/2)\partial_a \partial_{\tilde{a}}} H_a(z^\ast, z)
\]

\[
= c \tilde{I} + \int_{\mathbb{R}^3} d^3x \left( (-\Delta a) \ast a + [a, a] \ast [a, a] + (1/4) a \ast a \right),
\]

\[ \triangleright \]

Let \( e_i \) be a basis for \( \text{Ad}(\mathbb{G}) \) with \( e_i \ast e_j = \delta_{ij} \).

\[
[e_i, e_j] = c_{ij}^k e_k, \quad c_{ij}^k = [e_i, e_j] \ast e_k,
\]

(62)

so that the structure constants \( c_{ij}^k \) are antisymmetric not just in \( i, j \) but in all \( i, j, k \).

Thus, if \( a = \alpha_i e_i \in \mathbb{C}\text{Ad}(\mathbb{G}) \) then the matrix of \( \text{ad} a \) in the basis \( e_i \)

\[
(\text{ad} a)^k_j = \alpha_i c_{ij}^k
\]

(63)

is antisymmetric, so that

\[
a \ast a = \overline{\alpha} c_{ij}^k d^i c_{kj}^l,
\]

(64)
where overlines denote the complex conjugation. Since
\[
[a,a]^* \star [a,a] = \overline{d'}d'd''m_1[e_1,e_2] \star [e_1,e_2] = \overline{d'}d'd''m_1^k e_1^k e_2^k,
\] (65)
we get
\[
\frac{1}{2} \left( \frac{1}{2} \partial_{a'} \partial_a ([a,a]^* \star [a,a]) \right) \overset{64}{=} \frac{1}{4} a \star a.
\] (66)
(This computation is prompted by SIMON [29, page 217].)

The quadratic form \((-\Delta a) \star a\) is positive definite, so that the expectation of the operator with the normal symbol
\[
\int \mathbb{R}^3 d^3x (-\Delta a) \star a = 0.
\] (67)
In the same way the expectation
\[
e^{\partial_z \Psi^*} \left( \int \mathbb{R}^3 d^3x [a,a] \star [a,a] \right) e^{\partial_z^* \Psi} \geq 0.
\] (68)
Now these expectations inequalities imply
\[
\langle \hat{H} \rangle \geq \langle \hat{1}' \rangle + k
\] (69)
The inclusion relation presents one-one correspondence between magnetic \(n\)-particle states and \(n\)-particle states, so that the von Neumann dimensions with respect to \(\hat{1}'\) and \(\hat{1}\) in \(\mathcal{B}^\infty\) are the same.

Inequality \(69\) implies infinity and discreteness of the von Neumann spectrum of the energy-mass operator \(\hat{H}\); in particular, the usual mini-max spectral gap at the bottom of it.

The normal symbol \(61\) of the operator \(\hat{H}\) shows that on the vacuum state \(\Omega_0\)
\[
\hat{H}(\Omega_0) = k\mathbb{1} = (\hat{1}' + c\mathbb{1})(\Omega_0).
\] (70)
both are equal to the same constant \(k\). Since \(\hat{H} \geq \hat{1}' + c\mathbb{1}\), this implies that the first spectral values of of both operators are equal to \(k\); and that the second spectral value of \(\hat{H}\) is not less than the second spectral value of \(\hat{1}' + c\mathbb{1}\). We have arrived to

**Theorem 5.2** In any Lorentz coordinates the mass gap of the quantum energy–mass operator \(\hat{H}\) is not less than 1.
If we invoke a mass unit and a dimensionless gauge coupling constant, then, as a physical quantity, the mass gap is not less than the running dimensionful coupling constant in all Lorentz coordinates.

The minimal mass gap should be interpreted as the *quantum rest mass* of Yang–Mills fields.

## 6 Discussion

Two basic Higgsless mathematical approaches to the Cay Institute Yang–Mills problem have been proposed. (Note that the status report DOUGLAS [11] is mostly about the related mass gap problem for Seiberg–Witten gauge fields.)

Both approaches look for the mass gap in the Casimir energy spectrum of a postulated unitary representation of the Poincare group on a virtual Hilbert space where quantized Yang–Mills fields are supposed to act as operators.

- **Constructive methods of axiomatic quantum field theory** as in GLIMM-JAFFE [17] (see JAFFE-WITTEN [23]).

  Actually, various "no-go theorems" suggest that Gårding-Wightman axioms may be too strong even for scalar quantum fields with quartic self-interaction. In such a case

  1. If $d > 3$, then the quartic self-interaction is impossible (cf. BAUMANN [1]).
  2. For any $d$, interaction picture (and, therefore, its perturbative version) is incompatible with the second quantization (cf. HAAG [20]).
  3. For $d > 1$, the energy spectrum a QFT with a quartic self-interaction has no lower limit so that a cut-off is needed (cf. GLIMM-JAFFE [16]).

- **Perturbative quantization** (aka *Feynman integral*). Since the classical Yang-Mills theory is conformally invariant it has no mass parameter. To have a mass gap in quantum Yang-Mills theory the invariance must be broken. By FADDEEV [13] this may be done by renormalization of divergencies in the perturbative quantization.

  On the other hand, "Most people, however, believe that a solution of this [mass gap] problem will require non-perturbative techniques." (From Math review MR2206769 (2007c:81121) of L. Faddeev’s article).

  Indeed, our solution is obtained via non-perturbative techniques.

A choice of three natural fundamental physical units for a quantum Yang–Mills theory are the light velocity $c$, the Planck constant $\hbar$, and an energy-mass unit which may be viewed as a deformation parameter of the theory.
Usually, the strength of the Yang-Mills interaction is described by a dimensionless coupling constant. A change of the mass unit rescales the energy-mass functional. Rescaling the coupling constant instead, one converts the coupling constant into a physical quantity of the mass dimension (and simultaneously of the inverse length dimension). This may be viewed as a “dimension transmutation” of the dimensionless coupling constant into running dimensionful coupling constant.

Since the mass gap is proportional to the dimensionful coupling constant, it tends to zero together with increase of the energy–mass unit, or, equivalently with decrease of the length unit. This phenomenon may be viewed as asymptotic freedom at higher energies and/or smaller distances.

The following properties of the proposed quantum Yang–Mills theory correspond to Gårding-Whightman axioms (cf. STREATER-WIGHTMAN [30]).

**Relativistic covariance** Operators of creation and annihilation and their commutators are relativistically covariant in the Bargmann–Hida triple of Yang–Mills Cauchy data at different times.

**Stability** The spectrum of the energy–mass operator is bounded from below.

**Vacuum** The lowest eigenvalue of the energy–mass operator corresponds to a unique translationally invariant eigenstate $\Omega_0$.

**Locality** If $z \in \mathbb{C}\mathcal{C}^\infty$ and $w \in \mathbb{C}\mathcal{C}^-\infty$ have disjoint supports in $\mathbb{R}^3$, then $\hat{z}$ and $\hat{w}^*\dagger$ commute.

7 Appendix. Quantum dynamics of Yang–Mills fields

Interactions that are physically important in quantum field theory are so violent that they will kick any Schrödinger state vector out of Hilbert space in the shortest possible time interval. (DIRAC[12, p.5])

Quantum dynamics of Yang–Mills Cauchy data $z_t \in \mathcal{C}_0^\infty$ at variable time $t$ is governed by antinormal Schrödinger–Schwinger equation for quantum Yang–Mills transition amplitudes $\langle z_t|z_0 \rangle$, $z_0, z_t \in \mathcal{C}_0^\infty$ (see SCHWINGER[26])

$$\partial_t \langle z_t|z_0 \rangle = e^{z_t^*}(-iH(z_t^*z_t^\dagger))e^{z_0}. \quad (71)$$

(We use the one–one correspondence between the Cauchy data $z$ on $\mathbb{R}^3$ and the exponential functionals $e^z(\zeta^*) \equiv e^{z^*\zeta}$ on $\mathbb{C}\mathcal{C}^-\infty$.)
Let \( \{ p_n \} \) be a flag of finite dimensional orthogonal projectors in \( \mathcal{C}_n \) (i.e., an increasing sequence of projectors which are orthogonal in \( \mathcal{C}_0 \) and strongly converging to the unit operator in \( \mathcal{C}_0 \).

They naturally define the flag of finite dimensional projectors in the Gelfand triple \( \mathcal{C} \equiv \mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}_0^* \) and its complexification \( \overline{\mathcal{C}} \).

The quantization produces the corresponding flag of infinite dimensional orthogonal projectors \( \hat{\rho}_n \) in the Gelfand triple \( \mathcal{B} \equiv \mathcal{B}_0 \subset \mathcal{B}_0^* \subset \mathcal{B}_0^\infty \).

The contractions \( \tilde{H}_n \equiv \tilde{p}_n \mathcal{H}(z, \bar{z}^\dagger) \tilde{p}_n \) are operators in the Bargmann–Hida triples \( \mathcal{B}_n \equiv \tilde{p}_n \mathcal{B} \) over the finite-dimensional Hermitian spaces \( \mathcal{C}_n \equiv p_n \mathcal{C} \).

Moreover \( H(p_n z, p_n z^\dagger) \) is the antinormal symbol of \( \tilde{H}_n \). (By (49), this is straightforward for normal symbols, and then, by (55) for antinormal as well.)

Note, we have identified the finite-dimensional spaces \( \mathcal{C}_n \) with the Gelfand triples \( \mathcal{C}_n \subset \mathcal{C}_n \subset \mathcal{C}_n \). There Minlos Gauss measure is the standard Gauss measure on \( \mathcal{C}_n \) with the dense domains \( \mathcal{B}_n \).

Thus \( \tilde{H}_n \) are positive definite symmetric operators on the Hilbert spaces \( \mathcal{B}_n^0 \) with the dense domains \( \mathcal{B}_n^\infty \). They have Friedrichs selfadjoint extensions which are denoted again as \( \tilde{H}_n \).

Now the transition amplitudes in \( \mathcal{C}_n^\infty \) are
\[
\langle p_n z_1 | \tilde{p}_n \hat{z}_0 \rangle = e^{p_n z_1^*} e^{-i \tilde{H}_n} e^{p_n \hat{z}_0}, \tag{72}
\]
As in KLAUDER-SCAGERSTAM [11] (pp. 69-70), consider the strongly differentiable family of operators \( \hat{A}_{n, \tau}, 0 \leq \tau \leq t \), in \( \mathcal{B} \)
\[
[\hat{A}_{n, \tau}] \Psi(z_0^*) = \int dz^* d\bar{z} e^{-\bar{z}^* \bar{z}} e^{p_n \hat{z}_0} e^{-i \tilde{H}_n(z, \bar{z}) \tau} \Psi(z_0^*) \tag{73}
\]
Since \( |e^{-i \tilde{H}_n(z, \bar{z}) \tau}| = 1 \), the operator norms \( ||\hat{A}_{n, \tau}|| \leq 1 \) in \( \tilde{H}_n \).

Besides, the strong \( t \)-derivative \( (d/dt) \hat{A}_{n, \tau}(0) = \tilde{H}_n \) on the exponential states. Then, by the Chernoff’s product theorem (see CHERNOFF[8]), the evolution operator
\[
e^{-i \tilde{H}_n} = \lim_{N \to \infty} [\hat{A}_{n, \tau}]^N. \tag{74}
\]
Its kernel is the kernel contraction of the kernels of the factors
\[
\int I^N dz^* d\bar{z} \exp \sum_{j=0}^N \left[ (z_{j+1} - z_j)^* z_j - i \tilde{H}_n(z_j^*, z_j) / N \right], \tag{75}
\]
where \( z_{N+1} = z_t, z_0 = z_0 \).

Thus the amplitude \( e^{p_n z_0^*} e^{-i \tilde{H}_n} e^{p_n \hat{z}_0} \) is the \( N \)-iterated Gaussian integral over \( \mathcal{C} \) which, by the Fubini’s theorem, is equal to the \( N \)-multiple Gaussian integral over \( \mathcal{C}_N \).
In the notation \( \tau_j = j t/N \), \( z_{\tau_j} = z_j \), \( j = 0, 1, 2, \ldots, N \), and \( \Delta \tau_j = \tau_{j+1} - \tau_j \), the multiple integral is
\[
\int \prod_{j=1}^{N} d\tau_j^* d\tau_j \exp i \sum_{j=0}^{N} \Delta \tau_j \left[ -i (\Delta z_{\tau_j}/\Delta \tau_j)^* z_{\tau_j} - H_n(z_{\tau_j}^*, z_{\tau_j}) \right].
\]
(76)

Its limit at \( N = \infty \) is a rigorous mathematical definition of the heuristic antinormal Feynman integral
\[
\int_{z_0}^{z_t} \prod_{0 < \tau < t} d\tau_j^* d\tau_j \exp \int_{0}^{t} d\tau \left[ (\partial \tau_j^* z_{\tau_j^*} - iH_n(z_{\tau_j^*}^*, z_{\tau_j^*}) \right]
\]
(77)

over classical histories \( z_{\tau} \) between \( z_0 \) and \( z_t \) in \( C_n \).

Since the quantum Yang–Mills amplitude
\[
\langle z_t^* | z_0 \rangle = \lim_{n \to \infty} \langle p_n z_t^* | p_n z_0 \rangle,
\]
(78)

it is equal to the iterated limit of (75) as \( N \to \infty \) is followed by \( n \to \infty \). That iterated limit is a rigorous mathematical definition of the heuristic antinormal Feynman integral for the amplitude \( \langle z_t^* | z_0 \rangle \)
\[
\int_{z_0}^{z_t} \prod_{0 < \tau < t} d\tau_j^* d\tau_j \exp \int_{0}^{t} d\tau \left[ (\partial \tau_j^* z_{\tau_j^*} - iH_n(z_{\tau_j^*}^*, z_{\tau_j^*}) \right]
\]
(79)

over all classical histories \( z_{\tau} \) between \( z_0 \) and \( z_t \) in \( C_n \).

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