Existence in critical spaces for the magnetohydrodynamical system in 3D bounded Lipschitz domains

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Received: 6 March 2021 / Accepted: 19 August 2021 / Published online: 29 August 2021
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Abstract
Existence of mild solutions for the 3D MHD system in bounded Lipschitz domains is established in critical spaces with the absolute boundary conditions.

Keywords Magnetohydrodynamical system · Well-posedness · Lipschitz domains · Critical spaces

Mathematical Subject Classification 35A01 · 35Q35

1 Introduction

The magnetohydrodynamical system in a domain $\Omega \subset \mathbb{R}^3$ on a time interval $(0, T)$ ($0 < T \leq \infty$) as considered in Ref. [1] (with all constants equal to 1) reads

$$\begin{aligned}
\partial_t u - \Delta u + \nabla \pi + (u \cdot \nabla)u &= (\text{curl}\ b) \times b \quad \text{in} \ (0, T) \times \Omega \\
\partial_t b - \Delta b &= \text{curl}\ (u \times b) \quad \text{in} \ (0, T) \times \Omega \\
\text{div}\ u &= 0 \quad \text{in} \ (0, T) \times \Omega \\
\text{div}\ b &= 0 \quad \text{in} \ (0, T) \times \Omega,
\end{aligned}$$

(MHD)

where $u : (0, T) \times \Omega \to \mathbb{R}^3$ denotes the velocity of the (incompressible homogeneous) fluid, the magnetic field (in the absence of magnetic monopole) is denoted by $b : (0, T) \times \Omega \to \mathbb{R}^3$ and $\pi : (0, T) \times \Omega \to \mathbb{R}$ is the pressure of the fluid. The first equation of (MHD) corresponds to Navier–Stokes equations subject to the Laplace force $(\text{curl}\ b) \times b$ applied by the magnetic field $b$. Actually, the divergence-free condition on the magnetic field $b$ comes from the fact that $b$ is in the range of the curl.
operator. The second equation of (MHD) describes the evolution of the magnetic field following the so-called induction equation.

This system (MHD) (with $T = \infty$ and $\Omega = \mathbb{R}^3$) is invariant under the scaling $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$, $b_\lambda(t, x) = \lambda b(\lambda^2 t, \lambda x)$ and $\pi_\lambda(t, x) = \lambda^2 \pi(\lambda^2 t, \lambda x)$, $\lambda > 0$. This suggests that a critical space for $(u, b)$ is $\mathcal{C}([0, \infty); L^3(\mathbb{R}^3)^3) \times \mathcal{C}([0, \infty); L^3(\mathbb{R}^3)^3)$.

The purpose of this paper is to prove existence of solutions of this system in this critical space in a bounded Lipschitz domain under the so-called absolute boundary conditions, denoted by (BC1) below. This is investigated in Theorems 3.3 and 3.4 in Sect. 3. The methods used here come from the theory developed in Ref. [2] for the absolute boundary conditions.

In Sect. 2 are collected results on potential operators (similar to the famous Bogovskiñ operator), the Stokes operators with Dirichlet boundary conditions and Hodge boundary conditions, as well as properties of the Hodge Laplacian in bounded Lipschitz domains. Section 3 is devoted to the existence of mild solutions of the system (MHD) under absolute boundary conditions on a bounded Lipschitz domain in critical spaces.

2 Tools

In this section are recalled some results proved in Ref. [2] which will be useful in the following. See also Refs. [3, 4].

Notation 2.1 For an (unbounded) operator $A$ on a Banach space $X$, we denote by $\mathcal{D}(A)$ its domain, $\mathcal{R}(A)$ its range and $N(A)$ its null space.

2.1 Differential forms, potential operators

We consider the exterior derivative $d := \nabla \wedge = \sum_{j=1}^{n} \partial_j e_j \wedge$ and the interior derivative (or co-derivative) $\delta := -\nabla \cdot = -\sum_{j=1}^{n} \partial_j e_{j\cdot} \cdot$ acting on differential forms on a domain $\Omega \subset \mathbb{R}^n$, i.e. acting on functions from $\Omega$ to the exterior algebra $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^n$ of $\mathbb{R}^n$.

We denote by $\{e_S; S \subset \{1, \ldots, n\}\}$ the basis for $\Lambda$. The space of $\ell$-vectors $\Lambda^\ell$ is the span of $\{e_S; |S| = \ell\}$, where

$$e_S = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_\ell} \quad \text{for} \quad S = \{e_{j_1}, \ldots, e_{j_\ell}\} \quad \text{with} \quad j_1 < j_2 < \cdots < j_\ell.$$ 

Remark that $\Lambda^0$, the space of complex scalars, is the span of $e_\emptyset$ ($\emptyset$ being the empty set). We set $\Lambda^{\ell} = \{0\}$ if $\ell < 0$ or $\ell > n$.

On the exterior algebra $\Lambda$, the basic operations are

(\begin{itemize}
\item[(i)] the exterior product $\wedge : \Lambda^k \times \Lambda^\ell \rightarrow \Lambda^{k+\ell}$,
\item[(ii)] the interior product $\cdot : \Lambda^k \times \Lambda^\ell \rightarrow \Lambda^{\ell-k}$,
\item[(iii)] the Hodge star operator $\star : \Lambda^\ell \rightarrow \Lambda^{n-\ell}$,
\item[(iv)] the inner product $\langle \cdot, \cdot \rangle : \Lambda^{\ell} \times \Lambda^{\ell} \rightarrow \mathbb{R}$.
\end{itemize}
If \( a \in \Lambda^1, u \in \Lambda^\ell \) and \( v \in \Lambda^{\ell+1} \), then
\[
\langle a \wedge u, v \rangle = \langle u, a \lrcorner v \rangle.
\]

For more details, we refer to, e.g., [5, Sect. 2, 6, Sect. 2], noting that both these papers contain some historical background (and being careful that \( \delta \) has the opposite sign in Ref. [5]). In particular, we note the relation between \( d \) and \( \delta \) via the Hodge star operator:
\[
\star \delta u = (-1)^\ell d(\star u) \quad \text{and} \quad \star du = (-1)^{\ell-1} \delta(\star u) \quad \text{for an } \ell\text{-form } u. \tag{1}
\]

In dimension \( n = 3 \), this gives (see [6, Sect. 2]) for a vector \( a \in \mathbb{R}^3 \) identified with a 1-form
\[
\begin{align*}
&- \ u \text{ scalar, interpreted as } 0\text{-form: } a \wedge u = ua, a \lrcorner u = 0; \\
&- \ u \text{ scalar, interpreted as } 3\text{-form: } a \wedge u = 0, a \lrcorner u = ua; \\
&- \ u \text{ vector, interpreted as } 1\text{-form: } a \wedge u = a \times u, a \lrcorner u = a \cdot u; \\
&- \ u \text{ vector, interpreted as } 2\text{-form: } a \wedge u = a \cdot u, a \lrcorner u = -a \times u.
\end{align*}
\]

The domains of the differential operators \( d \) and \( \delta \), denoted by \( D(d) \) and \( D(\delta) \) are defined by
\[
D(d) := \{ u \in L^2(\Omega, \Lambda); du \in L^2(\Omega, \Lambda) \} \quad \text{and} \quad D(\delta) := \{ u \in L^2(\Omega, \Lambda); \delta u \in L^2(\Omega, \Lambda) \}.
\]

Similarly, the \( L^p \) versions of these domains read
\[
D^p(d) := \{ u \in L^p(\Omega, \Lambda); du \in L^p(\Omega, \Lambda) \} \quad \text{and} \quad D^p(\delta) := \{ u \in L^p(\Omega, \Lambda); \delta u \in L^p(\Omega, \Lambda) \}.
\]

The differential operators \( d \) and \( \delta \) satisfy \( d^2 = d \cdot d = 0 \) and \( \delta^2 = \delta \cdot \delta = 0 \). We will also consider the adjoints of \( d \) and \( \delta \) in the sense of maximal adjoint operators in a Hilbert space: \( \delta^* := d^* \) and \( d^* := \delta^* \). They are defined as the closures in \( L^2(\Omega, \Lambda) \) of the closable operators \((d^*, \mathcal{C}_c^\infty(\Omega, \Lambda))\) and \((\delta^*, \mathcal{C}_c^\infty(\Omega, \Lambda))\).

The following proposition has been proved in Ref. [2, Proposition 4.1] in a slightly more general framework than needed here (see also [6, Theorems 1.1, 4.6, 7, Theorem 1.5, Remark 4.12]). In these references, the authors based their construction on an integral formula for the operators \( R_\Omega, S_\Omega \) and \( K_\Omega \) below in domains star-shaped with respect to a ball and a partition of unity.

**Proposition 2.2** Suppose \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^n \). Then the following (Bogovskiï type) potential operators \( R_\Omega, S_\Omega \) and \( K_\Omega \) satisfy for all \( p \in (1, \infty) \), with the convention \( p^* = \frac{np}{n-p} \) if \( p < n \), \( p^* = +\infty \) if \( p > n \) and \( p^* \in [n, +\infty) \) if \( p = n \),

\[ \text{Existence in critical spaces for the magnetohydrodynamical...} \]
As direct consequence we obtain that \(dR_\Omega\) and \(d^*S_\Omega\) are projections from \(L^p(\Omega, \Lambda)\) onto the ranges of \(d\) and \(d^*, R^p(d)\) and \(R^p(d^*)\), for all \(p \in (1, \infty)\).

2.2 Hodge–Laplacian and Hodge–Stokes operators in Lipschitz domains

Definition 2.3 The Hodge–Dirac operator on \(\Omega\) with tangential boundary conditions is

\[ D_\parallel := d + d^*. \]

Note that \(-\Delta_\parallel := D_\parallel^2 = dd^* + d^*d\) is the Hodge–Laplacian with absolute (generalised Neumann) boundary conditions.

For a scalar function \(u : \Omega \to \Lambda^0\) we have that \(-\Delta_\parallel u = d^*du = -\Delta_N u\), where \(\Delta_N\) is the Neumann Laplacian.

Following [8, Sect. 4], we have that the operator \(D_\parallel\) is a closed densely defined operator in \(L^2(\Omega, \Lambda)\), and that

\[ L^2(\Omega, \Lambda) \cong \mathbb{R}(d) \oplus \mathbb{R}(d^*) \oplus \mathbb{N}(D_\parallel), \quad (H_2) \]

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where \(\mathbb{N}(D_\parallel) = \mathbb{N}(d) \cap \mathbb{N}(d^*) = \mathbb{N}(\Delta_\parallel)\), is finite dimensional. The orthogonal projection from \(L^2(\Omega, \Lambda)\) onto the null space of \(d^*, \mathbb{N}(d^*)\) (see (2)), restricted to 1-forms, is the well-known Helmholtz (or Leray) projection denoted by \(\mathbb{P}\). Restricted to 2-forms, the orthogonal projection from \(L^2(\Omega, \Lambda)\) onto the range of \(d, \mathbb{R}(d)\), will be denoted in the sequel by \(\mathbb{Q}\).

The \(p\) version of the previous Hodge decompositions can be found in Ref. [2, Theorem 4.3]: there exist Hodge exponents \(p_H, p^H = p_H'\) with \(1 \leq p_H < 2 < p^H \leq \infty\) such that

\[ L^p(\Omega, \Lambda) = \mathbb{R}^p(d) \oplus \mathbb{R}^p(d^*) \oplus \mathbb{N}(D_\parallel), \quad (H_p) \]
\[ = \mathbb{R}^p(d) \oplus \mathbb{N}^p(d^*) \quad , \]
\[ = \mathbb{N}^p(d) \oplus \mathbb{R}^p(d^*) \quad , \]
for all \( p \in (p_H, p^H) \) and the projections \( \mathbb{P} : L^p(\Omega, \Lambda^1) \to \mathbb{N}^p(d^*)_\Lambda^1 \), and \( \mathbb{Q} : L^p(\Omega, \Lambda^2) \to \mathbb{R}^p(d)_\Lambda^2 \) extend accordingly. Here, \( \mathbb{R}^p(A) \) (resp. \( \mathbb{N}^p(A) \)) denotes the range (resp. the null space) in \( L^p \) of the operator \( A \).

**Remark 2.4** If the domain is smooth or has a Lipschitz boundary, the following estimates on the Hodge exponents \( p_H \) and \( p^H \) hold.

1. If \( \Omega \subset \mathbb{R}^n \) is smooth, then \( p_H = 1 \) and \( p^H = \infty \) (see [9, Theorems 2.4.2, 2.4.14]).
2. In the case of a bounded Lipschitz domain, \( p_H < \frac{2n}{n+1} \) and consequently \( p^H > \frac{2n}{n-1} \), which gives in dimension \( n = 3 \): \( p_H < \frac{3}{2} \) and \( p^H > 3 \) (see [2, Sect. 7]).

**Remark 2.5** Proposition 2.2 and the projections \( \mathbb{P} \) and \( \mathbb{Q} \) yield

\[
\mathbb{P}(R_\Omega du + K_\Omega u) = u \quad \text{for } u \in \mathbb{N}^p(d^*)_\Lambda^1 ,
\]
\[ \mathbb{Q}(S_\Omega d^*b + K^*_\Omega b) = b \quad \text{for } b \in \mathbb{R}^p(d)_\Lambda^2 \quad ,
\]
for all \( p_H < p < p^H \). The second equation comes from the fact that \( \mathbb{R}^p(d) \subset \mathbb{N}^p(d) \), using (5).

The following results can be found partly in Ref. [3, Theorem 7.3] (sectoriality) and in Ref. [2, Sect. 8] (improvement of the interval of \( p \) for the Hodge–Stokes operator and bounded holomorphic functional calculus):

**Theorem 2.6** Suppose \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^n \). Define \( -\Delta = D^2_\Omega \) in \( L^2(\Omega, \Lambda) \). If \( p_H < p < p^H \), then the Hodge–Laplacian with absolute boundary conditions \( -\Delta \) is sectorial of angle 0 in \( L^p(\Omega, \Lambda) \) and for all \( \mu \in (0, \frac{\pi}{2}) \), \( -\Delta \) admits a bounded \( S^\mu_{p^+} \) holomorphic functional calculus in \( L^p(\Omega, \Lambda) \).

Define the Hodge–Stokes operator by \( S_\Omega := D^2_\Omega = d^*d \) in \( \mathbb{N}^2(d^*) \), restricted to 1-forms. If \( \max\{1, \frac{n p_H}{n+p_H} \} < p < p^H \), then \( S_\Omega \) is sectorial of angle 0 in \( \mathbb{N}^p(d^*)_\Lambda^1 \) and for all \( \mu \in (0, \frac{\pi}{2}) \), \( S_\Omega \) admits a bounded \( S^\mu_{p^+} \) holomorphic functional calculus in \( \mathbb{N}^p(d^*)_\Lambda^1 \). In particular, the semigroup \( (e^{-tS_\Omega})_{t \geq 0} \) is bounded on \( \mathbb{N}^p(d^*)_\Lambda^1 \) with norm denoted by \( K_{p,S} \).

Define the Hodge–Maxwell operator \( M_\Omega := dd^* \) in \( \mathbb{N}(d) \), restricted to 2-forms. If \( \max\{1, \frac{n p_H}{n+p_H} \} < p < p^H \), then \( M_\Omega \) is sectorial of angle 0 in \( \mathbb{N}^p(d)_\Lambda^2 \) and for all \( \mu \in (0, \frac{\pi}{2}) \), \( M_\Omega \) admits a bounded \( S^\mu_{p^+} \) holomorphic functional calculus in \( \mathbb{N}^p(d)_\Lambda^2 \).
particular, the semigroup \( (e^{-tM_\|})_{t \geq 0} \) is bounded on \( \mathbb{R}^p(d)_{1,1} \) with norm denoted by \( K_{p, M} \).

Using the results stated in Remark 2.5, one can prove \( L^r - L^u \) bounds for the operator \( S_\| \) (resp. \( M_\| \)) (see [4, Theorems 3.1, 4.1] for the dimension 3 and [10, Theorem 1.1] for the Riesz transform like estimates (7) and (9)).

**Theorem 2.7** Let \( p \in \left( \max\{1, \frac{n p_H}{n + p_H} \}, p^H \right) \) and \( q \in [p, p^H) \) such that \( \frac{1}{p} - \frac{\beta}{n} = \frac{1 - \gamma}{q} \) for some \( \beta \in [0, 1] \). Then the semigroup \( (e^{-tS_\|})_{t \geq 0} \) in \( \mathbb{N}^p(d^*)_{1,1} \) satisfies the estimates

\[
c_{p,q}^S := \sup_{t \geq 0} \| t^\frac{\beta}{2} e^{-tS_\|} \|_{\mathcal{N}^p(d^*)_{1,1} \to L^q} \left[ \sup_{t \geq 0} \| t^\frac{\gamma}{2} d^* e^{-tS_\|} \|_{\mathcal{N}^p(d^*)_{1,1} \to L^q} \right] < \infty,
\]

and

\[
\gamma_{p,q}^S := \| S_\|^{-\frac{\gamma}{2}} \|_{\mathcal{N}^p(d^*)_{1,1} \to L^q} < \infty.
\]

The semigroup \( (e^{-tM_\|})_{t \geq 0} \) in \( \mathbb{R}^p(d)_{1,2} \) satisfies the estimate

\[
c_{p,q}^M := \sup_{t \geq 0} \| t^\frac{\beta}{2} e^{-tM_\|} \|_{\mathcal{R}^p(d)_{1,2} \to L^q} \left[ \sup_{t \geq 0} \| t^\frac{\gamma}{2} d^* e^{-tM_\|} \|_{\mathcal{R}^p(d)_{1,2} \to L^q} \right] < \infty,
\]

and

\[
\gamma_{p,q}^M := \| M_\|^{-\frac{\gamma}{2}} \|_{\mathcal{R}^p(d)_{1,2} \to L^q} < \infty.
\]

**3 Existence in the case of absolute boundary conditions**

Thanks to the formula

\[
(u \cdot \nabla) u = \frac{1}{2} \nabla |u|^2 + u \times (\text{curl } u),
\]

for a sufficiently smooth vector field \( u \), the system (MHD) can be reformulated as follows:

\[
\begin{aligned}
\partial_t u - \Delta u + \nabla \pi_1 - u \times (\text{curl } u) &= (\text{curl } b) \times b \quad \text{in } (0, T) \times \Omega \\
\partial_t b - \Delta b &= \text{curl} (u \times b) \quad \text{in } (0, T) \times \Omega \\
\text{div } u &= 0 \quad \text{in } (0, T) \times \Omega \\
\text{div } b &= 0 \quad \text{in } (0, T) \times \Omega
\end{aligned}
\]

where the pressure \( \pi \) has been replaced by the so-called *dynamical pressure* \( \pi_1 = \pi + \frac{1}{2} |u|^2 \). This formulation can be translated in the language of differential forms: \( \pi_1 \) is a scalar function, interpreted as 0-form, \( u \) is a vector field interpreted as 1-form and \( b \) is a vector field interpreted as 2-form. Following Sect. 2 one can rewrite (10) in terms of differential forms:
The terms in the first equation are all 1-forms, in the second equation the terms are all 2-forms. The absolute boundary conditions associated with the previous system (MHD1) are defined by the term $d^* u$ in $- \Delta = (dd^* + d^*d)$:

$$
\begin{align*}
\begin{cases}
\nu \cdot u &= \nu \cdot u = 0 & \text{on } (0, T) \times \partial \Omega \\
- \nu \times \text{curl} \ u &= \nu \times du = 0 & \text{on } (0, T) \times \partial \Omega \\
- \nu \times b &= \nu \times db = 0 & \text{on } (0, T) \times \partial \Omega \\
v \text{div} b &= \nu \times db = 0 & \text{on } \text{amp}^4; (0, T) \times \partial \Omega
\end{cases}
\end{align*}
$$

(BC1)

This formulation can be used, for instance, to study the magnetohydrodynamical system in dimensions greater than or equal to 2 with the same theoretical tools. Let us point out that these boundary conditions are different to those usually investigated in magnetohydrodynamical problems, starting with the paper [1]; see also [11]. The boundary conditions (BC1) in the case of Navier–Stokes equations (i.e. for $b = 0$) have been studied in Ref. [4]; see also [12, 13].

**Remark 3.1** The last condition in (BC1) is void since $b \in R(d)_{\bar{\Lambda}_2}$; $db = 0$ in all $\Omega$.

**Definition 3.2** Let $\Omega \subseteq \mathbb{R}^3$. A mild solution of the system (MHD1) with absolute boundary conditions (BC1) and initial conditions $u_0 \in N(d^*)_{\bar{\Lambda}_1}$ and $b_0 \in R(d)_{\bar{\Lambda}_2}$ is a pair $(u, b)$ of vector fields satisfying

$$
\begin{align*}
\begin{cases}
\frac{d}{ds} u(s) u_0 + \int_0^s e^{-(t-s)S_1} \mathcal{P}(-u(s) \cdot du(s)) \, ds + \int_0^s e^{-(t-s)S_1} \mathcal{P}(d^* b(s) \cdot b(s)) \, ds, \\
\frac{d}{ds} b(s) b_0 + \int_0^s e^{-(t-s)M_1} (-d(u(s) \cdot db(s))) \, ds.
\end{cases}
\end{align*}
$$

(11) (12)

From now on, we assume the following technical (Leibniz rule-like) property on the domain $\Omega \subseteq \mathbb{R}^3$: for all $q \in [3, p^H]$, there exists a constant $C_q > 0$ such that

$$
\|d(\omega_1 \wedge \omega_2)\|_q \leq C_q \left( \left\|D\|_{\omega_1}\|_{L_q} \|\omega_2\|_{L_q} + \left\|\omega_1\|_{L_q} \|D\|_{\omega_2}\|_{L_q} \right\| \right),
$$

(13)

for all $\omega_1 \in D^q(D^q) \cap L^q(\Omega, \Lambda^1)$ and all $\omega_2 \in D^q(D^q) \cap L^q(\Omega, \Lambda^2)$. This is the case if the domain $\Omega$ is smooth since in that case $\|D\|_{\omega} \|_{L_q}$ dominates $\sum_j \|\partial_j \omega\|_{L_q}$. To the author knowledge, the validity of condition (13) is not known for general Lipschitz (bounded) domains.

The following theorem is about the global existence of mild solutions with small initial data.
Theorem 3.3 (Global existence) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain satisfying (13) or $\Omega = \mathbb{R}^3$. Then there exists $\varepsilon > 0$ such that for all $u_0 \in \mathfrak{N}^3(d^\ast)_{|_{\Lambda^1}}$ and $b_0 \in \mathbb{R}^3(d)_{|_{\Lambda^2}}$ with $\|u_0\|_3 + \|b_0\|_3 \leq \varepsilon$, the system (MHD1) with the boundary conditions (BC1) and $T = \infty$ admits a mild solution $u, b \in \mathcal{C}([0, \infty); \mathcal{L}^3(\Omega)^3)$.

The next result states local existence of mild solutions with no restriction on the size of the initial data.

Theorem 3.4 (Local existence) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain satisfying (13) or $\Omega = \mathbb{R}^3$. Then for all $u_0 \in \mathfrak{N}^3(d^\ast)_{|_{\Lambda^1}}$ and $b_0 \in \mathbb{R}^3(d)_{|_{\Lambda^2}}$ there exists $T > 0$ such that the system (MHD1) with the boundary conditions (BC1) admits a mild solution $u, b \in \mathcal{C}([0, T); \mathcal{L}^3(\Omega)^3)$.

The methods to prove these two theorems are classical based on a fixed point theorem, already used for the Navier–Stokes equations in the paper by Fujita and Kato [14] (see also Refs. [15–17]) for the Boussinesq system. Most of the tools used here appeared in the paper [4]; see also [2].

Let $q \in (3, \min\{p^H, 6\})$ and $\alpha \in (0, 1)$ such that $\frac{1}{q} = \frac{1}{3} - \frac{\alpha}{3}$. For $0 < T \leq \infty$, we define the following spaces

\[
\mathcal{U}_T := \left\{ u \in \mathcal{C}((0, T); \mathfrak{N}^3(d^\ast)_{|_{\Lambda^1}}); du \in \mathcal{C}((0, T); \mathcal{L}^q(\Omega, \Lambda^2)) : \sup_{0 < t < T} \left( t^\frac{1}{2} \| u(t) \|_q + t^\frac{1}{2} \| du(t) \|_q \right) < \infty \right\},
\]

and

\[
\mathcal{B}_T := \left\{ b \in \mathcal{C}((0, T); \mathfrak{B}^3(d)_{|_{\Lambda^2}}); db \in \mathcal{C}((0, T); \mathcal{L}^q(\Omega, \Lambda^1)) : \sup_{0 < t < T} \left( t^\frac{\alpha}{q} \| b(t) \|_q + t^\frac{1}{2} \| db(t) \|_q \right) < \infty \right\},
\]

endowed with the norms

\[
\| u \|_{\mathcal{U}_T} := \sup_{0 < t < T} \left( t^\frac{1}{2} \| u(t) \|_q + t^\frac{1}{2} \| du(t) \|_q \right),
\]

and

\[
\| b \|_{\mathcal{B}_T} := \sup_{0 < t < T} \left( t^\frac{\alpha}{q} \| b(t) \|_q + t^\frac{1}{2} \| db(t) \|_q \right).
\]

Lemma 3.5 For $u_0 \in \mathfrak{N}^3(d^\ast)_{|_{\Lambda^1}}$ and $b_0 \in \mathfrak{B}^3(d)_{|_{\Lambda^2}}$, we have

1. $a_1 : t \mapsto e^{-iS_t}u_0 \in \mathcal{U}_T$,
2. $a_2 : t \mapsto e^{-iM_t}b_0 \in \mathcal{B}_T$. 
for all $T > 0$. Moreover, for all $\varepsilon > 0$, there exists $T > 0$ such that

$$\|a_1\|_{\mathcal{V}_T} + \|a_2\|_{\mathcal{B}_T} \leq \varepsilon.$$  \hfill (18)

**Proof** By Theorem 7, the following bound holds for all $T > 0$:

$$\|a_1\|_{\mathcal{V}_T} + \|a_2\|_{\mathcal{B}_T} \leq c_{s,T}^{\mathcal{S}} \|u_0\|_3 + c_{3,q}^{\mathcal{M}} \|b_0\|_3.$$  \hfill (19)

Therefore, if $\|u_0\|_3$ and $\|b_0\|_3$ are small enough, (18) holds for every $T > 0$.

For any $u_0$ and $b_0$ (not necessarily small in the $L^3$ norm), for $\varepsilon > 0$, let $u_{0,\varepsilon} \in \mathbb{N}^q(d^n)_{\Lambda_1}$ and $b_{0,\varepsilon} \in \mathbb{R}^q(d)_{\lambda_2}$ such that

$$\|u_0 - u_{0,\varepsilon}\|_3 + \|b_0 - b_{0,\varepsilon}\|_3 \leq \varepsilon.$$

We denote by $a_{1,\varepsilon}$ and $a_{2,\varepsilon}$ the quantities $a_{1,\varepsilon}(t) = e^{-tS_1}u_{0,\varepsilon}$ and $a_{2,\varepsilon}(t) = e^{-tM_1}b_{0,\varepsilon}$. By (19), there holds

$$\|a_1 - a_{1,\varepsilon}\|_{\mathcal{V}_T} + \|a_2 - a_{2,\varepsilon}\|_{\mathcal{B}_T} \leq \varepsilon \left( c_{3,q}^{\mathcal{S}} + c_{3,q}^{\mathcal{M}} \right).$$  \hfill (20)

Applying (6) with $p = q$ and $\beta = 0$, we obtain

$$\|a_{1,\varepsilon}\|_{\mathcal{V}_T} \leq c_{q,q}^{\mathcal{S}} T^{\frac{q}{2}} \|u_{0,\varepsilon}\|_q.$$  \hfill (21)

The same reasoning applying (8) with $p = q$ and $\beta = 0$ yields

$$\|a_{2,\varepsilon}\|_{\mathcal{B}_T} \leq c_{q,q}^{\mathcal{M}} T^{\frac{q}{2}} \|b_{0,\varepsilon}\|_q.$$  \hfill (22)

Now choosing $\varepsilon > 0$ small enough and $T > 0$ small enough, we find that (18) holds. \hfill $\square$

Next, we define the operators

$$B_1(u,v)(t) = \int_0^t e^{-(t-s)S_1} \mathbb{P}(-u(s) \cdot dv) \, ds, \quad t \in [0,T), u, v \in \mathcal{V}_T,$$  \hfill (21)

$$B_2(b,b')(t) = \int_0^t e^{-(t-s)S_1} \mathbb{P}(-d^n b(s) \cdot b'(s)) \, ds, \quad t \in [0,T), b, b' \in \mathcal{B}_T,$$  \hfill (22)

$$B_3(u,b)(t) = \int_0^t e^{-(t-s)M_1} (-d(u(s) \cdot b(s))) \, ds, \quad t \in [0,T), u \in \mathcal{V}_T, b \in \mathcal{B}_T.$$  \hfill (23)

The next lemma gives a precise statement about the boundedness of the bilinear operators $B_1, B_2$ and $B_3$.

**Lemma 3.6** The bilinear operators $B_1, B_2$ and $B_3$ are bounded in the following spaces:
1. \( B_1 : \mathcal{U}_T \times \mathcal{U}_T \to \mathcal{U}_T \),
2. \( B_2 : \mathcal{B}_T \times \mathcal{B}_T \to \mathcal{U}_T \),
3. \( B_3 : \mathcal{U}_T \times \mathcal{B}_T \to \mathcal{B}_T \)

with norms independent from \( T > 0 \).

**Proof**

1. For \( u, v \in \mathcal{U}_T \), by definition of \( \mathcal{U}_T \) we have that \( s \mapsto s^{\frac{1}{2} + a} u(s) \) with \( \Omega(s) = \mathcal{U}_T \) with norm less than or equal to \( \|u\|_{\mathcal{U}_T} \|v\|_{\mathcal{U}_T} \). Since \( \frac{3}{2} < \frac{q}{2} < 3 \), \( \mathcal{P} \) is bounded from \( L^2(\Omega, \Lambda^1) \) to \( N_2(d^*)_{\Lambda^1} \). Moreover, \( e^{-(t-s)S_\|} \) maps \( N_2(d^*)_{\Lambda^1} \) to \( N^q(d^*)_{\Lambda^1} \) with norm \( c_s^{\frac{q}{2}} q \) thanks to (6) with \( p = \frac{q}{2} \). Therefore, we have for all \( t \in (0, T) \)

\[
\|B_1(u, v)(t)\|_q \leq \left( \int_0^t s^{-\frac{1}{2} - a} (t-s)^{-\frac{1}{2} - \frac{a}{2}} \, ds \right) \|u\|_{\mathcal{U}_T} \|v\|_{\mathcal{U}_T} \leq t^{-\frac{1}{2}} \|u\|_{\mathcal{U}_T} \|v\|_{\mathcal{U}_T}.
\]

This gives the first estimate for \( B_1(u, v) \in \mathcal{U}_T \). For the second estimate, we note that \( d^* e^{-(t-s)S_\|} \) maps \( N_2^2(d^*)_{\Lambda^1} \) to \( L^2(\Omega, \Lambda^1) \) with norm \( c_s^{\frac{q}{2}} q (t-s)^{-1+\frac{a}{2}} \) thanks to (6) with \( p = \frac{q}{2} \). Therefore, we have for all \( t \in (0, T) \)

\[
\|dB_1(u, v)(t)\|_q \leq \left( \int_0^t s^{-\frac{1}{2} - a} (t-s)^{-1+\frac{a}{2}} \, ds \right) \|u\|_{\mathcal{U}_T} \|v\|_{\mathcal{U}_T} \leq t^{-\frac{1}{2}} \|u\|_{\mathcal{U}_T} \|v\|_{\mathcal{U}_T},
\]

which gives the second estimate for \( B_1(u, v) \in \mathcal{U}_T \).

2. The proof that for \( b, b' \in \mathcal{B}_T, B_2(b, b') \in \mathcal{U}_T \) with norm independent from \( T > 0 \) follows the lines of the previous point. We omit the details here.

3. Thanks to the property (19), the proof that for \( u \in \mathcal{U}_T \) and \( b \in \mathcal{B}_T, B_3(u, b) \in \mathcal{B}_T \) with norm independent from \( T > 0 \) can be copied from the proof of point 1, using the fact that \( d^* e^{-(t-s)M_\|} \) maps \( R^2_\| (d)_{\Lambda^1} \) to \( L^2(\Omega, \Lambda^1) \) with norm \( c_s^{\frac{q}{2}} q (t-s)^{-1+\frac{a}{2}} \) thanks to (8) with \( p = \frac{q}{2} \).

This proves Lemma 3.6. \( \Box \)

**Lemma 3.7** Let \( T > 0 \). Assume that \((u, b) \in \mathcal{U}_T \times \mathcal{B}_T \) is a mild solution of (MHD1) with absolute boundary conditions (BC1) with initial conditions \( u_0 \in N^q(d^*)_{\Lambda^1} \) and \( b_0 \in R^3(d)_{\Lambda^2} \). Then \( u \in C_b([0, T];N^q(d^*)_{\Lambda^1}) \) and \( b \in C_b([0, T); R^3(d)_{\Lambda^2}) \).

**Proof** To prove this lemma, first observe that if \( u_0 \in N^q(d^*)_{\Lambda^1} \) and \( b_0 \in R^3(d)_{\Lambda^2} \), then for all \( T > 0, t \mapsto e^{-tM_\|} u_0 \in C_b([0, T];N^q(d^*)_{\Lambda^1}) \) and \( t \mapsto e^{-tM_\|} b_0 \in C_b([0, T); R^3(d)_{\Lambda^2}) \). It remains to show that if \( u \in \mathcal{U}_T \) and \( b \in \mathcal{B}_T \), then \( B_1(u, u) \in \mathcal{U}_T \); \( N^q(d^*)_{\Lambda^1} \), \( B_2(b, b) \in \mathcal{B}_T([0, T);N^q(d^*)_{\Lambda^1}) \) and \( B_3(u, b) \in \mathcal{B}_T([0, T); R^3(d)_{\Lambda^2}) \). The continuity is straightforward. To prove boundedness, it suffices to reproduce the proof of the previous lemma (recall that \( a = 1 - \frac{2}{q} \)) to obtain
where
\[ \|B_1(u, u)(t)\|_3 \lesssim \left( \int_0^t s^{-\frac{1}{2} - a} (t - s)^{-\frac{1}{2} + a} \, ds \right) \|u\|_{\mathcal{U}_T}^2 \lesssim \|u\|_{\mathcal{U}_T}^2, \]
using the fact that \( e^{-(t-s)S_\|} \) maps \( H^2(\mathbb{R}^3) \) to \( L^3(\Omega, \Lambda^1) \) with norm controlled by \( c_s^3 (t-s)^{-\frac{3}{4} + \frac{3}{2}} \) thanks to (6). The terms \( B_2 \) and \( B_3 \) can be treated similarly. \( \square \)

**Proof of Theorems 3.3 and 3.4** The system
\[ u = a_1 + B_1(u, u) + B_2(b, b) \quad \text{and} \quad b = a_2 + B_3(u, b), \quad (u, b) \in \mathcal{U}_T, \quad (24) \]
can be reformulated as
\[ u = a + B(u, u), \quad (25) \]
where \( u = (u, b) \in \mathcal{U}_T \times \mathcal{B}_T \), \( a = (a_1, a_2) \) and \( B(u, v) = (B_1(u, v) + B_2(b, b'), B_3(u, b')) \)
if \( u = (u, b) \) and \( v = (v, b') \). On \( \mathcal{U}_T \times \mathcal{B}_T \) we choose the norm \( \|(u, b)\|_{\mathcal{U}_T \times \mathcal{B}_T} := \|u\|_{\mathcal{U}_T} + \|b\|_{\mathcal{B}_T} \). One can easily check, using Lemma 3.6, that
\[ \|B(u, v)\|_{\mathcal{U}_T \times \mathcal{B}_T} \leq C \|u\|_{\mathcal{U}_T} \|v\|_{\mathcal{U}_T \times \mathcal{B}_T}, \]
where \( C \) is a constant independent from \( T > 0 \). We can then apply Picard’s fixed point theorem to prove that for \( u_0 \in \mathcal{N}^3(d^*)_{1, \Lambda} \) and \( b_0 \in \mathcal{N}^3(d)_{1, \Lambda^2} \), with \( T \leq \infty \) such that (18) holds for \( \epsilon = \frac{1}{4C} \), the system (25) admits a unique solution \( u = (u, b) \in \mathcal{U}_T \times \mathcal{B}_T \). By Lemma 3.7, this provides a mild solution \( (u, b) \in \mathcal{C}^1([0, T); \mathcal{N}^3(d^*)_{1, \Lambda} \times \mathcal{C}_B([0, T); \mathcal{R}^3(d)_{1, \Lambda^2}) \) of (MHD1) with boundary conditions (BC1). \( \square \)

**Funding** The funding was partially provided by ANR INFAMIE (ANR-15-CE40-001).

**Declarations**

**Conflict of interest** I hereby declare that the content of this manuscript is original and the result of my own work. This manuscript has not been submitted anywhere else and has not been published in any other journal.

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