1234-avoiding permutations and Dyck paths

Marilena Barnabei, Flavio Bonetti, and Matteo Silimbani

Dipartimento di Matematica, Università di Bologna
P.zza di Porta San Donato 5, 40126 Bologna, Italy

Abstract. We define a map $\nu$ between the symmetric group $S_n$ and the set of pairs of Dyck paths of semilength $n$. We show that the map $\nu$ is injective when restricted to the set of 1234-avoiding permutations and characterize the image of this map.

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1 Introduction

We say that a permutation $\sigma \in S_n$ contains a pattern $\tau \in S_k$ if $\sigma$ contains a subsequence that is order-isomorphic to $\tau$. Otherwise, we say that $\sigma$ avoids $\tau$. Given a pattern $\tau$, denote by $S_n(\tau)$ the set of permutations in $S_n$ avoiding $\tau$.

The sets of permutations that avoid a single pattern $\tau \in S_3$ have been completely determined in last decades. More precisely, it has been shown \cite{10} that, for every $\tau \in S_3$, the cardinality of the set $S_n(\tau)$ equals the $n$-th Catalan number, which is also the number of Dyck paths of semilength $n$ (see e.g. \cite{10}). Many bijections between $S_n(\tau), \tau \in S_3$, and the set of Dyck paths of semilength $n$ have been described (see \cite{4} for a fully detailed overview).

The case of patterns of length 4 appears much more complicated, due both to the fact that the patterns $\tau \in S_4$ are not equidistributed on $S_n$, and the difficulty of describing bijections between $S_n(\tau), \tau \in S_4$, and some set of combinatorial objects.
In this paper we study the case $\tau = 1234$. An explicit formula for the cardinality of $S_n(1234)$ has been computed by I. Gessel (see [2] and [5]), but there is no bijection (up to our knowledge) between $S_n(1234)$ and some set of combinatorial objects.

We present a bijection between $S_n(1234)$ and a set of pairs of Dyck paths of semilength $n$. More specifically, we define a map $\nu$ from $S_n$ to the set of pairs of Dyck paths, prove that every element in the image of $\nu$ corresponds to a single element in $S_n(1234)$, and characterize the set of all pairs that belong to the image of the map $\nu$.

2 Dyck paths

A Dyck path of semilength $n$ is a lattice path starting at $(0,0)$, ending at $(2n,0)$, and never going below the $x$-axis, consisting of up steps $U = (1,1)$ and down steps $D = (1, -1)$. A return of a Dyck path is a down step ending on the $x$-axis. A Dyck path is irreducible if it has only one return. An irreducible component of a Dyck path $P$ is a maximal irreducible Dyck subpath of $P$.

A Dyck path $P$ is specified by the lengths $a_1, \ldots, a_k$ of its ascents (namely, maximal sequences of consecutive up steps) and by the lengths $d_1, \ldots, d_k$ of its descents (maximal sequences of consecutive down steps), read from left to right. Set $A_i = \sum_{j=1}^{i} a_j$ and $D_i = \sum_{j=1}^{i} d_j$. If $n$ is the semilength of $P$, we have of course $A_k = D_k = n$, hence the Dyck path $P$ is uniquely determined by the two sequences $A = A_1, \ldots, A_{k-1}$ and $D = D_1, \ldots, D_{k-1}$. The pair $(A, D)$ is called the ascent-descent code of the Dyck path $P$.

Obviously, a pair $(A, D)$, where $A = A_1, \ldots, A_{k-1}$ and $D = D_1, \ldots, D_{k-1}$, is the ascent-descent code of some Dyck path of semilength $n$ if and only if

- $0 < k \leq n - 1$;
- $1 \leq A_1 < A_2 < \ldots < A_{k-1} \leq n - 1$;
- $1 \leq D_1 < D_2 < \ldots < D_{k-1} \leq n - 1$;
- $A_i \geq D_i$ for every $1 \leq i \leq k - 1$.

It is easy to check that the returns of a Dyck path are in one-to-one correspondence with the indices $1 \leq i \leq k$ such that $A_i = D_i$. Hence, a Dyck
path is irreducible whenever we have \( A_i > D_i \) for every \( 1 \leq i \leq k - 1 \).

For example, the ascent-descent code of the Dyck path \( P \) in Figure 1 is \((A, D)\), where \( A = 3, 6 \) and \( D = 2, 3 \). Note that \( A_1 > D_1 \) and \( A_2 > D_2 \). In fact, \( P \) is irreducible.

![Figure 1](image.png)

We describe an involution \( L \) due to Kreweras (a description of this bijection, originally defined in [7], can be found in [3] and discussed by Lalanne (see [8] and [9]) on the set of Dyck paths. Given a Dyck path \( P \), the path \( L(P) \) can be constructed as follows:

- if \( P \) is the empty path \( \epsilon \), then \( L(P) = \epsilon \);
- otherwise:
  - flip the Dyck path \( P \) around the \( x \)-axis, obtaining a path \( E \);
  - draw northwest (respectively northeast) lines starting from the midpoint of each double descent (resp. ascent);
  - mark the intersection between the \( i \)-th northwest and \( i \)-th northeast line, for every \( i \);
  - \( L(P) \) is the unique Dyck path that has valleys at the marked points (see Figure 2).

We define a further involution \( L' \) on the set of Dyck paths, which is a variation of the involution \( L \), as follows:

- if \( P \) is the empty path \( \epsilon \), then \( L(P) = \epsilon \);
- consider a Dyck path \( P \) and flip it with respect to a vertical line;
- decompose the obtained path into its irreducible components $U \ P_i \ D$;
- replace every component $U \ P_i \ D$ with $U \ L(P_i) \ D$ in order to get $L'(P)$ (see Figure 3).

We point out that the map $L'$ appears in a slightly modified version in the paper [3].

We now give a description of the map $L'$ in terms of ascent-descent code. Obviously, it is sufficient to consider the case of an irreducible Dyck path $P$. Let $(A, D)$ be the ascent-descent code of an irreducible path $P$ of semilength $n$, with $A = A_1, \ldots, A_h$ and $D = D_1, \ldots, D_h$. Straightforward arguments show that the ascent-descent code $(A', D')$ of $L'(P)$ can be described as follows:

- set $\bar{A}_i = A_i - 1$ and set $\hat{A} = [n-2] \setminus \{\bar{A}_1, \ldots, \bar{A}_h\} = \{\hat{A}_1, \ldots, \hat{A}_{n-2-h}\}$, where the $\hat{A}_i$’s are written in decreasing order. Then, $A'_i = n - \hat{A}_i$.

- consider the set $[n-2] \setminus \{D_1, \ldots, D_h\} = \{\hat{D}_1, \ldots, \hat{D}_{n-2-h}\}$, where the $\hat{D}_i$’s are written in decreasing order. Then, $D'_i = n - 1 - \hat{D}_i$.

Finally, we introduce an order relation $\leq$ on the set of Dyck paths of the same semilength. This order relation will be defined in three steps:
Consider two irreducible Dyck paths $P$ and $Q$ of semilength $n$. Let $(A, D)$ be the ascent-descent code of $P$, with $A = A_1, \ldots, A_k$ and $D = D_1, \ldots, D_k$. We say that $Q$ covers $P$ in the relation $\leq$ if the ascent code of $Q$ is obtained by removing an integer $A_i$ from $A$ and the descent code of $Q$ is obtained by removing an integer $D_j$ for $D$, with $j \geq i$.

Roughly speaking, $Q$ covers $P$ if it can be obtained from $P$ by “closing” the rectangles corresponding to an arbitrary collection of consecutive valleys of $P$;

- the desired order relation $\leq$ on the set of irreducible Dyck paths is the transitive closure of the above covering relation;

- the relation $\leq$ is extended to the set of all Dyck path of a given semilength as follows: if $P$ and $Q$ are two arbitrary Dyck paths and $P = P_1P_2\ldots P_r$ and $Q = Q_1Q_2\ldots Q_s$ are their respective decompositions into irreducible parts, then $P \leq Q$ whenever $r = s$ and $P_i \leq Q_i$ for every $i$. 

Figure 3. The map $L'$. 

\[ P \xrightarrow{\quad} P_1 = \varepsilon \quad \xrightarrow{\quad} \quad P_2 \quad P_3 \quad \xrightarrow{\quad} \quad L'(P) \]
3 LTR minima and RTL maxima of a permutation

Some of the well known bijections between $S_n(\tau)$, $\tau \in S_3$, and the set of Dyck paths of semilength $n$ (see [1], [6], and [10]) are based on the two notions of left-to-right minimum and right-to-left maximum of a permutation $\sigma = x_1 x_2 \ldots x_n$:

- the value $x_i$ is a left-to-right minimum (LTR minimum for short) at position $i$ if $x_i < x_j$ for every $j < i$;
- the value $x_i$ is a right-to-left maximum (RTL maximum) at position $i$ if $x_i > x_j$ for every $j > i$.

For example, the permutation
$$\sigma = 5 \ 3 \ 4 \ 8 \ 2 \ 1 \ 6 \ 7$$
has the LTR minima 5, 3, 2, and 1 (at positions 1, 2, 5, and 6) and RTL maxima 7 and 8 (at positions 8 and 4).

We denote by $vmin(\sigma)$ and $pmin(\sigma)$ the sets of values and positions of the LTR minima of $\sigma$, respectively. Analogously, $vmax(\sigma)$ and $pmax(\sigma)$ denote the sets of values and positions of the RTL maxima of $\sigma$.

Recall that the reverse-complement of a permutation $\sigma \in S_n$ is the permutation defined by
$$\sigma^{rc}(i) = n + 1 - \sigma(n + 1 - i).$$
For example, consider the permutation
\[ \sigma = 247318956. \]

Then:
\[ \sigma^{rc} = 451297368. \]

Note that the sets \( S_n(123) \) and \( S_n(1234) \) are closed under reverse-complement, namely, \( \sigma \in S_n(123) \) (respectively, \( \sigma \in S_n(1234) \)) if and only if \( \sigma^{rc} \in S_n(123) \) (resp. \( \sigma^{rc} \in S_n(1234) \)).

The first assertion in the next proposition goes back to the seminal paper [10], while the second one is an immediate consequence of the straightforward fact that \( x \) is a LTR minimum of a permutation \( \sigma \) at position \( i \) if and only if \( n + 1 - x \) is RTL maximum of \( \sigma^{rc} \) at position \( n + 1 - i \):

**Theorem 1** A permutation \( \sigma \in S_n(123) \) is completely determined by the two sets \( \text{vmin}(\sigma) \) and \( \text{pmin}(\sigma) \) of values and positions of its left-to-right minima. A permutation in \( S_n(123) \) is completely determined, as well, by the two sets \( \text{vmax}(\sigma) \) and \( \text{pmax}(\sigma) \) of values and positions of its right-to-left maxima.

Also 1234-avoiding permutations can be characterized in terms of LTR minima and RTL maxima. This characterization can be found in [2] and is based on an equivalence relation on \( S_n \) defined as follows: \( \sigma \equiv \sigma' \iff \sigma \text{ and } \sigma' \text{ share the values and the positions of LTR minima and RTL maxima.} \)

For example,
\[ 1234 \equiv 1324. \]

Straightforward arguments lead to the following result stated in [2]:

**Theorem 2** Every equivalence class of the relation \( \equiv \) contains exactly one 1234-avoiding permutation. In this permutation, the values that are neither LTR minima nor RTL maxima appear in decreasing order.
4 The maps $\lambda$ and $\mu$

We define two maps $\lambda$ and $\mu$ between $S_n$ and the set $D_n$ of Dyck paths of semilength $n$. Given a permutation $\sigma \in S_n$, the path $\lambda(\sigma)$ is constructed as follows:

- decompose $\sigma$ as $\sigma = m_1 w_1 m_2 w_2 \ldots m_k w_k$, where $m_1, m_2, \ldots, m_k$ are the left-to-right minima in $\sigma$ and $w_1, w_2, \ldots, w_k$ are (possibly empty) words;
- set $m_0 = n + 1$;
- read the permutation from left to right and translate any LTR minimum $m_i$ ($i > 0$) into $m_{i-1} - m_i$ up steps and any subword $w_i$ into $l_i + 1$ down steps, where $l_i$ denotes the number of elements in $w_i$.

The statement of Theorem 1 implies that the map $\lambda$ is a bijection when restricted to $S_n(123)$.

Note that the ascent-descent code $(A, D)$ of the path $\lambda(\sigma)$ is obtained as follows:

- $A = n + 1 - m_1, n + 1 - m_2, \ldots, n + 1 - m_{k-1}$;
- $D = p_2 - 1, p_3 - 1, \ldots, p_k - 1$, where $p_i$ is the position of $m_i$.

We define a further map $\mu : S_n \to D_n$:

- decompose $\sigma$ as $\sigma = u_h M_h u_{h-1} M_{h-1} \ldots u_1 M_1$, where $M_1, M_2, \ldots, M_h$ are the right-to-left maxima in $\sigma$ and $u_1, u_2, \ldots, u_k$ are (possibly empty) words;
- set $M_0 = 0$;
- associate with $M_i$ ($i > 0$) the steps $U^{m_i - m_{i-1}} D$
- associate with each entry in $u_i$ a $D$ step.

Also in this case, the map $\mu$ is a bijection when restricted to $S_n(123)$.

The ascent-descent code $(A^*, D^*)$ of the path $\mu(\sigma)$ is obtained as follows:

- $A^* = M_1, M_2, \ldots, M_{h-1}$;
\[ D^* = n - P_2, n - P_3, \ldots, n - P_h, \] where \( P_i \) is the position of \( M_i \).

In Figure 5 the two paths \( \lambda(\sigma) \) and \( \mu(\sigma) \) corresponding to \( \sigma = 6231754 \) are shown.

\[
\begin{align*}
\lambda(\sigma) & \\
\mu(\sigma) &
\end{align*}
\]

Figure 5. The Dyck paths corresponding to \( \sigma = 6231754 \).

We can now define a map \( \nu : S_n \to D_n \times D_n \), setting

\[ \nu(\sigma) = (\lambda(\sigma), \mu(\sigma)). \]

The statement of Theorem 2 implies that the map \( \nu \) is injective when restricted to \( S_n(1234) \).

Note that the map \( \nu \) behaves properly with respect to the reverse-complement and the inversion operators:

**Proposition 3** Let \( \sigma \) be a permutation in \( S_n \). We have:

- \( \nu(\sigma) = (L, R) \iff \nu(\sigma^{rc}) = (R, L) \), hence, the permutation \( \sigma \) is \( rc \)-invariant if and only if \( L = R \).

- \( \nu(\sigma) = (L, R) \iff \nu(\sigma^{-1}) = (\text{rev}(L), \text{rev}(R)) \), where \( \text{rev}(P) \) is the path obtained by flipping \( P \) with respect to a vertical line. Hence, the permutation \( \sigma \) is an involution if and only if both \( L \) and \( R \) are symmetric with respect to a vertical line.

\( \diamond \)

For example, consider \( \sigma = 6231754 \). The two paths associated with \( \sigma \) are shown in Figure 5. The permutation \( \sigma^{rc} = 4317562 \) is associated with the two paths in Figure 6, while the permutation \( \sigma^{-1} = 4237615 \) corresponds
to the two paths in Figure 7.

Moreover, the map $\nu$ has the following further property that will be crucial in the proof of our main result. Recall that a permutation $\sigma \in S_n$ is said right-connected if it does not have a suffix $\sigma'$ of length $k < n$, that is a permutation of the symbols $1, 2, \ldots, k$. For example, the permutation

$$\tau = 61275348$$

is right-connected, while

$$\sigma = 86457213$$

is not right-connected.

According to this definition, we can split every permutation into right-connected components:

$$\sigma = 86457213.$$

Note that, if a permutation $\sigma$ is not right-connected, $\sigma$ is the juxtaposition of a permutation $\sigma''$ of the set $\{t + 1, \ldots, n\}$ and the permutation $\sigma'$ of the set $\{1, \ldots, t\}$. 

Figure 6. The Dyck paths corresponding to $\sigma^{rc} = 4317562$.

Figure 7. The Dyck paths corresponding to $\sigma^{-1} = 4237615$. 

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Proposition 4 Let \( \sigma \) be a non right-connected permutation in \( S_n \), with \( \sigma = \sigma_1\sigma_2 \), where \( \sigma_1 \) is a permutation of the set \( \{ t + 1, \ldots, n \} \) and \( \sigma_2 \) is a permutation of set of the set \( \{ 1, \ldots, t \} \). Then:

\[
\lambda(\sigma) = P_1P_2 \quad \mu(\sigma) = Q_1Q_2,
\]

with \( P_i = \lambda(\sigma_i) \) and \( Q_i = \mu(\sigma_i) \), \( i = 1, 2 \).

The order relation on Dyck paths defined in Section 2 can be exploited to define two order relations on the set \( S_n \) as follows:

- \( \sigma \leq_\lambda \tau \) if and only if \( \lambda(\sigma) \leq \lambda(\tau) \);
- \( \sigma \leq_\mu \tau \) if and only if \( \mu(\sigma) \leq \mu(\tau) \).

These order relations can be intrinsically described as follows:

Proposition 5 Let \( \sigma, \tau \in S_n \). We have \( \sigma \leq_\lambda \tau \) whenever:

- \( v_{\min}(\tau) \subseteq v_{\min}(\sigma) \);
- \( p_{\min}(\tau) \subseteq p_{\min}(\sigma) \);
- setting:
  - \( v_{\min}(\sigma) = \{ m_1, \ldots, m_h \} \) (written in decreasing order),
  - \( v_{\min}(\sigma) \setminus v_{\min}(\tau) = \{ m_{i_1}, m_{i_2}, \ldots, m_{i_r} \} \) (in decreasing order),
  - \( p_{\min}(\sigma) \setminus p_{\min}(\tau) = \{ p_{j_1}, p_{j_2}, \ldots, p_{j_i} \} \) (in increasing order),
  - then \( i_k < j_k \) for every \( k \).

Similarly, \( \sigma \leq_\mu \tau \) whenever:

- \( v_{\max}(\tau) \subseteq v_{\max}(\sigma) \);
- \( p_{\max}(\tau) \subseteq p_{\max}(\sigma) \);
- setting:
  - \( v_{\max}(\sigma) = \{ M_1, \ldots, M_t \} \) (written in increasing order),
  - \( v_{\max}(\sigma) \setminus v_{\max}(\tau) = \{ M_{i_1}, M_{i_2}, \ldots, M_{i_q} \} \) (in increasing order),
  - \( p_{\max}(\sigma) \setminus p_{\max}(\tau) = \{ P_{j_1}, P_{j_2}, \ldots, P_{j_q} \} \) (in decreasing order),
  - then \( i_k < j_k \) for every \( k \).
For example, consider the permutation
\[ \sigma = 6 \ 8 \ 7 \ 3 \ 2 \ 5 \ 9 \ 1 \ 4. \]
We have \( v_{\min}(\sigma) = \{6, 3, 2, 1\} \), \( p_{\min}(\sigma) = \{1, 4, 5, 8\} \), \( v_{\max}(\sigma) = \{4, 9\} \), and \( p_{\max}(\sigma) = \{9, 7\} \). The permutation
\[ \tau = 3 \ 4 \ 9 \ 2 \ 6 \ 8 \ 7 \ 1 \ 5 \]
is such that \( v_{\min}(\tau) = \{3, 2, 1\} \) and \( p_{\min}(\tau) = \{1, 4, 8\} \), hence, \( \sigma \leq_{\lambda} \tau \). Moreover, the permutation
\[ \rho = 2 \ 7 \ 1 \ 3 \ 4 \ 6 \ 5 \ 8 \ 9 \]
is such that \( v_{\max}(\rho) = \{9\} \) and \( p_{\max}(\rho) = \{9\} \), hence, \( \sigma \leq_{\mu} \rho \).

5 Main results

We say that a pair of Dyck paths \((P, Q)\) is admissible if there exists a permutation \(\alpha\) such that \(P = \lambda(\alpha)\) and \(Q = \mu(\alpha)\). Needless to say, the set of admissible pairs is in bijection with the set of 1234-avoiding permutations.

We want to show that the operator \(L'\) on Dyck paths allows us to characterize the set of admissible pairs. We begin with a preliminary result concerning the pairs of Dyck paths corresponding to 123-avoiding permutations:

**Theorem 6** For every \(\sigma \in S_n(123)\), we have:
\[ \mu(\sigma) = L'(\lambda(\sigma)). \]

**Proof** Proposition 4, together with the definition of the map \(L'\), allows us to restrict our attention to the right-connected case.

Recall (see [10]) that a permutation \(\sigma\) avoids 123 if and only if the set \(v_{\min}(\sigma) \cup v_{\max}(\sigma) = [n]\). It is simple to check that, if \(\sigma\) is right-connected, the sets of LTR minima and RTL maxima are disjoint.

Consider now a permutation \(\sigma\) with LTR minima \(m_1, \ldots, m_{k-1}, m_k = 1\) and RTL maxima \(M_1, \ldots, M_{h-1}, M_h = n\). Denote by \((A, D)\) the ascent-descent code of the path \(P = \lambda(\sigma)\) and by \((A^*, D^*)\) the ascent-descent code of the path \(\mu(\sigma)\).
As noted before, the ascent code $A'$ of $L'(P)$ is obtained by computing the integers $\bar{A}_i = A_i - 1$ and then considering the set $\hat{A} = [n-2] \setminus \{\bar{A}_1, \ldots, \bar{A}_{k-1}\}$, which can be written as
\[
\hat{A} = \{n - (n-1), n - (n-2), \ldots, n-2\} \setminus \{n-m_1, \ldots, n-m_{k-1}\}.
\]
Since $\{m_1, \ldots, m_{k-1}\} \cup \{M_1, \ldots, M_{h-1}\} = \{2, 3, \ldots, n-1\}$, we have
\[
\hat{A} = \{n-M_1, \ldots, n-M_{h-1}\}.
\]
Hence, $A' = A^\ast$.

Similarly, the descent code $D'$ of $L'(P)$ is obtained by considering the set
\[
\hat{D} = [n-2] \setminus \{D_1, \ldots, D_{k-1}\} = [n-2] \setminus \{p_2-1, \ldots, p_k-1\}.
\]
Since $\{p_1, \ldots, p_{k-1}\} \cup \{P_1, \ldots, P_{h-1}\} = \{2, 3, \ldots, n-1\}$, we have
\[
\hat{D} = \{P_2-1, \ldots, P_{h-1}-1\}.
\]
Hence, $D' = D^\ast$.

For example, the 123-avoiding permutation $\sigma = 859762431$ corresponds to the pair of Dyck paths $(P, L'(P))$ in Figure 3.

We are now in position to state our main result:

**Theorem 7** A pair $(P, Q)$ is admissible if and only if $P \geq L'(Q)$ and $Q \geq L'(P)$.

**Proof** Consider a permutation $\sigma \in S_n(1234)$ and let $\sigma'$ be the unique permutation in $S_n(123)$ with the same LTR minima as $\sigma$, at the same positions. Obviously, $\sigma' \leq_{\mu} \sigma$, since in $\sigma'$ every element that is not a LTR minimum is a RTL maximum (see Proposition 5). Recalling that $\mu(\sigma') = L'(\lambda(\sigma)) = L'(P)$, we get the first inequality. The other inequality follows from the fact that the pair $(P, Q)$ is admissible whenever the pair $(Q, P)$ is admissible.

Consider now a pair of Dyck paths $(P, Q)$ such that $P \geq L'(Q)$ and $Q \geq L'(P)$. Proposition 4 allows us to restrict to the case $P, Q$ irreducible. Denote by $\sigma$ and $\tau$ the permutations in $S_n(123)$ corresponding via $\nu$ to the pairs $(P, L'(P))$ and $(L'(Q), Q)$, respectively. Since $P \geq L'(Q)$ and $Q \geq L'(P)$, we have $\tau \leq_{\lambda} \sigma$ and $\sigma \leq_{\mu} \tau$.

We define a permutation $\alpha \in S_n$ as follows:
\• \( \alpha(x) = \sigma(x) \) if \( x \in pmin(\sigma) \);
\• \( \alpha(x) = \tau(x) \) if \( x \in pmax(\tau) \);
\• if \( x \notin pmin(\sigma) \cup pmax(\tau) \), we have \( x \in pmin(\sigma) \setminus pmax(\tau) \).

The permutation \( \alpha \) is obtained as the concatenation of three decreasing sequences. Hence, \( \alpha \) avoids 1234. We have to prove that \( v_{min}(\sigma) = v_{min}(\alpha) \) and \( v_{max}(\tau) = v_{max}(\alpha) \).

It is immediate that \( v_{min}(\sigma) \subseteq v_{min}(\alpha) \). In order to prove that \( v_{min}(\sigma) = v_{min}(\alpha) \) it remains to show that the values \( m_{i_1}, m_{i_2}, \ldots, m_{i_r} \) are not LTR minima of \( \alpha \).

In fact, for every \( k \), consider \( \alpha(p_{j_k}) = m_{i_k} = \tau(p_{i_k}) \). Consider the sets \( A = \{p_1, p_2, \ldots, p_{i_k}\} \), \( B = \{m_1, m_2, \ldots, m_{i_k}\} \), and their subsets \( A' = \{p_1, p_2, \ldots, p_{i_k}\} \) and \( B' = \{m_1, m_2, \ldots, m_{i_k}\} \). The \( k \) elements in \( B' \) do not belong to \( v_{min}(\sigma) \) (and hence, the \( i_k - k \) elements in \( B \setminus B' \) are the largest elements in \( v_{min}(\sigma) \)).

Proposition 5 ensures that each of them occupies in \( \alpha \) a position that is strictly greater than the position occupied in \( \tau \). This implies that \( p_{j_k} < p_{i_k} \) and that at most \( k - 1 \) elements in \( B' \) occupy in \( \tau \) a position that belongs to \( A \). Hence, in \( \alpha \), at least \( i_k - k + 1 \) positions in \( A \) are occupied by entries belonging to \( v_{min}(\sigma) \). This implies that there is in \( \alpha \) a position preceding \( p_{j_k} \) occupied by a value less than \( m_{i_k} \). Hence, \( m_{i_k} \) is not a LTR minimum of \( \alpha \).

Analogous arguments can be used to prove that \( v_{max}(\sigma) = v_{max}(\tau) \). Hence, \( \nu(\alpha) = (P, Q) \), as desired.

\begin{itemize}
  \item For example, consider the pair of Dyck paths in Figure 8.
  \item It can be checked that \( P \geq L'(Q) \) and \( Q \geq L'(P) \). The permutations \( \sigma = \nu^{-1}((P, L'(P))) \) and \( \tau = \nu^{-1}((L'(Q), Q)) \) are as follows:
  \[ \sigma = 498271653 \quad \tau = 759432816. \]
\end{itemize}
We have \( v_{\text{min}}(\sigma) = \{ 4, 2, 1 \} \), \( p_{\text{min}}(\sigma) = \{ 1, 4, 6 \} \), \( v_{\text{min}}(\tau) = \{ 7, 5, 4, 3, 2, 1 \} \),
\( p_{\text{min}}(\tau) = \{ 1, 2, 4, 5, 6, 8 \} \), \( v_{\text{max}}(\sigma) = \{ 3, 5, 6, 7, 8, 9 \} \), \( p_{\text{max}}(\sigma) = \{ 9, 8, 7, 5, 3, 2 \} \),
\( v_{\text{max}}(\tau) = \{ 6, 8, 9 \} \), and \( p_{\text{max}}(\tau) = \{ 9, 7, 3 \} \).
The permutation \( \alpha = \nu^{-1}((P, Q)) \) is
\[
\alpha = 4 7 9 2 5 1 8 3 6.
\]
As expected, \( v_{\text{min}}(\alpha) = v_{\text{min}}(\sigma) \), \( p_{\text{min}}(\alpha) = p_{\text{min}}(\sigma) \), \( v_{\text{max}}(\alpha) = v_{\text{max}}(\tau) \),
and \( p_{\text{max}}(\alpha) = p_{\text{max}}(\tau) \).

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