Tensor products of $\mathfrak{psl}(2|2)$ representations

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Abstract

The aim of this work is to study finite dimensional representations of the Lie superalgebra $\mathfrak{psl}(2|2)$ and their tensor products. In particular, we shall decompose all tensor products involving typical (long) and atypical (short) representations as well as their so-called projective covers. While tensor products of long multiplets and projective covers close among themselves, we shall find an infinite family of new indecomposables in the tensor products of two short multiplets. Our note concludes with a few remarks on possible applications to the construction of $AdS_3$ backgrounds in string theory.

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1 Introduction

This short note is devoted to the representation theory of the Lie superalgebra $\mathfrak{psl}(2|2)$. The latter describes symmetries of several important physical systems, ranging from strings moving in an $AdS_3$ background [1] [2] to the quantum Hall effect (see e.g. [3]). The questions we address here, however, are purely mathematical and we shall only comment very briefly on our motivation from physics, leaving more concrete applications to a future publication.

The so-called A-series of simple Lie superalgebras [4] consists of $\mathfrak{sl}(n|m)$ with $n \neq m$ and $\mathfrak{psl}(n|n)$.

Finite dimensional representations of these Lie superalgebras with diagonal Cartan elements have been constructed and investigated extensively (see e.g. [5] [6] [7] for some early papers). Most of this work focused on irreducible representations, disregarding that even the very simplest Lie superalgebras from the A-series possess plenty of non-trivial indecomposables (see e.g. [6] [8]). In fact, the Lie superalgebras $\mathfrak{sl}(n|m)$ with $n, m > 1$ admit so many of them that they cannot even be classified [9]. Irreducible representations, on the other hand, are rather easy to list (see e.g. [10] and references therein). These fall into two different classes known as typical (long) and atypical (short) representations [7]. More general indecomposables may be regarded as composites of the latter.

Investigations of tensor products for the A-series cannot avoid dealing with indecomposable representations. In fact, it is well known that the product of two irreducibles is often not fully reducible [6] [11] and what is even worse: indecomposables of Lie superalgebras do not form an ideal in the fusion ring. Consequently, it would e.g. not be possible to determine how many times a given irreducible representation appears in a higher tensor product if we only knew the number of irreducibles in the two-fold products of irreducible representations, simply because the fusion paths leading to irreducible representations may pass through indecomposables. In other words, the study of tensor products for representations of Lie superalgebras cannot be consistently truncated to irreducible representations. General results show, however, that there is a preferable class of so-called projective representations which gives rise to an ideal in the fusion ring. This contains the typical representations along with certain maximal indecomposable composites of atypicals. The latter are known as projective covers of atypical representations.

1 The Lie superalgebras $\mathfrak{psl}(n|n)$ are obtained from $\mathfrak{sl}(n|n)$ when we remove the 1-dimensional center.
The indecomposable representations that emerge in products of irreducibles add a lot of interesting novel structure to the study of tensor products, but they certainly also bring a lot of extra difficulties. Even though tensor products of Lie superalgebras from the A-series have been investigated in the literature (see e.g. [11, 12] for some early work), most of the existing work excludes degenerate cases in which indecomposables appear as one of the factors or in the decomposition of the product. More extensive results on degenerate products seem to be restricted to a few simple examples, including $\mathfrak{sl}(2|1)$ [11] and $\mathfrak{gl}(1|1)$ (see e.g. [13]). And even in these cases, a complete treatment was only found recently [14].

In this note we shall study all finite dimensional tensor products for $\mathfrak{psl}(2|2)$ between typical and atypical representations as well as their projective covers. We shall see explicitly how the products involving at least one projective representation can be decomposed into a sum of typicals and projective covers (propositions 1, 2 and 3 below). On the other hand, a new family of indecomposables arises in tensor products of atypical representations (see proposition 4). The next section contains an introduction to the three most important types of $\mathfrak{psl}(2|2)$ representations. Section 3 is then devoted to our new results on tensor products of these representations. A complete list of our findings is provided in subsection 3.1 along with a few additional comments. Our claims are supported by two central ideas which are explained in subsection 3.2 before we demonstrate how they work together in our computation of tensor products. Since the full calculations are quite cumbersome and not very illuminating, we shall illustrate the key steps in two representative examples rather than attempting to present a general proof. In the concluding section, we will discuss at least briefly our motivations from physics and some potential applications.

2 Representations of $\mathfrak{psl}(2|2)$

In this first section we shall mainly review known results about some finite dimensional representations of $\mathfrak{g} = \mathfrak{psl}(2|2)$. We shall provide a complete list of irreducible representations, both typical and atypical, explain how they are constructed from the so-called Kac modules and we shall describe the projective covers of all atypical representations.
2.1 The Lie superalgebra $\mathfrak{psl}(2|2)$

The Lie superalgebra $\mathfrak{g} = \mathfrak{psl}(2|2)$ possesses six bosonic generators $K^{ab} = -K^{ba}$ with $a, b = 1, \ldots , 4$. In addition, there are eight fermionic generators that we denote by $S^a_\alpha$ with the new index $\alpha = 1, 2$ and where $a$ assumes the same values as for the bosonic generators. The relations of $\mathfrak{g}$ are given by

$$
\left[ K^{ab}, K^{cd} \right] = i \left[ \delta^{ac} K^{bd} - \delta^{bc} K^{ad} - \delta^{ad} K^{bc} + \delta^{bd} K^{ac} \right] \\
\left[ K^{ab}, S^c_\gamma \right] = i \left[ \delta^{ac} S^b_\gamma - \delta^{bc} S^a_\gamma \right] \\
\left[ S^a_\alpha , S^b_\beta \right] = \frac{1}{2} \epsilon_{\alpha\beta} \epsilon^{abcd} K^{cd} .
$$

Here, $\epsilon_{\alpha\beta}$ and $\epsilon^{abcd}$ denote the usual $\epsilon$-symbols with two and four indices, respectively, and a summation over repeated indices is implied.

Note that the even subalgebra $\mathfrak{g}^{(0)} = \mathfrak{psl}(2|2)^{(0)}$ is isomorphic to $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. The odd part $\mathfrak{g}^{(1)}$ of $\mathfrak{psl}(2|2)$ is spanned by the eight fermionic generators. We split the latter into two sets of four generators

$$
\mathfrak{g}^{(1)}_+ = \text{span}\{ S^a_2 \} , \quad \mathfrak{g}^{(1)}_- = \text{span}\{ S^a_1 \} .
$$

As indicated by the subscript $\pm$, we shall think of the fermionic generators $S^a_1$ as annihilation operators and of $S^a_2$ as creation operators.

Let us furthermore recall that the group $\text{SL}(2, \mathbb{C})$ acts on $\mathfrak{g}$ through outer automorphisms. For an element $u = (u^\alpha_{\beta}) \in \text{SL}(2, \mathbb{C})$ the latter read

$$
\gamma_u(K^{ab}) = K^{ab} , \quad \gamma_u(S^a_\alpha) = u^\alpha_{\beta} S^a_\beta ,
$$

Consistency with the defining relations of our Lie superalgebra is straightforward to check. It only uses the fact that $\det(u) = 1$.

2.2 Finite dimensional irreducible representations

The irreducible finite dimensional representations of $\mathfrak{g}$ are labeled by pairs $j_1, j_2$ with $j_i = 0, 1/2, 1, \ldots$. All these representations are highest weight representations and they are uniquely characterized by the highest weights $(j_1, j_2)$ of the corresponding even subalgebra $\mathfrak{g}^{(0)} \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. We shall see in a moment how such representations may be constructed explicitly.
For reasons that we shall understand below, the irreducible representations of \( g \) fall into two classes. So-called \textit{typical representations} appear for \( j_1 \neq j_2 \). We shall denote them as \([j_1, j_2]\). Their dimension is given by

\begin{equation}
\dim [j_1, j_2] = d_{[j_1, j_2]} = 16 (2j_1 + 1)(2j_2 + 1) . 
\end{equation}

(2.3)

Representations with labels \( j = j_1 = j_2 \) are \textit{atypical}, since their dimension is smaller than a naive application of formula (2.3) would suggest. For these irreducible representations we shall employ the symbol \([j]\) and one finds that

\begin{equation}
\dim [j] = d_{[j]} = 16j(j + 1) + 2 .
\end{equation}

The formula holds for \( j \neq 0 \). The representation \([0]\) is the trivial one-dimensional representation. Let us also point out that the 14-dimensional atypical representation \([1/2]\) is the adjoint representation of \( \mathfrak{psl}(2|2) \).

The irreducible representations \([i, j]\) and \([j]\) of the Lie superalgebra \( g \) can be restricted to the even subalgebra \( g^{(0)} \). With respect to this restricted action they decompose according to

\begin{equation}
[j]_{g^{(0)}} \cong (j + \frac{1}{2}, j - \frac{1}{2}) \oplus 2(j, j) \oplus (j - \frac{1}{2}, j + \frac{1}{2}) \quad (\text{for } j > 0) ,
\end{equation}

(2.4)

\begin{equation}
[i, j]_{g^{(0)}} \cong (i, j) \otimes [2(0, 0) \oplus 2(\frac{1}{2}, \frac{1}{2}) \oplus (0, 1) \oplus (1, 0)] .
\end{equation}

(2.5)

Here and in the following, the pairs \((i, j)\) denote irreducible representations of the even subalgebra. Note that these decomposition formulas are consistent with our expressions for the dimension of irreducible representations.

The irreducible representations of \( g \) possess one property that will become very important later on: They admit an implementation of the outer automorphisms \( \gamma_u \), see (2.2). To make this more precise, let us introduce the symbol \( \pi \) for the representation that sends elements \( X \in \mathfrak{psl}(2|2) \) to linear maps on the representation space \( \mathcal{H}_\pi \). If we compose a representation \( \pi \) with an automorphism \( \gamma \) of the Lie superalgebra, we obtain a new representation \( \pi \circ \gamma \) on the same graded vector space. In general, this new representation differs from \( \pi \). If \( \pi \) is one of our finite dimensional irreducible representations \([i, j]\) or \([j]\) and \( \gamma \) one of the automorphisms in (2.2), however, then the new representation is equivalent to the original one, i.e. for every \( u \in \text{SL}(2, \mathbb{C}) \) there exists an invertible linear map \( U_\pi : \mathcal{H}_\pi \longrightarrow \mathcal{H}_\pi \) such that

\begin{equation}
\pi \circ \gamma_u(X) = U_\pi \pi(X) U_\pi^{-1} \quad \text{for all } X \in \mathfrak{psl}(2|2) .
\end{equation}

(2.6)
Let us stress that the map \( u \rightarrow U_u \) defines a representation of the subgroup SL(2,C) of outer automorphisms on the representation space \( H_\pi \).

### 2.3 Kac modules and irreducible representations

It is useful for us to discuss briefly how the irreducible representations we have listed above can be constructed. The idea is rather standard: We begin with an irreducible highest weight representation \((j_1, j_2)\) of the even subalgebra \(\mathfrak{g}^{(0)}\). We declare that the corresponding representation space \(V_{(j_1, j_2)}\) is annihilated by \(S^a_1\) and then generate a so-called Kac module \([j_1, j_2]\) by application of the raising operators \(S^a_2\),

\[
[j_1, j_2] := \text{Ind}^{\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}}_{\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}} \mathcal{U}(\mathfrak{g}) \otimes \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} V_{(j_1, j_2)} .
\]

Here, we have extended the \(\mathfrak{g}^{(0)}\) module \(V_{(j_1, j_2)}\) to a representation of \(\mathfrak{g}^{(0)} \oplus \mathfrak{g}^-\) by setting \(S^a_1 V_{(j_1, j_2)} = 0\). Note that we can apply at most four fermionic generators to the states in \(V_{(j_1, j_2)}\). Therefore, the dimension of this Kac module is given by

\[
\dim [j_1, j_2] = 16(2j_1 + 1)(2j_2 + 1) .
\]

The Kac module \([j_1, j_2]\) is irreducible whenever \(j_1 \neq j_2\). In these generic cases it agrees with the typical representation. For \(j_1 = j_2 = j\), however, the associated Kac module turns out to be reducible but not fully reducible, i.e. it cannot be written as a direct sum of irreducible representations. If \(j \geq 1\) the structure of the Kac module can be encoded in the following chain

\[
[j, j] : [j] \rightarrow [j + \frac{1}{2}] \oplus [j - \frac{1}{2}] \rightarrow [j] , \tag{2.7}
\]

or, equivalently, in a planar diagram in which one direction refers to the spin \(j\) of the atypical constituents,

\[
[j, j] : [j] \rightarrow [j + \frac{1}{2}] \rightarrow [j] \rightarrow [j - \frac{1}{2}] . \tag{2.8}
\]

Since pictures of this type will appear frequently throughout this text, let us pause here for a moment and explain carefully how to decode their information. We read the diagram
from right to left. The rightmost entry in our chain contains the so-called socle of the indecomposable representation, i.e. the largest fully reducible invariant submodule we can find. In the case of our Kac module, the socle happens to be irreducible and it is given by the atypical representation \([j]\). If we divide the Kac module by the submodule \([j]\), we obtain a new indecomposable representation of our Lie superalgebra. Its diagram is obtained from the one above by removing the last entry and arrow. The socle of this quotient is a direct sum of the two atypical representations \([j \pm 1/2]\). It is rather obvious how to iterate this procedure until the entire indecomposable representation is split up into floors with only direct sums of irreducible representations appearing on each floor.

There are two special cases for which the decomposition of the Kac module does not follow the generic pattern. These are the cases \(j = 0\) and \(j = 1/2\),

\[
[0, 0] : [0] \to [\frac{1}{2}] \to [0] , \quad (2.9)
\]

\[
[\frac{1}{2}, \frac{1}{2}] : [\frac{1}{2}] \to [1] \to [0] \oplus [0] \to [\frac{1}{2}] . \quad (2.10)
\]

Let us note that our formula for the dimension of atypical representations follows directly from the decomposition of the corresponding Kac modules.

### 2.4 Projective covers of atypical representations

As we have seen in the last subsection, atypical representations can be extended into larger indecomposables. Kac modules are only one example of such composites and we shall indeed see several others as we proceed. Among them, however, there is one special class, the so-called projective covers \(P_g(j)\). By definition, these are the largest indecomposables whose socle consists of a single atypical representation \([j]\). General results imply that such a maximal indecomposable extension of \([j]\) exists and is unique. In case of \(j \geq 3/2\), the structure of \(P_g(j)\) is encoded in the following diagram

\[
P_g(j) : \quad [j] \to 2[j + \frac{1}{2}] \oplus 2[j - \frac{1}{2}] \to [j + 1] \oplus 4[j] \oplus [j - 1] \to \quad (2.11)
\]

\[
\to 2[j + \frac{1}{2}] \oplus 2[j - \frac{1}{2}] \to [j] .
\]

Note that \(P_g(j)\) contains an entire Kac module as proper submodule. In this sense, the Kac modules are extendable. We also observe one rather generic feature of projective covers: they are built up from different Kac modules in a way that resembles the pattern in which Kac modules are constructed out of irreducibles (see eq. (2.8)). One may see this
even more clearly if $P_g(j)$ is displayed as a 2-dimensional diagram in which the additional direction keeps track of the spin $j$ of the atypical constituents $[j]$, 

\[ P_g(j) : \]

\[
\begin{array}{c}
[0] \rightarrow 3\left[\frac{1}{2}\right] \rightarrow 2\left[1\right] \oplus 6\left[0\right] \rightarrow \left[\frac{1}{2}\right] \rightarrow [0] .
\end{array}
\]

We will continue to switch between such planar pictures and diagrams of the form (2.11).

The remaining cases $j = 0, 1/2, 1$ have to be listed separately. When $j = 1$ the picture is very similar only that we have to insert $2[0]$ in place of $[j - 1]$, 

\[ P_g(1) : [1] \rightarrow 2\left[\frac{3}{2}\right] \oplus 2\left[\frac{1}{2}\right] \rightarrow [2] \oplus 4[1] \oplus 2[0] \rightarrow 2\left[\frac{3}{2}\right] \oplus 2\left[\frac{1}{2}\right] \rightarrow [1] . \]

The projective cover of the atypical representation $[1/2]$ is obtained from the generic case by the formal substitution $2[j - 1/2] \rightarrow 3[0]$, 

\[ P_g(\frac{1}{2}) : \left[\frac{1}{2}\right] \rightarrow 2[1] \oplus 3[0] \rightarrow \left[\frac{3}{2}\right] \oplus 4\left[\frac{1}{2}\right] \rightarrow 2[1] \oplus 3[0] \rightarrow \left[\frac{1}{2}\right] . \]

Finally, the projective cover $P_g(0)$ of the trivial representation is given by, 

\[ P_g(0) : [0] \rightarrow 3\left[\frac{1}{2}\right] \rightarrow 2[1] \oplus 6[0] \rightarrow \left[\frac{1}{2}\right] \rightarrow [0] . \]
3 Tensor products of $\mathfrak{psl}(2|2)$ representations

This section contains the central results of this work, i.e. explicit formulas for the decomposition of tensor products between all the finite dimensional irreducible representations and projectives we have introduced in the previous section. We begin by stating our main results. Then we shall explain the two key ideas that are needed in the proof. Finally, we analyze two very representative examples.

3.1 Tensor products of irreducible representations

Before we start listing our results, we would like to introduce one object that will enable us to determine the contributions from typical representations in most of the tensor products below. We shall denote by $\pi_\mathfrak{g}$ a map which associates a direct sum of typical representations to any finite dimensional representation of the bosonic subalgebra. On the irreducible representations of $\mathfrak{g}^{(0)}$ it gives

$$\pi_\mathfrak{g}(i, j) := \begin{cases} [i, j] & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases} \quad (3.1)$$

This prescription is extended to any direct sum of irreducibles by linearity. $\pi_\mathfrak{g}$ enters e.g. in the following formula for the tensor product of typical and atypical representations.

**Proposition 1 (Tensor product of typical and atypical representations).** The tensor product of an atypical representation $[n]$ with a typical representation $[i, j]$ is given by

$$[n] \otimes [i, j] = \pi_\mathfrak{g}([n]|_{\mathfrak{g}^{(0)}} \otimes (i, j)) \quad \text{for } i + j \notin \mathbb{Z}.$$  

When $i + j \in \mathbb{Z}$, on the other hand, the tensor product can also contain projective covers,

$$[n] \otimes [i, j] = \pi_\mathfrak{g}([n]|_{\mathfrak{g}^{(0)}} \otimes (i, j)) \oplus \bigoplus_{l=p}^{\min(n+i, n+j)} \mathcal{P}_\mathfrak{g}(l) \quad (3.2)$$

where $p = \max(|n - i|, |n - j|)$ and $q = \min(n + i, n + j)$. The decomposition of the irreducible representation $[n]$ into representations of the even subalgebra $\mathfrak{g}^{(0)}$ was spelled out in eq. (2.4).

It is interesting to observe that the tensor product of the representation $[n]$ with the typical representation $[0, 2n]$ contains a single projective cover $\mathcal{P}_\mathfrak{g}(n)$. Proposition 2
may be employed to determine tensor products between typical and any indecomposable representation. The prescription requires introducing a new map \( S_g \) which sends an indecomposable representation \( \mathcal{R} \) to a sum of atypicals,

\[
S_g(\mathcal{R}) = \bigoplus_j [\mathcal{R} : [j]] \cdot [j].
\]  

(3.3)

Here, the symbol \([\mathcal{R} : [j]]\) denotes the total number of atypical representations \([j]\) in the decomposition series of \( \mathcal{R} \). In the case where \( \mathcal{R} \) is one of the projective covers, for example, these numbers can be read off from eqs. (2.11)-(2.15). With this notation, the tensor product between a typical representation \([i, j]\) and the indecomposable \( \mathcal{R} \) reads,

\[
\mathcal{R} \otimes [i, j] \cong S_g(\mathcal{R}) \otimes [i, j].
\]  

(3.4)

As we have anticipated, we may now employ proposition 1 to decompose the tensor product on the right hand side. Thereby, we can determine e.g. the fusion of typical representations and projective covers.

Proposition 2 (Tensor product of typical representations). The tensor product of two atypical representations \([i_1, j_1]\) and \([i_2, j_2]\) is given by

\[
[i_1, j_1] \otimes [i_2, j_2] = \pi_g([i_1, j_1]|_{g(0)} \otimes (i_2, j_2)) \quad \text{for} \quad i_1 + i_2 + j_1 + j_2 \notin \mathbb{Z}.
\]

If \( i_1 + i_2 + j_1 + j_2 \in \mathbb{Z} \), on the other hand, projective covers may appear in the decomposition,

\[
[i_1, j_1] \otimes [i_2, j_2] = \pi_g([i_1, j_1]|_{g(0)} \otimes (i_2, j_2)) \oplus 2 \cdot \bigoplus_{m=2p}^{2q} \mathcal{P}_g(m) \quad (3.5)
\]

\[
\oplus \left\{ \begin{array}{ll}
\delta_{p+1, q} \mathcal{P}_g(q + \frac{1}{2}) & , \ p > q \\
(1 - \delta_{j_1+j_2}^{i_1+i_2}) \cdot \mathcal{P}_g(q + \frac{1}{2}) \oplus (1 - \delta_{j_1-j_2}^{i_1-i_2}) \cdot \mathcal{P}_g(p - \frac{1}{2}) \oplus \delta_{p, 0} \cdot \mathcal{P}_g(0) & , \ p \leq q
\end{array} \right.
\]

The decomposition of typical representations \([i, j]\) into irreducibles of \( g^{(0)} \) appears in eq. (2.5). We have also introduced the parameters \( p \) and \( q \) by \( p = \max(|i_1 - i_2|, |j_1 - j_2|) \) and \( q = \min(i_1 + i_2, j_1 + j_2) \). Note that the last term in the first line subtracts one copy of the projective cover \( \mathcal{P}_g(0) \) whenever \( p \) vanishes.

At this point we are able to decompose all tensor products which involve at least one typical factor. Our next task is to analyze the products in which at least one factor is a projective cover \( \mathcal{P}_g(j) \).
Proposition 3 (Tensor product of atypical representations and projective covers). The tensor product of an atypical representation \([n]\) with a projective cover \(\mathcal{P}_g(j)\) with \(j \neq 0\) and \(n \neq 0\) is given by

\[
[n] \otimes \mathcal{P}_g(j) = \pi_g([n]|_{\mathfrak{g}^{(0)}} \otimes H(j)) \oplus 2 \cdot \bigoplus_{m=2|n-j|} \mathcal{P}_g(\frac{m}{2}) \oplus \delta_{n,j} \cdot \mathcal{P}_g(0)
\]

(3.6)

where \(H(j) = 2(j, j) \oplus (j + \frac{1}{2}, j + \frac{1}{2}) \oplus (j - \frac{1}{2}, j - \frac{1}{2})\).

(3.7)

The decomposition of atypical representations into modules of \(\mathfrak{g}^{(0)}\) can be found in eq. (2.4). In the special case of \(j = 0\) one obtains

\[
[n] \otimes \mathcal{P}_g(0) = \pi_g([n]|_{\mathfrak{g}^{(0)}} \otimes H(0)) \oplus 4 \cdot \mathcal{P}_g(n)
\]

Here we have to insert the representation \(H(0) = 2(0, 0) \oplus 2(\frac{1}{2}, \frac{1}{2})\) of the bosonic subalgebra in the argument of \(\pi_g\).

From proposition 3 we may compute the tensor product of a projective cover with any other indecomposable through the following extension of formula (3.4) to projective covers of atypical representations,

\[
\mathcal{R} \otimes \mathcal{P}_g(j) \cong \mathcal{S}_g(\mathcal{R}) \otimes \mathcal{P}_g(j)
\]

(3.8)

for all indecomposables \(\mathcal{R}\). The symbol \(\mathcal{S}\) has been introduced in equation (3.3) above such that it associates to \(\mathcal{R}\) the direct sum of irreducibles that appear in its decomposition series.

Note that so far we were able to express all tensor products through a direct sum of typical representations and projective covers. This is consistent with the before-mentioned general result that projective representations form an ideal in the fusion ring of representations. There remains, however, one more family of tensor products to be determined. These are the tensor products between two atypical representations. Not surprisingly, we shall encounter a new set of indecomposables in such tensor products. These possess the following form

\[
\pi_{i \otimes j}^{\text{indecomposable}} = \bigoplus_{k=|i-j|}^{i+j-1} [k + \frac{1}{2}] \longrightarrow \bigoplus_{k=|i-j|}^{i+j} 2[k] \longrightarrow \bigoplus_{k=|i-j|}^{i+j-1} [k + \frac{1}{2}] (i \neq j)
\]

(3.9)

\[
\pi_{j \otimes j}^{\text{indecomposable}} = \bigoplus_{k=0}^{2j-1} [k + \frac{1}{2}] \longrightarrow 3[0] \oplus \bigoplus_{k=1}^{2j} 2[k] \longrightarrow \bigoplus_{k=0}^{2j-1} [k + \frac{1}{2}].
\]

(3.10)
Note that the second line is essentially a special case of the first except that we replaced the term $2[j - j]$ by $3[0]$. When rewritten in terms of our 2-dimensional diagrams, these representations read,

\[
\begin{align*}
\pi_{i \otimes j}^{\text{indec}} : & & \begin{pmatrix} 2[i + j] \\ \vdots \\ 2[|i - j|] \end{pmatrix} \\
\pi_{j \otimes i}^{\text{indec}} : & & \begin{pmatrix} 2[2j] \\ \vdots \\ 3[0] \end{pmatrix}
\end{align*}
\]

**Proposition 4:** (Tensor product of atypical representations) The tensor product of two atypical representations $[i]$ and $[j]$ is given by

\[
[i] \otimes [j] = \pi_{\mathfrak{g}}((i, i) \otimes (j, j)) \oplus \delta_{i,j}[0] \oplus \pi_{i \otimes j}^{\text{indec}} .
\]

The formula holds for $i, j = 1/2, 1, 3/2, \ldots$. Tensor products of the trivial representation $[0]$ with any other atypical representation are obvious.

Let us note that for $i = j$ one copy of the atypical representation $[0]$ appears as a summand, i.e. it is not part of the single indecomposable representation that contains all the other atypical building blocks. We have also calculated a few tensor products between the new indecomposable representations $\pi_{i \otimes j}$ and atypicals. Such products turn out to generate further indecomposables with a structure that is similar to the one of $\pi_{i \otimes j}$ but involves different multiplicities. It seems within reach to fully analyze the fusion ring that is generated by irreducibles, but since the applications we have in mind do not require such an exhaustive study, we have not investigated these issues much further.

### 3.2 Proof of the decomposition formulas - general ideas

There are two main ideas that enter into the proof of the above formulas. To begin with, we shall exploit systematically that the outer automorphisms are implementable not only in the irreducibles of $\mathfrak{g}$ but also in all their tensor products. This will ultimately organize all the typical subrepresentations and, more importantly, the elements of the composition series of indecomposables into multiplets of the group $\text{SL}(2, \mathbb{C})$ of outer automorphisms.
In a second step we then restrict the action of \( g \) to the action of an embedded \( \mathfrak{h} = \mathfrak{sl}(2|1) \). Knowledge about the representation theory of the latter \([8, 11]\) (see also \([10]\)) along with a few new results from \([14]\) will then allow us to uniquely determine the structure of indecomposable \( g \) representations that appear in the tensor products of irreducibles and projectives. The consistency of our results has also been verified by checking associativity of the tensor products involving three representations of irreducible or projective type with labels equal or below 4 on a computer.

**Implementation of outer automorphisms.** We have stated above that the action of the \( \text{SL}(2,\mathbb{C}) \) outer automorphisms of \( g \) can be implemented in all of its irreducible representations. Now we will argue that implementability of outer automorphisms is respected by the operation of tensor products, by restriction to the socle and by quotients. This implies that outer automorphisms may be implemented separately in all representations that appear in the decomposition of tensor products of irreducible representations. In this sense, such representations are rather special.

**Lemma 1.** Suppose that the action of outer automorphisms can be implemented in two representations \( \pi \) and \( \pi' \) of the Lie superalgebra \( g \) and that these implementations respect the \( \mathbb{Z}_2 \) gradings of the representation spaces. Then it can also be implemented in the tensor product \( \pi \otimes \pi' \).

**Proof:** We denote the implementations of the outer automorphism \( \gamma \) in the representations \( \pi \) and \( \pi' \) by \( U_\pi(\gamma) \) and \( U_{\pi'}(\gamma) \), respectively. Then the implementation in the tensor product is trivially given by \( U_{\pi \otimes \pi'}(\gamma) = U_\pi(\gamma) \otimes U_{\pi'}(\gamma) \). \( \square \)

**Lemma 2.** Let \( \pi \) be a representation of \( g \) in which the outer automorphism \( \gamma \) is implemented by \( U_\pi(\gamma) \). Let furthermore \( \pi' \) be a subrepresentation that is invariant under the action of \( U_\pi(\gamma) \), i.e. \( U_\pi(\gamma)H_{\pi'} \subset H_{\pi'} \). Then the action of \( \gamma \) is implementable in the factor representation \( \pi/\pi' \).

**Proof:** The statement is obvious. \( \square \)

**Lemma 3.** Suppose that the action of an outer automorphism \( \gamma \) can be implemented in a representation \( \pi \) of the Lie superalgebra \( g \) by the implementation map \( U_\pi(\gamma) : H_\pi \to H_\pi \) and let \( \pi' \) be the socle of \( \pi \). Then \( U_\pi(\gamma)|_{H_{\pi'}} \) is an implementation of \( \gamma \) in \( \pi' \), i.e. \( U_\pi(\gamma)H_{\pi'} \subset H_{\pi'} \).
Proof: By definition, the socle $H_{\pi'}$ is the maximal semisimple submodule of $H_{\pi}$. Its image $U_{\pi}(\gamma)H_{\pi'} \subset H_{\pi}$ carries an action of the Lie superalgebra. The corresponding subrepresentation is given by $\pi' \circ \gamma$ and hence it decomposes into a sum of irreducibles just like $\pi'$ itself. In other words, the subspace $U_{\pi}(\gamma)H_{\pi'}$ is a semisimple submodule of $H_{\pi}$. From the maximality of the socle we therefore conclude $H_{\pi'} = U_{\pi}(\gamma)H_{\pi'}$. \qed

While the previous lemmas hold in full generality for all Lie superalgebras, the next one relies on special properties of the Lie superalgebra $\mathfrak{psl}(2|2)$ and its class of $\text{SL}(2,\mathbb{C})$ automorphisms $\gamma_u$ as defined in (2.2).

**Lemma 4.** Let $\pi$ be a representation of $\mathfrak{g} = \mathfrak{psl}(2|2)$ in which an outer automorphism $\gamma_u$ of the form (2.2) is implemented by $U_{\pi}$ and let $\pi'$ be an irreducible subrepresentation on the space $H_{\pi'}$. Then $\pi$ defines a subrepresentation on the space $U_{\pi}H_{\pi'}$ and the latter is isomorphic to $\pi'$.

*Proof.* For any vector $v \in H_{\pi'}$, we find $\pi(x)U_{\pi}v = U_{\pi}\pi \circ \gamma^{-1}(x)v \subset U_{\pi}H_{\pi'}$, hence $U_{\pi}H_{\pi'}$ is invariant. To see that the resulting representation is isomorphic to the original one we just need to realize a) that it is also irreducible, b) that the dimension agrees and c) that the weights and weight multiplicities coincide. While b) is obvious, c) holds because $\gamma_u$ acts trivially on the bosonic generators and especially on the Cartan elements. Assume now that there exists a proper subrepresentation on the space $H_{\pi''} \subset U_{\pi}H_{\pi'}$. Then, using the arguments of the first line above, we find a proper subrepresentation of the original representation $\pi'$ on the space $U_{\pi}^{-1}H_{\pi''} \subset H_{\pi'}$, contradicting its irreducibility. \qed

As simple as these statements are, they will be rather useful for our analysis of tensor products. They imply in particular, that all the indecomposable representations that arise in tensor products of irreducibles, and hence all the representations we discuss in this note, allow for an implementation of the outer automorphisms. This insight is particularly useful when we analyze the internal structure of indecomposable composites. In fact, if the action of outer automorphisms is implementable in an indecomposable, then it is implementable in its socle and the associated factor representation. Hence, the transformation properties under the group $\text{SL}(2,\mathbb{C})$ respect the structure of such indecomposable representations, i.e. they organize each floor of their decomposition diagram into $\text{SL}(2,\mathbb{C})$ multiplets of irreducible representations. For the indecomposables that appear in this text, the multiplicities are displayed in appendix A.
Decomposition with respect to $\mathfrak{sl}(2|1)$. In this paragraph we shall study a particular embedding of $\mathfrak{h} = \mathfrak{sl}(2|1)$ into $\mathfrak{g}$ and explain how the irreducible representations of $\mathfrak{g}$ decompose into representations of $\mathfrak{h}$. We do not intend to present a complete introduction into $\mathfrak{sl}(2|1)$ here, but restrict to a short list of relevant notations. More details can be found in the standard literature (see e.g. [10]) and in [14].

The even subalgebra $\mathfrak{h}^{(0)}$ of $\mathfrak{sl}(2|1)$ is given by the sum $\mathfrak{gl}(1) \oplus \mathfrak{sl}(2)$. Consequently, the Kac modules are labeled by pairs $(b, j)$ where $b \in \mathbb{C}$ and $j = 1/2, 1, \ldots$. We shall denote these representations by

$$\{b, j\} := \text{Ind}_{\mathfrak{h}^{(0)} \oplus \mathfrak{h}^{(1)}}^{\mathfrak{h}(0) \oplus \mathfrak{h}(1)} V_{(b-1/2, j-1/2)} = U(\mathfrak{h}) \otimes U(\mathfrak{h}) V_{(b-1/2, j-1/2)}.$$ 

The shift of the labels in our notations will turn out to be rather convenient in the following. We denote the Kac modules of $\mathfrak{h}$ by brackets $\cdot \cdot$ to distinguish them from those of $\mathfrak{g}$. For $b \neq \pm j$, the Kac modules $\{b, j\}$ give rise to irreducible representations of dimension $d_{\{b, j\}} = 8j$. Kac modules $\{b, j\}$ with $b = \pm j$ are indecomposable. Their structure can be encoded in the following short diagram

$$\{\pm j, j\} : \{j\} \pm \longrightarrow \{j - \frac{1}{2}\} \pm.$$ 

Here, $\{j\}_\pm$ denote the $(4j + 1)$-dimensional irreducible atypical representations of $\mathfrak{h}$.

The last type of representations that we shall need below are the projective covers $\mathcal{P}^\pm_{\mathfrak{h}}(j)$ of the atypical representations $\{j\}_\pm$. For $j \neq 0$, their structure is encoded in the following picture

$$\mathcal{P}^\pm_{\mathfrak{h}}(j) : \{j\}_\pm \longrightarrow \{j + \frac{1}{2}\}_\pm \oplus \{j - \frac{1}{2}\}_\pm \longrightarrow \{j\}_\pm.$$ 

(3.12)

These spaces are $16j + 4$ dimensional as one can easily check by adding up the dimensions of the atypical composition series. The projective cover of the trivial representation $\{0\} = \{0\}_\pm$ is an 8-dimensional representation that is given by

$$\mathcal{P}_{\mathfrak{h}}(0) : \{0\} \longrightarrow \{\frac{1}{2}\}_+ \oplus \{\frac{1}{2}\}_- \longrightarrow \{0\}.$$ 

(3.13)

Results for tensor products of these representations are spelled out in appendix B. Tensor products of irreducible representations were originally computed in [11] and results for tensor products of irreducible with the projective representations $\mathcal{P}^\pm_{\mathfrak{h}}(n)$ can be found in [14].
What we shall need in the following are explicit formulas for the decomposition of irreducible \( g = \mathfrak{psl}(2|2) \) representations with respect to the subalgebra \( h = \mathfrak{sl}(2|1) \). Obviously, these depend on the explicit choice of the embedding. Here we shall consider the case in which the central element \( Z \in h^{(0)} \) of the even part in \( \mathfrak{sl}(2|1) \) is identified with the Cartan element of the first \( \mathfrak{sl}(2) \) subalgebra of \( g^{(0)} \). With this choice we find

\[
[n]_h \cong \{n\}_+ \oplus \{n\}_- \oplus \mathcal{E}([n])
\]

where

\[
\mathcal{E}([n]) = \bigoplus_{b=-n+1/2}^{n-1/2} \{b, n + \frac{1}{2}\} .
\]

Below we shall think of \( \mathcal{E} \) as a map that sends any sum of atypical \( g \)-representations to a sum of typical representations for \( h \). For typical representations \([i, j]\) of \( g \) we shall first assume that \( j > i \). The other case will be treated below.

\[
[i, j]_h \cong \bigoplus_{b=-i}^{i} \left( \{b, j + 1\} \oplus (1 - \delta_{j,0})\{b, j\} \oplus \{b - \frac{1}{2}, j + \frac{1}{2}\} \oplus \{b + \frac{1}{2}, j + \frac{1}{2}\} \right) .
\]

Note that with our assumption \( i < j \) only typical representations occur on the right hand side. The formulas also holds true for \( i > j \) and \( i + j \) a half-integer. When \( i > j \) and \( i + j \) integer, on the other hand, the formula must be modified and the decomposition turns out to contain some of the projective covers \( \mathcal{P}^{\pm}_h(j) \) of atypical representations,

\[
[i, j]_h \cong \bigoplus_{b=-i}^{i} \left( \{b, j + 1\} \oplus (1 - \delta_{j,0})\{b, j\} \oplus \{b - \frac{1}{2}, j + \frac{1}{2}\} \oplus \{b + \frac{1}{2}, j + \frac{1}{2}\} \right) \oplus \bigoplus_{\nu=\pm} \left( \mathcal{P}_h^{\nu}(j) \oplus \mathcal{P}_h^{\nu}(j + 1/2) \right) .
\]

where the symbol \( \oplus' \) instructs us to omit all terms that would formally be associated with atypical representations. In addition, we shall agree throughout this paper to replace the sum \( \mathcal{P}^{+}_h(0) \oplus \mathcal{P}^{-}_h(0) \) by \( \mathcal{P}_h(0) \) whenever it appears. This is purely formal and has no meaning in terms of representations. Note that the left hand side only contains projective representations. Let us finally spell out the decomposition formulas for the projective covers and the indecomposables \( \pi^{\text{indec}}_{i \otimes j} \) which we defined in eq. (3.9, 3.10),

\[
\mathcal{P}_h(j)|_h \cong \bigoplus_{\nu=\pm} \left( \mathcal{P}_h^{\nu}(j + \frac{1}{2}) \oplus 2 \cdot \mathcal{P}_h^{\nu}(j) \oplus \mathcal{P}_h^{\nu}(|j - \frac{1}{2}|) \right) \oplus \mathcal{E} \circ \mathcal{S}(\mathcal{P}_h(j)) .
\]

\[
\pi^{\text{indec}}_{i \otimes j}|_h \cong \bigoplus_{\nu=\pm} \left( \{|i - j|\}_\nu \oplus \{i + j\}_\nu \oplus \bigoplus_{p=|i - j| + \frac{1}{2}} \mathcal{P}_h(p) \right) \oplus \mathcal{E} \circ \mathcal{S}(\pi^{\text{indec}}_{i \otimes j}) .
\]

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These formulas also hold for the special cases of \( j = 0 \) and \( i = j \) if we agree to replace \( \{0\}_+ \oplus \{0\}_- \) by \( \{0\} \). Once more such a replacement is purely formal and has no meaning in terms of representations. We note that it is rather easy to infer the projective covers \( P_\mathfrak{h} \) in the two decomposition formulas from our planar pictures for \( P_\mathfrak{h} \) and the indecomposables \( \pi^{\text{inde}} \). This concludes the presentation of the background material that is needed in the proof of our decomposition formulas for tensor products of \( \mathfrak{g} \).

### 3.3 Proof of the decomposition formulas - examples

Rather than trying to go through the general proof of our formulas we would like to illustrate the main ideas in two rather representative examples. These are the tensor product of the atypical representation \([1]\) with itself and with \([0, 2]\).

**The tensor product** \([1] \otimes [1]\). Let us begin by collecting a few results on the 34-dimensional atypical representation \([1]\) of \( \mathfrak{g} \). With respect to the embedded \( \mathfrak{h} \), this representation decomposes as follows

\[
[1]_\mathfrak{h} \cong \{1\}_+ \oplus \{1\}_- \oplus \{\frac{1}{2}, \frac{3}{2}\} \oplus \{-\frac{1}{2}, \frac{3}{2}\}
\]

Using standard results (see appendix B) about tensor products of irreducible representations of \( \mathfrak{h} \), we obtain the following decomposition formula for the tensor product \([1] \otimes [1]\) in terms of representations of \( \mathfrak{h} \),

\[
([1] \otimes [1])_\mathfrak{h} \sim 2 \cdot \{0\} \oplus 2 \cdot P_\mathfrak{h}(0) \oplus \bigoplus_{\nu=\pm} (\{2\}_\nu \oplus 3 \cdot P_\mathfrak{h}(\frac{1}{2}) \oplus P_\mathfrak{h}(1) \oplus 2 \cdot P_\mathfrak{h}(\frac{3}{2})) + \ldots \tag{3.19}
\]

where the dots stand for a sum of typical \( \mathfrak{h} \) representations. The latter will not play any role for the following analysis.

It is also easy to find the typical \( \mathfrak{g} \) representations in the tensor product of \([1]\) with itself,

\[
([1] \otimes [1])^{\text{typ}} = [0, 1] \oplus [0, 2] \oplus [1, 0] \oplus [1, 2] \oplus [2, 0] \oplus [2, 1]
\]

\[
([1] \otimes [1])^{\text{typ}}_\mathfrak{h} \sim 2 \cdot P_\mathfrak{h}(0) \oplus \bigoplus_{\nu=\pm} (2 \cdot P_\mathfrak{h}(\frac{1}{2}) \oplus P_\mathfrak{h}(1) \oplus P_\mathfrak{h}(\frac{3}{2})) + \ldots \tag{3.20}
\]

When we passed to the second line, we have inserted the results from our decomposition formulas for typical \( \mathfrak{g} \) representations with respect to the embedded \( \mathfrak{h} \). Once more, we have omitted all typical \( \mathfrak{h} \) representations.
A comparison of eqs. (3.19) and (3.20) gives the following intermediate result,

\[
\left( [1] \otimes [1] - ([1] \otimes [1])^{\text{typ}} \right) \bigg|_{\mathfrak{h}} \sim 2 \cdot \{0\} \oplus \bigoplus_{\nu = \pm} \left\{ \{2\}_\nu \oplus \mathcal{P}_\nu^{(1/2)}(\mathfrak{h}) \oplus \mathcal{P}_\nu^{(3/2)}(\mathfrak{h}) \right\} + \ldots
\]

The atypical $\mathfrak{h}$ representations that remain must come from the decomposition of the $\mathfrak{g}$ indecomposables that appear in the tensor product of [1]. A short glance on our decomposition formulas (3.14) and (3.18) confirms that this is indeed the case.

We can actually use this example to derive the structure of the indecomposable representation $\pi_{1 \otimes 1}$ from the information on its restriction to $\mathfrak{h}$. In fact, it is rather easy to see that the composition series of $\pi_{1 \otimes 1}$ contains the following list of atypical representations, each displayed with a multiplicity that refers to its transformation properties under the action of the SL(2, C) outer automorphisms$^2$

\[
\{0\}_1 , \{0\}_3 , \{2\}_{[\frac{1}{2}]}_1 , \{1\}_2 , \{2\}_{[\frac{3}{2}]}_1 , \{2\}_2 .
\]

A moment of thought reveals that all but the $\{0\}_1$ representation must be part of one single indecomposable in order to be able to recover our knowledge about the decomposition with respect to $\mathfrak{h}$. It is at this point where our assignment of multiplicities becomes crucial. In fact, our knowledge from the $\mathfrak{h}$ embedding would e.g. have been consistent with including two of the four trivial $\text{psl}(2|2)$ representations into the indecomposable. But since there is no doublet, we were forced to include the triplet. Hence, there is only a single irreducible representation $[0]$ left. In other words, the tensor product $[1] \otimes [1]$ contains only one true invariant. The presence of additional states which transform trivially but sit in the indecomposables has to be contrasted with the case of ordinary simple Lie algebras where the tensor product of a representation with its conjugate contains precisely one such state.

**The tensor product $[1] \otimes [0, 2]$.** With respect to the embedded $\mathfrak{h} \subset \mathfrak{g}$, the typical $\mathfrak{g}$ representation $[0, 2]$ decomposes as follows

\[
\{0, 2\} \big|_{\mathfrak{h}} \cong \{0, 3\} \oplus \{0, 2\} \oplus \{\frac{1}{2}, \frac{5}{2}\} \oplus \{-\frac{1}{2}, \frac{5}{2}\} .
\]

Using again standard results about tensor products of irreducible representations of $\mathfrak{h}$ we obtain the following decomposition formula for the tensor product $[1] \otimes [0, 2]$ in terms of

$^2$The subscripts may be determined from the SL(2, C) transformation properties of the bosonic multiplets in the involved tensor factors. The latter are listed in appendix A.
representations of $\mathfrak{h}$,

$$
([1] \otimes [0, 2])|_{\mathfrak{h}} \sim \bigoplus_{\nu = \pm} (\mathcal{P}_{\mathfrak{h}}^\nu(\frac{1}{2}) \oplus 2 \cdot \mathcal{P}_{\mathfrak{h}}^\nu(1) \oplus \mathcal{P}_{\mathfrak{h}}^\nu(\frac{3}{2})) + \ldots
$$

(3.21)

where the dots stand for a sum of typical $\mathfrak{h}$ representations as in the last example. One can easily find the typical $\mathfrak{g}$ representations in the tensor product of $[1]$ with $[0, 2]$,

$$
([1] \otimes [0, 2])^{\text{typ}} = [\frac{1}{2}, \frac{1}{2}] \oplus [\frac{1}{2}, \frac{3}{2}] \oplus [\frac{1}{2}, \frac{5}{2}] \oplus 2[1, 2] \oplus 2[1, 3] \oplus [\frac{3}{2}, \frac{3}{2}] .
$$

(3.22)

We observe that none of these representations contributes any indecomposable upon restriction to the embedded $\mathfrak{h}$. Hence, we obtain

$$
\left( [1] \otimes [0, 2] - ([1] \otimes [0, 2])^{\text{typ}} \right)|_{\mathfrak{h}} \sim \bigoplus_{\nu = \pm} (\mathcal{P}_{\mathfrak{h}}^\nu(\frac{1}{2}) \oplus 2 \cdot \mathcal{P}_{\mathfrak{h}}^\nu(1) \oplus \mathcal{P}_{\mathfrak{h}}^\nu(3/2)) + \ldots
$$

(3.23)

According to eq. (3.17), this particular sum of projective $\mathfrak{h}$ representations comes from the decomposition of the projective cover $\mathcal{P}_\mathfrak{g}(1)$. Hence, we conclude that the latter appears in the tensor product of $[1]$ with $[0, 2]$, in agreement with our proposition 1.

Before we conclude, we would like to point out that the we can read off the internal structure of $\mathcal{P}_\mathfrak{g}(1)$ from the information on its restriction to $\mathfrak{h}$. In fact one can see rather easily that the composition series of the indecomposables in the tensor product of $[1]$ and $[0, 2]$ contains the following list of atypical representations, each displayed with a multiplicity that refers to its transformation properties on the action of the outer automorphisms,$^3$

$$
[0]_2 , 2[\frac{1}{2}][2] , [1]_3 , 3[1]_3 , 2[\frac{1}{2}][2] , [2]_1 .
$$

Our planar picture for $\mathcal{P}_\mathfrak{g}(1)$ provides the unique pattern in which we can form a composite from these constituents that is consistent with the information (3.23) on the restriction to the subalgebra $\mathfrak{h}$.

### 4 Conclusions and Outlook

In this note we have succeeded to decompose all tensor products between finite dimensional irreducible and projective representations of $\mathfrak{psl}(2|2)$. Whereas tensor products involving at least one projective representation were shown explicitly to stay within the class of projectives, we have constructed a new family of finite dimensional indecomposable

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$^3$The relevant data that allow to determine the multiplicities can be found in appendix A.
representations that appear in the tensor product of atypicals. Preliminary investigations show that tensor products of these new indecomposables $\pi_i \otimes \pi_j$ with atypicals generate yet another family of representations whose structure resembles the one of $\pi_i \otimes \pi_j$, though with different multiplicities of the involved atypical building blocks. Since the applications we have in mind only require tensor products in which at least one factor is projective, we have not pushed our investigations further into this direction. We believe, however, that results can be obtained using the techniques we have developed above. Even though indecomposables of $\mathfrak{psl}(2|2)$ cannot be classified, it may well be possible to classify all those representations that arise in multiple tensor products of irreducibles. In fact, according to section 3.2, the latter admit an implementation of the $\text{SL}(2,\mathbb{C})$ outer automorphisms and therefore they form a rather distinguished sub-class of representations.

The techniques we have used here may also be applied to other Lie superalgebras of the A-series, in particular to $\mathfrak{psl}(n|n)$. A promising approach would be to address $\mathfrak{sl}(n|1)$ first and then to advance to non-trivial second label. Obviously, the structure of the representation theory, becomes much richer for larger Lie superalgebras, in particular because multiply atypical representations can occur (see, e.g., [15] and references therein). Some partial results in this direction will be published elsewhere.

We finally want to sketch at least one concrete physics problem to which we hope to apply the rather mathematical results of this note. During recent years, non-linear $\sigma$-models on supergroups and supercosets have surfaced in a variety of distinct problems and in particular through studies of string theory in certain RR backgrounds [16, 1, 2]. Many specific and important properties of these models, such as e.g. the possible existence of conformal invariance even in the absence of a Wess-Zumino term, originate from peculiar features of the underlying Lie superalgebra [17, 18].

Since the isometries of $AdS_3 \times S^3$ are generated by two copies of the even subalgebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2)$ of the non-compact real form $\mathfrak{psu}(1, 1|2)$ of $\mathfrak{psl}(2|2)$, it is not hard to believe that the corresponding $\sigma$-models enter the description of strings in an $AdS_3$ background [1, 2]. In fact, for strings moving in the presence of a pure NSNS background field the physics is described by a WZW model for $\mathfrak{psl}(2|2)$. This theory possesses a holomorphic and antiholomorphic $\mathfrak{psl}(2|2)$ current symmetry and it may be solved exactly after decoupling [19, 2] the bosons and the fermions, using results established in [20, 21, 22, 23, 24] and references therein. After the marginal deformation which arises from turning on a RR
background field, however, the local (worldsheet) symmetries of the system are reduced drastically \cite{17} and so far no solution has been found, in spite of the significant interest in such models (see also e.g. \cite{3}).

In a first step one may hope to determine the exact spectra of theories with $AdS_3 \times S^3$ target as a function of the strength of the RR flux. Results in \cite{17} imply that states which transform according to the same representation of the remaining global $\mathfrak{psl}(2|2) \oplus \mathfrak{psl}(2|2)$ symmetry experience the same energy shift as we switch on the RR background field. This motivates to classify all string states according to their behavior under the action of $\mathfrak{psl}(2|2)$. Such states arise by application of creation operators on certain “ground states” in the theory. At the WZW-point, these creation operators are the negative modes of the $\mathfrak{psl}(2|2)$ currents. The latter transform in the adjoint representation of $\mathfrak{psl}(2|2)$. Hence, in order to determine the transformation properties of excited states, we must control tensor powers (and the symmetric parts therein) of the adjoint representation. This is exactly where the results of the present note feed into studies of strings in $AdS_3$. Such an analysis is beyond the scope of this work but we plan to come back to these issues in a forthcoming publication.

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Appendix: SL(2, C) multiplicities

As it is mentioned in the main text, the implementation of the outer automorphisms helps a lot in understanding the decomposition of tensor products. In this appendix we would like to explain how the SL(2, C) action organizes representations into multiplets and list explicit results for all the indecomposables we are interested in.

Let us recall that the SL(2, C) automorphisms act trivially on the bosonic subalgebra. Hence, each representation in which these automorphisms are implemented may be decomposed into a sum of

$$(i, j) \otimes V^J \cong (i, j)_n$$

where $n = 2J + 1$,

$(i, j)$ are representations of the even subalgebra and $V^J$ carries an action of the automorphism group SL(2, C). Since all our representations are assumed to be finite dimensional, the same must be true for $V^J$. This means that only SL(2, C) representations with half-integer spin $J$ and dimension $n = 2J + 1$ can arise.

With the previous remarks in mind we can now move ahead and analyse how the various representations that appeared in the main text decompose into the building blocks $(i, j)_n$. We shall restrict our explicit lists here to the irreducible representations. Let us begin with the generic typical irreducible representations for $i, j > 1/2$:

$$[i, j] = (i, j)_3 = (i + 1, j)_1$$

$$(i + \frac{1}{2}, j + \frac{1}{2})_2 (i + 1, j)_1$$

$$(i + \frac{1}{2}, j - \frac{1}{2})_2 (i, j + 1)_1$$

$$(i - \frac{1}{2}, j + \frac{1}{2})_2 (i, j)_1$$

$$(i - \frac{1}{2}, j - \frac{1}{2})_2 (i - 1, j)_1$$

$$(i, j - 1)_3$$

(A.1)

In case one of the labels $i, j$ is equal to $1/2$, the generic decomposition gets reduced. When $j = 1/2$ and $i > 1/2$ we find

$$(i + \frac{1}{2}, 1)_2 (i + 1, \frac{1}{2})_1$$

$$(i + \frac{1}{2}, 0)_2 (i, \frac{3}{2})_1$$

$$(i - \frac{1}{2}, 1)_2 (i, \frac{1}{2})_1$$

$$(i - \frac{1}{2}, 0)_2 (i - 1, \frac{1}{2})_1$$

(A.2)
Obviously, the case of \( i = 1/2 \) and \( j > 1/2 \) is analogous. The series of representations with \( j = 0, i > 1/2 \), possesses an even shorter picture

\[
[i, 0] = (i, 0)_3 \quad (i + \frac{1}{2}, \frac{1}{2})_2 \quad (i, 1)_1 \quad (i - \frac{1}{2}, \frac{1}{2})_2 \quad (i - 1, 0)_1 .
\]

The last typical representation that remains to be treated is the case of \( i = 1/2 \) and \( j = 0 \) for which one finds

\[
\left[ \frac{1}{2}, 0 \right] = \left( \frac{1}{2}, 0 \right)_3 \quad (1, \frac{1}{2})_2 \quad (\frac{1}{2}, 0)_1 \quad (0, \frac{1}{2})_2 \quad (\frac{1}{2}, 1)_1 .
\]

Now we can turn to the irreducible atypical representations with \( j \geq 1/2 \),

\[
[j] = (j, j)_2 \quad (j + \frac{1}{2}, j - \frac{1}{2})_1 \quad (j - \frac{1}{2}, j + \frac{1}{2})_1 .
\]

The representation \([0]\) is trivially given by \((0, 0)_1\). This concludes our list of irreducible representations. Similarly, we could now analyse all those indecomposables which allow for an implementation of the \( \text{SL}(2, \mathbb{C}) \) automorphisms. Let us stress, that the requirement of implementability is not fulfilled for the Kac modules \([j, j]\). Hence, the above formulas should only be used for \( i \neq j \).

In subsection 3.2 we argued that the atypical constituents of the indecomposables \( P_g(j) \) and \( \pi_{i \otimes j} \) are organized in multiplets of \( \text{SL}(2, \mathbb{C}) \). For such multiplets we shall employ the symbol \([j]_m\). Note that the decomposition of \([j]_m\) in terms of \( g^{(0)} \oplus sl(2) \) representations is obtained from the corresponding decomposition of \([j]\) (see above) by tensoring with the \( \text{SL}(2, \mathbb{C}) \) representation \( V^I \) where \( m = 2I + 1 \). In case of the projective covers \( P(j), j \neq 0 \),
one finds the following structure,

\[ \mathcal{P}_g(j) : \]

\[
\begin{array}{c}
\downarrow \quad [j]_{1} \\
\downarrow \quad [j + 1/2]_{2} \quad [j + 1/2]_{2} \\
\downarrow \quad [j]_{3,1} \quad [j]_{1} \quad [j]_{1} \\
\downarrow \quad [j - 1/2]_{2} \quad [j - 1/2]_{2} \\
\downarrow \quad [j - 1]_{1}
\end{array}
\]

By \([j]_{3,1}\) we mean that there is one triplet \([j]_{3}\) and one singlet \([j]_{1}\). This result can be verified by the explicit decomposition of the tensor product between \([1/2]\) and \([j + 1/2, j - 1/2]\) in which, up to typicals, exactly one projective cover \(\mathcal{P}(j)\) appears.

\(\mathcal{P}(0)\) has to be treated separately. Its structure is encoded in a picture of the form

\[ \mathcal{P}_g(0) : \]

\[
\begin{array}{c}
\downarrow \quad [1]_{2} \\
\downarrow \quad [1/2]_{3} \\
\downarrow \quad [0]_{1} \quad [1/2]_{3} \\
\downarrow \quad [0]_{5,1} \\
\downarrow \quad [0]_{1}
\end{array}
\]

Finally, we also want to list the multiplicities for the indecomposables \(\pi_{i \otimes j}\) that arise in tensor products of atypicals. For these representations we find

\[ \pi^{\text{indec}}_{i \otimes j} : \]

\[
\begin{array}{c}
\downarrow \quad [i + j]_{2} \\
\downarrow \quad [i + j - 1/2]_{1} \\
\downarrow \quad [i - j + 1/2]_{1} \\
\downarrow \quad [i - j]_{2}
\end{array} \quad \begin{array}{c}
\downarrow \quad [i + j - 1/2]_{1} \\
\downarrow \quad [i - j + 1/2]_{1} \\
\downarrow \quad [i - j]_{2} \\
\downarrow \quad [i - j]_{2}
\end{array}
\]

\[ \pi^{\text{indec}}_{j \otimes j} : \]

\[
\begin{array}{c}
\downarrow \quad [2j]_{2} \\
\downarrow \quad [2j - 1/2]_{1} \\
\downarrow \quad [1/2]_{1} \\
\downarrow \quad [0]_{3}
\end{array}
\]

\[ \pi^{\text{indec}}_{i \otimes j} : \]

\[
\begin{array}{c}
\downarrow \quad [i + j]_{2} \\
\downarrow \quad [i + j - 1/2]_{1} \\
\downarrow \quad [i - j + 1/2]_{1} \\
\downarrow \quad [i - j]_{2}
\end{array} \quad \begin{array}{c}
\downarrow \quad [i + j - 1/2]_{1} \\
\downarrow \quad [i - j + 1/2]_{1} \\
\downarrow \quad [i - j]_{2} \\
\downarrow \quad [i - j]_{2}
\end{array}
\]

\[ \pi^{\text{indec}}_{j \otimes j} : \]

\[
\begin{array}{c}
\downarrow \quad [2j]_{2} \\
\downarrow \quad [2j - 1/2]_{1} \\
\downarrow \quad [1/2]_{1} \\
\downarrow \quad [0]_{3}
\end{array}
\]

23
B Appendix: Tensor products for $\mathfrak{sl}(2|1)$

In order to list results on the tensor products of $\mathfrak{sl}(2|1)$ representations, we would like to introduce a map $\pi_\mathfrak{h}$ which sends representations of the bosonic subalgebra $\mathfrak{h}^{(0)}$ to typical representations of $\mathfrak{h}$. Its action on irreducibles is given by

$$\pi_\mathfrak{h}(b - \frac{1}{2}, j - \frac{1}{2}) = \begin{cases} \{b, j\} & \text{for } b \neq \pm j , \\ 0 & \text{for } b = \pm j . \end{cases} \quad (B.8)$$

The map $\pi_\mathfrak{h}$ may be extended to a linear map on the space of all finite dimensional representations of $\mathfrak{h}^{(0)}$.

The first tensor product we would like to display is the one between two typical representations $[11]$. In our new notations, the decomposition is given by

$$\{b_1, j_1\} \otimes \{b_2, j_2\} = \pi_\mathfrak{h}((b_1 - \frac{1}{2}, j_1 - \frac{1}{2}) \otimes \{b_2, j_2\} = \begin{cases} \mathcal{P}_\mathfrak{h}(\pm|b_1 + b_2| \mp \frac{1}{2}) & \text{for } b_1 + b_2 = \pm j_1 + j_2 \\
\mathcal{P}_\mathfrak{h}^\pm(|b_1 + b_2|) \oplus \mathcal{P}_\mathfrak{h}^\pm(|b_1 + b_2| - \frac{1}{2}) & \text{for } b_1 + b_2 \in \pm\{|j_1 - j_2| + 1, \ldots, j_1 + j_2 - 1\} \\
\mathcal{P}_\mathfrak{h}(\pm|b_1 + b_2|) & \text{for } b_1 + b_2 = \pm|j_1 - j_2| . \end{cases} \quad (B.9)$$

Note that neither $j_1$ nor $j_2$ can vanish so that the three cases listed above are mutually exclusive. The first term computes all the typical representations that appear in the tensor product. All it requires is the decomposition of typical $\mathfrak{h}$ representations into irreducibles of the bosonic subalgebra,

$$\{b, j\} = \begin{cases} (b, j) \oplus (b + \frac{1}{2}, j - \frac{1}{2}) \oplus (b - \frac{1}{2}, j - \frac{1}{2}) \oplus (b, j - 1) . \end{cases}$$

and a computation of tensor products for representations of $\mathfrak{h}^{(0)} = \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)$ which presents no difficulty. The outcome is then converted into a direct sum of typical representations through our map $\pi_\mathfrak{h}$.

Tensor products of typical with atypical representations can also be found in Marcu’s paper. The results are

$$\{b_1, j_1\} \otimes \{j_2\} = \pi_\mathfrak{h}((b_1 - \frac{1}{2}, j_1 - \frac{1}{2}) \otimes \{j_2\} = \begin{cases} \mathcal{P}_\mathfrak{h}^\pm(|b_1 \pm j_2| - \frac{1}{2}) & \text{for } b_1 \pm j_2 \in \pm\{|j_1 - j_2| + 1, \ldots, j_1 + j_2\} \\
\mathcal{P}_\mathfrak{h}^\pm(|b_1 \pm j_2|) & \text{for } b_1 \pm j_2 \in \pm\{|j_1 - j_2|, \ldots, j_1 + j_2 - 1\} . \end{cases}$$

24
This formula may be used to determine the tensor product of typical representations with projective covers and other indecomposables. These tensor products are simply given by

$$\{b, j\} \otimes \mathcal{R} \cong \{b, j\} \otimes S_h(\mathcal{R}) \quad .$$

(B.11)

Here, the maps $S_h$ is defined such that it sends the indecomposable representation $\mathcal{R}$ to a direct sum of its atypical building blocks, i.e.

$$S_h(\mathcal{R}) = \bigoplus_{\pm j} [\mathcal{R} : \{j\}_\pm] \{j\}_\pm$$

(B.12)

Here, $[\mathcal{R} : \{j\}_\pm]$ counts how many times the atypical representation $\{j\}_\pm$ appears in the composition series of $\mathcal{R}$. In the special case of $\mathcal{R} = \mathcal{P}_h^\pm(j)$, these numbers may be read off from the diagrams (3.12) and (3.13).

Having gone through the entire list of products which involve at least one typical factor, we would like to turn to the fusion of projective covers with any other type of representation. The tensor product between a projective cover $\mathcal{P}_h^\pm(j)$ and an atypical representation $\{n\}_\pm$ with $n > 0$ is given by

$$\mathcal{P}_h^\pm(j) \otimes \{n\} = \pi_h(H_j^\pm \otimes \{n\}_{h(0)}) \oplus \mathcal{P}_h(\pm j + n)$$

where

$$H_j^\pm = (\pm j - \frac{1}{2}, j - \frac{1}{2}) \oplus (\pm (j + \frac{1}{2}) - \frac{1}{2}, j)$$

(B.13)

and $H_0^\pm = H_0 = (0, 0) \oplus (-1, 0)$. We also set $\{|n\}_\pm = \{|n|\}_\pm$. As before, we can exploit this formula further to determine all tensor products between a projective cover and an indecomposable composite of atypical representations. The corresponding formula employs our symbol $S_h$ (see eq. (B.12)),

$$\mathcal{P}_h^\pm(j) \otimes \mathcal{R} \cong \mathcal{P}_h^\pm(j) \otimes S_h(\mathcal{R}) \quad .$$

(B.14)

As an application, we can spell out the tensor product between two projective covers $\mathcal{P}_h^\pm(j_1), j_1 \geq 0$, and $\mathcal{P}_h(j_2) = \mathcal{P}_h^{\text{sign} j_2(|j_2|)},$

$$\mathcal{P}_h^\pm(j_1) \otimes \mathcal{P}_h(j_2) = \pi_h(H_{j_1}^\pm \otimes \mathcal{P}_h(j_2)|_{h(0)}) \oplus$$

$$\oplus \mathcal{P}_h(\pm j_1 + j_2 + \frac{1}{2}) \oplus 2 \cdot \mathcal{P}_h(\pm j_1 + j_2) \oplus \mathcal{P}_h(\pm j_1 + j_2 - \frac{1}{2})$$

(B.15)

The $h^{(0)}$ modules $H_j^\pm$ were defined in eq. (B.13). In the argument of $\pi_h$ the product $\otimes$ refers to the fusion between representations of the bosonic subalgebra $h^{(0)} = \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)$. 25
We are finally lacking a formula for the tensor product of two atypical representations. According to \cite{11}, such products are given by

\begin{equation}
\{j_1\}_\pm \otimes \{j_2\}_\pm = \{j_1 + j_2\}_\pm \oplus \bigoplus_{j = |j_1 - j_2|}^{j_1 + j_2 - 1} \{\pm(j_1 + j_2 + \frac{1}{2}), j + \frac{1}{2}\}, \quad (B.16)
\end{equation}

\begin{equation}
\{j_1\}_+ \otimes \{j_2\}_- = \{|j_1 - j_2|\}_{\text{sign}(j_1 - j_2)} \oplus \bigoplus_{j = |j_1 - j_2| + 1}^{j_1 + j_2} \{j_1 - j_2, j\}. \quad (B.17)
\end{equation}

Proofs for all these formulas can be found in \cite{11} and in our recent paper \cite{14}.

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