Predictions for Nonabelian Fine Structure Constants from Multicriticality

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Abstract

In developing a model for predicting the nonabelian gauge coupling constants we argue for the phenomenological validity of a “principle of multiple point criticality”. This is supplemented with the assumption of an “(grand) anti-unified” gauge group $SMG^{N_{gen.}} \sim U(1)^{N_{gen.}} \times SU(2)^{N_{gen.}} \times SU(3)^{N_{gen.}}$ that, at the Planck scale, breaks down to the diagonal subgroup. Here $N_{gen}$ is the number of generations which is assumed to be 3. According to this “multiple point criticality principle”, the Planck scale experimental couplings correspond to multiple point couplings of the bulk phase transition of a lattice gauge theory (with gauge group $SMG^{N_{gen.}}$). Predictions from this principle agree with running nonabelian couplings (after an extrapolation to the Planck scale using the assumption of a “desert”) to an accuracy of 7%. As an explanation for the existence of the multiple point, a speculative model using a glassy lattice gauge theory is presented.
1 Introduction

In the present article we describe a model that uses bulk phase transition values for the plaquette action parameters at the multiple point in the phase diagram for a Yang-Mills lattice gauge theory to calculate the values of the nonabelian gauge coupling constants for the standard model group ($SMG$) at the Planck scale. In our model, the $SMG$ comes about as the diagonal subgroup resulting from the Planck scale breakdown of our “anti-unified” gauge group. This “anti-unified” gauge group, on which the plaquette action of the lattice gauge theory is defined, is the cartesian product of a number $N_{gen}$ group factors each of which is the standard model group ($SMG$). The integer $N_{gen}$ designates the number of quark and lepton generations and is taken to have the value 3 in accord with experimental results. Hence our “anti-unified” gauge group is

\[
SMG \times SMG \times \cdots \times SMG \overset{\text{def.}}{=} SMG^{N_{gen}} = SMG^3.
\] (1)

So according to our model, the physical values of the $SMG$ nonabelian gauge couplings at the Planck scale are equal to the diagonal subgroup couplings corresponding to the multiple point action parameters of the Yang Mills lattice gauge theory having as the gauge group the “anti-unified” gauge group $SMG^3$.

The connection between our model and the accepted standard model description of fundamental physics is made at the Planck scale where we assume that our “anti-unified” gauge group $SMG^3$ is broken down to its diagonal subgroup

\[
(SMG^3G^3)_{\text{diag. subgr.}} \overset{\text{def.}}{=} \{(g, g, g) \mid g \in SMG\}
\] (2)

which is of course isomorphic to the $SMG$. The breakdown can come about due to ambiguities that arise under group automorphic symmetry operations. This is usually referred to as the “confusion” mechanism\[3, 4\].

In order to compare our Planck scale predictions for gauge coupling constants with experiment, we of course need to extrapolate experimental values to the Planck scale. We do this with a renormalization group extrapolation in which a “desert” scenario is assumed in going to the Planck scale.

It should be emphasized that our model for the prediction of gauge couplings using the “anti-unified” gauge group $SMG^3$ is incompatible with the currently popular $SU(5)$ or $SU(5)$ supersymmetric grand unified models and is therefore to be regarded as a rival to these.

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1°The choice of the Planck scale for the breaking of the (grand) “anti-unified” gauge group $SMG^3$ to its diagonal subgroup is not completely arbitrary insofar as gravity may in some sense be critical at the Planck scale. Also, our predictions are rather insensitive to variations of up to several orders of magnitude in the choice of energy at which the Planck scale is fixed.

2° The representation spectrum of the standard model suggests\[2\] using as the standard model group $SMG$ 

\[
SMG \overset{\text{def.}}{=} S(U(2) \times U(3)) \overset{\text{def.}}{=} \left\{ \begin{pmatrix} U_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & U_3 \end{pmatrix} \mid U_2 \in U(2), U_3 \in U(3), \det U_2 \cdot \det U_3 = 1 \right\}
\]

which has the same Lie algebra as the more commonly used covering group $U(1) \times SU(2) \times SU(3)$. 


Our suggestion that Nature seeks out the multiple point values of plaquette action parameters is formulated as “the principle of multiple point criticality”. The multiple point (critical) parameter values in a phase diagram for a Yang-Mills lattice gauge theory is the special point where all or at least a maximum number of the phases of the lattice gauge theory meet. The multiple point critical parameter values to which we refer are those that correspond to the bulk phase transition of a (Euclideanized) Yang-Mills lattice gauge theory with gauge group $SMG^3$. Drawing upon a suggestive analogy, the idea of a multiple point could be likened to the triple point of water which of course is unique in being the only point that is shared by the phase boundaries of all the phases of water (solid, liquid, and vapor) as depicted in the parameter space spanned by temperature and pressure.

By “phases”, we here refer to different physical behaviors that can be distinguished on the basis of the physics as viewed at a particular scale. We can think of the mean field approximation (MFA) which inherently makes phase determinations on the basis of the physics as seen from a very local point of view (e.g., the lattice scale)\[1]. Typically, the “phases” (at a scale) might be described in MFA in terms such as “Higg’s breakdown of a gauge group to a subgroup $K$, confinement-like (strong coupling) behavior w. r. t. an invariant subgroup $H \lhd K$ and Coulomb-like behavior w. r. t. the factor group $K/H$”; the symbol $\lhd$ denotes “is invariant subgroup of”. In this paper, we do not consider degrees of freedom with Higgs-like behavior (we limit ourselves to a few qualitative comments). This restriction amounts to the special case $SMG^{N_{gen.}} = K$ with one possible phase with confinement-like behavior for each invariant subgroup $H$ of the gauge group. In this special case, the Yang-Mills degrees of freedom corresponding to the remaining Lie subalgebra(s) are cosets of the factor group $SMG^{N_{gen.}}/H$ and behave as if in the “Coulomb phase”. Such a phase is said to be a “partially confined phase”; i.e., “confined” w. r. t. the invariant subgroup $H$.

In order to “see” all the possible partially confining phases (i.e., one for each invariant subgroup), we need to consider a class of plaquette actions general enough to provoke all these possible partially confining phases. Such a class of actions could, for example, consist of plaquette actions that can be expressed as a certain type of truncated character expansion. An alternative approach corresponds to the action ansatz used in this paper: we consider plaquette actions $S_\square$ that lead to distributions $e^{S_\square}$ consisting of narrow maxima centered at elements $p$ belonging to certain discrete subgroups of the center of the gauge group. The action at these peaks is then expressed as truncated Taylor expansions around the elements $p$. With this action ansatz, it is possible to provoke confinement-like or Coulomb-like behavior independently (approximately at least) for the 5 “constituent” invariant subgroups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $U(1)$, $SU(2)$, and $SU(3)$ of the $SMG$ which “span” the set of “all” invariant subgroups\[2] of the $SMG$. The mutually uncoupled variation in the distributions along these 5 “constituent” invariant subgroups is accomplished using 5 action parameters: 3 parameters $\beta_1$, $\beta_2$, and $\beta_3$ that allow adjustment of peak widths in the $U(1)$, $SU(2)$, and $SU(3)$ directions and 2 parameters $\xi_2$ and $\xi_3$ that provide for adjusting the relative heights of the peaks centered at elements $p \in \text{span}\{\mathbb{Z}_2, \mathbb{Z}_3\}$.

The phase transitions we are interested in are first order. Hence our model is a priori plagued by non-universality. However, our restriction on the form of the plaquette action nurtures the hope of at least an approximate universality.

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\[1\]We do not in this paper consider the infinity of invariant subgroups $\mathbb{Z}_N \subset U(1)$ for $N > 3$. 

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In order to explain why the multiple point should be realized in Nature, we propose a mechanism for the stability of this point that assumes a nonlocal lattice gauge glass model with a random plaquette action (e.g., we can assume quenched random values for character expansion parameters). By nonlocality we mean that action terms for Wilson loops of extent $A$ very large compared to the lattice constant $a$ are effectively nonlocal as seen from scales in the intermediate length range $[a, A]$. We define gauge couplings that run due to the inclusion of successively larger Wilson loops in going towards the infrared. The effect of these terms on the running of the couplings is describable by an extra term in the Callan-Symanzik $\beta$-function. The inclusion of these glassy nonlocal action terms in such a generalized $\beta$-function (really a multicomponent vector of generalized $\beta$-functions) is in addition to but opposite in sign to the normal Yang-Mills renormalization group contribution. We argue that rapid variations in the generalized $\beta$-functions at the multiple point can easily lead to zeros of these $\beta$-functions close to the multiple point. It is estimated that already at energies near the Planck scale the running plaquette action parameter values are presumably very close to those of the “infrared stable” fixed point zeros of the generalized beta functions which in turn are close to the multiple point.

An interesting suggestion arising from our principle of multiple point criticality is that it might offer an explanation for the smallness of the Weinberg-Salam Higgs mass compared say to the Planck scale because a principle of multiple point criticality, taken in an extended form, would suggest that the partially confining phases that are accessible at the multiple point by making infinitesimal changes in the multiple point coupling values should not only include Coulomb-like and confinement-like “phases”, but also Higgs-like “phases”. The sign change of the squared mass $m_{\text{Higgs}}^2$ of a Higgs particle at a phase boundary would imply that $m_{\text{Higgs}}^2$ is quite small (compared to the Planck scale) in the neighborhood of such a boundary (probably even zero if the transition is second order). The hierarchy problem might then be said to be Nature’s way of telling us that the Yang-Mills vacuum is very close to a Higgs phase boundary!

In section 2.1 we describe the lattice gauge group $SMG_{gen}^N = SMG^3$ that we use in our model. This so-called “anti-unified” gauge group consists of a cartesian product of a number $N_{gen}$ of replica of the standard model group. The integer $N_{gen}$ is the number of quark and lepton generations. The connection with the phenomenologically well-established standard model group is that the $SMG^3$ breaks down to the diagonal subgroup of $SMG^3$ which is isomorphic to the usual standard model group. In section 2.2, we present some theoretical considerations that support the assumption of the gauge group $SMG^3$. Included here is an explanation of our proposed “confusion” mechanism for the breakdown of $SMG^3$ to the diagonal subgroup. Actually, any Higgs field mechanism or other breaking machinery will do for our purpose as long as we get the breakdown of the gauge group $SMG^3$ to the diagonal subgroup $SMG_{diag}$. In section 2.3, we demonstrate the additivity in the inverse squared couplings of the group factors in the cartesian product $SMG^3$ in going to the diagonal subgroup.

The pivotal assumption, presented in section 3.1, is that Nature seeks out values for the plaquette action parameters that correspond to the multiple point values in the phase diagram of a Yang-Mills lattice gauge theory (with the gauge group $SMG_{gen}^N = SMG^3$). At the multiple point, all (or a maximal number) of the so-called partially confining phases (i.e., one for each invariant subgroup) of the non-simple gauge group $SMG^3$ come together. It is the multiple point values of the plaquette action parameters for the diagonal subgroup of the gauge group $SMG^3$ that, in the continuum limit, are equal to the experimental
gauge coupling values that have been extrapolated to the Planck scale. Section 3.2 is devoted to a short resume of earlier work. In section 3.3 we define partially confining phases. In section 3.4 we write down a simple plaquette action that can provoke first order phase transitions between the possible partially confining phase combinations (we ignore some discrete subgroups of the $U(1)$ groups).

In Section 4 we evaluate the continuum limit of the multiple point plaquette action parameter values corresponding to the diagonal subgroup of the gauge group $SMG^3$. This entails essentially corrections due to relative quantum fluctuations in the corresponding degrees of freedom in each of the $N_{\text{gen}}$ copies of the $SMG$ contained in the diagonal subgroup of $SMG^3$.

In Section 5 we present our calculated predictions for the nonabelian fine-structure constants in Tables 3 and 4.

In Section 6 we outline a rather speculative model which in essence explains the multiple point in our generalized phase diagram as a Planck scale fixed point corresponding to zeros of the components of a multicomponent vector of generalized $\beta$-functions.

Section 7 addresses the question of the consistency of our “anti-unified multiple point model” with established results. We point out that there are only rather limit possibilities to test our model by means other than predictions of standard model parameters such as the fine structure constants. Alternative testable predictions of our model, which would require going far beyond the standard model, are rather heavy cosmological strings with branches and a very long proton lifetime (if there is proton decay at all).

In Section 8, we supplement some concluding remarks with a short discussion of the accuracy of our numerical predictions. We end with some speculative comments.

2 The anti-unified gauge group

2.1 The gauge group $SMG^3$

In this paper we make two major assumptions: we assume the validity of the principle of multiple point criticality (see next section) and we assume our “anti-unified” gauge group $SMG^{N_{\text{gen}}} = SMG^3$ which breaks down at the Planck scale to the diagonal subgroup $SMG_{\text{diag}}$. The gauge group $SMG^3$ is described and discussed in this section.

In the models we and others have developed, the standard model group $SMG = S(U(2) \times U(3))$ arises as the diagonal subgroup $SMG_{\text{diag.}} \overset{\text{def}}{=} \{(u, u, u) | u \in SMG\}$ remaining after the Planck scale breakdown of this more fundamental (“anti-unified”) cartesian product gauge group $(SMG)^{N_{\text{gen}}}$ where $N_{\text{gen}}$ is the number of quark and lepton families (generations).

As far as the Lie algebra is concerned, we define

$$SMG^{N_{\text{gen}}} \overset{\text{def}}{=} \left( U(1) \times \left( U(1) \times \cdots \times U(1) \right) \right) \times SU(2) \times SU(2) \times \cdots \times SU(2) \times SU(3) \times \left( SU(3) \times \cdots \times SU(3) \right).$$

For the group definition of $SMG^3$, see formula (1) and the accompanying footnote 2 in the introduction. We take $N_{\text{gen}} = 3$ in accord with experiment. The identification of the number of $SMG$ factors in the cartesian product with the number of families $N_{\text{gen}}$ allows the possibility of having different gauge quantum numbers for the $N_{\text{gen}}$ different families.
2.2 Motivations for the $SMG^3$ gauge group

Before presenting various helpful theoretical arguments that lend credibility to the assumption of this Planck scale nonsimple gauge group, we point out that the most important argument is that our “fit” of fine structure constants is so good as to justify the claim of phenomenological evidence for the gauge group $SMG^3$ as the relevant one among the, after all, countable number of candidates for more fundamental gauge groups that could lead to the $SMG$.

A possible theoretical motivation for the “anti-unified gauge group $SMG^3$ and its subsequent breakdown to the diagonal subgroup could start with a scenario from “random dynamics”\[6\]: at an energy a little above the Planck scale, one has a multitude of gauge symmetries resulting from the FNNS exactification \[7\] of chance occurrences of approximate gauge symmetries. This collection of symmetries can be expected to be dominated by low-dimensional groups as such symmetries are most likely to occur by chance. We envision that the symmetry embodied by this collection of groups is broken down by a succession of steps the last of which, before the Weinberg-Salam breakdown, is the breakdown of the “anti-unified” $SMG^3$ to its diagonal subgroup. This succession of symmetry breakdowns is pictured as occurring for decreasing energies within a range of a few orders of magnitude at the Planck scale.

The succession of breakdowns envisioned coincides with gauge groups that are more and more depleted of group automorphisms\[4\]. We (and others) have proposed a breakdown mechanism\[3, \ 4\] called “confusion” that is active when gauge groups possess automorphisms.

We speculate that confusion breaking - that can be called into play by different types of automorphisms - can successively break very general groups with many cartesian product factors down to a collection of groups with especially few automorphisms as is characteristic of the $SMG$ itself. It is noteworthy that the $SMG$ has been shown in a certain sense \[8\] to be the group of rank 4 (and dimension less than 19) that is maximally deficient in automorphisms. We propose the group $SMG^3$ as the last intermediate step on the way to the $SMG$.

2.2.1 Gauge group breakdown by confusion

We now briefly explain how the confusion breakdown mechanism functions for gauge groups with outer automorphisms. First it is argued that, in the spirit of assuming a fundamental physics that can be taken as random, one is forced to allow for the possibility of having quenched random “confusion surfaces” in spacetime. The defining property of these surfaces is that (e.g. gauge) fields obey modified continuity conditions at such surfaces; for example, the permutation of a gauge field with an automorphic image of the field can occur. A nonsimply connected spacetime topology is essential for the presence of nontrivial confusion surfaces; a discrete spacetime structure such as a lattice is inherently nonsimply connected because of the “holes” in the structure.

\[4\]A group automorphism is defined as a bijective map of the group onto itself that preserves the group composition law. The set of all group automorphisms is itself a group some of the elements of which are inner automorphisms (i.e., just equivalence transformations within the group). There can also be outer automorphisms (essential for confusion) which are defined as factor groups of the group of automorphisms modulo inner automorphisms.
The essential feature of the “confusion” breakdown mechanism is that, in the presence of “confusion surfaces”, the distinct identities of a field and its automorphic image can be maintained **locally** but not **globally**. To see how this ambiguity arises, imagine taking a journey along a closed path on the lattice that crosses a confusion surface at which the labels of a gauge group element and its automorphic image are permuted. Even if one could, at the start of the journey, unambiguously assign say the names “Peter” and “Paul” to two gauge fields related by an automorphism, our careful accounting of the field identities as we travel around the loop would not, upon arriving back at the starting point, necessarily be in agreement with the names assigned when we departed on our journey. So an attempt to make independent global gauge transformations of Peter and Paul (sub)groups would not succeed. Therefore, for the action at confusion surfaces, there is not invariance under global gauge transformations of the whole gauge group but only under transformations of the subgroup left invariant by the automorphism.

The ambiguity under the automorphism caused by confusion (surfaces) is removed by the breakdown of the gauge group to the maximal subgroup which is left invariant under the automorphism. The diagonal subgroup of the cartesian product of isomorphic groups is the maximal invariant subgroup of the permutation automorphism(s); i.e., because the diagonal subgroup is the subgroup left invariant by the automorphism, it has the symmetry under gauge transformations generated by constant gauge functions (corresponding to the global part of a local gauge transformation) that survives after the ambiguity caused by the automorphism is removed by breakdown to the diagonal subgroup.

For the purpose of illustrating a possible origin of the “anti-unified” gauge group and its subsequent breakdown to the standard model group, we describe two important examples of group automorphisms - **examples 1 and 2** below - that call the confusion mechanism into play:

**Example 1.** Many groups have a charge conjugation-like automorphism corresponding in the $SU(N)$ case to complex conjugation of the matrices element by element. While for $SU(2)$ this is an inner automorphism, it is for higher $SU(N)$ groups an outer automorphism. According to the speculated confusion mechanism, such a group should break down to the subgroup consisting of only the real matrices which is the largest subgroup that is invariant under the automorphism. If the group is provided with C-breaking chiral fermions, the automorphism can be broken in this way thereby thwarting the “attack” from the confusion mechanism.

**Example 2.** There can be automorphisms under the permutation of identical group factors in a cartesian product group: we argue that the symmetry reduction (at the Planck scale) from $(SMG)^{N_{gen}}$ to $SMG = S(U(2) \times U(3))$ is triggered by the symmetry under the automorphism that permutes the $N_{gen}$ SMG factors in $(SMG)^{N_{gen}}$.

Elaborating briefly on example 1 above, we point out that with the exception of the semisimple groups such as $SU(2)$, $SO(3)$, the odd $N$ spin or $SO(N)$-groups and the symplectic groups, all groups have outer automorphisms of the complex conjugation- or charge conjugation-like type. Following a series of confusion breakdowns activated by charge conjugation-like automorphisms, we expect that the (intermediate) surviving gauge symmetry (i.e., that of $(SMG)^{N_{gen}}$) must have matter fields that break charge conjugation-like symmetries. In other words, the presence of such matter fields serves to protect the surviving symmetry from further breakdown by eliminating the possibility.
for further confusion of the surviving group with its automorphic image under charge conjugation.

In particular, we expect that a necessary condition for the survival of gauge groups like $U(1)$ and $SU(3)$ is the presence of some matter fields not invariant under charge conjugation. Protection against this sort of breakdown can be provided by chiral fields that break the charge conjugation symmetry of the gauge fields. In the case of the Standard Model, left- and right-handed fermions always appear in different representations so that confusion breakdown by way of a charge conjugation automorphism is not possible. In fact, the number of particles in a single generation in combination with the rather intricate way these are represented in the Standard Model can be shown to be the simplest possible manner in which gauge anomalies can be avoided\[9, 10\].

As mentioned in example 2 above, we assume that the final breakdown of gauge symmetry by confusion (at the Planck scale) is activated by the automorphism that permutes the $N_{gen}$ isomorphic cartesian product factors $SMG$ in $(SMG)^{N_{gen}}$. The elimination of the ambiguities that can arise in trying to keep track of the identities of a group element and its automorphic image under such permutations coincides with the breakdown to the standard model group $SMG = S(U(2) \times U(3))$ which, being the diagonal subgroup of $(SMG)^{N_{gen}}$, is invariant under the automorphism that permutes the $SMG$ group factors in $(SMG)^{N_{gen}}$. In order for this final confusion breakdown to work effectively, the cartesian product factors of $(SMG)^{N_{gen}}$ must presumably be truly isomorphic - i.e., the matter field content of each factor must essentially have the same structure. This combined with the fact that one usual fermion generation is known to provide the least complicated arrangement of particles that avoids gauge anomalies would strongly suggest that the $N_{gen}$ factors of $(SMG)^{N_{gen}}$ are simply dull repetitions of the Standard Model Group each one of which can have its own generation. Each of the $N_{gen}$ factors is the “ancestor” to one of the $N_{gen}$ generations of the diagonal subgroup identified with the usual Standard Model Group.

It should be emphasized that all the confusion breakdowns - those utilizing a series of charge conjugation automorphisms leading to $(SMG)^{N_{gen}}$ as well as the final confusion breakdown of the $SMG^3$ to the diagonal subgroup that is caused by the permutation automorphism - are assumed to take place within a rather narrow range of energies at the Planck scale.

Before leaving the confusion breakdown mechanism, we should point out that any mechanism that breaks the $SMG^3$ down to the diagonal subgroup would suffice for our model. A Higgs field mechanism could for example provide an alternative to the confusion mechanism of breakdown.

We end the discussion of the motivation for the “anti-unified” gauge group $SMG^{N_{gen}}$ with some remarks on some recent work\[18\] of a slightly different nature. In this work it is argued that the hierarchical mass spectrum of quarks and leptons - especially the large mass gaps between generations - calls for the association of different gauge quantum numbers to different generations. The feature that appears to be required is that, for example, the (gauge) quantum number difference between left and right $\mu$ is not the same as that between the left and right $e$. This suggests gauge group extensions (at the fundamental scale) having many features in common with $SMG^3$ although the large top quark mass presents a problem for the gauge group $SMG^3$. 
2.3 Additivity of the inverse squared couplings

The breakdown of the group $SMG^3$ to the diagonal subgroup has consequences for the gauge couplings that we now briefly describe. Recalling that the diagonal subgroup of $SMG^3$ corresponds by definition to identical excitations of the $N_{gen} = 3$ isomorphic gauge fields (with the gauge couplings absorbed) and using the names Peter, Paul, etc. as indices that label the $N_{gen}$ different isomorphic cartesian product factors of $(SMG)^{N_{gen}}$, one has}

$$
(gA_{\mu}(x))_{Peter} = (gA_{\mu}(x))_{Paul} = \cdots = (gA_{\mu}(x))_{N_{gen.}} \overset{def.}{=} (gA_{\mu}(x))_{diag.};
$$

(4)

this has the consequence that the common $(gF_{\mu\nu})_{diag}^2$ in each term of the lagrangian density for $(SMG)^{N_{gen.}$ can be factored out:

$$
\mathcal{L} = -1/(4g_{Peter}^2)(gF_{\mu\nu}(x))^2_{Peter} - 1/(4g_{Paul}^2)(gF_{\mu\nu}(x))^2_{Paul} - \cdots - 1/(4g_{N_{gen.}}^2)(gF_{\mu\nu}(x))^2_{N_{gen.}}
$$

(5)

$$
= (-1/(4g_{Peter}^2) - 1/(4g_{Paul}^2) - \cdots - 1/(4g_{N_{gen.}}^2)) \cdot (F_{\mu\nu}(x))^2_{diag} = -1/(4g_{diag}^2) \cdot (gF_{\mu\nu}(x))^2_{diag}.
$$

(6)

The inverse squared couplings for the diagonal subgroup is the sum of the inverse squared couplings for each of the $N_{gen.}$ isomorphic cartesian product factors of $(SMG)^3$. Additivity in the inverse squared couplings in going to the diagonal subgroup applies separately for each of the invariant Lie subgroups[4] $i \in \{SU(2), SU(3)\} \subset SMG$. However, for $U(1)$ it is possible to have terms in the lagrangian density for $(SMG)^{N_{gen.}$ that label the fields (with the gauge couplings absorbed) and using the names Peter, Paul, etc. which are equal in the diagonal subgroup. Therefore it becomes more complicated as to how one should generalize this notion of additivity. Terms of this type can directly influence the density for $(SMG)^{N_{gen.}$.

The inverse squared couplings for a given $i$ but different labels $\{Peter, Paul, \cdots, N_{gen.}\}$ are all driven to the multiple point in accord with the principle of multiple point criticality (see next section), these $\{Peter, Paul, \cdots, N_{gen.}\}$ couplings all become equal to the multiple point value $g_{i,multi.\ point}$; i.e.,:

$$
\frac{1}{g_{i,diag}^2} = \frac{1}{g_{i,Peter}^2} + \frac{1}{g_{i,Paul}^2} + \cdots + \frac{1}{g_{i,N_{gen}}^2} \quad (i \in \{SU(2), SU(3)\}).
$$

(7)

Assuming that the inverse squared couplings for a given $i$ but different labels $\{Peter, Paul, \cdots, N_{gen.}\}$ are all driven to the multiple point in accord with the principle of multiple point criticality (see next section), these $\{Peter, Paul, \cdots, N_{gen.}\}$ couplings all become equal to the multiple point value $g_{i,multi.\ point}$; i.e.,:

$$
\frac{1}{g_{i,Peter}^2} = \frac{1}{g_{i,Paul}^2} = \cdots = \frac{1}{g_{i,N_{gen}}^2} = \frac{1}{g_{i, multi.\ point}}.
$$

(8)

\footnote{As it is $gA_{\mu}$ rather than $A_{\mu}$ that appears in the (group valued) link variables $u \propto e^{igA_{\mu}}$, it is the quantities $(gA_{\mu})_{Peter}$, $(gA_{\mu})_{Paul}$, etc. which are equal in the diagonal subgroup.}

\footnote{For $U(1)$, a modification is required.}

\footnote{In seeking the multiple point for $SMG^3$, one is lead to seek criticality separately for the cartesian product factors as far as the nonabelian groups are concerned. For $U(1)$, recent progress suggests that one should seek the multiple point for the whole group $U(1)^3$ rather than for each of the $N_{gen.} = 3$ factors $U(1)$ separately. The reason for this complication concerning abelian groups (continuous or discrete) is that these have subgroups and thereby invariant subgroups (infinitely many for continuous abelian groups) that cannot be regarded as being a subgroup of one of the $N_{gen.}$ factors of $SMG^3$ or a cartesian product of such subgroups. Work on this problem is in progress and appears at this time to lead to a phenomenologically desirable factor “6” for the ratio $\alpha_{\mu\nu}(\alpha_{\mu\nu}^{\prime\prime} U(1))$ (where $\alpha_{\mu\nu}$, $U(1)$ is the critical coupling for the gauge group $U(1)$ instead of the factor “3” (from $N_{gen.}$) that would naively be expected for this ratio by analogy to the predictions for the nonabelian couplings.}
We see that the inverse squared coupling $1/g_i^{2, \text{diag}}$ for the $i$th subgroup of the diagonal subgroup is enhanced by a factor $N_{\text{gen}}$ relative to the corresponding subgroup $i$ of each of the $N_{\text{gen}}$ cartesian product factors $\text{Peter}, \text{Paul}, \cdots, N_{\text{gen}}$ of $(\text{SMG})^{N_{\text{gen}}}$:

$$\frac{1}{g_i^{2, \text{diag}}} = \frac{N_{\text{gen}}}{g_i^{2, \text{multi. point}}}.$$ (9)

It is this weakening of the coupling for each of the subgroups $i \in \{SU(2), SU(3)\}$ of the diagonal subgroup (i.e., the $\text{SMG}$) that constitutes the main role of the anti-unification scheme in our model. Anticipating the discussion of the role of the multiple point in the next section, we point out prematurely that while it is the $g_i, \text{multi. point}$ (i.e., $i = SU(2)$ or $SU(3)$) which are to be identified with the critical values (at the multiple point) of coupling constants for the bulk phase transition of a lattice Yang-Mills theory with gauge group $i$, it is the $g_i, \text{diag.} = g_i, \text{multi. point}/\sqrt{N_{\text{gen}}}$ that, in the continuum limit, are to be identified with the corresponding experimentally observed couplings extrapolated to the Planck scale\cite{16, 17}.

In summary, section 2.1 describes the “anti-unified” gauge group $\text{SMG}^{N_{\text{gen}}} = \text{SMG}^3$. Section 2.2 is devoted to an attempt to motivate the assumption of the “anti-unified” gauge group and includes a description of the “confusion” mechanism of breakdown of this group to the diagonal subgroup (which is isomorphic to the standard model group $\text{SMG}$). Other breakdown mechanisms for the gauge group $\text{SMG}^3$ would also do as long as they lead to the diagonal subgroup.

Assuming that Nature seeks out parameter values at the point in the phase diagram of a Yang-Mills lattice gauge theory (with gauge group $\text{SMG}^3$) where a maximum number of phases come together (i.e., the multiple point; see next section), it is seen in section 2.3 that the breakdown to the diagonal subgroup of $\text{SMG}^3$ results in an $N_{\text{gen}}$-fold decrease in the critical values of the fine-structure constants corresponding to the bulk phase transition values at the multiple point of $\text{SMG}^3$.

3 The principle of multiple point criticality

3.1 Statement of principle

The main point of the present work is to put forward the idea which we state as a principle - the principle of multiple point criticality:

\emph{At the fundamental scale (taken to be the Planck scale), the actual running gauge coupling constants correspond to the multiple point critical values in the phase diagram of a lattice gauge theory.}

The multiple point is a point in the phase diagram of the lattice gauge theory at which all - or at least many - partially confining phases meet. This point corresponds to critical values for the parameters used to describe the form of the action. In the rather crude

\footnote{Partially confining phases are explained in section 3.3}
mean field approximation that we consider, there is one partially confining phase for each invariant subgroup of the gauge group - including discrete invariant subgroups.

In section 3.2 we comment briefly on related earlier work. This is followed by a definition of partially confining phases in section 3.3. Section 3.4 is used to explain how we choose enough parameters in the plaquette action so as to: 1) have the existence of the partially confining phases we want to consider and 2) be able to make these phases come together at a point - the multiple point - in the space of these parameters.

### 3.2 Relation to earlier work

In earlier work, critical values for gauge couplings were calculated using an incomplete set of critical action parameters. We used this restricted form of criticality of gauge couplings to fit the number of generations which, at the time, was not known. The idea behind the principle of multiple point criticality is to calculate critical gauge couplings from a set of (multiple point) critical action parameters that is complete in the sense that a confinement phase for any invariant subgroup of the gauge group can be obtained for some choice of the parameters of this set. This has several advantages compared to earlier work.

1) A weak point in our previous procedure for calculating “critical” couplings using an incomplete set of critical action parameters is that, when applied to the groups $SU(2)$ and $SU(3)$ (the groups to which we have primarily applied this procedure), these phase transitions strictly speaking do not exist with the simplest type of action. There is no singularity in the partition function - in particular, not in a corrected mean field approximation - but rather only a peak or kink indicating behavior that presumably is qualitatively similar to a phase transition. With the multiple point criticality principle, we avoid this embarrassing situation because the multiple point for the groups $SU(2)$ and $SU(3)$ corresponds to a proper phase transition although it is possible to circumvent this transition by going around the critical point at the terminus of a phase boundary. That it is possible to go continuously from one phase to the other is due to having extra characters in the action (e.g., corresponding to the adjoint representation).

2) If we restrict ourselves to simply searching for parameter values that yield critical gauge couplings for the nonsimple gauge group $SMG^3$, it is necessary to supplement this restricted “criticality principle” with a statement that it is to be applied separately to all $N_{gen} = 3$ isomorphic $SU(2)$ and $N_{gen} = 3$ isomorphic $SU(3)$ invariant subgroup factors of $SMG^3$. This is because the prediction of the three continuum gauge couplings from our model requires that each of the $N_{gen}$ cartessian product factors $SU(2)$ as well as $SU(3)$.

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9In reality, such phases are not necessarily separated by a phase boundary everywhere in the action parameter space; e.g., phase boundaries that end at a critical point can be circumvented in going from one phase to another.

10In earlier work, we claimed that couplings should be critical (in the incomplete sense) at the fundamental scale (presumably the Planck scale) using several types of arguments: 1) couplings exceeding critical values at the fundamental scale would lead to confinement “already” at the Planck scale and thereby leave no trace of the corresponding degrees of freedom at lower energies; 2) for some reason - perhaps as protection from “Higgsning” - Nature seeks out the strongest allowed couplings; we referred to this tendency as “saturation” of the inequality to be fulfilled in order to avoid the Planck scale confinement mentioned in 1).

11In previous work, these phase transitions were “found” only in the context of our very crude $MFA$ approximation.
each of the $N_{\text{gen}}$ factors $SU(3)$ have critical coupling values. However, according to the principle of multiple point criticality, critical values are sought out for all invariant subgroups (including isomorphic copies of the same invariant subgroup and discrete invariant subgroups).

3) We think that almost any mechanism that explains why Nature seeks parameter values corresponding to a phase boundary for some group would almost certainly be applicable to invariant subgroups; this would amount to the prediction that Nature seeks out a phase boundary point such that infinitesimal changes in the parameters of the theory would give access to the partially confining phases corresponding to all invariant subgroups of the group in question. Assuming that such a point exists, it is, by definition, just the multiple point.

In summary, we have pointed out in this section that, from the standpoint of earlier work, the postulate of multiple point criticality has several advantages.

### 3.3 Classification of phases

Since we postulate that Nature seeks a special point in plaquette action parameter space - the multiple point - where many phases come together, we need now to develop an idea about what phases exist. This is the purpose of this subsection.

First it should be made clear that when we talk about phases of nonabelian groups at the Planck scale, we mean *first order* phase transitions between “phases” that are lattice artifacts. All these “phases” will, for sufficiently long distances, turn out to be confining with no long range correlations (corresponding to finite glueball masses) when, as is the case in this section, matter fields are ignored.

So for us, the interesting phases are separated by *first order* phase transitions. Such phases are governed by which *micro* physical fluctuation patterns yield the maximum value of $\log Z$. Qualitatively different short distance physics could consist of different distributions of group elements along various subgroups or invariant subgroups of the gauge group for different regions of (bare) plaquette action parameter space. It is therefore the physics at the scale of the lattice that is of interest because it is lattice scale physics that dominates the different $\log Z$ ansatzes that prevail (i.e., are maximum) in different parameter space regions separated by first order transitions. However, this does not mean that longer distance behavior is unchanged in passing from one “phase” to another. As an example, consider the string tension at the transition between two different *lattice scale phases* that both really correspond to confining phases in the usual sense: in what we designate as confining in the mean field approximation ($MFA$) or “confinement at the lattice scale”, the string tension has an order of magnitude given by dimensional arguments. On the other hand, in what we call the “Coulomb phase at the lattice scale”, or the Coulomb phase in the $MFA$ approximation, the string tension is much smaller; i.e., smaller by an exponential factor.

Since we want to relate these lattice artifact first order phase transitions to experimental observations, we are obliged to take the point of view that these artifacts have - one way or another - physical reality.

Now we want to assign names to the different lattice artifact phases, i.e., qualitatively different physical behaviors of the vacuum of a lattice gauge theory at the lattice scale. We use a classification according to whether or not there is spontaneous breakdown of the gauge symmetry remaining after an incomplete choice of gauge that we here take to
be the (latticized) Lorentz gauge. We therefore consider the transformation properties of the vacuum under gauge transformations that leave the Lorentz gauge condition (i.e., $\prod_{\text{emanating from } \bullet} U(\overrightarrow{-}) = 1$ for all sites $\bullet$) intact.

Examples of such gauge transformations are those that can be generated either by a constant gauge coordinate function $\alpha^a(x) = \alpha^a$ or a linear gauge coordinate function $\alpha^a(x) = \alpha^a x^\mu$ corresponding to gauge transformations designated respectively as $\Lambda_{\text{Const.}}(x) = e^{i\alpha^a t_a^\mu}$ and $\Lambda_{\text{Linear}}(x) = e^{i\alpha^a t_a x^\mu}$. The gauge coordinate function $\alpha^a(x)$ is the $a$th “color” coordinate of a Lie algebra vector $\alpha^a(x) t^a$. Here $t^a$ denotes an element of the Lie algebra satisfying the commutation relations $[t^a, t^b] = i e^{c^b} t^c$ where the $e^{c^b}$ are the structure constants.

On the basis of the transformation properties of the vacuum under gauge transformations of the types $\Lambda_{\text{Const.}}(x) = e^{i\alpha^a t_a^\mu}$ and $\Lambda_{\text{Linear}}(x) = e^{i\alpha^a t_a x^\mu}$, we want to classify the different possible “partially confining phases” of the vacuum. In our classification, there can be one “partially confining phase” for each possible combination of a subgroup $K$ and an invariant subgroup $H$ such that $K \subset \text{SMG}^3$ and $H \lhd K$. The subgroup $K$, which we refer to as the “unHiggsed” subgroup, is defined as the group of constant gauge transformations $\Lambda_{\text{Const.}}$ that leaves the vacuum invariant. The degrees of freedom corresponding to the homogeneous space $\text{SMG}^3/K$ are accordingly “Higgsed”. The invariant subgroup $H$, referred to as the confined subgroup, is defined as the invariant subgroup $H \lhd K$ of elements $h = e^{i\alpha^a t_a}$ such that the gauge transformations with linear gauge function $\Lambda_{\text{Linear}}$ exemplified by $\Lambda_{\text{Linear}} \overset{df}{=} h x^a$ leave the vacuum invariant.

For the degrees of freedom associated with the factor group $K/H$, there is invariance of the vacuum expectation value under the corresponding constant gauge transformations, while there is spontaneous breakdown under the linear gauge transformations corresponding to these degrees of freedom. Such degrees of freedom will be said to demonstrate “Coulomb-like” behavior.

The use of the adjectives “Higgs-like”, “Coulomb-like” and “confinement-like” in classifying qualitatively different physical behaviors at a given scale (here the Planck scale) is motivated from results obtained using the mean field approximation (MFA) which intrinsically distinguishes phases on the basis of the qualitative differences in the physics that are discernible at the scale of the lattice constant.

Assuming as we do here that all vacuum Yang-Mills degrees of freedom of the $\text{SMG}^3$ are “unHiggsed” (i.e., all $\Lambda_{\text{Const.}} \in \text{SMG}^3$ leave the vacuum invariant), we can then define the Yang-Mills degrees of freedom corresponding to an invariant subgroup $H \lhd \text{SMG}^3$ as being “confined” if $H$ is the maximal subgroup of $\text{SMG}^3$ such that linear gauge transformations $\Lambda_{\text{Linear}} \in H$ leave the vacuum invariant. The remaining Yang-Mills degrees of freedom take values in the set of cosets belonging to the factor group $\text{SMG}^3/H$ and behave as if in a “Coulomb phase”. We call this the “partially confining phase” that is confining w.r.t. $H$. In seeking the multiple point, we seek the point or surface in parameter space where “all” (or many) partially confining phases “touch” one

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12 The vacuum invariance referred to really means the invariance of the coefficients $\langle D_{ij}^{(\mu)}(U(\overrightarrow{x})) \rangle_{\text{vac}}$ in an expansion in (matrix elements of) continuous unitary irreducible representations $D_{ij}^{(\mu)}(U(\overrightarrow{x}))$. The expansion referred to is that corresponding to some link variable probability density function $P(U(\overrightarrow{x})) = \int \prod_{\alpha} d^{H_{\text{matter}}} U(\overrightarrow{\alpha}) \rho^S$.

13 In the quantity $x^a/a$, $a$ denotes the lattice constant; modulo lattice artifacts, rotational invariance allows the (arbitrary) choice of $x^1$ as the axis $x^a$ that we use.
3.4 Generalized action is required

We have seen that there are many possible phases according to the classification of the foregoing section. If we want to let the principle of multiple point criticality guide us in calculating the values of gauge couplings found in Nature, we need to find a set of action parameters that distinguishes all these phases from each other and also allows all these phases to come together at a point in the phase diagram. This of course requires the inclusion of a sufficient number of suitable parameters in the ansatz for the plaquette action used. This is what we seek to set up in this section: a sufficiently extensive class of parametrized plaquette actions.

To find out which action parameters to use, we have developed a technique for the construction of crude phase diagrams for gauge groups such as $SMG$, $SMG^3$ and $U(N)$ (collectively referred to by the symbol “$G$”); $U(N)$ is useful because it has many features in common the $SMG$ and $SMG^3$ while, for expository purposes, being simpler to deal with. As these groups are nonsimple, there are nontrivial partially confining phases corresponding to “confining” behavior for a number of invariant subgroups that can include discrete invariant subgroups. All these groups have $U(1)$ as an invariant subgroup. However, we do not in the present work consider the partially confining phases corresponding to all the discrete (invariant) subgroups of the abelian $U(1)$ subgroup; rather we restrict our considerations to those discrete invariant subgroups of $U(1)$ which are also discrete invariant subgroups of nonabelian Lie subgroup factors (i.e., $Z_2$ and $Z_3$ for $SMG$ or $Z_N$ for $U(N)$). A consequence of this restriction is that we are dealing with a finite number $n$ of partially confining phases.

Having a multiple point for $n$ such phases presupposes an appropriate choice of the action parameter space: for the parameter set chosen, it must be possible to bring all $n$ phases together at a point. That we want all $n$ phases to meet at the (multiple) point further requires that the parameter set choice can separate all $n$ phases (at least near the multiple point). This means that we need a parameterization that can make the various invariant subgroups go more or less independently from a confinement-like to a Coulomb-like behavior.

The most general form for the action requires an infinite number of parameters that could, in principle at least, be taken as the coefficients of an expansion in group characters corresponding to each imaginable way of associating group characters with Wilson loops. For such a very general action, the infinite dimensional phase diagram for the theory with a gauge group having $n$ invariant subgroups would have $n$ phases that meet along an infinite-dimensional “multiple point” critical surface of codimension $n−1$. The boundaries between the various phases - if first order - would be characterized by singularities in the first derivative of $\log Z = \max\{\log Z_L|L \triangleleft G\}$. At the boundary delineating a phase for which an invariant subgroup $H_i$ is realized as being confinement-like, $\log Z_{H_i}$ would dominate in a part of the neighborhood of such a boundary. We define the quantity $\log Z_H$ as being $\log Z$ when the latter is calculated as if we have a field configuration distribution for which the lattice gauge theory is in a phase for which $H$ is “confining” and $G/H$ is “Coulomb-like”\(^{14}\).

\(^{14}\) i.e., Bianchi identities are ignored for degrees of freedom along $H$ and MFA is used w. r. t. $G/H$ with link averages $\langle U(\,\rangle_{G/H} \rangle \neq 0$.\footnote{14}
In practice, we use an ansatz action having a finite, manageable number $d$ of $\beta$-parameters that span a $d$-dimensional subspace of the most general (infinite dimensional) parameter space. In this subspace, it is often possible to find a set of parameter values $\vec{\beta}_{\text{crit.}} \overset{\text{def.}}{=} \{\beta_{1, \text{crit.}}, \ldots, \beta_{d, \text{crit.}}\}$ lying in the infinite dimensional multiple point critical surface of codimension $n - 1$ that is embedded in the most general parameter space as described in the preceeding paragraph. In the $d$-dimensional subspace, such a $\vec{\beta}_{\text{crit.}}$ coincides with the multiple point or, more generally, lies in a manifold of multiple points that formally can be defined as follows:

$$\{\vec{\beta} \forall H_i \triangleleft G \ni \vec{\epsilon}_{\text{infiniteesimal}} = (\epsilon_1, \ldots, \epsilon_d) \ [\log Z_{H_i}(\vec{\beta} + \vec{\epsilon}) = \log Z = \max \{\log Z_K | K \triangleleft G\}]\}.$$  

(10)

If $d = n - 1$ for the $d$-dimensional subspace of the infinite dimensional action parameter space, multiple points are referred to as generic multiple points and can be found systematically for many parameter choices. If $d > n - 1$, then we would generally expect to have a multiple point critical surface of codimension $n - 1$. If $d < n - 1$, there can be what we call nongeneric multiple points but only for judicious choices of the $d$-dimensional parameterization.

In order to see the difference between a generic and a non-generic multiple point, let us consider a journey in action parameter space that starts at a random point and subsequently seeks out phase boundaries for which a successively greater number of phases are accessible by making infinitesimal changes in the action parameters at the points along the journey. In the generic case, the codimension $\text{codim}$ of the boundary goes up by one each time an additional phase is encountered so that a phase boundary surface/curve is in contact with $\text{codim} + 1$ phases. In the non-generic case, the number of phases accessible goes up faster than the codimension - at least once in the course of the journey in parameter space. So for the non-generic case, it is possible to have points in phase space (e.g., the multiple point) forming a surface at which infinitesimal variations in action parameters give access to a number of phases exceeding $\text{codim} + 1$ where $\text{codim}$ is the codimension of this surface.

### 3.4.1 The modified Manton action

We find nongeneric multiple points using an ansatz that restricts the class of actions to those that lead to distributions $e^{S_{\text{peak}}} \rho_{\text{peak}}$ of plaquette variables consisting of narrow "peaks" centered at the elements $p$ belonging to discrete subgroup(s) of the center of the gauge group. For expositional purposes, we can think of the gauge group $U(N)$ and the discrete invariant subgroup $Z_N \subset U(N)$. Formally we can characterize the simplest of this restricted class of actions as the projection of the class of all possible actions onto the subspace of actions having the form of second order Taylor expansions in group variables with one Taylor expansion at each element $p \in Z_N$. Assuming for simplicity that the

\[15\] Our crude approximation\[16\] used to extract qualitative features of the phase diagram implies that the continuum couplings at different points on the multiple surface have the same values. Variations in the continuum couplings along the multiple point surface cannot be seen in the approximation where we use the truncated Taylor expansion of (11) and (12).

\[16\] We are assuming that a procedure using a weak coupling approximation is valid even in a neighborhood at the critical $\beta$'s. The approximation in which $\frac{1}{2} \beta$ is considered to be very small is discussed below and in reference [13].
peaks expanded around each element $p \in \mathbb{Z}_N$ are symmetric (so that odd-order derivatives vanish), the simplest of this restricted class of plaquette actions (neglecting zero order terms) would then be of the form

$$\left. \frac{\partial^2 S_{\text{act.}}}{\partial \omega_a \partial \omega_b} \right|_{\omega(p)} (\omega_a - \omega^{(p)}_a)(\omega_b - \omega^{(p)}_b)$$  \hspace{1cm} (11)

Here the $\omega_a$ are coordinates on the group manifold in a neighborhood of an element $p \in \mathbb{Z}_N$. The coordinates at $p$ are denoted $\vec{\omega}^{(p)} = (\omega^{(p)}_1, \omega^{(p)}_2, \cdots, \omega^{(p)}_{\dim(U(N))})$.

From the assumption of a distribution $e^{S^{(1)}}$ of narrow peaks centered at the elements $p \in \mathbb{Z}_N$, a group element not close to some $p \in \mathbb{Z}_N$ leads to a vanishing value of $e^{S^{(1)}}$. This means that a nonvanishing value of $e^{S^{(1)}}$ at a given group element $\vec{\omega} = (\omega_1, \omega_2, \cdots, \omega_{\dim(U(N))})$ gets its value in our ansatz action solely from the Taylor expansion centered at just one element $p \in \mathbb{Z}_N$ (i.e., the one for which the quantity $\sum a,b g^{ab}(\vec{\omega}(p))(\omega_a - \omega^{(p)}_a)(\omega_b - \omega^{(p)}_b)$ is minimum). Here $g^{ab}$ is the metric tensor defined by requiring invariance under left and right group multiplication supplemented with normalization conventions. We define the quantities $\beta_i$ (at the point $p$) by

$$\frac{\partial^2 S_{\text{act.}}}{\partial \omega_a \partial \omega_b} \big|_{\vec{\omega}(p)} \overset{\text{def. of } \beta_i}{=} \sum_i \beta_i g^{ab}_i(p)$$  \hspace{1cm} (12)

where the index $i$ labels the Lie subgroups of the gauge group invariant w.r.t. the algebra. For the group $U(N)$, $i \in \{U(1), SU(N)\}$ and we take $p \in \mathbb{Z}_N$. So the action we are considering - from now on we call it the modified Manton action - leads to Gaussian peaks at each $p$; $\beta_i$ is the action parameter that determines the width of the Gaussian distribution along the $i$th Lie subgroup. We assume that $\beta_i$ is the same\footnote{In a more sophisticated ansatz, this need not be assumed.} at all elements $p \in \mathbb{Z}_N$. A full specification of our modified Manton action also requires parameters that specify the relative height of peaks at the different elements $p \in \mathbb{Z}_N$. For $N = 2$ or 3, one parameter - denote it by $\xi_N$ - is sufficient.

Using the parameters of the modified Manton action (e.g., for $U(N)$, the parameters are $\beta_i$ and $\xi_N$ described above), nongeneric multiple points are readily found. The selection of subgroups to which these parameters correspond make up what we call the “constituent” invariant subgroups\footnote{By constituent invariant subgroups we refer essentially to the cartesian product factors of the covering group together with a selection of the discrete subgroups of the center - namely the ones of special importance in obtaining the gauge group as a factor group (e.g., $\mathbb{Z}_2$ is of special importance in obtaining the factor group $U(2)$ from the covering group $U(1) \times SU(2)$ because $U(2)\cong (U(1) \times SU(2))/\mathbb{Z}_2$.} (e.g., for $U(N)$, the “constituent invariant subgroups are $U(1)$, $SU(N)$, and $\mathbb{Z}_N$). The corresponding set of parameters has the advantage that they are essentially independent: variation of one of these parameters leads to a change in the width of the distribution $e^{S_0}$ along the corresponding “constituent” invariant subgroup that is approximately uncoupled from the distributions of group elements along other “constituent” invariant subgroups. Also, all possible invariant subgroups $H$ of the gauge groups considered here can be constructed as factor groups associated with a subset of the set of the “constituent” invariant subgroups.

For the SMG, the “constituent” invariant subgroups are $U(1)$, $SU(2)$, $SU(3)$, $\mathbb{Z}_2$, and $\mathbb{Z}_3$. The $\beta_i$ ($i \in \{U(1), SU(2), SU(3)\}$) constitute three of the five required parameters for the modified Manton action. The remaining two parameters we shall designate as $\xi_2$ and

$$\xi_3$.
\( \xi_3 \). These are associated with the discrete invariant subgroups \( Z_2 \) and \( Z_3 \) and determine the relative heights of peaks of the distribution \( e^{S_\square} \) at various elements \( p \in \text{span}\{Z_2, Z_3\} \).

The modified Manton action for the \( SMG \) is

\[
S_\square(U(\square)) = \begin{cases} 
\sum_{i \in \{U(1), SU(2), SU(3)\}} \beta_i \text{dist}^2_i(U(\square), p) + \log \xi(p) & \text{for } U(\square) \text{ near } p \in \text{span}\{Z_2, Z_3\} \\
\approx -\infty & \text{for } U(\square) \text{ not near any } p \in \text{span}\{Z_2, Z_3\}
\end{cases}
\]

(13)

where the symbol \( \text{dist}^2_i(U(\square), p) \) denotes the component of the distance \( \bullet \) between the group element \( U(\square) \) and the nearest element \( p \in \text{span}\{Z_2, Z_3\} \) along the \( i \)th invariant Lie subgroup and

\[
\log \xi(p) = \left\{ \begin{array}{ll}
0 & \text{for } p \in Z_3 \\
\log \xi_2 & \text{for } p \notin Z_3
\end{array} \right\} + \left\{ \begin{array}{ll}
0 & \text{for } p \in Z_2 \\
\log \xi_3 & \text{for } p \notin Z_2
\end{array} \right\}.
\]

(14)

This action gives rise to a distribution \( e^{S_\square} \) having 6 Gaussian peaks at the elements \( \bullet \)

\[
p_r \overset{\text{def.}}{=} \begin{pmatrix}
e^{\frac{i\beta_1 r}{4}}1^{2 \times 2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & e^{\frac{-i\beta_3 r}{4}}1^{3 \times 3} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \in \text{span}\{Z_2, Z_3\} = Z_0 \quad \text{with } r = 0, 1, \cdots, 5.
\]

(15)

all having widths \( (2\beta_1)^{-1/2}, (2\beta_2)^{-1/2}, \) and \( (2\beta_3)^{-1/2} \) in respectively the \( U(1), SU(2), \) and \( SU(3) \) subgroup directions. For \( r = 0 \) (i.e., at the group identity), the peak height is 1; for \( r = 3, \) (i.e., at the nontrivial element of \( Z_2 \)) the peak height is \( \xi_2; \) for \( r = 2, 3 \) (i.e., the nontrivial elements of \( Z_3 \)), the peak heights are \( \xi_3; \) for \( r = 1, 5 \), the peak heights are \( \xi_3\xi_3 \).

Note that the assumption of very sharp peaks at the elements \( p \in \text{span}\{Z_2, Z_3\} \) means that actions of this class are in essence completely specified by parameters corresponding to the coefficients of second order terms in their Taylor expansions about these elements \( p \). Higher order terms in the Taylor expansion are irrelevant. Roughly speaking, this also means that physical quantities such as continuum couplings and the \( \log Z_H \) for various invariant subgroups \( H \) (and therefore the realizable phases) can only depend on the coefficients of second order action terms. Hence, the multiple point critical surface can be expected to be approximately parallel to surfaces of constant continuum coupling values thereby yielding an approximate universality.

3.4.2 Factorizing the free energy

The critical coupling values for the transitions on the lattice which we are interested in here happen to be so relatively weak that a weak coupling approximation is not meaningless. We therefore make use of such an approximation in conjunction with a mean field approximation (MFA) in our very crude exploratory studies of the qualitative features

\footnote{19}Left-right invariance of a Riemann space metric \( ds^2 = g_{ab}\omega^a d\omega^b \) specifies \( \text{dist}^2_i \) for each simple invariant subalgebra \( i \) up to a normalization factor. The decomposition of \( \text{dist}(p, u) = \int_p^u ds \) into components \( \text{dist}_i \) along the \( i \)th invariant subgroup is at least well defined for small distances.

\footnote{20}The notation used here is that of the footnote in the beginning of the Introduction where the \( SMG \) is embedded into \( SU(5) \).
to be expected for the phase diagram of a lattice gauge theory. A sensible weak coupling approximation requires that $\beta^{-1}$ is small compared to the squared extension of the group where of course $\beta$ is the coefficient of the real part of the trace of a plaquette variable in the action. Because we assume weak coupling, we use the approximation of a flat space measure in the evaluation of functional integrals with the limits of integration extended to $\pm\infty$.

It is natural to enquire as to how such a weak coupling approximation can have a chance of being sensible in dealing with the onset of a confinement-like phase at the lattice phase transition. Recall first that we use as the defining feature of a confinement-like phase that Bianchi identities can be ignored to a good approximation. But the variables to which the Bianchi identities apply are not plaquette variables but rather the variables taken by 3-dimensional volumes enclosed by plaquettes - in the simplest case, just the cubes bounded by six plaquettes. Now the distribution of such cube variables is the 6-fold convolution of the distribution of plaquette variables were it not for Bianchi identities. So if the distribution of plaquette variables has a width proportional to $\beta^{-1}$, the width of the distribution of group elements taken by cubes enclosed by six plaquettes is proportional to $(\beta/6)^{-1}$. Therefore, the question of the validity of using a weak coupling approximation at the phase transition on a lattice is really a question of whether the number 6 can be regarded as large compared to unity. Accepting this as true, we can conclude that even when $\beta^{-1}$ is small compared to the squared extension of the group as required for a meaningful weak coupling approximation, the quantity $(\beta/6)^{-1}$ is large enough compared to the squared extension of the group so as to justify the the use of the Haar measure distribution for the Bianchi-relevant cube variables obtained as the convolution of six plaquette variable distributions. This amounts to ignoring the Bianchi identities. A phase for which Bianchi identities can be ignored is, according to our ansatz, a confinement-like phase.

In this approximation we obtain an expression for the free energy $\log Z$ in terms of quantities $Vol(L) \overset{def.}{=} \frac{\text{vol}(L)}{\text{fluctuation vol.}} \cdot (\pi e)^{\frac{\text{dim}(L)}{2}}$ where $L$ is a factor group/invariant subgroup of the $SMG$ and $\text{vol}(L)$ is the volume of $L$. The “fluctuation volume” is defined as the width of the flat distribution that yields the same entropy as the original distribution; i.e.,

$$\Delta S_{\text{ent}} = - \log(e^{\beta \text{dist}^2}) - \log(\text{"flat distribution"}) > 0.$$ 

For large $\beta$ (weak coupling approximation), we have the approximation $\text{Vol}(L) \approx \beta^{\frac{\text{dim}(L)}{2}} \text{vol}(L)$ which we shall use in the sequel.

For the partially confining phase that is confined w. r. t. the invariant subgroup $H$, the free energy per active link is

$$\log Z_{\text{per active link}} = \max_H \{ \log Z_H | H \subset G \}$$

\[21\]This is consistent with the definition in section 3.3: when fluctuations are so strong that gauge symmetry is not broken by a gauge transformation with a linear gauge function $A_{\text{Linear}}$ (leading to a translation of the gauge potential $A_{\mu}$ by a constant), then the fluctuations can also be assumed to be so strong that the effect of Bianchi identities are washed out.

\[22\]Recall that Bianchi identities impose a constraint (e.g., modulo $2\pi$ for $U(1)$) on the divergence of flux from a volume enclosed by plaquettes.

\[23\]Active link means a link not fixed by a gauge choice in say the axial gauge.
where

\[ \log Z_H = \log \left[ \frac{\pi/6}{\dim(G/H)} \right] ^{\frac{\dim(H)}{2}} + 2 \log \left[ \frac{\pi}{\dim(H)} \right] \]

(17)

\[ = \log \left[ \frac{\pi/6}{\dim(G)} \right] + \log \left[ \frac{6\pi}{\dim(H)} \right] \]

where it is understood that \( \log Z_H \) is calculated using an ansatz that results in confinement for the invariant subgroup \( H \) and Coulomb-like behavior for the factor group \( SMG/H \).

In our approximation (17), it can be shown that at the multiple point, any two invariant subgroups \( H_1 \) and \( H_2 \) of the gauge group must satisfy the condition

\[ \log(\sqrt{6\pi})^{\dim(H_2) - \dim(H_1)} = \log \frac{\text{Vol}(H_2)}{\text{Vol}(H_1)} \]

(18)

where it is recalled that the quantity \( \text{Vol}(H_j) \) is essentially the volume of the subgroup \( H_j \) measured in units of the critical fluctuation volume. In special case where \( H_1 = 1 \), we get

\[ \frac{\log \text{Vol}(H_j)}{\dim(H_j)} = \log(\sqrt{6\pi}); \]

i.e., for any invariant subgroup \( H_j \) the quantity \( \text{Vol}(H_j) \) per Lie algebra dimension must be equal to the same constant ("\( \sqrt{6\pi} \)" at the multiple point.

This factorization property can be realized for all the 13 invariant subgroups \( H < SMG \) (or all the 5 invariant subgroups \( H < U(N) \)) using the previously defined set of constituent invariant subgroups. That is, it is possible to factorize \( \text{Vol}(H) \) into a product of some subset of a common set of 5 factors \( (1/p_i)\text{Vol}(K_i) \) (with \( p_i \in \mathbb{N}^+ \)) (3 such factors for \( U(N) \)) corresponding to the constituent invariant subgroups \( K_i \in \{Z_2, Z_3, U(1), SU(2), SU(3)\} \) for the SMG (for \( U(N) \), the constituent invariant subgroups are \( K_i \in \{SU(N), U(1), Z_N\} \)). Then it is possible by adjustment of the parameters \( \beta_1, \beta_2, \beta_3, \xi_2, \xi_3 \) to make the quantities \( \log Z_{H \text{ per active link}} \) equal for each invariant subgroup \( H < SMG \) (the same applies to each invariant subgroup \( H < U(N) \) using the parameters \( \beta_1, \beta_N, \xi_N \)). This is equivalent to finding a nongeneric multiple point (because \( 5 < n_{SMG} - 1 = 13 - 1 \) for the SMG and \( 3 < n_{U(N)} - 1 = 5 - 1 \) for \( U(N) \).

We explicitly demonstrate the factorizability of \( \text{Vol}(H) \) in the sense that we show that it is of the form \( \text{Vol}(H) = \text{product of some factors}(1/p_i)\text{Vol}(K_i) \) for each invariant subgroup \( H \) of both \( SMG \) and \( U(N) \). To do this, we use a calculational trick in which we replace each \( H \) by a cartesian product group related to \( H \) by a homomorphism that is locally bijective. This (to \( H \)) locally isomorphic cartesian product group consists of the covering Lie (sub)groups corresponding to the gauge degrees of freedom that the invariant subgroup \( H \) involves supplemented by the discrete constituent invariant subgroups contained in these Lie subgroups. For all the invariant subgroups \( H \), the cartesian product group replacement can be obtained by simply omitting factors in the cartesian product group replacement \( Z_2 \times Z_3 \times U(1) \times SU(2) \times SU(3) \) for the whole SMG. Of course such a cartesian product group in general differs in global structure from the invariant subgroup \( H \) that it replaces. However, as we are only interested in the quantity \( \text{Vol}(H) \)
for invariant subgroups \( H \), we can use a correction factor \( 1/p_H \) to adjust the quantity \( \text{Vol} \) of the cartesian product group replacement for \( H \) so as to make it equal to \( \text{Vol}(H) \).

As an example, consider the invariant subgroup \( H = U(2) \subset SMG \) which is locally isomorphic to the cartesian product group \( U(1) \times SU(2) \times \mathbb{Z}_2 \times \mathbb{Z}_3 \). By this we mean that, assuming the modified Manton action \([13]\) and a weak coupling approximation, the cartesian product group \( U(1) \times SU(2) \times \mathbb{Z}_2 \times \mathbb{Z}_3 \) simulates the subgroup \( U(2) \subset SMG \) in the sense that the regions on the group manifold of \( U(2) \subset SMG \) in which the probability distribution \( e^{S_{SM}} \) is concentrated can be brought into a one to one correspondence with centers of fluctuation sharply peaked around points in the cartesian product group. In other words, for \( U(2) \subset SMG \), the region of correspondence with the cartesian product group is the composite of 6 small neighborhoods around the elements \( p \in \text{span}\{\mathbb{Z}_2, \mathbb{Z}_3\} \). Even though the cartesian product group in this example contains \( 2 \cdot 2 \cdot 3 \) elements for each element in \( U(2) \), the action on the cartesian product group is defined so as to be \(-\infty\) everywhere except at one of the these 12 elements where this action then has the same value as the action at corresponding element of \( U(2) \).

In order to make the quantity \( \text{Vol}(U(1) \times SU(2) \times \mathbb{Z}_2 \times \mathbb{Z}_3) \) equal to \( \text{Vol}(U(2)) \) (for \( U(2) \subset SMG \)), the former must be reduced by a factor \( p_{U(2)} \) obtained as follows: Remember that the \( U(1) \) embedded in the \( SMG \) has a length \( 6 \cdot 2\pi \) so that the \( U(2) \) subgroup lying in the \( SMG \) is \( (U(1)_{2\pi} \times SU(2))/\mathbb{Z}_2 \). Comparing the quantity \( \text{Vol}(U(2)) = \text{Vol}(U(1)_{2\pi}) \cdot \text{Vol}(SU(2))/\mathbb{Z}_2 \) and the quantity \( \text{Vol} \) for the locally isomorphic cartesian product group: \( \text{Vol}(U(1)_{2\pi}) \times \text{Vol}(SU(2)) \times \text{Vol}(\mathbb{Z}_2) \times \text{Vol}(\mathbb{Z}_3) \), it is seen that, relative to \( \text{Vol} \) for the cartesian product group, the quantity \( \text{Vol}(U(2)) \) is down by \( (\#\mathbb{Z}_2) \cdot (\#\mathbb{Z}_2) \cdot (\#\mathbb{Z}_3) = 2 \cdot 2 \cdot 3 = 12 \) \( \text{def} \) \( p_{U(2)} \).

In Table 1, we demonstrate explicitly that the volume correction factors \( 1/p_H \) for all the invariant subgroups \( H \subset SMG \) can be factored into a subset of 5 factors \( 1/p_i \) associated with each of the “constituent” invariant subgroups \( K_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, U(1), SU(2), SU(3)\} \). For a given \( i \), \( p_i \) is always the same in any \( p_H \) in which \( p_i \) is a factor. Listed in the first column of Table 1 are the quantities \( \text{Vol}(H) \) for all 13 invariant subgroups \( H \) of the \( SMG \); listed in the second column are the quantities \( \text{Vol} \) for the corresponding, locally isomorphic cartesian product groups. The third column consists of the volume correction factors \( 1/p_H \) by which the quantities \( \text{Vol} \) for the cartesian product group in the second column must be multiplied in order to get the corresponding quantity \( \text{Vol}(H) \) in the first column. In the next five columns, we give the factorization of the correction factors \( 1/p_H \) into subsets of five rational quantities \( 1/p_i \) with \( i \in \{\mathbb{Z}_2, \mathbb{Z}_3, U(1), SU(2), SU(3)\} \) that are associated with the five quantities \( \text{Vol}(\mathbb{Z}_2), \text{Vol}(\mathbb{Z}_3), \text{Vol}(U(1)), \text{Vol}(SU(2)), \text{Vol}(SU(3)) \). Table 2 is constructed in an analogous fashion for the 5 invariant subgroups of \( U(N) \) using \( \mathbb{Z}_N, U(1), \) and \( SU(N) \) as the constituent invariant subgroups. For both the \( SMG \) and \( U(N) \), the important point is that, for any invariant subgroup \( H \), the factorization

\[
\text{Vol}(H) = \prod_i \left( \frac{\text{Vol}(K_i)}{p_i} \right) \quad (i \text{ runs over a subset of constituent invariant subgroups})
\]

\( ^{24}\) We define the quantity \( p_H = \#(H \cap D) \) where \( D \) (which has \( \#(D) = 36 \)) is the discrete subgroup of the center that must be divided out of the cartesian product group \( \mathbb{Z}_2 \times \mathbb{Z}_3 \times U(1) \times SU(2) \times SU(3) \) in order to get the \( SMG \); i.e., \((\mathbb{Z}_2 \times \mathbb{Z}_3 \times U(1) \times SU(2) \times SU(3))/D \cong SMG \overset{\text{def}}{=} S(U(2) \times U(3)) \).
is such that the correction factor $1/p_i$ corresponding to a given constituent invariant subgroup $K_i$ is always the same (unless the quantity $\text{Vol}(K_i)$ is absent in the product (21) in which case there is no entry in the column headed by $1/p_i$) in Tables 1 and 2.

The meeting of 13 partially confining phases at the (nongeneric) multiple point in the phase diagram for the $SMG$ in the 5-dimensional action parameter space is virtually impossible to depict clearly in a figure. However, the group $U(N)$, which has many features in common with the $SMG$, has a phase diagram with a nongeneric multiple point in 3 dimensions when we use an action ansatz analogous to that used for the $SMG$: Gaussian peaks at elements of $\mathbb{Z}_N \subset U(N)$. The phase diagram for $U(2)$ seen in Figures 1 and 2 shows, in our approximation, the 5 partially confining phases (corresponding to the 5 invariant subgroups of a $U(N)$ group) that meet at the multiple point.

### 3.4.3 Need for discrete subgroup parameters

It is instructive to answer the question: Why do we need the discrete group parameters? Recall that in our modified Manton action, there can be sharp “peaks” in the distribution $e^{S_0}$ of plaquette variables centered at nontrivial elements of discrete subgroups. However, to motivate the answer to our question, we revert for a moment to an action $S_0$ leading to a distribution $e^{S_0}$ of plaquette variables with just one “peak” (at the identity) - this is just the normal Manton action. Then the quantities $\text{Vol}$ corresponding to the *same Lie algebra ideals* in the Lie Algebra of the $SMG$ - i.e., volumes measured in units proportional to the fluctuation volume - obey the relations

\begin{align}
\text{Vol}(U(1)_{\text{subgr.}}) &= 6\text{Vol}(SMG/(SU(2) \times SU(3))) \\
\text{Vol}(SU(2)_{\text{subgr.}}) &= 2\text{Vol}(SMG/U(3)) \\
\text{Vol}(SU(3)_{\text{subgr.}}) &= 3\text{Vol}(SMG/U(2)) \\
\text{Vol}(U(2)_{\text{subgr.}}) &= 3\text{Vol}(SMG/SU(3)) \\
\text{Vol}(U(3)_{\text{subgr.}}) &= 2\text{Vol}(SMG/SU(2)) \\
\text{Vol}(SU(2) \times SU(3))_{\text{subgr.}} &= 6\text{Vol}(SMG/U(1)).
\end{align}

(21)

The important feature of this list is that it relates the volumes of subgroups and factor groups that correspond to the same Lie algebra ideals (in the Lie algebra of the $SMG$). It is seen that the volume of a subgroup with a given Lie algebra is larger than a factor group with the same Lie algebra by an integer factor equal to the number of center elements of the subgroup that are identified in the factor group.

Without the extra “peaks” of the modified Manton action, it is not possible to vary the volume on the left hand side of one of the equations (21) independently of the volume on the right hand side. Because of the (nontrivial) integer factor disparity in the volumes on the two sides of the equations (21), it is not possible to have, for example, $\text{Vol}(U(2)_{\text{subgr.}}) = \text{Vol}(SMG/SU(3))$ at the same point in the $\beta$ parameter space. In particular, two such volumes can never have critical values for the same values of the $\beta$’s which means that the two corresponding partially confining phases (i.e., confinement w.r.t. $U(2)$ and $SU(3)$) cannot meet at a multiple point.

This feature is seen in Fig. 2 which shows the phase diagram for the gauge group $U(2)$. In the plane defined by $\log\text{Vol}(Z_2) = \log 2$ (the maximum value of $\log\text{Vol}(Z_2)$), we have the phase diagram corresponding to the normal Manton action (one “peak” in the distribution of plaquette variables, centered at the identity). Due to the fact that say $\text{Vol}(SU(2)_{\text{subgr.}}) = 2\text{Vol}(U(2)/U(1))$ (measured in the same unit of volume which is
the fluctuation volume which in turn is proportional to \( \prod_{\beta \in \{U(1), SU(2), SU(3)\}} (2 \beta - \text{dim}(i)/2) \), it is impossible to have \( \text{Vol}(SU(2)_{\text{subgr}}) = \text{Vol}(U(2)/U(1)) \) for the same values of the \( \beta \) parameters; i.e., because the volumes of the subgroup and factor group corresponding to the same Lie algebra differ by a factor two, these two volumes cannot be critical for the same set of \( \beta \) parameters. This in turn precludes phases partially confined w.r.t. \( SU(2) \) and \( U(1) \) from coming together. In the plane \( \log \text{Vol}(\mathbb{Z}_2) = \log 2 \) of Fig. 2, it is indeed seen that these partially confining phases do not touch; the maximum number of phases that come together in this plane is three (i.e., not all four possible phases) where three is a generic number of phases that can meet in two dimensions.

In order to succeed in having, for example, \( \text{Vol}(SU(2)_{\text{subgr}}) = \text{Vol}(U(2)/U(1)) \) in the case of the group \( U(2) \), it is necessary to introduce a parameter that allows us to change the volume \( \text{Vol}(SU(2)_{\text{subgr}}) \) without changing the volume of the Lie algebra-identical factor group \( \text{Vol}(U(2)/U(1)) \). In this \( U(2) \) case, this is what is achieved by introducing the parameter \( \log \text{Vol}(\mathbb{Z}_2) \) which allows the variation of the relative heights of the distribution "peaks" centered at the two elements of \( \mathbb{Z}_2 \).

By introducing an action giving rise to extra "peaks" in \( e^{\text{Sc}} \) that are centered at elements that are (by definition) identified in a factor group but which can contribute to the volume of the subgroup having the same Lie algebra as the factor group, we gain a way of varying the volume of the subgroup \textit{without} varying the volume of the factor group. In the example of \( U(2) \) referred to above, the possibility of a peak in \( e^{\text{Sc}} \) also at the nontrivial element of \( \mathbb{Z}_2 \) means that the volume of the subgroup \( SU(2) \) can be made up of contributions from both peaks, whereas the volume of the Lie algebra-identical factor group \( U(2)/U(1) \) can only come from the peak at the identity since both elements of \( \mathbb{Z}_2 \) are identified in the factor group.

Relative to the approximate \( U(2) \) phase diagram of Fig. 2, the variation of the parameter \( \log \text{Vol}(\mathbb{Z}_2) \) can be described roughly as follows. Recall from above that in the plane defined by \( \log \text{Vol}(\mathbb{Z}_2) = \log 2 \), there is only one "peak" in the distribution \( e^{\text{Sc}} \) (centered at the identity). In this plane, the quantity \( \text{Vol} \) of the subgroup is identically twice that of the factor group since, by definition, there are only half as many elements (i.e., cosets) in the factor group as there are elements in the subgroup \( SU(2) \) with the same Lie algebra. Now as the value of the parameter \( \log \text{Vol}(\mathbb{Z}_2) \) is reduced, the pattern of fluctuations for the subgroup changes in such a way that a progressively larger proportion of the fluctuations are centered at the nontrivial element of \( \mathbb{Z}_2 \). These latter are not "noticed" by the factor group because fluctuations about the identity and fluctuations about the nontrivial element of \( \mathbb{Z}_2 \) correspond to fluctuations about the same coset of the factor group - namely the identity of the factor group. In our approximation, the parameter \( \log \text{Vol}(\mathbb{Z}_2) \) decreases until the two peaks of \( e^{\text{Sc}} \) (one at each element of \( \mathbb{Z}_2 \)) have the same volume corresponding to half the volume of the subgroup \( SU(2) \) (this happens for \( \log \text{Vol}(\mathbb{Z}_2) = 0 \)); i.e., the volume of \( SU(2) \) is distributed equally between the two peaks. At just this point in the progression to smaller values of \( \log \text{Vol}(\mathbb{Z}_2) \) (i.e., when \( \log \text{Vol}(\mathbb{Z}_2) = 0 \)), the multiple point is encountered. This coincides with the confinement of the \( \mathbb{Z}_2 \subset SU(2) \) and thereby, to the identification of the two peaks each of which contributes half the volume of \( SU(2) \) just prior to the confinement breakdown of the latter to \( SO(3) \). This identification means that the volume of the subgroup \( SU(2) \) is reduced by a factor 2 which is just the factor by which the volume of this subgroup is larger than the Lie algebra-identical factor group \( U(2)/U(1) \) in the absence of the extra parameter \( \log \text{Vol}(\mathbb{Z}_2) \) (i.e., for the normal Manton action with just one "peak" at the
identity). In particular, with the help of the extra parameter \( \log Vol(\mathbb{Z}_2) \), the quantities \( Vol(SU(2))_{\text{subgr}} \) and \( Vol(U(2)/U(1)) \) both can have critical values for the same set of \( \beta \) parameters including the set at the multiple point. This happens because, at the multiple point, the volume of the subgroup \( SU(2) \) is reduced to that of the Lie algebra-identical factor group \( U(2)/U(1) \). More generally, the critical volume associated with a given Lie subalgebra is, at the multiple point, that of the factor group obtained by identifying all elements of discrete subgroups of the center of the covering (sub)group corresponding to this Lie subalgebra.

In summary, this section 3.4 about the generalized action deals with the need for more than the usual number of parameters in the plaquette action if one wants to make the phases corresponding to confinement of the various invariant subgroups - including discrete (invariant) subgroups - share a common point (i.e., the multiple point) in the phase diagram. With our plaquette action parameterization, we can show the existence of and also the coincidence in one multiple point of phases corresponding to all invariant subgroups of the nonabelian components of the \( SMG \). The invariant subgroups that we do not consider here correspond solely to additional discrete (invariant) subgroups of \( U(1) \). The defining feature of a confinement-like phase for an invariant subgroup is the assumption that Bianchi identity constraints can be neglected for such a phase in a crude weak coupling approximation using a mean field approximation.

At the multiple point, we are dealing with first order phase transitions; therefore, a priori at least, our multiple point principle suffers from lack of universality. However, the fact that a weak coupling approximation is at least approximately applicable - even for the determination of critical couplings - leads to the irrelevance of terms greater than second order in Taylor expansions of the action and consequently fosters the hope of an approximative universality.

4 Correction due to quantum fluctuations

In our model, the \( SMG \) gauge coupling constants are to be identified with the couplings for the diagonal subgroup that results from the Planck scale breakdown of \( SMG^3 \). While in the naive continuum limit, the diagonal subgroup field configurations consist (by definition) of excitations that are identical for the \( N_{\text{gen.}} = 3 \) copies (labelled by names “Peter”, “Paul”, · · ·) of any \( SMG \) gauge degree of freedom \( A_{\mu}^b \); a more realistic view must take into account that the \( N_{\text{gen.}} \) copies of \( A_{\mu}^b \) in \( SMG^3 \): \( A_{\mu, \text{Peter}}^b, A_{\mu, \text{Paul}}^b, \cdots, A_{\mu, N_{\text{gen.}}}^b \) undergo quantum fluctuations relative to each other. In this section this correction is first estimated for a confinement-like phase (hereby justifying a disregard of Bianchi identities) and subsequently corrected so as to be approximately correct for a Coulomb-like phase.

Including the effect of fluctuations of a general quantum field \( \theta \) in the continuum limit is done using the effective action \( \Gamma[\theta_{c.d.}] \):

\[
\Gamma[\theta_{c.d.}] = S[\theta_{c.d.}] - \frac{1}{2} \text{Tr}(\log'(S''[\theta_{c.d.}])).
\] (22)

The correction to the continuum couplings that we calculate below consists in identifying the classical continuum action \( \int d^4x \frac{1}{4g}(gF_{\mu\nu})^2 \) with the effective action \( \Gamma \) - instead of with the lattice action \( S \) - in the naive continuum limit approximation. In calculating
this correction, we ignore nonabelian effects and assume that the action $S_{\text{Monte Carlo}}$ used in the literature\cite{20, 21, 22} deviates only slightly from the Manton action for which the Tr log correction is simply a constant. The $S_{\text{Monte Carlo}}$ could for example be the popular cosine action in the $U(1)$ case. First, however, we note that a change in the functional form of the action by $\delta S(\theta_{ct})$ leads to a functional change in the effective action $\Gamma(\theta_{ct})$ that differs from $\delta S(\theta_{ct})$ by a term proportional to $\frac{\text{Tr} \delta S''[\theta_{ct}]}{S''[\theta_{ct}]}$:

$$\delta \Gamma[\theta_{ct}] = \delta S[\theta_{ct}] + \frac{1}{2} \text{Tr}(\delta \log(S''[\theta_{ct}])) = \delta S[\theta_{ct}] + \frac{1}{2} \text{Tr} \left( \frac{\delta S''[\theta_{ct}]}{S''[\theta_{ct}]} \right)$$  \hspace{1cm} (23)

But as we are assuming that the variation $\delta S[\theta_{ct}]$ is done relative to the Manton action, we have

$$\frac{1}{2} \text{Tr} \left( \frac{\delta S''[\theta_{ct}]}{S''_{\text{Manton}}[\theta_{ct}]} \right) = \frac{1}{2} \text{Tr} \left( \delta S'[\theta_{ct}]/(\theta - \theta_{ct})^2 \right)$$ \hspace{1cm} (24)

where we have used that $S''_{\text{Manton}}[\theta_{ct}] \propto (\theta - \theta_{ct})^{-1} = \text{const.}$ and that, up to a constant, $\delta(S''[\theta_{ct}]) = S''[\theta_{ct}]$ (modulo a constant).

Neglecting nonabelian effects, we generalize this result to nonabelian gauge groups and write it more concretely using $U = e^{i\theta a e^i}$ and $S[U] = \sum_{\alpha} S_{\alpha}(U(\Box))$:

$$\Gamma[U_{ct}] = S[U_{ct}] + \frac{1}{2} \Delta S_{\Box}(U_{ct}(\Box))/((\theta_{\alpha}^a(\Box) - \theta_{\alpha}^a(\Box))^2) \hspace{1cm} \text{(summation over } a)$$ \hspace{1cm} (25)

We have for the Laplace-Beltrami operator

$$\frac{1}{2} \Delta S(U(\Box)) \overset{\text{def.}}{=} \lim_{\epsilon \to 0^+} \int d^{N^2-1}f \frac{\exp(-\frac{1}{2} f_a^2)S(U, e^{i\epsilon f_a}) - S(U)}{\int d^{N^2-1}f \exp(-\frac{1}{2} f_a^2) f_c^2} \hspace{1cm} \text{(sum over } a, b, d, e).$$ \hspace{1cm} (26)

where $f$ and $t$ denote respectively the $a$th Lie algebra component and Lie algebra generator. Upon expanding (in the representation $r$) the exponential $\exp(i f_b T_{b,r})$ representing $\exp(i f_a t_b)$ the argument of which is assumed to be small inasmuch as the $f_a$ are assumed to be small, one obtain

$$= \lim_{\epsilon \to 0^+} \sum_{r} \int \frac{\beta_r \exp(-\frac{1}{2} f_a^2) \text{Tr}(U \cdot (-\frac{1}{2} f_b T_{b,r} T_{c,r}))}{\int \exp(-\frac{1}{2} f_a^2) f_c^2} \hspace{1cm} \text{(sum over } a, b, c, d, e)$$ \hspace{1cm} (27)

$$= \sum_{r} \frac{\beta_r}{d_r} \text{Tr}_r(U \cdot (-\frac{1}{2} (T_{b,r})^2)) \hspace{1cm} \text{(sum over } b)$$ \hspace{1cm} (28)

where we have expanded the plaquette action in characters: for the representation $r$ of dimension $d$ the character $\chi_r$ is given by $\chi_r = \text{Tr}_r(U(\Box))$ and

$$S(U(\Box)) = \sum_r \frac{\beta_r}{d_r} \text{Tr}_r(U(\Box)).$$ \hspace{1cm} (29)

We have

$$\frac{1}{2} \Delta S(U(\Box)) = \sum_r -\frac{1}{2} \frac{\beta_r}{d_r} \text{Tr}_r(U(\Box)) \frac{C_r(2)}{N^2 - 1}$$ \hspace{1cm} (30)
where $C_r^{(2)}$ is the quadratic Casimir for the representation $r$. The Casimir is defined as $C_r^{(2)} \equiv \sum_b (T_{b,r})^2$. For the groups $SU(2)$ and $SU(3)$, the Lie algebra bases in the fundamental (defining) representations are taken respectively as $T_{b,r=2} = \frac{a^b}{2}$ and $T_{b,r=2} = \frac{\lambda^b}{2}$. The subscript $f$ denotes the fundamental representation; $\sigma^b$ and $\lambda^b$ are the Pauli and Gell-Mann matrices with the normalization $\text{Tr} (\sigma^a \sigma^b) = \frac{\delta^b}{2}$ and $\text{Tr} (\lambda^a \lambda^b) = \frac{\delta^b}{2}$. With this basis convention, and with the left-handed quark doublet field as an example, the covariant derivative is

$$D_{\mu}^{\ j} \beta = \partial_{\mu} \delta^{\ j}_{\alpha} \delta_{\beta}^{\ \alpha} - ig_2 A^b_{\mu} (\sigma^b)^{\ j}_{\alpha} \delta_{\beta}^{\ \alpha} - ig_3 A^b_{\mu} (\lambda^b)^{\ j}_{\alpha} \delta_{\beta}^{\ \alpha} - ig_1 \frac{1}{6} A^b_{\mu} \delta^{\ j}_{\beta} \delta^{\ \alpha}_{\alpha}. \quad (31)$$

where the index $b$ labels Lie algebra components, the indices $i, j$ label matrix elements of the (2-dimensional) fundamental representation of $SU(2)$, and the indices $\alpha, \beta$ label the matrix elements of the (3-dimensional) fundamental representation of $SU(3)$. The factor $\frac{1}{6}$ in the last term is the $U(1)$ quantum number $\frac{y}{2}$ where $y$ is weak hypercharge; the convention used is $Q = \frac{y}{2} + I_W$.

The above convention for the generators of $SU(2)$ and $SU(3)$ in the fundamental representation $f$ leads to a Casimir $C_f^{(2)} = \frac{N^2 - 1}{2N^2}$ for an $SU(N)$ group. From this it follows that, for the adjoint representation (denoted by $\text{adj}$.), the Casimir $C_{\text{adj}}^{(2)}$ for an $SU(N)$ group is given by $C_{\text{adj}}^{(2)} = N$.

Ignoring Bianchi identities, we get for the deviations

$$\langle (\theta_{\text{Peter}} - \theta_{\text{diag}})_{\alpha} \rangle = \frac{N^2 - 1}{2} \left( \frac{1}{2} \sum_r \beta_r \frac{C_r^{(2)}}{N^2 - 1} \right)^{-1} \text{(sum over } a) \text{ (confinement phase)} \quad (32)$$

$$= - \left( \frac{\beta_f}{d_f} \text{Tr}_f (U_{\text{diag}} (\Box)) C_f^{(2)} + \frac{\beta_{\text{adj}}}{d_{\text{adj}}} C_{\text{adj}}^{(2)} \text{Tr}_{\text{adj}} (U_{\text{diag}} (\Box)) \right) \frac{N^2 - 1}{2\beta_f C_f^{(2)} + \beta_{\text{adj}} C_{\text{adj}}^{(2)}}.$$ 

Letting the sum over representations run only over the fundamental (=defining) and adjoint representations labelled respectively by the subscripts $f$ and $\text{adj}$ (the only representations used in the Monte Carlo runs of references [20, 21, 22]), we get for the effective action [23]

$$\Gamma (U_{\text{diag}} (\Box)) = \quad (33)$$

$$= \frac{\beta_f}{d_f} \text{Tr}_f (U_{\text{diag}} (\Box)) \left( 1 - \frac{C_f^{(2)} (N^2 - 1)}{2(\beta_f C_f^{(2)} + \beta_{\text{adj}} C_{\text{adj}}^{(2)})} \right) + \frac{\beta_{\text{adj}}}{d_{\text{adj}}} \text{Tr}_{\text{adj}} (U_{\text{diag}} (\Box)) \left( 1 - \frac{C_{\text{adj}}^{(2)} (N^2 - 1)}{2(\beta_f C_f^{(2)} + \beta_{\text{adj}} C_{\text{adj}}^{(2)})} \right).$$

So with the continuum correction we have to make the replacement

$$\beta_r \rightarrow \beta_r (1 - \frac{C_r^{(2)} (N^2 - 1)}{2 \sum_r \beta_r C_r^{(2)}}) \text{ (for } \text{“confinement phase”}) \quad (34)$$

This expression for the effective action has been obtained using the approximation that all plaquette variables can be regarded as independent (i.e., Bianchi identities have been disregarded). This approximation is appropriate for the confinement phase. However, as we are interested in criticality as approached from the Coulomb phase (i.e., Coulomb
phase in our scale dependent sense), we want the quantum fluctuation correction in this phase where Bianchi identities must be respected. These identities reduce the number of degrees of freedom per plaquette that can fluctuate independently by a factor 2. In going to the Coulomb phase, the continuum-corrected $\beta_r$ is modified as follows:

$$\beta_r(1 - \frac{C_r}{2} \beta_r C_r^2)_{\text{confinement}} \rightarrow \beta_r(1 - \frac{C_r}{4} \beta_r C_r^2)_{\text{Coul. phase}}$$

Without the continuum correction, we have for the fine structure constants at the multiple (i.e., triple ) point

$$\frac{1}{\alpha_{\text{triple point}, \text{no cont.}}} = 4\pi \sum_r C_r^2 \beta_{\text{r, triple point}} N_r - 1$$

(36)

With the continuum-corrected $\beta_r$ in the Coulomb phase we have for the fine structure constants at the triple point

$$\frac{1}{\alpha_{\text{triple point, cont.}}} = 4\pi \sum_r C_r^2 \beta_{\text{r, triple point}} (1 - \frac{C_r^2}{4} \beta_{\text{r, triple point}} C_r^2)$$

(37)

5 Calculation of critical couplings at Planck scale

It can be argued that at the multiple point of the phase diagram for the whole $SMG^3 = SMG \times SMG \times SMG$, the nonabelian (plaquette) action parameters for each of the three cartesian product factors take the same values as at the multiple point for a single gauge group $SMG$. This allows us to determine the multiple point action parameters for the gauge group $SMG^3$ from a knowledge of the multiple point action parameters for just one of the $SMG$ factors of $SMG^3$. Accordingly, we can calculate the multiple point critical couplings from the couplings for the isolated $SU(2)$ and $SU(3)$ groups. To this end, we have used figures from the literature [20, 21, 22] to graphically extract the coordinates $(\beta_f, \beta_{\text{adj.}})_{\text{triple point}}$ of the triple point:

For $SU(2)$: $(\beta_f, \beta_{\text{adj.}})_{\text{triple point}} = (0.54, 2.4)$

For $SU(3)$: $(\beta_f, \beta_{\text{adj.}})_{\text{triple point}} = (0.8, 5.4)$

The calculation of $\alpha_2^{-1}$ and $\alpha_3^{-1}$ are presented in Tables 3 and 4 respectively. In these tables, the subscripts $\text{adj.}$ and $f$ denote respectively the adjoint and fundamental representations of the groups considered.

In order to get an idea of the order of magnitude of the error involved in estimating the average over the Laplace-Beltrami of the plaquette action only to next to lowest order, we make use of the fact that we can calculate this average to all orders in the case of a $\cos \theta$ action for a $U(1)$ gauge theory. In this case the averaging is readily performed and leads to an exponential for which the first terms of a Taylor expansion coincide with the terms we calculated using [4]. This suggests that also in the nonabelian cases it might be quite reasonable to “exponentiate” our “continuum” corrections and subsequently use the change made by such a procedure as a crude estimate of the error due to our omission of the second order perturbative terms. By exponentiated continuum corrections we mean by definition that, instead of the replacements [34] and [35], we use respectively
\[
\beta_r \rightarrow \beta_r \exp\left(-\frac{C_r^{(2)}(N^2 - 1)}{2 \sum \hat{\beta}_r C_r^{(2)}}\right) \quad \text{(for “confinement”)}
\]

(38)

and

\[
\beta_r \rightarrow \beta_r \exp\left(-\frac{C_r^{(2)}(N^2 - 1)}{4 \sum \hat{\beta}_r C_r^{(2)}}\right) \quad \text{(for “Coulomb” phase)}
\]

(39)

As evidenced by Tables 1 and 2, this exponentiation yields a change of the order of one unit in \(1/\alpha_{\text{crit.}, \text{cont.}} \approx 20\) from which we can estimate the uncertainty due to the neglect of higher order terms as being of the order of say 5%. While we are on the subject of uncertainty, we should mention that we have even ignored a relatively little term\(^{27}\) having the same order in perturbation in \(1/\beta\) as the above continuum correction. In the abelian case, this term is \(1/4\) of the continuum correction. Because this term vanishes for \(SU(N)\) in the large \(N\) limit, it is even smaller for the couplings for \(SU(2)\) and \(SU(3)\).

Since our deviations from the experimental couplings extrapolated to the Planck scale\(^{16, 17}\) are of the same order of magnitude as the uncertainty in the Monte Carlo data and the uncertainty due to chopping off the higher order continuum corrections, a calculation of the next order corrections and increased accuracy in the calculations are called for in order to determine if our deviations are significant.

6 Proposed model for the stability of the multiple point

We propose a mechanism for the stability of the multiple point that is based on a model which could be called a “nonlocal gauge glass model”\(^25\), which is very much inspired by the project of “random dynamics”\(^6\). The essential feature is the influence of a bias effect that can occur in the presence of a plaquette (or multi-plaquette) action the functional form of which is taken to be quenched random. This could mean that for each Wilson loop \(\Gamma\), the coefficients (called the “\(\beta\)’s”) in say a character expansion of the Wilson loop action are fixed at the outset as random values and remain fixed during the evaluation of the functional integral. While translational invariance is broken at least at small scales because a different set of random \(\beta\)’s is associated with each Wilson loop, it is presumably regained at least approximately in going to large distances inasmuch as it is assumed that the statistical distribution of quenched random variables is translationally invariant.

Randomly weighted terms in the action from the different Wilson loops would on the average contribute nothing to the inverse squared coupling were it not for the bias: the vacuum dominant value of a Wilson loop variable (a point in the gauge group) is correlated with the values of the quenched random coefficients for the Wilson loop under consideration. This correlation comes about because the vacuum field configuration\(^26\)

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\(^{25}\)The term “gauge glass” was appropriately coined by Jeff Greensite by analogy to a spin glass which is so named because the “frozen in” structure is reminiscent of that of glass.

\(^{26}\)Note that we envision a relatively complicated vacuum state in which the link or rather plaquette variables fluctuate around other elements than the unit element. However, these “other elements” must necessarily be elements of the center if “collapse” (\(\approx\) Higgs-like behavior) is to be avoided; this may require a connected center\(^{23}\) for the group that extends almost densely over the group.
adjustments resulting from the tendency to approximately maximize the exponential of the action $\exp(S)$ as a function of link variables will concurrently tend to make the second derivative w. r. t. Wilson loop variables of $\exp(S)$ more negative.

In the simplest model, the gauge glass we use is rather strongly nonlocal because we assume that the quenched random contributions to the action are not restricted to contributions from elementary plaquettes, but in principle include all Wilson loops. If this should lead to problems with locality, we can postulate that only loops up to some finite size are present in the action since the most crucial prerequisite for the bias mechanism is the inclusion of many Wilson loops with the size distribution being of only secondary importance.

The bias effect can be formulated as an additional term in the Callan-Symanzik $\beta$-functions (in addition to the normal renormalization group contribution). To see this, envision a series of calculations of the effective couplings $g(\mu)$ for successively larger inverse energies $\mu^{-1}$. For each value of $\mu^{-1}$, Wilson loops of size up $\mu^{-1}$ are included in computing $g(\mu)$; therefore a calculation of $g(\mu)$ includes more and more Wilson loops in going towards the infrared. The inclusion of progressively longer and longer loops takes place in a background field made up of contributions from the already included smaller loops that are approximately described as a background continuum Lagrange density $-\frac{1}{4g^2(\mu)}F_{\mu\nu}^2$. This process, in which the coupling $g(\mu)$ becomes smaller and smaller the more loops it accounts for, culminates in $g(\mu)$ attaining the critical value whereupon the influence of additional loops on the vacuum configuration is drastically diminished because the transition to a $g(\mu)$ corresponding to the Coulomb phase leads to a vacuum configuration that is much less readily influenced than in the confinement phase. Contributions from larger Wilson loops are no longer correlated with the vacuum dominant field configuration that is almost solely determined by the Wilson loops of smaller spatial extent. Without the “protection” of the bias effect, the contributions from these larger loops cancel out on the average because of the assumed randomness in the signs of action terms with the result that the effective couplings will no longer be modified much by the inclusion of larger Wilson loops that show up in going to larger length scales.

The variation of the effective coupling due to the bias effect might formally be included in a generalized Callan-Symanzik $\beta$-function. (actually we mean a multicomponent vector of generalized $\beta$-functions with one component for each parameter of a single plaquette action of a coarse-grained lattice at the scale $\mu$). These generalized $\beta$-functions (i.e., the components of the vector of generalized $\beta$-functions) contain contributions that take into account that the part of the Lagrangian of the theory that is used to define gauge couplings $g(\mu)$ is changing as we go to larger length scales. That this change has a nonvanishing average effect on the couplings is due to the bias effect. These extra contributions to the $\beta$-functions, which are in addition to the normal renormalization group effects, make the generalized $\beta$-functions explicitly scale dependent. Specifically, we envision rapid variations of the $\beta$-functions as the bias effect is drastically weakened at the transition to a Coulomb-like phase. If the $\beta$-functions become zero, this would result in an infrared attractive fixed point near the phase transitions at the multiple point.

An important point is that multiple point criticality is implied by almost any mechanism that drives a gauge coupling to a critical value because a mechanism that seeks out the critical coupling for some gauge group will probably function in the same way for all invariant subgroups of a gauge group. But this is tantamount to seeking out the multiple point which by definition is the point or surface in the phase diagram at the bor-
derline between confining and Coulomb phases for all invariant subgroups. In particular, our model as outlined above would imply that Wilson loop contributions $\prod_{q \in \Gamma} U(q)$ depending only on the cosets in $G/H$ w.r.t. some invariant subgroup $H$ would become very ineffective in bringing about a further increase in the inverse squared couplings (for the degrees of freedom corresponding to the factor group) once it is only the invariant subgroup $H$ of the group $G$ that remains “confining”; in other words, the couplings for $G/H$ stop falling (in the crudest approximation) once $G/H$ “reaches the Coulomb phase”.

Several alternatives to the nonlocal gauge glass explanation for the stability of the multiple point (assuming it exists) have been considered. We have, for example, in previous work, prior to the advent of the principle of multiple point criticality, used the entropy as the quantity to be maximized in the predictions of gauge couplings from criticality. In this earlier work, we have for the nonabelian Lie subgroups of the $SMG^3$ considered criticality only w.r.t. the Lie subgroups $SU(2)$ and $SU(3)$ and not for criticality w.r.t. the $Z_2$ and $Z_3$ discrete invariant subgroups of, respectively, $SU(2)$ and $SU(3)$. We found that the entropy, calculated to lowest order, was to first approximation constant on an interface of finite extent that separated the totally Coulomb-like and totally confinement phases in the parameter space of the $SMG$ phase diagram. In effect, this interface prohibited other partially confining phases from meeting at a multiple point. We now find that the addition of action parameters that also allow the discrete invariant subgroups to become critical results in the shrinking of this interface into a point that coincides with the multiple point. This can be expected to affect the entropy because, at the multiple point, we are also on the verge of confinement for the groups $Z_2$ and $Z_3$. This means that a small change in the appropriate action parameters can bring about a transition from a Coulomb-like phase to a confinement-like phase with the difference between the two being, for example, defined by the respective perimeter and area law decay of Wilson loops (for charges $1/2$ or $1/3$ in the case of $Z_2$ and $Z_3$ respectively).

In the action parameter space that includes parameters that can be adjusted so as to have criticality w.r.t. the discrete invariant subgroups $Z_2$ and $Z_3$, the entropy is constant to lowest order along a (hyper)surface separating the totally Coulomb from totally confined phases. But a calculation to next order appears to lead to the conclusion that the entropy is not maximum at the multiple point thereby obviating the idea of maximum entropy as an explanation for the multiple point.

However, it can be claimed that the multiple point is such a characteristic “corner” of the phase diagram that it is extremely likely that there is some relevant physical quantity or property that is extremized at this point. A possible scenario that might in part rescue the maximum entropy idea is that, at the multiple point, there are strong fluctuations along the discrete subgroup directions of the gauge group that, for given entropy, might be very effective in preventing potential Higg’s fields from bringing the model into a Higg’s phase. In other words, the entropy that comes from the “discretized” lack of knowledge (as to which element of the discrete invariant subgroups in the neighborhood of which the plaquette variable takes a value) may function better in suppressing the tendency for “Higgsning” than the same amount of entropy arising from fluctuations within the individual Gaussian distributions $e^{S_{Manton}}$ centered at the elements of $Z_2$ and $Z_3$. If this were true, one might use Higgs suppression as the property to be optimized at the multiple point.

Yet another admittedly rather speculative approach to explaining the multiple point suggest that the functional integral for the partition function in baby universe theory
should have a maximal value at the multiple point[24].

Meaningful continuum couplings for lattice gauge theories do not exist for couplings that exceed the critical values[15, 25]. This is corroborated by the observation[25] that Mitrushkin[26] only formally obtains a strong continuum coupling in the Coulomb phase.

In summary, we have in this section supplemented the postulate of the principle of multiple point criticality with proposals as to how a stable Planck scale multiple point might be realized. To this end, we described a gauge glass model inspired by random dynamics. This model, which uses a quenched random action, has a bias causing weaker couplings that is discontinuously diminished at the multiple point. This leads to a zero of a generalized Callan-Zymanzik $\beta$ function thereby establishing the multiple point as an approximate “infrared stable” fixed point.

The speculative nature of these arguments in no way detracts from the most important justification for the principle which is the noteworthy phenomenological success.

7 Is our “anti-unified multiple point model” consistent with established results?

Since the breakdown of our “anti-unified” $SMG^3$ to the diagonal subgroup (that is identified with the standard model group) occurs at a scale close to the Planck scale, we predict pure standard model results with an accuracy that is even greater than $SU(5)$ G.U.T. with or without supersymmetry (both of which have (on a log scale) a slightly lower unification scale than the Planck scale).

Since our good agreement for coupling constant values would not tolerate the survival of supersymmetry down to experimentally accessible scales, we accordingly predict that supersymmetric partners, if they exist at all, cannot, in practice at least, be observed. What could be tolerated in our model, which a priori assumes a desert in the renormalization group extrapolation to the Planck scale, are fields that would not appreciably affect the renormalization group development of the finestructure constants.

As alluded to in the Introduction, our model also predicts, as a consequence of the multiple point criticality, light (compared to Planck scale) scalars provided that this principle is extended to mean that Nature seeks parameter values that also are on the boundaries of Higgs phases of various types. Assuming weak first order (i.e., approximately second order) phase transitions at these boundaries, the predicted masses would presumably in most cases be a few orders of magnitude under the Planck mass. If some of the phase transitions happen to be very nearly second order, the corresponding scalars might turn out to have experimentally accessible masses. For the $\beta$-function as it pertains to the gauge couplings, the contributions from a scalar is smaller than from a fermion or a Yang-Mills particle in the same representation. Therefore, these scalars are not expected to influence our predictions appreciably provided only few of them have very small masses compared to the Planck scale.

Proton decay at the $SU(5)$ unification scale is predicted to be absent in our model because we simply do not have $SU(5)$ in our model. Above the $SMG^3$ breakdown scale (still approximately the Planck scale) where also in our model almost anything can happen, baryon violation may take place. The fact that baryon violation would first occur at the Planck scale would imply a long proton lifetime compared to present experimental limits if in fact it decays at all.
With the exception of predictions of parameters of the standard model, our model has only very limited possibilities for conflicts with present day experimental results because present experiments agree with the same pure standard model (including the desert!) which we essentially predict almost up to the Planck scale. However, from the viewpoint of astrophysics, our model may make some predictions: since the homotopic group

\[ \Pi_1(SMG^3/SMG_{diag}) \cong \Pi_1(U(1)^3/U(1)_{diag}) \cong \Pi_1(U(1)^2) \cong \mathbb{Z}^2 \]

of the space of Higgs vacuum states could lead to three types of stable cosmic strings that can branch into each other, a scenario with galaxy formation caused by cosmic strings would have some details modified in a welcomed way\[28\].

Although still suppressed compared to a Planck scale energy density, the cosmic strings of our model are expected to have high energy density compared to strings related to an \(SU(5)\)-scale.

New physical predictions may follow from our more general plaquette actions \(S_\square\) having parameters that can be adjusted so as to have distributions \(e^{eS_\square}\) of varying width that are concentrated within say an invariant subgroup \(\mathbb{Z}_N\) of the \(SMG\) (independent of the distribution widths along other invariant subgroups). Roughly our multiple point principle means that both discrete and Lie subgroups are simultaneously on the border between confinement and Coulomb. Now discrete gauge theories are really string theories in the sense that - in the Coulomb phase - magnetic field lines are (dual) strings. In the confinement phase, the magnetic field lines condense in the vacuum and electric field lines appear as strings. Even for first order phase transitions, the string tension of the magnetic/electric flux lines is at a minimum infinitesimally close to the transition (but still in Coulomb/confinement phase) because it approaches zero in the opposite phase (i.e, the confinement/Coulomb phase). If the discrete group deconfinement transition should happen to be second order or very weak first order (e.g., this might be the case for a \(\mathbb{Z}_N\) with sufficiently large \(N\)), the string tensions at the transition would be zero (or small).

Thinking of a \(\mathbb{Z}_N\) discrete subgroup, magnetic flux concentrated in flux tubes having units of \(2\pi/N\) could have very low energy. These flux tubes would appear as strings between monopoles of charge \(2\pi/N\) or form closed loops that could conceivably be found at energies aspired to experimentally.

A speculative connection between our model with the gauge group \(SMG^3\) and experimentally observable data can be established by adding the assumption that the mass hierarchy of quarks and leptons is due to an approximate conservation of the quantum numbers of our gauge model. This amounts to having small expectation values for the Higgs fields used to break \(SMG^3\) to \(SMG\). A recently performed\[18\] analysis of this type lead to the conclusion that the high top quark mass poses a problem for our model unless the model is endowed with an additional conserved quantum number (for an extra \(U(1)\)).

8 Conclusion

In this paper, we predict the values of the fine structure constants realized by Nature at the Planck scale for the nonabelian gauge groups \(SU(2)\) and \(SU(3)\) that agree with experiment to within our calculational accuracy. The prediction is that these coincide with the continuum limit of the diagonal subgroup couplings corresponding to multiple point values of the action parameters of a lattice gauge theory with the nonsimple gauge
group $SMG^{N_{gen.}} = SMG^3 = SMG \times SMG \times SMG$. Here $N_{gen.} = 3$ is the number of quark and lepton generations and $SMG$ denotes the standard model group. It is assumed in our model that the more fundamental gauge group $SMG^3$ breaks down at the Planck scale to the diagonal subgroup which is by definition identical with the standard model group. The multiple point in the phase diagram of a lattice gauge theory having a nonsimple gauge group is a point at which there are critical values for a maximum number of the action parameters; at this point, infinitesimal variations of these parameters can provoke “confinement-like phases” corresponding to each (or at least many) of the invariant subgroups of the gauge group. The criticality referred to here is that for the bulk phase transition of a (Euclideanized) lattice gauge theory. The experimental values for the W-couplings and the QCD gluon couplings after extrapolation to the Planck scale\cite{16, 17} are compared with multiple point couplings obtained by correcting data from Monte Carlo results\cite{20, 21, 22}. When these corrections to the critical couplings at the multiple point are taken into account, we find that, to within a 7% accuracy, there is a desired factor three deviation between the values for extrapolated continuum couplings for $SU(2)$ and $SU(3)$ and the triple point couplings for single $SU(2)$ and $SU(3)$ gauge groups. It is remarkable that this is true with an accuracy that is comparable to the uncertainty that comes about due to our having only performed the corrections connecting the lattice action $\beta$-parameters with the continuum couplings to next to lowest order in perturbation theory. That is to say that, to lowest order perturbation in $\beta^{-1}$ for continuum corrections, the exact validity of the factor 3 between the extrapolated experimental coupling values and the multiple point values cannot be excluded. As regards the factor 3, it should also be mentioned that our accuracy of 7% is on the verge of being good enough to be taken as an indication that the ratio $\alpha_{multi.\ point}/\alpha$ has an integer value. Having this “3” amounts to saying that the extrapolated fine structure constants are just three times as small as the couplings corresponding to the multiple point action parameter values of the lattice gauge theory with a single $SMG$. But the factor 3 is “explained” by our postulate of the Planck scale breakdown of the “anti-unified” gauge group $SMG^3$ to the diagonal subgroup. This follows because the diagonal subgroup has inverse squared couplings that are the sum of the inverse squared couplings for each of the $N_{gen.} = 3$ $SMG$ factors in $SMG^3$. A possible mechanism for this breakdown is that of “confusion”\cite{3, 4} which causes the gauge group $SMG^3$ to break down to the diagonal subgroup $SMG_{diag}$; the latter consists of those group elements of $SMG^3$ that are left invariant under the automorphism group (the latter being the group of permutations of the $SMG$ factors in the cartesian product $SMG^3$).

It can be argued that at the multiple point of the phase diagram for the whole $SMG^3 = SMG \times SMG \times SMG$, the nonabelian (plaquette) action parameters for each of the three cartesian product factors take the same values as at the multiple point point for a single $SMG$. This allows us to determine the multiple point action parameters for the gauge group $SMG^3$ from a knowledge of the multiple point action parameters for just one of the $SMG$ factors of $SMG^3$. We mention in passing that this is not so for the abelian couplings; here it is necessary to seek the value at the multiple point for the whole $U(1)^3$. The reason for this is related to the possibility that, for $U(1)$, there can be “mixed” terms $F_{\mu \nu}^{Peter} F^{\mu \nu}^{Paul}$ even in the continuum lagrangian as opposed to the case for nonabelian degrees of freedom where only quadratic terms $F_{\mu \nu}^{Peter} F^{\mu \nu}^{Peter}$ appear (the names “Peter” and “Paul” label a pair of the $SMG$ factors of $SMG^3 = SMG_{Peter} \times SMG_{Paul} \times SMG_{Maria}$). This complication may well lead to a phenomenologically desired factor 6 for the abelian coupling instead of the factor 3 obtained for the nonabelian
couplings.

We think that our accuracy of 7% or so for the (inverse) fine structure constants at the Planck scale is already so impressive that it strongly suggests that there may be some truth behind the part of our model that is relevant for obtaining this result. Furthermore, the validity of our arguments is presumably rather insensitive to whether we use precisely the gauge group $SMG^3$; indeed it is presumably sufficient that the nonabelian groups $SU(2)$ and $SU(3)$ are imbedded as diagonal subgroups in whatever the unifying or anti-unifying group might be. As long as the nonabelian subgroups appear as the diagonal subgroup of three isomorphic subgroups, it would not affect our result if there were other cartesian product factors involving again the same or other subgroups in the anti-unifying group. However, if the gauge group were embedded in a larger simple group - such as Georgi-Glashow $SU(5)$, it would impair the agreement with our model as would supersymmetry if it were not broken close to the Planck scale so as not to invalidate our use of the “desert” extrapolation of experimental couplings to the Planck scale.

We offer a speculative theoretical explanation for the stability of the multiple point the essence of which is that the inclusion of more and more Wilson loops with quenched random coefficients in the action makes the continuum coupling decrease rapidly as long as the theory is in the “confinement-like” phase. However, at the phase transition (i.e., multiple point), we speculatively predict that this tendency for couplings to continue to become weaker is so rapidly attenuated that it suddenly can be compensated by normal renormalization group effects already at energies of the order of the Planck scale. This could lead to a zero of the effective $\beta$-function (actually one should think of a $\beta$-function vector with one component for each of the action parameters) for coupling values close to those at the multiple point which then effectively functions as an “infrared stable” fixed point (really an “infrared” attractive surface).

While we purport to have demonstrated the phenomenological relevance of the principle of multiple point criticality for the nonabelian couplings, we point out that there are as yet unresolved problems with the group $U(1)$ because of the infinite number of invariant subgroups and corresponding phases expected for $U(1)^3$. However, we have made recent progress suggesting that, while it is presumably not possible to find a multiple point corresponding to an action parameter set with critical values for all invariant subgroups of $U(1)^3$, a candidate for a “maximal multiple point”, i.e., a multiple point in contact with a maximal number of phases, may exist in terms of a set of coupling parameters that comes about by requiring a “tightest packing” of points in a $N_{gen} = 3$ dimensional identification lattice embedded in a space with a metric that is used to define the Manton action. At first sight, this set of coupling parameters indeed appears to represent a point on the boundary between a totally Coulomb phase and a “large” number of phases that are confined w. r. t. various invariant subgroups. However, upon closer inspection, this candidate for a “maximal multiple point” turns out to lie in a totally confined region of the action parameter space. This problem may however be resolvable by supplementing the Manton action with extra terms in such a way as to relocate the phase boundaries. If this is feasible and if such a “maximal multiple point” is realized by Nature at the Planck scale in the same way as we argue for the multiple point criticality for the nonabelian groups, then, in the context of the anti-unification scheme described in section 2, the diagonal subgroup inverse fine structure constant would be related to the simple critical

\footnote{It is an amusing coincidence\cite{10} however that the value of the SUSY $SU(5)$ coupling at the unifying scale say is not so far from being at the multiple point for a single $SU(5)$.}
coupling \( \alpha_{U(1) \text{crit}} \) approximately through the relation

\[
\frac{1}{\alpha_{\text{diag}}} = \frac{1}{\alpha_{\text{crit}}} \left( N_{\text{gen}} + \left( \frac{N_{\text{gen}}}{2} \right) \right).
\]  

(40)

Taking the number of \( U(1) \) factors to be \( N_{\text{gen}} = 3 \), this relation yields the number “6” for the ratio \( \alpha_{\text{diag}}^{-1}/\alpha_{\text{crit}}^{-1} \) which, as pointed out in earlier work\[11\], is just what is needed to get agreement with the Planck scale extrapolation\[16, 17\] of the experimental data for the \( U(1) \) fine structure constant!

In connection with our ongoing endeavor to apply the multiple point criticality idea to the \( U(1) \) part of the standard model symmetry, we would like to mention what may be a noteworthy result. In considering the cartesian product of a large number \( L \) of \( U(1) \) factors, it appears that the maximization of the number of distinguishable phases that meet at the multiple point (in accord with the principle of multiple point criticality) requires a plaquette action form that implements a subdivision of the \( L \) factors of \( U(1) \) into “bunches” of \( U(1) \) factors with each bunch containing \( N_{\text{gen}} = 3 \) \( U(1) \) factors in the following sense: there is no interaction between the \( U(1) \) gauge fields from different bunches whereas \( U(1) \) factors within a bunch interact in a manner imposed by the assumption of a modified Manton action given by an identification lattice with hexagonal symmetry.

In an attempt to extend the idea that many phases come together at the multiple point to also include Higgs phases, it is speculated that this principle might suggest a system of Higgses in which each Higgs field only couples to gauge fields in one bunch. Within each bunch, the principle of multiple point criticality might then point to the “orthogonal” arrangement of \( N_{\text{gen}} \) Higgs (meaning that these Higgs fields can independently be Higgsed or not Higgsed). A Higgsing of a subset of these \( N_{\text{gen}} \) Higgs could, for each bunch, subsequently break the \( SMG_{N_{\text{gen}}} \) gauge symmetry to the diagonal subgroup. If a remaining Higgs field happens to have a Higgsed-unHiggsed phase transition that is almost second order, its vacuum expectation value at the multiple point might be small enough so that this Higgs field could be identified with the Weinberg-Salam Higgs.

With the above mechanism, it can be claimed that any large collection of \( U(1) \) factors would be broken down so as to effectively show up as if there were only \( N_{\text{gen}} \) \( U(1) \) factors with the rest - i.e. the \( L/N_{\text{gen}} - 1 \) other bunches - representing decoupled matter (dark matter ?) reminiscent of the two \( E_8 \) groups in \( SUSY \)-string theories of the heterotic type. We hope to deal with these problems in connection with our ongoing work on the \( U(1) \) coupling.

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Tables

Table 1 The quantity $Vol(H)$ for any one of the 13 invariant subgroups $H$ of the SMG (these are listed in left column), can be written as a product of some subset of the set of five quantities $(1/p_i)Vol(K_i)$. The common set of factors $1/p_{z_2} = 1, 1/p_{z_3} = 1, 1/p_{U(1)} = 1/6, 1/p_{SU(2)} = 1/2, and 1/p_{SU(3)} = 1/3$, some subset of which make possible the factorization of all the $Vol(H)$ into the product of corresponding subsets of the quantities $(1/p_{z_2})Vol(Z_2), (1/p_{z_3})Vol(Z_3), (1/p_{U(1)})Vol(U(1)), (1/p_{SU(2)})Vol(SU(2))$, and $(1/p_{SU(3)})Vol(SU(3))$, are given in the last five columns.

| $Vol(H)$       | locally isomorphic cart. prod. gr. | $1/p_H$ | $1/p_{z_2}$ | $1/p_{z_3}$ | $1/p_{U(1)}$ | $1/p_{SU(2)}$ | $1/p_{SU(3)}$ |
|----------------|-----------------------------------|---------|-------------|-------------|-------------|-------------|-------------|
| $Vol(1)$       | $Vol(Z_2)$                        | 1       | 1           |             |             |             |             |
| $Vol(Z_2)$     | $Vol(Z_3)$                        | 1       | 1           |             |             |             |             |
| $Vol(Z_2 \times Z_3)$ | $Vol(Z_2 \times Z_3)$          | 1/2     | 1           | 1           |             |             |             |
| $Vol(SU(2))$   | $Vol(SU(2) \times Z_2)$          | 1/3     | 1           |             |             |             |             |
| $Vol(SU(3))$   | $Vol(SU(3) \times Z_2)$          | 1/3     | 1           |             |             |             |             |
| $Vol(SU(2) \times SU(3))$ | $Vol(SU(2) \times SU(3) \times Z_2 \times Z_3)$ | 1/6     | 1           | 1           | 1/2         |             |             |
| $Vol(U(1))$    | $Vol(U(1) \times Z_2 \times Z_3)$ | 1/6     | 1           | 1           | 1/6         |             |             |
| $Vol(U(2))$    | $Vol(U(1) \times SU(2) \times Z_2 \times Z_3)$ | 1/12    | 1           | 1           | 1/6         | 1/2         |             |
| $Vol(SMG)$     | $Vol(U(1) \times SU(2) \times SU(3) \times Z_2 \times Z_3)$ | 1/36    | 1           | 1           | 1/6         | 1/2         | 1/3         |

Table 2 In a manner analogous to that of Table 1, the quantities $Vol(H)$ for the 5 invariant subgroups $H$ of $U(N)$ (listed in left column) factorize into products of subsets of the constituent quantities $(1/p_i)Vol(K_i)$. The coefficient $1/p_i$ of any corresponding $Vol(K_i)$ is, as seen in the last three columns, the same for all the $Vol(H)$ in which such a $Vol(K_i)$ contributes in the factorization of $Vol(H)$. Figures 1 and 2, which depict the phase diagram for $U(2)$, illustrate how the 5 partially confining phases of a $U(N)$ group meet at the multiple point in our approximation.

| $Vol(H)$       | locally isomorphic cart. prod. gr. | $1/p_H$ | $1/p_{z_N}$ | $1/p_{U(1)}$ | $1/p_{SU(N)}$ |
|----------------|-----------------------------------|---------|-------------|-------------|---------------|
| $Vol(1)$       | $Vol(Z_N)$                        | 1       |             |             |               |
| $Vol(Z_N)$     | $Vol(U(1) \times Z_N)$           | 1/N     | 1/1         | 1/1         |               |
| $Vol(U(1))$    | $Vol(U(1) \times SU(2) \times Z_N)$ | 1/N     | 1           |             |               |
| $Vol(U(N))$    | $Vol(U(1) \times SU(N) \times Z_N)$ | 1/N^2   | 1           | 1/1         | 1/1           |

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### Table 3: SU(2) Gauge Coupling

| Prediction for continuum limit coupling estimate, $1/\alpha_2$, triple point, cont., using | 14.2 \( \pm \) 1.5 |
| 1. not exponentiated: | \( \frac{0.71 \cdot 20 + 0.89 \cdot 1.7}{15.0} = 15.7 \pm 1 \) |
| 2. exponentiated: | \( \frac{0.75 \cdot 20 + 0.89 \cdot 1.7}{15.0} = 16.5 \pm 1 \) |

**Experimental value** [16, 17] for $1/\alpha_2$ reduced by a factor 3: “desert extrapolation” [16, 17] to Planck scale with one Higgs:

- not exponentiated:
  \[ \frac{1}{3} \cdot \alpha^{-1}_2(M_Z) = \frac{1}{3} \cdot (29.7 \pm 0.2) = 9.9 \pm 0.07 \]
- exponentiated:
  \[ \frac{1}{3} \cdot \alpha^{-1}_2(\mu_{Pl.}) = \frac{1}{3} \cdot 49.5 = 16.5 \]

| $\beta_{adj.}$, triple point (i.e., at triple point) | 2.4 (ca. 5% uncertainty from MC) |
| $\beta_f$, triple point (i.e., at triple point) | 0.54 (ca. 10% uncertainty from MC) |

| $\beta_{adj.}$-contribution to $1/\alpha_2$, triple point (without continuum correction) | $4\pi \frac{C^{(2)}_{adj}}{(2^2-1)} \beta_{adj, triple point} = 4\pi \cdot (2/3) \cdot 2.4 = 20$ |

| $\beta_f$-contribution to $1/\alpha_2$, triple point (without continuum correction): | $4\pi \frac{C^{(2)}_f}{(2^2-1)} \beta_{f, triple point} = 4\pi \cdot (\frac{3}{4}/3) \cdot 0.54 = 1.7$ |

| Full $1/\alpha_2$, triple point (without continuum correction): | $1/\alpha_2, triple point, full, no cont. = 20 + 1.7 = 21.7$ |

| Continuum correction factor for $\beta_{adj.}$-contribution: | |
| 1. not exponentiated (using (3)): | $1 - C^{(2)}_{adj} \pi \alpha_2, triple point, full, no. cont. = 1 - 2\pi/21.7 = 1 - 0.290 = 0.71$ |
| 2. exponentiated: | $\exp(-C^{(2)}_{adj} \pi \alpha_2, triple point, full, no. cont.) = \exp(-2\pi/21.7) = \exp(-0.290) = 0.75$ |

| Continuum correction factor for $\beta_f$-contribution: | |
| 1. not exponentiated (using (3)): | $1 - C^{(2)}_f \pi \alpha_2, triple point, full, no. cont. = 1 - (\frac{3}{4})\pi/21.7 = 1 - 0.109 = 0.89$ |
| 2. exponentiated: | $\exp(-C^{(2)}_f \pi \alpha_2, triple point, full, no. cont.) = \exp(-\frac{3}{4}\pi/21.7) = \exp(-0.109) = 0.90$ |
Table 4: SU(3) Gauge Coupling

| Description                                                                 | Value                                                                 |
|-----------------------------------------------------------------------------|----------------------------------------------------------------------|
| Prediction for continuum limit coupling estimate, $1/\alpha_3$, triple point, cont., using 1. not exponentiated: | $16.3 \sqrt{0.65 \cdot 25 + 0.84 \cdot 1.7} = 17.7 \pm 1$ |
| 2. exponentiated:                                                           | $17.5 \sqrt{0.70 \cdot 25 + 0.85 \cdot 1.7} = 18.9 \pm 1$ |
| Experimental value\[16, 17\] for $1/\alpha_3$ reduced by a factor 3:        | $\frac{1}{3} \cdot \alpha_3^{-1}(M_Z) = \frac{1}{3} \cdot (8.47 \pm 0.5) = 2.8 \pm 0.2$ |
| “desert extrapolation\[14, 17\]” to Planck scale with one Higgs:            | $\alpha_3^{-1}(\mu_{PL}) = \frac{1}{3} \cdot 53 \pm 0.7 = 17.7 \pm 0.3$ |
| $\beta_{adj.}$, triple point (i.e., at triple point)                        | 5.4 (ca. 5% uncertainty)                                             |
| $\beta_{f}$, triple point (i.e., at triple point)                           | 0.8 (ca. 20% uncertainty)                                            |
| $\beta_{adj.}$-contribution to $1/\alpha_3$, triple point (without continuum correction) | $4\pi \frac{C^{(2)}_{adj}}{(3^2-1)} \beta_{adj, triple point} = 4\pi \cdot (3/8) \cdot 5.4 = 25$ |
| $\beta_{f}$-contribution to $1/\alpha_3$, triple point (without continuum correction) | $4\pi \frac{C^{(2)}_{f}}{(3^2-1)} \beta_{f, triple point} = 4\pi \cdot (\frac{4}{3}^2/8) \cdot 0.8 = 1.7$ |
| Full $1/\alpha_3$, triple point (without continuum correction):             | $1/\alpha_3$, triple point, full, no cont. = 25 + 1.7 = 26.7          |
| Continuum correction factor for $\beta_{adj.}$-contribution:               |                                                                     |
| 1. not exponentiated (using \[4\]):                                        | $1 - C^{(2)}_{adj} \pi \alpha_3$, triple point, full, no cont. = 1 - $\frac{3\pi}{26.7} = 1 - 0.35 = 0.65$ |
| 2. exponentiated:                                                           | $\exp(-C^{(2)}_{adj} \pi \alpha_3$, triple point, full, no cont.) = $\exp(-3\pi/26.7) = \exp(-0.35) = 0.70$ |
| Continuum correction factor for $\beta_{f}$-contribution:                  |                                                                     |
| 1. not exponentiated (using \[4\]):                                        | $1 - C^{(2)}_{f} \pi \alpha_3$, triple point, full, no cont. = 1 - $\left(\frac{4}{3}\right)\pi/26.7 = 1 - 0.16 = 0.84$ |
| 2. exponentiated:                                                           | $\exp(-C^{(2)}_{f} \pi \alpha_3$, triple point, full, no cont.) = $\exp(-\left(\frac{4}{3}\right)\pi/26.7) = \exp(-0.16) = 0.85$ |
**Figure Captions**

Fig. 1 The region of allowed parameters ($\log Vol(SU(2)), \log Vol(U(1)), \log Vol(Z_2)$) for the modified Manton action: $\log(\pi e)^{3/2} \leq \log Vol(SU(2)) \approx \frac{3}{2} \log \beta_2 + \log vol(SU(2)) < \infty$, $\log(\pi e)^{1/2} \leq \log Vol(U(1)) \approx \frac{1}{2} \log \beta_1 + \log vol(U(1)) < \infty$, $0 \leq \log Vol(Z_2) \leq \log 2$. These intervals reflect our having used $VolH$ that, up to a factor $(\pi e)^{\frac{\text{dim}(H)}{2}}$, are measured in units of the fluctuation volume. The cube with the chopped off corner represents the region of total confinement. Walls that extend to $+\infty$ are terminated in the drawing with irregular wavy boundaries.

Fig. 2 Phase diagram for lattice gauge theory with gauge group $U(2)$ in our weak coupling approximation with modified Manton action. We have drawn the figure with positive effective dimension for the discrete constituent invariant subgroup $Z_2$. Rectangular signs on signposts are marked with the confining invariant subgroup $H$ and indicate the regions corresponding to the 5 possible phases; these 5 phases are seen to meet at the multiple point. The oval signs lie in the phase boundaries and specify the factor group $L = H_1/H_2$ formed from the two invariant subgroups $H_1$ and $H_2$ that are confined on the two sides of the boundary. It is these groups $L$ that change behavior in crossing the phase boundary in question. Unbroken shading lines indicate phase boundaries as seen from within the totally Coulomb-like phase.
This figure "fig1-1.png" is available in "png" format from:

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