Holomorphic isometric embeddings of the complex two-plane Grassmannian into quadrics

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Abstract
The present article studies holomorphic isometric embeddings of the complex two-plane Grassmannian into quadrics. We discuss the moduli space of these embeddings up to gauge and image equivalence using a generalisation of do Carmo–Wallach theory.

Keywords Moduli spaces · Holomorphic isomorphic embeddings · Grassmannian · Complex quadric · Vector bundles

Mathematics Subject Classification Primary 32H02 · Secondly 53C07

1 Introduction
The description of the moduli of holomorphic isometric embeddings of Kähler manifolds into quadrics can be dealt with using gauge theory [17, 18]. The description of the moduli in this sense is known, for instance, in the cases where the embedded manifold is $\mathbb{C}P^1$ [14, 15] and, more recently in the general case of $\mathbb{C}P^n$ [19]. The next desideratum is a study of embeddings of general Grassmannians into quadrics. The aim of this article is to report the results of such an investigation. In the present piece, we examine the question for a restricted family of Grassmannians: the complex Grassmann manifolds $\text{Gr}_m(\mathbb{C}^{m+2})$ of all two-codimensional linear subspaces in $\mathbb{C}^{m+2}$. This Hermitian symmetric space has a noteworthy geometrical structure as the unique compact, Kähler, quaternionic Kähler manifold with positive scalar curvature [3, 20, 22]. The embedding of arbitrary complex Grassmannians into quadrics will be analysed in a forthcoming paper [16].

Before our goal can be achieved, and according to the general theory in [17], it is necessary to investigate the representation theoretic description of certain modules associated with the relevant vector bundles appearing in the construction. In this sense, the central

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technical result proved in the paper is the determination of the decomposition of the space GS(mV_0, V_0) --to be defined in Sect. 2.5-- into irreducible representations in Sect. 4. The operation of determining the multiplicities of a decomposition into irreducible representations appearing in the tensor product of representations goes under the name of plethysm [10], or Cartan composite in the literature [5]. From the plethysm viewpoint, though Young tableaux and Littlewood–Richardson rules allow to systematically determine the decomposition of tensor products of SU(n) representations [9], Littlewood did not lay a straightforward method to determine which terms of the decomposition belong to the symmetric or skewsymmetric parts [13], and although nowadays powerful algebro–combinatoric techniques do exist which succeed in this pursue [1, 7], we are able to overcome this difficulty by explicitly working out the action of the centre of the isotropy subgroup on the highest and lowest weight vectors. The climax is achieved at the end of Sect. 4 where this result is used to describe the moduli up to gauge equivalence M_k. The special case of the complex two-plane Grassmannian in C^4, aka (complexified, compactified) Minkowski space, being of particular interest, is worked out as an example.

For convenience of the reader, Sect. 2 contains a summary of the basic definitions and theorems of the gauge theoretical approach to the study of moduli spaces of maps, that is, the generalisation of do Carmo–Wallach theory. This article being a sequel of sorts to [14, 15] and [19], this outline has been reduced to the bare minimum. It is nonetheless sufficient to understand the remaining sections of the paper without constant resource to these antecedents, nor to [17, 18] where detailed proofs could be found if desired.

Standard arguments in Sect. 3, where we prove the adequacy of the complex two-plane Grassmannian to the hypothesis of the generalisation of do Carmo–Wallach theory, have been improved and simplified using Kodaira embeddings by line bundles ([18], Theorem 6.1).

2 Preliminaries

2.1 Evaluations and natural identifications

Every manifold in this article is assumed to be connected. The space of sections of the vector bundle V → M, will be denoted by Γ(V) and it determines the evaluation map ev : M × Γ(V) → V : ev(x, t) = t(x). We do not make distinction between ev and its restriction to M × W where W is a subspace of Γ(V). Let the triple (V → M, h_V, ∇^V) or (V, h_V, ∇^V) denote a vector bundle equipped with a fibre metric and a compatible connection. Two of these triples are said to be isomorphic if there exists a bundle map between them preserving the fibre metrics and connections. Such a bundle map is called a bundle isomorphism. When considering holomorphic vector bundles over Kähler manifolds, the previous description given by the triple (V, h, ∇) represents the Hermitian bundle (V, h) equipped with the unique compatible Hermitian connection. In this context, the notion of bundle isomorphism between triples (V, h, ∇) is equivalent to holomorphic isomorphism between Hermitian bundles. The trivial vector bundle M × W → M will be denoted by W or W → M.

Let W be a real N-dimensional vector space and Gr_p(W) the real Grassmannian of oriented p-planes in W. Let S → Gr_p(W) denote the tautological bundle. Then, the universal quotient bundle Q → Gr_p(W) is defined by the exactness of the sequence 0 → S → W → Q → 0 of vector bundles over Gr_p(W). The natural projection W → Q
allows to regard $W$ as a subspace of $\Gamma(Q)$ and is thus identified with the evaluation map. Fixing an inner product on $W$ has the effect of inducing fibre metrics and canonical connections on $S$, $Q \to \text{Gr}_p(W)$.

Let $f : M \to \text{Gr}_p(W)$ be a map of a manifold $M$ into a Grassmannian, and denote by $f^*Q \to M$ the pull-back of the universal quotient bundle. Then, $f$ is said to be full if the induced linear map $W \to \Gamma(f^*Q)$ is a monomorphism.

A real oriented vector bundle $V \to M$ of rank $q$ is globally generated by the $N$-dimensional subspace of sections $W \subset \Gamma(V)$ if the evaluation map is surjective. Moreover, if $W$ has an orientation, this defines a map $f : M \to \text{Gr}_p(W)$,

$$f(x) = \text{Ker} \ ev_x = \{t \in W \mid t(x) = 0\}, \quad \dim \text{Ker} \ ev_x = p = N - q$$

where $\text{Gr}_p(W)$ is a real oriented Grassmannian and the orientation of $\text{Ker} \ ev_x$ is inherited from those of $V_x$ and $W$. The map $f$ is said to be induced by the pair $(V \to M, W)$. The induced map determines a natural identification of $V \to M$ with $f^*Q \to M$ (cf. [17]), and thus, every smooth map $f : M \to \text{Gr}_p(W)$ can be regarded as the induced map defined by $(f^*Q \to M, W)$.

If $W$ has an inner product, the adjoint of the evaluation map, $ev^* : V \to W$, determines a natural identification $V \cong f^*Q$, as vector bundles over $M$, when the image of $ev^*$ is restricted to $f^*Q \to M$. Therefore, $ev^* : V \to f^*Q$ is also termed a natural identification.

### 2.2 Image and gauge equivalence

Two mappings $f_1, f_2 : M \to \text{Gr}_p(W)$ are image equivalent, or congruent, if there is an isometry $\phi$ of $\text{Gr}_p(W)$ such that $f_2 = \phi f_1$. Each isometry $\phi$ of $\text{Gr}_p(W)$ defines a bundle automorphism $\hat{\phi}$ of $Q \to \text{Gr}_p(W)$ covering the isometry $\phi$. Besides, if the maps $f_1, f_2$ are induced by $(V, W)$, they are gauge equivalent if they are image equivalent under $\phi$ and, additionally, $ev^*_1 = \hat{\phi} ev^*_2$, where $ev^*_i : V \to f_i^*Q (i = 1, 2)$ are natural identifications. Under these circumstances, the connections on $V \to M$ induced by $ev^*_i$ are in the same orbit under gauge transformations. In Sect. 4, this will make possible to identify the moduli space up to image equivalence $\mathcal{M}_k$ as a quotient of the moduli space up to gauge equivalence $\mathcal{M}_k$ by the centraliser of the holonomy subgroup of the structure group of $f^*Q$.

### 2.3 Vector bundles over Hermitian symmetric spaces

The complex two-plane Grassmannian is a compact Hermitian symmetric space associated with the Hermitian symmetric pair of compact type $(\mathbb{G}, K) = (\text{SU}(\mathbb{N} + 2), \text{S}(\mathbb{U}(\mathbb{N}) \times \mathbb{U}(2)))$ or, in the diagrammatic notation of [2],

```
\begin{tikzpicture}
  \node (G) at (0,0) {$\mathbb{G}$};
  \node (K) at (1,0) {$\mathbb{K}$};
  \node (M) at (2,0) {$\mathbb{M}$};
  \node (L) at (3,0) {$\mathbb{L}$};
  \node (J) at (4,0) {$\mathbb{J}$};
  \node (P) at (5,0) {$\mathbb{P}$};
  \node (Q) at (6,0) {$\mathbb{Q}$};
  \node (R) at (7,0) {$\mathbb{R}$};
  \node (S) at (8,0) {$\mathbb{S}$};
  \node (T) at (9,0) {$\mathbb{T}$};
  \node (U) at (10,0) {$\mathbb{U}$};
  \node (V) at (11,0) {$\mathbb{V}$};
  \node (W) at (12,0) {$\mathbb{W}$};
  \node (X) at (13,0) {$\mathbb{X}$};
  \node (Y) at (14,0) {$\mathbb{Y}$};
  \node (Z) at (15,0) {$\mathbb{Z}$};

  \draw[-latex] (G) -- (K);
  \draw[-latex] (K) -- (M);
  \draw[-latex] (M) -- (L);
  \draw[-latex] (L) -- (J);
  \draw[-latex] (J) -- (P);
  \draw[-latex] (P) -- (Q);
  \draw[-latex] (Q) -- (R);
  \draw[-latex] (R) -- (S);
  \draw[-latex] (S) -- (T);
  \draw[-latex] (T) -- (U);
  \draw[-latex] (U) -- (V);
  \draw[-latex] (V) -- (W);
  \draw[-latex] (W) -- (X);
  \draw[-latex] (X) -- (Y);
  \draw[-latex] (Y) -- (Z);
\end{tikzpicture}
```

In general terms, if we regard $G \to G/K$ as a principal $K$-bundle, the canonical connection is such that the horizontal subspace is given by the left translation of $\mathfrak{m}$, where $\mathfrak{m}$ is defined by the standard decomposition of Lie algebras $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{m}$, $\mathfrak{g}$ and $\mathfrak{f}$ being the Lie algebras of $G$ and $K$.

Let $L \to G/K$ be an associated, complex line bundle $L = G \times_K V_0$, where $V_0$ is a complex 1-dimensional $K$-module. The complex line bundle $L \to G/K$ is actually holomorphic, by effect of the canonical connection. Moreover, there is an invariant Hermitian
metric $h$ on $L$ (indeed turning $L$ into an Einstein–Hermitian bundle), unique up to a positive constant multiple, for which the canonical connection is the Hermitian connection. (cf, [11, p.121 Proposition 6.2]).

2.4 Standard maps

If the space $W = H^0(G/K, L)$ of holomorphic sections of $L \to G/K$ is nontrivial, then, by the Bott–Borel–Weil Theorem, it is an irreducible complex $G$-module and globally generates $L$. Moreover, $W$ inherits a $G$–invariant $L^2$–Hermitian inner product, from the Riemannian structure on $G/K$ and the Hermitian structure of $(L, h)$. The celebrated Kodaira embedding of $G/K$ is then the map induced by $(L \to G/K, W)$. The importance of the existence of a $G$–invariant Hermitian inner product on $W$ in our theory can not be stressed enough, since it allows us to obtain the pull-back connection. In fact, the pull-back metric induces a Hermitian–Yang–Mills connection.

The real vector bundle obtained from $L \to G/K$ by restriction of the field of scalars to $\mathbb{R}$ is equipped with a fibre metric $\text{Re}(h)$ and an orientation induced by the complex structure. Furthermore, regard $W$ as a real, oriented vector space with a $G$–invariant $L^2$ inner product $(\cdot, \cdot)_W$. The new induced map into a real oriented Grassmannian is said to be standard. In this case, we obtain a totally geodesic holomorphic embedding of complex projective space into a real oriented Grassmannian, and the standard map is the composition of this last map with the Kodaira embedding. Thus, the induced connection is also the Hermitian–Yang–Mills connection.

2.5 do Carmo–Wallach theory

Let $W$ be an orthogonal $G$–module with respect to the inner product $(\cdot, \cdot)_W$. Regard $W$ as a $K$–module where $K$ is a subgroup of $G$ and let $V_0$ be a complex $K$–submodule of $W$, following [17, Lemma 5.17]. The orthogonal splitting $W = U_0 \oplus V_0$ allows the definition of a $G$–equivariant standard map $f_0 : M \to \text{Gr}_p(W)$, with $p = \dim U_0$, such that $f_0([g]) = gU_0 \subset W$, for all $[g] \in G/K$, $g \in G$.

Let $S(W)$ denote the set of symmetric endomorphisms of $W$, equipped with the $G$–invariant inner product $(A, B) = \text{trace} AB$, for $A, B \in S(W)$. Given $U, V$ subspaces of $W$, define the real subspace $S(U, V) \subset S(W)$ spanned by

$$S(u, v) := \frac{1}{2}\left\{ u \otimes (\cdot, v)_{w} + v \otimes (\cdot, u)_{w} \right\},$$

where $u \in U$, $v \in V$. Analogously, $\text{GS}(U, V) \subset S(W)$ is spanned by $gS(u, v)$, with $g \in G$.

Before we can proceed, it is necessary to define what means for a map to satisfy the gauge condition for a given Hermitian line bundle: Let $f : M \to \text{Gr}_n(\mathbb{R}^{n+2})$ be a full holomorphic map, and let $Q \to \text{Gr}_n(\mathbb{R}^{n+2})$ the universal quotient bundle, equipped with its canonical metric, connection, and almost complex structure. Moreover, let $(L, h)$ be a fixed Hermitian line bundle over $M$, regarded as a real, rank two differentiable vector bundle with complex structure $J_L$. Then, we say that $f$ satisfies the gauge condition for $(L, h)$ if the pull-back bundle $f^*Q \to M$ together with the pull-back metric, connection and complex structure is isomorphic (in the sense of Sect. 2.1) to $L \to M$ with its Hermitian metric, connection and complex structure.
Then, the generalisation of the do Carmo–Wallach Theorem for holomorphic maps of $G/K$ into a quadric reads:

**Theorem 2.1** Let $(L, h)$ be a fixed Hermitian line bundle, and let $f : G/K \to \text{Gr}_n(\mathbb{R}^{n+2})$ be a full holomorphic map satisfying the gauge condition for $(L, h)$. Let $W$ denote $H^0(G/K, L)$ regarded as a real vector space equipped with a complex structure. Then, there is a positive semi-definite symmetric endomorphism $T \in S(W)$ such that the pair $(W, T)$ satisfies the following three conditions:

(a) The vector space $\mathbb{R}^{n+2}$ is a subspace of $W$ with the inclusion $i : \mathbb{R}^{n+2} \to W$ preserving the inner products, and $L \to M$ is globally generated by $\mathbb{R}^{n+2}$.

(b) As a subspace, $\mathbb{R}^{n+2} = \text{Ker} T^\perp$ and the restriction of $T$ is a positive symmetric endomorphism of $\mathbb{R}^{n+2}$.

(c) The endomorphism $T$ satisfies

$$\left( T^2 - \text{Id}_W, \text{Gr}(V_0, W) \right)_S = 0, \quad \left( T^2, \text{Gr}(mV_0, W) \right)_S = 0.$$  

$t^* : W \to \mathbb{R}^{n+2}$ denotes the adjoint linear map of $i : \mathbb{R}^{n+2} \to W$, then $f : G/K \to \text{Gr}_n(\mathbb{R}^{n+2})$ is realised as

$$f([g]) = (i^* T_i)^{-1}(f_0([g]) \cap \text{Ker} T^\perp),$$  

where the orientation of $(i^* T_i)^{-1}(f_0([g]) \cap \text{Ker} T^\perp)_{[g]}$ is given by the orientation of $L_{[g]}$ and $\mathbb{R}^{n+2}$. Moreover, if the orientation of $\text{Ker} T$ is fixed, then we have a unique holomorphic totally geodesic embedding of $\text{Gr}_n(\mathbb{R}^{n+2})$ into $\text{Gr}_{n'}(W)$ by $i(\mathbb{R}^{n+2}) = \text{Ker} T^\perp$, where $n' = n + \dim \text{Ker} T$ and a bundle isomorphism $(\text{ev} \circ (i^* T_i))^* : L \to f^* Q$ as the natural identification by $f$. Such two maps $f_i, (i = 1, 2)$ are gauge equivalent if and only if $i^* T_i = i^* T_2 i$, where $T_i$ and $i$ correspond to $f_i$ in (2.1), respectively.

Conversely, suppose that a vector space $\mathbb{R}^{n+2}$ with an inner product and an orientation, and a positive semi-definite symmetric endomorphism $T \in \text{End}(W)$ satisfying conditions (a),(b),(c) are given. Then, we have a unique holomorphic totally geodesic embedding of $\text{Gr}_n(\mathbb{R}^{n+2})$ into $\text{Gr}_{n'}(W)$ after fixing the orientation of $\text{Ker} T$ and the map $f : G/K \to \text{Gr}_n(\mathbb{R}^{n+2})$ defined by (2.1) is a full holomorphic map into $\text{Gr}_n(\mathbb{R}^{n+2})$ satisfying the gauge condition with bundle isomorphism $L \cong f^* Q$ as the natural identification.

**Remark 1** Notice that although the standard map corresponding to the Kodaira embedding is $G$–equivariant, this need not be the case for the other maps in the moduli.

### 2.6 Motivation

While the preceding Sects. 2.1–2.5 provide an introduction to the language and techniques of the generalisation of do Carmo–Wallach theory, an account of the motivation seems necessary to bring the theory together. It seems that this is the right place for such a motivation, once the technical elements of the theory have been previously defined.

Suppose that a vector bundle $V \to M$ and a finite-dimensional subspace $W$ of the space of sections $\Gamma(V)$ is given. If the evaluation homomorphism $ev : M \times W \to V$ is surjective (so that $V \to M$ is globally generated by $W$), assigning to each $x \in M$ the $p$–plane $\text{Ker} ev_x \subset \text{Gr}_p(W)$ defines an induced map $f : M \to \text{Gr}_p(W)$. Using it to pull the natural exact sequence over $\text{Gr}_p(W)$ back to $M$, it can be proved that $V \to M$ is naturally identified.
with \( f^*Q \to M \) where \( Q \to \text{Gr}_p(W) \) stands for the universal quotient bundle. In case that one deals with vector bundles equipped with metrics and connections, additional differential conditions need to be considered (e.g., the gauge condition which states that the connection on \( V \to M \) must be gauge equivalent to the connection on \( f^*Q \to M \)). In the present article, \( M \) stands for a compact, irreducible, Hermitian symmetric space, and \( V \to M \) will be substituted by a complex homogeneous line bundle. The subspace of sections \( W \subset \Gamma(V) \) will be the space of holomorphic sections \( H^0(V) \subset \Gamma(V) \) since in this context this is the space which determines the standard maps (the general definition for standard map being those induced by sets of sections with the same eigenvalue for the Laplace operator acting on sections). Regard \( W \) as a real vector space and fix a \( \mathbb{R}^{n+2} \) subspace of \( W \), with inclusion map \( \iota \). This being a real space of sections \( \mathbb{R}^{n+2} \) also generates \( V \to M \) globally. The generalisation of the theory of do Carmo–Wallach shows that this \( \mathbb{R}^{n+2} \) can be regarded as the orthogonal complement of the kernel of a certain positive semi-definite symmetric endomorphism \( T \) of \( W \), which is indeed positive definite on \( \mathbb{R}^{n+2} \). Theorem 2.1, originally proved in [17] by generalising results from [8] and [21], gives the representation theoretic conditions that such endomorphism \( T \) must satisfy (condition (c) in the theorem). Each such \( T \) determines a new holomorphic embedding \( \text{Gr}_n(\mathbb{R}^{n+2}) \to \text{Gr}_n(W) \) where \( n' = n + \dim \ker T \). The theoretical expression for the holomorphic isometric embedding \( f \) constructed from the standard map \( f_0 \), the symmetric endomorphism \( T \), and the inclusion map \( \iota \) is given by Eq. 2.1. Each symmetric endomorphism \( T \) determines a unique gauge equivalence class of maps. Therefore, knowledge of the representation spaces where \( T \) lives determines all the relevant differential geometric information of the moduli space up to gauge equivalence \( \mathcal{M}_k \), in particular its dimension, and the meaning of the boundary points in the compactification.

### 3 Characterisation of Holomorphic isometric embeddings of \( \text{Gr}_m(\mathbb{C}^{m+2}) \) into quadrics

The objective of this section is to determine if the complex two-plane Grassmannian satisfies the hypothesis of Theorem 2.1, and how this special manifold fits with the general theory summarised in Sect. 2.

Note that, the real, oriented Grassmannian \( \text{Gr}_n(\mathbb{R}^{n+2}) \) can be regarded as a complex quadric hypersurface \( Q_n \) in \( \mathbb{C}P^{n+1} \) (cf. [12, pp. 278–282]). The universal quotient bundle over the quadric \( Q_n \) has a holomorphic bundle structure induced by the canonical connection. Notice that the curvature two-form \( R_Q \) of the canonical connection on the universal quotient bundle \( Q \to Q_n \) is a constant multiple of the Kähler two-form \( \omega_Q \) on \( Q_n \)

\[
R_Q = -\sqrt{-1}\omega_Q.
\]

In the case of the complex two-plane Grassmannian, the holomorphic line bundle \( \mathcal{O}(1) \to \text{Gr}_m(\mathbb{C}^{m+2}) \) is related to the universal quotient bundle \( \tilde{Q} \to \text{Gr}_m(\mathbb{C}^{m+2}) \) by \( \mathcal{O}(1) \cong \Lambda^2\tilde{Q} \). Therefore, the relation between the curvature two-forms \( R_{\mathcal{O}(1)} \), \( R_{\tilde{Q}} \) of these bundles over the complex two-plane Grassmannian is \( \text{Tr}R_{\tilde{Q}} = R_{\mathcal{O}(1)} \), and, again, the curvature is proportional to the Kähler form \( \omega : R_{\mathcal{O}(1)} = -\sqrt{-1}\omega \).

Let \( f : \text{Gr}_m(\mathbb{C}^{m+2}) \to Q_n \) be a holomorphic embedding. Then, \( f \) is called an isometric embedding of degree \( k \) if \( f^*\omega_Q = k\omega \). Note that this implies that \( k \in \mathbb{N} \). By definition, \( f : \text{Gr}_m(\mathbb{C}^{m+2}) \to Q_n \) is a holomorphic isometric embedding of degree \( k \) if and only if the pull-back of the canonical connection on the universal quotient bundle over \( Q_n \) is the canonical connection on \( \mathcal{O}(k) \to \text{Gr}_m(\mathbb{C}^{m+2}) \), namely, if and only if \( f \) satisfies the gauge condition for
Since the canonical connection on \((O(k), h_k)\) is the Hermitian connection, Theorem 2.1 allows us to determine the moduli space \(M_k\) of holomorphic isometric embeddings of degree \(k\) modulo the gauge equivalence of maps.

Note that, by virtue of the identification \(Q_n = \text{Gr}_n(\mathbb{R}^{n+2})\), switching simultaneously the orientation of \(\mathbb{R}^{n+2}\) and of the subspace induces an holomorphic isometry \(\tau\) of \(Q_n\). In what follows, we do not distinguish maps \(f : \text{Gr}_m(\mathbb{C}^{m+2}) \to Q_n\) which differ by composition with \(\tau\).

Let \((L, h)\) be a Hermitian holomorphic line bundle over a Kähler manifold \(M\). Then, the complex structure \(J_L\) of \(L \to M\) induces a complex structure \(J\) on \(H^0(M, L)\) such that \(J_L \circ \text{ev} = \text{ev} \circ J\), where \(\text{ev} : H^0(M, L) \to L\) is the evaluation map. If \(W = \mathbb{C}^{l+1} \subset H^0(M, L)\) is a complex subspace of holomorphic sections, it can also be regarded as the real vector space \((\mathbb{R}^{n+2}, J)\), where \(n = 2l\), equipped with the complex structure \(J\). Let the inner product on \(\mathbb{R}^{n+2}\) be defined as the real part of the complex structure \(J\). Then, the characterisation of the moduli space of full holomorphic isometric embeddings \(f : M \to Q_n = \text{Gr}_n(\mathbb{R}^{n+2})\) induced by \((L \to M, \mathbb{R}^{n+2})\). If \(f\) is a holomorphic isometric immersion, because of \(J_L \circ \text{ev} = \text{ev} \circ J\) it can also be regarded as a holomorphic isometric immersion into a totally geodesic \(\mathbb{C}^l \subset Q_n\).

Denote by \(S(\mathbb{R}^{n+2})\) the space of symmetric endomorphisms of \((\mathbb{R}^{n+2}, J)\), and let \(H(\mathbb{R}^{n+2}) \subset S(\mathbb{R}^{n+2})\) be the subspace of Hermitian endomorphisms. The \((l + 1)\)-structure of \(\mathbb{R}^{n+2}\) allows us to define the orthogonal complement of \(H(\mathbb{R}^{n+2}) \subset S(\mathbb{R}^{n+2})\). The orthogonal complementary subspace \(H(\mathbb{R}^{n+2})^\perp = \{ A \in S(\mathbb{R}^{n+2}) \mid JA = -AJ\}\) can be identified with the \(U(l + 1)\)-module \(S^2 \mathbb{C}^{l+1}\).

It is well-known that the Kodaira embedding into a complex projective space induced by \((O(k) \to \text{Gr}_m(\mathbb{C}^{m+2}), H^0(\text{Gr}_m(\mathbb{C}^{m+2}), O(k)))\) is rigid as a holomorphic isometric embedding: Calabi [6] proved that there is a unique class of congruent holomorphic isometric embeddings. When the bundle and space of sections are regarded as being real and oriented, the Kodaira embedding can also be viewed as a holomorphic isometric embedding into a quadric. Then, the characterisation of the moduli space of full holomorphic isometric embeddings \(\text{Gr}_m(\mathbb{C}^{m+2}) \to Q_n, Q_n\) follows from Theorem 6.10 of [18], which we recall here:

**Theorem 3.1** ([18, Theorem 6.10]) Let \((L, h)\) be a Hermitian holomorphic line bundle over a Kähler manifold \(M\). Let \(M\) be the moduli space of full holomorphic isometric immersions of \(M\) into a quadric \(\text{Gr}_n(\mathbb{R}^{n+2})\) with the gauge condition for \((L, h)\) modulo gauge equivalence relation of maps.

Suppose that \(\mathbb{R}^{n+2}\) is a complex subspace of \(H^0(M, L)\) and the inner product on \(\mathbb{R}^{n+2}\) is compatible with the complex structure. If there exists \(f \in M\) such that the evaluation map \(ev : \mathbb{R}^{n+2} \to L\) by \(f\) satisfies \(J_L \circ ev = ev \circ J\), then \(M\) has the induced complex structure and is an open submanifold of a complex subspace of \(H(\mathbb{R}^{n+2})^\perp\).

Therefore, the moduli up to gauge equivalence of maps, of full holomorphic isometric embeddings \(\text{Gr}_m(\mathbb{C}^{m+2}) \to Q_n\) satisfying the gauge condition for \((O(k) \to \mathbb{C}P^m, h_k)\) is an open submanifold of a complex subspace of \(H(\mathbb{R}^{n+2})^\perp\), where \(\mathbb{R}^{n+2} = H^0(\text{Gr}_m(\mathbb{C}^{m+2}), O(k))\).

### 4 Moduli spaces

The gauge theoretic generalisation of the do Carmo–Wallach theory requires a thorough understanding of the symmetric endomorphisms of the space \(W = H^0(G/K, L)\) of holomorphic sections of certain line bundles over \(G/K\).
Consider the symmetric pair \((G, K) = (SU(m + 2), SU(m) \times U(2))\), where an element in the isotropy subgroup \(K = SU(m) \times U(2)\) is of the form:

\[
\begin{pmatrix}
A & 0 \\
O & B
\end{pmatrix}, \quad A \in U(m), \ B \in U(2), \quad |A||B| = 1.
\]

This corresponds to the description of the complex two-plane Grassmannian as a Hermitian symmetric space \(Gr_m(C^{m+2}) = \frac{SU(m+1)}{SU(m) \times U(2)}\).

Several \(U(n)\) representations will be used in this section. Let us fix a maximal torus inside \(U(n)\) defined by the set of diagonal matrices, which will be denoted by \(\text{diag}(x_1, \ldots, x_n)\), and denote by \(V_n(\lambda_1, \lambda_2, \ldots, \lambda_n)\) an irreducible complex representation of \(U(n)\) with the highest weight \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\), where \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\) are integers. Then, the maximal torus acts on the highest weight vector \(\hat{\mathbf{w}} \in V_n(\lambda)\) as

\[
\text{diag}(x_1, \ldots, x_n) \hat{\mathbf{w}} = x_1^{\lambda_1} \ldots x_n^{\lambda_n} \hat{\mathbf{w}}.
\]

We can also regard \(V_n(\lambda)\) as a \(SU(n)\) representation, and in this case, it will satisfy \(V_n(\lambda_1, \ldots, \lambda_n) = V_n(\lambda_1 + t, \ldots, \lambda_n + t), t \in \mathbb{Z}\), so by convention we will choose \(t = -\lambda_n\) so the last weight becomes zero. The basis of dominant integral weights of \(SU(n)\) will be denoted by \(\pi_i\) \((1 \leq i \leq n - 1)\) and are defined by \(\lambda_1 = \cdots = \lambda_i = 1, \lambda_{i+1} = \cdots = \lambda_n = 0\). The \(SU(n)\) irreducible representation with highest weight \(\pi = \sum_i k_i \pi_i\) will be denoted by \(F_n(\pi)\). In diagrammatic notation, the fundamental representations \(F_n(\pi_i)\) are given by labelled Dynkin diagrams.

\[
F_n(\pi_i) = \bullet \cdots \bullet.
\]

The relation between \(SU(n)\) representations \(F_n(\pi)\) and \(V_n(\lambda)\) is

\[
F_n \left( \sum_{i=1}^{n-1} k_i \pi_i \right) = V_n \left( \sum_{i=1}^{n-1} k_i, \sum_{j=2}^{n-1} k_j, \ldots, k_{m-2} + k_{n-1}, k_{n-1}, 0 \right).
\]

The subindex \(n\) in \(F_n, V_n\) will be neglected when it is clear enough. Let \(\pi = \sum_i k_i \pi_i\) a dominant integral weight, associated with the \(SU(m+2)\) representation \(F(\pi)\). An element of the maximal torus of \(U(m+2)\) acts on the highest weight vector \(\hat{\mathbf{w}} \in F(\pi)\) as \(\text{diag}(x_1, \ldots, x_{m+2}) \hat{\mathbf{w}} = x_1^{\lambda_1} \ldots x_{m+2}^{\lambda_{m+2}} \hat{\mathbf{w}}\). Since the dual representation of \(V(\lambda)\) has lowest weight equal to \(-\lambda\), the torus of \(U(m+2)\) acts on the lowest weight vector \(\hat{\mathbf{w}} \in F(\pi)\) as it does on the highest weight vector of \(F(-\pi')\) with

\[
\pi' = - \sum_{i=1}^{m+1} k_i \pi_{m+2-i} = - \sum_{j=1}^{m+1} k_{m+2-j} \pi_j.
\]

Thus, we obtain

\[
\text{diag}(x_1, \ldots, x_{m+2}) \hat{\mathbf{w}} = x_1^{-\sum_{j=1}^{m+1} k_j \pi_{m+2-j}} x_2^{-\sum_{j=1}^{m+1} k_j \pi_{m+2-j}} \ldots x_m^{-\sum_{j=1}^{m+1} k_j \pi_{m+2-j}} x_{m+1}^{\lambda_1} \hat{\mathbf{w}}.
\]

\[
= x_1^{-\sum_{j=1}^{m+1} k_j \pi_{m+2-j}} x_2^{-\sum_{j=1}^{m+1} k_j \pi_{m+2-j}} \ldots x_m^{-\sum_{j=1}^{m+1} k_j \pi_{m+2-j}} x_{m+1}^{-\sum_{j=1}^{m+1} k_j \pi_{m+2-j}} \hat{\mathbf{w}}.
\]

We take an element of the centre \(U(1)\) of \(K = SU(m) \times U(2)\) is \(SU(m) \times SU(2) \times U(1)\):
\[ Y = \text{diag}(\frac{1}{m}, \ldots, \frac{1}{m}, y, y^{-\frac{1}{m}}) \].

Consequently, \( Y \) acts on \( \tilde{w} \) as
\[
Y \cdot \tilde{w} = y^{-\frac{1}{m} \sum_{i=1}^{k} b_i} y^{-\frac{1}{m} \sum_{i=1}^{k} b_i} \cdots y^{-\frac{1}{m} (k_2+k_1)} y^{-\frac{1}{m} (k_2+k_1)} y^{\frac{1}{m} k_1} \tilde{w}.
\]
\[
= y^{-\frac{1}{m} k_1} \frac{1}{m} \sum_{i=1}^m (m+2-i) k_i \tilde{w}.
\]

Since \( H^0(M; \mathcal{O}(k)) = F(\pi_2) \) by the Bott–Borel–Weil Theorem [4], the moduli space is contained in \( S^2(F(\pi_2)) \). This last space was denoted by \( H(W)^2 \) in Sect. 2. Firstly, by steady application of Littlewood–Richardson rules on Young tableaux with two rows and \( k \) columns [9], we have that

**Lemma 4.1**

\[
F(\pi_2) \otimes F(\pi_2) = \bigoplus_{i=0}^{k} \bigoplus_{j=i}^{k} V(2k - i, 2k - j, i, 0, \cdots, 0)
\]
\[
= \bigoplus_{i=0}^{k} \bigoplus_{j=i}^{k-i} V(2k - i, 2k - i - j, i + j, 0, \cdots, 0)
\]
\[
= \bigoplus_{i=0}^{k} \bigoplus_{j=0}^{k-i} F(j \pi_1 + (2k - 2i - 2j) \pi_2 + j \pi_3 + i \pi_4).
\]

Note that \( k \geq i + j \).

In order to achieve the description of the moduli, Theorems 2.1 and 3.1 require the specification of \( \text{GS}(mV_0, V_0) \cap H(W)^2 \).

**Lemma 4.2** The lowest weight of \( V(2k - i, 2k - i - j, i, 0, \cdots, 0) \) is

\[-2k + \left(1 + \frac{2}{m}\right)i + \left(\frac{1}{2} + \frac{1}{m}\right)j.\]

**Proof** If \( \tilde{w}_i \) is the lowest weight vector in \( V(2k - i, 2k - i - j, i, 0, \cdots, 0) \), then
\[
Y \cdot \tilde{w}_i = y^{-\frac{1}{m} j} y^{-\frac{1}{m} (m(2k-2i-2j)+(m-1)+m-2i)} \tilde{w}_i
\]
\[
= y^{-2k + (1 + \frac{2}{m})i + (\frac{1}{2} + \frac{1}{m})} \tilde{w}_i
\]
\[
= y^{-2k + (1 + \frac{2}{m})i + (\frac{1}{2} + \frac{1}{m})} \tilde{w}_i.
\]

Finally, we proceed to detail the moduli up to gauge equivalence of maps of holomorphic isometric embeddings of \( \text{Gr}_m(C^{m+2}) \rightarrow Q_p = \text{Gr}_p(W) \) of degree \( k \). If \( V_0 = C_k \) denotes
the one-dimensional representation of U(1) generated by Y with weight k, then the homogeneous bundle SU(m + 2) ×_{SU(m) × U(2)} C_{−k} is the complex line bundle O(k) → Gr_{m}(C^{m+2}) of degree k. Therefore, by the generalisation of do Carmo–Wallach, cf Theorem 2.1, W is identified with H^0(Gr_{m}(C^{m+2}), O(k)).

Lemma 4.3

Proof Let C^m denote the standard representation of U(m), that is, C^m = V_m(1, 0, ⋯, 0). The tautological vector bundle S → Gr_{m}(C^{m+2}) can be identified with the homogeneous bundle SU(m + 2) ×_{SU(m) × U(2)} C^m. Together with the universal quotient bundle Q → Gr_{m} C^{m+2} defined by the exact sequence 0 → S → C^{m+2} → Q → 0, they define the holomorphic tangent bundle, T → Gr_{m}(C^{m+2}), T = S^* ⊗ Q, which is a homogeneous bundle with standard fibre

\[
\left( C^m \otimes C_{\frac{1}{m}} \right)^* \otimes \left( C^2 \otimes C_{-\frac{1}{2}} \right) = C^{m*} \otimes C^2 \otimes C_{-\frac{1}{m} + \frac{1}{2}},
\]

and T^* → Gr_{m}(C^{m+2}) where

\[
T^* = C^m \otimes C^{2*} \otimes C_{\frac{1}{m} + \frac{1}{2}}.
\]

Since T^c ≃ m^c, and V_0 is spanned by the lowest weight vector of H^0(Gr_{m}(C^{m+2}), O(k)), then

\[
T^c \otimes V_0 = C^m \otimes C^{2*} \otimes C_{\frac{1}{m} + \frac{1}{2}} \otimes C_{-k} = C^m \otimes C^{2*} \otimes C_{-k + \frac{1}{m} + \frac{1}{2}}.
\]

Lemma 4.4

\[
\text{GS}(mV_0, V_0) \cap H(W)^{\perp} = V(2k, 2k, 0, ⋯, 0).
\]

Proof The lowest weight of GS(mV_0, V_0) with respect to Y is

\[
-k - k + \frac{1}{m} + \frac{1}{2} = -2k + \frac{1}{m} + \frac{1}{2}.
\]

Hence, if i ≥ 1, or i = 0 and j > 1, then

\[
-2k + \left( 1 + \frac{2}{m} \right) i + \left( \frac{1}{2} + \frac{1}{m} \right) j > -2k + \frac{1}{m} + \frac{1}{2}.
\]

It follows that

\[
\text{GS}(mV_0, V_0) \subset V(2k, 2k, 0, ⋯, 0) \oplus V(2k, 2k - 1, 1, 0, ⋯, 0) \quad (i = 0, j = 0)
\]

\[
\oplus \ V(2k, 2k - 1, 1, 0, ⋯, 0) \quad (i = 0, j = 1).
\]

Let ӯ denote as usual the highest weight vector of F(kπ_2), and denote by ӯ_{\perp} the vector in F(kπ_2) with weight just below that of ӯ. It is clear that ӯ \otimes ӯ_{\perp} is contained in S^2 F(kπ_2) and that it is the highest weight vector in the irreducible component V(2k, 2k, 0, ⋯, 0). Therefore,

\[
V(2k, 2k, 0, ⋯, 0) \subset S^2(F(k\pi_2)).
\]

On the other hand, ӯ \wedge ӯ_{\perp} belongs to \wedge^2(F(k\pi_2)), and it is the highest weight vector in the irreducible component V(2k, 2k - 1, 1, 0, ⋯, 0), so
\[ \text{V}(2k, 2k - 1, 1, 0, \ldots, 0) \subset \wedge^2(F(k \pi_2)), \]

thus, we have

\[ \text{GS(mV}_0, V_0) \cap H(W)^2 = \text{V}(2k, 2k, 0, \ldots, 0). \]

\[ \square \]

Let us denote by \( V_k \) the orthogonal complement to \( \text{GS(mV}_0, V_0) \) in \( H(W)^2 \). It follows from Theorem 2.1 that the moduli of holomorphic isometric embeddings of degree \( k \) up to gauge equivalence is an open subset of \( V_k \). Accordingly, applying the previous Lemma to the decomposition of \( H(W)^2 = S^2 F(k \pi_2) \) leads to

\[ S^2(F(k \pi_2)) = V_k \oplus \text{V}(2k, 2k, 0, \ldots, 0). \quad (4.1) \]

**Remark 2** A straightforward computation using hook length formulae on Young tableaux leads to the explicit, combinatorial expression for the (real) dimension of \( V_k \)

\[ \dim V_k = \dim S^2 F(k \pi_2) - \dim \text{V}(2k, 2k, 0, \ldots, 0) \]

which equals

\[ \frac{1}{(k + 1)^2} \left( \begin{array}{c} m - 1 + k \\ m - 1 \end{array} \right) \left( \begin{array}{c} m - 2 + k \\ m - 2 \end{array} \right) \left( k + 1 + \left( \begin{array}{c} m - 1 + k \\ m - 1 \end{array} \right) \left( \begin{array}{c} m - 2 + k \\ m - 2 \end{array} \right) \right) \]

\[ - \frac{2}{2k + 1} \left( \begin{array}{c} m - 1 + 2k \\ m - 1 \end{array} \right) \left( \begin{array}{c} m - 2 + 2k \\ m - 2 \end{array} \right) . \]

**Theorem 4.5** If \( f : \text{Gr}_m(C^{m+2}) \to \text{Gr}_N(R^{n+2}) \) is a full holomorphic isometric embedding of degree \( k \), then \( n + 2 \leq \dim V_k \).

Let \( \mathcal{M}_k \) be the moduli space of full holomorphic isometric embeddings of degree \( k \) of \( \text{Gr}_m(C^{m+2}) \) into \( \text{Gr}_N(R^{N+2}) \) by the gauge equivalence of maps, where \( N + 2 = \dim V_k \). Then, \( \mathcal{M}_k \) can be regarded as an open bounded convex body in \( V_k \).

Let \( \overline{\mathcal{M}_k} \) be the closure of the moduli \( \mathcal{M}_k \) by topology induced from the inner product. Every boundary point of \( \overline{\mathcal{M}_k} \) distinguishes a subspace \( R^{p+2} \) of \( R^{N+2} \) and describes a full holomorphic isometric embedding into \( \text{Gr}_p(R^{p+2}) \) which can be regarded as totally geodesic submanifold of \( \text{Gr}_N(R^{N+2}) \). The inner product on \( R^{N+2} \) determines the orthogonal decomposition of \( R^{N+2} : R^{N+2} = R^{p+2} \oplus R^{p+2\perp} \). Then, the totally geodesic submanifold \( \text{Gr}_p(R^{p+2}) \) can be obtained as the common zero set of sections of \( Q \to \text{Gr}_N(R^{N+2}) \), which belongs to \( R^{p+2\perp} \).

**Proof** The constraint \( n \leq N \) is a consequence of (a) in Theorem 2.1 and Bott–Borel–Weil theorem.

It follows from (c) in Theorem 2.1 that \( \text{GS(mV}_0, V_0)^1 \) is a parametrisation of the space of full holomorphic isometric embeddings \( f : \text{Gr}_m(C^{m+2}) \to \text{Gr}_N(R^{N+2}) \) of degree \( k \). Since the standard map into \( \text{Gr}_N(R^{N+2}) \) is the composite of the Kodaira embedding \( \text{Gr}_m(C^{m+2}) \to \mathbb{C}P^{\frac{m}{2}} \) and the totally geodesic embedding \( \mathbb{C}P^{\frac{m}{2}} \subset \text{Gr}_N(R^{N+2}) \), we can apply Theorem 3.1 and Eq. 4.1 to conclude that \( \mathcal{M}_k \) is a bounded connected open convex body in \( V_k \) with the topology induced by the \( L^2 \) scalar product.
Under the natural compactification in the $L^2$-topology, the boundary points correspond to endomorphisms $T$ which are not positive definite, but positive semi-definite. It follows from Theorem 2.1 that each of these endomorphisms defines a full holomorphic isometric embedding $\text{Gr}_m(\mathbb{C}^{m+2}) \to \text{Gr}_p(\mathbb{R}^{p+2})$, of degree $k$ with $p = 2k - \dim \text{Ker} T$, whose target embeds in $\text{Gr}_N(\mathbb{R}^{N+2})$ as a totally geodesic submanifold. The image of the embedding $\text{Gr}_p(\mathbb{R}^{p+2}) \to \text{Gr}_N(\mathbb{R}^{N+2})$ is determined by the common zero set of sections in $\text{Ker} T$. (See also the Remark after Proposition 5.14 in [17] for the geometric meaning of the compactification of the moduli space.)

\[ \square \]

**Remark 3** It follows from Corollary 5.18 in [18] that the first condition in (1) is automatically satisfied. Alternatively, using the same techniques as in Lemma 4.4, it can be shown that

\[ \text{GS}(V_0, V_0) \cap H(W) = V(2k, 2k, 0, \ldots, 0). \]

The centraliser $S^1 \cong U(1)$ of the holonomy subgroup $K = S(U(m) \times U(2))$ of the structure group of the line bundle acts on $\mathcal{M}_k$. For the general theory, see [18]. Therefore, the same argument and proof as the one of Theorem 8.1 in [14] applies leading to

**Corollary 4.6** The moduli space $\mathcal{M}_k$ of image equivalence classes of holomorphic isometric embeddings $\text{Gr}_m(\mathbb{C}^{m+2}) \to \text{Gr}_N(\mathbb{R}^{N+2})$, $N + 2 = \dim V_k$, of degree $k$, is

\[ \mathcal{M}_k = \mathcal{M}_k / S^1. \]

**Remark 4** There is a natural, induced complex structure defined on $\mathcal{M}_k$ from its embedding in $V_k$. It is also equipped with a compatible metric induced from the inner product, so $\mathcal{M}_k$ is a Kähler manifold. The aforesaid $S^1$-action preserves the Kähler structure on $\mathcal{M}_k$. The moment map $\mu : \mathcal{M}_k \to \mathbf{R} : |Id - T|^2$ induces the Kähler quotient, and $\mathcal{M}_k$ has a foliation whose general leaves are the complex projective spaces.

**Example** (Complexified, compactified) Minkowski Space, $\text{Gr}_2(\mathbb{C}^4) :$ Let $m = 2$ in the previous discussion. We have that

\[ \bigotimes^2 F(k\pi_2) = \bigoplus_{i=0}^{k} \bigoplus_{j=0}^{k-i} V(2k - i, 2k - i - j, i + j, i) \]

\[ = \bigoplus_{j=0}^{k} V(2k, 2k - j, j, 0) \oplus \bigoplus_{j=0}^{k-1} V(2k - 1, 2k - j - 1, j + 1, 1) \]

\[ \oplus \cdots \oplus \bigoplus_{j=0}^{1} V(k + 1, k - j + 1, k - 1 + j, k - 1) \oplus V(k, k, k) \]

Regarded as an $\text{SU}(4)$-module,

\[ V(2k - i, 2k - i - j, i + j, i) = F(j\pi_1 + (2k - 2i - 2j)\pi_2 + j\pi_3), \quad (k \geq i + j). \]

Let $\tilde{w}_l \in V(2k - 2i, 2k - 2i - j, i + j, i)$ be the lowest weight vector, then

\[ Y \cdot \tilde{w}_l = y^{-1/2} (2(2k - 2i - 2j) + j) \tilde{w}_l = y^{-2k + 2i + j} \tilde{w}_l, \]

so, the lowest weight of $V(2k - 2i, 2k - 2i - j, j, 0)$ is $-2k + 2i + j$. 

\[ \text{ Springer} \]
If \( V_0 = \mathbb{C}_- \) denotes the standard fibre of \( \mathcal{O}(k) \to \text{Gr}_2(\mathbb{C}^4) \), Lemma 4.3 shows that
\[
mV_0 = \mathbb{C}^2 \otimes \mathbb{C}^{2*} \otimes \mathbb{C}^1_{\frac{1}{2} + \frac{1}{2}} \otimes \mathbb{C}_{-k} = \mathbb{C}^2 \otimes \mathbb{C}^{2*} \otimes \mathbb{C}_{-k+1},
\]
therefore \( \text{GS}(mV_0, V_0) \) has the weight \(-k - k + 1 = -2k + 1\). Hence, if \( i \geq 1 \) or \( i = 0 \) and \( j > 1 \),
\[-2k + 2i + j > -2k + 1.\]

It follows that
\[
\text{GS}(mV_0, V_0) \subset \mathbf{V}(2k, 2k, 0, 0) \oplus \mathbf{V}(2k, 2k - 1, 1, 0)
\]

Since \( \mathbf{V}(2k, 2k, 0, 0) \subset S^2(\mathbf{F}(k\pi_2)) \) and \( \mathbf{V}(2k, 2k - 1, 1, 0, 0) \subset \wedge^2(\mathbf{F}(k\pi_2)) \), we conclude
\[
\text{GS}(mV_0, V_0) \subset \mathbf{V}(2k, 2k, 0, 0).
\]

and therefore the real dimension of \( V_k \) is
\[
\frac{2}{3}(1 + k)^2(1 + 2k)(3 + 2k) + \frac{1}{144}(1 + k)^5(2 + k)^2(3 + k)(24 + k(4 + k)(7 + k(4 + k))).
\]

If \( k = 1 \), we obtain \( \dim V_1 = 2 \), and the moduli space \( \mathcal{M}_1 \) is regarded as an open disc in \( \mathbb{C} \).
Therefore, for the specific example of \( \text{Gr}_2(\mathbb{C}^4) \), we obtain the following

**Corollary 4.7** There exists a one-parameter family \( \{ f_t \} \) with \( t \in [0, 1] \) of \( \text{SU}(4) \)-equivariant non-congruent holomorphic isometric embeddings of degree 1 of \( \text{Gr}_2(\mathbb{C}^4) \) into complex quadrics. The mapping \( f_0 \) corresponds to the Kodaira embedding, while \( f_1 \) is the identification \( \text{Gr}_2(\mathbb{C}^4) \cong \text{Gr}_4(\mathbb{R}^6) \).

**Proof** If \( k = 1 \), then \( W = \mathbf{F}(\pi_2) \cong \wedge^2(\mathbb{C}^4) \), which is naturally equipped with an invariant real structure \( \sigma \) such that \( (\sigma, \sigma J) \) spans \( V_1 \cong \mathbb{C} \). From this point, the proof is equal to that of Corollary 7.2 of [14]. The real invariant subspace of \( \wedge^2(\mathbb{C}^4) \) is \( \mathbb{R}^6 \), so that for the case \( t = 1 \) we obtain the identification \( \text{Gr}_2(\mathbb{C}^4) \cong \text{Gr}_4(\mathbb{R}^6) \).

Explicitly, for \( k = 1 \), the irreducible representation with weight \(-1\), denoted by \( \mathbb{C}_{-1} \), is contained in \( \wedge^2 \mathbb{C}^4 \) regarded as a real space. Hence, the standard map is defined as
\[
f_0([g]) = g\mathbb{C}_{-1}^4
\]
and the other maps in the moduli up to image equivalence \( \mathcal{M}_k \) are given by
\[
f_t([g]) = (1 + t\sigma)f_0([g]) = (1 + t\sigma)g\mathbb{C}_{-1}^4, \quad \forall t \in [0, 1].
\]

The kernel of \( 1 + \sigma \) is the orthogonal complement of the invariant real subspace \( \mathbb{R}^6 \) of \( \wedge^2 \mathbb{C}^4 \cong \mathbb{R}^{12} \). From Theorem 2.1, we can see that it corresponds to the totally geodesic submanifold \( \text{Gr}_4(\mathbb{R}^6) \) of \( \text{Gr}_{10}(\mathbb{R}^{12}) \) and the induced map by \( (\mathcal{O}(1), \mathbb{R}^6) \). □

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