Nonexistence results of global solutions for fractional order integral equations on the Heisenberg group

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Abstract— We consider the fractional order integral equation with a time nonlocal nonlinearity
\[ cD_{0,t}^\beta (u) + (\Delta_H)^m (u) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\omega)^{\alpha-1} |u(\omega)|^p \, d\omega, \]
posed in \((.,t) \in \mathbb{H} \times (0, \infty)\), supplemented with an initial data \(u(.,0) = u_0(.)\), where \(m > 1\), \(p > 1\), \(0 < \beta < 1\), \(0 < \alpha < 1\), and \(cD_{0,t}^\beta\) denotes the Caputo fractional derivative of order \(\beta\), and \(\Delta_H\) is the Laplacian operator on the \((2N+1)\)-dimensional Heisenberg group \(\mathbb{H}\). Then, we prove a blow up result for its solutions.

Index Terms—Riemann-Liouville, Heisenberg group, Laplace operator, Hilbert, space

1. Introduction

In this paper, we investigate the higher-order semilinear parabolic equation with nonlocal in time nonlinearity of the following form:
\[
\begin{cases}
cD_{0,t}^\beta (u) + (\Delta_H)^m (u) = I_{0,t}^\alpha |u(t)|^p, \\
\eta = (x,y,\tau) \in \mathbb{H}, \; t > 0
\end{cases}
\]
subject to the initial data
\[ u(\eta,0) = u_0(\eta), \]
Where \(I_{0,t}^\alpha \psi\) is the Riemann–Liouville fractional integral of order \((0 < \alpha < 1)\) defined for a continuous function \(\psi(t), \; t > 0\),
\[ (I_{0,t}^\alpha \psi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\omega)^{\alpha-1} \psi(\omega) \, d\omega, \]
Here, \(\Gamma(.)\) stands for the gamma function.
First, for the sake of the reader, we give some known facts about the Heisenberg group \(\mathbb{H}\) and the operator \(\Delta_H\). For their proof and more information, we refer
for example to \([1, 4, 5, 11, 19]\). The Heisenberg group \(\mathbb{H}\), whose elements are 
\[\eta = (x, y, \tau)\] is the Lie group \((\mathbb{R}^{2N+1}, \circ)\) with the group operation \(\circ\) defined by
\[\eta \circ \tilde{\eta} = (x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2(<x, \tilde{y}> - <\tilde{x}, y>)),\]
where \(<\cdot, \cdot>\) is the usual inner product in \(\mathbb{R}^N\). The laplacian \(\Delta_{\mathbb{H}}\) over \(\mathbb{H}\) is obtained from the vector fields \(X_i = \partial_{x_i} + 2y_i\partial_{\tau}\) and \(Y_i = \partial_{y_i} + 2x_i\partial_{\tau}\), by
\[\Delta_{\mathbb{H}} = \sum_{i=1}^{N} (X_i^2 + Y_i^2),\]
explicitly, we have
\[\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right),\]
A natural group of dilatations on \(\mathbb{H}\) is given by
\[\delta_\gamma (\eta) = (\gamma x, \gamma y, \gamma^2 \tau), \gamma > 0,\]
whose Jacobian determinant is \(\gamma^Q\) where
\[Q = 2N + 2\]
is the homogeneous dimension of \(\mathbb{H}\). The operator \(\Delta_{\mathbb{H}}\) is a degenerate elliptic operator. It is invariant with respect to the left translation of \(\mathbb{H}\) and homogeneous with respect to the dilatations \(\delta_\gamma\). More precisely, we have
\[\Delta_{\mathbb{H}} (u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}} u) (\eta \circ \tilde{\eta}), \Delta_{\mathbb{H}} (u \circ \delta_\gamma) = \gamma^2 (\Delta_{\mathbb{H}} u) \circ \delta_\gamma, \eta, \tilde{\eta} \in \mathbb{H}.\]
The natural distance from \(\eta\) to the origin is
\[|\eta|_{\mathbb{H}} = \left( \tau^2 + \left( \sum_{i=1}^{N} (x_i^2 + y_i^2) \right)^2 \right)^{\frac{1}{4}}.\]
Now, we call sub-elliptic gradient
\[\nabla_{\mathbb{H}} = (X, Y) = (X_1, \ldots, X_N, Y_1, \ldots, Y_N),\]
A remarkable property of the Kohn Laplacian is that a fundamental solution of \(-\Delta_{\mathbb{H}}\) with pole at zero is given by
\[\Gamma(\eta) = \frac{C_\Lambda}{|\eta|_{\mathbb{H}}^{\Lambda^*}},\]
where \(C_\Lambda\) is a suitable positive constant.
A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality
\[\|v\|_{L^2} = c\|\nabla_{\mathbb{H}} v\|_2, \forall v \in C_0^\infty (\mathbb{H}^N),\]
where \(\Lambda^* = \frac{2\Lambda}{N-2}\) and \(c\) is a positive constant.
This inequality ensures in particular that for every domain \(\Omega\) the function
\[\|v\| \leq \|\nabla_{\mathbb{H}} v\|_2,\]
is a norm on \( C^\infty_0(\Omega) \). We denote by \( S^1_0(\Omega) \) the closure of \( C^\infty_0(\Omega) \) with respect to this norm; \( S^1_0(\Omega) \) becomes a Hilbert space with the inner product

\[
< u, v >_{S^1_0} = \int_\Omega \nabla H u, \nabla H v >,
\]

**Fractional powers of sub-elliptic Laplacians.** Here, we recall a result on fractional powers of sub-Laplacian in the Heisenberg group. Let \( N(t, x) \) be the fundamental solution of \( \Delta_H + \frac{\partial}{\partial t} \). For all \( 0 < \beta < 4 \), the integral

\[
R_\beta(x) = \frac{1}{\Gamma\left(\frac{\beta}{2}\right)} \int_0^{+\infty} t^{\frac{\beta}{2} - 1} N(t, x) dt,
\]

converges absolutely for \( x \neq 0 \). If \( \beta < 0, \beta \neq 0, -2, -4, \ldots \), then

\[
\tilde{R}_\beta(x) = \frac{\beta}{\Gamma\left(\frac{\beta}{2}\right)} \int_0^{+\infty} t^{\frac{\beta}{2} - 1} N(t, x) dt,
\]

defines a smooth function in \( \mathbb{H} - \{0\} \), since \( t \mapsto N(t, x) \), vanishes of infinite order as \( t \to 0 \) if \( x \neq 0 \). In addition, \( \tilde{R}_\beta \) is positive and \( \mathbb{H} \)-homogeneous of degree \( \beta - 4 \).

**Theorem:**

For every \( v \in S(\mathbb{H}) \) (Schwartz’s class), we have \( (-\Delta_H)^s \in L^2(\mathbb{H}) \) and

\[
(-\Delta_H)^s = \int_\mathbb{H} (v(x \circ y) - v(x) - \nabla_H v(x), y >) \tilde{R}_{-2s}(y) dy,
\]

where \( \chi \) is the characteristic function of the unit ball \( B_\rho(0, 1) \), \( \rho(x) = R^{\frac{1}{2} - \alpha}(x) \), \( 0 < \alpha < 2 \), \( \rho \) is an \( \mathbb{H} \)-homogeneous norm in \( \mathbb{H} \) smooth outside the origin).

2. **Preliminaries**

2.1. **Definition.** (Riemann-Liouville fractional derivatives)

Let \( f \in AC[a, b], -\infty < a < b < +\infty \). The Riemann-Liouville left- and right-sided fractional derivatives of order \( \alpha \in (0, 1) \) are, respectively, defined by

\[
D^\alpha_{a+} f(t) = \frac{d}{dt} \mathcal{I}^{1-\alpha}_{a+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t - \tau)^{-\alpha} f(\tau) d\tau, \quad t > a \quad (2.1)
\]

and

\[
D^\alpha_{b-} f(t) = -\frac{d}{dt} \mathcal{I}^{1-\alpha}_{b-} f(t)
\]

1Let \( AC[a, b] \) be the space of functions \( f \) which are absolutely continuous on \( [a, b] \).

\( AC^n[a, b] = \left\{ f : [a, b] \to \mathbb{C} \text{ and } (D^n f)(x) \in AC[a, b] \right\} \).

In particular, \( AC^1[a, b] = AC[a, b] \).
\[= - \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_t^b (\tau - t)^{-\alpha} f(\tau) d\tau, \quad t < b \quad (2.2)\]

2.2. **Definition.** (Riemann-Liouville fractional integrals)

Let \( f \in L^1(a, b), -\infty < a < b < +\infty \), The Riemann-Liouville left- and right-sided fractional integrals of order \( \alpha \in (0, 1) \) are, respectively, defined by

\[ I_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{-(1-\alpha)} f(\tau) d\tau, \quad t > a \quad (2.3) \]

and

\[ I_{b-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{-(1-\alpha)} f(\tau) d\tau, \quad t < b \quad (2.4) \]

2.3. **Definition.** For \( 0 < \alpha < 1 \), the Caputo derivative of order \( \alpha \) for a differentiable function \( f : [0, \infty) \rightarrow \mathbb{R} \) can been written as

\[ {^c}D_{a+}^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_a^t (t - \tau)^{-\alpha} f'(\tau) d\tau, \quad t > a \quad (2.5) \]

It is clear that

\[ {^c}D_{a+}^\alpha f(t) = D_{a+}^\alpha [f(t) - f(0)], \]

Finally, taking into account the following integration by parts formula:

\[ \int_a^b f(t) D_{a+}^\alpha g(t) dt = \int_a^b D_{a+}^\alpha f(t) g(t) dt. \]

2.4. **Proposition.** For \( 0 < \alpha < 1, -\infty < a < b < +\infty \), we have the following identities

\[ D_{a+}^\alpha I_{a+}^\beta f(t) = f(t), \quad t \in (a, b) \]

for all \( f \in L^r(a, b), 1 \leq r \leq \infty \)

and

\[ -DD_{a+}^\alpha f = D_{a+}^{1+\alpha} f, \]

for all \( f \in AC^2[a, b], \) where \( D = \frac{d}{dt}. \)

For \( \rho \gg 1 \) and \( 0 < \alpha < 1 \), Let

\[ f(t) = \begin{cases} \left(1 - \frac{t}{T}\right)^\rho, & 0 < t \leq T, \\ 0, & t \geq T, \end{cases} \quad (2.6) \]

\[ D_{a+}^\alpha I_{a+}^\beta f(t) = \frac{(1 - \alpha + \rho)\Gamma(\rho - 1)}{\Gamma(2 - \alpha - \rho)} T^{-\alpha} \left(1 - \frac{t}{T}\right)^{\rho - \alpha}, \]

and

\[ \int_0^T f(t)^{\frac{\rho'}{r}} |D_{a+}^\alpha f(t)|^{\rho'} = CT^{1-\rho' \alpha}. \]
3. Nonexistence results

3.1. Definition. (Weak solution). Let $T > 0$, a locally integrable function $u \in C \left([0,T], L^1_{loc}(Q_T) \cap L^p_{loc}(Q_T)\right)$ is called a weak solution of (1.1) in $Q_T (Q_T = \mathbb{R} \times [0,T])$ subject to the initial data $u_0 \in L^1_{loc}(\mathbb{H})$ if the equality

$$\int_{Q_T} u_0 \partial_t u \phi d\omega + \int_{Q_T} u |\nabla u|^p d\omega = \int_{Q_T} \nabla H \cdot \nabla \phi d\omega$$

is satisfied for any $\phi$ be a smooth test function $\phi \in C^\infty_0(\mathbb{H})$ with $\phi(\cdot, T) = 0, \; \phi \geq 0, \; d\omega = d\eta dt$

and the solution is called global if $T = +\infty$.

3.2. Theorem. Let $p > 1$, and

$$p < p_c = \frac{(2N + 2)\beta + 2m}{(2N + 2)\beta + 2m(1 - \alpha)}.$$

(c for critical)

Then, (1.1) does not have a nontrivial global weak solution.

3.3. Proposition. Consider a convex function $F \in C^2(\mathbb{R})$. Assume that $\varphi \in C^\infty_0(\mathbb{R}^{2N+1})$, then

$$F'(\varphi)(-\Delta H)^m \varphi \geq (-\Delta H)^m F(\varphi).$$

In particular, if $F(0) = 0$ and $\varphi \in C^\infty_0(\mathbb{R}^{2N+1})$, then

$$\int_{\mathbb{R}^{2N+1}} F'(\varphi)(-\Delta H)^m \varphi d\eta \geq 0.$$

Let us mention that hereafter we will use inequality (2.1) for $F(\varphi) = \varphi^l, \; l \gg 1, \; \varphi \geq 0$, in this case it reads

$$(3.1) \quad l\varphi^{l-1}(-\Delta H)^m \varphi \geq (-\Delta H)^m \varphi^l.$$

We need the following Lemma taken from [32].

3.4. Lemma. Let $f \in L^1(\mathbb{R}^{2N+1})$ and $\int_{\mathbb{R}^{2N+1}} f d\eta \geq 0$. Then there exists a test function $0 \leq \varphi \leq 1$, such that

$$(3.2) \quad \int_{\mathbb{R}^{2N+1}} f \varphi d\eta \geq 0.$$

Proof of theorem :

The proof is done by contradiction. Suppose that $u$ is a global bounded weak solution. First we Choose the test function. For this aim, we shall use a nonnegative smooth function $\phi$ which was constructed in [20].

$$(3.3) \quad \phi(\xi) = \begin{cases} 1 & \text{if } 0 \leq \xi \leq 1, \\ \xi & \text{if } 1 \leq \xi \leq 2, \\ 0 & \text{if } \xi \geq 2, \end{cases}$$
\[ \varphi_1(\eta) = \phi\left(\frac{r^2 + |x|^2 + |y|^2}{R^4}\right), \quad \eta = (x, y, \tau) \in \mathbb{H} \]

\[ \varphi_2(t) = \begin{cases} 
(1 - \frac{t}{T})^\rho, & 0 < t \leq T, \ 
0, & t \geq T,
\end{cases} \quad \rho \gg 1 \]

\[ \varphi(\eta, t) = D^\alpha_{t|TR^{\frac{2m}{p}}}\varphi(\eta, t) = \varphi_1^l(\eta) D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi_2 \left(\frac{t}{R^{\frac{2m}{p}}}\right), \quad R > 0 \]

Let, \( Q = \mathbb{H} \times \left[0, TR^{\frac{2m}{p}}\right] \).

Using the Definition 3.1, we obtain

\[ \int_Q u_0 D^\beta_{t|TR^{\frac{2m}{p}}} D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi(\eta, t) \, d\eta \, dt + \int_Q D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi(\eta, t) \, I_0^\alpha |u|^p \, d\eta \, dt \]

\[ = \int_Q u D^\alpha_{t|TR^{\frac{2m}{p}}} D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi(\eta, t) \, d\eta \, dt + \int_Q u(-\Delta_{\mathbb{H}})^m D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi(\eta, t) \, d\eta \, dt, \]

A simple computation yields \( D^\beta_{t|TR^{\frac{2m}{p}}} \left(D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi\right) = D^{\alpha + \beta}_{t|TR^{\frac{2m}{p}}} \varphi \), we obtain

\[ c(TR^{\frac{2m}{p}})^{1-(\alpha + \beta)} \int_\mathbb{H} u_0 \varphi_1^l(\eta) \, d\eta + \int_Q \varphi |u|^p \, d\eta \, dt \]

\[ = \int_Q u \varphi_1^l(\eta) D^{\alpha + \beta}_{t|TR^{\frac{2m}{p}}} \varphi_2 \left(\frac{t}{R^{\frac{2m}{p}}}\right) \, d\eta \, dt + \int_Q u(-\Delta_{\mathbb{H}})^m \varphi_1^l(\eta) D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi_2 \left(\frac{t}{R^{\frac{2m}{p}}}\right) \, d\eta \, dt, \]

The application of inequality (3.1)

\[ l \varphi_1^{l-1}(-\Delta_{\mathbb{H}})^m \varphi_1 \geq (-\Delta_{\mathbb{H}})^m \varphi_1, \]

implies that

\[ \int_Q |u|^p \varphi_1 \, d\eta \, dt \]

\[ \leq l \int_Q u \varphi_1^{l-1}(\eta)(-\Delta_{\mathbb{H}})^m \varphi_1(\eta) D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi_2 \left(\frac{t}{R^{\frac{2m}{p}}}\right) \, d\eta \, dt + \int_Q u \varphi_1^l(\eta) D^{\alpha + \beta}_{t|TR^{\frac{2m}{p}}} \varphi_2 \left(\frac{t}{R^{\frac{2m}{p}}}\right) \, d\eta \, dt, \]

For estimating the second member of the above inequality, we write

\[ \int_Q u \varphi_1^{l-1}(\eta)(-\Delta_{\mathbb{H}})^m \varphi_1(\eta) D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi_2 \left(\frac{t}{R^{\frac{2m}{p}}}\right) \, d\eta \, dt \]

\[ = \int_Q u \varphi_1^{l-1}(\eta)(-\Delta_{\mathbb{H}})^m \varphi_1(\eta) D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi_2 \left(\frac{t}{R^{\frac{2m}{p}}}\right) \, d\eta \, dt. \]

According to \( \varepsilon \)-Young inequality

\[ XY \leq cX^p + C(\varepsilon)Y^{p'}, \quad p + p' = pp', \]

we have

\[ \int_Q u \varphi_1^{l-1}(\eta)(-\Delta_{\mathbb{H}})^m \varphi_1(\eta) D^\alpha_{t|TR^{\frac{2m}{p}}} \varphi_2 \left(\frac{t}{R^{\frac{2m}{p}}}\right) \, d\eta \, dt \]
\[
\leq \epsilon \int_Q |u|^p \varphi \, d\eta dt + C_1(\epsilon) \int_Q \varphi_1(t-1, r, \theta) \left( -\Delta_\mathbb{H} \right)^m \varphi_1(\eta) D_{\eta}^{\alpha} \varphi_2(\frac{t}{R^{2m}}) \varphi^{-\frac{\beta}{p'}} \, d\eta dt.
\]

In the same way, we get
\[
\int_Q u \varphi_1(\eta) D^{\alpha + \beta} \varphi_2(\frac{t}{R^{2m}}) \, d\eta dt
\]
\[
\leq \epsilon \int_Q |u|^p \varphi \, d\eta dt + C_2(\epsilon) \int_Q |\varphi_1(\eta) D^{\alpha + \beta} \varphi_2(\frac{t}{R^{2m}})\varphi^{-\frac{\beta}{p'}} \, d\eta dt
\]

Now, when \( \epsilon \) is small, and \( C = \max \{ C_1(\epsilon), C_2(\epsilon) \} \) we obtain
\[
\int_Q |u|^p \varphi \, d\eta dt \leq C \left\{ \int_Q \varphi_1(t-1, r, \theta) \left( -\Delta_\mathbb{H} \right)^m \varphi_1(\eta) D^{\alpha} \varphi_2(\frac{t}{R^{2m}}) \varphi^{-\frac{\beta}{p'}} \, d\eta dt
\]
\[
+ \int_Q |\varphi_1(\eta) D^{\alpha + \beta} \varphi_2(\frac{t}{R^{2m}})\varphi^{-\frac{\beta}{p'}} \, d\eta dt \right\},
\]
as
\[
\varphi^{-\frac{\beta}{p'}}(\eta, t) = \varphi_1^\frac{1}{p'}(\eta) \varphi_2(\frac{t}{R^{2m}}), \quad p' = \frac{p}{p-1}
\]
we have
\[
\int_Q |u|^p \varphi \, d\eta dt \leq C \left\{ \int_Q \varphi_1(t-1, r, \theta) \left( -\Delta_\mathbb{H} \right)^m \varphi_1(\eta) D^{\alpha} \varphi_2(\frac{t}{R^{2m}}) \varphi^{-\frac{\beta}{p'}} \, d\eta dt
\]
\[
+ \int_Q \varphi_1(\eta) D^{\alpha} \varphi_2(\frac{t}{R^{2m}}) \varphi^{-\frac{\beta}{p'}} \, d\eta dt \right\},
\]

We apply the change of next variables \( \tilde{\tau} = \frac{x}{R^{2m}}, \quad \tilde{\xi} = \frac{\xi}{R^{2m}}, \quad \tilde{\eta} = \frac{\eta}{R^{2m}}, \quad \tilde{t} = \frac{t}{R^{2m}}, \)
then we put
\[
\Omega = \{ \eta = (\tilde{\xi}, \tilde{\eta}) \in \mathbb{H}; \quad 0 \leq \tilde{\tau}^2 + |\tilde{\xi}|^4 + |\tilde{\eta}|^4 \leq 2 \}
\]
if we put
\[
A = \int_{Q_1} \varphi_1(t-1, r, \theta) \varphi_2(\frac{t}{R^{2m}}) \left( -\Delta_\mathbb{H} \right)^m \varphi_1(\eta) D^{\alpha} \varphi_2(\frac{t}{R^{2m}}) \varphi^{-\frac{\beta}{p'}} \, d\eta dt,
\]
\[
B = \int_{Q_1} \varphi_1(\eta) \varphi_2(\frac{t}{R^{2m}}) D^{\alpha + \beta} \varphi_2(\frac{t}{R^{2m}}) \varphi^{-\frac{\beta}{p'}} \, d\eta dt,
\]
we get
\[
A = \int_0^{TR^{2m}} \varphi_2(\frac{t}{R^{2m}}) D^{\alpha} \varphi_2(\frac{t}{R^{2m}}) \varphi^{-\frac{\beta}{p'}} \, dt \int_{\Omega} \varphi_1(t-1, r, \theta) \left( -\Delta_\mathbb{H} \right)^m \varphi_1(\eta) \varphi^{-\frac{\beta}{p'}} \, d\eta
\]
\[
= R^{2m} \varphi_2(\frac{t}{R^{2m}}) \int_0^T \varphi_2(\tilde{t}) \varphi^{-\frac{\beta}{p'}}(\tilde{t}) \, dt
\]
\[
\times R^{2N+2} \int_{\Omega} \varphi(t-1, r, \theta) \left( \tilde{\tau}^2 + |\tilde{\xi}|^4 + |\tilde{\eta}|^4 \right) \left( -\Delta_\mathbb{H} \right)^m \varphi_1(t-1, r, \theta) \varphi^{-\frac{\beta}{p'}} \, d\tilde{\xi} d\tilde{\eta} d\tilde{\tau},
\]
\[
= CT^{1-\frac{1}{2m}} R^{2N+2} \frac{2m}{p} - \frac{2m}{p} \]
\[
\times \int_{\Omega} \phi^{(l-p')} (\tilde{\tau}^2 + |\tilde{x}|^4 + |\tilde{y}|^4) \left| (-\Delta_H)^m \phi (\tilde{\tau}^2 + |\tilde{x}|^4 + |\tilde{y}|^4) \right|^p \, d\tilde{x}d\tilde{y}d\tilde{r},
\]

and

\[
B = \int_{0}^{TR} \frac{2m}{R^{2m}} \left( \frac{t}{R^{2m}} \right) |D_{l}^{\alpha+\beta} \varphi_2 \left( \frac{t}{R^{2m}} \right)|^p \, dt \int_{\mathbb{H}} \varphi_1^l (\eta) \, d\eta
\]

\[
= R^{2m} - \frac{2m\alpha+\beta}{\beta(p-1)} \int_{0}^{T} \frac{2m}{R^{2m}} \left( \frac{l}{R^{2m}} \right) |D_{l}^{\alpha+\beta} \varphi_2 \left( \frac{l}{R^{2m}} \right)|^p \, dl
\]

\[
\times R^{2N+2} \int_{\Omega} \phi^l \left( \tilde{\tau}^2 + |\tilde{x}|^4 + |\tilde{y}|^4 \right) d\tilde{x}d\tilde{y}d\tilde{r},
\]

\[
= CT^{1-\frac{2(\alpha+\beta)}{p}} R^{2N+2+\frac{2m}{\beta(p-1)}} \times \int_{\Omega} \phi^l \left( \tilde{\tau}^2 + |\tilde{x}|^4 + |\tilde{y}|^4 \right) d\tilde{x}d\tilde{y}d\tilde{r},
\]

in the last

\[
(3.4) \quad \int_{Q} |u|^p \, d\eta dt \leq C \{A + B\} \leq CR^{2N+2+\frac{2m}{\beta(p-1)}}
\]

Now, if

\[
2N + 2 + \frac{2m}{\beta} - \frac{2mp\alpha}{\beta(p-1)} < 0 \iff p < p_c
\]

by letting \( R \to +\infty \) in (3.4), we obtain

\[
\int_{Q} |u|^p \, d\eta dt = 0 \Rightarrow u \equiv 0,
\]

this is a contradiction. \( \square \)
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