Harnack Inequality for Semilinear SPDE with Multiplicative Noise*

Zhang Shao-Qin

School of Math. Sci. and Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China
Email: zhangsq@mail.bnu.edu.cn

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Abstract

By a new approximate method, dimensional free Harnack inequalities are established for a class of semilinear stochastic differential equations in Hilbert space with multiplicative noise. These inequalities are applied to study the strong Feller property for the semigroup and some properties of invariant measure.

AMS subject Classification (2000): 60J60.

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1 Introduction and main results

The main aim of this paper is to prove Harnack inequality for semilinear stochastic equations on Hilbert spaces with multiplicative noise. This type of inequality, which was proved for the first time in [15], has became a powerful tool in infinite dimensional stochastic analysis. There are many papers prove this type of inequality for SPDE with additive noise, see [3, 5, 7, 9, 10, 11, 12, 16, 17, 18, 19] and reference therein. In [14], the log-Harnack inequality for semilinear SPDE with non-additive noise was proved for the first time, but by the gradient estimate method used there, only determine and time independent coefficient was treated. A new method to deal with the case of general coefficients for SDE was introduced in [17]. This method has been generalized to functional stochastic differential equations, see [20]. In this paper, we generalized this method to the case of semilinear SPDE. There are some disadvantages for finite dimension approximate method here, see Remark 1.3, therefore we use the coupling argument again as in [17] with a slight modification. Since it seems not so clear to solves the similar equation of process $Y_t$ (see equation (2.3) in [17] ) in infinite dimension, we turn to a new process which plays the role as the difference of the coupling processes, we get it as a local strong solution of a SPDE and solve the equation

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by truncation in the same sprite in [2]. By this process and Girsanov theorem, we get a coupling in a new probability space. On the other hand, we get Harnack inequality by another type of approximation. We perturb the linear term by a suitable linear operator which closely relates to diffusion term. It’s different from finite dimensional approximate and Yosida approximate, by this perturbation, we get a stronger linear term and it makes us to prove the inequality for the perturbed equation more easy.

Let $H$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, consider the following stochastic differential equation on $H$:

\begin{equation}
\begin{aligned}
dx_t &= -Ax_tdt + F(t,x_t)dt + B(t,x_t)dW_t \\
W &= W(t), t \geq 0 \text{ is a cylindrical Brownian motion on } H \text{ with covariance operator } I \text{ on filtered probability space } (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0}), \text{ and the coefficients satisfy the following hypotheses:}
\end{aligned}
\end{equation}

(H1) $A$ is a negative self adjoint operator with discrete spectrum:

\begin{equation}
0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty,
\end{equation}

$\{\lambda_n, n \in \mathbb{N}\}$ are the eigenvalues of $A$, and $\{e_n\}_{n=1}^{+\infty}$ are the corresponding eigenvectors, the compact $C_0$ semigroup generated by $-A$ denoted by $S(t)$.

(H2) $F : [0, \infty) \times \Omega \times H \to H$ and $B : [0, \infty) \times \Omega \times H \to L(H)$ are $\mathcal{P}_\infty \times \mathcal{B}(H)$ measurable, here $\mathcal{P}_\infty$ is predictable $\sigma$-algebra on $[0, \infty) \times \Omega$ and $L(H)$ is all the bounded operators on $H$, and there exists an increasing function $K_1 : [0, +\infty) \to [0, \infty)$, such that

\begin{equation}
||F(t,x) - F(t,y)|| + ||B(t,x) - B(t,y)||_{HS} \leq K_1(t)||x - y||,
\end{equation}

for all $t \geq 0, x \in H, \mathbb{P}$-a.s, here $\| \cdot \|_{HS}$ denote the Hilber-Schmidt norm, and there exists $r > 1$, such that for all $t > 0$,

\begin{equation}
\mathbb{E} \left( \int_0^t ||F(s,0)|| ds \right)^r < \infty,
\end{equation}

\begin{equation}
\sup_{u \in [0,t]} \int_0^u (\mathbb{E} ||S(u - s)B(s,0)||^r_{HS})^{\frac{1}{r}} ds < \infty,
\end{equation}

(H3) There exist a decreasing function $\rho : [0, \infty) \to (0, \infty)$, and a bounded self adjoint operator $B_0$ satisfying that there exists $\{b_n > 0|n \in \mathbb{N}\}$ such that $B_0 e_n = b_n e_n$ and

\begin{equation}
B(t,x)B(t,x)^\ast \geq \rho(t)^2 B_0^2, \forall x \in H, t \geq 0, \mathbb{P}$-a.s.,
\end{equation}

(H4) $\text{Ran}(B(t,x) - B(t,y)) \subset \mathcal{D}(B_0^{-1})$ holds for all $(t,x) \in [0, \infty) \times H, \mathbb{P}$-a.s., and there exists an increasing function $K_2 : [0, \infty) \to \mathbb{R}$ such that

\begin{equation}
2\langle F(t,x) - F(t,y), B_0^{-2}(x - y) \rangle + ||B_0^{-1}(B(t,x) - B(t,y))||^2_{HS} \leq K_2(t)||B_0^{-1}(x - y)||^2
\end{equation}

holds for all $x,y \in \mathcal{D}(B_0^{-2})$ and all $t \geq 0, \mathbb{P}$-a.s.,
holds for all $T > K$ theorem, we need some lemmas, and denote $P(1.7)$

Theorem 1.2. If (H1)-(H4) hold, then

\begin{equation}
(1.8) \quad (P) \quad \text{(inequality)}
\end{equation}

(H5) There exists an increasing function $K_3 : [0, \infty) \rightarrow (0, \infty)$, such that $||(B(t, x)^* - B(t, y)^*)B_0^{-2}(x - y)|| \leq K_3(t)||x - y||H_0$ holds for all $x, y \in H$, $t \geq 0$ and $x - y \in \mathcal{D}(B_0^{-1})$ almost surely.

Remark 1.1. (1) Under (H1), we can replace $\mathcal{D}(B_0^{-2})$ in (H4) by $\bigcup_n H_n$, where $H_n = \text{span}\{e_1, \cdots, e_n\}$.

(2) (H3) equals to that $\text{Ran}(B(t, x)) \supset \text{Ran}B_0$ and $||(B(t, x)^{-1})z|| \leq \rho(t)^{-1}||B_0^{-1}z||$, for all $z \in \mathcal{D}(B_0^{-1})$, $t \geq 0$, $\mathbb{P}$-a.s.,

(3) (H5) will be used as a condition in addition to get Harnack inequality, and by (H4), see remark after theorem 1 in [1], and it seems not easy to get similar estimate for some estimates.

first lemma prove the existence and uniqueness of mild solution of the equation (1.1), and give Fixed a time $T > 0$, we focus our discussion on the interval $[0, T)$. In order to prove the main theorem, we need some lemmas, and denote $K_i(T)$ by $K_i$, $i = 1, 2, 3$, for simplicity’s sake. The first lemma prove the existence and uniqueness of mild solution of the equation (1.1), and give some estimates.

For the proof of Remark 1.1 see Appendix. We state our main result of this paper

Theorem 1.2. If (H1)-(H4) hold, then

\begin{equation}
P_T \log f(y) \leq \log P_T f(x) + \frac{K_2(T)||x - y||H_0}{2(1 - e^{K_2T})}, \quad \forall f \in \mathcal{B}_b(H), f \geq 1, x, y \in H, T > 0.
\end{equation}

If, in addition, (H5) holds, then for $p > (1 + \frac{K_3(T)}{\rho(T)})^2$, $\delta_{p,T} = K_3 \vee \frac{\rho(T)}{2}(\sqrt{p} - 1)$, the Harnack inequality

\begin{equation}
(P_T f(y))^p \leq (P_T f^p(x)) \exp \left[ \frac{K_2(T)\sqrt{p}(\sqrt{p} - 1)||x - y||H_0^2}{4\delta_{p,T}(\sqrt{p} - 1)\rho(T) - \delta_{p,T}(1 - e^{K_2T})} \right],
\end{equation}

holds for all $T > 0, x, y \in H$ and $f \in \mathcal{B}_b^+(H)$, where $||x||_{H_0}^2 = \sum_{n=0}^{+\infty} b_n^{-1}(x, e_n)^2$, $H_0 = \mathcal{D}(B_0^{-1})$.

Remark 1.3. One may use the finite dimension approximate method to get the Harnack inequalities, but here we mention that there are difficulties to overcome and it may not be better than the method used here. Let $\pi_n$ be the projection form $H$ to $H_n$, then get the following equation on $H_n$

\begin{equation}
\begin{aligned}
dx_t^n &= -A_n x^n_t dt + F_n(t, x^n_t) dt + B_n(t, x^n_t) dW^n_t,
\end{aligned}
\end{equation}

where,

\begin{equation}
A_n = \pi_n A, \quad F_n = \pi_n F|_{H_n}, \quad B_n = \pi_n B|_{H_n}, \quad W^n = \pi_n W,
\end{equation}

one may find that after projecting to lower dimension, an invertible operator may become degenerate, for example, an operator has the matrix form, \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]
under the orthonormal basis $\{e_1, e_2\}$. It’s easy to find that it’s degenerate after projecting to the subspace generated by $e_1$. By (H5), one may replace $B$ by its symmetrization $\sqrt{BB^*}$, but constant may become worse in (H4) and (H5), see remark after theorem 1 in [1], and it seems not easy to get similar estimate for $\sqrt{BB^*}$ as in (H5).

2 Proof of Theorem 1.2

Fixed a time $T > 0$, we focus our discussion on the interval $[0, T]$. In order to prove the main theorem, we need some lemmas, and denote $K_i(T)$ by $K_i$, $i = 1, 2, 3$, for simplicity’s sake. The first lemma prove the existence and uniqueness of mild solution of the equation (1.1), and give some estimates.
Lemma 2.1. Under the condition (H1) and (H2), equation (1.1) has a pathwise unique mild solution and

\[
\sup_{t \in [0, T]} \mathbb{E}\|x_t\|^r \leq C(r, T)(1 + \mathbb{E}\|x_0\|^r).
\]

Proof. The existence part goes along the same lines as that of Theorem 7.4 in [4], if we can prove that there exists \( p \geq 2 \), such that

\[
\sup_{t \in [0, T]} \mathbb{E}\left\| \int_0^t e^{-(t-s)A}F(s, x_s)ds \right\|^p < \infty,
\]

and

\[
\sup_{t \in [0, T]} \mathbb{E}\left\| \int_0^t e^{-(t-s)A}B(s, x_s)dW_s \right\|^p < \infty
\]

for all \( H \)-valued predictable processes \( x \) defined on \([0, T]\) satisfying

\[
\sup_{t \in [0, T]} \mathbb{E}\|x_t\|^p < \infty.
\]

In fact, for \( r \) in \((\mathbb{H}_2, \mathbb{H}_1)\),

\[
\sup_{t \in [0, T]} \mathbb{E}\left\| \int_0^t e^{-(t-s)A}B(s, x_s)dW_s \right\|^r
\]

\[
\leq \sup_{t \in [0, T]} \mathbb{E}\left\| \int_0^t e^{-(t-s)A}(B(s, x_s) - B(s, 0))dW_s \right\|^r + \sup_{t \in [0, T]} \mathbb{E}\left\| \int_0^t e^{-(t-s)A}B(s, 0)dW_s \right\|^r
\]

\[
\leq C(r, T)(1 + \mathbb{E}\|x_t\|^r) + \left( \frac{r}{2} (r - 1) \right)^{\frac{r}{2}} \sup_{t \in [0, T]} \left( \int_0^t \mathbb{E}\|S(t-s)B(s, 0)\|_{HS}^r \right)^{\frac{2}{r}} < \infty.
\]

\( F \) is treated similarly, we omit it. Estimate (2.1) follows from Gronwall’s lemma. For the uniqueness part. If \( x^1_t, x^2_t \) are mild solutions of equation (1.1), then

\[
\begin{align*}
\mathbb{E} \sup_{u \in [0, t]} \|x^1_u - x^2_u\|^r & \leq 2rT \mathbb{E} \sup_{u \in [0, t]} \int_0^u \|S(u-s) (F(s, x^1_s) - F(s, x^2_s))\|^r ds \\
& \quad + 2r \mathbb{E} \sup_{u \in [0, t]} \left\| \int_0^u S(u-s) (B(t, x^1_s) - B(t, x^2_s))dW_s \right\|^r
\end{align*}
\]

\[
\leq 2rT \int_0^t \mathbb{E}\|x^1_u - x^2_u\|^r ds + C(r, T) \mathbb{E} \int_0^t \|x^1_s - x^2_s\|^r ds
\]

\[
\leq C(r, T) \int_0^t \mathbb{E} \sup_{u \in [0, s]} \|x^1_u - x^2_u\|^r ds,
\]

by the second inequality, \( \mathbb{E} \sup_{u \in [0, t]} \|x^1_u - x^2_u\|^r < \infty \), then by Gronwall’s lemma, \( x^1_t = x^2_t \), \( \forall t \in [0, T] \), \( \mathbb{P} \)-a.s. \( \square \)
Denote $A_\epsilon = A + \epsilon B_0^{-2}$, $\mathcal{D}(A_\epsilon) = \mathcal{D}(A) \cap \mathcal{D}(B_0^{-2}) \subset \mathcal{D}(B_0^{-2})$, it is a self adjoint operator, the eigenvalues of $A_\epsilon$ are $\{\lambda_{n,\epsilon} := \lambda_n + \epsilon b_n^{-2} | n \in \mathbb{N}\}$ and the eigenvectors remain $\{e_n | n \in \mathbb{N}\}$. In fact, one can define a self adjoint operator $\tilde{A}$ by

\[
\mathcal{D}(\tilde{A}) = \left\{ x \in H \mid \sum_{n=0}^{+\infty} (\lambda_n + \epsilon b_n^{-2})^2 \langle x, e_n \rangle^2 < +\infty \right\},
\]

\[
\tilde{A}x = \sum_{n=0}^{+\infty} (\lambda_n + \epsilon b_n^{-2}) \langle x, e_n \rangle e_n,
\]

then by basic inequality and spectral decomposition of $A$ and $B_0^{-2}$, it is easy to see that $\tilde{A} = A_\epsilon$.

**Lemma 2.2.** For the mild solution of equation

\[
dx_t^\epsilon = -(A + \epsilon B_0^{-2})x_t^\epsilon dt + F(t, x_t^\epsilon)dt + B(t, x_t^\epsilon)dW_t, \quad x_0^\epsilon = x,
\]

we have

\[
\lim_{\epsilon \to 0^+} \mathbb{E}||x_t - x_t^\epsilon||^2 = 0, \forall t \in [0, T].
\]

**Proof.** Since

\[
x_t = e^{-tA}x + \int_0^t e^{-(t-s)A}F(s, x_s)ds + \int_0^t e^{-(t-s)A}B(s, x_s)dW_s,
\]

\[
x_t^\epsilon = e^{-t(A+\epsilon B_0^{-2})}x + \int_0^t e^{-(t-s)(A+\epsilon B_0^{-2})}F(s, x_s^\epsilon)ds + \int_0^t e^{-(t-s)(A+\epsilon B_0^{-2})}B(s, x_s^\epsilon)dW_s,
\]

then

\[
||x_t - x_t^\epsilon||^2 \leq 3||e^{-tA}x||^2 + 3||\int_0^t (e^{-(t-s)A}F(s, x_s) - e^{-(t-s)(A+\epsilon B_0^{-2})}F(s, x_s^\epsilon))ds||^2
\]

\[
+ 3||\int_0^t (e^{-(t-s)A}B(s, x_s) - e^{-(t-s)(A+\epsilon B_0^{-2})}B(s, x_s^\epsilon))dW_s||^2
\]

\[
=: I_1 + I_2 + I_3.
\]

It’s clear that $\lim_{\epsilon \to 0^+} I_1 = 0$. For $I_2$, we have

\[
I_2 \leq 6T \int_0^t ||e^{-(t-s)A} - e^{-(t-s)(A+\epsilon B_0^{-2})}||^2||F(s, x_s)||^2 ds
\]

\[
+ 6T \int_0^t ||e^{-(t-s)(A+\epsilon B_0^{-2})}(F(s, x_s) - F(s, x_s^\epsilon))||^2 ds =: I_{2,1} + I_{2,2},
\]

Since

\[
||(e^{-(t-s)A} - e^{-(t-s)(A+\epsilon B_0^{-2})})F(s, x_s)|| \leq C(1 + ||x_s||),
\]

\[
\lim_{\epsilon \to 0^+} ||(e^{-(t-s)A} - e^{-(t-s)(A+\epsilon B_0^{-2})})F(s, x_s)|| = 0.
\]
By domain convergence theorem, \( \lim_{t \to 0^+} \mathbb{E} I_{2,1} = 0 \). On the other hand,

\[
I_{2,2} \leq 6T \int_0^t ||e^{-(t-s)(A+tB_0^2)}(F(s,x_s) - F(s,x_s^e))||^2 ds \\
\leq 6T \int_0^t ||F(s,x_s) - F(s,x_s^e)||^2 ds \leq 6TK_1 \int_0^t ||x_s - x_s^e||^2 ds.
\]

(2.16)

For \( I_3 \),

\[
\mathbb{E} I_3 \leq 6\mathbb{E} \left| \int_0^t (e^{-(t-s)A} - e^{-(t-s)(A+tB_0^2)})B(s,x_s) dW_s \right|^2 \\
+ 6\mathbb{E} \left| \int_0^t e^{-(t-s)(A+tB_0^2)}(B(s,x_s) - B(s,x_s^e)) dW_s \right|^2 = I_{3,1} + I_{3,2},
\]

and

\[
\mathbb{E} I_{3,1} \leq 12T\mathbb{E} \int_0^t ||(I - e^{-(t-s)tB_0^2})(e^{-(t-s)A}B(s,0))||^2_H dW_s ds \\
\leq 12T\mathbb{E} \int_0^t \left| \int_0^t (e^{-(t-s)A} - e^{-(t-s)(A+tB_0^2)})(B(s,x_s) - B(s,0)) dW_s \right|^2_H ds \\
+ 12T\mathbb{E} \int_0^t \left| (I - e^{-(t-s)tB_0^2})(e^{-(t-s)A}(B(s,x_s) - B(s,0))) \right|^2_H ds = I_{3,1,1} + I_{3,1,2},
\]

(2.17)

since

\[
||(I - e^{-(t-s)tB_0^2})e^{-(t-s)A}B(s,0)||^2 = \sum_{n=1}^{+\infty} \left| ||(e^{-(t-s)tB_0^2} - I)e^{-(t-s)A}B(s,0)e_n|| \right|^2
\]

(2.18)

and

\[
\lim_{t \to 0^+} \left| ||(e^{-(t-s)tB_0^2} - I)e^{-(t-s)A}B(s,0)e_n|| \right| = 0
\]

(2.19)

\[
||e^{-(t-s)tB_0^2}e^{-(t-s)A}B(s,0)e_n|| \leq ||e^{-(t-s)A}B(s,0)e_n||
\]

(2.20)

By dominate convergence theorem, \( \lim_{t \to 0} I_{3,1,1} = 0 \). Note that \( B(s,x_s) - B(s,0) \in L_{HS}(H) \), and

\[
||e^{-(t-s)tB_0^2}||^2 \leq \int_0^t \left| \int_0^t (e^{-(t-s)A} - e^{-(t-s)(A+tB_0^2)})(B(s,x_s) - B(s,0)) dW_s \right|^2_H ds < \infty.
\]

(2.21)

By dominate convergence theorem, \( \lim_{t \to 0} I_{3,1,1} = 0 \). Note that \( B(s,x_s) - B(s,0) \in L_{HS}(H) \), and

\[
\sum_{n=1}^{+\infty} \left| ||(I - e^{-(t-s)tB_0^2})e^{-(t-s)A}(B(s,x_s) - B(s,0))e_n||^2
\]

(2.22)

and by (H2)

\[
\mathbb{E} \int_0^t \sum_{n=1}^{+\infty} ||e^{-(t-s)A}B(s,0)e_n||^2 ds = \mathbb{E} \int_0^t ||e^{-(t-s)A}B(s,0)||^2_{HS} ds < \infty.
\]

(2.23)

By dominate convergence theorem, \( \lim_{t \to 0} I_{3,1,1} = 0 \). Note that \( B(s,x_s) - B(s,0) \in L_{HS}(H) \), and

\[
||I - e^{-(t-s)tB_0^2})e^{-(t-s)A}(B(s,x_s) - B(s,0)||^2_{HS}
\]

(2.24)
and
\[(2.25) \quad ||(I - e^{-(t-s)}e^{B_{0,2}^{-2}}(e^{-(t-s)}A(B(s,x_s) - B(s,0)))e_n||^2 \leq ||(B(s,x_s) - B(s,0))e_n||^2\]
\[(2.26) \quad \mathbb{E}\int_0^t \sum_{n=1}^{+\infty} ||(B(s,x_s) - B(s,0))e_n||^2 ds \leq \mathbb{E}\int_0^t ||x_s||^2 ds < \infty,\]
by dominate convergence theorem, \(\lim_{\epsilon \to 0} \mathbb{E}I_{3,1} = 0.\) Finally,
\[(2.27) \quad \mathbb{E}I_{3,2} \leq 6T \mathbb{E}\int_0^t ||B(s,x_s) - B(s,x_s')||^2_{HS} ds \leq 6TK_2 \mathbb{E}\int_0^t ||x_s - x_s'||^2 ds.\]
Now, we have
\[(2.28) \quad \mathbb{E}||x_t - x_t'||^2 \leq \psi(t) + C(T, K_2) \mathbb{E}\int_0^t ||x_s - x_s'||^2 ds\]
for some \(\psi(t),\) which satisfies \(\lim_{\epsilon \to 0} \psi(t) = 0,\) then by Gronwall’s lemma,
\[(2.29) \quad \lim_{\epsilon \to 0} \mathbb{E}||x_t - x_t'||^2 = 0, \forall t \in [0, T].\]

Firstly, we shall consider the following equation, \(\xi_t = \frac{2}{K_2}(1 - e^{K_2(t-T)}),\)
\[(2.30) \quad dz_t = -A_t z_t dt + (F(t, x_t) - F(t, x_t - z_t))dt + (B(t, x_t) - B(t, x_t - z_t))dW_t\]
\[-\frac{1}{\xi_t}(B(t, x_t - z_t) - B(t, x_t))B(t, x_t)^{-1}z_t dt - \frac{1}{\xi_t}z_t dt, \ z_0 = z.\]
Note that, by (H2)-(H4),
\[(2.31) \quad F(t, x_t) - F(t, x_t - z_t) \in H, \ (B(t, x_t) - B(t, x_t - z_t)) \in L_{HS}(H, H_0),\]
\[(2.32) \quad (B(t, x_t - z_t) - B(t, x_t))B(t, x_t)^{-1} \in L(H_0, H_0),\]
it’s natural to solve the equation in \(H_0,\) we shall search a suitable Gelfand triple. To this end, we should restrict the operator \(A_t\) to \(H_0.\)

**Lemma 2.3.** Define \(A_{0,\epsilon}\) as follows
\[(2.33) \quad \mathcal{D}(A_{0,\epsilon}) = B_0(\mathcal{D}(A_t)), \ A_{0,\epsilon} x = A_t x, \forall x \in B_0(\mathcal{D}(A_t)),\]
then, \(A_{0,\epsilon}\) is well defined and \(A_{0,\epsilon}, B_0(\mathcal{D}(A_t)) = (B_0A_tB_0^{-1}, B_0(\mathcal{D}(A_t))).\)

**Proof.** It’s well defined. In fact for all \(x \in B_0(\mathcal{D}(A_t)),\)
\[(2.34) \quad \sum_{n=1}^{+\infty} \lambda^{n, \epsilon}_n(x, e_n)^2 = \sum_{n=0}^{+\infty} \lambda^{n, \epsilon}_n b_n^2(B_0^{-1} x, e_n)^2 \leq ||B||^2_H \sum_{n=1}^{+\infty} (\lambda^{n, \epsilon}_n)^2(B_0^{-1} x, e_n)^2 < +\infty,\]
then \(x \in \mathcal{D}(A_t),\) and
\[(2.35) \quad \sum_{n=1}^{+\infty} b_n^2(A_t x, e_n)^2 = \sum_{n=1}^{+\infty} \lambda^{n, \epsilon}_n(B_0^{-1} x, e_n)^2 < +\infty,\]
then \( A_\varepsilon x \in \mathcal{D}(B_0^{-1}) \), \( \forall x \in B_0(\mathcal{D}(A_\varepsilon)) \), i.e. \( A_\varepsilon x \in H_0 \). Finally, for all \( x \in B_0(\mathcal{D}(A)) \),

\[
B_0 A_\varepsilon B_0^{-1} x = A_\varepsilon B_0^{-1} x = A_\varepsilon x = A_{0,\varepsilon} x.
\]

Now, we can define our Gelfand triple. Let

\[
(V, ||\cdot||_V) = (\mathcal{D}(A_{0,\varepsilon}^{\frac{1}{2}}), ||A_{0,\varepsilon}^{\frac{1}{2}} \cdot ||_{H_0}),
\]

then \((V^*, ||\cdot||_{V^*})\) is the complete of \((H_0, ||A_{0,\varepsilon}^{-\frac{1}{2}} \cdot ||_{H_0}), V^* \supseteq H_0 \supseteq V\) is the triple we need. Since \(\mathcal{D}(A_\varepsilon) \subset \mathcal{D}(B_0^{-2})\), \(\mathcal{D}(A_{0,\varepsilon}) \subset \mathcal{D}(B_0^{-3})\), we have the following relationship moreover

\[
V^* \supseteq H \supseteq H_0 \supseteq \mathcal{D}(B_0^{-2}) \supseteq V.
\]

**Lemma 2.4.** If conditions (H1)-(H4) hold, equation (2.36) has a unique strong solution up to the explosion time \( \tau \).

**Proof.** Let

\[
G_n(t, v) = \begin{cases} 
B(t, x_t)^{-1}v, & \|v\|_{H_0} \leq n, \\
B(t, x_t)^{-1} \frac{nv}{\|v\|_{H_0}}, & \|v\|_{H_0} > n,
\end{cases}
\]

and for simplicity’s sake, we denote

\[
F(t, x_t - v_1) - F(t, x_t - v_2), \ G_n(t, v_1) - G_n(t, v_2), \ B(t, x_t) - B(t, x_t - z_t)
\]

by \( F(t, v_2, v_1), \ G_n(t, v_1, v_2), \ \hat{B}(t, z_t) \) respectively. We consider the following equation firstly,

\[
dz_t = -A_{0,\varepsilon} z_t dt + F(t, z_t, 0)dt - \frac{1}{\xi_t} z_t dt + \frac{1}{\xi_t} \hat{B}(t, z_t)G_n(t, z_t)dt + \hat{B}(t, z_t)dW_t
\]

\[
=: A_{n,\varepsilon}(t, z_t)dt + \hat{B}(t, z_t)dW_t
\]

It’s clearly that the hemicontinuous holds, since \( G_n(t, \cdot) \) remains a Lipschitz mapping from \( H_0 \) to \( H \). By the direct calculus, see Appendix, we get that, for all \( v, v_1, v_2 \in V \),

(A1) **Local monotonicity**

\[
2V^* \langle A_{n,\varepsilon}(t, v_1) - A_{n,\varepsilon}(t, v_2), v_1 - v_2 \rangle_V + \|\hat{B}(t, v_2) - \hat{B}(t, v_1)\|^{2}_{L_{HS}(H,H_0)} \\
\leq \left[ K_{2} + \frac{2n\sqrt{K_{2}} - 2}{\xi_t} + \frac{n^2 K_{1} \|B_0\|^{2}}{\epsilon^2 \xi_t^2 \delta^2} + \frac{2}{\xi_t} (\sqrt{K_2} |v_2|^2_{H_0} + \sqrt{\frac{2K_1}{\epsilon} \|B_0\| \cdot \|v_2\|_{V}^{2}}) \right] \times \\
\times \|v_1 - v_2\|^2_{H_0} - 2(1 - \delta^2)\|v_1 - v_2\|^2_{V}, \ \forall \delta \in (0, 1).
\]

(A2) **Coercivity**

\[
2V^* \langle A_{n,\varepsilon}(t, v), v \rangle_V + \|\hat{B}(t, v)\|^{2}_{L_{HS}(H,H_0)} \\
\leq -2(1 - \delta^2)\|v\|^2_{V} + (\frac{n\sqrt{K_{2}} - 2}{\xi_t} + \frac{n^2 K_{1}}{\epsilon^2 \xi_t^2 \delta^2})\|v\|^2_{H_0}, \ \forall \delta \in (0, 1).
\]
(A3) Growth

\[ ||A_{n,\epsilon}(t, v)||_{V^*}^2 \leq \left( \frac{||B_0||^2}{\epsilon \xi_t} K_2 + \left( 1 + \frac{||B_0||^4 K_1}{\epsilon \xi_t^2} \right) ||v||_{V^*}^2 \right) (1 + ||v||_{H_0}^2). \]

Since

(2.41) \[ ||\hat{B}(t, v)||_{L^{HS}}^2 = ||B_0^{-1} \hat{B}(t, v)||_{H^S}^2 \leq K_2 ||v||_{H_0}^2 + \frac{2K_1}{\epsilon} ||B_0||^3 ||v||_V ||v||_{H_0}. \]

does not satisfies the condition (1.2) in [6], but by the basic inequality one can check that the proof in Lemma 2.2 goes on well, see Appendix B. By the estimates above and Theorem 1.1 in [6] for any \( T_0 < T \), equation (2.31) has unique strong solution \((z_t^n)_{t \in [0, T_0]}\), one can extends the solution to the interval \([0, T]\) by the pathwise uniqueness and continuous. Next we shall let \( n \) goes to infinite. Let, \( m > n \),

(2.42) \[ \tau_m^n = \inf\{ t \in [0, T) \mid ||z_t^n||_{H_0} > n \}, \]
definite \( \inf \emptyset = T \), then

\begin{equation}
(2.43)
\begin{aligned}
z_t^m &= z_0 + \int_0^t (-A_{0,\epsilon}z_s^m + F(s, z_s^m, 0) - \frac{1}{\xi_s} z_s^m) ds \\
&\quad - \int_0^t \frac{1}{\xi_s} \hat{B}(s, z_s^m)B(s, x_s)^{-1} z_s^m ds + \int_0^t \hat{B}(s, z_s^m) dW_s, \quad t < \tau_m^n,
\end{aligned}
\end{equation}

by Itô’s formula and (A1), for \( t < \tau^n_n \land \tau^n_m \), we have

\[ d||z_t^n - z_t^m||_{H_0}^2 \leq 2\hat{B}(t, z_t^n) - \hat{B}(t, z_t^m))dW_t, \quad z_t^n - z_t^m \in H_0 \]

\[ = 2V_s(A_{n,\epsilon}(t, z_t^n) - A_{n,\epsilon}(t, z_t^m), z_t^n - z_t^m)_V + ||\hat{B}(t, z_t^n) - \hat{B}(t, z_t^m)||_{L^{HS}(H, H_0)} dt \]

\[ \leq \left( K_2 + \frac{2}{\xi_t} (n \sqrt{K_1} + \sqrt{K_2} ||z_t^n||_{H_0}^2 + \sqrt{\frac{2K_1}{\epsilon}} ||B_0||_V ||z_t^n||_{V^*}^2 + \frac{n^2 K_1}{\epsilon} ||B_0||^2 + \frac{n^2 K_1}{\epsilon^2 \xi_t^2 \delta^2} ||B_0||^2) \right) ||z_t^n - z_t^m||_{H_0}^2 \]

define

(2.44) \[ \Psi_s = K_2 + \frac{2}{\xi_s} (\sqrt{K_2} ||z_s^n||_{H_0}^2 + n \sqrt{K_1} + \sqrt{\frac{2K_1}{\epsilon}} ||B_0||_V ||z_s^n||_{V^*}^2 + \frac{n^2 K_1}{\epsilon^2 \xi_s^2 \delta^2} ||B_0||^2), \]

then

\begin{equation}
(2.45)
\begin{aligned}
&\exp \left[ - \int_0^t \Psi_s ds \right] ||z_t^n - z_t^m||_{H_0}^2 \\
&\leq \int_0^t 2 \exp \left[ - \int_0^r \Psi_s ds \right] ((\hat{B}(r, z_r^n) - \hat{B}(r, z_r^m))dW_r, z_r^n - z_r^m)_{H_0},
\end{aligned}
\end{equation}

therefore

(2.46) \[ \mathbb{E} \left\{ \exp \left[ - \int_0^{t \land \tau^n_n \land \tau^n_m} \Psi_s ds \right] ||z_t^{n \land \tau^n_n \land \tau^n_m} - z_t^{m \land \tau^n_n \land \tau^n_m}||_{H_0}^2 \right\} = 0. \]
Note that
\[(2.47) \quad \mathbb{E} \int_0^t ||z^n||_V^2 ds < \infty, \quad \forall t < T\]
implies
\[(2.48) \quad \int_0^t ||z^n||_V^2 ds < \infty, \quad \forall t \in [0, T), \quad \mathbb{P}\text{-a.s.},\]
then
\[(2.49) \quad z_{\tau_n \wedge \tau^n_m} = z^{\tau} = z_{\tau_n \wedge \tau_n}^m, \quad \forall t \in [0, T), \quad \mathbb{P}\text{-a.s.},\]
let \(t \uparrow T\), by the continuity, we have
\[(2.50) \quad z_{\tau_n \wedge \tau^n_m} = z_{\tau_n \wedge \tau_n}^m, \quad \mathbb{P}\text{-a.s.}\]
If \(\tau^n < \tau^n_n, z_{\tau_n} = z_{\tau^n_n} \in \partial B^H_{\mathbb{P}}(0)\), by the definition of \(\tau^n_n\), it’s a contradictory. Thus \(\tau^n > \tau^n_n\), similarly, \(\tau_n \leq \tau^n_n\), so \(\tau^n = \tau^n_n, \mathbb{P}\text{-a.s.}\) and \(z_{\tau^n_n} = z_{\tau^n_n}^m\). Therefore, we can definite
\[(2.51) \quad z_t = z^n_n, \quad t \leq \tau^n_n, \quad \tau = \sup_{n} \tau^n_n,\]
\((z, \tau)\) is a strong solution of equation (2.30). By the same method, we can prove the uniqueness easily.

**Proof of Theorem 1.2** Let
\[d\tilde{W}_s = dW_s + \frac{1}{\xi_s} B(s, x_s)^{-1} z_s ds, \quad s \leq T \wedge \tau\]
\[R_s = \exp \left[ - \int_0^s \xi_t^{-1} B(t, x_t)^{-1} z_t dW_t - \frac{1}{2} \int_0^s \frac{||B(t, x_t)^{-1} z_t||^2}{\xi_t} dt \right], \quad s \leq T \wedge \tau,\]
\[\tau_n = \inf \{ t \in [0, T) \mid ||z_t||_H_0 > n \}, \quad Q := R_{T \wedge \tau} \mathbb{P},\]
write the equation of \(z\) in the form of \(\tilde{W}\):
\[(2.52) \quad dz_t = -A_{0,\varepsilon} z_t dt + F(t, z_t, 0) dt + \hat{B}(t, z_t) d\tilde{W}_t - \frac{1}{\xi_t} z_t dt,\]
By Itô’s formula and (H4), for \(s \in [0, T)\), and for \(t < \tau_n \wedge s\),
\[(2.53) \quad d||z_t||^2_{H_0} = -2 ||z_t||^2_{H_0} dt + 2 \langle F(t, z_t, 0), z_t \rangle_{V} dt - \frac{2||z_t||^2_{H_0}}{\xi_t} dt\]
\[+ ||\hat{B}(t, z_t)||^2_{L_{HS}(H, H_0)} dt + 2 ||\hat{B}(t, z_t) d\tilde{W}_t, z_t||_{H_0}\]
\[\leq 2 \langle F(t, z_t, 0), z_t \rangle_{V} dt + ||\hat{B}(t, z_t)||^2_{L_{HS}(H, H_0)} dt\]
\[= - \frac{2||z_t||^2_{H_0}}{\xi_t} dt + 2 ||\hat{B}(t, z_t) d\tilde{W}_t, z_t||_{H_0} + K_2 ||z_t||^2_{H_0} dt,\]
and
\[
\frac{d||z_t||^2_{H_0}}{\xi_t} \leq -\frac{2||z_t||^2_{H_0}}{\xi_t^2} \, dt + \frac{K_2}{\xi_t}||z_t||^2_{H_0} \, dt - \frac{\xi'_t}{\xi_t^2}||z_t||^2_{H_0} \, dt + \frac{2}{\xi_t} \langle \hat{B}(t, z_t) d\hat{W}, z_t \rangle_{H_0}
\]
(2.54)
\[
= 2 - K_2 \xi_t + \frac{\xi'_t}{\xi_t^2}||z_t||^2_{H_0} \, dt + \frac{2}{\xi_t} \langle \hat{B}(t, z_t) d\hat{W}, z_t \rangle_{H_0}
\]
\[
= \frac{\theta}{\xi_t}||z_t||^2_{H_0} \, dt + \frac{2}{\xi_t} \langle \hat{B}(t, z_t) d\hat{W}, z_t \rangle_{H_0},
\]
by Girsanov theorem, \((\hat{W})_{t<\tau_n}\) is a Wiener process under the probability \(Q_{s,n} := R_{s\wedge \tau_n} P\), and
\[
\int_0^{s \wedge \tau_n} ||z_t||^2_{H_0} \, dt \leq \frac{||z_0||^2_{H_0}}{\theta \xi_0} + \int_0^{s \wedge \tau_n} \frac{2}{\theta \xi_t} \langle \hat{B}(t, z_t) d\hat{W}, z_t \rangle_{H_0},
\]
(2.55)
then
\[
\mathbb{E}_{Q_{s,n}} \int_0^{s \wedge \tau_n} ||z_t||^2_{H_0} \, dt \leq \frac{||z_0||^2_{H_0}}{\theta \xi_0},
\]
(2.56)
Since, by \((H3)\)
\[
\log R_u = - \int_0^u \xi^{-1}_t \langle B(t, x_t)^{-1} z_t, d\hat{W}_t \rangle + \frac{1}{2} \int_0^u \frac{||B(t, x_t)^{-1} z_t||^2}{\xi_t} \, dt
\]
(2.57)
\[
\leq - \int_0^u \xi^{-1}_t \langle B(t, x_t)^{-1} z_t, d\hat{W}_t \rangle + \frac{1}{2 \rho(T)^2} \int_0^u \frac{||z_t||^2_{H_0}}{\xi_t} \, dt, u \leq s \wedge \tau_n,
\]
(2.58)
As in \([17]\), we can prove that \(\{R_{s \wedge \tau} \mid s \in [0, T]\}\) is a martingale. Since
\[
\mathbb{E}_{Q} 1_{[\tau_n \leq t]} \frac{||z_{t \wedge \tau_n}||^2_{H_0}}{\xi_{t \wedge \tau_n}} \leq \mathbb{E}_{Q} \frac{||z_{t \wedge \tau_n}||^2_{H_0}}{\xi_{t \wedge \tau_n}} \leq \frac{||z_0||^2_{H_0}}{\xi_0},
\]
(2.59)
and
\[
\mathbb{E}_{Q} 1_{[\tau_n \leq t]} \frac{||z_{t \wedge \tau_n}||^2_{H_0}}{\xi_{t \wedge \tau_n}} \geq \frac{n Q(\tau_n \leq t)}{\xi_0}
\]
(2.60)
let \(n\) goes to infinite, we have \(Q(\tau_n \leq t) = 0, \forall t \in [0, T]\), then \(Q(\tau = T) = 1\). Now, since \(\tau = T\), \(Q\)-a.s., equation \(2.52\) can be solved up to time \(T\). Let
\[
\zeta = \inf \{t \in [0, T] \mid ||z_t||_{H_0} = 0\},
\]
we shall prove that \(\zeta \leq T\), here we assume \(\inf \theta = +\infty\). Otherwise, there exists a set \(\Omega_0\), such that \(P(\Omega_0) > 0\), and for any \(\omega \in \Omega_0\), \(\zeta(\omega) > T\), then by the continuity of path, we have
\[
\inf_{t \in [0, T]} ||z_t(\omega)||_{H_0} > 0,
\]
(2.62)
so
\begin{equation}
(2.63) \quad \int_0^T \frac{||z_t||^2_{\mathcal{H}_0}}{\xi_t^2} dt = +\infty,
\end{equation}
but
\begin{equation}
(2.64) \quad \mathbb{E}_Q \int_0^T \frac{||z_t||^2_{\mathcal{H}_0}}{\xi_t^2} dt \leq \frac{||z_0||^2_{\mathcal{H}_0}}{2\rho(T)^2\theta \xi_0} < +\infty,
\end{equation}
hence, \( \zeta \leq T, \mathbb{Q}\)-a.s., by the uniqueness of solution of equation (2.52), we have
\begin{equation}
(2.65) \quad z_t \equiv 0, \ t > \zeta, \mathbb{Q}\)-a.s.
\end{equation}
Thus, \( z_T = 0, \mathbb{Q}\)-a.s.

Next, we shall construct the coupling. Since under the probability space \((\Omega, \mathcal{F}, R_{T\wedge T}\mathbb{P}), (\tilde{W}_t)_{t\in[0,T]}\) is a Wiener process, let \( y \) be the unique mild solution of the following equation
\begin{equation}
(2.66) \quad dy_t = -A_y y_t dt + F(t, y_t) dt + B(t, y_t) d\tilde{W}_t, \ y_0 = y,
\end{equation}
for \( x_t \), it’s the unique solution of the following equation
\begin{equation}
(2.67) \quad dx_t = -A_x x_t dt + F(t, x_t) dt - \frac{z_t}{\xi_t} dt + B(t, x_t) d\tilde{W}_t, \ x_0 = x.
\end{equation}
For the process \( x_t - y_t \), it’s the mild solution of the following equation
\begin{equation}
(2.68) \quad du_t = -A_u u_t dt + F(t, u_t, 0) dt + \hat{B}(t, u_t) d\tilde{W}_t - \frac{z_t}{\xi_t} dt,
\end{equation}
note that \( z_t \) is a solution of equation
\begin{equation}
(2.69) \quad dz_t = -A_0 z_t dt + F(t, z_t, 0) dt + \hat{B}(t, z_t) d\tilde{W}_t - \frac{z_t}{\xi_t} dt.
\end{equation}
Similar to equation (1.41), one can prove that equation (2.68) has a strong solution in \( \mathcal{H}_0 \), since \( \mathcal{V}^* \supset H \supset \mathcal{H}_0 \) and \( A_0 \) is the restriction of \( A_x \) to \( \mathcal{H}_0 \), by the relation of variational solution and mild solution and the pathwise uniqueness, then \( z_t = x_t - y_t, \ \forall t \in [0, T], \mathbb{Q}\)-a.s.

By the method used in [17], we have log-Harnack inequality for equation (2.8):
\begin{equation}
(2.70) \quad P_T \log f(y) = \mathbb{E}_Q \log f(y_T) = \mathbb{E} R_{T\wedge T} \log f(x_T^*) \leq \mathbb{E} R_{T\wedge T} \log R_{T\wedge T} + \log \mathbb{E} f(x_T^*)
\leq \log P_T f(x) + \frac{||x - y||_{\mathcal{H}_0}}{2\rho(T)^2\theta \xi_0} = \log P_T f(x) + \frac{K_2 ||x - y||_{\mathcal{H}_0}}{2\rho(T)^2\theta(2 - \theta)(1 - e^{K_2 T})},
\end{equation}
then by lemma 1.2, let \( \epsilon \to 0 \), and choose \( \theta = 1 \), for \( f \in \mathcal{B}_b^+(H) \) and \( f \geq 1 \),
\begin{equation}
(2.71) \quad P_T \log f(y) \leq \log P_T f(x) + \frac{K_2 ||x - y||_{\mathcal{H}_0}}{2\rho(T)^2(1 - e^{K_2 T})}.
\end{equation}
If (H1) holds in addition, by inequality (2.55), we have
\begin{equation}
(2.72) \quad \mathbb{E}_{s,n} \exp \left( \frac{h}{\theta} \int_0^{\tau_n} \frac{||z_t||^2_{\mathcal{H}_0}}{\xi_t^2} dt \right)
\leq \exp \left( \frac{h||x - y||^2_{\mathcal{H}_0}}{\theta \xi_0} \right) \mathbb{E}_{s,n} \exp \left( \frac{2h}{\theta} \int_0^{\tau_n} \frac{1}{\xi_t} \langle \hat{B}(t, z_t) d\tilde{W}, z_t \rangle \right)
\leq \exp \left( \frac{h||x - y||^2_{\mathcal{H}_0}}{\theta \xi_0} \right) \mathbb{E}_{s,n} \left( \exp \left( \frac{8h^2 K_2^2}{\theta^2} \int_0^{\tau_n} \frac{||z_t||^2_{\mathcal{H}_0}}{\xi_t^2} dt \right) \right)^{\frac{1}{2}},
\end{equation}
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for \( h = \frac{\vartheta^2}{8K_3^2} \), and

\[
(2.73) \quad \mathbb{E}_{n,\epsilon} \exp \left[ \frac{\theta^2}{8K_3^2} \int_0^{\tau(n)} \left\| z_t \right\|^2_{H_0} \, dt \right] \leq \exp \left[ \frac{\theta K_2 \| x - y \|^2_{H_0}}{4K_3^2(2 - \theta)(1 - e^{-K_2T})} \right],
\]

Similar to [17], we get that

\[
(2.74) \quad \sup_{s \in [0,T]} \mathbb{E} R_{s,T}^{1+r} \leq \exp \left[ \frac{\theta K_2(2K_3 + \theta \rho(T)) \| x - y \|^2_{H_0}}{8K_3^2(2 - \theta)(K_3 + \theta \rho(T))(1 - e^{-K_2T})} \right]
\]

and for \( p > (1 + K_3)^2 \), \( \delta_{p,T} = K_3 \vee \frac{\rho(T)}{2} (\sqrt{p} - 1) \), \( f \in \mathcal{D}_b^+(H) \), choose \( \theta = \frac{2K_3\rho(T)}{\sqrt{p} - 1} \),

\[
(2.75) \quad (P_T^t f(y))^p \leq (P_T^t f^p(x))^p \exp \left[ \frac{K_2(T) \sqrt{p}(\sqrt{p} - 1) \| x - y \|^2_{H_0}}{4\delta_{p,T}(\sqrt{p} - 1)\rho(T) - \delta_{p,T}(1 - e^{-K_2T})} \right],
\]

by lemma 1.2, let \( \epsilon \downarrow 0 \), we have

\[
(2.76) \quad (P_T f(y))^p \leq (P_T^t f^p(x))^p \exp \left[ \frac{K_2(T) \sqrt{p}(\sqrt{p} - 1) \| x - y \|^2_{H_0}}{4\delta_{p,T}(\sqrt{p} - 1)\rho(T) + \delta_{p,T}(1 - e^{-K_2T})} \right],
\]

for \( x, y \in H, x - y \in \mathcal{D}(B_{0}^{-1}) \).

\[\square\]

3 Application

In this section, we give some simple applications of Theorem 1.2.

**Corollary 3.1.** Assume that \( F, B \) are determined and independent of \( t \) and \((H1) \) to \((H5) \) hold. If \( \lambda_0 > 0, \lambda_0 > K_1^2 + 2K_1 \) and \( B(0) \in L_{HS}(H) \), then

1. \( P_t \) has uniqueness invariant measure \( \mu \) and has full support on \( H, \mu(V) = 1 \).
2. If \( \sup_x \| B(x) \| < \infty \), then \( \mu(e^{\epsilon_0\| \cdot \|^2_{H_0}}) < \infty \) for some \( \epsilon_0 > 0 \).
3. If there exists \( q > 0 \) such that \( \inf_n b_n^2 \lambda_n^{-1} > 0 \), then \( \mu \) has full support on \( H_0 \).

**Proof.** Let \((V, \| \cdot \|_V) = (\mathcal{D}(A_{\frac{1}{2}}^\frac{1}{2}), \| A_{\frac{1}{2}}^{\frac{1}{2}} \cdot \|). \) Since \( \lambda_0 > 0 \) and \( B(0) \in L_{HS}(H) \), by \((H1) \), equation \((1.1) \) has strong solution and \( P_t \) is Feller semigroup. By Ito’s formula and \( \lambda_0 > K^2 - 2K_1 \), there exists a constant \( c > 0 \) such that

\[
d\| x_t \|^2 \leq \left( c - 2\left(1 - \frac{K_1^2 + 2K_1}{\lambda_0}\right)\right) \| x_t \|^2_{V} + 2\| F(0) \| \cdot \| x_t \| \right) \, dt + 2\langle B(x_t) dW_t, x_t \rangle
\]

and

\[
d e^{\epsilon \| x_t \|^2} \leq \epsilon e^{\epsilon \| x_t \|^2} \left( c - 2\left(1 - \frac{K_1^2 + 2K_1}{\lambda_0}\right)\right) \| x_t \|^2_{V} + \frac{\epsilon^2}{4} \| B^*(x_t) x_t \|^2 + 2\| F(0) \| \cdot \| x_t \| \right) \, dt
\]

+ \epsilon e^{\epsilon \| x_t \|^2} \langle B(x_t) dW_t, x_t \rangle,
Corollary 3.2. Assume (H1) to (H5) hold, for any small $\epsilon > 0$.

\[
    \text{Proof. It follows the proof of [16, 14, 19].}
\]

(3.2) $d||x_t(x) - x||^2 \leq -||x_t(x) - x||^2 dt + (c_1 + c_2||x_t(x)||^2)dt + 2(B(x_t)dW_t, x_t - x)$

here we denote $x_t(x)$ for the process starts from $x$, $c_1, c_2$ are constants depend on $x$. Using Harnack inequality (1.8), (3) can be proved following the line of [19].

\[ \square \]

Corollary 3.2. Assume (H1) to (H5) hold, $F$ and $B$ are determined and time independent, then for any $t > 0$, $P_t$ is $H_0$-strong Feller. Let $\mu$ be the $P_t$-subinvariant probability with full support on $H_0$ as in [14], then the transition density $p_t(x, y)$ w.r.t. $\mu$ satisfies

\[
    \|p_t(x, \cdot)\|_{L^p(\mu)} \leq \left\{ \int_{H_0} \exp \left[ -\frac{K_2 \sqrt{q}(\sqrt{q} - 1)||x - y||_{H_0}^2}{4\delta_q[(\sqrt{q} - 1)\rho - \delta_q](1 - e^{K_2 t})} \right] \mu(dy) \right\}^{-\frac{1}{q}}
\]

for all $1 < p < \frac{(K_2 + \rho)^2}{(K_2 + \rho)^2 - 1}$, here $q = \frac{p}{p - 1}$.

\[ \text{Proof. It follows the proof of [16, 14, 19].} \]

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Appendix

A. Proof of Remark 1.1

Proof of (1): since $\bigcup_n H_n$ is a core of $B_0^{-2}$, for any $x \in D(B_0^{-2})$, choose $\{x_n\}$ such that $x_n \to x$ and $B_0^{-2}x_n \to B_0^{-2}x$, hence $B_0^{-1}x_n \to x$, as $n \to +\infty$. Similarly, a sequence $\{y_n\}$ with the same property. Therefore

\[
    \begin{align*}
        &\|B_0^{-1}[B(t, x_n) - B(t, y_n)] - (B(t, x_m) - B(t, y_m))\|_{HS}^2 \\
        \leq &2K_2(||B_0^{-1}(x_n - x_m)||^2 + ||B_0^{-1}(y_n - y_m)||^2) - 4\langle F(t, x_n) - F(t, x_m), B_0^{-2}(x_n - x_m) \rangle \\
        &- 4\langle F(t, y_n) - F(t, y_m), B_0^{-2}(y_n - y_m) \rangle,
    \end{align*}
\]

by the continuous of $F$, we have that $\{B(t, x_n) - B(t, y_n)\}$ forms a Cauchy sequence in $L_{HS}(H, H_0)$. Note that $B(t, x_n) - B(t, y_n)$ convergent to $B(t, x) - B(t, y)$ in $L_{HS}(H)$, and $B_0^{-1}$ is closed, we have $B(t, x) - B(t, y) \in L_{HS}(H, H_0)$,

\[
    \lim_{n \to +\infty} (B(t, x_n) - B(t, y_n)) = B(t, x) - B(t, y),
\]

and

\[
    2\langle F(t, x) - F(t, y), B_0^{-2}(x - y) \rangle + ||B_0^{-1}(B(t, x) - B(t, y))||_{HS}^2 \leq K_2||B_0^{-1}(x - y)||^2.
\]
Proof of (2): we assume \( \rho(t) = 1 \), by definition, it’s clear that \( B_0 \) is one to one and has dense range.

\[
B(t,x)B(t,x)^* \geq B_0^2 \iff ||B(t,x)^*y|| \geq ||B_0y||, \forall y \in H,
\]
implies that \( \text{Ran}B(t,x) \supset \text{Ran}B_0 \) by Proposition B.1 in [4], and

\[
||z|| \geq ||B_0(B(t,x)^*)^{-1}z||, \forall z \in \text{Ran}(B(t,x)^*).
\]

Since for any \( z \in \text{Ran}(B(t,x)^*) \), \( y \in \text{Ran}(B(t,x)) \), we have

\[
(B(t,x)^{-1}y, z) = (B(t,x)B(t,x)^{-1}y, (B(t,x)^*)^{-1}z) = (y, (B(t,x)^*)^{-1}z),
\]
then

\[
z \in \mathcal{D}((B(t,x)^{-1})^*), (B(t,x)^{-1})^*z = (B(t,x)^*)^{-1}z.
\]

On the other hand, for any \( z \in \mathcal{D}((B(t,x)^{-1})^*) \), there exists \( z^* \) such that

\[
(B(t,x)^{-1}y, z) = (y, z^*), \forall y \in \mathcal{D}((B(t,x)^{-1}))
\]
let \( u = B(t,x)^{-1}y \), then \( (u, z) = (B(t,x)u, z^*) \), we have \( z = B(t,x)^z^* \) and

\[
(B(t,x)^*)^{-1}z = z^* = (B(t,x)^{-1})^*z,
\]
hence \( \mathcal{D}((B(t,x)^{-1})^*) = \mathcal{D}((B(t,x)^*)^{-1}) \). Therefore, \( ||z|| \geq ||B_0(B(t,x)^{-1})^*z|| \), for all \( z \in \text{Ran}B(t,x)^* \). Since \( \text{Ran}(B(t,x)^*) \) is dense in \( H \), \( B_0(B(t,x)^{-1})^* \) can be extended to be a bounded operator on \( H \), and for all \( z \in H, y \in H \), there is \( \{z_n\}_{n=1}^{+\infty} \), \( \lim_n z_n = z \), such that \( \lim_n B_0(B(t,x)^{-1})^*z_n = B_0(B(t,x)^{-1})^*z \), then

\[
(B_0(B(t,x)^{-1})^*y, z) = \lim_n (B_0(B(t,x)^{-1})^*z_n, y)
= \lim_n (z_n, (B(t,x)^{-1})B_0y) = (z, (B(t,x)^{-1})B_0y),
\]
hence \( ||(B(t,x)^{-1})B_0y|| \leq ||y|| \), for all \( y \in H \), let \( z = B_0y \), then \( ||(B(t,x)^{-1})z|| \leq ||B_0^{-1}z|| \), for all \( z \in \mathcal{D}(B_0^{-1}) \). By Proposition B.1 in [4], and the proof above, the converse is easy.


B. For Lemma 2.4

(1) For local monotonicity. For any \( v_1, v_2 \in V \),

\[
-2V \cdot \langle A_0, e(v_1 - v_2), v_2 \rangle_V = -2\sqrt{\langle A_0, e(v_1 - v_2) \rangle_H^2} = -2\|v_1 - v_2\|^2_V,
\]

\[
2V \cdot \langle F(t, v_1, v_2), v_1 - v_2 \rangle_V + || \hat{B}(t, v_1) - \hat{B}(t, v_2) ||^2_{L_{HS(H, H_0)}}
\]
\[
= 2 \langle F(t, v_1, v_2), B^{-1}_0(v_1 - v_2) \rangle + || B^{-1}_0(\hat{B}(t, v_1) - \hat{B}(t, v_2)) ||^2_{HS}
\]
\[
\leq K_2 \|v_1 - v_2\|^2_{H_0}
\]
and

\[
\frac{1}{\xi_t} \langle \hat{B}(t, v_1)G_n(t, v_1) - \hat{B}(t, v_2)G_n(t, v_2), v_1 - v_2 \rangle_V \\
= \frac{1}{\xi_t} \langle (\hat{B}(t, v_1) - \hat{B}(t, v_2))G_n(t, v_1) - \hat{B}(t, v_2)G_n(t, v_1, v_2), B_0^{-2}(v_1 - v_2) \rangle \\
\leq \frac{1}{\xi_t} \|B_0^{-1}(\hat{B}(t, v_1) - \hat{B}(t, v_2))\| \cdot \|G_n(t, v_1)\| \cdot \|v_1 - v_2\|_{H_0} \\
+ \frac{1}{\xi_t} \|B_0^{-1}\hat{B}(t, v_2)\| \cdot \|G_n(t, v_1, v_2)\| \cdot \|v_1 - v_2\|_{H_0},
\]

(3.10)

note that, by (H1),

\[
\|B_0^{-1}(\hat{B}(t, v_1) - \hat{B}(t, v_2))\|_{H_0}^2 \leq K_2 \|v_1 - v_2\|_{H_0}^2 - 2\langle F(t, v_1, v_2), B_0^{-2}(v_1 - v_2) \rangle \\
\leq K_2 \|v_1 - v_2\|_{H_0}^2 + 2K_1 \|B_0^{-1}A_0,eA_0,e(v_1 - v_2)\|_{H_0} \cdot \|B_0^{-1}A_0,eA_0,e(v_1 - v_2)\|_{H_0} \\
\leq K_2 \|v_1 - v_2\|_{H_0}^2 + 2K_1 \left( \sup_n \frac{b_n}{\lambda_n + \epsilon b_n^2} \right) \left( \sup_n \frac{1}{b_n \lambda_n + \epsilon b_n^2} \right) \|v_1 - v_2\|_V^2 \\
\leq K_2 \|v_1 - v_2\|_{H_0}^2 + \frac{2}{\epsilon} K_1 \|B_0\| \|v_1 - v_2\|_V^2,
\]

hence

\[
\frac{1}{\xi_t} \|B_0^{-1}(\hat{B}(t, v_1) - \hat{B}(t, v_2))\| \cdot \|G_n(t, v_1)\| \cdot \|v_1 - v_2\|_{H_0} \\
\leq \frac{n}{\xi_t} \left( \sqrt{K_2} \|v_1 - v_2\|_{H_0} + \sqrt{\frac{2}{\epsilon} K_1 \|B_0\| \cdot \|v_1 - v_2\|_V} \right) \|v_1 - v_2\|_{H_0} \\
\leq \left( \frac{n}{\xi_t} \sqrt{K_2} + \frac{n^2 K_1 \|B_0\|^2}{\epsilon \xi_t^2 \delta^2} \right) \|v_1 - v_2\|_{H_0} + \delta^2 \|v_1 - v_2\|_V^2,
\]

and

\[
\frac{1}{\xi_t} \|B_0^{-1}\hat{B}(t, v_2)\| \cdot \|G_n(t, v_1, v_2)\| \cdot \|v_1 - v_2\|_{H_0} \\
\leq \frac{1}{\xi_t} \left( \sqrt{K_2} \|v_2\|_{H_0} + \sqrt{\frac{2 K_1}{\epsilon} \|B_0\| \cdot \|v_2\|_V} \right) \|v_1 - v_2\|_{H_0}^2,
\]

(3.11)

hence, we have

\[
\frac{1}{\xi_t} \langle A_n,e(t, v_1) - A_n,e(t, v_2), v_1 - v_2 \rangle_V + \|\hat{B}(t, x_t - v_2) - \hat{B}(t, x_t - v_1)\|_{L_{H_0}(H, H_0)}^2 \\
\leq \left[ K_2 + \frac{2n \sqrt{K_2} - 2}{\xi_t} + \frac{n^2 K_1 \|B_0\|^2}{\epsilon^2 \xi_t^2 \delta^2} \right] \left( \sqrt{K_2} \|v_2\|_{H_0}^2 + \sqrt{\frac{2 K_1}{\epsilon} \|B_0\| \cdot \|v_2\|_V^2} \right) \times \\
\times \|v_1 - v_2\|_{H_0}^2 - 2(1 - \delta^2) \|v_1 - v_2\|_V^2.
\]

(2) For coercivity:

\[
-2V^* \langle A_0,e, v, v \rangle_V = -2\|v\|_V^2, \quad \|B_0^{-1}\hat{B}(t, v)\|_{H_0}^2 + 2\langle F(t, v, 0), B_0^{-2}v \rangle \leq K_2 \|v\|_V^2,
\]

(3.13)
\[
\frac{2}{\xi_t} V^* \langle \hat{B}(t, v) G_n(t, v), v \rangle_V \leq \frac{2}{\xi_t} \| B_0^{-1} \hat{B}(t, v) \| \cdot \| G_n(t, v) \| \cdot \| v \|_{H_0}
\]
\[
\leq \frac{2n}{\xi_t} (K_2 \| v \|^2 + \frac{2K_1}{\epsilon} \| B_0 \|_2^2 \| v \|_{V^2}^2) \| v \|_{H_0}
\]
\[
\leq \frac{2n}{\xi_t} (\sqrt{K_2} \| v \|_{H_0} + \sqrt{\frac{2K_1}{\epsilon}} \| B_0 \| \cdot \| v \|_{V^2}) \| v \|_{H_0}
\]
\[
\leq \left( \frac{2n \sqrt{K_2}}{\xi_t} + \frac{2n^2 K_1}{\epsilon \xi_t^2 \delta^2} \right) \| v \|^2_{H_0} + \delta^2 \| v \|^2_{V^2},
\]

hence
\[
2 V^* \langle A_n(t, v), v \rangle_V + \| \hat{B}(t, v) \|^2_{L^2(H, H_0)}
\leq - 2(1 - \delta^2) \| v \|_{V^2}^2 + \left( \frac{n^2 K_2}{\xi_t} - \frac{2}{\epsilon \xi_t^2 \delta^2} \right) \| v \|^2_{H_0}.
\]

(3) For Growth:
\[
\| A_{0, v} \|^2_{V^2} = \| v \|^2_{V^2}, \quad \| \frac{1}{\xi_t} v \|_{V^*} = \frac{1}{\xi_t} \| v \|_{V^*}, \quad \| F(t, v, 0) \|_{V^*} \leq \frac{K_1}{\sqrt{\epsilon}} \| v \|,
\]
since, by (H1),
\[
| V^* \langle F(t, v, 0), z \rangle_V | = | \langle F(t, v, 0), B_0^{-2} z \rangle | \leq K_1 \| v \| \cdot \| B_0^{-2} z \| \leq \frac{K_1}{\sqrt{\epsilon}} \| v \| \cdot \| z \|_{V}.
\]
And
\[
\| \frac{1}{\xi_t} \hat{B}(t, v) G_n(t, v) \|_{V^*} \leq \frac{\| B_0 \|}{\sqrt{\epsilon \xi_t}} \| \hat{B}(t, v) G_n(t, v) \|_{H_0}
\]
\[
\leq \frac{\| B_0 \|}{\sqrt{\epsilon \xi_t}} \| B_0^{-1} \hat{B}(t, v) \| \cdot \| G_n(t, v) \|_{L^2(H_0, H)}
\]
\[
\leq \left( \frac{\| B_0 \|}{\sqrt{\epsilon \xi_t}} \right) \| v \|_{H_0} \sqrt{K_2} + \frac{2K_1}{\epsilon} \| v \|_{V^2} \| B_0 \|_{H_0} \| v \|_{H_0},
\]

we have
\[
\| A_{n, v}(t, v) \|^2_{V^*} \leq \left( \frac{\| B_0 \| K_2}{\epsilon \xi_t} + \left( 1 + \frac{\| B_0 \|^4 K_1}{\epsilon \xi_t^2} \right) \| v \|^3_{V^2} \right) \left( 1 + \| v \|^4_{H_0} \right).
\]

(4) For the Lemma 2.2 of [6]: We give new estimates to replace inequalities (2.3) and (2.4) there. For convenience, we use the notations there. In (2.3), we only have to replace \( f_s \cdot \| X_s^{(n)} \|_{H}^{p-2} \) by \( \| X_s^{(n)} \|_{V} \cdot \| X_s^{(n)} \|_{H} \cdot \| X_s^{(n)} \|_{H}^{p-2} \) and use the basic inequality
\[
a \cdot b \leq \frac{a^2}{2\delta} + \frac{\delta b^2}{2}, \quad \forall \delta > 0,
\]
\[
2 V^* \langle A_n(t, v), v \rangle_V + \| \hat{B}(t, v) \|^2_{L^2(H, H_0)}
\leq - 2(1 - \delta^2) \| v \|_{V^2}^2 + \left( \frac{n^2 K_2}{\xi_t} - \frac{2}{\epsilon \xi_t^2 \delta^2} \right) \| v \|^2_{H_0}.
\]
and note that in our case $\alpha = 2$. For (2.4), one can use the following estimate,

$$
\mathbb{E} \left( \int_0^{\tau_n^R} \left( \|X_s^{(n)}\|_{H}^{2p-2} \|B(s, X_s^{(n)})\|_2^2 \right) \frac{1}{2} \right) \\
\leq \mathbb{E} \left( \int_0^{\tau_n^R} C \|X_s^{(n)}\|_{H}^{2p-2} (\|X_s^{(n)}\|_V \|X_s^{(n)}\|_H + \|X_s^{(n)}\|_H^2) \right) \frac{1}{2} \\
\leq C(\delta_1) \mathbb{E} \left( \int_0^{\tau_n^R} \|X_s^{(n)}\|_{H}^{2p-2} \|X_s^{(n)}\|_H^2 \right) \frac{1}{2} + \sqrt{\delta_1} \mathbb{E} \left( \int_0^{\tau_n^R} \|X_s^{(n)}\|_{H}^{2p-2} \|X_s^{(n)}\|_V^2 \right) \frac{1}{2} \\
\leq \delta_2 \mathbb{E} \sup_{s \in [0, \tau_n^R]} \|X_s^{(n)}\|_H^p + C(\delta_1, \delta_2) \mathbb{E} \int_0^{\tau_n^R} \|X_s^{(n)}\|_H^p ds \\
+ \sqrt{\delta_1} \mathbb{E} \sup_{s \in [0, \tau_n^R]} \|X_s^{(n)}\|_H^p \left( \int_0^{\tau_n^R} \|X_s^{(n)}\|_V^{p-2} \|X_s^{(n)}\|_V^2 \right) \frac{1}{2} \\
\leq (\delta_2 + \delta_3) \mathbb{E} \sup_{s \in [0, \tau_n^R]} \|X_s^{(n)}\|_H^p + \frac{\delta_1}{4 \delta_3} \mathbb{E} \int_0^{\tau_n^R} \|X_s^{(n)}\|_H^{p-2} \|X_s^{(n)}\|_V^2 ds + C(\delta_1, \delta_2) \mathbb{E} \int_0^{\tau_n^R} \|X_s^{(n)}\|_H^p ds,
$$

choose $\delta_2, \delta_3$ small enough and $\delta_1$ such that $\frac{\delta_1}{4 \delta_3}$ small enough, using $\alpha = 2$ again, then Gronwall’s lemma can be applied as in [6].

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