Entropy of Three–Dimensional
Asymptotically Flat Cosmological Solutions

Glenn Barnich

Vienna, Preprint ESI 2382 (2012) August 24, 2012

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at http://www.esi.ac.at
Entropy of three-dimensional asymptotically flat cosmological solutions

Glenn Barnich

Physique Théorique et Mathématique
Université Libre de Bruxelles
and
International Solvay Institutes
Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium

Abstract. The thermodynamics of three-dimensional asymptotically flat cosmological solutions that play the same role than the BTZ black holes in the anti-de Sitter case is derived and explained from holographic properties of flat space. It is shown to coincide with the flat-space limit of the thermodynamics of the inner black hole horizon on the one hand and the semi-classical approximation to the gravitational partition function associated to the entropy of the outer horizon on the other. This leads to the insight that it is the Massieu function that is universal in the sense that it can be computed at either horizon.
1 Introduction

In order to test holographic ideas in gravitational theories [1, 2] and go beyond the context of the AdS/CFT correspondence [3], it seems useful to first try to extend the AdS\(_3\) results to the flat case. Indeed, for the former case, there is complete control on symmetries, charges and central extensions [4], on solution space [5, 6], including black holes [7, 8] and a compelling conformal field theory interpretation [9].

Asymptotically flat gravity in three dimensions at null infinity [10] is arguably as simple and interesting a model: again, there is complete control on symmetries [11], charges, central extensions [12], solution space with a conformal field theory interpretation [13], and the precise relation to the AdS\(_3\) case [14]. In particular, the flat-space limit of the BTZ black holes are simple cosmological solutions.

The purpose of this paper is to try push this project to a level of understanding similar to that achieved in the AdS\(_3\) case by deriving the thermodynamics of the Cauchy horizon of the cosmological solutions and providing a holographic derivation of their entropy in the grand canonical ensemble through an appropriate Cardy-like formula.

We then point out that this thermodynamics is the flat-space limit of the one of the inner BTZ horizon recently considered in [15, 16]. The semi-classical approximation to the logarithm of the partition function of the AdS\(_3\) case is a Massieu function for the entropy of the outer horizon and has a good flat-space limit that coincides with the one directly computed in the flat case.

As a by-product and a case in point, it is then readily seen that the Massieu function is universal in the sense that it can be computed from the entropy at either BTZ horizon.

2 Thermodynamics of cosmological solutions

In BMS form, the cosmological solutions are explicitly given by

\[ ds^2 = 8GMdu^2 - 2dudr + 8GJdu\phi + r^2d\phi^2, \]  
(2.1)

with \( M > 0 \), while their ADM form is

\[ ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\phi + N^\phi dt)^2, \]

\[ N^2 = -8MG + \frac{16G^2J^2}{r^2}, \quad N^\phi = \frac{4GJ}{r^2}, \]  
(2.2)

where, again, \( t = u + f(r) \), \( \phi = \phi + g(r) \), with \( f' = N^{-2}, g' = -N^\phi f' \). In the discussions below, we explicitly assume \( J \neq 0 \) most of the time. In ADM coordinates, it follows that the null hypersurface

\[ r_C = \sqrt{\frac{2GJ^2}{M}}, \]  
(2.3)

is special, but in BMS coordinates, which now play the role of outgoing Eddington-Finkelstein coordinates, this hypersurface is regular.

As discussed in [14], let \( \alpha = \sqrt{8GM} \) and consider the coordinate changes

\[
X = \phi - \Omega_c t, \quad \text{so that} \quad (\phi, X) \sim (\phi + 2\pi, X + 2\pi), \quad T^2 = \frac{1}{\alpha^2} (r^2 - r_c^2) \quad \text{for} \quad r > r_c, \quad \text{and} \quad \bar{r}^2 = -\frac{1}{\alpha^2} (r^2 - r_c^2) \quad \text{for} \quad r < r_c.
\]

The metric then becomes

\[
ds^2 = \begin{cases} 
-dT^2 + r_c^2 dX^2 + \alpha^2 T^2 d\phi^2, & r > r_H, \\
\bar{r}^2 + r_c^2 dX^2 - \alpha^2 \bar{r}^2 d\phi^2, & r < r_H. 
\end{cases} (2.4)
\]

In the outer region, it thus describes a cosmology with spatial section a torus with radii \( r_c \) and \( \alpha T \). Furthermore, the curves \( \phi = \lambda = X, \quad T = \text{cte} \) respectively \( \bar{r} = \text{cte} \) are closed geodesics that are spacelike in the outer region and time-like in the inner region when \( \bar{r} > \frac{r_c}{\alpha} \). It follows that the hypersurface is a Cauchy horizon and that one may decide to cut the space-time at \( \bar{r} = \frac{r_c}{\alpha} \) and thus at \( r = 0 \), as in the BTZ case. The proof in [8] on the absence of closed time-like curves using ADM coordinates can then directly be applied to this case as well, the only difference being that the outer region (I) of the BTZ black hole has disappeared as the outer horizon is pushed to infinity in the flat limit.

From the point of view of identifications of Minkowski spacetime, this geometry has been studied previously in [17, 18].

The Cauchy horizon is also a Killing horizon. The generator is \( \xi = \partial_u + \Omega_c \partial_{\phi} = \partial_t + \Omega_c \partial_{\phi}, \) while the surface gravity is determined through \( \xi^\nu D_\nu \xi^\mu = \kappa C \xi^\mu, \) with

\[
\mu_C = \Omega_C = -\frac{2M}{J}, \quad T_C = \beta_C^{-1} = \frac{\Omega_C^2}{2\pi} = \frac{\Omega_c^2}{2\pi} r_c = \frac{2}{\pi} \sqrt{\frac{2GM^3}{J^2}}. (2.5)
\]

Let \( \Phi_C = \beta_C \mu_C = -\frac{2\pi |J|}{J} \). In terms of the Bekenstein-Hawking entropy

\[
S_C = \frac{2\pi r_c}{4G} = \frac{\pi^2}{G \beta C \mu^2_C} = \frac{\pi^2 \beta C}{G \Phi^2_C}, (2.6)
\]

the first law takes the form

\[
dM = -T_C dS_C - \Omega_C dJ. (2.7)
\]

Define \( \ln Z_C(\beta C, \Phi C) \) as the Legendre transform of the entropy \( S_C(M, J) \) that satisfies \( M = -\frac{\partial \ln Z_C}{\partial \beta C}, J = -\frac{\partial \ln Z_C}{\partial \Phi_C} \). It is a (generalized) Massieu function for the entropy and the semi-classical approximation of the partition function (see e.g. [19] and also [20, 21] for considerations in Euclidean quantum gravity).

\[\text{Note that there is a sign mistake in equation (61) of this reference, the first term on the right hand side should read } (\alpha dt + \frac{4GJ}{\pi} d\phi)^2\]
Taking into account the unusual form of the first law, this Legendre transform involves the opposite sign for $S_C$ as compared to the case of a standard first law,

$$\ln Z_C(\beta_C, \Phi_C) = -S_C - \beta_C M - \Phi_C J, \quad \ln Z_C = -\frac{\pi^2 \beta_C}{2G\Phi_C^2} = -\frac{\pi^2}{2G\beta_C \mu_C^2}. \quad (2.8)$$

Note that if one takes the Hawking temperature to be negative, $T_C = \beta_C^{-1} = -2\pi \sqrt{\frac{2GM^3}{J^2}}$, the first law comes with the usual sign, which implies that it is $S_C$ rather than $-S_C$ that is involved in the Legendre transform. The final expression for $\ln Z_C$ is unchanged.

### 3 Euclidean solution

When letting $t = -it_E, J = iJ_E, \alpha = i\alpha_E, r_{EC}^2 = \frac{16G^2 J_E^2}{\alpha_E^2}$, the corresponding Euclidean solution is

$$ds^2_E = \frac{\alpha_E^2 (r^2 - r_{EC}^2)}{r^2} dt_E^2 + \frac{r^2}{\alpha_E^2 (r^2 - r_{EC}^2)} dr^2 + r^2 (d\varphi + \frac{4G J_E}{r^2} dt_E)^2. \quad (3.1)$$

Let $\epsilon_J$ denote the sign of $J$. The change of coordinates

$$\begin{cases}
R_E = \frac{1}{\alpha_E} (r^2 - r_{EC}^2), \\
\varphi_E = -\epsilon_J \alpha_E \varphi, \\
Z_E = r_{EC} \varphi + \Omega_{EC} r_{EC} t_E
\end{cases} \iff \begin{cases}
r^2 = \frac{\alpha_E^2 R_E^2}{\alpha_E^2} + r_{EC}^2, \\
\varphi = -\frac{\epsilon_J}{\alpha_E} \varphi_E, \\
t_E = \frac{1}{\alpha_E \Omega_{EC}} \varphi_E + \frac{1}{\Omega_{EC} r_{EC}} Z_E,
\end{cases} \quad (3.2)$$

where $\Omega_{EC} = -\frac{2M}{J_E} = i\Omega_C$ brings the metric explicitly to the flat form

$$ds^2_E = dZ_E^2 + dR_E^2 + R_E^2 d\varphi_E^2. \quad (3.3)$$

Absence of conical singularities requires $\varphi_E \sim \varphi_E + 2\pi$ and implies $\varphi \sim \varphi + \Phi_{EC}, t_E \sim t_E + \beta_{EC}$, where $\Phi_{EC} = -\epsilon_J \frac{2\pi}{\alpha_E} = i\Phi_C$ and $\beta_{EC} = \epsilon_J \frac{2\pi}{\Omega_{EC}} \alpha_E = \beta$, in agreement with section 2.

### 4 3d flat space holography

#### 4.1 BMS3 algebra and group

Let $Y = Y(\phi), T = T(\phi)$. Following [11, 13, 22], the bms3 algebra can be represented in terms of vector fields in one, two and three dimensions, namely as

1. the semi-direct sum of algebra of vector fields $y = Y \partial_\phi$ on the circle with the abelian ideal of tensor densities of degree $-1, t = T d\phi^{-1}$, where $[y, t] = Y \partial_\phi T + \partial_\phi Y T$,
2. the Lie algebra of vector fields \( \xi = (T + u\partial_\phi Y)\partial_u + Y\partial_\phi \) on \( \mathcal{I}^+ = S^1 \times \mathbb{R} \) with coordinates \((u, \phi)\).

3. the algebra of vector fields describing the symmetries of asymptotically flat three-dimensional spacetimes at null infinity equipped with the Lie algebroid bracket.

The basis elements \( j_m \leftrightarrow (Y = e^{im\phi}, T = 0) \) and \( p_m \leftrightarrow (Y = 0, T = e^{im\phi}) \) satisfy the commutation relations

\[
\begin{align*}
i[j_m, j_n] &= (m - n)j_{m+n}, \\
i[j_m, p_n] &= (m - n)p_{m+n}, \\
i[p_m, p_n] &= 0.
\end{align*}
\]

The abstract BMS\(_3\) group \([11]\) is the semi-direct product of the group of diffeomorphisms \( \text{Diff}(S^1) \) of the circle with the abelian normal subgroup of tensor densities \( \mathcal{F}_1(S^1) \). If \( \phi' = f(\phi) \) denotes an element of the former and \( a = \alpha(\phi)d\phi^{-1} \) an element of the latter, whose component transforms as \( \alpha'(\phi') = (\frac{\partial \phi'}{\partial \phi})\alpha(\phi) \), the action of the diffeomorphisms on the tensor densities is given by

\[
(f \cdot a)(\phi) = (\alpha \frac{\partial f}{\partial \phi})(f^{-1}(\phi))d\phi^{-1}.
\]

The BMS\(_3\) group law is

\[
(f, a) \cdot (g, b) = (f \circ g, a + f \cdot b),
\]

with a realization as coordinate transformations of \( S^1 \times \mathbb{R} \) of the form

\[
\phi' = \phi'(\phi), \quad u' = \frac{\partial \phi'}{\partial \phi}(u + \alpha(\phi)).
\]

### 4.2 Gravitational results on the Minkowskian cylinder

We summarize here results of [13]. The BMS gauge consists in the metric ansatz

\[
ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2e^{2\beta} du dr + r^2 (d\phi - Ud\phi)^2,
\]

for three arbitrary functions \( \beta, V, U \). In the flat case, assuming \( \beta = o(1) = U \), the general solution to the equations of motion is

\[
ds^2 = \Theta(\phi) du^2 - 2dud\phi + 2\left[ \Xi(\phi) + \frac{u}{2} \partial_\phi \Theta(\phi) \right] dud\phi + r^2 d\phi^2.
\]

See e.g. [23], [24], [25], [26] for similar considerations on the globally well-defined BMS\(_4\) group.
The action of the asymptotic symmetries on solution space is given by

\[-\delta \Theta = Y \partial_\phi \Theta + 2 \partial_\phi Y \Theta - 2 \partial^3_\phi Y,\]
\[-\delta \Xi = Y \partial_\phi \Xi + 2 \partial_\phi Y \Xi + \frac{1}{2} T \partial_\phi \Theta + \partial_\phi T \Theta - \partial^3_\phi T.\] (4.7)

The conserved surface charges computed at \(\mathcal{J}^+\) with respect to the null orbifold \(\Theta = 0 = \Xi\) are

\[Q_{T,Y} = \frac{1}{16\pi G} \int_0^{2\pi} d\phi \left[ T \Theta + 2Y \Xi \right],\]
\[P^\text{cyl}_m = \frac{1}{16\pi G} \int_0^{2\pi} d\phi \, e^{im\phi} \Theta, \quad J^\text{cyl}_m = \frac{1}{8\pi G} \int_0^{2\pi} d\phi \, e^{im\phi} \Xi,\] (4.8)
\[\Theta = 8G \sum_m P^\text{cyl}_m e^{-im\phi}, \quad \Xi = 4G \sum_m J^\text{cyl}_m e^{-im\phi}.\]

The Dirac bracket charge algebra of the surface charge generators is then given by

\[i\{ J^\text{cyl}_m, J^\text{cyl}_n \} = (m - n) J^\text{cyl}_{m+n},\]
\[i\{ J^\text{cyl}_m, P^\text{cyl}_n \} = (m - n) P^\text{cyl}_{m+n} + \frac{c_2}{12} m^3 \delta^0_{m+n}, \quad c_2 = \frac{3}{G}\] (4.9)
\[i\{ P^\text{cyl}_m, P^\text{cyl}_n \} = 0.\]

The cosmological solutions are characterized by \(P^\text{cyl}_m = \delta^0_m M \geq 0\) and \(J^\text{cyl}_m = \delta^0_m J \in \mathbb{R}\) while \(P^\text{cyl}_m = -\frac{c_2}{24} \delta^0_m, J^\text{cyl}_m = 0\) for Minkowski space-time. In particular,

\[H = Q_{\partial_u} = P^\text{cyl+}_0, \quad J = Q_{\partial_\phi} = J^\text{cyl+}_0,\] (4.10)

and thus, for the cosmological solutions, \(H = M, J = J\).

From the way they are constructed as surface integrals at \(\mathcal{J}^+\), the quantities \(Q_{\partial_u}, Q_{\partial_\phi}\) associated with the null vector \(\partial_u\) and the space-like vector \(\partial_\phi\) are respectively the Bondi mass and angular momentum, which are conserved in three dimensions due to the absence of news. As discussed before, in the particular case of the cosmological solutions, \(\partial_u\) and \(\partial_\phi\) are in addition Killing vectors.

To recover more standard relations, it is useful to introduce the normalized variables

\[P_{++}(\phi) = -\frac{1}{8G} \Theta, \quad J_{++}(\phi) = -\frac{1}{4G} \Xi.\]

### 4.3 Mapping to the Euclidean plane

Let \(z = e^{i\phi}\). To go to the Euclidean plane, we also need \(u = -it_E\), but this is irrelevant for the charges which are time independent. Infinitesimally, one has \(z = \phi + Y(\phi)\) and works to first order in \(Y\), so that \(Y = e^{i\phi} - \phi\) and \(T = 0\).
Following the same computation as for the energy momentum tensor of a conformal field theory (see e.g. Section 5.4.1 of [27]), the finite version of the first relation of (4.7) implies that \( P_{++} \) transforms with the Schwarzian derivative,

\[
P(z) = \left( \frac{dz}{d\phi} \right)^{-2} P_{++}(\phi) + \frac{c_2}{12} \{ \phi; z \}, \quad \{ \phi; z \} = \frac{d^3 \phi}{d\phi dz} - \frac{3}{2} \left( \frac{d^2 \phi}{d\phi dz} \right)^2,
\]

so that

\[
P_{++}(\phi) = -z^2 P(z) + \frac{c_2}{24}.
\]

The second equation of (4.7) for \( T = 0 \) then implies that there is no Schwarzian derivative term in the transformation of \( J_{++}(\phi) \).

\[
J_{++}(\phi) = -z^2 J(z).
\]

For \( Y(\phi) \), because we are dealing with a vector field, we get \( Y(\phi) = \varepsilon(z)(iz)^{-1} \). The geometrical object is the tensor density \( t = T(\phi)d\phi^{-1} \), so that \( T(\phi) = \theta(z)(iz)^{-1} \), we then have

\[
-\frac{\delta}{\delta\varepsilon} P = \varepsilon \frac{\partial P}{\partial \varepsilon} + 2 \frac{\partial \varepsilon}{\partial P} + \frac{c_2}{12} \frac{\partial^3 \varepsilon}{\partial P},
\]

\[
-\frac{\delta}{\delta\varepsilon} J = 2 \frac{\partial J}{\partial \varepsilon} + \theta \frac{\partial P}{\partial \theta} + 2 \frac{\partial \theta P}{\partial \theta} + \frac{c_2}{12} \frac{\partial^3 \theta}{\partial P},
\]

\[
Q_{\varepsilon,\theta} = -\frac{1}{2\pi i} \oint_{|z| = 1} dz \left[ \theta(P - z^{-2} \frac{c_2}{24}) + \epsilon J \right],
\]

\[
P_m = \frac{1}{2\pi i} \oint_{|z| = 1} dz z^{m+1} P, \quad J_m = \frac{1}{2\pi i} \oint_{|z| = 1} dz z^{m+1} J,
\]

\[
P(z) = \sum_m P_m z^{-m-2}, \quad J(z) = \sum_m J_m z^{-m-2},
\]

\[
P_{\text{cyl}}^m = P_m - \frac{c_2}{24} \delta_m^0, \quad J_{\text{cyl}}^m = J_m.
\]

In terms of \( P_m, J_m \), the algebra is as in (4.9) with the central term changed from \( \frac{c_2}{12} m^3 \delta_m^{m+n} \) to \( \frac{c_2}{12} m (m^2 - 1) \delta_m^{m+n} \).

Minkowsi space-time corresponds to \( P_m = 0 = J_m \). The assumption is now that this solution corresponds to the vacuum state, then there is a mass gap and the cosmological solutions correspond to the other relevant states. In terms of \( P_0 \), the vacuum state is at zero eigenvalue and then the other relevant states with eigenvalues greater or equal to \( \frac{c_2}{24} \).

### 4.4 Cardy-like formula for the flat-space partition function

Consider the partition function on the torus defined as

\[
Z(\beta, \mu) = \text{Tr}_H e^{-\beta (H + \mu J)}.
\]

The considerations of this section have been elaborated on the basis of an argument that will appear in joint work by S. Detournay, T. Hartman and D. Hofman.
By introducing a temperature $\beta$, one introduces a length scale into the system, which can be taken to be $l$, so that $\tilde{\beta} = \frac{\beta}{l}$ is dimensionless. At this stage, $l$ has nothing to do with a cosmological radius. Consider then the complex plane with $z = \phi + i\tau$ and the cylinder defined by the periods $\omega_1, \omega_2 \in \mathbb{C}$. The BMS$_3$ transformation $\phi' = \frac{1}{\omega_1} \phi$, $\alpha(\phi) = 0$ implies $z' = \frac{1}{\omega_1} z$ so that one can set $\omega_1$ to 1 and work in terms of the modular parameter $\tau = \frac{\omega_2}{\omega_1}$. The question is then whether a PSL(2, $\mathbb{Z}$) transformation $(a \ b \ c \ d)$ of the torus can be induced from a BMS$_3$ transformation, or in other words, whether

$$\phi' + i\frac{u'}{l} = \frac{az + b}{cz + d} = \frac{(a\phi + b)(c\phi + d) + l^{-2}u'^2ac + i\frac{u'}{l}}{(c\phi + d)^2 + l^{-2}u'^2c^2}. \quad (4.16)$$

When $c = 0$, this is possible by choosing $\phi' = \phi$, $\alpha(\phi) = il(\phi - a(a\phi + b))$. When $c \neq 0$, this is possible only up to terms of order $l^{-2}$, in which case $\phi' = \frac{a\phi + b}{c\phi + d}, \alpha(\phi) = 0$.

It follows from section 3 that the modular parameter relevant for the cosmological solution is $\tau = \frac{1}{2\pi} (\Phi_E + i\beta) = \frac{\beta}{2\pi} (\mu_E + i\frac{1}{2})$ with $\mu = -i\mu_E$. Invariance under the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ would imply that

$$Z(\beta, \mu_E) = Z\left(\frac{4\pi^2}{\beta(1^{-2} + \mu_E^2)}\right) \rightarrow -\frac{4\pi^2}{\beta(1^{-2} + \mu_E^2)}.$$

In terms of $\Phi_E = \beta \mu_E$, this invariance takes the form

$$\beta \rightarrow \frac{4\pi^2}{\beta\left(\frac{\Phi_E^2}{\beta^2} + 1^{-2}\right)}, \quad \Phi_E \rightarrow -\frac{4\pi^2}{\Phi_E(1 + l^{-2}\frac{\beta^2}{\Phi_E^2})}. \quad (4.18)$$

It thus follows that the partition function of a BMS$_3$-invariant theory is expected to satisfy

$$Z(\beta, \mu_E) = Z\left(\frac{4\pi^2}{\beta\mu_E^2}, -\frac{4\pi^2}{\beta^2 \mu_E}\right),$$

or in terms of $\Phi_E$,

$$\beta \rightarrow \frac{4\pi^2 \beta}{\Phi_E^2}, \quad \Phi_E \rightarrow -\frac{4\pi^2}{\Phi_E}. \quad (4.20)$$

Taking into account the mass gap, one finds in the high-temperature limit $\beta \rightarrow 0$,

$$\ln Z_{\text{Cardy}}(\beta, \mu_E) = \frac{4\pi^2}{\beta \mu_E^2} \frac{c_2}{24} = \frac{\pi^2}{2G \beta \mu_E},$$

which agrees with (2.8).

## 5 Flat-space limit of AdS$_3$ results

### 5.1 Symmetries and charges

In order to compare flat-space and AdS$_3$ results, it is useful to present both in the same BMS gauge rather than using the more usual Fefferman-Graham gauge for the latter. The correct scaling of space-time coordinates that gives the limit then turns out to be
a modified Penrose limit. On the level of the algebra of symmetries and charges, this approach shows in detail and in spacetime terms how the contraction, identified previously on purely algebraic grounds in [12], comes about: in a first step, the two copies of the Virasoro algebra \( L^\pm_m \) with equal central charges \( c^\pm = \frac{3l}{2G} = c \) in the gravitational case, is presented in terms of the redefined generators

\[
P_m = \frac{1}{l} (L^+_m + L^-_m), \quad J_m = L^+_m - L^-_m,
\]

and reads

\[
i\{J_m, J_n\} = (m - n)J_{m+n} + \frac{c^+ - c^-}{12} m(m^2 - 1) \delta^0_{m+n},
\]

\[
i\{J_m, P_n\} = (m - n)P_{m+n} + \frac{c^+ + c^-}{12l} m(m^2 - 1) \delta^0_{m+n},
\]

\[
i\{P_m, P_n\} = \frac{1}{l^2} ((m - n)J_{m+n} + \frac{c^+ - c^-}{12} m(m^2 - 1) \delta^0_{m+n}),
\]

In the second step, at fixed generators, the limit \( l \to \infty \) is taken and reduces to the flat-space result.

When normalized with respect to the \( M = 0 = J \) BTZ black hole, the Hamiltonian is \( H = \frac{1}{l} (L_0 + L_0 - \frac{c}{12}) \). The mass gap with AdS_3 spacetime is the same than the one in flat space between the cosmological solutions and Minkowski spacetime and given by \( \frac{c}{12} = \frac{c}{24} = \frac{1}{8G} \), independently of \( l \).

### 5.2 Thermodynamics

For the BTZ black holes, the standard ADM form is

\[
ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2,
\]

\[
N^2 = \frac{r^2}{l^2} - 8MG + \frac{16G^2 J^2}{r^2}, \quad N^\phi = \frac{4GJ}{r^2}.
\]

Defining

\[
r^2 = 4GMl^2 \left[ 1 \pm \sqrt{1 - \frac{J^2}{M^2l^2}} \right], \quad M = \frac{r^2_+ + r^2_-}{8Gl^2}, \quad J = \frac{r^2_+ r^-}{4Gl},
\]

temperature, angular velocity and Bekenstein-Hawking entropy are given by

\[
T_H = \frac{1}{\beta} = \frac{r^2_+ - r^2_-}{2\pi l^2 r_+}, \quad \mu = \Omega_H = -\frac{r_-}{r_+ l}, \quad S_{BH} = \frac{2\pi r_+}{4G},
\]

with a first law of the form

\[
dM = T_H dS_{BH} - \Omega_H dJ.
\]
The Bekenstein-Hawking entropy, and thus also the matching Cardy formulas for the entropy (5.15) below, can obviously not be obtained as the limit $l \to \infty$ of the AdS case since $r_+(M, J, G; l)$ is pushed out to infinity and does not have a good limit.

Left and right temperatures are defined through

$$T_+ = \frac{T_H}{1 + l \Omega_H} = \frac{r_+ + r_-}{2\pi l^2}, \quad T_- = \frac{T_H}{1 - l \Omega_H} = \frac{r_+ - r_-}{2\pi l^2}. \quad (5.7)$$

Inverting the relations in terms of inverse temperature and chemical potential, one gets

$$r_+ = \frac{2\pi}{\beta(l^{-2} - \mu^2)}, \quad r_- = -\frac{2\pi l \mu}{\beta(l^{-2} - \mu^2)}, \quad S_{BH}(\beta, \mu) = \frac{\pi^2}{G\beta(l^{-2} - \mu^2)}. \quad (5.8)$$

In this case, due to the standard form of the first law, the Massieu function for the Bekenstein-Hawking entropy is

$$\ln Z_{BH}(\beta, \Phi) = S_{BH} - \beta M - \Phi J,$$

$$\ln Z_{BH} = \frac{\pi^2 \beta}{2G(l^{-2} - \Phi^2)} = \frac{\pi^2}{2G\beta(l^{-2} - \mu^2)}. \quad (5.9)$$

Its flat space-limit agrees with the one for the cosmological solution (2.8).

The thermodynamics of the inner (Cauchy) horizon of the BTZ black holes has also been discussed recently [16] and is given by

$$T_H^- = \frac{1}{\beta_-} = \frac{r_+^2 - r_-^2}{2\pi l^2 r_-}, \quad \mu^- = \Omega_H^- = -\frac{r_+}{r_- l}, \quad S^- = \frac{2\pi r_-}{4G^-}, \quad (5.10)$$

with a first law of the form

$$dM = -T_H^- dS^- - \Omega_H^- dJ. \quad (5.11)$$

As a side remark, note that when introducing $T_H^+ = T_H$, we have $\frac{1}{T_H^+} = \frac{1}{2}(\frac{1}{T_+} \pm \frac{1}{T_-})$.

Inverting gives in this case

$$r_- = \frac{2\pi}{\beta^-((\mu^-)^2 - l^{-2})}, \quad r_+ = -\frac{2\pi l \mu^-}{\beta^-((\mu^-)^2 - l^{-2})},$$

$$S^-(\beta^-, \mu^-) = \frac{\pi^2}{G\beta^-((\mu^-)^2 - l^{-2})}. \quad (5.12)$$

Again, taking into account the unusual form of the first law, the Massieu function is

$$\ln Z^-(\beta^-, \Phi^-) = -S^- - \beta^- M - \Phi^- J,$$

$$\ln Z^- = \frac{\pi^2 \beta^-}{2G(l^{-2}(\beta^-)^2 - (\Phi^-)^2)} = \frac{\pi^2}{2G\beta^-((\mu^-)^2 - l^{-2})}. \quad (5.13)$$
If one chooses the negative sign for the Hawking temperature, \( T_H = \frac{1}{\beta} = \frac{r^2 - r_+^2}{2\pi r_+^2} \), the signs in the first law and the Legendre transform become standard, \( r_+ (\beta_-, \mu_-) \), \( r_-(\beta_-, \mu_-) \) change sign, while the final expression for \( \ln Z^- \) is unchanged.

In all cases, the Massieu function is universal in the sense that it does not depend on whether one derives it from the Bekenstein-Hawking entropy of the inner or the outer BTZ horizon.

The horizon of the cosmological solution is the limit of the inner horizon of the BTZ black hole,

\[
r_C = \lim_{l \to \infty} r_- (M, J, G; l).
\]  

(5.14)

Furthermore, all thermodynamic variables of the cosmological solutions are precisely the flat-space limit \( l \to \infty \) of the variables of the inner horizon of the BTZ black hole.

### 5.3 Cardy-like formula in the grand canonical ensemble

As discussed in [9, 28, 29], when using the Cardy formulas

\[
S_{\text{Cardy}} = \pi \sqrt{\frac{2c^+ L_0^{\text{cycl}+}}{3}} + \pi \sqrt{\frac{2c^- L_0^{\text{cycl}-}}{3}} = \frac{\pi^2 l}{3} (c^+ T_+ + c^- T_-),
\]

(5.15)

one gets agreement,

\[
S_{\text{Cardy}} = S_{BH}.
\]

(5.16)

As a side remark, we also notice that \( S_{BH}^- = \frac{\pi^2 l}{3} (c^+ T_+ - c^- T_-) \).

Instead of a Cardy formula for the entropy, one can derive an equivalent formula for the partition in the standard way. Indeed, the partition function can be written as

\[
Z[\beta, \mu] = \text{Tr}_\mathcal{H} e^{-\beta (H + \mu \mathcal{J})} = \text{Tr}_\mathcal{H} q^{L_0^+} \bar{q}^{\bar{L}_0^+} q^{L_0^-} \bar{q}^{\bar{L}_0^-} = Z[\tau, \bar{\tau}],
\]

(5.17)

where \( q = e^{2\pi i \tau} \) and

\[
\tau = \frac{\beta}{2\pi} \left( \mu_E + i \frac{1}{l} \right),
\]

(5.18)

with

\[
\mu = -i \mu_E.
\]

(5.19)

Modular invariance now implies (4.17) for all values of \( l \), the flat-space limit being (4.19).

In the high temperature limit \( \beta \to 0 \) one then finds

\[
\ln Z_{\text{Cardy}} (\beta, \mu_E) = \frac{4\pi^2}{\beta (1 - \frac{\mu_E^2}{l^2}) \frac{c}{12l}},
\]

(5.20)

which agrees with (5.9) when using (5.19). In addition, its flat-space limit gives (4.21), as it should.
6 Discussion

In the context of non-relativistic versions of the AdS/CFT correspondence in three dimensions, the two dimensional Galilean conformal algebra $\mathfrak{gca}_2$ algebra plays a prominent role [30, 31, 32]. Based on [12] and the holographic interpretation of the flat space asymptotic structure in [13] in terms of the first two representations discussed in section 4 it has been pointed out in [33] that the $\mathfrak{bms}_3$ algebra is isomorphic to $\mathfrak{gca}_2$. Referring to this kind of symmetry based flat space holography as a BMS/GCA correspondence is thus misleading. This is so not only from a chronological but also from a physical point of view. Indeed, contrary to what the wording of [34] might suggest, the correct scaling of coordinates that implements the algebraic contraction [12] of the two copies of the Virasoro algebras to the $\mathfrak{bms}_3$ algebra in either bulk or boundary space-time terms, has nothing to do with a non-relativistic limit but rather, as shown through a detailed bulk analysis in [14], with a modified Penrose limit.

Of course, the isomorphism of algebras means that group theoretic results on $\mathfrak{gca}_2$ are very relevant for flat-space holography. Other results may be transposed as well. For instance, the remnant of modular invariance and the resulting Cardy-like formula for the partition function that comes from the non-relativistic contraction discussed in [35] corresponds to exchanging the role of $\beta$ and $\Phi_E$ followed by changing the signs in (4.20). This is consistent with the different roles played by $M_0, L_0$ and $P_0, J_0$ in both contexts.

Apart from the interest for asymptotically flat three-dimensional gravity and holography in this context, one of the more intriguing points of the analysis is the universality of the Massieu function with respect to the inner and outer BTZ horizons. The natural question that arises is whether this universality holds in more general cases and for other horizons as well. It is straightforward to check that it holds also for instance for the BTZ black hole in topological massive gravity or, with a bit more work, for the Kerr black hole. The results on the universality of the form of the first law at the inner horizon [15] should thus really be understood as the proof, for these cases, of the universality of the Massieu function. What this implies at the quantum level, maybe not quite so surprisingly, is that it is really the partition function that is universal.

Even though one gets the same semi-classical partition function whether one considers the temperature to be positive of negative at the Cauchy horizon of the cosmological solutions or of the BTZ black holes, one may wonder what the correct interpretation might be. A clue comes from the four-dimensional Kerr black hole, where one is required to take a negative temperature at the inner horizon in order to obtain the same functional form for the semi-classical partition function than at the outer horizon. This is in-line with the choice and the thermodynamic interpretation made in the original papers [36, 37]. We plan to address these issues more systematically elsewhere [38].
Acknowledgements

The author is grateful to S. Detournay for an illuminating discussion. He thanks R. Argurio, M. Bañados, A. Gomberoff, H. González, P.-H. Lambert, B. Oblak, D. Tempo, C. Troessaert and R. Troncoso for extensive collaborations on and discussions of relevant background material. This work is supported in part by the Fund for Scientific Research-FNRS (Belgium), by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P6/11, by IISN-Belgium, by “Communauté française de Belgique - Actions de Recherche Concertées” and by Fondecyt Projects No. 1085322 and No. 1090753.

References

[1] G. ’t Hooft, “Dimensional reduction in quantum gravity,” gr-qc/9310026

[2] L. Susskind, “The world as a hologram,” J. Math. Phys. 36 (1995) 6377–6396, hep-th/9409089

[3] J. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231–252, arXiv:hep-th/9711200

[4] J. D. Brown and M. Henneaux, “Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity,” Commun. Math. Phys. 104 (1986) 207.

[5] M. Banados, “Three-Dimensional Quantum Geometry and Black Holes,” in Trends in Theoretical Physics II, H. Falomir, R. E. Gamboa Saravi, and F. A. Schaposnik, eds., vol. 484 of American Institute of Physics Conference Series, pp. 147–169. 1999. hep-th/9901148

[6] K. Skenderis and S. N. Solodukhin, “Quantum effective action from the AdS/CFT correspondence,” Phys. Lett. B472 (2000) 316–322, hep-th/9910023

[7] M. Banados, C. Teitelboim, and J. Zanelli, “The black hole in three-dimensional space-time,” Phys. Rev. Lett. 69 (1992) 1849–1851, hep-th/9204099

[8] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, “Geometry of the (2+1) black hole,” Phys. Rev. D48 (1993) 1506–1525, gr-qc/9302012

[9] A. Strominger, “Black hole entropy from near-horizon microstates,” JHEP 02 (1998) 009, arXiv:hep-th/9712251
[10] E. Witten, “Talk given at Strings ’98.” available at
http://online.kitp.ucsb.edu/online/strings98/witten/.

[11] A. Ashtekar, J. Bicak, and B. G. Schmidt, “Asymptotic structure of symmetry
reduced general relativity,” Phys. Rev. D55 (1997) 669–686, gr-qc/9608042.

[12] G. Barnich and G. Compèrè, “Classical central extension for asymptotic
symmetries at null infinity in three spacetime dimensions,” Class. Quant. Grav. 24
(2007) F15. gr-qc/0610130. Corrigendum: ibid 24 (2007) 3139.

[13] G. Barnich and C. Troessaert, “Aspects of the BMS/CFT correspondence,” JHEP
05 (2010) 062, 1001.1541.

[14] G. Barnich, A. Gomberoff, and H. A. González, “Flat limit of three dimensional
asymptotically anti-de Sitter spacetimes,” Phys. Rev. D 86 (2012) 024020,
1204.3288.

[15] A. Castro and M. J. Rodriguez, “Universal properties and the first law of black hole
inner mechanics,” 1204.1284.

[16] S. Detournay, “Inner Mechanics of 3d Black Holes,” 1204.6088.

[17] K. Ezawa, “Transition amplitude in (2+1)-dimensional Chern-Simons gravity on a
torus,” Int.J.Mod.Phys. A9 (1994) 4727–4746, hep-th/9305170.

[18] L. Cornalba and M. S. Costa, “Time dependent orbifolds and string cosmology,”
Fortsch.Phys. 52 (2004) 145–199, hep-th/0310099.

[19] H. Callen, Thermodynamics and an introduction to thermostatistics. John Wiley &
Sons, second ed., 1985.

[20] G. W. Gibbons and S. W. Hawking, “Action integrals and partition functions in
quantum gravity,” Phys. Rev. D15 (1977) 2752–2756.

[21] J. D. Brown, G. L. Comer, E. A. Martinez, J. Melmed, B. F. Whiting, and J. W.
York, “Thermodynamic ensembles and gravitation,” Classical and Quantum
Gravity 7 (1990), no. 8, 1433–1444.

[22] G. Barnich and C. Troessaert, “Supertranslations call for superrotations,”
Proceedings of Science CNCFG 010 (2010) 1102.4632.

[23] V. Cantoni, “A Class of Representations of the Generalized Bondi-Metzner
Group,” Journal of Mathematical Physics 7 (1966), no. 8, 1361–1364.

[24] R. P. Geroch and E. Newman, “Application of the semidirect product of groups,”
J.Math.Phys. 12 (1971) 314.
[25] P. J. McCarthy, “Representations of the Bondi-Metzner-Sachs Group. I. Determination of the Representations,” *Royal Society of London Proceedings Series A* **330** (Nov., 1972) 517–535.

[26] P. J. McCarthy, “Structure of the Bondi-Metzner-Sachs Group,” *Journal of Mathematical Physics* **13** (1972), no. 11, 1837–1842.

[27] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal field theory*. Springer Verlag, 1997.

[28] J. M. Maldacena and A. Strominger, “AdS(3) black holes and a stringy exclusion principle,” *JHEP* **9812** (1998) 005, [hep-th/9804085](https://arxiv.org/abs/hep-th/9804085).

[29] S. Carlip, “What we don’t know about BTZ black hole entropy,” *Class. Quant. Grav.* **15** (1998) 3609–3625, [hep-th/9806026](https://arxiv.org/abs/hep-th/9806026).

[30] A. Bagchi and I. Mandal, “On Representations and Correlation Functions of Galilean Conformal Algebras,” *Phys. Lett.* **B675** (2009) 393–397, [0903.4524](https://arxiv.org/abs/0903.4524).

[31] A. Bagchi and R. Gopakumar, “Galilean Conformal Algebras and AdS/CFT,” *JHEP* **07** (2009) 037, [0902.1385](https://arxiv.org/abs/0902.1385).

[32] A. Bagchi, R. Gopakumar, I. Mandal, and A. Miwa, “GCA in 2d,” *JHEP* **08** (2010) 004, [0912.1090](https://arxiv.org/abs/0912.1090).

[33] A. Bagchi, “Correspondence between asymptotically flat spacetimes and nonrelativistic conformal field theories,” *Phys. Rev. Lett.* **105** (Oct, 2010) 171601.

[34] A. Bagchi and R. Fareghbal, “BMS/GCA Redux: Towards Flatspace Holography from Non-Relativistic Symmetries,” [1203.5795](https://arxiv.org/abs/1203.5795).

[35] K. Hotta, T. Kubota, and T. Nishinaka, “Galilean Conformal Algebra in Two Dimensions and Cosmological Topologically Massive Gravity,” *Nucl.Phys.* **B838** (2010) 358–370, [1003.1203](https://arxiv.org/abs/1003.1203).

[36] A. Curir, “Spin entropy of a rotating black hole,” *Il Nuovo Cimento B* (1971-1996) **51** (1979) 262–266. 10.1007/BF02743435.

[37] A. Curir and M. Francaviglia, “Spin thermodynamics of a Kerr black hole,” *Il Nuovo Cimento B* (1971-1996) **52** (1979) 165–176. 10.1007/BF02739031.

[38] G. Barnich and H. A. González, “Universality of the gravitational Massieu function.” in preparation.