REALIZATION OF LEVEL ONE REPRESENTATIONS OF $U_q(\hat{g})$
AT A ROOT OF UNITY

VYJAYANTHI CHARI AND NAIHUAN JING

Abstract. Using vertex operators, we construct explicitly Lusztig's $\mathbb{Z}[q,q^{-1}]$-lattice for the level one irreducible representations of quantum affine algebras of ADE type. We then realize the level one irreducible modules at roots of unity and show that the $q$-dimension is still given by the Weyl-Kac character formula. As a consequence we also answer the corresponding question of realizing the affine Kac-Moody Lie algebras of simply laced type at level one in finite characteristic.

0. Introduction

In [L3, L1] Lusztig proved that a quantum Kac–Moody algebra $U_q$ defined over $\mathbb{Q}(q)$ admits an $\mathcal{A} = \mathbb{Z}[q,q^{-1}]$-lattice $U_{\mathcal{A}}$ and that any irreducible highest weight integrable representation $V$ of $U$ admits a corresponding $\mathcal{A}$-lattice, say $V_{\mathcal{A}}$. This allows us to specialize $q$ to a non–zero complex number $\zeta$ and we let $U_{\zeta}, W_{\zeta}$ denote the corresponding objects. If $\zeta$ is not a root of unity, Lusztig proved that $W_{\zeta}$ is irreducible and its character is the same as that of the corresponding classical representation. On the other hand, when $\zeta$ is a primitive $l$th root of unity, the situation is more interesting, even for finite–dimensional Kac–Moody algebras. In that case, $W_{\zeta}$ is not always irreducible: a sufficient condition for irreducibility [APW] is that the highest weight $\Lambda$ of $V$ should be “small” in the sense that $(\Lambda, \alpha) < l$ for all positive roots $\alpha$. The corresponding question for infinite–dimensional Kac–Moody algebras at roots of unity is open, and in this paper we answer it in the case of level one representations of quantum affine algebras of ADE type. Note that the condition $(\Lambda, \alpha) < l$ never holds in this case; nevertheless, we find that $W_{\zeta}$ is irreducible provided that $l$ is coprime to the Coxeter number of the underlying finite–dimensional Lie algebra.

The level one representations of an affine Lie algebra of ADE type can be explicitly constructed in the tensor product of a symmetric algebra and a twisted group algebra [FKS]. Essentially, these representations are built from the canonical representation of an infinite–dimensional Heisenberg algebra. Later, in [FJ] this construction was extended to the case of the basic representations of the quantum affine algebras of ADE type. Again, the representations are built from the representation of a suitable quantum Heisenberg algebra. In this paper, we identify the natural lattice $V_{\mathcal{A}}$ of the level one representation explicitly as the tensor product of the lattice of Schur functions tensored with the obvious $\mathcal{A}$-lattice in the twisted group algebra (see also [12]). We also describe the action of the divided powers of the Chevalley (and Drinfeld) generators on an $\mathcal{A}$-basis of $V_{\mathcal{A}}$ and this allows us to

N.J. is partially supported by NSF grant DMS-9970493.
realize the level one irreducible representations \( W_\xi \) explicitly and prove that they are irreducible.

Our methods also apply to the study of highest weight representations of affine Lie algebras in characteristic \( p \), and the corresponding results are also new in that situation. In particular we give an explicit realization of the \( \mathbb{Z} \)-form \([B]\) of the vertex representation of the affine Lie algebras.

1. The algebras \( \mathbf{U}, \mathbf{U}_A \).

Throughout this paper \( \mathfrak{g} \) will denote a simply-laced, finite-dimensional complex simple Lie algebra and \( (a_{ij})_{i,j \in I}, I = \{1, \ldots , n\} \), will denote its Cartan matrix. Let \( (a_{ij})_{i,j \in I}, \hat{I} = I \cup \{0\} \), be the extended Cartan matrix of \( \mathfrak{g} \) and let \( \hat{\mathfrak{g}} \) be the corresponding affine Lie algebra. Let \( \mathbf{U} \) be the Hopf algebra generated as \( \mathfrak{g} \)-module by elements \( E_{\alpha_i}, F_{\alpha_j}, K_i^{\pm1} \) \((i \in \hat{I})\), \( D^{\pm1} \) with the following defining relations:

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \\
K_i D = D K_i, \quad DD^{-1} = D^{-1} D = 1, \\
DE_{\alpha_i} D^{-1} = q^{\delta_{ij}} E_{\alpha_i}, \quad DF_{\alpha_i} D^{-1} = q^{-\delta_{ij}} F_{\alpha_i}, \\
K_i E_{\alpha_j} K_i^{-1} = q^{a_{ij}} E_{\alpha_j}, \quad K_i F_{\alpha_j} K_i^{-1} = q^{-a_{ij}} F_{\alpha_j}, \\
[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} \frac{K_i - K_j^{-1}}{q - q^{-1}}, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right] (E_{\alpha_i})^r E_{\alpha_j} (E_{\alpha_i})^{1-a_{ij}-r} = 0 \quad \text{if } i \neq j, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1-a_{ij} \\ r \end{array} \right] (F_{\alpha_j})^r F_{\alpha_j} (F_{\alpha_j})^{1-a_{ij}-r} = 0 \quad \text{if } i \neq j.
\]

The comultiplication of \( \mathbf{U} \) is given on generators by

\[
\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes 1 + K_i \otimes E_{\alpha_i}, \quad \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes K_i^{-1} + 1 \otimes F_{\alpha_i}, \\
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(D) = D \otimes D,
\]
for \( i \in \hat{I} \).

Let \( U^+ \) (resp. \( U^-; U^0 \)) be the \( \mathbb{Q}(q) \)-subalgebras of \( U \) generated by the \( E_{\alpha_i} \) (resp. \( F_{\alpha_i}; K^\pm_i \) and \( D^\pm \)) for \( i \in \hat{I} \). The following result is well-known, see [L3] for instance.

**Lemma 1.1.** \( U \cong U^- \otimes U^0 \otimes U^+ \) as \( \mathbb{Q}(q) \)-vector spaces.

It is convenient to use the following notation:

\[
E^{(r)}_{\alpha_i} = \frac{E^{r}_{\alpha_i}}{[r]!}.
\]

The elements \( F^{(r)}_{\alpha_i} \) are defined similarly. Let \( U_A \) denote the \( A \)-subalgebra of \( U \) generated by \( E^{(r)}_{\alpha_i}, F^{(r)}_{\alpha_i}, K^\pm_i \) (\( i \in \hat{I} \)) and \( D^\pm \). The subalgebras \( U^\pm_A \) are defined in the obvious way.

For \( i \in \hat{I}, r \geq 1, m \in \mathbb{Z} \), define elements

\[
\begin{align*}
[K, m]_r &= \prod_{s=1}^{r} \frac{K_i q^{m-s+1} - K_i^{-1} q^{-m+s-1}}{q^r - q^{-r}}, \\
[D, m]_r &= \prod_{s=1}^{r} \frac{D q^{m-s+1} - D^{-1} q^{-m+s-1}}{q^r - q^{-r}}.
\end{align*}
\]

Let \( U^0_A \) be the \( A \)-subalgebra of \( U_A \) generated by \( K^\pm_i, D^\pm, [K, m]_r \) and \( [D, m]_r \), \( i \in \hat{I}, r \geq 1 \) and \( m \in \mathbb{Z} \). The following is well-known (see [L3]).

**Lemma 1.2.** We have \( U_A \cong U^- A U^0_A U^+ \).

We shall also need another realization of \( U \), due to [D1, B, J1].

**Theorem 1.** There is an isomorphism of \( \mathbb{Q}(q) \)-Hopf algebras from \( U \) to the algebra with generators \( x^\pm_{i,r} \) (\( i \in I, r \in \mathbb{Z} \)), \( K^\pm_i \) (\( i \in I \)), \( h_{i,r} \) (\( i \in I, r \in \mathbb{Z}\backslash\{0\} \)) and \( C^\pm \).
and the following defining relations:

\[ C^{±1} \text{ are central,} \]
\[ K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad CC^{-1} = C^{-1} C = 1, \]
\[ K_i K_j = K_j K_i, \quad K_i h_{j,r} = h_{j,r} K_i, \]
\[ K_i x_{j,r}^± K_i^{-1} = q^{±a_{ij}} x_{j,r}^±, \]
\[ DD^{-1} = D^{-1} D = 1, \quad DK_i = K_i D, \]
\[ D h_{j,r} D^{-1} = q^r h_{j,r}, \quad D x_{j,r}^± D^{-1} = q^r x_{j,r}^±, \]
\[ [h_{i,r}, h_{j,s}] = δ_{r,−s} \frac{1}{r} [ra_{ij}] \frac{C^r - C^{-r}}{q - q^{-1}}, \]
\[ [h_{i,±r}, x_{j,s}^±] = ±\frac{1}{r} [ra_{ij}] x_{j,s}^±, \quad r > 0, \]
\[ [h_{i,±r}, x_{j,s}^±] = ±C^r [ra_{ij}] x_{j,s}^±, \quad r > 0, \]
\[ x_{l,r+1}^± x_{j,s}^± - q^{a_{ij}x_{l,r}x_{j,s}^±} x_{l,r+1}^± = q^{a_{ij}x_{l,r}x_{j,s}^±} x_{l,r+1}^± - x_{l,r+1}^± x_{j,s+1}^±, \]
\[ [x_{i,r}^±, x_{j,s}^±] = δ_{i,j} \frac{C^{-s}ψ_{i,r+s}^± - C^{−r}ψ_{i,s+r}^±}{q - q^{-1}}, \]
\[ \sum_{m} \sum_{k=0}^{m} (-1)^k \left[ \frac{m}{k} \right] x_{l,r_1}^± x_{l,r_2}^± x_{l,r_3}^± \ldots x_{l,r_m}^± = 0, \quad \text{if} \quad i \neq j, \]

for all sequences of integers \( r_1, \ldots, r_m \), where \( m = 1 - a_{ij} \), \( Σ_m \) is the symmetric group on \( m \) letters, and the \( ψ_{i,r}^± \) are determined by equating powers of \( u \) in the formal power series

\[ \sum_{r=0}^{∞} ψ_{i,±r}^± u^±r = K_i^{±1} exp \left( ±(q - q^{-1}) \sum_{s=1}^{∞} h_{i,±s} u^±s \right). \]

Following [CP] Section 3], we define elements \( P_{k,i} \) and \( \tilde{P}_{k,i} \) via the generating functions

\[ P_{i,k}^±(u) = \sum_{k=0}^{∞} P_{i,k}^± u^k = \exp \left( −\sum_{k=1}^{∞} \frac{h_{i,±k}}{k} u^k \right) = \exp \left( −\sum_{k=1}^{∞} \frac{\tilde{h}_{i,±k}}{k} u^k \right), \]

\[ \tilde{P}_{i,k}^±(u) = \sum_{k=0}^{∞} \tilde{P}_{i,k}^± u^k = \exp \left( \sum_{k=1}^{∞} \frac{h_{i,±k}}{k} u^k \right) = \exp \left( \sum_{k=1}^{∞} \frac{\tilde{h}_{i,±k}}{k} u^k \right), \]

where \( \tilde{h}_{i,k} = \frac{k h_{i,k}}{k} \). Notice that these formulas are exactly those that relate the elementary symmetric functions (resp. complete symmetric functions) to the power sum symmetric functions [M]. For a vertex operator approach to this and to Schur functions, see [Z].

The following result was proved in [CP], Section 5].

**Lemma 1.3.** For all \( i \in I, k \in Z, k \geq 0 \), we have

\[ P_{i,k}^±, \quad \tilde{P}_{i,k}^± \in U_A. \]
Let $\tilde{U}_A$ (resp. $\hat{U}_A$) be the $\mathcal{A}$–subalgebra generated by $(x_i^\pm)^{(r)}$, $r,n \in \mathbb{Z}$, $r \geq 0$, $i \in I$, (resp. $n \in \mathbb{Z}$, $\pm n \geq 0$) and $U^0_A$. The following result is proved in [BCP, Section 2].

**Proposition 1.2.** We have, $U_A = \tilde{U}_A$, $U_A^\pm \subset \hat{U}_A$. □

Finally, let $\tilde{U}(0)$ (resp. $U_A(0)$) be the subalgebra of $U$ (resp. the $\mathcal{A}$–subalgebra of $U_A$) generated by the elements $h_i,n$, $i \in I$, $n \in \mathbb{Z}$, $\pm n \geq 0$), $C^{\pm 1}$. The subalgebras $U_A^\pm(0)$ and $U_A^\pm(0)$ are defined in the obvious way.

**Proposition 1.3.** (i) The algebra $U(0)$ is defined by the relations

$$[h_i,n,h_j,m] = \frac{1}{n} \delta_{m,-n}[na_{ij}] C^n - C^{-n} / q - q^{-1},$$

$$C^{\pm 1} h_i,n = h_i,n C^{\pm 1},$$

for all $i,j \in I$ and $m,n \in \mathbb{Z}$. In particular, $U(0)$ is commutative.

(ii) For all $i \in I$, $k > 0$, we have

$$P_{i,k}^\pm = -\frac{1}{k} \sum_{m=0}^{k} h_{i,m} P_{i,k-m}^\pm,$$

$$\tilde{P}_{i,k}^\pm = \frac{1}{k} \sum_{m=0}^{k} h_{i,m} \tilde{P}_{i,k-m}^\pm.$$

In particular, $\tilde{h}_{i,k}, \tilde{P}_{i,k}^\pm \in U_A(0)$ and as $\mathbb{Q}(q)$–spaces we have

$$U(0) \cong \mathbb{Q}(q) \otimes \mathcal{A} U_A(0),$$

$$U^\pm(0) \cong \mathbb{Q}(q) \otimes \mathcal{A} U_A^\pm(0).$$

(iii) Monomials in $P_{i,n}^\pm$ (resp. $\tilde{P}_{i,n}^\pm$), $i \in I$, $n > 0$, form a basis for $U_A^\pm(0)$.

**Proof.** Part (i) is a consequence of the PBW theorem for $U$ proved in [B]. Parts (ii) and (iii) follow from the definition of the elements $\tilde{P}_{i,k}^\pm$ (see [BCP] for details). □

2. **The Level One Representations of $U$ and $U_A$**

We begin this section by recalling the natural irreducible representation of $U(0)$ and we construct a natural $U_A(0)$–lattice in this representation. We then recall the definition of the highest weight representations $V_q(\Lambda)$ of $U$ and the lattice $V_A(\Lambda)$ of $U_A$, see [E2]. Finally, we recall the explicit construction of the level one representations given in [E2] and state and prove the main theorem of the paper.

Consider the left ideal $\mathcal{I}$ in $U(0)$ generated by $C^{\pm 1} - q^{\pm 1}$ and $U^+(0)$. Then, $U(0)/\mathcal{I}$ is a left $U(0)$–module through left multiplication. It is easy to see that as $\mathbb{Q}(q)$–spaces, we have

$$U^-(0) \cong U(0)/\mathcal{I}.$$
given a weight $\Lambda = (\theta, n)$ consider only type 1 representations. Writing

\begin{align*}
\pi(h_{i,n})h_{j,-m} &= \delta_{n,m} \frac{[na_{ij}]_n}{n},
\end{align*}

**Proposition 2.1.** (i) $\pi$ is an irreducible representation of $U(0)$. (ii) For $i, j \in I$, we have,

\begin{align*}
\pi(\hat{P}^+_i(u))\hat{P}^+_j(v) &= f_{i,j}(u,v)\hat{P}^+_j(v), \\
\pi(\hat{P}^+_i(u))\hat{P}^-_j(v) &= f_{i,j}(u,v)\hat{P}^-_j(v), \\
f_{i,j}(u,v)\pi(\hat{P}^+_i(u))\hat{P}^-_j(v) &= \hat{P}^-_j(v),
\end{align*}

where the power series $f_{i,j}$ is defined by

\begin{align*}
f_{i,j}(u,v) &= 1 \quad \text{if } a_{ij} = 0, \\
&= (1 - uv) \quad \text{if } a_{ij} = -1, \\
&= (1 - quv)^{-1} (1 - q^{-1}uv)^{-1} \quad \text{if } a_{ij} = 2.
\end{align*}

(iii) $\pi(U_A(0))U_A^- (0) \subset U^- _A(0)$.

**Proof.** Part (i) is well-known. For (ii), notice that the relations in Proposition 1.3 imply that

\begin{align*}
\pi(\hat{P}^+_i(u))\hat{P}^-_j(v) &= \exp \left( \sum_{k=1}^\infty \frac{\pi(h_{i,k})}{k} u^k \right) \exp \left( \sum_{k=1}^\infty \frac{\hat{h}_{j,-k}}{k} u^k \right) \\
&= \exp \left( \sum_{k=1}^\infty \frac{[ka_{ij}]}{k|k|} u^k \right) \exp \left( \sum_{k=1}^\infty \frac{\hat{h}_{j,-k}}{k} u^k \right) \exp \left( \sum_{k=1}^\infty \frac{\pi(h_{i,k})}{k} u^k \right).
\end{align*}

The second equality above follows by using the Campbell–Hausdorff formula. The calculation of $f_{i,j}(u,v)$ is now straightforward. The other equations are proved similarly. Part (iii) follows immediately from (ii).

By a weight, we mean a pair $(\mu, n) \in \mathbb{Z}^{|I|} \times \mathbb{Z}$. If $n = 0$, we shall denote the pair $(\mu, 0)$ as $\mu$. A representation $W$ of $U$ is said to be of type 1 if

\begin{align*}
W &= \bigoplus_{(\mu,n)} W_{\mu,n},
\end{align*}

where $W_{\mu,n} = \{ w \in W | K_i w = q^{n_1} w, \ D w = q^\mu w \}$. If $W_{\mu,n} \neq 0$, then $W_{\mu,n}$ is called the weight space of $W$ with weight $(\mu, n)$. Throughout this paper we will consider only type 1 representations. Writing $\theta = \sum_{i \in I} d_i \alpha_i$, we define the level of $(\mu, n)$ to be $\sum_{i \in I} d_i \mu_i + \mu_0$.

For $i \in I$, let $\Lambda_i$ be the $I$-tuple with one in the $i$th place and zero elsewhere. Given a weight $\Lambda = \sum_i n_i \Lambda_i$, $n_i \geq 0$, let $V_q (\Lambda)$ be the irreducible highest weight $U$-module with highest weight $\Lambda$ and let $v_\Lambda$ be the highest weight vector. Thus, $V_q (\Lambda)$ is generated by $v_\Lambda$ with relations,

\begin{align*}
E_{\alpha_i} v_\Lambda &= 0, \quad K_i v_\Lambda = q^{n_i} v_\Lambda, \quad D v_\Lambda = v_\Lambda, \quad F_{\alpha_i} v_\Lambda = 0,
\end{align*}
for \( i \in \hat{I} \). Clearly \( V_q(\Lambda) \) is of type 1. We say that \( V_q(\Lambda) \) has level one if \( \Lambda \) has level one.

Set

\[ V_\mathcal{A}(\Lambda) = U_\mathcal{A} \cdot v_\Lambda. \]

By Lemma 1.2 we see that \( V_\mathcal{A} = U_\mathcal{A} \cdot v_\Lambda \). The following result is now an immediate consequence of Proposition 1.2.

**Lemma 2.1.** We have

\[ V_\mathcal{A}(\Lambda) = \tilde{U}_\mathcal{A} \cdot v_\Lambda = \tilde{U}_{-\mathcal{A}} \cdot v_\Lambda. \]

The following result is due to Lusztig [L3].

**Proposition 2.2.** \( V_\mathcal{A}(\Lambda) \) is a \( U_\mathcal{A} \)-submodule of \( V_q(\Lambda) \) such that

\[ V_q(\Lambda) \cong V_\mathcal{A}(\Lambda) \otimes_\mathcal{A} Q(q). \]

Further,

\[ V_\mathcal{A}(\Lambda) = \bigoplus_{\mu, n} V_\mathcal{A}(\Lambda) \cap V_q(\Lambda)_{\mu, n}, \]

and

\[ \dim_\mathcal{A}(V_\mathcal{A}(\Lambda) \cap V_q(\Lambda)_{\mu, n}) = \dim_{Q(q)} V_q(\Lambda)_{\mu, n}. \]

We turn now to the realization of the level one representations of \( U \). In fact we shall restrict ourselves to constructing the basic representation of \( \hat{\mathfrak{g}} \), i.e. the representation corresponding to \( \Lambda_0 \). The construction of the other level one representations is identical except that one adjoins \( v_\Lambda \) to the twisted group algebra (see [FJ]).

Fix a bilinear map \( \epsilon : Q \times Q \to \{\pm 1\} \) such that for all \( i \in I, \alpha, \beta, \gamma \in Q \), we have,

\[
\begin{align*}
\epsilon(\alpha, 0) &= \epsilon(0, \alpha) = 1, \\
\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) &= \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma), \\
\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) &= (-1)^{|\alpha||\beta|}.
\end{align*}
\]

Let \( Q(q)[Q] \) be the twisted group algebra over \( Q(q) \) of the weight lattice of \( \mathfrak{g} \). Thus, \( Q(q)[Q] \) is the algebra generated by elements \( e^n, \eta \in Q \), subject to the relation,

\[ e^n e^{n'} = \epsilon(\eta, \eta')e^{n+n'}. \]

Set

\[ V_q = U^- (0) \otimes Q(q)[Q]. \]

Let \( z^0 : V_q \to V_q[z, z^{-1}] \) be the \( Q(q) \)-linear map defined by extending

\[ z^0(v \otimes e^n) = (v \otimes e^n)z^{\eta_i - |\alpha_i|}, \quad v \in U^- (0), \quad \eta \in Q. \]
Define operators $X_{i,n}^\pm$ on $V_q$ by means of the following generating series:

$$X_i^+(z) = \pi(\tilde{P}_i^-(z))\pi(P_i^+(q^{-1}z^{-1}))e^{\alpha_i z}.$$  
$$= \sum_{n \in \mathbb{Z}} X_{i,n}^+ z^{-n-1},$$  
$$X_i^-(z) = \pi(P_i^-(qz))\pi(\tilde{P}_i^+(z^{-1}))e^{-\alpha_i z^{-1}}.$$  
$$= \sum_{n \in \mathbb{Z}} X_{i,n}^- z^{-n-1}.$$  

The following result was proved in [FJ].

**Theorem 2.** The assignment $x_{i,n}^\pm \to X_{i,n}^\pm$, $h_{i,n} \to \pi(h_{i,n}) \otimes 1$ defines a representation of $U$ on $V_q$. In fact as $U$–modules we have

$$V_q(\Lambda_0) \cong V_q.$$  

Further, for all $i \in I$, $u \in U^-(0)$, $\eta \in Q$, we have

$$K_i(u \otimes e^{\eta}) = q^{|\eta| |\alpha_i|} u \otimes e^{\eta}, \quad C(u \otimes e^{\eta}) = u \otimes e^{\eta}.$$  

The highest weight vector in $V_q(\Lambda_0)$ maps to $1 \otimes 1$ under this isomorphism. 

Let $V_A$ be the image of $V_q(\Lambda_0)$ under this isomorphism. Clearly $V_A$ is a $U_A$–submodule of $V_q(\Lambda_0)$ and $V_q(\Lambda_0) \cong Q(q) \otimes_A V_A$.

Set

$$L = U_A(0) \otimes A[Q],$$

where $A[Q]$ is the $A$–span in $Q(q)[Q]$ of the elements $e^{\eta}$. It follows from Proposition 1.3 that

$$V_q \cong Q(q) \otimes_A L.$$

We now state our main result.

**Theorem 3.** The lattice $L$ is preserved by $U_A$, and

$$L \cong V_A$$

as $U_A$–modules.

**Remark.** The case $\mathfrak{g} = sl_2$ was studied in [J2]. In that paper, the author worked over $A = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ and proved that the corresponding lattice $L$ was preserved by $U_A$ and gave the action of the divided powers of the Drinfeld generators on the Schur functions.

The rest of the section is devoted to proving Theorem 3.

We begin with the following two lemmas which are easily deduced from the definition of $X_i^\pm(z)$ and Proposition 2.1.

**Lemma 2.2.** Let $i \in I$, $\eta \in Q$ and $m = |\eta| |\alpha_i|$. Then,

$$x_{i,-m-1}^+(1 \otimes e^{\eta}) = \epsilon(\alpha_i, \eta) \otimes e^{\alpha_i + \eta},$$  
$$x_{i,m-1}^-(1 \otimes e^{\eta}) = \epsilon(-\alpha_i, \eta) \otimes e^{-\alpha_i + \eta}. \quad \square$$
Lemma 2.3. Let \( r, l \in \mathbb{Z} \), \( r, l \geq 0 \), and let \( i, j_1, j_2 \cdots j_l \in I \). We have,

\[
X_i^+(z_1)X_i^+(z_2) \cdots X_i^+(z_r) \left( \hat{P}_{j_1}^-(w_1)\hat{P}_{j_2}^-(w_2) \cdots \hat{P}_{j_l}^-(w_l) \otimes e^n \right)
\]

\[
= \epsilon \cdot \prod_{k=1}^{r} z_k^{r-k+|\eta|+|\alpha_i|} \prod_{1 \leq k \leq s \leq r} (f_{i,i}((qz_k)^{-1}, z_s))^{-1} \prod_{1 \leq k \leq r, 1 \leq s \leq l} f_{i,j_s}((qz_k)^{-1}, w_s) \times \hat{P}_i^-(z_1)\hat{P}_i^-(z_2) \cdots \hat{P}_i^-(z_r)\hat{P}_{j_1}^-(w_1)\hat{P}_{j_2}^-(w_2) \cdots \hat{P}_{j_l}^-(w_l) \otimes e^{r\alpha_i+\eta}
\]

where \( \epsilon = \epsilon(r\alpha_i, \eta) \prod_{k=1}^{r-1} \epsilon(\alpha_i, k\alpha_i) \).

Proof. Observe that the left-hand side of the equation is an antisymmetric polynomial in \( z_1, z_2, \cdots, z_r \) and hence is divisible by the right hand side. Hence, by comparing degrees, we can write,

\[
\sum_{\sigma \in \mathfrak{S}_r} (-1)^{|\sigma|} \prod_{k<s} (z_{\sigma(k)} - q^{-2}z_{\sigma(s)}) = q^{-r(r-1)/2} [r]! \prod_{k<s} (z_k - z_s).
\]

But it is easy to see that the coefficient of \( z_1^{r-1}z_2^{r-2} \cdots z_{r-1} \) on the left hand side is

\[
\sum_{\sigma \in \mathfrak{S}_r} q^{-2l(\sigma)} = q^{-r(r-1)/2} [r]!,
\]

thus proving the proposition.

Lemma 2.4. We have,

\[
\sum_{\sigma \in \mathfrak{S}_r} (-1)^{|\sigma|} \prod_{k<s} (z_{\sigma(k)} - q^{-2}z_{\sigma(s)}) = C(q) \prod_{k<s} (z_k - z_s).
\]

Proof. Observe that the left-hand side of the equation is an antisymmetric polynomial in \( z_1, z_2, \cdots, z_r \) and hence is divisible by the right hand side. Hence, by comparing degrees, we can write,

\[
\sum_{\sigma \in \mathfrak{S}_r} (-1)^{|\sigma|} \prod_{k<s} (z_{\sigma(k)} - q^{-2}z_{\sigma(s)}) = C(q) \prod_{k<s} (z_k - z_s).
\]

But it is easy to see that the coefficient of \( z_1^{r-1}z_2^{r-2} \cdots z_{r-1} \) on the left hand side is

\[
\sum_{\sigma \in \mathfrak{S}_r} q^{-2l(\sigma)} = q^{-r(r-1)/2} [r]!,
\]

thus proving the proposition.

Lemma 2.5. (i) Let \( \delta = (\delta_1, \delta_2, \cdots, \delta_r) \in \mathbb{Z}^r \) be the \( r \)-tuple \((r-1, r-2, \cdots, 1, 0)\).

We have,

\[
\prod_{j<k} (z_j - z_k)^2 = \left( \sum_{\mu} a_{\mu} \sum_{\rho \in \mathfrak{S}_r} z_1^{\mu(1)} z_2^{\mu(2)} \cdots z_r^{\mu(r)} \right),
\]

where the sum is over \( \{\delta + \tau(\delta) : \tau \in \mathfrak{S}_r\} \) and \( a_{\mu} = (-1)^{|\tau|} \), if \( \mu = \delta + \tau(\delta) \).

(ii) Let \( R \) be a commutative ring and let \( G \in R[[z_1^{\pm 1}, z_2^{\pm 1}, \cdots, z_r^{\pm 1}]] \) be invariant under the action of the symmetric group \( \mathfrak{S}_r \). Then, for all \( n \in \mathbb{Z} \), the coefficient of \( z_1z_2 \cdots z_r \) in \( \prod_{j<k} (z_j - z_k)^2 G \) is divisible by \( r! \).

Proof. Since,

\[
\prod_{j<k} (z_j - z_k) = \sum_{\sigma \in \mathfrak{S}_r} (-1)^{|\sigma|} z_1^{\delta_{\sigma(1)}} z_2^{\delta_{\sigma(2)}} \cdots z_r^{\delta_{\sigma(r)}},
\]
we get
\[ \prod_{j<k} (z_j - z_k)^2 = \sum_{\sigma, \tau \in \mathcal{S}_\nu} (-1)^{(l+1)(l+\tau)} z_{\sigma(1)}^{\delta_{\sigma(1)} + \delta_{\tau(1)}} z_{\sigma(2)}^{\delta_{\sigma(2)} + \delta_{\tau(2)}} \cdots z_{\sigma(n)}^{\delta_{\sigma(n)} + \delta_{\tau(n)}}, \]
which becomes the formula in (i) on putting \( \rho = \sigma \tau \). Part (ii) follows trivially. \( \square \)

**Proof of Theorem 3.** Using Lemma 2.3 and Lemma 2.4, we get
\[
\sum_{\sigma \in \mathcal{S}_\nu} X_1^+(z_{\sigma(1)}) X_1^+(z_{\sigma(2)}) \cdots X_1^+(z_{\sigma(n)}) \big( P_{j_1}^-(w_1) P_{j_2}^-(w_2) \cdots P_{j_l}^-(w_l) \otimes e^\eta \big)
\]
\[
= q^{-r(r-1)/2} [r]^\epsilon \cdot (z_1 \cdots z_r)^{|\eta| |\alpha|} \prod_{k<s} (z_k - z_s)^{2} \prod_{1 \leq k \leq r, 1 \leq s \leq l} f_{i,j,s} \big( (qz_k)^{1-1}, w_s \big)
\]
\[
\times \big( \tilde{P}_{i_1}^-(z_1) \tilde{P}_{i_2}^-(z_2) \cdots \tilde{P}_{i_l}^-(z_l) \tilde{P}_{j_1}^-(w_1) \tilde{P}_{j_2}^-(w_2) \cdots \tilde{P}_{j_l}^-(w_l) \big) \otimes e^{r \alpha + \eta}, \quad (*)
\]
where the constant \( \epsilon \) is defined in Lemma 2.3.

Set \( F = \prod_{k<s} (z_k - z_s)^2 \) and let \( G \) be the right hand side of (*) divided by \( F \). Then Lemma 2.3 applies, and by collecting the coefficient of \((z_1 z_2 \cdots z_r)^{-n-1}\) on both sides of (*), we find that
\[
(x_{i,n}^+(r)) \tilde{L} \subset \mathcal{L},
\]
for all \( \mu_1, \mu_2, \ldots, \mu_l \in \mathbb{Z} \), \( \eta \in Q \), or equivalently that
\[
(x_{i,n}^+(r)) \mathcal{L} \subset \mathcal{L}.
\]
One proves similarly that \((x_{i,n}^-)^{(r)}\) preserves \( \mathcal{L} \). In particular, by Proposition 1.2, \( \mathcal{L} \) is preserved by \( U_A \). To complete the proof of the theorem we must prove that
\[
\mathcal{L} = V_A.
\]
Since \( 1 \otimes 1 \in V_A \), it follows from Lemma 2.2 and a simple induction that \( e^\eta \in V_A \) for all \( \eta \in Q \). Next, from Theorem 3, we see that for \( i \in I, k > 0, \)
\[
\tilde{P}_{i,k}^-(1 \otimes e^\eta) = \tilde{P}_{i,k}^- \otimes e^\eta.
\]
Since by Proposition 1.3, the monomials in the \( \tilde{P}_{i,k}^- \)'s span \( U_A(0) \), we see that \( \mathcal{L} \subset V_A \). The reverse inclusion \( V_A \subset \mathcal{L} \) is now clear, for
\[
V_A = U_A(1 \otimes 1) \subset \mathcal{L}
\]
since \( 1 \otimes 1 \in \mathcal{L} \). \( \square \)

3. Specialization to a Root of Unity

Throughout this section, we let \( N \) denote the Coxeter number of \( \mathfrak{g} \). It is well-known [8d] that \( N = n+1 \) (resp. \( 2n-2, 12, 18, 30 \)) if \( \mathfrak{g} \) is of type \( A_n \) (resp. \( D_n, E_6, E_7, E_8 \)). Let \( \zeta \in \mathbb{C}^* \) denote a primitive \( l^{th} \) root of unity, where \( l \) is a non-negative integer coprime to \( N \). Set \( n = |I| \). Finally, for any \( g \in A \) we let \( g_\zeta \in \mathbb{C}^* \) be the element obtained by setting \( q = \zeta \).
Lemma 3.1. Let $[A]$ denote the $n \times n$–matrix with coefficients in $\mathbb{A}$ whose $(i,j)$–th entry is $[a_{ij}]$. Then,

$$\det[A] = [n + 1], \quad \text{if } g \text{ is of type } A_n,$$

$$= [2](q^{n-1} + q^{n-1}), \quad \text{if } g \text{ is of type } D_n,$$

$$= (q^4 + q^{-4} - 1)(q^2 + q^{-2} + 1), \quad \text{if } g \text{ is of type } E_6,$$

$$= [2](q^6 + q^{-6} - 1), \quad \text{if } g \text{ is of type } E_7,$$

$$= q^8 + q^6 + q^{-6} + q^{-8} - q^2 - 1 - q^{-2} \quad \text{if } g \text{ is of type } E_8.$$

Further for all $k > 0$, we have

$$(\det[A])_{\zeta^k} = \det[A]_{\zeta^k} \neq 0.$$

Proof. The calculation of the determinant is straightforward. If $g$ is of type $A_n$, then it is easy to see that for all $k > 0$,

either $\zeta^{2k} = 1$, or $\zeta^{2k(n+1)} \neq 1$.

This proves the second statement of the Lemma for $g$ of type $A_n$. The other cases are proved by a similar analysis: in the hardest case $E_8$ one checks that

$q^8 + q^6 + q^6 - q^{-6} - q^{-8} - q^2 - 1 - q^{-2}$ divides $q^{60} - 1$ in $\mathbb{A}$. The result follows.

Let $C_{\zeta}$ be the one–dimensional $\mathbb{A}$–module defined by sending $q \to \zeta$. Let $U_{\zeta}$ be the algebra over $C$ defined by,

$U_{\zeta} = U_{\mathbb{A}} \otimes_{\mathbb{A}} C_{\zeta}.$

The subalgebras $U_{\zeta}^{\pm}$ and $U_{\zeta}^{\pm}(0)$ of $U_{\zeta}$ are defined in the obvious way and we have

$U_{\zeta} = U_{\zeta}^- U_{\zeta}^0 U_{\zeta}^+.$

Given an element $u \in U_{\mathbb{A}}$, we denote by $u$ the element $u \otimes 1$ in $U_{\zeta}$. It follows from Proposition 2.7 that we have a representation $\pi_{\zeta} : U_{\zeta}(0) \to \text{End}(U_{\zeta}(0))$.

Proposition 3.1. (i) For all $i \in I$, $k \in \mathbb{Z}$, $k > 0$ there exist elements $h^{i,k} \in U_{\zeta}(0)$ such that

$$[h^{i,k}, \tilde{h}_{j,m}] = \delta_{k,-m}\delta_{i,j},$$

$$[h^{i,k}, \tilde{P}_{j,m}] = \delta_{k,m}\delta_{i,j}.$$ 

(ii) $\pi_{\zeta}$ is an irreducible representation of $U_{\zeta}(0)$.

Proof. For $k \in \mathbb{Z}$, $k > 0$, we know by Lemma 3.1 that the matrix $[A]_{\zeta^k}$ is invertible. Let $b_{ij}(k)$ denote the inverse of this matrix.

For $i \in I$, $k \in \mathbb{Z}$, $k > 0$, set

$$h^{i,k} = \sum_{j \in I} b_{ij}(k)\tilde{h}_{j,k}.$$ 

Clearly $h^{i,k}$ satisfies

$$[h^{i,k}, \tilde{h}_{j,m}] = \delta_{k,-m}\delta_{i,j}.$$ 

The second formula in (i) is now clear from Proposition 1.3.
To prove (ii), assume that $W$ is a submodule of $U_{\zeta}^-(0)$ and let $0 \neq w \in W$. By Proposition 1.3, we can choose $i \in I$, $k \in \mathbb{Z}$, $k > 0$, such that

$$w = \sum_{r=0}^{n} (\tilde{P}_{i,k})^r w_r$$

where $w_r$ is a polynomial in the elements $\tilde{P}_{j,l}$, $j \neq i$, $1 \leq l \leq k$ and $\tilde{P}_{i,l}$, $1 \leq l < k$. Applying $h_{i,k}^r$ to $w$ repeatedly we find that $w_n \in W$. Repeating the argument we find that $1 \in W$ thus proving the Proposition.

We now turn to the representations of $U_{\zeta}^-$. Given $\Lambda = \sum_{i \in \hat{I}} n_i \Lambda_i$, $n_i \geq 0$, set,

$$W_{\zeta}^-(\Lambda) = \mathcal{V}_{\Lambda} \otimes_A C_{\zeta}.$$

It follows from Proposition 2.2 that $W_{\zeta}^-(\Lambda)$ is a representation of $U_{\zeta}^-$. Again, for $v \in \mathcal{V}_{\Lambda}$, we let $v \in W_{\zeta}^-(\Lambda)$ be the element $v \otimes 1$. Clearly $U_{\zeta}^+.v = 0$ and

$$W_{\zeta}^-(\Lambda) = U_{\zeta}^- v_{\Lambda}.$$ 

Set $W_{\zeta}^-(\Lambda)_{\mu,n} = (\mathcal{V}_{\Lambda} \cap V_q(\Lambda)_{\mu,n}) \otimes_A C_{\zeta}$. Then one knows from [L3, L2] that,

$$W_{\zeta}^-(\Lambda) = \bigoplus_{\mu,n} W_{\zeta}^-(\Lambda)_{\mu,n}, \quad \dim_C W_{\zeta}^-(\Lambda) = \dim_{Q} V_q(\Lambda),$$

and $w \in W_{\zeta}^-(\Lambda)_{\mu,n}$ iff

$$K_i.v = \zeta^{\mu_i} v, \quad D.v = \zeta^{\nu} v,$$

where $\mu_i = \mu'_i + l \mu''_i$, $0 \leq \mu'_i < l$, and $\nu'$ and $\nu''$ are defined similarly.

Turning now to the level one basic representation, we see from Theorem 3 that $W_{\zeta}^-(\Lambda_0) \cong U_{\zeta}^-(0) \otimes C_{\zeta}[Q]$.

The main result of this section is:

**Theorem 4.** $W_{\zeta}^-(\Lambda_0)$ is irreducible.

**Proof.** Set $W = W_{\zeta}^-(\Lambda_0)$ and let $0 \neq W'$ be a submodule of $W$. Then $W'$ contains a non-zero vector $w \in W_{\mu,n}$ such that

$$U_{\zeta}^+.w = 0.$$ 

It is clear from Theorem 3 that $w$ must be of the form $w_\mu \otimes e^\mu$ for some $w_\mu \in U_{\zeta}^- (0)$ with

$$U_{\zeta}^+(0).w_\mu = 0.$$ 

By Proposition 3.1 we see that this forces $w_\mu = 1$ and hence that $1 \otimes e^\mu \in W'$. Proposition 2.2 now shows that $1 \otimes e^\mu$ for all $\nu \in Q$ and hence finally that $W' = W$. \qed
REFERENCES

[APW] H. H. Andersen, P. Polo and K. Wen, Representations of quantum algebras, Invent. Math. 104 (1991), 1-59.
[B] J. Beck, Braid group action and quantum affine algebras, Commun. Math. Phys. 165 (1994), 555-568.
[BCP] J. Beck, V. Chari and A. Pressley, An algebraic characterization of the affine canonical basis, Duke Math. J., to appear.
[BFJ] J. Beck, I. B. Frenkel and N. Jing, Canonical basis and Macdonald polynomials, Adv. in Math. 140 (1998), 95-127.
[Br] R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA, 83 (1986), 3068-3071.
[Bo] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 4, 5, 6, Hermann, Paris, 1968.
[CP] V. Chari and A. Pressley, Quantum affine algebras at roots of unity, Representation Theory 1 (1997), 280-328.
[Da] I. Damiani, A basis of type Poincare-Birkhoff-Witt for the quantum algebra of $\hat{sl}(2)$, J. Algebra 161 (1993), 291-310.
[Dr] V. G. Drinfeld, A new realization of Yangians and quantum affine algebras. Soviet Math. Dokl. 36 (1988), 212-216.
[FJ] I. B. Frenkel and N. Jing, Vertex representations of quantum affine algebras, Proc. Natl. Acad. Sci. USA 85 (1988), 9373-9377.
[FK] I. B. Frenkel and V. G. Kac, Basic representations of affine algebras and dual resonance models, Invent. Math. 62 (1980), 23-66.
[J1] N. Jing, On Drinfeld realization of quantum affine algebras. The Monster and Lie algebras (Columbus, OH, 1996), pp. 195-206, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, 1998.
[J2] N. Jing, Symmetric polynomials and $U_q(\hat{sl}_2)$, math.QA/9902109.
[L1] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, Adv. in Math. 70 (1988), 237-249.
[L2] G. Lusztig, Quantum groups at roots of 1, Geom. Dedicata, 35 (1990), 89-113.
[L3] G. Lusztig, Introduction to quantum groups, Progress in Mathematics 110, Birkhäuser, Boston, 1993.
[M] I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd ed., Oxford University Press, New York, 1995.
[S] G. Segal, Unitary representations of some infinite-dimensional groups, Commun. Math. Phys. 80 (1981) 301-342.

VYJAYANTHI CHARI, UNIVERSITY OF CALIFORNIA, RIVERSIDE
E-mail address: chari@math.ucr.edu

NAIHUAN JING, NORTH CAROLINA STATE UNIVERSITY, RALEIGH
E-mail address: jing@math.ncsu.edu