Symmetries and Hamiltonian formalism
for complex materials

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Abstract
Preliminary results toward the analysis of the Hamiltonian structure
of multifield theories describing complex materials are mustered: we in-
volve the invariance under the action of a general Lie group of the balance
of substructural interactions. Poisson brackets are also introduced in the
material representation to account for general material substructures. A
Hamilton-Jacobi equation suitable for multifield models is presented. Fi-
nally, a spatial version of all these topics is discussed without making use
of the notion of paragon setting.

1 Lagrangian and Hamiltonian descriptions of
elastic bodies with substructure

In standard continuum mechanics, each material element of a body is “col-
lapsed” into the place occupied by its centre of mass; let $X$ be that place in
the reference placement; the set of all $X$ is taken to be a fit region $B_0$ of the
Euclidean space.

Sometimes such simplistic model of physical reality is insufficient; then, to
render the picture adequate, the material element must be portrayed as a system
and at least some coarse grained descriptor $\nu$ (an order parameter) enters the
picture.

Here, as in [1] (see for other details and additional results [2-5]), we take
$\nu$ as an element of some differentiable manifold $M$, and presume that physical
circumstances impose a single choice of metric and of connection for $M$. 
We also assume that the region occupied by the body in the current placement be obtained through a sufficiently smooth mapping \( x : \mathcal{B}_0 \rightarrow \mathcal{E} \); so that the current place of a material element at \( X \) in \( \mathcal{B}_0 \) is given by \( x(X) \); and \( \mathcal{B} = x(\mathcal{B}_0) \) is also fit; we denote as usual with \( F \) the placement gradient. We presume also that another sufficiently smooth mapping \( \nu : \mathcal{B}_0 \rightarrow \mathcal{M} \) shows the present value of the order parameter at \( X \). A motion is a pair of time-parametrized families \( x_t(X) = x(X, t) \) and \( \nu_t(X) = \nu(X, t) \), twice differentiable with respect to time. Rates in the material representation are indicated with \( \dot{x} \) and \( \dot{\nu} \).

We restrict here our attention to bodies for which a Lagrangian density \( L \) exists, so that the total Lagrangian \( L \) of the body is given by

\[
L_{\mathcal{B}_0} = \int_{\mathcal{B}_0} \mathcal{L}(X, x, \dot{x}, F, \nu, \dot{\nu}, \nabla \nu).
\]

the covariant gradient \( \nabla \nu \) being based on the mandatory connection. We presume that \( \mathcal{L} \) be of the form

\[
\mathcal{L}(X, x, \dot{x}, F, \nu, \dot{\nu}, \nabla \nu) = \frac{1}{2} \rho_0 \| \dot{x} \|^2 + \rho_0 \chi(\nu, \dot{\nu}) - \rho_0 e(X, F, \nu, \nabla \nu) - \rho_0 w(x, \nu),
\]

where \( \rho_0 \) is the referential mass density (conserved during the motion), \( \chi \) the kinetic co-energy (see [1], p. 19) associated with the substructure, \( e \) the elastic energy density and \( w \) the density of the potential of external actions, all per unit mass. Below we use the notation \( b = -\partial_x w \) for the density standard external actions and \( \beta = -\partial_\nu w \) for the substructural ones. The kinetic energy density \( \rho_0 \kappa(\nu, \dot{\nu}) \) pertaining to the substructure is the partial Legendre transform of \( \chi \) with respect to \( \dot{\nu} \).

If \( \mathcal{L} \) is sufficiently smooth, we may apply standard procedures to derive Lagrange equations for the functional \( L_{\mathcal{B}_0} \):

\[
\overline{\partial_x \mathcal{L}} = \partial_x \mathcal{L} - \text{Div} \partial_F \mathcal{L},
\]

\[
\overline{\partial_\nu \mathcal{L}} = \partial_\nu \mathcal{L} - \text{Div} \partial_\nabla \mathcal{L}.
\]

where \( \text{Div} \) is the divergence calculated with respect to \( X \), i.e. \( \text{Div} = \text{tr} \nabla \).

Put

\[
\delta \mathcal{H} = \dot{x} \cdot \partial_x \mathcal{L} + \dot{\nu} \cdot \partial_\nu \mathcal{L} - \mathcal{L}.
\]

Clearly, \( \delta \mathcal{H} \) is the density of the total energy. In fact, since \( \partial_\nu \mathcal{L} = \rho_0 \partial_\nu \chi \), the term \( \dot{\nu} \cdot \partial_\nu \chi - \chi \) in \( \delta \mathcal{H} \) coincides with the substructural kinetic energy density \( \kappa(\nu, \dot{\nu}) \) (hence the presence of \( \chi \) rather than \( \kappa \) in the expression of \( \mathcal{L} \)), then

\[
\delta \mathcal{H} = \frac{1}{2} \rho_0 \| \dot{x} \|^2 + \rho_0 \kappa(\nu, \dot{\nu}) + \rho_0 e(X, F, \nu, \nabla \nu) + \rho_0 w(x, \nu),
\]

as asserted.
The balance of energy can be expressed in terms of $\mathcal{H}$ as follows

$$\dot{\mathcal{H}} - \text{Div} (\dot{x} \mathbf{P} + \dot{\nu} \mathbf{S}) = 0,$$

(7)

where $\mathbf{P}$ and $\mathbf{S}$ are respectively the Piola-Kirchhoff stress and the referential microstress

$$\mathbf{P} = -\partial_{\xi} \mathcal{L}, \quad \mathbf{S} = -\partial_{\mathbf{v}} \mathcal{L}.$$  

(8)

That (7) is true follows from direct computation.

Equations (8) and (9) lead us to an appropriate version of Noether theorem (see [4] p. 29); here we follow the program of [6] (p. 284). We consider some virtual motion of our system, by assigning two one-parameter families $f_i^1$ of sufficiently smooth point valued diffeomorphisms, $i = 1, 2$, acting respectively on $B_0$ and $\mathcal{E}$, and a Lie group $G$ of transformations of $\mathcal{M}$. We indicate with a prime the derivative with respect to the relevant $s$.

1. At each $s_1$, $f_i^1$ acts on $B_0$ so that $X \mapsto f_i^1(X) \in \mathcal{E}$, and is isocoric (no virtual change of density), i.e. $\text{Div} f_i^1 = 0$; $f_0^1$ is the identity. We put $f_i^1(X) = w(X)$, then $w = 0$.

2. At each $s_2$, $f_2^2$ is a diffeomorphism that transforms $\mathcal{E}$ into itself. We assume that $f_0^2$ is the identity and put $f_0^2(x) = v(x)$.

3. A Lie group $G$, containing $SO(3)$, acts on $\mathcal{M}$ and the infinitesimal generator of this action at $\nu$ is indicated with $\xi_\mathcal{M}(\nu)$ (see [7], p. 256); $\nu_0$ is the value of $\nu$ after the action of $g \in G$. If we consider a one-parameter trajectory $s_3 \mapsto g_{s_3} \in G$ such that $g_0$ is the identity, we have also $s_3 \mapsto \nu_{s_3}$ and $\xi_\mathcal{M}(\nu) = \frac{d\nu}{ds_3}|_{s_3=0}$. When $G$ coincides with the special orthogonal group $SO(3)$, we identify $\xi_\mathcal{M}(\nu)$ with $A\mathbf{q}$, being $\mathbf{q}$ the characteristic vector of a rotational rigid velocity and $A$ a linear operator mapping vectors into elements of the tangent space of $\mathcal{M}$, namely, if $\nu_\mathbf{q}$ is the value of the order parameter measured by an observer after a rotation $\mathbf{q}$, then $A = \frac{d\nu}{d\mathbf{q}}|_{\mathbf{q}=0}$.

Henceforth, to simplify notations, we use $f_1$, $f_2$ and $\nu_0$ to indicate $f_i^1(X)$, $f_2^2(x)$, $\nu_{s_3}(X)$, and write $|0$ for $|s_1=0, s_2=0, s_3=0$. Moreover, grad indicates the gradient with respect to $x$.

We say that $\mathcal{L}$ is invariant with respect to $f_i$'s and $G$ when

$$\mathcal{L}(X, x, \dot{x}, F, \nu, \dot{\nu}, \nabla \nu) =$$

$$= \mathcal{L} \left( f_1, f_2, (\text{grad} f_2^2) \dot{x}, (\text{grad} f_2^2) F (\nabla f_2^2)^{-1}, \nu_0, \dot{\nu}_0, \nabla \nu_0, (\nabla f_1^1)^{-1} \right).$$

(9)

Let us define

$$Q = \partial_{\xi} \mathcal{L} \cdot (v - Fw) + \partial_{\mathbf{v}} \mathcal{L} \cdot (\xi_\mathcal{M}(\nu) - (\nabla \nu) w)$$

(10)
\[ \mathcal{F} = \mathcal{L} w + (\partial_{\mathbf{F}} \mathcal{L})^T (\mathbf{v} - \mathbf{F} w) + (\partial_{\nabla \mathbf{v}} \mathcal{L})^T (\xi_M (\nu) - (\nabla \nu) w), \]  

where \( \mathbf{v}, \mathbf{w} \) and \( \xi_M (\nu) \) are as mentioned in items 1, 2, 3.

**Theorem 1** (Nöther-like theorem for complex materials). If the Lagrangian density \( \mathcal{L} \) is invariant under \( f^1, f^2 \) and \( G \), then

\[ \dot{Q} + \text{Div} \mathcal{F} = 0. \]  

**Proof.** To prove the theorem, as a first step we note that (9) implies

\[ \frac{d}{ds} \mathcal{L} \left( f^1, f^2, (\text{grad} f^2) \dot{x}, (\text{grad} f^2) \mathbf{F} (\nabla f^1)^{-1}, \nu, \dot{\nu}, (\nabla \nu) (\nabla f^1)^{-1} \right) |_0 = 0, \]  

which lead to

\[ \frac{d}{ds} \mathcal{L} \left( f^1, f^2, (\text{grad} f^2) \dot{x}, (\text{grad} f^2) \mathbf{F} (\nabla f^1)^{-1}, \nu, \dot{\nu}, (\nabla \nu) (\nabla f^1)^{-1} \right) |_0 = 0, \]  

as a consequence of the properties listed under items 1, 2 and 3 above. Then, we calculate the time rate of the scalar \( Q \), the divergence of the vector \( \mathbf{F} \) and, by using the equations (3) and (4), identifying \( s_3 \) with \( t \), we recognize that

\[ \dot{Q} + \text{Div} \mathcal{F} = \frac{d}{ds_1} \mathcal{L} |_0 + \frac{d}{ds_2} \mathcal{L} |_0 + \frac{d}{ds_3} \mathcal{L} |_0, \]  

which proves the theorem. □

**Remark 1.** As a first special case, we require that \( f^2 \) alone acts on \( \mathcal{L} \) leaving \( \mathbf{v} \) arbitrary. By using (17) we obtain from (12)

\[ \left( \frac{\partial}{\partial t} \partial_{\dot{x}} \mathcal{L} - \partial_{\mathbf{x}} \mathcal{L} + \text{Div} \partial_{\mathbf{F}} \mathcal{L} \right) \cdot \mathbf{v} = 0, \]  

i.e.

\[ \rho_0 \ddot{x} = \rho_0 \mathbf{b} + \text{Div} \mathbf{P}, \]  

4
which is the standard equation of balance of momentum.

**Remark 2.** With $G$ arbitrary, we consider its action alone on $\mathcal{L}$; by using (18), with the identification $s_3 = t$, we obtain from (12) that

$$
\left( \frac{\partial}{\partial t} \partial_\nu \mathcal{L} - \partial_\nu \mathcal{L} + \text{Div} \partial_\nu \mathcal{L} \right) \cdot \xi_{\mathcal{M}} (\nu) = 0
$$

or

$$\rho_0 \left( \frac{\partial}{\partial t} \partial_\nu \chi - \partial_\nu \chi \right) + z - \rho_0 \beta - \text{Div} S = 0.
$$

$z = -\rho_0 \partial_\nu e$ is called *self-force* in the terminology of [1]. This result assures the covariance of the balance of substructural interactions. When $G$ coincides with $SO(3)$, the co-vector into parentheses in (22), namely the term multiplying $\xi_{\mathcal{M}} (\nu)$, must be an element of the null space of $A^T$ (see for details of this special case [1-4]).

**Remark 3.** As a second special choice, let $f^2$ be such that $v = \dot{q} \times (x - x_0)$ (with $\dot{q}$ a rigid rotational velocity $x_0$ a fixed point in space) and $G = SO(3)$; $\dot{q} \times$ is an element of its Lie algebra, thus $\xi_{\mathcal{M}} (\nu) = A\dot{q}$. If $\mathcal{L}$ is independent of $x$ and we assume that only $f^2$ and $G$ (in the form just defined) act on $\mathcal{L}$, we have

$$skw \left( F^T \partial_\nu F + \left( \nabla A \right)^T \partial_\nu L \right) = 0.
$$

where $e$ is Ricci’s alternating tensor and $skw (\cdot)$ extracts the skew-symmetric part of its argument.

**Remark 4.** If we require that $f^1$ alone acts on $\mathcal{L}$, with $w$ arbitrary (but satisfying 1), by using (16) we obtain from (12) that

$$
\left( F^T \partial_\nu F + \nabla \nu^T \dot{\nu} \right) - \text{Div} \left( \frac{1}{2} \rho_0 \| \dot{\nu} \|^2 + \chi (\nu, \dot{\nu}) \right) I - \partial_\nu \mathcal{L} = 0.
$$

where $\mathcal{P} = eI - F^T \mathcal{P} - \nabla \nu^T \dot{\nu} S$ is the modified Eshelby tensor for continua with substructure (see [4] for a similar result in a non-conservative setting, where the elastic potential $e$ is substituted by the free energy). $I$ is the second-order unit tensor and the product $\overline{z}$ is defined by $(\nabla \nu^T \overline{z} S) n \cdot u = S n \cdot (\nabla \nu) u$ for any pair of vectors $n$ and $u$.

**Remark 5.** Let us assume as special choices that $f^1$ is such that $w = \dot{q} \times (X - X_0)$ (with $\dot{q}$ a rigid rotational velocity $X_0$ a fixed point in space) and that $G = SO(3)$, being $\dot{q} \times$ an element of its Lie algebra, thus $\xi_{\mathcal{M}} (\nu) = A\dot{q}$. If the material is homogeneous, and we assume that $f^1$ and $G$ alone (in the form just defined) act on $\mathcal{L}$, we have

$$skw \left( F^T \partial_\nu F + \nabla \nu^T \dot{\nu} \right) = 0.
$$

**Remark 6.** The action of $f^1$ can be interpreted as a special virtual mutation of a possibly existing smooth distribution of inhomogeneities throughout the body, in the sense of [8]. In other words, we may say that (25) is the balance of interactions arising when the body mutates its inhomogeneous structure. This interpretation has been also suggested in [9] in non-conservative setting.
1.1 Hamilton equations

Define \( p \) and \( \mu \), respectively the canonical momentum and the canonical substructural momentum, by

\[
p = \partial_x \mathcal{L}, \quad \mu = \partial_\nu \mathcal{L}.
\]

The Hamiltonian density \( \mathcal{H} \),

\[
\mathcal{H}(\mathbf{X}, \mathbf{x}, \mathbf{p}, \mathbf{F}, \mathbf{\nu}, \mathbf{\mu}, \nabla \nu) = p \cdot \dot{x} + \mu \cdot \dot{\nu} - \mathcal{L}(\mathbf{X}, \mathbf{x}, \dot{x}, \mathbf{F}, \mathbf{\nu}, \dot{\nu}, \nabla \nu).
\]

has partial derivatives with respect to its entries; some of them are the opposite of the corresponding derivatives of \( \mathcal{L} \) so that (3), (4) can be also written respectively as

\[
\begin{align*}
\dot{p} &= -\partial_x \mathcal{H} + \text{Div} \partial_\mathbf{F} \mathcal{H}, \\
\dot{x} &= \partial_p \mathcal{H};
\end{align*}
\]

\[
\begin{align*}
\dot{\mu} &= -\partial_\nu \mathcal{H} + \text{Div} \partial_\mathbf{\nu} \mathcal{H}, \\
\dot{\nu} &= \partial_\mu \mathcal{H}.
\end{align*}
\]

2 Canonical Poisson brackets in multifield theories

We now consider a general boundary value problem where the following boundary conditions are associated with (28) and (29)

\[
\begin{align*}
\mathbf{x}(\mathbf{X}) &= \bar{x} \quad \text{on } \partial^{(1)} \mathcal{B}_0, \\
\partial_\mathbf{F} \mathcal{H} \mathbf{n} &= \mathbf{t} \quad \text{on } \partial^{(2)} \mathcal{B}_0, \\
\mathbf{\nu}(\mathbf{X}) &= \bar{\nu} \quad \text{on } \partial^{(1)} \mathcal{B}_0, \\
\partial_\mathbf{\nu} \mathcal{H} \mathbf{n} &= \mathbf{t} \quad \text{on } \partial^{(2)} \mathcal{B}_0;
\end{align*}
\]

\( \bar{x}, \mathbf{t}, \bar{\nu} \) and \( \mathbf{t} \) are prescribed on the relevant parts of the boundary and \( \text{Cl}(\partial \mathcal{B}_0) = \text{Cl}(\partial^{(1)} \mathcal{B}_0 \cup \partial^{(2)} \mathcal{B}_0) \), with \( \partial^{(1)} \mathcal{B}_0 \cap \partial^{(2)} \mathcal{B}_0 = \emptyset \), where \( \text{Cl} \) indicates closure and \( \mathbf{n} \) is the outward unit normal to \( \partial \mathcal{B}_0 \) at all points in which it is well defined.

We assume that there exist two surface densities \( \bar{U}(\mathbf{x}) \) and \( U(\mathbf{\nu}) \) such that

\[
\begin{align*}
t &= \rho_0 \partial_x \bar{U}, \quad \mathbf{t} = \rho_0 \partial_\nu U,
\end{align*}
\]

where \( \bar{U} \) and \( U \) plays here the rôle of surface potentials.
Then the Hamiltonian $H$ of the whole body is given by

$$H(X, x, p, \nu, \mu) = \int_{B_0} \mathcal{H}(X, x, p, \nu, \mu) - \int_{\partial(2)B_0} (\bar{U}(x) - U(\nu)).$$

(35)

Notice that we write $\mathcal{H}(X, x, p, \nu, \mu)$ instead of $\mathcal{H}(X, x, p, F, \nu, \mu, \nabla \nu)$ because below we consider directly variational derivatives.

**Theorem 2.** The canonical Hamilton equation

$$\dot{F} = \{F, H\}$$

(36)

is equivalent to the Hamiltonian system of balance equations (28)-(29) for a continuum with substructure where $F$ is any functional of the type $\int_{B_0} f(X, x, p, \nu, \mu)$, with $f$ a sufficiently smooth scalar density, and the Poisson bracket $\{\cdot, \cdot\}$ for a complex material is given by

$$\{F, H\} = \int_{B_0} \left( \frac{\delta f}{\delta x} \cdot \frac{\partial \mathcal{H}}{\partial p} - \frac{\delta f}{\delta p} \cdot \frac{\partial \mathcal{H}}{\partial x} \right) +$$

$$+ \int_{\partial(2)B_0} \left( \frac{\delta f}{\delta x} \cdot \frac{\partial \mathcal{H}}{\partial p} |_{\partial(2)B_0} - \frac{\delta f}{\delta p} \cdot \frac{\partial \mathcal{H}}{\partial x} |_{\partial(2)B_0} \right) +$$

$$+ \int_{B_0} \left( \frac{\delta f}{\delta \nu} \cdot \frac{\partial \mathcal{H}}{\partial \mu} - \frac{\delta f}{\delta \mu} \cdot \frac{\partial \mathcal{H}}{\partial \nu} \right),$$

(37)

where the variational derivative $\frac{\delta H}{\delta x}$ is obtained fixing $p$ and allowing $x$ to vary$^1$; an analogous meaning is valid for the variational derivative with respect to the order parameter.

The proof can be developed by direct calculation. Clearly, $\{\cdot, \cdot\}$ is bilinear and skew-symmetric, and one can check easily that it satisfies the Jacobi’s identity. We note that

$$\{F, H\} = \int_{B_0} \left( \frac{\delta f}{\delta x} \cdot \frac{\partial \mathcal{H}}{\partial p} - \frac{\delta f}{\delta p} \cdot \left( \partial_x \mathcal{H} - \text{Div} \bar{V} \mathcal{H} \right) \right) +$$

$$+ \int_{\partial(2)B_0} \left( \frac{\delta f}{\delta x} \cdot \frac{\partial \mathcal{H}}{\partial p} |_{\partial(2)B_0} - \frac{\delta f}{\delta p} \cdot \left( \partial_x \bar{U} - \partial_x \mathcal{H} \right) |_{\partial(2)B_0} \right) +$$

$$+ \int_{B_0} \left( \frac{\delta f}{\delta \nu} \cdot \frac{\partial \mathcal{H}}{\partial \mu} - \frac{\delta f}{\delta \mu} \cdot \left( \partial_\nu \mathcal{H} - \text{Div} \nabla \mathcal{H} \right) \right) +$$

$$+ \int_{\partial(2)B_0} \left( \frac{\delta f}{\delta \nu} \cdot \frac{\partial \mathcal{H}}{\partial \mu} |_{\partial(2)B_0} - \frac{\delta f}{\delta \mu} \cdot \left( \partial_\nu U - \partial_\nu \mathcal{H} \right) |_{\partial(2)B_0} \right) \right) \right),$$

(38)

$^1$See relevant remarks in [10].
and, in terms of functional partial derivatives,

\[
\dot{F} = \int_{B_0} \left( \frac{\delta f}{\delta x} \dot{x} + \frac{\delta f}{\delta p} \dot{p} + \frac{\delta f}{\delta \mu} \dot{\mu} \right) + \int_{\partial^{(2)} B_0} \frac{\delta f}{\delta x} \dot{x} |_{\partial^{(2)} B_0} + \int_{\partial^{(2)} B_0} \frac{\delta f}{\delta p} \dot{p} |_{\partial^{(2)} B_0}.
\]  

(39)

By identifying analogous terms in (38) and (39), we obtain both the Hamiltonian system (28)-(29) and the boundary conditions (30)-(33).

When we put \( F = H \), (36) coincides with the equation of conservation of energy. We have, in fact,

\[
\dot{H} = \{ H, H \} = 0.
\]

(40)

Geometrical properties of the Poisson brackets for direct models of rods, plates and complex fluids have been discussed in [10], [11].

3 A formal approach toward an Hamilton-Jacoby theory with gradient effects

Let \( h \) be a smooth diffeomorphism

\[
h : (X, x, p, F, \nu, \mu, \nabla \nu) \mapsto (X, x', \dot{x'}, p, \nu, \mu, \nabla \nu).
\]

(41)

The transformation \( h \) generates a new Hamiltonian density

\[
\mathcal{H}_s (X, x, p_s, F_s, \nu_s, \mu_s, \nabla \nu_s),
\]

(42)

with corresponding Lagrangian density

\[
\mathcal{L}_s = p_s \cdot \dot{x}_s + \mu_s \cdot \dot{\nu}_s - \mathcal{H}_s.
\]

(43)

If \( h \) were such that \( \mathcal{H}_s = 0 \), then an immediate integration of the system (28), (29) could be achieved. To this aim we choose \( h \) to be such that the integral of the difference \( \mathcal{L} - \mathcal{L}_s \) between two instants, say \( t_1 \) and \( t_2 \), be equal to the time derivative of a generating function \( S \) of the type \( S = S(t, X, x, p_s, \nu, \mu_s) \), i.e.

\[
\int_{t_1}^{t_2} \mathcal{L} - \mathcal{L}_s = S |_{t = t_2} - S |_{t = t_1}.
\]

(44)

Then, from (41) we would have

\[
(p \cdot \dot{x} + \mu \cdot \dot{\nu} - \mathcal{H}) - (p_s \cdot \dot{x}_s + \mu_s \cdot \dot{\nu}_s - \mathcal{H}_s) =
\]

\[
= \dot{S} = \partial_t S + \partial_x S \cdot \dot{x} + \partial_p S \cdot \dot{p} + \partial_{\nu} S \cdot \dot{\nu} + \partial_{\mu} S \cdot \dot{\mu},
\]

(45)

and hence

\[
p = \partial_x S, \quad \mu = \partial_{\nu} S.
\]

(46)
\[ x_* - x_0 = \partial p_* S, \quad \nu = \partial \mu_* S, \quad \text{(47)} \]

\[ \partial_t S + \mathcal{H} = \mathcal{H}_*, \quad \text{(48)} \]

To obtain (47) one makes use of the fact that

\[ \delta \int_{t_1}^{t_2} \left( p \cdot (x - x_0) + \mu \cdot \nu \right) dt = 0 \]

for variations vanishing at \( t_1 \) and \( t_2 \) (in the sense that \( \delta (\mu \cdot \nu) \mid_{t_1}^{t_2} = 0 \) and \( \delta (p \cdot (x - x_0)) \mid_{t_1}^{t_2} = 0 \) so that \( p \cdot \dot{x} = \dot{p} \cdot (x - x_0) \) and \( \mu \cdot \dot{\nu} = \dot{\mu} \cdot \nu \).

A necessary and sufficient condition to assure that \( \mathcal{H}_* = 0 \) is

\[ \partial_t S + \mathcal{H} (X, x, \partial_x S, F, \nu, \partial_\nu S, \nabla \nu) = 0, \quad \text{(49)} \]

which is a Hamiltonian-Jacobi like equation. Since \( \mathcal{H}_* = 0, \ p_* \) and \( \mu_* \) are constant in time, the time derivative of \( S \) reduces to

\[ \dot{S} = \partial_t S + \partial_x S \cdot \dot{x} + \partial_\nu S \cdot \dot{\nu} = -\mathcal{H} + p \cdot \dot{x} + \mu \cdot \dot{\nu} = \mathcal{L}. \quad \text{(50)} \]

The relation (50) allows us to determine \( S \) to within a constant, namely

\[ S = \int \mathcal{L} dt + \text{const.} \quad \text{(51)} \]

4 The spatial form

Circumstances in which the notion of reference placement is wanting, as in the case of fluids or granular flows, render the choice of a material or spatial representation not matter of form only (see, e.g., [12] for standard bodies). Here, having in mind the study of complex fluids, we provide a spatial variational derivation of the balance equations free of any concept of reference place or paragon setting and without even formal recourse to an inverse motion. So, in the present section \( x \in B \) is just a point in space. The notation \( u(x,t) \) is used for the velocity field over \( B \). The order parameter is now \( \nu(x,t) \) and we indicate with \( \upsilon(x,t) \) its actual rate. The symmetric tensor \( g \) is the spatial metric characterizing the present state of the body; it plays a prominent rôle because in this case the counterpart of (2) of the Lagrangian density is of the form

\[ \mathcal{L} (x, u, g, \nu, \upsilon, \nu, \text{grad} \nu) = \frac{1}{2} \rho \| \upsilon \|^2 + \rho \chi(\nu, \upsilon) - \rho e (g, \nu, \text{grad} \nu) - \rho w (x, \nu), \quad \text{(52)} \]

with some slight abuse of notation. We then find balance equations as conditions verifying the relation

\[ \delta \left( \int_0^t \int_B \mathcal{L} (x, u, g, \nu, \upsilon, \nu, \text{grad} \nu) \right) = 0, \quad \text{(53)} \]

where \( \dot{\delta} \) denotes the total variation.
To define the variation of the relevant fields, we make use of $f^2$ introduced at point 2 of Section 1 and identify $\delta x$ with $v$. We consider a special (though wide) subclass of possible vector fields $x \mapsto v(x)$ characterized by the circumstance that they are purely deformative; in other words, we choose $v$ such that $skewgradv = 0$.

We then define
\[
\hat{\delta} g = \frac{d}{ds} f^{2*} g \mid_{s=0} = L_v g = 2symgradv = 2gradv, \tag{54}
\]
where $f^{2*}$ means pull back and $L_v$ is thus the autonomous Lie derivative following the flow $v$. In analogous way, we put
\[
\hat{\delta} \nu = \delta \nu + (grad \nu)v, \tag{55}
\]
\[
gradv\hat{\delta} \nu = gradv\delta \nu + (grad \nu) gradv. \tag{56}
\]

As an intermediate step we notice that
\[
\hat{\delta} \int_B e(g, \nu, grad \nu) = \int_B \hat{\delta} e = \int_B \left(2\partial_\nu e \cdot grad \nu + \partial_\nu \cdot \hat{\delta} v + \partial_{grad \nu} e \cdot (grad \delta \nu + (grad \nu) gradv)\right). \tag{57}
\]

By developing the variation of $\delta e$, making use of $\delta \delta e$ and Gauss theorem, we recognize that appropriate balances in the bulk are
\[
\overline{\partial_a L} - \partial_x L + div\left(2\partial_\nu L - (grad \nu)^T \partial_{grad \nu} L\right) = 0, \tag{58}
\]
\[
\overline{\partial_v L} - \partial_\nu L + div(\partial_{grad \nu} L) = 0. \tag{59}
\]
Cauchy stress $T$ is then given by
\[
T = -2\partial_\nu L - (grad \nu)^T S_a, \tag{60}
\]
where the actual microstress $S_a$ is defined by
\[
S_a = -\partial_{grad \nu} L. \tag{61}
\]

In the case of simple bodies, (60) reduces to the well known Doyle-Ericksen formula.

**Remark 7.** A requirement of invariance of $e$ under the action of $SO(3)$ implies that
\[
skew(2\partial_\nu L) = e\left(\mathcal{A}^T z_a + (grad\mathcal{A}^T) S_a\right), \tag{62}
\]
where $z_a = -\rho \partial_\nu e$ is the actual self-force and $e$ Ricci’s alternating tensor.
4.1 Spatial Hamilton equations

To find appropriate spatial Hamilton equations, we follow the pattern of Section 1.1. To this end we define spatial canonical standard and substructural momenta ($\bar{p}$ and $\bar{\mu}$ respectively) through

$$\bar{p} = \partial_u L, \quad \bar{\mu} = \partial_\nu L. \quad (63)$$

Consequently, the spatial Hamiltonian density is given by

$$H(x, \bar{p}, g, \nu, \bar{\mu}, \text{grad}\nu) = \bar{p} \cdot u - \bar{\mu} \cdot v - L(x, u, g, \nu, u, \text{grad}\nu), \quad (64)$$

(with some slight abuse of notation) and has partial derivatives with respect to its entries. By evaluating the variation of $H$, taking into account (54) and (56), and comparing the result with the variation of $L$, after making use of the balances (58) and (59), we obtain the spatial form of the Hamilton equations:

$$\begin{align*}
\dot{\bar{p}} &= -\partial_x H + \text{div} \left( 2\partial_g H - (\text{grad}\nu)^T \partial_{\text{grad}\nu} H \right), \\
\dot{u} &= \partial_p H; \quad (65)
\end{align*}$$

$$\begin{align*}
\dot{\bar{\mu}} &= -\partial_\nu H + \text{div} (\partial_{\text{grad}\nu} H), \\
\dot{v} &= \partial_\mu H. \quad (66)
\end{align*}$$

4.2 Spatial Hamilton-Jacobi form

We may obtain the spatial counterpart of (49) by considering a smooth diffeomorphism

$$\bar{h} : (x, \bar{p}, g, \nu, \bar{\mu}, \text{grad}\nu) \mapsto (x^*, \bar{p}^*, g^*, \nu^*, \bar{\mu}^*, \text{grad}\nu^*), \quad (67)$$

which generates a new Hamiltonian density

$$\mathcal{H}^* (x^*, \bar{p}^*, g^*, \nu^*, \bar{\mu}^*, \text{grad}\nu^*) \cdot \quad (68)$$

Now, we may use a generating function $S = S(t, x, \bar{p}^*, \nu^*, \bar{\mu}^*)$, and, following the same procedure of Section 3, we find that a necessary and sufficient condition to assure that $\mathcal{H}^* = 0$ is

$$\partial_t S + \mathcal{H} (x, \partial_x S, g, \nu, \partial_\nu S, \text{grad}\nu) = 0. \quad (69)$$

4.3 A spatial form of Poisson brackets

For the spatial Hamiltonian in equations (65)-(66), taking into account (54)-(56), we define a new variational derivative $\frac{\partial H}{\partial x}$ through the relation

$$\frac{\partial H}{\partial x} (x, \bar{p}, \nu, \bar{\mu}) \cdot v = \left( -\partial_x H + \text{div} \left( 2\partial_g H - (\text{grad}\nu)^T \partial_{\text{grad}\nu} H \right) \right) \cdot v, \quad (70)$$
holding \( p \) fixed and allowing \( x \) to vary, for any \( v \) of the kind used in (54)-(56).

Consider a boundary value problem of the type

\[
\left( 2\partial_g \mathcal{H} - (\text{grad} \nu)^T \partial_{\text{grad} \nu} \mathcal{H} \right) n = \partial_x \bar{u} (x), \quad (\partial_{\text{grad} \nu} \mathcal{H}) n = \partial_{\nu} u (\nu), \quad \text{on } \partial B,
\]

(71)

(where \( \bar{u} (x) \) and \( u (\nu) \) are the counterparts of the surface potentials \( \bar{U} (x) \) and \( U (\nu) \)).

The total Hamiltonian is now given by \( \mathcal{H} (x, \bar{p}, \nu, \bar{\mu}) = \int_B \mathcal{H} \) (with some slight abuse of notation) and we list only the entries \( (x, \bar{p}, \nu, \bar{\mu}) \) because we consider the variational derivative (65) below. We consider also arbitrary functionals \( F \) of the type \( \int_B f (x, \bar{p}, \nu, \bar{\mu}) \), with \( f \) a sufficiently smooth scalar density.

Theorem 3. The canonical Hamilton equation

\[
\dot{F} = \{F, H\}_a
\]

is equivalent to the Hamiltonian system of balance equations (65)-(66) with

\[
\{\cdot, \cdot\}_a \text{ is bilinear, skew-symmetric and satisfies Jacobi’s identity.}
\]

5 Final remarks

To illustrate possible uses of Theorem 2, we list below some special cases. Analogous results accrue from Theorem 3.

Remark 8. If we choose \( f = p \cdot v \), with \( v \) an arbitrary vector, equation (28) and the boundary condition (31) follow immediately from (36).

Remark 9. Let \( f = \mu \cdot \xi_M (\nu) \), then from (36) we get (29) and the boundary condition (33).

Remark 10. Let \( f \) be of the form

\[
f = p \cdot (\dot{q} \times (x - x_0)) + \mu \cdot \dot{A} \dot{q},
\]

(74)

with \( \dot{q} \) arbitrary as in previous sections. Consider also, for the sake of simplicity, absence of external bulk interactions (the ones accounted for \( w (x, \nu) \)). By using (28) and (29), we obtain from (36)

\[
e (\partial_F \mathcal{H} \mathcal{F}^T) = \mathcal{A}^T \partial_{\nu} \mathcal{H} + (\nabla \mathcal{A}^T)^T \partial_{\nu} \mathcal{H}.
\]

(75)
These remarks are the Hamiltonian counterparts of Remarks 1, 2, 3. Of course, Poisson parentheses not only allow one to write in a concise form balance equations, but generate articulated geometric structures over the infinite-dimensional manifold of mappings showing placements and order parameters, and properties of these structures depend also on the geometric properties of $\mathcal{M}$.

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