On the Asserted Clash
between the Freud and the Bianchi Identities

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Summary.- Through a constructive method it is shown that the claim advanced in recent times about a clash that should occur between the Freud and the Bianchi identities in Einstein’s general theory of relativity is based on a faulty argument.

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In an article published in this Journal [1] it is asserted that in Einstein’s general theory of relativity the contracted Bianchi identity and the Freud identity [2] “clash and lead to a mathematical overdetermination which creates insurmountable internal difficulties for the curved-space-time theory of gravitation as a whole”. The present paper retries the steps of the argument that led to the quoted assertion, and shows that some of them are either wrong or not proven.

We shall deal with tensorial entities as well as with pseudo-tensorial ones, i.e. with geometric objects that transform tensorially only under the group of the affine transformations, and eventually with pseudo-tensorial entities that vanish everywhere in some coordinate system. The latter objects are called non-tensorial in ref. [1]. The same notation will be used both for the tensorial and for the pseudo-tensorial entities, in particular boldface letters will be used to denote both tensor and pseudo-tensor densities.

Let \( g_{ik} \) be a pseudo-Riemannian metric tensor; \( g^{ik} = (-g)^{1/2} g_{ik} \) is the associated contravariant tensor density, and \( \Gamma^i_{km} \) represents the Christoffel symbol built with \( g_{ik} \). The Ricci tensor \( R_{ik}(\Gamma) \), the scalar density \( R = g^{ik} R_{ik} \) and the stress-energy-momentum tensor \( T_{ik} \) are then defined, in keeping with the conventions chosen by Schrödinger [3]. We contemplate the pseudo-scalar density

\[
L = g^{ik} (\Gamma^b_{ak} \Gamma^a_{ib} - \Gamma^b_{ab} \Gamma^a_{ik});
\]

one can write [3]

\[
T_{km} = \frac{\delta R}{\delta g_{km}} = -\frac{\delta L}{\delta g_{km}} = \frac{\partial}{\partial x^a} \left( \frac{\partial L}{\partial g_{km}, a} \right) - \frac{\partial L}{\partial g_{km}},
\]
where the comma means the ordinary derivative. When $T^{km}$ is substituted in it, the contracted Bianchi identity comes to read:

$$T_{i;ik} = T_{i,k,i} - \frac{1}{2} T^{pq} g_{pq,k} = 0,$$

(3)

where the semicolon indicates the covariant derivative done with respect to $\Gamma_{km}^i$. But then

$$T^{pq} g_{pq,k} = \frac{\partial \partial L}{\partial x_a} \left( \frac{\partial L}{\partial g_{pq,i}} g_{pq,k} \right) - \frac{\partial L}{\partial x_k}$$

(4)

and, if we define the pseudo-tensor density of Einstein as

$$u_{ik} = \frac{1}{2} \left( \delta_{ik} L - \frac{\partial L}{\partial g_{pq,i}} g_{pq,k} \right) = \frac{1}{2} \left( \delta_{ik} L - \frac{\partial L}{\partial g_{pq,i}} g_{pq,k} \right),$$

(5)

the contracted Bianchi identity (3) takes the form

$$\left( T_{i,k} + u_{ik} \right)_{;i} = 0.$$

(6)

These well known developments are recalled here for clearness; we note in passing that the additional claim of ref. [1], that in general relativity $u_{ik,i} = 0$, already entails that $T_{i,k,i} = 0$, i.e. the very conclusion that is later reached in that paper through the argument of the clash between the Bianchi and the Freud identities. This conclusion is contradicted by a simple example. In fact, if $T_{ik} = \lambda g_{ik}$, where $\lambda$ is a constant, then

$$T_{i,k,i} = \lambda (-g)^{1/2},$$

(7)

and the vanishing of the right-hand side of (7) can only be ensured in particular systems of coordinates.

In ref. [1] $u_{ik}$ is assumed to split into a tensor density and some entity that can be made to vanish everywhere in some system of coordinates. Let us provide a construction that actually performs such a splitting. We first recall [4] that

$$\frac{\partial L}{\partial g_{pq,i}} = -\Gamma_{pq}^i + \frac{1}{2} \delta_q^i \Gamma_{pa}^a + \frac{1}{2} \delta_p^i \Gamma_{qa}^a$$

(8)

and since $g_{pq,k} = 0$, one readily gets

$$u_{ik} = \frac{1}{2} \left[ \delta_k^i g_{pq} (\Gamma_{aq}^b \Gamma_{pb}^a - \Gamma_{ab}^b \Gamma_{pq}^a) \right.$$  

$$- (\Gamma_{pq}^i - \frac{1}{2} \delta_q^i \Gamma_{pa}^a - \frac{1}{2} \delta_p^i \Gamma_{qa}^a) \cdot (g_{bp} \Gamma_{bk}^p + g_{bk} \Gamma_{bp}^q - g_{pq} \Gamma_{kb}^b) \right].$$

(9)

We register that

$$u_{ik} = g^{ik} (\Gamma_{ak}^b \Gamma_{ib}^a - \Gamma_{ab}^b \Gamma_{ik}^a) = L.$$  

(10)

Through eq. (9) a constructive approach to the splitting of $u_{ik}$ in the above-mentioned way becomes apparent. Besides the pseudo-Riemannian metric $g_{ik}$, let us consider a.
Minkowskian metric $s_{ik}$, whose Christoffel symbols $\Sigma^i_{km}$ can be transformed into zero everywhere in some coordinate system. One writes [5]

$$\Gamma^i_{km} = \Sigma^i_{km} + \Delta^i_{km}, \quad \Delta^i_{km} = \frac{1}{2} g^{ip}(g_{pk|m} + g_{pm|k} - g_{km|p}),$$

(11)

where “$|$” indicates the covariant derivative done with respect to $\Sigma^i_{km}$. Therefore, $\Delta^i_{km}$ is a tensor, but we shall not leave unnoticed that its definition in by no means unique since, for a given $g_{ik}$, the Minkowskian metric $s_{ik}$ can be so chosen as to ensure the overall vanishing of the connection $\Sigma^i_{km}$ in whatever coordinate system one likes, and the tensor $\Delta^i_{km}$ extracted from $\Gamma^i_{km}$ will depend on that choice. By substituting the $\Gamma^i_{km}$ given by eq. (11) in eq. (9) the latter can be rewritten as

$$u^i_k = t^i_k + w^i_k,$$

(12)

i.e. as the sum of a non-uniquely defined tensor density $t^i_k$ and of $w^i_k$, which by construction vanishes in the coordinate system in which $\Sigma^i_{km}$ is everywhere zero. Hence in that system $t^i_i = u^i_i = L$. But, since in that coordinate system the metric tensor $g_{ik}$ can be so chosen as to ensure that $L$ does not vanish, it turns out that the scalar density $t^i_i$ does not necessarily vanish, and the same occurrence happens to $t^i_k$. The assertion to the contrary contained in ref. [1] is thereby disproved.

Let us define now the pseudo-tensor density

$$U^i_k = T^i_k + u^i_k.$$  

(13)

Freud has shown [2] that one can write

$$U^i_k = \frac{\partial V^{ia}_k}{\partial x^a},$$

(14)

where the pseudo-tensor density $V^{ia}_k$ is skew in the upper indices and can be defined by the expansion of a determinant:

$$V^{ia}_k = \frac{1}{2} \begin{vmatrix} \delta^i_k & \delta^a_k & \delta^b_k \\ g^{is}_{bs} & g^{as}_{bs} & g^{bs}_{bs} \\ \Gamma^i_{bs} & \Gamma^a_{bs} & \Gamma^b_{bs} \end{vmatrix}.$$  

(15)

We introduce now another Minkowskian metric tensor, $l_{ik}$, whose Christoffel symbols $\Lambda^i_{km}$ will vanish in a coordinate system that in general differs from the one in which the $\Sigma^i_{km}$ vanish. We can write

$$\Gamma^i_{km} = \Lambda^i_{km} + \Theta^i_{km}, \quad \Theta^i_{km} = \frac{1}{2} g^{ip}(g_{pk;m} + g_{pm;k} - g_{km;p}),$$

(16)

where $\Theta^i_{km}$ is a tensor built with the covariant derivative “$|$” done with respect to $\Lambda^i_{km}$, and is not uniquely defined, due to the freedom of choice for $l_{ik}$. An elementary property of determinants ensures that, by substituting (16) in (15), one can write

$$V^{ia}_k = X^{ia}_k + Z^{ia}_k,$$

(17)

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i.e. one can split $V^i_k$ into a skew tensor density $X^i_{ka}$ and a skew object $Z^i_{ka}$ that vanishes when the $\Lambda^i_{km}$ are zero. The splitting is once more not unique: it depends on the choice of $l_{ik}$. We aim at achieving the same sort of splitting for $U^i_k$, i.e. at writing:

$$U^i_k = x^i_k + z^i_k, \quad (18)$$

where $x^i_k$ is a tensor density, while $z^i_k$ is zero in the coordinate system in which $\Lambda^i_{km}$ vanishes. Since

$$X^i_{ka} = X^i_{ka} + X^i_{ka} \Lambda^a_n, \quad (19)$$

is not a tensor density, we cannot pose $x^i_k = X^i_{ka}$. A position that fulfils our requirements is:

$$x^i_k = X^i_{ka}; \quad z^i_k = X^i_{ka} \Lambda^a_n + Z^i_{ka}, \quad (20)$$

but, of course, it does not allow to conclude that $x^i_{k,i}$ and $z^i_{k,i}$ must vanish separately due to the play of the indices.

The constructive method leads eventually to rewrite eq. (13) as follows:

$$T^i_k = U^i_k - u^i_k = x^i_k - t^i_k + z^i_k - w^i_k, \quad (21)$$

where $z^i_k - w^i_k$ is a non-vanishing tensor density, that can be annihilated through the particular choice $s_{ik} = l_{ik}$ for the two Minkowskian metrics. But this choice is by no means mandatory; therefore $U^i_k - u^i_k$ shall not be generally split into two tensorial parts only, as asserted in ref. [1].

In conclusion: a constructive method allows to appreciate that the splitting of $U^i_k - u^i_k$ into tensorial parts is arbitrary and generally threefold, that $t^i_k$ is not generally vanishing and that $x^i_{k,i}$ is not necessarily zero due to symmetry reasons. The argument offered in ref. [1] for proving that in general relativity a clash between the Freud and the Bianchi identities leads to the vanishing of $T^i_{k,i}$ requires instead a twofold splitting of $U^i_k - u^i_k$, a vanishing $t^i_k$ and a zero $x^i_{k,i}$. That argument therefore avails of either wrong or unproven premises.

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