Long-range order between the planets in the Solar system

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ABSTRACT

The Solar system is investigated for positional correlations between the planets using a logarithmic distance scale. The pair correlation function for the logarithm of the semimajor axis shows a regular distribution with five to seven consecutive peaks, and the Fourier transform hereof shows reciprocal peaks of first and second order. A procedure involving random permutations for the shuffling of the inter-logarithmic distances is employed. This probes for the presence of correlations of longer range than neighbouring planets. The use of permutations is, in particular, a helpful analysis when the number of data points is small. The pair correlation function of the permuted planets lacks the sequence of equidistant peaks, and its Fourier transform has no second-order peak. This analysis demonstrates the existence of longer ranged correlations in the Solar system.

Key words: history and philosophy of astronomy – planets and satellites: general – Solar system: general.

1 INTRODUCTION

Historically, humankind has been looking for regularities in the positions of the planets in the Solar system, and to this day it remains an open question whether the positions are random or not. The observed Titius–Bode rule is an example of an apparent regularity, which for nearly two-and-a-half centuries has been a subject for debate. In the present analysis, we do not ask the corresponding question of the existence of a linear regression between the positions; rather we analyse whether the correlations between the planets exist beyond the nearest planet, e.g. are there positional correlations between the next nearest planets and so on.

The Titius–Bode rule is an approximate algebraic relation between the semimajor axes of a planet and its planet number counting outwardly from the Sun, which was formulated in its original form by Johann Daniel Titius von Wittenberg in 1766 (see Nieto 1972). In short, the early formulation states that $a_n \approx (4 + 3 \times 2^n)/10$, where $n = \infty$ for Mercury and $0, 1, 2, \ldots$ for the succeeding planets and $a_n$ is the semimajor axis given in astronomical units. This empirical insight has been significant in the discovery of Uranus in 1781 and of Ceres, of the asteroid belt, in 1801. With the later discoveries of Neptune in 1846 and Pluto in 1930, it was found that the observed positions of the most recent planetary objects deviated somewhat more from the predictions of the original rule than what the positions of the previously known planets did. Subsequently, the enthusiasm for the rule began to fade. Now, some scientists find the rule to be nearly anachronistic. One textbook on planetary science advocates that the rule may be a probabilistic incident or at least without significance (Murray & Dermott 1999), while other textbooks do not describe the rule (e.g. de Pater & Lissauer 2001). Yet, many books do include a discussion of the rule; in one book the rule is considered to be a reflection of theories suggesting that the distances between planets are larger the farther the planets are from the Sun (see Jones 2007).

In our opinion, two additional factors have contributed to the uncertainty in how to appraise the significance of the Titius–Bode rule. For a number of years, improvement in predictions and understanding of Titius–Bode-like descriptions have focused on the number schemes being used for indexing the planets. Perhaps this left the impression on many readers that laws were being made to fit the data, strengthening the impression that any relationship was accidental (even when the authors thought otherwise). Another key factor contributing to the doubt about the Titius–Bode rule is the lack of a widely accepted theoretical framework for the rule despite the fact that it is now nearly two-and-a-half centuries old. On the other hand, some scientists remain fond of the rule and some have tried to prove and model it in various ways. In a summary, discussing the rule published in the beginning of this decade, Chapman (2001) wrote 'My scientific intuition tells me that there is something behind the Titius–Bode rule'.

Nieto (1972), in his book, captivatingly describes the history of thoughts on the progression of the planets from the book Mysterium Cosmographicum by Kepler, and onwardly, and the book diligently reviews the research on the Titius–Bode rule up to 1972, including work considering the satellite systems of the major planets. Basano & Hughes (1979) have suggested to modify the original
Titius–Bode law in an attempt to improve its agreement with data; Filippov (1991) has argued that resonances in the protoplanetary cloud can initiate planetary formation which are in agreement with a Titius–Bode-like behaviour. Dubrulle & Graner (1994a) suggested that scale invariance is involved, and Dubrulle & Graner (1994b) argued that it is easy to construct an endless collection of theoretical models predicting the Titius–Bode rule. With the use of two different criteria for random planetary systems, Hayes & Tremaine (1998) are unable to unambiguously evaluate the significance of the fits to Titius–Bode laws. Murray & Dermott (1999) have argued on the basis of a large set of stochastically generated satellite systems that Titius–Bode-like laws are likely to be insignificant. Using different ways to generate planet populations, Lynch (2003) has argued that this result is ambiguous and that he cannot rule out the possibility that the Titius–Bode rule could be significant. The scheme of Lynch has further been pursued by Neslušan (2004) who suggested that the Earth’s orbit is peculiar. Using an equation akin to Schrödinger’s wave equation, Pefinova et al. (2007) have suggested that this kind of quantization can index the observed positions in the Solar system. Very recently, Poveda & Lara (2008) have applied a Titius–Bode type of analyses to the 55 Cancri system and suggested positions for two further exoplanets. In this context, Kotiljarov (2008) has discussed a structural law of planetary systems, Butusov’s law.

In this Letter, we take a different view and analyse the correlations between the logarithmic positions of the planets. It is an important feature of our approach that one avoids making use of a planet-numbering scheme – instead correlations will reveal themselves if they are there. By applying a Fourier analysis, the range of correlations can be investigated. Do the correlations arise from a preference for the nearest-neighbour relationship (short-range order) or do part of the correlations involve planets more distant from each other (longer range)?

2 THE SOLAR SYSTEM

Are the positions of the planets correlated with each other? In order to address this question we calculate the pair correlation function, sometimes also called the auto-correlation function:

\[ P(\Delta) = \int_{-\infty}^{\infty} \rho(x + \Delta) \rho(x) \, dx, \]

where \( \rho(x) \) is a distribution akin to the likelihood for finding a planet at position \( x \) (it should not be confused with the density distribution for finding a kilo of matter). The pair correlation function, \( P(\Delta) \), then becomes related to the probability for finding a planet orbit at a given distance from another planet orbit. If the order is longer ranged, the pair correlation function will have several peaks, indicating probable distances for the nearest neighbour, next nearest neighbour etc. Also note that the pair correlation function is always a symmetrical function and of course has a central (self-correlation) peak at \( \Delta = 0 \).

In order to calculate the pair correlation function, we use the specific function \( \rho(x) \) depicted in Fig. 1. Here \( \rho(x) \) is described as a sum of Gaussian peaks with full-width-at-half-maximum \( w_\rho \). The function \( \rho(x) \) is not a density, but an indicator of the positions at which planets are placed:

\[ \rho(x) = \sum_{\text{planets}} \alpha_i \exp \left( - \frac{(x - x_i)^2}{w_\rho^2 / 2 \ln 2} \right), \]

where \( x_i \) is the logarithmic position of planet \( i \) measured in units of \( 10^6 \) km, i.e. \( x_i = \ln (a_i / 10^6 \text{ km}) \). The values \( x_i \) for the individual planets can be found in Table 1. The width \( w_\rho = 0.25 \) is chosen as \( w_\rho \simeq (x_N - x_1)/20 \). The scalefactors are chosen as \( \alpha_i = (r_i 10^6/a_i)^{1/8} \), where \( r_i \) is the radius of object \( i \) and \( a \) is the semimajor axis of the planet. The exponent 1/8 was chosen such that the weights would be approximately equal. A desired feature of this choice of \( \alpha_i \) is that an infinitesimally small object has zero weight, while larger objects have approximately equal weights. We have not chosen the mass as the weight since we are not studying correlations between the masses in the Solar system, but correlations between positions. In Fig. 1, the height of the 10 peaks is equal to \( \alpha_i r_i \).

The corresponding pair correlation function, \( P(\Delta) \), is shown in Fig. 2; it is observed that there are five clearly discernible nearly equidistant peaks to the right of the self-correlation peak at \( \Delta = 0 \). Furthermore, there are two additional weaker equidistant peaks. The positions of these seven peaks together with the central peak are depicted in Fig. 3. There is an almost perfect linear relationship between the positions of the peaks in the pair correlation function demonstrating the presence of a long-range order. The coefficient of determination (see Nolan 1994) is \( R^2 = 0.999434 \). The peaks appear equidistant.
there are five clearly visible equidistant peaks and two weaker equidistant
neighbour correlations are given by the first peak to the right of the central
peak; the finite resolution makes it easier for the eye to spot the
larger peaks at
$q\approx 1/(x_i - x_j)$, where $N = 10$ for the Solar system, i.e. $w_\rho = 0.2$. In Fig. 4, we have plotted the graph of the Fourier transform $H(q)$ introduced above; the main features, besides the peak at $q = 0$, are the first and second fundamental peaks at $q_1 \approx 1.57$ and $q_2 \approx 3.05$, respectively.

3 SHUFFLED SOLAR SYSTEMS

To illustrate the difference between short and longer range order, two simple cases are given. One example of the short-range order between the positions $x_i$ is given by

$$x_i = x_0 + \sum_{i=1}^{N-1} \delta_i(j),$$

where $\delta_i$ follows some stochastic distribution and $x_0$ is a constant. An example of a perfect long-range order, on the other hand, is of the form

$$x_i = x_1 + (i - 1)d_a + \delta_i(t),$$

where $d_a$ is a constant (the mean inter-distance) and $\delta_i$ a stochastic distribution.

With regard to the uniqueness of the order in the Solar system, we investigate its stochastic nature. If there is only short-range order, by which we mean that the order between the values of $d_i$ arises from a preference for a nearest-neighbour relationship between the planets with some stochastic nature, then the correlations should be preserved upon interchanging the $d_i$ values. We now generate an artificial Solar system using the data from our Solar system by the introduction of a random permutation for shuffling the $d_i$. In Figs 5–7, we investigate the results of such a permutation, 491 526 387, on the Solar system. The planet distribution, $\rho_{\text{Solar}}(x_i)$, and the pair correlation function, $P_{\text{Solar}}(\Delta)$, for the shuffled Solar system are shown in Figs 5 and 6, respectively. It becomes clear as one can see that the shuffled Solar system has lost the equidistant nature of the peaks in the pair correlation function. The random shuffling of the $d_i$ has destroyed the longer range order thereby underscoring its existence in the Solar system. The corresponding Fourier transform $H_{\text{sh}}(q)$ is depicted in Fig. 7.

Figure 2. Pair correlation function $P(\Delta)$ in equation (1) for the Solar system. $P(\Delta)$ is a symmetric function of $\Delta$, i.e. $P(-\Delta) = P(\Delta)$. Nearest-neighbour correlations are given by the first peak to the right of the central peak as well as by the first peak to the left of the central peak. On each side, there are five clearly visible equidistant peaks and two weaker equidistant peaks. The pair correlation function is plotted for $w_\rho = 0.25$. Choosing a larger $w_\rho$ would further smooth out the graph of $P(\Delta)$, while the positions of the principal peaks would remain the same.

Figure 3. Location, $\Delta$, of peaks in the pair correlation function for the Solar system as a function of the peak number. There is an approximative linear relationship of these peaks, which is shown by the straight line. The statistical coefficient of determination is $R^2 = 0.999434$; for a perfect linear fit, it would be $+1$.

Figure 4. Fourier transform, $H(q)$, of the pair correlation function $P$ of the Solar system. The first fundamental peak is at $q_1 \approx 1.57$ and the second one at $q_2 \approx 3.05$, which is fairly close (within 3 per cent) to $2q_1$. We include a Gaussian convolution with width $w_q$ to remove the rapid oscillations due to the finite size of the planetary system; such rapid oscillations are always present in Fourier analyses of finite size systems. The width is therefore chosen as a reciprocal measure of the size of the system, $w_q \approx 1/(x_N - x_1)$, where $N = 10$ for the Solar system, i.e. $w_q = 0.2$. In Fig. 4, we have plotted the graph of the Fourier transform $H(q)$ introduced above; the main features, besides the peak at $q = 0$, are the first and second fundamental peaks at $q_1 \approx 1.57$ and $q_2 \approx 3.05$, respectively.
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**Figure 5.** $\rho_{Sh}(x)$ for the shuffled Solar system. This system is obtained from the Solar system by permuting the distances between the first nine planets with the permutation 491 526 387.

**Figure 6.** Pair correlation function, $P_{Sh}(\Delta)$, of the shuffled Solar system obtained by a random permutation 491 526 387. We observe that the correlations tend to be destroyed compared to that of the Solar system. The six to seven major peaks on either side are not equidistant in contrast to the data for the Solar system.

**Figure 7.** Fourier transform $H_{Sh}(q)$ of the shuffled Solar system, given by pair correlation function $P_{Sh}$.

**Figure 8.** Scatter plot for 1000 random permutations. Here $|q_2 - 2q_1|$ is plotted as a function of the order, $n$, of the permutation, where $q_1$ is the position of the first fundamental peak and $q_2$ the position of the second. The order of a permutation is its cycle length, i.e. the number of successive times one needs to apply the permutation to obtain the trivial permutation. The trivial permutation is also shown on the plot, $n = 1$. The figure demonstrates that for the majority of permutations of interplanetary distances, the longer ranged order is destroyed. For the permutation discussed in Figs 5–7, 491 526 387, the order $n = 7$ and $|q_2 - 2q_1| = 0.702$.

### 4 CONCLUSION

We have demonstrated the presence of positional correlations between distant planets in the Solar system. A particular strong proof of this is the lack of similar correlations in permuted Solar systems. The shuffling result demonstrates that the relatively long-ranged correlations between the planets do not appear by chance but is a fundamental feature of the Solar system. In solid state physics the difference between long-range positional order and absence thereof distinguishes the crystalline state from the amorphous state (see Guinier 1994). The relationship between the Titius–Bode rule and long-range order is not bijective. A linear relationship on a logarithmic plot is not sufficient to reveal longer ranged correlations (see Fig. 9) since the shuffled Solar system is more accurately described ($R^2 = 0.99687$) by a linear fit than the Solar system itself ($R^2 = 0.99355$). Therefore, caution should be executed when linear plots are applied.

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Models for planetary system formation must include mechanisms that reach farther than the nearest neighbour. It can arise directly from the involved planet formation steps as well as from planetary system stability criteria. Some early theories are the electromagnetic theory of Birkeland (1912), the nebular theories of Berlage (1932) and the resonance theory of Filippov (1991). The recent surge in extrasolar planetary systems discovered has furthered systematic investigations of the criteria for stability of planetary systems. It has been suggested to be a common feature of the known planetary systems that they are in relatively close proximity to instability boundaries (see Barnes & Quinn 2004). Currently, there is significant ongoing progress in understanding gravitational scattering/planet-planet scattering (see Laakso et al. 2006; Ford & Rasio 2008), in understanding the Hill and Lagrange boundaries in resonant planetary systems (see Barnes & Greenberg 2007) and in understanding possible planet migration (see Zhou & Sun 2005). Perhaps, future work will show if similar effects can also have contributed to the shaping of the features responsible for the correlations in the Solar system.

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