Edge-connectivity in regular multigraphs from eigenvalues

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Abstract

Let $G$ be a $d$-regular multigraph, and let $\lambda_2(G)$ be the second largest eigenvalue of $G$. In this paper, we prove that if $\lambda_2(G) < d - \frac{1 + \sqrt{9d^2 - 10d + 17}}{4}$, then $G$ is 2-edge-connected. Furthermore, for $t \geq 2$ we show that $G$ is $(t + 1)$-edge-connected when $\lambda_2(G) < d - t$, and in fact when $\lambda_2(G) < d - t + 1$ if $t$ is odd.

1 Introduction

A simple graph is a graph without loops or multiple edges. In this paper, a multigraph is a graph that can have multiple edges but does not contain loops. A simple graph or multigraph $G$ is $k$-connected if $G$ has more than $k$ vertices and every subgraph obtained by deleting fewer than $k$ vertices is connected; the connectivity of $G$, written $\kappa(G)$, is the maximum $k$ such that $G$ is $k$-connected. The adjacency matrix $A(G)$ of $G$ is the $n$-by-$n$ matrix in which the entry $a_{i,j}$ is the number of edges in $G$ with endpoints $\{v_i, v_j\}$, where $V(G) = \{v_1, ..., v_n\}$. The eigenvalues of $G$ are the eigenvalues of its adjacency matrix $A(G)$. Let $\lambda_1(G), ..., \lambda_n(G)$ be its eigenvalues indexed in nonincreasing order. The Laplacian matrix of $G$ is $D(G) - A(G)$, where $D(G)$ is the diagonal matrix of degrees. Let $\mu_1(G), ..., \mu_n(G)$ be its eigenvalues indexed in nondecreasing order. Note that if $G$ is a $d$-regular graph, then $\lambda_i(G) = d - \mu_i(G)$ for $1 \leq i \leq n$.

A lot of research in graph theory over the last 40 years was stimulated by a classical result of Fiedler [4], stating that

$$\kappa(G) \geq \mu_2(G)$$

for a non-complete simple graph $G$. In 2002, Kirkland, Molitierno, Neumann, and Shader [6] characterized when equality in the inequality [1] holds.

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A simple graph or multigraph $G$ is $t$-edge-connected if every subgraph obtained by deleting fewer than $t$ edges is connected; the edge-connectivity of $G$, written $\kappa'(G)$, is the maximum $t$ such that $G$ is $t$-edge-connected. Note that $\kappa(G) \leq \kappa'(G)$. Chandran \cite{2} proved that if $G$ is an $n$-vertex $d$-regular simple graph with

$$\lambda_2(G) < d - 1 - \frac{d}{n-d},$$

then $\kappa'(G) = d$ and if $|[S, \overline{S}]| = d$, where for vertex sets $S$ and $T$, $[S, T]$ is the set of edges from $S$ to $T$, then $S$ equals a single vertex or $V(G - v)$ for a vertex $v \in V(G)$. Krivelevich and Sudakov \cite{7} slightly improved Chandran’s result and showed that if $G$ is a $d$-regular simple graph with $\lambda_2(G) \leq d - 2$, then $\kappa'(G) = d$. In 2010, Cioabă \cite{3} proved that if

$$\lambda_2(G) < d - \frac{2t}{d+1},$$

then $\kappa'(G) \geq t + 1$. When $t$ equals 1 or 2, he proved stronger results.

**Theorem 1.1.** \cite{3} Let $d$ be an odd integer at least 3 and let $\pi(d)$ be the largest root of $x^3 - (d-3)x^2 - (3d-2)x - 2 = 0$. If $G$ is a $d$-regular simple graph such that $\lambda_2(G) < \pi(d)$, then $\kappa'(G) \geq 2$.

This result is best possible in a sense that there exists a $d$-regular simple graph $H$ with $\lambda_2(H) = \pi(d)$ and with $\kappa'(H) = 1$ (See \cite{3}).

**Theorem 1.2.** \cite{3} If $G$ is a $d$-regular simple graph such that $\lambda_2(G) < \frac{d-3+\sqrt{(d+3)^2-16}}{2}$, then $\kappa'(G) \geq 3$.

This result is also best possible in a sense that there exists a $d$-regular simple graph $K$ with $\lambda_2(K) = \frac{d-3+\sqrt{(d+3)^2-16}}{2}$ and with $\kappa'(K) = 2$ (See \cite{3}). To guarantee higher edge-connectivity, the author conjectured in his thesis \cite{8}.

**Conjecture 1.3.** \cite{8} Let $\rho(d, t) = \begin{cases} \frac{d-4+\sqrt{(d+4)^2-8t}}{2}, & \text{when } t \text{ is odd} \\ \frac{d-3+\sqrt{(d+3)^2-8t}}{2}, & \text{when } t \text{ is even} \end{cases}$. For $t \geq 3$, if $G$ is a $d$-regular simple graph such that $\lambda_2(G) < \rho(d, t)$, then $\kappa'(G) \geq t + 1$.

In fact, the author \cite{8} proved that for a $d$-regular simple graph $G$ with $\kappa'(G) \leq t$, if there exists a vertex subset $S \subseteq V(G)$ with $|[S, \overline{S}]| = \kappa'(G)$ and both $|S|$ and $|\overline{S}|$ are at least $d + 4$ when $d$ is odd and at least $d + 3$ when $d$ is even, then $\lambda_2(G) \geq \rho(d, t)$. This gives a partial positive answer to Conjecture 1.3. For each positive integer $t \geq 2$, the author constructed a $t$-connected $d$-regular simple graph $G$ with $\lambda_2(G) = \rho(d, t)$, so the bound on the second largest eigenvalue is best possible if Conjecture 1.3 is true.
In this paper, we extend some of these results to multigraphs. The main results of this paper are Theorem 1.4 and Theorem 1.5. In fact, we obtain a best possible condition on the second largest eigenvalue \( \lambda_2(G) \) for a \( d \)-regular multigraph \( G \) to guarantee that \( \kappa'(G) \geq t + 1 \) for any positive integer \( t \). We prove separately when \( t = 1 \) in Section 3 and when \( t \geq 2 \) in Section 4.

**Theorem 1.4.** If \( G \) is a connected \( d \)-regular multigraph with \( \lambda_2(G) < \frac{d-1+\sqrt{9d^2-10d+17}}{4} \), then \( \kappa'(G) \geq 2 \).

In Section 2, for any positive integer \( d \geq 3 \), we construct an example of a \( d \)-regular multigraph \( H_{d,1} \) having \( \lambda_2(H_{d,1}) = \frac{d-1+\sqrt{9d^2-10d+17}}{4} \) and \( \kappa'(H_{d,1}) = 1 \). Our construction shows that Theorem 1.4 is best possible. To guarantee higher edge-connectivity, we need a smaller upper bound on \( \lambda_2 \) in terms of \( d \) and \( t \).

**Theorem 1.5.** For \( t \geq 2 \), if \( G \) is a connected \( d \)-regular multigraph with \( \lambda_2(G) < d - t \), then \( \kappa'(G) \geq t + 1 \). Furthermore, if \( t \) is odd and \( G \) is a connected \( d \)-regular multigraph with \( \lambda_2(G) < d - t + 1 \), then \( \kappa'(G) \geq t + 1 \).

We can rephrase Theorem 1.5 as follows: if \( G \) is a connected \( d \)-regular multigraph with \( \lambda_2(G) < d - 2l \) for \( l \geq 1 \), then \( \kappa'(G) \geq 2l + 2 \). Our result implies that as the upper bound on \( \lambda_2(G) \) for a \( d \)-regular graph \( G \) decreases by \( 2 \), the lower bound on \( \kappa'(G) \) increases by \( 2 \). However, decreasing the bound on \( \lambda_2(G) \) by \( 1 \) does not increase the lower bound on \( \kappa'(G) \) by \( 1 \); this is a sense in which our result is sharp. For example, \( \lambda_2(G) < d - 2 \) implies \( \kappa'(G) \geq 4 \), and \( \lambda_2(G) < d - 4 \) implies \( \kappa'(G) \geq 6 \), but \( \lambda_2(G) < d - 3 \) does not imply \( \kappa'(G) \geq 5 \). In Section 2, for any positive integer \( d \geq 3 \) and for even positive integer \( t \), we construct a \( d \)-regular multigraph \( H_{d,t} \) with \( \kappa'(H_{d,t}) = t \) and \( \lambda_2(H_{d,t}) = d - t \). Thus Theorem 1.5 is best possible for all \( t \geq 2 \) regardless of whether \( t \) is odd or even.

For undefined terms, see West [9] or Godsil and Royle [5].

## 2 The Construction

Cioabă [3] presented examples for sharpness in the upper bound on \( \lambda_2(G) \) for a \( d \)-regular simple graph \( G \) to guarantee that \( G \) is either 2-edge-connected or 3-edge-connected. In this section, we present a smallest \( d \)-regular multigraph \( H_{d,1} \) with \( \kappa'(H_{d,1}) = 1 \) and with \( \lambda_2(H_{d,1}) = \frac{d-1+\sqrt{9d^2-10d+17}}{4} \), and a smallest \( d \)-regular multigraph \( H_{d,t} \) with \( \kappa'(H_{d,t}) = t \) and with \( \lambda_2(H_{d,t}) = d - t \) for every even positive integer \( t \geq 2 \).

**Observation 2.1.** For a positive integer \( d \geq 3 \), there exists only one multigraph with three vertices such that every vertex has degree \( d \), except only one vertex with degree \( d - 1 \).
Proof. Let \( G \) be a multigraph with three vertices, say \( v_1, v_2, \) and \( v_3 \) such that \( v_1 \) and \( v_2 \) have degree \( d \), and \( v_3 \) has degree \( d - 1 \). Let \( a, b, \) and \( c \) be the number of edges between \( v_1 \) and \( v_2 \), between \( v_2 \) and \( v_3 \), and between \( v_3 \) and \( v_1 \), respectively. Since \( v_1 \) and \( v_2 \) have degree \( d \), and \( v_3 \) has degree \( d - 1 \), we have \( a + c = a + b = d \), \( c + b = d - 1 \), and \( 2(a + b + c) = 3d - 1 \). Thus, we have \( a = \frac{d + 1}{2} \) and \( b = c = \frac{d - 1}{2} \), which gives the desired result.

Construction 2.2. Let \( B_{d,1} \) be the 3-vertex multigraph guaranteed by Observation 2.1. Let \( H_{d,1} \) be the graph obtained from two copies of \( B_{d,1} \) by adding one edge between the two vertices with degree \( d - 1 \) in the copies. Note that \( H_{d,1} \) is the smallest \( d \)-regular multigraph with \( \kappa'(H_{d,1}) = 1 \).

Let \( t \) be an even positive integer less than \( d - 1 \), and let \( B_{d,t} \) be the multigraph with two vertices, say \( x \) and \( y \), such that every vertex has degree \( d - \frac{t}{2} \). Let \( H_{d,t} \) be the graph obtained from two copies of \( B_{d,t} \) by adding \( \frac{t}{2} \) edges between two \( x \) and between two \( y \) in the copies. Note that \( H_{d,t} \) is a smallest \( d \)-regular multigraph with \( \kappa'(H_{d,t}) = t \).

When \( d = 3 \), see Figure 1 and 2 for \( t = 1 \) and for \( t = 2 \), respectively.

Now, we determine the second largest eigenvalue of \( H_{d,t} \). First, we introduce some definitions. Consider a partition \( V(G) = V_1 \cup \cdots \cup V_s \) of the vertex set of \( G \) into \( s \) non-empty subsets. For \( 1 \leq i, j \leq s \), let \( b_{i,j} \) denote the average number of neighbours in \( V_j \) of the vertices in \( V_i \). The quotient matrix of this partition is the \( s \times s \) matrix whose \((i,j)\)-th entry equals \( b_{i,j} \).

Theorem 2.3. (see Corollary 2.5.4 in [1], Lemma 9.6.1 in [5]) The eigenvalues of the quotient matrix interlace the eigenvalues of \( G \).

This partition is equitable if for each \( 1 \leq i, j \leq s \), any vertex \( v \in V_i \) has exactly \( b_{i,j} \) neighbours in \( V_j \).

Theorem 2.4. (see Lemma 2.3.1 in [1], Theorem 9.3.3 in [5]) The set of the eigenvalues of \( G \) includes the eigenvalues of the quotient matrix for an equitable partition of \( G \).

Theorem 2.5. (see Proposition 3.4.1 in [1], Theorem 8.8.2 in [5]) A graph is bipartite if and only if its spectrum is symmetric about the origin.
Figure 2: A cubic multigraph with $\lambda_2 = 1$ and $\kappa' = 2$

**Theorem 2.6.** The second largest eigenvalues of $H_{d,1}$ and $H_{d,t}$ are \( \frac{d-1+\sqrt{9d^2-10d+17}}{4} \) and \( d-t \), respectively.

**Proof.** Partition the vertex set of $H_{d,t}$ into two parts: two copies of $V(B_{d,t})$ in $H_{d,t}$. Note that this partition is equitable, and its quotient matrix is

\[
A = \begin{pmatrix}
\frac{d}{2} & \frac{t}{2} & \frac{t}{2} \\
\frac{t}{2} & \frac{d}{2} & \frac{t}{2} \\
\frac{t}{2} & \frac{t}{2} & \frac{d}{2}
\end{pmatrix}.
\]

The characteristic polynomial of the matrix is \((x-d)(x-d+t)\). By Theorem 2.4, the numbers $d$ and $d-t$ are eigenvalues of $H_{d,t}$, and by Theorem 2.5, the eigenvalues of $H_{d,t}$ are $-d, -(d-t), d-t$, and $d$. Thus $\lambda_2(H_{d,t}) = d-t$.

The adjacency matrix of $H_{d,1}$ is

\[
\begin{pmatrix}
0 & \frac{d+1}{2} & \frac{d-1}{2} & 0 & 0 & 0 \\
\frac{d+1}{2} & 0 & \frac{d-1}{2} & 0 & 0 & 0 \\
\frac{d-1}{2} & \frac{d-1}{2} & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & \frac{d-1}{2} & \frac{d-1}{2} \\
0 & 0 & 0 & \frac{d+1}{2} & 0 & \frac{d+1}{2} \\
0 & 0 & 0 & \frac{d+1}{2} & \frac{d+1}{2} & 0
\end{pmatrix}.
\]

The characteristic polynomial of this matrix is

\[
(x-d) \left\{ x^2 - \left( \frac{d-1}{2} \right)x - \left( \frac{d^2 - d + 2}{2} \right) \right\} (x + \frac{d-3}{2})(x + \frac{d+1}{2})^2,
\]

so the eigenvalues of $H_{d,1}$ are \( \frac{d-1+\sqrt{9d^2-10d+17}}{4} \), \( -\frac{d+1}{2} \), \(-\frac{d+1}{2}\), and \(-\frac{d-3}{2}\). Thus, we have $\lambda_2(H_{d,1}) = \frac{d-1+\sqrt{9d^2-10d+17}}{4}$.

\[\square\]

### 3 Proof of Theorem 1.4

In this section, we prove Theorem 1.4 by finding appropriate partitions of a $d$-regular multigraph $G$ with $\kappa'(G) = 1$ such that the quotient matrices of these partitions have their
second largest eigenvalues greater than equal to $\frac{d-1+\sqrt{9d^2-10d+17}}{2}$. By Theorem 2.3, we have $\lambda_2(G) \geq \frac{d-1+\sqrt{9d^2-10d+17}}{2}$. Now, we prove Theorem 1.4.

**Proof of Theorem 1.4.** Assume to the contrary that $\kappa'(G) = 1$. Then there exists a vertex subset $S$ such that $|[S, \overline{S}]| = 1$. Let $a = |S|$ and let $b = |\overline{S}|$, so we have $a + b = n$. We may assume that $a \leq b$. Note that $d$, $a$, and $b$ must be odd by degree-sum formula, and $a$ is at least 3. The quotient matrix of the partition $S$ and $\overline{S}$ is

$$A = \begin{pmatrix} d - \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & d - \frac{1}{b} \end{pmatrix}.$$

The characteristic polynomial of this matrix equals

$$(x - d + \frac{1}{a}) (x - d + \frac{1}{b}) - \frac{1}{ab},$$

Thus the eigenvalues of the matrix $A$ are $d$ and $d - \frac{1}{a} - \frac{1}{b}$.

**Case 1.** $a \geq 5$. By Theorem 2.3 we have

$$\lambda_2(G) \geq d - \frac{1}{a} - \frac{1}{b} \geq d - \frac{1}{5} - \frac{1}{5} = d - \frac{2}{5}. \quad (3)$$

Since $(3d - \frac{3}{5})^2 - (9d^2 - 10d + 17) > \frac{32}{5}d - 17 > 0$ for $d \geq 3$, we have

$$d - \frac{2}{5} - \frac{d - 1 + \sqrt{9d^2 - 10d + 17}}{4} = \frac{3d - \frac{3}{5} - \sqrt{9d^2 - 10d + 17}}{4} > 0.$$

**Case 2.** $a = 3$ and $b \geq 11$. By Theorem 2.3 we have

$$\lambda_2(G) \geq d - \frac{1}{a} - \frac{1}{b} \geq d - \frac{1}{3} - \frac{1}{11} = d - \frac{14}{33}. \quad (4)$$

Since $(3d - \frac{23}{33})^2 - (9d^2 - 10d + 17) > \frac{64}{11}d - 17 > 0$ for $d \geq 3$, we have

$$d - \frac{14}{33} - \frac{d - 1 + \sqrt{9d^2 - 10d + 17}}{4} = \frac{3d - \frac{23}{33} - \sqrt{9d^2 - 10d + 17}}{4} > 0.$$

**Case 3.** $a = 3$ and $5 \leq b \leq 9$. Since $a = 3$ and there exists only one edge between $S$ and $\overline{S}$, we have $G[S] = B_{d,1}$ by Observation 2.1. Thus there exists only one vertex, say $x$, in
with a neighbor in $S$. Partition the vertex set of $G$ into three parts: $V(B_d, 1)$, $\{x\}$, and $V(G) - V(B_d, 1) - \{x\}$. The quotient matrix of this partition is

$$A = \begin{pmatrix} d - \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & 0 & d - 1 \\ 0 & c & d - c \end{pmatrix}.$$ 

where $c = \frac{d - 1}{b - 1}$. The characteristic polynomial of the matrix is

$$(x - d + \frac{1}{3})\{x(x - d + c) + c(1 - d)\} - \frac{1}{3}(x - d + c)$$

$$= (x - d + \frac{1}{3})\{x^2 - (d - c)x + c - cd\} - \frac{1}{3}(x - d + c)$$

$$= (x - d)\{x^2 + (c - d)x + c - cd\} + \frac{1}{3}\{x^2 + (c - d)x + c - cd - x + d - c\}$$

$$= (x - d)\{x^2 + (c - d)x + c - cd\} + \frac{1}{3}\{x^2 + (c - d - 1)x + (1 - c)d\}$$

$$= (x - d)\{x^2 + (c - d)x + c - cd\} + \frac{1}{3}(x - d)(x + c - 1)$$

$$= (x - d)\{x^2 + (c - d)x + c - cd + \frac{1}{3}(x + c - 1)\}$$

$$= (x - d)\{x^2 + (c - d + \frac{1}{3})x + \frac{4}{3}c - cd - \frac{1}{3}\}$$

The second largest root of the polynomial is

$$d - c - \frac{1}{3} + \frac{\sqrt{(d - c - \frac{1}{3})^2 - \frac{16}{3}c + 4cd + \frac{4}{3}}}{2}.$$ 

Now, it suffices to show that for $b \in \{5, 7, 9\}$, we have

$$d - c - \frac{1}{3} + \frac{\sqrt{(d - c - \frac{1}{3})^2 - \frac{16}{3}c + 4cd + \frac{4}{3}}}{2} > \frac{d - 1 + \sqrt{9d^2 - 10d + 17}}{4}.$$ 

Subcase 3-1. $b = 9$. By replacing $c$ with $\frac{d - 1}{8}$, we have

$$d - c - \frac{1}{3} + \frac{\sqrt{(d - c - \frac{1}{3})^2 - \frac{16}{3}c + 4cd + \frac{4}{3}}}{2} = \frac{21d - 5 + \sqrt{729d^2 - 882d + 1177}}{48}.$$ 

By subtracting $\frac{d - 1 + \sqrt{9d^2 - 10d + 17}}{4}$ from $\frac{21d - 5 + \sqrt{729d^2 - 882d + 1177}}{48}$, we have

$$\frac{9d + 7 + \sqrt{729d^2 - 882d + 1177} - 12\sqrt{9d^2 - 10d + 17}}{48}.$$ 

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Since both \(9d + 7 + \sqrt{729d^2 - 882d + 1177}\) and \(12\sqrt{d^2 - 10d + 17}\) are positive, it suffices to show that \(0 < (9d + 7 + \sqrt{729d^2 - 882d + 1177})^2 - (12\sqrt{d^2 - 10d + 17})^2\)

\[
= 2(9d + 7)\sqrt{729d^2 - 882d + 1177} - (486d^2 - 684d + 790).
\]

Since both \(2(9d + 7)\sqrt{729d^2 - 882d + 1177}\) and \(486d^2 - 684d + 790\) are positive for \(d \geq 1\), it suffices to show that \(0 < (2(9d + 7)\sqrt{729d^2 - 882d + 1177})^2 - (486d^2 - 684d + 790)^2\)

\[
= 236196d^4 + 81648d^3 + 79704d^2 + 420336d + 230692
- (236196d^4 - 664848d^3 + 1235736d^2 - 1080720d + 624100)
= 746496d^3 - 1156032d^2 + 1501056d - 393408.
\]

By comparing each coefficient of \(d^i\) for \(i \in \{0, 1, 2, 3\}\), we have this inequality

\[
746496d^3 - 1156032d^2 + 1501056d - 393408 > 746496d^3 - 1194393.6d^2 + 1492992d - 447897.6
\]

\[
= 746496(d^3 - 1.6d^2 + 2d - 0.6) = 746496\{d^2(d - 1.6) + 2(d - 1.6) + 2.6\}
> 746496(d^2 + 2)(d - 1.6) > 0
\]

for \(d \geq 3\).

**Subcase 3-2.** \(b = 7\). By plugging 7 into \(b\), we have

\[
\frac{d - \frac{d-1}{6} - \frac{1}{3} + \sqrt{(d - \frac{d-1}{6} - \frac{1}{3})^2 - \frac{16}{3} d - \frac{1}{6} + 4 d - \frac{1}{6} d + \frac{4}{3}}}{2} = \frac{5d - 1 + \sqrt{49d^2 - 66d + 81}}{12}.
\]

By subtracting \(\frac{d-1+\sqrt{9d^2-10d+17}}{4}\) from \(\frac{5d-1+\sqrt{49d^2-66d+81}}{12}\), we have

\[
\frac{2d + 2 + \sqrt{49d^2 - 66d + 81} - 3\sqrt{9d^2 - 10d + 17}}{12}.
\]

Since both \(2d + 2 + \sqrt{49d^2 - 66d + 81}\) and \(3\sqrt{9d^2 - 10d + 17}\) are positive, it suffices to show that \(0 < (2d + 2 + \sqrt{49d^2 - 66d + 81})^2 - (3\sqrt{9d^2 - 10d + 17})^2\)

\[
= 4(d + 1)\sqrt{49d^2 - 66d + 81} - (28d^2 - 32d + 68).
\]

Since both \(4(d + 1)\sqrt{49d^2 - 66d + 81}\) and \(28d^2 - 32d + 68\) are positive for \(d \geq 1\), it suffices to show that \(0 < (4(d + 1)\sqrt{49d^2 - 66d + 81})^2 - (28d^2 - 32d + 68)^2\)

\[
= 784d^4 + 512d^3 - 32d^2 + 1536d + 1296 - (784d^4 - 1792d^3 + 4832d^2 - 4352d + 4624)
= 2304d^3 - 4864d^2 + 5888d - 3328.
\]
By comparing each coefficient of $d^i$ for $i \in \{0, 1, 2, 3\}$, we have this inequality

$$2304d^3 - 4864d^2 + 5888d - 3328 > 2304d^3 - 5068.8d^2 + 4608d - 10137.6$$

$$= 2304(d^3 - 2.2d^2 + 2d - 4.4) = 2304\{d^2(d - 2.2) + 2(d - 2.2)\}$$

$$> 746496(d^2 + 2)(d - 2.2) > 0$$

for $d \geq 3$.

**Subcase 3-3.** $b = 5$. If we plug 5 into $b$, then we have

$$d - \frac{d-1}{4} - \frac{1}{3} + \frac{\sqrt{(d - \frac{d-1}{4} - \frac{1}{3})^2 - \frac{16}{9}d^2 + 4d + \frac{4}{9}}}{2} = \frac{9d - 1 + \sqrt{225d^2 - 354d + 385}}{24}.$$

By subtracting $\frac{d-1+\sqrt{9d^2-10d+17}}{4}$ from $\frac{9d-1+\sqrt{225d^2-354d+385}}{24}$, we have

$$\frac{3d + 5 + \sqrt{225d^2 - 354d + 385} - 6\sqrt{9d^2 - 10d + 17}}{24}.$$

Since both $3d + 5 + \sqrt{225d^2 - 354d + 385}$ and $6\sqrt{9d^2 - 10d + 17}$ are positive, it suffices to show that $0 < (3d + 5 + \sqrt{225d^2 - 354d + 385})^2 - (6\sqrt{9d^2 - 10d + 17})^2$

$$= 2(3d + 5)\sqrt{225d^2 - 354d + 385} - (90d^2 - 36d + 202).$$

Since both $2(3d + 5)\sqrt{225d^2 - 354d + 385}$ and $90d^2 - 36d + 202$ are positive for $d \geq 1$, it suffices to show that $0 < (2(3d + 5)\sqrt{225d^2 - 354d + 385})^2 - (90d^2 - 36d + 202)^2$

$$= 8100d^4 + 14256d^3 - 6120d^2 + 10800d + 38500 - (8100d^4 - 6480d^3 + 37656d^2 - 14544d + 40804)$$

$$= 20736d^3 - 43776d^2 + 25344d - 2304.$$

By comparing each coefficient of $d^i$ for $i \in \{0, 1, 2, 3\}$, we have this inequality

$$20736d^3 - 43776d^2 + 25344d - 2304 > 20736d^3 - 47692.8d^2 + 20736d - 20733.7$$

$$= 20736(d^3 - 2.3d^2 + d - 2.3) = \{20736(d^2 + 1)(d - 2.3)\} > 0$$

for $d \geq 3$.

**Case 4.** $a = 3$ and $b = 3$. By Observation [2.1] the only graph with $a = 3$ and $b = 3$ is $H_{d,1}$, which completes the proof.
4 Proof of Theorem 1.5

If the edge-connectivity of a regular multigraph $G$ is even, then there exists a vertex subset $S$ such that $|[S, V(G) - S]|$ is even. Since $G$ is multigraph, the size of $S$ may be 2, so we have much simpler case than the ones of Theorem 1.4.

Proof of Theorem 1.5 Assume to the contrary that $\kappa'(G) \leq t$. Then there exists a vertex subset $S$ such that $l = |[S, \overline{S}]| \leq t$ for some positive integer $l$. Let $a = |A|$ and let $b = |\overline{A}|$, so we have $a + b = n$. We may assume that $a \geq b$. Note that $a$ is at least 2. The quotient matrix of the partition $S$ and $\overline{S}$ is

$$A = \begin{pmatrix} d - \frac{l}{a} & \frac{l}{a} \\ \frac{l}{b} & d - \frac{l}{b} \end{pmatrix}. $$

The characteristic polynomial of this matrix equals

$$(x - d + \frac{l}{a}) (x - d + \frac{l}{b}) - \frac{l^2}{ab},$$

$$= (x - d)^2 + \left( \frac{1}{a} + \frac{1}{b} \right) (x - d) = (x - d) \left( x - d + \frac{1}{a} + \frac{1}{b} \right).$$

Thus the eigenvalues of the matrix $A$ are $d$ and $d - \frac{l}{a} - \frac{l}{b}$.

Since $b \geq a \geq 2$, by Theorem 2.3 we have

$$\lambda_2(G) \geq d - \frac{l}{a} - \frac{l}{b} \geq d - \frac{l}{2} - \frac{l}{2} = d - l \geq d - t. \quad (5)$$

Assume that $t$ is odd. Then we have $t \geq 3$. If $l = t$, then since $b \geq a \geq 3$,

$$\lambda_2(G) \geq d - \frac{1}{a} - \frac{1}{b} \geq d - \frac{2}{3} = d - \frac{2t}{3} \geq d - t + 1. \quad (6)$$

If $l < t$, then since $b \geq a \geq 2$,

$$\lambda_2(G) \geq d - \frac{l}{a} - \frac{l}{b} \geq d - l \geq d - t + 1. \quad (7)$$

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