ON THE PROPER USE OF THE REDUCED SPEED OF LIGHT APPROXIMATION

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ABSTRACT

I show that the reduced speed of light (RSL) approximation, when used properly (i.e., as originally designed—only for local sources but not for the cosmic background), remains a highly accurate numerical method for modeling cosmic reionization. Simulated ionization and star formation histories from the “Cosmic Reionization on Computers” project are insensitive to the adopted value of the RSL for as long as that value does not fall below about 10% of the true speed of light. A recent claim of the failure of the RSL approximation in the Illustris reionization model appears to be due to the effective speed of light being reduced in the equation for the cosmic background too and hence illustrates the importance of maintaining the correct speed of light in modeling the cosmic background.

Key words: cosmology: theory – galaxies: formation – large-scale structure of universe

The primary challenge of simulating radiative transfer in astrophysics is in the high value of the speed of light. The high dimensionality of the problem is a less severe technical challenge, since the dynamics of dark matter (“N-body”) is also six-dimensional, but from a technical point of view it is a solved problem.

One way to cope algorithmically with the extremely high value for the speed of light is a reduced speed of light (RSL) approximation (Gnedin & Abel 2001). The idea behind the RSL approximation is simple: Since most of astrophysical dynamics is modeled in the Newtonian limit (i.e., the leading term in the Taylor series expansion over powers of $1/c$) anyway, it is only important that the higher-order terms be small compared to the leading one. For a system with the characteristic velocity $v$ the subsequent terms are of the order of $v/c$, and as long as $v/c$ is much less than unity, the Newtonian limit is valid. Hence, it does not matter which value of $c$ to take as long as $v$ remains much less than the modified, “reduced” value, which I will label $\tilde{c}$ hereafter.

One, of course, has to be careful, because the idea presented in the previous paragraph applies only to the dynamics of nonrelativistic matter, and there are many other processes in physics, including the dynamics of photons themselves, where the specific value of $c$ actually matters. Unfortunately, this concept is occasionally not readily grasped, and the RSL approximation is used incorrectly. Hence, the purpose of this short paper is to clarify when one can and cannot use the RSL approximation in cosmological simulations.

It is instructive to start with the cosmological radiative transfer equation for the monochromatic radiation energy density $I_\nu(t, x)$ (measured in erg per cm$^3$ per Hz) as a function of cosmic time $t$ and comoving position $x$,

$$\frac{\partial I_\nu}{\partial t} + H \left( \nu \frac{\partial I_\nu}{\partial \nu} - 3 I_\nu \right) + n_c \frac{\partial I_\nu}{\partial x} = -\kappa_\nu I_\nu + S_\nu, \quad (1)$$

where $a$ is the cosmological scale factor, $H$ is the Hubble parameter, $\kappa_\nu$ is the absorption coefficient (per unit time), and $S_\nu$ is the source function. The absorption coefficient is usually a sum over various absorption processes,

$$\kappa_\nu = \sigma \sum_j \nu \sigma_j n_j,$$

where $n_j$ is the number density of some absorbing species $j$ and $\sigma_j$ is the cross-section of the appropriate atomic process; both of them are not affected by the RSL approximation in any way.

In order to introduce the RSL approximation as it was originally designed, it is instructive to split the full radiation energy density into two components: the mean cosmic background $I_\nu(t) \equiv \langle I_\nu \rangle$ and the fluctuation around the mean $\delta I_\nu(t, x) \equiv I_\nu - \bar{I}_\nu$. The equation for the cosmic background is easily derivable by spatially averaging Equation (1):

$$\frac{\partial \bar{I}_\nu}{\partial t} + H \left( \nu \frac{\partial \bar{I}_\nu}{\partial \nu} - 3 \bar{I}_\nu \right) = -\bar{\kappa}_\nu \bar{I}_\nu + \bar{S}_\nu, \quad (2)$$

where the mean absorption coefficient $\bar{\kappa}_\nu \equiv \langle \kappa_\nu \rangle / \bar{I}_\nu$ is radiation energy density weighted.

In some circumstances the equation for the fluctuations in the radiation energy density can be simplified. For example, when modeling cosmic reionization (the actual specific application considered in this paper) and while restricting the radiation under consideration to ionizing radiation only (with the mean free path much shorter than the cosmic horizon), one can neglect the cosmological expansion and redshift over the time a photon crosses the mean free path distance. Hence, terms proportional to the Hubble parameter can be omitted. The equation for the fluctuation then becomes (after it is multiplied by $a/\nu$)

$$\frac{a}{\nu} \frac{\partial}{\partial t} \delta I_\nu + a \frac{\partial}{\partial x} \delta I_\nu = -\frac{1}{\lambda_\nu} \delta I_\nu + \psi_\nu, \quad (3)$$

where $\lambda_\nu(t, x)$ is the local comoving photon mean free path,
which is independent of the speed of light, and a new source function $\psi_v$ is defined as

$$\psi_v = \frac{a}{c} \sum_k L_k \left( n_k - \bar{n}_k \right) + \left( \bar{n}_v - \kappa_v \right) \bar{I}_v,$$  

where the sum is over all sources with luminosities $L_k$ and number densities $n_k$ (galaxies of different masses, quasars of different luminosities, etc.).

It is only Equation (3) that can be solved in a Newtonian limit in some circumstances—namely, if all the sources evolve on timescales longer than the light crossing time of the photon mean free path and all cosmic ionization fronts move much slower than the speed of light.

In this case one can introduce the RSL approximation by replacing $c$ in the first term in Equation (3) with the effective speed of light $\hat{c}$ (circled):

$$a \frac{\partial}{\partial t} \delta I_v + \mathbf{n} \cdot \frac{\partial}{\partial \mathbf{x}} \delta I_v = -\frac{1}{\lambda_v} \delta I_v + \psi_v.$$  

It is important to emphasize that this is the only place where the speed of light should be reduced. In particular, the speed of light should not be reduced in the equation for the cosmic background, as that would make background evolution incorrect and is also not needed, since it is trivial to solve Equation (2). Nor should the $c$ that enters the definition for $\psi_v$ in Equation (4) be reduced, as it would result in an incorrect photon production rate.

In order to illustrate how the RSL approximation performs when used properly, I use CROC simulations of cosmic reionization (Gnedin 2014). CROC simulations are performed with the adaptive refinement tree (ART) code (Kravtsov 1999; Kravtsov et al. 2002; Rudd et al. 2008). In the ART code, Equation (5) is implemented in a further transformed form and is solved with the optically thin variable Eddington tensor method of Gnedin & Abel (2001). A complete description of the CROC radiative transfer solver, down to the finite difference operator and accuracy tests, is presented in Appendix C of Gnedin (2014).

In particular, the ART implementation of the RSL approximation does not adopt any specific value for $\hat{c}$, but instead imposes a fixed ratio of the hydrodynamic timestep $\Delta t_H$ and the radiative transfer timestep $\Delta t_{RT}$:

$$\Delta t_{RT} = \frac{\Delta t_H}{N_{RT}},$$

where $N_{RT}$ is the number of times the radiative transfer solver is “subcycled” (i.e., makes a timestep) for one hydrodynamic timestep. The hydrodynamic timestep is set by the hydrodynamic Courant–Friedrichs–Lewy condition:

$$\Delta t_H = C_{ CFL} \frac{\Delta r}{v_{MAX}},$$

where $C_{ CFL} = 0.5$ is the Courant–Friedrichs–Lewy number (a property of the hydrodynamic solver), $\Delta r$ is the spatial resolution, and $v_{MAX}$ is the maximum total (i.e., bulk plus sound) velocity on the grid (for an AMR code this condition is more complicated, accounting for different cell sizes at different refinement levels, but conceptually it is equivalent to a simple uniform grid). The radiative transfer solver sets its timestep as

$$\Delta t_{RT} = \frac{\Delta r}{\hat{c}};$$

hence, in CROC simulations, there is a relationship between $\hat{c}$ and the true speed of light,

$$\hat{c} = \left( \frac{N_{RT}}{C_{ CFL}} \right) \left( \frac{v_{MAX}}{c} \right) c.$$  

Typically, in simulations with a box size of $20 h^{-1}$ Mpc (used in this paper for testing), during the peak of reionization $v_{MAX} \approx 500 \, \text{km} \, \text{s}^{-1}$, and I use $N_{RT} = 30$ as the fiducial number (based on tests presented in Gnedin 2014), so in the CROC production runs, $\hat{c} \approx 0.1c$. The value of $\hat{c}$ gradually increases as the simulation proceeds, since gravitational clustering and stellar feedback drive gas to progressively higher velocities. It is also higher in larger-box simulations, which include more massive galaxies with higher escape
velocities. By varying $N_{\text{RT}}$ I can implement a different ratio of $\dot{c}/c$ in the simulations.

Figure 1 demonstrates the accuracy of the RSL approximation as implemented in the CROC simulations. The two panels show the ionization histories and star formation histories for three runs, all in $20h^{-1}\text{Mpc}$ boxes (larger box sizes would be too expensive for such a purely technical test), with the effective speed of light $\dot{c}$ set to approximately 3%, 10%, and 30% of $c$ ($N_{\text{RT}} = 10$, 30, and 100, respectively). Clearly, production CROC simulations (with $\dot{c} \gtrsim 0.1c$) are not compromised by the use of the RSL approximation.

This conclusion stands in conflict with the recent claim by Bauer et al. (2015), who found a large difference in the ionization history of the Illustris simulation when they changed the effective speed of light from $c$ to $0.1c$. At face value, that result is surprising—such a change would imply that most ionization fronts in the Illustris simulation propagate much faster than $0.1c$ (otherwise, there would not be any difference). This is in conflict with most studies of reionization, which find that reionization proceeds over a range of redshifts, with duration comparable to the Hubble time $t_{\text{H}}$. In the latter case the typical speed of ionization fronts would be $R_{\text{H}}/t_{\text{H}} = (R_{\text{H}}/R_{\text{H}}) \ll c$, where $R_{\text{H}}$ is the typical size of an ionized bubble and $R_{\text{H}}$ is the Hubble radius. At $z = 6$ the Hubble radius $R_{\text{H}} \sim 3000\text{Mpc}$ in comoving units, so for $R_{\text{H}} \sim 30\text{Mpc}$ (around the largest comoving size fitting into the Illustris simulation volume) the ionization front would move with about $3000\text{km} \text{s}^{-1}$, a speed typically found in many reionization simulations.

In order to explore the potential reasons for that discrepancy, I have also implemented a version of the RSL approximation which also modifies the equation for the cosmic background. Specifically, rewriting Equation (2) with the rhs in a form similar to Equation (5),

$$a \frac{\partial \mathcal{L}_{\nu}^l}{\partial t} + \frac{a}{c} \mathcal{H} \left( \nu \frac{\partial \mathcal{L}_{\nu}^l}{\partial \nu} - 3 \mathcal{L}_{\nu}^l \right) = - \frac{1}{\lambda_{\nu}} \mathcal{L}_{\nu}^l + \mathcal{L}_{\nu}^l,$$  

(6)

I replace both circled $c$ with $0.1c$ similarly to how the RSL approximation is used in Equation (5) (labeled $\dot{c}_{\text{bgr}} = 0.1c$). The ionization history for the so modified background equation is shown in Figure 2 (red line) together with the simulation with the correctly implemented RSL approximation (green line, the same as in Figure 1). Such an (improperly) modified RSL approximation makes a large error in the ionization state of the gas after the overlap, though the effect is still somewhat less significant than the one found by Bauer et al. (2015). Hence, this test emphasizes the importance of using the correct speed of light in the background equation and may also serve as an explanation for the difference between CROC and Illustris reionization models.

The reason for the large error appearing after reionization can be understood as follows. When the mean free path is short, the two terms on the rhs of Equation (2) or (6) are both significantly larger than the lhs and are approximately equal to each other, $S_{\nu} \approx R_{\text{H}}$, with the difference between them (the rhs of (2)) scaling as $\lambda_{\nu}/R_{\text{H}}$, where $R_{\text{H}} = c/H$ is the Hubble radius. Immediately after reionization the mean free path is of the order of 10 physical Mpc (Fan et al. 2006; Songaila & Cowie 2010), and the Hubble radius in physical units is of the order of $400\text{Mpc}$, so, indeed, $\lambda_{\nu}/R_{\text{H}} \ll 1$. However, reducing the speed of light by, say, a factor of 10 in the background Equation (6) is equivalent to reducing the Hubble radius by the same amount, to about 40 proper Mpc; at such a low value $\lambda_{\nu}/R_{\text{H}} \approx 0.25$ is no longer small enough (especially given that the actual ratio between the last term on the lhs and the first term on the rhs is exactly $3\lambda_{\nu}/R_{\text{H}}$), and the equilibrium solution $S_{\nu} \approx R_{\text{H}}/\lambda_{\nu}$ becomes significantly less accurate.

In conclusion, the RSL approximation, when used correctly, remains a robust and accurate numerical trick to lower the computational expense of an explicit momentum-based radiative transfer solver. It does break during the initial stages of a rapidly expanding ionized bubble, as shown by Rosdahl et al. (2013), but even that test is artificial—a strong source does not switch on suddenly in a perfectly neutral IGM.

The process of cosmic reionization is driven by the gradual increase in the production of ionizing photons, and the size of the ionized bubble is determined by the total amount of ionizing photons produced inside—the primary reason why numerous semi-analytical models based on the barrier-crossing formalism of Furlanetto et al. (2004) work so well. Thus, the rate of the propagation of ionization fronts in the bulk of the IGM is controlled by the photon production rate in the sources, not by the photon propagation speed.

One application where the RSL approximation may indeed fail is the rapid turn-on of a bright quasar. Even for that case, failure is not obvious, as a quasar turns on in a pre-existing ionized bubble, but any RSL-based code used for modeling that process needs to be specifically tested in a manner similar to the test shown in Figure 1.

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