STRINGS IN CURVED SPACETIMES

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ABSTRACT

Progress on the physics of strings in curved spacetime are comprehensively reviewed. We start by showing through renormalization group arguments that a meaningful quantum theory of gravity must be finite and must include all particle physics. Then, we review classical and quantum string propagation in curved spacetimes. We start by the general expansion method proposed by de Vega and Sánchez in 1987. The particle transmutation phenomena in asymptotically flat spacetimes are detailed including fermion-boson transitions in supergravity backgrounds. The next chapters review the exactly solvable cases of string propagation: shock waves, singular plane waves, conical spacetimes and de Sitter cosmological spacetime. The calculation of various physical quantities like the string mass and the energy-momentum tensor shows that classical and quantum string propagation in shock-waves and singular plane waves is physically meaningful and full of interesting new phenomena. The important phenomenon of string stretching that takes place when strings fall into spacetime singularities and in expanding universes is analyzed. We conclude by reporting on strings in de Sitter spacetime, where the string equations are integrable and reduce to the sinh-Gordon equation and to integrable generalizations of it.
1. Quantum Gravitation at the light of the Renormalization Group.

The construction of a sensible quantum theory of gravitation is probably the greatest challenge in today’s theoretical physics. Deep problems arise when (second) quantization is combined with general relativity. Statistical phenomena show up (Hawking’s radiation) when free fields are quantized in black-holes backgrounds. This entails a lack of quantum coherence even keeping the gravitational field classical.

Another problem (the most often discussed in this connection) is the one of renormalizability of the Einstein theory (or its various generalizations) when quantized as a local quantum field theory. Actually, even deeper conceptual problems arise when one tries to combine quantum concepts with General Relativity. That is, it may be very well that a quantum theory of gravitation needs new concepts and ideas. Of course, this future theory must have the today’s General Relativity and Quantum Mechanics (and QFT) as limiting cases. In some sense, what everybody is doing in this domain (including string theories approach) may be to the real theory what the old quantum theory in the 10’s was compared with quantum mechanics.

The main drawback to develop a quantum theory of gravitation is clearly the total lack of experimental guides for the theoretical development. Just by dimensional reasons, physical effects combining gravitation and quantum mechanics are relevant only at energies of the order of $M_{\text{Planck}} = \frac{\hbar c}{G} = 1.22 \times 10^{16}$ TeV. Such energies were available in the Universe at times $t < t_{\text{Planck}} = 5.4 \times 10^{-44}$ sec. Anyway, as a question of principle, to construct a quantum theory of gravitation is a problem of fundamental relevance for theoretical physics. In addition, one cannot rule out completely the possibility of some “low energy” ($E \ll M_{\text{Planck}}$) physical effect that could be experimentally tested.

Let us discuss now from a conceptual point of view the renormalizability question for gravitation. What is a renormalizable QFT? It is a theory with a domain
of validity characterized by energies $E$ such that [1]

$$E < \Lambda$$

Here, the scale $\Lambda$ is proper to the model under consideration (e.g., $\Lambda \simeq 1$ GeV for QED, $\Lambda \simeq 100$ GeV for the standard model of strong and electroweak interactions, etc.). One always applies the QFT in question till infinite energy (or zero distance) for virtual processes and finds usually ultra-violet infinities. These divergences reflect the fact that the model becomes unphysical for energies $\Lambda \ll E \leq \infty$. In a renormalizable QFT these infinities can be absorbed in a finite number (usually a few) parameters like coupling constants and mass ratios. Since these parameters (usually called renormalized masses and couplings) are not predicted by the model in question, one has to fit them to their experimental values. One needs a more general theory valid at energies beyond $\Lambda$ in order to compute these renormalized parameters (presumably from more fundamental constants). For example, $M_W/M_Z$ is calculable in a GUT whereas it must be fitted to its experimental value in the standard electroweak model.

Let us now see what are the consequences of Heisenberg’s principle in quantum mechanics combined with the notion of gravitational (Schwarzschild) radius in General Relativity. Assume we make two measurements at a very small distance $\Delta x$. Then,

$$\Delta p \sim \Delta E \sim 1/\Delta x$$

where we set $\hbar = c = 1$. For sufficiently large $\Delta E$, particles with masses $m \sim 1/\Delta x$ will be produced. The gravitational radius of such particles are of the order

$$R_G \sim Gm \sim \frac{(l_{\text{Planck}})^2}{\Delta x} \quad (1.1)$$

where $l_{\text{Planck}} \sim 10^{-33}$ cm. Now, General Relativity allows measures at a distance
\( \Delta x \), provided

\[
\Delta x > R_G \, \rightarrow \, \Delta x > \frac{(l_{\text{Planck}})^2}{\Delta x}.
\]

That is,

\[
\Delta x > l_{\text{Planck}} \, \text{ or } \, m < M_{\text{Planck}}
\]

(1.2)

This means that no measurements can be made at distances smaller than the Planck length and that no particle can be heavier than \( M_{\text{Planck}} \). This is a simple consequence of relativistic quantum mechanics combined with General Relativity.

In addition, the notion of locality and hence of spacetime becomes meaningless at the Planck scale. Notice that the equality in eq.(1.2) corresponds when the Compton length equals the Schwarzschild radius of a particle. Since \( M_{\text{Planck}} \) is the heaviest possible particle scale, a theory valid there (necessarily involving quantum gravitation) will also be valid at any lower energy scale. One may ignore higher energy phenomena in a low energy theory, but not the opposite. In other words, it will be a ‘theory of everything’. We think that this is the key point on the quantization of gravity. A theory that holds till the Planck scale must describe all what happens at lower energies including all known particle physics as well as what we do not know yet (that is, beyond the standard model). Notice that this conclusion is totally independent of the use of string models. A direct important consequence of this conclusion, is that it does not make physical sense to quantize pure gravity. A physically sensible quantum theory cannot contain only gravitons. To give an example, a theoretical prediction for graviton-graviton scattering at energies of the order of \( M_{\text{Planck}} \) must include all particles produced in a real experiment. That is, in practice, all existing particles in nature, since gravity couples to all matter.

Let us now come to the renormalizability problem for gravitation. As is clear from the preceding discussion, we have \( \Lambda \simeq M_{\text{Planck}} \) for gravitation. There cannot be any theory of particles beyond it. Therefore, if ultraviolet divergences appear in a quantum theory of gravitation, there is no way to interpret them as coming from
a higher energy scale as it is usually done in QFT (see above). That is, no physical understanding can be given to such UV infinities. The only logically consistent possibility would be to find a finite theory of quantum gravitation which is a TOE.

These simple arguments, based on the renormalization group lead us to an important conclusion: a consistent quantum theory of gravitation must be a finite theory and must include all other interactions. That is, it must be a theory of everything (TOE). This is a very ambitious project. In particular it needs the understanding of the present desert between 1 and $10^{16}$ TeV. There is an additional dimensional argument about the inference Quantum Theory of Gravitation $\rightarrow$ TOE. There are only three dimensional physical magnitudes in nature: length, energy and time and correspondingly only three dimensional constants in nature: c, h and G. All other physical constants like $\alpha = 1/137, 04..., M_{\text{proton}}/m_{\text{electron}}, \theta_{WS} ,...$ etc. are pure numbers and they must be calculable in a TOE. This is a formidable, but extremely appealing problem. From the theoretical side, the only serious candidate for a TOE is at present string theory. This is why we think that strings deserve a special attention in order to quantize gravity.

As a first step on the understanding of quantum gravitational phenomena in a string framework, we started in 1987 a programme of string quantization on curved spacetimes [2,3]. The investigation of strings in curved spacetimes is currently the best framework to study the physics of gravitation in the context of string theory, in spite of its limitations. First, the use of a continuous Riemanian manifold to describe the spacetime cannot be valid at scales of the order of $l_{\text{Planck}}$. More important, gravitational backgrounds effectively provide classical or semiclassical descriptions even if the matter backreaction to the geometry is included through semiclassical Einstein equations (or stringy corrected Einstein equations) by inserting the expectation value of the string energy-momentum tensor in the r.h.s. One would want a full quantum treatment for matter and geometry. However, to find a formulation of string theory going beyond the use of classical backgrounds is a very difficult (but fundamental!) problem. One would like to derive the spacetime
geometry as a classical and low energy ($\ll M_{\text{Planck}}$) limit from the solution of (quantum) string theory.

2. Strings in Curved Spacetimes. Introduction.

Let us consider bosonic strings (open or closed) propagating in a curved $D$-dimensional spacetime defined by a metric $G_{AB}(X), 0 \leq A, B \leq D - 1$. The action can be written as

$$S = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{g} g_{\alpha\beta}(\sigma, \tau) G_{AB}(X) \partial^\alpha X^A(\sigma, \tau) \partial^\beta X^B(\sigma, \tau)$$

(2.1)

Here $g_{\alpha\beta}(\sigma, \tau)$ ( $0 \leq \alpha, \beta \leq 1$ ) is the metric in the worldsheet, $\alpha'$ stands for the string tension. As in flat spacetime, $\alpha' \sim (M_{\text{Planck}})^{-2} \sim (l_{\text{Planck}})^2$ fixes the scale in the theory. There are no other free parameters like coupling constants in string theory. Besides eq. (2.1) which is the curved spacetime version of the Brink-DiVecchia-Howe-Polyakov action [12], one can also start from the Goto-Nambu action [13] which is classically equivalent to (2.1).

The string action (2.1) classically enjoys Weyl invariance on the world sheet

$$g_{\alpha\beta}(\sigma, \tau) \rightarrow \lambda(\sigma, \tau) g_{\alpha\beta}(\sigma, \tau)$$

(2.2)

plus the reparametrization invariance

$$\sigma \rightarrow \sigma' = f(\sigma, \tau) , \quad \tau \rightarrow \tau' = g(\sigma, \tau)$$

(2.3)

The dynamical variables being here the string coordinates $X_A(\sigma, \tau), (0 \leq A \leq D - 1)$ and the world-sheet metric $g_{\alpha\beta}(\sigma, \tau)$. Extremizing $S$ with respect to them
yields the classical equations of motion:

$$\partial^a [\sqrt{g} G_{AB}(X) \partial_a X^B(\sigma, \tau)] = (1/2) \sqrt{g} \partial_A G_{CD}(X) \partial_a X^C(\sigma, \tau) \partial^a X^D(\sigma, \tau)$$

$$0 \leq A \leq D - 1$$

(2.4)

$$T_{\alpha\beta} \equiv G_{AB}(X)[\partial_\alpha X^A(\sigma, \tau) \partial_\beta X^B(\sigma, \tau)$$

$$-(1/2) g_{\alpha\beta}(\sigma, \tau) \partial_\gamma X^A(\sigma, \tau) \partial^\gamma X^B(\sigma, \tau)] = 0 \ , \ 0 \leq \alpha, \beta \leq 1.$$  

(2.5)

Eqs.(2.5) contain only first derivatives and are therefore a set of constraints. Classically, we can always use the reparametrization freedom (2.3) to recast the worldsheet metric on diagonal form

$$g_{\alpha\beta}(\sigma, \tau) = \exp[\phi(\sigma, \tau)] \ \text{diag}(-1, +1)$$

In this conformal gauge, eqs.(2.4) - (2.5) take the simpler form:

$$\partial_{++} X^A(\sigma, \tau) + \Gamma^A_{BC}(X) \partial_{+} X^B(\sigma, \tau) \partial_{-} X^C(\sigma, \tau) = 0 \ , \ 0 \leq A \leq D - 1,$$

(2.6)

$$T_{\pm\pm} \equiv G_{AB}(X) \partial_{\pm} X^A(\sigma, \tau) \partial_{\pm} X^B(\sigma, \tau) \equiv 0, \ T_{+-} \equiv T_{-+} \equiv 0 \ \ (2.7)$$

where we introduce light-cone variables $x_{\pm} \equiv \sigma \pm \tau$ on the world-sheet and where $\Gamma^A_{BC}(X)$ stand for the Christoffel symbols associated to the metric $G_{AB}(X)$.

The string boundary conditions in curved spacetimes are identical to those in Minkowski spacetime. That is,

$$X^A(\sigma + 2\pi, \tau) = X^A(\sigma, \tau) \hspace{1cm} \text{closed strings}$$

$$\partial_\sigma X^A(0, \tau) = \partial_\sigma X^A(\pi, \tau) = 0 \hspace{1cm} \text{open strings}$$

(2.8)

In flat spacetime eqs.(2.6) are linear and one can solve them explicitly as well as the quadratic constraint (2.7) [14]. The solution of eqs.(2.6) in Minkowski spacetime
is usually written for closed strings as

\[ X^A(\sigma, \tau) = q^A + 2p^A \alpha' \tau + i \sqrt{\alpha'} \sum_{n \neq 0} \left\{ \alpha_n^A \exp[in(\sigma - \tau)] + \tilde{\alpha}_n^A \exp[-in(\sigma + \tau)] \right\} / n \]  

(2.9)

where \( q^A \) and \( p^A \) stand for the string center of mass position and momentum and \( \alpha_n^A \) and \( \tilde{\alpha}_n^A \) describe the right and left oscillator modes of the string, respectively. This resolution is no more possible in general for curved spacetime where the equations of motion (2.6) are non-linear. In that case, right and left movers interact with themselves and with each other. For some spacetimes eqs.(2.6) - (2.7) are exactly solvable [see secs. 4-6]. In addition, the question of how to define particle states in curved spacetime appears since no preferred frames exist in General Relativity. These questions already appeared in field theory (for the string case, see ref.[10]). In all our treatment we consider a given geometry \( G_{AB}(X) \) where our strings propagate. That is, we are dealing with test strings and we neglect for the moment their backreaction on the metric. Up to now, the string propagation has been solved in the following spacetimes thanks to specific features proper to each of them:

1) Linear graviton [9].
2) Shock-wave spacetimes (Aichelburg-Sexl metrics and generalizations) [4-5].
3) Non-linear gravitational plane waves [6-21].
4) Conical spacetimes (the geometry around a cosmic string)[11].
5) De Sitter cosmological spacetime [7].

Let us describe now the general scheme proposed in ref.[2] to solve the string equation of motion and constraints both classically and quantum mechanically. The principle is the following, we start from an exact solution of eq.(2.4) and develop in perturbations around it. We set

\[ X^A(\sigma, \tau) = q^A(\sigma, \tau) + \eta^A(\sigma, \tau) + \xi^A(\sigma, \tau) + ..... \]  

(2.10)

Here \( q^A(\sigma, \tau) \) is an exact solution of eq.(2.4) and \( \eta^A(\sigma, \tau) \) obeys the linearized
perturbation around $q^A(\sigma, \tau)$:

$$D_B^A \eta^B(\sigma, \tau) = 0 \quad (2.11)$$

where

$$D_B^A = \delta_B^A \partial^2 + \Gamma^A_{BC}(q)(\partial_- q^C \partial_+ + \partial_+ q^C \partial_-) + \partial_B \Gamma^A_{CD}(q) \partial_+ q^C \partial_- q^D \quad (2.12)$$

Here $\partial^2 \equiv \partial_{--}$ and $\xi_A(\sigma, \tau)$ is a solution of the second order perturbation around $q_A(\sigma, \tau)$

$$D_B^A \xi^B(\sigma, \tau) = -\eta^D \partial_D \Gamma^A_{BC}(q)(\partial_- q^C \partial_+ + \partial_+ q^C \partial_-) \eta^B - \Gamma^A_{BC}(q) \partial_+ \eta^B \partial_- \eta^C - (1/2) \eta^D \eta^E \partial^2 \Gamma^A_{DE}(q) \partial_+ q^B \partial_- q^C \quad (2.13)$$

Higher order perturbations can be considered systematically, but we will restrict here to first and second orders. Notice that eq.(2.11) is homogeneous whereas eq.(2.13) is inhomogeneous with the r.h.s. quadratic in the $\eta$’s, solutions of eq.(2.11).

The choice of the starting solution is upon physical insight. Usually we start from the solution describing the center of mass motion of the string $q^A(\tau)$ where

$$\ddot{q}^A(\tau) + \Gamma^A_{BC}(q) q^B(\tau) \dot{q}^C(\tau) = 0 \quad (2.14)$$

The world-sheet time variable is here identified with the proper time of the center of mass trajectory. It must be noticed that we are treating the space-time geometry exactly and taking the string oscillations around $q^A(\sigma, \tau)$ [for example the centre of mass solution $q^A(\tau)$ of eq.(2.14)] as perturbations. So, our expansion corresponds to low energy excitations of the string as compared with the energy associated to the geometry. In a cosmological or black hole metric, our method corresponds to an expansion in $\omega/M$, where $\omega$ is the string frequency mode and $M$ is the universe mass or the black hole mass respectively. This can be equivalently considered as
an expansion in powers of $\sqrt{\alpha'}$. Actually, since $\alpha' \sim (l_{Planck})^2$, the expansion parameter turns out to be the dimensionless constant

$$g \equiv l_{Planck}/R_c \simeq 1/(l_{Planck} M) \simeq \omega/M$$ (2.15)

where $R_c$ characterizes the curvature of the space-time under consideration. In most of the interesting situations one clearly has $g \ll 1$.

The constraint equations (2.5) must also be expanded in perturbations. We find up to terms of order higher than the second

$$2\pi \alpha' T_{\pm\pm} = G_{AB}(q) \partial_{\pm} q^A \partial_{\pm} q^B + \eta^C \partial_C G_{AB}(q) \partial_{\pm} q^A \partial_{\pm} q^B$$

$$+ 2 G_{AB}(q) \partial_{\pm} q^A \partial_{\pm} \eta^B + (1/2) \eta^C \eta^D \partial_{CD} G_{AB}(q) \partial_{\pm} q^A \partial_{\pm} q^B$$

$$+ G_{AB}(q) \partial_{\pm} \eta^A \partial_{\pm} \eta^B + 2 G_{AB}(q) \partial_{\pm} q^A \partial_{\pm} \xi^B$$

$$+ 2 \eta^C \partial_C G_{AB}(q) \partial_{\pm} q^A \partial_{\pm} \eta^B + \xi^C \partial_C G_{AB}(q) \partial_{\pm} q^A \partial_{\pm} q^B = 0$$

(2.16)

From this, we define the Virasoro generators $L_n$ as

$$T_{\pm\pm} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} L_n \exp[in(\sigma \pm \tau)]$$ (2.17)

It must be noticed that one must solve the linear equation (2.13), expressing the solution $\xi^A(\sigma, \tau)$, as a bilinear functional of $\eta^A(\sigma, \tau)$. Then, one must insert the $\eta^A(\sigma, \tau)$ and $\xi^A(\sigma, \tau)$ into the constraints (2.16) in order to obtain the $L_n$. In this approximation the $L_n$ turn to be bilinear in the $\eta^A(\sigma, \tau)$.

Let us now consider the case where the zeroth-order (exact) solution $q_A(\tau)$ describes the center of mass motion. In this case the fluctuation equations simplify to a set of ordinary differential equations [2]. That is, setting

$$\eta^A(\sigma, \tau) = \sum_{n \in \mathbb{Z}} e^{in\sigma} g_n^A(\tau)$$ (2.18)

the $g_n^A(\tau)$ satisfy a system of $D$ coupled ordinary differential equations:

$$\left[ \delta^A_B \left( \frac{d^2}{d\tau^2} + n^2 \right) + 2 \Gamma^A_{BC}(q) \dot{q}^C(\tau) + \dot{q}^C(\tau) \dot{q}^D(\tau) \partial_B \Gamma^A_{CD}(q) \right] g_n^B(\tau) = 0$$ (2.19)

Notice that the different n-modes are decoupled. If we apply our method to
Minkowski spacetime, the zero order solution of eq.(2.14), \( q^A(\tau) \), plus the first fluctuations \( \eta^A(\sigma, \tau) \) provide the exact solution of the string equations.

The use of this expansion method applied to cosmological spacetimes showed the appearance of unstabilities in the case of de Sitter [2,8]. That is, exponentially growing modes \( \eta^A(\sigma, \tau) \). In principle, one cannot believe anymore an expansion like this when the first order blows up. However, it turns out that these unstabilities far to be artifacts of the approximation used, revealed a true physical phenomenon: string stretching. That is, the string size grows indefinetely with time. This phenomenon, typical of strings falling into a singularity or in an inflationary universe is confirmed by the exact resolution of the string equations (see secs.4 and 6 and ref.[6,7]).

3. Strings in asymptotically flat spacetimes. Particle Transmutation.

When the metric \( G_{AB}(X) \) admits flat spacetime regions, say (+) and (-) (i.e. \( G_{AB}(X) \to \eta_{AB} \) ), as it is the case for black-holes, for shock-wave spacetimes and generally for asymptotically flat geometries, then the equations of motion (2.6) become in these regions

\[
\partial^2 X^A(\sigma, \tau)_\pm = 0
\]

In particular, \( \partial^2 U(\sigma, \tau) = 0 \) (where \( U = X^0 - X^{D-1} \)), and the light-cone gauge identification can be chosen

\[
U = 2 \alpha' p^U \tau
\]

This enables us to define ingoing and outgoing solutions for \( \tau \to -\infty \) and \( \tau \to +\infty \), respectively. That is,

\[
X^A(\sigma, \tau)_\pm = q^A_{\pm} + 2p^A_{\pm} \alpha' + i\sqrt{\alpha'} \sum_{n \neq 0} \{ \alpha^A_{n\pm} \exp[in(\sigma - \tau)] + \tilde{\alpha}^A_{n\pm} \exp[-in(\sigma + \tau)] \} / n
\]

The connection between the out operators \( \{ \alpha^A_{n+}, \tilde{\alpha}^A_{n+}, n \in \mathbb{Z} (n \neq 0), q^A_-, p^A_-, 0 \leq \}

\( A \leq D - 1 \) and the in operators \( \{ \alpha^A_{n-}, \tilde{\alpha}^A_{n-}, n \in \mathbb{Z} \ (n \neq 0), q^A, p^A; 0 \leq A \leq D - 1 \} \)
depends on the interaction with the geometry in the non-flat region \( s \). In general, the exact relation between \((+)\) and \((-)\) operators is highly non-trivial, involving all the in operators through a non-linear transformation

\[
\alpha^A_{n+} = F^A_n(\alpha^B_{m-}, \tilde{\alpha}^B_{m-}, q^B, p^B; m \epsilon \mathbb{Z} \ (m \neq 0), 0 \leq B \leq D - 1) \\
\tilde{\alpha}^A_{n+} = G^A_n(\alpha^B_{m-}, \tilde{\alpha}^B_{m-}, q^B, p^B; m \epsilon \mathbb{Z} \ (m \neq 0), 0 \leq B \leq D - 1)
\]

(3.4)

This transformation contains in principle all the information about the scattering of the string by the geometry. However, it is very difficult to obtain the functions \( F^A_n \) except for the exactly solvable cases (see secs.4-5). It is possible to make a detailed analysis in the context of the linearization method described in sec.II. [eqs.(2.11) - (2.14)]. In this approximation eq.(3.4) becomes a linear, e.g. a Bogoliubov transformation. We have in this scheme,

\[
\partial^2 \eta^A(\sigma, \tau)_{\pm} = 0
\]

(3.5)

We can define a basis of solutions \( f^{AB}_{n\pm}(\tau) \) of eq.(2.19) by selecting their asymptotic behavior:

\[
\lim_{\tau \to \pm\infty} f^{AB}_{n\pm}(\tau) = \delta^{AB} \exp(-in\tau)
\]

(3.6)

The choice of a positive frequency factor \( \exp(-in\tau) \) corresponds to in or out particle states for \( \tau \to -\infty \) or \( \tau \to +\infty \) respectively. For each value \((A, B)\) we have two basis \((-)\) and \((+)\). Since we have free oscillators in both asymptotic limits \( \tau \to \pm\infty \),

\[
\lim_{\tau \to \mp\infty} f^{AB}_{n\pm}(\tau) = A^{AB}_{n\pm} \exp(-in\tau) + B^{AB}_{n\pm} \exp(+in\tau)
\]

(3.7)

where the constant coefficients \( A^{AB}_{n\pm} \) and \( B^{AB}_{n\pm} \) depend on the detailed time evolution for all \( \tau \). We can expand the string fluctuations \( \eta^A(\sigma, \tau) \) in the basis \( f^{AB}_{n\pm}(\tau) \) and
in the basis $f_{AB}^{n-}(\tau)$ as:

$$
\eta^A(\sigma, \tau) = i \sqrt{\alpha'} \sum_{n \neq 0} \sum_{B=0}^{D-1} f_{AB}^{n}\{\alpha_{n+}^B \exp[i n \sigma] + \tilde{\alpha}_{n+}^B \exp[-i n \sigma]\} / n
$$

Then, from eqs.(3.6) -(3.8), we have

$$
\eta^A(\sigma, \tau)_- = \lim_{\tau \to -\infty} \eta^A(\sigma, \tau) =
\eta^A(\sigma, \tau)_+ = \lim_{\tau \to +\infty} \eta^A(\sigma, \tau) =
$$

$$
\begin{align*}
\eta^A(\sigma, \tau)_- &= i \sqrt{\alpha'} \sum_{n \neq 0} \exp[-i n \tau] \{\alpha_{n-}^A \exp[i n \sigma] + \tilde{\alpha}_{n-}^A \exp[-i n \sigma]\} / n \\
\eta^A(\sigma, \tau)_+ &= i \sqrt{\alpha'} \sum_{n \neq 0} \exp[-i n \tau] \{\alpha_{n+}^A \exp[i n \sigma] + \tilde{\alpha}_{n+}^A \exp[-i n \sigma]\} / n
\end{align*}
$$

and

$$
\begin{align*}
\alpha_{n+}^A &= \sum_{B=0}^{D-1} \left[ A_{n-}^{AB} \alpha_{n-}^B - B_{n-}^{AB} (\tilde{\alpha}_{n-}^B)\right] \\
\tilde{\alpha}_{n+}^A &= \sum_{B=0}^{D-1} \left[ A_{n-}^{AB} \tilde{\alpha}_{n-}^B - B_{n-}^{AB} (\alpha_{n-}^B)\right]
\end{align*}
$$

This is a Bogoliubov transformation, giving the outgoing operators $\alpha_{n+}^B, \tilde{\alpha}_{n+}^B$ as a linear superposition of the creation and annihilation ingoing operators $\alpha_{n-}^B, \tilde{\alpha}_{n-}^B$.

We see that for fixed $n$, transitions take place between the internal oscillatory modes of the string. The scattering of the string by the curved geometry involves two main effects:

(i) Polarization changes in the modes (without changing their right or left character).

(ii) A mixing of particle and antiparticle modes, changing at the same time their right or left character.

In other words: if the ingoing string has a right (or left) excited mode with a given polarization $B$, there will be in the outgoing state: (i) an amplitude $A_{n\pm}^{AB}$
for a right (or left) mode polarized in the A-direction and (ii) an amplitude $B_{n\pm}^{AB}$ for a left (or right) antiparticle mode polarized in the A-direction.

In field theory, mixing of particle and antiparticle modes usually means pair particle production. This is not the case here. A string always describes a single particle. Indeed, this particle is not of a fixed kind, but depends on the excitation state of the string. A scalar for the ground state, a vector particle for a one quantum excitation (in open strings), a tensor particle (graviton) for a two mode excitation (in closed strings), etc. [14]. Here, the mixing (ii) of modes and antimodes (3.10) implies inelastic processes changing the string excitation state due to the interaction with the geometry. In this process the initial particle of mass $m$ and spin $s$ transmutes into a different final particle of mass $m'$ and spin $s'$. We also find elastic processes (i) in which initial and final states describe the same particle (although the momentum and the spin polarization may change) [16]. Concrete examples of Bogoliubov transformations of the type (3.10) and the ensuing particle transmutations are reported below in secs. (4) and (5). It should be noticed that the particle transmutation phenomena appears at the zero genus level without introducing string loop corrections.

The Bogoliubov transformation (3.10) can be written as

$$
\alpha_{n+}^A = \exp(\mathcal{G}) \alpha_{n-}^A \exp(-\mathcal{G}) \quad \tilde{\alpha}_{n+}^A = \exp(\mathcal{G}) \tilde{\alpha}_{n-}^A \exp(-\mathcal{G})
$$

(3.11)

with $\mathcal{G} = -\mathcal{G}^\dagger$. The in and out vacua verify

$$
\alpha_{n-}^A |O_- > = \tilde{\alpha}_{n-}^A |O_- > = 0 \quad , \quad \alpha_{n+}^A |O_+ > = \tilde{\alpha}_{n+}^A |O_+ > = 0,
$$

(3.12)

for all $n \geq 1$. They are related by

$$
|O_+ > = \exp(\mathcal{G}) |O_- >
$$

(3.13)

Since the coefficients of the transformation eq.(3.10) are not necessarily real, $\mathcal{G}$ in
eq.(3.10) has the form:

\[ \mathcal{G} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{A,B=0}^{D-1} \mathcal{D}_{n}^{AB} \alpha_{n-}^{A} \tilde{\alpha}_{n-}^{B} - (\mathcal{D}_{n}^{AB})^*(\alpha_{n-}^{A})^{\dagger}(\tilde{\alpha}_{n-}^{B})^{\dagger} + \]

\[ + \left[ \mathcal{C}_{n}^{AB} - (\mathcal{C}_{n}^{AB})^* \right] \left[ \alpha_{n-}^{A}(\alpha_{n-}^{B})^{\dagger} + \tilde{\alpha}_{n-}^{A}(\tilde{\alpha}_{n-}^{B})^{\dagger} \right] \] (3.14)

At the first order in $g^{2}$, the coefficients $\mathcal{D}_{n}^{AB}$ and $\mathcal{C}_{n}^{AB}$ are related with the $\mathcal{A}_{n-}^{AB}$ and $\mathcal{B}_{n-}^{AB}$ through

\[ \mathcal{A}_{n-}^{AB} = \delta^{AB} + \mathcal{C}_{n}^{AB} - (\mathcal{C}_{n}^{AB})^* \]

\[ \mathcal{B}_{n-}^{AB} = (\mathcal{D}_{n}^{AB})^* \] (3.15)

Therefore, from eqs.(3.13) - (3.15), the in-out vacuum transition amplitude is given at first order by

\[ < O_+ | O_{-} > = < O_{-} | 1 - \mathcal{G} | O_{-} > = 1 - 4i \sum_{A=0}^{D-1} \sum_{n=1}^{\infty} \text{Im} \mathcal{C}_{n}^{AB} \] (3.16)

We see that the out-vacuum follows from the in-vacuum by filling it with ingoing mode pairs (here a mode pair is formed by a right and a left mode). Conversely, the in vacuum as seen by out observers is a superposition of all kind of out particle states. In particular, eq.(3.16) gives the amplitude to find the lightest scalar particle in the out state, when the ingoing state was precisely this lightest scalar. This effect is also a manifestation of the composite character of the strings. The infinite set of oscillator modes, constituting the string, become excited during the scattering by the influence of the gravitational field. In fact, any localized external field (gravitational or not) would lead to qualitatively similar effects.

The symmetric and traceless part of $\mathcal{B}_{1-}^{AB}$ gives the transition amplitude from the ground state $|O_{-} >$ to an outgoing graviton. The transition amplitude from $|O_{-} >$ to a final massless scalar (dilaton) is given by the trace of $\mathcal{B}_{1-}^{AB}$ and the antisymmetric part of $\mathcal{B}_{1-}^{AB}$ gives the transition amplitude from $|O_{-} >$ to a massless scalar.
antisymmetric tensor. The transition from \(|O_-\rangle\) to a higher massive state

\[
K \prod_{s=1}^{K} (\alpha_{n_-}^i)^\dagger (\tilde{\alpha}_{n_-}^B)^\dagger |0_-\rangle
\]

is in general non-zero and proportional to \((\mathcal{B}_{n_-}^{ij})^K\).

Here, we have considered bosonic strings. Particle transmutation takes also place for superstrings where the present discussion has been generalized in ref.[16]. We find that the massless particles cannot transmute among themselves for bosonic backgrounds (see below for fermionic backgrounds). Only elastic transitions between ingoing and outgoing massless particles are allowed. We observe that this is a general property exact to all orders in \(\sqrt{\alpha'}\), due to the fact that the transformation between the fermion oscillator operators \(S_{n_-}^A\) preserve the boson-fermion parity. The transitions from the massless states to a state with an even number of fermion operators \(S_{n_-}^A\) and an even number of boson operators \(\alpha_{m_-}^B\) are non-zero. Transitions from the ground states (massless) to high massive states

\[
K \prod_{s=1}^{K} (\alpha_{n_-}^i)^\dagger (\tilde{\alpha}_{n_-}^B)^\dagger \prod_{r=1}^{L} (S_{n_-}^A)^\dagger (\tilde{S}_{n_-}^B)^\dagger |0_-\rangle
\]

are non-zero and proportional to \((\mathcal{B}_{n_-}^{ij})^K(\mathcal{H}_{n_-}^{AB})^L\), where \(\mathcal{H}_{n_-}^{AB}\) stand for the fermionic Bogoliubov coefficients mixing \(S_{n_-}^A\) with \(\tilde{S}_{n_-}^B\). It must be noticed that several features, like the index structure of the Bogoliubov coefficients follow directly from the symmetry properties of the spacetime geometry. However, to find their explicit form, one must solve the string equations in the interaction region, close to the scattering center. In refs.[3] we solved the string equations to second order in \(g\) using the expansion method (eq.(2.10)) around the center of mass solution for the Schwarzschild black hole. We found in this way explicit formulas for \(A_{n_-}^{AB}\) and \(B_{n_-}^{AB}\).

In ref.[17], we propose the first to introduce a background containing also fermionic degrees of freedom, solution of the N=1 supergravity equations in D=8
dimensions. We found in [17] an explicit $N = 1$ linearized supergravity shockwave solution giving the spin-3/2 Rarita-Schwinger field (gravitino), and an exact (full non-linear) gravitational shockwave bosonic part. We studied then a Green-Schwartz superstring propagating in such supergravity background. We wrote and solved the superstring equations. Contrarily to the purely bosonic shockwave case (in which spinor-string propagation is free [4]), spinor-string coordinates couple here non-trivially to the transverse bosonic coordinates through the fermionic background. We found that outgoing fermionic-modes are mixed with ingoing bosonic (and fermionic) modes. This leads to a new feature of particle transmutation between bosons and fermions described by the superstring ground states (and also by the excited states). The presence of the bosonic-fermionic background provides in string theory a natural physical mechanism to transform bosons into fermions (and vice versa).

A particular consequence of this fact is the existence of non-zero transitions among the superstring ground states ; this means we have transmutation of massless fermions into massless bosons and vice versa. (This massless super Yang-Mills $N=1$, $D=10$ multiplet describes physical particles upon supersymmetry breaking at energies much lower than the Planck and grand unification scales). In ref.[18], we analysed and computed explicitly the more relevant and new particle transmutations that take place when an open superstring propagates through the supergravitational shock-wave previously found in [17]. Transition amplitudes among the ground states, first and second excited states are explicitly given in ref.[18].
4. Exactly solvable cases of strings propagating on curved geometries: shock-waves and nonlinear plane waves.

We consider in this section geometries where the full non-linear string equations (2.6) and constraints (2.7) can be exactly solved in closed form. We start by considering shock-waves. They describe the geometry around an ultrarelativistic source propagating in a fixed direction. The string equations are solvable both for point-like sources and for transversely extended sources[4]. Point-like sources correspond to an infinitely boosted black-hole and it is known as the Aichelburg-Sexl metric. The solvability of field[19] and string equations[4] in such geometries follow from the everywhere flatness of the spacetime except on the shock-wave plane $U \equiv X - T = 0$. An interesting non-trivial dynamics is described by the string solution which is obtained by matching flat spacetime string solutions across this plane.

The shock-wave spacetimes are described by the metric

$$(ds)^2 = -dUdV + \sum_{i=1}^{D-2} (dX^i)^2 - \delta(U) f_D(\vec{X}) (dU)^2 \quad (4.1)$$

where $U$ and $V$ are null coordinates, $U \equiv X^0 - X^{D-1}$, $V \equiv X^0 + X^{D-1}$, $X^i$, $1 \leq i \leq D-2$ are transverse spatial coordinates and $\vec{X} = (X^1, ..., X^{D-2})$. The function $f_D(\vec{X})$ obeys the equation

$$\partial_i^2 f_D(\vec{X}) = 16\pi G \tilde{p} \mu(\vec{X}) \quad (4.2)$$

whose solution can be written as

$$f_D(\vec{X}) = -16\pi G \tilde{p} \int \frac{d^{D-2}k}{k^2} \hat{\mu}(\vec{k}) \exp i\vec{k}.\vec{X} \quad (4.3)$$

where

$$\hat{\mu}(\vec{k}) = \int \frac{d^{D-2}X}{(2\pi)^{D-2}} \mu(\vec{X}) \exp -i\vec{k}.\vec{X}$$

and $G$ is the gravitational constant. This metric describes the gravitational field of an ultrarelativistic source moving along the $X^{D-1}$ axis with momentum $\tilde{p} \cdot \mu(\vec{X})$.
stands for the mass density as a function of the transverse coordinates. For a point-like source \((\mu(\vec{X}) = \delta(\vec{X}))\) we find the Aichelburg-Sexl metric (AS):

\[
f_D(\rho) = K \rho^{4-D}, \quad \text{for } D > 4 \quad \text{and} \quad f_4(\rho) = 8G\tilde{\rho} \log \rho,
\]

where \(\rho \equiv \sqrt{\vec{X} \cdot \vec{X}}\), \(K \equiv -\frac{8\pi^{2-D/2}}{D-4} \Gamma(D/2 - 2) G\tilde{\rho}\). Notice that the space-time is everywhere flat except on the plane \(U = X^0 - X^{D-1} = 0\), where a shock wave is located.

The Riemann tensor follows from the metric (4.1), with the result

\[
R^{UiUj} = (1/2) \delta(U) \partial_i \partial_j f_D(\vec{X})
\]

The string equations of motion (2.6) take the following form for the shock-wave (4.1)

\[
U'' - \ddot{U} = 0
\]

\[
V'' - \ddot{V} + f_D(\vec{X}) \left[(U')^2 - (\dot{U})^2\right] \delta'(U) + 2\partial_i f_D(\vec{X}) \left[U'^i - \ddot{U} \dot{X}^i\right] \delta(U) = 0
\]

\[
X'^i - \ddot{X}^i + (1/2) \partial_i f_D(\vec{X}) \left[(U')^2 - (\dot{U})^2\right] \delta(U) = 0
\]

where \(\cdot'\) and \(\cdot\) stand for \(\frac{\partial}{\partial \sigma}\) and \(\frac{\partial}{\partial \tau}\), respectively.

Since \(U(\sigma, \tau)\) obeys the d’Alembert equation, we can always make a conformal transformation \(\sigma \pm \tau \rightarrow F_{\pm}(\sigma \pm \tau)\), such that

\[
U = 2 \alpha' p^U \tau
\]

where \(p^U\) is constant. Therefore, the equations of motion (4.6) become

\[
V'' - \ddot{V} - f_D(\vec{X}(\sigma, 0)) \delta'(\tau) + \dot{f}_D(\vec{X}(\sigma, \tau)) \delta(\tau) = 0
\]

\[
X'^i - \ddot{X}^i - \alpha' p^U \partial_i f_D(\vec{X}(\sigma, 0))
\]

The string coordinates satisfy the d’Alembert equation for all \((\sigma, \tau)\) except at \(\tau = 0\), where \(\dot{X}^i(\sigma, \tau)\), \(V(\sigma, \tau)\) and \(\ddot{V}(\sigma, \tau)\) are discontinuous. One finds from eq. (4.8)
Here \( X^i_<(\sigma, \tau) \) and \( X^i_> (\sigma, \tau) \) stand for the string coordinates before \((\tau < 0)\) and after \((\tau > 0)\) the collision with the shock wave. Notice that \( U_> = U_< = 2\alpha' p^U \tau \) is continuous at \( \tau = 0 \). The last term in the first equation in (4.8) is indeed ambiguous due to the discontinuity of \( \dot{X}^i(\sigma, \tau) \) at \( \tau = 0 \). This ambiguity has been solved in ref.[5].

Since \( X^i_<(\sigma, \tau) \) and \( X^i_>(\sigma, \tau) \) obey the flat space-time (d’Alembert) equations, we can write the string solutions as in Minkowski spacetime [cfr. eq.(2.9)]

\[
X^A_<(\sigma, \tau) = q^A_< + 2p^A_< \alpha' \tau + \\
+ i\sqrt{\alpha'} \sum_{n \neq 0} \{\alpha^A_< n \exp[in(\sigma - \tau)] + \tilde{\alpha}^A_< n \exp[-in(\sigma + \tau)]\}/n \quad \text{for} \ \tau < 0
\]

\[
X^A_>(\sigma, \tau) = q^A_> + 2p^A_> \alpha' \tau + \\
+ i\sqrt{\alpha'} \sum_{n \neq 0} \{\alpha^A_> n \exp[in(\sigma - \tau)] + \tilde{\alpha}^A_> n \exp[-in(\sigma + \tau)]\}/n \quad \text{for} \ \tau > 0
\]

Inserting eqs.(4.10) in the discontinuity relations (4.9) yields the following matching relations:

\[
q^i_< - q^i_> = 0 \ , \ p^i_< - p^i_> = \frac{p^U \tilde{K}}{2\pi} \int_0^{2\pi} d\sigma D^i(\sigma) \ , 
q^V_< - q^V_> = -\frac{K}{2\pi} \int_0^{2\pi} d\sigma C^4-D(\sigma)
\]

\[
\alpha^A_< n - \alpha^A_> n = \frac{\sqrt{\alpha'} p^U \tilde{K}}{2\pi} \int_0^{2\pi} d\sigma D^i(\sigma) \exp(in\sigma) ,
\]

\[
\tilde{\alpha}^A_< n - \tilde{\alpha}^A_> n = \frac{\sqrt{\alpha'} p^U \tilde{K}}{2\pi} \int_0^{2\pi} d\sigma D^i(\sigma) \exp(-in\sigma) ,
\]

where

\[
C^i(\sigma) = q^i + i\sqrt{\alpha'} \sum_{n \neq 0} \{\alpha^i_< n - \tilde{\alpha}^A_< n \} \exp[-in\sigma]/n \ , \ \tilde{K} \equiv \frac{D-4}{2} K
\]
\[ C(\sigma) \equiv \sqrt{[C^i C^i]} \] and

\[ D^i(\sigma) = -\frac{1}{2k} \frac{\partial f_D(C)}{\partial C^i} \] (4.13)

The constraint equations simply read here

\[ \pm \partial_{\pm} V_{<} = \frac{1}{\alpha' \mu} \left( \partial_{\pm} X_{<}^i \right)^2 \quad \text{for } \tau < 0 \quad \text{and} \quad \pm \partial_{\pm} V_{>} = \frac{1}{\alpha' \mu} \left( \partial_{\pm} X_{>}^i \right)^2 \quad \text{for } \tau > 0. \] (4.14)

Hence, the coordinate \( V(\sigma, \tau) \) is not an independent variable and can be obtained from the transverse coordinates \( X^i(\sigma, \tau) \) through eqs.(4.14) up to the center of mass position \( q^V \). \( q^V \) is an independent dynamical variable, canonically conjugate to \( p^V \) (see [5] for details).

We give a well defined meaning to the transformation (4.11) at the quantum level by using the integral representation [15,5,20]

\[ D^i(\sigma) = -\frac{2i\pi^{D/2-1}}{\Gamma(D/2-1)} \int \frac{d^{D-2}k}{k^2} \mu(\tilde{k}) : \exp i\tilde{k}.\tilde{C}(\sigma) : k^i \] (4.15)

where : ... : stands for normal ordering with respect to the operators \( \alpha^i_\mu <, \tilde{\alpha}^A_\mu < \). Using this formula, the transformation from operators \( < \) to operators \( > \) can be generated by \( \exp(G) \) where

\[ G = -2iGp^U \int_0^{2\pi} d\sigma \int \frac{d^{D-2}k}{k^2} : \exp i\tilde{k}.\tilde{C} : \mu(\tilde{k}) \] (4.16)

As shown in ref.[20], \( \exp(G) \) is unitary provided the integrals \( \int d^{D-2}k \) are taken in principal value. In this framework, we have been able to compute physical quantities like the string squared mass [15,20],

\[ \alpha'(M^>)^2 = -\frac{D - 12}{12} + \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \left[ (\alpha^i_\mu^>)^\dagger \alpha^i_\mu^> + (\tilde{\alpha}^i_\mu^>)^\dagger \tilde{\alpha}^i_\mu^> \right], \] (4.17)

the number of modes \( N \) and the energy-momentum tensor [20]. All these quantities turn to be \textbf{finite} in shock-wave space-times. That is, there is no need of renormalization in contrast with the case in quantum field theory. The expectation value of
the string mass \((M_\sigma)^2\) after the collision with the shock-wave happens to be calculable in closed form. We found for the string ground-state \(|O_\sigma\rangle\) (the tachyon) [15,20]:

\[
\frac{<0_\sigma|(M_\sigma)^2|0_\sigma>}{<0_\sigma|0_\sigma>} = \mu_0^2 - 32\mu_0^2 L^2 D G^2 \pi^{3/2} \int d^{D-2}k |\mu(k)|^2 \frac{\tan[\pi \alpha'(k^2)] \Gamma(\alpha'(k^2))}{k^{2D-2} \alpha'(k^2) \Gamma(1/2 + \alpha'(k^2))}
\]

where \(\mu_0^2 = -\frac{D-12}{12\alpha'}\) is the tachyon squared mass. This term is the initial mass of the string. The second term in eq.(4.18) describes the the change of the string squared mass after the interaction with the shockwave. It can be interpreted as the infinite superposition of particle states that form \(|0_\sigma\rangle\) as seen by the outgoing (+) observers. Notice that this term is proportional to \(L^{D-2}\). That is the transverse size of the shockwave. Analogous formulae can be derived for excited states.

It should be noticed that the integrand in eq.(4.18) possess real singularities which we integrate in principal value following the prescription that makes \(\exp(G)\) unitary. These equally spaced real poles in \(k^2\) are characteristic of the tree level string spectrum (real mass resonances)[14]. The presence of such poles is not related at all to the structure of the spacetime geometry (which may be or not singular). It should be noticed that the claim in ref.[21] that \(<0_\sigma|(M_\sigma)^2|0_\sigma>\) is infinite in shock-wave spacetimes was incorrect as shown in refs.[15]. The only divergences that may eventually appear are related to the infinite transverse size of the shockwaves, as the factor \(L^{D-2}\) in eq.(4.18).

The exact expectation value of all the components of the energy-momentum tensor in shock-wave spacetimes of the type (4.1) were computed in ref.[20]. We expressed all of them in terms of explicit integral representations. All these physical magnitudes turned to be finite except at \(\tau = 0\). That is, they are finite except if one sits exactly at the singularity. The finiteness of these string calculations should be contrasted with QFT where the energy-momentum tensor always need regularization and renormalization [23].

In ref.[5] the two point amplitude, \(A_2(k_2, k_1)\), describing the scattering of the
lowest string excitation (the tachyon) by a gravitational shock wave of the type (4.1) is computed. For this purpose, we used the appropriate vertex operator in this background. We explicitly evaluated in ref.[5], $A_2(\vec{k}_2, \vec{k}_1)$ for large impact parameters $q$. It is given by the Coulombian amplitude plus string corrections of order $s/q$ \[ s = (k_1 + k_2)^2 \], for large $q$. These string corrections produce an infinite sequence of imaginary poles in $s$, the semi-infinite sequence of Coulomb poles noticed by 't Hooft [22] remaining always present.

Let us now consider strings propagating in gravitational plane-wave space-times. These are sourceless gravitational fields described by the metric

\[
(ds)^2 = -dU dV + \sum_{i=1}^{D-2} (dX_i)^2 - \left[ W_1(U) (X^2 - Y^2) + 2 W_2(U) XY \right] (dU)^2
\]

where $X \equiv X^1$, $Y \equiv X^2$. These space-times are exact solutions of the vacuum Einstein equations for any choice of the profile functions $W_1(U)$ and $W_2(U)$. In addition they are exact string vacua [24]. The case when $W_2(U) = 0$ describes waves of constant polarization. When both $W_1(U) \neq 0$ and $W_2(U) \neq 0$, eq.(4.19) describes waves with arbitrary polarization. If $W_1(U)$ and/or $W_2(U)$ are singular functions, space-time singularities will be present. The singularities will be located on the null plane $U = \text{constant}$. We consider profiles which are nonzero only on a finite interval $-T < U < T$, and which have power-type singularities [6],

\[
W_1(U) = \frac{\alpha_1}{|U|^\beta_1}, \quad W_2(U) = \frac{\alpha_2}{|U|^\beta_2}
\]

The spacetimes (4.19) share many properties with the shockwaves (4.1). In particular, $U(\sigma, \tau)$ obeys the d’Alembert equation and we can choose the light-cone gauge (4.7). The string equations of motion (2.6) become then in the metric (4.19):

\[
\begin{align*}
V'' - \ddot{V} + (2\alpha' p U)^2 \left[ \partial_U W_1 (X^2 - Y^2) + 2 \partial_U W_2 XY \right] \\
+ 8\alpha' p U \left[ W_1(X\dot{X} - Y\dot{Y}) + W_2(X\dot{Y} + Y\dot{X}) \right] &= 0 \\
X'' - \ddot{X} + (2\alpha' p U)^2 [W_1 X - W_2 Y] &= 0 \\
Y'' - \ddot{Y} + (2\alpha' p U)^2 [W_2 X - W_1 Y] &= 0
\end{align*}
\]
and the constraints (2.7) take the form:

\[
\pm \partial_\pm V_\perp = \frac{1}{\alpha' p^U} \left\{ (\partial_\pm X)^2 + (\partial_\pm Y)^2 + \sum_{i=3}^{D-2} (\partial_\pm X_i)^2 \right\} + \alpha' p^U \left[ W_1 (X^2 - Y^2) + 2 W_2 X Y \right]
\]

(4.22)

Let us analyse now the solutions of the string equations (4.21) and (4.22) for a closed string. The transverse coordinates obey the d’Alembert equation, with the solution

\[
X^i(\sigma, \tau) = q^i + 2p^i \alpha' \tau + i\sqrt{\alpha'} \sum_{n \neq 0} \{ \alpha_n^i \exp[-in(\sigma-\tau)] + \tilde{\alpha}_n^i \exp[-in(\sigma+\tau)] \}/n, \ 3 \leq i \leq D-2
\]

(4.23)

For the X and Y components it is convenient to Fourier expand as

\[
X(\sigma, \tau) = \sum_{n=-\infty}^{+\infty} \exp(in\sigma) \ X_n(\tau) \quad , \quad Y(\sigma, \tau) = \sum_{n=-\infty}^{+\infty} \exp(in\sigma) \ Y_n(\tau)
\]

(4.24)

Then, eqs. (4.21) for X and Y yield

\[
\ddot{X}_n + n^2 X_n - (2\alpha' p^U)^2 [W_1 X_n - W_2 Y_n] = 0
\]
\[
\ddot{Y}_n + n^2 Y_n - (2\alpha' p^U)^2 [W_2 X_n - W_1 Y_n] = 0
\]

(4.25)

where we consistently set \( U = 2\alpha' p^U \). Formally, these are two coupled one-dimensional Schrödinger-like equations with \( \tau \) playing the rôle of a spatial coordinate.

We study now the interaction of the string with the gravitational wave. For \( 2\alpha' p^U \tau < -T \), \( W_{1,2}(\tau) = 0 \) and therefore X, Y are given by the usual flat-space expansions

\[
X(\sigma, \tau) = q^X + 2p^X_\perp \alpha' \tau + i\sqrt{\alpha'} \sum_{n \neq 0} \{ \alpha_n^X \exp[-in\sigma] - \tilde{\alpha}_n^X \exp[i\sigma] \}/n
\]
\[
Y(\sigma, \tau) = q^Y + 2p^Y_\perp \alpha' \tau + i\sqrt{\alpha'} \sum_{n \neq 0} \{ \alpha_n^Y \exp[-in\sigma] - \tilde{\alpha}_n^Y \exp[i\sigma] \}/n
\]

(4.26)

These solutions define the initial conditions for the string propagation in \( \tau \geq -\tau_0 \equiv -\frac{T}{2\alpha' p^U} \). In the language of the Schrödinger-like equations we have a two
channel potential in the interval $-\tau_0 < \tau < +\tau_0$. We consider the propagation of the string when it approaches the singularity at $U = 0 = \tau$ from $\tau < 0$. When $W_1$ is more singular at $U = 0$ than $W_2$, i.e. $\beta_1 > \beta_2$ in eq.(4.20), the string behaviour is determined by $W_1$. Let us consider the case when both singularities are of the same type; i.e. $\beta_1 = \beta_2 \equiv \beta$. This case is actually generic since the case when $W_2(W_1)$ is more singular than $W_1(W_2)$ can be obtained by setting $\alpha_1 = 0$ ($\alpha_2 = 0$) in the $\beta_1 = \beta_2$ solution.

Eq.(4.25) can be approximated near $\tau = 0^-$ as

\[
\ddot{X}_n - \frac{(2\alpha'p^U)^2 - \beta}{|\tau|^\beta} [\alpha_1 X_n + \alpha_2 Y_n] = 0
\]

\[
\ddot{Y}_n - \frac{(2\alpha'p^U)^2 - \beta}{|\tau|^\beta} [\alpha_2 X_n - \alpha_1 Y_n] = 0
\]

(4.27)

The behaviour of the solutions $X_n(\tau)$ and $Y_n(\tau)$ for $\tau \to 0$ depends crucially on the value range of $\beta$. Namely, i) $\beta > 2$, ii) $\beta = 2$, iii) $\beta < 2$. For simplicity, we start with the case $\beta = 2$ where the solution is [6]:

\[
X_n(\tau) \overset{\tau \to 0^-}{\approx} c |\tau|^{\lambda_1} , \quad Y_n(\tau) \overset{\tau \to 0^-}{\approx} D |\tau|^{\lambda_2}
\]

(4.28)

Which has four solutions:

\[
\lambda_{1,2} = \frac{1}{2}[1 \pm \sqrt{1 - 4\tilde{\alpha}}] , \quad \lambda_{3,4} = \frac{1}{2}[1 \pm \sqrt{1 + 4\tilde{\alpha}}]
\]

(4.29)

Here, $\tilde{\alpha} \equiv \sqrt{\alpha_1^2 + \alpha_2^2}$. Notice that for any real value $\alpha_1$ and $\alpha_2$, we have $\lambda_4 < 0$, $\lambda_3 > 0$, $Re\lambda_{1,2} > 0$. The solution associated with $\lambda_4$ diverges at $\tau = 0$ as

\[
X_n(\tau)_4 \overset{\tau \to 0^-}{\approx} c |\tau|^{\lambda_4} (\alpha_1 + \tilde{\alpha}) , \quad Y_n(\tau)_4 \overset{\tau \to 0^-}{\approx} c \alpha_2 |\tau|^{\lambda_4}
\]

(4.30)

where $c$ is an arbitrary constant. The solutions associated with $\lambda_{1,2}$ and $\lambda_3$ vanish.
for $\tau \to 0^-$.

$$X_n(\tau) \sim \tau^{\alpha_1} d |\tau|^{\lambda_3} (\alpha_1 + \tilde{\alpha}) , \quad Y_n(\tau) \sim \tau^{\alpha_2} d |\tau|^{\lambda_3}$$

$$X_n(\tau)_{1,2} \sim \tau^{\alpha_3} k_{1,2} |\tau|^{\lambda_{1,2}} (\alpha_1 - \tilde{\alpha}) , \quad Y_n(\tau)_{1,2} \sim \tau^{\alpha_2} k_{1,2} |\tau|^{\lambda_{1,2}}$$

(4.31)

Here $d$, $k_1$ and $k_2$ are arbitrary constants. For $\tilde{\alpha} > 1/4$, the solutions $X_n(\tau)_{1,2}$ and $Y_n(\tau)_{1,2}$ approach the singularity oscillating with decreasing amplitudes. As it is clear, for generic initial conditions, the string behaviour near $\tau \to 0^-$ is dominated by the $(X_n(\tau)_{1,2}, Y_n(\tau)_{1,2})$ solution. The fact that $(X_n(\tau)_{1,2}, Y_n(\tau)_{1,2})$ diverges when $\tau \to 0^-$, means that the string goes to infinity as it approaches the singularity plane. From eqs. (4.30), we see that the string goes to infinity in a direction forming an angle $\alpha$ with the X-axis in the $X, Y$ plane. The string escape angle $\alpha$ is given by

$$\tan \alpha = \frac{\alpha_2}{\alpha_1} \quad \text{i.e.} \quad \tan 2\alpha = \frac{\alpha_2}{\alpha_1}$$

(4.32)

(see fig.1). When $\alpha_2 = 0$, then $\alpha = 0$ and the string escapes to infinity in the $X$-direction. For $\alpha_1 = 0$, then $\tan \alpha = \text{sign}(\alpha_2) = \pm 1$ and the string goes to infinity with an angle $(\pi/4)\text{sign}(\alpha_2)$. For $\alpha_1 > 0$, and arbitrary $\alpha_2$, the allowed directions are within the cone $|\alpha| < \pi/4$, as depicted in fig.1. If $\alpha_1 < 0$ the string escape angle is within the cone $|\alpha - \pi/2| < \pi/4$. In summary, for arbitrary values of $\alpha_1, \alpha_2$, the string escape angle can take any value between 0 and $2\pi$.

Let us now consider the case $\beta > 2$. We have [6]

$$X_n(\tau) \sim \tau^{\alpha_1} \text{C}[\alpha_1 \pm \tilde{\alpha}] |\tau|^{\beta/2} \exp[K|\tau|^{1-\beta/2}] , \quad Y_n(\tau) \sim \tau^{\alpha_2} \text{C}\alpha_2 |\tau|^{\beta/2} \exp[K|\tau|^{1-\beta/2}]$$

(4.33)

where $C$ is an arbitrary constant $K$ can take the four following values:

$$K_{1,2} = \pm \sqrt{\left(\frac{2\alpha'pU}{\beta/2 - 1}\right)^{1-\beta/2}} \sqrt{\alpha} , \quad K_{3,4} = \pm \sqrt{\left(\frac{2\alpha'pU}{\beta/2 - 1}\right)^{1-\beta/2}}$$

(4.34)

As for the $\beta = 2$ case, for $\beta \geq 2$ we have a divergent solution for $\tau \to 0^-$ associated
to $K_4$:

\[
X_n(\tau) \overset{\tau \to 0^-}{=} C [\alpha_1 + \tilde{\alpha}] |\tau|^{\frac{\alpha}{2}} \exp \left[ \frac{(2\alpha' p U |\tau|)^{1-\beta/2}}{\beta/2 - 1} \sqrt{\tilde{\alpha}} \right],
\]

\[
Y_n(\tau) \overset{\tau \to 0^-}{=} C \alpha_2 |\tau|^{\frac{\beta}{4}} \exp \left[ \frac{(2\alpha' p U |\tau|)^{1-\beta/2}}{\beta/2 - 1} \sqrt{\tilde{\alpha}} \right].
\]  

(4.35)

The other three solutions vanish for $\tau \to 0^-$. The solutions (1,2) oscillate for $\tau \to 0^-$ with decreasing amplitude. As in the $\beta = 2$ case, and for generic initial conditions, the string behaviour for $\tau \to 0^-$ is dominated by the $(X_n(\tau), Y_n(\tau))$ solution. The string goes to infinity in the same way as for the $\beta = 2$ case and with the same escape angle given by eq.(4.32).

Finally, let us discuss now the situation when $\beta < 2$. In this case, the solution for $\tau \to 0^-$ behaves as

\[
X_n(\tau) \overset{\tau \to 0^-}{=} A_1 + A_2 \tau + O(|\tau|^{2-\beta}) \quad , \quad Y_n(\tau) \overset{\tau \to 0^-}{=} B_1 + B_2 \tau + O(|\tau|^{2-\beta}) \quad , \quad \beta \neq 1
\]  

(4.36)

[In the special case $\beta = 1$ one should add a term $0(\tau \ln |\tau|)$]. For $\beta < 2$, the string coordinates $X$, $Y$ are always regular indicating that the string propagates smoothly through the gravitational-wave singularity $U = 0$. (Nevertheless, the velocities $\dot{X}$ and $\dot{Y}$ diverge at $\tau = 0$ when $1 \leq \beta < 2$).

Let us now summarize the string behaviour near the singularity $\tau \to 0^-$ for $\beta \geq 2$. For generic initial conditions, we see from eqs.(4.30) and (4.35) that the string behaves as

\[
(X(\sigma, \tau), Y(\sigma, \tau)) \overset{\beta = 2}{=} \overset{\tau \to 0^-}{\mathcal{A}(\sigma)} \sqrt{|\tau|}^{1-\sqrt{1+4\alpha}} (\alpha_1 + \tilde{\alpha}, \alpha_2),
\]

\[
(X(\sigma, \tau), Y(\sigma, \tau)) \overset{\beta > 2}{=} \overset{\tau \to 0^-}{\mathcal{B}(\sigma)} |\tau|^{\beta/4} \exp \left[ \frac{(2\alpha' p U |\tau|)^{1-\beta/2}}{\beta/2 - 1} \sqrt{\tilde{\alpha}} \right] (\alpha_1 + \tilde{\alpha}, \alpha_2)
\]  

(4.37)

The functions $\mathcal{A}(\sigma), \mathcal{B}(\sigma)$ depend on the initial conditions. The above solutions imply that the string **does not cross** the $U = 0$ singularity plane. The string goes off to infinity in the $(X,Y)$ plane, grazing the singularity plane $U = 0$ (therefore
never crossing it). Here, the string escapes to infinity with an angle $\alpha$ given by eq. (4.32) that depends upon the strengths of the profile singularities, $\alpha_1$ and $\alpha_2$, but not on $\beta$. (This means that the escape angle is solely determined by the polarization of the nonlinear gravitational wave (4.19)). At the same time, the presence of the oscillatory modes (4.31) and (4.33) with $K_{1,2}$ imply that the string oscillates in the $XY$ plane perpendicularly to the escape direction, with an amplitude vanishing for $\tau \rightarrow 0^-$. The non-oscillatory modes in (4.31) and (4.33) with $K_3$ are negligible since they are in the same direction as the divergent solutions (4.30) and (4.35).

The spatial string coordinates $X^i(\sigma, \tau) [3 \leq i \leq D-2]$ behave freely [eq.(4.23)]. The longitudinal coordinate $V(\sigma, \tau)$ follows from the constraint eqs.(4.22) and the solutions (4.37) for $X(\sigma, \tau), Y(\sigma, \tau)$ and $X^j(\sigma, \tau) [3 \leq j \leq D-2]$. We see that for $\tau \rightarrow 0^-$, $V(\sigma, \tau)$ diverges as the square of the singular solutions (4.37).

Let us consider the spatial length element of the string, i.e. the length at fixed $U = 2\alpha'p^U\tau$, between two points $(\sigma, \tau)$ and $(\sigma + d\sigma, \tau)$,

$$ds^2 = dX^2 + dY^2 + \sum_{j=3}^{D-2} (dX^j)^2$$

(4.38)

For $\tau \rightarrow 0^-$ eqs.(4.37) yield

$$ds^2 \underset{\tau \rightarrow 0^-}{=} \left[ (\alpha_1 + \tilde{\alpha})^2 + \alpha_2^2 \right] B'(\sigma)^2 d\sigma^2 \left| \tau \right|^{1-\sqrt{1+4\tilde{\alpha}}} \text{ for } \beta = 2,$$

$$ds^2 \underset{\tau \rightarrow 0^-}{=} \left[ (\alpha_1 + \tilde{\alpha})^2 + \alpha_2^2 \right] B'(\sigma)^2 d\sigma^2 \left| \tau \right|^{\beta/2} \exp \left[ \frac{(4\alpha'p^U|\tau|)^{1-\beta/2}}{\beta/2 - 1} \sqrt{\alpha} \right] \text{ for } \beta \geq 2.$$  

(4.39)

That is, the proper length between $(\sigma_0, \tau)$ and $(\sigma_1, \tau)$ is given by

$$\Delta s \underset{\tau \rightarrow 0^-}{=} \sqrt{\left[ (\alpha_1 + \tilde{\alpha})^2 + \alpha_2^2 \right] [B(\sigma_1) - B(\sigma_2)] \left| \tau \right|^{1-\sqrt{1+4\tilde{\alpha}}} \text{ for } \beta = 2},$$

$$\Delta s \underset{\tau \rightarrow 0^-}{=} \sqrt{\left[ (\alpha_1 + \tilde{\alpha})^2 + \alpha_2^2 \right] [B(\sigma_1) - B(\sigma_2)] \left| \tau \right|^{\beta/4} \exp \left[ \frac{(2\alpha'p^U|\tau|)^{1-\beta/2}}{\beta/2 - 1} \sqrt{\alpha} \right] \text{ for } \beta \geq 2}.$$  

(4.40)

We see that $\Delta s \rightarrow \infty$ for $\tau \rightarrow 0^-$. That is, the string stretches infinitely when
it approaches the singularity plane. This stretching of the string proper size also occurs for $\tau \to 0$ in the inflationary cosmological backgrounds (see sec.6).

Another consequence of eqs.(4.37) is that the string reaches infinity in a finite time $\tau$. In particular, for $\sigma$-independent coefficients, eqs.(4.37) describe geodesic trajectories. The fact that for $\beta \geq 2$, a point particle (as well as a string) goes off to infinity in a finite $\tau$ indicates that the space-time is singular.

What we have described is the $\tau \to 0^-$ behaviour for generic initial data. In particular, there is a class of solutions where $A(\sigma) \equiv 0$ and $B(\sigma) \equiv 0$, whereas the coefficients of the regular modes are arbitrary. In this class of solutions, X and Y vanish for $\tau \to 0^-$. However, when continued to $\tau > 0$, these solutions are complex. The real valued physical solution is $X = Y = 0$ for all $\tau \geq 0$. It means that the string gets trapped in the singularity plane $U = 0$ at the point $X = Y = 0$, where the gravitational forces are zero.

Finally, we would like to remark that the string evolution near the space-time singularity is a collective motion governed by the nature of the gravitational field. The (initial) state of the string fixes the overall $\sigma$-dependent coefficients $A(\sigma), B(\sigma)$ [see eqs.(4.37)], whereas the $\tau$-dependence is fully determined by the space-time geometry. In other words, the $\tau$-dependence is the same for all modes $n$. In some directions, the string collective propagation turns to be an infinite motion (the escape direction), whereas in other directions, the motion is oscillatory, but with a fixed (n-independent) frequency. In fact, these features are not restricted to singular gravitational waves, but are generic to strings in strong gravitational fields [see sec.(6) and refs.(4,7,8)].

For sufficiently weak spacetime singularities ($\beta_1 < 2$ and $\beta_2 < 2$), the string crosses the singularity and reaches the region $U > 0$. Therefore, outgoing scattering states and outgoing operators can be defined in the region $U > 0$. We explicitly found in ref.[6] the transformation relating the ingoing and outgoing string mode operators. For the particles described by the quantum string states, this relation implies two types of effects as described in sec.3 for generic asymptotically flat
spacetimes: (i) rotation of spin polarization in the (X,Y) plane, and (ii) trans-
mutation between different particles. We computed in [6] the expectation values
of the outgoing mass $M^2_>$ operator and of the mode-number operator $N_>$, in the
ingoing ground state $|O_<\rangle$. As for shockwaves (cfr. eq.(4.18)) , $M^2_>$ and $N_>$
have different expectation values than $M^2_<$ and $N_<$. This difference is due to the
excitation of the string modes after crossing the space-time singularity. In other
words, the string state is not an eigenstate of $M^2_>$, but an infinity superposition
of one-particle states with different masses. This is a consequence of the particle
transmutation which allows particle masses different from the initial one ($\mu^2_o$).

5. Strings in Conical Spacetimes (the
genometry around a cosmic string)

Conical spacetimes describe the geometry around a cosmic string. The space-
time surrounding such object is locally flat and there is a defect angle $\alpha$ propor-
tional to the mass density of the cosmic string. Therefore, the interaction of strings
(or particles) with such a geometry comes from the unusual periodicity requirement
by an angle $2\pi - \alpha$ around the cosmic string [11].

A conical space-time in D-dimensions is defined by the metric

$$(ds)^2 = -(dX^0)^2 + (dR)^2 + R^2 (d\Phi)^2 + \sum_{i=3}^{D-1} (dX^i)^2$$

where $R = \sqrt{X^2 + Y^2}$ and $\Phi = \arctan(Y/X)$ are cilindrical coordinates, but with
the range

$$0 \leq \Phi < 2\pi \alpha \quad , \quad \alpha \equiv 1 - 4G \mu$$

$\mu$ is the cosmic string tension ($G\mu \approx 10^{-6}$ for the Grand Unified theories cosmic
strings), $(dX^i)^2$ is a flat (D - 3) dimensional space and $X^i, 3 \leq i \leq D - 1$, are cartesian coordinates. The spatial points $(R, \Phi, X^i)$ and $(R, \Phi + 2\pi \alpha, X^i)$ are
identified. The space-time is locally flat for $R \neq 0$ but has a conelike singularity at $R = 0$ with azimuthal deficit angle

$$2\Delta = 2\pi (1 - \alpha) = 8\pi G\mu$$  \hspace{1cm} (5.3)

This geometry describes a straight cosmic string of zero thickness. It is a good approximation for very thin cosmic strings with large curvature radius. Globally, it has a non-trivial (multiply-connected) topology. The string equations of motion (2.6) are free equations in the coordinates $X^0, X, Y, X^i$ but with the condition that the points $(R, \Phi, X^i)$ and $(R, \Phi + 2\pi\alpha, X^i)$ are identified. Therefore, we can choose the light-cone gauge

$$U = 2\alpha' p^U \tau,$$

where $U \equiv X^0 - X^{D-1}$ \hspace{1cm} (5.4)

The constraints (2.7) in the light-cone gauge (5.4) completely determine the longitudinal $V \equiv X^0 + X^{D-1}$ coordinate of the string in terms of the transverse coordinates, as it should be.

Before solving the string equations, let us consider the point particle propagation. Since the equations of motion are locally those of Minkowski spacetime, the trajectories are straight lines. The non-trivial point is to impose the $2\pi\alpha$ periodicity condition on the angle $\Phi$. The solution is depicted in fig. 2 in coordinates where half of the defect angle ($\Delta$) is taken to the right and half to the left of the conical singularity at $O$. Therefore, particles passing on the right (left) of $O$ are deflected by $+\Delta(-\Delta)$. Notice that this deflection angle is independent from the impact parameter as well as from the energy of the ingoing particle. This shows the purely topological nature of this infinite range interaction. The solution of the string equations of motion follows analogously [11]. There appear essentially two different situations for the interaction between the fundamental string and the conical spacetime:
(i) The string does not touch the scatterer body (here represented by the cosmic string); in this case the string only suffers a deflection at the origin. We refer to this situation as elastic scattering.

(ii) The string collides against the scattering center. In this case, the string gets its normal modes excited besides being deflected. This happens each time a point of the string collides with the center. We refer to this process as inelastic scattering.

Let us first consider the case (i) (elastic scattering). For $\tau \to -\infty$, the ingoing-solution $X_A^<$ is just the free solution without deflection. For $\tau \to +\infty$, the outgoing solution $X_A^>$ is the free solution after deflection. We have

$$ (X_>^Y_>, Y_>^Y_>) = (X_<^Y_<, Y_<^Y_<)^R(\pm \Delta), \quad (5.5) $$

$R(\Delta)$ being the rotation matrix,

$$ R(\Delta) = \begin{pmatrix} \cos \Delta & \sin \Delta \\ -\sin \Delta & \cos \Delta \end{pmatrix}. \quad (5.6) $$

$\Delta$ is the deflection angle given by eq.(5.3) and the $+ (-)$ sign refers to a string passing to the right (left) of the cosmic string. The $(D-3)-X^i$ components are not affected by the scattering. From eqs.(5.5) and (5.6) we find that the outgoing zero-modes and the oscillators are given by the respective in operators rotated by $R(\pm \Delta)$. In other words, there is a rotation $\pm \Delta$ in the $(X,Y)$ polarization of the string modes after passing the cosmic string. That is, there is no creation or excitation of modes after passing the scattering center [$\alpha_n^A$ and $(\alpha_n^A)^\dagger$ are not mixed]. In this elastic case, there is no particle transmutation (see sec.3).

Let us discuss now the inelastic scattering (case ii). The string evolution is described by the free equations except at the collision point $(\sigma_0, \tau_0)$ with the cosmic string. We take the origin at the cosmic string, thus we have

$$ X(\sigma_0, \tau_0) = 0 , \quad Y(\sigma_0, \tau_0) = 0 . \quad (5.7) $$

The values of $\sigma_0$ and $\tau_0$ indeed depend on the string state before the collision, that
is, on the dynamical variables $q^X, q^Y, p^X, p^Y, \alpha^X_n, \alpha^Y_n, \tilde{\alpha}^X_n, \tilde{\alpha}^Y_n$ and [see below eq.(5.8)].

Since the deflection angles to the right and to the left of the cosmic string are different, we have now different matching conditions between the $<$ and the $>$ solutions for $0 \leq \sigma \leq \sigma_0$ and for $\sigma_0 \leq \sigma \leq 2\pi$, respectively. That is,

\[
X^A_>(\sigma, \tau_0) = X^B_<(\sigma, \tau_0) R^A_B(-\Delta)
\]
\[
\partial_\tau X^A_>(\sigma, \tau_0) = \partial_\tau X^B_<(\sigma, \tau_0) R^A_B(-\Delta), \quad 0 \leq \sigma \leq \sigma_0, A \equiv 1, 2.
\]  
\[
X^A_>(\sigma, \tau_0) = X^B_<(\sigma, \tau_0) R^A_B(+\Delta)
\]
\[
\partial_\tau X^A_>(\sigma, \tau_0) = \partial_\tau X^B_<(\sigma, \tau_0) R^A_B(+\Delta), \quad \sigma_0 \leq \sigma \leq 2\pi, A \equiv 1, 2.
\]

Here $X^1 \equiv X$ and $X^2 \equiv Y$. This guarantees continuity of the string coordinates and its derivatives at $\tau = \tau_0$. The string solutions $X^A_<(\sigma, \tau)$ and $X^A_>(\sigma, \tau)$ admit the usual expansion (4.10). It should be also noticed that eqs.(5.8) reproduce eqs.(5.5) with signs $+$ or $-$ when $\sigma_0 = 0$ or $\sigma_0 = 2\pi$, respectively, as it must be.

We recall that $\sigma_0$ and $\tau_0$ depend on the initial data of the string through the constraint (5.7) as

\[
0 = q^A_< + 2p^A_< \alpha' \tau_0 + i\sqrt{\alpha'} \sum_{n \neq 0} \{ \alpha^A_n < \exp[in(\sigma_0 - \tau_0)] + \tilde{\alpha}^A_n < \exp[-in(\sigma_0 + \tau_0)] \}/n, \quad A = 1, 2
\]

(5.9)

Imposing eqs.(5.8) at $\tau = \tau_0$ yields linear relations between the operators $>$ and $<$ appearing in the expansions (4.10). We find for the zero modes

\[
p^A_> = p^B_< R^A_B(\Delta) + \frac{\sigma_0 + \tau_0}{2\pi} p^B_< L^A_B + \frac{1}{4\pi \alpha'} q^A_< L^A_B + \frac{i}{4\pi \sqrt{\alpha'}} \sum_{n \neq 0} \frac{\exp[-in\tau_0]}{n} \{ \alpha^A_{n <} - \tilde{\alpha}^A_{n <}(1 - 2 \exp[-in\sigma_0]) \} L^A_B
\]

\[
q^A_> = q^B_< R^A_B(\Delta) + \frac{\sigma_0 - \tau_0}{2\pi} p^B_< L^A_B - \frac{\alpha' \tau_0}{\pi} p^B_< L^A_B + \frac{\sqrt{\alpha'}}{2\pi} \sum_{n \neq 0} \frac{\exp[-in\sigma_0]}{n^2} \{ \alpha^A_{n <} (\exp[in\sigma_0] - 1 - in\tau_0) + \tilde{\alpha}^A_{n <}(1 - \exp[-in\sigma_0] + in\tau_0(1 - 2 \exp[-in\sigma_0])) \} L^A_B, \quad A = 1, 2
\]

(5.10)
where $L^A_B \equiv R^A_B(-\Delta) - R^A_B(\Delta)$. We get for the oscillator modes:

\[
\alpha^A_{n>} = \alpha^B_{n<} \cdot R^A_B(+\Delta) - \frac{i \sqrt{\alpha}}{2\pi n} \exp[in\tau_0] \left[ 1 - \exp(-in\sigma_0) \right] \left[ p^B_<(1 - in\tau_0) - \frac{in}{2\alpha^B_\alpha^B_\sigma_0} \right] L^A_B
\]

\[
- \frac{i}{4\pi} \sum_{m \neq 0} \frac{m + n}{m(m - n)} \exp[i(n - m)\tau_0] \left\{ \exp[-i(n - m)\sigma_0] - 1 \right\} \alpha^B_m \cdot L^A_B
\]

\[
+ \frac{i}{4\pi} \sum_{m \neq 0} \left( \frac{1}{m} \right) \exp[i(n - m)\tau_0] \left\{ \exp[-i(n + m)\sigma_0] - 1 \right\} \tilde{\alpha}^B_m \cdot L^A_B,
\]

\[
(5.11)
\]

\[
\tilde{\alpha}^A_{n>} = \tilde{\alpha}^B_{n<} \cdot R^A_B(+\Delta) - \frac{i \sqrt{\alpha}}{2\pi n} \exp[in\tau_0] \left[ 1 - \exp(in\sigma_0) \right] \left[ p^B_<(1 + in\tau_0) + \frac{in}{2\alpha^B_\alpha^B_\sigma_0} \right] L^A_B
\]

\[
- \frac{i}{4\pi} \sum_{m \neq 0} \frac{m + n}{m(m - n)} \exp[i(n - m)\tau_0] \left\{ \exp[i(n - m)\sigma_0] - 1 \right\} \tilde{\alpha}^B_m \cdot L^A_B
\]

\[
+ \frac{i}{4\pi} \sum_{m \neq 0} \left( \frac{1}{m} \right) \exp[i(n - m)\tau_0] \left\{ \exp[i(n + m)\sigma_0] - 1 \right\} \alpha^B_m \cdot L^A_B,
\]

where the terms $m = n$ in the sums are defined as their limiting values, $m \to n$.

The Bogoliubov transformations eqs.(5.11) are interpreted as usual (sec.3). In this inelastic case in which the string collides against the cosmic string, besides the change of polarization, the modes become excited and then particle transmutation take place. As we have seen in sec.3 this phenomenon appears in general for strings propagating in curved (static as well as time dependent) spacetimes. Particle states get here transmuted at the classical (tree) level as a consequence of the interaction with the geometry. In the present case, this is a topological defect in the space-time.

It can be shown that the matching relations (5.10)-(5.11) yield for the Virasoro generators $L^A_n = L^\alpha_n$ [11]. In addition, the mass operators $M^2_<$ and $M^2_>$ have identical spectra although they are not identical : $M^2_\alpha^A_\alpha^B_\sigma_0 \neq M^2_\alpha^B_\alpha^B_\sigma_0$. The critical dimension is the same as in Minkowski spacetime.

In the second reference under [11] the superstring scattering by a cosmic string is solved. That is, we exactly quantized NSR and Green-Schwarz superstrings in the spacetime (5.1). As for the purely bosonic string, if the superstring touches the cosmic string, the scattering is inelastic, since the internal superstring modes
become excited and mixing of particle- and antiparticle-modes takes place leading to particle transmutation. We also generalized all these results to the case when the cosmic string is spinning. The metric (5.1) when the cosmic string possesses a spin $S$ takes the form

$$(ds)^2 = -(dX^0 + \frac{GS}{\alpha}d\Phi)^2 + (dR)^2 + R^2 (d\Phi)^2 + \sum_{i=3}^{D-1} (dX^i)^2 \quad (5.12)$$

Introducing the coordinate:

$$\tilde{X}^0 = X^0 + \frac{GS}{\alpha} \Phi$$

the metric becomes locally Minkowski and the points $(\tilde{X}^0, R, \Phi, X^i)$ and $(\tilde{X}^0 + 2\pi GS, R, \Phi + 2\pi \alpha, X^i)$ are identified. As a consequence, besides the deficit angle $8\pi G\mu$, the spacetime has a time-helical structure in the coordinates $(\tilde{X}^0, R, \Phi, X^i)$. That is, the spin introduces a shift $2\pi GS$ on $\tilde{X}^0$, besides the rotation $2\pi \alpha$ for two identical points. For the bosonic string, the spatial coordinates are given by the solution (4.10) as before. The new features when $S \neq 0$ appear in the coordinate $X^0$. We can choose here the light-cone gauge

$$X^0 + \frac{GS}{\alpha} \Phi + X^1 = 2 \alpha' p^U \tau$$

For the fermionic coordinates, the propagation is not affected by the spin of the cosmic string (fermion coordinates are invariant under space-time translations).
6. Exact Integrability of strings in D-dimensional de Sitter spacetime

Let us consider the D-dimensional de Sitter space-time with metric given by

\[ ds^2 = -dt_0^2 + \exp[2Ht_0] \sum_{i=1}^{D-1} (dX^i)^2. \]  
(6.1)

Here \( t_0 \) is the so called cosmic time. In terms of the conformal time \( \eta \),

\[ \eta \equiv -\frac{\exp[-Ht_0]}{H} , \quad -\infty < \eta \leq 0 , \]  
(6.2)

the line element becomes

\[ ds^2 = \frac{1}{H^2\eta^2}[-d\eta^2 + \sum_{i=1}^{D-1} (dX^i)^2] . \]  
(6.3)

The de Sitter spacetime can be considered as a D-dimensional hyperboloid embed-
ded in a D+1 dimensional flat Minkowski spacetime with coordinates \((q^0, ..., q^D)\):

\[ ds^2 = \frac{1}{H^2}[-(dq^0)^2 + \sum_{i=1}^{D} (dq^i)^2] \]  
(6.4)

where

\[ q^0 = \sinh Ht_0 + \frac{H^2}{2} \exp[Ht_0] \sum_{i=1}^{D-1} (X^i)^2 , \]

\[ q^1 = \cosh Ht_0 - \frac{H^2}{2} \exp[Ht_0] \sum_{i=1}^{D-1} (X^i)^2 , \]

\[ q^{i+1} = H \exp[Ht_0] X^i , \quad 1 \leq i \leq D-1 , \quad -\infty < t_0 , \quad X^i < +\infty . \]  
(6.5)

The complete de Sitter manifold is the hyperboloid

\[ -(q^0)^2 + \sum_{i=1}^{D} (q^i)^2 = 1. \]  
(6.6)

The coordinates \((t_0, X^i)\) and \((\eta, X^i)\) cover only the half of the de Sitter manifold \(q^0 + q^1 > 0\).
We will consider a string propagating in this D-dimensional space-time. The string equations of motion (2.6) in the metric (6.4) take the form:

\[ \partial_+ q + (\partial_+ q, \partial_- q) q = 0 \quad \text{with} \quad q.q = 1, \quad (6.7) \]

where \(\cdot\) stands for the Lorentzian scalar product \(a.b \equiv -a_0 b_0 + \sum_{i=1}^{D} a_i b_i, \quad x_\pm \equiv \frac{1}{2}(\tau \pm \sigma)\) and \(\partial_\pm q = \frac{\partial q}{\partial x_\pm}\). The string constraints on the world sheet are

\[ T_{\pm \pm} = \frac{\partial q}{\partial x_\pm} \cdot \frac{\partial q}{\partial x_\pm} = 0. \quad (6.8) \]

Eqs. (6.7) describe a non compact O(D,1) non-linear sigma model in two dimensions. In addition, the (two dimensional) energy-momentum tensor is required to vanish by the constraints eqs. (6.8). This system of non-linear partial differential equations can be simplified by choosing an appropriate basis for the string coordinates in the (D+1)-dimensional Minkowski space-time \((q^0, ..., q^D)\). The construction of this basis is analogous to the reduction of the O(N) non-linear sigma model [25]. We choose as a basis the vectors

\[ e_i = (q, \partial_+ q, \partial_- q, b_i, ..., b_{D+1}) \quad (6.9) \]

where the \(b_i, 1 \leq i \leq D + 1\) form an orthonormal set

\[ b_i.b_j = \delta_{ij} \quad \text{and} \quad b_i.q = b_i.q_\pm = 0 \quad (6.10) \]

We define

\[ \exp[a(\sigma, \tau)] = -\frac{\partial q}{\partial x_-} \cdot \frac{\partial q}{\partial x_+} \quad (6.11) \]

It is easy to show from eqs. (6.7) and (6.8) that

\[ q.\partial_\pm q = 0 \quad \partial_\pm q.\partial_\pm q = 0 \quad (6.12) \]
In the basis (6.9), the second derivatives of \( q \) expresses as [7]

\[
\partial_{++} q = \partial_+ \alpha \partial_+ q + \sum_{i=4}^{D+1} u_i b_i, \quad \partial_{--} q = \partial_- \alpha \partial_- q + \sum_{i=4}^{D+1} v_i b_i.
\] (6.13)

Here,

\[
u_i \equiv b_i.\partial_{++} q, \quad v_i \equiv b_i.\partial_{--} q.
\]

We find from eqs.(6.11) - (6.13) [7]

\[
\frac{\partial^2 \alpha}{\partial \tau^2} - \frac{\partial^2 \alpha}{\partial \sigma^2} - \exp(\alpha) + \exp(-\alpha) \sum_{i=4}^{D+1} u_i v_i = 0
\] (6.14)

This is the evolution equation for the function \( \alpha(\sigma, \tau) \) determining the scalar product \( q_+ . q_- \), for all \( D \). This is a generalization of the Sinh-Gordon equation. It remains to find now the evolution equations for the fields \( u_i \) and \( v_i \). In order to find such equations we express the derivatives of the basis vectors eq.(6.9) in terms of the basis itself (Gauss-Weingarten equations):

\[
\partial_+ e_i = A_{ij}(\sigma, \tau)e_j, \quad \partial_- e_i = B_{ij}(\sigma, \tau)e_j
\] (6.15)

The compatibility condition for eqs.(6.15) expresses as

\[
\partial_- A - \partial_+ B + [A, B] = 0
\] (6.16)

We want to stress that \( \exp[\alpha(\sigma, \tau)] \) has a clear physical interpretation. The invariant interval between two points on the string computed with the spacetime metric (6.4) is given by

\[
ds^2 = \frac{1}{H^2}dq.dq = \frac{1}{2H^2} \exp[\alpha(\sigma, \tau)] (d\sigma^2 - d\tau^2)
\] (6.17)

We see that the factor \( \exp[\alpha(\sigma, \tau)] \) determines the string size.
Let us start studying the case $D = 2$. In this case, a complete basis is formed by

$$ e_i = (q, \partial_+ q, \partial_- q) . $$

Therefore, eqs.(6.13) and (6.14) become :

$$ \partial_{++} q = \partial_+ \alpha \partial_+ q \quad , \quad \partial_{--} q = \partial_- \alpha \partial_- q $$

and

$$ \frac{\partial^2 \alpha}{\partial \tau^2} - \frac{\partial^2 \alpha}{\partial \sigma^2} - \exp \alpha = 0 . \quad (6.18) $$

This is the Liouville equation whose general solution is given by

$$ \alpha(\sigma, \tau) = \log \frac{2f'(x_+)}{f(x_+) + g(x_-)} , \quad (6.19) $$

where $f$ and $g$ are arbitrary functions of the indicated variables.

The $D = 2$ case can be solved directly from the equations of motion (6.7) and the constraints eqs.(6.8) in the coordinates $(u, v)$ defined as follows:

$$ q_0 = \sinh u \quad , \quad q_1 = \cosh u \cos v \quad , \quad q_2 = \cosh u \sin v $$

$$ -\infty < u < +\infty \quad , \quad 0 \leq v < 2\pi . \quad (6.20) $$

The constraints eqs.(6.8) take the form

$$ (\partial_{\pm} q)^2 = (\partial_+ v \cosh u)^2 - (\partial_- u)^2 = 0 \quad (6.21) $$

Therefore, we have

$$ \partial_+ v \cosh u = \pm \partial_+ u \quad , \quad \partial_- v \cosh u = \pm \epsilon \partial_- u \quad , \quad (6.22) $$

where $\epsilon^2 = 1$ . In addition,

$$ \partial_+ q \partial_- q = (\cosh u)^2 \partial_+ v \partial_- v - \partial_+ u \partial_- u = (\epsilon - 1) \partial_+ u \partial_- u \quad (6.23) $$
The general solution of eq.(6.22) is given by

\[ v = \pm 2 \arctan[\exp(u)] + G(x-) \quad , \quad v = \pm 2\epsilon \arctan[\exp(u)] + F(x+) \quad , \quad (6.24) \]

where \( F \) and \( G \) are arbitrary functions of the indicated variables. We have here two different cases, depending on whether \( \epsilon = -1 \) or \( \epsilon = +1 \).

**Case A : \( \epsilon = -1 \)**

Here, we find

\[ v = \frac{1}{2}[F(x+) + G(x-)] \quad , \quad u = \log[\pm \tan(\frac{F-G}{4})] \quad (6.25) \]

This corresponds to the previous solution eq.(6.19) with

\[ f(x+) = -\exp[iF(x+) ] \quad , \quad g(x-) = \exp[iG(x-)] \quad . \]

Using the world sheet conformal invariance, we can always choose the gauge where

\[ v = \sigma \quad , \quad 0 < \sigma \leq 2\pi \quad , \quad u = \log(\tan(\frac{\tau}{2})) \quad , \quad 0 < \tau \leq \pi , \]

\[ \sinh u = -\cot \tau \quad , \quad -\infty < u < +\infty . \quad (6.26) \]

This describes a string winded around the de Sitter universe and evolving with it. A half of string evolution \( \pi/2 < \tau < \pi \) corresponds to the expansion time \( 0 \leq u < \infty \) of the de Sitter universe. Similarly, for the first half \( 0 < \tau < \pi/2 \), which corresponds to the contraction phase \( -\infty < u \leq 0 \). (see fig.3). Eq.(6.26) describes a string winded once around de Sitter space (here a circle). More generally, we may have

\[ v = n\sigma \quad , \quad 0 < \sigma \leq 2\pi \quad , \quad u = \log(\tan(\frac{n\tau}{2})) \quad , \quad (6.27) \]

where \( n \) is an integer number. This solution describes a string winded \( n \) times around the de Sitter space.
Let us consider the invariant interval (6.4) between two points on the string using coordinates \((u, v)\),

\[
ds^2 = \frac{1}{H^2} [-du^2 + \cosh^2 u \, dv^2].
\]

For the solution eq.(6.27), we have

\[
ds^2 = \left(\frac{n}{H \sin n\tau}\right)^2 (d\sigma^2 - d\tau^2).
\]

In the asymptotic regions \(\tau \to 0^+\) and \(\tau \to \pi^-/n\), the conformal factor blows up. The proper length of the string stretches infinitely as

\[
\Delta s = \frac{1}{H\tau} \Delta \sigma \quad \text{for} \quad \tau \to 0^+
\]

\[
\Delta s = \frac{1}{H(\pi/n - \tau)} \Delta \sigma \quad \text{for} \quad \tau \to \pi^-/n.
\] (6.28)

This is analogous to the unstable behaviour found in D-dimensional inflationary backgrounds (see below and refs.[8]), as well as for strings falling into space-time singularities (see sec.4 and refs.[6]).

**Case B : \(\epsilon = +1\)**

In this case,

\[
\partial_+ q, \partial_- q = 0 \quad \partial_{+-} q = 0
\] (6.29)

and therefore, the parametrization in terms of the field \(\alpha(\sigma, \tau)\) breaks down. Eqs.(6.24) yields \(F = G = \text{constant} \equiv C\). Then,

\[
u = \log[\pm \tan(v - \frac{C}{2})]
\] (6.30)

Now, we consider the equation of motion (6.29) to find the dependence on \(x_+\) and
Using also eqs.(6.20) and (6.30), we find

\begin{align*}
q_0 &= \mp \cot(v - C), \quad q_1 = \pm \cot(v - C) \cos C \mp \sin C, \\
q_2 &= \pm \cot(v - C) \sin C \pm \cos C
\end{align*}

Therefore, we obtain

\begin{align*}
v &= C - \pi/2 + \arctan[R(x_+) + S(x_-)], \quad \cosh u = \sqrt{1 + [R(x_+) + S(x_-)]^2} \quad (6.32)
\end{align*}

where \(R(x_+\) and \(S(x_-\) are arbitrary functions of the indicated variables.

Using the world sheet conformal invariance, we can always choose

\begin{align*}
v &= C - \pi/2 \pm \arctan \tau \quad \cosh u = \sqrt{1 + \tau^2} \quad (6.33)
\end{align*}

Therefore, the string solution eqs.(6.31) yields

\begin{align*}
q_0 &= \tau, \quad q_1 = -\tau \cos C - \sin C, \quad q_2 = -\tau \sin C + \cos C \quad (6.34)
\end{align*}

This solution actually describes a particle trajectory since it was possible to gauge out the parameter \(\sigma\). Eq.(6.34) describes a geodesic in the two dimensional de Sitter space-time, that is the trajectory of a massless particle. Since transverse dimensions are absent, only massless states appear in this two-dimensional case. The solution eq.(6.33) or (6.34) is a particular case of the center of mass solution described in ref.[2] when \(D = 2\) and \(m = 0\). When \(\tau\) goes from \(-\infty\) to \(+\infty\), the light rays go from \(q^0 = -\infty\) to \(q^0 = +\infty\). At the same time the angle \(v\) varies through an interval of \(\pi\) : \(v(-\infty) = v(+\infty) \pm \pi\). In eq.(6.33), the signs \(\pm\) correspond to a motion in the positive or negative direction of the de Sitter spatial circle (see fig.4). It should be noticed that travelling from \(q^0 = \tau = -\infty\) to \(+\infty\), the particle goes over half of the de Sitter circle. The solutions described in the A and B cases, contain all the string solutions in the two-dimensional de Sitter space-time.
Let us now consider strings in the 2+1-dimensional de Sitter spacetime. There, we have a four dimensional embedding Minkowski space-time where the antisymmetric Levi-Civita tensor allows us to construct a vector \( b \equiv b_4 \) orthogonal to the vectors \( q, \partial_+ q \) and \( \partial_- q \), namely

\[
b_{a} \equiv \exp(-\alpha) \epsilon_{abcd} q_b (\partial_+ q)_c (\partial_- q)_d \tag{6.35}
\]

The vectors \( (q, \partial_+ q, \partial_- q, b) \) form a basis. In addition, \( b.b = 1 \). Here, the compatibility condition eq.(6.16) yields

\[
u \equiv u_4 = u(x_+), \quad v \equiv v_4 = v(x_-) \tag{6.36}
\]

\[
\partial^2 \alpha \over \partial x- \partial x+ - \exp \alpha+ \tag{6.36}
\]

Upon a conformal transformation \( x_\pm \rightarrow \Phi(x_\pm), \alpha \rightarrow \alpha + \log[\Phi'_+ \Phi'_-] \) with \( (\Phi'_+)^2 = u(x_+) \) and \( (\Phi'_-)^2 = v(x_-) \), eq.(6.36) takes the Sinh-Gordon form

\[
\frac{\partial^2 \alpha}{\partial \tau^2} - \frac{\partial^2 \alpha}{\partial \sigma^2} - \exp \alpha + \exp -\alpha = 0 \tag{6.37}
\]

Notice that for closed strings \( q(\sigma, \tau) \) and hence \( \alpha(\sigma, \tau) \) are periodic functions of \( \sigma \) with period \( 2\pi \). Therefore, to find string solutions in de Sitter spacetime we can start from a periodic solution of eq.(6.37), and insert it on the field equations (6.7):

\[
[\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \exp \alpha(\sigma, \tau)]q(\sigma, \tau) = 0 \tag{6.38}
\]

Once this linear equation in \( q(\sigma, \tau) \) is solved, it remains to impose the constraints (6.8) and eq.(6.11).

The energy density for the sinh-Gordon model (6.37) reads here

\[
\mathcal{H} = \frac{1}{2}[(\frac{\partial \alpha}{\partial \tau})^2 + (\frac{\partial \alpha}{\partial \sigma})^2] - 2 \cosh \alpha(\sigma, \tau) \tag{6.39}
\]

That means a potential unbounded from below

\[
V_{eff} = -2 \cosh \alpha \tag{6.40}
\]

with absolute minima at \( \alpha = +\infty \) and at \( \alpha = -\infty \). As the time \( \tau \) evolves, \( \alpha(\sigma, \tau) \)
will generically approach these infinite minima. The first minimum corresponds to
an infinitely large string whereas the second describes a collapsed situation. That
means that strings in de Sitter spacetime will generically tend either to inflate at
the same rate as the universe (when $\alpha \to +\infty$) or to collapse to a point (when
$\alpha \to -\infty$). As we shall see below these general trends are confirmed by the explicit
string solutions.

Let us start by studying solutions where $\alpha = \alpha(\tau)$. Then the energy is

$$\frac{1}{2} \alpha^2 - 2 \cosh \alpha = E = \text{constant} \geq -2 \quad (6.41)$$

$\alpha(\tau)$ describes the position of a non-relativistic particle with unit mass rolling
down the effective potential $V_{eff} = -2 \cosh \alpha$. A particularly interesting situation
is the critical case $E = -2$ when one starts to roll down from the maximum of
$V_{eff}$. That is, the initial speed is zero and the 'time' $\tau$ to reach either minimum
($\alpha = +\infty$ or $-\infty$) is infinity. The corresponding solutions are

$$\alpha_-(\tau) = \log[\coth^2 \frac{\tau}{\sqrt{2}}] \quad \text{and} \quad \alpha_+(\tau) = \log[\tanh^2 \frac{\tau}{\sqrt{2}}] \quad (6.42)$$

$\alpha_-(\tau)$ starts at $\alpha = 0$ for $\tau = -\infty$ and rolls down to the right reaching $\alpha = +\infty$
for $\tau \to 0^-$. The solution $\alpha_+(\tau)$ also starts at $\alpha = 0$ for $\tau = -\infty$ but rolls down to
the left reaching $\alpha = -\infty$ for $\tau \to 0^-$. Notice that $\alpha_+(\tau) = -\alpha_-(\tau)$. In addition
we have the trivial (but exact) solution $\alpha^0(\tau) \equiv 0$.

Now that the function $\alpha(\tau)$ is known, we proceed to solve eq.(6.38) for $q(\sigma, \tau)$
with the constraints (6.8) and (6.11). Since $q_0$ is a time-like coordinate, we shall
assume $q_0 = q_0(\tau)$. A natural ansatz is then

$$q = (q_0(\tau), q_1(\tau), f(\tau) \cos \sigma, f(\tau) \sin \sigma) \quad (6.43)$$

Then, eqs.(6.7), (6.8) and (6.11) require

$$q_0(\tau)^2 = q_1(\tau)^2 + f(\tau)^2 \quad (6.44)$$
\[ \left[ \frac{dq_0}{d\tau} \right]^2 = \left[ \frac{dq_1}{d\tau} \right]^2 + \left[ \frac{df}{d\tau} \right]^2 + f^2 \]  
\[ (6.45) \]

\[ e^{\alpha(\tau)} = \left[ \frac{dq_0}{d\tau} \right]^2 - \left[ \frac{dq_1}{d\tau} \right]^2 - \left[ \frac{df}{d\tau} \right]^2 + f^2 \]  
\[ (6.46) \]

and

\[ \frac{d^2}{d^2\tau} q_0 - e^{\alpha(\tau)} q_0(\tau) = 0 \]
\[ \frac{d^2}{d^2\tau} q_1 - e^{\alpha(\tau)} q_1(\tau) = 0 \]  
\[ (6.47) \]

\[ \frac{d^2}{d^2\tau} f(\tau) + f(\tau) - e^{\alpha(\tau)} f(\tau) = 0 \]

In addition, it seems reasonable to choose the time coordinate \( q_0(\tau) \) to be an odd function of \( \tau \). Remarkably enough, eqs.\((6.44)\) - \((6.47)\) admit consistent solutions for \( \alpha(\tau) = \alpha_+(\tau) \), \( \alpha(\tau) = \alpha_-(\tau) \) and \( \alpha(\tau) = 0 \). We find for \( \alpha(\tau) = \alpha^0(\tau) = 0 \)

\[ q^0(\sigma, \tau) = \frac{1}{\sqrt{2}} (\sinh \tau, \cosh \tau, \cos \sigma, \sin \sigma) \]  
\[ (6.48) \]

For \( \alpha(\tau) = \alpha_-(\tau) \), we have

\[ q_-(\sigma, \tau) = (\sinh \tau - \frac{1}{\sqrt{2}} \cosh \tau \coth[\frac{1}{\sqrt{2}} \tau], \cosh \tau - \frac{1}{\sqrt{2}} \sinh \tau \coth[\frac{1}{\sqrt{2}} \tau], \frac{1}{\sqrt{2}} \cos \sigma \coth[\frac{1}{\sqrt{2}} \tau], \frac{1}{\sqrt{2}} \sin \sigma \coth[\frac{1}{\sqrt{2}} \tau]) \]  
\[ (6.49) \]

And for \( \alpha(\tau) = \alpha_+(\tau) \) we find

\[ q_+(\sigma, \tau) = (\sinh \tau - \frac{1}{\sqrt{2}} \cosh \tau \tanh[\frac{1}{\sqrt{2}} \tau], \cosh \tau - \frac{1}{\sqrt{2}} \sinh \tau \tanh[\frac{1}{\sqrt{2}} \tau], \frac{1}{\sqrt{2}} \cos \sigma \tanh[\frac{1}{\sqrt{2}} \tau], \frac{1}{\sqrt{2}} \sin \sigma \tanh[\frac{1}{\sqrt{2}} \tau]) \]  
\[ (6.50) \]

These string solutions are given for a fixed de Sitter frame. Applying the de Sitter group to them yields a multi-parameter family of solutions. As it is clear, we can study them in the frame corresponding to eqs.\((6.48)\) - \((6.50)\) without loss of generality. Let us now discuss the physical interpretation of these solutions.
We recall that for a given time \( q_0 = q_0(\tau) \), the de Sitter space is a sphere \( S^2 \) with radius \( R(\tau) = \frac{1}{\pi} \sqrt{1 + q_0(\tau)^2} \). \( q^0(\sigma, \tau) \) [eq.(6.48)], describes a string of constant size in a de Sitter universe that inflates for \( \tau \to \infty \) since for this solution \( R(\tau) = \frac{1}{\pi} \sqrt{1 + \frac{\sinh^2 \tau}{2}} \). This solution is probably unstable under small perturbations.

The solution \( q^-(\sigma, \tau) \) is more interesting. One should distinguish four domains in it:

(i) \(-\infty < \tau < -\tau_0\), (ii) \(-\tau_0 < \tau < 0\), (iii) \(0 < \tau < \tau_0\), (iv) \(\tau_0 < \tau < +\infty\),

where \( q(\tau_0) = q(-\tau_0) = 0 \).

From eq.(6.49) we find \( \tau_0 = 1.489... \). In the intervals (i) and (iii) \( R(\tau) \) decreases (the universe contracts), whereas for (ii) and (iv) \( R(\tau) \) grows, (the universe expands). The string size is given here by

\[
S^-(\tau) = \frac{1}{\sqrt{2}H} \coth\left[\frac{1}{\sqrt{2}}|\tau|\right]
\]  \hspace{1cm} (6.51)

That is, the string size increases for \( \tau < 0 \) and decreases for \( \tau > 0 \) with a singular behaviour \( \frac{1}{|\tau|} \) for \( \tau \to 0 \). We see that the string size grows monotonically in intervals (i) and (ii), this growing becoming explosive for \( \tau \to 0 \) when the size of the de Sitter space diverges. Actually, the string grows there at the same rate as the whole space \( (\approx \frac{1}{|\tau|}) \).

For large \( |\tau| \) the de Sitter space is also very large with

\[
R^-(\tau) \quad |\tau| \to \infty \quad \frac{\sqrt{2} - 1}{2\sqrt{2}H} e^{\sqrt{2}|\tau|} \left[1 + O(e^{-|\tau|})\right]
\]  \hspace{1cm} (6.52)

whereas the string size tends to a constant

\[
S^-(\tau) \quad |\tau| \to \infty \quad \frac{1}{\sqrt{2}H} + O(e^{-\sqrt{2}|\tau|})
\]  \hspace{1cm} (6.53)

The behaviour for small \( |\tau| \) confirms the asymptotic results found in refs.[2 - 7 - 8].
It is interesting to study this string solution in another set of de Sitter coordinates. The cosmic time $t_0$ and the conformal time $\eta$ [eq.(6.2)] take for $q_-(\sigma, \tau)$ the form:

$$e^{Ht_0} = -\frac{1}{H\eta} = [1 - \frac{1}{\sqrt{2}} \coth(\frac{1}{\sqrt{2}}\tau)] e^\tau, \quad \rho = \frac{e^{-\tau}}{[1 - \sqrt{2} \tanh(\frac{1}{\sqrt{2}}\tau)]} \quad (\tau < 0).$$

(6.54)

where

$$\rho \equiv \frac{\sqrt{(q_2)^2 + (q_3)^2}}{H(q_0 + q_1)}. \quad (6.55)$$

Therefore, for $\tau \to 0$ (large universe and inflating string),

$$\eta = \frac{\tau}{\pi} \to 0, \quad \rho = 1 + O(\tau^2), \quad t_0 = -\frac{1}{H} \log |\tau| + O(\tau) \to \infty \quad (6.56)$$

whereas for $\tau \to -\infty$ (large universe but fixed string size),

$$t_0 = \frac{\tau}{\pi} + \frac{1}{H} \log(1+\sqrt{2}) \to -\infty, \quad \eta = -\frac{e^{-\tau}}{H(1 + \frac{1}{\sqrt{2}})} \to -\infty, \quad \rho = \frac{e^{-\tau}}{(1 + \frac{1}{\sqrt{2}})H} \to \infty. \quad (6.57)$$

We see that $\tau$ interpolates between the cosmic and the conformal time. Notice that this confirms the asymptotic behavior (6.56) discussed in previous works[2, 7, 8].

Let us now discuss the solution $q_+(\sigma, \tau)$ [eq.(6.50)]. There are here two phases:

(i) $\tau < 0$ : contraction phase, $R(\tau)$ decreases,

(ii) $\tau > 0$ : expansion phase, $R(\tau)$ grows.

The string size is here

$$S_+(\tau) = \frac{1}{\sqrt{2}H} \tanh\left[\frac{1}{\sqrt{2}}|\tau|\right]. \quad (6.58)$$

Therefore, the string contracts from a fixed size $\frac{1}{\sqrt{2}H}$ at $\tau = -\infty$ during (i) till the colapse at $\tau = 0$. At this point the de Sitter space has a minimum size $\frac{1}{H\pi}$. For
τ > 0 , the string size grows from zero till the fixed value \( \frac{1}{\sqrt{2H}} \), while the de Sitter space radius tends to infinity as

\[
R_+(\tau) \xrightarrow{\tau \to \infty} (\sqrt{2} - 1) \frac{1}{2\sqrt{2H}} e^\tau
\]

(6.59)

This behaviour is quite different from \( q_- (\sigma, \tau) \) and was not noticed before. Additional solutions follow by replacing

\[
\sigma \to n\sigma, \quad \tau \to n\tau, \quad n \in \mathbb{Z}
\]

(6.60)

in eqs.(6.48) - (6.50) . In these solutions the string is winded \( n \) times around the \( q_1 \) axis.

In addition, eq.(6.41) leads to elliptic solutions for \( E > -2 \). However, as shown in [7], the string constraints select periodic solutions of the sinh-Gordon equation associated to the lower boundary of the allowed zone, therefore excluding elliptic solutions. More precisely, we found (real) elliptic solutions of the sigma model field equations (6.7) for \( 0 \geq E > -2 \) but not solutions of the string equations since \( T_{\pm\pm} \) turned to be a non-vanishing constant for \( E > -2 \) [7].

As last topic, let us consider strings propagating in the 3 + 1 dimensional de Sitter spacetime. Here the reduced model (6.14) contains in principle four fields \( (u_4, u_5, v_4 \text{ and } v_5) \) besides \( \alpha(\sigma, \tau) \). Using conformal invariance and some O(3,1) invariance, we eliminate all these fields except one : \( \beta(\sigma, \tau) \). \( \beta(\sigma, \tau) \) is defined through:

\[
U(x_+)V(x_-) \cos \beta(\sigma, \tau) = u_4v_4 + u_5v_5
\]

where \( U(x_+) = V(x_-) = 1 \) in an appropriate conformal frame [7]. The equations of motion for \( \alpha(\sigma, \tau) \) and \( \beta(\sigma, \tau) \) can be derived from the (reduced) Lagrangian

\[
\mathcal{L} = -\frac{1}{2}[\left(\partial_\tau \alpha\right)^2 - \left(\partial_\sigma \alpha\right)^2] + \frac{1}{2}[\left(\partial_\tau \beta\right)^2 - \left(\partial_\sigma \beta\right)^2] - V(\alpha, \beta) \quad , \quad V(\alpha, \beta) = -e^\alpha - e^{-\alpha} \cos \beta.
\]

(6.61)

This potential can be related to the \( B_2 \) Toda field theory upon changing \( \beta \to i\beta \).
It is a hyperbolic Toda model connected with the O(3,1) group instead of the O(4) group.

The potential (6.61) is unbounded from below (as (6.39)-(6.40) in 2+1 dimensional de Sitter) and indicates that the string time evolution will tend to the absolute minima at $\alpha = +\infty$ or at $\alpha = -\infty$ with $|\beta| < \pi/2$ [7]. In ref.7 we derive the string behaviour near such points. We see from eq.(6.14) that $\alpha = +\infty$ and $\alpha = -\infty$ are present as strongly attractive points for any dimension D. In other words, strings in D-dimensional de Sitter spacetime tend generically to inflate and sometimes to collapse as we have seen explicitly in concrete 2+1 dimensional solutions.

7. Concluding Remarks

Unexpected and deep similarities appear between the string behaviour near spacetime singularities in singular plane waves (Sec.4) and for expanding (non-singular) universes (sec. 6). The $U = 2\alpha'pr \to 0$ string behaviour in the $\beta = 2$ singular plane wave case [eqs.(4.37)] is similar to the inflating string solution $q^-(\sigma, \tau)$ in de Sitter spacetime [eqs.(6.49) and (6.56)] for $\tau \to 0^-$. $U$ plays in the first case a similar rôle than the conformal time $\eta$ in the de Sitter case. Both $U$ and $\eta$ are proportional to $\tau$ in these regimes. Moreover, the string blows up in both cases with its size growing as a power of $1/\tau$ for $\tau \to 0$. We have $|\tau^{1-\sqrt{1+4\alpha}}|$ and $|\tau|^{-1}$, for singular plane waves ($\beta = 2$) and de Sitter respectively. In addition, for FRW universes with conformal factor proportional to $(\eta)^{-\gamma}$, ($\gamma > 0$), the string size blows up as $|\tau|^{-\gamma/2}$ for $\tau \to 0$ [8]. This entails a continuous exponent, as for singular plane waves.

The fact that similar features appear in such different geometries strongly indicates that the string stretching phenomenon is a generic property for strings on strong gravitational fields.

The aim of these lectures is to present the basic notions about strings in curved
spacetime. This is the beginning of a vast and relevant domain. We think that it will be helpful and necessary for the quantum understanding of gravity.

REFERENCES

1. See for example: K. G. Wilson, Rev. Mod. Phys. 47, 773 (1975) and Rev. Mod. Phys. 55, 583 (1983).

2. H. J. de Vega and N. Sánchez, Phys. Lett. 197B, 320 (1987).

3. H. J. de Vega and N. Sánchez, Nucl. Phys. B309, 552 and 577(1988).

4. H. J. de Vega and N. Sánchez, Nucl. Phys. B317, 706 (1989).
   D. Amati and K. Klimčík, Phys. Lett. 210B, 92 (1988), see also : ref.[5].

5. M. Costa and H. J. de Vega, Ann. Phys. (N.Y.) 211, 223 and 235 (1991).

6. H. J. de Vega and N. Sánchez, Phys. Rev. D 45, 2783 (1992).
   H. J. de Vega, M. Ramón Medrano and N. Sánchez, LPTHE Paris preprint 92-13.

7. H. J. de Vega and N. Sánchez, LPTHE Paris preprint 92-31, to appear in Phys. Rev. D, H. J. de Vega A. V. Mikhailov and N. Sánchez, LPTHE Paris preprint 92-32 to appear in Theor. Math. Phys., special volume in the memory of M. C. Polivanov

8. N. Sánchez and G. Veneziano, Nucl. Phys. B333, 253 (1990), M. Gasperini, N. Sánchez and G. Veneziano, Int. J. Mod. Phys. A6, 3853 (1991) and Nucl. Phys. B364, 365 (1991).

9. M. Ademollo, A. D’Adda, R. D’Auria, E. Napolitano, S. Sciuto, P. Di Vecchia, F. Gliozzi, R. Musto and F. Nicodemi, Nuovo Cimento 21A, 77(1974).

10. H. J. de Vega and N. Sánchez, Nucl. Phys. B299, 818(1988).
11. H. J. de Vega and N. Sánchez, *Phys. Rev. D* **42**, 3969 (1992) and
H. J. de Vega, M. Ramón Medrano and N. Sánchez, *Nucl. Phys.* **B374**, 405 (1992)

12. L. Brink, P. Di Vecchia and P. Howe, *Phys. Lett.* **65B**, 471 (1976).
A.M. Polyakov, *Phys. Lett.* **103B**, 207 and 211 (1981).

13. T. Goto, *Prog. Theor. Phys.* **46**, 1560(1971). O. Hara, *Prog. Theor. Phys.* **46**, 1549(1971). Y. Nambu, Lectures at the Copenhagen Symposium.

14. See for example: M. Green, J. S. Schwarz and E. Witten, Superstrings vol. I,
J. Scherk, *Rev. Mod. Phys.* **47**, 123(1975).

15. H. J. de Vega and N. Sánchez, *Phys. Lett.* **244B**, 215(1990).
H. J. de Vega and N. Sánchez, *Phys. Rev. Lett.* **65** (C), 1517(1990).

16. H. J. de Vega, M. Ramón Medrano and N. Sánchez, *Nucl. Phys.* **B351**, 277 (1991).

17. H. J. de Vega, M. Ramón Medrano and N. Sánchez, *Nucl. Phys.* **B374**, 425 (1992).

18. H.J.de Vega, M.Ramón Medrano and N. Sánchez, *Phys. Lett.* **285B**, 206(1992).

19. H. J. de Vega and N. Sánchez, *Nucl. Phys.* **B317**, 731 (1989).

20. H. J. de Vega and N. Sánchez, *Int. J. Mod. Phys.* **A7**, 3043 (1992).

21. G. Horowitz and A.R. Steif, *Phys. Rev. Lett.* **64**, 260(1990) and
*Phys. Rev. D* **42**, 1950 (1990).

22. G. ’t Hooft, *Phys. Lett.* **198B**, 61 (1987).

23. See for example, N. D. Birrell and P. C. W. Davies, Quantum Fields in curved spaces,Cambridge University Press, 1982

24. R. Güven, *Phys. Lett.* **191B**, 275 (1987).

25. K. Pohlmeyer *Comm. Math. Phys.* **46**, 207 (1976)
H. Eichenherr and J. Honerkamp, *J. Math. Phys.* **22**, 374 (1981).
8. Figure Captions

Fig. 1.

Escape string directions. Horizontal lines: $\alpha_1 > 0$, vertical lines: $\alpha_1 < 0$.

Fig. 2.

Particle trajectories in a conical spacetime with defect angle $2\pi(1 - \alpha)$.

Fig. 3.

The one-sheet hyperboloid represents the 1+1-dimensional de Sitter spacetime embedded in a three-dimensional space. The closed circle represents a string solution A [eq.(6.26)] at a given time.

Fig. 4.

Same as in fig. 3 but now the string solution B [eq.(6.34)] is drawn. This is in fact a geodesic.