Dynamical lattice computation of the Isgur-Wise functions $\tau_{1/2}$ and $\tau_{3/2}$

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We perform a two-flavor dynamical lattice computation of the Isgur-Wise functions $\tau_{1/2}$ and $\tau_{3/2}$ at zero recoil in the static limit. We find $\tau_{1/2}(1) = 0.297(26)$ and $\tau_{3/2}(1) = 0.528(23)$ fulfilling Uraltsev’s sum rule by around 80%. We also comment on a persistent conflict between theory and experiment regarding semileptonic decays of $B$ mesons into orbitally excited $P$ wave $D$ mesons, the so-called “$1/2$ versus $3/2$ puzzle”, and we discuss the relevance of lattice results in this context.

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1. Introduction

We are concerned with semileptonic decays of $B$ mesons ($B$ and $B^*$) into orbitally excited $P$ wave $D$ mesons (collectively denoted as $D^{**}$'s): $B^{(*)} \rightarrow D^{**} l \nu$. These decays are of particular interest, because there is a persistent conflict between theory and experiment, the so-called “1/2 versus 3/2 puzzle”: while experimental results indicate that a decay into “1/2 $P$ wave $D^{**}$'s” is more likely, theory favors the decay into “3/2 $P$ wave $D^{**}$'s” (for recent reviews cf. [1, 2]).

1.1 Heavy-light mesons

A heavy-light meson is made from a heavy quark ($b$, $c$) and a light quark ($u$, $d$), i.e. $B = \{ \bar{b} u, \bar{b} d \}$ and $D = \{ \bar{c} u, \bar{c} d \}$.

In the static limit ($m_b, m_c \rightarrow \infty$) there are no interactions involving the static quark spin. Therefore, it is appropriate to classify states according to parity $P$ and the total angular momentum of the light quarks and gluons $j$ (cf. the left column of Table 1).

If $m_b, m_c$ are finite, $j$ is not a good quantum number anymore. States have to be classified according to parity $P$ and total angular momentum $J$ (cf. the right column of Table 1). Although $j$ is not a “true quantum number” anymore, it is still an approximate quantum number justifying the notation $D^j$. The above mentioned $P$ wave $D^{**}$'s are $\{ D_0^0, D_1^1, D_1^2, D_2^2 \} = \{ D_0^{1/2}, D_1^{1/2}, D_1^{3/2}, D_2^{3/2} \}$.

| $j^P$ | $j^P$ |
|---|---|
| $(1/2)^-$ | $S$ |
| | $0^- \equiv B, D$ |
| | $1^- \equiv B^*, D^*$ |
| $(1/2)^+$ | $P^-$ |
| | $0^+ \equiv D_0^0 \equiv D_0^{1/2}$ |
| | $1^+ \equiv D_1^1 \equiv D_1^{1/2}$ |
| $(3/2)^+$ | $P^+$ |
| | $1^+ \equiv D_1^1 \equiv D_1^{1/2}$ |
| | $2^+ \equiv D_2^2 \equiv D_2^{3/2}$ |

Table 1: Classification of heavy-light mesons (left: static limit; right: finite heavy quark masses).

1.2 The 1/2 versus 3/2 puzzle

Experiments (ALEPH, BaBar, BELLE, CDF, DELPHI, DØ), which have studied the semileptonic decay $B \rightarrow X_c l \nu$ (where $X_c$ is some hadronic part containing a $c$ quark), find the following composition of $X_c$:

- $\approx 75\%$ $D$ and $D^*$, i.e. $S$ wave states (which is in agreement with theory).
- $\approx 10\%$ $D_1^{3/2}$ and $D_2^{3/2}$, i.e. $j = 3/2 P$ wave states (which is in agreement with theory).
- For the remaining $\approx 15\%$ the situation is rather vague: a natural candidate would be $D_0^{1/2}$ and $D_1^{1/2}$, i.e. $j = 1/2 P$ wave states. This, however, would imply $\Gamma(B \rightarrow D_0^{1/2} l \nu) > \Gamma(B \rightarrow D_1^{1/2} l \nu)$, which is in conflict with theory. This conflict between experiment and theory is called the 1/2 versus 3/2 puzzle.
On the theory side most statements are made in the static limit \( m_{\mu}, m_c \to \infty \). In this limit the eight matrix elements relevant for decays \( B \to D^{*+} l \nu \) can be parameterized by two form factors, the Isgur-Wise functions \( \tau_{1/2} \) and \( \tau_{3/2} \) \([3]\). Here we only list two of these matrix elements:

\[
\begin{align*}
\langle D_{0}^{1/2}(v') | \bar{c} \gamma_5 \gamma_{\mu} b | B(v) \rangle & \propto \tau_{1/2}(w)(v-v')_{\mu} \\
\langle D_{2}^{3/2}(v') | \bar{c} \gamma_5 \gamma_{\mu} \gamma_5 b | B(v) \rangle & \propto \tau_{3/2}(w)(w+1)\varepsilon_{\mu \alpha \nu}^* v^\alpha - \varepsilon_{\alpha \beta}^* v^\alpha v^\beta v'_{\nu},
\end{align*}
\]

(1.1) (1.2)

where \( v \) and \( v' \) are the four velocities associated with the \( B \) and the \( D \) meson respectively, \( w = (v' \cdot v) \) and \( \varepsilon \) is the polarization tensor of the \( D \) meson.

By means of operator product expansion (OPE) a couple of sum rules has been derived in the static limit \([4, 5]\). The most prominent in this context is the Uraltsev sum rule,

\[
\sum_n \left( |\tau_{3/2}^{(n)}(1)|^2 - |\tau_{1/2}^{(n)}(1)|^2 \right) = \frac{1}{4},
\]

(1.3)

where \( \tau_{1/2} \equiv \tau_{1/2}^{(0)} \), \( \tau_{3/2} \equiv \tau_{3/2}^{(0)} \) and the sum is over all \( 1/2 \) and \( 3/2 \) \( P \) wave states respectively. From experience with sum rules one expects approximate saturation from the ground states, i.e.

\[
|\tau_{3/2}^{(0)}(1)|^2 - |\tau_{1/2}^{(0)}(1)|^2 \approx \frac{1}{4},
\]

(1.4)

which implies \( |\tau_{1/2}(1)| < |\tau_{3/2}(1)| \). This in turn strongly suggests

\[
\Gamma(B \to D_{0}^{1/2} l \nu) < \Gamma(B \to D_{2}^{3/2} l \nu),
\]

which, as already mentioned, is in conflict with experiment.

Phenomenological models \([6, 7]\) give the same qualitative picture, even when considering finite heavy quark masses \([8]\).

Possible explanations to resolve the 1/2 versus 3/2 puzzle include the following:

- The experimental signal for the remaining 15% of \( X_c \) is rather vague; therefore, only a small part might actually be \( D_{0}^{1/2} \) and \( D_{1}^{1/2} \).
- Sum rules like (1.3) might not be saturated by the ground states.
- Sum rules derived by OPE hold in the static limit and might change for finite heavy quark masses.
- Sum rules make statements about the zero recoil situation \( (w = 1) \), where the \( B \) and the \( D \) meson have the same velocity; to obtain decay rates, however, one has to integrate over \( w \).

With a dynamical lattice computation of \( \tau_{1/2}(1) \) and \( \tau_{3/2}(1) \) in the static limit, which is presented in the following section, we attempt to shed some light on this puzzle.

2. Lattice computation of \( \tau_{1/2} \) and \( \tau_{3/2} \)

For a more detailed presentation of this computation we refer to \([9]\). We use a method, which was proposed and tested in the quenched case in \([10]\).
Since the “Isgur-Wise relations” (1.1) and (1.2) are not directly useful to compute \( \tau_{1/2}(1) \) and \( \tau_{3/2}(1) \) (the right hand sides vanish at zero recoil), they have to be rewritten as shown in (1.1):

\[
\langle D_0^{1/2}(v) | \bar{e} \gamma_5 \gamma_j D_k b | B(v) \rangle = -i g_{jk} \left( m(D_0^{1/2}) - m(B) \right) \tau_{1/2}(1) \tag{2.1}
\]

\[
\langle D_2^{3/2}(v, \bar{v}) | \bar{e} \gamma_5 \gamma_j D_k b | B(v) \rangle = +i \sqrt{3} e_{jk} \left( m(D_2^{3/2}) - m(B) \right) \tau_{3/2}(1). \tag{2.2}
\]

We compute \( \tau_{1/2} \) by means of (2.1) and an “effective form factor”:

\[
\tau_{1/2}(1) = \lim_{t_0 - t_1 \to \infty, t_1 - t_2 \to \infty} \tau_{1/2, \text{effective}}(t_0 - t_1, t_1 - t_2) \tag{2.3}
\]

\[
= \frac{1}{Z_\partial} \left| \frac{N(P_+)N(S) \left\langle \left( \bar{Q}(P-) \right)^{\dagger} \gamma_5 \gamma_j D_k \bar{Q}(t_1) \right\rangle \left( \bar{Q}(S) \right)^{(S)}(t_2) \rangle}{(m(P_+) - m(S)) \left\langle \left( \bar{Q}(P+) \right)^{(P+)}(t_0) \right\rangle \left( \bar{Q}(P-) \right)^{(P-)}(t_1) \rangle \left\langle \left( \bar{Q}(S) \right)^{(S)}(t_1) \right\rangle \left( \bar{Q}(S) \right)^{(S)}(t_2) \rangle} \right| \tag{2.4}
\]

To this end we need static-light meson creation operators \( \bar{Q}^{(S)}, \bar{Q}^{(P-)} \) and \( \bar{Q}^{(P+)} \), static-light meson masses \( m(S), m(P_+) \) and \( m(P_+) \), 2-point and 3-point functions, and norms \( N(S), N(P_-) \) and \( N(P_+) \). \( Z_\partial \) is a perturbatively computed renormalization constant, whose derivation is explained in detail in [12, 9]. The computation of \( \tau_{3/2} \) is analogous. Explicit formulae can be found in [9].

### 2.1 Simulation setup

We use \( L^3 \times T = 24^3 \times 48 \) gauge configurations produced by the European Twisted Mass Collaboration (ETMC). The gauge action is tree-level Symanzik improved and the fermionic action \( N_f = 2 \) Wilson twisted mass at maximal twist yielding automatic \( \bar{Q}(a) \) improvement of physical quantities. The lattice spacing is \( a = 0.0855 \text{fm} \). To be able to extrapolate our results to physical light quark masses, we consider three different bare quark masses \( \mu_q \) corresponding to “pion masses” \( m_{PS} \), which are listed in Table 2. For more details regarding these gauge configuration we refer to [13, 14].

| \( \mu_q \) | \( m_{PS} \) in MeV | number of gauge configurations |
|---|---|---|
| 0.0040 | 314(2) | 1400 |
| 0.0064 | 391(1) | 1450 |
| 0.0085 | 448(1) | 1350 |

**Table 2**: Bare quark masses, pion masses and number of gauge configurations.

### 2.2 Static-light meson creation operators

The meson creation operators we use are latticized versions of the continuum expression

\[
\bar{Q}^{(T)}(x) = \bar{Q}(x) \int d\hat{n} \Gamma(\hat{n}) U(x; x + r \hat{n}) \psi^{(u)}(x + r \hat{n}), \tag{2.5}
\]

where \( \bar{Q}(x) \) creates a static antiquark at position \( x \), \( \psi^{(u)}(x + r \hat{n}) \) creates a light quark separated by a distance \( r \) from the static antiquark, \( U \) is a gauge covariant parallel transporter and \( \Gamma \) a combination of spherical harmonics and \( \gamma \) matrices yielding well defined parity \( \mathcal{P} \) and total angular momentum of the light degrees of freedom \( j \). The operators are collected in Table 3.
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| $\Gamma(\hat{n})$ | $j^{\mathcal{D}}$ | $j^{\mathcal{P}}$ | $O_h$ | lattice $j^{\mathcal{D}}$ | notation |
|------------------|------------------|------------------|-------|--------------------------|----------|
| $\gamma_5$       | 0$^-$            | (1/2)$^-$        | $A_1$ | $(1/2)^-, (7/2)^-, ...$  | $S$      |
| 1                | 0$^+$            | (1/2)$^+$        |       | $(1/2)^+, (7/2)^+, ...$  | $P_-$    |
| $\gamma_1 \hat{n}_1 - \gamma_2 \hat{n}_2$ (cyclic) | 2$^+$           | (3/2)$^+$        | $E$   | $(3/2)^+, (5/2)^+, ...$  | $P_+$    |
| $\gamma_5 (\gamma_1 \hat{n}_1 - \gamma_2 \hat{n}_2)$ (cyclic) | 2$^-$           | (3/2)$^-$        |       | $(3/2)^-, (5/2)^-, ...$  | $D_{\pm}$|

Table 3: $J$: total angular momentum; $j$: total angular momentum of the light degrees of freedom; $\mathcal{D}$: parity.

2.3 2-point functions, static-light meson masses, norms of meson states

With meson creation operators (2.5) at hand it is straightforward to compute the 2-point functions

$$C(\Gamma(t)) = \langle \langle \mathcal{O}^{(\Gamma)}(t)^\dagger \mathcal{O}^{(\Gamma)}(0) \rangle \rangle, \quad \Gamma \in \{\gamma_5, 1, \gamma_1 \hat{n}_1 - \gamma_2 \hat{n}_2\}. \quad (2.6)$$

From these 2-point functions we extract the meson masses $m(S)$, $m(P^-)$ and $m(P^+)$ via effective mass plateaus. To illustrate the quality of our data we show effective masses for $\mu_q = 0.0040$ in Figure 1. For details regarding the computation of the low lying static-light meson spectrum within our twisted mass setup we refer to [15, 16].

![Figure 1](image-url) Figure 1: Effective masses for $S$, $P_-$ and $P_+$ for $\mu_q = 0.0040$.

Moreover, we obtain the ground state norms $N(S)$, $N(P_-)$ and $N(P_+)$ by fitting exponentials to the 2-point functions (2.6) at large temporal separations.

2.4 3-point functions

The computation of the 3-point functions is again straightforward. We chose to represent the covariant derivative inside the heavy-heavy current in a symmetric way by a single spatial link in positive and negative direction.

2.5 Results

In Figure 2a we show the effective form factors $\tau_{1/2, \text{effective}}$ (eqn. (2.4)) and $\tau_{3/2, \text{effective}}$ for $t_0 - t_2 = 10$ as functions of $t_0 - t_1$ for $\mu_q = 0.0040$ (plots for the other two quark masses look
qualitatively identical. We extract $\tau_{1/2}$ and $\tau_{3/2}$ by fitting constants to the central three data points as indicated by the dashed lines. Results are collected in Table 4.

$$\tau_{1/2} = 0.30 \pm 0.01, \quad \tau_{3/2} = 0.52 \pm 0.01$$

![Figure 2: a) Effective form factors $\tau_{1/2, \text{effective}}$ and $\tau_{3/2, \text{effective}}$ for $t_0 - t_2 = 10$ and $\mu_q = 0.0040$. b) Linear extrapolation of $\tau_{1/2}$ and $\tau_{3/2}$ in $(m_{\pi S})^2$ to the physical $u/d$ quark mass.](image)

Table 4: $\tau_{1/2}$ and $\tau_{3/2}$ and their contribution to the Uraltsev sum rule.

| $\mu_q$  | $\tau_{1/2}(1)$          | $\tau_{3/2}(1)$          | $(\tau_{3/2})^2 - (\tau_{1/2})^2$ |
|----------|------------------------|------------------------|----------------------------------|
| 0.0040   | 0.300 (14)             | 0.521 (13)             | 0.181 (16)                       |
| 0.0064   | 0.313 (10)             | 0.540 (13)             | 0.194 (13)                       |
| 0.0085   | 0.309 (12)             | 0.524 (8)              | 0.178 (9)                        |

As expected from sum rules $\tau_{3/2}$ is significantly larger than $\tau_{1/2}$. Moreover, we find that the ground states fulfill the Uraltsev sum rule (1.3) by around 80%.

We use our results at three different values of the pion mass to linearly extrapolate $\tau_{1/2}$ and $\tau_{3/2}$ in $(m_{\pi S})^2$ to the physical $u/d$ quark mass ($m_{\pi S} = 135$ MeV; cf. Figure 2b). Our final result is

$$\tau_{1/2}^{m_{\text{phys}}} (1) = 0.297 (26), \quad \tau_{3/2}^{m_{\text{phys}}} (1) = 0.528 (23).$$

3. Conclusions

Our result (2.7) confirms the sum rule expectation that $\tau_{3/2} (1) \gg \tau_{1/2} (1)$ in the static limit. When comparing to the experimentally measured form factors ($\tau_{1/2}^{\text{exp}} (1) = 1.28$ and $\tau_{3/2}^{\text{exp}} (1) = 0.75$ [17]) we find fair agreement for $\tau_{3/2}$ but a strong discrepancy for $\tau_{1/2}$.

In our opinion this discrepancy calls for action both on the theoretical and the experimental side: it would be highly desirable to have a first principles lattice computation of $\tau_{1/2}$ and $\tau_{3/2}$ beyond the zero recoil situation and also for finite heavy quark masses; on the other hand a thoroughly refined experimental analysis of the decay into $1/2 D^{**}$'s, for which the signal is rather faint, seems to be necessary.
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