Domain scaling and marginality breaking in the random-field Ising model

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A scaling description is obtained for the $d$-dimensional random field Ising model from domains in a bar geometry. Wall roughening removes the marginality of the $d = 2$ case, giving the $T = 0$ correlation length \( \xi \sim \exp\left(A h^{\gamma}\right) \) in $d = 2$, and for $d = 2 + \epsilon$ power law behaviour with $\nu = 2/\epsilon \gamma$, $h^* \sim \epsilon^{1/\gamma}$. Here, $\gamma = 2, 4/3$ (lattice, continuum) is one of four rough wall exponents provided by the theory. The analysis is substantiated by three different numerical techniques (transfer matrix, Monte Carlo, ground state algorithm). These provide for strips up to width $L = 11$ basic ingredients of the theory, namely free energy, domain size, and roughening data and exponents.

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1. INTRODUCTION

The phase diagram and critical properties of the d-dimensional random field Ising model has long been a subject of great interest. This is partly because of its role in describing the striking behaviour of diluted antiferromagnets in a uniform field. However, the model has become celebrated in its own right because of the way it captures in an extreme way the effects of randomness and frustration; and because of continuing questions concerning its lower critical dimension, the applicability of dimensional reduction, and the nature of the transition and critical behaviour. As is well known, the lower critical dimension turns out to be 2, which originally suggested by domain wall arguments, namely \( d_L = 2 \). This implies a marginal behaviour at dimension \( d = 2 \), and it has been argued that here domain roughening effects are important. This present paper provides a detailed study based on this point of view. We explore the consequences of a roughened domain wall picture, and at the same time confirm its validity by observing characteristics of the basic ingredients as well as consequent predicted behaviour in data from numerical studies.

The domain analysis is most conveniently made in a bar geometry. The dependence on a bar width \( L \) may be used to arrive at phenomenological scaling transformations. These in turn give the phase diagram and critical properties. In the particular case of low random field strength \( h \) and temperature \( T \), the domains span the bar width, and are well separated along the bar, so the basic “flat wall” description (without roughening) is very simple. As expected, it results, for \( d = 2 \), in a zero temperature fixed point at which the field scaling is marginal. Adding domain wall roughening breaks this marginality, and this ingredient is essential at and near \( d = 2 \). Consequences are: (i) the bulk correlation length behaviour \( \xi \sim \exp (A/h^\gamma) \) at \( T = 0 \) in \( d = 2 \), and (ii) for \( d = 2 + \epsilon \) the phase boundary joining the \( T \neq 0 \), \( h = 0 \) fixed point to one at \( T = 0 \), \( h = h^* \sim \epsilon^{1/\gamma} \) where \( \xi \sim [h - h^*]^{-2/\gamma} \). \( \gamma \) is one of four exponents occurring in the \( h \) and \( L \) dependence of the wall roughening free energy and characteristic scale, and is predicted to be \( \gamma = 2 \) in the two dimensional lattice. These results have been briefly reported elsewhere.

This description has been here tested by direct numerical investigations in the bar geometry for \( d = 2 \) (strips). This is carried out by transfer matrix techniques (another reason for using the bar geometry) and by Monte Carlo analysis using a new thermalization technique. In addition, we use data from a max-flow algorithm for constructing ground states which has been adapted for strips.

The domain description involves free energy \( \mathcal{F} \) and domain size \( \Xi \) as well as roughening characteristics. The largest eigenvalue of the transfer matrix gives accurate numerical data for \( \mathcal{F} \), and both \( \mathcal{F} \) and \( \Xi \) are available from the Monte Carlo analysis, for comparison with the theory. The wall roughening affects both \( \mathcal{F} \) and \( \Xi \), but its characteristics are most directly seen in measures of the wall profile itself. These and their associated exponents are best provided by the ground state algorithm. The resulting comparisons of theory and numerical data give a very complete test of the theoretical description and convincing support for its validity.

The outline of the paper is as follows. In section 1.A the domain picture is introduced and the flat wall theory is provided at \( T = 0 \) for the free energy, domain size, and correlation length of bars, and via phenomenological scaling, for the bulk criticality. Section 1.B generalises the flat wall description to low temperatures, and section 1.C describes the numerical approaches and the comparison of free energy and domain size data with flat wall theory. Section 2.A describes the domain wall roughening at \( T = 0 \), for lattice and continuum models via a simple approach and a field theory. First, a single “decoration” is discussed, then decorations on all scales, to provide the modified free energy and domain size, and hence the scaling and criticality. This description is generalised to low temperatures in section 2.B and compared with the numerical domain size and roughening data, including exponents, from the ground state algorithm in section 2.C. Section 3 states the main conclusions of the paper.

II. FLAT WALL

A. Zero Temperature Theory

1. Introduction

In this section, we establish the domain wall description of the RFIM by taking the zero temperature case and assuming flat walls. This is the springboard for all the further analytic work. Surprisingly perhaps, the results of even this very simply case are very good in certain ranges of dimensionality, but they indicate also where a more comprehensive picture involving roughening is required, and this is provided in section II.

The procedure introduced here is to obtain the dispositions of walls by minimizing an energy which is discussed below. The resulting domain size is then used in a phenomenological (finite size) scaling scheme to provide the RG transformation and hence the critical properties. Both these steps are most easily carried out in the bar geometry discussed below.

2. The Domain Picture

We suppose, following the prescription of Imry and Ma, that the contribution of the field to the energy of a domain goes like \( h \sqrt{V} \) where \( V \) is the number of spins.
enclosed within a domain and $h$ is the standard deviation of the field distribution $P$, assumed to have zero mean. This is plausible for any distribution of fields that tends towards a Gaussian distribution in the sense of the central limit theorem. Also, the energy change per domain due to the exchange interaction is exactly proportional to the perimeter of the domain $A$; so

$$\mathcal{F}(T = 0) = n_d(2J A - c_0 h \sqrt{V})$$  \hspace{1cm} (1)$$

where $n_d$ is the number of domains. In this equation $A$ and $V$ represent an average perimeter and volume measurement for all the domains. As it is a constant, the basic ferromagnetic energy $-2JNq$ (where $q$ is the coordination number of the lattice) is neglected here and throughout this paper. The constant $c_0$ is of order unity and represents both the statistical fluctuations and the selection effects of the domains. We describe the origins of this constant, and determine its value, in section III.

We have chosen the strip geometry in which to apply the zero temperature free energy for the following reasons: (i) convenience for application of finite-size scaling procedures, (ii) amenability to domain wall arguments, and (iii) we wish to later make contact with transfer matrix calculations. With these considerations in mind, we choose a $d$ dimensional cubic system of finite extent in $d - 1$ directions and infinite in the remaining dimension. Our analysis gives a length scale $\Xi_L$ which diverges like the correlation length $\xi_L(h)$ at the fixed point $h^*$. We can then determine the bulk critical properties via an RG equation of the form $h \to h' = R(h)$ which arises from the phenomenological finite-size scaling ansatz

$$\frac{\xi_L(R(h))}{L} = \frac{\xi_{bL}(h)}{bL}.$$  \hspace{1cm} (2)$$

Standard RG procedures provide from $R_b(h)$ the critical condition and exponents (from the fixed points and the eigenvalues, respectively).

3. The Zeroth Order Theory

Given the strip geometry described in the preceding section, the leading order modification of the ferromagnetic ground state where all the spins are aligned is a splitting of the system into $n_d$ domains. These domains are taken to have flat walls that span the width $L$ of the strip and are of typical length $\Xi_L$, as shown in Figure 1. For this picture to be consistent we need $\Xi_L \gg L$, which will turn out to hold if $h$ is sufficiently small. In this calculation, and in the rest of this paper, we set our units so that the lattice constant is unity. Using the example of equation (1) we may write the excess energy of this domain state as

$$\mathcal{F}(T = 0) = n_d \left(2J L^{d-1} - c_0 h \sqrt{\Xi_L L^{d-1}}\right)$$  \hspace{1cm} (3)$$

where $n_d = N/\Xi_L$. At zero temperature, this energy is equal to the Helmholtz free energy of the system. So, to find the equilibrium free energy we extremalize with respect to $\Xi_L$ yielding the solution

$$\Xi_L = L^{d-1} \left(\frac{c_0 h}{4J}\right)^{-2}.$$  \hspace{1cm} (4)$$

Applying the phenomenological scaling equation (2) gives an RG equation for $h$ of the form

$$h' = hh^{2d-2}.$$  \hspace{1cm} (5)$$

This implies that there is a fixed point at $h = 0$ which is unstable for $d < 2$ and stable for $d > 2$. The canonical RG prescription combined with (3) gives the critical exponent $\nu = 2/(2-d)$ for the unstable fixed point at $h = 0$. This result holds for all $d < 2$. However, it also warns us that $d = 2$ is a special case where the scaling field $h$ is marginal. This is connected with the fact that the lower critical dimension of the RFIM is $d_{c1} = 2$. Higher order terms in the free energy, however, can break this marginality. The wall roughening decorations described in section IV achieve this.

4. Relation of the Domain Size to Correlation Length

In the previous section we introduced the average domain size, $\Xi_L$. This is related, as we now show, to the correlation length $\xi_L$ of the flat wall model.

Consider the configurationally averaged correlation function $G(r)$ between spins in two columns separated by $r$ lattice units. Suppose that the flat domain walls are independently randomly distributed, so that $p$ is the probability that a domain wall lies between two arbitrarily chosen adjacent columns. Then, since all the spins in a given domain are aligned with each other,

$$G(r) = (1 - p)G(r - 1) - pG(r + 1)$$

$$= (1 - 2p)G(r - 1)$$  \hspace{1cm} (6)$$

and $G(0) = 1$. It follows that

$$G(r) = (1 - 2p)^r \equiv e^{-r/\xi_L},$$  \hspace{1cm} (7)$$

where the last relation defines the correlation length $\xi_L$.

However the definition of $p$ shows that its inverse is the average domain size $\Xi_L$:

$$p = \Xi_L^{-1}.$$  \hspace{1cm} (8)$$

So we conclude that the correlation length is related to the domain size by

$$\xi_L = \left[\ln \left(\frac{1}{1 - \frac{1}{\Xi_L}}\right)\right]^{-1}.$$  \hspace{1cm} (9)$$

In the limit of large $\Xi_L$ this gives $\xi_L = \Xi_L/2$, which is the same as provided directly by a continuum version of the analysis given above.
5. The Correlation Length at $T = 0$

If we assume a flat wall picture, then we can actually find the zero temperature correlation length of the RFIM on a strip exactly, without employing the more intuitive energetic arguments presented above. Using this result, we can check the results of both section 4.3 and section 4.4. This calculation generalizes the work done by Farhi and Guttmann for the one dimensional RFIM.

The key is to consider the connected correlation function

$$\chi_{il} = \langle \sigma_i \sigma_{i+l} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+l} \rangle$$  \hspace{1cm} (10)$$

where we use angled brackets to denote the usual thermodynamic average and $\sigma_i$ and $\sigma_{i+l}$ are any two spins in columns $i$ and $i + l$, respectively. We remark that when $T = 0$ the thermodynamic average becomes an average over the possible ground states of the system. The trick to finding the dominant behaviour of this function arises from a desire to devise an algorithm that will solve for the ground state of the RFIM. In considering the ground state, one notes that even if the field is small, the energy to create a domain wall, $2JL$, can be accumulated from fluctuations in the random field over large domains. This is the reason why there is no long–range order for finite $h$, even at zero temperature, in two dimensions. Note that the work of Imry and Ma demonstrates that in $d \leq 2$ the $L$ dependencies are such that this result persists in the $L \rightarrow \infty$ limit.

With this in mind, let us define the quantity

$$S(k, l) = \sum_{j=k}^{k+l} \sum_{i=1}^{L} h_{ij}$$ \hspace{1cm} (11)$$

where we temporarily adopt the notation that $i$ labels the vertical coordinate on the strip, and $j$ the horizontal one. Then, whenever $|S(k, l)| \geq 4JL$, it is favourable to insert a domain wall at positions $k$ and $k + l$. When we attempt to employ this algorithm on a variety of field configurations, one immediately notices that the ground state is still not completely determined.

A region between two opposite random fields, and with endpoints $k, j$ such that $S(k, j) = 0$ (and $j$ is the site closest to $k$ when this is so) is a “floppy domain” (FD) in the sense that in an average over the ground states, the spins in these domains may point up or down, and are not fixed like the spins in other regions (“rigid domains” (RD)).

One also notices that there is increased degeneracy if $2JL/h$ is an integer. This is related to the fact that the conditions for both $\uparrow \downarrow$ and $\downarrow \uparrow$ FD can be simultaneously satisfied. To avoid the additional difficulties involved in the analysis of this case, we shall assume that $2JL/h$ is not an integer.

It is the degeneracy in the ground state that gives rise to non–vanishing connected correlations. Indeed, it is the FD’s that contribute exclusively to $\chi_{il}$. For instance, if either spin $\sigma_i$ or spin $\sigma_{i+l}$ is in a RD then $\langle \sigma_i \sigma_{i+l} \rangle = \langle \sigma_i \rangle \langle \sigma_{i+l} \rangle$. Furthermore, if the two spins are in different FD, then they are uncorrelated random variables, and their contribution to $\chi_{il}$ also vanishes. Thus, one need only consider spins that are in the same FD.

Thus, we are left with a combination of three effects to consider

- The probability $W_1(l)$ that there exists a FD of size $l > 1$.
- Given that spin $\sigma_i$ is in a FD, the probability $W_2$ that the spin $\sigma_{i+l}$ is also in the FD.
- The thermal average and the average over random field configurations subject to the constraint that the spins $\sigma_i$ and $\sigma_{i+l}$ are both in the same FD. We label this probability $W_3$.

We also introduce the variable

$$L = \left[ \frac{2JL}{h} \right] + 1$$

where $[x]$ denotes the integer part of the non–integer $x$.

We can view the random field configuration on a strip of width $L$ as a random walk in $1 + 1$ dimensions. At each site along the horizontal direction of the strip, the walk changes by a height equal to the columnar sum of random field values at that point. For instance, if we have a binary distribution of fields, then for $L = 3$ the height changes by $\pm 3, \pm 1, \text{etc.}$ Using this view, $S(0, l)$ represents the height of the random walk at $x = l$ (see Figure 2).

If the aggregated change in height of the random walk ever reaches $L$ then we have accumulated enough fields in either the up or down direction to form a domain. If there are regions between these peaks where the change in height is zero, then these are FD. Figure 2 should also make clear the added ambiguity that arises if $2JL/h$ is an integer.

This random walk picture is very useful in evaluating the three previously introduced probabilities. For instance, $W_1(l)$ is the probability that the random walk returns to the origin after taking $l$ steps. This is known to fall off exponentially with $l$,

$$W_1(l) \sim \exp \left[ -\frac{l}{\xi_L(L)} \right].$$ \hspace{1cm} (12)$$

where $\xi_L(L)$ is a characteristic decay length. We shall return to an exact determination of this in a moment.

From simple geometrical considerations, $W_2 \sim 1/l$. For the third probability, we note that the average over ground states corresponds to an average over domain wall positions. Since the number of places for a domain wall to be inserted between two spins separated by a distance $l$ goes like $l$, we could estimate the conditional probability $W_3$ as $\sim 1/l^2$. These are only rough estimates, the probabilities depend also upon the ratio $h/J$. But we only
need here the \( l \) dependence, and the important feature of \( \mathcal{W}_1 \) and \( \mathcal{W}_3 \) is their power law decay in \( l \). We conclude that the only exponential contribution is from \( \mathcal{W}_1(l) \), and this gives the exponential decay of \( \chi \), so then the \( \xi_L(L) \) defined by (12) is actually the correlation length.

Now \( \mathcal{W}_1(l) \) is the probability that a directed random walker in 1 + 1 dimensions returns to the origin after \( \tilde{l} \) steps, without hitting walls at \( \pm L \). We solve this problem by reference to the transfer matrix formalism of random walks. This involves writing down a transfer matrix \( T(n, m) \) with \( n, m \in \{1, 2, \ldots, \mathcal{L}\} \) which represents the probability that the random walker makes a step from position \( m \) to \( n \). In terms of the RFIM, this means that

\[
T(n, m) = \text{Prob}(\sum_{i=1}^{L} h_{ij} = (n - m)h). \tag{13}
\]

The leading eigenvalue of the transfer matrix, \( \lambda(L) \) is related to the survival probability by

\[
\mathcal{W}_1(l) \sim \lambda(L)^l \tag{14}
\]

so that the correlation length of equation (12) is

\[
\xi_L(L) = \frac{-1}{\ln \lambda(L)}. \tag{15}
\]

For \( L = 1 \), the transfer matrix is (using (13)) tridiagonal and symmetric, with zeros on the diagonal and \( 1/2 \) on the neighboring off diagonals. We can generalize this result for all \( L \), by noting a general pattern in a row through the middle of the transfer matrix:

\[
\begin{array}{cccccccc}
(i-j) & \ldots & 4 & -3 & -2 & -1 & 0 & +1 & +2 & +3 & +4 & \ldots \\
L = 1 & \ldots & 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \ldots \\
L = 2 & \ldots & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \ldots \\
L = 3 & \ldots & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \ldots \\
L = 4 & \ldots & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \ldots \\
\vdots & & & & & & & & & & & \\
\end{array}
\]

The numerators of the elements follow the same pattern as the numbers in Pascal’s triangle, and the denominators are simply \( 2^L \). This is due to the Gaussian nature of the random walk whereby \( T_L(n, m) = [T_1(n, m)]^L \). These matrices are all diagonalized by the same type of vectors:

\[
\Phi_j = \sin (j \pi q). \tag{16}
\]

where the wavenumber \( q \) is chosen to be consistent with the boundary conditions of the matrix, \( i.e.\ q = n/(\mathcal{L}+1) \). The pattern of entries in the transfer matrix gives the eigenvalues of these vectors as

\[
\lambda = \cos^2(q \pi). \tag{17}
\]

Clearly, the largest eigenvalue is the one with \( q = q_1 = 1/(\mathcal{L}+1) \). If \( h \) is small, then \( \mathcal{L} \) is large and \( q_1 \) is small so that equation (13) gives the correlation length as

\[
\xi_L = \frac{1}{L \ln \cos \left( \frac{\pi}{2J/h+2} \right)} = 2L \left( \frac{2J}{\pi h} \right)^2 + \frac{16\pi^2}{h} + O(1). \tag{18}
\]

It is easy to generalize these arguments to higher dimensions via the replacement \( L \to L^{d-1} \). Then (13) agrees to highest order in \( h \) with our earlier result for the \( T = 0 \) small \( h \) correlation length (from combining (8) with (11)). However, it is important to note that this result came from an analysis of the connected correlation function, whereas our prior analysis applied to the disconnected correlation function. Although the correlation lengths derived from these two quantities should diverge in the same manner, they may differ by a multiplicative constant (\( e.g. c_0 \)). Indeed, this turns out to be the case in section \( \Pi C \).

Although this calculation also assumes a flat wall scenario, it should be accurate when \( \xi_L \gg L \) and is exact for \( L = 1 \).

6. Conclusions

In this section, finite-size scaling in the strip geometry has been used to study the zero temperature fixed point of the RFIM. We found the critical exponent \( \nu = 2/(2-d) \) for \( d < 2 \). This result indicates the need for a more sophisticated analysis when \( d = 2 \), and this follows in the subsequent chapters.

The procedure involved the determination of the free energy and the typical domain size and its relationship to the correlation length. These results are confirmed by a direct calculation of the correlation length of the RFIM on a strip, assuming flat domain walls, which agrees perfectly up to the coefficient \( c_0 \).

This provides a consistent treatment of the RFIM at zero temperature and low random field strength away from the lower critical dimension. However, when \( d = 2 \) fluctuations are very important to the critical behaviour. So, the next section studies the effects of fluctuations in the domain wall shape, and this will complete the description of the zero temperature RFIM near the critical point. Comparisons with numerical work and investigation of thermal effects will be provided in later sections.

B. Low Temperature Theory

The extension of the domain wall analysis to non–zero temperature requires the construction of an entropy. The procedure will then be to minimize the free energy \( F = U - TS \), and so determine the characteristic lengths for exploitation via phenomenological scaling. By working in the low field and low temperature limit, \( i.e. low \, H \, and \, T \), we can use the well separated strip–spanning domain
pictures developed in the previous section. For the zeroth order (flat wall) description, the entropy then turns out to be quite trivial, corresponding to the number of ways of laying down flat domain walls with an average spacing defined by $\Xi_L = N/n_d$. Thus, the entropy is

$$S_0(\Xi_L) = \ln \left( \frac{N}{n_d} \right) = N \left[ \frac{\ln(\Xi_L - 1)}{\Xi_L} - \ln(1 - \Xi_L^{-1}) \right]$$

(19)

choosing units so that $k_B = 1$ and using Stirling’s approximation in the thermodynamic limit. The free energy generalizing $\xi$ to $T \neq 0$ is then

$$F = n_d \left( 2JL^{d-1} - \frac{c_0h}{2} \sqrt{\Xi_L L^{d-1}} \right) - TS_0(\Xi_L).$$

(20)

Extremalizing with respect to $\Xi_L$ gives the following equation for the domain size

$$-2JL^{d-1} + \frac{c_0h}{2} L^{d-1/2} \Xi_L^{-1/2} + T \ln(\Xi_L - 1) = 0.$$  

(21)

The various limits of this equation are

$$\Xi_L = \left( \frac{c_0}{c_1} \right)^2 L^{d-1}, \quad T = 0$$

$$\Xi_L = 1 + \exp \left( \frac{2JL^{d-1}}{c_1} \right), \quad h = 0.$$  

(22)

As a check on equation (22), we can use it to investigate the low temperature behaviour of the pure Ising model (zero field). By using this form for $\Xi_L$ in the phenomenological renormalization equation (4) we get the following RG equation for $T/J$

$$\left( \frac{T}{J} \right)' = \frac{2L^{d-1} \left( \frac{T}{J} \right)}{2(bL)^{d-1} - (\ln b) \left( \frac{T}{J} \right)}.$$  

(23)

This has a fixed point at $T = 0$ which is unstable for $d \leq 1$; it is stable for $d > 1$, implying the existence of a finite critical temperature which we know the pure Ising model should have in $d > 1$. In $d = 1$, (22) gives the usual pure Ising result $\xi \sim \exp(2J/T)$.

Thus, in $d = 1$ the two scaling variables (scaling like a length) are $\exp(2J/T)$ and (from (3)) $b^{-2/(2-d)}$. It follows that in $d = 1$ the correlation length can be written in the crossover form

$$\xi = c_a e^{2J/T} \phi(h^2 e^{2J/T})$$  

(24)

where $\phi(x) \to c_a$ or $c_b/x$ for $x \to 0$ or $\infty$, respectively, with $c_a, c_b$ constants.

For $d > 1$ the stability with respect to temperature of the $(T,h) = (0,0)$ fixed point means that no crossover form then applies at low $T, h$, but instead $h^{2/(2-d)}T^{1/(d-1)}$ is invariant under scaling.

Because of the attractive thermal nature of this $T = 0$ fixed point at small $h$ and $d > 1$, the thermal scaling has little influence on the low $T$ low $h$ critical behaviour for $d > 1$. Nevertheless, thermal effects contained in (20) and (21) can substantially modify the behaviour of the free energy and domain size in the low $T, low h$ regime for $d > 1$, masking the field dependences more significant for the RG scaling in this regime. So (20) and (21) will be required for interpretations in the next section, where numerical techniques are used to confirm the basic scaling picture so far discussed.

### C. Numerical Evaluation

#### 1. Introduction

We wish to test the results developed in the previous section by comparing them to independent numerical investigations. If we find reasonable agreement between the two, then one can feel confident about the veracity of both results.

The structure of the analytical approach is such that the numerical tests can be applied at various stages. The most obvious test of such a theory is on its final scaling predictions for the bulk system criticality. However, this is difficult since it requires the investigation of a system large enough to show the bulk critical behaviour. It is much easier to apply the numerical techniques to the finite size strip geometry, as is done here, and to investigate the non-critical ingredients from which the scaling transformation is built, and hence establish the applicability of the basic flat wall procedure.

Three techniques have been found to be successful, and their results are reported in this paper. There are a transfer matrix calculation, a modified Monte Carlo simulation, and a ground state algorithm.

The transfer matrix method is appropriate to the strip geometry, and can in principle provide the phenomenological RG transformations directly and very precisely, as has been exemplified for many low dimensional lattice systems, particularly those with non-random transfer matrices.

Very accurate results have also been obtained for random bond and site diluted Ising models and a modification of those techniques is used here. Because of the quenched disorder present in such models (and in the RFIM), the first (dominant) Lyapunov exponent of the transfer matrix product gives the average free energy of the strip, which is one of the two key ingredients in our domain wall analysis which we would like to check by the transfer matrix and other techniques. The other ingredient is the domain wall size or correlation length. Typically, the correlation length is provided by the TM approach via

$$\xi_L = \frac{-1}{\ln(\lambda_1/\lambda_0)}$$  

(25)

where $\lambda_0$ and $\lambda_1$ are the largest and second largest eigenvalues, respectively. However, this relation gives the most
probable correlation decay, which need not be the same as the average correlation decay. In fact, the correlations may follow quite complex distribution functions, and the difference between “most probable” and “average” decays becomes especially important when the effective interactions may differ in sign, such as in spin glasses or random field problems. For this reason, the comparison with transfer matrix data will be restricted to the free energy.

Monte Carlo simulation is the second technique used. The usual equilibration problems set by critical slowing down are here compounded by the effects of very large energy barriers in situations \( d \leq 2 \) where the criticality is at \( T = 0 \). The problems are partly alleviated by the strip geometry, particularly in \( d = 2 \), where characteristic lengths of order \( h^{-2} \) (see (34)) replace exponentially large correlation lengths of the bulk system (see (77)). However the equilibration times remain excessively long, and a new algorithm (described in more detail in (42)) was devised and employed to markedly improve thermalization times. Briefly, this algorithm continues the ideas of multigrid methods with self adjusting renormalisations of parameters to allow for changes to be made on all length scales via moves of blocked spins. This permits the calculation of average energies, domain sizes, and correlation lengths, for comparison with the analytic predictions.

The final numerical technique used was an exact algorithm44 to construct the ground states for both binary and Gaussian random field configurations. Ground states for models with arbitrary random fields and arbitrary but not frustrated exchange interactions may be found in polynomial time by mapping the optimization problem to a min–weighted–cut problem on an associated graph44. A new min–cut max–flow algorithm44 was first implemented by Ogielski44 to demonstrate the practicability of this method. We here apply the algorithm to the strip geometry, and hereafter refer to this method as the “max–flow algorithm”. From these data it was possible to obtain \( T = 0 \) properties including average energy, domain size, as well as domain wall roughening characteristics which will be discussed later.

We now proceed to the numerical data which test the flat wall picture discussed so far. The transfer matrix determinations of the free energy are first described and compared with the analytic work, followed by the Monte Carlo results and comparisons, then max–flow results and comparisons.

2. Transfer Matrix

The simulations were done on strips of width \( L = 2,\ldots,7 \) and length \( N = 10^5 \) and a binary random field distribution. Periodic boundary conditions were applied in the “finite” direction whereas all possible trial state vectors were applied to termini of the strip at each end of the “infinite” direction. This procedure was then repeated for three different realizations of the random field configuration, and the resulting free energy

\[
\mathcal{F}/N + 2 = -kT \ln \lambda_0
\]

averaged over these realizations and over the \( 2^L \) trial vectors. The average was calculated in the root-mean-square manner, and the associated error in the free energy was calculated as the rms error in the free energy at each random field configuration.

We wish to test the predictions of the zeroth order flat wall theory (section 1A), in which the free energy at very low temperature and \( d = 2 \) is

\[
\mathcal{F}_0 = \frac{-Nc_0^2h^2}{8LJ}.
\]

To this end, we have generated data for \( L \in (2,7) \) at \( T = 0.1J \). For lower temperatures, the elements of the transfer matrix become larger than the computer can handle. If the strip width is much larger than 7 the run times become unreasonably long (over a week). Nevertheless, these strip widths are adequate for our purposes.

As a test of this functional form, we plot \( \ln(\mathcal{F}_{TM}/J) \) versus \( \ln(h/J) \) for various strip widths \( L \), where \( \mathcal{F}_{TM} \) represents the free energy determined by the transfer matrix method. Straight lines of slope 2 are expected, and this is indeed the case, as demonstrated in Figure 3.

Straight line fits to these plots yield the slopes given in Table 1. We include the value of the \( Q \) statistic, which is a measure of the probability that a value of \( \chi^2 \) as poor as the value we have found should occur by chance. \( Q \) is always in the range \((0,1)\) and a value of \( Q > 0.1 \) usually denotes a reliable fit.

The agreement with the predicted slope of 2 is acceptable, with increasing accuracy at larger \( L \). However, there appear to be systematic errors, as one can see from a close inspection of Figure 3. This is an indication of the importance of higher order decoration effects and finite–temperature effects, occurring even at these low temperatures and high fields.

We now attempt to obtain a best value for the constant \( c_0 \), so that we can compare the predicted free energy with the numerics, and extract subdominant features noted above.

A simple fit of the numerical free energy to equation (27) is statistically unreliable, and gives residual \( L \) dependence in \( c_0 \), and other artifacts. We concluded that this was due to the data being at \( T = 0.1J \) and not zero temperature. Indeed, for small \( h \), the leading temperature dependence can be very significant. The zeroth order finite temperature theory of (21) and (22) implies that for large \( \Xi_L \) in a two–dimensional lattice

1 As with all the numerical data presented in this paper, these calculations were performed on DEC Alpha workstations.
\[
\mathcal{F} = -N\frac{c_0^2h^2}{8LJ} \left[ 1 + \mathcal{O} \left( \frac{T}{LJ} \ln \left[ L \left( \frac{4J}{c_0h} \right)^2 \right] \right) \right].
\] (28)

Since the logarithm diverges for small \( h \), the temperature dependence becomes increasingly important in this regime.

To fit the free energy data properly, we need a free energy form applying for all \( h, T \) as long as \( \Xi_L \) is large. This can be obtained by reparameterizing the relations (21) and (23) giving the finite temperature flat wall theory, provided the theory is correct for large \( \Xi_L \), as we have argued. Using \( h, T \) small, \( \Xi_L \) large, \( d = 2 \), the approximate forms to reparameterise are the excess free energy

\[
\frac{\mathcal{F}}{NL} = f = -\frac{c_0h}{2\sqrt{\Xi}L}.
\] (29)

and the extremalizing equation

\[
\left( \frac{c_0hL^{1/2}}{4T} \right)^{1/2} \Xi_L^{1/2} + \frac{1}{2} \ln \left( \Xi_L \exp \left( \frac{2JL}{T} \right) \right) = 0.
\] (30)

We now define a new variable \( y = \tilde{c}(T)\sqrt{\Xi_L} \), where \( \tilde{c}(T) \equiv \exp(-LJ/T) \). Then, the intensive free energy is

\[
f = -\frac{c_0^2h^2}{8\beta yT}.
\] (31)

and the minimizing equation for \( \Xi_L \) has the simple form

\[
\ln y = -\tilde{\beta}y
\] (32)

where

\[
\tilde{\beta} = \frac{c_0\sqrt{L}}{4c}. \tag{33}
\]

We may then eliminate \( y \) between (31) and (32) to arrive at an equation for \( f \) involving \( \beta \). Inserting its explicit form leads to

\[
\frac{LJ}{T} = \ln \left( \frac{-c_0h}{2f\sqrt{L}} \right) - \frac{c_0^2h^2}{8fT}.
\] (34)

This is an implicit equation for \( f \) as a function of \( h, T, L \). We will use it to solve for \( f \) numerically and so determine \( c_0 \). The usual \( \chi^2 \) statistic was calculated

\[
\chi^2 = \sum_i \left( \frac{f(h_i, T_i, L_i) - F_{TM}}{\sigma_i} \right)^2.
\] (35)

where \( \sigma_i \) is the estimated error on \( F_{TM} \). This statistic was minimized by a bisection routine, which also calculates the \( Q \) statistic.

As well as obtaining \( c_0 \), we can test whether non–Gaussian effects occur for small \( L \), or perhaps the flat wall description breaks down for large \( L \) by varying the range of the fit. The values of \( T \) used were \( 0.1J, 0.2J \) and \( 0.3J \), with \( h \in \{0.05, 0.25\} \). Over this range of \( T \) and \( h \), the zeroth order theory predicts large correlation lengths at least > \( 10L \) for even a liberal estimate of \( c_0 \), which should make (24) applicable. This assumption was also demonstrated to be self–consistently true by using the fitted value of the selection constant to predict \( \Xi_L \). Over this range of fields, then, the results are presented in Table I.

One notices the striking improvement to the fits when the lower \( L \) values are dropped (\( Q = 0.686 \) is remarkably good). We attribute this to non–Gaussian selection effects at lower \( L \) values, where the domains do not sample over as many field configurations as a domain of the same size at large \( L \). Since the number of field configurations for a column of length \( L \) is \( 2^L \), we would expect a marked decrease in this effect as we increase \( L \).

The value \( c_0 \approx 1.85 \), found by fits of the theory to the free energy data, does not agree with that \( (c_0 = \pi) \) predicted in section IIA 3 by a random walk analysis of the disconnected correlation function. As explained in section IIA 5, this is due to the different correlation functions used in obtaining this (non–universal) constant.

The large \( Q \) statistic gives much confidence in the form of \( \mathcal{F} \) predicted from the flat–wall ansatz.

Figure 4 illustrates the quality of the agreement between the predicted form of the free energy and that predicted by equation (34) using \( c_0 = 1.85 \). One should remark that, while the fits were done for \( h \in \{0.05, 0.25\} \), the predicted free energy seems to lie within error for a much larger regime, especially for larger strip widths. Indeed, for \( L = 8, 9 \) the predicted free energy lies within error of the transfer matrix data all the way out to \( h = 1.0J \).

Transfer matrix data were also taken for \( h/J \in (1, 6) \) and low \( T \), to determine the effect of this increasing field strength. Typical results are shown in Figure 5. One notices that there are three separate regimes in the graphs:

- \( h/J \in (0, \approx 2) \). In this regime, the free energy seems to go roughly like the square of the field. As emphasized above, this dependence is close to that predicted by the flat wall theory, with increasing accuracy as the strip width, \( L \), increases, which is important for our scaling discussion of critical effects.

- \( h/J \in (\approx 2, 4) \). Here, the free energy crosses over from quadratic to linear dependence on \( h \), due to a rapidly decreasing domain size.

\[2\]The more efficient Newton–Raphson type routine could only be used if we had numerical derivatives of \( f \). These are not easily found from equation (34).

\[3\]The data at \( T = 0.2J \), while it was used in the fits, displays no new features.
h/J > 4. The field strength is greater than the energy cost of flipping a spin, and so all the spins align with their local field. Thus the free energy is linear in $h$, with unit slope.

To sum up: the transfer matrix approach has provided a rigorous test of the energetic arguments of section II A and II B. It provided excellent agreement over the range tested, giving confidence in the applicability of the theory.

However, a comparison of quantities like correlation length, domain size, etc., would be a more rigorous check and would also provide a better picture of the physics underpinning the results. Secure results on these lengths are not so far available from the transfer matrix method and so we turn to the Monte Carlo analysis, which can provide them.

3. Monte Carlo

Using the Block Monte Carlo algorithm described in reference 11, we now investigate other physical quantities which are directly accessible via our theoretical framework, but not easily obtained by transfer matrix analysis.

a. The Average Energy: First, we compare the configurationally averaged energy found by the Monte Carlo routine to that predicted by the flat wall results of chapter 11. This quantity is the first term in equation 19 and involves $\Xi_L$ which is given by numerical solution of equation 20. The theoretical and Monte Carlo data points are shown for $L = 1$ and 4 in Figure 8.

Similar results are obtained from the data for $L = 2$. Fitting the data to the theoretical form 15 gives $c_0 = 1.75 \pm 0.05$, in reasonable agreement with $c_0 = 1.85$ needed to fit the transfer matrix free energy data (section II C 2).

The theory agrees quite well with the Monte Carlo data up to a temperature, that diminishes with increasing field strength, at which $\Xi_L \approx L$ and the criterion for the validity of the theory breaks down. Yet, there is agreement within error up to $T = 0.7 J$ even for $h = 0.5 J$ when $L = 4$. This is a further indication that the zeroth order theory captures the essential flavour of the RFIM.

b. The Domain Distribution and Correlation Length–Domain Size Relationship: A key assumption in our derivation of the zeroth order entropy (section II B) and of a relation between domain size $\Xi_L$ and correlation length $\xi_L$ (section II A 4) was that the domain walls were randomly placed on the strip. This can be tested either by investigating the domain size distribution, which should then be Poisson, or by exploring the predicted relation between $\Xi_L$ and $\xi_L$.

Since the standard deviation and mean of a Poisson distribution are the same, we have evaluated these two quantities for $L = 1$, 2 and 4 for $h \leq 0.5 J$ and $T \leq J$. In every case they are found to be the same within error (never more than 5%). While this result is not definitive, it is very suggestive of a Poisson distribution.

Equation 15 expresses the relation between the correlation length $\xi_L$, and the domain size $\Xi_L$ when on a lattice, for randomly distributed walls. Figure 8 shows a test of this relationship for $L = 2$. Very similar plots arise for $L = 1$ and 4. In each case, the agreement with the predicted form is remarkable, and gives us confidence that, at least for large correlation lengths, the methods used in section II A 4 are correct and the domain walls really are Poisson distributed.

The final support for this conclusion comes from direct analysis of the probability distribution of domain sizes itself for $L = 1, 2, 4; h = 0, 0.2 J$ and $T$ up to $J$. The results for $L = 1$ are shown in Figure 8. The resulting size distribution is very close to exponential (providing strong evidence that the domains really are Poisson distributed) for $h = 0$, at least for smaller $L$’s and larger temperatures. However, when $h = 0.2 J$ and $T = 0.4 J$ the distribution seems to decrease at lower $\Xi$. This is evidence for domain wall repulsion, which is expected at higher fields when the domain walls begin to approach and undergo appreciable roughening (see section III). For $h < J$, however, this effect is only noticeable over about the lower 2% of the domain size distribution. Otherwise, the distribution is still exponential. So, the Poisson assumption appears to be approximately correct, and gives the correct behaviour at large domain sizes. Thus, the flat wall entropy should still be a valid starting point, although a more complete description is obviously required for a full understanding, particularly in the large size regime of interest for scaling.

c. The Domain Size Itself: Having justified assumptions implicit in the flat wall theory, we wish to test its predictions further. In this section, the domain size $\Xi_L$, predicted by equation 21, is compared with that measured by the Monte Carlo routine.

The results are shown for $L = 1$ and 4 in Figure 8. The data for $L = 2$ looks similar. The flattening of the data at low $T$’s is due the finite length of the simulated system: data in the flattening regime are discarded in quantitative comparisons. The constant $c_0$ was determined by a best fit to the data at $T = 0.1 J$, where we might expect the flat wall theory to be accurate. The result, $c_0 = 1$, contrasts with $c_0 = 1.75 \pm 0.05$ given by the average energy fits presumably because selection effects act differently for length scales and energetic quantities.

Contrary to the previous convincing agreements with theory, the comparison of the predictions of equation 21 to the domain size data is relatively poor. The reason for this poor agreement is not well understood, but may be due to non–Gaussian effects at smaller strip widths. However, we do get reasonable agreement (discarding flattened data) at the lowest of temperatures, and for larger $L$. These are the conditions that matter most for the scaling constructions, where the correlation length and domain size at $T = 0$ and large $L$ is of particular importance. These are discussed below using data from the
ground state algorithm.

4. Max-Flow algorithm

We have used the max-flow algorithm 4 to find ground states and hence ground state properties for many random field configurations generated from binary or Gaussian distributions. In the large scale limit, these two distributions are expected to give the same results, but there are differences, discussed in section III C, for small system or domain sizes. These differences were not as pronounced in the finite $T$ data obtained from the transfer matrix and Monte Carlo routines described above. In the present case, it becomes advantageous to use results from the Gaussian distribution as we do now.

These data were generated on strips of width $L = 2, \ldots, 11$ and length $N = 10^3$. These results were averaged over 100 independent random field configurations and the error bars shown represent the statistical variations in the data when averaged over these runs (as opposed to the width of the distribution of a variable quantity like the domain size). As the computational time required to generate a ground state via this algorithm grows like $n^3$, and the run time for an $n = 11 \times 10^3$ system was on order 10 days, it was not possible to significantly increase the system size beyond this limit.

The typical domain size $\Xi_L$ is shown in Figure 10. Also included in the Figure is a comparison with the theoretical $\Xi_L$ obtained from the random walk analysis (section III C) of the correlation length $\xi_L$ using a simple fitted constant of proportionality to convert this to a domain size. A quantitative assessment of the quality of the agreement is provided by the correlation coefficient $r = 0.9968$ and the statistic $\chi^2 = 0.426$. The fit gives a value of $c_0 = 1.98 \pm 0.02$, in reasonable agreement with the values obtained by both the transfer matrix and block Monte Carlo results at finite $T$.

Also considered was a fit to just the $h^{-2}$ leading term in the small $h$ expansion of the theoretical $\Xi_L$. This corresponds to 4 (i.e. to the basic $T = 0$ theory and differs from the full $\Xi_L$ by $O(1/h)$, significant except in the low $h$ limit. The comparison of fits to this reduced form with those to the full $\Xi_L$ show that if fields up to $h = 2.0 J$ are included, the fit to the reduced form gives a $\chi^2 = 4.215$. This shows a much poorer quality of fit than that to the full theoretical form. The neglect of the $O(1/h)$ terms is one of the sources of the reduced quality fit observed in the basic theory analysis of the Monte Carlo domain size data (section III C). However, it is difficult to incorporate these higher order terms in a consistent finite temperature theory. This shortcoming disappears in the small $h$ regime important for scaling.

This completes the comparisons of numerical analyses of (transfer matrix, block Monte Carlo, ground state algorithm) with the basic theory. The conclusion is that this theory provides an excellent account of the free energy and average energy; and an account of the domain size and correlation length which differs from the numerical data by effects whose origin is becoming clear, particularly so at low temperature. These effects are the domain wall repulsion due to non flat wall effects (wall roughening), seen in the domain distribution of section III C, and the (understood) $O(1/h)$ and other terms by which the basic theory differs from the random walk one. For these reasons, and for those given in section III A it is necessary to extend the theory by including wall roughening effects and to verify their importance, which is done in the next section.

III. DOMAIN WALL ROUGHENING

A. Theory

1. Introduction

The effect of domain wall roughening on the free energy, and hence the correlation length is estimated here by a decoration method applicable in general dimensionality. Lattice and continuum cases will be studied, and these turn out to have different roughening characteristics. We first outline a simple yet effective calculation (given previously in 4), first on a lattice and then in the continuum. Then, a functional field theory in the continuum is devised which provides a quantitative description of the effects of wall roughening. The varying results of these three approaches will then be compared. These results are next used to perform wall roughening on all length scales. This provides an RG transformation including the terms which break the marginality of $d = 2$. Hence, we can derive the scaling behaviour in $d = 2$, and the critical point and critical exponents of the RFIM in $d = 2 + \epsilon$. All these calculations are carried out at zero temperature for simplicity, and will be later generalized to finite temperature.

2. The Lattice Theory

On a lattice, the basic perturbation of the domain wall (“decoration”), to be later introduced on all scales, can be taken to be a rectangular shape (see Figure 11), of width $a$ and height $b$. The associated excess volume $\delta V$ and the excess area $\delta A$ are, in $d$ dimensions

$$\delta V = ba^{d-1}, \quad \delta A = 2(d-1)ba^{d-2}. \quad (36)$$

We assume that the domains are sufficiently large and the domain wall fluctuations sufficiently small (i.e. $\Xi_L \gg b$) that the fluctuations in the wall may be considered as uncorrelated from those in the body of the domain. Thus, we may treat the decoration as a separate entity from the bulk of the domain and apply the free energy 4 independently to it.
Substituting the values into equation (1) gives the zero temperature excess free energy
\[ \delta F(T = 0) = 4J(d - 1)ba^{d-2} - c_0hh^{1/2}a^{(d-1)/2}. \] (37)

We minimize this free energy with respect to \( b \) to arrive at the relation
\[ b = \left( \frac{c_0h}{8J(d-1)} \right)^2 a^{3-d}. \] (38)

This relationship is of the generic form
\[ b = c(h/J)^\kappa a^\zeta \] (39)

which defines a wandering exponent \( \zeta \), and a further “field” exponent \( \kappa \). We see that \( \zeta = 3 - d, \kappa = 2 \) in this simple description.

Inserting (38) into (37) gives the equilibrium value of the excess free energy and in particular its power law dependence on field and on the scale \( a \),
\[ \delta F(a) = -\frac{c_0h^2}{J} \frac{a^{3-d}}{16(d-1)}. \] (40)

3. The Simple Ansatz for the Continuum

We now perform a similar calculation to that in the above section but in a continuum rather than a lattice. It turns out that these two cases have different exponents.

By analogy to the previous section, the basic decoration of the domain wall is described by the typical height \( b \) and the typical width \( a \) (see Figure 1). These will again be related in the manner \( \xi \). Assuming axial symmetry, we denote the excursion of the domain wall from its mean position as \( z = bf(r/a) \) where \( f(0) = 1 \). Then, the excess surface area in the free energy (1) is
\[ \delta S(a, b) = \int_0^a dr r^{d-2} \left( \sqrt{1 + \left( \frac{b}{\sqrt{J}} f \left( \frac{r}{a} \right) \right)^2} - 1 \right) \]
\[ \rightarrow \begin{cases} \frac{b^2a^{d-3}}{d} & \text{if } b \ll a \\ \frac{ba^{d-2}}{d} & \text{if } b \gg a \end{cases} \]

Constants of proportionality have been disregarded here, and will be elsewhere in this section, since they do not affect the exponents in which we are interested.

Similarly, the excess volume is
\[ \delta V(a, b) = \int_0^a dr r^{d-2}bf \left( \frac{r}{a} \right) \sim ba^{d-1}. \] (41)

Using the above forms in the free energy (1) and minimizing with respect to \( b \) yields the following relations
\[ b \sim (\frac{b}{a})^{2/3} a^{5-d}, \text{ if } b \ll a \] (42)
\[ b \sim (\frac{h}{J})^2 a^{3-d}, \text{ if } b \gg a. \] (43)

For self-consistency, we must require that (42) holds at large scales \( a \) only for \( d > 2 \) and that \( h/J \) is not much larger than unity, or that \( d = 2 \) and \( h/J \ll 1 \). Similarly, equation (43) only holds if \( d < 2 \). We now check the assumption that the wall decorations are decoupled from the bulk of the domains, \( i.e. \) that \( b(L) \ll \Xi_L \). Using the form of \( \Xi_L \) given by the zeroth order decorations (4) and the solutions for \( b(L) \) given in equations (42) and (43) above we conclude that \( h/J \) must satisfy
\[ h/J \ll \begin{cases} \frac{L^{d-2}}{a^{d-2}} & \text{if } b \ll a \\ \frac{L^{d-2}}{b^{d-2}} & \text{if } b \gg a. \end{cases} \] (44)

In either case then, for large \( L \) and \( d > 2 \) there is a generous regime for \( h/J \) which satisfies the assumption. When \( d < 2 \), if \( L \) is finite we can always find \( h/J \) sufficiently small that (14) is satisfied, but the allowed regime for \( h/J \) decreases as \( L \) increases.

Thus, these simple arguments give the wandering exponent of the domain wall, \( \zeta \),
\[ \zeta = \begin{cases} \frac{5-d}{3} & \text{if } d \geq 2 \\ 3-d & \text{if } d \leq 2 \text{ and } h \ll 1 \end{cases} \] (45)

and the free energy
\[ \delta F(a) \sim \begin{cases} -\frac{h^{5/3}}{b^{d-2}} & \text{if } d \geq 2 \\ -\frac{h^{2}}{b^{d-2}} & \text{if } d \leq 2 \text{ and } h \ll 1. \end{cases} \] (46)

4. The Full Field Theory

A more complete treatment of this problem allows the entire shape of the interface to be determined by the system, and not just the height to width ratio as in the previous treatment. This will provide a check on the results of the previous section, and will provide more detailed information on the average equilibrium shape of the domain walls, and on proportionality constants.

In order to achieve this, we will establish a functional equation for the shape of the interface by rewriting the free energy of the domain wall as a functional of the wall profile \( P \). The equilibrium wall profile is then found by extremalizing this free energy via the Euler–Lagrange equation. The results of this section only hold in \( d \geq 2 \).

We use a \( d-1 \) dimensional cartesian coordinate system placed on the hyperplane where the flat interface would have been. Then, the deviation of the domain wall from this hyperplane is given by the function \( P(\vec{x}) \). Using the model of equation (1), this decoration has a free energy at \( T = 0 \) of
\[ F[P(\vec{x}), \nabla P(\vec{x})] = 2J \int \sqrt{1 + (\nabla P)^2} d^{d-1} \vec{x} - c_0h \sqrt{\int P(\vec{x})d^{d-1} \vec{x}} \] (47)
where both integrals are over the hyperplane defined by \( x_i \in [0, L] \). This free energy is minimized via the generalized Euler–Lagrange equation:

\[
\sum_{i=1}^{d-1} \frac{\partial}{\partial x_i} \frac{\delta F}{\delta (\partial P(x)/\partial x_i)} = \frac{\delta F}{\delta P(x)}.
\]  

(48)

Note that the RHS is a constant, and define it as

\[
N_0 J = \frac{-c_0 h}{4 \sqrt{\int P(x) dx}}.
\]  

(49)

This leads to a rather complicated \( d \)-dimensional differential equation.

\[
N_0 = \sum_{i=1}^{d-1} \left\{ \frac{(d-2)}{2} P'(r) \left( 1 + P'(r)^2 \right) + P''(r) \right\} \left( 1 + (\nabla P)^2 \right)^{3/2}.
\]  

(50)

Here, \( \partial_i \equiv \frac{\partial}{\partial x_i} \).

To simplify things we consider the isotropic solution, which only depends on the radial coordinate \( r^2 = \sum_{i=1}^{d-1} x_i^2 \).

Then, with \( \partial_r P(x) \equiv P'(r) \) the isotropic equation becomes

\[
N_0 = \frac{(d-2)}{2} \frac{P'(r) (1 + P'(r)^2) + P''(r)}{(1 + P'(r)^2)^{3/2}}.
\]  

(51)

Note that this equation may alternatively be obtained by extremalizing an isotropic free energy functional,

\[
\delta F[P(r), P'(r)] = 2J \Omega \int \sqrt{1 + P'(r)^2} r^{d-2} dr - c_0 h \sqrt{\Omega \int P(r) r^{d-2} dr}.
\]  

(52)

Here

\[
\Omega = \frac{\pi^{d-1/2}}{(d-2)!}(d-1)
\]  

(53)

is the area of the \( d-1 \) dimensional unit ball. However, in the more general case the isotropic solution is a member of the larger set of optimal solutions.

It may be checked by direct substitution that the solution to the wall decoration problem encapsulated in equations \([49]\) and \([51]\) is a hypersphere of radius \( R \)

\[
P(r) = \sqrt{R^2 - r^2} - P_o.
\]  

(54)

where \( R \) satisfies the requirement \(-1/R = N_0 \). The constant \( P_0 \) is determined by the choice of the boundary conditions \( P(r = L/2) = 0 \)

\[
P_0 = \sqrt{R^2 - \left( \frac{L}{2} \right)^2}.
\]  

(55)

In addition, the constraint \([53]\) imposes the self-consistency relation

\[
R^2 = \frac{J^2 (d-1)^2 4(\delta V)}{c_0^2 h^2}.
\]  

(56)

which we use to determine \( R(h) \). Here \( \delta V \) is the extra volume added by the decoration \( \delta V = \Omega \int_0^{L/2} P(r) r^{d-2} dr \).

Thus, \([54]\) is difficult to solve for \( R(h) \). However, we are interested in the regime \( h/J < 1 \) where we expect the decorations to be small \( (R \gg L) \) and we may expand in powers of \( L/R \). To first order in \( 1/R \) the excess volume and area are

\[
\delta V = \frac{\Omega}{R(d^2 - 1)} \left( \frac{L}{2} \right)^{d+1},
\]  

(57a)

\[
\delta A = \frac{\Omega}{2R^2(d + 1)} \left( \frac{L}{2} \right)^{d+1}.
\]  

(57b)

Then, equation \([56]\) gives the equilibrium radius as

\[
R = \left( \frac{c_0 h}{J} \right)^{-\frac{2}{d}} L^{\frac{d+1}{d}} \left( \frac{4\Omega(d-1)}{2(d+1)} \right)^\frac{1}{2}.
\]  

(58)

Inserting the first order forms \([53]\) evaluated at the equilibrium radius \([58]\) into \([17]\) then gives the first order free energy fluctuation

\[
\mathcal{F}_{wall}(L) = -\tilde{\mathcal{c}} \left( \frac{c_0 h}{J} \right)^{4/3} L^{2b+\frac{1}{3}} + O(h^{8/3}).
\]  

(59)

Here \( \tilde{\mathcal{c}} \) is the numerical constant

\[
\tilde{\mathcal{c}} = \left[ \frac{\Omega^{1/3}}{(d+1)^{1/3}(d-1)^{2/3}} \right] \left[ \frac{-2^{-1/3} + 2^{8/3}}{2(d+1)^{1/3}} \right]
\]  

(60)

and is positive for all \( d > 1 \). This shows that wall roughening is always favourable for \( d \geq 2 \).

Finally, the wall wandering height is

\[
b \equiv P(r = 0) \sim h^{2/3} L^{\frac{d+1}{3}} + O(h^{8/3}).
\]  

(61)

We now compare these three approaches to the wall decoration problem on the basis of their predicted wall behaviours of the wall wandering height \( b \) and the free energy of the wall \( \delta F \). We define the wandering exponent \( \zeta \), the free energy fluctuation exponent \( \omega \), and associated exponents \( \gamma \) and \( \kappa \) by the following equations

\[
\delta F \sim h^{\gamma} a^\kappa,
\]  

(62a)

\[
b \sim h^{\omega} a^\zeta.
\]  

(62b)

where \( a \) is the base length \( (L \text{ in the field theoretic description}) \). One sees that all the exponents of the simple continuum ansatz and the full field theory agree for \( d \geq 2 \). The complete comparison between all three treatments is given in Table \( IV \) and the results are shown graphically in Figure \( 12 \).
The predicted exponents in the continuum model agree with those of other extensions of the Imry-Ma type argument done by Natterman\cite{30} and Binder\cite{31} for the lattice. Grinstein\cite{44} finds the same values for $\gamma$, $\kappa$, and $\zeta$, as we do, but finds $\theta = 1$. It is also worth noting that the replica symmetric approach of Parisi and Sourlas\cite{45} for the continuous model gave $\zeta = (5 - d)/2$, but the inclusion of replica symmetry breaking gives $\zeta = (5 - d)/2\log(n)$. Finally, the lattice result is consistent with the results of transfer matrix calculations done in two dimensions by Fernandez \textit{et al.}\cite{46} and with numerical ground state calculations reported in section III C.

5. Decorations upon Decorations

The decoration to the flat domain walls described in section III A is (to lowest order in $h/J$), the most favorable twice differentiable form of the domain walls. However, discontinuities may be a feature of the “true” solution. We may incorporate such discontinuities by allowing the decoration to be repeated on all length scales, following a procedure of Binder\cite{42}. Basically, having made a basic change in the shape of the domain wall on scales of length $L$, one then looks at the domain wall on a length scale $L/n$ where $n > 1$ is some arbitrary integer, whose choice, we hope, should not strongly affect the results. If the random field is small, the wall will be slowly varying on such a scale and may be viewed as a hyperplane, or to preserve the simplicity of the discussion by taking $d = 2$, as a straight line. This straight line is then decorated in the same manner as the original wall. This process is repeated on length scales of order $a_i = L/n^i, i = 2, 3, \ldots$ until a cutoff length scale is reached (see Figure 4). Reversing to the case of general $d$, this cutoff occurs at a stage $k$ where the smallest length scale of decorations is the lattice constant (unity), \textit{i.e.}

$$\min (a_k, b_k) = 1. \quad (63)$$

From the results of the previous section, we know that $b_k = c_1(h/J)^{k}a_k^{\zeta}$ where $c_1$ is a constant (equal to $(c_0/8(d - 1))^2$ for the lattice case). Now $\kappa$ is greater than zero for both the lattice and the continuum. Thus, for small $h$, (63) becomes $b_k = 1$ so the final level of decorations, $k$, is determined by

$$a_k = L/n^k = c_1^{-1/\zeta}h^{-\kappa/\zeta} \quad (64)$$

This gives the cutoff $k$ as

$$k = \frac{\ln L + \frac{1}{2}\ln c_1 h^{\kappa}}{\ln n}. \quad (65)$$

This result should be approximately valid for large $L$.

Finally, all the changes to the free energy from all the various decorations are summed to arrive at a total decoration. At stage $i$ we have decimated the length scale by a factor of $n^i$ so there are $n^{i(d - 1)}$ decorations at this stage, and each gives a free energy contribution $\delta F(L/n^i)$. Hence, the total free energy of all the decorations will be

$$\delta F^{\text{Tot}}(L) = \sum_{i=0}^{k} n^{i(d - 1)}\delta F \left( \frac{L}{n^i} \right) \quad (66)$$

with $k$ as above.

In this way, then, we can estimate the effect of wall roughening on all length scales, and remove the restriction that the wall shape must be smooth. Such generalizations are crucial to get the correct marginality breaking free energy, as we shall see.

6. Scaling and criticality in $d = 2$ and $d = 2 + \epsilon$

The preceding results can now be used to construct the RG transformation in $d = 2 + \epsilon$. This allows us to investigate the marginality breaking effect of domain wall roughening in two dimensions, and to investigate the behaviour of the $h = 0$ fixed point as we move away from $d = 2$. So we obtain the two dimensional critical behaviour and also an idea of the critical properties in higher dimensionalities.

Using (63) and the free energy forms found in the previous sections, a full decoration of the domain walls may be done in general dimension $d \geq 2$. For the sake of generality we will use the general form of the wall free energy given by (62), though with restored generic constants as well as exponents, which can be read off from earlier results for specific cases. Thus, the general form of the wall free energy is

$$\delta F^{\text{Tot}} = -\tilde{c} \left( \frac{c_0 h}{J} \right)^{\gamma} L^\theta \sum_{i=0}^{k} n^{i(d - 1) - \theta}. \quad (67)$$

From the values of $\theta$ given in Table IV the summand is

$$n^{i(d - 1) - \theta} = \begin{cases} n^{(d-1)\theta} = 1 + i\frac{\phi}{2}\ln n, & \text{continuum} \\ n^{(2d-4)} = 1 + i2\epsilon\ln n, & \text{lattice} \end{cases} \quad (68)$$

for $d = 2 + \epsilon$. To further simplify matters we define the coefficient

$$\phi = \begin{cases} \frac{\theta}{2}, & \text{continuum} \\ \frac{\theta}{2}, & \text{lattice} \end{cases} \quad (68)$$

Then, the total free energy of the wall decorations may be concisely written as

$$\frac{\delta F^{\text{Tot}}}{J} = -\tilde{c} \left( \frac{c_0 h}{J} \right)^{\gamma} L^\theta \left[ (k + 1) + \epsilon \phi \ln n \frac{k(k + 1)}{2} \right]. \quad (69)$$

This result may be compared both with that of Villain\cite{8} and Grinstein and Ma\cite{48} who find $\delta F^{\text{Tot}} \sim h^{4/3} L \ln L$ for
If we attempt to let $\epsilon \to 0$ to extrapolate back to the $h = 0$ fixed point, we see that $\nu$ diverges. This is an indication that the divergence of the correlation length of the infinite two dimensional system is faster than any power law. This divergence can be obtained from equation (72) at $\epsilon = 0$:

$$h' = h \left(1 + ch^\gamma \frac{\ln b}{\ln n}\right).$$

(75)

This is equivalent (at small $h$) to

$$\exp \left(\frac{\ln n}{\gamma c(h')^\gamma}\right) = \frac{1}{b} \exp \left(\frac{\ln n}{\gamma ch^\gamma}\right).$$

(76)

This is the same form as the standard rescaling of a correlation length $\xi(h') = b^{-1}\xi(h)$, so the correlation length is

$$\xi \sim \exp \left(\frac{A}{h^\gamma}\right)$$

(77)

close to the fixed point $h^* = 0$, where $A$ is the non-universal constant $\ln n/(c\gamma)$.

The results of these calculations are summarized in Table III. Here the exponent is $\gamma = 4/3$ for the continuum or $\gamma = 2$ for the lattice. The exponential growth of $\xi$ was also found by Binder. His calculation was done explicitly on a 2D lattice and he found a form for $\xi$ identical to ours with $\gamma = 2$.

7. Conclusions

This concludes the $T = 0$ investigation of the effects of fluctuations in the domain wall shape. The analysis of the RFIM on a lattice and in a continuum has led to a difference in the critical behaviour of these two cases which arises from the difference of their wandering exponents. This has been seen previously in other works referred to above. One may speculate that this difference is due to the different effect of surface tension in the lattice versus the continuum. Interfacial roughening effectively acts to reduce the surface tension in the domains and is the crucial factor in breaking the marginality of $h$ in $d = 2$. This surface tension is coupled to the $d - 1$ dimensional area of the domain wall and this is clearly different between lattice and continuum cases because of the stepped character necessary on a lattice. Exactly how this mechanism comes into play in the RFIM is not clear, however.

The procedure was to perform decorations on all length scales to find the effect of domain wall roughening on the free energy and domain size. These determine the phenomenological renormalization group equation and hence the behaviour near the lower critical dimension. This revealed the evolution of the fixed point as it moves from $h = 0$ to $h = h_c \neq 0$ for $d > 2$. 

$d = 2$ using a continuum interface model, and with the result of Binder who finds $\delta F^{\text{Tot}} \sim h^2 L \ln L$ in a 2D lattice calculation. Both these results agree with (69) with the appropriate choice of the exponents $\gamma$ and $\theta$ (see Table IV).

The wall decorations have the effect of reducing the effective surface tension of the domains. As a result, the average domain size will decrease. We can quantify this result by rewriting the free energy of the entire strip as

$$F = \left(\frac{N}{\Xi_L}\right) \left(2JL^{d-1} - c_0 h \sqrt{\Xi_L L^{d-1} + \delta F^{\text{Tot}}}\right).$$

(70)

Here, we used the flat wall result [4] as a model, but noted the fact that there is, on average, a further contribution of $\delta F^{\text{Tot}}$ per domain wall from the decorations.

Extremalizing this free energy with respect to $\Xi_L$ gives the average domain size as

$$\Xi_L = \left(\frac{c_0 h}{J}\right)^{-2} L^{d-1} \left(1 + \frac{\delta F^{\text{Tot}}}{JL^{d-1}}\right)^{2}$$

(71)

As expected, one sees that the domain size is reduced by the effect of interface roughening.

This correction to the correlation length $\Xi_L$ also has implications for the phenomenological RG equation. This equation obtained from (71) with the use of (71) and (4) is now quite complex, but it can be simplified by linearizing in $\epsilon$ and using small $h/J$. After some tedious algebra, one finds the following RG equation:

$$h' = h \left(1 + ch^\gamma \frac{\ln b}{\ln n} - \frac{\epsilon}{2} \ln b\right)$$

(72)

where $c \equiv c_0 \gamma$ and we have let $J = 1$ for simplicity. Besides the trivial fixed point at $h = 0$, (72) has an unstable fixed point at

$$h^* = \left(\frac{\epsilon \ln n}{2\epsilon}\right)^{1/\gamma} + O(\epsilon^2)^{1/\gamma}$$

(73)

which remains in the small $h$ region of validity of our description since $\epsilon$ is regarded as small. The eigenvalue of (72) is given by

$$\frac{\partial h'}{\partial h} \bigg|_{h = h^*} = 1 + \frac{1}{2} \gamma h \ln b + O(\epsilon^2) = b^{1/\nu}.$$

This yields the critical exponent

$$\nu = \frac{2}{\epsilon \gamma}$$

(74)

which characterizes the scaling of the bulk correlation length, $\xi \sim |h - h^*|^{-\nu}$. Notice that the arbitrary scale variable $n$ disappears from the critical exponent $\nu$, which is a universal quantity, though it remains through a weak logarithmic dependence in $h^*$ and other non-universal quantities.
As with the flat wall analysis, both the resulting scaling and the ingredients which lead to it (size and field dependences of free energy and domain size, and now also roughening) can be directly checked by numerical work, and this will be done in section III C.

In the next section, we extend our results to include low temperature thermal effects.

B. Low Temperature Theory

The effect of the zero temperature wall roughening on the surface tension is (by (76)) equivalent to the replacement of $J$ by

$$j = J + \frac{\delta F_{\text{Tot}}}{2L^{d-1}}.$$  \hspace{1cm} (78)

Consequently, the low temperature analyses given in sections II B and II C 2 have to be modified by the replacement $J \rightarrow j$, in equations (20)–(24) and (30)–(34).

The roughening also affects the entropy since it increases the number of possible configurations. The entropy change can be obtained crudely by a slight generalisation of the counting used to derive (24), or by a more complete analysis given elsewhere.

As remarked in section II B the zeroth order thermal scaling (28) is not marginal in $d > 1$, and so the modifications just discussed do not change the RG thermal flow directions for $h$ small and $d$ near 2. So the $\epsilon$-expansion results of section II A 6 can be combined with those of section II B to infer the flow diagrams shown in Figure 14 for $d = 1, 2, 2 + \epsilon$. For $d = 2 + \epsilon$ this implies a second order phase transition at the phase boundary shown, inside which (i.e. at small $h, T$) long range order occurs for $h \neq 0$. This is consistent with previous renormalization group discussion.

The extended theory outlined in the first two paragraphs above provides the low temperature free energy modifications produced by rough walls.

This suggests the possibility of distinguishing wall roughening effects by comparing numerical data on free energy with this theory extended as just outlined. This comparison has been made with the transfer matrix free energy data discussed in section II C 2. Despite the excellent quality of the data, and its remarkably good agreement with the theory in zeroth order (flat walls), it was not found possible to extract wall roughening effects from it. This is because the statistical fluctuations in the data, although extremely small, are nevertheless greater than the roughening contributions in the ranges of parameters appropriate for the validity of the theory (ensuring $\xi \gg L$). So, we turn to other more sensitive comparisons.

C. Numerics - Correlation and roughening characteristics from the Max-flow ground state algorithm

This section describes the extraction of domain wall roughening characteristics from data obtained using the max–flow algorithm for constructing ground states.

One comparison uses domain size correlation length data, which is much more sensitive than the free energy to domain wall roughening, particularly in wide strips because of the crucial role of roughening in the marginal dimension $d = 2$. The other involves direct measures of the wall wandering. These measures are the average over many walls, (for a given $h$ and base scale $L$) of the maximum height (wandering excursion) $b_{\text{max}}$, or of the root mean square height $b_{\text{rms}}$.

Both binary and Gaussian random field distributions were used. As expected, the Gaussian distribution gives smoother dependences, and has provided the most useful data. But, the binary results are interesting. Figure 13 gives numerical data for the configurationally averaged domain size. A “stepping” of the data as a function of $h$ is apparent, particularly for smaller $L$. This is predicted by the flat wall description (section II A 3, equations (15), (17)) since in (17), $q = (2 + ([2JL/h])^{-1}$ involves the integer part function $\lfloor \cdot \rfloor$. There is also a certain discreteness implied in the use of the transfer matrix. For comparison, the flat wall theory is shown in the Figure. The flat wall random walk analysis also compares very satisfactorily with corresponding domain size data (Figure 15) from the Gaussian distribution. Here, the data contain no steps. If one repeats the random walk analysis with a Gaussian random field distribution, the “integer part” function does not occur, and the transfer matrix becomes a convolution operator. Since the eigenfunctions are again plane waves, the final result for the correlation length is the same, except that it has no “integer part” function, and thus no “stepping”. But, though the Gaussian distribution must therefore set the large length scale behaviour, in energy (Figure 16), domain size (above) and probably in time scales for relaxing on given characteristic scales (“dynamics”), there can be real differences for these quantities between binary and Gaussian cases at smaller scales.

The domain size data can be fitted to the flat wall theory or to the analysis including wall roughening (sections II A 3, II A 6). Equation (4) gives the basic flat wall description, in which for strips ($d = 2$), $\Xi_L \sim Lh^{-2}$, and one fitting coefficient ($c_0$) occurs. The fuller random walk version of the flat wall theory (section II A 3) is equivalent, for small $h/L$, but more generally it gives a $\Xi_L$ different by a “discreteness factor” $(1 + O(h/L))$. The wall roughening analysis instead corrects the flat wall $\Xi_L$ by a factor $(1 + O(h^7L^{9-d+11} \ln L))$. This should be amalgamated with the discreteness factor evident from the RW analysis. Then the domain wall roughening theory gives the best quality of fit to the max–flow data on $\Xi_L$. 

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For instance, considering a Gaussian random field distribution, Table [3] shows the decreased statistical error $\chi^2$ associated with the addition of this domain roughening term. Even though this extra term provides an extra free constant to the fit ($c = \tilde{c} \sigma^0$), it does not necessarily decrease $\chi^2$. Indeed, it worsens the fit to the zero'th order theory since the higher order “discreteness term” has been neglected. When this term is ignored, the roughening only has a correlation coefficient of $r = 0.218$. However, when this discreteness term has been included from the more complete random walk analysis, the wall roughening is seen to decrease the value of $\chi^2$. It then has a much more significant correlation coefficient of $r = 0.816$.

The max–flow data in the wall decoration variables $b_{rms}$, $b_{max}$ give much more conclusive evidence for the roughening effects and provide quantitative estimates of exponents. Data for the $h$–dependence of $b_{rms}$ for $L = 8$ are shown in Figure [15] for both binary and Gaussian distributions. Again, the jagged character of the binary result is evident, even at this moderately large $L$. So the remaining discussion is confined to the Gaussian case. To compare with the prediction (62b), a log–log plot of the Gaussian $b_{rms}$ versus $h$ is given for $L = 2, \ldots , 11$ in Figure [16]. A very similar plot is obtained also for $b_{max}$, though the absolute size of $b_{max}$ is larger, by up to a factor of 4. The departure from linearity of the log–log plot at small $h$ is almost certainly a lattice effect. That requires $b \gtrsim 1$, and using e.g. (33) (with $c_0 = 1.8$) to estimate where $b \approx 1$ suggests that (62b) should break down for $h/J \lesssim 5L^{-1/2}$, which is in qualitative agreement with what is seen. Similarly, a breakdown of the simple theory giving (62b) is expected when $b$ (strictly $b_{max}$) becomes comparable to $\Xi_L$, i.e. (using (30) and (1)) for $h/J \gtrsim 1.5$. This $L$–independent cutoff is consistent with the departures from linearity seen at the largest fields in the log–log plot. The scaling window between the upper and lower $h$ departures is quite wide for the largest $L$’s, and consequently they should give the most accurate field exponent $\kappa$ from power law fits to (62b) within the scaling window. The results, from both $b_{rms}$ and $b_{max}$ are shown in Figure [17] and are consistent with $\kappa \approx 2.1 \pm 0.3$. Figure [17] shows the $L$–dependent prefactor of $h^\kappa$ in $b_{rms}$ and $b_{max}$. In each case, the prefactor is very close to linear in $L$ in the larger $L$ scaling regime, consistent with $\zeta = 1$ in (62b). The behaviours seen in Figure [18] and in the corresponding plot for $b_{max}$, as well as the associated exponents $\kappa$, $\zeta$ are therefore consistent with the analysis in section III A.

IV. CONCLUSIONS

In this paper we have developed a domain scaling description for the random field Ising model by exploiting a bar geometry. As in earlier work of Villain, Grinstein and Ma, and Binder, the marginality of the basic flat wall picture at the lower critical dimension $d_l = 2$ is removed by wall roughening effects (Section II). The resulting phenomenological renormalization group transformation has been used to obtain the critical properties. In particular, the special critical behaviour of the two–dimensional correlation length and the phase diagram in $d = 2 + \epsilon$ have been given (Section II A).

The wall roughening ingredient has its own scaling characteristics (the exponents $\gamma, \theta, \kappa, \zeta$) which have been here discussed using both a simple analytic method (following Natterman and a field theoretic approach. These exponents have a direct bearing upon the RG transformation and hence on the scaling properties.

An essential element to this study has been the support that numerical studies have provided for the analytic description. The comparison between numerical data and analytic predictions was made in the strip geometry so that the critical ingredients of the theory could be directly investigated. These ingredients include the domain size, the free energy, and the roughening characteristics, as functions of $h$, $T$ and strip width $L$.

The numerical approaches employed were transfer matrix and Monte Carlo techniques, and a ground state (max–flow) algorithm. The first two give free energy and domain size data confirming the basic flat wall picture through: (i) the $h^2/L$ dependence of the $T = 0$ flat wall energy at low $h$, and its low temperature generalisation; (ii) the domain size distribution and the correlation length–domain size relationship.

The numerical ground state data, obtained via the max–flow algorithm, directly verify the basic validity of the domain size predicted by the (flat wall) random walk analysis, but they also provide evidence for the roughening effects in the domain size. The max–flow data on the wall decoration variables $b_{rms}$, $b_{max}$ are the most conclusive evidence for roughening effects. They give power law scaling in a window of the size predicted by the theory, sufficient at large $L$ to give quite accurate values for roughening exponents, in good agreement with the (lattice based) theory.

The conclusion is that the theory contains the correct ingredients, and the numerical data provide quantitative confirmations of the way they enter into the theory.

We have not made corresponding numerical investigations of the $d = 3$ case, where it is of course more difficult to obtain data for the range of $L$’s we needed here. However, we hope the present work may stimulate such an effort, which (e.g. using the ground state algorithm) could sort out the so far unresolved critical behaviour. Another extension, presently under consideration, is to the kinetic behaviour, for which the free energy scaling discussed here is an essential ingredient.
FIG. 1. The bar geometry, showing the flat domain walls that separate regions of up and down spins. The average domain size \( \Xi_L \) is the average of the \( \Xi \)'s.

FIG. 2. Illustration of how the random field configuration may be viewed as a random walk in 1+1 dimensions. We use \( L = 3 \) here.

FIG. 3. Log–log plots of the excess free energy at \( T = 0.1J \).

FIG. 4. A comparison of the free energy as determined by the transfer matrix data, and that determined by our low temperature theory using the fitted constant \( c_0 = 3.406 \), at \( T = 0.1J \) and \( T = 0.3J \).

FIG. 5. The free energy as a function of \( h \) for \( T/J = 0.1 \) and \( L \in (2, 7) \). These data were determined by the transfer matrix method, and illustrate the three regimes of \( h \) dependence

FIG. 6. The average energy of the \( L = 1 \) and \( L = 4 \) RFIM, as determined by both the Monte Carlo algorithm (data points) and the zeroth order theory (lines). The symbol \( \Diamond \) corresponds to \( h = 0 \), \( - \) to \( h = 0.1J \), \( \Box \) to \( h = 0.2J \) and \( \times \) to \( h = 0.5J \).

FIG. 7. The correlation length \( \xi_L \) plotted against the domain size \( \Xi_L \) for \( L = 2 \). We also include the predicted relation between these quantities for comparison. Plots for \( L = 1, 4 \) are essentially the same.

FIG. 8. The distribution of domain sizes according to the measure \( \Xi_L \) as found by the Monte Carlo routine when \( L = 1 \).
FIG. 9. The domain size $\Xi_L$ plotted as a function of temperature for $L = 1$ and $L = 4$. The data points represent the MC data whereas the lines show the result of solving the flat wall theory. The symbol $\Diamond$ corresponds to $h = 0$, $\Box$ to $h = 0.05J$, $\times$ to $h = 0.1J$, and $\triangle$ to $h = 0.5J$. The flattening of the data at low $T$ is due to the finite length of the simulated system.

FIG. 10. The typical domain size at $T = 0$ from the max–flow algorithm. The points represent the data from the max–flow algorithm, and the solid lines represent the theoretical form with a fitted multiplicative constant.

FIG. 11. First order decorations to a domain wall on a lattice (left) and in a continuum (right). Here, $d = 3$. The decorations have height $b$ and width $a$.

FIG. 12. Plot of the free energy exponent $\theta$ and the wandering exponent $\zeta$ for the wall decorations on a lattice and in a continuum.

FIG. 13. The process of placing decoration upon decorations, such that decorations occur on all length scales. This process has a cut–off length scale when the size of the decorations become on order of the lattice constant.

FIG. 14. The RG flow diagram for the RFIM as predicted by our finite size scaling analysis.

FIG. 15. The domain size as a function of $h/J$ at $T = 0$ determined by the max–flow algorithm (data points), and the flat wall analysis (lines) for a binary field distribution.

FIG. 16. The wall roughening exponent $\kappa$ as determined from power law fits to $b_{rms}$ and $b_{max}$. For large $L$, these tend towards 2.

FIG. 17. The wall roughening term $a\zeta$ as determined from power law fits to $b_{rms}$ and $b_{max}$. For large $L$, these tend towards a linear dependence on $L$, consistent with $\zeta = 1$, as shown by the lines of best fit.

FIG. 18. The root–mean–square domain wall width as a function of $h/J$ for $L = 8$. This plot emphasizes the difference between the binary and Gaussian random field distributions, showing the stepped nature of the former as more excitations become favourable at lower $h/J$.

TABLE I. Fits to the free energy data from the transfer matrix

| $L$ | Slope | Error | $Q$ |
|-----|-------|-------|-----|
| 2   | 1.792 | 0.007 | 0.0004 |
| 3   | 1.848 | 0.009 | 0.33 |
| 4   | 1.857 | 0.009 | 0.79 |
| 5   | 1.885 | 0.011 | 0.96 |
| 6   | 1.901 | 0.012 | 0.95 |
| 7   | 1.904 | 0.012 | 1.00 |
| 8   | 1.945 | 0.015 | 1.00 |
| 9   | 1.958 | 0.014 | 1.00 |

TABLE II. Fits of the free energy to the transfer matrix data at finite temperature

| $L$  | $N$  | $c_0$ | $\chi^2$ | $Q$ |
|------|------|-------|----------|-----|
| 2 - 9| 120  | 1.774 | 457      | $10^{-41}$ |
| 2 - 7| 90   | 1.761 | 375      | $10^{-37}$ |
| 2 - 5| 60   | 1.745 | 288      | $10^{-32}$ |

TABLE III. Main results of the Flat Wall analysis

| Dimensions | Fixed Point | Bulk Correlation Length |
|------------|-------------|------------------------|
| 2          | $h^* = 0$   | $\xi \sim \exp \left( A/h^* \right)$ |
| 2 + $\epsilon$ | $h^* = \left( \frac{4b_{rms} - 2c}{2c} \right)^{1/\gamma}$ | $\xi \sim \left( h - h^* \right)^{\frac{\gamma}{2}}$ |
### TABLE IV. Comparison of Approaches to Wall Decoration Problem

|       | $d < 2$                  | $d \geq 2$                  |
|-------|--------------------------|-----------------------------|
|       | Lattice | Simple | Continuum | Field | Theory | Lattice | Simple | Continuum | Field | Theory |
| $\zeta$ | $3 - d$ | $3 - d$ | N.A. | $3 - d$ | $(5 - d)/3$ | N.A. | $3 - d$ | $(d + 1)/3$ |
| $\theta$ | $3 - d$ | 1 | N.A. | $3 - d$ | $(d + 1)/3$ | N.A. | $3 - d$ | $(d + 1)/3$ |
| $\gamma$ | 2 | 2 | N.A. | 2 | $4/3$ | 2 | $4/3$ |
| $\kappa$ | 2 | 2 | N.A. | 2 | $2/3$ | 2 | $2/3$ |

### TABLE V. Results of fits to the $T = 0$ domain size data

|                        | Flat Domain Walls | Roughened Domain Walls |
|------------------------|-------------------|------------------------|
|                        | Zeroth Order | Random Walk | Zeroth Order | Random Walk |
| $\chi^2$              | 4.215          | 0.607                | 4.447          | 0.404                |