ON THE VOEVODSKY MOTIVE OF THE MODULI STACK OF VECTOR BUNDLES ON A CURVE

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Abstract

We define and study the motive of the moduli stack of vector bundles of fixed rank and degree over a smooth projective curve in Voevodsky’s category of motives. We prove that this motive can be written as a homotopy colimit of motives of smooth projective Quot schemes of torsion quotients of sums of line bundles on the curve. When working with rational coefficients, we prove that the motive of the stack of bundles lies in the localising tensor subcategory generated by the motive of the curve, using Białynicki-Birula decompositions of these Quot schemes. We conjecture a formula for the motive of this stack, and we prove this conjecture modulo a conjecture on the intersection theory of the Quot schemes.

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1. Introduction

Let $C$ be a smooth projective geometrically connected curve of genus $g$ over a field $k$. We denote the moduli stack of vector bundles of rank $n$ and degree $d$ on $C$ by $\text{Bun}_{n,d}$; this is a smooth algebraic stack of dimension $n^2(g-1)$. The cohomology of $\text{Bun}_{n,d}$ has been studied using a wide array of different techniques, and together with Harder–Narasimhan stratifications, these results are used to study the cohomology of moduli spaces of semistable vector bundles. In this paper, we study the motive of $\text{Bun}_{n,d}$ in the sense of Voevodsky.

Let us start with a chronological survey of the various results on the cohomology of $\text{Bun}_{n,d}$. One of the first calculations was a stacky point count of $\text{Bun}_{n,d}$ over a finite field $\mathbb{F}_q$ due to Harder [19]; the formula (cf. Theorem 5.2) is remarkably simple and involves the point count of the Jacobian of $C$ and a product of Zeta functions. These point counting methods enabled Harder and Narasimhan [20] to compute inductive formulae for the Betti numbers of moduli spaces of semistable vector bundles.

More recent approaches to studying the cohomology of $\text{Bun}_{n,d}$ utilise the ideas of [8] and [6] to give a closed formula (cf. Theorem 5.2) which is remarkably simple and involves the point count of the Jacobian of $C$ and a product of Zeta functions. These point counting methods enabled Harder and Narasimhan [20] to compute inductive formulae for the Betti numbers of moduli spaces of semistable vector bundles.

In this paper, we study the motive of $\text{Bun}_{n,d}$ in DM($k$). The first calculation was a stacky point count of $\text{Bun}_{n,d}$ over a finite field $\mathbb{F}_q$ due to Harder [19]; the formula (cf. Theorem 5.2) is remarkably simple and involves the point count of the Jacobian of $C$ and a product of Zeta functions. These point counting methods enabled Harder and Narasimhan [20] to compute inductive formulae for the Betti numbers of moduli spaces of semistable vector bundles.

Theorem 1.1. Assume that $C(k) \neq \emptyset$. For any effective divisor $D_0 > 0$, we have

$$M(\text{Bun}_{n,d}) \simeq \hocolim_{t \in \mathbb{N}} M(\text{Div}_{n,d}(ID_0)).$$

In particular, the motive $M(\text{Bun}_{n,d})$ is pure, in the sense that it lies in the heart of the Chow weight structure on DM($k$).
To describe the motives of the varieties $\text{Div}_{n,d}(lD_0)$, we use a $\mathbb{G}_m$-action and a Bialynicki-Birula decomposition as in \cite{8}. The varieties $\text{Div}_{n,d}(lD_0)$ come with natural actions of $\text{GL}_n$, and the morphisms $\text{Div}_{n,d}(lD_0) \to \text{Div}_{n,d}((l + 1)D_0)$ are equivariant with respect to the action. If we restrict the action to a generic one-parameter subgroup $\mathbb{G}_m \subset \text{GL}_n$, then the connected components of the fixed point locus can be identified with products of symmetric powers of $C$. We can then apply the Bialynicki-Birula decomposition (cf. Theorem 3.2) and its motivic counterpart \cite{10, 12, 27} (cf. Theorem 3.3) to deduce the following result.

**Theorem 1.2.** Assume that $R$ is a $\mathbb{Q}$-algebra; then $M(\text{Bun}_{n,d})$ lies in the localising tensor triangulated category of $\text{DM}(k, R)$ generated by $M(C)$. Hence, $M(\text{Bun}_{n,d})$ is an abelian motive.

The assumption that $R$ is a $\mathbb{Q}$-algebra is used to show that the motive of a symmetric power of $C$ is a direct factor of the motive of a power of $C$. For $C = \mathbb{P}^1$, as symmetric products of $\mathbb{P}^1$ are projective spaces, we deduce that $M(\text{Bun}_{n,d})$ is a Tate motive for any coefficient ring $R$.

We then conjecture the following formula for the motive of $\text{Bun}_{n,d}$.

**Conjecture 1.3.** Suppose that $C(k) \neq \emptyset$; then in $\text{DM}(k, R)$, we have

$$M(\text{Bun}_{n,d}) \simeq M(\text{Jac}(C)) \otimes M(\text{BG}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, R(i)[2i]).$$

where $Z(C, R(i)[2i]) := \bigoplus_{j=0}^{\infty} M(X(j)) \otimes R(ij)[2ij]$ denotes the motivic Zeta function.

To prove this formula, one needs to understand the behaviour of the transition maps in the inductive system given in Theorem 1.2 with respect to the motive Bialynicki-Birula decompositions. We formulate a conjecture (cf. Conjecture 1.1) on the behaviour of these transitions with respect to these decompositions; this is equivalent to a conjecture concerning the intersection theory of the the smooth projective Quot schemes $\text{Div}_{n,d}(D)$ (cf. Conjecture 4.12 and Remark 4.13). In Theorem 4.20 we prove that Conjecture 4.11 implies Conjecture 1.3; hence, it suffices to solve the conjecture on the intersection theory of these Quot schemes, which is ongoing work of the authors.

The assumption that $C$ has a rational point is needed so that Abel-Jacobi maps from sufficiently large symmetric powers of $C$ to $\text{Jac}(C)$ are projective bundles (cf. Remark 4.15). In fact, these projective spaces then contribute to the motive of $\text{BG}_m$ (cf. Example 2.20).

Finally let us state some evidence to support Conjecture 1.3 as well as some consequences of this conjecture. First, using Poincaré duality for smooth stacks (cf. Proposition 2.34), we can deduce a formula for the compactly supported motive of $\text{Bun}_{n,d}$ (cf. Theorem 5.1), which is better suited to comparisons with the results concerning the topology of $\text{Bun}_{n,d}$ mentioned above. In §5.2 we explain how this conjectural formula for $M_c(\text{Bun}_{n,d})$ is compatible with the Behrend–Dhillon formula \cite{6} by using a category of completed motives inspired by work of Zargar \cite{45} (cf. Lemma 5.4). In fact, in \cite{6}, it is also implicitly assumed that $C$ has a rational point in order to use the same argument involving Abel-Jacobi maps. In §5.3 we deduce from Conjecture 1.1 formulae for the motive (and compactly supported motive) of the substack $\text{Bun}^L_{n,d}$ of vector bundles with fixed determinant $L$ and the stack $\text{Bun}_{SL_n}$ of principal $\text{SL}_n$-bundles over $C$.

The structure of this paper is as follows: in §2 we summarise the key properties of motives of schemes and, after proving some elementary results about homotopy (co)limits, we define motives of smooth exhaustive stacks and prove some analogous properties. In §3 we explain the geometric and motivic Bialynicki–Birula decompositions. Then in §4 we prove Theorems 1.1 and 1.2 state Conjecture 1.3 and prove that this conjecture follows from a conjecture concerning the intersection theory of the smooth projective Quot schemes $\text{Div}_{n,d}(D)$ (cf. Theorem 4.20). In §5 we deduce from Conjecture 1.3 a formula for the compactly supported motive of $\text{Bun}_{n,d}$, which we compare with previous formulae in §6 and we deduce formulae for the motives of $\text{Bun}_{n,d}$ and $\text{Bun}_{SL_n}$ from Conjecture 1.1 (cf. Theorems 5.6 and 5.7). Finally, in Appendix A we explain and compare alternative approaches for defining étale motives of stacks.
1.1. Notation and conventions. Throughout we fix a field $k$ and all schemes and stacks are assumed to be defined over $k$. By an algebraic stack, we mean a stack $\mathcal{X}$ for the fppf topology with an atlas given by a representable, smooth surjective morphism from a scheme that is locally of finite type. For a closed substack $\mathcal{Y}$ of an algebraic stack $\mathcal{X}$, we define the codimension of $\mathcal{Y}$ in $\mathcal{X}$ to be the codimension of $\mathcal{Y} \times_\mathcal{X} U$ in $U$ for an atlas $U \to \mathcal{X}$; this is independent of the choice of atlas.

For $n \in \mathbb{N}$ and a quasi-projective variety $X$, the symmetric group $\Sigma_n$ acts on $X^{\times n}$ and the quotient is representable by a quasi-projective variety $X^{(n)}$, the $n$-th symmetric power of $X$.

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2. Motives of schemes and stacks

Let $k$ be a base field and $R$ be a commutative ring of coefficients. If the characteristic $p$ of $k$ is positive, then we assume either that $p$ is invertible in $R$ or that $k$ is perfect and admits the resolution of singularities by alterations. We let $\text{DM}(k, R) := \text{DM}^{\text{eff}}(k, R)$ denote Voevodsky’s category of (Nisnevich) motives over $k$ with coefficients in $R$; this is a monoidal triangulated category. For a separated scheme $X$ of finite type over $k$, we can associate both a motive $M(X) \in \text{DM}(k, R)$, which is covariantly functorial in $X$ and behaves like a homology theory, and a motive with compact supports $M^c(X) \in \text{DM}(k, R)$, which is covariantly functorial for proper morphisms and behaves like a Borel-Moore homology theory.

Without going into the details of the construction, we recall that objects in $\text{DM}(k, R)$ can be represented by (symmetric) $T$-spectra in complexes of Nisnevich sheaves with transfers (i.e., with additional contravariant functoriality for finite correspondences) of $R$-modules on the category of smooth $k$-schemes, where

$$T := \text{Cone}(R^\text{eff}_{tr}(\text{Spec } k) \to R^\text{eff}_{tr}(\mathbb{G}_m))[-1].$$

Here $R^\text{eff}_{tr}(X)$ denotes the sheaf of finite correspondences into $X$ with $R$-coefficients for $X \in \text{Sm}_k$. We write $R_{tr}(X)$ for the suspension spectrum $\Sigma_n^R R^\text{eff}_{tr}(X)$.

Remark 2.1. The main results of this paper hold in the category $\text{DM}^{\text{eff}}(k, R)$ of effective motives. We refrained from writing everything in terms of $\text{DM}^{\text{eff}}(k, R)$ for two reasons.

- Under our assumptions on $k$ and $R$, the functor $\text{DM}^{\text{eff}}(k, R) \to \text{DM}(k, R)$ is fully faithful [33, 38], so that results in $\text{DM}^{\text{eff}}(k, R)$ follow immediately from their stable counterparts.
- The motive with compact support $M^c(\mathcal{X})$ of an Artin stack, however it is defined, is almost never effective (see Section 2.7).

2.1. Properties of motives of schemes. The category $\text{DM}(k, R)$ was originally constructed in [42] and its deeper properties were established under the hypothesis that $k$ is perfect and satisfies resolution of singularities (with no assumption on $R$). They were extended to the case where $k$ is perfect by Kelly in [25], using Gabber’s refinement of de Jong’s results on alterations. Finally, the extension of scalars of $\text{DM}$ from a field to its perfect closure was shown to be an equivalence in [14, Proposition 8.1.(d)].

The motive $M(\text{Spec } k) := R_{tr}(\text{Spec } k)$ of the point is the unit for the monoidal structure, and there are Tate motives $R(n) \in \text{DM}(k, R)$ for all $n \in \mathbb{Z}$. For any motive $M$ and $n \in \mathbb{Z}$, we write $M(n) := M \otimes R(n)$, and we write $M \{n\} := M(n)[2n]$.

Let us list the main properties of motives that will be used in this paper.

- (Künneth formula): for schemes $X$ and $Y$, we have
  $$M(X \times_k Y) \simeq M(X) \otimes M(Y) \quad \text{and} \quad M^c(X \times_k Y) \simeq M^c(X) \otimes M^c(Y).$$

---

1 This definition is slightly different from the standard one where the atlas is allowed to be an algebraic space, and a condition on the diagonal is enforced; however this definition is sufficient for our purposes.
• (A¹-homotopy invariance): by construction of DM(k, R), for any Zariski-locally trivial affine bundle \( Y \to X \) with fibre \( A^n \), the induced morphisms
\[
M(Y) \to M(X) \quad \text{and} \quad M^c(Y)\{r\} \to M^c(X)
\]
are isomorphisms.
• (Projective bundle formula): for a vector bundle \( E \to X \) of rank \( r + 1 \), there are isomorphisms
\[
M(\mathbb{P}(E)) \simeq \bigoplus_{i=0}^{r} M(X)\{i\} \quad \text{and} \quad M^c(\mathbb{P}(E)) \simeq \bigoplus_{i=0}^{r} M^c(X)\{i\}.
\]
• (Gysin triangles): for a closed immersion \( i : Z \to X \) of codimension \( c \) between smooth \( k \)-schemes, there is a functorial distinguished triangle
\[
M(X - Z) \to M(X) \xrightarrow{\text{Gy}(i)} M(Z)\{c\} \xrightarrow{\delta}.
\]
• (Flat pullbacks for \( M^c \)): for a flat morphism \( f : X \to Y \) of relative dimension \( d \), there is a pullback morphism
\[
f^* : M^c(Y)\{d\} \to M^c(X).
\]
• (Localisation triangles): for a separated scheme \( X \) of finite type and \( Z \) any closed subscheme, there is a functorial distinguished triangle
\[
M^c(Z) \to M^c(X) \to M^c(X - Z) \xrightarrow{\delta}.
\]
• (Motives with and without compact supports): for a separated finite type scheme \( X \), there is a morphism \( M^c(X) \to M(X) \), which is an isomorphism if \( X \) is proper.
• (Internal homs and duals): the category \( \text{DM}(k, R) \) has internal homomorphisms, which can be used to define the dual of any motive \( M \in \text{DM}(k, R) \) as
\[
M^\vee := \text{Hom}(M, R(0)).
\]
• (Poincaré duality): for a smooth scheme \( X \) of pure dimension \( d \), there is an isomorphism
\[
M^c(X) \simeq M(X)^\vee\{d\}.
\]
• (Algebraic cycles): for a smooth scheme \( X \) (say of pure dimension \( d \) for simplicity), a separated scheme \( Y \) of finite type and \( i \in \mathbb{N} \), there is an isomorphism
\[
\text{CH}_i(X \times Y)_R \simeq \text{Hom}_{\text{DM}}(M(X), M^c(Y)\{d - i\})
\]
where \( \text{CH}_i \) denotes the Chow groups of cycles of dimension \( i \).
• (Compact generators): The triangulated category \( \text{DM}(k, R) \) is compactly generated, with a compact family of generators given by motives of the form \( M(X)(-n) \) for \( X \) smooth and \( n \in \mathbb{N} \). We denote by \( \text{DM}_{\text{gm}}(k, R) \) the triangulated subcategory consisting of compact objects. For any separated finite type scheme \( X \), both \( M(X) \) and \( M^c(X) \) are compact. We recall that an object \( M \in \text{DM}(k, R) \) is compact if and only if for all families \( (N_\alpha)_{\alpha \in A} \in \text{DM}(k, R)^A \) indexed by a set \( A \), the natural map \( \bigoplus_{\alpha \in A} \text{Hom}(M, N_\alpha) \to \text{Hom}(M, \bigoplus_{\alpha \in A} N_\alpha) \) is an isomorphism.

**Remark 2.2.** If we instead work with the category of étale motives \( \text{DM}^{\text{ét}}(k, R) \), this would lead to weaker results (as this category does not capture the information about integral and torsion Chow groups), but defining motives of stacks is technically simpler as we explain in Appendix [A].
2.2. **Homotopy (co)limits.** As $\text{DM}(k, R)$ is a compactly generated triangulated category, it admits arbitrary direct sums and arbitrary direct products \cite[Proposition 8.4.6]{33}; hence, one can define arbitrary homotopy colimits and homotopy limits for $\mathbb{N}$-indexed systems in $\text{DM}(k, R)$ using only the triangulated structure together with direct sums and products as follows.

**Definition 2.3.** The homotopy colimit of an inductive system $F_* : \mathbb{N} \to \text{DM}(k, R)$ is

$$\text{holim}_n F_n := \text{Cone} \left( \bigoplus_{i \in \mathbb{N}} F_i \xrightarrow{id - \sigma} \bigoplus_{i \in \mathbb{N}} F_i \right)$$

and the homotopy limit of a projective system $G^* : \mathbb{N}^{op} \to \text{DM}(k, R)$ is

$$\text{holim}_{n \in \mathbb{N}} G^n := \text{Cone} \left( \prod_{i \in \mathbb{N}} G^i \xrightarrow{id - \sigma} \prod_{i \in \mathbb{N}} G^i \right)[-1],$$

where we write $\sigma$ for any of the maps $F_i \to F_{i+1}$ (resp. $G^{i+1} \to G^i$) in the diagram.

Note that, by construction, for any given choice of such a cone, there is a compatible system of maps, i.e. an element of $\lim_i \text{Hom}(F_i, \text{holim}_n F_*)$ (resp. $\lim_j \text{Hom}(\text{holim}_n G^*, G^j)$).

**Remark 2.4.** Using \cite[Lemma 1.7.1]{33}, it is easy to extend this definition to homotopy colimits indexed by filtered partially ordered sets $I$ such that there exists a cofinal embedding $\mathbb{N} \to I$.

The reader unfamiliar with this definition should compare it with the definition of the limit and its derived functor $R^1 \lim$ for $\mathbb{N}$-indexed diagrams of abelian groups in \cite[Definition 3.5.1]{44}. Since homotopy (co)limits are defined by the choice of a cone, they are only unique up to the a non-unique isomorphism. However, in the case where the $\mathbb{N}$-indexed system actually comes from the underlying model category (that is, it can be realised as a system of $T$-spectra of complexes of Nisnevich sheaves with transfers and morphisms between them), then homotopy colimits can be realised in a simple, canonical way as follows.

**Lemma 2.5.** Let $S_0 \to S_1 \to S_2 \ldots$ be an inductive system of $T$-spectra of complexes of Nisnevich sheaves with transfers on $\text{Sm}_k$ with values in the category of $R$-modules. Then

$$\text{holim}_n S_n \simeq \text{colim}_n S_n$$

with the colimit being computed in the abelian category of $T$-spectra of complexes of Nisnevich sheaves with transfers.

**Proof.** We have a short exact sequence

$$0 \to \bigoplus_{n \geq 0} S_n \xrightarrow{id - \sigma} \bigoplus_{n \geq 0} S_n \to \text{colim}_m S_n \to 0$$

where the injectivity of the first map follows from the fact that the abelian category of $T$-spectra of complexes of Nisnevich sheaves with transfers is a Grothendieck abelian category, so in particular has exact filtered colimits. This exact sequence provides a distinguished triangle in the derived category, and so also in $\text{DM}(k, R)$, which exhibits the colimit as a cone of the map $id - \sigma$, and thus as a homotopy colimit. \hfill $\square$

Nevertheless homotopy colimits are functorial in the following relatively weak sense.

**Lemma 2.6.** For two $\mathbb{N}$-indexed inductive (resp. projective) systems $F_*, \tilde{F}_*$ (resp. $G^*, \tilde{G}^*$) in $\text{DM}(k, R)$, fix a choice of homotopy (co)limits $F, \tilde{F}$ (resp. $G, \tilde{G}$); that is, a specific choice of cones of the morphisms in Definition 2.3. Then, modulo these choices, there is a uniquely determined short exact sequence

$$0 \to R^1 \text{Hom}(F_n[1], \tilde{F}) \to \text{Hom}(F, \tilde{F}) \to \text{lim}_n \text{Hom}(F_n, \tilde{F}) \to 0$$

(resp. $0 \to R^1 \text{lim}_{n} \text{Hom}(G, \tilde{G}^n[-1]) \to \text{Hom}(G, \tilde{G}) \to \text{lim}_n \text{Hom}(G, \tilde{G}^n) \to 0$). In particular, for a morphism $f_* : F_* \to \tilde{F}_*$ of inductive systems, we can choose a morphism $f : F \to \tilde{F}$ such
that for any \( n \in \mathbb{N} \) the diagram

\[
\begin{array}{ccc}
F_n & \xrightarrow{f_n} & \tilde{F}_n \\
\downarrow & & \downarrow \\
F & \xrightarrow{f} & \tilde{F}
\end{array}
\]

commutes, and \( f \) is uniquely determined up to the \( R^1 \text{lim} \) term appearing in the above exact sequence (and there is a similar statement for homotopy limits). By abuse of notation, we sometimes denote such a morphism by \( M(f_*) \).

**Proof.** This follows directly from the definition of the functor \( R^1 \text{lim} \). \( \square \)

The compatibility between the weak functoriality and the triangulated structure is as follows.

**Lemma 2.7.** Consider an \( \mathbb{N} \)-indexed system of distinguished triangles \( F'_n \to F_n \to F''_n \xrightarrow{+} \) in \( \text{DM}(k, R) \). For any choice homotopy colimits of those systems and compatible morphisms between them as in Lemma 2.6, the triangle

\[
\text{hocolim}_n F'_n \to \text{hocolim}_n F_n \to \text{hocolim}_n F''_n \xrightarrow{+}
\]

is distinguished. The analogous statement holds for homotopy limits.

**Proof.** This follows directly from the fact that direct sums (resp. direct products) of distinguished triangles are distinguished and the nine lemma. \( \square \)

Let us state some results about simple homotopy (co)limits.

**Lemma 2.8.** For an inductive system \( F : \mathbb{N} \to \text{DM}(k, R) \) and \( A \in \text{DM}(k, R) \), there is an isomorphism

\[
\text{hocolim}_n (F_n \otimes A) \simeq (\text{hocolim}_n F_n) \otimes A.
\]

**Proof.** This follows from the definition of the homotopy colimit, as the tensor product commutes with direct sums in a tensor triangulated category. \( \square \)

**Lemma 2.9.** Let \( I \) be a set and let \( F_i^* : \mathbb{N} \to \text{DM}(k, R) \) be an inductive system for all \( i \in I \). Then there is an isomorphism

\[
\text{hocolim}_n \bigoplus_{i \in I} F^i_n \simeq \bigoplus_{i \in I} \text{hocolim}_n F^i_n.
\]

**Proof.** This follows from the definition of the homotopy colimit, as a direct sum of distinguished triangles is distinguished. \( \square \)

2.3. **Vanishing results for homotopy (co)limits.** In order to compute homotopy (co)limits, we will frequently rely on various vanishing results which we collect together in this section.

**Definition 2.10.** The dimensional filtration on \( \text{DM}(k, R) \) is the \( \mathbb{Z} \)-indexed filtration

\[
\cdots \subset \text{DM}(k, R)_{m-1} \subset \text{DM}(k, R)_m \subset \text{DM}(k, R)_{m+1} \subset \cdots
\]

where \( \text{DM}(k, R)_m \) denotes the smallest localising subcategory of \( \text{DM}(k, R) \) containing \( M^c(X)(n) \) for all separated schemes \( X \) of finite type over \( k \) and all integers \( n \) with \( \dim(X) + n \leq m \).

Note that analogous filtrations appear in the literature dealing with classes of stacks in the Grothendieck ring of varieties (for example, see [1]). Totaro proves the following result.

**Proposition 2.11 ([40, Corollary 8.4]).** The dimension filtration satisfies

\[
\bigcap_{m \leq 0} \text{DM}(k, R)_m \simeq 0.
\]

Moreover, for a projective system \( G^* : \mathbb{N}^\text{op} \to \text{DM}(k, R) \) and a sequence of integers \( a_n \to -\infty \) such that \( G^n \in \text{DM}(k, R)_{a_n} \) for all \( n \), it follows that \( \text{holim}_n G^n \simeq 0 \).
Corollary 2.12. Let \((M_n)_{n \in \mathbb{N}} \in \text{DM}(k, R)^N\) be a family of motives and \((a_n)_{n \in \mathbb{N}}\) be a sequence of integers such that \(a_n \to -\infty\) and \(M_n \in \text{DM}(k, R)_{a_n}\) for all \(n\). Then the natural morphism
\[
\bigoplus_{n \in \mathbb{N}} M_n \to \prod_{n \in \mathbb{N}} M_n
\]
is an isomorphism.

Proof. The cone of this morphism lies in \(\text{DM}(k, R)_{a_n}\) for all \(n \in \mathbb{N}\), thus is zero by Proposition 2.11 (for further details, see the proof of [10, Lemma 8.7]).

We will also need a vanishing result for homotopy colimits. However, the dual result to Proposition 2.11 does not hold in \(\text{DM}(k, R)\): if \(k\) has infinite transcendence dimension, there is an inductive system \((\ldots R(n)[n] \to R(n+1)[n+1] \ldots)\) whose homotopy colimit is non-zero (cf. [4, Lemma 2.4]). Hence, the intersection \(\cap_{n \geq 0} \text{DM}^{\mathbb{H}}(k, R)(n)\) is non-zero. Fortunately, with some control over the Tate twists and shifts, we can prove the following vanishing result.

Proposition 2.13. Let \(U_n \subset X_n\) be an inductive system of open immersions of smooth finite type \(k\)-schemes; that is, we have inductive systems \(U_n, X_n : \mathbb{N} \to \text{Sm}_k\) and a morphism \(U_n \to X_n\) such that \(U_n \to X_n\) is an open immersion for all \(n \in \mathbb{N}\). Let \(c_n\) be the codimension of the complement of \(X_n - U_n\) in \(X_n\). If \(c_n \to \infty\), then the morphism
\[
\colim_n R_{\text{tr}}(U_n) \to \colim_n R_{\text{tr}}(X_n)
\]
is an \(A^1\)-weak equivalence.

Proof. Let \(C_n\) be the level-wise mapping cone (as a \(T\)-spectrum of complexes of sheaves with transfers) of the morphism \(R_{\text{tr}}(U_n) \to R_{\text{tr}}(X_n)\). There are induced maps \(C_n \to C_{n+1}\) and we get a distinguished triangle
\[
\colim_n R_{\text{tr}}(U_n) \to \colim_n R_{\text{tr}}(X_n) \to \colim_n C_n \xrightarrow{\sim} \text{in DM}(k, R)\). Thus it suffices to show that \(\colim_n C_n\) is \(A^1\)-equivalent to 0.

Let \(W_n = X_n - U_n\), and consider a finite locally closed stratification \((W_i^j)_{n \in \mathbb{N}}\) with each \(W_i^j\) smooth over \(k\) (which exists because \(W_n\) is geometrically reduced and of finite type over \(k\)). For each \(1 \leq j \leq m_n\), we have a Gysin distinguished triangle
\[
M(X - (W^{m_n}_n \cup \ldots \cup W^1_n)) \to M(X - (W^{m_n}_n \cup \ldots \cup W^j_n)) \to M(W^{j-1}_n)\{\codim_X W^j_n\}
\]
By inductively applying the octahedral axiom to these distinguished triangles, we conclude that \(C_n\) is a successive extension of the motives \(M(W^j_n)\{\codim_X W^j_n\}\).

Since the category \(\text{DM}(k, R)\) is compactly generated by motives of the form \(M(X)(a)\), with \(X \in \text{Sm}_k\) and \(a \in \mathbb{Z}\), it suffices to show for all \(X \in \text{Sm}_k\) and \(a, b \in \mathbb{Z}\) that
\[
\text{Hom}(M(X)(a)[b], \colim_n C_n) = 0.
\]
As \(M(X)(a)[b]\) is compact and filtered colimits are homotopy colimits in \(\text{DM}(k, R)\), we have
\[
\text{Hom}(M(X)(a)[b], \colim_n C_n) = \colim_n \text{Hom}(M(X)(a)[b], C_n).
\]
Since each cone \(C_n\) is a successive extension of motives of the form \(M(Z)\{c\}\) for \(Z \in \text{Sm}_k\) and \(c \geq c_n\), it suffices to show for \(n \gg 0\) that for all \(Z \in \text{Sm}_k\) and \(c \geq c_n\)
\[
\text{Hom}(M(X)(a)[b], M(Z)\{c\}) = 0.
\]
Equivalently, by Poincaré duality, it suffices to show
\[
\text{Hom}(M(X) \otimes M^c(Z), R(c + \dim(Z) - a)(2c + 2\dim(Z) - b)) = 0.
\]
As \(c_n \to \infty\), we have \(c_n - b + a - \dim(X) > 0\) for \(n \gg 0\). Thus, for \(n \gg 0\) and \(c \geq c_n\),
\[
(2c + 2\dim(Z) - b) - ((c + \dim(Z) - a) + \dim(Z) + \dim(X)) = c - b + a - \dim(X)
\]
is strictly positive. Then we deduce that (1) holds by using Lemma 2.14 below. \(\square\)
Lemma 2.14. Let $X$ (resp. $Z$) be a variety of dimension at most $d$ (resp. $e$). For $k, l \in \mathbb{Z}$ with $l > k + d + e$, we have
\[ \text{Hom}(M(X) \otimes M^c(Z), R(k)[l]) = 0. \]

Proof. By our standing assumption on $k$ and $R$ and [14 Proposition 8.1.(d)], we can assume that $k$ is a perfect field. Let us first prove the claim when $Z$ is proper. Then $M^c(Z) \otimes M(X) \simeq M(Z \times_k X)$, as $M^c(Z) \simeq M(Z)$. We have
\[ \text{Hom}(M(Z \times_k X), R(k)[l]) = H^l_{cdh}(Z \times_k X, R(k)_{cdh}). \]
where $R(k)_{cdh}$ is the cdh-sheafification of the Suslin-Voevodsky motivic complex $\mathbb{Z}$. The cohomological dimension of $(Z \times_k X)_{cdh}$ is at most $d + e$ by [37 Theorem 5.13], and the complex $R(k)_{cdh}$ is $0$ in cohomological degrees greater than $k$, which implies the result.

We now turn to the general case. Let $\bar{Z}$ be any compactification of $Z$ (not necessarily smooth). From the localisation triangle for the closed pair $(\bar{Z} - Z, \bar{Z})$, we obtain a long exact sequence
\[ \rightarrow \text{Hom}(M(X) \otimes M^c(\bar{Z}), R(k)[l]) \rightarrow \text{Hom}(M(X) \otimes M^c(Z), R(k)[l]) \rightarrow \text{Hom}(M(X) \otimes M^c(\bar{Z} - Z), R(k)[l - 1]) \rightarrow \]
As both $\bar{Z}$ and $\bar{Z} - Z$ are proper with dimensions bounded by $e$ and $e - 1$ respectively, we deduce that the outer terms vanish by the proper case, and so we obtain the desired result. ☐

2.4. Motives of stacks. The definition of motives of stacks in general is complicated by the fact that the category $\text{DM}(k, R)$ does not satisfy descent for the étale topology (as this is already not the case for Chow groups); hence naive approaches to defining motives of even Deligne-Mumford stacks in terms of an atlas do not work. In Appendix A we explain and compare alternative approaches for defining étale motives of stacks, provide references to the literature, and define motives with compact supports.

To define the motive of a quotient stack $\mathfrak{X} = [X/G]$ independently of the presentation of $\mathfrak{X}$ as a quotient stack, we need an appropriate notion of “algebraic approximation of the Borel construction $X \times^G EG$”. We use a variant of the definition of compactly supported motives of quotient stacks given by [10] and extend this to more general stacks. More precisely, we will define the motive of certain smooth stacks over $k$, possibly not of finite type, which are exhaustive in the following sense.

Definition 2.15. Let $\mathfrak{X}_0 \xrightarrow{i_0} \mathfrak{X}_1 \xrightarrow{i_1} \ldots \subset$ be a filtration of an algebraic stack $\mathfrak{X}$ by increasing open substacks $\mathfrak{X}_i \subset \mathfrak{X}$ which are quasi-compact and cover $\mathfrak{X}$; we will simply refer to this as a filtration. Then an exhaustive sequence of vector bundles on $\mathfrak{X}$ with respect to this filtration is a pair $(V_\bullet, W_\bullet)$ given by a sequence of vector bundles $V_m$ over $\mathfrak{X}_m$ together with injective maps of vector bundles $f_m : V_m \rightarrow V_{m+1} \times_{\mathfrak{X}_{m+1}} \mathfrak{X}_m$ and closed substacks $W_m \subset V_m$ such that
\begin{enumerate}
  \item the codimension of $W_m$ in $V_m$ tends towards infinity,
  \item the complement $U_m := V_m - W_m$ is a separated finite type $k$-scheme, and
  \item we have $f_m^{-1}(V_{m+1} \times_{\mathfrak{X}_{m+1}} \mathfrak{X}_m) \subset W_m$ (so that $f_m(U_m) \subset U_{m+1} \times_{\mathfrak{X}_{m+1}} \mathfrak{X}_m$).
\end{enumerate}

An exhaustive stack is a stack which admits such an exhaustive sequence with respect to some filtration.

For this paper, we have in mind two important examples of exhaustive stacks: i) quotient stacks (cf. Lemma [2,10]), and ii) the stack of vector bundles on a curve (cf. Proposition [1,2]).

Lemma 2.16. Let $\mathfrak{X} = [X/G]$ be a quotient stack of a quasi-projective scheme $X$ by an affine algebraic group $G$ such that $X$ admits a $G$-equivariant ample line bundle; then $\mathfrak{X}$ is exhaustive.

Proof. Let $\mathfrak{X} = [X/G]$ be a quotient stack; then there is an exhaustive sequences of vector bundles over $\mathfrak{X}$ with respect to the constant filtration built from a faithful $G$-representation $G \rightarrow GL(V)$ such that $G$ acts freely on an open subset $U \subset V$. More precisely, we let $W := V - U \subset V$ and for $m \geq 1$ consider the $G$-action on $V^m$, which is free on the complement of $W^m$. First, we construct an exhaustive sequence $(B_\bullet, C_\bullet)$ of vector bundles over $BG$ (with
respect to the constant filtration) by taking $B_m = V^m$ with the natural transition maps and $C_m = W^m$. We then form the exhaustive sequence 

$$([(X \times B_\bullet)/G], [(X \times C_\bullet)/G])$$

on $[X/G]$; this works as the open complement $(X \times (B_m - C_m))/G$ is a quasi-projective scheme by Proposition 7.1 due to the existence of a $G$-equivariant ample line bundle on $X$. \hfill $\Box$

**Definition 2.17.** Let $\mathfrak{X}$ be a smooth exhaustive stack; then for an exhaustive sequence $(V_\bullet, W_\bullet)$ of vector bundles with respect to a filtration $\mathfrak{X} = \cup_m \mathfrak{X}_m$, we define the motive of $\mathfrak{X}$ as

$$M(\mathfrak{X}) := \text{colim}_m R_{tr}(U_m)$$

with transition maps induced by the compositions of $f_m : U_m \to U_{m+1} \times_{\mathfrak{X}_{m+1}} \mathfrak{X}_m$ (by Definition 2.15(ii)) and transition morphisms $f_m$ restricts to such a morphism) with projection $U_{m+1} \times_{\mathfrak{X}_{m+1}} \mathfrak{X}_m \to U_{m+1}$, and the colimit is taken in the category of $T$-spectra of complexes of sheaves with transfers.

By Lemma 2.16 this definition implies the following lemma.

**Lemma 2.18.** In $\text{DM}(k, R)$, for $\mathfrak{X}$ and $(V_\bullet, W_\bullet)$ as in Definition 2.17 we have an isomorphism

$$M(\mathfrak{X}) \simeq \text{hocollim} M(U_m).$$

Before proving this definition is independent of the choice of filtration and exhaustive sequence, we first note that if $\mathfrak{X} = X$ is a scheme, then this definition coincides with the usual definition of the motive of $X$. Indeed $X$ is an exhaustive stack, as we can take the constant filtration and let $V_m = U_m = X$.

**Lemma 2.19.** The motive of a smooth exhaustive stack $\mathfrak{X}$ does not depend (up to a canonical isomorphism) on the choice of the filtration or the exhaustive sequence of vector bundles.

**Proof.** We first fix a filtration of $\mathfrak{X} = \cup_m \mathfrak{X}_m$ and prove the resulting object does not depend on the exhaustive sequence. Let $(V_\bullet, W_\bullet)$ and $(V'_\bullet, W'_\bullet)$ be two exhaustive sequence of vector bundles on $\mathfrak{X}$. As in the proof of Theorem 8.5, we introduce a new sequence $(V''_\bullet, W''_\bullet)$ with

$$(V''_m, W''_m) := (V_m \times_{\mathfrak{X}_m} V'_m, W_m \times_{\mathfrak{X}_m} W'_m).$$

and transition morphisms $f'_m := f_m \times f'_m : V_m \times_{\mathfrak{X}_m} V'_m \to (V_{m+1} \times_{\mathfrak{X}_{m+1}} \mathfrak{X}_m) \times_{\mathfrak{X}_m} (V'_{m+1} \times_{\mathfrak{X}_{m+1}} \mathfrak{X}_m)$, which are injective vector bundle homomorphisms. Let us prove that this sequence satisfies properties (ii) and (iii) in Definition 2.17 (in fact, it satisfies property (i), but this is not needed below). For (ii) we note that $U''_m := V''_m - W''_m = U_m \times_{\mathfrak{X}_m} V'_m \cup V_m \times_{\mathfrak{X}_m} U'_m$ is a separated scheme of finite type, as $U_m \times_{\mathfrak{X}_m} V'_m$ (resp. $V_m \times_{\mathfrak{X}_m} U'_m$) is a vector bundle over the scheme $U_m$ (resp. $U'_m$). For (iii) we have

$$(f''_m)^{-1}(W''_{m+1} \times_{\mathfrak{X}_{m+1}} \mathfrak{X}_m) \subseteq (f'_m)^{-1}(W'_{m+1} \times_{\mathfrak{X}_{m+1}} \mathfrak{X}_m) \subseteq W'_m.$$ Given these two properties, one can define a system of $T$-spectra $\ldots \to R_{tr}(U''_m) \to R_{tr}(U''_{m+1}) \to \ldots$ as in Definition 2.17 and to complete the proof, it suffices by symmetry to show that $	ext{colim}_m R_{tr}(U''_m) \simeq \text{colim}_m R_{tr}(U_m)$ in $\text{DM}(k, R)$.

We note that $U_m \times_{\mathfrak{X}_m} V'_m$ is open in $U'_m$ (thus a smooth scheme) and that the codimension of the complement satisfies

$$(\text{codim}_{U'_m}(U_m \times_{\mathfrak{X}_m} V'_m)) \geq \text{codim}_{U'_m}(U_m \times_{\mathfrak{X}_m} V'_m) = \text{codim}_{U'_m}(W_m)$$

where we have used that $U''_m$ is dense open in $V''_m$ and that $V'_m \to \mathfrak{X}_m$ is a vector bundle. In particular, these codimensions tend to infinity with $m$. Moreover, $U_m \times_{\mathfrak{X}_m} V'_m \to U_m$ is a vector bundle. Hence we have the following two morphisms of inductive systems of $T$-spectra

$$\ldots \xrightarrow{\text{tr}} R_{tr}(U''_m) \xrightarrow{\text{tr}} R_{tr}(U''_{m+1}) \xrightarrow{\text{tr}} \ldots$$

$$\ldots \xrightarrow{\text{tr}} R_{tr}(U_m \times_{\mathfrak{X}_m} V'_m) \xrightarrow{\text{tr}} R_{tr}(U_{m+1} \times_{\mathfrak{X}_{m+1}} V''_{m+1}) \xrightarrow{\text{tr}} \ldots$$

$$\xrightarrow{\mathbb{A}^1 \text{-w.e.}} \ldots$$

$$\xrightarrow{\mathbb{A}^1 \text{-w.e.}} \ldots$$

$$\ldots \xrightarrow{\text{tr}} R_{tr}(U_m) \xrightarrow{\text{tr}} R_{tr}(U_{m+1}) \xrightarrow{\text{tr}} \ldots$$
A filtered colimit of $A^1$-weak equivalences is an $A^1$-weak equivalence, so that the bottom morphism of systems induces an $A^1$-weak equivalence on colimits. It is thus enough to show that the top morphism of systems also induces an $A^1$-weak equivalence; by (3), this follows from Proposition 2.13.

Now let us prove that the definition is independent of the filtration. Let $(X_n)_{n \geq 0}$ and $(X'_n)_{n \geq 0}$ be two such filtrations of $X$. Note that by quasi-compactness of the stacks $X_n$ and the fact that $X = \cup_n X'_n$, for $n \in \mathbb{N}$, there exists $n' \in \mathbb{N}$ such that $X_n \subset X'_{n'}$. Hence, we can find strictly increasing sequences $(n_i)_{i \geq 0}$, $(n'_i)_{i \geq 0}$ such that $X_{n_i} \subset X'_{n'_i} \subset X_{n_2} \subset X'_{n'_2}$ and so on. Since colimits are stable under passing to a cofinal sequence, we obtain the result.

Example 2.20. The compactly supported motive of the classifying space $BG_m$ of the multiplicative group $G_m$ is computed by Totaro [40, Lemma 8.7], by using the family of representations $\rho_n : G_m \to \mathbb{A}^n$ given by $t \mapsto \text{diag}(t, \ldots, t)$. More precisely, we have an exhaustive sequence over $BG_m$ given by $(V_n := [\mathbb{A}^n/G_m], W_n := \{0\}/G_m)_{n \in \mathbb{N}}$, and we can also use this to compute the motive of $BG_m$. The open complement is $U_n := [(\mathbb{A}^n - \{0\})/G_m] \cong \mathbb{P}^{n-1}$ and so we have

$$M(BG_m) = \text{colim}_n Rtr(\mathbb{P}^{n-1}) \simeq \text{hocolim}_n M(\mathbb{P}^{n-1}) \simeq \bigoplus_{j \geq 0} R\{j\}.$$ 

Proposition 2.21. Let $\mathcal{X}$ be a smooth exhaustive stack. Fix an exhaustive sequence $(V_\bullet, W_\bullet)$ of vector bundles with respect to a filtration $\mathcal{X} = \cup_m \mathcal{X}_m$ so that $M(\mathcal{X}) = \text{colim}_m Rtr(U_m)$. Let $N \in DM_0(k, R)$ be any compact motive. Then we have isomorphisms

(i) $\text{Hom}(N, M(\mathcal{X})) \cong \text{colim}_m \text{Hom}(N, M(U_m))$

(ii) $\text{Hom}(M(\mathcal{X}), N) \cong \text{lim}_m \text{Hom}(M(U_m), N)$.

Proof. Part (i) holds, as filtered colimits in $DM(k, R)$ are homotopy colimits and $N$ is compact. For (ii), by Lemma 2.16 it suffices to show that $R^1 \text{lim} \text{Hom}(M(U_n)[1], N) = 0$. We actually show that the map $\text{Hom}(M(U_{n+1})[1], N) \to \text{Hom}(M(U_n)[1], N)$ is an isomorphism for $n$ large enough, which implies that the corresponding $R^1 \text{lim}$ term vanishes.

Since $N$ is compact, there exists $m \in \mathbb{Z}$ such that $N \in DM(k, R)_m$ (see Definition 2.11). By the argument in the proof of Proposition 2.13 for $n$ large enough, the cone of $M(U_n) \to M(V_n)$ is in the triangulated subcategory generated by motives of the form $M(Z)[i]$ with $Z \in \text{Sm}_k$ and $c > m$. Then by [40, Lemma 8.1], this implies that $\text{Hom}(\text{Cone}(M(U_n) \to M(V_n)), N[i]) = 0$ for all $i \in \mathbb{Z}$, thus in particular $\text{Hom}(M(U_n)[1], N) \cong \text{Hom}(M(V_n)[1], N)$.

By definition, the transition maps in the system $\text{Hom}(M(V_n)[1], N)$ are induced by the maps $M(V_n) \to M(V_{n+1})$ where $i_n$ is the open immersion $X_n \hookrightarrow X_{n+1}$. Both $V_n$ and $i^n_n V_{n+1}$ are vector bundles over the same stack $X_n$, and $V_n \to i^n_n V_{n+1}$ is a map of vector bundles, so $M(V_n) \to M(i^n_n V_{n+1})$ is an isomorphism. On the other hand, $i^n_n V_{n+1} \to V_{n+1}$ is an open immersion, and $\text{codim}_{n+1}(V_{n+1} - i^n_n V_{n+1}) = \text{codim}_{X_{n+1}}(X_{n+1} - X_n)$ which tends to infinity with $n$ by assumption. To complete the proof, we use the same argument as in the paragraph above to deduce that $\text{Hom}(M(V_{n+1})[1], N) \cong \text{Hom}(M(V_n)[1], N)$ for $n$ large enough.

This shows, in particular, that the following definition is not unreasonable (since it can be computed via any given exhaustive sequence).

Definition 2.22. Let $\mathcal{X}$ be a smooth exhaustive stack, and $p, q \in \mathbb{Z}$. The motivic cohomology of $\mathcal{X}$ is defined as

$$H^p(\mathcal{X}, R(q)) := \text{Hom}(M(\mathcal{X}), R(q)[p]).$$

It is not immediately clear that the definition of the motive of an exhaustive stack is functorial, but it is relatively simple to prove a weak form of functoriality for certain representable morphisms. To have more systematic forms of functoriality it is better to work with other definitions of motives of stacks as explained in Appendix A.

Lemma 2.23. Let $g : \mathcal{X} \to \mathcal{Y}$ be a flat finite type representable morphism of smooth algebraic stacks such that $\mathcal{Y}$ is exhaustive, with an exhaustive sequence of vector bundles $(V_\bullet, W_\bullet)$ on
\[ Y \text{ with respect to a filtration } Y = \cup_m Y_m. \text{ Then } X \text{ is exhaustive, with the exhaustive sequence } (g^*V, g^{-1}(W)) \text{ with respect to the filtration } X = \cup_m g^{-1}(Y_m). \text{ Moreover, } g \text{ induces a morphism } M(g) : M(X) \to M(Y). \]

**Proof.** Since \( g \) is flat, we have that \( \text{codim}_{g^*V_n}(g^{-1}(W_n)) \geq \text{codim}_{V_n}(W_n) \) which tends to infinity with \( n \). By the fact that \( g \) is representable and of finite type, \( g^*V_n - g^{-1}(W_n) \simeq (V_n - W_n) \times_Y X \) is a separated finite type \( k \)-scheme. We leave the verification of Property [iii] in Definition 2.15 to the reader. The morphism \( M(g) \) is then defined by taking colimits in the morphism of systems of \( T \)-spectra \( R_{tr}(g^{-1}(U_n)) \to R_{tr}(U_n). \)

**Remark 2.24.** The flatness condition was imposed only in order to prove the codimension condition on \( (g^*V, g^{-1}(W)) \). There are many other cases in which the definition of \( M(f) \) above works; for instance, for any finite type representable morphism between quotient stacks.

One can also prove Künneth isomorphisms and \( A^1 \)-homotopy invariance.

**Proposition 2.25.** Let \( X \) and \( Y \) be smooth exhaustive smooth stacks over \( k \).

(i) The stack \( X \times_k Y \) is smooth exhaustive and there is a Künneth isomorphism \( M(X \times_k Y) \simeq M(X) \otimes M(Y) \).

(ii) If \( E \to X \) is a vector bundle, then \( E \) is a smooth exhaustive stack and \( M(E) \simeq M(X) \).

**Proof.** For part (i), let \( X_n \) (resp. \( Y_n \)) be filtrations with respect to which \( X \) (resp. \( Y \)) admits an exhaustive sequence of vector bundles \( (V, W) \) (resp. \( (V', W') \)). Then we leave the reader to easily verify that on \( X \times_k Y \) there is an exhaustive sequence compatible with the filtration \( X \times_k Y = \cup_n X_n \times Y_n \) given by \( (V \times V', W \times_k V' \cup V \times_k W') \). We thus get

\[
M(X \times_k Y) = \colim_n R_{tr}(U_n \times U'_n) \simeq \colim_n R_{tr}(U_n) \otimes R_{tr}(U'_n) \simeq \colim_n \colim_m R_{tr}(U_n) \otimes R_{tr}(U'_m) \\
simeq \colim_n \colim_m R_{tr}(U_n) \otimes R_{tr}(U'_m) = M(X) \otimes M(Y)
\]

where we have used the commutation of tensor products of \( T \)-spectra with colimits and the fact that the diagonal \( N \to N \times N \) is cofinal. The proof of part (ii) follows by a similar argument using the pullback of any exhaustive sequence from \( X \) to \( E \) and homotopy invariance for schemes.

**Remark 2.26.** In fact, the argument used in Proposition 2.25 (i) enables us to define a morphism

\[
M(\Delta) : M(X) \to M(X) \otimes M(X)
\]

for any smooth exhaustive stack \( X \). Indeed this morphism is defined as the colimit over \( n \) of the morphisms \( R_{tr}(U_n) \to R_{tr}(U_n \times U_n) \simeq M(U_n) \otimes M(U_n) \), and one can check that this morphism is independent of the presentation.

### 2.5. Chern classes of vector bundles on stacks.

For a vector bundle \( E \) over a smooth \( k \)-scheme \( X \), one has motivic incarnations of Chern classes given by a morphism

\[
c_j(E) : M(X) \to R\{j\}.
\]

Let us extend this notion to vector bundles on smooth exhaustive stacks.

**Definition 2.27.** Let \( E \to X \) be a vector bundle on a smooth exhaustive stack and let \( (V, W) \) be an exhaustive system of vector bundles on \( X \) with respect to a filtration \( X = \bigcup_n X_n \); then the pullback \( E_n \) of \( E \) to the smooth scheme \( U_n := V_n - W_n \to V_n \to X_n \to X \) determines a morphism \( c_j(E_n) : M(U_n) \to R\{j\} \). Since these morphisms are compatible, this determines an element of \( \lim_n \text{Hom}(M(U_n), R\{j\}) \), which corresponds to a morphism

\[
c_j(E) : M(X) \to R\{j\}
\]

by Proposition 2.21 as \( R\{j\} \) is compact.

**Lemma 2.28.** This definition does not depend on any of the above choices.

**Proof.** We omit the details as the proof is very similar to that of Lemma 2.19. \[\square\]
We will need some basic functoriality results for these Chern classes.

**Lemma 2.29.** Let \( g : \mathcal{X} \to \mathcal{Y} \) be a flat finite type representable morphism of smooth algebraic stacks such that \( \mathcal{Y} \) is exhaustive (then \( \mathcal{X} \) is exhaustive by Lemma 2.23). For a vector bundle \( \mathcal{E} \to \mathcal{Y} \) and \( j \in \mathbb{N} \), we have a commutative diagram

\[
\begin{array}{ccc}
M(\mathcal{X}) & \xrightarrow{M(f)} & M(\mathcal{Y}) \\
c_1(f^*\mathcal{E}) & \downarrow & c_1(\mathcal{E}) \\
R\{j\} & = & R\{j\}.
\end{array}
\]

**Proof.** This follows from the proof of Lemma 2.23 and the functoriality of Chern classes for vector bundles on smooth schemes. \( \square \)

2.6. **Motives of \( \mathbb{G}_m \)-torsor.** In this section, we prove a result for the motive of a \( \mathbb{G}_m \)-torsor, which will be used to compute the motive of the stack of principal \( \text{SL}_n \)-bundles. We recall that a \( \mathbb{G}_m \)-torsor over a stack \( \mathcal{X} \) is a morphism \( \mathcal{X} \to B\mathbb{G}_m \), or equivalently a cartesian square

\[
\begin{array}{ccc}
\mathcal{Y} & \to & \text{Spec} \mathcal{k} \\
\downarrow & & \downarrow \\
\mathcal{X} & \to & B\mathbb{G}_m.
\end{array}
\]

**Proposition 2.30.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be smooth exhaustive stacks and suppose that \( \mathcal{Y} \to \mathcal{X} \) is a \( \mathbb{G}_m \)-torsor. Let \( \mathcal{L} := [\mathcal{Y} \times \mathbb{A}^1/\mathbb{G}_m] \) be the associated line bundle over \( \mathcal{X} \). Then there is a distinguished triangle

\[
(4) \quad M(\mathcal{Y}) \to M(\mathcal{X}) \to M(\mathcal{X})\{1\} \to
\]

where the morphism \( M(\mathcal{Y}) \to M(\mathcal{X})\{1\} \) is the following composition

\[
\varphi_L : M(\mathcal{X}) \xrightarrow{M(\Delta)} M(\mathcal{X}) \otimes M(\mathcal{X}) \xrightarrow{id \otimes c_1(\mathcal{L})} M(\mathcal{X})\{1\}.
\]

**Proof.** Let us first verify the statement for schemes. Let \( X \) be a scheme and \( Y \to X \) be a \( \mathbb{G}_m \)-torsor. If we let \( L = Y \times_{\mathbb{G}_m} \mathbb{A}^1 \) denote the associated line bundle; then \( Y = L - X \). By \( \mathbb{A}^1 \)-homotopy invariance, we have \( M(L) \cong M(X) \) and \( M^n(L) \cong M^n(X)\{1\} \). If \( X \) is smooth, then so is \( L \) and we can consider the Gysin triangle associated to the closed immersion \( X \to L \) as the zero section:

\[
M(Y) \to M(X) \simeq M(X) \to M(X)\{1\} \to.
\]

Then the map \( M(X) \to M(X)\{1\} \) is given by the first Chern class of \( L \) by [16, Example 1.25].

Now suppose we are working with smooth exhaustive stacks. We note that the morphism \( M(\Delta) \) is defined in Remark 2.26. Let \( (V_\bullet, W_\bullet) \) be an exhaustive sequence on \( \mathcal{X} \) with respect to a filtration \( \mathcal{X} = \bigcup_n \mathcal{X}_n \) and let \( U_n := V_n - W_n \) as usual. Then we let \( L_n \) denote the pullback of the line bundle \( \mathcal{L} \to \mathcal{X} \) to \( U_n \) and let \( Y_n \subset L_n \) denote the complement to the zero section. Since \( U_n \) are schemes, we have for all \( n \), distinguished triangles

\[
(5) \quad M(Y_n) \to M(U_n) \to M(U_n)\{1\} \to.
\]

Since the fibre product \( U_n \cong L_n \times_{L_{n+1}} U_{n+1} \) is transverse, the Gysin morphisms for \( (U_n, L_n) \) and \( (U_{n+1}, L_{n+1}) \) are compatible, and so by taking the homotopy colimit of the distinguished triangles \( \square \), we obtain a distinguished triangle of the form \( \square \) and one can check that the morphism \( M(\mathcal{X}) \to M(\mathcal{X})\{1\} \) is \( \varphi_L \) by definition of \( M(\Delta) \) and the first Chern class. \( \square \)

**Example 2.31.** Let us consider the universal \( \mathbb{G}_m \)-torsor \( \text{Spec} k \to B\mathbb{G}_m \); then the associated line bundle is \( \mathcal{L} = [\mathbb{A}^1/\mathbb{G}_m] \to B\mathbb{G}_m \). We recall that in Example 2.20 we used the exhaustive sequence \( (V_n = [\mathbb{A}^n/\mathbb{G}_m], W_n = [\{0\}/\mathbb{G}_m]) \) for the trivial filtration on \( B\mathbb{G}_m \) to show that

\[
M(B\mathbb{G}_m) = \text{hocolim}_n M(\mathbb{P}^{n-1}) \simeq \bigoplus_{n \geq 0} R\{n\}.
\]
Using the notation of the proof above, the line bundle $L_n$ on $U_n = \mathbb{P}^{n-1}$ is the tautological line bundle $O_{\mathbb{P}^n}(-1)$, whose first Chern class $c_1(L_n) : M(\mathbb{P}^{n-1}) \simeq \bigoplus_{j=0}^{n-1} R\{j\} \rightarrow R\{1\}$ is just the projection onto this direct factor. It follows from this that the morphism
\[
\varphi_L : M(BG_m) \simeq \bigoplus_{n \geq 0} R\{n\} \rightarrow M(BG_m)\{1\} \simeq \bigoplus_{n \geq 1} R\{n\}
\]
is the natural projection.

2.7. Compactly supported motives and Poincaré duality. One can define the compactly supported motive of an exhaustive stack as follows.

**Definition 2.32.** Let $\mathfrak{X}$ be an exhaustive algebraic stack. For an exhaustive sequence $(V_\bullet, W_\bullet)$ of vector bundles on $\mathfrak{X}$ with respect to a filtration $\mathfrak{X} = \bigcup_m \mathfrak{X}_m$, we define the compactly supported motive of $\mathfrak{X}$ by
\[
M^c(\mathfrak{X}) := \text{holim} M(U_m)\{-\text{rk}(V_m)\}
\]
with transition maps given by the composition
\[
M^c(U_{m+1})\{-\text{rk}(V_{m+1})\} \rightarrow M^c(U_{m+1} \times \mathfrak{X}_{m+1} \mathfrak{X}_m)\{-\text{rk}(V_{m+1})\} \rightarrow M^c(U_m)\{-\text{rk}(V_m)\}
\]
where the first map is the flat pullback for the open immersion $U_{m+1} \times \mathfrak{X}_{m+1} \mathfrak{X}_m \rightarrow U_m$ and the second map is the localisation map for the immersion $f_m : U_m \hookrightarrow U_{m+1} \times \mathfrak{X}_{m+1} \mathfrak{X}_m$ of relative dimension $\text{rk}(V_{m+1}) - \text{rk}(V_m)$.

**Remark 2.33.** One can prove that the definition of the compactly supported motive of an exhaustive stack is independent of the choice of filtration and exhaustive sequence similarly to Lemma 2.19 (see also [40, Theorem 8.5]). Furthermore, unlike in the motive case, we do not have to assume that $\mathfrak{X}$ is smooth (as the argument uses Proposition 2.11 instead of Lemma 2.14).

We can show one part of the statement of Poincaré duality for exhaustive smooth stacks follows from Poincaré duality for schemes.

**Proposition 2.34** (Poincaré duality for exhaustive stacks). Let $\mathfrak{X}$ be a smooth exhaustive stack of dimension $d$; then there is an isomorphism
\[
M(\mathfrak{X})^\vee \simeq M^c(\mathfrak{X})\{-d\}
\]
in $\text{DM}(k, R)$.

**Proof.** This follows directly from the definitions of $M(\mathfrak{X})$ and $M^c(\mathfrak{X})$ and Poincaré duality for schemes, as the dual of a homotopy colimit is a homotopy limit. \(\square\)

Note that, because the dual of an infinite product is not in general an infinite sum, it is not clear in general that the duality works the other way.

3. Motivic Białynicki-Birula decompositions

3.1. Geometric Białynicki-Birula decompositions. Let $X$ be a smooth projective $k$-variety equipped with a $\mathbb{G}_m$-action. By a result of Białynicki-Birula [7], there exists a decomposition of $X$, indexed by the connected components of the fixed locus $X^{\mathbb{G}_m}$, with very good geometric properties. In fact, this decomposition exists in the following slightly more general context.

**Definition 3.1.** A $\mathbb{G}_m$-action on a smooth quasi-projective $k$-variety $X$ is semi-projective if

- $X^{\mathbb{G}_m}$ is proper (and thus projective), and
- for every point $x \in X$ (not necessarily closed), the action map $f_x : \mathbb{G}_m \rightarrow X$ given by $t \mapsto t \cdot x$ extends to a map $\tilde{f}_x : \mathbb{A}^1 \rightarrow X$. Since $X$ is separated, the extension is unique and we write $\lim_{t \rightarrow 0} t \cdot x$ for the limit point $\tilde{f}_x(0) \in X$.

In particular, any $\mathbb{G}_m$-action on a smooth projective variety is semi-projective. Note that the limit point $\lim_{t \rightarrow 0} t \cdot x$ is necessarily a fixed point of the $\mathbb{G}_m$-action if it exists.
Theorem 3.2 (Białynicki-Birula). Let $X$ be a smooth quasi-projective variety over $k$ with a semi-projective $\mathbb{G}_m$-action. Then the following statements hold.

(i) The fixed locus $X^{\mathbb{G}_m}$ is smooth and projective. Write $\{X_i\}_{i \in I}$ for its set of connected components and $d_i$ for the dimension of $X_i$.

(ii) For $i \in I$, write $X_i^+$ for the attracting set of $X_i$, i.e., the set of all points $x \in X$ such that $\lim_{t \to 0} t \cdot x \in X_i$. Then $X_i^+$ is a locally closed subset of $X$ and $X = \bigsqcup_{i \in I} X_i^+$.

(iii) For every $i \in I$, the map of sets $X_i^+ \to X_i$ given by $x \mapsto \lim_{t \to 0} t \cdot x$ is a morphism of schemes $p_i^+: X_i^+ \to X_i$, which is a Zariski locally trivial fibration in affine spaces. For each $i \in I$, we have
\[ \dim(X) = d_i + c_i^+ + r_i^+ \]
where $c_i^+ = \text{codim}_X(X_i^+)$ and $r_i^+$ is the rank of $p_i^+$.

(iv) The tangent space $T_x X$ of a fixed point $x \in X_i$ admits a $\mathbb{G}_m$-action, hence a weight space decomposition $T_x X = \bigoplus_{k \in \mathbb{Z}} (T_x X)_k$. Then we have $T_x X_i = (T_x X)_0$ and $N_{X_i/X_i^+} \simeq \bigoplus_{k>0} (T_x X)_k$ and $(N_{X_i/X_i^+})_{X_i} \simeq \bigoplus_{k<0} (T_x X)_k$.

(v) Let $n := |I|$; then there is a bijection $\varphi : \{1, \ldots, n\} \to I$ and a filtration of $X$ by closed subschemes
\[ \emptyset = Z_n \subset Z_{n-1} \subset \ldots \subset Z_0 = X \]
such that, for all $1 \leq k \leq n$, we have that $Z_{k-1} - Z_k = X_{\varphi(k)}^+$ is a single attracting set (and thus, in particular, is smooth).

Proof. Points (i)-(iv) are all established in [7, Theorem 4.1] under the assumption that $k$ is algebraically closed (the hypothesis that $X$ is smooth and quasi-projective is used to ensure the existence of an open covering by $\mathbb{G}_m$-invariant affine subsets). The hypothesis that $k$ is algebraically closed was then removed by Hasselink in [23].

As the proof of (v) is scattered through [21], we recapitulate their argument. Let $L$ be a very ample line bundle on the quasi-projective variety $X$. By [26, Theorem 1.6] applied to the smooth (hence normal) variety $X$, there exists an integer $n \geq 1$ such that $L^\otimes n$ admits a $\mathbb{G}_m$-linearisation. In particular, this provides a projective space $\mathbb{P}$ with a linear $\mathbb{G}_m$-action and a $\mathbb{G}_m$-equivariant immersion $\iota : X \to \mathbb{P}$. Let $\{P_j\}_{j \in J}$ be the connected components of $\mathbb{P}^{\mathbb{G}_m}$ with corresponding attracting sets $P_j^+$ for each $j \in J$; then by equivariance of $\iota$, there is a (not necessarily injective) map $\tau : I \to J$ such that $\iota(X_i^+) \subset P_{\tau(i)}^+$ for all $i \in I$.

As each $X_i$ is connected, the group $\mathbb{G}_m$ acts on $L|_{X_i}$ via a character $\omega_i \in \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$. For the partial order on $I$ given by $i < i' \iff \omega_i > \omega_{i'}$, we claim that for $i \neq i' \in I$
\[ X_i^+ \cap X_i'^+ \neq \emptyset \text{ only if } i' > i. \]
Indeed, we can similarly define a partial order on $J$ such that $i < i'$ if and only if $\tau(i) < \tau(i')$ by equivariance of $\iota$; then one can easily deduce that (8) holds for $\mathbb{P}$ from the linearity of the $\mathbb{G}_m$-action on $\mathbb{P}$. We now deduce (8) for $X$ from the corresponding ambient property for $\mathbb{P}$, the only non-trivial case to consider is when $i \neq i'$ have the same image $j$ under $\tau$, so that $X_i^+$ and $X_i'^+$ are both contained in $P_j^+$. In this case, if $x \in X_i^+ \cap X_i'^+$, then by passing to an algebraic closure of $k$ if necessary, we can assume that there is a connected curve $S \subset X$ with $x \in S$ and $S - x \subset X_i^+$; then $S \subset P_j^+$ and as the action on $X$ is semi-projective, $p_j^+(S) \subset X^{\mathbb{G}_m}$ and this connects $X_i$ and $X_i'$, contradicting $i \neq i'$.

Finally to prove the filterability of $X$, we choose any total ordering of $I$ extending the above partial order and we view this ordering as a bijection $\varphi : \{1, \ldots, n\} \to I$. Then for $0 \leq k \leq n$,
\[ Z_k := \bigcup_{i \in I : \varphi^{-1}(i) > k} X_i^+ \]
is closed in $X$ by (8) with $Z_n = \emptyset$ and $Z_0 = X$. 

Remark 3.3. Let $X$ be smooth projective with a fixed $\mathbb{G}_m$-action $(t, x) \mapsto t \cdot x$. The opposite $\mathbb{G}_m$-action $(t, x) \mapsto t^{-1} \cdot x$ has the same fixed point locus as the original action, but the associated
Białynicki-Birula decomposition is different. We write $X_i^-$ for the associated strata, $c_i^-$ (resp. $r_i^-$) for their codimension (resp. their rank as affine bundles), etc. By Theorem 3.2(iv) we see that $c_i^- = r_i^+$ and $r_i^- = c_i^+$, and that the strata $X_i^+$ and $X_i^-$ intersect transversally along $X_i$.

3.2. Motivic consequences. Let $X$ be a smooth quasi-projective variety with a semi-projective $\mathbb{G}_m$-action. The geometry exhibited in the previous sections implies a decomposition of the motive of $X$. There are in fact two natural such decompositions, one for the motive $M(X)$ and one for the motive with compact support $M^c(X)$. These motivic decompositions have been studied in [10, 12, 22]; we explain and expand upon their results in this section. Recall that for two smooth $k$-schemes $X$ and $Y$ with $X$ of dimension $d$ and an integer $i \in \mathbb{N}$, there is an isomorphism

$$\text{CH}_i(X \times Y)_R \simeq \text{Hom}_{DM}(M(X), M^c(Y)\{d-i\})$$

with $\text{CH}_i$ the Chow groups of cycles of dimension $i$; when this does not lead to confusion, we use the same notation for a cycle and the corresponding map of motives.

Theorem 3.4. Let $X$ be a smooth quasi-projective variety with a semi-projective $\mathbb{G}_m$-action. With the notation of Theorem 3.2, for each $i \in I$, we let $\gamma_i^+$ be the class of the algebraic cycle given by the closure $\Gamma_{p_i}$ of the graph of $p_i^+ : X_i^+ \to X_i$ in $X \times X_i$, and we let $(\gamma_i^+)^!$ be the class of the transposition of this graph closure. Then we have the following motivic decompositions (where we use without comment that $M(X_i) \simeq M^c(X_i)$ as $X_i$ is projective).

(i) (Białynicki-Birula decomposition for the motive): There is an isomorphism

$$M(X) \simeq \bigoplus_{i \in I} M(X_i)\{c_i^+\}.$$

induced by the morphisms $M(X) \to M^c(X_i) \simeq M(X_i)\{c_i^+\}$ given by the classes $\gamma_i^+$ for each $i \in I$.

(ii) (Białynicki-Birula decomposition for the compactly supported motive): There is an isomorphism

$$\bigoplus_{i \in I} M(X_i)\{r_i^+\} \simeq M^c(X).$$

induced by the morphisms $M(X_i)\{r_i^+\} \to M^c(X)$ given by the classes $(\gamma_i^+)^!$ for each $i \in I$.

(iii) The Poincaré duality isomorphism

$$M^c(X) \simeq M(X)^!\{d\}$$

identifies the motivic Białynicki-Birula decomposition of $M^c(X)$ from (ii) with the dual of the motivic Białynicki-Birula decomposition of $M(X)$ from (i). In other words, for every $(i, j) \in I^2$, the composite map

$$M(X_i)\{r_i^+\} \xrightarrow{(\gamma_i^+)^!} M^c(X) \xrightarrow{M(X_i)^!} M(X_j)^!\{d - c_j^+\}$$

is $0$ if $i \neq j$ and is a twist of the Poincaré duality isomorphism for the motive of the smooth projective variety $X_i$ if $i = j$ (noting the equality $d - c_j^+ - r_i^+ = d_i$).

Proof. We first prove (i). By Theorem 3.2 there is a filtration $\emptyset = Z_n \subset Z_{n-1} \subset \ldots \subset Z_0 = X$ by closed subvarieties such that, for all $1 \leq k \leq n$, we have that $Z_k - Z_{k-1} = X_k^+$ is an attracting cell. Let $U_k := X - Z_k$, which is an open subset and so in particular is smooth. For $1 \leq i \leq k$, let us write $\gamma_{i,k}$ for the closure of $\Gamma_{p_i}$ in $U_k \times X_i$ (this makes sense since $X_i^+ \subset U_i \subset U_k$) so that $\gamma_i^+ = \gamma_{i,n}^+$. We will prove, by induction on $1 \leq k \leq n$, that the map

$$\bigoplus_{i=1}^k \gamma_{i,k}^+ : M(U_k) \to \bigoplus_{1 \leq i \leq k} M(X_i)\{c_i^+\}$$

is an isomorphism. For $k = 1$, the statement holds trivially as $U_0 = \emptyset$ and so $M(U_1) = M(X_1^+) \simeq M(X_1)$ via $p_1^+$. Assume that the statement is true for $k - 1$. We have a closed
immersion \( i_k : X_k^+ \to U_k \) between smooth schemes with codimension \( c_k^+ \) and open complement \( U_{k-1} \); hence, there is a Gysin triangle

\[
M(U_{k-1}) \to M(U_k) \xrightarrow{\text{Gy}(i_k)} M(X_k^+ \{c_k^+ \}) \xrightarrow{\tau} 
\]

for \( 1 \leq k \leq n \). Since \( (\gamma_{i,k}^+)\mid_{U_{k-1} \times X_k} = \gamma_{i,k-1}^+ \), the following diagram commutes

\[
\begin{array}{ccc}
M(U_{k-1}) & \xrightarrow{\phi_{i=1}^{k-1} \gamma_{i,k-1}^+} & M(U_k) \\
\bigoplus_{i=1}^{k-1} M(X_i) & \xrightarrow{\phi_{i=1}^{k-1} \gamma_{i,k-1}^+} & \bigoplus_{i=1}^{k-1} M(X_i) \\
\end{array}
\]

where the left vertical map is an isomorphism by induction. This shows that the triangle splits.

As \( p_k^+ : X_k^+ \to X_k \) is a Zariski locally trivial fibration of affine spaces, \( M(p_k^+ \circ \text{Gy}(i_k) : M(U_k) \to M(X_k)\{c_k^+ \}) \) coincides with the map \( M(U_k) \to M(X_k)\{c_k^+ \} \) induced by \( \gamma_{k,k}^+ \). Let us write \( \gamma_{0,k}^+ \) for the graph of \( p_k^+ \) considered as a subscheme of \( X_k^+ \times X_k \).

Let us recall the functoriality of the Poincaré duality isomorphism with respect to algebraic cycles. Let \( Y_1, Y_2 \) be smooth projective varieties of dimensions \( d_1 \) and \( d_2 \), and \( \gamma \in \text{CH}^r(Y_1 \times Y_2) \), which induces morphisms \( \gamma : M(Y_1) \to M(Y_2)\{c-d_2 \} \) and \( \gamma^t : M(Y_2) \to M(Y_1)\{c-d_1 \} \). Then the following diagram is commutative

\[
\begin{array}{ccc}
M(Y_1) & \xrightarrow{\gamma} & M(Y_2)\{c-d_2 \} \\
\downarrow & & \downarrow \\
M(Y_1)^{\vee}\{d_1 \} & \xrightarrow{(\gamma^t)^{\vee}\{c \}} & M(Y_2)^{\vee}\{c \} \\
\end{array}
\]

From this commutativity, it suffices to show that \( \gamma_{0,k}^+ \circ \text{pr}_1^* \text{Gy}(i_k) : M(U_k \times X_k) \to R(c_k^+ + d_k) \) coincides with the map \( \gamma_{k,k}^+ : M(U_k \times X_k) \to R(c_k^+ + d_k) \). Let us denote by \( a_k : X_k^+ \times_k X_k \) and \( b_k : U_k \times X_k \) the closed immersions, so that \( b_k = (i_k \times X_k) \circ a_k \). Consider the diagram

\[
\begin{array}{ccc}
M(U_k \times X_k) & \xrightarrow{\gamma_{k,k}^+} & R(c_k^+ + d_k) \\
\downarrow & & \downarrow \\
\text{Gy}(U_k \times X_k) & \xrightarrow{\text{Gy}(i_k \times X_k)} & M(X_k^+ \times X_k)\{c_k^+ \} \\
\downarrow & & \downarrow \\
\text{Gy}(a_k)^*\{c_k^+ \} & \xrightarrow{\text{Gy}(a_k)\{c_k^+ \}} & M(\gamma_{0,k}^+ \to \text{Spec}(k)) \xrightarrow{R(c_k^+ + d_k)} R(c_k^+ + d_k) \\
\end{array}
\]

in \( \text{DM}(k, R) \). The left triangle commutes because of the general behaviour of Gysin maps with respect to composition \cite{16} Theorem 1.34]. The outer square and the bottom quadrilateral commute because of the compatibility of Gysin maps with fundamental classes of cycles of smooth subvarieties in motivic cohomology \cite{15} Lemma 3.3]. This implies that the top triangle commutes. Since \( \text{pr}_1 \) is a smooth morphism, we have \( \text{pr}_1^* \text{Gy}(i_k) = \text{Gy}(i_k \times X_k) \) by \cite{16} Proposition 1.19 (1)] and the commutation of the top triangle is exactly the equality we want. This concludes the proof of (i).

Statements (ii) and (iii) are deduced from (i) by applying Poincaré duality and using the functoriality of the Poincaré duality isomorphism with respect to algebraic cycles recalled above in the proof of (i). □

In the smooth projective case, one has \( M^e(X) \simeq M(X) \) and one would like compare this decomposition with the decomposition obtained for the opposite \( \mathbb{G}_m \)-action.
Question 3.5. Let $X$ be a smooth projective variety with a $\mathbb{G}_m$-action. Then, via the isomorphism $M^c(X) \simeq M(X)$, do the motivic Białynicki-Birula decompositions of $M(X)$ in Theorem 3.3(i) and of $M^c(X)$ in Theorem 3.3(ii) for the opposite $\mathbb{G}_m$-action coincide? In other words, for every $(i, j) \in I^2$, is the composition

$$M(X_i)(r_i)[j_i] \xrightarrow{(\gamma_i)^j} M^c(X) \xrightarrow{\gamma_j} M(X_j)(c_j^+)$$

zero if $i \neq j$ and the identity if $i = j$ (noting the equality $r_i^+ = c_i^+$)?

4. THE MOTIVE OF THE STACK OF VECTOR BUNDLES

Throughout this section, we let $C$ be a smooth projective geometrically connected curve of genus $g$ over a field $k$. We fix $n \in \mathbb{N}$ and $d \in \mathbb{Z}$ and let $\text{Bun}_{n,d}$ denote the stack of vector bundles over $C$ of rank $n$ and degree $d$; this is a smooth stack of dimension $n^2(g-1)$. In this section, we give a formula for the motive of $\text{Bun}_{n,d}$ in $\text{DM}(k, R)$, by adapting a method of Bifet, Ghione and Letizia [8] to study the cohomology of $\text{Bun}_{n,d}$ using matrix divisors. This argument was also used by Behrend and Dhillon [6] to give a formula for the virtual motivic class of $\text{Bun}_{n,d}$ in (a completion of) the Grothendieck ring of varieties (see [47,2]).

We can define the motive of $\text{Bun}_{n,d}$, as it is exhaustive: to explain this, we filter $\text{Bun}_{n,d}$ using the maximal slope of all vector subbundles.

**Definition 4.1.** The slope of a vector bundle $E$ over $C$ is $\mu(E) := \frac{\deg(E)}{\text{rk}(E)}$. We define

$$\mu_{\text{max}}(E) = \max\{\mu(E') : 0 \neq E' \subset E\}.$$

The maximal slope $\mu_{\text{max}}(E)$ is equal to the slope of the first vector bundle appearing in the Harder–Narasimhan filtration of $E$. For $\mu \in \mathbb{Q}$, we let $\text{Bun}_{n,d}^{\leq \mu}$ denote the substack of $\text{Bun}_{n,d}$ consisting of vector bundles $E$ with $\mu_{\text{max}}(E) \leq \mu$; this substack is open by upper semi-continuity of the Harder–Narasimhan type [34]. Any sequence $(\mu_i)_{i \in \mathbb{N}}$ of increasing rational numbers tending to infinity defines a filtration of $(\text{Bun}_{n,d}^{\leq \mu_i})_{i \in \mathbb{N}}$ of $\text{Bun}_{n,d}$. The stacks $\text{Bun}_{n,d}^{\leq \mu}$ are all quasi-compact, as they are quotient stacks. Indeed, all vector bundles over $C$ of rank $n$ and degree $d$ with maximal slope less than or equal to $\mu$ form a bounded family (cf. [26, Theorem 3.3.7]), and so can be parametrised by an open subscheme $Q^{\leq \mu}$ of a Quot scheme, and then $\text{Bun}_{n,d}^{\leq \mu} \simeq [Q^{\leq \mu}/\text{GL}_n]$ (for further details, see for example [29, Théorème 4.6.2.1]).

We will construct an exhaustive sequence of vector bundles on $\text{Bun}_{n,d}$ by using matrix divisors.

4.1. Matrix divisors. A matrix divisor (after Weil) of rank $n$ and degree $d$ on $C$ is a locally free subsheaf of $\mathcal{K}^\oplus n$ of rank $n$ and degree $d$, where $\mathcal{K}$ denotes the constant $\mathcal{O}_C$-module equal to the function field of $C$.

Let $D$ be an effective divisor on $C$ and let $\text{Div}_{n,d}(D)$ denote the scheme parametrising subsheaves of $\mathcal{O}_C(D)^{\oplus n}$ of rank $n$ and degree $d$. Equivalently,

$$\text{Div}_{n,d}(D) := \text{Quot}_{\mathcal{O}_C}^{n \deg D - d}(\mathcal{O}_C(D)^{\oplus n}),$$

can be thought of as the Quot scheme parametrising torsion quotient sheaves of $\mathcal{O}_C(D)^{\oplus n}$ of degree $n \deg D - d$. The Quot scheme $\text{Div}_{n,d}(D)$ is a projective variety, and it is also smooth, as it parametrises torsion quotient sheaves on a curve (for example, see [26 Proposition 2.2.8]); moreover, it has dimension $n^2 \deg D - nd$ by the Riemann-Roch formula.

For effective divisors $D' \geq D \geq 0$ on $C$, there is a natural closed immersion

$$i_{D,D'} : \text{Div}_{n,d}(D) \to \text{Div}_{n,d}(D'),$$

compatible with the forgetful morphisms to $\text{Bun}_{n,d}$, and so we can construct an ind-variety $\text{Div}_{n,d} := (\text{Div}_{n,d}(D))_D$ of matrix divisors of rank $n$ and degree $d$ on $C$.

We will use the forgetful map $\text{Div}_{n,d} \to \text{Bun}_{n,d}$ to study the motive of $\text{Bun}_{n,d}$ in terms of that of $\text{Div}_{n,d}$ and, in particular, to define an exhaustive sequence of vector bundles on $\text{Bun}_{n,d}$.
Proposition 4.2. Let $D$ be a non-zero effective divisor on $C$ and let $\mu_l := \deg(D) - 2g + 1 - \frac{1}{l}$, $l \in \mathbb{Q}$ for $l \in \mathbb{N}$; then there is an exhaustive sequence of vector bundles $(V_l \to \text{Bun}^{\leq \mu_l}_{n,d})_{l \in \mathbb{N}}$. Furthermore, we have

$$M(\text{Bun}_{n,d}) \simeq \colim_l R_{\text{tr}}(\text{Div}^{\leq \mu_l}_{n,d}(ID))$$

where $\text{Div}^{\leq \mu_l}_{n,d}(ID)$ is the open subvariety of $\text{Div}_{n,d}(ID)$ consisting of rank $n$ degree $d$ matrix divisors $E \to \mathcal{O}_C(ID)^{\oplus n}$ with $\mu_{\text{max}}(E) \leq \mu_l$.

**Proof.** For all vector bundles $E$ with $\mu_{\text{max}}(E) \leq \mu_l$, as $\mu_{\text{max}}(E) < \deg(ID) - 2g + 2$, we have

$$H^1(E^\vee \otimes \mathcal{O}_C(ID)^{\oplus n}) = 0.$$

Indeed if this vector space was non-zero then by Serre duality, there would exist a non-zero homomorphism $\mathcal{O}_C(ID)^{\oplus n} \otimes \omega_C^{-1} \to E$, but one can check that this is not possible by using the Harder-Narasimhan filtration and standard results about homomorphisms between semistable bundles of prescribed slopes (see [20 Proposition 1.2.7] and also [8 §8.1]). Hence, there is a vector bundle $V_l := R^0p_{1*}(\mathcal{E}^\vee_{\text{uni}} \otimes p_2^*\mathcal{O}_C(ID)^{\oplus n})$ over $\text{Bun}^{\leq \mu_l}_{n,d}$, whose fibre over $E$ is the space $\text{Hom}(E, \mathcal{O}_C(ID)^{\oplus n})$. Let $U_l \subset V_l$ denote the open subset of injective homomorphisms; then we have

$$U_l \cong \text{Div}^{\leq \mu_l}_{n,d}(ID),$$

and the closed complement $W_l$ of non-injective homomorphisms has codimension greater than or equal to $l \deg(D) - K$ for a constant $K$ independent of $l$ and $D$ by Lemma 4.3 below, and so $\text{codim}(W_l, V_l) \to \infty$ as $l \to \infty$. Let

$$i_l : \text{Bun}^{\leq \mu_l}_{n,d} \hookrightarrow \text{Bun}^{\leq \mu_{l+1}}_{n,d}$$

denote the open immersion; then the injective sheaf homomorphism $\mathcal{O}_C(ID) \to \mathcal{O}_C((l+1)D)$ determines an injective homomorphism $i_l : V_l \to i_l^*V_{l+1}$. Moreover $f_l^{-1}(W_{l+1}) \subset W_l$, as if $E \to \mathcal{O}_C(ID)^{\oplus n} \to \mathcal{O}_C((l+1)D)^{\oplus n}$ is not injective, then the homomorphism $E \to \mathcal{O}_C(ID)^{\oplus n}$ is not injective. In particular, $(V_l, W_l)_{l \in \mathbb{N}}$ is an exhaustive sequence of vector bundles over $\text{Bun}_{n,d} = \cup_{l \in \mathbb{N}} \text{Bun}^{\leq \mu_l}_{n,d}$ and so we have

$$M(\text{Bun}_{n,d}) = \colim_l R_{\text{tr}}(U_l) \simeq \colim_l R_{\text{tr}}(\text{Div}^{\leq \mu_l}_{n,d}(ID))$$

as required. \hfill $\square$

It remains to prove Lemma 4.3 which is a slight modification of [8 Lemma 8.2] with weaker assumptions. In fact, in the proof of [8 Proposition 8.1], the stronger assumptions of [8 Lemma 8.2] are not satisfied when this result is applied; however, one can instead apply the following lemma.

**Lemma 4.3.** Let $E$ and $F$ be locally free $\mathcal{O}_C$-modules of rank $n$. There exists $K \in \mathbb{N}$ such that, for any effective divisor $D$ on $C$, the codimension $c_D$ of the closed subset of non-injective sheaf homomorphisms in $\text{Hom}(E, F(D))$ satisfies $c_D \geq \deg(D) - K$.

**Proof.** From the exact sequence of $k$-vector spaces

$$0 \to \text{Hom}(E, F) \to \text{Hom}(E, F(D)) \xrightarrow{\varphi} \text{Hom}(E, \mathcal{O}_D^{\oplus n}) \to \text{Ext}^1(E, F)$$

we deduce

$$\dim \text{Hom}(E, \mathcal{O}_D^{\oplus n}) \leq \dim \text{Hom}(E, F(D)) + \dim \text{Ext}^1(E, F).$$

Let $\text{Hom}_{\text{ni}}(E, F(D)) \subset \text{Hom}(E, F(D))$ be the closed subset of non-injective homomorphisms, and $\text{Hom}_{<n}(E, \mathcal{O}_D^{\oplus n}) \subset \text{Hom}(E, \mathcal{O}_D^{\oplus n})$ be the closed subset of homomorphisms which factor through a surjection $E \to G \subset \mathcal{O}_D^{\oplus n}$ such that for every $y \in \text{supp}(D)$, the rank of $G$ at $y$ is strictly less than $n$. As in the proof of [8 Lemma 8.2], we have that the codimension $e_D$ of $\text{Hom}_{<n}(E, \mathcal{O}_D^{\oplus n})$ in $\text{Hom}(E, \mathcal{O}_D^{\oplus n})$ satisfies

$$e_D \geq \deg(D)$$

(8)
and \( \Phi(\text{Hom}_{\text{fin}}(E, F(D))) \subset \text{Hom}_{<n}(E, \mathcal{O}_D^{\boxplus n}) \). Hence,

\[
(9) \quad \dim \text{Hom}_{\text{fin}}(E, F(D)) \leq \dim \text{Hom}_{<n}(E, \mathcal{O}_D^{\boxplus n}) + \dim \text{Hom}(E, F).
\]

For a smooth irreducible variety \( X \) and a closed subset \( Y \), recall that we have \( \text{codim}_X(Y) = \dim(X) - \dim(Y) \). Therefore

\[
c_D \geq \dim \text{Hom}(E, F(D)) - \dim \text{Hom}_{<r}(E, \mathcal{O}_D^{\boxplus n}) - \dim \text{Hom}(E, F) \geq e_D - \dim \text{Hom}(E, F) - \dim \text{Ext}^1(E, F) \geq \deg(D) - \dim \text{Hom}(E, F) - \dim \text{Ext}^1(E, F).
\]

where the first inequality follows from (9), the second inequality follows from (7) and the final inequality follows from (8). This concludes the proof. \( \square \)

**Theorem 4.4.** In \( \text{DM}(k, R) \), for any non-zero effective divisor \( D \) on \( C \), we have

\[
M(\text{Bun}_{n,d}) \simeq \colim_i R_\text{tr}(\text{Div}_{n,d}(lD)) \simeq \hocolim M(\text{Div}_{n,d}(lD)).
\]

**Proof.** The right isomorphism follows from Lemma 2.5 By Proposition 4.2 we have

\[
M(\text{Bun}_{n,d}) \simeq \colim_i R_\text{tr}(\text{Div}_{n,d}(lD)).
\]

We then obtain the left isomorphism by applying Proposition 2.13 to the inductive system of open immersions \( (\text{Div}_{n,d}^{\leq \mu_l}(lD) \hookrightarrow \text{Div}_{n,d}(lD)) \in \mathbb{N} \). To apply this corollary, we need to check that the closed complements \( \text{Div}_{n,d}(lD) - \text{Div}_{n,d}^{\leq \mu_l}(lD) \) have codimensions tending to infinity with \( l \).

Let \( E \) be a vector bundle with \( \mu_{\max}(E) > \mu_l \); then \( \mu_{\max}(E) \geq \mu_l + \frac{1}{m} \) since slopes are rational numbers with denominator at most \( n \). Thus \( \mu_{\max}(E) > 2g - 1 \geq \mu_l + \frac{1}{m} + 2g - 1 = \deg(lD) \), and so by [8, Proposition 5.2 (4)], we have

\[
\text{codim}(\text{Div}_{n,d}(lD) - \text{Div}_{n,d}^{\leq \mu_l}(lD)) \geq l \deg D - c
\]

for a constant \( c \) independent of \( l \), which completes the proof. \( \square \)

We recall that a motive is pure if it lies in the heart of Bondarko’s Chow weight structure on \( \text{DM}(k, R) \) defined in [9]. In particular, the motive of any smooth projective variety is pure and, as \( M(\text{Bun}_{n,d}) \) is described as a homotopy colimit of motives of smooth projective varieties, we deduce the following result.

**Corollary 4.5.** The motive \( M(\text{Bun}_{n,d}) \) is pure.

This corollary sits well with the fact that the cohomology of \( \text{Bun}_{n,d} \), and more generally the cohomology of moduli stacks of principal bundles on curves, is known to be pure in various contexts; for instance, if \( k = \mathbb{C} \), the Hodge structure is pure by [39, Proposition 4.4], and over a finite field, the \( \ell \)-adic cohomology is pure by [24, Corollary 3.3.2].

**4.2. The Białynicki-Birula decomposition for matrix divisors.** We recall that the Quot scheme \( \text{Div}_{n,d}(D) \) is a smooth projective variety of dimension \( n^2 \deg D - nd \). The group \( \text{GL}_n \) acts on \( \text{Div}_{n,d}(D) \) by automorphisms of \( \mathcal{O}_C(D)^{\boxplus n} \). If we fix a generic 1-parameter subgroup \( \mathbb{G}_m \subset \text{GL}_n \) of the diagonal maximal torus \( T = \mathbb{G}_m^n \), then the fixed points of this \( \mathbb{G}_m \)-action agree with the fixed points for the \( T \)-action. These actions and their fixed points were studied by Strømme [35]; the fixed points are matrix divisors of the form

\[
\bigoplus_{i=1}^n \mathcal{O}_C(D - F_i) \hookrightarrow \mathcal{O}_C(D)^{\boxplus n}
\]

for effective divisors \( F_i \) such that \( \sum_{i=1}^{n} \deg F_i = n \deg D - d \). By specifying the degree \( m_i \) of each \( F_i \) we index the connected components of this torus fixed locus; more precisely, the components indexed by a partition \( m = (m_1, \ldots, m_n) \) of \( n \deg D - d \) is the following product of symmetric powers of \( C \)

\[
C^m := C^{(m_1)} \times \cdots \times C^{(m_n)}.
\]
Strømme also studied the associated Białynicki-Birula decomposition (for $C = \mathbb{P}^1$) and this was later used by Bifet, Ghione and Letizia \cite{[8]} (for $C$ of arbitrary genus) to study the cohomology of moduli spaces of vector bundles. Using the same ideas, del Baño showed that the Chow motive of $\text{Div}_{n,d}(D)$ (with $\mathbb{Q}$-coefficients) is the $(n \deg D - d)$-th symmetric power of the motive of $C \times \mathbb{P}^{n-1}$; see \cite{[17]} Theorem 4.2.

In order to consider such a Białynicki-Birula decomposition, let us fix $G_m \hookrightarrow \text{GL}_n$ of the form $t \mapsto \text{diag}(t^{w_1}, \ldots, t^{w_n})$ with decreasing integral weights $w_1 > \cdots > w_n$. The action of $t \in G_m$ on $\text{Div}_{n,d}(D)$ is given by precomposition with the corresponding automorphism of $O_C(D)^{\otimes n}$. The Białynicki-Birula decomposition for this $G_m$-action gives a stratification of

$$\text{Div}_{n,d}(D) = \bigsqcup_{m \vdash n \deg D - d} \text{Div}_{n,d}(D)^+_{m}$$

where $\text{Div}_{n,d}(D)^+_{m}$ is a smooth locally closed subvariety of $\text{Div}_{n,d}(D)$ consisting of points whose limit as $t \to 0$ under the $G_m$-action lies in $C^{(m)}$. In fact, we can give a more precise modular description of the strata. First, let us introduce some notation.

**Definition 4.6.** Let $M = \bigoplus_{i=1}^n M_i = O_C(D)^{\otimes n}$ and $M^{\leq i} = \bigoplus_{j \leq i} M_j$ for $1 \leq i \leq n$. For a sub sheaf $E \subset M$ with torsion quotient $T$, let $E^{\leq i} := E \cap M^{\leq i}$ and $T^{\leq i} := M^{\leq i}/E^{\leq i}$, and let $E_i := E^{\leq i}/E^{\leq i-1}$ and $T_i := T^{\leq i}/T^{\leq i-1}$. We note that $E_i$ is invertible and $T_i$ is torsion.

For the universal sequence $0 \to E \to \pi_C^* M \to T \to 0$ on $\text{Div}_{n,d}(D) \times C$, we define $E^{\leq i}$, $E_i$, $T^{\leq i}$, $T_i$ as above; however, in general, $E^{\leq i}$ and $T^{\leq i}$ are no longer flat over $\text{Div}_{n,d}(D)$. For any family $F$ over $S$ of rank $n$ subsheaves of $M$, we define $F^{\leq i}$ and $F_i$ as above.

**Lemma 4.7.** For a partition $m = (m_1, \ldots, m_n)$ of $n \deg D - d$, the following statements hold.

(i) The BB stratum $\text{Div}_{n,d}(D)^+_{m}$ has the following universal property: a morphism $f : S \to \text{Div}_{n,d}(D)$ factors via $\text{Div}_{n,d}(D)^+_{m}$ if and only if, for the corresponding family $F \hookrightarrow M_S$ over $S$, the families $F_i$ are flat over $S$ of degree $(\deg D - m_i$ for $1 \leq i \leq n$.

(ii) The retraction $\text{Div}_{n,d}(D)^+_{m} \to C^{(m)}$ is a Zariski locally trivial affine space fibration of rank $r^+_{m} := \sum_{i=1}^n (n - i)m_i$.

(iii) Let $N^+_{m}$ denote the normal bundle to $\text{Div}_{n,d}(D)^+_{m} \subset \text{Div}_{n,d}(D)$. Then we have

$$N^+_{m}|_{C^{(m)}} \cong \bigoplus_{i > j} (\pi_{C^{(m)}}, \text{Hom}(E_i|_{C^{(m)}} \times C, T_j|_{C^{(m)}} \times C))$$

In particular, the stratum $\text{Div}_{n,d}(D)^+_{m}$ has codimension $c^+_{m} := \sum_{i=1}^n (i - 1)m_i$.

**Proof.** The description of the strata is given in \cite{[8]} §3 and the normal spaces to the strata at fixed points are described in \cite{[8]} §6, from which one can compute the ranks $r^+_{m}$ and the codimensions, as $c^+_{m} + r^+_{m} + \sum_{i=1}^n m_i = \dim \text{Div}_{n,d}(D)$. To prove (iii), one can argue as in \cite{[8]} Proposition 5.2 (3)] by identifying the strata with locally closed subschemes of certain flag Quot schemes (encoding the natural flag given by the $G_m$-action). The tangent sheaf to $\text{Div}_{n,d}(D)$ is

$$T \text{Div}_{n,d}(D) \cong (\pi_{\text{Div}_{n,d}(D)})_* \text{Hom}(E, T)$$

and the tangent sheaf of the flag Quot scheme is given by replacing $\text{Hom}$ with the filtration preserving homomorphisms $\text{Hom}_\ldots$. After applying $(\pi_{\text{Div}_{n,d}(D)})_*$ to the short exact sequence

$$0 \to \text{Hom}_\ldots(E, T) \to \text{Hom}(E, T) \to \text{Hom}_+(E, T) \to 0$$

we also obtain a short exact sequence, as $\text{Ext}^1(E, T) = 0$ for a torsion sheaf $T \to C$. Then, as in \cite{[8]}, one deduces $N^+_{m} \cong (\pi_{\text{Div}_{n,d}(D)})_* (\text{Hom}_+(E, T)|_{\text{Div}_{n,d}(D)^+_{m} \times C})$, from which (iii) follows. \hfill \Box

We can now state the motivic Białynicki-Birula decomposition in this special case.

**Corollary 4.8.** In $\text{DM}(k, R)$, we have a direct sum decomposition

$$M(\text{Div}_{n,d}(D)) \cong \bigoplus_{m \vdash n \deg(D) - d} M(C^{(m)})(c^+_{m}).$$
4.3. A description of the motive of the stack of bundles. From the above results, we deduce the following description of \( M(\text{Bun}_{n,d}) \).

**Theorem 4.9.** Assume that \( R \) is a \( \mathbb{Q} \)-algebra. Then the motive \( M(\text{Bun}_{n,d}) \) of the stack of rank \( n \) degree \( d \) vector bundles on \( C \) is contained in the smallest localising tensor triangulated category of \( \text{DM}(k,R) \) containing the motive of the curve \( C \).

**Proof.** Since \( R \) is a \( \mathbb{Q} \)-algebra, for any motive \( M \in \text{DM}(k,R) \) and any \( i \in \mathbb{N} \), there exists a symmetric power \( \text{Sym}^i M \) which is a direct factor of \( M^\otimes i \) and such that, for any \( X \) smooth quasi-projective variety, we have \( M(X^{(i)}) \sim \text{Sym}^i M(X) \) [11 Proposition 2.4]. Therefore the result follows from Theorem 4.4 and Corollary 4.8. \( \square \)

Since the motive of a curve is an abelian motive (that is, it lies in the localising subcategory generated by motives of abelian varieties), we immediately deduce the following result.

**Corollary 4.10.** Assume that \( R \) is a \( \mathbb{Q} \)-algebra; then \( M(\text{Bun}_{n,d}) \) is an abelian motive.

A similar result was obtained by Del Baño for the motive of the moduli space of stable vector bundles of fixed rank and degree [17 Theorem 4.5].

4.4. A conjecture on the transition maps. In order to obtain a formula for this motive, we need to understand the functoriality of these motivic BB decompositions for the closed immersions \( i_{D,D'} : \text{Div}_{n,d}(D) \to \text{Div}_{n,d}(D') \) for divisors \( D' \geq D \geq 0 \). More precisely, the map \( i_{D,D'} \) and the decompositions of Corollary 4.8 induce a commutative diagram

\[
\begin{array}{ccc}
M(\text{Div}_{n,d}(D)) & \xrightarrow{M(i_{D,D'})} & M(\text{Div}_{n,d}(D')) \\
\downarrow & & \downarrow \\
\bigoplus_{m \in \mathbb{N}} n \deg(D) \cdot d \cdot M(C^{(m)}) \{c_m\} & \xrightarrow{\bigoplus_{m \in \mathbb{N}} k_{m,m'}} & \bigoplus_{m \in \mathbb{N}} n \deg(D') \cdot d \cdot M(C^{(m')}) \{c_{m'}\}
\end{array}
\]

with induced morphisms \( k_{m,m'} : M(C^{(m)})\{c_m\} \to M(C^{(m')})\{c_{m'}\} \) between the factors.

Although we have \( i_{D,D'}(C^{(m)}) \subset C^{(m'+\delta)} \) for \( \delta := \deg(D' - D) \), the morphisms \( k_{m,m'} \) are not induced by these fixed loci inclusions, as the closed subscheme \( \text{Div}_{n,d}(D) \hookrightarrow \text{Div}_{n,d}(D') \) does not intersect the BB strata in \( \text{Div}_{n,d}(D') \) transversally. Indeed, we note that \( c_m \neq c_{m' + \delta} \).

However, we have \( c_m = c_{m'} \), when \( m' = m + (n\delta,0,\ldots,0) \). In this case, there is a morphism

\[
f_{m,m'} := a_{D' - D} \times \text{Id}_{C^{(m_1)}} \times \cdots \times \text{Id}_{C^{(m_n)}} : C^{(m)} \to C^{(m')}
\]

where \( a_{D' - D} : C^{(m)} \to C^{(m')} \) corresponds to adding \( n(D' - D) \). Based on some small computations, we make the following conjectural description for the morphisms \( k_{m,m'} \).

**Conjecture 4.11.** Let \( D' \geq D \) be effective divisors on \( C \); then the morphisms

\[
k_{m,m'} : M(C^{(m)})\{c_m\} \to M(C^{(m')})\{c_{m'}\}
\]

fitting into the commutative diagram [11] are the morphisms \( M(i_{D,D'}^*) \) when \( m' = m + (n \deg(D' - D),0,\ldots,0) \) and are zero otherwise.

One can also formulate this conjecture on the level of Chow groups, as there is also a BB decomposition for Chow groups. This leads to the following conjecture on the intersection theory of the Quot schemes \( \text{Div}_{n,d}(D) \).

**Conjecture 4.12.** For effective divisors \( D' \geq D \) on \( C \), we have a commutative diagram

\[
\begin{array}{ccc}
\text{CH}^*(\text{Div}_{n,d}(D')) & \xrightarrow{i_{D,D'}^*} & \text{CH}^*(\text{Div}_{n,d}(D)) \\
\downarrow & & \downarrow \\
\bigoplus_{m' + n \deg(D') \cdot d} \text{CH}^{*-c_{m'}}(C^{(m')}) & \xrightarrow{\bigoplus_{m' + n \deg(D') \cdot d} k_{m,m'}} & \bigoplus_{m + n \deg(D) \cdot d} \text{CH}^{*-c_m}(C^{(m)})
\end{array}
\]

where the vertical maps are the BB isomorphisms and \( k_{m,m'} : \text{CH}^{*-c_{m'}}(C^{(m')}) \to \text{CH}^{*-c_m}(C^{(m)}) \) is given by \( f_{m,m'}^* \) when \( m' = m + (n \deg(D' - D),0,\ldots,0) \) and is zero otherwise.
Remark 4.13. Since \( \text{Div}_{n,d}(D) \) are smooth varieties, their motives encode their Chow groups. In particular, this means Conjecture \([4.11]\) implies Conjecture \([4.12]\). In fact, at least with rational coefficients, Conjecture \([4.12]\) for all field extensions of \( k \) is equivalent to Conjecture \([4.11]\) for all field extensions of \( k \) by \([25, \text{Lemma 1.1}]\).

In Theorem \([4.120]\) below, we deduce from Conjecture \([4.11]\) a conjectural formula for \( M(\text{Bun}_{n,d}) \) (cf. Conjecture \([4.19]\)). In fact, as we explain in \([5.2]\) our conjectural formula for \( M(\text{Bun}_{n,d}) \) (or strictly speaking the formula for the compactly supported motive \( M(\text{Bun}_{n,d}) \) that follows by Poincaré duality) resembles many other classical formulas for invariants of \( \text{Bun}_{n,d} \), such as Harder’s stacky point count of \( \text{Bun}_{n,d} \) over a finite field and the Behrend-Dhillon formula for the class of \( \text{Bun}_{n,d} \) in a dimensional completion of the Grothendieck ring of varieties.

4.5. The motive of the stack of line bundles. Throughout this section, we assume that \( C(k) \neq \emptyset \) and we prove a formula for the motive of the stack of line bundles (cf. Corollary \([4.16]\)) by proving a result about the motive of an inductive system of symmetric powers of curves (cf. Lemma \([4.14]\)); the proof of this lemma is probably well-known, at least in cohomology, but we nevertheless include a proof as a generalisation of this argument is used in Theorem \([4.20]\).

Lemma 4.14. Suppose that \( C(k) \neq \emptyset \). For \( d \in \mathbb{Z} \) and an effective divisor \( D_0 \) on \( C \) of degree \( d_0 > 0 \), consider the inductive system \((C^{(l_0-d)})_{l \geq |d|} \) with \( a_{D_0} : C^{(l_0-d)} \to C^{((l+1)l_0-d)} \) given by sending a degree \( ld_0 - d \) effective divisor \( D \) to \( D + D_0 \). Then in \( \text{DM}(k, R) \) we have

\[
\text{hocolim}_{l \geq |d|} M(C^{(l_0-d)}) \simeq M(\text{Jac}(C)) \otimes M(\mathbb{BG}_m).
\]

Proof. For notational simplicity, we prove the statement for \( d = 0 \); the proof is the same in general. We consider the Abel-Jacobi maps \( A_{l_1} : C^{(l_0-d)} \to \text{Jac}(C) \) defined using \( lD_0 \) which are compatible with the morphisms \( a_{D_0} : C^{(l_0-d)} \to C^{((l+1)l_0-d)} \). For \( ld_0 > 2g - 2 \) as \( C(k) \neq \emptyset \), the Abel-Jacobi map is a \( \mathbb{P}^{ld_0-g} \)-bundle: we have \( C^{(l_0-d)} \cong \mathbb{P}(p_*p_l) \), where \( p_l \) is a Poincaré bundle on \( \text{Jac}(C) \times C \) of degree \( ld_0 \) and \( p : \text{Jac}(C) \times C \to \text{Jac}(C) \) is the projection. In fact, we can assume that \( p_{l+1} = p_l \otimes q^*(\mathcal{O}_{\text{Jac}(D_0)}) \) for the projection \( q : \text{Jac}(C) \times C \to C \). Then \( p_l \) is a subbundle of \( p_{l+1} \), and the induced map between the projectivisations is \( a_{D_0} \).

By the projective bundle formula, for \( ld_0 > 2g - 2 \), we have

\[
M(C^{(l_0-d)}) \simeq M(\mathbb{P}^{ld_0-g}) \otimes M(\text{Jac}(C))
\]

such that the transition maps \( M(C^{(l_0-d)}) \to M(C^{((l+1)l_0-d)}) \) induce the identity on \( M(\text{Jac}(C)) \). Hence, by Lemma \([4.8]\) we can pullout the motive of \( \text{Jac}(C) \) from this homotopy colimit

\[
\text{hocolim}_{l} M(C^{(l_0-d)}) \simeq M(\text{Jac}(C)) \otimes \text{hocolim}_{l} M(\mathbb{P}^{ld_0-g}).
\]

By Example \([2.20]\) we have \( M(\mathbb{BG}_m) \simeq \text{hocolim}_{l} M(\mathbb{P}^r) \). As the inductive system \( (M(\mathbb{P}^{ld_0-g}))_l \) is a cofinal subsystem of this system, we conclude the result using \([33, \text{Lemma 1.7.1}]\). \( \square \)

Remark 4.15. If \( C(k) = \emptyset \), then the Abel-Jacobi map from a sufficiently high symmetric power of \( C \) is not a projective bundle in general, but rather a Brauer-Severi bundle \([30]\).

From this result we obtain a formula for the motive of the stack of line bundles.

Corollary 4.16. Suppose that \( C(k) \neq \emptyset \). Then in \( \text{DM}(k, R) \), there is an isomorphism

\[
M(\text{Bun}_{1,d}) \simeq M(\text{Jac}(C)) \otimes M(\mathbb{BG}_m).
\]

In particular, if \( C \) has a rational point or \( d = 0 \), then \( M(\text{Bun}_{1,d}) \simeq M(\text{Jac}(C)) \otimes M(\mathbb{BG}_m) \).

Proof. By Theorem \([4.4]\) we have \( M(\text{Bun}_{1,d}) \simeq \text{hocolim}_l M(D_{1,d}(lD_0)) \) for any effective divisor \( D_0 \) on \( C \) of degree \( d_0 > 0 \). Since \( D_{1,d}(lD_0) \cong C^{(l_0-d)} \), we have

\[
M(\text{Bun}_{1,d}) \simeq \text{hocolim}_l M(D_{1,0}(lD_0)) \simeq \text{hocolim}_l M(C^{(l_0-d)}) \simeq M(\text{Jac}(C)) \otimes M(\mathbb{BG}_m)
\]

by Lemma \([4.14]\). \( \square \)
Remark 4.17. Alternatively, we can deduce this result as $\text{Bun}_{1,d} \to \text{Pic}^d(C) \cong \text{Jac}(C)$ is a trivial $\mathbb{G}_m$-gerbe (or equivalently, $\text{Pic}^d(C)$ is a fine moduli space, which is true as by assumption $C(k) \neq \emptyset$ and thus the Poincaré bundle gives a universal family).

4.6. A conjectural formula for the motive. Let us explain how to deduce a formula for the motive of $\text{Bun}_{n,d}$ from Theorem 4.4 and Conjecture 4.11

Throughout this section, we continue to assume that $C(k) \neq \emptyset$.

Definition 4.18. Let $X$ be a quasi-projective $k$-variety and $N \in \text{DM}(k, R)$. The motivic zeta function of $X$ at $N$ is

$$Z(X, N) = \sum_{i=0}^{\infty} M(X^{(i)}) \otimes N^{\otimes i} \in \text{DM}(k, R).$$

We can now state our main conjecture; for evidence supporting this conjecture, see [5,2]

Conjecture 4.19. Suppose that $C(k) \neq \emptyset$; then in $\text{DM}(k, R)$, we have

$$M(\text{Bun}_{n,d}) \simeq M(\text{Jac}(C)) \otimes M(\text{BG}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, R\{i\}).$$

Theorem 4.20. Conjecture 4.11 implies Conjecture 4.19.

Proof. By Theorem 4.4 for any non-zero effective divisor $D_0$ on $C$, we have

$$M(\text{Bun}_{n,d}) \simeq \text{hocolim}_l M(\text{Div}_{n,d}(lD_0))$$

with transition morphisms induced by $i_l : \text{Div}_{n,d}(lD_0) \to \text{Div}_{n,d}((l+1)D_0))$. Let $d_0 = \text{deg}(D_0)$; then the decomposition from Corollary 4.8 for the divisor $D = lD_0$ is

$$M(\text{Div}_{n,d}(lD_0)) \simeq \bigoplus_{m \in \mathbb{N}^{n-1}} M(C_{\{m\}}\{c_m\})$$

We can write the inductive system $l \mapsto M(\text{Div}_{n,d}(lD_0))$ as a direct sum of inductive systems as follows. For $m^b = (m^b_2, \ldots, m^b_n) \in \mathbb{N}^{n-1}$ and $l \in \mathbb{N}$, we write $m_1(l) := nl_0d - d - \sum_{i=2}^{n-1} m^b_i$ and $m(l) := (m_1(l), m^b) \in \mathbb{Z} \times \mathbb{N}^{n-1}$. Notice that $c_m(l) = \sum_i (i-1)m^b_i$ only depends on $m^b$, and we will also use the notation $c_{m^b}$. For $m^b \in \mathbb{N}^{n-1}$, we define an inductive system $P_{m^b, l} : \mathbb{N} \to \text{DM}(k, R)$ as follows:

$$P_{m^b, l} := \begin{cases} 0 & \text{if } m_1(l) < 0, \\ M(C_{\{m(l)\}}\{c_{m(l)}\}) & \text{if } m_1(l) \geq 0 \end{cases}$$

where the map $P_{m^b, l} \to P_{m^b, l+1}$ is zero if $m_1(l+1) < 0$ and the morphism

$$k_{m^b, l} := M(a_{nD_0}) \otimes \text{id}_{M(C_{\{m\}}\{c_{m}\})}.$$ 

if $m_1(l+1) \geq 0$. Assuming Conjecture 4.11, we have an isomorphism

$$M(\text{Div}_{n,d}(lD_0)) \simeq \bigoplus_{m^b \in \mathbb{N}^{n-1}} \text{hocolim}_l P_{m^b, l}$$

as inductive systems of motives indexed by $l \in \mathbb{N}$. By Corollary 4.8 and Lemma 2.9, we deduce

$$M(\text{Bun}_{n,d}) \simeq \bigoplus_{m^b \in \mathbb{N}^{n-1}} \text{hocolim}_l P_{m^b, l}.$$ 

For each $m^b$ and $l$, we have a generalised Abel-Jacobi map

$$AJ_{m^b, l} : C_{\{m(l)\}} \to \text{Pic}^{nl_0d - d}(C) \times C^{(m_2)} \times \cdots \times C^{(m_n)} \cong \text{Jac}(C) \times C^{(m^b)}$$

sending $(F_1, \ldots, F_n)$ to $(\mathcal{O}_C(\sum_{i=1}^n F_i), F_2, \ldots, F_n)$. In fact, if $m_1(l) > 2g-2$, this morphism is a $\mathbb{P}^{m_1(l)-g}$-bundle: we have that $C_{\{m(l)\}} \cong \mathbb{P}(p_*\mathcal{F})$ where $p : \text{Jac}(C) \times C^{(m^b)} \to \text{Jac}(C) \times C^{(m^b)}$ is the projection and $\mathcal{F}$ is the tensor product of the pullback of the degree $nl_0d - d$ Poincaré
bundle $P \to \text{Jac}(C) \times C$ with the pullbacks of the duals of the universal line bundles $L_i \to C^{(m_i)} \times C$ for $2 \leq i \leq n$. In this case, by the projective bundle formula, we have

$$M(C^{(m(l))}) \simeq M(\text{Jac}(C)) \otimes M(C^{(m)}) \otimes M(\mathbb{P}^{m_1(l)-g})$$

Then by Lemma 2.8 and Lemma 4.14, we have

$$\text{hocolim}_{l} P^b_{m,l} \simeq \text{hocolim}_{l: m_1(l)>2g-2} M(C^{(m(l))})\{c_m\}$$

$$\simeq \text{hocolim}_{l: m_1(l)>2g-2} M(\mathbb{P}^{m_1(l)-g}) \otimes M(\text{Jac}(C)) \otimes M(C^{(m)})\{c_m\}$$

$$\simeq M(B\mathbb{G}_m) \otimes M(\text{Jac}(C)) \otimes M(C^{(m)})\{c_m\}.$$ 

Hence, using the commutation of sums and tensor products, we have

$$M(\text{Bun}_{n,d}) \simeq \bigoplus_{m^b \in \mathbb{N}^{n-1}} M(B\mathbb{G}_m) \otimes M(\text{Jac}(C)) \otimes M(C^{(m)})\{c_m\}$$

$$\simeq M(B\mathbb{G}_m) \otimes M(\text{Jac}(C)) \otimes \bigoplus_{m^b \in \mathbb{N}^{n-1}} \bigotimes_{i=2}^{n} M(C^{(m_i)})\{(i-1)m_i\}$$

$$\simeq M(B\mathbb{G}_m) \otimes M(\text{Jac}(C)) \otimes \bigotimes_{i=1}^{n-1} Z(C, R\{i\}),$$

which completes the proof of the theorem. \hfill \Box

**Remark 4.21.** Since the morphism $\text{det} : \text{Bun}_{n,d} \to \text{Bun}_{1,d}$ is not representable or of finite type, one needs to carefully define the induced morphism of motives $M(\text{det}) : M(\text{Bun}_{n,d}) \to M(\text{Bun}_{1,d})$: we do not present this additional construction here. However, modulo this extra work, the proof of Theorem 4.20 implies the following compatibility between the formula of Conjecture 4.19 and the isomorphism $M(\text{Bun}_{1,d}) \simeq M(\text{Jac}(C)) \otimes M(B\mathbb{G}_m)$ of Corollary 4.16. First, note that there is a canonical direct factor $R\{0\}$ in the motive $\bigotimes_{i=1}^{n-1} Z(C, R\{i\})$ and thus a direct factor $M(\text{Jac}(C)) \otimes M(B\mathbb{G}_m)$ in the formula of Conjecture 4.19. Then the map $M(\text{det})$ is precisely the projection onto that direct factor.

5. Consequences and Comparisons with Previous Results

As above, we let $C$ be a smooth projective geometrically connected curve of genus $g$ over a field $k$. We fix $n \in \mathbb{N}$ and $d \in \mathbb{Z}$ and let $\text{Bun}_{n,d}$ denote the stack of vector bundles over $C$ of rank $n$ and degree $d$.

5.1. **The compactly supported motive.** Let $M^c(\text{Bun}_{n,d}) \in \text{DM}(k, R)$ denote the compactly supported motive of $\text{Bun}_{n,d}$ (as defined in [2,7]). By Theorem 4.3 and the fact that the dual of an infinite sum is an infinite product, we have for any fixed $D_0 > 0$

$$M^c(\text{Bun}_{n,d}) \simeq \text{hocolim}_{l} M^c(\text{Div}_{n,d}(lD_0)).$$

However homotopy limits rarely commute to inclusions of localising subcategories of compactly generated triangulated categories, and it is not difficult to deduce much from this formula.

**Theorem 5.1.** Assume that Conjecture 4.19 holds and that $C(k) \neq \emptyset$; then, in $\text{DM}(k, R)$, we have

$$M^c(\text{Bun}_{n,d}) \simeq M^c(B\mathbb{G}_m)\{(n^2-1)(g-1)\} \otimes M^c(\text{Jac } C) \otimes \bigotimes_{i=2}^{n} Z(C, R\{-i\}).$$

**Proof.** We apply Poincaré duality for smooth stacks (cf. Proposition 2.34) to the formula for $M(\text{Bun}_{n,d})$ in Conjecture 4.19.
Let us first calculate the duals of the motives of the Jacobian of $C$ and of the classifying space $BG_m$, and of the motivic zeta function. As $\text{Jac}(C)$ is smooth and projective of dimension $g$, we have by Poincaré duality

$$M(\text{Jac}(C))^\vee \simeq M^c(\text{Jac}(C))\{g\} \simeq M(\text{Jac}(C))\{-g\}.$$ 

As $BG_m$ is a smooth quotient stack of dimension $-1$, we have by Proposition 2.34 that

$$M(BG_m)^\vee \simeq M^c(BG_m)\{1\}.$$ 

As the dual of an infinite sum of motives is the infinite product of the dual motives, we have

$$Z(C, R\{i\})^\vee = \left( \bigoplus_{j=0}^\infty M(C^{(j)})\{ij\} \right)^\vee = \prod_{j=0}^\infty \left( M(C^{(j)})\{ij\} \right)^\vee \simeq \prod_{j=0}^\infty M(C^{(j)})\{-ij\}$$

as the symmetric power $C^{(j)}$ of the curve $C$ is a smooth projective variety of dimension $j$.

By Corollary 2.12, we have that for $l \geq 2$, the natural morphism in $\text{DM}(k, R)$

$$\bigoplus_{j=0}^\infty M(C^{(j)})\{-lj\} \to \prod_{j=0}^\infty M(C^{(j)})\{-lj\}$$

is an isomorphism. Hence, $Z(C, R\{i\})^\vee = Z(C, R\{-l+1\})$ for $l \geq 1$.

As $\text{Bun}_{n,d}$ is a smooth stack of dimension $n^2(g-1)$, Poincaré duality gives

$$M^c(\text{Bun}_{n,d}) \simeq M(\text{Bun}_{n,d})^\vee \{n^2(g-1)\}$$

$$\simeq M(BG_m)^\vee \otimes M(\text{Jac}(C))^\vee \otimes \bigotimes_{i=1}^{n-1} Z(C, \mathbb{Q}\{i\})^\vee \{n^2(g-1)\}$$

$$\simeq M^c(BG_m)\{(n^2-1)(g-1)\} \otimes M^c(\text{Jac} C) \otimes \bigotimes_{i=2}^n Z(C, \mathbb{Q}\{-i\})$$

which gives the above formula. \qed

5.2. Comparison with previous results. In this section, we compare our conjectural formula for the (compactly supported) motive of $\text{Bun}_{n,d}$ (cf. Theorem 5.1) with other results concerning topological invariants of $\text{Bun}_{n,d}$. One of the first formulae to appear in the literature, was a computation of the stacky point count of $\text{Bun}_{n,d}$ over a finite field $\mathbb{F}_q$, which is defined as

$$|\text{Bun}_{n,d}(\mathbb{F}_q)|_{st} := \sum_{E \in \text{Bun}_{n,d}(\mathbb{F}_q)} \frac{1}{|\text{Aut}(E)|}.$$ 

Theorem 5.2 (Harder, Siegel). Over a finite field $\mathbb{F}_q$, we have

$$|\text{Bun}_{n,d}(\mathbb{F}_q)|_{st} = \frac{q^{(n^2-1)(g-1)}}{q-1} |\text{Jac}(C)(\mathbb{F}_q)| \prod_{i=2}^n \zeta_C(q^{-i})$$

where $\zeta_C$ denotes the classical Zeta function of $C$.

We note that our conjectural formula for $M_c(\text{Bun}_{n,d})$ is a direct translation of this formula, when one replaces a variety (or stack) by its (stacky) point count, $q^k$ by $R\{k\}$, and the classical Zeta function by the motivic Zeta function.

Behrend and Dhillon [4] give a conjectural description for the class of the stack of principal $G$-bundles over $C$ in a dimensional completion $\tilde{K}_0(\text{Var}_k)$ of the Grothendieck ring of varieties when $G$ is a semisimple group and, moreover, they prove their formula for $G = \text{SL}_n$ by following the geometric arguments in [8]. By a minor modification of their computation, one obtains the following formula for the class of $\text{Bun}_{n,d}$.
Theorem 5.3 (Behrend–Dhillon). In $\hat{K}_0(\text{Var}_k)$, the class of $\text{Bun}_{n,d}$ is given by
\[
[\text{Bun}_{n,d}] = \mathbb{L}^{(n^2-1)(g-1)} [BG_m][\text{Jac}(C)] \prod_{i=2}^n Z(C, L^{-i})
\]
where $\mathbb{L} := [k^1]$ and $Z(C, t) := \sum_{j \geq 0} [\mathcal{C}^{(j)}]t^j$.

It is possible to compare their formula with our conjectural formula for $M^c(\text{Bun}_{n,d})$ if one passes to the Grothendieck ring of a certain dimensional completion of $\text{DM}(k, R)$. We sketch this comparison here. First note that it does not make sense to work with the Grothendieck ring of $\text{DM}(k, R)$, as this category is cocomplete and so by the Eilenberg swindle, $K_0(\text{DM}(k, R)) \simeq 0$. Instead, we will follow the lines of Zargar \cite{Zagar} §3 and work with a completion of $\text{DM}(k, R)$; however, note that Zargar works with effective motives and completes with respect to the slice filtration and we will instead complete with respect to the dimensional filtration.

For this completion process, one would like to take limits of projective systems of triangulated categories, but the category of triangulated categories is unsuitable for this task; hence, in the rest of this section, we let $\text{DM}(k, R)$ denote the symmetric monoidal stable $\infty$-category underlying Voevodsky’s category. As in Definition 2.10 given $m \in \mathbb{Z}$, we write $\text{DM}_{gm}(k, R)_r$ (resp. $\text{DM}(k, R)_r$) for the full sub-$\infty$-category (resp. presentable sub-$\infty$-category) of $\text{DM}(k, R)$ generated by motives of the form $M^c(X)(r)$ with $X$ being a separated finite type $k$-scheme with $\dim(X) + r \leq n$. The localisations $\text{DM}_{gm}(k, R)/\text{DM}_{gm}(k, R)_r$ are symmetric monoidal $\infty$-categories (presentable in the non-geometric case), as this filtration is symmetric monoidal.

One can then define
\[
\text{DM}_{gm}^\wedge(k, R) := \varprojlim_{r \in \mathbb{N}} \text{DM}_{gm}(k, R)/\text{DM}_{gm}(k, R)_r
\]
as symmetric monoidal $\infty$-categories. By localisation, the functor $M^c : \text{Var}_k \to \text{DM}_{gm}(k, R)$ induces a ring morphism $\chi_c : \hat{K}_0(\text{Var}_k) \to K_0(\text{DM}_{gm}^\wedge(k, R))$.

One can then adapt the argument in \cite{Voevodsky} Lemma 3.2 to show that the two functors
\[
\text{DM}_{gm}(k, R) \to \text{DM}_{gm}^\wedge(k, R) \to \text{DM}^\wedge(k, R)
\]
are fully faithful. There is also a functor $(-)^\wedge : \text{DM}(k, R) \to \text{DM}^\wedge(k, R)$. The following result is then clear from comparing the two formulas and using that $M(C^{(i)})\{-ij\}$ lies in $\text{DM}_{gm}(k, R)_{j(1-i)} \subset \text{DM}_{gm}(k, R)_{-j}$ for $i \geq 2$.

Lemma 5.4. Assume that $C(k) \neq \emptyset$ and that Conjecture \cite[17]{Voevodsky} holds. Then $M^c(\text{Bun}_{n,d})^\wedge$ lies in $\text{DM}_{gm}^\wedge(k, R)$ and we have $\chi_c[\text{Bun}_{n,d}] = [M^c(\text{Bun}_{n,d})^\wedge]$ in $K_0(\text{DM}_{gm}^\wedge(k, R))$.

Note that, as in \cite{Voevodsky}, it is not clear how small $K_0(\text{DM}_{gm}^\wedge(k, R))$ is, and in particular if the natural map $K_0(\text{DM}_{gm}(k, R)) \to K_0(\text{DM}_{gm}^\wedge(k, R))$ is injective, which limits the interest of such a comparison. To go further, one could try to introduce an analogue of the ring $\mathcal{M}(k, R)$ in \cite{Voevodsky}; however, we do not pursue this here.

5.3. Vector bundles with fixed determinant and $\text{SL}_d$-bundles. For a line bundle $L$ on $C$ of degree $d$, we let $\text{Bun}^L_{n,d}$ denote the stack of rank $n$ degree $d$ vector bundles over $C$ with determinant isomorphic to $L$, and we let $\text{Bun}^\wedge_{n,d}$ be the stack of pairs $(E, \phi)$ with $E$ a rank $n$ degree $d$ vector bundle and $\phi : \text{det}(E) \simeq L$. We have the following diagram of algebraic stacks with cartesian squares
\[
\begin{array}{ccc}
\text{Bun}^L_{n,d} & \longrightarrow & \text{Spec}(k) \\
& \downarrow & \\
\text{Bun}^\wedge_{n,d} & \longrightarrow & \text{BG}_m \\
& \downarrow & \\
\text{Bun}_{n,d} & \overset{\text{det}}{\longrightarrow} & \text{Bun}_{1,d} \\
& \downarrow & \\
& \text{Pic}^d(C) & \\
\end{array}
\]
where the morphism $\det$ is smooth and surjective. In fact, the bottom left square is cartesian by the See-saw Theorem. We see that $\Bun^L_{n,d}$ is thus a closed smooth substack of $\Bun_{n,d}$ of codimension $g$, and that $\Bun^L_{n,d} \to \Bun_{n,d}$ is a $\mathbb{G}_m$-torsor. Moreover, when $d = 0$ and $L = \mathcal{O}_C$, the stack $\Bun^L_{n,0}$ is actually isomorphic to the stack $\Bun_{SL_n}$ of principal $SL_n$-bundles; indeed, the vector bundle associated to an $SL_n$-bundle via the standard representation has its determinant bundle canonically trivialised. For more details on this picture, in the more general case of principal bundles, see \cite{[5]} [§1-2]

We will use these facts to compute the motive of $\Bun_{SL_n}$ assuming Conjecture \ref{MMCL}. First, we claim that the smooth stacks $\Bun^L_{n,d}$ and $\Bun^L_{n,d}$ are exhaustive; the claim for the former follows from Proposition 2.23, as the determinant bundle canonically trivialised. For more details on this picture, in the more general case of principal bundles, see \cite{[5]} [§6].

Proposition 5.5. For an effective divisor $D$ on $C$, the closed immersion $j_D : \Div^L_{n,d}(D) \to \Div_{n,d}(D)$ is transverse to the BB strata in $\Div_{n,d}(D)$. Hence, Conjecture \ref{MMCL} implies the analogous statement for the Quot schemes of matrix divisors with fixed determinant.

Proof. The fixed loci for the $\mathbb{G}_m$-action on $\Div^L_{n,d}(D)$ are smooth closed subvarieties of $C(\omega)$, which we denote by $\Div^L_{n,d}(D)$ and consist of $(D_1, \ldots, D_n) \in C(\omega)$ such that $nD - \sum_{i=1}^n D_i \in [L]$. It follows from \cite{[5]} [§6] that

$$\text{codim}_{\Div^L_{n,d}(D)}(\Div^L_{n,d}(D)\mid_m^+) = \text{codim}_{\Div_{n,d}(D)}(\Div_{n,d}(D)\mid_m^+) = c_m^+.$$ 

Hence, it suffices to verify that the pullbacks of the normal bundles of $\Div^L_{n,d}(D) \to \Div_{n,d}(D)$ and $\Div_{n,d}(D)\mid_m^+ \to \Div^L_{n,d}(D)$ are isomorphic; this follows from the description of the normal bundle of the former given in \cite{[5]} [§6] and of the latter in Lemma 4.7.

Since $j_D$ is transverse to the BB strata, Conjecture \ref{MMCL} for the transition maps in the motivic BB decompositions of $\Div$ implies the analogous statement for $\Div^L$, as we obtain the analogous morphisms by intersecting with the classes of $\Div^L_{n,d}(D) \times C^L(\omega) \to \Div_{n,d}(D) \times C^L(\omega)$.

\[\square\]

Theorem 5.6. Assume that Conjecture \ref{MMCL} holds and $C(k) \neq \emptyset$. In $\DM(k, R)$, we have

$$M(\Bun^L_{n,d}) \simeq M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, R\{i\})$$

and

$$M^c(\Bun^L_{n,d}) \simeq M^c(B\mathbb{G}_m) \{(n^2 - 1)(g - 1)\} \otimes \bigotimes_{i=2}^n Z(C, R\{-i\}).$$

Proof. First one restricts $\Div_{n,d} \to \Bun_{n,d}$ to $\Div^L_{n,d} \to \Bun^L_{n,d}$ and, analogously to Theorem \ref{MMCL}, one proves that for any non-zero effective divisor $D_0$ on $C$, there is an isomorphism

$$M(\Bun^L_{n,d}) \simeq \hocolim_i M(\Div^L_{n,d}(D_0))$$

in $\DM(k, R)$; for the relevant codimension estimates, one can use \cite{[5]} [§6] together with Lemma 4.3. We then consider the Białynicki-Birula decompositions for the smooth closed $\mathbb{G}_m$-invariant subvarieties $\Div^L_{n,d}(D_0) \subset \Div_{n,d}(D_0)$, whose fixed loci is the disjoint union of $C^L(\omega)$ for partitions $m$ on $n \deg(D_0) - d$. By Proposition 5.5, the codimensions of the BB strata in $\Div^L_{n,d}(D_0)$ equal the codimensions of the BB strata in $\Div_{n,d}(D_0)$ as $j_D$ is transverse to the BB strata.

By the final statement of Proposition 5.5 we can follow the argument of the proof of Theorem \ref{MMCL} and define $\mathbb{P}_{m, l}$ as in the proof of Theorem \ref{MMCL}, but by replacing $C^L(\omega)$ by $C^L(\omega)$; then

\[\square\]
analogously to (13) we have a corresponding isomorphism with superscript \(L\) inserted on both sides. For \(m_1(l) > 2g - 2\), the projection morphism
\[
C^{(m_1)}_L \rightarrow C^{(m_2)} \times \ldots \times C^{(m_n)}
\]
is a \(\mathbb{P}^{m_1-g}\)-bundle (as we have fixed the determinant and \(C(k) \neq \emptyset\)). Hence
\[
\text{hocolim}_{l} P^L_{m_1} \simeq \text{hocolim}_{l} P^L_{m_1(l)>2g-2} \simeq \text{hocolim}_{l} M(\mathbb{P}^{m_1(l)-g}) \otimes M(C(m_1)) \{c_{m_1}\}
\]
\[
\simeq M(B\mathbb{G}_m) \otimes M(C(m_1)) \{c_{m_1}\}
\]
where we have used Lemma 2.8 and Example 2.20. Then the remainder of the proof for the formula for \(M_\ast(B\mathbb{G}_m)\) follows that of Theorem 4.20 verbatim, and the formula for \(\mathcal{M}_\ast(C, R)\) follows by Poincaré duality, as the codimension of \(B\mathbb{G}_m\) is \(g\).

**Theorem 5.7.** Assume that Conjecture 4.11 holds and \(C(k) \neq \emptyset\). Then, in \(\text{DM}(k, R)\), we have
\[
M(B\mathbb{G}_m, n) \simeq \bigotimes_{i=1}^{n-1} Z(C, R\{i\})
\]
and
\[
\mathcal{M}_\ast(B\mathbb{G}_m, n) \simeq \bigotimes_{i=2}^{n} Z(C, R\{i\}) \otimes R\{(n^2 - 1)(g - 1)\}.
\]

**Proof.** By Poincaré duality and the same argument as in the proof of Theorem 5.1, the formula for \(\mathcal{M}_\ast\) follows from the formula for \(M_\ast\).

As \(B\mathbb{G}_m\) is a \(\mathbb{G}_m\)-torsor equal to the pullback of the universal \(\mathbb{G}_m\)-torsor on \(B\mathbb{G}_m\) via the morphisms \(\det : Bun^G_{\mathbb{G}_m} \rightarrow B\mathbb{G}_m\), the idea is to use Proposition 2.30 and Example 2.31. Indeed, by Proposition 2.30 we have a distinguished triangle
\[
M(B\mathbb{G}_m, n) \rightarrow M(Bun^G_{\mathbb{G}_m}, n) \rightarrow M(Bun^G_{\mathbb{G}_m}, n) \rightarrow M(B\mathbb{G}_m, n)
\]
where \(\varphi := \varphi_n\) is defined using the line bundle \(L_n \rightarrow Bun^G_{\mathbb{G}_m}\) associated to the above \(\mathbb{G}_m\)-torsor. In fact, if \(U_n \rightarrow Bun^G_{\mathbb{G}_m} \times C\) denotes the universal bundle and \(\pi_1 : Bun^G_{\mathbb{G}_m} \times C \rightarrow Bun^G_{\mathbb{G}_m}\) denotes the projection, then \(\det(U_n) \cong \pi_1^*\big((L_n)\big)\) by the See-saw Theorem, as \(C\) is compact, and we are working on the stack of bundles with fixed determinant.

By Theorem 5.6, we have \(M(Bun^G_{\mathbb{G}_m}, n) \simeq Z \otimes M(B\mathbb{G}_m)\), where \(Z := \bigotimes_{i=1}^{n-1} Z(C, R\{i\})\).

By using the projection \(M(B\mathbb{G}_m) \rightarrow R\{0\}\), we can construct a morphism
\[
s : M(B\mathbb{G}_m, n) \rightarrow M(Bun^G_{\mathbb{G}_m}, n) \simeq Z \otimes M(B\mathbb{G}_m) \rightarrow Z \otimes R\{0\} \simeq Z,
\]
which we claim is an isomorphism. Let \(\tilde{\varphi} : M(B\mathbb{G}_m) \rightarrow M(B\mathbb{G}_m, \{1\})\) be the morphism induced by the universal line bundle \(\mathcal{O}(-1) = \mathbb{A}^1/\mathbb{G}_m\) on \(B\mathbb{G}_m\) in the sense of Proposition 2.30. This morphism was computed in Example 2.31. Then we have the following diagram
\[
\begin{tikzcd}
M(B\mathbb{G}_m, n) \arrow[r, \varphi] \arrow[d, s] & M(Bun^G_{\mathbb{G}_m}, n) \arrow[r, \varphi] \arrow[d, i] & M(Bun^G_{\mathbb{G}_m}, \{1\}) \arrow[d, i] \arrow[r, \varphi] & M(B\mathbb{G}_m, \{1\}) \arrow[d, \varphi] \\
Z \otimes M(B\mathbb{G}_m) \arrow[r, \text{id} \otimes \tilde{\varphi}] & Z \otimes M(B\mathbb{G}_m) \arrow[r, \text{id} \otimes \tilde{\varphi}] & Z \otimes M(B\mathbb{G}_m, \{1\}) & \end{tikzcd}
\]
where both rows are distinguished triangles and the left hand square commutes by definition of \(s\). We claim that the right hand square also commutes, from which it follows that \(s\) is an isomorphism.

To prove the claim, by definition of \(\varphi\) and \(\tilde{\varphi}\) in terms of first Chern classes in Proposition 2.30, it is enough to show that the map \(c_1(L_n) : M(Bun^G_{\mathbb{G}_m}, n) \rightarrow R\{1\}\) is equal to the composition
\[
M(Bun^G_{\mathbb{G}_m}, n) \simeq Z \otimes M(B\mathbb{G}_m) \rightarrow M(B\mathbb{G}_m) \rightarrow M(B\mathbb{G}_m, \{1\}) \rightarrow R\{1\}.
\]
We will show this via the presentation of $\text{Bun}^{O_C}_{n,0}$ in terms of matrix divisors with determinant $O_C$ which was used to construct the isomorphism in Theorem 5.5.

Fix an effective divisor $D_0$ of degree $d_0 > 0$ and let $\mathcal{L}_{n,l}$ be the pullback of $\mathcal{L}_n$ along $\text{Div}_{n,0}(lD_0) \rightarrow \text{Bun}^{O_C}_{n,0}$. By definition of Chern classes of exhaustive stacks, we have to prove that, for all $l \in \mathbb{N}$ large enough, the map $c_1(\mathcal{L}_{n,l}) : \text{M}(\text{Div}^{O_C}_{n,0}(lD_0)) \rightarrow R\{1\}$ is equal to the composition

\begin{equation}
M(\text{Div}^{O_C}_{n,0}(lD_0)) \rightarrow M(\text{Bun}^{O_C}_{n,0}) \simeq Z \otimes M(BG_m) \rightarrow M(BG_m) c_1(O(-1)) R\{1\}.
\end{equation}

There is a natural determinant morphism $\text{det} : \text{Div}^{O_C}_{n,0}(lD_0) \rightarrow \text{Div}^{O_C}_{1,0}(nlD_0)$. Again by the See-saw theorem, we have $\pi_1^*(\mathcal{L}_{n,l}) \cong \text{det}(\mathcal{E}_{n,l})$, where $\mathcal{E}_{n,l} \rightarrow \text{Div}^{O_C}_{n,0}(lD_0) \times C$ is the universal bundle on this projective space. We have $(\text{det} \times \text{id}_C)^*(\mathcal{E}_{1,nl}) \cong \text{det}(\mathcal{E}_{n,l})$ and again by the See-saw Theorem, there is a line bundle $\mathcal{L}_{1,nl} \rightarrow \text{Div}^{O_C}_{1,0}(nlD_0)$ such that $\pi_1^*(\mathcal{L}_{1,nl}) \cong \mathcal{E}_{1,nl}$.

We now have to delve into the proof of Theorem 5.6. In particular, for $nlD_0 > 2g - 2$, we see that $\text{Div}^{O_C}_{1,0}(nlD_0)$ is isomorphic to $\mathbb{P}^{nlD_0-g}$. Moreover, via this isomorphism, $\mathcal{L}_{1,nl}$ is identified with the tautological bundle $O(-1)$ on this projective space. By the proof of Theorem 5.6 we have a commutative diagram

\[
\begin{array}{ccc}
M(\text{Div}^{O_C}_{n,0}(lD_0)) & \rightarrow & M(\text{Bun}^{O_C}_{n,0}) \\
M(\text{det}) & & \sim \quad Z \otimes M(BG_m) \\
M(\text{Div}^{O_C}_{1,0}(nlD_0)) & \rightarrow & M(\mathbb{P}^{nlD_0-g}) \rightarrow M(BG_m).
\end{array}
\]

Hence the composition (17) is equal to

\[
M(\text{Div}^{O_C}_{n,0}(lD_0)) \xrightarrow{M(\text{det})} M(\text{Div}^{O_C}_{1,0}(lD_0)) \rightarrow M(BG_m) c_1(O(-1)) R\{1\}
\]

or in other words to the first Chern class of the pullback of $O(-1)$ along $\text{Div}^{O_C}_{n,0}(lD_0) \rightarrow BG_m$. As observed above, this pullback is equal to $\text{det}^*(\mathcal{L}_{1,nl}) \simeq \mathcal{L}_{n,l}$, which concludes the proof. \qed

APPENDIX A. ÉTALE MOTIVES OF STACKS

Let us explain and compare some definitions of motives of stacks (see also [11, 40, 24]).

A.1. Definitions. Let $DA^{\acute{e}t}(k, R)$ denote the triangulated category of \'{e}tale motives without transfers over $k$ with coefficients in $R$ [3, §3]. Recall that objects in $DA^{\acute{e}t}(k, R)$ can be represented by (symmetric) $T$-spectra in complexes of \'{e}tale sheaves of $R$-modules on the category of smooth $k$-schemes, where

\[ T := \text{Cone}(R^{\text{eff}}(\text{Spec } k) \rightarrow R^{\text{eff}}(G_m))[−1]. \]

Here $R^{\text{eff}}(X)$ denotes the \'{e}tale sheaf of $R$-modules freely generated by $X$ for $X \in \text{Sm}_k$. We write $R(X)$ for the suspension spectrum $\Sigma^\infty_+ R^{\text{eff}}(X)$. Compare this definition with the one for $DM(k, R)$ at the beginning of Section 2. The definition of $DM(k, R)$ also yields an \'{e}tale variant $DM^{\acute{e}t}(k, R)$ by replacing the Nisnevich topology by the \'{e}tale topology. The three categories are related by two adjunctions

\[
\mathbb{L}_{a_{tr}} : DA^{\acute{e}t}(k, R) \rightleftarrows DM^{\acute{e}t}(k, R) : \mathbb{R}\alpha_{tr}
\]

and

\[
\mathbb{L}_{a_{\acute{e}t}} : DM^{\text{Nis}}(k, R) \rightleftarrows DM^{\acute{e}t}(k, R) : \mathbb{R}\alpha_{\acute{e}t}.
\]

With our standing hypotheses on $(k, R)$, the functor $\mathbb{L}_{a_{tr}}$ is always an equivalence [3, Theorem B.1], while the functor $\mathbb{L}_{a_{\acute{e}t}}$ is an equivalence if $R$ is a $\mathbb{Q}$-algebra [41, Theorem 14.30].

In $DA^{\acute{e}t}(k, R)$, there are two natural ways to define the motive of an algebraic stack $\mathfrak{X}$: first using the Čech hypercover associated to an atlas and second via a nerve construction.

Definition A.1. Let $\mathfrak{X}$ be an algebraic stack. We define the following motives.
Remark A.2.
(i) For an atlas $A \to \mathcal{X}$, we let $A^{[n]} := A \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} A$ be the $n+1$-fold self-fibre product of $A$ in $\mathcal{X}$ or equivalently the $n$th Čech simplicial scheme associated to this atlas.\footnote{$A^{[n]}$ is a scheme itself, as $A$ is assumed to be a scheme and the morphism $A \to \mathcal{X}$ is representable} We define the étale motive of $\mathcal{X}$ with respect to this atlas as

$$M_{\text{ét}}(\mathcal{X},A) := \Sigma_T^\infty R^{\text{ét}}(A[^*]),$$

where $R^{\text{ét}}(A[^*])$ denotes the complex of étale sheaves associated to the simplicial sheaf of $R$-modules $A[^*]$.

(ii) Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of stacks; then for any atlas $A_{\mathcal{Y}} \to \mathcal{Y}$, the map $A_{\mathcal{X}} := A_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ is an atlas for $\mathcal{X}$ and thus there is a morphism

$$M_{\text{ét}}(f) : M_{\text{ét}}(\mathcal{X},A_{\mathcal{X}}) \to M_{\text{ét}}(\mathcal{Y},A_{\mathcal{Y}}).$$

(iii) We define the nerve-theoretic motive of $\mathcal{X}$ by taking the functor $\mathcal{X} : \text{Sm}_{k} \to \text{Gpds}$ valued in the category of groupoids and composing with the nerve $N : \text{Gpds} \to \text{sSets}$ to the category of simplicial sets and then finally composing with the singular complex $\text{sing} : \text{sSets} \to C_*\text{(abgp)}$ which takes values in the category of complexes of abelian groups; this gives a complex of étale sheaves of abelian groups $a_{\text{ét}}\text{sing} \circ N \circ \mathcal{X}$ on $\text{Sm}_{k}$, which we tensor with our coefficient ring $R$ and put

$$M_{\text{nerve}}(\mathcal{X}) := \Sigma_T^\infty (a_{\text{ét}}(\text{sing} \circ N \circ \mathcal{X}) \otimes R).$$

Remark A.2.
(i) By definition, the motive of an algebraic stack $\mathcal{X}$ with respect to an atlas $U \to \mathcal{X}$ and the nerve-theoretic motive of $\mathcal{X}$ are both effective motives, as both definitions take place in $\text{DA}^{\text{eff, ét}}(k,R)$.

(ii) If $\mathcal{X}$ is a scheme $X$, then clearly we have $M(X) \simeq M_{\text{atlas},X}(X)$ and in fact this also agrees with the nerve-theoretic definition by Lemma A.3 below.

Lemma A.3. In $\text{DA}^{\text{ét}}(k,R)$ for an algebraic stack $\mathcal{X}$ and for any atlas $A \to \mathcal{X}$, we have

$$M_{\text{atlas},A}(\mathcal{X}) \simeq M_{\text{nerve}}(\mathcal{X}).$$

In particular, the isomorphism class of $M_{\text{atlas},A}(\mathcal{X})$ is independent of the atlas.

Proof. This follows as $\text{DA}^{\text{ét}}(-,R)$ satisfies cohomological descent with respect to $h$-hypercoverings \cite[Theorem 14.3.4.(2)]{13}, thus in particular with respect to Čech hypercoverings induced by smooth surjective maps. For a more detailed argument, see \cite{11} which treats the case of a Deligne-Mumford stack $\mathcal{X}$ in $\text{DM}^{\text{ét}}(k,\mathbb{Q})$ (replacing étale descent by smooth descent). $\square$

For an exhaustive stack $\mathcal{X}$, one can define the motive of $\mathcal{X}$ in $\text{DA}^{\text{ét}}(k,R)$ (or in fact, for any choice of topology) exactly as in Definition \ref{def:motive_exhaustive} by using an exhaustive sequence of vector bundles with respect to a filtration of $\mathcal{X}$. In this appendix, to distinguish the motive in Definition \ref{def:motive_exhaustive} (resp. its étale, without transfers analogue) from other definitions, we denote it by $M_{\text{mono}}(\mathcal{X})$ (resp. $M_{\text{mono}}^{\text{ét}}(\mathcal{X})$) and say $\mathcal{X}$ is mono-exhaustive (rather than just exhaustive).

In fact, dual to Definition \ref{def:injective_sequence} of an (injective) exhaustive sequence is the following notion of a surjective exhaustive sequence as in \cite{10} [8].

Definition A.4. For a smooth stack $\mathcal{X}$, a surjective exhaustive sequence of vector bundles on $\mathcal{X}$ with respect to a filtration $\mathcal{X}_0 \xrightarrow{i_0} \mathcal{X}_1 \xrightarrow{i_1} \cdots \subset \mathcal{X}$ is a pair $(V_\bullet,W_\bullet)$ given by a sequence of vector bundles $V_m$ over $\mathcal{X}_m$ together with surjective maps of vector bundles $f_m : V_{m+1} \times_{\mathcal{X}_{m+1}} \mathcal{X}_m \to V_m$ and closed substacks $W_m \subset V_m$ such that

(i) the codimension of $W_m$ in $V_m$ tend towards infinity,
(ii) the complement $U_m := V_m - W_m$ is a separated finite type $k$-scheme, and
(iii) we have $W_{m+1} \times_{\mathcal{X}_{m+1}} \mathcal{X}_m \subset f_m^{-1}(W_m)$.\footnote{\cite{10}[8]}
If \( \mathcal{X} \) admits such a surjective sequence, we say \( \mathcal{X} \) is an epi-exhaustive stack. Then we define the motive of \( \mathcal{X} \) (resp. the étale motive of \( \mathcal{X} \)) with respect to such a surjective exhaustive sequence \((V_*, W_*)\) of vector bundles on \( \mathcal{X} \) by
\[
M_{\text{epi}}(\mathcal{X}) := \operatorname{holim}_m M(U_m) \in \text{DM}(k, R) \quad \text{(resp. } M_{\text{epi}}^\text{ét}(\mathcal{X}) := \operatorname{holim}_m M(U_m) \in \text{DM}(k, R))
\]
with transition maps given by the composition
\[
M(U_m) \cong M(V_{m+1} \times_{X_{m+1}} X_m - f_m^{-1}(W_m)) \to M(U_{m+1} \times_{X_{m+1}} X_m) \to M(U_{m+1})
\]
where the first isomorphism follows as \( V_{m+1} \times_{X_{m+1}} (X_m - f_m^{-1}(W_m)) \to (V_m - W_m) = U_m \) is a vector bundle, and the final maps come from the open immersions \( V_{m+1} \times_{X_{m+1}} X_m - f_m^{-1}(W_m) \subset U_{m+1} \times_{X_{m+1}} X_m \) and \( U_{m+1} \times_{X_{m+1}} X_m \to U_{m+1} \) (resp. the analogous maps for étale motives).

Similarly, we define the compactly supported motive of \( \mathcal{X} \) with respect to this sequence by
\[
M_{\text{c}}(\mathcal{X}) := \operatorname{holim}_m M(U_m)\{-r_m\}
\]
where \( r_m := \operatorname{rk}(V_m) \), with transition maps given by the composition
\[
\begin{align*}
M^c(U_{m+1})\{-r_{m+1}\} & \quad \longrightarrow \quad M^c(U_{m+1} \times_{X_{m+1}} X_m)\{-r_{m+1}\} \\
& \quad \downarrow \\
M^c(V_{m+1} \times_{X_{m+1}} X_m - f_m^{-1}(W_m))\{-r_{m+1}\} & \quad \cong \quad M^c(U_m)\{-r_{m}\}
\end{align*}
\]
where the first two morphisms are flat pullbacks associated to open immersions and the final isomorphism follows from \( \mathcal{A}^1 \)-homotopy invariance.

**Remark A.5.** Unlike in Definition 2.17 since we have homotopy (co)limits, Definition A.4 depends on the choice of a cone, hence is only defined up to a non-unique isomorphism.

Totaro proves his definition of the compactly supported motive of a quotient stack (defined using surjective exhaustive sequences) is independent of the choice of such sequence [40, Theorem 8.5]; in fact, his proof can also be adapted following the lines of Lemma 2.19 to show the above definitions for the motive of an epi-exhaustive smooth stack is independent of the filtrations and the choices of surjective exhaustive sequences.

The construction implies immediately the following isomorphisms.

**Lemma A.6.** Let \( \mathcal{X} \) be a mono-exhaustive stack (resp. an epi-exhaustive stack). Then
\[
\mathbb{L}_{\text{at}} M^\text{ét}_{\text{mono}}(\mathcal{X}) \cong \mathbb{L}_{\text{at}} M^\text{ét}_{\text{mono}}(\mathcal{X}) \quad \text{(resp. } \mathbb{L}_{\text{at}} M^\text{c}_{\text{epi}}(\mathcal{X}) \cong \mathbb{L}_{\text{at}} M^\text{c}_{\text{epi}}(\mathcal{X}))
\]
in \( \text{DM}^\text{ét}(k, R) \).

**A.2. Main comparison result.**

**Proposition A.7.** Let \( \mathcal{X} \) be a mono-exhaustive (resp. epi-exhaustive) smooth stack; then we have
\[
\mathbb{L}_{\text{at}} M^\text{ét}_{\text{atlas}}(\mathcal{X}) \cong \mathbb{L}_{\text{at}} M^\text{ét}_{\text{mono}}(\mathcal{X}) \quad \text{(resp. } \mathbb{L}_{\text{at}} M^\text{c}_{\text{atlas}}(\mathcal{X}) \cong \mathbb{L}_{\text{at}} M^\text{c}_{\text{epi}}(\mathcal{X}))
\]
in \( \text{DM}^\text{ét}(k, R) \) for any choice of atlas \( \mathcal{A} \to \mathcal{X} \). In particular, when \( R \) is a \( \mathbb{Q} \)-algebra, the motive \( M^\text{mono}_0(\mathcal{X}) \) (resp. \( M^\text{c}_{\text{epi}}(\mathcal{X}) \)) can be computed from the étale motive.

**Proof.** We present the proof in the mono-exhaustive case, the epi-exhaustive case is similar. By the first isomorphism in Lemma A.6 and the fact that \( \mathbb{L}_{\text{at}} \) is an equivalence of categories, we see that it suffices to construct an isomorphism \( M^\text{ét}_{\text{atlas}}(\mathcal{X}) \cong M^\text{c}_{\text{mono}}(\mathcal{X}) \).

Let \( \mathcal{A} \to \mathcal{X} \) be any atlas of \( \mathcal{X} \). In particular, \( \mathcal{A} \) is a smooth \( \mathcal{A} \)-scheme, not of finite type in general. Recall that by hypothesis, there is a filtration \( \mathcal{X} = \bigcup_{m \in \mathbb{N}} X_m \) by increasing quasi-compact open substacks and an exhaustive sequence of vector bundles \((V_*, W_*)\) with respect to this filtration. For any \( m \in \mathbb{N} \), because \( X_m \) is quasi-compact there exists a union \( \mathcal{A}_m \) of finitely many connected components of \( \mathcal{A} \times \mathcal{X} X_m \) (so that \( \mathcal{A}_m \) is a smooth finite type \( \mathcal{A} \)-scheme) such that \( \mathcal{A}_m \to X_m \) is an atlas; we can assume furthermore that \( \mathcal{A}_m \subset \mathcal{A}_{m+1} \) and that \( \mathcal{A} = \bigcup_m \mathcal{A}_m \). This implies that the étale sheaf \( R^\text{ét}(\mathcal{A}) \) is the colimit of \( (R^\text{ét}(\mathcal{A}_m))_{m \in \mathbb{N}} \). For \( m, k \geq 0 \), we let
\[
\tau_{k,m} : R(\mathcal{A}_m^{(k)}) := \Sigma^k_{\leq k} \left( \cdots \to 0 \to R(\mathcal{A}_m^{[k-1]}) \to \cdots \to R(\mathcal{A}_m) \to 0 \to \cdots \right)
\]
In the category of $T$-spectra of complexes of étale sheaves, we have
\[
R(A^{[n]}) \simeq \text{colim}_k \tau_{\leq k}(R(A^{[k]})) \simeq \text{colim}_m \text{colim}_k \tau_{\leq k}(R(A^{[k]})) \\
\simeq \text{colim}_m \text{colim}_k \tau_{\leq k}(R(A^{[k]})) \simeq \text{colim}_m R(A^{[m]}).
\]

For $m \in \mathbb{N}$, let $\tilde{V}_m := V_m \times_k A_m$ and define $\tilde{W}_m$ and $\tilde{U}_m$ analogously; as $A \to X$ is representable, these are (smooth, finite type) $k$-schemes. The maps $V_m \to V_{m+1}$ lift to maps $\tilde{V}_m \to \tilde{V}_{m+1}$. We have
\[
\text{colim}_m R(V_m \times_X A^{[m]}) \simeq \text{colim}_m R(A^{[m]}),
\]
which is a filtered colimit of $\mathbb{A}^1$-equivalences, thus an $\mathbb{A}^1$-equivalence. We have $V_m \times_X A^{[n]} \simeq \tilde{V}_m \times_k A_m^{[n-1]}$ for all $n \geq 1$, thus each term is a smooth finite type $k$-scheme and
\[
\text{codim}(V_m \times_X A^{[n]}) \geq \text{codim}_{\tilde{V}_m} W_m
\]
tends to $\infty$ uniformly in $n$. Proposition 2.13 and a truncation argument as above imply that
\[
\text{colim}_m R(U_m \times_X A^{[m]}) \to \text{colim}_m R(V_m \times_X A^{[m]})
\]
is an $\mathbb{A}^1$-equivalence. For every $m \geq 1$, the morphism $R(U_m \times_X A^{[m]}) \to R(U_m)$ is an $\mathbb{A}^1$-equivalence by cohomological descent with respect to $h$-hypercoverings [13, Theorem 14.3.4.(2)]. We finally conclude that the map
\[
\text{colim}_m R(V_m \times_X A^{[m]}) \to \text{colim}_m R(U_m)
\]
is an $\mathbb{A}^1$-equivalence since it is a filtered colimit of $\mathbb{A}^1$-equivalences. Putting everything together, we get an isomorphism in $\text{DA}^\text{et}(k, R)$
\[
M^{\text{ét}}_{\text{atlas, } A}(X) \simeq \text{colim}_m R(U_m) =: M^{\text{ét}}_{\text{mono}}(X)
\]
which concludes the proof. \qed

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