\( \mathcal{N} = 1/2 \) Super Yang-Mills Theory on Euclidean \( AdS_2 \times S^2 \)

Ali Imaanpur\(^a,b\) and Shahrokh Parvizi\(^a\)

\(^a\) Institute for Studies in Theoretical Physics and Mathematics (IPM)
P.O. Box 19395-5531, Tehran, Iran

\(^b\) Department of Physics, School of Sciences
Tarbiat Modares University, P.O. Box 14155-4838, Tehran, Iran

aimaanpu, parvizi@theory.ipm.ac.ir

Abstract

We study D-branes in the background of Euclidean \( AdS_2 \times S^2 \) with a graviphoton field turned on. As the background is not Ricci flat, the graviphoton field must have both self-dual and antself-dual parts. This, in general, will break all the supersymmetries on the brane. However, we show that there exists a limit for which one can restore half of the supersymmetries. Further, we show that in this limit, the \( \mathcal{N} = 1/2 \) SYM Lagrangian on flat space can be lifted on to the Euclidean \( AdS_2 \times S^2 \) preserving the same amount of supersymmetries as in the flat case. We observe that without the \( C \)-dependent terms present in the action this lift is not possible.
1 Introduction

Recent studies on string theory in the background of a graviphoton field have revealed new structures on the worldvolume of the corresponding D-branes. In fact, it is found that in such a background the odd coordinates of the superspace turn out to be nonanticommuting. There are, though, two different approaches to the problem. Either, one could insist on preserving the whole $\mathcal{N} = 1$ supersymmetry, as in the work of Ooguri-Vafa [1]. Or, as in [2, 3], one could assume that the anticommutation relations between the odd coordinates on superspace survive the field theory limit. In the latter case, however, one loses half of the supersymmetries, and so it is called $\mathcal{N} = 1/2$ super Yang-Mills theory. Superspace deformations of this kind have been studied in some earlier works [4, 5, 6, 7, 8]. But the fact that it arises in a natural way from string theory was the consequence of recent works [9, 1, 2]. Different aspects of $\mathcal{N} = 1/2$ supersymmetric models have further been studied in [10]-[17]. While the instanton solutions and some nonperturbative effects [18]-[22], along with the generalizations to $\mathcal{N} = 2$, and other interesting features of noncommutative superspace have been explored in [23]-[35].

So far, the study of D-branes in the presence of a graviphoton field has been restricted to flat space times. Roughly speaking, the supergravity field equations imply that for having a flat background, one has to choose a graviphoton field which has a zero energy momentum tensor. In Euclidean signature, on the other hand, one way of getting a zero energy momentum tensor is to choose the graviphoton field to be (anti)self-dual. For a typical graviphoton field, however, the energy momentum tensor is not necessarily vanishing. And hence, the supergravity equations would imply that the spacetime cannot be flat. In Lorentzian signature, a well known example of this type is $AdS_2 \times S^2$ together with a “self-dual” graviphoton field. In this article, we will be studying the Euclidean version of this solution.

Upon rotation to Euclidean space, the graviphoton field will have both self-dual and antself-dual parts. From the standard arguments in string theory it then follows that the anticommutation relations for both right-handed and left-handed odd coordinates on the D-brane get deformed. Furthermore, this also affects the supersymmetry algebra breaking all the supercharges on the brane. In this article, we take a scaling limit where the self-dual part becomes very large, and at the same time the antself-dual part goes to zero. The limiting process, however, leaves the energy momentum tensor unaffected and so the $AdS_2 \times S^2$ background is left unchanged. As a result of this limiting half of the supersymmetries can be restored, just as in the flat case. Having the $\mathcal{N} = 1/2$ supersymmetry algebra, we go on to define the corresponding Lagrangian on the $AdS_2 \times S^2$ background. In doing so, we will make two interesting observations. Firstly, we observe that for having a supersymmetric Lagrangian in this background one actually does need the extra $\mathcal{C}$-dependent terms present in the Seiberg action. In other words, it is not possible to have a pure $\mathcal{N} = 1$ SYM action on $AdS_2 \times S^2$. Secondly, even for having an $\mathcal{N} = 1/2$ supersymmetric Lagrangian in this background we need to modify the supersymmetry transformations. However, we show that the modified supersymmetry transformations continue to be a realization of $\mathcal{N} = 1/2$ supersymmetry algebra.

At the end, we comment on the plane wave limit of $AdS_2 \times S^2$ in the Lorentzian signature. In this limit the deformation parameter $\mathcal{C}$ will have a zero determinant, which
allows rotation to a frame where half of the odd coordinates become anticommuting. However, making the corresponding supercharges anticommute is not straightforward. In fact, we find a linear combination of supercharges that anticommute, but then they will not act as derivations.

2 String Theory on $AdS_2 \times S^2$ with RR Graviphoton Background

Consider type-II string theory compactified on a 6-dimensional Calabi-Yau $M$. Besides the NS-NS fields, we are also considering form fields which come from the compactification of RR form fields on the Calabi-Yau cycles. Among these fields there is a one form field, the graviphoton field, which can be obtained from wrapping RR-forms on cycles of $M$. In pure spinor notation the field strength of this graviphoton can be shown as follows:

$$P^{\alpha\beta} = \frac{1}{2} P^{\mu\nu} (\sigma_{\mu\nu})^{\alpha\beta}$$

$$P^{\dot{\alpha}\dot{\beta}} = \frac{1}{2} P^{\mu\nu} (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} ,$$

where by construction $P^{\alpha\beta}$ and $P^{\dot{\alpha}\dot{\beta}}$ are self-dual and antiself-dual parts of the graviphoton field, respectively. Here $(\alpha, \dot{\alpha}, \hat{\alpha}, \dot{\hat{\alpha}})$ show the left-handed and right-handed indices of $\mathcal{N} = 2$ fermions.

In Euclidean signature, the energy momentum tensor of a self-dual (antiself-dual) graviphoton field is zero, and thus there is no back reaction on the geometry. However, if we consider a typical graviphoton field with both self-dual and antiself-dual parts, there will be a back reaction. In this article, we will be interested in the specific example of Euclidean $AdS_2 \times S^2$. We are going to examine the branes in this background, and in the next section write down their effective SYM Lagrangian. In Lorentzian signature, this background together with a “self-dual” graviphoton field is a well-known supergravity solution, on which the string theory is already studied [36, 37].

Let us start with the string action on flat space, and in the hybrid or pure spinor formalism [37]:

$$S_{flat} = \frac{1}{\alpha'} \int d^2 z \left( \frac{1}{2} \partial_z x^\mu \partial_z x_\mu + p_\alpha \partial_z \theta^\alpha + \bar{p}_{\dot{\alpha}} \partial_z \bar{\theta}^{\dot{\alpha}} + p_{\dot{\alpha}} \partial \theta^{\dot{\alpha}} + \bar{p}_{\hat{\alpha}} \partial \bar{\theta}^{\hat{\alpha}} \right)$$

where $p$’s are the momentum conjugate to $\theta$’s. Following [37], we introduce a new set of variables (and similar definitions for $\bar{d}, \bar{d}, \bar{q}$ and $\bar{q}$):

$$y^\mu = x^\mu + i \theta^\alpha \sigma_{\alpha a} \bar{\theta}^{\bar{\alpha}} + i \bar{\theta}^{\dot{\alpha}} \sigma_{a \dot{\alpha}} \bar{\theta}^{\dot{\bar{\alpha}}}$$

$$\bar{q}_{\bar{a}} = \bar{p}_{\bar{a}} - i \theta^\alpha \sigma_{\alpha \bar{a}} \partial_z x_\mu - \theta \bar{\theta} \partial_z \bar{\theta}_{\bar{a}} + \frac{1}{2} \bar{\theta}_{\bar{a}} \partial_z (\theta \bar{\theta})$$

$$d_\alpha = - p_\alpha - i \sigma_{\alpha \bar{a}} \bar{\theta}^{\bar{\alpha}} \partial_z x_\mu + \frac{1}{2} \theta \bar{\theta} \partial_z \theta_\alpha - \frac{3}{2} \partial_z (\theta_\alpha \bar{\theta})$$

$$q_\alpha = - p_\alpha + i \sigma_{\alpha \bar{a}} \bar{\theta}^{\bar{\alpha}} \partial_z x_\mu - \bar{\theta} \partial_z \theta_\alpha + \frac{1}{2} \theta_\alpha \partial_z (\bar{\theta} \theta)$$

$$\bar{d}_{\bar{\alpha}} = \bar{p}_{\bar{\alpha}} + i \theta^\dot{\alpha} \sigma_{a \dot{\alpha}} \partial_z x_\mu + \frac{1}{2} \theta \bar{\theta} \partial_z \bar{\theta}_{\dot{\alpha}} - \frac{3}{2} \partial_z (\bar{\theta}_{\dot{\alpha}} \theta) .$$

(2.4)
By adding the vertex operator of the graviphoton field, we find the following action:

$$S_{flat} = \frac{1}{\alpha'} \int d^2z \left( \frac{1}{2} \partial_z y^\mu \partial y_\mu - d_\alpha \partial z \theta^\alpha + \bar{q}_\bar{\alpha} \partial z \bar{\theta}^\bar{\alpha} - d_\bar{\alpha} \partial \bar{\theta}^\bar{\alpha} + \bar{q}_\bar{\alpha} \partial \theta^\alpha \right) + \alpha' P^{\alpha \dot{\beta}} d_\alpha d_{\dot{\beta}} + \alpha' \bar{P}^{\dot{\alpha} \dot{\beta}} \bar{d}_{\dot{\alpha}} \bar{d}_{\dot{\beta}} ) . \quad (2.5)$$

It is now straightforward to covariantize the above action, and define it on $AdS_2 \times S^2$ background, as follows

$$S = \frac{1}{\alpha'} \int \left[ \frac{1}{2} \Pi^\alpha \Pi_\alpha + \bar{d}_{\bar{\alpha}} \Pi^\bar{\alpha} + d_\alpha \Pi_\bar{\alpha} + \bar{d}_{\dot{\alpha}} \Pi_{\dot{\alpha}} \right] + \alpha' d_\alpha P^{\alpha \dot{\beta}} d_{\dot{\beta}} + \alpha' \bar{d}_{\bar{\alpha}} \bar{P}^{\dot{\alpha} \dot{\beta}} \bar{d}_{\dot{\beta}} \right] , \quad (2.6)$$

where we have defined $\Pi^\alpha_A \equiv E^A_M \partial_z Z^M$, with $E^A_M$ representing the supervierbein and $Z^M = (x^\mu, \theta^\alpha, \bar{\theta}^{\bar{\alpha}}, \dot{\theta}^\beta, \dot{\bar{\theta}}^{\dot{\bar{\alpha}}})$. The index $A$ indicates the tangent superspace indices ($c, \alpha, \dot{\alpha}, \dot{\beta}, \dot{\bar{\alpha}}$). The lowest component of $E^\alpha_\mu$ is the vierbein, and the lowest components of $E^\alpha_M$ and $E^\dot{\dot{\alpha}}_M$ are the gravitini which are set to zero in our background (no gravitino). The equations of motion for $d_\alpha$ and $d_{\dot{\alpha}}$ fields read,

$$\alpha' P^{\alpha \dot{\beta}} d_{\dot{\beta}} + \Pi^\alpha = 0 ,$$
$$\alpha' P^{\dot{\alpha} \beta} d_{\beta} - \Pi^\dot{\alpha} = 0 . \quad (2.7)$$

Using these equations of motions, we can integrate out the corresponding fields (as well as $\bar{d}_{\bar{\alpha}}$ and $\bar{d}_{\dot{\bar{\alpha}}}$) to find the following action:

$$S = \frac{1}{\alpha'} \int dxdz \left[ \frac{1}{2} \Pi^\alpha \Pi_\alpha + \frac{1}{\alpha'} P^{\alpha \beta} \Pi^\beta + \frac{1}{\alpha'} P^{\dot{\alpha} \dot{\beta}} \Pi^\dot{\beta} \right] , \quad (2.8)$$

where $P^{\alpha \beta}$ and $\bar{P}^{\dot{\alpha} \dot{\beta}}$ are the inverses of $P^{\dot{\alpha} \beta}$ and $\bar{P}^{\dot{\alpha} \beta}$, respectively.

In the present set up, D-branes can be introduced by a set of consistent open string boundary conditions as follows

$$\theta_\alpha = \theta_{\bar{\alpha}} , \quad d_\alpha = d_{\bar{\alpha}} . \quad (2.9)$$

Putting the second equation above in (2.7) implies that

$$\Pi^\alpha = - \Pi^\dot{\alpha} . \quad (2.10)$$

To see the consequence of the above equation, let us first introduce the explicit forms of $\Pi^A_j$’s. Since the background is fixed, in the expression for the supervierbeins we set the gravitino field to zero, i.e., $\psi^\alpha_\mu = \psi^\dot{\alpha}_\mu = 0$, which therefore results to

$$\Pi^\alpha = \partial_z \theta^\alpha ,$$
$$\Pi_\alpha = \partial_\bar{\alpha} \theta^\alpha ,$$
$$\Pi^\dot{\beta} = c^\beta_\mu \partial z x^\mu + i \partial z \theta^\alpha \sigma^\mu_{\alpha \dot{\alpha}} \theta^{\dot{\alpha}} + i \bar{q}_{\dot{\alpha}} \sigma^\alpha_{\alpha \dot{\alpha}} \partial z \bar{\theta}^{\dot{\alpha}} + i \bar{q}_{\dot{\alpha}} \sigma^\alpha_{\alpha \dot{\alpha}} \partial z \bar{\theta}^{\dot{\alpha}} = c^\gamma_\mu \partial z y^\mu ,$$
$$\Pi^\beta = c^\beta_\mu \partial z x^\mu + i \partial z \theta^\alpha \sigma^\mu_{\alpha \dot{\alpha}} \theta^{\dot{\alpha}} + i \bar{q}_{\dot{\alpha}} \sigma^\alpha_{\alpha \dot{\alpha}} \partial z \bar{\theta}^{\dot{\alpha}} + i \bar{q}_{\dot{\alpha}} \sigma^\alpha_{\alpha \dot{\alpha}} \partial z \bar{\theta}^{\dot{\alpha}} = c^\gamma_\mu \partial z y^\mu . \quad (2.11)$$
Further, equation (2.10) now implies the boundary conditions on the odd coordinates

$$\partial_\alpha \theta^\alpha = -\partial_\bar{\alpha} \bar{\theta}^\alpha ,$$  \hspace{1cm} (2.12)

which, together with the bosonic boundary conditions, defines a D-brane filling the whole 4-dimensional space.

At this stage, by plugging (2.11) into (2.8), we can write the action in terms of $$(y^\mu, \theta^\alpha, \bar{\theta}^\bar{\alpha}, \tilde{\theta}^\hat{\alpha})$$. In these coordinates, it is easy to calculate the two point function of $\theta$'s:

$$\langle \theta^\alpha(z, \bar{z}) \theta^\beta(w, \bar{w}) \rangle = \frac{\alpha'^2 P^{\alpha \beta}}{2\pi i} \log \frac{\bar{z} - w}{z - \bar{w}} .$$  \hspace{1cm} (2.13)

Following the standard argument about the non-commutativity on the brane, we find a set of non(anti)commutative coordinates on the brane as follows:

$$\{ \theta^\alpha, \theta^\beta \} = \alpha'^2 P^{\alpha \beta}$$  \hspace{1cm} (2.14)

$$\{ \bar{\theta}^\bar{\alpha}, \bar{\theta}^\bar{\beta} \} = \alpha'^2 P^{\bar{\alpha} \bar{\beta}} .$$  \hspace{1cm} (2.15)

In the field theory limit, where the low energy dynamics of D-branes is studied, we have to consider the limit of $\alpha' \to 0$. However, the nonanticommuting characteristics of D-branes observed above can survive the limit if at the same time we take the limit $P^{\alpha \beta} \to \infty$, holding $C^{\alpha \beta} \equiv \alpha'^2 P^{\alpha \beta}$ fixed. Therefore in such a limit, we expect that the decoupled theory on the worldvolume of the brane to be a deformed SYM theory on a noncommutative curved superspace. However, a simple analogy with the supersymmetry algebra in the flat case shows that the above deformation will break all the supersymmetries.

Although the superspace deformation breaks all the supersymmetries, in the following, we show that there exists a limit where we can restore half of the supersymmetries. To see this, first note that in Euclidean signature the energy momentum tensor of the graviphoton field reads

$$T_{\mu \nu} = \frac{1}{2} C^+_{\mu \lambda} C^-_{\nu \rho} g^{\lambda \rho} ,$$  \hspace{1cm} (2.16)

where $C^+ (C^-)$ indicate the self-dual (antiself-dual) part of the graviphoton field. Therefore, if the metric is not Ricci flat, the supergravity equations imply that the graviphoton field must have both self-dual and antiself-dual parts. Let us, though, introduce a real parameter $k$ and write $T_{\mu \nu}$ differently

$$T_{\mu \nu} = \frac{1}{2} \left( k C^+_{\mu \lambda} \right) \left( \frac{1}{k} C^-_{\nu \rho} \right) g^{\lambda \rho} .$$  \hspace{1cm} (2.17)

Writing $T_{\mu \nu}$ in this way, suggests that we can define a new graviphoton field

$$\tilde{C}_{\mu \nu} = \tilde{C}^+_{\mu \nu} + \tilde{C}^-_{\mu \nu} = k C^+_{\mu \nu} + \frac{1}{k} C^-_{\mu \nu} ,$$  \hspace{1cm} (2.18)

with the same energy momentum tensor as before (i.e., when $k = 1$). Furthermore, this allows us to take a limit where $k \to \infty$, without disturbing the equations of motion or the
AdS$_2 \times S^2$ background. The point, however, is that in this limit the antiself-dual part of the graviphoton field goes to zero. Hence, recalling the definitions

$$C^{\alpha\beta} = \frac{1}{2} (\sigma_{\mu\nu})^{\alpha\beta} C^{\mu
u+},$$

$$C^{\dot{\alpha}\dot{\beta}} = \frac{1}{2} (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} C^{\mu
u-},$$

Eq. (2.15) implies that the usual anticommutation relation between $\bar{\theta}$ coordinates is restored in this limit, i.e. we will have

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} \quad (2.19)$$

$$\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0, \quad (2.20)$$

or in components we will have $C^{\alpha\beta} \sim k/R$, and $C^{\dot{\alpha}\dot{\beta}}$ scales to zero for large $k$. In the next section, we will further show that in the large $k$ limit, the $\mathcal{N} = 1/2$ SYM Lagrangian on flat Euclidean space [2] can be lifted on to the curved background AdS$_2 \times S^2$.

In the last part of this section, we examine the corresponding Euclidean solution of a given solution of supergravity equations of motion in Lorentzian signature. Specifically, it is well known that a “self-dual” graviphoton field

$$C = C_{\mu\nu} dx^\mu \wedge dx^\nu = 2R (\cosh \rho d\tau \wedge d\rho + \cos \theta d\psi \wedge d\theta), \quad (2.21)$$

together with a metric of AdS$_2 \times S^2$

$$ds^2 = R^2(-\cosh^2 \rho d\tau^2 + d\rho^2 + \cos^2 \theta d\psi^2 + d\theta^2) \quad (2.22)$$

solve the supergravity equations. Obviously if we rotate to a Euclidean signature by sending $\tau \rightarrow i\tau$, the rotated metric and graviphoton field continue to solve the supergravity equations in the Euclidean signature.† Upon Wick rotation, the graviphoton field will have both self-dual and antiself-dual parts

$$C = C^+ + C^-, \quad (2.23)$$

where

$$C^+ = (1 + i) R k \ (\cosh \rho d\tau \wedge d\rho + \cos \theta d\psi \wedge d\theta)$$

$$C^- = -(1 - i) \frac{R}{2k} \ (\cosh \rho d\tau \wedge d\rho - \cos \theta d\psi \wedge d\theta),$$

†The graviphoton field becomes complex-valued upon Wick rotation. Nevertheless, we could have instead worked with the following real-valued field

$$C^+ = 2Rk \ (\cosh \rho d\tau \wedge d\rho + \cos \theta d\psi \wedge d\theta)$$

$$C^- = -\frac{R}{k} \ (\cosh \rho d\tau \wedge d\rho - \cos \theta d\psi \wedge d\theta),$$

which also solves the supergravity field equations.
and we have included the $k$ factor for the limiting purposes. It is easy to explicitly check that the above field configuration, together with

$$ds^2 = R^2(\cosh^2 \rho d\tau^2 + d\rho^2 + \cos^2 \theta d\psi^2 d\theta^2), \quad (2.24)$$

is a solution to the supergravity equations in Euclidean signature.\footnote{We note that the field $C$ is covariantly constant, and thus satisfies both the Maxwell equations and the Bianchi identity.}

Now that we have a consistent Euclidean background solution, we can go on to discuss the construction of supersymmetric action in this background.

### 3 \( \mathcal{N} = 1/2 \) SYM action on Euclidean \( AdS_2 \times S^2 \)

We have chosen to study D-branes on Euclidean \( AdS_2 \times S^2 \) for the following reasons. Firstly, this background, with the corresponding graviphoton field, is a maximally supersymmetric solution admitting maximal number of Killing spinors. Secondly, in the Euclidean version, the self-dual and ant-self-dual parts of the graviphoton field are independent of each other, and thus can be scaled differently. For constructing the action, we choose the minimal coupling to the background metric. So basically we define the \( \mathcal{N} = 1/2 \) SYM action of Seiberg \[2\] on Euclidean \( AdS_2 \times S^2 \) simply by covariantizing the ordinary derivatives both in the action and in the supersymmetry transformations. This will then reduce to the action on flat space in the large \( R \) limit. The invariance of the action under the \( Q \) supersymmetry, however, is not obvious and we are going to check it in detail. Along the way, we make two important observations. Firstly, we observe that this lift is not possible for pure \( \mathcal{N} = 1 \) SYM action, and in fact one does need the extra \( C \) terms in the action for having invariance under \( Q \). Secondly, even in the case of \( C \)-deformed theory, we need to deform the supersymmetry variation of \( \bar{\lambda} \) to have a supersymmetric action.

To begin with, let us consider the deformed action of Seiberg \[2\] on a curved D-brane which has filled the Euclidean \( AdS_2 \times S^2 \) space:

$$S_{(C \neq 0)} = \int \sqrt{g} d^4 x \mathrm{Tr} \left[ -\frac{1}{2} F_{\mu \nu} F^{\mu \nu} - 2 i \lambda \sigma^\mu D_\mu \bar{\lambda} - i C^{\mu \nu} F_{\mu \nu} \bar{\lambda} \lambda + \frac{1}{4} |C|^2 (\bar{\lambda} \lambda)^2 \right], \quad (3.1)$$

where \( D_\mu \equiv \nabla_\mu + [A_\mu, ] \), and \( \nabla_\mu \) is the covariant derivative on the curved background \( AdS_2 \times S^2 \). We now show that the above action is invariant under the following deformed supersymmetry transformations

$$\begin{align*}
\delta \lambda &= \sigma^{\mu \nu} \epsilon (F_{\mu \nu} + \frac{i}{2} C_{\mu \nu} \bar{\lambda} \lambda) \\
\delta A_\mu &= -i \bar{\lambda} \sigma_\mu \epsilon \\
\delta F_{\mu \nu} &= i \epsilon (\sigma_\nu D_\mu - \sigma_\mu D_\nu) \bar{\lambda} + i (\nabla_\mu \epsilon \sigma_\nu - \nabla_\nu \epsilon \sigma_\mu) \bar{\lambda} \\
\delta \bar{\lambda} &= \frac{4 \epsilon}{\alpha'},
\end{align*} \quad (3.2)$$

We note that the field $C$ is covariantly constant, and thus satisfies both the Maxwell equations and the Bianchi identity.
where now, $\epsilon$ and $\bar{\epsilon}$ are the Killing spinors on Euclidean $AdS_2 \times S^2$ satisfying the Killing equations,

\[
\nabla_\mu \epsilon^\alpha = \frac{1}{\alpha'} C^{\alpha\beta} \sigma_{\mu\beta\dot{\alpha}} \epsilon^{\dot{\alpha}}
\]

\[
\nabla_\mu \bar{\epsilon}^{\dot{\alpha}} = \frac{1}{\alpha'} C_{\dot{\alpha}\dot{\beta}\dot{\mu}} \sigma_{\mu\dot{\alpha}} \epsilon^{\dot{\alpha}} .
\]

Note that, for the last transformation in (3.2) to make sense, only the $U(1)$ part of the $\lambda$ field is meant to transform. So if we let $a = 1, 2, 3$ and $\bar{a} = 4$ denote the $SU(N)$ and $U(1)$ gauge indices, respectively, then by the last transformation we mean $\delta \lambda^{\bar{a}} = 4 \delta a^4 / \alpha'$. Further note that, in the scaling limit of large $k$, if we choose $\epsilon \sim 1$ then the above equations imply that $\bar{\epsilon} \sim 1/k$. It is now straightforward to check that the action (3.1) is invariant under the transformations (3.2). Let us see this in some more details. The variation of terms present in the action are as follows:

\[
\delta \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right) = -2i \epsilon F_{\mu\nu} \sigma_\nu D_\mu \bar{\lambda} - 2i F_{\mu\nu} \nabla_\mu \epsilon \sigma_\nu \bar{\lambda} \tag{3.4}
\]

\[
\delta \left( -2i \lambda \sigma_\mu D_\mu \bar{\lambda} \right) = -2i(\sigma_{\mu \nu}) (F_{\mu\nu} + \frac{i}{2} C_{\mu\nu} \bar{\lambda} \lambda) \sigma_\rho D_\rho \bar{\lambda} - \frac{8i}{\alpha'} \lambda \sigma_\mu \nabla_\mu \bar{\epsilon} \tag{3.5}
\]

\[
\delta \left( -i C_{\mu\nu} F_{\mu\nu} \bar{\lambda} \lambda \right) = 2i C_{\mu\nu} \sigma_\nu D_\mu \bar{\lambda} (\lambda \lambda) + 2 C^{\mu\nu} \nabla_\mu \epsilon \sigma_\nu \bar{\lambda} (\lambda \lambda) - \frac{8i}{\alpha'} C_{\mu\nu} F_{\mu\nu} (\epsilon \bar{\lambda}) \tag{3.6}
\]

\[
\delta \left( \frac{1}{4} |C|^2 (\lambda \lambda) \right)^2 = \frac{4}{\alpha'} |C|^2 (\lambda \lambda) (\epsilon \bar{\lambda}) . \tag{3.7}
\]

Upon using the Killing spinor Eqs. (3.3), we see that the two terms on the RHS of (3.4), and the first term on the RHS of (3.5) can be combined into

\[
-2i \epsilon F_{\mu\nu} \sigma_\nu D_\mu \bar{\lambda} - 2i F_{\mu\nu} \nabla_\mu \epsilon \sigma_\nu \bar{\lambda} - 2i F_{\mu\nu} (\sigma_{\mu \nu}) \sigma_\rho D_\rho \bar{\lambda} = \frac{8i}{\alpha'} C_{\mu\nu} F_{\mu\nu} (\epsilon \bar{\lambda}) , \tag{3.8}
\]

which cancels the last term in Eq. (3.6). The second term of (3.5) cancels the first term in (3.6)

\[
C_{\mu\nu} \sigma_{\mu \nu} \epsilon \sigma_\rho D_\rho \bar{\lambda} (\lambda \lambda) + 2 \epsilon C_{\mu\nu} \sigma_\nu D_\mu \bar{\lambda} (\lambda \lambda) = 0 . \tag{3.9}
\]

And finally, using (3.3) again, the second term of Eq. (3.6) cancels the one in (3.7),

\[
2 C^{\mu\nu} \nabla_\mu \epsilon \sigma_\nu \bar{\lambda} (\lambda \lambda) + \frac{4}{\alpha'} |C|^2 (\lambda \lambda) (\epsilon \bar{\lambda}) = 0 . \tag{3.10}
\]

Also we note that the last term in (3.5) goes to zero in the limit $k \to \infty$. This completes the proof of invariance of the action under $Q$ supersymmetry.

The modification we made in the transformation of $\bar{\lambda}$ is necessary for two reasons. Firstly, although it is of order $1/k$ and vanishes when $k \to \infty$, we have to keep it as it gives rise to some finite terms when acted on, for example, the third term in the action which is of order $k$. Secondly, this modification makes $Q^2$ vanish on shell. On the other hand, in the limit of $k \to \infty$, one gets back the usual $\mathcal{N} = 1/2$ algebra.
4 Plane wave limit of $\text{AdS}_2 \times S^2$

In this section, we study the plane wave limit of our supergravity setup. To consider this limit, we return to the Lorentzian version of $\text{AdS}_2 \times S^2$ in the presence of graviphoton field, and take the limit for both metric and RR field. The $\text{AdS}_2 \times S^2$ metric reads:

$$ds^2 = R^2(-\cosh^2 \rho \, dr^2 + d\rho^2 + \cos^2 \theta \, d\psi^2 + d\theta^2). \quad (4.1)$$

We now switch to the light cone coordinates $\tilde{x}^\pm = (\tau \mp \psi)/2$, and do the following rescaling,

\begin{align*}
x^+ &= \tilde{x}^+, \quad r = \rho R, \\
x^- &= \tilde{x}^-/R^2, \quad y = \theta R. \quad (4.2)
\end{align*}

Taking the large limit of $R$, we arrive at the plane-wave limit of the metric and the RR-form field,

\begin{align*}
ds^2 &= -4 dx^+ dx^- - (r^2 + y^2)(dx^+)^2 + dr^2 + dy^2 \\
C &= dx^+ \wedge dr + dx^+ \wedge dy. \quad (4.3)
\end{align*}

In the vierbeins basis $e^a_\mu$, the graviphoton field can be written as,

$$C^{\alpha\beta} = \frac{1}{2} C^{\mu\nu} (\sigma_{ab})^{\alpha\beta} e^a_\mu e^b_\nu. \quad (4.4)$$

Substituting the corresponding vierbeins, we find

\begin{align*}
C^{\alpha\beta} &= \frac{1}{2}(\sigma_{20} + \sigma_{30} + \sigma_{12} + \sigma_{13})^{\alpha\beta} \\
C^{\dot{\alpha}\dot{\beta}} &= \frac{1}{2}(\sigma_{20} + \sigma_{30} + \sigma_{12} + \sigma_{13})^{\dot{\alpha}\dot{\beta}}. \quad (4.5)
\end{align*}

which, as shown in section 2, give rise to the non-(anti)commutativity relations between the odd coordinates

\begin{align*}
\{\theta^\alpha, \theta^\beta\} &= C^{\alpha\beta} \\
\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} &= C^{\dot{\alpha}\dot{\beta}}. \quad (4.6)
\end{align*}

At first sight, it seems that the above anti-commutation relations break all the supersymmetries. However, note that in the plane wave limit (4.5) the determinant of $C^{\alpha\beta}$ (and $C^{\dot{\alpha}\dot{\beta}}$) vanishes, which means that there exists a linear combination of $\theta^\alpha$’s for which some of the anticommutators are zero. Therefore, in principle, it should be possible to restore part of the supersymmetry algebra. In fact, it is possible to redefine the supercharges as

\begin{align*}
\hat{Q}_1 &= A Q_1 + B Q_2 \\
\hat{Q}_2 &= B Q_1 + A Q_2, \quad (4.7)
\end{align*}

with some coefficients $A$ and $B$, such that the algebra of supercharges is changed into a more attractive one,

\begin{align*}
\{\hat{Q}_2, \hat{Q}_2\} &= \{\hat{Q}_1, \hat{Q}_2\} = 0 \\
\{\hat{Q}_1, \hat{Q}_1\} &\neq 0. \quad (4.8)
\end{align*}
The above algebra can be interpreted as the Lorentzian $\mathcal{N} = 1/2$ SUSY algebra for which $\hat{Q}_1$ and $\hat{Q}_1$ are broken while $\hat{Q}_2$ and $\hat{Q}_2$ are survived. Unfortunately, since $A$ and $B$ in (4.7) depend on derivatives, the new charges are not linear in derivatives, and hence not obeying the Leibnitz’s rule of derivation.

5 Summary and Conclusion

In trying to extend the $\mathcal{N} = 1/2$ supersymmetric theory from flat space to a curved space as an effect of a graviphoton field, we observed that a graviphoton field with both self-dual and antiself-dual parts breaks the supersymmetry completely. On the other hand, taking a self-dual graviphoton field is not a solution to the supergravity equations in a curved background. However, we introduced a limit in which we could keep the background as Euclidean $AdS_2 \times S^2$ while the antiself-dual part of the graviphoton is approaching zero. We showed in this limit, the $C$-deformed SYM theory, as introduced in [2] for flat space, can be lifted on the $AdS_2 \times S^2$ background. Further, by a small modification of the SUSY transformations, we proved the $\mathcal{N} = 1/2$ invariance of the theory.

Acknowledgment

Authors are very grateful to G. Mandal and S. Trivedi for fruitful discussions around the subject. S. P. would like to thank the hospitality of the department of theoretical physics in Tata Institute of Fundamental Research, Mumbai, and also the Center for Advanced Mathematical Sciences, Beirut, where part of this work was done. The work of S. P. was supported by Iranian TWAS chapter Based at ISMO.

References

[1] H. Ooguri, and C. Vafa, The C-Deformation of Gluino and Non-planar Diagrams, Adv. Theor. Math. Phys. 7, 53 (2003), [arXiv:hep-th/0302109].

[2] N. Seiberg, Noncommutative Superspace, N=1/2 Supersymmetry, Field Theory and String Theory, JHEP 0306 (2003) 010, [arXiv:hep-th/0305248].

[3] N. Berkovits, and N. Seiberg, Superstrings in Graviphoton Background and N=1/2+3/2 Supersymmetry, JHEP 0307 (2003) 010, [arXiv:hep-th/0306226].

[4] R. Casalbuoni, Relativity And Supersymmetries, Phys. Lett. B 62, 49 (1976), On The Quantization Of Systems With Anticommutating Variables, Nuovo Cim. A 33, 115 (1976), The Classical Mechanics For Bose-Fermi Systems, Nuovo Cim. A 33, 389 (1976).

[5] J.H. Schwarz, and P. van Nieuwenhuizen, Speculations Concerning a Fermionic Structure of Space-time, Lett. Nuovo Cim. 34 (1982) 21.
[6] P. Bouwknegt, J. G. McCarthy and P. van Nieuwenhuizen, *Fusing the coordinates of quantum superspace*, Phys. Lett. B 394, 82 (1997), [arXiv:hep-th/9611067].

[7] S. Ferrara, M.A. Lledo, *Some Aspects of Deformations of Supersymmetric Field Theories*, JHEP 0005 (2000) 008, [arXiv:hep-th/0002084].

[8] D. Klemm, S. Penati, and L. Tamassia, *Non(anti)commutative Superspace*, Class. Quant. Grav. 20 (2003) 2905, [arXiv:hep-th/0104190].

[9] J. de Boer, P. Grassi, and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, Phys. Lett. B 574, 98 (2003), [arXiv:hep-th/0302078].

[10] R. Britto, B. Feng, and S.J. Rey, *Deformed Superspace, N=1/2 Supersymmetry and (Non)Renormalization Theorems*, JHEP 0307 (2003) 067, [arXiv:hep-th/0306215].

[11] R. Britto, B. Feng, and S.J. Rey, *Non(anti)commutative Superspace, UV/IR Mixing, and Open Wilson Lines*, JHEP 0308 (2003) 001, [arXiv:hep-th/0307091].

[12] M. T. Grisaru, S. Penati, and A. Romagnoni, *Two-loop Renormalization for Nonanti-commutative N=1/2 Supersymmetric WZ Model*, JHEP 0308 (2003) 003, [arXiv:hep-th/0307099].

[13] R. Britto, and B. Feng, *N=1/2 Wess-Zumino model is renormalizable*, Phys. Rev. Lett. 91, 201601 (2003), [arXiv:hep-th/0307165].

[14] A. Romagnoni, *Renormalizability of N=1/2 Wess-Zumino model in superspace*, JHEP 0310, 016 (2003), [arXiv:hep-th/0307209].

[15] O. Lunin, and S.J. Rey, *Renormalizability of Non(anti)commutative Gauge Theories with N=1/2 Supersymmetry*, JHEP 0309, 045 (2003), [arXiv:hep-th/0307275].

[16] D. Berenstein, and S.J. Rey, *Wilsonian Proof for Renormalizability of N=1/2 Supersymmetric Field Theories*, Phys. Rev. D 68, 121701 (2003), [arXiv:hep-th/0308049].

[17] M. Alishahiha, A. Ghodsi and N. Sadooghi, *One-loop perturbative corrections to non(anti)commutativity parameter of N = 1/2 supersymmetric U(N) gauge theory*, arXiv:hep-th/0309037.

[18] A. Imaanpur, *On Instantons and Zero Modes of N = 1/2 SYM Theory*, JHEP 0309, 077 (2003), [arXiv:hep-th/0308171].

[19] A. Imaanpur, *Comments on Gluino Condensates in N = 1/2 SYM Theory*, JHEP 0312, 009 (2003), [arXiv:hep-th/0311137].

[20] P. A. Grassi, R. Ricci and D. Robles-Llana, *Instanton calculations for N = 1/2 super Yang-Mills theory*, arXiv:hep-th/0311155.

[21] R. Britto, B. Feng, O. Lunin and S. J. Rey, *U(N) instantons on N = 1/2 superspace: Exact solution and geometry of moduli space*, arXiv:hep-th/0311275.
[22] M. Billo, M. Frau, I. Pesando and A. Lerda, *N = 1/2 gauge theory and its instanton moduli space from open strings in R-R background*, arXiv:hep-th/0402160.

[23] T. Araki, K. Ito, and A. Ohtsuka, *Supersymmetric Gauge Theories on Noncommutative Superspace*, Phys. Lett. B 573, 209 (2003), [arXiv:hep-th/0307076].

[24] S. Ferrara, and E. Sokatchev, *Non-anticommutative N=2 super-Yang-Mills theory with singlet deformation*, Phys. Lett. B 579, 226 (2004), [arXiv:hep-th/0308021].

[25] E. Ivanov, O. Lechtenfeld, and B. Zupnik, *Nilpotent deformations of N=2 superspace*, JHEP 0402, 012 (2004), [arXiv:hep-th/0308012].

[26] S. Terashima, and J. Yee, *Comments on Noncommutative Superspace*, JHEP 0312, 053 (2003), [arXiv:hep-th/0306237].

[27] R. Abbaspur, *Scalar Solitons in Non(anti)commutative Superspace*, arXiv:hep-th/0308050.

[28] M. Chaichian, and A. Kobakhidze, *Deformed N=1 supersymmetry*, arXiv:hep-th/0307243.

[29] I. Bars, C. Deliduman, A. Pasqua and B. Zumino, *Superstar in noncommutative superspace via covariant quantization of the superparticle*, Phys. Rev. D 68, 106006 (2003), [arXiv:hep-th/0308107].

[30] A. Sako and T. Suzuki, *Ring structure of SUSY $\ast$ product and 1/2 SUSY Wess-Zumino model*, Phys. Lett. B 582, 127 (2004), [arXiv:hep-th/0309076].

[31] B. Chandrasekhar and A. Kumar, *D = 2, N = 2, supersymmetric theories on non(anti)commutative superspace*, JHEP 0403, 013 (2004), [arXiv:hep-th/0310137].

[32] D. Mikulovic, *Seiberg-Witten map for superfields on canonically deformed N = 1, d = 4 superspace*, JHEP 0401, 063 (2004), [arXiv:hep-th/0310065].

[33] S. Iso, and H. Umetsu, *Gauge Theory on Noncommutative Supersphere from Supermatrix Model*, arXiv:hep-th/0311005.

[34] T. Araki, K. Ito and A. Ohtsuka, *N = 2 supersymmetric U(1) gauge theory in noncommutative harmonic superspace*, JHEP 0401, 046 (2004), [arXiv:hep-th/0401012].

[35] C. Saemann and M. Wolf, *Constraint and super Yang-Mills equations on the deformed superspace $R^{[14]}_h$, arXiv:hep-th/0401147.*

[36] J. G. Zhou, *Super 0-brane and GS superstring actions on AdS$_2 \times$ S$^2$, Nucl. Phys. B 559, 92 (1999), [arXiv:hep-th/9906013].

[37] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, *Superstring theory on AdS$_2 \times$ S$^2$ as a coset supermanifold, Nucl. Phys. B 567, 61 (2000), [arXiv:hep-th/9907200].