Categoricity of Shimura Varieties

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Let $F$ be an algebraically closed field of characteristic 0 with an embedding $F_0 \rightarrow F$. This defines a structure in the language $\mathcal{L}_{F_0} = \{+, \cdot, F_0\}$ which is the extension of the language of rings by a set of constants for $F_0$. 

Let $T$ be the complete first-order theory of $V$ in this language. As $V(\mathbb{C})$ is bi-interpretable with $F$, then $T$ has the same model-theoretic properties as $\text{ACF}_0$, in particular, it is uncountably categorical.
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The set $V(F)$ (consisting of the $F$-points of $V$) can be interpreted over the $\mathcal{L}_{F_0}$-structure $F$. For every $m \geq 1$, every subvariety of $V(F)^m$ definable over $F_0$ is definable in the language $\mathcal{L}_{F_0}$.
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Quotient Varieties

Let $U \subseteq \mathbb{C}^n$ be a complex domain, and suppose that there is an action of a group $\Gamma$ on $U$ such that $V(\mathbb{C}) = \Gamma \backslash U$ is an algebraic variety.
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**Example**

Let $\mathbb{H} \subset \mathbb{C}$ denote the upper half-plane. The group $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ through Möbius transformations. The quotient $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ is an algebraic variety isomorphic to $\mathbb{C}$. 

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Quotient varieties can have more interesting structures than just being algebraic varieties. In order to witness this with model theory, it helps to expand the language and make it two-sorted: $q : U \rightarrow V(\mathbb{C})$

- $V$ has the language of algebraic varieties,
- $U$ at least has the structure of a $\Gamma$-action,
- $q$ is a function symbol invariant under $\Gamma$. 
Let $X^+$ denote a complex domain and suppose that there is an algebraic group $G$ defined over $\mathbb{Q}$ such that $G(\mathbb{R})^+$ acts on $X^+$ through biholomorphisms. Under certain axioms on this data, one can find discrete subgroups $\Gamma < G(\mathbb{Q})^+$ such that $\Gamma \backslash X^+$ is a variety.
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Choosing the axioms to be the axioms of *Shimura data*, the quotient $S(\mathbb{C}) := \Gamma \backslash X^+$ is called a *(connected)* Shimura variety.
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By some very deep theorems, Shimura varieties are canonically defined over a corresponding number field $E = E(G, X^+)$ called the *reflex field*. 
Suppose $V \subseteq S$ is a subvariety such that one can find a subdomain $X_V^+ \subseteq X^+$ and an algebraic subgroup $H \leq G$ defined over $\mathbb{Q}$, so that $(H, X_V^+)$ also satisfies the axioms of Shimura data.
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Let $\Gamma^H = \Gamma \cap H(\mathbb{Q})$.

If $V(\mathbb{C}) = \Gamma^H \backslash X^+_V$, then we call $V$ a special subvariety of $S$. We also call $X^+_V$ a special domain for $V$. 
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A 0-dimensional special subvariety of $S$ is called a *special point*.
Let $p : X^+ \to S(\mathbb{C})$ be a Shimura variety. Choose $g_1, \ldots, g_n \in G(\mathbb{Q})^+$. Define:

$$p_g : X^+ \to S(\mathbb{C})^n$$

$$x \mapsto (p(g_1x), \ldots, p(g_nx)).$$

The image of $p_g$ is a special subvariety of $S^n$ which we denote $Z_g$. 
Let \( p : X^+ \to S(\mathbb{C}) \) be a Shimura variety. We interpret this as a Shimura structure \( q : D \to S(F) \) using the language \( \mathcal{L} \) consisting of:

1. \( S(F) \) is interpreted as an algebraic variety over \( F_0 = E(\Sigma) \), where \( \Sigma \) is the set of coordinates of all special points of \( S(\mathbb{C}) \).
2. \( D \) is a set with an action of \( G(\mathbb{Q})^+ \) and also predicates \( D^V \subseteq D^m \) (for all \( m \geq 1 \)) interpreted as the special domains of a special subvariety \( V \).
3. \( q \) is a function.

Let \( \text{Th}(p) \) be the complete first-order theory of \( p : X^+ \to S(\mathbb{C}) \) in this language.

Remark: Every special subvariety of \( S_m \) is definable over \( E(\Sigma) \).
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Shimura Structures

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Remark

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S. Eterović (Oxford)  Categoricity of Shimura Varieties  June 2019  7 / 10
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**Remark**

Every special subvariety of \( S^m \) is definable over \( E(\Sigma) \).
$\text{Th}(p)$ is not uncountably categorical because there is no restriction on the sizes of the fibres of $q$. From construction, the fibres of $p$ are all of size $\Gamma$, but as $\Gamma$ is a countably infinite group, this condition cannot be stated in a first-order way.
Standard Fibres

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Let SF be the $\mathcal{L}_{\omega_1,\omega}$-sentence:

$$\forall x, y \in D \left( q(x) = q(y) \implies \bigvee_{\gamma \in \Gamma} x = \gamma y \right).$$

Let $\text{Th}_{\text{SF}}(p) = \text{Th}(p) \cup \text{SF}$. 
Let $\Gamma'$ be a normal finite-index subgroup of $\Gamma$. This induces a natural map:

$$\Gamma' \backslash X^+ \xrightarrow{\psi} \Gamma \backslash X^+$$

whose fibres have a simply transitive action of $\Gamma / \Gamma'$. 

Let $z' \in \Gamma' \backslash X^+$ and $z \in \Gamma \backslash X^+$ be such that $\psi(z') = z$. Let $L$ be a finitely generated field extension of $E(\Sigma)$ over which $z$ is defined. THEN $\text{Aut}(C / L)$ also acts on $\psi^{-1}(z)$ in a way that is compatible with the action of $\Gamma / \Gamma'$. Thus we get a homomorphism:

$$\rho_{z'} : \text{Aut}(C / L) \to \Gamma / \Gamma'.$$
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Repeat the construction of the homomorphism $\rho_{z'}$ for all finite-index normal subgroups of $\Gamma$, and choose the $z'$ in a compatible way. We then get a homomorphism:

$$\text{Aut}(\mathbb{C}/L) \to \overline{\Gamma} := \lim_{\Gamma' \to \Gamma} \Gamma / \Gamma'.$$
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This last question is of great interest in number theory, and had already been independently considered by Richard Pink (2006), and in more specific cases, it is known as the Mumford-Tate conjecture (still, mostly open).