Finding the ways of efficient mixing in liquids is one of the very popular subjects in physics of fluids because of numerous important technological applications ranging from the industrial production of blends and alloys to the percolation of turbulent flows of viscous liquids through porous media. In connection with that the experimental and theoretical study of the chaotic motion of passively advected particles in two-dimensional flows in presence of moving rods ("stirrers") become very popular during the last decade after the key work [1] generalized and developed later in [2,3]. The most attention in these works is paid to the following question: how should the stirrers (playing the role of topological obstacles in the liquid) move to produce the strong chaos in passively advected particles in viscous liquid. This question is known as finding the best "stirring protocol". The presence of the strong chaotic behavior due to motion of stirrers is guaranteed by topological arguments based on the Thurston–Nielsen theory [4,5] which postulates the existence of regions in the flow with pseudo–Anosov dynamics when every point of the flow demonstrates the exponential stretching with positive Lyapunov exponent.

On the basis of topological arguments it has been pointed out in [4] that the chaotic behavior in the flow of advected particles can be quantitatively characterized by measuring the Lyapunov exponents of the product of matrices representing the sequential permutation of neighboring stirrers. Each generator of the braid group, $\sigma_i$ ($1 \leq i \leq n$) representing the permutation of stirrers $i$ and $i + 1$ in the comb can be written in the matrix (Buran) representation (see, for example, [12]). Thus, the word $W_T$ is an $n \times n$ square matrix $W_T$ containing the full information about the "stirring protocol". Computing the Lyapunov exponents $\lambda(W_T)$ for different sequences of permutations and looking at their dependence $\lambda(T)$ one can conclude about the properties of the mixing regime.

In this letter we formulate the mixing problem in a slightly different manner paying the most attention to its statistical aspects. Briefly, the outline of our consideration is as follows. We suppose that the matrix $W_T$ is the time–ordered product of $T$ generators of the braid group taken in a random order with the uniform distribution over the set generators $\{\sigma_1, \sigma_2, ..., \sigma_n\}$. In terms of the works [1,3,5] this procedure of mixing can be regarded as a random stirring protocol. For each generated sequence $W_T$ (for fixed $T$) we compute the associated Lyapunov exponent, $\lambda(W_T)$. Averaging $\lambda(W_T)$ over the ensemble of all possible sequences $W_T$ we find the mean value, $\overline{\lambda(T)}$: also we compute the variance, $u^2(T) = (\lambda(W_T) - \overline{\lambda(T)})^2$. Our main result consists in the following. We show that in the broad interval of values $T$ and $n$, the variance $u(T, n)$ demonstrates the Kardar–Parisi–Zhang (KPZ) scaling (see, for review [13]), i.e. $u(T, n) = n^{1/3}g(T/n^{1/2})$ with $g(u) \sim u^{1/3}$ for $u \ll 1$ and $g(u) \sim \text{const}$ for $u \gg 1$. Such a behavior is typical for many non-stationary growth problems. It is believed now that KPZ universality class for correlated stochastic processes is as typical as the Gaussian statis-
tics for uncorrelated ones.

To make the content of this letter as selfconsistent as possible, it is instructive to introduce the basic definitions and precisely formulate the model under consideration. We begin with the definition of the braid group, \( B_n \). The group \( B_{n+1} \) has \( n \) generators \( \sigma_1, \ldots, \sigma_n \) (and their inverses) with the commutation relations:

\[
\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\
\sigma_i \sigma_k &= \sigma_k \sigma_i & |i - k| &\geq 2
\end{align*}
\]

Any arbitrary word written in terms of letters–generators from the set \( \{\sigma_1, \ldots, \sigma_n, \sigma_1^{-1}, \ldots, \sigma_n^{-1}\} \) gives a particular braid. The graphic representation of the braid generators is shown in Fig.1. The length of the braid is the total number of used letters. Diagrammatically the braid can be represented by a set of crossing strings going from the top to the bottom after subsequent gluing the braid generators.

\[
\begin{array}{cccccccc}
& & & & & & & \\
1 & \cdots & i-1 & i & i+1 & \cdots & n \\
\sigma_i &=& & & & & \\
1 & \cdots & i-1 & i & i+1 & \cdots & n
\end{array}
\]

\[
\begin{array}{cccccccc}
& & & & & & & \\
1 & \cdots & i-1 & i & i+1 & \cdots & n \\
\sigma_i^{-1} &=& & & & & \\
1 & \cdots & i-1 & i & i+1 & \cdots & n
\end{array}
\]

Figure 1: Braid group generator \( \sigma_i \) and \( \sigma_i^{-1} \).

The generators of the groups \( B_n \) admit the matrix representations. For our purposes it is more convenient to use for the braid group generator \( \sigma_i \) the Marnus representation instead of the Bureau one \[12\]. Thus, we have for \( \sigma_i \):

\[
\sigma_i = \begin{pmatrix}
1 & 0 & \cdots \\
0 & \ddots & \\
\vdots & \ddots & A \\
& \ddots & \\
& & 0 & 1
\end{pmatrix}
\]

where \( \lambda \) is the row; \( A = \begin{pmatrix} 1 & 0 & 0 \\ u & -u & 1 \\ 0 & 0 & 1 \end{pmatrix} \) \hspace{1cm} \( \text{(2)} \)

For different \( i \)’s \((2 \leq i \leq n - 1)\) the \( 3 \times 3 \) blocks \( A \) in \( \text{(2)} \) "slides" along the diagonal, while for \( i = 1 \) and \( i = n \) the corner blocks are correspondingly \( A' \) and \( A'' \), where

\[
A' = \begin{pmatrix}
-u & 1 \\
0 & 1 \\
1 & 0 \\
\end{pmatrix}; \quad A'' = \begin{pmatrix}
1 & 0 \\
u & -u
\end{pmatrix}
\]

\hspace{1cm} \( \text{(3)} \)

Note, that the matrices \( A' \) and \( A'' \) are the generators of the simplest nontrivial group \( B_3 \). It is known \[12\] that \( B_3 \) for \( u = -1 \) coincides with the modular group \( PSL(2, Z) \), i.e. the group \( B_3 \) is the central extension of \( PSL(2, Z) \). To speak rigorously, from this point of view, the matrix representation of \( B_3 \) used in \[1\] and in the consecutive works deals with the group \( PSL(2, Z) \), but not with \( B_3 \).

Now we are in position to precisely formulate the model under consideration and the main question of our interest. In what follows we consider for simplicity the semigroup \( B^+_{n+1} \), i.e. we construct the words of letters (generators) from the set \( \{\sigma_1, \ldots, \sigma_n\} \) only (without the inverses). The extension of our consideration on the full group and related difficulties are discussed briefly at the end of this letter.

Define the random walk on a semigroup \( B^+_{n+1} \). Let \( \mu = \frac{1}{n} \) be the uniform measure on the sets of generators of \( B^+_{n+1} \). Consider the (right–hand side) random walk on \( B^+_{n+1} \) with fixed framings, i.e. regard the Markov chain with the following transition probabilities: the word \( W \) transforms into \( W \sigma \), where \( \sigma \) is one of the generators from the set \( \{\sigma_1, \ldots, \sigma_n\} \) taken with the uniform probability \( \mu \). For any particular \( T \)-step random walk on the semigroup \( B^+_{n} \) we construct the word (the matrix) \( W_T(B^+_{n+1}) \) and compute the associated largest eigenvalue \( \Lambda_T(u) \) which in the terminology of the work \[6\] is called the braiding factor. Then we extract the maximal Lyapunov ("braiding") exponent \( \lambda_T \) by taking the limit

\[
\lambda_T = \lim_{u \to \infty} \frac{\ln |\Lambda_T(u)|}{\ln u}
\]

Note the difference between \( \lambda_T \) and the definition of the braiding factor in \[6\]. Actually, to find the maximal Lyapunov exponent we take the limit with respect to the dummy variable \( u \) which enters in the definition of the matrix representation of the braid group generator \[2\], while in \[6\] the limit is taken with respect to the "time" i.e. the length of the word, \( T \). Below we present the arguments which justify our definition making it topologically and physically transparent.

Since the word \( W_T \) is random, the associated Lyapunov exponent \( \lambda_T \) is also a random value. Now we proceed as it has been stated above. We average \( \lambda(W_T) \) over the ensemble of all possible sequences \( W_T \) and find the mean value, \( \Lambda(T) \). Then we compute the variance, \( u^2(T) = (\lambda(W_T) - \Lambda(T))^2 \) and consider its dependence against \( T \). The function \( u(T) \) is the central result of our letter.

The matrix representation \[2\] introduced above permits us to link the topological problems with the "Tetris-like" models models of ballistic deposition. Here the notion of the random walk on the semigroup \( B^+_{n} \) becomes very useful. On one hand, we see from Fig.1 that the generator \( \sigma_i \) has the clear topological meaning, thus the information about the degree of entanglement of threads (i.e. about the braiding factor) is encoded in the word \( W_T \). On the other hand, the word written in terms of generators of \( B^+_{n} \) can be interpreted as a heap (or pile) obtained by a 1+1 sequential ballistic deposition process in a bounding box. The matrix representation \[2\] allows to determine the corresponding height profile. Let us discuss this connection in more details.
A standard one-dimensional ballistic deposition model with next-nearest-neighboring (NKN) interactions is formulated as follows (for more details, see Refs. [14–15]). Consider a box divided in $n$ columns (of unit width each) enumerated by an index $i$ ($i = 1, 2, ..., n$). The free conditions are assumed for left and right boundaries. At the initial time moment, ($t = 0$), the system is empty. Then, at each tick of the clock, $t = 1, 2, ..., T$, we deposit an elementary cell ("particle") of unit height and width in a randomly chosen column, $i$. Suppose that the distribution on the set of columns is uniform. Define the height, $h_i(t)$, in the column $i$ at time moment $t$. Assume now, as it is depicted in Fig. 2, that the cells in the nearest-neighboring columns interact in such a way that they can only touch each other by corners, but never by their vertical sides. This implies that after having deposited a particle to the column $i$, the height of this column is modified according to the following rule:

$$h_i(t + 1) = \max\{h_{i-1}(t), h_i(t), h_{i+1}(t)\} + 1 \quad (5)$$

If at the time moment $t$ nothing is added to the column $i$, its height remains unchanged: $h_i(t + 1) = h_i(t)$. A set of deposited particles forms a heap (a pile) as shown in Fig. 2.

![Figure 2: The heap created by sequential deposition of blocks. The interactions of two sequential blocks (1st and 2nd) are shown.](image)

The height $h_i(t)$ ($1 \leq i \leq n$) in the column $i$ at time $t$ can be easily computed using the matrix representation similar to (2). The deposition event in the column $i$ means the application of the generator $g_i$ with the following matrix representation:

$$g_i = \begin{pmatrix} 1 & 0 & \ldots \\ 0 & \ddots & \\ \vdots & \ddots & B \\ \vdots & 0 & \ddots \\ 0 & \ldots & 0 & 1 \end{pmatrix} \quad \text{row } i; \quad B = \begin{pmatrix} 1 & 0 & 0 \\ u & u & u \\ 0 & 0 & 1 \end{pmatrix} \quad (6)$$

Graphically the deposition process is associated with a sequence of "dropping events" of elementary blocks—generators $g_i$ ($1 \leq i \leq n$). Since we rise a heap by sequential multiplication of matrices (6), we arrive at the matrix $\hat{U}(n,u) = \prod_{t=0}^{n} g_t$: ($1 \leq t \leq T$). Each matrix element of $\hat{U}(n,u)$ is a polynomial of the variable $u$. Take now a vector column $a(t = 0) = (a_1, ..., a_n)$ where the components of $a(t = 0)$ are distinct nonzero values. The local heights $h(T) = (h_1(T), ..., h_n(T))$ we extract as follows:

$$h(T) = \lim_{u \to \infty} \frac{\ln[\hat{U}(n,u) a(t = 0)]}{\ln u} \quad (7)$$

Let us demonstrate that the definition of heights in (7) is consistent with the updating rules (5) for the NNN-ballistic deposition process. Using (6) we can write the recursion relation for the dynamics of the vector $a(t)$:

$$a(t + 1) = g_i a(t); \quad (i \in [1, n]) \quad (8)$$

If $i = i$, then

$$a_i(t + 1) = u a_{i-1}(t) + u a_i(t) + u a_{i+1}(t) \quad (9)$$

Supposing that $a_i(t) = u^{h_i(t)}$ and substituting this ansatz into (5), we get:

$$h_i(t + 1) = \frac{e^{(h_{i-1}(t+1)) \ln u} + e^{(h_i(t)+1) \ln u} + e^{(h_{i+1}(t)+1) \ln u}}{\ln u} \quad (10)$$

In the limit $u \to \infty$ Eq. (10) coincides with Eq. (7).

The similarity between the representations (2) and (6) allows us to expect that the sequential multiplication of braided semigroup generators can also be interpreted as a sort of ballistic deposition process. To see that let us first make the following temporary replacement for the $3 \times 3$ block $A$ in (2):

$$A = \begin{pmatrix} 1 & 0 & 0 \\ u & u & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \to \quad \tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & u \\ 0 & 0 & 1 \end{pmatrix} \quad (11)$$

By this exchange we ensure for $\tilde{A}$ the absence of any cancellations of terms of polynomials in $u$ in course of matrix multiplications since all matrix elements stay positive. We show later that even for the true block $A$ such cancellations are exponentially rare do not change the obtained results. Now the multiplication of matrices (2) with the replacement $A \to \tilde{A}$ has the meaning of the heap construction shown in Fig. 3 with slightly modified local interactions. One can imagine that every elementary block has "sticky" right-hand side as it is depicted in Fig. 3.

The corresponding height $H_i(t)$ after depositing a particle to the column $i$ at time $t$ is modified as follows:

$$H_i(t + 1) = \max\{H_{i-1}(t), H_i(t) + 1, H_{i+1}(t) + 1\} \quad (12)$$
If at the time moment \( t \) nothing is added to the column \( i \), its height remains unchanged: \( H_i(t + 1) = H_i(t) \).

Our first result concerns the computation of the height profiles for deposition processes depicted in Fig.2 and Fig.3. The main quantity of interest is the variance \( w_h(T, n) \) defined for the standard NNN-deposition process as follows:

\[
w_h(T, n) = \frac{1}{n^{1/2}} \left[ \sum_{i=1}^{n} \left( \overline{H} - h_i \right)^2 \right]^{1/2} = \frac{1}{n} \sum_{i=1}^{n} h_i
\]

(13)

The mean value \( \overline{H} \) and the variance \( w_h(T, n) \) for the ballistic deposition of one-sided-sticky particles (see Fig.3) are defined in the same way. We plot in Fig.4 the dependence \( w_H(t) \) in the coordinates \((w/n^{1/2}, t/n^{3/2})\) implying the known KPZ scaling \( w_H = n^{1/2} f(t/n^{3/2}) \) with \( f(u) \sim u^{1/3} \) for \( u \ll 1 \) and \( f(u) \sim \text{const} \) for \( u \gg 1 \).

Comparing figures Fig.2 and Fig.3 we can expect that the modifications of local NNN-interactions would not affect the scaling behavior of the width of the height distribution in a growing heap defined by (12). This is actually so and as it can be seen by comparing curves (1,2) with (3,4) in Fig.4. Thus, we recover the KPZ scaling for the variance \( w_H(t, n) \) (as the function of "time" \( t \)).

Now we return to the random multiplication of matrices (2) - generators of the braid semigroup, \( W_T =: \prod_{i=1}^{T} \sigma_u \), and numerically compute the associated "height", \( Y = (y_1(T), \ldots, y_n(T)) \), defined similarly to (4):

\[
Y(T) = \lim_{u \to \infty} \frac{\ln|\hat{W}_T(n, u) a(t = 0)|}{\ln u}
\]

(14)

The interpretation of \( Y \) as a height is still valid however some care should be taken to check the absence of statistically significant amount of cancellations due to the "−" sign in the block A in (2). The possible cancellations are due to the commensurability effects and are negligible for generic initial set \( a(t = 0) = (a_1, \ldots, a_n) \) (for instance, if \( a_i, 1 \leq i \leq n \), are distinct real values from the interval \([0,1] \)). The corresponding scaling dependence is shown in Fig.4 (curves 5 and 6).

The obtained KPZ scaling for the "heights" depicted in Fig.4 associated with the random multiplication of braiding matrices indicates the possibility of similar behavior for the braiding exponent \( \lambda_T \) defined in (4). To check that we have multiplied sequentially \( T \) randomly taken (with uniform distribution \( \mu = \frac{1}{n} \)) matrices — generators of the braid semigroup \( B_n^+ \) — from the set \( \{\sigma_1, \ldots, \sigma_{n-1}\} \) and obtained the associated "word" in the matrix representation \( \hat{W}_T \). Then we have computed numerically the highest eigenvalue \( \Lambda(\hat{W}_T) \) and extracted the braiding exponent by taking a limit \( u \to \infty \) as indicated in (4). The results are summarized in Fig.5 where the KPZ scaling is clearly seen.
Few words should be said about the procedure of numerical computation of the resulting matrix $\hat{W}_T$ and the extraction of the highest eigenvalue $\Lambda\{\hat{W}_T\}$. The matrix multiplication is performed exactly in a symbolic form and each matrix element is the polynomial of some degree of $u$. Then we have left in each matrix element, $W_{ij}$, only the monomial of highest degree of $u$. Since later we have taken the limit $u \to \infty$ only such monomials dominate. By Gershgorin theorem [17] we have found the disk of radius $r < r(j) = \sum_{i=1}^{n} |W_{ij}|$, where the highest eigenvalue is located and we have localized the position of this highest eigenvalue. We have also checked each matrix $\hat{W}_T$ on the existence of zero's rows. No such rows have been detected at least for $T < 2 \times 10^6$ and $n < 400$.

The model described here can be extended to the case of a ”symmetric random stirring protocol”, when the symmetric random walk on the full braid group $B_{n+1}$ is considered. For this situation we expect the same KPZ scaling for the variance of the braiding exponent, however the description of this process is much more involved because the random walk on the full braid group is not a Markovian process. Namely, the cancellation in the word two sequentially added opposite generators at times $t$ and $t + 1$, like $\sigma_i(t) \sigma_{i-1}(t+1) = \hat{e}$ demands the knowledge of the full prehistory since the initial time moment $t = 0$. This makes the exact analytic approaches to the random walks on the full braid group rather doubtful. Nevertheless, we hope that some approximative methods like the one developed in [18] should still be valid and can be used for the investigation of random mixing.

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