Degenerate Plebanski Sector
and its Spin Foam Quantization

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Abstract

We show that the degenerate sector of Spin(4) Plebanski formulation of four-dimensional gravity is exactly solvable and describes covariantly embedded SU(2) BF theory. This fact provides its spin foam quantization and allows to test various approaches of imposing the simplicity constraints. Our analysis suggests a unique method of imposing the constraints which leads to a consistent and well defined spin foam model.
1 Introduction

The path integral quantization is a powerful method of quantizing dynamical systems. However, in the case of gravity the usual path integral is not well-defined due to the well-known non-renormalizability problem. In this situation the only hope is to use some non-perturbative methods. A particularly interesting possibility to construct a well-defined path integral for quantum gravity is suggested by the spin foam approach [1, 2]. In its framework, transition amplitudes between quantum states of spatial geometry are represented as sums over spin foams, i.e., two-dimensional cellular complexes colored by certain group theoretic data.

In three dimensions this approach is realized by two spin foam models: Ponzano-Regge model [3] and Turaev-Viro model [4], which provide quantization for three-dimensional gravity with a vanishing and a positive cosmological constant, respectively. These models are well-grounded since they can be connected either to the loop quantization [5] or to the Chern-Simons quantization [6, 7], which has a solid basis based on the canonical analysis. Thus, in the three-dimensional case the spin foam quantization is the full-fledged approach to quantum gravity.

In four dimensions the status of the spin foam quantization is much more controversial. The only models which are affordable to the direct spin foam quantization are the so-called BF
theories [8] which are described by the following action

\[ S_{\text{BF}}[\omega, B] = \int_M d^4x \text{Tr} (B \wedge F(\omega)), \quad (1.1) \]

where \( \omega \) is the connection one-form valued in the Lie algebra \( \mathfrak{g} \) of a certain group \( G \), \( F(\omega) \) is its curvature two-form, \( B \) is a two-form also valued in \( \mathfrak{g} \), and \( \text{Tr} \) is evaluated using the Killing form on \( \mathfrak{g} \). These theories are closed cousins of three-dimensional gravity. First of all, the action of the latter has a form similar to (1.1) with the only difference that \( B \) should be taken as a one-form and interpreted as a triad. And second, all these theories are actually topological. As a result, as in the case of three-dimensional gravity, their spin foam quantization can be obtained on a fixed triangulation on which it does not depend [9] (for recent developments, see also [10]). In contrast, for non-topological theories, the dependence on the triangulation should be removed by summing over all of them or over their dual two-complexes.

Besides these topological spin foam models of BF theories, there have been suggested several models supposed to describe gravity. Among them the most prominent ones are the Barrett-Crane (BC) model [11, 12] and the so-called new models, which are due to Engle, Pereira, Rovelli, Livine (EPRL) [13] and Freidel, Krasnov (FK) [14, 15]. However, all of them have been derived using certain assumptions and simplifications. Their common starting point is the so-called Plebanski theory, which represents gravity as a constrained BF theory with \( G = \text{Spin}(4) \) in the Riemannian or \( G = \text{SL}(2, \mathbb{C}) \) in the Lorentzian case. Namely, it realizes the simple fact that, if the action (1.1) is supplemented by a constraint ensuring that

\[ B = \star(e \wedge e), \quad (1.2) \]

it reproduces the usual Hilbert-Palatini formulation. The idea leading to the spin foam models mentioned above is that these constraints, called commonly simplicity, can be incorporated at the quantum level. Thus, they are supposed to be imposed on the spin foam representation of the quantum BF theory and should convert the trivial dynamics of a topological theory into that of quantum gravity. This strategy, which can be summarized as “first quantize, then constrain”, is now the usual approach to four-dimensional spin foam models, and what distinguishes various models is only the way the simplicity constraints are incorporated.

However, although this strategy seems to be well motivated and leads to interesting results, it does not agree with the rules of quantization of constraint systems. The simplicity constraints are known to be second class and therefore affect the symplectic structure to be quantized, which can be evaluated using the Dirac bracket. It has been argued that this and some other effects are not taken properly into account in the spin foam approach based on the above strategy [16, 17]. In particular, relying on the consistency with the canonical quantization, it has been suggested a certain modification of the vertex amplitude [18], which is the most important quantity in spin foam models encoding their dynamics. If one follows the usual strategy, the vertex coincides with the one of BF theory, but restricted to a set of representations and intertwiners, assigned to the elements of the cellular decomposition, satisfying the simplicity constraints. Equivalently, it can be represented as the boundary state associated with a four-simplex evaluated on a flat connection, or as

\[ A_v = \left( \prod_\tau \int_G \mathcal{D}g_\tau \right) \Psi \left[ g_{u(f)}^{-1}g_{d(f)} \right], \quad (1.3) \]

where \( \Psi[g_f] \) is the boundary state depending on the group elements assigned to triangles of the four-simplex, the product goes over its tetrahedra, \( u(f)/d(f) \) denotes the upper/down tetrahedron sharing triangle \( f \), and the integration measure should be taken to be the usual Haar
measure on the group, $Dg = dg$. In [18] it has been argued that the correct vertex for the constrained theory should be given by the same formula, but with the measure which involves the delta-function of secondary second class constraints, conjugate to the simplicity and ensuring that they are of second class.

Given this situation when there are several proposals for the spin foam quantization of quantum gravity, it is desirable to have some simplified models which allow to test various features of these proposals. Moreover, since most problems and ambiguities arising in four-dimensional spin foam models come from the difficulties in imposing the simplicity constraints, such a model should mimic the structure of Plebanski formulation. In other words, it should be of the following form

$$S_{\text{Th2}} = S_{\text{Th1}} + \text{constraints},$$

where the constraints convert the theory Th1 into the theory Th2. Finally, to be a good test-ground, both theories, given by the actions $S_{\text{Th1}}$ and $S_{\text{Th2}}$, should have known spin foam representations. Then we can verify which of the methods to impose the constraints reduces the spin foam quantization of Th1 to that of Th2. If a method does not allow to recover the known quantization of Th2, this strongly suggests that it is not applicable also in the case of Plebanski theory. On the other hand, if we find a quantization procedure which passes through our test, one can hope that it will work for gravity as well.

In four dimensions the above requirements suggest that Th1 and Th2 should be of BF type (1.1) because these are the only theories with a well-established spin foam quantization. Then the constraints can be used to reduce a gauge group $G$ to its subgroup $H \subset G$. Thus, one arrives at an essentially unique model suitable for all our purposes: it should represent BF theory with the gauge group, for example, Spin(4) reduced by means of some constraints to the SU(2) BF theory.

In [18] it has been noticed that such a reduction does take place if the $B$-field and the spin connection are required to satisfy

$$B^{IJ}x_J = 0, \quad J^{IJ}_{KL}(x)\omega^{KL} = 2x^{[I}\mathrm{d}x^{J]},$$

where $x^I$ is a normal vector describing the embedding of SU(2) into Spin(4) and $J^{IJ}_{KL}(x)$ is the projector on the orthogonal completion of the $\mathfrak{su}(2)$ subalgebra stabilizing $x^I$ (see appendix A.1). Moreover, it was demonstrated that one recovers the known spin foam representation of the SU(2) BF theory from the known spin foam quantization of the Spin(4) BF theory only if one incorporates the constraints on the spin connection into the definition of the vertex amplitude in the way described below (1.3). However, the important drawback of this consideration was that the constraints (1.5) have been imposed by hand and not derived from a classical action of Plebanski type.

An analogous model, but based on a solid canonical analysis, was proposed recently in [19] (see also [20]). It also represents the reduction of Spin(4) BF to SU(2) BF, but this time in three dimensions where the resulting theory is nothing else but three-dimensional gravity with vanishing cosmological constant. The constraints performing this reduction are a direct generalization of the simplicity constraints of the Plebanski formulation so that one could test whether one of the standard methods of imposing the simplicity reproduces the Ponzano-Regge model. It was shown that, as in four dimensions [18], it becomes possible to recover the Ponzano-Regge vertex amplitude only if the simplicity constraints are supplemented by the secondary second class constraints restricting the holonomies of the spin connection. Furthermore, in [20], the same model was reanalyzed and it was shown how it can be quantized so that the result
is consistent with the canonical quantization approach and reproduces all ingredients of the Ponzano-Regge model, including face, edge and vertex amplitudes.

Although these results are already very suggestive, they may be argued to be not directly applicable to the four-dimensional spin foam models. The reason for this is that in three dimensions, for spin foams dual to a simplicial discretization, all edges are three-valent. As a result, the space of gauge invariant intertwiners assigned to these edges is one-dimensional and the most interesting part of the simplicity constraints, the so-called cross simplicity, is automatically satisfied. On the other hand, it is the treatment of the cross simplicity that distinguishes the spin foam models mentioned above, so that one could think that this part of the constraints is particularly crucial for the whole construction.

To avoid such critics, in this paper we return to the original four-dimensional model of [18] and fill there all the gaps both at classical and quantum level. Namely, in section 2 we propose a classical action which represents the Spin(4) BF theory reduced down to SU(2) BF, where the constraints (1.5) appear as primary and secondary constraints, respectively. Remarkably, this action represents a system very close to the physical system we are interested in — it describes the degenerate sector of Plebanski theory. Thus, the latter is exactly solvable and can be seen as a covariant embedding of the well known topological theory. We provide a thorough canonical analysis of this degenerate sector both with a partial gauge fixing and without it. In particular, we show that adding the degeneracy condition to the usual simplicity constraints, one generates constraints for constraints. They are responsible for the well known fact [21, 22] that the phase space of degenerate configurations is larger than its non-degenerate version. It is also possible to include the Immirzi parameter into the model which does not lead to any complications.

Then in section 3 we consider the spin foam quantization of this model. First, we apply the usual quantization strategy employed in the EPRL and FK approaches. Since our classical action differs from Plebanski theory only by the presence of the degeneracy condition, to get its spin foam quantization, it is sufficient to extract the degenerate sector of the new spin foam models. If it is done, following the usual ideas, by restricting the boundary or kinematical data, the result strongly disagrees with the known vertex amplitude for the four-dimensional SU(2) BF theory. On the other hand, if the degeneracy condition is represented as a constraint on the group elements appearing in the integral formula (1.3) for the vertex and is added to the integration measure, one does get the right result. Since this constraint can be equally seen as a discretization of the secondary second class constraints, this modification of the measure is in the full agreement with our proposal for the vertex amplitude spelled above.

At the same time, the analysis of the constraint imposition in the framework of the new models shows that the constraint on holonomies ensuring the correct vertex amplitude erases all information about the solution of the original simplicity constraints of the EPRL and FK models. Thus, the main ingredients of these models appear to be irrelevant for getting the right dynamics, which calls for a reconsideration of these approaches. As an alternative, we suggest another quantization procedure, which summarizes the analysis of [18, 20] and is consistent with the canonical approach by construction. Being applied to the model under consideration, it gives precisely the right result: the Crane-Yetter model with the SU(2) structure group. Moreover, in the course of evaluation of the partition function, we clarify the role of different constraints in the spin foam quantization. In particular, we observe that the vertex amplitude is completely determined by the secondary second class constraints putting restrictions on holonomies of the spin connection, whereas the primary simplicity constraints affect only the gluing of different vertex contributions and are not relevant for dynamics. A discussion of these and other issues can be found in the concluding section.
Our conventions are explained in appendix A. We restrict ourselves to the Riemannian case to not bother the reader with signs which otherwise would pop out here and there. However, all the presented results are easily generalized to the Lorentzian case as well. In appendix B we present the details of the canonical analysis of the degenerate Plebanski sector without a partial gauge fixing, whereas the last appendix C makes explicit the constraints for constraints.

2 Degenerate sector of Plebanski formulation

2.1 The action and constraints

Let us start from the usual Plebanski action

\[ S_{\text{dPl}}[\omega, B, \lambda] = \frac{1}{2} \int_M d^4x \left[ \varepsilon^{\mu\nu\rho\sigma} \text{Tr}(B_{\mu\nu}F_{\rho\sigma}) + \frac{1}{2} \lambda^{\mu\nu\rho\sigma} \text{Tr}(\star B_{\mu\nu}B_{\rho\sigma}) \right], \tag{2.1} \]

where the Lagrange multiplier field \( \lambda \) is chosen to be a spacetime pseudo-tensor satisfying the following symmetry properties: it is antisymmetric in the first and second pair of indices, \( \lambda^{\mu\nu\rho\sigma} = \lambda^{[\mu[\nu[\rho\sigma]} \), and is symmetric under their exchange, \([\mu\nu] \leftrightarrow [\rho\sigma] \). Usually, one also adds the tracelessness condition \( \varepsilon^{\mu\nu\rho\sigma} \lambda^{\mu\nu\rho\sigma} = 0 \) which we however omit. As a result, the variation with respect to the Lagrange multiplier generates the following constraints

\[ \Phi_{\mu\nu\rho\sigma} = \frac{1}{2} \varepsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = 0. \tag{2.2} \]

They represent the usual 20 simplicity constraints supplemented by an additional condition which forces the \( B \)-field to belong to the degenerate sector, i.e. to give a vanishing four-dimensional volume \( \mathcal{V} = \frac{1}{12} \varepsilon^{\mu\nu\rho\sigma} \Phi_{\mu\nu\rho\sigma} \). However, in contrast to the usual case, not all of these 21 constraints are independent. It turns out that there are 6 constraints for constraints, which we display explicitly in appendix C. As a result, one remains only with 15 independent constraints which thus reduce the number of independent components of the \( B \)-field from 36 to 21.

It is easy to see that there are two sectors of solutions to the simplicity constraints (2.2),

\[ \begin{align*}
\text{“deg-gravitational” : } & B^{IJ}_{\mu\nu} = \frac{1}{2} \varepsilon^{IJ}_{\ K\ L} x^K b^L_{\mu\nu}, \\
\text{“deg-topological” : } & B^{IJ}_{\mu\nu} = x^{[I} b^J_{\mu\nu]},
\end{align*} \tag{2.3a} \tag{2.3b} \]

where the vector \( x^I \) is supposed to be normalized as \( x^I x_I = 1 \). In analogy with the usual case, we call them “degenerate gravitational” and “degenerate topological”\textsuperscript{1}. The reason for this will be clear from what follows. Both solutions (2.3) contain 21 independent degrees of freedom, i.e. the same as the solution space of the simplicity constraints: 3 degrees of freedom are contained in \( x^I \) and 18 are described by \( b^I_{\mu\nu} \) because the latter field can be chosen to satisfy the linear constraints

\[ x_I b^I_{\mu\nu} = 0. \tag{2.4} \]

They fix uniquely the ambiguity in \( b^I_{\mu\nu} \) and will be always assumed to hold in the following analysis.

\textsuperscript{1}There might be also a “twice degenerate” sector because the proof in appendix C that there are only 6 constraints for constraints relies on the assumption that a certain matrix constructed from the \( B \)-field is invertible. But we do not consider here this possibility and assume that the field \( b^I_{\mu\nu} \) is generic so that it ensures certain non-degeneracy conditions appearing in the course of our analysis.
If one fixes the vector $x^I$, thereby reducing the gauge symmetry from Spin(4) to the subgroup SU(2) which preserves this vector, the two sectors of solutions (2.3) can be equivalently characterized by linear simplicity constraints

\[ \text{"deg-gravitational" : } \Phi^{(gr)}_{\mu\nu} = x_J B^{IJ}_{\mu\nu} = 0, \]  
\[ \text{"deg-topological" : } \Phi^{(top)}_{\mu\nu} = \varepsilon^{IJ} K_L x_J B^{KL}_{\mu\nu} = 0, \]  

again in direct analogy with the usual non-degenerate case [23, 24, 14, 25]. However, the difference is that here the linear constraints (2.5) exhaust all simplicity constraints, whereas in the non-degenerate case they should be supplemented by the volume constraint. Indeed, since both constraints (2.5) satisfy $x^I \Phi^{(gr)}_{\mu\nu} = x^I \Phi^{(top)}_{\mu\nu} = 0$, they give rise to 18 independent equations. This reduces the number of independent components of the $B$-field to 18, which coincides with the number of independent degrees of freedom described by $b^{I\mu\nu}$.

To understand the meaning of the two sectors, let us assume for simplicity that $x^I = \text{const}$. Then one can easily extract an equation without derivatives from the equation of motion obtained by variation of (2.1) with respect to the spin connection $\varepsilon^{\mu\nu\rho\sigma} D^\rho \varepsilon^{KL} x_J = 0 \Rightarrow x^I F^{IJ}_{\mu\nu} = 0$. (2.8)

This result indicates that only the part of the connection describing the SU(2) subgroup survives, whereas the orthogonal part vanishes. Now we can plug the solution of the simplicity constraints (2.3) into the original action (2.1). It is immediate to see that, due to (2.8), in the deg-topological sector the resulting action identically vanishes, whereas in the deg-gravitational sector it becomes

\[ S_{gr}[\omega, b] = \frac{1}{4} \int_M d^4x \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{IJ} K_L x^I b^{KL}_{\mu\nu} F^{IJ}_{\rho\sigma}. \]  

This is nothing else but the action of the four-dimensional SU(2) BF theory covariantly embedded into Spin(4). Fixing the time gauge $x^I = \delta^I_0$, one recovers its usual form written in terms of the $\mathfrak{su}(2)$-valued 2-form $b^I_{\mu\nu}$ and the $\mathfrak{su}(2)$-connection $\omega^{ij}_\mu$. The other components of the original fields vanish due to (2.4) and (2.8).

Since in the deg-topological sector we do not obtain any meaningful theory, we will be mainly concentrated on the deg-gravitational sector. Remarkably, it is given by a well known topological theory so that we know its exact classical as well as quantum description. For our purposes it will be however important to understand also the canonical structure of the original Spin(4)-invariant theory (2.1) which we present in the next subsection. Then in subsection 2.3 we will see how the reduction to the SU(2) BF theory described here is established in the Hamiltonian formalism. The reader who is not interested in details of this canonical analysis can proceed directly to section 2.4.

\[ \text{Note however that in the non-degenerate case the roles of } \Phi^{(gr)} \text{ and } \Phi^{(top)} \text{ are exchanged, i.e. the former corresponds to the topological sector and the latter to the gravitational.} \]
2.2 Canonical analysis

Here we present the Hamiltonian formulation of the degenerate gravitational sector of Plebanski theory in a partially fixed gauge. Namely, we fix the normal field \( x^I(x) \) to be a given function on spacetime. The choice of the normal allows to replace the quadratic simplicity constraints (2.2) by their linearized version (2.5), which can be done directly in the action and gives the possibility to restrict to the particular sector we are interested in from the very beginning. An analogous formulation of the gravitational sector of Plebanski theory has been considered in [25] (see also [26]). However, the important difference of our approach is that the normal \( x^I \) is considered as a fixed non-dynamical variable. This is motivated by the following application of these results to the spin foam quantization of our model. This quantization is implemented via a path integral where the gauge freedom generated by boost transformations should be fixed by a gauge choice. The most convenient way to do this is precisely to fix the normal \( x^I \). This gauge fixing is analogous to what is done in the standard loop quantum gravity where one imposes the so-called time gauge corresponding to a particular choice of \( x^I = \delta^I_0 \). Here we could also restrict ourselves to this simple gauge, in which case the following derivation considerably simplifies. We however prefer to keep \( x^I \) an arbitrary function to show that in this general case one obtains nice covariant structures.

We emphasize that the results presented here can also be derived going through the complete canonical analysis of the original action (2.1) carried out without imposing any gauge fixing. We provide such analysis in appendix [3] where the canonical structure of both solution sectors (2.3) is elucidated. They possess a very intricate constraint structure which however reduces to the one of this subsection upon restricting to the deg-gravitational sector, fixing \( x^I \), and solving auxiliary constraints.

Thus, our starting point is the following action

\[
S_{\text{deg-gr}}[\omega, B, \lambda; x] = \frac{1}{2} \int_M d^4x \left[ \varepsilon^{\mu\nu\rho\sigma} B_{\mu\nu} F_{\rho\sigma}^{IJ} + 4\lambda_{I}^{\mu\nu} x_J B_{\mu\nu}^{IJ} \right], \tag{2.10}
\]

where \( x^I \) is just a parameter and not a dynamical variable. After the 3+1 decomposition of this action, one can recognize that the phase space is parametrized by \( \omega_{IJ}^a, B_{IJ}^0 a \) and \( \lambda_{I}^{\mu\nu} \) appear without time derivatives and therefore play the role of Lagrange multipliers. However, not all of them generate constraints. Whereas the variation with respect to \( \omega_{IJ}^a \) and \( \lambda_{I}^{\mu\nu} \) does give rise to the primary constraints

\[
\mathcal{G}_{IJ} = D_a \tilde{P}_{IJ}^a \approx 0, \quad \Phi_I^a = x^J \tilde{P}_{IJ}^a \approx 0, \tag{2.13a}
\]

the variation with respect to \( \lambda_{I}^{0a} \) leads to a condition on the Lagrange multipliers,

\[
x_J B_{IJ}^{0a} = 0. \tag{2.14}
\]

At the same time, \( B_{IJ}^{0a} \) gives the following equation

\[
\varepsilon^{abc} F_{bc}^{IJ} + 4\lambda_{[I}^{\mu\nu} x_J^{\ ]} \approx 0. \tag{2.15}
\]
It can be split into two parts: one gives a condition on the Lagrange multipliers

$$\lambda^0_I = -\frac{1}{2} \varepsilon^{abc} F_{bc}^{IJ} x_J, \tag{2.16}$$

and the second is a primary constraint

$$C^a_I = \varepsilon_{IJKL} x^J \varepsilon^{abc} F_{bc}^{KL} \approx 0. \tag{2.17}$$

Taking into account the condition (2.14), the Hamiltonian is given by a linear combination of the primary constraints introduced above

$$-H = \frac{1}{2} \varepsilon^{IJ} K_L x_J B^{KLC}_{0a} C^a_I + \omega^0_{IJ} G_{IJ} + \varepsilon_{abc} \lambda^a_I \phi^c_I. \tag{2.18}$$

Now one should find the conditions under which the primary constraints are preserved in time. Since their evolution is generated by the Hamiltonian (2.18), this boils down to the study of the constraint algebra. Introducing the smeared constraints

$$G(n) = \int d^3x n^{IJ} G_{IJ}, \quad C^a(v) = \int d^3x v^l C^a_l, \tag{2.19}$$

the non-trivial commutators are given by

$$\{G(n), G(m)\} = G([n, m]),$$

$$\{G(n), C^a(v)\} = C^a(n \cdot v) - 4 \int d^3x \varepsilon^{IJ} K_L x_J v^K n^{LM} x_M,$$

$$\{G(n), \Phi^a_I\} = -n^{IJ} \Phi^a_J + \tilde{P}^a_{IJ} n^{JK} x_K,$$

$$\{C^a(v), \Phi^b_I\} = -2 \varepsilon_{IJKL} x^J x^K \varepsilon^{abc} D_c x^L,$$

where we used (2.16). To apply these results, we start from the primary constraint $\Phi^a_I$. Its conservation leads to the condition

$$\dot{\Phi}^a_I = 2 \varepsilon^{abc} B^{IJ}_{0b} D_c x_J + \tilde{P}^a_{IJ} (\omega^0_{JK} x_K + \dot{x}^J) \approx 0. \tag{2.21}$$

Combined with the three equations following from the Gauss constraint

$$x^J G_{IJ} = D_a \Phi^a_I - \tilde{P}^a_{IJ} D_a x^J \approx 0, \tag{2.22}$$

it gives rise to the condition on the Lagrange multipliers $\omega^0_{IJ}$

$$D_0 x^I = 0 \tag{2.23}$$

and to the secondary constraints

$$\Psi^a_I = D_a x^I \approx 0. \tag{2.24}$$

Then the conservation of the Gauss constraint $G_{IJ}$ gives a relation between the Lagrange multipliers $\lambda^\mu_J$

$$\dot{G}_{IJ} \approx -4 x^I D^J_{0a} \lambda^a_K - \varepsilon_{abc} x^I \tilde{P}^a_{IJ} \lambda^b_K = 0 \quad \implies \quad B^{IJ}_{\mu \nu} \lambda^\mu_J = 0, \tag{2.25}$$

whereas the conservation of $C^a_I$ does not generate new conditions.
The next step is to study the secondary constraints $\Psi^I_a$ (2.24). In fact, due to the identity $x^j \Psi^I_a = 0$, they give only 9 independent equations. These constraints satisfy the following commutation relations

$$\{ C^a(v), \Psi^I_b \} = 0, \quad \{ G(n), \Psi^I_b \} = x_J D_b n^{IJ}, \quad \{ \Phi^a, \Psi^I_b \} = -\frac{1}{2} \delta^n_a (\delta^I_J - x_I x^J) \delta(x, y) \quad (2.26)$$

Taking into account the restriction (2.23), the conservation of $\Psi^I_a$ therefore amounts to vanishing of all Lagrange multipliers $\lambda^{ab}_I = 0$. (Recall that these multipliers can be chosen to satisfy $x^I \lambda^{ab}_I = 0$ from the very beginning so that the number of their independent components equals the number of independent secondary constraints.) Moreover, it is easy to check that

$$\varepsilon^{abc} F_{bc}^{IJ} x_J = 2 \varepsilon^{abc} D_b \Psi^I_c \implies \lambda^{0a}_I = 0, \quad (2.27)$$

which fixes the last undetermined part of the Lagrange multipliers $\lambda^{ab}_I$. Thus, the stabilization procedure stops at this point.

Once we found all the constraints, we can study whether they are of first or second class. Due to (2.22), we remain only with four types of constraints:

$$C^a_I, \quad \hat{G}_I \equiv \varepsilon_{IJ}^{KL} x^J G_{KL}, \quad \Phi^a_I, \quad \Psi^I_a. \quad (2.28)$$

Furthermore, due to the Bianchi identity, the first ones additionally satisfy

$$D_a C^a_I = 2 \Psi^I_a C^a_{IJKL} + 2 \varepsilon_{IJ}^{KL} x^J \varepsilon^{abc} \Psi^K_a D_b \Psi^L_c. \quad (2.29)$$

Using the commutation relations (2.20) and (2.26), it is trivial to check that the constraints $C^a_I$ and $\hat{G}_I$ are first class, whereas $\Phi^a_I$ and $\Psi^I_a$ are second class. As a result, the $18 + 18 = 36$ configuration variables are restricted by $(9 - 3) + 3 = 9$ first class and $9 + 9 = 18$ second class constraints, which leaves us with a zero dimensional physical phase space. This confirms that this theory is topological, i.e. it does not contain propagating degrees of freedom.

### 2.3 Reduction to SU(2) BF theory

The meaning of the constraints (2.28) is quite transparent: the second class constraints, $\Phi^a_I$ and $\Psi^I_a$, fix the off-diagonal (boost) degrees of freedom in the chiral decomposition of the $\mathfrak{so}(4)$ algebra or, more precisely, the configuration variables from the orthogonal completion to the $\mathfrak{su}(2)_x$ subalgebra. The remaining constraints, which describe the dynamics of the variables from this subalgebra, are nothing else but the usual constraints of the SU(2) BF theory [27]. Namely, $C^a_I$ gives the flatness condition for an SU(2)-connection and $\hat{G}_I$ is the corresponding Gauss constraint generating SU(2) gauge transformations. This becomes especially clear in the time gauge $x^I = \delta^I_0$ where the second class constraints simply mean that $\hat{P}^{(+)} a = \hat{P}^{(-)} a$ and $\omega^{(+)}_a = \omega^{(-)}_a$. For a constant $x^I$ these relations get rotated by a constant Spin(4) transformation mapping $\delta^I_0$ into $x^I$. It is however amusing to see how they generalize to the case of arbitrary gauge where $x^I$ can vary in spacetime.

Let us introduce the projections of our fields on the $\mathfrak{su}(2)_x$ subalgebra

$$\tilde{P}_{IJ}^a = I_{IJ}^{KL}(x) \hat{P}_{KL}^a, \quad b_{0a}^{IJ} = I_{KL}^{IJ}(x) B_{0a}^{KL}, \quad (2.30)$$

where $I_{IJ}^{KL}(x)$ is the projector given in (A.2), so that the new fields solve (2.13b) and (2.14), respectively. Besides, we define the following connection

$$A_{IJ}^\mu = I_{IJ}^{KL}(x) \omega_{KL}^\mu + 2 x^I \partial_\mu x^J, \quad (2.31)$$
where the last term takes care about variations of the normal field. This connection coincides with the original spin connection $\omega_{\mu}^{IJ}$ on the surface of (2.23) and (2.24) and satisfies the constraint

$$J_{KL}^{IJ}(x)A_{\mu}^{KL} = 2x^{[I}\partial_{\mu}x^{J]}, \quad (2.32)$$

which is identical to the one appearing in the Lorentz covariant formulation of loop quantum gravity [28, 29]. The characteristic feature of this connection is that its holonomies map a vector from $\mathfrak{su}(2)_{x_1}$ to a vector in $\mathfrak{su}(2)_{x_2}$ subalgebra [16], i.e. for constant $x^I$ they belong to the SU(2)$_x$ subgroup.

In terms of the new variables and taking into account all conditions on the Lagrange multipliers, the 3+1 decomposed action (2.10) can be written as

$$S_{\text{deg-gr}} = \int \mathbb{R} \int_{\Sigma} d^3x \left[ \tilde{\mathcal{P}}_{IJ}^a \partial_0 A_{\mu}^{IJ} + 2\Phi_a^a \partial_0 \Psi^J + \varepsilon^{abc}b_{0a}^{I} \left( F_{BC}^{IJ}(A) - 2\Psi_b^I \Psi^J \right) 
+ A_{0}^{IJ} \left( \mathcal{G}_{IJ}^{(A)} + 2\Phi_a^a \Psi^J \right) \right], \quad (2.33)$$

where $F_{ab}^{IJ}(A)$ is the curvature of the connection $A_{\alpha}^{IJ}$ and

$$\mathcal{G}_{IJ}^{(A)} = \partial_0 \mathcal{P}_{IJ} + [A_0, \mathcal{P}_{IJ}]. \quad (2.34)$$

Setting the second class constraints to zero, one obtains the usual BF action where all variables are projected down to SU(2)$_x$. It provides a covariant embedding of the SU(2) BF theory into the Spin(4) formalism.

### 2.4 Summary

Let us summarize what we have found. The theory (2.1) describing the degenerate sector of Plebanski formulation of general relativity has two sectors of solutions of the simplicity constraints (2.2). In the sector which we called “degenerate gravitational”, it reduces to the four-dimensional SU(2) BF theory covariantly embedded into Spin(4) gauge group. The embedding is characterized by the normal vector $x^I$. In the partially fixed gauge where $x^I$ is a fixed function, the theory possesses two types of second class constraints conjugate to each other:

$$\text{primary} \quad \Phi_0^a = x^J \tilde{P}_{IJ}^a, \quad \text{secondary} \quad \Psi_0^I = D_a x^I. \quad (2.35)$$

The remaining constraints are first class and generate the gauge symmetries of the SU(2) BF theory.

The presence of the second class constraints, as usual, leads to a modification of the symplectic structure: the Poisson bracket (2.12) has to be replaced by the appropriate Dirac bracket. The latter can be easily calculated and is given by

$$\{\omega_0^I(x), \tilde{P}_{KL}^b(y)\}_D = \delta^b_a \delta^{IJ}_{KL}(x) \delta(x, y) \quad (2.36)$$

with all other commutators being vanishing. This result again demonstrates that only the SU(2)$_x$ part of the configuration variables is dynamical and implies that $\tilde{P}_{IJ}^a$ and $\omega_0^I$ are not canonically conjugate anymore.
2.5 Inclusion of the Immirzi parameter

It is easy to include the Immirzi parameter \[30\] into our model. To this end, one makes the usual replacement of the \(B\)-field in the BF part of the action by the combination \(B + \frac{1}{\gamma} \star B\). It leads to a mixing of the two solution sectors (2.3). Now both of them reduce to the SU(2) BF theory, just in the deg-topological case the resulting action is multiplied by the factor \(1/\gamma\).

In the Hamiltonian formulation of the deg-gravitational sector described by the action

\[
S_{\text{deg-gr}}^{(\gamma)}[\omega, B, \lambda; x] = \frac{1}{2} \int_{\mathcal{M}} d^4x \left[ \varepsilon^{\mu\nu\rho\sigma} \left( B_{\mu}^{IJ} + \frac{1}{2\gamma} \varepsilon^{IJ} KL B_{\mu}^{KL} \right) F_{\rho\sigma}^{IJ} + \frac{4}{\gamma} \lambda_{\mu}^I x_J B_{\mu}^{IJ} \right],
\]

(2.37)

the presence of the Immirzi parameter affects the Poisson symplectic structure so that the new canonical variables are

\[
\omega_a^{IJ} \quad \text{and} \quad \tilde{P}_a^{IJ} = (1 + \gamma^{-1}x) \tilde{P}_a^{IJ}.
\]

Nevertheless, all the remaining structure does not change and there are just slight modifications in the stabilization procedure of section 2.2. In particular, the final constraints acting on the phase space (2.38) comprise \(C^*_i\) and \(\tilde{G}_I\) which are first class and \(\Phi^a_I, \Psi^a_I\) which are second class, and all these constraints are given by the same expressions as without \(\gamma\). Moreover, due to the second class constraints, the \(\gamma\)-dependent symplectic structure given by Poisson brackets is replaced by the \(\gamma\)-independent one described by the same Dirac brackets (2.36) as before.

3 Spin foam quantization

In this section we consider the spin foam quantization of our model. Since after implementing the second class constraints (in the deg-gravitational sector) it coincides with the SU(2) BF theory, we know what the final result should be: it is given by the Crane-Yetter model \[31, 9\] with the structure group SU(2) represented by the following spin foam state sum

\[
Z_{\text{CY}}^{\text{SU}(2)}(\Delta^*) = \sum_{j_f} \sum_{i_e} \prod_{f \in \Delta^*} (2j_f + 1) \prod_{v \in \Delta^*} A_{v}^{\text{SU}(2)},
\]

(3.1)

where \(\Delta^*\) is a two-complex dual to a simplicial triangulation \(\Delta\) of the spacetime manifold \(\mathcal{M}\), \(j_f\) labels irreducible representations of SU(2) attached to the faces of \(\Delta^*\), \(i_e\) are SU(2) invariant intertwiners assigned to its edges, and \(A_v^{\text{SU}(2)}\) is the vertex amplitude given by the SU(2) \(\{15j\}\) symbol. The latter is obtained by evaluation of the SU(2) spin network represented by the pentagon graph, which is dual to the boundary of a 4-simplex \(\sigma\in\Delta\) dual to the vertex \(v \in \Delta^*\),

\[
A_v^{\text{SU}(2)}(\vec{j}, \vec{i}) = \{15j\} \equiv (3.2)
\]
The normalization of the intertwiners used in this evaluation is defined in appendix A.2.

However, we would like to proceed in a different way which would avoid quantizing the degrees of freedom on the constraint surface only. Our aim is to find a quantization of the original theory with constraints, (2.1) or (2.10) (or even (2.37)), such that it reproduces the SU(2) Crane-Yetter model (3.1). In particular, we would like to check whether the quantization strategies used to get the EPRL or the FK model are able to do this. Since our model is only slightly different from the gravitational sector of Plebanski theory, our study represents a very serious test on the validity of these quantization approaches.

### 3.1 Discretization

Before we start discussing the quantization, let us put our model on a cellular complex and discretize its variables. As usual, the $B$-field is discretized by associating Lie algebra valued elements $B_f \in \mathfrak{so}(4)$ to the faces of the dual complex $\Delta^*$ and can be obtained as integrals of the $B$-field over the dual triangles

$$B^{IJ}_f = \int_{t_f} B^{IJ}.$$  

(3.3)

The spin connection $\omega^{IJ}$ gives rise to group elements $g_e$ which coincide with its holonomies along the edges $e \in \Delta^*$. However, it is also convenient to introduce the holonomies $g_{ve}$, going from vertex $v$ to the center of edge $e$, and $g_{fe}$, going from the center of face $f$ also to the center of the edge. $g_{ve}$ and $g_{ef}$ will denote their inverse. The former provide a refined version of our basic dynamical variables which are obtained as $g_e = g_{ve} g_{ev'}$, where $v$ and $v'$ are the two vertices joined by $e$. On the other hand, $g_{fe}$ are needed to bring the bivectors (3.3) to the reference frame where $g_{ve}$ are acting. In particular, we define

$$B_{ef} = g_{ef} B_f g_{fe}.$$  

(3.4)

In this sense, these group elements may be considered as non-dynamical auxiliary variables completing the definition of the discrete $B$-field. Altogether, $g_{ve}$ and $g_{fe}$ provide the discretization of the spin connection on the two-complex obtained by subdivision of $\Delta^*$ into wedges, which are in one-to-one correspondence with pairs $(vf)$, as shown on Fig. 1 [32].

Finally, we should discretize the normal field $x^I$ appearing explicitly in the gauge fixed action (2.10) and in the linearized simplicity constraints (2.5). This field is analogous to a similar field appearing in the canonical formulation of the usual Holst and Plebanski actions [33] where it describes the normal to the three-dimensional spacelike slices. In spin foam models it usually gives rise to the normal vectors $x_e$, which can be viewed as elements of the factor space $X = \text{Spin}(4)/\text{SU}(2)$ [23, 18, 25, 34]. Such a vector is interpreted geometrically as the normal to the tetrahedron dual to edge $e$. However, in the degenerate case it is more natural to associate such normal vectors to 4-simplices. Thus, at the discrete level the field $x^I$ will be represented by a set of unit vectors $x_v$.

With these definitions we can now discretize the simplicity constraints. Their quadratic form (2.2) is discretized as usual giving rise to diagonal, cross and volume simplicity obtained by averaging the two bivectors over the same triangle, different triangles belonging to the same tetrahedron, or non-intersecting triangles of the same 4-simplex, respectively. The only difference with the usual case is the form of the volume constraint which requires that the geometric volume of the 4-simplex vanishes.

On the other hand, the discretization of the linear simplicity constraints (2.5a) involves an extra ingredient. Indeed, they relate the fields, $x^I$ and $B^{IJ}$, which after discretization live at
different elements of the cellular decomposition: at vertices and faces, respectively. Due to this, the bivectors should be transported to the reference frame of a vertex using the holonomies introduced above. As a result, the discrete simplicity constraints read as follows

$$g_{ve}B_{ef}g_{ev} \cdot x_v = 0, \quad \forall f \supset e \supset v. \quad (3.5)$$

It is important to notice that since there are two ways to connect a face to a vertex (by going either through $e = u(f)$ or $e = d(f)$), the conditions (3.5) constrain not only the bivectors, but also the holonomies. This shows that at the discrete level the primary and secondary constraints are not well distinguished from each other.

To make contact with the new spin foam models, it is convenient to change a bit the point of view and to write the simplicity constraints in the reference frame of a tetrahedron or its dual edge. To this end, we define

$$x_e(v) = g_{ev}x_v$$

so that the condition (3.5) becomes

$$B^{IJ}_{ef}(x_e(v))_J = 0, \quad \forall f \supset e \supset v. \quad (3.7)$$

Up to insertion of the Hodge operator, this is the usual form of the linear simplicity constraints used in the new spin foam models. In our case it should be supplemented by the additional requirement that the normals $x_e(v)$ originate from the same vector $x_v$ and therefore must satisfy

$$g_{ve}x_e(v) = g_{ve'}x_{e'}(v), \quad \forall e, e' \supset v. \quad (3.8)$$

Then we turn to the secondary second class constraints (2.24). They restrict the holonomies of the spin connection and at the discrete level read as

$$x_v = g_{e'}x'_v, \quad \forall e \supset v, v' \quad (3.9)$$
Being combined with (3.6), they can be equivalently rewritten as
\[ x_e(v) = x_e(v'). \] (3.10)

This shows that, provided the secondary constraints are imposed, one can drop the \( v \)-dependence of the normals \( x_e \). Moreover, this suggests to introduce the normal vectors associated to all elements of the spin foam cellular complex: \( x_v, x_e \) and \( x_f \). This pluralistic point of view allows to formulate all constraints in a simple and uniform way. Indeed, they become equivalent to the following relations

\[ B_{IJ}^f(x_f)_J = 0, \] (3.11a)
\[ x_e = g_e v x_v, \quad x_f = g_f e x_e. \] (3.11b)

These relations represent the straightforward discretization of the primary and secondary second class constraints (2.35), respectively. We emphasize that it is crucial to consider the primary and secondary constraints on equal footing. For example, taken alone, (3.11a) is not sufficient to generate the degeneracy condition for a 4-simplex. On the other hand, altogether the conditions (3.11) provide an elegant and amazingly simple discrete formulation of all second class constraints of the continuous theory.

To facilitate the use of the constraints in the discretized path integral, let us rewrite them using the chiral decomposition of Spin(4) (see appendix A.1). To this end, we note that each normal vector \( x \in X \) gives rise to an element \( x \in SU(2) \) defined by
\[ x = g_x^- (g_x^+)^{-1}, \] (3.12)
where \( g_x \in \text{Spin}(4) \) is a representative of \( x \) which choice does not affect the definition of \( x \). Then it is clear that (3.11) is equivalent to

\[ B_f^- = x_f B_f^+ x_f^{-1}, \] (3.13a)
\[ g_{ve}^- = x_v g_{ve}^+ x_v^{-1}, \quad g_{fe}^- = x_f g_{fe}^+ x_f^{-1}. \] (3.13b)

The other constraints appearing above like (3.7) and (3.8) have a similar representation.

### 3.2 The usual strategy

Our first aim is to apply to our model the quantization strategy employed in the EPRL and FK spin foam models and based on the idea “first quantize, then constrain”. This implies that one should start from the unconstrained Spin(4) BF theory and incorporate the simplicity constraints at quantum level. The spin foam quantization of the unconstrained theory is provided by the Crane-Yetter model \([9]\) with the structure group Spin(4) represented by the following state sum
\[ Z_{\text{CY}}^{\text{Spin}(4)}(\Delta^*) = \sum_{\lambda=(j^+,j^-) \to f} \sum_{I=(i^+,i^-) \to e} \prod_{f \in \Delta^*} (2j_f^+ + 1)(2j_f^- + 1) \prod_{v \in \Delta^*} \{15j^+_v\} \{15j^-_v\}, \] (3.14)

where the sum goes over all Spin(4) irreducible representations \( \lambda_f \) and all Spin(4) invariant intertwiners \( I_e \). The simplicity constraints in this approach are supposed to restrict the allowed set of representations and intertwiners such that the resulting state sum provides the discretized path integral for the constrained theory. In our case, the result must reproduce the partition function (3.1).
However, it is clear that if the effect of the constraint imposition is only the reduction of the admissible group theoretic data, the state sum \((3.14)\) will never reduce to \((3.1)\). Indeed, the two partition functions have different vertex amplitudes and it is impossible to reduce one to another by restricting the kinematical data.

This approach can be realized explicitly by proceeding as follows. As is well known, the asymptotic analysis of both EPRL and FK models reveals that they contain a degenerate sector \([22, 35]\). Its geometric interpretation precisely corresponds to the classical geometries described by our model. Thus, a simple way to get a spin foam quantization of \((2.37)\) is to extract the degenerate sector from the EPRL or FK state sum. This can be achieved by expressing this state sum as an integral over coherent states \([14, 15]\). It has been shown \([35]\) that each quasiclassical Regge geometry contributing to the asymptotics of this integral can be uniquely reconstructed from the boundary data consisting of the \(SU(2)\) representations associated to triangles of \(\Delta\) (or faces of \(\Delta^*\)) and coherent states assigned to each pair \((ef)\). The degenerate sector is then extracted by restricting only to those boundary data which lead to degenerate Regge geometries. Since this restriction is imposed only on the boundary data, it affects only the set of representations and intertwiners which survive in the partition function \((3.14)\) after imposing the simplicity constraints. It does not affect the general form of the vertex amplitude and therefore cannot reproduce the desired result \((3.2)\).

This method of imposing of the degeneracy condition has a clear drawback: it ensures the vanishing of the geometric volume of 4-simplices only in the quasiclassical limit. On the other hand, this condition appears as a part of our simplicity constraints which are expected to hold at the full quantum level. This suggests that one should look for an alternative approach. Such an approach does exist and moreover it leads to the correct vertex amplitude. But to realize it, we should be ready to go beyond the usual strategy and to constrain not only the kinematical data in \((3.14)\), but also the group elements entering the definition of the vertex amplitude.

In the previous section we showed that the simplicity constraints can be represented as a combination of two conditions, \((3.7)\) and \((3.8)\). The former are the usual linear simplicity constraints in the non-degenerate sector of Plebanski formulation. The new spin foam models provide a way to implement them at quantum level so that this step can be considered as being already accomplished. Thus, it remains only to incorporate the second condition, which restricts us precisely to the degenerate sector by requiring that the normals to all tetrahedra of a 4-simplex, transferred to the same frame, coincide.

Rather than a constraint on the normals \(x_e(v)\), the condition \((3.8)\) can be viewed as a constraint on the holonomies \(g_{ve}\). These holonomies coincide with the group elements appearing in the integral formula for the vertex amplitude of the new models

\[
A^{(\gamma)}_v(j_f, k_{ef}, i_e) = \int \prod_{e \supseteq v} dg_{ve} S_{(\Gamma_v, \lambda^f(j_f), k_{ef}, i_e)} \left[ g_{vu(f)}^{-1} g_{vd(f)}, x_e(v) \right],
\]

where \(u(f)\) and \(d(f)\) denote the two edges belonging to the face \(f\) and sharing the vertex \(v\) (one of them is considered as “up” and the other as “down”, see Fig. 11), and \(S_{(\Gamma_v, \lambda_f, k_{ef}, i_e)}[g_f, x_e]\) is the so-called projected spin network \([36]\) defined on the graph \(\Gamma_v\) dual to the boundary of a 4-simplex (this is the same graph as the one appearing in \((3.2)\)). The projected spin network is labeled by \(\text{Spin}(4)\) representations \(\lambda_f\) attached to the links of the graph, \(SU(2)\) representations \(k_{ef}\) assigned to the ends of the links, and \(SU(2)\) invariant intertwiners \(i_e\) associated to the nodes. In both EPRL and FK models, the imposition of the simplicity constraints \((3.7)\) leads to that
the Spin(4) representations $\lambda_f$ are defined in terms of the SU(2) representations $j_f$ as\footnote{One should take into account that the constraints (3.7) differ from the linear simplicity constraints describing the gravitational sector in the new spin foam models by the absence of the Hodge operator, i.e. they are actually analogous to the constraints specifying the topological sector (see footnote 2). This difference can be accounted by replacing the Immirzi parameter by its inverse.}

$$\lambda(\gamma)(j) = \left( \frac{1}{2\gamma} (1 + \gamma) j, \frac{1}{2\gamma} |1 - \gamma| j \right). \tag{3.16}$$

On the other hand, the representations $k_{ef}$ are treated differently: in the FK model they can be arbitrary, whereas in the EPRL model they are fixed to be $k_{ef} = j_f$. (For $\gamma = 0$, the EPRL prescription gives $\lambda^{(0)}(j) = (j, j)$, $k_{ef} = 0$ and reproduces the BC model.)

Given the integral representation (3.15), the natural idea to incorporate the condition (3.8) is to insert it into the measure. This amounts to adding the factor $\delta ((g_{\bar{e}e})^{-1} x_u g_{\bar{e}e} x_e^{-1}(v))$ and it is straightforward to evaluate the resulting vertex amplitude. To this end, let us recall the explicit expression for the projected spin network

$$S_{(\Gamma_v, \lambda_f, k_{ef}, i_e)} [g_f, x_e] = \bigotimes_{e \supset v} i_e \cdot \bigotimes_{f \supset v} C^j_j f u(f) f D^{(\lambda_f)} \left( g_{\bar{e}u(f)}^{-1} g_f g_{d(f)} \right) C^{j_1 j_2 f d(f)f}, \tag{3.17}$$

where $D^{(\lambda_f)}(g)$ is the image of $g \in \text{Spin}(4)$ in representation $\lambda_f = (j^+_f, j^-_f)$ and $C^{j_1 j_2 j_3}$ is the invariant map whose matrix elements are given by the Clebsch-Gordan coefficients. Then using the following property of the SU(2) matrix elements

$$\sum_{m, m', n, n'} C^{j^+_j j^-_j j_1}_{m m' \ell_1} C^{j^+_j j^-_j j_2}_{m' n' \ell_2} = \delta_{j_1 j_2} d_{j_1}^{-1} D^{j_1}_{\ell_1 \ell_2}(h), \tag{3.18}$$

where $d_j = 2j + 1$, one can easily show that

$$A^{(\text{deg})}_v(j_f, k_{ef}, i_e) = \left. \int \prod_{e \supset v} \left[ d g_{\bar{e}e} g_{\bar{e}e}^{-1} \delta \left( (g_{\bar{e}e})^{-1} x_u g_{\bar{e}e} x_e^{-1}(v) \right) \right] \right|_{(\Gamma_v, \lambda^{(\gamma)} f, k_{ef}, i_e)} S_{(\Gamma_v, \lambda^{(\gamma)} f, k_{ef}, i_e)} \left( g_{w(u)}^{-1} g_{d(f)} x_e(v) \right),$$

where the $\{15j\}$ symbol is constructed out of ten representations $k_f = k_{u(f)} f = k_{d(f)} f$ and five representations characterizing the intertwiners $i_e$, exactly as in (3.2). Thus, up to a normalization factor, we reproduced the correct vertex amplitude of the SU(2) Crane-Yetter model (3.1)! This beautiful result seems to indicate that the quantization strategy realized by the new spin foam models passes our consistency check. However, its close inspection raises several questions. First of all, a striking feature of (3.19) is that it does not depend at all on the restrictions on the representations obtained by imposing the first part of the simplicity constraints (3.7): it holds for any set of $\lambda_f$ and $k_{ef}$. On the other hand, these restrictions are at the core of the new spin foam models and it is very puzzling that, after imposing the remaining part of the simplicity given by (3.8), all information about them is completely erased. The only case where some information remains corresponds to $\gamma = 0$ in the EPRL model. But it is even worse. In this case $k_f$ are fixed to be zero and we do not get the correct vertex at all. Furthermore, although the final result (3.19) is perfectly fine for all values of the Immirzi parameter, the intermediate step (3.15) is not defined for irrational $\gamma$.\footnote{One should take into account that the constraints (3.7) differ from the linear simplicity constraints describing the gravitational sector in the new spin foam models by the absence of the Hodge operator, i.e. they are actually analogous to the constraints specifying the topological sector (see footnote 2). This difference can be accounted by replacing the Immirzi parameter by its inverse.}
All these issues demonstrate that the constraints put forward by the EPRL and FK models are not really relevant for getting the correct spin foam dynamics, and even generate some strange unphysical effects like the quantization of the Immirzi parameter. On the other hand, the correct dynamics is obtained by imposing the constraints missing in the usual approach. In fact, there is a crucial difference between (3.7) and (3.8): although they both needed to discretize the primary simplicity constraints, the latter are better seen as a discretization of the secondary constraints (2.24). (Let us recall that at the discrete level there is no a clear distinction between the two types of constraints such that exists in the continuum theory.) Thus, the insertion of these constraints into the integration measure is exactly what has been suggested in the introduction (see below (1.3)), in our previous works [18, 16], and in [19]. There it was claimed that the measure over holonomies should include the delta function of the secondary class constraints. Here we see quite explicitly that this insertion is indeed necessary.

As a result, we arrive at the following situation. It is indeed possible to extract from the EPRL and FK models the correct dynamics of the degenerate sector provided we incorporate the secondary second class constraints directly into the definition of the vertex amplitude. However, this modification of the vertex makes the imposition of the primary simplicity constraints completely irrelevant. This questions the ability of these approaches to capture the right dynamics in the gravitational sector where the secondary constraints have been ignored so far.

In fact, as we will show in the next subsections, there is a consistent way of quantizing the theory (2.37) which leads to the full correct result (3.1), including not only the vertex, but also the edge and face amplitudes. It requires to take into account all constraints (3.11). As a consequence, one can use any version of the primary simplicity constraints: either at vertices (3.5), or at edges (3.7), or even at faces (3.11a) — they all become equivalent. However, these constraints become important only for gluing the contributions of different simplices, whereas the vertex amplitude associated with a given simplex turns out to be completely determined by the secondary constraints restricting holonomies. These results clearly show that the usual strategy to the spin foam quantization, based on the use of only the primary constraints, is not satisfactory.

### 3.3 Canonically inspired quantization

In this subsection we provide the rules to construct the partition function of a constrained theory of Plebanski type. These rules summarize the results obtained for the vertex amplitude in [18] and the quantization procedure for the three-dimensional model of [19] developed in [20]. Here we formulate them in a coherent way, which can be applied in quite generic situations. In particular, in the next subsection these quantization rules are applied to our model describing the degenerate sector of Plebanski theory, and shown to reproduce the correct quantization given by the partition function (3.1).

We assume that the theory to be quantized has the structure of Plebanski formulation of general relativity, i.e. it is represented as topological BF theory supplemented by primary constraints \( \phi \), whose time evolution generates secondary constraints \( \psi \). Furthermore, we impose the partial gauge fixing of the boost gauge freedom, as we did in section 2.2. Then the quantization procedure we propose involves the following steps:

1. First, we need to discretize the primary and secondary second class constraints. We assume that their discrete versions give certain restrictions

\[
\phi_{\text{discr}}(B, x) = 0, \quad \psi_{\text{discr}}(g, x; B) = 1
\]
on the bivectors and the holonomies, respectively. Both constraints are expected to depend on the normals $x \in X$ assigned to the elements of the two-complex $\Delta^*$ and we allowed the secondary constraints to depend on the bivectors. In the model we consider here this dependence will be absent, which significantly simplifies its spin foam representation. However, it is expected to arise in the constraints describing the gravitational sector of Plebanski theory.

2. Using the discrete constraints \((3.20)\), we construct the measures

$$D^{(x)}[B] = \Delta(B, x) \delta(\phi_{\text{discr}}(B, x)) dB,$$
$$D^{(x;B)}[g] = \delta(\psi_{\text{discr}}(g, x; B)) dg,$$

where the first one includes the factor which represents a discretization of the determinant of the Dirac matrix, $|\text{det} \{\phi, \psi\}|$. Since typically it does not depend on the spin connection, it can be expressed through the bivectors and the normals and therefore attributed to the first measure only.

3. Given the measure for holonomies, we evaluate the following quantity, which we interpret as vertex amplitude,

$$A_v(\lambda_f, k_{ef}, i_e) = \int \prod_{e \supset v} D^{(x;B)}[g_{ve}] S_{(\Gamma_v, \lambda_f, k_{ef}, i_e)} \left[ g^{-1}_{vu(f)} g_{vd(f)}, x_e \right]. \quad (3.22)$$

This is the same formula as \((1.3)\) given in the introduction, where the boundary state is taken to be the projected spin network $S_{(\Gamma_v, \lambda_f, k_{ef}, i_e)}[g_f, x_e]$, and generalizes \((3.15)\) and \((3.19)\). Due to the gauge invariance of projected spin networks and to the following covariance property of the measure

$$D^{(x;B)}[g_{ve}] = D^{(g; x; gBg^{-1})}[g_{ve}], \quad g \in \text{Spin}(4), \quad (3.23)$$

the vertex amplitude \((3.22)\) is independent of the normals. In contrast, if the measure depends on bivectors, this dependence propagates to $A_v$ and, as a result, it cannot be viewed as a true vertex amplitude in the spin foam representation. If however the measure is $B$-independent, as it happens in our model, the formula \((3.22)\) does provide the spin foam vertex.

4. To achieve the correct gluing of the vertex contributions, one has to perform several additional steps. The first of them is to evaluate what can be called the vertex amplitude in the “connection” representation

$$A_v[g_f, x_e] = \sum_{\lambda_f, k_{ef}, i_e} \left( \prod_{f \supset v} d\lambda_f \right) \left( \prod_{(e, f) \supset v} d_{k_{ef}} \right) A_v(\lambda_f, k_{ef}, i_e) S_{(\Gamma_v, \lambda_f, k_{ef}, i_e)}[g_f, x_e]. \quad (3.24)$$

In generic case this name is not quite precise because this quantity carries a dependence on the bivectors, which we did not indicate explicitly, originating from the measure in the definition of $A_v(\lambda_f, k_{ef}, i_e)$.  

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5. The quantity (3.24) can be already used to glue several vertex contributions. However, such gluing can be written more elegantly if one first passes to the $B_f$-representation. To this end, one defines

$$A_v[\hat{B}_f, x_f] = \int \prod_{f \supset v} \left[ dg_f \exp \left\{ \text{Tr} \left( (1 + \gamma^{-1}) B_f \cdot g_{fu(f)} g_f g^{-1}_{fu(f)} \right) \right\} \right] A_v[g_f, x_e], \quad (3.25)$$

where $\gamma$ is the Immirzi parameter and we used the fact that the amplitude (3.25) depends on bivectors only in the combination $\hat{B}_f \equiv g^{-1}_{fu(f)} B_f g_{fu(f)}$. The group elements $g_{fe}$ are restricted to satisfy the same type of the second class constraints $\psi_{\text{disc}}$ as $g_{ve}$. We do not integrate over them since they drop out from the final partition function. Alternatively, one could insert integrals $\int D^{(x;B)}[g_{fe}]$ which explicitly put the constraints on these holonomies. Note in contrast that the group elements $g_f$ are integrated with the standard Haar measure and not the one involving the second class constraints.

6. Finally, the total partition function is formed by multiplying the vertex amplitudes (3.25) using the non-commutative star-product \[37, 38\] and integrating the result over the bivectors with the measure (3.21)

$$Z = \int \prod_f D^{(x)}[B_f] \left( \star_v A_v[\hat{B}_f, x_f] \right). \quad (3.26)$$

The order in the non-commutative product is dictated by the orientation of faces and by the choice in each face of a “reference tetrahedron” [34]. As above, the covariance of the measure ensures the independence of the partition function on the normals, which expresses the independence of the quantization on the gauge fixing.

Several comments concerning this construction are in order:

- These rules to construct the partition function have been derived by discretizing the canonical path integral of the original theory and by splitting it into contributions associated with different vertices [18]. Therefore, the consistency with the canonical quantization is in a sense built-in to this approach. In particular, we will see below how the quantum amplitudes introduced above allow to recover various elements of loop quantum gravity.

- The key element of this construction is the formula for the vertex amplitude (3.22). This is a straightforward generalization of the usual prescription for the evaluation of the vertex in the spin foam models considered in the literature. The only difference is that the measure in (3.22) is supposed to be non-trivial and, in particular, to include the secondary second class constraints. However, this difference has drastic consequences. Without including the constraints, the integral defining the vertex is equivalent to the evaluation of a simplex boundary state on a flat connection. This recipe has its origin in the topological BF theory which describes the dynamics of flat connections. On the other hand, in the presence of the secondary constraints in the measure, such correspondence does not work anymore and the dynamics becomes more complicated.

---

4 This product is defined on the plane waves $e_g(x) = e^{i \text{Tr}(g \cdot x)}$, with $x \in g$ and momentum given by group element $g$, as $e_{g_1} \star e_{g_2} = e_{g_1 g_2}$. 

19
• While $A_v(\lambda_f, k_f, i_e)$ encodes the dynamics, the vertex amplitude in the “connection” representation $A_v[g_f, x_e]$ describes the kinematical Hilbert space of the theory. It shows that the kinematical states always appear as linear combinations of projected spin networks [23, 18], which is indeed the case for all spin foam models of four-dimensional gravity considered in the literature. But not all linear combinations are physically relevant. The space of all projected spin networks is too huge and only the states given by (3.24) contribute to the path integral.

• The non-commutative star-product is introduced in the final formula (3.26) in order to combine different exponential factors into the exponential of the discrete BF action. This construction is similar to the one introduced recently in the study of a non-commutative flux representation [37, 38, 34].

• It should be emphasized that at none of the steps one integrates over the normal vectors. This is consistent with their interpretation as gauge fixing parameters. Moreover, they completely drop out of the partition function, so that in the Lorentzian case an integral over these normals would produce an overall infinite factor.

• An important feature of the proposed quantization is the difference in the roles of the primary and secondary constraints. Whereas the latter enter the definition of the vertex amplitude and affect the dynamics, the former become relevant only when one glues different simplices together, i.e. at the very last stage. This is in a drastic contrast with the usual spin foam approach where the primary constraints play the central role and the secondary constraints have been ignored at all so far.

3.4 It works!

Let us now apply the quantization procedure of the previous subsection to our model which, after partial gauge fixing, is given by the action (2.37). To make it more understandable, we will follow the procedure step by step.

1. The second class constraints in our case are given by $\Phi_f^I$ and $\Psi_a^I$ (2.35). They have been already discretized in section 3.1 (see (3.13)), so that in the chiral notations we have

$$\phi_{\text{discr}}(B, x) = B_f - x_f B_f^+ x_f^{-1}, \quad \psi_{\text{discr}}(g, x) = (g_{ve})^{-1} x_e g_{ve}^+ x_e^{-1}. \quad (3.27)$$

It is useful to note that, besides the second class constraints on the canonical variables, the gauge fixed action leads to conditions on the Lagrange multipliers. These conditions, provided by (2.14) and (2.23), are naturally combined with the second class constraints, which makes possible to extend the latter to spacetime covariant conditions.

2. The measures (3.21) take the form

$$\mathcal{D}^{(x)}[B_f] = \delta \left( B_f - x_f B_f^+ x_f^{-1} \right) \, dB_f, \quad \mathcal{D}^{(x)}[g_{ve}] = \delta \left( (g_{ve})^{-1} x_e g_{ve}^+ x_e^{-1} \right) \, dg_{ve}, \quad (3.28)$$

where we took into account that the factor $\Delta(B, x)$ is trivial due to the last commutation relation in (2.26). Note also that the measure over holonomies remains independent of the bivectors.
3. The vertex amplitude given by the general formula (3.22) is evaluated in exactly the same way as in section 3.2. Similarly to (3.19), one finds

\[ A_v(\lambda_f, k_{ef}, i_e) = \{15j\} \prod_{f \ni v} d_{k_f}^{-1} \delta_{h_{u(f)}h_{d(f)}}, \tag{3.29} \]

Thus, up to a normalization factor, we again reproduce the correct vertex amplitude of the SU(2) Crane-Yetter model (3.1).

4. The rest of the construction will restore the correct face and edge amplitudes. But not only that. It will also teach us several important lessons. In particular, let us evaluate the vertex amplitude in the “connection” representation (3.24). Since the vertex (3.29) does not depend on \(\lambda_f\), the sum over this label can be done explicitly. Indeed, it is easy to prove the following identity

\[ \sum_{j^+, j^-} d_{j^+} d_{j^-} \sum_{m, m', n, n'} C_{m m' n n'}^{j^+ j^-} D_{m m'}^{j^+} (g^+) D_{n n'}^{j^-} (g^-) C_{n n' \ell_1 \ell_2}^{j^+ j^-} = \delta (g^- (g^+)^{-1} D_{\ell_1 \ell_2}^{j} (g^+), \tag{3.30} \]

due to which one finds

\[ A_v[g_f, x_f] = \sum_{k_f, i_e} \{15j\} \left[ \prod_{f \ni v} d_{k_f} \delta (g_f (x) (g_f (x))^{-1}) \right] \tilde{S}_{\Gamma_v, k_f, i_e} [g_f^+ (x)], \tag{3.31} \]

where we introduced \(g_f (x) = g^{-1}_{x u(f)} g_f x_{d(f)}\) and \(\tilde{S}_{\Gamma_v, k_f, i_e}\) is the usual SU(2) spin network associated with the boundary graph \(\Gamma_v\). This result has two remarkable features. First, the delta function on the r.h.s. imposes the same condition as the second class constraint (3.27) on the holonomy \(g_{u(f)} d(f)\). Second, the amplitude (3.31) is represented as a sum over SU(2) spin networks, which shows that the latter span the kinematical Hilbert space of our theory in full agreement with the SU(2) Crane-Yetter model. From this one concludes that the sum over the auxiliary representation labels, which the vertex amplitude does not depend on, has two important effects: on the one hand, it restores the secondary second class constraints on the arguments of the boundary states and, on the other hand, it provides the reduction from the space of all projected spin networks to the kinematical Hilbert space of the theory. We emphasize that this reduction does not involve the primary simplicity constraints at all and is achieved due to the secondary constraints encoded in the form of the vertex amplitude!

5. The next step is to substitute (3.31) into (3.25). This gives

\[ A_v[B_f, x_f] = \sum_{k_f, i_e} \{15j\} \left[ \prod_{f \ni v} d_{k_f} \int_{\text{SU}(2)} dh_f e^{i \text{tr} (b_{f}^{(\gamma)} h_{f})} \right] \tilde{S}_{\Gamma_v, k_f, i_e} [h_f], \tag{3.32} \]

where we used the chiral decomposition of the \(\text{so}(4)\) trace (A.7), the gauge invariance of spin networks, and denoted

\[ \text{su}(2) \ni b_{f}^{(\gamma)} = \frac{1}{2} (1 + \gamma^{-1}) B_{f}^+ + \frac{1}{2} (1 - \gamma^{-1}) x_{f}^{-1} B_{f} x_{f}. \tag{3.33} \]
6. Finally, we glue the amplitudes (3.32) together by means of the formula (3.26). To distinguish the representations and the group elements associated to different vertices, we put the index \( v \) on them, i.e. in (3.32) one should make the replacements \((k_f, i_e) \mapsto (k_{vf}, i_{ve})\) and \(h_f \mapsto h_{vf}\). Then the non-commutative star-product ensures that each face comes with the factor \(\exp \left\{ i \operatorname{tr} (b_f^{(\gamma)} H_f) \right\} \) where \(H_f = \prod_{v \subset f} h_{vf}\). Since the geometric meaning of \(h_{vf}\) is the positive chiral part of the curvature around the wedge \((vf)\) (see Fig. 1), the group element \(H_f\) gives (the positive chiral part of) the full curvature around the face \(f\). On the other hand, due to the primary simplicity constraints entering the measure on the bivectors, one has \(b_f^{(\gamma)} = B_f^+\) so that the integrals in (3.26) generate \(\delta(H_f)\) imposing the flatness condition. Expanding the \(\delta\)-function in the sum over representations, it is easy to see that the full partition function is given by

\[
Z = \sum_{j_f} \sum_{k_{vf}, i_{ve}} \prod_{v,f} \left[ d_{k_{vf}} \int_{\text{SU}(2)} dh_{vf} \right] \prod_f \left[ d_{j_f} \chi_{j_f} \left( \prod_{v \subset f} h_{vf} \right) \right] \prod_v \left[ \{15j\} \mathcal{S}(\gamma, k_{vf}, i_{ve})[h_{vf}] \right],
\]

(3.34)

where \(\chi_j\) is the \(\text{SU}(2)\) character of representation \(j\). It is immediate to check that, doing the remaining integration and contracting all indices, one reproduces the \(\text{SU}(2)\) Crane-Yetter state sum (3.1) with the same face, edge and vertex amplitudes.

Thus, we conclude that the quantization rules given above lead to the correct spin foam quantization of the degenerate sector of Plebanski theory and therefore provide the correct implementation of all the constraints.

3.5 Why does it work?

In the previous subsection we went through a long way to get the partition function of the constrained theory. In fact, in the particular case of our model, there is a shorter way to arrive at the same result, which partially explains the origin of the proposed quantization rules. It relies on the observation that the vertex amplitude in the “connection” representation \(A_v[g_f, x_e]\) has a much simpler expression. Indeed, in [18] it has been proven that its spin foam like representation (3.24) follows from

\[
A_v[g_f, x_e] = \int \prod_{e \supset v} D^{(x: B)}[g_{ve}] \prod_{f \supset v} \delta \left( g_{ru(f)} g_{g_f g_{vd(f)}^{-1}} \right).
\]

(3.35)

As a result, one immediately obtains

\[
\tilde{A}_v[\tilde{B}_f, x_f] = \int \prod_{e \supset v} D^{(x: B)}[g_{ve}] \prod_{f \supset v} \exp \left( i \operatorname{Tr} \left[ (1 + \gamma^{-1} \gamma \star) B_f \cdot G_{vf} \right] \right),
\]

(3.36)

where \(G_{vf} = g_{fu(f)} g_{ud(f)} g_{vd(f)} g_{df(f)}\) is the curvature around the wedge \((vf)\). Then the partition function (3.26) produces the exponential of the unconstrained Spin(4) BF action integrated with the measure (3.21) involving the primary and secondary second class constraints. This is nothing else but a discretization of the standard canonical path integral for the constrained theory. In our case the discrete constraints (3.27) can be solved explicitly, which simply reduces the path integral to the \(\text{SU}(2)\) sector, and therefore the coincidence with the reduced phase space quantization (3.1) is guaranteed.
One can ask: why did we do all the above complicated calculations if they are not required to get the final spin foam model? The point is that the shorter way is available only if it is possible to explicitly find the reduced phase space at the discrete level. Although this can be done for our simple model, this seems to be out of reach in more complicated situations such as the gravitational sector of Plebanski theory. On the other hand, the representation disentangles the vertex contributions and the gluing of different vertices, and importantly this is done before implementing the constraints. This can be viewed as the first step towards the spin foam representation of the partition function, which should follow after integrating out the remaining geometric variables.

Furthermore, the derivation of the previous subsection clarified many subtle issues such as the imposition of constraints, gauge invariance, the role of the Immirzi parameter, etc. Some of them have been already discussed above, and we will summarize once more our main conclusions and observations in the next section.

4 Discussion

In this paper we studied the classical and quantum descriptions of the degenerate sector of Plebanski formulation of general relativity. We have shown that one of its subsectors, analogous to the gravitational sector of Plebanski theory, provides an interesting and useful model to test the ideas of the spin foam quantization. Classically, it represents a constrained four-dimensional Spin(4) BF theory which, upon elimination of the constraints, reduces to the SU(2) BF theory. Since both these theories are of BF type, their spin foam quantization is well known and can be used to find the correct way of implementing the constraints. As a result, we formulated a general procedure to build the partition function of any constrained theory of Plebanski type. In particular, we showed that, being applied to our model, this procedure works perfectly, giving rise to the right kinematical Hilbert space and generating the right dynamics, i.e. those which agree with the SU(2) Crane-Yetter model.

One of the main results of this analysis is the clarification of the role of the primary and secondary second class constraints in the construction of the spin foam partition function. As has been already argued before \[18, 16, 19, 20\], the secondary constraints affect the measure for holonomy variables and determine the form of the vertex amplitude. On the other hand, the primary constraints enter only at the very last step of the construction when the contributions of different simplices are glued together. In fact, this is a very natural result. The primary constraints is a simple consequence of the choice of our basic variables \(B_f\), which live on the boundary of 4-simplices. At the same time, the secondary constraints appear as a commutator of the primary ones with the Hamiltonian and therefore contain information about the dynamics of the theory. Moreover, they constrain the variables \(g_{ve}\) living “inside” 4-simplices. Giving these observations, it should not be surprising that the main quantity responsible for the dynamics in the spin foam approach is governed by the secondary constraints, whereas the primary ones play only some minor role at the boundary. Furthermore, even the kinematical Hilbert space of the constrained theory turns out to be completely determined by the secondary constraints because the kinematical boundary states appear as projected spin networks weighted by the vertex amplitudes (see \(3.24\)).

All these statements are in a drastic contrast with the usual constructions performed in

\[^5\text{A similar modification of the measure for holonomies has been found also in recent group field theory constructions} \[34\].\]
the four-dimensional spin foam models of general relativity where the principal role is given to
the primary constraints, whereas the secondary constraints are not considered at all. In the
quasiclassical limit these models do provide the right dynamics since in this limit one sets on
shell where the secondary constraints are effectively induced. However, beyond the limit the
primary constraints are not sufficient to suppress the quantum fluctuations of the degrees of
freedom constrained by the secondary ones. This is why the latter should be taken into account
and are crucial to get the right dynamics at quantum level. This is clearly demonstrated by our
model as well as by its three-dimensional analogue [19].

It is worth also to note that our derivation confirmed once more that the closure constraint
of Regge calculus should not be imposed in spin foam models. This constraint requires the
invariance of intertwiners which is achieved by integration over the normal vectors $x_e$ associated
to tetrahedra of the simplicial decomposition. However, our model, in agreement with previous
claims [23, 18, 39], clearly shows that such integration would be inconsistent and the normal
vectors should be kept fixed, which is nothing else but the usual gauge fixing of the boost gauge
freedom in the gravity path integral.

Another interesting point is that our quantization has been done in the presence of the
Immirzi parameter. Nevertheless, it did not have any effect on the constrained theory both at
classical and quantum level. After imposition of the constraints, it completely drops out of the
partition function. This should be compared with the claims that its appearance in loop quantum
gravity is a consequence of an unfortunate choice of variables (the Ashtekar-Barbero connection
which is not a pull-back of a spacetime connection) and with a right choice it disappears from
physical results [40, 28, 17].

Finally, let us comment on the extension of our construction to the physical case of the
gravitational sector of Plebanski theory. The main difference distinguishing it from our model is
the form of the constraints which cannot be written anymore in the simple form (2.35) or (3.27)
after discretization. In particular, the secondary constraints become explicitly dependent on the
$B$-field [41, 42]. Although the construction of section 3.3 is still well defined in the presence
of such dependence, it gives rise to many complications. The most important one is that the
quantity (3.22) starts to depend on the bivectors $B_f$ and its interpretation as a vertex amplitude
is not viable anymore. It is not clear whether this is a serious problem or just a minor obstacle.
In principle, the $B$-dependence can make impossible to integrate out the bivectors because the
resulting integrals are not of $BF$ type anymore. However, given that in our procedure this
integration appears only as a way to glue the simplex contributions, one may still hope that it
will be possible to factorize the $B$-dependence and to extract the spin foam vertex. In any case,
this issue certainly deserves a further study.

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A  Conventions

A.1  \( \mathfrak{so}(4) \) algebra and the chiral decomposition

Our conventions for indices are the following: \( \mu, \nu, \ldots \) denote spacetime indices, \( a, b, \ldots \) are spatial indices, \( I, J, \ldots \) are the \( \mathfrak{so}(4) \) indices in the tangent space, and \( i, j, \ldots \) refer to the \( \mathfrak{su}(2) \) subalgebra. \( \varepsilon^{IJKL} \) and \( \varepsilon^{ijk} \) are the invariant fully anti-symmetric tensors in four and three dimensions, respectively, normalized by \( \varepsilon^{0123} = \varepsilon^{123} = 1 \). Besides, we use \( (\cdot) \) and \( [\cdot] \) to denote symmetrization and anti-symmetrization, respectively, with weight 1/2. Since all tangent space indices are contracted with a unit metric, we do not follow the rule, that the contracted indices should be in opposite positions, very strictly.

The canonical basis of \( \mathfrak{so}(4) \) is composed of the rotation and boost generators, \( L_i \) and \( K_i \), which have the following commutation relations

\[
[L_i, L_j] = \varepsilon_{ij}^k L_k, \quad [K_i, K_j] = \varepsilon_{ij}^k L_k, \quad [K_i, L_j] = \varepsilon_{ij}^k K_k. \tag{A.1}
\]

The rotation generators form the canonically embedded \( \mathfrak{su}(2) \) subalgebra. On the other hand, given a four-dimensional normal vector \( x^I \), one can introduce a boosted subalgebra \( \mathfrak{su}(2)_x \). It is formed by the generators which leave the vector invariant. If one introduces the covariant notation for the \( \mathfrak{so}(4) \) generators \( T^{ij} = \frac{1}{2} \varepsilon^{ijk} L_k \), \( T^{0i} = \frac{1}{2} K^i \), then

\[
I^{IJ}_{KL}(x) = \frac{1}{2} \varepsilon^{IJM} \varepsilon_{KLP} x^M x_N = \delta^{IJ}_{KL} - 2 x^I \delta^K_L x^J, \quad J^{IJ}_{KL}(x) = 2 x^I \delta^K_L x^J, \tag{A.2}
\]

are the projectors on \( \mathfrak{su}(2)_x \) and its orthogonal completion, respectively, i.e. acting on the generators in the vector representation, they satisfy \( I^{IJ}_{KL}(x) T^{KL} \cdot x^N = 0 \).

The antisymmetric bivectors \( B^{IJ} \) form the adjoint representation of the \( \mathfrak{so}(4) \) algebra. On this representation we define the action of the Hodge operator as \( (\ast B)^{IJ} = \frac{1}{2} \varepsilon^{IJ}_{KL} B^{KL} \). Since the Hodge operator squares to one, \( \ast^2 = \text{id} \), it splits the space of bivectors into the direct sum of two eigenspaces with eigenvalues \( \pm 1 \),

\[
B = B^{(+)}_i T^{(+)}_i + B^{(-)}_i T^{(-)}_i, \quad \ast T^{(\pm)} = \pm T^{(\pm)}. \tag{A.3}
\]

This corresponds to the chiral decomposition \( \text{Spin}(4) = \text{SU}(2) \times \text{SU}(2) \), and the generators of the Lie algebras of the two chiral subgroups are given by \( T^{(\pm)}_i = \frac{1}{2} (L_i \pm K_i) \) with

\[
[T^{(\pm)}_i, T^{(\pm)}_j] = \varepsilon_{ij}^k T^{(\pm)}_k, \quad [T^{(\pm)}_i, T^{(\mp)}_j] = 0. \tag{A.4}
\]

In the adjoint representation we normalize the trace such that

\[
\text{Tr}(T^{IJ} T^{KL}) = \delta^{IJ}_{KL} \quad \Longrightarrow \quad \text{Tr}(AB) = A^{IJ} B_{IJ}, \tag{A.5}
\]

where \( \delta^{IJ}_{KL} = \delta^K_L \varepsilon^{IJ} \). Then the chiral decomposition \( (A.3) \) implies that

\[
\text{Tr}(T^{(+)}_i T^{(+)}_j) = \delta^{(+)}_{ij}, \quad \Longrightarrow \quad \text{Tr}(AB) = A^{(+)}_i B^{(+)}_i + A^{(-)}_i B^{(-)}_i. \tag{A.6}
\]

On the other hand, it is useful to remember that the \( \mathfrak{su}(2) \) generators \( L_i \) have a different normalization, namely, \( \text{tr} (L_i L_j) = 2 \delta_{ij} \). Due to this, in terms of the \( \mathfrak{su}(2) \) elements \( B^{(\pm)} = B^{(\pm)}_i L^i \), the \( \mathfrak{so}(4) \) trace reads

\[
\text{Tr}(AB) = \frac{1}{2} \text{tr} (A^{(+) B^{(+)})} + \frac{1}{2} \text{tr} (A^{(-) B^{(-)})}. \tag{A.7}
\]
\subsection*{A.2 Clebsch-Gordan coefficients}

Our conventions for SU(2) invariant intertwiners follow \cite{14}. A generic invariant intertwiner is denoted by $i$ and is supposed to be normalized as

$$
\sum_{m_1 \cdots m_L} i_{m_1 \cdots m_L} \overline{i_{m_1 \cdots m_L}} = 1.
$$

(A.8)

In the particular case of three coupled representations, the matrix elements of the intertwiners are given by the Clebsch-Gordan coefficients $C_{j_1 j_2 j_3}^{j_1 j_2 j_3}$. It is convenient also to define the two invariant maps based on these intertwiners

$$
C_{j_1 j_2 j_3} : \mathcal{H}^{(j_1)} \otimes \mathcal{H}^{(j_2)} \rightarrow \mathcal{H}^{(j_3)} \quad \text{and} \quad \overline{C_{j_1 j_2 j_3}} : \mathcal{H}^{(j_3)} \rightarrow \mathcal{H}^{(j_1)} \otimes \mathcal{H}^{(j_2)}.
$$

(A.9)

Then the following properties are satisfied

$$
C_{j_1 j_2 j_3} C_{j_1 j_2 j_3} = d_j^{-1} \delta_{j_j' \delta_{m n} \delta_{m n}'}
$$

(A.10)

where $d_j = 2j + 1$ is the dimension of the SU(2) representation. In particular, the first property ensures that one gets the right normalization (A.8). Finally, we fix the normalization of Wigner matrices by requiring

$$
\int_{SU(2)} \mathcal{D}^{(j)}_m \overline{\mathcal{D}^{(j')}_{m'}} (h) = d_j^{-1} \delta_{j_j'} \delta_{m n} \delta_{m n}'.
$$

(A.11)

With these normalizations, the matrix elements are recoupled as follows

$$
\mathcal{D}^{(j_1)}_{m_1 n_1} (h) \mathcal{D}^{(j_2)}_{m_2 n_2} (h) = \sum_{j_1 + j_2} \sum_{m, n} d_j C_{j_1 j_2 j}^{j_1 j_2 j} C_{j_1 j_2 j}^{j_1 j_2 j} \mathcal{D}^{(j)}_{m n} (h).
$$

(A.12)

These properties are sufficient to prove (3.18) and (3.30).

\section*{B Canonical analysis of the degenerate Plebanski sector}

The canonical analysis of the gravitational sector of Spin(4) Plebanski theory has been carried out for the first time in \cite{41}, and further elaborated in \cite{42, 43}. The degenerate sector described by the action (2.1) can be analyzed along a similar way. Here we follow the original method of \cite{41}.

One starts as usual from the 3+1 decomposition. One immediately observes that $\omega_{a}^{IJ}$ and $\tilde{P}_{aIJ} = \varepsilon^{abc} B_{bcIJ}$ appear as conjugate variables, whereas the action does not contain time derivatives of $\lambda_{\mu \nu \sigma}$, $\omega_{a}^{0J}$ and $B_{0aIJ}$ which therefore are expected to play the role of Lagrange multipliers. However, the latter variables appear quadratically in the term generating the simplicity constraints which leads to certain complications in considering them directly as Lagrange multipliers. To avoid these complications, one adds new non-dynamical variables $\mu_{a}^{IJ}$ and $\pi_{a}^{IJ}$ which enforce the vanishing of the momenta conjugate to $B_{0aIJ}$ by means of the following additional term

$$
\int_{\mathcal{M}} d^4 x \left( \text{Tr}(\pi_{a}^{0} \partial_{0} B_{0a}) - \text{Tr}(\mu_{a} \pi_{a}) \right).
$$

(B.1)
Then the phase space is spanned by $\omega^I_a$, $B^I_{\mu\nu}$ and $\pi^I_a$ with the symplectic structure given by

$$\{\omega^I_a(x), \pi^J_b(y)\} = \delta^I_b \delta^J_a \delta(x, y),$$

$$\{B^I_{0a}(x), \pi^J_{KL}(y)\} = \delta^I_b \delta^J_a \delta(x, y),$$

(B.2)

and the total action leads to the following primary constraints

$$\pi^I_a \approx 0 \quad \text{(B.3a)}$$

$$G^I_I = D^a_a \pi^I_a \approx 0, \quad \text{(B.3b)}$$

$$\Phi(B, B)_{ab} = 1$$

$$\Phi(\tilde{P}, B)_{ab} = 1$$

$$\Phi(\tilde{P}, \tilde{P})_{ab} = 1$$

(B.3c, B.3d, B.3e)

The last three are nothing else but the various components of the simplicity constraints (2.2). But as is shown in appendix C, the six constraints $\Phi(B, B)_{ab}$ are in fact linear combinations of $\Phi(\tilde{P}, B)_{ab}$ and $\Phi(\tilde{P}, \tilde{P})_{ab}$, and do not require a special attention. The Hamiltonian can be written in the following form

$$-H = \varepsilon^{abc} B^I_{0d} F^J_{bd} + \omega^I_a G^J_I + \lambda^{0ab} \Phi(B, B)_{ab} + \frac{1}{2} \lambda^{0abc} \varepsilon_{bcd} \Phi(\tilde{P}, B)^d_a$$

$$+ \frac{1}{16} \lambda^{abcd} \varepsilon_{abf} \varepsilon_{cdg} \Phi(\tilde{P}, \tilde{P})_{fg} - \mu^I_a \pi^I_a.$$  

(B.4)

The next step is to study the conditions imposed by the conservation of the primary constraints. Let us compute their time derivatives by commuting them with the Hamiltonian (B.4). First, one finds

$$\dot{\pi}^I_a = \varepsilon^{abc} F^I_{bc} + 2 \lambda^{0ab} (\ast B)^I_{0b} + \frac{1}{2} \lambda^{0abc} \varepsilon_{bcd} (\ast \tilde{P})^d_I \approx 0.$$  

(B.5)

These 18 equations split into two sets. By contracting them with $\tilde{P}^b_{IJ}$, the last terms produce the simplicity constraints so that one remains with 9 conditions

$$C^I = \varepsilon^{abcd} F^I_{cd} \tilde{P}^b_{IJ} \approx 0,$$

(B.6)

which should be interpreted as secondary constraints. The remaining 9 equations following from (B.5) can be written as

$$\varepsilon^{bcd} \text{Tr}(\tilde{P}^a \ast F_{cd}) + 2 \lambda^{0abc} \text{Tr}(\tilde{P}^a B_{0c}) + \frac{1}{2} \lambda^{0bcd} \varepsilon_{cdg} \text{Tr}(\tilde{P}^a \tilde{P}^g) = 0$$

(B.7)

and fix the Lagrange multipliers $\lambda^{0bcd}$.

The conservation of the Gauss constraint $G^I_I$ does not generate new conditions provided it is shifted as follows

$$G^I_I = G^I_I + [B_{0a}, \pi^I_a].$$

(B.8)

The shift ensures that $G^I_I$ is a generator of Spin(4) gauge transformations and since the Hamiltonian is gauge invariant, $G^I_I$ is preserved under evolution.

Next it is convenient to consider $\Phi(\tilde{P}, \tilde{P})_{ab}$. Its commutator with the Hamiltonian generates 6 new conditions

$$\Psi^I_{ab} = \varepsilon^{ed(a} \text{Tr}(B_{0d} \ast D^b_{e} \tilde{P}^b)) \approx 0,$$

(B.9)
which give rise to additional secondary constraints. Having obtained these constraints, we can now prove a very useful Lemma which facilitates a lot the following analysis.

**Lemma:** Let \( b^I_{\mu\nu} \) be defined by the solution of the simplicity constraints in any of the two sectors (see (2.3)). Assume that i) \( \tilde{p}_{a}^{I} = \varepsilon^{abc}b_{\mu}^{I} \) is invertible in the sense that there exists \( \tilde{p}_{a}^{I} \) such that
\[
\tilde{p}_{a}^{I}\tilde{p}_{a}^{I} = \delta_{a}, \quad \tilde{p}_{a}^{I}\tilde{p}_{a}^{J} = \delta_{J} - x^I x^J; \tag{B.10}
\]

ii) the matrix
\[
Q^{ab,cd} = \varepsilon^{IJKL}x^I_{P_0} x^J_{P_0} \tilde{p}_{a}^{I} \tilde{p}_{b}^{I} D_{c} x_{L}, \tag{B.11}
\]
in invertible. Then \( D_{a} x^I \) can be expressed as a linear combination of \( \Psi^{ab} \), \( \tilde{G}^{a} = \text{Tr}(\tilde{P}^{a} \ast \mathcal{G}) \) and the simplicity constraints, and therefore it weakly vanishes.

**Proof:** First, it is easy to see that in both solution sectors (2.3), i.e. on the surface of the simplicity constraints, one has
\[
\tilde{G}^{a} = -\frac{1}{2} \varepsilon^{IJKL}x^I_{P_0} x^J_{P_0} \tilde{p}_{a}^{I} D_{b} x_{L}, \\
\Psi^{ab} = -\frac{1}{2} \varepsilon^{cd(a}(\varepsilon^{IJKL}x^I_{P_0} x^J_{P_0}) \tilde{p}_{a}^{I} D_{c} x_{L}. \tag{B.12}
\]

Then one can check that
\[
Q^{ab,cd} p_{(c}^I D_{d)} x_I = 2\Psi^{ab} - \tilde{G}^{(a}(\varepsilon^{b)cd} p_{b}^{I} D_{c} x_{L}. \tag{B.13}
\]

Since the matrix \( Q^{ab,cd} \) is assumed to be invertible, this implies that \( p_{(a}^I D_{b)} x_I \) is weakly vanishing. Finally, one verifies that
\[
D_{a} x^I = \tilde{p}_{a}^{I} (p_{(a}^I D_{b)} x_I) - \varepsilon^{IJKL} x^I_{P_0} x^J_{P_0} \tilde{p}_{a}^{I} \tilde{G}^{b} \approx 0. \tag{B.14}
\]

We still need to analyze the stability of two simplicity constraints, (B.3c) and (B.3d). However, since the former is expressible through the latter and (B.3c), only \( \Phi(\tilde{P}, B)^a_b \) remains to be considered. Its time derivative leads to the following condition
\[
\dot{\Phi}(\tilde{P}, B)^a_b = -\text{Tr}([\omega_0, B_{0b}] \ast \tilde{P}^a) + \text{Tr}(\mu_b \ast \tilde{P}^a) \approx 0, \tag{B.15}
\]

where we neglected the term \( 2\varepsilon^{abcd} \text{Tr}(B_{0d} \ast D_{c} B_{0b}) \) because, on the surface of the simplicity constraints, it is proportional to \( D_{c} x^I \) and vanishes weakly by the above Lemma. The resulting equation fixes 9 of the 18 components of the Lagrange multiplier \( \mu_{a}^{IJ} \) and does not lead to new constraints.

This completes the analysis of the primary constraints. But the appearance of the secondary constraints \( \mathcal{O}^{ab} \) and \( \Psi^{ab} \) requires to repeat the stabilization procedure. However, at this step it works rather differently for the deg-gravitational and deg-topological sectors. Due to this reason, we consider them separately.

### B.1 Degenerate gravitational sector

First, let us consider the conservation of the secondary constraints \( \Psi^{ab} \). This gives the following equation
\[
\dot{\Psi}^{ab} = 2\varepsilon^{cd(a}(\varepsilon^{b)g} \text{Tr}(B_{0d} \ast D_{c} D_{f} B_{0g}) - \varepsilon^{cd(a} \text{Tr}([\omega_0, B_{0d}] \ast D_{c} \tilde{P}^b) + \varepsilon^{cd(a} \text{Tr}(\mu_d \ast D_{c} \tilde{P}^b)) \\
+ \lambda^{0cd(a} \text{Tr}(\tilde{P}^b) B_{0c, B_{0d}}) + \frac{1}{8} \lambda^{fgp} \varepsilon_{cfg} \varepsilon_{rqp} \varepsilon^{cd(a} \text{Tr}(\hat{P}^b B_{0c, B_{0d}}) \approx 0. \tag{B.16}
\]
However, it is easy to see that the first term weakly vanishes due to the Lemma, whereas the next two terms produce the equation \( (B.15) \) plus contributions proportional to \( D_c x^f \). As a result, the stability condition reduces to the vanishing of the last two terms. The crucial question for us is the form of the matrix in front of the Lagrange multiplier \( \lambda^{fgpq} \) in the last term, which arises from the commutator of \( \Psi^{ab} \) with the primary constraints \( \Phi(\tilde{P}, \tilde{P})^{cd} \). In the deg-gravitational sector, where the \( B \)-field is given by \( (2.3a) \), it is found to be

\[
\{\Psi^{ab}, \Phi(\tilde{P}, \tilde{P})^{cd}\} = 4 \tilde{P}_{ij}^a \varepsilon^{b(c)} \tilde{P}^{d)k}_{1k} B_{0q}^{ij} \approx Q^{ab,cd}.
\]  

(B.17)

Since this is the same matrix which appears in the Lemma and generically it is invertible, the condition \( (B.16) \) fixes the six Lagrange multipliers \( \lambda^{fgpq} \). At the same time, it shows that \( \Phi(\tilde{P}, \tilde{P})^{ab} \) and \( \Psi^{ab} \) are mutually non-commuting.

Before we proceed further, we prove an additional useful result that the curvature of the spin connection is weakly vanishing. Indeed, using \( (2.3a) \) and the inverse field \( p^I_\gamma \) introduced above, one has

\[
\varepsilon^{abc} F^I_{bc} = \varepsilon^{abc} f^{I}_{KL}(x) F^K_{bc} + \varepsilon^{abc} J^I_{KL}(x) F^K_{bc} = \varepsilon^{IJ} f^{I}_{KL} p^L_a C^{ab} + 4 \tilde{x}^I [\varepsilon^{abc} D_b D_c x^I] \approx 0,
\]

where in the second term we represented the curvature as a commutator of two covariant derivatives. Evaluating now the time derivative of the secondary constraint \( C^{ab} \)

\[
\dot{C}^{ab} = 2 \varepsilon^{acd} \varepsilon^{bgf} \text{Tr}(B_{0g} D_f F_{cd}) - 2 \lambda^{gacr} \text{Tr}(\tilde{P}^b \ast D_c B_{og}) - \frac{1}{4} \lambda^{fgpq} \varepsilon^{efg} \varepsilon^{rpq} \varepsilon^{cdar} \text{Tr}(\tilde{P}^b \ast D_d \tilde{P}^r) \approx 0,
\]

one immediately concludes that the stability condition is satisfied due to the above Lemma and the vanishing of the curvature proven in \( (B.18) \).

As a result, the stabilization procedure finishes at this point and the list of all constraints is given by \( \pi^{a}_{I,J}, G_{IJ}, \Phi(\tilde{P}, B)^a_b, \Phi(\tilde{P}, \tilde{P})^{ab}, \Psi^{ab}, \) and \( C^{ab} \). Note however that the last constraints are reducible. Namely, due to the Bianchi identity, they satisfy

\[
D_a (C^{ab}_0 p^I_\gamma) = \frac{1}{2} \varepsilon^{IJKL} \varepsilon^{acd} F^K_{cd} D_a x^J,
\]

(B.20)

where the r.h.s., as we know, vanishes on the surface of the other constraints. Thus, only six components of \( C^{ab} \) are independent. To split the resulting constraints into first and second class, we introduce

\[
\pi^{ab}_{(1)} = \text{Tr}(\pi^a \tilde{P}^b), \quad \pi^{ab}_{(2)} = \text{Tr}(\pi^a \ast \tilde{P}^b).
\]

(B.21)

Then, using the weak vanishing of the curvature and of the covariant derivative of the normal vector, it is straightforward to verify that \( \pi^{a}_{(1)}, G_{IJ} \) and \( C^{ab} \) are first class, whereas \( \pi^{ab}_{(2)}, \Phi(\tilde{P}, B)^a_b, \Phi(\tilde{P}, \tilde{P})^{ab} \) and \( \Psi^{ab} \) are second class. This implies the following counting of degrees of freedom. The original phase space is \( 4 \times 18 = 72 \) dimensional. The second class constraints remove \( 9 + 9 + 6 + 6 = 30 \) degrees of freedom, whereas the first class constraints together with the corresponding gauge fixing conditions fix \( 2 \times (9 + 6 + (9 - 3)) = 42 \) of them. This leaves us with a zero-dimensional phase space confirming that we are describing a topological theory.

If one partially fixes the gauge, taking the normal vector \( x^I(x) \) to be a given function \( x^I(x) \), this gauge fixing condition can be combined with \( \Phi(\tilde{P}, \tilde{P})^{ab} \) to get the simplicity constraints \( (2.13b) \). Moreover, the part of the Gauss constraint \( \tilde{G}^a \) is conjugate to the gauge fixing condition and therefore becomes second class. As shown by our Lemma, it can be combined with \( \Psi^{ab} \) to get
the secondary constraints \((2.23)\). Besides, the constraints \(\Phi(\tilde{P}, B)^a_b\) become identical to the condition \((2.14)\) on the Lagrange multipliers, whereas \(C^{ab}\) and the remaining part of \(G_{IJ}\) are equivalent to \(C^I_I\) and \(G_I\) defined in \((2.17)\) and \((2.28)\), respectively. After integrating out the auxiliary variables \(\pi^I_{IJ}\), one recovers the same phase space and the same constraint structure as in section \(2.2\) where the canonical analysis was performed for the gauge fixed action \((2.10)\).

Furthermore, it is possible to show the following equality

\[
\Delta^D_D | \det \{x, \tilde{G}\} | \delta(\pi_{(1)}) \delta(\pi_{(2)}) \delta(\Phi(\tilde{P}, \tilde{G})) \delta(\Phi(\tilde{P}, B)) \delta(\Psi) \delta(x - x(x)) \\
\sim \delta(\pi^a_{IJ}) \delta(x J B^{Ia}_0) \delta(x^J \tilde{F}^{ab}_I) \delta(D_a x^I),
\]

where

\[
\Delta_D \sim (\det \tilde{p})^6 (\det Q)^2
\]

is the determinant of the Dirac matrix of the commutators of the second class constraints, the second factor is a part of the Faddeev-Popov determinant corresponding to the gauge fixing of the normal \(x^I\), and all equations are given up to numerical factors. The meaning of this result is twofold: first, it demonstrates explicitly the recombination of the constraints mentioned above and, second, it confirms the triviality of the factor \(\Delta\) from \((3.21)\) (see \((3.28)\)). Thus, all results derived from the gauge fixed action \((2.10)\) are in the full agreement with the complete canonical analysis of the initial gauge invariant theory.

### B.2 Degenerate topological sector

Now we turn to the deg-topological sector described by the solution \((2.3b)\). The conservation of the secondary constraints \(\Psi^{ab}\) still generates the condition \((B.16)\) where the first line can be dropped. However, in contrast to the previous case, the matrix appearing in the last term vanishes

\[
\{\Psi^{ab}, \Phi(\tilde{P}, \tilde{P})^{cd}\} = \tilde{P}^a_{IJ} \varepsilon^{b)g(c} \tilde{P}^{d)I_K} B_{0g}^I \approx 0
\]

so that \(\Phi(\tilde{P}, \tilde{P})^{ab}\) and \(\Psi^{ab}\) are now mutually commuting. Moreover, the forth term also vanishes in the deg-topological sector so that the stability of \(\Psi^{ab}\) does not generate any new conditions.

The second crucial difference arising in this sector is that the secondary constraints \(C^{ab}\) \((B.6)\) turn out to be linearly dependent with other constraints. Indeed, using \((2.3b)\) and the same trick as in \((B.18)\), one obtains

\[
C^{ab} = -2 \varepsilon^{acde} \tilde{P}^d D_e D_d x^f.
\]

Then the above Lemma ensures that this quantity can be expressed as a linear combination of the Gauss constraint, simplicity and \(\Psi^{ab}\). Due to this, \(C^{ab}\) does not require a separate consideration and its stability follows from the above analysis.

As a result, the independent set of constraints is provided by \(\pi^a_{IJ}\), \(G_{IJ}\), \(\Phi(\tilde{P}, B)^a_b\), \(\Phi(\tilde{P}, \tilde{P})^{ab}\), and \(\Psi^{ab}\). To split them into first and second class, besides \((B.21)\), we define

\[
\hat{\Psi}^{ab} = \Psi^{ab} - \{\Psi^{ab}, \pi^c_{(2)}\} (\text{Tr}(\tilde{P} \tilde{P})^{-1})_{df} \Phi(\tilde{P}, B)^f_c.
\]

Then one can check that \(\pi^{ab}_{(1)}, G_{IJ}, \Phi(\tilde{P}, \tilde{P})^{ab}\) and \(\hat{\Psi}^{ab}\) are first class, whereas \(\pi^{ab}_{(2)}\) and \(\Phi(\tilde{P}, B)^a_b\) are mutually non-commuting and therefore give second class constraints. The counting of degrees of freedom works as above. The second class constraints give \(9 + 9 = 18\) conditions, whereas the first class constraints and their gauge fixing conditions produce \(2 \times (9 + 6 + 6 + 6) = 54\) more. Altogether they fix all degrees of freedom of the original 72-dimensional phase space, also showing that this sector describes a topological theory.
C  Constraints for constraints

Let us consider the simplicity constraints split according to the 3+1 decomposition as in (B.3) and written in terms of the chiral variables. They take the following form

\[
\Phi(B, B)_{ab} = B^{(+)}_{0a} B^{(+)}_{0b} - B^{(-)}_{0a} B^{(-)}_{0b},
\]
\[
\Phi(\tilde{P}, B)_{ab} = \tilde{P}^{(+a)}_{i} B^{(+)}_{0b} - \tilde{P}^{(-a)}_{i} B^{(-)}_{0b},
\]
\[
\Phi(\tilde{P}, \tilde{P})_{ab} = \tilde{P}^{(+a)}_{i} \tilde{P}^{(+b)}_{i} - \tilde{P}^{(-a)}_{i} \tilde{P}^{(-b)}_{i}, \tag{C.1}
\]

Then it is straightforward to check that the following linear combination of constraints

\[
\Upsilon_{ab} = \varepsilon_{c_1 c_2 c_3} \varepsilon_{d_1 d_2 d_3} \left[ \frac{1}{2} \left( \frac{1}{3} \tilde{P}^{(+c_1)}_{i_1} \tilde{P}^{(-d_1)}_{i_1} \Phi(B, B)_{ab} - B^{(+)}_{0a} B^{(-)}_{0b} \Phi(\tilde{P}, B)_{c_1 b} \right)
        - \tilde{P}^{(+c_1)}_{i_1} B^{(-)}_{0b} \Phi(\tilde{P}, B)_{d_1 a} + B^{(+)}_{0a} B^{(-)}_{0b} \Phi(\tilde{P}, \tilde{P})_{d_1 d_1} \right]
\]

identically vanishes. This can be done either expanding explicitly all sums over repeated indices or assuming that \( \tilde{P}^{(\pm)a} \) are two invertible matrices which allows to write, for instance,

\[
\varepsilon_{c_1 c_2 c_3} \tilde{P}^{(+c_2)}_{i_2} \tilde{P}^{(c_3)}_{i_3} = \left( \det \tilde{P}^{(+)} \right) \varepsilon_{i_1} \varepsilon_{i_2} \varepsilon_{i_3} \left( \tilde{P}^{(+c_1)} \right)_{i_1} \tag{C.3}
\]

Using this property in each term in (C.2), the vanishing of \( \Upsilon_{ab} \) follows trivially.

The nine quantities \( \Upsilon_{ab} \) encode constraints for the simplicity constraints (2.2). However, they themselves are not all independent. If one considers their antisymmetric part \( \Upsilon_{[ab]} \), it is possible to show that the coefficients in front of the constraints vanish on the constraint surface. Indeed, assuming again the invertibility of \( \tilde{P}^{(\pm)}_{i} \), one finds

\[
6 \Upsilon_{[ab]} = \left( B^{(+)}_{0b} \left( \tilde{P}^{(+)} \right)_{i} \tilde{P}^{(-1)}_{i} - B^{(-)}_{0b} \left( \tilde{P}^{(-)} \right)_{i} \tilde{P}^{(-1)}_{i} \right) \left( \Phi(\tilde{P}, B)_{a}^{c} - B^{(-)}_{0a} \left( \tilde{P}^{(-)} \right)_{c} \Phi(\tilde{P}, \tilde{P})^{cd} \right), \tag{C.4}
\]

where the prefactor can be rewritten as a linear combination of the original constraints (C.1)

\[
B^{(+)}_{0b} \left( \tilde{P}^{(+)} \right)_{i} \tilde{P}^{(-1)}_{i} - B^{(-)}_{0b} \left( \tilde{P}^{(-)} \right)_{i} \tilde{P}^{(-1)}_{i} =
2 \left( \tr (\tilde{P}^{(+)} \tilde{P}^{(-)})^{d}_{c} \right) \left( \Phi(\tilde{P}, B)^{d}_{b} - B^{(-)}_{0b} \left( \tilde{P}^{(-)} \right)_{g} \Phi(\tilde{P}, \tilde{P})^{gd} \right). \tag{C.5}
\]

Thus, only the symmetric part of (C.2) generates constraints for constraints and allows to express \( \Phi(B, B)_{ab} \) in terms of other 15 simplicity constraints.

One can also notice that the vanishing of (C.2) continues to hold even if one replaces there one or two of the fields \( B_{0a} \) by \( \tilde{P}^{a}_{i} \). For example, replacing only one field, one obtains the following combination of constraints

\[
\tilde{\Upsilon}^{a}_{b} = \varepsilon_{c_1 c_2 c_3} \varepsilon_{d_1 d_2 d_3} \left[ \frac{1}{2} \left( \frac{1}{3} M^{c_1 d_1} \Phi(\tilde{P}, B)_{b}^{a} - M^{a d_1} \Phi(\tilde{P}, B)^{c_1}_{b} - \tilde{P}^{(+c_1)}_{i} B^{(-)}_{0b} \Phi(\tilde{P}, \tilde{P})_{a}^{d_1} \right)
        + \tilde{P}^{(+a)}_{i} \Phi(\tilde{P}, \tilde{P})_{b}^{c_1 d_1} \right] M^{c_2 d_2} M^{c_3 d_3} - \tilde{P}^{(+c_2)}_{i} B^{(-)}_{0b} M^{a d_2} M^{c_3 d_3} \Phi(\tilde{P}, \tilde{P})_{a}^{c_1 d_1}, \tag{C.6}
\]

where we denoted \( M^{ab} = \frac{1}{2} \tr (\tilde{P}^{(a)} \tilde{P}^{(-b)})^{ab} \). This combination is also vanishing, so that one could think that these are additional constraints for constraints. However, in contrast to (C.2), \( \tilde{\Upsilon}^{a}_{b} \) vanish for any \( \Phi(\tilde{P}, B)^{a}_{b} \) and symmetric \( \Phi(\tilde{P}, \tilde{P})^{ab} \), not necessarily of the form (C.1). Again this can be easily proven by assuming the invertibility of the matrix \( M^{ab} \). Due to this, they do not reduce the number of independent constraints which remains to be 15.
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