Geometric momentum generates rotations

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As a nonrelativistic particle constrained to remain on an \(N - 1\) \((N \geq 2)\) dimensional hypersurface embedded in an \(N\) dimensional Euclidean space, two different components \(p_i\) and \(p_j\) \((i, j = 1, 2, 3, \ldots N)\) of the Cartesian momentum of the particle are not commuting with each other. In fact, the so-called fundamental quantum conditions \([p_i, p_j]\) \((\neq 0)\) depend on products of positions and momenta in uncontrollable ways, thus their physical consequences are little known. The local analysis technique is for the first time utilized to crack this long-standing problem. The first finding is that this noncommutativity on a local point of the hypersurface can be fully understood and the quantum conditions include many novel properties including, e.g., operator-ordering free, dependent on the local curvature of the surface, and classifiable into two simple sets. The second finding is that, for a small circle lying on each of the tangential planes and covering the local point, the noncommutativity leads to a rotation operator and the amount of the rotation is an angle anholonomy; and for a short arc length in each of the intersecting curves and centering the given point the noncommutativity leads to a translation plus an additional rotation and the amount of the rotation is one half of the tangential angle change of the arc.

I. INTRODUCTION

In quantum mechanics there are so-called fundamental quantum conditions that include as the vital part the commutation relations between any pair of different components of momentum [1], and the momentum operators in flat space are well studied. However, once the commutation relations for momentum yield non-trivial results, e.g., for the particle in curved space, the systems are far beyond fully understood because no reliable fundamental quantum conditions for the momentum can even be obtained. The most simple model of curved spacetime is the curved hypersurface. Though Dirac proposed in 1950’ the standard procedure of constructing the commutation relations for momentum, and the quantization of the motion for the particle on the surface has been studied for more than five decades, the essential understanding of the commutation relations for momentum is still lacking. In order to make some significant progress, let us recall the importance of local analysis in the general relativity, which tells that the small region of globally curved spacetime is approximately flat. In fact, the local analysis is a powerful technique in differential geometry and non-linear differential equations, etc. In present study, such a technique is utilized to investigate the long-lasting noncommutative commutation relations for momentum.

For a nonrelativistic particle constrained to remain on an \((N - 1)\)-dimensional smooth curved surface \(\Sigma^{N-1}\) in flat space \(R^N\) \((N \geq 2)\), one can define the cartesian momentum \(p_i\) (hereafter \(i, j, l = 1, 2, 3, \ldots N\)) corresponding to the corresponding cartesian coordinate \(x_i\). In classical mechanics, we know that two different components of the momentum \(p\) do not commute with each other [2, 6],

\[
[p_i, p_j] = \Pi_{ij} \equiv \sum_{l=1}^{N} (n_j n_{i,l} - n_i n_{j,l}) p_l \neq 0, (i \neq j),
\]

where subscript \(D\) in the square bracket denotes the Dirac bracket, and \(n_i\) is the \(i\)-th component of the normal vector \(n\) at a point of the surface \(\Sigma^{N-1}\) and symbol "", \(l\)" in the subscript stands for the derivative with respect to the coordinate \(x_l\), and so forth. Although derived more than five decades [2, 6], this noncommutativity remains less understood except for very special case such as the spherical surface and the flat plane. For a particle on the spherical surface of radius \(R\) whose origin is at the center of the sphere, we have well-defined results as [2, 13],

\[
[p_i, p_j] = -KL_{ij},
\]

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where $K = R^{-2}$ is the curvature, and $L_{ij} \equiv x_ip_j - x_jp_i$ is the $ij$-th component of the angular momentum in $x_i, x_j$ ($i \neq j$) plane. To note that once the curvature $K$ is zero, $p_i$ and $p_j$ in (1) is mutually commutable: $[p_i, p_j] = 0$.

Therefore we can make a general conjecture that $(n_ip_{j,1} - n_jn_{i,1})$ in (1) might depend on the curvature of the surface in some ways. It is worthy of pointing out that all previous explorations of quantum version of (1) have focused on the form of the momentum $p_i$ itself, and its relation to the form of Hamiltonian $\hat{H}$.

In classical mechanics, $p \equiv P - n \cdot P$ with $P$ being the ordinary linear momentum in $R^N$, thus $p$ is the projection of $P$ onto the tangential planes of the surface [6, 7]. In quantum mechanics, the commutator relation between $p_i$ and $p_j$ is hypothesized to be given by,

$$[\hat{p}_i, \hat{p}_j] = \frac{i\hbar}{2} \Pi_{ij} \neq 0, \ (i \neq j),$$

where the operator $\hat{\Pi}_{ij}$ can take infinitely many different forms [8,11] because $\Pi_{ij}$ involves non-commuting operators and there is no controllable manner to distributing them, c.f., $\Pi_{ij}$ defined in (1). Here the caret-shaped symbol $\wedge$ placed on top of variables is used to emphasize that it is an operator for avoiding possible confusion, which is usually omitted.

In quantum mechanics, there are many ways to give the same form of the momentum operator which satisfies $n \cdot p = 0$, the so-called geometric momentum [2,11,12],

$$p = -i\hbar(\nabla S + \frac{Mn}{2}),$$

where $M = -\sum_{i=1}^{N} n_{i,i}$ is the mean curvature which is sum of all principal curvatures. We note that some prefers to call the mean curvature part $-i\hbar Mn/2$ of (1) as geometric momentum [13], which is similar to call $-qA$ in the kinematical (or mechanical) momentum $p = p_\epsilon - qA$ the electromagnetic momentum, where $p_\epsilon \equiv -i\hbar \nabla$ is the canonical momentum.

It is recently demonstrated that the geometric momentum [14] is indispensable when dealing with the propagation of surface plasmon polaritons on metallic wires [20]. The commutators between any two different components of the momentum $p$ satisfy [17],

$$[p_i, p_j] = \frac{i\hbar}{2} \sum_{l=1}^{N} \left((n_jn_{i,l} - n_in_{j,l})p_l + p_l(n_jn_{i,l} - n_in_{j,l})\right).$$

However, if one attempts to start from the noncommutative relation (1) to reach its quantum mechanical version for a small area of surface around a given point, only the right-handed side of the equation (1) matters. Fortunately, the results are operator-ordering free, which is in sharp contrast to what the noncommutativity might suggest. Results in section II show that the leading contribution of $(n_jn_{i,1} - n_in_{j,1})p_l$ can be categorized into two classes. In section III we construct two geometrically infinitesimal displacement operators (GIDOs), and demonstrate that these two GIDOs can be divided into two groups of operators, in which one is purely rotational and another is translational plus rotational. Section IV presents conclusions and discussions.

### II. MOMENTUM NONCOMMUTATIVITY AND CURVATURES

Let us consider the surface equation $f(x) = 0$, where $f(x)$ is some smooth function of position $x = (x_1, x_2, ... x_N)$ in $R^N$, whose normal vector is $n \equiv \nabla f(x)/|\nabla f(x)|$. We can always choose the equation of the surface such that $|\nabla f(x)| = 1$, so that $n \equiv \nabla f(x)$. This is because physics depends on what are independent of form of the surface equations, and a simple and convenient form suffices. In the following, we will prove that $\Pi_{ij}$ is locally either, hereafter $a, b = 1, 2, ..., N - 1$,

$$\Pi_{ab} \approx -K_{ab}L_{ab}, \ \text{or} \ \Pi_{ab} \approx -k_a p_a, \ \ (a, \neq b),$$

where $k_a$ is the $a$-th principal curvature, and $K_{ab} \equiv k_ak_b$ is the $ab$-th sectional curvature.

**Proof.** At any point of the surface, there are, respectively, $(N - 1)(N - 2)/2$ mutually perpendicular two-dimensional tangential planes and $N - 1$ normal sectional curves whose tangential vectors at the point are also mutually perpendicular. Let us put the origin of $N$ dimensional cartesian coordinates at the given point of the surface. In a sufficiently small region covering the origin, we construct a system of orthogonal coordinates $(x_1, x_2, ... x_{N-1}, x_N)$ tangent and normal to the hypersurface, and the surface equation around the origin can be $f(x) \equiv x_N - w(x_1, x_2, ... x_{N-1})$ where $w(x_1, x_2, ... x_{N-1})$ is Monge’s form of the hypersurface. For convenience, we can always choose the coordinates such that the normal direction $n$ is along the $x_N$-axis and principal directions are along $x_a$ respectively, and the hypersurface is asymptotically represented by the generalization of the two-dimensional Dupin indicatrix [21, 22],

$$x_N \approx \frac{1}{2} \sum_{a=1}^{N-1} \kappa_a x_a^2,$$
where $k_a$ is the $a$-th principal curvature of the curve formed by the intersection of the $x_a x_N$-plane and the hypersurface $\Sigma^{N-1}$ at the origin, and the intersection is in fact the normal section, and there are in total $N - 1$ normal sections. Evidently, at the origin, there are $N - 1$ distinct tangential vectors along intersecting curves, respectively.

As two verifications of the asymptomatic form (6) of the surface around the origin, we check the mean curvature $\Sigma^{N-1}$ at the origin, and the finite area $\Delta S$ of the surface $\Sigma^{N-1}$ is vanishing, the angle anholonomy is zero. If the surface is locally a saddle, the infinitesimal angle anholonomy is negative. If it is a cylinder whose gaussian curvature is vanishing, the angle anholonomy is zero.

By a trivial result $[p_N, p_N] = 0$ is excluded. Q.E.D.

Three immediate remarks on these local relations (8) and (10) follow. 1) They depend on the local geometric invariants of the surface such as $K_{ab}$, $k_a$, $L_{ab}$ and $p_a$ etc., so they hold irrespective of coordinates chosen. 2) The brackets (8) and (10) are zero once $K_{ab}$ and $k_a$ are zero respectively, as expected. 3) In quantum mechanics, we have the local commutation relations,

$$[p_a, p_{N}] \approx -i\hbar K_{ab} L_{ab}, \quad [p_a, p_N] \approx -i\hbar k_a p_a.$$  

Apparently, these two groups (10) of commutation are completely different. However, a surprising fact is, both share a rotation operator in common, which is going to be pointed out in detail shortly.

III. GIDOS AND ROTATIONS

Now we investigate the physical significance of the commutation relations (10).

First, we construct a GIDO along a small circle which is approximated by a small square in the $x_a x_b$-plane; and let the small square be formed by four points at $A(-\delta x_a/2, -\delta x_b/2)$, $B(\delta x_a/2, -\delta x_b/2)$, $C(\delta x_a/2, \delta x_b/2)$ and $D(\delta x_a/2, -\delta x_b/2)$, with center at the origin $O(0, 0)$ with $\delta x_a = |\delta x_a|$. The initial and final points of the displacements coincide at point $A(-\delta x_a/2, -\delta x_b/2)$, and order of the displacement is $A \to B \to C \to D \to A$. We have a GIDO along a small square $\Box$ABCD,

$$G_\Box \equiv e^{i\frac{\delta x_b p_b}{\hbar}} e^{i\frac{\delta x_a p_a}{\hbar}} e^{-i\frac{\delta x_b p_b}{\hbar}} e^{-i\frac{\delta x_a p_a}{\hbar}} \approx e^{\frac{i}{\hbar}\int \delta x_a \delta x_b K_{ab} L_{ab}} \approx e^{\frac{i}{\hbar} \oint (\delta x_a \delta x_b K_{ab}) L_{ab}}.$$  

In calculation, the Baker-Campbell-Hausdorff formula for two possibly noncommutative operators $X$ and $Y$ as $e^{X} e^{Y} \approx e^{X+Y} e^{[X,Y]/2}$ is used. We see that the GIDO $G_\Box$ (12) is a rotational operator on the $x_a x_b$-plane, and the angle of the rotation is $(\delta x_a \delta x_b K_{ab})$ which is the sectional anholonomy. It is originally defined by the angle of rotation of the vector as it is accumulated during parallel transport of the vector on a the hypersurface along the small circle. The angle anholonomy formed by a loop covering an finite area $\Delta S$ on the hypersurface is given by,

$$\sum_{a,b=1}^{N-1} \int_{\Delta S} K_{ab} dx_a \wedge dx_b,$$  

where the finite area $\Delta S$ is formed by infinitely many flat pieces covering the area, and $\sum_{a,b=1}^{N-1} \oint K_{ab} dx_a \wedge dx_b = 2\pi \chi$, where $\chi$ is the Chern number.

If the hypersurface is a two-dimensional spherical surface, the angle anholonomy is equal to the solid angle subtended by loop. If the surface is locally a saddle, the infinitesimal angle anholonomy is negative. If it is a cylinder whose gaussian curvature is vanishing, the angle anholonomy is zero.
There is similarity between the geometric momentum \(1\) and the kinematical momentum \(p = (-i\hbar \nabla - qA)\) in the presence of magnetic field \(B = \nabla \times A\). We also have noncommutative relations \(23\),

\[
[p_i, p_j] = i\hbar \sum_{k=1}^{N} q_{i,j,k} B_k.
\]

The GIDO now produces a purely phase factor,

\[
G_{AB} \equiv e^{i\frac{\delta x_a p_a - \delta x_N p_N}{\hbar}} e^{-i\frac{\delta x_a p_a + \delta x_N p_N}{\hbar}} \approx e^{\frac{2\delta x_a p_a}{\hbar}} \equiv e^{\frac{\hbar}{2} q \phi_{xy}} \exp(-\frac{i \delta x_a p_a}{\hbar} k_a p_a),
\]

where \(\delta \phi_{xy} \equiv \delta x \delta y B_z\) denotes the magnetic flux though the small area \(\delta x \delta y\) formed by the small square and \(\delta \Omega_{xy} \equiv q \delta \phi_{xy}\) is clearly the geometric phase \(24\). For the kinematical momentum, this GIDO \(12\) predicts a totally different action: rotating the quantum state. Thus, the external field and space-time background play different roles on the physical states.

Secondly, considering the small arc length from \(E(\delta x_a, -\delta x_N)\) via \(O(0, 0)\) to \(G(\delta x_a, -\delta x_N)\) along the intersection of the \(x_a x_N\)-plane and the hypersurface around the origin, i.e., along the small portion of the normal sectional curve, we immediately find that the commutator \([p_a, p_N] \approx -i\hbar k_a p_a\) leads to a displacement plus an additional rotation. To see it, we construct following GIDO which shifts a quantum state along the arc from point \(E \to O \to G\),

\[
G_\approx \equiv \exp(-i\frac{\delta x_a p_a - \delta x_N p_N}{\hbar}) \exp(-i\frac{\delta x_a p_a + \delta x_N p_N}{\hbar}) \approx \exp(-i\frac{2\delta x_a p_a}{\hbar}) \exp(-\frac{i \delta x_a \delta x_N}{\hbar^2} [p_a, p_N]).
\]

In right-handed side of this equation, we see two parts, and one is a simple translational operator \(\exp(-i\frac{2\delta x_a p_a}{\hbar})\) and another is,

\[
\exp(-\frac{i \delta x_a \delta x_N}{\hbar^2} [p_a, p_N]) = \exp\left(i\frac{i \delta x_a \delta x_N}{\hbar^2} k_a p_a\right).
\]

The physical significance becomes evident. The arc length element of along \(E \to O \to G\) is \(ds = 2\sqrt{\delta x_N^2 + \delta x_a^2} \approx 2\delta x_a\) \((\delta x_N \approx k_a x_a \delta x_a\) from \(17\) and \(1 \ll (k a x_a)^2\) due to the locally approximated flatness of the surface). The change of the tangential vector along the arc is \(-\delta \theta \equiv k_a ds \equiv 2k_a \sqrt{\delta x_N^2 + \delta x_a^2} \approx 2k_a \delta x_a\), and we have from above equation \(17\),

\[
\exp\left(i\frac{i \delta x_a \delta x_N}{\hbar^2} k_a p_a\right) \approx \exp\left(-\frac{i \delta \theta}{2} L_{N_a}\right),
\]

where an angular momentum operator defined by a torque of momentum \(p_a\) with respective to point \((0, -\delta x_N)\) is \(L_{N_a} \equiv \delta x_N p_a\). Let us move a quantum state along closed curves formed by piecewise smooth normal sectional lines, the rotation operator gives an accumulation of the rotational angle is \(\sum \delta \theta = \pi\) with addition of planar angles between neighboring short lines. Specially, when the surface is a two-dimensional spherical surface, the normal sectional curves are great circles and the GIDO \(G_\approx\) for a great circle leads to that the total angular change is \(\pi\).

So far, we have demonstrated a surprising result. Two seemingly different kinds of noncommutativity, given by \(11\), have the same crucial parts: rotation operators given by \(G_\square\) \(12\) and \(18\) in \(G_\approx\) \(10\), respectively. We can now safely say that geometric momentum generates rotations.

**IV. CONCLUSIONS AND DISCUSSIONS**

For a nonrelativistic particle constrained to remain on a hypersurface, Dirac brackets for two different components of momentum are not mutually commuting with each other. The noncommutativity on a local point of the hypersurface is examined and results show that the noncommutativity is due to the local curvature of the surface. At the point, there are, respectively, \((N - 1)(N - 2)/2\) mutually perpendicular two-dimensional tangential planes and \(N - 1\) mutually perpendicular normal sectional curves. In quantum mechanics, with GIDOs constructed on the base of the geometric momentum, we find that, at the point, for a small circle lying on each of the tangential planes and covering the point the noncommutativity leads to a rotation operator and the amount of the rotation is an angle anholonomy, and for a short arc length in each of the intersecting curves and centering the given point the noncommutativity leads to a translation plus an additional rotation and the amount of the rotation is one half of the tangential angle change of the arc.
The key and novel findings of the present study are two-fold. 1) Once the complicated Dirac brackets (1) of momentum are examined around a local point of the surface, they become much simpler and reduce to two seemingly different groups (6). 2) With two kinds of quantum mechanical GIDO constructed in (12) and (16), we show that these two groups of commutation share the same essence in common, causing rotations of the quantum state on the surface.

In many aspects our results are in sharp contrast to what the intuition suggests. For instance, the locally approximated flatness of the surface suggests that the geometric momentum might reduce to the usual one, but it does not for the geometric momentum generates rotations. The noncommutativity of commutation relations for momentum operators on a local point remains, but the heavy operator-ordering difficulty is got rid of. There is no angular momentum operator in the commutation relations \( [p_a, p_N] \approx -i\hbar k_a p_a \), but they can certainly make quantum states on the surface be angularly shifted.

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