Activation Confinement Inside Complex Networks Communities

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In this work it is described how to enhance and generalize the equivalent model (arXiv:0802.0421) of integrate-and-fire dynamics in order to treat any complex neuronal networks, especially those exhibiting modular structure. It has been shown that, though involving only a handful of equivalent neurons, the modular equivalent model was capable of providing impressive predictions about the non-linear integrate-and-fire dynamics in two hybrid modular networks. The reported approach has also allowed the identification of the causes of transient spiking confinement within the network communities, which correspond to the fact that the little activation sent from the source community to the others implies in long times for reaching the nearly-simultaneous activation of the concentric levels at the other communities and respective avalanches. Several other insights are reported in this work, including the smoothing of the spiking functions, the consideration of intra-ring connections and its effects, as well as the identification of how the weights in the equivalent model change for different source nodes. This work has paved the way for a number of promising developments, which are identified and discussed. Preliminary results are also described which reveal waves induced by the integrate-and-fire dynamics along the steady-state regime.

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‘Eutropia is not one, but all these cities together; only one is inhabited at a time, the others are empty...’ (Invisible Cities, I. Calvino)

I. INTRODUCTION

Complex networks (e.g. [1, 2, 8, 4, 5]) are interesting and useful because they exhibit structured topology departing significantly from uniformly random connectivity. While the famous scale free networks (e.g. [3]) present degree heterogeneity enhancing the probability of hubs, several real-world networks exhibit modular organization, i.e. they include communities. Informally speaking, communities are portions of the network (nodes and respective edges) which are more intensely connected one another than with the remainder of the network, e.g. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Networks including communities are very important in theory and practice exactly because of their modularity, which implies strong effects on the respective topological and dynamical features. For instance, the edges implementing the connections between two communities tend to receive particularly intense activation during dynamics (e.g. information transfer or transportation), corresponding to functional bottlenecks. Interestingly, several of the real-world networks present modular organization. For instance, the urban/land transportation system is underlain by a complex network including the communities (towns and cities) and intercommunity connections (roads). Other especially important systems exhibiting modularity includes the brain, collaborations, and economic systems, amongst many other examples.

Because of the special importance of modularity for the structure and dynamics of complex networks, much attention has been focused on the problem of identifying the respective communities (e.g. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). More recently, investigations targeting transient non-linear dynamics in integrate-and-fire complex neuronal networks indicated that the transient activation of neurons by a source placed at one of the nodes tend to be confined during a transient period in the community to which the source node belongs [13, 20]. These results have important implications both for neuroscience and complex network research. In the former case, the fact that topological localization (i.e. communities) tend to localize neuronal activation in time, seems to be compatible with the fact that, normally, the cognitive tasks have to be performed within limited intervals of time inside specific functional modules (e.g. [21, 22, 23]). Most other traditional types of dynamics (e.g. traditional random walks or self-avoiding random walks) are unable to yield such a confinement of the dynamics inside the communities. So, it could be possible that the non-linear combination of the integration and fire elements in neuronal cells is a necessary requisite for time-space compartmentalization of information processing in neuronal systems.

It is interesting to observe that many other real-world systems and problems also involve similar threshold-based dynamics, including disease spreading, transportation, production (where the output goods depends on the arrival of all parts), information processing and computing, amongst many others. Therefore, studies of the relationship between the integrate-and-fire dynamics and the topological communities have potential for applications in all these real-world problems. Several are the implications of such studies also for complex network research. In particular, this issue lies at the very heart of the important structure-and-dynamics paradigm, in which relationships between the topological and functional properties of networks are sought and investigated.
In a work preceding the identification of the spiking confinement in integrate-and-fire complex neuronal networks, another remarkable effect had been observed. More specifically, the activation, through a single node, of several types of integrate-and-fire complex systems was found to lead to avalanches of activation and spikings. Such a phenomenon has been observed for the Erdős-Rényi and Barabási-Albert models, as well as for two types of knitted networks (the path-regular and path-transformed BA structures). However, no avalanches have been observed for geographical integrate-and-fire networks. This interesting phenomenon was later found to be strongly defined by the hierarchical (or concentrical) organization of the respective complex networks. More specifically, the number of nodes in each of the concentric levels, defined by the selection of the source node as a topological reference, play a fundamental role in defining not only the timing of the activation of each concentric level, but also the instant and intensity of the main avalanches. In particular, the main avalanche tends to be produced by the almost simultaneous spiking of all neurons in the concentric level containing the largest number of neurons.

Assuming that the networks exhibit a reasonable level of degree regularity, such a strong relationship between the concentric structure and activation allowed a simple but effective equivalent model to be defined which captures the intrinsic dynamics of avalanche formation. By using this equivalent model, which consists simply of a chain integrate-and-fire network with weights and varying thresholds, it is possible to make reasonably accurate predictions about the intensity and time of the avalanches even for non-regular degree networks such as the Barabási-Albert model and the real-world C. elegans neuronal network.

Interestingly, it was subsequently identified that the dissemination of the activation emanating from the source node in integrate-and-fire complex neuronal networks exhibiting modular structure tends to be constrained inside the community to which the source node belongs. This phenomenon is illustrated in Figure 1 with respect to a hybrid network (Fig. 1b) composed of 4 communities of different type, namely Erdős-Rényi (nodes 1 to 50), Barabási-Albert (nodes 51 to 100), Watts-Strogatz (nodes 101 to 150) as well as a geographical structure (nodes 151 to 200). Node 29, which belongs to the Erdős-Rényi community, was selected as the source of activation, which was kept constant at intensity 1. The respectively obtained activogram (namely the activation of each neuron along the transient time) and spikegram (the spikes generated by each neuron along time) are shown in Figures 1(b) and (c), respectively. The total activation and number of spikes along time are shown in (d) and (e), respectively.

The activogram (Fig. 1b) makes it clear that community 1 (ER), extending from node 1 to 50, was activated first and remained so along nearly 120 time steps, with the whole network undergoing an avalanche of activation subsequently. A similar behavior was observed for the spikes (Fig. 1c). A few neurons from other communities exhibited earlier activations and spiking. These neurons correspond to those implementing the inter-community connections. As expected, the total activation, shown in Figure 1(d), increases linearly, reflecting the facts that activation is constantly fed into the system through the source node and the conservation of activation allowed by the adopted integrate-and-fire dynamics. As depicted in Figure 1 a completely different evolution was obtained for the total number of spikes, which shows a clear surge after nearly 120 time steps, corresponding to the main avalanche of spikes taking place for most of the complex neuronal network.

Because of the close relationship between the communities and the beginning activation times and the average activation observed during the transient regime, it was possible to use the patterns of spiking along the first time steps in order to identify with accuracy the original communities. More specifically, the activation is placed at each node and the pattern of beginning activation times or average activation are obtained and then decorrelated by using the Principal Component Analysis (e.g. [5, 31]) methodology. Well-defined clusters are obtained in the projected space which corresponds to the original communities, while the points between the clusters corresponds to the edges which implement the intercommunities connections. Such a result has important implications for both neuroscience and complex networks research. In the former case, the identified relationship between communities and integrate-and-fire transient dynamics suggests that the integrate-and-fire dynamics of biological neuronal cells is critical for confining the neuronal activation inside the spatially clustered functional processing modules of the brain (e.g. [21, 22]). Actually, this relationship may play a key role in defining the functional modules during the growth of the neural system as a consequence of correlations between the presented stimuli. The implications for complex networks research are also many. For instance, the method seems to produce well-defined clusters even for relatively low modularity indices, suggesting that it may be an effective computational choice for community detection. In addition, the relationship between community structure and integrate-and-fire dynamics lies at the heart of the important current investigations of the relationship between complex networks structure and dynamics (e.g. [1, 16, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]). Indeed, the confinement phenomenon has been observed (19, 20) to depend on the type of network topology, with geographical structures being particularly poor in constraining activation. For all such reasons, it is important to investigate in more detail the origin and properties of this interesting phenomenon. This is precisely the main objective of the current work.

It has been conjectured that the confinement of the activation/spiking in integrate-and-fire complex neuronal networks is related to the avalanches of acti-
FIG. 1: A hybrid complex network containing four communities of different types (ER, BA, WS, and GG) is shown in (a). The excitation of the network through one of its nodes $i$ tends to activate first the community to which this node belongs, confining the activation during a transient period of time. For instance, the excitation of neuron $i = 29$ induced an early activation of community 1 in which it is included, as shown in the respective activogram (b) and spikegram (c). The total activation and spikes along time are shown in (d) and (e), respectively.

vations which take place during the transient regime in the greatest part of the analysed networks. This seems to be supported by the fact that the geographical structures, which do not produce avalanches, are less likely to constrain the activation. In the present work, we extend the methodology for relating the avalanches to the hierarchical organization of complex networks in order to address the activation confinement inside communities.
The equivalent model described in [26] is enhanced to include the intra-level activations and then extended to represent modular complex networks. This has been accomplished by obtaining the hierarchical structure for the network as a whole and then separating it into modules (see also [42]), on which the modular equivalent model was based) corresponding to each original community. By using a combination of theoretical and experimental approaches involving the above concepts, it has been possible to obtain decisive insights and proper understanding of the origin and properties of the activation confinement phenomenon in integrate-and-fire networks.

The article starts by reviewing the basic concepts and follows by presenting the enhanced and extended equivalent model. By considering several hybrid networks, several key effects underlying the confinement phenomenon are identified and discussed. The article concludes by identifying its main contributions and prospects for future developments.

II. BASIC CONCEPTS

A. Complex Network Representation and Measurements

In this work we consider undirected (theoretical models used to obtain the hybrid networks), directed (the respective extension to complex neuronal networks), as well as unweighted and weighted (the equivalent models) networks. Weighted and directed networks are the more general case from which the above specific structures can be obtained [9]. Such a type of network can be fully represented in terms of its weight matrix \( W \), so that each edge extending from node \( i \) to node \( j \) with respective weight \( w \) implies \( W(j, i) = w \). The absence of edges from node \( i \) to node \( j \) is expressed as \( W(j, i) = 0 \). The topology of a complex network described in terms of its weight matrix can be immediately obtained by setting to one all non-zero weights, which yields the adjacency matrix. Undirected networks can be obtained by symmetrizing all the directed connections, so that each undirected edge incorporates two directed edges with opposite directions.

The immediate neighbors of a node \( i \) are those nodes which receive an edge from it. The outdegree of a node \( i \) is the number of its immediate neighbors. The indegree of a node \( i \) is the number of nodes which send edges to \( i \). If the network is undirected, the indegree and outdegree are identical and called simply degree. Two nodes are adjacent if they share one edge. Two edges are adjacent if they share one node. A sequence of adjacent edges define a walk. A walk without repetition of nodes or edges is a path. The length of a walk or path is equal to the number of edges they contain.

B. Hierarchical Organization of Undirected and Directed Networks

Given a undirected network and one of its nodes (or even a subgraph [29,42]) as a reference, it is possible to obtain its respective hierarchical organization with respect to the chosen reference. Such an organization is defined by the hierarchical (or concentric) neighborhoods of the reference node. Let us call the reference node \( i \). Its first hierarchical neighborhood includes the nodes which are directly attached to \( i \), i.e. the nodes which are adjacent to it. The second hierarchical neighbors are those which are at shortest path length 2 from \( i \), and so on. The hierarchical structure of a network can be obtained by flooding the network from the reference node [27]. More specifically, the reference is labeled with value 1. Then, the nodes which are adjacent to the reference are labeled 2, and so on.

Given the hierarchical organization of a network with respect to one of its nodes, it is possible to obtain a series of hierarchical measurements (e.g. [27,28,29,30]) which provide information not only about the immediate neighborhood of a node, but about the whole set of topological scales from the immediate node level to the complete network level. The properties of the hierarchical organization of several theoretical and real-world networks have been systematically investigated in previous works (e.g. [27,28,29,30]). The generalization of the hierarchical approach to consider subgraph references has been described in [29].

Because in this work we will be dealing with directed networks, it is important to consider the generalization of the hierarchical concepts to this type of networks. This can be immediately achieved by considering the flooding algorithm outlined above. Given the reference node, the hierarchies along the directed network is identically obtained by the same algorithm. However, while the outgoing connections in the obtained hierarchical structure are always found between successive concentric levels (i.e. extending from level \( h \) to level \( h+1 \), with necessarily no connections between \( h \) and any other concentric level), the backward connections can take place between any concentric levels (e.g. it is possible to have directed edges extending from level \( h \) to any of the previous levels).

C. Hybrid Modular Networks

In this work we will consider 4 theoretical complex networks models (e.g. [1,2,3,4,5]) as corresponding to communities: Erdõ-Rényi (ER, see also [44]), Barabási-Albert (BA), Watts-Strogatz (WS), as well as a simple geographical model (GG). The ER structures are obtained by connecting pair of edges with constant probability. A BA network is obtained [3] by starting with \( m_0 \) nodes and incorporating new nodes with \( m \) edges each, which are attached to the previous nodes with probability proportional to their respective degrees. The WS
structures considered in this work starts as linear regular lattices of suitable degree, followed by rewiring of 10% of the edges. The GG networks are obtained by distributing nodes along a two-dimensional space and connecting all nodes which are closer than a maximum distance. In all cases, only the largest connected component was used. Similar intra-community degrees are adopted henceforth.

Informally speaking, a community in a network is one of its subgraphs characterized by the fact that its nodes are more intensely connected one another than with the remainder of the network. A complex network incorporating communities is henceforth called a modular network. A hybrid modular network involves communities of different topological types. In addition to specifying the type of each community, it is also necessary to specify the inter and intra-community degrees. Figure 1(a) illustrates a hybrid network composed of 4 communities of respective ER, BA, WS and GG types. Each of these communities have 50 nodes. The inter-community degree is 1 (i.e. each two nodes are connected, in the average, with one node from another community) and intra-community degree equal to 6 (i.e. each node connects to 6 other nodes, in the average, inside its respective community). Hybrid modular networks are particularly useful for community studies because they allow the identification of the effects of their respective topologies on the community structure and separation.

D. Integrate-and-Fire Complex Neuronal Networks

Figure 2 illustrates the basic integrate-and-fire model of neuron adopted in this work. It includes a set of weights, an integrator, a memory \( S(i) \) containing the internal activation, and a non-linear transfer function, namely a hard-limit function with threshold \( T(i) \).

![FIG. 2: The integrate-and-fire model of neuronal cell adopted in this work.](image)

Because the original hybrid modular networks obtained as described in Section II C are undirected, it is necessary to transform them into respective directed versions in order to be used as complex neuronal networks. This can be immediately achieved by splitting each undirected edge into two directed edges, corresponding to a dendrite and an axon. Observe that all the nodes in the resulting directed complex neuronal networks consequently have indegree equal to the outdegree. All hybrid complex networks considered in this article have weights equal to 1. The equivalent models, however, use varying weights in order to implement the proportional splitting of activation between hierarchical levels. Also, while all thresholds in the hybrid complex neuronal networks are identical to 1, the thresholds in the equivalent models are set so as to reproduce the inertia of the number of nodes associated to each equivalent node.

The source of activation is placed at a specific node of the network (it is also possible to consider activations emanating from groups of nodes). The activation fed by the source into the complex neuronal networks are constant and equal to 1, implying linear increase of the total activation along time (see Figure 1(b)). All activations arriving at a given neuron are integrated (remember all weights are equal to 1) along time (facilitation [44]), being stored into the internal memory \( S(i) \). Once this integrated value exceeds the threshold, i.e. \( S(i) \geq T(i) \), the neuron fires, liberating the stored activation, which is equally distributed among the axons. Such an operation ensures conservation of the total activation. Though the liberation of the total activation is not biologically realistic (but can be emulated by setting adequate thresholds), similar phenomena of avalanches and activation confinement inside communities have been identified for complex neuronal networks where the outgoing activation is limited to the threshold value, with the difference that the avalanches express themselves as peaks of spiking, instead of abrupt increases as in Figure 1(e).

III. THE FIRST EQUIVALENT MODEL

The equivalent model of activation of integrate-and-fire complex neuronal networks described in [26] was allowed because the activation of nodes in such systems is strongly determined by the hierarchical organization of the respective structures. After choosing the source node \( i \), this model is obtained as follows. By using the methodology described in [27, 28, 29, 30] we calculate: the hierarchical number of nodes \( n_h(i) \) (i.e. the total number of connections from the neurons in level \( h \) to the neurons in level \( h + 1 \)) as well as the hierarchical degree \( k_h(i) \) (i.e. the number of nodes within the concentric level \( h \)). The equivalent nodes are defined as subsuming the set of nodes at each respective concentric level of the original network (see also [12]). Because the networks originally considered had symmetric connections (the undirected edges were symmetrized by splitting them into two directed edges), the only possible connections between the equivalent nodes take place between adjacent concentric levels, i.e. the node at level \( h \) connects only to nodes \( h - 1 \) and \( h + 1 \). Such a connectivity implies the equivalent model to be a chain of the equivalent nodes. The weights of the connections between adjacent nodes are defined as
\begin{align*}
W(h + 1, h) &= k_h(i)/d \\
W(h - 1, h) &= k_{h-1}(i)/d
\end{align*}

where \( d = k_h(i) + k_h h - 1 \) for proper normalization. Thus, the weights represent the fraction of the activation which is sent to the left and right neighbors of the equivalent nodes in the chain model. Such a model is then activated from the source node \( i \).

Assuming a reasonable level of degree regularity in the network, the neurons at each hierarchical level will tend to fire almost simultaneously after a period of activation integration which is proportional to the sum of the number of nodes from the hierarchical level to the current level. So, the beginning activation times of the neurons at each concentric levels can be predicted from such sums of nodes. The main avalanche tends to take place when the neurons in the concentric level with the largest number of nodes spike. Because the activation liberated by this event is very large, it tends to fire all neurons along the adjacent layers (in both directions) within a brief period of time. The intensity of the main avalanche can be estimated from the number of neurons at the level with the largest number of nodes and at its adjacent levels. The time at which the avalanche takes place can be predicted by adding the number of neurons from the concentric level 1 up to the critical level.

IV. ENHANCING THE EQUIVALENT MODEL

In this section we describe how to enhance the chain equivalent model, previously suggested in [26], in order to incorporate the intra-ring edges within each concentric level. This can be immediately done by incorporating self-connections in the chain equivalent model, in addition to the weights respective to the connections from level \( h \) to levels \( h + 1 \) and \( h - 1 \), given by the hierarchical degrees \( k_h(h) \) and \( k_h h - 1 \). In order to properly normalize the distribution of activation from each level \( h \), the respective weights are now redefined as

\begin{align*}
W(h + 1, h) &= k_h(i)/d \\
W(h - 1, h) &= k_{h-1}(i)/d \\
W(h, h) &= a_h(i)/d
\end{align*}

where \( d = k_h(i) + k_h h - 1 + a_h(i) \), \( k_h(i) \) is the hierarchical degree of level \( h \) with respect to reference node \( i \) and \( a_h(i) \) is the intra-ring degree of level \( h \) with respect to node \( i \). The latter measurement expresses the number of edges within the concentric level \( h \).

The incorporation of the self-connections into each concentric level is expected to provide a more complete model of the dynamics unfolding along the hierarchical organization of the original network. More specifically, as hinted in [26], the intra-ring edges tend to reduce the distribution of its current activation to the other levels by deviating a portion \( W(h, h) \) of activation towards the same level. By doing so, the intra-ring connections tend to delay the nearly-simultaneous spiking of the respectively concentric levels. In case the intra-ring edges are not distributed uniformly between the nodes of a given concentric level, the neurons of the next layer will tend to spike in a less simultaneous way, implying a smoothing of the number of spikings and avalanches.

Figure 3 illustrates the effect of incorporating the intra-ring degrees while modeling a traditional (i.e. without communities) ER network with 100 nodes and average degree equal to 8. Figure 3(a) shows the total number of spikes obtained for the whole original network, while Figure 3(b) shows the respectively smoothed version. Because the curves showing the number of spikes tend to be jagged as a consequence of the saw oscillations discussed in [26], it is interesting to smooth the activation/spiking signals. Figure 3(b) was obtained by convolving the curve in Figure 3(a) with a Gaussian function with standard deviation equal to 6 time steps. The total number of spikes produced by the respective chain equivalent model without consideration of the intra-ring degrees (as adopted in [26]) is shown in Figure 3(c), with its respective smooth version depicted in (d). As expected, the elimination of the intra-ring connections tends to enhance the simultaneity of the spikings, producing a more definite and abrupt avalanche. The results obtained by the enhanced chain equivalent model incorporating self-connections are shown in Figures 3(e) and (f) respectively to the original signal and its smoothed version. It is clear that the incorporation of the intra-ring degrees as the weights of the self-connections in the chain equivalent model led to a substantially more precise estimation of the original spiking dynamics (Fig. 3a and b).

The effect of the intra-ring connections in reducing the simultaneity of the spiking at each concentric level and dispersing the avalanche (i.e. implying a smoother transition along time) suggests that it would be possible to enhance the simultaneous activation of the communities while the network is fed from the source node, possibly enhancing further the ability of the method to detect communities. Such a possibility is explored further in Section VII.

V. GENERALIZING THE EQUIVALENT MODEL TO NON-SYMMETRIC CONNECTIVITY

As mentioned in Section III, the first equivalent model assumed the edges of the original network to be symmetric in the sense of one incoming directed edge matching every outgoing directed edge. Such a connectivity was immediately implied by the splitting of the originally undirected edges into two directed edges. In this sec-
FIG. 3: The total number of spikes along time obtained for a complete original ER network with 100 nodes and average degree equal to 6 (a), and its respectively smoothed version (b). The total number of spikes as estimated by the chain equivalent model while not taking into account the intra-ring connections (c), and its respectively smoothed version. The consideration of the intra-ring connections enhanced the estimation of the number of spikes as shown in (e) and the respectively smoothed version (f). The smoothings were performed by convolving the original curves with a Gaussian function with standard deviation equal to 8 time steps.

In this section we generalize the equivalent model in order to allow the consideration of asymmetric directed connections, i.e. any directed complex neuronal network, by considering the hierarchical organization of directed complex networks described in Section III B. Such a generalization is explained with the help of Figure 4, which also illustrates the previously introduced concepts of self-connections.

Figure 4(a) shows a simple directed network involving several asymmetric connections. Let us consider node 1 as the reference node. By flooding the network from this node, the hierarchical layers are defined by the nodes which are simultaneously flooded at each time, yielding the hierarchical organization shown in Figure 4(b). Alternatively, the nodes at hierarchical level $h$ are those which are at shortest directed path distance $h$ from the reference node $i = 1$. Observe that the forward directed edges (i.e. those edges connecting from level $i$ onwards) are only allowed between subsequent levels, i.e. the nodes...
FIG. 4: Example of how to obtain the generalized equivalent model from an asymmetrically directed complex neuronal network. The weights are obtained from the generalized hierarchical degrees. For instance, the weight of the connection from equivalent node $e_4$ to the equivalent node $e_1$ is obtained by dividing the number of edges departing from $e_4$ towards $e_1$ (i.e. 1 edge) by the total number of edges departing from $e_4$ towards any other equivalent node except itself (i.e. 8 edges).

in hierarchical level $h$ can send forward connections only to the immediately adjacent level $h + 1$. Contrariwise, the backward connections are allowed from each level $h$ to any of the previous levels (e.g. the connection from node 8 to node 1). Observe also the existence of the intra-ring connections at levels 3 and 4. Because of such a more generalized connectivity, the hierarchical degrees and intra-ring degrees are redefined as the generalized hierarchical degrees $w(e_g, e_h)$, expressing the number of edges sent from the equivalent node $e_g$ to the equivalent node $e_h$. The generalized hierarchical degrees of the directed network in Figure 4(a) are shown in Figure 4(c).

VI. THE MODULAR EQUIVALENT MODEL

Having enhanced and generalized the original chain equivalent model, we are now in a position to extend it to modular networks, i.e. complex networks which incorporate communities. This will be done with respect to an example considering the simple modular network in Figure 5. This network includes two respectively identified communities, with 4 inter-community connections. The first step is to select the reference node so that the hierarchical organization of the whole network can be determined as discussed before. We select node 1 as reference and source of activation. The hierarchical organization of the network in Figure 5 is shown in Figure 6. A total of 6 concentric levels were identified for this modular network.

FIG. 5: A simple modular network composed of two communities.
Thus far, we have simply adopted the typical approach for the hierarchical analysis of the network with respect to a given reference node, without considering its community structure. We do that now by defining each equivalent node $e_{c,h}$ as subsuming the nodes of the hierarchical level $h$ which belong to community $c$. For instance, in the case of the network in Figure 5 we have that $e_{1,3}$ is equal to 3, because there are 3 nodes of community $c = 1$ at the concentric level $h = 3$. The generalized hierarchical degrees of the equivalent nodes, henceforth expressed as $k(e_{d,g}, e_{c,h})$, are defined as the number of edges in the original network going from the nodes corresponding to the equivalent node $e_{c,h}$ to $e_{d,g}$. The weights of the connections between the equivalent nodes can now be calculated as

$$W(e_{d,g}, e_{c,h}) = k(e_{d,g}, e_{c,h})/d$$  \hspace{1cm} (1)$$

where $d = \sum_{g \in \Omega(h)} k(e_{d,g}, e_{c,h})$ and $\Omega(h)$ is the set of equivalent nodes which receive a directed edge from $e_{c,h}$. The thresholds of each equivalent node are given by their respective number of original nodes. The specification of the thresholds completes the definition of the modular equivalent model.

Figure 7 shows the modular equivalent model obtained for the network in Figure 5. Observe that because of finite-size effect and statistical fluctuations, the GG networks have degrees smaller than 6.

Let us choose node 1, which belongs to the ER community, as the reference node. The generalized hierarchical number of nodes (i.e. the number of nodes inside each of the obtained equivalent nodes) are given in Table I.

**VII. ACTIVATION CONFINEMENT IN HYBRID MODULAR NETWORKS**

In this section we illustrate how the modular equivalent model can be used in order to better understand the transient confinement of integrate-and-fire activation inside the communities. In order to do so, we consider two hybrid modular networks, each composed by ER, BA, WS and GG modules with 50 nodes each (except GG, which has 46 nodes), intra-community degree equal to 6 and inter-community degree equal to 1 and 2. These two hybrid modular networks, henceforth abbreviated as Net 1 and Net 2, are shown in Figure 8. Observe that, because of finite-size effect and statistical fluctuations, the GG networks have degrees smaller than 6.

Let us choose node 1, which belongs to the ER community, as the reference node. The generalized hierarchical number of nodes (i.e. the number of nodes inside each of the obtained equivalent nodes) are given in Table I.

The weights obtained for each of the two hybrid modular networks in Figure 8 are respectively shown in Figure 9. It is clear from this figure that most of the activa-
FIG. 8: The two hybrid modular networks considered in this work. Each network has 4 communities (ER, BA, WS and GG as indicated in Fig 1), with intra-community degree equal to 6 and respective inter-community degrees equal to 1 (a) and 2 (b). These two networks are henceforth abbreviated as Net 1 and Net 2.

tion transfers occur between the adjacent levels (i.e. the weights near the main diagonal are higher). The second network exhibits more intense off-diagonal weights as a consequence of its higher inter-community degree (twice as much as for the first network). Such longer range connections are expected to intensify the lack of simultaneity between the neurons belonging to each equivalent node.

Figure 10 shows the weights which are obtained by selecting node 50 (also in the community ER) as the reference in Net 1 (Fig. 10a) and Net 2 (Fig. 10b). While little changes can be observed for Net 1 (with respect to the weights in Fig. 9), slightly more accentuated differences are noticed for Net 2. Similar trends have been observed for other nodes, suggesting that different choices of reference node have relatively little effect for networks with smaller inter-community degree.

Figure 11 shows the weights which are obtained by selecting node 170, now in the WS community as the reference node for Net 1 (a) and Net 2 (b). Interestingly, the weights implied by this new reference node for Net 1 are similar to those obtained for node 50 as reference. As in the previous example, larger differences of weights were obtained for Net 2.

As discussed in [26], the timing of the main avalanches in each community can be estimated by adding the number of nodes from the concentric level 1 to the level exhibiting the largest number of nodes inside each community. Thus, for the ER community, which has maximum number of nodes at the fifth concentric level of Net 1, we expect the main avalanche to take place after approximately $1 + 5 + 28 + 16 + 2 + 13 + 4 + 24 + 2 + 18 = 113$ time steps.

Figure 12 shows the smoothed total number of spikes obtained for each community in each of the original and equivalent models of Net 1 in Figure 8(a). The total number of spikes were smoothed through convolution with Gaussian functions with standard deviations 8 (original net) and 16 (equivalent model) time steps. The different degrees of smoothing are necessary in order to account for the fact that the equivalent model involves considerably fewer neurons which spike simultaneously, leaving greater gaps along the number of spikes curve. Except for the plateaux in the curves obtained for the modular equivalent models, an impressive agreement can be verified between the results obtained considering the whole network and the respective predictions of the dynamics by the modular equivalent model. Particularly precise estimations have been obtained for the first community in both networks. The adherence of the model also increased with the inter-community degree, implying more accurate predictions to be obtained for Net 2. The timing of the respective avalanches have been predicted with remarkable precision in all cases.

The impressive agreement between the results obtained by the modular equivalent model, which incorporates only 30 nodes against the 196 nodes in the original networks, corroborates the fact that the integrate-and-fire dynamics can be to its greatest extent captured by considering the hierarchical organization of the original com-
plex neuronal networks. Such a modeling and simulations make it clear that the transient confinement of the spikes within the community where the source is placed (this community is henceforth called the source community) is a direct consequence of the integrate-and-fire mechanism acting inside each equivalent node. More specifically, the source community activates first because most of the activation coming from the source node goes into neurons belonging to that community. This is an immediate consequence of the three following features: (i) the source node sends, in the average, most of its axons towards nodes in its own community; (ii) low off-diagonal weights of the modular equivalent model are typically obtained for modular networks; and (ii) longer times are required to induce nearly simultaneous spiking of equivalent nodes related to a large number of nodes in the original network. Indeed, the communities which do not contain the source node will only start spiking after integrating for a long time the typically the activation received from the source community. This time is particularly long because these communities initially receive only a fraction of the activation in the source community, combined with the large threshold required for these communities to become activate, which typically takes place only after the main avalanche. Indeed, the timing of the main avalanches obtained for the non-source communities tended to be substantially larger than that of the source community.

In order to learn more about the integrate-and-fire dynamics in modular complex neuronal networks, it is interesting to compare the total number of spikes obtained by comparing the dynamics obtained for the whole original network and respective versions in which the intra-ring
connections have been removed [27]. Such modified networks, henceforth called open networks, can be obtained by flooding the original network, starting from the reference node, and removing all edges which are flooded in the same flooding stage. Figure 11 shows the smoothed total number of spikes obtained for the open versions of Net 2 and its respective modular equivalent model. Interestingly, except for a small reduction in the values of total number of spikes, the curves obtained for the original network and respectively smoothed version are remarkably similar. Actually, a more careful comparison between the figures reveals that the removal of the intra-ring connections implies a slight increase in the derivative of the avalanche transition. This was indeed expected, but not to such a small degree, because the intra-ring connections are known [26] to disperse the avalanche transition by implying the neurons related to each equivalent node to spike in a less simultaneous fashion.

VIII. CONCLUDING REMARKS

Since its modern origins in the late 90’s, and through its impressive development ever since, the research area known as complex networks has been characterized by special interest in the structural properties of networks as well as with relationships with the respectively obtained dynamics. Because of the interesting structural features implied by the presence of communities in networks, great attention has been focused on studies of modular networks as well as on community finding algorithms. The present work lies at the heart of the intersection of these two important issues in complex networks research, namely the identification of communities and the interplay between community structure and activation confinement, being also related to fundamental aspects of neuroscience research. More specifically, in this work we have investigated in detail the interesting effect of spiking confinement inside the communities of a networks. This has been accomplished with the help of an extension of the chain equivalent model [26]. The main contributions of the present work are listed and discussed as follows.

Incorporation of Intra-Ring Connections in the Chain Equivalent Model: When first proposed [26], the chain equivalent model did not take into account the connections within the same concentric level along the hierarchical organization of the networks. By incorporating self-connections at the equivalent nodes of the chain model, it became possible to achieve a more precise model of the integrate-and-fire dynamics unfolding along the hierarchical structure of the complex neuronal networks. The incorporation of the intra-ring connections can lead to more precise estimation of the original dynamics. In addition, the role of the intra-ring edges in undermining the simultaneity of the spiking at each level, as well as the respective smoothing of the avalanche transitions, became clear when these connections were removed. Such an interesting effect suggests that the removal of the intra-ring connections in the original structure may lead to slight enhancements while mapping the spiking patterns into clusters by means of the PCA methodology [19, 20].

Smoothing of Activation Curves: The curves of number of spikes typically obtained for complex neuronal networks involve intense saw oscillations [26], i.e. alternating high and low number of spikes along subsequent time steps. In the present work, we have shown that the smoothing of the number of spike curves can yield smoothed curves which are much more sensible and provide more meaningful description of the integrate-and-fire dynamics in complex neuronal networks.

Generalization of the Chain Equivalent Model for Asymmetric Connectivity: The chain equivalent model has been extended to consider complex neuronal
networks with asymmetric connections, i.e. not every directed edge needs to have the respective counter-directed edge as previously adopted. Such a generalization implied that while connections extending from the source node onwards always extend from an equivalent node in one concentric level to an equivalent node in the next level, backward connections can take place between an equivalent level and equivalent nodes in any of the previous levels. Though the two hybrid modular networks considered in this work were directed but had symmetric connections, such a generalization was fundamental for allowing the extension of the equivalent model to cope with modular complex neuronal networks.

**Development of the Modular Equivalent Model:**
In this work, a modular equivalent model was developed where the equivalent nodes are determined with respect not only to the concentric levels, but also taking into account the original communities. The modular equivalent model is a weighted complex neuronal network itself, suitable for modeling any complex neuronal networks, directed/undirected, modular or not, and with or without weights. By considering two hybrid modular networks, it has been presently shown that the modular equivalent model, though containing only a handful of equivalent nodes and connections, can predict with surprising accuracy the dynamics of sophisticated complex neuronal networks, even in the presence of hubs (BA) or lack of the small-world property (GG). In addition, the timing of the main avalanches in each community could be accurately predicted by taking into account the number of nodes associated to the concentric levels of the network. Interestingly, different choices of the reference node seem to have relatively little impact over the weights of the equivalent model.

**Removal of Intra-Ring Edges:** The contribution of the intra-ring connections, namely the edges within the same concentric level, was investigated by removing such nodes from the two considered hybrid modular networks. The respectively obtained results yielded similar curves of total number of spikes along time, though with slightly smaller values and with moderate increase of the derivative of the avalanche transitions. Such an effect suggests that the removal of the intra-ring connections may contribute to improving the efficiency of the
FIG. 13: The smoothed total number of spikes obtained for each community in Net 2 by using the original network (a, c, e, g) and respective equivalent models (b, d, f, h).

Integration-and-Fire Methods for Community Identification reported in [19, 20].

**Full Understanding of the Confinement Effect:** The development of the completely general modular equivalent model, as well as its application to the modeling and accurate prediction of the integrate-and-fire dynamics in two hybrid modular complex networks, allowed a comprehensive understanding of the reason why the spiking activation tends to remain confined, during a transient period of time, inside the community to which the source node belongs. Three are the reasons leading to the confinement phenomenon: (i) the tendency that most of the axons emanating from the source node are sent to neurons in the same community; (ii) the fact that most of the connections inside the source community are sent to nodes inside itself; and (iii) the combined effect of the integrate-and-fire dynamics, in the sense that the equivalent nodes associated to large number of original nodes require much activation in order to reach the respective threshold. It should be observed that the increase of the inter-community degree tends to reduce the confinement effect because such an increase tends to undermine the effect in (iii). Therefore, such results confirmed the hypotheses advanced in [19, 20] about the origin and properties of the confinement effect.

The relevance of the reported methodology and results is corroborated by the several respectively implied prospects for future developments, which are listed and briefly discussed as follows.

**Dynamics After Removal of Activation:** It has been preliminary verified that the removal of the external activation after a period of time can contribute to emphasizing the identification of communities by using the integrate-and-fire approach. Such transient activations may also lead to oscillation and pattern formation inside the complex neuronal networks. Further related investigations, facilitated by the possibility of using the modular equivalent model in order to treat a large number of networks in a reasonable period of time, are required in order to fully assess the potential of such a strategy.

**Networks with Decaying Activations:** As shown in [20], the incorporation of the biologically realistic effect of activation decay with time (during facilitation [44])
can contribute to enhancing the activation confinement inside the topological communities. The modular equivalent model can be immediately adapted to incorporate such an additional dynamical feature, allowing the investigation of the effect of the activation decay on the overall dynamics for a large number of complex neuronal networks.

Networks with Limited Activation Transfers: Another modification of the integrate-and-fire dynamics which can be easily accommodated in the modular equivalent model is the limitation of the output activation. More specifically, while in the present work we considered the internally stored activation to be integrally distributed among the outgoing axons, it is also possible to limit such activations to a fixed value (the mostly stereotyped action potentials [43]). Such a kind of dynamics is much more biologically realistic than the currently considered model. Results obtained by considering this type of activation transfer limitation are shown in Figure 15. Additional investigations indicate that avalanches as well as the confinement effects occur even when such a limitation is imposed and can be predicted by using a respectively adapted modular equivalent model. The limitation of the activation transfer through the axons also led to moderate improvements in community identification in several situations.

Equilibrium Dynamics: Yet another promising future work would be to characterize the equilibrium dynamics of the complex neuronal networks after the activation source has been removed. Particularly worth of attention would be the identification of attractors of the dynamics, as well as their relationship with the specific topological properties and modularity of each complex neuronal networks. It has been preliminary observed that the equilibrium dynamics of complex neuronal networks tend to exhibit oscillation patterns defined by groups of nodes, not necessarily belonging to the same communities, which acquire identical patterns of oscillation. It would be particularly interesting to relate such synchronization with the topological and non-linear features of the integrate-and-fire complex neuronal networks.

Activation-Induced Waves: Figure 15(a) illustrates the number of spikes obtained for a longer period of time (the transfer of activation was limited to 1 at
each axon in this case and continuous external activation with intensity 1 was fed into the system), with the transient dynamics taking place along approximately the initial 500 time steps. The respective power spectrum obtained by considering the number of spikes from time 1001 to 1500, shown in Figure 15(b), suggests that the number of spikes exhibits a pronounced regular oscillation at frequency 52, corresponding to a basic oscillation with period of 68 time steps in the original signal in (a). Observe also that the DC component (i.e. average of the signal), not shown in the power spectra, has value close to the amplitude of the waves.

This result, as well as other preliminary investigations, indicates that the steady-state spiking of complex neuronal networks being continuously fed, combined with limitation of the activation transfers, tend to present such relatively well-defined oscillations. Such oscillations may also be related to successive occurrences of avalanches. Given that we have illustrated that the choice of the source node seems to have a relatively little effect on the weights of the equivalent model, such waves would have similar properties irrespectively of the source position. It would be interesting to investigate to which point such waves induced by the integrate-and-fire oscillations are possibly related to brain waves, such as the alpha and beta rhythms, by considering the stimuli activation. In addition, how would the topological properties of complex neuronal networks affect such induced waves? Because the intensity and timing of the avalanches are determined by the number of nodes at the concentric levels, it is possible that the frequency of the induced waves are defined by this hierarchical measurement. Thus, the progressive activation of the communities are expected to induce variations of the wave frequencies. Systematic additional investigations are required in order to address such interesting perspectives.

**Analytical Extensions of the Modular Equivalent Model:** The current work, as well as the previous investigation of the avalanches phenomenon [20], were strongly based on the development of progressively more general equivalent models, which immediately allowed the identification of how the avalanches and activation confinement arise in complex neuronal networks. It would be interesting to consider additional theoretical investigations, especially based on non-linear systems theory, founded on the simplified representation in terms of the equivalent models.

**Comparison Between Several Integrate-and-Fire Methods for Community Detection:** It has become clear that a series of modifications of the integrate-and-fire dynamics can be implemented, many of which have been found to contribute to improvements in community finding. It is now necessary to investigate systematically, by considering several combinations of such modifications, which particular parameters and integrate-and-fire specific dynamics tend to yield the best community identification.

**Comparison with other Community Finding Algorithms:** Though the present study was not particularly concerned with the effectiveness of the integrate-and-fire method for community identification, it would be interesting to use the equivalent model, as well as the whole networks, in order to investigate further how this method compares with more traditional community finding algorithms.

**Topology Arising as a Consequence of the Integrate-and-Fire Dynamics:** So far, we have studied the dynamics given the topology of the network. The opposite is equally interesting, namely to change the topology as a consequence of the dynamics. Such investigations are particularly relevant in neuroscience because they are intrinsically associated to the all-important phenomenon of memory. More specifically, it would be particularly exciting to incorporate Hebbian-like mechanisms, where the connectivity would be reinforced between nodes (or equivalent nodes) which tend to spike at similar times. Would the communities be emphasized by such a kind of topological dynamics? How about the potential of such an approach for pattern recognition, where patterns are trained by enhancing both the intensity and number of the edges (each pattern would be trained into a community)? In particular, the identification of new patterns to be trained, possibly requiring additional neurons or connections, could be performed by taking into account the beginning activation times of the neurons in each previous community.

**Activation from Sets of Nodes:** Though attention has been focused on the activation of the complex networks from individual nodes, corresponding to the activation sources, it would be interesting to consider the activation emanating from sets of nodes placed or not within the same community. Among the many related possibilities, it would be especially worth investigating whether the consideration of more than one source can lead to improved community detection. Another possibility would be to quantify the effect of adding source nodes on the overall activation and spiking.

**Applications to Real-World Networks:** Because of its theoretical emphasis, only models of hybrid modular networks have been considered in the present article. This was especially critical because we needed to know the communities a priori in order to be able to construct the modular equivalent models. However, many interesting studies can be obtained by considering real-world networks. More specifically, given a real-world network, its modular organization could be identified by using a community finding algorithm (including the integrate-and-fire method), allowing the construction of the respective modular equivalent model, which would capture in a simplified and explicit way the main aspects of the respective non-linear dynamics, paving the way for further theoretical or applied investigations.

**Implications for Biology and Complex Systems:** For a long time the ‘divide-and-conquer’ strategy has been acknowledge as a particularly effective means for taming complexity. The processing of information in the
brain constitutes no exception, involving spatial and temporal modularization of the respective processing [21, 22]. The close relationship between the transient confinement of spiking within topological modules in complex neuronal networks is intrinsically related to this fundamental aspect of the brain organization. Because of its simplicity and efficiency in capturing the non-linear integrate-and-fire dynamics, the modular equivalent model allows the development of dynamic models of the functional architecture of the brain by considering only the available interconnectivity between the involved modules and information about the number of neurons and processing layers in each of such modules. Such a simplified but effective model could be used to make all sorts of investigations and predictions about the dynamics of brain activation, as well as to study a number of neuronal diseases involving avalanches of brain activation (e.g. Alzheimer’s disease) and/or topological changes in the connectivity.

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