Eigensensitivity analysis of large-scale structures by substructuring method

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ABSTRACT: In sensitivity-based model updating, the eigensolutions and the associated derivatives of the analytical model need to be calculated iteratively to achieve an optimal solution, which is very time-consuming for a large-scale structure. To overcome this shortcoming, this paper proposes a new method to calculate the eigensensitivity via substructuring approach. When the design parameters in one particular substructure need to update, the sensitivity matrices to the updating parameters at this substructure are calculated while they are zeros at other substructures. The eigensensitivity at the global level is assembled by performing some constraints on the sensitivity matrices at the substructure level. The proposed method is applied in a frame and a practical bridge structure. Numerical results show that the present substructuring approach is more efficient than the traditional global method, at the satisfactory precision.

1. INTRODUCTION

The numerical results from a finite element (FE) model often differ from the experimental results of real structures. FE model updating is often required to identify and correct the uncertain parameters of FE model and is usually posed as an optimization process. During the sensitivity-based model updating process, the eigensolutions and the eigensensitivity of the analytical model need to be calculated in each iteration (Brownjohn et al., 2001). Although the development of the FE method, together with the growing computational speed and storage capabilities, has made it possible to solve some large and complex problems, the sensitivity-based model updating of large-scale structures are usually time-consuming or even prohibited.

The eigensensitivity is used to indicate the variation trends of the design parameters, and thus accelerate the convergence of the optimization process. Nevertheless, the cost of calculating the eigensensitivity is always the dominant contributor to the total cost in many model updating procedures. Fox and Kapoor (1968) proposed the modal method to determine the eigenvalue and eigenvector derivatives. Such procedure is sometimes computationally expensive, since it takes account of all modes of the system to calculate the required eigenvalue and eigenvector derivatives.
Nelson (1976) proposed an exact analytical method for eigensensitivity analysis, which calculated eigenvector derivatives of one mode by just using the modal parameters of that mode only. Many researchers strive to accelerate the calculation of eigensensitivity, either by reducing the required modes (Wang, 1991, Alvin, 1997), or by reducing the degree of freedom (DOF) of the structure (Lin, 1995). However, most of them require to recalculate the eigensensitivity in the global structure level, even if only one design parameter is changed. It is time-consuming. This paper intends to accelerate the calculation of eigensensitivity by considering only particular substructures instead of the global structure.

Weng and Xia (2007, 2008) proposed a modal truncation technique to improve the efficiency of the Kron’s substructuring method to obtain the eigensolutions. Only some lower modes of the substructures are retained in the technique, while the higher modes are discarded and compensated with the first-order residual flexibility. This paper extends this improvement to calculate the eigensensitivity via substructuring approach. The derivative matrices of particular substructures are calculated to assemble the eigensensitivity formula of the global structure. Since the substructures are always much smaller than the global structure, the computation efficiency is improved significantly. The derived formula is applied into a frame structure and a highway bridge, to verify the effectiveness and accuracy of the proposed method.

2. BASIC THEORY

The substructuring method for eigensolutions calculation mainly includes three steps (Kron, 1963): first, the global structure with \( N \) DOFs is divided into \( NS \) substructures according to some criteria; second, each substructure is analyzed independently. For example, the \( j \)th substructure has \( n(j) \) DOFs \((j = 1, 2, \ldots, NS)\) and \( n(j) \) eigenpairs as:

\[
\begin{bmatrix}
\lambda_1^{(j)} & \lambda_2^{(j)} & \cdots & \lambda_{n(j)}^{(j)}
\end{bmatrix}
\quad \text{and}
\begin{bmatrix}
\phi_1^{(j)} & \phi_2^{(j)} & \cdots & \phi_{n(j)}^{(j)}
\end{bmatrix}
\]

third, the divided substructures are reconnected by virtual work and geometric compatibility to recover the global structure. Due to the orthogonality properties, the eigenequation of the assembled global structure is transformed into (Sehmi, 1989):

\[
\begin{bmatrix}
\Lambda^p & -\Gamma \\
-\Gamma^T & 0
\end{bmatrix}
\begin{bmatrix}
\tau
\end{bmatrix}
= \lambda \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\tau
\end{bmatrix}

\text{and}
\Gamma = \left[ C\Phi^p \right]^T
\]

In this equation, \( \Lambda^p = \text{Diag} \left[ \Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(NS)} \right] \), \( \Phi^p = \text{Diag} \left[ \Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(NS)} \right] \), and \( C \) is a rectangular connection matrix, which constrains the interface DOFs in different substructures to move jointly. \( \tau \) is the internal connection forces, and \( \lambda \) is the \( i \)th eigenvalue of the global structure. \( z \) is the mode participation factor, and indicates the contribution of each mode of the substructures to the modes of the global structure by \( \Phi = \Phi^p \{ z \} \).

In Kron’s substructuring method, the complete eigensolutions of all substructures are required to assemble the primitive form of \( \Lambda^p \) and \( \Phi^p \). Weng and Xia (2007, 2008) proposed the first-order residual flexibility based substructuring (FRFS) method to improve the calculation of eigensolutions in both accuracy and efficiency. If there are \( m(j) \) eigensolutions in the \( j \)th substructure retained as ‘master’ modes. The residual higher modes are discarded as ‘slave’ variables and compensated by the first-order residual flexibility. For the \( i \)th mode, the eigenequation has (Weng and Xia, 2007, 2008):

\[
\begin{bmatrix}
\Lambda^m + \Gamma_m \left( \Lambda^{(i)} \right)^{-1} \Gamma_m \\
-\Gamma_m^T & 0
\end{bmatrix}
\begin{bmatrix}
\tau
\end{bmatrix}
= \lambda \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\tau
\end{bmatrix}

\text{and}
\Gamma_m = \left[ C\Phi^m \right]^T \cdot \left( \Lambda^{(i)} \right)^{-1} \cdot \Gamma_m = C\Phi^p \left( \Lambda^{(i)} \right)^{-1} \left[ \Phi^p \right]^T \cdot C^T
\]

In this equation, the subscript ‘m’ and ‘s’ represent the master and slave variables respectively. \( \Lambda^m \) and \( \Phi^m \) are diagonal assembly of master eigenvalues and eigenvectors of all substructures. \( \Phi^p \left( \Lambda^{(i)} \right)^{-1} \left[ \Phi^p \right]^T \) is the first-order residual flexibility, which can be represented by diagonally
assembling the residual flexibility of all substructures. The size of the reduced eigenequation (Eq. (2))
equals to the number of master modes \( p \), which is much smaller than that of the original
eigenequation (Eq. (1)). Based on the reduced eigenequation, the eigenvalue \( \lambda \) and mode
participation factor \( \{z_i\} \) can be obtained with traditional method, such as Lanczos method or
Subspace Iteration method (Bathe, 1976).

3. EIGENVALUE AND EIGENVECTOR DERIVATIVES

To extend this simplified eigenequation for the eigensensitivity analysis, Eq. (2) is differentiated
with respect to an elemental parameter \( r \) (for example flexural rigidity) as (Xia et al, 2008):

\[
\begin{align*}
\frac{\partial}{\partial r} \left[ \Lambda_m^p + \Gamma_m \left( \Lambda^e \right)^{-1} \Gamma \right]^{-1} \left[ \Gamma_m \right]^T 
\end{align*}
\]

Pre-multiplying \( \{z_i\}^T \) on both sides of Eq. (3) and rearranging the equation, the derivative of
eigenvalue \( \lambda \) with respect to the designed parameter \( r \) is:

\[
\frac{\partial \lambda}{\partial r} = \left[ \frac{\partial}{\partial r} \left[ \Lambda_m^p + \Gamma_m \left( \Lambda^e \right)^{-1} \Gamma \right]^{-1} \left[ \Gamma_m \right]^T \right] \left[ \frac{\partial}{\partial r} \left[ \{z_i\} \right]^T \right] \{z_i\} \quad (4a)
\]

In this equation,

\[
\frac{\partial}{\partial r} \left[ \Lambda_m^p + \Gamma_m \left( \Lambda^e \right)^{-1} \Gamma \right]^{-1} \left[ \Gamma_m \right]^T = \frac{\partial \Lambda_m^p}{\partial r} + \frac{\partial \Gamma_m}{\partial r} \left( \left( \Lambda^e \right)^{-1} \Gamma \right)^{-1} \left( \frac{\partial \left( \Lambda^e \right)^{-1} \Gamma \right)}{\partial r} + \frac{\partial \left( \Lambda^e \right)^{-1} \Gamma \right)}{\partial r} \quad (4b)
\]

\[
\frac{\partial}{\partial r} \left( \Gamma^T \right) = \left( \frac{\partial}{\partial r} \left( \Gamma^T \right) \right) = \left( \frac{\partial}{\partial r} \left( \left( \Lambda^e \right)^{-1} \right) \right) \left( \frac{\partial}{\partial r} \Gamma \right) \quad (4c)
\]

\[
\frac{\partial}{\partial r} \left[ \left( \Lambda^e \right)^{-1} \right] = C \times \text{Diag} \left[ \frac{\partial}{\partial r} \left( \left( \Lambda^e \right)^{-1} \right) \right] C^T \times \text{Diag} \left[ C \right] \times C^T \quad (4d)
\]

\[
\frac{\partial}{\partial r} \left[ \Gamma \right] = C \times \text{Diag} \left[ \frac{\partial}{\partial r} \left[ \Gamma \right] \right] C^T \quad (4e)
\]

where \( \frac{\partial \Lambda_m^p}{\partial r} \) and \( \frac{\partial \Phi_m^e}{\partial r} \) are diagonal assembly of the eigenvalue and eigenvector derivatives of the
substructures. The eigenvalue and eigenvector derivatives \( \frac{\partial \lambda_i}{\partial r} \) and \( \frac{\partial \Phi_i^e}{\partial r} \) are only calculated in the
particular substructure which contains the elemental parameter \( r \), and they are zeros in other
substructures.

Since the \( i \)th eigenvector of the global structure can be recovered by:

\[
\Phi_i = \Phi_m^e \{z_i\} \quad (5)
\]

the eigenvector derivative of the \( i \)th mode to the structural parameter \( r \) can be differentiated as:
\[
\frac{\partial \Phi}{\partial r} = \frac{\partial \Phi_m^p}{\partial r} \{z_i\} + \Phi_m^p \left\{ \frac{\partial z_i}{\partial r} \right\}
\] (6)

Mathematically, if the eigenvectors \(\Phi_m^p\) and eigenvector derivatives \(\frac{\partial \Phi_m^p}{\partial r}\) of the substructures are regarded as the vector space, the \(\left\{ \frac{\partial z_i}{\partial r} \right\}\) and \(\{z_i\}\) act as the weight. So far, once the item \(\left\{ \frac{\partial z_i}{\partial r} \right\}\) is available, the eigenvector sensitivity of the global structure is obtainable.

Separating the item \(\left\{ \frac{\partial z_i}{\partial r} \right\}\) into the sum of a particular part and a homogeneous part as:
\[
\left\{ \frac{\partial z_i}{\partial r} \right\} = \{v_i\} + c_i \{z_i\}
\] (7)

where \(c_i\) is a participation factor. Substituting Eq. (7) into Eq. (3), the eigenequation for the \(i\)th mode is transformed into:
\[
\left[ A_m^i + \Gamma_m \left( A^r \right)^{-1} \Gamma_i \right] \left\{ z_i \right\} - \chi I \left\{ v_i \right\} = - \frac{\partial \left[ A_m^i + \Gamma_m \left( A^r \right)^{-1} \Gamma_i \right]^{-1}}{\partial r} \left( A_m^i - \chi I \right) \left\{ z_i \right\}
\] (8)

and the vector \(\{v_i\}\) can be solved from the above equation (Nelson, 1976).

Regarding the reduced eigenequation Eq. (2), the orthogonal condition of eigenvector satisfies:
\[
\{z_i\}^T \{z_i\} = 1
\] (9)

Differentiating Eq. (9) with respect to \(r\) gives:
\[
\frac{\partial \{v_i\}^T}{\partial r} \{z_i\} + \{z_i\}^T \frac{\partial \{v_i\}}{\partial r} = 0
\] (10)

Substituting Eq. (7) into Eq. (10), the participation factor \(c_i\) is obtained as:
\[
c_i = - \frac{1}{2} \left( \{v_i\}^T \{z_i\} + \{z_i\}^T \{v_i\} \right)
\] (11)

Finally, the first-order derivative of \(\{z_i\}\) with respect to the structural parameter \(r\) is:
\[
\left\{ \frac{\partial z_i}{\partial r} \right\} = \{v_i\} - \frac{1}{2} \left( \{v_i\}^T \{z_i\} + \{z_i\}^T \{v_i\} \right) \{z_i\}
\] (12)

\(\left\{ \frac{\partial z_i}{\partial r} \right\}\) is derived based on the reduced eigenequation (Eq. (2)), which has a much smaller size than that of the global structure. Consequently, the computation efficiency can be improved significantly. The detailed application of the proposed substructuring method and its efficiency will be discussed in the following examples.

4. CASE STUDIES

4.1 A frame structure
The detailed process on performing the proposed substructuring method is illustrated by a frame structure in this example. The frame is modeled by 160 two-dimensional beam elements with 2.5m long as in Fig. 1. The material constants are chosen as: bending rigidity \((EI) = 170 \times 10^6 \text{ N m}^2\), axial rigidity \((EA) = 2500 \times 10^6 \text{ N}\), mass per unit length \((\rho A) = 110 \text{ kg/m}\), and Poisson’s ratio = 0.3.

The Young’s module of one column element in Substructure 2 (denoted in Fig. 1) is chosen as the design parameter \(r_1\), and the eigensensitivity of the first 20 modes of the global structure with respect to this design parameter can be calculated with the following three main steps:

(1) The frame is divided into three substructures (NS = 3) as shown in Fig. 1.
(2) Each substructure is analyzed individually. Following the author’s previous discussion (Weng and Xia, 2008), the first 50 modes in each substructure are retained as master modes. The derivative matrices regarding the first 50 modes are calculated as:

(a) For Substructure 2 (includes the design parameter $r_1$), the derivative matrices of eigensolutions are calculated as: $\frac{\partial \lambda^{(2)}_m}{\partial r_1}$, $\frac{\partial \Phi^{(2)}_m}{\partial r_1}$, (m=50) with traditional method, and the derivative matrix of residual flexibility is: $\frac{\partial F^{(2)}_r}{\partial r_1}$ according to Eq. (4d);

(b) For other substructures (excludes the design parameter $r_1$), the derivative matrices of eigensolutions and residual flexibility are zeros as: $\frac{\partial \lambda^{(i)}_m}{\partial r_1} = 0$, $\frac{\partial \Phi^{(i)}_m}{\partial r_1} = 0$, $\frac{\partial F^{(i)}_r}{\partial r_1} = 0$, ($i$=1,3).

(3) The substructures are reassembled into the global structure. The primitive form of the derivative matrices are constructed as:

$$
\frac{\partial \Lambda^p}{\partial r_1} = \text{Diag} \left( \frac{\partial \lambda^{(1)}_m}{\partial r_1}, \frac{\partial \lambda^{(2)}_m}{\partial r_1}, \frac{\partial \lambda^{(3)}_m}{\partial r_1} \right), \frac{\partial \Phi^p}{\partial r_1} = \text{Diag} \left( \frac{\partial \Phi^{(1)}_m}{\partial r_1}, \frac{\partial \Phi^{(2)}_m}{\partial r_1}, \frac{\partial \Phi^{(3)}_m}{\partial r_1} \right), \frac{\partial F^p}{\partial r_1} = \text{Diag} \left( \frac{\partial F^{(1)}_r}{\partial r_1}, \frac{\partial F^{(2)}_r}{\partial r_1}, \frac{\partial F^{(3)}_r}{\partial r_1} \right)
$$

According to Eq. (12), the item $\frac{\partial z}{\partial r_1}$ can be computed. Up to now, the eigensensitivity of the global structure with respect to parameter $r_1$ can be formed according to Eq. (4a) and Eq. (6).

Following the three steps, the eigensensitivity of the first 20 modes with respect to the parameter $r_1$ are calculated. For comparison, the eigensensitivity are computed with traditional global method (Nelson, 1976), which are regarded as accurate ones. The results from the proposed substructuring method and the traditional global method are compared in Table 1. In this table, Correlation of Eigenvector Derivatives (COED) indicates the similarity of the eigenvector derivatives between the Nelson’s method and the proposed substructuring method, and given by:
in which, \( \frac{d\Phi}{dr_t} \) and \( \frac{d\Phi}{dr_t} \) represents the eigenvector derivatives obtained by Nelson’s method and by the proposed substructuring method respectively.

Table 1. The eigensensitivity with respect to the design parameter \( r_t \)

| Mode | Eigenvectors | Eigenvector | Eigenvalues | Eigenvector | Eigenvalues | Eigenvector |
|------|--------------|-------------|-------------|-------------|-------------|-------------|
|      | Global method | Substructuring method | Relative error | COED | Global method | Substructuring method | Relative error | COED |
| 1    | 0.8756       | 0.8756      | 0.00%       | 0.9958      | 65.563      | 66.5113      | 1.45%         | 0.9974       |
| 2    | 3.6014       | 3.6014      | 0.00%       | 0.9894      | 46.987      | 47.0576      | 0.15%         | 0.9995       |
| 3    | 3.2932       | 3.2939      | 0.02%       | 0.9964      | 93.8384     | 94.6951      | 0.83%         | 0.9998       |
| 4    | 47.1077      | 47.1103     | 0.01%       | 0.9967      | 207.5821    | 207.6526     | 0.03%         | 0.9996       |
| 5    | 67.9608      | 68.0655     | 0.15%       | 0.9988      | 151.7001    | 151.6071     | -0.06%        | 0.9998       |
| 6    | 274.9365     | 275.6273    | 0.25%       | 0.9984      | 374.4167    | 374.854      | 0.12%         | 0.9998       |
| 7    | 205.7972     | 205.8071    | 0.00%       | 0.9973      | 178.3376    | 178.0631     | -0.15%        | 0.9995       |
| 8    | 715.4812     | 720.8485    | 0.75%       | 0.9972      | 111.3421    | 110.4803     | -0.77%        | 0.9991       |
| 9    | 633.383      | 637.1267    | 0.59%       | 0.9979      | 121.8211    | 122.157      | 0.28%         | 0.9995       |
| 10   | 523.1332     | 526.1615    | 0.58%       | 0.9983      | 160.7281    | 162.7171     | 1.24%         | 0.9992       |

Table 1 demonstrated that, the errors of most eigenvalue derivatives are less than 1%, and the similarity of eigenvector derivatives are above 0.98. It means that, when the first 50 eigensolutions in each substructure are chosen as master, the proposed substructuring method can reach a good precision in calculating the eigensensitivity for the first 20 modes. Furthermore, in the second step of eigensensitivity calculation, the derivative matrices are required in particular substructure which includes the design parameter. The eigensensitivity of the global structure is formed on the particular substructure and a reduced eigenequation. The computation efficiency might be improved, and will be investigated in the following example.

4.2 A practical bridge

To further investigate the property of the proposed substructuring method, such as efficiency and division formation, a practical bridge over the Balla Balla River in Western Australia is discussed here. The FE model of this bridge (Fig. 2) has 907 elements, 947 nodes, and 5420 DOFs in total (Xia et al, 2008).

The designed parameters include the Young’s modulus of 10 elements, which are randomly chosen from the 907 elements. The computation time on calculating the eigensensitivity of the first 20 modes with respect to the ten elemental parameters are denoted as the indicator of efficiency. Certainly, the division formation of the substructures affects the computation efficiency. To investigate the effect of division formation, the bridge is averagely divided into 5 substructures, 8 substructures, 11 substructures and 15 substructures along the longitudinal direction.
Selecting different master modes in each substructure will influence the accuracy of the proposed substructuring method, and undoubtedly consuming different computation time. If the errors of eigenvalue derivatives are required below 3%, there are 80 master modes retained in each substructure with the division formation of 5 substructures, 60 master modes with 8 substructures, 50 master modes with 11 substructures, and 50 master modes with 15 substructures. The total computation time in calculating the eigensensitivity of the first 20 modes of the global structure with the four division schemes are compared in Fig. 3.

From Fig. 3, it can be found that:

1. Since the eigensensitivity can be formed based on only particular substructure and the reduced eigenequation, the proposed method can improve the computation efficiency. For example, with the traditional global method, one has to obtain the eigensensitivity of the global structure from a large system matrix which has the size of 5420×5420. However, with the proposed substructuring method, when the global structure is divided into 11 substructures, the eigensensitivity can be instead by analyzing a substructure with about 500 DOFs and a reduced eigenequation with the size of 550×550.

2. The computation efficiency heavily depends on the substructure division and the master modes selection. More master modes undoubtedly contribute to higher precision, but consume more computation time. The master modes depend on the precision requirement. The influence of division formation is more complicated. For example, dividing the global structure into 5 substructures or 8 substructures take longer computation time than that of 11 substructures. This is because handling large substructures take up much computation resource. However, the smaller substructures are not always favorable. The division formation of 15 substructures is not as efficient as that of 11 substructures, since much excessive substructures lead to large connection matrix \( C \) and the primitive matrices of the substructures. In that case, the assembly of the ‘substructures’ to the ‘global’ structure will take longer time. One should trade off the number of substructures and the size of each...
substructure, and it is expected that some kind of additional trial and error approach is used before applying it to model updating optimization.

5. CONCLUSION
In this paper, a substructuring scheme is utilized to calculate the first-order derivatives of eigensolutions. Following three main steps, a frame structure and a practical bridge are analyzed, which verify that the proposed method can achieve a good accuracy when proper master modes are retained.

When an elemental parameter is changed within a substructure, only that particular substructure needs to be reanalyzed, and other substructures are untouched. The eigensensitivity of the global structure is assembled by constraining on the substructures. Therefore, the computation efficiency is improved. Additionally, the division formation heavily influences the computation efficiency, which should be considered cautiously in advance.

The advantages of this substructuring method will be more prominent in the practical model updating process when the eigensensitivity need to be calculated repeatedly. These merits further investigations.

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