Comments on “Generalization of Shannon-Khinchin axioms to nonextensive systems and the uniqueness theorem for the nonextensive entropy”

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Abstract—Recently, Suyari has proposed a generalization of Shannon-Khinchin axioms, which determines a class of entropies containing the well-known Tsallis and Havrda-Charvat entropies [H. Suyari, IEEE Trans. Inf. Theory, vol. 50, pp. 1783-1787, Aug. 2004]. In this comment we show that the class of entropy functions determined by Suyari’s axioms is wider than the one proposed by Suyari and give a counterexample. Additionally, we generalize Suyari’s axioms characterizing recently introduced class of entropies obtained by averaging pseudoadditive information content introduced in [V. Ilić and M. Stanković,”Comments on “Nonextensive Entropies derived from Form Invariance of Pseudoadditivity”” Submitted, 2012].

Index Terms—Information measure, nonextensive entropy, nonextensive system, pseudoadditivity, Shannon- additivity, Tsallis entropy, Weierstrass function

I. GENERALIZED SHANNON-KHINCHIN AXIOMS

In this section we review generalized Shannon-Khinchin axioms. We discuss the proof for unique class satisfying the axioms proposed by Suyari [1] and give a counterexample.

A. Suyari’s theorem

Let $\Delta_n$ be the $n$-dimensional simplex,

$$\Delta_n = \left\{ (p_1, \ldots, p_n) \mid p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}$$

and let $\mathbb{R}^+$ denote the set of positive real numbers.

For a function $S_q : \Delta_n \to \mathbb{R}^+$, $q \in \mathbb{R}^+$, $n \in \mathbb{N}$, we define the following Shannon-Khinchin (SK) axioms [SK1]~[SK4]:

[SK1] continuity: $S_q$ is continuous in $\Delta_n$ and with respect to $q \in \mathbb{R}^+$;
[SK2] maximality: for any $n \in \mathbb{N}$ and any $(p_1, \ldots, p_n) \in \Delta_n$

$$S_q(p_1, \ldots, p_n) \leq S_q\left(\frac{1}{n}, \ldots, \frac{1}{n}\right);$$

[SK3] expansability: $S_1(p_1, \ldots, p_n, 0) = S_1(p_1, \ldots, p_n);$

[SK4] generalized Shannon additivity: if

$$p_{ij} \geq 0, \quad p_i = \sum_{j=1}^{m_i} p_{ij}, \quad p(j|i) = \frac{p_{ij}}{p_i},$$

$$\forall i = 1, \ldots, n, \quad \forall j = 1, \ldots, m_i,$$

then the following equality holds:

$$S_q(p_{11}, \ldots, p_{mn}) = S_q(p_1, \ldots, p_n) + \sum_{i=1}^{n} p_i S_q(p(1|i), \ldots, p(m_i|i)).$$

Remark 1: [SK1][SK4] reduces to [SK1]-[SK4] for $q = 1$. As shown in [2], $S_1$ is Shannon entropy, i.e.

$$S_1 = -k \sum_{i=1}^{n} p_i \ln p_i,$$

where $k > 0$. Because of [SK1], we have

$$\lim_{q \to 1} S_q = S_1 = -k \sum_{i=1}^{n} p_i \ln p_i.$$

Theorem 1: Let $S_q : \Delta_n \to \mathbb{R}^+$, $q \in \mathbb{R}^+$, $n \in \mathbb{N}$ be a function which is not identically equals to zero for $n > 1$ and satisfies [SK1]-[SK3] and the following generalized Shannon additivity axiom for $q \in \mathbb{R}^+$. Then, $S_q : \Delta_n \to \mathbb{R}^+$, $n \in \mathbb{N}$, is uniquely determined with

$$S_q(p_1, \ldots, p_n) = \frac{1 - \sum_{i=1}^{n} p_i^q}{\phi(q)},$$

for $q \in \mathbb{R}^+ \setminus \{1\}$ and $(p_1, \ldots, p_n) \in \Delta_n$, and $\phi(q)$ satisfies the following properties i-iv):

i) $\phi(q)$ is continuous and has the same sign as $q - 1$;
ii) $\lim_{q \to 1} \phi(q) = 0$, and $\phi(q) \neq 0$ for $q \neq 1$;
iii) there exists an interval $(a, b) \in \mathbb{R}^+$ such that $a < 1 < b$ and $\phi(q)$ is differentiable on the interval $(a, 1) \cup (1, b)$;
iv) there exists a positive constant $k$ such that $\lim_{q \to 1} \frac{\phi(q)}{q^k} = 1/k$.

According to Suyari’s proof, the form [3] is uniquely determined by solving functional equation [5] from [SK4] and by using the continuity with respect to $p$ as assumed in [SK1]. Property (i) follows from maximality condition [SK2]. The continuity with respect to $q$ from [SK1] implies condition (ii). [SK4] is used only for proof of the basic case $q = 1$, at which [SK1]~[SK4] reduce to Shannon-Khinchin axioms.
To satisfy \( q \), Suyari, using the fact that numerator
\( 1 - \sum_{i=1}^{n} p_i^q \) is differentiable with respect to \( q \), concludes that properties (iii) and (iv) must be satisfied for \( \phi(q) \). However, \( q \) will actually be satisfied iff:

\[
\lim_{q \to 1} S_q(p_1, \ldots, p_n) = \lim_{q \to 1} \frac{q-1}{\phi(q)} \cdot \lim_{q \to 1} \frac{1 - \sum_{i=1}^{n} p_i^q}{q-1} = -k \cdot \sum_{i=1}^{n} p_i \ln p_i
\]

i.e. iff

\[
\lim_{q \to 1} \frac{\phi(q) - \phi(1)}{q-1} = \frac{1}{k} \quad (10)
\]

(We have used L’Hospital’s rule to show that \( \lim \frac{1 - \sum_{i=1}^{n} p_i^q}{q-1} = -\sum_{i=1}^{n} p_i \ln p_i \))

Accordingly, properties (iii) and (iv) should be replaced with iii’ \( \phi(q) \) is differentiable in \( q = 1 \) and

\[
\frac{d\phi(q)}{dq} \bigg|_{q=1} = \frac{1}{k}
\]

B. Counterexample

The Weierstrass function is a well known example of nowhere differentiable continuous function \( [3] \). It is defined with:

\[
W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x),
\]

where \( 0 < a < 1 \), \( b \) is a positive odd integer, \( ab > 1 + 3\pi/2 \) and \( x \in \mathbb{R} \). The Weierstrass function is bounded, since

\[
|W(x)| \leq \sum_{k=0}^{\infty} a^k \leq W(0) < \infty,
\]

where \( W(0) = 1/(1-a) \). Using the Weierstrass function we construct \( \phi(q) \), which satisfies properties (i), (ii) and (iii’), but not properties (iii) and (iv).

Let

\[
\phi(q) = \frac{q-1}{k} \cdot W(q-1) + 2 \cdot W(0) \quad (14)
\]

Since \( W(x) \) is continuous and \( W(x) + 2W(0) > 0 \) according to \( [13] \), \( \phi(q) \) satisfies properties (i) and (ii). Moreover,

\[
\left. \frac{d\phi(q)}{dq} \right|_{q=1} = \lim_{q \to 1} \frac{\phi(q) - \phi(1)}{q-1} = \lim_{q \to 1} \frac{\phi(q)}{q-1} = \frac{1}{k}
\]

and function \( \phi(q) \) satisfies property (iii’).

However, function \( \phi(q) \) does not satisfy property (iii) from Suyari’s theorem since it is differentiable only in \( q = 1 \). Oppositely, the function

\[
\frac{1}{q-1} \cdot \phi(q) = \frac{1}{k} \cdot W(q-1) + 2 \cdot W(0) \quad (16)
\]

should be differentiable for some \( q \neq 1 \) as a product of differentiable functions, further implying differentiability of \( W(q-1) \), which is impossible since the Weierstrass function is nowhere differentiable.

II. NEW AXIOMATIC SYSTEM

In this section we review the class entropies obtained as averaging of pseudoadditive information content introduced in \( [4] \). After that we generalize Suyari’s axiomatic system, which uniquely determines the class of entropies derived in \( [4] \).

A. Entropy as expected information content

Theorem 2: Let \( I_q(p) \) be a function of two variables \( q \in \mathbb{R}^+ \) and \( p \in (0, 1] \), which satisfies the following axioms

[S0] \( I_1(p) = -k \ln p \), \( k > 0 \),
[S1] \( I_q(p) \) is continuous with respect to \( p \in (0, 1] \) and \( q \in \mathbb{R}^+ \),
[S2] \( I_q(p) \) is convex with respect to \( p \in (0, 1] \) for any fixed \( q \in \mathbb{R}^+ \),
[S3] There exists a function \( \phi : \mathbb{R}^+ \to \mathbb{R} \) such that

\[
I_q(p_1p_2) = I_q(p_1) + I_q(p_2) + \phi(q) \cdot I_q(p_1) \cdot I_q(p_2)
\]

for any \( p_1, p_2 \in (0, 1] \), \( \phi(q) \neq 0 \) for \( q \neq 1 \) and \( \phi(q) \) is continuous.

1The axiom [S3] from \( [4] \) is actually not the same [S3] from the present paper, but it is equivalent. It can be obtained if we set \( \phi(q) = \varphi(q)/k \), where \( \varphi : \mathbb{R}^+ \to \mathbb{R} \) is continuous.
Then, the unique nontrivial solution is given by
\[ I_q(p) = \frac{1}{\phi(q)} \cdot \left( p^{\alpha(q)} - 1 \right), \]
where \( k \) is a positive constant and
(a) \( \alpha(q) \) is continuous with respect to any \( q \in \mathbb{R}^+ \),
\( \alpha(1) = 0, \alpha(q) \neq 0 \) for \( q \neq 1 \) and
\[ \lim_{q \to 1} \frac{\alpha(q)}{\phi(q)} = -k. \]
(b) it holds that
\[ \alpha(q) \in \begin{cases} (-\infty, 0] & \text{for } \phi(q) > 0 \\ [0, 1] & \text{for } \phi(q) < 0. \end{cases} \]

**Remark 2:** Note that condition \( \alpha(q) \neq 0 \) for \( q \neq 1 \) ensures that information content is not identically equal zero for some \( q \).

Nonextensive entropy of distribution \( p, S_q(p) \), is defined as the appropriate expectation value of \( I_q \),
\[ S_q(p) \equiv E_{q,p}[I_q]. \]
The expectation is chosen so that the maximality principle is satisfied:
\[ S_q(p_1, \ldots, p_n) \leq S_q\left(\frac{1}{n}, \ldots, \frac{1}{n}\right). \]
The simplest case is the trace form expectation:
\[ E_{q,p}[I_q] \equiv \sum_{i=1}^{n} e_q(p_i) \cdot I_q(p_i) \]
In this case the maximality condition is satisfied if \( e_q(p) \cdot I_q(p) \) is concave as shown in [5]. For the information content [17] one possible choice is \( e_q(p) = p^{-\alpha(q)+1} \) in which case we obtain
\[ S_q(p) = \frac{1 - \sum_{i=1}^{n} p_i^{-\alpha(q)+1}}{\phi(q)}. \]
Note that the trace form expectation operator [20] represents the generalization of expectation operator which is used in axiom [GSK4] for \( \alpha(q) = 1 - q \). In the following subsection we will give the axiomatization of entropy [22] by generalization of the axiomatic system [GSK1]-[GSK4] from section IIA based on expectation operator [20].

**B. New axiomatic system**

**Theorem 3:** Let \( S_q : \Delta_n \to \mathbb{R}^+ \), \( q \in \mathbb{R}^+, n \in \mathbb{N} \) be a function which is not identically equal zero for \( n > 1 \) and satisfies [GSK1]-[GSK3] and the following generalized Shannon additivity axiom for \( q \in \mathbb{R}^+ \).

[gsk4] **generalized Shannon additivity:** if
\[ p_{ij} \geq 0, \quad p_i = \sum_{j=1}^{m_i} p_{ij}, \quad p(j|i) = \frac{p_{ij}}{p_i}, \quad \forall i = 1, \ldots, n, \quad \forall j = 1, \ldots, m_i, \]
and \( \alpha : \mathbb{R}^+ \to \mathbb{R} \) is a continuous function, then the following equality holds:
\[ S_q(p_1, \ldots, p_n) = S_q(p_1) + \sum_{i=1}^{n} p_i^{-\alpha(q)+1} S_q(p(1|i), \ldots, p(m_i|i)), \]
where \( \alpha(q) \) is continuous, \( \alpha(1) = 0 \) and \( \alpha(q) \neq 0 \) for \( q \neq 1 \).
Then, \( S_q : \Delta_n \to \mathbb{R}^+ \), \( n \in \mathbb{N} \), is uniquely determined with
\[ S_q(p_1, \ldots, p_n) = \frac{1 - \sum_{i=1}^{n} p_i^{-\alpha(q)+1}}{\phi(q)}, \]
where (p_1, \ldots, p_n) \in \Delta_n, \( q \in \mathbb{R}^+ \) and
(a) \( \phi(q) \) is continuous with respect to any \( q \in \mathbb{R}^+ \), \( \phi(q) \neq 0 \) for \( q \neq 1 \) and \( \phi(1) = 0 \) [OP] and
\[ \lim_{q \to 1} \frac{\alpha(q)}{\phi(q)} = -k. \]
(b) it holds that
\[ \alpha(q) \in \begin{cases} (-\infty, 0] & \text{for } \phi(q) > 0 \\ [0, 1] & \text{for } \phi(q) < 0. \end{cases} \]

**Remark 3:** The axiom [GSK4] reduces to [GSK4] if we choose \( \alpha(q) = 1 - q \).

**Remark 4:** The conditions for \( \alpha(q) \) and \( \phi(q) \) are identical to the conditions from subsection IIA. Accordingly, the class of entropy functionals (25) is the same as the class (22).

**Proof:** By straightforward repetition of steps from to Suyari’s proof, the form (25) is uniquely determined by solving functional equation (24) from [GSK4] and by using the continuity with respect to \( p \) as assumed in [GSK1].

The properties of \( \phi(q) \) given by (a) straightforwardly follow from continuity of \( S_q \) and \( \alpha(q) \). Before proving the equality (26), we make the following note. The condition (26) is necessary and sufficient for satisfaction of the limit property (7).
\[ \lim_{q \to 1} \frac{1 - \sum_{i=1}^{n} p_i^{-\alpha(q)+1}}{\phi(q)} = -\sum_{i=1}^{n} k \cdot p_i \ln p_i. \]

To prove this, let us introduce \( \gamma(q) = p^{-\alpha(q)} - 1 \). Using \( \gamma(q) \to 0 \) when \( q \to 1 \) and \( (1 + t)^+ \to e \) when \( t \to 0 \), we have
\[ \lim_{q \to 1} \frac{1 - p^{-\alpha(q)}}{\phi(q)} = \lim_{q \to 1} \frac{\alpha(q)}{\phi(q)} \cdot 1 - p^{-\alpha(q)} = \lim_{q \to 1} \frac{\alpha(q)}{\phi(q)} \cdot \frac{\ln p}{\ln (1 + \gamma(q))^{\gamma(q)}} = \lim_{q \to 1} \alpha(q) \cdot \phi(q) \cdot \ln p. \]

For \( p_i = p = 1/n \), (28) reduces to
\[ \lim_{q \to 1} \frac{1 - p^{-\alpha(q)}}{\phi(q)} = -k \cdot \ln p, \]
and equality holds.
and according to (29), the condition (26) is necessary. On the other hand,
\[
\lim_{q \to 1} \frac{1 - \sum_{i=1}^{n} p_i^{-\alpha(q)+1}}{\phi(q)} = \lim_{q \to 1} \sum_{i=1}^{n} p_i \cdot \frac{1 - p_i^{-\alpha(q)}}{\phi(q)} \tag{31}
\]
and (29) implies the sufficiency of condition (26).

Property (b) can be proven by taking the second derivative of \( I_q(p) \) with respect to \( p \), which should be nonnegative for any fixed \( q \in \mathbb{R}^+ \), since \( I_q(p) \) is convex by [T2]. Thus, we can derive a constraint
\[
\frac{\alpha(q)}{\phi(q)} \cdot (\alpha(q) - 1) \geq 0 \tag{32}
\]
for any \( q \in \mathbb{R}^+ \). The constraint (32) is satisfied if
\[
\alpha(q) \in \begin{cases} (-\infty, 0] \cup [1, \infty) & \text{for } \phi(q) > 0 \\ [0, 1] & \text{for } \phi(q) < 0. \end{cases} \tag{33}
\]
As shown in [4], if \( \alpha(q) \) and \( \phi(q) \) are continuous and the equality (26) holds, then \( \alpha(q) \notin [1, \infty) \), and the equality (27) follows, which proves and theorem.

III. Conclusion

In this paper, we reviewed generalized Shannon-Khinchin axioms proposed by Suyari in [1]. We discussed Suyari’s proof of a unique class of functions satisfying those axioms, pointed out the oversight, supported it with a counterexample and gave the correction of the proof. Suyari’s paper has been widely cited and a similar oversight has been noticed in some of them. For example, in [6] the author follows the same procedure in generalizations of Hobson’s axioms, which leads to the similar incorrectness.

In addition, we generalize Suyari’s axioms characterizing the recently introduced class of entropies obtained by averaging pseudoadditive information content introduced in [4].

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