Symbolic powers of planar point configurations

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May 1, 2014

Abstract

Let $Z$ be a finite set of points in the projective plane and let $I = I(Z)$ be its homogeneous ideal. In this note we study the sequence $\alpha(I^{(m)})$, $m = 1, 2, 3, \ldots$, of initial degrees of symbolic powers of $I$. We show how bounds on the growing order of elements in this sequence determine the geometry of $Z$.

1 Introduction

There has been considerable interest in symbolic powers of ideals motivated by various problems in Algebraic Geometry, Commutative Algebra and Combinatorics, see e.g. [4], [11], [15] and references therein. Our motivation originates in the theory of linear systems, more exactly in postulation or interpolation problems. The unsolved up to date conjectures of Nagata and Segre-Harbourne-Gimigliano-Hirschowitz are prominent examples of such problems. We refer to [2] for precise formulation, motivation and more open problems. These problems can be formulated in the language of symbolic powers. For example, the Nagata conjecture predicts that if $Z$ is a set of $r \geq 10$ general points in the projective plane $\mathbb{P}^2$ and $I$ is the saturated ideal of $Z$, then the initial degree $\alpha(I^{(m)})$ of the $m$–th symbolic power of $I$ is bounded from below by $m\sqrt{r}$.

In the present paper we look at a fixed finite set $Z$ of arbitrary points in the projective plane $\mathbb{P}^2$ with defining saturated ideal $I = I(Z)$ and we consider the increasing sequence

$$\alpha(I) < \alpha(I^{(2)}) < \alpha(I^{(3)}) < \ldots$$

of initial degrees of symbolic powers of $I$. The differences in this sequence will be denoted by

$$\alpha_{m,n}(Z) := \alpha(I^{(m)}) - \alpha(I^{(n)})$$

for $m > n$. Bocci and Chiantini initiated in [3] the study of the relationship between the numbers $\alpha_{m,n}(Z)$ and the geometry of the set $Z$ by studying the difference $\alpha_{2,1}(Z)$. In particular, they proved that if this number is the minimal possible i.e. $\alpha_{2,1}(Z) = 1$, then the points in $Z$ are either collinear or they are all the intersection points of $\alpha(2Z)$ general lines. It is natural to look at this problem more generally and consider also other numbers $\alpha_{m,n}(Z)$. Our main result is the following characterization

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*The second named author was partially supported by NCN grant UMO-2011/01/B/ST1/04875*
Theorem. If
\[ \alpha_{2,1}(Z) = \cdots = \alpha_{k+1,k}(Z) = d, \]
then
i) for \( d = 1 \) and \( k \geq 2 \) the set \( Z \) is contained in a line, i.e. \( \alpha(Z) = 1 \);
ii) for \( d = 2 \) and \( k \geq 4 \) the set \( Z \) is contained in a conic, i.e. \( \alpha(Z) = 2 \).

Moreover, both results are sharp, i.e. there are examples showing that one cannot relax the assumptions on \( k \).

This result is proved in the text in Theorems 3.1 and 4.14.

Remark. Naively one could expect that assuming sufficiently many equalities
\[ \alpha_{2,1}(Z) = \cdots = \alpha_{k+1,k}(Z) = d, \quad (2) \]
one could always conclude \( \alpha(Z) = d \). Example 4.17 shows that this is false for all \( d \geq 4 \) even if the equalities in (2) hold for all \( k \geq 1 \).

We expect that the Theorem holds also for \( d = 3 \) but it seems that a proof requires some new ideas. We hope to come back to this question in the future.

We work over an algebraically closed field of characteristic zero.

## 2 Symbolic powers and initial degrees

Let \( Z = \{P_1, \ldots, P_r\} \) be a fixed finite set of points in the projective plane. Let \( I(P_i) \) be the maximal ideal of the point \( P_i \) in the graded ring \( R = \mathbb{K}[x_0, x_1, x_2] \), where \( \mathbb{K} \) is an algebraically closed field of characteristic zero. Then the ideal of \( Z \) is
\[ I = I(Z) = I(P_1) \cap \cdots \cap I(P_r). \]

For a positive integer \( k \), the \( k \)–th symbolic power of \( I \) is defined by
\[ I^{(k)} := I(P_1)^k \cap \cdots \cap I(P_r)^k. \quad (3) \]

The definition of the symbolic power of an arbitrary ideal is more complicated and involves associated primes, see [16, Chapter IV.12, Definition], but in our context (3) suffices. The subscheme of \( \mathbb{P}^2 \) defined by the ideal \( I^{(k)} \) will be denoted by \( kZ \).

It is convenient to study slightly more general subschemes, the so called fat points. To this end let \( m = (m_1, \ldots, m_r) \in \mathbb{Z}^r \) be a vector of non-negative integers. We write
\[ I(mZ) = I(m_1 P_1 + \cdots + m_r P_r) = I(P_1)^{m_1} \cap \cdots \cap I(P_r)^{m_r}. \]

For an arbitrary homogeneous ideal
\[ I = \bigoplus_{n \geq 0} I_n, \]
we define the initial degree of \( I \) to be the number
\[ \alpha(I) = \min \{ n \in \mathbb{Z} : I_n \neq 0 \}. \]
If $C$ is a divisor defined by some polynomial $f \in I_n$, then we abuse a little bit the notation and write $C \in I_n$. As the explicit equations defining a divisor will never play a role in this paper, this should cause no confusion.

Finally, we write $\mathbb{1}$ for the vector $(1, \ldots, 1) \in \mathbb{Z}^r$ and we extend the definition in (II) to the case of inhomogeneous multiplicities by putting

$$\alpha_{m,n}(Z) := \alpha(I(mZ)) - \alpha(I(nZ)).$$

Thus $\alpha_{m,n}(Z) = \alpha_{m,n}(Z)$ for $m = (m, \ldots, m)$ and $n = (n, \ldots, n) \in \mathbb{Z}^r$.

We begin by the following observation which is crucial in the sequel. Roughly speaking, it amounts to saying that if a divisor $C$ is of minimal degree and some difference $\alpha$ is given, then every effective subdivisor of $C$, in particular every irreducible component of $C$, is again of minimal degree for an appropriate subscheme and the difference at this subscheme is at most that at $Z$.

**Lemma 2.1.** Let $Z$ be a fixed set of points $P_1, \ldots, P_r \in \mathbb{P}^2$ and let $m > n$ be positive integers. Let $\beta = \alpha(I(mZ))$, $\gamma = \alpha(I(nZ))$ and $\alpha = \alpha_{m,n}(Z) = \beta - \gamma$. Let $C \in I(mZ)\beta$ be an effective divisor. Furthermore let

$$C = C_1 + C_2$$

be a sum of two integral non-zero divisors. Let $\beta_j = \deg(C_j)$, $m_i^{(j)} = \text{ord}_P C_j$, $m^{(j)} = (m_1^{(j)}, \ldots, m_r^{(j)})$, $n_i^{(j)} = \max \left\{ m_i^{(j)} - (m - n), 0 \right\}$

and $n^{(j)} = (n_1^{(j)}, \ldots, n_r^{(j)})$. Then

i) $\beta_j = \alpha(I(m^{(j)}Z))$ and

ii) $\alpha_{m^{(j)}, n^{(j)}} \leq \alpha$.

for $j = 1, 2$.

**Proof.** We prove the statement for $C_1$, the claim for $C_2$ follows by symmetry.

i) Assume to the contrary that there is a divisor $C_1'$ of degree $\deg(C_1') = \beta_1' < \beta_1$ with vanishing vector $m^{(1)}$. Then $C_1' + C_2$ has vanishing vector $m^{(1)} + m^{(2)} = m$ and degree $\beta_1' + \beta_2 < \beta_1 + \beta_2 = \beta$, a contradiction.

ii) If $n^{(1)} = 0$, then let $\Gamma_1 = 0$, otherwise let $\Gamma_1$ be a non-zero divisor of least degree $\gamma_1$ in $I(m^{(1)}Z)$. Let $D_1 = \Gamma_1 + C_2$. Then

$$\text{ord}_{P_i} D_1 = \text{ord}_{P_i} \Gamma_1 + \text{ord}_{P_i} C_2 = \max \left\{ m_i^{(1)} - (m - n), 0 \right\} + m_i^{(2)} \geq n.$$ 

Indeed, if $m_i^{(1)} - (m - n) = \max \left\{ m_i^{(1)} - (m - n), 0 \right\}$, then

$$m_i^{(1)} - (m - n) + m_i^{(2)} = m_i^{(1)} + m_i^{(2)} - m + n \geq n.$$ 

If it is $0 = \max \left\{ m_i^{(1)} - (m - n), 0 \right\}$, then

$$0 \geq m_i^{(1)} - (m - n) \geq n - m_i^{(2)}$$

implies

$$n \leq m_i^{(2)} = \text{ord}_{P_i} C_2 = \text{ord}_{P_i} D_1.$$
Hence it must be
\[ \deg(D_1) = \gamma_1 + \beta_2 \geq \gamma \]
by assumption, which implies
\[ a_{m^{(1)}_j, n^{(1)}_j} = \beta_1 - \gamma_1 = \beta - (\beta_2 + \gamma_1) \leq \beta - \gamma = \alpha \]
as asserted.

Note that it might happen that some entries in \( n^{(j)} \) are zero or even that both vectors \( n^{(1)} \) and \( n^{(2)} \) are zero.

**Corollary 2.2.** Keeping the notation from the above Lemma, let \( C \) be a divisor in \( I(mZ)_\beta \) with the decomposition
\[ C = \sum a_j C_j \]
into distinct irreducible components with \( \gamma_j = \deg(C_j) \) and \( m^{(j)}_i = \text{ord}_P C_j \). Let \( m^{(j)} = (m^{(j)}_1, \ldots, m^{(j)}_r) \). Then
\begin{enumerate}
  \item \( \gamma_j = \alpha(I(m^{(j)} Z)) \) and
  \item for \( n^{(j)}_i = \max \{ m^{(j)}_i - (m - n), 0 \} \) we have \( a_{m^{(j)}_i, n^{(j)}_i}(Z) \leq \alpha \).
\end{enumerate}

**Proof.** Simply apply Lemma 2.1 to every component \( C_j \) of \( C \) and its residual divisor \( C - C_j \). \( \square \)

### 3 Differences equal 1

We begin by comparing \( \alpha(kZ) \) and \( \alpha(Z) \), which is the first natural generalization of the situation studied by Bocci and Chiantini. Since a derivative of a polynomial having \( k \)-fold zeroes at \( Z \) has zeroes of order at least \( k - 1 \), we have
\[ \alpha_{k,1}(Z) \geq k - 1. \tag{4} \]

Now, we describe all cases when there is the equality in (4), the case \( k = 2 \) being settled by Bocci and Chiantini.

**Theorem 3.1.** Let \( Z \subset \mathbb{P}^2 \) be a finite set of points and let \( k \geq 3 \) be an integer. If
\[ \alpha_{k,1}(Z) = k - 1, \]
then the points in \( Z \) are collinear.

**Proof.** The idea is to reduce the statement to the classification of Bocci and Chiantini and to exclude the configuration of intersection points of lines.

Let \( C \) be a divisor of degree \( d = \alpha(kZ) \) vanishing at \( kZ \), defined by the polynomial \( f \). Let \( D \) be the divisor given by some derivative of \( f \) of order \( (k - 2) \). Then \( D \) is a divisor of degree \( d - (k - 2) \) vanishing along \( 2Z \). Our assumption
\[ \alpha(Z) = \alpha(kZ) - \alpha_{k,1}(Z) = d - (k - 1) \]
implies that \( \alpha_{k,1}(Z) = 1 \). Thus \( \alpha_{k,1}(Z) = 1 \) and either all points in \( Z \) are collinear in which case we are done, or they are intersection points of \( d - (k - 2) \) general lines. In the later case, let \( L \) be one of these lines. Then \( L \cdot C = d \) and
a) either \( L \cdot C \geq (d - (k - 2) - 1)k \),

b) or \( L \) is a component of \( C \).

In case a) we see immediately that \( k \geq d \).

Since \( C \) contains points of multiplicity \( k \), it follows, that in fact \( k = d \). But then \( Z \) consists of a single point and we are done.

We may assume that the case b) holds for all lines in the arrangement. Subtracting all these lines from \( C \), we obtain a curve \( C' \) of degree \( d' = d - (d - (k - 2)) = k - 2 \) passing through \((k - 2)Z\). Hence \( d' \geq k - 2 \) which gives again the multiplicity equal to the degree and we are done.

\[ \square \]

Remark 3.2. One could revoke a result of Wüstholz [5, page 76] in order to establish the above theorem. We preferred however to present an elementary proof. Note also for the future reference that the results along the lines of [5] do not lead to a proof of Theorem 4.14.

Another possible approach to a generalization of the result of Bocci and Chiantini is to replace the number \( \alpha_{2,1} \) by \( \alpha_{k,k-1} \). This leads to some new configurations.

We begin by the following general Lemma.

Lemma 3.3. Let \( Z = \{ P_1, \ldots, P_r \} \) be a set of \( r \) points in the projective plane \( \mathbb{P}^2 \) and let \( m_1, \ldots, m_r \) be positive integers. Suppose that the minimal degree of a divisor passing through points \( P_1, \ldots, P_r \) with multiplicities at least \( m_1, \ldots, m_r \) is \( d \), and for multiplicities \( (m_1 - 1), \ldots, (m_r - 1) \), it is \( d - 1 \). Moreover we assume that there is a reduced and irreducible curve

\[ C \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{I}(mZ)). \]

Then \( C \) must be a line (and consequently \( d = m_1 = \cdots = m_r = 1 \)).

Proof. Assume to the contrary that the degree \( d \) of \( C \) is at least 2. Let \( C' \) be a divisor defined by a first order derivative of the equation of \( C \). Then

\[ C' \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d - 1) \otimes \mathcal{I}(1 - 1)Z) \]

and \( C \) and \( C' \) have no common components. It follows that

\[ d(d - 1) = C \cdot C' \geq \sum_{i=1}^{r} m_i(m_i - 1). \tag{5} \]

On the other hand, there is by assumption no curve of degree \( (d - 2) \) passing through \( P_1, \ldots, P_r \) with multiplicities \( (m_1 - 1), \ldots, (m_r - 1) \). A simple dimension count gives

\[ d(d - 1) \leq \sum_{i=1}^{r} m_i(m_i - 1). \tag{6} \]

It follows that

\[ d(d - 1) = \sum_{i=1}^{r} m_i(m_i - 1). \tag{7} \]

In particular this means that the divisors \( C \) and \( C' \) meet only in points \( P_1, \ldots, P_r \).
Note that (7) implies that there are at least \(d\) independent sections in \(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1) \otimes \mathcal{I}(m-1)Z)\). Indeed,

\[
h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1) \otimes \mathcal{I}(m-1)Z) \geq \left(\frac{d+1}{2}\right) - \sum_{i=1}^{r} \left(\frac{m_i}{2}\right) = d.
\]

Hence there is a section \(s'' \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1) \otimes \mathcal{I}(m-1)Z)\) vanishing at a point \(R \in C\) different from \(P_1, \ldots, P_r\). By (7) the divisor \(C''\) defined by \(s''\) must have a common component with \(C\). As \(C\) is irreducible and the degrees do not agree, this is clearly impossible. 

The key idea in the proof of the next theorem is to observe that the above Lemma applies to reducible curves component by component.

**Theorem 3.4.** Let \(Z = \{P_1, \ldots, P_r\} \subset \mathbb{P}^2\) be a finite set of points. If 
\[\alpha_{k,k-1}(Z) = 1\]
for some \(k \geq 2\) then

a) either the points in \(Z\) are collinear,

b) or \(Z\) consists of all intersection points of some arrangement of lines (which might be non reduced and degenerate i.e. containing points through which more than two lines pass).

**Proof.** Let \(C\) be a divisor of minimal degree \(d\) vanishing at \(kZ\) and let
\[C = \sum_{j=1}^{s} a_jC_j\]
be its decomposition with irreducible and reduced components \(C_j\). Let \(\gamma_j\) be the degree of \(C_j\) and \(m_{(j)} = \text{ord}_{P_i} C_j\) its order of vanishing at the point \(P_i\).

It follows from Corollary 2.2 that \(\gamma_j\) is the least number \(\gamma\) such that there exists a curve \(D\) of degree \(\gamma\) vanishing at \(P_1, \ldots, P_r\) with multiplicities \(m_{(1)}^{(j)}, \ldots, m_{(r)}^{(j)}\).

The same Corollary implies that \(\gamma_j - 1\) is the least number \(\gamma\) such that there exists a curve \(D\) of degree \(\gamma\) vanishing at \(P_1, \ldots, P_r\) with multiplicities \((m_{(1)}^{(j)} - 1), \ldots, (m_{(r)}^{(j)} - 1)\).

Thus we can apply Lemma 3.3 to every component \(C_j\) and we are done as we get case a) for \(s = 1\) and case b) otherwise. 

We conclude this section with the following corollary, strengthening the statement of Theorem 3.1.

**Corollary 3.5.** Assume that \(k \geq 3\) and \(Z\) is a collection of points such that 
\[\alpha_{k,k-1}(Z) = \alpha_{k-1,k-2}(Z) = 1\]
Then \(Z\) consists of collinear points.

**Proof.** Let \(C \in I(kZ)\) be a divisor of degree \(d = \alpha(kZ)\). By Theorem 3.4 \(C\) consists of \(d\) lines. If among these lines there are \(3\) (or more) in general position, i.e. not all passing through the same point, then we can subtract these \(3\) lines from \(C\) obtaining a curve vanishing along \((k-2)Z\) of degree \(d-3\), which contradicts our assumptions. Thus all lines in \(C\) must pass through the same point. Assume that \(C_{\text{red}}\) the support of \(C\) consists of \(a\) lines. Subtracting all of them once from \(C\) produces a curve of degree \(d-a\) vanishing along \((k-1)Z\). This implies \(a = 1\) and we are done.
Note that the following example shows that it is essential that there are two consecutive differences of 1 in the sequence $\alpha(kZ)$.

**Example 3.6.** Let $Z = \{P_1, P_2, P_3\}$ be 3 general points. Then

$$\alpha(kZ) = \begin{cases} 
3m - 1 & \text{for } k = 2m - 1 \text{ odd} \\
3m & \text{for } k = 2m \text{ even}
\end{cases}$$

In particular, infinitely many differences $\alpha_{k,k-1}$ are equal 1.

**Proof.** This is obvious for $m = 1$ and we proceed by induction on $m$. Assuming the theorem for $2m - 1$ and $2m$, we want to prove it for $2(m + 1) - 1 = 2m + 1$ and $2(m + 1)$. In the even case we apply Bezout theorem in order to show that a divisor vanishing along $Z$ to order $2(m + 1)$ must be the $(m + 1)$–fold union of the 3 lines determined by pairs of points in $Z$. Then we have

$$\alpha((2m-1)Z) = 3m-1, \  \alpha(2mZ) = 3m, \ \alpha((2m+1)Z) = X, \ \alpha(2(m+1)Z) = 3(m+1).$$

This show that $X$ is either $3m + 1$ or $3m + 2$. The first case is excluded by Corollary 2.2 as the points are not collinear. The remaining case is our claim. 

\[\square\]

### 4 Differences equal 2

In this section we study subschemes $Z$ with differences equal to 2. We begin by the following general statement, which restricts possible configurations. It parallels the result of Theorem 3.4 in the present setting.

**Theorem 4.1.** Let $Z = \{P_1, \ldots, P_r\} \subset \mathbb{P}^2$ be a finite set of points such that

$$\alpha_{k,k-1}(Z) = 2$$

for some $k \geq 2$. Moreover, assume that $C$ is a divisor of degree $d = \alpha(kZ)$ vanishing along $kZ$. Then every irreducible component of $C$ is a rational curve;

**Proof.** Let $\tilde{C}$ be a component of $C = D + \tilde{C}$. Let $\tilde{d} = \deg \tilde{C}, \ \tilde{m}_j = \text{mult}_{P_j} \tilde{C}$. We want to show that $\tilde{C}$ is rational. If $\tilde{d} = 1$ or 2 then we are done. Similarly, if $\tilde{d} = 3$ and $\tilde{C}$ is singular, then we are also done. If $\tilde{d} = 3$ and $C$ is smooth, then $D$ is a curve of degree $d - 3$ vanishing along $Z$ to order $(k - 1)$, which contradicts our assumption. Thus, we can assume that $\tilde{d} \geq 4$.

Lemma 2.1 implies that $h^0((\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(\tilde{d} - 3) \otimes J((\tilde{m} - 1)Z)) = 0$. Counting conditions, we have

$$(\tilde{d} - 3)\tilde{d} < \sum_{j=1}^{r}(\tilde{m}_j - 1)\tilde{m}_j.$$

On the other hand, $\tilde{C}$ is irreducible, hence by the genus formula

$$(\tilde{d} - 1)(\tilde{d} - 2) \geq \sum_{j=1}^{r}(\tilde{m}_j - 1)\tilde{m}_j.$$

Observe that the above implies

$$(\tilde{d} - 1)(\tilde{d} - 2) - 1 \leq \sum_{j=1}^{r}(\tilde{m}_j - 1)\tilde{m}_j \leq (\tilde{d} - 1)(\tilde{d} - 2),$$
and since the terms in the middle and on the right hand side are always even, while
the term on the left is odd, we have the equality in the genus formula. This implies
that $\tilde{C}$ has geometric genus 0. Moreover, all singularities of $\tilde{C}$ are ordinary multiple
points.

Some examples of subschemes $Z$ with $\alpha_{2,1} = 2$ have been discussed in [3]. We
repeat here some of them and discuss a few new ones. The first one must be expected
after the statement of Corollary 3.5.

**Example 4.2.** Let $Z = \{P_1, \ldots, P_r\}$ be a set of $r \geq 5$ points on a smooth conic $C$. Then

$$\alpha(kZ) = 2k$$

for all positive integers $k$.

Heading for examples of irreducible curves satisfying Theorem 4.1 we have the

**Example 4.3.** Let $d \geq 2$, $k \geq 2$, $r \geq 1$ be integers satisfying

$$(d - 1)(d - 2) = rk(k - 1), \quad 2(d - 1) < rk.$$ 

Assume that there exists an irreducible curve $C \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{I}(kZ))$. If $Z$ is
the set of singular points of $C$ then

$$\alpha_{k,k-1}(Z) = 2.$$ 

**Proof.** We show that $\alpha(kZ) = d$ and $\alpha((k - 1)Z) = d - 2$. To this end it suffices to show that

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{I}(kZ)) \neq \emptyset,$$

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d - 1) \otimes \mathcal{I}(kZ)) = \emptyset,$$

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d - 2) \otimes \mathcal{I}((k - 1)Z)) \neq \emptyset,$$

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d - 3) \otimes \mathcal{I}((k - 1)Z))) = \emptyset.$$ 

The first claim is satisfied by assumption. To prove the second, assume that there
exists a curve $D$ of degree $d - 1$ vanishing at $kZ$. Then, since $C$ is irreducible, by
Bezout we have

$$d(d - 1) \geq rk^2 = rk(k - 1) + rk > (d - 1)(d - 2) + 2(d - 2) = (d + 1)(d - 2),$$

which is false. To prove the third claim, compute conditions

$$(d - 1)(d + 2) - rk(k - 1) = (d - 1)(d + 2) - (d - 1)(d - 2) > 0.$$ 

To prove the last claim, again use Bezout

$$d(d - 3) \geq rk(k - 1) = (d - 1)(d - 2),$$

which gives absurd.

Examples 4.4 and 4.5 below show that curves satisfying numerical conditions of
Example 4.3 exist for $k = 2$ and $k = 3$. We do not know if such curves exist for
$k \geq 4$. Possibly examples to consider would be

- a curve of degree 13 with 11 points of multiplicity 4;
• a curve of degree 17 with 12 points of multiplicity 5;
• a curve of degree 22 with 14 points of multiplicity 6 and so on.

**Example 4.4.** Let $C$ be an irreducible curve of degree $d \geq 5$ with the maximal possible number of nodes, i.e. \((d-1)(d-2)\) of them. Such a curve exists by a classical result of Severi, see [8] for a modern approach and much more. Let $Z$ be the set of all nodes of $C$. Then
$$\alpha(2Z) = d \quad \text{and} \quad \alpha(Z) = d-2.$$ The construction of an example with $k = 3$ is a way more involved. We thank Joaquim Roé for making us aware of papers [6] and [7] and explaining to us their content.

**Example 4.5.** Let $C$ be a smooth plane cubic. There are 9 inflection points on $C$ and there are 12 lines arranged so that each inflection point is contained in exactly 4 lines and each line contains exactly 3 points. This is the Hesse configuration, see [1] for more details and interesting background.

The dual configuration consists of 9 lines $L_1, \ldots, L_9$ and 12 points arranged so that each line passes through exactly 3 lines and each point is contained in exactly 3 lines. The divisor $D' = L_1 + \cdots + L_9$ has degree 9 and exactly 12 triple points. Let $L$ be an additional line intersecting $D'$ transversally, in particular not passing through any of the triple points. The divisor $D = D' + L$ has degree 10 and contains 12 triple points and 9 double points $L_i \cap L$. All of these singularities are ordinary, as they arise as intersection points of reduced lines. Let $W = W(10; 3 \times 12, 2 \times 9)$ be the variety parameterizing all plane curves of degree 10 with 12 triple points and 9 double points. Let $V \subset W$ be a component containing $D$. Then [6] Proposition 1.2 implies that $V$ has a good dimension (for precise definition we refer to [6] and [7]).

The divisor $D$ considered as an element of $W(10; 3 \times 12)$ is virtually connected i.e. remains connected after removing arbitrary number of triple points. This is guaranteed by the line $L$ intersecting all components of $D'$. Then [7] Theorem 4.3 implies that a general member of $W(10; 3 \times 12)$ is irreducible. Let $C$ be such a member and let $Z$ be the set of triple points on $C$. Then
$$\alpha(2Z) = 8 \quad \text{and} \quad \alpha(3Z) = 10.$$ We can derive new examples combining curves discussed in Example 4.4. For the purpose of this example we allow a smooth conic as a nodal rational curve (with no nodes). A line however is excluded.

**Example 4.6.** Let $C_1, C_2$ be two nodal rational curves of degree $d_1 \geq 2$ and $d_2 \geq 2$ respectively, intersecting transversally. Let
$$Z = \text{sing}(C_1) \cup \text{sing}(C_2) \cup (C_1 \cap C_2).$$ Then
$$\alpha(2Z) = d_1 + d_2 \quad \text{and} \quad \alpha(Z) = d_1 + d_2 - 2.$$ **Proof.** Assume that $\alpha(2Z) = \gamma \leq d_1 + d_2 - 1$ and let $\Gamma$ be a divisor in $|\mathcal{O}_{\mathbb{P}^2}(\gamma) \otimes \mathcal{I}(2Z)|$. We show to the contradiction that $C_1$ and $C_2$ are components of $\Gamma$. Indeed, if not, then we have
$$(d_1 + d_2 - 1)d_1 \geq \Gamma \cdot C_1 \geq 2(d_1 - 1)(d_1 - 2) + 2d_1d_2.$$
This is equivalent to
\[ d_1(d_1 + d_2 - 5) + 4 \leq 0, \]
which is never satisfied for \( d_1, d_2 \geq 2 \). The same argument applies to \( C_2 \).

The second assertion is proved in the similar way.

Finally, we show that the irreducible curves appearing in Theorem 4.1 can all be lines. This implies in particular, that the converse to Theorem 3.4 b) cannot be true.

**Example 4.7.** Let \( L_1, \ldots, L_d \) be a general arrangement of \( d \geq 3 \) lines in \( P_2 \) and let \( Z \) be the set of points consisting of all intersection points \( P_{ij} = L_i \cap L_j \) but \( P_{12} \).

Then \( \alpha(2Z) = d \) and \( \alpha(Z) = d - 2 \).

*Proof.* The proofs of both claims are the same. If there were a divisor of a lower degree in either case, then we show, intersecting it with configuration lines, that it must contain all lines in the first case and the lines \( L_3, \ldots, L_d \) in the second case.

The next example shows that there is no straightforward generalization of Corollary 3.5, there might be two consecutive differences of initial degree equal to 2 without the points being forced to lie on a conic. Lemma 4.15 exhibits another example of this kind with 3 consecutive differences equal 2, but the argument there is less explicit.

**Example 4.8.** Let \( Z = \{P_1, \ldots, P_6\} \) be a set of 6 general points in the plane. Let \( C_i \) be the conic passing through all points in \( Z \) but \( P_i \), for \( i = 1, \ldots, 6 \). Then

\[ \alpha(5Z) = 12, \quad \alpha(4Z) = 10 \quad \text{and} \quad \alpha(3Z) = 8. \]

*Proof.* Again, the argument is the Bezout’s Theorem. We discuss only the last case. Assume to the contrary that there is a divisor \( \Gamma_1 \) of degree at most 7 vanishing along \( 3Z \). If \( C_1 \) is not a component of \( \Gamma_1 \), then intersecting with \( C_1 \) we have

\[ 14 \geq \Gamma_1 \cdot C_1 \geq 5 \cdot 3 = 15, \]

a contradiction. Thus \( C_1 \) is contained in \( \Gamma_1 \) and there is a divisor \( \Gamma_2 = \Gamma_1 - C_1 \) of degree at most 5 with vanishing vector \( (3, 2, 2, 2, 2, 2) \) at \( P_1, \ldots, P_6 \). If \( C_2 \) is not a component of \( \Gamma_2 \), then we have

\[ 10 \geq \Gamma_2 \cdot C_2 \geq 3 + 4 \cdot 2 = 11, \]

a contradiction. Thus \( C_2 \) is a component of \( \Gamma_2 \) and we get a new divisor \( \Gamma_3 = \Gamma_2 - C_2 \) of degree at most 3 and vanishing vector \( (2, 2, 1, 1, 1, 1) \). This divisor must split the line \( L_{12} \) through the points \( P_1 \) and \( P_2 \). Then the residual curve \( \Gamma_4 = \Gamma_3 - L_{12} \) is a conic containing all the points \( P_1, \ldots, P_6 \), which contradicts our assumption that the points are general.

Note, that in this example \( \alpha(2Z) \leq 5 \). Indeed, it is enough to take the conics \( C_1 \) and \( C_2 \) and the line \( L_{12} \) to obtain a divisor with double points along \( Z \).

It is nevertheless natural to wonder if there is a result along the lines of Corollary 3.5 in the case of degree jumping by 2.
Conjecture 4.9. Assume that $k \geq 5$ and $Z$ is a collection of points such that 
$$\alpha_{k,k-1}(Z) = \alpha_{k-1,k-2}(Z) = \alpha_{k-2,k-3}(Z) = \alpha_{k-3,k-4}(Z) = 2.$$ 
Then $Z$ is contained in a single conic.

We were not able to prove this conjecture, yet there is a strong supporting evidence, steaming partly from the next lemmata and Theorem 4.14.

Now, we want to investigate more closely rational curves appearing as components in Theorem 4.1. Taking into account Corollary 2.2, we arrive at the following statement.

Lemma 4.10. Let $C$ be an irreducible curve of degree $d$ with multiplicities $m_1 \geq 2$, $m_2, \ldots, m_r$ at points $P_1, \ldots, P_r$. Let $\underline{n} = (m_1 - 2, m_2 - 1, \ldots, m_r - 1)$. Then 
$$\alpha(\underline{n}Z) \leq d - 3,$$
where as usually $Z = \{P_1, \ldots, P_r\}$.

Proof. It follows from the genus formula that 
$$(d - 1)(d - 2) \geq \sum_{i=1}^{r} m_i(m_i - 1).$$

The existence of a divisor of degree $d - 3$ with vanishing vector $\underline{n}$ follows from the inequalities 
$$(m_1 - 2)(m_1 - 1) + \sum_{i=2}^{r} m_i(m_i - 1) = \sum_{i=1}^{r} m_i(m_i - 1) - 2(m_1 - 1) \leq d(d - 3) + 2(2 - m_1) \leq d(d - 3)$$
and we are done.

As a consequence we derive the following statement.

Lemma 4.11. Let $Z = \{P_1, \ldots, P_r\}$ be a reduced 0–dimensional subscheme of $\mathbb{P}^2$. Let $C = \sum a_i C_j$ be a divisor of degree $d = \alpha(kZ)$ vanishing at $kZ$ for some $k \geq 2$. Assume that $\alpha((k-1)Z) = d - 2$ and that for each point $P_i$ there exists at least one component $C_j(i)$ singular at $P_i$, i.e. $\text{ord}_{P_i}C_j(i) \geq 2$. Then 
$$\text{ord}_{P_i} C = k$$
for all $i = 1, \ldots, r$.

Proof. Suppose to the contrary that, after possible renumbering of the points, $\text{ord}_{P_i} C \geq k + 1$ and, after possible renumbering of the components, $\text{ord}_{P_i} C_1 \geq 2$. Let $m_i = \text{ord}_{P_i} C_1$. For the divisor $D = C - C_1$ we have 
$$\text{ord}_{P_i} D \geq k - m_i \quad \text{for} \quad i \geq 2 \quad \text{and} \quad \text{ord}_{P_i} D \geq k + 1 - m_1.$$ 

Lemma 4.10 applies to the curve $C_1$, so that there exists a divisor $\Gamma$ of degree $\text{deg}(\Gamma) = \text{deg}(C_1) - 3$ with 
$$\text{ord}_{P_i} \Gamma \geq m_i - 1 \quad \text{for} \quad i \geq 2 \quad \text{and} \quad \text{ord}_{P_1} \Gamma \geq m_1 - 2.$$ 

Then $\Gamma + D$ has degree $d - 3$ and vanishes along $(k-1)Z$, which contradicts our assumptions.
A straightforward useful consequence of the above two lemmata is the following.

**Corollary 4.12.** Let $Z = \{P_1, \ldots, P_r\}$ be a reduced 0–dimensional subscheme of $\mathbb{P}^2$. Let $C = \sum a_jC_j$ be a divisor of degree $d = \alpha(kZ)$ vanishing at $kZ$ for some $k \geq 2$. Assume that for a fixed $i \in \{1, \ldots, r\}$ there is a component $C_i$ of $C$ singular at $P_i$ and that $\text{ord}_{P_i} C \geq k + 1$. Then $\alpha((k-1)Z) \leq d - 3$.

Now we are heading for a result paralleling Theorem 3.1 in the present setting. First we need some preparations. The following Lemma and its proof are motivated by a result of Lazarsfeld [13, Proposition 2.5].

**Lemma 4.13.** Let $D \subset \mathbb{P}^2$ be a reduced divisor of degree $d$ with multiplicities $m_1 \geq \ldots \geq m_r \geq 2$ in some points $P_1, \ldots, P_r$. Let $m = (m_1, \ldots, m_r)$ be the multiplicities vector and $Z = \{P_1, \ldots, P_r\}$. Then $(\frac{m}{BD} - 1)Z$ imposes independent conditions on curves of degree $k \geq d - 2$, i.e.

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k) \otimes \mathcal{J}(\frac{m}{BD} - 1)Z) = 0$$

for all $k \geq (d - 2)$.

**Proof.** Let $\Gamma$ be a reduced divisor of degree $\gamma$ vanishing along $Z$ with no common components with $D$. For the $\mathbb{Q}$–divisor

$$D' = (1 - \varepsilon)D + m_1 \varepsilon \Gamma,$$

we have

$$\text{ord}_{P_i} D' = (1 - \varepsilon) \text{ord}_{P_i} D + m_1 \varepsilon \text{ord}_{P_i} \Gamma \geq m_i$$

and

$$\text{deg}(D') = (1 - \varepsilon)d + m_1 \varepsilon \gamma < d + 1$$

for $\varepsilon$ sufficiently small. For the line bundle $L = \mathcal{O}_{\mathbb{P}^2}(k + 3)$, the difference $(L - D')$ is big and nef for all $k \geq d - 2$ and we have

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k) \otimes \mathcal{J}(D')) = 0$$

by Nadel vanishing theorem [12, Theorem 9.4.8]. On the other hand, the multiplier ideal $\mathcal{J}(D')$ vanishes on $(\frac{m}{BD} - 1)Z$ because of [12, Proposition 9.3.2] and it is cosupported on a finite set since no component of $D'$ has a coefficient $\geq 1$.

**Theorem 4.14.** Let $Z \subset \mathbb{P}^2$ be a finite set of $r$ points and let $k \geq 5$ be an integer. If $p$ is an integer such that

$$\alpha(Z) = p, \quad \alpha(2Z) = p + 2, \quad \ldots, \quad \alpha(kZ) = p + 2(k - 1),$$

then $p = 2$, i.e. the points in $Z$ are contained in a single conic.

**Proof.** If $r \leq 5$, then there is nothing to prove. We postpone the case $r = 6$ until Lemma 4.15. So we assume from now on

$$r \geq 7. \quad (8)$$

It follows from the proof below that in this case one can relax the assumption on $k$ to $k \geq 4$.

Let $C_i$ denote a divisor of minimal degree vanishing along $iZ$ for $i = 1, \ldots, k$. From [3, Proposition 4.4] we know that for $C_2$ there are the following possibilities:
a) $C_2$ is a double conic and $p + 2 = 4$, or

b) $C_2$ contains a double line $2L$ and the set $Z' = Z \setminus L$ is a non-empty configuration of all intersection points of $p$ general lines $L_1, \ldots, L_p$, or

c) $C_2$ is reduced.

In case a) we are immediately done.

In case b) we observe first that there are $s \geq p$ points from $Z$ on the line $L$. Indeed, as there is no curve of degree $(p - 1)$ containing $Z$, it must be

$$0 \geq \binom{p+1}{2} - \binom{p}{2} - s = p - s.$$

Next, let $D = C_4$ be a divisor of degree $p + 6$ vanishing along $4Z$. We have

$$p + 6 = D \cdot L \geq 4 \cdot s \geq 4 \cdot p,$$

which implies either that $p = 2$ in which case we are done, or that $L$ is a component of $D$. Similarly, step by step, we show that $L$ is a component of $(D - L)$ and that $(D - 2L)$ contains all lines from the arrangement. Thus

$$D - 2L - L_1 - \cdots - L_p$$

is a divisor of degree 4 vanishing along $2Z$. It follows again, that $p = 2$ and we are done.

For the rest of the proof we assume that $C_2$ is a reduced divisor.

Then Lemma 4.13 implies that the points $P_1, \ldots, P_r$ impose independent conditions on curves of degree $p$. Hence at most $(p+1)$ of them lie on a line and we have the following bounds on their number

$$\binom{p+1}{2} \leq r \leq \binom{p+2}{2} - 1. \quad (9)$$

Case $C_3$ reduced.

If $C_3$ is reduced, we derive from Lemma 4.13 and the existence of $C_2$ that

$$\binom{p+4}{2} - 3r \geq 1, \quad (10)$$

which, taking into account (9), implies $p \leq 3$. If $p \leq 2$, we are done. If $p = 3$, then (10) implies $r \leq 6$, which contradicts (9).

Case $C_3$ non-reduced.

Let

$$C_3 = a\Gamma + R$$

be a decomposition of $C_3$ with $a = 2$ or $a = 3$ (note that $a > 3$ is immediately excluded as then one could replace $C_3$ by a lower degree divisor $(a - 1)\Gamma + R$). We assume that $\Gamma$ and $R$ have no common components. We write $Z' = Z \setminus \Gamma$ and $Z'' = Z \setminus Z'$, so that $Z'$ is contained entirely in $R$ and $Z''$ is contained in $\Gamma$. Corollary 4.12 implies that $\Gamma$ is smooth in points of $Z$. Hence $(a - 1)\Gamma + R$ vanishes along $2Z$. This shows that $\Gamma$ is either a line or a (possibly singular) conic.
Subcase $C_3$ contains a triple component, i.e. $a = 3$.

If $R = 0$, then we are done. Otherwise $R$ is a divisor of degree $p + 4 - 3\deg(\Gamma)$ vanishing to order $3$ along $Z'$. Taking derivatives of the equation of $R$, we see that there is another divisor $R''$ of degree $p + 2 - 3\deg(\Gamma)$ vanishing along $Z'$. The union $R'' \cup \Gamma$ has degree $p + 2 - 2\deg(\Gamma)$ and vanishes along $Z$. This shows that $\deg(\Gamma) = 1$ in this case. Then the divisor $R$ has degree $p + 1$ and vanishes to order $3$ on $Z'$. Since there is no divisor of degree $\leq p - 2$ vanishing on $Z'$ (otherwise its union with $\Gamma$ would have degree $p - 1$ and would vanish on $Z$), we have

$$\alpha(Z') = p - 1, \quad \alpha(2Z') = p, \quad \alpha(3Z') = p + 1.$$  

Theorem 3.1 implies that $Z'$ is contained in a line, hence $Z$ is contained in a conic.

Subcase $C_3$ contains a double component, i.e. $a = 2$.

We study now the cases $\deg(\Gamma) = 1$ and $\deg(\Gamma) = 2$ separately.

**Subsubcase $\Gamma$ is a conic.**

In this case we have $\deg(R) = p$ and $R$ has multiplicity $3$ along $Z'$. Taking twice a derivative of the equation of $R$, we see that there exists a curve of degree $p - 2$ vanishing along $Z'$. On the other hand, there cannot exist a curve of lower degree, as its union with $\Gamma$ would be of degree less than $p$ and would vanish along $Z$. Thus Corollary 3.5 implies that $Z'$ consists of collinear points. Let $L$ be a line containing $Z'$. Then $\Gamma + L$ vanishes along $Z$. Hence $p = 3$.

The divisor $R$ has degree $3$, vanishes with multiplicity $3$ along $Z'$ and it vanishes along $Z''$. If there are at least $2$ points in the set $Z'$, then $R = 3L$ and it cannot vanish along $Z''$ (if $Z''$ is empty, then $Z = Z'$ is contained in a line and we are done). Hence $Z'$ is a single point. There must be at least $6$ points in $Z''$, as there are at least $7$ points altogether. On the other hand, the divisor $R$ consists of $3$ lines $L_1, L_2, L_3$ whose union vanishes in all points in $Z''$. This shows that there are at most $6$ points in $Z'' \subset \Gamma$.

So there are exactly $6$ points from $Z$ on the conic $\Gamma$, the intersection points of $\Gamma$ with the lines $L_1, L_2, L_3$. We show that then $\alpha(4Z) \geq 10$, contradicting the assumptions of the theorem. Suppose to the contrary that there exists a divisor $D$ of degree $9$ vanishing along $4Z$. Then, by the standard Bezout argument, we show step by step that $D$ must split off the curves $2\Gamma, L_1, L_2$ and $L_3$. The residual divisor has than degree $2$ and it vanishes in all points of $Z$, a contradiction.

**Subsubcase $\Gamma$ is a line.**

In this situation we have $\deg(R) = p + 2$ and $R$ has multiplicities $3$ in points of the set $Z'$. Note that there is no divisor of degree less or equal $p - 2$ vanishing along $Z'$, since its union with $\Gamma$ would be of degree $p - 1$ and would vanish along $Z$. This gives the following possibilities for $\alpha$’s of $Z'$:

| Case | $\alpha(Z')$ | $\alpha(2Z')$ | $\alpha(3Z')$ |
|------|--------------|----------------|----------------|
| A)   | $p$          | $p + 1$        | $p + 2$        |
| B)   | $p - 1$      | $p$            | $p + 1$        |
| C)   | $p - 1$      | $p$            | $p + 2$        |
| D)   | $p - 1$      | $p + 1$        | $p + 2$        |

In cases A) and B), Theorem 3.1 implies that $Z'$ is contained in a line, hence $Z$ is contained in a union of two lines and we are done.

In case C), the main result in [3] implies that either the points in $Z'$ are all collinear and we are done, or that $Z'$ consists of all intersection points of a general
arrangement of \( p \) lines \( L_1, \ldots, L_p \). In the later case, there are exactly \( \binom{p}{2} \) points in \( Z' \). Hence by \( (9) \) there are at least

\[
\left( \frac{p+1}{2} \right) - \left( \frac{p}{2} \right) = p
\]

points in \( Z'' \). The divisor \( D = \Gamma + R \) is reduced and has multiplicities 2 on \( Z'' \) and multiplicities 3 on \( Z' \). Lemma 4.13 implies that \( Z'' + 2Z' \) imposes independent conditions on curves of degree \( (p+1) \). Since

\[\Gamma \cup L_1 \cup \cdots \cup L_p\]

is a divisor of degree \( p + 1 \) vanishing on \( Z'' + 2Z' \), it must be

\[
\left( \frac{p+3}{2} \right) - p - 3\left( \frac{p}{2} \right) \geq 1.
\]

This is possible only if \( p = 3 \), so that there are exactly 3 points in \( Z' \) and at least 4 points on \( \Gamma \). Let \( L_1, L_2, L_3 \) be the lines determined by the pairs of points in \( Z' \). Consider \( C_4 \) of degree 9 passing through \( 4Z \). It follows (by repeated use of Bezout) that \( C_4 = 3\Gamma + 2L_1 + 2L_2 + 2L_3 \), but it is impossible, since then at most three points on \( \Gamma \) can attain the multiplicity 4.

In case D) \( R \) splits into \( (p+2) \) lines by Theorem 3.4. None of these lines is a double line, as this case was already covered by (b2). This arrangement of lines has triple points in \( Z' \) and vanishes in \( Z'' \). It follows that there are at most \( (p+2) \) points in \( Z'' \) (intersection points of \( \Gamma \) with the arrangement lines) and at most \( \frac{1}{3}\binom{p+2}{2} \) points in \( Z' \) (we use here a very rough estimate on the number of possible triple points in a configuration of lines). Taking into account (9), we must have

\[
p + 2 + \frac{1}{3}\left( \frac{p+2}{2} \right) \geq \left( \frac{p+1}{2} \right).
\]

This is possible only for \( p = 3 \) or \( p = 4 \). If \( p = 3 \), then there are at most 2 points in \( Z' \), hence \( \alpha(Z') = 1 = p - 2 \), a contradiction. Similarly, if \( p = 4 \), then there are at most 3 points in the set \( Z' \) and \( \alpha(Z') = 2 = p - 2 \) again.

Now we turn to the case of 6 points. This Lemma shows in particular that the assumptions in Theorem 4.14 are sharp.

**Lemma 4.15.** If \( Z \) consists of 6 points and

\[
\alpha(Z) = p, \ \alpha(2Z) = p + 2, \ \alpha(3Z) = p + 4, \ \alpha(4Z) = p + 6,
\]

then either \( p = 2 \) and \( Z \) lies on a conic or \( p = 3 \) and \( Z \) is a configuration of intersection points \( A, B, C \) of three general lines, and additional points \( D, E, F \), each of those lying on exactly one of the lines, one for each line. This is indicated on the picture below.

\[\text{Diagram of lines and points, similar to previous diagrams.}\]
If moreover
\[ \alpha(5Z) = p + 8, \]
then \( Z \) is contained in a conic, i.e. \( p = 2 \).

**Proof.** Since 6 points always lie on a cubic, we have \( p \leq 3 \) and the only interesting case is \( p = 3 \).

We say that two sets \( Z = \{P_1, \ldots, P_6\}, Z' = \{P'_1, \ldots, P'_6\} \), of points in \( \mathbb{P}^2 \) are equivalent if they have the same Hilbert functions for any set of multiplicities, i.e.
\[
\dim(I(m_1P_1 + \ldots + m_6P_6))_t = \dim(I(m'_1P'_1 + \ldots + m'_6P'_6))_t
\]
for all non-negative integers \( t, m_1, \ldots, m_6 \). An equivalence class is called a type.

Obviously, equivalent sets of points have the same initial degree for any multiplicity sequence.

From \[9\] we know that for 6 points there are exactly 11 types. Moreover, there are algorithms to find Hilbert functions for a given type and multiplicities \[10\]. Using them we verify that in order to have \( \alpha(4Z) = 9 \), the only possibility is to have \( Z \) of type 9 in \[10\], which is equivalent to the configuration described in the Lemma.

The picture below gives examples of divisors vanishing along 2\( Z \), 3\( Z \) and 4\( Z \).

![Examples of divisors vanishing along 2Z, 3Z, and 4Z](image)

Further computations using the above mentioned algorithms show that for this type \( \alpha(5Z) = 12 \). This concludes the proof of the Lemma and thus also the proof of Theorem 4.14. \( \square \)

**Remark 4.16.** Note that one could prove Lemma 4.15 along the lines of Example 4.8 without calling to a computer aided argument. However, this would take several pages of dull computations which we preferred to avoid.

Now we show that one cannot expect a result along lines of Theorem stated in the Introduction for \( d \geq 4 \).

**Example 4.17.** Let \( Z \) be a set of 16 very general points in the projective plane. Then \[14\] Proposition implies that for all \( k \geq 1 \) there is no divisor of degree \( 4k \) vanishing to order \( k \) along \( kZ \). On the other hand, an elementary conditions counting shows that there is always a divisor of degree \( 4k + 1 \) vanishing along \( kZ \). These two facts together imply that
\[ \alpha_{k+1,k} = 4 \]
for all \( k \geq 1 \) but \( Z \) is not contained in a curve of degree 4 (since \( \alpha(Z) = 5 \)).

With a little more care the same idea can be used to produce similar examples for arbitrary \( d \geq 4 \).
Acknowledgement. We would like to thank Thomas Bauer, Joaquim Roé and Stefan Müller-Stach for helpful remarks and comments. Parts of this paper were written while the second author was a visiting professor at the University Mainz as a member of the program Schwerpunkt Polen. The nice working conditions and financial support are kindly acknowledged.

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