Remarks on additivity of the Holevo channel capacity and of the entanglement of formation

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(16th May 2003)

Abstract
The purpose of this article is to discuss the relation between the additivity questions regarding the quantities (Holevo) capacity of a quantum channel $T$ and entanglement of formation of a bipartite state $\rho$. In particular, using the Stinespring dilation theorem, we give a formula for the channel capacity involving entanglement of formation. This can be used to show that additivity of the latter for some states can be inferred from the additivity of capacity for certain channels.

We demonstrate this connection for some families of channels, allowing us to calculate the entanglement cost for many states, including some where a strictly smaller upper bound on the distillable entanglement is known. Group symmetry is used for more sophisticated analysis, giving formulas valid for a class of channels. This is presented in a general framework, extending recent findings of Vidal, Dür and Cirac.

We also discuss the property of superadditivity of the entanglement of formation, which would imply both the general additivity of this function under tensor products and of the Holevo capacity (with or without linear cost constraints).

1 Introduction
Quantum information theory has progressed considerably over the last decade: today we understand much better the information transmission properties of quantum channels, and entanglement has turned from an oddity first into a valuable effect and then into a quantifiable resource, as shown by the many

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well-motivated entanglement measures that have been put forward. Almost all of them are operationally grounded as some optimal performance parameter, and can be written as solutions to various high-dimensional or even asymptotic optimisation problems.

All of these capacities and entanglement measures raise the natural problem of *additivity* under tensor products, i.e. the question, if the independent supply of two specimens of the resource has as its performance the sum of the performances of the individual objects (be they channels or states). For some of the current measures of entanglement additivity has been disproved by counterexamples (for the so-called *relative entropy of entanglement* in [36]), for others, like the distillable entanglement [7] it is claimed improbable [28]. For some, however, additivity is still widely conjectured, most notably for a bound on the distillable entanglement by Rains [24], and for the *entanglement of formation* [7].

The literature on the subject is vast and increasing rapidly, and in the present paper we will only make a small contribution. We shall be concerned with the entanglement of formation, and with the aforementioned classical capacity of quantum channels, pointing out a connection between the two that also relates their additivity problems.

We outline briefly the content of the rest of the paper: in sections 2 and 3 the classical capacity of a channel and the entanglement of formation of a state are reviewed. In section 4 a simple observation on the Stinespring dilation of a completely positive map provides the link between the two quantities, which is exploited in a number of examples in section 6; group symmetry is introduced in section 7 adding another example, and to supply formulas valid for a class of channels which includes examples discussed in section 4 as special cases. And in section 8 some of these results are used to demonstrate a gap between entanglement cost and distillable entanglement.

In section 5 we discuss *superadditivity* of entanglement of formation as a (conjectured) property which would unify the additivity questions considered here: it implies additivity of entanglement of formation, of channel capacity, and of channel capacity with a linear cost constraint. We conclude with a discussion of our observations and related works.

## 2 Holevo capacity

We consider block coding of classical information via the quantum channel

\[ T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}_2), \]

where \( \mathcal{H} \) and \( \mathcal{H}_2 \) are Hilbert spaces. If the encoding is restricted to product states it is known [17, 26] that the capacity is given by

\[ C(T) = \sup \{ I(p; T(\pi)) : \{ p_i, \pi_i \} \text{ pure state ensemble on } \mathcal{H} \}, \]

where the *Holevo mutual information* of an ensemble \( \{ p_i, \rho_i \} \) is given by

\[ I(p; \rho) = S \left( \sum_i p_i \rho_i \right) - \sum_i p_i S(\rho_i). \]
Here $S(\omega) = -\text{Tr}\omega \log \omega$ is the von Neumann entropy of a state. For finite dimensional $\mathcal{H}_2$ the sup in eq. (1) is indeed a max, attained for an ensemble of at most $(\dim \mathcal{H}_2)^2$ states.

It is conjectured that for a product of channels making use of entangled input states does not help to increase the capacity:

$$C(T_1 \otimes T_2) = C(T_1) + C(T_2).$$  \hspace{1cm} (2)

(The question is implicit in [16] and the above references, and made explicit in [8], where it was speculated that the answer may be negative.)

This would imply that for a product of channels making use of entangled input states does not help to increase the capacity:

$$C(T_1 \otimes T_2) = C(T_1) + C(T_2).$$  \hspace{1cm} (2)

(3)

(4)

Observe that, as in the case of the Holevo capacity, ”$\leq$” follows easily from the fact that the right hand side is achieved by product state ensembles. If this would turn out to be true, the entanglement cost $E_c(\rho)$ of $\rho$, i.e. the asymptotic rate of EPR pairs to approximately create $n$ copies of $\rho$ is given by $E_f(\rho)$: in [13] it was proved rigorously that

$$E_c(\rho) = \lim_{n \to \infty} \frac{1}{n} E_f(\rho^\otimes n).$$

3 Entanglement of formation

Let $\rho$ be a state on $\mathcal{H}_1 \otimes \mathcal{H}_2$. The entanglement of formation of $\rho$ is defined as

$$E_f(\rho) := \inf \left\{ \sum_i p_i E(\pi) : \{p_i, \pi_i\} \text{ pure state ens. with } \sum_i p_i \pi_i = \rho \right\},$$  \hspace{1cm} (3)

where the (entropy of) entanglement for a pure state $\pi$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined as

$$E(\pi) := S(\text{Tr}_{\mathcal{H}_2} \pi) = S(\text{Tr}_{\mathcal{H}_1} \pi).$$

If the rank of $\rho$ is finite the inf is in fact a min, achieved for an ensemble of at most $(\text{rank } \rho)^2$ elements.

This quantity was proposed in [7] as a measure of how costly in terms of entanglement the creation of $\rho$ is.

It is conjectured (but only in a few cases proved: the only published examples are in [33]) that $E_f$ is an additive function with respect to tensor products:

$$E_f(\rho_1 \otimes \rho_2) = E_f(\rho_1) + E_f(\rho_2).$$  \hspace{1cm} (4)

Observe that, as in the case of the Holevo capacity, ”$\leq$” follows easily from the fact that the right hand side is achieved by product state ensembles. If this would turn out to be true, the entanglement cost $E_c(\rho)$ of $\rho$, i.e. the asymptotic rate of EPR pairs to approximately create $n$ copies of $\rho$ is given by $E_f(\rho)$: in [13] it was proved rigorously that

$$E_c(\rho) = \lim_{n \to \infty} \frac{1}{n} E_f(\rho^\otimes n).$$
Note that the function $E_f$ has the property of being a convex roof:

$$E_f(\rho) = \inf \left\{ \sum_i p_i E_f(\rho_i) \colon \{p_i, \rho_i\} \text{ ensemble with } \sum_i p_i \rho_i = \rho \right\}. \quad (5)$$

The cases in which $E_f$ is known are arbitrary states of $2 \times 2$–systems [41], isotropic states in arbitrary dimension [31], Werner and OO–symmetric states [36], and some other highly symmetric states [33].

4 Stinespring dilations: linking $C(T)$ and $E_f(\rho)$

Due to a theorem of Stinespring [30] the completely positive and trace preserving map $T$ can be presented as the composition of an isometric embedding of $\mathcal{H}$ into a bipartite system with a partial trace:

$$T : \mathcal{B}(\mathcal{H}) \xrightarrow{U} \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \xrightarrow{\text{Tr}_{\mathcal{H}_1}} \mathcal{B}(\mathcal{H}_2). \quad (6)$$

See [25] for a discussion on how to construct this from the so–called Kraus (operator sum) representation [23], $T(\rho) = \sum_i A_i \rho A_i^*$ with $\sum_i A_i^* A_i = 1$, of $T$. We shall use this construction later on in the examples 5 and 6.

By embedding into larger spaces we can present $U$ as restriction of a unitary, which often we silently assume done. Denote $\mathcal{K} := U\mathcal{H} \subset \mathcal{H}_1 \otimes \mathcal{H}_2$, the image subspace of $U$. Then we can say that $T$ is equivalent to the partial trace channel, with inputs restricted to states on $\mathcal{K}$. This entails:

**Theorem 1**

$$C(T) = \sup \{ S(\text{Tr}_{\mathcal{H}_1} \rho) - E_f(\rho) : \rho \text{ state on } \mathcal{K} \}. \quad (7)$$

**Proof.** Very simple: choosing an input ensemble for $T$ amounts by our above observation to choosing an ensemble $\{p_i, \pi_i\}$ on $\mathcal{K}$. Denoting $\rho = \sum_i p_i \pi_i$, the average output state of $T$ in eq. (1) is just $\text{Tr}_{\mathcal{H}_1} \rho$, while the individual output states are the $\text{Tr}_{\mathcal{H}_1} \pi_i$. Hence the second term in eq. (1), the average of output entropies, has as its infimum $E_f(\rho)$ when we vary over ensembles with fixed $\rho$.

Note that if we choose the dimension of $\mathcal{H}_1$ large enough, every channel from $\mathcal{H}$ to $\mathcal{H}_2$ corresponds to a subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$ (though not uniquely) and vice versa.

**Remark 2** The quantity $S(\text{Tr}_{\mathcal{H}_1} \rho) - E_f(\rho)$ in the optimisation problem in theorem 1 equals the entropy of the subalgebra $\mathcal{B}(\mathcal{H}_2)$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, as defined by Connes, Narnhofer and Thirring [11]: this was observed by Benatti, Narnhofer and Uhlmann [6].

This has interesting consequences: for each subspace $\mathcal{K}$ of the tensor product there is a convex set $\mathcal{O}_T$ of states $\rho$ supported on it which maximise eq. (7).
The reason for convexity is again very simple: let $\rho, \rho' \in S(\mathcal{K})$. Then
\[
S(p\text{Tr}_{\mathcal{H}_1}\rho + (1 - p)\text{Tr}_{\mathcal{H}_1}\rho') \geq pS(\text{Tr}_{\mathcal{H}_1}\rho) + (1 - p)S(\text{Tr}_{\mathcal{H}_1}\rho'),
\]
\[
E_f(p\rho + (1 - p)\rho') \leq pE_f(\rho) + (1 - p)E_f(\rho'),
\]
by concavity (convexity) of $S(E_f)$. Hence the aim function in eq. (7) is concave, which implies that the set of $\rho$ for which it is at least $R$ is a convex set, for any real $R$.

Observe that by this argument both $S(\text{Tr}_{\mathcal{H}_1}\rho)$ and $E_f(\rho)$ are constants for $\rho \in \mathcal{O}_T$. Indeed, one can show (see the discussion below, in this section) that even all $\text{Tr}_{\mathcal{H}_1}\rho, \rho \in \mathcal{O}_T$, are identical.

For such states the additivity of $E_f$ is implied by the additivity of $C$ for the corresponding channels: indeed, assume that for two channels $T, T'$ that optimal input states in the sense of eq. (7) are $\rho \in \mathcal{O}_T, \rho' \in \mathcal{O}_{T'}$, respectively, with reduced states $\rho_2$ and $\rho'_2$. Then, assuming additivity we get
\[
S(\rho_2) - E_f(\rho) + S(\rho'_2) - E_f(\rho') = C(T) + C(T')
\]
\[
= C(T \otimes T')
\]
\[
\geq S(\rho_2 \otimes \rho'_2) - E_f(\rho \otimes \rho'),
\]
which by our earlier remarks implies additivity. Thus we have proved

**Theorem 3** If for any two channels $T$ and $T'$, each with a Stinespring dilation chosen as in eq. (6), $C(T \otimes T') = C(T) + C(T')$, then
\[
\forall \rho \in \mathcal{O}_T, \rho' \in \mathcal{O}_{T'} \quad E_f(\rho \otimes \rho') = E_f(\rho) + E_f(\rho').
\]

Most interesting is the case when we know $C(T^\otimes n) = nC(T)$, because then we can conclude $E_f(\rho^\otimes n) = nE_f(\rho)$, thus determining the entanglement cost of $\rho$ (see section 5). For example, King [19, 20] proved this for unital qubit–channels, Shor [29] for entanglement–breaking channels, and King [21] for arbitrary depolarising channels, giving rise to a host of states for which we thus know that the entanglement cost equals $E_f$. Examples are discussed in section 6 below and the following two sections.

It is natural to consider ways to implement an implication of additivity going the other way than theorem 3 from entanglement of formation to Holevo capacity.

Indeed, in another look at eq. (7), let us focus on the other quantity of interest in the optimisation: this is the von Neumann entropy of the output state. In general, while there can be many ensembles maximising eq. (1) (let us assume for the moment that the output space is finite dimensional), and in fact many averages $\sum_i p_i \pi_i$ (the set $\mathcal{O}_T$ of optimal input states introduced
above), the average output state of such an optimal ensemble, $\omega = \sum_i p_i T(\pi_i)$, is unique: the reason is the strict concavity of the von Neumann entropy, so if we had two ensembles with different output states, mixing the ensembles would strictly increase the Holevo mutual information. Let us denote this optimal output state $\omega(T)$.

It is clear that the additivity conjecture eq. (2) implies that

$$\omega(T \otimes T') = \omega(T) \otimes \omega(T'),$$

but the reverse seems not obvious. Still, eq. (9) might be a reasonable first step towards proving additivity of $C(T)$ in general.

Unfortunately, even assuming additivity of the entanglement of formation, we have not been able to derive additivity of the channel capacity from eq. (9).

However, let us assume that for the product channel $T \otimes T'$ an optimal input state in eq. (7) is a product (due to the non–uniqueness of optimal input states there might also be entangled ones!), $\rho \otimes \rho'$, say. Then clearly, $E_f(\rho \otimes \rho') = E_f(\rho) + E_f(\rho')$ implies $C(T \otimes T') = C(T) + C(T')$, in a reversal of the argument from the proof of theorem 3.

5 Superadditivity: unifying $C(T)$ and $E_f(\rho)$

Looking at eq. (7), and trying to find a unifying reason why both of the above discussed additivity conjectures should hold, we are led to speculate that $E_f$ might not only be additive with respect to tensor products (eq. (4)), but have even a superadditivity property for arbitrary states on a composition of two bipartite systems:

Let $\rho$ be a state on $\mathcal{H} \otimes \mathcal{H}'$, where $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}' = \mathcal{H}_1' \otimes \mathcal{H}_2'$. Then superadditivity means that

$$E_f(\rho) \geq E_f(\text{Tr}_{\mathcal{H}'} \rho) + E_f(\text{Tr}_{\mathcal{H}} \rho),$$

where all entanglements of formation are understood with respect to the 1–2–partition of the respective system. (This relation was apparently first considered in [36], and called strong superadditivity there. We call it just “superadditivity” here in simple analogy to, e.g., subadditivity of the von Neumann entropy.)

Note that this implies additivity of $E_f$ when applied to $\rho_1 \otimes \rho_2$ since we remarked in section 3 that the other inequality is trivial.

Note on the other hand that it also implies additivity of $C(T)$ by eq. (7) in section 4 by replacing a supposedly optimal $\rho$ on $\mathcal{K} \otimes \mathcal{K}'$ (for two channels $T$ and $T'$, and corresponding Stinespring dilations which give rise to the subspaces $\mathcal{K}$ and $\mathcal{K}'$ in respective bipartite systems) by the tensor product of its marginals, we can only increase the entropy (subadditivity), and only decrease the entanglement of formation (superadditivity).

As an extension, let us show that it even implies an additivity formula for the classical capacity under linear cost constraints (see [18]): in this problem, there is given a selfadjoint operator $A$ on the input system, and a real number
α, additional to the channel \( T \). As signal states we allow only such states \( \sigma \) on \( \mathcal{H}^\otimes n \) for which \( \text{Tr}(\sigma \hat{A}) \leq n\alpha + o(n) \), with
\[
\hat{A} = \sum_{k=1}^{n} \mathbb{1}^{(k-1)} \otimes A \otimes \mathbb{1}^{(n-k)}.
\]
(i.e., their average cost is asymptotically bounded by \( \alpha \)). Then it can be shown \([18, 39]\) that the capacity \( C(T; A, \alpha) \) in the thus constrained system and using product states is given by a maximisation as in eq. (1), only that the ensembles \( \{p_i, \pi_i\} \) are restricted by \( \sum_i p_i \text{Tr}(\pi_i A) \leq \alpha \). (The same treatment applies if there are several linear cost inequalities of this kind. It is only for simplicity of notation that we stick to the case of a single one.) Because of the linearity of this condition in the states this yields a formula for \( C(T; A, \alpha) \) very similar to theorem 1:
\[
C(T; A, \alpha) = \sup \{ S(\text{Tr}_j \rho) - E_f(\rho) : \rho \text{ state on } \mathcal{K}, \text{Tr}(\rho A) \leq \alpha \}.
\]
By the general arguments given in previous sections we can conclude that this function is concave in \( \alpha \). The question of course is again, if entangled inputs help to increase the capacity, or if
\[
C(T^\otimes n; \hat{A}, n\alpha) \geq nC(T; A, \alpha).
\]
We shall show that this indeed follows from the superadditivity, by showing the following: for channels \( T, T' \), cost operators \( A, A' \), and cost threshold \( \tilde{\alpha} \):
\[
C(T \otimes T'; A \otimes \mathbb{1} + \mathbb{1} \otimes A'; \tilde{\alpha}) = \sup_{\alpha + \alpha' = \tilde{\alpha}} \{ C(T; A, \alpha) + C(T'; A', \alpha') \}.
\]
(Then, by induction and using the concavity, the equality in eq. (12) follows.) Indeed, “\( \geq \)” is obvious by choosing, for \( \alpha + \alpha' = \tilde{\alpha} \), optimal states \( \rho, \rho' \) in the sense of eq. (11), and considering \( \rho \otimes \rho' \). In the other direction, assume any optimal \( \omega \) for the product system, with marginal states \( \rho, \rho' \); by definition,
\[
\text{Tr}(\rho \otimes \rho')(A \otimes \mathbb{1} + \mathbb{1} \otimes A') = \text{Tr}(\omega(A \otimes \mathbb{1} + \mathbb{1} \otimes A')) \leq \tilde{\alpha},
\]
so also the product \( \rho \otimes \rho' \) is admissible, and since there exist \( \alpha, \alpha' \) summing to \( \tilde{\alpha} \) such that \( \text{Tr}(\rho A) \leq \alpha, \text{Tr}(\rho' A') \leq \alpha' \), the claim follows in exactly the same way as for the unconstrained capacity.

We have thus proved:

**Theorem 4** Superadditivity of \( E_f \), eq. (10), implies additivity of entanglement of formation, of the Holevo capacity and of the Holevo capacity with cost constraint under tensor products.

Observe the strong intuitive appeal of the superadditivity property: it says that by measuring the entanglement via \( E_f \), a system can only appear less entangled if judged by looking at its subsystems individually. Note that this
is almost trivially true (by definition) for the **distillable entanglement**, while wrong for the **relative entropy of entanglement** \(^3\) because this would make it an additive quantity, which we know it isn’t \(^3\). The superadditivity also bears semblance to a distributional property of the so-called **tangle** \(^1\).

Superadditivity is thus a very strong property. If there is one “nice” underlying mathematical structure to the additivity of \(E_f\), it should indeed be this. Note that it is true if one of the marginal states, say \(\text{Tr}_H'\), is separable: because then its \(E_f\) is 0, and eq. (10) simply expresses the monotonicity of \(E_f\) under local operations (in this case: partial traces). This was previously noted in \(^3\).

Observe that it is sufficient to prove superadditivity for a pure state \(\rho = |\psi\rangle \langle \psi|\), as then we can apply it to an optimal decomposition of \(\rho\), together with the convex roof property, eq. (4). This was apparently considered by Benatti and Narnhofer \(^5\), who even conjectured “good decompositions” of the reduced states \(\text{Tr}_H|\psi\rangle \langle \psi|\) and \(\text{Tr}_H'|\psi\rangle \langle \psi|\). This latter conjecture however was refuted by Vollbrecht and Werner \(^35\) who constructed a counterexample.

On the other hand, there is limited positive evidence in favour of superadditivity: In \(^35\), eq. (16), it is actually proved if the partial trace in one of the subsystems is entanglement-breaking. We observed (following \(^36\)) that it is trivially true if one of the reduced states is separable. Some of our examples yield more cases of superadditivity. E.g. in example \(6\) we constructed the subspaces \(K_\lambda\): for every pure state \(\psi \in K_\lambda \otimes \cdots \otimes K_\lambda\), with reduced density operators \(\rho_1, \ldots, \rho_n\) we get (using the additivity of the minimal output entropy proved in \(^21\))

\[
E(\psi) \geq S_{\text{min}}(T_1) + \ldots + S_{\text{min}}(T_n) = E_f(\rho_1) + \ldots + E_f(\rho_n),
\]

the second line by the insight of example \(6\) that all states supported on \(K_\lambda\) have the same entanglement of formation.

Similarly, our other examples yield certain pure states for which we obtain superadditivity.

It seems to us that this question most elegantly sums up the two most prominent additivity question in quantum information theory, and we would like to pose it as a challenge: either to prove superadditivity (thus proving additivity of \(E_f\) and of \(C\)), or to find a counterexample.

### 6 Examples

In this and the following two sections we want to demonstrate how theorem \(^3\) can be used to construct nontrivial states for which we can compute the entanglement cost, to reproduce some known results of this sort, and even exhibit “irreversibility of entanglement”.

**Example 5** Consider the generalised depolarising channels of qubits:

\[
T : \rho \mapsto \sum_{s=0,x,y,z} p_s \sigma_s \rho \sigma_s^1,
\]
with \( \sigma_0 = \mathbb{1} \), the familiar Pauli matrices
\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
and a probability distribution \( (p_x)_{x=0,x,y,z} \). For these channels additivity of the capacity under tensor product with an arbitrary channel was proved in [20].

Note that up to unitary transformations on input and output system each unital qubit channel has this form, by the classification of qubit maps of King and Ruskai [22], and Fujiwara and Algoet [12]. By this result we also can assume that
\[
p_0 + p_z - p_x - p_y \geq |p_0 + p_y - p_x - p_z|, |p_0 + p_x - p_y - p_z|. \tag{13}
\]

It is easy to see that for such a channel the capacity is given by \( C(T) = 1 - S_{\min}(T) \), with the minimum output entropy achieved at the eigenstates \( |0\rangle, |1\rangle \) of \( \sigma_z \): \( S_{\min}(T) = S(T(|0\rangle\langle 0|)) = S(T(|1\rangle\langle 1|)) \). An optimal ensemble is the uniform distribution on these states.

It is easy to construct a Stinespring dilation for this map, by an isometry \( U : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \otimes \mathbb{C}^4 \), in block form:
\[
U = \begin{pmatrix} \sqrt{p_0} \sigma_0 \\ \sqrt{p_x} \sigma_x \\ \sqrt{p_y} \sigma_y \\ \sqrt{p_z} \sigma_z \end{pmatrix},
\]
and the corresponding subspace \( \mathcal{K} \subset \mathbb{C}^2 \otimes \mathbb{C}^4 \) is spanned by
\[
|\psi_T\rangle = \sqrt{p_0} |0\rangle \otimes |0\rangle + \sqrt{p_x} |1\rangle \otimes |x\rangle + i \sqrt{p_y} |1\rangle \otimes |y\rangle + \sqrt{p_z} |0\rangle \otimes |z\rangle,
\]
\[
|\psi_T^+\rangle = \sqrt{p_0} |1\rangle \otimes |0\rangle + \sqrt{p_x} |0\rangle \otimes |x\rangle - i \sqrt{p_y} |0\rangle \otimes |y\rangle - \sqrt{p_z} |1\rangle \otimes |z\rangle.
\]
The optimal input state corresponds to the equal mixture \( \rho_T \) of these two pure states.

From these observations, together with theorem [8] we obtain that
\[
E_f(\rho_T) = S_{\min}(T) = H(p_0 + p_z, 1 - p_0 - p_z),
\]
and \( E_f(\rho_T \otimes \sigma) = E_f(\rho_T) + E_f(\sigma) \) for any \( \sigma \in \mathcal{O}_T \), with arbitrary channel \( T' \). In particular,
\[
E_c(\rho_T) = E_f(\rho_T) = H(p_0 + p_z, 1 - p_0 - p_z).
\]

In fact, we proved that the decomposition of \( \rho_T^{\otimes n} \) into the \( 2^n \) equally weighted tensor products of \( |\psi_T\rangle\langle \psi_T| \) and \( |\psi_T^+\rangle\langle \psi_T^+| \) is formation–optimal. By the convex roof property of \( E_f \) this implies that any convex combination of these states is a formation–optimal decomposition (this argument was also used in [20] to extend the domain of states with known entanglement of formation). In particular, we can conclude that any mixture \( \rho \) of \( |\psi_T\rangle\langle \psi_T| \) and \( |\psi_T^+\rangle\langle \psi_T^+| \) has
\[
E_c(\rho) = E_f(\rho) = H(p_0 + p_z, 1 - p_0 - p_z). \tag{14}
\]
The case of equal $p_x, p_y, p_z$ leads to the usual unitarily covariant depolarising channel. This is contained in the following:

**Example 6** Consider the $d$–dimensional depolarising channel with parameter $\lambda$:

$$ T : \rho \mapsto \lambda \rho + (1 - \lambda) \frac{1}{d} \mathbb{1}, $$

with $-\frac{1}{d^2-1} \leq \lambda \leq 1$ for complete positivity, to ensure that $T$ can be represented as a mixture of generalised Pauli actions:

$$ T(\rho) = p_0 \rho + (1 - p_0) \sum_{i=1}^{d^2-1} \frac{1}{d^2-1} \sigma_i \rho \sigma_i^\dagger, $$

with an orthogonal set of unitaries (a “nice error basis”, see e.g. [37] for constructions) $\sigma_i$, i.e.

$$ \sigma_0 = \mathbb{1}, \quad \text{Tr}(\sigma_i^\dagger \sigma_j) = d \delta_{ij}, $$

and $p_0 = \lambda + (1 - \lambda)/d^2$.

For this channel, [21] proves the additivity of $C(T)$ and $S_{\text{min}}(T)$, and it is quite obvious that

$$ C(T) = \log d - S_{\text{min}}(T) = \log d - S(T(|\psi\rangle \langle \psi|)), $$

for arbitrary $|\psi\rangle \in \mathbb{C}^d$, optimal input ensembles being those mixing to $\frac{1}{d} \mathbb{1}$. It is easy to evaluate this latter von Neumann entropy:

$$ S(T(|\psi\rangle \langle \psi|)) = H \left( \lambda + \frac{1 - \lambda}{d}, \frac{1 - \lambda}{d}, \ldots, \frac{1 - \lambda}{d} \right) $$

$$ = H \left( \left( 1 - \frac{1}{d} \right)(1 - \lambda), 1 - \left( 1 - \frac{1}{d} \right)(1 - \lambda) \right) + \left( 1 - \frac{1}{d} \right)(1 - \lambda) \log(d - 1). $$

Again, it is easy to construct a Stinespring dilation $U : \mathbb{C}^d \rightarrow \mathbb{C}^d \otimes \mathbb{C}^{d^2}$ in block form:

$$ U = \begin{pmatrix}
\sqrt{p_0} \mathbb{1} \\
\sqrt{\frac{1- p_0}{d^2-1}} \sigma_1 \\
\vdots \\
\sqrt{\frac{1- p_0}{d^2-1}} \sigma_{d^2-1}
\end{pmatrix}, $$

such that the subspace of interest is $K_\lambda := U \mathbb{C}^d$, its maximally mixed state denoted $\rho_\lambda$. Then theorem [3] allows us to conclude that $E_f(\rho_\lambda \otimes \sigma) = E_f(\rho_\lambda) + E_f(\sigma)$ for any $\sigma \in \mathcal{O}_T$. In particular

$$ E_c(\rho_\lambda) = E_f(\rho_\lambda) = S_{\text{min}}(T). $$

By the argument familiar from example 5 we can conclude even that any mixture of product states on $K_\lambda^{\otimes n}$ has entanglement of formation $n S_{\text{min}}(T)$, in particular for every state $\rho$ supported on $K_\lambda$ we obtain

$$ E_c(\rho) = E_f(\rho) = S_{\text{min}}(T). $$

10
In the following section we will study some other examples, involving symmetry, which allows evaluation of the entanglement of formation in some cases, and also the entanglement cost.

7 Group symmetry

Imposing a group symmetry via representation on the involved (sub-)spaces as follows, we obtain another example, such as Vidal, Dür and Cirac [33], and formulas valid for a class of channels. Note that the symmetry is used principally for simplifying computations.

Assume that a compact group $G$ (with Haar measure $dg$) acts irreducibly both on $K$ and $H_2$ by a unitary representation (which we denote by $V_g$ and $U_g$), which commutes with the map $T$ (partial trace):

$$\text{Tr}_{H_1}(V_g \sigma V_g^\dagger) = U_g(\text{Tr}_{H_2} \sigma) U_g^\dagger.$$  \hspace{1cm} (15)

For example let there also be a unitary representation of $G$ on $H_1$, denoted $\tilde{U}_g$, such that $K$ is an irreducible subspace of the representation $V_g = \tilde{U}_g \otimes U_g$. We call this the Product Case.

In the general, non–product case of eq. (15), it is an easy exercise to show that, with $P$ denoting the projection onto $K$ in $H_1 \otimes H_2$,

$$C(T) = \log \text{dim} H_2 - E_f \left( \frac{1}{\text{Tr} P} P \right),$$  \hspace{1cm} (16)

$$E_f \left( \frac{1}{\text{Tr} P} P \right) = \min \{ E(\psi) : |\psi\rangle \in K \}.$$  \hspace{1cm} (17)

Indeed, in the second equation, “$\geq$” is trivially true, and for the opposite direction choose a minimum entanglement pure state $|\psi_0\rangle \in K$, and consider the decomposition $\{ V_g |\psi_0\rangle |\psi_0\rangle V_g^\dagger : g \in G \}$ of $(\frac{1}{\text{Tr} P} P)$ (by Schur’s lemma!): all these states $V_g |\psi_0\rangle |\psi_0\rangle V_g^\dagger$ have the same entanglement,

$$E(V_g |\psi_0\rangle) = S(\text{Tr}_1 (V_g |\psi_0\rangle |\psi_0\rangle V_g^\dagger))$$

$$= S(U_g \text{Tr}_1 |\psi_0\rangle |\psi_0\rangle U_g^\dagger)$$

$$= S(\text{Tr}_1 |\psi_0\rangle |\psi_0\rangle) = E(\psi_0),$$  \hspace{1cm} (18)

using eq. (15). As for the capacity, in the light of eq. (7) and using eq. (17), the “$\leq$” is trivial, and the argument just given proves equality.

Moreover, for all states $\rho$ spanned by $\{ V_g |\psi_0\rangle |\psi_0\rangle V_g^\dagger : g \in G \}$, where $|\psi_0\rangle$ is a pure state with $E(|\psi_0\rangle) = \min \{ E(|\psi\rangle) : |\psi\rangle \in K \}$, we can conclude that

$$E_f(\rho) = \min \{ E(\psi) : |\psi\rangle \in K \}.$$
We even obtain the entanglement cost of all the $\rho$ spanned by $\{V_g \rho_0 V_g^* : g \in G\}$, in the cases where we know that $E_c(\rho) = E_f(\rho)$. consider the chain of inequalities

$$E_f\left(\left(\frac{P}{\text{Tr}P}\right)^\otimes n\right) \leq \int d^n g E_f(V_{g_1} \otimes \cdots \otimes V_{g_n} \rho \otimes^n V_{g_1}^\dagger \otimes \cdots \otimes V_{g_n}^\dagger)$$

$$= \int d^n g \sum_{k_i} E_f(V_{g_k} \rho V_{g_k}^\dagger)$$

$$= nE_f\left(\left(\frac{P}{\text{Tr}P}\right)^\otimes n\right).$$

Here the first inequality is due to the convexity (see the definition) of $E_f$, applied to the family $V_g \rho V_g^\dagger$ with Haar measure, and the others are by subadditivity of $E_f$ and the assumption. But the right hand side in the first line equals $E_f(\rho \otimes^n)$, since any decomposition of $\rho \otimes^n$ translates into a decomposition of $V_{g_1} \otimes \cdots \otimes V_{g_n} \rho \otimes^n V_{g_1}^\dagger \otimes \cdots \otimes V_{g_n}^\dagger$ of the same entanglement, and vice versa.

Hence

$$E_c(\rho) = E_f(\rho) = \min\{E(\psi) : |\psi\rangle \in \mathcal{K}\}. \quad (19)$$

(Note that in [33] this was argued by making use of being in the “product case”, in which case the group action on $\mathcal{K}$ is performable by LOCC; then the first inequality above was argued by nonincrease of $E_f$ under LOCC transformations.)

In particular, if in addition the action of $G$ in $\mathcal{K}$ is transitive, we can conclude (19) for all the state supported on $\mathcal{K}$, because (18) implies that $E(|\psi\rangle)$ takes the same value for any pure state $|\psi\rangle$ in $\mathcal{K}$.

This group symmetry argument simplifies the analysis of unital qubit channels and generalised depolarising channels. In the former case, $G$ is chosen to be $SU(d)$, while in the latter, we consider the group $G = \{1, R, R^2, R^3\}$, with

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

In both cases, we define representations $V_g, U_g$ of $G$ by $V_g = U_g U^*$ and $U_g = g$. They are irreducible, and satisfy the condition eq. (15). Hence, general arguments in this section, directly implies results about these examples in the previous section.

The following example is constructed using group symmetry.

**Example 7** Vidal, Dür and Cirac [33] consider the subspace $\mathcal{K}$ of $\mathbb{C}^3 \otimes \mathbb{C}^6$.
spanned by
\begin{align*}
|0\rangle_s &= \frac{1}{2} (|1\rangle|2\rangle + |2\rangle|1\rangle + \sqrt{2} |0\rangle|3\rangle), \\
|1\rangle_s &= \frac{1}{2} (|2\rangle|0\rangle + |0\rangle|2\rangle + \sqrt{2} |1\rangle|4\rangle), \\
|2\rangle_s &= \frac{1}{2} (|0\rangle|1\rangle + |1\rangle|0\rangle + \sqrt{2} |2\rangle|5\rangle).
\end{align*}
By using the isomorphism $|j\rangle \leftrightarrow |j\rangle_s$ between $\mathbb{C}^3$ and $\mathcal{K}$, it is easily checked that $\text{Tr}_{\mathcal{C}^s}$ implements the channel map

$$T : \rho \mapsto \frac{1}{4} (\mathds{1} + \rho^\top),$$

hence we are in the transitive covariant case, with $U \in \text{SU}(3)$ and $V = \mathds{1}$. It is straightforward to check that this channel is entanglement-breaking (see [33]): hence [29] tells us that its capacity is additive, and we can apply theorem 3.

By our general observations above we can conclude that for any state $\rho$ supported on $\mathcal{K}$, $E_c(\rho) = E_f(\rho) = 3/2$.

Following [33], we can introduce (for $j = 0, 1, 2$)

$$|j\rangle_t = |\Phi_3\rangle \otimes |j\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3,$$
and form the superpositions

$$|\tilde{j} \rangle := c|j\rangle_s \oplus s|j\rangle_t \in \mathbb{C}^3 \otimes (\mathbb{C}^6 \oplus \mathbb{C}^9)$$
in the direct sum of the respective supporting spaces, with $|c|^2 + |s|^2 = 1$. This obviously retains the covariant nature, and allows us to implement the mixtures of $T$ with the constant map onto $\frac{1}{3} \mathds{1}$, so we get every channel

$$T_p : \rho \mapsto p \frac{1}{3} \mathds{1} + (1 - p) \rho^\top,$$
for $3/4 \leq p \leq 1$, all of which are clearly entanglement-breaking, so the same technique applies, and we find subspaces on which every state has $E_c = E_f = \text{const.} \in [3/2, \log 3]$.

In [33], by implementing other entanglement-breaking channels (and using Shor’s result [29] on capacity additivity), other, and more general results of this type were obtained.

**Example 8** The “$U \otimes U$”–representation of SU(3) on $\mathbb{C}^3 \otimes \mathbb{C}^3$ decomposes into two irreducible parts, the symmetric subspace of dimension 6 and the antisymmetric subspace $\mathcal{A}$ of dimension 3. The latter has a nice basis given by

$$|0\rangle_a = \frac{1}{\sqrt{2}} (|1\rangle|2\rangle - |2\rangle|1\rangle),$$

$$|1\rangle_a = \frac{1}{\sqrt{2}} (|2\rangle|0\rangle - |0\rangle|2\rangle),$$

$$|2\rangle_a = \frac{1}{\sqrt{2}} (|0\rangle|1\rangle - |1\rangle|0\rangle),$$

13
which we use to identify \( \mathcal{A} \) with \( \mathbb{C}^3 \).

Notice that the partial trace over the first factor (say) implements a unital channel with symmetry \( (U \in SU(3) \text{ on } \mathbb{C}^3 \text{ and } V = U \otimes U \text{ on } \mathcal{A}) \), which is even transitive (hence all states \( \rho_a \) supported on \( \mathcal{A} \) have the same entanglement of formation \( E_f(\rho_a) = 1 \)), but it is neither depolarising nor entanglement-breaking: in the above identification it reads

\[
T_{\text{VDC}} : \rho_a \mapsto \frac{3}{2} \left( \frac{1}{3} I \right) - \frac{1}{2} \rho^T.
\]

Notice that this is one of the very channels used in [38] to disprove the general multiplicativity conjecture for the maximal output \( p \)-norm of a channel. Incidentally, this property is the main tool in King’s proofs of the additivity of channel capacities [19, 20, 21].

Denoting the maximally mixed state on \( \mathcal{A} \) by \( \sigma_{\mathcal{A}} \), it was shown in [27] that

\[
E_f(\sigma_{\mathcal{A}} \otimes \sigma_{\mathcal{A}}) = 2 E_f(\sigma_{\mathcal{A}}) = 2. 
\]

Subsequently, Yura [42] has shown that for all \( n \),

\[
E_f(\sigma_{\mathcal{A}}^n) = n,
\]

showing that the entanglement cost of this state is indeed 1.

The above examples show that using covariance one can often evaluate the entanglement of formation. By carefully choosing the supporting subspace of the state we can use our main theorem yielding even the entanglement cost.

8 Gap between \( E_c \) and \( E_D \)

Returning to example 5, let us demonstrate that the states discussed there exhibit a gap between the entanglement cost and distillable entanglement for some of these states, by use of the log–negativity bound \( \log \| \rho_T^F \|_1 \) on distillable entanglement [34].

We use the notation of example 5 in particular we assume the channel \( T \) to be a mixture of Pauli rotations, with probability weights according to eq. (13). The partial transpose \( \rho_T^\Gamma \) of the optimal state \( \rho_T \) decomposes into a direct sum of two \( 4 \times 4 \)-matrices, which turn out to have the same characteristic equation

\[
f(2z) = 0,
\]

where

\[
f(z) = z^4 - z^3 + 4(p_0p_xp_y + p_0p_zp_y + p_0p_yp_z + p_xp_yp_z)z - 16p_0p_xp_yp_z.
\]

Since \( f(2z) = 0 \) has only one negative root \( z_0 \) and \( f \) is decreasing in a neighbourhood of it, \( \log \| \rho_T^F \|_1 < E_c(\rho_T) \) is equivalent to

\[
f \left( - \frac{2E_c(\rho_T) - 1}{2} \right) = f \left( - \frac{2H(p_0 + p_z, 1 - p_0 - p_z)}{2} - 1 \right) > 0,
\]

using \( \| \rho_T^\Gamma \|_1 = 1 - 4z_0 \).

That is, if \( p_0, p_x, p_y, p_z \) satisfy this inequality, there is a gap between the entanglement cost of \( \rho_T \), and its distillable entanglement; figure 1 shows a plot
Figure 1: Plots in a \((p_x, p_y, p_z)\)–frame of the admissible parameters according to eq. (13) and of the region for which eq. (20) holds (between the two surfaces).

of the region of these \((p_x, p_y, p_z)\). By continuity, also for a mixture of \(|\psi_T\rangle\langle\psi_T|\) and \(|\psi_T^⊥\rangle\langle\psi_T^⊥|\) which is sufficiently close to \(\rho_T\), we observe a similar gap.

Especially, for \(p_0 = 1/2, p_x = p_y = p_z = 1/6\), a short calculation reveals that \(\|\rho_T^F\|_1 = 5/3\), so \(E_D(\rho_T) \leq \log(5/3) \approx 0.737\), which is smaller than the entanglement cost \(E_c(\rho_1) = H(1/3, 2/3) \approx 0.918\).

If \(p_0 + p_z = p_x + p_y = 1/2\) and \(p_0 \neq p_z, p_x \neq p_y\), we can even prove for all true mixtures \(\rho_{T,s} = s|\psi_T\rangle\langle\psi_T| + (1-s)|\psi_T^⊥\rangle\langle\psi_T^⊥|\) of \(|\psi_T\rangle\langle\psi_T|\) and \(|\psi_T^⊥\rangle\langle\psi_T^⊥|\), that \(E_D(\rho_{T,s}) < E_c(\rho_{T,s})\) holds; by eq. (14) the latter is 1 for all these \(\rho_{T,s}\), and the key observation is that \(\log \|\rho_T^F\|_1\) is strictly smaller than \(E_c(\rho_{T,s})\) in this case, for the condition (20) is always satisfied. Hence \(\|\rho_T^F\|_1 < 2\).

The convexity of trace norm and the observation \(\|\psi_T^\Gamma\|_1 = 2\) leads, for \(1/2 \leq s < 1\) (which we may assume by symmetry), to

\[
\|\rho_{T,s}^F\|_1 \leq (2s - 1) \|\psi_T\psi_T^\Gamma\|_1 + (2 - 2s) \|\rho_T^F\|_1
\]

\[
< (2s - 1) \cdot 2 + (2 - 2s) \cdot 2 = 2,
\]

and consequently we have \(\log \|\rho_T^F\|_1 < 1 = E_c(\rho_T)\).

This fact is also proven by noting that the negativity is strictly convex for mixings of \(|\psi_T\rangle\langle\psi_T|\) and \(|\psi_T^⊥\rangle\langle\psi_T^⊥|\), i.e.

\[
\left\| (s|\psi_T\rangle\langle\psi_T| + (1-s)|\psi_T^⊥\rangle\langle\psi_T^⊥|) \right\|_1 < s \left\|\psi_T\psi_T^\Gamma\right\|_1 + (1-s) \left\|\psi_T^⊥\psi_T^⊥\right\|_1
\]
if $0 < s < 1$. This is proved by finding eigenvectors with nonzero overlap of the two partial transposes such that one has a negative, the other a positive eigenvalue.

9 Conclusion

We demonstrated a link between the additivity problems for classical capacity of quantum channels and entanglement of formation, resulting in the additivity of the latter for many states, by invoking recent additivity results for the former. This allows us to establish in particular a gap between distillable entanglement and entanglement cost for many of these states. By exploiting the fact that $E_f$ is a convex roof, this additivity can be extended to even more states, though it is not clear how far this would get us, even taking the general additivity conjecture for granted.

It is obvious that we only probed the scope of the method, and it is clear that other examples of the same sort can be constructed, adding to the list of states for which the entanglement cost is known. Each channel for which additivity of its capacity is established will add to this list.

The method generalises part of the argument found in the recent work of Vidal, Dur and Cirac [33], but for the case of entanglement breaking channels their method is more general.

On the side of general insights, the attempt to link the two additivity conjectures considered here led us to consider the superadditivity of entanglement of formation as a relation which integrates them neatly. We were even able to exhibit a few cases where it is known to hold, providing modest evidence in favour of it.

Since completion of this work, subsequent research has further clarified the picture presented here: Ruskai [quant-ph/0303141] showed that not all bipartite states can be associated with a channel such as to make use of theorem 3 to prove additivity. Audenaert and Braunstein [quant-ph/0303045] have re-expressed the superadditivity of entanglement of formation using tools from convex analysis, and showed that the multiplicativity conjecture for maximal output $p$-norms [1], for $p$ close to 1, of filtering operations implies superadditivity. Shor [quant-ph/0305035] has complemented our theorem 4 by showing that the general conjectures of superadditivity of $E_f$, additivity of $E_f$ under tensor products, and additivity of $C$ are in fact equivalent to each other and to the additivity of minimal output entropy of a channel.

Acknowledgements

We thank K. G. Vollbrecht and R. F. Werner for conversations about the superadditivity conjecture, and A. Uhlmann for pointers to the literature.

KM and TS are supported by the Japan Science and Technology Corporation, AW is supported by the U.K. Engineering and Physical Sciences Research
Council, and gratefully acknowledges the hospitality of the ERATO Quantum Computation and Information project, Tokyo, on the occasion of a visit during which part of the present work was done.

References

[1] G. G. Amosov, A. S. Holevo, R. F. Werner, “On the additivity hypothesis in quantum information theory” (Russian), Problemy Peredachi Informatsii, vol. 36, no. 4, pp. 25–34, 2000. English translation in Probl. Inf. Transm., vol. 36, no. 4, pp. 305–313, 2000.

[2] G. G. Amosov, A. S. Holevo, “On the multiplicativity conjecture for quantum channels”, Theor. Probab. Appl., vol. 47, no. 1, pp. 143–146, 2002.

[3] K. Audenaert, J. Eisert, E. Jane, M.B. Plenio, S. Virmani, B. De Moor, “The asymptotic relative entropy of entanglement”, Phys. Rev. Letters, vol. 87, 217902, 2001.

[4] K. Audenaert, B. De Moor, K. G. H. Vollbrecht, R. F. Werner, “Asymptotic Relative Entropy of Entanglement for Orthogonally Invariant States”, Phys. Rev. A, vol. 66, 032310, 2002.

[5] F. Benatti, H. Narnhofer, “On the Additivity of the Entanglement of Formation”, Phys. Rev. A, vol. 63, 042306, 2001.

[6] F. Benatti, H. Narnhofer, A. Uhlmann, “Decompositions of Quantum States with Respect to Entropy”, Rep. Math. Phys., vol. 38, no. 1, pp. 123–141, 1996.

[7] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, W. K. Wootters, “Mixed-state entanglement and quantum error correction”, Phys. Rev. A, vol. 54, no. 5, pp. 3824–3851, 1996.

[8] C. H. Bennett, C. A. Fuchs, J. A. Smolin, “Entanglement–Enhanced Classical Communication on a Noisy Quantum Channel”, in: Quantum Communication, Computing, and Measurement (O. Hirota, A. S. Holevo, C. M. Caves eds.), pp. 79–88, Plenum, New York, 1997.

[9] D. Bruss, L. Faoro, C. Macchiavello, M. Palma, “Quantum entanglement and classical communication through a depolarising channel”, J. Mod. Optics, vol. 47, no. 2, pp. 325–331, 2000.

[10] V. Coffman, J. Kundu, W. K. Wootters, “Distributed Entanglement”, Phys. Rev. A, vol. 61, 052306, 2000.

[11] A. Connes, H. Narnhofer, W. Thirring, “Dynamical entropy of C∗–algebras and von Neumann algebras”, Comm. Math. Phys., vol. 112, no. 4, pp. 691–719, 1987.

[12] A. Fujiwara, P. Algoet, “One–to–one parametrization of quantum channels”, Phys. Rev. A, vol. 59, pp. 3290–3294, 1999.

[13] A. Fujiwara, T. Hashizume, “Additivity of the capacity of depolarizing channels”, Phys. Lett. A, vol. 299, no. 5/6, pp. 469–475, 2002.
[14] P. M. Hayden, M. Horodecki, B. M. Terhal, “The asymptotic entanglement cost of preparing a quantum state”, J. Phys. A: Math. Gen., vol. 34, no. 35, pp. 6891–6898, 2001.

[15] A. S. Holevo, “Some estimates for the amount of information transmittable by a quantum communications channel” (Russian), Problemy Peredachi Informatsii, vol. 9, no. 3, pp. 3–11, 1973. English translation: Probl. Inf. Transm., vol. 9, no. 3, pp. 177–183, 1973.

[16] A. S. Holevo, “Problems in the mathematical theory of quantum communication channels”, Rep. Mathematical Phys., vol. 12, no. 2, pp. 273–278, 1979.

[17] A. S. Holevo, “The capacity of the quantum channel with general signal states”, IEEE Trans. Inf. Theory, vol. 44, no. 1, pp. 269–273, 1998.

[18] A. S. Holevo, “On Quantum Communication Channels with Constrained Inputs”, e–print quant-ph/9705054, 1997.

[19] C. King, “Maximization of capacity and $\ell_p$ norms for some product channels”, J. Math. Phys., vol. 43, no. 3, pp. 1247–1260, 2002.

[20] C. King, “Additivity for unital qubit channels”, J. Math. Phys., vol. 43, no. 10, pp. 4641–4653, 2002.

[21] C. King, “The capacity of the quantum depolarizing channel”, e–print quant-ph/0204172, 2002.

[22] C. King, M. B. Ruskai, “Minimal Entropy of States Emerging from Noisy Quantum Channels”, IEEE Trans. Inf. Theory, vol. 47, pp. 192–209, 2001.

[23] K. Kraus, States, Effect and Operations: Fundamental Notions of Quantum Theory, Springer Verlag, Berlin 1983.

[24] E. M. Rains, “A Semidefinite Program for Distillable Entanglement”, IEEE Trans. Inf. Theory, vol. 47, no. 7, pp. 2921–2933, 2001.

[25] M. B. Ruskai, “Inequalities for Quantum Entropy: A Review with Conditions for Equality”, J. Math. Phys., vol. 43, pp. 4358–4375, 2002.

[26] B. Schumacher, M. D. Westmoreland, “Sending classical information via noisy quantum channels”, Phys. Rev. A, vol. 56, no. 1, pp. 131–138, 1997.

[27] T. Shimono, “Lower bound for entanglement cost of antisymmetric states”, e–print quant-ph/0203039, 2002.

[28] P. W. Shor, J. A. Smolin, B. M. Terhal, “Nonadditivity of Bipartite Distillable Entanglement follows from Conjecture on Bound Entangled Werner States”, Phys. Rev. Lett., vol. 86, pp. 2681–2684, 2001.

[29] P. W. Shor, “Additivity of the Classical Capacity of Entanglement–Breaking Quantum Channels”, e–print quant-ph/0201149, 2002.

[30] W. F. Stinespring, “Positive functions on $C^*$–algebras”, Proc. Amer. Math. Soc., vol. 6, pp. 211–216, 1955.
[31] B. M. Terhal, K. G. H. Vollbrecht, “The Entanglement of Formation for Isotropic States”, Phys. Rev. Letters, vol. 85, pp. 2625–2628, 2000.

[32] V. Vedral, M. B. Plenio, “Entanglement measures and purification procedures”, Phys. Rev. A, vol. 57, no. 3, pp. 1619–1633, 1998.

[33] G. Vidal, W. Dür, J. I. Cirac, “Entanglement cost of mixed states”, Phys. Rev. Letters, vol. 89, no. 2, 027901, 2002.

[34] G. Vidal, R. F. Werner, “A computable measure of entanglement”, Phys. Rev. A, vol. 65, 032314, 2002.

[35] K. G. H. Vollbrecht, R. F. Werner, “A counterexample to a conjectured entanglement inequality”, e–print quant-ph/0006046, 2000.

[36] K. G. H. Vollbrecht, R. F. Werner, “Entanglement Measures under Symmetry”, Phys. Rev. A, vol. 64, 062307, 2001.

[37] R. F. Werner, “All teleportation and dense coding schemes”, J. Phys. A, vol. 34, no. 35, pp. 7081–7094, 2001.

[38] R. F. Werner, A. S. Holevo, “Counterexample to an additivity conjecture for output purity of quantum channels”, J. Math. Phys., vol. 43, no. 9, pp. 4353–4357, 2002.

[39] A. Winter, “Coding Theorem and Strong Converse for Quantum Channels”, IEEE Trans. Inf. Theory, vol. 45, no. 7, pp. 2481–2485, 1999.

[40] A. Winter, “Scalable programmable quantum gates and a new aspect of the additivity problem for the classical capacity of quantum channels”, e–print quant-ph/0108066, 2001.

[41] W. K. Wootters, “Entanglement of Formation of an Arbitrary State of Two Qubits”, Phys. Rev. Letters, vol. 80, no. 10, pp. 2245–2248, 1998.

[42] F. Yura, “Entanglement cost of three–level antisymmetric states”, J. Phys. A: Math. Gen., vol. 36, no. 15, pp. L237–L242, 2003.