The Drinfel’d Double versus the Heisenberg Double for Hom-Hopf Algebras

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Abstract. In this paper we mainly construct the Drinfel’d double \( (H^{op} \Join H^*, \alpha \otimes (\alpha^{-1})^*) \) for finite-dimensional Hom-Hopf algebra \( (H, \alpha) \) by generalizing the Majid’s bicrossproduct. Such a Drinfel’d double \( (H^{op} \Join H^*, \alpha \otimes (\alpha^{-1})^*) \) has a quasitriangular structure \( R \) satisfying the quantum Hom-Yang-Baxter equations. Finally, we find that the deformation of the product in \( (H^{op} \Join H^*, \alpha \otimes (\alpha^{-1})^*) \) is related to the Heisenberg double \( (H^{op} \# H^*, \alpha \otimes (\alpha^{-1})^*) \).

Keywords: Hom-Hopf algebra; Drinfeld double; Majid’s bicrossproduct; Heisenberg double.

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0. INTRODUCTION

Algebraic deformation has been well developed recently into a new broad class of non-associative algebras, and its theory has been applied in modules of quantum phenomena, as well as in analysis of complex systems. Hom-type algebras appeared first in physical contexts, in connection with twisted, deformed derivatives and corresponding generalizations, deformations of vector fields (see [2],[3], for example). Then the deformation has been investigated by Makhlouf and Silvestrov in [11]. In their construction of Hom-Lie algebra, the Jacobi identity is replaced by the so-called Hom-Jacobi identity, namely

\[
[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0,
\]

where \( \alpha \) is an endomorphism of the Lie algebra. The main initial goal of this investigation was to create a unified general approach to examples of \( q \)-deformations of Witt and Virasoro algebras. Particularly it was observed that in these examples some \( q \)-deformations of
ordinary Lie algebra Jacobi identities hold. Motivated by these new interesting examples, quasi-Hom-Lie algebra and Hom-Lie algebra were introduced in [3].

The idea of construction by adapting associativity-like conditions via endomorphisms was applied to other algebraic structures. For example, Hom-algebras and Hom-coalgebra was introduced for the first time in [9] and [10], respectively. Here the Hom-associativity and Hom-coassociativity were obtained by twisting the original associativity and coassociativity by endomorphisms. Hom-module and Hom-comodule were naturally developed in the same way. Then Hom-bialgebras were defined in a rational way. These objects are slightly different from what we will study (called monoidal Hom-Hopf algebra) in this paper, as introduced in [1].

In [4], the authors constructed Radford’s biproduct on monoidal Hom-Hopf algebras. In [13], the author introduced the quasitriangular Hom-bialgebras (not monoidal Hom-bialgebras), as a generalization of the ordinary quasitriangular bialgebras and the quantum Hom-Yang-Baxter equation (QHYBE) of the form

\[
R^{12}(R^{13}R^{23}) = (R^{13}R^{23})R^{12},
\]

\[
(R^{12}R^{13})R^{23} = R^{23}(R^{13}R^{12}).
\]

The construction of Majid’s bicrossproduct Hopf algebras was motivated by the search for examples of self-dual algebraic structures, which means in the first place to find a category with a dualising endofunctor such that some kind of Pontryagin duality theorem holds. The bicrossproduct Hopf algebras provide numerous examples of non-commutative and non-cocommutative Hopf algebras, and moreover they turn out be closely related to the Drinfeld double.

In this paper, we will construct the Majid’s bicrossproduct for Hom-Hopf algebras and then in the framework of Hom-Hopf algebras, we will consider the Drinfeld double associated to a dual pair of Hom-Hopf algebras, which means that the underlying space need not be finite dimensional.

This paper is organized as follows.

In Section 1, we will recall the definitions and results of Hom-Hopf algebras, such as Hom-algebras, Hom-coalgebras, Hom-modules, Hom-comodules and the Hom-smash products.

In Section 2, we will introduce the notion of bicrossproduct \((A \# H, \alpha_A \otimes \alpha_H)\), and we will give the conditions for \((A \ltimes H, \alpha_A \otimes \alpha_H)\) to form a Hom-Hopf algebra, generalizing the Majid’s bicrossproduct defined in [5]. And a class of bicrossproduct Hom-bialgebras is constructed. Besides double crossed product in the Hom-context is constructed and a class of quasitriangular Hom-Hopf algebras is given.

In Section 3, we will construct the Drinfeld double associated to a dual pair of Hom-Hopf algebras, which is a generalization of the Drinfeld double of finite dimensional Hom-
Hopf algebra.

In section 4, we will establish the relation between Drinfeld double and Heisenberg double.

Throughout this article, all the vector spaces, tensor product and homomorphisms are over a fixed field $k$. We use the Sweedler’s notation for the terminologies on coalgebras. For a coalgebra $C$, we write comultiplication $\Delta(c) = \sum c_1 \otimes c_2$ for any $c \in C$.

1. PRELIMINARIES

In what follows, we will recall the definitions in [7] on the Hom-associative algebras, Hom-coassociative coalgebras, Hom-modules and Hom-comodules.

A unital Hom-associative algebra is a triple $(A, \mu, \alpha)$ where $\alpha : A \rightarrow A$ and $\mu : A \otimes A \rightarrow A$ are linear maps, with notation $\mu(a \otimes b) = ab$ such that for any $a, b, c \in A$,

$$\alpha(ab) = \alpha(a)\alpha(b), \quad \alpha(1_A) = 1_A,$$

$$1_Aa = \alpha(a) = a1_A, \quad \alpha(a)(bc) = (ab)\alpha(c).$$

A linear map $f : (A, \mu_A, \alpha_A) \rightarrow (B, \mu_B, \alpha_B)$ is called a morphism of Hom-associative algebra if $\alpha_B \circ f = f \circ \alpha_A, \quad f(1_A) = 1_B$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

A counital Hom-coassociative coalgebra is a triple $(C, \Delta, \varepsilon, \alpha)$ where $\alpha : C \rightarrow C$, $\varepsilon : C \rightarrow k$, and $\Delta : C \rightarrow C \otimes C$ are linear maps such that

$$\varepsilon \circ \alpha = \varepsilon, \quad (\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha,$$

$$(\varepsilon \otimes \text{id}) \circ \Delta = \alpha = (\text{id} \otimes \varepsilon) \circ \Delta,$$

$$(\Delta \otimes \alpha) \circ \Delta = (\alpha \otimes \Delta) \circ \Delta.$$ 

A linear map $f : (C, \Delta_C, \alpha_C) \rightarrow (D, \Delta_D, \alpha_D)$ is called a morphism of Hom-coassociative coalgebra if $\alpha_D \circ f = f \circ \alpha_C, \quad \varepsilon_D \circ f = \varepsilon_C$ and $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$.

In what follows, we will always assume all Hom-algebras are unital and Hom-coalgebras are counital.

A Hom-bialgebra is a quadruple $(H, \mu, \Delta, \alpha)$, where $(H, \mu, \alpha)$ is a Hom-associative algebra and $(H, \Delta, \alpha)$ is a Hom-coassociative coalgebra such that $\Delta$ and $\varepsilon$ are morphisms of Hom-associative algebra.

A Hom-Hopf algebra $(H, \mu, \Delta, \alpha)$ is a Hom-bialgebra $H$ with a linear map $S : H \rightarrow H$ such that

$$S \circ \alpha = \alpha \circ S,$$

$$\sum S(h_1)h_2 = \sum h_1S(h_2) = \varepsilon(h)1,$$
for any \( h \in H \).

As a consequence, we have the following properties for \( S \):

\[
\sum S(h_1) \otimes S(h_2) = \sum S(h_2) \otimes S(h_1),
\]

\[
S(gh) = S(h)S(g), \quad \varepsilon \circ S = \varepsilon.
\]

Let \((A, \alpha_A)\) be a Hom-associative algebra, \( M \) a linear space and \( \alpha_M : M \rightarrow M \) a linear map. A left \( A \)-module structure on \((M, \alpha_M)\) consists of a linear map \( A \otimes M \rightarrow M \), \( a \otimes m \mapsto a \cdot m \), such that

\[
1_A \cdot m = \alpha_M(m),
\]

\[
\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m),
\]

\[
\alpha_A(a) \cdot (b \cdot m) = (ab) \cdot \alpha_M(m),
\]

for any \( a, b \in A \) and \( m \in M \).

Similarly we can define the right \((A, \alpha)\)-modules. Let \((M, \mu)\) and \((N, \nu)\) be two left \((A, \alpha)\)-modules, then a linear map \( f : M \rightarrow N \) is a called left \( A \)-module map if \( f(am) = af(m) \) for any \( a \in A \), \( m \in M \) and \( f \circ \mu = \nu \circ f \).

Let \((C, \alpha_C)\) be a Hom-coassociative coalgebra, \( M \) a linear space and \( \alpha_M : M \rightarrow M \) a linear map. A right \( C \)-comodule structure on \((M, \alpha_M)\) consists a linear map \( \rho : M \rightarrow M \otimes C \) such that

\[
(id \otimes \varepsilon_C) \circ \rho = \alpha_M,
\]

\[
(\alpha_M \otimes \alpha_C) \circ \rho = \rho \circ \alpha_M,
\]

\[
(\rho \otimes \alpha_C) \circ \rho = (\alpha_M \otimes \Delta) \circ \rho.
\]

Let \((M, \mu)\) and \((N, \nu)\) be two right \((C, \gamma)\)-comodules, then a linear map \( g : M \rightarrow N \) is a called right \( C \)-comodule map if \( g \circ \mu = \nu \circ g \) and \( \rho_N(g(m)) = (g \otimes \text{id})\rho_M(m) \) for any \( m \in M \).

Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra. A Hom-associative algebra \((A, \mu_A, \alpha_A)\) is called a left \( H \)-module Hom-algebra if \((A, \alpha_A)\) is a left \( H \)-module, with action denoted by \( H \otimes A \rightarrow A \), \( h \otimes a \mapsto h \cdot a \), such that

\[
\alpha^2_H(h) \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b),
\]

\[
h \cdot 1_A = \varepsilon(h)1_A,
\]

for any \( h \in H \) and \( a, b \in A \).

**Proposition 1.1** [12] Let \((H, \mu_H, \Delta_H)\) be a bialgebra and \((A, \mu_A)\) is called a left \( H \)-module algebra in the usual sense, with action denoted by \( H \otimes A \rightarrow A \), \( h \otimes a \mapsto h \cdot a \). Let \( \alpha_H : H \rightarrow H \) be a bialgebra endomorphism and \( \alpha_A : A \rightarrow A \) an algebra endomorphism, such that \( \alpha_A(h \cdot a) = \alpha_H \cdot \alpha_A(a) \) for all \( h \in H, a \in A \). If we consider
the Hom-bialgebra $H_{\alpha_H} = (H, \alpha_H \circ \mu_H, \Delta_H \circ \alpha_H, \alpha_H)$ and the Hom-associative algebra $A_{\alpha_A} = (A, \alpha_A \circ \mu_A, \alpha_A)$, then $A_{\alpha_A}$ is a left $H_{\alpha_H}$-module Hom-algebra in the above sense, with action $H_{\alpha_H} \otimes A_{\alpha_A} \rightarrow A_{\alpha_A}$, $h \triangleright a = \alpha_A(h \cdot a) = \alpha_H(h) \triangleright \alpha_A(a)$.

When $A$ is a left $H$-module Hom-algebra, in [7] the Hom-smash product $A \# H$ is defined as follows:

$$(a \# h)(b \# k) = \sum a(\alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(b)) \# \alpha_H^{-1}(h_2)k,$$

for any $a, b \in A$ and $h, k \in H$.

2. MAJID’S BICROSSPRODUCTS FOR HOM-HOPF ALGEBRAS

In this section, we will construct the bicrossproducts on Hom-bialgebras, which generalize the Majid’s bicrossproducts in the usual Hopf algebras. Then we can get Drinfel’d double for Hom-Hopf algebras in the setting of the Majid’s bicrossproducts.

Definition 2.1. Let $(C, \Delta_C, \alpha_C)$ and $(D, \Delta_D, \alpha_D)$ be Hom-coassociative coalgebras. A linear map $\Phi : C \otimes D \rightarrow D \otimes C$ is called a Hom-twisting map between $C$ and $D$ if the following conditions hold:

$$\begin{align*}
(\Delta_D \otimes \alpha_C) \circ \Phi &= (id_D \otimes \Phi) \circ (\Phi \otimes id_D) \circ (\alpha_C \otimes \Delta_D), \\
(\alpha_D \otimes \Delta_C) \circ \Phi &= (\Phi \otimes id_C) \circ (id_C \otimes \Phi) \circ (\Delta_C \otimes \alpha_D), \\
(\alpha_D \otimes \alpha_C) \circ \Phi &= \Phi \circ (\alpha_C \otimes \alpha_D), \\
(\varepsilon_C \otimes id) \circ \Phi &= id \otimes \varepsilon_C, \\
(id \otimes \varepsilon_D) \circ \Phi &= \varepsilon_D \otimes id.
\end{align*}$$

If we denote the element $\Phi(c \otimes d)$ in $D \otimes C$ by $\Phi(c \otimes d) = \sum d^\Phi \otimes c^\Phi = \sum d^\Phi \otimes c^\Phi$ for any $c \in C, d \in D$, then the above identities can be rewritten as

$$\begin{align*}
\sum (d_1^\Phi) \otimes (d_2^\Phi) \otimes \alpha_C(c^\Phi) &= \sum (d_1^\Phi \otimes d_2^\Phi) \otimes \alpha_C(c^\Phi), \\
\sum \alpha_D(d^\Phi) \otimes (c_1^\Phi) \otimes (c_2^\Phi) &= \sum \alpha_D(d^\Phi) \otimes (c_1^\Phi \otimes (c_2^\Phi), \\
\sum \alpha_D(d^\Phi) \otimes \alpha_C(c^\Phi) &= \sum \alpha_D(d^\Phi) \otimes \alpha_C(c^\Phi), \\
\sum \varepsilon_C(c^\Phi)d^\Phi &= \varepsilon_C(c)d, \\
\sum \varepsilon_D(d^\Phi)c^\Phi &= \varepsilon_D(d)c.
\end{align*}$$
Proposition 2.2. Let $(C, \Delta_C, \alpha_C)$ and $(D, \Delta_D, \alpha_D)$ be Hom-coassociative coalgebras and $\Phi : C \otimes D \to D \otimes C$ be the Hom-twisting map. Define $\Delta : C \otimes D \to (C \otimes D) \otimes (C \otimes D)$ by

$$
\Delta(c \otimes d) = \sum c_1 \otimes d_1^{\Phi} \otimes c_2^{\Phi} \otimes d_2,
$$

and $\varepsilon : C \otimes D \to k$ by

$$
\varepsilon(c \otimes d) = \varepsilon_C(c) \varepsilon_D(d),
$$

for any $c \in C, d \in D$. Then $(C \otimes D, \Delta, \alpha_C \otimes \alpha_D)$ is a Hom-coassociative coalgebra.

Proof The counit is straightforward to verify. For any $c \in C, d \in D$, firstly

$$(\Delta \circ (\alpha_C \otimes \alpha_D))(c \otimes d) = \sum \Delta(\alpha_C(c) \otimes \alpha_D(d)) = \sum \alpha_C(c_1) \otimes \alpha_D(d_1)^{\Phi} \otimes \alpha_C(c_2)^{\Phi} \otimes \alpha_D(d_2)$$

$$= \sum \alpha_C(c_1) \otimes \alpha_D(d_1^{\Phi}) \otimes \Delta(c_2^{\Phi} \otimes d_2)$$

$$= \sum \alpha_C(c_1) \otimes \alpha_D(d_1^{\Phi}) \otimes (c_2^{\Phi})_1 \otimes (d_2)_1^{\Phi} \otimes ((c_2^{\Phi})_2)^{\Phi} \otimes d_2$$

$$= \sum \alpha_C(c_1) \otimes \alpha_D(d_1^{\Phi}) \otimes (c_2^{\Phi})_1 \otimes (d_2)_1^{\Phi} \otimes (c_2^{\Phi})_2 \otimes d_2 \otimes \alpha_D(d_2)$$

$$= \sum \alpha_C(c_1) \otimes \alpha_D(d_1^{\Phi}) \otimes (c_2^{\Phi})_1 \otimes (d_2)_1^{\Phi} \otimes (c_2^{\Phi})_2 \otimes \alpha_C(c_2^{\Phi}) \otimes \alpha_D(d_2)$$

$$= \Delta \circ (\alpha_C \otimes \alpha_D) \circ \Delta(c \otimes d).$$

By the definition, the proof is completed.

Definition 2.3. Let $(H, \alpha_H)$ be a Hom-bialgebra. A Hom-coassociative coalgebra $(C, \alpha_C)$ is called a right $H$-comodule Hom-coalgebra if $(C, \alpha_C)$ a right $H$-comodule, with the comodule structure map $\rho : C \to C \otimes H$, $c \mapsto \sum c(0) \otimes c(1)$, such that the following conditions hold

$$
\sum \varepsilon_C(c(0)) c(1) = \varepsilon_C(c) 1_H,
$$

$$
\sum c(0)_1 \otimes c(0)_2 \otimes \alpha^2_H(c(1)) = \sum c(0)_1 \otimes c(2)_1 \otimes c(1)_1 \alpha(1),
$$

for any $c \in C$.

Example 2.4. Let $H$ be a bialgebra and $C$ a right $H$-comodule coalgebra in the usual sense. The coaction is denoted by $C \to C \otimes H$, $c \mapsto \sum c[0] \otimes c[1]$. Let $\alpha_H$ be a bialgebra
endomorphism on \( H \), and \( \alpha_C \) a coalgebra endomorphism such that
\[
\sum \alpha_C(c)_0 \otimes \alpha_C(c)_1 = \sum \alpha_C(c_0) \otimes \alpha_H(c_1),
\]
for any \( c \in C \). Consider the Hom-bialgebra \( H_{\alpha_H} = (H, \alpha_H \circ \mu_H, \Delta_H \circ \alpha_H, \alpha_H) \) and the Hom-coassociative coalgebra \( C_{\alpha_C} = (C, \Delta_C \circ \alpha_C, \alpha_C) \). Then \( C_{\alpha_C} \) is a right \( H_{\alpha_H} \)-comodule Hom-coalgebra with the coaction \( C_{\alpha_C} \to C_{\alpha_C} \otimes H_{\alpha_H}, c \mapsto \sum c_0 \otimes c_1 = \sum \alpha_C(c_0) \otimes \alpha_H(c_1). \)

**Proposition 2.5.** Let \( (A, S_A, \alpha_A) \) and \( (H, S_H, \alpha_H) \) be Hom-Hopf algebras. Assume \( A \) is a left \( H \)-module Hom-algebra with the action \( H \otimes A \to A, h \otimes a \mapsto h \cdot a \) and \( H \) is a right \( A \)-comodule Hom-coalgebra with the coaction \( H \to H \otimes A, h \mapsto \sum h_0 \otimes h_1 \). If the following conditions are satisfied

1. For any \( h \in H, b \in A, \)
\[
\Delta(h \cdot b) = \sum \alpha_H^{-1}(h_1_0)(h_1_1) \alpha_A^{-1}(b_0)(b_1),
\]
\[
\varepsilon_A(h \cdot b) = \varepsilon_A(b) \varepsilon_H(h),
\]

2. For any \( h, g \in H, \)
\[
\sum (hg)_0 \otimes (hg)_1 = \sum \alpha_H^{-1}(h_0_0)(g_0) \alpha_A^{-1}(h_1_1)(g_1),
\]

3. For any \( h \in H, b \in A, \)
\[
\sum h_2_0 \otimes (h_1 \cdot b)_2_1 = \sum h_1_0 \otimes (h_2 \cdot b),
\]

then \( (A \otimes H, \alpha_A \otimes \alpha_H) \) is a Hom-Hopf algebra under the Hom-smash product, Hom-smash coproduct, that is,
\[
(a \otimes h)(b \otimes g) = \sum a \alpha_H^{-2}(h_1) \alpha_A^{-1}(b) \otimes \alpha_H^{-1}(h_2) g,
\]
\[
\Delta(a \otimes h) = \sum a_1 \otimes \alpha_H^{-1}(h_1_0) \alpha_A^{-1}(a_2) \alpha_A^{-2}(h_1_1) \otimes h_2,
\]

with the antipode \( S : A \ltimes H \to A \ltimes H \) given by
\[
S(a \ltimes h) = \sum (1 \ltimes S_H(\alpha_H^{-2}(h_0_0)))(S_A(\alpha_A^{-2}(a)) \alpha_A^{-3}(h_1)) \times 1,
\]

for any \( h, g \in H, a, b \in A. \)

We will call this Hom-Hopf algebra Hom-bicrossproduct, and denote it by \( (A \ltimes H, \alpha_A \otimes \alpha_H). \)
Proof Firstly we need to prove $\Delta$ and $\varepsilon$ is a morphism of Hom-associative algebra. Indeed for any $a, b \in A$ and $g, h \in H$,

$$\Delta((a \otimes h)(b \otimes g))$$

$$= \Delta(\sum a(\alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(b_1)) \otimes \alpha_H^{-1}(g_1) \otimes \alpha_H^{-1}(h_2)g)$$

$$= \sum a(\alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(b_1)) \otimes \alpha_H^{-1}(h_2(1)g_1(0))$$

$$\otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-1}((\alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(b_2))_2)\alpha_A^{-2}((\alpha_H^{-1}(h_2(1))g_1(1)) \otimes \alpha_H^{-1}(h_2)g_2$$

$$= \sum a(\alpha_H^{-3}(h_1(0)) \cdot \alpha_A^{-1}(b_1)) \otimes \alpha_H^{-3}(h_2(1)(0)) \alpha_H^{-1}(g_1(0)) \otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \}$$

$$\otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \otimes \alpha_H^{-1}(h_2(2))g_2$$

$$= \sum a(\alpha_H^{-3}(h_1(0)) \cdot \alpha_A^{-1}(b_1)) \otimes \alpha_H^{-3}(h_2(1)(0)) \alpha_H^{-1}(g_1(0)) \otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \}$$

$$\otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \otimes \alpha_H^{-1}(h_2(2))g_2$$

$$= \sum a(\alpha_H^{-3}(h_1(0)) \cdot \alpha_A^{-1}(b_1)) \otimes \alpha_H^{-3}(h_2(1)(0)) \alpha_H^{-1}(g_1(0)) \otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \}$$

$$\otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \otimes \alpha_H^{-1}(h_2(2))g_2$$

$$= \sum a(\alpha_H^{-3}(h_1(0)) \cdot \alpha_A^{-1}(b_1)) \otimes \alpha_H^{-3}(h_2(1)(0)) \alpha_H^{-1}(g_1(0)) \otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \}$$

$$\otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \otimes \alpha_H^{-1}(h_2(2))g_2$$

$$= \sum a(\alpha_H^{-3}(h_1(0)) \cdot \alpha_A^{-1}(b_1)) \otimes \alpha_H^{-3}(h_2(1)(0)) \alpha_H^{-1}(g_1(0)) \otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \}$$

$$\otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \otimes \alpha_H^{-1}(h_2(2))g_2$$

$$= \sum a(\alpha_H^{-3}(h_1(0)) \cdot \alpha_A^{-1}(b_1)) \otimes \alpha_H^{-3}(h_2(1)(0)) \alpha_H^{-1}(g_1(0)) \otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \}$$

$$\otimes \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \{ \alpha_A^{-1}(a_2)\alpha_A^{-3}(h_1(1)) \} \otimes \alpha_H^{-1}(h_2(2))g_2$$
\[
\sum a_1 (\alpha_H^{-3}(h_{1(0)1}) \cdot \alpha_A^{-1}(b_1)) \otimes \alpha_H^{-2}(h_{1(0)2}) \alpha_A^{-1}(g_{1(0)}) \otimes \{\alpha_A^{-1}(a_2)\alpha_A^{-2}(h_{1(1)})\}
\]
\[
\{\alpha_H^{-2}(h_{21}) \cdot \alpha_A^{-2}(b_2)\alpha_A^{-3}(g_{1(1)})\} \otimes \alpha_H^{-1}(h_{22}) g_2
\]
\[
= \Delta(a \otimes h) \Delta(b \otimes g).
\]

It is straightforward to check that \(\varepsilon\) is a morphism of Hom-associate algebra. For the antipode \(S\) we have

\[
\sum (a \times h)_1 S((a \times h)_2)
\]
\[
= \sum (a_1 \times \alpha_H^{-1}(h_{1(0)}))[1 \times S_H(\alpha_H^{-2}(h_{2(0)}))
\]
\[
(S_A((\alpha_A^{-3}(a_2)\alpha_A^{-4}(h_{1(1)})))\alpha_A^{-3}(h_{2(1)})) \times 1)
\]
\[
= \sum [a_1 \times \alpha_H^{-2}(h_{1(0)})] S_H(\alpha_H^{-2}(h_{2(0)}))
\]
\[
[S_A(\alpha_A^{-1}(a_2)(\alpha_A^{-3}(h_{1(1)})))\alpha_A^{-3}(h_{2(1)})) \times 1]
\]
\[
= \sum [a_1 \times \alpha_H^{-2}(h_{0(1)})] S_H(\alpha_H^{-2}(h_{0(2)})) [S_A(\alpha_A^{-1}(a_2)(\alpha_A^{-1}(h_{1}))) \times 1]
\]
\[
= \sum [a_1 \times \varepsilon(h_{0})] 1 [S_A(\alpha_A^{-1}(a_2)(\alpha_A^{-1}(h_{1}))) \times 1]
\]
\[
= \varepsilon(a) \varepsilon(h) 1 \times 1.
\]

Similarly we have \(\sum S(a \times h)_1 (a \times h)_2 = \varepsilon(a) \varepsilon(h) 1 \times 1\). Thus \(A \times H\) is a Hom-Hopf algebra. The proof is completed.

**Example 2.6.** Let \(A = \text{span}\{1, x\}\) over a fixed field \(k\) with \(\text{char} k \neq 2\). Then define \(\beta\) as a \(k\)-linear automorphism of \(A\) by

\[
\beta(1_A) = 1_A, \quad \beta(x) = -x.
\]

Define the multiplication on \(A\) by

\[
1_A 1_A = 1_A, \quad 1_A x = -x, \quad x^2 = 0,
\]

then it is not hard to check that \((A, \beta)\) is a Hom-associative algebra.

For \(A\), define the comultiplication and the counit by

\[
\Delta(1_A) = 1_A \otimes 1_A, \quad \Delta(x) = (-x) \otimes 1 + 1 \otimes (-x),
\]
\[
\varepsilon(1_A) = 1, \quad \varepsilon(x) = 0,
\]
\[
S_A(1_A) = 1_A, \quad S_A(x) = -x,
\]
then it is easy to check that \((A, \beta)\) is a Hom-Hopf algebra.

Let \(H = \text{span}\{1, g | g^2 = 1\}\) be the group algebra. Then obviously \((H, id)\) is a Hom-Hopf algebra.
Define the left action of $H$ on $A$ by $\cdot : H \otimes A \to A$ by

$$1_H \cdot 1_A = 1_A, \quad 1_H \cdot x = -x, \quad g \cdot 1_A = 1_A, \quad g \cdot x = x.$$ 

Then it is easy to check that $(A, \beta)$ is a left $(H, \text{id})$-module algebra.

Define the right coaction of $A$ on $H$ by $\rho : H \to H \otimes A$ by

$$\rho(1_H) = 1_H \otimes 1_A, \quad \rho(g) = g \otimes 1_A,$$

then we can verify that $(H, \text{id})$ is a right $(A, \beta)$-comodule coalgebra, and the conditions in the above proposition can be satisfied. Hence we have a Hom-Hopf algebra $(A \# H, \beta \# \text{id})$ by the bicrossproduct.

The product:

|       | $1_A \# 1_H$ | $1_A \# g$ | $x \# 1_H$ | $x \# g$ |
|-------|--------------|-------------|------------|----------|
| $1_A \# 1_H$ | $1_A \# 1_H$ | $1_A \# g$ | $-x \# 1_H$ | $-x \# g$ |
| $1_A \# g$ | $1_A \# g$ | $1_A \# 1_H$ | $x \# g$ | $x \# 1_H$ |
| $x \# 1_H$ | $-x \# 1_H$ | $-x \# g$ | $0$ | $0$ |
| $x \# g$ | $-x \# g$ | $-x \# 1$ | $0$ | $0$ |

coproduct:

$$\Delta(1_A \# 1_H) = 1_A \# 1_H \otimes 1_A \# 1_H, \quad \Delta(1_A \# g) = 1_A \# g \otimes 1_A \# g,$$

$$\Delta(x \# 1_H) = -x \# 1_H \otimes 1_A \# 1_H - 1_A \# 1_H \otimes x \# 1_H,$$

$$\Delta(x \otimes g) = 1_A \# g \otimes x \# g - x \# g \otimes 1_A \# g,$$

and the antipode:

$$S(1_A \# 1_H) = 1_A \# 1_H, \quad S(1_A \# g) = 1_A \# g,$$

$$S(x \# 1_H) = -x \# 1_H, \quad S(x \# g) = x \# g.$$

**Example 2.7.** Let $A$ and $H$ be two Hopf algebras, and $A \# H$ the bicrossproduct of $A$ and $H$ with $A$ a left $H$-module algebra and $H$ a right $A$-comodule coalgebra. Assume that $\alpha_A$ and $\alpha_H$ are the automorphisms of Hopf algebras of $A$ and $H$ respectively, and satisfy the conditions in Proposition 1.1 and Example 2.5 such that $A_{\alpha_A}$ is a left $H_{\alpha_H}$-module algebra and $H_{\alpha_H}$ is a right $A_{\alpha_A}$-comodule coalgebra. Then we have a Hom-type bicrossproduct $A_{\alpha_A} \# H_{\alpha_H}$.

Next we will construct a class of Hom-Hopf algebras.
Corollary 2.8. For any Hom-Hopf algebra \((H, S, \alpha)\), there exists a bicrossproduct structure on the space \((H \otimes H^\text{op}, \alpha \otimes \alpha)\).

Proof Firstly \((H, \alpha)\) is a left \((H^\text{op}, \alpha)\)-module Hom-algebra under the action 
\[
h \cdot a = \sum (S(\alpha^{-2}(h_1))\alpha^{-1}(a))\alpha^{-1}(h_2),
\]
for any \(a, h \in H\). The verification of module condition is straightforward and is left to the reader. For any \(h, a, b \in H\),
\[
\sum (h_1 \cdot a)(h_2 \cdot b)
= \sum (S(\alpha^{-2}(h_{11}))\alpha^{-1}(a))\alpha^{-1}(h_{12})[[S(\alpha^{-2}(h_{21}))\alpha^{-1}(b))\alpha^{-1}(h_{22})]
= \sum (S(\alpha^{-2}(h_{11}))\alpha^{-1}(a))(\alpha^{-2}(h_{12})S(\alpha^{-2}(h_{21}))[ba^{-1}(h_{22})]
= \sum (S(\alpha^{-1}(h_{11}))\alpha^{-1}(a))(\alpha^{-3}(h_{211})S(\alpha^{-2}(h_{212}))[ba^{-1}(h_{22})]
= \sum (S(h_1)a)(bh_2) = \sum (S(h_1)\alpha^{-1}(ab))\alpha(h_2)
= \alpha^2(h) \cdot (ab).
\]
That is, \((H, \alpha)\) is a left \((H^\text{op}, \alpha)\)-module Hom-algebra.

Secondly \((H^\text{op}, \alpha)\) is a right \((H, \alpha)\)-comodule Hom-coalgebra with the right coaction 
\[
\rho(h) = \sum \alpha^{-1}(h_{12}) \otimes S(\alpha^{-2}(h_{11}))\alpha^{-1}(h_2),
\]
for any \(h \in H\). The fact that \((H^\text{op}, \alpha)\) is a right \((H, \alpha)\)-comodule is obvious, and for \(h \in H\),
\[
\sum h_{(0)1} \otimes h_{(0)2} \otimes \alpha^2(h_1)
= \sum \alpha^{-1}(h_{121}) \otimes \alpha^{-1}(h_{122}) \otimes S(h_{11})\alpha(h_2)
= \sum h_{12} \otimes \alpha^{-1}(h_{212}) \otimes [S(\alpha^{-1}(h_{11}))\alpha^{-4}(h_{2111})S(\alpha^{-4}(h_{2112})))h_{22}
= \sum \alpha^{-1}(h_{112}) \otimes \alpha^{-1}(h_{212}) \otimes [S(\alpha^{-2}(h_{111}))\alpha^{-2}(h_{12})S(\alpha^{-3}(h_{211})))]h_{22}
= \sum \alpha^{-1}(h_{112}) \otimes \alpha^{-1}(h_{212}) \otimes [S(\alpha^{-2}(h_{111}))\alpha^{-1}(h_{12})][S(\alpha^{-2}(h_{212}))\alpha^{-1}(h_{22})]
= \sum h_{1(0)} \otimes h_{2(0)} \otimes h_{1(1)}h_{2(1)}.
\]
Then \((H^\text{op}, \alpha)\) is a right \((H, \alpha)\)-comodule Hom-coalgebra.
Finally for any \( h, g, a \in H \),
\[
\sum \langle h \cdot a \rangle_1 \otimes \langle h \cdot a \rangle_2
= \sum (S(\alpha^{-2}(h_{12}))\alpha^{-1}(a_1))\alpha^{-1}(h_{21}) \otimes (S(\alpha^{-2}(h_{11}))\alpha^{-1}(a_2))\alpha^{-1}(h_{22})
= \sum (S(\alpha^{-3}(h_{121}))\alpha^{-1}(a_1))\alpha^{-2}(h_{122}) \otimes [S(\alpha^{-2}(h_{11}))(\alpha^{-4}(h_{211})S\alpha^{-4}(h_{212}))]
\alpha^{-1}(a_2)\alpha^{-2}(h_{22})
= \sum (S(\alpha^{-4}(h_{1211}))\alpha^{-1}(a_1))\alpha^{-3}(h_{1122}) \otimes [S(\alpha^{-3}(h_{1111}))\alpha^{-2}(h_{12})]
[(S\alpha^{-3}(h_{21}))\alpha^{-2}(a_2))\alpha^{-2}(h_{22})]
= \sum (S(\alpha^{-3}(h_{11(0)1}))\alpha^{-1}(a_1))\alpha^{-2}(h_{1(0)2}) \otimes \alpha^{-1}(h_{1(1)})[(S\alpha^{-3}(h_{21}))\alpha^{-2}(a_2))\alpha^{-2}(h_{22})]
= \sum \alpha^{-1}(h_{1(0)}) \cdot a_1 \otimes \alpha^{-1}(h_{1(1)})((\alpha^{-1}(h_2) \cdot \alpha^{-1}(b_2))
\]
and
\[
\sum \langle gh \rangle_0 \otimes \langle gh \rangle_1 = \sum \alpha^{-1}(g_{12}h_{12}) \otimes S(\alpha^{-2}(g_{11}h_{11}))\alpha^{-1}(g_{2}h_{2})
= \sum \alpha^{-1}(g_{12})\alpha^{-1}(h_{12}) \otimes [S(\alpha^{-2}(h_{11}))S(\alpha^{-2}(g_{11}))]\alpha^{-1}(g_2)\alpha^{-1}(h_2)
= \sum \alpha^{-1}(g_{12})\alpha^{-1}(h_{12}) \otimes [S(\alpha^{-3}(g_{11}))\alpha^{-2}(g_2))\alpha^{-1}(h_2)]
= \sum \alpha^{-1}(g_{12})\alpha^{-2}(h_{122}) \otimes [S(\alpha^{-3}(h_{1111}))\alpha^{-2}(h_{12})]
[(S\alpha^{-3}(h_{2121}))S(\alpha^{-4}(g_{11}))\alpha^{-3}(g_2))\alpha^{-2}(h_{22})]
= \sum g_{(0)}\alpha^{-1}(h_{1(0)}) \otimes \alpha^{-1}(h_{1(1)})((\alpha^{-1}(h_2) \cdot \alpha^{-1}(g_{11}))
\]
For the last identity,
\[
\sum h_{2(0)} \otimes (h_1 \cdot a)h_{2(1)}
= \sum \alpha^{-1}(h_{212}) \otimes [(S(\alpha^{-2}(h_{11}))\alpha^{-1}(a))\alpha^{-1}(h_{12})][S(\alpha^{-2}(h_{211}))\alpha^{-1}(h_{22})]
= \sum \alpha^{-1}(h_{212}) \otimes [(S(\alpha^{-2}(h_{11}))\alpha^{-1}(a))\alpha^{-2}(h_{12})S(\alpha^{-3}(h_{211}))]h_{22}
= \sum \alpha^{-1}(h_{221}) \otimes [(S(\alpha^{-2}(h_{11}))\alpha^{-1}(a))\alpha^{-4}(h_{1211})S(\alpha^{-4}(h_{1212})))\alpha(h_2)
= \sum h_{12} \otimes (S(\alpha^{-1}(h_{11}))a)\alpha(h_2)
= \sum h_{12} \otimes S(h_{11})(ah_2)
= \sum h_{12} \otimes [S(\alpha^{-1}(h_{11}))(a^{-3}(h_{211})S(\alpha^{-3}(h_{212})))\alpha^{-1}(h_{22})]
= \sum h_{1(0)} \otimes h_{1(1)}(h_2 \cdot a).
\]
The proof is completed.

Therefore by Proposition 2.5, we have a bicrossproduct structure on \( H \otimes H^{op} \) with multiplication and comultiplication as follows:
\[
(a \triangleright h)(b \triangleright k) = \sum a[(S(\alpha^{-4}(h_{11}))\alpha^{-2}(b))\alpha^{-3}(h_{12})] \triangleright k\alpha^{-1}(h_2),
\]
12
\[ \Delta(a \ltimes h) = \sum a_1 \ltimes \alpha^{-2}(h_{112}) \otimes \alpha^{-1}(a_2) (S(\alpha^{-4}(h_{111})) \alpha^{-3}(h_{12})) \ltimes h_2, \]

for any \( a, b, h, k \in H \).

**Definition 2.9.** Let \((H, \alpha_H)\) be a Hom-bialgebra and \((C, \alpha_C)\) a Hom-coassociative coalgebra. Then \( C \) is called a left \( H \)-module Hom-coalgebra if \( C \) is a left \( H \)-module with the action \( H \otimes C \to C, \ h \otimes c \mapsto h \cdot c \), such that

\[ \Delta(h \cdot c) = \sum h_1 \cdot c_1 \otimes h_2 \cdot c_2, \]

for any \( h \in H, \ c \in C \).

**Definition 2.10.** Let \((A, \alpha_A)\) and \((H, \alpha_H)\) be two Hom-bialgebras. \((A, H)\) is called a matched pair if there exist linear maps

\( \triangleleft : H \otimes A \to H, \triangleright : H \otimes A \to A, \)

turning \( A \) into a left \( H \)-module Hom-coalgebra and turning \( H \) into a right \( A \)-module Hom-coalgebra such that the following conditions are satisfied:

\[ (hg) \triangleleft a = \sum (h \triangleleft (\alpha^{-2}_H(g_1) \triangleright \alpha^{-3}_A(a_1)) (\alpha^{-1}_H(g_2) \triangleleft \alpha^{-2}_A(a_2))), \quad (2.5) \]

\[ h \triangleright (ab) = \sum (\alpha^{-2}_H(h_1) \triangleright \alpha^{-1}_A(a_1)) (\alpha^{-3}_H(h_2) \triangleleft \alpha^{-2}_A(a_2) \triangleright b), \quad (2.6) \]

\[ \sum h_1 \triangleleft a_1 \otimes h_2 \triangleright a_2 = \sum h_2 \triangleleft a_2 \otimes h_1 \triangleright a_1. \quad (2.7) \]

**Proposition 2.11.** Let \((A, S_A, \alpha_A)\) and \((H, S_H, \alpha_H)\) be two Hom-Hopf algebras, and \((A, H)\) is a matched pair. There exists a unique Hom-bialgebra structure on the vector space \( A \otimes H \) such that the multiplication is given by

\[ (a \otimes h)(b \otimes g) = \sum a(\alpha^{-2}_H(h_1) \triangleright \alpha^{-2}_A(b_1)) \otimes (\alpha^{-2}_H(h_2) \triangleleft \alpha^{-2}_A(b_2)) g, \]

the comultiplication is given by

\[ \Delta(a \otimes h) = \sum a_1 \otimes h_1 \otimes a_2 \otimes h_2, \]

and the antipode is given by

\[ S(a \otimes h) = (1_A \otimes S_H \alpha^{-1}_H(h))(S_A \alpha^{-1}_A(a) \otimes 1_H), \]

for any \( a, b \in A \) and \( g, h \in H \).

Equipped with this Hopf algebra structure, \( A \otimes H \) is called double crossed product of \( A \) and \( H \) denoted by \( A \ltimes H \).
Proof Define the linear map $R : H \otimes A \to A \otimes H$ by

$$R(h \otimes a) = \sum a_H^{-2}(h_1) \triangleright \alpha_A^{-2}(b_1) \otimes \alpha_H^{-2}(h_2) \triangleleft \alpha_A^{-2}(b_2),$$

for any $a \in A$ and $h \in H$.

By Proposition 2.6 in [7], in order to prove the multiplication is Hom-associative, we need only to verify that $R$ is a Hom-twisting between $H$ and $A$. Indeed, first easy to see that

$$R(\alpha_H \otimes \alpha_A) = (\alpha_A \otimes \alpha_H)R.$$

Then for any $a, b \in A$ and $h \in H$,

$$\sum (ab)_R \otimes \alpha_H(h)_R$$

$$= \sum a_H^{-1}(h_1) \triangleright \alpha^{-2}(a_1b_1) \otimes \alpha^{-1}_H(h_2) \triangleleft \alpha^{-2}(a_2b_2)$$

$$= \sum a_H^{-1}(h_1) \triangleright \alpha^{-2}(a_1b_1) \otimes \alpha^{-1}_H(h_2) \triangleleft \alpha^{-2}(a_2b_2)$$

$$= \sum (\alpha_H^{-3}(h_{11}) \triangleright \alpha^{-3}(a_{11}))((\alpha_H^{-4}(h_{12}) \triangleright \alpha^{-4}(a_{12})) \triangleright \alpha^{-2}(b_1)) \otimes \alpha^{-1}(h_2) \triangleleft \alpha^{-2}(a_2b_2)$$

$$= \sum (\alpha_H^{-2}(h_1) \triangleright \alpha^{-2}(a_1))((\alpha_H^{-4}(h_2) \triangleleft \alpha^{-4}(a_2)) \triangleright \alpha^{-2}(b_1))$$

$$\otimes \alpha^{-1}_H(h_2) \triangleleft \alpha^{-3}(a_{22}) \triangleleft \alpha^{-1}(h_2)$$

$$= \sum a_Rb_r \otimes \alpha_H(h_{Rr}).$$

Similarly for any $a \in A$ and $g, h \in H$, we have

$$\sum a_A(a)_R \otimes (hg)_R = \sum a_A(a_{Rr}) \otimes h_{r}g_{Rr}.$$

Now $A \otimes H$ is a Hom-associative algebra. Next for any $a, b \in A, h, g \in H$,

$$\Delta((a \otimes h)(b \otimes g))$$

$$= \sum \Delta(a(\alpha_H^{-2}(h_1) \triangleright \alpha^{-2}(b_1)) \otimes \alpha_H^{-2}(h_2) \triangleleft \alpha^{-2}(b_2))g$$

$$= \sum a_1(\alpha_H^{-2}(h_{11}) \triangleright \alpha^{-2}(b_{11})) \otimes (\alpha_H^{-2}(h_{21}) \triangleleft \alpha^{-2}(b_{21}))g_1$$

$$\otimes a_2(\alpha_H^{-2}(h_{12}) \triangleright \alpha^{-2}(b_{12})) \otimes (\alpha_H^{-2}(h_{22}) \triangleleft \alpha^{-2}(b_{22}))g_2$$

$$= \sum a_1(\alpha_H^{-2}(h_{11}) \triangleright \alpha^{-2}(b_{11})) \otimes (\alpha_H^{-2}(h_{12}) \triangleleft \alpha^{-2}(b_{12}))g_1$$

$$\otimes a_2(\alpha_H^{-2}(h_{21}) \triangleright \alpha^{-2}(b_{21})) \otimes (\alpha_H^{-2}(h_{22}) \triangleleft \alpha^{-2}(b_{22}))g_2$$

$$= \Delta(a \otimes h)\Delta(b \otimes g).$$

Therefore $A \otimes H$ is a Hom-bialgebra. It is straightforward to check that $S$ is the antipode. The proof is completed.

Let $(H, \alpha)$ be a Hom-bialgebra. In [8], the authors have defined the Hom-associative algebra $(H^*, (\alpha^{-1})^*)$, the linear dual of $H$, where the multiplication is given by

$$(f \bullet g)(h) = \sum f(\alpha^{-2}(h_1))g(\alpha^{-2}(h_2)),$$
for any \( f, g \in H^* \), and \( h \in H \).

Now we need to define the comultiplication on \( H^* \) by

\[
f(gh) = \sum f_1(\alpha^2(h))f_2(\alpha^2(g)),
\]

for any \( f \in H^* \) and \( g, h \in H \). In other words, \( \langle \Delta(f), h \otimes k \rangle = f(\alpha^{-2}(hk)) \).

**Proposition 2.12.** Let \((H, S, \alpha)\) be a Hom-Hopf algebra. With the multiplication and comultiplication defined on \( H^* \) as above, \((H^*, S^*, (\alpha^{-1})^*)\) is a Hom-Hopf algebra.

**Proof** First we need to prove that \((H^*, (\alpha^{-1})^*)\) is a Hom-coassociative coalgebra. Indeed for any \( g, h, k \in H \) and \( f \in H^* \),

\[
f((gh)\alpha(k)) = \sum f_1(\alpha^2(gh))f_2(\alpha^3(k))
\]

\[
= \sum f_{11}(\alpha^4(g))f_{12}(\alpha^4(h))f_2(\alpha^3(k)),
\]

and

\[
f(\alpha(g)(hk)) = \sum f_1(\alpha^3(g))f_2(\alpha^2(hk))
\]

\[
= \sum f_1(\alpha^3(g))f_{21}(\alpha^4(h))f_{22}(\alpha^4(k)),
\]

therefore \( \sum(\alpha^{-1})^*(f_1) \otimes f_2 \otimes f_{22} = \sum f_{11} \otimes f_{12} \otimes (\alpha^{-1})^*(f_2). \)

Furthermore

\[
\langle \Delta \circ (\alpha^{-1})^*(f), h \otimes k \rangle = (\alpha^{-1})^*(f)(\alpha^{-2}(hk)) = f(\alpha^{-3}(hk)),
\]

and

\[
\sum(\alpha^{-1})^*(f_1)(h)(\alpha^{-1})^*(f_2)(k) = \sum f_1(\alpha^{-1}(h))f_2(\alpha^{-1}(k)) = f(\alpha^{-3}(hk)).
\]

That is, \( \Delta \circ (\alpha^{-1})^* = ((\alpha^{-1})^* \otimes (\alpha^{-1})^*) \circ \Delta \). Then \((H^*, (\alpha^{-1})^*)\) is a Hom-coassociative coalgebra.

Finally for any \( f, g \in H^* \) and \( h, k \in H \),

\[
\langle \Delta(f \bullet g), h \otimes k \rangle = (f \bullet g)(\alpha^{-2}(hk))
\]

\[
= \sum f(\alpha^{-4}(h_1k_1))g(\alpha^{-4}(h_2k_2))
\]

\[
= \sum f_1(\alpha^{-2}(h_1))f_2(\alpha^{-2}(k_1))g_1(\alpha^{-2}(h_2))g_2(\alpha^{-2}(k_2))
\]

\[
= \sum (f_1 \bullet g_1 \otimes f_2 \bullet g_2, h \otimes k).
\]

Therefore \((H^*, (\alpha^{-1})^*)\) is a Hom-bialgebra. It is not hard to verify that \( S^* \) is the antipode.

**Proposition 2.13.** Let \((C, \alpha_C)\) be a Hom-coassociative coalgebra, and \((M, \alpha_M)\) is a right \( C \)-comodule. Then \((M, \alpha_M)\) is a left \((C^*, (\alpha_C^{-1})^*)\)-module with the action
Proposition 2.14. Let \((A, \alpha_A)\) and \((H, \alpha_H)\) be two Hom-bialgebras, \((A \otimes H, \alpha_A \otimes \alpha_H)\) the bicrossproduct of \(A\) and \(H\). Define the left action \(\triangleright : A^* \otimes H \to H\) of \(A^*\) on \(H\) by

\[
f \triangleright h = \sum f(h(1))h(0),
\]

and the right action \(\triangleleft : A^* \otimes H \to A^*\) of \(H\) on \(A^*\) by

\[
\langle f \triangleleft h, a \rangle = \langle f, h \cdot \alpha_A^{-2}(a) \rangle,
\]

for any \(f \in A^*, h \in H\) and \(a \in A\).

Then \((A^*, (\alpha_A^{-1})^*)\) and \((H, \alpha_H)\) is a matched pair.

Proof First, we need to verify that \(H\) is a left \(A^*\)-module Hom-coalgebra. By Proposition 4.4, \(H\) is a left \(A^*\)-module. Then for any \(h \in H\) and \(f \in A^*\),

\[
\Delta(f \triangleright h) = \sum f(h(1))h(0) \otimes h(0)
\]

So \(H\) is a left \(A^*\)-module Hom-coalgebra.

Now for any \(g, h \in H, f \in A^*\) and \(a, b \in A\)

\[
\langle (\alpha_A^{-1})^*(f) \triangleleft (hg), a \rangle = \langle f, \alpha_A^{-1}(hg) \cdot \alpha_A^{-3}(a) \rangle
\]

and obviously

\[
(\alpha_A^{-1})^*(f \triangleleft h) = (\alpha_A^{-1})^*(f) \triangleleft \alpha_A(h).
\]

Then

\[
\langle \Delta(f \triangleleft h), a \otimes b \rangle = \langle f \cdot h, \alpha_A^{-2}(ab) \rangle = \langle f, h \cdot \alpha_A^{-2}(ab) \rangle
\]

Therefore \(A^*\) is a right \(H\)-module Hom-coalgebra.
Next we will verify the compatible conditions in Definition 4.1. For any \( f, g \in A^* \), \( a \in A \) and \( h, k \in H \),

\[
\langle (f \bullet g) \triangleleft h, a \rangle = \langle f \bullet g, h \cdot \alpha_A^{-2}(a) \rangle
\]

\[
= \langle f, \alpha_A^{-2}((h \cdot \alpha_A^{-2}(a))_1) \rangle \langle g, \alpha_A^{-2}((h \cdot \alpha_A^{-2}(a))_2) \rangle
\]

\[
= \langle f, \alpha_H^{-3}(h_1(0)) \cdot \alpha_A^{-4}(a_1) \rangle \langle g, \alpha_H^{-3}(h_1(1)) \rangle \langle g_2, \alpha_H^{-1}(h_2) \rangle \langle g_2, \alpha_A^{-3}(a_2) \rangle
\]

\[
= \langle f, ((\alpha_A^*)^2(g_1) \triangleright \alpha_H^{-3}(h_1)) \rangle \langle g, \alpha_A^{-2}(a_1) \rangle \langle g_2, \alpha_H^{-2}(h_2) \rangle \langle g_2, \alpha_A^{-2}(a_2) \rangle
\]

\[
= \langle f, \triangleright ((\alpha_A^*)^2(g_1) \triangleright \alpha_H^{-3}(h_1)) \rangle \langle g, \alpha_A^{-2}(a_1) \rangle \langle g_2, \alpha_H^{-2}(h_2), \alpha_A^{-2}(a_2) \rangle
\]

\[
= \langle [f \triangleright ((\alpha_A^*)^2(g_1) \triangleright \alpha_H^{-3}(h_1))] \rangle \langle [\alpha_A^*(g_2) \triangleright \alpha_H^{-2}(h_2)], a \rangle,
\]

thus we have the condition (4.1). And by

\[
f \triangleright (hk) = \sum f((hk)(1))(hk)(0)
\]

\[
= \sum \langle f, \alpha_A^{-1}(h_1(1)) \cdot \alpha_H^{-1}(h_2) \cdot \alpha_A^{-1}(k_1(1)) \rangle \alpha_H^{-1}(h_1(0)) \rangle k(0)
\]

\[
= \sum \langle f_1, \alpha_A(h_1(1)) \rangle \langle f_2, \alpha_H(h_2) \cdot \alpha_A(k_1(1)) \rangle \alpha_H^{-1}(h_1(0)) \rangle k(0)
\]

\[
= \sum \langle (\alpha_A^*)^3(f_1), \alpha_A^{-1}(h_1(1)) \rangle \langle (\alpha_A^*)^3(f_2) \triangleright \alpha_H^{-2}(h_2), k_1(1) \rangle \alpha_H^{-1}(h_1(0)) \rangle k(0)
\]

\[
= \sum \langle (\alpha_A^*)^3(f_1) \triangleright \alpha_H^{-1}(h_1) \rangle \langle (\alpha_A^*)^3(f_2) \triangleright \alpha_H^{-2}(h_2) \rangle \triangleright k,
\]

we get the condition (4.2). As for the condition (4.3),

\[
\sum \langle f_1 \triangleleft h_1, a \rangle \langle g, f_2 \triangleright h_2 \rangle = \sum \langle f_1, h_1 \cdot \alpha_A^{-2}(a) \rangle \langle g, h_2(0) \rangle \langle f_2, h_2(1) \rangle
\]

\[
= \sum \langle f, \alpha_A^{-2}((h_1 \cdot \alpha_A^{-2}(a))_2(1)) \rangle \langle g, h_2(0) \rangle
\]

\[
= \sum \langle f, \alpha_A^{-2}(h_1(1)(h_2 \cdot \alpha_A^{-2}(a))) \rangle \langle g, h_2(0) \rangle
\]

\[
= \sum \langle f_1, h_1(1) \rangle \langle f_2, h_2 \cdot \alpha_A^{-2}(a) \rangle \langle g, h_1(0) \rangle
\]

\[
= \sum \langle f_2 \triangleleft h_2, a \rangle \langle g, f_1 \triangleright h_1 \rangle.
\]

Therefore \((A^*, (\alpha_A^{-1})^*)\) and \((H, \alpha_H)\) is a matched pair. The proof is completed.

By this result, we have the double crossed product \((H \bowtie A^*, \alpha_H \otimes (\alpha_A^{-1})^*)\) with the multiplication given by

\[
(h \otimes f)(k \otimes g) = \sum h \alpha_H^{-2}(k_1(0)) \otimes \langle f, \alpha_A^{-2}(k_1(1)) \rangle \alpha_H^{-2}(k_2) \cdot \alpha_A^{-2}(a) \rangle \rangle g,
\]

the comultiplication given by

\[
\Delta(h \otimes f) = \sum h_1 \otimes f_1 \otimes h_2 \otimes f_2,
\]

for any \( h, k \in H \) and \( f, g \in A^* \).
Corollary 2.15. Let \((H, \alpha)\) be a Hom-Hopf algebra. We have the Drinfeld double \((H^{op} \bowtie H^*, \alpha \otimes (\alpha^{-1})^*)\) with the tensor comultiplication and multiplication given by
\[
(h \otimes f)(k \otimes g) = \sum \alpha^{-2}(k_{21})h \otimes [\alpha^{-3}(k_{22}) \mapsto ((\alpha^*)^2(f) \leftarrow S\alpha^{-3}(k_1))]g,
\]
for any \(h, k \in H\) and \(f, g \in H^*\), where \(\langle f \leftarrow h, k \rangle = \langle f, h\alpha^{-2}(k) \rangle\) and \(\langle h \mapsto f, k \rangle = \langle f, \alpha^{-2}(k)h \rangle\).

Example 2.16. Let \(G\) be a finite group and \(\phi\) an automorphism of \(G\). Then \((kG, \phi)\) with the following structure map is a Hom-Hopf algebra:
\[
g \cdot h = \phi(gh), \ \Delta(g) = \phi(g) \otimes \phi(g), \ \varepsilon(g) = 1, \ S(g) = g^{-1}.
\]
Let \(\{e_g\}_{g \in G}\) be the dual basis of the basis of \(kG\). Then we have the Hom-Hopf algebra \(k^G\), dual of \(kG\), with the multiplication
\[
e_g e_h = \delta_{g,h} e_{\phi(g)},
\]
and the comultiplication, counit and antipode
\[
\Delta^*(e_g) = \sum_{uv = \phi(g)} e_u \otimes e_v, \ \varepsilon^*(e_g) = \delta_{g,1}, \ S^*(e_g) = e_{g^{-1}},
\]
for any \(g, h \in G\).

By Corollary 2.15, the multiplication in \(D(kG)\) is given by
\[
(g \otimes e_h)(p \otimes e_q) = \phi(pg) \otimes \delta_{\phi(p)h\phi(p^{-1}),q} e_{\phi(q)}.
\]

Definition 2.17. A quasitriangular monoidal Hom-Hopf algebra is a Hom-Hopf algebra \((H, \alpha_H)\) with an element \(R \in H \otimes H\) satisfying
1. \(\Delta^{op}(x)R = R\Delta(x)\) for any \(x \in H\),
2. \((\Delta \otimes \alpha)R = R^{13}R^{23}\),
3. \((\alpha \otimes \Delta)R = R^{13}R^{12} \).

Example 2.18. Let \(H\) be the monoidal Hom-algebra generated by the elements \(1_H, g, x\) satisfying the following relations:
\[
1_H1_H = 1_H, \ 1_Hg = g1_H = g, \ 1_Hx = x1_H = -x,
\]
\[
g^2 = 1_H, \ x^2 = 0, \ gx = -xg.
\]
The automorphism \(\alpha : H \rightarrow H\) is defined by
\[
\alpha(1_H) = 1_H, \ \alpha(g) = g, \ \alpha(x) = -x, \ \alpha(gx) = -gx.
\]
Then \((H, \alpha)\) is a Hom-associative algebra, and \(\alpha^2 = id\).

Define
\[
\Delta(1_H) = 1_H \otimes 1_H, \quad \Delta(g) = g \otimes g, \\
\Delta(x) = (-x) \otimes g + 1 \otimes (-x), \\
\varepsilon(1_H) = 1, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0, \\
S(1_H) = 1_H, \quad S(g) = g, \quad S(x) = -gx.
\]

Then \((H, \alpha)\) is a monoidal Hom-Hopf algebra. Let \(R = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)\). It is easy to check that \((H, \alpha, R)\) is quasitriangular.

**Proposition 2.19.** The Drinfeld double \((H^{op} \bowtie H^*, \alpha \otimes (\alpha^{-1})^*)\) has the quasitriangular structure
\[
R = \sum 1 \bowtie (\alpha^{-1})^*(e_i) \otimes S^{-1}(e_i) \bowtie \varepsilon,
\]
where \(\{e_i\}\) and \(\{e_i^*\}\) are a base of \(H\) and its dual base in \(H^*\) respectively.

**Proof** For any \(h, k, k' \in H\) and \(f, g, g' \in H^*\), on one hand,
\[
\langle \Delta^{op}(h \otimes f)R, g \otimes k \otimes g' \otimes k' \rangle \\
= \sum \langle (h_2 \otimes f_2)(1 \otimes (\alpha^{-1})^*(e_i)), g \otimes k \rangle \langle (h_1 \otimes f_1)(S^{-1}(e_i) \otimes \varepsilon), g' \otimes k' \rangle \\
= \sum \langle g, \alpha(h_2) \rangle \langle f_2, \alpha^{-2}(k_1) \rangle \langle g', S^{-1}(\alpha^{-5}(k_{221}))h_1 \rangle \\
\quad \langle f_1, \alpha^{-5}(k_{2222}) \rangle \langle \alpha^{-3}(k')S^{-1}(\alpha^{-5}(k_{21})) \rangle \langle f_2, \alpha^{-3}(k_{11}) \rangle \\
= \sum \langle g, \alpha(h_2) \rangle \langle g', S^{-1}(\alpha^{-4}(k_{21}))h_1 \rangle \\
\quad \langle f_1, \alpha^{-4}(k_{22}) \rangle \langle \alpha^{-3}(k')S^{-1}(\alpha^{-5}(k_{12})) \rangle \langle f_2, \alpha^{-3}(k_{11}) \rangle \\
= \sum \langle g, \alpha(h_2) \rangle \langle g', S^{-1}(\alpha^{-4}(k_{21}))h_1 \rangle \\
\quad \langle f, [\alpha^{-6}(k_{22})] \alpha^{-5}(k')S^{-1}(\alpha^{-7}(k_{12})) \rangle \alpha^{-5}(k_{11}) \\
= \sum \langle g, \alpha(h_2) \rangle \langle g', S^{-1}(\alpha^{-4}(k_{21}))h_1 \rangle \\
\quad \langle f, [\alpha^{-6}(k_{22})] \alpha^{-4}(k') \rangle [S^{-1}(\alpha^{-6}(k_{12})) \alpha^{-6}(k_{11})] \\
= \sum \langle g, \alpha(h_2) \rangle \langle g', S^{-1}(\alpha^{-3}(k_1))h_1 \rangle \langle f, [\alpha^{-4}(k_2) \alpha^{-3}(k')] \rangle.
On the other hand,

\[
\langle R\Delta(h \otimes f), g \otimes k \otimes g' \otimes k' \rangle \\
= \sum \langle (1 \otimes (\alpha^{-1})^*(e'))(h_1 \otimes f_1), g \otimes k \rangle \langle (S^{-1}(\alpha(e))) \otimes \varepsilon)(h_2 \otimes f_2), g' \otimes k' \rangle \\
= \sum \langle g, \alpha^{-1}(h_{112}) \rangle \langle e^i, S\alpha^{-3}(h_{111})(\alpha^{-5}(k_1)\alpha^{-3}(h_{122}))(f_1, \alpha^{-2}(k_2)) \rangle \\
\langle g', h_2 S^{-1}(\alpha(e))) \rangle \langle f_2, \alpha^{-1}(k') \rangle \\
= \sum \langle g, \alpha^{-1}(h_{112}) \rangle \langle f, \alpha^{-4}(k_2)\alpha^{-3}(k') \rangle \langle g', h_2[S^{-1}(\alpha^{-5}(k_1)\alpha^{-3}(h_{122}))(\alpha^{-3}(h_{111}))] \rangle \\
= \sum \langle g, h_{12} \rangle \langle f, \alpha^{-4}(k_2)\alpha^{-3}(k') \rangle \langle g', [S^{-1}(\alpha^{-3}(k_1))h_1] \rangle \\
\]

Hence \(\Delta^{op}(h \otimes f)R = R\Delta(h \otimes f)\). Similarly we have \((\Delta \otimes \alpha)R = R^{13}R^{23}, (\alpha \otimes \Delta)R = R^{13}R^{12}\). This completes the proof.

**Example 2.20.** In the Example 2.16, the Drinfeld double of \((kG, \phi)\) is given where \(G\) is a finite group and \(\phi\) is a group automorphism of \(G\). Then by the above proposition, the quasitriangular structure of \(D(kG)\) is

\[
R = \sum_{g \in G} 1_G \otimes e_{\phi(g)} \otimes g^{-1} \otimes 1_{kG}.
\]

3. DUAL PAIRS OF HOM-HOPF-ALGEBRAS

In this section, we will consider dual pairs of Hom-Hopf algebras and then give the Drinfel’d double associated to a pairing of Hom-Hopf algebras.

**Definition 3.1.** Let \((A, \alpha_A)\) and \((B, \alpha_B)\) be two Hom-Hopf algebras and \((-,-): A \otimes B \to k\) a non-degenerate bilinear form. \((A, B)\) is called a dual pair if

\[
(\alpha_A(a), \alpha_B(b)) = (a, b), \quad (aa', b) = (\alpha_A^2(a), b_1)(\alpha_A^2(a'), b_2),
\]

\[
(a, bb') = (a_1, \alpha_B^2(b))(a_2, \alpha_B^2(b')), \quad (a, 1) = \varepsilon(a), \quad (1, b) = \varepsilon(b),
\]

\[
(S_A(a), b) = (\alpha, S_B^{-1}(b)).
\]

**Lemma 3.2** Let \((A, \alpha_A)\) and \((B, \alpha_B)\) be two Hom-Hopf algebras and \((A, B)\) a dual pair. Under the following actions:

\[
b \triangleright a = (a_2, b)a_1, \quad a \triangleleft b = (a_1, b)a_2,
\]

\[
\]
\[ a \triangleright b = (a, b_1) b_2, \quad b \triangleleft a = (a, b_1) b_2, \]

\[ A \text{ is } B\text{-bimodule, and } B \text{ is } A\text{-bimodule.} \]

**Proof** The proof is straightforward.

Then we have two linear maps \( R_1, R_2 : A \otimes B \to A \otimes B \) given by

\[
R_1(a \otimes b) = (\alpha_A(a_2), b_1) \alpha_A^{-1}(a_1) \otimes \alpha_B^{-1}(b_2),
\]

\[
R_2(a \otimes b) = (\alpha_A(a_1), b_2) \alpha_A^{-1}(a_2) \otimes \alpha_B^{-1}(b_1).
\]

Easy to check that both are invertible with the inverses

\[
R_1^{-1}(a \otimes b) = (S^{-1} \alpha_A(a_2), b_1) \alpha_A^{-1}(a_1) \otimes \alpha_B^{-1}(b_2),
\]

\[
R_2^{-1}(a \otimes b) = (S^{-1} \alpha_A(a_1), b_2) \alpha_A^{-1}(a_2) \otimes \alpha_B^{-1}(b_1).
\]

**Proposition 3.3.** Define the linear map \( T = R_1 \circ R_2^{-1} \circ \tau : B \otimes A \to A \otimes B \), where \( \tau \) is the flip map. Then \( T \) is a Hom-twisting map.

**Proof** Obviously \( (\alpha_A \otimes \alpha_B) \circ T = T \circ (\alpha_B \otimes \alpha_A) \). Then for any \( a, a' \in A \) and \( b \in B \),

\[
(\mu_A \otimes \alpha_B)(id \otimes T)(T(b \otimes a) \otimes a')
\]

\[
= \alpha_A^{-2}(a_2 a_2') \otimes \alpha_B^{-2}(b_2 b_2') (S^{-1} \alpha_A^{-1}(a_1 a_1'), \alpha_B^{-1}(b_2 b_2')) (a_2 b_2', \alpha_B^{-1}(b_2 b_2'))
\]

\[
= \alpha_A^{-2}(a_2 a_2') \otimes \alpha_B^{-2}(b_2 b_2') (S^{-1} \alpha_A^{-1}(a_1 a_1'), \alpha_B^{-1}(b_2 b_2')) (a_2 b_2', \alpha_B^{-1}(b_2 b_2'))
\]

\[
= \alpha_A^{-2}(a_2 a_2') \otimes \alpha_B^{-2}(b_2 b_2') (S^{-1} (a_1 a_1'), b_2) (a_2 b_2', b_2)
\]

\[
= T(\alpha_B(b) \otimes a a').
\]

Similarly we have \( T \circ (\mu_B \otimes \alpha_A) = (\alpha_A \otimes \mu_B) \circ (T \otimes id)(id \otimes T) \). Therefore \( T \) is a Hom-twisting map.

Hence we have the Hom-associative algebra \( A \Join B \) with the multiplication

\[
(a \Join b)(a' \Join b') = (S^{-1} \alpha_A(a_1'), b_2)(a_2 b_2', \alpha_B^{-1}(b_2 b_2')) a a \alpha_A^{-2}(a_2 b_2', b_2) b_2'.
\]

**Theorem 3.4.** If we define \( \Delta : A \Join B \to A \Join B \otimes A \Join B \) by

\[
\Delta(a \Join b) = a_1 \Join b_2 \otimes a_2 \Join b_1,
\]

\[ 21 \]
the antipode by

\[ S = T \circ \tau \circ (S_A \otimes S_B^{-1}). \]

Then \( A \bowtie B \) is a Hom-Hopf algebra, which is called the Drinfeld double of \( A \) and \( B \).

**Remark 3.5.** Note that in the our construction, we donnot need the Hom-Hopf algebra finite dimensional. In fact for any finite dimensional Hom-Hopf algebra \((H, \alpha)\), \((H^{op}, H^*)\) is a dual pair if for any \( h \in H \) and \( f \in H^* \), \((a, f) = \langle f, a \rangle \). One can see that the Drinfeld double in this section is really a generalization of that in Corollary 2.15.

## 4. THE DRINFELD’ DOUBLE VERSUS THE HEISENBERG DOUBLE FOR HOM-HOPF ALGEBRAS

In this section, the relation between the Drinfeld double and Heisenberg double is established.

**Definition 4.1.** Let \((H, \alpha)\) be a Hom-Hopf algebra. The linear map \(\sigma : H \otimes H \rightarrow k\) is called a left Hom-2-cocycle if the following conditions are satisfied

\[ \sigma \circ (\alpha \otimes \alpha) = \sigma, \quad (4.1) \]

\[ \sigma(l_1, k_1)\sigma(\alpha^2(h), l_2k_2) = \sigma(h_1, l_1)\sigma(h_2l_2, \alpha^2(k)), \quad (4.2) \]

for any \( h, l, k \in H \). \(\sigma\) is normal if \(\sigma(1, h) = \sigma(h, 1) = \varepsilon(h)\).

Similarly if the condition (4.2) is replaced by

\[ \sigma(\alpha^2(h), l_1k_1)\sigma(l_2, k_2) = \sigma(h_1l_1, \alpha^2(k))\sigma(h_2, l_2), \]

then \(\sigma\) is a right Hom-2-cocycle.

**Proposition 4.2.** Let \((H, \alpha)\) be a Hom-Hopf algebra. If \(\sigma\) is a left Hom-2-cocycle, for any \( h, k \in H \), define multiplication

\[ h \cdot_\sigma k = \sigma(h_1, k_1)\alpha^{-1}(h_2k_2), \]

then \((H, \cdot_\sigma, \alpha)\) is a Hom-associative algebra, called the left twist of \(H\).

And if \(\sigma\) is a right Hom-2-cocycle, define multiplication

\[ h_\cdot k = \alpha^{-1}(h_1k_1)\sigma(h_2, k_2), \]

then \((H, \cdot_\sigma, \alpha)\) is also a Hom-associative algebra, called the right twist of \(H\).
Proof We only prove for the case of left Hom-2-cocycle. The right twist can be proven similarly. First of all, for any $h, k, l \in H$,

$$1 \cdot_\sigma h = h \cdot_\sigma 1 = \alpha(h),$$

and

$$\alpha(h \cdot_\sigma k) = \sigma(h_1, k_1)\alpha(\alpha^{-1}(h_2 k_2)) = \alpha(h) \cdot_\sigma \alpha(k).$$

Then

$$\alpha(h) \cdot_\sigma (k \cdot_\sigma l) = \alpha(h) \cdot_\sigma (\sigma(k_1, l_1)\alpha^{-1}(k_2 l_2))$$
$$= \sigma(k_{11}, l_{11})\alpha^2(h_1), k_{12}l_{12})h_2 \alpha^{-1}(k_2 l_2)$$
$$= \sigma(h_{11}, k_{11})\sigma(h_{12}k_{12}, \alpha^2(l_1))\alpha^{-1}(h_2 k_2)l_2$$
$$= (h \cdot_\sigma k) \cdot_\sigma \alpha(l).$$

We have constructed the Drinfeld double $D(H)$ of $(H, \alpha_H)$, that is the space $H^{op} \otimes H^*$ with the multiplication

$$(h \otimes f)(k \otimes g) = \sum \alpha^{-2}_H(k_{21})h \otimes [\alpha^{-3}_H(k_{22}) \rightarrow ((\alpha^*_H)^2(f) \leftarrow S\alpha^{-3}_H(k_1))]g,$$

for any $h, k \in H$ and $f, g \in H^*$.

**Proposition 4.3.** Assume that $\sigma$ is a left Hom-2-cocycle on the Hom-Hopf algebra $(H, \alpha)$, then the comultiplication $\Delta$ of $H$ makes $\sigma H$ into a right $H$-comodule Hom-algebra. Similar results hold for the algebra $H_\sigma$ if $\sigma$ is a right Hom-2-cocycle.

**Lemma 4.4.** Define $\sigma : D(H) \otimes D(H) \to k$ by

$$\sigma(h \otimes f, k \otimes g) = \varepsilon(h)g(1)\langle f, \alpha(g) \rangle.$$

Then $\sigma$ is a left 2-cocycle on $D(H)$.

**Proof** The proof is straightforward.

The Heisenberg double of a Hom-Hopf algebra $(A, \alpha_A)$, denoted by $H(A)$, is the smash product $A \# A^*$ with respect to the left regular action of $A^*$ on $A$, that is, for any $a, b \in A$ and $f, g \in A^*$,

$$(a \# f)(b \# g) = a(f_1 \circ \alpha^2_A \leftarrow \alpha^{-1}_A(b))\#(f_2 \circ \alpha_A)g.$$
We need to check that $D(A)$ and $H(A^\text{op})$ share the same multiplication. Indeed for any $f, g \in A^*$ and $a, b \in A$,
\[
(a \otimes f) \cdot_\sigma (b \otimes g) = \sigma(a_1 \otimes f_1, b_1 \otimes g_1)(\alpha^{-1} \otimes \alpha^*)((a_2 \otimes f_2)(b_2 \otimes g_2))
\]
\[
= \varepsilon(a_1)\alpha^{-3}(b_{221})\alpha^{-1}(a_2)\langle f_1, \alpha(b_1) \rangle
\]
\[
\otimes [\alpha^{-4}(b_{222}) \rightarrow ((\alpha^3)^*(f_2) \leftarrow S^{-1}\alpha^{-4}(b_{21})))g
\]
\[
= \alpha^{-1}(b_1)a \otimes \langle f, \alpha^{-1}(b_2) \rangle g,
\]
which is exactly the multiplication in $H(A^\text{op})$.

In [8] we have another form of the Drinfeld double $D(A)$, which is the space $(A^\text{op})^* \otimes A$ with the multiplication
\[
(f \otimes a)(g \otimes b) = f[(\alpha_A^{-3}(a_1) \rightarrow (\alpha_A^2(g))) \leftarrow S^{-1}\alpha_A^{-3}(a_{222})] \otimes \alpha_A^{-2}(a_{21})b,
\]
for any $a, b \in A$ and $f, g \in A^*$.

**Lemma 4.6.** Define $\eta : D(A) \otimes D(A) \rightarrow k$ by
\[
\eta(f \otimes a, g \otimes b) = \varepsilon(b)f(1)\langle g, \alpha(a) \rangle.
\]
Then $\sigma$ is a right 2-cocycle on $D(H)$.

**Proof** The proof is straightforward.

**Theorem 4.7.** The Heisenberg double $H(A^*)$ of $A^*$ is the right twist of the Drinfeld double $D(A)$ by the right 2-cocycle given above.

**Proof** Just as in Theorem 4.5, we just compute the multiplication. For any $f, g \in A^*$ and $a, b \in A$,
\[
(f \otimes a) \cdot_\eta (g \otimes b) = (\alpha^* \otimes \alpha^{-1})((f_1 \otimes a_1)(g_1 \otimes b_1))\eta(f_2 \otimes a_2, g_2 \otimes b_2)
\]
\[
= f[(\alpha^{-4}(a_{11}) \rightarrow (\alpha^3)^*(g_2)) \leftarrow S^{-1}\alpha^{-4}(a_{122})] \otimes \alpha^{-3}(a_{121})b\langle f_1, \alpha(a_2) \rangle
\]
\[
= f\langle g, \alpha^{-1}(a_1) \rangle \otimes \alpha^{-1}(a_2)b,
\]
which is exactly the multiplication in $H(A^*)$.

The next corollary follows easily from Proposition 4.3.
Corollary 4.8. (1) The comultiplication of $D(A)$ in Theorem 4.5, considered as a map from $H(A^{op})$ to $H(A^{op}) \otimes D(A)$ makes $H(A^{op})$ into a right $D(A)$-comodule Hom-algebra.
(2) The comultiplication of $D(A)$ in Theorem 4.7, considered as a map from $H(A^*)$ to $D(A) \otimes H(A^*)$ makes $H(A^*)$ into a left $D(A)$-comodule Hom-algebra.

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