Comparing the number of infected vertices in two symmetric sets for Bernoulli percolation (and other random partitions)

Thomas Richthammer

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Abstract

For Bernoulli percolation on a given graph $G = (V, E)$ we consider the cluster of some fixed vertex $o \in V$. We aim at comparing the number of vertices of this cluster in the set $V_+$ and in the set $V_-$, where $V_+, V_- \subset V$ have the same size. Intuitively, if $V_-$ is further away from $o$ than $V_+$, it should contain fewer vertices of the cluster. We prove such a result in terms of stochastic domination, provided that $o \in V_+$, and $V_+, V_-$ satisfy some strong symmetry conditions, and we give applications of this result in case $G$ is a bunkbed graph, a layered graph, the 2D square lattice or a hypercube graph. Our result only relies on general probabilistic techniques and a combinatorial result on group actions, and thus extends to fairly general random partitions, e.g. as induced by Bernoulli site percolation or the random cluster model.

1 Introduction

We consider Bernoulli (bond) percolation on a graph $G = (V, E)$ with parameter $p \in (0, 1)$. This stochastic process is given in terms of a family $(Z_e)_{e \in E}$ of independent $\{0, 1\}$-valued random variables such that $\mathbb{P}_p(Z_e = 1) = p$. Here the index in the notation for probabilities indicates the $p$-dependence. We may think of percolation as the random subgraph of $G$ with vertex set $V$ and edge set $E_Z = \{e \in E : Z_e = 1\}$. The connectedness relation in this random subgraph usually is denoted by $\leftrightarrow$, and the connected component of a vertex $v \in V$ is called the cluster of $v$ and denoted by $C_v$. Thus

$$C_v = \{u \in V : \exists n \geq 0, v_0, \ldots, v_n \in V : v_0 = v, v_n = u, \forall 0 \leq i < n : Z_{v_i v_{i+1}} = 1\}.$$ 

In the case that $G$ is an infinite graph with some regularity, e.g. $G = \mathbb{Z}^d$, Bernoulli percolation is very well studied and there still are various interesting open problems (e.g. see [G1]). We will consider the cluster $C_o$ of a fixed vertex $o \in V$. $o$ could be thought of as the origin of an infection that can be transmitted via edges $e \in E$ such that $Z_e = 1$, so that $C_o$ represents the set of infected vertices. In case of $G$ being a regular lattice such as $\mathbb{Z}^d$ there are classical results on the size of $C_o$ (e.g. see [G1]) and the asymptotic shape of $C_o$ (e.g. see [ACC]). Our focus is on monotonicity properties of the shape.
To motivate our interest, let us consider the cluster of $o = (0, 0)$ for Bernoulli percolation on $G = \mathbb{Z}^2$, and let $W_n$ denote the number of infected vertices in the layer $V_n := \{n\} \times \mathbb{Z}$, i.e. $W_n := |C_o \cap V_n|$. Following the intuition that vertices closer to $o$ can be infected more easily than vertices further away, it is reasonable to expect that $W_n$ decreases with the distance $|n|$ of the layer to the origin $o$. Of course, one has to make sense of the comparison of the number of infected vertices in different layers: One might compare average values and thus expect that $\mathbb{E}_p(W_m) \leq \mathbb{E}_p(W_n)$ for all $0 \leq n \leq m$. Alternatively one might compare the preference of the distributions to take greater values (in the sense of stochastic domination) and thus even expect that $W_m \preceq W_n$ w.r.t. $\mathbb{P}_p$ for all $0 \leq n \leq m$, which is defined to mean $\mathbb{E}_p(f(W_m)) \leq \mathbb{E}_p(f(W_n))$ for all increasing, bounded functions $f : \{0, 1, 2, \ldots\} \cup \{\infty\} \to \mathbb{R}$. It turns out that both of these conjectures are open problems. There is an obvious link to the monotonicity of connection probabilities. E.g. in the above example it should be expected that $\mathbb{P}_p(o \leftrightarrow o + (n, 0))$ is decreasing in $n$ for $n \geq 0$. So far this is only known if $p$ is sufficiently small, see [LPS] and the references therein.

Another famous example of this kind of monotonicity property is the bunkbed conjecture, which will be stated in the following section. Again, not much is known rigorously, see [HL] and the references therein. It seems that these kind of monotonicity properties in general are difficult to establish, which at first glance might seem somewhat surprising. The contribution of this paper is the comparison of the infected number of vertices in two subsets $V_+, V_- \subset V$ in the special case that $o$ is contained in $V_+$ and assuming that $V_+, V_-$ are highly symmetric.

We will state this result in the following section and provide several applications, namely in case that $G$ is a bunkbed graph, a layered graph, the 2D square lattice or a hypercube graph. While our interest is mainly in percolation, the proof of our main result does not use any specific properties of this model, but relies on general probabilistic techniques such as conditioning, a symmetrization procedure similar to the one used in [HL] to prove the bunkbed conjecture for the complete symmetric graph, and a combinatorial argument concerning group actions on pairs of sets. Thus in Section 3 we will briefly revisit some aspects of actions of groups on sets and state and proof our combinatorial result. In Section 4 we will state and prove our generalized result on random partitions that allows for an application not only in Bernoulli bond percolation, but in many other processes that produce random partitions such as Bernoulli site percolation or the random cluster model (see [G2]). In Section 5 we will return to the setting of Bernoulli percolation and infer the results presented in Section 2 from our generalized results.

2 Results

Let us start by reviewing concepts and introducing notations concerning graphs that will be used throughout the paper. Let $G = (V, E)$ be a fixed (simple) graph. We will assume $G$ to be connected and locally finite, i.e. for every $v \in V$ the degree $\deg(v)$ is finite. Note that here finite graphs are considered to be
Let \( C \) a subgroup of \( \text{Aut}(G) \) denote the orbit of vertex \( v \) i.e. the number of edges of the shortest path connecting \( v \) and \( w \). For \( A \subset V \) and \( k \geq 1 \) let
\[
\partial_k A := \{ v \in A : \exists w \in A^c : d(v, w) \leq k \}
\]
denote the (inner) boundary of \( A \) of thickness \( k \). The set of graph automorphisms \( \text{Aut}(G) \) is the set of all bijections on \( V \) such that the edges are preserved in that for all \( v, w \in V \) we have \( vw \in E \) iff \( \varphi(v)\varphi(w) \in E \). A set \( \Gamma \subset \text{Aut}(G) \) is said to act transitively on a set \( V' \subset V \) iff
\[
\forall \varphi \in \Gamma : \varphi(V') = V' \quad \text{and} \quad \forall v, w \in V' \exists \varphi \in \Gamma : \varphi(v) = w.
\]
Let
\[
\Gamma_v(w) = \{ \varphi(w) : \varphi \in \Gamma \text{ s.t. } \varphi(v) = v \}
\]
denote the orbit of vertex \( w \in V \) under all graph automorphisms in \( \Gamma \) fixing \( v \in V \). We are now ready to state the main result of our paper:

**Theorem 1** We consider Bernoulli percolation with parameter \( p \in (0, 1) \) on a connected, locally finite graph \( G = (V, E) \). Let \( C := C_o \) denote the cluster of a fixed vertex \( o \in V \). Let \( V_+, V_- \subset V \) be two disjoint sets of vertices and let \( \Gamma \) be a subgroup of \( \text{Aut}(G) \) such that we have
\[
\text{(Γ1)} \quad \forall \varphi \in \Gamma : \varphi(V_+), \varphi(V_-) \in \{V_+, V_-\},
\]
\[
\text{(Γ2)} \quad \Gamma \text{ acts transitively on } V_+ \cup V_- \quad \text{and}
\]
\[
\text{(Γ3)} \quad \forall v \in V_+, w \in V_- : |\Gamma_v(w)| = |\Gamma_w(v)|
\]
Let \( C_+ := C \cap V_+ \), \( C_- := C \cap V_- \) and \( C_\pm := C_+ \cup C_- \). If \( o \in V_+ \), then we have
\[
|C_-| \leq |C_+| \quad \text{w.r.t. } \mathbb{P}_p(\cdot | |C_\pm| < \infty) \quad \text{and in particular}
\]
\[
\mathbb{E}_p(C_- | |C_\pm| < \infty) \leq \mathbb{E}_p(C_+ | |C_\pm| < \infty) \quad \text{and} \quad \mathbb{E}_p(C_-) \leq \mathbb{E}_p(C_+).
\]

**Remark 1** Some comments on the above theorem:

- As already remarked in the introduction, this result is really about rather general random partitions of \( V_+ \cup V_- \) under suitable symmetry assumptions on \( V_+, V_- \), see Theorem \( \mathbb{[2]} \) in Section \( \mathbb{[4]} \). This generalization implies corresponding results for other interesting stochastic processes such as independent bond percolation with varying edge probabilities (e.g. in \( \mathbb{Z}^2 \) one might choose different percolation parameters \( p, p' \in (0, 1) \) for horizontal and vertical edges respectively), site percolation, see \( \mathbb{[G1]} \), or the random cluster model, see \( \mathbb{[G2]} \).

- Given \( V_+, V_- \) it often makes sense to choose \( \Gamma \) as large as possible, i.e. \( \Gamma := \{ \varphi \in \text{Aut}(G) : \varphi(V_+), \varphi(V_-) \in \{V_+, V_-\} \} \). Then (Γ1) holds trivially and we have the best chance for (Γ2) to hold. Indeed, the only advantage for a different choice for \( \Gamma \) is that it may be easier to verify (Γ3).
• The conditions (Γ1) - (Γ3) should be considered a symmetry assumption on \( V_+ \cup V_- \). The existence of \( \tau \in \text{Aut}(G) \) such that \( \tau(V_+) = V_- \), \( \tau(V_-) = V_+ \) implies that \( V_+, V_- \) are symmetric to each other, and \( \varphi \in \text{Aut}(G) \) such that \( \varphi(V_+) = V_+ \), \( \varphi(V_-) = V_- \) describes internal symmetries of \( V_+ \) and \( V_- \). (Γ1) and (Γ2) are rather strong assumptions. Thus for some graphs there may be only trivial choices for \( V_+, V_- \) for which a suitable \( \Gamma \) exists. However for graphs that are sufficiently symmetric there are interesting choices for \( V_+, V_- \) that allow an application of the theorem, as seen in the following examples.

• (Γ3) on the other hand is a weak assumption. While it is possible to engineer graphs and sets \( V_+, V_- \) so that for all choices of \( \Gamma \) (Γ3) fails, whereas all other conditions are met, in most applications (Γ3) is easily verified. In Section 2 we present various conditions that imply (Γ3) and are indeed easy to check, whenever they hold.

• The conclusion of the theorem contains both a comparison of conditional and unconditional expectations. In case of infinite sets \( V_+, V_- \), in the supercritical phase of percolation only the former is interesting, while in the subcritical phase there is no difference between them.

The following applications are meant to illustrate how to make use of the above theorem. We concentrate on graph classes that appear elsewhere in the literature with some connection to monotonicity questions. These graphs turn out to be Cartesian products: For graphs \( G_1 = (V_1, E_1), \ldots, G_n = (V_n, E_n) \) their Cartesian product \( G_1 \square \ldots \square G_n \) is the graph \( (V, E) \) with \( V = V_1 \times \ldots \times V_n \) and \( E = \{(v_1, \ldots, v_n) : \exists i \in \{1, \ldots, n\} : v_iw_i \in E_i, \forall j \neq i : v_j = w_j \in V_j \} \).

We note that any choice of \( \varphi_i \in \text{Aut}(G_i) \) \((1 \leq i \leq n)\) gives an automorphism \((\varphi_1, \ldots, \varphi_n)\) on the product graph. Our first two applications are quite general, and we need the following graph properties: A graph \( G = (V, E) \) is called

• vertex-transitive if \( \forall v, w \in V \exists \varphi \in \text{Aut}(G) : \varphi(v) = w \).

• vertex-swap-transitive if \( \forall v, w \in V \exists \varphi \in \text{Aut}(G) : \varphi(v) = w, \varphi(w) = v \).

• \(*\)-amenable, if for all \( k \geq 1 \) there is a sequence of finite sets \( V_n \subset V \) such that \( \frac{|\varphi(V)|}{|V|} \to 0 \). (In particular finite graphs are \(*\)-amenable.)

**Corollary 1** Bunkbed graphs. Let \( G = (V, E) \) be a connected, locally finite graph, and \( L_2 \) the line graph on two vertices, i.e. \( L_2 = (\{0, 1\}, \{\{0, 1\}\}) \). We consider the bunkbed graph \( G \square L_2 \) and \( o = (0', 0) \) for some \( 0' \in V \). If \( G \) is either a \(*\)-amenable vertex-transitive graph or a vertex-swap-transitive graph, then the conclusion of Theorem 3 holds for \( V_+ = V \times \{0\} \) and \( V_- = V \times \{1\} \).

Bunkbed graphs are best known from the bunkbed conjecture, which states that for any bunkbed graph \( G \square L_2 \), where \( G = (V, E) \) is a connected finite graph, for all \( p \in (0, 1) \) the corresponding Bernoulli percolation satisfies

\[
\forall 0', v \in V : P_p((0', 0) \leftrightarrow (v, 0)) \geq P_p((0', 0) \leftrightarrow (v, 1)).
\]
Indeed, there are stronger versions of the conjecture, but even in this basic form it is known to be true only for a few special cases (see [R] and the references therein). If true, the bunkbed conjecture would immediately imply $\mathbb{E}_p(\mathcal{C}_+) \geq \mathbb{E}_p(\mathcal{C}_-)$ for $V_+ = V \times \{0\}$ and $V_- = V \times \{1\}$, since $\mathcal{C}_+ = \sum_{v \in V} 1_{\{o+1(v,0)\}}$ and similarly for $\mathcal{C}_-$. Thus the conjecture would imply the expectation part of the above corollary; on the other hand, there is no direct link between the assertion $\mathcal{C}_- \leq \mathcal{C}_+$ and connection probabilities. Finally, we note that the above corollary can easily be generalized: E.g. we could consider graphs with two symmetric layers, where the two layers are connected by edges in a different (but still sufficiently symmetric) way. In case that $G$ is highly symmetric, there might also be other interesting choices of $V_+\!, V_-\!$.

**Corollary 2** Layered graphs. Let $G = (V, E)$ be a connected, locally finite graph, and $L_\infty$ the bi-infinite line graph, i.e. $L_\infty = (\mathbb{Z}, \{\{i, i+1\} : i \in \mathbb{Z}\})$. We consider the layered graph $G \sqcap L_\infty$ and $o = (o', 0)$ for some $o' \in V$. If $G$ is either a $\ast$-amenable, vertex transitive graph or a vertex-swap-transitive graph, then the conclusion of Theorem 1 holds for the following choices of $V_+, V_-:\!$

(a) $V_+ = V \times \{0\}$ and $V_- = V \times \{k\}$ for $k \neq 0$,

(b) $V_+ = V \times n\mathbb{Z}$ and $V_- = V \times (n\mathbb{Z} + k)$ for $1 \leq k < n$.

(c) $V_+ = V \times (2n\mathbb{Z} \cup (2n\mathbb{Z} + k))$ and $V_- = V \times ((2n\mathbb{Z} + n) \cup (2n\mathbb{Z} + n + k))$ for $0 < k < 2n$ with $k \neq n$.

Here it can also be conjectured that for any layered graph $G \sqcap L_\infty$, where $G = (V, E)$ is a finite graph, for all $p \in (0, 1)$ and arbitrary $o := (o', 0)$ the corresponding Bernoulli percolation satisfies

$$\forall v \in V \forall m > n \geq 0 : \mathbb{P}_p(o \leftrightarrow (v, n)) \geq \mathbb{P}_p(o \leftrightarrow (v, m))$$

and thus also

$$\forall m > n \geq 0 : \mathbb{E}_p(|\mathcal{C}_o \cap V \times \{n\}|) \geq \mathbb{E}_p(|\mathcal{C}_o \cap V \times \{m\}|).$$

The latter would immediately imply the expectation part of the above corollary in case of (a). For some related monotonicity results for layered graphs we refer to [KR]. Also we would like to point out that the special case of $G$ being a regular tree (as e.g. considered in [GN]) is covered in the above Corollary, since regular trees are vertex-swap-transitive. Again the above corollary can easily be generalized (similarly to the preceding one). Also the two above examples could easily be generalized to general product graphs $G = G_1 \sqcap ... \sqcap G_n$, where the $G_i$ satisfy suitable conditions. Instead of pursuing possible generalizations, we now turn to a concrete graph, namely the graph that is most prominently featured in Bernoulli percolation.

**Corollary 3** 2D square lattice $\mathbb{Z}^2$. We consider the 2D square lattice $\mathbb{Z}^2 := L_\infty \sqcap L_\infty$ and $o = (0, 0)$. The conclusion of Theorem 1 holds for

(a) $V_+ = \{o, v\}$, $V_- = \{w, v + w\}$ or $V_+ = \{o, v + w\}$, $V_- = \{w, v\}$ for arbitrary $v, w \in \mathbb{Z}^2 \setminus \{o\}$ with $v \perp w$.

(b) $V_+ = \mathbb{Z}v$, $V_- = w + \mathbb{Z}v$ for arbitrary $v \neq o, w \notin \mathbb{Z}v$. 

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(c) $V_+ = \mathbb{Z}v + \mathbb{Z}u, V_- = w + V_+$, for linearly independent $v, u \in \mathbb{Z}^2$ and $w \notin V_+$.

In fact there are many more classes of choices of suitable $V_+, V_-$ satisfying our symmetry assumptions; some of them are illustrated in Figure 1. Instead of attempting to list all of these choices, we would like to point out that in each case the expectation part of the above corollary gives interesting relations between connection probabilities. Writing $c_v(p) := \mathbb{P}_p(o \leftrightarrow v)$, e.g. a particular choice in (a) gives $c_{0,0} + c_{1,1} \geq c_{1,0} + c_{0,1}$, i.e. $1 + c_{1,1} \geq 2c_{1,0}$, and similarly we have $1 + c_{2,0} \geq 2c_{1,1}$. Unless $p$ is small, even these simple relations seem to be nontrivial. It is certainly not feasible to obtain these relations by estimating connection probabilities; it seems they are only obtainable via symmetry considerations like the one underlying our main theorem.

Figure 1: 20 classes of possible choices of $V_+, V_-$ in Corollary 3, where every class is illustrated by a typical representative. Here $\mathbb{Z}^2$ is embedded into $\mathbb{R}^2$ in the canonical way. The vertices of $V_+$ are marked in black, the vertices of $V_-$ are marked in white, and the shown pattern extends periodically. The origin $o$ is in one of the vertices of $V_+$. We note that the classes are not disjoint, and for some of the classes the set $V_\pm = V_+ \cup V_-$ is the same: for classes 1-2 the set consists consists of 4 vertices, for 3-4 all vertices are on a single line, for 5-6 and for 7-11 all vertices are on two parallel lines, for 12,13-15 and 16-20 the vertices are on infinitely many parallel lines. Case (a) of the corollary corresponds to the classes 1,2, case (b) to 4,5, and case (c) to 12,14.

Our last application concerns hypercube graphs. Here $\mathcal{C}_\pm$ is finite. Furthermore we will disregard the conclusion of the theorem concerning stochastic
domination, and we will formulate the conclusion concerning the expectation in terms of relations between connection probabilities as above.

**Corollary 4** Hypercube graphs. For $d \geq 1$ we consider the $d$-dimensional hypercube graph $L_2^d = L_2 \square \ldots \square L_2$ and $o = (0,\ldots,0)$. Let $c_i(p) := \mathbb{P}_p(o \leftrightarrow v)$ where $d(v,o) = i$. For all $k, l \geq 0$ such that $k + l \leq d$ we have

$$\sum_{i=0}^{k} \sum_{j=0}^{l} c_{i+j}(-1)^i \binom{k}{i} \binom{l}{j} \geq 0 \quad \text{and} \quad |\sum_{i=0}^{k} c_i(-1)^i \binom{k}{i}| \geq |\sum_{i=0}^{k} c_{i+l}(-1)^i \binom{k}{i}|.$$  

In particular for the discrete derivatives (i.e. the finite differences) of the sequence at 0. We note that similarly to the above corollary we can obtain relations between connection probabilities in other highl symmetric graphs such as the graphs of (quasi-) regular polyhedra, symmetric complete bipartite graphs or the Petersen graph and its relatives.

### 3 Group actions on pairs of sets

A major ingredient of the proof of our main theorem is a combinatorial result concerning group actions on pairs of sets, which will be presented in this section. For some basics and results on group actions in the context of graph theory see [GR]. We start by reviewing properties of group actions and corresponding notations. Let $\Gamma$ be a group acting on a set $X$. Our notation for the group action will be $g(x)$ for $g \in \Gamma$ and $x \in X$. The action of $\Gamma$ on $X$ naturally extends to subsets via $g(A) := \{g(x) : x \in A\}$ for $g \in \Gamma, A \subset X$. The action of $\Gamma$ on two sets $X, X'$ naturally extends to pairs via $g(x, x') = (g(x), g(x'))$ for $g \in \Gamma, x \in X, x' \in X'$. We will use standard notation for orbits and stabilizers: For a subset $\Gamma' \subset \Gamma$ and for $x \in X$ we write $\Gamma'(x) = \{g(x) : g \in \Gamma'\}$ and similar for sets and pairs of sets. Also we write $\Gamma_x \Gamma_x,A = \{g \in \Gamma : g(x) = x, g(A) = A\}$. We say that a group $\Gamma$ acts transitively on $X$ iff $\forall x,y \in X \exists g \in \Gamma : g(x) = y$.

**Theorem 2** Let $X, X'$ be sets, and let the set of pairs of finite subsets of $X$ and $X'$ be denoted by $\mathcal{M} = \{(A, A') : A \subset X, A' \subset X', |A|, |A'| < \infty\}$. Let $\Gamma$ be a group acting transitively both on $X$ and on $X'$, and suppose that

$$\forall x \in X, x' \in X' : |\Gamma_x(x')| = |\Gamma_{x'}(x)|.$$  

(3.1)
Then for every $\bar{A} = (A, A') \in \hat{M}$ and all $o \in X$, $o' \in X'$

$$[\bar{A}]_{o} := \{ \bar{B} \in \Gamma(\bar{A}) : o \in B \} \text{ and } [\bar{A}]_{o'} := \{ \bar{B} \in \Gamma(\bar{A}), o' \in B' \}$$

we have $||[\bar{A}]_{o}|| \cdot |A'| = ||[\bar{A}]_{o'}|| \cdot |A|$.

If additionally the orbits in (3.1) are finite, then also $||[\bar{A}]_{o}||, ||[\bar{A}]_{o'}|| < \infty$.

**Proof:** Let us first assume that $|A|, |A'| > 0$. We observe that

$$[\bar{A}]_{o} = \bigcup_{x \in A} [A]_{o}^{x} \text{ for } [A]^{x}_{o} := \{ g(\bar{A}) : g \in \Gamma, g(x) = o \}.$$ 

This union is not disjoint. However it is easy to see that $[\bar{A}]_{o} = [A]^{y}_{o}$ for $y \in \Gamma_\bar{A}(x)$, and $[A]^{x}_{o} \cap [A]^{y}_{o} = \emptyset$ otherwise. Noting that $\Gamma_\bar{A}(x) \subset A$ is finite, we thus obtain

$$||[\bar{A}]_{o}|| = \sum_{x \in A} \frac{||[A]^{x}_{o}||}{|\Gamma_\bar{A}(x)|}.$$ 

Next for every $x \in A$ we choose $g_{x} \in \Gamma$ such that $g_{x}(x) = o$ (using the transitivity) and note that $\{ g \in \Gamma : g(x) = o \} = g_{x}\Gamma$. This gives $[A]^{x}_{o} = g_{x}(\Gamma_\bar{A}(\bar{A}))$, which implies $||[A]^{x}_{o}|| = |\Gamma_\bar{A}(\bar{A})|$.

In order to obtain a suitable expression for the last term, we fix some $x' \in A'$. The following lemma (used twice) implies

$$|\Gamma_\bar{A}(\bar{A})| \cdot |\Gamma_{x, \bar{A}}(x')| = |\Gamma_{x}(\bar{A})| = |\Gamma_{x}(x')| \cdot |\Gamma_{x,x'}(\bar{A})|.$$ 

Noting that $\Gamma_{x, \bar{A}}(x') \subset A'$ is also finite, we may combine the above equalities we get

$$||[\bar{A}]_{o}|| = \sum_{x \in A} \frac{|\Gamma_\bar{A}(\bar{A})|}{|\Gamma_{x}(x')|} = \sum_{x \in A} \frac{|\Gamma_{x}(x')| \cdot |\Gamma_{x,x'}(\bar{A})|}{|\Gamma_{x}(x')| \cdot |\Gamma_{x,x'}(\bar{A})|}.$$ 

Again by the following lemma we have $|\Gamma_\bar{A}(x')| \cdot |\Gamma_{x, \bar{A}}(x')| = |\Gamma_{A}(x, x')|$. Summing both sides over all possible choices of $x' \in A'$ we obtain

$$||[\bar{A}]_{o}|| \cdot |A'| = \sum_{x \in A, x' \in A'} \frac{|\Gamma_{x}(x')| \cdot |\Gamma_{x,x'}(\bar{A})|}{|\Gamma_{A}(x, x')|}, \text{ and similarly we have}$$

$$||[\bar{A}]_{o'}|| \cdot |A| = \sum_{x \in A, x' \in A'} \frac{|\Gamma_{x}(x')| \cdot |\Gamma_{x,x'}(\bar{A})|}{|\Gamma_{A}(x, x')|}.$$

Since we have assumed that $|\Gamma_{x}(x')| = |\Gamma_{x}(x')|$, the above sums are equal, which gives the first part of the claim. If the orbits of stabilizers are finite, then also

$$|\Gamma_{x,x'}(\bar{A})| \leq |\Gamma_{x,x'}(A)| \cdot |\Gamma_{x,x'}(A')| \leq |\Gamma_{x}(A)| \cdot |\Gamma_{x}(A')|$$

$$\leq \prod_{y \in A} |\Gamma_{x}(y)| \prod_{y' \in A'} |\Gamma_{x}(y')| < \infty,$$

and thus the above sums are finite, which gives the second part of the claim.

Finally, in case of $|A| = 0$ or $|A'| = 0$ we note that $|A| = 0$ implies that $|A| = 0$ (and similarly for $A'$), so in this case the first claim holds trivially. We also note that $||[\bar{A}((A', A'))_{o}|| \leq ||[\bar{A}(A'), A')]_{o}||$ for all finite $A \subset X, A' \subset X'$ (and similarly for $A'$), so in this case the second claim follows from the finiteness in the main case. \[\Box\]
Lemma 1 Let $\Gamma$ be a group acting on the sets $X$ and $X'$. For $x \in X$, $x' \in X'$ we have
\[
|\Gamma((x,x'))| = |\Gamma(x)| \cdot |\Gamma_x(x')|.
\]

**Proof:** Let $y \in \Gamma(x)$, i.e. $y = g_0(x)$ for some $g_0 \in \Gamma$. We have
\[
\{g(x') : g \in \Gamma \text{ s.t. } g(x) = y\} = \{g(x') : g \in g_0\Gamma_x\} = g_0(\Gamma_x(x')),
\]
which implies
\[
|\Gamma((x,x'))| = \sum_{y \in \Gamma(x)} |\{g(x') : g \in \Gamma \text{ s.t. } g(x) = y\}| = |\Gamma(x)| \cdot |\Gamma_x(x')|
\]
as claimed. \(\Box\)

Remark 2 Concerning condition (3.1).

- **Condition (3.1)** is natural in that it directly gives the claim for the special case of $\bar{A} = (A, A')$ with $|A| = |A'| = 1$. Indeed, for $\bar{A} = (\{x\}, \{x'\})$ we have (using transitivity) $|[\bar{A}]_\varnothing| = |[\bar{A}]_x| = |\Gamma_x(x')|$ and $|[\bar{A}]_{x'}| = |[\bar{A}]_{x'}| = |\Gamma_{x'}(x)|$.

- **Condition (3.1)** is not very explicit. In the following we will see that it trivially holds if $|X| = |X'| < \infty$. We note that under this stronger assumption the proposition can be shown with a less complicated double counting argument, but this assumption is too strong for our purposes.

Lemma 2 Let $\Gamma$ be a group acting transitively both on the set $X$ and on the set $X'$. If $|X| = |X'| < \infty$, then (3.1) holds.

**Proof:** Let $x \in X$, $x' \in X'$. Applying Lemma 1 twice we obtain
\[
|\Gamma(x)| \cdot |\Gamma_x(x')| = \Gamma((x,x')) = |\Gamma(x')| \cdot |\Gamma_{x'}(x)|.
\]
By the given transitivity $\Gamma(x) = X$ and $\Gamma(x') = X'$. combining this with $|X| = |X'| < \infty$ and the above equality we obtain $|\Gamma_x(x')| = |\Gamma_{x'}(x)|$. \(\Box\)

4 Random partitions

Here we formulate and prove a more general version of our main result for random partitions of a set. We first introduce the setting. Let $X$ be a countable set. Let $P_X$ denote the set of all partitions of $X$ into disjoint subsets. For $x, y \in X$ let $\{x \leftrightarrow y\}$ denote the set of all partitions such that $x, y$ belong to the same set of the partition. $\mathcal{F}_X := \sigma(\{x \leftrightarrow y\})$ is a suitable $\sigma$-algebra on $P_X$. A probability distribution $\mathbb{P}$ on $(P_X, \mathcal{F}_X)$ will be called a random partition of $X$. For $x \in X$ and $P \in P_X$ let $\mathcal{C}_x(P)$ denote the set of the partition $P$ containing $x$, so that $y \in \mathcal{C}_x(P)$ iff $P \in \{x \leftrightarrow y\}$. $\mathcal{C}_x$ is a random variable on $(P_X, \mathcal{F}_X)$, called the cluster of $x$. Suitable symmetry assumptions will be formulated via the action of a group $\Gamma$ on $X$. This action naturally extends to $P_X$ via $g(P) = \{g(A) : A \in P\}$ for $P \in P_X$. We note that every $g \in \Gamma$ may thus be considered a measurable map from $(P_X, \mathcal{F}_X)$ to $(P_X, \mathcal{F}_X)$. 

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Theorem 3 Let $X_+, X_-$ be disjoint countable sets and let $\Gamma$ be a group acting on $X := X_+ \cup X_-$ such that

\[(\Gamma'') \forall g \in \Gamma : g(X_+), g(X_-) \in \{X_+, X_-\},\]

\[(\Gamma''') \forall x_+, x_- \in X_+ \times X_- : |\Gamma_{x_+}(x_-)| = |\Gamma_{x_-}(x_+)| < \infty.\]

Let $\mathcal{P}$ be a random partition of $X$, let $\mathcal{C} := \mathcal{C}_o$ denote the cluster of some fixed $o \in X_+$ and let $\mathcal{C}_+ := \mathcal{C} \cap X_+$ and $\mathcal{C}_- := \mathcal{C} \cap X_-$. Suppose that $\mathcal{P}$ is invariant under every $g \in \Gamma$. Then for every bounded $f : \mathbb{N} \to \mathbb{R}$ we have

$$E\left((f(|\mathcal{C}_+|) - f(|\mathcal{C}_-|))1_{(|\mathcal{C}| < \infty)}\right) = \frac{E\left(\left(|\mathcal{C}_+| - |\mathcal{C}_-|\right)(f(|\mathcal{C}_+|) - f(|\mathcal{C}_-|))\right)}{|\mathcal{C}_+| + |\mathcal{C}_-|}1_{(|\mathcal{C}| < \infty)}.$$

In particular this implies that $|\mathcal{C}_-| \leq |\mathcal{C}_+|$ w.r.t. $\mathbb{P}(.||\mathcal{C}| < \infty)$.

Proof: We have $X_+ \neq \emptyset$ and w.l.o.g. also $X_- \neq \emptyset$. By $(\Gamma')$

$$\Gamma = \Gamma_+ \cup \Gamma_-,$$

where

$$\Gamma_+ = \{g \in \Gamma : g(X_+) = X_+, g(X_-) = X_-\}$$

and

$$\Gamma_- = \{g \in \Gamma : g(X_+) = X_-, g(X_-) = X_-.\}.$$

$(\Gamma'')$ implies $\Gamma_- \neq \emptyset$, and fixing some $\tau \in \Gamma_-$ we have $\Gamma_+ = \tau \Gamma_+$. Let

$$M := \{\bar{A} \subset X : |\bar{A}| < \infty\}, \quad M_o := \{A \in M : o \in A\}$$

and for $\bar{A} \in M_o$ let

$$[\bar{A}]_+ := \Gamma_+(\bar{A}) \cap M_o, \quad [\bar{A}]_- := \Gamma_-(\bar{A}) \cap M_o$$

so that $[\bar{A}] = [\bar{A}]_+ \cup [\bar{A}]_-$. We now would like to apply Theorem 2 to the group action $\Gamma_+$ on the sets $X_+$ and $X_-$. By $(\Gamma'')$ both the action on $X_+$ and the action on $X_-$ is transitive, and since $\Gamma_x = (\Gamma_+)_x$ for all $x \in X$, condition \(3.1\) is an immediate consequence of $(\Gamma'')$. Identifying $\bar{A} = (A_+, A_-) \in \bar{M}$ with $\bar{A} = A_+ \cup A_- \in M$ (where $A_+ \subset X_+, A_- \subset X_-$), we note that

$$[\bar{A}]_+ = \{B_+ \cup B_- : (B_+, B_-) \in [\bar{A}]_o\} \quad \text{and} \quad [\bar{A}]_- = \{\tau(B_+ \cup B_-) : (B_+, B_-) \in [\bar{A}]_{\tau^{-1}(o)}\}.$$ 

In particular $|[\bar{A}]_+| = |[\bar{A}]_o|$ and $|[\bar{A}]_-| = |[\bar{A}]_{\tau^{-1}(o)}|$, thus Theorem 2 gives

$$|[\bar{A}]_+| \cdot |A_-| = |[\bar{A}]_-| \cdot |A_+|$$

and

$$|[\bar{A}]_+|, |[\bar{A}]_-| < \infty$$

for every $A \in M_o$.

We now observe that for bounded $f : \mathbb{N} \to \mathbb{R}$ we have

$$E((f(|\mathcal{C}_+|) - f(|\mathcal{C}_-|))1_{(|\mathcal{C}| < \infty)}) = \sum_{B \in M_o} (f(|B_+|) - f(|B_-|))\mathbb{P}(\mathcal{C} = \bar{B})$$

$$= \sum_{\bar{A} \in M_o} \sum_{\bar{B} \in [\bar{A}]} \mathbb{P}(\mathcal{C} = \bar{A}) \frac{f(|B_+|) - f(|B_-|)}{|[A]|} \mathbb{P}(\mathcal{C} = \bar{B})$$

$$= \sum_{\bar{A} \in M_o} \mathbb{P}(\mathcal{C} = \bar{A}) \sum_{\bar{B} \in [\bar{A}]} \frac{f(|B_+|) - f(|B_-|)}{|[A]|}.$$
Here in the first step we have used that on $|\mathcal{C}| < \infty$ we have $\mathcal{C} \in M_o$ and that $M_o$ is countable, since $X$ is countable; in the second step we have used that $[\bar{B}]$ is finite for all $\bar{B} \in M_o$ by the above; in the third step we have used that $\mathbb{P}$ is $\Gamma$-invariant, and that for all $\bar{A}, \bar{B} \in M_o$

$$\bar{A} \in [\bar{B}] \iff [\bar{A}] = [\bar{B}] \iff \bar{B} \in [\bar{A}],$$

which follows from the observation that non-disjoint orbits have to coincide. We have also used Fubini’s theorem, which is justified since $f$ is bounded. We continue to simplify the inner sum: We note that for all $\bar{B} \in [\bar{A}]_+$ we have $|B_+| = |A_+|$ and $|B_-| = |A_-|$, whereas for $\bar{B} \in [\bar{A}]_-$ we have $|B_+| = |A_-|$ and $|B_-| = |A_+|$. In particular for $|A_+| = |A_-|$ the inner sum equals 0. In case of $|A_+| \neq |A_-|$ the sets $[\bar{A}]_+, [\bar{A}]_-$ are disjoint, and we may rewrite the inner sum

$$\sum_{B \in [\bar{A}]} \frac{f(|B_+|) - f(|B_-|)}{|[\bar{A}]|} = \sum_{B \in [\bar{A}]_+} \frac{f(|A_+|) - f(|A_-|)}{|[\bar{A}]|} + \sum_{B \in [\bar{A}]_-} \frac{f(|A_-|) - f(|A_+|)}{|[\bar{A}]|}$$

$$= (f(|A_+|) - f(|A_-|)) \frac{|\bar{A}|_+ - |\bar{A}|_-}{|\bar{A}|_+ + |\bar{A}|_-} = (f(|A_+|) - f(|A_-|)) \frac{|A_+| - |A_-|}{|A_+| + |A_-|}$$

In the last step we have used $\frac{|\bar{A}|_-}{|[\bar{A}]|} = \frac{|A_-|}{|A_+| + |A_-|}$ from above. We have thus shown

$$\mathbb{E}\left((f(|\mathcal{C}_+|) - f(|\mathcal{C}_-|))1_{|\mathcal{C}| < \infty}\right) = \sum_{\bar{A} \in M_o} \frac{|A_+| - |A_-|)(f(|A_+|) - f(|A_-|))}{|A_+| + |A_-|} \mathbb{P}(\mathcal{C} = \bar{A}),$$

which equals the claimed expectation. If $f$ is increasing, the sum is $\geq 0$, which gives $\mathbb{E}(f(|\mathcal{C}_+|)1_{|\mathcal{C}| < \infty}) \geq \mathbb{E}(f(|\mathcal{C}_-|)1_{|\mathcal{C}| < \infty})$ and thus the claimed domination. \hfill $\square$

**Remark 3** Some remarks on the above theorem on random partitions:

- In view of the assumed $\Gamma$-invariance of $\mathbb{P}$, the conditions $(\Gamma_1')$ - $(\Gamma_3')$ on the action of $\Gamma$ should be considered a symmetry assumption on $X_+, X_-$. One should think of $X_-$ as a copy of $X_+$ with both sets exhibiting a high degree of symmetry.

- We note that we can obtain similar results in the same way. E.g. the same proof shows that under the given conditions we have

$$\mathbb{E}\left(|\mathcal{C}_-|1_{|\mathcal{C}| < \infty}\right) = \mathbb{P}(|\mathcal{C}| < \infty, |\mathcal{C}_-| > 0).$$

Assumption $(\Gamma_3')$ is not very explicit; in the following lemma we give a second natural condition which implies the equality in $(\Gamma_3')$.

**Lemma 3** Let $\Gamma$ be a group acting on $X$ and $x, y \in X$ such that for some $g \in \Gamma$ we have $g(x) = y, g(y) = x$, then $|\Gamma_y(x)| = |\Gamma_x(y)|$.

**Proof:** The claim follows from $\Gamma_y(x) = (g^{-1}\Gamma_x g)(x) = g^{-1}(\Gamma_x(y))$. \hfill $\square$
5 Independent bond percolation

We now return to the setting of independent percolation from the first section and proof the results stated there. In order to be able to infer Theorem 3 from Theorem 4 we need some auxiliary results.

Lemma 4 We consider independent bond percolation with parameter \( p \in (0, 1) \) on a connected, locally finite graph \( G = (V, E) \). Let \( C_o \) denote the cluster of a fixed vertex \( o \in V \). Let \( V_+, V_\subset V \) be two disjoint sets of vertices such that for some \( \tau \in \text{Aut}(G) \) we have \( \tau(V_+) = V_- \) and \( \tau(V_-) = V_+ \). Then we have \( \mathbb{E}_p(|C_o \cap V_+|) = \infty \) iff \( \mathbb{E}_p(|C_o \cap V_-|) = \infty \).

Proof: Suppose that \( \mathbb{E}_p(|C_o \cap V_+|) = \infty \). For \( o' := \tau(o) \) we have
\[
\mathbb{E}_p(|C_o \cap V_-|) = \mathbb{E}_p(|C_o' \cap V_+|) \geq \mathbb{E}_p(|C_o \cap V_+|1_{\{o\not\leftrightarrow o'\}})
\]
\[
= \sum_{n \geq 1} \mathbb{P}_p(|C_o \cap V_+| \geq n, o \leftrightarrow o') \geq \sum_{n \geq 1} \mathbb{P}_p(|C_o \cap V_+| \geq n) \mathbb{P}_p(o \leftrightarrow o')
\]
\[
= \mathbb{E}_p(|C_o \cap V_+|) \mathbb{P}_p(o \leftrightarrow o') = \infty.
\]
In the first step we have used the \( \tau \)-invariance of \( \mathbb{P}_p \), in the second step we have used that \( C_o' = C_o \) on \( \{o \not\leftrightarrow o'\} \), in the third and fifth step we used that \( \mathbb{E}(X) = \sum_{n \geq 1} \mathbb{P}(X \geq n) \) for any random variable with values in \( \{0, 1, 2, \ldots \} \cup \{\infty\} \), in the fourth step we have used the FKG-inequality (e.g. see [G]) on the increasing events \( \{\|C_o \cap V_+| \geq n\} \) and \( \{o \leftrightarrow o'\} \), and the last step follows from \( \mathbb{E}_p(|C_o \cap V_+|) = \infty \) and \( \mathbb{P}_p(o \leftrightarrow o') > 0 \). (We have not used the FKG inequality directly on the expectation, since it is usually formulated for random variables that are bounded or have second moments.) We have thus shown that \( \mathbb{E}_p(|C_o \cap V_+|) = \infty \) implies \( \mathbb{E}_p(|C_o \cap V_-|) = \infty \), and the other implication is obtained similarly.

Lemma 5 Let \( G = (V, E) \) be a connected locally finite graph, then \( V \) is countable, and for all \( \Gamma \in \text{Aut}(G) \) and \( x, y \in V \) the orbit \( \Gamma_x(y) \) is finite.

Proof: For \( x \in V \) and \( k \geq 0 \) let \( S_k(x) := \{y \in V : d(x, y) = k\} \). We have \( |S_0(x)| = 1 \) and \( |S_{k+1}(x)| \leq |S_k(x)| \cdot \max\{\text{deg}(v) : v \in S_k(x)\} \). Since \( G \) is locally finite this inductively implies that \( |S_k(x)| \) is finite for all \( k \). In combination with the connectedness of \( G \) this shows that \( V = \bigcup_{k \geq 0} S_k(x) \) is countable and that for all \( x, y \in V \) we have \( |\Gamma_x(y)| \leq |S_{d(x,y)}(x)| < \infty \), noting that graph automorphisms preserve the graph distance.

Proof of Theorem 3: By Lemma 5 \( V \) is countable. We have \( V_+ \neq \emptyset \) and w.l.o.g. we may assume \( V_- \neq \emptyset \). We note that \( \Gamma \) induces a group action on \( V_\pm \). The properties (Γ1') - (Γ3') follow directly from the corresponding properties (Γ1) - (Γ3) and Lemma 5. We note that for Bernoulli percolation on \( G = (V, E) \) the cluster decomposition of \( V \) induces a random partition of \( V_\pm := V_+ \cup V_- \). Its distribution \( \mathbb{P} \) is \( \varphi \)-invariant for every \( \varphi \in \Gamma \), because the distribution of Bernoulli percolation is invariant under all graph automorphisms. We have thus verified all assumptions of Theorem 3 and therefore
we get the stochastic domination as claimed. From this we immediately get the inequality of conditional expectations, since any \(\{0,1,2,\ldots\}\)-valued random variable \(X\) we have \(E(X) = \sum_{n \geq 0} P(X \geq n)\). For the inequality of the unconditional expectations we first note that in case of \(E_p(|\xi|) = \infty\) Lemma 4 implies that \(E_p(|\xi|) = \infty = E_p(|\xi|)\), since (\(\Gamma_1\)), (\(\Gamma_2\)) imply the existence of some \(\tau \in \Gamma\) with \(\tau(V_+) = V_+\) and \(\tau(V_-) = V_+\). In case of \(E_p(|\xi|) < \infty\) we have \(|\xi| < \infty\) \(P_p\)-a.s., i.e. conditional and unconditional expectations are equal.

Before we show how to infer the corollaries from the theorem we collect several sufficient conditions for \((\Gamma_3)\).

**Proposition 1** Let \(G = (V,E)\) be a connected, locally finite graph, let \(V_+,V_- \subset V\) be disjoint, \(V_+ := V_+ \cup V_-\), and let \(\Gamma\) be a subgroup of \(\text{Aut}(G)\) satisfying \((\Gamma_1)\), \((\Gamma_2)\). If one of the following conditions is satisfied, then \(\Gamma\) also satisfies \((\Gamma_3)\):

(a) **Finiteness condition**: \(V_\pm\) is finite.

(b) **Amenability condition**: For all \(k \geq 1\) there is a sequence \(V_n \subset V\) such that 
\[
\frac{|V_+ \cap V_n|}{|V_+ \cap V_n|} \to \frac{1}{2} \quad \text{and} \quad \frac{|V_- \cap V_n|}{|V_- \cap V_n|} \to 0 \quad \text{for} \quad n \to \infty.
\]

(c) **Transitivity condition**: \(\forall v \in V_+, w \in V_- \exists \varphi \in \Gamma : \varphi(v) = w, \varphi(w) = v\).

**Proof**: (a) This follows from Lemma 2. Indeed, by \((\Gamma_1)\) and \((\Gamma_2)\) the group \(\Gamma_+ := \{\varphi \in \Gamma : \varphi(V_+) = V_+, \varphi(V_-) = V_-\}\) acts transitively both on \(V_+\) and \(V_-\), and there is a \(\tau \in \Gamma\) such that \(\tau(V_+) = V_-\) and \(\tau(V_-) = V_+\), which implies \(|V_+| = |V_-|\) and by assumption both are finite. Thus Lemma 2 gives \((\Gamma_+) w(v) = (\Gamma_+) w(v)\) for all \(v \in V_+\) and \(w \in V_-\). Since \((\Gamma_+) u = \Gamma u\) for all \(u \in V_+ \cup V_-\), this gives the claim.

(b) This follows similarly as in the proof of Lemma 2 by a double counting argument. Let \(v \in V_+, w \in V_-\) and set \(k := d(v,w)\). Using \((\Gamma_2)\), for every \(x \in V_+ \cap V_n\) we can choose \(\varphi_x \in \Gamma\) such that \(\varphi_x(v) = x\) and set \(x' := \varphi_x(w)\).

We note that \(\varphi_x(\Gamma_+ w) = \Gamma_+(x')\), which implies \(|\Gamma_+ w| = |\Gamma_+ x'|\), and we note that \((x,y) \in \Gamma_+(v,w) \iff (x,y) \in \Gamma_+(x',x') \iff y \in \Gamma_+ x'(x')\), which gives
\[
\Gamma_+(v,w) \cap V_n = \{(x,y) \in V_n^2 : x \in V_+, y \in \Gamma_+(x')\}, \quad \text{and thus}
|\Gamma_+ \cap V_n| - |\Gamma_+ (v,w) \cap V_n^2| = \sum_{x \in \Gamma_+ \cap V_n} (|\Gamma_+(x')| - |\Gamma_+(x') \cap V_n|).
\]

The last sum is \(\geq 0\), and noting that for all \(u \in \Gamma_+(x')\) we have \(d(x,u) = d(x,x') = d(v,w) = k\), we see that \(\Gamma_+(x') \subset V_n\) for \(x \in V_+ \cap (V_n \setminus \partial V_n)\). Thus the above sum is also
\[
\leq \sum_{x \in \Gamma_+ \cap V_n} |\Gamma_+(x')| \leq |\Gamma_+ \cap \partial V_n| |\Gamma_+(w)| \leq |V_\pm \cap \partial V_n| |\Gamma_+(w)|.
\]

Similarly we obtain
\[
0 \leq |V_+ \cap V_n| |\Gamma_+(w)| \leq |V_\pm \cap \partial V_n| |\Gamma_+(w)|.
\]
Using both of these estimates we see that
\[
\left| \frac{|V_+ \cap V_n||\Gamma_v(w)| - |V_- \cap V_n||\Gamma_w(v)|}{|V_+ \cap V_n|} \right| \leq \left| \frac{|V_+ \cap \partial_k V_n|}{|V_+ \cap V_n|} (|\Gamma_v(w)| + |\Gamma_w(v)|) \right|.
\]
Letting \( n \to \infty \) on both sides and using the amenability condition, this gives
\[
\frac{1}{2}|\Gamma_v(w)| - \frac{1}{2}|\Gamma_w(v)| = 0, \text{ i.e. } |\Gamma_v(w)| = |\Gamma_w(v)| \text{ as desired.}
\]
(c) This follows from Lemma 3.

Finally we will infer the corollaries of Section 2 from Theorem 1. We simply need to check conditions (Γ1)-(Γ3) for a suitable choice of Γ. In fact, we always set
\[
\Gamma := \{ \varphi \in Aut(G) : \varphi(V_+), \varphi(V_-) \in \{V_+, V_-\} \},
\]
so that (Γ1) is trivially satisfied, we check (Γ2) by describing suitable graph automorphisms, and we use the above proposition to check (Γ3).

**Proof of Corollary 1:** For the given choice of \( V_+ = V \times \{0\}, V_- = V \times \{1\} \) let Γ as above. \( G \square L_2 \) is still connected and locally finite. (Γ1) is satisfied. We note that \( (\varphi, \tau) \in \Gamma \) for all \( \varphi \in Aut(G), \tau \in Aut(L_2) \). Thus the vertex-transitivity of \( G \) and \( L_2 \) imply the vertex-transitivity of \( G \square L_2 \). To check (Γ3) we note that for any \( V' \subset V \) we have
\[
\frac{|V_+ \cap (V' \times \{0,1\})|}{|V_+ \cap (V' \times \{0,1\})|} = 1 \quad \text{and} \quad \frac{|\partial_d V' \times \{0,1\}|}{|V' \times \{0,1\}|} = \frac{|(\partial_d V') \times \{0,1\}|}{|V' \times \{0,1\}|} = \frac{|\partial_d V'|}{|V'|},
\]
and thus the *-amenability of \( G \) implies the amenability condition from Proposition 1. We also note that the vertex-swap-transitivity of \( G \) implies the vertex-swap-transitivity of \( G \square L_2 \). Thus in both cases (Γ3) follows from Proposition 1.

**Proof of Corollary 2:** This can be done analogously as the proof of Corollary 1. For any of the given choices of \( V_+, V_- \) let Γ as above. In any case \( V_+ = V \times A_+, V_- = V \times A_- \) for some \( A_+, A_- \subset \mathbb{Z} \) and thus \( V_\pm = V \times A_\pm \) for \( A_\pm = A_+ \cup A_- \). Let \( \Gamma' := \{ \tau \in Aut(L_\infty) : \tau(A_+), \tau(A_-) \in \{A_+, A_-\} \} \). We note that \( (\varphi, \tau) \in \Gamma \) for all \( \varphi \in Aut(G), \tau \in \Gamma' \). It is easy to check that in any case \( \Gamma' \) acts vertex-swap-transitively on \( A_\pm \): For (a) consider the reflection in \( \frac{1}{2} \), for (b) and (c) consider the translations by \( nl \) and the reflections in \( nl + \frac{1}{2} \) for \( l \in \mathbb{Z} \). In particular this implies (Γ2) and if \( G \) is vertex-swap-transitive this also implies (Γ3). To check (Γ3) in case of amenability, it suffices to note that for finite \( V' \subset V \) and \( I_N := \{-N, \ldots, N-1\} \) we have
\[
\frac{|V_+ \cap (V' \times I_N)|}{|V_\pm \cap (V' \times I_N)|} = \frac{|A_+ \cap I_N|}{|A_\pm \cap I_N|} \to \frac{1}{2} \quad \text{for } N \to \infty \quad \text{and}
\]
\[
\frac{|V_+ \cap \partial_d (V' \times I_N)|}{|V_\pm \cap (V' \times I_N)|} \leq \frac{|\partial_d V' \times (A_\pm \cap I_N)|}{|V' \times (A_\pm \cap I_N)|} + \frac{|V' \times (A_\pm \cap \partial_d I_N)|}{|V' \times (A_\pm \cap I_N)|} = \frac{|\partial_d V'|}{|V'|} + \frac{|A_\pm \cap \partial_d I_N|}{|A_\pm \cap I_N|},
\]
and here the first term gets small for appropriate $V'$ by $*$-amenability of $G$ and the second term gets small for $N \to \infty$. (In order to check the above assertions for $N \to \infty$ one needs to consider the cases (a),(b),(c) separately.) This shows the amenability condition of Proposition 1 and thus $(\Gamma 3)$. 

Proof of Corollary 3: This can be done analogously. For any of the given choices of $V_+,V_-$ let $\Gamma$ as above. It is easy to check $(\Gamma 2)$: In case of (a) we have $V_\pm = \{a,v,w,v+w\}$ and $\Gamma$ contains two distinct reflections. In case of (b) $\Gamma$ contains all translations by $nv,n \in \mathbb{Z}$ and all reflections in the points $\frac{n}{2}v + \frac{1}{2}w$, $n \in \mathbb{Z}$. In case of (c) $G$ contains all translations by $nv+mu$, $n,m \in \mathbb{Z}$ and all reflections in the points $\frac{n}{2}v + \frac{m}{2}u + \frac{1}{2}w$, $n,m \in \mathbb{Z}$. $(\Gamma 3)$ follows from the amenability condition of the above proposition. In case of (a) it is trivial, and in cases (b),(c) we choose $V_N = \{-N,\ldots,N\}^2$. In case of (b) $|V_+ \cap V_N|$ grows linearly in $N$ and $|V_+ \cap V_N| - |V_- \cap V_N|$ remains bounded, and $|V_+ \cap \partial_d V_N|$ also remains bounded. In case of (c) $|V_+ \cap V_N|$ grows quadratic in $N$ and $|V_+ \cap V_N| - |V_- \cap V_N|$ grows at most linearly, and $|V_+ \cap \partial_d V_N|$ also grows linearly.

Proof of Corollary 4: We first note that $\mathbb{P}_p(o \leftrightarrow v)$ indeed only depends on the graph distance $d(v,o)$, since for every permutation $\sigma$ of $\{1,\ldots,d\}$ the map $\varphi_{\sigma}$ define by $f_{\sigma}(v_1,\ldots,v_d) = (v_{\sigma(1)},\ldots,v_{\sigma(d)})$ is a graph automorphism of $L^d_G$ with $f_{\sigma}(o) = o$. Let $k,l \ge 0$ such that $k + l \le d$. W.l.o.g. we may assume $k \ge 1$, since otherwise the assertions are trivial. To obtain the first relation we consider the vertex sets $V_+ = \{v \in V : \forall i > k + l : v_i = 0\}$, $V_- = \{v \in V_+ : \sum_{i=1}^k v_i \text{ is even}\}$ and $V_- = \{v \in V_+ : \sum_{i=1}^l v_i \text{ is odd}\}$. Let $\Gamma := \{\varphi \in \text{Aut}(G) : \varphi(V_+),\varphi(V_-) \in \{V_+,V_-\}\}$. Let $\tau_i : V \to V$ denote the reflection of the $i$-th component, i.e. $\tau_i(v) = w$ iff $w_j = v_j$ for all $j \neq i$ and $w_i = 1 - v_i$. For all $i \le k + l$ we have $\tau_i \in \Gamma$. This implies $(\Gamma 2)$. $(\Gamma 3)$ is trivially satisfied by the above proposition, since $L^d_G$ is finite. Thus Theorem 1 implies $\mathbb{E}(\mathbb{C}_+ - \mathbb{E}(\mathbb{C}_-) \ge 0$. Letting $A_{i,j} = \{v \in V_+ : \sum_{m=1}^k v_m = i,\sum_{m=k+1}^l v_m = j\}$ we have

$$\mathbb{E}_p(\mathbb{C}_+) - \mathbb{E}_p(\mathbb{C}_-) = \sum_{v \in V_+} \mathbb{P}_p(o \leftrightarrow v) - \sum_{v \in V_-} \mathbb{P}_p(o \leftrightarrow v)$$

$$= \sum_{i,j} \left( \sum_{v \in V_+ \cap A_{i,j}} \mathbb{P}_p(o \leftrightarrow v) - \sum_{v \in V_- \cap A_{i,j}} \mathbb{P}_p(o \leftrightarrow v) \right)$$

$$= \sum_{i,j} (-1)^i c_{i+j} |A_{i,j}| = \sum_{i,j} (-1)^i c_{i+j} \binom{k}{i} \binom{l}{j}.$$ 

This gives the first relation. For the second relation we choose $V_+ := V_0 \cup V_1$, where $V_0 := \{v \in V : \forall i > k : v_i = 0\}$ and $V_1 := \{v \in V : \forall k < i \le k + l : v_i = 1,\forall i > k + l : v_i = 0\}$. We first set $V_+ = \{v \in V_0 : \sum_{i=1}^k v_i \text{ is even}\} \cup \{v \in V_1 : \sum_{i=1}^k v_i \text{ is odd}\}$ and $V_- = \{v \in V_0 : \sum_{i=1}^k v_i \text{ is odd}\} \cup \{v \in V_1 : \sum_{i=1}^k v_i \text{ is even}\}$. Let $\Gamma := \{\varphi \in \text{Aut}(G) : \varphi(V_+),\varphi(V_-) \in \{V_+,V_-\}\}$. Let $\tau : V \to V$ denote the reflection of the components $k+1,\ldots,k+l$, i.e. $\tau(v) = w$ iff $w_{i} = 1 - v_i$ for all $i \in \{k+1,\ldots,k+l\}$ and $w_i = v_i$ for all $i \notin \{k+1,\ldots,k+l\}$. 

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We have $\tau \in \Gamma$ and $\tau_i \in \Gamma$ for all $i \leq k$. This implies (Γ2). (Γ3) is satisfied, since $L^2$ is finite. Thus Theorem implies $\mathbb{E}(C_+) - \mathbb{E}(C_-) \geq 0$. Letting $A_i = \{v \in V_\pm : \sum_{m=1}^k v_m = i\}$ we have

$$
\mathbb{E}_p(C_+) - \mathbb{E}_p(C_-) = \sum_i \left( \sum_{v \in V_+ \cap A_i} \mathbb{P}_p(o \leftrightarrow v) - \sum_{v \in V_- \cap A_i} \mathbb{P}_p(o \leftrightarrow v) \right)
$$

$$
= \sum_i \left( (-1)^i c_i + (-1)^{i+1} c_{i+l} \right) |A_i| = \sum_i (-1)^i c_i \binom{k}{i} - \sum_i (-1)^i c_{i+l} \binom{k}{i}.
$$

Similarly choosing $V_+ = \{v \in V_0 : \sum_{i=1}^k v_i \text{ is even}\} \cup \{v \in V_1 : \sum_{i=1}^k v_i \text{ is even}\}$ and $V_- = \{v \in V_0 : \sum_{i=1}^k v_i \text{ is odd}\} \cup \{v \in V_1 : \sum_{i=1}^k v_i \text{ is odd}\}$ we get

$$
0 \leq \mathbb{E}_p(C_+) - \mathbb{E}_p(C_-) = \sum_i (-1)^i c_i \binom{k}{i} + \sum_i (-1)^i c_{i+l} \binom{k}{i}.
$$

In combination this gives the second relation. Noting $\Delta[c](m) = c_{m+1} - c_m$ and $\Delta^{k+1}[c] = \Delta[\Delta^k[c]]$, it is easy to see that $\Delta^k[c](m) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} c_{m+i}$ by induction. Thus the first relation in case of $l = 0$ and the second relation give the two assertions on the discrete derivatives. □

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