Toeplitz operators with pluriharmonic symbol

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Let \( m \geq 1 \) be an integer and let \( H_m(\mathbb{B}) \) be the analytic functional Hilbert space on the unit ball \( \mathbb{B} \subset \mathbb{C}^n \) given by the reproducing kernel \( K_m(z, w) = (1 - \langle z, w \rangle)^{-m} \). We prove that Toeplitz operators with pluriharmonic symbol on \( H_m(\mathbb{B}) \) can be characterized by an algebraic identity which extends the classical Brown-Halmos characterization of Toeplitz operators on the Hardy space of the unit disc as well as corresponding results of Louhichi and Olofsson for Toeplitz operators with harmonic symbol on weighted Bergman spaces of the disc.

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§1 Introduction

A result of Brown and Halmos [2] from 1963 shows that an operator \( T \in L(H^2(\mathbb{T})) \) on the Hardy space of the unit disc is a Toeplitz operator \( T_f = P_{H^2(\mathbb{T})}M_f|_{H^2(\mathbb{T})} \) with \( L^\infty \)-symbol \( f \in L^\infty(\mathbb{T}) \) if and only if the operator \( T \) satisfies the algebraic identity \( M_z^*TM_z = T \). In 2008 Louhichi and Olofsson [7] proved that an operator \( T \in L(A_m(\mathbb{D})) \) on the standard weighted Bergman space

\[
A_m(\mathbb{D}) = \{ f \in \mathcal{O}(\mathbb{D}); \| f \|^2 = \frac{m^2 - 1}{\pi} \int_\mathbb{D} |f(z)|^2(1 - |z|^2)^{m-2}dz < \infty \}
\]

is a Toeplitz operator with harmonic symbol \( f \) on \( \mathbb{D} \) if and only if \( T \) satisfies the algebraic identity

\[
M_z^*TM'_z = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} M_z^kTM_z^k.
\]

Here \( M'_z = M_z(M_z^*M_z)^{-1} \in L(A_m(\mathbb{D})) \) is the Cauchy dual of the multiplication operator \( M_z : A_m(\mathbb{D}) \to A_m(\mathbb{D}), g \mapsto zg \).

The weighted Bergman space \( A_m(\mathbb{D}) \) is the analytic functional Hilbert space with reproducing kernel \( K_m(z, w) = (1 - z\overline{w})^{-m} \). The aim of the present note is to extend the above result from [7] to the analytic functional Hilbert spaces \( H_m(\mathbb{B}) \) on the unit ball \( \mathbb{B} \subset \mathbb{C}^n \) given by the reproducing kernels \( K_m(z, w) = (1 - \langle z, w \rangle)^{-m} \), where \( m \geq 1 \) is an arbitrary positive integer. We show that an
operator $T \in L(H_m(\mathbb{B}))$ is a Toeplitz operator with pluriharmonic symbol $f$ on $\mathbb{B}$ if and only if the operator $T$ satisfies the algebraic identity

$$M_z^* TM_z' = P_{\text{im}M_z} \left( \sum_{k=0}^{m-1} (-1)^k \binom{m}{k+1} \sigma_{M_z}(T) \right) P_{\text{im}M_z'}.$$ 

Here $M_z^*$ is the adjoint of the row operator $M_z : H_m(\mathbb{B})^n \to H_m(\mathbb{B})$, $(g_i) \mapsto \sum_{1 \leq i \leq n} z_i g_i$. $M_z^*$ is a suitably defined Cauchy dual of the multiplication tuple $M_z$ and the operator $\sigma_{M_z} \in L(H_m(\mathbb{B}))$ acts as $\sigma_{M_z}(T) = \sum_{1 \leq i \leq n} M_z TM_z^*$.

For $m \in \{1, \ldots, n-1\}$, the multiplication tuple $M_z$ is not subnormal and it is not immediately obvious how to define Toeplitz operators with non-analytic symbols in these cases (see e.g. [1, 12]). Using the fact that pluriharmonic functions on $\mathbb{B}$ admit a decomposition $f = g + \overline{h}$ with analytic functions $g, h \in \mathcal{O}(\mathbb{B})$, we suggest a natural definition of Toeplitz operators with pluriharmonic symbol on analytic functional Hilbert spaces over $\mathbb{B}$.

For $m = 1$, the space $H_1(\mathbb{B})$ is the Drury-Arveson space on the unit ball and the above algebraic characterization reducing the one-dimensional results of Louhichi and Olofsson from [7].

$\S 2$ Results

Let $H_m(\mathbb{B})$ be the functional Hilbert space with reproducing kernel

$$K_m : \mathbb{B} \times \mathbb{B} \to \mathbb{C}, \quad K_m(z, w) = \frac{1}{(1 - \langle z, w \rangle)^m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \langle z, w \rangle^k,$$

where $m > 0$ is a positive integer. Then

$$H_m(\mathbb{B}) = \{ f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}) ; \| f \|^2 = \sum_{\alpha \in \mathbb{N}^n} |f_\alpha|^2 \mu_m(\alpha) < \infty \}$$
with \( \rho_m(\alpha) = \frac{(m+|\alpha|-1)!}{\alpha(m-1)!} \).

The tuple
\[
M_z = (M_{z_1}, \ldots, M_{z_n}) \in L(H_m(\mathbb{B}))^n
\]
consisting of the multiplication operators \( M_{z_i} : H_m(\mathbb{B}) \to H_m(\mathbb{B}) \), \( f \mapsto z_i f \), with the coordinate functions is well defined and its Koszul complex
\[
K^\cdot(M_z, H_m(\mathbb{B})) \xrightarrow{\ell} \mathbb{C} \longrightarrow 0
\]
augmented by the point evaluation \( \epsilon_\lambda : H_m(\mathbb{B}) \to \mathbb{C} \), \( f \mapsto f(\lambda) \), is exact. In particular, the last map in the Koszul complex
\[
H_m(\mathbb{B})^n \xrightarrow{M_z} H_m(\mathbb{B}), \quad (f_i)_{i=1}^n \mapsto \sum_{i=1}^n z_i f_i
\]
has closed range \( M_z H_m(\mathbb{B})^n = \{ f \in H_m(\mathbb{B}) ; \ f(0) = 0 \} \). The above properties of the functional Hilbert spaces \( H_m(\mathbb{B}) \) are well known and follow for instance from the results in Section 4 of [4].

In the following we write \( M_z : H_m(\mathbb{B})^n \to H_m(\mathbb{B}) \) for the row multiplication defined as above, and we write \( M_z^* : H_m(\mathbb{B}) \to H_m(\mathbb{B})^n \), \( f \mapsto (M_z^* f)_{i=1}^n \), for its adjoint. Since the row operator \( M_z \) has closed range, the operator \( M_z^* M_z : \text{Im} \ M_z^* \to \text{Im} \ M_z^* \) is invertible. We use the notation \( (M_z^* M_z)^{-1} \) for its inverse. The space \( H_m(\mathbb{B}) \) admits the orthogonal decomposition
\[
H_m(\mathbb{B}) = \bigoplus_{k=0}^\infty \mathbb{H}_k
\]
into the spaces \( \mathbb{H}_k \subset \mathbb{C}[z] \) consisting of all homogeneous polynomials of degree \( k \). For each function \( f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha \in H_m(\mathbb{B}) \), its homogeneous expansion
\[
f = \sum_{k=0}^\infty f_k \quad \text{with} \quad f_k = \sum_{|\alpha| = k} f_\alpha z^\alpha,
\]
coincides with the decomposition of \( f \) as an element of the orthogonal direct sum \( H_m(\mathbb{B}) = \bigoplus_{k=0}^\infty \mathbb{H}_k \). Let us denote by \( \delta, \Delta : H_m(\mathbb{B}) \to H_m(\mathbb{B}) \) the invertible diagonal operators acting as
\[
\delta(\sum_{k=0}^\infty f_k) = f_0 + \sum_{k=1}^\infty \frac{m+k-1}{k} f_k
\]
and \( \Delta (\sum_{k=0}^\infty f_k) = \sum_{k=0}^\infty \frac{m+k}{k+1} f_k \). One can show that
\[
M_z^* \delta f = (M_z^* M_z)^{-1}(M_z^* f)
\]
for all \( f \in H_m(\mathbb{B}) \) and that the row operator \( \delta M_z : H_m(\mathbb{B})^n \to H_m(\mathbb{B}) \) defines a continuous linear extension of the operator
\[
M_z (M_z^* M_z)^{-1} : \text{Im} \ M_z^* \to H_m(\mathbb{B}).
\]
(Lemma 1 in [3]). The diagonal operators \( \delta \) and \( \Delta \) satisfy the intertwining relation \( \delta M_z = M_z (\oplus \Delta) \) and \( \Delta \) admits the representation
\[
\Delta = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sum_{|\alpha| = j} \gamma_\alpha M_z^\alpha M_z^{m-\alpha} = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma^j_M (1_{H_m(\mathbb{B})})
\]
(Lemma 3 in [3]). Here \( \gamma_{\alpha} = |\alpha|! / \alpha! \) for \( \alpha \in \mathbb{N}^n \) and \( \sigma_{M_z} \in L(H_m(\mathbb{B})) \) is defined by
\[
\sigma_{M_z}(X) = \sum_{i=1}^{n} M_{z_i} X M_{z_i}^*.
\]

In dimension \( n = 1 \), the operator
\[
M'_z = \delta M_z : H_m(\mathbb{B})^n \to H_m(\mathbb{B})
\]
is usually called the Cauchy dual of \( M_z \) (see e.g. [11]).

We begin by showing that the powers of \( M_z \) and \( M_z^* \) satisfy a Brown-Halmos type condition.

**1 Lemma.** For each multiindex \( \gamma \in \mathbb{N}^n \), the identity
\[
M_z^{\gamma} M_z^* = P_{\text{Im}M_z^*} \left( \bigoplus_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j \end{array} \right) \sigma^j_{M_z} (M_z^\gamma) \right) P_{\text{Im}M_z^*}
\]
holds.

**Proof.** Note that \( \text{Ker} M_z = (\text{Im} M_z^*)^\perp \). Hence we obtain
\[
M_z^\gamma M_z^* = P_{\text{Im}M_z^*} \left( M_z^\gamma \delta M_z^* \delta M_z \right) P_{\text{Im}M_z^*}
\]
\[
= P_{\text{Im}M_z^*} \left( (M_z^\gamma M_z)^{-1} M_z^* M_z^\gamma M_z (\oplus \Delta) \right) P_{\text{Im}M_z^*}
\]
\[
= P_{\text{Im}M_z^*} \left( (M_z^\gamma M_z)^{-1} (M_z^* M_z) (\oplus M_z \Delta) \right) P_{\text{Im}M_z^*}
\]
\[
= P_{\text{Im}M_z^*} \left( \bigoplus_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j \end{array} \right) \sigma^j_{M_z} (M_z^\gamma) \right) P_{\text{Im}M_z^*}.
\]

Here we have used the identity \( (M_z^\gamma M_z)^{-1} (M_z^* M_z) = P_{\text{Im}M_z^*} \). \( \square \)

By passing to adjoints we find that also the powers of \( M_z^* \) satisfy the identity
\[
M_z^{\gamma *} M_z^* = P_{\text{Im}M_z^*} \left( \bigoplus_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j \end{array} \right) \sigma^j_{M_z} (M_z^{* \gamma}) \right) P_{\text{Im}M_z^*}.
\]

Let us denote by \( \mathcal{T}_{BH} \subset L(H_m(\mathbb{B})) \) the set of all bounded linear operators on \( H_m(\mathbb{B}) \) that satisfy the Brown-Halmos type condition
\[
M_z^{* \gamma} T M_z' = P_{\text{Im}M_z^*} \left( \bigoplus_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j \end{array} \right) \sigma^j_{M_z} (T) \right) P_{\text{Im}M_z^*}.
\]

Our aim is to show that \( \mathcal{T}_{BH} \) consists precisely of all Toeplitz operators with pluriharmonic symbol. To show that every operator \( T \) in \( \mathcal{T}_{BH} \) is a Toeplitz operator we decompose \( T \) into its homogeneous components.
For this purpose, we denote by $U : \mathbb{R} \to L(H_m(\mathbb{B}))$ the strongly continuous unitary operator group acting as $(U(t)f)(z) = f(e^{ikt}z)$. Then $H_m(\mathbb{B})$ admits the orthogonal decomposition

$$H_m(\mathbb{B}) = \bigoplus_{k \in \mathbb{Z}} H_k,$$

where the spaces $H_k$ are the images of the orthogonal projections $P_k \in L(H_m(\mathbb{B}))$ defined by

$$P_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} U(t) dt.$$

Here the integrand is regarded as a continuous function with values in the locally convex space $L(H_m(\mathbb{B}))$ equipped with the strong operator topology and the integral is defined as a weak integral (see Theorem 3.27 in [10] and Section 20.6(3) in [6]). All operator-valued integrals used in the following should be understood in this sense.

An application of Cauchy’s integral theorem yields that $P_k = 0$ for $k < 0$. For $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, the identity $U(t)P_k = e^{ikt}P_k$ holds. The space $H_k$ consists precisely of all functions $f \in H_m(\mathbb{B})$ with

$$f(e^{ikt}z) = e^{ikt}f(z)$$

for all $t \in \mathbb{R}$ and all $z \in \mathbb{B}$. Thus $H_k = \mathbb{H}_k$ for $k \geq 0$. For a bounded operator $T \in L(H_m(\mathbb{B}))$ and $k \in \mathbb{Z}$, we define its $k$th homogeneous component $T_k \in L(H_m(\mathbb{B}))$ by

$$T_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} U(t) T U(t)^* dt.$$

Let $k \in \mathbb{Z}$. The $k$th homogeneous component of the adjoint $T^*$ of $T$ is given by $(T^*)_k = (T_{-k})^*$. To check this identity it suffices to observe that

$$\langle (T^*)_k f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle e^{-ikt} U(t) T^* U(t)^* f, g \rangle dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \langle f, e^{ikt} U(t) T U(t)^* g \rangle dt = \langle f, T_{-k} g \rangle$$

for all $f, g \in H_m(\mathbb{B})$. Since $U(t)^*H_j = e^{-ijt}1_{H_j}$, we obtain that

$$T_k f = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(k+j)t} U(t) T f dt = P_{k+j} T f \in H_{j+k}$$

for $f \in H_j$. Thus $T_k H_j \subset H_{j+k}$ for all $j \in \mathbb{Z}$. Any bounded operator on $H_m(\mathbb{B})$ with this property will be called homogeneous of degree $k$.

A standard argument using the Fejér kernel $K_N(t) = \sum_{\lvert j \rvert \leq N} (1 - \frac{\lvert j \rvert}{N+1}) e^{ikt}$ as summability kernel (Lemma I.2.2 in [5]) shows that, for each $f \in H_m(\mathbb{B})$,

$$T f = \lim_{N \to \infty} \frac{1}{2\pi} \int_0^{2\pi} K_N(t) U(t) T U(t)^* f dt = \lim_{N \to \infty} \sum_{\lvert j \rvert \leq N} (1 - \frac{\lvert j \rvert}{N+1}) T_k f$$

5
2 Lemma. Let $T \in \mathcal{T}_{BH}$ be given. Then $T_k \in \mathcal{T}_{BH}$ for all $k \in \mathbb{Z}$.

Proof. A straightforward calculation shows that

$$U(t)^* \delta M_z = e^{-it} \delta M_z (\oplus U(t)^*)$$

and hence $M_z^* \delta U(t) = e^{it} (\oplus U(t)) M_z^* \delta$ for $t \in \mathbb{R}$. But then

$$M_z^* \delta T_k \delta M_z = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} (\oplus U(t)) (M_z^* \delta T \delta M_z) (\oplus U(t)^*) dt$$

for $k \in \mathbb{Z}$. Since $U(t)M_z = e^{it} M_z U(t)$ for $t \in \mathbb{R}$ and $j = 1, \ldots, n$, the space $\text{Im} M_z^*$ is reducing for $\oplus U(t)$ and

$$U(t)M_z^* T M_z^{*\alpha} U(t)^* = M_z^* U(t) T U(t)^* M_z^{*\alpha} \quad (t \in \mathbb{R}, \alpha \in \mathbb{N}^n).$$

Using the hypothesis that $T \in \mathcal{T}_{BH}$, we find that $M_z^* \delta T_k \delta M_z$ is given by

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} (\oplus U(t)) P_{\text{Im} M_z^*} \left( \oplus \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma_{M_z}(T) P_{\text{Im} M_z^*} (\oplus U(t)^*) dt \right)$$

$$= P_{\text{Im} M_z^*} \left( \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} (\oplus \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma_{M_z}(U(t)TU(t)^*)) dt \right) P_{\text{Im} M_z^*}$$

$$= P_{\text{Im} M_z^*} \left( \oplus \sum_{j=0}^{m-1} (-1)^j \binom{m}{j+1} \sigma_{M_z}(T_k) \right) P_{\text{Im} M_z^*}.$$

Thus we have shown that $T_k \in \mathcal{T}_{BH}$ for every $k \in \mathbb{Z}$. □

All operators $T \in \mathcal{T}_{BH}$ that are homogeneous of non-negative degree act as multiplication operators.

3 Theorem. Let $T \in \mathcal{T}_{BH}$ be homogeneous of degree $r \in \mathbb{N}$. Then $T$ acts as the multiplication operator

$$Tf = (T1)f \quad (f \in H_m(\mathbb{B})).$$

Proof. Define $q = T1 \in \mathcal{H}_r$. We write $M_q : H_m(\mathbb{B}) \to H_m(\mathbb{B})$, $f \mapsto qf$, for the operator of multiplication with $q$ and show by induction on $k$ that $T = M_q$ on $\mathcal{H}_k$ for all $k \in \mathbb{N}$.

For $k = 0$, this is obvious. Suppose that the assertion has been proved for $j = 0, \ldots, k$ and fix a polynomial $p \in \mathcal{H}_{k+1}$. By Lemma 1 in [3] we have

$$M_z^* \delta T \delta M_z (M_z^* p) = M_z^* \delta T M_z (M_z^* M_z)^{-1} M_z^* p = M_z^* \delta TP_{\text{Im} M_z^*} p$$

$$= M_z^* \delta T p = \frac{m + k + r}{k + r + 1} M_z^* (Tp).$$
Here we ave used that \( p \in \mathbb{H}_{k+1} \subset \mathbb{C}^{\perp} = \text{Im} M_z \).

Using the induction hypothesis and Lemma 3 from [3], we find that

\[
\begin{align*}
P_{\text{Im}M_z}^* \left( \oplus \sum_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j + 1 \end{array} \right) \sigma_j M_z^*(T) \right) P_{\text{Im}M_z}^*(M_z^*p) &= P_{\text{Im}M_z}^* \left( \oplus \sum_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j + 1 \end{array} \right) \sum_{|\alpha|=j} \gamma_{\alpha} M_z^{\alpha} TM_z^{\alpha} \right) (M_z^*p) \\
&= P_{\text{Im}M_z}^* \left( \oplus M_q \sum_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j + 1 \end{array} \right) \sigma_j^* M_z (1_{H_m(\mathbb{B})}) \right) (M_z^*p) \\
&= P_{\text{Im}M_z}^* \left( \oplus M_q \Delta \right) (M_z^*p) = \frac{m+k}{k+1} P_{\text{Im}M_z}^* (\oplus M_q)(M_z^*p).
\end{align*}
\]

By hypothesis

\[
\frac{m+k+r}{k+r+1} M_z^* Tp = \frac{m+k}{k+1} P_{\text{Im}M_z}^* (\oplus M_q) M_z^* p.
\]

By applying the operator \( M_z(M_z^* M_z)^{-1} = \delta M_z|_{\text{Im} M_z^*} \) to both sides of this equation, and by using the identities

\[
(M_z^* M_z)^{-1}(M_z^* M_z) = P_{\text{Im}M_z}^*, \quad M_z(M_z^* M_z)^{-1} M_z^* = P_{\text{Im}M_z},
\]

we find that

\[
\frac{m+k+r}{k+r+1} Tp = M_z(M_z^* M_z)^{-1}\left( \frac{m+k+r}{k+r+1} M_z^* Tp \right)
\]

\[
= M_z(M_z^* M_z)^{-1}\left( \frac{m+k}{k+1} P_{\text{Im}M_z}^* (\oplus M_q) M_z^* p \right)
\]

\[
= \frac{m+k}{k+1} \delta M_z(M_z^* M_z)^{-1}(M_z^* M_z) (\oplus M_q) M_z^* p
\]

\[
= \frac{m+k}{k+1} \delta M_z(\oplus M_q) M_z^* p
\]

\[
= \frac{m+k}{k+1} M_q M_z M_z^* p.
\]

Since \( M_z M_z^* = \sum_{j=1}^{\infty} \frac{j}{m+j-1} P_{\mathbb{H}_j} \) (Proposition 4.3 in [3]), we conclude that

\[
Tp = \frac{m+k}{k+1} M_q \frac{k+1}{m+k} p = M_q p.
\]

This observation completes the induction and hence the whole proof. \( \square \)

In general it is not obvius how to define Toeplitz operators with non-analytic symbols on the full range of all analytic Besov-Sobolev spaces \( H_m(\mathbb{B}) \) \((m \geq 1)\). However, for the case of pluriharmonic symbols, this problem can easily be overcome. To begin with, let us fix a function \( f \in H_m(\mathbb{B}) \). Then the set \( D_f = \{ u \in H_m(\mathbb{B}); \; fu \in H_m(\mathbb{B}) \} \subset H_m(\mathbb{B}) \) is a dense linear subspace which contains \( \mathbb{C}[z] \), and

\[
T_f : D_f \to H_m(\mathbb{B}), \; u \mapsto fu
\]
is a densely defined closed linear operator. For \( g \in H_m(\mathbb{B}) \), let us denote by \( g_\alpha = g^{\alpha}(0)/\alpha! \) its Taylor coefficients at \( z = 0 \). Then for \( \alpha \in \mathbb{N}^n \) and \( u \in D_f \),

\[
\langle fu, z^\alpha \rangle_{H_m(\mathbb{B})} = \frac{(fu)_\alpha}{\rho_m(\alpha)} = \sum_{0 \leq \beta \leq \alpha} f_\beta u_{\alpha-\beta} \frac{\rho_m(\alpha-\beta)}{\rho_m(\alpha)} \rho_m(\alpha).
\]

Therefore the polynomials are contained in the domain of the adjoint \( T_f^* \) of \( T_f \) and

\[
T_f^* z^\alpha = \sum_{0 \leq \beta \leq \alpha} \frac{\rho_m(\alpha-\beta)}{\rho_m(\alpha)} f_\beta z^{\alpha-\beta}
\]

for \( \alpha \in \mathbb{N}^n \). In particular, for any fixed polynomial \( p \in \mathbb{C}[z] \), the mapping

\[
H_m(\mathbb{B}) \to H_m(\mathbb{B}), \ f \mapsto T_f^* p
\]

is conjugate linear and continuous.

We call a bounded linear operator \( T \in L(H_m(\mathbb{B})) \) a Toeplitz operator with pluriharmonic symbol \( f \) if there are functions \( g, h \in H_m(\mathbb{B}) \) with \( f = g + h \) and

\[
T_p = T_g p + T_h^* p \quad \text{for all } p \in \mathbb{C}[z].
\]

An elementary argument shows that, although the representation \( f = g + h \) is only unique up to an additive constant, the right-hand side of the above equation is independent of the choice of \( g \) and \( h \). Furthermore, the function \( f = g + h \) is uniquely determined by \( T \). Indeed, if \( g, h \in H_m(\mathbb{B}) \) satisfy the above relation with \( T = 0 \), then

\[
g + \overline{h}(0) = (T_g + T_h^*)(1) = 0.
\]

But then \( T_h^* z^\alpha = -T_g z^\alpha = \overline{h}(0) z^\alpha \) for all \( \alpha \in \mathbb{N}^n \). Hence \( h_\beta = 0 \) for all \( \beta \in \mathbb{N}^n \setminus \{0\} \) and \( g + h = 0 \). In the following we use the notation \( T_f \) for the Toeplitz operator on \( H_m(\mathbb{B}) \) with pluriharmonic symbol \( f \).

4 Theorem. Let \( T \in \mathcal{T}_{BH} \) be given. Define

\[
g = (T - T_0)(1) \quad \text{and} \quad h = T^*(1).
\]

Then \( T = T_f \) is a Toeplitz operator with pluriharmonic symbol \( f = g + h \).

Proof. For \( k \in \mathbb{Z} \), let as before

\[
T_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} U(t)TU(t)^* dt
\]

denote the \( k \)th homogeneous component of \( T \). Define \( g, h \in H_m(\mathbb{B}) \) and the pluriharmonic function \( f \) as in the statement of the theorem. Our aim is to show that \( T = T_f \). Set

\[
q_k = T_k1 \quad \text{for } k \geq 0, \quad q_k = (T_k)^*1 \quad \text{for } k < 0.
\]
Then
\[
T1 = \lim_{N \to \infty} \sum_{|k| \leq N} (1 - \frac{|k|}{N+1})T_k 1 = \lim_{N \to \infty} \sum_{k=0}^{N} (1 - \frac{k}{N+1})q_k
\]
and
\[
T^*1 = \lim_{N \to \infty} \sum_{k=0}^{N} (1 - \frac{k}{N+1})(T^*_k) 1 = \lim_{N \to \infty} \sum_{k=-N}^{0} (1 - \frac{|k|}{N+1})(T_k)^* 1
\]
\[
= \lim_{N \to \infty} \sum_{k=-N}^{0} (1 - \frac{|k|}{N+1})\bar{q}_k,
\]
where all sequences converge in $H_m(\mathbb{B})$. Since $T, T^* \in T_{BH}$, it follows from Lemma 2 and Theorem 3 that
\[
T_k = T_{T_k(1)} = T_{q_k}, \quad k \geq 0 \quad \text{and} \quad (T^*)_{-k} = T_{(T^*)_{-k}(1)} = T_{(T_k)^* 1} = T_{\bar{q}_k}
\]
for $k < 0$. Let $p \in \mathbb{C}[z]$ be a polynomial. Because of $T_k = (T^*)_{-k}$ we find that
\[
T^*_p = \lim_{N \to \infty} \sum_{|k| \leq N} (1 - \frac{|k|}{N+1})T_k^* p
\]
\[
= \lim_{N \to \infty} \sum_{k=0}^{N} (1 - \frac{k}{N+1})q_k p - T_0(1)p + \sum_{k=-N}^{0} (1 - \frac{|k|}{N+1})T^*_k q_k p
\]
\[
= T_{g^p} + \lim_{N \to \infty} \sum_{k=-N}^{0} (1 - \frac{|k|}{N+1})T^*_k q_k p.
\]
Since the mapping $H_m(\mathbb{B}) \to H_m(\mathbb{B}), u \mapsto T^* u$, is conjugate linear and continuous, we conclude that
\[
T^*_p = T_{g^p} + T_{h^p}.
\]
Thus we have shown that $T = T_f$ with $f = g + \overline{h}$. □

To prove that conversely each Toeplitz operator with pluriharmonic symbol satisfies the Brown-Halmos condition, we use Lemma 1 and an approximation argument.

5 Theorem. Let $T = T_f \in L(H_m(\mathbb{B}))$ be a Toeplitz operator with pluriharmonic symbol $f = g + \overline{h}$, where $g, h \in H_m(\mathbb{B})$. Then $T \in T_{BH}$.

Proof. Let $f = g + \overline{h}$ with $g, h \in H_m(\mathbb{B})$. Let us denote by $g = \sum_{k=0}^{\infty} g_k$ and $h = \sum_{k=0}^{\infty} h_k$ the homogeneous expansions of $g$ and $h$. Again using the continuity of the maps
\[
H_m(\mathbb{B}) \to H_m(\mathbb{B}), f \mapsto T^*_f p \quad (p \in \mathbb{C}[z])
\]
and the fact that $\delta M_z \mathbb{C}[z]^n \subset \mathbb{C}[z]$, we obtain as an application of Lemma 1 that
\[
M^*_z T_f M_z^*(p) = \sum_{k=0}^{\infty} M^*_z \delta T_{g_k} \delta M_z(p_k) + \sum_{k=0}^{\infty} M^*_z \delta T_{h_k} \delta M_z(p_k)
\]
projection of \( L \) with \( f \) for all polynomials \( p, q \) defines an isometric isomorphism between the Banach space of all bounded boundary map \( \phi \) for each tuple \( (p_i) \in \mathbb{C}[z]^n \). Using Lemma 1 in \[3\] we find that
\[
(\oplus M_z^{*\alpha}) P_{\text{Im}M_z} = (\oplus M_z^{*\alpha})(M_z^* M_z)^{-1}(M_z^* M_z) = (\oplus M_z^{*\alpha}) M_z^* \delta M_z.
\]
This identity shows that the operator \((\oplus M_z^{*\alpha}) P_{\text{Im}M_z}\) maps \( \mathbb{C}[z]^n \) into itself. Thus by reversing the above arguments we obtain that
\[
M_z^* T f M_z^*(p_i) = P_{\text{Im}M_z} \left( \oplus \sum_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j+1 \end{array} \right) \sigma^j_M(T_g) \right) P_{\text{Im}M_z}(p_i)
\]
\[
+ P_{\text{Im}M_z} \left( \oplus \sum_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j+1 \end{array} \right) \sigma^j_M(T_h) \right) P_{\text{Im}M_z}(p_i)
\]
\[
= P_{\text{Im}M_z} \left( \oplus \sum_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m \\ j+1 \end{array} \right) \sigma^j_M(T_f) \right) P_{\text{Im}M_z}(p_i)
\]
for each tuple \((p_i) \in \mathbb{C}[z]^n\). Since the polynomials are dense in \( H_m(\mathbb{B}) \), the proof is complete. \( \square \)

We finish by indicating that our definition of Toeplitz operators with pluriharmonic symbol coincides with the usual one on the Hardy space and the weighted Bergman spaces. Let \( m \geq n \) be an integer and let \( f : \mathbb{B} \to \mathbb{C} \) be a bounded pluriharmonic function. Then there are functions \( g, h \in H_m(\mathbb{B}) \) with \( f = g + \overline{h} \) (see for instance Proposition 6.1 in \[13\]). Suppose first that \( m \geq n + 1 \). Let \( T_f = P_{H_m(\mathbb{B})} M_f |_{H_m(\mathbb{B})} \), where \( P_{H_m(\mathbb{B})} \) denotes the orthogonal projection of \( L^2(\mathbb{B}, \mu_m) \) onto \( H_m(\mathbb{B}) \) and \( M_f \) is the operator of multiplication with \( f \) on \( L^2(\mathbb{B}, \mu_m) \) (cf. Section 1). Then
\[
\langle T_f p, q \rangle = \langle f p, q \rangle = \langle g p, q \rangle + \langle p, h q \rangle = \langle T_g p + T_h^* p, q \rangle
\]
for all polynomials \( p, q \in \mathbb{C}[z] \). Next let us consider the case \( m = n \). The boundary map
\[
h^{\infty}(\mathbb{B}) \to L^\infty(S), \quad \varphi \mapsto \varphi^*
\]
defines an isometric isomorphism between the Banach space of all bounded \( \mathcal{M} \)-harmonic functions on \( \mathbb{B} \) equipped with the supremum norm and \( L^\infty(S) \) formed with respect to the normalized surface measure on \( S = \partial \mathbb{B} \). The inverse of this map is given by the Poisson transform (see Chapter 5 in \[9\])
\[
L^\infty(S) \to h^{\infty}(\mathbb{B}), \quad \varphi \mapsto P[\varphi].
\]
For \( \varphi \in h^{\infty}(\mathbb{B}) \), the Toeplitz operator \( T_\varphi : H^2(\mathbb{B}) \to H^2(\mathbb{B}) \) is defined by
\[
T_\varphi(u) = C[\varphi^* u^*],
\]
where the right-hand side denotes the Cauchy integral of $\varphi^*u^* \in L^2(S)$. For $f, g, h$ as above and any pair of polynomials $p, q \in \mathbb{C}[z]$, we obtain

$$\langle T_fp, q \rangle_{H^2(B)} = \langle C[(gp)^*], q \rangle_{H^2(B)} + \langle C[(\overline{hp})^*], q \rangle_{H^2(B)}.$$ 

By Theorem 5.6.8 in [9], we have

$$\langle C[(gp)^*], q \rangle_{H^2(B)} = \langle gp, q \rangle_{H^2(B)}$$

and as an application of Theorem 5.6.9 in [9] we obtain

$$\langle C[(\overline{hp})^*], q \rangle_{H^2(B)} = \langle (\overline{hp})^*, q^* \rangle_{L^2(S)}$$

$$= \langle P_{H^2(S)}(\overline{hp})^*, q^* \rangle_{L^2(S)} = \langle (\overline{hp})^*, q^* \rangle_{L^2(S)}$$

$$= \langle p^*, (h^q)^* \rangle_{L^2(S)} = \langle p, h^q \rangle_{H^2(B)}.$$ 

Thus for $m \geq n$, it follows that

$$\langle T_fp, q \rangle = \langle T_g p + T^*_h p, q \rangle$$

for all polynomials $p, q \in \mathbb{C}[z]$. Hence $T_f = T_f$ on $H_m(B)$ for $m \geq n$. In the particular case $n = 1 = m$ each Toeplitz operator $T_f \in L(H^2(T))$ with $L^\infty$- symbol $f \in L^\infty(S)$ coincides up to unitary equivalence with a Toeplitz operator with harmonic symbol on $\mathbb{D}$, more precisely, $T_f \cong T_{P[f]}$. Thus the results contained in Theorem 4 and Theorem 5 reduce to the classical Brown-Halmos characterization [2] of Toeplitz operators on the Hardy space of the unit disc in this case.

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