STABLE DECOMPOSITIONS OF CERTAIN REPRESENTATIONS OF
THE FINITE GENERAL LINEAR GROUPS

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Abstract. We prove that the irreducible decomposition of the permutation representation
of $\text{GL}_n(F[q])$ on $\text{GL}_n(F[q])/\text{GL}_{n-m}(F[q])$ stabilizes for large $n$. We deduce, as a consequence, a
representation stability theorem for finitely generated $\text{VIC}$-modules.

1. Introduction

Let $k$ be an algebraically closed field of characteristic zero and $F[q]$ a finite field of order $q$. Denote by $G_n$ the finite general linear group $\text{GL}_n(F[q])$. For $r < n$, we consider $G_r$ as a
subgroup of $G_n$ in the standard way, and write $k[G_n]/G_r$ for the permutation representation
of $G_n$ on $G_n/G_r$. In this paper, we show that the collection of multiplicities which occur
in the irreducible decomposition of $k[G_n]/G_{n-m}$ as a representation of $G_n$ is independent of
$n$ for $n \geq 3m$. We prove, in fact, a more precise theorem that describes how the irreducible
decomposition is independent of $n$ for $n \geq 3m$. To state the theorem, we need to recall the
parametrization of the irreducible representations of $G_n$ over $k$.

The irreducible representations of $G_n$ were found by Green [6]. Let $C_n$ be the set of
cuspidal irreducible representations of $G_n$ (up to isomorphism), and let $C = \sqcup_{n \geq 1} C_n$. We
set $d(\rho) = n$ if $\rho \in C_n$. By a partition, we mean a non-increasing sequence of non-negative
integers $(\lambda_1, \lambda_2, \ldots)$ where only finitely many terms are non-zero. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a
partition, we set $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. Let $P$ be the set of partitions. For any function
$\mu : C \rightarrow P$, let

$$\|\mu\| = \sum_{\rho \in C} d(\rho)|\mu(\rho)|.$$ 

It is well-known (see, for example, [9] or [11]) that there is a natural parametrization of the
isomorphism classes of irreducible representations of $G_n$ by functions $\mu : C \rightarrow P$ such that
$\|\mu\| = n$. We write $\varphi(\mu)$ for the irreducible representation of $G_n$ parametrized by $\mu$.

Let $\iota \in C_1$ be the trivial representation of $G_1$. Suppose $\lambda : C \rightarrow P$ is a function and
$\lambda(\iota) = (\lambda_1, \lambda_2, \ldots)$. If $n$ is an integer $\geq \|\lambda\| + \lambda_1$, we define the function $\lambda[n] : C \rightarrow P$ with
$|\lambda[n]| = n$ by

$$\lambda[n](\rho) = \begin{cases} 
(n - \|\lambda\|, \lambda_1, \lambda_2, \ldots) & \text{if } \rho = \iota, \\
\lambda(\rho) & \text{if } \rho \neq \iota.
\end{cases}$$

It is plain that for each function $\mu : C \rightarrow P$ with $\|\mu\| < \infty$, there exists a unique function
$\lambda : C \rightarrow P$ such that $\mu = \lambda[n]$ where $n = \|\mu\|$.

Definition 1. Suppose $V_m, V_{m+1}, \ldots$ is a sequence where each $V_n$ is a representation of $G_n$
over $k$. We say that the sequence $\{V_n\}$ is multiplicity stable if there exists an integer $N \geq m$
such that in the irreducible decomposition
\[ V_n = \bigoplus_{\lambda} \varphi(\lambda[n])^{c_n(\lambda)} \] (where \(0 \leq c_n(\lambda) \leq \infty\)),
the multiplicities \(c_n(\lambda)\) do not depend on \(n\) for \(n \geq N\); in particular, for any \(\lambda\) such that \(\lambda[N]\) is not defined, one has \(c_n(\lambda) = 0\) for every \(n \geq N\). We call the smallest such \(N\) the multiplicity stability degree of \(\{V_n\}\).

**Theorem 2.** Fix a non-negative integer \(m\), and let \(V_n = k[G_n/G_{n-m}]\) for \(n \geq m\). The sequence \(\{V_n\}\) is multiplicity stable with multiplicity stability degree \(\leq 3m\).

Our motivation to consider the question of multiplicity stability for the sequence in the above theorem comes from representation stability theory, in which a similar phenomenon for representations of symmetric groups is studied in [1] and [2]; see [3] for a survey. In particular, an analog of Theorem 2 for symmetric groups was proved by Hemmer [7, Theorem 2.4] using Pieri’s rule. Our proof of Theorem 2 uses a branching rule for the finite general linear groups due to Thoma [10, Satz 2] and Zelevinsky [11, Corollary 13.8].

In more detail, suppose \(V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_2} \cdots\) is a sequence where each \(V_n\) is a representation of \(G_n\) over \(k\) and each \(\phi_n\) is a \(k\)-linear map. We say that \(\{V_n, \phi_n\}\) is a consistent sequence if, for every non-negative integer \(n\) and for every \(g \in G_n\), the following diagram commutes:

\[
\begin{array}{ccc}
V_n & \xrightarrow{\phi_n} & V_{n+1} \\
g \downarrow & & \downarrow g \\
V_n & \xrightarrow{\phi_n} & V_{n+1}
\end{array}
\]

(where \(g\) acts on \(V_{n+1}\) by considering it as an element of \(G_{n+1}\)). In analogy with [2, Definition 2.6], we made the following definition in [5, Definition 1.5].

**Definition 3.** A consistent sequence \(\{V_n, \phi_n\}\) is representation stable if the following conditions are satisfied:

1. (RS1) For all \(n\) sufficiently large, the map \(\phi_n : V_n \to V_{n+1}\) is injective.
2. (RS2) For all \(n\) sufficiently large, the span of the \(G_{n+1}\)-orbit of \(\phi_n(V_n)\) is all of \(V_{n+1}\).
3. (RS3) The sequence \(\{V_n\}\) is multiplicity stable.

Let \(\mathcal{V}\) be the category whose objects are the finite dimensional vector spaces over \(\mathbb{F}_q\) and whose morphisms are the pairs \((f, K)\) where \(f\) is an injective linear map and \(K\) is a complementary subspace to the image of \(f\). More precisely, a morphism \(X \to Y\) in the category \(\mathcal{V}\) is the data of an injective linear map \(f : X \to Y\) and a subspace \(K\) of \(Y\) such that \(Y\) is the internal direct sum of \(f(X)\) and \(K\). The composition of morphisms is defined in the natural way: \((g, L) \circ (f, K) = (g \circ f, g(K) \oplus L)\). Observe that every endomorphism in the category \(\mathcal{V}\) is an automorphism, and the group of automorphisms of \(\mathbb{F}_q^n\) as an object of \(\mathcal{V}\) is the group \(G_n\).

A \(\mathcal{V}\)-module over a commutative ring \(R\) is, by definition, a functor from \(\mathcal{V}\) to the category of \(R\)-modules. We shall recall the notion of finite generation for \(\mathcal{V}\)-modules in Section 3. It should be noted that \(\mathcal{V}\)-modules were first introduced and studied by Putman and Sam in [5]; they proved, in particular, that finitely generated \(\mathcal{V}\)-modules
over a noetherian ring has an inductive description named central stability (see [8, Theorem E]). For a finitely generated VIC-module \( V \) over \( k \), our next result gives a description of the asymptotic behaviour of the sequence \( \{ V(F_n^q) \} \) as representations of the groups \( G_n \). A VIC-module \( V \) over \( k \) defines a consistent sequence \( \{ V_n, \phi_n \} \) where \( V_n = V(F_n^q) \) and \( \phi_n \) is induced by the morphism \( (f_n, K_n) \) where \( f_n \) is the standard inclusion \( F_n^q \hookrightarrow F_{n+1}^q \) and \( K_n \) is the subspace of vectors in \( F_{n+1}^q \) whose first \( n \) coordinates are 0.

**Theorem 4.** Let \( V \) be a VIC-module over \( k \) and \( \{ V_n, \phi_n \} \) the consistent sequence obtained from \( V \). Then \( V \) is finitely generated if and only if \( \{ V_n, \phi_n \} \) is representation stable and \( \dim(V_n) < \infty \) for every \( n \). Moreover, if \( V \) is finitely generated, then there exists a polynomial \( P \in \mathbb{Q}[T] \) and an integer \( N \) such that \( \dim(V_n) = P(q^n) \) for all \( n \geq N \).

As we shall explain in Section 3, Theorem 4 generalizes the main results of [5]. However, the proof of Theorem 4 depends crucially on the key propositions in [5] and also uses Theorem 2.

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2. **Branching rule for finite general linear groups**

2.1. **Branching rule.** The branching rule of Thoma [10] for the groups \( G_n \) gives a formula for the multiplicity of an irreducible representation of \( G_{n-1} \) in the restriction of an irreducible representation of \( G_n \) to \( G_{n-1} \). We shall use an equivalent combinatorial description of the multiplicity due to Zelevinsky [11].

First, we introduce some notations.

**Notation 5.** Suppose that \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \) are partitions.

We write \( \lambda \vdash \mu \) if

\[
\lambda_i \leq \mu_i \leq \lambda_i + 1 \quad \text{for every } i,
\]
or equivalently, if the Young diagram of \( \mu \) can be obtained by adding at most one box to each row in the Young diagram of \( \lambda \).

We write \( \mu \dashv \lambda \) if

\[
\mu_i - 1 \leq \lambda_i \leq \mu_i \quad \text{for every } i,
\]
or equivalently, if the Young diagram of \( \lambda \) can be obtained by removing at most one box from each row in the Young diagram of \( \mu \).

Obviously, one has \( \lambda \vdash \mu \) if and only if \( \mu \dashv \lambda \).

**Notation 6.** Suppose we have functions \( \lambda: \mathcal{C} \rightarrow \mathcal{P} \) and \( \mu: \mathcal{C} \rightarrow \mathcal{P} \). We write \( \lambda \vdash \mu \) if \( \lambda(\rho) \vdash \mu(\rho) \) for each \( \rho \in \mathcal{C} \). Similarly, we write \( \mu \dashv \lambda \) if \( \mu(\rho) \dashv \lambda(\rho) \) for each \( \rho \in \mathcal{C} \).

Recall that we write \( \varphi(\mu) \) for the irreducible representation of \( G_n \) parametrized by a function \( \mu: \mathcal{C} \rightarrow \mathcal{P} \) such that \( ||\mu|| = n \). The following branching rule was proved by Zelevinsky [11, Corollary 13.8]; it was also proved by Thoma [10, Satz 2] in another form.
**Fact 7** (Branching rule). If $\mu : C \rightarrow P$ is a function such that $\|\mu\| = n$, and $\nu : C \rightarrow P$ is a function such that $\|\nu\| = n - 1$, then the multiplicity of $\varphi(\nu)$ in the restriction of $\varphi(\mu)$ to $G_{n-1}$ is equal to the number of functions $\lambda : C \rightarrow P$ such that $\nu \rightarrow \lambda$ and $\lambda \rightarrow \mu$.

We shall need the branching rule for restriction of an irreducible representation of $G_n$ to $G_{n-m}$ where $n \geq m \geq 1$.

**Definition 8.** Suppose $\mu : C \rightarrow P$ is a function such that $\|\mu\| = n$, and $\nu : C \rightarrow P$ is a function such that $\|\nu\| = n - m$, where $n \geq m \geq 1$. By a zigzag path from $\nu$ to $\mu$, we mean two sequences of functions

$$
\lambda^{(r)} : C \rightarrow P \quad \text{where } r = 1, \ldots, m,
$$

$$
\mu^{(s)} : C \rightarrow P \quad \text{where } s = 1, \ldots, m,
$$

such that $\|\mu^{(s)}\| = n - m + s$ for every $s$, and one has:

$$
(9) \quad \mu^{(m)} = \mu. \quad \lambda^{(m)} \quad + \quad \mu^{(m-1)} \quad - \quad \lambda^{(m)} \quad + \quad \mu^{(m-1)} \quad - \quad \cdots
$$

We define $ZZ(\nu, \mu)$ to be the set of all zigzag paths from $\nu$ to $\mu$.

The following theorem follows easily from Fact 7.
Theorem 10. Suppose $\mu : \mathcal{C} \to \mathcal{P}$ is a function such that $\|\mu\| = n$, and $\nu : \mathcal{C} \to \mathcal{P}$ is a function such that $\|\nu\| = n - m$, where $n \geq m \geq 1$. Then the multiplicity of $\varphi(\nu)$ in the restriction of $\varphi(\mu)$ to $G_{n-m}$ is equal to $|ZZ(\nu, \mu)|$.

Proof. We use induction on $m$. The base case $m = 1$ is precisely Fact 7.

Suppose $m > 1$. Let $S$ be the set of all functions $\mu' : \mathcal{C} \to \mathcal{P}$ such that $\|\mu'\| = n - 1$. By Fact 7, we have:

$$\text{Res}_{G_{n-1}}^{G_n} (\varphi(\mu)) = \bigoplus_{\mu' \in S} \left( \bigoplus_{Z \in ZZ(\mu', \mu)} \varphi(\mu') \right).$$

Hence, by the induction hypothesis, the multiplicity of $\varphi(\nu)$ in $\text{Res}_{G_{n-m}}^{G_n}$ is equal to:

$$\sum_{\mu' \in S} \left( \sum_{Z \in ZZ(\mu', \mu)} |ZZ(\nu, \mu')| \right) = |ZZ(\nu, \mu)|.$$

□

2.2. Proof of Theorem 2. We shall need the following simple observation.

Lemma 11. Suppose we have a zigzag path as shown in (9). Fix $\rho \in \mathcal{C}$. If

$$\nu(\rho) = (\nu_1, \nu_2, \ldots),$$
$$\lambda^{(r)}(\rho) = (\lambda_1^{(r)}, \lambda_2^{(r)}, \ldots) \text{ for } r = 1, \ldots, m,$$
$$\mu^{(s)}(\rho) = (\mu_1^{(s)}, \mu_2^{(s)}, \ldots) \text{ for } s = 1, \ldots, m,$$

then

$$\nu_i - r \leq \lambda_i^{(r)} \leq \nu_i + r - 1,$$
$$\nu_i - s \leq \mu_i^{(s)} \leq \nu_i + s,$$

for every $i = 1, 2, \ldots$ and $r, s = 1, \ldots, m$.

Proof. To go from $\nu$ to $\lambda^{(r)}$ along the zigzag path, we need to pass through $r$ arrows of type $\rightarrow$ and $r - 1$ arrows of type $\leftarrow$. To go from $\nu$ to $\mu^{(s)}$ along the zigzag path, we need to pass through $s$ arrows of type $\rightarrow$ and $s$ arrows of type $\leftarrow$.

We keep track of how the Young diagram of $\nu(\rho)$ changes as we move along the zigzag path. When we move along an arrow of type $\rightarrow$, the number of boxes in a row of the Young diagram will not decrease and will increase by at most 1. When we move along an arrow of type $\leftarrow$, the number of boxes in a row of the Young diagram will not increase and will decrease by at most 1. The inequalities follow. □

We now prove Theorem 2.

Fix a non-negative integer $m$ and set $V_n = k[G_n/G_{n-m}]$ for each $n \geq m$. Observe that:

$$V_n \cong \text{Ind}_{G_{n-m}}^{G_n}(i_{n-m}),$$
Let us write \( \nu_n : G \rightarrow \mathcal{P} \) be the function defined by

\[
\nu_n = \begin{cases} 
(n-m, 0, 0, \ldots) & \text{if } \rho = \iota, \\
(0, 0, \ldots) & \text{if } \rho \neq \iota.
\end{cases}
\]

Then by \([11, \text{Proposition 9.6}]\) one has \( \iota_n = \phi(\nu_n) \).

Suppose \( \mu : G \rightarrow \mathcal{P} \) is a function such that \( \| \mu \| = n \). Then one has:

- multiplicity of \( \phi(\mu) \) in \( V_n \)
- \( \dim \text{Hom}_{G_n}(\text{Ind}^{G_n}_{G_{n-m}}(\iota_{n-m}), \phi(\mu)) \)
- \( \dim \text{Hom}_{G_{n-m}}(\iota_{n-m}, \text{Res}^{G_n}_{G_{n-m}}(\phi(\mu))) \) (by Frobenius reciprocity)
- multiplicity of \( \phi(\nu_n) \) in \( \text{Res}^{G_n}_{G_{n-m}}(\phi(\mu)) \)

\[= \left| \text{ZZ}(\nu_{n-m}, \mu) \right| \quad \text{(by Theorem 10)} \]

Let \( \lambda : G \rightarrow \mathcal{P} \) be the function such that \( \mu = \lambda[n] \), and write \( \lambda[n] = (n-\| \lambda \|, \lambda_1, \lambda_2, \ldots) \).

Suppose that \( \phi(\lambda[n]) \) is an irreducible component of \( V_n \). Then there exists a zigzag path from \( \nu_{n-m} \) to \( \lambda[n] \). By Lemma \([11]\), one has \( n - \| \lambda \| \geq (n - m) - m \) and \( \lambda_1 \leq m \), so \( 3m - \| \lambda \| \geq \lambda_1 \).

Therefore, the function \( \lambda[3m] : G \rightarrow \mathcal{P} \) is well-defined. It suffices to prove that:

\[
\text{ZZ}(\nu_{\ell-m}, \lambda[\ell]) = \text{ZZ}(\nu_{\ell+1-m}, \lambda[\ell+1]) \quad \text{for every } \ell \geq 3m.
\]

To this end, define a map

\[ h : \mathcal{P} \rightarrow \mathcal{P}, \quad (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_1 + 1, \alpha_2, \alpha_3, \ldots). \]

For any function \( \alpha : G \rightarrow \mathcal{P} \), define the function \( \tilde{\alpha} : G \rightarrow \mathcal{P} \) by \( \tilde{\alpha}(\iota) = h(\alpha(\iota)) \) and \( \tilde{\alpha}(\rho) = \alpha(\rho) \) if \( \rho \neq \iota \). Note that \( \nu_{\ell-m} = \nu_{\ell+1-m} \) and \( \lambda[\ell] = \lambda[\ell+1] \). Moreover, for any functions \( \alpha : G \rightarrow \mathcal{P} \) and \( \beta : G \rightarrow \mathcal{P} \), if \( \alpha \rightarrow \beta \) or \( \alpha \rightarrow \beta \), then \( \tilde{\alpha} \rightarrow \tilde{\beta} \) or \( \tilde{\alpha} \rightarrow \tilde{\beta} \), respectively. Fix \( \ell \geq 3m \) and define the map

\[ h : \text{ZZ}(\nu_{\ell-m}, \lambda[\ell]) \rightarrow \text{ZZ}(\nu_{\ell+1-m}, \lambda[\ell+1]), \quad \left( \{ \lambda^{(r)} \}, \{ \mu^{(s)} \} \right) \mapsto \left( \{ \lambda^{(r)} \}, \{ \mu^{(s)} \} \right). \]

We claim that the map \( h \) is bijective. The injectivity of \( h \) is clear from the injectivity of \( h \).

To see the surjectivity of \( h \), note that the image of \( h \) is the set of all partitions \( (\alpha_1, \alpha_2, \ldots) \) such that \( \alpha_1 - 1 \geq \alpha_2 \). Suppose we have a zigzag path

\[ \left( \{ \lambda^{(r)} \}, \{ \mu^{(s)} \} \right) \in \text{ZZ}(\nu_{\ell+1-m}, \lambda[\ell+1]). \]

Let us write \( \alpha^{(r)}(\iota) = (\alpha_1^{(r)}, \alpha_2^{(r)}, \ldots) \) and \( \beta^{(s)}(\iota) = (\beta_1^{(s)}, \beta_2^{(s)}, \ldots) \). By Lemma \([11]\), we have:

- \( \alpha_1^{(r)} \geq (\ell + 1 - m) - r \) and \( \alpha_2^{(r)} \leq r - 1; \)
- \( \beta_1^{(s)} \geq (\ell + 1 - m) - s \) and \( \beta_2^{(s)} \leq s. \)
Since $\ell \geq 3m$ and $r, s \leq m$, it follows that:
\[
\alpha_1^{(r)} - 1 \geq m > \alpha_2^{(r)};
\]
\[
\beta_1^{(s)} - 1 \geq m \geq \beta_2^{(s)}.
\]
Hence, the zigzag path $\{\alpha^{(r)}\}, \{\beta^{(s)}\}$ is in the image of $h$. The map $h$ is thus bijective and we have proven (12). This completes the proof of Theorem 2.

3. Representation stability for VIC-modules

3.1. Notations. For any finite group $G$ and representation $\pi$ of $G$, we write $\pi_G$ for the quotient space of $G$-coinvariants of $\pi$.

Let $n$ be any non-negative integer. For any non-negative integers $m$ and $r$ such that $n = m + r$, let $H_{m,r}$ be the subgroup of $G_n$ consisting of all matrices of the form:

\[
(13) \quad g = \begin{pmatrix}
1_m & g_{12} \\
0 & g_{22}
\end{pmatrix},
\]

and $1_m$ is the identity element of $G_m$ and $g_{22} \in G_r$. Let $L_{m,r}$ be the subgroup of $H_{m,r}$ consisting of all matrices $g$ of the form (13) such that $g_{12} = 0$.

For any non-negative integer $m$, we define a VIC-module $P(m)$ by

\[
P(m)(-) = k \Hom_{VIC}(\mathbb{F}_q^m, -),
\]

that is, $P(m)$ is the composition of the functor $\Hom_{VIC}(\mathbb{F}_q^m, -)$ followed by the free $k$-module functor.

3.2. Generalities. We recall some basic definitions on VIC-modules.

A homomorphism $F : U \to V$ of VIC-modules is a natural transformation from the functor $U$ to the functor $V$. If $U$ and $V$ are VIC-modules such that $U(X)$ is a $k$-submodule of $V(X)$ for every object $X$ of VIC, and the collection of inclusion maps $U(X) \hookrightarrow V(X)$ defines a homomorphism $U \to V$ of VIC-modules, then we say that $U$ is a VIC-submodule of $V$. A VIC-module $V$ is said to be generated by a subset $S \subset \bigsqcup_{n \geq 0} V(\mathbb{F}_q^n)$ if the only VIC-submodule of $V$ containing $S$ is $V$; we say that $V$ is finitely generated if it is generated by a finite subset $S$.

It is plain (see, for example, [4, Lemma 2.14]) that a VIC-module $V$ is finitely generated if and only if there exists a surjective homomorphism

\[
P(m_1) \oplus \cdots \oplus P(m_k) \twoheadrightarrow V \quad \text{for some } m_1, \ldots, m_k \geq 0.
\]

It is also easy to see that one has:

**Proposition 14.** Let $V$ be a VIC-module and let $\{V_n, \phi_n\}$ be the consistent sequence obtained from $V$. Then $V$ is finitely generated if and only if $\{V_n, \phi_n\}$ satisfies condition (RS2) and $\dim(V_n) < \infty$ for every $n$.

**Proof.** See [4, Proposition 5.2].

Let us mention that it was also shown in [8, Theorem E] that if $V$ is finitely generated, then $\{V_n, \phi_n\}$ satisfies condition (RS2).
3.3. Weak stability and weight boundedness. Suppose \( \{V_n, \phi_n\} \) is a consistent sequence of representations of \( G_n \).

For any non-negative integers \( n, m \) and \( r \) such that \( n = m + r \), the map \( \phi_n : V_n \to V_{n+1} \) descends to a map
\[
\phi_{m,r} : (V_n)_{H_{m,r}} \to (V_{n+1})_{H_{m,r+1}}.
\]

**Definition 16.** We say that \( \{V_n, \phi_n\} \) is:

- weakly stable if for each \( m \in \mathbb{Z}_+ \), there exists \( s \geq 0 \) such that for each \( r \geq s \), the map \( \phi_{m,r} \) of (15) is bijective.
- weight bounded if there exists a non-negative integer \( a \) such that for every \( n \geq 0 \) and every irreducible subrepresentation \( \varphi(\lambda[n]) \) of \( V_n \), one has \( \|\lambda\| \leq a \).

The following key proposition is proved in [5].

**Proposition 17.** Let \( \{V_n, \phi_n\} \) be a consistent sequence of representations of \( G_n \). Suppose that \( \{V_n, \phi_n\} \) is weakly stable and weight bounded. Then:

(i) \( \{V_n, \phi_n\} \) satisfies conditions (RS1) and (RS3).
(ii) If \( \dim(V_n) < \infty \) for every \( n \geq 0 \), then there exists a polynomial \( P \in \mathbb{Q}[T] \) and an integer \( N \) such that \( \dim(V_n) = P(q^n) \) for all \( n \geq N \).

**Proof.** (i) See [5] Proposition 4.1, Proposition 4.3.
(ii) Immediate from condition (RS3) and [5] Proposition 5.2. \( \square \)

3.4. Proof of Theorem 4. We now prove Theorem 4.

Let \( V \) be a finitely generated VIC-module and \( \{V_n, \phi_n\} \) the consistent sequence obtained from \( V \). By Proposition 14 and Proposition 17, it suffices to prove that \( \{V_n, \phi_n\} \) is weakly stable and weight bounded. We need the following lemma.

**Lemma 18.** For any non-negative integer \( m \), the VIC-module \( P(m) \) is weakly stable.

**Proof.** Write \( \{P(m), \phi_m\} \) for the consistent sequence obtained from \( P(m) \).

Fix a non-negative integer \( \ell \) and set \( s = m + \min(m, \ell) \). We shall prove that for \( r \geq s \), the map
\[
\phi_{\ell,r} : (P(m))_{H_{\ell,r}} \to (P(m))_{H_{\ell,r+1}}
\]
(\( \text{where } n = \ell + r \)) is surjective. This would imply that \( \phi_{\ell,r} \) is bijective for all \( r \) sufficiently large.

Observe that for \( n \geq m \), the group \( G_n \) acts transitively on \( \text{Hom}_{VIC}(\mathbb{F}_q^m, \mathbb{F}_q^n) \) and \( L_{m,n-m} \) is the stabilizer of the pair \( (f, K) \) where \( f \) is the standard inclusion \( \mathbb{F}_q^m \hookrightarrow \mathbb{F}_q^n \) and \( K \) is the subspace of vectors in \( \mathbb{F}_q^n \) whose first \( m \) coordinates are 0. Hence, it suffices to prove that for \( r \geq s \), the map
\[
H_{\ell,r}/G_n \to H_{\ell,r+1}/G_{n+1}
\]
induced by the standard inclusion \( G_n \hookrightarrow G_{n+1} \) is surjective. We shall prove the stronger statement that for \( r \geq s \), the map
\[
L_{\ell,r}/G_n \to L_{\ell,r+1}/G_{n+1}
\]
is surjective. Thus, suppose that \( r \geq s \) and choose any \( x \in G_{n+1} \). We need to show that for some \( g \in L_{\ell,r+1} \) and \( g' \in L_{m,n+1-m} \), the only nonzero entry in the last row or last column of \( g'xg' \) is the entry in position \( (n+1, n+1) \) and it is equal to 1.
First, we may choose \( g_1 \in L_{\ell,r+1} \) and \( g_1' \in L_{m,n+1-m} \) such that the entries of \( g_1xg_1' \) in positions \((i,j)\) are 0 if \( i > m + \ell \) and \( j \leq m \), or if \( i \leq \ell \) and \( j > m + \ell \). Indeed, we may first perform row operations on the last \( r + 1 \) rows of \( x \) to change the entries in positions \((i,j)\) to 0 for \( i > m + \ell \) and \( j \leq m \); then, we may perform column operations on the last \( n+1-m \) columns to change the entries in positions \((i,j)\) to 0 for \( i \leq \ell \) and \( j > m + \ell \). Let \( x_1 = g_1xg_1' \).

Next, since \( x_1 \) has rank \( n+1-m+\ell+\min(m,\ell) \), there exists \( i,j > m + \ell \) such that the entry of \( x_1 \) in position \((i,j)\) is nonzero. By first interchanging row \( i \) with row \( n+1 \), and then interchanging column \( j \) with column \( n+1 \), we obtain a matrix whose entry in position \((n+1,n+1)\) is nonzero. It is now easy to see that for some \( g_2 \in L_{\ell,r+1} \) and \( g_2' \in L_{m,n+1-m} \), the matrix \( g_2x_1g_2' \) lies in \( G_n \).

We can now prove that the consistent sequence \( \{V_n,\phi_n\} \) is weakly stable. Since \( V \) is finitely generated, there is a surjective homomorphism \( P \rightarrow V \) where \( P \) is a VIC-module of the form \( P(m_1) \oplus \cdots \oplus P(m_k) \) for some \( m_1,\ldots,m_k \geq 0 \). Let \( \{P_n,\phi_n\} \) be the consistent sequence obtained from \( P \). By Lemma [17] \( \{P_n,\phi_n\} \) is weakly stable. Fix \( m \geq 0 \). Then for all \( r \) sufficiently large, the top horizontal map in the following commuting diagram is bijective:

\[
\begin{array}{ccc}
(P_{m+r})_{H_{m,r}} & \xrightarrow{\phi_{m,r}} & (P_{m+r+1})_{H_{m,r+1}} \\
\downarrow & & \downarrow \\
(V_{m+r})_{H_{m,r}} & \xrightarrow{\phi_{m,r}} & (V_{m+r+1})_{H_{m,r+1}}
\end{array}
\]

Since the two vertical maps in the above diagram are surjective, it follows that the bottom horizontal map is surjective for all \( r \) sufficiently large, and hence bijective for all \( r \) sufficiently large.

Since \( V \) is a quotient of \( P(m_1) \oplus \cdots \oplus P(m_k) \), to see that \( \{V_n,\phi_n\} \) is weight bounded, it suffices to show that \( P(m) \) is weight bounded for each \( m \geq 0 \). As we noted in the proof of Lemma [17] for each \( n \geq m \), the \( G_n \) representation \( P(m)(\mathbb{F}_q^n) \) is isomorphic to the permutation representation of \( G_n \) on \( G_n/L_{m,n-m} \). But \( L_{m,n-m} \) is conjugate to \( G_{n-m} \) in \( G_n \). Therefore, the permutation representation of \( G_n \) on \( G_n/L_{m,n-m} \) is isomorphic to the permutation representation of \( G_n \) on \( G_n/G_{n-m} \). We deduce from Theorem [2] that the consistent sequence obtained from \( P(m) \) is weight bounded. This completes the proof of Theorem [4]

3.5. Remarks. We end with some remarks on the connection of this paper to other works.

(i) It is a difficult result [3] Theorem C] of Putman and Sam that any finitely generated VIC-module over a noetherian ring is noetherian. An easier proof for finitely generated VIC-modules over a field of characteristic zero is given in [4] Example 3.12. It was shown in [4] Proposition 5.1] and [3] Theorem E] that condition (RS1) is also a simple consequence of the noetherian property.

(ii) Let VI be the category whose objects are the finite dimensional vector spaces over \( \mathbb{F}_q \) and whose morphisms are the injective linear maps. In [5] Theorem 1.6], the authors proved a representation stability theorem for finitely generated VI-modules over \( k \). This result can also be deduced from Theorem [4] there is a forgetful functor from the category VIC to the category VI, and the pullback of any finitely generated VI-module is a finitely generated
VIC-module; see [8, Remark 1.13]. The proof of [5, Theorem 1.6], however, does not require the use of the branching rule (Fact 7) for the finite general linear groups.

(iii) We should mention that the categories VI and VIC are analogues of the category FI of finite sets and injective maps. FI-modules were introduced and studied by Church, Ellenberg and Farb in [1]. One of the main results of their paper is a representation stability theorem for finitely generated FI-modules over a field of characteristic zero. The proof of Proposition [17(i)] (given in [5, Proposition 4.1 and Proposition 4.3]) is an adaptation of their arguments for consistent sequences of representations of the symmetric groups to our situation of consistent sequences for representations of the finite general linear groups.

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