GENERALIZATION AND VARIATIONS OF PELLET’S THEOREM
FOR MATRIX POLYNOMIALS

A. Melman
Department of Applied Mathematics
School of Engineering, Santa Clara University
Santa Clara, CA 95053
e-mail : amelman@scu.edu

Abstract

We derive a generalized matrix version of Pellet’s theorem, itself based on a gen-
eralized Rouché theorem for matrix-valued functions, to generate upper, lower, and
internal bounds on the eigenvalues of matrix polynomials. Variations of the theorem
are suggested to try and overcome situations where Pellet’s theorem cannot be applied.

Key words : matrix polynomial, Pellet, Cauchy, zero, root, eigenvalue, bound

AMS(MOS) subject classification : 12D10, 15A18, 30C15

1 Introduction

Polynomial eigenvalue problems have been investigated for quite some time ([7], [12], [13])
and have important applications in a wide range of engineering fields such as vibration
analysis, acoustics, and fluid mechanics - to name just a few ([19]). It is, in general, costly
to compute polynomial eigenvalues for large problems, but bounds on such eigenvalues are
relatively easy to obtain. They provide information on the location of eigenvalues that can
be used by iterative methods for computing them and are also useful in the computation
of pseudospectra.

The polynomial eigenvalue problem is to find a nonzero eigenvector \( v \), corresponding
to an eigenvalue \( z \) satisfying \( P(z)v = 0 \), where

\[
P(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_0,
\]

with \( A_j \in \mathbb{C}^{m \times m} \) for \( j = 0, \ldots, n \). We will assume throughout that \( \det(P) \) is not identically
zero. If \( A_n \) is singular then \( P \) has infinite eigenvalues and if \( A_0 \) is singular then zero is
an eigenvalue. There are \( nm \) eigenvalues, including possibly infinite ones. The finite
eigenvalues are the solutions of \( \det(P) = 0 \).

To explain the aims of this work, we start with the scalar polynomial with complex
coefficients

\[
p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,
\]

and \( a_n a_0 \neq 0 \). If \( \zeta \) is a zero of \( p \), then for \( a_k \neq 0 \) with \( 1 \leq k \leq n - 1 \) we have

\[-a_k \zeta^k = a_n \zeta^n + a_{n-1} \zeta^{n-1} + \cdots + a_{k+1} \zeta^{k+1} + a_{k-1} \zeta^{k-1} + \cdots + a_0,
\]

and an analogous equality for \( k = 0, n \), which implies that

\[
|a_k| |\zeta|^k = \left| a_n \zeta^n + a_{n-1} \zeta^{n-1} + \cdots + a_{k+1} \zeta^{k+1} + a_{k-1} \zeta^{k-1} + \cdots + a_0 \right|
\leq |a_n| |\zeta|^n + |a_{n-1}| |\zeta|^{n-1} + \cdots + |a_{k+1}| |\zeta|^{k+1} + |a_{k-1}| |\zeta|^{k-1} + \cdots + |a_0| \quad (1)
\]
Applying the equivalent of inequality (1) with $k = n$, we obtain that $|\zeta|$ must satisfy

$$|a_n| |\zeta|^n - |a_{n-1}| |\zeta|^{n-1} - \cdots - |a_1| |\zeta| - |a_0| \leq 0.$$ 

This means that $|\zeta|$ can be no larger than the unique positive root of

$$|a_n|x^n - |a_{n-1}|x^{n-1} - \cdots - |a_1|x - |a_0| = 0.$$ 

Analogously, one finds that $|\zeta|$ can be no smaller than the unique positive root of

$$|a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x - |a_0| = 0.$$ 

These results are due to Cauchy ([4], [14, Theorem (27,1), p. 122]).

When we apply inequality (1) with $1 \leq k \leq n - 1$ and $a_k \neq 0$, we obtain

$$|a_n| |\zeta|^n + |a_{n-1}| |\zeta|^{n-1} + \cdots + |a_k+1| |\zeta|^{k+1} - |a_k| |\zeta|^k + |a_{k-1}| |\zeta|^{k-1} + \cdots + |a_0| \geq 0. \quad (2)$$

Now, the equation

$$|a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_k+1|x^{k+1} - |a_k|x^k + |a_{k-1}|x^{k-1} + \cdots + |a_0| = 0$$

has, by Descartes’ rule, either two or no positive roots. If it has no positive roots or if the two positive roots coincide, then inequality (2) always satisfied and provides no information on $|\zeta|$. However, if it has two distinct positive roots, say $x_1$ and $x_2$ with $x_1 < x_2$, then inequality (2) implies that either $|\zeta| \leq x_1$ or $|\zeta| \geq x_2$ and that no zero of $p$ has a modulus in $(x_1, x_2)$. As a straightforward consequence of Rouché’s theorem, Pellet’s theorem makes this result more precise. It is stated as follows.

**Theorem 1.1 (Pellet, [7], [14, Th.(28,1), p.128])** Given the polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ with complex coefficients, $a_0a_k \neq 0$, and $n \geq 3$, let $1 \leq k \leq n - 1$, and let the polynomial

$$f_k(x) = x^n + |a_{n-1}|x^{n-1} + \cdots + |a_k+1|x^{k+1} - |a_k|x^k + |a_{k-1}|x^{k-1} + \cdots + |a_0|$$

have two distinct positive roots $x_1$ and $x_2$ with $x_1 < x_2$. Then $p$ has exactly $k$ zeros in or on the circle $|z| = x_1$ and no zeros in the annular ring $x_1 < |z| < x_2$.

Cauchy’s results can be considered as a special limit case of Pellet’s theorem.

We have two main goals. The first is to generalize Pellet’s theorem (and therefore also Cauchy’s results) to matrix polynomials, with matrix norms replacing absolute values. Such a generalized theorem for the spectral norm (2-norm) was recently derived in [3], where it was used to determine initial approximations for iterative methods like the Ehrlich-Aberth method ([1], [2], [3]). Using a different proof, we obtain a generalization valid for any subordinate norm, not just the spectral norm.

Our second goal is to find a way to, at least sometimes, overcome situations where $f_k$ in Theorem 1.1 (or an analogous polynomial for the generalized version of that theorem) does not have positive roots for a particular value of $k$. To do this, we derive a variation of the
A generalized Pellet theorem that relies on the existence of positive roots for a polynomial of roughly half the degree of $f_k$.

Only polynomials (or matrix polynomials) with very special coefficients would have more than a few values of $k$ for which Pellet’s theorem can be applied and even when there exists a significant gap between groups of zeros (or eigenvalues), it frequently occurs that the theorem is unable to detect it. Unfortunately, there exist no results for Pellet’s theorem that would allow one to predict if the function $f_k$ has positive roots or not, and the same problem naturally carries over to our variation of that theorem. This makes it impossible to predict if and when our result improves over Pellet’s theorem. On the other hand, it does provide an alternative that currently does not exist in cases where Pellet’s theorem is not applicable and we will present extensive numerical examples illustrating its usefulness.

We also suggest an idea applicable to scalar polynomials, whereby the zeros of a polynomial are computed as the eigenvalues of an appropriate matrix polynomial. Perhaps paradoxically, this can lead (like the variation mentioned above) to an improvement of the original (scalar) version of Pellet’s theorem by using its generalized (matrix) version. At the same time, this approach can also improve the upper and lower bounds resulting from Cauchy’s result. Its potential will be demonstrated by numerical experiments for a restricted class of polynomials.

We will not dwell on the numerical problem of finding the positive roots of functions like $f_k$ in Theorem 1.1 if they exist. An efficient way to compute them can be found in [15], while bounds on such roots were derived in [8]. The number of arithmetic operations that this requires is typically dwarfed by the much more costly computation of the eigenvalues of a matrix polynomial. Moreover, this work focuses on Pellet’s theorem if and when it is used. It does not focus on whether or not it should be used in the first place, as this depends very much on external factors.

The organization of the paper is as follows. In Section 2 we review a few preliminaries that will be needed in the later sections. In Section 3 we derive the generalized Pellet theorem, in Section 4 we propose a variation of that theorem, and in Section 5 we present the aforementioned idea for scalar polynomials.

## 2 Preliminaries

Since it is mentioned several times, we begin by stating Rouché’s well-known theorem.

**Theorem 2.1** (Rouché, [15], Theorem 1.6) Let $f$ and $g$ be analytic in the interior of a simple closed curve $\Gamma$ and continuous on $\Gamma$, and let $|g(z)| < |f(z)|$ for all $z \in \Gamma$. Then $f + g$ and $f$ have the same number of zeros in the interior of the curve.

All matrix norms throughout this work are assumed to be vector-induced (or **subordinate**). The main matrix norms we will use are the 1-Norm, $\infty$-Norm, and the 2-Norm (or **spectral** Norm).
norm), defined ([10], p. 294-295) for $A \in \mathfrak{C}^{n \times n}$ with elements $a_{ij}$ by

$$
||A||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = ||A^*||_{\infty},
$$

$$
||A||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| = ||A^*||_1,
$$

$$
||A||_2 = \max \left\{ \sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A \right\} = ||A^*||_2.
$$

The zeros of a monic polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with complex coefficients are the eigenvalues of its $n \times n$ companion matrix

$$
C(p) = \begin{pmatrix}
0 & -a_0 \\
1 & -a_1 \\
\ddots & \ddots \\
1 & -a_{n-1}
\end{pmatrix}.
$$

Likewise, the eigenvalues of the monic matrix polynomial

$$
P(z) = Iz^n + A_{n-1}z^{n-1} + \cdots + A_1 z + A_0,
$$

where $A_j \in \mathfrak{C}^{m \times m}$ for $j = 0, \ldots, n-1$, are given by the eigenvalues of the $nm \times nm$ block companion matrix

$$
C(P) = \begin{pmatrix}
0 & -A_0 \\
I & -A_1 \\
\ddots & \ddots \\
I & -A_{n-1}
\end{pmatrix},
$$

with $I$ the $m \times m$ identity matrix. Since the size of $I$ will usually be clear from the context, it will be omitted from the notation.

The reciprocal monic polynomial of $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with $a_0 \neq 0$ is defined by $p_r(z) = z^n p(1/z)/a_0$. Its zeros are the reciprocals of the zeros of $p$. Likewise, the reciprocal matrix polynomial of $P(z) = Iz^n + A_{n-1}z^{n-1} + \cdots + A_1 z + A_0$ with $A_j \in \mathfrak{C}^{m \times m}$ for $j = 0, \ldots, n-1$ and $A_0$ nonsingular is defined by $P_r(z) = z^n A_0^{-1}P(1/z)$. Its eigenvalues are the reciprocals of the eigenvalues of $P$.

3 Generalized Rouché and Pellet theorems

The standard proof of Pellet’s theorem uses Rouché’s theorem for analytical functions, which suggests that a generalized version of Rouché’s theorem for analytical matrix-valued functions will be needed. Such a theorem can be derived from the generalized Rouché theorem for bounded linear operators in [8, Theorem 9.2, p. 206] (see also [6]). To do this, we need a few definitions for which we have used similar notation and style as in [8, Chapter XI]. We denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from $X$
Hilbert space is separable if and only if it has a countable orthonormal basis.

Theorem 3.1 (Generalized Rouché theorem for operators.) Let \( W, S : \Omega \to \mathcal{L}(H) \) be analytic operator functions, where \( \Omega \) is an open connected subset of \( \mathcal{C} \) and \( H \) is a separable Hilbert space, and assume that \( W \) is normal with respect to the simple closed curve \( \Gamma \subseteq \Omega \).

If \( \|W(z)^{-1}S(z)\| < 1 \) for all \( z \in \Gamma \), then \( W + S \) is also normal with respect to \( \Gamma \) and \( W + S \) and \( W \) have the same number of finite eigenvalues inside \( \Gamma \), counting multiplicities.

The norm used in the previous theorem can be any norm, induced by a norm on \( H \). We obtain the following generalization of Rouché theorem for analytic matrix-valued functions as an immediate consequence of Theorem 3.1. We note that matrix polynomials are special cases of such functions.

Theorem 3.2 (Generalized Rouché theorem for matrices.) Let \( A, B : \Omega \to \mathcal{C}^{m \times m} \) be analytic matrix-valued functions, where \( \Omega \) is an open connected subset of \( \mathcal{C} \) and assume that \( A(z) \) is nonsingular for all \( z \) on the simple closed curve \( \Gamma \subseteq \Omega \).

If \( \|A(z)^{-1}B(z)\| < 1 \) for all \( z \in \Gamma \), then \( \det(A + B) \) and \( \det(A) \) have the same number of zeros inside \( \Gamma \), counting multiplicities.

Proof. The proof follows directly from Theorem 3.1. In this case, \( H \equiv \mathcal{C}^m \), a finite dimensional Hilbert space which is (trivially) separable and \( \mathcal{L}(H) \equiv \mathcal{C}^{m \times m} \). Furthermore, since \( A(z) \) is a matrix, it is a bounded linear operator that is normal with respect to \( \Gamma \) because \( A(z) \) is invertible for all \( z \in \Gamma \) and, since \( \mathcal{C}^m \) is finite dimensional, \( A(z) \) is (trivially) a Fredholm operator for any \( z \) in the interior of \( \Gamma \). By Theorem 3.1, \( A + B \) is then also normal with respect to \( \Gamma \) and has the same number of finite eigenvalues inside \( \Gamma \) as \( A \), which means that \( \det(A + B) \) and \( \det(A) \) have the same number of zeros inside \( \Gamma \), counting multiplicities. \( \square \)

Remarks. (1) The norm in Theorem 3.2 can be any matrix norm, induced by a norm on \( \mathcal{C}^m \).

(2) Since \( \|A^{-1}(z)B(z)\| \leq \|A^{-1}(z)\| \cdot \|B(z)\| \), it is sufficient that \( \|B(z)\| < \|A^{-1}(z)\|^{-1} \) for the condition \( \|A^{-1}(z)B(z)\| < 1 \) to be satisfied.

Finally, Theorem 3.2 leads to the following generalization of Pellet’s theorem (Theorem 1.1).
Theorem 3.3 (Generalized Pellet theorem.) Let

\[ P(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0 \]

be a matrix polynomial with \( n \geq 2 \), \( A_j \in \mathbb{C}^{m \times m} \) for \( j = 0, \ldots, n \), and \( A_0 \neq 0 \). Let \( A_k \) be invertible for some \( k \) with \( 1 \leq k \leq n-1 \), and let the polynomial

\[ f_k(x) = ||A_n|| x^n + ||A_{n-1}|| x^{n-1} + \cdots + ||A_{k+1}|| x^{k+1} - ||A_k^{-1}||^{-1} x^k + ||A_{k-1}|| x^{k-1} + \cdots + ||A_1|| x + ||A_0|| \]

have two distinct positive roots \( x_1 \) and \( x_2 \) with \( x_1 < x_2 \). Then \( \det(P) \) has exactly \( km \) zeros in or on the disk \( |z| = x_1 \) and no zeros in the annular ring \( x_1 < |z| < x_2 \).

Proof. By Theorem 3.2 if

\[ \left| \left( A_k z^k \right)^{-1} \left( A_n z^n + A_{n-1} z^{n-1} + \cdots + A_{k+1} z^{k+1} + A_{k-1} z^{k-1} + \cdots + A_1 z + A_0 \right) \right| < 1 \]

for \( |z| = x \), then \( P \) and \( A_k z^k \) both have the same number of eigenvalues in the open disk \( |z| < x \), namely, \( km \). From Remark (2) following the proof of Theorem 3.2, a sufficient condition for inequality (3) to be satisfied is

\[ \left| A_n z^n + A_{n-1} z^{n-1} + \cdots + A_{k+1} z^{k+1} + A_{k-1} z^{k-1} + \cdots + A_1 z + A_0 \right| < \left| \left( A_k z^k \right)^{-1} \right|^{-1} \]

Since

\[ \left| A_n z^n + A_{n-1} z^{n-1} + \cdots + A_{k+1} z^{k+1} + A_{k-1} z^{k-1} + \cdots + A_1 z + A_0 \right| \]

\[ \leq ||A_n|| |z|^n + ||A_{n-1}|| |z|^{n-1} + \cdots + ||A_{k+1}|| |z|^{k+1} + ||A_{k-1}|| |z|^{k-1} + \cdots + ||A_1|| |z| + ||A_0|| , \]

inequality (4) will be satisfied for \( |z| = 1 \) whenever \( x \) is such that

\[ ||A_n|| |x|^n + ||A_{n-1}|| |x|^{n-1} + \cdots + ||A_{k+1}|| |x|^{k+1} + ||A_{k-1}|| |x|^{k-1} + \cdots + ||A_1|| |x| + ||A_0|| < ||A_k^{-1}||^{-1} |x|^k , \]

i.e., when \( f_k(x) < 0 \), where \( f_k \) was defined in the statement of the theorem. By Descartes’ rule, \( f_k \) has either two or no positive roots. Since it was assumed that \( f_k \) has two positive roots \( x_1 \) and \( x_2 \) and since \( f_k(0) > 0 \), inequality (5) will be satisfied for any \( x \) such that \( x_1 < x < x_2 \). This concludes the proof. \( \square \)

Remarks. (1) Applying Theorem 3.3 to the matrix polynomial \( A_k^{-1} P \) yields better values for \( x_1 \) and \( x_2 \) since \( ||A_k^{-1} A_j|| \leq ||A_k^{-1}|| \cdot ||A_j|| \). The disadvantage is that all the coefficients need to be multiplied by \( A_k^{-1} \), which could be costly.

(2) A version of Theorem 3.3 (applied to \( A_k^{-1} P \)) was also obtained in [3] by using a different generalization of Rouché’s theorem from [16] and [20], which limits that result to the spectral norm. Because it is based on Theorem 3.2, Theorem 3.3 holds for any matrix norm induced by a vector norm. This becomes more important as the size of the coefficient matrices increases because the spectral norm can be costly to evaluate, whereas, e.g., the 1-Norm and \( \infty \)-Norm can be computed cheaply. Of course, what constitutes a
costly computation is a matter of opinion and there are undoubtedly cases where using the spectral norm is appropriate. (3) Whether the polynomial $f_k$ will have two positive roots depends on the relative sizes of its coefficients. Generally speaking, this will happen when the coefficient of $x^k$ is large compared to most of the other coefficients, but there exist no results that predict how large it needs to be. As in the scalar case, if $f_k$ does not have positive zeros, then Theorem 3.3 provides no information about a possible gap between groups of eigenvalues.

The following theorem, which is a generalization of Cauchy’s result, can be considered as a special limit case of the generalized Pellet theorem just as in the scalar case. Its proof is entirely analogous and will therefore be omitted.

**Theorem 3.4 (Generalized Cauchy theorem.)** All eigenvalues of the matrix polynomial

$$P(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0,$$

where $A_j \in \mathbb{C}^{m \times m}$, for $j = 0, \ldots, n$, lie in $|z| \leq R$ when $A_n$ is nonsingular, and lie in $|z| \geq r$ when $A_0$ is nonsingular, where $R$ and $r$ are the unique positive roots of

$$||A_n^{-1}||^{-1} x^n - ||A_{n-1}|| x^{n-1} - \cdots - ||A_1|| x - ||A_0|| = 0$$

and

$$||A_n|| x^n + ||A_{n-1}|| x^{n-1} + \cdots + ||A_1|| x - ||A_0^{-1}||^{-1} = 0,$$

respectively.

This theorem was also derived in a different way as Lemma 3.1 in [9]. Again, it can be applied to $A_n^{-1} P$ and $A_0^{-1} P$ to yield tighter bounds, but at the expense of adding $n$ multiplications of two $m \times m$ matrices.

### 4 Variation of Pellet’s theorem

The following theorem is a variation of Pellet’s theorem, obtained by squaring the block companion matrix and repartitioning the result, which leads to additional Cauchy-type and Pellet-type bounds.

**Theorem 4.1** The squares of the eigenvalues of the monic matrix polynomial

$$P(z) = I z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0,$$

where $n$ is a positive even integer and $A_j \in \mathbb{C}^{m \times m}$ for $j = 0, \ldots, n-1$, are given by the eigenvalues of the monic matrix polynomial

$$Q(z) = I z^{n/2} + B_{n/2-1} z^{n/2-1} + \cdots + B_1 z + B_0,$$

where $B_j \in \mathbb{C}^{2m \times 2m}$ for $j = 0, \ldots, n/2-1$, with

$$B_0 = \begin{pmatrix} A_0 & -A_0 A_{n-1} \\ A_1 & -A_1 A_{n-1} + A_0 \end{pmatrix}; \quad B_j = \begin{pmatrix} A_{2j} & -A_{2j} A_{n-1} + A_{2j-1} \\ A_{2j+1} & -A_{2j+1} A_{n-1} + A_{2j} \end{pmatrix}, \quad j = 1, 2, \ldots, n/2-1.$$
Moreover, let $B_k$ be invertible for some $k$ with $1 \leq k \leq n/2 - 1$, and let the polynomial
\[ g_k(y) = y^{n/2} + ||B_{n/2-1}||y^{n/2-1} + \cdots + ||B_k||y^k + \cdots + ||B_1||y + ||B_0|| \]
have two distinct positive roots $y_1$ and $y_2$ with $y_1 < y_2$. Then $P$ has exactly $2km$ eigenvalues in or on the disk $|z| = \sqrt{y_1}$ and no eigenvalues in the annular ring $\sqrt{y_1} < |z| < \sqrt{y_2}$.

In addition, all eigenvalues of $P$ lie in $|z| \leq \sqrt{\rho}$ and, when $B_0$ is nonsingular, in $z \geq \sqrt{\tau}$, where $\rho$ and $\tau$ are the unique positive roots of
\[ y^{n/2} - ||B_{n/2-1}||y^{n/2-1} - \cdots - ||B_1||y - ||B_0|| = 0 \]
and
\[ y^{n/2} + ||B_{n/2-1}||y^{n/2-1} + \cdots + ||B_1||y - ||B_0^{-1}||^{-1} = 0 , \]
respectively.

**Proof.** The companion matrix $C(P)$ of $P$ and its square $C^2(P)$, whose eigenvalues are the squares of those of $P$, are given by
\[
C(P) = \begin{pmatrix}
0 & -A_0 & A_0A_{n-1} \\
I & -A_1 & A_1A_{n-1} - A_0 \\
& \ddots & \ddots \\
& & I & -A_{n-1} \nend{pmatrix} \quad \text{and} \quad C^2(P) = \begin{pmatrix}
0 & -A_0 & A_0A_{n-1} \\
0 & -A_1 & A_1A_{n-1} - A_0 \\
& \ddots & \ddots \\
& & I & -A_{n-1} & A_{n-2} \nend{pmatrix}.
\]

Since $n$ is even, $C^2(P)$ can be repartitioned into $n/2$ blocks of size $2m \times 2m$ as follows:
\[
C^2(P) = \begin{pmatrix}
0 & 0 & -A_0 & A_0A_{n-1} \\
I & 0 & -A_1 & A_1A_{n-1} - A_0 \\
& \ddots & \ddots & \ddots \\
& & I & 0 & -A_{n-2} & A_{n-2}A_{n-1} - A_{n-3} \\
& & & I & 0 & -A_{n-1} & A_{n-1}^2 - A_{n-2} \nend{pmatrix}.
\]

In other words, $C^2(P)$ can be written as
\[
C^2(P) = \begin{pmatrix}
0 & -B_0 & -B_1 \\
& \ddots & \ddots \\
& & I & -B_{n/2-1} \nend{pmatrix},
\]
where
\[
B_0 = \begin{pmatrix} A_0 & -A_0A_{n-1} \\
A_1 & -A_1A_{n-1} + A_0 \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} A_{2j} & -A_{2j}A_{n-1} + A_{2j-1} \\
A_{2j+1} & -A_{2j+1}A_{n-1} + A_{2j} \end{pmatrix}, \quad j = 1, 2, \ldots, n/2 - 1 .
\]
and where \( I \) now stands for the \( 2m \times 2m \) identity matrix. The expression for \( C^2(P) \) in (6) is the block companion matrix \( C(Q) \) of the matrix polynomial

\[
Q(z) = I z^{n/2} + B_{n/2-1} z^{n/2-1} + \cdots + B_1 z + B_0,
\]

whose eigenvalues, being the same as those of \( C^2(P) \), are the squares of the eigenvalues of \( P \). The remainder of the proof follows directly from applying Theorem 3.3 and Theorem 3.4 to the matrix polynomial \( Q \) and the fact that its eigenvalues are the squares of those of \( P \). \( \Box \)

Theorem 4.1 can be used to potentially improve over Theorem 3.3 for a given matrix polynomial \( P \). This may take the form of improved upper and/or lower bounds, but it may also be the case that a gap between groups of eigenvalues can now be computed that would otherwise not have been detected, which is often more important. Furthermore, the degree of \( Q \) is only half that of \( P \) which has a generally beneficial effect on computations involving it. Of course, the cost of squaring \( C(P) \), while still orders of magnitude lower than that of actually computing the eigenvalues, may not be negligible, depending on the size of \( m \).

Additional bounds can be derived by applying Theorem 4.1 to the reciprocal matrix polynomial \( P_r \), which yields a matrix polynomial that we designate by \( Q_R \). This matrix polynomial is, in general, different from \( Q_r \) which is the reciprocal polynomial of \( Q \), defined in (6). The reciprocal of the Cauchy upper bound for \( P_r \) or \( Q_r \) is equal to the Cauchy lower bound for \( P \) or \( Q \), respectively. However, the reciprocal of the Cauchy upper bound for \( Q_R \), while being a lower bound on the moduli of the eigenvalues of \( P \), is, in general, different from the Cauchy lower bound for \( Q \). An analogous situation exists for the reciprocal of the Cauchy lower bound for \( Q_R \). We point out that \( P, P_r, Q, Q_r, \) and \( Q_R \) here are monic polynomials.

When \( n \) is odd we consider \( zP(z) \) instead of \( P(z) \), which simply has \( m \) extra zero eigenvalues. In such a case, \( A_0 \) is the zero matrix, which makes \( B_0 \) in Theorem 4.1 singular so that \( Q \) cannot be used to obtain a lower bound. To remedy this, one can use \( zP_r(z) \): the reciprocal of the upper bound on its largest eigenvalue provides the desired lower bound on the smallest eigenvalue of \( P \).

Many different additional bounds can be derived by considering \( MP \) or \( NQ \) instead of \( P \) or \( Q \) for any nonsingular matrices \( M \) and \( N \), or by using different scalings of the companion matrices. It would therefore be impractical to compare all possible variants. It is also difficult if not impossible to predict which particular bound will outperform other bounds for any given situation. Instead, we will present several numerical examples to illustrate that our bounds can be valuable complements to existing ones.

The bounds’ complexity for a particular value of \( k \) is linear in the degree \( n \) and further depends on the coefficient matrices’ properties. If, for instance, the coefficients exhibit structure, such as symmetry or sparsity, then the various matrix manipulations involved in their computation generally require fewer operations. The choice of bounds therefore ultimately depends on the situation, which is outside the scope of our discussion.

We observe that Theorem 4.1 leads to the slightly counterintuitive situation where the generalized matrix version of the theorem is applied to a scalar polynomial.
Theorem 4.1 always provides upper and lower bounds but it does have limitations since the $2m \times 2m$ matrix coefficients of $Q$ are twice the size of those of $P$. This leads to the computation of gaps between groups of eigenvalues that contain an even multiple of $m$, instead of just a multiple of $m$.

Finally, we remark that higher powers of $C(P)$ could similarly be used, although the computational cost involved may render the resulting bounds inefficient.

We now consider three examples to compare Theorem 3.3 and Theorem 3.4 with Theorem 4.1. In Example 1 we compare the upper and lower bounds resulting from Theorem 3.4 with those from Theorem 4.1 for both scalar and matrix polynomials. In Example 2 we compare the number of gaps between groups of eigenvalues detected by Theorem 3.3 with the number of gaps detected by Theorem 4.1 for matrix polynomials and in Example 3 we do the same for scalar polynomials. We also compared the gap width, although the ability to detect a gap is usually more important than improving the gap width, and it is our first priority. These examples focus somewhat more on Pellet bounds than on Cauchy bounds since they are less frequently mentioned in the literature. Although $n$ is an even number in the examples, this is not essential and similar conclusions can be drawn for odd numbers.

**Example 1**

For this example we have generated matrix polynomials of degree $n = 10$ with $m = 1, 2, 10, 25$, where $m = 1$ corresponds to a scalar polynomial. They take the form

$$P(z) = Iz^{10} + A_9 z^9 + \cdots + A_0,$$

where each of the $m \times m$ matrices $A_j$, for $j = 0, \ldots, 9$, has complex elements whose real and imaginary parts are uniformly distributed in $[-1, 1]$, multiplied by a random number which is uniformly distributed in $[0, 10]$. To each $P$ corresponds a polynomial $Q$, given by

$$Q(z) = Iz^5 + B_4 z^4 + \cdots + B_0,$$

where the $2m \times 2m$ matrices $B_j$, for $j = 0, \ldots, 4$, are defined as in Theorem 4.1. For each value of $m$, one thousand such random matrix polynomials were generated. We then compared the ratios of the bounds to the moduli of their largest and smallest eigenvalues, respectively. The following upper and lower bounds were compared:

**Upper bounds:**
- $P$: Theorem 3.4 with $P$
- $Q$: Theorem 4.1 with $Q$

**Lower bounds:**
- $P$: Lower bound in Theorem 3.4 with $P$
- $Q$: Lower bound in Theorem 4.1 with $Q$
- $A_0^{-1} P$: Lower bound in Theorem 3.4 with $A_0^{-1} P$ instead of $P$
- $B_0^{-1} Q$: Lower bound in Theorem 4.1 with $B_0^{-1} Q$ instead of $Q$
- $Q_R$: Reciprocal of upper bound in Theorem 4.1 with $Q_R$.

Preceding each bound’s description is its corresponding designation in the tables below.
We recall that the bound designated by $Q_R$ is obtained by forming $Q_R$ from $P$, just as $Q$ is formed from $P$. This notation will also be used for all subsequent examples.

In Table 1, we have listed as percentages the means of the ratios of the upper bound to the modulus of the largest eigenvalue with their standard deviation between parentheses. For instance, a ratio of 3/2 corresponds to 150%. The closer the number is to 100, the better it is, both for upper and lower bounds. The bounds were computed with the 1-Norm, $\infty$-Norm, and the 2-Norm. In Table 2, we have listed the number of times each bound was the better bound. In Table 3 and Table 4, the same was done for the lower bounds.

As Table 1 shows, using Theorem 4.1 produces better upper bounds on average for all three norms, except for scalar polynomials. This advantage seems to increase with increasing $m$. The standard deviations are generally comparable, except for the higher values of $m$ with the 2-Norm, where they are larger for the $Q$ bound. The 2-Norm gives the best results but is more costly to compute. It is also clear from Table 2 that even when a bound is worse on average, there is still a non-negligible number of cases where that bound prevails over the other one.

For the lower bounds, Table 3 and Table 4 show that the advantage goes to $Q_R$ for $m > 1$, while $P$ dominates for $m = 1$. As expected, the bounds $A_0^{-1}P$ and $B_0^{-1}Q$ are better than $P$ and $Q$, respectively. There were no instances where the latter two delivered the best bound. As for the upper bounds, the 2-Norm gives the best results.

Summarizing, we can say that, for $m \geq 2$, the upper bounds were, on average, improved by the use of Theorem 4.1 with $Q$, whereas for the lower bounds, the same is true for Theorem 4.1 with $Q_R$ instead of $Q$. The quality of the bounds improved with higher values of $m$. Although this is but one class of examples, the matrices were randomly generated without any effort at special selection.

| $m$ | $P$ | $Q$ | $m$ | $P$ | $Q$ | $m$ | $P$ | $Q$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 116 (17) | 131 (18) | 1   | 116 (17) | 122 (18) | 1   | 116 (17) | 118 (16) |
| 2   | 161 (32) | 162 (29) | 2   | 162 (33) | 147 (30) | 2   | 140 (28) | 133 (23) |
| 10  | 316 (45) | 247 (47) | 10  | 317 (46) | 216 (50) | 10  | 167 (19) | 161 (31) |
| 25  | 481 (62) | 312 (67) | 25  | 481 (62) | 271 (71) | 25  | 177 (14) | 174 (39) |

Table 1: Comparison of the upper bounds with $n = 10$.

| $m$ | $P$ | $Q$ | $m$ | $P$ | $Q$ | $m$ | $P$ | $Q$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 939 | 61  | 1   | 759 | 241 | 1   | 557 | 443 |
| 2   | 583 | 417 | 2   | 275 | 725 | 2   | 342 | 658 |
| 10  | 152 | 848 | 10  | 136 | 864 | 10  | 317 | 683 |
| 25  | 93  | 907 | 25  | 90  | 910 | 25  | 340 | 660 |

Table 2: Best bound frequencies for the upper bounds with $n=10$. 
Example 2

In this example we compare the ability to produce a gap between the moduli of two groups of eigenvalues for Theorem 3.3 and Theorem 4.1. To do so we have generated a class of matrix polynomials of degree $n = 14$ with $m \times m = 25 \times 25$ matrix coefficients of the form

$$P(z) = I z^{14} + A_{13} z^{13} + \cdots + A_0.$$  

The elements of the matrices $A_j$ have real and imaginary parts that are uniformly distributed in $[-50^2/2, 50^2/2]$, $[-200^2/2, 200^2/2]$, and $[-2, 2]$ for $j = 11$, $j = 12$, and $j \neq 11, 12, 13$, respectively. One thousand such polynomials were generated for which the real and imaginary parts of the elements of $A_{13}$ are in the intervals $[-\eta, \eta]$, where $\eta = 0, 0.25, 0.5, 1$. For each matrix polynomial, we verified if Theorem 3.3 and Theorem 4.1 were applicable and if they were, computed the ratio of the gap between the moduli of the two groups of eigenvalues to the actual gap. The theorems were applied to both $P$ and $Q$, and also to both $A_{k-1}^1 P$ and $B_{k-1}^1 Q$, with $k = 12$.

Throughout this example, only the 1-Norm was used to limit the number of tables. Very similar results are obtained for the $\infty$-Norm. Better results for all bounds are achieved for the 2-Norm (spectral norm), but the relative performance of the bounds follows the

| $m$ | $P$ | $Q$ | $A_{0}^{-1} P$ | $B_{0}^{-1} Q$ | $Q_R$ |
|-----|-----|-----|----------------|----------------|------|
| 1   | 87  (9) | 33 (21) | 87 (9) | 43 (24) | 81 (9) |
| 2   | 49 (13) | 16 (13) | 65 (13) | 25 (18) | 67 (11) |
| 10  | 10 (6) | 4 (4) | 27 (10) | 8 (10) | 39 (9) |
| 25  | 4 (3) | 2 (2) | 16 (7) | 5 (6) | 29 (8) |

| $m$ | $P$ | $Q$ | $A_{0}^{-1} P$ | $B_{0}^{-1} Q$ | $Q_R$ |
|-----|-----|-----|----------------|----------------|------|
| 1   | 87  (9) | 38 (23) | 87 (9) | 45 (25) | 87 (8) |
| 2   | 61 (14) | 21 (17) | 71 (13) | 29 (20) | 78 (12) |
| 10  | 25 (10) | 7 (9) | 39 (12) | 11 (13) | 54 (12) |
| 25  | 15 (7) | 4 (6) | 26 (9) | 6 (9) | 42 (11) |

Table 3: Comparison of the lower bounds with $n = 10$.

| $m$ | $A_{0}^{-1} P$ | $B_{0}^{-1} Q$ | $Q_R$ |
|-----|----------------|----------------|------|
| 1   | 761           | 9              | 230  |
| 2   | 395           | 19             | 586  |
| 10  | 85            | 17             | 898  |
| 25  | 48            | 11             | 941  |

| $m$ | $A_{0}^{-1} P$ | $B_{0}^{-1} Q$ | $Q_R$ |
|-----|----------------|----------------|------|
| 1   | 720           | 22             | 258  |
| 2   | 277           | 19             | 704  |
| 10  | 99            | 7              | 894  |
| 25  | 56            | 3              | 941  |

| $m$ | $A_{0}^{-1} P$ | $B_{0}^{-1} Q$ | $Q_R$ |
|-----|----------------|----------------|------|
| 1   | 87            | 34             | 87 (9) |
| 2   | 49 (13)       | 16 (13)        | 65 (13) |
| 10  | 9 (6)         | 4 (5)          | 27 (10) |
| 25  | 4 (3)         | 2 (3)          | 16 (7) |

Table 4: Best bound frequencies for the lower bounds with $n = 10$.  

1-Norm

$\infty$-Norm

2-Norm
same trends. The 2-Norm’s computational cost is higher than the other two norms, but we do not advocate the use of any particular norm as this choice depends very much on the situation.

On the left in Table 5 are listed the number of times (out of 100 000 cases) that a gap was computed between the \( mk = 300 \) smallest and \( m(n - k) = 50 \) largest eigenvalues for each value of \( \eta \) when applying Theorem 3.3 and Theorem 4.1 to the matrix polynomials \( P \) and \( Q \). The table contains the total number of computed gaps for each matrix polynomial as well as the number of times each matrix polynomial was the only one of the two for which a gap could be computed. On the right in Table 5 are listed, as percentages, the means of the ratios of the gap from Theorem 3.3 and Theorem 4.1 to the actual gap, with the standard deviations between parentheses. The rightmost column lists the percentage of cases where the gap for \( Q \) was larger than the one for \( P \) when a gap was produced for both. Table 6 is the analogous table obtained by applying the theorems to \( A_{k}^{-1}P \) and \( B_{k/2}^{-1}Q \) instead of \( P \) and \( Q \), respectively.

Overall, the results obtained by applying Theorem 4.1 improve as \( ||A_{13}|| \) becomes smaller, which is understandable since it tends to keep \( ||B_{j}|| \) \((j = 0, \ldots, n/2 - 1)\) of the same order of magnitude as \( ||A_{j}|| \) \((j = 0, \ldots, n - 1)\). This is more important here than for mere upper and lower bounds as in Example 1, because the existence of positive roots for \( f_{k} \) in Theorem 3.3 and \( g_{k} \) in Theorem 4.1 is heavily dependent on the relative magnitudes of their coefficients. The results show clearly that, as \( ||A_{13}|| \) becomes smaller, the application of Theorem 4.1 improves the number of times a gap can be detected when compared to Theorem 3.3 and there was a large number of cases in which a gap could be computed for \( Q \), but not for \( P \). The average gap width is generally better for \( P \) and \( A_{k}^{-1}P \) and becomes better for \( Q \) only when \( \eta \approx 0 \).

For this example we found that when \( \eta \) is small, the number of gaps detected for \( Q \) is not much lower than for \( B_{k/2}^{-1}Q \), which is more costly to compute.

| \( \eta \) | \( P \) TOTAL | \( Q \) TOTAL | \( P \) ONLY | \( Q \) ONLY |
|---|---|---|---|---|
| 1 | 210 | 17 | 193 | 0 |
| 0.5 | 314 | 535 | 0 | 221 |
| 0.25 | 379 | 865 | 0 | 486 |
| 0 | 433 | 907 | 0 | 474 |

| \( \eta \) | \( P \) | \( Q \) | \% Gap(Q) \(_1\) Gap(P) |
|---|---|---|---|
| 1 | 16 (7) | 5 (3) | 0 |
| 0.5 | 18 (8) | 12 (5) | 22 |
| 0.25 | 21 (8) | 19 (6) | 64 |
| 0 | 22 (8) | 26 (8) | 100 |

Table 5: Gap frequencies and gap ratios for \( P \) and \( Q \) when \( n = 14 \), \( m = 25 \), and \( k = 12 \).
Table 6: Gap frequencies and gap ratios for $A_k^{-1}P$ and $B_{k/2}^{-1}Q$ when $n=14$, $m=25$, and $k=12$.

**Example 3**

This example illustrates how Theorem 4.1, applied to scalar polynomials ($m=1$), can sometimes be used to provide gaps between the moduli of zeros of a polynomial when the classical scalar version of Pellet’s theorem cannot. To this end, we have generated 1000 scalar polynomials of the form

$$p(z) = z^{20} + a_{19}z^{19} + \cdots + a_0,$$

where the coefficients $a_j$ are random real numbers, uniformly distributed on $[-2, -1] \cup [1, 2]$ for $j = 3, 5, 11, 13$, on $[-10, -8] \cup [8, 10]$ for $j = 4$, on $[-16, -14] \cup [14, 16]$ for $j = 12$, and on $[-1, 1]$ for all $j \neq 3, 4, 5, 11, 12, 13$. Theorem 3.3 was applied for $k=4, 12$.

Because the matrices involved are all $2 \times 2$, we have used the 2-Norm, which can easily be computed in this case and we have applied Theorem 4.1 only to $B_{k/2}^{-1}Q$ since here the matrix multiplications are not costly and the results are better than when the theorem is applied to $Q$.

The results are displayed in Table 7 in the same format as for the previous example. The designation $A_k^{-1}P$ is replaced by $p$ since for the scalar version of Pellet’s theorem there is no difference between its application to $p$ and to $a_k^{-1}p$. The numbers of times that a gap could be computed for both $k=4$ and $k=12$ for the same polynomial are listed in Table 8.

For $k=4$, more gaps are detected by the classical Pellet theorem than with Theorem 4.1, although there are still 21 cases where the latter does provide a gap when the classical Pellet theorem cannot and that gap is also wider in 29% of the cases when both provide a gap. For $k=12$, the situation is reversed: using the generalized Pellet theorem for $B_{k/2}^{-1}Q$ delivers significantly better results than using the scalar Pellet theorem for $p$. There seems to be no particular reason why the classical Pellet theorem is better for $k=4$, but not for $k=12$.

As Table 8 shows, $B_{k/2}^{-1}Q$ delivered a gap for both $k=4$ and $k=12$ in 30 cases more than the 4 cases in which the classical Pellet theorem did.
Table 7: Gap frequencies and gap ratios for $p$ and $B_{k/2}^{-1}Q$ when $n = 20$, $m = 1$, and $k = 4, 12$.

![Table 7](image)

Table 8: Gap frequencies when $n = 20$ and $m = 1$ for both $k = 4$ and $k = 12$.

![Table 8](image)

Remark. The reported average gap widths are over all cases that a gap was produced for a particular theorem and not over only those cases where both theorems produced a gap. The difference in gap width is usually not by an order of magnitude, which would be unexpected. The only time this happens is in limiting cases, where one theorem barely manages to detect a gap and the other detects a gap of average width. Our interest was less in the gap width than in detecting gaps that could previously not be detected.

5 Scalar polynomials as matrix polynomials

In the remainder of this work, we suggest an idea for an alternative way to treat scalar polynomials. More specifically, we propose expressing a scalar polynomial as the determinant of a matrix polynomial to which the generalized Pellet theorem can then be applied, instead of applying the regular (scalar) version of Pellet’s theorem to the original scalar polynomial. The intention is, as before, to try and overcome situations where the real polynomial $f_k$ in Theorem 1.1 does not have positive roots, and to create new upper and lower bounds on the moduli of the zeros with the generalized Cauchy result (Theorem 3.4).

There are infinitely many ways to write a scalar polynomial as the determinant of a matrix polynomial and the best way to proceed is, a priori, not clear. As an example, consider the polynomial $z^4 - z^2 + 3z - 2$. It can be written as

$$\det \left( \begin{array}{cc} z^2 - \sqrt{2} & z \\ z - 3 & z^2 + \sqrt{2} \end{array} \right) = \det \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) z^2 + \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) z + \left( \begin{array}{cc} -\sqrt{2} & 0 \\ -3 & \sqrt{2} \end{array} \right),$$

but it can also be written as, e.g.,

$$\det \left( \begin{array}{cc} z^2 & -1 \\ 3z - 2 & z^2 - 1 \end{array} \right) = \det \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) z^2 + \left( \begin{array}{cc} 0 & 0 \\ 3 & 0 \end{array} \right) z + \left( \begin{array}{cc} 0 & -1 \\ -2 & -1 \end{array} \right).$$

Furthermore, one is not limited to using $2 \times 2$ matrices, and matrices of larger size can also be used. Although this approach is not applicable for every value of $k$ in Theorem 1.1 it does provide new upper and/or lower bounds on the moduli of the zeros.
It is difficult to predict which of the equivalent matrix polynomial representations of a general scalar polynomial will yield the best results, but numerical experiments seem to indicate that better results are obtained if the coefficients are, in a sense, equally distributed among the elements of the matrix polynomial. This distribution therefore needs to be tailored to the particular polynomial under consideration, which makes it difficult to formulate a general method. Consequently, it is a conjecture that our idea might lead to a useful general method.

To strengthen this conjecture, we present the following lemma, where we apply it to a class of lacunary polynomials, followed by an illustrative numerical example.

**Lemma 5.1** Let

\[ p(z) = az^n + bz^{n-1} + cz^{n-2} + \alpha z^2 + \beta z + \gamma , \]  

where \( a, b, c, \alpha, \beta, \gamma \in \mathbb{C} \) with \( a\alpha \neq 0 \), and let

\[
A = \begin{pmatrix}
\sqrt{a} & 0 \\
0 & \sqrt{a}
\end{pmatrix},
B = \begin{pmatrix}
\frac{b}{2\sqrt{a}} + i \left(c - \frac{b^2}{4a}\right)^{1/2} & 0 \\
0 & \frac{b}{2\sqrt{a}} - i \left(c - \frac{b^2}{4a}\right)^{1/2}
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
\sqrt{a} & 0 \\
0 & 0
\end{pmatrix},
D = \begin{pmatrix}
\frac{b}{2\sqrt{a}} + i \left(c - \frac{b^2}{4a}\right)^{1/2} & 0 \\
0 & \sqrt{a}
\end{pmatrix},
E = \begin{pmatrix}
0 & 0 \\
0 & \frac{b}{2\sqrt{a}} - i \left(c - \frac{b^2}{4a}\right)^{1/2}
\end{pmatrix},
\]

\[
V = \begin{pmatrix}
0 & \beta \\
\frac{\beta}{2\sqrt{\alpha}} - i \left(\gamma - \frac{\beta^2}{4\alpha}\right)^{1/2} & 0
\end{pmatrix},
W = \begin{pmatrix}
0 & \gamma - \frac{\beta^2}{4\alpha} \\
\frac{\beta}{2\sqrt{\alpha}} - i \left(\gamma - \frac{\beta^2}{4\alpha}\right)^{1/2} & 0
\end{pmatrix}.
\]

Then the zeros of \( p \) are the eigenvalues of the matrix polynomial

\[ Q_{\text{even}}(z) = Az^{n/2} + Bz^{n/2-1} + Vz + W \]  

when its degree \( n \) is even, and they are the finite eigenvalues of

\[ Q_{\text{odd}}(z) = Cz^{(n+1)/2} + Dz^{(n-1)/2} + Ez^{(n-3)/2} + Vz + W \]  

when its degree \( n \) is odd.

**Proof.** The polynomial \( p \) can be written as follows:

\[
p(z) = az^n + bz^{n-1} + cz^{n-2} + \alpha z^2 + \beta z + \gamma = az^{n-2} \left(z^2 + \frac{b}{a}z + \frac{c}{a}\right) + \alpha \left(z^2 + \frac{\beta}{\alpha}z + \frac{\gamma}{\alpha}\right)
\]

\[
= z^{n-2} \left(a \left(z + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}\right) + \left(\alpha \left(z + \frac{\beta}{2\alpha}\right)^2 + \gamma - \frac{\beta^2}{4\alpha}\right).
\]
Now assume that \( n \) is even. After factoring the quadratic expressions, we obtain

\[
p(z) = \det \left( \begin{pmatrix} \alpha z + \frac{b}{2\sqrt{a}} + i \left( c - \frac{b^2}{4a} \right)^{1/2} \\
\sqrt{\alpha} z + \frac{\beta}{2\sqrt{\alpha}} - i \left( \gamma - \frac{\beta^2}{4\alpha} \right)^{1/2} \end{pmatrix} \right)
\]

which is easily seen to be equivalent to \( \det(Q_{\text{even}}(z)) \), with

\[
Q_{\text{even}}(z) = Az^{n/2} + Bz^{n/2-1} + Vz + W,
\]

where the \( 2 \times 2 \) matrices \( A, B, V, \) and \( W \) are defined in the statement of the lemma.

When \( n \) is odd, we can similarly write \( p \) as

\[
p(z) = \det \left( \begin{pmatrix} \alpha z + \frac{b}{2\sqrt{a}} + i \left( c - \frac{b^2}{4a} \right)^{1/2} \\
\sqrt{\alpha} z + \frac{\beta}{2\sqrt{\alpha}} - i \left( \gamma - \frac{\beta^2}{4\alpha} \right)^{1/2} \end{pmatrix} \right)
\]

which is equivalent to \( \det(Q_{\text{odd}}(z)) \), with

\[
Q_{\text{odd}}(z) = Cz^{(n+1)/2} + Dz^{(n-1)/2} + Ez^{(n-3)/2} + Vz + W,
\]

where the \( 2 \times 2 \) matrices \( C, D, E, V, \) and \( W \) are defined in the statement of the lemma.

Lemma 5.1 can now be used in conjunction with Theorem 8.3 and Theorem 8.4 to obtain additional and/or improved results for the scalar versions of these theorems, as long as the appropriate matrices are nonsingular. In this regard we note that the matrices \( A \) and \( V \) are always nonsingular since we assumed that \( a\alpha \neq 0 \), and the matrices \( C \) and \( E \) are always singular. As with Theorem 4.1 not all of the scalar bounds can be improved, but enough can to make it worthwhile. When applying a similar process as in Lemma 5.1 to a general polynomial, the bounds, for a particular value of \( k \), require \( O(n) \) operations since the coefficient matrices of the equivalent matrix polynomial are always of size \( 2 \times 2 \). Similarly to what we had before, the degrees of the new matrix polynomials \( Q_{\text{even}} \) and \( Q_{\text{odd}} \) are half and roughly half that of \( P \), respectively.

Remarks. Although we will not pursue the matter further, Lemma 5.1 allows for many variations on its theme. For instance, a polynomial of the form \( az^n + bz^{n-1} + cz^{n-2} + \alpha z + \beta \) with \( a\alpha \neq 0 \) becomes of the form of a polynomial as in Lemma 5.1 with \( \gamma = 0 \) after multiplication by \( z \), so that the lemma becomes applicable to such polynomials as well. Moreover, the lacunary polynomial \( p \) in Lemma 5.1 is a special case for \( r = s = 1 \) of the polynomial

\[
az^{n+2r-2} + bz^{n+r-2} + cz^{n-2} + \alpha z^s + \beta z^r + \gamma,
\]

where \( r \geq 1 \) and \( s \geq 1 \) are integers. Lemma 5.1 can easily be modified to include these more general polynomials, which can be further generalized by replacing \( z \) with a polynomial in \( z \).
The following example illustrates how Lemma 5.1 can be used to improve upper and lower bounds on the moduli of the zeros of a polynomial and also to detect gaps between groups of zeros where the classical scalar Pellet theorem cannot, or to improve the width of the gaps that it does detect.

**Example 4**

For this example, we have generated 1000 polynomials of the form (7), with real random coefficients \( a, b, c, \alpha, \beta, \) and \( \gamma \) uniformly distributed in \([-50, 50]\) and for \( n = 20, 40, 80 \). Both the upper and lower bounds from Theorem 3.4 and its scalar version were computed, as well as the bounds from Theorem 3.3 and its scalar version for \( k = 2 \) and \( k = n - 2 \), i.e., bounds on the gap between 2 and \( n - 2 \) zeros. These bounds were computed for both \( p \) and \( Q_{even} \). The matrices \( B \) and \( W \) defining \( Q_{even} \) were nonsingular in all cases. Again, as before, because all matrices involved are \( 2 \times 2 \), we used only the 2-Norm. We have computed means and standard deviations of ratios and expressed them as percentages and we have listed gap frequencies with the same conventions as in the previous examples. The results were collected in the tables below.

Table 9 lists the ratios of the upper and lower bounds to the moduli of the largest and smallest zeros, respectively. The designation ”scalar” and ”matrix” refers to the use of \( p \) from (7) and \( Q_{even} \) from (8), respectively. The last three columns list the percentage of cases for which using \( Q_{even} \) yielded a better result than using \( p \) for the upper, lower and both upper and lower bounds on the moduli of the zeros of \( p \). The upper and lower bound ratios using \( p \) are quite unaffected by the degree of \( p \), whereas using \( Q_{even} \) yields better ratios and smaller standard deviations that improve with increasing degree \( n \), a property that seems to be independent of the distribution of the coefficients.

Table 10 lists the number of times a gap could be computed with both the scalar and generalized (matrix) version of Theorem 3.3 for \( k = 2 \) and \( k = n - 2 \). There is a clear advantage to using \( Q_{even} \) that becomes more significant with increasing degree \( n \). The same is true for the number of times both gaps could be computed, which can be found in Table 11. To pick but one instance, for \( n = 40 \), the classical scalar version of Pellet’s theorem detects a gap between the 38 smallest and 2 largest zeros 102 times. The generalized Pellet theorem with \( Q_{even} \) adds another 55 times to that and also detects more than twice as many gaps (32 vs. 14) for both \( k = 2 \) and \( k = 38 \).

Table 12 contains the means and standard deviations of the gaps and the percentage of times \( Q_{even} \) produced a larger gap than \( p \) for those cases in which both produced a gap. This was close to half of all cases, while producing a larger average ratio of computed gap to exact gap. The computed gap may be larger for one or the other, although the situation could be reversed for the bounds defining that gap. In other words, \( Q_{even} \) may produce a smaller gap than \( p \), but its lower gap bound or its upper gap bound may be better. This means that combining the bounds from \( Q_{even} \) and \( p \) would improve the gap ratio even more.

18
Similar results are obtained for odd powers and it was also observed that Lemma 5.1 produces better results for this class of polynomials than Theorem 4.1.

References

[1] Aberth O. *Iteration methods for finding all zeros of a polynomial simultaneously.* Math. Comput., 27 (1973), 339–344.

[2] Bini, D.A. *Numerical computation of polynomial zeros by means of Aberth's method.* Numer. Algorithms, 13 (1996), 179–200.

[3] Bini, D.A., Noferini, V., and Sharify, M. *Locating the eigenvalues of matrix polynomials.* arXiv:1206.3632v2 (2 August 2012).

[4] Cauchy, A.L. *Exercises de mathématique.* Oeuvres (2) Vol. 9 (1829), p.122.
[5] Ehrlich, L.W. A modified Newton method for polynomials. Comm. of ACM, 10 (1967), 107–108.

[6] Gohberg, I. and Sigal, E. I. An operator generalization of the logarithmic residue theorem and Rouch’s theorem. (Russian) Mat. Sb. (N.S.), 84(126) (1971), 607-629. English translation in Math. USSR-Sb., 13 (1971), 603-625.

[7] Gohberg, I., Lancaster, P., and Rodman, L. Matrix polynomials. Computer Science and Applied Mathematics. Academic Press, Inc., New York-London, 1982.

[8] Gohberg, I., Goldberg, S., and Kaashoek, M. A. Classes of linear operators. Vol. I. Operator Theory: Advances and Applications, 49. Birkhuser Verlag, Basel, 1990.

[9] Higham, N.J., Tisseur, F. Bounds for eigenvalues of matrix polynomials. Lin. Alg. Appl., 358 (2003), 5-22.

[10] Horn, R. A. and Johnson, C. R. Matrix Analysis. Cambridge University Press, Cambridge, 1988.

[11] Lang, S. Complex Analysis. Springer-Verlag, New York, 1999.

[12] Lancaster, P. Inversion of lambda-matrices and application to the theory of linear vibrations. Arch. Rational Mech. Anal., 6 (1960), 105-114.

[13] Lancaster, P. Lambda Matrices and Vibrating Systems. Pergamon Press, Oxford, 1966.

[14] Marden, M. Geometry of polynomials. Mathematical Surveys, No. 3, American Mathematical Society, Providence, R.I., 1966.

[15] Melman, A. Implementation of Pellet’s Theorem. Submitted manuscript, 2012.

[16] Monden, Y. and Arimoto, S. Generalized Rouché’s theorem and its application to multivariate autoregressions. IEEE trans. on Accoustics, Speech, and Signal Processing, 28 (1980), 733-738.

[17] Pellet, M.A. Sur un mode de séparation des racines des équations et la formule de Lagrange. Bull. Sci. Math., 5 (1881), 393–395.

[18] Rouché, E. Mémoire sur la série de Lagrange. J. École Polytech., 22 (1862), 217–218.

[19] Tisseur, F. and Meerbergen, K. The quadratic eigenvalue problem. SIAM Rev., 43 (2001), 235-286.

[20] Vaidyanathan, P.P and Mitra, S.K. A unified structural interpretation of some well-known stability test procedures for linear systems. Proceedings of the IEEE, 75 (1984), 478–497.