Lag synchronization in coupled chaotic systems via intermittent control

Junjian Huang*, Pengcheng Wei

Department of Computer Science, Chongqing Education College,
9 Xuefu Road, Chongqing 400067, PR China

Abstract

This paper deals with the lag synchronization of two coupled identical chaotic systems by intermittent control. Via intermittent control with periodically intervals, we can obtain the lag synchronization. Some sufficient conditions for the stabilization and synchronization of a large class of coupled identical chaotic systems will be derived by using Lyapunov stability theory. Compared to exiting results, some less conservative conditions are derived to guarantee the stabilization of nonlinear system. The analytical results are confirmed by numerical simulations.

1. Introduction

After the seminal works of Pecora and Carroll [1], the idea of synchronization of chaotic systems has received a great deal of interest among researchers from various fields. Over the past decade, several different regimes of chaos synchronization have been investigated, i.e., complete synchronization [1, 2], generalized synchronization [3], projective synchronization [4], phase synchronization [5], lag synchronization [6], and anticipating synchronization [7, 8]. In the complete synchronization, the master’s state $x(t)$ and the slave’s state $y(t)$ are identical, i.e., $y(t) \rightarrow x(t)$ [i.e., $y(t)$ approaches to $x(t)$ as $t$ approaches to infinite similarly hereinafter]. Generalized synchronization is defined as the presence of some function relation between the slave’s states and the master’s, i.e., $y(t) \rightarrow g(x(t))$. In the projective synchronization, there exists a scale factor in the amplitude of the master’s state variable and that of the slave’s, i.e., $y(t) \rightarrow \alpha x(t)$. Lag synchronization appears as a coincidence of shifted-in-time states of two systems, $y(t) = x(t-\tau)$ with positive $\tau$, and has been studied in between symmetrically coupled nonidentical oscillators [6].

Some important control methods including continuous control and discontinuous control have been developed for stabilizing and synchronizing chaotic systems, such as state feedback control [9, 10], adaptive control [11], switching control [12, 13], impulsive control [14-17], and intermittent control [18-22], etc.

Recently, intermittent control of nonlinear system has drawn receiving increasing interests in process control, ecosystem management, windshield wipers intermittent control, and communication and so on. As a special form of switching control, intermittent control is also divided into two classes: state-dependent switching rule and time-switching rule. The former implies the control operation is activated only when the states come into the certain region which is often pre-given; while the later activates the control only in some finite time intervals, the system evolves freely when the time goes

* Corresponding author. Tel.: +86-023-62658269.
E-mail address: hnomu@sina.com.

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out of those intervals. Therefore, these intermittent control systems are open-loop. Compared with continuous control method, intermittent control method is advantageous for its efficiency. In this paper, we have formulated the lag synchronization problem for chaotic systems by means of periodically intermittent control. Our interest focuses on the class of intermittent control with time duration, namely, the control is activated in certain nonzero time intervals, and off in other time intervals. Also, we will remove this limitation and design a general periodically-intermittent controller for chaotic systems.

The rest of the paper is organized as follows. In Section II, the problem to be considered is formulated and some necessary preliminaries are restated. In Section III, studies the lag synchronization of chaotic systems, together with its simplified version, is derived by rigorous theoretical analysis. An algorithm for determining the control law is then given by virtue of the theoretical results. In Section IV, a lag synchronization error system will be stabilized numerically by periodically intermittent control, which is followed by the conclusions in Section V.

2. Theoretical analysis

Consider a drive system given by

$$\dot{x} = f(x)$$  \hspace{1cm} (1)

Where \( x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n \), \( f(x) = (f_1(x), f_2(x), \cdots, f_n(x))^T \) is a nonlinear vector function. Let \( \Omega \subset \mathbb{R}^n \) be a chaotic bounded set of \( (1) \), which is globally attractive. As for vector function \( f(x) \), assume that for any \( x, y \in \Omega \) we have

$$|f_i(x) - f_i(y)| \leq L_j |x - y|, i, j = 1, 2, \cdots, n$$  \hspace{1cm} (2)

The above condition is considered as the uniform Lipschitz condition, and \( L_j > 0 \) refers to the uniform Lipschitz constant.

In what follows, the coupled response system with feedback control is given by

$$\dot{y} = f(y) + u(t)$$  \hspace{1cm} (3)

Where \( y = (y_1, y_2, \cdots, y_n)^T \in \mathbb{R}^n \) is the response state. \( u(t) \) is the intermittent control gain defined by:

$$u(t) = k(y - x(t - \tau)), \quad nT \leq t \leq nT + \sigma T;$$
$$u(t) = 0, \quad nT + \sigma T \leq t \leq (n + 1)T.$$  \hspace{1cm} (4)

Where \( \tau > 0 \) is the propagation delay, \( k \) denotes control strength, \( 0 < \sigma < 1 \) denotes switching rate and \( T \) denotes control period. Let \( e(t) = y(t) - x(t - \tau) \) be the lag synchronization error between the systems \( (1) \) and \( (3) \), then yields the error system

$$\dot{e}(t) = y(t) - \dot{x}(t - \tau) = f(y) - f(x(t - \tau)) + u(t).$$  \hspace{1cm} (5)

Under the control of the form (4), the lag synchronization error (5) can be rewritten as

$$\dot{e}(t) = f(y) - f(x(t - \tau)) + ke(t), \quad nT \leq t \leq nT + \sigma T;$$
$$\dot{e}(t) = f(y) - f(x(t - \tau)), \quad nT + \sigma T \leq t \leq (n + 1)T.$$  \hspace{1cm} (6)

Lemma 1, [23] Given any real matrices \( \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4 \) of appropriate dimensions and a scalar \( \varepsilon > 0 \) such that \( 0 < \Sigma_1 = \Sigma_3^T \). Then the following inequality holds:

$$\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_3 \leq \varepsilon \Sigma_1^T \Sigma_1 + \varepsilon^{-1} \Sigma_2^T \Sigma_2 + \Sigma_4.$$  \hspace{1cm} (7)

3. Main results

In this section, by using intermittent control techniques, several theoretical results are developed to realize lag synchronization between the drive and the response systems \( (1) \) and \( (3) \), several appropriate comparisons are provided to illustrate the effectiveness of our results.

**Theorem 1:** Suppose that positive constants exist \( \varepsilon_i > 0 \) and \( g_i > 0 \) such that the following hold.

1) \( 2k l + \varepsilon_i l + \varepsilon_i^{-1} l, l + g_i l \leq 0; \)

$$\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_3 \leq \varepsilon \Sigma_1^T \Sigma_1 + \varepsilon^{-1} \Sigma_2^T \Sigma_2 + \Sigma_4.$$  \hspace{1cm} (8)

$$\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_3 \leq \varepsilon \Sigma_1^T \Sigma_1 + \varepsilon^{-1} \Sigma_2^T \Sigma_2 + \Sigma_4.$$  \hspace{1cm} (9)
2) \( \epsilon_1 I + \epsilon_2^{-1}I - g_1 I \leq 0 \);
3) \( g_1 \sigma T - g_2 (1 - \sigma) > 0 \).

Then, the lag synchronization error dynamical of the system (6) is globally exponentially stable, and moreover,

\[
\|e(t)\| = \|e(t_0)\| \exp(-\frac{1}{2}(g_1 \sigma T - g_2 (1 - \sigma))(t - \sigma T)),
\]

implying that the two systems (1) and (3) are globally asymptotically lag synchronized.

**Proof:** Consider the following Lyapunov-like function:

\[
V(e(t)) = e^T(t) e(t)
\] (7)

Which implies that \( V(e(t)) = \|e(t)\|^2 \).

When \( nT \leq t \leq nT + \sigma T \), with the help of Lemma 1, the derivative of Eq. (7) with respect to time \( t \) along the trajectories of the first subsystem of the system (6) is calculated and estimated as follows:

\[
\dot{V}(e(t)) = 2e^T(t) e(t)
\]

\[
\leq e^T(t) [2I + \epsilon_1 I + \epsilon_2^{-1}I] e(t) - g_1 e^T(t) e(t)
\]

\[
\leq -g_2 V(e(t)).
\]

Thus, we have

\[
\dot{V}(e(t)) \leq -g_2 V(e(t)), nT \leq t \leq nT + \sigma T
\]

Which implies that when \( nT \leq t \leq nT + \sigma T \)

\[
V(e(t)) \leq V(e(nT)) \exp(-g_2 (t - nT)).
\]

Similarly, when \( nT + \sigma T \leq t \leq (n+1)T \), we have

\[
\dot{V}(e(t)) \leq e^T(t) [\epsilon_1 I + \epsilon_2^{-1}I - g_2 I] e(t) + g_2 e^T(t) e(t)
\]

\[
\leq g_2 V(e(t)).
\]

Therefore, we derive that when \( nT + \sigma T \leq t \leq (n+1)T \)

\[
\dot{V}(e(t)) \leq g^2_2 V(e(t))
\]

and

\[
V(e(t)) \leq V(e(nT + \sigma T)) \exp(g_2 (t - nT - \sigma T)).
\]

Therefore,

\[
\dot{V}(e(t)) \leq V(e(nT)) \exp(-g_2 (t - nT)), nT \leq t \leq nT + \sigma T;
\]

\[
\dot{V}(e(t)) \leq V(e(nT + \sigma T)) \exp(g_2 (t - nT - \sigma T)), nT + \sigma T \leq t \leq (n+1)T
\]

Following the same line of argument of the proof of Theorem 1 of [21], we can obtain:

for any \( t \geq 0 \).

\[
\|e(t)\|^2 = V(e(t)) \leq V(e(t_0)) \exp(-(g_1 \sigma T - g_2 (1 - \sigma))(t - \sigma T)),
\]

This implies that the origin of system (6) is globally exponentially stable, and the following estimate holds:

\[
\|e(t)\| = \|e(t_0)\| \exp(-(g_1 \sigma T - g_2 (1 - \sigma))(t - \sigma T)/2), t \geq 0.
\] (8)

Hence, the two systems (1) and (3) are globally asymptotically lag synchronized.

The proof is thus completed.

**Remark 1.** If we replace the first two conditions in Theorem 1 by the scalar equalities

\[
g_1^* = -2k - \epsilon_1 - \epsilon_2^{-1}L, \quad g_2^* = \epsilon_1 + \epsilon_2^{-1}L,
\]

where \( g_1^* \geq g_1 \) and \( g_2^* \leq g_2 \), then, Theorem 1 also holds.

Furthermore, letting \( \epsilon_1 = \epsilon_2 \), Theorem 1 will reduce the following corollary.

**Corollary 1.** If there exist constants \( k, 0 < \sigma < 1, \epsilon_1 \) such that \( g_1^* \sigma T - g_2^* (1 - \sigma) > 0 \), where

\[
g_1^* = -2k - \epsilon_1 - \epsilon_2^{-1}L, \quad g_2^* = \epsilon_1 + \epsilon_2^{-1}L,
\]

the lag synchronization error system (6) is globally
exponentially stable, and the lag synchronization between the systems (1) and (3) is achieved.

4. Numerical example

In this section, Chua’s oscillator given by Eq. (9) is presented to verify the theoretical results in this paper [24]. The program dde23.m in MATLAB is used to integrate the differential equations with the initial conditions $x(0) = [0.1, 0.2, 0.2]$ and $y(0) = [0.2, -0.1, 0.1]$,

$$\begin{align*}
\dot{x}_1 &= \alpha(x_2 - x_1 - g(x_1)), \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta x_2,
\end{align*}$$

where $\alpha$, $\beta$ are parameters and $g(x_i)$ is the piecewise-linear characteristics of Chua’s diode, which is defined by

$$g(x_i) = bx_i + 0.5(a - b)(|x_i| + 1 - |x_i|),$$

Where $a < b < 0$ are two constants. In this section, we always choose the system parameters as $\alpha = 9.2156$, $\beta = 15.9946$ and $a = -1.24905$, $b = -0.75735$, which make Chua’s circuit (9) chaotic.

The waves of state variables are showed in Fig.1.

In what follows, the corresponding response system is taken as

$$\begin{align*}
\dot{y}_1 &= \alpha(y_2 - y_1 - g(y_1)) + h_1(t)(y_1 - x_1(t - \tau)), \\
\dot{y}_2 &= y_1 - y_2 + y_3 + h_1(t)(y_2 - x_2(t - \tau)), \\
\dot{y}_3 &= -\beta y_2 + h_1(t)(y_3 - x_3(t - \tau)),
\end{align*}$$

Where

$$
h_1(t) = k, nT \leq t \leq nT + \sigma T;$$

\[h_1(t) = 0, nT \leq t \leq nT + \sigma T.\]

For numerical simulation, we select $T=8$, $\sigma = 0.7$ and $k = -5$. The solution of the three-variable autocatalator model with intermittent control is synchronization, as shown in Fig. 1. Fig. 2 shows the time response curves of the lag synchronization error in case of $\tau = 0.02$.

![Fig. 1. Lag synchronization for two coupled chaotic chua’s systems with time delay $\tau = 0.02$.](image)

5. Conclusions

In this paper, we have formulated the lag synchronization problem for chaotic systems by means of periodically intermittent control and design a general periodically-intermittent controller for chaotic systems. Lag synchronization criteria are established based on Lyapunov stability theory and linear matrix inequality techniques. Numerical simulations have showed the validity of theoretical result.

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Fig. 2. The time-response curves of the lag synchronization error when $\tau = 0.02$