HIGHER ARITHMETIC INTERSECTION THEORY

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ABSTRACT. We give a new definition of higher arithmetic Chow groups for smooth projective varieties defined over an arithmetic field, which is similar to Gillet and Soulé’s definition of arithmetic Chow groups. We also give a compact description of the intersection theory of such groups. A consequence of this theory is the definition of a height pairing between two higher algebraic cycles, of complementary dimensions, whose real regulator class is zero. This description agrees with Beilinson’s height pairing for the classical arithmetic Chow groups. We also give examples of the higher arithmetic intersection pairing in dimension zero that, assuming a conjecture by Milnor on the independence of the values of the dilogarithm, are non-zero.

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1. Introduction

1.1. Higher arithmetic Chow groups. Let $X$ be a smooth and projective variety of dimension $d$, defined over an arithmetic field $F$. In [GS90], Gillet and Soulé defined the arithmetic Chow groups of $X$, denoted by $\hat{\text{CH}}^p(X)$. In fact, in loc. cit. the arithmetic Chow groups are defined in much greater generality for regular quasi-projective schemes over an arithmetic ring, but in this paper we will only treat the case of smooth and projective varieties defined over an arithmetic field $F$. The elements of $\hat{\text{CH}}^p(X)$ are classes of pairs $(Z,g_Z)$, with $Z$ a codimension $p$ subvariety of $X$ and $g_Z$ a Green current for $Z$. The groups $\hat{\text{CH}}^p(X)$ satisfy the following properties (see [BG97] for details):

- They fit into an exact sequence

\begin{equation}
\text{CH}^p(X,1) \xrightarrow{\rho_{\text{nc}}} \tilde{\mathcal{D}}^{2p-1}(X,p) \xrightarrow{\alpha} \hat{\text{CH}}^p(X) \xrightarrow{\zeta} \text{CH}^p(X) \to 0,
\end{equation}

where the group $\text{CH}^p(X,1)$ is Bloch’s higher Chow group [Blo86a], the map $\rho_{\text{nc}}$ is Beilinson’s regulator, $\mathcal{D}^*(X,p)$ is the Deligne complex computing the real Deligne cohomology $H^*_D(X,\mathbb{R}(p))$ of $X$ and $\tilde{\mathcal{D}}^{2p-1}(X,p) = \mathcal{D}^{2p-1}(X,p)/\text{Im } d_\mathcal{D}$.

- There is an intersection pairing

$$\hat{\text{CH}}^p(X) \otimes \hat{\text{CH}}^q(X) \to \hat{\text{CH}}^{p+q}(X)$$
turning $\bigoplus_{p \geq 0} \widehat{CH}^p(X)$ into a commutative graded unitary algebra. Note that, since we are working with varieties over a field, there is no need to tensor with $\mathbb{Q}$ to have a well defined product.

- If $f : X \to Y$ is a morphism of smooth projective varieties, then there exists a pullback
  
  $$f^* : \widehat{CH}^p(Y) \to \widehat{CH}^p(X),$$

  and if $f$ is smooth, since it is also proper, there exists a pushforward map
  
  $$f_* : \widehat{CH}^p(X) \to \widehat{CH}^{p-l}(Y),$$

  where $l = \dim(X) - \dim(Y)$. The inverse image is an algebra homomorphism and, together with the direct images, it satisfies the projection formula.

- The intersection product, the direct image and the inverse image are compatible with the corresponding operations in the classical Chow groups.

In parallel with Deligne and Soulé’s proposal for a higher arithmetic $K$-theory, Goncharov [Gon05] introduced the higher arithmetic Chow groups. Temporarily, in this introduction, we denoted them as $\widehat{CH}^p(X, n)_{Go}$. These groups satisfy $\widehat{CH}^p(X, 0)_{Go} = \widehat{CH}^p(X)$ and were constructed to extend the exact sequence 1.1 to a long exact sequence

$$\cdots \to \widehat{CH}^p(X, n)_{Go} \xrightarrow{\zeta} CH^p(X, n) \xrightarrow{\rho} H^{2p-n}(X, \mathbb{R}(p)) \xrightarrow{\alpha} \cdots \to CH^p(X, 1) \xrightarrow{\rho_{n+1}} \widehat{D}^{2p-1}(X, p) \xrightarrow{\alpha} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0.$$

The construction of Goncharov uses a morphism of complexes

$$Z^p(X, \ast)_0 \xrightarrow{P} \widehat{D}^{2p-*}(X, p),$$

where $Z^p(X, \ast)_0$ is the normalized cubical Bloch complex, whose homology computes the higher Chow groups $CH^p(X, \ast)$, and $\widehat{D}_D(X, p)$ is the Deligne complex of currents, which computes the real Deligne cohomology of $X$. Due to the use of the Deligne complex of currents, this construction left open the following questions:

1. Does the composition of the isomorphism $K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p \geq 0} CH^p(X, n)_{\mathbb{Q}}$ with the morphism induced by $P$ agree with the Beilinson regulator?
2. Can one define a product structure on $\bigoplus_{p,n} \widehat{CH}^p(X, n)_{Go}$?
3. Are there well-defined pull-back morphisms?

All these questions were answered positively by the first author together with Felic [BGF12] and with Felic and Takeda [BGFT11]. Namely, in [BGF12], there is a new construction of higher arithmetic Chow groups, that we temporarily denote $\widehat{CH}^p(X, n)_{BF}$. These groups satisfy $\widehat{CH}^p(X, 0)_{BF} = \widehat{CH}^p(X)$ and fit in the same long exact sequence as the groups $\widehat{CH}^p(X, n)_{Go}$, but they are defined using only differential forms with logarithmic singularities.
This allows to define inverse images and products. Latter in [BGFT11] they proved the existence of a natural isomorphism
\[ \hat{CH}^p(X, n)_{BF} \cong \hat{CH}^p(X, n)_{Go} \]
and the fact that the morphism induced by \( P \) agrees with Beilinson’s regulator.

Following Takeda’s alternative definition of higher arithmetic \( K \)-theory ([Tak05]), one can envision a different kind of higher arithmetic Chow groups, that we denote by \( \hat{CH}^p(X, n) \). These new higher arithmetic Chow groups should fit in exact sequences of the form
\[
\begin{align*}
\text{CH}^p(X, n + 1) & \xrightarrow{\rho_{n+1}} \tilde{D}^{2p-n-1}(X, p) \xrightarrow{\omega} \hat{CH}^p(X, n) \xrightarrow{\zeta} \text{CH}^p(X, n) \to 0.
\end{align*}
\]
In particular, for this groups the map \( \hat{CH}^p(X, n) \xrightarrow{\zeta} \text{CH}^p(X, n) \) is surjective. Such a definition was already proposed in Elisenda Feliu’s thesis ([Fel07]), but was not developed further. The definition given there is based on homological algebra constructions and is not very well suited for concrete computations.

The relationship between the groups \( \hat{CH}^p(X, n) \) and the previously defined \( \hat{CH}^p(X, n)_{BF} \) or \( \hat{CH}^p(X, n)_{Go} \), is as follows. There is a map
\[ \omega: \hat{CH}^p(X, n) \to \mathcal{D}^{2p-n}(X). \]
Writing \( \hat{CH}^p(X, n) = \ker(\omega) \), we get
\[ \hat{CH}^p(X, n)_{Go} = \hat{CH}^p(X, n)_{BF} = \begin{cases} 
\hat{CH}^p(X, 0), & \text{for } n = 0, \\
\hat{CH}^p(X, n), & \text{for } n > 0.
\end{cases} \]

The aim of this paper is to give a new presentation of higher arithmetic Chow groups that is closer to the original definition of arithmetic Chow groups, and to develop an intersection theory which can be seen as a natural generalization of the intersection theory of arithmetic Chow groups.

We emphasize that the use of Bloch’s cubical complex for higher Chow groups restricts us to varieties over an arithmetic field. The question whether one can develop a theory for higher arithmetic Chow groups for arithmetic varieties over general arithmetic rings, is still open.

1.2. The main definitions. We now give a brief description of the construction of the higher arithmetic Chow groups, as presented in this paper. For more details the reader is referred to the main text.

There are several complexes that compute Deligne cohomology of \( X \). Using differential forms, we will consider the complexes \( \mathcal{D}, \mathcal{D}_t, \mathcal{D}_{TW} \) described in Section 4. The first is the simplest one and was introduced in [Del85], the second is the closest to the original definition of Deligne cohomology, while the third, called the Thom-Whitney version of the Deligne complex, is the more involved, but has the advantage of having a product that is graded commutative and associative at the level of complexes. In order to define the groups \( \hat{CH}^p(X, n)^0 \) it does not matter which complex one uses, but the groups \( \hat{CH}^p(X, n) \) depend on the complex of forms used to define them. Since we want an intersection product that is associative and graded commutative we will mainly use the complex \( \mathcal{D}_{TW} \).
There are variants of the three complexes computing Deligne cohomology that are defined using currents. They are denoted denoted $D$, $D_{t,D}$ and $D_{TW,D}$. In particular, $D_{TW,D}$ is a module over the algebra $D_{TW}$.

The cubical version of Goncharov regulator can be modified easily to give a morphism of complexes

$$P: Z^p(X,*)_0 \to D^{2p-n}_{TW,D}(X,p).$$

Using this map we can extend the notion of Green current for higher cycles.

**Definition 1.1.** Let $Z \in Z^p(X,n)_0$ be a cycle, that is $\delta Z = 0$, then a $D_{TW}$-Green current for $Z$ is a current $g_Z \in D^{2p-n-1}_{TW,D}(X,p)$ such that

$$P(Z) + dg_Z = [\omega],$$

with $\omega \in D^{2p-n}(X,n)$ a smooth form. The class of $g_Z$ in

$$\hat{D}^{2p-n-1}_{TW,D}(X,p) = D^{2p-n-1}_{TW,D}(X,p)/\text{Im } d_D$$

is denoted by $\hat{g}_Z$ and is called a class of Green currents. Given a Green current $g_Z$ we write $\omega(g_Z)$ for the only form such that

$$P(Z) + dg_Z = [\omega(g_Z)].$$

**Definition 1.2.** A codimension $p$ higher arithmetic cycle is a pair $(Z, \hat{g}_Z)$ with $Z \in Z^p(X,n)_0$ a cycle and $\hat{g}_Z$ a class of Green currents for $Z$. The group of codimension $p$ higher arithmetic cycles will be denoted by $\hat{Z}^p(X,n)_0$.

A higher arithmetic cycle is called rationally equivalent to zero if it is of the form $(\delta T, -\hat{P}(T))$ for $T \in Z^p(X,n+1)_0$. The subgroup of cycles rationally equivalent to zero is denoted $\hat{Z}^p_{rat}(X,n)_0$. Finally the higher arithmetic Chow groups are defined as

$$\hat{CH}^p(X,n) = \hat{Z}^p(X,n)_0/\hat{Z}^p_{rat}(X,n)_0.$$

As in the theory of arithmetic Chow groups $\hat{CH}^p(X)$, we show that these groups are functorial and have a product structure. The definition above is very well suited to define direct images and one can also define inverse images for smooth maps. But in order to define products and inverse images we need to define the analogue of a Green form with logarithmic singularities. The main difficulty here is that the cycle $Z$ and the current $g_Z$ live in different spaces.

Let $\square = \mathbb{P}^1 \setminus \{1\}$ and write $\square^n = \square \times \cdots \times \square$. The varieties $\square$ form a cocubical scheme (see Section 2.2). In particular we have face maps

$$\delta^j_i: \square^n \to \square^{n+1},$$

for $j = 0, 1$ and $i = 1, \ldots, n + 1$.

An element $Z \in Z^p(X,n)_0$ is an algebraic cycle in $X \times \square^n$ that intersects properly all the faces of $X \times \square^n$ with the extra condition that $Z \in \cap_{i=1}^{n+1}\ker(\delta^1_i)^*$, see §3 for more details. We write $|Z| \subset X \times \square^n$ for its support and let $|Z|_k \subset X \times \square^k$ be the codimension $p$ subset given as

$$|Z|_k = \bigcup_{i_1, \ldots, i_{n-k}} (\delta^1_{i_1})^{-1} \cdots (\delta^1_{i_{n-k}})^{-1}|Z|.$$
For any quasi-projective variety \( Y \) we denote as \( \mathcal{D}_{\text{TW,log}}^*(Y, p) \) the Thom-Whitney version of the Deligne complex using differential forms with logarithmic singularities at infinity.

A Green form with logarithmic singularities for a cycle \( Z \in Z^p(X, n)_0 \) is a staircase that allows us to go from the current \( \delta_Z \) of integration along the cycle \( Z \) in \( X \times \square^n \) to the smooth form \( \omega_Z \) on \( X \).

**Definition 1.3.** Given a cycle \( Z \in Z^p(X, n)_0 \), a \( \mathcal{D}_{\text{TW}} \)-Green form (with logarithmic singularities) for \( Z \) is an \( n \)-tuple \( g_Z := (g_n, g_{n-1}, \ldots, g_0) \in \bigoplus_{k=n}^0 \mathcal{D}_{\text{TW,log}}^{2p-n+k-1}(X \times \square^k \setminus |Z|, k, p) \), with

\[
(\delta^i_1)^* g_k = 0, \quad i = 1, \ldots, k,
\]

and such that, if \( n > 0 \),

1. \( \delta_Z + d[g_n] = 0 \), where \( \delta_Z \) is viewed as an element in \( \mathcal{D}_{\text{TW,D}}(X, p) \) (see Example 4.16).
2. \( (-1)^n k \delta g_k + dg_{k-1} = 0 \), \( k = 2, \ldots, n \).
3. \( (-1)^n \delta g_1 + dg_0 \in \mathcal{D}_{\text{TW}}^{2p-n}(X, p) \).

where in the previous equations \( \delta \) is the differential in the cocubical direction

\[
\delta g_k = \sum_{i=1}^k (-1)^i (\delta^i_1)^* g_k.
\]

In this case we write \( \omega(g_Z) = (-1)^n \delta g_1 + dg_0 \).

While, if \( n = 0 \) the previous conditions collapse to the classical condition

1. \( \delta_Z + d[g_n] \in [\mathcal{D}_{\text{TW}}^{2p}(X, p)] \),

and write \( [\omega(g_Z)] = \delta_Z + d[g_n] \).

Further, we call \( g_Z \) a basic Green form if \( g_n \) is a basic Green form for the algebraic cycle \( Z \) [BG94b, 4.6].

Every Green form \( g_Z \) for a cycle \( Z \in Z^p(X, n)_0 \) gives rise to a Green current for \( Z \), denoted by \([g_Z]\), see Proposition 6.13. Moreover, each class of Green currents \( g_Z \) contains a representative of the type \([g_Z]\) for a basic Green form \( g_Z \), see Proposition 6.14.

If \( Z \in Z^p(X, n)_0 \) and \( W \in Z^q(X, m)_0 \) are two cycles that intersect properly (Definition 3.10), \( g_Z \) is a Green current for \( Z \) and \( g_W \) a basic Green form with logarithmic singularities for \( W \), one can define in a formal way the current \( \mathcal{P}(Z) \cdot [g_W] \), see Definition 6.18, and the *-product between the Green current and the Green form is defined as

\[
g_Z \ast g_W = \mathcal{P}(Z) \cdot [g_W] + g_Z \cdot \omega(g_W).
\]

This product is well defined on the class of Green currents, see Definition 6.21. Finally, using a moving lemma suited for our purpose (Lemma 3.12), the *-product above, and the product of higher Chow groups, in Theorem 7.12 we define a product for higher arithmetic Chow groups which is graded commutative and associative.

A fallout of this theory of higher arithmetic Chow groups, is a definition of height pairing between higher cycles with trivial real regulators. This has
been developed in §7.5 (Definition 7.18). It is a very interesting direction, and will be the subject of a future project.

1.3. Examples of intersection pairing in dimension zero. In order to produce the first examples of non-zero higher arithmetic products, we study the case of dimension zero. That is, we consider $X = \text{Spec}(F)$ as a smooth projective variety over $F$.

Higher Chow groups of number fields have been extensively studied. They are the subject of Goncharov’s programme [Gon05]. For instance in [Pet09] there are many concrete examples of computations of higher Chow groups for particular fields.

The case of dimension zero is special because, in dimension zero, there is no difference between forms and currents. Therefore for cycle $Z \in Z^p(F,n)_0$, the current $P(Z)$ is already a form and 0 is a Green current for the cycle $Z$. This gives us many examples of higher arithmetic cycles. One has to be careful that, if $Z_1$ and $Z_2$ are rationally equivalent, that is they define the same class in $\text{CH}^p(F,n)_0$, then $(Z_1,0)$ and $(Z_2,0)$ do not need to be rationally equivalent in the arithmetic setting.

In consequence we can define a pairing

$$(\cdot, \cdot)_{2p-1,2q-1}: Z^p(F,2p-1)_0 \times Z^q(F,2q-1)_0 \to \frac{H^1_d(F,\mathbb{R}(p+q))}{\text{Im}(\rho_{Be})}, \quad p, q \geq 1,$$

given, for two cycles $\alpha \in Z^p(F,2p-1)_0$ and $\beta \in Z^q(F,2q-1)_0$, by

$$(\alpha, \beta)_{2p-1,2q-1} := \pi_*([(\alpha,0)] \cdot [(\beta,0)]).$$

We note that this pairing is defined at the level of cycles but does not descend to the level of Chow groups.

A crucial observation needed to compute this pairing is the fact that $[(\alpha,0)] \cdot [(\beta,0)] = [(\alpha \cdot \beta,0)]$ and that the cycle $\alpha \cdot \beta$ is torsion. For instance, if $p = q = 1$ and $p = 1, q = 2$, such intersection pairing is given by the Bloch-Wigner polylogarithm functions. This is the subject of our study in the last section.

1.4. Layout of the paper. Now we briefly describe the layout of this paper. Sections two to five are largely preliminary in nature. Here we fix notations and state results that we need for the rest of the paper. In section 2 we recall the zig-zag diagrams of complexes. Such diagrams are useful in two different ways. First, to define Deligne cohomology and second, to define higher arithmetic Chow groups. We also explain the Thom Whitney simple of such diagrams, that will be used to give a graded commutative and associative differential graded algebra that computes Deligne cohomology. In this section we also recall the theory of cubical abelian groups and complexes.

In Section 3 we recall the cubical version of Bloch’s Higher Chow groups as well as the moving lemmas that we will need in the paper.

Section 4 is devoted to recalling Deligne-Beilinson cohomology and the different complexes that we can use to compute it. Of particular importance for us is the Thom-Whitney complex because the product is graded commutative and associative at the level of complexes and not just at the
level of cohomology. We include a key lemma (Lemma 4.22), which is used in later sections to prove different properties of Green currents.

In Section 5 we recall the cycle class map from higher cycles to Deligne cohomology through the cubical version of Goncharov regulator.

In Section 6 we develop the theory of higher Green current and form in detail. We show the functorial properties of Green currents, and develop a product of Green currents for two cycles intersecting properly.

Section 7 is devoted towards developing the theory of higher arithmetic Chow groups, and an intersection theory which generalizes the one developed by Gillet and Soulé for arithmetic Chow groups. It also includes a definition of a higher height pairing which can be seen as a generalization of Beilinson’s height pairing, for higher Chow groups.

Finally, in section 8, we give examples of intersection product in case of dimension zero. This section gives the reader a recipe to compute the higher arithmetic intersection pairing for a particular choice of Green current, and it is given using Goncharov’s regulator at the level of complexes.

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2. Preliminaries on homological algebra

2.1. Complexes and simples. In this paper we use the standard conventions on (co)-chain complexes and on differential graded algebras (dg-algebras for short). If $\mathcal{A}$ is an abelian category, a cochain complex in $\mathcal{A}$ is a pair $A = (A^\bullet, d_A)$, where $A^\bullet = \bigoplus_{n \in \mathbb{Z}} A^n$ is a graded object and $d_A: A^\bullet \to A^\bullet$ is a homogeneous map of degree 1 such that $d_A^2 = 0$. The cochain complex $(A^\bullet, d_A)$ will usually be denoted by the letter $A$ without decoration unless we want to stress the fact that it is a complex or we want to emphasize the degree.

Given a cochain complex $A = (A^\bullet, d_A)$ in an abelian category, the shift by an integer $m$, denoted by $A[m] = (A[m]^\bullet, d_A[m])$, is the shifted graded object $A[m]^n = A^{n+m}$ with differential $d_A[m] = (-1)^m d_A$.

The simple of a cochain map $f: A \to B$ is the cochain complex $s(f) = (s(f)^\bullet, d_s)$ with

$$s(f)^n = A^n \oplus B^{n-1}, \quad d_s(a, b) = (d_Aa, f(a) - dBb).$$

Note that this is same as the cone of $-f$ shifted by $-1$. There are maps

$$b: B \to s(f)[1] \quad a: s(f) \to A$$

$$b \mapsto (0, -b), \quad (a, b) \mapsto a.$$
such that
\[ s(f) \xrightarrow{a} A \xrightarrow{f} B \xrightarrow{b} s(f)[1] \]
is a distinguished triangle in the derived category. Therefore there is an
associated long exact sequence
\[ \cdots \to H^n(s(f)) \xrightarrow{a} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{b} H^{n+1}(s(f)) \to \cdots \]
If \( f \) is injective, there is a quasi-isomorphism
\[ s(f)[1] \xrightarrow{\pi} \text{coker}(f) = B/A, \quad (a, b) \mapsto [-b], \]
while if \( f \) is surjective, there is a quasi-isomorphism
\[ \ker(f) \xrightarrow{\iota} s(f), \quad a \mapsto (a, 0). \]

Let \( n \in \mathbb{Z} \). The canonical truncation of \( A \) at degree \( n \) is the cochain complex \( \tau_{\leq n} A = (\tau_{\leq n} A^*, d_{\tau_{\leq n} A}) \) given by
\begin{equation}
(\tau_{\leq n} A)^r = \begin{cases} 
A^r & r < n, \\
\ker(d_A: A^n \to A^{n+1}) & r = n, \\
0 & r > n, 
\end{cases}
\end{equation}
and differential induced by \( d_A \). It follows from the definition that
\[ H^r(\tau_{\leq n} A) = \begin{cases} 
H^r(A), & r \leq n, \\
0, & r > n. 
\end{cases} \]

**Definition 2.1.** ([BGF12], definition 1.2.2) A \( k \)-iterated cochain complex \( A^* = (A^*, d_1, \ldots, d_k) \) of abelian groups is a \( k \)-graded object together with \( k \) endomorphisms \( d_1, \ldots, d_k \) of multidegrees \( l_1, \ldots, l_k \), such that, for all \( i, j, d_id_i = 0 \) and \( d_id_j = d_jd_i \). The endomorphism \( d_i \) is called the \( i \)-th differential of \( A^* \).

A cochain morphism between two \( k \)-iterated cochain complexes is a morphism of \( k \)-graded objects that respects the degrees and that commutes with all the differentials.

When dealing with \( k \)-iterated complexes we will use the multi-index notation
\[ j = (j_1, \ldots, j_k), \quad |j| = \sum_{i=1}^k j_i, \quad |j|_l = \sum_{i=1}^l j_i, \]
in particular \( |j|_0 = 0 \).

Generalizing the case of a double complex, from a \( k \)-iterated complex we can construct a simple complex.

**Definition 2.2.** (definition 1.2.4 of [BGF12]) Let \( (A^*, d_1, \ldots, d_k) \) be a \( k \)-iterated cochain complex. The simple complex of \( A^* \) is a cochain complex \( s(A)^* \) whose graded groups are
\[ s(A)^n = \bigoplus_{|j| = n} A^j, \]
and whose differential $s(A)^n \xrightarrow{d_i} s(A)^{n+1}$ is defined by, for every $a_j \in A^j$ with $|j| = n$

$$d_s(a_j) = \sum_{i=1}^{k} (-1)^{|j_i|-1} d_{l}(a_j) \in s(A)^{n+1}.$$ 

If $I^k$ denote the category of all $k$-iterated cochain complexes, then the simple associated as above describes a functor $s(-): I^k \to I^1$. Likewise, one can also define the category $I_k$ of $k$-iterated chain complexes using a similar methodology.

**Example 2.3.** Let $f: (A^*, d) \to (B^*, d)$ be a morphism of complexes, then $f$ determines a 2-iterated complex, also denoted by $f$, that is given by

$$f^{0,n} = A^n, f^{1,n} = B^n, d_1 = f, d_2 = d.$$ 

with this convention the simple of $f$ as a cochain morphism of complexes described in (2.1), agrees with its simple as a 2-iterated complex.

### 2.2. Cubical abelian groups

Cubical objects on a category are modeled on the cube as simplicial objects are modeled on a simplex (see [GNAPGP88] for details). We describe the holomogical version (with chain complexes) because the higher Chow groups form a homology theory.

Let $C_\cdot = \{C_n\}_{n \geq 0}$ be a cubical abelian group with face maps $\delta^j_i: C_n \to C_{n-1}$, for $i = 1, \ldots, n$ and $j = 0, 1$, and degeneracy maps $\sigma_i: C_n \to C_{n+1}$, for $i = 1, \ldots, n+1$. Let $D_n \subset C_n$ be the subgroup of degenerate elements of $C_n$, and let $\tilde{C}_n = C_n / D_n$.

Let $C_\cdot$ denote the associated chain complex, that is, the chain complex whose $n$-th graded piece is $C_n$ and whose differential is given by

$$\delta = \sum_{i=1}^{n} \sum_{j=0,1} (-1)^{i+j} \delta^j_i.$$ 

Thus $D_\cdot$ is a subcomplex and $\tilde{C}_\cdot$ is a quotient complex. We fix the normalized chain complex associated to $C_\cdot$, $NC_\cdot$, to be the chain complex whose $n$-th graded group is

$$NC_n := \bigcap_{i=1}^{n} \ker \delta^1_i,$$ 

and whose differential is $\delta = \sum_{i=1}^{n} (-1)^i \delta^0_i$. It is well-known that there is a decomposition of chain complexes

$$(2.3) \quad C_\cdot \cong NC_\cdot \oplus D_\cdot$$

giving an isomorphism $NC_\cdot \cong \tilde{C}_\cdot$. In general the complex $D_\cdot$ is not acyclic and the homology of the complex $C_\cdot$ does not agree with the homology of $NC_\cdot$. But it turns out that the homology of $NC_\cdot$ is the “right one”. See for instance [Mas91, Chapter VII].

For certain cubical abelian groups, the normalized chain complex can be further simplified, up to homotopy equivalence, by considering the elements which belong to the kernel of all faces but $\delta^0_1$. 
Definition 2.4. Let $C_\cdot$ be a cubical abelian group. The \textit{refined normalized complex}, denoted $N_0 C_\cdot$, is the complex defined by

\begin{equation}
N_0 C_n = \bigcap_{i=1}^n \ker \delta_i^1 \cap \bigcap_{i=2}^n \ker \delta_i^0, \quad \text{and differential } \delta = -\delta_1^0.
\end{equation}

The proof of the next proposition is analogous to the proof of Theorem 4.4.2 in [Blo]. The result is proved there only for the cubical abelian group defining the higher Chow complex (see §3.1 below). We give here the abstract version of the statement whose proof can be found in [BGF12, Prop.1.20].

Proposition 2.5. Let $C_\cdot$ be a cubical abelian group. Assume that it comes equipped with a collection of maps $h_j : C_n \to C_{n+1}$, $j = 1, \ldots, n$, such that, for any $l = 0, 1$, the following identities are satisfied:

\begin{equation}
\begin{aligned}
\delta_1^j h_j &= \delta_{j+1}^1 h_j = s_j \delta_j^1, \\
\delta_0^j h_j &= \delta_{j+1}^0 h_j = \text{Id}, \\
\delta_l^j h_j &= \begin{cases} h_{j-1} \delta_l^i & i < j, \\
h_j \delta_l^i & i > j + 1. \end{cases}
\end{aligned}
\end{equation}

Then, the inclusion of complexes

\[ i : N_0 C_\cdot \hookrightarrow NC_\cdot \]

is a homotopy equivalence. More precisely, there is a map $\pi : NC_\cdot \to N_0 C_\cdot$ such that $\pi \circ i = \text{Id}$ and $i \circ \pi$ is homotopically equivalent to the identity. In particular $N_0 C_\cdot$ is a direct summand of $NC_\cdot$.

Remark 2.6. To every cubical abelian group $C_\cdot$ there are associated four chain complexes: $C_\cdot$, $NC_\cdot$, $N_0 C_\cdot$ and $\tilde{C}_\cdot$. In some situations it will be necessary to consider the cochain complexes associated to these chain complexes. In this case we will denote them, respectively, by $C^\cdot$, $NC^\cdot$, $N_0 C^\cdot$ and $\tilde{C}^\cdot$.

2.3. Cubical cochain complexes. Let $X^\cdot_\cdot$ be a cubical cochain complex, where $\cdot$ denotes the cochain degree and $\cdot$ the cubical degree. Then, for every $m$, we have defined the cochain complexes $NX^m_\cdot$, $N_0 X^m_\cdot$ and $\tilde{X}^m_\cdot$.

Proposition 2.7. Let $X^\cdot_\cdot, Y^\cdot_\cdot$ be two cubical cochain complexes and let $f : X^\cdot_\cdot \to Y^\cdot_\cdot$ be a morphism. Assume that for every $m$, the cochain morphism

\[ X^m_\cdot \xrightarrow{f_m} Y^m_\cdot \]

is a quasi-isomorphism. Then, the induced morphisms

\[ NX^m_\cdot \xrightarrow{f_m} NY^m_\cdot \]

and $\tilde{X}^m_\cdot \xrightarrow{f_m} \tilde{Y}^m_\cdot$ are quasi-isomorphisms.

Proof. See [BGF12, Prop. 1.24].

A similar result holds true for the refined normalized complex.
Proposition 2.8. Let $X^*, Y^*$ be two cubical cochain complexes and $f : X^* \to Y^*$ a morphism of complexes such that for every $m$, the cochain morphism

$$X^*_m \xrightarrow{f_m} Y^*_m$$

is a quasi-isomorphism. Assume furthermore that, for each $n$, there are morphisms of complexes $h_j : X^*_n \to X^*_{n+1}$, $h'_j : Y^*_n \to Y^*_{n+1}$, $j = 1, \ldots, n$ satisfying the identities (2.5) and the commutativity $f_n \circ h_j = h'_j \circ f_n$. Then the induced morphisms

$$N_0 X^*_n \to N_0 Y^*_n$$

are quasi-isomorphisms.

Proof. Since the maps $h_j$ and $h'_j$ are morphisms of complexes, then $N_0 X^*_n$ is a direct summand as a complex of $N X^*_n$. Similarly for $N_0 Y^*_n$. By the commutation of the $h$ with $f$, the induced morphism $N X^*_n \to N Y^*_n$ respects these direct factors. Thus this proposition follows from Proposition 2.7. □

Let $X^*$ be a cubical cochain complex, then normalization and cohomology commute with each other. More precisely, if we normalize with respect to the cubical direction, for each $m$, we obtain a cochain complex $N X^*_m$ with $r$-th cohomology $H^r(N X^*_m)$. On the other hand, if we first take $r$-th cohomology in the cochain direction we obtain a cubical object $H^r(X^*)$, whose normalization is given, in degree $m$ by $N H^r(X^*)_m$.

Proposition 2.9. With the above setting, the natural morphism

$$H^r(N X^*_m) \xrightarrow{f} N H^r(X^*)_m$$

is an isomorphism for all $m \geq 0$.

Proof. See [BGF12, Prop. 1.25] □

Raising the cubical degree and taking normalization we obtain a 2-iterated cochain complex

$$N X^r,n = N X^r_{n-n}.$$ and an associated simple cochain complex $sN X^r$. In this situation we will denote by $d$ the cochain (or chain differential) and by $\delta$ the cubical differential. We will always put the cochain differential as the first one, therefore the differential in the simple complex, denoted $d_s$ is given, for $\omega \in N X^r,n$ by

$$d_s \omega = d \omega + (-1)^r \delta \omega.$$

3. Higher Chow groups

We recall here the definition and main properties of the higher Chow groups defined by Bloch in [Blo86a]. Initially, they were defined using the chain complex associated to a simplicial abelian group. However, since we are interested in the product structure, it is more convenient to use the cubical presentation, as given by Levine in [Lev94].
3.1. The cubical Bloch complex for higher Chow groups. Fix a base field \( k \) and let \( \mathbb{P}^1 \) be the projective line over \( k \). Let \( \square = \mathbb{P}^1 \setminus \{1\} (\cong \mathbb{A}^1) \). The cartesian product \((\mathbb{P}^1)^n\) has a cocubical scheme structure. For \( i = 1, \ldots, n \), we denote by \( t_i \in (k \cup \{\infty\}) \setminus \{1\} \) the absolute coordinate of the \( i \)-th factor. Then the coface and codegeneracy maps are defined as
\[
\delta^0_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n),
\delta^1_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \infty, t_i, \ldots, t_n),
\sigma^i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, t_i+1, \ldots, t_n).
\]
Then, \( \square \cdot \) inherits a cocubical scheme structure from that of \((\mathbb{P}^1)^n\). An \( r \)-dimensional face of \( \square^n \) is any subscheme of the form \( \delta^0_i \cdot \cdots \cdot \delta^0_r \cdot \sigma^i \cdot \cdots \cdot \sigma^r(\square^n) \). According to the convention used in [KL08], we regard \( \square^n \) as a face.

In the definition of higher Chow groups, it is customary to represent \( \mathbb{A}^1 \) as \( \mathbb{P}^1 \setminus \{1\} \), so that the face maps are represented by the inclusion at zero and the inclusion at infinity as in [Lev94]. In this way the cubical structure of \( \square \) is compatible with the cubical structure of \((\mathbb{P}^1)^n\) in [BGW98]. In the literature, the usual representation \( \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\} \) is used sometimes. One can translate from one definition to the other by using the involution
\[
x \mapsto \frac{x}{x-1}.
\]
This involution has the fixed points \( \{0, 2\} \) and interchanges the points 1 and \( \infty \).

Let \( X \) be an equidimensional quasi-projective scheme of dimension \( d \) over the field \( k \). Let \( Z^p(X, n) \) be the free abelian group generated by the codimension \( p \) closed irreducible subvarieties of \( X \times \square^n \), which intersect properly all the faces of \( \square^n \). The pull-back by the coface and codegeneracy maps of \( \square \) endow \( Z^p(X, \cdot) \) with a cubical abelian group structure, given by
\[
\delta^i = (\delta^i)^*, \quad \sigma^i = (\sigma^i)^*.
\]
Note that the indexes are raised or lowered to reflect the change from cocubical to cubical structures.

Let \( (Z^p(X, \cdot), \delta) \) be the associated chain complex (see §2.2) and consider the normalized chain complex associated to \( Z^p(X, \cdot) \),
\[
Z^p(X, n)_0 := NZ^p(X, n) = \bigcap_{i=1}^n \ker \delta^i_1.
\]
An element \( Z \in Z^p(X, n)_0 \) will be called a pre-cycle, and a (higher) cycle if it also satisfies \( \delta(Z) = 0 \). We have also the identification
\[
Z^p(X, n)_0 = \tilde{Z}^p(X, n) := Z^p(X, n)/DZ^p(X, n),
\]
where \( DZ^p(X, n) \) is the group of degenerate pre-cycles.

**Definition 3.1.** Let \( X \) be a quasi-projective equidimensional scheme over a field \( k \). The higher Chow groups defined by Bloch are
\[
\text{CH}^p(X, n) := H_n(Z^p(X, \cdot)_0).
\]
Let $N_0$ be the refined normalized complex of Definition (2.4), and let $\mathbb{Z}^p(X,\ast)_0$ be the complex with
\begin{equation}
\mathbb{Z}^p(X, n)_{00} := N_0 \mathbb{Z}^p(X, n) = \bigcap_{i=1}^n \ker \delta_i^1 \cap \bigcap_{i=2}^n \ker \delta_i^0. 
\end{equation}

Fix $n \geq 0$. For every $j = 1, \ldots, n$, we define a map
\begin{equation}
\square^{n+1} \xrightarrow{h^j} \square^n \\
(t_1, \ldots, t_{n+1}) \mapsto (t_1, \ldots, t_{j-1}, 1 - (t_j - 1)(t_{j+1} - 1), t_{j+2}, \ldots, t_{n+1}).
\end{equation}
The maps $h^j$ are smooth, hence flat, so they induce pull-back maps
\begin{equation}
h_j : \mathbb{Z}^p(X, n) \to \mathbb{Z}^p(X, n+1), 
\end{equation}
that satisfy the conditions of Proposition 2.5. Therefore we obtain

**Lemma 3.2.** The inclusion
\begin{equation}
\mathbb{Z}^p(X, n)_{00} := N_0 \mathbb{Z}^p(X, n) \to \mathbb{Z}^p(X, n)_0
\end{equation}
is a homotopy equivalence.

For more details see [Blo, §4.4]. Note that in *loc. cit.* the refined normalized complex is given by considering the elements in the kernel of all faces but $\delta_1^1$, instead of $\delta_1^0$ like here and $\square$ is defined as $k^1$. After applying the involution (3.1), the map $h^j$ agrees with the map denoted $h^{n-j}$ in *loc. cit.*

### 3.2. Moving lemmas

We gather some moving lemmas that we will need for further discussions. The master moving lemma is Theorem 3.5 below.

**Definition 3.3.** Let $X$ be an equi-dimensional $k$-scheme of finite type and denote by $X_{\text{sm}}$ the subscheme of smooth points. Let $C$ be a finite set of locally closed irreducible subsets of $X_{\text{sm}}$ and let $e: C \to \mathbb{Z}_{\geq 0}$ be a function.

We define $\mathbb{Z}^p(X, n)_C, e \subset \mathbb{Z}^p(X, n)$ to be the subgroup generated by integral closed subschemes $Z \subset X \times \square^n$ such that, for each $C \in C$ and each face $F$ of $\square^n$ (including the case $F = \square^0$),
\[
\text{codim}_{C \times F}(Z \cap (C \times F)) \geq p - e(C).
\]

**Lemma 3.4.** For each $p \geq 0$, the groups $\mathbb{Z}^p_C, e(X, \ast)$ form a sub-cubical group of $\mathbb{Z}^p(X, \ast)$. Moreover, this cubical subgroup is stable under the pull-back morphisms $h_j$ of (3.4).

**Proof.** Let $Z \in \mathbb{Z}^p_C, e(X, n)$. Then for a face $F$ of $\square^{n-1}$, each $F_i := \delta_i^1(F)$ is a face of $\square^n$, where $\epsilon = 0, 1$. Now the first part follows from the fact
\[
\text{codim}_{C \times F}(\delta_i^1(Z) \cap (C \times F)) = \text{codim}_{C \times F_i}(Z \cap (C \times F_i)),
\]
for any $C \in C$. For the second part, we use a similar argument, now noticing the fact that for each face $F$ of $\square^{n+1}$, $h_j(F)$ gives a face in $\square^n$ (including possibly the total space $\square^n$ if either $t_j$ or $t_{j+1}$ equals zero) and that the induced map $F \to h_j(F)$ is flat.

As a consequence of Lemma 3.4 we can define the complexes $\mathbb{Z}^p_C, e(X, \ast)_0$ and $\mathbb{Z}^p_C, e(X, \ast)_{00}$.
Theorem 3.5. Let $X$ be a quasi-projective scheme over $k$, $C$ a finite subset of locally closed irreducible subsets of $X_{\text{sm}}$ and $\epsilon: C \to \mathbb{Z}_{\geq 0}$ a function. Then the injective morphism of complexes

$$Z^p_{C,\epsilon}(X, \cdot)_0 \to Z^p(X, \cdot)_0$$

is a quasi-isomorphism.

Proof. This is [KL08, Theorem 1.10]. □

Corollary 3.6. With the hypothesis of Theorem 3.5, the morphism of complexes

$$Z^p_{C,\epsilon}(X, \cdot)_{00} \to Z^p(X, \cdot)_{00}$$

is a quasi-isomorphism.

Proof. Since the morphisms $h_j$ respect the subcomplex $Z^p_{C,\epsilon}(X, \cdot)$, the argument that shows that $Z^p(X, \cdot)_{00}$ and $Z^p(X, \cdot)_0$ are quasi-isomorphic, also shows that $Z^p_{C,\epsilon}(X, \cdot)_{00}$ and $Z^p_{C,\epsilon}(X, \cdot)_0$ are quasi-isomorphic. Therefore the corollary is a consequence of Theorem 3.5. □

We next want to specialize Theorem 3.5 to the particular cases we will use in the sequel.

Definition 3.7. Let $f: X \to Y$ be an arbitrary map between two smooth quasi-projective varieties $X,Y$ over $k$. Let $Z^p_f(Y,n) \subset Z^p(Y,n)$ be the subgroup generated by the codimension $p$ irreducible subvarieties $Z \subset Y \times \Box^n$, such that, for every face $F$ of $\Box^n$

$$\text{codim}_{X \times F} ((f \times \text{Id})^{-1}(Z) \cap (X \times F)) \geq p.$$  

Lemma 3.8. Let $f: X \to Y$ be a morphism of smooth quasi-projective varieties. Then there is a finite subset $C$ of locally closed irreducible subsets of $Y$ and a function $\epsilon: C \to \mathbb{Z}_{\geq 0}$ such that $Z^p_f(Y, \cdot) = Z^p_{C,\epsilon}(Y, \cdot)$.

Proof. This is proved in [Lev98, Part I, Chapter II, Lemma 3.5.2]. □

Corollary 3.9. The inclusions $Z^p_f(Y, \cdot)_{00} \to Z^p(Y, \cdot)_{00}$ and $Z^p_f(Y, \cdot)_{00} \to Z^p(Y, \cdot)_{00}$ are quasi-isomorphisms.

Definition 3.10. Let $X$ be a smooth quasi-projective scheme over $k$, and let $p,q,n,m \geq 0$ be positive integers. If $Z \in Z^p(X,n)$, $W \in Z^q(X,m)$ are pre-cycles. We say that $Z$ and $W$ intersect properly if, for any face $F$ of $\Box^{n+m}$,

$$\text{codim}_{X \times F} (p_{12}^{-1}[Z] \cap p_{13}^{-1}[W] \cap (X \times F)) \geq p + q,$$

where

$$p_{12}: X \times \Box^n \times \Box^m \to X \times \Box^n, \quad p_{13}: X \times \Box^n \times \Box^m \to X \times \Box^n$$

are the projections.

Remark 3.11. Suppose $q = 0$. For $m \geq 0$ all the elements of $Z^0(X,m)$ are degenerate. Therefore $Z^0(X,m)_0 = 0$ and the case $m = 0$ is the only non-trivial situation. If $W \in Z^0(X,0)_0$, then it is a component of $X$ and if $Z$ intersects properly all the faces of $X \times \Box^n$, then $Z$ necessarily intersects $W$ properly.
Let $W \in Z^q(X, m)$ be a pre-cycle. We denote by $Z^p_W(X, n) \subset Z^p(X, n)$ be the subgroup generated by the codimension $p$ irreducible subvarieties $Z \subset X \times \square^m$, such that $Z$ and $W$ intersect properly.

**Lemma 3.12.** Let $X$ be a smooth quasi-projective scheme over $k$ and $W \in Z^q(X, m)$ a pre-cycle. Then there is a finite subset $C$ of locally closed irreducible subsets of $X$ and a function $e : C \to \mathbb{Z}_{\geq 0}$ such that $Z^p_W(X, *) = Z^p_{C, e}(X, *)$.

**Proof.** For simplicity of the discussion we assume $W$ to be irreducible. Being a pre-cycle, it intersects all the faces of $X \times \square^m$ properly. For every face $H$ of $\square^m$ (including the total face $\square^m$), we define

$$C_{H,i} := \left\{ x \in X \mid \dim \left( \pi^{-1}_m(x) \cap W \cap (X \times H) \right) = i \right\},$$

where $\pi_m : X \times \square^m \to X$ is the projection. Then, $C_{H,i}$ is a constructible subset of $X$. Let $C^\ell_{H,i}$ be locally closed, irreducible subsets of $X$, such that $C_{H,i} = \bigcup \ell C^\ell_{H,i}$. Observe that, for every face $F$ of $\square^m$, $\bigcup \ell C^\ell_{H,i} = X$. From the condition of proper intersection, for every face $H$ of $\square^m$,

$$\text{codim}_{X \times H} (W \cap (X \times H)) \geq q \implies \dim (W \cap (X \times H)) \leq d + h - q,$$

where $h = \dim (H)$. Let $d^\ell_{H,i} = \dim (C^\ell_{H,i})$. Then

$$d^\ell_{H,i} + i \leq \dim (W \cap (X \times H)) \leq d + h - q.$$

Let $C = \{ C^\ell_{H,i} \}$, and $e(C^\ell_{H,i}) = d + h - q - d^\ell_{H,i} - i \geq 0$.

We claim that $Z_{C, e}(X, n) = Z_W(X, n)$. Notice that, any face $F$ of $\square^{n+m}$ is of the form $G \times H$, where $G$ is a face of $\square^n$ and $H$ is a face of $\square^m$. Write $g = \dim (G)$ and $f = \dim (F)$, so $f = g + h$. For any face $H$ of $\square^m$ we have the decomposition $X \times \square^n = (\bigcup \ell C^\ell_{H,i}) \times \square^n$. Let now $Z \in Z^p(X, n)$ be a pre-cycle. Then

\begin{align*}
\text{(3.5)} \quad p^0_{12} |Z| \cap p^{-1}_{13} |W| \cap (X \times F) \\
= p^0_{12} (|Z| \cap (X \times G) ) \cap p^{-1}_{13} (|W| \cap (X \times H)) \\
= \bigcup_{i, \ell} p^0_{12} (|Z| \cap C^\ell_{H,i} \times G) \cap p^{-1}_{13} (|W| \cap (X \times H)).
\end{align*}

Now

$$\text{codim}_{C^\ell_{H,i} \times G} \left( |Z| \cap (C^\ell_{H,i} \times G) \right) \geq p - e(C^\ell_{H,i})$$

**$$\iff$$**

$$\dim \left( |Z| \cap (C^\ell_{H,i} \times G) \right) \leq d^\ell_{H,i} + g - p + e(C^\ell_{H,i}).$$

Finally, $Z$ belongs to $Z^p_{C, e}(X, n)$ if and only if, for every face $F = G \times H$ and every $i, \ell$, the condition

\begin{align*}
\text{(3.6)} \quad \dim (|Z| \cap C^\ell_{H,i} \times G) \leq d^\ell_{H,i} + g - p + e(C^\ell_{H,i})
\end{align*}

holds. By the definition of $C^\ell_{H,i}$, the condition (3.6) is equivalent to the condition

$$\dim p^0_{12} (|Z| \cap C^\ell_{H,i} \times G) \cap p^{-1}_{13} (|W \cap X \times H)$$

$$\leq d^\ell_{H,i} + g - p + e(C^\ell_{H,i}) + i = d + f - p - q,$$
where, in the last equality we have used the definition of \( e(C_{H,i}) \). Therefore, using the decomposition (3.5) we deduce that condition (3.6) is satisfied for every face \( F \) and indices \( \ell, i \) if and only if
\[
\text{codim}_{X \times F} \left( (p_{12}^{-1}|Z| \cap p_{13}^{-1}|W| \cap (X \times F)) \right) \geq p + q.
\]
In other words, \( Z \in Z_{C,\ell}^p(X,n) \) if and only if \( Z \in Z_{W}^p(X,n) \) proving the result.

\[ \square \]

**Corollary 3.13.** The inclusions \( Z_{H}^0(Y,*)_0 \to Z^p(Y,*)_0 \) and \( Z_{W}^0(Y,*)_0 \to Z^p(Y,*)_00 \) are quasi-isomorphisms.

**Remark 3.14.** Lemma 3.12 and Corollary 3.13 can easily be generalized to a finite family of pre-cycles.

### 3.3. Functoriality.

It follows easily from the definition that the complex \( Z^p(Y,*)_0 \) is covariant with respect to proper maps (with a shift in the grading) and contravariant for flat maps.

Let \( f: X \to Y \) be a morphism between smooth quasi-projective schemes over \( k \). Then, \( Z^p_f(Y,*)_0 \) is a chain complex and, by Corollary 3.9 the inclusion of complexes
\[
Z^p_f(Y,*)_0 \subseteq Z^p(Y,*)_0
\]
is a quasi-isomorphism. Moreover, by the definition of \( Z^p_f(Y,*) \), the pull-back by \( f \) is defined for algebraic cycles in \( Z^p_f(Y,*)_0 \) and hence there is a well-defined pull-back morphism
\[
CH^p(Y,n) \xrightarrow{f^*} CH^p(X,n).
\]

### 3.4. Product structure.

Let \( X \) be a quasi-projective scheme over \( K \) and \( \alpha \in CH^p(X,n) \) and \( \beta \in CH^q(X,m) \). We can represent \( \beta \) by an element \( W \in Z^q(X,m)_0 \) and, thanks to Corollary 3.13, \( Z \) by an element \( Z \in Z^p_{H}(X,n)_0 \), which in turn implies that \( W \in Z^p_{F}(X,m) \). Let \( p_{12} \) and \( p_{13} \) be as in Definition 3.10. Since \( Z \) and \( W \) intersect properly, the intersection product
\[
p_{12}^*Z \cdot p_{23}^*W \in Z^{p+q}(X,n+m)
\]
is well defined. This defines a pairing
\[
(3.7) \quad \alpha \cdot \beta = [p_{12}^*Z \cdot p_{23}^*W].
\]

**Proposition 3.15.** Let \( X \) be a smooth quasi-projective scheme over \( k \). The pairing (3.7) defines an associative product on
\[
CH^*(X,*) := \bigoplus_{p,n} CH^p(X,n).
\]

This product is graded commutative with respect to the degree given by \( n \). That is, if \( \alpha \in CH^p(X,n) \) and \( \beta \in CH^q(X,m) \), then
\[
\alpha \cdot \beta = (-1)^{nm} \beta \cdot \alpha.
\]

**Proof.** We first show that the intersection product is well defined. If \( W' \) is another representative of \( \beta \) with \( \delta T = W - W' \), we can represent \( \alpha \) by a cycle \( Z' \) that intersects properly \( T, W \) and \( W' \) (see Remark 3.14). Since
\[
\delta(T \cdot Z') = W \cdot Z' - W \cdot Z',
\]
the product is independent on the choice of the representative $W$. A similar argument shows the independence on the choice of the representative $Z$.

The associativity follows from the fact that the intersection product of cycles intersecting properly is associative already at the level of cycles.

The commutativity is more involved because writing

$$X \times □^{n+m} \longrightarrow X \times □^m \times □^n$$

it is evident that in general,

$$p_{12}^* Z \cdot p_{23}^* W \neq (-1)^{nm}(p_{12}')^* W \cdot (p_{23}')^* Z.$$ 

Thus, it is not graded commutative at the level of cycles but only at the level of Chow groups. The proof of the graded commutativity in [Lev94] uses an explicit homotopy. By technical reasons, this homotopy is only defined for cycles in the refined normalized complex $Z^p(X, *)_{00}$. This is harmless thanks to Lemma 3.2.

We recall briefly the construction of the homotopy $H$. For details, the reader is encouraged to consult §5.4 of [BGF12]. Let

$$h^*: Z^p(X, n + m)_{00} \rightarrow Z^p(X, n + m + 1)_{00}$$

denote the morphism induced by

$$h_{n+m}: (3.8) \quad X \times □^{n+m+1} \rightarrow X \times □^{n+m}$$

$$(p, x_1, \cdots, x_{n+m+1}) \mapsto (p, x_2, \cdots, x_{n+m}, x_1 + x_{n+m+1} - x_1 x_{n+m+1}).$$

and let $\tau$ be the automorphism of $X \times □^n$, given by

$$(p, x_1, \cdots, x_n) \mapsto (p, x_2, \cdots, x_n, x_1).$$

Finally, for each $n, m \geq 0$, the morphism

$$H_{n,m}: Z^p(X, n + m)_{00} \rightarrow Z^p(X, n + m + 1)_{00}$$

is defined by (see Proposition 5.35 of [BGF12])

$$(3.9) H_{n,m}(Z) = \left\{ \begin{array}{ll} \sum_{i=0}^{n-1} (-1)^{(m+i)(n+m-i)} h_{n+m}^*((\tau)^{m+i}(Z)), & n \neq 0, \\
0, & n = 0, \end{array} \right.$$ 

for $Z \in Z^p(X, n + m)_{00}$. By [BGF12, Lemma 5.35], if $Z \in Z^p(X, n)_{00}$ and $W \in Z^q(X, m)_{00}$ are cycles intersecting properly, then

$$(3.10) \quad \delta H_{n,m}(Z \cdot W) = Z \cdot W - (-1)^{nm} W \cdot Z,$$

showing the graded commutativity. \qed
4. Deligne-Beilinson cohomology

In this section we will recall the definition real Deligne-Beilinson cohomology and describe several complexes to compute it. Since we will work mainly with smooth projective varieties, real Deligne cohomology agrees with real Deligne-Beilinson cohomology and with absolute Hodge cohomology in the range of interest [Be˘ı83]. Note that to extend the constructions given here to quasi-projective varieties it may be useful to work directly with real absolute Hodge cohomology instead of real Deligne-Beilinson cohomology.

As a reference for mixed Hodge structures we will use [PS08].

4.1. Conventions on differential forms and currents. When dealing with differential forms, currents and cohomology classes, one can use the topologist convention, where the emphasis is put on having real or integral valued classes. For instance, in this convention the first Chern class of a line bundle will have integral coefficients. In algebraic geometry, the fact that rational de Rham classes are not rational in the topological, the ubiquitous appearance of the period $2\pi i$, and the fact the choice of a particular square root of $-1$ is non canonical makes it useful to use a different convention. In fact, let $X$ be a projective smooth variety of dimension $d$ defined over $\mathbb{Q}$ and $X(\mathbb{C})$ be the associated complex manifold. Then there is a canonical isomorphism in top degree

$$H_{Zar}^{2d}(X, \Omega^*) = H^{2d}_{sing}(X(\mathbb{C}), (2\pi i)^d \mathbb{Q})$$

given by integration of differential forms. One has to take care that integration of differential forms depends on the choice of a global orientation and the standard choice of global orientation n a complex manifold depends on the choice of the square root of $1$.

The algebro-geometric convention aims to control these obvious powers of $2\pi i$ and make the above isomorphism canonical. For instance, in this convention the first Chern class of a line bundle has coefficients in $(2\pi i)\mathbb{Z}$.

Of course using one convention or the other is a matter of taste and one can go easily from one to the other by a normalization factor. In this paper we will follow the algebraic geometry convention. Therefore, it is useful to incorporate different powers of $2\pi i$ in the standard operations regarding forms and currents as in [BGKK07, §5.4]. We summarize here the conventions used because they differ from commonly used notations.

Let $X$ be a complex manifold. We will denote by $E^*_{X}$ the differential graded algebra of complex valued differential forms on $X$, by $E^*_{X,R}$ the subalgebra of real valued forms and by $E^*_{X,c}$ and $E^*_{X,R,c}$ the subalgebras of differential forms with compact support. The complexes of currents are defined as the topological dual of the latter ones. Namely $E^n_{X}$ and $E^n_{X,R}$ are the topological dual of $E^*_{X,c}$ and $E^*_{X,R,c}$ respectively, with differential given by

$$dT(\eta) = (-1)^{n+1}T(d\eta).$$

Assume that $X$ is equidimensional of dimension $d$. We then write

$$D^n_X = E^{n-2d}_{X,c}, \quad D^n_{X,R} = (2\pi i)^{-d}E^{n-2d}_{X,R,c}.$$ 

In other words

$$D_X = E_X[-2d]|(-d),$$
where the symbol \((-d)\) refers to the above change of real structures. This implies that
\[
D^n_{X,R} = \{ T \in D^n_X \mid \forall \eta \in E^{2d-n}_{X,R}, \ T(\eta) \in (2\pi i)^{-d}\mathbb{R}\}.
\]

To be consistent with these choices we need to adjust the definition of the current associated to a locally integrable form and a cycle. Given a locally integrable differential form \(\omega\) of degree \(n\), we will denote by \([\omega] \in D^n_X\) the current defined by
\[
(4.1) \quad [\omega](\eta) = \frac{1}{(2\pi i)^d} \int_X \omega \wedge \eta.
\]

With this convention, the morphism of complexes \([\cdot] : E^*_X \to D^*_X\) sends \(E^*_X, R\) to \(D^*_X, R\).

If \(f : X \to Y\) is a proper map of complex manifolds, of dimensions \(d, d'\) and relative dimension \(e = d - d'\), then the push-forward of currents \(f_* : D^n_X \to D^{n-2e}_Y\) is defined, for \(T \in D^n_X\) and \(\eta \in E^{2d-n}_Y\) by
\[
f_* T(\eta) = T(f^* \eta).
\]

Then \(f_*\) sends \(D^n_X, R\) to \((2\pi i)^{-e}D^n_Y, R\).

Finally, assume that \(X\) is algebraic and \(Y \subset X\) is a codimension \(p\) subvariety of \(X\). Let \(\iota : \tilde{Y} \to X\) be a resolution of singularities of \(Y\). Then the current integration along \(Y\) is defined as \(\delta_Y = \iota_* [1_{\tilde{Y}}]\), where \(1_{\tilde{Y}}\) is the constant function 1 on \(\tilde{Y}\). Therefore
\[
(4.2) \quad \delta_Y(\eta) = \frac{1}{(2\pi i)^{d-p}} \int_{\tilde{Y}} \iota^* \eta.
\]

Then \(\delta_Y \in (2\pi i)^p D^{2p}_{X,R}\). Given any cycle \(\zeta \in Z^p(X)\) we define \(\delta_{\zeta}\) by linearity.

Remark 4.1. We stress the fact that the sign of the integral depends on the choice of an orientation. If \(z_1, \ldots, z_d\) are local complex coordinates with \(z_j = x_j + iy_j\), then the standard orientation is given by the volume form
\[
\text{Vol} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_d \wedge dy_d.
\]

If we change the choice of the square root of \(-1\) from \(i\) to \(-i\) then \(\text{Vol}\) is sent to \((-1)^d \text{Vol}\), which is the same change of sign suffered by \((2\pi i)^d\). Therefore the symbols \([\omega]\) and \(\delta_Y\) as used here do not depend on a particular choice of \(\sqrt{-1}\).

Example 4.2. To see how this conventions work in practice, we review the classical example of the logarithm. Consider the case \(X = \mathbb{P}^1(\mathbb{C})\) with absolute coordinate \(t\). So \(\text{div}(t) = [0] - [\infty]\). Then
\[
\partial \bar{\partial} [\log |t|] = -\delta_{\text{div}} t = \delta_{\infty} - \delta_0
\]
\[
(4.3) \quad d \left[ \frac{dt}{|t|} \right] = \delta_{\text{div}} t = \delta_0 - \delta_{\infty},
\]
\[
d \left[ \frac{d\bar{t}}{|t|} \right] = -\delta_{\text{div}} t = \delta_{\infty} - \delta_0.
\]
More generally, if $X$ is a complex manifold, $L$ is a line bundle provided with a smooth hermitian metric $\| \cdot \|$ and $s$ is a nonzero rational section of $L$, then the Poincaré-Lelong formula reads

$$\partial \bar{\partial} \log \|s\|^2 = [c_1(L, \| \cdot \|)] - \delta_{\text{div} s},$$

where $c_1(L, \| \cdot \|) \in (2\pi i)E_X^{1,0}$ is the first Chern form of $L$.

4.2. Dolbeault complexes. We recall from [BG97] the notion of Dolbeault complex.

**Definition 4.3.** A Dolbeault complex $A = (A^*_{\mathbb{R}}, d_A)$ is a graded complex of real vector spaces, which is bounded from below and equipped with a bigrading on $A_{\mathbb{C}} = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, i.e.,

$$A^n_{\mathbb{C}} = \bigoplus_{p+q=n} A^{p,q},$$

satisfying the following properties:

(i) The differential $d_A$ can be decomposed as the sum $d_A = \partial + \bar{\partial}$ of operators $\partial$ of type $(1,0)$, resp. $\bar{\partial}$ of type $(0,1)$.

(ii) It satisfies the symmetry property $A^{p,q}_{\mathbb{C}} = A^{q,p}_{\mathbb{C}}$, where $\bar{\alpha}$ denotes the complex conjugation induced by the real structure of $A_{\mathbb{C}}$. In other words, if $\omega \in A_{\mathbb{R}}$ and $\alpha \in \mathbb{C}$ then $\omega \otimes \alpha = \omega \otimes \bar{\alpha}$.

**Notation 4.4.** Given a Dolbeault complex $A = (A^*_{\mathbb{R}}, d_A)$, we will use the following notations. The Hodge filtration $F$ of $A$ is the decreasing filtration of $A_{\mathbb{C}}$ given by

$$F^p A^n := F^p A^n_{\mathbb{C}} := \bigoplus_{p' \geq p} A^{p',n-p'}_{\mathbb{C}},$$

The filtration $\overline{F}$ of $A$ is the complex conjugate of $F$, i.e.,

$$\overline{F}^p A^n = \overline{F}^p A^n_{\mathbb{C}} = \overline{F}^p A^n_{\mathbb{C}}.$$

For an element $x \in A_{\mathbb{C}}$, we write $x^{i,j}$ for its component in $A^{i,j}$. For $k, k' \geq 0$, we define an operator $F^{k,k'} : A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$ by the rule

$$F^{k,k'}(x) := \sum_{l \geq k, l' \geq k'} x^{l,l'}.$$

We note that the operator $F^{k,k'}$ is the projection of $A^n_{\mathbb{C}}$ onto the subspace $F^k A^* \cap \overline{F}^{k'} A^*$. We will write $F^k = F^{k,-\infty}$.

We denote by $A^p_{\mathbb{R}}(p)$ the subgroup $(2\pi i)^p A^p_{\mathbb{R}} \subseteq A^p_{\mathbb{C}}$, and we define the operator

$$\pi_p : A_{\mathbb{C}} \rightarrow A^p_{\mathbb{R}}(p)$$

by setting $\pi_p(x) := \frac{1}{2}(x + (-1)^p \bar{x})$.

**Definition 4.5.** A Dolbeault algebra $A = (A^*_{\mathbb{R}}, d_A, \wedge)$ is a Dolbeault complex equipped with an associative and graded commutative product

$$\wedge : A^*_R \times A^*_R \rightarrow A^*_R$$

such that the induced multiplication on $A^*_{\mathbb{C}}$ is compatible with the bigrading, i.e.,

$$A^{p,q} \wedge A^{p',q'} \subseteq A^{p+p',q+q'}.$$
If $A$ is a Dolbeault algebra, the a Dolbeault module over $A$ is a Dolbeault complex $B = (B^*_R, d_B)$ together with an $A$-module structure
\[
\wedge : A^*_R \times B^*_R \to B^*_R
\]
compatible with the bigrading,

**Example 4.6.** Let $X$ be a smooth variety over $\mathbb{C}$ of dimension $d$.

1. Let $E^n_{X, R}$ denote the complex of smooth real valued differential forms on $X^{\text{an}} = X(\mathbb{C})$ while $E^n_X$ denote the one of smooth complex valued differential forms. It is a Dolbeault complex with the usual bigrading
\[
E^n_X = \bigoplus_{p+q=n} E^{p,q}_X.
\]
This Dolbeault complex will be denoted $E_X$. It is a Dolbeault algebra. When $X$ is projective, $E^n_X$ coputes the cohomology of $X$ with its real structure and Hodge filtration.

2. Let $D^n_{X, R}$ be the complex of currents on $X(\mathbb{C})$ with the real subcomplex $D^n_{X, R}$ as in section 4.1. The complex of currents has also a bigrading and is another example of a Dolbeault complex that we denote $D_X$. It is a Dolbeault module over $E_X$.

3. Assume that $X$ is projective and let $Y \subset X$ be a simple normal crossing divisor on $X$. Let $E^n_{X, R}(\log Y)$ denote the complex of real valued smooth differential forms on $U = X(\mathbb{C}) \setminus Y(\mathbb{C})$ with logarithmic singularities along $Y$ as in [BG94a]. This complex is also a Dolbeault complex and computes the cohomology of $U$ with its real structure and its Hodge filtration. In fact it also has a weight filtration that allow us to compute the real mixed Hodge structure of the cohomology of $X$. Some times it is useful to have a complex that depends only on $U$ and not in a particular compactification. To this end we will write
\[
E^n_{U, \log} = \lim_{\to} E^n_{X'}(\log Y'),
\]
were the limit is taken for all smooth compactifications $U \hookrightarrow X'$ with $Y' = X' \setminus U$ simple normal crossing divisor.

### 4.3. Deligne cohomology.

We recall now the definition of Deligne cohomology.

**Definition 4.7.** Let $A$ be a Dolbeault complex. For each $p \in \mathbb{Z}$, consider the obvious inclusions morphisms $t_p : F^pA^*_C \to A^*_C$ and $\alpha_p : A^*_R(p) \to A^*_C$.

The total real Deligne complex of $A$ twisted by $p$ is the simple complex associated to the map $(t_p, -\alpha_p)$:
\[
\mathcal{D}_t(A, p) = s \left( F^pA^*_C \oplus A^*_R(p) \xrightarrow{(t_p, -\alpha_p)} A^*_C \right).
\]
More explicitly,
\[
\mathcal{D}_t^n(A, p) = (2\pi i)^p A^*_R \oplus F^pA^*_C \oplus A^{n-1}_C,
\]
with differential
\[
d(r, f, \omega) = (dr, df, t_p(f) - \alpha_p(r) - d\omega).
\]
The Deligne cohomology of $A$ twisted by $p$ is the cohomology of the complex $\mathcal{D}_t(A, p)$,

$$H^n_\mathcal{D}(A, p) = H^n(\mathcal{D}_t(A, p)).$$

By its construction as the simple of a diagram, Deligne cohomology fits in a long exact sequence.

**Proposition 4.8.** Let $A$ be a Dolbeault complex. Then there is a long exact sequence

$$\cdots \rightarrow H^{n-1}(A^*_\mathbb{R}(p)) \oplus H^{n-1}(F^pA^*_\mathcal{C}) \rightarrow H^{n-1}(A^*_\mathcal{C}) \rightarrow$$

$$H^n_\mathcal{D}(A, p) \rightarrow H^n(A^*_\mathbb{R}(p)) \oplus H^n(F^pA^*_\mathcal{C}) \rightarrow H^n(A^*_\mathcal{C}) \rightarrow \cdots$$

**Proof.** The result follows from the standard long exact sequence associated to the simple of a morphism. \[\square\]

**Remark 4.9.** If the spectral sequence associated to the Hodge filtration $F$ degenerates at the term $E_1$ as happens with the Dolbeault complexes $E_X$ and $D_X$ for $X$ projective, the above exact sequence can be written as

$$\cdots \rightarrow H^{n-1}(A^*_\mathbb{R}(p)) \oplus F^pH^{n-1}(A^*_\mathcal{C}) \rightarrow H^{n-1}(A^*_\mathcal{C}) \rightarrow$$

$$H^n_\mathcal{D}(A, p) \rightarrow H^n(A^*_\mathbb{R}(p)) \oplus F^pH^n(A^*_\mathcal{C}) \rightarrow H^n(A^*_\mathcal{C}) \rightarrow \cdots$$

**Example 4.10.** Let $X$ be a smooth projective variety over $\mathbb{C}$.

1. The complex $\mathcal{D}_t(E_X, p)$ will be denoted as $\mathcal{D}_t(X, p)$ and its cohomology by $H^n_\mathcal{D}(X, \mathbb{R}(p))$. It is called the Deligne cohomology of $X$.

2. The complex $\mathcal{D}_t(D_X, p)$ will be denoted $\mathcal{D}_t(D, X, p)$. This complex is quasi-isomorphic to $\mathcal{D}_t(X, p)$, thus it also computes the Deligne cohomology of $X$.

3. Let $Y$ be a simple normal crossing divisor of $X$ and $U = X \setminus Y$. Then the cohomology of the complex $\mathcal{D}_t(E_X, \log Y, p)$ only depends on $U$ and not on the compactification $X$. We will denote it as $H^n_\mathcal{D}(U, \mathbb{R}(p))$. This cohomology is called Deligne-Beilinson cohomology of $U$. In fact,

$$H^n_\mathcal{D}(U, \mathbb{R}(p)) = H^*(\mathcal{D}_t(E_U, \log, p)),$$

where $E_U, \log$ was introduced in Example 4.6 (3). We will write $\mathcal{D}_{t, \log}(U, p) := \mathcal{D}_t(E_U, \log, p)$.

4. Let $Z \subset X$ be a proper closed subset and write $V = X \setminus Z$. Then there is a morphism of Dolbeault complexes $E_X \rightarrow E_V, \log$ that induces morphisms $\mathcal{D}_t(X, p) \rightarrow \mathcal{D}_{t, \log}(V, p)$. We denote

$$\mathcal{D}_{t, Z}(X, p) = s(\mathcal{D}_t(X, p) \rightarrow \mathcal{D}_{t, \log}(V, p)),$$

the simple complex associated to these morphisms. We define the Deligne-Beilinson cohomology of $X$ with support on $Z$ as

$$H^n_{\mathcal{D}, Z}(X, \mathbb{R}(p)) = H^*(\mathcal{D}_{t, Z}(X, p)).$$

5. Let $\mathcal{Z}^p$ denote the directed set of algebraic closed subsets of $X$ of codimension at least $p$ ordered by inclusion. Then we will write

$$\mathcal{D}_{t, \mathcal{Z}^p}(X, p) = \lim_{Z \in \mathcal{Z}^p} s(\mathcal{D}_t(X, p) \rightarrow \mathcal{D}_{t, \log}(X \setminus Z, p))$$

and

$$H^n_{\mathcal{D}, \mathcal{Z}^p}(X, \mathbb{R}(p)) = H^*(\mathcal{D}_{t, \mathcal{Z}^p}(X, p)).$$
Proposition 4.11. Let $X$ be a smooth complex variety.

1. Let $Z \subset X$ be an irreducible subvariety of codimension $p$. Then
$$H_{\mathbb{D},Z}^n(X, \mathbb{R}(p)) = \begin{cases} 0, & \text{if } n < 2p, \\ \mathbb{R} \cdot [Z], & \text{if } n = 2p, \end{cases}$$
where $[Z]$ is the fundamental class of $Z$.

2. Moreover
$$H_{\mathbb{D},Zp}^n(X, \mathbb{R}(p)) = \begin{cases} 0, & \text{if } n < 2p, \\ Z^p(X) \otimes \mathbb{R}, & \text{if } n = 2p, \end{cases}$$
where $Z^p(X)$ is the group of codimension $p$ cycles on $X$.

Proof. By purity, the cohomology of a smooth variety with support on an irreducible subvariety of codimension $p$ satisfies
$$H^n_{\mathbb{D}}(X, \mathbb{R}(p)) = \begin{cases} 0, & \text{if } n < 2p, \\ \mathbb{R} \cdot [Z], & \text{if } n = 2p, \end{cases}$$
Moreover, the fundamental class satisfies
$$[Z] \in F^pH_2^p(X), \quad [Z] \in H_2^p(X, (2\pi i)^p\mathbb{R}).$$
Therefore statement (1) follows from the long exact sequence of Proposition 4.8.

Statement (2) follows from statement (1). $\square$

Remark 4.12. The Deligne-Beilinson cohomology groups in degree bigger that $2p$ can be seen as pathological (see [Be˘ı83]). It is convenient to truncate the total Deligne complex in degree $2p$. Let $\tau_{\leq 2p}$ be the canonical truncation given by (2.2).

We will use the notation
$$\tau_{\mathcal{D}}(X, p) = \tau_{\leq 2p} \mathcal{D}(X, p), \quad \tau_{\mathcal{D},Z}(X, p) = \tau_{\leq 2p} \mathcal{D}_{t,Z}(X, p).$$

Example 4.13. Assume that $X$ is projective. Thus we can use the complex $\mathcal{D}_{t,D}(X, p)$ to compute the Deligne cohomology of $X$. Let $Z$ be a closed subvariety of $X$ of codimension $p$. Recall that, following the notation (4.2), the current $\delta_Z$ already incorporates a twist. Therefore it belongs at the same time to $(2\pi i)^pD^p_{X,\mathbb{R}}$ and to $F^pD^p_{X}$ and it represents the image of the fundamental class $[Z]$ in $H^{2p}(X, \mathbb{C})$. The triple
$$(\delta_Z, \delta_Z, 0) \in \mathcal{D}_{t,D}(X, p).$$
is closed and represents a class in $H^{2p}_{\mathbb{D}}(X, p)$ that is called the fundamental class of $Z$ and is denoted as $[Z]$.

By abuse of language, the triple (4.5) will also be denoted $\delta_Z$.

4.4. The Thom-Whitney simple for Deligne cohomology. Assume that $A$ is a Dolbeault algebra. This is the case for the Hodge complexes (1) and (3) of Example 4.6. Then the complex $\mathcal{D}_t(A, p)$ has several product structures [Be˘ı83], [EV88], one for each $\beta \in \mathbb{R}$. All of them are homotopically equivalent. The one for $\beta = 1/2$ is graded commutative while the ones with $\beta = 0, 1$ are associative, but none of them is graded commutative and
associative at the same time. To have a graded commutative and associative algebra we use the Thom-Whitney simple as in [BGW98, §6].

Denote by \( L_a = (L_a^0, d) \) the algebraic de Rham complex of \( A_1^* \), that is,

\[
L_a^0 = \mathbb{R}[\varepsilon], \quad L_a^1 = \mathbb{R}[\varepsilon] d\varepsilon,
\]

where \( \varepsilon \) is an indeterminate.

**Definition 4.14.** Let \( A \) be a Dolbeault complex and \( p \) an integer. Then the Thom-Whitney Deligne complex of \( A \) twisted by \( p \), \( D_{TW}(A,p) \) is the subcomplex of

\[
(2\pi i)^p A^*_a \oplus F^p A^*_C \oplus L_a^* \otimes A^*_C
\]

given by

\[
D_{TW}(A,p) = \left\{ (r,f,\omega) \middle| \begin{align*}
\omega \mid_{\varepsilon=0} &= \alpha(r), \\
\omega \mid_{\varepsilon=1} &= \iota(f)
\end{align*} \right\},
\]

where \( \iota_p \) and \( \alpha_p \) are as in Definition 4.7. Since in a Dolbeault complex, the maps \( \alpha_p \) and \( \iota_p \) are injective, the information conveyed by \( r \) and \( f \) is redundant. Therefore we will simplify the notation by writing

\[
D_{TW}(A,p) = \left\{ \omega \in L_a^* \otimes A^*_C \mid \begin{align*}
\omega \mid_{\varepsilon=0} &= A^*_a(p), \\
\omega \mid_{\varepsilon=1} &= F^p A^*_C \end{align*} \right\}
\]

**Example 4.15.** Let \( X \) be a smooth projective variety over \( \mathbb{C} \). The Thom-Whitney simple \( D_{TW}(E_X,p) \) will be denoted by \( D_{TW}(X,p) \). The Thom-Whitney simple \( D_{TW}(D_X,p) \) will be denoted by \( D_{TW,D}(X,p) \). Then

\[
D_{TW}^\bullet(X,\ast) = \bigoplus_p D_{TW}^\bullet(X,p)
\]

is a bigraded associative algebra. It is graded-commutative with respect to the first degree. That is, if \( \omega \in D_{TW}^n(X,p) \) and \( \omega' \in D_{TW}^m(X,q) \), then

\[
\omega \cdot \omega' = (-1)^{nm} \omega' \cdot \omega.
\]

The Thom-Whitney simple \( D_{TW,D}(X,\ast) \) is a module over \( D_{TW}(X,\ast) \).

Following the notation introduced in Remark 4.12 we will use the notation

\[
\tau D_{TW}^\bullet(X,p) = \tau \leq 2p D_{TW}^\bullet(X,p), \quad \tau D_{TW}^\bullet(X,\ast) = \bigoplus_p \tau D_{TW}^\bullet(X,p).
\]

Clearly \( \tau D_{TW}^\bullet(X,\ast) \) is still an associative, graded commutative algebra.

**Example 4.16.** Let \( X \) be a smooth projective variety over \( \mathbb{C} \) and \( Z \) a codimension \( p \) subvariety. Then, using the shorthand (4.6), the class \([Z]\) is represented in \( D_{TW,D}(X,p) \) by

\[
\delta_Z = 1 \otimes \delta_Z + d\varepsilon \otimes 0.
\]
4.5. The Deligne complex. Following [Del85], in [BG97] a concise complex that computes Deligne cohomology was introduced. If $A$ is a Dolbeault complex, the associated Deligne complex is denoted $D(A,p)$. It is given by

$$D^n(A,p) = \begin{cases} A_{\mathbb{R}}^{n-1}(p-1) \cap \bigoplus_{p' < p, q' < p} A_{\mathbb{C}}^{p', q'}, & \text{if } n < 2p, \\ A_{\mathbb{R}}^n(p) \cap \bigoplus_{p' \geq p, q' \geq p} A_{\mathbb{C}}^{p', q'}, & \text{if } n \geq 2p. \end{cases}$$

with differential $d$ given, for $x \in D^n(A,p)$ by

$$dx = \begin{cases} -\pi(dx), & \text{if } n < 2p - 1, \\ -2\partial\bar{\partial}x, & \text{if } n = 2p - 1, \\ d(x), & \text{if } n > 2p - 1, \end{cases}$$

where, for $n < 2p - 1$, $\pi: A^n(\mathbb{C}) \to D^n(A,p)$ is the projection $\pi = \pi_{p-1} \circ F^{p-1,p-1}$. There are homotopy equivalences

$$\begin{array}{c} D_t(A,p) \xrightarrow{H} D(A,p) \\ \xleftarrow{G} \end{array}$$

given, for $(r, f, \omega) \in D_t^n(A,p)$, by

$$H(r, f, \omega) = \begin{cases} \pi(\omega), & \text{if } n \leq 2p - 1, \\ F^{p,p} + 2\pi_p(\partial\omega^{p-1,n-p+1}), & \text{if } n \geq 2p, \end{cases}$$

and, for $x \in D^n(A,p)$, by

$$G(x) = \begin{cases} (\partial x^{p-1,n-p} - \partial x^{n-p,1-p}, 2\partial x^{p-1,n-p}, x), & \text{if } n \leq 2p - 1, \\ (x, x, 0), & \text{if } n \geq 2p. \end{cases}$$

If $A$ is a Dolbeault algebra, then $D(A, \ast)$ has a graded commutative product given by

$$(4.8) \quad x \cdot y = H(G(x) \ast_{1/2} G(y)).$$

This product is only associative up to homotopy. See [BG97, §2] for more details.

The construction above can be applied to the Dolbeault complexes $E_X$, $E_X(\log Y)$, $E_{U,\log}$ and $D_X$. We will use the notation

$$\begin{array}{c} \mathfrak{D}(X,p) = D(E_X,p), \\ \mathfrak{D}_{\log}(U,p) = D(E_{U,\log},p), \\ \mathfrak{D}_D(X,p) = D(D_X,p). \end{array}$$

Example 4.17. Summarizing, let $A$ be a Dolbeault complex. Then we have at our disposal the following diagram of complexes and morphisms

$$\begin{array}{c} \mathfrak{D}_{TW}(A,p) \xrightarrow{I} \mathfrak{D}_t(A,p) \xrightarrow{H} \mathfrak{D}(A,p) \end{array}$$

where the arrows are homotopy equivalences. The leftmost complex has the advantage that, when $A$ is a Dolbeault algebra, it is an associative and graded commutative algebra. On the middle complex, we have several product structures, but none is at the same time graded commutative and associative. The rightmost complex is the smallest one and gives a more...
concise description of Deligne cohomology but again has the disadvantage that the product (4.8) is only associative up to homotopy.

In particular, if $X$ be a smooth projective variety over $\mathbb{C}$, we can specialize diagram (4.9) to the case $A = E^*_X$ to obtain a diagram

\[(4.10)\quad \mathcal{D}_{TW}(X,p) \to \mathcal{D}_t(X,p) \to \mathcal{D}(X,p).\]

**Example 4.18.** There are several variants of the diagram in Example 4.17. First we can use currents instead of differential forms, in this case we obtain the diagram

\[(4.11)\quad \mathcal{D}_{TW,D}(X,p) \to \mathcal{D}_{t,D}(X,p) \to \mathcal{D}(X,p).\]

The complexes of this diagram are covariant with respect to proper morphisms.

Similarly, if $U$ is quasi-projective, we can use differential forms with logarithmic singularities at infinity as in Example 4.10 2 to obtain a diagram

\[(4.12)\quad \mathcal{D}_{TW,\log}(U,p) \to \mathcal{D}_{t,\log}(U,p) \to \mathcal{D}_{\log}(U,p)\]
of complexes that compute the Deligne-Beilinson cohomology of $U$.

We can also define complexes that compute Deligne-Beilinson cohomology with support in a subvariety or in a family of supports using examples 4.10 3 and 4.10 4.

### 4.6. An analytic lemma.

We gather in this section some analytic formulas that will allow us later to make formal computations with respect differential forms and currents.

First we adapt the notion of basic Green form from [BG94b, 4.6] to the Thom-Whitney complex. Note that this is a variant of the notion of Green form of logarithmic type in [GS90, 1.3.2].

**Definition 4.19.** Let $X$ be a complex projective manifold, $D$ a simple normal crossings divisor and $U = X \setminus D$. Let $Y$ be a codimension $p$ cycle on $U$. A **Green form** for $Y$ is an element $g_Y \in \mathcal{D}^{2p-1}_{TW,\log}(U \setminus |Y|, p)$ of the form

\[
(\varepsilon + 1) \otimes \partial g + (\varepsilon - 1) \otimes \bar{\partial} g + d\varepsilon \otimes g
\]
such that, on $U$,

\[
d[g_Y] + \delta_Y = [\omega_Y],
\]
where $\omega_Y \in \mathcal{D}^{2p}_{TW,\log}(U \setminus |Y|, p)$. We say that the Green form $g_Y$ is a **basic Green form** if there exists a resolution of singularities $(\tilde{U}, E)$ of $(U, Y)$ such that, locally in any coordinate neighborhood with coordinates $(z_1, \ldots, z_n)$ where $E$ is given by $z_1 \ldots z_k = 0$, then $g$ can be written as

\[
g = \sum_{i=1}^k -\alpha_i \log |z_i| + \beta
\]
where $\alpha_i$, $i = 1, \ldots, k$ and $\beta$ are smooth forms and $\alpha_i|_{z_i=0}$ is $\partial$-closed and $\bar{\partial}$-closed.

We also define the product of a form with logarithmic singularities and the current integration along a cycle.
**Definition 4.20.** Let $X, D, U$ as in Definition 4.19. Let $Y$ be a prime cycle on $U$ and $g \in \mathcal{D}^n_{TW, \log}(U, p)$. Let $\tilde{Y}$ be the closure of $Y$ on $X$ and $\tilde{Y}$ a resolution of singularities of $\overline{\tilde{Y}}$. Let $\eta: \tilde{Y} \to X$ the induced map. If $\eta^*g$ is locally integrable in $\tilde{Y}$ then we define the product

$$\delta_Y \cdot g = \eta_*[\eta^*g].$$

Note that, even if $Y$ is a cycle on $U$, the current $\delta_Y \cdot g$ is a current in the whole $X$. If $g$ is also locally integrable on $X$, then this product is also denoted as $\delta_Y \cdot [g]$. We extend this product to arbitrary cycles by linearity. Since $\delta_Y$ is of even degree, we write

$$g_1 \cdot \delta_Y \cdot g_2 = \delta_Y \cdot g_1 \cdot g_2.$$

We use a similar notation for inverse images.

**Definition 4.21.** Let $X, D, U$ and $g$ be as before. Let $f: Y \to X$ be a morphism such that $f^{-1}(D)$ is a simple normal crossings divisor. If $g$ and $f^*g$ are locally integrable, we write

$$f^*[g] = [f^*g].$$

**Lemma 4.22.** Let $X, D$ and $U$ be as before. Let $Y_0, \ldots, Y_r$ be cycles on $U$ intersecting properly of codimension $p_0, \ldots, p_r$ respectively, $g_i \in \mathcal{D}^{n_i}_{TW, \log}(U \setminus \{Y_i, q_i\})$ and $1 \leq s \leq r$. Assume that, for $1 \leq i < s$, the condition $n_i < 2p_i - 1$ hold, while for $s \leq i \leq r$, $g_i$ is a basic Green form for the cycle $Y_i$, satisfying the equation $d(g_i) + \delta_{Y_i} = \omega_i$. Assume furthermore that, for each irreducible component $D_j$ of $D$ there is an index $i(j)$ such that $g_{i(j)}$ is smooth in a neighbourhood of $D_j$ and $g_{i(j)}|_{D_j} = 0$. Then the following statements hold:

1. The form $g_1 \cdots g_r$, and the forms $g_1 \cdots dg_i \cdots g_r$ for $i = 1, \ldots, r$ are locally integrable on $X$, and

$$d[g_1 \cdots g_r] = \sum_{i=1}^{s-1} (-1)^{n_1 + \cdots + n_i - 1} [g_1 \cdots dg_i \cdots g_r]$$

$$+ \sum_{i=s}^{r} (-1)^{n_1 + \cdots + n_i - 1} ([g_1 \cdots \omega_i \cdots g_r] - g_1 \cdots \delta_{Y_1} \cdots g_r).$$

2. Let $f: Y \to X$ be a morphism of projective complex manifolds such that $f^{-1}(D)$ is a simple normal crossings divisor. Write $V = Y \setminus f^{-1}(D)$ and $f_0: V \to U$ for the restriction of $f$ to $V$. Assume furthermore that $\text{codim}_V f_0^{-1}Y_i \geq \text{codim}_U Y_i$, $i = 1, \ldots, r$ and that the subvarieties $f_0^{-1}Y_i$ intersect properly in $V$. In particular there are well defined cycles $f_0^{-1}Y_i$. Then the forms

$$f^*(g_1 \cdots g_r) \quad \text{and} \quad f^*(g_1 \cdots dg_i \cdots g_r), \ i = 1, \ldots, r$$

are locally integrable on $Y$ and

$$df^*[g_1 \cdots g_r] = \sum_{i=1}^{s-1} (-1)^{n_1 + \cdots + n_i - 1} f^*[g_1 \cdots dg_i \cdots g_r]$$

$$+ \sum_{i=s}^{r} (-1)^{n_1 + \cdots + n_i - 1} (f^*[g_1 \cdots \omega_i \cdots g_r] - f^*g_1 \cdots \delta_{f_0^{-1}Y_i} \cdots f^*g_r).$$
(3) Let \( \widetilde{Y}_0 \) be a resolution of singularities on the closure \( Y_0 \) and \( \eta_0: \widetilde{Y}_0 \to X \) as in Definition 4.20. Then the forms 
\[ \eta_0^*(g_1 \cdots g_r) \text{ and } \eta_0^*(g_1 \cdots dg_i \cdots g_r), \quad i = 1, \ldots, r \]
are locally integrable on \( \widetilde{Y}_0 \) and 
\[
d\delta_{Y_0} \cdot [g_1 \cdots g_r] = \sum_{i=1}^{s-1} (-1)^{n_1 + \cdots + n_{i-1}} \delta_{Y_0} \cdot [g_1 \cdots dg_i \cdots g_r] 
+ \sum_{i=s}^{k} (-1)^{n_s + \cdots + n_{i-1}} (\delta_{Y_0} \cdot [g_1 \cdots \omega_i \cdots g_r] - [g_1 \cdots \delta_{Y_0} \cdot Y_i \cdots g_r]),
\]
where the intersection \( Y_0 \cdot Y_i \) means the intersection of cycles in \( U \).

**Proof.** To prove the lemma, first one unwraps the definition of the Thom-Whitney complex to obtain statements about usual differential forms. Then the proof that all the involved differential forms are locally integrable uses the techniques of [BG94b, Proposition 3.3]. Note that an arbitrary form with logarithmic singularities does not need to be locally integrable. The key point in the proof of local integrability is that wherever one of the forms may have arbitrary logarithmic singularities (any component of \( D \)), another of the forms vanishes. Thus the vanishing condition of some of the forms along the divisor \( D \) is a necessary condition for the local integrability. The proof of the equations involving the differential use the techniques of the proof of [GS90, 2.1.4], [GS90, 2.2.2]. Notationally the proof is more involved because it deals with a product of an arbitrary number of functions while in loc. cit. it involves at most three functions. Conceptually is simpler because here we are assuming that all the intersections are proper. \( \square \)

5. **The cycle class map**

The aim of this section is to recall two explicit incarnations of the cycle class map from higher Chow groups to Deligne cohomology. The first is a cubical version of Goncharov cycle class map defined in [Gon05], while the second is a minor variant of a construction by Bloch in [Blo86b], that was used in [BGF12]. After composing with the isomorphism between \( K \) theory and higher Chow groups, both maps induce Beilinson’s regulator.

5.1. **Wang forms and Goncharov regulator.** Consider the inclusion \( \mathbb{G}_m(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C}) \), with absolute coordinate \( t \) and write \( D = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{G}_m(\mathbb{C}) = \{0, \infty\} \). Let \( \lambda \in (L_{\mathbb{C}} \otimes E_{\mathbb{P}^1}(\log D))^1 \) be the element given by

\[
\lambda = \frac{-1}{2} \left( (\varepsilon + 1) \otimes \frac{dt}{t} + (\varepsilon - 1) \otimes \frac{dt}{t} + d\varepsilon \otimes \log t \right).
\]

Then
\[
\lambda|_{\varepsilon=0} = \frac{-1}{2} \left( \frac{dt}{t} - \frac{dt}{t} \right) \in (2\pi i)E_{\mathbb{P}^1,\mathbb{R}}^1(\log D),
\]

\[
\lambda|_{\varepsilon=1} = \frac{dt}{t} \in F^1E_{\mathbb{P}^1}^1(\log D).
\]
Therefore, following the shorthand (4.6), we deduce that
\[ \lambda \in \mathcal{D}^1_{TW}(E_{P^1}(\log D), 1). \]

The form \( \lambda \) satisfies the properties
\begin{align*}
(5.2) \quad & \lambda |_{t=1} = 0 \\
(5.3) \quad & d\lambda = 0.
\end{align*}

But we are more interested in the associated current. Write
\[ [\lambda] = -\frac{1}{2} \left( (\varepsilon + 1) \otimes \left[ \frac{dt}{t} \right] + (\varepsilon - 1) \otimes \left[ \frac{d\bar{t}}{\bar{t}} \right] + d\varepsilon \otimes [\log tt] \right). \]

One easily verifies that \([\lambda] \in \mathcal{D}^1_{TW}(\mathbb{P}^1, 1)\) and, using equations (4.3), that it satisfies
\begin{align*}
(5.4) \quad & d[\lambda] = -1 \otimes \delta_{\text{div}} = \delta_\infty - \delta_0. \\
(5.5) \quad & W_n |_{\mathbb{P}^1 \setminus \Box^n} = 0, \\
(5.6) \quad & dW_n = 0,
\end{align*}

while the associated current \([W_n] \in \mathcal{D}_{TW,D}(\mathbb{P}^1)^n, n)\) satisfies
\begin{equation}
(5.7) \quad d[W_n] = \sum_{i=1}^{n} \sum_{j=0,1} (-1)^{i+j} (\delta^i_j)_{\ast}[W_n_{i-1}].
\end{equation}

This formula follows from the fact that \( \lambda \) is a basic Green form for the cycle \([0] - [\infty]\) and Lemma 4.22. Alternatively, the currents \([W_n]\) can be defined using the Cartesian product (also called tensor product or direct product) of currents
\[ [W_n] = [W_1] \times \cdots \times [W_1]. \]

Then equation (5.7) follows from (5.4) and the fact that the Cartesian product satisfies Leibnitz rule [Fed69, 4.1.8].

Let \( \mathfrak{A} \) be any of the complexes of diagram (4.9). Then we will denote by \( W_n|_{\mathfrak{A}} \) the image of \( W_n \) in the corresponding complex of forms in (4.10) and \([W_n]|_{\mathfrak{A}}\) the image of \([W_n]\) in the complex of currents in (4.11). For instance
\[ [W_n]|_{\mathfrak{A}} \in \mathcal{D}_{\mathfrak{A}}^n((\mathbb{P}^1)^n, n) \]
is the cubical version of the current originally used by Goncharov. For further reference we note that
\[
(W_1)_D = \left[ -\frac{1}{2} \log t \right].
\]

All the different families of currents \([W_n]_\lambda\) satisfy the analogue of equation (5.7). For shorthand, if there is no decoration, we will mean that Thom-Whitney version. That is
\[
[W_n] = [W_n]_{\text{TW}}.
\]

We can now describe the cubical version of Goncharov’s regulator for projective varieties.

**Definition 5.1.** Let \(X\) be a smooth projective complex variety of dimension \(d\). We denote by \(p_1, p_2\) the two projections from \(X \times (\mathbb{P}^1)^n\) to the first and second factor. Let \(Z \subset X \times \square^n\) be a codimension \(p\) subvariety that meets properly all the faces. We define the current \(P(Z) \in D_{TW,D}^{2p-n}(X,p)\) as
\[
P(Z) = (p_1)_* \delta_Z \cdot [p_2^* W_n].
\]

This current is well defined thanks to Lemma 4.22. The map \(P\) is extended by linearity to define the **cubical Goncharov regulator map** \(P : Z^p(X,*) \to D_{TW,D}^{2p-n}(X,p)\).

Definition 5.1 can be adapted to any of the complexes of currents of Diagram (4.11). We denote any such complex as \(\mathcal{A}_D\) and \(P_{\mathcal{A}}(Z)\) will be the image of \(Z\) in any \(\mathcal{A}_D\). For instance
\[
P_D(Z) \in D_D^{2p-n}(X,p)
\]
is the cubical version of the original Goncharov regulator.

**Proposition 5.2.**

1. The map \(P : Z^p(X,*) \to D_{TW,D}^{2p-n}(X,p)\) is a morphism of complexes.
2. If \(\zeta \in Z^p(X,*)\) is a degenerate pre-cycle then \(P(\zeta) = 0\).
3. \(P\) induces a morphism of complexes
\[
\tilde{Z}^p(X,*) \to D_{TW,D}^{2p-n}(X,p)
\]
that agrees with the restriction of \(P\) to \(NZ^p(X,*)\).

**Proof.** The first statement follows from equations Lemma 4.22 and the fact that \(\lambda\) is a basic Green form for the cycle \([0]-[\infty]\) that vanishes at the point 1. Formally it follows from follows from equations (5.5) and (5.7). The second statement is a direct check while the third statement is consequence of the other two. \(\square\)

**Remark 5.3.** The regulator \(P_e\) used in the paper [BGFT11] (in section 7.2) agrees with \(P_D\).

**Remark 5.4.** The Goncharov regulator is very concrete and well suited for computations. Nevertheless it has some drawbacks. In particular, due to the use of currents, it is difficult to establish its contravariant functoriality and its multiplicativity. In the paper [BGF12], E. Feliu and the first author have introduced another map that uses only differential forms and not currents, making it easier to establish the contravariance and multiplicative properties.
of the regulator. In the paper [BGFT11], in joint work with Takeda they have used this new map to prove that the cubical version of Goncharov’s regulator agrees after composing with the isomorphism with $K$-theory with Beilinson’s regulator. The map of Burgos-Feliu is a minor modification of a map introduced by Bloch [Blo86b].

Goncharov regulator is compatible with direct images with respect to proper maps.

**Proposition 5.5.** Let $f : X \to Y$ be a morphism between smooth projective varieties over $\mathbb{C}$ of relative dimension $e$. Thus the map is proper and, for integers $p, n$, there are direct image maps

\[ f_* : \mathcal{A}^{2p-n}_D(X, n) \to \mathcal{A}^{2p-n-2e}_D(X, p - e). \]

Then the compatibility

\[ \mathcal{P}_\mathcal{A} \circ f_* = f_* \circ \mathcal{P}_\mathcal{A} \]

holds.

5.2. **Differential forms and affine lines.** Let $X$ be a smooth complex projective variety and let $\mathcal{A}$ denote any complex appearing in diagram (4.9). Then for every $n, p \geq 0$, let $\mathcal{A}^r_{\log}(X \times \square^n, p)$ denote the construction $\mathcal{A}$ applied to the Dolbeault complex $E_{X \times \square^n, \log}$. Let $\tau \mathcal{A}^r_{\log}(X \times \square^n, p)$ be its canonical truncation in degree $2p$. That is

\[ \tau \mathcal{A}^r_{\log}(X \times \square^n, p) = \tau_{\leq 2p} \mathcal{A}^r(E_{X \times \square^n, \log}, p). \]

The structural maps of the cocubical scheme $\square$ induce a cubical structure on $\tau \mathcal{A}^r_{\log}(X \times \square^*, p)$ for every $r$ and $p$.

Following the convention of §2.3, we consider the 2-iterated cochain complex

\[ \tau \mathcal{A}^r_{\log}(X, p) := \tau \mathcal{A}^r_{\log}(X \times \square^n, p), \]

with differentials $(d, \delta) = \sum_{i=1}^n (-1)^i (\delta^0 - \delta^1)$. So $d$ is the differential in the complex $\mathcal{A}$, while $\delta$ denotes the differential in the cubical complex. Let

\[ \tau \mathcal{A}^r_{\log}(X, p) = \tau \mathcal{A}^r_{\log}(X, p) \]

be the associated simple complex. Its differential $d_*$ is given, for every $\alpha \in \tau \mathcal{A}^r_{\log}(X, p)$, by $d_*(\alpha) = d(\alpha) + (-1)^r \delta(\alpha)$. Since we are using cubical structures, this complex does not compute the right cohomology and we have to normalize it.

For every $r, n$, we write

\[ \tau \mathcal{A}^{r-n}_{\log}(X, p) = \tau \mathcal{A}^r_{\log}(X \times \square^n, p) : = N \tau \mathcal{A}^r_{\log}(X \times \square^n, p). \]

Hence $\tau \mathcal{A}^r_{\log}(X, p) = \text{the normalized 2-iterated complex and we denote by } \tau \mathcal{A}^r_{\log}(X, p)$ the associated simple complex. The next result is proved in [BGF12, Proposition 2.8] for Deligne-Beilinson cohomology. The proof applies mutatis mutandis to any of the complexes of diagram (4.9).

**Proposition 5.6.**

1. For every $n > 0$ the complex $\tau \mathcal{A}^{r-n}_{\log}(X, p)$ is acyclic.

2. The natural morphism of complexes

\[ \tau \mathcal{A}^r_{\log}(X, p) := \tau \mathcal{A}^{r,0}_{\log}(X, p) = \tau \mathcal{A}^r_{\log}(X, p) \to \tau \mathcal{A}^r_{\log}(X, p) \]

is a quasi-isomorphism.
5.3. Cycle class map with differential forms. Let $X$ be a smooth projective variety over $\mathbb{C}$ as in the previous section. Let $Z^p_{n,X}$ be the set of all Zariski closed subsets of $W \subset X \times \square^n$ of codimension $\geq p$ such that, for every face $F$ of $\square^n$,
\[
\text{codim}_{X \times F}(W \cap X \times F) \geq p.
\]
We order $Z^p_{n,X}$ by inclusion. It is a directed set and the subsets of codimension exactly equal to $p$ is a cofinal subset. When there is no source of confusion, we simply write $Z^p_n$ or even $Z^p$ for $Z^p_{n,X}$.

Consider the cubical abelian group
\[
\mathcal{H}^p(X, *) := \lim_{Z \in \mathbb{Z}^p} \mathcal{H}^{2p}_{\square^n}(X \times \square^n, \mathbb{R}(p)),
\]
with faces and degeneracies induced by those of $\square^n$. Let $\mathcal{H}^p(X, *)_0$ be the associated normalized complex.

Lemma 5.7. Let $X$ be a smooth projective complex variety. For every $p \geq 0$, there is a natural isomorphism of chain complexes,
\[
Z^p(X, *)_0 \otimes \mathbb{R} \cong \mathcal{H}^p(X, *)_0
\]
sending $Z$ to $\text{cl}(Z) = [Z]$.

Proof. Follows from Proposition 4.11. \qed

Remark 5.8. Observe that the complex $\mathcal{H}^p(X, *)_0$ has the same functorial properties as $Z^p(X, *)_0 \otimes \mathbb{R}$.

For $n \geq 0$ let $\mathfrak{A}^{*,-n}_{\mathbb{A},Z^n}(X, p)$ be the complex
\[
\mathfrak{A}^{*,-n}_{\mathbb{A},Z^n}(X, p) = \lim_{Z \in \mathbb{Z}^p} s \left( \mathfrak{A}^{*,-n}_{\mathbb{A}}(X, p) \to \mathfrak{A}(E^*_X \times \square^n \setminus \text{cl}(Z), \log P) \right)
\]
and write
\[
\tau \mathfrak{A}^{*,-n}_{\mathbb{A},Z^n}(X, p) = \tau \mathfrak{A}^{*,-n}_{\mathbb{A},Z^n}(X, p).
\]
Note that we are truncating after taking the simple. Varying $n$ the complexes (5.11) form a cubical truncating cochain complex and its normalization is a 2-iterated cochain complex,
\[
\tau \mathfrak{A}^{*,-n}_{\mathbb{A},Z^n}(X, p)_0 = N \tau \mathfrak{A}^{*,-n}_{\mathbb{A},Z^n}(X, p).
\]
Following the convention of §2.3, the differential will be denoted by $(d, \delta)$ where $d$ is the differential coming from the complex direction and $\delta$ the differential coming from the cubical direction.

We denote by $\tau \mathfrak{A}^{*,-n}_{\mathbb{A},Z^n}(X, p)_0$ the associated simple complex and by $d_s$ its differential.

We shift from a cochain complex to a chain complex and denote by $\tau \mathfrak{A}^{2p-*}_{\mathbb{A},Z^n}(X, p)_0$ the chain complex whose $n$-graded piece is $\tau \mathfrak{A}^{2p-n}_{\mathbb{A},Z^n}(X, p)_0$.

The fact that we are using the truncation implies that an element $\beta \in \tau \mathfrak{A}^{2p-n}_{\mathbb{A},Z^n}(X, p)_0$ is of the form
\[
\beta = ((\omega_n, g_n), \ldots, (\omega_0, g_0)),
\]
with $(\omega_i, g_i) \in \mathfrak{A}^{2p-n+i-i}_{\mathbb{A},Z^n}(X, p)$ and the top element $(\omega_n, g_n) \in \mathfrak{A}^{2p-n}_{\mathbb{A},Z^n}(X, p)$ satisfies $d(\omega_n, g_n) = 0$, thus is a cycle in the complex $\mathfrak{A}^{*,-n}_{\mathbb{A},Z^n}(X, p)$.
Proposition 5.9. For every $p \geq 0$, the family of morphisms
\[
\tau A_{2p}^n(X, p) \rightarrow H^p(X, n)\]
defines a surjective quasi-isomorphism of chain complexes.

Proof. Except for the surjectivity, this statement for the complex $D$ is [BGF12, Proposition 2.13]. The proof in loc. cit. works for any of the complexes of Example 4.17 and shows the surjectivity. The key ingredients for the proof are Proposition 4.11 and Proposition 2.9. □

We now consider the map of forgetting supports
\[
\rho: \tau A_{2p}^n(X, p) \rightarrow \tau A_{2p}^n(X, p)\]
\[
((\omega_n, g_n), \ldots, (\omega_0, g_0)) \mapsto (\omega_n, \ldots, \omega_0).
\]

Theorem 5.10. The map induced in cohomology by the zig-zag diagram
\[
\begin{array}{ccc}
\mathcal{H}^p(X, *) & \xrightarrow{\gamma_1} & \tau A_{2p}^n(X, p) \\
Z^p(X, *) & \sim & \tau A_{2p}^n(X, p)
\end{array}
\]
agrees with one induced by the map $P_A$ of Definition 5.1 (Goncharov regulator).

Proof. This is essentially the content of [BGFT11, Theorem 7.7].

Since the tools used in the proof of Theorem 5.10 will be used later we summarize then here. Consider the diagram of complexes
\[
\begin{array}{ccc}
\mathcal{H}^p(X, *) & \xrightarrow{\gamma_1} & \tau A_{2p}^n(X, p) \\
\tau A_{2p}^{n-*}(X, p) & \sim & \tau A_{2p}^{n-*}(X, p)
\end{array}
\]
and denote
\[
\tau A_{2p}^{n-*}(X, p) = s(\rho - \gamma_1)[-1].
\]

Composing the map $P_A$ with the inverse of the isomorphism $\gamma_1$ of Lemma 5.7, we obtain a morphism
\[
P_A: \mathcal{H}^p(X, *) \rightarrow \tau A_{2p}^{n-*}(X, p).
\]

There is a morphism that also represents the regulator map
\[
\rho: Z^p(X, *) \rightarrow \tau A_{2p}^{n-*}(X, p)
\]
given by $\rho(Z) = (0, \gamma_1(Z), 0)$.

Finally there is a morphism of complexes ([BGFT11, Theorem 7.7])
\[
\psi: \tau A_{2p}^{n-*}(X, p) \rightarrow \tau A_{2p}^{n-*}(X, p)
\]
given by
\[
\psi((\omega_{n-1}, g_{n-1}), \ldots, (\omega_0, g_0), Z, (\alpha_n, \ldots, \alpha_0)) = \mathcal{P}(Z) + \sum_{i=0}^{n} (p_1)_* [\alpha_i \cdot W_{i, A} - g_i \cdot W_{i, A}].
\]

The fact that the forms appearing in (5.13) are locally integrable follows from Lemma 4.22.

Putting together the above morphisms we obtain a commutative diagram, from which the theorem is derived,
\[
\begin{array}{c}
Z^p(X, *)_0 \xrightarrow{\rho} \tau \mathfrak{A}^{2p-s}_D(X, p)_0 \\
\downarrow \rho \\
\tau \mathfrak{A}^{2p-s}_{k, H}(X, p) \xrightarrow{\psi} \tau \mathfrak{A}^{2p-s}_D(X, p).
\end{array}
\]

Remark 5.11. In [BGF12], it is the complex $\mathfrak{D}^*(X, *)$ that is used. When discussing the product we will adopt the Thom-Whitney complex.

6. Green currents and Green forms

In this section we will introduce Green currents and Green forms of logarithmic type for higher cycles and study their basic properties.

Notation: The notations in this section will be the same that has been used in §5. Let $\mathfrak{A}$ be any of the complexes that appear in Example 4.17 and Example 4.18. To fix ideas we will be thinking mainly in the complex $\mathfrak{D}_{TW}$ because it has better multiplicative properties. We will use the notation $\mathfrak{A}_{D}$ for the corresponding complex with currents (Diagram (4.11)).

The differential of any such complex (irrespective of the definition by forms or by currents) will be denoted by $d$. The elements of $\mathfrak{A}^*(X, \bullet)$ will be called forms while the elements of $\mathfrak{A}^*_D(X, \bullet)$ will be called currents.

We recall also the map
\[
\mathcal{P}_{\mathfrak{A}}: Z^p(X, *) \to \mathfrak{A}^{2p-s}_D(X, p)
\]
given in Definition 5.1. We will also denote by $\mathcal{P}_{\mathfrak{A}}$ the induced morphism $\mathcal{P}: Z^p(X, *)_0 \to \mathfrak{A}^{2p-s}_D(X, p)$ (see Proposition 5.2).

For any cochain complex $(A^*, d_A)$, we will denote
\[
(6.1) \quad \tilde{A}^n = A^n / \text{Im} d_A.
\]

6.1. Green currents. Let $X$ be a smooth and projective variety of dimension $d$ defined over $\mathbb{C}$.

Definition 6.1. Let $Z \in Z^p(X, n)_0$ be a higher cycle (that is a pre-cycle $Z$ such that $\delta Z = 0$). Then an $\mathfrak{A}$-Green current for $Z$ is a current
\[
g_Z \in \mathfrak{A}^{2p-n-1}_D(X, p)
\]
such that
\[(6.2) \quad \mathcal{P}_A(Z) + dg_Z = [\omega_Z], \text{ for } \omega_Z \in \mathcal{A}^{2p-n}(X, p).\]

In other words $\mathcal{P}_A(Z) + dg_Z$ is the image of a smooth form. A class of Green currents is the class of a Green current in the quotient
\[\mathfrak{A}^{2p-n-1}_D(X, p) = \mathfrak{A}^{2p-n-1}_D(X, p)/\text{Im } d.\]

If $g_Z$ is a Green current, its class will be denoted $\tilde{g}_Z$. If $g_Z$ is a Green current for the cycle $Z$ and $\tilde{g}_Z$ is its class, we denote $\omega(g_Z) = \omega(\tilde{g}_Z)$ the form in $\mathfrak{A}^{2p-n}(X, p)$ such that
\[[\omega(g_Z)] = \mathcal{P}_A(Z) + dg_Z.\]

Note the parallel between this definition and [GS90, Definition 1.2.3]. Observe also that equation (6.2) implies that $\omega_Z$ represents the cycle class of $Z$ in $H^{2p-n}_A(X, \mathbb{R}(p))$.

**Remark 6.2.** From the Green current alone, it is not always possible to recover the cycle $Z$ and the smooth form $\omega_Z$. For instance in Remark 8.1 we will see that 0 can be a Green current for a nonzero cycle. Therefore the cycle $Z$ is part of the data of the Green current and will always be either explicit or implicit.

**Remark 6.3.** If $Z \in Z^p(X, n)_0$ is a higher cycle, it is also a cycle in $X \times \Box^n$. Therefore a Green current for $Z$ could also mean a current $g$ on $X \times \Box^n$ satisfying $dg + \delta g$ smooth. To distinguish between both cases, the second will be called a Green current for $Z$ on $X \times \Box^n$. This ambiguity will cause more problems for Green forms latter, and we will use the same method to distinguish both meanings of Green form.

**Lemma 6.4.** Let $Z \in Z^p(X, n)_0$ be a cycle. Then an $\mathfrak{A}$-Green current for $Z$ exists. Moreover if $\tilde{g}_Z$ and $\tilde{g}'_Z$ are two classes of $\mathfrak{A}$-Green currents for $Z$ then the difference
\[\tilde{g}_Z - \tilde{g}'_Z \in \mathfrak{A}^{2p-n-1}(X, p),\]
is the class of a smooth form.

**Proof.** Since $\mathcal{P}_A$ is a morphism of complexes, the condition $\delta Z = 0$ implies $d\mathcal{P}_A(Z) = 0$. Since the inclusion $[\cdot] : \mathfrak{A}^*(X, p) \to \mathfrak{A}^*_D(X, p)$ is a quasi-isomorphism, the cohomology class of $\mathcal{P}_A(Z)$ can be represented by en element of $\mathfrak{A}^*(X, p)$. In other words, there exist elements $g_Z \in \mathfrak{A}^{2p-n-1}_D(X, p)$ and $\omega_Z \in \mathfrak{A}^{2p-n}(X, p)$ such that $\mathcal{P}_A(Z) + dg_Z = [\omega_Z]$, showing the existence of Green currents.

Let now $\tilde{g}_Z$ and $\tilde{g}'_Z$ be two classes of Green currents for $Z$. Write $\omega = \omega(\tilde{g}_Z)$ and $\omega' = \omega(\tilde{g}'_Z)$ and let $g_Z$ and $g'_Z$ be two representatives of the given classes. Then
\[d(g_Z - g'_Z) = [\omega - \omega'].\]

Thus $[\omega - \omega']$ is exact in the complex $\mathfrak{A}^{2p-n}_D(X, p)$, therefore $\omega - \omega'$ is exact in the complex $\mathfrak{A}^{2p-n}(X, p)$, and there exists $u_1 \in \mathfrak{A}^{2p-n-1}(X, p)$ with
\[d(g_Z - g'_Z) = d[u_1].\]
The current $g_Z - g'_Z - [u_1]$ is closed, so its cohomology class can be represented by a form, and there are elements $u_2 \in \mathcal{A}^{2g-2}(X, p)$ and $v \in \mathcal{A}^{2g-1}(X, p)$ with

$$g_Z - g'_Z - [u_1] = [u_2] + dv$$

proving the second statement. □

6.2. Green forms. As we will see latter, the definition of Green current is well suited to define direct images. Nevertheless, as was the case for the classical arithmetic Chow groups, in order to define inverse images and products we need to represent a Green current by a differential form with singularities only along the cycle. The difficulty here is that a higher cycle lives in $X \times \square^n$. In order to go from $X \times \square^n$ to $X$ we need a “staircase” of differential forms.

Let $Z \in \mathcal{Z}^p(X, n)_0$ be a pre-cycle. We denote by $|Z|$ the support of $Z$ and, for each $k = 0, \ldots, n$ we denote by $|Z|_k \subset X \times \square^k$ the codimension $p$ subset given as

$$|Z|_k = \bigcup_{i_1, \ldots, i_{n-k}} (\delta_{i_0}^{i_1})^{-1} \ldots (\delta_{i_0}^{i_{n-k}})^{-1}|Z|.$$

In particular $|Z| = |Z|_n$. By abuse of notation, if $W \in \mathcal{Z}^p_n$ is a closed Zariski subset, we will denote

$$|W|_k = \bigcup_{i_1, \ldots, i_{n-k}} (\delta_{i_0}^{i_1})^{-1} \ldots (\delta_{i_0}^{i_{n-k}})^{-1}W.$$

For each $k = 0, \ldots, n$, let $H_k \subset X \times \square^k$ be the normal crossing divisor

$$H_k = \bigcup_{i=1}^k X \times \delta_i^0(\square^{k-1}).$$

In particular $H_0 = 0$.

We denote by

$$\mathcal{A}_{log}^r(X \times \square^k \setminus |Z|_k, p)_0 \subset \mathcal{A}_{log}^r(X \times \square^k \setminus |Z|_k, p)$$

the subcomplex of elements $x$ whose restriction $x|_{H_k} = 0$. As in the case of a normalized complex, there is a differential

$$\delta: \mathcal{A}_{log}^r(X \times \square^k \setminus |Z|_k, p)_0 \to \mathcal{A}_{log}^r(X \times \square^{k-1} \setminus |Z|_{k-1}, p)_0$$

given by

$$\delta x = \sum_{i=1}^k (-1)^i (\delta_{i_0}^{i_1})^* x,$$

that turns $\mathcal{A}_{log}^r(X \times \square^k \setminus |Z|_k, p)_0$ into a 2-iterated complex.

Analogously, we denote by

$$\mathcal{A}_{Z|k}^r(X \times \square^k, p)_0 \subset \mathcal{A}_{Z|k}^r(X \times \square^k, p),$$

the subcomplex of elements $x$ whose restriction $x|_{H_k} = 0$. Again it has a differential $\delta$ turning it into a 2-iterated complex.
Definition 6.5. Given a cycle \( Z \in Z^p(X, n)_0 \), an \( \mathfrak{A} \)-Green form (or just a Green form is \( \mathfrak{A} \) is understood) for \( Z \) is an \( n \)-tuple

\[
\mathfrak{g}_Z := (g_n, g_{n-1}, \cdots, g_0) \in \bigoplus_{k=n}^{0} \mathfrak{A}_{\log}^{2p-n+k-1}(X \times \square^k \setminus |Z|_k, p)_0,
\]

such that, if \( n > 0 \),

1. \( \delta_Z + d[g_n] = 0 \), where \( \delta_Z \) is viewed as an element in \( \mathfrak{A}_D \) (see example 4.16).
2. \( (-1)^{n-k+1} \delta g_k + dg_{k-1} = 0, \quad k = 2, \cdots, n \).
3. \( (-1)^n \delta g_1 + dg_0 \in \mathfrak{A}^{2p-n}(X, p) \). In words, the form \( (-1)^n \delta g_1 + dg_0 \) extends to a smooth form on the whole \( X \).

While, if \( n = 0 \) the previous conditions collapse to the condition

\[
\delta_Z + d[g_n] \in [\mathfrak{A}^{2p}(X, p)].
\]

The case \( n = 0 \) will be implicitly understood in the sequel and will not be treated separately. As was the case for Green currents, we will use the notation

\[
\omega(\mathfrak{g}_Z) := (-1)^n \delta g_1 + dg_0 \in \mathfrak{A}^{2p-n}(X, p).
\]

If \( Z \in Z^p(X, n)_0 \) is a cycle in the refined normalized complex, then a refined Green form is defined as a Green form satisfying the stronger condition

\[
(6.5) \quad \mathfrak{g}_Z \in \bigoplus_{k=0}^{n} \mathfrak{A}_{\log}^{2p-n+k-1}(X \times \square^k \setminus |Z|_k, p)_0.
\]

A Green form or a refined Green form will be called basic if \( g_n \) is a basic Green form for \( Z \) on \( X \times \square^n \) in the sense of Definition 4.19 (see Remark 6.3).

For defining inverse images and products it is important to have control on where the singularities of a Green form are. This is the reason we ask in the definition of a Green form that

\[
g_k \in \mathfrak{A}_{\log}^{2p-n+k-1}(X \times \square^n \setminus |Z|_n, p)_0,
\]

but in intermediate steps, it is useful to relax this condition.

Definition 6.6. Given a cycle \( Z \in Z^p(X, n)_0 \), a loose \( \mathfrak{A} \)-Green form (or just a loose Green form is \( \mathfrak{A} \) is understood) for \( Z \) is an \( n \)-tuple

\[
\mathfrak{g}_Z := (g_n, g_{n-1}, \cdots, g_0) \in \bigoplus_{k=n}^{0} \mathfrak{A}_{\log}^{2p-n+k-1}(X \times \square^k \setminus |Z|_n, p)_0,
\]

satisfying conditions (1), (2) and (3).

Let \( \mathfrak{g}_Z = (g_n, \cdots, g_0) \) be a loose Green form for \( Z \). By the definition of \( \mathfrak{A}_{\log}^{2p-n+k-1}(X \times \square^k \setminus |Z|_n, p)_0 \) as direct limit, for each \( k \geq 0 \), we can choose a Zariski closed subset \( W_k \in Z^p_k \) such that

\[
g_k \in \mathfrak{A}_{\log}^{2p-n+k-1}(X \times \square^k \setminus W_k, p)_0.
\]

Adding to \( W_n \) sets of the form \( \pi^* W_k \), where \( \pi : X \times \square^n \to X \times \square^k \) is the projection, we can find a subset \( W \in Z^p_n \) such that \( W_k \subset |W|_k \).
Consider now the element
\[ g_k \in \mathfrak{A}_{\log}^{2p-n+k-1}(X \times \square^k \setminus |W|, p)_0, \quad k = 1, \ldots, n. \]

**Proposition 6.8.** Let \( Z \in Z^p(X, n)_0 \) be a cycle (that is, \( \delta Z = 0 \)). Then a loose Green form for \( Z \) exists.

**Proof.** Consider the class \( \text{cl}(Z) \in H^0(X, n) \). Since \( Z \) is a cycle, by Proposition 5.9, there is an element
\[ \beta = ((\omega_0, g_0), \ldots, (\omega_n, g_n)) \in \tau \mathfrak{A}_{\log, Z}^{2p-n}(X, p)_0 \]
with \( d_s \beta = 0 \) and
\[ (6.6) \quad \text{cl}(\omega_n, g_n) = \gamma_1(\beta) = \text{cl}(Z). \]

By [BG94b, Theorem 4.4 (1)] (see also [BG97, Theorem 5.9]), the equation (6.6) implies that
\[ \delta \gamma + d[g_n] = [\omega_n]. \]

Consider now the element
\[ \rho(\beta) = (\omega_n, \ldots, \omega_0) \in \tau \mathfrak{A}_{\log}^{2p-n}(X, p)_0. \]

Since \( \beta \) is a cycle, \( \rho(\beta) \) is a cycle. By Proposition 5.6 (2) there is an element
\[ (\eta_{n+1}, \ldots, \eta_0) \in \mathfrak{A}_{\log}^{2p-n-1}(X, p)_0 \]
such that
\[ d_s(\eta_{n+1}, \ldots, \eta_0) = (\omega_n, \ldots, \omega_0) - (0, \ldots, 0, \omega). \]

In particular \( d\eta_{n+1} = 0 \). Since the complex \( \tau \mathfrak{A}_{\log, Z}^{2p-n}(X, p)_0 \) is acyclic for all \( n > 0 \) (Proposition 5.6) we can find an element \( \bar{\eta}_{n+1} \in \mathfrak{A}_{\log}^{2p-n-1}(X, p)_0 \) such that \( d\bar{\eta}_{n+1} = \eta_{n+1} \). Then, one can verify that
\[ \beta' := \beta - d_s((\eta_{n+1}, \bar{\eta}_{n+1}), (\eta_n, 0), \ldots, (\eta_0, 0)) = ((0, g'_n), \ldots, (0, g'_1), (\omega, g'_0)). \]

Since \( \bar{\eta}_{n+1} \) is a smooth form, \( \text{cl}(0, g'_n) = \text{cl}(\omega_n, g_n) = \text{cl}(Z) \). Therefore
\[ \delta Z + d[g'_n] = 0. \]

This equation, plus the condition that \( d_s \beta' = 0 \) implies that \( (g'_n, \ldots, g'_0) \) is a loose Green form for \( Z \).

**Proposition 6.9.** Let \( Z \in Z^p(X, n)_0 \) be a cycle and \( g = (g_n, \ldots, g_0) \) a loose Green form for \( Z \). Assume that \( g \) has singular support in \( W \in Z^p_n \). Then there are elements \( u_n, \ldots, u_0 \), with
\[ u_k \in \mathfrak{A}_{\log}^{2p-n+k-2}(X \times \square^k \setminus |W|, p)_0 \]
such that
\[ (6.7) \quad g'_n := g_n + du_n \in \mathfrak{A}_{\log}^{2p-n-1}(X \times \square^n \setminus |Z|, p)_0, \]
\[ (6.8) \quad g'_k := g_k + (-1)^{n-k-1}du_{k+1} + du_k \in \mathfrak{A}_{\log}^{2p-n+k-1}(X \times \square^k \setminus |Z|, p)_0, \]
for \( k = 0, \ldots, n-1 \). In consequence \( g' = (g'_n, \ldots, g'_0) \) is a Green form for \( Z \).
Proof. We first note that the cohomology of the complex $\mathfrak{A}_{|Z|}^n (X \times \Box^k, p)\circ$
fits in a long exact sequence
$$\rightarrow H^r(\mathfrak{A}_{|Z|}^n (X \times \Box^k, p)\circ) \rightarrow H^r_{\Delta_{\Box}} (X \times \Box^k, p) \rightarrow H^r_{\Delta_{\Box}|Z| \cap H_k} (H_k, p) \rightarrow \cdots$$
We now prove the existence of $u_n$. Since $Z \in Z^p(X, n)_0$ the cohomology class of $Z$ with supports
$$[Z] \in H^{2p}_{\Delta_{\Box}|Z|} (X \times \Box^n, p)$$
satisfies $[Z] \mid_{H_n} = 0$. Therefore it lifts to a class in
$$H^{2p}_{\Delta_{\Box}|Z|} (X \times \Box^n, p)\circ$$
This lifting is unique because $H^{2p-1}_{\Delta_{\Box}|Z| \cap H_n} (H_n, p) = 0$ by semipurity and the hypothesis that $Z$ intersects properly all the faces.

The pair $(0, g_n)$ represents the class of $Z$ in the cohomology of the complex
$$\mathfrak{A}_{|W|}^n (X \times \Box^n, p)\circ$$
and one readily checks that

$$d(g_n + (-1)^{n-k+1} \delta u_k) \in \mathfrak{A}'_{\log} (X \times \Box^k \setminus |Z|, p)\circ$$
Using semi-purity of Deligne-Beilinson cohomology and Proposition 5.6 one can prove that the map
$$H^r(\mathfrak{A}_{\log}^* (X \times \Box^k \setminus |Z|, p)) \rightarrow H^r(\mathfrak{A}_{\log}^* (X \times \Box^k \setminus |W|, p))$$
is injective for $r = 2p - 1$ and an isomorphism for $r < 2p - 1$. In particular, since $k \neq n - 1$, it is injective for $r = 2p - n + k$ and surjective for $r = 2p - n + k - 1$. Thus the existence of $u_k$ follows from Lemma 6.10 below.

The last statement is clear form the definition of the $g'_k$. □

Lemma 6.10. Let $A^* \rightarrow B^*$ be an injective morphism of complexes such that the map $H^r(A^*) \rightarrow H^r(B^*)$ is injective for $r = n$ and surjective for $r = n - 1$. Let $x \in B^{n-1}$ such that $dx \in A^n$. Then there is an element $y \in B^{n-2}$ such that $x + dy \in A^{n-1}$.

Proof. The element $dx \in A^n$ is closed in the complex $A^*$ and exact in the complex $B^*$. By the injectivity condition it is exact in $A^*$. So there is $z \in A^{n-1}$ with $dx = dz$. The element $x - z \in B^{n-1}$ is closed. By the surjectivity condition its cohomology class lies in the image of the cohomology of $A^*$. Thus there is an element $y \in B^{n-2}$ such that $x - z + dy \in A^{n-1}$ proving the lemma. □
Proposition 6.11. Let \( Z \in Z^n(X, n)_0 \) be a cycle and \( g = (g_n, \ldots, g_0) \) a Green form for \( Z \). Then there is an element
\[
u \in \mathcal{A}^{2p-2}(X \times \Box^n \setminus |Z|, p)_0
\]
such that \((g_n + du, g_{n-1} + \delta u, g_{n-2}, \ldots, g_0)\) is a basic Green form for \( Z \).

Proof. By simplicity we assume that \( Z \) is a prime cycle. The general case follows from this one by linearity. By [BG94b, Lemma 4.6] a basic Green form \( g' \) for \( Z \) on \( X \times (\mathbb{P}^1)^n \) exists. By [BG97, Proposition 5.5 (1)] there exist \( u \in \mathcal{A}^{2p-2}(X \times \Box^n \setminus |Z|, p) \) and \( \alpha \in \mathcal{A}^{2p-1}(X \times \Box^n, p) \) such that, on \( X \times \Box^n \),
\[
g_n + du = g' + \alpha.
\]
Since \( g' + \alpha \) is a basic Green form for \( Z \) on \( X \times \Box^n \), the proposition is proved. \( \square \)

Theorem 6.12. Let \( Z \in Z^n(X, n)_0 \) be a cycle (that is, \( \delta Z = 0 \)). Then a basic Green form for \( Z \) exists.

Proof. Follows from propositions 6.8, 6.9 and 6.11. \( \square \)

Given a Green form \( g_Z \), we next want to extract a Green current corresponding to it.

Proposition 6.13. Let \( Z \in Z^n(X, n)_0 \) be a cycle and \( g_Z = (g_n, \ldots, g_0) \) a loose Green form for \( Z \). Then the current
\[
[g_Z] := \sum_{i=0}^{n} (p_1)_*[g_i \cdot W_i]
\]
is a Green current for \( Z \).

Proof. For simplicity we compute on the Thom-Whitney simple. Strictly speaking, Lemma 4.22 does not apply here because \( g_Z \) is not a basic Green form. Nevertheless [BG94b, Proposition 3.3] still implies that all the forms \( g_i \cdot W_i \) are locally integrable. Let \( W \) be the singular support of \( g_Z \). Arguing as in Proposition 6.11, there is an element
\[
u \in \mathcal{A}^{2p-2}(X \times \Box^n \setminus |W|, p)_0
\]
such that \( g'_n := g_n + du \) is a basic Green form for \( Z \) on \( X \times \Box^n \). Write \( g'_{n-1} = g_{n-1} + \delta u \) and \( g'_k = g_k \) for \( k \leq 2 \). Since, using Lemma 4.22 and (5.7),
\[
d(p_1)_*[u \cdot W_n] = (p_1)_*[du \cdot W_n] + (p_1)_*[\delta u \cdot W_{n-1}],
\]
it is enough to prove the result for \( g'_Z = (g'_n, \ldots, g'_0) \). Thus from now on we assume that \( g_n \) is a basic Green form for \( Z \) on \( X \times \Box^n \).

By Lemma 4.22 again we have the residue relation
\[
d[g_n \cdot W_n] = -\delta Z \cdot W_n - [\delta g_n \cdot W_{n-1}],
\]
and for \( r < n \), the relations
\[
d[g_r \cdot W_r] - [d g_r \cdot W_r] = (-1)^{n-r+1}[\delta g_r \cdot W_{r-1}].
\]
Using the above residue relations, and the step relations defining \( g_Z \), we compute

\[
d[g_Z] = \sum_{i=0}^{n} d(p_1)_*[g_i \cdot W_i]
\]

\[
= \sum_{i=0}^{n} (p_1)_*[d[g_i \cdot W_i]]
\]

\[
= -(p_1)_*(\delta_Z \cdot W_n) + \sum_{i=1}^{n} (-1)^{n-i+1} (p_1)_*[\delta g_i \cdot W_{i-1}]
\]

\[
+ \sum_{i=0}^{n-1} (-1)^{n-i-1} (p_1)_*[\delta g_{i+1} \cdot W_i] + \omega(g_Z)
\]

\[
= \omega(g_Z) - (p_1)_*(\delta_Z \cdot W_n)
\]

\[
= \omega(g_Z) - \mathcal{P}(Z),
\]

since \( u \) and \( v \) have opposite signs, they cancel. This proves the statement.

\[ \square \]

**Proposition 6.14.** Let \( Z \in Z^p(X,n)_0 \) be a cycle, and \( \tilde{g}_Z \) a class of \( \mathfrak{A} \)-Green currents for \( Z \). Then there exists a basic \( \mathfrak{A} \)-Green form \( g_Z \) such that

\[ [g_Z] \in \tilde{g}_Z. \]

**Proof.** By Theorem 6.12, a basic Green form \( g'_Z = (g_n, \ldots, g_0) \) for \( Z \) exists. By Proposition 6.13 the current \( [g'_Z] \) is a Green current for \( Z \). By Lemma 6.4, there is a current \( u \in \mathfrak{A}^{2p-n-2}(X,p) \) and a form \( v \in \mathfrak{A}^{2p-n-1}(X,p) \) such that

\[ g_Z - [g'_Z] = v + du. \]

Writing \( g_Z = (g_n, \ldots, g_0 + v) \) we obtain the result. \[ \square \]

We next discuss when two Green forms give rise to the same Green current.

**Proposition 6.15.** Let \( Z \in Z^p(X,n)_0 \) be a cycle and let \( g = (g_n, \ldots, g_0) \) and \( g' = (g'_n, \ldots, g'_0) \) be two loose Green forms for the cycle \( Z \) with singular support contained in a subset \( W \in Z^p_p \). Then

\[ [g] = [g'] \]

if and only if there are elements \( u_n, \ldots, u_0 \), with

\[ u_k \in \mathfrak{A}^{2p-n-k-2}(X \times \square^k \setminus |W|_k, p)_0 \]

such that

\[ g'_n - g_n = du_n \]

\[ g'_k - g_k = (-1)^{n-k-1} \delta u_{k+1} + du_k, \quad k = 0, \ldots, n - 1. \]
If $g$ and $g'$ are Green forms for $Z$ then we can choose
\[ u_k \in \mathfrak{A}^{2p-n+k-2}_{\log}(X \times \Box^k \setminus |Z|, p) \].

**Proof.** Assume that such elements exist. Write
\[ u = \sum_{i=0}^n (p_i)_*[u_i \cdot W_i, \mathfrak{A}] \].

A computation similar to the one in the proof of Proposition 6.13 shows that
\[ [g'] - [g] = du \]
so $[g']$ and $[g]$ represent the same class of Green currents.

Conversely, assume that equation (6.10) is satisfied. Write $g'' = g' - g$. Then $g''$ is a loose Green form for the cycle zero ans singular support contained in $W$. Moreover $\omega(g'') = 0$. Applying Proposition 6.9 to $g''$ we find elements $u''_n, \ldots, u''_0$ such that
\[ \alpha_n := g_n - g'_n + du''_n \in \mathfrak{A}^{2p-1}_{\log}(X \times \Box^n, p) \]
\[ \alpha_k := g_k - g'_k + (-1)^{n-k-1} \delta u''_{k+1} + du''_k \in \mathfrak{A}^{2p-n+k-1}_{\log}(X \times \Box^k, p) \]
for $k = 0, \ldots, n-1$. Therefore
\[ \alpha := (0, \alpha_n, \ldots, \alpha_0) \in \tau \mathfrak{A}^{2p-n-1}(X, p) \].

The properties of Green form imply that $\alpha$ is closed. By Proposition 5.6 (1) and induction over $n-k$, we can find elements $u''_n, \ldots, u''_0$ such that
\[ 0 = \alpha_n + du''_n, \]
\[ 0 = \alpha_k + (-1)^{n-k-1} \delta u''_{k+1} + du''_k, \quad k = 1, \ldots, n-1. \]

So $\alpha$ is cohomologous to the element
\[ (0, \ldots, 0, \alpha_0 + (-1)^{n-1} \delta u''_1) \]
Now equation (6.10) implies that $\alpha_0 + (-1)^{n-1} \delta u''_1$ is a boundary in the complex $\mathfrak{A}^*(X, p)$, so there is a $u''_0$ such that
\[ \alpha_0 + (-1)^{n-1} \delta u''_1 + du''_0 = 0. \]
Writing $u_k = u'_k + u''_k$ we obtain the elements needed to prove the converse statement.

The last statement is the particular case when $W = |Z|$. \qed

### 6.3. Functorial properties of Green currents and Green forms

We next study the functorial properties of Green currents and Green forms. For the direct image it is better to work with Green currents, while for inverse images it is necessary to work with Green forms.

As in the classical case, for the direct image to exists at the level of Green currents, we need a smooth proper map.

**Proposition 6.16.** Let $f$ be a map between smooth projective complex varieties. Let $Z \in Z^p(X, n)$ be a cycle and $g_Z$ a Green current for $Z$. Then, if $f$ is smooth of $\omega(g_Z) = 0$, then the current $f_*g_Z$ is a Green current for the cycle $f_*Z$. 

Since \( Z \) with the previous notation, the current Proposition 6.17.

because \(| f_* \omega(g_Z)| = f_* \mathcal{P}(Z) = [f_* \omega(g_Z)] - \mathcal{P}(f_*Z)\)

proving the result in this case. If \( f \) is not smooth but \( \omega(g_Z) = 0 \), then

\[
df_*g_Z = f_*dg_Z = f_*[\omega(g_Z)] - f_*\mathcal{P}(Z) = -f_*\mathcal{P}(f_*Z)
\]

proving the result in the remaining case. \( \square \)

We next study the inverse image. The inverse image is only defined for classes of Green currents and not for Green currents. Let \( f: X \to Y \) be a morphism of smooth projective varieties and \( Z \in \mathcal{Z}_Y^n(X, n)_0 \) be a cycle that is in good position with respect to \( f \). Let \( \tilde{g}_Z \) be a class of Green currents for \( Z \). Choose a basic Green form \( g_Z = (g_n, \ldots, g_0) \) for \( Z \) such that \( [g_Z] \in \tilde{g}_Z \).

We write

\[
f^*g := ((f \times \text{Id}_{\mathbb{R}^n})^*g_n, \ldots, f^*g_0).
\]

For shorthand write \( g'_k = (f \times \text{Id}_{\mathbb{R}^k})^*g_k \). By the functoriality of all the operations involved and Lemma 4.22, it is easy to check that \( f^*g_Z \) satisfies the conditions

1. \( \delta f^*_Z + d[g'_n] = 0 \),
2. \( (-1)^{n-k+1}\delta g'_k + d[g'_{k-1}] = 0, \quad k = 2, \ldots, n. \)
3. \( (-1)^n\delta g'_1 + d[g'_0] = f^*(\omega(\tilde{g}_Z)). \)

Since \( Z \) is in good position with respect to \( f \), \( f^{-1}[Z]_k \) has codimension \( p \) and \( f^*g_Z \) a loose Green form for \( f^*Z \). In general it is not a Green form because \( [f^*Z]_k \) may not agree with \( f^{-1}[Z]_k \).

**Proposition 6.17.** With the previous notation, the current \([f^*g_Z]\) is a Green current for \( f^*Z \). Moreover, \( g'_Z \) is another choice of Green form for \( Z \) with \([g_Z] = [g'_Z] \), then

\[
[f^*g_Z] = [f^*g'_Z].
\]

That is, the class of \([f^*g_Z]\) does not depend on the choice of \( g_Z \).

**Proof.** The first statement is proved using the same argument as in the proof of Proposition 6.13. To prove the second statement Let \( g \) and \( g' \) be two Green forms for \( Z \) representing the same class of Green currents. By Proposition 6.15, there are elements \( u_k \in \mathfrak{g}_k^{2p-n+k-2}(X \times \square^k \setminus [Z]_k, p)_0 \), \( k = 0, \ldots, n \) satisfying the conditions in the proposition. Then the elements \( f^*u_k \) satisfy the same conditions for the Green forms \( f^*g_Z \) and \( f^*g'_Z \), so they represent the same class of Green currents.

From the last proposition, and for cycles \( Z \in \mathcal{Z}_Y^n(Y, n)_0 \) we can define an inverse image for a class of Green currents as

\[
f^*\tilde{g}_Z := [f^*g_Z],
\]

for any choice of green forms of logarithmic type \( g_Z \) of \( Z \), such that \([g_Z] \in \tilde{g}_Z \).
6.4. The $\ast$-product. In this section we will give the definition of the product of Green currents. Since we want a graded-commutative and associative product we will work in the Thom-Whitney complex. So in this section $\mathfrak{A} = \mathcal{D}_{TW}$.

We start by giving the definition of the $\ast$-product of Green forms and Green currents.

**Definition 6.18.** Let $Z \in \mathcal{Z}^p(X,n)_0$ and $W \in \mathcal{Z}^q(X,m)_0$ be cycles in the normalized complex, intersecting properly as in Definition 3.10. Let $g_Z$ be a Green current for the cycle $Z$ and let $g_W$ be a basic Green form for the cycle $W$ with components $g_W = (g'_m, \ldots, g'_0)$.

Let $\omega_Z = \omega(g_Z)$ and $\omega_W = \omega(g_W)$. We denote formally the product of currents

$$P(Z) \cdot [g_W] := \sum_{j=0}^m (p_1)_* (\delta Z \cdot W_n \cdot g'_j \cdot W_j),$$

where $\delta Z \cdot W_n \cdot g'_j \cdot W_j$ is seen as a current in $X \times (\mathbb{P}^1)^{n+m}$ as in Definition 4.20 and $p_1 : X \times (\mathbb{P}^1)^{n+m} \to X$ is the projection. Then the $\ast$-product of the Green current $g_Z$ and the Green form $g_W$ is the current defined as

$$g_Z \ast g_W := (-1)^n P(Z) \cdot [g_W] + g_Z \cdot \omega_W$$

**Proposition 6.19.** Let $Z$, $W$, $g_Z$ and $g_W$ be as in Definition 6.18. Then the current $g_Z \ast g_W$ satisfies

$$P(Z \cdot W) + d(g_Z \ast g_W) = [\omega_Z \cdot \omega_W].$$

Hence is a Green current for the product $Z \cdot W$.

**Proof.** We compute, using Lemma 4.22, the fact that $Z$ is a cycle, the properties defining Green forms and the properties of the Wang forms (5.7)

$$d(g_Z \ast g_W) = (-1)^n \sum_{j=0}^m (p_1)_* (d(\delta Z \cdot W_n \cdot g'_j \cdot W_j)) + d(g_Z \cdot \omega_W)$$

$$= (-1)^n \sum_{j=0}^m (p_1)_* (\delta_{ij} \cdot W_{n-1} \cdot g'_j \cdot W_j) - (p_1)_* (\delta_{ij} \cdot W_n \cdot W_{n+m})$$

$$+ \sum_{j=0}^{m-1} (p_1)_* (\delta Z \cdot W_n \cdot d g'_j \cdot W_j)$$

$$+ \sum_{j=1}^m (-1)^{m-j-1}(p_1)_* (\delta Z \cdot W_n \cdot \delta g'_j \cdot W_{j-1})$$

$$+ [\omega_Z \cdot \omega_W] - P(Z) \cdot \omega_W$$

$$= -P(Z \cdot W) + (p_1)_* (\delta Z \cdot W_n \cdot \omega_W) + [\omega_Z \cdot \omega_W] - P(Z) \cdot \omega_W$$

$$= -P(Z \cdot W) + [\omega_Z \cdot \omega_W].$$

$\square$
Proposition 6.20. If \( g'_Z \) is another Green current for \( Z \) and \( g'_W \) is another Green form for \( W \) with

\[
g_Z = \tilde{g}_Z, \quad [g_W]^\sim = [g'_W]^\sim,
\]

then

\[
(g_Z \ast g_W)^\sim = (g'_Z \ast g'_W)^\sim.
\]

**Proof.** Write \( g_Z - g'_Z = dc \). Then

\[
g_Z \ast g_W - g'_Z \ast g_W = (dc) \cdot \omega(g_W) = d(c \cdot \omega(g_W)).
\]

Thus \( (g_Z \ast g_W)^\sim = (g'_Z \ast g'_W)^\sim \). Applying Proposition 6.15 to \( g'_W \) and \( g'_W \)

we obtain a tuple \( u = u_m, \ldots, u_0 \). Define

\[
P(Z) \cdot [u] = \sum_{j=0}^{m} (\delta_Z \cdot W_n \cdot u_j \cdot W_j)
\]

then, using Lemma 4.22, one can verify that

\[
g'_Z \ast g'_W - g'_Z \ast g_W = d(P(Z) \cdot [u]).
\]

Thus \( (g'_Z \ast g'_W)^\sim = (g'_Z \ast g_W)^\sim \). □

In view of the previous proposition the following definition makes sense.

**Definition 6.21.** Let \( Z \in Z^p(X, n)_0 \) and \( W \in Z^q(X, m)_0 \) be cycles intersecting properly and \( \tilde{g}_Z \) and \( \tilde{g}_W \) classes of Green currents for \( Z \) and \( W \).

Choose any representative \( g_Z \) of \( \tilde{g}_Z \) and a Green form \( g_W \) with \( [g_W] \in \tilde{g}_W \). Then the \( \ast \)-product of \( \tilde{g}_Z \) and \( \tilde{g}_W \) is defined as

\[
\tilde{g}_Z \ast \tilde{g}_W = ((-1)^nP(Z) \cdot [g_W] + g_Z \cdot \omega(g_W))^\sim.
\]

**Theorem 6.22.** Let \( Z \in Z^p(X, n)_0 \), \( W \in Z^q(X, m)_0 \) and \( T \in Z^r(X, \ell)_0 \) cycles such that \( W \) intersects properly \( Z \) and \( T \), \( Z \) intersect properly \( W \cdot T \) and \( T \) intersects properly \( Z \cdot W \). Let \( \tilde{g}_Z \), \( \tilde{g}_W \) and \( \tilde{g}_T \) be classes of Green currents for \( Z \), \( W \) and \( T \) respectively. Then

1. \( \tilde{g}_Z \ast \tilde{g}_W = (-1)^{nm} \tilde{g}_W \ast \tilde{g}_Z \).
2. \( \tilde{g}_Z \ast (\tilde{g}_W \ast \tilde{g}_T) = (\tilde{g}_Z \ast \tilde{g}_W) \ast \tilde{g}_T \).

**Proof.** We start by proving the commutativity. Choose \( g_Z = (g_0, \ldots, g_0) \)

and \( g_W = (g'_m, \ldots, g'_0) \) basic Green forms for \( Z \) and \( W \) with \( g_Z := [g_Z] \in \tilde{g}_Z \)

and \( g_W := [g_W] \in \tilde{g}_W \). We define

\[
[g_Z] \cdot P(W) := \sum_{i=0}^{n} (p_1)_* (g_i \cdot W_i \cdot \delta_W \cdot W_m),
\]

and

\[
g_Z \ast' g_W := [g_Z] \cdot P(W) + (-1)^n \omega(g_Z) \cdot g_W.
\]

We also define the product of currents

\[
[g_Z] \cdot [g_W] = \sum_{i=0}^{n} \sum_{j=0}^{m} (p_1)_* (g_i \cdot W_i \cdot g'_j \cdot W_j).
\]

Then, using Lemma 4.22,

\[
d([g_Z] \cdot [g_W]) =
\]

\[
\omega_Z \cdot g_W - P(Z) \cdot [g_W] + (-1)^{n-1} g_Z \cdot \omega_Z + (-1)^n [g_Z] \cdot P(W)).
\]
Therefore
\begin{equation}
(g_Z \ast g_W)^\sim = (g_Z \ast' g_W)^\sim.
\end{equation}

So we are left to compare $g_Z \ast' g_W$ and in $(-1)^{nm}g_W \ast g_Z$. But inspecting all the integrals that appear in both terms, one realizes that they are equal except for a reordering of the variables of integration. Since the value of an integral does not depend on the name of the integration variables we deduce that

$$(g_Z \ast g_W)^\sim = (-1)^{nm}(g_W \ast g_Z)^\sim.$$ 

We next prove the associativity. We choose basic Green forms for the cycles as follows

\[g_Z = (g_n, \ldots, g_0), \quad g_Z := [g_Z] \in \tilde{g}_Z, \quad \omega_Z := \omega(\tilde{g}_Z),\]

\[g_W = (g'_n, \ldots, g'_0), \quad g_W := [g_W] \in \tilde{g}_W, \quad \omega_W := \omega(\tilde{g}_W),\]

\[g_T = (g''_n, \ldots, g''_0), \quad g_T := [g_T] \in \tilde{g}_T, \quad \omega_T := \omega(\tilde{g}_T).\]

We define the current

\[[g_Z] \cdot \mathcal{P}(W) : [g_T] = \sum_{i=0}^{n} \sum_{k=0}^{\ell} g_i \cdot W_i \cdot \gamma_W \cdot g_k \cdot W_k.\]

Then, using again Lemma 4.22,

\[d((-1)^m[g_Z] \cdot \mathcal{P}(W) : [g_T] + [g_Z] : [g_W] \cdot \omega_T) = (-1)^m (\omega_Z \cdot \mathcal{P}(W) : [g_T] - \mathcal{P}(Z \cdot W) : [g_T]) + (-1)^n (-[g_Z] \cdot \mathcal{P}(W) : \omega_T + [g_Z] : \mathcal{P}(W \cdot T)) + (\omega_Z \cdot [g_W] \cdot \omega_T - \mathcal{P}(Z) : [g_W] \cdot \omega_T) + (-1)^n (-[g_Z] \cdot \omega_W \cdot \omega_T + [g_Z] : \mathcal{P}(W) \cdot \omega_T) = (-1)^n (g_Z \ast' (g_W \ast g_T) - (g_Z \ast g_W) \ast g_T).\]

Applying equation (6.11) we obtain the second statement of the theorem. 

\[\square\]

The product of Green forms is compatible with inverse images and there is a projection formula that relates the product, the inverse image and the direct image. The next result can be verified formally thanks to Lemma 4.22 and the corresponding properties of forms and currents.

**Proposition 6.23.** Let $f : Y \to X$ be a morphism of complex projective manifolds, Let $Z \in Z^p(X, n)_0$ and $W \in Z^q(X, m)_0$ cycles in good position with respect to $f$ that intersect properly and such that $Z \cdot W$ is also in good position with respect to $f$ and $T \in Z^r(Y, \ell)_0$ a cycle that intersects properly $f^*Z$. Let $\tilde{g}_Z, \tilde{g}_W$ and $\tilde{g}_T$ be classes of Green currents for $Z, W$ and $T$ respectively. Then

1. $f^*(\tilde{g}_Z \ast \tilde{g}_W) = f^*\tilde{g}_Z \ast f^*\tilde{g}_W$;
2. $f_* (f^* \tilde{g}_Z \ast \tilde{g}_T) = \tilde{g}_Z \ast f_* \tilde{g}_T$. 

7. Higher arithmetic Chow groups

In this section we present a definition of higher arithmetic Chow groups for a smooth and projective variety $X$ of dimension $d$, defined over an arithmetic field $F$.

Adapting Deligne-Soulé proposal for higher arithmetic $K$-theory to higher arithmetic Chow groups, we can define them as the relative homology of the cycle class map between Bloch cycle complex and a complex that computes Deligne-Beilinson cohomology. We will denote by $\widehat{\text{CH}}^\ast(X, \ast)_{0}$ the groups obtained following this idea. They fit in a long exact sequence

$$\cdots \to H^{2p-n-1}_D(X, \mathbb{R}(p)) \to \widehat{\text{CH}}^p(X, n)_{0} \to \text{CH}^p(X, n) \to H^{2p-n}_D(X, \mathbb{R}(p)) \to \cdots$$

This method is followed by Goncharov [Gon05] and by Burgos and Feliu [BGF12].

Takeda [Tak05] has proposed a new definition of higher arithmetic $K$-groups that map surjectively over the higher $K$ groups. Translating his idea to higher arithmetic Chow groups, one sought to find groups $\widehat{\text{CH}}^\ast(X, \ast)$ that fit into an exact sequence of the form

$$\text{CH}^p(X, n+1) \to D^{2p-n-1}(X, p)/\text{Im } d_D \to \widehat{\text{CH}}^p(X, n) \to \text{CH}^p(X, n) \to 0,$$

where $D^n(X, \ast)$ is a complex that computes real Deligne-Beilinson cohomology (or absolute Hodge cohomology). For each $n, p$, the group $\widehat{\text{CH}}^p(X, n)_{0}$ would be recovered as a special subgroup of $\widehat{\text{CH}}^p(X, n)$. Clearly such a definition would depend on the choice of the complex $D^n(X, \ast)$. A definition along these lines was already proposed in Elisenda Feliu’s thesis ([Fel07], §3.10), where it was referred to as the modified higher arithmetic Chow groups.

Below we give a definition of such higher arithmetic Chow groups which is analogous to the one given by Gillet and Soulé.

7.1. Higher arithmetic Chow groups. Throughout this section, we will work over an arithmetic field. An arithmetic field is a triple $(F, \Sigma, F_{\infty})$, where $F$ is a field, $\Sigma$ is a non-empty set of complex embeddings $F \hookrightarrow \mathbb{C}$ of $F$ and $F_{\infty}$ is a conjugate-linear $\mathbb{C}$-algebra automorphism of $\mathbb{C}^\Sigma$ that leaves the image of $F$ under the diagonal immersion invariant. Primarily, we will use the notation $F$, and the triple will be understood. Typical examples of an arithmetic field are number fields, $\mathbb{R}$ and $\mathbb{C}$. Let $\mathfrak{A}$ be any of the complexes that appear in Example 4.17 and let $X$ be a smooth and projective variety of dimension $d$, defined over an arithmetic field $F$.

Real varieties. Since $X$ is defined over an arithmetic field $F$, we consider the associated complex space

$$X_{\mathbb{C}} := \bigsqcup_{\sigma} X_{\sigma} \in \Sigma; \quad \sigma: F \hookrightarrow \mathbb{C}.$$ 

Then $F_{\infty}$ induces an antilinear involution on $X_{\mathbb{C}}$ also denoted $F_{\infty}$. We will denote by $\sigma_{F_{\infty}}$, the involution of $\mathfrak{A}^n(X_{\mathbb{C}}, p)$ given, for any element $\eta \in \mathfrak{A}^n(X_{\mathbb{C}}, p)$, by

$$\sigma_{F_{\infty}}(\eta) = F_{\infty} \eta.$$ 

Then, the real Deligne-Beilinson cohomology of $X$ is defined as
\[ H^n_D(X, \mathbb{R}(p)) := H^n_D(X, \mathbb{R}(p))^{\sigma_{F,\infty}}, \]
where the superscript $\sigma_{F,\infty}$ means the fixed part under $\sigma_{F,\infty}$. The real Deligne cohomology of $X$ is the cohomology of the real complex
\[ A^n(X, p) := A^n(X, \mathbb{R}(p))^{\sigma_{F,\infty}}, \]
i.e. there is an isomorphism
\[ H^n_D(X, \mathbb{R}(p)) \cong H^n(A^*(X, p), d). \]
We will consider this setting for the rest of the article.

**Definition 7.1.** We define the group of *higher arithmetic cycles*, denoted $\hat{Z}_p(X, n, A)$, as the subgroup of $\mathbb{Z}_p(X, n, 0) \oplus A^{2p-n-1}(X, p)$ consisting of pairs $(Z, \tilde{g}_Z)$ such that $Z$ is a cycle and $\tilde{g}_Z$ is a class of $A$-Green current for $Z$.

We will frequently omit the symbol $\tilde{g}$ over a Green current in an arithmetic cycle as it is understood.

Let $\hat{Z}_p^{rat}(X, n, A)$ be the subgroup of $\hat{Z}_p(X, n, A)$ generated by elements of the form
\[ (\delta Z, -\tilde{g}_{\text{top}}(Z)), \text{ for } Z \in Z^p(X, n+1). \]

We define the *higher arithmetic Chow groups* to be the quotient
\[ \hat{CH}_p^p(X, n, A) := \hat{Z}_p(X, n, A)/\hat{Z}_p^{rat}(X, n, A). \]

Again when $A$ is understood we omit it from the notation.

**Remark 7.2.** Similarly, we can define $\hat{Z}_p^p(X, n, A)_00$ and $\hat{Z}_p^{rat}(X, n, A)_00$ using cycles in the refined normalized complex and obtain
\[ \hat{CH}_p^p(X, n, A) = \hat{Z}_p^p(X, n, A)_00/\hat{Z}_p^{rat}(X, n, A)_00. \]

An arithmetic field $F$ is a particular case of an arithmetic ring. Therefore we have at our disposal the arithmetic Chow groups $\hat{CH}_p^p(X)$ defined in [GS90]. We can recover these groups using the Deligne complex $\mathcal{D}$. We recall that $d$ is the dimension of $X$.

**Proposition 7.3.** For each $p \geq 0$, the assignment
\[ (Z, g_Z) \mapsto (Z, 2(2\pi i)^{d-p+1} g_Z) \]
defines an isomorphism
\[ \hat{CH}_p^p(X, 0, \mathcal{D}) \cong \hat{CH}_p^p(X). \]

**Proof.** One has to take care of the different normalization used here and in [GS90]. Let $Z \in Z^p(X)$ be a cycle of codimension $p$ and denote by $\delta_Z^{\text{top}}$ the current integration along $Z$ with the topologist convention. Then
\[ \delta_Z = \frac{1}{(2\pi i)^{d-p}} \delta_Z^{\text{top}}. \]
Recall the real operator $d^c = i/(4\pi)(\bar{\partial} - \partial)$. The differential
\[ d_{\mathcal{D}} : \mathcal{A}^{2p-1}(X, p) \to \mathcal{A}^{2p}(X, p) \]
is given by
\[ d_{\mathcal{D}} = -2\bar{\partial}\partial = 2(2\pi i)dd^c. \]
The condition for \( g \) to be a Green current for \( Z \) in the sense of loc. cit. is that

\[
ddf^c g + \delta_Z^{\text{top}} \text{ is a smooth form.}
\]

Since we are in the case \( n = 0 \), we have that \( \mathcal{P}(Z) = \delta_Z \). Therefore \( g \) is a Green current for \( Z \), in the sense of the present paper, if

\[
-2\partial \bar{\partial} g + \delta_Z \text{ is a smooth form.}
\]

Using (7.2) and (7.3) we deduce that \( g \) is a Green current for \( Z \) in the sense of this paper, if and only if \( 2(2\pi i)^{d-p+1}g \) is a Green current in the sense of loc. cit. Therefore the groups of arithmetic cycles are the same in both cases.

Now we have to check that the notion of rational equivalence in both theories are equivalent. The main idea is to follow the proof of [Ful84, Proposition 1.6].

Observe that, if \( Z \in Z^p(X,1) \) is degenerate, then \( \delta Z = 0 \) and \( \mathcal{P}(Z) = 0 \). Thus in order to discuss arithmetic rational equivalence at the level of \( \tilde{Z}^p(X,0,\mathfrak{m}) \) we can use the group \( Z^p(X,1) \) instead of \( Z^p(X,1)_0 \).

In [GS90] the group of arithmetic cycles rationally equivalent to zero is generated by cycles of the form \( (\text{div} f, (2\pi i)^{d-p+1}i_*[-\log |f|^2]) \), where \( i: W \to X \) is the inclusion of a codimension \( p-1 \) subvariety and \( f \in K(W)^X \) is a nonzero rational function. The factor \( (2\pi i)^{d-p+1} \) comes from (4.1).

Consider \( \Gamma_f \subset W \times \mathbb{P}^1 \), the graph of \( f \). By sending it to \( X \times \mathbb{P}^1 \) and restricting to \( X \times \square^1 \) we obtain a subvariety \( Z_f \subset X \times \square^1 \). Since it is defined as the graph of a nonzero rational function, it intersect properly the faces of \( X \times \square^1 \) and determines a pre-cycle also denoted by \( Z_f \).

Now we compute, using (5.8) and (5.9), and the notation therein,

\[
-\mathcal{P}(Z_f) = \frac{1}{2}(p_1)_*i_*[\log t \log t] = \frac{1}{2}i_*[\log |f|^2].
\]

while

\[
\delta Z_f = Z_f \cap X \times \infty - Z_f \cap X \times 0 = - \text{div}(f).
\]

In conclusion, the assignment (7.1) sends \((-\delta Z_f, \mathcal{P}(Z_f))\) to the arithmetic cycle \((\text{div} f, (2\pi i)^{d-p+1}i_*[-\log |f|^2])\) and every arithmetic cycle rationally equivalent to zero in loc. cit. is also rationally equivalent to zero here.

Conversely, let \( Z \) be an irreducible subvariety of \( X \times \square^1 \) intersecting properly all the faces. The restriction of the projection \( p_2: X \times \square^1 \to \square^1 \) determines a nonzero rational function \( f' \) on \( Z \). Let \( W \subset X \) be the closure of the image of \( Z \) by the first projection and let \( p: Z \to W \) be the induced map. If \( \text{codim} W > p-1 \) then one verifies easily that \( \delta Z = 0 \) and \( \mathcal{P}(Z) = 0 \).

If \( \text{codim} W = p-1 \), write \( f = N_{K(Z)/K(W)}f' \), where \( N_{K(Z)/K(W)} \) is the norm of the field extension \( K(Z)/K(W) \). Then, by the discussion before [Ful84, Proposition 1.6]

\[
\delta Z = - \text{div} f.
\]

Let \( i': Z \to X \times \square^1 \) and \( i: W \to X \) denote the inclusions. Then, using the definition of the map \( \mathcal{P} \) and of the norm of a field extension,

\[
\mathcal{P}(Z) = p_1^*i'_*[-\log |f'|^2] = \frac{1}{2}i_*[-\log |f|^2].
\]
In consequence the notions of rational equivalence agree and we obtain the result.

**Notation 7.4.** Let $F$ be an arithmetic field and $X$ a smooth projective variety of dimension $d$ defined over $F$. Let $\mathcal{A}$ be any of the complexes in Diagram (4.9). When $\mathcal{A}$ has been fixed, we will often drop the suffix/prefix $\mathcal{A}$ from the definition. For example, the higher arithmetic Chow groups for a fixed complex $\mathcal{A}$ will simply be denoted by $\hat{\text{CH}}^p(X, \bullet)$, and the Goncharov regulator by $P$.

**Remark 7.5.** Let

\[
\mathcal{B} \xrightarrow{\varphi} \mathcal{A}
\]

be two of the complexes of Example 4.17 with the corresponding morphisms. Then, for each $n, p$ there are induced morphisms

\[
\hat{\text{CH}}^p(X, n, \mathcal{B}) \xrightarrow{\varphi} \hat{\text{CH}}^p(X, n, \mathcal{A}).
\]

**7.2. Exact sequences.** Like the usual arithmetic Chow groups, the higher arithmetic version comes equipped with the following natural maps:

\[
\tilde{\mathcal{A}}^{2p-n-1} \xrightarrow{\omega} \hat{\text{CH}}^p(X, n, \mathcal{A}), \quad b \mapsto [(0, [b])],
\]

\[
\hat{\text{CH}}^p(X, n, \mathcal{A}) \xrightarrow{\omega} \hat{\text{CH}}^p(X, n), \quad [(Z, g)] \mapsto [Z],
\]

\[
\hat{\text{CH}}^p(X, n, \mathcal{A}) \xrightarrow{\omega} \tilde{\mathcal{A}}^{2p-n}, \quad [(Z, gZ)] \mapsto \omega(gZ).
\]

We will denote

\[
\hat{\text{CH}}^p(X, n, \mathcal{A})^0 = \ker(\omega) \subset \hat{\text{CH}}^p(X, n, \mathcal{A}),
\]

and write

\[
\hat{\text{CH}}^p(X, n, \mathcal{A})^0 \xrightarrow{\omega} \hat{\text{CH}}^p(X, n, \mathcal{A}), \quad [(Z, gZ)] \mapsto [(Z, gZ)],
\]

\[
\tilde{\mathcal{A}}^{2p-n-1} \xrightarrow{\omega} \hat{\text{CH}}^p(X, n, \mathcal{A})^0, \quad b \mapsto [(0, b)].
\]

**Remark 7.6.** In a later sequel, we will show that for $n > 0$ the groups $\hat{\text{CH}}^p(X, n, \mathcal{A})^0$ are isomorphic to the higher arithmetic Chow groups defined in [BGF12]. Hence they don’t depend on the complex $\mathcal{A}$.

Let $\rho: \text{CH}^p(X, n) \to H^{2p-n}_D(X, \mathbb{R}(p))$ denote the cubical Goncharov regulator. Recall that it is represented at the level of complexes by the map $P$, and agrees, after composing with the isomorphism between $K$-theory and higher Chow groups with Beilinson’s regulator.

Denote also by $\rho: \text{CH}^p(X, n) \to \tilde{\mathcal{A}}^{2p-n}$ the composition

\[
\text{CH}^p(X, n) \xrightarrow{P} H^{2p-n}_D(X, \mathbb{R}(p)) \xrightarrow{\omega} \tilde{\mathcal{A}}^{2p-n}.
\]

This map is given by $[Z] \mapsto \tilde{\omega}(gZ)$, for a choice of Green current $gZ$ for $Z$, because $P(Z)$ is cohomologous to $\omega(gZ)$.

Analogous to the exact sequences that arise for arithmetic Chow groups, we have the following exact sequences:
Theorem 7.7. The following sequences are exact

\[ \begin{align*}
(1) & \quad \text{CH}^p(X, n + 1) \xrightarrow{\rho} \tilde{\mathbb{A}}^{2p-n-1} \xrightarrow{\alpha} \tilde{\text{CH}}^p(X, n, \mathfrak{A}) \xrightarrow{\zeta} \text{CH}^p(X, n) \to 0. \\
(2) & \quad \text{CH}^p(X, n + 1) \xrightarrow{\rho} H^2_{\mathbb{D}}(X, \mathbb{R}(p)) \xrightarrow{\alpha} \tilde{\text{CH}}^p(X, n, \mathfrak{A}) \xrightarrow{\zeta} \text{CH}^p(X, n) + Z\tilde{\mathbb{A}}^{2p-n} \to H^2_{\mathbb{D}}(X, \mathbb{R}(p)) \to 0. \\
(3) & \quad \text{CH}^p(X, n + 1) \xrightarrow{\rho} \tilde{\mathbb{A}}^{2p-n-1} \xrightarrow{\alpha} \tilde{\text{CH}}^p(X, n, \mathfrak{A})^0 \xrightarrow{\zeta} \text{CH}^p(X, n)^0 \to 0,
\end{align*} \]

where for (3), \( \text{CH}^p(X, n)^0 = \ker(\rho) \).

Proof. Since the ideas involved in all the three sequences are similar in nature, we show the proof for (1), leaving the rest to the reader. We first show assumed to be projective, the morphism \( f \).

\[
\begin{align*}
\text{Theorem 7.8.} \quad & \text{Direct and inverse images.}\end{align*}
\]

We fix an arithmetic field \( F \) projective varieties over \( F \) of relative dimension \( e \). Since \( X \) and \( Y \) are assumed to be projective, the morphism \( f \) is proper.

(1) Let \( Z \in \text{CH}^p(X, n)^0 \) be a cycle, \( g Z \) a Green current for \( Z \) and \( \omega Z = \omega(g Z) \). If either \( \omega Z = 0 \) or the map \( f \) is smooth, then \( f_* g Z \) is a Green current for the cycle \( f_* Z \).

(2) There is a morphism

\[
f_* : \tilde{\text{CH}}^p(X, n)^0 \to \tilde{\text{CH}}^{p-e}(Y, n)^0,
\]

given by \( (Z, g Z) \mapsto (f_* Z, f_* g Z) \).
If $f$ is smooth, there is a morphism
\[ f_* : \widehat{\text{CH}}^p(X, n) \to \widehat{\text{CH}}^{p_e}(Y, n), \]
given by $(Z, g_Z) \mapsto (f_* Z, f_* g_Z)$.

The direct image of higher arithmetic Chow groups is compatible with the direct image of higher Chow groups, with the direct image in Deligne-Beilinson cohomology and, when $f$ is smooth with the direct image of differential forms. In particular, all the exact sequences of Theorem 7.7 give rise to commutative diagrams.

The direct image is functorial, that is, if $g : Y \to Z$ is another morphism then $(f \circ g)_* = f_* \circ g_*$. 

Proof. Note that (1) is essentially the content of Proposition 6.16. Again by the compatibility of direct images with the differential of currents and Proposition 5.5, we see that the maps in (2) and 3 are well defined. The use of $(f_* Z, f_* g_Z)$, instead of $(f_* Z, f_* \tilde{g}_Z)$ is justified in this definition from the commutativity relation $f_* du = d(f_* u)$, for any current $u$.

The compatibility with the other direct images and the functoriality can be verified easily. □

7.3.2. Inverse images. The next result is the existence of inverse images. As in the previous section the field $F$ and the complex $\mathfrak{A}$ are fixed.

Let $f : X \to Y$ be a morphism of smooth projective varieties over $F$. The higher arithmetic Chow groups can be defined using cycles in good position with respect to $f$. Denote by $\hat{Z}_f^p(Y, n, \mathfrak{A})$ the subgroup of $\hat{Z}_f^p(Y, n, \mathfrak{A})$ consisting of pairs $(Z, \tilde{g}_Z)$ with $Z \in Z_f^p(Y, n)_{0}$, and by $\hat{Z}_{f, \text{rat}}^p(Y, n, \mathfrak{A})$ the subgroup of $\hat{Z}_{f, \text{rat}}^p(Y, n, \mathfrak{A})$ consisting of elements of the form
\[ (\delta Z, -\tilde{P}_\mathfrak{A}(Z)), \text{ for } Z \in Z_f^p(Y, n + 1)_{0}. \]

Lemma 7.9. The natural map
\[ \hat{Z}_f^p(Y, n, \mathfrak{A}) \to \hat{\text{CH}}^p(Y, n, \mathfrak{A}) \]
is an isomorphism.

Proof. Repeating the proof of Theorem 7.7 (1) and using Corollary 3.9 one sees that there is a commutative diagram with exact rows
\[
\begin{array}{cccccc}
\text{CH}_f^p(X, n + 1) & \rightarrow & \hat{\mathfrak{A}}^{2p - n - 1} & \rightarrow & \hat{Z}_f^p(Y, n, \mathfrak{A}) & \rightarrow & \text{CH}_f^p(X, n) & \rightarrow & 0 \\
\| & & & & & & | & & |
\end{array}
\]
\[
\begin{array}{cccccc}
\text{CH}_f^p(X, n + 1) & \rightarrow & \hat{\mathfrak{A}}^{2p - n - 1} & \rightarrow & \hat{\text{CH}}^p(X, n, \mathfrak{A}) & \rightarrow & \text{CH}_f^p(X, n) & \rightarrow & 0 \\
\| & & & & & & | & & |
\end{array}
\]
from which the lemma follows. □

We use the previous lemma to define inverse images. Let $(Z, \tilde{g}_Z) \in \hat{Z}_f^p(Y, n, \mathfrak{A})$. By Proposition 6.17, the pair $(f^* Z, f^* \tilde{g}_Z)$ is an arithmetic cycle on $X$. 

Theorem 7.10. Let \( f: X \to Y \) be a morphism of smooth projective varieties over \( F \).

1. The previous construction induces a morphism
   \[ f^*: \widehat{\text{CH}}^p(Y, n) \to \widehat{\text{CH}}^p(X, n) \]
   that sends \( \widehat{\text{CH}}^p(Y, n)^0 \) to \( \widehat{\text{CH}}^p(X, n)^0 \).

2. The inverse image of higher arithmetic Chow groups is compatible with the inverse image of higher Chow groups, with the inverse image in Deligne-Beilinson cohomology and with the inverse image of differential forms. In particular, all the exact sequences of Theorem 7.7 give rise to commutative diagrams.

3. The inverse image is functorial, that is, if \( g: Y \to Z \) is another morphism then \( (f \circ g)^* = g^* \circ f^* \).

Proof. Let \( \alpha \in \widehat{\text{CH}}^p(Y, n) \). From Lemma 7.9, we can write \( \alpha = [(Z, \tilde{g}Z)] \), with \( Z \in Z^p(Y, n)_0 \). Then \( f^*(\alpha) = [(f^*Z, f^*\tilde{g}Z)] \). In view of Lemma 7.9, to see that \( f^* \) is well defined as a map between higher arithmetic Chow groups, we need to show that, for any precycle \( Z' \in Z^p(Y, n+1)_0 \), the relation
   \[ f^*(-\overline{P}(Z')) = -\overline{P}(f^*Z') \]
holds, where the inverse image in the left hand side is the inverse image of classes of Green currents. To compute the inverse image on the left side we need a basic Green form for \( \delta(Z') \) representing the class \( -\overline{P}(Z') \).

Since \( Z' \in Z^p(Y, n+1)_0 \), its class in \( \text{CH}^p(Y \times \square^{n+1}) \) is zero, because, \((\delta^1)^*Z' = 0 \) and \((\delta^1)^* \) induces an isomorphism between \( \text{CH}^p(Y \times \square^n) \) and \( \text{CH}^p(Y \times \square^n) \). Therefore there exists a basic Green form \( g_{n+1} \) for \( Z' \) on \( Y \times \square^{n+1} \) (Definition 4.19) such that \( dg_{n+1} = 0 \). Then
   \[ (7.4) \quad g_{\delta(Z')} = (\delta g_{n+1}, 0, 0, \cdots, 0) \]
is a basic Green form for \( \delta(Z') \). Hence, from Proposition 6.13 we know that the corresponding current is a Green current for \( \delta(Z') \). We compute, using Lemma 4.22
   \[ d((p_1)_*[g_{n+1} \cdot W_{n+1}]) = -(p_1)_*[\delta Z' \cdot W_{n+1}] = -(p_1)_*[\delta g_{n+1} \cdot W_n] = -P(Z') - [g_{\delta(Z')}], \]
which implies that \( g_{\delta(Z')} \) represents the class \( -\overline{P}(Z') \). So \( f^*(-\overline{P}(Z')) = [f^*g_{\delta(Z')}^-] \) and \( f^* \) is well defined. The compatibility with other inverse images and the functoriality are left to the reader. □

7.4. Product structure. The intersection theory of higher Chow groups and the *-product of Green currents of higher algebraic cycles described in the last section, combines to give an intersection theory at the level of higher arithmetic Chow groups. To this end we first use Corollary 3.13 to see that the higher arithmetic Chow groups can be represented using cycles intersecting properly a given cycle.

Let \( X \) be a smooth projective variety over \( F \) and \( W \in Z^q(X, m) \) a precycle. Denote by \( \widehat{Z}_W^p(X, n, \mathfrak{A}) \) the subgroup of \( \widehat{Z}_W^p(X, n, \mathfrak{A}) \) consisting of
pairs \((Z, \tilde{g}_Z)\) with \(Z \in Z^p_W(X, n)\), and by \(\hat{Z}^p_{W, \text{rat}}(X, n, \mathfrak{A})\) the subgroup of \(\hat{Z}^p_{\text{rat}}(X, n, \mathfrak{A})\) consisting of elements of the form
\[
(\delta Z, -\tilde{P}_A(Z)), \quad \text{for} \quad Z \in Z^p_W(Y, n + 1).
\]

The following result is proved as Lemma 7.9.

**Lemma 7.11.** The natural map
\[
\frac{\hat{Z}^p_W(X, n, \mathfrak{A})}{\hat{Z}^p_{W, \text{rat}}(X, n, \mathfrak{A})} \to \hat{CH}^p(X, n, \mathfrak{A})
\]
is an isomorphism.

We now fix the complex \(\mathfrak{D}_{TW}\) and denote \(\hat{CH}^p(X, n) = \hat{CH}^p(X, n, \mathfrak{D}_{TW})\).

**Theorem 7.12.** There is a pairing
\[
\hat{CH}^p(X, n) \otimes \hat{CH}^q(X, m) \to \hat{CH}^{p+q}(X, n + m),
\]
which is associative and graded commutative with respect to \(n\), and that satisfies the following properties:

1. The maps
   \[
   \zeta: \bigoplus_{p, n \geq 0} \hat{CH}^p(X, n) \to \bigoplus_{p, n \geq 0} CH^p(X, n),
   \]
   and
   \[
   \omega: \bigoplus_{p, n \geq 0} \hat{CH}^p(X, n) \to \bigoplus_{p, n \geq 0} Z^{2p-n} \mathfrak{D}_{TW}^p(X, p),
   \]
   are multiplicative.

2. For a morphism \(f: X \to Y\), the pullback
   \[
f^*: \hat{CH}^p(Y, n) \to \hat{CH}^p(X, n)
   \]
is multiplicative, i.e., given \(\alpha \in \hat{CH}^p(Y, n)\) and \(\beta \in \hat{CH}^q(Y, m)\), we have
   \[
f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta).
   \]
   Further if \(f\) is smooth and proper, the pushforward morphism satisfies the projection formula: Let \(\alpha \in \hat{CH}^p(Y, n)\) and \(\beta \in \hat{CH}^q(X, m)\). Then
   \[
f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta).
   \]

**Proof.** Let \(\alpha \in \hat{CH}^p(X, n)\) and \(\beta \in \hat{CH}^q(X, m)\) be two elements in the higher arithmetic Chow groups. Choose a representative \((W, \tilde{g}_W)\) of \(\beta\). From Lemma 7.11, \(\alpha\) can be represented as \((Z, \tilde{g}_Z)\) with \(Z\) intersecting \(W\) properly as in Definition 3.10. We define
\[
\alpha \cdot \beta := [(Z \cdot W, \tilde{g}_Z \ast \tilde{g}_W)].
\]
We have to show that this product does not depend on the chosen representatives.
Lemma 7.13. Let \((W, \tilde{g}_W)\) as before. Let \(Z' \in Z^p_W(X, n+1)_0\) be a pre-cycle. Then
\[
(-1)^{nm} \tilde{g}_W * (\tilde{\mathcal{P}}(Z')) = -\tilde{\mathcal{P}}(Z') * \tilde{g}_W = -\tilde{\mathcal{P}}(Z' \cdot W),
\]
where the \(*\)-products are defined using that \(-\mathcal{P}(Z')\) is a Green current for \(\delta Z'\).

Proof of the lemma. By Theorem 6.22 (1), \(\tilde{\mathcal{P}}(Z') * \tilde{g}_W = (-1)^{nm} \tilde{g}_W * \tilde{\mathcal{P}}(Z')\), so we only need to compute \(\tilde{g}_W * \tilde{\mathcal{P}}(Z')\). Let \(g_{\delta(Z')}\) be as in (7.4). Since \(W\) and \(\delta(Z')\) intersect properly, we compute using Lemma 4.22, the fact that \(W\) is a cycle and that \(dg_{n+1} = 0\),
\[
d(p_1)_*(\delta_W \cdot W_m \cdot g_{n+1} \cdot W_{n+1})
= (-1)^{m+1} (p_1)_* \left( (\delta_W \cdot Z_m) \cdot W_{m+n} + (\delta_W \cdot W_m \cdot g_{n+1} \cdot W_n) \right)
= (-1)^m (-\mathcal{P}(W \cdot Z') - g_W * g_{\delta(Z')}).
\]
This shows that
\[
\tilde{g}_W * [g_{\delta(Z')}'^*] = -\tilde{\mathcal{P}}(W \cdot Z').
\]
Since the integrals defining \(\mathcal{P}(W \cdot Z')\) and \((-1)^{nm} \mathcal{P}(Z' \cdot W)\) are the same except for the ordering of the variables, we obtain the lemma. \(\square\)

Thanks to Lemma 7.13 the product is well defined. Since the product of higher cycles intersecting properly is associative, by (2) of Theorem 6.22 we deduce that, if \(Z, W \) and \(T\) are as in Theorem 6.22,
\[
((Z, g_Z) \cdot (W, g_W)) \cdot (T, g_T) = ((Z \cdot W) \cdot T, (g_Z \cdot g_W) * g_T)
= (Z \cdot (W \cdot T), g_Z \cdot (g_W * g_T)) = (Z, g_Z) \cdot ((W, g_W) \cdot (T, g_T)).
\]

We show the graded commutativity in more details. Let \(\alpha\) and \(\beta\) be as above, with representatives \([[Z, g_Z]]\) and \([[W, g_W]]\) respectively, such that \(Z \in Z^p(X, n)_{00}\) and \(W \in Z^q(X, m)_{00}\) intersect properly. Let \(H_{n,m}\), given in (3.9), be the homotopy that proves the graded commutativity of the product in the higher Chow groups. Then, using (1) of Theorem 6.22 and (3.10),
\[
\alpha \cdot \beta - (-1)^{nm} \beta \cdot \alpha = [[Z \cdot W, g_Z * g_W]] - ((-1)^{nm}) [[W \cdot Z, g_W * g_Z]]
= [[Z \cdot W - (-1)^{nm} W \cdot Z, g_Z * g_W - (-1)^{nm} g_W * g_Z]]
= [[(\delta(H_{n,m}(Z \cdot W)), dv)]]
\]
If we show that \(\mathcal{P}(H_{n,m}(Z \cdot W)) = 0\), then
\[
[[(\delta(H_{n,m}(Z \cdot W)), dv)]
= [[(\delta(H_{n,m}(Z \cdot W)), -\mathcal{P}(H_{n,m}(Z \cdot W))) + [(0, dv)] = 0,
\]
and the graded commutativity follows.

Lemma 7.14. The equation
\[
\mathcal{P}(H_{n,m}(Z \cdot W)) = 0
\]
holds.
Proof of the lemma. In view of the definition (3.9) of $H_{n,m}$, it is enough to show that given a pre-cycle $T \in Z^n(X, n)_0$, and $h_n^*$ the morphism given by (3.8), we have $\mathcal{P}(h_n^*(T)) = 0.$

For $n$ fixed, denote by $p_1 : X \times \square^n \to X$ and $p_1' : X \times \square^{n+1} \to X$ the first projection. Then

$$\mathcal{P}(h_n^*(T)) = (p)_* \left( \delta_{h_n^*(T)} \cdot W_{n+1} \right).$$

Since $p'_1 = p_1 \circ h_n$, we rewrite

$$\mathcal{P}(h_n^*(T)) = (p_1)_* h_n^* \left( \delta_{h_n^*(T)} \cdot W_{n+1} \right).$$

Let

1. $T' = h_n^{-1}(T) \subset X \times \square^{n+1}$.
2. $T' \to T'$ a resolution of singularities of $T'$.
3. $\bar{T}$ a smooth compactification of $T$,

such that there is a map

$$\bar{\tau}_n : \bar{T} \to X \times (\mathbb{P}^1)^n,$$

extending $h_n$. We write $\eta_n : \bar{T} \to X \times \square^{n+1}$. Then, by definition,

$$h_n^* \left( \delta_{h_n^*(T)} \cdot W_{n+1} \right) = \bar{T}_{n*}[\eta_n^* W_{n+1}].$$

So it is enough to prove that $\bar{T}_{n*}[\eta_n^* W_{n+1}] = 0$. Let $\omega$ be an arbitrary smooth test form in $X \times (\mathbb{P}^1)^n$. We have to show that

$$\bar{T}_{n*}[\eta_n^* W_{n+1}](\omega) = \int_{\bar{T}} \eta_n^* W_{n+1} \cdot \bar{T}_n \omega = 0.$$

The right hand side is an improper integral that can be computed in an open dense subset of $\bar{T}$.

There are open dense subsets $U_T \subset T$ and $W_T \subset \bar{T}$, such that the induced map

$$\bar{T}_n : W_T \to U_T,$$

is a fibration. Locally in analytic topology, one can write $U_T = \cup_i U_i, T$, such that

$$\bar{T}_{n*}^{-1} (U_i, T) \cap W_T = U_i, T \times C.$$

Here $C$ is the fibre, a smooth curve inside $\square^2$ corresponding to the first and the last coordinate of $\square^{n+1}$. Recall that

$$W_{n+1} \cdot \bar{T}_n \omega = \pi_1^* \lambda \cdot \pi_2^* \lambda \cdots \pi_{n+1}^* \lambda \cdot \bar{T}_n \omega,$$

where $\pi_i : X \times \square^{n+1} \to \square$ is the projection to the $i$-th coordinate, and

$$\lambda = -\frac{1}{2} \left( (\varepsilon + 1) \otimes \frac{dt}{t} + (\varepsilon - 1) \otimes \frac{dt}{t} + d\varepsilon \otimes \log t \right),$$

were $t$ is the coordinate of $\square$. From the definition of $h_n$, and the fact that it is locally a fibration, $\{ \pi_i^* \lambda \}_{i=2}^n$ and $\bar{T}_n \omega$ do not depend on the coordinate of $C$. Thus we can write

$$\bar{T}_{n*}[\eta^*_n W_{n+1} \cdot \bar{T}_n \omega]_{h_{n*}} = \pm (\pi_2^* \lambda \cdots \pi_n^* \lambda \cdot \omega)_{h_{n*}} \int_{C} \pi_1^* \lambda \cdot \pi_{n+1}^* \lambda.$$

Let $\bar{C} \subset (\mathbb{P}^1)^2$ be the closure of $C$ inside $((\mathbb{P}^1))^2$, and $p : \bar{C} \to \text{Spec}(F)$ be the structural morphism. Then, the integral on the right hand side is the same as $\mathcal{P}(C) = p_* [W_{2|C}]$. Thus Lemma 7.14 follows from Lemma 7.15 below. □
Lemma 7.15. Let \( C \subset \Box^2 \) be a curve that intersects properly all the faces of \( \Box^2 \) and \( \overline{C} \subset (\mathbb{P}^1)^2 \) its closure. Let \( p: \overline{C} \to \text{Spec } F \) be the structural morphism. Then
\[
p_*[W_2 |c] = 0.
\]

Proof. Let \( x, y \) be the coordinates of \( \Box^2 \). Then, the part of type \((1, 1)\) of \( W_2 \) is given by
\[
(W_2)^{(1,1)} = \frac{1}{4}(e^2 - 1) \otimes \left( \frac{dx}{x} \wedge \frac{dy}{y} - \frac{dy}{y} \wedge \frac{d\bar{x}}{x} \right)
\]
\[
= \frac{1}{4}(e^2 - 1) \otimes d \left( \log x \bar{x} \left( \frac{dy}{y} + \frac{d\bar{y}}{y} \right) \right).
\]
The coordinates \( x \) and \( y \) restricted to \( C \) give rational functions on \( C \) that we denote also by \( x \) and \( y \). The fact that \( C \) intersects properly the faces of \( \Box^2 \) implies that \( \text{div } x \) and \( \text{div } y \) are disjoint. Denote by \( f = \log x \bar{x} \). As is customary, if \( \text{div}(y) = \sum n_p p \), then we write
\[
f(\text{div}(y)) = \sum n_p f(p).
\]
Since
\[
\frac{1}{2\pi i} \oint_C d \left( f \frac{dy}{y} \right) = -f(\text{div } y), \quad \frac{1}{2\pi i} \oint_C d \left( f \frac{d\bar{y}}{y} \right) = f(\text{div } y),
\]
we obtain the lemma. \( \Box \)

We have proved that the product is graded commutative. The compatibility of the product with the morphisms \( \zeta \) and \( \omega \) is a direct computation.

We leave to the reader to check that the product is compatible with inverse images and the projection formula. \( \Box \)

Remark 7.16. Following Remark 7.5, let \( \mathcal{B} \) be another of the complexes of Example 4.17 and let
\[
\widehat{\text{CH}}^p(X, n, \mathcal{B}) \xrightarrow{\varphi} \text{CH}^p(X, n, \mathfrak{D}_{\text{TW}}) \cdot \widehat{\text{CH}}^p(X, n, \mathcal{B})
\]
be the morphisms of Remark 7.5. We can define a product in \( \widehat{\text{CH}}^*(X, *, \mathcal{B}) \) by the rule
\[
\alpha \cdot \beta = \psi(\varphi(\alpha) \cdot \varphi(\beta)).
\]
This product is still graded commutative, but in general, it is not associative. Using \( \mathcal{B} = \mathfrak{D} \) and \( n = 0 \) we recover the original arithmetic intersection product of [GS90] through Proposition 7.3.

7.5. Height pairing of cycles with trivial real regulator. In this subsection, we assume that the arithmetic field \( F \) is a number field.

We denote by \( Z^*(X, \bullet)_0 \) the subgroup of cycles whose image by the real regulator to Deligne cohomology is zero. That is
\[
Z^*(X, \bullet)_0 = \{ Z \in Z^*(X, \bullet)_0 | \delta(Z) = 0, \mathcal{P}(Z) \text{ is a boundary} \}
\]
if \( Z \in Z^p(X, n)_0 \), since \( \mathcal{P}(Z) \) is a boundary, there exists a current \( g_Z \) such that \( \mathcal{P}(Z) + d(g_Z) = 0 \). This current is a Green current for \( Z \). Then \( [(Z, g_Z)] \in \text{CH}^p(X, n)_0 \), since \( \omega(Z, g_Z) = 0 \).
For two elements \( Z \in Z^p(X, n)_0^0 \) and \( W \in Z^q(X, m)_0^0 \), we define
\[
(7.5) \quad Z \bullet W := [(Z, g_Z)] \cdot [(W, g_W)],
\]
where \( g_Z \) and \( g_W \) are Green currents satisfying
\[
(7.6) \quad \mathcal{P}(Z) + d(g_Z) = 0, \quad \mathcal{P}(W) + d(g_W) = 0.
\]
If \( Z \) and \( W \) intersect properly, then
\[
Z \bullet W = [(Z \cdot W, g_Z \ast g_W)].
\]

To see that this product is well defined we have to show that this definition is independent of the choice of Green currents for the cycles \( Z \) and \( W \). It boils down to the following lemma.

**Lemma 7.17.** Let \( \eta \in Z\mathcal{D}_{T^1 \mathcal{W}}^{2p-n-1}(X, p) \) be a closed form, so we can see the current \([\eta] \) as a Green current for the cycle \( 0 \in Z^p(X, n)_0 \), and \([(W, g_W)] \in \widehat{\text{CH}}^q(X, m)^0 \), so \( \omega(g_W) = 0 \). Then the relations
\[
(7.7) \quad ([\eta] \ast g_W)^\sim = (g_W \ast [\eta])^\sim = 0
\]
\[
(7.8) \quad a(\eta) \cdot [(W, g_W)] = [(W, g_W)] \cdot a(\eta) = 0
\]
hold true.

**Proof.** Since the cycle zero intersects properly any other cycle, the second equation follows from the first.

We compute the product in one direction, choosing a Green form \( g_W \) such that \([g_W] \in \tilde{\omega}_W \). We obtain
\[
[\eta] \ast g_W = \mathcal{P}(0) \cdot [g_W] + [\eta] \cdot \omega(g_W),
\]
up to a boundary. By definition the formal product \( \mathcal{P}(0) \cdot [g_W] \) is zero. Moreover \( \omega(g_W) = 0 \) concluding that \(([\eta] \ast g_W)^\sim = 0 \). On the other direction, since \(([\eta] \ast g_W)^\sim = (-1)^{nm} (g_W \ast [\eta])^\sim \), we conclude that \((g_W \ast [\eta])^\sim = 0 \). \( \square \)

We can now define an analogue of Beilinson’s height pairing for higher cycles. We take the following notational liberty: \( \text{CH}^*(\text{Spec}(F), \bullet) \) will be denoted simple by \( \text{CH}^*(F, \bullet) \) when there is no cause for confusion. For the higher arithmetic Chow groups and Deligne cohomology of \( F \), we will follow the same convention.

**Definition 7.18.** Assume that \( X \) is equidimensional of dimension \( d \). Let \( p, q, m, n \geq 0 \) be integers satisfying the relation \( 2(p + q - d - 1) = n + m \). Let \( \pi: X \to \text{Spec} F \) be the structural morphism. For cycles \( Z \in Z^p(X, n)_0^0 \) and \( W \in Z^q(X, m)_0^0 \), the height pairing is defined as
\[
\langle Z, W \rangle = \pi_!(Z \bullet W) \in \widehat{\text{CH}}^{p+q-d}(F, n + m)^0.
\]

Assume that \( Z \) and \( W \) intersect properly. Then we can give a formula for the class of the above height pairing in \( \widehat{\text{CH}}^{p+q-d}(F, n + m)^0 \otimes \mathbb{Q} \). Since \( n + m = 2(p + q - d - 1) \) is even, we know that the group \( \text{CH}^{p+q-d}(F, n + m) \) is torsion. We have the exact sequence
\[
(7.9) \quad 0 \to \frac{H^1_D(F, \mathbb{R}(p + q - d))}{\text{im}(\rho_{\text{Be}})} \to \widehat{\text{CH}}^{p+q-d}(F, n + m) \to \text{CH}^{p+q-d}(F, n + m) \to 0,
\]
where $\rho_{\text{Be}}$ denotes Beilinson’s regulator. The pairing $\langle Z, W \rangle$ is given, at the level of higher arithmetic cycles, by $[\langle \pi_*(Z \cdot W), \pi_*(gz * gw) \rangle]$. Since the cycle $\pi_*(Z \cdot W) \in \text{CH}^{p+q-d}(F, n + m)$ is torsion, some integral multiple of it is the boundary of a pre-cycle. Formally, let $N$ be the order of $\pi_*(Z \cdot W)$, then there is a $T \in Z^{p+q-d}(F, n + m + 1)$ such that

$$N(\pi_*(Z \cdot W)) = \delta(T).$$

Consider the element

$$N[\langle \pi_*(Z \cdot W), \pi_*(gz * gw) \rangle] - [\delta(T), \rho(T)].$$

This element represents the same class as $N(Z, W)$, but is of the form

$$a([N \pi_*(gz * gw) - \rho(T)]).$$

The class of the current $(N \pi_*(gz * gw) - \rho(T))^\sim$ belongs to the group $\tilde{D}_{1}^{*}(F, p + q - d)$. Since $D_{1}^{*}(F, p + q - d) = 0$, this current is closed and gives us a class

$$[N \pi_*(gz * gw) - \rho(T)] \in H_{\delta}^{1}(F, \mathbb{R}(p + q - d)).$$

Thus, we the rational height pairing is given by

$$\langle Z, W \rangle_{Q} = \frac{1}{N} \left( N \pi_*(gz * gw) - \rho(T) \right) \in \frac{H_{\delta}^{1}(F, \mathbb{R}(p + q - d))}{(\text{Im}(\rho_{\text{Be}}))} \otimes \mathbb{Q} = \text{CH}^{p+q-d}(F, n + m)_{Q}.$$

Here $-$ denotes the class in $H_{\delta}^{1}(F, \mathbb{R}(p + q - d))/\text{Im}(\rho_{\text{Be}})$.

The height pairing defined above has two components, the one coming from intersection theory, $\pi_*(Z \cdot W)$, that we have been able to write in term of $\rho(T)$ and the one coming from the Green currents $\pi_*(gz * gw)$. The fact that we have written the intersection theoretical part as $\rho(T)$ used that some higher Chow groups of the ground field are torsion. Of course, if the ground field is not a number field, this may not be possible. However, the component $\pi_*(gz * gw)$ can be defined for any arithmetic field. For instance when $F = \mathbb{C}$, since all that we have used to define it involves complex geometry. In fact the term $\pi_*(gz * gw)$ resembles the Archimedean component of Beilinson’s height pairing. By the same reasons as before, the current $\pi_*(gz * gw)$ is closed.

**Definition 7.19.** Let $Z \in Z^{p}(X, n)_{0}$ and $W \in Z^{q}(X, m)_{0}$ be cycles intersecting properly, then the **Archimedean component of the higher height pairing** is defined as

$$\langle Z, W \rangle_{\infty} := (\pi_*(gz * gw))^\sim \in H_{\delta}^{1}(F, p + q - d),$$

for any choice of Green current $gz$ for $Z$ and a Green current $gw$ for $W$.

**Proposition 7.20.** The Archimedean component $(Z, W)_{\infty}$ of the height pairing, does not depend on the choice of Green currents $gz$ and $gw$ satisfying $\omega(gz) = 0$ and $\omega(gw) = 0$.

**Proof.** This proposition is a direct consequence of equation (7.7) in Lemma 7.17, and the details are omitted. \qed
Remark 7.21. The higher height pairing $\langle Z, W \rangle$ only depends on the classes $[Z], [W] \in \text{CH}^*(X, \bullet)^0$. Indeed, if $W' = W + \delta T$, then $g_{W'} = g_W - P(T)$ is a Green current for $W'$ satisfying the analogue of conditions (7.6). Thus

$$Z \cdot W' = [(Z, g_Z)] \cdot [(W', g_{W'})] = [(Z, g_Z)] \cdot [(W, g_W)] = Z \cdot W.$$  

By contrast the Archimedean component of the height pairing $\langle Z, W \rangle_\infty$ depends on the cycles $Z$ and $W$.

Remark 7.22. The Archimedean part of Beilinson’s height pairing has two other incarnations. One via the product in real-Deligne cohomology, and another via the theory of biextensions of mixed Hodge structures of weight $-1$ (see [Hai90] for details). It would be interesting to study if such incarnations are possible for the Archimedean component of the higher height pairing. This will be the subject of a future project.

8. Example: the case of dimension zero

Let $F$ be a number field and $X$ a smooth projective variety over $F$. For each of the complexes $\mathfrak{A}$ in diagram (4.9) we have defined higher arithmetic Chow groups $\widehat{\text{CH}}^p(X, n, \mathfrak{A})$ equipped with direct images, inverse images and products. In the case of $\widehat{\text{CH}}^p(X, n, \mathfrak{A}_{TW})$ the product is associative and graded commutative. For the other complexes, the product fails to be associative in general. In this section we specialize to the case $X = \text{Spec}(F)$ and derive the first non trivial examples of a higher arithmetic intersection pairing. For simplicity in the notation, when $X = \text{Spec}(F)$ we will write

$$H^*_D(F, \mathbb{R}(*) := H^*_D(X, \mathbb{R}(*)},$$

and similarly, $\text{CH}^p(F, n)$ and $\widehat{\text{CH}}^p(F, n)$.

8.1. Higher Arithmetic Chow groups of a number field. We now look for the groups $\widehat{\text{CH}}^p(F, n, \mathfrak{A})$ that may be non-zero. To simplify the discussion, we will neglect torsion and work after tensoring with $\mathbb{Q}$.

Let $r_1$ and $r_2$ denote the number of real and (non-conjugate) complex embeddings of $F$ respectively.

We start by writing down the exact sequence of Theorem 7.7 (1) in case of $X = \text{Spec}(F)$ and after tensoring with $\mathbb{Q}$:

$$\text{CH}^p(F, n + 1)_\mathbb{Q} \xrightarrow{\text{BCH}} \mathfrak{A}^{2p-n}(F, p)/\text{Im}(d) \xrightarrow{a} \widehat{\text{CH}}^p(F, n, \mathfrak{A})_\mathbb{Q} \xrightarrow{\zeta} \text{CH}^p(F, n)_\mathbb{Q} \rightarrow 0.$$  

By Borel’s Theorem (see for instance [BG02, Theorem 9.9]), after applying the isomorphism between higher Chow groups and motivic cohomology, we have that

$$\text{CH}^p(F, n)_\mathbb{Q} = \begin{cases} \mathbb{Q}, & \text{if } p = n = 0, \\ F^\times \otimes \mathbb{Q}, & \text{if } p = n = 1, \\ \mathbb{Q}^{r_1+r_2}, & \text{if } 2p - n = 1, p \text{ odd}, \\ \mathbb{Q}^{r_2}, & \text{if } 2p - n = 1, p \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$
As an example, for $F$ map. They are given by the kernel of the Beilinson regulator (8.5) $0 \to \hat{D}(8.4)$ that

In particular, $D_{\mathcal{A}}$ consists of all elements $(F,n,\sigma)$ such that

We now turn our attention to $\mathcal{A} = \hat{D}$. This case is more complicated because

In this case we obtain that $D_{\mathcal{A}}(F,n) \neq \{0\} \iff p = n = 0$, or $p > 0, n = 0, 1$.

The groups $\hat{\text{CH}}^p(F,n) \neq 0$ do not depend on the complex used to compute the cohomology. They are given by the kernel of the Beilinson regulator map.

As an example, for $F = \mathbb{Q}(i)$, where $i$ is a solution of the equation $x^2 + 1 = 0$, $\hat{\text{CH}}^1(F,1)^0$ consists of all elements $\frac{a}{c} + i\frac{b}{c} \in \mathbb{Q}(i)$, such that $a^2 + b^2 = c^2$ (Pythagorean triplets).
Remark 8.1. Let $Z \in Z^p(F, 2p - 1)_{0}$. Since $X = \text{Spec}(F)$ has dimension zero, then $\mathfrak{D}_{1,W,D}(F, p) = \mathfrak{D}_{1,W}(F, p)$. Therefore $\mathcal{P}(Z)$ already belongs to $\mathfrak{D}_{1,W}(F, p)$, and, in consequence, we can take 0 as the Green current for $Z$, with the result that $\omega_{Z} = \mathcal{P}(Z)$. We note that, although it is the zero Green current it is not trivial, in the sense that the corresponding Green form may be non-zero. We caution the reader also that this in not a splitting of the exact sequence (8.5) because, if $Z_{1}, Z_{2} \in Z^p(F, 2p - 1)_{0}$ are two cycles with the same class in $CH^p(F, 2p - 1)$, then $(Z_{1}, 0)$ and $(Z_{2}, 0)$ may give different classes in $\mathbf{CH}^0(F, 2p - 1, \mathfrak{D}_{TW})$.

8.2. Examples of intersection products in dimension zero. For $X = \text{Spec}(F)$, the diagonal map $X \rightarrow X \times X$ is an isomorphism and there is no difference between the external and the intersection product. In particular all (admissible) pre-cycles intersect properly.

Remark 8.2. From the computations on the previous section we observe that the only non-trivial products among higher arithmetic Chow groups in the case of number fields are the product that involve $\mathbf{CH}^0(F, 0)$ and the products of the form

$$\mathbf{CH}^p(F, 2p - 1, \mathfrak{D}_{TW})_{\mathbb{Q}} \otimes \mathbf{CH}^q(F, 2q - 1, \mathfrak{D}_{TW})_{\mathbb{Q}} \rightarrow \mathbf{CH}^{p+q}(F, 2(p + q) - 2, \mathfrak{D}_{TW})_{\mathbb{Q}}.$$ 

If we start with two classes, one in $\mathbf{CH}^p(F, 2p - 1)_{0}$ and one in $\mathbf{CH}^q(F, 2q - 1)_{0}$ then the intersection product in $\mathbf{CH}^{p+q}(F, 2(p + q) - 2, \mathfrak{D}_{TW})_{\mathbb{Q}}$ is always zero. Indeed, for $p > 1$ or $q > 1$, we know that at least one of the groups $\mathbf{CH}^p(F, 2p - 1)_{0}$ or $\mathbf{CH}^q(F, 2q - 1)_{0}$ is torsion, and so will be the intersection product. While for $p = q = 1$, the groups $\mathbf{CH}^1(F, 1)_{0}$ are not torsion. Since $\mathbf{CH}^1(F, 1)_{0} \subset F^{\chi}$ is the kernel of the logarithm map, we can consider $\{\alpha\}, \{\beta\} \in \mathbf{CH}^1(F, 1)_{0}$, with $\alpha, \beta \in F^{\chi}$ and $\alpha, \beta$ in the kernel of the regulator. Choose and embedding $\sigma : F \rightarrow \mathbb{C}$. After enlarging $F$ if needed, we can assume that $F$ is stable under complex conjugation and let $h$ be the automorphism of $F$ induced by complex conjugation. Since $|\sigma(\alpha)| = |\sigma(\beta)| = 1$, then

$$h^{*}\{\alpha\} = \{\alpha^{-1}\}, \quad h^{*}\{\beta\} = \{\beta^{-1}\}.$$ 

Identifying $\{\alpha^{-1}\}$ with $-\{\alpha\}$ and $\{\beta^{-1}\} = -\{\beta\}$ in $CH^1(F, 1)$ (which of course, amounts to adding a boundary $\delta T$). But as we will see in Lemma 8.4, all such boundaries have trivial image under $\mathcal{P}$ we deduce

$$h^{*}(\{\alpha\}, \{\beta\}) = (\{\alpha\}, \{\beta\}) \in \mathbf{CH}^2(F, 2).$$

Since any element $\gamma \in H_{D}^1(F, \mathbb{R}(2))$ satisfies $h^{*}(\gamma) = -\gamma$, for parity reasons, the product $(\{\alpha\}, \{\beta\}) = 0$.

By Remark 8.2, if we want to have a non-zero higher arithmetic intersection product, we need to look at cycles whose image by the regulator map is non-zero.

As observed in Remark 8.1, the zero dimensional case is special because for a cycle $\alpha \in Z^p(F, 2p - 1)_{0}$ we can always choose the Green current 0.
Lemma 8.3. Let \( \alpha \in Z^p(F,2p-1)_0 \) and \( \beta \in Z^q(F,2q-1)_0 \), then the product \([(\alpha,0)] \cdot [(\beta,0)]\) is given by \([(\alpha \cdot \beta,0)]\).

**Proof.** Let \( g_\beta = \{g_{2q-1}', \ldots, g_0'\} \) be a Green form for \( \beta \) such that the corresponding Green current satisfies \([g_\beta] = 0\). In other words
\[
\sum_{j=0}^{2q-1} (p_j)_* (g_j' \cdot W_j) = 0.
\]
The lemma amounts to showing that, writing \( g_\alpha = 0 \) and \( \omega_\beta = \mathcal{P}(\beta) \),
\[
g_\alpha \cdot g_\beta = (-1)^{2p-1} \mathcal{P}(\alpha) \cdot [g_\beta] + g_\alpha \cdot \omega_\beta = 0.
\]
Clearly, \( g_\alpha \cdot \omega_\beta = 0 \). We now denote by
\[
p_k : \Delta^k \to \text{Spec}(F)
\]
the structural morphism. From the definition
\[
\mathcal{P}(\alpha) \cdot [g_\beta] = \sum_{j=0}^{2q-1} (p_{2p+j-1})_* (p_{2p-1}^* \cdot W_{2p-1}) \cdot p_j^* (g_j' \cdot W_j).
\]
Since \( \mathcal{P}(\alpha) \) is smooth, using the projection formula for forms and currents, we can write
\[
\mathcal{P}(\alpha) \cdot [g_\beta] = \mathcal{P}(\alpha) \cdot \left( \sum_{j=0}^{2q-1} (p_j)_* (g_j' \cdot W_j) \right) = 0.
\]
This concludes the proof. \(\square\)

Fix \( \alpha \) and \( \beta \) as in the previous lemma. We want to devise a method to compute \((\alpha,\beta)_{p,q}\). Since the group \( \text{CH}^{p+q}(F,2(p + q) - 2) \) is torsion, there exist an integer \( N > 0 \) and a pre-cycle \( T \in Z^{p+q}(F,2(p + q) - 1)_0 \), satisfying \( N(\alpha \cdot \beta) = \delta(T) \). Therefore, in the group \( \text{CH}^{p+q}(F,2(p + q) - 2)_Q \), by Lemma 8.3 we derive the equation
\[
(\alpha,0) \cdot (\beta,0) = (\alpha \cdot \beta,0) = \frac{1}{N} (\delta(T),0) = \frac{1}{N} (0,-\mathcal{P}(T)).
\]
Using the isomorphism
\[
\text{CH}^{p+q}(F,2(p + q) - 2)_Q \cong [H_{2d}^1(F,\mathbb{R}(p + q))/\text{Im}(\rho_{Be,Q})] \otimes \mathbb{Q},
\]
we obtain a formula for the intersection product
\[
(8.6) \quad (\alpha,\beta)_{p,q} = (\alpha,0) \cdot (\beta,0) = \frac{1}{N} (-\mathcal{P}(T)).
\]
Here $-\mathcal{P}(T)$ means the class of $-\mathcal{P}(T)$ in $[H^1_D(F, \mathbb{R}(p + q))/\text{Im}(\rho_{Be})] \otimes \mathbb{Q}$.

The pre-cycle $T$ is equation (8.6) can be computed in many cases as a linear combination of pre-cycles that explain known relations in higher Chow groups, like the Totaro pre-cycle that explains the Steinberg relations. Among them, there are the pre-cycles that explain the multilinearity relations. The next lemma shows that we do not need to worry about them.

**Lemma 8.4.** Let $\alpha, \beta \in F^\times \setminus \{1\}$ and $Z \in Z^p(F, 2p - 1)_0$ a cycle. Let $T \in Z^{p+1}(F, 2p + 1)_0$ be a pre-cycle such that

$$\delta T = \{\alpha \beta\} \times Z - \{\alpha\} \times Z - \{\beta\} \times Z. \tag{8.7}$$

Then $\mathcal{P}(T) = 0$.

**Proof.** Let $T'$ be another pre-cycle satisfying (8.7). Then $T - T'$ is a cycle, and

$$\mathcal{P}(T) - \mathcal{P}(T') = \mathcal{P}(T - T') \in \text{Im}(\rho_{Be, Q}).$$

Thus if we prove the lemma for a particular choice of pre-cycle, it is also true for any other pre-cycle satisfying (8.7). Consider the pre-cycle $T = C \times Z \subset \square^{2p+1}$, where $C \subset \square^2$ is the curve parameterized by

$$t \mapsto \left( t, \frac{(t - \alpha)(t - \beta)}{(t - 1)^2} \right).$$

It is straightforward to check that, being $Z$ a cycle in the normalized complex, then $T \in Z^{p+1}(F, 2p + 1)_0$ satisfies (8.7). Using the multiplicativity of the Thom-Whitney complex we deduce

$$\mathcal{P}(T) = \mathcal{P}(C) \cdot \mathcal{P}(Z).$$

Finally, $\mathcal{P}(C) = 0$ by Lemma 7.15, and we obtain this result. \qed

**Remark 8.5.** The pairing $(\ , \ )_{p, q}$ above is a pairing between cycles and does not descend to a pairing of higher Chow groups. There is an obvious problem: Suppose $Z \in Z^p(F, 2p - 1)_0$ is a boundary, i.e., there exists a (pre)-cycle $Z' \in Z^p(F, 2p)_0$, such that $Z = \delta(Z')$. Now, by assigning the 0 Green current to $Z$, the height pairing for any cycle $W \in Z^q(F, 2q - 1)_0$ is given by

$$(Z, W)_{p, q} = \mathcal{P}(Z' \cdot W),$$

which for appropriate choices of $W$, may be non-trivial (even after taking the class mod the image of the regulator).

At the level of cycles, we can compute specific examples of this intersection pairing $(\ , \ )_{p, q}$. We do so in two situations: the case $p = q = 1$ and the case $p = 1, q = 2$.

**Case** $p = q = 1$. We want to compute examples of the pairing

$$(\ , \ )_{1, 1} : Z^1(F, 1)_0 \otimes Z^1(F, 1)_0 \to \frac{H^1_D(F, \mathbb{R}(2))}{\text{Im} (\rho_{Be})} \otimes \mathbb{Q},$$

given by

$$(\alpha, \beta)_{1, 1} = (\alpha, 0) \cdot (\beta, 0)$$
We start with the case $\beta = 1 - \alpha$. As we will see later, this is the main ingredient of the general case.

It is well known and easily verifiable that $\alpha \cdot (1 - \alpha) = \delta(C_\alpha)$, where $C_\alpha := \left( z, 1 - \frac{\alpha}{z}, 1 - z \right)$, is the Totaro pre-cycle. Thus the height pairing is given by

$$(\alpha, 1 - \alpha)_{1,1} = -\mathcal{P}(C_\alpha).$$

We can identify $H^1_D(F, \mathbb{R}(2)) = \mathbb{R}(1)^{r_2}$, where $r_2$ is the number of non-equivalent complex immersions of $F$. Let $\mathcal{L}_2$ be the Bloch-Wigner dilogarithm function. By [Gon95, Theorem 3.6] (see also §2 and §3 of loc. cit.)

$$P(C_\alpha) = -\frac{1}{2\pi} (i\mathcal{L}_2(\sigma_1(\alpha)), \ldots, i\mathcal{L}_2(\sigma_{r_2}(\alpha))).$$

We specialize to the case $F = \mathbb{Q}(i)$. As we will recall in the next section, by [Pet09], we know that $i\mathcal{L}_2(i)$ (up to factors of $\pi$) generates $\text{Im}(\rho_{BC,\mathbb{Q}}) : \text{CH}^2(F,3)_{\mathbb{Q}} \to H^1_D(F,\mathbb{R}(2))_{\mathbb{Q}}$.

It will be interesting to prove that, for certain $\alpha \in \mathbb{Q}(i)^\times \setminus \{1\}$, the value $\mathcal{L}_2(\alpha)$ is not a (rational) multiple of $\mathcal{L}_2(i)$. This would be an example of the non-triviality of the defined intersection pairing. By a conjecture of Milnor (stated in [Mil82], end of page 21), such a $\alpha$ exists.

Now, let $\alpha, \beta$ in $Z^1(F, 1)$. We want to reduce the computation of $(\alpha, \beta)_{1,1}$ to the previous case. Using that $K_2(F) = F^\times \otimes F^\times / \sim$, where $\sim$ is generated by the Steinberg relations, is torsion and that $\text{CH}^2(F,2)_{\mathbb{Q}} = K_2(F)_{\mathbb{Q}}$, we know that there is an integer $N > 0$ such that $\alpha \times \beta \in Z^2(F,2)_{\mathbb{Q}}$ can be written as

$$\alpha \times \beta = \frac{1}{N} \sum_i \gamma_i \times (1 - \gamma_i) + \sum_j \delta(T_j),$$

where the pre-cycles $T_j$ are of the form discussed in Lemma 8.4. By Lemma 8.4, the height pairing $(\alpha, \beta)_{1,1}$ is given by

$$(\alpha, \beta)_{1,1} = \sum_i -\mathcal{P}(C_{\gamma_i}).$$

To simplify the computation latter we need another lemma.

**Lemma 8.6.** Let $\alpha, \beta \in F^\times \setminus 1$.

(1) For any pre-cycle $T \in Z^2(F,3)_0$ with

$$\delta T = \{\alpha\} \times \{-\alpha\},$$

we have $\mathcal{P}(T) = 0$.

(2) For any pre-cycle $T \in Z^2(F,3)_0$ with

$$\delta T = \{\alpha\} \times \{\beta\} + \{\beta\} \times \{\alpha\},$$

we have $\mathcal{P}(T) = 0$. 
Proof. By the same argument as in the proof of Lemma 8.4, it is enough to prove the result for one single $T$ for each equation. For the first statement we use that
\[
\alpha \otimes (-\alpha) = \alpha \otimes (1 - \alpha) + \alpha^{-1} \otimes 1 - \alpha^{-1},
\]
thanks to the identity $-\alpha = (1 - \alpha)/(1 - \alpha^{-1})$. Therefore
\[
\{\alpha\} \times \{-\alpha\} = \delta C_\alpha + \delta C_{\alpha^{-1}} + \delta T,
\]
where $T$ is a linear combination of pre-cycles of the form considered in Lemma 8.4. Using Lemma 8.4, equation (8.8) together with the fact that $L_2(u^{-1}) = -L_u(u)$, we obtain the first statement.

The second statement follows from the first statement and Lemma 8.4, using the identity
\[
\alpha \otimes \beta + \beta \otimes \alpha = \alpha \beta \otimes (-\alpha \beta) - \alpha \otimes (-\alpha) - \beta \otimes (-\beta).
\]

We illustrate this procedure with a prototypical example. Again we consider the case $F = \mathbb{Q}(i)$ and let $\alpha = 2 + 3i$ and $\beta = 1 - 2i$. We want to compute $(\alpha, \beta)_{1,1}$. Using the identities
\[
1 - (-1 - i)^6 = (2 + i)(2 + 3i),
\]
\[
1 - (2 + 3i) = i(-1 - i)(1 - 2i),
\]
one can check that, in $F^\times \otimes F^\times$, the following relation holds true.
\[
12 \left(\alpha \otimes \beta\right) = -12 \left((-1 - i) \otimes (1 - (-1 - i))\right) + 2 \left((-1 - i)^6 \otimes (1 - (-1 - i)^6)\right) + 12 \left((2 + 3i) \otimes (1 - (2 + 3i))\right),
\]
where ... contains terms of the form $u \otimes v + v \otimes u$. Using lemmas 8.4, 8.6, and equation (8.8), we deduce that
\[
\frac{1}{i} (\alpha, \beta)_{1,1} = -\frac{1}{2\pi} L_2(-1 - i) + \frac{1}{12\pi} L_2((-1 - i)^6) + \frac{1}{2\pi} L_2(2 + 3i),
\]
modulo of course, the image of the regulator.

Case $p = 1, q = 2$, $F = \mathbb{Q}(i)$. We consider the higher cycles
\[
i \in Z^1(F, 1), \quad Z_i \in Z^2(F, 3),
\]
where
\[
Z_i := \left(z, 1 - \frac{i}{z}, 1 - z\right) - \left(z, \frac{(z - i)^4}{(z - 1)^4}, 1 - z\right).
\]
By [Pet09], the element $[Z_i]$ is a generator of the free part of $\text{CH}^2(F, 3)$. Therefore, $P(Z_i) = iL_2(i)$ generates
\[
\text{Im} (p_{\text{Be}, \mathbb{Q}}: \text{CH}^2(F, 3)_\mathbb{Q} \to H^1_D(F, \mathbb{R}(2))_\mathbb{Q}).
\]
Further, the intersection $i \cdot Z_i$ is given by
\[
i \cdot Z_i = \delta \left(\frac{4(C'_i - C''_i) - \Xi_i}{\Xi_i}\right),
\]
where $C'_i, C''_i, \Xi_i$ depend on $i$. This completes the proof.
where

\[ C'_i = \left( z_2, 1 - i \frac{z_1}{z_2}, z_1, 1 - \frac{z_2}{z_1}, 1 - z_1 \right), \]

\[ C''_i = \left( z_1, 1 - i \frac{z_2}{z_1}, z_2, 1 - \frac{z_1}{z_2}, 1 - z_2 \right), \]

and

\[ \Xi_i = \left( z_2, 1 - i \frac{z_1}{z_2}, z_1, \frac{(z_1 - i)^4}{(z_1 - 1)^4}, 1 - z_2 \right). \]

The cycles \( C'_i, C''_i \) are higher analogues of the Totaro cycle. They are closely related to Goncharov’s programme of associating the Bloch group to the higher Chow group of points (see [Gon05] for details).

Hence, the intersection pairing is given by

\[ (i, Z_i)_{1,2} = -(4P(C'_i - C''_i) - P(\Xi_i)). \]

Using Lemma 7.15, now for the curve \( C = \left( z_1, \frac{(z_1 - i)^4}{(z_1 - 1)^4} \right) \), and the multiplicativity of the Thom-Whitney complex, it is easy to show that \( P(\Xi_i) = 0 \). Thus we get the pairing

\[ (i, Z_i)_{1,2} = -4P(C'_i - C''_i) = \frac{2}{\pi^2} (L_3(i)), \]

where \( L_3 \) denotes the (Bloch-Wigner) tri-logarithm. The last equality follows from Theorem 3.6 of [Gon95] once again.

**Remark 8.7.** Based on Goncharov’s programme ([Gon05]) and Zagier’s conjecture [Zag91], one can ask whether the intersection pairings \((\alpha, \beta)_{p,q}\) can be written as a combination of polylogarithms.

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