Quantum Theory Allows Measurement of Non-Hermitian Operators

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In quantum theory, a physical observable is represented by a Hermitian operator as it admits real eigenvalues. This stems from the fact that any measuring apparatus that is supposed to measure a physical observable will always yield a real number. However, reality of eigenvalue of some operator does not mean that it is necessarily Hermitian. There are examples of non-Hermitian operators which may admit real eigenvalues under some symmetry conditions. One may wonder if there is any way to measure a non-Hermitian operator, for example, the average of a non-Hermitian operator in a quantum state. We show that quantum theory allows direct measurement of any non-Hermitian operator via the weak measurement. The average of a non-Hermitian operator in a pure state is a complex multiple of the weak value of the positive semi-definite part of the non-Hermitian operator. We also prove a new uncertainty relation for any two non-Hermitian operators and illustrate this for the creation and annihilation operators, and the Kraus operators.

Introduction.– One of the basic postulates of quantum mechanics limits the possible quantum mechanical observables to be the Hermitian ones [1]. The Hermiticity of the quantum mechanical observables seems to be a compelling and plausible postulate as the eigenvalues of Hermitian operators are real. Moreover, Hermitian Hamiltonian yields a unitary evolution leading to the conservation of the probability. But the reality of the spectrum of the quantum mechanical observables does not imply that the observables must be Hermitian. In fact, there are certain class of observables that are not Hermitian yet their spectrum is real. The reason for the reality of such operators is argued to be the underlying symmetry of the operators with certain other restrictions. This has resulted in the attempts to lift the postulate of Hermiticity and allow for more general operators. It is known that there are non-Hermitian operators which possess real eigenvalues if one imposes some symmetry conditions, namely the $PT$–symmetry, which is unbroken. $PT$–symmetry is said to be not spontaneously broken if the eigenfunctions of the non-Hermitian operator are itself $PT$–symmetric. Such kind of operators which respect the unbroken $PT$–symmetry are ingredients of $PT$–symmetric quantum mechanics [2–3].

In quantum theory, the concept of weak measurement was introduced by Aharonov-Albert-Vaidman [4–6] to study the properties of a quantum system in a pre and post-selected ensembles. In this formalism the measurement of an observable leads to a weak value of the observable with unexpectedly strange properties. In fact, the weak value is shown to be complex, in general, and can take values outside the spectrum of the observable. In particular, if one measures an observable $O$ of the system weakly, with a preselected state $|ψ_i⟩$ at time $t_i$ and postselected state $|ψ_f⟩$ at time $t_f$, the value of the observable measured at time $t_i \leq t \leq t_f$ is given by the weak value of observable, which is $ψ_f ⟨O⟩^w_{ψ_i} = ⟨ψ_f |O |ψ_i⟩ / |⟨ψ_f |ψ_i⟩|$. The concept of weak measurements has been generalized further beyond its original formulation [7–12]. In recent years, weak values have found numerous applications. For example, the Panchratnam geometric phase is nothing but the phase of a complex valued weak value that arises in the context of weak measurements [15]. It has been shown that weak measurements can be used for interrogating quantum systems in a coherent manner [16–17]. In addition, it plays important role in understanding the uncertainty principle in the double-slit experiment [18–19], resolving Hardy’s paradox [20], analyzing tunneling time [21–22], protecting quantum entanglement from decoherence [23–24], modifying the decay law [25]. Remarkably, it is possible to express the wavefunction as a weak value of a projector and this paved the way to measure the wavefunction of single photon directly [26–27]. Similarly, in quantum metrology the phase sensitivity of a quantum measurement is given by the variance of the imaginary parts of the weak values of the generators over the different measurement outcomes [28]. For a very recent review on weak measurements one can look at Ref. [29].

Despite having complex spectrum, in general, the non-Hermitian operators have found applications in theoretical work as a mathematical model for studying open quantum systems in nuclear physics [30] and quantum optics [31], among others to name a few. In these fields, the non-Hermitian Hamiltonian appears as an effective description for the subsystem of the full system. Non-Hermitian operators that can be expressed as a product of two non commuting Hermitian operators do appear in the formalism that describes quantum states using quasiprobability distribution such as the Dirac distribution [32–35], the Moyal distribution [36–37], etc. But, in general, if we allow for non-Hermitian operators which may posses complex eigenvalues, there is no experimental way to measure the expectation value of such operators in a given quantum state. This is one of the reasons that the non-Hermitian operators have often not found a place in the realm of quantum mechanics. Therefore, having a method to measure the expectation values of non-Hermitian operators is required in order to place these operators on a firm footing. Here we propose an experimentally verifiable procedure to measure...
the complex expectation value of a general non-Hermitian operator. The key to measurement of non-Hermitian operators is the notion of the polar decomposition of any operator and the process of weak measurement. In this letter, we show that quantum theory allows direct measurement of any non-Hermitian operator via the weak measurement. The average of a non-Hermitian operator is a complex multiple of the weak value of the positive semi-definite part of the non-Hermitian operator. We illustrate our main result with examples. Our method opens up the possibility of measuring the average of creation and annihilation operators in any state. We prove a new uncertainty relation for any two non-Hermitian operators and show that they can also be measured experimentally. As an application, we show that the uncertainty in the Kraus operators govern the fidelity of the output state for a quantum channel. If the total uncertainties in the Kraus operators is less, then the fidelity will be more. In the appendix, we have provided an example that displays how to find the average of a $PT$-symmetric Hamiltonian, which is non-Hermitian, for a qubit.

**Expectation value of Non-Hermitian operator.**-- Let us consider a non-Hermitian operator $A$. The expectation value of such an operator in a quantum state $|\psi\rangle$, given by $\langle \psi | A | \psi \rangle$, is in general a complex number. This makes it unobservable in a laboratory experiment. But here we present a formalism to get around this problem. To present the main idea, we need the polar decomposition of a matrix. Let $A \in \mathbb{C}^{m \times n}$, $m \geq n$. Then, there exists a matrix $U \in \mathbb{C}^{m \times m}$ and a unique Hermitian positive semi-definite matrix $R \in \mathbb{C}^{m \times m}$ such that $A = UR$, with $U^\dagger U = I$. The positive semi-definite matrix $R$ is given by $R = \sqrt{A^\dagger A}$, even if $A$ is singular. If rank $(A) = n$ then $R$ is positive definite and $U$ is uniquely determined.

Let us consider a quantum system initially in the state $|\psi\rangle \in \mathcal{H} = \mathbb{C}^d$. Suppose we are interested in measuring the average of a non-Hermitian operator $A$ in the state $|\psi\rangle$. Consider the polar decomposition of an operator $A \in \mathbb{C}^{d \times d}$, given by $A = UR$, where $R$ is a positive semi-definite operator and $U$ is a unitary operator. The average of a non-Hermitian operator in the pure state $|\psi\rangle$, is given by

$$
(A) = \langle \psi | A | \psi \rangle = \langle \psi | UR | \psi \rangle = \frac{\langle \phi | R | \phi \rangle}{\langle \phi | \phi \rangle} \langle \phi | \phi \rangle = \phi(R)_{\psi}^{w} \langle \phi | \phi \rangle,
$$

where $|\phi\rangle = U^\dagger |\psi\rangle$ and $\phi(R)_{\psi}^{w}$ is the weak value of positive semi-definite operator $R$, given by $\langle \phi | R | \phi \rangle/\langle \phi | \phi \rangle$. Now, given a non-Hermitian operator $A$, we first find out $R = \sqrt{A^\dagger A}$ and the corresponding unitary $U$. The measurement of the expectation value of $A$ in a quantum state $|\psi\rangle$ can be carried out as follows. We start with a system which is preselected in the state $|\psi_0\rangle = |\psi\rangle$ and weakly measure the positive semi-definite operator $R$ in the preselected state $|\psi\rangle$. The weak measurement can be realized using the interaction between the system and the measurement apparatus which is governed by the interaction Hamiltonian

$$
H_{\text{int}} = g\delta(t - t_0)R \otimes P,
$$

where $g$ is the strength of the interaction that is sharply peaked at $t = t_0$. $R$ is an observable of the system and $P$ is that of the apparatus. Under the action of the interaction Hamiltonian, the system and apparatus evolve as

$$
|\psi\rangle \otimes |\Phi\rangle \rightarrow e^{-i\frac{g}{\hbar}R \otimes P} |\psi\rangle \otimes |\Phi\rangle.
$$

After the weak interaction, we postselect the system in the state $|\phi\rangle = U^\dagger |\psi\rangle$ with the postselection probability given by $|\langle \phi | \psi \rangle|^2$. This yields the desired weak value of $R$, i.e.,

$$
\phi(R)_{\psi}^{w} = \frac{\langle \phi | R | \psi \rangle}{\langle \phi | \phi \rangle}.
$$

Therefore, multiplying $\langle \phi | \psi \rangle$ to $\phi(R)_{\psi}^{w}$, gives us $\langle \phi | A | \psi \rangle$. To this end we have provided a procedure to measure the expectation value of a non-Hermitian operator. Equivalently, one can also write $A = SU$, where $S = URU^\dagger = \sqrt{AA^\dagger}$ is a positive semi-definite operator. In this case the average of $A$ in a pure state $|\psi\rangle$ is given by

$$
\langle A \rangle = \langle \psi | A | \psi \rangle = \langle \psi | SU | \psi \rangle = \frac{\langle \psi | S | \chi \rangle}{\langle \psi | \chi \rangle} \langle \chi | \phi \rangle = \phi(S)_{\psi}^{w} \langle \chi | \phi \rangle,
$$

where $|\chi\rangle = U |\psi\rangle$ and $\phi(S)_{\psi}^{w}$ is the weak value of positive semi-definite operator $S$, given by $\langle \psi | S | \chi \rangle/\langle \psi | \chi \rangle$. Following the same procedure as above one can measure the weak value of $S$ with preselection in the state $|\chi\rangle = U |\psi\rangle$ and postselection in the state $|\psi\rangle$. Now the weak value, $\phi(S)_{\psi}^{w}$, multiplied by $\langle \psi | \chi \rangle$ yields $\langle \psi | A | \psi \rangle$. Furthermore, we have

$$
\langle \psi | A | \psi \rangle = \phi(S)_{\psi}^{w} \langle \chi | \phi \rangle = \phi(R)_{\psi}^{w} \langle \phi | \psi \rangle.
$$

Interestingly, our method can also be applied to measure the weak value of any non-Hermitian operator $A$ in a preselected state $|\psi\rangle$ and postselected state $|\psi'\rangle$. Using polar decomposition of $A = UR$, the weak value of $A$ is given by

$$
\psi''(A)_{\psi}^{w} = \frac{\langle \psi' | UR | \psi \rangle}{\langle \psi' | \psi \rangle} = \psi''(R)_{\psi}^{w}.z,
$$

where $\psi''(R)_{\psi}^{w} = \langle \psi'' | R | \psi \rangle/\langle \psi'' | \psi \rangle$ is the weak value of $R$ and $z = \langle \psi'' | \psi \rangle$. Thus, the weak value of any non-Hermitian operator $A$ with the preselection in the state $|\psi\rangle$ and postselection in the states $|\psi'\rangle$ is equal to the weak value of $R$ with the preselection and the postselection in the states $|\psi\rangle$ and $|\psi''\rangle$, respectively, multiplied by the complex number $z$.

**Measurement of product of two non-commuting Hermitian operators.--** Next we consider some examples to supplement our procedure of measuring expectation value of non-Hermitian operators. Consider measurement of $\Pi_i(B) \Pi_j(C)$, with $\Pi_i(B) = |\psi_i\rangle \langle \psi_i|$ and $\Pi_j(C) = |\phi_j\rangle \langle \phi_j|$, where $|\psi_i\rangle$ and $|\phi_j\rangle$ $(i, j = 1, 2, \ldots, d)$ are eigenstates of two non-commuting Hermitian operators $B$ and $C$, respectively. The product $\Pi_i(B) \Pi_j(C)$ is a non-Hermitian operator. In fact,
the average of this operator in a quantum state is nothing but the discrete version of the Dirac distribution [32,35]. We will show that our method can be applied to measure the expectation value of $\Pi_j(B)\Pi_j(C)$ using the polar decomposition and the weak measurement. For this non-Hermitian operator, i.e., $A = (\psi_i|\psi_j)\langle\psi_j|\psi_i\rangle$, let the polar decomposition be denoted by $A = UR$, where $R = (\langle\psi_i|\psi_j\rangle|\psi_j\rangle)\langle\psi_j|$ and $U$ is determined by relation $U(\psi_i) = e^{-i\eta}|\psi_i\rangle$, where $\eta = \frac{\langle\psi_i|\psi_j\rangle}{\langle\psi_j|\psi_j\rangle}$. Such a unitary operator is given by

$$U = U(m) = e^{i\eta} \sum_{k=0}^{d-1} |\psi_k \oplus m\rangle \langle\phi_k|,$$

(7)

with $j \oplus m = i$ and $\oplus$ denotes the addition modulo $d$. Now the expectation value of $\Pi_j(B)\Pi_j(C)$ in a state $|\psi\rangle$ is given by

$$\langle\psi|\Pi_j(B)\Pi_j(C)|\psi\rangle = \langle\psi|U(m)R|\psi\rangle = \psi'(R)_{\psi'} : \langle\psi'|\psi\rangle,$$

(8)

where $|\psi'\rangle = U^\dagger |\psi\rangle$. Thus, the expectation value of $\Pi_j(B)\Pi_j(C)$ in the state $|\psi\rangle$ is given by the weak value $\psi'(R)_{\psi'} = \frac{\psi'(R)_{\psi'}}{\langle\psi'|\psi\rangle}$ of $R$ multiplied by a complex number $\langle\psi'|\psi\rangle$.

It is claimed in [27] that the weak average, without post-selection, of a non-Hermitian operator $A$ in a state $\rho$ is equal to, in general complex, expectation value of $A$ in the state $\rho$, i.e., $\langle A^\omega \rangle_\rho = \text{Tr}[A]\rho$. Following [35,40], one can devise an experimental method to measure this complex expectation value of $A$. The method is shown only for non Hermitian operators which are product of non commuting Hermitian operators. In this method the interaction Hamiltonian for system and apparatus is designed to be $H = g \sum_{i=1}^{N} A_i \otimes P_i$ in order to measure expectation value $\langle\Pi_j A_i\rangle$. Unlike our method, which can be used to measure expectation value of any non-Hermitian operator, the earlier method is applicable only to the cases of non-Hermitian operators which are product of non-commuting Hermitian operators. Next, we consider an example of non-Hermitian operator which is not product of Hermitian operators.

**Measurement of creation operator.** – Let us consider the creation and the annihilation operators for a single mode electromagnetic field. It is known that the polar decomposition of creation and annihilation operators for the radiation field has problems related to the unitary part of the decomposition [41,43]. This problem is addressed as the non-existence of Hermitian phase operator for the infinite dimensional Hilbert space [43,45]. The problem is resolved by taking Hilbert space to be finite dimensional and taking the limit at the end of all the calculations. In the finite dimensional Hilbert space there is a well defined Hermitian phase operator called as the Pegg-Barnett phase operator [46,47]. This leads to the polar decompositions of the creation and the annihilation operators, which are given by

$$\hat{a} = e^{i\phi_\theta} \sqrt{N},$$

(9)

$$\hat{a}^\dagger = \sqrt{N}e^{-i\phi_\theta},$$

(10)

where $\phi_\theta = \sum_{m=0}^{s} \theta_m \langle\theta_m|\langle\theta_m, \theta_m = \theta_0 + 2m\pi/(s + 1)$ and the orthonormal phase states $|\theta_m\rangle$ are given by $(s + 1)^{-1/2} \sum_{n=0}^{s} e^{im\theta_n}|n\rangle$. The phase states satisfy $e^{i\phi_\theta} |\theta_m\rangle = e^{i\phi_\theta} |\theta_m\rangle$. Now consider the expectation value of creation operator in a general state from the $(s+1)$ dimensional Hilbert space. This is given by

$$\langle\psi|\hat{a}^\dagger |\psi\rangle = \langle\psi|\sqrt{N}e^{-i\phi_\theta}|\psi\rangle$$

$$= \langle\psi|\sqrt{N}|\chi\rangle \langle\psi|\chi\rangle$$

$$= \psi(\sqrt{N}w)_{\chi} : \langle\psi'|\chi\rangle,$$

(11)

where $|\chi\rangle = e^{-i\phi_\theta} |\psi\rangle$. Thus, by measuring the weak value of square root of number operator with preselection in the state $|\chi\rangle$ and postselection in the state $|\psi\rangle$, Consider a general state $|\psi\rangle = \sum_{m=0}^{s} c_m |m\rangle$ with $\sum_{m=0}^{s} |c_m|^2 = 1$. Here $|\chi\rangle$ and $\psi(\sqrt{N})_{\chi}$ are given by

$$|\chi\rangle = e^{-i\phi_\theta} |\psi\rangle = \sum_{m=1}^{s} c_{m-1} |m\rangle + c_s e^{-i(s+1)\theta_0} |0\rangle,$$

(12)

and

$$\psi(\sqrt{N})_{\chi} = \frac{\sum_{m=1}^{s} c_{m-1} c_s^{*} \sqrt{m}}{c_s c_0 e^{-i(s+1)\theta_0} + \sum_{m=1}^{s} c_{m-1} c_s^{*} \sqrt{m}}.$$

(13)

Therefore, we have $\langle\psi|\hat{a}^\dagger |\psi\rangle = \sum_{m=1}^{s} c_{m-1} c_s^{*} \sqrt{m}$. For equally superposed number state, i.e., $c_m = e^{im\phi}/\sqrt{(s+1)}$, we have

$$\langle\psi|\hat{a}^\dagger |\psi\rangle = \frac{e^{-i\nu}}{s+1} \sum_{m=1}^{s} \sqrt{m}.$$  

(14)

Using the result that for any real number $r$ with $r \geq 1$ and positive integer $n$

$$\sum_{m=1}^{s} m^{-r} = \frac{r}{(s+1)^{r+1} - 2(s+1)^{r}} - \Phi_s(r),$$

(15)

where $\Phi_s(r)$ is a function of $r$ with $s$ as a parameter and is bounded between $0$ and $1/2$ [48,49]. Putting $r = 2$ in the above formula, we get $\sum_{m=1}^{s} \sqrt{m} = \frac{2}{3}(s+1)^{3/2} - \frac{1}{2}(s+1)^{1/2} - \Phi_s(2)$, where $0 \leq \Phi_s(2) \leq 1/2$. Therefore, we have

$$\langle\psi|\hat{a}^\dagger |\psi\rangle = \frac{e^{-i\nu}}{s+1} \frac{2}{3}(s+1)^{3/2} - \frac{1}{2}(s+1)^{1/2} - \Phi(1/2).$$

(16)
Interestingly, one can invert Eq. (14) to get

$$\sum_{m=1}^{s} \sqrt{m} = \frac{e^{i\nu}}{s+1} \langle \psi | a^\dagger | \psi \rangle. \quad (17)$$

Therefore, one can use the experimentally obtained expectation value of the creation operator to estimate the sum of the square roots of the first s natural numbers and then this result can be compared to the Ramanujan’s formula [49] for the above series.

**Uncertainty Relation for Non-Hermitian Operators.**— For any Hermitian operator, if we measure it in an arbitrary state, there will always be a finite uncertainty, unless the state is an eigenstate of the observable (Hermitian operator) that is being measured. Similarly, one can ask if there is an uncertainty associated to the measurement of any non-Hermitian operator. The variance of a non-Hermitian operator A in a state |ψ⟩ is defined as [50]

$$\Delta A^2 := \langle \psi | (A^\dagger - \langle A^\dagger \rangle)(A - \langle A \rangle) | \psi \rangle,$$  

where (A) = ⟨ψ|A|ψ⟩ and ⟨A⟩ = ⟨ψ|A|ψ⟩. Also, ΔA^2 can be written as

$$\Delta A^2 = \langle \psi | A^\dagger A | \psi \rangle - \langle \psi | A^\dagger | \psi \rangle \langle \psi | A | \psi \rangle = \langle f|f⟩,$$  

where |f⟩ = (A − ⟨A⟩)|ψ⟩. Even though A is non-Hermitian, |f⟩ is a valid quantum state as using the polar decomposition of A = SAUA renders |f⟩ as a linear combination of |ψ⟩ and SA(UA|ψ⟩). Similarly, we can define the uncertainty for the non-Hermitian operator B as

$$\Delta B^2 = \langle \psi | B^\dagger B | \psi \rangle - \langle \psi | B^\dagger | \psi \rangle \langle \psi | B | \psi \rangle = \langle g|g⟩,$$  

where |g⟩ = (B − ⟨B⟩)|ψ⟩. Now we have

$$\Delta A^2 \Delta B^2 = \langle f|f⟩ \langle g|g⟩ \geq |\langle f|g⟩|^2,$$  

whereby in the last line we have used the Cauchy-Schwarz inequality. This gives us

$$\Delta A \Delta B \geq |\langle A^\dagger B⟩ - \langle A^\dagger \rangle \langle B \rangle|.$$  \hspace{1cm} (22)

The above equation is true only when the domain of A contains the range of B. Now let us use the polar value decompositions of A and B, namely, A = SAUA and B = SBUB. Using these, we can simplify Eq. (22) as

$$\Delta A \Delta B \geq |\langle \psi | U_A^\dagger S_A S_B U_B | \psi \rangle - \langle \psi | U_A^\dagger S_A | \psi \rangle \langle S_B U_B | \psi \rangle| = |\phi | S_A P S_B | \chi \rangle \cdot |\langle \phi | \chi \rangle|,$$  \hspace{1cm} (23)

where P = (I − |ψ⟩⟨ψ|), |ϕ⟩ = U_A|ψ⟩ and |χ⟩ = U_B|ψ⟩. Thus, we have the generalized uncertainty relation for any two non-Hermitian operators as

$$\Delta A \Delta B \geq |\phi | S_A P S_B | \chi \rangle | \cdot |\langle \phi | \chi \rangle|.$$  \hspace{1cm} (24)

This means that the product of uncertainties in measurement of two non Hermitian operators is greater than or equal to the product of the absolute weak value of the non Hermitian operator S_A P S_B and absolute value of ⟨ϕ|χ⟩, where |ϕ⟩ = U_A|ψ⟩, |χ⟩ = U_B|ψ⟩, A = S_A U_A and B = S_B U_B. |ϕ| S_A P S_B |χ⟩ can be obtained experimentally following the procedure to measure the weak value of a non-Hermitian following Eq. (6).

For the case of Hermitian operators A and B, we have S_A = A, U_A = I, S_B = B and U_B = I. This leads to |ϕ⟩ = |ψ⟩ and |χ⟩ = |ψ⟩. Therefore, Eq. (24) becomes

$$\Delta A \Delta B \geq |\frac{1}{2} |\langle A, B \rangle | | | \psi \rangle |.$$  \hspace{1cm} (25)

Thus, for the usual Hermitian operators A and B, we have the Robertson uncertainty relation.

**The uncertainty relation for the creation and annihilation operators**.— Consider the uncertainty relation for creation and annihilation operators in the phase state |θ_m⟩. This reads as

$$\Delta \hat{a} \Delta \hat{a} \geq |(\hat{a}^\dagger)^2 - \langle \hat{a}^\dagger \rangle^2|,$$  \hspace{1cm} (26)

where A = s √⟨θ_m|A|θ_m⟩ − ⟨θ_m|A|θ_m⟩ ⟨θ_m|A|θ_m⟩ with A = a, a^\dagger. Using the expressions ⟨θ_m|a^\dagger a|θ_m⟩ = s^\frac{1}{2} = ⟨θ_m|a^\dagger a|θ_m⟩, ⟨θ_m|a^\dagger|θ_m⟩ = (s + 1)^{-1} e^{-θ_m} \sum_{n=0}^{s} \sqrt{n}, and ⟨θ_m|(a^\dagger)^2|θ_m⟩ = (s + 1)^{-2} e^{-2θ_m} \sum_{n=0}^{s} \sqrt{n(n-1)}, we have

$$\Delta \hat{a} \Delta \hat{a} = (\Delta \hat{a}^\dagger)^2 = |\frac{s}{2} - (s + 1)^{-1} \sum_{n=0}^{s} \sqrt{n(n-1)}|^2.$$  \hspace{1cm} (27)

and

$$|(\hat{a}^\dagger)^2 - \langle \hat{a}^\dagger \rangle^2| = (s + 1)^{-1} \sum_{n=0}^{s} \sqrt{n(n-1)}.$$  \hspace{1cm} (28)

![FIG. 1. This figure clearly shows that the uncertainty relation given for non Hermitian operators is satisfied by the creation and annihilation operators in phase states.](image-url)
Uncertainty for Kraus operator and fidelity of state.– One may ask if the uncertainty in the non-Hermitian operator can have some real physical meaning. Here we provide an example, which will show that uncertainties in the Kraus operators have a clear meaning. Suppose that an initial state of a system is $|\psi\rangle$ and it passes through a quantum channel. Let the Kraus elements of the channel be denoted by $\{E_k\}$. The state of the system after passing through the quantum channel is given by

$$|\psi\rangle \rightarrow \rho = \mathcal{E}(|\psi\rangle \langle \psi|) = \sum_k E_k |\psi\rangle \langle \psi| E_k^\dagger.$$  

(29)

Now, the fidelity between the pure initial state and the mixed final state is given by

$$F = \langle \psi | \rho | \psi \rangle = \sum_k (|\langle \psi | E_k | \psi \rangle|^2).$$  

(30)

This shows that, by measuring the average of the non-Hermitian operators $E_k$ in the state $|\psi\rangle$ one can find the fidelity between the input and the output state. Now consider the variance of $E_k$ in the state $|\psi\rangle$. This is given by

$$\Delta E_k^2 = \langle \langle \psi | E_k^\dagger E_k | \psi \rangle - \langle \langle \psi | E_k^\dagger | \psi \rangle \langle \psi | E_k | \psi \rangle \rangle.$$  

(31)

Summing over $k$ on both the sides and using the relation $\sum_k E_k^\dagger E_k = I$, we have

$$F + \sum_k \Delta E_k^2 = 1.$$  

(32)

This relation gives us a physical meaning to the uncertainties in the Kraus operators. This shows that if the total uncertainties in the Kraus operators is less, then the fidelity between the input and output states will be more. Thus, the fidelity and the uncertainty play a complementary role in the quantum channel. Hence, to preserve a state more efficiently, one should have less uncertainties in the Kraus operators.

Let us consider a specific example of qubit channel, namely, amplitude damping channel. The Kraus operators for this channel are given by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad \text{and} \quad E_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$  

(33)

If we pass an arbitrary state, $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$, of the qubit through the amplitude damping channel, then the output state is given by

$$\rho = \sum_{k=1}^{2} E_k |\psi\rangle \langle \psi| E_k^\dagger$$

$$= \frac{1}{2}[e_1 |0\rangle \langle 0| + e_2 |1\rangle \langle 1| + e_3 |0\rangle \langle 1| + e_4^* |1\rangle \langle 0|],$$  

(34)

where $e_1 = 1 + p + (1-p) \cos \theta$, $e_2 = (1-p) - (1-\cos \theta)$ and $e_3 = e^{-i\phi} \sqrt{1-p} \sin \theta$. The fidelity $F = \langle \psi | \rho | \psi \rangle$ is given by

$$F = \frac{1}{4} [3 + \sqrt{1-p} - p + 2p \cos \theta + (1-p - \sqrt{1-p}) \cos 2\theta].$$  

(35)

The product of the uncertainties in the Kraus operators for the amplitude damping channel has an upper bound, which is given by $\Delta E_1 \Delta E_2 \leq \frac{1-\rho}{2}$. Thus, we have, for a channel with two Kraus operators

$$\frac{1-F}{2} \geq \Delta E_1 \Delta E_2 \geq |\phi [S_1 \psi^\perp \langle \psi^\perp | S_2 \nu^\perp \rangle \cdot \langle \phi | \chi \rangle|,$$  

(36)

where $E_1 = S_1 U_1$, $E_2 = S_2 U_2$, $|\phi\rangle = U_1 |\psi\rangle$, $|\chi\rangle = U_2 |\psi\rangle$, and $|\psi\rangle + |\psi^\perp\rangle \langle \psi^\perp | = I$. Here, $S_1 = E_1$, $U_1 = I$, $S_2 = \sqrt{p} |0\rangle \langle 0|$, and $U_2 = \sigma_x$. We have

$$|\phi [S_1 \psi^\perp \langle \psi^\perp | S_2 \nu^\perp \rangle \cdot \langle \phi | \chi \rangle|$$

$$= 2 \cos \phi \cos^2 \frac{\theta}{2} \sin^4 \frac{\theta}{2} \sqrt{p} (1 - \sqrt{1-p}).$$  

(37)

$\Delta E_1 = \cos \left(\frac{\theta}{2}\right) \sin \left(\frac{\theta}{2}\right) \sqrt{p} (1 - \sqrt{1-p})$ and $\Delta E_1 = \sqrt{p} \cos^2 \left(\frac{\theta}{2}\right)$. Fig. (2) shows the bounds on $\Delta E_1 \Delta E_2$ as a function of $p$ at fixed value of $\theta = \pi/2$ and $\phi = \pi/4$, which validates Eq. (30).

**Conclusion.–** Even though quantum theory has been around since the last century, usual text books have always restricted to discussions of measurement of Hermitian operators as they possess real eigenvalues. In this paper, we have addressed the question of experimental feasibility of measuring the expectation value of any non-Hermitian operator in a pure quantum state. Surprisingly, quantum theory does allow measurement of any non-Hermitian operator. We have shown that the expectation value of a non-Hermitian operator in a quantum state is equal to the weak value of the positive semi-definite part of the operator, modulo a complex number. Due to the experimental viability of the weak measurements, it will be possible to measure the expectation value of the non-Hermitian operator. We have provided some examples to illustrate this technique. In particular, we have applied our method to measure the expectation value of creation operator in a general state. This leads to an interesting link between the sum of the square roots of first $s$ natural numbers and the expectation value of the creation operator. Since, this expectation value can be measured experimentally, one can estimate the value.
of the sum of the square roots of first $s$ natural numbers experimentally and compare it with the Ramanujan’s formula for the same. Furthermore, we have proved an uncertainty relation for any two non-Hermitian operators. This uncertainty relation takes the usual form once we consider the operators to be Hermitian. We have provided a relation between the average of variances of the Kraus elements of a channel, which can be Hermitian or non-Hermitian, and the channel fidelity. Our paper may open up the possibility of considering the non-Hermitian operators not only as a mathematical tool but also as an experimental arsenal in quantum theory, quantum information and almost all areas of physics.

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**Appendix**

**Measurement of $\mathcal{PT}$ symmetric Hamiltonian**

There exists a class of Hamiltonians which are non-Hermitian and yet they possess real eigenvalues when they respect $\mathcal{PT}$ symmetry. However, in general they possess non-normalizable eigenstates and complex eigenvalues, so in usual quantum world one may think that we cannot measure their expectation values. But using our formalism one can in principle measure them directly.

The simplest example of a general $\mathcal{PT}$ symmetric Hamiltonian in $2 \times 2$, is given by

$$H = \begin{pmatrix} re^{i\theta} & t \\ s & re^{-i\theta} \end{pmatrix},$$

(38)

where $r, s, t, \theta$ are the real parameters. The eigenvalues are given by $\epsilon_{\pm} = r \cos \theta \pm \sqrt{s t - r^2 \sin^2 \theta}$ and corresponding eigenstates of this Hamiltonian are given by

$$|\epsilon_+\rangle = \frac{1}{\sqrt{\cos \alpha}} \left( e^{i\alpha/2} \right), \quad |\epsilon_-\rangle = \frac{1}{\sqrt{\cos \alpha}} \left( e^{-i\alpha/2} \right),$$

where $\alpha$ is defined by the relation $\sin \alpha = \frac{r}{\sqrt{s t}} \sin \theta$. Let the polar decomposition of $H$ be $H = UR$, where $R = \sqrt{H^* H}$.
and for \( r^2 \neq st, U = HR^{-1} \). Here,

\[
R^2 = \begin{pmatrix}
  r^2 + s^2 & r(s + t)e^{-i\theta} \\
  r(s + t)e^{i\theta} & r^2 + t^2
\end{pmatrix}.
\]

(39)

Therefore, the positive semidefinite operator \( R \) for the non-Hermitian operator \( H \) is given by

\[
R = \frac{1}{2\sqrt{2}A} \begin{pmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{pmatrix},
\]

(40)

where \( R_{11} = (A - (s - t))B_- + (A + (s - t))B_+ \), \( R_{12} = 2r(B_+ - B_-)e^{-i\theta} \), \( R_{21} = 2r(B_+ - B_-)e^{i\theta} \) and \( R_{22} = (A + (s - t))B_- + (A - (s - t))B_+ \) with

\[
A = \sqrt{4r^2 + (s - t)^2}
\]

(41)

\[
B_\pm = \sqrt{2r^2 + s^2 + t^2 \pm (s + t)A}.
\]

(42)

Now, we have

\[
R^{-1} = \frac{1}{2\sqrt{2}AB_+B_-} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},
\]

(43)

where \( S_{11} = (A + (s - t))B_- + (A - (s - t))B_+ \), \( S_{12} = -2r(B_+ - B_-)e^{-i\theta} \), \( S_{21} = -2r(B_+ - B_-)e^{i\theta} \), and \( S_{22} = (A - (s - t))B_- + (A + (s - t))B_+ \). Using \( U = HR^{-1} \), we have

\[
U = \frac{1}{\sqrt{2AB_+B_-}} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},
\]

(44)

where \( U_{11} = [(A + (s - t))B_- + (A - (s - t))B_+ - 2t(B_+ - B_-)]re^{i\theta}, U_{12} = t[(A - (s - t))B_- + (A + (s - t))B_+ - 2r^2(B_+ - B_-), U_{21} = s[(A + (s - t))B_- + (A - (s - t))B_+ - 2r^2(B_+ - B_-)]re^{-i\theta} \). For the special case of \( s = t \), \( r \neq \pm s \) and \( r > s \), we have \( H = UR \), where

\[
R = \begin{pmatrix} r & s e^{-i\theta} \\ s e^{i\theta} & r \end{pmatrix},
\]

(45)

\[
U = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.
\]

(46)

For other special case of \( s = t, r \neq \pm s \) and \( r < s \), we have \( H = UR \), where

\[
R = \begin{pmatrix} s & r e^{-i\theta} \\ r e^{i\theta} & s \end{pmatrix},
\]

(47)

\[
U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Now the expectation value of \( A \) in a general single qubit state \(|\psi\rangle = \cos(\eta/2)|0\rangle + e^{i\xi} \sin(\eta/2)|1\rangle \) is given by

\[
\langle \psi | H | \psi \rangle = r \cos \theta + s \cos \xi \sin \eta + i r \sin \theta \cos \eta.
\]

(47)

The above expectation value of \( A \) in the state \(|\psi\rangle \) can be realized experimentally, if we measure \( R \), which is Hermitian, weakly with the preselection in the state \(|\psi\rangle \) and postselection in the state \( U^\dagger |\psi\rangle \) and then multiply the weak value, thus obtained, with the complex number \( \langle \psi | U | \psi \rangle \).