On the Comparison between the Reliability of Units Produced by Different Production Lines

Mahmoud M. M. Mansour | Rashad M. EL-Sagheer | Mohamed A. W. Mahmoud | Mohamed S. Aboshady

Faculty of Engineering, Department of Basic Science, The British University in Egypt, El-Sherouk City, Egypt
Faculty of Science, Department of Mathematics, Al-Azhar University, Cairo, Egypt
High Institute of Computer and Management Information System, First Statement, New Cairo, Cairo, Egypt

Correspondence
Mohamed S. Aboshady, Faculty of Engineering, Department of Basic Science, The British University in Egypt, El-Sherouk City, Egypt.
Email: Mohamed.Aboshady@bue.edu.eg

Abstract
The paper discusses how to evaluate the reliability of units produced by different production lines. The procedure is based on selecting independent random samples of units produced by different production lines and then evaluating reliability functions for each group of units. The comparison between these reliability functions at a given time allows manufacturing experts to evaluate the effectiveness of production lines. A statistical methodology has been taken based on the assumption that the lifetime of units produced by each product line has a Weibull Gamma distribution. Then, real-world data are used to illustrate the study’s contribution to reliability theory applications.

Keywords
Accelerated bias-corrected confidence interval (Boot-BCa), bias-corrected confidence interval (Boot-BC), comparing the reliability between \( k \) production lines, joint progressive Type-II censoring, MCMC technique

1 | INTRODUCTION

In the manufacturing sectors, automation has increased plant capacity in the process industries. As a result, productivity and product quality both increase. However, significant expenditure is also the main issue for all automated factories. Therefore, it is also expected that operating systems or production lines inside this plant will function properly and for a long time. The question is how to know the ability of these systems or production lines will perform their missions successfully without failure or at least predict the failure time of one of these product lines. This knowledge may give manufacturers a great opportunity to work on the analysis of failure and hence follow the procedures of maintenance policy at a suitable time. Most studies focus on the reliability analysis of various systems in different process industries like the sugar industry, and thermal power plants by checking the performance of the operating equipment. This checking can be implemented via one of the following approaches, fault tree analysis, failure mode and effect analysis, supplementary variable technique, fuzzy Lambda–Tau technique, availability, maintainability, and dependability analysis. For more details, see Kumar et al.\(^1\) and Soltanali.\(^2\) The practical previous approaches stated may require some effort and cost and may be subjected to measurement errors. In this work, the reliability of production lines will be estimated based on the lifetime of the units produced by these production lines from an inferential statistical view. The authors in this paper try to get benefits from statistical inference approaches via applying these approaches through practical situations interested in estimating the reliability.

The progressive Type-II censoring acquires its importance from giving the experimenter the flexibility to remove some units from the test at non-terminal time points. This feature leads to reduce the cost of carrying out such experiments and
accelerates the lifetime of the test. Life testing aims to obtain information about the population parameters through testing some units, where knowing this information or data help statisticians to estimate, for instance, the reliability function after fitting these data to the appropriate population. Several authors have studied the estimation problems based on progressive Type-II censored samples, for example, Wu and Chang,1 Abdel-Hamid and AL-Hussaini,4 Ahmed,5 El-Sagheer and Ahsanullah,6 and Abdel-Hamid.7 The joint censoring scheme has been previously developed in the literature. For example, Balakrishnan and Rasouli8 discussed exact inference for two exponential populations when Type-II censoring is implemented on the two samples in a combined manner. Balakrishnan and Feng9 generalized their work to the case of $k$ samples situation. Recently, statisticians turn their attention to a new development of progressive Type-II censoring, which is called joint progressive Type-II censoring (JPROG-II-C). Balakrishnan and Rasouli10 subsequently extended their work to the case of two exponential populations when JPROG-II-C is implemented on the two samples. Doostparast et al.11 estimated the parameters of two Weibull populations based on JPROG-II-C samples using the Bayesian estimation under LINEX loss function. Balakrishnan et al.12 generalized the work of Balakrishnan and Rasouli10 to $k$ exponential populations. The JPROG-II-C samples can be observed mostly in experimental situations. Assume that units are being manufactured by $k$ different lines within the same facility. From each production line $h$, a random sample of size $n_h$, $h = 1, 2, … , k$, is selected where the samples are independent. Now, these independent samples are simultaneously put to life testing. The test may be expensive or time-consuming, so the experimenter will proceed to apply the progressive Type-II strategy. Accordingly, the experimenter will terminate the experiment when the $r$th failure occurs. In this paper, $k$ independent samples from WG populations are put under life testing based on JPROG-II-C. The remainder of this paper is organized as follows: Section 2 provides a description of WG distribution (WGD) and the test steps. In Section 3, the maximum likelihood estimates (MLEs) of the parameters under consideration are obtained in addition to the corresponding ACIs and four types of bootstrap confidence intervals are also computed. The comparison between the reliability of $k$ production lines is also developed in Section 3. Section 4 is devoted to the Bayesian approach that uses the famed MCMC technique. An illustrative example is presented to explain the theoretical results at $k = 2$ in Section 5. In addition to computing the expected values of the number of failures before the test procedure, the exact values of the number of failures are also observed after the test procedure. A simulation study is performed to compare the expected values with the exact values in Section 6 at $k = 2$ in addition to applying on a real data example. Finally, the conclusion is placed in Section 7.

# 2 MODEL DESCRIPTION

This section contains two subsections. The first is devoted to giving a brief description of WGD. The likelihood function under $k$ populations is presented based on JPROG-II-C in the second subsection.

## 2.1 Weibull gamma distribution

The WG is a suitable distribution for the phenomenon of loss of signals in telecommunications, which is called fading, when multipath is superimposed on shadowing, see Bithas.13 The WGD had been introduced by Nadarajah and Kotz.14 A random variable $X$ is said to have WGD with scale parameter $\alpha$ if its probability density function (PDF) is given by

$$f(x; \alpha, \theta, \beta) = \frac{\theta \beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} \left(1 + \left(\frac{x}{\alpha}\right)^{\beta}\right)^{-(\beta+1)}, \quad x > 0; \alpha, \theta, \beta > 0, \tag{1}$$

and the reliability function at any time $t$ is

$$R(t) = P(X > t) = \left(1 + \left(\frac{t}{\alpha}\right)^{\beta}\right)^{-\beta}, \quad t > 0. \tag{2}$$

For more details about WGD and its properties, see Molenberghs and Verbeke15 and Mahmoud et al.16 The estimation of WGD parameters is discussed by EL-Sagheer17 based on progressive Type-II censored samples.
2.2 Implementation of JPROG-II-C

Suppose that \( X_{h1}, X_{h2}, \ldots, X_{hn_h} \) are the lifetimes of \( n_h \) units from line \( A_h \), and assume that they are independent and identically distributed (iid) variables from a population with cumulative distribution function (CDF) \( F_h(x) \) and PDF \( f_h(x) \), \( h = 1, 2, \ldots, k \). Let \( N = \sum_{h=1}^{k} n_h \) denotes the total sample size, \( r \) denotes the total number of the observed failures and \( \xi_1 \leq \xi_2 \leq \ldots \leq \xi_N \) represent the order statistics of the \( N \) random variables \( \{X_{h1}, X_{h2}, \ldots, X_{hn_h}\} \). When JPROG-II-C is implemented on \( k \) samples, the observable data will consist of \((\delta, \xi)\), where \( \xi = (\xi_1 \leq \xi_2 \leq \ldots \leq \xi_r) \), \( \xi_i \in \{X_{h1}, X_{h2}, \ldots, X_{hn_h}\} \) for, \( h_1 = 1, 2, \ldots, k \), \( i = 1, 2, \ldots, r \), and depending on \((h_1 \leq h_2 \leq \ldots \leq h_r)\), \( \delta \) can be defined as \( \delta = (\delta_1(h), \delta_2(h), \ldots, \delta_r(h)) \), where

\[
\delta_i(h) = \begin{cases} 1, & \text{if } h = h_i \\ 0, & \text{elsewhere} \end{cases},
\]

with \( \sum_{h=1}^{k} \delta_i(h) = 1 \), for \( i = 1, 2, \ldots, r \). After the first failure occurrence at time \( \xi_1 \), \( R_1 \) units are randomly removed from the remaining \( N - 1 \) surviving units. At time \( \xi_2 \), the second failure happened and \( R_2 \) units are randomly removed from the remaining \( N - R_1 - 2 \) surviving units. Finally, at time \( \xi_r \), \( R_r = N - r - \sum_{i=1}^{r-1} R_i \) surviving units are removed from the test when the \( r \)th failure happened. A detailed plan for PROG-II-C is discussed by Balakrishnan and Aggarwala. Let \( s_j(h) \) represent the number of the removed units at the occurrence time of the \( i \)th failure from sample \( h \). Hence \( R_i = \sum_{h=1}^{k} s_i(h) \), which is previously determined by the experimenter, for \( i = 1, 2, \ldots, r \). According to the JPROG-II-C, it is possible to assume \( r = \sum_{h=1}^{k} M_r(h) \) where \( M_r(h) = \sum_{i=1}^{r} \delta_i(h) \) represents the number of failures occurred in sample \( h \).

3 Maximum Likelihood Estimation

The log-likelihood functions are the basis for deriving estimators of parameters, given data. MLEs are characterized by different advantages such as asymptotically normally distributed, asymptotically minimum variance, asymptotically unbiased, and satisfy the invariant property, see Azzalini and Royall for more information on likelihood theory. Balakrishnan et al. introduced the likelihood function in a general form for \( k \) samples drawn from any PDF under JPROG-II-C as

\[
L(\alpha_1, 2, \ldots, k, \beta_1, 2, \ldots, k, \delta, s, \xi) = c_r \prod_{i=1}^{r} \prod_{h=1}^{k} (f_h(\xi_i))^{\delta_i(h)} \prod_{i=1}^{r} \prod_{h=1}^{k} (F_h(\xi_i))^{s_i(h)},
\]

where \( s = (s_1(h), s_2(h), \ldots, s_r(h)) \) with \( F_h(\xi_i) = 1 - F_h(\xi_i) \) and \( c_r = D_1 D_2 \), with

\[
D_1 = r \prod_{i=1}^{r} \left[ \sum_{h=1}^{k} \left( n_h - M_{\alpha_{i-1}}(h) - \sum_{j=1}^{i-1} s_j(h) \right) \delta_i(h) \right],
\]

and

\[
D_2 = \prod_{i=1}^{r-1} \left[ \prod_{h=1}^{k} \left( \frac{n_h - M_{\alpha_{i-1}}(h) - \sum_{j=1}^{i-1} s_j(h)}{s_i(h)} \right) \right].
\]

If the removal of units occur only at the terminal point, then we will obtain the joint Type-II censoring scheme as a special case of JPROG-II-C, that is, \( s_i(h) = 0 \), therefore \( R_i = 0 \) when \( i = 1, 2, \ldots, r - 1 \) and \( s_r(h) = n_h - M_r(h) \), for all \( h = 1, 2, \ldots, k \), hence \( R_r = \sum_{h=1}^{k} s_r(h) = N - r \). For \( F_h(\xi_i) = (1 + (\xi_i/\alpha_h)^{\beta_h})^{-\beta_h} \) and \( f_h(\xi_i) = \frac{\beta_h}{\alpha_h^{\beta_h+1}} (\xi_i/\alpha_h)^{\beta_h-1} (1 + (\xi_i/\alpha_h)^{\beta_h})^{-\beta_h+1} \), \( i = 1, 2, \ldots, r \), \( h = 1, 2, \ldots, k \), representing the survival function and PDF for the test units respectively, Equation (4) can be
written as follows:

\[
L(\alpha_1,\ldots,k,\delta_{1,\ldots,k},\beta_{1,\ldots,k},\delta,s,\zeta) = c \prod_{h=1}^{k} \left( \frac{\theta_h \beta_h}{\alpha_h} \right)^{M_r(h)} \times \exp \left\{ (\theta_h - 1) \left[ \sum_{l=1}^{r} \delta_l(h) \log \zeta_l - M_r(h) \log \alpha_h \right] \right\} 
\]

\[
\times \exp \left\{ - \sum_{l=1}^{r} \left[ (\beta_h(\delta_l(h) + s_l(h)) + \delta_l(h)) \log \left( 1 + \left( \frac{\zeta_l}{\alpha_h} \right)^{\beta_h} \right) \right] \right\}.
\]

(5)

The log-likelihood function may then be written as

\[
\log L = \log c_r + \sum_{h=1}^{k} M_r(h)(\log \theta_h + \log \beta_h - \theta_h \log \alpha_h)
\]

\[
+ \sum_{h=1}^{k} \sum_{i=1}^{r} \left[ (\theta_h - 1)\delta_i(h)\log \zeta_i - (\beta_h(\delta_i(h) + s_i(h)) + \delta_i(h)) \log \left( 1 + \left( \frac{\zeta_i}{\alpha_h} \right)^{\beta_h} \right) \right],
\]

and thus we have the likelihood equations for \(\alpha_h, \theta_h\) and \(\beta_h\), \(h = 1, 2, \ldots, k\), respectively, as

\[
\frac{M_r(h)\theta_h - \theta_h}{\alpha_h} \frac{\theta_h + 1}{\sum_{i=1}^{r} \frac{\beta_i(h)(\delta_i(h) + s_i(h)) + \delta_i(h)}{1 + \left( \frac{\zeta_i}{\alpha_h} \right)^{\beta_i(h)}}} = 0,
\]

(6)

\[
(1 - \theta_h \log \alpha_h) \frac{M_r(h)}{\beta_h} + \sum_{i=1}^{r} \left[ \delta_i(h) \log \zeta_i - \frac{1}{\alpha_h} \left( \frac{\beta_i(h)(\delta_i(h) + s_i(h)) + \delta_i(h)}{1 + \left( \frac{\zeta_i}{\alpha_h} \right)^{\beta_i(h)}} \right) \right] = 0,
\]

(7)

and

\[
\frac{M_r(h)}{\beta_h} - \sum_{i=1}^{r} (\delta_i(h) + s_i(h)) \log \left( 1 + \left( \frac{\zeta_i}{\alpha_h} \right)^{\beta_i(h)} \right) = 0.
\]

(8)

Since the above Equations (6)–(8) are nonlinear simultaneous equations in \(3k\) unknown variables \(\alpha_h, \theta_h, \beta_h\) for \(h = 1, 2, \ldots, k\), it is obvious that an exact solution is not easy to obtain. Therefore, a numerical method such as Newton Raphson can be used to find approximate solution. The algorithm has been implemented using the following steps:

(1) Use the method of moments or some other proper estimates of the parameters as initial points for iteration, denote the initials as \((\alpha_{h(0)}, \theta_{h(0)}, \beta_{h(0)})\) for the parameters \((\alpha_h, \theta_h, \beta_h)\) and set \(j = 0\).

(2) Calculate \((\frac{\partial \log L}{\partial \alpha_h}, \frac{\partial \log L}{\partial \theta_h}, \frac{\partial \log L}{\partial \beta_h})\) at \((\alpha_{h(j)}, \theta_{h(j)}, \beta_{h(j)})\) and the observed asymptotic Fisher information matrix \(I^{-1}(\alpha_h, \theta_h, \beta_h)\), given in the next section.

(3) Update \((\alpha_h, \theta_h, \beta_h)\) as

\[
(\alpha_{h(j+1)}, \theta_{h(j+1)}, \beta_{h(j+1)}) = (\alpha_{h(j)}, \theta_{h(j)}, \beta_{h(j)}) - \left( \frac{\partial \log L}{\partial \alpha_h}, \frac{\partial \log L}{\partial \theta_h}, \frac{\partial \log L}{\partial \beta_h} \right)_{(\alpha_{h(j)}, \theta_{h(j)}, \beta_{h(j)})} \times I^{-1}(\alpha_h, \theta_h, \beta_h).
\]

(4) Put \(j = j + 1\), and then return to step 2.

(5) Continue the consecutive steps until

\[
| (\alpha_{h(j+1)}, \theta_{h(j+1)}, \beta_{h(j+1)}) - (\alpha_{h(j)}, \theta_{h(j)}, \beta_{h(j)}) | \leq \varepsilon \rightarrow 0.
\]

The final estimates of \((\alpha_h, \theta_h, \beta_h)\) are the MLEs of the parameters, denoted as \((\hat{\alpha}_h, \hat{\theta}_h, \hat{\beta}_h)\) for \(h = 1, 2, \ldots, k\).

In the next two subsections, ACIs and four different bootstrap confidence intervals for the parameters \((\alpha_h, \theta_h, \beta_h)\) are deduced based on the MLEs.
3.1 Approximate confidence intervals

The $(1 - \vartheta)100\%$ ACIs for the parameters $\alpha_h, \theta_h, \beta_h$, $h = 1, 2, \ldots, k$, can be written as

$$(\hat{\alpha}_{hL}, \hat{\alpha}_{hU}) = \hat{\alpha}_h \pm z_{1 - \frac{\vartheta}{2}} \sqrt{\text{var}(\hat{\alpha}_h)},$$

$$(\hat{\theta}_{hL}, \hat{\theta}_{hU}) = \hat{\theta}_h \pm z_{1 - \frac{\vartheta}{2}} \sqrt{\text{var}(\hat{\theta}_h)},$$

$$(\hat{\beta}_{hL}, \hat{\beta}_{hU}) = \hat{\beta}_h \pm z_{1 - \frac{\vartheta}{2}} \sqrt{\text{var}(\hat{\beta}_h)},$$

where $z_{1 - \frac{\vartheta}{2}}$ is the percentile of the standard normal distribution with left-tail probability $1 - \frac{\vartheta}{2}$, and $\text{var}(\hat{\alpha}_h), \text{var}(\hat{\theta}_h), \text{var}(\hat{\beta}_h)$ represent the asymptotic variances of the MLEs, which can be calculated using the inverse of the Fisher information matrix. Let $I(\Omega_1, \Omega_2, \Omega_3) = I(\Omega_1^1, \ldots, \Omega_1^k, \Omega_2^1, \ldots, \Omega_2^k, \Omega_3^1, \ldots, \Omega_3^k)$ denote the asymptotic Fisher information matrix of the parameters $\Omega_1 = \alpha_h, \Omega_2 = \theta_h$, and $\Omega_3 = \beta_h$, $h = 1, 2, \ldots, k$, where

$$I(\Omega_1, \Omega_2, \Omega_3) = -\left(\frac{\partial^2 \log L}{\partial \Omega_i \partial \Omega_j}\right), i, j = 1, 2, 3,$$

for each $h$.

The asymptotic variance–covariance matrix for the MLEs can then be calculated as follows:

$$I^{-1} = \left[-\left(\frac{\partial^2 \log L}{\partial \Omega_i \partial \Omega_j}\right)\right]^{-1}_{(\hat{\Omega}_1, \hat{\Omega}_2, \hat{\Omega}_3)},$$

(9)

for more details see Cohen.21

3.2 Bootstrap confidence intervals

The Bootstrap confidence intervals are proposed based on the parametric bootstrap methods where the parametric model for the data is known, $f(\xi; \cdot)$, up to the unknown parameters $(\alpha_h, \theta_h, \beta_h)$, such that the bootstrap data are sampled from $f(\xi; \hat{\alpha}_h, \hat{\theta}_h, \hat{\beta}_h)$, where $(\hat{\alpha}_h, \hat{\theta}_h, \hat{\beta}_h)$ are the MLEs from the original data. A lot of papers dealt only with percentile bootstrap method (Boot-p) based on the idea of Efron22 and bootstrap-t method (Boot-t) based on the idea of Hall,23 such as Soliman et al.,24 El-Sagheer,25 and others. In this article, two additional types of Bootstrap CIs, Boot-BC and Boot-BCa, based on the idea of DiCiccio and Efron,26 are discussed. The following algorithm is followed to obtain bootstrap samples for the four methods:

(1) Based on the original JPROG-II-C sample, $\zeta_1 \leq \zeta_2 \leq \ldots \leq \zeta_r$, compute $\hat{\alpha}_h, \hat{\theta}_h, \hat{\beta}_h$ for $h = 1, 2, \ldots, k$.

(2) Use $\hat{\alpha}_h, \hat{\theta}_h$, and $\hat{\beta}_h$ to generate a bootstrap sample $\zeta^*$ with the same values of $R_i$, $i = 1, 2, \ldots, r$ using algorithm presented in Balakrishnan and Sandhu.27

(3) As in step (1) and based on $\zeta^*$ compute the bootstrap sample estimates for $\hat{\alpha}_h^*, \hat{\theta}_h^*$, and $\hat{\beta}_h^*$, say $\hat{\alpha}_h^*, \hat{\theta}_h^*$, and $\hat{\beta}_h^*$.

(4) Repeat the previous steps (2) and (3) $B$ times and arrange all $\hat{\alpha}_h^*, \hat{\theta}_h^*$ and $\hat{\beta}_h^*$ in ascending order to obtain the bootstrap sample $(\hat{\Omega}_h^*[1], \hat{\Omega}_h^*[2], \ldots, \hat{\Omega}_h^*[B])$, $j = 1, 2, 3$, where $\hat{\Omega}_h^* = (\hat{\Omega}_h^*[1], \hat{\Omega}_h^*[2], \hat{\Omega}_h^*[3], \hat{\beta}_h^*)$.

3.2.1 Bootstrap-p confidence interval

Let $\Phi(z) = P(\hat{\Omega}_h^* \leq z)$ be the CDF of $\hat{\Omega}_h^*$. Define $\hat{\Omega}_h^{\text{Boot}}(\frac{\vartheta}{2}) = \Phi^{-1}(z)$ for a given $z$. The approximate bootstrap-p 100(1 - $\vartheta$)% confidence interval of $\hat{\Omega}_h^*$ is given by

$$\left[\hat{\Omega}_h^{\text{Boot}}(\frac{\vartheta}{2}), \hat{\Omega}_h^{\text{Boot}}(1 - \frac{\vartheta}{2})\right].$$
3.2.2 Bootstrap-t confidence interval

Consider the order statistics $\mu_{j_h}^{[1]} < \mu_{j_h}^{[2]} < \ldots < \mu_{j_h}^{[B]}$ where

$$\mu_{j_h}^{[p]} = \frac{\sqrt{B} (\hat{\Omega}_{j_h}^{[p]} - \hat{\Omega}_{j_h})}{\sqrt{Var(\hat{\Omega}_{j_h}^{[p]})}}, \quad p = 1, 2, \ldots, B; \quad j = 1, 2, 3; \quad \text{for } h = 1, 2, \ldots, k.$$ 

where $\hat{\Omega}_{j_h} = \hat{\alpha}_h$, $\hat{\Omega}_{2_h} = \hat{\delta}_h$, and $\hat{\Omega}_{3_h} = \hat{\beta}_h$ while $Var(\hat{\Omega}_{j_h}^{[p]})$ is obtained using the inverse of the Fisher information matrix as done before in Equation (9). Let $W(z) = P(\mu_{j_h}^{*} < z)$, $j = 1, 2, 3$ be the CDF of $\mu_{j_h}^{*}$. For a given $z$, define

$$\hat{\Omega}_{j_h,\text{Boot} - t}^* = \hat{\Omega}_{j_h} + B^{-1/2} \sqrt{Var(\hat{\Omega}_{j_h}^{[p]})} W^{-1}(z).$$

Thus, the approximate bootstrap-t $100(1 - \vartheta)%$ confidence interval of $\hat{\Omega}_{j_h}^{*}$ is given by

$$\left[ \hat{\Omega}_{j_h,\text{Boot} - t}^* \left( \frac{\vartheta}{2} \right), \quad \hat{\Omega}_{j_h,\text{Boot} - t}^* \left( 1 - \frac{\vartheta}{2} \right) \right].$$

3.2.3 Bootstrap bias-corrected confidence interval

Let $\Phi(z) = \vartheta$ be the standard normal CDF, with $z_\vartheta = \Phi^{-1}(\vartheta)$. Define the bias-correction constant $z_o$ from the following probability $P(\hat{\Omega}_{j_h}^{*} \leq \hat{\Omega}_{j_h}) = G(z_o), \quad j = 1, 2, 3$, where $G(.)$ is the CDF of the bootstrap distribution and

$$P(\hat{\Omega}_{j_h}^{*} \leq \hat{\Omega}_{j_h}) = \frac{\# \left\{ \hat{\Omega}_{j_h}^{[p]} < \hat{\Omega}_{j_h} \right\}}{B}, \quad p = 1, 2, \ldots, B; \quad j = 1, 2, 3; \quad \text{for } h = 1, 2, \ldots, k.$$ 

thus

$$z_o = \Phi^{-1} \left( \frac{\# \left\{ \hat{\Omega}_{j_h}^{[p]} < \hat{\Omega}_{j_h} \right\}}{B} \right), \quad p = 1, 2, \ldots, B; \quad j = 1, 2, 3; \quad \text{for } h = 1, 2, \ldots, k. \quad (10)$$

For a given $\vartheta$, and the bias-correction constant $z_o$, then

$$\hat{\Omega}_{j_h,\text{Boot} - BC}^* = G^{-1}[\Phi(2z_o + z_\vartheta)]. \quad (11)$$

Thus, the approximate bootstrap-BC $100(1 - \vartheta)%$ confidence interval of $\hat{\Omega}_{j_h,\text{Boot} - BC}^*$ is given by

$$\left[ \hat{\Omega}_{j_h,\text{Boot} - BC}^* \left( \frac{\vartheta}{2} \right), \quad \hat{\Omega}_{j_h,\text{Boot} - BC}^* \left( 1 - \frac{\vartheta}{2} \right) \right].$$

3.2.4 Bootstrap bias-corrected accelerated confidence interval

Let $\Phi(z) = \vartheta$ be the standard normal CDF, with $z_\vartheta = \Phi^{-1}(\vartheta)$ and the bias-correction constant $z_o$, which is defined in Equation (10). Then

$$\hat{\Omega}_{j_h,\text{Boot} - BCA}^* = G^{-1} \left[ \Phi \left( z_o + \frac{z_o + z_\vartheta}{1 - a_{j_h}(z_o + z_\vartheta)} \right) \right], \quad j = 1, 2, 3; \quad \text{for } h = 1, 2, \ldots, k. \quad (12)$$
where $a_{jh}$ is called the acceleration factor, which is estimated by a simple jack-knife method. Let $y_j$ represent the original data with the $i$th point omitted, say $y_{−i,j} = y_{1:j} < y_{2:j} < \ldots < y_{r:j}$, and $\hat{\Omega}_{jh} = \hat{\Omega}_j(y_{−i,j})$ be the estimate of $\Omega_{jh}$ constructed from this data, $\Omega_{1h} = \alpha h$, $\Omega_{2h} = \theta h$, and $\Omega_{3h} = \beta h$. Let $\bar{\Omega}_{jh}$ be the mean of the $\hat{\Omega}_{jh}$'s. Then $a_{jh}$ is estimated by

$$a_h = \frac{\sum_{i=1}^{r} (\bar{\Omega}_{jh} - \hat{\Omega}_{jh}^i)^3}{6 \left[ \sum_{i=1}^{r} (\bar{\Omega}_{jh} - \hat{\Omega}_{jh}^i)^2 \right]^2}, \quad j = 1, 2, 3; \quad \text{for } h = 1, 2, \ldots, k.$$  

For more details see Refs.28,29 If $a_h = 0$, Equation (12) reduces to Equation (11). Then, the approximate bootstrap-BC $100(1 - \delta)$% confidence interval of $\hat{\Omega}_{kBoo}^{*} \cap BC_a$ is given by

$$\left[ \hat{\Omega}_{jhBoo}^{*} - BC_a \left( \frac{\delta}{2} \right), \hat{\Omega}_{jhBoo}^{*} - BC_a \left( 1 - \frac{\delta}{2} \right) \right].$$

### 3.3 Comparison between the reliability of $k$ production lines

To compare the reliability of the units from $k$ production lines, the following lemma is needed.

**Lemma 1.** If the random variable $X \sim WG(\alpha, \theta, \beta)$, then $Y = (\frac{T}{\alpha})^\beta \sim WG(1, 1, \beta)$.

**Proof.** The proof is easy to obtain. If a unit $U_i$ is selected randomly from the production line $i$ and another one $U_j$ is selected randomly from the production line $j$, for $i, j = 1, 2, \ldots, k$ and $i \neq j$. Based on Equation (2), the invariance property of the MLEs of the parameters and using Lemma 1, the reliability function for any two production lines $i$ and $j$ at time $t$ are

$$R_i(t) = (1 + t)^{-\hat{\beta}_i} \quad \text{and} \quad R_j(t) = (1 + t)^{-\hat{\beta}_j}; i, j = 1, 2, \ldots, k, \quad i \neq j.$$  

(13)

According to the transformation stated in the previous lemma, we can prove the existence and uniqueness of the MLEs, for $\hat{\beta}_h$ only, graphically, see Balakrishnan and Kateri30 for more details. Equation (8) can be rewritten as follows:

$$\frac{1}{\hat{\beta}_h} = \frac{\sum_{i=1}^{r} (\delta_i(h) + s_i(h)) \log \left( 1 + \left( \frac{c_i}{\alpha_i} \right)^{\frac{\alpha h}{\beta h}} \right)}{M_r(h)} = \text{Constant.}$$  

(14)

**Corollary 1.** For any two production lines $i$ and $j$ and $\hat{\beta}_i \leq \hat{\beta}_j$, then $R_i(t) \geq R_j(t)$ at any time $t$, as illustrated in Figure 1.

### 4 Bayesian Estimation Based on Different Loss Functions

Let the prior knowledge of the parameters $\alpha_h, \theta_h, \text{and} \beta_h$ be described by the following prior distributions:

$$\pi_1(\alpha_h) \propto \alpha_h^{\mu_h-1} e^{-\alpha_h\lambda_h}, \quad \alpha_h > 0,$$

$$\pi_2(\theta_h) \propto \theta_h^{\nu_h-1} e^{-\theta_h\omega_h}, \quad \theta_h > 0,$$

$$\pi_3(\beta_h) \propto \beta_h^{\rho_h-1} e^{-\beta_h\upsilon_h}, \quad \beta_h > 0,$$

(15)

and $\alpha_h, \theta_h, \text{and} \beta_h$ are independent random variables for all $h = 1, 2, \ldots, k$. Many authors like Kundu and Howlader,31 Dey and Dey32 and Dey et al.33 established the Bayesian estimation for their parameter models based on informative gamma priors. Mansour and Ramadan34 studied parameters estimation of the modified extended exponential distribution based...
on a Hybrid Type-II censoring scheme, they obtained the Bayesian estimates for noninformative priors and Gamma priors, and they concluded that Bayesian estimates for Gamma priors have the best performance.

Hence, the joint prior of the parameters $\alpha_h, \theta_h,$ and $\beta_h$ can be written as follows:

$$\pi(\alpha_h, \theta_h, \beta_h) \propto \alpha_h^{\mu_h-1} \theta_h^{\nu_h-1} \beta_h^{\rho_h-1} e^{-(\alpha_h \lambda_h + \theta_h \omega_h + \beta_h \upsilon_h)}.$$  \hfill (16)

The joint posterior density function of $\alpha_h, \theta_h,$ and $\beta_h$, denoted by $\pi^*(\alpha_h, \theta_h, \beta_h | \delta, s, \zeta)$ can be written as

$$\pi^*(\alpha_h, \theta_h, \beta_h | \delta, s, \zeta) = \frac{L(\alpha_h, \theta_h, \beta_h | \delta, s, \zeta) \times \pi(\alpha_h, \theta_h, \beta_h)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\alpha_h, \theta_h, \beta_h | \delta, s, \zeta) \times \pi(\alpha_h, \theta_h, \beta_h) d\alpha_h d\theta_h d\beta_h}. \hfill (17)$$

The joint posterior distribution, which combines the information in both the prior distribution and the likelihood, leads to containing more accurate information and getting a narrower range of possible values for the parameters. The Bayes estimate of any function of the parameters, say $g(\alpha_h, \theta_h, \beta_h), h = 1, 2, \ldots, k,$ using SEL function is given by

$$\hat{g}_{BS}(\alpha_h, \theta_h, \beta_h) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty g(\alpha_h, \theta_h, \beta_h) \times L(\alpha_h, \theta_h, \beta_h) \times \pi(\alpha_h, \theta_h, \beta_h) d\alpha_h d\theta_h d\beta_h}{\int_0^\infty \int_0^\infty \int_0^\infty L(\alpha_h, \theta_h, \beta_h) \times \pi(\alpha_h, \theta_h, \beta_h) d\alpha_h d\theta_h d\beta_h}, \hfill (18)$$

while the Bayes estimate of $g(\alpha_h, \theta_h, \beta_h)$, for all $h = 1, 2, \ldots, k$, using LINEX loss function is given by

$$\hat{g}_{BL}(\alpha_h, \theta_h, \beta_h) = -\frac{1}{c} \log \left[ \frac{\int_0^\infty \int_0^\infty \int_0^\infty e^{-c g(\alpha_h, \theta_h, \beta_h)} \pi(\alpha_h, \theta_h, \beta_h) d\alpha_h d\theta_h d\beta_h}{\int_0^\infty \int_0^\infty \int_0^\infty L(\alpha_h, \theta_h, \beta_h) \times \pi(\alpha_h, \theta_h, \beta_h) d\alpha_h d\theta_h d\beta_h} \right], \ c \neq 0, \hfill (19)$$

where

$$\frac{\int_0^\infty \int_0^\infty \int_0^\infty e^{-c g(\alpha_h, \theta_h, \beta_h)} \times L(\alpha_h, \theta_h, \beta_h) \times \pi(\alpha_h, \theta_h, \beta_h) d\alpha_h d\theta_h d\beta_h}{\int_0^\infty \int_0^\infty \int_0^\infty L(\alpha_h, \theta_h, \beta_h) \times \pi(\alpha_h, \theta_h, \beta_h) d\alpha_h d\theta_h d\beta_h}.$$  \hfill (20)

It is noticed that the ratio of the two integrals given by Equations (18) and (20) cannot be obtained in an explicit form. In this case, the MCMC technique is used to generate samples from the posterior distributions and then the Bayes estimates of the parameters $\alpha_h, \theta_h,$ and $\beta_h,$ for all $h = 1, 2, \ldots, k,$ will be computed. The main theme of the MCMC technique is to compute approximate values for the integrals in Equations (18) and (20). An important sub-class of MCMC methods is Gibbs sampling and more general Metropolis within Gibbs samplers. The Metropolis algorithm is a random walk that uses an acceptance/rejection rule to converge to the target distribution. The Metropolis algorithm was first proposed by Metropolis et al.\textsuperscript{35} and was then generalized by Hastings.\textsuperscript{36} From Equations (5), (16), and (17), the joint posterior density...
function of $\alpha_h, \theta_h,$ and $\beta_h$ can be written as

$$\pi^*(\alpha_h, \theta_h, \beta_h|\delta, s, \zeta) \propto \prod_{h=1}^{k} \frac{\theta^M_r(h) + q_h - 1}{\alpha^\mu_h - \mu_h + 1} \times \exp \{- (\alpha_h \lambda_h + \theta_h \omega_h + \beta_h \psi_h)\} \times \exp \left\{ - \sum_{i=1}^{r} \left[ (\beta_h (\delta_i(h) + s_i(h)) + \delta_i(h)) \log \left( 1 + \left( \frac{\zeta_i}{\alpha_h} \right)^\delta_h \right) \right]\right\} \times \exp \left\{ (\theta_h - 1) \left[ \sum_{i=1}^{r} \delta_i(h) \log \xi_i - M_r(h) \log \alpha_h \right]\right\}. \quad (21)$$

The conditional posterior densities of $\alpha_h, \theta_h,$ and $\beta_h$ can also be written as

$$\pi^*_1(\alpha_h|\theta_h, \beta_h, \delta, s, \xi) \propto \prod_{h=1}^{k} \frac{\theta^\mu_h - M_r(h)}{\alpha^\mu_h - \mu_h + 1} \times \exp \{- (\theta_h - 1)[M_r(h) \log \alpha_h] - \alpha_h \lambda_h\} \times \exp \left\{ - \sum_{i=1}^{r} \left[ (\beta_h (\delta_i(h) + s_i(h)) + \delta_i(h)) \log \left( 1 + \left( \frac{\zeta_i}{\alpha_h} \right)^\delta_h \right) \right]\right\} \times \exp \left\{ (\alpha_h - 1) \left[ \sum_{i=1}^{r} \delta_i(h) \log \xi_i - M_r(h) \log \alpha_h \right]\right\} \quad (22)$$

$$\pi^*_2(\theta_h|\alpha_h, \beta_h, \delta, s, \xi) \propto \prod_{h=1}^{k} \frac{\theta^\mu_h - M_r(h)}{\alpha^\mu_h - \mu_h + 1} \times \exp \{- (\alpha_h - 1)[M_r(h) \log \alpha_h] - \alpha_h \lambda_h\} \times \exp \left\{ - \sum_{i=1}^{r} \left[ (\beta_h (\delta_i(h) + s_i(h)) + \delta_i(h)) \log \left( 1 + \left( \frac{\zeta_i}{\alpha_h} \right)^\delta_h \right) \right]\right\} \times \exp \left\{ (\theta_h - 1) \left[ \sum_{i=1}^{r} \delta_i(h) \log \xi_i - M_r(h) \log \alpha_h \right]\right\} \quad (23)$$

$$\pi^*_3(\beta_h|\alpha_h, \theta_h, \delta, s, \zeta) \equiv \text{gamma} \left[ M_r(h) + p_h, v_h + \sum_{i=1}^{r} \left[ (\delta_i(h) + s_i(h)) \log \left( 1 + \left( \frac{\zeta_i}{\alpha_h} \right)^\delta_h \right) \right]\right]\quad (24)$$

Now, the following steps illustrate the method of the Metropolis–Hastings algorithm within Gibbs sampling to generate the posterior samples as suggested by Tierney, and then the Bayes estimates and the corresponding CRI$s$ can be obtained

1. Start with $(\alpha_h^{(0)} = \hat{\alpha}_h, \theta_h^{(0)} = \hat{\theta}_h$ and $\beta_h^{(0)} = \hat{\beta}_h)$.
2. Put $l = 1$.
3. Generate $\beta_h^{(l)}$ from

$$\text{gamma} \left[ M_r(h) + p_h, v_h + \sum_{i=1}^{r} \left[ (\delta_i(h) + s_i(h)) \log \left( 1 + \left( \frac{\zeta_i}{\alpha_h} \right)^\delta_h \right) \right]\right]\quad (24)$$

4. Using the following Metropolis–Hastings method, generate $\alpha_h^{(l)}$ and $\theta_h^{(l)}$ from Equations (22) and (23) with the suggested normal distributions

$$N(\alpha_h^{(l-1)}, var(\alpha_h)) \text{ and } N(\theta_h^{(l-1)}, var(\theta_h), \text{ respectively)}$$

where $var(\alpha_h)$ and $var(\theta_h)$ can be obtained from the main diagonal in asymptotic inverse Fisher information matrix (9).

(i) Generate a proposal $\alpha_h^*$ from $N(\alpha_h^{(l-1)}, var(\alpha_h))$ and $\theta_h^*$ from $N(\theta_h^{(l-1)}, var(\theta_h)).$
(ii) Evaluate the acceptance probabilities

\[ \rho_{\alpha_h} = \min \left[ 1, \frac{\pi_1^* (\alpha_h^{(l-1)}, \beta_h^{(l)}, \delta, s, \zeta)}{\pi_1^* (\alpha_h^{(l-1)}, \beta_h^{(l)}, \delta, s, \zeta)} \right], \]

\[ \rho_{\theta_h} = \min \left[ 1, \frac{\pi_2^* (\theta_h^{(l-1)}, \alpha_h^{(l)}, \beta_h^{(l)}, \delta, s, \zeta)}{\pi_2^* (\theta_h^{(l-1)}, \alpha_h^{(l)}, \beta_h^{(l)}, \delta, s, \zeta)} \right]. \] (25)

(iii) Generate \( u_1 \) and \( u_2 \) from a Uniform (0,1) distribution.

(iv) If \( u_1 \leq \rho_{\alpha_h} \), then accept the proposal and set \( \alpha_h^{(l)} = \alpha_h^{*}, \) else set \( \alpha_h^{(l)} = \alpha_h^{(l-1)}. \)

(v) If \( u_2 \leq \rho_{\theta_h} \), then accept the proposal and set \( \theta_h^{(l)} = \theta_h^{*}, \) else set \( \theta_h^{(l)} = \theta_h^{(l-1)}. \)

(5) Compute \( \alpha_h^{(l)} \) and \( \theta_h^{(l)} \).

(6) Put \( l = l + 1. \)

(7) Repeat steps 3–6 \( Q \) times.

(8) In order to guarantee the convergence and to remove the influence of the initial values selection, the first \( M \) simulated points are ignored. The selected samples are \( \alpha_h^{(l)} \) and \( \theta_h^{(l)}, \) \( l = M + 1, \ldots, Q, \) for sufficiently large \( Q. \) The approximate Bayes estimates for \( \alpha_h, \theta_h, \) and \( \beta_h \) based on SEL are

\[ \alpha_{hBS} = \frac{1}{Q-M} \sum_{l=M+1}^{Q} \alpha_h^{(l)}, \]

\[ \theta_{hBS} = \frac{1}{Q-M} \sum_{l=M+1}^{Q} \theta_h^{(l)}, \] \hspace{1cm} (26)

\[ \beta_{hBS} = \frac{1}{Q-M} \sum_{l=M+1}^{Q} \beta_h^{(l)}. \]

and the estimates for the same parameters under LINEX loss function are

\[ \alpha_{hBL} = -\frac{1}{c} \log \left[ \frac{1}{Q-M} \sum_{l=M+1}^{Q} e^{-c \alpha_h^{(l)}} \right], \]

\[ \theta_{hBL} = -\frac{1}{c} \log \left[ \frac{1}{Q-M} \sum_{l=M+1}^{Q} e^{-c \theta_h^{(l)}} \right], \] \hspace{1cm} (27)

\[ \beta_{hBL} = -\frac{1}{c} \log \left[ \frac{1}{Q-M} \sum_{l=M+1}^{Q} e^{-c \beta_h^{(l)}} \right]. \]

(9) To calculate the credible intervals (CRI) of \( \Omega_{j,h}, \) where \( \Omega_{1,h} = \alpha_h, \Omega_{2,h} = \theta_h, \) and \( \Omega_{3,h} = \beta_h, \) the quantiles of the sample are assumed to be the endpoints of the intervals. Sort \( \{ \Omega_{1,h}^{M+1}, \Omega_{1,h}^{M+2}, \ldots, \Omega_{1,h}^{Q} \} \) as \( \{ \Omega_{1,h}^{(1)}, \Omega_{1,h}^{(2)}, \ldots, \Omega_{1,h}^{(Q-M)} \}, j = 1, 2, 3; \) for \( h = 1, 2, \ldots, k. \) Hence the 100(1 – \( \vartheta \))% symmetric CRI of \( \Omega_{j,h} \) is given by

\[ \left[ \Omega_{j,h} \left( 1 - \frac{\vartheta}{2} \right), \Omega_{j,h} \left( 1 + \frac{\vartheta}{2} \right) \right]. \] \hspace{1cm} (28)

5 | SIMULATION STUDY

The prior knowledge of the number of failures before the test procedure plays a meaningful role in determining the appropriate sampling plans and developing good estimators for parameters. In this section, a simulation study is performed
under different JPROG-II-C schemes where 1000 JPROG-II-C samples are generated from the two WG populations, based on distinct sample sizes, for calculating the approximation of the expected values of the number of failures from the first production line before the test procedure (A.E.B). The mean of the exact number of failures after the test procedure (M.E.A) is also computed from the first production line. The comparison between the A.E.B and M.E.A is necessary to judge the effectiveness of the approximation introduced by Parsi and Bairamov38 throughout our model. Parsi and Bairamov38 used their approximation in the case of JPROG-II-C samples generated from two Weibull Gamma and two Pareto populations. They found that the results of A.E.B and M.E.A were close to each other in most cases. After applying this approximation to the simulated JPROG-II-C samples generated from two WG populations, we reached the same results introduced by Parsi and Bairamov and found that they are close together in most cases, as shown in Table 1. In addition to these close results, the A.E.B and M.E.A. values seem to be the same for large values of $N$. The calculations for our model are performed based on the following assumptions, $X_1$ is chosen from $WG(6, 3, 4)$ and $X_2$ from $WG(2, 5, 1.5)$, $p = 0.13$. Once again $X_1$ is chosen from $WG(2, 5, 1.5)$ and $X_2$ from $WG(6, 3, 4)$, $p = 0.87$, where $p = P(X_1 < X_2) = \int_0^\infty G(x; \alpha_2, \theta_2, \beta_2) dF(x; \alpha_1, \theta_1, \beta_1)$. $X_1$ and $X_2$ represent the lifetime of the first production line units and the second production line units, respectively. Also $G(x; \alpha_2, \theta_2, \beta_2)$ is the survival function of $X_2$, while $F(x; \alpha_1, \theta_1, \beta_1)$ is the CDF of $X_1$.

It is noted that the A.E.B and M.E.A. values are close together in all schemes except for schemes no. 4–6. Also, for large values of $N$, the A.E.B and M.E.A. values seem to be the same.

6 | APPLICATIONS

In this section, our model is applied in the case of two WG populations. First, we will show the results through a simulated example, then we will apply on a real data example.

6.1 | Simulated example

An approximation is used for calculating the expected values of the number of failures before the test procedure. The exact values of the number of failures are also observed after the test procedure. The approximated values are compared with the exact values through a simulated example. A simulated data is generated in the case of $k = 2$ and the following algorithm shows how the simulated JPROG-II-C data are generated from the two production lines.

1. Generate a random sample with size $n_1$ from $WG(\alpha_1, \theta_1, \beta_1)$.
2. Generate another random sample with size $n_2$ from $WG(\alpha_2, \theta_2, \beta_2)$, such that the two samples, with sizes $n_1$ and $n_2$ are independent.
3. Combine the two samples in one sample with size $N = n_1 + n_2$.
4. Sort the joint sample.
5. Specify the censoring size $r$ and the censoring scheme $R_i$, which represent the number of the removed units at the time of the $i$th failure from the joint sample, $i = 1, 2, ..., r$.
6. Specify $s_1(1)$ and $s_1(2)$, which represent the number of the removed units at the time of the $i$th failure from the sample nos. 1 and 2, respectively. So, $R_i = \sum_{h=1}^2 s_i(h), i = 1, 2, ..., r$.
7. Using the famed algorithm described in Balakrishnan and Sandhu39 for generating progressive Type-II samples, the ordered JPROG-II-C is

$$\xi_{1;r,N}^R < \xi_{2;r,N}^R < ... < \xi_{r;r,N}^R.$$ 

8. Define the two indicators $\delta_1(1)$ and $\delta_1(2)$ that determine if the $i$th failure belongs to sample no. 1 or 2, respectively, such that $\sum_{h=1}^2 \delta_i(h) = 1$, for $i = 1, 2, ..., r$.
9. Use the JPROG-II-C sample to compute the MLEs for the model parameters. The Newton–Raphson method is applied for solving the nonlinear system to obtain the MLEs of the parameters.
10. Compute the 95% bootstrap confidence intervals for the model parameters, using the steps described in Section 3.
11. Compute the Bayes estimates of the model parameters based on MCMC algorithm described in Section 4.
The comparison between A.E.B and M.E.A for the first production line.

| Scheme no. | N  | r      | Scheme (R)          | \((n_1, n_2)\) | \(p = 0.13\) | \(p = 0.87\) |
|------------|----|--------|---------------------|----------------|-------------|-------------|
|            |    |        |                     |                | A.E.B       | M.E.A       | A.E.B       | M.E.A       |
| 1          | 30 | 10     | \((0,\ldots,0,20)\) | (20, 10)       | 3.091       | 3.028       | 9.142       | 9.109       |
| 2          | (15, 15) | 1.667 | 1.664               | 8.332          | 8.319       |
| 3          | (10, 20) | 0.858 | 0.930               | 6.909          | 6.974       |
| 4          | (20, 10) | 5.629 | 4.807               | 9.334          | 7.832       |
| 5          | (20, 0, 0, 0, 0, 0, 0) | (15, 15) | 9.799 | 3.477               | 8.255          | 6.656        |
| 6          | (10, 20) | 2.908 | 2.198               | 14.192         | 5.250       |
| 7          | (20, 0, \ldots, 0) | (15, 15) | 6.348 | 6.506               | 6.789          | 6.776        |
| 8          | (10, 20) | 4.742 | 4.800               | 5.258          | 5.137       |
| 9          | (15, 15) | 3.211 | 3.243               | 3.652          | 3.479       |
| 10         | (20, 10) | 5.829 | 5.825               | 13.492         | 13.582      |
| 11         | 15 | (0,\ldots,0,15) | (15, 15) | 3.056 | 2.926               | 11.944         | 12.140      |
| 12         | (10, 20) | 1.508 | 1.391               | 9.171          | 9.137       |
| 13         | (20, 10) | 9.785 | 9.846               | 10.556         | 10.056      |
| 14         | (15, 15) | 7.177 | 7.337               | 7.823          | 7.605       |
| 15         | (10, 20) | 4.444 | 4.841               | 5.215          | 5.165       |
| 16         | (20, 10) | 10.039 | 10.153 | 17.455         | 17.532      |
| 17         | 20 | (0,\ldots,0,10) | (15, 15) | 5.508 | 5.683               | 14.492         | 14.347      |
| 18         | (10, 20) | 2.545 | 2.421               | 9.961          | 9.809       |
| 19         | (20, 10) | 13.184 | 13.216 | 10.137         | 13.331      |
| 20         | (10, 0, \ldots, 0) | (15, 15) | 9.866 | 9.886               | 10.134         | 10.113      |
| 21         | (10, 20) | 9.863 | 6.615               | 6.816          | 6.796       |
| 22         | (30, 20) | 5.296 | 4.980               | 17.560         | 17.669      |
| 23         | 50 | 20     | (0,\ldots,0,30) | (25, 25)       | 3.624       | 3.391       | 16.376         | 16.546      |
| 24         | (25, 25) | 2.440 | 2.389               | 14.704         | 14.974      |
| 25         | (30, 20) | 11.749 | 11.897 | 12.321         | 12.073      |
| 26         | (25, 25) | 9.761 | 9.826               | 10.239         | 10.033      |
| 27         | (20, 30) | 7.679 | 7.875               | 8.251          | 8.202       |
| 28         | (30, 20) | 7.631 | 7.457               | 21.587         | 21.690      |
| 29         | 25 | (0,\ldots,0,25) | (25, 25) | 5.154 | 4.884               | 19.846         | 20.210      |
| 30         | (20, 30) | 3.413 | 3.156               | 17.369         | 17.576      |
| 31         | (30, 20) | 14.788 | 14.842 | 15.211         | 15.138      |
| 32         | (25, 0, \ldots, 0) | (25, 25) | 12.316 | 12.428         | 12.684         | 12.560      |
| 33         | (20, 30) | 9.789 | 9.941               | 10.212         | 10.070      |
| 34         | (40, 30) | 7.437 | 7.058               | 25.742         | 25.906      |
| 35         | 70 | 30     | (0,\ldots,0,40) | (35, 35)       | 5.651       | 5.306       | 24.349         | 24.734      |
| 36         | (30, 40) | 4.258 | 4.129               | 22.563         | 22.976      |
| 37         | (40, 30) | 16.909 | 17.100 | 17.327         | 17.209      |
| 38         | (35, 35) | 14.787 | 14.763 | 15.213         | 15.083      |
| 39         | (30, 40) | 12.673 | 12.761 | 13.091         | 12.824      |
| 40         | (40, 30) | 9.612 | 9.224               | 29.583         | 29.918      |
| 41         | 35 | (0,\ldots,0,35) | (35, 35) | 7.251 | 6.916               | 27.749         | 28.146      |
| 42         | (30, 40) | 5.417 | 5.033               | 25.388         | 25.779      |
| 43         | (40, 30) | 19.795 | 19.896 | 20.165         | 20.123      |
| 44         | (35, 35) | 17.313 | 17.391 | 17.687         | 17.576      |
| 45         | (30, 40) | 14.835 | 14.858 | 15.205         | 15.050      |

Table 2 shows the simulated failure times from the first production line where the initial values for the parameters are chosen as follows: \( \alpha_1 = 7.2, \theta_1 = 9, \beta_1 = 5.5 \) with sample size \( n_1 = 24 \).

Table 3 shows the simulated failure times from the production line no. 2 where the initial values for the parameters are chosen in this case as follows: \( \alpha_2 = 2, \theta_2 = 3, \beta_2 = 0.06 \) with sample size \( n_2 = 23 \). It is clear that the value of \( \beta_2 \) is less than the value of \( \beta_1 \). So from Corollary 1, the units manufactured by the second production line have higher reliability than those manufactured by the first production line. Also, the difference in Tables 2 and 3 is easily noticed, in addition to the big difference between failure times in Table 3 and the small difference between failure times in Table 2 due to the
Effect of $\beta_1$ and $\beta_2$ values on the units' reliability in each production line, and the random withdrawal of the removed units, which is a specific identity for progressive censoring.

Table 4 shows the simulated JPROG-II-C data from the two production lines with joint progressive censoring schemes $N = n_1 + n_2$, $r = 10$. $R = (2, 2, 0, 2, 2, 1, 1, 2, 0, 25)$ is chosen based on the choice of $s(1)$ and $s(2)$, where $s(1) = (2, 0, 0, 2, 0, 1, 2, 0, 9)$ and $s(2) = (0, 2, 0, 2, 1, 0, 0, 0, 16)$, such that $R_i = \sum_{h=1}^2 s_i(h)$, $i = 1, 2, \ldots, 10$. While Tables 5 and 6 show different point estimates for $\alpha_1, \alpha_2, \beta_1, \beta_2$ and 95% confidence intervals for the same parameters, respectively.

Also, graphical representations that show the existence and uniqueness of $\beta_1$ and $\beta_2$ based on Equation (14) are shown in Figure 2A, B, respectively.

In the MCMC approach, we run the chain 52,000 times and discard the first 2000 values as “burn-in” while the prior knowledge parameters are chosen to be $\mu_1 = \mu_2 = q_1 = q_2 = p_1 = p_2 = 0.01$ and $\lambda_1 = \lambda_2 = \omega_1 = \omega_2 = v_1 = v_2 = 2$.

**Table 5** Different point estimates for $\alpha_1, \beta_1, \beta_1; \alpha_2, \beta_2$.

|                 | (\hat{\alpha}_1)^{\text{ML}} | (\hat{\alpha}_1)^{\text{Boot}} | (\hat{\alpha}_1)^{\text{Boot}} | (\hat{\alpha}_1)^{\text{ML}} | (\hat{\alpha}_1)^{\text{Boot}} | (\hat{\alpha}_1)^{\text{Boot}} |
|----------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $\alpha_1$     | 6.3647           | 6.3200           | 6.1146           | 6.4491           | 6.4491           | 6.4491           |
| $\beta_1$      | 5.735            | 5.5335           | 4.2986           | 2.2711           | 2.2711           | 2.2711           |
| $\beta_2$      | 1.4707           | 1.7095           | 1.0192           | 1.4551           | 1.4551           | 1.4551           |
| $\beta_2$      | 3.9635           | 3.3810           | 3.6432           | 3.9849           | 3.9849           | 3.9849           |
| $\beta_2$      | 0.0297           | 0.2316           | 0.0122           | 0.0290           | 0.0290           | 0.0290           |

**Figure 2** Graphical illustration for the existence and uniqueness of the MLEs in the simulated data example. MLEs, maximum likelihood estimates.
TABLE 6 Ninety-five percent confidence intervals for $\alpha_1, \theta_1, \beta_1; \alpha_2, \theta_2, \beta_2$.

| Method     | $\alpha_1$               | Length | $\alpha_2$               | Length |
|------------|--------------------------|--------|--------------------------|--------|
| ACI        | [1.6889, 11.0405]        | 9.3516 | [0.0480, 2.8935]         | 2.8456 |
| Boot-p CI  | [5.1195, 7.3432]         | 2.2235 | [0.5345, 5.0974]         | 4.5625 |
| Boot-t CI  | [5.6166, 6.2762]         | 0.6594 | [−2.2214, 1.4165]        | 3.6376 |
| Boot-BC CI | [5.2265, 7.3862]         | 2.1596 | [0.8681, 5.4262]         | 4.5578 |
| Boot-BCa CI| [5.1067, 7.1822]         | 2.0754 | [0.7644, 5.1064]         | 4.3413 |
| CRI        | [6.3907, 6.4825]         | 0.0912 | [1.4395, 1.4702]         | 0.0305 |

| Method     | $\theta_1$               | Length | $\theta_2$               | Length |
|------------|--------------------------|--------|--------------------------|--------|
| ACI        | [3.6357, 17.9132]        | 14.5475| [−5.7082, 13.6357]       | 19.3432|
| Boot-p CI  | [7.5863, 15.3836]        | 7.7973 | [0.0014, 5.8277]         | 5.8263 |
| Boot-t CI  | [8.7421, 10.2884]        | 1.5463 | [3.2635, 3.9634]         | 0.6999 |
| Boot-BC CI | [7.1542, 15.1540]        | 7.9998 | [0.0018, 5.9530]         | 5.9511 |
| Boot-BCa CI| [7.5863, 15.3641]        | 7.7777 | [0.0015, 5.8277]         | 5.8262 |
| CRI        | [10.4955, 10.6692]       | 0.1735 | [3.7968, 4.2318]         | 0.4349 |

| Method     | $\beta_1$               | Length | $\beta_2$               | Length |
|------------|--------------------------|--------|--------------------------|--------|
| ACI        | [−35.2902, 46.7603]      | 82.0509| [−0.0613, 0.1215]        | 0.1816 |
| Boot-p CI  | [0.8481, 10.4994]        | 9.6514 | [0.0058, 1.9442]         | 1.9382 |
| Boot-t CI  | [0.3656, 5.5876]         | 5.2219 | [−0.1031, 0.0292]        | 0.1318 |
| Boot-BC CI | [0.7926, 9.7427]         | 8.9501 | [0.0091, 2.4553]         | 2.4457 |
| Boot-BCa CI| [0.4894, 9.1255]         | 8.6361 | [0.0072, 2.0050]         | 1.9979 |
| CRI        | [0.9100, 4.2419]         | 3.3319 | [0.0061, 0.0699]         | 0.0639 |

TABLE 7 Data Set 1.

|               |                |                |                |                |
|---------------|----------------|----------------|----------------|----------------|
|               | 693.73         | 704.66         | 323.83         | 778.17         |
|               | 123.06         | 637.66         | 383.43         | 151.48         |
|               | 108.94         | 50.16          | 622.90         | 350.70         |
|               | 671.49         | 183.16         | 257.44         | 727.23         |
|               | 291.27         | 101.15         | 303.90         | 163.40         |
|               | 141.38         | 700.74         | 765.14         | 187.13         |
|               | 350.70         | 145.96         | 350.70         | 145.96         |

TABLE 8 Data Set 2.

|               |                |                |                |                |
|---------------|----------------|----------------|----------------|----------------|
|               | 578.62         | 756.70         | 594.29         | 166.49         |
|               | 99.72          | 707.36         | 765.14         | 187.13         |
|               | 145.96         | 350.70         | 177.25         |                |
|               | 547.44         | 116.99         | 375.81         | 581.60         |
|               | 119.86         | 48.01          | 200.16         | 36.75          |
|               | 244.53         | 83.55          |                |                |

After applying the approximation of Parsi and Bairamov, and to assess the effectiveness of that approximation, we get the expected values of the number of failures for the first production line which is 8.13025, and the expected value of the number of failures for the second production line is 1.86975. While the exact number of failures after the test procedure is 7 for the first production line and 3 for the second production line.

6.2 Real-data example

In this subsection, the data of Xia et al. are used as an application of the JPROG-II-C model. This data represent the ordered breaking strengths of jute fiber at gauge lengths 10 and 20 mm. Jute Fiber has many uses in different sectors especially in the Engineering field such as the construction sector, automobile sector, textile sector, and so forth. The following references show the uses of Jute Fiber in different industries and its relation to the Engineering field: Shelar and Uttamchand, Zakaria et al., Das et al., and Chakraborty et al.

The Kolmogorov–Smirnov (K–S) distance between the empirical distribution of Data Set 1, shown in Table 7, and CDF of WGD is 0.105828 with a p-value equals 0.855287. Hence, the WGD fits well to Data Set 1.

While the K–S distance between the empirical distribution of Data Set 2, shown in Table 8, and CDF of WGD is 0.149029 with a p-value equals 0.473037. Hence, the WGD also fits well to Data Set 2.
Table 9 shows the JPROG-II-C data from the two Data Sets 1 and 2 with joint progressive censoring schemes \( N = n_1 + n_2, r = 20 \). \( R = (2, 2, 0, 0, 2, 1, 0, 0, 0, 2, 0, 0, 2, 1, 0, 0, 2, 0, 0, 2, 0, 0, 2) \) is chosen based on the choice of \( s(1) \) and \( s(2) \), where \( s(1) = (2, 0, 0, 0, 2, 1, 0, 0, 0, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 2, 0, 11) \) and \( s(2) = (0, 2, 0, 0, 2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 11) \), such that \( R_i = \sum_{h=1}^{2} s(h), i = 1, 2, \ldots, 20 \). Different point estimates for \( \alpha_1, \theta_1, \beta_1; \alpha_2, \theta_2, \beta_2 \) and 95% confidence intervals for the same parameters are shown in Tables 10 and 11, respectively.

The effectiveness of the proposed methods can be checked by comparing the results of these different methods. One can see that the results of Bayesian and MLEs are close together through the simulated and real examples. In addition, the authors showed the existence and uniqueness of the MLEs.

A graphical representations that show the existence and uniqueness of \( \beta_1 \) and \( \beta_2 \) based on Equation (14) are shown in Figure 3A,B, respectively.

In the MCMC approach, we run the chain for 52,000 times and discard the first 2000 values as “burn-in” while the prior knowledge parameters are chosen to be \( \mu_1 = \mu_2 = q_1 = q_2 = p_1 = p_2 = 0.01 \) and \( \lambda_1 = \lambda_2 = w_1 = w_2 = v_1 = v_2 = 2 \).
The approximation of Parsi and Bairamov\(^{38}\) shows that the expected value of the number of failures for the first production line is 7.0759 and the expected value of the number of failures for the second production line is 12.9241. While the exact number of failures after the test procedure is 8 for the first production line and 12 for the second production line.

7 CONCLUSIONS

Using the JPROG-II-C samples strategy, the estimation of the 3\(k\) parameters of \(k\) WG populations are performed based on Bayes and non-Bayes methods. Four types of bootstrap confidence intervals are used to obtain 95% confidence intervals for the unknown parameters. The importance of the MCMC technique was noticeable in Bayesian estimation using the Metropolis–Hastings method. An illustrative example is presented to show how the MCMC and parametric bootstrap methods work. Also, the reliability of units that are manufactured by \(k\) production lines is compared based on the invariance property of the MLEs of the parameters. The prior experimental knowledge of the number of failures for the first production line is computed approximately before the test procedure and compared with the exact number of failures after the test procedure. The proposed model can be applied to compare the reliability among different production lines through the failure times of the units produced by these production lines. These production lines may be located either at the same factory or in another place. The proposed model is recommended to be applied for evaluating the quality of the production lines and it can be a good reference for Engineering studies that are interested in the field of failure analysis. It also gives more reliable results whenever the size of selected samples for the study was relatively large. Since the proposed model depends on a parametric model, it is recommended to apply it to real data sets that are characterized by high fitting criteria, measured basically through \(p\)-value and K–S distance for the WGD.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

ORCID

Mahmoud M. M. Mansour https://orcid.org/0000-0002-0184-3427
Rashad M. EL-Sagheer https://orcid.org/0000-0002-7828-3536
Mohamed S. Aboshady https://orcid.org/0000-0001-9003-8122

REFERENCES

1. Kumar J, Bansal SA, Mehta M, Singh H. Reliability analysis in process industries—an overview. GIS Sci J. 2020;7(3):151-168.
2. Soltanali H, Hojjatshpour M, Torres Farinha J. An improved risk and reliability framework-based maintenance planning for food processing systems. Qual Technol Quant Manag. 2023;20(2):256-278.
3. Wu SJ, Chang CT. Inference in the Pareto distribution based on progressive type II censoring with random removals. J Appl Stat. 2003;30(2):163-172.
4. Abdel-Hamid AH, Al-Hussaini EK. Inference for a progressive stress model from Weibull distribution under progressive type-II censoring. *J Comput Appl Math.* 2011;235(17):5259-5271.

5. Ahmed EA. Bayesian estimation based on progressive type-II censoring from two-parameter bathtub-shaped lifetime model: an Markov chain Monte Carlo approach. *J Appl Stat.* 2014;41(4):752-768.

6. El-Sagheer RM, Ahsanullah M. Statistical inference for a step-stress partially accelerated life test model based on progressively type-II censored data from Lomax distribution. *J Appl Stat Sci.* 2015;21(4):307.

7. Abdel-Hamid AH. Constant-partially accelerated life tests for Burr type-XII distribution with progressive type-II censoring. *Comput Stat Data Anal.* 2009;53(7):2511-2523.

8. Balakrishnan N, Rasouli A. Exact likelihood inference for two exponential populations under joint type-II censoring. *Comput Stat Data Anal.* 2008;52(5):2725-2738.

9. Balakrishnan N, Su F. Exact likelihood inference for k exponential populations under joint type-II censoring. *Comm Stat Simul Comput.* 2015;44(3):591-613.

10. Rasouli A, Balakrishnan N. Exact likelihood inference for k exponential populations under joint progressive type-II censoring. *Commun Stat Theory Methods.* 2010;39(12):2172-2191.

11. Doostparast M, Ahmadi MV, Ahmadi J. Bayes estimation based on joint progressive type II censored data under LINEX loss function. *Commun Stat - Simul Comput.* 2013;42(8):1865-1886.

12. Balakrishnan N, Su F, Liu KY. Exact likelihood inference for k exponential populations under joint progressive type-II censoring. *Commun Stat - Simul Comput.* 2015;44(4):902-923.

13. Bithas PS. Weibull-gamma composite distribution: alternative multipath/shadowing fading model. *Electron Lett.* 2009;45(14):749-751.

14. Nadarajah S, Kotz S. A class of generalized models for shadowed fading channels. *Wirel Pers Commun.* 2007;43:1113-1120.

15. Molenberghs G, Verbeke G. On the Weibull-Gamma frailty model, its infinite moments, and its connection to generalized log-logistic, logistic, Cauchy, and extreme-value distributions. *J Stat Plan Inference.* 2011;141(2):861-868.

16. Mahmoud M, Abdel-Aty Y, Mohamed N, Hamedani G. Recurrence relations for moments of dual generalized order statistics from Weibull gamma distribution and its characterizations. *J Stat Appl Probab.* 2014;3:189-199.

17. EL-Sagheer RM. Estimation of parameters of Weibull–Gamma distribution based on progressively censored data. *Stat Pap.* 2018;59(2):725-757.

18. Balakrishnan N, Balakrishnan N, Aggarwala R. *Progressive Censoring: Theory, Methods, and Applications.* Springer Science & Business Media; 2000.

19. Azzalini A. *Statistical Inference based on the Likelihood.* Vol 68. CRC Press; 1996.

20. Royall R. *Statistical Evidence: A Likelihood Paradigm.* Vol 71. CRC press; 1997.

21. Cohen AC. Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples. *Technometrics.* 1965;7(4):579-588.

22. Efron B. *The Jackknife, the Bootstrap and Other Resampling Plans.* SIAM; 1982.

23. Hall P. Theoretical comparison of bootstrap confidence intervals. *Ann Stat.* 1988:927-953.

24. Soliman AA, Abd-Elah AH, Abou-Elheggag NA, Ahmed EA. Modified Weibull model: a Bayes study using MCMC approach based on progressive censoring data. *Reliab Eng Syst Safe.* 2012;100:48-57.

25. EL-Sagheer RM. Inferences in constant-partially accelerated life tests based on progressive type-II censoring. *Bull Malays Math Sci Soc.* 2018;41:609-626.

26. DiCiccio TJ, Efron B. Bootstrap confidence intervals. *Stat Sci.* 1996;11(3):189-228.

27. Balakrishnan N, Sandhu R. A simple simulational algorithm for generating progressive type-II censored samples. *Am Stat.* 1995;49(2):229-230.

28. Efron B, Tibshirani RJ. *An Introduction to the Bootstrap.* CRC Press; 1994.

29. Davison AC, Hinkley DV. *Bootstrap Methods and their Application.* No. 1. Cambridge University Press; 1997.

30. Balakrishnan N, Kateri M. On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data. *Stat Probab Lett.* 2008;78(17):2971-2975.

31. Kundu D, Howlader H. Bayesian inference and prediction of the inverse Weibull distribution for type-II censored data. *Comput Stat Data Anal.* 2010;54(6):1547-1558.

32. Dey S, Dey T. On progressively censored generalized inverted exponential distribution. *J Appl Stat.* 2014;41(12):2557-2576.

33. Dey S, Singh S, Tripathi YM, Asgharzadeh A. Estimation and prediction for a progressively censored generalized inverted exponential distribution. *Stat Methodol.* 2016;32:185-202.

34. Mansour MMM, Ramadan DA. Statistical inference to the parameters of the modified extended exponential distribution under the type-II hybrid censoring scheme. 2020;15:19-44.

35. Metropolis N, Rosenbluth AW, Rosenbluth MN, Teller AH, Teller E. Equation of state calculations by fast computing machines. *J Chem Phys.* 1953;21(6):1087-1092.

36. Hastings WK. Monte Carlo sampling methods using Markov chains and their applications. *Biometrika.* 1970;57:97-109.

37. Tierney L. Markov chains for exploring posterior distributions. *Ann Stat.* 1994;22:1701-1728.

38. Parsi S, Bairamov I. Expected values of the number of failures for two populations under joint type-II progressive censoring. *Comput Stat Data Anal.* 2009;53(10):3560-3570.

39. Balakrishnan N, Sandhu R. A simple simulational algorithm for generating progressive type-II censored samples. *Am Stat.* 1995;49(2):229-230.
40. Xia Z, Yu J, Cheng L, Liu L, Wang W. Study on the breaking strength of jute fibres using modified Weibull distribution. Compos - A: Appl Sci Manuf. 2009;40(1):54-59.
41. Shelar PB, Narendra Kumar U. A short review on jute fiber reinforced composites. Mater Sci Forum. 2021;1019:32-43.
42. Zakaria M, Ahmed M, Hoque MM, Islam S. Scope of using jute fiber for the reinforcement of concrete material. Text Cloth Sustain. 2017;2:1-10.
43. Das SC, Paul D, Fahad MM, Das MK, Rahman GS, Khan MA. Effect of fiber loading on the mechanical properties of jute fiber reinforced polypropylene composites. Chem Eng Sci. 2018;8(4):215-224.
44. Chakraborty S, Kundu SP, Roy A, Adhikari B, Majumder SB. Effect of jute as fiber reinforcement controlling the hydration characteristics of cement matrix. Ind Eng Chem Res. 2013;52(3):1252-1260.

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AUTHOR BIOGRAPHIES

Mahmoud M.M. Mansour is an Associate Professor of Mathematical Statistics at the Basic Science Department, Faculty of Engineering, The British University in Egypt. He received his Bachelor of Science in Mathematics from Faculty of Science, Al-Azhar University, in 2009, where he also got his Master of Science in Statistical Testing in 2014. He received his PhD from the Faculty of Science AL Azhar University Egypt in 2017. His research interests include the Theory of Reliability, Censored Data, Life Testing, and Distribution Theory.

Rashad M. EL-Sagheer is an Assistant Professor of Statistics at Mathematics Department, Faculty of Science, Al-Azhar University/ High Institute of Computer and Management Information System, First Statement, New Cairo, Egypt. Received his B. A. Degree in Mathematics from Faculty of Science AL-Azhar University in 2004. He received Master’s and PhD degrees in Mathematical Statistics from Faculty of Science Sohag University in 2011 and 2014, respectively. He has several publications in the international journals and conferences include: Statistical Inference, Theory of Estimation, Bayesian Inference, Order Statistics, Records, Theory of Reliability, Censored Data, Life Testing and Distribution Theory, Statistical Computing, and Reliability Analysis. He published more than 83 papers in reputed international journals. He supervised 17 MSc and PhD students.

Mohamed A.W. Mahmoud is a Professor of Mathematical Statistics in Department of Mathematics and a former Dean of Faculty of Science, Al-Azhar University, Cairo, Egypt. He received his PhD in Mathematical statistics in 1984 from Assiut University, Egypt. His research interests include: Theory of Reliability, Ordered Data, Characterization, Statistical Inference, Distribution Theory, Discriminant Analysis, and Classes of Life Distributions. He published and co-authored more than 100 papers in reputed international journals. He supervised more than 62 MSc thesis and more than 75 PhD thesis.

Mohamed S. Aboshady is an Assistant Professor of Engineering Mathematics at the Basic Science Department, Faculty of Engineering, The British University in Egypt. Received his Bachelor degree in Electrical Engineering from the Military Technical College in 2004. He received MSc degree in Engineering Mathematics from the Military Technical College in 2012. He received a PhD in Engineering Mathematics from the faculty of Engineering, Ain Shams University, Egypt in 2019. His research interests include Graph Theory, Graph Labeling, Number Theory, Reliability Analysis, and Mathematical Statistics.