ROBUST ATTRACTORS FOR A KIRCHHOFF-BOUSSINESQ TYPE EQUATION

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Abstract. The paper studies the existence of the pullback attractors and robust pullback exponential attractors for a Kirchhoff-Boussinesq type equation:

\[ u_{tt} - \Delta u_t + \Delta^2 u = \text{div} \left\{ \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right\} + \Delta g(u) + f(x,t). \]

We show that when the growth exponent \( p \) of the nonlinearity \( g(u) \) is up to the critical range:

\[ 1 \leq p \leq p^* \equiv \frac{N+2}{(N-2)}, \]

(i) the IBVP of the equation is well-posed, and its solution has additionally global regularity when \( t > \tau \); (ii) the related dynamical process \( \{U_f(t,\tau)\} \) has a pullback attractor; (iii) in particular, when \( 1 \leq p < p^* \), the process \( \{U_f(t,\tau)\} \) has a family of pullback exponential attractors, which is stable with respect to the perturbation \( f \in \Sigma \) (the sign space).

1. Introduction. In this paper, we are concerned with the existence of the pullback attractors and robust pullback exponential attractors for a Kirchhoff-Boussinesq type equation:

\[ u_{tt} - \Delta u_t + \Delta^2 u = \text{div} \left\{ \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right\} + \Delta g(u) + f(x,t), \]  

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) (1 \( \leq N \leq 6 \)) with the smooth boundary \( \partial \Omega \), and the assumptions on the nonlinear feedback force \( g(u) \) and the time-dependent external force \( f(x,t) \) will be specified later.

Chueshov and Lasiecka [6, 8] proposed 2D Kirchhoff-Boussinesq model:

\[ u_{tt} + ku_t + \Delta^2 u = \text{div}[g_0(\nabla u)] + \Delta [g_1(u)] - g_2(u), \]  

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where $k > 0$ is the damping parameter, the mapping $g_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and the sufficiently smooth functions $g_1$ and $g_2$ represent nonlinear feedback forces acting upon the plate. They showed that the model (3) arises as the limit of the Midlin-Timoshenko equations which describe the dynamics of a plate that accounts for transverse shear effects (cf. [21, 20]). When the terms $\text{div}[g_0(\nabla u)]$ and $g_2(u)$ are absent, Eq. (3) becomes “good” Boussinesq equation with viscous damping (cf. [1, 22, 27]). While the nonlinear restoring force $g_0(\nabla u)$ naturally arises in the Kirchhoff plate model (cf. [7]), this is why Eq. (3) is called Kirchhoff-Boussinesq model.

One typical example of physical interests is $g_0(\nabla u) = |\nabla u|^{p-1} \nabla u$ (p-Laplacian) (cf. [16, 17, 23]); another is $g_0(\nabla u) = \nabla u \sqrt{1+|\nabla u|^2}$ (cf. [16, 18, 24, 31]).

By taking $g_0(\nabla u) = |\nabla u|^2 \nabla u, g_1(u) = u^2 + u, (4)$ Chueshov and Lasiecka [6, 8] studied the well-posedness of the corresponding 2D model (3), and they also established the existence of global attractor provided that the damping parameter $k$ is sufficiently large. These investigations are of particular difficulties because both the Kirchhoff nonlinearity $g_0(\nabla u)$ and the Boussinesq nonlinearity $\Delta [g_1(u)]$ appear at the same time, which leads to that the techniques used to deal with the longtime dynamics of both Kirchhoff plate model (cf. [23, 29]) and the Boussinesq model (cf. [14, 15, 30, 32, 34, 35]) fail.

Replacing the weak damping $ku_t$ in Eq. (1) by the strong structural damping $-\Delta u_t$, which plays a regularizing effect for the solutions and may make that the corresponding equation is of some parabolic-like properties, taking $g_0(\nabla u) = \nabla u \sqrt{1+|\nabla u|^2}$ and taking account of the effect of the time-dependent external force, we get model equation (1).

Pullback attractor and pullback exponential attractor are two key concepts to study the longtime dynamics of non-autonomous infinite dimensional dynamical system, the theory on them have been extensively developed and applied to many model equations arising from mathematical physics (cf. [2, 3, 26, 28] and references therein). In the concrete, a process acting on the Banach space $E$ is a two-parametrical family of operators $\{U(t,\tau) : E \to E| t \geq \tau, t, \tau \in \mathbb{R}\}$ (or $t, \tau \in \mathbb{Z}$ for discrete time) satisfying

$U(t,s)U(s,\tau) = U(t,\tau), \quad U(\tau,\tau) = I$ (identity operator), \quad $t, s, \tau \in \mathbb{R}, \quad t \geq s \geq \tau$.

A family of nonempty compact subsets $\{A(t)\}_{t \in \mathbb{R}}$ in $E$ is said to be a pullback attractor of the process $U(t,\tau)$ if it is invariant, i.e., $U(t,s)A(s) = A(t), \quad t \geq s$, and it pullback attracts all bounded subsets of $E$, i.e., for every bounded subset $D \subset E$ and $t \in \mathbb{R},$

$$\lim_{s \to +\infty} \text{dist}_E\{U(t,t-s)D, A(t)\} = 0. \quad (5)$$

However, the pullback attractor may have two drawbacks: (i) the rate of convergence in (5) may be slow, which leads to that it is difficult to estimate the pullback attracting rate in term of the physical parameters of the system; (ii) in many situations, one cannot show the finite dimensionality of the sections of the pullback attractor, which results in that the pullback attractor may be unobservable in experiments or in numerical simulations.

In order to overcome these drawbacks, Efendiev, Miranville and Zelik [12] further proposed the concept of pullback exponential attractor (see Def. 4.1 below)
and established its existence and stability criterion for the discrete dynamical process. Later, Langa, Miranville and Real [19], Czaja and Efendiev [10] extended the existence results in [13] to the continuous process. Carvalho and Sooner [4, 5] further improved the results in [10, 19] and give some applications to the hyperbolic equations. More recently, Yang and Li [33] further established two criteria on the existence of robust pullback exponential attractors.

In the present paper, the Kirchhoff type nonlinearity $g_0(\nabla u) = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ is another kind of nonlinear restoring force, whose role is different from that in (4) for it can not provide useful $L^4$-estimate for the gradient of the weak solutions (cf. [6, 8]), so considering this kind of Kirchhoff nonlinearity is also challenging.

By assuming strong damping instead of weak one, several subtle issues (e.g. uniqueness of solutions, the sufficiently large damping parameter $k$, the restriction $N = 2$ for the space dimension) dealt in the past for this kind of model (cf. [6, 8]) are no longer obstacle because the strong damping provides the helpful regularity effects and increases the expectation of nice results for this kind of model.

But even in this case, the techniques of dealing with the damped Boussinesq type equations used in [30, 32, 34] are still useless for the appearance of the Kirchhoff type nonlinearity. However, we can use the delicate multiplier technology to get the Lipschitz stability of the weak solutions in weaker space, and then concentrate our attention on the robust attractors of the nonautonomous dynamical systems. The adding of nonautonomous term makes that the Kirchhoff-Boussinesq model is of more suitability and stimulates us to investigate the robustness of the related pullback exponential attractors on its perturbations, which is completely different from the upper semicontinuity of the global attractors on the damping parameters (dissipative index) as done for damped Boussinesq model in [32, 34].

The research on the well-posedness and longtime dynamics of this mix-and-match model is interesting not only for its physical background but also for its complexity of overcoming both the the Kirchhoff nonlinearity and the Boussinesq nonlinearity. Though the model is originally a specifically 2D plate model, it is interesting to investigate it in more general ND ($N \geq 2$) case just as done by mathematicians dealing with the Kirchhoff plate model [9]:

$$u_{tt} - \phi(\|\nabla u\|^2)\Delta u - \sigma(\|\nabla u\|^2)\Delta u_t + f(u) = g(x).$$  \hspace{1cm} (6)

Indeed, it is just Chueshov’s work on the ND Kirchhoff plate model (6) and creatively using the multiplier method which contains many delicate techniques that make him find a supercritical index of the nonlinear source term $f(u)$: $p_{**} = \frac{N+4}{(N+2)^2}$ and break though the longtime existed restriction of the critical index $p^* = \frac{N+4}{(N-2)^2}$ and show that when the growth exponent $p$ of the source term is up to the supercritical range $p^* < p < p_{**}$ ($N \geq 3$), the IBVP of Eq. (6) is still well-posed and the related solution semigroup has a partially strong global attractor (cf. [9]). These new interesting phenomena are interested by PDE/dynamical systems researchers and readers and stimulate then a further study on the longtime dynamics for this model, one can see [11, 28, 33] and references therein.

It is just relying on the continuous research from various angles for a model equation (e.g. Kirchhoff wave model, Boussinesq model, Kirchhoff-Boussinesq model and so on) that promotes the appearance of good mathematical idea and technology and deepens the understanding for the science laws behind it.

The present paper devotes to investigate the well-posedness and the existence of pullback attractor and robust pullback exponential attractors for problem (1)-(2).
We show that that when the growth exponent \( p \) of the nonlinearity \( g(u) \) is up to the critical range: \( 1 \leq p \leq p^* = \frac{N+2}{(N-2)} \), 
(i) problem (1)-(2) is well-posed, and its solution has additionally global regularity when \( t > \tau \); 
(ii) the related dynamical process \( \{U_f(t, \tau)\} \) has a pullback attractor; 
(iii) in particular, when \( 1 \leq p < p^* \), by using an abstract criteria recently established in [33] we show that the process \( \{U_f(t, \tau)\} \) has a family of pullback exponential attractors, which is stable with respect to the perturbation \( f \in \Sigma \) (the sign space).

The paper is organized as follows. In Section 2, we discuss the well-posedness and some parabolic-like properties of the weak solutions. In Section 3, we investigate the existence of pullback attractor. In Section 4, we study the existence and stability of pullback exponential attractors.

2. Well-posedness. We begin with the following abbreviations:

\[ L^p = L^p(\Omega), \quad H^k = H^k(\Omega), \quad V_2 = H^2 \cap H^1_0, \quad \| \cdot \| = \| \cdot \|_{L^2}, \quad \| \cdot \|_p = \| \cdot \|_{L^p}, \]

with \( p \geq 1 \), \( H^k \) are the \( L^2 \)-based Sobolev spaces and \( H^k_0 \) are the completion of \( C_\infty^0(\Omega) \) in \( H^k \) for \( k > 0 \). The notation \((\cdot, \cdot)\) for the \( L^2 \)-inner product will also be used for the notation of duality pairing between dual spaces, \( C(\cdots) \) denotes positive constants depending on the quantities appearing in the parenthesis.

We define the operator \( A : V_2 \rightarrow V_{-2} \) (the dual space of \( V_2 \)),

\[ (Au, v) = (\Delta u, \Delta v) \quad \text{for any} \quad u, v \in V_2. \]

Then, \( A \) is self-adjoint and strictly positive on \( V_2 \), and we can define the power \( A^s \) of \( A \) (\( s \in \mathbb{R} \)), and the spaces \( V_s = D(A^s) \) are Hilbert spaces with the scalar products and the norms

\[ (u, v)_s = (A^{\frac{s}{2}}u, A^{\frac{s}{2}}v), \quad \| u \|_s = \| A^{\frac{s}{2}}u \|. \]

We define the operator \( B : V_1 \rightarrow V_{-1} \),

\[ (Bu, v) = \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \nabla v \right) \quad \text{for any} \quad u, v \in V_1. \]

Obviously, the operator \( B \) is bounded, \( \| Bu \|_{V_{-1}} \leq |\Omega|, \forall u \in V_1. \)

We denote the phase spaces

\[ X_2 = V_2 \times L^2, \quad X_1 = V_1 \times V_{-1}, \quad X_0 = L^2 \times V_{-2}, \]

which are equipped with the usual graph norms, for example,

\[ \|(u, v)\|^2_{X_1} = \| u \|^2_{V_1} + \| v \|^2_{V_{-1}}. \]

Obviously, they are the Hilbert spaces and

\[ X_2 \hookrightarrow X_1 \hookrightarrow X_0. \]

Rewriting Eq. (1) as an operator equation and applying \( A^{-\frac{1}{2}} \) to the resulting expression, we get the equivalent problem:

\[ A^{-\frac{1}{2}}u_{tt} + A^{\frac{1}{2}}u + u_t + A^{-\frac{1}{2}}Bu + g(u) = A^{-\frac{1}{2}}f, \quad (7) \]

\[ u(t) = u_0^\tau, \quad u_t(t) = u_1^\tau. \quad (8) \]
Assumption 2.1. (i) \( g \in C^1(\mathbb{R}) \), \( g(0) = 0 \),
\[
\liminf_{|s| \to \infty} g'(s) > -\sqrt{\lambda_1}, 
\]
where \( \lambda_1(> 0) \) is the first eigenvalue of the operator \( A \), and
\[
|g'(s)| \leq C(1 + |s|^{p-1}), \quad s \in \mathbb{R},
\]
where \( 1 \leq p < \infty \) as \( N = 1, 2 \); \( 1 \leq p \leq p^* \equiv \frac{N+2}{N-2} \) as \( 3 \leq N \leq 6 \).
(ii) \( f \in L^2_{loc}(\mathbb{R}; V_{-1}), (u_0^*, u_1^*) \in X_1 \), with \( \|(u_0^*, u_1^*)\|_{X_1} \leq R \).

Remark 2.2. Formula (9) implies that \( g'(s) > -C, \forall s \in \mathbb{R} \), and there exists a positive constant \( \theta > 0 \) such that
\[
G(s) \geq -\frac{\theta}{2} s^2 - C(\theta), \quad g(s)s \geq G(s) - \frac{\theta}{2} s^2 - C(\theta), \quad s \in \mathbb{R},
\]
where \( G(s) = \int_0^s g(r)dr \).

Theorem 2.3. Let Assumptions 2.1 be valid. Then problem (7)-(8) admits a unique weak solution \( u \), with \( \varphi_u = (u, u_t) \in C(\mathbb{R}_+; X_1) \cap L^2(\tau, T; X_2) \), and it possesses the following properties:
(i) (Dissipativity)
\[
\|u(t)\|_{V_1}^2 + \|u_t(t)\|_{V_{-1}}^2 \leq Ce^{-\gamma(t-\tau)}(\|u_0^*\|_{V_1}^2 + \|u_1^*\|_{V_{-1}}^2 + \|u_0^*\|_{V_1}^2) + C\left(\int_\tau^t e^{-\gamma(t-s)}\|f(s)\|_{V_2}^2 ds + 1\right), \quad t \geq \tau,
\]
\[
\int_\tau^{t+1} \left(\|u(r)\|_{V_1}^2 + \|u_t(r)\|^2 + \|u_{tt}(r)\|_{V_{-2}}^2\right)dr \leq K, \quad t \in [\tau, T],
\]
where \( \mathbb{R}_+ = [\tau, +\infty) \), \( K = C(R, T) \), \( \gamma \) is a small positive constant.
(ii) (Lipschitz stability)
\[
\|(z(t), z_t(t))\|_{X_1}^2 + \int_\tau^t \|z_t(s)\|^2 ds \leq K_1(\|z(\tau), z_t(\tau)\|_{X_1}^2), \quad t \in [\tau, T],
\]
where \( K_1 = C(R, T, T - \tau) > 0 \), \( z = u - v \), \( u \) and \( v \) are two weak solutions of problem (7)-(8) corresponding to initial data \( (u_0^*, u_1^*), (v_0^*, v_1^*) \in X_1 \).
(iii) (Global regularity when \( t > \tau \)) For any \( a \in (\tau, T) \), \( (u, u_t) \in L^\infty(a; T; X_2) \), and
\[
\|u(t)\|_{V_2}^2 + \|u_t(t)\|^2 \leq K_1\left(\frac{1}{a - \tau} + 1\right), \quad t \in (a, T].
\]
(iv) (Energy identity) The following energy identity
\[
E(\varphi_u(t)) + \int_s^t \left(\|u_t(r)\|^2 + (A^{-\frac{1}{2}}Bu, u_t)\right)dr = E(\varphi_u(s)) + \int_s^t (A^{-\frac{1}{2}}f, u_t)dr
\]
holds for \( t \geq s \geq \tau \), where \( \varphi_u = (u, u_t) \) and
\[
E(\varphi_u) = \frac{1}{2}(\|A^{-\frac{1}{2}}u\|^2 + \|A^{\frac{1}{2}}u\|^2) + \int_\Omega G(u)dx.
\]
Proof. We first formally obtain some a priori estimates to the solutions of problem (7)-(8). Using the multiplier \( u_t + \gamma u \) in Eq. (7), we have
\[
\frac{d}{dt} H_1(\varphi_u) + I_1(\varphi_u) = 0, \quad t > \tau,
\]
where
\[ H_1(\varphi_u) = E(\varphi_u) + \gamma \left( A^{-\frac{1}{2}} u, A^{-\frac{1}{2}} u_t \right) + \frac{1}{2} \| u \|^2. \]
\[ I_1(\varphi_u) = \| u \|^2 - \gamma \| A^{-\frac{1}{2}} u \|^2 + (A^{-\frac{1}{2}} B u - A^{-\frac{1}{2}} f, u_t + \gamma u) + \gamma \left[ \| A^\frac{1}{2} u \|^2 + (g(u), u) \right]. \]

Obviously,
\[ |(A^{-\frac{1}{2}} B u - A^{-\frac{1}{2}} f, u_t + \gamma u)| \leq (\| B u \|_{V^{-\frac{1}{2}}} + \| f \|_{V^{-\frac{1}{2}}}) \| u_t + \gamma u \|_{V^{-\frac{1}{2}}} \leq \frac{1}{4} (\| u_t \|_{V^{-\frac{1}{2}}}^2 + \gamma^2 \| u \|^2_{V^{-\frac{1}{2}}}) + \| f \|^2_{V^{-\frac{1}{2}}} + C, \]
\[ \gamma \left[ (g(u), u) - \int_\Omega G(u) dx \right] \geq -\frac{\gamma \theta}{2} \| u \|^2 - C \geq -\frac{\gamma \theta}{2\sqrt{\lambda_1}} \| u \|^2_{V^{-\frac{1}{2}}} - C. \]

Taking \( \gamma : 0 < \gamma \leq 2 \left( 1 - \frac{\theta}{\sqrt{\lambda_1}} \right) \), we have
\[ I_1(\varphi_u) - \gamma H_1(\varphi_u) \geq \frac{1}{4} \| u_t \|^2 - \| f \|^2_{V^{-\frac{1}{2}}} - C, \]
\[ C_0 \| \varphi_u \|^2_{V^{-\frac{1}{2}}} - C \leq H_1(\varphi_u) \leq C_1 [\| u \|^2_{V^{-\frac{1}{2}}} + \| u_t \|^2_{V^{-\frac{1}{2}}} + \| u \|^{p+1}_{V^1}] + C. \]

Inserting (17) into (16) gives
\[ \frac{d}{dt} H_1(\varphi_u) + \gamma H_1(\varphi_u) + \frac{1}{4} \| u_t \|^2 \leq C(1 + \| f \|^2_{V^{-\frac{1}{2}}}), \quad t > \tau. \]

Applying the Gronwall lemma to (18) and making use of (17) yields (10).

Letting \( \gamma = 0 \) in (18), integrating the resulting expression over \((t, t+1)\) and using (10) we obtain
\[ \int_t^{t+1} \| u_t(s) \|^2 ds \leq K, \quad t \in [\tau, T]. \]

Using the multiplier \( A^{\frac{1}{2}} u \) in Eq. (7) gives
\[ \frac{d}{dt} \left[ (u, u_t) + \frac{1}{2} \| u \|^2_{V^1} \right] + \| u \|^2_{V^1} + \int_{\Omega} \frac{\| \nabla u \|^2}{\sqrt{1 + \| \nabla u \|^2}} dx + \int_{\Omega} g'(u) \| \nabla u \|^2 dx = \| u_t \|^2 + (f, u). \]

Integrating (20) over \((t, t+1)\) and making use of (10), (19) yields
\[ \int_t^{t+1} \| u(s) \|^2_{V^1} ds \leq K, \quad t \in [\tau, T]. \]

It follows from Eq. (7) and estimates (19), (21) that
\[ \int_t^{t+1} \| u_{tt}(s) \|^2_{V^{-\frac{1}{2}}} ds \leq C \int_t^{t+1} \left( \| u(s) \|^2_{V^2} + \| u_t(s) \|^2 \right. \]
\[ \left. + \| B u \|^2_{V^{-\frac{1}{2}}} + \| g(u) \|^2 + \| f(s) \|^2_{V^{-\frac{1}{2}}} \right) ds \leq K, \quad t \in [\tau, T], \]

where we have used the estimate
\[ \int_t^{t+1} \| g(u) \|^2 ds \leq C \int_t^{t+1} (1 + \| u \|^{2p}_{2p}) ds \leq C \int_t^{t+1} (1 + \| u \|^{2p(1-\mu)}_{p+1}) \| u \|^{2\mu}_{V^2} ds \leq C \int_t^{t+1} (1 + \| u \|^2_{V^2}) ds, \]
and where
\[ \mu = \frac{N(p-1)}{p(4(p+1) - N(p-1))}, \quad \mu \leq 1 \text{ for } 1 \leq N \leq 6. \] (23)

The combination of (19), (21)-(22) gives (11).

Let \( u, v \) be two weak solutions of problem (7)-(8) corresponding to initial data \((u_0^0, u_0^1), (v_0^0, v_0^1) \in X_1 \). Then \( z = u - v \) solves
\[ A^{-\frac{1}{2}} z_{tt} + A^{\frac{1}{2}} z + g(u) - g(v) + A^{-\frac{1}{2}} B u - A^{-\frac{1}{2}} B v = 0, \] (24)
\[ z(t) = u_0^0 - v_0^0 \equiv z_0^0, \quad z_t(t) = u_0^1 - v_0^1 \equiv z_0^1. \]

Using the multiplier \( z_t \) in Eq. (24) gives
\[ \frac{1}{2} \frac{d}{dt} (\|z\|^2_{V_1} + \|z_t\|^2_{V_{-1}}) + \|z_t\|^2 + (g(u) - g(v), z_t) + (A^{-\frac{1}{2}} B u - A^{-\frac{1}{2}} B v, z_t) = 0. \] (25)

Obviously,
\[ |(A^{-\frac{1}{2}} B u - A^{-\frac{1}{2}} B v, z_t)| \leq \|B u - B v\|_{V_{-1}} \|z_t\|_{V_{-1}}, \]
\[ \leq \left\| \int_0^1 \left[ \frac{\nabla z}{\sqrt{1 + |\nabla z\eta|^2}} - \frac{\nabla z\eta \cdot \nabla z_0}{(1 + |\nabla z\eta|^2)^{\frac{3}{2}}} \right] \, d\eta \right\| \|z_t\|_{V_{-1}} \]
\[ \leq 2 \int_0^1 \left\| \frac{\nabla z}{\sqrt{1 + |\nabla z\eta|^2}} \right\| d\eta \|z_t\|_{V_{-1}}, \]
\[ \leq 2 \left( \int_0^1 \right) \|z\|_{V_1} \|z_t\|_{V_{-1}}, \]
\[ \leq \|z_t\|^2_{V_{-1}} + \|z\|^2_{V_1}, \]
where \( z_\eta = \eta u + (1 - \eta) v, \) \( 0 < \eta < 1. \) By the interpolation inequality,
\[ \|u\|_{V_1}^{p-1} \leq \|u\|_{(p+1)\frac{p}{2(p+1)+1}}^{p-1} \|u\|_{V_2}^{p-1} \leq K \|u\|_{V_2}^{p-1}, \] (27)
where \( \theta = \frac{N}{2N - (N-4)(p+1)}, \) \( \theta(p-1) \leq 1 \) for \( 1 \leq N \leq 6, \) so
\[ |(g(u) - g(v), z_t)| \leq C(1 + \|u\|_{(p+1)\frac{p}{2(p+1)+1}}^{p-1} + \|v\|_{(p+1)\frac{p}{2(p+1)+1}}^{p-1}) \|z\|_{(p+1)\frac{p}{2(p+1)+1}} \|z_t\| \]
\[ \leq K(1 + \|u\|^0_{V_2} + \|v\|^0_{V_2}) \|z\|_{V_1} \|z_t\| \]
\[ \leq K(1 + \|u\|^2_{V_2} + \|v\|^2_{V_2}) \|z\|^2_{V_1} + \frac{1}{4} \|z_t\|^2. \] (28)

Inserting (26) and (28) into (25) yields
\[ \frac{d}{dt} (\|z\|^2_{V_1} + \|z_t\|^2_{V_{-1}}) + \frac{1}{4} \|z_t\|^2 \leq K(1 + \|u\|^2_{V_2} + \|v\|^2_{V_2}) (\|z\|^2_{V_1} + \|z_t\|^2_{V_{-1}}), \]
\[ \|z(t), z_t(t)\|^2_{X_1} + \int_\tau^t \|z_s(s)\|^2 ds \]
\[ \leq \exp \left\{ \int_\tau^t K(1 + \|u(r)\|^2_{V_2} + \|v(r)\|^2_{V_2}) dr \right\} \|z(t), z_t(t)\|^2_{X_1} \]
\[ \leq K_1 \|z_\tau(z_\tau), z_\tau(z_\tau)\|^2_{X_1}, \quad \tau < t \leq T. \]

That is, estimate (12) holds.

Based on estimates (10)-(12) (which obviously hold for the Galerkin approximations), by using the Galerkin method one easily obtains that problem (7)-(8) admits a unique weak solution \( u, \) with \( \varphi_u = (u, u_t) \in C_w(\mathbb{R}_T; X_1) \cap L^2(\tau; T; X_2), \) and estimates (10)-(12) hold for the solution \( u, \) we omit the proving process here.
Using the multiplier $A^{\frac{1}{2}}u_t$ in Eq. (7) yields
\[
\frac{d}{dt}H_2(t) + \|u_t\|_{V_1}^2 = (g'(u)\nabla u, \nabla u_t) + (A^{-\frac{1}{2}}f, A^{\frac{1}{2}}u_t) \\
\leq C(1 + \|u\|_{L^p(\Omega_T)}^{p-1})\|\nabla u\|_{L^p(\Omega_T)}\|\nabla u_t\| + \frac{1}{4}\|u_t\|_{V_1}^2 + 2\|f\|_{L^2(\Omega_T)}^2 \\
\leq K(1 + \|u\|_{L^p(\Omega_T)}^{p-1})\|u\|_{V_2}\|u_t\|_{V_1} + \frac{1}{4}\|u_t\|_{V_1}^2 + 2\|f\|_{L^2(\Omega_T)}^2 \\
\leq \frac{1}{2}\|u_t\|_{V_1}^2 + K(1 + \|u\|_{V_2})H_2(t) + 2\|f\|_{L^2(\Omega_T)}^2,
\]
where we have used (27) and \( \tau \) holds for \( t \geq \tau \).

Integrating (30) for \( t \geq \tau \) and energy equality (14) holds for \( t \geq s \geq \tau \).

Then by estimates (19) and (21), we have
\[
\int_{t}^{t+r} K(1 + \|u(s)\|_{V_2})ds \leq K_1, \quad \int_{t}^{t+r} 2\|f(s)\|_{V_2}^2 ds \leq K_1, \\
\int_{t}^{t+r} H_2(s)ds \leq K_1, \quad t \in (\tau, T).
\]

Taking \( \tau < t \leq t_1 < t + r \), multiplying (29) by \( e^{-\int_{t}^{t+r} K(1 + \|u(s)\|_{V_2})ds} \) and integrating the resulting expression over \( (t_1, t + r) \) yields
\[
H_2(t + r) \leq H_2(t_1) e^{\int_{t}^{t+r} K(1 + \|u(s)\|_{V_2})ds} + \int_{t}^{t+r} 2\|f(s)\|_{V_2}^2 e^{\int_{t+r}^{t} K(1 + \|u(r)\|_{V_2})dr}dr ds \\
\leq e^{K_1} \left( H_2(t_1) + K_1 \right).
\]

Integrating (30) for \( t_1 \) over \( (t, t + r) \) gives
\[
H_2(t + r) \leq e^{K_1} K_1 \left( \frac{1}{r} + 1 \right) \leq K_1 \left( \frac{1}{a - \tau} + 1 \right), \quad t > \tau.
\]

Since \( t > \tau \) is equivalent to \( t + r > a \), we obtain formula (13) by variable transformation.

Since \((u, u_t) \in L^2(\tau, T; V_2 \times L^2)\) (see (11)), we can use the multiplier \( u_t \) in Eq. (7) and energy equality (14) holds for \( t \geq s \geq \tau \).

By energy identity (14), we have that for any \( t_0 \geq \tau \),
\[
\limsup_{t \to t_0} \frac{1}{2} \left( \|A^{-\frac{1}{2}}u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 \right) + \liminf_{t \to t_0} \int_{\Omega} G(u(t_0))dx \\
\leq \lim_{t \to t_0} \int_{\Omega} \left[ \frac{1}{2} \left( \|A^{-\frac{1}{2}}u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 \right) + \int G(u(t))dx \right] \\
= \frac{1}{2} \left( \|A^{-\frac{1}{2}}u_t(t_0)\|^2 + \|A^{\frac{1}{2}}u(t_0)\|^2 \right) + \liminf_{t \to t_0} \int_{\Omega} G(u(t_0))dx \\
\leq \liminf_{t \to t_0} \frac{1}{2} \left( \|A^{-\frac{1}{2}}u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 \right) + \liminf_{t \to t_0} \int_{\Omega} G(u(t))dx,
\]
where we have used in the last inequality the facts: \((u, u_t) \in C_w(\mathbb{R}_r; X_1), \) Remark 2.2 and the Fatou Lemma. Therefore,
\[
\lim_{t \to t_0} \|(u(t), u_t(t))\|_{X_1} = \|(u(t_0), u_t(t_0))\|_{X_1}.
\]
By the uniform convexity of the Banach space \(X_1, (u, u_t) \in C(\mathbb{R}_r; X_1).\)

**Remark 2.4.** We see from formulas (23) and (27) that the restriction for the space dimension: \(1 \leq N \leq 6\) is indispensable for the proof of Theorem 2.3.

3. **Pullback attractor.** Under Assumption 2.1, we define the operator \(U_f(t, \tau) : X_1 \to X_1,\)
\[
U_f(t, \tau) \varphi_\tau = \varphi(t) = (u(t), u_t(t)) \quad \text{for every} \quad \varphi_\tau \in X_1, \quad t \geq \tau,
\]
where \(u\) is a weak solution of problem (7)-(8), with initial data \(\varphi(\tau) = \varphi_\tau.\) By Theorem 2.3, the family of operators \(\{U_f(t, \tau)| t \geq \tau, \tau \in \mathbb{R}\}\) forms a continuous process on \(X_1.\)

**Theorem 3.1.** Let Assumptions 2.1 be valid, and
\[
\text{either} \quad \|f\|_{L^2(-\infty; t; V_{-1})} < \infty \quad \text{or} \quad \|f\|_{L^\infty(-\infty; t; V_{-1})} < \infty, \quad \forall t \in \mathbb{R}. \tag{31}
\]
Then the process \(\{U_f(t, \tau)\}\) possesses a pullback attractor \(\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}},\) where
\[
A(t) = \bigcap_{s \geq 0} \left[ \bigcup_{\tau \geq s} U_f(t, t - \tau)D(t - \tau) \right]_{X_1}, \quad t \in \mathbb{R}, \tag{32}
\]
is bounded in \(X_2\) for each \(t \in \mathbb{R},\) and where the set \(D(t)\) is as shown in (35).

**Proof.** Let the ball \(B(t) = \{\xi \in X_1 | \|\xi\|_{X_1} \leq R(t)\},\) with
\[
R^2(t) = \begin{cases} 2C\left(1 + \|f\|_{L^2(-\infty; t; V_{-1})}^2\right), & \text{if} \quad \|f\|_{L^2(-\infty; t; V_{-1})} < \infty, \\ 2C\left(1 + \frac{1}{\gamma} \|f\|_{L^\infty(-\infty; t; V_{-1})}^2\right), & \text{if} \quad \|f\|_{L^\infty(-\infty; t; V_{-1})} < \infty. \end{cases} \tag{33}
\]
Estimate (10) implies that the family \(\mathcal{B} = \{B(t)\}_{t \in \mathbb{R}}\) is a pullback absorbing family of \(\{U_f(t, \tau)\}\) in \(X_1.\) So for any bounded set \(B \subset X_1, U_f(t, t - \tau)B \subset B(t)\) for all \(\tau \geq T_1 = T_1(t, B).\) Due to \(R(t) \leq R(s)\) for \(t \leq s,\) we see from (10) that
\[
\|U_f(t, t - \tau)\varphi_\tau\|_{X_1}^2 \leq e^{-\tau t} Q(R(t - \tau)) + \frac{1}{2} R^2(t)
\leq e^{-\tau t} Q(R(t)) + \frac{1}{2} R^2(t),
\]
where \(\varphi_\tau \in B(t - \tau), \) \(Q(R) = C(R^2 + R^{{p+1}}).\) By the arbitrariness of \(\varphi_\tau \in B(t - \tau),\) we have that for any \(t \in \mathbb{R},\) there exists a \(T_1 > 0\) such that
\[
U_f(t, t - \tau) B(t - \tau) \subset B(t), \quad U_f(t - 1, t - \tau) B(t - \tau) \subset B(t - 1), \quad \tau \geq T_1. \tag{34}
\]
Let
\[
D(t) = \left[ \bigcup_{\tau \geq T_1} U_f(t, t - \tau)B(t - \tau) \right]_{X_1} \subset B(t). \tag{35}
\]
By the pullback absorbing property of \(\mathcal{B} = \{B(t)\}_{t \in \mathbb{R}},\) for any bounded set \(B \subset X_1,\) there exists a \(\tau_B > 0\) such that
\[
U_f(t - T_1 - 1, t - \tau)B \subset B(t - T_1 - 1), \quad \forall \tau \geq \tau_B.
\]
Therefore,
\[
U_f(t, t - \tau)B = U_f(t, t - T_1 - 1)U_f(t - T_1 - 1, t - \tau)B \\
\subset U_f(t, t - T_1 - 1)B(t - T_1 - 1) \subset D(t), \ \forall \tau \geq \tau_B,
\]
which means that \( \mathcal{D} = \{D(t)\}_{t \in \mathbb{R}} \) is a pullback absorbing family of the process \( \{U_f(t, \tau)\} \). Moreover,
\[
U_f(t, t - \tau)D(t - \tau) \subset U_f(t, t - \tau)B(t - \tau) \subset D(t), \ \tau \geq T_t,
\]
that is, the family \( \mathcal{D} \) is also pullback \( \mathcal{D} \)-absorbing. By formula (34),
\[
U_f(t, t - \tau)B(t - \tau) = U_f(t, t - 1)U_f(t - 1, t - \tau)B(t - \tau) \\
\subset U_f(t, t - 1)B(t - 1), \ \tau \geq T_t.
\]
Hence,
\[
D(t) \subset [U_f(t, t - 1)B(t - 1)]_{\tau=1}, \ \forall t \in \mathbb{R}.
\]
By estimate (13) (taking \( \tau = t - 1, a = t - 1/2 \) there), \( U_f(t, t - 1)B(t - 1) \) is bounded in \( X_2 \). So the set \( D(t) \) is bounded in \( X_2 \) for each \( t \in \mathbb{R} \). Therefore, for any sequences \( \tau_n \to \infty \) and \( x_n \in D(t - \tau_n) \), the sequence \( \{U_f(t, t - \tau_n)x_n\}_{n \in \mathbb{N}} \) is precompact in \( X_1 \) for \( X_2 \leftrightarrow X_1 \). That is, the process \( \{U_f(t, \tau)\} \) is pullback \( \mathcal{D} \)-asymptotically compact in \( X_1 \). Therefore, it has a pullback attractor \( \mathcal{A} = \{A(t)\}_{t \in \mathbb{R}} \), the set \( A(t) \) is as shown in (32) (cf. Theorem 4.2 in [28]), and \( A(t) \) is bounded in \( X_2 \) for \( A(t) \subset D(t) \) for each \( t \in \mathbb{R} \).

**Remark 3.2.** (i) In Theorem 3.1, condition (31) can be replaced by \( f \in L^1_0(\mathbb{R}; V_{-1}) \) (see Remark 4.6: (i)).

(ii) We see from formula (35) that
\[
\bigcup_{t \leq 0} A(t) \subset \bigcup_{t \leq 0} D(t) \subset B(0).
\]

4. Robust pullback exponential attractors.

**Definition 4.1.** [12] Let \( E \) be a Banach space with the norm \( \| \cdot \|_E \), \( M \) be a subset of \( E \), which is a metric space equipped with the distance \( d(x, y) = \|x - y\|_E \), the family \( \{U(t, \tau)\} \) be a process acting on \( M \). Then the triple \( \{U(t, \tau), M, E\} \) is said to be a non-autonomous dynamical system, \( M \) and \( E \) are said to be the phase space and the universal space, respectively.

**Definition 4.2.** [13] A family \( \{\mathcal{M}(t)\}_{t \in \mathbb{R}} \) of subsets of \( M \) is said to be an exponential attractor of the non-autonomous dynamical system \( \{U(t, \tau), M, E\} \), if
1. it is semi-invariant, i.e., \( U(t, s)\mathcal{M}(s) \subset \mathcal{M}(t), \ t \geq s; \)
2. each section \( \mathcal{M}(t) \) is a compact set of \( E \) and its fractal dimension in \( E \) is uniformly bounded, i.e.,
\[
\sup_{t \in \mathbb{R}} \dim_f(\mathcal{M}(t), E) < \infty;
\]
3. it pullback attracts every bounded subset \( B \) in \( M \) at an exponential rate, i.e.,
\[
\text{dist}_E\{U(t, t - s)B, \mathcal{M}(t)\} \leq C(B)e^{-\beta s}, \ t \in \mathbb{R}, \ s \geq T(B),
\]
for some \( \beta > 0 \).

Obviously, when \( M = E \), Def. 4.1-4.2 coincide with the standard ones.
Lemma 4.3. [33] Let $\Sigma$ be a symbol space or an index set, $M$ be a bounded closed subset of the Banach space $E$, which is equipped with the distance $d(x, y) = \|x - y\|_E$, and $(U_\sigma(t, \tau), M, E)$ be a non-autonomous dynamical system for each $\sigma \in \Sigma$. Assume that:

(i) there exist constants $T > 0$, $L_T > 0$, such that, for any $\tau \in \mathbb{R}$, $x_1, x_2 \in M$,

$$\bigcup_{\sigma \in \Sigma} \sup_{t \in [0, T]} \|U_\sigma(t + \tau, \tau)x_1 - U_\sigma(t + \tau, \tau)x_2\|_E \leq L_T \|x_1 - x_2\|_E;$$

(ii) there exist a mapping $K_\sigma^n : M \to Z$ for each $\sigma \in \Sigma$, $n \in \mathbb{Z}$ such that for any $x, y \in M$,

$$\sup_{\sigma \in \Sigma} \sup_{n \in \mathbb{Z}} \|K_\sigma^n x - K_\sigma^n y\|_Z \leq L \|x - y\|_E,$$

$$\|U_\sigma((n + 1)T, nT)x - U_\sigma((n + 1)T, nT)y\|_E \leq \eta \|x - y\|_E + n_Z(K_\sigma^n x - K_\sigma^n y),$$

where $\eta : 0 < \eta < 1$, $L > 0$ are constants independent of $\sigma$ and $n$.

Then, for each $\theta \in (\eta, 1)$, $\sigma \in \Sigma$, the dynamical system $(U_\sigma(t, \tau), M, E)$ possesses a pullback exponential attractor $\{M_\sigma^\theta(t)\}_{t \in \mathbb{R}}$. Moreover, the map $\sigma \to M_\sigma^\theta(t)$ is stable in the following sense: for any $\sigma_0 \in \Sigma$, if $\sigma \in \Sigma$ satisfies

$$\Gamma(\sigma, \sigma_0) \equiv \sup_{t \in [0, T]} \sup_{\tau \in \mathbb{R}} \sup_{x \in M} \|U_\sigma(t + \tau, \tau)x - U_{\sigma_0}(t + \tau, \tau)x\|_E < 1,$$

then

$$\sup_{t \in \mathbb{R}} \operatorname{dist}_E(M_\sigma^\theta(t), M_{\sigma_0}^\theta(t)) \leq C[\Gamma(\sigma, \sigma_0)]^\lambda,$$

where $C > 0$, $0 < \lambda < 1$ are constants independent of $\sigma$.

Let

$$L_2^2(\mathbb{R}; V_{-1}) = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}; V_{-1}) \mid \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s)\|_{V_{-1}}^2 ds < +\infty \right\},$$

equipped with the norm

$$\|f\|_{L_2^2(\mathbb{R}; V_{-1})} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s)\|_{V_{-1}}^2 ds.$$
where $\tilde{K}_1 = C(R, t - \tau, \|f_0\|_{L^2_2(\mathbb{R}; V_{-1})})$, which is monotone on the second variable $t - \tau, z = u - v, u$ and $v$ are two weak solutions of problem (7)-(8) corresponding to initial data $(v_0^u, u_0^t)$, $(v_0^v, v_1^t) \in X_1$ and symbols $f_1, f_2 \in \Sigma$.

In particular, when $1 \leq p < p^*$, the following Lipschitz stability and quasi-stability hold in weaker space $X_0$,

$$
\|(z(t), z(t))\|_{X_0}^2 + \int_0^t\|(z(s), z_t(s))\|_{X_1}^2 \, ds
\leq \tilde{K}_1\left(\|(z(\tau), z_t(\tau))\|_{X_0}^2 + \|f_1 - f_2\|_{L^2_2(\tau, t; V_{-1})}^2\right),
$$

(38)

$$
\|(z(t), z_t(t))\|_{X_0}^2 \leq C\varepsilon^{-\kappa(t-\tau)}\|(z(\tau), z_t(\tau))\|_{X_0}^2 + \tilde{K}_1 \int_\tau^t e^{-\kappa(t-r)} \left(\|z(r)\|_{V_{-1}}^2 + \|z_t(r)\|_{V_{-1}}^2 + \|f_1 - f_2\|_{V_{-1}}^2\right) \, dr
$$

(39)

for all $t \geq \tau$, where $\tilde{K} = C(R, \|f_0\|_{L^2_2(\mathbb{R}; V_{-1})})$, and $\kappa$ is a small positive constant.

Proof. Obviously, the function $z$ solves

$$
A^{-\frac{1}{2}} z_{tt} + A^{-\frac{1}{2}} v + z_t + g(u) - g(v) + A^{-\frac{1}{2}} Bu - A^{-\frac{1}{2}} Bv = A^{-\frac{1}{2}} f_1 - A^{-\frac{1}{2}} f_2, \tag{40}
$$

$$
z(\tau) = u_0^\tau - v_0^\tau \equiv z_0^\tau, \quad z_t(\tau) = u_1^\tau - v_1^\tau \equiv z_1^\tau.
$$

Similar to the proof on the stability of weak solutions in $X_1$ in Theorem 2.3, one easily obtains estimate (37) by using the multiplier $z_t$ in Eq. (40). Thus we only prove (38)-(39) here.

Using the multiplier $A^{-\frac{1}{2}} z_t + \varepsilon z$ in Eq. (40) and making use of the Sobolev embedding $V_{1-\delta} \hookrightarrow L^{p+1}$ for $\delta : 0 < \delta \ll 1$ and the interpolation theorem, we have

$$
\frac{d}{dt} H_3(z, z_t) + \varepsilon \|A^{\frac{1}{2}} z\|^2 + (1 - \varepsilon) \|A^{-\frac{1}{2}} z_t\|^2
\leq (\|Bu - Bv\|_{V_{-1}} + \|f_1 - f_2\|_{V_{-1}}) \|A^{-\frac{1}{2}} z_t + \varepsilon z\|_{V_{-1}}
\leq (\|Bu - Bv\|_{V_{-1}} + \|f_1 - f_2\|_{V_{-1}}) + (1 + \|\eta\|_{p+1} + \|\eta\|_{p+1}^2) (\|z\|_{V_{-1}}^2 + \|z_t\|_{V_{-1}}^2)
$$

$$
\leq \tilde{K} \left[\left(\int_0^1 \frac{\nabla z \cdot \eta}{\sqrt{1 + |\nabla z|^2}} \, d\eta + \|f_1 - f_2\|_{V_{-1}}\right) (\|z\|_{V_{-1}} + \|z_t\|_{V_{-1}})
+ \|z\|_{V_{-1}}^2 + \|z_t\|_{V_{-1}}^2\right]
\leq \frac{\varepsilon}{4} (\|z\|_{V_{-1}}^2 + \|z_t\|_{V_{-1}}^2) + \tilde{K}(\|z\|_{V_{-1}}^2 + \|z_t\|_{V_{-1}}^2 + \|f_1 - f_2\|_{V_{-1}}^2),
$$

where $z_\eta$ is as shown in (26) and

$$
H_3(z, z_t) = \frac{1}{2} \left[\|A^{-\frac{1}{2}} z_t\|^2 + \|z\|^2 + \varepsilon \left(\|z\|^2 + 2(z, A^{-\frac{1}{2}} z_t)\right)\right] \sim \|z\|^2 + \|z_t\|_{V_{-1}}^2
$$

for $\varepsilon > 0$ suitably small. Hence,

$$
\frac{d}{dt} H_3(z, z_t) + \kappa H_3(z, z_t) + \kappa (\|z\|_{V_{-1}}^2 + \|z_t\|_{V_{-1}}^2) \leq \tilde{K}(\|z\|_{V_{-1}}^2 + \|z_t\|_{V_{-1}}^2 + \|f_1 - f_2\|_{V_{-1}}^2), \tag{41}
$$
Applying the Gronwall inequality to (41) gives (39) and (38) (by using the fact that \( f \) is a uniformly (w.r.t. \( f \)) continuous, and the following translation identity holds: \( U_{T(h)}(t, \tau) = U_f(t + h, \tau + h), \ h \in \mathbb{R}, \ t \geq \tau, \ t, \tau \in \mathbb{R}. \) \( \square \)

Theorem 4.4 shows that the family of processes \( \{U_f(t, \tau)\}, f \in \Sigma, \) is \( (X_1 \times \Sigma, X_1) \) continuous, and the following translation identity holds:

\[
U_{T(h)}(t, \tau) = U_f(t + h, \tau + h), \ h \in \mathbb{R}, \ t \geq \tau, \ t, \tau \in \mathbb{R}. \tag{42}
\]

**Theorem 4.5.** Let Assumptions 2.1 be valid, with \( f \in \Sigma \) and \( 1 \leq p < p^* \). Then

(i) (Existence) for each \( f \in \Sigma \), the non-autonomous dynamical system \( \{U_f(t, \tau), X_1\} \) has a pullback exponential attractor \( \mathcal{M}^1 = \{M^1(t)\}_{t \in \mathbb{R}} \), and the sections \( M^1(t) \) are uniformly (w.r.t. \( f \in \Sigma \) and \( t \in \mathbb{R} \)) bounded in \( X_2 \);

(ii) (Stability) there exists a \( \delta > 0 \) such that for any \( f \in \Sigma \), with \( \|f - f_0\|_{L^2(\mathbb{R}; V_{-1})} \leq \delta \),

\[
\sup_{t \in \mathbb{R}} \text{dist}^*_{X_1}\{M^1(t), M^{f_0}(t)\} \leq C\|f - f_0\|_{L^2(\mathbb{R}; V_{-1})}^\nu, \tag{43}
\]

where \( C > 0, \ 0 < \nu < 1 \) are some constants independent of \( f \).

**Proof.** By estimate (10),

\[
\|(u(t), u_t(t))\|_{X_1}^2 \leq Ce^{-\gamma(t-\tau)}\left(\|u_0\|_{V_1}^2 + \|u_\tau\|_{V_{-1}}^2 + \|u_0\|_{V_1}^{p+1} + C\left(1 + \frac{\gamma}{\gamma - 1}\|f_0\|_{L^2(\mathbb{R}; V_{-1})}\right)^\frac{1}{2}\right), \tag{44}
\]

We see from (44) that the ball \( B_0 = \{\xi \in X_1 | \|\xi\|_{X_1} \leq R_0\} \), with

\[
R_0 = \left[2C\left(1 + \frac{\gamma}{\gamma - 1}\|f_0\|_{L^2(\mathbb{R}; V_{-1})}\right)^\frac{1}{2}\right], \tag{45}
\]

is a uniformly (w.r.t. \( f \in \Sigma \) and \( \tau \in \mathbb{R} \)) pullback absorbing ball of the process \( \{U_f(t, \tau)\} \). So for every bounded set \( B \subset X_1 \), there exists a \( T_1 = T(B) > 0 \) such that

\[
\bigcup_{f \in \Sigma} \bigcup_{t \geq T_1} U_f(t + \tau, \tau)B \subset B_0, \ t \geq T_1.
\]

In particular, there exists a \( T_0 > 0 \) such that

\[
\bigcup_{f \in \Sigma} U_f(t_0)B_0 \subset B_0, \ t \geq T_0.
\]

Let

\[
B = \left[\bigcup_{f \in \Sigma} \bigcup_{t \geq T_0 + 1} U_f(t_0)B_0\right]_{X_0} (\subset B_0).
\]

For every \( \tau \in \mathbb{R}, f \in \Sigma \), translation identity (42) implies that there exist \( f_1, f_2 \in \Sigma \) such that

\[
U_f(t + \tau, \tau) = U_{f_1}(t, \tau), \ U_{f_1}(t, T_1) = U_{f_2}(t - T_1, 0), \ t \geq T_1.
\]

So for any bounded set \( B \subset X_1 \),

\[
U_f(t + \tau, \tau)B = U_{f_1}(t, 0)B = U_{f_1}(t, T_1)U_{f_1}(T_1, 0)B \subset U_{f_2}(t - T_1, 0)B_0 \subset B, \ t \geq T_1 + T_0 + 1 \equiv t_0, \tag{46}
\]

that is, \( B \) is a uniformly (w.r.t. \( f \in \Sigma \) and \( \tau \in \mathbb{R} \)) absorbing set of the dynamical system \( \{U_f(t, \tau), X_1\} \). Due to \( B \subset [B_0]_{X_0} = B_0 \), we have

\[
\bigcup_{f \in \Sigma} \bigcup_{\tau \in \mathbb{R}} U_f(t + \tau, \tau)B \subset \bigcup_{f \in \Sigma} \bigcup_{\tau \in \mathbb{R}} U_f(t + \tau, \tau)B_0 = \bigcup_{f \in \Sigma} U_f(t, 0)B_0 \subset B, \ t \geq T_0 + 1.
\]
For every \( f \in \Sigma \),
\[
U_f(t,0)B_0 = U_f(t,t-1)U_f(t-1,0)B_0 \subset U_f(t,t-1)B_0, \quad t \geq T_0 + 1.
\]
Therefore,
\[
\bigcup_{f \in \Sigma} \bigcup_{t \geq T_0 + 1} U_f(t,0)B_0 \subset \bigcup_{f \in \Sigma} \bigcup_{t \in \mathbb{R}} U_f(t,t-1)B_0.
\]
By estimate (13) (taking \( \tau = t-1, a = t-1/2 \) there),
\[
\bigcup_{f \in \Sigma} \bigcup_{t \in \mathbb{R}} U_f(t,t-1)B_0 \text{ is bounded in } X_2.
\]
By the lower semi-continuity of weak limit we obtain that \( \mathcal{B} \) is bounded in \( X_2 \).
Obviously, \( \mathcal{B} \) is a metric space equipped with the distance \( d(x,y) = \|x - y\|_{X_0} \), we show that the dynamical system \((U_f(t,\tau),\mathcal{B},X_0)\) has a robust pullback exponential attractor.

By estimate (44),
\[
\sup_{f \in \Sigma} \sup_{\varphi \in \mathcal{B}} \|U_f(t+\tau,\varphi)\|^2_{X_1} \leq C(R_0), \quad t \geq 0, \quad \tau \in \mathbb{R}. \tag{47}
\]
For every \( \varphi, \psi \in \mathcal{B}, f_1, f_2 \in \Sigma \) and \( \tau \in \mathbb{R} \), let
\[
(z(t+\tau), z_t(t+\tau)) = U_{f_1}(t+\tau,\tau)\varphi - U_{f_2}(t+\tau,\tau)\psi = (u(t+\tau), u_t(t+\tau)) - (v(t+\tau), v_t(t+\tau)), \quad t \geq 0.
\]
Then \( z = u - v \) solves
\[
A^{-\frac{1}{2}}z_{tt} + A^\frac{1}{2}z + z_t + g(u) - g(v) + A^{-\frac{1}{2}}(Bu - Bv) = A^{-\frac{1}{2}}f_1 - A^{-\frac{1}{2}}f_2, \tag{48}
\]
\[
(z(\tau), z_t(\tau)) = \varphi - \psi.
\]
It follows from (38)-(39) that
\[
\|(z, z_t)(t+\tau)\|^2_{X_0} + \int^t_{t+\tau} \|(z, z_t)(s)\|^2_{X_1} ds
\]
\[
\leq C(T) \left( \|\varphi - \psi\|^2_{X_0} + \|f_1 - f_2\|^2_{L^2(t,\tau+t;V_{-1})} \right), \quad t \in [0,T], \tag{49}
\]
\[
\|(z, z_t)(t+\tau)\|^2_{X_0}
\]
\[
\leq Ce^{-\alpha t} \|\varphi - \psi\|^2_{X_0} + \tilde{K} \left( \|(z, z_t)\|^2_{L^2(t,\tau+t;V_{-1} \times V_{-1})} + \|f_1 - f_2\|^2_{L^2(t,\tau+t;V_{-1})} \right). \tag{50}
\]
Taking \( r > \max\{N/2 + 2, 4\} \) and making use of the Sobolev embedding
\[
L^2 \hookrightarrow V_{-r+4}, \quad V_{-2} \hookrightarrow V_{-r+2}, \quad V_{-r-2} \hookrightarrow L^\infty, \quad L^1 \hookrightarrow V_{-r+2},
\]
we infer from Eq. (48) and estimate (49) that
\[
\int^t_{t+\tau} \|z_{tt}(s)\|^2_{V_{-r}} ds
\]
\[
\leq C \int^t_{t+\tau} \left( \|z\|^2_{V_{-r+4}} + \|z_t\|^2_{V_{-r+2}} + \|g(u) - g(v)\|^2_{V_{-r+2}}
\]
\[
+ \|Bu - Bv\|^2_{V_{-r}} + \|f_1 - f_2\|^2_{V_{-r}} \right) ds
\]
\[
\leq C \int^t_{t+\tau} \left( \|z\|^2 + \|z_t\|^2_{V_{-r}} + \|g(u) - g(v)\|^2_{L^2} + \|Bu - Bv\|^2_{V_{-1}} + \|f_1 - f_2\|^2_{V_{-1}} \right) ds
\]
\[
\leq C(T)(\|\varphi - \psi\|^2_{X_0} + \|f_1 - f_2\|^2_{L^2(t,\tau+t;V_{-1})}), \quad t \in [0,T],
\]
where we have used the estimates (see (26))
\[
\|Bu - Bu\|_{L^2}^2 \leq 2\|\zeta_i\|_{l^2},
\]
\[
\|g(u) - g(v)\|_{L^2}^2 \leq C\left(\int_{\Omega} (1 + |u|^p + |v|^p)|z|dx \right)^2
\leq C(1 + \|u\|_{p+1}^{2p} + \|v\|_{p+1}^{2p})\|z\|_{p+1}^2 \leq C\|z\|_{l^2}^2.
\]

Estimate (49) means that\n
\[\sup_{t \in [0,T]} \sup_{f \in \Sigma} \|U_t(t + \tau, \tau)\varphi - U_t(t, \tau)\psi\|_{X_0} \leq L_T \|\varphi - \psi\|_{X_0}, \quad t \in [0, T],\]
\[
\Gamma(f_1, f_2) = \sup_{t \in [0,T]} \sup_{\tau \in \mathbb{R}} \|U_{f_1}(t + \tau, \tau)\varphi - U_{f_2}(t + \tau, \tau)\varphi\|_{X_0}
\leq L_T \sqrt{T + 1}\|f_1 - f_2\|_{L^2(\mathbb{R}, V_{-r})},\]
where \(L_T^2 = C(T)\). Let the set\n\[
Z = \{(u, u_t) \in L^2(0, T; X_0) | u_{tt} \in L^2(0, T; V_{-r})\},
\]
which is equipped with the norm\n\[
\|(u, u_t)\|_Z^2 = \int_0^T (\|u(t)\|^2 + \|u_t(t)\|^2_{V_{-2}} + \|u_{tt}(t)\|^2_{V_{-r}})dt.
\]

Obviously, \(Z\) is a Banach space. Define the mapping, for each \(n \in \mathbb{N}, \ f \in \Sigma,\)
\[
K_n^f : \mathcal{B} \rightarrow Z, \ K_n^f \varphi = U_f(t, nT, nT)\varphi = (u(\cdot + nT), u_t(\cdot + nT)),
\]
where \(u(\cdot + nT)\) means \(u(t + nT), t \in [0, T]\). By estimates (49) and (51),
\[
\|K_n^f \varphi - K_n^f \psi\|_Z^2 = \int_{nT}^{(n+1)T} (\|z(s)\|^2 + \|z_t(s)\|^2_{V_{-2}} + \|z_{tt}(s)\|^2_{V_{-r}})ds \leq L_T^2 \|\varphi - \psi\|_{X_0}^2
\]
for all \(\varphi, \psi \in \mathcal{B}, \ f \in \Sigma, \) where \(L_T^2 = (T + 1)C(T)\). Therefore, by (50) (with \(f_1 = f = f_2\) there),
\[
\|U_f((n+1)T, nT)\varphi - U_f((n+1)T, nT)\psi\|_{X_0} \leq \eta_T \|\varphi - \psi\|_{X_0} + n_Z(K_n^f \varphi - K_n^f \psi),
\]
where \(T : \eta_T = C^{-\kappa_T} < 1, \) and
\[
n_Z^2(\varphi_u) = L_T^2 \int_0^T \|\varphi_u(t)\|_{V_{-1} \times V_{-s}}^2 dt, \quad \varphi_u = (u, u_t) \in Z.
\]

Obviously, \(n_Z(\cdot)\) is a compact seminorm on \(Z\) for \(Z \hookrightarrow L^2(0, T; V_{-1} \times V_{-3})\) (cf. [25]). Then, by Lemma 4.3, the dynamical system \((U_f(t, \tau), \mathcal{B}, X_0)\) has a pullback exponential attractor \(\mathcal{M}^f = \{M_\theta^f(t)\}_{t \in \mathbb{R}}\) for each \(f \in \Sigma\) and \(\theta \in (\eta_T, 1)\).

Moreover, for any \(f \in \Sigma\) satisfying \(\|f - f_0\|_{L^2(\mathbb{R}, V_{-1})} < 1/(L_T \sqrt{T + 1})\), we infer form (52) that \(\Gamma(f, f_0) < 1\), so by Lemma 4.3, there exist constants \(C > 0\) and \(\nu : 0 < \nu < 1\) such that
\[
\sup_{t \in \mathbb{R}} \text{dist}_{X_0}^\text{symm} \{M_\theta^f(t), M_\theta^{f_0}(t)\} \leq C\|f - f_0\|_{L^2(\mathbb{R}, V_{-1})}^{\nu}.
\]

Now, we recover the topology of original phase space \(X_1\). Define the mapping
\[
I : \mathcal{B} \rightarrow X_1, \quad I \varphi = \varphi, \quad \forall \varphi \in \mathcal{B}.
\]
For every \(\varphi \in \mathcal{B}\), taking account of the boundedness of \(\mathcal{B}\) in \(X_2\) and using the interpolation theorem, we have
\[
\|\varphi\|_{X_1} \leq \|\varphi\|_{X_2}^{\frac{1}{2}} \|\varphi\|_{X_0}^{\frac{1}{2}} \leq C \|\varphi\|_{X_0}^{\frac{1}{2}}, \quad (53)
\]
which means that the mapping \( I \) is 1/2-Hölder continuous. So \( M^I_0(t) = IM^I_{0}(t) \) is bounded in \( X_2 \) and compact in \( X_1 \) for \( X_2 \hookrightarrow \hookrightarrow X_1 \), and

\[
\sup_{t \in \mathbb{R}} \dim_f(M^I_0(t), X_1) \leq 2 \sup_{t \in \mathbb{R}} \dim_f(M^I_0(t), X_0) < \infty.
\]

Moreover, by formulas (53), (46) and Def. 4.2, we have that for any bounded set \( B \subset X_1 \),

\[
dist_{X_1} \{ U_I(t, t-s)B, M^I_0(t) \} \\
\leq C \left( \text{dist}_{X_0} \{ U_I(t, t-s/2)U_I(t-s+t-s/2, t-s)B, M^I_0(t) \} \right)^{\frac{1}{2}} \\
\leq C \left( \text{dist}_{X_0} \{ U_I(t, t-s/2)B, M^I_0(t) \} \right)^{\frac{1}{2}} \\
\leq C(B)e^{-\frac{\beta s}{2}}, \quad s \geq 2t_0 + 2T(B),
\]

where \( \beta > 0 \) is a constant and \( t_0 = t_0(B) \) is as shown in (46). Thus \( \mathcal{M}^I = \{ M^I_0(t) \} \in \mathbb{R} \) is the desired pullback exponential attractor, and

\[
\sup_{t \in \mathbb{R}} \text{dist}^{symm}_{X_1} \{ M^I_0(t), M^I_{0}(t) \} \leq C \sup_{t \in \mathbb{R}} \left[ \text{dist}^{symm}_{X_0} \{ M^I_0(t), M^I_{0}(t) \} \right]^{\frac{1}{2}} \\
\leq C||f - f_0||^2_{L^2(B; V)}.
\]

Remark 4.6. (i) The fact that “\( B_0 \) is a uniformly (w.r.t. \( f \in \Sigma \) and \( \tau \in \mathbb{R} \)) pullback absorbing ball of the process \( \{ U_I(t, \tau) \} \) (see (45))” implies that the family of pullback absorbing balls \( \mathcal{B} = \{ B(t) \} \in \mathbb{R} \) as shown in formula (33) degenerates to be a ball \( B_0 \) (replacing radius \( \mathcal{R}(\bar{t}) \) there (see (33)) by \( R_0 \)), and the set \( D(t) \) as shown in (35) becomes \( D(t) = \bigcup_{\tau \geq T} U_I(t, t-\tau)B_0 \subset B_0 \) (see (35)), and the family \( \mathcal{D} = \{ D(t) \} \in \mathbb{R} \) is still a pullback absorbing family of the process \( \{ U_I(t, \tau) \} \) in \( X_1 \), which is also pullback \( \mathcal{D} \)-absorbing, and the set \( D(t) \) is bounded in \( X_2 \) for each \( t \in \mathbb{R} \).

(ii) Under the assumptions of Theorem 4.5, by Remark 3.2: (i), the process \( \{ U_I(t, \tau) \} \) has a pullback attractor \( \mathcal{A} = \{ A^I(t) \} \in \mathbb{R} \). By Remark 3.2: (ii),

\[
dist_{X_1} \{ A^I(t), M^I(t) \} = \lim_{s \to +\infty} \dist_{X_1} \{ U_I(t, t-s)A^I(t-s), M^I(t) \} = 0
\]

which means that \( A^I(t) \subset M^I(t), t \in \mathbb{R} \), so

\[
\sup_{t \in \mathbb{R}} \dim_f(A^I(t), X_1) \leq \sup_{t \in \mathbb{R}} \dim_f(M^I(t), X_1) < +\infty.
\]

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