Numerical study of temperature dependent thermal conductivity and homogeneous-heterogeneous reactions on Williamson fluid flow

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Abstract
Understanding the flow nature of non-Newtonian fluids need more studies in recent days in terms of their various aspects. Further, heat transfer analysis is an inspiring topic in non-Newtonian flows because of its dominant role in technology. With these incentives, a theoretical investigation is performed to study the time-dependent flow of a non-Newtonian fluid model with focus on heat and mass transfer. In this study, we proposed the mathematical model for the flow of Williamson fluid by incorporating the impact of infinite shear rate of viscosity. With the assistance of rheological expression of Williamson fluid, we construct the governing field equations. For physical relevance we analyzed the impacts of homogeneous-heterogeneous reactions on the flow field. The system of governing partial differential equations is transformed into a set of ordinary differential equations by adopting the non-dimensional variables. This non-dimensional ruling problem together with physical boundary conditions are tackled numerically by utilizing Runge–Kutta Fehlberg scheme via MATLAB software. The physical behavior of friction factor, reduced Nusselt number, dimensionless velocity, temperature and concentration distributions for distinct estimations of leading parameters is examined with the assistance of graphs.

Nomenclature

| Symbol | Description |
|--------|-------------|
| \( u \) | Velocity components |
| \( v \) | |
| \( x, y \) | Cartesian coordinates |
| \( Ec \) | Eckert number |
| \( T \) | Fluid temperature |
| \( T_w \) | Surface temperature |
| \( T_\infty \) | Ambient temperature |
| \( s \) | Mass flux parameter |
| \( C \) | Fluid concentration |
| \( C_w \) | Surface volume fraction |
| \( C_\infty \) | Ambient nanoparticle volume fraction |
| \( U_w \) | Stretching velocity |
| \( k \) | Thermal conductivity |
| \( B_0 \) | Magnetic parameter |
| \( f \) | Dimensionless stream function |
| \( U_\infty \) | Free stream velocity |
| \( \alpha_m \) | Thermal diffusivity |
| \( c_p \) | Specific heat |
We  Local Weissenberg number
A  Unsteadiness parameter
$\beta^*$  Ratio of viscosities
$v_m$  Mass flux velocity
$a^*, b^*, c$  Constants
Pr  Prandtl number
$C_f$  Skin friction coefficient
Nu  Local Nusselt number
Re  Local Reynolds number
$\Gamma$  Relaxation time
$\rho$  Fluid density
$\mu$  Generalized Newtonian viscosity
$\mu_0$  Zero shear viscosity
$\mu_\infty$  Infinite shear viscosity
$\dot{\gamma}$  Magnitude of deformation rate
$\nu$  Kinematic viscosity
$\psi$  Stream function
$\tau_w$  Surface shear stress
$\theta$  Dimensionless temperature
$\eta$  Dimensionless similarity variable
$(\rho c_p)_{\text{p}}$  Effective heat capacity of a nanoparticle
$(\rho c_p)_{\text{f}}$  Heat capacity of the base fluid
$\tau$  Parameter defined by the ratio $\frac{(\rho c_p)_{\text{p}}}{(\rho c_p)_{\text{f}}}$
$\phi$  Dimensionless concentration
$k_r$, $k_s$  Rate constants
$A$, $B$  Chemical species

Introduction

The Williamson fluid is an essential class of pseudoplastic fluid models. In the previous couple of decades, a substantial number of research achievements concerning Williamson fluid model have been published. Among these literary works, numerous researchers have put their effort to describe the rheological practices of Williamson fluid for pondering the effects of Weissenberg number on flow and pumping characteristics. The Williamson fluid flow towards a stretchable surfaces allured distinct researchers due to its vital applications in many aspects such as polymer extrusion, plastic films, metal spinning, and metallurgical processes. In 1929, Williamson [1] proposed a model to assign the flow of pseudoplastic liquids and experimentally substantiated the outcomes. Later on, Nadeem and Akram [2] obtained the analytical solution peristaltic flow of a Williamson fluid in an asymmetric channel and reported that pressure gradient is a decreasing function of Weissenberg number. In another paper, Nadeem et al [3] solved analytically the boundary layer flow of Williamson fluid model towards a stretched surface. Akbar et al [4] studied numerically the peristaltic flow of a Williamson fluid in an asymmetric channel. Their results show that the nanoparticle concentration reduces with the increment in Brownian motion parameter. The peristaltic flow of electrically conducting Williamson fluid through a porous medium by considering viscous dissipation and Joule heating effects are analyzed by Eldabe et al [5]. They revealed that concentration profile reduces with higher magnetic field parameter. Krishnamurthy et al [6] demonstrated numerical solution of Williamson nanofluid flow and melting heat transfer past a horizontal plate in the presence of chemical reaction. They illustrated that chemical reaction parameter declines the solutal boundary layer thickness. Additionally, Bhatti and Rashidi [7] addressed the combined effects of thermal radiation and thermo-diffusion on Williamson nanofluid. Their investigations show that concentration profile is reduced by higher Lewis number. Hayat et al [8] deliberated the melting effects on MHD flow of Williamson fluid induced by a nonlinear variable thickness surface. They indicated that heat transfer rate grows with higher
values of thermophoresis parameter. Recently, Kumaran and Sandeep [9] have explored the influences of parabolic flow of Williamson nanofluid with cross diffusion effect. Hayat et al [10] determined the numerical investigations of Williamson fluid with an endoscope by considering modified Darcy law. They found that higher values of Hartmann number reduces the axial velocity. Very recently, Hamid et al [11] examined the unsteady flow of Williamson fluid induced by a wedge shape geometry. They proposed that the shear stress at the surface is higher for large wedge angle parameter. Hashim et al [12] analyzed the impacts of Williamson fluid flow driven by a wedge geometry and illustrate that the rate of heat transfer rises with larger unsteady parameter. In another study, Hashim et al [13] considered the mixed convection flow of Williamson nanofluid past a radially stretched surface with magnetic field and variable thermal conductivity. They accomplished that the temperature of the fluid and the thickness of thermal boundary layer increases by higher values of thermal conductivity parameter. Bahiraei et al [14] illustrated the mechanism of fluid flow and energy efficiency of a non-Newtonian liquid in the presence of nanoparticles. Makinde et al [15] analytically studied the behavior of electrically conducting nanofluid induced by a nonlinear stretched surface with the presence of heat generation/absorption and chemical reacting effects. Mabood et al [16] numerically investigated the phenomenon of heat transport of water-based nanofluid in a rotating system with thermophoresis and Brownian motion parameter. Recently, the theory of Newtonian/non-Newtonian MHD fluid flow has been reviewed by many researchers [17–19].

The homogeneous and heterogeneous interactions are the natural processes of chemical reactions. The heterogeneous reaction parameter has the ability to speed the concentration of the catalyst close to the surface and to decline the concentration of bulk liquid. Some investigators at present described the stretchable flows by considering homogeneous and heterogeneous effects. Merkin [20] initially examined the laminar flow of a viscous fluid with homogeneous-heterogeneous reactions along a flat plate. He reported homogeneous reaction for cubic autocatalysis and heterogeneous reaction on the catalyst surface. It can be seen that near the leading edge of the flat plate reaction is dominant. Homogeneous-heterogeneous reactions with equal diffusivities have been proposed by Chaudhary and Merkin [21]. Ziabakhsh et al [22] discussed the feature of flow and diffusion of chemically reactive species along a nonlinear stretching surface immersed in a porous medium. Khan and Pop [23] studied the outcomes of homogeneous-heterogeneous reactions in a viscoelastic fluid past a stretched surface. Their study reveals that the concentration at the surface has been reduced by the large viscoelastic parameter. Kameswaran et al [24] analyzed the impact of homogeneous-heterogeneous reactions in a nanofluid embedded in a porous stretching surface. They found that the strength of heterogeneous reaction diminishes the concentration profile. In the recent past, Hayat et al [25] reported the characteristics of carbon nano-tubes in the stagnation point flow induced by a nonlinear stretched surface with variable thickness. They concluded that the drag surface force decreases with the increment of ratio parameter. Chen et al [26] obtained the numerical solution of homogeneous-heterogeneous interaction on the homogeneous ignition feature in hydrogen-fueled catalytic micro-reactors in the submillimeter to millimeter range. Recently, Hashim and Khan [27] discussed numerically the flow and heat transfer analysis for Carreau fluid with homogeneous-heterogeneous reactions. Khan et al [28] addressed the flow over a non-linear stretched surface with Joule heating and homogeneous-heterogeneous reactions. Moreover, Khan et al [29] examined the characteristics of homogeneous-heterogeneous processes in the 3D flow of Sisko fluid over a bidirectional stretching surface. Few investigations about homogeneous-heterogeneous reactions are reported in the [30–34].

Based on literature review, the intention of this paper is to formulate the basic conservation equations for the boundary layer flow of non-Newtonian Williamson fluid model by employing the Boussinesq approximations. These types of flows are usually assumed to be steady, nevertheless, in various engineering and industrial applications, unsteadiness turn into an integral part of the physical problem where flow becomes time-dependent. The second objective is the numerical investigation of heat transfer mechanism of Williamson fluid flow past a stretching surface with variable thermal conductivity and homogeneous-heterogeneous effects in the presence of infinite shear rate viscosity. The numerical solutions for the dimensionless stream function, dimensionless temperature and concentration is deduced with the assistance of Runge–Kutta Fehlberg method alongside the shooting technique. The impacts of several included physical parameters on dimensionless profiles of the velocity, temperature and concentration, the local skin friction and local Nusselt number are illustrated with the help of graphs.

Problem development

In current review, we assume a time-dependent laminar flow of Williamson fluid past a stretching surface. The stretching surface has the velocity $U_0(x, t)$, which changes with time and distance along the surface $x$ (see figure 1). It is also mentioned that the temperature of the sheet is $T_w(x, t)$ which is higher than the ambient temperature $T_\infty$ ($T_w > T_\infty$). The boundary layer flow with homogeneous and heterogeneous processes
comprising two chemical species $A$ and $B$ is assumed in this study and reported by Merkin [20] and Chaudhary and Merkin [21] and is given as follows:

$$A + 2B \rightarrow 3B, \ \text{rate} = k_{2}a^{b}b^{2},$$

$$A \rightarrow B, \ \text{rate} = k_{1}a^{b}.$$  \hspace{1cm} (1)

Under the above assumptions the regulating flow equations can be expressed in the following manner [19, 35];

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$  \hspace{1cm} (3)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^{2}u}{\partial y^{2}} \left[ \beta^{a} + (1 - \beta^{a}) \left( 1 - \Gamma \frac{\partial u}{\partial y} \right)^{-1} \right]$$

$$+ \nu \Gamma \frac{\partial u}{\partial y} \left[ (1 - \beta^{a}) \left( 1 - \Gamma \frac{\partial u}{\partial y} \right)^{2} \right],$$  \hspace{1cm} (4)

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{(\rho c)_{T}} \frac{\partial}{\partial y} \left( K(T) \frac{\partial T}{\partial y} \right),$$  \hspace{1cm} (5)

$$\frac{\partial a^{b}}{\partial t} + u\frac{\partial a^{b}}{\partial x} + v\frac{\partial a^{b}}{\partial y} = \frac{D_{A}}{\partial y^{2}} a^{b} - k_{a}a^{b}b^{2},$$  \hspace{1cm} (6)

$$\frac{\partial b^{b}}{\partial t} + u\frac{\partial b^{b}}{\partial x} + v\frac{\partial b^{b}}{\partial y} = \frac{D_{B}}{\partial y^{2}} b^{b} + k_{b}a^{b}b^{2},$$  \hspace{1cm} (7)

with the boundary conditions

$$u = U_{w}(x, t), \ v = 0, \ T = T_{w}(x, t), \frac{\partial a^{b}}{\partial y} = k_{a}a^{b}, \frac{\partial b^{b}}{\partial y} = -k_{b}a^{b} \ \text{at} \ y = 0$$

$$u \rightarrow 0, \ v \rightarrow 0, \ T \rightarrow T_{\infty}, \ a^{b} \rightarrow a_{0}, \ b^{b} \rightarrow 0 \ \text{asy} \rightarrow \infty.$$  \hspace{1cm} (8)

In the above expressions $\nu$ represents the kinematic viscosity, $K(T)$ is the variable thermal conductivity and it is of the form, $K(T) = k_{x} \left[ 1 + \varepsilon \frac{T - T_{0}}{T_{\infty} - T_{0}} \right]$ with $\varepsilon$ being a small parameter and $C_{p}$ the specific heat, $D_{A}$ and $D_{B}$ are the respective diffusion species coefficients of $A$ and $B$, $a_{0}$ is a positive constant.

Further, we assumed that the wall stretching velocity $U_{w}(x, t)$ and the temperature $T_{w}(x, t)$ are of the following form:

$$U_{w}(x, t) = \frac{ax}{1 - ct}, \ T_{w}(x, t) = T_{\infty} + \frac{T_{0}U_{w}x}{\nu(1 - ct)^{2}},$$  \hspace{1cm} (10)

The following suitable transformations are used in the present case:

$$\eta = \sqrt{\nu U_{w}x}, \ \psi(x, y, t) = \sqrt{\nu U_{w}x} f(\eta), \ \theta(\eta) = \frac{T - T_{\infty}}{T_{w} - T_{\infty}},$$  \hspace{1cm} (11)

where $\psi$ represents stream function and is defined as $u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x}$. The concentrations of the chemical species $A$ and $B$ are expressed as
\[ a^* = a_0g(\eta) \text{ and } b^* = a_0h(\eta), \]

where \( a_0 \) is a constant, \( g(\eta) \) and \( h(\eta) \) is the dimensionless concentration.

By introducing the above transformations the governing flow equations of this problem are reduced as follows:

\[
\begin{align*}
(1 + \varepsilon \theta) \theta'' + \varepsilon (\theta')^2 + \text{Pr} (f f'' - 2f' \theta) - \text{Pr} \frac{A}{2} (\theta' + 3 \theta) &= 0, \\
\frac{1}{\text{Sc}} g'' + fg' - Kgh^2 - \frac{J}{2} Ag' &= 0, \\
\frac{1}{\text{Sc}} h'' + fh' + Kgh^2 - \frac{J}{2} Ah' &= 0,
\end{align*}
\]

\begin{align*}
 f(0) &= 0, & f'(0) &= 1, & \theta(0) &= 1, & f'(\infty) &= 0, & \theta(\infty) &= 0, \\
g'(0) &= K_c g(0), & \delta h'(0) &= -K_c g(0), & g(\infty) &= 1, & h(\infty) &= 0,
\end{align*}

where \( \text{We} = \sqrt{\frac{\rho C_p \nu^{1/3}}{k}} \) depicts the local Weissenberg number, \( \text{Pr} = \frac{1\mu C_C \nu}{k} \) is a parameter indicates the Prandtl number, \( A = \frac{c_1}{a} \) is the unsteadiness parameter, \( \text{Sc} = \frac{\nu}{k} \) denotes the Schmidt number, \( K = \frac{k_i(a - c)}{a} \) is the strength of the homogeneous reaction, \( K_c = \frac{k_i}{a} \text{Pr} \) is the strength of the heterogeneous reaction, \( \delta = \frac{2b_{21}}{h} \) corresponds to the ratio of diffusion-coefficient and \( \beta^* = \frac{\nu_k}{\nu_0} \) denotes the viscosity ratio parameter \( (0 < \beta^* < 1) \).

These are respectively defined as in most applications, we expect the diffusion-coefficients of chemically species \( (A) \) and \( (B) \) are of comparable sizes, which leads us to further assumption that the diffusion-coefficients \( (D_0) \) and \( (D_0) \) are equal, i.e., to take \( \delta = 1 \) (Chaudhary and Merkin [21]) This assumption gives us;

\[ g(\eta) + h(\eta) = 1. \]

Thus equations (15) and (16) become

\[ \frac{1}{\text{Sc}} g'' + fg' - \frac{J}{2} Ag' - K_c (1 - g)^2 = 0, \]

subject to the boundary conditions

\[ g'(0) = K_c g(0), \quad g(\eta) \rightarrow 1 \text{ as } \eta \rightarrow \infty. \]

The skin friction coefficient \( C_f \) and the local Nusselt number \( N_{th} \) are defined as;

\[
C_f = \frac{\tau_w}{\rho U_w^2}, \quad N_{th} = \frac{\chi q_w}{k_\infty (T_w - T_\infty)},
\]

where the wall shear stress \( \tau_w \) and the wall heat flux \( q_w \) are given by

\[ \tau_w = \mu \frac{\partial u}{\partial y} \left[ \beta^* + (1 - \beta^*) (1 - \frac{1}{2} \frac{\partial u}{\partial y})^{-1} \right], \quad q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}. \]

In view of equations (22) and (23) the following expressions takes the form

\[ Re^{1/2} C_f = \frac{f''(0)}{[\beta^* + (1 - \beta^*) (1 - \text{We} f''(0))^{-1}]}, \quad Re^{-1/2} N_{th} = -\theta'(0), \]

where \( Re \left(= \frac{\nu}{\nu} \right) \) specify the local Reynolds number.

**Numerical method**

In this study, we employ an efficient numerical technique Runge–Kutta Fehlberg method with shooting scheme as discussed by Pal and Shivakumara [36] to examine the flow model for distinct values of leading parameters. In this method, choosing an appropriate finite value of \( \eta \rightarrow \infty \) is a critical step. Consequently, in order to obtain \( \eta \rightarrow \infty \) for the boundary value problem described, we start with initial estimated values for the set of a particular physical parameters, so as to obtain \( f(0), \theta(0) \) and \( g'(0) \). The first step towards this is to discretize the governing system into a system of five simultaneous differential equations of first order. For this purpose we introduce the new variables.

\[ f = Y_1, f' = Y_2, f'' = Y_3, \quad \theta = Y_4, \quad \theta' = Y_5, \quad g = Y_6, \quad g' = Y_7. \]
In this section, we explore the effects of time dependent flow of Williamson fluid past a stretching surface with variable thermal conductivity and homogeneous-heterogeneous interactions. The system of nonlinear ordinary differential equations (26)–(28) together with boundary conditions (29) and (30) are numerically elucidated by utilizing RK-45 with Newton shooting technique. The impact of some pertinent physical parameters on dimensionless velocity, temperature gradient and concentration profiles along with the skin friction coefficient and local Nusselt number is analyzed and illustrated graphically.

Validation of numerical scheme

In order to verify the accuracy of our numerical results, a comparison is made between our results and previously published literature. Table 1 indicates a comparison value of the skin friction coefficient with variation in A when \( \text{We} = \beta^* = \text{K_c} = 0 \) and \( \text{Sc} = 0 \).

| A   | Sharidan et al [37]  | Chamkha et al [38] | Khan and Azam [39] | Present study | Percent error |
|-----|----------------------|--------------------|-------------------|---------------|---------------|
| 0.8 | -1.261 042           | -1.261 512         | -1.261 043        | -1.261 042    | -3.33 * 10^-7 |
| 1.2 | -1.377 722           | -1.378 052         | -1.377 724        | -1.377 7237   | -2.50 * 10^-7 |

The boundary conditions now become

\[ Y_1(0) = 0, \quad Y_2(0) = 1, \quad Y_3(0) = s_1, \]
\[ Y_4(0) = 1, \quad Y_5(0) = s_2, \quad Y_6(0) = -K_c Y_6(0), \quad Y_6(0) = s_3. \]

In this system, four initial conditions are known and the other three unknown conditions \( s_1, s_2 \) and \( s_3 \) are first guessed and afterword fixed with Newton-Raphson’s method for each given set of parameters. Later on, a finite value for \( \eta_{bc} \) is selected so that the far field boundary conditions hold at highest value of \( \eta_{bc} \). We took \( \eta_{bc} = 10 \) to perform our computations. The absolute convergence criteria is assume to be \( 10^{-6} \) to get the desired degree of accuracy.

Results and discussion

In this section, we explore the effects of time dependent flow of Williamson fluid past a stretching surface with variable thermal conductivity and homogeneous-heterogeneous interactions. The system of nonlinear ordinary differential equations (26)–(28) together with boundary conditions (29) and (30) are numerically elucidated by utilizing RK-45 with Newton shooting technique. The impact of some pertinent physical parameters on dimensionless velocity, temperature gradient and concentration profiles along with the skin friction coefficient and local Nusselt number is analyzed and illustrated graphically.

Figure 2 highlights the variation of velocity components \( f'(\eta) \), temperature gradient \( \theta'(\eta) \) and concentration profile \( g'(\eta) \) against multiple values of unsteadiness parameter. From figure 2, one can observe that an enhancement in the unsteadiness parameter \( A \) results in a decrease in the velocity and temperature profile. We also perceived a rise in the concentration profile for higher unsteadiness parameter. From graphical illustration we also observed that momentum and thermal boundary layer thickness increases with augmented unsteadiness parameter but quiet opposite behavior is seen for solutal concentration. Physically, it is stated that higher unsteadiness parameter relates to a smaller stretching rate in the x-direction and causes in a slight reduction in the velocity field. Further, less amount of heat and mass is transferred from the fluid to the surface, therefore a reduction occurs in the temperature profile while an opposite pattern is seen for concentration distribution.

The impact of the \( \beta^* \) on the \( f'(\eta) \) and \( \theta'(\eta) \) is sketched in the figure 3. Higher values of viscosity ratio parameter causes a growth in the velocity of the fluid and decline in the temperature profile. However, momentum boundary layer thickness expands and thermal boundary layer thickness diminishes with the augmented values of viscosity ratio parameter \( \beta^* \).
Figure 2. Velocity, temperature and concentration profiles for different values of $A$.

Figure 3. Velocity and temperature profiles for different values of $\beta^*$. 
The influence of the Weissenberg number $We$ on the velocity $f^\prime(\eta)$ and temperature distributions $\theta(\eta)$ are displayed in the figure 4. It can be shown that an increase in $We$ reduces the velocity of the fluid as well as momentum boundary layer thickness. For the physical point of view, Weissenberg number is the dimensionless parameter that is used to compare the elastic to viscous forces. It is due to the fact that higher values of $We$ reduces the viscous forces which causes a fact that velocity of the fluid decreases. A noteworthy increment in the temperature profile is marked when the $We$ is increased.

Figure 5(a) demonstrates the effect of Prandtl number $Pr$ on the temperature profile $\theta(\eta)$ It is observed that temperature of the fluid and associated thermal boundary layer thickness decreases with larger Prandtl number. Physically means that the Prandtl number possesses low thermal conductivity and fluid particles needs more time to transferred the heat to its surrounding consequently we observed a reduction in the temperature profile $\theta(\eta)$.

Figure 5(b) depicts the influence of thermal conductivity parameter $\varepsilon$ on dimensionless temperature field $\theta(\eta)$. A significant growth in the temperature gradient is noticed when $\varepsilon$ is enhanced. It is also observed that thermal boundary layer thickness rises significantly when thermal conductivity parameter is increased. This is due to the fact that when $\varepsilon$ rises then considerable heat transfers from plate to the material which grow the temperature of Williamson fluid.
Figure 6(a) is illustrated to examine the characteristics of strength of homogeneous reaction parameter $K$ on solute concentration $g(\eta)$. For higher values of $K$ concentration of the fluid and solute boundary layer thickness reduces. Physically, it can be stated that diffusion coefficients is dominated by the reaction rates.

Figure 6(a) is sketched to interpret the behavior of the strength of heterogeneous reaction $K_s$ on concentration distribution $g(\eta)$. From this plot, one can notice that higher values of $K_s$ depreciates the concentration of the fluid and associated boundary layer thickness.

To exhibit the effects of $Sc$ on concentration gradient $g(\eta)$ against $\eta$ is displayed in figure 7. On can observe that as $Sc$ rises then there is an enhancement in the concentration profile $g(\eta)$. As the ratio of momentum to mass diffusivity is defined as Schmidt number. This means that high momentum diffusivity as compared to mass diffusivity increases Schmidt number. In the physical point of view, an increment in the Schmidt number diminishes the molecular diffusivity and that causes in a decrease concentration boundary layer thickness.

The variation of skin friction coefficient $Re^{1/2}C_{fp}$ against $We$ and $\beta^*$ is illustrated through figure 8. From this plot, it is noticed that higher values of $We$ and $\beta^*$ decreases the drag force at the surface of plate.

The influence of $Pr$ and $\varepsilon$ on Nusselt number $Re^{-1/2}Nu_\varepsilon$ is represented in figure 9. It can be inferred that the heat flux on the boundary experiences a decreasing influence due to Prandtl number $Pr$ and thermal...
On the other hand, since the increase in Prandtl number leads to decrease in thermal boundary layer thickness, therefore the Nusselt number decreases with increase of Prandtl number.

Figure 10 is sketched to exhibit the comparison of two numerical techniques bvp4c and shooting method for velocity, temperature and concentration profiles. Where dots are representing the solution calculated by bvp4c and the solution obtained by shooting method is drawn through straight lines. The solution profiles of both numerical techniques are in a remarkable assertion for velocity, temperature and concentration distribution.

Conclusions

In this article, the numerical solutions have been obtained for homogeneous and heterogeneous reaction on an unsteady, laminar and two-dimensional flow of Williamson non-Newtonian fluid model through a stretching sheet in the presence variable thermal conductivity. The well-known Runge–Kutta Fehlberg Method with
A shooting scheme is utilized to solve the reduced momentum and energy equations of the current fluid flow model. Moreover, a comparison between the results obtained by Runge–Kutta method and the Matlab routine bvp4c is also presented for non-dimensional velocity and temperature profiles. Therefore, a very good agreement is seen between these results. The key outcomes of this analysis are summarized as

- The dimensionless velocity profiles \( f'(\eta) \) as well as temperature profiles \( \theta(\eta) \) are decrease with an increment in the unsteadiness parameter \( A \).
- The augmented values of \( We \) decreases the fluid velocity. However, an opposite behavior is noticed for temperature profile.
- Viscosity ratio parameter \( \beta^* \) increases the momentum boundary layer thickness and decreases the thermal boundary layer thickness.
- The temperature and thermal boundary layer thickness are decreasing function for Prandtl number \( Pr \).
- The variable thermal conductivity parameter \( \varepsilon \) increases the temperature of the fluid.
- Higher strength of heterogeneous and homogeneous reactions lead to enlarge the concentration of the catalyst at the surface.
- The skin friction coefficient decreases with the increasing of viscosity ratio parameter.
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