Quantum corrections to energy of short spinning string in $AdS_5$

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Abstract

Motivated by a desire to shed light on the strong coupling behaviour of dimensions of “short” gauge theory operators we consider the famous example of folded spinning string in $AdS_5$ in the limit of small semiclassical spin parameter $S = S \sqrt{\lambda}$. In this limit the string becomes short and is moving in a near-flat central region of $AdS_5$. Its energy scales with spin as $E = \lambda^{1/4} \sqrt{2S} [a_0 + a_1S + a_2S^2 + ...]$. We explicitly compute the leading 1-loop quantum $AdS_5 \times S^5$ superstring corrections to the short-string limit coefficients $a_0$ and $a_1$ and show, in particular, that $a_1$ receives a contribution containing $\zeta(3)$. 

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1 Introduction

The remarkable progress achieved recently in uncovering the integrable structure underlying the spectrum of planar $\mathcal{N} = 4$ SYM theory or the free $AdS_5 \times S^5$ superstring theory was largely limited to a sector of gauge theory operators with large number of fields/derivatives or strings with large values of quantum numbers like spins. It is important to try to learn more about dimensions/energies of “short” operators/strings and a step in that direction is to study quantum corrections to energies of strings carrying parametrically small values of spins.

With this motivation in mind here we revisit the computation of the 1-loop quantum correction to the energy of the prototypical example of rotating string – folded rotating string located at the center of $AdS_5$ [1, 2].

The classical energy of this string is proportional to string tension, i.e. $E_0 = \sqrt{\lambda} \mathcal{E}(S) = \frac{S}{\sqrt{\lambda}}$ and in the limit of large $S$ one finds [2]: $E_0 = S + \frac{2}{\pi} \ln S + ...$. In general, the radial coordinate $\rho$ of the global $AdS_5$ space ($ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2$) is expressed in terms of an elliptic function of the spatial string coordinate $\sigma$ and thus finding the explicit form of the 1-loop correction [3] to the energy $E_1$ of this soliton solution of 2d string sigma model appears to be technically challenging. The analytic form of the quantum correction can be found in the limit of large $S$ when the ends of the string reach the boundary of the $AdS_5$. Then the solution drastically simplifies ($\rho$ becomes linear in $\sigma$) [3, 4] and one finds that $E_1 = c_1 \ln S + ...$, $c_1 = -\frac{3}{2} \ln \frac{2}{\pi}$.

Since rotation of the string balances the contracting effect of its tension, smaller values of the spin correspond to smaller values of the length of the string whose center of mass is at $\rho = 0$: $S$ essentially measures the length of the string. Since the $AdS_5$ space is nearly flat at the vicinity of $\rho = 0$, the slowly rotating (i.e. small) string with $S \ll 1$ should have essentially the same classical energy as in flat space [2], i.e. $E_0 = \sqrt{2\sqrt{\lambda}S} + ...$.

Below we shall expand the general expression for the 1-loop correction to the energy of the spinning string in [3] (given by a sum of logarithms of determinants of the 2d second order differential operators depending on the string background) in the “short string” limit $S \ll 1$ and find explicitly the coefficients of the first two leading terms in the small spin expansion of the 1-loop energy.

Our results can be summarized as follows. Given the energy $E(S, \lambda)$ of the corresponding state in the AdS/CFT spectrum we may expand it at large $\lambda$ with $S = \frac{S}{\sqrt{\lambda}}$ fixed, i.e. in the semiclassical string limit. Expanding then in the limit $S \ll 1$, i.e. $S \ll \sqrt{\lambda}$, and re-expressing $E$ as a function of $S$ and $\lambda$ one is to find

$$E(S, \lambda) = \lambda^{1/4} \sqrt{2S} \left[ h_0(\lambda) + h_1(\lambda)S + h_2(\lambda)S^2 + ... \right],$$

$$h_n = \frac{1}{(\sqrt{\lambda})^n} (a_{n0} + \frac{a_{n1}}{\sqrt{\lambda}} + \frac{a_{n2}}{(\sqrt{\lambda})^2} + ...), \quad \lambda \gg 1, \quad \frac{S}{\sqrt{\lambda}} = \text{fixed} \ll 1.$$  \hspace{1cm} (1.1)

In the classical string theory limit

$$a_{00} = 1, \quad a_{10} = \frac{3}{8}, \quad a_{20} = -\frac{21}{128}, ....$$  \hspace{1cm} (1.2)

In the quantum string theory limit
while our 1-loop string computation gives
\[ a_{01} = 3 - 4 \ln 2 \approx 0.227 , \quad a_{11} = -\frac{1219}{576} + \frac{3}{2} \ln 2 + \frac{3}{4} \zeta(3) \approx -0.175 . \] (1.4)

The leading \( \sqrt{2S} \) term has the same form as in the flat-space string theory, but its coefficient gets renormalized from its classical value 1. Classically, a short string in the middle of \( AdS_5 \) does not feel the curvature so its energy is the same as in flat space. In flat space the string fluctuations are essentially quadratic and massless (as happens in the in light-cone gauge). They thus decouple from the rotating string background (as we shall discuss explicitly for the Green-Schwarz string in covariant gauge in Appendix A below) and do not change the classical \( E = \sqrt{4\pi T S} \) expression (here \( T = \frac{1}{2\pi\alpha'} \)). In curved space the bosonic fluctuations feel the curvature and as a result get mass depending on the string background; the fermionic fluctuations get similar mass due to their coupling to the RR 5-form background. While most of the resulting contributions to the leading \( \sqrt{2S} \) term in the energy cancel between the bosonic and fermionic terms, there is a nontrivial residue (proportional to the \( \sigma \)-derivative of the fermionic mass term) leading to the non-zero value of the 1-loop \( a_{01} \) coefficient.

Explicitly, (1.1) can be written as
\[
E(S, \lambda) = \lambda^{1/4} \sqrt{2S} \left[ (1 + \frac{a_{01}}{\sqrt{\lambda}} + ...) + (a_{10} + \frac{a_{11}}{\sqrt{\lambda}} + ...) \frac{S}{\sqrt{\lambda}} + (a_{20} + \frac{a_{21}}{\sqrt{\lambda}} + ...) \frac{S^2}{(\sqrt{\lambda})^2} + ... \right].
\] (1.5)

In contrast to the large spin (or “long string”) limit where the limits of large \( \lambda \) and large \( S \) appear to commute and thus one finds the same \( S \) dependence of the gauge theory anomalous dimension and string theory energy at both weak and strong coupling, \( E = S + f(\lambda) \ln S + ... \), with \( f(\lambda \ll 1) = c_1\lambda + c_2\lambda^2 + ... \), \( f(\lambda \gg 1) = \sqrt{\lambda}(b_0 + \frac{b_1}{\sqrt{\lambda}} + ...) \) here one cannot directly continue (1.1) to small \( \lambda \) and small \( S \).

Indeed, the anomalous dimensions of low-twist gauge-theory operators like \( \text{tr}(\Phi D^2 \Phi) \) computed for small \( \lambda \) and fixed \( S \) (see, e.g., [5]) and then formally expanded in small \( S \) limit scale as
\[
E(\lambda, S) = q_0(\lambda) + q_1(\lambda) S + q_2(\lambda) S^2 + O(S^3) , \quad \lambda \ll 1 , \quad S = \text{fixed} ,
\] (1.6)
where
\[
q_n(\lambda) = d_{n0} + d_{n1}\lambda + d_{n2}\lambda^2 + ... , \quad n = 0, 1, 2, ... .
\] (1.7)

To relate the “small-spin” string theory (1.1) and the gauge theory (1.6) expansions one would need to resum the series in both arguments \( (\lambda, S) \), e.g., first sum up the weak-coupling expansion in (1.6) and then re-expand the result first in large \( \lambda \) for fixed \( S = \frac{S}{\sqrt{\lambda}} \) and then in small \( S \).

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1 This contribution was missed in the original version of this paper.

2 The perturbative string theory and perturbative gauge theory limits are actually different as limits of functions on the two-parameter space \( (\lambda, S) \): in string theory one assumes \( \lambda \gg 1 \) with \( S = \frac{S}{\sqrt{\lambda}} \) fixed and then takes \( S \) large; in gauge theory one assumes \( \lambda \ll 1 \) with \( S \) fixed and then takes \( S \) large. However, this appears not to matter for the leading \( \ln S \) term which can be described by a single universal interpolating function of \( \lambda \) (cusp anomaly).

3 If the twist two operator in question is assumed to be from the \( \mathfrak{sl}(2) \) sector then it is BPS for \( S = 0 \) so that \( q_0 = 2 \).
In view of the need for this resummation which is, in fact, a generic situation in comparing the semiclassical string theory and the perturbative gauge theory expansions\(^4\) it is not clear at the moment how to directly interpret our result (1.5) as a strong coupling limit of a gauge-theory anomalous dimension.

We shall start in section 2 with a review of the folded spinning string solution and its small spin expansion [2].

In section 3 we shall first recall the general expression for the quadratic fluctuation Lagrangian \(\tilde{L}[3]\) of the \(AdS_5 \times S^5\) superstring [11] near the folded spinning string solution. We will then expand the coefficients in \(\tilde{L}\) in the small spin or short string parameter \(\epsilon = \sqrt{2S} + \ldots\). This expansion may be viewed as a particular case of a near flat space expansion of the quantum \(AdS_5 \times S^5\) superstring. We will then compute the leading \(O(\epsilon)\) term in the 1-loop string energy determining the coefficient \(a_{01}\) in (1.4), (1.5).

In section 4 we shall expand the 2d determinants that enter the expression for the 1-loop partition function to first two leading orders in \(\epsilon\) and compute the value of the coefficient \(a_{11}\) in (1.4), (1.5).

In Appendix A we shall present the flat-space Green-Schwarz string analog of this computation showing explicitly (in a covariant \(\kappa\)-symmetry gauge) why the classical \(E = \sqrt{4\pi T S}\) expression is not renormalized by quantum fluctuations.

In Appendix B we shall briefly discuss how to generalise our computation to the case of the short string expansion of the folded spinning string solution which also carries a momentum \(J\) in \(S^5[3]\) (details of this case are worked in the follow-up paper [16]).

In Appendix C we shall mention a curious regularization scheme ambiguity which appears, in particular, when interchanging a sum with an integral in certain 1-loop terms.

## 2 Short string limit of folded spinning string solution

Let us start with a review of the classical solution for the folded string spinning in the \(AdS_3\) part of \(AdS_5\),

\[
\begin{align*}
t &= \kappa \tau, \quad \phi = w \tau, \quad \rho = \rho(\sigma), \quad ds^2 &= -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2, \\
\rho^2 &= \kappa^2 \cosh^2 \rho - w^2 \sinh^2 \rho.
\end{align*}
\]

where

\[
\coth^2 \rho_* = \frac{w^2}{\kappa^2} \equiv 1 + \frac{1}{\epsilon^2}.
\]

\(^4\)Analogous resummation is needed to compare the weak coupling gauge theory expansion for anomalous dimensions of sl(2) sector operators in the limit \(\lambda \ll 1\) with \(J \gg 1\), \(S \gg 1\), \(j = \frac{J}{\ln S}\) = fixed and \(j < 1\) with the strong-coupling string theory expansion in the limit \(\lambda \gg 1\) with \(\mathcal{J} = \frac{J}{\sqrt{\lambda}}\), \(S = \frac{\sqrt{\lambda}}{\sqrt{\lambda}}\), \(\ell = \frac{-j}{\sqrt{\lambda}}\) = fixed and \(\ell < 1\) (see [3, 6, 7, 4, 8, 9, 10]).
Thus $\epsilon$ measures the length of the string. The solution of the differential equation (2.2), i.e.

$$\rho' = \pm \kappa \sqrt{1 - \epsilon^{-2} \sinh^2 \rho}, \quad \rho(0) = 0$$  \hspace{1cm} (2.4)$$
can be written in terms of the Jacobi function $sn$

$$\sinh \rho = \epsilon \, sn(\kappa^{-1} \sigma, -\epsilon^2).$$  \hspace{1cm} (2.5)$$
The periodicity in $\sigma$ implies the following condition on the parameters \[2\]

$$\kappa = \epsilon \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2\right).$$  \hspace{1cm} (2.6)$$
The classical energy $E_0 = \sqrt{\lambda} E_0$ and the spin $S = \sqrt{\lambda} S$ are found to be

$$E_0 = \epsilon \, _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2\right), \quad S = \frac{\epsilon^2}{2} \sqrt{1 + \epsilon^2} \, _2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; -\epsilon^2\right)$$  \hspace{1cm} (2.7)$$
Here we will be interested in the short string limit $0 < \epsilon \ll 1$ in which

$$\rho_* = \epsilon - \frac{1}{6} \epsilon^3 + O(\epsilon^5).$$  \hspace{1cm} (2.8)$$
In the strict limit $\epsilon = 0$ or $\kappa = 0$ we get $\rho = \rho_* = 0$, so that the string shrinks to a point with $E = 0.1$

From (2.7) we obtain in the small $\epsilon$ or the small $S$ limit

$$\epsilon = \sqrt{2S} - \frac{1}{4\sqrt{2}} S^{3/2} + ... , \quad E_0 = \sqrt{2S} + \frac{3}{4\sqrt{2}} S^{3/2} + ... ,$$  \hspace{1cm} (2.9)$$
so the short string limit corresponds to $S \ll 1$ and the expansion of the energy looks like

$$E_0(S, \lambda) = \lambda^{1/4} \sqrt{2S} + \frac{3}{4\sqrt{2}} \lambda^{-1/4} S^{3/2} + O(S^{5/2}) .$$  \hspace{1cm} (2.10)$$
For the purpose of computing the 1-loop correction to the energy to order $O(S^{3/2})$ we will need the expression for $\rho(\sigma)$ to order $\epsilon^4$. Expanding the exact solution (2.5) in powers of $\epsilon$ we obtain

$$\sinh \rho = \epsilon \sin \sigma - \frac{\epsilon^3}{4} \sin \sigma \cos^2 \sigma + O(\epsilon^5)$$  \hspace{1cm} (2.11)$$
Other useful expansions are

$$\kappa = \epsilon(1 - \frac{\epsilon^2}{4} + ...), \quad w = 1 + \frac{\epsilon^2}{4} + ... , \quad \rho' = \epsilon \cos \sigma - \frac{\epsilon^3}{4} \cos^3 \sigma + ... ,$$  \hspace{1cm} (2.12)$$
$$\kappa \sinh \rho = \epsilon^2 \sin \sigma - \frac{\epsilon^4}{8} (3 + \cos 2\sigma) \sin \sigma + ... ,$$  \hspace{1cm} (2.13)$$
$$w \cosh \rho = 1 + \frac{\epsilon^2}{4} (1 + 2 \sin^2 \sigma) - \frac{\epsilon^4}{64} (8 - \cos 4\sigma) + ... .$$  \hspace{1cm} (2.14)$$

\[1\]Note that in this limit the string disappears instead of reducing to a massless point particle with non-zero momentum moving along null geodesic. This corresponds in flat space to considering a massive string state in the rest frame (which is possible in covariant quantization). In contrast to the flat space case where adding a non-zero center of mass momentum can be achieved by a Lorentz boost, adding a motion of the spinning string center of mass in curved $AdS_5 \times S^5$ space is a nontrivial operation (different parts of the string move along different geodesics) which leads in general to a new nontrivial configuration.
The above small spin expansion is an example of a near flat space expansion: the leading-order in $\epsilon$ solution can be identified with the folded spinning string solution in the flat space

$$t = \epsilon \tau , \quad \rho = \epsilon \sin \sigma , \quad \phi = \tau , \quad ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 ,$$

(2.15)

where $\epsilon$ is an arbitrary constant amplitude. The energy and the spin then satisfy the usual flat-space Regge relation (we use string tension $T = \sqrt{\lambda}/2\pi$)

$$E_0 = \epsilon \sqrt{\lambda} , \quad S = \frac{\epsilon^2}{2} \sqrt{\lambda} , \quad \text{i.e.} \quad \mathcal{E}_0 = \lambda^{3/4} \sqrt{2S} .$$

(2.16)

In the flat space case this is the exact expression for any value of $S$ (cf. (2.10)) which also does not receive quantum corrections (see Appendix A).

### 3 1-loop correction to $\sqrt{S}$ term in short string energy

Following [3] and expanding the $AdS_5 \times S^5$ string action [11] in conformal gauge to quadratic order in fluctuations near the folded spinning string one finds $\tilde{S} = -\frac{\sqrt{\lambda}}{4\pi} \int dt f^{2\pi} d\sigma \tilde{L}$ with the bosonic part ($a = 0, 1$)

$$\tilde{L}_B = -\partial_a \tilde{t} \partial^a \tilde{t} - \mu_1^2 \tilde{t}^2 + \partial_a \tilde{\phi} \partial^a \tilde{\phi} + \mu_2 \tilde{\phi}^2 + 4\tilde{\rho} (\kappa \sinh \rho \partial_0 \tilde{t} - w \cosh \rho \partial_0 \tilde{\phi}) + \partial_a \tilde{\phi} \partial^a \tilde{\phi} + \mu_2 \tilde{\phi}^2 + \partial_a \beta_a \partial^a \beta_a + \partial_a \phi \partial^a \phi + \partial_a \chi_s \partial^a \chi_s ,$$

(3.1)

where

$$\mu_1^2 = 2\rho^2 - \kappa^2 , \quad \mu_2 = 2\rho^2 - w^2 , \quad \mu_3 = 2\rho^2 - w^2 - \kappa^2 , \quad \mu_4 = 2\rho^2 .$$

(3.2)

Here $\beta_a (u = 1, 2)$ are two $AdS_5$ fluctuations transverse to the $AdS_3$ subspace in which the string is moving, while $\phi, \chi_s (s = 1, 2, 3, 4)$ are fluctuations in $S^5$.

The fermionic part of the quadratic fluctuation Lagrangian can be put into the form [3]

$$\tilde{L}_F = 2\partial \bar{D}_F \bar{\vartheta} , \quad D_F = i(\Gamma^a \partial_a - \mu_F \Gamma_{234}) , \quad \mu_F = \rho' ,$$

(3.3)

and can be interpreted as describing a system of 4+4 2d Majorana fermions with $\sigma$-dependent mass $\mu_F$. Let us briefly recall the derivation [3] of this expression. One starts with the quadratic fermionic term in the $AdS_5 \times S^5$ action and fixing the conformal gauge $\sqrt{-g} g^{ab} = \eta^{ab}$ and the $\kappa$-symmetry gauge $\theta^1 = \theta^2 = \theta$ (where $\theta$ is a MW 10-d spinor) one gets

$$\tilde{L}_F = 2i \bar{\vartheta} \bar{g}^a (D_a - i \frac{1}{2} \Gamma_\ast g_a) \vartheta , \quad \Gamma_\ast = i \Gamma_{01234} ,$$

(3.4)

where

$$D_a = \partial_a + i \frac{1}{4} \partial_a X^M \omega^M_{AB} \Gamma_{AB} , \quad g_a = \partial_a X^M E^A_M \Gamma_A .$$

Identifying $(t, \rho, \phi)$ with the directions $M = 0, 1, 2$ and introducing $\vartheta$ as

$$\vartheta = \sqrt{\rho} \ e^{-\frac{1}{2} \alpha \Gamma_{02}} \theta , \quad \cosh \alpha = \frac{\kappa \cosh \rho}{|\rho|} ,$$

(3.5)
one ends up with (3.3) where $\Gamma^a = \eta^{ab}\Gamma_b = (-\Gamma_0,\Gamma_1)$ and $\tilde{\partial} = \psi^T \Gamma^0$ with $\Psi$ being real 16-component (Weyl) spinor.$^1$ Note that since $\rho(\sigma)$ is periodic function, the same applies to $\alpha$, i.e. the rotated fermions are periodic in $\sigma$ just like the original ones. Note also that the fermionic mass term has its origin in the RR 5-form coupling term in the quadratic fermionic Lagrangian.$^2$

Since $\ln \det(\Gamma^0 D_\rho) = \ln \det(D_\rho) = \frac{1}{2} \ln \det(D_\rho)^2$ we conclude that the fermionic contribution to the 1-loop string partition function is determined by the following second-order differential operator

$$\Delta_\rho \equiv (D_\rho)^2 = -\partial^a \partial_a + \mu_\tau' \Gamma_{1234} + \mu_\rho^2 ,$$

where we used that $\Gamma_{(a\Gamma_b)} = \eta_{ab}$, $\{\Gamma_a,\Gamma_{234}\} = 0$, $(\Gamma_{234})^2 = -1$ and that $\mu_\tau$ depends only on $\sigma$ ($\mu_\tau' \equiv \partial_1 \mu_\tau$). Furthermore, since $(\Gamma_{1234})^2 = 1$ we can diagonalize this operator so we will end up with the following contribution to the 1-loop 2d effective action coming from $4+4=8$ effective fermionic degrees of freedom.$^3$

$$-\frac{1}{2} \left( 4 \ln \det \Delta_\rho + 4 \ln \det \Delta_\rho^2 \right),$$

where

$$\Delta_\rho \equiv -\partial^a \partial_a + \mu_\tau^2 \equiv \pm \mu_\tau^2 + \mu_\rho^2 \equiv \pm \rho'' + \rho'^2.$$  

Next, we expand the coefficients in the fluctuation Lagrangian in $\epsilon$ as discussed in the previous section. To leading order in $\epsilon$ we get

$$\mu_i^2 = \epsilon^2 \cos 2\sigma + ... , \quad \mu_\phi^2 = -1 + (\cos 2\sigma + \frac{1}{2})\epsilon^2 + ... ,$$

$$\mu_\rho^2 = -1 + (\cos 2\sigma - \frac{1}{2})\epsilon^2 + ... , \quad \mu_\beta^2 = 2\epsilon^2 \cos 2\sigma + ... ,$$

$$\tilde{\mu}_{\tau\pm} = \mp \epsilon \sin \sigma + \epsilon^2 \cos 2\sigma + ... ,$$

$$4\tilde{\rho}(\kappa \sinh \rho \partial_0 \tilde{\ell} - \cosh \rho \partial_0 \tilde{\phi}) = \tilde{\rho}\{4\epsilon^2 \sin \sigma \partial_0 \tilde{\ell} - [4 + 2(1 + 2 \sin^2 \sigma)]\partial_0 \tilde{\phi}\}.$$  

If we set $\epsilon$ to zero we are back to the flat space case (see Appendix A): indeed, the only two coupled modes that are not massless are then described by

$$\tilde{L}_0 = \partial_0 \tilde{\phi} \partial^0 \tilde{\phi} - \tilde{\phi}^2 - 4\tilde{\phi} \partial_0 \tilde{\phi} + \partial_0 \bar{\partial}^a \rho - \tilde{\rho}^2 ,$$

which becomes the Lagrangian for two massless modes after a $\tau$-dependent rotation

$$\rho = \eta_1 \cos \tau + \eta_2 \sin \tau, \quad \tilde{\phi} = -\eta_1 \sin \tau + \eta_2 \cos \tau.$$  

If we perform this rotation also at order $\epsilon^2$ we get $\tilde{L}_B = \tilde{L}_0 + \epsilon^2 \tilde{L}_1 + O(\epsilon^4)$ where $\tilde{L}_0$ is the same as in flat space and a nontrivial part of the subleading term is

$$\tilde{L}_1 = - \cos 2\sigma \tilde{t}^2 + (\sin^2 \tau + \cos^2 \sigma)\eta_1^2 + (\cos^2 \tau + \cos^2 \sigma)\eta_2^2 + 2(\eta_1 \cos \tau + \eta_2 \sin \tau)\tilde{t} \sin \sigma - 2(\eta_1 \sin \tau + \eta_2 \cos \tau)\tilde{t} \sin \sigma - \eta_1 \eta_2 \sin 2\tau - \eta_1 \eta_2 (1 + 2 \sin^2 \sigma).$$

$^1$For a discussion of spinor notation see, e.g., Appendix A in $^{[17]}$.

$^2$This structure of the fermionic contribution was understood in collaboration with M. Beccaria.

$^3$We shall not use this $\tau$-dependent form of the fluctuation Lagrangian for explicit computations below.
The order $\epsilon$ contribution coming from $\rho''$ term in the effective fermionic mass in (3.8) will cancel out in the sum of the $4+4$ fermionic contributions but there will be an additional $\epsilon^2$ term coming from the double insertion of this term. Let us first ignore this extra $\epsilon^2$ contribution coming from the presence of the $\mu' = \rho''$ term in the fermionic masses. Then one may argue on general grounds that the leading $\epsilon^2$ part of 1-loop correction to string energy should vanish. Indeed, then the 1-loop correction to string energy will look like (assuming all propagators were diagonalized)\(^4\)

\[\Gamma_1 = \frac{1}{2} \sum_i (-1)^{n_i} \ln \frac{\det[\partial_0^2 - \partial_1^2 + \epsilon^2 M_i^2]}{\det[\partial_0^2 - \partial_1^2]} \sim \epsilon^2 \int d\tau d\sigma \; \text{Tr} \sum_i (-1)^{n_i} M_i^2 + O(\epsilon^4) . \tag{3.16}\]

Since $t = \kappa \tau$, $\kappa = \epsilon + \ldots$ the 1-loop correction to string energy is given by

\[E_1 = \frac{\Gamma_1}{\kappa T}, \quad T \equiv \int d\tau \to \infty . \tag{3.17}\]

In general, $M_i^2$ may be non-trivial matrices which depend on $\tau, \sigma$. Let us now recall that the 1-loop logarithmic UV divergencies in the $AdS_5 \times S^5$ superstring action expanded near an arbitrary string solution manifestly cancel in the conformal gauge [12, 3]. The nontrivial UV logarithmic divergencies have as their coefficient precisely the sum of the mass squared terms in the r.h.s. of (3.16)\(^5\) it vanishes for a generic on-shell string background, thus implying the absence of the $\epsilon^2$ term in the 1-loop string partition function (again, modulo the additional $\epsilon^2$ contribution coming from $\rho''$ part that we are temporarily ignoring).

Let us now verify this cancellation by direct computation. For the contribution of the $\beta_u$ fields we get (rotating to euclidean time, $\tau \to i\tau$, and factorizing the infinite time interval $T$)

\[\det[-\partial_1^2 - \partial_0^2 + 2\epsilon^2 \cos^2 \sigma] = T \int \frac{d\omega}{2\pi} \det[-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos^2 \sigma] . \tag{3.18}\]

We can now use perturbation theory in $\epsilon^2$, i.e.

\[\ln \frac{\det[A + \epsilon^2 B]}{\det A} = \epsilon^2 \text{Tr}[A^{-1} B] + O(\epsilon^4) . \tag{3.19}\]

Then to order $\epsilon^2$ (here $\sigma \in (0, 2\pi)$)

\[\ln \frac{\det[-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos^2 \sigma]}{\det[-\partial_1^2 + \omega^2]} \approx \epsilon^2 \sum_n \frac{2}{n^2 + \omega^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2 \sigma = \epsilon^2 \sum_n \frac{1}{n^2 + \omega^2} . \tag{3.20}\]

Similarly, the $\epsilon^2$ contribution of the fermionic modes coming from $\rho^2$ term in (3.8) is proportional to

\[\ln \frac{\det[-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos^2 \sigma]}{\det[-\partial_1^2 + \omega^2]} \approx \epsilon^2 \sum_n \frac{1}{n^2 + \omega^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2 \sigma = \frac{\epsilon^2}{2} \sum_n \frac{1}{n^2 + \omega^2} . \tag{3.21}\]

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\(^4\)To cancel the leading flat space term, i.e. to ensure that the total number of effective degrees of freedom is zero, one of course is to include also the conformal gauge ghost contribution.

\(^5\)In general, the additional $\pm \rho''$ terms in the fermionic mass contributions of course cancel against each other.
The nontrivial part of the euclidean partition function contributing to the $\epsilon^2$ term under consideration is

$$Z = \frac{\det^2[-\partial_0^2 - \partial_1^2 + \epsilon^2 \cos 2\sigma] \det^2[-\partial_0^2 - \partial_1^2]}{\det^2[-\partial_0^2 - \partial_1^2 + 2\epsilon^2 \cos 2\sigma] \det^2[-\partial_0^2 - \partial_1^2] \det^2 Q}$$

(3.22)

involves the operator $Q$ on the space of the three mixed fluctuations $\rho, \bar{\phi}, \bar{t}$ in (3.1),

$$Q = \begin{pmatrix} \partial_0^2 + \partial_1^2 - \epsilon^2 \cos 2\sigma & 0 & -2i\epsilon^2 \sin \sigma \partial_0 \\ 0 & -\partial_0^2 - \partial_1^2 - 1 + \epsilon^2 \left(\frac{1}{2} + \cos 2\sigma\right) & 2i\partial_0 + i\epsilon^2 \left(\frac{1}{2} + \sin^2 \sigma\right) \partial_0 \\ 2i\epsilon^2 \sin \sigma \partial_0 & -2i\partial_0 - i\epsilon^2 \left(\frac{1}{2} + \sin^2 \sigma\right) \partial_0 & -\partial_0^2 - \partial_1^2 - 1 - \epsilon^2 \left(\frac{1}{2} - \cos 2\sigma\right) \end{pmatrix}$$

Since there is no explicit $\tau$ dependence in the functional determinants we can write the relevant part of the 1-loop correction as

$$\tilde{\Gamma}_1 = -\ln Z_1 = -\frac{T}{4\pi} \int_{-\infty}^{\infty} d\omega \ln \frac{\det^2[-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos 2\sigma]}{\det^2[-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos 2\sigma] \det^2[-\partial_1^2 + \omega^2] \det[Q_\omega]}$$

(3.23)

where $Q_\omega = Q(\partial_0 \rightarrow i\omega)$.

Let us now expand: $Q_\omega = Q^{(0)} + \epsilon^2 Q^{(2)} + \ldots$, where

$$Q^{(0)} = \begin{pmatrix} -\left(\partial_1^2 + \omega^2\right) & 0 & 0 \\ 0 & -\partial_1^2 + \omega^2 - 1 & -2\omega \\ 0 & 2\omega & -\partial_1^2 + \omega^2 - 1 \end{pmatrix},$$

(3.24)

$$Q^{(2)} = \begin{pmatrix} -\cos 2\sigma & 0 & 2\omega \sin \sigma \\ 0 & \cos 2\sigma + \frac{1}{2} & -\omega \left(\frac{1}{2} + \sin^2 \sigma\right) \\ -2\omega \sin \sigma & \omega \left(\frac{1}{2} + \sin^2 \sigma\right) & \cos 2\sigma - \frac{1}{2} \end{pmatrix}.$$  

(3.25)

Defining

$$P_\omega = \begin{pmatrix} -\left(\partial_1^2 + \omega^2\right) & 0 & 0 \\ 0 & -\partial_1^2 + \omega^2 & 0 \\ 0 & 0 & -\partial_1^2 + \omega^2 \end{pmatrix},$$

(3.26)

the remaining part of $\tilde{\Gamma}_1$ may be written as

$$\frac{T}{4\pi} \int d\omega \left( \ln \frac{\det[Q_\omega]}{\det[Q^{(0)}]} - \ln \frac{\det[P_\omega]}{\det[Q^{(0)}]} \right).$$

(3.27)

The second term here vanishes for the same reason why the rotation in (3.14) lead to the standard massless kinetic terms for the two originally coupled modes and thus to the trivial flat-space partition function. Indeed, the “mixed” 2 by 2 block contribution to $\ln \det[Q^{(0)}]$, can be written as $\ln \det[-\partial_1^2 + (\omega + i)^2] + \ln \det[-\partial_1^2 + (\omega - i)^2]$. Under the integral over $\omega$ one can then shift $\omega$ by $-i$ in one term and by $+i$ in another to get the cancellation against other massless determinants. These separate shifts are thus consistent with the trivial (supersymmetric) result.

---

6Here we choose not to rotate $\bar{t} \rightarrow i\bar{t}$ to make all fluctuations having physical norm but this can be easily done at any stage of what follows; we shall assume this rotation in the free (flat) part of the partition function.
for $\Gamma_1$ in flat space, and we shall perform similar shifts of the corresponding terms in what follows (in particular in $\det[Q_{\omega}^{(0)}]$ contribution of the first term in (3.27)).

To compute the first term in (3.27) we expand in $\epsilon$ as in (3.19)

$$\ln \det[Q_{\omega}^{(0)}] = \epsilon^2 \text{Tr}[(Q_{\omega}^{(0)})^{-1}Q_{\omega}^{(2)}] + ... = \epsilon^2 \sum_n \int_0^{2\pi} \frac{d\sigma}{2\pi} (Q_{\omega}^{(0)})^{-1}_{ij}(Q_{\omega}^{(2)})_{ji} + ... .$$ (3.28)

The momentum-space propagator corresponding to $Q_{\omega}^{(0)}$ is

$$(Q_{\omega}^{(0)})^{-1} = \begin{pmatrix} -\frac{1}{n^2+\omega^2} & 0 & 0 \\ 0 & \frac{1}{n^2+2n^2(\omega^2-1)+(\omega^2+1)^2} & 0 \\ 0 & 0 & \frac{1}{n^2+2n^2(\omega^2-1)+(\omega^2+1)^2} \end{pmatrix}.$$ (3.29)

It can be diagonalized by a rotation

$$M^{-1}(Q_{\omega}^{(0)})^{-1} M \equiv D_{\omega}^{(0)} = \begin{pmatrix} -\frac{1}{n^2+\omega^2} & 0 & 0 \\ 0 & \frac{1}{n^2+(\omega+i)^2} & 0 \\ 0 & 0 & \frac{1}{n^2+(\omega-i)^2} \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$ (3.30)

$Q_{\omega}^{(2)}$ gets rotated into

$$M^{-1}Q_{\omega}^{(2)}M \equiv D_{\omega}^{(2)} = \begin{pmatrix} -\cos 2\sigma & i\omega(1 - \frac{1}{2}\cos 2\sigma) + \cos 2\sigma \\ -2\omega \sin \sigma & -\frac{1}{2} & -i\omega(1 - \frac{1}{2}\cos 2\sigma) + \cos 2\sigma \\ -2\omega \sin \sigma & -\frac{1}{2} & -i\omega(1 - \frac{1}{2}\cos 2\sigma) + \cos 2\sigma \end{pmatrix}$$

and the $\epsilon^2$ term in (3.28) becomes

$$\epsilon^2 \sum_n \int_0^{2\pi} \frac{d\sigma}{2\pi} D_{\omega}^{(0)}D_{\omega}^{(2)} = \epsilon^2 \sum_n \left[ \frac{i\omega}{n^2 + (\omega+i)^2} - \frac{i\omega}{n^2 + (\omega-i)^2} \right].$$ (3.31)

Thus finally (using that $\kappa = \epsilon + ..$, see (2.11))

$$\tilde{E}_1 = \frac{\tilde{\Gamma}_1}{\kappa T} = -\frac{\epsilon}{4\pi} \int_{-\infty}^{\infty} d\omega \sum_n \left[ \frac{2}{n^2 + w^2} - \frac{i\omega}{n^2 + (\omega+i)^2} + \frac{i\omega}{n^2 + (\omega-i)^2} \right] + O(\epsilon^3).$$ (3.32)

Doing the opposite shifts of $\omega$ in each of the last two terms we conclude that the order $\epsilon = \sqrt{2S} + ...$ term in $\tilde{E}_1$ indeed vanishes, i.e.

$$\tilde{E}_1 = 0 + O(\epsilon^3).$$ (3.33)

The formal argument leading to (3.33) overlooked an important subtlety of IR divergences that we have so far postponed to discuss but which will become crucial below. Indeed, if the sum over $n$ in (3.31) runs over all values from $-\infty$ to $+\infty$ one may get different results by interchanging the order of integration over $\omega$ and summation over $n$: the integral over $\omega$ has an IR divergence at $n = 0$. 

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In fact, as in the usual perturbative expansion near a soliton, there is an issue of possible IR singularities due to a zero mode associated to the translational symmetry \( \sigma \rightarrow \sigma + \sigma_0 \). In the present case of expansion in \( \epsilon \) the “free” propagator is essentially the massless one on \( R \times S^1 \) and thus the zero mode that is not damped in the path integral corresponds to \( n = 0 \). Its contribution can be either regularized by introducing a small mass or \( i \epsilon \) in the propagator as in [13] or by isolating the modes constant in \( \sigma \) in the path integral and thus not including the \( n = 0 \) contributions in the propagators (as is done, e.g., in quantizing a sigma model on a compact 2d space). This is the prescription we shall adopt here, i.e. the sums over \( n \) in (3.20), (3.21), (3.31) and (3.32) will be understood not to include the \( n = 0 \) term.

Let us now include the \( \epsilon^2 \) contribution to the effective action coming from the \( \rho'' \) term in the fermionic mass that we so far ignored, i.e. compute \( \delta E_1 \) giving

\[
E_1 = \frac{\Gamma_1}{\kappa T} = \tilde{E}_1 + \delta E_1 + O(\epsilon^3), \quad \delta E_1 = O(\epsilon^2),
\]  

(3.34)

where (to the leading order of expansion of masses in \( \epsilon \))

\[
\Gamma_1 = -\frac{T}{4\pi} \int_{-\infty}^{\infty} d\omega \ln \frac{\det^4[-\partial_1^2 + \omega^2 + \epsilon^2 \cos^2 \sigma - \epsilon \sin \sigma] \det^4[-\partial_1^2 + \omega^2 + \epsilon^2 \cos^2 \sigma + \epsilon \sin \sigma]}{\det^2[-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos^2 \sigma] \det^3[-\partial_1^2 + \omega^2] \det[Q_\omega]}
\]

At order \( \epsilon^2 \) the \( \rho'' = -\epsilon \sin \sigma + ... \) part of the fermionic mass contributes to \( \Gamma_1 \) the following term

\[
\delta \Gamma_1 = \frac{T}{\pi} \epsilon^2 \int d\omega \sum_{n_1, n_2} \frac{1}{n_1^2 + \omega^2} \frac{1}{n_2^2 + \omega^2} \int \frac{d\sigma_1}{2\pi} \frac{d\sigma_2}{2\pi} \sin \sigma_1 \sin \sigma_2 \, e^{i(n_1-n_2)(\sigma_1-\sigma_2)}
\]

\[
= \frac{T}{\pi} \epsilon^2 \int d\omega \left[ \sum_{n\neq0,1} \frac{1}{n^2 + \omega^2 (n-1)^2 + \omega^2} + \sum_{n\neq0,-1} \frac{1}{n^2 + \omega^2 (n+1)^2 + \omega^2} \right]
\]

(3.35)

Summing over \( n \) gives

\[
\delta \Gamma_1 = -\frac{T \epsilon^2}{4\pi} \int d\omega \left[ \frac{4}{\omega^2 (\omega^2 + 1)} - \frac{4\pi \coth \pi \omega}{\omega (4\omega^2 + 1)} \right],
\]  

(3.36)

and thus finally

\[
\delta E_1 = -\frac{\epsilon}{4\pi} \int d\omega \left[ \frac{4}{\omega^2 (\omega^2 + 1)} - \frac{4\pi \coth \pi \omega}{\omega (4\omega^2 + 1)} \right] = \epsilon (3 - 4 \ln 2),
\]  

(3.37)

which leads to the value of \( a_{01} \) quoted in (1.4)\(^7\)

\( ^7 \)As one can check, the same value is found by doing first the integral and then the sum in (3.35) (this also applies to (3.33), (3.32)). We thank M. Beccaria for this observation.
4 1-loop correction to the $S^{3/2}$ term in the string energy

Let us now compute the next 1-loop correction to the short string energy: the coefficient $a_{11}$ of the $S^{3/2}$ term in (1.1) or (1.2). For that we shall consider the next order of the near flat space or $\epsilon \to 0$ expansion of the fluctuation Lagrangian (3.1),(3.3). As in the previous section, we shall treat separately the contributions coming from the $\rho''$ terms in the effective fermionic mass terms in (3.8) (the reason is that while the expansion of $\rho'^2$ and similar mass terms contains only even powers of $\epsilon$, the expansion of $\rho''$ contains both even and odd powers of $\epsilon$).

Let us first ignore the contributions coming from the $\rho''$ terms and add them later. As in (3.19) we shall use that

$$\ln \frac{\det[A + \epsilon^2 B + \epsilon^4 C]}{\det A} = \epsilon^2 \text{Tr}[A^{-1} B] - \frac{\epsilon^4}{2} \text{Tr}[A^{-1} B A^{-1} B] + \epsilon^4 \text{Tr}[A^{-1} C] + O(\epsilon^6)$$  \hspace{1cm} (4.1)

Expanding the fluctuation Lagrangian in $\epsilon$ using (2.11), etc., we get

$$\tilde{L} = \tilde{L}_0 + \epsilon^2 \tilde{L}_1 + \epsilon^4 \tilde{L}_2 + ...$$  \hspace{1cm} (4.2)

where the $\epsilon^4$ terms in the masses and the mixing terms are

$$\delta \mu_\tilde{t}^2 = \epsilon^4 \left(\frac{1}{2} - \cos^4 \sigma\right), \quad \delta \mu_\tilde{\phi}^2 = \epsilon^4 \left(\frac{5}{32} - \cos^4 \sigma\right), \quad \delta \mu_\tilde{\rho}^2 = \epsilon^4 \left(\frac{21}{32} - \cos^4 \sigma\right),$$

$$\delta \mu_\tilde{\beta}^2 = -\epsilon^4 \cos^4 \sigma, \quad \delta \mu_\tilde{\rho}^2 = -\frac{1}{2} \epsilon^4 \cos^4 \sigma,$$  \hspace{1cm} (4.3)

$$\delta[4 \tilde{\rho}(\kappa \sinh \rho \partial_0 \tilde{t} - \omega \cosh \rho \partial_0 \tilde{\phi})] = -\frac{1}{2} \epsilon^4 \rho[(3 + \cos 2\sigma)\sin \sigma \partial_0 \tilde{t} - (1 - \frac{1}{8} \cos 4\sigma)\partial_0 \tilde{\phi}]$$

Let us first compute the $\epsilon^4$ contribution to 1-loop effective action coming from the terms like $\epsilon^4 \text{Tr}[A^{-1} C]$ in (1.1). Using the Fourier representation in the (Euclidean) world-sheet time direction ($\partial_0 \to i\omega$) the operator $Q$ acting on the $\tilde{t}, \tilde{\rho}, \tilde{\phi}$ subspace can be expanded as (cf. (3.22), (3.23))

$$Q_\omega = Q_\omega^{(0)} + \epsilon^2 Q_\omega^{(2)} + \epsilon^4 Q_\omega^{(4)} + ...$$  \hspace{1cm} (4.4)

$$Q_\omega^{(4)} = \begin{pmatrix} -\left(\frac{1}{2} - \cos^4 \sigma\right) & 0 & -\frac{\omega}{4}(3 + \cos 2\sigma)\sin \sigma \\ 0 & \frac{5}{32} - \cos^4 \sigma & -\frac{\omega}{32}(\cos 4\sigma - 8) \\ -\omega(3 + \cos 2\sigma)\sin \sigma & \frac{\omega}{32}(\cos 4\sigma - 8) & \frac{21}{32} - \cos^4 \sigma \end{pmatrix}$$  \hspace{1cm} (4.5)

As in (3.30) we rotate this to $M^{-1} Q_\omega^{(4)} M = D_\omega^{(4)}$ whose diagonal elements are

$$\text{diag}[D_\omega^{(4)}] = \left\{-\frac{1}{2} + \cos^4 \sigma; \frac{1}{32}(13 - 8i\omega - 32 \cos^4 \sigma + i\omega \cos 4\sigma); \frac{1}{32}(13 + 8i\omega - 32 \cos^4 \sigma - i\omega \cos 4\sigma)\right\}$$

The computation of the $\epsilon^4$ term in (4.1) coming from the coupled bosonic part gives

$$\text{Tr}[(Q_\omega^{(0)})^{-1} Q_\omega^{(4)}] = \sum_n \int_0^{2\pi} \frac{d\sigma}{2\pi} \text{Tr}[(Q_\omega^{(0)})^{-1} Q_\omega^{(4)}] = \sum_n \int_0^{2\pi} \frac{d\sigma}{2\pi} \text{Tr}[D_\omega^{(0)} D_\omega^{(4)}]$$

$$= \frac{1}{32} \sum_n \left[\frac{4}{n^2 + w^2} + \frac{1 - 8i\omega}{n^2 + (\omega + i)^2} + \frac{1 + 8i\omega}{n^2 + (\omega - i)^2}\right].$$  \hspace{1cm} (4.6)
The $\epsilon^4$ contribution of the decoupled modes $\beta_u$ coming from the single insertion of the $\epsilon^4$ perturbation, i.e. an $\epsilon^4\text{Tr}[A^{-1}C]$ type term is
\[
\frac{\det[-\partial^2 + \omega^2 + 2\epsilon^2 \cos^2 \sigma - \epsilon^4 \cos^4 \sigma]}{\det[-\partial^2 + \omega^2]} \rightarrow - \epsilon^4 \sum_n \frac{1}{n^2 + \omega^2} \int_0^{2\pi} d\sigma \frac{\cos^4 \sigma}{2\pi^4} = -\epsilon^4 \frac{3}{8} \sum_n \frac{1}{n^2 + \omega^2}. \quad (4.7)
\]

The single fermionic field gives just half of this contribution (up to the sign).

Putting together all of the contributions of the type $\epsilon^4\text{Tr}[A^{-1}C]$ we get
\[
\tilde{\Gamma}_1 \rightarrow -\frac{T \epsilon^4}{4\pi} \int_{-\infty}^{\infty} d\omega \sum_n \left[ - \frac{7}{8} \frac{28}{n^2 + \omega^2} - \frac{1}{32} \frac{1 - 8i\omega}{(\omega + i)^2} - \frac{1}{32} \frac{1 + 8i\omega}{(\omega - i)^2} \right]. \quad (4.8)
\]

Now let us compute the contributions of the type $\frac{1}{2} \epsilon^4\text{Tr}[A^{-1}BA^{-1}B]$ in (4.1). Let us start with the decoupled fields $\beta_u$. Using the form of the $O(\epsilon^4)$ correction to the corresponding mass we get
\[
\left( \frac{\epsilon^4}{2} \text{Tr}[A^{-1}BA^{-1}B] \right)_\beta = \frac{\epsilon^4}{2} \sum_{n_1, n_2} \frac{1}{n_1^2 + \omega^2} \frac{1}{n_2^2 + \omega^2} \times 4 \int_0^{2\pi} d\sigma_1 d\sigma_2 \frac{\cos^2 \sigma_1 e^{i\sigma_1(n_1-n_2)} \cos^2 \sigma_2 e^{-i\sigma_2(n_1-n_2)}}{2\pi} \quad (4.9)
= \frac{\epsilon^4}{2} \sum_n \frac{1}{n^2 + \omega^2} \left[ \frac{1}{n^2 + \omega^2} + \frac{1}{4[(n-2)^2 + \omega^2]} + \frac{1}{4[(n+2)^2 + \omega^2]} \right]
\]

As discussed at the end of the previous section, to project out the zero mode contribution the sums over $n$ in the massless propagators should not include the $n = 0$ point. Thus the sum in (4.8) should be over all $n \neq 0$. In computing the integrals over $\sigma$ in (4.9) we have formally shifted $n$ by $\pm 2$, so the last line in the above equation should be understood as a combination of the three sums where in the first sum $n \neq 0$, in the second $n \neq 0, 2$ and in the third $n \neq 0, -2$.

The corresponding fermionic contribution is essentially $\frac{1}{4}$ of (4.8), as $\mu^2$ (without $\pm \rho^n$ part) is half of $\mu^2_{\beta}$, but here there are two mass insertions. Putting together such contributions from the decoupled bosons and the fermions we observe that they cancel each other.

Next, let us find the $\epsilon^4\text{Tr}[A^{-1}BA^{-1}B]$ type contribution of the coupled set of fluctuations. It can be written as (see (3.24),(3.25))
\[
\frac{\epsilon^4}{2} \text{Tr}[(Q^{(0)}_\omega)^{-1}Q^{(2)}_\omega(Q^{(0)}_\omega)^{-1}Q^{(2)}_\omega] \quad (4.10)
= \frac{\epsilon^4}{2} \sum_{n_1, n_2} \int_0^{2\pi} d\sigma_1 d\sigma_2 \text{Tr}[(Q^{(0)}_\omega)^{-1}(n_1)Q^{(2)}_\omega(n_2)(Q^{(0)}_\omega)^{-1}(n_2)Q^{(2)}_\omega(n_1)] e^{i(n_1-n_2)(\sigma_1-\sigma_2)}
\]

To compute this expression we again first diagonalize the propagator matrix and then integrate over $\sigma$. Putting together all the contributions from the two insertions of the $\epsilon^2$ perturbations and
adding the contribution with single $\epsilon^4$ insertion (4.8) we get the following result for the 1-loop effective action to order $\epsilon^4$ (without yet including the $\rho''$ fermionic mass term contributions)

$$
\bar{\Gamma}_1(\epsilon^4) = -\frac{T \epsilon^4}{4\pi} \int_{-\infty}^{\infty} d\omega \left\{ \sum_n \left[ -\frac{7}{8} \frac{1}{n^2 + w^2} - \frac{1}{32} \frac{1 - 8i \omega}{n^2 + (\omega + i)^2} - \frac{1}{32} \frac{1 + 8i \omega}{n^2 + (\omega - i)^2} \right] 
+ \frac{1}{2} \sum_n \left[ -\frac{\omega^2}{n^2 + (\omega + i)^2} - \frac{\omega^2}{n^2 + (\omega - i)^2} \right]
+ \frac{1}{4} \frac{1}{n^2 + w^2} \left( \frac{1}{(n-2)^2 + \omega^2} + \frac{1}{(n+2)^2 + \omega^2} \right) + \frac{1}{2} \frac{1}{[n^2 + (\omega + i)^2]} \frac{1}{n^2 + (\omega - i)^2} \right]
+ \omega^2 \left( \frac{1}{(n+1)^2 + \omega^2} + \frac{1}{(n-1)^2 + \omega^2} \right) \left( \frac{1}{n^2 + (\omega + i)^2} + \frac{1}{n^2 + (\omega - i)^2} \right)
+ \frac{(1 + i \omega)^2}{4} \frac{1}{n^2 + (\omega - i)^2}\left( \frac{1}{(n-2)^2 + (\omega + i)^2} + \frac{1}{(n+2)^2 + (\omega - i)^2} \right)
+ \frac{(1 - i \omega)^2}{4} \frac{1}{n^2 + (\omega + i)^2} \left( \frac{1}{(n-2)^2 + (\omega + i)^2} + \frac{1}{(n+2)^2 + (\omega + i)^2} \right) \right\} \right) \right) (4.11)

Again, this expression should be understood as a combination of sums over $n$ where the values of $n$ for which the effective (shifted) value of $n$ vanishes should be projected out as it came from the original $n_i$ in the propagator after doing the integral over $\sigma$ and shifting the summation index. For example, we have

$$
\sum_{n_1 \neq 0, n_2 \neq 0} \frac{1}{n_1^2 + \omega^2} \frac{1}{n_2^2 + \omega^2} \int_{0}^{2\pi} d\sigma_1 d\sigma_2 \cos 2\sigma_1 \cos 2\sigma_2 \epsilon^{(n_1, n_2)}(\sigma_1 - \sigma_2)
= \frac{1}{4} \sum_{n \neq 0, 2} \frac{1}{n^2 + \omega^2} \frac{1}{(n-2)^2 + \omega^2} + \frac{1}{4} \sum_{n \neq 0, -2} \frac{1}{n^2 + \omega^2} \frac{1}{(n+2)^2 + \omega^2}. \quad (4.12)
$$

The first three terms in (4.11) can be simplified as in (3.32) by doing separate shifts of $w$ by $\pm i$ in the last two terms; this gives

$$
-\frac{1}{32} \sum_{n \neq 0} \int_{-\infty}^{\infty} d\omega \left[ \frac{28}{n^2 + w^2} + \frac{1 - 8i \omega}{n^2 + (\omega + i)^2} + \frac{1 + 8i \omega}{n^2 + (\omega - i)^2} \right] = -\frac{7}{16} \sum_{n \neq 0} \int_{-\infty}^{\infty} d\omega \frac{1}{n^2 + \omega^2}. \quad (4.13)
$$

Similar separate shifts of $w$ under the integral $\int_{-\infty}^{\infty} d\omega$ can be used to transform some other terms in (4.11). For example, we get

$$
\frac{\omega^2}{[n^2 + (\omega + i)^2]^2} + \frac{\omega^2}{[n^2 + (\omega - i)^2]^2} \rightarrow 2 \frac{\omega^2 - 1}{(n^2 + \omega^2)^2}, \quad (4.14)
$$

$$
\frac{1}{[n^2 + (\omega + i)^2][n^2 + (\omega - i)^2]} = i \frac{1}{2\omega} \left[ \frac{1}{n^2 + (\omega + i)^2} - \frac{1}{n^2 + (\omega - i)^2} \right] \rightarrow -\frac{1}{2(\omega^2 + 1)} \frac{1}{n^2 + \omega^2}
$$

Using the identity $\frac{1}{ab} = (\frac{1}{a} - \frac{1}{b})\frac{1}{b-a}$ with $a, b$ being $(n + k)^2 + (\omega + v)^2$, $(k = 0, \pm 2, v = 0, \pm i)$
and shifting $\omega$ in terms containing only propagator factors with $(\omega \pm i)$ one finds that

$$\sum_{n\neq 0, -1} \frac{\omega^2}{[(n+1)^2 + \omega^2][n^2 + (\omega + i)^2]} + \sum_{n\neq 0, 1} \frac{\omega^2}{[(n-1)^2 + \omega^2][n^2 + (\omega + i)^2]} + \text{c.c.}$$

$$\rightarrow \sum_{n\neq 0, -1} \frac{\omega^2(n-1)}{[(n-1)^2 + \omega^2][(n-1)^2 + \omega^2]} - \sum_{n\neq 0, -1} \frac{\omega^2(n+1)}{[(n+1)^2 + \omega^2][(n+1)^2 + \omega^2]}$$

$$- \sum_{n\neq 0, -1} \frac{n - \omega^2(n+2)}{(n^2 + \omega^2)^2} - \sum_{n\neq 0, 1} \frac{-n + \omega^2(n-2)}{(n^2 + \omega^2)^2}$$

$$\rightarrow -\frac{2}{(\omega^2 + 1)^2} + \sum_{n\neq 0} \frac{4\omega^2}{(n^2 + \omega^2)^2}.$$  \hspace{1cm} (4.15)

The second line above comes from the unshifted terms, while the third line from the $\omega$-shifted terms. Let us mention that to arrive to the result in the last line in (4.15) we have assumed the prescription in which the sums over $n$ (in infinite limits) are computed before doing the integral over $\omega$ so that one is allowed to do shifts of the summation index $n$. If one would instead assume that the integral over $\omega$ (in infinite limits) is done before the evaluation of the sums the result would be different\(^1\). We shall discuss the origin of this ambiguity in Appendix C.

Performing similar shifts of $\omega$ and $n$ in the last two lines in (4.11) we get

$$\sum_{n\neq 0, -1} \frac{\omega^2}{[(n+1)^2 + \omega^2][n^2 + (\omega - i)^2]} + \sum_{n\neq 0, 1} \frac{\omega^2}{[(n-1)^2 + \omega^2][n^2 + (\omega - i)^2]} + \text{c.c.}$$

$$\rightarrow \sum_{n\neq 0, -1} \frac{\omega^2(n-1)}{[(n-1)^2 + \omega^2][(n-1)^2 + \omega^2]} - \sum_{n\neq 0, -1} \frac{\omega^2(n+1)}{[(n+1)^2 + \omega^2][(n+1)^2 + \omega^2]}$$

$$- \sum_{n\neq 0, -1} \frac{n - \omega^2(n+2)}{(n^2 + \omega^2)^2} - \sum_{n\neq 0, 1} \frac{-n + \omega^2(n-2)}{(n^2 + \omega^2)^2}$$

$$\rightarrow -\frac{2}{(\omega^2 + 1)^2} + \sum_{n\neq 0} \frac{4\omega^2}{(n^2 + \omega^2)^2}.$$  \hspace{1cm} (4.16)

where the final term should be summed over $n \neq 0, 2$.

Collecting the above expressions we get for (4.11)

$$\bar{\Gamma}_1(\epsilon^4) = -\frac{T \epsilon^4}{4\pi} \int_{-\infty}^{\infty} d\omega \left( C_0 + C_1 + C_2 + \sum_{n=3}^{\infty} S_n \right),$$  \hspace{1cm} (4.17)

where

$$C_0 = -\frac{1}{(\omega^2 + 1)^2}, \quad C_1 = \frac{7\omega^4 + 84\omega^2 + 93}{8(\omega^2 + 1)(\omega^2 + 9)}, \quad C_2 = \frac{8\omega^6 + 137\omega^4 - 89\omega^2 - 308}{8(\omega^2 + 4)(\omega^4 + 17\omega^2 + 16)}$$  \hspace{1cm} (4.18)

$$S_n = \frac{1}{8(n^2 + \omega^2)^2} \left[ 9\omega^2 - 7n^2 + 16 - \frac{2(n^2 + \omega^2)}{\omega^2 + 1} \right.$$

$$\left. - (n^2 + \omega^2)(\omega^2 - 3) \left( \frac{1}{(n+2)^2 + \omega^2} + \frac{1}{(n-2)^2 + \omega^2} \right) \right].$$  \hspace{1cm} (4.19)

\(^1\)We are grateful to M. Beccaria for pointing out this ambiguity to us.
The result is UV finite as expected \cite{3}. It is also IR finite (which would not be the case if the zero mode contributions were not properly projected out).

The integrals over $\omega$ give

\[
\int_{-\infty}^{\infty} d\omega \ C_0 = -\frac{\pi}{2}, \quad \int_{-\infty}^{\infty} d\omega \ C_1 = \frac{9\pi}{8}, \quad \int_{-\infty}^{\infty} d\omega \ C_2 = \frac{17\pi}{128}. \quad (4.20)
\]

As discussed above we need first to compute the sum and then the integral. Remarkably, the sum over $n$ of $S_n$ in (4.19) can be performed exactly and we obtain

\[
C_3(\omega) \equiv \sum_{n=3}^{\infty} S_n = \frac{\pi^2(\omega^2 + 1)\text{csch}^2 \pi \omega}{2\omega^2} + \frac{\pi(5\omega^2 + 4) \coth \pi \omega}{8\omega^3(\omega^2 + 1)} - \frac{53}{48(\omega^2 + 1)} - \frac{27}{32(\omega^2 + 4)} + \frac{3}{16(\omega^2 + 9)} + \frac{19}{96(\omega^2 + 16)} - \frac{5}{8\omega^2} - \frac{1}{4(\omega^2 + 1)^2} + \frac{6}{(\omega^2 + 4)^2} - \frac{1}{\omega^4}. \quad (4.21)
\]

To compute $\int_{-\infty}^{\infty} d\omega C_3(\omega)$ it is convenient to decompose $C_3 = C_{30} + C_{31}$ as

\[
C_{30}(\omega) = \frac{\pi^2 \text{csch}^2 \pi \omega}{2\omega^2} + \frac{\pi(5\omega^2 + 4) \coth \pi \omega}{8\omega^3(\omega^2 + 1)} - \frac{53}{48(\omega^2 + 1)} - \frac{27}{32(\omega^2 + 4)} + \frac{3}{16(\omega^2 + 9)} + \frac{19}{96(\omega^2 + 16)} - \frac{5}{8\omega^2} - \frac{1}{4(\omega^2 + 1)^2} + \frac{6}{(\omega^2 + 4)^2} - \frac{1}{\omega^4}, \quad (4.22)
\]

\[
C_{31}(\omega) = \frac{\pi^2}{2} \text{csch}^2 \pi \omega - \frac{1}{2\omega^2}. \quad (4.23)
\]

The first integral can be performed using residues theorem on a contour that includes the real axis and a semi-circular loop going to infinity in the upper half plane; the simple poles are at $\omega = in$, $n > 0$. We obtain

\[
\int_{-\infty}^{\infty} d\omega \ C_{30} = -\frac{133}{128} \pi + \pi \zeta(3). \quad (4.24)
\]

Noticing that $C_{31} = \frac{d}{d\omega}(\frac{1}{2\omega} - \frac{\pi}{2} \coth \pi \omega)$ the other integral is just

\[
\int_{-\infty}^{\infty} d\omega \ C_{31} = -\pi. \quad (4.25)
\]

Collecting these expressions we obtain the following result for (4.11)

\[
\tilde{\Gamma}_1(\epsilon^4) = \frac{T}{4} \left[ \frac{41}{32} - \zeta(3) \right] \epsilon^4. \quad (4.26)
\]

Let us now include the extra contributions due to the $\rho''$ terms in the fermionic masses in (3.8): according to (3.7) the non-trivial part of the full 1-loop fermionic contribution is (after Fourier transform in time direction)

\[
-\frac{1}{2} \ln \left( \frac{\det^4[-\partial_t^2 + \omega^2 + \rho^2 + \rho'']}{\det^4[-\partial_t^2 + \omega^2]} \right) \cdot \frac{\det^4[-\partial_t^2 + \omega^2 + \rho^2 - \rho'']}{\det^4[-\partial_t^2 + \omega^2]} \cdot \frac{\det^4[-\partial_t^2 + \omega^2 + \rho^2 - \rho'']}{\det^4[-\partial_t^2 + \omega^2]}. \quad (4.27)
\]

\footnote{We thank M. Beccaria for this observation and correcting our original result for $a_{11}$.}
Recalling that
\[
\rho'^2 = \epsilon^2 \cos^2 \sigma - \frac{\epsilon^4}{2} \cos^4 \sigma + ..., \quad \rho'' = -\epsilon \sin \sigma + \frac{3\epsilon^3}{4} \sin \sigma \cos^2 \sigma + ... ,
\]
and since the full expression is symmetric under \(\rho'' \to -\rho''\) we conclude that \(\rho''\) terms contribute only at even orders in \(\epsilon\). The extra contributions that we need to compute at order \(\epsilon^4\) are the following
\[
\ln \frac{\det[A + B\epsilon + C\epsilon^2 + D\epsilon^3 + E\epsilon^4]}{\det A} \to \epsilon^4 Tr[(A^{-1}B)^2 A^{-1}C] - \epsilon^4 Tr[A^{-1}BA^{-1}D] - \frac{\epsilon^4}{4} Tr[(A^{-1}B)^4] = \epsilon^4 (I + II + III),
\]
where in our case we have
\[
B = -\sin \sigma, \quad C = \cos^2 \sigma, \quad D = \frac{3}{4} \sin \sigma \cos^3 \sigma, \quad E = -\frac{1}{2} \cos^4 \sigma
\]
The additional contribution to (4.27) is then
\[
\delta \Gamma_1(\epsilon^4) = -\frac{T \epsilon^4}{4\pi} \int d\omega \, 8 (I + II + III).
\]
The explicit computation of these terms gives
\[
I = \frac{1}{8} \left[ \sum_{n \neq 0,1} \frac{1}{(n^2 + \omega^2)^2} \frac{1}{(n - 1)^2 + \omega^2} + \sum_{n \neq 0,-1} \frac{1}{(n^2 + \omega^2)^2} \frac{1}{(n + 1)^2 + \omega^2} \right]
- \frac{1}{16} \left[ \sum_{n \neq 0,1,2} \frac{1}{n^2 + \omega^2} \frac{1}{(n - 1)^2 + \omega^2} + \frac{1}{(n + 1)^2 + \omega^2} \right]
+ \sum_{n \neq 0,-1,-2} \frac{1}{n^2 + \omega^2} \frac{1}{(n + 1)^2 + \omega^2} \frac{1}{(n + 2)^2 + \omega^2}
= \frac{\pi (12\omega^4 + 23\omega^2 + 2)}{16\omega^3(4\omega^2 + 1)^2(\omega^2 + 1)} \coth \pi \omega + \frac{\pi^2}{8\omega^2(4\omega^2 + 1)^2} \sinh^2 \pi \omega - \frac{1}{8\omega^4(\omega^2 + 1)^2(\omega^2 + 4)},
\]
\[
II = \frac{3}{64} \left( \sum_{n \neq 0,1} \frac{1}{n^2 + \omega^2} \frac{1}{n - 1)^2 + \omega^2} + \sum_{n \neq 0,-1} \frac{1}{n^2 + \omega^2} \frac{1}{(n + 1)^2 + \omega^2} \right)
= \frac{3}{64} \left( \frac{4\pi}{\omega(1 + 4\omega^2)} \coth \pi \omega - \frac{4}{\omega^2(1 + \omega^2)} \right),
\]
and

\[
\text{III} = -\frac{1}{64} \left[ \sum_{n \neq 0, -1, 2} \frac{1}{n^2 + \omega^2} \left( \frac{1}{(n + 1)^2 + \omega^2} - \frac{1}{(n - 2)^2 + \omega^2} \right) \right.
\]
\[
+ \sum_{n \neq 0, 1, 2} \frac{1}{n^2 + \omega^2} \left( \frac{1}{(n - 1)^2 + \omega^2} - \frac{1}{(n - 2)^2 + \omega^2} \right)
\]
\[
+ \sum_{n \neq 0, 1} \frac{1}{(n^2 + \omega^2)^2} \left( \frac{1}{(n - 1)^2 + \omega^2} + 2 \sum_{n \neq 0, 1, -1} \frac{1}{(n^2 + \omega^2)^2} \left( \frac{1}{(n - 1)^2 + \omega^2} - \frac{1}{(n - 2)^2 + \omega^2} \right) \right)
\]
\[
= -\frac{\pi^2}{32} \frac{8 \omega^4 + 10 \omega^2 + 2}{4 \omega^2 (4 \omega^2 + 1)^3 (\omega^2 + 1)} \frac{1}{\sinh^2 \pi \omega} - \frac{\pi}{32} \frac{60 \omega^4 + 35 \omega^2 + 2}{4 \omega^2 (4 \omega^2 + 1)^3 (\omega^2 + 1)} \coth \pi \omega
\]
\[
+ \frac{\omega^2 + 2}{4 \omega^4 (\omega^2 + 1)^2 (\omega^2 + 4)}.
\] (4.34)

According to the prescription already used above we have performed the sums first and then the integral over \( \omega \). Collecting the three contributions we obtain for (4.31)

\[
\delta \Gamma_1(\epsilon^4) = -\frac{T \epsilon^4}{4 \pi} \int d\omega \left[ \frac{\pi^2 (8 \omega^2 + 1)}{2 (4 \omega^2 + \omega^2)^2 \sinh^2 \pi \omega} \right.
\]
\[
+ \frac{\pi (96 \omega^8 + 240 \omega^6 + 202 \omega^4 + 33 \omega^2 + 2)}{4 (\omega^2 + 1)(4 \omega^2 + \omega^2)^3} \coth \pi \omega + \frac{3 \omega^6 + 17 \omega^4 + 32 \omega^2 + 8}{2 \omega^4 (\omega^2 + 1)^2 (\omega^2 + 4)} \right] (4.35)

Performing the integral over \( \omega \) we find

\[
\delta \Gamma_1(\epsilon^4) = -\frac{T}{4} \left[ \frac{559}{72} + 6 \ln 2 + \frac{5}{2} \zeta(3) \right] \epsilon^4 (4.36)
\]

To obtain the total 1-loop energy we need to add the contributions in (4.26) and (4.36), \( \tilde{\Gamma}_1(\epsilon^4) + \delta \Gamma_1(\epsilon^4) \), and also to express \( \kappa \) and \( \epsilon \) in terms of the spin. As a result, we finally obtain the following 1-loop correction

\[
E_1 = \sqrt{2} \sqrt{3} (3 - 4 \ln 2) + \frac{1}{\sqrt{2}} \left[ \frac{-1219 + 864 \ln 2 + 432 \zeta(3)}{288} \right] S^{3/2} + O(S^{5/2}) , (4.37)
\]

which leads to the value of the coefficient \( a_{11} \) quoted in (1.4).

**Note Added**

A different calculation of the 1-loop correction to the folded string energy in the small spin limit was recently carried out, partly using numerical evaluation, by N. Gromov (private communication). It led to the same structure of the expansion of the energy (1.1) as found here but apparently with somewhat different coefficients than in (1.4). This disagreement may be related to our prescription of projecting out the zero mode contribution.
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Appendix A: Vanishing of 1-loop correction to folded string energy in flat space

Here we shall show the vanishing of the 1-loop correction to the folded string energy in flat space by using the GS formalism in the covariant $\kappa$-symmetry gauge.

In the flat space
\[ ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 \]  
the folded string solution in the conformal gauge is (here $\epsilon$ is an arbitrary constant)
\[ \bar{t} = \epsilon \tau, \quad \bar{\rho} = \epsilon \sin \sigma, \quad \bar{\phi} = \tau. \]  
Equivalently, $x_1 = \rho \cos \phi = \epsilon \sin \sigma \cos \tau, \quad x_2 = \rho \sin \phi = \epsilon \sin \sigma \sin \tau$. The bosonic part of quadratic fluctuation Lagrangian in the conformal gauge is
\[ \tilde{L}_B = \ddot{\bar{t}}^2 - \ddot{\bar{\rho}}^2 - \ddot{\bar{\phi}}^2 - \ddot{\phi}^2 (\dot{\bar{\phi}}^2 - \dot{\phi}^2) - 4 \ddot{\bar{\rho}} \dot{\bar{\phi}} \dot{\bar{\phi}} \]  
After the rescaling $\tilde{\bar{\rho}} \dot{\bar{\phi}} = \dot{\phi}$ this becomes
\[ \tilde{L} = \ddot{\bar{t}}^2 - \ddot{\bar{\rho}}^2 - \ddot{\phi}^2 - \ddot{\phi}^2 + \ddot{\phi}^2 - \ddot{\phi}^2 - 4 \ddot{\bar{\rho}} \dot{\bar{\phi}} \]  
Performing further the rotation $\tilde{\bar{\rho}} = \eta_1 \cos \tau + \eta_2 \sin \tau, \quad \tilde{\bar{\phi}} = -\eta_1 \sin \tau + \eta_2 \cos \tau$, this becomes the Lagrangian for free massless bosons
\[ \tilde{L}_B = -\partial_a \bar{t} \partial^a \bar{t} + \partial_a \eta_1 \partial^a \eta_1 + \partial_a \eta_2 \partial^a \eta_2 \]  
Starting with the quadratic part of the GS superstring action in fermions in flat space in general coordinates
\[ L_F = (\sqrt{-gg^{ab} s^{IJ} - \epsilon^{ab} s^{IJ} \bar{\theta}^I \bar{\theta}^J}) \partial_a \theta^b \theta^I \theta^J, \]  
where $s^{IJ} = \text{diag}(1,-1)$, $\partial_a = \Gamma_A E^a \partial \phi X^\mu$, $D_a = \partial_a + \frac{1}{4} \partial_a X^M \omega^A_M \Gamma_{AB}$, and fixing the $\kappa$-symmetry gauge as
\[ \theta_1 = \theta_2 = \theta \]  
we get
\[ L_F = 2i \sqrt{-gg^{ab} \bar{\theta} \theta_D \theta}. \]  
The induced metric for the above classical solution is, of course, conformally flat
\[ ds^2 = \epsilon^2 \cos^2 \sigma (-d\tau^2 + d\sigma^2). \]
Then labeling the coordinates as \((X^0, X^1, X^2) = (t, \rho, \phi)\) we get

\[
\begin{align*}
\varrho_0 &= \epsilon (\Gamma_0 + \sin \sigma \Gamma_2), \\
\rho_1 &= \epsilon \cos \sigma \Gamma_1 \\
D_0 &= \partial_0 - \frac{1}{2} \Gamma_{12}, \\
D_1 &= \partial_1.
\end{align*}
\] (A.10)

and thus

\[
\begin{align*}
L_F &= 2 \tilde{\theta} D_F \theta, \\
D_F &= i(-\varrho_0 D_0 + \varrho_1 D_1) = i \epsilon \left[ - (\Gamma_0 + \sin \sigma \Gamma_2) \partial_0 + \Gamma_1 \cos \sigma \partial_1 + \frac{1}{2} (\Gamma_{012} - \Gamma_1 \sin \sigma) \right]
\end{align*}
\] (A.11)

After the rotation

\[
\tilde{\theta} = e^{-\frac{1}{8\alpha} \alpha_3 \Gamma_2} \theta, \quad \sinh \alpha = \tan \sigma,
\] (A.12)

the fermionic operator becomes

\[
\tilde{D}_F = i \epsilon \left( - \Gamma_0 \cos \sigma \partial_0 + \Gamma_1 \cos \sigma \partial_1 - \frac{1}{2} \Gamma_1 \sin \sigma \right).
\] (A.13)

Finally, rescaling \(\tilde{\theta} = \frac{1}{\sqrt{\epsilon \cos \sigma}} \theta\) we end up with the action for \(\vartheta\) with the free massless Dirac operator

\[
D_F = i (-\Gamma_0 \partial_0 + \Gamma_1 \partial_1). \quad (A.14)
\]

Thus both the bosonic and the fermionic fluctuations decouple from the background and cannot contribute to the classical relation between the energy and the spin, \(E = \sqrt{\frac{2}{3}} S\). In fact, as is well known, the 1-loop shift of the GS superstring vacuum energy is zero because of the balance of the number of bosonic and fermionic degrees of freedom (assuming we include also the conformal gauge ghost contribution).

### Appendix B: Generalization to non-zero \(S^5\) angular momentum

The above discussion of the spinning string in \(AdS_5\) can be generalized to the case of the \((S, J)\) string which is spinning with spin \(S\) in \(AdS_5\) but also moving with momentum \(J\) around big circle in \(S^5\) [3]. This generalization is potentially important as it allows one to relate the corresponding string states to operators like \(\text{tr}(D_S^* \Phi^J)\) in the closed \(sl(2)\) sector of the SYM theory (with \(J\) having the interpretation of the length of the corresponding spin chain [14]).

The relations in section 2 have straightforward generalization to the case when the string also moves along the \(S^1\) in \(S^5\):

\[
\begin{align*}
\varphi &= \nu \tau, \\
J &= \sqrt{\lambda} \nu, \\
\rho^2 &= \kappa^2 \cosh^2 \rho - w^2 \sinh^2 \rho - \nu^2, \\
0 &\leq \rho \leq \rho_*, \\
\coth^2 \rho_* &= \frac{w^2 - \nu^2}{\kappa^2 - \nu^2} \equiv 1 + \frac{1}{\epsilon^2}, \quad \rho_* = \epsilon - \frac{1}{6} \epsilon^3 + \ldots.
\end{align*}
\] (B.1)

Here \(\nu \equiv J\) plays the role of the semiclassical \(S^5\) momentum parameter and \(\epsilon\) again measures the length of the string. To include nonzero \(\nu\) one is to shift \(w \rightarrow \sqrt{w^2 - \nu^2}, \quad \kappa \rightarrow \sqrt{\kappa^2 - \nu^2}.\)
We get (cf. (2.6), (2.7))

\[
\sqrt{\kappa^2 - \nu^2} = \epsilon^2 F_1\left(\frac{1}{2}; 1; -\epsilon^2\right), \quad \mathcal{E}_0 = \frac{\kappa}{\sqrt{\kappa^2 - \nu^2}} \epsilon^2 F_1\left(-\frac{1}{2}; 1; -\epsilon^2\right),
\]

(B.3)

\[
S = \frac{w}{\sqrt{\kappa^2 - \nu^2}} \epsilon^3 2 F_1\left(\frac{1}{2}; \frac{3}{2}; -\epsilon^2\right).
\]

(B.4)

To consider the short string limit we should expand in small $\epsilon$ while keeping $\nu$ arbitrary. Then we find

\[
\mathcal{E}_0^2 = \nu^2 + \epsilon^2 (1 + \nu^2) + \frac{\epsilon^4}{2} (1 + \nu^2) + O(\epsilon^6), \quad S^2 = \frac{\epsilon^4}{4} (1 + \nu^2) + \frac{\epsilon^6}{16} (1 - \nu^2) + O(\epsilon^8)
\]

i.e.

\[
\epsilon^2 = \frac{2S}{\sqrt{1 + \nu^2}} + O(S^2), \quad \mathcal{E}_0^2 = \nu^2 + 2S\sqrt{1 + \nu^2} + O(S^2).
\]

(B.5)

The short string limit $\epsilon \ll 1$ can thus be achieved by, e.g., considering a slowly spinning string $S \ll 1$ or by assuming large momentum in $S^5$, i.e. $\nu \gg 1$. The latter is the fast string or BMN-like limit while the former may be called a near flat space limit in which $\nu$ may be kept arbitrarily small.

Below we shall concentrate on the short string limit $\epsilon \ll 1$. If we further assume that $\epsilon \ll \nu$ then the classical energy will be

\[
\mathcal{E}_0 = \nu + \frac{S}{\nu} \sqrt{\nu^2 + 1} + O(S^2).
\]

(B.7)

If we then expand in large $\nu \gg 1$ that will correspond to the usual fast short string limit where one takes $\nu$ large at fixed $\frac{S}{\nu} = \frac{S}{\hat{\nu}}$ and then expands in $\frac{S}{\nu} \ll 1$

\[
\mathcal{E}_0 = \nu + S + \frac{S}{2\nu^2} + \ldots, \quad \nu \gg 1, \quad \frac{S}{\nu} \ll 1.
\]

(B.8)

In the slow short string limit we have $\epsilon \ll 1$, $S \ll 1$; if we assume in addition that the $S^5$ rotational energy is smaller than the spinning one, then $\nu \ll \sqrt{S} \ll 1$. In this case $\nu \ll \epsilon$ which is opposite to the above assumption that led to (B.7). Here we get $\epsilon = \sqrt{2S} - \frac{1}{4\sqrt{2}} S^{3/2} (1 + \frac{2\nu^2}{S}) + \ldots$ so that the classical energy has a “near flat space” expansion form

\[
\mathcal{E}_0 = \sqrt{2S} \left(1 + \frac{\nu^2}{4S} + \ldots\right) + \frac{3}{4\sqrt{2}} S^{3/2} \left(1 + \frac{5\nu^2}{12S} + \ldots\right) + \ldots, \quad \nu \ll \sqrt{S} \ll 1.
\]

(B.9)

The fluctuation Lagrangian will now have 4 of $S^5$ fields having mass $\nu^2$ and while the masses of the other fluctuation fields become (cf. (3.1), (3.2), (3.3)): $\mu_\eta = 2\rho^2 - \kappa^2 + \nu^2$, $\mu_\phi = 2\rho^2 - \omega^2 + \nu^2$, $\mu_\rho = 2\rho^2 - \omega^2 - \kappa^2 + 2\nu^2$, $\mu_\beta = 2\rho^2 + \nu^2$, $\mu_\psi = \sqrt{\rho^2 + \nu^2}$.

(B.10)
One can then compute the 1-loop correction to string energy by expanding in the short string limit, i.e. in \( \epsilon \ll 1 \) while keeping \( \nu \) fixed.

Expanding the masses and the coefficients in the mixing term in the fluctuation Lagrangian we get the following expression for the 1-loop effective action (cf. (3.23)–(3.27))

\[
\Gamma_1\left( \epsilon^2 \right) = - \frac{T}{4\pi} \int_{-\infty}^{\infty} d\omega \left( 8 \ln \frac{\det[\Delta_0 + \nu^2 + \epsilon^2 \cos^2 \sigma]}{\det[\Delta_0 + \nu^2]} - 2 \ln \frac{\det[\Delta_0 + \nu^2 + 2\epsilon^2 \cos^2 \sigma]}{\det[\Delta_0 + \nu^2]} \right) \\
- \ln \frac{\det[Q_\omega]}{\det[Q_\omega^{(0)}]} - \ln \frac{\det[P_\omega]}{\det[Q_\omega^{(0)}]},
\]

where now

\[
P_\omega = \begin{pmatrix}
-\Delta_0 & 0 & 0 \\
0 & \Delta_0 + \nu^2 & 0 \\
0 & 0 & \Delta_0 + \nu^2
\end{pmatrix}, \quad \Delta_0 \equiv -\partial_1^2 + \omega^2
\]

and the mixing term operator \( Q_\omega \) is given to order \( \epsilon^2 \) by the following matrix (\( i = 1, 2, 3 \))

\[
(Q_\omega)_{1i} = \{ -(\Delta_0 + \epsilon^2 \cos 2\sigma); 0; 2\epsilon w \sin \sigma \sqrt{\nu^2 + \epsilon^2} \}
\]

\[
(Q_\omega)_{2i} = \{ 0; \Delta_0 - 1 + \epsilon^2(\cos 2\sigma + \frac{1}{2}); -2\omega(1 + \frac{1}{2} \epsilon^2 \sin^2 \sigma) \sqrt{\nu^2 + 1 + \frac{1}{2} \epsilon^2} \}
\]

\[
(Q_\omega)_{3i} = \{ -2\epsilon w \sin \sigma \sqrt{\nu^2 + \epsilon^2}; 2\omega(1 + \frac{1}{2} \epsilon^2 \sin^2 \sigma) \sqrt{\nu^2 + 1 + \frac{1}{2} \epsilon^2}; \Delta_0 - 1 + \epsilon^2(\cos 2\sigma - \frac{1}{2}) \}
\]

So far we considered \( \epsilon \ll 1 \) with \( \nu \) arbitrary. Next, we may specify either to the fast short string case (\( \nu \gg \epsilon \)) or to the slow short string case (\( \nu \ll \epsilon \)). In the fast string case we get

\[
Q_\omega = Q_\omega^{(0)} + \epsilon Q_\omega^{(1)} + \epsilon^2 Q_\omega^{(2)} + \ldots
\]

where

\[
Q_\omega^{(0)} = \begin{pmatrix}
-\Delta_0 & 0 & 0 \\
0 & \Delta_0 - 1 & -2\omega \sqrt{1 + \nu^2} \\
0 & 2\omega \sqrt{1 + \nu^2} & \Delta_0 - 1
\end{pmatrix}, \quad Q_\omega^{(1)} = \begin{pmatrix}
0 & 0 & 2\omega \nu \sin \sigma \\
0 & 0 & 0 \\
-2\omega \nu \sin \sigma & 0 & 0
\end{pmatrix}
\]

\[
Q_\omega^{(2)} = \begin{pmatrix}
-\cos 2\sigma & 0 & \frac{\omega}{\sqrt{1 + \nu^2}} \left[ \frac{1}{2} + (1 + \nu^2) \sin^2 \sigma \right] \\
0 & \cos 2\sigma + \frac{1}{2} & -\frac{\omega}{\sqrt{1 + \nu^2}} \left[ \frac{1}{2} + (1 + \nu^2) \sin^2 \sigma \right] \\
0 & 0 & \cos 2\sigma - \frac{1}{2}
\end{pmatrix},
\]

We can again diagonalize the propagator matrix

\[
D_\omega^{(0)} = M^{-1}(Q_\omega^{(0)})^{-1} M = \begin{pmatrix}
-\frac{1}{n^2 + \omega^2} & 0 & 0 \\
0 & \frac{1}{n^2 + (\omega + i\nu \sqrt{1+\nu^2})^2 + \nu^2} & 0 \\
0 & 0 & \frac{1}{n^2 + (\omega - i\nu \sqrt{1+\nu^2})^2 + \nu^2}
\end{pmatrix},
\]

\[^{3}\text{Here we expanded to order } \epsilon^2 \text{ in small } \epsilon \text{ at fixed } \nu \text{ but in some terms formally kept } \epsilon^2 \text{ contributions under the square roots to allow for a smooth } \nu \rightarrow 0 \text{ limit.}\]
where $M$ is the same as in (3.30). Similarly,

$$D^{(1)}_\omega = M^{-1} Q^{(1)}_\omega M = \left( \begin{array}{ccc} 0 & \omega \nu \sin \sigma & \omega \nu \sin \sigma \\ -2\omega \nu \sin \sigma & 0 & 0 \\ -2\omega \nu \sin \sigma & 0 & 0 \end{array} \right), \quad (B.16)$$

$$D^{(2)}_\omega = M^{-1} Q^{(2)}_\omega M = \left( \begin{array}{ccc} -\cos 2\sigma & 0 & 0 \\ 0 & \cos 2\sigma + \frac{i\omega [\frac{1}{2} + (1+\nu^2)\sin^2 \sigma]}{\sqrt{1+\nu^2}} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \cos 2\sigma - \frac{i\omega [\frac{1}{2} + (1+\nu^2)\sin^2 \sigma]}{\sqrt{1+\nu^2}} \end{array} \right).$$

One can show that the last term in (B.11) vanishes. The leading term in the short-string limit of $\Gamma_1$ is of order $\epsilon^2$. To compute it we note that

$$\ln \frac{\det[-\partial^2_\omega + \omega^2 + \nu^2 + 2\epsilon^2 \cos^2 \sigma]}{\det[-\partial^2_\omega + \omega^2 + \nu^2]} \approx \epsilon^2 \sum_n \frac{2}{n^2 + \omega^2 + \nu^2} \cos^2 \sigma = \epsilon^2 \sum_n \frac{1}{n^2 + \omega^2 + \nu^2}. \quad (B.17)$$

and use the expansion

$$\ln \frac{\det[A + \epsilon B_1 + \epsilon^2 B_2]}{\det A} = \epsilon \text{Tr}[A^{-1} B_1] + \epsilon^2 \text{Tr}[A^{-1} B_2] - \frac{\epsilon^2}{2} \text{Tr}[A^{-1} B_1 A^{-1} B_1] + ... \quad (B.18)$$

in the third nontrivial term in (B.11). The order $\epsilon$ contribution vanishes. The $\epsilon^2$ terms come from $\text{Tr}[D^{(0)}_\omega D^{(2)}_\omega]$ and $\text{Tr}[D^{(0)}_\omega D^{(1)}_\omega D^{(0)}_\omega D^{(1)}_\omega]$. Summing them up we get for the $\epsilon^2$ term in the effective action

$$\Gamma_1(\epsilon^2) = \frac{T \epsilon^2}{4\pi} \int_{-\infty}^{\infty} d\omega \sum_n \left( -\frac{2}{n^2 + \omega^2 + \nu^2} + \frac{\nu^2 + 2}{2\sqrt{\nu^2 + 1}} \left[ \frac{i\omega}{n^2 + \nu^2 + (\omega + i\sqrt{\nu^2 + 1})^2} + \text{c.c.} \right] \right. \right.$$  

$$\left. - \frac{\nu^2 \omega^2}{2(n^2 + \omega^2)} \left[ \frac{1}{(n + 1)^2 + \nu^2 + (\omega + i\sqrt{\nu^2 + 1})^2} + \left( \frac{1}{(n - 1)^2 + \nu^2 + (\omega + i\sqrt{\nu^2 + 1})^2} + \text{c.c.} \right) \right] \right) \quad (B.19)$$

Performing separate shifts of $\omega$ under the integrals in various terms as discussed in sections 3 and 4 gives

$$\Gamma_1(\epsilon^2) = \frac{T \epsilon^2}{4\pi} \int_{-\infty}^{\infty} d\omega \sum_{n=-\infty}^{\infty} \nu^2 \left[ \frac{1}{n^2 + \omega^2 + \nu^2} \right. \right.$$

$$\left. + \frac{\omega^2(n - 1 - \nu^2) - (\nu^2 + 1)(n + 1 + \nu^2)}{[(n + 1)^2 + \nu^2 + \omega^2](n + 1 + \nu^2)^2 + \omega^2(\nu^2 + 1)]} \right]. \quad (B.20)$$

The sum over $n$ can be performed exactly and we get

$$\Gamma_1(\epsilon^2) = \frac{T \epsilon^2}{4\pi} \int_{-\infty}^{\infty} d\omega \frac{\pi \sin(2\pi \nu^2)}{\cos(2\pi \nu^2) - \cosh(2\pi \omega \sqrt{\nu^2 + 1})}, \quad (B.21)$$
or finally
\[ \Gamma_1 = \frac{T\epsilon^2}{4} \frac{2\nu^2 - 1}{\sqrt{\nu^2 + 1}} + O(\epsilon^4) . \] (B.22)

Recalling that \( E_1 = \frac{\Gamma_1}{\nu} \) and that in the “short fast string” limit under the consideration (i.e. \( \epsilon \ll 1, \epsilon \ll \nu \)) one has \( \kappa = \nu + \frac{\epsilon^2}{2\nu} + ... \), we finally obtain
\[ E_1 = \frac{S}{2\nu} \frac{2\nu^2 - 1}{\nu^2 + 1} + O(S^2) , \] (B.23)
where we have replaced \( \epsilon \) by \( S \) using (B.6). So far \( \nu \) here is arbitrary apart from the condition \( \nu \gg \epsilon \), i.e. \( 2S \ll \nu^2(1 + \nu^2) \), so that (B.23) is the 1-loop correction to the classical energy in (B.7).

Assuming further that \( \nu \gg 1 \) we get
\[ E_1 = \frac{S}{\nu} - \frac{3}{2\nu^3} + ... = \frac{S}{J}(1 - \frac{3}{2}\frac{\lambda}{J^2} + ...) + ... , \] (B.24)
which should be the correction to (B.8).

This expression may be compared to the 1-loop correction to the folded spinning string energy found by quantizing the \( sl(2) \) Landau-Lifshitz model in Appendix D of [15]
\[ E_1 = -\frac{S}{2\nu^3} + O(S^2) = -\frac{\lambda}{2J^2} S + O(S^2) . \] (B.25)

There one first have taken the large \( \nu \) limit with \( \frac{S}{\nu} \) kept fixed and then expanded in \( \frac{S}{\nu} \ll 1 \). Here the order of limits was different (we first expanded in \( \epsilon \) for fixed \( \nu \)) and that could be a possible reason for a disagreement between (B.24) and (B.25). To recover the standard fast string result one would need to start with the short string fluctuation operators in (B.13), where no assumption on \( \frac{S}{\nu} \) was made, use them and (B.17) without expanding in \( \epsilon \), compute the determinants needed in (B.11), then expand in large \( \nu \) with \( \frac{S}{\nu} \) kept fixed, and at the end take \( \frac{S}{\nu} \) to be small.

One can then consider the 1-loop correction in the small \( \nu \) region by taking \( \epsilon \) to zero while keeping the parameter \( x \equiv \frac{\nu}{\epsilon} \) fixed, i.e. scaling \( \nu \) to zero together with \( \epsilon \) so that \( \frac{\nu}{\sqrt{2S}} \approx x \) remains finite. We refer to the follow-up paper [16] for the details.

**Appendix C: A comment on regularization ambiguity**

Let us start with the expression in the second line in (4.15)
\[ X \equiv \sum_{n \neq 0,1} \frac{\omega^2(n - 1)}{((n - 1)^2 + \omega^2)^2} - \sum_{n \neq 0,-1} \frac{\omega^2(n + 1)}{((n + 1)^2 + \omega^2)^2} \]
\[ = \frac{2\omega^2}{(\omega^2 + 1)^2} + \sum_n R_n(\omega) , \] (C.1)

\[ \text{The presence of an unusual } \frac{S}{\nu} \text{ term in the 1-loop correction (B.24) may be an artifact of the limit of the above expansion procedure.} \]
where
\[ R_n(\omega) = \frac{\omega^2(n-1)}{(n-1)^2 + \omega^2} - \frac{\omega^2(n+1)}{(n+1)^2 + \omega^2} \] (C.2)

We need to compute \( \int d\omega \sum_n X \) and thus
\[ Y \equiv \int_{-\infty}^{\infty} d\omega \sum_{n=-\infty}^{\infty} R_n(\omega) . \] (C.3)

If we perform the sum over \( n \) first, as we did in the main text, then result is zero, i.e.
\[ Y = 0 . \] (C.4)

This can be seen right away, of course, by performing opposite \( n \)-shifts, i.e. \( n \to n \pm 1 \) in the two terms in \( R_n(\omega) \). Curiously, if instead we perform the integral over \( \omega \) first, and then do the sum we obtain
\[ Y = -2\pi . \] (C.5)

Looking in more detail at the origin of this ambiguity one discovers that it may be interpreted as a UV regularization anomaly. Indeed, if we replace the sum over \( n \) by an integral, and introduce cutoffs \( N \) and \( L \) for the integral over \( n \) and \( \omega \) respectively, we obtain
\[ Y \to Y(L, N) \equiv \int_{-L}^{L} \int_{-N}^{N} d\omega d\eta \ R_n(\omega) , \] (C.6)
\[ Y(L, N) = 2 \left[ -(N-1) \tan^{-1} \frac{L}{N-1} - (N+1) \tan^{-1} \frac{L}{N+1} \right] . \] (C.7)

In accord with the above remarks we find
\[ Y(L, \infty) = 0 , \quad Y(\infty, N) = -2\pi . \] (C.8)

More generally, we can take the limit \( N, L \to \infty \) with \( N = aL \) where \( a \) is a fixed constant. Then we get a finite result that depends on \( a \)
\[ Y(a) \equiv Y(L, aL)_{L\to\infty} = \frac{4a}{1 + a^2} - 4 \cot^{-1} a . \] (C.9)

The previous results in (C.8) correspond to the choice of \( a = \infty \) or \( a = 0 \): The two limits are now
\[ Y(a = \infty) = 0 , \quad Y(a = 0) = -2\pi . \] (C.10)

In this paper we have chosen to perform the sums first as this is is a natural prescription to dealt with the corresponding \( 2d \) functional determinants on \( R \times S^1 \).

In the absence of \( 2d \) Lorentz covariance (broken by our background and by the topology of the world sheet) it is not a priori clear which regularization should be preferred: that choice may be hidden in how one should implement the global symmetries of the superstring theory at the quantum level. One possibility is to demand that since this regularization ambiguity has
UV nature, the regularization on the world-sheet cylinder $R \times S^1$ should be the same as on $R^{1,1}$, i.e. on the infinite plane which appears in the long string limit. That would suggest that a UV cutoff should be imposed in the 2d Lorentz-invariant way, i.e. $\omega^2 + n^2 > \Lambda^2$, $\Lambda \rightarrow \infty$. Setting $\omega = p \cos \varphi$, $n = p \sin \varphi$ with $p < \Lambda$ and integrating first over $\varphi$ from 0 to $2\pi$ we get 0 (assuming analytic continuation from relevant region of large $p$). Thus we end up with the same result as in the regularization we have preferred above in the main text.

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