Low-Dimensional Pinned Distance Sets Via Spherical Averages

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Abstract
An inequality is derived for the average energies of pinned distance measures. This refines Mattila’s theorem on distance sets (Mattila in Mathematika 34:207–228, 1987) to pinned distance sets, and gives an analogue of Liu’s theorem (Liu in Geom Funct Anal 29:283–294, 2019) for pinned distance sets of dimension smaller than 1.

Keywords
Hausdorff dimension · Fourier transform

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1 Introduction
For \( n \geq 2 \) let \( d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty) \) be the Euclidean distance function \( (x, y) \mapsto |x - y| \), and for fixed \( x \in \mathbb{R}^n \) let \( d_x : \mathbb{R}^n \rightarrow [0, \infty) \) be the pinned distance function \( d_x(y) = d(x, y) \). For finite compactly supported Borel measures \( \mu \) and \( \nu \) on \( \mathbb{R}^n \), define \( \Delta(\mu, \nu) \) by

\[
\int \phi(s) d\Delta(\mu, \nu)(s) = \int s^{-(n-1)/2} \phi(s) d\#(\mu, \nu)(s)
= \int \int |x - y|^{-(n-1)/2} \phi(|x - y|) \, d\mu(x) \, d\nu(y),
\]

for all non-negative Borel measurable \( \phi : \mathbb{R} \rightarrow [0, \infty] \). Let

\[
\sigma(\mu, \nu)(r) = \int_{S^{n-1}} \frac{\mu(r\xi)}{\nu(r\xi)} d\sigma(\xi), \quad r \geq 0,
\]

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where $\sigma$ is the surface measure on the sphere $S^{n-1}$, and

$$\hat{\mu}(\xi) = \int e^{-2\pi i \langle x, \xi \rangle} d\mu(x).$$

The aim of this work is to give a sufficient condition for the inequality

$$\int It(\Delta(v, \delta_x)) d\mu(x) \lesssim_{t, \alpha, \gamma} c_\alpha(\mu) I_\gamma(v),$$

in terms of $t$, $\alpha$ and $\gamma$, via the $L^2$ spherical averages of $\hat{\mu}$. Here,

$$c_\alpha(\mu) = \sup_{x \in \mathbb{R}^n} \frac{\mu(B(x, r))}{r^\alpha},$$

the mutual $t$-energy of $\mu$ and $v$ is defined by

$$I_t(\mu, v) = \int \int |x - y|^{-t} d\mu(x) dv(y),$$

and $I_t(\mu) := I_t(\mu, \mu)$. The main result, Theorem 2.1, refines Mattila’s theorem (see [8, Theorems 4.16 and 4.17] or [9, Proposition 15.2]) to pinned distance measures and is the Hausdorff dimension analogue to Liu’s theorem [6], which gives a sufficient condition for the support of some $d_x#(\mu)$ to have positive Lebesgue measure. The proof consists of augmenting Mattila’s proof with a “Cauchy–Schwarz reversal” technique (also used in [6]). I do not know if the proof from [6] can be directly extended to lower-dimensional pinned distance sets. The two main tools used in [6] are Lemma 1.1 (stated below) and Liu’s identity [6, Theorem 1.9]:

$$\int_0^\infty |(\sigma_t \ast f)(x)|^2 t^{n-1} dt = \int_0^\infty |(\sigma_r \ast f)(x)|^2 r^{n-1} dr, \quad x \in \mathbb{R}^n,$$

for Schwartz $f$ on $\mathbb{R}^n$, where $\sigma_r$ is the pushforward of $\sigma$ under the map $\xi \mapsto r\xi$. One key part of the proof of Mattila’s theorem is the identity [8, Theorem 4.6]

$$\int_0^\infty \Delta(f, g)(t) \Delta(h, k)(t) dt = c_n \int_0^\infty \Sigma(f, g)(r) \Sigma(h, k)(r) dr,$$

for Schwartz $f, g, h, k$ on $\mathbb{R}^n$, where

$$\Sigma(f, g)(r) = r^{(n-1)/2} \sigma(f, g)(r), \quad r \geq 0.$$
in Mattila’s original proof should result in a pinned distance version, and this is the main idea guiding the proof of Theorem 2.1 below.

The case $\mu = \nu$ in (1.1) is related to the distance set problem, which asks whether the condition $\dim A > n/2$ for $A \subseteq \mathbb{R}^n$ implies that $d(A \times A)$ has positive Lebesgue measure. This is still open, as is the stronger pinned version, which asks whether $\dim A > n/2$ implies the existence of some $x \in A$ such that $d_x(A)$ has positive measure.

Throughout, given $\alpha \in [0, n]$ let $\beta(\alpha, S^{n-1})$ be the supremum over all $\beta \geq 0$ such that for all Borel measures $\mu$ with $\text{supp} \mu \subseteq B(0, 1)$,

$$\int |\hat{\mu}(r\xi)|^2 \, d\sigma(\xi) \leq \beta \, r^{-\beta} I_\alpha(\mu), \quad \forall r > 0.$$ 

The following “Cauchy–Schwarz reversal” lemma will be needed. In [9, p. 197] it is attributed to [1, Lemma C.1]; the version below is identical to Lemma 3.2 from [6].

**Lemma 1.1** Let $\mu$ be a Borel measure supported in the unit ball of $\mathbb{R}^n$. Then for any $\alpha \in [0, n]$ and any $\epsilon > 0$,

$$\int |(\hat{\sigma} \ast f)(x)|^2 \, d\mu(x) \lesssim_{\alpha, \epsilon} c_\alpha(\mu)r^{\epsilon-\beta(\alpha, S^{n-1})} \int |\hat{f}(r\xi)|^2 \, d\sigma(\xi), \quad \forall r \geq 1,$n

for all Schwartz $f$.

## 2 Average Energies of Pinned Distance Measures

The following theorem is the main result.

**Theorem 2.1** Let $n \geq 2$. If $\alpha, \gamma \in [0, n]$ and

$$0 < t < \gamma + \beta(\alpha, S^{n-1}) - n + 1 \leq 1, \quad (2.1)$$

then for any Borel measures $\mu$ and $\nu$ supported in the unit ball of $\mathbb{R}^n$,

$$\int I_t(d_\#(\nu, \delta_x)) \, d\mu(x) \lesssim_{t, \alpha, \gamma} c_\alpha(\mu)I_\gamma(\nu), \quad \gamma \leq (n-1)/2,$$

$$\int I_t(\Delta(\nu, \delta_x)) \, d\mu(x) \lesssim_{t, \alpha, \gamma} c_\alpha(\mu)I_\gamma(\nu), \quad \gamma > (n-1)/2. \quad (2.2)$$

One corollary of Theorem 2.1 is that for any Borel set $A$ with $(n-1)/2 < \dim A \leq n/2$,

$$\forall \epsilon > 0 \exists x \in A : \dim d_x(A) \geq \dim A - \frac{(n-1)}{2} - \epsilon. \quad (2.3)$$

This is originally due to Oberlin-Oberlin [10] and follows from (2.2), Frostman’s lemma and the lower bound $\beta(\alpha, S^{n-1}) \geq (n-1)/2$ for $\alpha \geq (n-1)/2$ [8].
lower bound in (2.3) for the full distance set was proved earlier by Falconer [5]. For 
\(n \in \{2, 3\}\), Shmerkin’s bound from [12] gives an improvement over (2.3), and therefore
(2.3) is not sharp for 
\(n \in \{2, 3\}\) and \((n - 1)/2 < \dim A \leq n/2\).

For \(\dim A > n/2\), another corollary of Theorem 2.1 is that

\[\forall \epsilon > 0 \exists x \in A : \dim d_x(A) \geq \min \left\{ 1, 2 \dim A - \frac{\dim A}{n} - (n - 1) \right\} - \epsilon, \quad (2.4)\]

which follows from (2.2) and the following inequality from [3]:

\[\beta(\alpha, S^{n-1}) \geq \frac{\alpha(n - 1)}{n}, \quad n/2 \leq \alpha \leq n.\]

When \(n = 2\), (2.4) is weaker than the combined results of [7,11,12] for all values of \(\dim A\). For even \(n \geq 4\), (2.4) is weaker than what would likely follow from the methods in [4,7]. For odd \(n \geq 3\), and

\[\frac{n}{2} < \dim A \leq \frac{n}{2} + \frac{1}{4} + \frac{1}{8n - 4},\]

(2.4) is new (as far as I am aware), with the exception that if \(n = 3\) and \(\dim A\) is sufficiently close to \(3/2\), the bound from [12] is better than (2.4) by a small absolute constant.

**Proof of Theorem 2.1** Assume that \(\gamma > (n - 1)/2\); the proof for the case \(\gamma \leq (n - 1)/2\) is virtually identical with one simplification. By scaling it may be assumed that \(\mu\) and \(\nu\) are probability measures. Let \(f\) and \(g\) be non-negative smooth compactly supported functions on \(\mathbb{R}^n\), and abbreviate \(\Delta(f, g) = \Delta\), which is a finite measure by [8, Lemma 4.3]. Then

\[I_t(\Delta) \lesssim |I_t(F)| + |I_t(\Delta, K)| \quad ([8, p. 221]). \quad (2.5)\]

The functions \(F, K\) and corresponding quantities \(I_t(F), I_t(\Delta, K)\) from [8] will not be redefined here; all that will be needed is that \(F\) satisfies

\[|I_t(F)| \lesssim \int_0^\infty r^{t+n-2} |\sigma(f, g)(r)|^2 \, dr \quad ([8, p. 221])\]

\[\lesssim \|f\|^2_1 \|g\|^2_1 + \int_1^\infty r^{t+n-2} |\sigma(f, g)(r)|^2 \, dr, \quad (2.6)\]

and that \(K\) satisfies both

\[|I_t(\Delta, K)| = \left| \int_0^\infty \int_0^\infty \Delta(s) K(x) |s - x|^{-t} \, ds \, dx \right| \quad ([8, p. 221]), \quad (2.7)\]

and

\[K(x) = x^{1/2} \int_0^\infty r^{n/2} R(rx) \sigma(f, g)(r) \, dr, \quad x > 0, \quad ([8, Lemma 4.14]), \quad (2.8)\]
where \( R : (0, \infty) \to \mathbb{C} \) is some Borel function with

\[
|R(x)| \lesssim \min \left\{ x^{-1/2}, x^{-3/2} \right\}, \quad \forall x > 0 \quad ([8, \text{Lemma 4.11}]). \tag{2.9}
\]

Interpolating the two bounds in (2.9) gives

\[
|R(x)| \lesssim x^{(t-3)/2}, \quad \forall x > 0.
\tag{2.10}
\]

Let \( \epsilon > 0 \) be small, to be chosen later. Using (2.7), (2.8) and then (2.10) gives

\[
|I_t(\Delta, K)| = \left| \int_0^\infty \Delta(s) \int_0^\infty r^{n/2} \sigma(f, g)(r) \int_0^\infty |s-x|^{-t} x^{1/2} R(r, x) \, dx \, dr \, ds \right|
\leq \int_0^\infty \Delta(s) \int_0^\infty r^{(n+t-3)/2} |\sigma(f, g)(r)| \int_0^\infty |s-x|^{-t} x^{(t-3)/2} \, dx \, dr \, ds
\sim \left( \int_0^\infty s^{-t/2} \Delta(s) \, ds \right) \cdot \left( \int_0^\infty r^{(n+t-3)/2} |\sigma(f, g)(r)| \, dr \right)
\leq \epsilon \|f\|_1^2 \|g\|_1^2 + \int_1^\infty r^{n+t-2+\epsilon} |\sigma(f, g)(r)|^2 \, dr.
\tag{2.13}
\]

To get from (2.11) to (2.12), the term \( \int_0^\infty s^{-t/2} \Delta(s) \, ds \) is equal to \( I_{(n+t-1)/2}(f, g) \) by definition, which equals a constant multiple of \( \int_0^\infty r^{(n+t-3)/2} \sigma(f, g)(r) \, dr \) by polar coordinates and the Fourier formula for the mutual energy [8, Eq. 3.5].

Substituting (2.6) and (2.13) into (2.5) gives

\[
I_t(\Delta(f, g)) \lesssim \epsilon \|f\|_1^2 \|g\|_1^2 + \int_1^\infty r^{n+t-2+\epsilon} |\sigma(f, g)(r)|^2 \, dr,
\tag{2.14}
\]

for any smooth, non-negative compactly supported functions \( f \) and \( g \). For each integer \( j \geq 1 \) let \( \phi_j(z) = j^n \phi(jz) \), where \( \phi \) is a non-negative radial bump function on \( \mathbb{R}^n \) which is compactly supported in the unit ball and satisfies \( \int \phi = 1 \). For fixed \( x \in \text{supp} \mu \), taking \( f = f_j = \nu * \phi_j \) and \( g = g_{j,x} = \delta_x * \phi_j \) in (2.14) gives

\[
I_t(\Delta(v, \delta_x)) \leq \lim_{j \to \infty} \inf I_t(\Delta(f_j, g_{j,x}))
\leq \epsilon + \lim_{j \to \infty} \int_1^\infty r^{n+t-2+\epsilon} |\sigma(f_j, g_{j,x})(r)|^2 \, dr.
\tag{2.15}
\]

The first inequality in (2.15) is justified by the following argument. If \( v_m := \nu \upharpoonright A_m \) where \( A_m = \{ y \in \text{supp} \nu : \int |y-z|^{-\gamma} \, dv(z) \leq m \} \), then \( c_{\gamma}(v_m) < \infty \) (see [9, p. 20]), so by using

\[
I_t(\Delta(v, \delta_x)) = \lim_{m \to \infty} I_t(\Delta(v_m, \delta_x)), \quad I_t(\Delta(v_m * \phi_j, g_{j,x})) \leq I_t(\Delta(f_j, g_{j,x})),
\]

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it may be assumed in proving the first part of (2.15) that $c_Y (v) < \infty$. By [2, Theorem 2.8], $f_j \times g_{j,x} \overset{\ast}{\rightharpoonup} v \times \delta_x$, and hence $d_\#(f_j \times g_{j,x}) \overset{\ast}{\rightharpoonup} d_\#(v \times \delta_x)$. Using the definition of $\Delta$, Fubini’s theorem, and the assumption that $c_Y (v) < \infty$, gives

$$\Delta\left( f_j \times g_{j,x} \right) [0, \varepsilon] \lesssim c_Y (v) \varepsilon^{\gamma - (n - 1)/2}, \quad \forall \varepsilon > 0,$$

(2.16)

and similarly

$$\Delta\left( v \times \delta_x \right) [0, \varepsilon] \lesssim c_Y (v) \varepsilon^{\gamma - (n - 1)/2}, \quad \forall \varepsilon > 0.$$

(2.17)

The condition $d_\#(f_j \times g_{j,x}) \overset{\ast}{\rightharpoonup} d_\#(v \times \delta_x)$ combined with (2.16) and (2.17) yields $\Delta(f_j \times g_{j,x}) \overset{\ast}{\rightharpoonup} \Delta(v \times \delta_x)$. Therefore the Fourier transform of $\Delta(f_j, g_{j,x})$ converges pointwise to the Fourier transform of $\Delta(v, \delta_x)$. Fatou’s lemma and the Fourier formula for energy ([8, Eq. 3.5]) then give the first part of (2.15).

Integrating the outer parts of (2.15) with respect to $\mu$ gives

$$\int I_t(\Delta(v, \delta_x)) \, d\mu(x) \lesssim \varepsilon + \liminf_{j \to \infty} \int_1^\infty \int r^{n+t-2+\varepsilon} |\sigma(f_j, g_{j,x}) (r)|^2 \, d\mu(x) \, dr.$$

(2.18)

The inequality $|\sigma(f_j, g_{j,x}) (r)| \leq |\sigma(v, \delta_x) (r)|$ holds for every $r > 0$ and $x \in \text{supp} \mu$, since $\phi$ is a radial bump function which integrates to 1. Moreover, $\sigma(f_j, g_{j,x}) (r) \to \sigma(v, \delta_x)(r)$ as $j \to \infty$, pointwise for every $r > 0$ and $x \in \text{supp} \mu$. Therefore, applying dominated convergence and then Lemma 1.1 to (2.18) gives

$$\int I_t(\Delta(v, \delta_x)) \, d\mu(x) \lesssim \varepsilon + \int_1^\infty \int r^{n+t-2+\varepsilon} |\sigma(v, \delta_x)(r)|^2 \, d\mu(x) \, dr$$

$$= 1 + \int_1^\infty \int r^{n+t-2+\varepsilon} |(\hat{\sigma}* \nu)(x)|^2 \, d\mu(x) \, dr$$

$$\lesssim \varepsilon + c_\alpha(\mu) \int_1^\infty r^{n+t-2+2\varepsilon - \beta(\alpha, S^{n-1})} \int |\vartheta(r \xi)|^2 \, d\sigma(\xi) \, dr$$

$$\lesssim c_\alpha(\mu) I_t(v),$$

by (2.1), provided $\varepsilon$ is small enough, again using polar coordinates and the Fourier formula for the energy ([8, Eq. 3.5]). This proves (2.2) when $\gamma > (n - 1)/2$. The only adjustment required in the case $\gamma \leq (n - 1)/2$ is that the first part of (2.15) must be replaced by $I_t(d_\#(v, \delta_x)) \lesssim \liminf_{j \to \infty} I_t\left( \Delta(f_j, g_{j,x}) \right)$.

\[\square\]

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