GLOBAL ROUGH SOLUTION FOR $L^2$-CRITICAL SEMILINEAR HEAT EQUATION IN THE NEGATIVE SOBOLEV SPACE

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Abstract. In this paper, we consider the Cauchy global problem for the $L^2$-critical semilinear heat equations
\[ \partial_t h = \Delta h \pm |h|^{4/d} h, \]
with $h(0, x) = h_0$, where $h$ is an unknown real function defined on $\mathbb{R}^+ \times \mathbb{R}^d$. In most of the studies on this subject, the initial data $h_0$ belongs to Lebesgue spaces $L^p(\mathbb{R}^d)$ for some $p \geq 2$ or to subcritical Sobolev space $H^s(\mathbb{R}^d)$ with $s > 0$. We here prove that there exists some positive constant $\varepsilon_0$ depending on $d$, such that the Cauchy problem is locally and globally well-posed for any initial data $h_0$ which is radial, supported away from origin and in the negative Sobolev space $\dot{H}^{-\varepsilon_0}(\mathbb{R}^d)$ including $L^p(\mathbb{R}^d)$ with certain $p < 2$ as subspace. Furthermore, unconditional uniqueness, and $L^2$-estimate both as time $t \to 0$ and $t \to +\infty$ were considered.

1. Introduction

Consider the initial value problem for a semilinear heat equation:
\[ \begin{cases} \partial_t h = \Delta h \pm |h|^\gamma h, \\ h(0, x) = h_0(x), \end{cases} \tag{1.1} \]
where $h(t, x)$ is an unknown real function defined on $\mathbb{R}^+ \times \mathbb{R}^d$, $d \geq 2$, $\gamma > 1$. The positive sign “+” in nonlinear term of (1.1) denotes focusing source, and the negative sign “−” denotes the defocusing one. The Cauchy problem (1.1) has been extensively studied in Lebesgue space $L^p(\mathbb{R}^d)$ by many peoples, see e.g. \cite{2,3,4,6,7,10,12,13,14,15,16,18,19,21,25,26} and so on. The equation enjoys an interesting property of scaling invariance

\[ h_\lambda(t, x) := \lambda^{2/(\gamma-1)} h(\lambda^2 t, \lambda x), \quad h_\lambda(0, x) := \lambda^{2/(\gamma-1)} h_0(\lambda x), \quad \lambda > 0, \]

that is, if $h(t, x)$ is the solution of heat equation (1.1), then $h_\lambda(t, x)$ also does with the scaling data $\lambda^{2/\gamma} h_0(\lambda x)$. An important fact is that Lebesgue space $L^{p_c}(\mathbb{R}^d)$ with $p_c = \frac{d(\gamma-1)}{2}$ is the only one invariant under the same scaling transform:

\[ h_0(x) \mapsto \lambda^{2/(\gamma-1)} h_0(\lambda x). \]

If we consider the initial data $h_0 \in L^p(\mathbb{R}^d)$, then the scaling index

\[ p_c = \frac{d(\gamma - 1)}{2} \]

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plays a critical role on the local/global well-posedness of (1.1). Roughly speaking, one can divide the dynamics of (1.1) into the following three different regimes: (A) the subcritical case $p > p_c$, (B) the critical case $p = p_c$, (C) the supercritical case $p < p_c$. Specifically, in cases (A) and (B), i.e. $p \geq p_c$, when $p > \gamma$, Weissler in [25] proved the local existence and uniqueness of solution $h \in C([0,T); L^q(\mathbb{R}^d)) \cap L^{\infty}_{loc}((0,T]; L^\infty(\mathbb{R}^d))$. Later, Brezis and Cazenave [2] proved the unconditional uniqueness of Weissler’s solution. In double critical case $p = p_c = \gamma$ (i.e. $p = \gamma = \frac{d}{d-2}$), the local conditional wellposedness of the problem (1.1) was due to Weissler in [26], but the unconditional uniqueness fails, see Ni-Sacks [16], Terraneo [22]. In the supercritical case (C), i.e. $p < p_c$, it seems that there exists no local solution in any reasonable sense for some initial data $h_0 \in L^p(\mathbb{R}^d)$. In particular, in focusing case, there exists a nonnegative function $h_0 \in L^p(\mathbb{R}^d)$ such that the (1.1) does not admit any nonnegative classical $L^p$-solution in $[0,T)$ for any $T > 0$, see e.g. Brezis and Cabré [1], Brezis and Cazenave [2], Haraux-Weissler [9] and Weissler [25, 26]. Also, one see book Quitnner-Souplet [17] for many related topics and references.

In this paper, we mainly concerned with the local and global existence of solution for some supercritical initial data $h_0 \in L^p(\mathbb{R}^d)$ by $p < p_c$ and more generally, initial data in $\dot{H}^{-\epsilon}$. For simplicity, we only consider the Cauchy problem for the $L^2$-critical semilinear heat equations,

$$
\begin{align*}
\partial_t h &= \Delta h + \mu |h|^\frac{4}{d} h, \\
h(0, x) &= h_0(x), \quad x \in \mathbb{R}^d,
\end{align*}
$$

That is, $p_c = 2$ (i.e. $\gamma = 1 + \frac{4}{d}$), we will prove that there exists some positive constant $\epsilon_0$ depending on $d$, such that the Cauchy problem is locally and globally wellposed for any initial data $h_0$ is radial, supported away from origin and in the negative Sobolev space $\dot{H}^{-\epsilon_0}(\mathbb{R}^d)$, which includes certain $L^p$-space with $p < p_c = 2$ as a subspace (see Remark 1.1 below). We remark that, at present the the range of $\epsilon_0$ in the following theorem may not be optimal to local and global existence of solution of the problem (1.2). On the other hand, we also mention that a result in Brezis and Freidman [3] implies that the problem (1.2) has no any solution (even weak one) with a Dirac initial data $\delta$, which is in $H^{-\epsilon}(\mathbb{R}^d)$ for any $s > d/2$.

**Theorem 1.1.** Let $\mu = \pm 1$ and

$$
\epsilon_0 \in \left[0, \frac{d-1}{d+2}\right), \quad d \geq 2.
$$

(1.3)
Suppose that \( h_0 \in \dot{H}^{-\varepsilon_0}(\mathbb{R}^d) \) is a radial initial data satisfying \( \text{supp } h_0 \subset \{ x : |x| \geq 1 \} \). Then there exists a time \( \delta = \delta(h_0) > 0 \) and a unique strong solution
\[
h \in C([0, \delta); L^2(\mathbb{R}^d) + \dot{H}^{-\varepsilon_0}(\mathbb{R}^d)) \cap L^{\frac{2(2+d)}{d}}_{\text{loc}}([0, \delta] \times \mathbb{R}^d)
\]
to the equation (1.2) with the initial data \( h_0 \). Moreover, the following two statements hold:

1. If \( d > 4 \), then the solution \( h \) is unique in the following sense that there exists a unique function \( w \) in \( C([0, \delta], L^2(\mathbb{R}^d)) \) such that
\[
h = e^{t\Delta}h_0 + w.
\]

2. If \( \|h_0\|_{\dot{H}^{-\varepsilon_0}(\mathbb{R}^d)} \) is small enough, then the solution is global in time and satisfies the following decay estimate for \( d \geq 4 \),
\[
\|h(t)\|_{L^2} \lesssim t^{-\frac{d}{2}}\|h_0\|_{\dot{H}^{-\varepsilon_0}}, \quad t > 0.
\]

Remark 1.1. If \( h_0 \in L^p \) for some \( p < 2 \), then there exists some \( \varepsilon_0 > 0 \) such that \( h_0 \in \dot{H}^{-\varepsilon_0}(\mathbb{R}^d) \) and
\[
\|h_0\|_{\dot{H}^{-\varepsilon_0}(\mathbb{R}^d)} \lesssim \|h_0\|_{L^p(\mathbb{R}^d)}
\]
by the Sobolev embedding estimate (see e.g. Lemma 3.1 below). Thus, Theorem 1.1 shows that the solution \( h \) of the equation (1.2) exists locally for any radial and supported away from zero initial datum \( h_0 \) in \( L^p(\mathbb{R}^d) \) as \( p \in \left( \frac{d^2+4d-2}{2d^2+2d}, 2 \right) \) and \( d \geq 2 \).

Remark 1.2. It seems that the restriction \( d > 4 \) is necessary for unconditional uniqueness. In fact, when \( d = 4 \), the uniqueness problem is related to the “double critical” case (i.e. \( p = p_c = \gamma = \frac{d}{d-2} = 2 \)). It was well-known that the unconditional uniqueness failed by Ni-Sacks [16] and Brezis and Cazenave [2].

Finally, it is worth mentioning that in the defocusing case, the smallness restriction on the initial datum in the statement (2) is not necessary for global existence. Indeed, we have \( h(\delta) \in L^2(\mathbb{R}^d) \), then it follows by considering the solution from \( t = \delta \). Moreover, it is easy to find a large class of \( h_0 \) satisfying the conditions of theorem above. As described in Remark 1.1 our result shows that the solution \( h \) of the equation (1.2) exists globally on \( \mathbb{R}^+ \), for any the initial datum \( h_0 \) in \( L^p(\mathbb{R}^d) \) with some \( p < 2 \), which is radial and supported away from zero.

The paper is organized as follows: In Section 2, we will list several useful lemmas about Littlewood-Paley theory, and space-time estimates for the solution of linear heat equation. Then in Section 3, we will give the proof of the main results, respectively.
2. Preliminary

2.1. Littlewood-Paley multipliers and related inequalities. Throughout this paper, we write \( A \lesssim B \) to signify that there exists a constant \( c \) such that \( A \leq cB \), while we denote \( A \sim B \) when \( A \lesssim B \lesssim A \). We first define the Littlewood-Paley projection multiplier. Let \( \phi(\xi) \) be a fixed real-valued radially symmetric bump function adapted to the ball \( \{ \xi \in \mathbb{R}^d : |\xi| \leq 2 \} \) which equals 1 on the ball \( \{ \xi \in \mathbb{R}^d : |\xi| \leq 1 \} \). Define a dyadic number to any number \( N \in 2^\mathbb{Z} \) of the form \( N = 2^j \) where \( j \in \mathbb{Z} \) (the integer set). For each dyadic number \( N \), we define the the Fourier multipliers

\[
\hat{P}_{\leq N}f(\xi) := \phi(\xi/N) \hat{f}(\xi), \quad \hat{P}_N f(\xi) := \phi(\xi/N) - \phi(2\xi/N) \hat{f}(\xi),
\]

where \( \hat{f} \) denotes the Fourier transform of \( f \). Moreover, define \( P_{>N} = I - P_{\leq N} \) and \( P_{<N} = P_{\leq N} - P_N \), etc. In particular, we have the telescoping expansion:

\[
P_{\leq N} = \sum_{M \leq N} P_M f; \quad P_{>N} = \sum_{M > N} P_M f
\]

where \( M \) ranges over dyadic numbers. It was well-known that the Littlewood-Paley operators satisfy the following useful Bernstein inequalities with \( s > 0 \) and \( 1 \leq p \leq q \leq \infty \) (see e.g. Tao [23]):

\[
\| P_{\geq N} f \|_{L^p_\xi(\mathbb{R}^d)} \lesssim N^{-s} \| \nabla |^s P_{\geq N} f \|_{L^p_\xi(\mathbb{R}^d)}, \quad \| \nabla |^s P_{\leq N} f \|_{L^p_\xi(\mathbb{R}^d)} \lesssim N^s \| P_{\leq N} f \|_{L^p_\xi(\mathbb{R}^d)};
\]

\[
\| \nabla |^s P_{\leq N} f \|_{L^p_\xi(\mathbb{R}^d)} \sim N^{\pm s} \| P_{\leq N} f \|_{L^p_\xi(\mathbb{R}^d)};
\]

\[
\| P_N f \|_{L^q(\mathbb{R}^d)} \lesssim N^{\left(\frac{d}{q} - \frac{d}{p}\right)} \| f \|_{L^p_\xi(\mathbb{R}^d)}, \quad \| P_{\leq N} f \|_{L^q(\mathbb{R}^d)} \lesssim N^{\left(\frac{d}{p} - \frac{d}{q}\right)} \| f \|_{L^p_\xi(\mathbb{R}^d)};
\]

Moreover, we also have the following mismatch estimate, see e.g. [11].

**Lemma 2.1** (Mismatch estimates). Let \( \phi_1 \) and \( \phi_2 \) be smooth functions obeying

\[
|\phi_j| \leq 1 \quad \text{and} \quad \text{dist}(\text{supp} \phi_1, \text{supp} \phi_2) \geq A,
\]

for some large constant \( A \). Then for \( m > 0 \), \( N \geq 1 \) and \( 1 \leq p \leq q \leq \infty \),

\[
\| \phi_1 P_{\leq N} (\phi_2 f) \|_{L^q_\xi(\mathbb{R}^d)} = \| \phi_1 P_{\geq N} (\phi_2 f) \|_{L^q_\xi(\mathbb{R}^d)} \lesssim_m A^{-m+\frac{d}{p} - \frac{d}{q}} N^{-m} \| \phi_2 f \|_{L^p_\xi(\mathbb{R}^d)}.
\]

2.2. Space-time estimates of linear heat equation. Let \( e^{t\Delta} \) denote the heat semigroup on \( \mathbb{R}^d \). Then for suitable function \( f \), \( e^{t\Delta} f \) solves the linear heat equation

\[
\partial_t h = \Delta h, \quad h(0, x) = f(x), \quad t > 0, \quad x \in \mathbb{R}^d,
\]

and the solution satisfies the following fundamental space-time estimates:
Lemma 2.2. Let \( f \in L^p(\mathbb{R}^d) \) for \( 1 \leq p \leq \infty \), then
\[
\| e^{t\Delta} f \|_{L_t^\infty L_x^p(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim \| f \|_{L^p(\mathbb{R}^d)}. \tag{2.1}
\]
Moreover, let \( I \subset \mathbb{R}^+ \), then for \( f \in L^2(\mathbb{R}^d) \) and \( F \in L_{tx}^{\frac{2(2+d)}{d}}(\mathbb{R}^+ \times \mathbb{R}^d) \),
\[
\| \nabla e^{t\Delta} f \|_{L_t^1 L_x^2(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim \| f \|_{L^2(\mathbb{R}^d)}; \tag{2.2}
\]
\[
\| e^{t\Delta} f \|_{L_{tx}^{\frac{2(2+d)}{d}}(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim \| f \|_{L^2(\mathbb{R}^d)}; \tag{2.3}
\]
\[
\left\| \int_0^t e^{(t-s)\Delta} F(s) \, ds \right\|_{L_\infty L_x^2 \cap L_{tx}^2(\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \| F \|_{L_{tx}^{\frac{2(2+d)}{d}}(\mathbb{R}^+ \times \mathbb{R}^d)}. \tag{2.4}
\]

We can give some remarks on the inequalities (2.1) – (2.4) above as follows:

(i). The estimate (2.1) is classical and immediately follows from the Younger inequality by the following heat kernel integral:
\[
(e^{t\Delta} f)(x) = (4\pi t)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y) \, dy, \quad t > 0.
\]
More generally, for all \( 1 \leq p \leq q \leq \infty \), the following (decay) estimates hold:
\[
\| e^{t\Delta} f \|_{L^q(\mathbb{R}^d)} \lesssim t^{\frac{d}{2} - \frac{d}{q}} \| f \|_{L^p(\mathbb{R}^d)}, \quad t > 0. \tag{2.5}
\]

(ii). The estimate (2.2) is equivalent to a kind of square-function inequality on \( L^2(\mathbb{R}^d) \), which can be reformulated as
\[
\left\| \left( \int_0^\infty \| \nabla e^{t\Delta} f \|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^d)} \lesssim \| f \|_{L^2(\mathbb{R}^d)},
\]
which follows directly by the Plancherel’s theorem, and also holds in the \( L^p(\mathbb{R}^d) \) for \( 1 < p < \infty \) (see e.g. Stein[20, p. 27-46]).

(iii). The estimate (2.3) can be obtained by interpolation between the (2.1) and (2.2):
\[
\| e^{t\Delta} f \|_{L_{tx}^{\frac{2(2+d)}{d}}(\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \| e^{t\Delta} f \|_{L_t^\infty L_x^2(\mathbb{R}^+ \times \mathbb{R}^d)} \left\| \nabla e^{t\Delta} f \right\|_{L_{tx}^2(\mathbb{R}^+ \times \mathbb{R}^d)}.
\]

(iv). The estimate (2.4) consists of the three same type inequalities with the different norms \( L_t^\infty L_x^2, L_{tx}^{\frac{2(2+d)}{d}} \) and \( L_t^1 \dot{H}_x^1 \) on the left side. As shown in (iii) above, the second norm \( L_{tx}^{\frac{2(2+d)}{d}} \) can be controlled by interpolation between \( L_t^\infty L_x^2 \) and \( L_t^1 \dot{H}_x^1 \). Because of similarity of their proofs, we can give a proof to the first one, which is the special case of the following lemma. It is worth to noting that when \( p < \infty \), the estimate is \( L^2 \)-subcritical.

Lemma 2.3. Let \( 2 \leq p \leq \infty \), and the pair \((p_1, r_1)\) satisfy
\[
\frac{2}{p_1} + \frac{d}{r_1} = \frac{d}{2} + \frac{2}{p}, \quad 1 \leq p_1 \leq 2, \quad 1 < r_1 \leq 2,
\]
then
\[ \left\| \int_0^t e^{(t-s)\Delta} F(s) ds \right\|_{L_t^p L_x^2(\mathbb{R}^+) \times \mathbb{R}^d} \lesssim \| F \|_{L_t^{p_1} L_x^{p_1}(\mathbb{R}^+ \times \mathbb{R}^d)}. \]

**Proof.** By Plancherel’s theorem, it is equivalent that
\[ \left\| \int_0^t e^{-(t-s)|\xi|^2} \hat{F}(\xi, s) ds \right\|_{L_t^p L_x^2(\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \| F \|_{L_t^{p_1} L_x^{p_1}(\mathbb{R}^+ \times \mathbb{R}^d)}. \] (2.6)

Since by the Young inequality of the convolution on \( \mathbb{R}^+ \), for any \( 1 \leq p_1 \leq p \leq \infty \),
\[ \left\| \int_0^t e^{-(t-s)|\xi|^2} \hat{F}(\xi, s) ds \right\|_{L^p(\mathbb{R}^+)} \lesssim \| \xi \|^{-\frac{2}{p} + \frac{2}{p_1}} \| \hat{F}(\xi, \cdot) \|_{L_t^{p_1}(\mathbb{R}^+)} \]

Note that \( p_1 \leq 2 \leq p \), thus by Minkowski’s inequality, Plancherel’s theorem, Sobolev’s embedding we obtain
\[ \left\| \int_0^t e^{-(t-s)|\xi|^2} \hat{F}(\xi, s) ds \right\|_{L_t^p L_x^2(\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \| \nabla \|^{-\frac{2}{p} + \frac{2}{p_1}} \| F \|_{L_t^{p_1} L_x^{p_1}(\mathbb{R}^+ \times \mathbb{R}^d)}, \]

which gives the desired estimate (2.6). \( \square \)

Finally, we also need the following maximal \( L^p \)-regularity result for the heat flow. See Lemarie-Rieusset’s book [5, P.64] for example.

**Lemma 2.4.** Let \( p \in (1, \infty), q \in (1, \infty), \) and let \( T \in (0, \infty] \), then the operator \( A \) defined by
\[ f(t, x) \mapsto \int_0^t e^{(t-s)\Delta} f(s, \cdot) ds \]
is bounded from \( L^p((0, T), L^q(\mathbb{R}^d)) \) to \( L^p((0, T), L^q(\mathbb{R}^d)) \).

3. **Proof of Theorem 1.1**

In this section, we will divide several subsection to finish the proof of Theorem 1.1. For the end, we first establish a supercritical estimate on the linear heat flow in the following subsection.

3.1. **A supercritical estimate on the linear heat flow.** Let us recall the following radial Sobolev embedding, see [24] for example.

**Lemma 3.1.** Let \( \alpha, q, p, s \) be the parameters which satisfy
\[ \alpha > -\frac{d}{q}; \quad \frac{1}{q} \leq \frac{1}{p} \leq \frac{1}{q} + s; \quad 1 \leq p, q \leq \infty; \quad 0 < s < d \]
with
\[ \alpha + s = d \left( \frac{1}{p} - \frac{1}{q} \right). \]
Moreover, let at most one of the following equalities hold:
\[ p = 1, \quad p = \infty, \quad q = 1, \quad q = \infty, \quad \frac{1}{p} = \frac{1}{q} + s. \]
Then the radial Sobolev embedding inequality holds:
\[ \| \cdot |x|^a u \|_{L^q(\mathbb{R}^d)} \lesssim \| \nabla |^s u \|_{L^p(\mathbb{R}^d)}. \]

**Lemma 3.2.** For any \( q > 2 \) and any \( \gamma \in \left( \frac{1}{2} - \frac{3}{q}, 1 - \frac{4}{q} \right) \), suppose that the radial function \( f \in H^\gamma(\mathbb{R}^d) \) satisfying
\[ \text{supp } f \subset \{ x : |x| \geq 1 \}, \]
then
\[ \| e^{t\Delta} f \|_{L^q_t(\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \| |\nabla|^\gamma f \|_{L^2(\mathbb{R}^d)}. \]

**Proof.** By Lemma 2.1 we have
\[ \| e^{t\Delta} f \|_{L^q_t(\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \| f \|_{L^\infty(\mathbb{R}^d)}. \]
Let \( \alpha = \frac{d}{2} - s > 0 \) and \( s \in \left( \frac{1}{2}, 1 \right) \), then by Lemma 3.1 we have
\[ \| f \|_{L^\infty(\mathbb{R}^d)} \lesssim \| |x|^\alpha f \|_{L^\infty(\mathbb{R}^d)} \lesssim \| |\nabla|^s f \|_{L^2(\mathbb{R}^d)}, \]
where the first inequality above has used the condition \( \text{supp } f \subset \{ x : |x| \geq 1 \} \). Thus we get that
\[ \| e^{t\Delta} f \|_{L^q_t(\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \| |\nabla|^\gamma f \|_{L^2(\mathbb{R}^d)}. \quad (3.1) \]
Interpolation between this last estimate and (2.2), gives our desired estimates. \( \Box \)

### 3.2. Local theory and global criterion.

We use \( \chi_{\leq a} \) for \( a \in \mathbb{R}^+ \) to denote the smooth function
\[ \chi_{\leq a}(x) = \begin{cases} 1, & |x| \leq a, \\ 0, & |x| \geq \frac{11}{10} a, \end{cases} \]
and set \( \chi_{\geq a} = 1 - \chi_{\leq a} \).

Now write
\[ h_0 = v_0 + w_0, \quad (3.2) \]
where
\[ v_0 = \chi_{\geq \frac{1}{2}}(P_{\geq N} h_0), \quad w_0 = h_0 - v_0. \]
Then we will first claim that \( w_0 \in L^2(\mathbb{R}^d) \), and
\[ \| w_0 \|_{L^2(\mathbb{R}^d)} \lesssim N^{s_0} \| h_0 \|_{\dot{H}^{-s_0}(\mathbb{R}^d)}. \quad (3.3) \]
Note that $w_0 = \chi_{\leq \frac{1}{2}}(P_{\geq N}h_0) + P_{<N}h_0$. Firstly, we give the following estimate on the first part, which is a consequence of Lemma 2.1.

**Lemma 3.3.** Let $h_0$ be the function satisfying the hypothesis in Theorem 1.1, then

$$
\| \chi_{\leq \frac{1}{2}}(P_{\geq N}h_0) \|_{L^2(\mathbb{R}^d)} \lesssim N^{-1} \| h_0 \|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. 
$$

**Proof.** By the support property of $h_0$, we may write

$$
\chi_{\leq \frac{1}{2}}(P_{\geq N}h_0) = \chi_{\leq \frac{1}{2}}(P_{\geq N}\chi_{\geq \frac{n}{10}}h_0)
= \chi_{\leq \frac{1}{2}}(P_{\geq N}\chi_{\geq \frac{n}{10}}P_{2N}h_0) + \sum_{M=4N}^{\infty} \chi_{\leq \frac{1}{2}} P_{\geq N}(\chi_{\geq \frac{n}{10}} P_M h_0). 
$$

By Lemma 2.1 and Bernstein’s inequality, we have

$$
\| \chi_{\leq \frac{1}{2}}(P_{\geq N}\chi_{\geq \frac{n}{10}}P_{2N}h_0) \|_{L^2(\mathbb{R}^d)} \lesssim N^{-10} \| P_{\leq 2N}h_0 \|_{L^2(\mathbb{R}^d)} 
\lesssim N^{-1} \| h_0 \|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. 
$$

Moreover, since $P_{\geq N} = I - P_{<N}$ and $M > 2N$, we obtain

$$
\chi_{\leq \frac{1}{2}} P_{\geq N}(\chi_{\geq \frac{n}{10}} P_M h_0) = -\chi_{\leq \frac{1}{2}} P_{<N}(\chi_{\geq \frac{n}{10}} P_M h_0)
= -\chi_{\leq \frac{1}{2}} P_{<N}(P_{\geq \frac{1}{5}M} (\chi_{\geq \frac{n}{10}}) P_M h_0),
$$

where $P_{\geq \frac{1}{5}M}(\chi_{\geq \frac{n}{10}})$ denotes the high frequency truncation of the bump function $\chi_{\geq \frac{n}{10}}$.

Note that

$$
\| \chi_{\leq \frac{1}{2}} P_{<N}(P_{\geq \frac{1}{5}M} (\chi_{\geq \frac{n}{10}}) P_M h_0) \|_{L^2(\mathbb{R}^d)} \lesssim \| P_{\geq \frac{1}{5}M} (\chi_{\geq \frac{n}{10}}) \|_{L^\infty(\mathbb{R}^d)} \| P_M h_0 \|_{L^2(\mathbb{R}^d)}
\lesssim M^{-2} \| \Delta P_{\geq \frac{1}{5}M} (\chi_{\geq \frac{n}{10}}) \|_{L^\infty(\mathbb{R}^d)} \| P_M h_0 \|_{L^2(\mathbb{R}^d)}
\lesssim M^{-1} \| \chi_{\geq \frac{n}{10}} P_M h_0 \|_{L^2(\mathbb{R}^d)}.
$$

Hence, we have

$$
\| \chi_{\leq \frac{1}{2}} P_{\geq N}(\chi_{\geq \frac{n}{10}} P_M h_0) \|_{L^2(\mathbb{R}^d)} \lesssim M^{-1} \| h_0 \|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. 
$$

Therefore, taking summation, we obtain

$$
\sum_{M=4N}^{\infty} \| \chi_{\leq \frac{1}{2}} P_{\geq N}(\chi_{\geq \frac{n}{10}} P_M h_0) \|_{L^2(\mathbb{R}^d)} \lesssim N^{-1} \| h_0 \|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. 
$$

Inserting (3.6) and (3.7) into (3.5), we prove the lemma.

Moreover, by the Bernstein estimate,

$$
\| P_{<N} h_0 \|_{L^2(\mathbb{R}^d)} \lesssim N^{\epsilon_0} \| h_0 \|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}.
$$

Then this last estimate combining with Lemma 3.3 gives (3.3).
Second, we claim that
\[ \|v_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)} \lesssim \|h_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. \] (3.8)

Indeed,
\[ \|v_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)} \lesssim \|h_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)} + \|\chi_{\leq \frac{1}{2}}(P_N h_0)\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. \]

Hence, we only consider the latter term. By Sobolev’s embedding and Hölder’s inequality, we have
\[ \|\chi_{\leq \frac{1}{2}}(P_N h_0)\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)} \lesssim \|\chi_{\leq \frac{1}{2}}(P_N h_0)\|_{L^2(\mathbb{R}^d)}. \]

Hence (3.8) follows from Lemma 3.3.

We denote
\[ v_L(t) = e^{t\Delta} v_0. \]

Then \( v_L \) is globally existence, and by Plancherel’s theorem and (3.8)
\[ \|v_L(t)\|_{L^\infty_t \dot{H}^{-\epsilon_0}(\mathbb{R}^{1+d})} \lesssim \|v_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)} \lesssim \|h_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}, \] (3.9)

Moreover, let \( \epsilon \) be a sufficiently small positive constant, then we claim that
\[ \|v_L(t)\|_{L^2_{t\geq \frac{2}{d+2}}(\mathbb{R}^{1+d})} \lesssim N^{-\frac{1}{d+2}+\epsilon_0+\epsilon} \|h_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. \] (3.10)

Indeed, let \( \gamma = -\frac{d-1}{d+2} + \epsilon \), then by Lemma 3.2,
\[ \|v_L(t)\|_{L^2_{t\geq \frac{2}{d+2}}(\mathbb{R}^{1+d})} \lesssim \|\nabla^\gamma \chi_{\leq \frac{1}{2}}(P_N h_0)\|_{L^2(\mathbb{R}^d)}. \]

Note that
\[ \|\nabla^\gamma \chi_{\leq \frac{1}{2}}(P_N h_0)\|_{L^2(\mathbb{R}^d)} \leq \|\nabla^\gamma (P_N h_0)\|_{L^2(\mathbb{R}^d)} + \|\nabla^\gamma \chi_{\leq \frac{1}{2}}(P_N h_0)\|_{L^2(\mathbb{R}^d)}. \]

For the former term, since \( \gamma < -\epsilon_0 \), by Bernstein’s inequality,
\[ \|\nabla^\gamma (P_N h_0)\|_{L^2(\mathbb{R}^d)} \lesssim N^{\gamma+\epsilon_0} \|h_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. \]

So we only need to estimate the latter term. Let \( q \) be the parameter satisfying
\[ \frac{1}{q} = \frac{1}{2} - \frac{\gamma}{d}, \]
then \( q > 1 \). Since \( \gamma < 0 \), by Sobolev’s and Hölder’s inequalities,
\[ \|\nabla^\gamma \chi_{\leq \frac{1}{2}}(P_N h_0)\|_{L^2(\mathbb{R}^d)} \lesssim \|\chi_{\leq \frac{1}{2}}(P_N h_0)\|_{L^q(\mathbb{R}^d)} \lesssim N^{1/q} \|h_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. \]

Furthermore, by Lemma 3.3
\[ \|\chi_{\leq \frac{1}{2}}(P_N h_0)\|_{L^2(\mathbb{R}^d)} \lesssim N^{-1} \|h_0\|_{\dot{H}^{-\epsilon_0}(\mathbb{R}^d)}. \]
Combining the last two estimates above, we obtain

\[ \left\| \nabla^\gamma \chi_{\leq \frac{1}{2}} (P_{\geq N} h_0) \right\|_{L^2(\mathbb{R}^d)} \lesssim N^{-1} \left\| h_0 \right\|_{H^{-\frac{2}{3}}(\mathbb{R}^d)}. \]

This gives (3.10).

Now we denote \( w = h - v_L \), then \( w \) is the solution of the following equation,

\[
\begin{aligned}
\partial_t w &= \Delta w \pm |h|^\frac{5}{4} h, \\
w(0, x) &= w_0(x) = h_0 - v_0.
\end{aligned}
\tag{3.11}
\]

The following lemma is the local well-posedness and global criterion of the Cauchy problem (3.11).

**Lemma 3.4.** There exists \( \delta > 0 \), such that for any \( h_0 \) satisfying the hypothesis in Theorem 1.1 and \( w_0 = h_0 - v_0 \), the Cauchy problem (3.11) is well-posed on the time interval \([0, \delta]\), and the solution

\[
w \in C_{t} L^2_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right) \cap \cap \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right) \cap \cap \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right).
\]

Furthermore, let \( T^* \) be the maximal lifespan, and suppose that

\[
w \in L^\infty_t L^2_x \left( \left[ 0, T^* \right] \times \mathbb{R}^d \right),
\]

then \( T^* = +\infty \). In particular, if \( \| h_0 \|_{H^{-\frac{2}{3}}(\mathbb{R}^d)} \ll 1 \), then \( T^* = +\infty \).

**Proof.** For local well-posedness, we only show that the solution \( w \in L^\infty_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right) \cap \cap \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right) \cap \cap \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right) \) for some \( \delta > 0 \). Indeed, the local well-posedness with the lifespan \([0, \delta]\) is then followed by the standard fixed point argument. By Duhamel’s formula, we have

\[
w(t) = e^{t\Delta} w_0 \pm \int_0^t e^{(t-s)\Delta} |h(s)|^{\frac{5}{4}} h(s) \, ds.
\]

Then by Lemma 2.2 for any \( t_* \leq \delta \),

\[
\left\| w \right\|_{L^\infty_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right)} \lesssim \left\| e^{t\Delta} w_0 \right\|_{L^\infty_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right)} + \left\| h \right\|_{L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right)}.
\]

Note that

\[
\left\| h \right\|_{L^\infty_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right)} \lesssim \left\| v_L \right\|_{L^\infty_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right)} + \left\| w \right\|_{L^\infty_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right)},
\]

let \( \eta_0 = \left( \frac{4}{3} + 1 \right) \left( \frac{d-1}{2} - \varepsilon_0 - \epsilon \right) > 0 \), then using (3.10), we obtain

\[
\left\| w \right\|_{L^\infty_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right)} \lesssim \left\| e^{t\Delta} w_0 \right\|_{L^\infty_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right)} + N^{-\eta_0} \left\| h_0 \right\|_{H^{-\frac{2}{3}}(\mathbb{R}^d)} + \left\| w \right\|_{L^\infty_t L^2_x \left( \left[ 0, \delta \right] \times \mathbb{R}^d \right)}.
\]
Noting that either \( \|h_0\|_{\dot{\mathcal{H}}^{-\epsilon_0}(\mathbb{R}^d)} \ll 1 \), or choosing \( \delta \) small enough and \( N \) large enough, we have
\[
\|e^{t\Delta} w_0\|_{L_{tx}^{2(2+d)}([0,\delta] \times \mathbb{R}^d)} + N^{-\eta_0} \|h_0\|_{\dot{\mathcal{H}}^{-\epsilon_0}(\mathbb{R}^d)}^{\frac{4}{2(2+d) - 2}} \ll 1,
\]
then by the continuity argument, we
\[
\|w\|_{L_{tx}^{2(2+d)}([0,\delta] \times \mathbb{R}^d)} \lesssim \|e^{t\Delta} w_0\|_{L_{tx}^{2(2+d)}([0,\delta] \times \mathbb{R}^d)} + N^{-\eta_0} \|h_0\|_{\dot{\mathcal{H}}^{-\epsilon_0}(\mathbb{R}^d)}^{\frac{4}{2(2+d) - 2}}.
\]
Further, by Lemma 2.2 again,
\[
\|w\|_{L_{x}^{2}H_{s}^{1}([0,\delta] \times \mathbb{R}^d)} + \sup_{t \in [0,\delta]} \|w\|_{L_{x}^{2}(\mathbb{R}^d)} \lesssim \|w_0\|_{L_{x}^{2}(\mathbb{R}^d)} + \|h\|_{L_{tx}^{2(2+d)}}^{\frac{4}{2(2+d) - 2}}([0,\delta] \times \mathbb{R}^d)
\]
\[
\lesssim \|w_0\|_{L_{x}^{2}(\mathbb{R}^d)} + \|w\|_{L_{tx}^{2(2+d)}([0,\delta] \times \mathbb{R}^d)} + \|w\|_{L_{tx}^{2(2+d)}([0,\delta] \times \mathbb{R}^d)}^{\frac{4}{2(2+d) - 2}}.
\]
Hence, using (3.10) and (3.12), we obtain
\[
\|w\|_{L_{x}^{2}H_{s}^{1}([0,\delta] \times \mathbb{R}^d)} + \sup_{t \in [0,\delta]} \|w\|_{L_{x}^{2}(\mathbb{R}^d)} \leq C,
\]
for some \( C = C(N, \|h_0\|_{\dot{\mathcal{H}}^{-\epsilon_0}(\mathbb{R}^d)}) > 0 \).

Suppose that
\[
w \in L_{tx}^{2(2+d)}([0,T^*) \times \mathbb{R}^d),
\]
then if \( T^* < +\infty \), we have
\[
\|w(T^*)\|_{L_{x}^{2}(\mathbb{R}^d)} \lesssim \|e^{T^*\Delta} w_0\|_{L_{tx}^{2(2+d)}([0,T^*) \times \mathbb{R}^d)} + \|h\|_{L_{tx}^{2(2+d)}([0,T^*) \times \mathbb{R}^d)}^{\frac{4}{2(2+d) - 2}}([0,T^*) \times \mathbb{R}^d)
\]
\[
\lesssim \|w_0\|_{L_{x}^{2}(\mathbb{R}^d)} + N^{-\eta_0} \|h_0\|_{\dot{\mathcal{H}}^{-\epsilon_0}(\mathbb{R}^d)}^{\frac{4}{2(2+d) - 2}} + \|w\|_{L_{tx}^{2(2+d)}([0,T^*) \times \mathbb{R}^d)}^{\frac{4}{2(2+d) - 2}}.
\]
Hence, \( w \) exists on \([0,T^*)\), and \( w(T^*) \in L_{x}^{2}(\mathbb{R}^d) \). Hence, using the local theory obtained before from time \( T^* \), the lifespan can be extended to \( T^* + \delta \), this is contradicted with the definition of the maximal lifespan \( T^* \). Hence, \( T^* = +\infty \).

3.3. **Uniqueness.** Here we adopt the argument in \([15]\), where the main tool is the the maximal \( L^p \)-regularity of the heat flow. Let \( h_1, h_2 \) be two distinct solutions of (1.2) with the same initial data \( h_0 \), and write
\[
h_1 = e^{s\Delta} h_0 + w_1; \quad h_2 = e^{s\Delta} h_0 + w_2.
\]
By the Duhamel formula, we have
\[
w_1(t) = \int_0^t e^{(t-s)\Delta} e^{s\Delta} h_0 + w_1 ds; \quad w_2(t) = \int_0^t e^{(t-s)\Delta} e^{s\Delta} h_0 + w_2 ds.
\]
Denote \( w = w_1 - w_2 \), then \( w \) obeys

\[
w(t) = \int_0^t e^{(t-s)\Delta} \left[ |e^{s\Delta}h_0 + w_1|^{\frac{4}{p}} (e^{s\Delta}h_0 + w_1) - |e^{s\Delta}h_0 + w_2|^{\frac{4}{p}} (e^{s\Delta}h_0 + w_2) \right] ds.
\]

Note that there exists an absolute constant \( C > 0 \) such that

\[
|e^{s\Delta}h_0 + w_1|^{\frac{4}{p}} (e^{s\Delta}h_0 + w_1) - |e^{s\Delta}h_0 + w_2|^{\frac{4}{p}} (e^{s\Delta}h_0 + w_2) \leq C \left( |e^{s\Delta}h_0|^{\frac{4}{p}} + |w_1|^{\frac{4}{p}} + |w_2|^{\frac{4}{p}} \right) |w|.
\]

Then by the positivity of the heat kernel, we have

\[
|w(t)| \leq C \int_0^t e^{(t-s)\Delta} \left( |e^{s\Delta}h_0|^{\frac{4}{p}} + |w_1(s)|^{\frac{4}{p}} + |w_2(s)|^{\frac{4}{p}} \right) |w(s)| ds.
\]

Then we get that for \( 2 \leq p < \infty, \tau \in (0, \delta] \),

\[
\|w\|_{L^p_t((0,\tau); L^2(\mathbb{R}^d))} \lesssim \left\| \int_0^t e^{(t-s)\Delta} |e^{s\Delta}h_0|^{\frac{4}{p}} |w(s)| ds \right\|_{L^p_t((0,\tau); L^2(\mathbb{R}^d))} + \left\| \int_0^t e^{(t-s)\Delta} \left( |w_1(s)|^{\frac{4}{p}} + |w_2(s)|^{\frac{4}{p}} \right) |w(s)| ds \right\|_{L^p_t((0,\tau); L^2(\mathbb{R}^d))}.
\]

For the first term in the right-hand side above, using Lemma 2.3 and choosing \( p \) large enough, we have

\[
\left\| \int_0^t e^{(t-s)\Delta} |e^{s\Delta}h_0|^{\frac{4}{p}} |w(s)| ds \right\|_{L^p_t((0,\tau); L^2(\mathbb{R}^d))} \lesssim \left\| |e^{s\Delta}h_0|^{\frac{4}{p}} |w(s)| \right\|_{L^{p_1}_t((0,\tau); L^{r_1}(\mathbb{R}^d))},
\]

where we have chose \((p_1, r_1)\) that

\[
\frac{1}{p_1} = \frac{2}{d + 2} + \frac{1}{p}; \quad \frac{1}{r_1} = \frac{2}{d + 2} + \frac{1}{2}.
\]

(Note that \( d > 4 \) and \( p \) is large, we have that \( p_1 \in (1, 2), r_1 \in (1, 2) \)). Hence, by Hölder’s inequality, we obtain that

\[
\left\| \int_0^t e^{(t-s)\Delta} |e^{s\Delta}h_0|^{\frac{4}{p}} |w(s)| ds \right\|_{L^p_t((0,\tau); L^2(\mathbb{R}^d))} \lesssim \left\| |e^{s\Delta}h_0|^{\frac{4}{p}} |w(s)| \right\|_{L^{p_1}_t((0,\tau); L^{r_1}(\mathbb{R}^d))} \lesssim \left\| e^{s\Delta}h_0 \right\|_{L^{2(d+2)/(d+4)}_{t_4}((0,\tau) \times \mathbb{R}^d)} \left\| w \right\|_{L^p_t((0,\tau); L^2(\mathbb{R}^d))}.
\]

For the second term in the right-hand side above, using Lemma 2.4

\[
\left\| \int_0^t e^{(t-s)\Delta} \left( |w_1(s)|^{\frac{4}{p}} + |w_2(s)|^{\frac{4}{p}} \right) |w(s)| ds \right\|_{L^p_t((0,\tau); L^2(\mathbb{R}^d))} \lesssim \left\| (-\Delta)^{-\frac{1}{2}} \left( |w_1(s)|^{\frac{4}{p}} + |w_2(s)|^{\frac{4}{p}} \right) |w(s)| \right\|_{L^p_t((0,\tau); L^2(\mathbb{R}^d))}.
\]
Since $d > 4$, by Sobolev’s embedding, we further have
\[
\| \int_0^t e^{(t-s)\Delta} \left( |w_1(s)|^{2/3} + |w_2(s)|^{2/3} \right) |w(s)| ds \|_{L^2_T((0,\tau); L^2(\mathbb{R}^d))} \\
\lesssim \left( \| w_1(s) \|^{2/3} + \| w_2(s) \|^{2/3} \right) \| w(s) \|_{L^2_T((0,\tau); L^2(\mathbb{R}^d))} \\
\lesssim \left( \| w_1 \|^{2/3}_{L^\infty_T((0,\tau); L^2(\mathbb{R}^d))} + \| w_2 \|^{2/3}_{L^\infty_T((0,\tau); L^2(\mathbb{R}^d))} \right) \| w \|_{L^2_T((0,\tau); L^2(\mathbb{R}^d))}.
\]
Collection the estimates above, we obtain that
\[
\| w \|_{L^2_T((0,\tau); L^2(\mathbb{R}^d))} \lesssim \rho(\tau) \cdot \| w \|_{L^2_T((0,\tau); L^2(\mathbb{R}^d))},
\]
where
\[
\rho(\tau) = \left\| e^{\tau \Delta} h_0 \right\|_{L^2_T(0,\tau) \times \mathbb{R}^d}^{2(d+2)} + \| w_1 \|^{2/3}_{L^\infty_T((0,\tau); L^2(\mathbb{R}^d))} + \| w_2 \|^{2/3}_{L^\infty_T((0,\tau); L^2(\mathbb{R}^d))}.
\]
By (3.10) and Lemma 2.2, we have
\[
\| e^{\tau \Delta} h_0 \|_{L^2_T(0,\tau) \times \mathbb{R}^d}^{2(d+2)} \to 0, \quad \text{when } \tau \to 0.
\]
Further, since $w_1, w_2 \in C([0, \delta], L^2(\mathbb{R}^d))$, we get
\[
\lim_{\tau \to 0} \rho(\tau) \to 0.
\]
Hence, choosing $\tau$ small enough and from (3.13), we obtain that $w \equiv 0$ on $t \in [0, \tau)$. By iteration, we have $w_1 \equiv w_2$ on $[0, \delta)$. This proves the first statement (1) in Theorem 1.1.

### 3.4. $L^2$-estimates

In this subsection, we prove the second statement (2) in Theorem 1.1.

Firstly, by Lemma 3.4, when $\| h_0 \|_{H^{-c_0}(\mathbb{R}^d)} \ll 1$, we immediately have the global existence of the solution for the both cases $\mu = \pm 1$. However, in the defocusing case ($\mu = 1$), the smallness of $\| h_0 \|_{H^{-c_0}(\mathbb{R}^d)} \ll 1$ can be cancelled. In fact, note that $h = v_L + w$ and
\[
\| v_L \|_{L^2(\mathbb{R}^d)} = \| e^{-t|\xi|^2} \hat{v}_0(\xi) \|_{L^2(\mathbb{R}^d)} \\
\lesssim \| e^{-t|\xi|^2} |\xi|^{c_0} \|_{L^\infty(\mathbb{R}^d)} \| v_0 \|_{H^{-c_0}} \lesssim t^{-c_0/2} \| h_0 \|_{H^{-c_0}}.
\]
Hence, from Lemma 3.1, we have $h(\delta) \in L^2(\mathbb{R}^d)$. Let $I = [0, T^*)$ be the maximal lifespan of the solution $h$ of the Cauchy problem (1.2). Then from the $L^2$ estimate of the solution (by inner producing with $h$ in (1.2)), we have
\[
\sup_{t \in I} \| h \|_{L^2}^2 + \| \nabla h \|_{L^2_{t,x}(I \times \mathbb{R}^d)}^2 \leq \| h_0 \|_{L^2}^2.
\]
This gives the uniform boundedness of $\| h \|_{L^2_{t,x}(I \times \mathbb{R}^d)}^{2(d+2)}$ and thus $\| w \|_{L^2_{t,x}(I \times \mathbb{R}^d)}^{2(d+2)}$. Then by the global criteria given in Lemma 3.4, we have $T^* = +\infty$. 

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Some text in the image was not legible, but the overall content is clear. The text refers to Sobolev’s embedding, estimates involving $L^2$ norms, and global criteria for solutions of heat equations. The proof involves limits as $\tau \to 0$, and conclusions are drawn about the global existence of solutions under certain conditions.
Secondly, we consider the time estimate of the solution ($\mu = \pm 1$). When $t \leq 1$, it follows from (3.14) and Lemma 3.4, that
\[ \| h(t) \|_{L^2} \lesssim t^{-\frac{d}{4}} \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} , \quad \text{for any } t \in (0, 1). \]
So it remains to show the decay estimate when $t > 1$. By Duhamel’s formula, we have
\[ \| h(t) \|_{L^2(\mathbb{R}^d)} \leq \| e^{t\Delta} h_0 \|_{L^2(\mathbb{R}^d)} + \left\| \int_0^t e^{(t-s)\Delta} |\dot{h}(s)|^{\frac{3}{2}} h(s) \, ds \right\|_{L^2(\mathbb{R}^d)}. \]
Similar as (3.14), we have
\[ \left\| e^{t\Delta} h_0 \right\|_{L^2(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}} \| h_0 \|_{\dot{H}^{-\frac{1}{2}}}. \]
Then using the estimate above and Lemma 2.5, we further have
\[
\begin{align*}
\| h(t) \|_{L^2(\mathbb{R}^d)} & \lesssim t^{-\frac{d}{2}} \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} + \int_0^t \left\| e^{(t-s)\Delta} |\dot{h}(s)|^{\frac{3}{2}} h(s) \right\|_{L^2(\mathbb{R}^d)} \, ds \\
& \lesssim t^{-\frac{d}{2}} \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} + \int_0^t |t-s|^{-1} \left\| h(s) \right\|_{L^2(\mathbb{R}^d)} \, ds \\
& \lesssim t^{-\frac{d}{2}} \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} + \int_0^t |t-s|^{-1} \| h \|_{L^2(\mathbb{R}^d)}^{\frac{2d}{d+4}} \, ds.
\end{align*}
\]
In the last step we have used the fact $d \geq 4$ such that $\frac{2d}{d+4} \geq 1$.

Now we denote
\[ \| h \|_{X(T)} = \sup_{t \in [0, T]} \left( t^{\frac{d}{2}} \| h(t) \|_{L^2(\mathbb{R}^d)} \right). \]
Fixing $T > 1$, then for any $t \in (1, T]$,
\[
\begin{align*}
\| h(t) \|_{L^2(\mathbb{R}^d)} & \lesssim t^{-\frac{d}{2}} \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} + \int_0^t |t-s|^{-1} s^{-\frac{d}{2(4+1)}} \, ds \| h(t) \|_{X(T)}^{\frac{4}{d+4}} \\
& \lesssim t^{-\frac{d}{2}} \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} + t^{-\frac{d}{2(4+1)}} \| h(t) \|_{X(T)}^{\frac{4}{d+4}} \\
& \lesssim t^{-\frac{d}{2}} \left( \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} + \| h(t) \|_{X(T)}^{\frac{4}{d+4}} \right).
\end{align*}
\]
Thus we obtain that
\[ \| h(t) \|_{X(T)} \lesssim \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} + \| h(t) \|_{X(T)}^{\frac{4}{d+4}}. \]
By the continuity argument, we get
\[ \| h(t) \|_{X(T)} \lesssim \| h_0 \|_{\dot{H}^{-\frac{1}{2}}}. \]
Since the estimate is independent on $T$, we give that
\[ \| h(t) \|_{L^2} \lesssim t^{-\frac{d}{2}} \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} , \quad \text{for any } t > 1. \]
Therefore, we obtain that
\[ \| h(t) \|_{L^2} \lesssim t^{-\frac{d}{2}} \| h_0 \|_{\dot{H}^{-\frac{1}{2}}} , \quad \text{for any } t > 0. \]
This proves the second statement (2) in Theorem 1.1.
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