On global solutions to a viscous compressible two-fluid model with unconstrained transition to single-phase flow in three dimensions

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Abstract
We consider the Dirichlet problem for a compressible two-fluid model in multi-dimensions. It consists of the continuity equations for each fluid and the momentum equations for the mixture. This model can be derived from the compressible two-fluid model with equal velocities (Bresch et al., in Arch Rational Mech Anal 196:599–629, 2010) and from a scaling limit of the Vlasov-Fokker-Planck/compressible Navier-Stokes (Mellet and Vasseur, Commun Math Phys 281(3):573–596, 2008) (see also the compressible Oldroyd-B model with stress diffusion (Barrett et al., Commun Math Sci 15:1265 –1323, 2017). Another interesting connection is that it is formally the equations of compressible magnetohydrodynamic (MHD) flows without resistivity in two dimensions under the action of vertical magnetic field (Li and Sun, J Differ Equ 267(6):3827–3851, 2019). Under weak assumptions on the initial data which can be discontinuous, unbounded and large as well as involve transition to pure single-phase points or regions, we show existence of global weak solutions with finite energy. The essential novelty of this work, compared with previous works on the same model, is that transition to each single-phase flow is allowed without any constraints between adiabatic constants or two densities. It means that one of the phases can vanish in a point while the other can persist. The lack of enough regularity for each densities brings up essential difficulties in the two-component pressure compared with the single-phase model, i.e., compressible Navier-Stokes equations. The key points to achieve the main result rely on the variables reduction technique for the pressure function, domain separation, and some new estimates. As a byproduct, we obtain the existence of global weak solutions to the compressible MHD system without resistivity in two dimensions under the action of non-negatively vertical magnetic field, which represents a step forward to the study of the global large solution to the compressible MHD system without resistivity.

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1 Introduction

1.1 Background and motivation

Multi-phase fluid models have various applications in different areas, such as the petroleum industry, nuclear, chemical-process, and cryogenics [2,3,9,11,23,34]. They also are quite relevant for the studies of some models like cancer cell migration model [10,13], MHD system [25], and compressible two-fluid Oldroyd-B model with stress diffusion [1]. In this paper, we consider the Dirichlet problem for a viscous compressible two-fluid model with one velocity and a pressure of two components in three spatial dimensions, i.e.,

$$
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho u_t + \text{div}(\rho u \otimes u) + \nabla P(n, \rho) &= \mu \Delta u + (\mu + \lambda) \nabla \text{div} u \quad \text{on } \Omega \times (0, \infty),
\end{align*}
$$

(1.1)

with the initial-boundary conditions

$$
\begin{align*}
n(x, 0) &= n_0(x), \quad \rho(x, 0) = \rho_0(x), \quad (\rho + n)u(x, 0) = M_0(x) \quad \text{for } x \in \overline{\Omega}, \\
u|_{\partial \Omega} &= 0 \quad \text{for } t \geq 0,
\end{align*}
$$

(1.2)

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, and $\rho$ and $n$, $u$, and $P$ represent the densities of two fluids, the velocity of the fluids, and the pressure, respectively. $\mu$ and $\lambda$ are the viscosity coefficients satisfying $\mu > 0$, $2\mu + 3\lambda \geq 0$. Here we assume that $\mu$ and $\lambda$ are constants. The pressure we study is given by

$$
P(n, \rho) = n^\Gamma + \rho^\gamma,
$$

(1.3)

or by

$$
\begin{align*}
P &= A_+(\rho_+)^\gamma = A_- (\rho_-)^\Gamma, \\
\rho \rho_- + n \rho_+ &= \rho_+ \rho_- ,
\end{align*}
$$

(1.4)

for constants $A_+, A_- > 0$ and $\gamma, \Gamma > 1$, where $\rho = \omega \rho_+, n = (1 - \omega) \rho_-$, and $\omega = \omega(x, t) \in [0, 1]$ denotes the volume fraction of the fluid $+\in$ in the mixture. For (1.4), one can use the implicit function theorem to define $\rho_+ = \rho_+(n, \rho)$ and $\rho_- = \rho_-(n, \rho)$ which represent

$$
\begin{align*}
\rho_+ &= \rho_+(n, \rho), \\
\rho_- &= \rho_-(n, \rho).
\end{align*}
$$
the densities of the fluids + and −, respectively (please see [3,6,29] for more details). Note that (1.3) is motivated by a limiting system derived from Vlasov-Fokker-Planck/compressible Navier-Stokes system [28], and by compressible MHD system for two-dimensional case [25], and by compressible Oldroyd-B model with stress diffusion [1], and that (1.4) is motivated by the compressible two-fluid model with possibly unequal velocities [3].

Our aim is to study the global existence of weak solution to (1.1) with large initial data in three dimensions. When $\rho \equiv 0$ or $n \equiv 0$, the system (1.1) reduces to compressible Navier-Stokes equations for isentropic flow. In this case, some pioneering works on this topics have been achieved. More specifically, Lions [26] obtained the first global existence result on weak solution with large initial data in multi-dimensions, where $P = R \rho^\gamma$ for some positive constant $R$ and any given $\gamma \geq \frac{9}{5}$ for three dimensions. The constraint for $\gamma$ was relaxed to $\gamma > \frac{3}{2}$ by Feireisl [15] and by Feireisl-Novotný-Petzeltová [18], and to $\gamma > 1$ by Jiang-Zhang [24] for spherically symmetric weak solutions. The pressure function in [18,24,26] is monotone and convex, which is very essential for the compactness of density. Feireisl [16] extended the result to the case for more general pressure $P(\rho)$ of monotonicity for $\rho \geq \rho_\ast$. Very recently, Bresch and Jabin [5] developed a new method to derive the compactness of the density which does not rely on any monotonicity assumptions on the pressure. It remains largely open\(^1\) whether the above results for three dimensions can be extended to the more physical case that the adiabatic constant $\gamma > 1$.

When the pressure is of two components like in (1.1), it will become more challenging. Some nice properties of one-component pressure are not available any more due to some cross products like $f_1(\rho)f_2(n)$ or even more implicitly $f_3(n, \rho)f_2(n)$ and $f_3(n, \rho)f_4(\rho)$ for some known scalar functions $f_i$, $i = 1, 2, 3, 4$. At the first glance, it seems that more regularity on the densities is required to handle the cross products in the context of passing to the limits. These extra regularity properties are, so far, out of reach for large solutions, and the classical techniques cannot be applied directly on (1.1).

We will give a brief overview for the relevant results on the model (1.1). In fact, the studies of the model have been very active for the past few years. Some global existence results are obtained, however, mostly subject to the case for the domination conditions\(^2\).

- For the one-dimensional case, Evje and Karlsen [11] obtained the global existence result on weak solution with large initial data subject to the domination conditions. The one-dimensional properties of the equations implies that the densities of the fluids are bounded for large initial data. This good property is essential to show strong convergence of the densities in the context of the approximation system. The domination condition was removed later by Evje, the author, and Zhu [14] by introducing a new energy equality, which allows transition to each single-phase flow. For the global existence of small solutions, please refer for instance to [9,12,32,33] and the references therein.

- For the multi-dimensional case, in particular for three dimensions, some new challenges arise due to the multi-dimensional nonlinearity. The boundedness of the densities can not be derived so far as the one-dimensional case with large initial data. However, with some smallness assumptions, the boundedness of the density and the derivatives of the other quantities arising in the equations can be derived to handle the cross products conveniently, and we refer the readers to [20,21,31]. In a recent work by Maltese et al. [27],

\(^1\) The problem has been solved by Jiang-Zhang [24] for spherically symmetric weak solutions in multi-dimensions.

\(^2\) It means that $n_0 \leq \overline{\tau}_0\rho_0$ or $\rho_0 \leq \overline{\tau}_1n_0$ for some positive constants $\overline{\tau}_0$ and $\overline{\tau}_1$, which implies that the two fluids are dominated by one of the fluids.
the authors considered another interesting model with the pressure of two components which can be transformed to the one with one-component pressure, i.e.,

\[
\begin{align*}
    \rho_t + \text{div}(\rho u) &= 0, \\
    Z_t + \text{div}(Zu) &= 0, \\
    (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla Z' &= \mu \Delta u + (\mu + \lambda) \nabla \text{div} u.
\end{align*}
\]  

(1.5)

Thus it makes the approach for compressible Navier-Stokes equations applicable to prove the global existence of weak solutions to (1.5) with large initial data. After that the authors obtained the equivalence between (1.5) and the original system for \( \gamma \geq \frac{9}{5} \). But it is not the case for the two-fluid system.

Very recently, with large initial data and the domination conditions or alternatively with \( \Gamma \) and \( \gamma \) close enough, Vasseur, the author, and Yu [30] obtained the global existence of weak solutions to (1.1) by decomposing the pressure function and deriving a new compactness theorem for transport equations with possible diffusion, where the pressure is determined by the explicit case (1.3) for \( \gamma > \frac{9}{5} \).

The result with the domination condition was later extended to the case that both \( \Gamma \) and \( \gamma \) can touch \( \frac{9}{5} \) by Novotný and Pokorný [29] where more general pressure laws covering the cases of both (1.3) and (1.4) were considered.

With large initial data but without any domination conditions in multi-dimensions, the global existence theory for weak solutions only holds for the two-fluid Stokes equations on the d-dimensional torus \( \mathbb{T}^d \) for \( d = 2, 3 \). We refer the readers to the seminal work by Bresch, Mucha, Zatorska [6] where the pressure is given by (1.4). The proof relies on the Bresch-Jabin’s new compactness tools for compressible Navier-Stokes equations and the reformulated system

\[
\begin{align*}
    R_t + \text{div}(Ru) &= 0, \\
    Q_t + \text{div}(Qu) &= 0, \\
    -(\lambda + 2\mu) \text{div} u + a^+ \left( Z(R, Q)^{\gamma^+} - \{Z(R, Q)^{\gamma^+}\} \right) &= 0, \\
    \text{rot} u = 0, \quad \int_{\mathbb{T}^d} u(x, t) \, dx = 0,
\end{align*}
\]  

(1.6)

where \( R = \rho = \alpha \rho_+ \), \( Q = n = (1-\alpha)\rho_- \), \( \{Z(R, Q)^{\gamma^+}\} = \left( \int_{\mathbb{T}^d} Z(R, Q)^{\gamma^+} \, dx \right)/|\mathbb{T}^d| \), \( a^+ = A_+ \), and \( \gamma^+ = \gamma \). Note that [6] does not need any domination conditions for \( \Gamma, \gamma > 1 \), although the nonlinear terms \( [\rho + n]u \) and \( \text{div}[(\rho + n)u \otimes u] \) in the momentum equations are ignored so that the momentum equations can be transformed to (1.6).3

The case without any domination conditions makes the system (1.1) more realistic in some physical situations and more “two fluids” properties from mathematical points of view. In this case, however, it is still open whether the global existence of weak solution exists for possibly large initial data in multi-dimensions. In this paper, we focus on the Dirichlet problem.
1.2 Main result

Note that for each cases of (1.3) and (1.4), the pressure $P(n, \rho)$ satisfies

$$\frac{1}{C_0} (n^\gamma + \rho^\gamma) \leq P(n, \rho) \leq C_0 (n^\gamma + \rho^\gamma)$$

(1.7)

for some positive constant $C_0$. In fact, (1.7) is naturally true for the case (1.3). For the second case (1.4), we only consider the case of $\gamma \geq \Gamma$, since for the other case, it is similar. More specifically, in view of (1.4), we obtain that

$$\rho_- = (1 - \alpha)\rho_- + \alpha \rho_- = n + \alpha \left( \frac{A_+}{A_-} \right)^\gamma \rho_+^\gamma = n + \left( \frac{A_+}{A_-} \right)^\gamma \rho_-^{\gamma - 1}$$

$$\geq n + \left( \frac{A_+}{A_-} \right)^\gamma \rho_-^{\gamma - 1},$$

(1.8)

and that

$$\rho_- = n + \left( \frac{A_+}{A_-} \right)^\gamma \rho_-^{\gamma - 1} = n + \left( \frac{A_-}{A_+} \right)^{-\gamma} \rho_-^{-1} \leq n + \frac{1}{2} \rho_- + c_0 \rho^\gamma.$$  

(1.9)

(1.8) and (1.9) imply (1.7).

In addition, for any smooth solution of system (1.1), the following energy equalities holds for any time $0 \leq t \leq T$:

$$\frac{d}{dt} \int_\Omega \left[ \frac{(\rho + n)|u|^2}{2} + G(\rho, n) \right] dx + \int_\Omega \left[ \mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 \right] dx = 0,$$  

(1.10)

where

$$G(\rho, n) = \begin{cases} 
\frac{n^\gamma}{\gamma - 1} + \frac{\rho^\gamma}{\gamma - 1}, & \text{if } P \text{ is given by (1.3)}, \\
\frac{P(n, \rho)}{\gamma - 1} + \frac{1 - \alpha}{\gamma - 1}, & \text{if } P \text{ is given by (1.4)}. 
\end{cases}$$

(1.11)

Here (1.11) is given in [30] and (1.11) follows from (3.14) with $\epsilon, \delta = 0$.

Motivated by (1.7) and (1.10), in order to make the initial energy is finite, we set the following conditions on the initial data, i.e.,

$$\inf_{x \in \Omega} \rho_0 \geq 0, \quad \inf_{x \in \Omega} n_0 \geq 0, \quad \rho_0 \in L^\gamma(\Omega), \quad n_0 \in L^\gamma(\Omega),$$

(1.12)

and

$$M_0 \sqrt{\rho_0 + n_0} \in L^2(\Omega) \quad \text{where} \quad \frac{M_0}{\sqrt{\rho_0 + n_0}} = 0 \text{ on } (x \in \Omega | \rho_0(x) + n_0(x) = 0).$$

(1.13)

where $M_0$ is the initial momentum of the mixture given in (1.2).

The definition of weak solution in the energy space is given in the following sense.

**Definition 1.1** (Global weak solution) We call $(\rho, n, u) : \Omega \times (0, \infty) \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3$ a global weak solution of (1.1)–(1.2) if for any $0 < T < +\infty$,

- $\rho \in L^\infty(0, T; L^\gamma(\Omega)), \ n \in L^\infty(0, T; L^\gamma(\Omega)), \ \sqrt{\rho + n} u \in L^\infty(0, T; L^2(\Omega)), \ u \in L^2(0, T; H^1_0(\Omega))$,
- $(\rho, n, u)$ solves the system (1.1) in $\mathcal{D}'(Q_T), \quad \text{where } Q_T = \Omega \times (0, T),$
- $(\rho, n, (\rho + n)u)(x, 0) = (\rho_0(x), n_0(x), M_0(x)), \quad \text{for a.e. } x \in \Omega,
• (1.1)_1 and (1.1)_2 hold in \( \mathcal{D}'(\mathbb{R}^3 \times (0, T)) \) provided \( \rho, n, u \) are prolonged to be zero on \( \mathbb{R}^3 / \Omega \),
• the equation (1.1)_1 and (1.1)_2 are satisfied in the sense of renormalized solutions, i.e.,
\[
\partial_t b(f) + \text{div}(b(f)u) + [b'(f)f - b(f)] \text{div}u = 0
\]
holds in \( \mathcal{D}'(Q_T) \), for any \( b \in C^1(\mathbb{R}) \) such that \( b'(z) \equiv 0 \) for all \( z \in \mathbb{R} \) large enough, where \( f = \rho, n \).

Now we are in the position to state our main result in the paper.

**Theorem 1.2** For any given \( \Gamma \geq \frac{9}{8} \) and \( \gamma \geq \frac{9}{5} \). Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) of class \( C^{2+v} \) for some \( v > 0 \). Under the conditions of (1.12)-(1.13), there exists a global weak solution \( (\rho, n, u) \) to (1.1)-(1.2).

**Remark 1.3** In Theorem 1.2, the global weak solution exists for \( \Gamma, \gamma \geq \frac{9}{5} \) without any domination conditions, which implies that transition to each single-phase flow is allowed. In addition, \( \Gamma \) and \( \gamma \) are independent within the interval \( [\frac{9}{5}, \infty) \), which indicates that it is not necessary for them to stay close to each other like
\[
\max \left\{ \frac{3\gamma}{4}, \gamma - 1, \frac{3(\gamma + 1)}{5} \right\} < \Gamma < \min \left\{ \frac{4\gamma}{3}, \gamma + 1, \frac{5\gamma}{3} - 1 \right\}
\]
as in [30] where \( \Gamma, \gamma > \frac{9}{5} \) and the pressure is given by the explicit pressure (1.3).

Theorem 1.2 provides the first result on the global solution to the compressible two-fluid system (1.1) in multi-dimensions without any domination conditions and smallness assumptions for the pressure given by (1.3) or by (1.4). Note that when \( \rho \equiv 0 \) or \( n \equiv 0 \), Theorem 1.2 perfectly matches the result of Lions [26] for compressible Navier-Stokes equation in a bounded domain of \( \mathbb{R}^3 \). Lemma 2.1 is very essential in the proof, which needs \( \rho, n \in L^2(0, T; L^2(\Omega)) \). As a consequence, we require that \( \Gamma + \theta_1 \geq 2 \) and \( \gamma + \theta_2 \geq 2 \) in Lemma 5.1, which yields \( \Gamma, \gamma \geq \frac{9}{5} \). Therefore it remains open whether both \( \Gamma \) and \( \gamma \) can get close to \( \frac{3}{2} \) in three dimensions even for the case with domination conditions.

**Remark 1.4** Note that Theorem 1.2 is also true for the two-dimensional case, and that the pressure function (1.3) for \( \Gamma = 2 \) is corresponding to the compressible MHD system without resistivity in two dimensions under the action of vertical magnetic field [25]. Thus as a byproduct, we obtain the existence of global weak solutions to the two-dimensional and non-resistive MHD system with non-negatively vertical magnetic field\(^3\). For the three-dimensional case with the pressure (1.3) and \( \Gamma = 2 \), it is motivated by compressible Oldroyd-B model with stress diffusion [1].

The main ingredients in the proof are stated as follows. As mentioned in the previous works [6,29,30], the main challenges focus on the pressure of two components which brings out some cross terms between the two densities. Section 5 is the main ingredient in the proof. In fact, in Sect. 5, the main point is to prove that \( P(n, \rho) = P(n, \rho) \) where \( P(n, \rho) \) is the weak limit of the approximate pressure \( P(n, \rho) \) as \( \delta \to 0^+ \). It suffices to establish the strong convergence of \( \rho_\delta \) and \( n_\delta \) as \( \delta \to 0^+ \). To achieve this, it is crucial to prove that
\[
\inf \left\{ \frac{T_k(\rho)}{T_k(n)} P(n, \rho), P(n, \rho) \right\} \leq \inf \left\{ \frac{T_k(\rho)}{T_k(n)} P(n, \rho), P(n, \rho) \right\},
\]
(1.15)

\(^3\) For the compressible MHD system with resistivity, the global existence of weak solutions with large initial data has been achieved by Hu, Wang [22]. However, for the case without resistivity, more essential challenges will arise due to the lack of regularity of the magnetic field.
a.e. on $Q_T$, where $T_k$ is a smooth cut-off function for $k = 1, 2, \ldots$. In Lions-Feireisl’s framework for compressible Navier-Stokes equations, the one-component pressure function with monotonicity and convexity gives rise to

$$
\overline{T_k}(\rho) \cdot P(\rho) \leq \overline{T_k}(\rho) P(\rho).
$$

But it is not the case for two-fluid system.

Compared with [30], the new challenge for the proof in the context of allowing unconstrained transition to single-phase flow is to remove (1.14) and allow the two indexes $\Gamma, \gamma$ to touch $\frac{9}{5}$. We state the main differences in the proof as below.

- First, to justify (1.15) without (1.14), we can not use the same decomposition of pressure by Vasseur, the author, and Yu ([30]) in the whole domain $Q_T$ any more, i.e.,

$$
P(n_\delta, \rho_\delta) = A^\Gamma (\rho_\delta + n_\delta)^\Gamma + B^\gamma (\rho_\delta + n_\delta)^\gamma + \text{remainder}
$$

a.e. on $Q_T$, where $(A, B) = (\frac{n}{\rho + n}, \frac{\rho}{\rho + n})$ if $\rho + n \neq 0$, since one can not even guarantee the integrability of $(\rho_\delta + n_\delta)^\gamma$ and $(\rho_\delta + n_\delta)^\Gamma$ in the whole domain without (1.14) based on the known estimates of $(\rho_\delta, n_\delta)$ in $L^{\gamma + \theta_2}(Q_T) \times L^{\Gamma + \theta_1}(Q_T)$ for $\theta_2 = \theta_2(\gamma)$, $\theta_1 = \theta_1(\Gamma)$ (see Lemma 5.1), and we do not even have $P(n, \rho) \leq \overline{P}(n, \rho)$ for the pressure (1.4). In this work we observe that the weighted functions $A$ and $B$ are able to cancel some possible oscillation of $\rho_\delta + n_\delta$. As a matter of fact, $A \rho_\delta$ and $B n_\delta$ are bounded in $L^{\Gamma + \theta_1}(Q_T')$ and in $L^{\gamma + \theta_2}(Q_T')$, respectively, for some domain $Q_T' \subset Q_T$ where the measure of $Q_T / Q_T'$ is small enough. This can be achieved by obtaining that $n_\delta - Ad_\delta \to 0$ and $\rho_\delta - Bd_\delta \to 0$ strongly in $L^1(Q_T)$ from (2.3) for $s = 1$. Thus we are able to justify (1.15) a.e. on $Q_T'$ by means of the decomposition of the pressure and the cut-off functions on $Q_T'$. Finally, by sending the measure of $Q_T / Q_T'$ to zero, we get (1.15). See Lemma 5.2 for more details. For the implicit pressure (1.4), we introduce a new non-decreasing function $G_{A,B}$ (see (5.37)), i.e.,

$$
G_{A,B}(z) := P(Az, Bz) - \frac{C_2}{\max\{\Gamma, \gamma\}} [(Az)^\Gamma + (Bz)^\gamma],
$$

to connect the implicit pressure with the explicit convex function $\frac{C_2}{\max\{\Gamma, \gamma\}} [(Az)^\Gamma + (Bz)^\gamma]$. The construction of $G_{A,B}$ is inspired by [16] for the compressible Navier-Stokes equations with non-mono pressure of one component.

- Second, to allow both $\Gamma$ and $\gamma$ to touch $\frac{9}{5}$, which represents a major step forward for the cases of transition to each single-phase flow and of more general pressure law compared with [30] where $\Gamma, \gamma \in (\frac{9}{5}, \infty)$ and the pressure (1.3) is considered only, it is important to prove that

$$
\| T_k(\rho) - \overline{T_k}(\rho) + T_k(n) - \overline{T_k}(n) \|_{L^{\Gamma}(Q_T / Q_T,k)} \to 0
$$
as $k \to \infty$, where $Q_{T,k}$ is given by (5.50). On the other hand, it is not difficult to justify

$$
\| T_k(\rho) - \overline{T_k}(\rho) + T_k(n) - \overline{T_k}(n) \|_{L^1(Q_T / Q_T,k)} \to 0
$$
as $k \to \infty$. Thus by means of the standard interpolation inequality, it suffices to get the upper bound of

$$
T_k(\rho) - \overline{T_k}(\rho) + T_k(n) - \overline{T_k}(n) \|_{L^{\Gamma_{\min} + 1}(Q_T / Q_T,k)} \quad (1.16)
$$
uniformly for $k$, where $\Gamma_{\min} + 1 = \min\{\Gamma, \gamma\} + 1 > 2$. In view of that $\rho_\delta^\Gamma$ and $n_\delta^\gamma$ might not be bounded in $L^{p_1}(Q_T)$ uniformly for $\delta$ where $p_1 > 1$, we derive a new estimate,
\[ \lim_{\delta \to 0} \| T_k(Ad_\delta) + T_k(Bd_\delta) - T_k(Ad) - T_k(Bd) \|_{L^{r_{min}+1}(Q_T')} \leq C_k \sigma^{\frac{K_{min}-1}{K_{min}}} + C \]  

(1.17)

where \( d_\delta = \rho_\delta + n_\delta \), \( K_{min} = \min\{ \Gamma_1^{\frac{\gamma_1}{\gamma_2}} + \theta_1, \gamma_2^{\frac{\gamma_1}{\gamma_2}} + \theta_2 \} \), and \( |Q_T'| \leq \sigma \). Here \( C \) is independent of \( \sigma, \delta \), and \( k \), and \( C_k \) is independent of \( \sigma \) and \( \delta \) but may depend on \( k \). With the new estimate (1.17), (1.16) can be bounded uniformly for \( k \). See Lemma 5.5 for more details.

The rest of the paper is organized as follows. In Sect. 2, we present some useful lemmas which will be used in the proof of Theorem 1.2. In Sect. 3, as usual we construct an approximation system with artificial viscosity coefficients in both continuity equations and with artificial pressure in the momentum equations. Then we explore a formal energy estimate due to the more complicated pressure (1.4) and sketch the proof of the global existence of the solution to the approximation system by virtue of the standard Faedo-Galerkin approach. In Sect. 4, we pass the quantities to the limits as the artificial viscosity coefficient goes to zero. With the artificial pressure, the pressure given by (1.3) or (1.4) has enough integrability. Then we only need to handle the difficulties arising in the implicit pressure (1.4) compared with our previous work [30]. In Sect. 5, we take the limits as the coefficient of artificial pressure, i.e., \( \delta \), go to zero. It is the last step for the proof. Some new estimates along with some new ideas are obtained in this section.

2 Some useful tools

Lemma 2.1 Let \( v_K \to 0 \) as \( K \to +\infty \), and \( v_K \geq 0 \). If \( \bar{\varrho}'_K \geq 0 \) for i = 1, 2, 3, ..., is a solution to

\[
(\bar{\varrho}'_K)_t + \text{div}(\bar{\varrho}'_K u_K) = v_K \Delta \bar{\varrho}'_K, \quad \bar{\varrho}'_K|_{t=0} = \bar{\varrho}'_0, \quad v_K \frac{\partial \bar{\varrho}'_K}{\partial \nu}|_{\partial \Omega} = 0, \tag{2.1}
\]

with \( C_0 \geq 1 \) independent of \( K \) such that

- \( \| \bar{\varrho}'_K \|_{L^2(0,T;L^2(\Omega))} + \| \bar{\varrho}'_K \|_{L^\infty(0,T;L^{\gamma'}(\Omega))} \leq C_0 \), \( \sqrt{v_K} \| \nabla \bar{\varrho}'_K \|_{L^2(0,T;L^2(\Omega))} \leq C_0 \).
- \( \| u_K \|_{L^2(0,T;H^1(\Omega))} \leq C_0 \).
- for any \( K > 0 \) and any \( t > 0 \):

\[
\int_{\Omega} \frac{(b^i_K)^2}{d_K} \, dx \leq \int_{\Omega} \frac{(b^i_0)^2}{d_0} \, dx, \tag{2.2}
\]

where \( b^i_K = \bar{\varrho}'_K, \quad d_K = \sum_{i=1}^{N} \bar{\varrho}'_K \) for any fixed integer \( N \geq 2 \), and \( \gamma' > 1 \).

Then, up to a subsequence, we have

\[
\bar{\varrho}'_K \to \varrho', \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^{\gamma'}(\Omega)),
\]

\[
u_K \to u \quad \text{weakly in } L^2(0,T;H^1(\Omega)),
\]

as \( K \to \infty \), and for any \( s \geq 1 \),

\[
\lim_{K \to +\infty} \int_0^T \int_{\Omega} d_K |a'_K - a|^s \, dx \, dt = 0, \tag{2.3}
\]
where \( a_K^i = \frac{b_K^i}{d_K} \) if \( d_K \neq 0 \), \( a^i = \frac{b^i}{a} \) if \( d \neq 0 \), and \( a_K^id_K = b^i_K \), \( a^id = b^i \) for \( i=1, 2, 3, \ldots \). Here \((b^i, d)\) is the weak limit of \((b^i_K, d_K)\) as \( K \to \infty \).

**Remark 2.2** For \( i = 1, 2 \), Lemma 2.1 can be found in [30]. It is not difficult to verify the more general case for \( i = 1, 2, 3, \ldots \), since (2.1) is a linear equation. The compactness conclusion here for the multi-equations with possible diffusion can be applied to study the multi-fluid system introduced in [29] where \( P = P(\rho_1, \rho_2, \ldots, \rho_N) \). Note that the proof in [30] relies on the DiPerna-Lions renormalized argument for transport equations [7,8]. Thus the \( L^2 \) bounds of the densities make it possible to use this theory for equations (2.1).

**Lemma 2.3** [30] Let \( \beta : \mathbb{R}^N \to \mathbb{R} \) be a \( C^1 \) function with \( |\nabla \beta(X)| \leq L^\infty(\mathbb{R}^N) \), and \( R \in (L^2(0, T; L^2(\Omega)))^N \), \( u \in L^2(0, T; H^1_0(\Omega)) \) satisfy

\[
\frac{\partial}{\partial t} R + \text{div}(u \otimes R) = 0, \quad R|_{t=0} = R_0(x)
\]

in the distribution sense. Then we have

\[
(\beta(R))_t + \text{div}(\beta(R)u) + [\nabla \beta(R) \cdot R - \beta(R)]\text{div}u = 0
\]

in the distribution sense. Moreover, if \( R \in L^\infty(0, T; L^\gamma(\Omega)) \) for \( \gamma > 1 \), then

\[ R \in C([0, T]; L^1(\Omega)) \]

and so

\[
\int_\Omega \beta(R)\, dx(t) = \int_\Omega \beta(R_0)\, dx - \int_0^t \int_\Omega [\nabla \beta(R) \cdot R - \beta(R)]\text{div}u\, dx\, dt.
\]

**Remark 2.4** \( N \) in Lemma 2.3 is specified to be 2 in the present paper (see Lemma 5.2).

**Lemma 2.5** ([19], Theorem 10.19) Let \( I \subset \mathbb{R} \) be an interval, \( Q \subset \mathbb{R}^N \) be a domain, and

\[
(P, G) \in C(I) \times C(I)
\]

be a couple of non-decreasing functions.

Assume that \( \varrho_n \in L^1(Q; I) \) is a sequence of functions such that

\[
\begin{align*}
P(\varrho_n) & \to \overline{P(\varrho)}, \\
G(\varrho_n) & \to \overline{G(\varrho)}, \\
P(\varrho_n)G(\varrho_n) & \to \overline{P(\varrho)G(\varrho)},
\end{align*}
\]

weakly in \( L^1(Q) \). Then

\[
\overline{P(\varrho)G(\varrho)} \leq \overline{P(\varrho)G(\varrho)}, \text{ a.e. in } Q.
\]

### 3 Existence of solutions to an approximate system

In this section, we construct a sequence of global weak solution \((\rho, n, u)\) to the following approximation system (3.1)–(3.3). Motivated by the work of [18,30], we consider the following approximation system

\[
\begin{align*}
n_t + \text{div}(nu) &= \epsilon \Delta n, \\
\rho_t + \text{div}(\rho u) &= \epsilon \Delta \rho, \\
[(\rho + n)u]_t + \text{div}[(\rho + n)u \otimes u] + \nabla P(n, \rho) \\
&\quad + \delta \nabla (\rho + n)^\beta + \epsilon \nabla u \cdot \nabla (\rho + n) \\
&= \mu \Delta u + (\mu + \lambda)\text{div}u
\end{align*}
\]

(3.1)
on $\Omega \times (0, \infty)$, with initial and boundary condition

$$
\begin{align*}
\left(\rho, n, (\rho + n)u\right)|_{t=0} &= (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}) \text{ on } \Omega, \\
\left(\frac{\partial \rho}{\partial v}, \frac{\partial n}{\partial v}, u\right) |_{\partial \Omega} &= 0,
\end{align*}
$$

(3.2) (3.3)

where $\epsilon, \delta > 0, \beta > \max\{\epsilon, \Gamma + 1, \gamma + 1\}, M_{0,\delta} = (\rho_{0,\delta} + n_{0,\delta})u_{0,\delta}$ and $n_{0,\delta}, \rho_{0,\delta} \in C^3(\Omega), u_{0,\delta} \in C^3_0(\Omega)$ satisfies

$$
\begin{align*}
&\begin{cases}
0 < \delta \leq \rho_{0,\delta}, n_{0,\delta} \leq \delta^{-\frac{1}{\eta}}, \left(\frac{\partial n_{0,\delta}}{\partial v}, \frac{\partial \rho_{0,\delta}}{\partial v}\right) |_{\partial \Omega} = 0, \\
lim_{\delta \to 0} \left(\|\rho_{0,\delta} - \rho_0\|_{L^\gamma(\Omega)} + \|n_{0,\delta} - n_0\|_{L^\gamma(\Omega)}\right) = 0, \\
u_{0,\delta} = \frac{\varphi_\delta}{\sqrt{\rho_{0,\delta} + n_{0,\delta}}} \eta_\delta \ast \left(\frac{M_0}{\sqrt{\rho_0 + n_0}}\right), \\
\sqrt{\rho_{0,\delta} + n_{0,\delta}}u_{0,\delta} \to \frac{M_0}{\sqrt{\rho_0 + n_0}} \text{ in } L^2(\Omega) \text{ as } \delta \to 0, \\
M_{0,\delta} \to M_0 \text{ in } L^1(\Omega) \text{ as } \delta \to 0,
\end{cases}
\end{align*}
$$

(3.4)

where $\delta \in (0, 1), \eta$ is a standard mollifier, $\varphi_\delta \in C_0^\infty(\Omega), 0 \leq \varphi_\delta \leq 1$ on $\Omega$ and $\varphi_\delta \equiv 1$ on $\{x \in \Omega | \text{dist}(x, \partial \Omega) > \delta\}$.

In order to simplify the presentation of the proof, we only consider the more complicated case of pressure, i.e., (1.4), in the rest of the paper.

3.1 A formal energy estimate

The main difference between the approximation system (3.1)–(3.3) and the one in [30] by Vasseur, the author, and Yu is that one of the pressure functions, i.e., (1.4), is more complicated. Therefore we will give a formal energy estimate in this part so that the Galerkin approach could work as in [30]. More specifically, we consider the pressure given by (1.4), and suppose that the solution to (3.1)–(3.3) is smooth enough.

Define $P_1(\rho_+) = A_+(\rho_+)^\gamma$ and $P_2(\rho_-) = A_-(\rho_-)^\Gamma$. Since $P(n, \rho) = P_1(\rho_+) = P_2(\rho_-)$, we decompose the pressure into two parts, i.e.,

$$
P(n, \rho) = \alpha P(n, \rho) + (1 - \alpha) P(n, \rho) = \alpha P_1(\rho_+) + (1 - \alpha) P_2(\rho_-),
$$

(3.5)

where $\alpha = \frac{\rho_+ - \rho_-}{\rho_+ - \rho_-}$. Actually, the idea for the decomposition (3.5) has been used by Evje, the author, Zhu [14] and by Bresch, Mucha, Zatorska [6] to study the one-dimensional case for the full compressible two-fluid equations with singular pressure gradient and multi-dimensional case for the compressible two-fluid Stokes equations, respectively. It is motivated by the full compressible two-fluid system with unequal velocities, see [3,4]. However, the Laplacian of $\rho$ and $n$ in (3.1) will make the estimates more complicated.

Multiplying (3.1)3 by $u$, integrating by parts over $\Omega$, and using (3.1)1 and (3.1)2, we have

$$
\begin{align*}
\frac{d}{dt} \int_{\Omega} \frac{1}{2} (\rho + n)|u|^2 \, dx &+ \int_{\Omega} \left(\mu|\nabla u|^2 + (\mu + \lambda)|\text{div}u|^2\right) \, dx \\
&= -\delta \int_{\Omega} u \cdot \nabla (\rho + n)^\beta \, dx - \int_{\Omega} u \cdot \nabla P(n, \rho) \, dx \\
&= I_1 + I_2.
\end{align*}
$$

(3.6)
For $I_1$, we have
\[
I_1 = \delta \int_{\Omega} \frac{\beta}{\beta - 1} (\rho + n)^{\beta - 1} \nabla \cdot [(\rho + n)u] \, dx
\]
\[
= -\delta \int_{\Omega} \frac{\beta}{\beta - 1} (\rho + n)^{\beta - 1} \rho + n \, dx + \delta \varepsilon \int_{\Omega} \frac{\beta}{\beta - 1} (\rho + n)^{\beta - 1} \Delta (\rho + n) \, dx
\]
\[
= -\frac{\delta}{\beta - 1} \frac{d}{dt} \int_{\Omega} (\rho + n)^{\beta - 1} \, dx - \delta \varepsilon \int_{\Omega} \beta (\rho + n)^{\beta - 2} |\nabla (\rho + n)|^2 \, dx.
\]  
(3.7)

For $I_2$, by virtue of the decomposition (3.5), we have
\[
I_2 = -\int_{\Omega} \alpha \nabla P_1(\rho_+) \cdot u \, dx - \int_{\Omega} (1 - \alpha) \nabla P_2(\rho_-) \cdot u \, dx
\]
\[
= I_{2,1} + I_{2,2}.
\]

For $I_{2,1}$, we have
\[
I_{2,1} = -\int_{\Omega} \frac{\gamma A_+}{\gamma - 1} (\rho u) \cdot \nabla \rho_+^{\gamma - 1} \, dx
\]
\[
= -\int_{\Omega} \frac{\gamma A_+}{\gamma - 1} \rho \rho_+^{\gamma - 1} \, dx + \varepsilon \int_{\Omega} \frac{\gamma A_+}{\gamma - 1} \rho_+^{\gamma - 1} \Delta \rho \, dx
\]
\[
= -\frac{d}{dt} \int_{\Omega} \frac{\gamma A_+}{\gamma - 1} \rho_+^{\gamma - 1} \, dx + \int_{\Omega} A_+ \gamma \rho \rho_+^{\gamma - 2}(\rho_+) \, dx - \varepsilon \int_{\Omega} A_+ \gamma \rho_+^{\gamma - 2} \nabla \rho \cdot \nabla \rho_+ \, dx
\]
\[
= -\frac{d}{dt} \int_{\Omega} \frac{\gamma A_+}{\gamma - 1} \rho_+^{\gamma - 1} \, dx + \int_{\Omega} \alpha (A_+ \rho_+^{\gamma - 1})_t \, dx - \varepsilon \int_{\Omega} A_+ \gamma \rho_+^{\gamma - 2} \nabla \rho \cdot \nabla \rho_+ \, dx.
\]  
(3.8)

Similarly, for $I_{2,2}$, we have
\[
I_{2,2} = -\frac{d}{dt} \int_{\Omega} \frac{\Gamma A_-}{\Gamma - 1} n \rho_-^{\Gamma - 1} \, dx + \int_{\Omega} (1 - \alpha)(A_- \rho_-^{\Gamma})_t \, dx - \varepsilon \int_{\Omega} A_- \Gamma \rho_-^{\Gamma - 2} \nabla n \cdot \nabla \rho_- \, dx.
\]  
(3.9)

(3.8) and (3.9) yield that
\[
I_{2,1} + I_{2,2} = -\frac{d}{dt} \int_{\Omega} \left( \frac{\gamma A_+}{\gamma - 1} \alpha \rho_+^{\gamma - 1} + \frac{\Gamma A_-}{\Gamma - 1} (1 - \alpha) \rho_-^{\Gamma} \right) \, dx + \int_{\Omega} (A_+ \rho_+^{\gamma})_t \, dx
\]
\[
- \varepsilon \int_{\Omega} \left( A_+ \gamma \rho_+^{\gamma - 2} \nabla \rho \cdot \nabla \rho_+ + A_- \Gamma \rho_-^{\Gamma - 2} \nabla n \cdot \nabla \rho_- \right) \, dx
\]
\[
= -\frac{d}{dt} \int_{\Omega} A_+ \rho_+^{\gamma} \left( \frac{\gamma}{\gamma - 1} \alpha + \frac{\Gamma}{\Gamma - 1} (1 - \alpha) - 1 \right) \, dx
\]
\[
- \varepsilon \int_{\Omega} \left( A_+ \gamma \rho_+^{\gamma - 2} \nabla \rho \cdot \nabla \rho_+ + A_- \Gamma \rho_-^{\Gamma - 2} \nabla n \cdot \nabla \rho_- \right) \, dx
\]
\[
= -\frac{d}{dt} \int_{\Omega} A_+ \rho_+^{\gamma} \left( \frac{\alpha}{\gamma - 1} + \frac{1 - \alpha}{\Gamma - 1} \right) \, dx
\]
\[
- \varepsilon \int_{\Omega} \left( A_+ \gamma \rho_+^{\gamma - 2} \nabla \rho \cdot \nabla \rho_+ + A_- \Gamma \rho_-^{\Gamma - 2} \nabla n \cdot \nabla \rho_- \right) \, dx,
\]  
(3.10)

where we have used $A_- \rho_-^{\Gamma} = A_+ \rho_+^{\gamma}$. We still need to analyze the last integral on the right hand side of (3.10). More specifically, substituting $\rho_- = \left( \frac{A_+}{A_-} \right)^{\frac{1}{\gamma}} \rho_+^{\gamma}$ into (1.4), and
differentiating the result with respect to $x$, we have
\[
(\frac{A_+}{A_-})^\frac{1}{\Gamma}\nabla\rho\rho_+^{\gamma} + (\frac{A_+}{A_-})^\frac{1}{\Gamma}\rho\rho_+^{\gamma-1}\nabla\rho_+ + \nabla\rho_+ + n\nabla\rho_+ = (\frac{A_+}{A_-})^\frac{1}{\Gamma}(\frac{\gamma}{\Gamma} + 1)\rho_+^{\gamma}\nabla\rho_+,
\]
which implies that
\[
\nabla\rho_+ = \left( (\frac{A_+}{A_-})^\frac{1}{\Gamma}\nabla\rho\rho_+^{\gamma} + \nabla\rho_+ \right) \left[ (\frac{A_+}{A_-})^\frac{1}{\Gamma}(\frac{\gamma}{\Gamma} + 1)\rho_+^{\gamma} - (\frac{A_+}{A_-})^\frac{1}{\Gamma}\rho_+^{\gamma-1} - n \right]^{-1}.
\]
Note that
\[
n = (1 - \alpha)\rho_- = (1 - \alpha)\left( \frac{A_+}{A_-} \right)^\frac{1}{\Gamma}\rho_+^{\gamma}.
\]
Hence we have
\[
\nabla\rho_+ = \left[ (\frac{A_+}{A_-})^\frac{1}{\Gamma}\nabla\rho\rho_+^{\gamma} + \nabla\rho_+ \right] \left[ (\frac{A_+}{A_-})^\frac{1}{\Gamma}(\frac{\gamma}{\Gamma} + 1)\rho_+^{\gamma} - (\frac{A_+}{A_-})^\frac{1}{\Gamma}\rho_+^{\gamma-1} - (1 - \alpha)\left( \frac{A_+}{A_-} \right)^\frac{1}{\Gamma}\rho_+^{\gamma} \right]^{-1}
\]
\[
= \frac{(\frac{A_+}{A_-})^\frac{1}{\Gamma}\nabla\rho\rho_+^{\gamma} + \nabla\rho_+}{(\frac{A_+}{A_-})^\frac{1}{\Gamma}\rho_+^{\gamma} \left[ \frac{\gamma}{\Gamma}(1 - \alpha) + \alpha \right]}
\]
\[
(3.11)
\]
Since
\[
\rho_- = \left( \frac{A_+}{A_-} \right)^\frac{1}{\Gamma}\rho_+^{\gamma},
\]
we have
\[
\nabla\rho_- = \left( \frac{A_+}{A_-} \right)^\frac{1}{\Gamma}\rho_+^{\gamma-1}\nabla\rho_+
\]
\[
= \frac{\gamma}{\Gamma}\rho_+^{\gamma-1}\nabla\rho_+ + \nabla n.
\]
(3.12)
where we have used (3.11).

Now we are in a position to evaluate the last integral on the right hand side of (3.10). In view of (3.11) and (3.12), we have
\[
A_+\rho_+^{\gamma-2}\nabla\rho \cdot \nabla\rho_+ + A_-\Gamma\rho_-^{\gamma-2}\nabla n \cdot \nabla\rho_-
\]
\[
= A_+\rho_+^{\gamma-2}\left[ \frac{\gamma}{\Gamma}\rho_+^{\gamma-1}\nabla\rho_+ + \nabla n \cdot \nabla\rho_- \right] + A_-\Gamma\rho_-^{\gamma-2}\left[ \frac{\gamma}{\Gamma}\rho_+^{\gamma-1}\nabla\rho_+ + \nabla n \cdot \nabla\rho_+ \right]
\]
\[
= \gamma A_+\rho_+^{\gamma-2}\left[ \rho_+^{\gamma-2}\rho_+^{\gamma-1}\nabla\rho_+ + \nabla\rho_+ \cdot \nabla n \right] + \rho_-^{\gamma-2}\left[ \frac{\gamma}{\Gamma}\rho_+^{\gamma-1}\nabla\rho_+ + \nabla n \cdot \nabla\rho_+ \right]
\]
\[
= \gamma A_+\rho_+^{\gamma-2}\left[ \rho_+^{\gamma-2}\nabla\rho_+ + \rho_+^{-1}\nabla\rho_+ \cdot \nabla n + (A_+)^{-\frac{1}{\gamma}}\rho_+^{\gamma-1}\nabla n \cdot \nabla\rho_+ + (A_+)^{-\frac{1}{\gamma}}\rho_+^{\gamma-1}\nabla n \cdot \nabla\rho_+ \right]
\]
\[
= \gamma A_+\rho_+^{\gamma-2}\left[ \rho_+^{\gamma-2}\nabla\rho_+ + (A_+)^{-\frac{1}{\gamma}}\rho_+^{\gamma-1}\nabla n \right].
\]
This combined with (3.10) yields
\[ I_2 = I_{2,1} + I_{2,2} \]
\[ = -\frac{d}{dt} \int_\Omega A_+ \rho_+^\gamma \left( \frac{\alpha}{\gamma - 1} + \frac{1 - \alpha}{\Gamma - 1} \right) dx \]
\[ - \epsilon \int_\Omega \frac{\gamma A_+ \rho_+^\gamma}{\gamma + 1} \left( \frac{1}{(1 - \alpha) + \alpha} \right) \rho_+^{-1} \nabla \rho + \left( \frac{A_+}{A_-} \right)^{-\frac{1}{2}} \rho_+^{-\frac{3}{2}} \nabla n \right)^2 dx. \]

Combining (3.6), (3.7) and (3.13), we have
\[
d \frac{dt}{\Omega} \left[ \frac{1}{2} \rho + n |u|^2 + \frac{\delta}{\beta - 1} (\rho + n)^\beta + A_+ \rho_+^\gamma \left( \frac{\alpha}{\gamma - 1} + \frac{1 - \alpha}{\Gamma - 1} \right) \right] dx
\[ + \int_\Omega \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 \right) dx
\[ = -\epsilon \int_\Omega \left[ \delta \beta (\rho + n)^{\beta - 2} |\nabla (\rho + n)|^2 + \frac{\gamma A_+ \rho_+^\gamma}{\gamma + 1} \left( \frac{1}{(1 - \alpha) + \alpha} \right) \rho_+^{-1} \nabla \rho + \left( \frac{A_+}{A_-} \right)^{-\frac{1}{2}} \rho_+^{-\frac{3}{2}} \nabla n \right)^2 dx. \]

(3.14)

### 3.2 Faedo-Galerkin approach

In this part, motivated by \[18, 26\] (see also \[30\]), we will use Faedo-Galerkin approach to construct a global weak solution to (3.1), (3.2) and (3.3). To begin with, we consider a sequence of finite dimensional spaces
\[ X_k = \{ \text{span} \{ \psi_j \}_{j=1}^k \}^3, \quad k \in \{ 1, 2, 3, \ldots \}, \]
where \( \{ \psi_i \}_{i=1}^\infty \) is the set of the eigenfunctions of the Laplacian:
\[
\begin{cases} 
-\Delta \psi_i = \lambda_i \psi_i & \text{on } \Omega, \\
\psi_i |_{\partial \Omega} = 0.
\end{cases}
\]

For any given \( \epsilon, \delta > 0 \), we shall look for the approximate solution \( u_k \in C([0, T]; X_k) \) (for any fixed \( T > 0 \) given by the following form:
\[
\int_\Omega (\rho_k + n_k) u_k(t) \cdot \psi dx - \int_0^T \int_\Omega m_{0,\delta} \cdot \psi dx = \int_0^T \int_\Omega \left[ \mu \Delta u_k + (\mu + \lambda) \nabla \text{div} u_k \right] \cdot \psi dx ds
\[ - \int_0^T \int_\Omega \left[ \text{div} (\rho_k + n_k) u_k \otimes u_k + \nabla P(n_k, \rho_k) + \delta \nabla (\rho_k + n_k)^\beta + \epsilon \nabla u_k \cdot \nabla (\rho_k + n_k) \right] \cdot \psi dx ds
\]
(3.15)

for \( t \in [0, T] \) and \( \psi \in X_k \), where \( \rho_k = \rho_k(u_k) \) and \( n_k = n_k(u_k) \) satisfy
\[
\begin{cases} 
\partial_t n_k + \text{div} (n_k u_k) = \epsilon \Delta n_k, \\
\partial_t \rho_k + \text{div} (\rho_k u_k) = \epsilon \Delta \rho_k, \\
n_k |_{t=0} = n_{0,\delta}, \quad \rho_k |_{t=0} = \rho_{0,\delta}, \\
\left( \frac{\partial n_k}{\partial \nu}, \frac{\partial \rho_k}{\partial \nu} \right) |_{\partial \Omega} = 0.
\end{cases}
\]

(3.16)

Due to Lemmas 2.1 and 2.2 in \[18\], the problem (3.15) can be solved on a short time interval \([0, T_k]\) for \( T_k \leq T \) by a standard fixed point theorem on the Banach space \( C([0, T_k]; X_k) \). To show that \( T_k = T \), as in \[18\] (see also \[30\]), we only need to get the energy estimate
(3.14) with \((\rho, n, u)\) replaced by \((\rho_k, n_k, u_k)\), which could be done by differentiating (3.15) with respect to time, taking \(\psi = u_k(t)\) and using (3.16). We refer the readers to [18] for more details. Thus, we obtain a solution \((\rho_k, n_k, u_k)\) to (3.15)–(3.16) globally in time with the following bounds

\[
0 < \frac{1}{c_k} \leq \rho_k(x, t), n_k(x, t) \leq c_k \text{ for a.e. } (x, t) \in \Omega \times (0, T),
\]

\[
\sup_{t \in [0, T]} \|\rho_+ k(t)\|_{L^r(\Omega)}^r \leq C(\rho_0, n_0, M_0),
\]

\[
\sup_{t \in [0, T]} \|\rho_- k(t)\|_{L^r(\Omega)}^r \leq C(\rho_0, n_0, M_0),
\]

\[
\delta \sup_{t \in [0, T]} \|\rho k(t) + n_k(t)\|_{L^p(\Omega)}^p \leq C(\rho_0, n_0, M_0),
\]

\[
\sup_{t \in [0, T]} \|\sqrt{\rho_k + n_k(t)} u_k(t)\|_{L^2(\Omega)}^2 \leq C(\rho_0, n_0, M_0),
\]

\[
\int_0^T \|u_k(t)\|_{H_0^1(\Omega)}^2 \, dt \leq C(\rho_0, n_0, M_0),
\]

\[
\epsilon \int_0^T (\|\nabla \rho_k(t)\|_{L^2(\Omega)}^2 + \|\nabla n_k(t)\|_{L^2(\Omega)}^2) \, dt \leq C(\beta, \delta, \rho_0, n_0, M_0),
\]

\[
\|\rho_k + n_k\|_{L^{\beta+1}(Q_T)} \leq C(\epsilon, \beta, \delta, \rho_0, n_0, M_0),
\]

where \(Q_T = \Omega \times (0, T)\) and \(\beta \geq 4\).

This yields the following Proposition by the analysis in [18] (see also [30]).

**Proposition 3.1** Suppose \(\beta > \max\{4, \Gamma + 1, \gamma + 1\}\). For any given \(\epsilon, \delta > 0\), there exists a global weak solution \((\rho, n, u)\) to (3.1), (3.2) and (3.3) such that for any given \(T > 0\), the following estimates

\[
\sup_{t \in [0, T]} \|\rho_+ (t)\|_{L^r(\Omega)}^r \leq C(\rho_0, n_0, M_0),
\]

\[
\sup_{t \in [0, T]} \|\rho_- (t)\|_{L^r(\Omega)}^r \leq C(\rho_0, n_0, M_0),
\]

\[
\delta \sup_{t \in [0, T]} \|(\rho, n)(t)\|_{L^p(\Omega)}^p \leq C(\rho_0, n_0, M_0),
\]

\[
\sup_{t \in [0, T]} \|\sqrt{\rho + nu(t)}\|_{L^2(\Omega)}^2 \leq C(\rho_0, n_0, M_0),
\]

\[
\int_0^T \|u(t)\|_{H_0^1(\Omega)}^2 \, dt \leq C(\rho_0, n_0, M_0),
\]

\[
\epsilon \int_0^T (\|\nabla \rho, \nabla n(t)\|_{L^2(\Omega)}^2) \, dt \leq C(\beta, \delta, \rho_0, n_0, M_0),
\]

and

\[
\|(\rho, n)(t)\|_{L^{\beta+1}(Q_T)} \leq C(\epsilon, \beta, \delta, \rho_0, n_0, M_0)
\]

hold, where the norm \(\|\cdot, \cdot\|\) denotes \(\|\cdot\| + \|\cdot\|\), and \(\rho, n \geq 0\) a.e. on \(Q_T\).

Finally, there exists \(r > 1\) such that \(\rho_t, n_t, \nabla^2 \rho, \nabla^2 n \in L^r(Q_T)\) and the equations (3.1)\textsubscript{1} and (3.1)\textsubscript{2} are satisfied a.e. on \(Q_T\).
4 The vanishing of the artificial viscosity

In this section, let $C$ denote a generic positive constant depending on the initial data, $\delta$ and some other known constants but independent of $\epsilon$.

4.1 Passing to the limit as $\epsilon \to 0^+$

The uniform estimates for $\epsilon$ resulting from (3.18), (3.19), and (3.20) are not sufficient to obtain the weak convergence of the artificial pressure $P(n_\epsilon, \rho_\epsilon) + \delta(\rho_\epsilon + n_\epsilon)^\beta$ which is bounded only in $L^1(Q_T)$. Thus we need to obtain higher integrability estimate of the artificial pressure uniformly for $\epsilon$.

In the rest of the section, we remove the subscript $\epsilon$ of the solutions for brevity.

Lemma 4.1 Let $(\rho, n, u)$ be the solution given by Proposition 3.1, then

$$ \int_0^T \int_\Omega (n^{\gamma+1} + \rho^{\gamma+1} + \delta \rho^{\beta+1} + \delta n^{\beta+1}) \, dx \, dt \leq C $$

for $\beta > \max\{4, \Gamma + 1, \gamma + 1\}$.

Proof The proof can be done by using (1.7) and the arguments similar to [18] where the test function $\psi(t)\mathcal{B}[\rho - \hat{\rho}]$ is replaced by $\psi(t)\mathcal{B}[\rho + n - \hat{\rho} + \hat{n}]$. Here

$$ \mathcal{B} : \left\{ f \in L^p(\Omega); \ |\Omega|^{-1} \int_\Omega f \, dx = 0 \right\} \mapsto W_0^{1,p}(\Omega), \quad 1 < p < \infty, $$

$$ \psi \in C_0^\infty(0, T), \quad 0 \leq \psi \leq 1, \text{ and } \hat{G} = \frac{1}{|\Omega|} \int_\Omega G \, dx $$

for $G = \rho, \ \rho + n$.

Due to the relation between $P$ and $(n, \rho)$, i.e., (1.7), we have the following corollary.

Corollary 4.2 Let $(\rho, n, u)$ be the solution given by Proposition 3.1, then

$$ \int_0^T \int_\Omega (\rho_+^{\gamma_1} + \rho_-^{\Gamma_1}) \, dx \, dt \leq C, $$

where $\gamma_1 = \gamma \min\{\frac{\gamma+1}{\gamma}, \frac{\Gamma+1}{\Gamma}\}$ and $\Gamma_1 = \Gamma \min\{\frac{\gamma+1}{\gamma}, \frac{\Gamma+1}{\Gamma}\}$. Note that $\gamma_1 > \gamma$ and $\Gamma_1 > \Gamma$.

With (3.18)–(3.23) and Lemma 4.1 and Corollary 4.2, we are able to pass to the limits as $\epsilon \to 0^+$.

Before doing this, we need to dress the approximate solution constructed in Proposition 3.1 in the lower subscript “$\epsilon$” for fixed $\delta > 0$, i.e., $(\rho_\epsilon, n_\epsilon, u_\epsilon)$. Then letting $\epsilon \to 0^+$ (taking a subsequence if necessary), we have

$$ \left\{ \begin{array}{l}
(\rho_\epsilon, n_\epsilon) \to (\rho, n) \text{ in } C([0, T]; L^1_{\text{weak}}(\Omega)) \cap C([0, T]; H^{-1}(\Omega)) \text{ and weakly in } L^{\beta+1}(Q_T), \\
(\epsilon \Delta \rho_\epsilon, \epsilon \Delta n_\epsilon) \to 0 \text{ weakly in } L^2(0, T; H^{-1}(\Omega)), \\
u_\epsilon \to u \text{ weakly in } L^2(0, T; H^1_0(\Omega)), \\
(\rho_\epsilon + n_\epsilon)u_\epsilon \to (\rho + n)u \text{ in } C([0, T]; L^2_{\text{weak}}(\Omega)) \cap C([0, T]; H^{-1}(\Omega)), \\
(\rho_\epsilon n_\epsilon, n_\epsilon u_\epsilon) \to (\rho u, nu) \text{ in } \mathcal{D}'(Q_T), \\
(\epsilon \Delta n_\epsilon + \epsilon \Delta u_\epsilon) \to (\Delta n + \Delta u) \text{ in } \mathcal{D}'(Q_T), \\
P(n_\epsilon, \rho_\epsilon) + \delta(\rho_\epsilon + n_\epsilon)^\beta \to P(n, \rho) + \delta(\rho + n)^\beta \text{ weakly in } L^\frac{\beta+1}{\beta}(Q_T), \\
\epsilon \nabla u_\epsilon \cdot \nabla (\rho_\epsilon + n_\epsilon) \to 0 \text{ in } L^1(Q_T),
\end{array} \right. $$

$$ (4.1) $$
and $\rho, n \geq 0$, where the limit $(\rho, n, u)$ solves the following system in the sense of distribution on $QT$ for any $T > 0$:

$$
\begin{align*}
\begin{cases}
{n_t + \text{div}(nu) = 0,} \\
{\rho_t + \text{div}(\rho u) = 0,}
\end{cases}
\end{align*}
$$

$$
\begin{align*}
[(\rho + n)u]_t + \text{div}[(\rho + n)u \otimes u] + \nabla P(n, \rho) + \delta(\rho + n)^\beta = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u
\end{align*}
$$

with initial and boundary condition

$$
\begin{align*}
(\rho, n, (\rho + n)u)|_{t=0} = (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}), \\
u|_{\partial \Omega} = 0,
\end{align*}
$$

where $f(t, x)$ denotes the weak limit of $f_\epsilon(t, x)$ as $\epsilon \to 0$.

To this end, we have to show that

$$
P(n, \rho) + \delta(\rho + n)^\beta = P(n, \rho) + \delta(\rho + n)^\beta.
$$

### 4.2 The weak limit of the pressure

This part is similar to [30], where it focuses on the more complicated pressure $P$, since the artificial pressure term $\delta(\rho + n_\epsilon)^\beta$ controls the possible oscillation for $(\rho_\epsilon + n_\epsilon)^\gamma$ and $(\rho_\epsilon + n_\epsilon)^\nu$ arising in one of the decomposition terms of the pressure, i.e., $P(A_\epsilon, B_\epsilon)$ where $d_\epsilon = \rho_\epsilon + n_\epsilon, (A, B) = \left(\frac{n}{\rho + n}, \frac{\rho}{\rho + n}\right)$ if $\rho + n \neq 0$, and $0 \leq A, B \leq 1, \left(A(\rho + n), B(\rho + n)\right) = (n, \rho)$.

**Claim**

$$
P(n, \rho) + \delta(\rho + n)^\beta = P(n, \rho) + \delta(\rho + n)^\beta
$$

a.e. on $QT$.

The proof of (4.5) relies on the following lemmas. In particular, the next lemma plays an essential role.

**Lemma 4.3** Let $(\rho_\epsilon, n_\epsilon)$ be the solution stated in Proposition 3.1, and $(\rho, n)$ be the limit in the sense of (4.1), then

$$
(\rho + n)P(n, \rho) \leq (\rho + n)P(n, \rho)
$$

a.e. on $\Omega \times (0, T)$.

**Proof** The idea is similar to [30] by Vasseur, the author, and Yu. However, since the pressure here is more complicated, we have to give a complete proof.

As in [30], the pressure and $n_\epsilon + \rho_\epsilon$ are decomposed as follows.

$$
\begin{align*}
P(n_\epsilon, \rho_\epsilon) &= P(A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) - P(A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) + P(A_\epsilon d_\epsilon, B_\epsilon d_\epsilon), \\
n_\epsilon + \rho_\epsilon &= (A_\epsilon + B_\epsilon)d_\epsilon = (A + B)d_\epsilon + (A_\epsilon - A + B_\epsilon - B)d_\epsilon.
\end{align*}
$$

where $d_\epsilon = \rho_\epsilon + n_\epsilon, d = \rho + n, (A_\epsilon, B_\epsilon) = \left(\frac{n}{\rho + n}, \frac{\rho}{\rho + n}\right)$ if $d_\epsilon \neq 0, (A, B) = \left(\frac{n}{\rho + n}, \frac{\rho}{\rho + n}\right)$ if $d \neq 0, 0 \leq A_\epsilon, B_\epsilon, A, B \leq 1, \text{and} (A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) = (n_\epsilon, \rho_\epsilon), (A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) = (n_\epsilon, \rho_\epsilon), (A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) = (n_\epsilon, \rho_\epsilon), (\rho, n)$ is the limit of $(\rho_\epsilon, n_\epsilon)$ in a suitable weak topology.
For any \( \psi \in C([0, t]), \phi \in C(\overline{\Omega}) \) where \( \psi, \phi \geq 0 \), we use (4.7) and obtain
\[
\int_0^t \psi \int_\Omega \phi P(n_\epsilon, \rho_\epsilon)(\rho_\epsilon + n_\epsilon)\,dx\,ds
= \int_0^t \psi \int_\Omega \phi P(Ad_\epsilon, Bd_\epsilon)(A + B)d_\epsilon\,dx\,ds
+ \int_0^t \psi \int_\Omega \phi P(Ad_\epsilon, Bd_\epsilon)(A_\epsilon - A + B_\epsilon - B)d_\epsilon\,dx\,ds
+ \int_0^t \psi \int_\Omega \phi \left[ P(A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) - P(Ad_\epsilon, Bd_\epsilon) \right](\rho_\epsilon + n_\epsilon)\,dx\,dt
= \sum_{i=1}^3 II_i. \tag{4.8}
\]

For \( II_2 \), we follow an argument similar to [30]. More specifically, there exists a positive integer \( k_0 \) large enough such that
\[
\max\{k_0\gamma/k_0 - 1, k_0\Gamma/k_0 - 1\} \leq \beta \tag{4.9}
\]
due to the assumption that \( \max\{\Gamma, \gamma\} < \beta \). Therefore (4.9) implies that
\[
\int_{Q_T} d_\epsilon^k \left| d_\epsilon^{k_0/\Gamma} + d_\epsilon^{k_0/\gamma} - 1 \right| \,dx\,dt \leq C \int_{Q_T} \left| d_\epsilon^{\beta+1} + 1 \right| \,dx\,dt \leq C \tag{4.10}
\]
where we have used Lemma 4.1 that \( d_\epsilon \) is bounded in \( L^{\beta+1}(Q_T) \) uniformly for \( \epsilon \).
Recalling (1.7), we have
\[
P(Ad_\epsilon, Bd_\epsilon) \leq C_0(A^{\Gamma}d_\epsilon^{\Gamma} + B^{\gamma}d_\epsilon^{\gamma}). \tag{4.11}
\]
This together with Hölder inequality and (4.10) yields
\[
|II_2| \leq C \left( \int_{Q_T} d_\epsilon |A_\epsilon - A|^{k_0} \,dx\,dt \right)^{1/k_0} \left( \int_{Q_T} d_\epsilon |A^{\Gamma}d_\epsilon^{\Gamma} + B^{\gamma}d_\epsilon^{\gamma}|^{k_0/\gamma - 1} \,dx\,dt \right)^{k_0-1/k_0}
+ C \left( \int_{Q_T} d_\epsilon |B_\epsilon - B|^{k_0} \,dx\,dt \right)^{1/k_0} \left( \int_{Q_T} d_\epsilon |A^{\Gamma}d_\epsilon^{\Gamma} + B^{\gamma}d_\epsilon^{\gamma}|^{k_0/\gamma - 1} \,dx\,dt \right)^{k_0-1/k_0}
\leq C \left( \int_{Q_T} d_\epsilon |A_\epsilon - A|^{k_0} \,dx\,dt \right)^{1/k_0} \left( \int_{Q_T} d_\epsilon |d_\epsilon^{k_0/\Gamma} + d_\epsilon^{k_0/\gamma} - 1| \,dx\,dt \right)^{k_0-1/k_0} \tag{4.12}
+ C \left( \int_{Q_T} d_\epsilon |B_\epsilon - B|^{k_0} \,dx\,dt \right)^{1/k_0} \left( \int_{Q_T} d_\epsilon |d_\epsilon^{k_0/\Gamma} + d_\epsilon^{k_0/\gamma} - 1| \,dx\,dt \right)^{k_0-1/k_0}
\leq C \left( \int_{Q_T} d_\epsilon |A_\epsilon - A|^{k_0} \,dx\,dt \right)^{1/k_0} + C \left( \int_{Q_T} d_\epsilon |B_\epsilon - B|^{k_0} \,dx\,dt \right)^{1/k_0}.
Choosing \( \nu_k := \nu = \epsilon \) in Lemma 2.1, we conclude that

\[
\left( \int_{\Omega_T} d_\epsilon |A_\epsilon - A|^{k_0} \, dx \, dt \right)^{\frac{1}{k_0}} \to 0,
\]

\[
\left( \int_{\Omega_T} d_\epsilon |B_\epsilon - B|^{k_0} \, dx \, dt \right)^{\frac{1}{k_0}} \to 0
\]

as \( \epsilon \) goes to zero. In fact, \( d_\epsilon \in L^\infty(0, T; L^\beta(\Omega)) \) for \( \beta > 4 \), and \( u_\epsilon \in L^2(0, T; H^1_0(\Omega)) \), and

\[
\sqrt{\epsilon} \|
abl\rho_\epsilon \|_{L^2(0, T; L^2(\Omega))} \leq C_0, \quad \sqrt{\epsilon} \|
abl n_\epsilon \|_{L^2(0, T; L^2(\Omega))} \leq C_0,
\]

and for any \( \epsilon > 0 \) and any \( t > 0 \):

\[
\int_{\Omega} \frac{b_\epsilon^2}{d_\epsilon} \, dx \leq \int_{\Omega} \frac{b_0^2}{d_0} \, dx \tag{4.14}
\]

where \( d_\epsilon = \rho_\epsilon + n_\epsilon, b_\epsilon = \rho_\epsilon, n_\epsilon \), and (4.14) is obtained in Remark 2.4, [30]. Thus, we are able to apply Lemma 2.1 to deduce (4.13). Hence we have \( II_2 \to 0 \) as \( \epsilon \to 0 \).

For \( II_3 \), the analysis becomes more complicated due to the pressure. First, we need the following estimate.

\[
P(A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) - P(Ad_\epsilon, Bd_\epsilon) = A_+ \gamma [\rho_+(\xi_1, \xi_2)]^{\gamma-1} \partial_{\xi_1} \rho_+(\xi_1, \xi_2) [A_\epsilon d_\epsilon - Ad_\epsilon]
\]

\[
+ A_+ \gamma [\rho_+(\xi_1, \xi_2)]^{\gamma-1} \partial_{\xi_2} \rho_+(\xi_1, \xi_2) [B_\epsilon d_\epsilon - Bd_\epsilon]
\]

\[
= A_+ \gamma \left( \frac{A_-}{A_+} \right)^{\gamma} [\rho_+(\xi_1, \xi_2)]^{\gamma-\frac{r}{\gamma}}
\]

\[
\times \left[ \frac{1}{2} (1 - \alpha(\xi_1, \xi_2)) + \alpha(\xi_1, \xi_2) \right] [A_\epsilon d_\epsilon - Ad_\epsilon]
\]

\[
+ \frac{A_+ \gamma [\rho_+(\xi_1, \xi_2)]^{\gamma-1}}{\frac{1}{2} (1 - \alpha(\xi_1, \xi_2)) + \alpha(\xi_1, \xi_2)} [B_\epsilon d_\epsilon - Bd_\epsilon], \tag{4.15}
\]

since

\[
\left\{ \begin{align*}
\frac{\partial \rho_+(n, \rho)}{\partial n} &= \left( \frac{A_-}{A_+} \right)^{\frac{1}{\gamma}} \frac{1}{\gamma} \rho_+^{\frac{r}{\gamma}}, \\
\frac{\partial \rho_+(n, \rho)}{\partial \rho} &= \frac{\frac{1}{2} (1 - \alpha) + \alpha}{\frac{1}{2} (1 - \alpha) + \alpha}
\end{align*} \right. \tag{4.16}
\]

which can be obtained similarly to (3.11). Here \( \xi_1 (\xi_2) \) varies between \( A_\epsilon d_\epsilon (B_\epsilon d_\epsilon) \) and \( Ad_\epsilon (Bd_\epsilon) \).

In view of (1.7), we have

\[
\rho_+(\xi_1, \xi_2) \leq C \left( \frac{\xi_1}{\xi_1} + \xi_2 \right) \leq C \left[ (\rho_\epsilon + n_\epsilon)^{\frac{r}{\gamma}} + \rho_\epsilon + n_\epsilon \right], \tag{4.17}
\]

where we have used

\[
0 \leq \xi_1, \xi_2 \leq \rho_\epsilon + n_\epsilon.
\]
By virtue of (4.15) and (4.17), and using Young inequality, we have

\[
|P(A_\varepsilon d_\varepsilon, B_\varepsilon d_\varepsilon) - P(A d_\varepsilon, B d_\varepsilon)| \leq C[(\rho_\varepsilon + n_\varepsilon)^\gamma + \rho_\varepsilon + n_\varepsilon]^{\gamma - \frac{1}{\gamma}} |A_\varepsilon d_\varepsilon - A d_\varepsilon + C[(\rho_\varepsilon + n_\varepsilon)^\gamma + \rho_\varepsilon + n_\varepsilon]^{\gamma - 1} |B_\varepsilon d_\varepsilon - B d_\varepsilon| \\
\leq C \left[ (\rho_\varepsilon + n_\varepsilon)^\gamma + (\rho_\varepsilon + n_\varepsilon)\gamma + 1 \right] \left( |A_\varepsilon d_\varepsilon - A d_\varepsilon| + |B_\varepsilon d_\varepsilon - B d_\varepsilon| \right) \\
\leq C d_\varepsilon^{\Gamma_m + 1} \left( |A_\varepsilon d_\varepsilon - A d_\varepsilon| + |B_\varepsilon d_\varepsilon - B d_\varepsilon| \right) , \tag{4.18}
\]

where \( \Gamma_m = \max \{ \Gamma, \gamma \} \).

Now we are in a position to estimate \( II_3 \). In fact, there exists a positive integer \( k_1 \) large enough such that

\[
(\Gamma_m + 2 - \frac{1}{k_1}) \frac{k_1}{k_1 - 1} < \beta + 1 \tag{4.19}
\]
due to the assumption \( \gamma + 1, \Gamma + 1 < \beta \).

In virtue of (4.18), we have

\[
|II_3| = \int_0^T \int_\Omega \psi \left[ P(A_\varepsilon d_\varepsilon, B_\varepsilon d_\varepsilon) - P(A d_\varepsilon, B d_\varepsilon) \right] (\rho_\varepsilon + n_\varepsilon) \, dx \, dt \\
\leq C \int_{Q_T} (d_\varepsilon^{\Gamma_m + 1} + 1) |A_\varepsilon d_\varepsilon - A d_\varepsilon| \, dx \, ds + C \int_{Q_T} (d_\varepsilon^{\Gamma_m + 1} + 1) |B_\varepsilon d_\varepsilon - B d_\varepsilon| \, dx \, ds \\
= C \int_{Q_T} (d_\varepsilon^{\Gamma_m + 2 - \frac{1}{k_1}} d_\varepsilon^{\frac{1}{k_1}} + d_\varepsilon^{\frac{1}{k_1}} d_\varepsilon^2) |A_\varepsilon - A| \, dx \, ds \\
+ C \int_{Q_T} (d_\varepsilon^{\Gamma_m + 2 - \frac{1}{k_1}} d_\varepsilon^{\frac{1}{k_1}} + d_\varepsilon^{\frac{1}{k_1}} d_\varepsilon^2) |B_\varepsilon - B| \, dx \, ds .
\]

Then applying Hölder inequality, we get

\[
|II_3| \leq C \left( \int_{Q_T} d_\varepsilon \left( d_\varepsilon^{\Gamma_m + 2 - \frac{1}{k_1}} \right)^{\frac{k_1}{k_1 - 1}} \, dx \, dt \right)^{\frac{k_1 - 1}{k_1}} \left( \int_{Q_T} d_\varepsilon |A_\varepsilon - A|^{k_1} \, dx \, dt \right)^{\frac{1}{k_1}} \\
+ C \left( \int_{Q_T} d_\varepsilon \, dx \, dt \right)^{\frac{1}{2}} \left( \int_{Q_T} d_\varepsilon |A_\varepsilon - A|^{2} \, dx \, dt \right)^{\frac{1}{2}} \\
+ C \left( \int_{Q_T} d_\varepsilon \left( d_\varepsilon^{\Gamma_m + 2 - \frac{1}{k_1}} \right)^{\frac{k_1}{k_1 - 1}} \, dx \, dt \right)^{\frac{k_1 - 1}{k_1}} \left( \int_{Q_T} d_\varepsilon |B_\varepsilon - B|^{k_1} \, dx \, dt \right)^{\frac{1}{k_1}} \\
+ C \left( \int_{Q_T} d_\varepsilon \, dx \, dt \right)^{\frac{1}{2}} \left( \int_{Q_T} d_\varepsilon |B_\varepsilon - B|^{2} \, dx \, dt \right)^{\frac{1}{2}} \\
\leq C \left( \int_{Q_T} d_\varepsilon |A_\varepsilon - A|^{k_1} \, dx \, dt \right)^{\frac{1}{k_1}} + C \left( \int_{Q_T} d_\varepsilon |A_\varepsilon - A|^{2} \, dx \, dt \right)^{\frac{1}{2}} \\
+ C \left( \int_{Q_T} d_\varepsilon |B_\varepsilon - B|^{k_1} \, dx \, dt \right)^{\frac{1}{k_1}} + C \left( \int_{Q_T} d_\varepsilon |B_\varepsilon - B|^{2} \, dx \, dt \right)^{\frac{1}{2}} \rightarrow 0
\]

as \( \varepsilon \to 0^+ \), where we have used (4.13), (4.19), Lemma 4.1, and Young inequality.
Combining (4.8), (4.12) and (4.20), we have
\[
\lim_{\epsilon \to 0^+} \int_0^t \int_{\Omega} \phi P(n, \rho) \rho \, dx \, ds = \int_0^t \int_{\Omega} \phi (A + B) P(Ad, Bd) \, dx \, ds \\
\geq \int_0^t \int_{\Omega} \phi (A + B) P(Ad, Bd) \, dx \, ds = \int_0^t \int_{\Omega} \phi (\rho + n) P(Ad, Bd) \, dx \, ds
\]
(4.21)
where we have used that \(A + B = 1\), and Lemma 2.5 such that
\[
P(Ad, Bd) \geq P(Ad, Bd)
\]
due to the fact that the functions \(z \mapsto P(Az, Bz)\) and \(z \mapsto z\) are non decreasing functions. Here \(\overline{\cdot}\) represents the weak limit of \(\cdot\) with respect to \(d\) as \(\epsilon \to 0^+\). Note that in this section \(P(Ad, Bd)\) and \(P(Ad, Bd)\) are bounded in \(L^{\frac{\beta+1}{m+1}}(Q_T)\) and in \(L^{\frac{\beta+1}{m}}(Q_T)\), respectively, due to Lemma 4.1. Moreover, both \(\frac{\beta+1}{m+1}\) and \(\frac{\beta+1}{m}\) are large than 1, which implies that \(P(Ad, Bd)\) and \(P(Ad, Bd)\) are well-defined.

We claim that
\[
\int_0^t \int_{\Omega} \phi (\rho + n) P(Ad, Bd) \, dx \, ds = \int_0^t \int_{\Omega} \phi (\rho + n) P(n, \rho) \, dx \, ds.
\]
(4.22)
In fact,
\[
\int_0^t \int_{\Omega} \phi (\rho + n) P(Ad, Bd) \, dx \, ds \\
= \lim_{\epsilon \to 0^+} \int_0^t \int_{\Omega} \phi (\rho + n) P(Ad, Bd) \, dx \, ds \\
= \lim_{\epsilon \to 0^+} \int_0^t \int_{\Omega} \phi (\rho + n) P(Ad, Bd) \, dx \, ds + \lim_{\epsilon \to 0^+} \int_0^t \int_{\Omega} \phi (\rho + n) \left[ P(Ad, Bd) - P(n, \rho) \right] \, dx \, ds
\]
(4.23)
Similar to \(II_3\), the last term on the right hand side of (4.23) converges to zero as \(\epsilon \to 0^+\). Hence we get (4.22).

In view of (4.21), (4.22), and the fact that the test functions \(\phi\) and \(\psi\) are arbitrary, we complete the proof of the lemma.

\[\square\]

\textbf{Lemma 4.4} Let \((\rho_\epsilon, n_\epsilon, u_\epsilon)\) be the solution stated in Proposition 3.1, and \((\rho, n, u)\) be the limit in the sense of (4.1), then
\[
\lim_{\epsilon \to 0^+} \int_{Q_T} \phi H_\epsilon (\rho_\epsilon + n_\epsilon) \, dx \, dt = \int_{Q_T} \phi H(\rho + n) \, dx \, dt,
\]
(4.24)
for any \( \psi \in C_0^\infty(0, T) \) and \( \phi \in C_0^\infty(\Omega) \), where

\[
H_\epsilon := P(n_\epsilon, \rho_\epsilon) + \delta(\rho_\epsilon + n_\epsilon)^\beta - (2\mu + \lambda)\text{div}_\epsilon,
\]

\[
H := P(n, \rho) + \delta(\rho + n)^\beta - (2\mu + \lambda)\text{div}.
\]

**Remark 4.5** The proof of (4.24) is motivated by [18] for Navier-Stokes equations. In fact, the lemma can be found in [30] where the pressure is given by (1.3). For the pressure (1.4), the proof is similar.

With Lemmas 4.3 and 4.4, it is not difficult to obtain the next lemma.

**Lemma 4.6** Let \((\rho_\epsilon, n_\epsilon)\) be the solution stated in Lemma 3.1, and \((\rho, n)\) be the limit in the sense of (4.1), then

\[
\int_0^t \int_\Omega (\rho + n)\text{div} u \, dx \, ds \leq \lim_{\epsilon \to 0^+} \int_0^t \int_\Omega (\rho_\epsilon + n_\epsilon)\text{div}_\epsilon \, dx \, ds
\]

for a.e. \( t \in (0, T) \).

By virtue of Lemma 4.4 in [30], we have

\[
\int_\Omega \left[ \rho_\epsilon \log \rho_\epsilon - \rho \log \rho + n_\epsilon \log n_\epsilon - n \log n \right](t) \, dx
\]

\[
\leq \int_0^t \int_\Omega (\rho + n)\text{div} u \, dx \, ds - \int_0^t \int_\Omega (\rho_\epsilon + n_\epsilon)\text{div}_\epsilon \, dx \, ds
\]

for a.e. \( t \in (0, T) \).

Passing both sides of (4.26) to the limits as \( \epsilon \to 0^+ \), and using (4.25), we have

\[
\int_\Omega \left[ \rho \log \rho - \rho \log \rho + n \log n \right](t) \, dx \leq 0.
\]

Thanks to the convexity of \( z \mapsto z \log z \), we have

\[
\overline{\rho \log \rho} \geq \rho \log \rho \quad \text{and} \quad \overline{n \log n} \geq n \log n
\]

ea.e. on \( QT \). This turns out that

\[
\int_\Omega \left[ \rho \log \rho - \rho \log \rho + n \log n \right](t) \, dx = 0.
\]

Hence we get

\[
\overline{\rho \log \rho} = \rho \log \rho \quad \text{and} \quad \overline{n \log n} = n \log n
\]

ea.e. on \( QT \), which implies that \((\rho_\epsilon, n_\epsilon) \to (\rho, n)\) a.e. in \( QT \). It combined with Lemma 4.1 yields strong convergence of \((\rho_\epsilon, n_\epsilon)\) in \( L^{\beta_1}(QT) \) for any \( \beta_1 < \beta + 1 \). Thus we complete the proof of (4.5).

To this end, we give a proposition as a summary for this section.

**Proposition 4.7** Suppose \( \beta > \max\{4, \Gamma + 1, \gamma + 1\} \). For any given \( \delta > 0 \), there exists a global weak solution \((\rho_\delta, n_\delta, u_\delta)\) to the following system over \( \Omega \times (0, \infty) \):

\[
\begin{aligned}
&n_t + \text{div}(nu) = 0, \\
&\rho_t + \text{div}(\rho u) = 0, \\
&[(\rho + n)u]_t + \text{div}[(\rho + n)u \otimes u] + \nabla P(n, \rho) + \delta \nabla(\rho + n)^\beta = \mu \Delta u + (\mu + \lambda)\nabla\text{div}u.
\end{aligned}
\]

\( \diamond \) Springer
with initial and boundary condition

\[
\begin{align*}
(\rho, n, (\rho + n)u)|_{t=0} &= (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}) \text{ on } \Omega, \\
\quad u|_{\partial\Omega} &= 0 \text{ for } t \geq 0, 
\end{align*}
\]

(4.28) (4.29)

such that for any given \( T > 0 \), the following estimates

\[
\begin{align*}
\sup_{t \in [0, T]} \| \rho_\delta(t) \|_{L^\gamma(\Omega)} &\leq C(\rho_0, n_0, M_0), \\
\sup_{t \in [0, T]} \| n_\delta(t) \|_{L^\Gamma(\Omega)} &\leq C(\rho_0, n_0, M_0), \\
\delta \sup_{t \in [0, T]} \| (\rho_\delta(t), n_\delta(t)) \|_{L^{\beta}(\Omega)} &\leq C(\rho_0, n_0, M_0), \\
\sup_{t \in [0, T]} \| \sqrt{\rho_\delta + n_\delta(t)} u_\delta(t) \|_{L^2(\Omega)}^2 &\leq C(\rho_0, n_0, M_0), \\
\int_0^T \| u_\delta(t) \|_{H^1_0(\Omega)}^2 \, dt &\leq C(\rho_0, n_0, M_0), \\
\end{align*}
\]

(4.30) (4.31) (4.32) (4.33) (4.34)

and

\[
\| (\rho_\delta(t), n_\delta(t)) \|_{L^{\beta+1}(\Omega_T)} \leq C(\beta, \delta, \rho_0, n_0, M_0) 
\]

(4.35)

hold, where the norm \( \| (\cdot, \cdot) \| \) denotes \( \| \cdot \| + \| \cdot \| \).

5 The vanishing of the artificial pressure

Let \( C \) be a generic constant depending only on the initial data and some other known constants but independent of \( \delta \), which will be used throughout this section.

5.1 Passing to the limit as \( \delta \rightarrow 0^+ \)

In this section, we will obtain the global existence of the weak solution to (1.1)–(1.2) by passing to the limit of \((\rho_\delta, n_\delta, u_\delta)\) as \( \delta \rightarrow 0^+ \). To begin with, we have to get the higher integrability estimates of the pressure \( P \) uniformly for \( \delta \) for the same reason as in the previous section.

In fact, as in [18] (see also [30]), we have the following lemma.

**Lemma 5.1** Let \((\rho_\delta, n_\delta, u_\delta)\) be the solution stated in Proposition 4.7, then we have

\[
\int_{\Omega_T} (n_\delta^\gamma + \rho_\delta^\gamma + \delta n_\delta^\gamma + \delta \rho_\delta^\gamma) \, dx \, dt \leq C(\theta_1, \theta_2)
\]

(5.1)

for any positive constants \( \theta_1 \) and \( \theta_2 \) satisfying

\[
\theta_1 < \frac{\Gamma}{3} \text{ and } \theta_1 \leq \min \left\{ 1, \frac{2\Gamma}{3} - 1 \right\}; \quad \theta_2 < \frac{\gamma}{3} \text{ and } \theta_2 \leq \min \left\{ 1, \frac{2\gamma}{3} - 1 \right\} \quad \text{if } \Gamma, \gamma \in \left( \frac{3}{2}, \infty \right).
\]
With (4.30), (4.31), (4.33), (4.34), and (5.1), letting $\delta \to 0^+$ (taking a subsequence if necessary), we have

$$\begin{align*}
\rho_\delta &\to \rho \text{ in } C([0, T]; L^\gamma_{\text{weak}}(\Omega)) \text{ and weakly in } L^{\gamma+\theta_2}(QT) \text{ as } \delta \to 0^+, \\
n_\delta &\to n \text{ in } C([0, T]; L^1_{\text{weak}}(\Omega)) \text{ and weakly in } L^{\Gamma+\theta_1}(QT) \text{ as } \delta \to 0^+, \\
u_\delta &\to u \text{ weakly in } L^2(0, T; H^1_0(\Omega)) \text{ as } \delta \to 0^+, \\
(\rho_\delta + n_\delta)u_\delta &\to (\rho + n)u \text{ in } C([0, T]; L^\min\{\gamma+\theta_1, \Gamma+\theta_1\}(\Omega)) \cap C([0, T]; H^{-1}(\Omega)) \text{ as } \delta \to 0^+, \\
(\rho_\delta u_\delta, n_\delta u_\delta) &\to (\rho u, nu) \text{ in } \mathcal{D}'(QT) \text{ as } \delta \to 0^+, \\
P(n_\delta, \rho_\delta) &\to P(n, \rho) \text{ weakly in } L^{\min\{\gamma+\theta_2, \Gamma+\theta_1\}}(QT) \text{ as } \delta \to 0^+, \\
\delta(\rho_\delta + n_\delta)^\beta &\to 0 \text{ in } L^1(QT) \text{ as } \delta \to 0^+,
\end{align*}$$

where the limit $(\rho, n, u)$ solves the following system in the sense of distribution over $\Omega \times [0, T]$ for any given $T > 0$:

$$\begin{cases}
n_t + \text{div}(nu) = 0, \\
\rho_t + \text{div}(\rho u) = 0, \\
[(\rho + n)u]_t + \text{div}[(\rho + n)u \otimes u] + \nabla P(\rho, n) = \mu \Delta u + (\mu + \lambda)\nabla \text{div} u,
\end{cases} \tag{5.3}$$

with initial and boundary condition

$$\begin{align*}
(\rho, n, (\rho + n)u)|_{t=0} &= (\rho_0, n_0, M_0) \text{ on } \partial \Omega, \tag{5.4} \\
u|_{\partial \Omega} &= 0 \text{ for } t \geq 0. \tag{5.5}
\end{align*}$$

Finally, we need to justify that $\bar{P}(\rho, n) = P(\rho, n)$. In fact, this has already been done by Vasseur, the author, and Yu in [30] for the pressure law (1.3) subject to the constraints

$$\max \left\{ \frac{3\gamma}{4}, \gamma - 1, \frac{3(\gamma + 1)}{5} \right\} < \Gamma < \min \left\{ \frac{4\gamma}{3}, \gamma + 1, \frac{5\gamma}{3} - 1 \right\} \tag{5.6}$$

and $\Gamma, \gamma > \frac{9}{5}$, which implies that $\Gamma$ and $\gamma$ have to stay not too far from each other. Thus to consider the case that $\Gamma, \gamma \geq \frac{9}{5}$ without any other constraints, some new ingredients will be needed in the following analysis.

### 5.2 The weak limit of the pressure

To obtain the global existence of weak solution to (1.1)–(1.2), we have to justify the following claim.

**Claim**

$$P(n, \rho) = P(n, \rho) \tag{5.7}$$

for any $\Gamma, \gamma \geq \frac{9}{5}$.

To prove (5.7), it suffices to derive the strong convergence of $\rho_\delta$ and $n_\delta$ as $\delta \to 0^+$. In this section, we need that $\rho_\delta$ and $n_\delta$ are bounded in $L^2(QT)$ for that it will be essential to employ Lemma 2.1. As a consequence, the restriction that $\gamma, \Gamma \geq \frac{9}{5}$ is needed in view of Lemma 5.1.
Lemmas 5.1 and 4.1 indicate that the uniform integrability of $\rho_\delta$ and $n_\delta$ is weaker when $\Gamma, \gamma < 3$. Thus some estimates such as (4.6) can not be obtained in this part. For this reason, we consider a family of cut-off functions introduced in [18] and references therein, i.e.,

$$T_k(z) = kT\left(\frac{z}{k}\right), \ z \in \mathbb{R}, \ k = 1, 2, \ldots$$

(5.8)

where $T \in C^\infty(\mathbb{R})$ satisfies

$$T(z) = \begin{cases} 
  z & \text{for } z \leq 1, \\
  2 & \text{for } z \geq 3,
\end{cases}$$

and $T$ is concave.

The first conclusion in this subsection plays a very important role, which is only subject to the constraint $\Gamma, \gamma \geq \frac{9}{5}$.

**Lemma 5.2** Let $(\rho_\delta, n_\delta)$ be the solutions constructed in Proposition 4.7, and $(\rho, n)$ be the limit, then

$$\begin{cases}
  T_k(\rho) P(n, \rho) \leq T_k(\rho) P(n, \rho), \\
  T_k(n) P(n, \rho) \leq T_k(n) P(n, \rho),
\end{cases}$$

(5.9)

a.e. on $\Omega \times (0, T)$, for any $\Gamma, \gamma \geq \frac{9}{5}$.

**Proof** In view of Lemma 2.1 with $\nu_K = 0$ and $s = 1$ (see (2.3)) where the condition (2.2) can be ensured by using Lemma 2.3 for $N = 2$ and $R = (n_\delta, \rho_\delta + n_\delta), (\rho_\delta, \rho_\delta + n_\delta)$, we have

$$\begin{cases}
  n_\delta - Ad_\delta \to 0 \text{ a.e. in } Q_T, \\
  \rho_\delta - Bd_\delta \to 0 \text{ a.e. in } Q_T,
\end{cases}$$

(5.10)

as $\delta \to 0^+$ (taking a subsequence if necessary), where $d_\delta = \rho_\delta + n_\delta$. (5.10) and Egrov theorem imply that for any small positive constant $\sigma$, there exists a domain $Q'_T \subset Q_T$, such that $|Q_T/Q'_T| \leq \sigma$ and that

$$\begin{cases}
  n_\delta - Ad_\delta \to 0 \text{ uniformly in } Q'_T, \\
  \rho_\delta - Bd_\delta \to 0 \text{ uniformly in } Q'_T
\end{cases}$$

(5.11)

as $\delta \to 0^+$ (taking the same sequence as in (5.10)).

In view of (5.11), we obtain that there exists a positive constant $\delta_0$ such that

$$\begin{cases}
  Ad_\delta \leq n_\delta + 1, \\
  Bd_\delta \leq \rho_\delta + 1
\end{cases}$$

(5.12)

for $\delta \leq \delta_0$ and any $(x, t) \in Q'_T$. Note that $\delta_0$ does not depend on $(x, t)$.

Therefore for $\delta \leq \delta_0$, $Ad_\delta$ and $Bd_\delta$ are bounded in $L^{\Gamma + \theta_1}(Q'_T)$ and in $L^{\gamma + \theta_2}(Q'_T)$, respectively. Note that when $\Gamma + \theta_1 > \gamma + \theta_2$ or $\Gamma + \theta_1 < \gamma + \theta_2$, one can not generally guarantee that

$$\begin{cases}
  d_\delta = \rho_\delta + n_\delta \in L^{\Gamma + \theta_1}(Q_T), \\
  d_\delta = \rho_\delta + n_\delta \in L^{\gamma + \theta_2}(Q_T),
\end{cases}$$
since the only useful information we have is
\[
\begin{cases}
  \rho_\delta \in L^\gamma+\theta_2(Q_T) \cap L^\infty(0, T; L^\gamma(\Omega)), \\
n_\delta \in L^{\Gamma+\theta_1}(Q_T) \cap L^\infty(0, T; L^\Gamma(\Omega)).
\end{cases}
\]

Thus it indicates that the weighted functions $A$ and $B$ can cancel some possible oscillation of $d_\delta$.

Without loss of generality, we only show the proof of (5.9)$_1$. In fact, the proof of (5.9)$_2$ is similar. To begin with, we divide an integral into a sum of two parts, i.e., Integrability Part + Small Region Part. More precisely, we have
\[
\int_{Q_T} \Phi T_k(\rho_\delta) P(n_\delta, \rho_\delta) \, dx \, dt = \int_{Q'_T} \Phi T_k(\rho_\delta) P(n_\delta, \rho_\delta) \, dx \, dt + \int_{Q_T/Q'_T} \Phi T_k(\rho_\delta) P(n_\delta, \rho_\delta) \, dx \, dt,
\]
for any $\Phi \in C(\overline{Q_T})$ where $\Phi \geq 0$.

- **Analysis of the Integrability Part.**

\[
\lim_{\delta \to 0^+} \int_{Q'_T} \Phi T_k(\rho_\delta) P(n_\delta, \rho_\delta) \, dx \, dt = \lim_{\delta \to 0^+} \int_{Q'_T} \Phi T_k(Bd_\delta) P(Ad_\delta, Bd_\delta) \, dx \, dt + \lim_{\delta \to 0^+} \int_{Q'_T} \Phi [T_k(\rho_\delta) - T_k(Bd_\delta)] P(Ad_\delta, Bd_\delta) \, dx \, dt + \lim_{\delta \to 0^+} \int_{Q'_T} \Phi T_k(\rho_\delta) [P(n_\delta, \rho_\delta) - P(Ad_\delta, Bd_\delta)] \, dx \, dt
\]
\[
= \sum_{i=1}^3 III_i.
\]

For $III_2$, in view of (5.11), the continuity of the map $z \mapsto T_k(z)$, and the boundedness of $P(Ad_\delta, Bd_\delta)$ in $L^{m_2}(Q_T)$ due to (1.7), (5.12), and (5.1), we have
\[
III_2 \to 0
\]
as $\delta \to 0^+$, where $\theta_m = \min\{\frac{\Gamma+\theta_1}{\gamma}, \frac{\gamma+\theta_2}{\gamma}\}$.

For $III_3$, similar to (4.15) and (4.17), we get
\[
\left| P(Ad_\delta, Bd_\delta) - P(Ad_\delta, Bd_\delta) \right|
\]
\[
\leq \frac{A_+ \gamma \left(\frac{\Gamma}{\Gamma+\theta_1}\right) [\rho_+(\eta_1, \eta_2)]^{\gamma-1}}{\Gamma \left[1-\alpha(\eta_1, \eta_2)\right]+\alpha(\eta_1, \eta_2)} \left| A_3d_\delta - Ad_\delta \right| + \frac{A_+ \gamma [\rho_+(\eta_1, \eta_2)]^{\gamma-1}}{\Gamma \left[1-\alpha(\eta_1, \eta_2)\right]+\alpha(\eta_1, \eta_2)} \left| B_3d_\delta - Bd_\delta \right|
\]
\[
\leq C [\rho_+(\eta_1, \eta_2)]^{\gamma-1} \left| A_3d_\delta - Ad_\delta \right| + C [\rho_+(\eta_1, \eta_2)]^{\gamma-1} \left| B_3d_\delta - Bd_\delta \right|
\]
\[
\leq C \left[ A_3d_\delta + Ad_\delta \right]^{\gamma-1} \left[ B_3d_\delta + Bd_\delta \right] \left| A_3d_\delta - Ad_\delta \right| + C \left[ A_3d_\delta + Ad_\delta \right]^{\gamma(1-\frac{1}{\gamma})} \left| B_3d_\delta - Bd_\delta \right|
\]
\[
+ C \left[ (A_3d_\delta + Ad_\delta)^{\Gamma(1-\frac{1}{\gamma})} + (B_3d_\delta + Bd_\delta)^{\gamma(1-\frac{1}{\gamma})} \right] \left| B_3d_\delta - Bd_\delta \right|,
\]
where we have used

\[
\begin{aligned}
\rho_+(\eta_1, \eta_2) &\leq C_0^+ \left( \eta_1^\gamma + \eta_2 \right), \\
\eta_1 &\leq A_\delta d_\delta + A d_\delta, \\
\eta_2 &\leq B_\delta d_\delta + B d_\delta.
\end{aligned}
\]

(5.16)

Therefore we obtain

\[
|III_3| \leq C_k \lim_{\delta \to 0^+} \int_{Q_T'} \left[ (A_\delta d_\delta + A d_\delta)^{\Gamma-1} + (B_\delta d_\delta + B d_\delta)^{\gamma (1-\frac{1}{\Gamma})} \right] n_\delta - A d_\delta\ dx\ dt
\]

\[
+ C_k \lim_{\delta \to 0^+} \int_{Q_T'} \left[ (A_\delta d_\delta + A d_\delta)^{1-\frac{1}{\Gamma}} + (B_\delta d_\delta + B d_\delta)^{\gamma - 1} \right] |\rho_\delta - B d_\delta|\ dx\ dt
\]

\[
\rightarrow 0
\]

(5.17)

as \( \delta \to 0^+ \), due to (5.1), (5.11), and (5.12).

In view of (5.14) and (5.17), (5.13) can be refined as follows.

\[
\lim_{\delta \to 0^+} \int_{Q_T'} \Phi T_k(\rho_\delta) P(n_\delta, \rho_\delta)\ dx\ dt = \int_{Q_T'} \Phi \overline{T_k(B d)} \overline{P(A d, B d)}\ dx\ dt
\]

\[
\geq \int_{Q_T'} \Phi \overline{T_k(B d)} \overline{P(A d, B d)}\ dx\ dt
\]

(5.18)

due to Lemma 2.5 and the fact that the maps \( z \mapsto T_k(B z) \) and \( z \mapsto P(A z, B z) \) are non-decreasing.

Note that

\[
\begin{aligned}
\int_{Q_T'} \Phi \overline{T_k(B d)} \overline{P(A d, B d)}\ dx\ dt
&= \lim_{\delta \to 0^+} \int_{Q_T'} \Phi T_k(B d_\delta) \overline{P(A d, B d)}\ dx\ dt \\
&= \lim_{\delta \to 0^+} \int_{Q_T'} \Phi T_k(\rho_\delta) \overline{P(A d, B d)}\ dx\ dt \\
&\quad + \lim_{\delta \to 0^+} \int_{Q_T'} \Phi \left[ T_k(B d_\delta) - T_k(\rho_\delta) \right] \overline{P(A d, B d)}\ dx\ dt \\
&= \int_{Q_T'} \Phi \overline{T_k(\rho)} \overline{P(A d, B d)}\ dx\ dt,
\end{aligned}
\]

where we have used (5.11), the continuity of the map \( z \mapsto T_k(z) \), and \( \overline{P(A d, B d)} \in L^\theta_m(Q_T) \) with \( \theta_m = \min\{ \frac{\Gamma+\theta_1}{\Gamma}, \frac{\gamma+\theta_2}{\gamma} \} > 1 \), such that

\[
\lim_{\delta \to 0^+} \int_{Q_T'} \Phi \left[ T_k(B d_\delta) - T_k(\rho_\delta) \right] \overline{P(A d, B d)}\ dx\ dt \to 0
\]
as $\delta \to 0^+$. Similarly, we have
\[
\int_{Q_T'} \Phi T_k(\rho) \overline{P(Ad, Bd)} \, dx \, dt = \lim_{\delta \to 0^+} \int_{Q_T} \Phi T_k(\rho) P(Ad_\delta, Bd_\delta) \, dx \, dt
\]
\[
= \lim_{\delta \to 0^+} \int_{Q_T} \Phi T_k(\rho) P(n_\delta, \rho_\delta) \, dx \, dt
\]
\[
+ \lim_{\delta \to 0^+} \int_{Q_T} \Phi T_k(\rho) \left[ P(Ad_\delta, Bd_\delta) - P(n_\delta, \rho_\delta) \right] \, dx \, dt
\]
\[
= \int_{Q_T'} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt.
\]  
(5.20)

Combining (5.19) and (5.20), we have
\[
\int_{Q_T'} \Phi \overline{T_k(Bd)} \overline{P(Ad, Bd)} \, dx \, dt = \int_{Q_T'} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt. 
\]  
(5.21)

Note that the left term of (5.21) is exactly the same as the right term of (5.18). Hence we obtain from (5.18) and (5.21) that
\[
\lim_{\delta \to 0^+} \int_{Q_T} \Phi T_k(\rho_\delta) P(n_\delta, \rho_\delta) \, dx \, dt \geq \int_{Q_T} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt.
\]  
(5.22)

• **Analysis of the Small Region Part.**

For fixed $k$, we have
\[
\lim_{\delta \to 0^+} \int_{Q_T' / Q_T} \Phi T_k(\rho_\delta) P(n_\delta, \rho_\delta) \, dx \, dt = \int_{Q_T' / Q_T} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt, 
\]  
(5.23)

since $T_k(\rho_\delta) P(n_\delta, \rho_\delta)$ is bounded in $L^0(\Omega_T)$ uniformly for $\delta > 0$, where $\theta_m = \min\{\frac{\Gamma + \theta_1}{\theta_1}, \frac{\varphi + \theta_2}{\gamma}\} > 1$.

• **Analysis of the whole Part.**

By virtue of (5.22) and (5.23), we have
\[
\int_{Q_T} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt
\]
\[
= \lim_{\delta \to 0^+} \int_{Q_T'} \Phi T_k(\rho_\delta) P(n_\delta, \rho_\delta) \, dx \, dt + \lim_{\delta \to 0^+} \int_{Q_T' / Q_T} \Phi T_k(\rho_\delta) P(n_\delta, \rho_\delta) \, dx \, dt
\]
\[
\geq \int_{Q_T'} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt + \int_{Q_T' / Q_T} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt
\]
\[
= \int_{Q_T} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt - \int_{Q_T' / Q_T} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt
\]
\[
+ \int_{Q_T' / Q_T} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt. 
\]  
(5.24)

Since $|Q_T / Q_T'| \leq \sigma$, letting $\sigma$ go to zero, we obtain that the last two terms on the right hand side of (5.24) will vanish. Hence we have
\[
\int_{Q_T} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt \geq \int_{Q_T} \Phi T_k(\rho) \overline{P(n, \rho)} \, dx \, dt.
\]
Since $\Phi$ is arbitrary, we get (5.9)$_1$. By using the arguments similar to the proof of (5.9)$_1$, we get (5.9)$_2$. Therefore we complete the proof of the lemma.

Lemma 5.3 Let $(\rho_\delta, n_\delta, u_\delta)$ be the solution stated in Proposition 4.7 and $(\rho, n, u)$ be the limit, then

$$
\lim_{\delta \to 0^+} \int_0^T \int \phi H_\delta [T_k(\rho_\delta) + T_k(n_\delta)] \, dx \, dt = \int_0^T \int \phi H \left[ T_k(\rho) + T_k(n) \right] \, dx \, dt,
$$

(5.25)

for any $\psi \in C_0^\infty(0, T)$ and $\phi \in C_0^\infty(\Omega)$, where

$$
\begin{align*}
H_\delta &:= P(n_\delta, \rho_\delta) - (2\mu + \lambda) \text{div} u_\delta, \\
H &:= P(n, \rho) - (2\mu + \lambda) \text{div} u.
\end{align*}
$$

(5.26)

Remark 5.4 Lemma 5.3 is motivated by [18,26]. The statement of the lemma for the two-fluid model can be found in [30].

To show the strong convergence of $\rho_\delta$ and $n_\delta$, motivated by [18,26] (see also [30]), we define

$$
L_k(z) = \begin{cases} 
\log z, & 0 \leq z \leq k, \\
\log k + z \int_k^z \frac{T_k(s)}{s^2} \, ds, & z \geq k,
\end{cases}
$$

satisfying

$$
L_k(z) = \beta_k z - 2k \text{ for all } z \geq 3k,
$$

where

$$
\beta_k = \log k + \int_k^{3k} \frac{T_k(s)}{s^2} \, ds + \frac{2}{3}.
$$

We denote $b_k(z) := L_k(z) - \beta_k z$ where $b_k'(z) = 0$ for all large $z$, and

$$
b_k'(z)z - b_k(z) = T_k(z). \tag{5.27}
$$

Note that $\rho_\delta, n_\delta \in L^2(Q_T), \rho, n \in L^2(Q_T),$ and $u_\delta, u \in L^2(0, T; H^1_0(\Omega)).$ Then using the same arguments as in [30] where Lemma 5.3 is used, we arrive at

$$
\int_\Omega \left[ \frac{L_k(\rho)}{L_k(n)} - L_k(\rho) + \frac{L_k(n)}{L_k(n)} \right] \, dx
$$

$$
= \frac{1}{2\mu + \lambda} \int_0^t \int_\Omega \left( \frac{T_k(\rho)}{T_k(n)} + \frac{T_k(n)}{T_k(n)} \right) P(n, \rho) \, dx \, ds
$$

$$
- \frac{1}{2\mu + \lambda} \int_0^t \int_\Omega \left( \frac{T_k(\rho)}{T_k(n)} + \frac{T_k(n)}{T_k(n)} \right) P(n, \rho) \, dx \, ds
$$

$$
+ \int_0^t \int_\Omega \left[ T_k(\rho) - \frac{T_k(\rho)}{T_k(n)} + T_k(n) - \frac{T_k(n)}{T_k(n)} \right] \text{div} u \, dx \, ds.
$$

This together with (5.9) yields

$$
\int_\Omega \left[ \frac{L_k(\rho)}{L_k(n)} - L_k(\rho) + \frac{L_k(n)}{L_k(n)} \right] \, dx
$$

$$
\leq \int_0^t \int_\Omega \left[ T_k(\rho) - \frac{T_k(\rho)}{T_k(n)} + T_k(n) - \frac{T_k(n)}{T_k(n)} \right] \text{div} u \, dx \, ds. \tag{5.28}
$$
In order to include the case that both $\gamma$ and $\Gamma$ can touch $\frac{\theta}{\Gamma}$, we need the following new estimate.

**Lemma 5.5** Let $(\rho_0, n_0)$ be the solution stated in Proposition 4.7 and $(\rho, n)$ be the limit, then

$$\lim_{\delta \to 0} \|T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) - T_k(Bd)\|_{L^{\Gamma_{min}+1}(Q_T')} \leq C_k \sigma \frac{K_{min}^{-1}}{K_{min}} + C$$

for any $\Gamma, \gamma \geq \frac{\theta}{\Gamma}$, and any given $k > 0$, where $C$ is independent of $\sigma, \delta$, and $k$, and $C_k$ is independent of $\sigma$ and $\delta$ but may depend on $k$. Here

$$\Gamma_{min} = \min\{\Gamma, \gamma\}, \quad K_{min} = \min\left\{\frac{2 + \theta_1}{\Gamma}, \frac{2 + \theta_2}{\gamma}, 2\right\}.$$  \hspace{1cm} (5.29)

**Proof** Note that

$$|T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) - T_k(Bd)|^{\Gamma_{min}+1}$$

$$\leq \left( |A|d_{\delta} - d| + |B|d_{\delta} - d| \right)^{\Gamma_{min}} |T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) - T_k(Bd)|$$

$$\leq C \left( |A|d_{\delta} - d|^\Gamma + |B|d_{\delta} - d|^\gamma + 1 \right) |T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) - T_k(Bd)|$$

$$\leq C \left[ (Ad_{\delta})^\Gamma - (Ad)^\Gamma + (Bd_{\delta})^\gamma - (Bd)^\gamma \right] |T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) - T_k(Bd)|$$

$$+ C |T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) - T_k(Bd)|$$

due to the fact that

$$|T'(x)| \leq 1$$

for any $x \geq 0$, and that

$$|x - y|^\Gamma \leq |x^\Gamma - y^\Gamma|, \quad |x - y|^\gamma \leq |x^\gamma - y^\gamma|$$

for any $x, y \geq 0$.

Therefore we have

$$\int_{Q_T'} |T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) - T_k(Bd)|^{\Gamma_{min}+1} dx \, dt$$

$$\leq C \int_{Q_T'} \left[ (Ad_{\delta})^\Gamma - (Ad)^\Gamma + (Bd_{\delta})^\gamma - (Bd)^\gamma \right] |T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) - T_k(Bd)| dx \, dt$$

$$+ C \int_{Q_T'} |T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) - T_k(Bd)| dx \, dt$$

$$:= IV_1^\delta + IV_2^\delta,$$

where $Q_T'$ is introduced in (5.11).

For $IV_1^\delta$, we have

$$IV_1^\delta = C \int_{Q_T'} \left[ (Ad_{\delta})^\Gamma + (Bd_{\delta})^\gamma \right] |T_k(Ad_{\delta}) + T_k(Bd_{\delta})| dx \, dt$$

$$- C \int_{Q_T'} \left[ (Ad_{\delta})^\Gamma + (Bd_{\delta})^\gamma \right] |T_k(Ad) + T_k(Bd)| dx \, dt$$

$$- C \int_{Q_T'} \left[ (Ad)^\Gamma + (Bd)^\gamma \right] |T_k(Ad_{\delta}) + T_k(Bd_{\delta})| dx \, dt$$

$$+ C \int_{Q_T'} \left[ (Ad)^\Gamma + (Bd)^\gamma \right] |T_k(Ad) + T_k(Bd)| dx \, dt.$$
Taking the limit as $\delta \to 0$ (taking a subsequence if necessary), we have

$$\lim_{\delta \to 0} IV_1^\delta = C \lim_{\delta \to 0} \int_{Q'_T} \left[ (Ad_\delta)^\gamma + (Bd_\delta)^\gamma \right] \left[ T_k(Ad_\delta) + T_k(Bd_\delta) \right] dx \, dt$$

$$- C \int_{Q'_T} \frac{(Ad)^\gamma + (Bd)^\gamma}{T_k(Ad) + T_k(Bd)} dx \, dt$$

$$+ C \int_{Q'_T} \left[ (Ad)^\gamma + (Bd)^\gamma - (Ad)^\gamma \right] dx \, dt$$

$$- (Bd)^\gamma \left[ \frac{T_k(Ad)}{T_k(Ad) + T_k(Bd)} - T_k(Ad) - T_k(Bd) \right] dx \, dt.$$ 

Due to the convexity of $z \mapsto (Bz)^\gamma + (Az)^\gamma$ and the concavity of $z \mapsto T_k(Az) + T_k(Bz)$ such that

$$\frac{(Ad)^\gamma + (Bd)^\gamma}{T_k(Ad) + T_k(Bd)} \geq (Ad)^\gamma + (Bd)^\gamma,$$

$$\frac{T_k(Ad)}{T_k(Ad) + T_k(Bd)} \leq T_k(Ad) + T_k(Bd),$$

we have

$$\lim_{\delta \to 0} IV_1^\delta \leq C \lim_{\delta \to 0} \int_{Q'_T} \left[ (Ad_\delta)^\gamma + (Bd_\delta)^\gamma \right] \left[ T_k(Ad_\delta) + T_k(Bd_\delta) \right] dx \, dt$$

$$- C \int_{Q'_T} \frac{(Ad)^\gamma + (Bd)^\gamma}{T_k(Ad) + T_k(Bd)} dx \, dt.$$ 

(5.32)

For $IV_2^\delta$, we apply Young inequality and obtain

$$IV_2^\delta \leq \frac{1}{2} \int_{Q'_T} \left| T_k(Ad_\delta) + T_k(Bd_\delta) - T_k(Ad) - T_k(Bd) \right|^{\Gamma_{min} + 1} dx \, dt + C_1. \quad (5.33)$$

Combining (5.31) with (5.32) and (5.33) yields

$$\lim_{\delta \to 0} \int_{Q'_T} \left| T_k(Ad_\delta) + T_k(Bd_\delta) - T_k(Ad) - T_k(Bd) \right|^{\Gamma_{min} + 1} dx \, dt$$

$$\leq C \lim_{\delta \to 0} \int_{Q'_T} \left[ (Ad_\delta)^\gamma + (Bd_\delta)^\gamma \right] \left[ T_k(Ad_\delta) + T_k(Bd_\delta) \right] dx \, dt$$

$$- C \int_{Q'_T} \frac{(Ad)^\gamma + (Bd)^\gamma}{T_k(Ad) + T_k(Bd)} dx \, dt + C_1.$$ 

(5.34)

On the other hand,

$$\frac{\partial P(Az, Bz)}{\partial z} = A + \rho_+^{\gamma - 1}(Az, Bz) \left[ \frac{\partial \rho_+(Az, Bz)}{\partial n} A + \frac{\partial \rho_+(Az, Bz)}{\partial \rho} B \right]. \quad (5.35)$$

Recalling (4.16), we have

$$\left\{ \begin{array}{c}
\frac{\partial \rho_+(Az, Bz)}{\partial n} = \frac{(\frac{A}{n_0})^\gamma}{\gamma (1-\alpha) + \alpha}, \\
\frac{\partial \rho_+(Az, Bz)}{\partial \rho} = \frac{1}{\gamma (1-\alpha) + \alpha}.
\end{array} \right.$$
This together with (5.35) gives

\[
\frac{\partial P(Az, Bz)}{\partial z} = A+\gamma \rho_{+}^{\gamma-1}(Az, Bz) \left[ \left( \frac{A}{A_{+}} \right)^{\gamma} \left( \rho_{+}(Az, Bz) \right)^{1-\gamma} + \frac{1}{\gamma} \frac{A_{+}}{\gamma} (1-\alpha) + \alpha \right]
\]

\[
= \frac{A_{+}+\gamma}{\gamma} \left[ \left( \frac{A}{A_{+}} \right)^{\gamma} \left( \rho_{+}(Az, Bz) \right)^{1-\gamma} + \frac{A_{+}}{\gamma} (1-\alpha) + \alpha \right] + \frac{A_{+}+\gamma}{\gamma} (1-\alpha) + \alpha)
\]

\[
\geq \frac{A_{+}+\gamma}{\max\{\gamma, 1\}} \left[ \left( \frac{A}{A_{+}} \right)^{\gamma} \left( \rho_{+}(Az, Bz) \right)^{1-\gamma} + \frac{A_{+}}{\gamma} (1-\alpha) + \alpha \right]
\]

\[
\geq C_{2} \left[ A^\gamma + B^\gamma \right],
\]

where we have used (1.7), and $C_{2} = C_{2}(A_{+}, A_{-}, \Gamma, \gamma) > 0$.

Thus, we introduce

\[
G_{A, B}(z):= P(Az, Bz) - \frac{C_{2}}{\max\{\Gamma, \gamma\}} \left[ (Az)^{\gamma} + (Bz)^{\gamma} \right],
\]

which is inspired by [16] for the single-phase flow where non-mono pressure of one component is studied.

In view of (5.36) and (5.37), we obtain

\[
\frac{d}{dz} G_{A, B}(z) = \frac{\partial P(Az, Bz)}{\partial z} - C_{2} \left[ \frac{\Gamma}{\max\{\Gamma, \gamma\}} A^\gamma + \frac{\gamma}{\max\{\Gamma, \gamma\}} B^\gamma \right] \geq 0,
\]

and thus $z \mapsto G_{A, B}(z)$ is a non-decreasing function over $[0, \infty)$.

Let’s return to (5.34), and make use of $G_{A, B}(z)$. Then we get

\[
\lim_{\delta \to 0} \int_{Q_{T}^d} |T_{k}(Ad) - T_{k}(Ad) - T_{k}(Bd) - T_{k}(Bd)|^{\Gamma_{min}+1} dx dt
\]

\[
\leq \frac{C_{\max\{\Gamma, \gamma\}}}{C_{2}} \lim_{\delta \to 0} \int_{Q_{T}^d} \frac{C_{2}}{\max\{\Gamma, \gamma\}} \left[ (Ad)^{\gamma} + (Bd)^{\gamma} \right] \left[ T_{k}(Ad) + T_{k}(Bd) \right] dx dt
\]

\[
\frac{C_{\max\{\Gamma, \gamma\}}}{C_{2}} \left[ \frac{\Gamma}{\max\{\Gamma, \gamma\}} A^\gamma + \frac{\gamma}{\max\{\Gamma, \gamma\}} B^\gamma \right] \left( \frac{\Gamma}{\max\{\Gamma, \gamma\}} A^\gamma + \frac{\gamma}{\max\{\Gamma, \gamma\}} B^\gamma \right) \right] dx dt + C_{1}
\]

\[
= \frac{C_{\max\{\Gamma, \gamma\}}}{C_{2}} \lim_{\delta \to 0} \int_{Q_{T}^d} P(Ad, Bd)[T_{k}(Ad) + T_{k}(Bd)] dx dt
\]

\[
- \frac{C_{\max\{\Gamma, \gamma\}}}{C_{2}} \int_{Q_{T}^d} \frac{P(Ad, Bd)}{T_{k}(Ad) + T_{k}(Bd)} \frac{\Gamma}{\max\{\Gamma, \gamma\}} A^\gamma + \frac{\gamma}{\max\{\Gamma, \gamma\}} B^\gamma dx dt
\]

\[
- \frac{C_{\max\{\Gamma, \gamma\}}}{C_{2}} \left[ \frac{\Gamma}{\max\{\Gamma, \gamma\}} A^\gamma + \frac{\gamma}{\max\{\Gamma, \gamma\}} B^\gamma \right] \left[ T_{k}(Ad) + T_{k}(Bd) \right] dx dt + C_{1}
\]

Note that

\[
\lim_{\delta \to 0} \int_{Q_{T}^d} G_{A, B}(d)[T_{k}(Ad) + T_{k}(Bd)] dx dt + \int_{Q_{T}^d} \frac{G_{A, B}(d)}{T_{k}(Ad) + T_{k}(Bd)} dx dt
\]

\[
= \int_{Q_{T}^d} \frac{G_{A, B}(d)}{T_{k}(Ad) + T_{k}(Bd)} dx dt - \int_{Q_{T}^d} \frac{G_{A, B}(d)}{T_{k}(Ad) + T_{k}(Bd)} dx dt (5.39)
\]

\[
\leq 0,
\]
due to Lemma 2.5 and the fact that $z \mapsto G_{A,B}(z)$ and $z \mapsto T_k(Az) + T_k(Bz)$ are non-decreasing functions.

By virtue of (5.39), (5.38) yields

$$\lim_{\delta \to 0} \int_{Q'_T} |T_k(Ad_\delta) + T_k(Bd_\delta) - T_k(Ad) - T_k(Bd)|^{\Gamma_{\min}+1} \, dx \, dt$$

$$\leq \frac{C \max\{\Gamma, \gamma\}}{C_2} \lim_{\delta \to 0} \int_{Q'_T} P(Ad_\delta, Bd_\delta) [T_k(Ad_\delta) + T_k(Bd_\delta)] \, dx \, dt$$

$$- \frac{C \max\{\Gamma, \gamma\}}{C_2} \int_{Q'_T} \frac{P(Ad, Bd)}{T_k(Ad) + T_k(Bd)} \, dx \, dt + C_1.$$  \hspace{1cm} (5.40)

By virtue of the uniform convergence (5.11), we rewrite (5.40) as

$$\lim_{\delta \to 0} \int_{Q'_T} |T_k(Ad_\delta) + T_k(Bd_\delta) - T_k(Ad) - T_k(Bd)|^{\Gamma_{\min}+1} \, dx \, dt$$

$$\leq \frac{C \max\{\Gamma, \gamma\}}{C_2} \lim_{\delta \to 0} \int_{Q'_T} P(n_\delta, \rho_\delta) [T_k(n_\delta) + T_k(\rho_\delta)] \, dx \, dt$$

$$- \frac{C \max\{\Gamma, \gamma\}}{C_2} \int_{Q'_T} \frac{P(Ad_\delta, Bd_\delta)}{T_k(Ad) + T_k(Bd)} \, dx \, dt + C_1$$

$$= \frac{C \max\{\Gamma, \gamma\}}{C_2} \lim_{\delta \to 0} \int_{Q'_T} P(n_\delta, \rho_\delta) [T_k(n_\delta) + T_k(\rho_\delta)] \, dx \, dt$$

$$- \frac{C \max\{\Gamma, \gamma\}}{C_2} \int_{Q'_T} \frac{P(n_\delta, \rho_\delta)}{T_k(Ad) + T_k(Bd)} \, dx \, dt + C_1.$$  \hspace{1cm} (5.41)

Similarly for the second term on the right hand side of (5.41), we have

$$\lim_{\delta \to 0} \int_{Q'_T} |T_k(Ad_\delta) + T_k(Bd_\delta) - T_k(Ad) - T_k(Bd)|^{\Gamma_{\min}+1} \, dx \, dt$$

$$\leq \frac{C \max\{\Gamma, \gamma\}}{C_2} \lim_{\delta \to 0} \int_{Q'_T} P(n_\delta, \rho_\delta) [T_k(n_\delta) + T_k(\rho_\delta)] \, dx \, dt$$

$$- \frac{C \max\{\Gamma, \gamma\}}{C_2} \int_{Q'_T} \frac{P(n_\delta, \rho_\delta)}{T_k(Ad) + T_k(Bd)} \, dx \, dt + C_1$$

$$= \frac{C \max\{\Gamma, \gamma\}}{C_2} \lim_{\delta \to 0} \int_{Q'_T} H_\delta [T_k(n_\delta) + T_k(\rho_\delta)] \, dx \, dt$$

$$+ \frac{C \max\{\Gamma, \gamma\}(2\mu + \lambda)}{C_2} \lim_{\delta \to 0} \int_{Q'_T} \text{div} \, \delta [T_k(n_\delta) + T_k(\rho_\delta)] \, dx \, dt$$

$$- \frac{C \max\{\Gamma, \gamma\}}{C_2} \int_{Q'_T} \frac{P(n_\delta, \rho_\delta)}{T_k(Ad) + T_k(Bd)} \, dx \, dt + C_1.$$  \hspace{1cm} (5.42)

In view of (5.25), we can take some appropriate test functions, for example,

$$\psi_j \in C^\infty_0(0, T), \quad \psi_j(t) \equiv 1 \text{ for any } t \in [\frac{1}{j}, T - \frac{1}{j}], \quad 0 \leq \psi_j \leq 1, \quad \psi_j \to 1.$$  \hspace{1cm} (5.43)
as \( j \to \infty \), and

\[
\phi_j \in C_0^\infty (\Omega), \quad \phi_j(x) \equiv 1 \text{ for any } x \in \left\{ x \in \Omega | \text{dist}(x, @\gamma) \geq \frac{1}{j} \right\},
\]

\[
0 \leq \phi_j \leq 1, \quad \phi_j \to 1,
\]

as \( j \to \infty \), such that

\[
\lim_{\delta \to 0^+} \int_{Q_T} H_\delta [T_k(\rho_\delta) + T_k(n_\delta)] \, dx \, dt = \int_{Q_T} H [\overline{T_k(\rho)} + \overline{T_k(n)}] \, dx \, dt.
\]

(5.45)

Then from (5.42) and (5.45), we obtain

\[
\lim_{\delta \to 0} \int_{Q_T'} |T_k(Ad_\delta) + T_k(Bd_\delta) - T_k(Ad) - T_k(Bd)|^{\Gamma_{\min} + 1} \, dx \, dt
\]

\[
\leq \frac{C \max\{\Gamma, \gamma\}}{C_2} \int_{Q_T} H [\overline{T_k(n)} + \overline{T_k(\rho)}] \, dx \, dt
\]

\[
\frac{C \max\{\Gamma, \gamma\}}{C_2} \lim_{\delta \to 0} \int_{Q_T/\delta} H_\delta [T_k(n_\delta) + T_k(\rho_\delta)] \, dx \, dt
\]

\[
+ \frac{C \max\{\Gamma, \gamma\}(2\mu + \lambda)}{C_2} \lim_{\delta \to 0} \int_{Q_T'} \text{div}u_\delta [T_k(n_\delta) + T_k(\rho_\delta)] \, dx \, dt
\]

\[
- \frac{C \max\{\Gamma, \gamma\}(2\mu + \lambda)}{C_2} \int_{Q_T'} P(n, \rho) \overline{T_k(n)} + \overline{T_k(\rho)} \, dx \, dt + C_1
\]

(5.46)

and thus

\[
\lim_{\delta \to 0} \int_{Q_T'} |T_k(Ad_\delta) + T_k(Bd_\delta) - T_k(Ad) - T_k(Bd)|^{\Gamma_{\min} + 1} \, dx \, dt
\]

\[
\leq \frac{C \max\{\Gamma, \gamma\}}{C_2} \int_{Q_T/\delta} P(n, \rho) \overline{T_k(n)} + \overline{T_k(\rho)} \, dx \, dt
\]

\[
- \frac{C \max\{\Gamma, \gamma\}}{C_2} \int_{Q_T/\delta} H [\overline{T_k(n)} + \overline{T_k(\rho)}] \, dx \, dt
\]

\[
+ \frac{C \max\{\Gamma, \gamma\}(2\mu + \lambda)}{C_2} \lim_{\delta \to 0} \int_{Q_T'} \text{div}u_\delta [T_k(n_\delta) + T_k(\rho_\delta)] \, dx \, dt
\]

\[
- \frac{C \max\{\Gamma, \gamma\}(2\mu + \lambda)}{C_2} \int_{Q_T'} \text{div}T_k(n_\delta) + T_k(\rho_\delta) \, dx \, dt + C_1
\]

\[
= \frac{C \max\{\Gamma, \gamma\}}{C_2} \int_{Q_T/\delta} \left[ P(n, \rho) \frac{T_k(n)}{2} + \frac{T_k(\rho)}{2} - H [\overline{T_k(n)} + \overline{T_k(\rho)}] \right] \, dx \, dt
\]

\[
+ \frac{C \max\{\Gamma, \gamma\}(2\mu + \lambda)}{C_2} \lim_{\delta \to 0} \int_{Q_T'} \text{div}u_\delta \left[ T_k(n_\delta) + T_k(\rho_\delta) - \overline{T_k(n)} + \overline{T_k(\rho)} \right] \, dx \, dt + C_1
\]

\[
:= V_1 + V_2 + C_1,
\]

since

\[
H_\delta:= P(n_\delta, \rho_\delta) - (2\mu + \lambda)\text{div}u_\delta,
\]

\[
\overline{H}:= \overline{P(n, \rho)} - (2\mu + \lambda)\text{div}u.
\]
For $V_1$, we apply Hölder inequality, (5.2), and (5.1), and then obtain

$$V_1 \leq \frac{C \max\{\Gamma, \gamma\}}{C_2} \left( \int_{Q_T/Q'T} \left| P(n, \rho) \frac{T_k(n) + T_k(\rho)}{T_k(n) + T_k(\rho)} - H[T_k(n) + T_k(\rho)] \right|^{K_{\min}} dx \, dt \right)^{\frac{1}{K_{\min}}}$$

where $K_{\min} = \min\{\Gamma + \theta_1, \gamma + \theta_2, 2\} > 1$, and $C_k^3$ is independent of $\sigma$ for $\sigma \in (0, 1)$ but may depend on $k$.

For $V_2$, by virtue of Hölder inequality and (4.34), we have

$$V_2 \leq \frac{C \max\{\Gamma, \gamma\}(2\mu + \lambda)}{C_2} \limsup_{\delta \to 0} \left( \int_{Q_T'} |\text{div}_{\delta}\tilde{u}|^2 dx \, dt \right)^{\frac{1}{2}}$$

$$\cdot \left( \int_{Q_T'} \left| T_k(n_{\delta}) + T_k(\rho_{\delta}) - T_k(n) + T_k(\rho) \right|^2 dx \, dt \right)^{\frac{1}{2}}$$

$$\leq C_4 \left( \limsup_{\delta \to 0} \int_{Q_T'} \left| T_k(n_{\delta}) + T_k(\rho_{\delta}) - T_k(n) + T_k(\rho) \right|^2 dx \, dt \right)^{\frac{1}{2}}$$

$$+ C_4 \left( \limsup_{\delta \to 0} \int_{Q_T'} \left| T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) + T_k(Bd) \right|^2 dx \, dt \right)^{\frac{1}{2}}$$

where $C_4$ is independent of $\sigma, \delta$, and $k$. This together with the lower semi-continuity of $L^2$ norm and Young inequality deduces that

$$V_2 \leq \frac{1}{2} \limsup_{\delta \to 0} \int_{Q_T'} \left| T_k(n_{\delta}) + T_k(\rho_{\delta}) - T_k(n) + T_k(\rho) \right|^\Gamma dx \, dt + C_5$$

$$= \frac{1}{2} \lim_{\delta \to 0} \int_{Q_T'} \left| T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) + T_k(Bd) \right|^\Gamma dx \, dt + C_5,$$

for some positive constant $C_5$ independent of $\sigma, \delta$, and $k$. Here we have applied (5.10) to the equality.

Thus we have

$$\lim_{\delta \to 0} \int_{Q_T'} \left| T_k(Ad_{\delta}) + T_k(Bd_{\delta}) - T_k(Ad) + T_k(Bd) \right|^\Gamma dx \, dt \leq 2C_k^3 \sigma \frac{K_{\min}^{-1}}{\Gamma_{\min}} + 2C_5 + 2C_1,$$

according to (5.47), (5.48), and (5.49).

The proof of the lemma is complete. □

**Corollary 5.6** Let $(\rho_{\delta}, n_{\delta})$ be the solution stated in Proposition 4.7 and $(\rho, n)$ be the limit, then

$$\lim_{\delta \to 0} \|T_k(n_{\delta}) + T_k(\rho_{\delta}) - T_k(n) + T_k(\rho)\|^\Gamma_{\min+1}_{L^\Gamma_{\min+1}(Q_T)} \leq C$$

for any given $k > 0$, where $C$ is independent of $\sigma, \delta$, and $k$. Here $\Gamma_{\min}$ and $K_{\min}$ are given by (5.29).
Proof In view of (5.11), we have

$$\lim_{\delta \to 0} \int_{Q_T} \left| T_k(n_\delta) + T_k(\rho_\delta) - T_k(n) - T_k(\rho) \right|^{\Gamma_{\min} + 1} dx \, dt$$

$$= \lim_{\delta \to 0} \int_{Q_T/\delta} \left| T_k(n_\delta) + T_k(\rho_\delta) - T_k(n) - T_k(\rho) \right|^{\Gamma_{\min} + 1} dx \, dt$$

$$+ \lim_{\delta \to 0} \int_{Q_T} \left| T_k(Ad_\delta) + T_k(Bd_\delta) - T_k(Ad) - T_k(Bd) \right|^{\Gamma_{\min} + 1} dx \, dt.$$ 

Similar to (5.48), the first term on the right hand side will tend to zero as $\sigma \to 0^+$. And for the second term, we use Lemma 5.5. Consequently, letting $\sigma \to 0^+$, we complete the proof of the corollary. □

Corollary 5.6 combined with the lower semi-continuity of the norm yields the following corollary.

Corollary 5.7 Let $(\rho_\delta, n_\delta)$ be the solution stated in Proposition 4.7 and $(\rho, n)$ be the limit, then

$$\left\| \overline{T_k(n)} + \overline{T_k(\rho)} - T_k(n) - T_k(\rho) \right\|_{L^{\Gamma_{\min} + 1}(Q_T)} \leq C$$

for any given $k > 0$, where $C$ is independent of $k$. □

Denote

$$Q_{T,k} = \left\{ (x, t) \in Q_T \mid \rho(x, t) \geq k, \text{ or } n(x, t) \geq k \right\}. \tag{5.50}$$

Here we are able to control the right-hand side of (5.28) in the following lemma.

Lemma 5.8 Let $(\rho_\delta, n_\delta, u_\delta)$ be the solution stated in Proposition 4.7 and $(\rho, n, u)$ be the limit, then

$$\lim_{k \to \infty} \int_{Q_T} \left[ T_k(\rho) - \overline{T_k(\rho)} + T_k(n) - \overline{T_k(n)} \right] \text{div} \, u \, dx \, dt = 0. \tag{5.51}$$

Proof Using Hölder inequality and Corollary 5.7, we have

$$\int_{Q_T} \left[ T_k(\rho) - \overline{T_k(\rho)} + T_k(n) - \overline{T_k(n)} \right] \text{div} \, u \, dx \, dt$$

$$= \int_{Q_{T,k}} \left[ T_k(\rho) - \overline{T_k(\rho)} + T_k(n) - \overline{T_k(n)} \right] \text{div} \, u \, dx \, dt$$

$$+ \int_{Q_{T}/Q_{T,k}} \left[ T_k(\rho) - \overline{T_k(\rho)} + T_k(n) - \overline{T_k(n)} \right] \text{div} \, u \, dx \, dt \tag{5.52}$$

$$\leq \left\| T_k(\rho) - \overline{T_k(\rho)} + T_k(n) - \overline{T_k(n)} \right\|_{L^2(Q_{T,k})} \left\| \text{div} u \right\|_{L^2(Q_{T,k})}$$

$$+ \left\| T_k(\rho) - \overline{T_k(\rho)} + T_k(n) - \overline{T_k(n)} \right\|_{L^2(Q_{T}/Q_{T,k})} \left\| \text{div} u \right\|_{L^2(Q_{T}/Q_{T,k})}$$

$$\leq C \left\| \text{div} u \right\|_{L^2(Q_{T,k})} + C \left\| T_k(\rho) - \overline{T_k(\rho)} + T_k(n) - \overline{T_k(n)} \right\|_{L^2(Q_{T}/Q_{T,k})}.$$ 

For the second term on the right hand side of (5.52), by virtue of the standard interpolation inequality and Corollary 5.7, we have

$$\left\| T_k(\rho) - \overline{T_k(\rho)} + T_k(n) - \overline{T_k(n)} \right\|_{L^2(Q_{T}/Q_{T,k})}$$
\[ \leq \| T_k(\rho) - \bar{T}_k(\rho) + T_k(n) - \bar{T}_k(n) \|_{L^1(Q_{T}/Q_{T,k})} \]

\[ \| T_k(\rho) - \bar{T}_k(\rho) + T_k(n) - \bar{T}_k(n) \|_{L^1(Q_{T}/Q_{T,k})} \]

\[ \leq C \| T_k(\rho) - \bar{T}_k(\rho) \|_{L^1(Q_{T}/Q_{T,k})} + C \| T_k(n) - \bar{T}_k(n) \|_{L^1(Q_{T}/Q_{T,k})}. \]  

(5.53)

Note that

\[ \lim_{k \to \infty} \| \text{div} u \|_{L^2(Q_{T,k})} = 0 \]

(5.54)

since the Lebesgue measure of \( Q_{T,k} \) converges to zero as \( k \to \infty \), due to

\[ \int_{Q_T} \left( n^{\alpha_1} + \rho^{\alpha_1} \right) \, dx \, dt \leq C \]

given by (5.1).

Therefore, to get (5.51), it suffices to prove

\[ \| T_k(\rho) - \bar{T}_k(\rho) \|_{L^1(Q_{T}/Q_{T,k})} + \| T_k(n) - \bar{T}_k(n) \|_{L^1(Q_{T}/Q_{T,k})} \to 0 \]

as \( k \to \infty \), according to (5.52) and (5.53).

Recalling that \( T_k(z) = z \) if \( z \leq k \), we have

\[ \| T_k(\rho) - \bar{T}_k(\rho) \|_{L^1(Q_{T}/Q_{T,k})} + \| T_k(n) - \bar{T}_k(n) \|_{L^1(Q_{T}/Q_{T,k})} \]

\[ = \| \rho - \bar{T}_k(\rho) \|_{L^1(Q_{T}/Q_{T,k})} + \| n - \bar{T}_k(n) \|_{L^1(Q_{T}/Q_{T,k})} \]

\[ \leq \liminf_{k \to \infty} \| \rho_\delta - T_k(\rho_\delta) \|_{L^1(Q_{T})} + \| n_\delta - T_k(n_\delta) \|_{L^1(Q_{T})} \]

\[ \leq C \liminf_{k \to \infty} \| \rho_\delta \|_{L^1(Q_{T})} + C \| n_\delta \|_{L^1(Q_{T})} \to 0 \]

as \( \delta \to 0 \), due to (5.1).

Therefore we are ready to prove (5.7). In fact, in view of (5.28) and (5.8), we have

\[ \lim_{k \to \infty} \int_{Q_T} [L_k(\rho) - L_k(\rho) + \bar{L}_k(n) - \bar{L}_k(n)] \, dx \leq 0. \]  

(5.56)

By the definition of \( L(\cdot) \), it is not difficult to justify that

\[ \left\{ \begin{array}{ll} \lim_{k \to \infty} \| L_k(\rho) - \rho \log \rho \|_{L^1(\Omega)} + \| L_k(n) - n \log n \|_{L^1(\Omega)} = 0, \\
\lim_{k \to \infty} \| L_k(\rho) - \rho \log \rho \|_{L^1(\Omega)} + \| L_k(n) - n \log n \|_{L^1(\Omega)} = 0. \end{array} \right. \]  

(5.57)

(5.56) and (5.57) yields

\[ \int_{\Omega} \left[ \rho \log \rho - \rho \log \rho + \bar{n} \log \bar{n} - n \log n \right] \, dx \leq 0. \]  

(5.58)

On the other hand, since \( \rho \log \rho \leq \bar{\rho} \log \rho \) and \( n \log n \leq \bar{n} \log \bar{n} \) due to the convexity of \( z \mapsto z \log z \), we have

\[ \rho \log \rho = \rho \log \rho \quad \text{and} \quad \bar{n} \log \bar{n} = n \log n. \]
It allows us to have the strong convergence of $\rho_\delta$ and $n_\delta$ in $L^{\gamma_1}(Q_T)$ and in $L^{\Gamma_1}(Q_T)$ for any $\gamma_1 \in [1, \gamma + \theta_2]$ and $\Gamma_1 \in [1, \Gamma + \theta_1]$, respectively. Therefore we proved (5.7).

Then the proof of Theorem 1.2 is complete.

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