Drinfeld-Manin Instanton and Its Noncommutative Generalization

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Abstract

The Drinfeld-Manin construction of $U(N)$ instanton is reformulated in the ADHM formulism, which gives explicit general solutions of the ADHM constraints for $U(N)$ ($N \geq 2k - 1$) $k$-instantons. For the $N < 2k - 1$ case, implicit results are given systematically as further constraints, which can be used to the collective coordinate integral. We find that this formulism can be easily generalized to the noncommutative case, where the explicit solutions are as well obtained.
1 Introduction

Instanton solutions in gauge field theory are of great physical and mathematical interest [1, 2, 3, 4, 5, 6]. Many significant achievements have been made in this region since their discovery in 1975 [7].

Because of the great significance of instanton solutions in various aspects of physics and mathematics, it is necessary to obtain all these solutions in gauge field theory. This task was almost accomplished in 1978, when Atiyah, Hitchin, Drinfeld and Manin (ADHM) established the famous construction of instantons for almost all gauge groups \[ SU(N) \] [8, 9]. This ADHM construction essentially reduces the problem of solving a set of nonlinear partial differential equations, which defines the instantons, to that of solving a set of quadratic algebraic equations, called the ADHM constraints. It gives the most general instanton configurations, and so provides the probability to learn the whole instanton moduli spaces.

But even algebraic equations are not always solvable, so the ADHM constraints remain a difficult problem. In other words, it is hard to attain satisfactory parametrization of instanton moduli spaces. For gauge group \( U(N) \), or essentially \( SU(N) \), during a rather long time since the presentation of ADHM construction, general solutions of the ADHM constraints are known only when \( k = 1 \) and \( N \) arbitrary or \( k \leq 3 \) and \( N = 2 \) [9, 10, 11] (except for the Drinfeld-Manin parametrization explained below), where \( k \) is the topological charge, or equivalently the instanton number [12], which is an integer classifying the instanton solutions. In 1999, Dorey et al. essentially rediscovered the Drinfeld-Manin parametrization for \( N \geq 2k \) [13], of which they seemed not aware.

In recent years the study of gauge field theory on noncommutative space time becomes an active research area [14, 15, 16], mostly due to its relevance with string theory [17]. An interesting phenomenon in noncommutative gauge field theory is that instanton solutions survive the space-time noncommutativity, and the moduli spaces of them get even better behaved [18]. Correspondingly, the ADHM construction has been generalized to the noncommutative case [19, 20].\(^2\) The noncommutative ADHM constraints

\(^1\)More precisely, the construction for exceptional groups is not known yet.

\(^2\)In fact, the ADHM constraints arise naturally as the D-flat condition of the worldvolume theory of the Dp-brane in Dp-brane-D\((p+4)\)-brane bound systems [21, 22]. When a constant NS-NS B-field is present in the worldvolume of the D\((p+4)\)-branes, the worldvolume theory of the D\((p+4)\)-branes becomes noncommutative, and a Fayet-Iliopoulos D-term appears in the worldvolume theory of the Dp-branes [23]. Corresponding to this term, one must add a constant term to the ADHM constraints.
seem even more difficult to solve: for gauge group $U(N)$, up to now only when $k = 1$ and $N$ arbitrary or $k = 2$ and $N = 1$ general solutions are known \[24\].

Drinfeld and Manin presented another construction of instantons \[25\] shortly after the ADHM construction, from a slightly different point of view. This construction explicitly gives parametrization of the $U(2k)$ $k$-instanton moduli space. In addition, all $U(N)$ $k$-instanton configurations can be indirectly obtained. Their original description of this construction was in a vector-bundle language. In this article we will translate it into the more familiar ADHM language and see how they give explicit general solutions of the ADHM constraints with gauge group $U(N)$ ($N \geq 2k - 1$) and topological charge $k$. For the $N < 2k - 1$ case, the further constraints are hard to solve explicitly, but our systematic discussion can offer an indirect way to the collective coordinate integral in this case. Moreover, fortunately, a noncommutative generalization of this ADHM formulation of Drinfeld-Manin instanton is straightforward.

This paper is organized as follow. In Sec.2 and Sec.3 we recall the definition of instantons and the ADHM construction, in the commutative case and the noncommutative case, respectively. In Sec.4 the Drinfeld-Manin construction is briefly reviewed and reformulated in the ADHM formulism. This construction is generalized to the noncommutative case in Sec.5. In the appendix, the conditions for a Hermitian matrix of restricted rank are given. These conditions are needed in the discussion of the $N < 2k$ case.

2 Instantons and (ordinary) ADHM construction

Instanton solutions in (Euclidean) gauge field theory were discovered by Belavin, Polyakov, Schwartz and Tyupkin (BPST) in 1975 \[7\]. They are defined by the so-called (anti-)self-dual equations:

$$\tilde{F}_{mn} = \pm F_{mn}, \quad (m, n = 1, 2, 3, 4) \quad (1)$$

and the solutions are known as self-dual (SD, for “+” sign) and anti-self-dual (ASD, for “−” sign) instantons. The definition of dual field $\tilde{F}_{mn}$ is familiar in electrodynamics, which is

$$\tilde{F}_{mn} = \frac{1}{2} \epsilon_{mnpq} F_{pq} \quad (2)$$

when the standard Euclidean metric $g_{mn} = \delta_{mn}$ is assumed. We note that the notions of SD and ASD are interchanged by a parity transformation. Without loss of generality we will consider only the ASD instantons.
All the (ASD) instanton solutions can be obtained by the ADHM construction [8, 9], as follows. In this construction we introduce the following ingredients (for $U(N)$ gauge theory with instanton number $k$):

- $k \times k$ matrix $B_{1,2}$, $k \times N$ matrix $I$ and $N \times k$ matrix $J$,
- the following quantities:

$$\mu_r = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - JJ^\dagger,$$  \hspace{1cm} (3)  
$$\mu_c = [B_1, B_2] + IJ.$$  \hspace{1cm} (4) 

The claim of ADHM is as follows:

- Given $B_{1,2}$, $I$ and $J$ such that $\mu_r = \mu_c = 0$, an ASD gauge field can be constructed;
- All ASD gauge fields can be obtained in this way.

It is convenient to introduce a quaternionic notation for the 4-dimensional Euclidean space-time indices:

$$x \equiv x^n \sigma_n, \quad \bar{x} \equiv x^n \bar{\sigma}_n = x^\dagger,$$  \hspace{1cm} (5) 

where $\sigma_n \equiv (i\vec{\tau}, 1)$ and $\tau^c$, $c = 1, 2, 3$ are the three Pauli matrices, and the conjugate matrices $\bar{\sigma}_n \equiv (-i\vec{\tau}, 1) = \sigma_n^\dagger$. Then the basic object in the ADHM construction is the $(N + 2k) \times 2k$ matrix $\Delta$ which is linear in the space-time coordinates:

$$\Delta = a + b\bar{x} \equiv a + b(\bar{x} \otimes 1_k),$$  \hspace{1cm} (6) 

where the constant matrices

$$a = \begin{pmatrix} I_1^\dagger & J \\ B_2^\dagger & -B_1 \end{pmatrix}, \quad b = \begin{pmatrix} 0_{N \times k} & 0_{N \times k} \\ 1_k & 0 \\ 0 & 1_k \end{pmatrix}.$$  \hspace{1cm} (7) 

It is easy to check that the ADHM constraints (3) and (4) are equivalent to the so-called factorization condition:

$$\Delta^\dagger \Delta = \begin{pmatrix} f^{-1} & 0 \\ 0 & f^{-1} \end{pmatrix},$$  \hspace{1cm} (8) 

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where $f(x)$ is a $k \times k$ Hermitian matrix. From the above condition we can construct a Hermitian projection operator $P$: \(^3\)

$$P = \Delta f \Delta^\dagger.$$  \hfill (9)

Obviously, the null space of $\Delta^\dagger(x)$ is of $N$ dimension for generic $x$. The basis vector for this null space can be assembled into an $(N + 2k) \times N$ matrix $U(x)$:

$$\Delta^\dagger U = 0,$$  \hfill (10)

which can be chosen to satisfy the following orthonormal condition:

$$U^\dagger U = 1.$$  \hfill (11)

The above orthonormal condition guarantees that $UU^\dagger$ is also a Hermitian projection operator. Now it can be proved (see \cite{20}) that the completeness relation

$$P + UU^\dagger = 1$$

holds if $U$ contains the whole null space of $\Delta^\dagger$. In other words, this completeness relation requires that $U$ consists of all the zero modes of $\Delta^\dagger$.

The (anti-Hermitian) gauge potential is constructed from $U$ by the following formula:

$$A_m = U^\dagger \partial_m U.$$  \hfill (13)

Then we get the corresponding field strength:

$$F_{mn} = \partial_{[m} A_{n]} + A_{[m} A_{n]} \equiv \partial_m A_n - \partial_n A_m + [A_m, A_n]$$

$$= \partial_{[m} (U^\dagger \partial_{n]} U) + (U^\dagger \partial_{[m} U)(U^\dagger \partial_{n]} U) = \partial_{[m} U^\dagger (1 - UU^\dagger) \partial_{n]} U$$

$$= \partial_{[m} U^\dagger \Delta f \Delta^\dagger \partial_{n]} U = U^\dagger \partial_{[m} \Delta f \partial_{n]} \Delta^\dagger U = U^\dagger b \tilde{\sigma}_{[m} \sigma_{n]} f b^\dagger U$$

$$= 2i \eta^c_{mn} U^\dagger b(\tau^c f) b^\dagger U.$$  \hfill (14)

Here $\eta^c_{mn}$ is the standard ’t Hooft $\eta$-symbol, which is anti-self-dual:

$$\frac{1}{2} \varepsilon_{mnpq} \eta^c_{pq} = -\tilde{\eta}^c_{mn}. $$  \hfill (15)

\(^3\)We use the following abbreviation for expressions with $f$:

$$\Delta f \Delta^\dagger \equiv \Delta \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \Delta^\dagger = \Delta(1_2 \otimes f) \Delta^\dagger.$$
3 Noncommutative ADHM construction

First let us recall briefly the gauge field theory on noncommutative Euclidean space (time)\(^4\). For a general noncommutative \(\mathbb{R}^4\) we mean a space with Hermitian-operator coordinates \(x^n, n = 1, \cdots, 4\), which satisfy the following relations:

\[
[x^m, x^n] = i\theta^{mn},
\]

where \(\theta^{mn}\) are real constants. If we assume the standard (Euclidean) metric for the noncommutative \(\mathbb{R}^4\), we can use the orthogonal transformation with positive determinant to change \(\theta^{mn}\) into the following standard form:

\[
(\theta^{mn}) = \begin{pmatrix}
0 & \theta^{12} & 0 & 0 \\
-\theta^{12} & 0 & 0 & 0 \\
0 & 0 & 0 & \theta^{34} \\
0 & 0 & -\theta^{34} & 0
\end{pmatrix},
\]

(17)

By using this form of \(\theta^{mn}\), the only non-vanishing commutators are

\[
[x^1, x^2] = i\theta^{12}, \quad [x^3, x^4] = i\theta^{34},
\]

and the other two obtained by using the anti-symmetric property of commutators.

The full noncommutative gauge field theory demands most of the abstract notions from noncommutative geometry, such as differential forms and vector bundles on noncommutative spaces [27, 28]. But for the \(U(N)\) gauge theory on noncommutative Euclidean space, things will be much simpler: in fact, the final effect is almost to replace all the coordinates in ordinary \(U(N)\) gauge theory with the above operator coordinates. However, a definition of derivatives in the noncommutative case are necessary for any gauge field theory. We define

\[
\partial_m f = -i\theta_{mn} [x^n, f],
\]

(19)

where \(\theta_{mn}\) is the matrix inverse of \(\theta^{mn}\). For our standard form (17) of \(\theta^{mn}\) we have

\[
\partial_1 f = \frac{i}{\theta^{12}} [x^2, f], \quad \partial_2 f = -\frac{i}{\theta^{12}} [x^1, f],
\]

(20)

and similar relations for \(\partial_{3,4}\).

Now we recall the noncommutative ADHM construction [19] briefly here. By introducing the same data as above but considering the coordinates as

\(^4\)For general reviews on noncommutative geometry and field theory, see, for example, [14, 15, 16, 26].
Noncommutative we see that the factorization condition still gives $\mu_c = 0$, but $\mu_r$ no longer vanishes. It is easy to check that the following relation holds:

$$\mu_r = \zeta \equiv 2\theta^{12} + 2\theta^{31}. \quad (21)$$

The form of ADHM constraints is invariant whether the space time is commutative or not.

The space-time noncommutativity brings nontrivial effects on the physics of gauge field theory. A remarkable example is the mixing between the infrared (IR) and the ultraviolet (UV) degrees of freedom [29]. Concerning the ADHM construction, in the noncommutative case the operator $\Delta^\dagger \Delta$ always has no zero mode (see [14]) and the moduli spaces of noncommutative instantons are better behaved than their commutative counterparts (see, for example, the lectures by H. Nakajima [18]). A related interesting fact is that noncommutative $U(1)$ gauge theory allows nonsingular instanton solutions [19, 30], while in the commutative case the simplest gauge group for which nonsingular instanton solutions exist is $SU(2)$.

Whether in the commutative case or in the noncommutative case, we can find that the above ADHM construction with $b$ in the canonical form is unaffected by the following transformations:

$$\Delta \rightarrow \begin{pmatrix} 1_N & 0 \\ 0 & 1_2 \otimes u \end{pmatrix} \Delta (1_2 \otimes u^\dagger), \quad (22)$$

where $u \in U(k)$. This is called the auxiliary symmetry of the ADHM construction, which acts on $a$, $f$ and $U$ as

$$B_1 \rightarrow uB_1u^\dagger, \quad (23)$$
$$B_2 \rightarrow uB_2u^\dagger, \quad (24)$$
$$I \rightarrow uI, \quad (25)$$
$$J \rightarrow Ju^\dagger, \quad (26)$$
$$f \rightarrow ufu^\dagger, \quad (27)$$
$$U \rightarrow \begin{pmatrix} 1_N & 0 \\ 0 & 1_2 \otimes u \end{pmatrix} U. \quad (28)$$

Now we can do a parameter counting for the (commutative or noncommutative) ADHM $U(N)$ $k$-instanton. $a$ in the form contains $4k^2 + 4Nk$ real parameters. The ADHM constraints impose $3k^2$ real conditions on them, and the auxiliary symmetry removes further $k^2$ real degrees of freedom. In total we have $4Nk$ real parameters left, which is expected according to physical analysis [31].
The above ADHM construction is also unaffected by the following transformations:

\[ \Delta \rightarrow \left( \begin{array}{cc} \mathcal{U} & 0 \\ 0 & 1_{2k} \end{array} \right) \Delta, \; \mathcal{U} \in SU(N), \] (29)

which can be regarded as the global gauge rotations of the instanton configuration. This global gauge symmetry leaves \( B_{1,2} \) and \( f \) unchanged and acts on \( I, J \) and \( U \) as

\[ I \rightarrow \mathcal{I} \mathcal{U}^\dagger \] (30)
\[ J \rightarrow \mathcal{U} J \] (31)
\[ U \rightarrow \left( \begin{array}{cc} \mathcal{U} & 0 \\ 0 & 1_{2k} \end{array} \right) U. \] (32)

If we wish to eliminate this global gauge symmetry from the \( 4Nk \) real parameters and retain the “purely” physical degrees of freedom, the number of independent real parameters will be \( 4Nk - N^2 + 1 \) for \( k \geq N/2 \), and \( 4Nk - N^2 + (N - 2k)^2 + 1 = 4k^2 + 1 \) for \( K \leq N/2 \) because in this case only \( N^2 - (N - 2k)^2 - 1 \) degrees of freedom in the \( SU(N) \) group act nontrivially on \( I \) and \( J \).

4 ADHM formulation of the Drinfeld-Manin construction

Shortly after the ADHM construction was established, Drinfeld and Manin successfully constructed all instanton solutions from a so-called “instanton bundle” point of view [25], which we call the Drinfeld-Manin construction. In this construction, the Euclidean space time is compactified by a point to \( S^4 \) and the instanton gauge potentials are considered as Levi-Civita connections on some nontrivial vector bundles, named instanton bundles, on this \( S^4 \). The instanton bundles are complex bundles (for the case of \( U(N) \) gauge group) orthogonally complementary, under some metrics, to a trivial vector bundle \( M \). The (anti-)self-duality of the Levi-Civita field strength imposes some conditions on the metric, which are actually the ADHM constraints.

We can always perform a complex linear transformation (on the basis vectors of the fibre space) to make the (Hermitian) metric standard. If we have done so, then the column vectors of \( \Delta \) in the ADHM construction constitute a basis of the section space of \( M \). So the matrix \( U \) consists of orthonormal basis vectors of the section space of the instanton bundle \( L \), and \( UU^\dagger \) is the projection operator corresponding to \( L \). As is familiar to us,
the gauge potential \( L \) is natural as the Levi-Civita connection on \( L \). The above statements briefly explain how the instanton bundle can be related to the familiar ADHM objects.

To formulate the Drinfeld-Manin construction in the ADHM language, we first concentrate on the \( U(2k) \) \( k \)-instanton case. Now

\[
h = \begin{pmatrix} b & a \end{pmatrix}
\]

is a \( 4k \times 4k \) square matrix, and

\[
\Delta = hX,
\]

where

\[
X \equiv \begin{pmatrix} \bar{x} \otimes 1_k \\ 1_{2k} \end{pmatrix}.
\]

(35)

Thus we have

\[
\Delta^\dagger \Delta = X^\dagger h^\dagger h X = X^\dagger \begin{pmatrix} 1_{2k} & a \\ a^\dagger a \end{pmatrix} X \equiv X^\dagger Q X,
\]

(36)

where

\[
a \equiv \begin{pmatrix} B_2^\dagger & -B_1 \\ B_1^\dagger & B_2 \end{pmatrix}
\]

(37)

is the lower blocks of \( a \).

In fact, the column vectors of \( X \) constitute a basis of the section space of \( M \) (before we perform the complex linear transformation mentioned above) and \( Q \) is the corresponding metric. From the ADHM point of view now, to make \( \Delta^\dagger \Delta \) of the factorized form \( (8) \), it is easy to see that \( Q \) must satisfy the following factorization condition:

\[
a^\dagger a = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix},
\]

(38)

where \( R \) is a \( k \times k \) constant Hermitian matrix. Using the auxiliary symmetry transformation \( (22) \), we can make \( R \) diagonalized:

\[
R = \text{diag}(r_1, r_2, \cdots, r_k), \quad r_1 \leq r_2 \leq \cdots \leq r_k.
\]

(39)

On the one hand, we can assume the above form of \( R \) to fix the auxiliary symmetry, which is nonphysical; on the other hand, even assuming this cannot completely fix the auxiliary symmetry: for generic \( R \) this residual
symmetry is $U(1)^x k/U(1)$, and if some of the $r_i$ are equal, this residual symmetry is even larger. Further, for generic $R$ this residual symmetry can be completely fixed by requiring $(k - 1)$ of the off-diagonal elements of $B_1$ or $B_2$, say $(B_1)_{ik} (i = 1, 2, \cdots, k - 1)$, to be real; special cases of coincident $r_i$ can be carefully treated as well.

To sum up, we can choose $\mathbf{a}$ and $\mathbf{R}$ of the form \( \text{(39)} \) as the collective coordinates of the $U(2k)$ $k$-instanton, while removing some of the degrees of freedom in $\mathbf{a}$. Obviously, the number of independent real parameters is $4k^2 + k - (k - 1) = 4k^2 + 1$, which coincides with the parameter counting in last section. Noting that

$$a^\dagger a = a^\dagger \mathbf{a} + K^\dagger K,$$

(40)

where

$$K \equiv \begin{pmatrix} I^\dagger & J \end{pmatrix}$$

(41)

is the upper blocks of $\mathbf{a}$, $\mathbf{a}$ and $\mathbf{R}$ must satisfy the condition that

$$S \equiv 1_2 \otimes R - a^\dagger a$$

(42)

is a positive semidefinite matrix. This condition introduces a boundary to the span of the parameters in $\mathbf{a}$ and $\mathbf{R}$. Thus we have obtained parametrization of the $U(2k)$ $k$-instanton moduli space, though the complicated boundary makes it a little imperfect, which is an inevitable consequence of the highly nontrivial topology of the instanton moduli space. This parametrization (also for the following $N > 2k$ case) is, in fact, rediscovered by Dorey et al. in 1999 \cite{13}, but they did not point out the relation between their work and \cite{25}.

Now the matrix $K$ can be expressed in terms of $\mathbf{a}$ and $\mathbf{R}$ due to

$$K^\dagger K = S.$$

(43)

Because in the present case $K$ is a square matrix, one may naturally take $K = K^\dagger = S^{1/2}$. This expression of $K$ seems simple and explicit, but it includes three steps: diagonalizing, extracting the square root, and undoing the diagonalization. In fact, to diagonalize $S$ needs to solve an equation of degree $k$, which we must avoid if we have better choices. Fortunately, a better choice does exist. We may have in remembrance the simplification of quadratic forms via congruent transformations in basic linear algebra:

$$B^T E B = A,$$

(44)
where $E$ is the canonical form of $A$. If $A$ is nonsingular, $E$ will be the identity matrix. Otherwise $E$ will have the form $\text{diag}(1, \cdots, 1, 0, \cdots, 0)$, which can be considered, in a different point of view, as $E$ always being the identity while allowing $B$ to be singular:

$$B^T B = A. \quad (45)$$

The transformation matrix $B$ can be easily obtained by completing squares or by simultaneous row and column transformations, without solving any nonlinear equations. Now $S$ here is a Hermitian form, not a quadratic one, but the method is similar.

Next we can consider the $N \neq 2k$ cases. These are very simple. If $N > 2k$, it is easy to find, as has been shown in many literatures, a natural embedding of the above $U(2k)$ solution $K$ in the $U(N)$ solution $K'$:

$$K' = \begin{pmatrix} 0_{(N-2k) \times 2k} \\ K \end{pmatrix}. \quad (46)$$

This gives the $4k^2 + 1$ “purely” physical degrees of freedom of the $U(N)$ $k$-instanton. To get all the “ADHM” degrees of freedom, i.e., including the global gauge rotations, we only need to perform the following transformations:

$$K' \rightarrow U K', \quad U \in \frac{U(N)}{U(1) \times U(N-2k)}. \quad (47)$$

which add $N^2 - (N - 2k)^2 - 1$ more parameters to the “purely” physical degrees of freedom and make the total number of real parameters $4Nk$.

If $N < 2k$, we can simply restrict the rank of $S$ not greater than $N$. Then from Eq. (43) it is easy to see that $K$ can take the following form:

$$K = \begin{pmatrix} K' \\ 0_{(2k-N) \times 2k} \end{pmatrix}, \quad (48)$$

where the $N \times 2k$ matrix $K'$ is the ADHM matrix for the $U(N)$ $k$-instanton. Linear algebra tells us that for an $l \times l$ Hermitian matrix $H$ the condition $\text{rank}(H) \leq l - r$ is equivalent to $r^2$ real conditions on the elements of $H$. So the number of “purely” physical parameters is $4k^2 + 1 - (2k - N)^2 = 4Nk - N^2 + 1$, which again coincides with the parameter counting in last section. The global gauge rotations are introduced as

$$K' \rightarrow U K', \quad U \in SU(N), \quad (49)$$
which supply the other $N^2 - 1$ real parameters for all the “ADHM” degrees of freedom. So far, everything seems well, but in fact the $(2k - N)^2$ real conditions become another trouble. The appendix of this paper will show how to explicitly write down these conditions on elements of $\mathfrak{a}$ and $R$. There we will see that for $N < 2k - 1$ they are too complicated to solve, so this formulism is not appropriate to give explicit solutions for this case. However, these systematic conditions can be useful to offer an indirect way to the instanton collective coordinate integral, which is left for future works.

Only the $N = 2k - 1$ case is simple. In this case there is only one condition:

$$\det(S) = 0, \quad (50)$$

which from Eq. (42) can be regarded as a quadratic equation of one of the $r_i$, say $r_k$. So we can take the same free parameters as in the $N = 2k$ case except $r_k$, and express $r_k$ in terms of the other parameters. The quadratic equation (50) has two roots. A little thought will make it clear that one of the eigenvalues of $S$ has been negative when we take the smaller root. Thus we can only take the greater one as $r_k$, which accomplishes parametrization of the $U(2k - 1)$ $k$-instanton moduli space.

5 Noncommutative Drinfeld-Manin instanton

How to establish the Drinfeld-Manin construction in the noncommutative case is an interesting problem. Appealing to the well-developed ADHM construction may be much easier than considering noncommutative instanton bundles. The commutative ADHM construction can be regarded as a special case ($\zeta = 0$) of the noncommutative ADHM construction. So we can anticipate that it is straightforward to generalize the ADHM formulism of the Drinfeld-Manin construction to the noncommutative case.

In fact, like Eq. (38), the factorization condition (8) in the noncommutative case gives the following condition on $a$:

$$a^\dagger a = \begin{pmatrix} R + \zeta & 0 \\ 0 & R \end{pmatrix} = \text{diag}(r_1 + \zeta, \cdots, r_k + \zeta, r_1, \cdots, r_k). \quad (51)$$

So we can similarly choose $a$ and $r_i$ ($i = 1, 2, \cdots, k$) as the collective coordinates of the noncommutative $U(2k)$ $k$-instanton (while removing some of the degrees of freedom in $\mathfrak{a}$ as in the commutative case). Now Eq. (12) becomes

$$S \equiv \begin{pmatrix} R + \zeta & 0 \\ 0 & R \end{pmatrix} - a^\dagger a. \quad (52)$$
and the following things are the same as in the commutative case.

To be more clear, our solution of the noncommutative ADHM $U(2k)$ $k$-instanton is

$$a = \left( \begin{array}{c} S^{1/2} \\ \mathbf{a} \end{array} \right),$$

(53)

where $S$ is defined in Eq.(52) and $\mathbf{a}$ defined in Eq.(37). And we must keep in mind that the square root here is understood in the sense of the simplification of Hermitian forms, as explained in last section. It is easy to check that this solution does satisfy the corresponding ADHM constraints, and it has the correct number of free parameters, as we have mentioned above.

The techniques to deal with the $N \neq 2k$ cases in the noncommutative case and that in the commutative case are completely the same. In fact the global gauge rotations in gauge field theory are unaffected by the space-time noncommutativity. Again the $N = 2k - 1$ case is simple enough to be solved. So we also obtain parametrization of the noncommutative $U(N)$ ($N \geq 2k - 1$) $k$-instanton moduli space.

To end this paper, let us focus on the two-instanton case. For $k = 2$, we essentially obtain explicit general solutions of the (commutative or noncommutative) ADHM constraints for $U(N)$ ($N \geq 3$) gauge groups. Counting the $U(2)$ two-instanton solution already known [9, 11], we have general solutions of all the commutative $U(N)$ two-instantons. However, the general solution of the noncommutative $U(2)$ two-instanton, which may be of much interest, is yet to be found.

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A Conditions for a Hermitian matrix of restricted rank

Consider an $l \times l$ Hermitian matrix $H$. We introduce the following decomposition of $H$:

$$H = \left( \begin{array}{cc} F_{r \times r} & C \\ C^\dagger & H_{(l-r) \times (l-r)} \end{array} \right),$$

(54)
and define an \((l - r + 1) \times (l - r + 1)\) matrix
\[
H'_{ij} = \begin{pmatrix} F_{ij} & C_i \\ C_i^\dagger & \mathbb{H} \end{pmatrix},
\]
(55)
where \(C_i\) is the \(i\)th row of \(C\). Assuming \(\det(H) \neq 0\), then the following two propositions are equivalent:

- \(\text{rank}(H) = l - r\);
- \(\det(H'_{ij}) = 0\) for all \(i, j = 1, 2, \ldots, r\).

It is apparent that the latter can be deduced from the former. Now we explain how the former can be deduced from the latter.

First, for a fixed \(j\), the \((l - r) \times (l - r + 1)\) matrix
\[
H' = \begin{pmatrix} C_j^\dagger & \mathbb{H} \end{pmatrix}
\]
(56)
is obviously of rank \(l - r\). Then \(\det(H'_{ij}) = 0\) means that the rank will not increase when we append a row \(C_i' \equiv \begin{pmatrix} F_{ij} & C_i \end{pmatrix}\) to \(H'\), so \(C_i'\) is a linear combination of the row vectors of \(H'\). This is the case for all \(i\), so we can conclude that the following matrix
\[
H_j = \begin{pmatrix} F_j & C \\ C_j^\dagger & \mathbb{H} \end{pmatrix}
\]
(57)
is of rank \(l - r\), where \(F_j\) is the \(j\)th column of \(F\).

Next, the \(l \times (l - r)\) matrix
\[
H' = \begin{pmatrix} C \\ \mathbb{H} \end{pmatrix}
\]
(58)
is again of rank \(l - r\). Thus \(\text{rank}(H_j) = l - r\) means that the rank will not increase when we append a column
\[
\hat{C}_j = \begin{pmatrix} F_j \\ C_j^\dagger \end{pmatrix}
\]
(59)
to \(H'\), so \(\hat{C}_j\) is a linear combination of the column vectors of \(H'\). Again this is the case for all \(j\), so we attain the desired result \(\text{rank}(H) = l - r\).

Because \(H\) is Hermitian, \(\det(H'_{ij}) = 0\) are in fact \(r^2\) real conditions. The combination of \(\det(H) \neq 0\) and these conditions is a sufficient condition.
for \( \text{rank}(H) \leq l - r \). Of course, it is not necessary. If \( \det(H) = 0 \) for the decomposition \( [54] \), we must take another \((l - r) \times (l - r)\) submatrix of \( H \) as \( H \) and obtain another \( r^2 \) real conditions. If \( H \) has no nonsingular \((l - r) \times (l - r)\) submatrix, the rank of \( H \) is less than \( l - r \). Altogether, the requirement \( \text{rank}(H) \leq l - r \) is achieved.

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