Two Gamma Quarkonium and Positronium Decays with Two-Body Dirac Equations of Constraint Dynamics

Horace W. Crater

The University of Tennessee Space Institute, Tullahoma, TN 37388

Cheuk-Yin Wong

Department of Physics, University of Tennessee, Knoxville, TN 37996

Two-Body Dirac equations of constraint dynamics provide a covariant framework to investigate the problem of highly relativistic quarks in meson bound states. This formalism eliminates automatically the problems of relative time and energy, leading to a covariant three dimensional formalism with the same number of degrees of freedom as appears in the corresponding nonrelativistic problem. It provides bound state wave equations with the simplicity of the nonrelativistic Schrödinger equation. Unlike other three-dimensional truncations of the Bethe-Salpeter equation, this covariant formalism has been thoroughly tested in nonperturbative contexts in QED, QCD, and nucleon-nucleon scattering. Here we continue the important studies of this formalism by extending a method developed earlier for positronium decay into two photons to tests on the sixteen component quarkonium wave function solutions obtained in meson spectroscopy. We examine positronium decay and then the two-gamma quarkonium decays of $\eta_c$, $\eta_c^\prime$, $\chi_{c0}$, $\chi_{c2}$, and $\pi^0$. The results for the $\pi^0$, although off the experimental rate by 13%, is much closer than the usual expectations from a potential model.

I. CONSTRAINT TWO-BODY DIRAC EQUATIONS FOR QED AND QCD

The Bethe-Salpeter equation (BSE) can be derived from QFT but its implementation invariably involves a non-unique choice from a very large class of three dimensional truncations, almost all of which work well in the perturbative context. We describe one such approach, the Two-Body Dirac Equations (TBDE), based on Dirac’s constraint dynamics that have been successfully applied to two-body bound state problems in QED [1] and QCD [2] and to two-body nucleon-nucleon scattering [3]. We will describe various nonperturbative tests we have applied to distinguish it from the others [2]. We then describe its most recent test in the treatment of two-photon decays of positronium and quarkonium [4].

The TBDE can be derived from the BSE but they had their origins in classical relativistic mechanics [3]. For spin-zero particles one constructs two generalized mass shell constraints $H_i \equiv p_i^2 + m_i^2 + \Phi_i \approx 0; \ i = 1, 2$ and guarantees their compatibility by requiring that the potentials satisfy a relativistic third-law condition $\Phi_1 = \Phi_2 = \Phi(x_1, p_1, p_2) \equiv \Phi_w$ and depend only on the interparticle separation component perpendicular to the total momentum,

$$x_{12 \perp}^\mu = (\eta^\mu + \hat{P}^\mu \hat{p}^\nu) (x_1 - x_2)_\nu; \ \hat{P}^\mu \equiv P^\mu / \sqrt{-P^2}; \ x_{12 \perp} \cdot \hat{P} = 0.$$  

(1.1)

Thus, interactions depend on the invariant $\sqrt{x_{12}^2} = \sqrt{x^2 + (x \cdot \hat{P})^2} \equiv r$ which reduces to the spatial separation $\sqrt{r^2}$ in the c.m frame, where the relative time cancels out covariantly. Calculating the difference of these compatible constraints, $H_{1} - H_{2} = -2P \cdot p \approx 0$, gives a covariant elimination of the c.m. relative energy, complimentary to the relative time restriction. The relative momentum $p^\mu = (\varepsilon_2 p_1^\mu - \varepsilon_1 p_2^\mu) / w$ is given in terms of the total c.m. energy $w = \varepsilon_1 + \varepsilon_2$ and the c.m. constituent energies $\varepsilon_1 - \varepsilon_2 = (m_1^2 - m_2^2) / w$. It is canonically conjugate to $x_{12 \perp}$ in a covariant three-dimensional way, $\{x_{12 \perp}^\mu, p^\mu\} = \eta^\mu_{\perp \nu}$. The other independent combination of the two constraints yields a dynamical equation $(\varepsilon_2 H_1 + \varepsilon_1 H_2) / w = p_{1 \perp}^\mu + \Phi_w - b^2(w) \approx 0$ whose quantization provides a covariant Schrödinger-like equation $(p_{1 \perp}^\mu + \Phi_w)\psi = b^2(w)\psi$ with the triangle function

$$b^2(w) = (w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2) / 4w^2 = \varepsilon_w^2 - m_w^2$$  

(1.2)

playing the role the eigenvalue. Its appearance signals exact two-body kinematics with effective particle motion displaying an Einstein relation between energy $\varepsilon_w = (w^2 - m_1^2 - m_2^2) / 2w$, mass $m_w = m_1 m_2 / w$, and momentum.

For quantum mechanical spinning particles one has two Dirac equations (here given for minimal vector and scalar interactions) instead of two generalized Klein-Gordon-like mass shell constraints.

$$S_i \psi \equiv \gamma_5 (\gamma_i \cdot (p_i - \hat{A}_i) + m_i + \hat{S}_i) \psi = 0; \ i = 1, 2.$$  

(1.3)

As in the spinless case they are compatible, $[S_1, S_2] \psi = 0$, provided that supersymmetry is added to the conditions that apply in the two body spinless case. One finds that the constituent scalar $\hat{S}_i = \hat{S}_i(S(r), A(r), p_{\perp}, w, \gamma_1, \gamma_2)$ and vector $\hat{A}_i^\mu = \hat{A}_i^\mu(A(r), p_{\perp}, w, \gamma_1, \gamma_2)$ potentials are spin-dependent with the dynamics arising from single invariant...
functions, one for each type of interaction.

One of the advantages this covariant three dimensional equation has over the Bethe Salpeter equation is its simplicity. We obtain an invariant three dimensional Schrödinger-like equation

\[ \{p^2 + \Phi_w(\sigma_1, \sigma_2, p_\perp, A(r), S(r))\} \psi = b^2(w)\psi \]  \hspace{1cm} (1.4)

in terms of a sixteen component wave function \( \psi = [\psi_1, \psi_2, \psi_3, \psi_4] \). Explicitly one finds that the equation has the following form exhibiting coupling between the upper-upper and lower-lower components in addition to the usual spin-dependent interactions symbolized by \( \Phi(\sigma_1, \sigma_2, p_\perp, A(r), S(r)) \),

\[ \{p^2 + 2m_wS + S^2 + 2\epsilon_{wA} - A^2 + \Phi_{11}\} \psi_1 + \Phi_{14} \psi_4 - b^2(w)\psi = 0. \] \hspace{1cm} (1.5)

This equation and the one for the lower-lower component are quantum mechanically well defined, allowing nonperturbative numerical solutions. A second advantage is that in the case of lowest order QED where \( A(r) = -\alpha/r \) they have an analytic Sommerfeld-like solution for singlet positronium with spectral results that agree with standard perturbative results with the same spectral agreement holding numerically with triplet states \([1]\). This implies that they are less likely to produce spurious physics when applied to QCD as may occur in formalisms that do not have this type of nonperturbative QED agreement. Without successful nonperturbative QED tests how can a candidate two-body formalism be trusted for QCD spectral results?

For QCD bound states we use a covariant version of the Adler-Piran static quark potential \([6]\) \( V_{AP}(r) = \Lambda(U(\Lambda r) + U_0) \) \( (= A + S) \) obtained from an effective non-linear Maxwell equation embodying QCD. It analytically displays both asymptotic freedom \( \Lambda U(\Lambda r << 1) \sim 1/r \ln \Lambda r \) and linear quark confinement, \( \Lambda U(\Lambda r > 2) \sim \Lambda^2 r \). We obtain a covariant reinterpretation of their static model by replacing \( r \) with \( x_\perp \) and apportioning the potential between our vector and scalar invariants \( A \) and \( S \) so that at short distance it is vector and long distance it is scalar \([2]\). Once the underlying vector and scalar invariants and quark masses are fixed so are all the spectral predictions from Eq. (1.4).

We obtain very good results for the entire meson spectrum from the light pion to the heavy upsilon states. The quality of our fit is about the same obtained by Godfrey and Isgur \([7]\), but with just 2 invariant functions instead of their 6 \([2]\). The relativistic coupling structure of our equations are equally important for positronium and \( \pi - \rho \) system. If we ignore the coupling to the lower-lower components \( \psi_4 \), we obtain poor hyperfine splitting results for the positronium system. Likewise we would obtain poor results for the hyperfine \( \pi - \rho \) splitting if we ignored that coupling \( (m_\pi \sim 850 \text{ MeV}; m_\rho \sim 1060 \text{ MeV}) \) instead of the fully coupled results \( m_\pi \sim 144 \text{ MeV}; m_\rho \sim 792 \text{ MeV} \) \([2]\).

The nonperturbative structures in our equations provide for chiral symmetry in that the pion (although not its excited states or \( \rho \) behaves like a Goldstone boson. With the coupling structure, the pion mass tends to zero as the quark mass tends to zero, but without the coupling to lower-lower component this does not occur. However, the interaction structure of the TBDE when restricted to vector and scalar interactions does not give the functional dependence of \( m_\pi \) on the quark mass of \( m_q^2 \sim m_q \) behavior dictated from the nonconserved axial current generator of chiral symmetry.

II. TWO-PHOTON DECAYS FROM TWO-BODY DIRAC EQUATION WAVE FUNCTIONS

We take advantage of this close relation between the pion and positronium to treat two photon meson and positronium decays equally. The on-shell Feynman amplitude for electron-positron annihilation in the singlet state is

\[ M_{\alpha\beta} = \frac{e^2}{(2\pi)^3 \sqrt{2}} \{ 5^{(s_+)}(p_+) [\gamma \cdot \epsilon^{(a_1)}] \cdot \frac{m - i\gamma \cdot (p_- - k_1)}{(p_- - k_1)^2 + m^2 - i\delta} \gamma \cdot \epsilon^{(a_2)} + (1 \Leftrightarrow 2) u^{(s_-)}(p_-) - (s_+ \Leftrightarrow s_-) \} \] \hspace{1cm} (2.1)

The two-step change from above amplitude to that for positronium annihilation in which constituents are off shell is indicated below

\[ M_{\alpha\beta} \rightarrow \int d^3p \bar{\psi}(p) M_{\alpha\beta} \equiv M_{1S_0 \rightarrow 2\gamma} \equiv \int d^3p \bar{\psi}(p) \tilde{\Gamma}(p,k) u(p) \]

\[ = \int d^3p \bar{\psi}(p) Tr \Gamma(p,k) u(p) \tilde{\bar{\psi}}(-p) \rightarrow \int d^3p Tr \Gamma(p,k) \Psi(p) \] \hspace{1cm} (2.2)

in which in the final step the 4x4 matrix wave function \( \Psi \) replaces the outer product of the free \( u, \bar{\psi} \) spinors.

In Schrödinger-like form the TBDE for singlet positronium becomes

\[ (-\nabla^2 - 2\epsilon_{w} \alpha/r - \alpha^2/r^2) \psi(r) = b^2(w)\psi(r), \] \hspace{1cm} (2.3)
with an exact solution wave function mildly singular at the origin. However, unlike standard approaches with rates proportional to $|\psi(0)|^2$ (which would give divergent results) the configuration space form of Eq. (2.2) below smears the annihilation amplitude over a Compton wave length so that singularities are rendered harmless:

$$M_{S_0 \rightarrow 2\gamma} = e^2 \sqrt{\pi/2} \int d^3r \exp(-i\mathbf{k} \cdot \mathbf{r}) Tr[\psi(\mathbf{r})[\gamma \cdot \sigma_1 (m+i\gamma \cdot \nabla) \gamma \cdot \sigma_2 + \gamma \cdot \sigma_2 (m+i\gamma \cdot \nabla) \gamma \cdot \sigma_1] \exp(-m\mathbf{r}/r)].$$  (2.4)

This gives rise to standard decay width formulae for positronium:

$$\Gamma(^1S_0 \rightarrow 2\gamma) = ma^5/2,$$
$$\Gamma(^3P_0 \rightarrow 2\gamma) = 3ma^7/256,$$  
$$\Gamma(^3P_2 \rightarrow 2\gamma) = ma^7/320,$$  
$$\Gamma(^3P_0 \rightarrow 2\gamma)/\Gamma(^3P_2 \rightarrow 2\gamma) = 15/4.$$  (2.5)

We recast the equations $S\psi = 0$ in terms of the mass and energy potentials

$$M_i = m_i \cosh L(S, A) + m_i \sinh L(S, A)$$
$$E_i = \epsilon_i \cosh G(A) - \epsilon_i \sinh G(A);$$  
$$G = \exp G,$$  (2.6)

for which

$$S_i = G\beta_i \Sigma_i \cdot p + E_i \beta_i \gamma_5 - i(2)G\Sigma_j \cdot \partial(G\beta_i + L\beta_2)\gamma_5 \gamma_2;$$  
$$i, j = 1, 2.$$  (2.7)

For the pion and other 2-\(\gamma\) meson decays we need to transform from 16 component wave function solutions of TBDE to their 4x4 matrix forms. The computation is simplified by first obtaining decoupled Schrödinger-like form of the TBDE through use of the plus/minus combinations $\phi_\pm = \psi_1 \pm \psi_4; \ \chi_\pm = \psi_2 \pm \psi_3$ of the original TBDE wave functions. A further spin-dependent scale transformation to $\psi_\pm, \eta_\pm$ by the c.m. transformation

$$\phi_\pm = \exp(F + K\sigma_1 \hat{\sigma} \sigma_2 \hat{\tau})\psi_\pm; \ \chi_\pm = \exp(F + K\sigma_1 \hat{\sigma} \sigma_2 \hat{\tau})\eta_\pm$$  (2.8)

gives a still simpler decoupled form (with no momentum-dependent interactions),

$$\{p^2 + 2m_wS(|r|) + S^2(|r|) + 2\Delta A(|r|) - \Delta^2(|r|) + \Phi_D(S, A, \nabla S, \nabla A, \nabla^2 S, \nabla^2 A) + L \cdot (\sigma_1 + \sigma_2) \Phi_{SO}(..) + L \cdot (\sigma_1 - \sigma_2) \Phi_{SO}(..) + iL \cdot \sigma_1 \times \sigma_2 \Phi_{SO}(..) + \sigma_1 \cdot \sigma_2 \Phi_{SS}(..) + \sigma_1 \cdot \hat{\sigma} \sigma_2 \cdot \hat{\tau} \Phi_{TT}(..) + \sigma_1 \cdot \hat{\sigma} \sigma_2 \cdot \hat{\tau} L \cdot (\sigma_1 + \sigma_2) \Phi_{SOT}(..)\} \psi_+ = b^2(w)\psi_+.$$  

This four-component Schrödinger-like form has familiar spin-dependent interactions depending on the underlying invariants $S$ and $A$ that govern the scalar and vector interactions. Each of the four-component spinor wave functions is replaced by a 2x2 matrix wave functions

$$\phi_\pm \equiv \phi_\pm 1 + \phi_\pm \sigma, \ \chi_\pm \equiv \chi_\pm 1 + \chi_\pm \sigma \quad (2.9)$$

given in terms of singlet (scalar) $(\phi_\pm 1, \chi_\pm 1, \psi_\pm 0, \eta_\pm 0)$ and triplet (vector) $(\phi_\pm \sigma, \chi_\pm \sigma, \psi_\pm \eta_\pm)$ wave functions.

The decay amplitude requires the full 4x4 matrix wave function

$$\psi(r) = \exp(F)[\cosh K\Psi(r) - \sinh K\Sigma \cdot \hat{\sigma} \Psi(r) \Sigma \cdot \hat{\tau}]$$
$$\Psi(r) = (\psi_+ q_1 + \psi_- iq_2 + \eta_+ q_0 + \eta_- q_3)/(2\sqrt{2});$$  
$$q_{ij} = \delta_{ij} q_0 + i\epsilon_{ijk} q_k.$$  (2.10)

Once we solve for four component spinor $\psi_\pm$ we can use the two-body Dirac equation to solve for the remaining 12 components, $\psi_0-, \psi_0+, \eta_{0+}, \eta_{0-}$, with the final connection to the numerical meson wave functions from writing $\Psi(r)$ in terms of scalar $Y_{jm}$ and vector spherical harmonics $Y_{j+m}, X_{jm}$,

$$\psi_+ = \psi_{0+} 1 + \psi_+ \sigma$$
$$\psi_{0+} = (u_{0j}^+(r)/r)Y_{jm}; \ \psi_+ = (u_{j+11j}^+(r)/r)Y_{jm+} + (u_{j-11j}^+(r)/r)Y_{jm-} + (u_{j+111j}^+(r)/r)X_{jm-}.$$  (2.11)

Of crucial importance is that the 16 component or 4x4 matrix form of TBD wave function must satisfy CM energy
dependent norm:

$$ \int d^3x [\psi^\dagger (1 + 4w^2\beta_1\beta_2\partial^2/\partial w^2) \psi] \equiv \int d^3x \psi^\dagger L \psi = 1, \quad (2.12) $$

with the matrix $\Delta(\mathbf{r}) = \gamma_5 \gamma_2 [L - \gamma_1 \cdot \gamma_2 \mathbf{G}] / 2$ displaying the core scalar and vector interactions. In 4x4 matrix form we have

$$ \psi = \exp(F) [\cosh K \Psi(\mathbf{r}) - \sinh K \mathbf{\Sigma} \cdot \mathbf{\hat{r}} \Psi(\mathbf{r}) \mathbf{\Sigma} \cdot \mathbf{\hat{r}}] \equiv K \Psi(\mathbf{r}), \quad (2.13) $$

giving us the norm in terms $\psi$ and the Pauli reduction solution $\Psi(\mathbf{r})$. Thus in place of the Naive Norm (NN),

$$ \int d^3x Tr \Psi(\mathbf{r})^\dagger \Psi(\mathbf{r}) = 1, $$

we have the Two-Body Dirac Norm (TBDN),

$$ \int d^3x Tr \psi^\dagger L \psi = \int d^3x Tr (K \Psi(\mathbf{r}))^\dagger L K \Psi(\mathbf{r}) = 1. \quad (2.14) $$

Using the (TBDN) norm has a significant effect on the decay amplitude and rate compared to that of the (NN) norm.

For pseudoscalar meson decay, the triplet wave function is zero ($\psi_+ = 0$) while for vector wave functions the singlet wave function ($\psi_+ = 0$) is zero. The final step of the decay formalism consists of substituting the resulting forms (2.10) into Eq.(2.4) for the matrix element and performing the numerical integration.

| Meson | Expt. | TBDE(TBDN) | TBDE(NN) |
|-------|-------|------------|----------|
| $\pi^0(1^3S_0 - 0.135)$ | 7.72±0.4 eV | 8.73 eV | 33.5 eV |
| $\eta_c(1^3S_0 - 2.976)$ | 7.4±1.0 keV | 6.20 keV | 6.18 keV |
| $\eta_c(2^1S_0 - 3.263)$ | 1.3±0.6 keV | 3.36 keV | 1.95 keV |
| $\chi_0(1^3P_0 - 3.415)$ | 2.6±0.65 keV | 3.96 keV | 3.34 keV |
| $\chi_2(1^3P_2 - 3.556)$ | 0.53±0.09 keV | 0.743 keV | 0.435 keV |

**TABLE I: Meson 2 $\gamma$ Decay Rates**

The results are for the $\pi^0$ is very encouraging being off by only 13%. This is in sharp contrast to the 3 order of magnitude errors in most potential model applications. If we use the naive norm (NN) and amplitude then we obtain 33.5 eV. The results for the $\eta_c$ meson is about the same degree of accuracy. Work by Ackleh and Barnes [8] and earlier ones by Haynes and Isgur [9] found it necessary to include results that appealed to effective field theories for the decay instead of the strictly microscopic approach we have taken here.

In summary, the constraint TBDE formalism gives a covariant two-body wave equation that a) provides comprehensive account for entire meson spectrum, b) is rigorously tested in QED, c) displays a remarkable connection between pion and singlet positronium that accounts for Goldstone boson-like behavior of pion, d) leads to a formalism that works well for light and heavy two-photon meson decays.

[1] H. W. Crater, R. Becker, C. Y. Wong and P. Van Alstine, Phys. Rev. D46, 5117 (1992)
[2] Details may be found in H. W. Crater and P. Van Alstine, Phys. Rev. D70, 034026 (2004) and in references contained therein.
[3] B. Liu and H. W. Crater, Phys. Rev. C67, 024001 (2003).
[4] H. Crater, C. Y. Wong, and P. Van Alstine,Phys. Rev. D74, 054028 (2006)
[5] H. Sazdjian, J. Math. Phys. 28 2618 (1987), H. W. Crater and P. Van Alstine, Ann. Phys. (N.Y.) 148 , 57 (1983).
[6] S. L. Adler and T. Piran, Phys. Lett., 117B, 91 (1982) and references contained therein.
[7] S. Godfrey and N. Isgur, Phys. Rev. D1 32, 189 (1985)
[8] E. S. Ackleh and T. Barnes, Phys. Rev. D 45, 232, (1992).
[9] C. Hayne and N. Isgur, Phys. Rev. D 25, 1944, (1982).