River channel networks created by Poisson Equation and Inhomogeneous Permeability Models (II):
Horton’s law and fractality

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Abstract. In the previous paper, we prepared the Poisson Equation and the Inhomogeneous Permeability Models (PEM & IPM) that can create tree-shaped networks under the conditions of homogeneous or inhomogeneous permeability. The driving force of channel network formation is derived from two-dimensional Poisson equations in both models, the solutions of which are supposed to represent a gravitational pressure field. Particularly important is the latter IPM that succeeds to simulate seemingly natural and realistic river channel basins, in which regional fluctuations of geographical properties concerning soil, precipitation and vegetation are reflected by inhomogeneous permeability. However, we did not refer to the relationship with the Horton’s law and the identification of fractal dimensions. This paper examines the consistency with the Horton’s law and the measurement of fractal dimensions in tree network systems generated by IPMs under the more improved resolution. Our numerical simulations show good accordance with the Horton’s law, however, the calculation of fractal dimensions using the bifurcation and length ratios is not satisfactory because of a large uncertainty. Then, we originally propose an alternative method, referred to as the “extended cluster dimension”, which makes possible to identify the exact value of fractal dimensions in river network systems.

Keywords: Extended cluster dimension, Hack’s law, Horton’s law, Horton-Strahler order, Inhomogeneous Permeability Model (IPM), Melton’s law
### Abbreviations

| Abbreviation | Description |
|--------------|-------------|
| PEM          | Poisson Equation Model |
| IPM          | Inhomogeneous Permeability Model |
| OCN          | Optimal Channel Network |
| FDM          | Finite Difference Method |

### Nomenclature

- $\omega$: Horton-Strahler stream order
- $\Omega$: Maximum Horton-Strahler stream order
- $R_B$: Bifurcation ratio
- $R_L$: Length ratio
- $D$: Fractal dimension calculated by $R_B$ and $R_L$
- $L_{max}$: Mainstream length or longest stream length
- $F_S$: Stream frequency
- $D_c$: Cluster dimension
- $D_L$: Extended cluster dimension
- $N$: Simulation area size
- $k_p$: Ratio of permeability in network cells to that in non-network cells
- $k_r$: Randomization parameter
- $\text{Max}$: Selection rule to choose maximum value
- $\text{Min}$: Selection rule to choose minimum value
- $\text{Av.}$: Average value
- $\overline{\text{Av.}}$: Average of $\text{Av.}$
- $\text{S.D.}$: Standard deviation
- $\text{C.C.}$: Correlation coefficient
- $\overline{S}$: Average slope of mainstream
- $\text{Rn}$: Randomization table
1. Introduction

In the history of natural science, various profound laws have discovered by careful observations of nature. Then, mathematical or physical theorists are tempted to construct simulation models to elucidate the mechanisms of these laws. The most persuasive test for the effect of these models is whether simulation results agree with these empirical laws or not. In the case of river channel formation, corresponding are the Horton’s law [1], the Hack’s law [2], the Melton’s law [3], and so on, among which of great importance is undoubtedly the Horton’s law.

A variety of mathematical models have been constructed for drainage network formation. These are, for example, the random walk model [4], the allometric growth model [5], the infinite topologically random channel network model [6,7], the Fibonacci tree model [8,9], etc. However, we do not refer to these simulation models because they are not based on optimization theories. We insist that the ultimate cause to create ubiquitous structures such as tree-shaped networks is optimization principles.

So far, we have referred to two mathematical models for network formation: one originates from the Optimal Channel Network (OCN) theory by Rodriguez-Iturbe, Rinaldo, et al. [10,11,12,13,14], and the other from the Constructal law by Bejan et al. [15,16,17,18]. Both models are based on the optimization principle that energy expenditure or flow resistance is minimal. For the sake of convenience, we call them the “OCN model” and the “Constructal model”, respectively. Each model possesses both strong and weak points together.

First, the OCN model can reproduce minute and realistic river channel images and the consistency with the Horton’s law is also guaranteed, which are its advantages. However, the OCN model does not seek for the cause of river channel formation. Furthermore, the OCN model does not refer to developing processes of channel network systems. What is important is whether the resultant network configuration is optimal or not. We suspect that the absence of explanation for developing processes is the essential disadvantage of the OCN model.

Compared with the OCN model, the Constructal model takes into consideration developing processes of river channel systems, which is the outstanding merit of the model. Namely, the Constructal model puts the Poisson equation at the core of the two-dimensional area-to-point flow model and deduces the driving force of river channel formation from solutions of this partial differential equation [15]. On the other hand, a serious defect is that river network patterns created by Constructal models seem coarse and not so fine as compared with those by OCN models [15]. Moreover, the comparison between the model and the Horton’s law is not pursued at all. From the beginning, the Constructal theory does not attach importance to fractal structures in river network systems. For example, Reis stated as follows [19]. In contrast with fractal geometry, self-similarity needs not to be alleged previously, but appears as a result of the Constructal optimization of river networks.
Previously, we presented an original model that unified two existing models, i.e., the OCN model and the Constructal model, and incorporated the advantages of both models [20]. This model referred to as the Inhomogeneous Permeability Model (IPM) is founded on the basis of Poisson equations as well as the Constructal model and characterized by following features, such as (1) clear mechanisms that promote tree network formation, (2) moderate performances that can create fine and realistic structures, and (3) simple and plain algorithms that can work even on usual PC systems.

However, two significant challenges were not resolved and remained to be explored as future issues. These are (1) the certification of the consistency with the Horton’s law and (2) the identification of accurate fractal dimensions. In this study, we would like to find solutions of two problems and contribute to establish the reliability of IPMs.

2. Methods

2.1. Horton-Strahler ordering system

Prior to the application of the Horton’s law to the model, it is necessary to introduce an appropriate ordering system that numbers each stream constituting river channel systems. The order of streams is usually measured with the Horton-Strahler ordering system [13,19,21]. Ordering rules in this system are summarized as follows:

(1) channels that originate from sources are defined as the first-order stream;
(2) when two streams of the order $\omega$ join, a stream of the order $\omega + 1$ is created;
(3) when two streams of the different order join, a stream of the higher order is created.

In our simulations, it can happen that three streams join at one site. Thus, the following four supplementary rules will be helpful. These are:

(4) when three streams of the order $\omega$ join, a stream of the order $\omega + 1$ is created;
(5) when two streams of the order $\omega$ and another stream of the order less than $\omega$ join, a stream of the order $\omega + 1$ is created;
(6) when one stream of the order $\omega$ and two other streams of the order less than $\omega$ join, a stream of the order $\omega$ is created;
(7) when three streams of the different order join, a stream of the highest order is created.

The number of streams $N_\omega$ of the Horton-Strahler order $\omega$ is specified according to these ordering rules in the present study. The maximum value of Horton-Strahler orders is particularly signified as $\Omega$. The sample of the river channel network is shown in Fig. 1.

It is also helpful to clarify several technical terms to describe tree network systems. First, the “node” is a junction where a specific link is started or ended. Usually, the “node” is the point where two or three links joins. However, the source, where no link joins from the upper stream, is regarded as a “node”. Next, the “link” is a straight or bended line within which no “node” exists. Meanwhile,
the “stream” is a line connected with one or several “links”, where the order of “links” consisting of the “stream” should be all the same. The distinction between the “link” and the “stream” is critically important. The “segment” is usually equal to the “link”.

The “segment” is usually equal to the “link”. The symbol “●” expresses the node in which a link or a stream is started or ended. The link is a straight or bended line that connects two nodes directly. No node is contained within a link. Meanwhile, the stream is a series of links with the same order. It is possible that a stream consists of either only one or more than two links. There exist 23 nodes, 22 links and 16 streams in this tree network system, which are classified by the stream order $\omega$, such as 12 streams of $\omega=1$, 3 streams of $\omega=2$ and 1 stream of $\omega=3$. Then, the maximum stream order $\Omega=3$. The stream order $\omega$ of each stream is differentiated by line widths.

Figure 1: Nodes, links, streams and Horton-Strahler stream orders. There are $5 \times 5 = 25$ sites in this schematic river network system, which are tentatively numbered from $n=0$ to 24. The outlet is located at the lower left corner, the number of which is $n=0$. The symbol “●” expresses the node in which a link or a stream is started or ended. The link is a straight or bended line that connects two nodes directly. No node is contained within a link. Meanwhile, the stream is a series of links with the same order. It is possible that a stream consists of either only one or more than two links. There exist 23 nodes, 22 links and 16 streams in this tree network system, which are classified by the stream order $\omega$, such as 12 streams of $\omega=1$, 3 streams of $\omega=2$ and 1 stream of $\omega=3$. Then, the maximum stream order $\Omega=3$. The stream order $\omega$ of each stream is differentiated by line widths.

The Horton-Strahler ordering system is now the most commonly used in hydrogeomorphology [13]. The essence of the ordering procedure is summarized in following two [22]:

1. the first-order streams of $\omega=1$ are the headwaters that start at sources, i.e., the streams that have no tributary;
2. the confluence of streams with the order less than $\omega$ gives no influence upon the number of the stream with the order $\omega$, that is, all the junctions with lower-order streams are ignored in assigning order numbers;

However, some other ordering systems are also known. For example, Horton originally had a different idea, in which the mainstream was denoted by the same number all the way from its headwaters to the outlet. Thus, the essence (1) mentioned above is not held [1,22]. Meanwhile, the order is numbered by the concept of the link magnitude according to Shreve [6,13]. Here, the link magnitude of headwaters is 1, and that of the link with two incoming links is the sum of two link
magnitudes. Thus, the magnitude of each link is exactly equal to the number of sources draining into
the corresponding link. In this case, the essence (2) is broken and the concept of “stream” becomes
meaningless. The ordering procedure using the link magnitude is similar to that by Rzhanitsyn, where
the source magnitude is not 1 but 2 [22]. Then, it is likely that the ordering system by Shreve is
referred to that by Rzhanitsyn.

2.2. Horton’s law

In the 1940s, Horton introduced an empirical law with respect to the number of streams \( N_\omega \) and the
average length \( \bar{L}_\omega \) of streams of the order \( \omega \) in river channel networks [1]. According to the Horton’s
law, two ratios, i.e., the bifurcation ratio \( R_B \) and the length ratio \( R_L \), are constant, which are
mathematically expressed as follows.

\[
\frac{N_\omega}{N_{\omega+1}} = R_B, \quad \frac{\bar{L}_{\omega+1}}{\bar{L}_\omega} = R_L. \tag{1}
\]

In natural river basins, \( R_B \) ranges between 3 and 5, typically 4, and \( R_L \) ranges between 1.5 and 3.5,
typically 2 [13,16]. Since then, the Horton’s law has functioned as a reliable criterion to testify the
legitimacy and rationality of mathematical models that simulate river channel networks.

How is the Horton’s law applied to the tree network system? We would like to offer a rough
explanation using a simple example of the river channel pattern illustrated in Fig. 1. As is explained
before, we can find 16 streams in this tree network system, which are classified into 12 streams of
\( \omega=1 \), 3 streams of \( \omega=2 \) and 1 stream of \( \omega=3 \). Thus, \( N_1=12 \), \( N_2=3 \) and \( N_3=1 \). Averaging two values
\( N_1/N_2=4.0 \) and \( N_2/N_3=3.0 \), we get the bifurcation ratio, \( R_B=(4.0+3.0)/2=3.5 \). The number of sources,
which is equal to \( N_1=12 \), is sometimes called the “magnitude” [23,24].

Meanwhile, the length of 12 streams of \( \omega=1 \) is all 1, which gives the average length of \( \omega=1 \),
\( \bar{L}_1=(1 \times 12)/12=1.0 \). Next, 3 streams of \( \omega=2 \) contain 1 stream of the length 1, 1 stream of the length 3
and 1 stream of the length 4, therefore, the average length is \( \bar{L}_2=(1 \times 1+3 \times 1+4 \times 1)/3=2.667 \). In the
same way, the average length of \( \omega=3 \) is \( \bar{L}_3=(4 \times 1)/1=4.0 \). Using the mean value of \( \bar{L}_2/\bar{L}_1=2.667 \) and
\( \bar{L}_3/\bar{L}_2=1.5 \), we obtain as a length ratio, such as \( R_L=(2.667+1.5)/2=2.083 \). These estimations are listed
in Table 1.

In the case of Fig. 1, two ratios, \( R_B=3.5 \) and \( R_L=2.083 \), are both included within reasonable ranges,
\( 3 \leq R_B \leq 5 \) and \( 1.5 \leq R_L \leq 3.5 \). Moreover, the fractal dimension defined by the formula

\[
D = \frac{\ln R_B}{\ln R_L} \tag{2}
\]

is 1.707, which is also appropriate because the value of \( D \) should be \( 1 < D < 2 \). Figure 1 is devised only
for the use of explanation, so that the size of the area, \( 5 \times 5=25 \), is too coarse to mimic natural river
basins. However, the way to calculate three key values of the Horton’s law, \( R_B \), \( R_L \) and \( D \), will be
confirmed. The simulation area of IPMs in this study is a square mesh further divided into \( N \times N \) small
squares, where $N=41$. Thus, the total number of cells is $41 \times 41 = 1681$.

Table 1: Calculation of bifurcation ratio ($R_B$) and length ratio ($R_L$) in Fig. 1.

| Order $(\omega)$ | Number $(N_{\omega})$ | Bifurcation Ratio | Mean Value ($R_B$) | Total Length | Average Length ($L_{\omega}$) | Length Ratio | Mean Value ($R_L$) |
|------------------|------------------------|-------------------|-------------------|--------------|-------------------------------|--------------|-------------------|
| 1                | 12                     | 4.0               | 1$\times$12=12    | 1.0          | 2.667                         | 1.5          | 2.803             |
| 2                | 3                      | 3.5               | 1$\times$1+3$\times$1+4$\times$1=8 | 2.667     |                               | 1.5          |                   |
| 3                | 1                      | 3.0               | 4$\times$1=4      | 4.0          |                               |              |                   |

It should be noted that the diagram in Fig. 1 is constructed arbitrarily for the use of explanation. Nevertheless, not only the values of $R_B$ and $R_L$ but the fractal dimension $D=1.707$ are natural and within reasonable ranges in accordance with the Horton’s law.

2.3. Hack’s law and Melton’s law

In addition to the Horton’s law, following two are also available. First, the Hack’s law expresses the relation between the mainstream length $L_{\omega}$ and the area $A_{\omega}$ of a river basin with streams up to the order $\omega$ [2,19].

$$L_{\omega} = \alpha (A_{\omega})^\beta.$$  \hspace{1cm} (3)

In the formula (3), $\alpha \sim 1.4$ and $\beta \sim 0.568$ are constant. As our simulation models assume that every block within the whole simulation area is occupied by some stream or other in the final state, so $\omega=\Omega$. Moreover, the area corresponding to the mainstream is $A_{\Omega}=A=41 \times 41 = 1680$, where the outlet is excepted. Assuming that the mainstream and the longest streams are equal, the theoretical value of the mainstream length $L_{\Omega}=L_{\text{max}}$ should be

$$L_{\text{max}} \sim 1.4 \times 1680^{0.586} \approx 95.083.$$  \hspace{1cm} (4)

The longest stream length when all the stream lengths are assumed 1 is also called the “topological diameter” [5,6].

On the other hand, the Melton’s law indicates the following relations.

$$D_{\omega} = \frac{L_T}{A}, \quad F_S = \frac{N_S}{A}, \quad F_S = 0.694(D_{\omega})^2.$$  \hspace{1cm} (5)

The drainage density $D_{\omega}$ and the stream frequency $F_S$ are defined by the total length of streams of all orders $L_T$, the number of streams of all orders $N_S$ and the total drainage area $A$ [3,19]. Considering that $L_T=A$ in our simulation settings, the drainage density $D_{\omega}=D_{\Omega}=1.0$ at the final state. Then,

$$F_S = 0.694.$$  \hspace{1cm} (6)

This is the theoretical value of $F_S$ expected by the Melton’s law.
2.4. Cluster dimension and extended cluster dimension

The cluster dimension or the mass dimension $D_c$ is one of the representative values frequently used to identify the fractal dimension, which is defined as follows [13].

$$N_r = r^{D_c}. \quad (7)$$

Here, $r$ represents the size such as the length or the radius from the origin, and $N_r$ does the number of masses or clusters contained in a corresponding area. In this calculation, a series of similar figures, such as squares or circles, are assumed, where the size of figures is magnified in a constant rate. Then, the number of clusters included within each figure $N_r$ is counted in order of the size $r$. Thus, $N_r$ is always the increasing function of $r$, i.e., $N_1 \leq N_2 \leq N_3 \leq \cdots$.

Assuming that similar figures are squares and that the origin is fixed at the lower left corner of the whole domain, i.e., at the lower left corner of the outlet. In usual calculations of the cluster dimension, the site that contains the stream of the network pattern is counted as 1, irrespective of the stream order. Then, $N_r$ is the integer and $N_1=1$. However, if this rule is applied strictly to our model, the result is always $D_c=2.0$ because all the site is occupied by any stream at the final state, as confirmed in Fig. 2. Thus, it is difficult to apply the cluster dimension to our simulation model for the calculation of fractal dimensions, just as it is.

![Figure 2: Calculation of cluster dimension. $N_1=1$, $N_2=4=2^2$, $N_3=9=3^2$, $N_4=16=4^2$, $N_5=25=5^2$. If the cluster dimension is applied strictly to our model, the result is always $D_c=2.0$.](image)

In order to avoid this kind of inconvenience, a manipulation is required to obtain appropriate fractal dimensions, which can be described as follows. In the first place, we use the length or the distance from the outlet $L$ instead of the square size $r$ in the definition (7) or Fig. 2. However, following the cluster dimension where $N_r=1$ ($r=1$), the length of the cell where the outlet is located is 1, thus $N_L=1$ ($L=1$). That is, the distance $L$ is replaced by the real length+1 practically. In cases of $L \geq 2$,
the number of cells that situate at the distance $L$ and less than $L$ are summed up together. For example, the number of cells whose distance from the outlet is 1 and 2 is counted in the case of $L=2$. The fractal dimension calculated by this method is named the “extended cluster dimension” and specified by $D_L$ in this article. In the definition of extended cluster dimensions, all the steams are unleashed and extended out of the square domain.

Rationality of $D_L$ will be confirmed by the following example. Figure 3 (a) is an infeasible snake-like river channel with no tributary. In this extreme case, the fractal dimension should be 1. Actually, $N_1=1^1$, $N_2=2^1$, $N_3=3^1$, $\cdots$, $N_L=L^1$, which lead to $D_L=1.0$, as shown in Fig. 3 (b).

![Figure 3](image)

**Figure 3:** Calculations of extended cluster dimension. (a) Snake-like river channel sample. (b) Calculation of $D_L$ for the pattern of (a). $D_L=1.0$. (c) Calculation of $D_L$ for the pattern of Fig. 1 or 2. $D_L=1.433$. In (b) and (c), $N_L$ is plotted on a double-logarithm squared paper, then, the least square method is employed.

The calculation of extended cluster dimensions regarding the general network pattern such as Fig. 1 or Fig. 2 is as follows, where the mesh size $N=5$. Considering that the longest stream length $L_{\text{max}}=9$, the variable $L$ is changed from 1 to 10 because the real stream length+1 is a substantial variable represented by $L$. Then, the total cluster number of cells corresponding to $L$ and less than $L$ are counted in ascending order.

To begin with, the smallest domain, where $L=1$, contains only one cell, which is the outlet whose cell number $n=0$ as shown in Fig. 1. Consequently, the cluster number $N_1=1$. With respect to the second smallest domain, $L=2$, there exist three cells, which are the cell contained in the smallest domain, $L=1$, plus newly added two cells, whose cell numbers are $n=1$ and 5 in Fig. 1. That is, $N_2=N_1+2=3$. In the same way, we can decide such as $N_3=N_2+1=4$, $N_4=N_3+2=6$, $N_5=N_4+4=10$, and $N_{10}=25$.

Thereafter, ten values $N_1\sim N_{10}$ are plotted on a double-logarithmic paper as in Fig. 3 (c). Drawing the regression line by means of the least square method, we can obtain $D_L=1.433$ from its inclination.
as the extended cluster dimension.

As mentioned in the section 2.1, there are several ordering systems other than the Horton-Strahler system. We would like to emphasize that the extended cluster dimension, as well as the original cluster dimension, is independent of ordering systems. Whatever system is adopted, the value of \( D_L \) is not affected at all. Irrelevance to the ordering system might be one of the biggest merits of this calculation method.

2.5. PC performance, resolution and simulation area

PC environments in this study are slightly improved compared with those in the previous study [20]. That is, the device is a newly purchased laptop PC (NEC: LAVIE, CPU: Intel(R) Core (TM) i7-8550U, OS: Windows 10). The programing language is Java, and all the programs are compiled using a free software (Borland: Turbo JBuilder 2007), which are the same as those of the previous study [20].

However, our PC performance is no better than those of preceding studies as before. While the resolution of our river channel images is increased from 37×37 to 41×41, it is still lower than those of these studies. For example, the resolution is 128×128 in the OCN model [12] and 51×51 in the Constructal model [15]. Taking into consideration that the performances of computer systems used in these preceding studies are much higher than ours, we would like to stress again that even domestic type PCs of reasonable prices are available for drawing sufficiently fine and naturalistic tree network images.

The simulation area is a square lattice as illustrated in Fig. 4 (a). The square area is further divided into \( N \times N \) small square cells, where \( N=41 \). Thus, the whole area is constituted of \( 41 \times 41 = 1681 \) cells. Our PC system requires about 110 minutes to complete each tree network pattern of \( N=41 \). Twice larger simulation time than that in the previous studies is due to the increase in resolution [20].

Figure 4 (a) shows the area when the outlet is located at the corner (vertex) of the domain. The outlet from which water is drained is shown by a black square. There are two candidate cells that can be the second network cell in the next step, which are also shown by gray squares in Fig. 4 (a). The outlet from which the network grow is, as it were, the seed of the tree-shaped river channel network.

Following the previous study [20], we continue to use some special terms also in this article. First, the “network cell” is defined as a cell that is already a component of the existing network. Next, the “candidate cell” means a cell that is adjacent to the “network cell”, which is expected to be a new “network cell” in the next step. The “non-network cell” contains all the cells that do not belong to the “network cell”, which includes the “candidate cell”. Further, the “critical cell” is the newest “network cell” just changed from the “candidate cell”. In the case of Fig. 4 (a), the outlet is the only “network cell”, and others are all “non-network cells”, within which two “candidate cells” are contained.
Figure 4: (a) Simulation area. A black square located at the lower left corner of the square mesh indicates the outlet, which is the only “network cell” at the initial state that can function as a seed of network. Two gray squares adjacent to the black square exhibit the “candidate cells”. (b) Initial distributions of $u$-values in IPM. The abscissa stands for the central position of each cell along the $x$-axis. Solid lines connect $u$-values corresponding to the same $y$-coordinate. The outline of the IPM will be described in the succeeding section.

2.6. Outline of Inhomogeneous Permeability Model (IPM)

Previously, we presented original mathematical models that simulate tree-shaped networks such as natural river channel basins [20]. These are the Poisson Equation Model (PEM) for homogeneous conditions and the Inhomogeneous Permeability Model (IPM) for inhomogeneous conditions. Only the IPM is available in the present study, however, this model is also based on the Poisson equation as well as the PEM. That is, the IPM should be called the extended version of the PEM corresponding to inhomogeneous conditions. We have already explained IPMs in detail in the previous article. Then, we present a rough sketch of the model here and explain the outline only.

As well as our previous study [20], we begin with the following two-dimensional Poisson equation, where two variables $(x,y)$ signify horizontal coordinates.

$$\frac{\partial(K \frac{\partial P}{\partial x})}{\partial x} + \frac{\partial(K \frac{\partial P}{\partial y})}{\partial y} + \frac{\dot{m}'' \nu}{W} = 0. \quad (8)$$

The pressure field $P(x,y)$ is induced by the uniform mass flow $\dot{m}''$ from the upper surface as a constant rainfall. In the initial state of IPMs, the whole area is filled with porous media of non-uniform permeability $K(x,y)$, which means that the area is covered with the different quality of soil and vegetation. Regarding other parameters, $\nu = \mu/\rho$ is the kinematic viscosity where $\mu$ and $\rho$ are the viscosity and the density of the fluid, respectively, and $W$ is the depth of the media. We modify (8) to the dimensionless form (9) for the practical use.
\[
\frac{\partial (k \frac{\partial u}{\partial x})}{\partial x} + \frac{\partial (k \frac{\partial u}{\partial y})}{\partial y} + f(x,y) = 0 ,
\]
\[
\frac{\partial k}{\partial x} + k \frac{\partial^2 u}{\partial x^2} + \frac{\partial k}{\partial y} \frac{\partial u}{\partial y} + k \frac{\partial^2 u}{\partial y^2} + f(x,y) = 0 , \tag{9}
\]
\[
\frac{\partial k}{\partial x} + \frac{\partial k}{\partial y} + k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + f(x,y) = 0 .
\]

The simplified Poisson equation (9) obtained by non-dimensionalization is extensively used in the analyses of IPMs.

The dimensionless function \(k(x,y)\) defines the initial distribution of inhomogeneous permeability, whose average value is almost equal to 1. It should be noted that \(k\)-values are given as a scalar in each site, the number of which is \(N\times N\). In the current circumstance, a certain amount of water is continuously and uniformly poured into the simulation area from the external environment, i.e., from the upper surface, which controls the amount of \(f(x,y)\). Then, it is reasonable to assume that \(f(x,y)\) is constant at the initial moment all over the area except for the outlet.

The IPM expressed by the Poisson equation (9) is a conceptual model, so that the absolute values of non-dimensional variables make no sense. Only relative \(u\)-values are significant. Considering that (9) is linear and that absolute \(u\)-values are meaningless, we can give a finite value to \(f(x,y)\) arbitrarily. Then, we postulate that \(f(x,y)=1\) at the starting point in all cells except for the outlet. That is, the total amount of water pouring into the whole simulation area is always \(N^2\) per unit time. IPMs represented by (9) are solved using the Finite Difference Method (FDM) assisted by the Cholesky decomposition method [20].

In the most part of boundaries, Neumann boundary conditions, \(\partial u/\partial n=0\), are imposed, which means that neither inflow nor outflow of water takes place through boundaries. However, as a \(u\)-value of the outlet is fixed at 0 throughout the simulation, the Dirichlet boundary condition, \(u=0\), is exceptionally imposed only in this site. The example of the initial \(u\)-value profile along the \(x\)-axis at the starting point is displayed in Fig. 4 (b), where the outlet locates at the lower left corner. All of \(u\)-values are positive, although \(u=0\) at the outlet.

As is already explained in the section 2.5, numerical simulations are performed within a square area, which is further divided into \(N\times N\) small square cells. In the case of IPMs, it is assumed that a total area is coated with inhomogeneous porous media with permeability \(k(x,y)\) in the initial state of simulations except for the outlet from which water is drained. Under the condition of a constant rainfall, the pressure field \(u(x,y)\) represented by the solution of the Poisson equation (9) continues to increase almost uniformly and proportionally with precipitation, i.e., time. For the first step, two cells that exist adjacent to the outlet are candidates that can be changed to the channel. As one of two is selected and connected with the outlet, the number of network cells becomes two. These two cells including the outlet constitute an initial form of the tree-shaped channel network.

After the first conversion, the cell previously occupied by initial media is replaced by new media
with higher permeability $K_P$ ($K_P > K$), meaning that water becomes easy to flow within the new channel network. Then, the Poisson equation is recalculated in new conditions. In this way, the number of cells with permeability $K_P$ continues to increase one by one, joining to the existing network. This process is repeated until all the cell is converted to the network cell, and a channel network covers the whole area.

Suppose that $k_p$ is the ratio of permeability in network cells to that in non-network cells, then, $k_p = K/K_p < 1$. Thus, $f(x,y) = k_p$ for network cells under the condition of constant rainfall. We adopt $k_p = 0.1$ for all simulations, which is used as a representative value in Constructal models [15,16]. Meanwhile, non-network cells continue to store rainfall in proportion to time because water is not drained in these regions. Hence, the values of $f(x,y)$ in non-network cells continue to increase until they are incorporated into the existing network. In our PC algorithm, $f(x,y) = t + 1$ is assumed for non-network cells, where the variable $t$ means elapsed time from the start.

The candidates expected to be next network cells must be adjoining the existing river network. Here, two selection rules can be conceivable, which are the “Max selection rule” and the “Min selection rule”. In the case of the Max selection rule, one of the candidate cells whose slope to the adjacent network cell is maximal is selected and changed as a critical cell, while the cell with the minimal slope is selected in the case of the Min selection rule. From the optimal viewpoint, the Max selection rule is preferable to the Min selection rule because the former system reaches the final optimal state more quickly than the latter. Thus, the Max selection rule should be adopted in principle. However, the simulations by the Min selection rule are also tested in this study.

Here, the slope, i.e., the gradient can be calculated from the Poisson equation (9) as $m = \Delta u/h$, where $h$ is the interval between adjacent cells. If the interval $h$ is constant and $h = 1$, the difference of $u$-values $\Delta u$ between a candidate cell and its neighboring network cell is exactly equal to the gradient. Thus, $m = \Delta u$, then, the gradient can be signified by $\Delta u$ with the denominator omitted.

Finally, the two-dimensional inhomogeneous permeability function $k(x,y)$ is set up using randomization tables in the whole simulation area. With respect to randomization of permeability, the simplest method is adopted as well as in the previous study [20]. Random numbers are scattered within a whole area, just as it is. Using the consecutive number $n$, the random number $r[n]$, instead of $r(n)$, and the one-dimensional array $k[n]$, instead of $k(x,y)$, $k[n]$ is specified, such that

$$k[n] = 1 + k_r r[n], \quad 0 < k_r < 1, \quad 1 < r[n] < 1,$$

$$n = 0, 1, \ldots, N^2 - 1.$$

The randomization parameter $k_r$ indicates the amplitude of randomization, hence, $k[n]$ ranges between $1-k_r$ and $1+k_r$. The variable $k[n]$ should be positive in any site, which requires $0 < k_r < 1$. The consecutive cell number $n$ varies from 0 to $N^2 - 1$, covering all the sites. The relation between the consecutive cell number $n$ and the coordinate $(x,y)$ are $x = n \% N$ and $y = n/N$, where “%” and “/” represent the remainder and the quotient, respectively. In our numerical simulations, $k_r$-values are varied within a range of $0.025 \leq k_r \leq 0.125$. 

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In general, large permeability means the increase in easiness to flow through the corresponding cell. On the contrary, small permeability suggests the decrease in easiness, i.e., the increase in difficulty to pass through the cell. It is supposed that permeability comprehensively reflects geographic properties of each site such as roughness of the soil, unevenness of the surface, vegetation on the surface, and so on.

Compared with the previous study [20], it is confirmed that unnatural and artificial structures appear and that unnaturally disturbed river basin images appear in large randomization parameter regions such as $0.15 \leq k_r \leq 0.2$, although the situation considerably depends on the randomization table in use. However, simulation results are sufficiently natural and realistic within the range of $0.025 \leq k_r \leq 0.125$. While the reason is not clear, discrepancies might be attributed to the increase in resolution from $N=37$ to 41.

3. Results

3.1. River channel patterns

To start with, we draw a variety of river channel patterns by means of IPMs. In this study, our numerical simulations are repeated three times using different random number tables as well as in the previous study [20], which are named Rn#0, Rn#1 and Rn#2 for the sake of convenience. Typical river network images for Rn#0, Rn#1 and Rn#2 are displayed in Fig. 5, where the randomization parameter is fixed at $k_r=0.1$. Among six images, (a), (b) and (c) are performed under the condition of the Max selection rule, while (d), (e) and (f) are under the condition of the Min selection rule. The mainstream lengths $L_{\text{max}}$ are recorded for reference.

The outlet placed at the lower left corner is emphasized by a comparatively large blue square. The streams of different Horton-Strahler orders are distinguished by the line width. That is, streams with the larger Horton-Strahler order are drawn by the thicker line. The line width of first order streams that start from sources is especially 1, i.e., the finest width.

Another parameter $k_p$ is fixed at 0.1, which is the ratio of permeability in network cells to that in non-network cells. Although every cell has different permeability in the non-network state, the conversion ratio $k_p$ is constant for all cells. Namely, the values of permeability $k(x,y)$ in non-network cells are proportionally reduced to permeability $k_p \times k(x,y)$ after converted to network cells.

The maximum Horton-Strahler orders $\Omega$ are also identified in Fig. 5. A total of 30 patterns drawn in this study show either $\Omega=5$ or $\Omega=6$, where the number of the former type is 9 and that of the latter type is 21. It is probably true that the maximum order $\Omega$ is increased with the increase in the size of the simulation area $N$. Thus, the positive correlation is expected between $\Omega$ and $N$. For example, when $N=37$ or $N=41$, $\Omega=5$ or $\Omega=6$, while $\Omega=3$ when $N=5$, as shown in Fig. 1. Furthermore, the ratio of $\Omega=6$ to $\Omega=5$ is increased from $20/28=0.714$ ($N=37$) in the previous study [20] to $21/9=2.333$ ($N=41$) in
this study. These observations suggest that the larger area size $N$, the larger maximum order $\Omega$.

Figure 5: River channel patterns by means of IPM. The Max selection rule is employed in (a), (b) and (c), while the Min selection rule is in (d), (e) and (f). Simulation settings are the same with each other except for the selection rule and the random number table in use. The Neumann boundary conditions ($\partial u/\partial n=0$) are imposed except for the outlet. The line widths are varied depending on the Horton-Strahler order as well as in Fig. 6. $k_p=0.1$, $k_r=0.1$. (a) Max, Rn#0, $L_{\text{max}}=97$, $\Omega=5$, (b) Max, Rn#1, $L_{\text{max}}=171$, $\Omega=5$, (c) Max, Rn#2, $L_{\text{max}}=82$, $\Omega=6$, (d) Min, Rn#0, $L_{\text{max}}=142$, $\Omega=6$, (e) Min, Rn#1, $L_{\text{max}}=144$, $\Omega=6$, (f) Min, Rn#2, $L_{\text{max}}=143$, $\Omega=6$.

3.2. Dependences on randomization parameter

Next, we examine the dependence of river channel patterns on the randomization parameter $k_r$ for both selection rules. The simulation results are shown in Fig. 6 for three randomization parameters, $k_r=0.025$, 0.05 and 0.075 in use of the same random number table Rn#0. Three figures (a), (b) and (c) correspond to the Max selection rule, whereas (d), (e) and (f) correspond to the Min selection rule. Figure 5 (a) and (d) are also referential, where $k_r=0.1$.

When the random number table Rn#0 and the Max selection rule are adopted, river network patterns seem to be altered sensitively depending on the randomization parameter $k_r$. Meanders of streams and disturbances of channel network patterns are gradually increased with the increase in the
parameter $k_r$. However, the mainstream lengths, i.e., the longest stream lengths $L_{\text{max}}$, are not so largely changed irrespective of $k_r$.

From the viewpoint of optimization principles such as the OCN theory or the Constructal law, the usage of the Max selection rule is rational because the fastest relaxation to the optimal stable state is expected. However, it is an interesting issue whether the Min selection rule generates river networks with a lot of bifurcation branches or not.

In fact, it is certified that the Min selection rule can promote the formation of realistic river network patterns as well as the Max selection rule. Figure 6 (d), (e) and (f) show the simulation results in the Min selection cases for three randomization parameters, $k_r=0.025$, 0.05 and 0.075. Other settings are not changed compared with the Max selection cases (a), (b) and (c). Comparison among three images clarifies that meandering or sinuosity as a whole disappear with the increase in the randomization parameter $k_r$. At the same time, the longest stream lengths $L_{\text{max}}$ are dropped rapidly, the tendencies of which are quite different from those in the Max cases.

Figure 6: River channel patterns by means of IPM (Rn#0). The Max selection rule is adopted in (a), (b) and (c), while the Min selection rule is in (d), (e) and (f). The same random number table Rn#0 is in use for all six patterns. $k_r=0.1$. (a) Max, $k_r=0.025$, $L_{\text{max}}=91$, (b) Max, $k_r=0.05$, $L_{\text{max}}=102$, (c) Max, $k_r=0.075$, $L_{\text{max}}=107$, (d) Min, $k_r=0.025$, $L_{\text{max}}=194$, (e) Min, $k_r=0.05$, $L_{\text{max}}=169$, (f) Min, $k_r=0.075$, $L_{\text{max}}=158$. The theoretical value predicted by the Hack’s law is $L_{\text{max}}=95.083$.

The general impressions received from simulation patterns are somewhat different between in the
Max and Min selection cases, which could be summarized as follows. The channel network patterns by the Max selection rule seem to be composed of comparatively long tributaries. Some of them branch off near the outlet, whose lengths are not so different from that of the mainstream. Meanwhile, a meandering mainstream is obviously specified in the Min selection cases, whose lengths are distinguishably long compared with other tributaries. These differences in appearance tend to be diminished with the increase in the randomization parameter $k_r$.

The differences of views in perceptual observations mentioned above are easily confirmed by comparing mainstream lengths, i.e., the longest stream lengths $L_{\text{max}}$. It is obvious that if the randomization parameter $k_r$ is the same, the values of $L_{\text{max}}$ by the Min rule are much larger than those by the Max rule, although the differences are reduced with the increase in $k_r$.

From these observations, we can expect that both types of river channel patterns could be substantiated in the natural world within a moderate randomization range. For example, the widespread tree-shaped patterns by the Max selection rule exist in comparatively flat plains, whereas the meandering slender patterns with a prolonged mainstream by the Min selection rule are dominated in steep mountainous regions. It is likely that the gradient of river channels is deeply concerned with the difference in appearances between the Max and Min selection rules.

### 3.3. Bifurcation ratio, length ratio and fractal dimension

As stated in the section 2.2, the Horton’s law is one of the standard criteria to testify the reliability of numerical models that reproduce river channel networks. According to the Horton’s law, the permissible range and the typical value are 3–5 and 4 for the bifurcation ratio $R_B$, 1.5–3.5 and 2 for the length ratio $R_L$, respectively [13,16]. It is self-evident that the permissible range of the fractal dimension is $1< D<2$.

Table 2 shows the values of $R_B$ and $R_L$ for all river network patterns obtained from our numerical simulations in the case of both Max and Min selection rules. The total number of 30 simulations are carried out, using three random number tables, Rn#0, Rn#1 and Rn#2. Fractal dimensions $D$ derived from $R_B$ and $R_L$ by the formula (2) are also listed. The values of $\overline{\text{Av.}}$-columns are averaged among three randomization cases with the randomization parameter $k_r$ fixed. Most of the values are included within permissible ranges. However, some of them are out of the ranges, which are emphasized by underlines.

Table 2 shows the values of $R_B$ and $R_L$ for all river network patterns obtained from our numerical simulations in the case of both Max and Min selection rules. The total number of 30 simulations are carried out, using three random number tables, Rn#0, Rn#1 and Rn#2. Fractal dimensions $D$ derived from $R_B$ and $R_L$ by the formula (2) are also listed. The values of $\overline{\text{Av.}}$-columns are averaged among three randomization cases with the randomization parameter $k_r$ fixed. Most of the values are included within permissible ranges. However, some of them are out of the ranges, which are emphasized by underlines.

Such values as $\overline{\text{Av.}}$, S.D., S.D./$\overline{\text{Av.}}$ and C.C. are also indicated in Table 2. $\overline{\text{Av.}}$ is the average of Av., S.D. is the standard deviation calculated using $\overline{\text{Av.}}$, and C.C. is the correlation coefficient between Av. and the parameter $k_r$. The ratio S.D./$\overline{\text{Av.}}$ is, as it were, the normalized standard deviation, by which we intend to compare the degree of dispersion or scattering among various physical quantities of different characteristics. For example, as the normalized dispersion S.D./$\overline{\text{Av.}}$ of $R_L$ is larger than that of $R_B$ in both selection rules, we infer that bifurcation ratios are extended relatively to a wider range
than length ratios.

Table 2: Simulation results of bifurcation ratios $R_B$, length ratios $R_l$ and fractal dimensions $D$.

| Randomization Parameter ($k_r$) | Bifurcation Ratio ($R_B$) | Length Ratio ($R_l$) | Fractal Dimension ($D$) |
|----------------------------------|---------------------------|----------------------|-------------------------|
|                                  | Rn$\#0$ Rn$\#1$ Rn$\#2$ | Av. Rn$\#0$ Rn$\#1$ Rn$\#2$ | Av. Rn$\#0$ Rn$\#1$ Rn$\#2$ Av. |
| Max selection rule               |                           |                      |                         |
| 0.025                            | 4.049 3.961 5.327         | 4.445 1.905 1.931    | 2.629 2.155 2.169 2.091 | 1.730 1.997         |
| 0.05                             | 5.265 5.220 4.005         | 4.830 2.799 2.776    | 2.234 2.603 1.614 1.619 | 1.726 1.653         |
| 0.075                            | 3.979 3.966 3.984         | 3.976 1.897 2.058    | 1.967 1.974 2.157 1.909 | 2.043 2.036         |
| 0.1                              | 5.235 5.186 4.083         | 4.835 2.738 3.069    | 2.024 2.610 1.644 1.468 | 1.995 1.702         |
| 0.125                            | 4.130 4.120 4.080         | 4.110 2.567 1.863    | 1.980 2.136 1.505 2.276 | 2.059 1.947         |
| Av.                              |                           | 4.439                | 2.296                   | 1.867               |
| S.D.                             |                           | 0.356                | 0.262                   | 0.158               |
| S.D./Av.                         |                           | 0.080                | 0.114                   | 0.085               |
| C.C.                             | -0.265                    | -0.017               | -0.046                  |
| Min selection rule               |                           |                      |                         |
| 0.025                            | 5.335 5.244 5.325         | 5.301 3.646 3.384    | 4.413 3.814 1.294 1.359 | 1.126 1.260         |
| 0.05                             | 3.956 3.988 5.288         | 4.411 3.142 2.453    | 3.124 2.906 1.201 1.542 | 1.462 1.402         |
| 0.075                            | 3.852 3.995 3.951         | 3.933 2.385 2.465    | 1.992 2.281 1.552 1.535 | 1.993 1.693         |
| 0.1                              | 3.853 4.059 4.014         | 3.975 3.011 2.370    | 4.077 3.153 1.224 1.623 | 0.989 1.279         |
| 0.2                              | 3.890 3.983 3.986         | 3.953 3.405 1.997    | 2.338 2.580 1.109 1.998 | 1.628 1.578         |
| Av.                              |                           | 4.315                | 2.947                   | 1.442               |
| S.D.                             |                           | 0.524                | 0.524                   | 0.169               |
| S.D./Av.                         |                           | 0.122                | 0.178                   | 0.117               |
| C.C.                             | -0.845                    | -0.599               | 0.429                   |

Av. shows the average values of $R_B$, $R_l$ and $D$ among three randomization cases, Rn$\#0$, Rn$\#1$ and Rn$\#2$, where the parameter $k_r$ is the same. Fractal dimensions $D$ are calculated according to the formula, $D=\ln R_B/\ln R_l$. Av., S.D. and C.C. mean the average of Av., the standard deviation and the correlation coefficient, respectively. Underlined figures indicate that these are out of permissible ranges.

Meanwhile, the correlation coefficient C.C. is introduced to examine the tendencies of quantitative change with the value of $k_r$. The plus or minus sign of C.C. means the positive or negative correlation with $k_r$, which increases linearly. C.C.-values in Table 2 suggest that the bifurcation ratio $R_B$, the length ratio $R_l$ and the fractal dimension $D$ of the Max selection cases are all decreased with the increase in $k_r$. However, the absolute values of C.C. are small particularly for $R_l$ and $D$, meaning that these two are almost constant within the range of $0.025 \leq k_r \leq 0.125$ in numerical simulations of the
Max selection rule. Further, we would like to stress that the average value of fractal dimension \( D = 1.867 \) in Max selection rules resides near 2, which is the dimension of a plane surface. This is probably related to the appearance of river channels especially to comparatively small \( L_{\text{max}} \)-values.

Simulation results by means of the Min selection rule are also reported in Table 2. Deviations from the permissible ranges are outstanding especially when \( k_r = 0.025 \). However, discrepancies tend to disappear with the increase in \( k_r \), within the randomization range of \( k_r \geq 0.05 \). Regarding the signs of correlation coefficients \( \text{C.C.} \), only the value for \( D \) is opposite to that of the Max selection rule. That is, the bifurcation ratio \( R_B \) and the length ratio \( R_L \) are decreased with the increase in \( k_r \), while the fractal dimension \( D \) is increased. In the case of the Min selection rule, three absolute values of \( \text{C.C.} \) are comparatively large as compared with the Max case, suggesting stronger correlations with \( k_r \) for three variables \( R_B, R_L \) and \( D \).

![Figure 7](image-url)

Figure 7: Dependences of (a), (d) bifurcation ratios \( R_B \), (b), (e) length ratios \( R_L \) and (c), (f) fractal dimensions \( D \) on randomization parameters \( k_r \). Three diagrams (a), (b) and (c) correspond to the Max selection rule, and (d), (e) and (f) to the Min selection rule. The ensemble averages of three kinds of randomization, \( R_n \#0, R_n \#1 \) and \( R_n \#2 \), are represented by crosses (\( \times \)), where the vertical lines show the maximum and the minimum among three.
It is notable that the average fractal dimension $D=1.442$ of the Min selection rule is much smaller than $D=1.867$ of the Max selection rule. In general, the shapes with the fractal dimension near 2 are close to plane surfaces while those with the fractal dimension near 1 are close to curved lines. The comparison between the fractal dimensions $D$ and the longest stream lengths $L_{\text{max}}$ will be discussed comprehensively in the following section.

Six diagrams of Fig. 7 exhibit the dependences of bifurcation ratios $R_B$, length ratios $R_L$ and fractal dimensions $D$ on the randomization parameter $k_r$, where the maximum, the minimum and the ensemble average of three randomization cases, Rn#0, Rn#1 and Rn#2, are marked for each $k_r$. Comparatively large fluctuations and differences between the maximum and the minimum through the whole range are outstanding for all three variables, $R_B$, $R_L$ and $D$.

### 3.4. Mainstream length and stream frequency

Besides the Horton’s law, the Hack’s law and the Melton’s law predict the mainstream length $L_{\text{max}}$ and the stream frequency $F_S$ of river channel systems, respectively. Mathematical modelers can evaluate their works also with reference to these empirical laws. According to the formulae (4) and (6), typical values derived from two laws are $L_{\text{max}}=95.083$ and $F_S=0.694$ in our simulation settings $N=41$, respectively. The values obtained from our numerical simulations by both the Max and Min selection rules are shown in Table 3. The data of the same $k_r$ are averaged among three randomization cases, Rn#0, Rn#1 and Rn#2. The meanings of Av., $\bar{\text{Av.}}$, S.D. and C.C. are the same as those in Table 2.

First, we inspect the simulation results of mainstream lengths $L_{\text{max}}$, where the longest stream is regarded as the mainstream. When the Max selection rule is employed, mainstream lengths, i.e., maximum $L$-values seem to show almost good accordance with the theoretical value. However, not only strong dependence on randomization tables is observed but obvious anomalies appear even within a same randomization table. The values of $L_{\text{max}}$ seem to increase with the increase in the parameter $k_r$ in the Max cases, which is possibly due to the increase in randomization.

In the Min selection cases, on the other hand, the values are generally larger than those in the Max cases and also the predicted value $L_{\text{max}}=95.083$ within the almost whole range $0.025 \leq k_r \leq 0.125$. However, these values tend to decrease with the increase in $k_r$, the tendency of which is opposite to that in the Max case. As a result, the differences of $L_{\text{max}}$ values in both cases gradually diminished and approach with each other as the $k_r$-value is increased. At the same time, the differences in appearances of river channel images are apt to become unclear between the two selection cases.

It is interesting that the signs of correlation coefficients C.C. of $L_{\text{max}}$ are different with each other between the Max and Min cases, which is clearly seen also in Fig. 8. The close correlation between $L_{\text{max}}$ and the fractal dimensions $D$ in Table 2 for the change of $k_r$ is one of the main themes in this study.
Table 3: Simulation results of mainstream lengths $L_{\text{max}}$ and stream frequencies $F_S$

| Randomization Parameter ($k_r$) | Max selection rule | Min selection rule | SF ($F_S$) |
|---------------------------------|--------------------|--------------------|------------|
|                                 | Rn#0   | Rn#1   | Rn#2   | Av.   | Rn#0   | Rn#1   | Rn#2   | Max | Min |
| 0.025                           | 91     | 111    | 82     | 94.7  | 194    | 206    | 150    | 183.3 | 0.553 0.536 |
| 0.05                            | 102    | 122    | 112    | 112.0 | 169    | 162    | 149    | 160.0 | 0.551 0.553 |
| 0.075                           | 107    | 104    | 86     | 99.0  | 158    | 150    | 109    | 139.0 | 0.539 0.558 |
| 0.1                             | 97     | 171    | 82     | 116.7 | 142    | 144    | 143    | 143.0 | 0.526 0.565 |
| 0.125                           | 134    | 127    | 88     | 116.3 | 155    | 126    | 130    | 137.0 | 0.521 0.566 |

| Av.                            | 107.7  | 152.5  | 0.538 0.556 |
| S.D.                           | 9.154  | 17.434 | 0.013 0.011 |
| S.D./Av.                      | 0.085  | 0.114  | 0.024 0.020 |
| C.C.                          | 0.742  | -0.890 | -0.978 | 0.940 |

The theoretical value of the mainstream length derived from the Hack’s law is $L_{\text{max}}=95.083$, and that of the stream frequency from the Melton’s law is $F_S=0.694$, respectively. As for stream frequencies $F_S$, only average values are listed because of extremely small fluctuations (S.D.=0.013 and 0.011).

![Graph](image1.png)

Figure 8: Dependences of mainstream lengths $L_{\text{max}}$ on randomization parameters $k_r$. (a) Max selection rule, (b) Min selection rule. The ensemble averages of three kinds of randomization, Rn#0, Rn#1 and Rn#2, are represented by crosses ($\times$) with vertical lines that show the maximum and the minimum.

Meanwhile, stream frequencies derived from the simulation data are somewhat smaller than the theoretical value, $F_S=0.694$, within the whole range in both the Max and Min cases. Moreover, the $\text{Av.}$-values of $F_S$ are not so different with each other between both cases. In our simulation settings, the drainage density $D_\Omega$ is assumed to be 1.0 because all the cell is converted to the network cell and occupied by water channels at the final stage, which could be the reason for comparatively small values of $F_S$. These conditions are inconceivable to be realized in natural river channel systems.
It should be noticed that the fluctuation of the stream frequency $F_S$ is very small, which is confirmed by extremely small values of S.D. (S.D/ $\bar{A}v.$) such as 0.013 (0.024) in the Max case and 0.011 (0.020) in the Min case. For these reasons, only average values are listed in Table 3, and dependences of $F_S$ are omitted in Fig. 8.

4. Discussion

4.1. Comparisons with real data

With respect to bifurcation ratios $R_B$, length ratios $R_L$ and fractal dimensions $D$, the accordance of simulation results in Table 2 with the Horton’s law seems satisfactory. It is certain that some of them lie out of the permissible ranges. However, most of them are within the ranges, which would guarantee the validity and rationality of our numerical model IPM.

Comparisons of simulation results with real data are the first priority for mathematical models. In order to reinforce the reliability of the IPM, we perform these comparisons with actual measurements of natural river basins. Table 4 exemplifies the data of $R_B$, $R_L$ and $D$ observed in real river basins. In the first set labelled as Basins (A), the data of 23 Southern Italy basins are collected by Claps et al. [24,25]. While some subdomains are included among them, they are dealt with equivalently. The second set, Basins (B), contains 12 natural stream networks in the Appalachian Plateau. These are taken from Liu [26], which originate further from Morisawa [27]. In both sets, fractal dimensions are calculated according to the definition derived from the Horton’s law, $D=\ln R_B/\ln R_L$. The average $\bar{A}v.$-values, which corresponds to $\bar{A}v.$ in Table 2, and standard deviations S.D. are also calculated individually in Table 4.

The details of geomorphic properties for Basins (A) are lacking in the literatures [24,25]. However, the topographical map shown in the literature implies that the most part of basins locates in the flat plain along the Adriatic Sea [24]. Meanwhile, topographical data of Basins (B) are described comparatively in detail in the literature [27]. The Appalachian Plateau or the Appalachian Mountains locates in eastern North America, which spans U.S. to Canada. The area consists of folded strata created about 480 million years ago during the Ordovician era, the average height of which is about one thousand meters. The long-term erosion has carved the plateau, forming a thousand of drainage valleys [27].

Table 4 also displays the average slopes of mainstreams $\bar{S}$ for two sets, however, calculations are performed in different manners between Basins (A) and Basins (B). The values of $\bar{S}$ in Basins (A) are obtained by the division of the mainstream length by the total elevation drop [25]. Meanwhile, $\bar{S}$-values of Basins (B) are simply averaged among the slopes of different order streams [27]. Due to a shortage of common data sets, we could not help adopt different calculation methods for $\bar{S}$. Nevertheless, much attention should be paid to the fact that the average value of $\bar{S}$ (=0.028) in Basins
(A) is smaller than that (=0.068) in Basins (B). It is possible to interpret that rivers of Basins (A) flow through the flat land, whereas those of Basins (B) through the steep mountainous area.

| Basins (A)            | $R_B$ | $R_L$ | $D$ | $\bar{S}$ | Basins (B)            | $R_B$ | $R_L$ | $D$ | $\bar{S}$ |
|-----------------------|-------|-------|-----|-----------|-----------------------|-------|-------|-----|-----------|
| Ofanto a S.Samuele    | 3.69  | 2.18  | 1.68|           | Tar Hollow, O.        | 4.13  | 3.55  | 1.12| 0.232     |
| Ofanto a Cairano      | 4.00  | 2.14  | 1.82|           | Home Creek, O.        | 2.90  | 2.17  | 1.37| 0.062     |
| Ofanto a Monteverde   | 4.06  | 2.25  | 1.73|           | Mill Creek, O.        | 4.67  | 2.66  | 1.58| 0.142     |
| Ofanto a Rocchetta    | 4.13  | 2.25  | 1.75|           | Green Lick, Pa.       | 4.41  | 3.32  | 1.24| 0.093     |
| Atella                | 3.78  | 2.18  | 1.71|           | Beech Creek, O.       | 3.80  | 2.61  | 1.39| 0.014     |
| Arcidiaconata         | 4.12  | 2.39  | 1.63| 0.029     | Piney Creek, Md.      | 4.12  | 2.64  | 1.46| 0.034     |
| Lapilloso             | 4.34  | 2.28  | 1.78| 0.034     | Casselman River, Md.  | 3.75  | 2.24  | 1.64| 0.040     |
| Venosa a p.te Ferroviario | 4.78  | 2.62  | 1.62|           | Emory River, Tenn.    | 3.82  | 1.92  | 0.05| 0.061     |
| Venosa a p.te S.Angelo | 5.09  | 2.87  | 1.54|           | Youghiogheny River, Md. | 4.57  | 2.24  | 1.88| 0.019     |
| Lione a p.te Brandi   | 3.10  | 1.18  | 1.91|           | Daddy’s Creek, Tenn.  | 4.13  | 2.18  | 1.82| 0.026     |
| Lione a p.te Canosa-Lavello | 3.19  | 1.81  | 1.96|           | Little Mahoning Creek, Pa. | 4.07  | 2.80  | 1.36| 0.042     |
| Carapelle             | 3.94  | 2.39  | 1.57|           | Allegheny River, Pa.  | 4.47  | 2.37  | 1.74| 0.052     |
| Cervaro               | 4.14  | 2.48  | 1.56|           |                      |       |       |    |           |
| Candelo a p.te 13 luci | 3.72  | 2.25  | 1.62|           |                      |       |       |    |           |
| Cellone a p.te S.Vincenzo | 3.83  | 2.73  | 1.34| 0.026     |                      |       |       |    |           |
| Cellone a p.te Foggia-Lucera | 4.10  | 2.74  | 1.40| 0.018     |                      |       |       |    |           |
| Cellone a p.te Foggia-S.Severo | 4.09  | 2.75  | 1.39|           |                      |       |       |    |           |
| Vulgano               | 3.79  | 2.26  | 1.63| 0.030     |                      |       |       |    |           |
| Salsola a Casanova    | 3.28  | 2.30  | 1.43| 0.036     |                      |       |       |    |           |
| Casanova a p.te Lucera-Motta | 3.44  | 2.55  | 1.32| 0.033     |                      |       |       |    |           |
| Salsola a p.te Foggia-S.Severo | 3.63  | 2.28  | 1.56|           |                      |       |       |    |           |
| Triolo                | 3.72  | 2.31  | 1.57|           |                      |       |       |    |           |
| Canale S.Maria        | 3.72  | 2.57  | 1.39| 0.015     |                      |       |       |    |           |
| Av.                   | 3.89  | 2.36  | 1.60| 0.028     | Av.                  | 4.07  | 2.56  | 1.55| 0.068     |
| S.D.                  | 0.45  | 0.27  | 0.17| 0.007     | S.D.                | 0.48  | 0.48  | 0.28| 0.060     |

Regarding Basins (A), the data of $R_B$, $R_L$, and $D$ are cited from the literature by Claps et al. [24], while $\bar{S}$ are calculated using the data by Claps et al. [25], respectively. Meanwhile, Basins (B) are according to Liu [26] and Morisawa [27]. Fractal dimension in Basins (A) are rounded off to the second decimal places. A datum with the underline exceeds the permissive range.

With respect to bifurcation ratios $R_B$, length ratios $R_L$ and fractal dimensions $D$, average values $\bar{AV}$ or Av. and standard deviations S.D. are compared between simulation results and real values in natural basins in Table 5. The values of $\bar{AV}$ instead of Av. are adopted as average values in simulation.
results of IPMs. Both average values and standard deviations are comparable with each other between numerical simulations, IPM (Max) and IPM (Min), and natural measurements, Basins (A) and Basins (B). The fact that two inequalities \( D_{\text{IPM (Max)}} > D_{\text{IPM (Min)}} \) and \( D_{\text{Basins (A)}} > D_{\text{Basins (B)}} \) simultaneously hold would suggest that IPM (Max) and IPM (Min) correspond to Basin (A) and Basin (B), respectively.

Our IPM is a two-dimensional model, so that simulation results include no information about the elevation of each site. If the three-dimensional model with the axis of the height is provided, the comparison with real data for the slope of streams such as \( S \) in Table 4 could be significant. In any case, the improvement of the model will belong to future issues.

Table 5: Comparisons of average values \( \bar{A}_{\text{Av.}} \) (Av.) and standard deviation S.D. between IPMs and real basins.

|       | IPM (Max) | IPM (Min) | Basins (A) | Basins (B) |
|-------|-----------|-----------|------------|------------|
| \( R_B \) \( \bar{A}_{\text{Av.}} \) (Av.) | 4.439 | 4.315 | 3.89 | 4.07 |
| S.D.  | 0.356 | 0.524 | 0.45 | 0.48 |
| \( R_L \) \( \bar{A}_{\text{Av.}} \) (Av.) | 2.296 | 2.947 | 2.36 | 2.56 |
| S.D.  | 0.262 | 0.524 | 0.27 | 0.48 |
| \( D \) \( \bar{A}_{\text{Av.}} \) (Av.) | 1.867 | 1.442 | 1.60 | 1.55 |
| S.D.  | 0.158 | 0.169 | 0.17 | 0.28 |

\( \bar{A}_{\text{Av.}} \) are used as average values in IPM (Max) and IPM (Min), while Av. are in Basins (A) and Basins (B).

Different from idealized mathematical simulation models such as the IPM in this article, real river systems are generally much more complex. For instance, one problem that geomorphologists frequently face lies in the identification of the first-order streams [22]. The locations of sources are easily moved by such kind of events as a sudden rainfall. The first-order stream in the past might be currently the second-order or the third-order stream. If these situations are possible to happen, the local change of stream orders near headwaters could affect the global distribution of stream order numbers in total watersheds. As a result, the application of the Horton-Strahler ordering system to real river systems could induce some kinds of ambiguity in identifying fractal dimensions of natural river systems. The fractal dimension is an intrinsic quantity accompanying with each watershed, so it should not be influenced by the choice of the ordering system. We think that this is one of the defects that is inherent in the identification of fractal dimensions by means of the definition (2), such that \( D = \ln R_B / \ln R_L \). How can we get rid of these demerits? It is clear that the dependence of fractal dimensions on ordering systems is not preferable, thus, should be eliminated.
4.2. Alternative definition of fractal dimension in IPM

The situation mentioned above is the same in the simulation world using mathematical models. Throughout the numerical experiments by means of our IPMs, one of the most disputable challenges is extremely large fluctuations and uncertainties in calculations of the fractal dimension $D$. The values of $D$ found by the definition (2) range from 0.989 to 2.276, some of which are obviously without the permissible range. The discrepancy must be solved in this article.

Here, we would like to propose an alternative calculation method. That is, the extended cluster dimension $D_L$ explained in the section 2.4 can be adopted instead of the fractal dimensions $D$ defined by the formula (2), which uses the bifurcation ratio $R_B$ and the length ratio $R_L$.

Figure 9 shows two representative samples for the calculation of the extended cluster dimensions $D_L$ in IPMs of $N=41$, which correspond to Fig. 5 (a) and (d), respectively. It is clearly recognized that $N_L$ values lie almost in a straight line on the double-logarithmic papers in both the Max (a) and the Min (b) selection cases. The linearity of the $N_L$ data is equally observed for other samples, which makes it possible to identify exact values for all 30 cases.

![](image)

**Figure 9:** Calculations of extended cluster dimensions in IPM (Rn#0). $k_p=0.1$, $k_r=0.1$. (a) Max selection rule, $D_L=1.678$. (b) Min selection rule, $D_L=1.496$.

Table 6 displays all 30 values of the extended cluster dimensions $D_L$ of IPMs, covering two selection rules, Max and Min, and three random number tables, Rn#0, Rn#1 and Rn#2. It turns out that values gather within a very narrow range and fluctuate little, compared with those obtained by the existing method such as the definition (2). Needless to say, two Horton’s numbers $R_B$ and $R_L$ have nothing to do through the whole calculation process of $D_L$. In other words, the extended fractal dimensions $D_L$ do not depend on the ordering systems. They are determined regardless of the Horton’s
ratios $R_B$ and $R_L$.

### Table 6: Extended cluster dimensions.

| Randomization Parameter ($k_r$) | Max Selection Rule | Min Selection Rule |
|---------------------------------|--------------------|--------------------|
|                                 | Rn#0   | Rn#1   | Rn#2 | Av. | Rn#0 | Rn#1 | Rn#2 | Av. |
| 0.025                           | 1.726  | 1.693  | 1.751| 1.723| 1.393| 1.404| 1.469| 1.422|
| 0.05                            | 1.677  | 1.632  | 1.661| 1.657| 1.409| 1.454| 1.530| 1.464|
| 0.075                           | 1.678  | 1.640  | 1.717| 1.678| 1.443| 1.515| 1.591| 1.516|
| 0.1                             | 1.678  | 1.527  | 1.705| 1.637| 1.496| 1.523| 1.527| 1.515|
| 0.125                           | 1.587  | 1.606  | 1.694| 1.629| 1.513| 1.549| 1.530| 1.531|

**Av.** 1.665 1.490

**S.D.** 0.034 0.041

**S.D./Av.** 0.020 0.027

**C.C.** -0.870 0.933

The values of $D_L$ are plotted summarily in Fig. 10, where ensemble averages are plotted by crosses ($\times$) together with vertical lines that show the maximum and the minimum among three kinds of randomization by Rn#0, Rn#1 and Rn#2. It should be noticed that the vertical scales in Fig. 10 are much finer than those in Fig. 7 (c) and (f).

![Figure 10: Dependences of cluster dimensions $D_L$ on randomization parameters $k_r$. (a) Max selection rule, (b) Min selection rule.](image)

### 4.3. Advantages of extended cluster dimension

In the end, we examine the applicability of extended cluster dimensions $D_L$ proposed in the present study. Figure 11 shows comparisons of $D_L$ with (a) usual fractal dimensions $D$ calculated by the
equation (2) and with (b) the mainstream, i.e., the longest stream lengths $L_{\text{max}}$. Correlation coefficients C.C. are also specified. The positive correlation is recognized in (a) between $D_L$ and $D$ although not so clear.

However, more interesting is a negative correlation between $D_L$ and $L_{\text{max}}$, which seems explicitly strict, for most of the data are concentrated near a straight line with a negative slope. The negative C.C. in Fig. 11 (b) means that $D_L$ is certainly decreased with the increase in $L_{\text{max}}$. In our simulation settings, the river basin area, i.e., the total stream number is constant and not varied whatever pattern is drawn. If so, the number of tributaries should be reduced as the value of $L_{\text{max}}$ becomes large. As a result, river channel patterns come close to a simple curved line with a few numbers of branches. Further, the extended fractal dimension $D_L$ also approach 1, which is a dimension of a line. It is necessary to remember the extreme case of Fig. 3 (a) and (b), where the extended fractal dimension is exactly 1.

Generally speaking, the fractal dimension of river channel systems is thought to represent the complexity of river channel networks. Then, it is a sufficiently reasonable conclusion that the extended cluster dimension introduced in this study can be substituted for the fractal dimension in river channel networks created by IPMs.

Another interesting feature in Fig. 11 is uneven distributions of data between the Max and Min selection cases. It is clearly recognized that the data of the Max selection rule are mainly situated on the right side, while those of the Min selectin rule are on the left side. Considering that the extended cluster dimensions $D_L$ of the left side are close to 1 and that those of the right side are close to 2, this result is consistent with observations described in the section 3.2. That is, a meandering mainstream characterizes river basin patterns by the Min selection rule, whose length is distinguishably long compared with other tributaries. On the other hand, an obscure mainstream with a lot of tributaries is characteristic of the Max selection patterns, some of which ramify almost near the outlet.

![Figure 11](image)

Figure 11: Correlations of extended cluster dimension $D_L$ with (a) fractal dimension $D$ and (b) longest stream length $L_{\text{max}}$. “×” and “●” express the data by the Max and Min selection rules, respectively.
5. Conclusions

(1) The Inhomogeneous Permeability Model (IPM) can serve as a useful simulation tool to draw a variety of natural and realistic river channel patterns. When the randomization parameter \(k_r\) is chosen within the range of \(0.025 \leq k_r \leq 0.125\), moderately disturbed and meandering tree-shaped network patterns can be generated under the conditions of both the Max and Min selection rules.

(2) Regarding a total of 30 cases examined in this article, most of the values such as bifurcation ratios \(R_B\), length ratios \(R_L\) and fractal dimensions \(D\) are included within permissible ranges predicted by the Horton’s law, while considerably fluctuated. However, the situation is almost the same also in natural river basins, thus, it is possible to insist that our numerical model IPM shows good accordance with the Horton’s law.

(3) With respect to the mainstream length, i.e., the longest stream length \(L_{\text{max}}\), the values of the Max selection rule are almost constant, while slightly increased with the increase in \(k_r\). Then, the agreements with the Hack’s law could be satisfactory, where the predicted value is \(L_{\text{max}} = 95.083\). Meanwhile, the values of \(L_{\text{max}}\) in Min selection cases are considerably larger than 95.083 especially in small \(k_r\) region, however, decreased with the increase in \(k_r\). As a result, the mainstream lengths in both cases approach with each other particularly in the larger \(k_r\) range. Inconsistency with the Hack’s law, that is, too large \(L_{\text{max}}\) values of Min selection cases in the small \(k_r\) region probably mean that the Min selection cannot be taken place in these conditions.

(4) Whichever selection rule is chosen, the values of stream frequencies \(F_S\) are somewhat smaller than those expected by the Melton’s law within the whole randomization range. We speculate that the reason is attributed to the simulation settings in our IPMs, where all the river channels are folded and stuffed in a square area at the final state. This kind of environment is difficult to be perfectly realized in the natural world.

(5) The fractal dimension \(D\) calculated using the bifurcation ratio \(R_B\) and the length ratio \(R_L\) shows large uncertainties. Then, an alternative method referred to as the extended fractal dimension \(D_L\) is devised in this study, modifying the well-known cluster dimension. We can obtain more precise values of fractal dimensions with little fluctuation by means of this newly proposed method, which is irrespective of ordering systems.

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