A new Hybrid Lattice Attack on Galbraith’s Binary LWE Cryptosystem

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Abstract. LWE-based cryptosystems are an attractive alternative to traditional ones in the post-quantum era. To minimize the storage cost of part of its public key - a 256 × 640 integer matrix, T - a binary version of T has been proposed. One component of its ciphertext, c₁, is computed as c₁ = Tu where u is an ephemeral secret. Knowing u, the plaintext can be deduced. Given c₁ and T, Galbraith’s challenge is to compute u with existing computing resources in 1 year. Our hybrid approach guesses and removes some bits of the solution vector and maps the problem of solving the resulting sub-instance to the Closest Vector Problem in Lattice Theory. The lattice-based approach reduces the number of bits to be guessed while the initial guess based on LP relaxation reduces the number of subsequent guesses to polynomial rather than exponential in the number of guessed bits. Further enhancements partition the set of guessed bits and use a 2-step application of LP. Given the constraint of processor cores and time, a one-time training algorithm learns the optimal combination of partitions yielding a success rate of 9% - 23% with 1000 - 100,000 cores in 1 year. This compares favourably with earlier work that yielded 2% success with 3000 cores.

Keywords: Learning With Errors, Closest Vector Problem, Galbraith’s binary LWE, Linear Programming, Integer Linear Programming

1 Introduction

Introduced by Regev [11] in 2005, Learning with Errors (LWE) is a problem in machine learning and is as hard to solve as certain worst-case lattice problems. Unlike most widely used cryptographic algorithms, it is known to be invulnerable to quantum computers. It is the basis of many cryptographic constructions including IND-CPA and IND-CCA secure encryption [11] [19], homomorphic encryption [21] [22], identity-based encryption [15] [16], oblivious transfer protocols [17], lossy-trapdoor functions [18] and many more.

The LWE cryptosystem performs bit by bit encryption. The private key, s, is a vector of length n where each element of s is randomly chosen over \( \mathbb{Z}_q \), q prime. The corresponding public key has two components. The first is a random \( m \times n \) matrix, T, with elements over \( \mathbb{Z}_q \) and with rows denoted \( a_i \). The second
component is a vector, \( b \), of length \( m \) where the \( i^{th} \) element of \( b \) is \( s^T a_i + e_i \) (mod \( q \)). The \( e_i \)'s are drawn from a discretized normal distribution with mean 0 and standard deviation \( \sigma \).

To encrypt a bit, \( x \), a random binary vector (nonce), \( u \), of length \( m \) is chosen. This is a per-message ephemeral secret. The ciphertext is \((c_1, c_2)\) where \( c_1 = T u \) (mod \( q \)) and \( c_2 = bu + x \lfloor q/2 \rfloor \) (mod \( q \)). A received message is decrypted to 0 or 1 depending on whether \( c_2 - c_1 s \) is closer to 0 or \( \lfloor q/2 \rfloor \).

To thwart various lattice-based attacks, Lindner et al. [20] suggested the values of 256, 640 and 4093 respectively for \( n \), \( m \) and \( q \) leading to a public key of size \( 640(256 + 1) \log_2(4093) = 246.7 \) Kbytes. However, this is far higher than the size of the RSA or ECC public keys which are less than 1 Kbyte.

To reduce the key size of original LWE problem, [1] proposed that the matrix \( T \) and secret vector \( s \) be binary. In addition, possible attacks on the nonce to recover the plaintext were discussed. Galbraith posed two challenges, the first challenge is to compute \( u \) given a random \( 256 \times 400 \) binary matrix \( T \) and \( c_1 = Tu \) in one day using an ordinary PC. The second challenge is to compute \( u \) with matrix \( T \) dimension \( 256 \times 640 \) in one year using “current computing facilities”.

The aim of this work is to address Galbraith’s second challenge with a much higher success rate than what was previously achieved [2],[5]. Ours is a hybrid approach using, both, Linear Programming (LP) and a lattice-based one.

In our approach, LP is initially applied on the given instance. Based on the LP output, an initial guess of \( p \) bits of the secret is made. A sub-instance is created after removing the \( p \) guessed bits. By using a lattice-based approach (rather than Integer Linear Programming (ILP) [5]) to solve the reduced instance, we require 20% fewer bits of the solution vector to be guessed and removed. This results in a 300% saving in computation time. The problem of obtaining the remaining bits of the secret is mapped to the Closest Vector Problem (CVP) in lattice theory. If the initial guess is unsuccessful, a fresh guess is made and the process is repeated until we run out of compute power or time. The substantial decrease in execution time is due to, both, a fewer number of bits to be guessed and, further, a considerable reduction in the expected number of guesses from exponential to polynomial in \( p \) due to the application of LP.

The second enhancement involves a two-step application of LP. After its first application, \( p \) bits are guessed and removed. LP is again applied on the reduced sub-instance after which \( 230 - p \) bits are guessed and removed. The lattice-based approach is then employed on the doubly reduced sub-instance. The key advantage of the two-step approach is that the expected number of errors in the \( 230 - p \) guessed bits is reduced compared to the case with a single application of LP. This translates to a substantial reduction in the number of guesses (and hence execution time).

For a given \( p \) and the initial guesses of the \( p \) and \( 230 - p \) bits (based on the LP output), we hypothesize that there are respectively \( e_1 \) and \( e_2 \) errors in those bits. Different hypotheses are defined by varying the values of \( p \), \( e_1 \), \( e_2 \), and \( c_2 \). Our strategy is to learn which hypotheses are satisfied by the largest number of the instances while simultaneously factoring the computation time.
to process the guesses (the number of guess is a function of $p, e_1, e_2$). In the training phase, we use a variant of the Budgeted Maximum Coverage Problem to iteratively select the best set of hypotheses subject to optimizing an objective function that incorporates the number of new instances added and the execution time. By combining multiple hypotheses, we obtain a substantial improvement in success rate or equivalently a considerable reduction in execution time for a given success rate.

The paper is organized as follows. Section 2 presents background material. Our contributions are presented as three strategies - the first two are presented in Section 3 and the third is in Section 4. Section 5 is a brief summary of related work. Section 6 concludes the paper.

| Symbol | Meaning |
|--------|---------|
| $u_s$  | LP output vector after sorting and round-off |
| $u'_s$ | Sorted, rounded-off $640 - p$ bit vector after applying LP twice |
| $u_{s1}, u_{s2}, u_{s3}$ | First 230, next 256 and last 154 bits of solution vector $u_s$ |
| $p$    | Number of bits removed after first application of LP |
| $e_1$  | Number of errors in $p$ bits. |
| $e_2$  | Number of errors in next $230 - p$ bits. |
| $t_{LP}$ | Computing time of Linear Programming (LP) |
| $t_{LR}$ | Computing time of Lattice Reduction (LR) |
| $t_B$  | Computing time of Babai’s Nearest Plane algorithm |
| $s(p, e_1, e_2)$ | Instances with $e_1$ errors in first $p$ bits and $e_2$ errors in $230 - p$ bits. |
| $t(p, e_1, e_2)$ | Total time to compute Algorithm 1 with input parameters $p, e_1, e_2$ |

2 Background

A Lattice $L$ is a discrete additive subgroup of $\mathbb{R}^n$. Equivalently, $L$ is comprised of integer linear combinations of a set of linearly independent vectors. A lattice is represented by a basis, a set of $m$ linearly independent integer vectors $(b_1, b_2, \cdots, b_m)$ each of size $n$ which generates the lattice

$$L(b_1, b_2, \cdots, b_m) = \left\{ \sum_{i=0}^{m} x_i b_i : x_i \in \mathbb{Z} \right\}$$

A lattice can have multiple bases and a basis is usually represented by $m \times n$ matrix where $m$ basis vectors are rows of the matrix.

One of the hard problems in lattice based cryptography is the Closest Vector Problem (CVP). The problem is to find a lattice vector $u$ given a basis $B$ of some lattice $L = L(B)$ and a non-lattice vector $v \in \mathbb{R}^n$ with minimum $\|u - v\|$.
One of the best known solutions to solving CVP is the Nearest Plane algorithm developed by Babai\textsuperscript{[7]}.

Before applying Babai’s Algorithm, lattice reduction is performed to obtain short and near-orthogonal basis vectors. Some of the different lattice reduction algorithms are LLL, BKZ and BKZ2.0. BKZ\textsuperscript{[12]} algorithms behave differently based on block size \( k \). In practice, run time of BKZ increases rapidly with block size and becomes practically infeasible for \( k > 30 \) or so. Chen and Nguyen\textsuperscript{[8]} presented an updated version of BKZ i.e. BKZ 2.0. It uses extreme pruning techniques of Gama-Nguyen-Regev\textsuperscript{[13]} that significantly decreases the running time of enumeration subroutine without degrading its output quality allowing much higher block size (\( \beta \geq 40 \)) in high dimension.

Implementations of LLL, BKZ and BKZ 2.0 are available in many software packages, notably in NTL\textsuperscript{[6]}, FLINT\textsuperscript{[9]} and fplll\textsuperscript{[10]}. In our implementation, we have used BKZ 2.0 from fplll library with block size \( k = 22 \).

For pre-processing and guessing some bits of the ephemeral secret, \( u \) we use Linear Programming (LP). LP is used to determine the best possible solution from a given list of requirements represented in the form of linear relationships.

The standard algorithm for solving LP is the Simplex Algorithm though it is not guaranteed to run in polynomial time. Later, it was shown that LP could be done in polynomial time by using the Ellipsoid Algorithm (but it tends to be fairly slow in practice). Karmarkar proposed a much faster polynomial-time algorithm - the first of a class of so-called interior-point methods.

In our implementation, we have used Matlab’s inbuilt linear programming function, linprog. There are three variants of linear programming algorithms available in Matlab viz dual-simplex, interior-point (default) and interior-point-legacy. Often, the dual-simplex and interior-point algorithms are fast and use least memory. The interior-point-legacy method is similar to the interior-point algorithm but uses more memory and is slower and less robust. We have used the default algorithm provided by Matlab to obtain the initial guess of \( u \).

### 3 Our Approach

We introduce Strategy 1 which involves a lattice-based approach in conjunction with LP. Strategy 2 which involves a 2-step application of LP is introduced next. We begin by reviewing Strategy 0 which uses ILP.

#### 3.1 Strategies 0, 1 and 2

As stated earlier, we attempt to obtain \( u \) given \( T \) and \( c_1 \) in the equation below

\[
Tu = c_1
\]  

Our approach is summarized in Flowchart 3.1. The pre-processing step creates an approximate solution to Equation 1. We then guess selected bits in \( u \) (Step 1) and create a reduced sub-instance (Step 2) by removing the guessed
bits in \( u \), deleting the corresponding columns of matrix \( T \) and re-computing the value of \( c_1 \). Formally, if the values of the guessed bits are \( u^{(i_1)}, u^{(i_2)}, \ldots, u^{(i_r)} \) in positions \( i_1, i_2, \ldots, i_r \), then those bits are removed from \( u \), the \( i_1^{th}, i_2^{th}, \ldots, i_r^{th} \) columns in \( T \) are removed and new value of \( c_1 \) is computed as

\[
c_1' = c_1 - \sum_{i=i_1, i_2, \ldots, i_r} u^{(i)}T^{(i)}
\]

We then solve the resulting sub-instance and verify (Step 3) whether the computed solution is binary and satisfies Equation 1. If not, we proceed with the remaining guesses until the secret is obtained or we run out of guesses or resources. While the approach is straightforward, several issues need to be addressed in its implementation.

1. Which bits of \( u \) may be guessed and what are the values of the guessed bits?
2. What method/technique should be used to solve the sub-instance (Step 2)?
3. If all the guessed bits are correctly guessed, then will the desired solution be obtained?

We next outline several implementation strategies - Strategy 0 was adopted in [5] while strategies 1-3 are newly introduced here.

**Strategy 0:** The LP formulation below from [2] is employed to solve Equation 1 (Step 0).

Optimization function: \( F(u) = 0 \)

Constraints: \( Tu = c_1, 0 \leq u_i \leq 1, 1 \leq i \leq m \)

For the size of matrix \( T \) under consideration, the elements of the solution vector are fractions between 0 and 1. These are sorted in order of increasing proximity to 0.5 and then rounded to 0 or 1. Let \( u_s \) denote the resulting vector and let \( T_s \) be the matrix obtained by re-arranging columns of \( T \) in the same order in which the bits of \( u \) are re-arranged to obtain \( u_s \) so that

\[
T_s u_s = c_1
\]

Extensive experiments conducted in [5] indicate that the bits in \( u_s \) differ from the corresponding bits in \( u \) in roughly 20% of the positions. Moreover, the probability that a bit in \( u_s \) is in error increases with its position (from left to right). To corroborate those findings, we generated 10,000 random instances and performed LP on each. We found that the average error probability in the first 100 bits is 0.02 while it is 0.44 for the last 100 bits.

[5] report that if all errors in the first 280 bits of \( u_s \) were corrected, then the sub-instance created after removing these bits could be solved using Integer Linear Programming (ILP) with almost 100% success rate. The first 280 bits in \( u_s \) is our initial guess. Subsequent guesses are obtained by flipping different combinations of at most \( e \) of those 280 bits. Thus, there are

\[
\sum_{i=0}^{e} \binom{280}{i}
\]

guesses to
be made. For each guess, we create a reduced sub-instance and solve it using ILP.

In our experiments, we found that less than 12% of the instances have 11 or fewer errors in the first 280 bits of $u_s$. The number of guesses for the partial secret is hence $\sum_{i=0}^{11} \binom{280}{i} \sim 1.8 \times 10^{19}$. Processing a guess involves an ILP computation. The computation time for ILP is input-dependent. [5] placed a limit of 30 seconds on an ILP instance. Hence, the total time to process all guesses is about a billion years using 3000 cores for a success rate of 12%. We next outline a strategy wherein it suffices to guess only 230 bits rather than 280 bits.

**Strategy 1:** As in Strategy 0, we compute $u_s$ and create a smaller sub-instance by removing 230 bits in $u_s$. Unlike Strategy 0, we use a lattice-based approach to solve the resulting sub-instance. Based on extensive experiments with 10,000 instances, we found that it suffices to remove only the first 230 bits from $u_s$ to guarantee a solution for the sub-instance with probability $\sim 1$.  

![Flowchart 3.1: Flowchart expounding the approach in Strategies 0 and 1](image-url)
For a given guess of the first 230 bits of $u$, the remaining bits of the secret are computed as follows. Let $(T_{s1}, T_{s2}, T_{s3})$ be a partitioning of $T_s$. Here $T_{s1}, T_{s2}$ and $T_{s3}$ are $256 \times 230$, $256 \times 256$ and $256 \times 154$ sub-matrices. Let $u_s = (u_{s1}, u_{s2}, u_{s3})$ be the corresponding partitioning of $u_s$ into sub-vectors of length 230, 256 and 154 respectively. Using Equation 3, we have

$$T_{s1}u_{s1} + T_{s2}u_{s2} + T_{s3}u_{s3} = c_1$$

(4)

Re-arranging and pre-multiplying by $T_{s2}^{-1}$,

$$T_{s2}^{-1}T_{s3}u_{s3} \equiv -u_{s2} + T_{s2}^{-1}(c_1 - T_{s1}u_{s1}) \mod \alpha$$

(5)

$T_{s2}^{-1}$ will, in general, have fractional values. So, we compute $T_{s2}^{-1}$ and all terms of Equation 5 modulo a large prime, $\alpha$. $u_{s2}$ is a binary vector and is of negligible norm compared to $T_{s2}^{-1}(c_1 - T_{s1}u_{s1})$. So

$$(T_{s2}^{-1}T_{s3})u_{s3} \approx T_{s2}^{-1}(c_1 - T_{s1}u_{s1})$$

(6)

The LHS of Equation 6 is a vector in the lattice with basis $T_{s2}^{-1}T_{s3}$ while the RHS is a non-lattice vector. Hence the problem of computing $u_{s3}$ maps to the classical Closest Vector Problem (CVP) in the theory of lattices.

Assuming $e$ of the first 230 bits of $u_s$ are in error, the time to discover the secret is

$$t_{LP} + t_{LR} + \sum_{i=0}^{e} \left(\frac{p}{i}\right)t_B$$

(7)

1 We experimented with different sizes and values of $\alpha$. The selected value of bit size is a trade-off between success probability and lattice reduction time. 26-bit primes yielded a success rate of 99%. 

Strategy 2: As in Strategy 1, we guess 230 bits of the solution vector but we do so in two steps. Algorithm 1 summarizes the procedure.

**Algorithm 1: Hybrid Lattice Approach**

**Input:** $T_{256 \times 640}$, $c_{256 \times 1}$, $p$, $e_1$ and $e_2$

**Output:** $u$ (with significant probability $Tu = c_1$)

**function:** ourLP($X$, $z$)

1. Create an LP instance for $Xy = z$
2. Use the LP solver to obtain $y$
3. Sort $y$, round it and call it $y_s$
4. // Sorting is in order of increasing proximity to 0.5
5. Arrange columns of $X$ in the same order in which elements of $y$ are arranged to obtain $y_s$ and call it $X_s$
6. return ($X_s, y_s$)

7. $(T_s, u_s) =$ ourLP($T, c_1$)
   // Let $T_s$, and $T_s$ respectively denote columns $231 - 486$ and $487 - 640$ of $T_s$.
8. Compute $T_s^{-1} \mod \alpha$ and perform lattice reduction on $T_s^{-1}T_s$
9. $v_1 =$ first $p$ bits of $u_s$
10. for each vector $v'_1$ such that $\|v'_1 - v_1\|_1 \leq e_1$ do
11.   $c' = c_1 - \sum_{i=1}^{p} v_1^{(i)}T_s^{(i)}$
12.   $T'_{256 \times (640 - p)}$ consists of last $(640 - p)$ columns of $T_s$
13.   $(T'_s, u'_s) =$ ourLP($T', c'$)
14.   $v_2 =$ First $(230 - p)$ bits of $u'_s$
15.   for each vector $v'_2$ such that $\|v'_2 - v_2\|_1 \leq e_2$ do
16.     Substitute $v'_1$ and $v'_2$ for $u_{s_1}$ in Equation 6
17.     Solve CVP using Babai's Algorithm
18.     if correct solution is obtained then
19.       Output $v'_1, v'_2$
20.     Output -1

The function, ourLP is invoked to create a sorted, rounded-off solution vector, $u_s$ as in Strategy 1. The first $p$ bits of $u_s$ is the initial guess of these bits. In each iteration of the outer loop, a fresh guess is made by flipping some combination of $e_1$ or fewer of those $p$ bits. The $p$ bits are removed and a sub-instance of size $640 - p$ is created. ourLP is invoked to obtain $u_s$, a partial solution vector of size $640 - p$. Each iteration of the inner loop involves guessing the first $230 - p$ bits of $u_s$ by flipping a different combination of $e_2$ bits. Thus, a total of $p + (230 - p) = 230$ bits are guessed to obtain $u_{s_1}$ in Equation 6.

A CVP instance is created and solved to obtain $u_{s_3}$. This procedure continues until it runs out of guesses or resources. Based on the above description of Algorithm 1, it is clear that LP and Lattice Reduction (LR) are applied once
per iteration of the outer loop while Babai’s algorithm is executed once per iteration of the inner loop. The total time to run Algorithm 1 is thus

\[ t(p, e_1, e_2) = t_{LP} + \sum_{i=0}^{e_1} \binom{p}{i} \left[ t_{LP} + t_{LR} + \sum_{j=0}^{e_2} \binom{230-p}{j} \times t_B \right] \] (8)

The notations and values of the execution times for the various operations are listed in Table 3.1. The times were measured on Intel i5 Gen 4, with 3.5 GHz clock and 8 GB DRAM running Ubuntu 16.04 64-bit LTS. The LP solver of Matlab 2015b was used. BKZ with block size=22 implemented in Sage and Babai’s Nearest Plane algorithm were used.

| Algorithm          | Implemented In | Time (seconds) |
|--------------------|----------------|----------------|
| Linear Programming | Matlab         | \( t_{LP} = 0.5 \) |
| Int. Linear Programming | Matlab | 30 (bound) |
| Babai’s NP Algo.   | Sage           | \( t_B = 8 \) |
| BKZ with \( \beta = 22 \) | Sage | \( t_{LR} = 10500 \) |

### 3.2 Results

To estimate success probability, we created a training set of \( n = 10,000 \) randomly generated instances. LP was applied on each instance, the LP output was sorted and rounded. The first \( p \) bits of the resulting solution vector (\( u_s \)) were compared with the corresponding bits of the actual secret to determine \( e_1 \), the number of bits in error. The \( p \) bits of \( u_s \) were corrected and removed to create a sub-instance over which LP was again applied. The reduced solution vector was sorted and rounded to obtain \( u_s' \). The first \( 230 - p \) bits of \( u_s' \) were compared with the corresponding bits of the true secret to determine the number of bits in error, \( e_2 \). The instance was then added to the “instance set”, \( s(p, e_1, e_2) \) - this is the set of instances with \( e_1 \) errors in the first \( p \) bits of \( u_s \) and \( e_2 \) errors in the first \( 230 - p \) bits of \( u_s' \). This was carried out for all \( n = 10,000 \) instances. \( \frac{|s(p, e_1, e_2)|}{n} \) is a reasonable estimate of the success probability of running Algorithm 1 with input parameters \( p, e_1 \) and \( e_2 \).

The computation times and success probabilities were computed for varying \( e_1 \) and \( e_2 \) and for \( p \) ranging from 0 to 230 in steps of 5. The total computation time assumes the values in Table 3.1 and the availability of 3000 cores. The value of \( p \) which maximizes success probability is shown in Table 3.2. For a fixed value of \( e_1 + e_2 \), this value of \( p \) increases with \( e_1 \).
(a) Components of Execution Time for $(e_1, e_2) = (3, 2)$

(b) Comparison of $e_1 + e_2 = 5$ ($e_1 = 3, e_2 = 2$) and $e_1 + e_2 = 6$ ($e_1 = 4, e_2 = 2$)

Fig.3.1: Success Probability and Computation Time as a function of $p$
Table 3.2: Value of $p$ which maximizes success probability for given $(e_1, e_2)$ pair with 3000 cores

| $e_1, e_2$ | $p$ | Success Probability | Time (days) |
|------------|-----|---------------------|-------------|
| 0, 5       | 0   | 5%                  | 162         |
| 1, 4       | 75  | 4%                  | 56          |
| 2, 3       | 150 | 4.7%                | 30          |
| 3, 2       | 175 | 6.1%                | 80          |
| 4, 1       | 195 | 6.4%                | 2,551       |
| 0, 6       | 0   | 9.4%                | 6,105       |
| 1, 5       | 75  | 7.7%                | 1,694       |
| 2, 4       | 130 | 8.2%                | 1,075       |
| 3, 3       | 160 | 10.6%               | 1,234       |
| 4, 2       | 175 | 11.8%               | 3,448       |
| 5, 1       | 195 | 12.4%               | 97,999      |

The maximum value of success probability increases with $e_1$ (beyond $e_1=0$). This is at the cost of sharply escalating computation times (beyond $e_1=1$). The superiority of Strategy 2 over Strategy 1 is also on display - $e_1=0$ corresponds to Strategy 1. Note that Strategy 2 provides a higher success rate of 6.1% (for $p = 175, e_1 = 3, e_2 = 2$) versus 5% (Strategy 1). Moreover the computation time of the former is only 80 days compared to 162 days for the latter.

![Fig. 3.2: Distribution of errors before and after second application of LP](image-url)
The variation of the execution time of Algorithm 1 denoted $t(p, e_1, e_2)$ with $p$, $e_1$ and $e_2$ can be better understood by examining the contribution of its various components. The computation time is dominated by the execution of the Babai’s Algorithm and Lattice Reduction (LR) (to a first-order approximation the time for Linear Programming may be ignored).

LR is executed $\sum_{i=0}^{p} \binom{p}{i}$ times while Babai’s Algorithm is executed $\sum_{i=0}^{e_1} \binom{p}{i} \times \sum_{j=0}^{e_2} \binom{230-p}{j}$ times. As shown in Figure 3.1a, the contribution of the latter to $t(p, 3, 2)$ peaks at $p \approx 140$ while that of the former increases with $p$. Overall, $t(p, 3, 2)$ increases monotonically. For $e_1 > 3$, LR dominates while for $e_1 < 3$, Babai’s algorithm dominates the computation time.

With Strategy 1, only a single execution of the compute-intensive LR operation is performed. However, the number of executions of Babai’s algorithm, $\sum_{i=0}^{e} \binom{i}{p}$, is significantly higher than that with Strategy 2. For example, to achieve success probabilities of 5% and 6% with Strategies 1 and 2, the number of executions of Babai’s algorithm are respectively $5.2 \times 10^9$ and $1.4 \times 10^9$ resulting in an execution time of 162 and 42.5 days with 3000 cores respectively. Thus, even though the time spent executing LR with Strategy 2 is 37.2 days, the overall time of Strategy 2 is less than 50% that of Strategy 1.

The difference in the number of executions of Babai’s algorithm in Strategies 1 and 2 is partially explained by examining the distribution of errors in the first $230 - p$ bits of $u_s$ before and after the second application of LP. The two distributions (Figure 3.2) have a similar shape but the latter is shifted left. Hence, the second application of LP reduces the errors which in turn necessitates fewer guesses and iterations of the inner loop of Algorithm 1.

Solving instances with 6 errors in the first 230 bits ($e_1 + e_2 = 6$) greatly increases the success probability but at the expense of vastly higher execution time (Figure 3.1b). The latter is because the number of guesses (and hence execution time) is exponential in the number of errors. Once again, Strategy 2 yields a much higher success rate (11.8% versus 9.4%) with only 60% of the execution time required by Strategy 1 (Table 3.2). The maximum success rate is 12.4% but at the cost of 3000 cores running continuously for about 300 years! In the next section, we unveil a strategy which achieves much higher success rate but executes in only 1 year.

### 4 Strategy 3

Strategy 2 attempted to run Algorithm 1 with the best possible parameter values $(p, e_1$ and $e_2)$ determined from experiments on a training set of 10,000 instances. The highest success probability obtained was 6.1% with 3000 cores in a year.

Our next strategy (Strategy 3) is to greatly improve on this success rate by running Algorithm 1 repeatedly with different parameter values subject to resource constraints. To illustrate this idea, consider the three parameter sets in Table 4.1. If Algorithm 1 is run in isolation with each of the parameter sets shown, the success rates are 3.9%, 4.2% and 5.9% with 3000 cores in 52, 50 and 80 days respectively. However if Algorithm 1 is run twice with the first two
parameter sets, a total of $|s(80, 1, 4) \cup s(120, 2, 3)| = 616$ instances are likely to succeed.

Table 4.1: Performance of Algorithm 1 with single and multiple parameter sets

| $(p, e_1, e_2)$  | $N(p, e_1, e_2)$ | $T(p, e_1, e_2)$ |
|-----------------|-----------------|-----------------|
| (80, 1, 4)      | 391             | 52              |
| (120, 2, 3)     | 422             | 50              |
| (170, 3, 2)     | 593             | 80              |
| Union           | 900             | 182             |

Fig. 4.1: Union of different instance sets appearing in Table 4.1

If Algorithm 1 is run a third time with the parameter set (170, 3, 2), the overall success rate increases to 9%. This is achieved with 3000 cores running continuously for about 6 months. The increases in success rates are best visualized with the Venn diagram in Figure 4.1.

To maximize success probability, it is necessary to identify the parameter sets with which Algorithm 1 should be run so that the union of the corresponding instance sets is maximized while constraining the total execution time to 1 year. However, this is not straightforward given that total number of instance sets is 276 (since $p$ is varied from 5 to 230 in steps of 5 and $e_1 + e_2 \leq 5$). Our problem maps to the “Budgeted Maximum Coverage Problem” known to be NP-hard.
[14] proposes a greedy heuristic for the above problem and shows that their solution is within \((1 - \frac{1}{\epsilon})\) of the optimal solution.

**Algorithm 2**: Picking optimal instance sets

Input: \(S = \{s_1, s_2 \cdots\}, T = \{t_1, t_2 \cdots\}, \tau, c\)

Output: \(S' = \{s_1', s_2' \cdots\}\)

1. \(U \leftarrow \phi, t \leftarrow 0, S' \leftarrow \phi\)

2. while true do

   3. \(I \leftarrow \phi\)

   4. for each \(s_i\) in \(S\) do

      5. if \(|s_i - U| \neq 0\) and \(\frac{t + t_i}{c} \leq \tau\) then

         6. \(I.append(s_i)\)

     if \(|I| \neq 0\) then

     7. \(k \leftarrow \arg \max_{s_i \in I} \frac{|s_i - U|}{t_i}\)

     8. \(U \leftarrow U \cup s_k\)

     9. \(t \leftarrow t + t_i\)

    10. \(S'.append(s_k)\)

    11. \(S.delete(s_k)\)

    else

   12. break

14. Output \(S'\)

Algorithm 2, based on [14], takes as input the set of all instances, \(S\), the set of corresponding execution times, \(T\), number of cores, \(c\) and the bound on total execution time, \(\tau\). For brevity, an instance set is denoted \(s_i\) and the corresponding execution time is denoted \(t_i\). It is assumed that the instance sets have already been computed as explained in the previous section. During each iteration, Algorithm 2 selects a new instance set. The instance set, \(s_{ij}\), selected in iteration \(j\) is that which maximizes \(\frac{|s_i - U_j|}{t_i}\) where \(U_j = \bigcup_{k=1}^{j} s_k\) and \(\sum_{k=1}^{j} t_k < \tau\).

Algorithm 2 terminates when no instance set can contribute a fresh instance to the set of instances so far accumulated in \(U\) or if adding any instance set causes the total computation time to exceed \(\tau\). An estimate of the success probability achievable with \(c\) cores in time, \(\tau\) is \(\frac{\left| \sum_{i=1}^{m} s_{ij} \right|}{m}\) where \(m\) is the total number of iterations executed by Algorithm 2. Given an arbitrary instance whose secret needs to be discovered, the output of Algorithm 2 is used as follows. Run Algorithm 1 repeatedly with input parameters corresponding to the \((p, e_1, e_2)\) parameters of instance sets \(s_{i_1}, s_{i_2}, \cdots s_{i_m}\).

In the training phase, Algorithm 2 was run with instance sets derived from 10,000 randomly generated instances. \(\tau\) was fixed to be 1 year but the number of cores was varied. Two cases were considered - (i) \(e_1 + e_2 = 5\) and (ii) \(e_1 + e_2 = 6\). The success probability for each case with varying number of cores was estimated and plotted (Figure 4.2). The success probability with 3000 cores is 13% and increases to over 15% with 10,000 cores for \(e_1 + e_2 = 5\). This is considerably better compared to Strategy 2 (6.1% success probability).
Table 3.2 showed that the computational resources required to execute Algorithm 1 are substantially higher assuming 6 rather than 5 errors in the first 230 bits of \( u_s \). Hence the success probability with a smaller number of cores is higher for \( e_1 + e_2 = 5 \) compared to the case with \( e_1 + e_2 = 6 \). Beyond 50,000 cores, the case of 6 errors has much higher success probability. With 100,000 cores, the success probability is 23% and increases to 33% with 1000,000 cores.

To test the efficacy of our approach, we generated 2000 random test instances. Based on the results obtained by applying Algorithm 2 on the training data, we computed the average success probabilities of the test instances. The results as a function of number of cores is shown by dashed lines in Figure 4.2. There is a very close match between the success rates obtained in the training and testing phases with a maximum discrepancy of around 4%.

5 Related Work

[1] attempt to learn some of the elements of \( u \) and then use CVP to solve the reduced sub-instance. The following observation was made. If the elements of \( u \) and \( A \) are randomly chosen, then the average value of an element in \( c_1 \) would be \( \frac{640}{4} = 160 \). If the \( i^{th} \) entry of \( c_1 \) is small and the Hamming weight of the \( i^{th} \) row of \( T \) is not especially low, then \( u \) is likely to be 0 in many of the positions corresponding to 1’s in the \( i^{th} \) row of \( T \). Using this idea, some of the bits in \( u \) may be guessed and CVP used to solve the reduced instance. However, [1] states
that this method does not seem to be particularly effective beyond number of columns of $T = 400$.

\cite{4} implemented parallel enumeration for the Bounded Distance Decoding (BDD) problem and used it to solve Galbraith’s first challenge. They solved Galbraith’s first challenge using Ruhr-University’s “Crypto Crunching Cluster” (C3) within 4.5 hours. However they did not report any results related to the solution of the second challenge.

Herold and May\cite{2} studied the application of LP and ILP to obtain $u$. They obtained results for $n = 256$ and $m$ ranging from 400 to 640 for 1000 instances. The execution of a particular instance using ILP was aborted if it failed to obtain a solution within 10 seconds. The success probability dropped from 100% at $m = 490$ to 1% at $m = 590$. Under certain mild assumptions, they also proved that the solution with LP relaxation for $m \leq 2n$ is unique. For any given instance they computed a score which quantifies the search space for the ILP. $2^{19}$ instances of GB-LWE were generated for $m = 640$. From this ensemble, 271 weak instances were identified. 16 of these were solved within half an hour each. Since ILP is NP-hard and has, in general, exponential running time they did not provide any time bound for solving an instance.

Herold and May’s work was extended by \cite{5} They presented an approach to classify an instance as easy, moderate or hard. Out of 100 easy instances from 1000 randomly generated instances they solved 5 instances in a day using 150 cores and 18 instances in 50 days using 3000 cores. They concluded that the increase in success rate could be achieved by exponential growth in the number of core-days.

6 Conclusion

We addressed Galbraith’s second challenge - recovery of the ephemeral key, $u$ given a $256 \times 640$ matrix $T$ and ciphertext $c_1 = Tu$. Our approach involved repeatedly guessing the first 230 bits of $u$ by modifying an initial guess based on the output produced by applying LP. Our first strategy was to create and solve the resulting sub-instance using CVP. The second strategy involved 2-step guessing of the 230 bits before and after the second application of LP. This enhancement resulted in a larger number of instances with fewer number of errors in the initial guess thereby increasing success probability. Also, while there were a larger number of LR operations with Strategy 2, the reduced number of Babai NP computations was greatly reduced resulting in much lower overall computation time. With Strategy 1, we achieved a success rate of 9.4% using about 50,000 cores in 1 year while the success probability with Strategy 2 increased to 11.8% using only 27,000 cores in 1 year.

Strategy 3 makes repeated invocations to Algorithm 1 with different input parameters. The problem of learning the optimal input parameters is mapped to a variant of the Budgeted Maximum Coverage Problem. The parameters learned in the training phase surprisingly exhibited substantial diversity. Also, there was considerable variance in the execution times of the multiple runs of Algorithm 1
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with the input parameters learned. This enhancement greatly increased the success probability to 16% with approximately 27,000 cores in 1 year. With 100,000 cores the success probability touched 23%. One further avenue of investigation is the application of LP three or more times and the use of ILP or another method to solve the resulting sub-instance.

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