NON-PERTURBATIVE EFFECTS IN 2-D STRING THEORY
OR
BEYOND THE LIOUVILLE WALL

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ABSTRACT

We discuss continuous and discrete sectors in the collective field theory of \( d = 1 \) matrix models. A canonical Lorentz invariant field theory extension of collective field theory is presented and its classical solutions in Euclidean and Minkowski space are found. We show that the discrete, low density, sector of collective field theory includes single eigenvalue Euclidean instantons which tunnel between different vacua of the extended theory. We further show that these “stringy” instantons induce non-perturbative effective operators of strength \( e^{-\frac{1}{g}} \) in the extended theory. The relationship of the world sheet description of string theory and Liouville theory to the effective space-time theory is explained. We also comment on the role of the discrete, low density, sector of collective field theory in that framework.
1. INTRODUCTION

Non-perturbative aspects of string theory are an essential piece of information needed to make the comparison between string theory predictions and the real observable world. Matrix models, and especially $d = 1$ matrix models [1], offer a unique opportunity to obtain some insight into non-perturbative string theory. Certain matrix models have associated with them very simple string theories with a low number of degrees of freedom, propagating in a low number of space-time dimensions.

The $d = 1$ matrix model is the most complicated matrix model which can be solved exactly. On the other hand, it describes the simplest space-time dynamics which is still interesting. In the double scaling limit, the $d = 1$ matrix model describes strings propagating in one time dimension and one spatial dimension. An equivalent description is given in terms of a bosonic collective field theory in $1 + 1$ dimensions of one massless field [3], [4]. Notable features of collective field theory is that the kinetic energy is not canonical and it is not Lorentz invariant. Yet another equivalent description is in terms of $1 + 1$ dimensional fermionic field theory [1–3]. This description is useful to obtain the general set of classical solutions of the field theory [2]. In the bosonic theory, scattering amplitudes were calculated in perturbation theory [4], [5]. Similar calculations were done in the matrix model formulation and in the fermionic field theory in refs.[4], [10], and ref. [11]. In this paper, we use the bosonic collective field theory because it has a more transparent space-time description.

As stated above, the $d = 1$ matrix models, or the equivalent field theories have the power to describe non-perturbative phenomena in the associated $1 + 1$ string theories. This is interesting by itself. However, there may well be general features of non-perturbative string theory that are common to all string theories, including more complicated theories in higher dimensions such as $d = 4$. By studying the generic features of non-perturbative behaviour in $1 + 1$ dimensional string theories, one may learn about more realistic 4-dimensional string theories. It is of interest to ask whether or not there are any indications in string theory of relevant, non-perturbative behaviour. The answer [12] is yes!

First recall that in quantum field theory there is a well known connection between the large order behaviour of amplitudes and non-perturbative effects. Typically, amplitudes grow as $G!$ where $G$ is the number of loops, while non-perturbative effects have strength $e^{-g^2}$, where $g$ is the coupling parameter of the theory. Both of these facts follow from the existence of non-trivial classical solutions of the equations of motion of the field theory.
(or related field theory) in Euclidean space, i.e., instantons. The magnitude of the non-perturbative effects due to non-trivial solutions in a field theory with one dimensionless coupling parameter $g$ can be estimated using a simple scaling argument. Since the coupling parameter in this case can be scaled away, the action can be written as $S(\phi, g) = \frac{1}{g^2} \tilde{S}(\tilde{\phi})$, where $\tilde{S}$ does not depend on $g$. Therefore, any classical Euclidean solution with finite action has an action of order $\frac{1}{g^2}$. The magnitude of large order terms in the perturbative expansion can also be estimated by counting Feynman diagrams. The number $G!$ basically comes from the number of diagrams.

Large order growth of perturbative amplitudes is a common feature of matrix models and more complicated string theories [12]. For a review of large order behaviour of matrix model amplitudes see ref. [13]. All matrix models, as well as the critical bosonic string theory in 26 dimensions, exhibit a strange phenomenon. The magnitude of $G$’th order amplitudes in perturbation theory grow like $(2G)!$. We discovered that this simple, unalarming fact has far reaching consequences, which we explain in this paper.

It turns out that, in much the same way as $G!$ behaviour corresponds to $e^{-\frac{1}{g^2}}$ non-perturbative effects in quantum field theory, in matrix models the large order $(2G)!$ behaviour would correspond to non-perturbative effects of strength $e^{-\frac{1}{g^2}}$. How do these peculiar effects arise? In matrix models, there is a new type of instanton, involving a single eigenvalue, that is responsible for these effects. For a discussion of one eigenvalue instantons see [12], [13]. They were also discussed in [14] and in the context of supersymmetric matrix models in refs. [15] and [16].

In view of the above scaling argument in quantum field theory, it is of interest to ask how an action of order $\frac{1}{g}$ can ever arise. The answer is that, in matrix models, the associated effective action does not obey the same scaling argument, $S(\phi, g) \neq \frac{1}{g^2} \tilde{S}(\tilde{\phi})$. Instead, one finds that $g$ cannot be completely scaled out of $\tilde{S}$ due to “scale breaking terms”. That is $S(\phi, g) = \frac{1}{g^2} \tilde{S}(\tilde{\phi}, g)$. It follows that a non-trivial solution can be a function of $g$. Furthermore, if for such a solution $\tilde{S} \sim g$, then $S \sim \frac{1}{g}$. This is exactly what happens for one eigenvalue instantons.

The questions that we set out answer in this paper are:

1. Is there a canonical, Lorentz invariant effective field theory that describes matrix models and collective field theory?

2. What is the relation of single eigenvalue instantons to such a field theory, and in what sense do they describe a tunneling phenomenon?
3. Do these instantons induce calculable non-perturbative operators in the effective field theory?

4. What is the relationship of all of the above to string theory?

The main line of this paper is built around the answers to these questions. In section 2 we discuss various facts about matrix models and collective field theory. In section 3, the double scaling limit is presented in the context of collective field theory. We then discuss the high density limit of this theory and the static solution of its equations of motion. Similarly, we define the low density, finite eigenvalue limit. We show that, in Euclidean space, there is a single eigenvalue instanton solution of the equations of motion which has action $\pi g$.

Section 4 is devoted to extending the collective field theory, which is a non-canonical, non-Lorentz invariant theory of a single field $\phi$, to a canonical, Lorentz invariant effective field theory of two fields $\zeta$ and $D$. This theory exhibits the scale breaking terms responsible for the unusual action, $S \sim \frac{1}{g}$, of the instantons. This answers question 1. above in the affirmative. In section 5 we discuss, in detail, the vacuum solutions of the effective field theory and how these include wormhole-like configurations. These configurations look like two identical Liouville vacua linked by a single eigenvalue instanton. The normalized action of these configurations is $S = \frac{\pi g}{2}$. Thus question 2. is answered. We then show that these configurations induce calculable operators in the $\zeta, D$ effective theory with strength $e^{-\frac{\pi}{g}}$, as conjectured by Shenker. This answers question 3. Until this point, we have been discussing matrix models only. In section 6 we review the relationship of these double scaled matrix models to 1 + 1 dimensional string theories, and argue that these string theories must include the new vacuum configurations, single eigenvalue instantons, and induced operators of strength $e^{-\frac{\pi}{g}}$. This answers question 4. and concludes the main content of the paper. Finally, in section 7 we present some possible directions in which our results can be extended.

This paper is an extended and detailed version of [17].

2. MATRIX MODEL AND COLLECTIVE FIELD THEORY

In this section we review a few well known facts and a few less well known facts about the $d = 1$ matrix model and collective field theory and present them in a form appropriate for the following discussion of string theory.
2.1. Matrix Model

The fundamental variables of a time dependent, hermitian matrix model are $N \times N$ hermitian matrices $M(t)$. Their dynamics is described by the Lagrangian

$$L(\dot{M}, M) = \frac{1}{2} Tr \dot{M}^2 - V(M)$$

where, in general, $V$ is a finite polynomial

$$V(M) = \sum_n g_n Tr M^n$$

and $g_n$ are real coupling parameters. Clearly the mass dimension of $M$ is $-\frac{1}{2}$ and, hence, the couplings $g_n$ have positive mass dimensions. The conjugate momenta to $M$ are the $N \times N$ hermitian matrices $\Pi_M(t) = \dot{M}$. It follows that the associated Hamiltonian is given by

$$H(\Pi_M, M) = \frac{1}{2} Tr \Pi_M^2 + V(M)$$

The partition function of the hermitian matrix model can be written in terms of a path integral

$$Z_N(g_n) = \int [dM][d\Pi_M] e^{i \int dt \{ \Pi_M \dot{M} - H(\Pi_M, M) \}}$$

One now notes that matrices $M$ remain hermitian under the transformation $M \rightarrow UMU^\dagger$. The Lagrangian is invariant under such transposition as long as $U \in U(N)$. Therefore, as far as the partition function is concerned, the matrices $M$ can always be expressed in terms of their $N$ real eigenvalues $\lambda_i$. Furthermore, only correlation functions of operators that are $U(N)$ singlets are considered, since it is these singlet operators that correspond to string theory. It follows that the entire singlet sector theory can be completely expressed in terms of the eigenvalues $\lambda_i$. We do this explicitly in the next section. Another, equivalent, formulation of the same theory is the collective field representation which we discuss in section (2.3).
2.2. Effective Theory for Eigenvalues

We proceed to evaluate the partition function (2.5) in terms of the eigenvalues $\lambda_i(t), i = 1, \ldots, N$ of the matrix. The change of variables from $M$ to its eigenvalues and angular variables is non-linear. It is, therefore, difficult to proceed directly from the partition function (2.5). It is simpler to return to expression (2.4) and to use the formalism developed in ref.[18]. The result is that, up to an unimportant normalization factor coming from the integration over angular variables

$$Z_N(g_n) = \int [d\lambda_i] [d\Pi_{\lambda_i}] e^{i \int dt \left\{ \sum_i \Pi_{\lambda_i} \dot{\lambda}_i - H_{eff}(\Pi_{\lambda_i}, \lambda_i) \right\}}$$ (2.6)

where $H_{eff}$ can be determined as follows. The effective Hamiltonian operator in the $\lambda_i$ representation is found to be

$$\hat{H}_{eff} = \mathcal{J}^{1/2} \hat{H} \mathcal{J}^{-1/2}$$ (2.7)

where $\mathcal{J}$ is the Jacobian for the change of variables from the matrix variables to the eigenvalue variables

$$\mathcal{J} = \prod_{i<j} (\lambda_i - \lambda_j)^2$$ (2.8)

and

$$\hat{H} = \sum_i -\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i)$$ (2.9)

Here $V(\lambda_i)$ is the potential (2.2) written in terms of the eigenvalues.

The effective Hamiltonian should be hermitian in the new variables so care should be exercised in interpreting Eq.(2.7). The Hamiltonian $\hat{H}_{eff}$ can be evaluated explicitly

$$\hat{H}_{eff} = \sum_i \left[ -\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + \frac{1}{2} \left( \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right)^2 + V(\lambda_i) \right]$$ (2.10)

Noting that in the $\lambda_i$ representation the operator $\Pi_{\lambda_i} = -i \frac{\partial}{\partial \lambda_i}$, it follows that the associated classical Hamiltonian in Eq.(2.6) is given by

$$H_{eff} = \sum_i \left[ \frac{1}{2} \Pi_{\lambda_i}^2 + \frac{1}{2} \left( \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right)^2 + V(\lambda_i) \right]$$ (2.11)
Inserting this expression into (2.6) and performing the Gaussian $\Pi_{\lambda_i}$ integration, we find that

$$Z_N(g_n) = \int [d\lambda_i] e^{i \int dt L_{\text{eff}}(\dot{\lambda}_i, \lambda_i)}$$

where

$$L_{\text{eff}}(\dot{\lambda}_i, \lambda_i) = \frac{1}{2} \sum_i \dot{\lambda}_i^2 - V_{\text{eff}}(\lambda_i)$$

and

$$V_{\text{eff}} = V_{\text{coll}}(\lambda_i) + V(\lambda_i)$$

The induced term in the effective potential, $V_{\text{coll}}$, is

$$V_{\text{coll}}(\lambda_i) = \frac{1}{2} \sum_i \left( \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)} \right)^2$$

whereas the original potential in terms of the eigenvalues, $V(\lambda_i)$ is

$$V(\lambda_i) = \sum_i \sum_n g_n \lambda_i^n$$

The classical equations of motion of the theory are then given by

$$\frac{d^2 \lambda_i}{dt^2} = - \frac{d}{d\lambda_i} V(\lambda_i) - \frac{d}{d\lambda_i} V_{\text{coll}}(\lambda_i) \quad i = 1, \ldots, N$$

In general the solutions of Equations (2.17) are very complicated, but there are some conditions on the density of eigenvalues that make them tractable.

We reiterate, for emphasis, that Eqs. (2.17) are the complete and unique equations of motion of the matrix model, restricted to the singlet sector. They have to be satisfied in any other representation of the theory.

### 2.3. Collective Field Theory

A useful step on the road from the matrix model to string theory is collective field theory [2]. Relevant references are [19] – [21] and [22]. The idea is to start from the matrix model and, by performing a series of changes of variables, arrive at a field theory representation of the matrix model. We review here the derivation of the collective field theory Lagrangian.
We restrict the range of the eigenvalues $\lambda_i$ to be finite and impose periodic boundary conditions. That is

\[-\frac{1}{2}L \leq \lambda_i \leq \frac{1}{2}L\]  

A continuous spatial variable, $x$, is now introduced which also satisfies

\[-\frac{1}{2}L \leq x \leq \frac{1}{2}L\]  

One can now define the eigenvalue density, $\phi$, by

\[
\phi(x, t) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - \lambda_i(t))
\]

Note that $\phi$ satisfies the constraint

\[
\int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx \phi(x, t) = 1
\]

The eigenvalue density is also called the collective field.

Since the number of degrees of freedom of the system is $N$, not all $\phi(x, t)$ are independent. The $N$ independent variables associated with $\phi$ are the Fourier components

\[
\phi_{k_n}(t) = \int dx e^{-ik_n x} \phi(x, t)
\]

where

\[
k_n = \frac{2\pi n}{L} \quad n = \pm 1, \pm 2 \ldots, \pm \frac{N}{2}
\]

The highest momentum is $k_{max} = \frac{\pi N}{L}$.

We proceed to evaluate the partition function (2.3) in terms of the collective field $\phi$. It is easier to return to expression (2.4) and to use the formalism developed in ref. [18]. The partition function is given by

\[
Z_N(g_n) = \int [d\phi][d\Pi]\phi e^{i \int dt dx \left\{ \Pi \dot{\phi} - H_{eff}(\Pi, \phi) \right\}}
\]

where $\Pi$ is the conjugate momentum of the field $\phi$ and $H_{eff}$ can be determined as follows.

Let $J[\phi]$ be the Jacobian associated with the change of variables from $\lambda_i$ to $\phi$. Since $J$
can be quite complicated to evaluate directly, it is more convenient to express it in terms of two functions \( \Omega(x, y, \phi), \omega(x, \phi) \) defined as follows.

\[
\Omega(x, y, \phi) = \sum_{i,j} \frac{\delta \phi(x)}{\delta \lambda_i} \frac{\delta \phi(y)}{\delta \lambda_j} = \frac{1}{N} \frac{\partial}{\partial x} \frac{\partial}{\partial y}[\delta(x - y)\phi(x)]
\]

(2.25)

\[
\omega(x, \phi) = -\sum_i \frac{\delta^2 \phi(x)}{\delta \lambda_i^2} = 2N \frac{\partial}{\partial x} [\phi(x) P \int dy \phi(y)]
\]

(2.26)

where \( P \) stands for the principal part. The Jacobian can be shown to satisfy

\[
\int dy \Omega(x, y, \phi) \frac{\delta}{\delta \phi(y)} \ln J = \frac{\delta \Omega}{\delta \phi} + \omega
\]

(2.27)

This equation can be solved for \( J \). The result is

\[
\ln[J] = N^2 \int dx dy \phi(x) \ln |x - y - i\epsilon| \phi(y)
\]

(2.28)

where \( \epsilon \) is a regulator that is taken to zero eventually.

The Hamiltonian density of the theory is then given by

\[
\mathcal{H}_{eff} = \frac{1}{2} \Pi_\phi \Omega \Pi_\phi + \frac{1}{8} \frac{\delta \ln[J]}{\delta \phi} \Omega \frac{\delta \ln[J]}{\delta \phi} + N(V(x) - \mu_F)\phi
\]

(2.29)

where \( \mu_F \) is a Lagrange multiplier introduced to enforce the constraint (2.21) and \( V \) is the potential obtained from the eigenvalue potential (2.10) expressed in the new variables, \( V(x) = \sum_n g_n x^n \).

Using Eqs. (2.25), (2.28), and the identity

\[
\frac{1}{2} \int dx \phi(x, t) P \int dy \frac{\phi(y, t)}{x - y} P \int dz \frac{\phi(z, t)}{x - z} = \frac{\pi^2}{6} \int dx \phi^3(x, t)
\]

(2.30)

This Hamiltonian density becomes

\[
\mathcal{H}_{eff} = \frac{1}{2N} \partial_x \Pi_\phi \phi \partial_x \Pi_\phi + \frac{N^3 \pi^2}{6} \left[ \phi^3 - \phi(x)\phi(y)\phi(z)\delta(x - y)\delta(x - z) \right]
\]

(2.31)

\[
+ N(V(x) - \mu_F)\phi
\]

The significance of the peculiar term \( N^3 \phi(x)\phi(y)\phi(z)\delta(x - y)\delta(x - z) \) can be understood by evaluating it using \( \phi(x, t) = \frac{1}{N} \sum_i \delta (x - \lambda_i(t)) \). The result is

\[
\int dx \phi(x)\phi(y)\phi(z)\delta(x - y)\delta(x - z) = N\delta^2(0)
\]

(2.32)
This term then can be thought of as classical counter term consistent with the underlying matrix model. It ensures that field configurations of the form \( \phi = \frac{1}{N} \sum_i \delta (x - \lambda_i(t)) \) have finite energy by subtracting classical self-energy contributions. It’s effect on smooth field configurations is negligible. We will stop writing this term down at this point. It is however important in discussing the low density limit in the next section.

It is useful to rescale the various quantities according to

\[
\phi \rightarrow \frac{1}{\sqrt{N}} \phi, \quad \Pi_\phi \rightarrow \frac{1}{N} \Pi_\phi, \quad x \rightarrow \sqrt{N} x
\]

\[
V \rightarrow NV, \quad \mu_F \rightarrow N \mu_F
\]

The Hamiltonian density then becomes

\[
H_{\text{eff}} = \frac{1}{2N^2} \partial_x \Pi_\phi \partial_x \Pi_\phi + \frac{N^2 \pi^2}{6} \phi^3 + N^2 (V(x) - \mu_F) \phi
\]

Inserting this expression into Eq.(2.24) and noting that the first term in the Hamiltonian can be written as

\[
\int dx \frac{1}{2N^2} \partial_x \Pi_\phi \partial_x \Pi_\phi = \int dx dy \Pi_\phi(x,t) \frac{1}{2N} \Omega(x,y,\phi) \Pi_\phi(y,t)
\]

we can perform the Gaussian \( \Pi_\phi \) integration. The result is

\[
Z_N(g_n) = \int [d\phi] \det \Omega \frac{1}{2} e^{iN^2 \int dt L_{\text{eff}}(\dot{\phi},\phi)}
\]

where

\[
L_{\text{eff}} = \int dx \left\{ \frac{1}{2} \int \frac{x}{\phi} \int \frac{\dot{x}}{\phi} - \frac{\pi^2}{6} \phi^3 - (V(x) - \mu_F) \phi \right\}
\]

Here the contact terms in Eq.(2.32) are omitted. Note the appearance of the factor \( \det \Omega - \frac{1}{2} \) which comes from doing the Gaussian integral over the conjugate momentum \( \Pi_\phi \).

A good check on the validity of Eq.(2.36) is to substitute the rescaled version of Eq.(2.20) \( \phi = \sum_i \delta (x - \lambda_i(t)) \) into it. The result should be identical to Eq.(2.12). Inserting the above rescaled version of (2.20) into (2.37) (including the contact terms) and using

\[
\frac{1}{2} \int dx \phi(x) P \int dy \frac{\phi(y)}{x-y} P \int dz \frac{\phi(z)}{x-z} = \frac{1}{2} \sum_i \left( \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)} \right)^2
\]
we find that $L_{\text{eff}}(\dot{\phi}, \phi)$ is equal to $L_{\text{eff}}(\dot{\lambda}_i, \lambda_i)$ in (2.13). Similarly, one can show that $\int [d\phi] \det \Omega^{-\frac{1}{2}} = \int [d\lambda_i]$ and, hence the two partition functions are equal, as they must be. We note for future use that one can use the identity (2.30) in the other direction and deduce that an equivalent form of the collective potential in Eq.(2.13) is

$$V_{\text{coll}}(\lambda_i) = \frac{\pi^2}{6} \sum_i \left( \sum_{j \neq i} \delta(\lambda_i - \lambda_j) \right)^2$$  \hspace{1cm} (2.39)

At this point, however, we must be careful. Recall that the eigenvalues $\lambda_i$ are independent variables, whose equation of motion follow from varying the action in Eq.(2.13) with respect to each of the $\lambda_i$'s. However, as discussed above, for finite $N$ the collective field $\phi(x, t)$ is highly constrained. It does not correspond to an infinite number of degrees of freedom. Hence, the $\phi$ equation of motion is not obtained by varying the action with respect to $\phi$. The correct procedure is the following. Start with the equations of motion for the eigenvalues $\lambda_j$

$$\frac{\delta}{\delta \lambda_j(t)} S_{\text{eff}}[\dot{\lambda}_i, \lambda_i] = 0$$  \hspace{1cm} (2.40)

where $S_{\text{eff}}[\dot{\lambda}_i, \lambda_i] = \int dt L_{\text{eff}}(\dot{\lambda}_i, \lambda_i)$. Then by using the rescaled version of (2.20) and the fact that $L_{\text{eff}}(\dot{\phi}, \phi) = L_{\text{eff}}(\dot{\lambda}_i, \lambda_i)$ convert (2.40) to an equation for $\phi$. The result is

$$\frac{\partial}{\partial y} \frac{\delta S_{\text{eff}}[\dot{\phi}, \phi]}{\delta \phi(y, t)} \bigg|_{y = \lambda_j(t)} = 0 \hspace{1cm} j = 1, \ldots, N$$  \hspace{1cm} (2.41)

Note that there are $N$ equations of motion since $y$ must be evaluated at $\lambda_j$ for all $j = 1, \ldots, N$. Furthermore, note that $\phi'$s satisfying the naive $\phi$ equation of motion

$$\frac{\delta S_{\text{eff}}[\dot{\phi}, \phi]}{\delta \phi(y, t)} = 0$$  \hspace{1cm} (2.42)

also satisfy Eq.(2.41). However there are solutions of Eq.(2.41) which do not satisfy Eq.(2.42). We return to this important point later.

3. LARGE $N$ LIMIT

In the previous section, we discussed matrix models, in both the $\lambda_i$ and $\phi$ representations for finite $N$. Furthermore, the $\lambda_i$'s and $x$ were restricted to satisfy $-\frac{1}{2}L \leq \lambda_i \leq \frac{1}{2}L$ and $-\frac{1}{2}L \leq x \leq \frac{1}{2}L$ respectively for finite $L$. In this section we discuss the limits $N \to \infty$ and $L \to \infty$. We find it convenient to take the limit $L \to \infty$ at the outset. Henceforth, $L \to \infty$ and $\lambda_i$ and $x$ satisfy $-\infty \leq \lambda_i \leq \infty$ and $-\infty \leq x \leq \infty$. We now want to let $N \to \infty$. However, as we proceed to show, there are several inequivalent ways in which the large $N$ limit can be taken.
3.1. Double scaling limit

Before taking the \( N \to \infty \) limit, it is essential to specify the dependence on \( N \) of the coupling parameters \( g_n \) in the potential (2.2). As usual, if we do not choose the \( N \) dependence of the couplings in any special way, the resulting large \( N \) limit is a free theory [23], [24]. The choice of \( N \) dependence of couplings (as \( N \to \infty \)) that turns out to be most relevant for string theory is called the double scaling limit. It involves specifying the exact \( N \) dependence of one coupling parameter. Which of the coupling parameters is specified is not important. The \( \lambda_i \) representation in subsection (2.1) and the \( \phi \) representation in subsection (2.2) are equivalent. Therefore the double scaling limit can be taken in either representation. We find it more convenient to define and take this limit in the \( \phi \) representation. Therefore, here and throughout the rest of the paper, we use the collective field representation.

To enable us to take the double scaling limit the potential, \( V(x) \), has to have a local maximum at some point \( x^* \). For every value of \( x^* \) one gets the same double scaling limit, so the position of the maximum is unimportant. Therefore, for simplicity, we set \( x^* = 0 \). Then \( V(x) = V(0) - \frac{1}{2}x^2 + \cdots \), where, without loss of generality, we have chosen the coefficient of the second term to be \( \frac{1}{2} \). There is a region \( |x| \leq x_{\text{max}} \) where the higher order terms in \( V(x) \) can be neglected. We restrict our attention to that region. Therefore, in that region \( V(x) = V(0) - \frac{1}{2}x^2 \). Inserting this expression into the Lagrangian (2.37), the combination \( V(0) - \mu_F \) appears. We denote \( V(0) - \mu_F \) by \( \frac{1}{2} \mu \) and assume that \( \mu > 0 \).

The double scaling limit is defined by specifying the \( N \) dependence of \( \mu \)

\[
N\mu = \frac{1}{g} \tag{3.1}
\]

so that \( g \) remains finite as \( N \to \infty \). The parameter \( g \) is related to the string coupling parameter, as we discuss later on. (For this reason this parameter is often denoted \( g_{\text{st}} \) in the literature.)

It is convenient at this point to again rescale

\[
\phi \to \frac{1}{\sqrt{N}}\phi, \quad x \to \sqrt{N}x \tag{3.2}
\]

This rescaling removes the factor \( N^2 \) in Eq. (2.36). Now take the \( N \to \infty \) limit. Then \( x = \sqrt{\mu} \to x = \sqrt{\frac{\mu}{g}} \) and the region \( |x| \leq x_{\text{max}} \) gets blown up to the whole real axis \(-\infty \leq x \leq \infty \).
The Lagrangian density in Eq. (2.37) now becomes

\[ L_{\text{eff}} = \frac{1}{2} \int \dot{\phi} \frac{x}{\phi} - \frac{\pi^2}{6} \phi^3 - \frac{1}{2} \left( \frac{1}{g} - x^2 \right) \phi \]  

(3.3)

The classical equations of motion derived using Eqs. (2.41) and (3.3) are

\[ \frac{\partial}{\partial x} \left( \int dy \partial_t \frac{\dot{\phi}}{\phi} - \frac{1}{2} \frac{\dot{\phi} \dot{\phi}}{\phi^2} - \frac{\pi^2}{2} \phi^2 - \frac{1}{2} \left( \frac{1}{g} - x^2 \right) \right) \big|_{x=\lambda_i(t)} = 0 \]  

(3.4)

where the index \( i \) now runs over \( i = 1, \ldots, N \to \infty \).

At this point we realize that to make sense of the equations of motion within the double scaling limit, it is necessary to specify more accurately the density structure of the eigenvalues \( \lambda_i \). The additional information lies in the nature of the limit of \( \frac{N}{L} \) as \( L \to \infty \), \( N \to \infty \), which has not been specified yet.

3.2. High Density Limit

The high density (HD) limit of collective field theory is defined as follows. Consider a region of \( x \), denoted \( I \), of length \( l(I) \). The number of eigenvalues in this region is \( N(I) \). Then take the the double scaling limit in such a way that

\[ \frac{N(I)}{l(I)} \to \infty \]  

(3.5)

It is clear that in this region of \( x \) there are an infinite number of eigenvalues. The collective field \( \phi \) now has an infinite number of degrees of freedom and the classical equations of motion (3.4) become simply

\[ \frac{\partial}{\partial x} \left( \int dy \partial_t \frac{\dot{\phi}}{\phi} - \frac{1}{2} \frac{\dot{\phi} \dot{\phi}}{\phi^2} - \frac{\pi^2}{2} \phi^2 - \frac{1}{2} \left( \frac{1}{g} - x^2 \right) \right) = 0 \]  

(3.6)

The static solution of these equations is very simple

\[ \phi_0 = \frac{1}{\pi} \sqrt{x^2 - \frac{1}{g}} \]  

(3.7)

where \( |x| \geq \sqrt{\frac{1}{g}} \). Note that \( \phi_0 \) actually makes the term inside the parenthesis in Eq. (3.6) vanish. That is, \( \phi_0 \) is a solution of the conventional field theory equations of motion Eq. (2.42).
In the following we want the HD regime of $x$ to possess this static solution. Hence we must take the HD limit only for $|x| \geq \sqrt{\frac{1}{g}}$. Also it is important to recall that $\phi_0$ is the average eigenvalue density. Clearly, for consistency $\phi_0 \gg 1$. Note, however that $\phi_0$ decreases and then vanishes as $|x|$ approaches $\sqrt{\frac{1}{g}}$. Therefore, the HD regime of $x$ is further restricted to $|x| \gg \sqrt{\frac{1}{g}}$.

Figure 1. The potential and classical solution.

What happens if one tries to ignore that restriction and tries to use the HD limit in the whole region $|x| \geq \sqrt{\frac{1}{g}}$? It comes back and shows itself in a different and interesting way. To explain the different disguises in which the same problem presents itself, we need to mention a few more facts about collective field theory. Namely, its perturbation expansion, and in particular the large order behaviour of that perturbation series.

To obtain the perturbation expansion one expands $\phi$ around the classical solution $\phi_0$

$$\phi = \phi_0 + \frac{1}{\sqrt{\pi}} \partial_x \zeta$$

(3.8)

Substituting this into the Lagrangian (3.3) yields

$$L_{\text{eff}} = \int dx \left\{ \frac{1}{2\pi \phi_0} \dot{\zeta}^2 - \frac{\pi}{2} \phi_0 (\partial_x \zeta)^2 - \frac{\sqrt{\pi}}{6} (\partial_x \zeta)^3 + \cdots \right\}$$

(3.9)

To obtain a canonical kinetic term for the field $\zeta$ change coordinates to the Liouville coordinate

$$\tau = \frac{1}{\pi} \int^x dy \frac{\phi_0}{\zeta}$$

(3.10)

$L_{\text{eff}}$ then becomes

$$L_{\text{eff}} = \int d\tau \left\{ \frac{1}{2} \dot{\zeta}^2 - \frac{1}{2} (\partial_\tau \zeta)^2 - \frac{1}{\pi^2 \phi_0^2(\tau)} \frac{1}{6} (\partial_\tau \zeta)^3 + \cdots \right\}$$

(3.11)
It follows from the cubic term that \( \frac{1}{\pi^3 \phi_0^2(\tau)} \) is to be identified as the coupling parameter of collective field theory. However, \( \frac{1}{\phi_0^2(\tau)} \) becomes large exactly where the classical solution \( \phi_0 \) becomes small, which is where the HD expansion breaks down. The field theory becomes strongly coupled or in other words the semi-classical expansion, the loop expansion, breaks down.

What happens if this fact is ignored? Perturbation series takes its revenge by growing too fast. In fact, the \( G' \)th order in perturbation expansion grows like \( (2G)! \). There are a few diagrams that become large [8]. The main contribution comes from the region (in \( x \) space) where the coupling parameter becomes large. This is very similar to the phenomenon of renormalons [25] in ordinary non-asymptotically free quantum field theory. There the coupling parameter grows (in momentum space), and there are a few diagrams that receive large contributions from that region of integration in momentum space.

The behavior of the perturbation series of collective field theory is different than the growth of perturbation series caused by instantons in ordinary field theory in two ways. The first is in the rate it grows ((2G)! vs. \( G! \)), and the other is the way it grows. In ordinary field theories with instantons the number of diagrams grows, and not their magnitude.

Note, however, that all these phenomena are related to an expansion around a "bad" static classical solution \( \phi_0 \). Therefore, like the situation in ordinary field theories with instantons, other classical solution may actually be responsible for that particular behaviour of the perturbation series.

### 3.3. Low Density Limit and One Eigenvalue Instantons

The low density (LD) limit of collective field theory is defined as follows. Consider a region of \( x \), denoted \( J \), of length \( l(J) \). The number of eigenvalues in this region is \( N(J) \). Then take the the double scaling limit in such a way that

\[
\frac{N(J)}{l(J)} \rightarrow \text{finite} \quad (3.12)
\]

It is clear that in this region of \( x \) there is a finite number of eigenvalues. The collective field \( \phi \) now has a finite number of degrees of freedom in that region and the classical equations of motion (3.14) become simply

\[
\frac{\partial}{\partial x} \left( \int^x dy \partial_t \left[ \frac{y}{\phi} - \frac{x}{2} \frac{\dot{\phi}^2}{\phi^2} - \frac{\pi^2}{2} \phi^2 - \frac{1}{2} \left( \frac{1}{g} - x^2 \right) \right] \right) \bigg|_{x = \lambda_i(t)} = 0 \quad (3.13)
\]

We thank M. Moshe for drawing our attention to this analogy.
where the index $i$ now runs over $i = 1, ..., N(J)$.

In this paper, we are particularly concerned with LD regions containing a single eigenvalue $\lambda^*(t)$. Using the relation

\[ \phi(x, t) = \delta(x - \lambda^*(t)) \] (3.14)

the equation of motion (3.13) (including the contact terms) simply becomes

\[ \frac{d^2\lambda^*}{dt^2} = \lambda^*(t) \] (3.15)

Note that this equation is identical to Eq.(2.17) evaluated for a single eigenvalue with $V(\lambda^*) = V(0) - \frac{1}{2}(\lambda^*)^2$. There are no non-trivial static solutions of Eq.(3.13). The only reason that we can get a static solution in the HD region collective field theory is due to the interactions between eigenvalues. The interaction energy balances the potential energy and the “particles” are at rest. The solutions that we are looking for are, therefore, time dependent.

The general solution of Eq.(3.15) is given by

\[ \lambda^*(t) = E^* \cosh t + F^* \sinh t \] (3.16)

where $E^*$ and $F^*$ are real constants. The energy of this solution is

\[ E = V(0) - \frac{1}{2} ((E^*)^2 - (F^*)^2) \] (3.17)

As discussed in the previous subsection, we have restricted the HD region to $|x| \geq \sqrt{\frac{1}{g}}$.

Therefore, in this paper, the LD region is in the complementary region $-\sqrt{\frac{1}{g}} \leq x \leq \sqrt{\frac{1}{g}}$.

It is clear that almost all solutions (3.16) leave the LD region after a finite time. The only exceptions are the two solutions corresponding to $|E^*| = |F^*$| which end up sitting on top of the potential at time $t \to \infty$. These do not have an obvious physical interest.

So far we have discussed classical solutions of the equations of motion in the LD region in real Minkowski time. These correspond to a finite number of eigenvalues moving in real time. More interesting are the classical solutions of the equations of motion in the LD region in Euclidean time. These are instantons that correspond to tunneling of eigenvalues across the potential barrier. Going to Euclidean time the resulting equation of motion becomes

\[ \frac{\partial}{\partial x} \left( -\int dy \int \phi \frac{\delta}{\delta \phi} + \frac{1}{2} \int \phi \frac{\delta}{\delta \phi} - \frac{\pi^2}{2} \phi^2 - \frac{1}{2} \left( \frac{1}{g} - x^2 \right) \right) |_{x=\lambda_i(\theta)} = 0 \] (3.18)
where $\theta$ is Euclidean time. Again we are particularly interested in a LD region containing a single eigenvalue $\lambda^*(\theta)$. Using the relation

$$\phi(x, \theta) = \delta(x - \lambda^*(\theta)) \quad (3.19)$$

the equation of motion (3.18) (including the contact terms) simply becomes

$$\frac{d^2\lambda^*}{d\theta^2} = -\lambda^*(\theta) \quad (3.20)$$

The general solution of Eq.(3.20) is given by

$$\lambda^*(\theta) = A^* \cos \theta + B^* \sin \theta \quad (3.21)$$

where $A^*$ and $B^*$ are real constants. As in Minkowski space, the Euclidean LD region is taken to be $-\sqrt{\frac{1}{g}} \leq x \leq \sqrt{\frac{1}{g}}$.

As an example consider the collective field (3.19) corresponding to a single eigenvalue $\lambda^*$ with the boundary conditions $\lambda^*(\theta_0) = \sqrt{\frac{1}{g}}$ and $\dot{\lambda}^*(\theta_0) = 0$. It follows from (3.21) that

$$\lambda^*(\theta) = \sqrt{\frac{1}{g}} \cos (\theta - \theta_0) \quad (3.22)$$

The solution is shown in Figure 2.

---

Figure 2. A single eigenvalue $\lambda^*$ in Euclidean space.

The corresponding collective field configuration is

$$\phi_{\text{inst}}(x, \theta) = \delta \left( x - \frac{1}{\sqrt{g}} \cos (\theta - \theta_0) \right) \quad (3.23)$$
This is an instanton that corresponds to tunneling of one eigenvalue across the barrier from $x = \sqrt{\frac{1}{g}}$ at $\theta = \theta_0$ to $x = -\sqrt{\frac{1}{g}}$ at $\theta = \theta_0 + \pi$. Note that the classical solution in Eq.(3.23) is not a solution of the Euclidean continuation of of the unconstrained field theory equations of motion Eq.(2.42).

The action of the instanton $\phi_{\text{inst}}$ can be computed from the Euclidean continuation of (3.3). The result is

$$S_{\text{eff}}[\phi_{\text{inst}}, \dot{\phi}_{\text{inst}}] = \int_0^\pi d\theta \int_{-\sqrt{\frac{1}{g}}}^{+\sqrt{\frac{1}{g}}} dx \delta \left( x - \frac{1}{\sqrt{g}} \cos(\theta) \right) \left\{ \frac{1}{2g} \sin^2(\theta) - \frac{1}{2g} \cos^2(\theta) + \frac{1}{g} \right\} = \frac{\pi}{g}$$

(3.24)

in agreement with the large order behavior of the perturbation series in $g$. We want to stress that the action of this solution is not infinite as one might have thought. The local interaction terms look infinite. For example, the $(\phi_{\text{inst}})^3$ term gives a contribution proportional to $(\delta(0))^2$. However, this contribution gets canceled, due to the self-energy subtraction terms. These terms also fix the normalization of the field configuration that corresponds to a single eigenvalue. If we tried a configuration of the form $\phi = A\delta(x - \lambda^*(t))$ where $A \neq 1$ it would have had infinite energy.

Now let us see why we obtained an action of $\frac{1}{g}$ and not $\frac{1}{g^2}$, as expected in ordinary field theory. Collective field theory has two ingredients that make this possible. The first is the presence of scale breaking terms mentioned in the introduction, i.e. the coupling parameter $g$ cannot be scaled out from the Lagrangian. The second and related ingredient is that the solution is a constrained solution that depends on the coupling parameter in a way which is of course different from what could be expected for a solution of the $\phi$ equations of motion.

We can now say that we understand why the perturbation expansion of collective field theory in the double scaling limit is growing as it grows. Besides the static solution, there are time dependent Euclidean (constrained) solutions that contribute to the path integral. The important configurations are singular configurations, but have finite action. Perturbation theory is smart enough and knows about these solutions. It lets us know of their existence by growing accordingly.
4. SPACE-TIME EFFECTIVE ACTION

The collective field theory Lagrangian density, expression (3.3), has two notable deficiencies. First, the kinetic energy term is not in canonical form. This means that we have not identified correctly the canonical field of the theory. Second, and more important, the coordinate $x$ appears in the potential energy and therefore Lorentz invariance seems to be broken explicitly. In this section we remove both deficiencies. The first, following ref. [2], by field and coordinate redefinitions. The second, following ref. [26], by enlarging the theory to include a new field. The non-trivial vacuum expectation value of this new field is responsible for the spontaneous breaking of Lorentz invariance.

4.1. Canonical Collective Field

As we demonstrated in section 3, the collective field theory divides naturally into three regions shown in Figure 3. Regions $I$ and $III$ admit the static solution $\phi_0$ given in Eq.(3.7). Region $II$, on the other hand, does not admit a static solution. It does, however, possess Euclidean time dependent solutions.

The two high density regions $|x| > \sqrt{\frac{1}{g}}$, are denoted by I and III in Figure 3. The low density region $|x| \leq \sqrt{\frac{1}{g}}$, is denoted II.

\begin{equation}
\phi = \phi_0 + \frac{1}{\sqrt{\pi}} \partial_x \zeta
\end{equation}

Figure 3. High and low density regions

We begin the discussion in this section with region $I$. The first step is to shift the double scaled collective field $\phi$ by the classical static solution $\phi_0$ in Eq.(3.7).
The resulting Lagrangian is \( L = L_\zeta + L_0 \) where
\[
L_\zeta = \int dx \left\{ \frac{1}{2} \frac{\dot{\zeta}^2}{\phi_0} + \frac{1}{6} \pi^2 \partial_x \zeta \right\} \cdot - \frac{1}{2} \pi \phi_0 (\partial_x \zeta)^2 - \frac{\sqrt{\pi}}{6} (\partial_x \zeta)^3 \right\}
\]
(4.2)
and
\[
L_0 = \int dx \left\{ -\frac{\pi^2}{6} (\phi_0)^3 - \frac{1}{2} \left( \frac{1}{g} - x^2 \right) \phi_0 \right\}
\]
(4.3)
To obtain a canonical kinetic term for the field \( \zeta \), change coordinates to the Liouville coordinate defined by
\[
\tau - \tau_0^I = \frac{1}{\pi} \int_{x_0}^{x} \frac{dy}{\phi_0} = \ln \left[ x + \sqrt{x^2 - \frac{1}{g}} \right] - \ln \left[ x_0 + \sqrt{x_0^2 - \frac{1}{g}} \right]
\]
(4.4)
For simplicity, we take \( x_0 = \sqrt{\frac{1}{g}} \) and \( \tau_0^I = \ln \sqrt{\frac{1}{g}} \). In this case
\[
x = \sqrt{\frac{1}{g}} \cosh (\tau - \ln \sqrt{\frac{1}{g}})
\]
(4.5)
and the range of the two coordinates is
\[
\sqrt{\frac{1}{g}} \leq x \leq \infty
\]
\[
\ln \left[ \sqrt{\frac{1}{g}} \right] \leq \tau \leq \infty
\]
(4.6)
The static solution in the new coordinate becomes
\[
\phi_0(\tau) = \frac{1}{\pi \sqrt{g}} \sinh (\tau - \ln \sqrt{\frac{1}{g}})
\]
(4.7)
Other choices of \( x_0, \tau_0^I \) in Eq.(4.4) would lead to an overall rescaling of \( \phi_0(\tau) \) and a rescaling of \( g \) in Eq.(4.7).
Rewritten in terms of the Liouville coordinate the classical Lagrangian, \( L_\zeta \), is given by
\[
L_\zeta = \int d\tau \left\{ \frac{1}{2} \frac{\dot{\zeta}^2}{\phi_0^2(\tau)} + \frac{1}{6} \pi^2 \phi_0^2(\tau) (\partial_\tau \zeta)^3 \right\}
\]
(4.8)
where \( \phi_0 \) is given by Eq.(4.7). Note that now the kinetic term of \( \zeta \) is indeed \( \frac{1}{2} \left( \dot{\zeta}^2 - (\partial_\tau \zeta)^2 \right) \) as it should be.
The pure $\phi_0$ Lagrangian, $L_0$, turns into

$$L_0 = \int d\tau \frac{\pi^3}{3} \phi_0^4$$

From the cubic interaction term in $L_\zeta$, it follows that the coupling parameter of collective field theory is

$$\frac{1}{\pi^2 \phi_0^2(\tau)} = 4\sqrt{\pi} \frac{e^{-2\tau}}{(1 - \frac{1}{g}e^{-2\tau})^2}$$

The coupling parameter vanishes as $\tau \to \infty$ and explodes at $\tau = \ln[\sqrt{\frac{1}{g}}]$.

4.2. Lorentz Invariance

We now describe a field theory that reduces to the collective field theory of region $I$ when the various fields obtain their expectation values. The idea was discussed for $\mu = 0$ in [26]. We limit ourselves here to flat target space, but $\mu \neq 0$. We note that the $\zeta$ theory is not Lorentz invariant. Our interpretation is that this is really a Lorentz invariant field theory of two fields, $\zeta$ and $D$, expanded around the vacuum expectation values of the two fields. The new field $D$ has a vacuum expectation value that breaks Lorentz invariance, and that is the reason that the $\zeta$ theory alone is not Lorentz invariant.

We also know that the true theory is defined for all $\zeta, D$ field configurations and should not be expressible just as an expansion around a particular solution. We therefore look for a field theory which has the appropriate solutions.

To find out the background independent field theory we have to identify the expectation value of the $\zeta$ and $D$ fields first. Motivated by the comparison between collective field theory and the Polyakov description of the related string theory (see section 6) we postulate that

$$< G_{\mu\nu} > = \eta_{\mu\nu}$$
$$< D > = -2\tau$$
$$< \zeta > = \frac{1}{g}$$

Here we added the expectation value of the metric as well. Our convention is $\eta_{\mu\nu} = \\
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}$. Note that the field $D$ has the non-translation invariant vacuum expectation value.

We list the background independent form of the different quantities.
\[
\pi^2 \phi_0^2(\tau) \rightarrow \frac{1}{4\sqrt{\pi}} e^{-D} \left( 1 - \frac{1}{g} e^D \right)^2
\]
(4.12)

\[
\partial_\tau \zeta \rightarrow \frac{1}{2} \nabla D \cdot \nabla \zeta
\]
(4.13)

\[
\dot{\zeta}^2 - (\partial_\tau \zeta)^2 \rightarrow \nabla \zeta \cdot \nabla \zeta
\]
(4.14)

The Lorentz non-invariant quantities on the left are obtained from the Lorentz invariant quantities on the right by letting \( D = \langle D \rangle \) and \( \zeta = \langle \zeta \rangle + \zeta' \).

Also, let
\[
\int dt d\tau = \int d^2 x
\]
(4.15)

We can now write the \( \zeta \) action using the previous dictionary of expressions. The result is
\[
S_\zeta = \int d^2 x \left\{ \frac{1}{2} \frac{\nabla \zeta \cdot \nabla \zeta}{1 + 2\sqrt{\pi} \frac{e^D}{(1 - \frac{1}{g} e^D)^2} \nabla \zeta \cdot \nabla D} - \frac{\sqrt{\pi}}{4} \frac{e^D}{(1 - \frac{1}{g} e^D)^2} \frac{1}{2 + 2\sqrt{\pi} \frac{e^D}{(1 - \frac{1}{g} e^D)^2} \nabla \zeta \cdot \nabla D} \right\}
\]
(4.16)

We also need the action for the pure \( D \) sector. This action is not supplied in such a clear way by collective field theory, although some hints are given in (4.3). We also know some of the lowest order terms through other methods of calculation.

Recall the pure \( \phi_0 \) terms given in Eq.(4.9). They can be reexpressed using the dictionary of expressions Eqs.(4.12)-(4.15) as
\[
S_0 = -\frac{1}{384\pi} \int d^2 x e^{-2D} \left[ 1 - \frac{1}{g} e^D \right]^4 [((\nabla D)^2 - 4]
\]
(4.17)

Since space-time is flat in this case we ignore curvature terms in the action. Higher derivative terms like \((\nabla D \cdot \nabla D)^2\) cannot be ruled out at this point. In what follows we treat the Lagrangian \((4.17)\) as if it were the exact Lagrangian. Therefore, the full space time action is
\[
S = \int d^2 x \times
\left\{ \frac{1}{2} \frac{\nabla \zeta \cdot \nabla \zeta}{1 + 2\sqrt{\pi} \frac{e^D}{(1 - \frac{1}{g} e^D)^2} \nabla \zeta \cdot \nabla D} - \frac{\sqrt{\pi}}{4} \frac{e^D}{(1 - \frac{1}{g} e^D)^2} \frac{1}{2 + 2\sqrt{\pi} \frac{e^D}{(1 - \frac{1}{g} e^D)^2} \nabla \zeta \cdot \nabla D} \right\}
\]
(4.18)
Let us now discuss the equations of motion derived from this action and check that our construction is indeed consistent. The equations of motion are complicated so we do not write them out explicitly. However, it is straightforward to verify that indeed

\[ \langle G_{\mu\nu} \rangle = \eta_{\mu\nu} \]
\[ \langle D \rangle = -2\tau \]
\[ \langle \zeta \rangle = \frac{1}{g} \]  
(4.19)

is an exact solution of these equations of motion. There is also agreement with the \( \sigma \)-model calculation (see section 6.) This is a non-trivial consistency check.

The solution (4.19) is not a unique solution of the equations of motion. The action (4.18) is Lorentz invariant. The solution (4.19) breaks that invariance and, therefore, there should be a family of solutions associated with the broken generators. Indeed the most general solution for \( D \) is (see [7],[26])

\[ \langle D \rangle = a(t - \bar{t}) + b(\tau - \bar{\tau}) \]
\[ b^2 - a^2 = 4 \]  
(4.20)

The solution (4.19) corresponds to \( a = 0, b = -2, \bar{\tau} = 0 \).

Because the field \( \zeta \) has only derivative couplings, the action (4.18) is also invariant under a \( \zeta \) shift symmetry

\[ \zeta \rightarrow \zeta + \text{constant} \]  
(4.21)

which is again reflected in the fact that any \( \zeta = \text{constant} \) is an allowed solution. We see now that the original identification of the relation between the coordinate \( \tau \) and \( x \) in Eq.(4.13) was not unique. However each definition corresponds to a single and particular choice of \( a, \bar{t}, \bar{\tau} \) in Eq.(4.20).

Note that the action (4.18) reduces to the one obtained in [26] in the limit \( \mu \rightarrow 0 \). To see that, replace \( \phi_0 \) in Eq.(1.7) by it’s \( \mu = 0 \) expression, \( \phi_0^{\mu=0} = \frac{1}{2\pi} e^\tau \).

There is a very important difference between the theories with \( \mu = 0 \) and with \( \mu \neq 0 \). It is the appearance of what we call “scale breaking terms”. For the case \( \mu = 0 \) the coupling parameter of the effective field theory is expressed as a function of \( D \) alone. When \( \mu \neq 0 \), this is not possible. The obstruction comes from the fact that the effective field theory coupling parameter, \( g \), depends on both \( g = \frac{1}{\mu} \) and \( e^D \), and not just on \( e^D \).

It is impossible to absorb \( g \) into a redefinition of \( D \). As explained in the introduction, this fact is closely related to the behaviour of large order perturbation series and to appearance of finite action configurations with action \( \frac{1}{g} \) as opposed to \( \frac{1}{g^2} \).
4.3. Extension to All of Space Time

At this point, we must recall that the effective Lagrangian (4.18) has been constructed only for region $I$ of collective field theory. Hence, the spatial parameter $\tau$ is restricted to satisfy $\ln[\sqrt{\frac{1}{g}}] \leq \tau \leq \infty$. It is essential that the effective field theory be defined for all $\tau$ in the range $-\infty \leq \tau \leq \infty$. It is possible to achieve this by appropriate modification of the previous discussion in regions $III$ and $II$. We find it easier, however, to simply postulate that effective field theory (4.18) is valid for all $\tau$, as it must be and then work our way backward and show that effective field theory reduces to collective fields theory in regions $III$ and $II$.

In region $III$ the extension is straightforward. Since region $III$ is also a high density region where the static solution $\phi_0$ exists, collective field theory Lagrangian (4.2) and (4.3) remain valid in that region. Similarly, the relationship between $x$ and $\tau$ is almost identical to that in region $I$. However, there are two trivial differences. Region $III$ runs from $-\infty \leq x \leq \sqrt{\frac{1}{g}}$. Therefore Eq.(4.4) must be modified in region $III$ to be

$$\tau - \tau_{0}^{III} = -\frac{1}{\pi} \int_{x_0}^{x} \frac{dy}{\phi_0}$$

(4.22)

where $x_0 = -\sqrt{\frac{1}{g}}$. The second difference is that, having chosen $\tau_0^I = \ln[\sqrt{\frac{1}{g}}]$ in region $I$, we are no longer free to choose $\tau_0^{III}$ in region $III$. It is determined by matching conditions with region $II$. Solving Eq.(4.22) for $x$ we find

$$x = -\frac{1}{\sqrt{g}} \cosh(\tau - \tau_{0}^{III})$$

(4.23)

The range of the two coordinates is

$$\infty \leq x \leq -\sqrt{\frac{1}{g}}$$

$$-\infty \leq \tau \leq \tau_{0}^{III}$$

(4.24)

The static solution in the new coordinate becomes

$$\phi_0(\tau) = -\frac{1}{\pi \sqrt{g}} \sinh(\tau - \tau_{0}^{III})$$

(4.25)
Collective field theory Lagrangian (4.8) and (4.9) remain valid in region III, except that \( \phi_0 \) is given by Eq.(4.25). Now assume that effective field theory (4.18) remains valid in region III. It is straightforward to show that

\[
\begin{align*}
< G_{\mu\nu} > &= \eta_{\mu\nu} \\
< D > &= 2 \left( \tau - \left[ \tau_0^{III} - 2 \ln\left(1 - g\right) \right] \right) \\
< \zeta > &= \frac{1}{g}
\end{align*}
\] (4.26)

is an exact solution of the associated equations of motion. Writing \( D = < D > \) and \( \zeta = < \zeta > + \zeta' \), and inserting this into effective field theory (4.18) yields collective field theory Lagrangian (4.8) and (4.9) with \( \phi_0 \) given by Eq.(4.25). Hence effective field theory (4.18) is indeed valid in region III.

Region II is more complicated. The static solution \( \phi_0 \) is not defined there, and hence, Eq.(4.1) must be modified to

\[
\phi = \partial_x \zeta
\] (4.27)

It follows that \( L = L_\zeta + L_0 \), where the \( \zeta \) Lagrangian is

\[
L_\zeta = \int dx \left\{ \frac{\dot{\zeta}^2}{6} (\partial_x \zeta)^3 - \frac{1}{2} \left( \frac{1}{g} - x^2 \right) \partial_x \zeta \right\}
\] (4.28)

and \( L_0 = 0 \). Furthermore, it is impossible to naively extend the relationship between \( \tau \) and \( x \) to region II because then the coordinate \( \tau \) would become imaginary. Following the usual analytic continuation in matrix model we define the relation between \( x \) and \( \tau \) to be what it would have been if \( \phi_0 \) was the analytic continuation of the classical solution, i.e. \( \sqrt{\frac{1}{g} - x^2} \).

The relation between \( x \) and \( \tau \) is therefore

\[
\tau - \tau_0^{II} = - \int_{x_0}^{x} \frac{dy}{\sqrt{\frac{1}{g} - y^2}}
\] (4.29)

where \( x_0 = \sqrt{\frac{1}{g}} \). Again \( \tau_0^{II} \) must be determined by matching conditions with region I. Solving Eq.(4.29) for \( x \) we find

\[
x = \sqrt{\frac{1}{g}} \cos \left( \tau - \tau_0^{II} \right)
\] (4.30)
The range of the two coordinates is

\[-\sqrt{\frac{T}{g}} \leq x \leq \sqrt{\frac{T}{g}}\]

\[\tau_{II}^0 - \pi \leq \tau \leq \tau_{II}^0\]  \hspace{1cm} (4.31)

At this point we can solve for \(\tau_{II}^0\) and \(\tau_{III}^0\). We demand that the three regions of \(\tau\) given in Eqs. (4.6), (4.24) and (4.31) continuously match onto one another at their common boundaries. This implies that

\[\tau_{II}^0 = \ln\left[\frac{\sqrt{T}}{g}\right]\]

\[\tau_{III}^0 = \ln\left[\frac{\sqrt{T}}{g}\right] - \pi\]  \hspace{1cm} (4.32)

The coordinate \(\tau\) is now defined continuously over the whole real axis \(-\infty \leq \tau \leq \infty\) as required.

It is clear however that to any other choice of \(\tau_I^0\) in Eq. (4.4) corresponds another set of matching conditions. The only difference is that regions \(I, II, III\) will be displaced by the appropriate amount. This is a reflection of the spontaneously broken translation invariance of the field theory.

In terms of the coordinate \(\tau\), the Lagrangian (4.28) becomes

\[L_\zeta = \int d\tau \left\{ \frac{1}{\varphi_0} \partial_\tau \zeta \frac{1}{2} \left( \dot{\zeta}^2 - (\partial_\tau \zeta)^2 \right) - \frac{\pi^2}{6\varphi_0^2} (\partial_\tau \zeta)^3 \right\}\]  \hspace{1cm} (4.33)

where

\[\varphi_0(\tau) = \sqrt{\frac{T}{g}} \sin(\tau - \tau_{II}^0)\]  \hspace{1cm} (4.34)

Comparing eq. (4.33) against \(L_\zeta\) in region \(I\) and \(III\), we see that \(L_\zeta\) in region \(II\) is of a completely different structure. This difference can be traced to two sources. The first is that \(\phi_0\) does not exist in region \(II\). The second is that the quantity \(\varphi_0\), which takes the place of \(\phi_0\) in region \(II\), is really an imaginary extension of \(\phi_0\) in the sense that \(\varphi_0 = -i\phi_0\).

We expect, therefore, that we will have trouble extracting the region \(II\) collective field theory from the effective field theory. Let us try anyway. Assume the effective field theory (4.18) remains valid in region \(II\). Then it is straight forward to show that if we expand

\[D = \langle D \rangle\]

\[\zeta = \langle \zeta \rangle + i\sqrt{\pi} \zeta'\]  \hspace{1cm} (4.35)
where
\[
\langle G_{\mu\nu} \rangle = -\eta_{\mu\nu}
\]
\[
\langle D \rangle = 2i(\tau - \tau_0^{II}) - 2\ln[\sqrt{\frac{T}{g}}]
\]
\[
\langle \zeta \rangle = \frac{i}{4g\sqrt{\pi}} \left[ \sin(\tau - \tau_0^{II}) - 2(\tau - \tau_0^{II}) \right]
\]
and go through the same steps as before, we do indeed reproduce collective field theory (4.28).

The role reversal between \( \tau \) and \( t \) expressed in Eq. (4.36) resembles what happens in certain analytically continued coordinate regions of a two dimensional black hole.

The expected trouble presents itself in Eqs. (4.35) and (4.36) in two ways. First, (4.36) is not a solution of the effective field theory equations of motion and, second, \( \langle D \rangle \) and \( \langle \zeta \rangle \) are imaginary. These are a direct consequence of the two sources of trouble mentioned above. In order to cancel the \( \phi_0 \) dependence in the collective field theory, which is implicit in the effective field theory (1.18), it is necessary for \( \langle \zeta \rangle \) to have the \( \tau \) dependence shown in Eq. (4.35). However, these terms do not satisfy the equations of motion. Furthermore, in order to obtain \( \varphi_0 \) instead of \( \phi_0 \), it is necessary for \( \langle \zeta \rangle \) and \( \langle D \rangle \) to be imaginary. We conclude from this that, although it is possible to derive the collective field theory (1.33) from the effective field theory (1.18), the relationship is strictly formal the effective field theory is not an efficient way of describing the dynamics.

The real reason for all these difficulties is that collective field theory in region II is a low density theory of a finite number of eigenvalues. We do not expect such theory to be described efficiently by a continuous effective field theory. If we insist on describing the low density regions in terms of the effective field theory then the scale breaking terms, discussed previously, allow for the peculiar instanton action \( \frac{1}{g} \) (see next section.).

5. STRINGY INSTANTONS

In the previous section we showed that a Lorentz invariant effective field theory can be associated with matrix models in the double scaling limit. In this section, we discuss solutions of this effective field theory, both in Minkowski and Euclidean space-time. We describe instantons, that are associated with tunneling between different solutions and present their effects as new effective terms in \( D, \zeta \) theory.
5.1. Solutions in Minkowski Space-Time

The action for the Minkowski space-time effective field theory is given in Eq. (4.18). We assume that, a priori, this action is valid everywhere in space and time. It is important to note that all interaction terms in (4.18) are proportional to

\[ g(D) = 4\sqrt{\pi e^D \left(1 - \frac{1}{g} e^D\right)^2} \]  

(5.1)

and, therefore \( g(D) \) is the effective coupling parameter of the theory. It is straightforward to derive the equations of motion associated with (4.18). However, these equations are complicated and, for that reason, will not be written down explicitly here. Of more interest is the general solution, given by

\[ \langle G_{\mu\nu} \rangle = \eta_{\mu\nu} \]

\[ \langle D \rangle = a(t - \bar{t}) + b(\tau - \bar{\tau}) \]  

(5.2)

\[ \langle \zeta \rangle = \frac{1}{g} + c \]

where \( a, b, c, \bar{t} \) and \( \bar{\tau} \) are real parameters, \( b^2 - a^2 = 4 \) and \( c, \bar{t}, \bar{\tau} \) are arbitrary.

Of particular interest in this section are the static solutions \( a = 0 \). In this case

\[ \langle G_{\mu\nu} \rangle = \eta_{\mu\nu} \]

\[ \langle D \rangle = \pm 2(\tau - \bar{\tau}) \]  

(5.3)

\[ \langle \zeta \rangle = \frac{1}{g} + c \]

First consider the solution where

\[ \langle D \rangle = -2(\tau - \bar{\tau}) \]  

(5.4)

Then the effective coupling parameter becomes

\[ g_-(\tau - \bar{\tau}) = 4\sqrt{\pi} \frac{e^{-2(\tau - \bar{\tau})}}{\left(1 - \frac{1}{g} e^{-2(\tau - \bar{\tau})}\right)^2} \]  

(5.5)

Note that \( g_- \) is a function of \( \tau - \bar{\tau} \) and, hence, changes in its value for different points in space. Furthermore, \( g_-(\tau - \bar{\tau}) \to \infty \) when \( \tau = \tau_0^I \) where

\[ \tau_0^I = \bar{\tau} + \ln \sqrt{-\frac{T}{g}} \]  

(5.6)
It is not hard to show that for \( \tau < \tau_0 \), the vacuum can be described by a \( < D > = +2(\tau - \bar{\tau}') \) solution where \( \bar{\tau}' = \bar{\tau} + 2 \ln \sqrt{\frac{1}{g}} \). Such solutions will be described later. Therefore we restrict \( \tau \) to satisfy \( \tau \geq \tau_0 \). There is no loss in generality by setting \( \bar{\tau} = 0 \). Then \( \tau_0 = \ln \sqrt{\frac{1}{g}} \) and the effective coupling becomes

\[
g_-(\tau) = 4\sqrt{\pi} \frac{e^{-2\tau}}{\left(1 - \frac{1}{g}e^{-2\tau}\right)^2} \quad (5.7)
\]

We plot this function in Figure 4.

---

Figure 4. Space dependent effective coupling \( g_-(\tau) \).

The physical interpretation of this vacuum state is the following. For spatial points \( \tau >> \ln \sqrt{\frac{1}{g}} \), the effective coupling is small and physics is well described by the effective field theory \( (4.18) \). However, as \( \tau \) approaches \( \ln \sqrt{\frac{1}{g}} \) from the right, the coupling parameter blows up and a region of strong coupling is encountered. Therefore, as \( \tau \to \ln \sqrt{\frac{1}{g}} \) the effective field theory ceases to adequately describe physics. The region to the left of the barrier, \( \tau < \ln \sqrt{\frac{1}{g}} \), is terra incognita. The effective field theory is not valid in this region. Perhaps new, previously unknown dynamics applies there. More of this shortly.

Now consider the solution where

\[
< D > = 2(\tau - \bar{\tau}'')
\quad (5.8)
\]

Then the effective coupling parameter becomes

\[
g_+(\tau - \bar{\tau}'') = 4\sqrt{\pi} \frac{e^{2(\tau - \bar{\tau}'')}}{\left(1 - \frac{1}{g}e^{2(\tau - \bar{\tau}'')}\right)^2} \quad (5.9)
\]
Note that $g_+(\tau - \bar{\tau}''') \to \infty$ when $\tau = \tau^{III}_0$ where

$$\tau^{III}_0 = \bar{\tau}'' - \ln \sqrt{\frac{1}{g}}$$

(5.10)

For $\tau > \tau^{III}_0$, the vacuum can be described by a $< D >= -2(\tau - \bar{\tau}''')$ solution. Since these solutions have been discussed above, we restrict $\tau$ to satisfy $\tau \leq \tau^{III}_0$. For the present, we will set $\bar{\tau}''' = 2\ln \sqrt{\frac{1}{g}} - \pi$. It follows that $\tau^{III}_0 = \ln \sqrt{\frac{1}{g}} - \pi$ and the effective coupling becomes

$$g_+(\tau) = 4\sqrt{\pi} \frac{e^{2(\tau - 2\ln \sqrt{\frac{1}{g}} + \pi)}}{(1 - \frac{1}{g}e^{2(\tau - 2\ln \sqrt{\frac{1}{g}} + \pi)})^2}$$

(5.11)

We plot this function in Figure 5.

Figure 5. Space dependent effective coupling $g_+(\tau)$.

The physical interpretation of this vacuum state is identical to the physical interpretation of the vacuum state above. However in this case, the small coupling region where the effective field theory is valid is $\tau << \ln \sqrt{\frac{1}{g}} - \pi$, whereas the strong coupling region is $\tau \sim \ln \sqrt{\frac{1}{g}} - \pi$. The region to the right of the barrier, $\tau > \ln \sqrt{\frac{1}{g}} - \pi$ is terra incognita. The solutions (5.4) and (5.8) represent two vacuum states of the effective field theory. There is, however, another possible vacuum structure which is the combination of these two solutions. That is, take

$$< D > = -2\tau$$

(5.12)

for $\tau \geq \tau^I_0 = \ln \sqrt{\frac{1}{g}}$, henceforth called region $I$, and

$$< D > = -2(\tau - 2\ln \sqrt{\frac{1}{g}} + \pi)$$

(5.13)
for $\tau \leq \tau_0^{III} = \ln \sqrt{\frac{1}{g}} - \pi$, henceforth called region $III$. The effective coupling parameter in region $I$ is $g_-$ given in Eq.(5.7) and in region $III$ is $g_+$ given in Eq.(5.11). The spatial interval $\ln \sqrt{\frac{1}{g}} - \pi \leq \tau \leq \ln \sqrt{\frac{1}{g}}$ is called region $II$. We plot the effective coupling parameters in Figure 6.

Figure 6. Effective coupling parameter for a combined solution.

The physical interpretation of this vacuum state is the following. In regions $I$ and $III$, away from the boundary points, the effective coupling parameter is small and physics is described by the effective field theory (4.18). As $\tau$ approaches $\ln \sqrt{\frac{1}{g}}$ from the right and as $\tau$ approaches $\ln \sqrt{\frac{1}{g}} - \pi$ from the left, the coupling parameter blows up and a region of strong coupling is encountered. Region $II$ is terra incognita. Perhaps new, previously unknown, dynamics applies there. We note in passing that we have chosen the width of region $II$ to be $\pi$. This choice agrees with our choice of matrix model potential. It is possible, of course, to make the width of region $II$ arbitrary by changing the value of $\bar{\tau}''$ in region $III$. This arbitrariness is discussed in the next section.

The three solutions just discussed are the only possible types of vacua of the effective field theory. Of particular interest is the last solution, shown graphically in Figure 6. If all we knew was the effective field theory, then we would have no interpretation of physics in region $II$. However, we know more than the effective field theory. We know matrix models, and that the effective field theory is the high density limit of a double scaled matrix model. Comparing the vacuum of Figure 6 to the matrix model solution in section 5, we know exactly how to describe physics in region $II$. Physics in that region is not described by effective field theory (4.18), but rather by the low density collective field theory (4.33). The vacuum of this low density theory is the state corresponding to the situation when there are no eigenvalues in the region $\ln \sqrt{\frac{1}{g}} \leq \tau \leq \ln \sqrt{\frac{1}{g}} - \pi$. In this case, the vacuum
solution of the collective field theory in region II is clearly

\[ < \phi > = 0 \]  \hspace{1cm} (5.14)

We want to emphasize that \( \phi = 0 \) is a solution of the true collective field theory equations of motion (2.41) even though it is not a solution of the naive equations of motion (2.42). The solution in eq.(5.14) matches continuously onto the vacua of regions I and III. To see this consider region I and note that the vacuum solution (5.12) and (5.3) in the \( D \) and \( \zeta \) variables is equivalent, using Eq.(4.1), to the collective field theory vacuum

\[ < \phi > = \phi_0(\tau) \]  \hspace{1cm} (5.15)

where \( \phi_0 \) is given in Eq.(4.7). It follows that

\[ \phi_0(\ln \sqrt{\frac{T}{g}}) = 0 \]  \hspace{1cm} (5.16)

which matches Eq.(5.14) continuously at the boundary \( \tau_0^I = \ln \sqrt{\frac{T}{g}} \). Similarly, in region III the \( D, \zeta \) variables vacuum solution (5.13) and (5.3) is equivalent to the collective field theory vacuum

\[ < \phi > = \phi_0(\tau) \]  \hspace{1cm} (5.17)

where \( \phi_0 \) is given by Eq.(4.25). It follows that

\[ \phi_0(\ln \sqrt{\frac{T}{g} - \pi}) = 0 \]  \hspace{1cm} (5.18)

which matches Eq.(5.14) continuously at the boundary \( \tau_0^I = \ln \sqrt{\frac{T}{g} - \pi} \). Furthermore, note that the conjugate momentum \( \Pi_{\phi} \) of the static solution \( \phi_0 \) vanishes in both regions I and III. Similarly, the conjugate momentum vanishes in region II, since there are no eigenvalues there. Hence, the momentum matches continuously across the boundaries.

The complete vacuum solution written in terms of the collective field \( \phi \) for all of \( \tau \) space, is shown in Figure 7.
Recall that the position of this vacuum solution in $\tau$-space was fixed by choosing $\bar{\tau} = 0$, thereby making $\tau^I_0 = \ln \sqrt{\frac{1}{g}}$. Furthermore, the width of region $II$ was chosen to be $\pi$ by letting $\bar{\tau}'' = \ln \sqrt{\frac{1}{g}} - \pi$. Note that, by adjusting $\bar{\tau}''$ to maintain a fixed width $\pi$, the entire vacuum solution can be translated anywhere in $\tau$-space by varying the value of $\bar{\tau}$.

Finally, let us recall the discussion of single eigenvalue solutions of the Minkowski space-time equations of motion in section 3.3. It follows from Eqs. (3.16) and (3.17) that any real solution satisfying the boundary condition $\lambda^*(t_0) = \sqrt{\frac{1}{g}}$, must have initial velocity $\dot{\lambda}(t_0) > -\sqrt{\frac{1}{g}}$ in order to overcome the potential barrier and connect region $I$ with region $III$. In this case, however, the conjugate momentum $\Pi_{\lambda^*} = \dot{\lambda}^*$ is non-vanishing at the boundary. Therefore, such single eigenvalue Minkowski solutions do not continuously connect the zero momentum static solution $\phi_0$ of regions $I$ and $III$.

5.2. Solutions in Euclidean Space-Time

The action for Euclidean space-time effective field theory is easily obtained from Eq.(4.18) by analytic continuation of the time variable $t$ to Euclidean time $\theta$. The exact form of the Euclidean action is not of importance in this section and therefore we do not write it down explicitly. What is important, is that the effective coupling parameter and the static solutions of the Euclidean equation of motion are still given by Eqs.(5.1) and (5.3) respectively. It follows that in Euclidean space, the vacuum structure given in Eqs.(5.12) and (5.13), and pictorially displayed in Figure 6, is also valid. Region $II$ is now described
by the analytic continuation of the low density collective field theory \((4.28)\) to Euclidean space. Unlike the situation in Minkowski space, there is now a non-trivial excitation of one eigenvalue in region \(II\) that connects the vacua of region \(I\) and region \(III\). This single eigenvalue excitation in Euclidean space was constructed in section 3.3, and presented in terms of the collective field theory in eq.(3.23). Rewriting this solution in the \(\tau\) coordinate, we find
\[
\phi_{\text{inst}}(\tau, \theta) = \frac{1}{\sin(\theta - \theta_0)} \delta \left( \tau - \left[ \ln \sqrt{\frac{1}{g}} - (\theta - \theta_0) \right] \right)
\]
(5.19)

This is an instanton which corresponds to the tunneling of a single eigenvalue across the barrier from \(\tau = \ln \sqrt{\frac{1}{g}}\) at \(\theta = \theta_0\) to \(\tau = \ln \sqrt{\frac{1}{g}} - \pi\) at \(\theta = \theta_0 + \pi\). Note that from Eq.(3.21) it follows that the velocity of the eigenvalue at either side of the barrier vanishes. Therefore, the Euclidean conjugate momentum of the instanton in region \(II\), matches continuously at the boundaries with the vanishing conjugate momentum of the static vacua \(\phi_0\) in regions \(I\) and \(III\). We represent this tunneling process in Figure 8.

Figure 8. Instanton connecting regions \(I\) and \(III\).

It is of some interest to reexpress this instanton in terms of the field \(\zeta\) using Eq.(4.27). In terms of \(\zeta\) the instanton is given by
\[
\zeta_{\text{inst}}(\tau, \theta) = \Theta \left( \tau - \left[ \ln \frac{1}{\sqrt{g}} - (\theta - \theta_0) \right] \right)
\]
(5.20)

Note that this solution matches continuously onto the vacuum value of \(\zeta\), \(< \zeta > = \frac{1}{g} + c_I\) and \(\zeta = \frac{1}{g} + c_I - 1\) in regions \(I\) and \(III\) respectively for the appropriate value of \(c_I\).
Therefore, the instanton field configuration is simply a kink moving in Euclidean time. The position of the kink is where the argument of $\Theta$ in Eq.(5.20) vanishes.

We want to stress that this configuration is not a solution of the $D, \zeta$ effective field theory. We have not included explicitly the terms in the field theory that correspond to self energy subtraction (see Eq.(2.31)). However once they are taken into account, the instanton configuration has a finite action $\pi g$, as shown in Eq.(3.24). Finally, we note that the initial tunneling time at which the instanton starts its journey across the barrier, $\theta_0$, is arbitrary.

5.3. Effective Operators

In this section we integrate over the instantons and represent their effects as effective terms in the $D, \zeta$ theory. Since $\zeta$ is the light field we restrict our attention to $\zeta$ operators. The effective operators are especially important. They provide the bridge between the discrete, low density, sector of the theory and the continuous sector. A full analysis is beyond the scope of the present paper. We will content ourselves with a discussion of how the procedure of integrating out instantons is implemented in this particular case and derive the dominant operator induced by instantons.

For a general discussion of the issue of integrating out instantons in field theory see ref.[27].

The integration over instantons is performed in the dilute gas approximation. This approximation is justified for small $g$. The first step in integrating out the instantons was done in the previous section by constructing the single instanton configuration. The second step is to identify on how many parameters the solution depends. The number of parameters is usually the number of broken generators of the full symmetry group of the theory. The parameters become collective coordinates and are integrated over.

The third step is to integrate out the instantons in the semiclassical approximation and present the result as a sum over local operators constructed from $\zeta$ and its derivatives. These are then inserted back into an effective action. Thus the effects of the instantons are included.

We start by identifying the parameters in the stringy instanton. They are:

1. The position of the instanton in Euclidean space $\bar{\tau}, \theta_0$.
2. The orientation of the instanton in Euclidean space $\alpha$.

The parameters $\bar{\tau}$ and $\theta_0$ were defined in Eq.(5.2) and (5.19). The parameter $\alpha$ is related to the parameter $a$ in the Euclidean space continuation of Eq.(5.2), $a = 2 \sin \alpha$. Changing $\alpha$ results in the rotation of the vacuum solution in $\tau - \theta$ space.
There are three zero modes corresponding to the three broken generators of the Euclidean group associated with $\tilde{\tau}, \theta_0, \alpha$. These have to be integrated and produce a volume factor

$$Vol \propto \int d\tilde{\tau} d\theta_0 d\alpha$$  \hspace{1cm} (5.21)$$

When we discussed the various types of vacua, the width of region $II$ was another parameter that appeared to characterize the vacuum solution. However, it does not correspond to a zero mode. In fact it corresponds to the constant mode of the scale factor of the metric $G_{\mu\nu}$, which is a massive mode. The width of region $II$ is arbitrary but fixed. It was chosen to be $\pi$ to agree with previous matrix model and collective field theory calculations.

The instanton solution also depends on $\frac{1}{g}$, the constant mode of the field $\zeta$. It corresponds to the generator of the spontaneously broken shift symmetry $\zeta \rightarrow \zeta + \text{const}$. Although this broken symmetry does have a zero mode associated with it, we do not integrate over it. The reason is that theories with different values of $g$ are really different theories. Their coupling parameters are different, and therefore also physical amplitudes.

We now proceed to the third and final step. The dilute gas summation over instantons induces effective terms in the $D,\zeta$ Lagrangian. The most general action induced by instantons is

$$\Delta S = \int d\tau d\theta \{ \sum_n C_n O_n(\tau, \theta) \}$$ \hspace{1cm} (5.22)$$

where $O_n$ are local operators built from $D$ and $\zeta$ and their derivatives. The coefficients $C_n$ can be computed by expanding the action around the instanton background. We find it convenient here to use the effective field theory and its formal, but accurate, relationship to the collective field theory in region $II$, given by Eq.(4.35) and Eq.(4.36).

Therefore we write

$$\zeta = <\zeta> + i\sqrt{\pi}(\zeta_{\text{inst}} + \zeta')$$ \hspace{1cm} (5.23)$$

Then

$$\int [d\zeta'] e^{-S(\tilde{\zeta}_{\text{inst}} + i\sqrt{\pi}\zeta', <D>)} = \int [d\zeta'] e^{-S_0(\tilde{\zeta}_{\text{inst}}, <D>) + \delta S(\tilde{\zeta}_{\text{inst}}, <D>, \zeta')}$$ \hspace{1cm} (5.24)$$

where $\tilde{\zeta}_{\text{inst}} = <\zeta> + i\sqrt{\pi}\zeta_{\text{inst}}$. Both $<D>$ and $<\zeta>$ are defined in Eq.(4.36), and $\zeta_{\text{inst}}$ is defined in Eq.(5.20). The remaining integrals in Eq.(5.24) are computed in the semiclassical approximation, i.e. expanding $\delta S$ to quadratic terms in $\zeta'$ only and performing the resulting Gaussian integral.
To extract particular $C_n$’s one has to compute appropriate expectation values and compare them to the expectation values in the trivial vacuum. In adapting this formalism to our theory we have to take special care because the instanton is not a solution of the naive $D, \zeta$ equations of motion. All the coefficients $C_n$ are proportional to the universal factor of the exponent of the instanton action and the remaining factor depends on the particular operator that is considered. Since the “size” of the instanton is $\sqrt{g}$ (recall Eq.(3.1)), the dimension of the operator determines the $g$ dependence of $C_n$.

$$C_n = \tilde{C}_n g^{d(n)} e^{-\frac{\pi g}{5}}$$ \hspace{1cm} (5.25)

where

$$d(n) = \text{dimension}(O_n) \frac{1}{2} - 1 \hspace{1cm} (5.26)$$

and $\tilde{C}_n$ is a numerical coefficient. The coefficient $\tilde{C}_n$ is not expected to be particularly large or particularly small. For example, the operator $\nabla \zeta \cdot \nabla \zeta$ has naive mass dimension 2, and therefore it’s coefficient is proportional to $g^0$. The unit operator has dimension 0, and therefore $C_0 \propto \frac{1}{g}$.

We are interested in large $\frac{1}{g}$ that corresponds to small $g$. In that case the dominant and most interesting operator is the unit operator. All other operators are suppressed by powers of $g$. The coefficient of the unit operator is given by

$$C_0 = \tilde{C}_0 \frac{1}{g} e^{-\frac{\pi g}{5}}$$ \hspace{1cm} (5.27)

The numerical coefficient $\tilde{C}_0$ is given by

$$\tilde{C}_0 = \lim_{g \to 0} g \int [d\zeta'] e^{\delta S_2(\zeta_{\text{inst}}, D_{\text{inst}}, \zeta')} \int [d\zeta'] e^{-S_2(\zeta')}$$ \hspace{1cm} (5.28)

where $\delta S_2$ and $S_2$ are the quadratic actions around the instanton and trivial vacuum respectively.

The result in Eq.(5.27) was obtained in the background of a constant field $< \zeta > = \frac{1}{g}$. Lorentz invariance then dictates that at least for slowly varying fields the effective operator depends on the full field $\zeta$ and not just its constant mode $\frac{1}{g}$. Therefore the final result for the induced operator is

$$\Delta L_0 = \tilde{C}_0 \zeta e^{-\pi \zeta}$$ \hspace{1cm} (5.29)

This operator breaks the $\zeta$ shift symmetry. It induces a runaway non-perturbative potential for the field $\zeta$. 

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6. STRING THEORY

In the previous sections we discussed the space-time effective theory associated with matrix models and collective field theory. We now review the connection between the world sheet description of 1+1 dimensional string theory and matrix models and collective field theory. We compare solutions of our effective space-time theory to solutions of the \( \beta \)-function equations. We also discuss the stringy instantons of subsection (5.2) from the world sheet point of view.

6.1. String Theory and Liouville Theory

The class of 1 + 1 dimensional string theories that we are interested in is described by the following two dimensional \( \sigma \)-model \[28],[29],

\[
I = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{g}} \left\{ \hat{g}^{\alpha\beta} G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + \hat{R} D(X) + 2T(X) \right\}
\]

(6.1)

where \( \hat{g}_{\alpha\beta} \) is the fixed world sheet metric with Euclidean signature and \( \hat{R} \) is the corresponding Ricci scalar. The sigma model field \( X_\mu \) stands for two scalar fields, \( X_0(z) \), and \( X_1(z) \). The field \( G_{\mu\nu}(X) \) is the target space metric, assumed here to have Euclidean signature, \( D(X) \) is the dilaton, and \( T(X) \) is the tachyon. The names of the fields are a little bit misleading. They originate from the form of the world sheet couplings of these fields. In 26 dimensional Minkowski space critical string theories, the tachyon is indeed tachyonic and the metric and dilaton correspond to massless fields. However, as we see shortly, in 1 + 1 dimensions, the physical tachyon is really massless and the metric and dilaton correspond to massive non-propagating fields.

Consistent string backgrounds are described by conformal field theories. The conditions for conformal invariance are determined in general by the equations \( \beta = 0 \), where \( \beta \) is the beta-function of the theory. Of course, the \( \beta \)-function equations can only be computed in some perturbative scheme. The lowest order equations for the theory described by Eq. (6.1) are

\[
R_{\mu\nu} + 2\nabla_\mu \nabla_\nu D = 0
\]

\[
-\frac{1}{2} \nabla^2 D + (\nabla D)^2 + 4 = 0
\]

\[
-\nabla^2 T + 2\nabla D \cdot \nabla T - 4T = 0
\]

(6.2)

We can compare Eqs. (6.2) to the lowest order equations of motion derived from (4.18). From this comparison we deduce that, to this order, the field \( D \) appearing in (4.18) is the
same as the dilaton $D$ in (6.1) and that the field $\zeta$ in (4.18) is related to the tachyon as follows

$$\zeta \propto Te^{-D} \quad (6.3)$$

This result is rather remarkable. It says that the equations of motion of the background fields for the class of string theories specified by the action (6.1) are, to lowest order, identical to the equations of motion of the effective field theory extension of matrix models. Can we extend this identification beyond lowest order? To do this, let us consider the solutions of the equations of motion (6.2). These solutions can be classified into two families. The first family consists of solutions with constant metric $G_{\mu\nu}$, and therefore vanishing curvature. The second family consists of curved space solutions. We do not discuss the second family of curved space solutions in this paper.

The flat space solutions are

$$G_{\mu\nu} = \delta_{\mu\nu},$$

$$D = a(X_0 - \bar{X}_0) + b(X_1 - \bar{X}_1),$$

$$T = me^D \quad (6.4)$$

where $a^2 + b^2 = 4$. Of particular interest is the static solution

$$G_{\mu\nu} = \delta_{\mu\nu},$$

$$D = -2X_1,$$

$$T = me^{-2X_1} \quad (6.5)$$

By substituting the static solution of Eq. (6.5) into the sigma model (6.1) and writing $X_1 = \varphi$, the following world sheet conformal field theory is obtained

$$I = \frac{1}{4\pi} \int d^2z \sqrt{g} \left\{ \hat{g}^{\alpha\beta} \partial_\alpha X_0 \partial_\beta X_0 + \hat{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - 2\hat{R}_\varphi + 2me^{-2\varphi} \right\} \quad (6.6)$$

which can be identified as the Liouville conformal field theory with $c_m = 1$ matter. To obtain a Minkowski signature string theory one has to analytically continue $X_0 \to it(z)$, which then becomes the time variable of target space. The field $\varphi$ corresponds to the spatial dimension of target space.

Liouville theory (6.6) was compared to matrix models and to collective field theory by a number of authors. Specifically, these authors studied the complete theory, going beyond the lowest order. The conclusion is that they describe of the same theory. Evidence to
this effect was obtained on many levels e.g., see [4],[10], and [30]. For a review and more comprehensive list of references see ref.[9]. In particular, the relation between $\zeta$ and $T$ in Eq.(6.3) as well as the linear relation between $\varphi$ and $\tau$ have been well documented.

More important, from our point of view, is that, by computing scattering amplitudes of fluctuations around the Liouville vacuum, one can determine the equations of the original string backgrounds $D$ and $T$ beyond the lowest order. When these are compared to the full equations of motion derived from Eq.(4.18), one finds that they are identical, as long as the two $D$ fields are identified, relation (6.3) holds and the parameter $m$ in (6.3) is chosen to be

$$m = \frac{1}{g}$$

(6.7)

We conclude, therefore, that

The string theories associated with the world sheet action (6.1) have the same equations of motion and effective action for their background fields as does

the effective field theory for the matrix model given in Eq.(4.18).

Furthermore, even the low density region of the matrix model discussed earlier is expected to describe physical aspects of these string theories, such as their non-perturbative behaviour. It follows that the discussion of vacua, single eigenvalue instantons and induced operators given in the previous section is, in fact, applicable to the string theories associated with (6.1).

6.2. Minkowski and Euclidean Space-Time Backgrounds

Based on our discussion of solutions of space-time effective theory of section 5 we can now discuss general flat space solutions of the non-linear extension of Eqs.(6.2). Since they are the same equations they have the same solutions as explained before. Each solution corresponds to a different conformal field theory. The world sheet description of the different possible conformal backgrounds were discussed in ref.[31]. The different backgrounds fall into three classes. All of them can be obtained from the conformal field theory (6.6) by a change of coordinates. The first class corresponds to the coordinate change

$$X'_0 = \gamma(X_0 - v\varphi)$$

$$\varphi' = \gamma(\varphi - vt)$$

(6.8)
where $\gamma = \frac{1}{\sqrt{1-v^2}}$. This change of coordinates transforms (6.6) into a theory with $c_m < 1$. The second class corresponds to the coordinate change as well as the analytic continuation $X_0 \to it$

$$t' = \gamma (t + v \varphi)$$

$$\varphi' = \gamma (\varphi - vt) \quad (6.9)$$

This change of coordinates transforms (6.6) into a theory with $1 < c_m < 25$. The third class of conformal field theories is obtained by interchanging the role of $\varphi$ and $X_0$ in (6.6). As we can see from the factor $\gamma$ in transformation rules (6.8),(6.9), Lorentz invariance creeps into there. It is impossible to understand that from the world sheet point of view. However, from the point of view of effective field theory this is just a reflection of it’s Lorentz invariance. Every class of theories corresponds to a different solution for the dilaton field. The first class and half of the third class corresponds to Euclidean solutions as described in section (5.2), the second class and the remaining half of the third class corresponds to Minkowski solutions described in section (5.1).

The coupling parameter of the string theory defined by the $\sigma$-model (6.1) is $e^D$, and therefore, for the static solution (5.3) is position dependent.

$$g_{st}(\varphi) = e^{-2\varphi} \quad (6.10)$$

In general $g_{st}$ is position and time dependent. Note that the string coupling parameter in Eq.(5.10) and effective field theory coupling parameter in Eq.(5.5) are different. In fact, the string coupling parameter remains finite for finite spatial coordinate, while the effective field theory coupling parameters blows up at the boundary of region $I$.

In most string theories discussed so far, it was believed that the string coupling parameter and the coupling parameter in the effective field theory are the same. There is a simple world sheet argument, based on the universal form of dilaton interactions, that actually “proves” that. However, in this particular example it fails! Another parameter $g$ appears, and enters into the expression for the coupling parameter.

The stringy instantons of section (5.3) can now be partially described in world sheet language. The two high density regions $I$ and $III$ can each be described by a separate Liouville theory. The boundary of each region where the effective coupling parameter becomes strong is known as the “Liouville wall”. Region $II$ in between region $I$ and $III$ cannot be described in terms of the world sheet theory. It would correspond to an infinitely strong coupling parameter. The signal for the break down of the world sheet description
in this case is the “wall”. The only hint that one can obtain from the world sheet theory, as to what happens behind the wall, is given by the growth of large order amplitudes of tachyons. Inside region II, behind the “Liouville wall”, new dynamics unaccessible to the Polyakov description of string theory takes over. It is described by a version of string theory capable of describing discrete physics, the matrix model and collective field theory. It allows for quantum mechanical amplitudes for tunneling processes where part of space enters the wall and emerges on the other side into a copy of the world that it left.

7. CONCLUSION and OUTLOOK

In this section we list some of the directions in which the results obtained in this paper can be extended.

An obvious extension, which we do not foresee any problems in performing and is currently under investigation [32], is to consider the supersymmetric extension of collective field theory and matrix models. The supersymmetric collective field theory is discussed in ref. [33] and the supersymmetric eigenvalue theory is discussed in ref. [15], where it is shown to be equivalent to the Marinari-Parisi model. In ref. [34] all these theories are shown to be equivalent to each other. Therefore there is a unique supersymmetric extension of matrix models and collective field theory.

This supersymmetric theory does have one eigenvalue instantons. Their action is again \( \frac{1}{g} \). In ref. [15] it is shown that these instantons break supersymmetry with the expected strength of \( e^{-\frac{1}{g}} \). It should be possible to write down a supersymmetric space-time effective action analogous to Eq.(4.18), identify it’s classical Euclidean solutions and obtain the corresponding stringy instantons as well as the effective operators that they induce. We expect that these operators break supersymmetry non-preturbatively.

The most interesting and crucial step in finding out whether or not our stringy instantons, or some of their higher dimensional relatives, play an important role in string theory is to find out their effects in 4-dimensional string theories. Since matrix models lose most of their power in 4 dimensions, it is unlikely, but not impossible, that a direct application of the same techniques would be useful to study that question. We can however look for space-time solutions that have the same general features of the solutions described in section 5. The most important of those features is that the space (or space-time) dependent effective coupling parameter is blowing up at a finite point in space.
There is in fact a class of 4-dimensional string solutions that have this property. They are described in refs. [35]–[37]. The main features of these cosmological solutions is that they possess a space dependent effective coupling parameter that becomes infinite at a finite space-time point. The supersymmetric counterparts of these solutions were constructed in [37].

It is tempting to conjecture that stringy instantons similar to our stringy instantons connect different regions of space-time and that they induce non-perturbative operators of the type discussed in section 5. In that case these operators are expected to be proportional to the universal factor

\[ e^{-\sqrt{S}} \]

Here \( S \) is a complex field that naturally appears in the effective low energy supergravity field theory obtained from superstring theory. The dilaton is related to the real part of \( S \).

\[ < \text{Re}S > \sim \frac{1}{g^2} \]

Note that the non-perturbative effects considered previously in the literature induced operators that were proportional to the universal factor

\[ e^{-S} \]

Since the coupling parameter \( g \) is expected to be small, the difference between these two universal factors is quite big. This may have important phenomenological consequences.

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References

[1] D. J. Gross, N. Miljkovic, Phys.Lett.B238 (1990) 217;
P. Ginsparg and J. Zinn-Justin, Phys. Lett. B240 (1990) 333;
E. Brezin, V. Kazakov and Al. B. Zamolodchikov, Nucl. Phys. B338 (1990) 673.
[2] S.R. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639.
[3] J. Polchinski, Nucl.Phys. B346 (1990) 253.
[4] D. J. Gross and I.R. Klebanov, Nucl.Phys. B352 (1991) 671.
[5] M. Sengupta and S. R. Wadia, Int.J.Mod.Phys. A6 (1991) 1961.
[6] A. Dhar, G. Mandal and S. R. Wadia, IAS Princeton preprint, IASSNS-HEP-91-89
(1992).
[7] J. Polchinski, Nucl.Phys. B362 (1991) 125.
[8] K. Demeterfi, A. Jevicki and J. P. Rodrigues, Nucl.Phys. B362 (1991) 173;
K. Demeterfi, A. Jevicki and J P. Rodrigues, Nucl.Phys. B365 (1991) 499.
[9] K. Demeterfi, A. Jevicki and J. P. Rodrigues, Mod. Phys. Lett. A6 (1991) 3199.
[10] D. J. Gross and I. R. Klebanov, Nucl. Phys. B359 (1991) 3.
[11] G. Moore, Nucl.Phys. B368 (1992) 557.
[12] S. Shenker, Cargese Workshop on Random Surfaces, Quantum Gravity and Strings,
Cargese, France (1990).
[13] P. Ginsparg and J. Zinn-Justin, Phys. Lett. B255 (1991) 189.
[14] O. Lechtenfeld, Int. J. Mod.Phys. A7 (1992).
[15] A. Dabholkar, Nucl.Phys. B368 (1992) 293.
[16] J. D. Cohn and H. Dykstra, Mod.Phys.Lett. A7 (1992) 1163.
[17] R. Brustein and B. Ovrut, Penn. preprint, UPR-522T (1992).
[18] B. Sakita, Quantum Theory of Many-Variable Systems and Fields, World Scientific,
Singapore 1985.
[19] A. Jevicki and B. Sakita, Nucl. Phys. B165 (1980) 511.
[20] J. D. Cohn and S.P. De Alwis, Nucl.Phys.B368 (1992) 79.
[21] D. Karabali and B. Sakita, Int. J. Mod. Phys. A6 (1991) 5079.
[22] A. Jevicki, Nucl.Phys. B376 (1992) 75.
[23] R. Brustein and S. De Alwis, Proceedings of the PASCOS-91 Int. Symp., Boston 1991,
P. Nath and S. Reucroft Ed., World Scientific.
[24] E. Brezin and J. Zinn-Justin, Ecole Normale preprint, LPTENS-92-19, 1992.
[25] G. Parisi, Phys. Lett. B76 (1978) 65.
[26] R. Brustein and S. De Alwis, Phys. Lett. B272 (1992) 285.
[27] M. A. Shifman, A. I. Vainshtein and V. I. Zhakharov, Nucl. phys. B165 (1980) 45.
[28] S. Elitzur, A. Forge and E. Rabinovici, Nucl. Phys. B359 (1991) 581.
[29] A. Tseytlin, Phys. Lett. B264 (1991) 311.
[30] P. Di Francesco and D. Kutasov, Nucl. Phys. B375 (1992), 119.
[31] D. Minic, J. Polchinski and Z. Yang, Nucl.Phys. B369 (1992) 324.
[32] R. Brustein, M. Faux and B. Ovrut, Penn Preprint, To Appear.
[33] A. Jevicki and J. P. Rodrigues, Phys. Lett. B268 (1991) 53.
[34] J. P. Rodrigues and J. van Tonder, Witwatersrand preprint, CNLS-92-02 (1992).
[35] R. Myers, Phys. Lett. B199 (1987) 371.
[36] I. Antoniadis, C. Bachas, J. Ellis and D. Nanopoulos, Phys. Lett. B211 (1988) 393.
[37] S. De Alwis, J. Polchinski and R. Schimmrigk, Phys. Lett. B218 (1988) 449.