Local Lie algebra determines base manifold *

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Abstract

It is proven that a local Lie algebra in the sense of A. A. Kirillov determines the base manifold up to a diffeomorphism provided the anchor map is nowhere-vanishing. In particular, the Lie algebras of nowhere-vanishing Poisson or Jacobi brackets determine manifolds. This result has been proven for different types of differentiability: smooth, real-analytic, and holomorphic.

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Dedicated to Hideki Omori

1 Introduction

The classical result of Shanks and Pursell [PS] states that the Lie algebra \( \mathfrak{X}_c(M) \) of all compactly supported smooth vector fields on a smooth manifold \( M \) determines the manifold \( M \), i.e., the Lie algebras \( \mathfrak{X}_c(M_1) \) and \( \mathfrak{X}_c(M_2) \) are isomorphic if and only if \( M_1 \) and \( M_2 \) are diffeomorphic. A similar theorem holds for other complete and transitive Lie algebras of vector fields [KMO1, KMO2] and for the Lie algebras of all differential and pseudodifferential operators [DS, GP].

There is a huge list of papers in which special geometric situations (hamiltonian, contact, group invariant, foliation preserving, etc., vector fields) are concerned. Let us mention the results of Omori [O1] (Ch. X) and [O2] (Ch. XII), or [Am, AG, FT, HM, Ry, G5], for which specific tools were developed in each case. There is however a case when the answer is more or less complete in the whole generality. These are the Lie algebras of vector fields which are modules over the corresponding rings of functions (we shall call them modular). The standard model of a modular Lie algebra of vector fields is the Lie algebra \( \mathfrak{X}(\mathcal{F}) \) of all vector fields tangent to a given (generalized) foliation \( \mathcal{F} \). If Pursell-Shanks-type results are concerned in this context, let us recall the work of Amemiya [Am] and our paper [G1], where the developed algebraic approach made it possible to consider analytic cases as well. The method of Shanks and Pursell consists of the description of maximal ideals in the Lie algebra \( \mathfrak{X}_c(M) \) in terms of the points of \( M \): maximal ideals are of the form \( \tilde{p} \) for \( p \in M \), where \( \tilde{p} \) consists of vector fields which are flat at \( p \). This method, however, fails in analytic cases, since analytic vector fields flat at \( p \) are zero on the corresponding component of \( M \). Therefore in [Am, G1] maximal finite-codimensional subalgebras are used instead of ideals. A similar approach is used in [GG] for proving that the Lie algebras associated with Lie algebroids determine base manifolds.

The whole story for modular Lie algebras of vector fields has been in a sense finished by the brilliant purely algebraic result of Skryabin [SI], where one associates the associative algebra of functions with the Lie algebra of vector fields without any description of the points of the manifold as ideals. This final result implies in particular that, in the case when modular Lie algebras of vector fields contain finite families of vector fields with no common zeros (we say that they are strongly non-singular),

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isomorphisms between them are generated by isomorphisms of corresponding algebras of functions, i.e., by diffeomorphisms of underlying manifolds.

On the other hand, there are many geometrically interesting Lie algebras of vector fields which are not modular, e.g. the Lie algebras of hamiltonian vector fields on a Poisson manifold etc. For such algebras the situation is much more complicated and no analog of Skryabin method is known in these cases. In [GG] a Pursell-Shanks-type result for the Lie algebras associated with Jacobi structures on a manifold has been announced. The result suggests that the concept of a Jacobi structure should be developed for sections of an arbitrary line bundle rather than for the algebra of functions, i.e., sections of the trivial line bundle. This is exactly the concept of local Lie algebra in the sense of A. A. Kirillov [K], which we will call also Jacobi-Kirillov bundle.

In the present note we complete the Lie algebroid result of [GG] by proving that the local Lie algebra determines the base manifold up to a diffeomorphism if only the anchor map is nowhere-vanishing (Theorem 7). The methods, however, are more complicated (due to the fact that the Lie algebra of Jacobi-hamiltonian vector fields is not modular) and different from those in [GG]. A part of these methods is a modification of what has been sketched in [G6]. However, the full generalization of [GG] for local Lie algebras on arbitrary line bundles, i.e., the description of isomorphisms of local Lie algebras, is much more delicate and we postpone it to a separate paper. Note also that in our approach we admit different categories of differentiability: smooth, real-analytic, and holomorphic (Stein manifolds).

2 Jacobi modules

What we will call Jacobi module is an algebraic counterpart of geometric structures which include Lie algebroids and Jacobi structures (or, more generally, local Lie algebras in the sense of Kirillov [Ki]). For a short survey one can see [G7], where these geometric structures appeared under the name of Lie QD-algebroids.

The concept of a Lie algebroid (or its pure algebraic counterpart – a Lie pseudoalgebra) is one of the most natural concepts in geometry.

Definition 1. Let $\mathcal{E}$ be a commutative and unitary ring, and let $\mathcal{A}$ be a commutative $R$-algebra. A Lie pseudoalgebra over $R$ and $\mathcal{A}$ is an $\mathcal{A}$-module $\mathcal{E}$ together with a bracket $[\cdot, \cdot] : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ on the module $\mathcal{E}$, and an $\mathcal{A}$-module morphism $\alpha : \mathcal{E} \to \text{Der}(\mathcal{A})$ from $\mathcal{E}$ to the $\mathcal{A}$-module $\text{Der}(\mathcal{A})$ of derivations of $\mathcal{A}$, called the anchor of $\mathcal{E}$, such that

(i) the bracket on $\mathcal{E}$ is $R$-bilinear, alternating, and satisfies the Jacobi identity:

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]].$$

(ii) For all $X, Y \in \mathcal{E}$ and all $f \in \mathcal{A}$ we have

$$[X, fY] = f[X, Y] + \alpha(X)(f)Y;$$

(iii) $\alpha([X, Y]) = [\alpha(X), \alpha(Y)]_c$ for all $X, Y \in \mathcal{E}$, where $[\cdot, \cdot]_c$ is the commutator bracket on $\text{Der}(\mathcal{A})$.

A Lie algebroid on a vector bundle $E$ over a base manifold $M$ is a Lie pseudoalgebra on the $(\mathbb{R}, C^\infty(M))$-module $\mathcal{E} = \text{Sec}(E)$ of smooth sections of $E$. Here the anchor map is described by a vector bundle morphism $\alpha : E \to TM$ which induces the bracket homomorphism from $(\mathcal{E}, [\cdot, \cdot])$ into the Lie algebra $(\mathcal{X}(M), [\cdot, \cdot]_{\mathcal{E}})$ of vector fields on $M$. In this case, as in the case of any faithful $\mathcal{A}$-module $\mathcal{E}$, i.e., when $fX = 0$ for all $X \in \mathcal{E}$ implies $f = 0$, the axiom (iii) is a consequence of (i) and (ii). Of course, we can consider Lie algebroids in the real-analytic or holomorphic (on complex holomorphic bundles over Stein manifolds) category as well.

Lie pseudoalgebras appeared first in a paper by Herz [He], but one can find similar concepts under more than a dozen of names in the literature (e.g. Lie modules, $(R, A)$-Lie algebras, Lie-Cartan pairs, Lie-Rinehart algebras, differential algebras, etc.). Lie algebroids were introduced by Pradines [Pr] as infinitesimal parts of differentiable groupoids. In the same year a book by Nelson [Ne] was published,
where a general theory of Lie modules together with a big part of the corresponding differential calculus can be found. We also refer to a survey article by Mackenzie [Ma2].

Note that Lie algebroids on a singleton base space are Lie algebras. Another canonical example is the tangent bundle \( TM \) with the canonical bracket \([\cdot, \cdot]_{\text{ef}}\) on the space \( \mathcal{X}(M) = \text{Sec}(TM) \) of vector fields.

The property \([\cdot, \cdot]_1\) of the bracket in the \( \mathcal{A}\)-module \( \mathcal{E}\) can be expressed as the fact that \( \text{ad}_X = [X, \cdot] \) is a quasi-derivation in \( \mathcal{E}\), i.e., an \( R\)-linear operator \( D \) in \( \mathcal{E}\) such that \( D(fY) = fD(Y) + \hat{D}(f)Y \) for any \( f \in \mathcal{A}\) and certain derivation \( \hat{D} \) of \( \mathcal{A}\) called the anchor of \( D\). The concept of quasi-derivation can be traced back to N. Jacobson [J1, J2] as a special case of his pseudo-linear endomorphism. It has appeared also in [Ne] under the name of a module derivation and used to define linear connections in the algebraic setting. In the geometric setting, for Lie algebroids, it has been studied in [Ma1], Ch. III, under the name covariant differential operator.

Starting with the notion of Lie pseudoalgebra we obtain the notion of Jacobi module when we drop the assumption that the anchor map is \( \mathcal{A}\)-linear.

**Definition 2.** Let \( R\) be a commutative and unitary ring, and let \( \mathcal{A}\) be a commutative \( R\)-algebra. A Jacobi module over \((R, \mathcal{A})\) is an \( \mathcal{A}\)-module \( \mathcal{E}\) together with a bracket \([\cdot, \cdot] : \mathcal{E} \times \mathcal{E} \to \mathcal{E}\) on the module \( \mathcal{E}\) and an \( R\)-module morphism \( \alpha : \mathcal{E} \to \text{Der}(\mathcal{A})\) from \( \mathcal{E}\) to the \( \mathcal{A}\)-module \( \text{Der}(\mathcal{A})\) of derivations of \( \mathcal{A}\), called the anchor of \( \mathcal{E}\), such that (i)-(iii) of Definition 1 are satisfied. Again, for faithful \( \mathcal{E}\), the axiom (iii) follows from (i) and (ii). This concept is in a sense already present in [He], although in [He] it has been assumed that \( \mathcal{A}\) is a field. It has been observed in [He] that every Jacobi module (over a field) of dimension \( > 1\) is just a Lie pseudoalgebra.

**Definition 3.** (cf. [G4]) A Lie QD-algebroid is a Jacobi module structure on the \((\mathbb{R}, C^\infty(M))\)-module \( \mathcal{E} = \text{Sec}(E)\) of sections of a vector bundle \( E\) over a manifold \( M\).

The case \( \text{rank}(E) = 1\) is special by many reasons and it was originally studied by A. A. Kirillov [K3]. For a trivial bundle, well-known examples are those given by Poisson or, more generally, Jacobi brackets (cf. [J3]). In [K3] such structures on line bundles are called local Lie algebras and in [Mr] – Jacobi bundles. We will refer to them also as to local Lie algebras or Jacobi-Kirillov bundles and to the corresponding brackets as to Jacobi-Kirillov brackets.

**Definition 4.** A Jacobi-Kirillov bundle (local Lie algebra in the sense of Kirillov) is a Lie QD-algebroid on a vector bundle of rank 1. In other words, a Jacobi-Kirillov bundle is a Jacobi module structure on the \((\mathbb{R}, C^\infty(M))\)-module \( \mathcal{E}\) of sections of a line bundle \( E\) over a smooth manifold \( M\). The corresponding bracket on \( \mathcal{E}\) we call Jacobi-Kirillov bracket and the values of the anchor map \( \alpha : \mathcal{E} \to \mathcal{X}(M)\) we call Jacobi-hamiltonian vector fields.

It is easy to see (cf. [G2]) that any Lie QD-algebroid on a vector bundle of rank \( > 1\) must be a Lie algebroid. Of course, we can consider Lie QD-algebroids and Jacobi-Kirillov bundles in real-analytic or in holomorphic category as well.

Since quasi-derivations are particular first-order differential operators in the algebraic sense, it is easy to see that, for a Jacobi module \( \mathcal{E}\) over \((R, \mathcal{A})\), the anchor map \( \alpha : \mathcal{E} \to \text{Der}(\mathcal{A})\) is also a first-order differential operator, i.e.,

\[
\alpha(fgX) = f\alpha(gX) + g\alpha(fX) - f\alpha(X) \tag{2}
\]

for any \( f, g \in \mathcal{A}\) and \( X \in \mathcal{E}\). Denoting the Jacobi-hamiltonian vector field \( \alpha(X) \) by \( \hat{X}\), we can write for any \( f, g \in \mathcal{A}\) and \( X, Y \in \mathcal{E}\),

\[
[gX, fY] = g\hat{X}(f)Y - f\hat{Y}(g)X + fg[X,Y] \tag{3}
\]

so that for the map \( \Lambda_X : \mathcal{A} \times \mathcal{A} \to \mathcal{A}\) defined by \( \Lambda_X(g, f) := g\hat{X}(f) - f\hat{X}(g)\) we have

\[
\Lambda_X(g, f)Y = -\Lambda_Y(f, g)X. \tag{4}
\]
The above identity implies clearly that, roughly speaking, \( \text{rank}_A \mathcal{E} = 1 \) ‘at points where \( \Lambda \) is non-vanishing’ (cf. \[G7\]) and that
\[
\Lambda_X (g,f) X = -\Lambda_X (f,g) X. \tag{5}
\]
The identity \[3\] does not contain much information about \( \Lambda_X \) if there is ‘much torsion’ in the module \( \mathcal{E} \). But if, for example, there is a torsion-free element in \( \mathcal{E} \), say \( X_0 \), (this is the case of the module of sections of a vector bundle), then the situation is simpler. In view of \[4\], \( \Lambda_{X_0} \) is skew-symmetric and, in turn, by \[4\] every \( \Lambda_X \) is skew-symmetric. Every \( \Lambda_X \) is by definition a derivation with respect to the second argument, so, being skew-symmetric, it is a derivation also with respect to the first argument. Since in view of \[4\],
\[
[gX, fX] = \left(g \widehat{X}(f) - f \cdot \widehat{X}(g) + \Lambda_X (f,g)\right) X,
\]
and since \( \Lambda_X \) and \( \widehat{X} \) respect the annihilator \( \text{Ann}(X) = \{ f \in A : fX = 0 \} \), we get easily the following.

**Proposition 1** If \( \mathcal{E} \) is a Jacobi module over \((R, A)\), then, for every \( X \in \mathcal{E} \), the map \( \Lambda_X : A \times A \to A \) induces a skew-symmetric bi-derivation of \( \mathcal{A}/\text{Ann}(X) \) and the bracket
\[
\{f, g\}_X = \Lambda_X (f,g) + f \cdot \widehat{X}(g) - g \cdot \widehat{X}(f), \tag{6}
\]
where \( \mathcal{f} \) denotes the class of \( f \in A \) in \( A/\text{Ann}(X) \), is a Jacobi bracket on \( \mathcal{A}/\text{Ann}(X) \) associated with the Jacobi structure \((\Lambda_X, \widehat{X})\). Moreover, \( \mathcal{A}/\text{Ann}(X) \ni \mathcal{f} \mapsto fX \in \mathcal{E} \) is a Lie algebra homomorphism of the bracket \( \{\cdot, \cdot\}_X \) into \([\cdot, \cdot]\).

For pure algebraic approaches to Jacobi brackets we refer to \[S2, S3, G4\].

**Corollary 1** If the \( A \)-module \( \mathcal{E} \) is generated by torsion-free elements, then for every \( X \in \mathcal{E} \), the map \( \Lambda_X : A \times A \to A \) is a skew-symmetric bi-derivation and the bracket
\[
\{f, g\}_X = \Lambda_X (f,g) + f \cdot \widehat{X}(g) - g \cdot \widehat{X}(f), \tag{7}
\]
is a Jacobi bracket on \( A \) associated with the Jacobi structure \((\Lambda_X, \widehat{X})\). Moreover, \( A \ni f \mapsto fX \in \mathcal{E} \) is a Lie algebra homomorphism of the bracket \( \{\cdot, \cdot\}_X \) into \([\cdot, \cdot]\).

For any torsion-free generated Jacobi module, e.g. a module of sections of a vector bundle, we have additional identities as shows the following.

**Proposition 2** If the \( A \)-module \( \mathcal{E} \) is generated by torsion-free elements, then for all \( f_1, \ldots, f_m \in A \), \( m \geq 2 \), and all \( X, Y \in \mathcal{E} \),
\begin{align*}
(a) \quad (m-1)[FX, Y] = & \sum_{i=1}^{m} [F_iX, f_iY] - [X, FY] \\
(b) \quad (m-2)[FX, Y] = & \sum_{i=1}^{m-1} [F_iX, f_iY] + [F_mY, f_mX],
\end{align*}
where \( F = \prod_{i=1}^{m} f_i \), \( F_k = \prod_{i \neq k} f_i \).

**Proof.** (a) We have (cf. \[9\])
\[
\sum_{i=1}^{m} [F_iX, f_iY] = \sum_{i=1}^{m} \left( F_i \widehat{X}(f_i) Y - f_i \widehat{Y}(F_i) X + F[X, Y] + \Lambda_X (F_i, f_i) Y \right) = \widehat{X}(F) Y - (m-1) \widehat{Y}(F) X + mF[X, Y] + \sum_{i \neq j} F_{ij} \Lambda_X (f_j, f_i) Y = [X, FY] + (m-1)[FX, Y] + \sum_{i \neq j} F_{ij} \Lambda_X (f_j, f_i) Y,
\]
where $F_{ij} = \prod_{k \neq i, j} f_k$. The calculations are based on the Leibniz rule for derivations:

$$\hat{X} \left( \prod_{i=1}^{m} f_i \right) = \sum_{i=1}^{m} F_i \hat{X}(f_i),$$

etc. Since, due to Corollary 1, $\Lambda_X$ is skew-symmetric and $F_{ij} = F_{ji}$, we have

$$\sum_{i \neq j} F_{ij} \Lambda_X(f_j, f_i)Y = 0$$

and (a) follows.

(b) In view of (a), we have

$$(m - 2)[F_X, Y] = \sum_{i=1}^{m} [F_i X, f_i Y] - [X, F_X] - [F_X, Y]$$

$$= \sum_{i=1}^{m-1} [F_i X, f_i Y] + [F_m X, f_m Y] - [X, F_m f_m Y] - [F_m f_m X, Y].$$

But

$$[F_m X, f_m Y] - [X, F_m f_m Y] - [F_m f_m X, Y] = [F_m Y, f_m X]$$

is a particular case of (a).

\[\square\]

### 3 Useful facts about associative algebras

In what follows, $\mathcal{A}$ will be an associative commutative unital algebra over a field $\mathbb{K}$ of characteristic 0. Our standard model will be the algebra $\mathcal{C}(N)$ of class $\mathcal{C}$ functions on a manifold $N$ of class $\mathcal{C}$, $\mathcal{C} = C^\infty, C^\omega, \mathcal{H}$. Here $C^\infty$ refers to the smooth category with $\mathbb{K} = \mathbb{R}$, $C^\omega$ – to the $\mathbb{R}$-analytic category with $\mathbb{K} = \mathbb{R}$, and $\mathcal{H}$ – to the holomorphic category of Stein manifolds with $\mathbb{K} = \mathbb{C}$ (cf. [G1, AG]). All manifolds are assumed to be paracompact and second countable. It is obvious what is meant by a Lie QD-algebroid or a Jacobi-Kirillov bundle of class $\mathcal{C}$. The rings of germs of class $\mathcal{C}$ functions at a given point are noetherian in analytic cases that is no longer true in the $C^\infty$ case. However, all the algebras $\mathcal{C}(N)$ are in a sense noetherian in finite codimension. To explain this, let us start with the following well-known observation.

**Theorem 1** Every maximal finite-codimensional ideal of $\mathcal{C}(N)$ is of the form $\mathfrak{p} = \{ f \in \mathcal{C}(N) : f(p) = 0 \}$ for a unique $p \in N$ and $\mathfrak{p}$ is finitely generated.

**Proof.** The form of such ideals is proven e.g. in [G1, Proposition 3.5]. In view of embedding theorems for all types of manifolds we consider, there is an embedding $f = (f_1, \ldots, f_n) : N \to \mathbb{K}^n$, $f_i \in \mathcal{C}(N)$. Then, the ideal $\mathfrak{p}$ is generated by $\{ f_i - f_i(p) \cdot 1 : i = 1, \ldots, n \}$. In the smooth case it is obvious, in the analytic cases it can be proven by means of some coherent analytic sheaves and methods parallel to those in [G2, Note 2.3].

\[\square\]

**Remark.** Note that in the case of a non-compact $N$ there are maximal ideals of $\mathcal{C}(N)$ which are not of the form $\mathfrak{p}$. They are of course of infinite codimension. It is not known if the above theorem holds also for manifolds which are not second countable (cf. [GS]).

For a subset $B \subset \mathcal{A}$, by $\text{Sp}(\mathcal{A}, B)$ we denote the set of those maximal finite-codimensional ideals of $\mathcal{A}$ which contain $B$. For example, $\text{Sp}(\mathcal{A}, \{ 0 \})$ is just the set of all maximal finite-codimensional ideals which we denote shortly by $\text{Sp}(\mathcal{A})$. Put $\overline{B} = \bigcap_{I \in \text{Sp}(\mathcal{A}, B)} I$. For an ideal $I \subset \mathcal{A}$, by $\sqrt{I}$ we denote the radical of $I$, i.e.,

$$\sqrt{I} = \{ f \in \mathcal{A} : f^n \in I, \text{ for some } n = 1, 2, \ldots \}.$$ 

The following easy observations will be used in the sequel.
Theorem 2  (a) If $I$ is an ideal of codimension $k$ in $\mathcal{A}$, then $\sqrt{T} = T$ and $(\overline{T})^k \subset I$.

(b) Every finite-codimensional prime ideal in $\mathcal{A}$ is maximal.

(c) If a derivation $D \in \text{Der}(\mathcal{A})$ preserves a finite-codimensional ideal $I$ in $\mathcal{A}$, then $X(\mathcal{A}) \subset \overline{T}$.

(d) If $I_1, \ldots, I_n$ are finite-codimensional and finitely generated ideals of $\mathcal{A}$, then the ideal $I_1 \cdot \ldots \cdot I_n$ is finite-codimensional and finitely generated.

Proof.- (a) The descending series of ideals

$$ I + \overline{T} \supset I + (\overline{T})^2 \supset \ldots $$

stabilizes at $k$th step at most, so $I + (\overline{T})^k = I + (\overline{T})^{k+1}$. Applying the Nakayama’s Lemma to the finite-dimensional module $(I + (\overline{T})^k)/I$ over the algebra $A/I$, we get $(I + (\overline{T})^k)/I = \{0\}$, i.e., $(\overline{T})^k \subset I$, thus $\overline{T} \subset \sqrt{I}$. Since for all $J \in \text{Sp}(A, I)$ we have $\sqrt{J} \subset \sqrt{I} = J$, also $\overline{J} \supset \sqrt{I}$.

(b) If $I$ is prime and finite-codimensional, $\sqrt{I} = I$ and $\sqrt{\overline{J}} = \overline{\sqrt{J}}$ by (a). But a finite intersection of maximal ideals is prime only if they coincide, so $\overline{J} = J$ for a single $J \in \text{Sp}(A, I)$.

(c) By Lemma 4.2 of [G1], $D(I) \subset I$ for a finite-codimensional ideal $I$ implies $D(A) \subset J$ for each $J \in \text{Sp}(A, I)$.

(d) It suffices to prove (d) for $n = 2$ and to use the induction. Suppose that $I_1, I_2$ are finite-codimensional and finitely generated by $\{u_i\}$ and $\{v_j\}$, respectively. It is easy to see that $I_1 \cdot I_2$ is generated by $\{u_i \cdot v_j\}$ and that $I_1 \cdot I_2$ is finite-codimensional in $I_1$. Indeed, if $c_1, \ldots, c_k \in \mathcal{A}$ represent a basis of $A/I_2$, then $\{c_i u_i\}$ represent a basis of $I_1/(I_1 \cdot I_2)$.

\[\square\]

Theorem 3  For an associative commutative unital algebra $\mathcal{A}$ the following are equivalent:

(a) Every finite-codimensional ideal of $\mathcal{A}$ is finitely generated.

(b) Every maximal finite-codimensional ideal of $\mathcal{A}$ is finitely generated.

(c) Every prime finite-codimensional ideal of $\mathcal{A}$ is finitely generated.

Proof.- (a) $\Rightarrow$ (b) is trivial, (b) $\Rightarrow$ (c) follows from Theorem 2(b), and (c) $\Rightarrow$ (a) is a version of Cohen’s Theorem for finite-dimensional ideals.

\[\square\]

Definition 5. We call an associative commutative unital algebra $\mathcal{A}$ noetherian in finite codimension if one of the above (a), (b), (c), thus all, is satisfied.

An immediate consequence of Theorem 4 is the following.

Theorem 4  The algebra $\mathcal{A} = \mathcal{C}(N)$ is noetherian in finite codimension.

4 Spectra of Jacobi modules

Let us fix a Jacobi module $(\mathcal{E}, [\cdot, \cdot])$ over $(\mathbb{K}, \mathcal{A})$. Throughout this section we will assume that $\mathcal{E}$ is finitely generated by torsion-free elements and that $\mathcal{A}$ is a noetherian algebra in finite codimension over a field $\mathbb{K}$ of characteristic 0. The $(\mathbb{K}, \mathcal{C}(N))$-modules of sections of class $\mathcal{C}$ vector bundles over $N$ can serve as standard examples.

For $L \subset \mathcal{E}$, by $\hat{L}$ denote the image of $L$ under the anchor map: $\hat{L} = \{\alpha(X) : X \in L\} \subset \text{Der}(\mathcal{A})$. The set $\hat{\mathcal{E}}$ is a Lie subalgebra in $\text{Der}(\mathcal{A})$ with the commutator bracket $[\cdot, \cdot]$ and we will refer to $\hat{\mathcal{E}}$ as to the Lie algebra of ‘Jacobi-hamiltonian vector fields’. The main difference with the ‘modular’ case (in particular, with that of Pursell and Shanks [PS]) is that $\hat{\mathcal{E}}$ is no longer, in general, an $\mathcal{A}$-module, so we cannot multiply by ‘functions’ inside $\hat{\mathcal{E}}$. However, we can still try to translate some properties of the Lie algebra $(\mathcal{E}, [\cdot, \cdot])$ into the properties of the Lie algebra $\hat{\mathcal{E}}$ of Jacobi-hamiltonian vector fields by
Applying the identity since a symplectic Poisson bracket on a compact manifold, since the Lemma 1 therein in no longer true for Jacobi modules. In fact, as easily shows the example of anchor map does not vanish. Note that the method developed in [GG] for Lie pseudoalgebras fails, in means of the anchor map and to describe some ‘Lie objects’ in \( E \) or \( \hat{E} \) by means of ‘associative objects’ in \( \mathcal{A} \).

The spectrum of the Jacobi module \( E \), denoted by \( \text{Sp}(E) \), will be the set of such maximal finite-codimensional Lie subalgebras in \( E \) that not contain finite-codimensional Lie ideals of \( E \). In nice geometric situations, \( \text{Sp}(E) \) will be interpreted as a set of points of the base manifold at which the anchor map does not vanish. Note that the method developed in [GG] for Lie pseudoalgebras fails, since the Lemma 1 therein in no longer true for Jacobi modules. In fact, as easily shows the example of anchor map does not vanish. Note that the method developed in [GG] for Lie pseudoalgebras fails, in means of the anchor map and to describe some ‘Lie objects’ in \( E \) or \( \hat{E} \) by means of ‘associative objects’ in \( \mathcal{A} \).

Let us fix some notation. For a linear subspace \( L \) in \( E \) and for \( J \subset \mathcal{A} \), denote

- \( N_L = \{ X \in E : [X, L] \subset L \} \) - the Lie normalizer of \( L \);
- \( U_L = \{ X \in E : [X, E] \subset L \} \);
- \( I(L) = \{ f \in \mathcal{A} : \forall X \in E \ [fX \in L] \} \) - the largest associative ideal \( I \) in \( \mathcal{A} \) such that \( IE \subset L \);
- \( E_J = \{ X \in E : \mathcal{X}(A) \subset J \} \).

It is an easy exercice to prove the following proposition (cf. [GG], Theorem 1.6).

**Proposition 3** If \( L \) is a Lie subalgebra in \( E \), then \( N_L \) is a Lie subalgebra containing \( L \), the set \( U_L \) is a Lie ideal in \( N_L \), and \( N_L(I(U_L)) \subset I(U_L) \).

Choose now generators \( X_1, \ldots, X_n \) of \( E \) over \( \mathcal{A} \). For a fixed finite-codimensional Lie subalgebra \( L \) in \( E \) put \( U_i = \{ f \in \mathcal{A} : fX_i \in U_L \} \) and \( U = \cap_{i=1}^n U_i \). Since \( U_L \) is clearly finite-codimensional in \( E \), all \( U_i \) are finite-codimensional in \( \mathcal{A} \), so is \( U \).

**Lemma 1**

(a) \( [U^m X_j, X_k] \subset L \) for all \( j, k = 1, \ldots, n \) and \( m \geq 3 \).

(b) \( [U^m X_j, U^l X_k] \subset L \) for all \( j, k = 1, \ldots, n \) and \( m, l \geq 1 \).

**Proof.** (a) Take \( f_1, \ldots, f_m \in U \). Since \( f_i X_k \in U_L \), Proposition 2(b) implies \( [f_1 \cdots f_m X_j, X_k] \in L \).

(b) The inclusion is trivial for \( l = 1 \), so suppose \( l \geq 2 \). Take \( f_1, \ldots, f_m \in U \), \( f_{m+1} \in U^l \) and put \( F = f_1 \cdots f_{m+1}, F_i = \prod_{r \neq i} f_r \). By Proposition 2(b)

\[
[f_1 \cdots f_m X_k, f_{m+1} X_j] = (m-1) [FX_j, X_k] - \sum_{i=1}^m [F_i X_j, f_i X_k].
\]

Since \( F \in U^{m+l} \), according to (a), \( [FX_j, X_k] \in L \) and \( [F_i X_j, f_i X_k] \subset [E, U_L] \subset L \), so the lemma follows.

**Theorem 5** The ideal \( I(U_L) \) is finite-codimensional in \( \mathcal{A} \) provided \( L \) is a finite-codimensional Lie subalgebra in \( E \).

**Proof.** Let \( U \) be the associative subalgebra in \( \mathcal{A} \) generated by \( U \). It is finite-codimensional and, in view of Lemma 1(b), \( [UX_j, UX_k] \subset L \). Being finite-codimensional in \( \mathcal{A} \), the associative subalgebra \( U \) contains a finite-codimensional ideal \( J \) of \( \mathcal{A} \) (cf. [GG], Proposition 2.1b)). Hence \( [JX_j, JX_k] \subset L \) and, since \( X_i \) are generators of \( E \), \( [JX_j, JX_k] \subset L \). Note that we do not exclude the extremal case \( U = \mathcal{A} = J \).

Applying the identity

\[
[f_1 f_2 X, Y] = [f_2 X, f_1 Y] + [f_1 X, f_2 Y] - [f_1 f_2 Y, X]
\]
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for $f_1, f_2 \in J$, $X \in U_L$, we see that $J^2 U_L \subset U_L$. In particular, $J^2 U_X \subset U_L$ for all $i = 1, \ldots, n$, so $J^2 U \subset U$ and hence $J^2 U \subset U$ and $J^3 \subset U_L$. Consequently $J^3 \subset I(U_L)$. Since $J$ is finite-codimensional and finitely generated, $J^3$ is finite-codimensional (Theorem 5(d)), so $I(U_L)$ is finite-codimensional.

Denote $\text{Sp}_E(A)$ the set of these maximal finite-codimensional ideals $I \subset A$ which do not contain $\tilde{E}(A)$, i.e., $E \not\subset \tilde{E}$. Geometrically, $\text{Sp}_E(A)$ can be interpreted as the support of the anchor map. Recall that $\text{Sp}(E)$ is the set of these maximal finite-codimensional Lie subalgebras in $E$ which do not contain finite-codimensional Lie ideals.

**Theorem 6** The map $J \mapsto E_J$ constitutes a bijection of $\text{Sp}_E(A)$ with $\text{Sp}(E)$. The inverse map is $L \mapsto \sqrt{I(L)}$.

**Proof.** Let us take $J \in \text{Sp}_E(A)$. In view of (a), $J^2 E \subset E_J$ which implies that $E_J$ is finite-codimensional, as $J^2$ is finite-codimensional and $E$ is finitely generated.

We will show that $E_J$ is maximal. Of course, $E_J \not\subset E$ and $E_J$ is of finite codimension, so there is a maximal Lie subalgebra $L$ containing $E_J$. We have

$$J^2 E \subset E_J \subset L \implies J^2 \subset I(L) \implies J \subset \sqrt{I(L)} \implies J = \sqrt{I(L)}.$$

Moreover, $I(L)$ is finite-codimensional, and since, due to (b), $\sqrt{I(L)} \subset I(L)$, then, by Theorem (c), $\tilde{L}(A) \subset J$, i.e., $L \subset E_J$ and finally $L = E_J$.

Finally, suppose $P$ is a finite-codimensional Lie ideal of $E$ contained in $E_J$. Then $U_P$ is a Lie ideal in $E$ of finite codimension and, according to Theorem (a), $I(U_P)$ is a finite-codimensional ideal in $A$. Since $\tilde{E}(I(U_P)) \subset I(U_P)$, and since $I(U_P) \subset I(U_L) \subset J$, we have $\tilde{E}(A) \subset J$, i.e., $E = E_J$; a contradiction.

Suppose now that $L \in \text{Sp}(E)$. Observe first that $N_L = L$, since otherwise $L$ would be a Lie ideal, that would, in turn, imply $U_L \subset L$ and $I(U_L) \subset I(L)$. Since $U_L$ is finite-codimensional, Theorem 5 shows that $I(L)$ is finite-codimensional. Exactly as above we show that $\tilde{L}(A) \subset \sqrt{I(L)}$, i.e., $L \subset E_J$, where $J = \sqrt{I(L)}$. By Theorem (a), $J^k \subset I(L)$ for some $k$, so if we had $E_J \not\subset E$, then $J^k : E$ would be a finite-codimensional Lie ideal contained in $L$. Thus $E_J \not\subset E$ and there is $I \in \text{Sp}(A, J)$ with $E_J \not\subset E$.

We know already that in this case $E_i$ is maximal. Since $L \subset E_J \subset E_i$ and $L$ is maximal, we have $L = E_i$ and $I = J = \sqrt{I(L)}$.

**Corollary 2** Let $(E, [\cdot, \cdot])$ be a Lie QD-algebroid of class $C$ (i.e., a Jacobi module over $(K, C(N))$ of class $C$ sections of a class $C$ vector bundle) over a class $C$ manifold $N$. Let $S \subset N$ be the open support of the anchor map, i.e., $S = \{p \in N : \hat{X}(p) \neq 0 \text{ for some } X \in E\}$. Then the map $p \mapsto p^* = \{X \in E : \hat{X}(p) = 0\}$ constitutes a bijection of $S$ with $\text{Sp}(E)$.

Let $\hat{E}$ be the image of the anchor map $\alpha : E \to \text{Der}(A)$. By definition of a Jacobi module, $\hat{E}$ is a Lie subalgebra in $(\text{Der}(A), [\cdot, \cdot])$. Since $\alpha : E \to \hat{E}$ is a surjective Lie algebra homomorphism, it induces a bijection of $\text{Sp}(E)$ onto $\text{Sp}(\hat{E})$, $L \mapsto \hat{L} = \alpha(L)$. Thus we get the following.

**Corollary 3** Let $(E, [\cdot, \cdot])$ be a Lie QD-algebroid of class $C$ over a class $C$ manifold $N$. Let $S \subset N$ be the open support of the anchor map, i.e., $S = \{p \in N : \hat{X}(p) \neq 0 \text{ for some } X \in E\}$. Then the map $p \mapsto \hat{p} = \{\xi \in \hat{E} : \xi(p) = 0\}$ constitutes a bijection of $S$ with $\text{Sp}(\hat{E})$.

## 5 Isomorphisms

It is clear that any isomorphism $\Psi : E_1 \to E_2$ of the Lie algebras associated with Jacobi modules $E_i$ over $(R_i, A_i)$, $i = 1, 2$, induces a bijection $\psi : \text{Sp}(E_2) \to \text{Sp}(E_1)$. Since the kernels $K_i$ of the anchor maps $\alpha_i : E_i \to \hat{E}_i$ are the intersections

$$K_i = \bigcap_{L \in \text{Sp}(E_i)} L, \quad i = 1, 2,$$
Proposition 4 If the Lie algebras $(\mathcal{E}_i, [\cdot, \cdot])$, associated with Jacobi modules $\mathcal{E}_i$, $i = 1, 2$, are isomorphic, then the Lie algebras of Jacobi-hamiltonian vector fields $\hat{\mathcal{E}}_i$, $i = 1, 2$, are isomorphic.

The following theorem describes isomorphisms of the Lie algebras of Jacobi-hamiltonian vector fields.

Theorem 7 Let $(\mathcal{E}_i, [\cdot, \cdot])$ be a Lie QD-algebroid of class $\mathcal{C}$, over a class $\mathcal{C}$ manifold $N_i$, and let $\mathcal{S}_i \subset N_i$ be the (open) support of the anchor map $\alpha_i : \mathcal{E}_i \to \mathcal{E}_i$, $i = 1, 2$. Then every isomorphism of the Lie algebras of Jacobi-hamiltonian vector fields $\Phi : \hat{\mathcal{E}}_1 \to \hat{\mathcal{E}}_2$ is of the form $\Phi(\xi) = \varphi_*(\xi)$ for a class $\mathcal{C}$ diffeomorphism $\varphi : S_1 \to S_2$.

Corollary 4 If the Lie algebras associated with Lie QD-algebroids $E_i$ of class $\mathcal{C}$, over class $\mathcal{C}$ manifolds $N_i$, $i = 1, 2$, are isomorphic, then the (open) supports $\mathcal{S}_i \subset N_i$ of the anchor maps $\alpha_i : \mathcal{E}_i \to \mathcal{E}_i$, $i = 1, 2$, are $\mathcal{C}$-diffeomorphic. In particular, $N_1$ and $N_2$ are $\mathcal{C}$-diffeomorphic if the anchors are nowhere-vanishing.

Proof of Theorem 7. According to Corollary 3 the isomorphism $\Phi$ induces a bijection $\varphi : S_1 \to S_2$ such that, for every $\xi \in \hat{\mathcal{E}}_1$ and every $p \in S_1$,

$$\xi(p) = 0 \iff \Phi(\xi)(\varphi(p)) = 0. \quad (9)$$

First, we will show that $\varphi$ is a diffeomorphism of class $\mathcal{C}$. For, let $f \in \mathcal{C}(N_1)$. Since the anchor map is a first-order differential operator, for every $X \in \mathcal{E}_1$ we have $\hat{f}^2 X = 2 f \cdot \hat{f} X - f^2 \cdot \hat{X}$. In particular, for any $p \in N_1$,

$$\hat{f}^2 X(p) - 2 f(p) \hat{f} X(p) + f^2(p) \hat{X}(p) = 0,$$

so that, due to (9),

$$\Phi(\hat{f}^2 X)(\varphi(p)) = 2 f(p) \Phi(\hat{f} X)(\varphi(p)) - f^2(p) \Phi(\hat{X})(\varphi(p)). \quad (10)$$

We can rewrite (10) in the form

$$\Phi(\hat{f}^2 X) = 2(f \circ \psi) \cdot \Phi(\hat{f} X) - (f \circ \psi)^2 \cdot \Phi(\hat{X}), \quad (11)$$

where $\psi = \varphi^{-1}$ and the both sides of (11) are viewed as vector fields on $S_2$. In a similar way one can get

$$\Phi(\hat{f}^3 X) = 3(f \circ \psi)^2 \cdot \Phi(\hat{f} X) - 2(f \circ \psi)^3 \cdot \Phi(\hat{X}). \quad (12)$$

To show that $f \circ \psi$ is of class $\mathcal{C}$, choose $q \in S_2$ and $X \in \mathcal{E}_1$ such that $\Phi(\hat{X})(q) \neq 0$. Then we can choose local coordinates $(x_1, \ldots, x_n)$ around $q$ such that $\Phi(\hat{X}) = \partial_1 = \frac{\partial}{\partial x_1}$. If $a$ is the first coefficient of the vector field $\Phi(\hat{f} X)$ in these coordinates, we get out of (11) and (12) that $(f \circ \psi)^2 - 2a(f \circ \psi)$ and $2(f \circ \psi)^3 - 3a(f \circ \psi)^2$ are of class $\mathcal{C}$ in a neighbourhood of $q$. But

$$(f \circ \psi)^2 - 2a(f \circ \psi) = (f \circ \psi - a)^2 - a^2 \quad (13)$$

and

$$2(f \circ \psi)^3 - 3a(f \circ \psi)^2 = 2(f \circ \psi - a)^3 + 3a(f \circ \psi - a)^2 - a^2, \quad (14)$$

so $(f \circ \psi - a)^2$ and $(f \circ \psi - a)^3$ are functions of class $\mathcal{C}$ in a neighbourhood of $q$, as the function $a$ is of class $\mathcal{C}$. Now we will use the following lemma which proves that $f \circ \psi - a$, thus $f \circ \psi$, is of class $\mathcal{C}$.
Lemma 2 If \( g \) is a \( K \)-valued function in a neighbourhood of \( 0 \in K^n \) such that \( g^2 \) and \( g^3 \) are of class \( C \), then \( g \) is of class \( C \).

Proof. In the analytic cases the lemma is almost obvious, since \( g = \frac{a^3}{a^2} \) is a meromorphic and continuous function. In the smooth case the Lemma is non-trivial and proven in [Jo].

To finish the proof of the theorem, we observe that \( f \circ \psi \) is of class \( C \) for all \( f \in C(N_2) \) implies that \( \psi \), thus \( \varphi = \psi^{-1} \), is of class \( C \) and we show that \( \Phi = \varphi \) or, in other words, that \( \hat{Y}(f \circ \psi) = \Phi(\hat{Y})(f \circ \psi) \) for all \( f \in C(N_1) \) and all \( Y \in \mathcal{E}_1 \). Indeed, for arbitrary \( f \in C(N_1) \) and \( X, Y \in \mathcal{E}_1 \), the bracket of vector fields \([\hat{Y}, f^2 \hat{X}]\) reads

\[
[\hat{Y}, f^2 \hat{X}] = [\hat{Y}, 2f \cdot \hat{f} \hat{X} - f^2 \cdot \hat{X}]
= 2\hat{Y}(f) \cdot \hat{f} \hat{X} - 2f \cdot \hat{Y}(f) \cdot \hat{X} + 2f[\hat{Y}, \hat{f} \hat{X}] - f^2[\hat{Y}, \hat{X}].
\]

Hence, similarly as in (11),

\[
\Phi([\hat{Y}, f^2 \hat{X}]) = 2(\hat{Y}(f) \circ \psi) \cdot \Phi(\hat{f} \hat{X}) - 2(f \circ \psi) \cdot (\hat{Y}(f) \circ \psi) \cdot \Phi(\hat{X}) + 2(f \circ \psi) \cdot \Phi([\hat{Y}, \hat{f} \hat{X}]) - (f \circ \psi)^2 \cdot \Phi([\hat{Y}, \hat{X}]).
\]

Comparing the above with

\[
[\Phi(\hat{Y}), \Phi(f^2 \hat{X})] = [\Phi(\hat{Y}), 2(f \circ \psi) \cdot \Phi(\hat{f} \hat{X}) - (f \circ \psi)^2 \cdot \Phi(\hat{X})],
\]

we get easily

\[
\left( \hat{Y}(f) \circ \psi - \Phi(\hat{Y})(f \circ \psi) \right) \left( \Phi(\hat{f} \hat{X}) - (f \circ \psi) \cdot \Phi(\hat{X}) \right) = 0.
\]

(15)

After polarizing with \( f = f + h \) and multiplying both sides by \( \hat{Y}(f) \circ \psi - \Phi(\hat{Y})(f \circ \psi) \), we get the identity

\[
\left( \hat{Y}(f) \circ \psi - \Phi(\hat{Y})(f \circ \psi) \right)^2 \left( \Phi(\hat{h} \hat{X}) - (h \circ \psi) \cdot \Phi(\hat{X}) \right) = 0,
\]

valid for all \( f, h \in C(N_1) \) and all \( X, Y \in \mathcal{E}_1 \). From (16) we get

\[
(\hat{Y}(f) \circ \psi)(q) = (\Phi(\hat{Y})(f \circ \psi))(q)
\]

for such \( q = \varphi(p) \in S_2 \) for which in no neighbourhood of them the anchor map is a differential operator of order 0, i.e., for \( q \) which do not belong to

\[
S_2^0 = \{ \varphi(p) \in S_2 : \hat{h} \hat{X}(p') = h(p') \hat{X}(p') \text{ for all } h \in C(N_1), \ X \in \mathcal{E}_1 \text{ and } p' \text{ close to } p \}.
\]

If, on the other hand, \( q \in S_2^0 \), then \( \Phi(\hat{h} \hat{X})(q') = (h \circ \psi)(q') \cdot \Phi(\hat{X})(q') \) for \( q' \) from a neighbourhood of \( q \), so that comparing in this neighbourhood

\[
\Phi([\hat{Y}, \hat{f} \hat{X}]) = (\hat{Y}(f) \circ \psi) \cdot \Phi(\hat{X}) + (f \circ \psi) \cdot \Phi([\hat{Y}, \hat{X}])
\]

we get

\[
(\hat{Y}(f) \circ \psi)(q) \cdot \Phi(\hat{X})(q) = \Phi(\hat{Y})(f \circ \psi)(q) \cdot \Phi(\hat{X})(q),
\]

thus

\[
(\hat{Y}(f) \circ \psi)(q) = \Phi(\hat{Y})(f \circ \psi)(q)
\]

also for \( q \in S_2^0 \). □

Remark.
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(a) For Jacobi-Kirillov bundles with all leaves of the characteristic foliation (i.e., orbits of $\tilde{E}$) of dimension $> 1$ there is much simpler argument showing that $\psi$ is smooth than the one using Lemma 2. The difficulty in the general case comes from singularities of the ‘bivector field’ part of the anchor map and forced us to use Lemma 2.

(b) Theorem 7 has been proven for Lie algebroids in [GG], so the new (and difficult) here is the case of Jacobi-Kirillov bundles with non-trivial ‘bivector part’ of the bracket. A similar result for the Lie algebras of smooth vector fields preserving a symplectic or a contact form up to a multiplicative factor has been proven by H. Omori [O1]. These Lie algebras are the Lie algebras of locally hamiltonian vector fields for the Jacobi-Kirillov brackets associated with the symplectic and the contact form, respectively.

Corollary 5

(a) If the Lie algebras $(\mathcal{C}(N_i), \{\cdot, \cdot\}_{\beta_i})$ of the Jacobi contact brackets, associated with contact manifolds $(N_i, \beta_i)$, $i = 1, 2$, of class $\mathcal{C}$, are isomorphic, then the manifolds $N_1$ and $N_2$ are $C$-diffeomorphic.

(b) If the Lie algebras associated with nowhere-vanishing Poisson structures of class $\mathcal{C}$ on class $\mathcal{C}$ manifolds $N_i$, $i = 1, 2$, are isomorphic, then the manifolds $N_1$ and $N_2$ are $C$-diffeomorphic.

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