Decomposition of Complete Tripartite Graphs Into 5-Cycles

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Abstract

A result of D. Sotteau on the necessary and sufficient conditions for decomposing the complete bipartite graphs into even cycles has been shown in many occasions, that it is a very important tool in the theory of graph decomposition into even cycles. In order to have similar tools in the case of odd cycle decomposition, obviously bipartite graphs are not suitable to be considered. Searching for such tools, we have considered decomposition of complete tripartite graphs, $K_{r,s,t}$, into 5-cycles. There are some necessary conditions that we have shown their sufficiency in the case of $r = t$, and some other cases. Our conjecture is that these conditions are always sufficient.

1 Introduction

A theorem of D. Sotteau [2] states that,

Necessary and sufficient conditions in order that $K_{m,n}$ admits a decomposition into $2k$-cycles are:

(i) $m \geq k$, $n \geq k$;
(ii) $m$ and $n$ both are even;
(iii) $2k | mn$.

This theorem is a very useful tool in recursive construction in cycle decomposition problems. But it may be used only when cycles are of even length. So if we want to

∗Research of this author is partially supported by Research Council of Sharif University of Technology.
work with odd cycles we should look at other alternatives, say complete tripartite graphs, \( K_{r,s,t} \).

Decomposition of \( K_{r,s,t} \) into triangles is easy:

**Proposition 1.** Necessary and sufficient conditions for \( K_{r,s,t} \) to be decomposed into 3-cycles is that \( r = s = t \).

**Proof.** Necessity of the conditions is trivial and to show the sufficiency one may use a Latin square of size \( r \).

Next case will be 5-cycle decomposition.

**Example 1.** The complete tripartite graph \( K_{4,2,2} \) may be decomposed into 5-cycles. Suppose the sets \( A = \{ a_1, a_2, a_3, a_4 \}, \ B = \{ b_1, b_2 \}, \) and \( C = \{ c_1, c_2 \} \) are three parts of this graph. Then a decomposition may be done as follows:

\( (b_1, a_1, c_2, a_3, c_1), (b_2, a_1, c_1, a_4, c_2), (c_2, a_2, b_2, a_3, b_1), \) and \( (c_1, a_2, b_1, a_4, b_2) \).

## 2 Necessary Conditions

To decompose \( K_{r,s,t} \) into 5-cycles some immediate necessary conditions follow.

**Theorem 1.** Let \( r \leq s \leq t \). If there exists a decomposition of \( K_{r,s,t} \) into 5-cycles, then the following conditions are satisfied:

(i) \( r, s, \) and \( t \) all are even or all are odd;

(ii) \( 5 \mid rs + rt + st \);

(iii) \( t \leq 4rs/(r + s) \).

**Proof.** Each cycle uses two edges from each vertex, thus the degree of each vertex must be even. In other words \( r + s, r + t, \) and \( s + t \) all are even. Thus (i) follows. To see (ii), note that \( rs + rt + st \) is the total number of edges in \( K_{r,s,t} \).

For (iii), we note that each 5-cycle uses at least one edge and at most three edges from between any two parts of \( K_{r,s,t} \). This results in six inequalities:

\[
\begin{align*}
2st/3(s + t) & \leq r \leq 4st/(s + t), \\
2rt/3(r + t) & \leq s \leq 4rt/(r + t), \\
2rs/3(r + s) & \leq t \leq 4rs/(r + s).
\end{align*}
\]
Since $r \leq s \leq t$, the inequality $t \leq 4rs/(r + s)$ implies other ones. 

**Conjecture.** Three conditions given in Theorem 1 are also sufficient.

We have proved this conjecture in the case when two parts have equal number of vertices and satisfy the necessary conditions with one exception of $K_{5x,5x,z}$ ($z$ not multiple of 5). Also we have some results for the remaining cases. For some small cases it has been shown to be true.

### 3 An Application

Before going further we show that how the decomposition of $K_{r,s,t}$ can be a tool for 5-cycle decompositions. In 1966 Alex Rosa [1] proved that,

A complete graph on $n$ vertices, $K_n$, admits a 5-cycle decomposition if and only if $n \equiv 1$ or $5 \pmod{10}$.

We may prove the sufficiency of conditions in Rosa’s theorem by applying the decomposition of $K_{r,s,t}$ into 5-cycles. (Necessity of the conditions is trivial as usual).

For $n$ we have two cases:

(i) $n = 10l + 1$,

(ii) $n = 10l + 5$.

(i) In this case we let $r = 10l' + 5$ and $s = 10l'' + 1$ where:\n\n\[ l' = \lfloor l/3 \rfloor \quad \text{and} \quad l'' = l - 2 \lfloor l/3 \rfloor - 1. \]

Then $n = r + r + s$. Now we proceed by induction, by decomposing $K_r$, $K_t$, $K_s$, and $K_{r,r,s}$ into 5-cycles and “patching” them together to obtain a decomposition for $K_n$.

(ii) In this case we let $r = 10l' + 5$ and $s = 10l'' + 5$, where $l'$ and $l''$ are as above.

And by a similar method we may decompose $K_n$.

### 4 Sufficiency of Conditions

In this section we prove some useful theorems and some results on the sufficiency of conditions. First an extension theorem:
**Theorem 2.** If $K_{r,s,t}$ admits a 5-cycle decomposition, then so does $K_{ar,as,at}$ for each positive integer $a$.

**Proof.** We apply Proposition 1 to the decomposition of $K_{a,a,a}$ into triangles. Indeed, each part in $K_{ar,as,at}$ may be partitioned into $a$ classes, each with equal number of vertices. Then each class of vertices in each part may be considered as a vertex. This will give a $K_{a,a,a}$ which, in turn, can be decomposed into triangles. Each of these triangles represents a $K_{r,s,t}$.

**Corollary 1.** $K_{r,r,r}$ admits a 5-cycle decomposition if and only if $5 \mid r$.

**Proof.** Necessary conditions of Section 2, in this case, imply $5 \mid r$. To show sufficiency, according to Theorem 2, we just need a decomposition of $K_{5,5,5}$. Suppose that the sets $A = \{a_1, a_2, a_3, a_4, a_5\}$, $B = \{b_1, b_2, b_3, b_4, b_5\}$, $C = \{c_1, c_2, c_3, c_4, c_5\}$ are three parts of this graph. Then a decomposition may be done with the base cycles as follows:

$$(a_2, b_5, c_1, b_1, c_5), (b_2, c_5, a_1, c_1, a_5), \text{ and } (c_2, a_5, b_1, a_1, b_3).$$

In other words, we may generate other cycles by adding the indices of the above base cycles (mod 5).

**Corollary 2.** For every positive integer $n$, $K_{2n,2n,4n}$ admits a 5-cycle decomposition.

**Proof.** In the Example 1 of Section 1, we gave a decomposition for $K_{2,2,4}$.

**Corollary 3.** For every positive integer $n$, $K_{m,3m,3m}$ admits a 5-cycle decomposition.

**Proof.** $K_{1,3,3}$ may be decomposed easily as follows. Suppose that the sets $A = \{a_1\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$ are three parts of this graph. Then a decomposition may be done with the base cycles as follows:

$$(a_1, b_i, c_i, b_{i+1}, c_{i+2}) \pmod{3},$$

where, $i = 1, 2, 3$.

Next we use the idea of decomposition of $K_{2n,2n,4n}$ and $K_{m,3m,3m}$, which were done by base cycles, in the following theorem which is our main theorem.

**Theorem 3.** Suppose that at least two parts in a complete tripartite graph have the same number of vertices, say $K_{r,r,s}$. And suppose that the triplet $(r, r, s)$ satisfies all three necessary conditions given in Theorem 1. Then $K_{r,r,s}$ has a 5-cycle decomposition except possibly when $r$ is a multiple of 5 but $s$ is not.
Proof. By simple arithmetics, from the necessary conditions we may deduce that: there exists \( m \) and \( n \) such that \( r = 3m + 2n \) and \( s = m + 4n \).

Now we give the following base cycles, assuming that three parts are \( A = \{a_1, \ldots, a_s\} \), \( B = \{b_1, \ldots, b_r\} \), and \( C = \{c_1, \ldots, c_r\} \).

- \( r \) is even, which implies that \( s \) is also even:
  \[
  (b_{i+2n-j}, a_{m+j}, c_{i+1}, a_{m+2n+2j-i+2[i/2]}, c_i)
  (c_{i+3m+n+j}, a_{m+n+j}, b_{i+1}, a_{m+2n+2j-i+2[i/2]}, b_i)
  (a_j, b_i, c_{i+3j}, b_{i+2}, c_{i+3j+1})
  \]
  \( i = 1, 2, \ldots, r; \ j = 1, 2, \ldots, m. \)

  The indices of \( b \)'s and \( c \)'s are all to be computed modulo \( r \).

- \( r \) and \( s \) are odd:
  \[
  (b_{i+2n-j}, a_{m+j}, c_{i+1}, a_{m+2n+2j-i+2[i/2]}, c_i)
  (b_{r+2n-j}, a_{m+j}, c_{r+1}, a_{m+n+j}, c_r)
  \]
  \( i = 1, 2, \ldots, r-1; \ j = 1, 2, \ldots, n; \)

  \[
  (c_{i+1}, a_{m+n+j}, b_{i+n+2-j}, a_{m+2n+2j-i+2[i/2]}, b_{i+n+1-j})
  (c_r, a_{m+2n+2j-1}, b_{r+n+1-j}, a_{m+2n+2j}, b_{r+n-j})
  (c_1, a_{m+2n+j}, b_{r+n+2-j}, a_{m+j+n}, b_{r+n+1-j})
  \]
  \( i = 1, 2, \ldots, r-2; \ j = 1, 2, \ldots, n; \)

  \[
  (a_j, b_i, c_{i+3j}, b_{i+2}, c_{i+3j+1})
  \]
  \( i = 1, 2, \ldots, r; \ j = 1, 2, \ldots, m. \)

  Again the indices of \( b \)'s and \( c \)'s are all to be computed modulo \( r \).

\[\blacksquare\]

5 Searching for a decomposition in other cases

One may get discouraged and conjecture that the condition \( r = t \) is also necessary!

But it is not true. We show this in the following:

Lemma 1. If \( K_{a,b,b} \), \( K_{a,c,c} \), and \( K_{b,c,c} \) each admits a 5-cycle decomposition, then so does \( K_{a+c,b+c,b+c} \).

Proof. Consider the following \( 2 \times 2 \) latin square:

\[
\begin{array}{cc}
  a & c \\
  b & b & c \\
  c & e & b
\end{array}
\]
Using this latin square we may decompose $K_{a+b+c+b+c}$ into $K_{b,a,b}$, $K_{b,c,c}$, $K_{c,a,c}$, and $K_{c,c,b}$ each of which, by assumption, admits a 5-cycle decomposition.

Also proofs of the following lemmas are immediate.

**Lemma 2.** If $K_{a,b,b}$ and $K_{c,b,b}$ each admits a 5-cycle decomposition, then so does $K_{a+c,2b,2b}$.

**Lemma 3.** If $K_{a,b,b}$ and $K_{b,a,a}$ each admits a 5-cycle decomposition, then so does $K_{a+b,2a,2b}$.

**Example 2.** $K_{30,20,40}$ admits a 5-cycle decomposition.

**Proof.** By Corollary 2, $K_{20,10,10}$ can be decomposed.

In the following we give two more decompositions:

- $K_{3,5,5}$, where its parts are $A = \{a, b, c\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{5, 6, 7, 8, 9\}$:
  
  a0615 a1726 a2837 a3948 a4509 b5291 c6b70 b9c80 c3b47 b2c18.

- $K_{4,10,10}$, where its parts are $A = \{a, b, c, d\}$, $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and $C = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$:
  
  a1122 a2314 a3143 a4215 a5261 a6336 a7760 a8608 b1624 b2537 b3405 b4653 b5566 b6478 b7291 b8102 c1709 c0186 c3993 c1579 c6954 c2788 c5832 d1948 d3094 d0596 d7047 d2083 d6758 d1730 d9289 d5482 d90a0 9b0c8 7c7a9 4d5c4.

Thus, $K_{6,10,10}$ can be decomposed by Corollary 2. And $K_{10,20,20}$ may be decomposed by Lemma 2 and using $K_{6,10,10}$ and $K_{4,10,10}$. Therefore, $K_{30,20,40}$ admits a decomposition by Lemma 3 and using $K_{10,20,20}$ and $K_{20,10,10}$.

**References**

[1] A. Rosa. O cyklických rozkladoch kompletného grafu na nepárnouholníky. Čas. Pěst. Mat., 91:53–63, 1966.

[2] D. Sotteau. Decomposition of $K(m, n)$ into cycles (circuits) of length $2k$. J. of Combinatorial Theory B, 30:75–81, 1981.