CHARACTERIZATION OF LOG-CONVEX DECAY IN NON-SELFADJOINT DYNAMICS

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Abstract. The short-time and global behavior are studied for an autonomous linear evolution equation, which is defined by a generator inducing a uniformly bounded holomorphic semigroup in a Hilbert space. A general necessary and sufficient condition is introduced under which the norm of the solution is shown to be a log-convex and strictly decreasing function of time, and differentiable also at the initial time with a derivative controlled by the lower bound of the generator, which moreover is shown to be positively accretive. Injectivity of holomorphic semigroups is the main technical tool.

1. Introduction

The subject of this note is the global and short-time behavior of the solutions to an autonomous linear evolution equation having a possibly non-selfadjoint generator $-A$.

It is assumed that $A$ is an accretive operator with domain $D(A)$ in a complex Hilbert space $H$, with norm $|\cdot|$ and inner product $(\cdot|\cdot)$, and that $-A$ generates a uniformly bounded, holomorphic $C_0$-semigroup $e^{-zA}$ for $z$ in an open sector of the form $\Sigma_\delta = \{ z \in \mathbb{C} \mid -\delta < \arg z < \delta \}$. Then the “height” function

$$ h(t) = |e^{-tA}u_0| $$

is studied for the solution $u(t) = e^{-tA}u_0$ to the following Cauchy problem, where only initial data $u_0 \neq 0$ are considered,

$$ \partial_t u + Au = 0 \quad \text{for } t > 0, \quad u(0) = u_0 \quad \text{in } H. $$

The intention is to investigate the algebraic conditions on $A$, which give a log-convex decay of $h(t)$.

In a recent article on final value problems by A.-E. Christensen and the author [1], cf. also [2], it was elucidated and proved (except for one remnant) that if $A$ is an elliptic variational operator and $A$ is hyponormal, cf. work of Janas [6], then in terms of the numerical range and the lower bound

$$ \nu(A) = \{ (Ax|x) \mid x \in D(A), |x| = 1 \}, \quad m(A) = \inf \Re \nu(A), $$

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there is a “nice” decay of the height function:

\[ h(t) \text{ is strictly positive, strictly decreasing, and strictly convex} \]

on the closed halfline \( t \geq 0 \), and \( h(t) \) is differentiable at \( t = 0 \), for \( |u_0| = 1 \) generally with \( h'(0) \leq -m(A) \), though with

\[ h'(0) = -\Re(Au_0 \mid u_0) \quad \text{if in addition} \quad u_0 \in D(A). \]  

(4)

First of all this shows how the short-time behavior at \( t = 0 \) via \( h'(0) \) is specifically controlled by \( \nu(A) \), the numerical range of \( A \), and not by its spectrum \( \sigma(A) \); whereas the crude decay estimate \( h(t) \leq Ce^{-tA} \) for \( t \to \infty \) is given by the spectral abscissa \( \sigma = \inf \Re \sigma(A) \) of \( A \), say in case \( A^* = A \geq 0 \).

Secondly, the global behavior of the height \( h(t) \) is expressed in its strict decrease and strict convexity: even if \( A \) has eigenvalues in \( \mathbb{C} \setminus \mathbb{R} \), as may be the case, they do not induce oscillations in the size of the solution \( e^{-tA}u_0 \) for such \( A \)—this is ruled out by strict convexity, which thus can be seen as a stiffness in the decay of \( h(t) \).

The present paper generalizes the above-mentioned results of [2, 1] in three ways: First the restriction to variational generators \( -A \) is completely removed.

Secondly, the additional assumption that \( A \) is hyponormal is replaced by the weaker condition that \( A \) satisfies the following for vectors \( x \in D(A^2) \) such that \( |x| = 1 \),

\[ 2(\Re(Ax \mid x))^2 \leq \Re(A^2x \mid x) + |Ax|^2. \]  

(5)

The third improvement is the stronger conclusion that \( h(t) \) actually is log-convex.\(^1\) In fact, condition (5) characterizes the \( A \) for which \( h(t) \) is log-convex; cf. Theorem 2.5 below.

Somewhat surprisingly, strict monotone decay \( h(t) \searrow 0 \) for \( t \to \infty \) results from (log-)convexity of \( h \) (since \( e^{-tA} \) is uniformly bounded), hence follows whenever the generator \( A \) fulfills (5). The convexity of \( h \) then implies existence of \( h'(0) = \inf h' < 0 \) and that (4) holds. The latter shows that \( A \) is barely better than accretive \((m(A) \geq 0)\) in the sense that its numerical range is contained in the open right half-plane,

\[ \nu(A) \subset \{ z \in \mathbb{C} \mid \Re z > 0 \} =: \mathbb{C}_+. \]  

(6)

It seems appropriate to call \( A \) a positively accretive operator, when it has the property (6). This is a milder condition on \( A \) than strict accretivity \((m(A) > 0)\), used by Kato [7]. The elliptic variational generators in [2, 1] are all strictly accretive, but as described, there is no need to find a substitute assumption for this, as any \( A \) satisfying criterion (5) automatically is positively accretive.

To shed more light on the log-convexity criterion (5), recall that every \( B \in \mathbb{B}(H) \) satisfies \( B = X + iY \) for uniquely determined selfadjoint operators \( X, Y \in \mathbb{B}(H) \), namely \( X = \frac{1}{2}(B + B^*) \) and \( Y = \frac{1}{2i}(B - B^*) \). Hence

\[ \Re(Bu \mid u) = \Re((X^2 - Y^2 + i(XY + YX))u \mid u) = |Xu|^2 - |Yu|^2, \]  

(7)

\[ |Bu|^2 = |Xu + iYu|^2 = |Xu|^2 + |Yu|^2 + 2\Im(Xu \mid Yu). \]  

(8)

\[ |Bu|^2 = |Xu + iYu|^2 = |Xu|^2 + |Yu|^2 + 2\Im(Xu \mid Yu). \]  

(9)

\(^1\)A fortunately inconsequential flaw in the argument given for the strict convexity in [1] is pointed out in Remark 5 below. A remedy of this is provided by means of the present more general results.
As the terms $\pm |Yu|^2$ cancel when the last two lines are added, (5) reduces for $B = X + iY$ in $\mathcal{B}(H)$ to
\[
(Xu \mathrel{|} u)^2 \leq (|Xu||u|)^2 + \Im(Xu \mathrel{|} Yu||u|)^2 \quad \text{for all } u \in H. \tag{10}
\]
Here it is noteworthy that $Y$ only appears in one term. In view of the Cauchy–Schwarz inequality, it is clear that the above is fulfilled when $X$ and $Y$ fit so together that the imaginary part is positive for all $u \in H$. In terms of the commutator $[Y, X] = YX - XY$, one may write (10) equivalently as
\[
(Xu \mathrel{|} u)^2 \leq (|Xu||u|)^2 + \frac{1}{2} \Im([Y, X]u \mathrel{|} u)^2 \quad \text{for all } u \in H. \tag{11}
\]
However, (10) and (11) are always violated for certain $B \in \mathcal{B}(H)$ when $\dim H \geq 2$; cf. Remark 4 below.

So, in other words, criterion (5) is not fulfilled for every operator $A$ in $H$, neither for bounded $A$, nor for $n \times n$-matrices, $n \geq 2$. It is therefore envisaged that (5) can give rise to interesting examples when $A$ is a suitable realization of a partial differential operator.

2. Discussion and Main Results

The reader is assumed familiar with semigroup theory, for which the book of Pazy [11] could be a reference; the simpler Hilbert space case is exposed, e.g., by Grubb [3, Ch. 14]. It is briefly mentioned that there is a bijective correspondence between the $C_0$-semigroups $e^{-tA}$ in $\mathcal{B}(H)$ that are uniformly bounded, i.e., $\|e^{-tA}\| \leq M$ for $t \geq 0$, and holomorphic in some sector $\Sigma_\delta \subset \mathbb{C}$ for $\delta \in [0, \frac{\pi}{2}]$, and the densely defined, closed operators $A$ in $H$ satisfying a resolvent estimate $|\lambda| \cdot \|(A + \lambda I)^{-1}\| \leq C$ for all $\lambda \in \{0\} \cup \Sigma_{\delta + \pi/2}$.

It is classical that, since $\sigma(A) \subset \{z \in \mathbb{C} \mid \Re z \geq \varepsilon \}$ for some $\varepsilon > 0$, there is a bound $\|e^{-tA}\| \leq M_\varepsilon e^{-\eta t}$ for $t \geq 0, 0 < \eta < \varepsilon$. This yields the crude decay estimate $h(t) \leq M_\varepsilon e^{-\eta t}|u_0|$. Log-convexity may be a new aspect in the context, so the discussion is begun with this. First it is recalled that for a strictly positive function $f : \mathbb{R} \to [0, \infty] =: \mathbb{R}_+$, log-convexity means that $\log f(t)$ is convex, that is, for all $r \leq t$ in $\mathbb{R}$ and $0 < \theta < 1$,
\[
f((1 - \theta)r + \theta t) \leq f(r)^{1-\theta}f(t)^{\theta}. \tag{12}
\]
As a slight extension, this makes sense for non-negative functions $f : \mathbb{R} \to [0, \infty]$ too.

A classical exercise shows for the intermediate point $s = (1 - \theta)r + \theta t$ that one has $\theta = (s - r)/(t - r)$. Explicitly log-convexity therefore means for the height function that, for $0 \leq r < s < t$,
\[
|e^{-sA}u_0| \leq |e^{-rA}u_0|^{1-\frac{s-r}{t-r}}|e^{-tA}u_0|^{\frac{s-r}{t-r}}. \tag{13}
\]
The operator $A$ is just a positive scalar if $\dim H = 1$, so (13) is then an identity because of the functional equation of the exponential function $e^{-tA}$ (whereas its slightly weaker property of strict convexity is expressed via a sharp inequality, oddly enough). For $\dim H > 1$ the inequality (13) is by no means obvious for the operator function $e^{-tA}$ in $\mathcal{B}(H)$; it is the main subject of this paper. It is noteworthy that the power function $t \mapsto t^\theta$ in (13) does not require its continuous extension to $t = 0$, for since $e^{-tA}u_0$ is holomorphic, the height function fulfills $h(t) > 0$, or equivalently $e^{-tA}u_0 \neq 0$, for $t \geq 0$. 


This follows from the restriction to $u_0 \neq 0$ and the crucial fact that $e^{-zA}$ is an injection for all $z \in \Sigma_\delta$:

**Lemma 2.1** ([1]). Whenever $-A$ generates a holomorphic semigroup $e^{-zA}$ in $\mathbb{B}(X)$ for some complex Banach space $X$, and $e^{-zA}$ is holomorphic in the open sector $\Sigma_\delta \subset \mathbb{C}$ given by $|\arg z| < \delta$ for some $\delta > 0$, then the operator $e^{-zA}$ is injective on $X$ for each such $z$.

The injectivity is for $t > 0$ clearly equivalent to the geometric property that two solutions $e^{-tA}v$ and $e^{-tA}w$ to the differential equation $u' + Au = 0$ cannot have any points of confluence in $X$ when $v \neq w$. One obvious consequence of this is the backward uniqueness of $u' + Au = 0$; i.e., $u(T) = 0$ implies $u(0) = 0$. But injectivity was seemingly first obtained in [1], cf. the elementary proof in Proposition 1 there, using unique analytic continuation. [14, Cor. 4.3.9] is analogous, but is given for the Laplacian $A = -\Delta$ on Euclidean space, though for local vanishing of $e^{t\Delta}u_0$ in an open set at a fixed time $t > 0$. (An early attempt to obtain Lemma 2.1 was made in a special case in [16], but it had flaws pointed out in [1].)

Injectivity of $e^{-tA}$ is also a crucial tool for the proof of the log-convexity in the present paper. Indeed, the fact that $h(t) > 0$ allows an application of the next result, that characterizes the log-convex $C^2$-functions as the solutions to a differential inequality:

**Lemma 2.2.** If $f$ is continuous $[0, \infty[ \to \mathbb{R}_+$ and $C^2$ for $t > 0$, the following properties are equivalent:

(I) For $0 < t < \infty$ it holds true that

$$f'(t)^2 \leq f(t)f''(t).$$

(II) $f(t)$ is log-convex on the open halfline $]0, \infty[$, that is,

$$f(s) \leq f(r) \frac{\log f(t)}{\log r} \quad \text{for } 0 < r < s < t < \infty.$$ 

In the affirmative case $f(t)$ is log-convex also on the closed halfline $[0, \infty[$.

**Remark 1.** It is classical that a $C^2$-function $f$ is convex if and only if $f'' \geq 0$. This positivity is fulfilled if $f$ satisfies (I), as $(f')^2 \geq 0$ and $f(t) > 0$ is assumed; and it is so in qualified way, equivalent to log-convexity by Lemma 2.2. Though the lemma is not mentioned, convexity is amply elucidated in [10].

**Proof.** By the assumptions $F(t) = \log f(t)$ is defined for $t \geq 0$ and $C^2$ for $t > 0$, as the Chain Rule gives

$$F''(t) = \left( \frac{f'(t)}{f(t)} \right)' = \frac{f''(t)f(t) - f'(t)^2}{f(t)^2}. $$

Hence (I) is equivalent to $F''(t) \geq 0$ for $t > 0$, which is the criterion for the $C^2$-function $F$ to be convex for $t > 0$; which is a paraphrase of the condition (II) for log-convexity of the positive function $f(t)$ for $t > 0$.

By letting $r \to 0^+$ for fixed $s < t$, it follows from the continuity of $f(r)$ and of $\exp(\frac{\log f(r)}{\log r})$, that the inequality in (II) is valid for $0 \leq r < s < t$. Consequently $f$ is log-convex on the closed halfline $[0, \infty[$.

To shed light on the lemma’s consequences for height functions, one may conveniently use differential calculus in Banach spaces as exposed, e.g., by Hörmander [5,
Hence a normalization to $x$ above, for obviously this condition is fulfilled for every $t > 0$ to be equivalent to the property $h_{\mathbb{R}} \mapsto \text{trivial initial data.}$ By passing to the limit for Lemma 2.2 that $u$ and to

The differential inequality in (I) of Lemma 2.2,

$$h'(t) = \frac{(u' \cdot u) + (u \cdot u')}{2 \sqrt{(u \cdot u^2)}} = -\frac{\Re(Au \cdot u)}{|u|};$$

and hence, since $u'' = (e^{-tA}u_0)' = A^2e^{-tA}u_0 = A^2u,$

$$h''(t) = \frac{(A^2u \cdot u) + 2(Au \cdot Au) + (u \cdot A^2u)}{2|u|} - \frac{(\Re(Au \cdot u))^2}{|u|^3}.$$ (18)

The differential inequality in (I) of Lemma 2.2,

$$(h'(t))^2 \leq h''(t)h(t),$$

is therefore equivalent to

$$\frac{\Re(Au \cdot u)^2}{|u|^2} \leq \Re(A^2u \cdot u) + (Au \cdot Au) - \frac{\Re(Au \cdot u)^2}{|u|^2};$$

and to

$$2\Re(Au \cdot u)^2 \leq \left(\Re(A^2u \cdot u) + |Au|^2\right)|u|^2.$$ (20)

Obviously this condition is fulfilled for every $t > 0$ when $A$ satisfies condition (5) above, for $u(t) = e^{-tA}u_0$ belongs to the subspace $D(A^n) \subset D(A^2)$ for every $n \geq 2,$ and all $u_0 \in H,$ when the semigroup is holomorphic. So in this case, it follows from Lemma 2.2 that $h(t) = |e^{-tA}u_0|$ is log-convex for $t \geq 0,$ for the continuity of $h(t)$ and of its derivatives given above entail that the $C^2$-condition is fulfilled.

Conversely, in case the height function $h(t)$ is known to be log-convex for all $u_0 \neq 0,$ then the generator $-A$ necessarily fulfills condition (5) above. Indeed, in view of the equivalence of (19) and (20), the former of these holds by the log-convexity of $h,$ and so does the latter. Especially it is seen by insertion of an arbitrary $u_0 \in D(A^2)$ in (21) and commutation of $A$ and $A^2$ with the semigroup that

$$2\Re(e^{-tA}Au_0 | e^{-tA}u_0)^2 \leq \left(\Re(e^{-tA}A^2u_0 | e^{-tA}u_0) + |e^{-tA}Au_0|^2\right)|e^{-tA}u_0|^2.$$ (22)

By passing to the limit for $t \to 0^+$ it follows for reasons of continuity that

$$2\Re(Au_0 | u_0)^2 \leq \left(\Re(A^2u_0 | u_0) + |Au_0|^2\right)|u_0|^2.$$ (23)

Hence a normalization to $x = \frac{1}{|u_0|}u_0$ yields (5) for every unit vector $x$ in $D(A^2).$

Altogether this shows that (5) characterizes the generators $-A$ of the uniformly bounded, holomorphic semigroups having log-convex height functions for all non-trivial initial data.

The log-convexity criterion (5) should be compared to the sufficient condition $h''(t) > 0$ for strict convexity. The latter is seen at once from the above arguments to be equivalent to the property

$$\Re(Ax \cdot x)^2 < \Re(A^2x \cdot x) + |Ax|^2,$$

for $x \in D(A^2), |x| = 1,$ (24) where in comparison to (5) the inequality is strict and a factor of 2 is absent on the left-hand side.
This clearly indicates that log-convexity is stronger than strict convexity for non-constant functions:

**Lemma 2.3.** When \( f : I \to [0, \infty) \) is log-convex on an interval or halfline \( I \subset \mathbb{R} \), then \( f \) is convex—and if \( f \) is not constant in any subinterval, then \( f \) is strictly convex on \( I \).

**Proof.** Convexity on \( I \) follows from (12) and Young’s inequality for the dual exponents \( 1/\theta \) and \( 1/(1 - \theta) \):

\[
f((1 - \theta)r + \theta t) \leq f(r)^{1-\theta} f(t)^{\theta} \leq (1 - \theta) f(r) + \theta f(t). \tag{25}
\]

In case \( f(r) \neq f(t) \), then the last inequality is strict, as equality holds in Young’s inequality if and only if the numerators are identical (cf. [10, p. 14]). This yields the inequality of strict convexity in this case.

If there is a common value \( C = f(r) = f(t) \) for some \( r < t \) in \( I \), there is by assumption a \( u \in ]r, t[ \) so that \( f(u) \neq f(r) \), and because of the convexity of \( f \) this entails that \( f(u) < f(r) = f(t) \): when \( r < s \leq u \) one may write \( s = (1 - \theta)r + \theta u \) and \( s = (1 - \omega)r + \omega t \) for suitable \( \theta, \omega \in ]0, 1[ \), so clearly

\[
f(s) \leq (1 - \theta)f(r) + \theta f(u) < (1 - \omega)f(r) + \omega f(t) = C = (1 - \omega)f(r) + \omega f(t); \tag{26}
\]

similarly for \( u \leq s < t \); so \( f \) is strictly convex. \( \square \)

For completeness it is noted that for example \( f(t) = e^t - 1 \) is convex, but not log-convex as \( (\log f)'' < 0 \). However, when \( f : I \to [0, \infty) \) is log-convex, so is the stretched function defined for \( a < b \) in \( I \) as

\[
f_{a,b}(t) = \begin{cases} f(t) & \text{for } t < a, \\ f(a) & \text{for } a \leq t < b, \\ f(t-b) & \text{for } b \leq t. \end{cases} \tag{27}
\]

This follows from the geometrically obvious fact that the convexity of \( \log f \) survives the stretching. Since \( f_{a,b} \) clearly is not strictly convex, the last assumption of Lemma 2.3 is necessary.

When \( A \) does satisfy condition (5), so that \( h(t) \) is log-convex on \( [0, \infty[ \) for every \( u_0 \neq 0 \) (cf. the last part of Lemma 2.2), then \( h(t) \) is necessarily strictly decreasing on \( [0, \infty[ \): the decay estimate \( h(t) \leq Ce^{-t\eta} \) and the mere convexity statement in Lemma 2.3 show that \( h \) then satisfies the assumptions in the following self-suggesting

**Lemma 2.4.** If \( f : [0, \infty[ \to \mathbb{R}_+ \) is convex and \( f(t) \to 0 \) for \( t \to \infty \), then \( f \) is strictly monotone decreasing.

**Proof.** Given \( r < s \) in \( [0, \infty[ \), then \( 0 < f(t_0) < \frac{1}{2} f(r) \) holds for some \( t_0 > s \). Taking \( \ell(t) = \alpha t + \beta \) so that \( \ell(r) = f(r) \) and \( \ell(t_0) = f(t_0) \), the fact \( \alpha < 0 \) and convexity on \( [r, t_0] \) yield \( f(s) \leq \ell(s) < \ell(r) = f(r) \). \( \square \)

Consequently \( h(t) = |e^{-tA} u_0| \) is strictly decreasing on \( [0, \infty[ \) (hence has \( h'(t) < 0 \) for \( t > 0 \)). Therefore \( h \) attains a unique global maximum at \( t = 0 \). Moreover, as \( h \) cannot be constant in any subinterval, \( h \) is a strictly convex function on \( [0, \infty[ \), according to Lemma 2.3 and the log-convexity.
By the convexity of $h$ one has that $h''(t) \geq 0$ for $t > 0$, so $h'(t)$ is monotone increasing on $[0, \infty]$. Consequently $\lim_{t \to 0^+} h'(t) = \inf_{t > 0} h'$ exists and belongs to $[\infty, 0]$, as $h' < 0$. By the Mean Value Theorem there is some $t' \in [0, t]$ so that

$$\frac{(h(t) - h(0))}{t} = h'(t') < 0. \quad (28)$$

This implies that $h(t)$ is (extended) differentiable from the right at $t = 0$, with $h'(0) = \inf h'$. Since the strong continuity and strict decrease of $h$ gives $|e^{-tA}u_0| \nearrow 1$ for $t \to 0^+$, an application of (17) yields

$$h'(0) = \inf h' \leq \limsup_{t \to 0^+} h'(t) \leq \limsup_{t \to 0^+} (-m(A)|e^{-tA}u_0|) \leq -m(A). \quad (29)$$

In case $u_0 \in D(A)$ one can exploit that $h'(0) = \lim_{t \to 0^+} h'(t)$ by commuting $A$ with the semigroup in (17), which in the limit gives, because of the strong continuity at $t = 0$ and the continuity of inner products,

$$h'(0) = \lim_{t \to 0^+} -\Re(e^{-tA}Au_0 | e^{-tA}u_0) = -\Re(Au_0 | u_0) \quad \text{for } u_0 \in D(A), \ |u_0| = 1. \quad (30)$$

In addition, it is seen from this that $h'(0)$ is a real number for $u_0 \in D(A)$, so $h \in C^1([0, \infty], \mathbb{R})$ for such $u_0$. For general $u_0 \in H$ it follows from the Chain Rule that $h \in C^\infty(\mathbb{R}_+, \mathbb{R})$.

It is also noteworthy that criterion (5) implies that $A$ is positively accretive; cf. (6). Indeed, as $h'(0) < 0$ was seen above, (30) gives $\Re(Au_0 | u_0) = -h'(0) > 0$ whenever $|u_0| = 1$ in $D(A)$; whence $\nu(A) \subset \mathbb{C}_+$.

The above discussion can now be summed up as the main result of this article:

**Theorem 2.5.** When $-A$ denotes a generator of a uniformly bounded, holomorphic $C_0$-semigroup $e^{-tA}$ in a complex Hilbert space $H$, then the following properties are equivalent:

(i) For every $x \in D(A^2)$ with $|x| = 1$,

$$2(\Re(Ax | x))^2 \leq \Re(A^2x | x) + |Ax|^2. \quad (31)$$

(ii) The height function $h(t) = |e^{-tA}u_0|$ is log-convex on $[0, \infty]$ for every $u_0 \neq 0$; that is, whenever $0 \leq r < s < t$,

$$|e^{-sA}u_0| \leq |e^{-rA}u_0| \cdot |e^{-tA}u_0| \frac{r-s}{t-s}. \quad (32)$$

In the affirmative case, the height function $h(t)$ is for $u_0 \neq 0$ moreover strictly decreasing (hence strictly convex) on the closed halfline $[0, \infty]$ and differentiable from the right at $t = 0$, with a derivative in $[-\infty, 0[$, which satisfies

$$h'(0) = \inf_{t > 0} h'(t) \leq -m(A) \quad \text{for } |u_0| = 1; \quad (33)$$

and if $u_0 \in D(A)$ with $|u_0| = 1$, then

$$h'(0) = -\Re(Au_0 | u_0) \quad (34)$$

$$h \in C^1([0, \infty], \mathbb{R}) \cap C^\infty(\mathbb{R}_+, \mathbb{R}). \quad (35)$$

Furthermore, when $A$ has the properties (i) and (ii), then $A$ is positively accretive, $\nu(A) \subset \mathbb{C}_+$.

**Remark 2.** If $A$ is strictly accretive, it is clear that (33) is stronger than the property $h'(0) \in [-\infty, 0]$. Otherwise, when $A$ is merely positively accretive, then $h'(0) < 0$ may be the stronger statement.
Returning to the case of hyponormal generators considered in [1], it is first recalled that a densely defined unbounded operator \( A \) in \( H \), following Janas [6], is said to be hyponormal if
\[
D(A) \subset D(A^*), \quad |Ax| \geq |A^*x| \quad \text{for all } x \in D(A).
\]  
(36)
Obviously this is fulfilled if \( A^* = A \), but the hyponormal operators extend the selfadjoint operators in another direction than symmetric ones do (as these have a full operator inclusion \( A \subset A^* \)). Since clearly \( A \) is normal if and only if both \( A \) and \( A^* \) are hyponormal, this operator class is quite general.

In case \( A \) is a hyponormal operator in \( H \), the inclusion \( D(A) \subset D(A^*) \) gives at once for \( x \in D(A) \) that
\[
2\Re(Ax \, | \, x) = (Ax \, | \, x) + (x \, | \, Ax) = ((A + A^*)x \, | \, x).
\]  
(37)
Invoking also the norm inequality from the definition of hyponormality, a similar reasoning shows for \( x \in D(A^2) \), since \( D(A^2) \subset D(A) \subset D(A^*) \), that
\[
|(A + A^*)x|^2 = |Ax|^2 + (Ax \, | \, A^*x) + (A^*x \, | \, Ax) + |A^*x|^2 \leq 2|Ax|^2 + 2\Re(A^2x \, | \, x).
\]  
(38)
Hence, by using the Cauchy–Schwarz inequality in the above identity, one finds
\[
2(\Re(Ax \, | \, x))^2 = \frac{1}{2}((A + A^*)x \, | \, x)^2
\]
\[
\leq \frac{1}{2}|(A + A^*)x|^2 |x|^2 \leq \left(|Ax|^2 + \Re(A^2x \, | \, x)\right)|x|^2.
\]  
(39)
After a normalization to \( |x| = 1 \), this shows that a hyponormal operator always fulfills condition (i) in Theorem 2.5, cf. (31). Therefore one has the following generalization of [1] to the case of hyponormal non-variational generators:

**Corollary 1.** Let \( -A \) generate a uniformly bounded holomorphic \( C_0 \)-semigroup \( e^{-tA} \) in a complex Hilbert space \( H \). If \( A \) is hyponormal, cf. (36), then \( A \) fulfills \( \nu(A) \subset \mathbb{C}_+ \) and the equivalent conditions (i) and (ii) in Theorem 2.5, and consequently the height function \( e^{-tA}u_0 \) has all the properties of log-convexity, strict convexity and strict decrease together with differentiability at \( t = 0 \) given in the theorem.

However, true hyponormality only exists outside the realm of matrices and Hilbert–Schmidt operators:

**Remark 3.** For \( A \in \mathcal{B}(H) \) hyponormality means that \( ((A^*A - AA^*)x \, | \, x) \geq 0 \) for all \( x \), i.e., the commutator is positive, \( [A^*, A] \geq 0 \). For such \( A \) the trace \( \text{tr}([A^*, A]) \) is defined, and if \( A \) is a Hilbert–Schmidt operator, so that \( A^*A \) and \( AA^* \) are of trace class, \( \text{tr}([A^*, A]) = \text{tr}(A^*A) - \text{tr}(AA^*) = 0 \). Since \( \|T\| \leq \text{tr}(T) \) when \( T \) is positive, it follows that \( [A^*, A] = 0 \) in \( \mathcal{B}(H) \). (Cf. [12, Section 3.4] for these facts.) Hence every hyponormal Hilbert–Schmidt operator is normal. Especially every hyponormal \( n \times n \)-matrix is normal.

It is instructive to review condition (31) in case the accretive operator \( A \) is variational: that is, for some Hilbert space \( V \subset H \) algebraically, topologically and densely and some sesquilinear form \( a : V \times V \to \mathbb{C} \), which is \( V \)-bounded and \( V \)-elliptic in the sense that (with \( \| \cdot \| \) denoting the norm in \( V \) for some \( C_0 > 0 \)
\[
\Re a(u, u) \geq C_0\|u\|^2 \quad \text{for all } u \in V,
\]  
(40)
it holds for \( A \) that \( (Au \, | \, v) = a(u, v) \) for all \( u \in D(A) \) and \( v \in V \). This framework and Lax–Milgram’s lemma on the properties of \( A \) is exposed in [3, Ch. 12]
and [4, Ch. 3]. It is classical that $-A$ generates a holomorphic semigroup $e^{-tA}$ in $B(H)$; an explicit proof is given in, e.g., [1, Lem. 4].

For such operators, $(A^2u | w) = a(Au, u)$ and $|Au|^2 = a(u, Au)$ clearly hold for every $u \in D(A^2)$. So with the usual convention for the “real” part, namely that $a_R(v, w) = \frac{1}{2}(a(v, w) + a(w, v))$ for $v, w \in V$, one has

$$\Re(A^2u | u) + |Au|^2 = \Re(a(Au, u) + \Re(a(u, Au) = 2\Re(a_R(Au, u)). \quad (41)$$

Thus the log-convexity criterion (31) can be stated for $V$-elliptic variational operators in the form of a comparison of sesquilinear forms,

$$(\Re(a(u, u))^2 \leq \Re(a_R(Au, u))(u | u) \quad (42)$$

**Example 1.** To see that variational operators need not be hyponormal, one may take $H = L_2(\alpha, \beta)$, with norm $\|f\| = (\int_\alpha^\beta |f(x)|^2 \, dx)^{1/2}$, for reals $\alpha < \beta$ and let $V = \{ v \in H^1(\alpha, \beta) | u(\alpha) = 0 \}$ be a subspace of the first Sobolev space with norm given by $\|f\|^2 = \int_\alpha^\beta (|f(x)|^2 + |f'(x)|^2) \, dx$ and the sesquilinear form

$$a(u, v) = \int_\alpha^\beta u'(x)v'(x) + u(x)v(x) \, dx. \quad (43)$$

This is clearly $V$-bounded, and also $V$-elliptic: using partial integration and taking the mean of the two expressions for $a(u, v)$, one finds $\Re(a(u, u) = \|u''\|^2 + \frac{1}{2}|u(\beta)|^2$, so that $\Re(a(u, u) \geq C_0\|u\|^2$ follows for all $u \in V$ and, e.g., $C_0 = \min(\frac{1}{2}, (\beta - \alpha)^{-2})$ by ignoring the last term and applying the Poincaré inequality (it is known that a standard proof of this, as in, e.g., [3, Thm. 4.29], applies to the functions in $V$).

The induced operator $A$ acts in the distribution space $D'(\alpha, \beta)$ of Schwartz [15] as $Au = -u''+u'$, which is the advection-diffusion operator having its domain given by a mixed Dirichlet and Neumann condition,

$$D(A) = \{ u \in V \mid u \in H^2(\alpha, \beta), u'(\beta) = 0 \}$$

$$= \{ u \in H^2(\alpha, \beta) | u(\alpha) = 0, u'(\beta) = 0 \}. \quad (44)$$

(The pure Dirichlet realization of $A = -u''+u'$ has been studied at length; cf. Chapter 12 in the treatise of Embree and Trefethen [17], where use of pseudospectra is the main tool.)

Since $A^*$ is induced by the adjoint form $a^*(u, v) = a(v, u)$, it is similarly seen that $A^*u = -u''-u'$, but here with the domain characterized by a mixed Dirichlet and Robin condition,

$$D(A^*) = \{ u \in H^2(\alpha, \beta) | u(\alpha) = 0, u'(\beta) + u(\beta) = 0 \}. \quad (45)$$

As both $D(A) \setminus D(A^*) \neq \emptyset$ and $D(A^*) \setminus D(A) \neq \emptyset$, it follows from (36) that neither $A$ nor $A^*$ is hyponormal. This is part of the motivation for the introduction of the general condition (31) in this paper.

### 3. ACCRETIVE SQUARES

The considerations in [2, 1] also dealt with variational operators $A$ that, instead of being hyponormal, have accretive squares,

$$\nu(A^2) \subset \mathbb{C}_+ = \{ z \in \mathbb{C} | \Re z \geq 0 \}. \quad (46)$$

The discussion in Section 2 also extends to such operators without the assumption that $A$ is variational, albeit only strict convexity is obtained for $h(t)$. 

Indeed, when \( m(A^2) \geq 0 \) holds, then it is seen from (18) and Cauchy–Schwarz’ inequality that

\[
 h''(t) = \frac{\Re(A^2 u | u)|u|^2 + |Au|^2|u|^2 - (\Re(Au | u))^2}{|u|^4} \geq \frac{(|Au||u|^2 - (\Re(Au | u))^2}{|u|^4} \geq 0. \tag{47}
\]

Of course the mere convexity of \( h \) for \( t > 0 \) is implied by the above inequality \( h''(t) \geq 0 \), so

\[
 h(s) \leq \frac{t-s}{t-r} h(r) + \frac{s-r}{t-r} h(t) \quad \text{for } 0 < r < s < t. \tag{48}
\]

As \( h \) is continuous on \([0, \infty[\), this extends to \( 0 \leq r < s < t \), so \( h \) is convex on \([0, \infty[\). Hence Lemma 2.4 also applies to \( h \), yielding its strict decrease. The arguments below Lemma 2.4 then apply verbatim, which leads to differentiability at \( t = 0 \) etc. of \( h(t) \) (skipping the reference to Lemma 2.3 here). Moreover, this also yields that \( A \) is positively accretive.

However, it remains to prove \( h(t) \) strictly convex on \([0, \infty[\) when \( A^2 \) is accretive (because of the factor 2 on the left-hand side of (31), this condition is hardly implied by \( m(A^2) \geq 0 \)). In the case \( \nu(A^2) \subset \mathbb{C}_+ \), clearly the first inequality in (47) is strict, so that \( h''(t) > 0 \) for \( t > 0 \). Thus \( h \) is strictly convex for such \( A \).

However, by inspection of the formula above, \( h''(t) = 0 \) is seen to imply that both \( \Re(A^2 u | u) \geq 0 \) and \( (|Au||u|^2 - (\Re(Au | u))^2 \geq 0 \) must hold with equality in the first numerator. But then the inequalities

\[
 |\Re(Au | u)| \leq (|Au||u|) \leq |Au||u| \tag{49}
\]

hold with equality. As Cauchy-Schwarz’ inequality is an identity only for proportional vectors, there is some \( \lambda = \mu + i \omega, \mu, \omega \in \mathbb{R} \), such that \( Au(t) = \lambda u(t) \).

Insertion of this into the equation \( h''(t) = 0 \) yields

\[
 \Re \lambda^2 |u|^4 + |\lambda|^2 |u|^4 - (\Re \lambda |u|^2)^2 = 0, \tag{50}
\]

which reduces to

\[
 \mu^2 = 0. \tag{51}
\]

Hence \( \lambda = i \omega \) is an eigenvalue of \( A \), as \( u(t) = e^{-tA}u_0 \neq 0 \) in view of the restriction to \( u_0 \neq 0 \) and injectivity of \( e^{-tA} \); cf. Lemma 2.1. But it was seen above that \( A \) is positively accretive, so it cannot have any eigenvalues on \( i \mathbb{R} \). Consequently \( h''(t) > 0 \) holds for all \( t > 0 \), so \( h(t) \) is strictly convex for \( t > 0 \).

To extend the strict convexity to the closed halfline where \( t \geq 0 \), one may conveniently take recourse to the slope function \( S(r, t) = (h(t) - h(r))/(t-r) \). Because of the Mean Value Theorem and the strict increase of \( h' \), this satisfies \( S(r, s) < S(s, t) \) whenever \( 0 < r < s < t \); which is a classical criterion for strict convexity of \( h \) on \([0, \infty[\). But this sharp inequality extends to the case \( r = 0 \), for by introducing some \( r' \) such that \( r = 0 < r' < s < t \), one finds from the convexity of \( h \) on \([0, \infty[\) obtained after (48) that

\[
 S(0, s) \leq S(r', s) < S(s, t). \tag{52}
\]

Indeed, the first of these inequalities is valid since the slope function \( S(s, t) \) is monotone increasing in both arguments separately for every convex function on an interval. Hence \( h \) is strictly convex on \([0, \infty[\).
Altogether this proves a result analogous to Theorem 2.5, but not quite as strong as this, for operators $A$ with accretive squares:

**Proposition 1.** If $-A$ generates a uniformly bounded, holomorphic $C_0$-semigroup $e^{-tA}$ in a complex Hilbert space $H$ and $A$ has an accretive square, that is

$$\nu(A^2) \subset \overline{C}_+,$$

then if $u_0 \neq 0$ the height function $h(t) = |e^{-tA}u_0|$ is strictly convex and strictly decreasing on $[0, \infty[$, even with $h'' > 0$ for $t > 0$, and it is differentiable from the right at $t = 0$, with a derivative in $[-\infty, 0]$ satisfying

$$h'(0) = \inf_{t > 0} h'(t) \leq -m(A) \quad \text{for } |u_0| = 1;$$

and if $u_0 \in D(A)$ with $|u_0| = 1$ it holds true that $h'(0) = -\Re(Au_0 | u_0)$ and that $h \in C^1([0, \infty[, \Re) \cap C^\infty([0, \infty[, \Re)$. Furthermore, $A$ is then positively accretive, that is, $\nu(A) \subset \mathbb{C}_+$.

Here $h''(t) > 0$ was mentioned explicitly, as not all strictly convex functions fulfill this (cf. $t^4$), whereas in Theorem 2.5 this property was straightforward from the differential inequality characterizing log-convexity.

The last fact in Proposition 1 that $A$ is positively accretive can *post festum* be much sharpened: for an accretive operator $A$ to have an accretive square, cf. (46), it is *necessary* that $A$ has semiangle $\delta \leq \frac{\pi}{2}$, that is, $|\Im z| \leq \Re z$ for every $z \in \nu(A)$. This was shown already by Showalter [16, Lem. 3], who gave the main lines in the proof of the following

**Lemma 3.1.** If $A$ is an operator in $H$ such that both $A$, $A^2$ are accretive and $\Re \mu < 0$ for some $\mu$ in the resolvent set $\rho(A)$, then $|\arg z| \leq \pi/4$ for all $z \in \nu(A)$.

**Proof.** First the claim is proved for every bounded operator $B \in \mathbb{B}(H)$; here $B = X + iY$ for selfadjoint $X$, $Y \in \mathbb{B}(H)$, as noted prior to (7). Since $B$ is accretive, $(Xu | u) = \Re(Bu | u) \geq 0$ holds for $u \in H$, as does

$$\Re(B^2u | u) \geq 0 \iff ((X^2 - Y^2)u | u) \geq 0 \iff |Yu|^2 \leq |Xu|^2 \quad \text{for all } u \in H.$$ 

By the polar decomposition, $Y = US$ holds for a partial isometry $U$ and $S = |Y| = (Y^*Y)^{1/2}$; the latter is positive and fulfills $|Yu| = |Su|$ and $|Yu|^2 \leq (Su | u)$ for all $u \in H$ [recall that as $S \geq 0$, one has $|Yu|^2 = |(Su | U^*u)|^2 \leq (Su | u)(SU^*u | U^*u) = (Su | u)^2$, for $USU^* = |Y| = Y$ as $Y$ is selfadjoint, cf. [12, 3.2.19]]. Exploiting the fact $|Yu| = |Su|$ in the above, $m(B^2) \geq 0$ is seen to imply $X^2 - S^2 \geq 0$, which by the well-known operator monotonicity of the square root on positive operators implies that $X \geq S$; cf. [12, E3.2.13]. When combined with the second fact on $S$, one finds $|Yu| \leq |Su| \leq |Xu|$, so that $z = (Bu | u)$ belongs to the closed sector $\Sigma_{\pi/4} \subset \mathbb{C}$ given by $|\arg z| \leq \pi/4$.

For general accretive $A$, the assumption on $\rho(A)$ implies, since $\nu(A) \subset \overline{C}_+$, that every $\lambda$ having $\lambda \in \rho(A)$ implies that $\forall \lambda < 0$ belongs to $\rho(A)$; cf. the proof of [11, Thm. 3.9]. Therefore the resolvent $B = (A + \varepsilon I)^{-1}$ is in $\mathbb{B}(H)$ for all $\varepsilon > 0$, and for $v = B^2u \in D(A^2)$,

$$\Re(B^2u | u) = \Re((A + \varepsilon I)^2v | v) = \varepsilon^2|v|^2 + 2\varepsilon\Re(Av | v) + \Re(A^2v | v).$$

Now, as $A$, $A^2$ are accretive, so is $B^2$ for $\varepsilon > 0$; whilst $(Bu | u) = \varepsilon|v|^2 + (Av | v)$ yields that $B$ is accretive. So by the above, $(Bu | u) \in \Sigma_{\pi/4}$ for any $\varepsilon > 0$; hence, by the formula, $(Av | v)$ must belong to $\Sigma_{\pi/4}$ too. \qed
As motivation for stating Lemma 3.1 and giving a concise proof (without the assumption, made in [16], that $-A$ should generate a $C_0$-semigroup), it should be mentioned that, contrary to the claim in [16], having semiangle $\delta \leq \pi/4$ does not suffice for $A^2$ to be accretive.

This inaccuracy was pointed out by means of the counter-example in [1, Rem. 9], which is slightly reformulated here for a better reading and in order to note explicitly that the constructed operator gives rise to a contraction semigroup by the Lumer–Philips theorem, cf. [3, Cor. 14.11] or [11, Thm. 4.3]:

**Example 2.** To obtain an operator $A$ so that $m(A^2) < 0 < m(A)$ and $\nu(A) \subset \Sigma_{\pi/4}$, it suffices to take $A$ in $\mathcal{B}(H)$ if $\dim H \geq 2$: As $A = X + iY$ for selfadjoint $X$, $Y \in \mathcal{B}(H)$, cf. (7), clearly $m(A) = m(X)$. Here $X$ can just be chosen to have two orthonormal eigenvectors $v_1, v_2$ with eigenvalues $\lambda_2 > \lambda_1 > 0$ and $X = I$ on $H \ominus \text{span}(v_1, v_2)$, if this is non-trivial. Then $m(X) = \min(1, \lambda_1) > 0$. Obviously $
u(A) \subset \Sigma_{\pi/4}$ means that $|(Yv | v)| \leq (Xv | v)$ for all $v \in H$, or that $-X \leq Y \leq X$.

This is achieved for $Y = \delta X + \lambda_2 U$ if $\delta > 0$ is small enough and $U$ is a partial isometry interchanging $v_1$ and $v_2$, with $U = 0$ on $H \ominus \text{span}(v_1, v_2)$. In fact, writing $v = c_1 v_1 + c_2 v_2 + v_\perp$ for $v_\perp \in H \ominus \text{span}(v_1, v_2)$, since $v_1 \perp v_2$, the inequalities for $Y$ are equivalent to $2\lambda_1 |\Re(c_1 c_2)| \leq \lambda_1 (1 - \delta)|c_1|^2 + (1 - \delta)\lambda_2 |c_2|^2 + (1 - \delta)|v_\perp|^2$, which by Young’s inequality is assured if $1/(1 - \delta) \leq (1 - \delta)^2$, that is if $0 < \delta \leq 1 - \sqrt{\lambda_2/\lambda_1}$.

Now, $m(A^2) \geq 0$ means that $|Xv|^2 \geq |Yv|^2$ for all $v \in H$, but this is always violated, as one can see from $|Yv|^2 = \delta^2 |Xv|^2 + \lambda_2^2 |Uv|^2 + 2\delta \lambda_1 \Re(Xv | Uv)$ by inserting $v = v_1$, whereby the last term vanishes as $v_1 \perp v_2 = Uv_1$: this leads to

$$|Xv_1|^2 - |Yv_1|^2 = \lambda_1^2 |v_1|^2 - (\delta^2 \lambda_2^2 |v_1|^2 + \lambda_1^2 |v_2|^2) = -\delta^2 \lambda_1^2 < 0.$$  \hspace{1cm} (57)

Specifically the symmetric, but non-normal matrices $A = \left( \begin{array}{cc} \delta & 0 \\ 0 & \lambda \end{array} \right) + i\lambda \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ are counter-examples in $\mathcal{B}(\mathbb{C}^2)$ for $\lambda > 0$, $0 < \delta \leq 1/2$.

**Remark 4.** The bounded operator $A$ in Example 2 is also useful in relation to hyponormality and the convexity criterion (31) in Theorem 2.5. Here it is shown explicitly that it does not have these properties (for hyponormality this also follows from Remark 3). The notation from Example 2 is continued here.

First $A = X + iY = (1 + i\delta)X + \lambda_1 U$ entails $[A^*, A] = 2i[X, Y] = 2\lambda_1 i[X, U]$, so $w = c_1 v_1 + c_2 v_2$ gives $[A^*, A]w = 2i\lambda_1 ((\lambda_1 - \lambda_2) c_2 v_1 + (\lambda_2 - \lambda_1) c_1 v_2)$. Hence $[A^*, A] \neq 0$, so $A$ is non-normal in $\mathcal{B}(H)$.

Now, $([A^*, A]w | w) = 4\lambda_1 (\lambda_1 - \lambda_2) \Im(c_1 c_2)$, and inserting $c_1 = \pm i c_2 = 2^{-1/2}$ yields that $\nu([A^*, A])$ contains $\pm 2\lambda_1 (\lambda_2 - \lambda_1) \neq 0$; hence $[A^*, A]$, $[A, A^*]$ are non-positive. So neither $A$ nor $A^*$ is hyponormal.

Furthermore, for $A$ the criterion for bounded operators in (10) is that, for $v \in H$,

$$2(Xv | v)^2 = (|Xv| |v|)^2 \leq (|Xv||v|)^2 + 3(Xv | \delta Xv + \lambda_1 Uv)v|v|^2$$

$$= (|Xv||v|)^2 + 3\Im(Xv | Uv)v|v|^2.$$ \hspace{1cm} (58)

Here it is obvious that $\delta$ is absent in the criterion. To show that the inequality is violated for any choice of $\lambda_2 > \lambda_1 > 0$ it suffices to insert vectors of the form $v = isv_1 + v_2$ for $s > 0$. Indeed, $|v|^2 = s^2 + 1$ due to the orthogonality, and $Xv = is\lambda_1 v_1 + 2\lambda_2 v_2$ while $Uv = v_1 + isv_2$, so the above gives for this $v$,

$$(s^2\lambda_1 + \lambda_2^2)^2 \leq (s^2 + 1)(s^2\lambda_1^2 + \lambda_2^2 + s\lambda_1 (\lambda_1 - \lambda_2)).$$ \hspace{1cm} (59)
As the fourth order term $\lambda_1^2 s^4$ cancels on both sides, the term of highest degree is $s^3 \lambda_1 (\lambda_1 - \lambda_2)$ on the right-hand side. After division by $s^3$ and passage to the limit $s \to \infty$, one therefore arrives at the false statement “$0 \leq \lambda_1 (\lambda_1 - \lambda_2)$.” Consequently the operator $A$ from Example 2 does not fulfill the log-convexity criterion in Theorem 2.5 for any of the considered values of the parameters. Especially this is so for the matrix given at the end of Example 2.

4. Final remarks

**Remark 5.** When $\nu(A^2) \subset \mathbb{C}_+$ as in Section 3, it is also illuminating to observe the possibility to depart from the cleaner expression of the derivatives of the squared height $h(t)^2 = |e^{-tA} u_0|^2$:

$$(h^2)'(t) = (-2 \Re(Au | u))' = 2 \Re(A^2 u | u) + 2|Au|^2 \geq |A e^{-tA} u_0|^2.$$ (60)

Here $e^{-tA}$ is injective, and $A$ is so as $\nu(A) \subset C_+$, whence $u_0 \neq 0$ yields $(h^2)'' > 0$. Similarly $(h^2)'' > 0$ can be seen from (39) to hold if $A$ is hyponormal. That is, $h^2$ is in both cases strictly convex for $t > 0$.

But as $\sqrt{t}$ is concave (not convex), strict convexity of $h^2$ is not simply carried over to $h$. As the task is to prove $h^1$ strictly increasing, the formula $h' = (h^2)'/(2\sqrt{h^2})$ looks convincing as there is strict decrease of the denominator while the numerator is increasing—but it does not lead to the desired conclusion because $(h^2)' < 0$.

This small point was overlooked in [1, Prop. 4], yet the statement given there is nevertheless correct. Indeed, [1, Prop. 4] is generalized to non-variational $A$ having accretive squares in Proposition 1, and to non-variational hyponormal generators $A$ in Corollary 1. A further generalization to generators $A$ satisfying the log-convexity criterion (31) is provided by Theorem 2.5.

**Remark 6.** For matrices $A$ in $\mathbb{B}(\mathbb{C}^n)$ the dynamical properties of (2) have been studied for decades, and, e.g., Perko [13, Ch. 1] gave a concise treatment with many explicit formulas for the exponential matrix $e^{-tA}$ and the resulting solution $u(t)$. However, most systems have eigenvalues that are complicated or even impossible to write down ($n \geq 5$), and this led Moler and Van Loan to review the possibilities in 1978 in “Nineteen dubious ways to calculate the exponential of a matrix,” with an update in 2003 [9].

The present results are closer in spirit to more recent work, a glimpse of which is given here, following the inspiring exposition of Embree and Trefethen [17, Ch. 14]. A main subject of interest has been the behavior of the operator norm $E(t) = \|e^{-tA}\|$, which has the advantage of being independent of any initial data $u_0$, thereby letting the influence of especially non-normal matrices shine through. At $t = 0$ it is a main result that the numerical range $\nu(A)$ controls the growth rate of $E$,

$$E'(0) = -m(A).$$ (61)

For $t \to \infty$ it is known that $\frac{1}{t} \log E(t) \to -\sigma$, where again $\sigma = \inf \Re \sigma(A)$ denotes the spectral abscissa of $A$; so the long-term behavior is controlled by $\sigma(A)$. For the transition phase there is the pseudospectral estimate $\sup_{t \geq 0} E(t) \geq \alpha_e(-A)/\varepsilon$, supplied with estimates from below of $\sup_{0 \leq s \leq t} E(s)$ that permit an exploration of the time $t_0$ at which $\sup_{t > t_0} E(t)$ is attained.

However, when $u_0$ is reintroduced, the inequality $|e^{-tA} u_0| \leq E(t)|u_0|$ is a crude estimate (a worst-case scenario), which does not suffice to settle whether $h(t) = |e^{-tA} u_0|$ has properties like strict decrease or log-convexity as in Theorem 2.5.
Moreover, (61) is often highly misleading for the short-time behavior of $h(t)$ itself. For example, for $A = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ one has $\nu(A) = [-1, 3]$ with $m(A) = -1$, so while $E'(0) = 1$ holds by (61), indicating a growth at $t = 0$, the choice $u_0 = e_2 = (0, 1)$ gives, because of (34),

$$h'(0) = -\Re(Ae_2 | e_2) = -3 < 1 = E'(0).$$

This sharp contrast between the properties of the solutions to $u' + Au = 0$ with $u(0) = u_0$ and those of $\|e^{-tA}\|$ also motivates the study of the height function $h(t) = |e^{-tA}u_0|$ in the present paper.

**Remark 7.** If the generator $-A$ of the uniformly bounded holomorphic semigroup is dissipative, i.e., $A$ is accretive, then $e^{-tA}$ is a classical contraction semigroup; cf. [3, Cor. 14.11]. That is, $\|e^{-tA}\| \leq 1$ holds for the operator norm for $t \geq 0$, whence $h(t) \leq \|e^{-tA}\| \cdot |u_0| \leq |u_0|$, i.e., an estimate by a constant. If $m(A) > 0$ it is also classical that $-(A - \varepsilon I)$ for $0 < \varepsilon < m(A)$ generate the contractions $e^{-t(A-\varepsilon I)} = e^{\varepsilon t}e^{-tA}$, so the sharper estimate $|e^{-tA}u_0| \leq e^{-\varepsilon t}|u_0|$ holds for $t \geq 0$ and any $\varepsilon \in [0, m(A)]$, hence also for $\varepsilon = m(A)$. But this exponential decay is just a crude estimate that requires $m(A) > 0$. For comparison it is observed that if $A$ just satisfies the log-convexity criterion (31), so that $m(A) = 0$ is possible and Theorem 2.5 applies, the log-convex and strictly decreasing behavior of the height function $|e^{-tA}u_0|$ constitutes a rather more precise dynamical property of the evolution problem $u' + Au = 0$, $u(0) = u_0$.

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