Riemannian Conjugate Gradient Descent Method for Third-Order Tensor Completion

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Abstract: The goal of tensor completion is to fill in missing entries of a partially known tensor under a low-rank constraint. In this paper, we mainly study low rank third-order tensor completion problems by using Riemannian optimization methods on the smooth manifold. Here the tensor rank is defined to be a set of matrix ranks where the matrices are the slices of the transformed tensor obtained by applying the Fourier-related transformation onto the tubes of the original tensor. We show that with suitable incoherence conditions on the underlying low rank tensor, the proposed Riemannian optimization method is guaranteed to converge and find such low rank tensor with a high probability. In addition, numbers of sample entries required for solving low rank tensor completion problem under different initialized methods are studied and derived. Numerical examples for both synthetic and image data sets are reported to demonstrate the proposed method is able to recover low rank tensors.

Keywords: tensor completion, manifold, tangent spaces, conjugate gradient descent method

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1 Introduction

This paper addresses the problem of low rank tensor completion when the rank is a priori known or estimated. Let $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ be a third-order tensor that is only known on a subset $\Omega$ of the complete set of entries. The low rank tensor completion problem consists of finding the tensor with the lowest rank that agrees with $A$ on $\Omega$:

$$\min_{Z} \text{rank}(Z) \quad \text{s.t.} \quad Z \in \mathbb{R}^{n_1 \times n_2 \times n_3}, \quad P_\Omega(Z) = P_\Omega(A),$$

where

$$P_\Omega : \mathbb{R}^{n_1 \times n_2 \times n_3} \rightarrow \mathbb{R}^{n_1 \times n_2 \times n_3}, \quad Z_{ijk} \rightarrow \begin{cases} Z_{ijk}, & \text{if } (i,j,k) \in \Omega, \\ 0, & \text{if } (i,j,k) \notin \Omega, \end{cases}$$

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denotes the projection onto $\Omega$. In particular, if $n_3 = 1$ the tensor completion problem [1] reduces to the well known matrix completion problem, see for instance [3, 4, 6, 8, 25] and references therein.

Tensor has many kinds of rank definitions which lead to different mathematical models for studying the low rank tensor completion problem. In order to well understand the model (1), we first summarize some most popular definitions of tensor rank. Let $X \in \mathbb{R}^{n_1 \times \cdots \times n_d}$. The CANDECOMP/PARAFAC (CP) rank of a tensor $X$ is defined as the minimal number of summations of rank-one tensors that generate $X$. CP rank is an exact analogue to the matrix rank, however, its properties are quite different. For example, calculating the tensor (CP) rank has been demonstrated to be NP-hard [18, 35]. The Tucker rank [26] (or multilinear rank) of $X$ is defined as $\text{rank}_n(X) = (\text{rank}(X_1), \text{rank}(X_2), \ldots, \text{rank}(X_d))$, where $X_i$ is derived by unfolding $X$ along its $i$-th mode into a matrix with size $n_i \times \prod_{k \neq i} n_k$. Although multilinear rank is perhaps the most widely adopted rank assumption in the existing tensor completion literature, a crucial drawback is pointed out that it only takes into consideration the ranks of matrices that are constructed based on the unbalanced matrixization scheme [37]. The Tensor Train (TT) rank [23] of $X$ is defined as a vector $r = [r_1, \ldots, r_{d-1}]$ with $r_j = \text{rank}(X^{<j>})$, where $X^{<j>}$ is the $j$-th unfolding of $X$ as a matrix with size $(n_1 \cdots n_j) \times (n_{j+1} \cdots n_d)$. TT-rank is generated by the TT decomposition using the link structure of each core tensor. Since the link structure, the TT rank is only efficient for higher order tensor completion problem.

Optimization technique on Riemannian manifold has gained increasing popularity in recent years. Meanwhile, some classical optimization algorithms that worked well in the Euclidean space have been extended to the smooth manifolds. For instance, gradients descent method, conjugate gradients method and Riemannian trust-region method can be considered, see [1, 7, 19, 28, 29, 30] and the references therein. For the low rank tensor completion problem, the Riemannian manifold theory can provide us an alternative way to consider the rank constraint condition. Note that the fixed tensor set can form a smooth manifold, then the model given in (1) can be equivalent rewritten as

$$\min_\frac{1}{2} ||P_\Omega(Z) - P_\Omega(A)||_F^2 \text{ s.t. } Z \in \mathcal{M}_r := \{Z \in \mathbb{R}^{n_1 \times n_2 \times n_3} | \text{rank}(Z) = r\},$$

with $P_\Omega$ is a projection to the sampling set $\Omega$, where $\Omega (|\Omega| = m)$ is a set of indices sampled independently and uniformly without replacement. When the rank in (2) is chosen as the Tucker rank, Kressner et al. [19] showed the fixed Tucker rank tensor set form a smooth embedded submanifold of $\mathbb{R}^{n_1 \times n_2 \times n_3}$, and proposed a number of basic tools from differential geometry for tensors of low Tucker rank. Moreover, they studied the low Tucker rank tensor completion problem by Riemannian CG method. With the manifold framework listed in [19], Heidel and Schulz [10] considered the low Tucker rank tensor completion problem by Riemannian trust-region methods. Some other results can be found in [11, 22]. Besides the fixed Tucker rank tensor manifold, other choices of the smooth manifold are also used such as hierarchical Tucker format [27] and fixed TT rank manifold [36] for handling high dimensional applications. Unfortunately, their results are not sufficient to derive the bounds on the number of sample entries required to recover low rank tensors.

Recently, Kilmer et al. [16, 17] proposed the tubal rank of a third-order tensor, which is based on tensor-tensor product (t-product) and its algebra framework, where the t-product
allows tensor factorizations like matrix cases. Jiang et al. [13] showed that one can recover a low tubal-rank tensor exactly with overwhelming probability by simply solving a convex program. Other results can be found in [32, 33] and the references therein. These approaches have been shown to yield good recovery results when applied to the tensors from various fields such as medical imaging, hyperspectral images and seismic data.

1.1 The Contribution

In this paper, we mainly consider the low rank tensor completion problem by Riemannian optimization methods on the manifold of low transformed rank tensor. Mathematically, it can be expressed as

\[
\min f(Z) = \frac{1}{2} \| R_\Omega(Z) - R_\Omega(A) \|_F^2 \quad \text{s.t. } \text{rank}_t(Z) = r.
\]  

(3)

Here, \( \text{rank}_t(Z) \) is the transformed multi-rank of \( Z \) given in [17] and \( R_\Omega \) denotes the sampling operator

\[
R_\Omega(Z) = \sum_{i,j,k=1}^{[\Omega]} \langle E_{ijk}, Z \rangle E_{ijk},
\]  

(4)

with \( E_{ijk} \) being a tensor whose \((i,j,k)\) position is 1 and zeros everywhere else. The \((i,j,k)\)-th component of \( R_\Omega(Z) \) is zero unless \((i,j,k) \in \Omega\). Here, we consider the sampling with replacement model instead of without replacement model. Then, for \((i,j,k) \in \Omega\), \( R_\Omega(Z) \) is equal to \( Z_{ijk} \) times the multiplicity of \((i,j,k) \in \Omega\). Sampling with replacement model can be viewed as a proxy for the uniform sampling model and the failure probability under the uniform sampling model is less than or equal to the failure probability under the sampling with replacement model [31].

The main contribution of this paper can be expressed as follows. We first establish the set of fixed transformed multi-rank tensors forms a Riemannian manifold, which is different from the well known fixed Tucker rank tensor manifold [19] and fixed tensor train rank manifold [36]. And then, we show that with suitable incoherence conditions the proposed Riemannian optimization method on the fixed transformed multi-rank tensor is guaranteed to converge to the underlying low rank tensor with a high probability. Moreover, numbers of sample entries required for solving low rank tensor completion problem under different initialized methods are studied and derived.

The outline of this paper is given as follows. In Section 2, we summarize the notations used throughout this paper. The preliminaries of tensor singular value decomposition theory as well as the differential geometric properties of transformed multi-rank tensor manifold are also presented. In Section 3, the tensor conjugate gradient descent algorithms based on the fixed multi-rank tensor manifold is presented and analysed. In Section 4, we provide the bounds on the number of sample entries required for tensor completion under different initialization methods. In Section 5, we present several synthetic data and imaging data sets to demonstrate the performance of the proposed algorithms. Finally, some concluding remarks are given in Section 6.
2 Preliminaries

In this section, some notations and notions relate to tensors and manifolds used throughout this paper are reviewed. We also refer the reader to [15, 34] and references therein for more details about tensors and manifolds, respectively.

2.1 Tensors

Throughout this paper, tensors are denoted by boldface Euler letters and matrices by boldface capital letters. Vectors are represented by boldface lowercase letters and scalars by lowercase letters. The field of real number is denoted as $\mathbb{R}$. For a third-order tensor $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, we denote its $(i,j,k)$-th entry as $A_{ijk}$ and use the MATLAB notation $A(:, :, :)$, $A(:, i, :)$, and $A(:, :, i)$ to denote the $i$-th horizontal, lateral and frontal slice, respectively. Specifically, the front slice $A(:, :, i)$ is denoted compactly as $A(i)$ and the horizontal slice $A(:, i, :)$ is denoted as $A[i]$. $A(i, j, :)$ denotes a tubal fiber oriented into the board obtained by fixing the first two indices and varying the third. Moreover, a tensor tube of size $1 \times 1 \times n_3$ is denoted as $\hat{a}$ and a tensor column of size $n_1 \times 1 \times n_3$ is denoted as $\vec{a}$.

The inner product of $A$ and $B$ in $\mathbb{R}^{n_1 \times n_2}$ is given by $\langle A, B \rangle = \text{Tr}(A^T B)$, where $A^T$ denotes the transpose of $A$ and $\text{Tr}(\cdot)$ denotes the matrix trace. The inner product of $A$ and $B$ in $\mathbb{R}^{n_1 \times n_2 \times n_3}$ is defined as

$$\langle A, B \rangle = \sum_{i=1}^{n_3} \langle A^{(i)}, B^{(i)} \rangle.$$  \hfill (5)

For a tensor $A$, we denote the infinity norm as $\|A\|_\infty = \max_{ijk} |A_{ijk}|$ and the Frobenius norm as $\|A\|_F = \sqrt{\sum_{ijk} |A_{ijk}|^2}$.

2.2 t_{c}-SVD

Based on applying the Fast Fourier Transform (FFT) along all the tubes of a tensor, Kilmer et al. [16, 17] introduced the tensor-tensor product (tt-product) and tensor singular value decomposition (t-svd) theory, respectively. Later, Kernfeld et al. [15] shown that the tt-product and t-svd can be implemented by using a discrete cosine transform, and the corresponding algebraic framework can also be derived. In signal processing, many real data sets satisfy reflexive Boundary Conditions rather than periodic boundary conditions, then better results can be derived by using Discrete Cosine Transform (DCT) instead of FFT. Moreover, DCT only produces real number for real input in the transform domain which is important in the Riemannian manifold structure analysis. For these reasons, in this paper, we mainly consider the tensor singular value decomposition theory based on the DCT.

For a third-order tensor $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, $\hat{A}$ represents the tensor obtained by taking the DCT of all the tubes along the third dimension of $A$, i.e.,

$$\text{vec}(\hat{A}(i, j, :)) = \text{dct}(\text{vec}(A(i, j, :))),$$

where $\text{vec}$ is the vectorization operator that maps the tensor tube to a vector, and $\text{dct}$ stands for the DCT. For compactness, we will denote $\hat{A} = \text{dct}(A, [:], 3)$. In the same fashion, one can also
compute $\mathbf{A}$ from $\hat{\mathbf{A}}$ via idct$(\hat{\mathbf{A}}, [], 3)$ using the inverse DCT operation along the third-dimension. For sake of brevity, we direct the interested readers to [16, 17].

**Definition 2.1 (Block diagonal form of third-order tensor [17]).** Let $\mathbf{A}$ be the block diagonal matrix of the tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ in the transform domain, namely,

$$
\mathbf{A} = \text{blockdiag}(\hat{\mathbf{A}}) = \begin{bmatrix}
\hat{\mathbf{A}}^{(1)} \\
\hat{\mathbf{A}}^{(2)} \\
\vdots \\
\hat{\mathbf{A}}^{(n_3)}
\end{bmatrix} \in \mathbb{R}^{n_1 n_3 \times n_2 n_3}.
$$

In addition, the block diagonal matrix can be converted into a tensor by the ‘fold’ operator:

$$
\text{fold} \left( \text{blockdiag}(\hat{\mathbf{A}}) \right) = \hat{\mathbf{A}}.
$$

The following fact will be used through out the paper. For any tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $\mathbf{B} \in \mathbb{R}^{n_2 \times n_4 \times n_3}$, the inner product of two tensors satisfies $\langle \mathbf{A}, \mathbf{B} \rangle = \langle \mathbf{A}, \mathbf{B} \rangle \in \mathbb{R}$. After introducing the tensor notation and terminology, we give the basic definitions on $t_c$-product and outline the associated algebraic framework which serves as the foundation for our analysis in next section.

**Definition 2.2 ([15]).** The $t_c$-product $\mathbf{A} * \mathbf{B}$ of $\mathbf{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $\mathbf{B} \in \mathbb{R}^{n_2 \times n_4 \times n_3}$ is a tensor $\mathbf{C} \in \mathbb{R}^{n_1 \times n_4 \times n_3}$ which is given by

$$
\mathbf{C} = \mathbf{A} * \mathbf{B} = \text{idct} \left[ \text{fold} \left( \text{blockdiag}(\hat{\mathbf{A}}) \times \text{blockdiag}(\hat{\mathbf{B}}) \right) \right],
$$

where “$\times$” denotes the usual matrix product.

Note that a third-order tensor of size $n_1 \times n_2 \times n_3$ can be regarded as an $n_1 \times n_2$ matrix with each entry as a tube lies in the third dimension. This new perspective has endowed multidimensional data arrays with an advantageous representation in real-world applications. Similar as Definition 2.2, it is convenient to rewrite Definition 4.4 in [15] as follows.

**Definition 2.3 ([15]).** The transpose of $\mathbf{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ with respect to dct is the tensor $\mathbf{A}^T \in \mathbb{R}^{n_2 \times n_1 \times n_3}$ obtained by

$$
\mathbf{A}^T = \text{idct} \left[ \text{fold} \left( \text{blockdiag}(\hat{\mathbf{A}})^T \right) \right].
$$

Next we would like to introduce the identity tensor with respect to DCT, which is also given in [15]. We construct a tensor $\mathcal{I} \in \mathbb{R}^{n \times n \times n_3}$ with each frontal slice $\mathcal{I}^{(i)} (i = 1, \ldots, n_3)$ being an $n \times n$ identity matrix.

**Definition 2.4.** ([13, Proposition 4.1]) The identity tensor $\mathcal{I}_{\text{dct}} \in \mathbb{R}^{n \times n \times n_3}$ (with respect to DCT) is defined to be a tensor such that $\mathcal{I}_{\text{dct}} = \text{idct}[\mathcal{I}]$.

Note that $\mathbf{A}$ is a diagonal tensor if and only if each frontal slice $\mathbf{A}^{(i)}$ is a diagonal matrix. The aforementioned notions allow us to propose the following tensor singular value decomposition theory ($t_c$-SVD).
Definition 2.5 ([15]). For \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \), the t\(_c\)-SVD of \( A \) is given as

\[
A = U * S * V^T,
\]

where \( U \in \mathbb{R}^{n_1 \times n_1 \times n_3} \) and \( V \in \mathbb{R}^{n_2 \times n_2 \times n_3} \) are orthogonal tensors, and \( S \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) is a diagonal tensor, respectively.

Then \( U \), \( V \) and \( S \) in the t\(_c\)-SVD can be computed by SVDs of \( \hat{A}(i) \), which is summarized in Algorithm 1.

**Algorithm 1** t\(_c\)-SVD for third-order tensors [15]

**Input:** \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \).

1. \( \hat{A} = \text{dct}[A] \);
2. for \( i = 1, \ldots, n_3 \) do
3. \( [U, S, V] = \text{SVD}(\hat{A}(i)) \);
4. \( \hat{U}(i) = U, \hat{S}(i) = S, \hat{V}(i) = V \);
5. end for
6. \( \hat{U} = \text{idct}[\hat{U}], S = \text{idct}[\hat{S}], V = \text{idct}[\hat{V}] \).

**Output:** \( U \in \mathbb{R}^{n_1 \times n_1 \times n_3}, S \in \mathbb{R}^{n_1 \times n_2 \times n_3}, V \in \mathbb{R}^{n_2 \times n_2 \times n_3} \).

2.3 Tensor Transformed Multi-rank

Based on t\(_c\)-SVD given in Definition 2.5 one can get the definitions of the transformed multi-rank and tubal rank, respectively.

**Definition 2.6** ([32]). The transformed multi-rank of a tensor \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) denoted as \( \text{rank}_{t}(A) \), is a vector \( r \in \mathbb{R}^{n_3} \) with its \( i \)-th entry as the rank of the \( i \)-th frontal slice of \( \hat{A} \), i.e.,

\[
\text{rank}_{t}(A) = r, \quad r_i = \text{rank}(\hat{A}(i)), \quad i = 1, \ldots, n_3.
\]

The tubal rank of a tensor, denoted as \( \text{rank}_{ct}(A) \), is defined as the number of nonzero singular tubes of \( S \), where \( S \) comes from the t\(_c\)-SVD of \( A = U * S * V^T \), i.e.,

\[
\text{rank}_{ct}(A) = \# \{i : S(i, i, :) \neq 0\} = \max_i r_i.
\]

For computational improvement, we will use the skinny t\(_c\)-SVD throughout the paper unless otherwise stated.

**Remark 2.1.** For \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) with \( \text{rank}_{t}(A) = (r_1, \ldots, r_{n_3}) = r \) and \( \text{rank}_{ct}(A) = r \), denote \( I_r = \text{idct}[I_d] \), where each frontal slice of \( I_d \) obeying \( I_d(i) = \begin{pmatrix} r_i & 0 \\ 0 & 0 \end{pmatrix} \) (\( i = 1, \ldots, n_3 \)). Then the skinny t\(_c\)-SVD of \( A \) is given as \( A = U * S * V^T \), where \( U \in \mathbb{R}^{n_1 \times r \times n_3}, V \in \mathbb{R}^{n_2 \times r \times n_3} \) satisfying

\[
U^T * U = I_r, \quad V^T * V = I_r, \quad \text{rank}_{t}(U) = \text{rank}_{t}(V) = r, \quad \text{rank}_{ct}(U) = \text{rank}_{ct}(V) = r,
\]

and \( S \in \mathbb{R}^{r \times r \times n_3} \) is diagonal tensor.
The spectral norm and the condition number of a tensor are defined as follows.

**Definition 2.7.** The tensor spectral norm of \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \), denoted as \( \|A\| \), is defined as \( \|A\| = \|\overline{A}\| \), where \( \overline{A} \) is the block diagonal matrix of \( A \) in the transform domain. The condition number of \( A \), denoted by \( \kappa(A) \) is defined as \( \kappa(A) = \frac{\sigma_{\text{max}}(A)}{\sigma_{\text{min}}(A)} \), where \( \sigma_{\text{max}}(A) \) and \( \sigma_{\text{min}}(A) \) are the largest and the smallest nonzero singular values of \( A \), respectively.

In other words, the tensor spectral norm of \( A \) equals to the matrix spectral norm of its block diagonal form \( \overline{A} \).

**Definition 2.8 ([32]).** Then the tensor operator norm is defined as \( \|L\|_{\text{op}} = \sup_{\|A\|_{F} \leq 1} \|L(A)\|_{F} \).

If \( L : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^{n_4 \times n_2 \times n_3} \) is a tensor operator mapping an \( n_1 \times n_2 \times n_3 \) tensor \( A \) to an \( n_4 \times n_2 \times n_3 \) tensor \( B \) via the \( t_c \)-product as \( B = L(A) = L \ast A \), where \( L \) is an \( n_4 \times n_1 \times n_3 \) tensor, we have \( \|L\|_{\text{op}} = \|L\| \). Now we need to introduce a new kind of tensor basis which is different from Definition 2.2 in [32]. It is worth noting that the new tensor basis plays an important role in tensor coordinate decomposition and defining the tensor incoherence conditions in the sequel.

**Definition 2.9.** The transformed column basis with respect to \( \text{dct} \), denoted as \( \vec{e}_i \), is a tensor of size \( n_1 \times 1 \times n_3 \) with the \( i \)-th tube of \( \text{dct}[\vec{e}_i] \) is equal to \( 1/\sqrt{n_3} \vec{1} \) (each entry in the \( i \)-th tube is \( 1/\sqrt{n_3} \)) and the rest equaling to 0. Its associated conjugate transpose \( \vec{e}_i^T \) is called transformed row basis with respect to \( \text{dct} \).

Moreover, some incoherence conditions on \( L_0 \) are needed to ensure that it is not sparse.

**Definition 2.10 (Tensor Incoherence Conditions).** Suppose that \( L_0 \in \mathbb{R}^{n_1 \times n_2 \times n_3} \), and \( \text{rank}_t(L_0) = r \) with \( \text{rank}_c(L_0) = r \). Its skinny \( t_c \)-SVD is \( L_0 = U \ast S \ast V^T \). Then \( L_0 \) is said to satisfy the tensor incoherence conditions with parameter \( \mu_0 > 0 \) if

\[
\max_{i=1,\ldots,n_1} \|U^T \ast \vec{e}_i\|_{F} \leq \sqrt{\frac{\mu_0 r}{n_1}}, \quad \max_{j=1,\ldots,n_2} \|V^T \ast \vec{e}_j\|_{F} \leq \sqrt{\frac{\mu_0 r}{n_2}}.
\]

(6)

**Definition 2.11 (Tensor Joint Incoherence Condition).** Suppose that \( L_0 \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) with \( \text{rank}_t(L_0) = r \) and \( \text{rank}_c(L_0) = r \). Assume there exist a positive numerical constant \( \mu_1 \) such that

\[
\|L_0\|_{\infty} \leq \mu_1 \sqrt{\frac{r}{n_1n_2n_3}} \|L_0\|.
\]

### 2.4 Manifolds

A smooth manifold with its tangent space, a proper definition of Riemannian metric for gradient projection and the retraction map are three essential settings for Riemannian optimization.
Moreover, the key idea of Riemannian gradient iteration method contains two step in each iteration: (1) perform a gradient step in the search space; (2) map the result back to the manifold by a proper retraction. In this subsection, we first show the fixed transformed multi-rank tensor set forms an embedded manifold of $\mathbb{R}^{n_1 \times n_2 \times n_3}$. And then, we also consider the tangents spaces, Riemannian metric as well as the retraction mapping relate to the fixed transformed multi-rank tensor manifold.

The following proposition shows that the fixed transform multi-rank tensor set is indeed a smooth manifold.

**Proposition 2.2.** Let

$$\mathcal{M}_r = \{ X \in \mathbb{R}^{n_1 \times n_2 \times n_3} : \text{rank}_i(X) = r \}$$

(7)

denote the set of fixed transformed multi-rank tensors with $r = (r_1, r_2, \ldots, r_{n_3})$. Then $\mathcal{M}_r$ is an embedded manifold of $\mathbb{R}^{n_1 \times n_2 \times n_3}$ and its dimension is $\sum_{i=1}^{n_3} ((n_1 + n_2) r_i - r_i^2)$.

While the existence of such a smooth manifold structure, the tangent space of a point on the manifold can be given as follows.

**Proposition 2.3.** Let $\mathcal{M}_r$ be given as (7) and $X_0 \in \mathcal{M}_r$ be arbitrary. Suppose that $X_0 = U_t \ast S_t \ast V_t^T$, then the tangent space of $\mathcal{M}_r$ at $X_0$ can be given as

$$T_{X_0} \mathcal{M}_r = \{ U_t \ast Z_1^T + Z_2 \ast V_t^T \},$$

(8)

where $Z_1 \in \mathbb{R}^{n_2 \times r \times n_3}$ and $Z_2 \in \mathbb{R}^{n_1 \times r \times n_3}$ are free parameters.

The tangent bundle of $\mathcal{M}_r$, denoted by $T \mathcal{M}_r$, is defined as the disjoint union of the tangent spaces at all points of $\mathcal{M}_r : T \mathcal{M}_r = \bigcup_{p \in \mathcal{M}_r} T_p \mathcal{M}_r$. In some sense, the tangent bundle can be seen as a collection of vector spaces. Note that when a smooth manifold $\mathcal{M}$ is endowed with a specific Riemannian metric $g$, then the pair $(\mathcal{M}, g)$ is called a Riemannian manifold. Here, for the smooth manifold $\mathcal{M}_r$, the Riemannian metric $g_{X_0}$ is defined as

$$g_{X_0}(\zeta, \eta) := \langle \zeta, \eta \rangle \quad \text{with} \quad X_0 \in \mathcal{M}_r \quad \text{and} \quad \zeta, \eta \in T_{X_0} \mathcal{M}_r.$$
2.4.1 Retraction

A Riemannian manifold is not a linear space in general, the calculations required for a continuous optimization method need to be performed in its tangent space. Therefore, in each step, a so-called retraction mapping is needed to project points from a tangent bundle $T\mathcal{M}$ to the manifold $\mathcal{M}$ to generate the new iteration. Known from Definition 1 in [2] that retractions are essentially first-order approximations of the exponential map of the manifold which is usually expensive to compute. If $\mathcal{M}$ is an embedded submanifold, then the orthogonal projection

$$P_\mathcal{M}(x + \xi) = \arg\min_{y \in \mathcal{M}} \|x + \xi - y\|$$

include the so called projective retraction

$$R : \mathcal{U} \to \mathcal{M}, \ (x, \xi) \to P_\mathcal{M}(x + \xi).$$

For the fixed rank matrix manifold case, mapping $R$ can be computed in closed-form by the SVD truncation [28]. In our setting, the Riemannian manifold $\mathcal{M}_r$ is a embedded manifold of $\mathbb{R}^{n_1 \times n_2 \times n_3}$, we can choose metric projection as a retraction:

$$R_X : \mathcal{U} \to \mathcal{M}_r, \ (X, \xi) \to P_{\mathcal{M}_r}(X + \xi) := \arg\min_{Z \in \mathcal{M}_r} \|X + \xi - Z\|_F,$$

(11)

where $\mathcal{U} \subseteq T_X \mathcal{M}_r$ is a suitable neighborhood around zero and $P_{\mathcal{M}_r}$ is the orthogonal projection onto $\mathcal{M}_r$. Under the tensor $t_c$-SVD framework, and recall the retraction operator defined in (11), we can get the follows.

**Proposition 2.4. ($t_c$-SVD truncation as Retraction)** Let $X \in \mathcal{M}_r$, with $r = (r_1, ..., r_{n_3})$. The map

$$R_X : T_X \mathcal{M}_r \to \mathcal{M}_r, \ (X, \xi) \to P_{\mathcal{M}_r}(X + \xi) := \mathcal{U} \ast \mathcal{S}_r \ast \mathcal{V}^T$$

(12)

where $\mathcal{U} \in \mathbb{R}^{n_1 \times r \times n_3}$, $\mathcal{V} \in \mathbb{R}^{n_2 \times r \times n_3}$, $\text{rank}_k(\mathcal{U}) = \text{rank}_k(\mathcal{V}) = r$, $\mathcal{U}^T \ast \mathcal{U} = I_r$, $\mathcal{V}^T \ast \mathcal{V} = I_r$, and $\mathcal{S}_r$ is a diagonal tensor with

$$\mathcal{S}_r(i, i) := \begin{cases} 
\hat{S}_r(i) & \text{if } i \leq r_i, \\
0 & \text{if } i > r_i,
\end{cases}$$

is a retraction on $\mathcal{M}_r$ around $X$.

In addition, vector transport is defined as a method to transport tangent vectors from one tangent space to another, which was introduced in [1, 19]. In the Riemannian conjugate gradient descent method discussed here, the search direction is a linear combination of the projected gradient direction and the past search direction projected onto the tangent space of the current estimate. Then we can define

$$T_{X \to Y} : T_X \mathcal{M}_r \to T_Y \mathcal{M}_r, \ \xi \to P_{T_Y \mathcal{M}_r}(\xi),$$

where $P_{T_Y \mathcal{M}_r}$ is defined as in [10].
3 The Convergence Analysis

3.1 Sampling with Replacement

For matrix or tensor completion problems, most of the existing work [4, 13, 19, 28] studied a Bernoulli sampling model as a proxy uniform sampling. It has been proved that the probability of failure when the set of observed entries is sampled uniform from the collection of set is bounded by 2 times of the probability of failure under the Bernoulli model [5]. However, it is bigger or equal to the probability that fails when the entries are sampled independently with replacement [24]. It is surprising that after changing the sampling model, most of the theorems from [4] came to be simple consequences of a noncommutative variant of Bernstein’s inequality [24].

In this paper, we consider the sampling with replacement model for $\Omega$ in which each index is sampled independently from the uniform distribution on $\{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \times \{1, \ldots, n_3\}$.

At first glance, sampling with replacement is not suitable for analyzing matrix or tensor completion problems, as there may be some duplicate entries. However, the maximum duplication of any entry can be derived. Then, this model can be viewed as a proxy for the uniform sampling model.

Different from the existing conclusions which based on sampling without replacement model, here $R_\Omega$ defined as (4) is not a unitary projection if there are duplicates in $\Omega$. Then, the tensor completion model (1) can be rewritten as

$$\min_{Z} f(Z) = \frac{1}{2} \|R_\Omega(Z - A)\|_F^2 \quad \text{s.t.} \quad \text{rank}_t(Z) = r.$$ 

Suppose that $m = |\Omega|$ and $n = \max\{n_1, n_2, n_3\}$, then for $\beta > 1$, we have the following lemma.

**Lemma 3.1.** With high probability at least $1 - n^{3-3\beta}$, the maximum number of repetitions of any entry in $\Omega$ is less than $\frac{10}{3} \log n$ for $n > 9$ and $\beta > 1$.

**Proof.** The proof follows the lines of [24, Proposition 5] in the matrix case. The tool used here is the standard Chernoff bound for the Bernoulli distribution. Note that for a fixed entry, the probability it is sampled more than $t$ times is equal to the probability of more than $t$ heads occurring in a sequence of $m$ tosses where the probability of a head is $\frac{1}{n_1n_2n_3}$. It follows from [9] that

$$P[\text{more that } t \text{ head in } m \text{ trials}] \leq \left( \frac{m}{n_1n_2n_3} \right)^t \exp \left( t - \frac{m}{n_1n_2n_3} \right).$$

Then if $n \geq 9$, applying the union bound over all of the $n_1n_2n_3$ entries we can get

$$P[\text{any entry is selected more than } \frac{10}{3} \beta \log n \text{ times}]$$

$$\leq n_1n_2n_3 \left( \frac{10}{3} \beta \log n \right) - \frac{10}{3} \beta \log n \exp \left( \frac{10}{3} \beta \log n \right)$$

$$\leq n_1n_2n_3 \left( \frac{10}{3} \beta \log n \log \left( \frac{10}{3} \beta \log n \right) \right) \exp \left( \frac{10}{3} \beta \log n \right) \leq n_1n_2n_3 \left( \frac{10}{3} \beta \log n \right) \log \left( \frac{10}{3} \beta \log n \right)^{-1} \leq n^{3-3\beta}.$$

It follows from Lemma 3.1 that $\|R_\Omega\| \leq \frac{10}{3} \beta \log n$ with high probability.
3.2 The Algorithm

We first identify a small neighborhood around the measured low rank tensor such that for any given initial guess in this neighborhood, the tensor Riemannian conjugate gradient descent (Algorithm 2) will converge linearly to the objective tensor. In Algorithm 2 (line 3), the search direction is a linear combination of the projected gradient descent direction and the past search direction projected onto the tangent space of the current estimate. The orthogonalization weight \( \beta \) in line 2 is selected in a way such that \( P_{T_l}(P_l) \) is conjugate orthogonal to \( P_{T_l}(P_{l-1}) \). In line 6, \( H_r(\cdot) \) denotes the \( tc \)-SVD truncation operator given in Proposition 2.4.

**Algorithm 2 Tensor Riemannian Conjugate Gradient Descent**

**Initialization:** \( X_0 = U_0 \ast S_0 \ast V_0^T \in \mathbb{R}^{n_1 \times n_2 \times n_3} \), \( \beta_0 = 0 \), and \( Q_{-1} = 0 \).

for \( l = 0, 1, \ldots \) do

1: \( G_l = R_{\Omega}(X - X_l) \);

2: \( \beta_l = -\frac{\langle P_{T_{l-1}}(G_l), R_{\Omega}P_{T_l}(Q_{l-1}) \rangle}{\| P_{T_{l-1}}(G_l) \|_F \| P_{T_l}(Q_{l-1}) \|_F} \);

3: \( Q_l = P_{T_l}(G_l) + \beta_l P_{T_l}(Q_{l-1}) \);

4: \( \alpha_l = \frac{\langle P_{T_l}(G_l), R_{\Omega}P_{T_l}(Q_l) \rangle}{\langle P_{T_l}(Q_l), R_{\Omega}P_{T_l}(Q_l) \rangle} \);

5: \( W_l = X_l + \alpha_l Q_l \);

6: \( X_l = H_r(W_l) \)

end for

In order to improve the robustness of the non-linear conjugate gradient descent methods, we introduce a restarted variant in Algorithm 4: that is, \( \beta_1 = 0 \) and restarting occurs as long as either of the following conditions is violated:

\[
\frac{\| \langle P_{T_l}(G_l), P_{T_l}(Q_{l-1}) \rangle \|}{\| P_{T_l}(G_l) \|_F \| P_{T_l}(Q_{l-1}) \|_F} \leq k_1, \quad \| P_{T_l}(G_l) \|_F \leq k_2 \| P_{T_l}(Q_{l-1}) \|_F.
\] (13)

The first restarting condition guarantees that the residual will be substantially orthogonal to the past search direction when projected onto the tangent space of current estimate so that the new search direction can be sufficiently gradient related. In the classical CG algorithm for linear systems, the residual is exactly orthogonal to all the past search directions. Roughly speaking, the second restarting condition implies that the projection of current residual cannot be too large when compared to the projection of the past residual since the search direction is gradient related by the first restarting condition. In our implementations, we take \( k_1 = 0.1 \) and \( k_2 = 1.0 \).

We first list some lemmas which will be used many times in the sequel. Their proofs can be found in Section 7.

**Lemma 3.2.** Suppose that \( X, X_l \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) with \( \text{rank}_t(X) = \text{rank}_t(X_l) = r \) and \( \text{rank}_{ct}(X_l) = \text{rank}_{ct}(X_l) = r \). Their skinny \( tc \)-SVDs are given as \( X = U \ast \Sigma \ast V^T \) and \( X_l = U_l \ast T_l \ast V_l^T \). Let \( T \) and \( T_l \) be the corresponding tangent spaces of the fixed transformed multi-rank manifold at \( X \) and \( X_l \).
Lemma 3.3. Let $R_{\Omega}$ be defined as \(3\), $X, X_i, T, T_i$ be defined as in Lemma 3.2 and $n = \max\{n_1, n_2, n_3\}$. Assume that

\[
\| R_{\Omega} \| \leq \frac{10}{3} \beta \log n, \quad (14) \\
\| P_{\Omega} - p^{-1} P_{\Omega} R_{\Omega} P_{\Omega} \| \leq \epsilon_0, \quad (15) \\
\frac{\| X_i - \mathcal{X} \|_F}{\sigma_{\min}(\mathcal{X})} \leq \frac{3p^\frac{1}{2} \epsilon_0}{20 \beta (1 + \epsilon_0) \log n}. \quad (16)
\]

for some $0 < \epsilon_0 < 1$ and $\beta > 1$. Then

\[
\| R_{\Omega} P_{\Omega} \| \leq \frac{10}{3} \beta \log(n)(1 + \epsilon_0)p^\frac{1}{2}, \quad (17) \\
\| P_{\Omega} - p^{-1} P_{\Omega} R_{\Omega} P_{\Omega} \| \leq 4 \epsilon_0. \quad (18)
\]

Lemma 3.4 (Lemma 4.4 in [30]). Let $\rho_1, \rho_2$ and $\gamma$ be positive constants satisfying $\rho_2 > \rho_1$. Define

\[
\gamma_1 = \rho_1 + \gamma, \gamma_2 = (\rho_2 - \rho_1) \gamma, \text{ and } v = \frac{1}{2} (\gamma_1 + \sqrt{\gamma_1^2 + 4 \gamma_2}).
\]

Let $\{c_l\}_{l \geq 0}$ be a non-negative sequence satisfying $c_1 \leq v c_0$ and

\[
c_{l+1} \leq \rho_1 c_l + \rho_2 \sum_{j=0}^{l-1} \gamma^{l-j} c_j, \quad \forall \ l \geq 1.
\]

Then if $\gamma_1 + \gamma_2 < 1$, we have $v < 1$ and $c_{l+1} \leq v^{l+1} c_0$.

In addition, we need to estimate $\alpha_l$ and $\beta_l$ in Algorithm 4 with the restarting conditions in [13]. Their proof can be found in Section 7.

Lemma 3.5. Assume that (18) is satisfied. When restarting occurs, $\beta_l = 0$, then the stepsize $\alpha_l$ in Algorithm 2 can be bounded as

\[
\frac{1}{1 + 4 \epsilon_0} \leq \alpha_l = \frac{\| P_{\Omega} (G_l) \|_F^2}{\langle P_{\Omega} (G_l), R_{\Omega} P_{\Omega} (G_l) \rangle} \leq \frac{1}{1 - 4 \epsilon_0}. \quad (18)
\]

Moreover, the spectral norm of $\| P_{\Omega} - \alpha_l P_{\Omega} R_{\Omega} P_{\Omega} \|$ can be bounded as

\[
\| P_{\Omega} - \alpha_l P_{\Omega} R_{\Omega} P_{\Omega} \| \leq \frac{8 \epsilon_0}{(1 - 4 \epsilon_0)}. \quad (18)
\]
Lemma 3.6. Assume that \((13)\) is satisfied. When restarting dose not occur, \(\beta_t \neq 0\), then we have
\[
|\beta_t| \leq \epsilon_\beta \quad \text{and} \quad |\alpha_t \cdot p - 1| \leq \epsilon_\alpha,
\]
where
\[
\epsilon_\beta = \frac{4k_2 \epsilon_0}{(1 - 4\epsilon_0)} + \frac{k_1 k_2}{(1 - 4\epsilon_0)}, \quad \epsilon_\alpha = \frac{4 \epsilon_0}{(1 - 4\epsilon_0) - k_1(1 + 4\epsilon_0)}.
\]
Moreover, the spectral norm of \(\|P_{\Omega_i} - \alpha_t P_{\Omega_i} R_{\Omega_i} P_{\Omega_i}\|\) can be bounded as
\[
\|P_{\Omega_i} - \alpha_t P_{\Omega_i} R_{\Omega_i} P_{\Omega_i}\| \leq 4\epsilon_0 + \epsilon_\alpha (1 + 4\epsilon_0).
\]

With the above tools in hand, we can get one of the main results of this paper.

Theorem 3.7. Suppose that \((14)-(16)\) are satisfied, where \(\beta > 1\), \(k_1, k_2\) are defined as in \((13)\), and \(\epsilon_0\) is a positive numerical constant satisfying \(\gamma_1 + \gamma_2 < 1\), where
\[
\gamma_1 = \frac{18\epsilon_0 - 10k_1 \epsilon_0(1 + 4\epsilon_0)}{(1 - 4\epsilon_0) - k_1(1 + 4\epsilon_0)} + \frac{4k_2 \epsilon_0 + k_1 k_2}{1 - 4\epsilon_0}, \quad \gamma_2 = \frac{8k_2 \epsilon_0 + 2k_1 k_2}{1 - 4\epsilon_0}.
\]
Then we have \(v_{cg} = \frac{1}{2}(\gamma_1 + \sqrt{\gamma_1^2 + 4\gamma_2}) < 1\) and the iterates \(X_t\) generated by Algorithm 2 satisfy
\[
\|X_t - X\|_F \leq v_{cg}^t \|X_0 - X\|_F.
\]
When \(k_1 = k_2 = 0\), \(v_{cg}\) is reduced to \(\frac{18\epsilon_0}{1 - 4\epsilon_0} < 1\). On the other hand, we have \(\lim_{\epsilon_0 \to 0}(\gamma_1 + \gamma_2) = 3k_1 k_2\). So if \(k_1 k_2 < \frac{1}{3}\), \(\gamma_1 + \gamma_2\) can be less than one when \(\epsilon_0\) is small. In particular, when \(k_1 = 0.1\) and \(k_2 = 1\), a sufficient condition for \(\gamma_1 + \gamma_2 < 1\) is \(\epsilon_0 \leq 0.01\).

Proof. We prove the results by induction. Firstly, assume that for all \(j \leq l\), we have
\[
\frac{\|X_j - X\|_F}{\sigma_{\min}(X)} \leq \frac{3p^2 \epsilon_0}{20 \beta \log (n + \epsilon_0)}.
\]
Since \((14)-(16)\) are satisfied, Lemma 3.3 implies,
\[
\|R_{\Omega_i} P_{\Omega_i}\| \leq \frac{10}{3} \beta \log (n + 1 + \epsilon_0) p^2 \quad \text{and} \quad \|P_{\Omega_i} - P_{\Omega_i} R_{\Omega_i} P_{\Omega_i}\| \leq 4\epsilon_0.
\]
Thus the assumptions in Lemma 3.6 are satisfied for all \(j \leq l\).

For the case \(k = l + 1\). Noting that \(W_1 = X_1 + \alpha_l Q_l\) in Algorithm 4, and \(X_{l+1} = \mathcal{H}_e(W_l)\), there holds
\[
\|X_l - X\|_F \leq \|X_l - W_l\|_F + \|W_l - X\|_F \leq 2\|W_l - X\|_F.
\]
It follows that
\[
\begin{aligned}
\|X_{l+1} - X\|_F &\leq 2\|X_l + \alpha_l Q_l - X\|_F \\
&= 2\|X_l - X - \alpha_l P_{\Omega_i} R_{\Omega_i}(X_l - X) + \alpha_l \beta_l P_{\Omega_l}(Q_{l-1})\|_F \\
&= 2\|(P_{\Omega_i} - \alpha_l P_{\Omega_i} R_{\Omega_i}) (X_l - X) + (I - P_{\Omega_i}) (X_l - X) - \alpha_l P_{\Omega_i} R_{\Omega_l}(I - P_{\Omega_l})(X_l - X)\|_F \\
&\leq 2\|(P_{\Omega_i} - \alpha_l P_{\Omega_i} R_{\Omega_i}) (X_l - X)\|_F + 2\|(I - P_{\Omega_i})(X_l - X)\|_F \\
&\quad + 2\|\alpha_l P_{\Omega_i} R_{\Omega_l}(I - P_{\Omega_l})(X_l - X)\|_F + 2\|\alpha_l \beta_l P_{\Omega_l}(P_{\Omega_l})\|_F \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
\]
When one of the conditions in (13) is violated, and the restarting occurs, then $\beta_l$ is set 0. In this case, Lemmas 3.2 and 3.5 imply that

$$I_1 = 2\| (P_{T_l} - \alpha_l P_{T_l} R_\Omega P_{T_l}) (X_l - X') \|_F \leq \frac{16\epsilon_0}{(1 - 4\epsilon_0)} \| X_l - X' \|_F,$$

$$I_2 = 2\| (I - P_{T_l}) (X_l - X') \|_F \leq \frac{\epsilon_0}{1 - 4\epsilon_0},$$

$$I_3 = 2\| \alpha_l P_{T_l} R_\Omega (I - P_{T_l}) (X_l - X') \|_F \leq \frac{\epsilon_0}{1 - 4\epsilon_0},$$

$$I_4 = 0.$$

Therefore, there holds

$$\| X_{l+1} - X' \|_F \leq \frac{18\epsilon_0}{1 - 4\epsilon_0} \| X_l - X' \|_F \leq \left( \frac{18\epsilon_0}{1 - 4\epsilon_0} \right)^l \| X_0 - X' \|_F.$$

When the conditions in (13) are satisfied, and the restarting does not occur and the following results can be derived. For $I_1$, by Lemma 3.5, we have

$$I_1 = 2\| (P_{T_l} - \alpha_l P_{T_l} R_\Omega P_{T_l}) (X_l - X') \|_F \leq (8\epsilon_0 + 2\epsilon_\alpha (1 + 2\epsilon_0)) \| X_l - X' \|_F.$$

For $I_2$, noting that $P_{T_l} (X_l) = X_l$, by the fifth inequality in Lemma 3.2, we have

$$I_2 = 2\| (I - P_{T_l}) (X_l - X') \|_F = 2\| (I - P_{T_l}) (X) \|_F \leq \frac{\| X_l - X' \|_F^2}{\sigma_{\min}(X)}$$

$$\leq \frac{3p^2 \epsilon_0}{10\beta \log n (1 + \epsilon_0)} \| X_l - X' \|_F \leq \frac{\epsilon_0}{1 - 4\epsilon_0} \| X_l - X' \|_F \leq (1 + \epsilon_\alpha) \epsilon_0 \| X_l - X' \|_F.$$

For $I_3$, noting the definition of $\epsilon_\alpha$ and the inequality (17) in Lemma 3.3, we have

$$I_3 = 2\| \alpha_l P_{T_l} R_\Omega (I - P_{T_l}) (X_l - X') \|_F \leq (1 + \epsilon_\alpha) \epsilon_0 \| X_l - X' \|_F.$$

For $I_4$, noting that

$$\beta_l P_{T_l} (P_{l-1}) = \sum_{j=1}^{l-1} (\Pi_{i=j+1}^{l-1} \beta_i) (\Pi_{k=j}^{l-1} P_{T_k}) (G_j),$$

therefore, there holds

$$\| X_{l+1} - X' \|_F \leq \frac{18\epsilon_0}{1 - 4\epsilon_0} \| X_l - X' \|_F \leq \left( \frac{18\epsilon_0}{1 - 4\epsilon_0} \right)^l \| X_0 - X' \|_F.$$
we have

\[ I_4 \leq 2|\alpha_l| \sum_{j=0}^{l-1} (\Pi^I_{l=j+1} \beta_j) \| (\Pi^{l-1}_{l=j}) P_{\Omega_i}(G_j) \|_F \]

\[ \leq 2|\alpha_l| \sum_{j=0}^{l-1} \epsilon^{l-j}_{\beta} \| P_T(G_j) \|_F = 2|\alpha_l| \sum_{j=0}^{l-1} \epsilon^{l-j}_{\beta} \| P_T R_{\Omega_i}(X_j - X) \|_F \]

\[ \leq 2|\alpha_l| \sum_{j=0}^{l-1} \epsilon^{l-j}_{\beta} (\| P_{\Omega_i} R_{\Omega_i} (X_j - X) \|_F + \| P_{\Omega_i} R_{\Omega_i} (I - P_T) (X_j - X) \|_F) \]

\[ \leq 2|\alpha_l| \sum_{j=0}^{l-1} \epsilon^{l-j}_{\beta} (\| P_{\Omega_i} R_{\Omega_i} (X_j - X) \|_F + \| R_{\Omega_i} P_T \| \frac{\| (X_j - X) \|_2^2}{\sigma_{min}(X)}) \]

\[ \leq \frac{1 + \epsilon_{\alpha}}{p} \sum_{j=0}^{l-1} \epsilon^{l-j}_{\beta} (2(1 + 4\epsilon_0) + p\epsilon_0) \| X_j - X \|_F \]

\[ = (1 + \epsilon_{\alpha}) \sum_{j=0}^{l-1} \epsilon^{l-j}_{\beta} (2(1 + 4\epsilon_0) + \epsilon_0) \| X_j - X \|_F, \]

where the second inequality follows from

\[ \| (\Pi^{l-1}_{l}} P_{\Omega_i}(G_j) \|_F \leq \| P_T(G_j) \|_F. \]

Taking \( I_1 - I_4 \) into \( \| \) yields

\[ \| X_{l+1} - X \|_F \]

\[ \leq (10\epsilon_0 + 2\epsilon_\alpha(1 + 5\epsilon_0)) \| X_l - X \|_F + (1 + \epsilon_\alpha)(2(1 + 4\epsilon_0) + \epsilon_0) \sum_{j=0}^{l-1} \epsilon^{l-j}_{\beta} \| X_j - X \|_F \]

\[ \leq (10\epsilon_0 + 2\epsilon_\alpha(1 + 5\epsilon_0)) \| X_l - X \|_F + (2 + 10\epsilon_0 + 2\epsilon_\alpha)(1 + 5\epsilon_0) \sum_{j=0}^{l-1} \epsilon^{l-j}_{\beta} \| X_j - X \|_F \]

\[ := \rho_1 \| X_l - X \|_F + \rho_2 \sum_{j=0}^{l-1} \epsilon^{l-j}_{\beta} \| X_j - X \|_F. \]

Define

\[ \gamma_1 = \rho_1 + \epsilon_\beta = \frac{18\epsilon_0 - 10\epsilon_0 k_1(1 + 4\epsilon_0)}{(1 - 4\epsilon_0) - k_1(1 + 4\epsilon_0)} + \frac{4k_2\epsilon_0 + k_1 k_2}{1 - 4\epsilon_0}, \]

\[ \gamma_2 = (\rho_2 - \rho_1)\epsilon_\beta = \frac{8k_2\epsilon_0 + 2k_1 k_2}{1 - 4\epsilon_0}, \]

\[ \nu_{cg} = \frac{1}{2}(\gamma_1 + \sqrt{\gamma_1^2 + \gamma_2^2}). \]
When $l = 0$, it is easy to get
\[ \|X_1 - X\|_F \leq \frac{18\varepsilon_0}{1 - 4\varepsilon_0} \|X_0 - X\|_F \leq v_{cg}\|X_0 - X\|_F. \]

It follows Lemma 3.4 that if $\gamma_1 + \gamma_2 < 1$, we have $v_{cg} < 1$, then
\[ \|X_{l+1} - X\|_F \leq v_{l+1}^{l+1}\|X_0 - X\|_F, \]
which completes the proof. \qed

We remark that the manifold $M_r$ is not closed, and the closure of $M_r$ are bounded by transformed multi-rank $r$ (point bounded) tensors. Then a sequence of tensors in $M_r$ may approach a tensor which the $i$th transformed tubal rank less than $r_i$, which is saying that the limit of Algorithm 2 may not be in the manifold $M_r$. In order to avoid this statements happens, we need the following results.

**Proposition 3.8** (Proposition 4.1 in [28]). Let $\{X_i\}$ be an infinite sequence of iterates generated by Algorithm 2. Then, every accumulation point $X_*$ of $\{X_i\}$ satisfies $P_T X_* R_\Omega(X_*) = P_T X_* R_\Omega(A)$.

### 4 The Initialization

In Theorem 3.7, there are three conditions (14)-(16) to guarantee the convergence of Alg. 2. The requirement for $R_\Omega$ to be bounded in (14) is just an requirement of the sampling model, and by Lemma 3.1 it can be satisfied with probability at least $1 - \frac{3n^3}{2}$. The second condition (15) plays a key role in nuclear norm minimization for tensor completion which have been proved in [8, 24, 29] under different sampling models. For tensor case, it also can be seen as a local restricted isometry property and has been established in [13] for the Bernoulli model. In our setting, we consider the sampling with replacement model instead of without replacement model, then by Lemma 4.2 we have (15) satisfied with probability at least $1 - 2(m(1) n_3) \frac{1}{\mu n(1) n_3}$, as long as $m \geq \frac{20}{\mu} \beta \mu r n_3 \log(n(1) n_3)$. Thus the only issue that remains to be addressed is how to produce an initial guess that is sufficiently close to the original tensor. In this section, we will consider two initialization strategies.

#### 4.1 Hard Thresholding

Based on the transformed tensor incoherence conditions given in (6), we can prove the following lemma which will be used many times in the proofs of Lemmas 4.2 and 4.3.

**Lemma 4.1.** Let $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ be an arbitrary tensor, and $T$ be given as (8). Suppose that the transformed tensor incoherence conditions (6) are satisfied, then
\[ \|P_T (E_{abc})\|_F^2 \leq \frac{2\mu_0 r}{n(2)}. \]
Proof. By the definition of $P_{T}$, it suffices to show $\|U^T \ast e_{abc}\|_F^2 \leq \mu_0 r/n_1$ and $\|e_{abc} \ast V\|_F^2 \leq \mu_0 r/n_2$. We only prove the first one and the second one can be proved similarly. Simple calculation shows that

$$\|U^T \ast e_{abc}\|_F^2 = \|U^T e_{abc}\|_F^2 = \sum_{k=1}^{n_3} \|\mu_k \cdot U_k e_a e_b\|_2^2 = \sum_{k=1}^{n_3} \|\mu_k \cdot U_k e_a\|_2^2,$$

where $[\mu_1, \cdots, \mu_{n_3}]^T$ is the $c$-th column of the DCT matrix. Noting that the magnitude of each entry of the DCT matrix is bounded by $2/\sqrt{n_3}$, $\|U^T \ast e_{abc}\|_F^2 \leq \mu_0 r/n_1$ follows immediately from the tensor incoherence condition.

Denote $n_{(1)} = \max\{n_1, n_2\}$, $n_{(2)} = \min\{n_1, n_2\}$. Jiang [13] derived some lemmas based on Bernoulli model sampling without replacement model. For sampling with replacement model, we can get the following results.

**Lemma 4.2.** Suppose $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is a fixed tensor, and $\Omega$ with $|\Omega| = m$ is a set of indices sampled independently and uniformly with replacement. Let $\mathcal{X} = \mathcal{U} \ast \mathcal{S} \ast \mathcal{V}^T$ be a tubal rank $r$ tensor which satisfy the incoherence conditions given in (6). Then for all $\beta > 1$,

$$\|\frac{n_1 n_2 n_3}{m} \mathcal{P}_T \mathcal{R}_\Omega \mathcal{P}_T - \mathcal{P}_T\|_{op} \lesssim \sqrt{\frac{\mu_0 r n_{(1)} n_3 \beta \log(n_{(1)} n_3)}{m}},$$

holds with high probability with the condition that $m \gtrsim \beta \mu_0 r n_{(1)} n_3 \log(n_{(1)} n_3)$.

**Lemma 4.3.** Suppose $\mathcal{Z} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is a fixed tensor, and $\Omega$ with $|\Omega| = m$ is a set of indices sampled independently and uniformly with replacement. Then for all $\beta > 1$

$$\|(I - \frac{n_1 n_2 n_3}{m} \mathcal{R}_\Omega) \mathcal{Z}\| \lesssim \sqrt{\frac{\beta n_{(1)}^2 n_{(2)} n_3 \log(n_{(1)} n_3)}{m}} \|\mathcal{Z}\|_\infty,$$

holds with high probability with the condition that $m \gtrsim \beta n_{(1)} n_3 \log(n_{(1)} n_3)$.

**Lemma 4.4.** Suppose that $|\Omega| = m$ is a set of indices sampled independently and uniformly with replacement. Let $\mathcal{X}_0 = \mathcal{H}_r(\mathcal{S}(\mathcal{R}_\Omega(\mathcal{X})))$. Then

$$\|\mathcal{X}_0 - \mathcal{X}\|_F \lesssim \sqrt{\frac{\mu_1^2 r^2 \beta n_{(1)} n_3 \log(n_{(1)} n_3)}{m}} \|\mathcal{X}\|$$

holds with high probability with the condition that $m \gtrsim \beta n_{(1)} \log(n_{(1)} n_3)$.

Then we can establish the following theorem.

**Theorem 4.5.** Let $\mathcal{X} = \mathcal{U} \ast \mathcal{S} \ast \mathcal{V}^T \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ with $\text{rank}_u(\mathcal{X}) = r$ and $\text{rank}_c(\mathcal{X}) = r$. Suppose that $|\Omega| = m$ is a set of indices sampled independently and uniformly with replacement. Let $\mathcal{X}_0 = \mathcal{H}_r(\mathcal{S}(\mathcal{R}_\Omega(\mathcal{X})))$. Then the iterates generated by Algorithm 2 are guaranteed to converge to $\mathcal{X}$ with high probability provided

$$m \gtrsim \max\left\{\frac{\mu_0 r}{c_0} \log \frac{1}{n_{(1)} n_3}, \frac{\mu_1 r (1 + \epsilon_0) \kappa \beta^{\frac{1}{2}}}{c_0} n_{(2)}^2 \log n\right\} \beta n_{(1)} n_3 \log \frac{1}{n_{(1)} n_3}.$$

where $\kappa$ is the condition number of $\mathcal{X}$.  

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Proof. It is readily seen that when (21) is satisfied, the assumptions in Lemma 3.3 hold. Thus this theorem follows immediately from Theorem 3.7.

4.2 Resampling and Trimming

In Theorem 4.5, the sampling numbers depends on \( n^5 \), however, in [13] the sampling numbers can be reduced to \( n^2 \). Then we need to find some new initialization scheme to reduce the sampling numbers theoretically. In this subsection, we generalize the trimming procedure which used in matrix case [29] to tensor case, and the specific details can be found in Algorithm 3. Moreover, when the resampling scheme breaks the dependence between the past iterate and the new sampling set, we apply the tensor trimming method (Algorithm 4) to project the estimate onto the set of \( \mu \)-incoherent tensors. After that, we need to prove the output of Algorithm 3 reaches a neighborhood of the original tensor where Theorem 3.7 is activated. First of all, we need the following lemmas.

Algorithm 3 Initialization via Tensor Resampled Riemannian Gradient Descent and Tensor Trimming

Partition \( \Omega \) into \( L + 1 \) equal groups: \( \Omega_0, \ldots, \Omega_L \); and the size of every group is denoted by \( m \).

Set \( Z_0 = \mathcal{H}_r(\frac{m}{n^3} R_{\Omega_0}(X)) \)
for \( l = 0, \ldots, L - 1 \) do
1: \( \hat{Z}_l = \text{trim}(Z_l) \);
2: \( Z_{l+1} = \mathcal{H}_r(\frac{n^3}{m} P_{\mathcal{T}_l} R_{\Omega_{l+1}}(X - \hat{Z}_l)) \);
end for
Output: \( X_0 = Z_L \)

Algorithm 4 Tensor Trimming

Input \( Z_l = U_l \ast S_l \ast V_l^T \).
Output: \( \hat{X}_l = A_l \ast S_l \ast B_l^T \), where,
\[
A_l^{[i]} = \frac{U_l^{[i]}}{\|U_l^{[i]}\|_F} \min\{\|U_l^{[i]}\|_F, \sqrt{\frac{m}{n_1}}\}, \quad B_l^{[i]} = \frac{V_l^{[i]}}{\|V_l^{[i]}\|_F} \min\{\|V_l^{[i]}\|_F, \sqrt{\frac{m}{n_2}}\}.
\]

Lemma 4.6. Let \( X, X_l \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) be two tensors with \( \text{rank}(X) = r \) and \( \text{rank}_{ct}(X) = \text{rank}_{ct}(X_l) = r \). Given their skinny \( t_c \)-SVD \( X_l = U_l \ast S_l \ast V_l^T \) and \( X = U \ast S \ast V^T \), let \( T \) and \( T_l \) be the tangent spaces of the fixed transformed multi-rank manifold at \( X, X_l \), respectively. Assume \( \Omega \) with \( |\Omega| = m \) is a set of indices sampled independently and uniformly with replacement. Suppose that
\[
\|P_{U_i} \ast \tilde{e}_i\|_F \leq \sqrt{\frac{\mu_0 r}{n_1}}, \quad \|P_{V_j} \ast \tilde{e}_j\|_F \leq \sqrt{\frac{\mu_0 r}{n_2}}, \quad \|P_{U_l} \ast \tilde{e}_i\|_F \leq \sqrt{\frac{\mu_0 r}{n_1}}, \quad \text{and} \quad \|P_{V_l} \ast \tilde{e}_j\|_F \leq \sqrt{\frac{\mu_0 r}{n_2}}
\]
hold for all \( 1 \leq i, j \leq n \). Then for any \( \beta > 1 \),
\[
\left\| \frac{n_1 n_2 n_3}{m} P_{\mathcal{T}_l} R_{\Omega}(P_{\mathcal{T}_l} - P_{U_i}) - P_{\mathcal{T}_l}(P_{\mathcal{T}_l} - P_{U_i}) \right\|_F \lesssim \sqrt{\frac{\beta \mu_0 m(1)n_3r \log(n_1)n_3}{m}}
\]
holds with high probability provided \( m \gtrsim \beta n(1)n_3 r \log n(1)n_3 \).

**Lemma 4.7.** Let \( \mathcal{U}, \mathcal{U} \in \mathbb{R}^{n_1 \times r \times n_3} \) be two orthogonal tensors. Then there exist an orthogonal tensor \( Q \in \mathbb{R}^{r \times r \times n_3} \) such that
\[
\| \mathcal{U} - \mathcal{U} * Q \|_F \leq \| \mathcal{U} \mathcal{U}^T - \mathcal{U} \mathcal{U}^T \|_F.
\]

**Lemma 4.8.** Suppose that \( Z_i \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) satisfies \( Z_i = \mathcal{U}_i \mathcal{S}_i \mathcal{V}_i^T \) and
\[
\| Z_i - \mathcal{X} \|_F \leq \frac{\sigma_{\text{min}}(\mathcal{X})}{10\sqrt{2}},
\]
where \( \text{rank}_i(\mathcal{U}_i) = \text{rank}_i(\mathcal{V}_i) = \text{rank}(Z_i) = r \). Then for \( 1 \leq i \leq n_1, 1 \leq j \leq n_2 \) the tensor \( \hat{Z}_i = \hat{\mathcal{U}}_i \hat{\mathcal{S}}_i \hat{\mathcal{V}}_i^T \) returned by Algorithm 3 satisfies
\[
\| \hat{\mathcal{U}}_i \hat{\mathcal{S}}_i \mathcal{V}_i^T \|_F \leq \frac{10}{9} \sqrt{\frac{\mu_0 r}{n_1}} \quad \text{and} \quad \| \hat{\mathcal{V}}_i \mathcal{V}_i^T \|_F \leq \frac{10}{9} \sqrt{\frac{\mu_0 r}{n_2}}.
\]

Moreover, letting \( \kappa = \frac{\sigma_{\text{max}}(\mathcal{X})}{\sigma_{\text{min}}(\mathcal{X})} \), then
\[
\| \hat{Z}_i - \mathcal{X} \|_F \leq 8\kappa \| Z_i - \mathcal{X} \|_F.
\]

With the tools in hand, we can prove the following lemma which plays a key role in deciding the bound on the number of sample entries required for tensor completion.

**Lemma 4.9.** Suppose \( \mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) with \( \text{rank}_t(\mathcal{X}) = r \) and \( \text{rank}_c(\mathcal{X}) = r \), \( \kappa \) is the condition number of \( \mathcal{X} \) and \( L \) is defined as in Algorithm 3. Then the output of Algorithm 3 satisfies
\[
\| \mathcal{X}_0 - \mathcal{X} \|_F \leq \left( \frac{5}{6} \right)^L \frac{\sigma_{\text{min}}(\mathcal{X})}{256\kappa^2}
\]
with high probability provided
\[
m \gtrsim \max\{\mu_0 3^r, \mu_1^2 r^2 \kappa^6 \} n(1)n_3 \log(n(1)n_3).
\]

**Proof.** Assume that
\[
\| Z_i - \mathcal{X} \|_F \leq \frac{\sigma_{\text{min}}(\mathcal{X})}{256\kappa^2}.
\]
It follows from Lemma 4.8 that \( \hat{\mathcal{Z}}_i \) is an incoherent tensor with incoherence parameter \( \frac{100}{81} \mu \) and
\[
\| \hat{\mathcal{Z}}_i - \mathcal{X} \|_F \leq 8\kappa \| Z_i - \mathcal{X} \|_F.
\]

The approximation error at the \((l + 1)\)th iteration can be decomposed as
\[
\| \mathcal{Z}_{l+1} - \mathcal{X} \|_F \leq 2\| (\mathcal{P}_{\mathcal{U}_i} - \frac{n^2}{m} \mathcal{P}_{\mathcal{U}_i} \mathcal{P}_{\mathcal{U}_{i+1}} \mathcal{P}_{\mathcal{T}_i})(\hat{\mathcal{Z}}_i - \mathcal{X}) \|_F
\]
\[
+ 2\| (I - \mathcal{P}_{\mathcal{T}_i})(\hat{\mathcal{Z}}_i - \mathcal{X}) \|_F + 2\frac{n^2}{m} \mathcal{P}_{\mathcal{T}_i} \mathcal{P}_{\mathcal{U}_{i+1}} (I - \mathcal{P}_{\mathcal{T}_i})(\hat{\mathcal{Z}}_i - \mathcal{X}) \|_F
\]
\[
:= I_5 + I_6 + I_7.
\]
It follows Lemma 3.2 that
\[ \|P_{\tilde{\Omega}} - \frac{n^2}{m} P_{\tilde{\Omega}} P_{\Omega_{i+1}}\| \lesssim \sqrt{\frac{\mu_0 r m n_3 \beta \log(n_1 n_3)}{m}} \]
with high probability. Thus
\[ I_5 \lesssim \kappa \sqrt{\frac{\mu_0 r m n_3 \beta \log(n_1 n_3)}{m}} \|Z_i - X\|_F. \]

By Lemma 3.2 we have
\[ I_6 \leq 2 \frac{\|\hat{Z}_i - X\|^2_F}{\sigma_{\min}(X)} \leq \frac{128 \kappa^2 \|Z_i - X\|^2_F}{\sigma_{\min}(X)} \leq \frac{1}{2} \|Z_i - X\|_F. \]

Note that \( \hat{Z}_i \) is independent of \( \Omega_{i+1} \) with the incoherence parameter \( \frac{100}{81} \kappa_0 \), then it follows Lemma 4.6 that
\[ \left\| \frac{n_1 n_2 n_3}{m} P_{\tilde{\Omega}} R_{\Omega_{i+1}} (P_{\tilde{\Omega}} - P_{\tilde{\Omega}_i}) \right\| \lesssim \sqrt{\frac{\beta \mu_0 n_3 r \log(n_1 n_3)}{m}} \]
with high probability. Moreover, due to \( X = U^T X = \hat{U}^T X = \hat{V}_i^T X \) and \( P_{\tilde{\Omega}}(\hat{Z}_i) = \hat{Z}_i \), we have
\[
(I - P_{\tilde{\Omega}})(\hat{Z}_i - X) = -(I - P_{\tilde{\Omega}})(X) \\
= -(U^T X + \hat{U}^T X + \hat{V}_i^T X + \hat{V}_i^T X = \hat{V}_i^T X = \hat{V}_i^T X + \hat{V}_i^T X = \hat{V}_i^T X + \hat{V}_i^T X = \hat{V}_i^T X + \hat{V}_i^T X) \\
= -(U^T - \hat{U} \hat{U}^T) X = (I - \hat{V}_i^T \hat{V}_i^T) \\
= (P_{\tilde{\Omega}} - P_{\tilde{\Omega}_i}) (\hat{Z}_i - X) = (I - P_{\tilde{\Omega}}). \]

Together with
\[ P_{\tilde{\Omega}}((P_{\tilde{\Omega}} - P_{\tilde{\Omega}_i}) (\hat{Z}_i - X) = (I - P_{\tilde{\Omega}}) = 0, \]
we can bound \( I_7 \) as follows,
\[ I_7 = 2 \left\| \frac{n_1 n_2 n_3}{m} P_{\tilde{\Omega}} R_{\Omega_{i+1}} (I - P_{\tilde{\Omega}})(\hat{Z}_i - X) \right\|_F \\
\leq 2 \left\| \frac{n_1 n_2 n_3}{m} P_{\tilde{\Omega}} R_{\Omega_{i+1}} (I - P_{\tilde{\Omega}})(\hat{Z}_i - X) - P_{\tilde{\Omega}}(I - P_{\tilde{\Omega}})(\hat{Z}_i - X) \right\|_F \\
\lesssim \left\| \frac{n_1 n_2 n_3}{m} (P_{\tilde{\Omega}} R_{\Omega_{i+1}} - P_{\tilde{\Omega}})(P_{\tilde{\Omega}} - P_{\tilde{\Omega}_i}) \right\| \|\hat{Z}_i - X\|_F \\
\lesssim \frac{\beta \mu_0 (\log(n_1 n_3))}{m} \|Z_i - X\|_F. \]

\[ \kappa \sqrt{\frac{\beta \mu_0 (\log(n_1 n_3))}{m}} \|Z_i - X\|_F. \]
Combining them together gives
\[ \|Z_{L+1} - X\|_F \leq \left( \frac{1}{2} + 96\kappa \sqrt{\frac{\beta \mu_0 n_1 n_3 r \log(n_1 n_3)}{m}} \right) \|Z_i - X\|_F \leq \frac{5}{6} \|Z_i - X\|_F \]
holds with high probability provided
\[ \hat{m} \gtrsim \beta \mu_0 \kappa^2 n_1 n_3 r \log(n_1 n_3). \] (23)

Noting that \( Z_0 = H_r \left( \frac{\hat{m}}{n_1 n_2 n_3} P_{\Omega_0}(X) \right) \) and
\[ \sigma_{\text{min}}(X) \frac{256}{12} = \frac{\|X\|}{\|X\|_\infty} \leq \mu_1 \sqrt{\frac{r}{n_1 n_2 n_3}} \|X\|, \]
by Lemma 4.3 we can obtain
\[ \mathbb{P} \left[ \|\rho^{-1} R_\Omega(X) - X\| > \frac{1}{256 \mu_1 \kappa^2} \sqrt{\frac{n_1 n_2}{r^2}} \|X\|_\infty \right] \leq 2n_1 n_3 \exp \left( -\frac{c n_1 n_2}{n_1 n_3} \|X\|_\infty^2 \right) = 2(n_1 n_3)^{1-c}, \]
with the proviso that
\[ \hat{m} \gtrsim \beta \mu_1^2 \kappa^6 r^2 n_1 n_3 \log(n_1 n_3). \] (24)

It follows that
\[ \|Z_0 - X\|_F \leq \sqrt{n_3} \|Z_0 - X\| \leq \sqrt{n_3} \frac{1}{256 \mu_1 \kappa^2} \sqrt{\frac{n_1 n_2}{r^2}} \|X\|_\infty \leq \sigma_{\text{min}}(X) \frac{256}{12} \]
Therefore taking a maximum of the right hand sides of (23) and (24) gives
\[ \|Z_L - X\|_F \leq \left( \frac{5}{6} \right)^L \frac{\sigma_{\text{min}}(X)}{256 \kappa^2} \]
with high probability provided \( \hat{m} \gtrsim \max\{\beta_0 r \kappa^2, \beta_1 r^2 \kappa^6\} \beta n_1 n_3 \log(n_1 n_3) \).

In Algorithm 3, we use fixed stepsize \( \frac{n_1 n_2 n_3}{\hat{m}} \) for ease of exposition, which can be replaced by the adaptive stepsize. Lemma 4.9 implies that if we take \( L \geq 6 \log \left( \frac{\beta n \log n}{24 \epsilon_0} \right) \), the condition (16) in Theorem 3.7 can be satisfied with probability at least
\[ 1 - 2(n_1 n_3)^{1-c} - 12 \log \left( \frac{\beta n \log n}{24 \epsilon_0} \right) ((n_1 n_3)^{1-\frac{c}{2}} + (n_1 n_3)^{1-\frac{c}{2}}). \]
Combining the above results, we can obtain the followings.

**Theorem 4.10.** Suppose \( X_l \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) with \( \text{rank}_k(X_l) = r \) and \( \text{rank}_{cl}(X_l) = r, \kappa \) is the condition number of \( X \). Let \( \Omega (|\Omega| = m) \) is a set of indices sampled independently and uniformly with replacement. Let \( X_0 \) be the output of Algorithm 2. Then the iterates generated by Alg. 3 is guaranteed to converge to \( X \) with high probability provided
\[ m \gtrsim \max\{\beta_0 r \kappa^2, \beta_1 r^2 \kappa^6\} \beta n_1 n_3 \log(n_1 n_3) \log \left( \frac{\beta n \log n}{24 \epsilon_0} \right). \]

**Proof.** This theorem follows from Lemma 3.3, Theorem 3.7, and Lemma 4.9. \( \square \)
5 Numerical Results

In this section, numerical results are presented to show the effectiveness of the proposed methods (Algorithms 2 and 3) for tensor completion. We also compare our methods with the t-svd method introduced in [32]. All the experiments are performed under Windows 7 and MATLAB R2018a running on a desktop (Intel Core i7, @ 3.40GHz, 8.00G RAM)

The relative error (Res) is defined by

$$\text{Res} = \frac{\|X - X_0\|_F}{\|X_0\|_F},$$

where $X$ is the recovered solution and $X_0$ is the ground-truth tensor. Moreover, in order to evaluate the performance for real-world tensors, the peak signal-to-noise ratio (PSNR) is used to measure the equality of the estimated tensors, which is defined as follows:

$$\text{PSNR} = 10 \log_{10} \frac{n_1 n_2 n_3 (X_{\text{max}} - X_{\text{min}})^2}{\|X - X_0\|_F^2},$$

where $X_{\text{max}}$ and $X_{\text{min}}$ are maximal and minimal entries of $X_0$, respectively. The stopping criterion of the algorithm is set to

$$\frac{\|X_{l+1} - X_l\|_F}{\|X_l\|_F} \leq 10^{-4}.$$

5.1 Synthetic Data

In this subsection, we first verify the correct recovery phenomenon of Algorithms 2 and 3 for synthetic datasets. For simplicity, we consider the tensors of size $n \times n \times n$ with dimension $n \leq 100$. We generate the clean tensor $X_0 = S \ast W$ with tubal rank $\text{rank}_t(X_0) = r$, where the entries of $S \in \mathbb{R}^{n \times r \times n}$ and $W \in \mathbb{R}^{r \times n \times n}$ are independently sampled from a standard Gaussian distribution $\mathcal{N}(0, 1)$. We check the recovery abilities of our Algorithms 2 and 3 as a function of tensor dimension $n$, the tubal rank $r$, and the sampling size $m$. We set $r$ to be different specified values, and vary $n$ and $m$ to empirically investigate the probability of recovery success. For each pair $n$ and $m$, we simulate 10 test instances and declare a trial to be successful if the recovered tensor $X_l$ satisfies $\frac{\|X_0 - X_l\|_F}{\|X_0\|_F} < 10^{-3}$. Figure 1 reports the fraction of perfect recovery for each pair (black = 0% and white = 100%).

We see clearly that the recovery is correct under the sampling sizes given in Theorems 4.5 and 4.10 for all the cases. Moreover, compared with Algorithm 2, the sampling sizes needed of Algorithm 3 are improved after applying the trimming method. In the following Tables and Figures, ‘Res’, ‘Ite’, ‘Time’ and ‘Sr’ denotes the relative error, the iteration steps, the CPU time and the sampling ratio, respectively. To further corroborate our theoretical results, we check the effects of different values $r$ in the hard thresholding operator proposed in Algorithm 2 on ‘Res’, ‘Ite’ and ‘Time’, respectively. For a given tensor $X$ with $n = 50$ and $\text{rank}_t(X) = 4$, we test the effects under different sampling ratios, and show the results in Figure 2. It can be seen that when $\text{rank}_t(X) = 4$, the ‘Res’, ‘Ite’ and ‘Time’ are better than the other settings under different sampling rates, respectively. In addition, we compare our Algorithm 2 with one well
known convex method, namely, t-svd method introduced in [32]. Once again, we fix $n = 50$, and generate random tensor $X$ as the prior experiments. For simplicity, we test $\text{rank}_{ct}(X) = 2$ and $\text{rank}_{ct}(A) = 4$ under different sampling ratios. As shown in Table 1 the ‘Res’, ‘Ite’ and ‘Time’ of our Algorithm 2 are much better than the t-svd method.

5.2 Color Image Recovery

It is well known that a $n_1 \times n_2$ color image with red, blue and green channels can be naturally regarded as a third-order tensor $A \in \mathbb{R}^{n_1 \times n_2 \times 3}$. Each frontal slice of $A$ corresponds to a channel of the color image. Actually, each channel of a color image may not be low-rank, but their top
singular values dominate the main information \cite{20, 21}. Hence, the image can be reconstructed into a low-tubal-rank tensor by truncated t-svd. We use the proposed Algorithm 2 to recover the sampled images under different sampling ratios and test the restoration performances of the proposed Algorithm by computing the PSNR on the sampled images. The original color image shown in the Figure 3 is a $481 \times 321 \times 3$ tensor with transformed tubal multi-rank $\mathbf{r} = (29, 5, 1)$. The sampled images and recovered images are listed in the first and second line of Figure 3, respectively. Similar to synthetic data case, we compare our proposed Algorithm 2 with the t-SVD method in \cite{32}. Both ‘PSNR’ and ‘Time’ are listed for the comparison of different methods which are shown in Table 2. We find that the corresponding performance of PSNR by the proposed Algorithm 1 is better than that by t-svd, and the running time by Algorithm 1 is less than that by t-svd when the sampling ratios is small ($sr \leq 0.6$).

Moreover, we also test the effects of the values of $\mathbf{r}$ in the hard thresholding operator for tensor completion by the proposed Algorithm 1. For the original color image with transformed tubal multi-rank $\mathbf{r} = (29, 5, 1)$, we test the effects on ‘PSNR’ and ‘Time’ of the values $\mathbf{r} + h\mathbf{b}$ with $h = -2, -1, \ldots, 5$, and $\mathbf{b} = [1, 1, 1]^T$, by applying Algorithm 1 under different sampling ratios, respectively. The testing results are shown in Figure 4. It can be seen that, when $h = 0$, the ‘PSNR’ and ‘Time’ are better than the other settings.

### 5.3 Video Data

We consider the given video data Carphone (144 $\times$ 176 $\times$ 180)\footnote{https://media.xiph.org/video/derf/} to test Algorithm 2. We display the visual comparisons of the testing data in tensor completion with sampling ratios 0.4, 0.5, 0.6, 0.7, 0.8, by Algorithm 1 and t-svd in Figure 5. We can see that the images recovered by Algorithm 2 are much better than t-svd when the sampling ratios are small. The Algorithm
Table 1: The comparison of Algorithm 2 and t-svd with different tubal ranks.

| Tubal rank | Sr   | Alg. 2 |       | t-svd [32] |       |
|------------|------|--------|-------|------------|-------|
|            |      | Time   | Ite   | Res        | Time  | Ite   | Res  |
| 0.4        | 1.5701 | 6     | 9.5524e-6 | 7.1069 | 117 | 1.6783e-5 |
| 0.5        | 1.4471 | 5     | 8.4042e-6 | 4.8378 | 80  | 1.1604e-5 |
| 0.6        | 1.2834 | 4     | 4.7529e-7 | 3.5759 | 58  | 7.5460e-5 |
| 0.7        | 1.3337 | 4     | 4.3528e-7 | 2.8086 | 43  | 6.1080e-5 |
| 0.8        | 1.1767 | 3     | 2.3870e-6 | 2.2422 | 32  | 4.7257e-5 |
| 0.4        | 2.0595 | 8     | 3.4762e-5 | 13.8184 | 234 | 3.8036e-5 |
| 0.5        | 1.6810 | 6     | 1.4283e-5 | 8.5930 | 141 | 2.3471e-5 |
| 0.6        | 1.5210 | 5     | 2.6949e-6 | 5.7455 | 95  | 4.3793e-5 |
| 0.7        | 1.3695 | 4     | 5.2723e-6 | 4.0027 | 66  | 4.5901-5 |
| 0.8        | 1.4065 | 4     | 1.3991e-6 | 2.8752 | 47  | 6.7191e-5 |

Figure 3: Recovery results under different sampling ratios by Algorithm 2 with \( r = (29, 5, 1) \).
Table 2: PSNR and Time by Algorithm 1 and t-svd \cite{32} for a color image recovery with $r = (29, 5, 1)$.

| Sr  | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|-----|-----|-----|-----|-----|-----|
| **Alg. 1** | PSNR | 41.42 | 41.64 | 41.69 | 41.73 | 41.76 |
|     | Time | 12.62 | 11.75 | 10.95 | 10.11 | 9.51 |
| **t-SVD** | PSNR | 30.01 | 31.12 | 31.73 | 32.12 | 32.96 |
|     | Time | 13.64 | 13.20 | 11.28 | 9.86  | 8.88  |

Figure 4: The effect of different setting of $r$ by Algorithm 1 for given tensor with $r = (29, 5, 1)$. $\text{Alg. 1}$ can keep more details than t-svd for the testing video. We also show the ‘PSNR’ and ‘Time’ by Algorithm 1 and t-svd for Carphone data with sampling ratios 0.4, 0.5, 0.6, 0.7, 0.8 in Table \ref{tab:3}. It can be seen that the PSNR values obtained by Algorithm 2 are higher than those by t-svd, and the CPU time of Algorithm 1 are less than those by t-svd.

Table 3: PSNR and Time by Algorithm 2 and t-svd \cite{32} for video data recovery.

| Sr  | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|-----|-----|-----|-----|-----|-----|
| **Alg. 2** | PSNR | 48.65 | 48.94 | 49.11 | 49.23 | 49.46 |
|     | Time | 126.88 | 105.11 | 91.84 | 76.88 | 70.65 |
| **t-SVD** | PSNR | 46.89 | 47.82 | 48.24 | 48.64 | 48.69 |
|     | Time | 145.21 | 139.50 | 131.58 | 127.90 | 121.50 |

6 Conclusion

In this paper, we mainly consider the low rank tensor completion problem by Riemannian optimization methods on the manifold, where the tensor rank is relate to the transformed tubal multi-rank which is based on tensor singular value decomposition. We discuss the convergence
properties of Riemannian conjugate gradient methods under different initialized methods, respectively. The minimum sampling sizes needed to recover the low transformed multi-rank tensor are also derived. Numerical simulation shows that the algorithms are able to recover the low transformed multi-rank tensor under different initialized methods.

7 Appendix

7.1 Proof of Proposition 2.2, 2.3 and 2.4

In order to prove Proposition 2.2, we need the following results which given in [34].

Lemma 7.1 (Lemma 5.5 in [34]). Let \( \mathcal{M} \) be a smooth manifold and \( \mathcal{N} \) is a subset of \( \mathcal{M} \). Suppose every point \( p \in \mathcal{N} \) has a neighborhood \( \mathcal{U} \subset \mathcal{M} \) such that \( \mathcal{U} \cap \mathcal{N} \) is an embedded submanifold of \( \mathcal{U} \). Then \( \mathcal{N} \) is an embedded submanifold of \( \mathcal{M} \).

Proof of Proposition 2.2. Suppose that \( \mathcal{E}_0 \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) with \( \text{rank}_{cl}(\mathcal{E}_0) = r \) and \( \text{rank}_t(\mathcal{E}) = r = (r_1, \ldots, r_{n_3}) \). By the element transformations it can be expressed as

\[
\mathcal{E}_0 = \begin{pmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{pmatrix}
\]

where \( A \in \mathbb{R}^{r \times r \times n_3} \) with \( \text{rank}(A) = r \), \( B \in \mathbb{R}^{r \times (n_2-r) \times n_3} \), \( C \in \mathbb{R}^{(n_1-r) \times r \times n_3} \), and \( D \in \mathbb{R}^{(n_1-r) \times (n_2-r) \times n_3} \). Set \( \phi: \mathbb{R}^{n_1 \times n_2 \times n_3} \rightarrow \mathbb{R}^{n_1 n_3 \times n_2 n_3} \) as:

\[
\phi(A) = \text{blockdiag}(\hat{A}) = \begin{bmatrix}
\hat{A}^{(1)} & & \\
& \hat{A}^{(2)} & \\
& & \ddots \\
& & & \hat{A}^{(n_3)}
\end{bmatrix},
\]
with \( \hat{A} = \text{dct}(A, [], 3) \). It is easy to see that \( \phi \) is a bijective. Let \( \mathcal{U} \) be the open set

\[
\mathcal{U} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{n_1 \times n_2 \times n_3} : \text{rank}(A) = r \right\}
\]

which contains \( \mathcal{E}_0 \). Define \( F : \mathcal{U} \to \mathbb{R}^{\sum_{i=1}^{n_3} (n_1 - r_i) \times \sum_{i=1}^{n_3} (n_2 - r_i)} \) as

\[
F\left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = (\hat{D}(1) - \hat{B}(1) (\hat{A}(1))^{-1} \hat{C}(1)) \oplus \cdots \oplus (\hat{D}(n_3) - \hat{B}(n_3) (\hat{A}(n_3))^{-1} \hat{C}(n_3)).
\]

Clearly, \( F \) is smooth. In order to show it is a submersion, we need to show \( F'(p) \) is surjective for each \( p \in \mathcal{U} \). Note that \( \mathbb{R}^{\sum_{i=1}^{n_3} (n_1 - r_i) \times \sum_{i=1}^{n_3} (n_2 - r_i)} \) is a vector space, the tangent vector at \( F(p) \) can be defined as \( \sum_{i=1}^{n_3} (n_1 - r_i) \times \sum_{i=1}^{n_3} (n_2 - r_i) \) matrices. Given

\[
\mathcal{E} = \text{idct} \left( \begin{pmatrix} \hat{A}(1) & \hat{B}(1) & \hat{C}(1) & \hat{D}(1) \\ \hat{A}(n_3) & \hat{B}(n_3) & \hat{C}(n_3) & \hat{D}(n_3) \end{pmatrix} \right) \in \mathcal{U},
\]

and any tensor

\[
\mathcal{X} = \text{idct} \left( \begin{pmatrix} \hat{A}(1) & \hat{X}(2) & \cdots & \hat{X}(n_3) \\ \hat{A}(n_3) & \hat{X}(n_3) & \cdots & \hat{X}(n_3) \end{pmatrix} \right) \in \mathbb{R}^{n_1 \times n_2 \times n_3}
\]

with

\[
\hat{X}(i) = \begin{pmatrix} 0_{r_i \times r_i} & 0_{r_i \times (n_2 - r_i)} \\ 0_{(n_1 - r_i) \times r_i} & X(i)_{(n_1 - r_i) \times (n_2 - r_i)} \end{pmatrix}, \ i = 1, \ldots, n_3.
\]

Define a curve \( \tau : (-\xi, \xi) \to \mathcal{U} \) by

\[
\tau(t) = \text{idct} \left( \begin{pmatrix} \hat{A}(1) & \hat{X}(2) & \cdots & \hat{X}(n_3) \\ \hat{A}(n_3) & \hat{X}(n_3) & \cdots & \hat{X}(n_3) \end{pmatrix} \right),
\]

where \( \hat{Y}(i) = \hat{D}_i + t \hat{X}(i)_{(n_1 - r_i) \times (n_2 - r_i)} - \hat{C}(i) (\hat{A}(i))^{-1} \hat{B}(i), \ i = 1, \ldots, n_3 \). We remark that \( \hat{Y}(i), \ i =
1, ..., $n_3$, does not need to possess the same sizes, respectively. Then we can get

$$F_\ast \tau' (0) = (F \circ \tau)'(t) = \left( \frac{d}{dt} \big|_{t=0} (\hat{Y}^{(1)}) \right) \cdots \left( \frac{d}{dt} \big|_{t=0} (\hat{Y}^{(n_3)}) \right)$$

$$= \begin{pmatrix} X_{(n_1-r_1) \times (n_2-r_1)}^{(1)} & \cdots & X_{(n_1-r_3) \times (n_2-r_3)}^{(n_3)} \end{pmatrix},$$

where $F_\ast$ is the push-forward projection relate $F$. Then $F$ is a submersion and so $M_\tau \cap U$ is an embedded submanifold of $U$. Next if $E'$ is an arbitrary tensor of rank$_i (E') = r$. Just note that it can be transformed to one in $U$ by a rearrangement along the first and second directions. Such a rearrangement $R$ is a linear isomorphism that preserves the tensor transformed multi-rank, so $U_0 = R^{-1}(U)$ is a neighborhood of $E'$ and $F \circ R : U_0 \rightarrow \mathbb{R}^{n_1n_3 \times n_2n_3}$ is a submersion whose zero level set is $M_\tau \cap U$. Thus every point in $M_\tau$ has a neighborhood $U$ such that $M_\tau \cap U_0$, so $M_\tau$ is an embedded submanifold. Moreover, note that rank$(F_\ast \tau' (0)) = \sum_{i=1}^{n_3}((n_1 + n_2)r_i - r_i^2)$ which is saying that $M_\tau$ possess dimension $\sum_{i=1}^{n_3}((n_1 + n_2)r_i - r_i^2)$.

**Proof of Proposition 2.3.** Suppose that $X_l \in M_\tau$, it follows from Definition 2.5 that $X_l$ can be expressed as

$$X_l = U_l \ast S \ast V_l^T,$$

where rank$_i (U_l) = \text{rank}_i (V_l) = \text{rank}_i (S) = r$, $U_l^T \ast U = \mathbb{I}_r$, $V_l^T \ast V_l = \mathbb{I}_r$, and $S$ is a diagonal tensor. Then define a curve $\gamma : (-\zeta, \zeta) \rightarrow M_\tau$ by setting $\gamma(t) = (U_l + tX) \ast S \ast ((V_l + tY))^T$ where $X$ and $Y$ are arbitrary tensors with proper sizes. It is easy to check that $\gamma$ is smooth and $\gamma(0) = X_l$. Then, $\gamma'(0) = X \ast S \ast V_l^T + U_l \ast S \ast Y^T$, and Proposition 2.3 can be derived by setting $S \ast Y^T$ as $Z_1^T$ and $X \ast S$ as $Z_2$.

A retraction is any smooth map from the tangent bundle $TM$ into $M$ that approximates the exponential map to the first order. In order to prove the $t_c$-SVD truncation is a retraction we need introduce the following definition.

**Definition 7.1 (Definition 1 in [2]).** Let $M$ be a smooth submanifold of $\mathbb{R}^{n_1 \times \cdots \times n_d}$, $0_x$ denote the zero element of $T_xM$. A mapping $R$ from the tangent bundle $TM$ into $M$ is said to be a retraction on $M$ around $x \in M$ if there exists a neighborhood $U$ of $(x, 0_x)$ in $TM$ such that the following properties hold:

(a) We have $U \subseteq \text{dom} (R)$ and the restriction $R : U \rightarrow M$ is smooth.

(b) $R(y, 0_y)$ for all $(y, 0_y) \in U$.

(c) With the canonical identification $T_0(T_xM \simeq T_xM)$, $R$ satisfies the local rigidity condition:

$$DR(x, \cdot)(0_x) = \text{id}_{T_xM} \text{ for all, } (x, 0_x) \in U,$$

where $\text{id}_{T_xM}$ denotes the identity mapping on $T_xM$. 

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Proof of Proposition 2.4. Let $H_{r_i}$ denote the SVD truncation of the $i$-th front slice of a tensor in the DCT transform domain. Then $P_{M_{r_i}}(X + \xi)$ given in (12) can be expressed as $\text{idct} \circ (H_{r_1} \circ \cdots \circ H_{r_{n_3}}) \circ \text{dct}(X + \xi)$, where $H_{r_i}$ are independent with each other. Suppose that $N_i$ denotes the open set of tensors that the $i$-th front slice if $\text{dct}(N_i)$ has a nonzero gap between the $r_i$-th and the $r_{i+1}$-th singular values. Then we can get the truncation operator $H_{r_i}$ is smooth and well-defined on $N_i$. Note that $X \in M_{r_i}$ means that $X$ contains in any open set $N_i$ and $\hat{X}$ is a fixed point of every $H_{r_i}$. Therefore, it is possible to construct an open neighborhood $N \in M_{r_i}$ of $X$ such that $\|X + t\xi - P_{M_{r_i}}(X + t\xi)\|_F = O(t)^2$ for $t \to 0$. Hence, $P_{M_{r_i}}(X + t\xi) = (X + t\xi) + O(s)^2$, which gives $\frac{d}{dt} P_{M_{r_i}}(X + t\xi)|_{t=0} = \xi$. In other words, $D P_{M_{r_i}}(X, \cdot)(0_X) = \text{id}_{T X M_{r_i}}$, which completes the proof.

7.2 Proof of Lemma 3.2

Proof of Lemma 3.2. The proof of (i) can be proceeded as follows:

$$
\|U_l U_l^T - U \ast U^T\| = \|U_l U_l^T - U \ast U^T\| \\
= \|U_l U_l^T - U \ast U^T\| \\
= \max_{i=1,\ldots,n_3} \|\hat{U}_l^{(i)} (\hat{U}_l^{(i)})^T - \hat{U}^{(i)} (\hat{U}^{(i)})^T\| \\
\leq \max_{i=1,\ldots,n_3} \frac{\|\hat{X}_l^{(i)} - \hat{X}^{(i)}\|_F}{\sigma_{\min}(\hat{X}^{(i)})} \\
\leq \frac{\max_{i=1,\ldots,n_3} \|\hat{X}_l^{(i)} - \hat{X}^{(i)}\|_F}{\sigma_{\min}(\hat{X})} \\
\leq \frac{\|X - \hat{X}\|_F}{\sigma_{\min}(\hat{X})},
$$

where the fourth line follows from Lemma 4.1 in [30]. We can similarly prove (ii).
Noting that \( \text{dct}(\cdot) \) is a unitary transform, we have
\[
\|\mathcal{U}_l \ast \mathcal{U}_l^T - \mathcal{U} \ast \mathcal{U}^T\|_F^2 = \|\mathcal{U}_l \ast \mathcal{U}_l^T - \mathcal{U} \ast \mathcal{U}^T\|_F^2 \\
= \|\mathcal{U}_l \mathcal{U}_l^T - \mathcal{U} \mathcal{U}^T\|_F^2 \\
= \sum_{i=1}^{n_3} \|\hat{\mathcal{X}}_i^{(i)} - \hat{\mathcal{X}}_i^{(i)}\|_F^2 \\
\leq \sum_{i=1}^{n_3} \frac{2\|\hat{\mathcal{X}}_i^{(i)} - \hat{\mathcal{X}}_i^{(i)}\|_F^2}{\sigma_{\min}^2(\hat{\mathcal{X}}_i^{(i)})} \\
\leq \frac{2}{\sigma_{\min}^2(\mathcal{X})} \sum_{i=1}^{n_3} \|\hat{\mathcal{X}}_i^{(i)} - \hat{\mathcal{X}}_i^{(i)}\|_F^2 \\
= \frac{2\|X - \tilde{X}\|_F^2}{\sigma_{\min}^2(\mathcal{X})},
\]
where the fourth line follows from Lemma 4.1 in [30]. Taking square roots on both sides yields (iii) and (iv) can be proved similarly.

Since \( P_T(\mathcal{X}) = \mathcal{X}, (I - P_{T_l})\mathcal{X} = P_T(\mathcal{X}) - P_{T_l}(\mathcal{X}) \). Thus,
\[
P_T(\mathcal{X}) - P_{T_l}(\mathcal{X}) = \mathcal{U} \ast \mathcal{U}^T \ast \mathcal{X} + \mathcal{X} \ast \mathcal{V} \ast \mathcal{V}^T - \mathcal{U} \ast \mathcal{U}^T \ast \mathcal{X} \ast \mathcal{V} \ast \mathcal{V}^T \\
= (\mathcal{U}_l \ast \mathcal{U}_l^T) \ast \mathcal{X} + \mathcal{X} \ast (\mathcal{V} \ast \mathcal{V}^T - \mathcal{V}_l \ast \mathcal{V}_l^T) \\
- \mathcal{U} \ast \mathcal{U}^T \ast \mathcal{X} \ast \mathcal{V} \ast \mathcal{V}^T + \mathcal{U} \ast \mathcal{U}^T \ast \mathcal{X} \ast \mathcal{V}_l \ast \mathcal{V}_l^T \\
- \mathcal{U} \ast \mathcal{U}^T \ast \mathcal{X} \ast \mathcal{V}_l \ast \mathcal{V}_l^T + \mathcal{U}_l \ast \mathcal{U}_l^T \ast \mathcal{X} \ast \mathcal{V}_l \ast \mathcal{V}_l^T \\
= (\mathcal{U} \ast \mathcal{U}^T - \mathcal{U}_l \ast \mathcal{U}_l^T) \ast (\mathcal{X} - \mathcal{X}_l) \ast (\mathcal{I}_r - \mathcal{V}_l \ast \mathcal{V}_l^T).
\]
Therefore, (v) can be established as follows:
\[
\|(I - P_{T_l})\mathcal{X}\|_F = \|(\mathcal{U} \ast \mathcal{U}^T - \mathcal{U}_l \ast \mathcal{U}_l^T) \ast (\mathcal{X} - \mathcal{X}_l) \ast (\mathcal{I}_r - \mathcal{V}_l \ast \mathcal{V}_l^T)\|_F \\
\leq \|(\mathcal{U} \ast \mathcal{U}^T - \mathcal{U}_l \ast \mathcal{U}_l^T)\| \|\mathcal{X} - \mathcal{X}_l\|_F \|\mathcal{I}_r - \mathcal{V}_l \ast \mathcal{V}_l^T\| \\
\leq \|\mathcal{X} - \mathcal{X}_l\|_F \|\mathcal{X} - \mathcal{X}_l\|_F \frac{\sigma_{\min}(\mathcal{X})}{\sigma_{\min}(\mathcal{X})} \\
\leq \|\mathcal{X} - \mathcal{X}_l\|_F^2 \frac{\sigma_{\min}(\mathcal{X})}{\sigma_{\min}(\mathcal{X})},
\]
where the third line follows from (i).

For any \( \mathcal{Z} \), we have
\[
(P_{T_l} - P_T)(\mathcal{Z}) = \mathcal{U}_l \ast \mathcal{U}_l^T \ast \mathcal{Z} + \mathcal{Z} \ast \mathcal{V}_l \ast \mathcal{V}_l^T - \mathcal{U}_l \ast \mathcal{U}_l^T \ast \mathcal{Z} \ast \mathcal{V} \ast \mathcal{V}^T - \mathcal{U} \ast \mathcal{U}^T \ast \mathcal{Z} \ast \mathcal{V} \ast \mathcal{V}^T \\
+ \mathcal{U} \ast \mathcal{U}^T \ast \mathcal{Z} \ast \mathcal{V} \ast \mathcal{V}^T \\
= (\mathcal{U}_l \ast \mathcal{U}_l^T - \mathcal{U} \ast \mathcal{U}^T) \ast \mathcal{Z} \ast (\mathcal{I}_r - \mathcal{V} \ast \mathcal{V}^T) + (\mathcal{I}_r - \mathcal{U}_l \ast \mathcal{U}_l^T) \ast \mathcal{Z} \ast (\mathcal{V}_l \ast \mathcal{V}_l^T - \mathcal{V} \ast \mathcal{V}^T).
\]
Consequently, taking the Frobenius norm on both sides of the above equality and utilizing (i) and (ii) yields (vi). \( \square \)
Moreover, the application of the triangle inequality yields that
\[ \| R_\Omega P_T(Z) \|_F^2 = \langle R_\Omega P_T(Z), R_\Omega P_T(Z) \rangle \leq \frac{10}{3} \beta \log(n) \langle P_T(Z), R_\Omega P_T(Z) \rangle \]
\[ = \frac{10}{3} \beta \log(n) \langle P_T(Z), P_T R_\Omega P_T(Z) \rangle \]
\[ \leq \frac{10}{3} \beta \log(n)(1 + \epsilon_0)p \| P_T(Z) \|_F^2. \]

It follows that \( \| R_\Omega P_T \| \leq \sqrt{\frac{10}{3} \beta \log(n)(1 + \epsilon_0)p} \)

\[ \| R_\Omega P_T \| \leq \| R_\Omega (P_T - P_T) \| + \| R_\Omega P_T \| \leq \frac{10}{3} \beta \log(n)(1 + \epsilon_0)p^{\frac{1}{2}}. \]

Moreover, the application of the triangle inequality yields that
\[ \| P_T - p^{-1} P_T R_\Omega P_T \| \]
\[ = \| P_T - P_T + P_T - p^{-1} P_T R_\Omega P_T + p^{-1} P_T R_\Omega P_T - p^{-1} P_T R_\Omega P_T - p^{-1} P_T R_\Omega P_T \| \]
\[ \leq \| P_T - P_T \| + p^{-1} \| P_T R_\Omega P_T - P_T R_\Omega P_T \| + \| P_T - p^{-1} P_T R_\Omega P_T \| \]
\[ \leq 4 \epsilon_0, \]

which completes the proof.

\[ \square \]

### 7.4 Proof of Lemmas 3.5 and 3.6

**Proof of Lemma 3.5** When \( \beta_i = 0 \) one has \( P_i = P_M G_i \). Thus \( \alpha_i \) can be expressed as
\[ \alpha_i = \frac{\| P_M G_i \|_F^2}{\langle P_M G_i, R_\Omega P_M G_i \rangle}. \]

Note that if \( \| P_M G_i \|_F \leq 4 \epsilon_0 \) is satisfied, then
\[ \| P_M G_i, R_\Omega P_M G_i \| \leq p \| P_M G_i \| + p \| R_\Omega P_M G_i \| \leq (1 + 4 \epsilon_0)p. \]

Consequently,
\[ \langle P_M G_i, R_\Omega P_M G_i \rangle = \langle P_M G_i, P_M G_i R_\Omega P_M G_i \rangle \leq (1 + 4 \epsilon_0)p \| P_M G_i \|_F^2. \]

On the other hand,
\[ \| P_M G_i \|_F^2 = \langle P_M G_i, (P_M - p^{-1} P_M R_\Omega P_M) G_i \rangle + \langle P_M G_i, p^{-1} P_M R_\Omega P_M G_i \rangle \]
\[ \leq 4 \epsilon_0 \| P_M G_i \|_F^2 + p^{-1} \langle P_M G_i, R_\Omega P_M G_i \rangle \]

Combining the above two inequalities together yields that
\[ \frac{1}{(1 + 4 \epsilon_0)p} \leq \alpha_i \leq \frac{1}{(1 - 4 \epsilon_0)p}. \]
It follows that
\[
\| P_{\bar{T}} - \alpha_l P_{\bar{T}} R_{\Omega} P_{\bar{T}} \| \leq \| P_{\bar{T}} - p^{-1} P_{\bar{T}} R_{\Omega} P_{\bar{T}} \| + (\alpha_l - p^{-1}) \| P_{\bar{T}} R_{\Omega} P_{\bar{T}} \|
\leq 4\epsilon_0 + \frac{4\epsilon_0(1 + 4\epsilon_0)p}{(1 - 4\epsilon_0)p} = \frac{8\epsilon_0}{1 - 4\epsilon_0}.
\]
This completes the proof. \(\square\)

**Proof of Lemma 3.6.** When \(\beta_l \neq 0\), one has \(P_l = P_{\bar{T}}(G_l) + \beta_l P_{\bar{T}}(Q_{l-1})\). Then the orthogonalization weight \(\beta_l\) can be bounded as follows

\[
|\beta_l| = \frac{\langle P_{\bar{T}}(G_l), R_{\Omega} P_{\bar{T}}(Q_{l-1}) \rangle}{\langle P_{\bar{T}}(Q_{l-1}), R_{\Omega} P_{\bar{T}}(Q_{l-1}) \rangle}.
\]

Thus there holds
\[
|\langle P_{\bar{T}}(Q_l), P_{\bar{T}}(G_l) \rangle| = |\langle P_{\bar{T}}(G_l) + \beta_l P_{\bar{T}}(Q_{l-1}), P_{\bar{T}}(G_l) \rangle| \geq \| P_{\bar{T}}(G_l) \|^2_F - |\langle \beta_l P_{\bar{T}}(Q_{l-1}), P_{\bar{T}}(G_l) \rangle| \geq \left(1 - \frac{k_1(1 + 4\epsilon_0)}{1 - 4\epsilon_0}\right) \| P_{\bar{T}}(G_l) \|^2_F.
\]

Moreover, by the Cauchy-Schwarz inequality we have
\[
\left(1 - \frac{k_1(1 + 4\epsilon_0)}{1 - 4\epsilon_0}\right) \| P_{\bar{T}}(G_l) \|^2_F \leq |\langle P_{\bar{T}}(Q_l), P_{\bar{T}}(G_l) \rangle| \leq \| P_{\bar{T}}(Q_l) \|_F \| P_{\bar{T}}(G_l) \|_F.
\]

Therefore,
\[
\| P_{\bar{T}}(G_l) \|_F \leq \frac{1}{1 - \frac{k_1(1 + 4\epsilon_0)}{1 - 4\epsilon_0}} \| P_{\bar{T}}(G_l) \|_F.
\]
Noting that
\[ \alpha_l = \frac{\langle P_{T_l}(G_l), P_{T_l}(Q_l) \rangle}{\langle P_{T_l}(Q_l), P_{T_l}(Q_l) \rangle} = p^{-1} + \frac{\langle P_{T_l}(G_l), (P_{T_l} - p^{-1} P_{T_l} P_\Omega P_{T_l})(Q_l) \rangle}{\langle P_{T_l}(Q_l), P_{T_l}(Q_l) \rangle}, \]
we have
\[ |\alpha_l \cdot p - 1| \leq p \left| \frac{\langle P_{T_l}(G_l), (P_{T_l} - p^{-1} P_{T_l} P_\Omega P_{T_l})(Q_l) \rangle}{\langle P_{T_l}(Q_l), P_{T_l}(Q_l) \rangle} \right| \leq \frac{4\epsilon_0}{(1 - 4\epsilon_0) - k_1(1 + 4\epsilon_0)}. \]
Thus the spectral norm of \( \| P_{T_l} - \alpha_l P_{T_l} R_\Omega P_{T_l} \| \) can be bounded as
\[ \| P_{T_l} - \alpha_l P_{T_l} P_\Omega P_{T_l} \| \leq \| P_{T_l} - p^{-1} P_{T_l} P_\Omega P_{T_l} \| + (\alpha_l - p^{-1}) \| P_{T_l} P_\Omega P_{T_l} \| \leq 4\epsilon_0 + (\alpha_l p - 1)\| p^{-1} P_{T_l} P_\Omega P_{T_l} \| \leq 4\epsilon_0 + 4\epsilon_0(1 + 4\epsilon_0). \]
This completes the proof. \( \square \)

### 7.5 Proof of Lemma 4.2

**Lemma 7.2** (Bernstein’s Inequality [24]). Let \( X_1, \ldots, X_L \in \mathbb{R}^{n \times n} \) be independent zero mean random matrices of dimension \( d_1 \times d_2 \). Suppose
\[ \rho_k^2 = \max \{ \| E[X_k X_k^T] \|, \| E[X_k^T X_k] \| \} \]
and \( \| X_k \| \leq B \) almost surely for all \( k \). Then for any \( \tau > 0 \),
\[ \mathbb{P}[\| \sum_{k=1}^L X_k \| > \tau] \leq (d_1 + d_2) \exp \left( \frac{-\tau^2 / 2}{\sum_{k=1}^L \rho_k^2 + B\tau / 3} \right). \]

**Proof of Lemma 4.2**. We begin the proof with the following decomposition:
\[ P_T(Z) = \sum_{abc} \langle P_T(Z), E_{abc} \rangle E_{abc} = \sum_{abc} \langle Z, P_T(E_{abc}) \rangle E_{abc}. \]
It follows that
\[ P_T R_\Omega P_T(Z) = \sum_{k=1}^m \langle Z, P_T(E_{a_k b_k c_k}) \rangle P_T(E_{a_k b_k c_k}), \]
where \((a_k, b_k, c_k)\) are indices sampled from \( \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \times \{1, \ldots, n_3\} \) independently and uniform with replacement.

Let \( T_{a_k b_k c_k} \) be the linear operator which maps \( Z \) to \( \langle Z, P_T(E_{a_k b_k c_k}) \rangle P_T(E_{a_k b_k c_k}) \). It is not hard to see that \( T_{a_k b_k c_k} \) is rank-1 linear operator with
\[ \| T_{a_k b_k c_k} \| = \| P_T(E_{a_k b_k c_k}) \|_F^2 \leq \frac{2\mu r}{n(2)}, \]
where
where the inequality follows from Lemma 4.1. Since \( \|P_T\| \leq 1 \), it follows that

\[
\left\| T_{a_k b_k c_k} - \frac{1}{n_1 n_2 n_3} P_T \right\| \leq \max \left\{ \|P_T (E_{a_k b_k c_k})\|_F^2, \frac{1}{n_1 n_2 n_3} \right\} \leq \frac{2 \mu r}{n(2)},
\]

where the first inequality uses the fact that if \( A \) and \( B \) are positive semidefinite matrices, then \( \|A - B\| \leq \max\{\|A\|, \|B\|\} \). In addition, we have

\[
\left\| E \left[ \left( T_{a_k b_k c_k} - \frac{1}{n_1 n_2 n_3} P_T \right)^2 \right] \right\| = \left\| E \left[ \|P_T (E_{a_k b_k c_k})\|_F^2 T_{a_k b_k c_k} \right] - \frac{2}{n_1 n_2 n_3} P_T E (T_{a_k b_k c_k}) + \frac{1}{n_1 n_2 n_3} P_T \right\| \leq \max \left\{ \frac{2 \mu r}{n(2)} \|E[T_{a_k b_k c_k}]\|, \frac{1}{n_1 n_2 n_3} \right\}, \tag{25}
\]

Noticing that \( P_T R_1 P_T = \sum_{k=1}^m T_{a_k b_k c_k} \), taking \( \tau = \frac{m}{n_1 n_2 n_3} \sqrt{c_4 \mu r (n_1 n_2 n_3) \log(n_1 n_2 n_3)} \) in Lemma 7.2 yields the result.

### 7.6 Proof of Lemma 4.3

**Proof of Lemma 4.3.** First note that

\[
\frac{n_1 n_2 n_3}{m} R_1(Z) - Z = \frac{1}{m} \sum_{k=1}^m n_1 n_2 n_3 Z_{a_k b_k c_k} E_{a_k b_k c_k} - Z.
\]

By Definition 2.7

\[
\left\| \frac{n_1 n_2 n_3}{m} R_1(Z) - Z \right\| = \left\| \frac{n_1 n_2 n_3}{m} R_1(Z) - Z \right\| = \left\| \frac{1}{m} \sum_{k=1}^m n_1 n_2 n_3 Z_{a_k b_k c_k} E_{a_k b_k c_k} - Z \right\|. \tag{26}
\]

Let

\[
\overline{E_{a_k b_k c_k}} = n_1 n_2 n_3 Z_{a_k b_k c_k} E_{a_k b_k c_k} - Z.
\]

It is easy to see that \( E[\overline{E_{a_k b_k c_k}}] = 0 \). By the definition of \( E_{a_k b_k c_k} \) and the fact DCT is a unitary transform, a simple calculation yields that

\[
\|\overline{E_{a_k b_k c_k}}\| \leq 1 \quad \text{and} \quad \|Z\| \leq \sqrt{n_1 n_2 n_3} \|Z\|_{\infty}.
\]

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Hence, \( \| \mathbf{c}_{akbkck} \| \leq \frac{3n_1n_2n_3}{\| \mathbf{z} \|_\infty} \) for \( n_3 \geq 2 \).

To bound \( \mathbb{E}[\mathbf{c}_{akbkck}^T \mathbf{c}_{akbkck}] \), we need to first bound \( \sum_{abc} \mathbf{z}_{abc}^2 \mathbf{c}_{abc}^T \mathbf{c}_{abc} \). To this end, let \( [\mu_{1abc}^ab, \cdots, \mu_{n3abc}^ab]^T \) be the DFT of the \((a, b)\)-th tube of \( \mathbf{c}_{abc} \). Then,

\[
\mathbf{c}_{abc}^T \mathbf{c}_{abc} = \begin{bmatrix}
(\mu_{1abc}^a)^2 \mathbf{e}_b \mathbf{e}_b^T \\
\vdots \\
(\mu_{n3abc}^a)^2 \mathbf{e}_b \mathbf{e}_b^T
\end{bmatrix},
\]

Thus we have

\[
\sum_{abc} \mathbf{z}_{abc}^2 \mathbf{c}_{abc}^T \mathbf{c}_{abc} = \sum_b \begin{bmatrix}
\sum_{ac} \mathbf{z}_{abc}^2 (\mu_{1abc}^a)^2 \mathbf{e}_b \mathbf{e}_b^T \\
\vdots \\
\sum_{ac} \mathbf{z}_{abc}^2 (\mu_{n3abc}^a)^2 \mathbf{e}_b \mathbf{e}_b^T
\end{bmatrix},
\]

which implies

\[
\| \sum_{abc} \mathbf{z}_{abc}^2 \mathbf{c}_{abc}^T \mathbf{c}_{abc} \| \leq \max_{b, i} \sum_{ac} \mathbf{z}_{abc}^2 (\mu_{iabc}^a)^2 \leq \| \mathbf{z} \|_\infty^2 \sum_{ac} (\mu_{iabc}^a)^2 \leq n_1 \| \mathbf{z} \|_\infty^2,
\]

where the last equality follows from \( \sum_{c}(\mu_{abc}^c)^2 = 1 \) due to that DCT is a unitary transform. It follows that

\[
\| \mathbb{E}[\mathbf{c}_{akbkck}^T \mathbf{c}_{akbkck}] \| = \| n_1^2 n_2^2 n_3^2 \mathbb{E}[\mathbf{z}_{akbkck}^2 \mathbf{c}_{abc}^T \mathbf{c}_{abc}] - \mathbf{z}^T \mathbf{z} \|
\leq \max \left\{ n_1^2 n_2^2 n_3^2 \| \mathbb{E}[\mathbf{z}_{akbkck}^2 \mathbf{c}_{abc}^T \mathbf{c}_{abc}] \|, \| \mathbf{z}^T \mathbf{z} \| \right\}
\leq n_1^2 n_2^2 n_3^2 \| \mathbf{z} \|_\infty^2.
\]

Moreover, \( \| \mathbb{E}[\mathbf{c}_{akbkck}^T \mathbf{c}_{akbkck}] \| \) can be bounded similarly.

Applying the Bernstein’s inequality to (26) concludes the proof.

7.7 Proof of Lemma 4.4

Proof of Lemma 4.4. Denote \( \mathcal{W}_0 = H_r(p^{-1} R_{11}(\mathcal{X})) \). Lemma 4.3 implies that

\[
\| \mathcal{X}_0 - \mathcal{X} \| = \| \mathcal{X}_0 - \mathcal{W}_0 + \mathcal{W}_0 - \mathcal{X} \| \leq 2 \| \mathcal{W}_0 - \mathcal{X} \| \leq \sqrt{ \frac{\beta n_1^2 n_2^2 n_3^2 \log(n_1 n_2 n_3)}{m} } \| \mathcal{X} \|_\infty
\]

holds with high probability. Consequently,

\[
\| \mathcal{X}_0 - \mathcal{X} \|_F = \| \mathcal{X}_0 - \mathcal{X} \|_F \leq \sqrt{n_3^2} \| \mathcal{X}_0 - \mathcal{X} \| \leq \sqrt{ \frac{\beta n_1^2 n_2^2 n_3^2 \log(n_1 n_2 n_3)}{m} } \| \mathcal{X} \|_\infty.
\]

Therefore, plugging the joint incoherence condition \( \| \mathcal{X} \|_\infty \leq \mu_1 \sqrt{ \frac{\| \mathcal{X} \|}{n_1 n_2 n_3} } \) into the above inequality yields that

\[
\| \mathcal{X}_0 - \mathcal{X} \|_F \leq \sqrt{ \frac{\mu_1^2 \beta n_1^2 n_2^2 n_3^2 \log(n_1 n_2 n_3)}{m} } \| \mathcal{X} \|,
\]

which completes the proof.
7.8 Proof of Lemma 4.6

Proof of Lemma 4.6 Noting that

$$ (P_U - P_{U_k})(Z) = \sum_{abc} \langle (P_U - P_{U_k})(Z), \mathcal{E}_{abc} \rangle \mathcal{E}_{abc} = \sum_{abc} \langle Z, (P_U - P_{U_k})(\mathcal{E}_{abc}) \rangle \mathcal{E}_{abc}, $$

we have

$$ P_{T_l} R_{\Omega}(P_U - P_{U_k})(Z) = \sum_{k=1}^m \langle Z, (P_U - P_{U_k})(\mathcal{E}_{a_k b_k c_k}) \rangle P_{T_l}(\mathcal{E}_{a_k b_k c_k}). $$

Let $T_{a_k b_k c_k}$ be the linear operator which maps $Z$ to $\langle Z, (P_U - P_{U_k})(\mathcal{E}_{a_k b_k c_k}) \rangle P_{T_l}(\mathcal{E}_{a_k b_k c_k})$. It follows that

$$ \| T_{a_k b_k c_k} \| \leq \| P_{T_l}(\mathcal{E}_{a_k b_k c_k}) \| F(\| P_U(\mathcal{E}_{a_k b_k c_k}) \| F + \| P_{U_k}(\mathcal{E}_{a_k b_k c_k}) \| F) \leq \frac{4 \mu r}{n(2)}. $$

Moreover, there holds

$$ \| \mathbb{E}[(T_{a_k b_k c_k} - \mathbb{E}(T_{a_k b_k c_k}))^T (T_{a_k b_k c_k} - \mathbb{E}(T_{a_k b_k c_k}))] \| \leq \frac{1}{n_1^2 n_2^2 n_3^3} \| (P_U - P_{U_k}) P_{T_l}(P_U - P_{U_k}) \| \lesssim \frac{\mu r}{n(1)^2 n(2)^2 n(3)}. $$

Similarly, we have

$$ \| \mathbb{E}[(T_{a_k b_k c_k} - \mathbb{E}(T_{a_k b_k c_k}))(T_{a_k b_k c_k} - \mathbb{E}(T_{a_k b_k c_k}))^T] \| \lesssim \frac{\mu r}{n(1)^2 n(2)^2 n(3)}. $$

Noting that $P_{T_l} R_{\Omega}(P_U - P_{U_k}) = \sum_{k=1}^m T_{a_k b_k c_k}$, the lemma follows immediately from the Bernstein’s inequality.

7.9 Proof of Lemma 4.7

Proof of Lemma 4.7 Similar to the proof for the matrix case in [30], we have

$$ \| U_l - U \ast Q \|^2_F = \langle U_l - U \ast Q, U_l - U \ast Q \rangle = 2r - 2 \langle U_l, U \ast Q \rangle, $$

$$ \| U_l \ast U^T - U \ast U^T \|^2_F = 2r - 2 \langle U_l \ast U^T, U \ast U^T \rangle. $$

Then it suffices to show there exists an orthogonal tensor $Q \in \mathbb{R}^{p \times r \times n_2}$ such that

$$ \langle U_l, U \ast Q \rangle \geq \langle U_l \ast U^T, U \ast U^T \rangle, $$

which is equivalent to $\langle U \ast U_l, Q \rangle \geq \langle U^T \ast U_l, U^T \ast U_l \rangle$. Noting that $U^T \ast U_l = Q_1 \ast \Sigma \ast Q_2^T$, we can choose $Q = Q_1 \ast Q_2^T$, which completes the proof.
7.10 Proof of Lemma 4.8

Proof of Lemma 4.8. Let $d = ||X_l - X||_F$. By Lemma 3.2 we have

$$||U_l * U_l^T - U * U^T||_F \leq \frac{\sqrt{2d}}{\sigma_{\min}(X)} \quad \text{and} \quad ||\nu_l * \nu_l^T - \nu * \nu^T||_F \leq \frac{\sqrt{2d}}{\sigma_{\min}(X)}.$$  

Moreover, by Lemma 4.7 there exists two unitary tensors $Q_u$ and $Q_v$ such that

$$||U_l - U * Q_u||_F \leq \frac{\sqrt{2d}}{\sigma_{\min}(X)}, \quad ||\nu_l - \nu * Q_v||_F \leq \frac{\sqrt{2d}}{\sigma_{\min}(X)}.$$  

Noting that $||X_l - X||_F \leq \frac{\sigma_{\min}(X)}{10\sqrt{2}} \leq \frac{\sigma_{\max}(X)}{10\sqrt{2}}$ and $||X_l|| \leq ||X + Z_l - X|| \leq \sigma_{\max}(X) + d$, we have

$$||S_l - Q_u^* * S * Q_v||_F = \|U_l^T * X_l * \nu_l - (U * Q_u)^T * X * (\nu * Q_v)||_F \leq ||U_l^T * X_l * \nu_l - (U * Q_u)^T * X * (\nu * Q_v)||_F \leq 4kd,$$

where $\kappa$ is the condition number of $X$. Recall that $A_l$ and $B_l$ are defined as

$$A_l[i] = \frac{U_l[i]}{||U_l[i]||_F} \min(\|U_l[i]\|_F, \sqrt{\frac{\mu_0 r}{n_1}}), \quad B_l[i] = \frac{\nu_l[i]}{||\nu_l[i]||_F} \min(\|\nu_l[i]\|_F, \sqrt{\frac{\mu_0 r}{n_2}}).$$

Together with

$$||U_l * Q_u[i]||_F \leq \sqrt{\frac{\mu_0 r}{n_1}}, \quad ||\nu_l * Q_v[i]||_F \leq \sqrt{\frac{\mu_0 r}{n_2}},$$

we have

$$||A_l[i] - (U * Q_u)[i]||_F \leq ||U_l[i] - (U * Q_u)[i]||_F, \quad ||B_l[i] - (\nu * Q_v)[i]||_F \leq ||\nu_l[i] - (\nu * Q_v)[i]||_F.$$  

It follows that

$$||A_l - U * Q_u||_F \leq ||U_l - U * Q_u||_F \leq \frac{\sqrt{2d}}{\sigma_{\min}(X)}, \quad ||B_l - U * Q_v||_F \leq ||\nu_l - U * Q_v||_F \leq \frac{\sqrt{2d}}{\sigma_{\min}(X)}.$$  

Thus together with $\tilde{X} = A_l * S_l * B_l^*$, we have

$$||\tilde{X} - X||_F = ||A_l * S_l * B_l^* - (U * Q_u) * (Q_u^T * S * Q_v) * (\nu * Q_v)^T||_F \leq \|A_l^T * S_l * B_l^T - (U * Q_u) * (Q_u^T * S * Q_v) * (\nu * Q_v)^T\|_F + \|S_l * B_l^T - (U * Q_u) * S_l * B_l^T\|_F + \|U_l * Q_u - (U * Q_u)\|_F + \|S_l - Q_u^T * S * Q_v\|_F + \|S_l * B_l - (U * Q_u) * B_l - (U * Q_u)\|_F \leq 8kd.$$
We also need to estimate the incoherence of $\hat{Z}_l$. Since $A_l$ and $B_l$ are not necessarily unitary, we consider their QR factorizations:

$$A_l = \tilde{U}_l \ast R_u, \quad B_l = \tilde{V}_l \ast R_v.$$  

Noting that

$$\sigma_{\min}(A_l) = \min(\sigma(\tilde{A}_l)) \geq 1 - \|A_l - U \ast Q_u\| \geq 1 - \frac{\sqrt{2d}}{\sigma_{\min}(X)} \geq \frac{9}{10},$$

$$\sigma_{\min}(B_l) = \min(\sigma(\tilde{B}_l)) \geq 1 - \|B_l - V \ast Q_v\| \geq 1 - \frac{\sqrt{2d}}{\sigma_{\min}(X)} \geq \frac{9}{10},$$

by the assumption $d \leq \sigma_{\min}(X)/\sqrt{2}$ and the Weyl inequality, we have $\|R_u^{-1}\| \leq \frac{10}{9}$ and $\|R_v^{-1}\| \leq \frac{10}{9}$. Consequently,

$$\|\hat{U}_l\|_F = \|\tilde{U}_l\|_F = \|A_l \ast R_u^{-1}\|_F \leq \frac{10}{9} \sqrt{\frac{\mu_0 r}{n_1}}, \quad \|\hat{V}_l\|_F = \|\tilde{V}_l\|_F = \|B_l \ast R_v^{-1}\|_F \leq \frac{10}{9} \sqrt{\frac{\mu_0 r}{n_2}}.$$  

This completes the proof.  

\[\blacksquare\]

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