The Golden Age of the Mathematical Finance

José Manuel Corcuera.*

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Abstract

This paper is devoted to show that the last quarter of the past century can be considered as the golden age of the Mathematical Finance. In this period the collaboration of great economists and the best generation of probabilists, most of them from the Strasbourg’s School led by Paul André Meyer, gave rise to the foundations of this discipline. They established the two fundamentals theorems of arbitrage theory, close formulas for options, the main modelling approaches and created the appropriate framework for the posterior development.

Key words: Financial asset pricing; Options; Arbitrage; Complete markets; Semimartingale; Utility indifference price; Fundamental theorems of asset pricing.

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1 Introduction

In the last quarter of the past century a new mathematical discipline emerged: the Mathematical Finance. It was the conjunction of great economists like F. Black, M. Scholes, R. Merton, at the United States, and many others, with great mathematicians, most of them belonging to, or following, the famous Séminaire de Probabilités de Strasbourg under the leadership of Paul André Meyer. The theory of stochastic integrals with respect to semimartingales developed in this Seminar was the mathematical basis to establish the main results of the new discipline. In this paper we are going to explain how arose and were proved the two main theorems and how they provided the framework to the Arbitrage Theory, the core of the Mathematical Finance. The number of results and new paths opened during this period was too huge to be described in one single paper. Issues like local and stochastic volatility models, Dupire et al. (1994), Heston (1993), the different approaches in credit risk, Duffie and Singleton (1999), the use of strict local martingales to describe bubbles, Loewenstein and Willard (2000), the Heath-Jarrow-Morton approach in interest rate models, Heath, Jarrow, and Morton (1992), the models under transaction costs, Leland (1985), and a long etc, are not treated here. However many excellent books on the matter, written before the end of the last century, already include them, for instance, Lamberton and Lapeyre (1996), Musiela and Rutkowski (1997), Björk (1998), Dana and Jeanblanc-Picqué (1996) or Shiryaev (1999), among others.

In the next section we explain the beginning, at the 70’s, and the precedents. In the third section we explain the two fundamental theorems and their proofs, showing this intertwining among economists and mathematicians. Finally we explain the arbitrage theory for pricing and some of the strategies to solve the problem of incompleteness and the multiplicity of (non) arbitrage prices.

*Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, E-08007 Barcelona, Spain. E-mail: jmcorcuera@ub.edu. The work of J. M. Corcuera is supported by the Spanish grant MTM2013-40782-P.
The inception

Probably everything started at 1973 when Black and Scholes published their famous paper, *The pricing of options and corporate liabilities*, see Black and Scholes (1973), where they give a formula for a call option based on an arbitrage argument and the Itô formula. They assume that the stock $S_t$ follows a geometric Brownian motion (a model proposed previously by Samuelson (1965)), that is

$$dS_t = S_t (\mu dt + \sigma dW_t), S_0 > 0.$$ 

with $B$ a Brownian motion, and that the value of the unit of money, say $B_t$, evolves as

$$dB_t = B_t r dt.$$ 

for $\mu, \sigma, r$ constants. The way that Black and Scholes obtained the formula is the following. Suppose that the price, at time $t$, of the call option, where the payoff is $(S_T - K)^+$, is a smooth function of the form

$$C_t := f(t, S_t),$$

and consider a portfolio with $\beta_t$ calls and $\alpha_t$ stocks, the cost of this portfolio is

$$\beta_t C_t + \alpha_t S_t := V_t,$$

and when it evolves, in a self-financed way its value changes as

$$dV_t = \beta_t dC_t + \alpha_t dS_t,$$

$$dV_t = \beta_t \left( \partial_t f dt + \partial_x f dS_t + \frac{1}{2} \partial_{xx} f S_t^2 \sigma^2 dt \right) + \alpha_t dS_t,$$

where we apply the Itô formula for stochastic differentials. Now if we take $\alpha_t = - \beta_t \partial_x f$ we have that the cost of this portfolio is

$$dV_t = \beta_t \left( \partial_t f + \frac{1}{2} \partial_{xx} f S_t^2 \sigma^2 \right) dt.$$

It behaves like a bank account! remember that if we put $V_t$ in the bank account then $dV_t = V_t r dt$, then if we assume an equilibrium situation in such a way that it is not possible to do profit without risk, we must have

$$\beta_t \left( \partial_t f + \frac{1}{2} \partial_{xx} f S_t^2 \sigma^2 \right) = r V_t = r (\beta_t C_t - \beta_t \partial_x f S_t) = r \beta_t (f - \partial_x f S_t).$$

So, the price of a call is the solution of the partial differential equation

$$\partial_t f + \sigma^2 x^2 \partial_{xx} f = r f, \quad (1)$$

with the boundary condition $f(T, x) = (x - K)^+$. By doing a change of variable it is easy to obtain

$$C_t = S_t \Phi(d_+) - Ke^{-r(T-t)} \Phi(d_-)$$

where $\Phi$ is the c.d.f. of a standard normal distribution and

$$d_{\pm} = \frac{\log(S_t/K) + (r \pm \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}.$$ 

If we apply the Feynman-Kac formula, it is easy to see that

$$C_t = \mathbb{E}_* \left( (S_T - K)^+ e^{-r(T-t)} \mid S_t \right).$$
where $E_*$ is an expectation assuming that
\[ dS_t = rS_t dt + \sigma S_t dW_t^*, \]
where $W^*$ is a Brownian motion as well. In fact by the integration by parts formula, the boundary condition and equation (1)
\[ (S_T - K)_+ e^{-r(T-t)} = f(T, S_T) e^{-r(T-t)} = f(t, S_t) + \int_t^T e^{-r(u-t)} \partial_x f \sigma S_u dW_u^* \]
\[ + \int_t^T e^{-r(u-t)} \left( \partial_u f + r S_u \partial_x f + \frac{1}{2} \sigma^2 S_u^2 \partial_{xx} f \right) du \]
\[ - \int_t^T e^{-r(u-t)} r f du \]
\[ = f(t, S_t) + \int_t^T e^{-r(u-t)} \partial_x f \sigma S_u dW_u^*. \]
Now taking the conditional expectation we obtain the result.

However Samuelson and Merton in 1969, see also Merton (1973) where the author extends the analytical method of Black and Scholes to price more complex options, wrote a paper where they give the following formula for the price of a call option with strike $K$
\[ f(t, S_t) = e^{-r(T-t)} \int_{\mathbb{R}}^\infty (zS_t - K) dQ_{T-t}(z) \] (2)
where
\[ \int_0^\infty zdQ_{T-t}(z) = e^{r(T-t)}. \]
Then, in the special case when $Q_t$ is a log-normal distribution with log-variance equal to $\sigma^2(T-t)$ we recover the Black-Scholes formula.

Nevertheless we have to move back until 1900 to see the real birth of the mathematical finance. At that time the French postgraduate student Louis Bachelier presented his thesis *Théorie de la Speculation*. As it is said in Courtault et al. (2000) this thesis, supervised by Henri Poincaré, contains ideas of enormous value in both finance and probability. Bachelier was the first to consider the stochastic processes that today we call Brownian or Wiener process. He use it to model the movements of stock prices. He consider two constructions of the process, as a limit of (today named) a random walk and as a solution of a Fourier equation (today heat equation). Even he finds the distribution of the supremum of the Brownian motion. From a financial point of view he introduces a new idea to price an option. He uses the principle of the mathematical expectation in games: a game is fair if the expectation of the profit is zero, but he calculate the price by taking the expectation with respect to what today we call a *risk-neutral probability*. He establishes, as a fundamental principle, that the mathematical expectation of the profit of a buyer or seller of a financial product has to be zero. This principle allows him to calculate the price of a call option and he obtains, translated in the usual nowadays notation, see Bachelier (1900, p. 50), that
\[ C_0 = \int_{S_0 - K}^\infty (z + S_0 - K) dQ_T(z) \] (3)
where $Q_T \sim N(0, \sigma^2 T)$. It is important to remark that for Bachelier the Gaussian distribution was the objective one, the historical one. Even the same seems to be true for the Black and Scholes model: they probably believed that the geometric Brownian motion was the real distribution of the stock and there were not distinction between the real and the risk-neutral probability. At least this is how Merton interprets the Black-Scholes formula. He says, in Merton 73, page 161: "...However, $dQ$ [in (2)] is a risk-adjusted ("util-prob") distribution, depending on both, risk-prefereces and aggregate supplies, while the distribution in (1) is the objective distribution of returns on the common stock". Moreover, note that (3) is an arithmetic
version of (2), except for the discount factor $e^{-r(T-t)}$. It seems that, at least in the French market, all payments were related to a single date, in this way the discount factor is not needed in (3), that is Bachelier was talking about forward values.

The way that Bachelor's thesis was appreciated at his time is a bit controversial. It is an extended legend that Henri Poincaré, his supervisor, undervalued it because according to Bernstein (1993) he says "The topic is somewhat remote from those our candidates are in the habit of treating". But this appreciation has been discussed in Courtault et al. (2000) where they present the Poincaré's report on the Bachelier thesis showing that Poincaré supported quite strongly the dissertation. Also they show that the reason why Bachelier did not get a position in Paris is not related with any bad opinion of Poincaré but because Bachelier enrolled in the army at the First World War. Moreover Bernstein tells a story about how the pioneers in mathematical finance became aware about Bachelor's thesis suggesting that his work was ignored for a long period, but there is evidence that Bachelor's work was more known than the above legend says, Kolmogorov, Keynes, Feller, Lévy, among others, knew it.

Following the works of Black, Scholes and Merton, Harrison and Kreps, Harrison and Kreps (1979) realize that the absence of arbitrage can be characterized by the existence of a risk-neutral probability and that, adding a notion of completeness, price of derivatives could be obtained as expectations of discounted payoffs with respect to the risk-neutral probability, they analyze the discrete time case and where the probability space is finite. Later Harrison and Pliska in Harrison and Pliska (1981) consider the continuous time case and where the stock process is a semimartingale. They show that the existence of a local martingale measure (or risk-neutral measure) implies absence of arbitrage. They also define what is a complete market and establish a relation with the theorems about martingale representation and the set of risk-neutral probabilities. They prove that if the market is complete the set of risk-neutral probabilities is a singleton. This latter issue is treated in more detail in their paper Harrison and Pliska (1983). These papers established what are known as the two fundamental theorems of asset pricing, the first one is the characterization of the arbitrage and the second that of completeness. These theorems open the way for extensions of the Black-Scholes model and new price formulas of new derivatives. Curiously the semimartingale theory, the only applicable since the semimartingale set of process is invariant under changes of equivalent probabilities, was being developed in these decades but without a direct relation with the issue. Harrison and Pliska say: "the parts of probability theory most relevant to the general question (about what processes for the stock yield a complete market) are those results, usually abstract in appearance and French in origin...". They are refereeing to the Strasbourg Seminar led by André Meyer. They also add "...We have started to feel that all the standard problems studied in martingale theory and all the major results must have interpretations and applications in our setting".

### 3 The fundamental theorems of asset pricing

The term Fundamental Theorems of Asset Pricing (FTAP) was coined by Phil Dybvig in 1987, to describe the work initiated by his thesis advisor, Steve Ross around 1978, see Ross (1978). The connection to martingale theory is in Harrison and Kreps (1979) with an important extension to the continuous time setting in Harrison and Pliska (1981). In fact there are two fundamental theorems as we will see.

Let $S = (S^0_t, S^1_t, \ldots, S^d_t)_{t \in I}$ be a non-negative $d+1$-dimensional semimartingale representing the price process of $d+1$ securities, here $I$ is either a discrete finite set $I = \{0, 1, \ldots, T\}$ or a compact interval $I = [0, T]$. This process is assumed to be defined in a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where $\mathcal{F} = (\mathcal{F}_t)_{t \in I}$ is a filtration representing the flow of information and satisfying the usual conditions. We suppose that $S^0_t > 0$ is $\mathcal{F}_{t-1}$-measurable, $t = 1, \ldots, T$, that is $S^0$ is predictable. We also assume that $\mathcal{F}_0$ is trivial (a.s.) and that $\mathcal{F}_T = \mathcal{F}$. We define the discounted price as

\[ \tilde{S}_t := \frac{S_t}{S^0_t}, \quad t \in I. \]
3.1 Discrete time setting

**Definition 1** A trading strategy is a predictable stochastic process $\phi = ((\phi_i^0, \phi_i^1, \ldots, \phi_i^d))_{1 \leq i \leq T}$ in $\mathbb{R}^{d+1}$, that is $\phi_i^t$ is $\mathcal{F}_{t-1}$-measurable, for all $1 \leq t \leq T$.

$\phi_i^t$ indicates the number of units of security $i$ in the portfolio at time $t$ and that $\phi$ is *predictable* means that the positions in the portfolio at $t$ is decided at $t-1$, using the information available in $\mathcal{F}_{t-1}$. Then we have also the following definitions.

**Definition 2** The value of the portfolio associated with a trading strategy $\phi$ is given by

$$V_t(\phi) = \phi_t \cdot S_t := \sum_{i=0}^d \phi_i^t S_i^t, \quad t = 1, \ldots, T, \quad V_0(\phi) = \phi_1 \cdot S_0.$$ 

and its discounted value

$$\tilde{V}_t(\phi) := \phi_t \cdot \tilde{S}_t$$

**Definition 3** A trading strategy $\phi$ is said to be self-financing if

$$V_t(\phi) = \phi_{t+1} \cdot S_{t+1}, \quad t = 1, \ldots, T - 1.$$ 

It is easy to see that the *self-financing* condition is equivalent to the equality

$$V_t(\phi) = \tilde{V}_0(\phi) + \sum_{i=1}^t \phi_i \cdot \Delta \tilde{S}_i, \quad t = 1, \ldots, T.$$ 

**Definition 4** An admissible trading strategy $\phi$ is a self-financing strategy satisfying the following constraint

$$V_t(\phi) \geq 0 \text{ a.s. for } t = 0, 1, \ldots, T$$

**Definition 5** An admissible trading strategy $\phi$ is an arbitrage opportunity if it satisfies

$$V_0(\phi) = 0, \text{ a.s.}$$

$$V_T(\phi) \geq 0, \text{ a.s.} \text{ and } P(V_T > 0) > 0.$$ 

In an analogous way we can say when a self-financing strategy is an arbitrage. In the discrete time setting we have the following lemma

**Lemma 1** The class of admissible trading strategies contains no arbitrage opportunities if and only if the class of self-financing strategies contains no arbitrage opportunities.

**Proof.** Let $\varphi$ be a self-financing strategy that is an arbitrage. Define

$$t = \inf\{u, \tilde{V}_u(\varphi) \geq 0 \text{ a.s. for all } u > t \},$$

note that $t \leq T - 1$ since $\tilde{V}_T(\varphi) \geq 0$. Let $A = \{\tilde{V}_t(\varphi) < 0\}$, define the predictable vector process $\theta$, such that for all $i = 1, \ldots, d$

$$\theta_i^u = \begin{cases} 
0 & u \leq t \\
1_{A \varphi_i^u} & u > t 
\end{cases}$$
Lemma 2
\[ Y \in \text{give a proof based in the following lemma for the case that} \] 
\[ d \quad \text{R \ theorem in} \] 
the finite case the standard proof is that of Harrison and Pliska (1981), based on the separation hyperplane \( \theta \) so

Before stating the second fundamental theorem of asset pricing (SFTAP) we need the following definitions.

Definition 6 Probabilities \( P \) and \( P^* \) defined on \( (\Omega, \mathcal{F}) \) are said to be equivalent, we write \( P \sim P^* \), if they have the same null-sets.

Now we can establish the first fundamental theorem of the asset pricing (FFTAP).

Theorem 1 There are not arbitrage opportunities if and only if there is a probability \( P^* \sim P \) under which the process \( \bar{S} \) is a martingale.

Proof. (Sufficiency) Suppose that \( \bar{S} \) is a martingale under some \( P^* \sim P \). Then, let \( \phi \) be an arbitrage opportunity, so, since it is admissible, \( V_t(\phi) \geq 0 \) a.s. \( P^* \) (because the equivalence between \( P^* \) and \( P \)) and \( E_{P^*}(V_t(\phi)) \) is well defined, although possibly infinite. Then we have

\[ E_{P^*}(\hat{V}_t(\phi)) = E_{P^*}(\phi_t \cdot \bar{S}_t) = E_{P^*}(\phi_t \cdot E_{P^*}(\bar{S}_t|\mathcal{F}_{t-1})) \]
\[ = E_{P^*}(\phi_t \cdot E_{P^*}(\bar{S}_t|\mathcal{F}_{t-1})) = E_{P^*}(\phi_t \cdot E_{P^*}(\bar{S}_t|\mathcal{F}_{t-1})) \]
\[ = E_{P^*}(\phi_t \cdot \bar{S}_{t-1}) = E_{P^*}(\phi_{t-1} \cdot \bar{S}_{t-1}) = ... = E_{P^*}(\hat{V}_0(\phi)) = 0. \]

where in the second line we use the predictability of \( \phi \) and the martingale property of \( \bar{S} \), and in the third line the self-financing condition.

(Necessity) Now, if the number of elements in \( \Omega \) is finite we have simpler proofs than in the general case. For the finite case the standard proof is that of Harrison and Pliska (1981), based on the separation hyperplane theorem in \( \mathbb{R}^{[10]} \). If the sample space is not finite Morton (1988) and Dalang, Morton, and Willinger (1990) give a proof based in the following lemma for the case that \( d = 1 \).

Lemma 2 Let \( Y \) be a bounded \( d \)-dimensional random vector defined on some \( (\Omega, \mathcal{F}, P) \) and assume that \( Y \in K \) for some compact set in \( \mathbb{R}^d \) then one of the following two conditions hold,

1. There exists \( \alpha \in \mathbb{R}^d \) with \( \alpha \cdot Y \geq 0 \) a.s and \( P(\alpha \cdot Y > 0) > 0 \).
2. There exists a positive \( g \in C(K) \) (the space of real continuous functions on \( K \)) with \( E(g(Y)Y) = 0 \).

For \( d > 1 \) the idea is patching together the conditional martingale measures corresponding to each period but this involves subtle arguments by using the Measurable Selection Theorem. For alternative proofs that try to simplify the above one, see Schachermayer (1992), Rogers (1994) and Jacod and Shiryaev (1998).

Before stating the second fundamental theorem of asset pricing (SFTAP) we need the following definitions.
Definition 7 Let $X$ be a claim, that is $X \geq 0$, and $X \in \mathcal{F}_T$. We say that $X$ is attainable, or replicable, if $X$ is equal to the final value of an admissible strategy.

Assume that the market model is free of arbitrage, then the set of risk-neutral probabilities is not empty. If we choose one, say $\mathbb{P}^*$, we have the following definition.

Definition 8 We say that the market model is complete if every claim $X \in \mathcal{F}_T$, such that $\mathbb{E}_{\mathbb{P}^*} \left( \tilde{X} \right) < \infty$, is attainable.

Then the SFTAP reads as follows.

Theorem 2 The market is complete if and only if the risk-neutral probability is unique

Proof. (The 'only if' part) Assume that the market is complete, then the claims of the form $S_0^A 1_A$ with $A \in \mathcal{F}_T$ are attainable, therefore we can write

$$1_A = \tilde{V}_0(\phi) + \sum_{t=1}^{T} \phi_t \cdot \Delta \tilde{S}_t$$

where $\phi$ is admissible. Now, if we assume that there are two risk-neutral probabilities, say $\mathbb{P}^*$ and $\mathbb{Q}^*$, we have, since $\tilde{S}$ is a martingale with respect to both probabilities $\mathbb{P}^*$ and $\mathbb{Q}^*$, that

$$\mathbb{E}_{\mathbb{P}^*} (1_A) = \mathbb{E}_{\mathbb{Q}^*} (1_A),$$

in such a way that $\mathbb{P}^*$ and $\mathbb{Q}^*$ are the same probability in $\mathcal{F}_T$.

(The 'if' part ) Assume first that $\Omega$ is finite. Let $H$ be the subset of random variables of the form

$$\tilde{V}_0(\phi) + \sum_{t=1}^{T} \phi_t \cdot \Delta \tilde{S}_t$$

with $\phi$ predictable. $H$ is a vector subspace of the vectorial space, say $E$, formed by all random variables. Moreover it is not a trivial subspace, in fact since the market is incomplete there will exist $h$ such that $\frac{\Delta h}{\Delta \tilde{S}_t} \notin H$ (note that if $h \geq 0$ can be replicated by a non-admissible strategy then the market cannot be viable). Let $\mathbb{P}^*$ be a risk-neutral probability in $E$, we can define the scalar product $\langle X, Y \rangle = \mathbb{E}_{\mathbb{P}^*} (XY)$. Let $X$ be an random variable orthogonal to $H$, and set

$$\mathbb{P}^{**}(\omega) = \left( 1 + \frac{X(\omega)}{2||X||_{\infty}} \right) \mathbb{P}^*(\omega).$$

Then we have an equivalent probability to $\mathbb{P}^*$:

$$\mathbb{P}^{**}(\omega) = \left( 1 + \frac{X(\omega)}{2||X||_{\infty}} \right) \mathbb{P}^*(\omega) > 0,$$

$$\mathbb{P}^{**}(\omega) = \mathbb{P}^*(\omega) + \frac{\mathbb{E}_{\mathbb{P}^*}(X)}{2||X||_{\infty}} = \sum \mathbb{P}^*(\omega) = 1.$$

Now, take $1_A$, with $A \in \mathcal{F}_{t-1}$. Then $1_A \Delta \tilde{S}^j_t \in H$ for all $j = 1, \ldots, d$ and $X$ is orthogonal to $H$, therefore

$$\mathbb{E}_{\mathbb{P}^{**}} \left( 1_A \Delta \tilde{S}^j_t \right) = \mathbb{E}_{\mathbb{P}^*} \left( 1_A \Delta \tilde{S}^j_t \right) + \frac{\mathbb{E}_{\mathbb{P}^*} \left( X 1_A \Delta \tilde{S}^j_t \right)}{2||X||_{\infty}} = 0,$$

in such a way that $\tilde{S}$ is a $\mathbb{P}^{**}$-martingale. The surprise for the general case is that if the risk-neutral probability is unique then $\Omega$ is essentially finite for $\mathbb{P}_1$, in the sense that $\mathcal{F}_T$ is purely atomic with respect to $\mathbb{P}$ with at most $(d+1)^N$ atoms, see Theorem 6 in Jacod and Shiryaev (1998).
3.2 Continuous time setting

Now we have that \( I = [0, T] \), \( S = (S_t^0, S_t^1, ..., S_t^d)_{t \in I} \) is a non-negative \( d + 1 \)-dimensional semimartingale representing the price process of \( d + 1 \) securities. This process is assumed to be defined in a complete filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) where \( \mathbb{F} = (\mathcal{F}_t)_{t \in I} \) is a filtration satisfying the usual conditions. We suppose that \( S_t^0 > 0 \). We also assume that \( \mathcal{F}_0 \) is trivial (a.s.) and that \( \mathcal{F}_T = \mathcal{F} \). We define the discounted value as

\[
\tilde{S}_t := \frac{S_t}{S_T^0}, \quad t \in I.
\]

Definition 9 A trading strategy is a predictable stochastic process \( \phi = (\phi_t^0, \phi_t^1, ..., \phi_t^d)_{1 \leq t \leq T} \) in \( \mathbb{R}^{d+1} \), that means that \( \phi_t^i \) is measurable with respect to the sigma-field generated by the càglàd adapted processes and that is integrable with respect to \( S \).

Then we have also the following definitions.

Definition 10 The discounted value of the portfolio associated with a trading strategy \( \phi \) is given by

\[
V_t(\phi) = \phi_t \cdot \tilde{S}_t = \sum_{i=0}^{d} \phi_t^i \tilde{S}_t^i,
\]

Definition 11 A trading strategy \( \phi \) is said to be self-financing if

\[
V_t(\phi) = \phi_0 \cdot \tilde{S}_0 + \int_0^t \phi_t \cdot d\tilde{S}_t, \quad t \in [0, T].
\]

Definition 12 An admissible trading strategy \( \phi \) is a self-financing strategy satisfying

\[
V_t(\phi) \geq -a, \quad \text{for all } t \in [0, T]
\]

for some \( a > 0 \).

According to Delbaen and Schachermayer (1994) we use the notation

\[
K_0 = \left\{ \int_0^T \phi_s \cdot d\tilde{S}_s, \phi \text{ admissible} \right\}
\]

\[
C_0 = K_0 - L_0^0
\]

\[
K = K_0 \cap L_\infty
\]

\[
C = C_0 \cap L_\infty
\]

\[
\overline{C} \text{ the closure of } C \text{ under } L_\infty
\]

Definition 13 We say that the model satisfies the No Free Lunch with Vanishing Risk condition (NFLVR) if

\[
\overline{C} \cap L_\infty^\infty = \{0\}
\]

We have the FFTAP in continuous time

Theorem 3 Let \( \tilde{S} \) be a locally bounded \( \mathbb{R}^d \)-valued semimartingale. There are not free lunches with vanishing risk if and only if there is probability \( \mathbb{P}^* \sim \mathbb{P} \) under which \( \tilde{S} \) is a local martingale.
Proof. (Sufficiency) Let $\phi$ be an admissible strategy with initial value equal to zero and let $\mathbb{P}^*$ be a probability such that $\tilde{S}$ is a $\mathbb{P}^*$-local martingale. Since $V(\phi) = \int_0^T \phi_t \cdot d\tilde{S}_t$ is bounded below it is a local martingale (see Ansel and Stricker (1994)) and in fact a supermartingale, by Fatou’s lemma. Then we have

$$\mathbb{E}_{\mathbb{P}^*}\left(\dot{V}_T(\phi)\right) \leq 0$$

and we obtain that $\mathbb{E}_{\mathbb{P}^*}(h) \leq 0$ for every $h$ in $\mathcal{C}$ and consequently for $h$ in $\bar{C}$ by the Lebesgue theorem. Therefore $\mathcal{C} \cap L^\infty = \{0\}$.

(Necessity) This is the difficult part, but essentially it consists in considering the weak* topology in $L^\infty$ and to show that, under the NFLVR condition, $\mathcal{C}$ is closed with this topology (the proof of this is very technical and the reference is Delbaen and Schachermayer (1994)). Then we can apply the separation theorem with this weak topology (e.g., 9.2 in Schaefer (1971)) and to show that there exist $\mathbb{P}^* \sim \mathbb{P}$ such that $\mathbb{E}_{\mathbb{P}^*}(h) \leq 0$ for each $h$ in $\mathcal{C}$ (see the details in Schachermayer (1994), 3.1). Now if we assume first that $\tilde{S}$ is a bounded semimartingale we have that for each $s < t$, $B \in \mathcal{F}_s$ and $\alpha \in \mathbb{R}$, $\alpha 1_B (\tilde{S}_t - \tilde{S}_s) \in \mathcal{C}$. Therefore $\mathbb{E}_{\mathbb{P}^*}(1_B (\tilde{S}_t - \tilde{S}_s)) = 0$ and $\mathbb{P}^*$ is a risk-neutral probability for $\tilde{S}$. If $\tilde{S}$ is locally bounded then a localization argument, and the result for the bounded case, allows us to obtain a locally martingale measure $\mathbb{P}^*$ for $\tilde{S}$.

Also we have a result for the case that $\tilde{S}$ is not locally bounded. The result involves the concept of $\sigma$-martingale, that is a process that can be obtained as a stochastic integral of a $\mathbb{P}^*$-martingale (see Ansel and Stricker (1994)) and in fact a supermartingale, by Fatou’s lemma. Then we have $\mathcal{E}_{\mathbb{P}^*} (\tilde{S})$ is not locally bounded. The result involves the concept of $\sigma$-martingale, that is a process that can be obtained as a stochastic integral of a $\mathbb{P}^*$-martingale (see Ansel and Stricker (1994)).

**Theorem 4** Let $\tilde{S}$ be a $\mathbb{R}^d$-valued semimartingale. There are no free lunches with vanishing risk if and only if there is probability $\mathbb{P}^* \sim \mathbb{P}$ under which $\tilde{S}$ is a $\sigma$-martingale.

We have the analogous definitions for completeness in the the continuous time case, thought the meaning of replication now is in terms of stochastic integrals.

**Definition 14** Let $X$ be a claim, that is $X \geq 0$, and $X \in \mathcal{F}_T$. We say that $X$ is attainable, or replicable, if $X$ is equal to the final value of an admissible strategy.

Assume that the market model satisfies the NFLVR condition, then the set of risk-neutral probabilities is not empty. If we choose one, say $\mathbb{P}^*$, we have the following definition.

**Definition 15** We say that the market model is complete if every claim $X \in \mathcal{F}_T$, such that $\mathbb{E}_{\mathbb{P}^*}(\tilde{X}) < \infty$, is attainable.

Then the SFTAP for the continuous time setting reads the same way as in the discrete time case.

**Theorem 5** The market is complete if and only if the risk-neutral probability is unique

Proof. (The ‘only if’ part) The proof is exactly the same as in the discrete time case.

(The ‘if’ part) Suppose that the risk neutral probability is unique. By Theorem 11.2 in Jacod (1979) $\tilde{S}$ has the representation property with respect to $\mathbb{P}^*$ (any $\mathbb{P}^*$-martingale can be written as an integral with respect to $\tilde{S}$) if and only $\mathbb{P}^*$ is an extremal measure in the set of all probability measures for which $\tilde{S}$ is a local martingale. Suppose that $\mathbb{P}^*$ is not extremal, then there exist $\mathbb{Q}$, $\mathbb{Q}'$ local martingale measures (not equivalent to $\mathbb{P}$) such that $\mathbb{P}^* = \lambda \mathbb{Q} + (1 - \lambda) \mathbb{Q}'$ for some $\lambda \in (0, 1)$. Also $\mathbb{Q}_\beta := \beta \mathbb{Q} + (1 - \beta) \mathbb{Q}'$ is a local martingale measure for any $\beta \in (0, 1)$, and since $\mathbb{Q} << \mathbb{P}^*$ and $\mathbb{Q}' << \mathbb{P}^*$ we obtain that $\mathbb{Q}_\beta \sim \mathbb{P}^*$ contradicting the uniqueness of $\mathbb{P}^*$. Now by the representation property, if $\mathbb{E}_{\mathbb{P}^*}(\tilde{X}) < \infty$ we can write

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{X} | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(\tilde{X}) + \int_0^t \phi_s \cdot d\tilde{S}_s, t \in [0, T]$$

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and, in particular, with \( \phi_0 \cdot \tilde{S}_0 = \mathbb{E}_{\mathbb{P}^*} \left( \tilde{X} \right) \),

\[
\tilde{X} = \phi_0 \cdot \tilde{S}_0 + \int_0^T \phi_t \cdot d\tilde{S}_t.
\]

Notice that the admissibility condition for \( \phi \) is trivially satisfied since \( X \geq 0 \).

### 4 The arbitrage theory

From the previous results we deduce that in a complete market model we can price any payoff \( X \) such that \( \mathbb{E}_{\mathbb{P}^*} \left( \tilde{X} \right) < \infty \) and the price at time \( t \) is obviously the price of the replicating portfolio, \( V_t(\phi) \), given by the strategy \( \phi \) and such that \( \tilde{V}_t(\phi) = \tilde{V}_T(\phi) \) is a \( \mathbb{P}^* \) martingale. Then since \( \tilde{X} = \tilde{V}_T(\phi) \) we have that

\[
V_t(\phi) = S_0^0 \mathbb{E}_{\mathbb{P}^*} \left( \tilde{X} \mid \mathcal{F}_t \right).
\]  (4)

Notice that we can have other admissible strategies, say \( \varphi \), replicating \( X \) but such that, the corresponding discounted portfolios, are local martingales and since they are bounded below they are supermartingales. Then

\[
\tilde{V}_t(\phi) = \mathbb{E}_{\mathbb{P}^*} \left( \tilde{X} \mid \mathcal{F}_t \right) = \mathbb{E}_{\mathbb{P}^*} \left( \tilde{V}_T(\varphi) \mid \mathcal{F}_t \right) \leq \tilde{V}_t(\varphi).
\]

Consequently they are more expensive and no one would pay for that.

If the model is not complete but satisfying NFLV, the formula (4) can be used for pricing if we take one risk-neutral probability since, if we do that, we will have a price market model free of arbitrage. But what is the correct risk-neutral probability? One way to select a risk-neutral probability is to ask an additional property to the risk-neutral probability. An alternative is to choose a risk-neutral probability, say \( \mathbb{P}^* \), close to \( \mathbb{P} \). In this sense if we take a strictly convex function we can try to minimize

\[
\mathbb{E} \left( V \left( \frac{d\mathbb{P}^*}{d\mathbb{P}} \right) \right).
\]  (5)

For instance if \( V(x) = x \log x \) we have the minimal entropy martingale measure, see Frittelli (2000), that coincides with the Esscher measure in the case we model the stock by a geometric Lévy model, see Chan (1999). Some risk-neutral measures, like the minimal martingale measure are also related to the minimization of the cost to replicate perfectly the contingent claims, see Föllmer and Schweizer (1991). It can be seen that, in certain cases, the minimal martingale measure minimizes (5) with \( V(x) = \frac{x^2}{2} \), see Chan (1999) and Schweizer (1995).

If we consider the (negative) Legendre transform of \( V \) we have

\[
U(x) = \inf_y \left( V(y) + xy \right),
\]

for the examples above we obtain \( U(x) = -e^{-x} \) if \( V(x) = x \log x \) and \( U(x) = -\frac{x^2}{2} \) if \( V(x) = \frac{x^2}{2} \), that can be interpreted as utility functions. Then by the duality relationship we have that, under appropriate conditions see (Schachermayer, 2000), the optimal wealth, say \( W_T^* \), when we try to maximize the expected utility of the final wealth (by using admissible strategies), satisfies

\[
\frac{U'(W_T^*)}{\mathbb{E}(U'(W_T))} \frac{d\mathbb{P}^*}{d\mathbb{P}}
\]

where \( \frac{d\mathbb{P}^*}{d\mathbb{P}} \) minimize (5). Then if we use this risk-neutral probability to price derivatives what we obtain is the marginal utility indifference price proposed by Mark H.A. Davis, see Davis (1997). In fact, let define, for an initial wealth \( x \),

\[
v(x) := \sup \mathbb{E} \left( U \left( W_{T,x} \right) \right)
\]
assuming $U : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave and strictly increasing $C^2$ function, with $U''(\infty) = 0$ and $U'(0^+) = \infty$. If we invest $\varepsilon$ in a claim $\xi$ with price $p$ we can look for the value of $p$, say $\hat{p}$, such that

$$\frac{d}{d\varepsilon} \sup E\left(U\left(W_{T,x-\varepsilon} + \frac{\varepsilon}{\hat{p}} \xi\right)\right)_{\varepsilon=0} = 0,$$

then, we have that

$$E\left(U\left(W_{T,x-\varepsilon} + \frac{\varepsilon}{\hat{p}} \xi\right)\right) - E\left(U\left(W_{T,x-\varepsilon}\right)\right) = \varepsilon E\left(U'\left(W_{T,x-\varepsilon}\right) \frac{\xi}{\hat{p}}\right) + o(\varepsilon)$$

and from here we obtain that

$$\hat{p} = \frac{E\left(U'\left(W^*_T,x\right) \xi\right)}{v'(x)},$$

since $v'(x) = E\left(U'\left(W^*_T,x\right)\right)$ we obtain that

$$\hat{p} = E_{P^*}(\xi),$$

with $\frac{dP^*}{dP} = \frac{U'(W^*_T,x)}{E(U'(W^*_T,x))}.$

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