INVERSION OF ADJUNCTION FOR RATIONAL AND DU BOIS PAIRS

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Abstract. We prove several results about the behavior of Du Bois singularities and Du Bois pairs in families. Some of these generalize existing statements about Du Bois singularities to the pair setting while others are new even in the non-pair setting. We also prove a new inversion of adjunction result for Du Bois and rational pairs. In the non-pair setting this asserts that if a family over a smooth base has a special fiber $X_0$ with Du Bois singularities and the general fiber has rational singularities, then the total space has rational singularities near $X_0$.

1. Introduction

Rational singularities have been the gold-standard for "mild" singularities in algebraic geometry for several decades. Whenever a new class of varieties with singularities is discovered, the first question usually asked is whether or not the new varieties have rational singularities. Varieties with rational singularities behave cohomologically as if they were smooth. However, for many purposes rational singularities are not broad enough. For instance, nodes are not rational singularities, and more generally, singularities appearing on stable varieties, that is mild degenerations of smooth ones that are necessary to consider in order to compactify moduli spaces are not always rational. The class of Du Bois (or DB) singularities is slightly more inclusive than rational singularities. Du Bois singularities behave cohomologically as if they had simple normal crossing singularities (i.e., a higher dimensional version of nodes).

Recently, Kollár and the first named author \[\text{Kov11, Kol13}\] have introduced the notions of rational and Du Bois pairs $(X, D)$ for a normal variety $X$ and a reduced divisor $D \subseteq X$. These notions are philosophically distinct from the singularities considered typically in the minimal model program since $(X, D)$ having rational (respectively Du Bois) singularities does not generally imply that the ambient space $X$ has rational (respectively Du Bois) singularities \[\text{Kol13 Remark 2.81(2)}\] (respectively \[\text{Example 2.10, Example 2.14}\]). Instead the singularities of $(X, D)$ measure the connection between the singularities of $X$ and $D$ (a notion obviously connected with problems related to inversion of adjunction).

Even though a priori rational and Du Bois singularities are not part of the class one usually associates with the minimal model program, these singularities play important roles in both the mmp and moduli theory via the fact that (semi-)log canonical singularities are Du Bois \[\text{KSS10, KK10}\]. In addition, Du Bois singularities have also played important roles in various other contexts recently. They are arguably the largest class of singularities for which we know that Kodaira vanishing holds \[\text{Pat13}\], they appear in proofs of extension and...
other vanishing theorems [GKKP11], positivity theorems [Sch12], categorical resolutions [Lun12], log canonical compactifications [HX13] and many other results more directly related to the minimal model program.

It is now a basic tenet of the minimal model program that the right way to study singularities is via pairs cf. [Kol97, Kol13]. This allows for more freedom in applications and makes inductive arguments easier. The same is true for rational and Du Bois singularities. The introduction of Du Bois pairs streamlined some existing proofs cf. [Kol13, Chapter 6] and extended the realm of applications. For instance, like Du Bois singularities, if \((X, Z)\) is a Du Bois then the ideal sheaf of \(Z\) satisfies various vanishing theorems which we hope will be useful in the future.

In this paper we extend several recent results on Du Bois singularities to the context of Du Bois pairs, notably the recent deformation of Du Bois singularities found in [KS11a] and the requisite injectivity theorem, a result of Kollár and Kovács on the behavior of depth in Du Bois families, and the characterization of Cohen-Macaulay Du Bois singularities of [KSS10].

Furthermore, we prove a new inversion of adjunction statement for rational and Du Bois pairs. This statement is new even in the non-pair setting. Roughly speaking, in the non-pair setting it says that if \(f : X \to B\) is a family over a smooth base such that the general fiber has rational singularities and the special fiber has Du Bois singularities, then \(X\) has rational singularities in a neighbourhood of the special fiber. See Theorem E below for the general statement.

We state each of these theorems below. We begin with the deformation statement.

**Theorem A** ([Theorem 4.4]). Let \(X\) be a scheme essentially of finite type over \(\mathbb{C}\), \(Z \subseteq X\) is a reduced subscheme and \(H\) a reduced effective Cartier divisor on \(X\) that does not contain any component of \(Z\). If \((H, Z \cap H)\) is a Du Bois pair, then \((X, Z)\) is a Du Bois pair near \(H\).

Just as in the non-pair setting, to prove this we first show an injectivity theorem.

**Theorem B** ([Theorem 3.2]). Let \(X\) be a variety over \(\mathbb{C}\) and \(Z \subseteq X\) a subvariety. Then the natural map

\[
\Phi^j : \mathcal{E}xt^j_{\mathcal{O}_X} (\mathcal{O}^0_{X,Z}, \omega^X) \hookrightarrow \mathcal{E}xt^j_{\mathcal{O}_X} (\mathcal{I}_Z, \omega^X)
\]

is injective for every \(j \in \mathbb{Z}\).

Here \(\mathcal{E}xt^j_{\mathcal{O}_X} (\_ , \omega^X)\) is shorthand to denote \(h^j (\mathcal{R}Hom_{\mathcal{O}_X} (\_ , \omega^X))\).

We also generalize some of the results of [KK10] for families, to the context of Du Bois pairs.

**Theorem C** ([Corollary 5.6]). Let \(f : (X, Z) \to B\) be a flat projective Du Bois family with \(\mathcal{O}_Z, \mathcal{I}_Z\) flat over \(B\) as well. Assume that \(V\) is connected and the generic fibers \((\mathcal{I}_Z)_{\text{gen}}\) are Cohen-Macaulay, then all the fibers \((\mathcal{I}_Z)_b\) are Cohen-Macaulay.

We have a multiplier ideal/module like characterization of Du Bois pairs.

**Theorem D** ([Theorem 6.3]). Let \(X\) be a normal variety and \(Z \subseteq X\) a divisor. Further let \(\pi : \widetilde{X} \to X\) be a log resolution of \((X, Z)\) with \(E = \pi^{-1}(Z)_{\text{red}} \vee \text{exc}(\pi)\). If \(\mathcal{I}_Z\) is Cohen-Macaulay then \((X, Z)\) is Du Bois if and only if

\[
\pi_* \omega^\widetilde{X} (E) \simeq \omega^X (Z).
\]

All of the above results are used in the proof of our inversion of adjunction result.
Theorem E (Theorem 7.1). Let \( f : X \to B \) be a flat projective family over a smooth connected base \( B \) with \( \dim B \geq 1 \), \( H = f^{-1}(0) \) the special fiber, and \( D \) a reduced codimension 1 subscheme of \( X \) which is flat over \( B \). Assume that \( (H, D|_H) \) is a Du Bois pair and that \( (X \setminus H, D \setminus H) \) is a rational pair. Then \((X, D)\) is a rational pair.

This last result is new even in the case \( D = 0 \), see Corollary 7.8. In the special case when \( X \setminus H \) is smooth and \( Z = 0 \), Theorem E follows from [Sch07, Theorem 5.1].

Statements similar to Theorem E have been proved in many related situations. Here is a non-exhaustive list of some of these results: [KSBB88, Theorem 5.1], [FW89, Proposition 2.13], [Kar00, Theorem 2.5], [Kaw07, Sch09], [Hac12] and [Lrt14].

2. Definitions and basic properties

Throughout this paper, all schemes are Noetherian and essentially\(^{\dagger}\) of finite type over \( \mathbb{C} \). Furthermore, all schemes are separated. Given divisors \( D = \sum a_iD_i \) and \( D' = \sum b_iD_i \) on a normal variety (possibly allowing \( a_i, b_j \) to be zero), we define \( D \vee D' = \sum \max(a_i, b_i)D_i \) and \( D \wedge D' = \sum \min(a_i, b_i)D_i \). Of course, if \( D \) and \( D' \) have no common components then \( D \vee D' = D + D' \) and \( D \wedge D' = 0 \). On a scheme \( X \) essentially of finite type over \( \mathbb{C} \), we use \( \text{D}(\_\_\_\_) = \mathcal{R}\mathcal{H}\text{om}_{\hat{\mathcal{O}}_X}(\_\_, \omega_X^{-1}) \) to denote the Grothendieck duality functor.

2.1. Rational pairs. First we recall the notion of rational pairs defined by the first named author and Kollár as described in [Kol13, Chapter 2]. Note that a similar notion was defined by the second named author and Takagi in [ST08]. The two notions are closely related but different. Their relationship is similar to how dlt singularities compare to klt singularities. Here we will discuss the former notion which in the dlt vs klt analogy corresponds to dlt.

Definition 2.1. Let \( X \) be a normal variety and \( D \subseteq X \) an integral Weil divisor on \( X \). A log resolution \((Y, D_Y) \xrightarrow{\pi} (X, D)\) is a resolution of singularities such that \( D_Y \) is the strict transform of \( D \), and such that \((D_Y)_{\text{red}} \cup \text{exc}(\pi)\) is a simple normal crossing divisor.

Definition 2.2. A reduced pair \((X, D)\) consists of a normal variety \( X \) and a reduced divisor \( D \) on \( X \). For the definition of an snc pair, the strata of an snc pair and other normal crossing conditions please refer to [Kol13, 1.7].

One frequently wants log resolutions that do not blowup unnecessary centers. One good way to achieve this is with a thrifty resolution.

Definition 2.3 (Thrifty resolution [Kol13, 2.79]). Let \((X, D)\) be a reduced pair. A thrifty resolution of \((X, D)\) is a resolution \( \pi : Y \to X \) such that:

(a) \( D_Y = \pi_*^{-1}D \) is a simple normal crossing divisor:
(b) \( \pi \) is an isomorphism over the generic point of every stratum of the snc locus of \((X, D)\) and \( \pi \) is an isomorphism at the generic point of every stratum of \((Y, D_Y)\).

Item (b) can also be replaced by:

(b') The exceptional set \( E \) of \( \pi \) does not contain any stratum of \((Y, D_Y)\) and \( \pi(E) \) does not contain any stratum of the simple normal crossing locus of \((X, D)\).

We can now define rational pairs [Kol13, Section 2.5].

Definition 2.4 (Rational pairs). A reduced pair \((X, D)\) is called a rational pair if there exists a thrifty resolution \( \pi : (Y, D_Y) \to (X, D) \) such that

(i) \( \mathcal{O}_X(-D) \cong \pi_* \mathcal{O}_Y(-D_Y) \).
(ii) \( \mathcal{R}^i \pi_* \mathcal{O}_Y(-D_Y) = 0 \) for all \( i > 0 \).

\(^{\dagger}\)

that is, a localization of a finite type scheme.
(iii) $\mathcal{R}^i\pi_*\omega_Y(D_Y) = 0$ for all $i > 0$.

If $(X, D)$ is a rational pair, and is in characteristic zero, then every thrifty resolution satisfies the properties (i), (ii), (iii) above [Kol13, Corollary 2.86]. Even better though, property (iii) always holds in characteristic zero as we point out below, whether or not $(X, D)$ is a rational pair.

Theorem 2.5. Let $(X, D)$ be a reduced pair and $\pi : (Y, D_Y) \to (X, D)$ a thrifty resolution, then $\mathcal{R}^i\pi_*\omega_Y(D_Y) = 0$ for all $i > 0$.

Proof. This follows from [Kol13, Theorem 10.39]. Alternatively, one can prove this directly:

Claim 2.6. Let $\pi : E \to D$ be a proper birational map between reduced equidimensional $\mathbb{C}$-schemes of finite type such that $E$ is a simple normal crossing divisor in some smooth ambient space. Assume that $\pi$ is birational on every irreducible components of $E$. Then $\mathcal{R}^i\pi_*\omega_E = 0$ for $i > 0$.

Proof of Claim 2.6. We proceed by induction on $\dim D$ and the number of irreducible components of $E$, and note that the base case is simply Grauert-Riemenschneider vanishing [GR70]. Write $E = E_0 \cup E'$ where $E_0$ is an irreducible component of $E$ and $E'$ are the remaining irreducible components. We have a short exact sequence

$$0 \to \mathcal{O}_E \to \mathcal{O}_{E_0} \oplus \mathcal{O}_{E'} \to \mathcal{O}_{E_0 \cap E'} \to 0.$$ 

Dualizing we obtain

$$0 \to \omega_{E_0} \oplus \omega_{E'} \to \omega_E \to \omega_{E_0 \cap E'} \to 0.$$ 

Applying $\mathcal{R}^i\pi_*$ and the inductive hypothesis to $E_0$, $E'$ and $E_0 \cap E'$ proves the claim. □

Alternative Proof of Theorem 2.5. Push forward the short exact sequence $0 \to \omega_Y \to \omega_Y(D_Y) \to \omega_{D_Y} \to 0$ via $\pi$ and apply the claim to $\omega_{D_Y}$ and $\omega_Y$. □

This gives us the following criterion.

Corollary 2.7. Let $(X, D)$ be a reduced pair and $\pi : (Y, D_Y) \to (X, D)$ a thrifty resolution. Then $(X, D)$ is a rational pair if and only if

$$\mathcal{R}\mathcal{H}^\bullet \mathcal{O}_X(-D), \omega_X) \simeq \mathcal{R}\pi_*\omega_Y(D_Y)[\dim X] \simeq \pi_*\omega_Y(D_Y)[\dim X]$$

for some thrifty resolution. Furthermore, in characteristic zero the second isomorphism is automatic.

Proof. Observe that the conditions (i) and (ii) of Definition 2.4 are equivalent to the isomorphism $\mathcal{R}\pi_*\mathcal{O}_Y(-D_Y) \simeq \mathcal{O}_X(-D)$. Applying Grothendieck duality and condition (iii) to this isomorphism yields the statement. The characteristic zero statement is simply Theorem 2.5. □

2.2. Du Bois pairs. The notion of Du Bois singularities is becoming more and more part of the basic knowledge in higher dimensional geometry. In particular, for the notion of the Deligne-Du Bois complex of a scheme of finite type over $\mathbb{C}$ and its degree zero associated graded complex, denoted by $\Omega^0_X$, we refer the reader to [Kol13, Section 6.1].

In contrast, the notion of Du Bois pairs is relatively new, defined, and so here we discuss some of their basic properties.

Given a (possibly non-reduced) subscheme $Z \subseteq X$ one has an induced map in $D^b_{\text{coh}}(X)$,

$$\Omega^0_Z \to \Omega^0_X,$$
noting that by definition $\Omega^0_Z = \Omega^0_{Z, \text{red}}$. Then $\Omega^0_{X, Z}$ is defined to be the object in the derived category making the following an exact triangle:

$$\Omega^0_{X, Z} \to \Omega^0_Y \to \Omega^0_Z \xrightarrow{+1}.$$  

If $\mathcal{I}_Z$ is the ideal sheaf of $Z$, then it is easy to see that there is a natural map $\mathcal{I}_Z \to \Omega^0_{X, Z}$, [Kov11, Section 3.D].

**Definition 2.8.** [Kov11, Definition 3.13] We say that $(X, Z)$ is a Du Bois pair (or simply a DB pair) if the canonical map $\mathcal{I}_Z \to \Omega^0_{X, Z}$ is a quasi-isomorphism.

In the original definition of a Du Bois pair in [Kov11] it was assumed that $Z$ is reduced. As it turns out, this is not a necessary hypothesis.

**Lemma 2.9.** If $(X, Z)$ is a Du Bois pair and $X$ is reduced, then $Z$ is reduced.

**Proof.** Note that $\Omega^0_Z = \Omega^0_{Z, \text{red}}$ and so $\Omega^0_{X, Z} \simeq \Omega^0_{X, Z, \text{red}}$. On the other hand, we also have an exact sequence:

$$0 \to h^0(\Omega^0_{X, Z}) \to h^0(\Omega^0_X) \to h^0(\Omega^0_Z) \xrightarrow{\simeq} 0 \to \mathcal{I}_{\text{Im}(Z^{\text{SN}}) \subseteq X^{\text{SN}}} \to \mathcal{O}^{\text{SN}}_{X} \to \mathcal{O}^{\text{SN}}_{Z, \text{red}}$$

where $X^{\text{SN}}, Z^{\text{SN}}$ are the seminormalizations of $X$ and $Z_{\text{red}}$ respectively, and the right two isomorphisms come from [Sai00]. Note that the scheme theoretic image of $Z^{\text{SN}}_{\text{red}}$ in $X^{\text{SN}}$ is reduced. The fact that the left-most map is an isomorphism implies that $h^0(\Omega^0_{X, Z})$ is a radical ideal in $\mathcal{O}^{\text{SN}}_X$. Since $(X, Z)$ is Du Bois, we see that $h^0(\Omega^0_{X, Z}) = \mathcal{I}_{Z \subseteq X}$ and hence $\mathcal{I}_{Z \subseteq X}$ is radical in $\mathcal{O}^{\text{SN}}_X$ and hence also in $\mathcal{O}_X = \mathcal{O}^{\text{SN}}_{X, \text{red}}$ as desired.

Frequently we will take the Grothendieck dual of $\Omega^0_{X, Z}$. Hence following the notation of [KST1a] we will write

$$\omega^\vee_{X, Z} := \mathcal{R}\text{Hom}_{\mathcal{O}_X}(\Omega^0_{X, Z}, \omega^\vee_X).$$

We refer to [KK10, Section 6.1] for basic properties of Du Bois pairs. It is important to remark that this notion of pairs is very different in flavor from the definition of $(X, Z)$ being log canonical or log terminal. The next example shows that $(X, Z)$ being Du Bois does not imply that $X$ is Du Bois.

**Example 2.10** (A Du Bois pair whose ambient space is not Du Bois). Let $R$ denote the pullback of the following diagram:

$$k[x] \simeq k[x, y]/(y) \leftarrow k[x, y] \phantom{\leftarrow} \uparrow$$

$$k[x^2, x^3] \leftarrow R$$

where the non-dotted arrows are induced in the obvious ways. It is easy to see that $R = k[x^2, x^3, y, xy]$. By construction $X = \text{Spec } R$ is not Du Bois since it is not seminormal. However, we claim that the pair $(\text{Spec } R, V((y, xy)_R))$ is Du Bois. Consider the following
diagram:

\[
\begin{array}{c}
0 \longrightarrow \langle y, yx \rangle_R \longrightarrow R \longrightarrow k[x^2, x^3] \longrightarrow 0 \\
\alpha \downarrow \quad \beta \quad \gamma \\
0 \longrightarrow \langle y \rangle k[x, y] \longrightarrow k[x, y] \longrightarrow k[x] \longrightarrow 0.
\end{array}
\]

The maps labeled \(\beta\) and \(\gamma\) are the seminormalizations but \(\alpha\) is an isomorphism. On the other hand, we know that \(\mathcal{O}^0_X = \mathcal{O}^0_{X, \text{sn}}\) in general since they have the same hyperresolution. Therefore, up to harmless identification of modules with sheaves on an affine scheme, we see \(\mathcal{O}_{\text{Spec } R}^0 \simeq k[x, y]\) and \(\mathcal{O}_{\text{Spec } k[x^2, x^3]}^0 \simeq k[x]\) and so

\[\langle y, yx \rangle_R = \langle y \rangle k[x, y] \simeq \mathcal{O}_{\text{Spec } R, V(\langle y, yx \rangle_R)}^0.\]

This proves that \((\text{Spec } R, V(\langle y, yx \rangle_R))\) is Du Bois and completes the example.

Next we will give an example of a normal Du Bois pair whose ambient space is not Du Bois. To this end we will use a criterion for a cone being a Du Bois pair. In order to do that we need to recall a definition [Kol13, III.3.8]:

Let \(X\) be a projective scheme and \(\mathcal{L}\) an ample line bundle on \(X\). The spectrum of the section ring of \(\mathcal{L}\) also called the (generalized) ample cone over \(X\) with co-normal bundle \(\mathcal{L}\) is

\[C_a(X, \mathcal{L}) := \text{Spec}_k \bigoplus_{p \geq 0} H^0(X, \mathcal{L}^p).\]

If no confusion is likely, in particular when \(\mathcal{L}\) is fixed, we will use the shorthand of \(C_X := C_a(X, \mathcal{L})\). Notice that for a subscheme \(Z \subseteq X\) there is a natural map \(\iota : C_a(Z, \mathcal{L}|_Z) \rightarrow C_a(X, \mathcal{L})\) which is a closed embedding away from the vertex \(P \in C_X\). By a slight abuse of notation we will also use \((C_X, C_Z)\) to denote the pair \((C_a(X, \mathcal{L}), \iota(C_a(Z, \mathcal{L}|_Z)))\).

Now we are ready to state the needed Du Bois criterion:

**Proposition 2.11.** ([GK14] cf. [Ma13]) Let \(X\) be a smooth projective variety, \(Z \subseteq X\) an snc divisor (possibly the empty set), and \(\mathcal{L}\) an ample line bundle on \(X\). Then \((C_X, C_Z)\) is a Du Bois pair if and only if

\[H^i(X, \mathcal{L}^m(-Z)) = 0\]

for all \(i, m > 0\).

**Proof.** In the case \(Z = \emptyset\) this follows from [Ma13 4.4]. The general case works similarly. For a direct proof see [GK14].

While the above is sufficient for our purposes, we also obtained independently a slightly different statement using similar methods.

**Lemma 2.12** (Du Bois pairs for graded rings). Let \(X\) be a projective variety with Du Bois singularities, \(\mathcal{L}\) an ample line bundle and \(D\) a reduced connected divisor on \(X\). Assume that \(D\) also has only Du Bois singularities. Form the corresponding section ring \(S = \bigoplus_{i \geq 0} \Gamma(X, \mathcal{L}^i)\) and \(I = \bigoplus_{i \geq 0} \Gamma(X, \mathcal{O}_X(-D) \otimes \mathcal{L}^i)\). Fix \(m = S_+\) to be the irrelevant ideal. Set \(Y = CX = \text{Spec } S\) and \(Z = CD = \text{Spec } (S/I)\). If

\[H^1(X, \mathcal{O}_X(-D) \otimes \mathcal{L}^i) = 0\]

for \(i \geq 0\) so that \(S/I \simeq \bigoplus_{i \geq 0} \Gamma(D, \mathcal{L}^p|_D)\) then for all \(i \geq 1\) we have

\[h^i_{Y, Z} \simeq \lvert H^i_m(I) \rvert_{>0} > 0.\]

Again under the hypothesis (2.12.1) we see immediately that \((Y, Z)\) is Du Bois if and only if \([H^i_m(I)]_{>0} = 0\) for every \(i > 0\).
Proof. First observe that both $Y$ and $Z$ are seminormal since they are saturated section rings over seminormal schemes. L. Ma showed in [Ma13, Equation (4.4.4) in the proof of Theorem 4.4] that

\[(2.12.2) \quad h^i \Omega^0_Y \simeq \left[H^i_m(S)\right]_{>0}\]

for $i > 0$. Likewise $h^i \Omega^0_Z = \left[H^i_m(S/I)\right]_{>0}$ for $i > 0$. Now we analyze $h^i \Omega^0_Y$ via a spectral sequence. Since $Y$ is Du Bois outside of the origin $V(m)$, we see that $h^j \Omega^0_Y$ is supported only at the origin for $j > 0$. It follows that the $E_2$-page of the spectral sequence

\[H^i_m(h^j \Omega^0_Y) \Rightarrow h^{i+j} \mathcal{R} \Gamma_m(\Omega^0_Y)\]

looks like

\[
\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
h^0 \Omega^0_Y & 0 & 0 & 0 & 0 & \cdots \\
h^2 \Omega^0_Y & 0 & 0 & 0 & 0 & \cdots \\
h^3 \Omega^0_Y & 0 & 0 & 0 & 0 & \cdots \\
0 & H^1_m(S) & H^2_m(S) & H^3_m(S) & H^4_m(S) & \cdots \\
\end{array}
\]

Here we are using the fact that $S$ is seminormal, and so $h^0 \Omega^0_Y = S$. It is not difficult to see that the unique nonzero map of the $(i - 1)$st page of this spectral sequence induces the isomorphism of [2.12.2], and so those unique non-zero maps are injective. Thus the spectral sequence contains the data of a long exact sequence

\[0 \to H^1_m(I) \to H^1_m(\Omega_Y^0) \to h^1 \Omega^0_Y \to H^2_m(S) \to H^2_m(\Omega_Y^0) \to h^2 \Omega^0_Y \to H^3_m(S) \to \ldots\]

Hence $\mathbb{H}^i_m(\Omega_Y^0) = [H^i_m(S)]_{>0}$ for $i \geq 2$ and $\mathbb{H}^1_m(\Omega_Y^0) \simeq H^1_m(S)$. Likewise $\mathbb{H}^i_m(\Omega_Z^0) = [H^i_m(S/I)]_{<0}$ for $i \geq 2$ and $\mathbb{H}^1_m(\Omega_Z^0) \simeq H^1_m(S/I)$. Furthermore, since $Y$ and $Z$ are seminormal we see that $h^0 \Omega^0_{Y,Z} = I$ and so the same spectral sequence argument implies that we have a long exact sequence

\[0 \to H^1_m(I) \to H^1_m(\Omega_{Y,Z}^0) \to h^1 \Omega_{Y,Z}^0 \to H^2_m(I) \to H^2_m(\Omega_{Y,Z}^0) \to h^2 \Omega_{Y,Z}^0 \to H^3_m(I) \to \ldots\]

We still have the labeled surjectivities by the Matlis dual of Theorem 3.2, which we will prove later (we assume it for now). Thus it is enough to see that the maps above make the identification $\mathbb{H}^i_m(\Omega_{Y,Z}^0) = [H^i_m(I)]_{>0}$ for $i \geq 2$.

We consider the diagram with distinguished triangles as rows

\[
\begin{array}{c}
I \to S \to S/I \to +1 \\
\Omega_{Y,Z}^0 \to \Omega_Y^0 \to \Omega_Z^0 \to +1.
\end{array}
\]
We will apply the functor $\mathcal{R}\Gamma_m(\_)$ and take cohomology $i \geq 1$ to obtain:

$$
\begin{array}{cccccc}
H^i_m(S) & \to & H^i_m(S/I) & \to & H^{i+1}_m(I) & \to & H^{i+1}_m(S) & \to & H^{i+1}_m(S/I) \\
\alpha & & \beta & & \gamma & & \delta & & \\
\mathbb{H}^i_m(\Omega^0_Y) & \to & \mathbb{H}^i_m(\Omega^0_Z) & \to & \mathbb{H}^{i+1}_m(\Omega^0_{Y,Z}) & \to & \mathbb{H}^{i+1}_m(\Omega^0_Y) & \to & \mathbb{H}^{i+1}_m(\Omega^0_Z)
\end{array}
$$

\[H^i_m(S)\leq_0 \quad [H^i_m(S/I)]\leq_0 \quad [H^{i+1}_m(S)]\leq_0 \quad [H^{i+1}_m(S/I)]\leq_0\]

Note that $\gamma$ is the map we already identified as surjective above. It is easy to see that the vertical maps $\alpha, \beta, \delta, \epsilon$ are the projections and so $[\alpha]_{\leq 0}, [\beta]_{\leq 0}, [\delta]_{\leq 0},$ and $[\epsilon]_{\leq 0}$ are isomorphisms. Thus $[\gamma]_{\leq 0}$ is also an isomorphism. But from the second row we see that $\mathbb{H}^i_m(\Omega^0_{Y,Z})$ is of non-positive degree so that $\mathbb{H}^i_m(\Omega^0_{Y,Z}) = [H^i_m(I)]_{\leq 0}$ for $i \geq 2$ which completes the proof.

\[\square\]

**Remark 2.13.** It would be natural to try to prove a common generalization of the (independently obtained) [Proposition 2.11] and [Lemma 2.12].

**Example 2.14** (A normal Du Bois pair whose ambient space is not Du Bois). Let $W$ be an arbitrary smooth canonically polarized variety, that is, $W$ is smooth and projective and $\omega_W$ is ample. Further let $n, a \in \mathbb{N}$ such that $1 \leq a \leq n, X = W \times \mathbb{P}^n$ and $\mathcal{L} = \pi_1^*\omega_W \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(a)$. Finally, let $K \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree $n + 1$, that is, $\mathcal{O}_{\mathbb{P}^n}(K) \simeq \omega_{\mathbb{P}^n}^{-1}$, and let $Z = W \times K$. We claim that, using the above notation, $(CX, CZ)$ is a Du Bois pair, while $CX$ itself is not. Note also, that by construction $CX$ is normal.

Consider $H^1(X, \mathcal{O}_X(-Z) \otimes \mathcal{L}^d)$ for $i \geq 0$ and observe that

$$
\mathcal{O}_X(-Z) \otimes \mathcal{L}^d = \pi_2^*\mathcal{O}_{\mathbb{P}^n}(-n - 1) \otimes \pi_1^*\omega_W \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(a) = \pi_2^*\mathcal{O}_{\mathbb{P}^n}(a - n - 1) \otimes \pi_1^*\omega_W.
$$

Since $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a - n - 1)) = H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a - n - 1)) = 0,$ it follows by the Künneth formula that $H^1(X, \mathcal{O}_X(-Z) \otimes \mathcal{L}^d) = 0$, so the hypotheses of [Lemma 2.12] are satisfied. (This paragraph is unnecessary if you instead apply [Proposition 2.11].)

Let $r = \dim W$ and consider $H^r(X, \mathcal{L})$. By the Künneth formula

$$
H^r(X, \mathcal{L}) \supseteq H^r(W, \omega_W) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a)) \neq 0,
$$

and hence by [Proposition 2.11] $CX$ is not Du Bois.

On the other hand by choice we have that

\[(2.14.1) \mathcal{L}(-Z) \simeq \pi_1^*\omega_W \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(a - n - 1)\]

where $-n - 1 < a - n - 1 < 0$ and hence $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a - n - 1)) = 0$ for all $q \in \mathbb{N}$, so, again by the Künneth formula it follows that $H^i(X, \mathcal{L}(-Z)) = 0$ for all $i > 0$.

In order to conclude that $(CX, CZ)$ is a Du Bois pair we need that $H^i(X, \mathcal{L}^m(-Z)) = 0$ for all $i, m > 0$. We have seen that this holds for $m = 1$. If $m > 1$ then $\mathcal{M} := \mathcal{L}^{m-1} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(a)$ is ample on $X$ and by (2.14.1) and Kodaira vanishing we have that

$$
H^i(X, \mathcal{L}^m(-Z)) \simeq H^i(X, \omega_X \otimes \mathcal{M}) = 0
$$

and hence it follows from [Proposition 2.11] that $(CX, CZ)$ is indeed a Du Bois pair.

We will find the following lemma useful, cf. [Esn90] [Sch07].

**Lemma 2.15.** Suppose $(X, Z)$ is a pair with $Z \subseteq X$ reduced schemes. Suppose further that $X \subseteq Y$ where $Y$ is smooth. Let $\pi : \tilde{Y} \to Y$ be a log resolution of both $X$ and $Z$.
in $Y$ and set $\overline{X}$ and $\overline{Z}$ to be the reduced preimages of $X$ and $Z$ in $\tilde{Y}$ respectively. Then $\Omega^{0}_{X,Z} \simeq \mathcal{R}\pi_{*}\mathcal{J}_{\mathcal{Z} \subseteq \mathcal{X}}$ where $\mathcal{J}_{\mathcal{Z} \subseteq \mathcal{X}}$ is the ideal of $\mathcal{Z}$ in $\mathcal{X}$.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
\mathcal{R}\pi_{*}\mathcal{J}_{\mathcal{Z} \subseteq \mathcal{X}} & \rightarrow & \mathcal{R}\pi_{*}\mathcal{O}_{\overline{X}} \\
\mathcal{R}\pi_{*}\Omega^{0}_{X,Z} & \rightarrow & \mathcal{R}\pi_{*}\Omega^{0}_{X} \\
\mathcal{R}\pi_{*}\Omega^{0}_{Z} & \rightarrow & \mathcal{R}\pi_{*}\Omega^{0}_{Z} + 1 \\
\Omega^{0}_{X,Z} & \rightarrow & \Omega^{0}_{X} + 1 \\
\end{array}
$$

The top vertical arrows are quasi-isomorphisms since $\overline{X}$ and $\overline{Z}$ are SNC and hence Du Bois. The lemma follows immediately. $\square$

There are some situations when a pair being Du Bois implies that the ambient space is also Du Bois. It is proved in [GK14] that this happens if $X$ is Gorenstein, but that $X$ being $\mathbb{Q}$-Gorenstein is not sufficient. Another simple situation in which this holds is the following:

**Lemma 2.16.** Let $X$ be a reduced $\mathbb{C}$-scheme essentially of finite type and $H$ a Cartier divisor. If $(X, H)$ is a Du Bois pair then $X$ (and hence $H$) is also Du Bois.

**Proof.** The statement is local and so we may assume that $X = \text{Spec } R$ is affine. We know that $\mathcal{O}_{X}(-H) \rightarrow \Omega^{0}_{X,H}$ is a quasi-isomorphism and thus so is $\mathcal{O}_{X} \rightarrow \Omega^{0}_{X,H} \otimes \mathcal{O}_{X}(H)$. We will show that this map factors through $\Omega^{0}_{X,H}$, which will complete the proof by [Kov99] 2.3.

Embed $X \subseteq Y$ as a smooth scheme and let $\pi : \tilde{Y} \rightarrow Y$ be a simultaneous log resolution of $(Y, X)$ and $(Y, H)$ with $\overline{X}, \overline{H}$ the reduced total transforms of $X$ and $H$ respectively. Then $\Omega^{0}_{X,H} = \mathcal{R}\pi_{*}\mathcal{J}_{\mathcal{X}}$ by Lemma 2.15. Fix $X'$ to be the components of $\overline{X}$ which are not also components of $\overline{H}$ and we see that $\mathcal{J}_{\mathcal{X}} \simeq \mathcal{O}_{X'}(-\overline{H}|_{X'})$. Thus $\Omega^{0}_{X,H} \otimes \mathcal{O}_{X}(H) \simeq \mathcal{R}\pi_{*}\mathcal{O}_{X'}((\pi^{*}H - \overline{H})|_{X'})$.

Since $\pi^{*}H - \overline{H}$ is effective, we obtain a map $\Omega^{0}_{X} \simeq \mathcal{R}\pi_{*}\mathcal{O}_{\overline{X}} \rightarrow \mathcal{R}\pi_{*}\mathcal{O}_{X'} \rightarrow \mathcal{R}\pi_{*}\mathcal{O}_{X'}((\pi^{*}H - \overline{H})|_{X'}) \simeq \Omega^{0}_{X,H} \otimes \mathcal{O}_{X}(H)$.

This map obviously factors the quasi-isomorphism $\mathcal{O}_{X} \rightarrow \Omega^{0}_{X,H} \otimes \mathcal{O}_{X}(H)$ and hence the proof is complete. $\square$

We recall properties of $\Omega^{0}_{X,Z}$ that we will need later.

**Lemma 2.17.** Let $X$ be a scheme over $\mathbb{C}$ and $Z \subseteq X$ is a closed subscheme and $j : U = X \setminus Z \hookrightarrow X$ the complement of $Z$. Then:

(a) If in addition $X$ is proper, then $H^{i}(X, \mathcal{I}_{Z}) \rightarrow H^{i}(X, \Omega^{0}_{X,Z})$ is surjective for all $i \in \mathbb{Z}$, [Kol11] Corollary 4.2, [Kol13] Theorem 6.22.

(b) If $H$ is a general member of a base point free linear system then $\Omega^{0}_{X,Z} \otimes \mathcal{O}_{H} \simeq \Omega^{0}_{H,H \cap Z}$, [Kol11] Proposition 3.18, [Kol13] Theorem 6.5(6)].

(c) If $X = U \cup V$ is a decomposition into closed subschemes and $Z \subseteq X$ is another closed subscheme, then we have a distinguished triangle:

$$
\Omega^{0}_{U \cup V, Z} \rightarrow \Omega^{0}_{U, Z \cap U} \oplus \Omega^{0}_{V, Z \cap V} \rightarrow \Omega^{0}_{U \cap V, Z \cap U \cap V} + 1
$$

cf. [Kol13] Theorem 6.5(11)].
(d) Let $X = U \cup V$ be a decomposition of $X$ into closed subschemes. Then

$$
\Omega^0_{U \cup V, V} \simeq \Omega^0_{U, U \cap V}.
$$

cf. [Kov11, 3.19], [Kol13, Theorem 6.17].

Proof. Parts (a) and (b) follow from the references in their statement. For (c), the included reference only states the triangle in the case that $Z = \emptyset$. However, this more general version follows easily from the following diagram:

\[
\begin{array}{c}
\Omega^0_{U \cup V, Z} \to \Omega^0_{U, Z \cap U} \oplus \Omega^0_{V, Z \cap V} \to \Omega^0_{U \cup V, Z \cap U \cap V} \to +1 \\
\Omega^0_{U \cup V} \to \Omega^0_U \oplus \Omega^0_V \to \Omega^0_{U \cup V} \to +1 \\
\Omega^0_Z \to \Omega^0_{Z \cap U} \oplus \Omega^0_{Z \cap V} \to \Omega^0_{Z \cap U \cap V} \to +1 \\
+1 \downarrow \quad +1 \downarrow \quad +1 \downarrow
\end{array}
\]

and the 9-lemma in triangulated categories [Kov13, B.1].

For (d) consider the distinguished triangle $\Omega^0_{U \cup V} \to \Omega^0_U \oplus \Omega^0_V \to \Omega^0_{U \cap V} +1$ of part (c) with $Z = \emptyset$ and [KK10, 2.1] implies that the left vertical arrow of the following diagram is an isomorphism:

\[
\begin{array}{c}
\Omega^0_{U \cup V, V} \to \Omega^0_{U \cup V} \to \Omega^0_V \to +1 \\
\Omega^0_{U \cup V} \to \Omega^0_U \to \Omega^0_{U \cap V} \to +1 \\
\Omega^0_{U \cup V, V} \to \Omega^0_{U \cup V} \to \Omega^0_V \to +1
\end{array}
\]

For more details see the proof in the references and replace $\Omega^X$ with $\Omega^0$. \qed

We have one more lemma which constructs a natural exact triangle for Du Bois pairs.

Lemma 2.18. Let $X$ be a variety and $W, Z \subseteq X$ subvarieties. Then there is a distinguished triangle

$$
\Omega^0_{X, W \cup Z} \to \Omega^0_{X, Z} \to \Omega^0_{W, Z \cap W} +1.
$$

In particular, because there is also a short exact sequence:

$$
0 \to \mathcal{I}_{W \cup Z \subseteq X} \to \mathcal{I}_{Z \subseteq X} \to \mathcal{I}_{Z \cap W \subseteq W} \to 0,
$$

if any two of $\{(X, W \cup Z), (X, Z), (W, Z \cap W)\}$ are Du Bois, so is the third.
Proof. We begin with a diagram of distinguished triangles as columns and rows (cf. [Kol13, 6.5.11] and [Kov13, Thm. B1]):

\[
\begin{array}{cccc}
\Omega_X^0 & \Omega_X^0 & \Omega_X^0 & +1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\Omega_X^0 & \Omega_X^0 & \Omega_X^0 & +1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\Omega_W^0 & \Omega_W^0 & \Omega_W^0 & +1 \\
\end{array}
\]

Then the octahedral axiom implies that there exists a diagram of distinguished triangles,

\[
\begin{array}{cccc}
\Omega_X^0 & \Omega_X^0 & \Omega_X^0 & +1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\Omega_X^0 & \Omega_X^0 & \Omega_X^0 & +1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\Omega_W^0 & \Omega_W^0 & \Omega_W^0 & +1 \\
\end{array}
\]

We need to identify \( K^* \). Notice that the bottom row also fits into another diagram of distinguished triangles (cf. [Kov13, Thm. B1]):

\[
\begin{array}{cccc}
\Omega_X^0 & \Omega_X^0 & \Omega_X^0 & +1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\Omega_X^0 & \Omega_X^0 & \Omega_X^0 & +1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\Omega_W^0 & \Omega_W^0 & \Omega_W^0 & +1 \\
\end{array}
\]

Hence \( K^* \simeq \Omega_{Z\cap W}^0 \) and the lemma follows. \( \square \)

Finally, note that being Du Bois is a direct generalization of being rational for pairs.

Theorem 2.19. ([Kov11, Corollary 5.6], cf. [Kol13, 6.25]) If \((X, D)\) is a rational pair then \((X, D)\) is also a Du Bois pair.

3. An injectivity theorem for Du Bois pairs

A key ingredient of the proof that Du Bois singularities are deformation invariant was an injectivity theorem [KS11a, Theorem 3.3]. In this section, we generalize that result to the context of pairs.

Lemma 3.1. (cf. [KS11a, Lemma 3.1]) Let \( X \) be a variety, \( Z \subseteq X \) is a subvariety and \( \mathcal{L} \) a semi-ample line bundle. Let \( s \in \mathcal{L}^n \) be a general global section for some \( n \gg 0 \) and take the \( n^{th} \)-root of this section as in [KM98, 2.50]:

\[
\eta : Y = \text{Spec} \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i} \to X.
\]
Set $W = \eta^{-1}(Z)$ (with the induced scheme structure). Note that the restriction satisfies $\eta|_W : W = \text{Spec} \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}|_Z \to Z$. Then as before, writing $\eta_* = R\eta_*$,

$$\eta_* \Omega^0_{Y,W} \simeq \Omega^0_{X,Z} \otimes \eta_* \mathcal{O}_Y \simeq \bigoplus_{i=0}^{n-1} (\Omega^0_{X,Z} \otimes \mathcal{L}^{-i}),$$

and this direct sum decomposition is compatible with the decomposition $\eta_* \mathcal{O}_Y = \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}$.

**Proof.** Although not explicitly stated, it is easy to see that [KST1a Lemma 3.1] is functorial in that it is compatible with the map $Z \to X$. Then by applying [Lemma 2.17 b), the result follows from the diagram:

$$\begin{array}{cccccc}
\eta_* \Omega^0_{Y,W} & \longrightarrow & \eta_* \Omega^0_{Z} & \longrightarrow & \eta_* \Omega^0_{W} & +1 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\Omega^0_{X,Z} \otimes \eta_* \mathcal{O}_Y & \longrightarrow & \Omega^0_{X} \otimes \eta_* \mathcal{O}_Y & \longrightarrow & \Omega^0_{Z} \otimes \eta_* \mathcal{O}_Y & \simeq \Omega^0_{Z} \otimes \mathcal{O}_Z \eta_* \mathcal{O}_Z +1.
\end{array}$$

□

Setting $\omega^*_i \mathcal{O}_X = R\mathcal{H}om_{\mathcal{O}_X}(\Omega^0_{X,Z}, \omega^*_X)$ as in (2.9.1) we easily obtain the following.

**Theorem 3.2.** Let $X$ be a variety over $\mathbb{C}$ and $Z \subseteq X$ a subvariety. Then the natural map

$$\Phi^j : h^j(\omega^*_X, Z) \hookrightarrow h^j(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega^*_X))$$

is injective for every $j \in \mathbb{Z}$.

**Proof.** The proof is essentially the same as in [KST1a Theorem 3.3] so we only sketch it briefly. First, since the question is local and compatible with restricting to an open subset, we may assume that $X$ is projective with ample line bundle $\mathcal{L}$. It follows from taking a cyclic cover with respect to a general section of $\mathcal{L}^n$, for $n \gg 0$, and applying [Lemma 2.17 a) and Lemma 3.1] that $H^j(X, \mathcal{I}_Z \otimes \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}) \to \mathbb{H}^j(X, \Omega^0_{X,Z} \otimes \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i})$ surjects for all $j \geq 0$. Therefore $H^j(X, \mathcal{I}_Z \otimes \mathcal{L}^{-i}) \to \mathbb{H}^j(X, \Omega^0_{X,Z} \otimes \mathcal{L}^{-i})$ surjects for all $i, j \geq 0$.

By an application of Serre-Grothendieck duality we obtain an injection

(3.2.1) $\mathbb{H}^j(X, \mathcal{I}_Z \otimes \mathcal{L}^i) \hookrightarrow \mathbb{H}^j(X, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega^*_X) \otimes \mathcal{L}^i)$

for all $i, j \geq 0$. But for $i \gg 0$, by Serre vanishing, we obtain that

(3.2.2) $H^0(X, h^j \mathcal{I}_Z \otimes \mathcal{L}^i) \hookrightarrow H^0(X, h^j R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega^*_X) \otimes \mathcal{L}^i)$

is injective as well (since the spectral sequence computing (3.2.1) degenerates). On the other hand, if $h^j \mathcal{I}_Z \to h^j R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega^*_X)$ is not injective, for some $i \gg 0$ neither is (3.2.2). This completes the proof. □

4. Deformation of Du Bois pairs

In [KST1a Corollary 4.2], we showed the following result: Let $f : X \to B$ be a flat proper family over a smooth curve $B$ with a fiber $X_0$, $0 \in B$, having Du Bois singularities. Then there is an open neighborhood $0 \in U \subseteq B$ such that the fibers $X_u$ have Du Bois singularities for $u \in U$. In this section, we generalize this result to Du Bois pairs. We mimic our previous approach as much as possible.

First we need a lemma which is presumably well known but for which we know no reference.
Lemma 4.1. Let $X$ be a reduced scheme and $Z \subseteq X$ a reduced subscheme with ideal sheaf $\mathcal{I}_Z$. Further let $H \subseteq X$ be a Cartier divisor with ideal sheaf $\mathcal{I}_H$ such that $H$ does not contain any irreducible components of either $X$ or $Z$. Then

$$\mathcal{I}_H \cap \mathcal{I}_Z = \mathcal{I}_H \cdot \mathcal{I}_Z.$$ 

Proof. The statement is local, so we may assume that $X = \text{Spec} \ A$. Let $I \subseteq A$ be the ideal of $Z$, i.e., $\mathcal{I}_Z = \sqrt{I}$. Since $Z$ is reduced, $I = \sqrt{I}$ and hence $I = \cap_{i=1}^r p_i$ with prime ideals $p_i \subset A$. Assume that this is an economic decomposition, i.e., the $p_i$ are minimal primes of $I$. Further let $f \in A$ be a local equation for $H$, i.e., $\mathcal{I}_H = (f)$. The assumption that $H$ does not contain any irreducible components of either $X$ or $Z$ implies that

(a) $f$ is not contained in any minimal primes of $A$, and
(b) $f$ is not contained in any of the $p_i$.

Claim 4.2. For any prime ideal $p \subseteq A$ such that $f \notin p$,

$$(f) \cap p = fp.$$ 

Proof. $(f) \cap p \supseteq fp$ trivially, so we only need to prove the opposite containment. Let $fg \in (f) \cap p$. Since $f \notin p$, it follows that $g \in p$, so $fg \in fp$ as desired. \hfill $\square$

Applying this to the $p_i$ we obtain that

$$\tag{4.2.1} (f) \cap I = (f) \cap \left(\cap_{i=1}^r p_i\right) = \bigcap_{i=1}^r ((f) \cap p_i) = \bigcap_{i=1}^r fp_i.$$ 

Claim 4.3. Assume that $f$ is not contained in any minimal primes of $A$. Then for any set of prime ideals $\{p_i \subseteq A\}$,

$$\tag{4.3.1} \bigcap_{i=1}^r fp_i = f \left(\cap_{i=1}^r p_i\right).$$ 

Proof. Let $x \in \bigcap_{i=1}^r fp_i$ and let $g_i \in p_i$ such that $x = fg_i$ for each $i$. We claim that $g_i = g_j$ for any $i, j$. Indeed, $fg_i = x = fg_j$ so

$$f(g_i - g_j) = 0 \in \bigcap_{p \subseteq A \text{ is a minimal prime}} p.$$ 

By assumption $f \notin p$ for any of the $p$, so we must have $g_i - g_j \in p$ for all $p$. However, since $X$ is reduced,

$$\bigcap_{p \subseteq A \text{ is a minimal prime}} p = 0,$$ 

so it follows that $g_i = g_j =: g$. Finally this implies that $x = fg \in f \left(\cap_{i=1}^r p_i\right)$. \hfill $\square$

Combining (4.2.1) and (4.3.1) implies that $(f) \cap I = f \cdot I$. \hfill $\square$

Now we prove that if a special fiber supports a Du Bois pair, so does the total space near that fiber. We begin by clarifying some notation: By a Cartier divisor on a possibly non-normal scheme we mean a subscheme locally defined by a single non-zero-divisor near every point.

Theorem 4.4. Let $X$ be a scheme essentially of finite type over $\mathbb{C}$, $Z \subseteq X$ a reduced subscheme and $H$ a reduced effective Cartier divisor on $X$ that does not contain any component of $Z$. If $(H, Z \cap H)$ is a Du Bois pair, then $(X, Z)$ is a Du Bois pair near $H$. It then follows from Lemma 2.18 that $(X, Z \cup H)$ is Du Bois near $H$. 

Proof. Next, choose a closed point $q$ of $X$ contained within $H$. It is sufficient to prove that $(X, Z)$ is Du Bois at $q$. Let $R$ denote the stalk $\mathcal{O}_{X,q}$ and replace $X$ by Spec $R$. Choose $f \in R$ to denote a defining equation of $H$ in $R$. Consider the following diagram whose rows are distinguished triangles in $D^b_{\text{coh}}(X)$:

\[ \begin{array}{cccccc}
I_Z & \xrightarrow{\times f} & I_Z & \to & I_Z/(f \cdot I_Z) & \to A^* \\
\Omega^0_{X,Z} & \xrightarrow{\times f} & \Omega^0_{X,Z} & \to & A^* & \to \Omega^0_{H,Z \cap H} \\
\end{array} \]

(4.4.1)

where $A^*$ is the term completing the second row to a distinguished triangle. We claim we have a map $\tau$ as above such that $\tau \circ \rho$ is a quasi-isomorphism. Certainly we have a diagram with distinguished triangles for rows and columns

\[ \begin{array}{cccccc}
\Omega^0_{X,Z} & \xrightarrow{\times f} & \Omega^0_{X,Z} & \to & A^* & \to \Omega^0_{H,Z \cap H} \\
\Omega^0_{X,Z} & \xrightarrow{\times f} & \Omega^0_{X,Z} & \to & B^* & \to \Omega^0_H \\
\Omega^0_{X,Z} & \xrightarrow{\times f} & \Omega^0_{X,Z} & \to & C^* & \to \Omega^0_{Z \cap H} \\
\end{array} \]

and the existence of $\tau$ follows immediately from the existence of $\kappa$ and $\mu$ whose existence follows from the proof of [KS11a, Theorem 4.1]. Note that the assumptions imply that $H|_Z = H \cap Z$ is a Cartier divisor on $Z$, so we may indeed use [KS11a, Theorem 4.1] for both $X$ and $Z$. Since $I_Z/(f \cdot I_Z) = I_Z/(f) \cap I_Z)$ by Lemma 4.1 and because $(H, Z \cap H)$ is a Du Bois pair, we see $\tau \circ \rho$ is an isomorphism as claimed.

Next we apply the Grothendieck duality functor $D(\_\_)$ $= \mathcal{R}\text{Hom}^*_R(\_\_, \omega^*_R)$ to (4.4.1) and take cohomology:

\[ \begin{array}{cccccc}
\cdots & \xrightarrow{h^i(D(I_Z)) \times f} & h^i(D(I_Z)) & \xrightarrow{\delta_i} & h^i(D(I_Z)/(f \cdot I_Z)) & \xrightarrow{\alpha_i} & h^{i-1}(D(I_Z)) & \xrightarrow{\times f} & h^{i-1}(D(I_Z)) & \cdots \\
\cdots & \xrightarrow{h^i(\omega^*_{X,Z}) \times f} & h^i(\omega^*_{X,Z}) & \xrightarrow{\gamma_i} & h^i(D(A^*)) & \xrightarrow{\beta_i} & h^{i-1}(\omega^*_{X,Z}) & \xrightarrow{\times f} & h^{i-1}(\omega^*_{X,Z}) \\
\end{array} \]

where the $\Phi^*$ are injective by Theorem 3.2 and $\gamma_i$, which was obtained from $\rho$, is surjective since $\tau \circ \rho$ is an isomorphism.

The proof now follows exactly as the main theorem of [KS11a], or dually of [Kov00, Theorem 3.2]. Fix $z \in h^{i-1}(D(I_Z))$. Pick $w \in h^i(D(A^*))$ such that $\alpha_i(z) = \gamma_i(w)$. Since $\delta_i(\alpha_i(z)) = 0$ and $\Phi^i$ is injective, it follows that there exists a $u \in h^{i-1}(\omega^*_{X,Z})$ such that $\beta_i(u) = w$. Therefore, $\alpha_i(\Phi^{i-1}(u)) = \alpha_i(z)$ and so

\[ z - \Phi^{i-1}(u) \in f \cdot h^{i-1}(D(I_Z)). \]
Now, fix $E_{i-1}$ to be the cokernel of $\Phi^{i-1}$ and set $z \in E_{i-1}$ to be the image of $z$. Equation (4.42) then guarantees that $z \in f \cdot E_{i-1}$. The multiplication map $E_{i-1} \xrightarrow{\times f} E_{i-1}$ is then surjective and so Nakayama’s lemma guarantees that $\Phi^{i-1}$ is also surjective. Therefore $\omega^*_{X,Z} \to D(I_Z)$ is a quasi-isomorphism which implies that $(X, Z)$ is a Du Bois pair. \hfill \square

We immediately obtain:

**Corollary 4.5.** Let $f : X \to B$ be a flat proper family of varieties over a smooth one-dimensional scheme $B$ essentially of finite type over $\mathbb{C}$ (for instance, a smooth curve). Further let $Z \subseteq X$ be a subscheme such that no component of $Z$ is contained in any component of any fiber of $f$ and $b \in B$ a closed point such that $(X_b, Z_b)$ is a Du Bois pair. Then there exists a neighborhood $b \in U \subseteq B$ such that

(a) $(X, Z)$ is Du Bois over $U$, and

(b) the fibers $(X_u, Z_u)$ are Du Bois for all $u \in U$.

**Proof.** The non-Du Bois locus $T$ of $(X, Z)$ is closed, and since $f$ is proper, $f(T)$ is also closed. Hence (a) follows from Corollary 4.4 and by replacing $B$ with an open set, we may assume that $(X, Z)$ is Du Bois. Then the Bertini type theorem (Lemma 2.17) implies that (b) follows after possibly shrinking $U$. \hfill \square

**Corollary 4.6.** Let $f : X \to B$ be a flat proper family of varieties over a smooth scheme $B$ essentially of finite type over $\mathbb{C}$. Further let $Z \subseteq X$ be a subscheme which is also flat over $B$ and $b \in B$ a closed point such that $(X_b, Z_b)$ is a Du Bois pair. Then there exists a neighborhood $b \in U \subseteq B$ such that $(X, Z)$ is Du Bois over $U$.

**Proof.** We may assume that $B$ is affine and let $d = \dim B$. We first show that $(X, Z)$ itself is Du Bois in a neighborhood of $(X_b, Z_b)$. Let $H_1, \ldots, H_d$ be general smooth subschemes going through $b$ whose local defining equations generate the maximal ideal of $b$ (i.e., locally analytically they are coordinate hyperplanes). The pair $(X_b, Z_b) = (X_{H_1 \cap \cdots \cap H_d}, Z_{H_1 \cap \cdots \cap H_d})$ is Du Bois by assumption, hence since $X_b = X_{H_1 \cap \cdots \cap H_d}$ is a hypersurface in $X_{H_2 \cap \cdots \cap H_d}$ it follows that the pair $(X_{H_2 \cap \cdots \cap H_d}, Z_{H_2 \cap \cdots \cap H_d})$ is Du Bois in a neighborhood of $X_b$ by Corollary 4.5. Let $W_1$ denote the non-Du Bois locus of $(X_{H_2 \cap \cdots \cap H_d}, Z_{H_2 \cap \cdots \cap H_d})$. Since $W_1$ is closed and $f$ is proper, we see that $f(W_1)$ is closed in $H_2 \cap \cdots \cap H_d$ and doesn’t contain $b$. Shrinking $B$ if necessary, we may assume that $W_1$ is empty. Next observe that $H_2 \cap \cdots \cap H_d$ is a hypersurface in $H_3 \cap \cdots \cap H_d$ and again we see that $(X_{H_3 \cap \cdots \cap H_d}, Z_{H_3 \cap \cdots \cap H_d})$ is Du Bois in a neighborhood of $X_{H_3 \cap \cdots \cap H_d}$ by Corollary 4.5. Set $W_2 = B$ to be the non-Du Bois locus of $(X_{H_3 \cap \cdots \cap H_d}, Z_{H_3 \cap \cdots \cap H_d})$ and note that $f(W_2)$ does not intersect $H_3 \cap \cdots \cap H_d$. We shrink $B$ again if necessary so that $W_2 = \emptyset$. Iterating this procedure proves the statement. \hfill \square

In order to extend Corollary 4.5(b) to families over arbitrary dimensional bases we need the following lemma.

**Lemma 4.7.** Let $f : X \to B$ be a flat proper family of varieties over a scheme $B$ essentially of finite type over $\mathbb{C}$. Further let $Z \subseteq X$ be a subscheme which is also flat over $B$ and assume that $(X, Z)$ is a Du Bois pair. Then

$$V = \{ b \in B \mid (X_b, Z_b) \text{ is a Du Bois pair} \}$$

is a constructible set in $B$. Furthermore, if $B$ is smooth, then $V$ is open.

**Proof.** We use induction on the dimension of $B$.

Let $\pi : B' \to B$ be a resolution of singularities and consider the base change $f' : X' = X_{B'} \to B'$, a flat proper family over $B'$ and with $Z' = Z_{B'} \subseteq X'$ subscheme flat over $B'$. The non-Du Bois locus

$$T' = \{ b' \in B' \mid (X_{b'}, Z_{b'}) \text{ is not Du Bois} \}$$

is constructible in $B'$. Therefore, taking the fiber product $X' \times_B X = X_{B'}$ over $B'$ we get:

$$\omega^*_{X', Z'} \to D(I_{Z'})$$

is a quasi-isomorphism which implies that $(X': Z')$ is a Du Bois pair. \hfill \square
Notice that all the fibers of $f' : X' \to B'$ appear as fibers of $f : X \to B$ (up to harmless field extension), so $b' \in V' = \pi^{-1}(V) \subseteq B'$ if and only if the fiber $(X'_{b'}, Z'_{b'})$ is a Du Bois pair. It follows from Corollary 4.6 that by replacing $B'$ with an open subset we may assume that $(X', Z')$ is a Du Bois pair. It also follows that it is enough to prove the statement over a smooth base. Indeed, that implies that $V'$ is open in $B'$ and hence $V = f(V')$ is constructible.

To simplify notation we will replace $B$ with $B'$ and assume that $B$ is smooth and irreducible, but use the inductive hypothesis without these additional assumptions.

The Bertini type statement [Lemma 2.17(b)] implies that there is a dense open subset $U \subseteq B$ contained in $V$. The case $\dim B = 1$ follows immediately via the fact that in a curve any set containing a dense open set is itself open.

In general, it follows that $\dim(B \setminus U) < \dim B$ so by induction $V \setminus U$ is a constructible set in $B \setminus U$ and hence $V$ is constructible in $B$. In the case of a smooth base [Corollary 4.6] implies that $V$ is stable under generalization and since we have just proved that it is constructible it follows that it is open. 

**Corollary 4.8.** Let $f : X \to B$ be a flat proper family of varieties over a smooth scheme $B$ essentially of finite type over $\mathbb{C}$. Further let $Z \subseteq X$ be a subscheme which is also flat over $B$ and $b \in B$ a closed point such that $(X_b, Z_b)$ is a Du Bois pair. Then there exists a neighborhood $b \in U \subseteq B$ such that $(X_u, Z_u)$ is a Du Bois pair for all $u \in U$.

**Proof.** Observe that the non-Du Bois locus $W$ of $(X, Z)$ is closed in $X$ and since $f$ is proper, $f(W)$ is also closed in $B$. Note that $f(W)$ does not contain $b$ so it also does not contain the generic point of $B$. Hence by replacing $B$ by a neighborhood $U \subseteq B$ of $b \in B$, we may assume that $(X, Z)$ is Du Bois. Then the statement follows from [Lemma 4.7].

**Remark 4.9.** One can recover special cases of inversion of adjunction for log canonicity [Kaw07] easily from [Theorem 4.4]. For instance, let $(X, D + H)$ be a pair with $K_X, D$ and $H$ Cartier and assume that $(H, D|_H)$ is slc or equivalently Du Bois [Kol13]. Then $(X, D + H)$ is Du Bois or equivalently lc by [Theorem 4.4].

## 5. A result of Kollár-Kovács for pairs

The following was shown in [KK10, Theorem 7.12]: Let $f : X \to B$ be a flat projective family of varieties with Du Bois singularities. Then if $B$ is connected and the general fiber is Cohen-Macaulay, then all the fibers are Cohen-Macaulay.

We would like to generalize this to the context of Du Bois pairs, at least in the case when $Z$ is a divisor. We recommend for the reader to have a copy of [KK10] available when reading this section as we refer to a number of lemmas therein. We begin by generalizing a result of Du Bois and Jarraud to pairs cf. [DJ74], [DB81].

**Theorem 5.1.** Let $f : X \to B$ be a flat proper morphism between schemes of finite type over $\mathbb{C}$. Assume that $B$ is smooth and let $Z \subseteq X$ be a subscheme that is flat over $B$. Further assume that the geometric fibers $(X_b, Z_b) \to b$ are Du Bois. Then $\mathcal{R}^i f_* \mathcal{I}_Z$ is locally of finite rank and furthermore compatible with base change for all $i$, in other words $(\mathcal{R}^i f_* \mathcal{I}_Z)_T \simeq \mathcal{R}^i f_* \mathcal{I}_{Z_T}$, for any morphism $T \to B$.

**Proof.** For some $b \in B$, let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{B,b}$, and $S = S_n = \text{Spec} \mathcal{O}_{B,b}/\mathfrak{m}^{n+1}$ for $n \in \mathbb{N}$. Further let $\mathcal{I}_{Z_b}$ resp. $\mathcal{I}_{Z_S}$ denote the ideal sheaf of $Z_b$ in $X_b$ resp. $Z_S$ in $X_S$. 


Consider the following commutative diagram:

$$
\begin{array}{ccc}
H^i_c((X \setminus Z)^{\text{an}}, \mathbb{C}) & \xrightarrow{\lambda} & H^i_c((X \setminus Z)^{\text{an}}, \mathbb{C}) \\
\downarrow & & \downarrow \\
H^i_c((X \setminus Z)^{\text{an}}, \mathbb{C}) & \xrightarrow{\gamma} & H^i_c((X \setminus Z)^{\text{an}}, \mathbb{C})
\end{array}
\quad
\begin{array}{ccc}
H^i(X, \mathcal{I}_Z) & \xrightarrow{\beta} & H^i(X, \mathcal{I}_Z) \\
\downarrow & & \downarrow \\
H^i(X_b, \mathcal{I}_{Z_b}) & \xrightarrow{\delta} & H^i(X_b, \mathcal{I}_{Z_b})
\end{array}
\quad
\begin{array}{cc}
\alpha & \beta & \gamma & \delta
\end{array}
$$

Observe that $\lambda$ is an isomorphism since $X_S$ and $X_b$ have the same support. By [Kov11 4.1] cf. [Kol13, Theorem 6.8] $\gamma$ is surjective, so $\gamma \circ \lambda = \mu \circ \alpha$ is surjective and hence $\mu$ is surjective. By Serre’s GAGA principle [Ser56] $\beta$ and $\delta$ are isomorphisms and hence $\nu$ is surjective. Finally, the statement follows by Cohomology and Base Change [Gro63 7.7].

Next we prove the analogue of the main flatness and base change result of Kollár-Kovács for Du Bois pairs [KK10 Theorem 7.9].

**Theorem 5.2.** Let $f : X \to B$ be a flat projective morphism between schemes of finite type over $\mathbb{C}$ and assume that $B$ is smooth. Let $Z \subseteq X$ be a closed subscheme that is flat over $B$ and $\mathcal{L}$ a relatively ample line bundle on $X$. Then

(a) the sheaves $h^{-i}(\mathcal{R} \text{Hom}_{\mathcal{O}_X}^*(\mathcal{I}_Z, \omega^*_f))$ are flat over $B$ for all $i$.

(b) the sheaves $f_* (h^{-i}(\mathcal{R} \text{Hom}_{\mathcal{O}_X}^*(\mathcal{I}_Z, \omega^*_f)) \otimes \mathcal{L}^q)$ are locally free and compatible with arbitrary base change for all $i > 0$ and $q \gg 0$.

(c) for any base change $\vartheta : T \to B$ and for all $i > 0$,

$$
(h^{-i}(\mathcal{R} \text{Hom}_{\mathcal{O}_X}^*(\mathcal{I}_Z, \omega^*_f)))|_T \cong h^{-i}(\mathcal{R} \text{Hom}_{\mathcal{O}_{T}}^*(\mathcal{I}_{Z_T}, \omega^*_{f_T})).
$$

**Proof.** We follow the proof of [KK10, Theorem 7.9]. We may assume that $B = \text{Spec } R$ is affine and hence that $\mathcal{L}^m$ is globally generated for $m \gg 0$. For such an $m \gg 0$, choose a general section $\sigma \in H^0(X, \mathcal{L}^m)$ and consider the cyclic cover induced by $\sigma$:

$$
\mathcal{A} = \bigoplus_{j=0}^{m-1} \mathcal{L}^{-j} \cong \bigoplus_{j=0}^{m-1} \mathcal{L}^{-j} \oplus (t^m - \sigma).
$$

Set $h : Y = \text{Spec}_X \mathcal{A} \to X$, and $Z_Y = h^{-1}Z$ with the induced reduced scheme structure. Then the geometric fibers of the composition $(Y, Z_Y) \to B$ are also Du Bois by [Kol13 Corollary 6.21]. Note that by construction $\mathcal{I}_{Z_Y} = \bigoplus_{j=0}^{m-1} \mathcal{I}_Z \otimes \mathcal{L}^{-j}$. Hence $\mathcal{R}^j h_* \mathcal{I}_{Z_Y}$ is locally free of finite rank and compatible with arbitrary base change by [Theorem 5.1]. It follows that the summands of these modules, the $\mathcal{R}^j f_* (\mathcal{I}_Z \otimes \mathcal{L}^{-j})$, are also locally free and compatible with base change. Since we may choose $m$ arbitrarily large, this holds for all $j \in \mathbb{N}$. It follows immediately that $\text{Hom}_{\mathcal{O}_B}(\mathcal{R}^j f_*(\mathcal{I}_Z \otimes \mathcal{L}^{-j}), \mathcal{O}_B)$ is also locally free and compatible with base change.

By Grothendieck duality and [KK10 Lemma 7.3] (cf. proof of [KK10 Lemma 7.2]) it follows that

$$
\text{Hom}_{\mathcal{O}_B}(\mathcal{R}^j f_*(\mathcal{I}_Z \otimes \mathcal{L}^{-q}), \mathcal{O}_B) \cong f_* h^{-i} \mathcal{R} \text{Hom}_{\mathcal{O}_X}^*(\mathcal{I}_Z, \omega^*_f \otimes \mathcal{L}^{-q}) \cong f_* (h^{-i} \mathcal{R} \text{Hom}_{\mathcal{O}_X}^*(\mathcal{I}_Z, \omega^*_f) \otimes \mathcal{L}^q)
$$
and hence (b) is proven. Just as in [KK10] Theorem 7.9, (a) follows from (b) by an argument similar to [Har67] Chapter III, Theorem 9.9.

Finally we prove (c). Since $f: X \to B$ is projective and $B$ is affine, we may factor $f$ as $X \xrightarrow{\pi} \mathbb{P}^n_B \xrightarrow{\pi} B$. It then suffices to show that

$$
\varphi^{-i} : h^{-i}(R\mathcal{H}om_{\mathcal{O}_B}(\mathcal{I}_Z, \omega_{\pi}[n])) \to \varphi^{-i}(R\mathcal{H}om_{\mathcal{O}_B}(\mathcal{I}_T, \omega_{\pi}[n]))
$$

is an isomorphism. As in [KK10] Theorem 7.9, we proceed by descending induction on $i$ (the base case where $i \geq 0$ is obvious). We observe that $\mathcal{I}_Z$ is flat since so are $\mathcal{O}_X$ and $\mathcal{O}_Z$ and assume that $\varphi^{-(i+1)}$ is an isomorphism by induction. Since $h^{-i}(R\mathcal{H}om_{\mathcal{O}_B}(\mathcal{I}_Z, \omega_{\pi}[n]))$ is flat by (a), we may apply [AK80] Theorem 1.9 which completes the proof. □

The following is the analog of [KK10] Theorem 7.11 for pairs.

**Theorem 5.3.** Let $f: X \to B$ be a flat projective morphism between schemes of finite type over $\mathbb{C}$. Assume that $B$ is smooth and let $Z \subseteq X$ be a subscheme that is flat over $B$. Let $x \in X$ be a closed point and let $b = f(x)$. Then $\mathcal{I}_{Z_b} \subseteq \mathcal{O}_{X_b}$ is $S_k$ at $x$ if and only if

$$(5.3.1) \quad \left( h^{-i}R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega^*_Z) \right)_y = 0$$

for $i < \min(k, \dim \{y\}, \dim_x X)$ for all $y \in X_b$ such that $x \in \{y\}$. In particular, $\mathcal{I}_{Z_b}$ is $S_k$ if and only if $(5.3.1)$ holds for $i < \min(k, \dim \{y\}, \dim_x X)$ for all $y \in X_b$ (not restricted to closed points).

First we prove a lemma.

**Lemma 5.4.** Let $X$ be a scheme that admits a dualizing complex $\omega_X^*$. Let $x \in X$ and $\mathcal{F}$ a coherent sheaf on $X$. Then $\mathcal{F}$ is $S_k$ at $x \in X$ if and only if

$$
\left( h^{-i}R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega^*_X) \right)_y = 0
$$

for $i < \min(k, \dim \mathcal{F}_y) + \dim \{y\}$ for all $y \in X$ such that $x \in \{y\}$.

**Proof.** This is a direct consequence local duality [Har67] and the cohomological criterion for depth, see for instance [Kov11] Proposition 3.2. □

**Proof of Theorem 5.3.** By the lemma, $\mathcal{I}_{Z_b} \subseteq \mathcal{O}_{X_b}$ is $S_k$ at $x$ if and only if

$$
\left( h^{-i}R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_{Z_b}, \omega^*_{X_b}) \right)_y = 0
$$

for $i < \min(k, \dim(\mathcal{I}_{Z_b})_y + \dim \{y\}) = \min(k, \dim(\mathcal{I}_Z)_y, \dim_x X)$ for all $y \in X_b$ such that $x \in \{y\}$. By Theorem 5.2

$$
\left( h^{-i}R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_{Z_b}, \omega^*_{X_b}) \right)_y \cong \left( h^{-i}R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega^*_Z) \right)_y \cong \left( h^{-i}R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega^{*}_{Z}) \right)_y.
$$

But notice that the right side is zero if and only if $h^{-i}R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega^*_Z)_y$ is zero by Nakayama’s lemma. This implies the desired statement. □

Finally, we describe how the $S_k$ condition behaves for pairs made up of Du Bois families.

**Theorem 5.5.** Let $f: (X, Z) \to B$ be a flat projective Du Bois family with $\mathcal{O}_Z$ (and hence $\mathcal{I}_Z$) flat over $B$ as well. Assume that $B$ is connected and the generic fibers $(\mathcal{I}_Z)_{\text{gen}}$ are $S_k$, then all the fibers $(\mathcal{I}_Z)_b$ are $S_k$. 
Proof. By working with one component of $B$ at a time, we may assume that $B$ is irreducible and hence that $X$ is equidimensional. If $(\mathcal{I}_Z)_b \simeq \mathcal{I}_Z$ (by flatness of $\mathcal{O}_Z$) is not $S_k$ at some point $y \in X_b$, then by Theorem 5.3 $h^{-i}R\text{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{I}_Z, \omega^*_X) \neq 0$ near $y$ for some $i < \min(k + \dim \{y\}, \dim X)$. Fix an irreducible component $W \subseteq \text{supp}(h^{-i}R\text{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{I}_Z, \omega^*_X))$ and observe that $\dim W_b$ is constant for $b \in B$ since $h^{-i}R\text{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{I}_Z, \omega^*_X)$ is flat by Theorem 5.2(a). However, in that case it follows that $h^{-i}R\text{Hom}_{\mathcal{O}_X}^\bullet(\mathcal{I}_Z, \omega^*_X)$ is non-zero near some point $\eta \in X_{\text{gen}}$ such that $\dim \{\eta\} = \dim \{y\}$ which contradicts the assumption that the generic fiber is $S_k$. □

We immediately obtain the following.

Corollary 5.6. Let $f : (X, Z) \to B$ be a flat projective Du Bois family with $\mathcal{O}_Z$ (and hence $\mathcal{I}_Z$) flat over $B$ as well. Assume that $B$ is connected and the generic fibers $(\mathcal{I}_Z)_{X_{\text{gen}}}$ are Cohen-Macaulay, then all the fibers $(\mathcal{I}_Z)_b$ are Cohen-Macaulay.

At this point it is natural to ask the next question.

Question 5.7. Suppose that $(X, Z)$ is a pair and that $H \subseteq X$ is a Cartier divisor such that $(H, Z \cap H)$ is a Du Bois pair. If $\mathcal{I}_Z|_{X \setminus H}$ is Cohen-Macaulay then is it true that $\mathcal{I}_Z$ is Cohen-Macaulay?

In the case that $Z = \emptyset$, the analogous result holds in characteristic $p > 0$ for $F$-injective singularities by [HMS12, Appendix by K. Schwede and A. K. Singh].

6. A result of Kovács-Schwede-Smith for pairs

The goal of this section is to prove the analog of the main result of [KSS10] for pairs $(X, Z)$.

Lemma 6.1. Let $X$ be a normal $d$-dimensional variety, $Z \subseteq X$ a reduced closed subscheme and $\Sigma \subseteq X$ a codimension $\geq 2$ subset containing the singular locus of $X$. Let $\pi : \widetilde{X} \to X$ be a log resolution of $(X, \Sigma \cup Z)$ with $E = \pi^{-1}(\Sigma \cup Z)_{\text{red}}$. Then

(a) $\mathcal{O}^0_X, \Sigma \cup Z \simeq \mathcal{R}_{\pi_*} \mathcal{O}_{\widetilde{X}}(-E)$, and

(b) $h^{-d}(-\omega^*_X, Z) \simeq \pi_* \omega^*_X(E)$.

Proof. First we claim that both $\mathcal{R}_{\pi_*} \mathcal{O}_{\widetilde{X}}(-E)$ and $\pi_* \omega^*_X(E)$ are independent of the choice of $\pi$. This was proved for $\mathcal{R}_{\pi_*} \mathcal{O}_{\widetilde{X}}(-E)$ on pages 67-68 in the proof of [KS11b, Theorem 6.4] and for $\pi_* \omega^*_X(E)$ in [KSS10, Lemma 3.12]. Therefore we are free to choose $\pi$ and hence we may assume that it is an isomorphism outside of $\Sigma \cup Z$. We have the distinguished triangle

$$
\begin{array}{ccc}
\mathcal{O}^0_X & \to & \mathcal{R}_{\pi_*} \mathcal{O}^0_{\widetilde{X}} \oplus \mathcal{O}^0_{\Sigma \cup Z} \\
\approx & & \approx \\
\mathcal{O}^0_{\widetilde{X}} & \to & \mathcal{R}_{\pi_*} \mathcal{O}^0_{\widetilde{X}} \oplus \mathcal{O}^0_{\Sigma \cup Z}
\end{array}
$$

$+1$ for $X$ and $E$ are Du Bois. In the next diagram the first two rows are distinguished triangles by definition cf. [2.7.1]. The third row is simply the pushforward of a natural short exact sequence from $X$. The previous diagram and [KK10, 2.1] implies that $\alpha$ is an isomorphism and . The other two isomorphisms again follow since $\widetilde{X}$ and $E$
are Du Bois. Note that the columns are not exact.

\[ \begin{array}{c}
\mathcal{O}_X^{0, \Sigma \cup Z} \\
\simeq \alpha
\end{array} \xrightarrow{\pi_*} \begin{array}{c}
\mathcal{O}_X^{0} \\
\simeq \mathcal{O}_{\Sigma \cup Z}^{0} \\
\mathcal{O}_X^{0} \\
\simeq \mathcal{O}_{\Sigma \cup Z}^{0}
\end{array} \xrightarrow{+1} \begin{array}{c}
\mathcal{O}_X^{0} \\
\mathcal{O}_{\Sigma \cup Z}^{0} \\
\mathcal{O}_X^{0} \\
\mathcal{O}_{\Sigma \cup Z}^{0}
\end{array} \]

It follows that the dotted arrow and hence its composition with \(\alpha\) are also isomorphisms. This proves (a).

In order to prove (b), consider the map \(\mathcal{O}_X^{0, \Sigma \cup Z} \to \mathcal{O}_{\Sigma \cup Z}^{0}\) obtained in

\[ \begin{array}{c}
\mathcal{O}_X^{0, \Sigma \cup Z} \\
\simeq \mathcal{O}_X^{0} \\
\mathcal{O}_X^{0} \\
\mathcal{O}_X^{0}
\end{array} \xrightarrow{\pi_*} \begin{array}{c}
\mathcal{O}_{\Sigma \cup Z}^{0} \\
\mathcal{O}_{\Sigma \cup Z}^{0} \\
\mathcal{O}_X^{0} \\
\mathcal{O}_X^{0}
\end{array} \xrightarrow{+1} \begin{array}{c}
\mathcal{O}_X^{0} \\
\mathcal{O}_{\Sigma \cup Z}^{0} \\
\mathcal{O}_X^{0} \\
\mathcal{O}_X^{0}
\end{array} \]

Now we have a distinguished triangle

\[ (6.1.1) \quad \mathcal{O}_X^{0, \Sigma \cup Z} \to \mathcal{O}_{\Sigma \cup Z}^{0} \to C^* \to +1. \]

**Claim 6.2.** With notation above, \(0 = h^{-d}(D(C^*)) = h^{-d+1}(D(C^*))\).

**Proof of claim.** Consider the following diagram with distinguished triangles as rows and columns:

\[ \begin{array}{c}
\mathcal{O}^{0}_{\Sigma \cup Z}[-1] \\
\simeq \mathcal{O}^{0}_{Z}[-1] \\
\simeq \mathcal{O}^{0}_{\Sigma \cup Z, Z} \\
\mathcal{O}^{0}_{X, \Sigma \cup Z} \\
\simeq \mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X}
\end{array} \xrightarrow{\pi_*} \begin{array}{c}
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X}
\end{array} \xrightarrow{+1} \begin{array}{c}
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X} \\
\mathcal{O}^{0}_{X}
\end{array} \xrightarrow{+1} \]

It follows from [Kov13, B.1] that \(C^* \simeq \mathcal{O}^{0}_{\Sigma \cup Z, Z}\). On the other hand, by Lemma 2.17(d) \(\mathcal{O}^{0}_{\Sigma \cup Z, Z} \simeq \mathcal{O}^{0}_{\Sigma \cup Z, \Sigma \cap Z}\) and hence \(C^* \simeq \mathcal{O}^{0}_{\Sigma \cup Z, \Sigma \cap Z}\).

Next recall that by Theorem 3.2 there exists a natural injective map

\[ (6.2.1) \quad h^{-j}D(\mathcal{O}^{0}_{\Sigma \cup Z}) \hookrightarrow h^{-j}RHom_{\mathcal{O}_X}^{\cdot} (\mathcal{O}_{\Sigma \cup Z, Z}, \mathcal{O}^{*}_{\Sigma}). \]

Since \(\dim \Sigma \leq d - 2\), the right hand side of (6.2.1) is zero for \(j \geq d - 1\). This completes the proof of the claim. \(\square\)

Grothendieck duality and part (a) implies that \(h^{-d}(\omega^{\cdot}_{X, \Sigma \cup Z}) \simeq \pi_\ast \omega_X(E)\) and it follows from Claim 6.2 that \(h^{-d}(\omega^{\cdot}_{X, \Sigma \cup Z}) \simeq h^{-d}(\omega^{\cdot}_{X, Z})\), which in turn implies part (b). \(\square\)
Theorem 6.3. Let $X$ be a normal variety and $Z \subseteq X$ a divisor. Let $\pi : \tilde{X} \to X$ be a log resolution of $(X, Z)$ with $E = \pi^{-1}(Z)_{\text{red}} \vee \text{exc}(\pi)$. If $\mathcal{I}_Z$ is Cohen-Macaulay then $(X, Z)$ is Du Bois if and only if $$\pi_*\omega_{\tilde{X}}(E) \simeq \omega_X(Z).$$

Proof. Since $\mathcal{I}_Z$ is Cohen-Macaulay, $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega_{\tilde{X}}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega_X)[\dim X]$ by the local dual of the local cohomology criterion for Cohen-Macaulayness. Because the map

$$\omega_{X,Z} \to R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega_X)$$

is injective on cohomology by [Theorem 3.2], it follows that $h^i(\omega_{X,Z}) = 0$ for $i \neq -\dim X$ and hence $(X, Z)$ is Du Bois if and only if $h^{-\dim X}(\omega_{X,Z}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Z, \omega_X) \simeq \omega_X(Z)$ is an isomorphism. But $h^{-\dim X}(\omega_{X,Z}) \simeq \pi_*\omega_{\tilde{X}}(E)$ by [Lemma 6.1], so the statement follows. \hfill \Box

7. An inversion of adjunction for rational and Du Bois pairs

In this final section of the paper, we will prove the following theorem.

Theorem 7.1. Let $f : X \to B$ be a flat projective family over a smooth connected base $B$, $A \subseteq B$ a smooth closed subscheme containing no component of $B$ and $H = f^{-1}(A)$. Assume that $X$ is geometrically integral and let $D$ be a reduced codimension 1 subscheme of $X$ which is flat over $B$. Assume that for every $s \in A$, $(X_s, D_s)$ is Du Bois and that $(X \setminus H, D \setminus H)$ is a rational pair. Then $(X, D)$ is a rational pair.

Remark 7.2. In the introduction, $A$ was assumed to be a closed point. This version is more general and more convenient for our proof.

Remark 7.3. The assumptions also imply the following auxiliary conditions:

(a) $\mathcal{I}_D$ is flat over $B$ and no component of $D$ contains a fiber of $f$. In particular $D$ and $H$ have no common components.

(b) As for any $s \in A$, $H_s = X_s$, it follows that $(H, D \cap H)$ is Du Bois by [Corollary 4.6].

(c) $X \setminus H$ is normal by the definition of a rational pair.

Before embarking on proving the theorem, we will first prove several lemmas that show that our situation is simpler than it might first appear.

First we show that we may assume that $A$ is a divisor in $B$.

Lemma 7.4. In order to prove [Theorem 7.1], it is sufficient to assume that $A$ is a smooth Cartier divisor in $B$.

Proof. The statement is local over the base so we may assume that $B$ is affine. Additionally, since we only need to work in a neighborhood of a point $a \in A$, we may assume that $(X, D)$ is Du Bois and all the fibers $(X_b, D_b)$ for all $b \in B$ are Du Bois by [Corollary 4.6] and [Corollary 4.8]. Choose a general hypersurface $G$ containing $A$ and note that since $A$ is smooth we may assume that $G$ is smooth. Then the hypotheses of the theorem are satisfied for $G$ replacing $A$ as well since $X \setminus f^{-1}(G) \subseteq X \setminus f^{-1}(A)$ and since we already assumed that all the fibers $(X_b, D_b)$ over all points $b \in B$ were Du Bois. \hfill \Box

From this point forward, we will assume that $B$ is a smooth affine scheme, $A$ is a smooth hypersurface in $B$ and $H = f^{-1}(A)$.

Next we show that under the assumptions of the theorem we have the following:

Lemma 7.5. $X$ is normal and thus $D$ is also a divisor.
Proof. Since $H$ is reduced, every point $\eta \in H$ have depth at least $\min(1, \dim \mathcal{O}_{H, \eta})$. Because $f : X \to B$ is flat, the local defining equation of $H$ is a regular element in $\mathcal{O}_X$, so any point $\eta \in X$ that lie in $H$ have depth at least $\min(2, \dim \mathcal{O}_{X, \eta})$, and thus $X$ is $S_2$ along $H$. Since $X \setminus H$ is normal it is $S_2$ and so $X$ is $S_2$ everywhere. Finally observe that $H$ is reduced, hence generically regular and $X \setminus H$ is $R_1$. As $H$ is Cartier, these imply that $X$ is also $R_1$ and therefore normal. \hfill $\square$

Now observe that the fact that $(H, D \cap H)$ is Du Bois (cf. Remark 7.3(b)) says something about the structure of $D$ on $X$.

**Lemma 7.6.** With notation as in Theorem 7.1, no stratum of the snc locus of $(X, D)$ can be contained inside $H$.

**Proof.** Assume to the contrary that there exists a stratum $Z$ of the snc locus of $(X, D)$ contained in $H$. Let $\eta$ be the generic point of $Z$. By assumption $\eta \in H$ and $(X, D)$ is snc at $\eta$, so $\mathcal{O}_{X, \eta}$ is a regular ring. Let $n = \dim \mathcal{O}_{X, \eta}$. Replace $X$ by Spec $\mathcal{O}_{X, \eta}$ and $H$ and $D$ by their pullbacks to this local scheme (in this step we lose projectivity, but we will not need that for now). Note that $D$ is now Cartier and in fact snc. Furthermore $D + H$ has $n + 1$ irreducible components containing $\eta$, so $(X, D + H)$ cannot be Du Bois (or equivalently log canonical since $X$ is Gorenstein). As we observed $(H, D \cap H)$ is a Du Bois pair and so since $D$ is Cartier, we see that $H$ is itself Du Bois by Lemma 2.16. Hence $D \cap H$ is also Du Bois and thus $(H, D \cap H)$ is Du Bois by Theorem 4.4. But then by Theorem 4.4 again we see that $(X, D + H)$ is Du Bois as well. This is a contradiction. \hfill $\square$

Next we setup the notation for the proof of Theorem 7.1. Let $\Sigma$ denote the non-snc locus of $(X, D)$. Observe that as $X$ is normal and $D$ is a reduced divisor by Lemma 7.5.

1. $\operatorname{codim}_X(\Sigma) \geq 2$

Additionally assume that $\pi : Y \to X$ is a log resolution of $(X, D \cup H \cup \Sigma)$ which simultaneously gives a thrifty resolution of $(X, D)$. To see such a $\pi$ exists, first take a thrifty resolution $(\bar{U}, D_U)$ of $(X, D)$ and then perform a log resolution of the scheme-theoretic preimages of $H$ and $\Sigma$ on $U$ (while keeping the strict transform $D_U$ snc). The result can be assumed to be a thrifty resolution of $(X, D)$ since the preimages of $\Sigma$ and $H$ do not contain any strata of $(U, D_U)$ by Lemma 7.6.

Set $\overline{H}$ to be the reduced total transform of $H$ and $\overline{D}$ the reduced total transform of $D$, set $D_Y$ to be the strict transform of $D$ and set $E$ to be $(\pi^{-1}(\Sigma))_{\text{red}}$.

**Proof of Theorem 7.1.** Clearly $(X, D)$ is a Du Bois pair and such that all the fibers $(X_b, D_b)$ are Du Bois by Corollary 4.6 and Corollary 4.8 (possibly after shrinking the base $B$ around $A$). By Corollary 5.6, we know that $\mathcal{O}_X(-D)$ is Cohen-Macaulay. Thus by the local dual version of the local-cohomological criterion for Cohen-Macaulayness, $R \mathcal{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{O}_X(-D), \omega_X^*)$ has cohomology only in one term. In particular,

$$(7.6.1) \quad R \mathcal{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{O}_X(-D), \omega_X^*) \cong \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-D), \omega_X)[\dim X] \cong \omega_X(D)[\dim X].$$

Therefore by Corollary 2.7, it suffices to show that $\omega_X(D) \cong \pi_\ast \omega_Y(D_Y)$.\hfill $\square$

Next observe that $(H, D|_H)$ is a Du Bois pair for the same reasoning and hence by Lemma 2.18 we see that $(X, D \cup H) = (X, D + H)$ is a Du Bois pair.

**Claim 7.7.** With notation as above, $\pi_\ast \omega_Y(D_Y \vee \overline{H} \vee E) \cong \pi_\ast \omega_Y(D_Y + \overline{H})$.

Note that $D_Y + \overline{H} = D_Y \vee \overline{H}$ since the divisors have no common components.

**Proof of Claim 7.7.** The containment $\supseteq$ is obvious since $D$ and $H$ do not share a component (cf. Remark 7.3(a)) so choose $f \in \pi_\ast \omega_Y(D_Y \vee \overline{H} \vee E)$. Thus we observe that $\operatorname{div}_Y(f) +$
$K_Y + D_Y \vee \overline{\mathcal{I}} \vee E = \text{div}_Y(f) + K_Y + D_Y + \overline{\mathcal{I}} \vee E \geq 0$. Working on $U = Y \setminus \overline{\mathcal{I}} = \pi^{-1}(X \setminus H)$ we see that $\text{div}_Y(f) + K_U + D_Y|_U + E|_U \geq 0$. But since $(X \setminus H, D \setminus H)$ is a rational pair,

$$\pi_* \omega_U(D_Y|_U) = \pi_* \omega_U(D_Y|_U + E) = \omega_{X \setminus H}(D|_{X \setminus H}),$$

so $\text{div}_U(f) + K_U + D_Y|_U + E|_U \geq 0$ is equivalent to $\text{div}_Y(f) + K_U + D_Y|_U \geq 0$. Because the components of $E$ that lie over $H$ are also components of $\overline{\mathcal{I}}$, it follows that $\text{div}_Y(f) + K_U + D_Y + \overline{\mathcal{I}} \geq 0$. This proves \textbf{Claim 7.7}. \hfill $\square$

By Lemma 6.1 we see that $h^{-\dim X} \omega_{X, D + H}|_X \cong \pi_* \omega_Y(D_Y \vee \overline{\mathcal{I}} \vee E)$ which agrees with $\pi_* \omega_Y(D_Y + \overline{\mathcal{I}})$ by the claim. Since $(X, D + H)$ is a Du Bois pair, $h^{-\dim X} \omega_{X, D + H}|_X \cong \omega_X(D + H)$ and so in conclusion we have that

$$\omega_X(D + H) \cong \pi_* \omega_Y(D_Y + \overline{\mathcal{I}}).$$

Twisting both sides by $H$ and using the projection formula we see that

$$\omega_X(D) \cong \pi_* \omega_Y(D_Y - (\pi^* H - \overline{\mathcal{I}})) \subseteq \pi_* \omega_Y(D_Y)$$

since $\pi^* H - \overline{\mathcal{I}}$ is effective. But $\pi_* \omega_Y(D_Y) \subseteq \omega_X(D)$ for any normal pair $(X, D)$ and so $\omega_X(D) \cong \pi_* \omega_Y(D_Y)$ as desired. \hfill $\square$

Setting $D = 0$ we obtain the following:

**Corollary 7.8.** Let $f : X \to B$ be a flat projective family over a smooth base $B$ and $H = f^{-1}(0)$ a special fiber. Assume that $H$ has Du Bois singularities and that $X \setminus H$ has rational singularities. Then $X$ has rational singularities.

There is a variant of our inversion of adjunction theorem that we would also like to prove (even in the $D = 0$ case).

**Conjecture 7.9.** Suppose $(X, D)$ is a pair with $D$ a reduced Weil divisor. Further suppose that $H$ is a Cartier divisor on $X$ not having any components in common with $D$ such that $(H, D \cap H)$ is Du Bois and such that $(X \setminus H, D \setminus H)$ is a rational pair. Then $(X, D)$ is a rational pair.

The only place where our proof above does not work in this situation is when we prove that $\mathcal{O}_X(-D)$ is Cohen-Macaulay. In particular, to accomplish this generalization, we would simply need a version of \textbf{Corollary 5.6} that is not tied to a projective or proper family.

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