Flavour-conserving oscillations of Dirac-Majorana neutrinos

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Abstract

We analyze both chirality-changing and chirality-preserving transitions of Dirac-Majorana neutrinos. In vacuum, the first ones are suppressed with respect to the others due to helicity conservation and the interactions with a (“normal”) medium practically does not affect the expressions of the probabilities for these transitions, even if the amplitudes of oscillations slightly change. For usual situations involving relativistic neutrinos we find no resonant enhancement for all flavour-conserving transitions. However, for very light neutrinos propagating in superdense media, the pattern of oscillations $\nu_L \rightarrow \nu^C_L$ is dramatically altered with respect to the vacuum case, the transition probability practically vanishing. An application of this result is envisaged.

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1 Introduction

In this paper we correct and generalize a previous one [1] in which we studied Pontecorvo neutrino-antineutrino oscillations. In ref. [1] we confined ourselves to the zeroth order in the ultrarelativistic limit, which is without doubt the most significative one; however, from a theoretical point of view, it is interesting to consider also the possibility of chirality-changing oscillations [2] (both in vacuum and in matter), and this will be done here going just a bit beyond the zeroth order. In particular in this paper we also correct some mistakes which brought in ref. [1] to a slightly incorrect formula for the survival probability in presence of matter oscillations (eq. (55) in [1]); however, the conclusions reached there remain unchanged.

The importance of considering Dirac-Majorana neutrinos lies in the fact that these are described by the most general mass term in the electroweak lagrangian:

\[ -\mathcal{L}_{DM}^{m} = \sum_{l,l'} \nu_{l'} R M_{P}^{l} \nu_{l} L + \frac{1}{2} \sum_{l,l'} \nu_{l'} R M_{P}^{l} \nu_{l} L + \frac{1}{2} \sum_{l,l'} \nu_{l'} L M_{P}^{l} \nu_{l} R + h.c. \]  

(1)

\((l, l' = e, \mu, \tau)\), \(M_{D}\), \(M_{1}\), \(M_{2}\) being the Dirac mass matrix and the two Majorana mass matrices, respectively. This scenario involves the existence of the known active states \(\nu_{e}, \nu_{\mu}, \nu_{\tau}\) (the superscript \(c\) denotes charge conjugation) as well as sterile states \(\nu_{\text{sterile}}\); the mass eigenstates are Majorana fields.

The mass term in (1) is predicted in many GUTs [3] and provides a simple framework for the so-called “see-saw” mechanism [4], which allows to give a very small mass to neutrinos in a very natural way.

The most impressive thing related to the mass term in (1) is, however, its rich phenomenology: neutrino flavour oscillations, the decays such as \(\mu \rightarrow e\gamma\), \(\mu \rightarrow 3e\), \(\tau \rightarrow e\pi^{0}\), the conversion \(\mu-e\) in presence of nuclei \(\mu^{-} + (Z, A) \rightarrow e^{-} + (Z, A)\), as well as neutrinoless double beta decay \((Z, A) \rightarrow (Z + 2, A) + 2e^{-}\) and neutrino-antineutrino oscillations can occur. All these phenomena are now studied experimentally [5].

In this paper we will concentrate our attention on the oscillation phenomena of Dirac-Majorana neutrinos, limiting ourselves to the case of flavour-conserving ones to put in evidence the salient features related to the non conservation of the total lepton number, which is a typical prediction for Majorana mass eigenstates. The most general case of flavour-changing oscillation phenomena will be considered elsewhere.

Since the mass term in (1) involves both \(\nu_{e}, \nu_{\mu}, \nu_{\tau}\), in general we can have transitions between all these states. So, active-active neutrino oscillations, such as \(\nu_{e} \rightarrow \nu_{\mu}\), can occur, as well as active-sterile ones, such as \(\nu_{e} \rightarrow \nu_{\text{sterile}}\). Note that even if helicity is always conserved in vacuum (this is related to Lorentz invariance), chirality is not in general conserved, and so we can have chirality-changing processes. However, this implies that for ultrarelativistic neutrinos \((m/k \ll 1)\), where \(m\) is a mass parameter and \(k\) the neutrino momentum) propagating in vacuum the chirality-changing transitions are suppressed with respect to the other ones [6], because in this limit chirality is almost coincident with helicity, and the “wrong” chirality state is only a small component of the physical neutrino field. This suppression remains also for propagation in a medium, unless
one considers helicity-flipping interactions, such as the case of neutrinos with magnetic moments interacting with a magnetic field [7]. In this work we don’t deal with this last possibility, since here we will not consider magnetic moment interactions.

Our main task is to study both vacuum and matter (flavour-conserving) oscillations of Dirac-Majorana neutrinos, given the relevant importance of neutrino oscillations in astrophysics [8] and cosmology [9]. In the following section we mainly review vacuum oscillations (see, for example, [10] and references therein) and give the expressions for the transition probabilities. In section 3 we develop matter oscillations in homogeneous media and particularly study two unusual limits, given the absence of resonances for flavour-conserving transitions. Finally, in the last section we discuss the obtained results and give our conclusions.

2 Vacuum Oscillations

Let us consider the propagation in vacuum of Dirac-Majorana neutrinos with 4-momentum $k^{\mu} = (\omega, \mathbf{k})$:

$$\mathcal{L} = \bar{\nu} \sigma \gamma^\mu \nu - \frac{1}{2} m_D (\bar{\nu} \nu + \bar{\nu}^c \nu^c) - \frac{1}{2} m_M (\bar{\nu} \nu^c + \bar{\nu}^c \nu)$$

(2)

where $m_D, m_M$ are the Dirac and Majorana mass, respectively. for the given flavour (for simplicity we assume that the mass matrices in (1) are diagonal and then consider one flavour at a time). We take equal Majorana masses for the left-handed and right-handed neutrino, assuming a minimal choice for the mass parameters. In the chiral Weyl basis for the Dirac gamma matrices, denoting

$$\nu = \begin{pmatrix} \nu_L \\ \nu_R \end{pmatrix} \quad \nu^c = \begin{pmatrix} \nu^c_L \\ \nu^c_R \end{pmatrix}$$

(3)

the equations of motion are given by

$$\begin{pmatrix} \omega + \lambda k & -m_+ \\ -m_+ & \omega - \lambda k \end{pmatrix} \begin{pmatrix} n_{1\pm} \\ n_{2\pm} \end{pmatrix} = 0$$

(4)

where $\lambda = \pm 1$ is the helicity eigenvalue and

$$m_{\pm} = \frac{1}{2} (m_D \pm m_M)$$

(5)

$$n_{1\pm} = \pm \nu_L + \nu^c_L$$

(6)

$$n_{2\pm} = \pm \nu_R + \nu^c_R$$

(7)

(for further reference, see [1]; note, however, that in the present notation $\nu^c_L = (\nu_R)^c$, $\nu^c_R = (\nu_L)^c$). For definiteness, let us consider the “+” states (the same will be valid for the “-” states). Eq. (4) can be written in the useful hamiltonian form

$$H_+ \begin{pmatrix} n_{1+} \\ n_{2+} \end{pmatrix} = \omega \begin{pmatrix} n_{1+} \\ n_{2+} \end{pmatrix}$$

(8)
where
\[
H_+ = \begin{pmatrix}
-\lambda k & m_+ \\
m_+ & \lambda k
\end{pmatrix}
\] (9)

This hamiltonian is diagonalized by the matrix
\[
U = \begin{pmatrix}
\cos \theta_+ & \sin \theta_+ \\
-\sin \theta_+ & \cos \theta_+
\end{pmatrix}
\] (10)

with
\[
\tan 2\theta_+ = -\frac{m_+}{\lambda k}
\] (11)

and the eigenvalues are given by
\[
E_{1+} = \sqrt{k^2 + m_+^2} \equiv E_+
\] (12)
\[
E_{2+} = -E_+
\] (13)

while the eigenvectors by
\[
\begin{pmatrix}
  n_1^+ \\
  n_2^+
\end{pmatrix} = U \begin{pmatrix}
  n_1^+ \\
  n_2^+
\end{pmatrix}
\] (14)

(the negative energy states are interpreted as usual in relativistic quantum theory). Similarly for the “-” states. So, for \(\lambda = -1\) the time evolution of the left-handed neutrino state is given by
\[
|\nu_L(t)\rangle = \frac{1}{2} \left( \begin{array}{c}
\cos \theta_+ e^{-iE_+t} |n_1^+(0)\rangle - \sin \theta_+ e^{iE_+t} |n_2^+(0)\rangle \\
-\cos \theta_- e^{-iE_-t} |n_1^-(0)\rangle + \sin \theta_- e^{iE_-t} |n_2^-(0)\rangle
\end{array} \right)
\] (15)

and substituting in this expression the relation (14) and (6), (7), we find that even if at \(t = 0\) we have created a \(\nu_L\), at further time \(t\) we can detect a \(\nu_L\), as well as a \(\nu_R\) or \(\nu^c_R\) or \(\nu^c_L\). The transition probabilities are given by
\[
P(\nu_L \rightarrow \nu_R) = |\langle \nu_R | \nu_L(t) \rangle|^2 = \frac{1}{4} \left( \frac{m_+}{E_+} \sin E_+ t + \frac{m_-}{E_-} \sin E_- t \right)^2
\] (16)
\[
P(\nu_L \rightarrow \nu^c_R) = |\langle \nu^c_R | \nu_L(t) \rangle|^2 = \frac{1}{4} \left( \frac{m_+}{E_+} \sin E_+ t - \frac{m_-}{E_-} \sin E_- t \right)^2
\] (17)
\[
P(\nu_L \rightarrow \nu^c_L) = |\langle \nu^c_L | \nu_L(t) \rangle|^2 = \sin^2 \frac{E_+ + E_-}{2} t \sin^2 \frac{E_+ - E_-}{2} t \\
+ \frac{1}{4} \left( \frac{k}{E_+} \sin E_+ t - \frac{k}{E_-} \sin E_- t \right)^2
\] (18)

while the survival probability is
\[
P(\nu_L \rightarrow \nu_L) = \cos^2 \frac{E_+ + E_-}{2} t \cos^2 \frac{E_+ - E_-}{2} t + \frac{1}{4} \left( \frac{k}{E_+} \sin E_+ t + \frac{k}{E_-} \sin E_- t \right)^2
\] (19)
In the limit of Dirac neutrinos \((m_M = 0)\), only total lepton number conserving processes take place \[10\]

\[
P(\nu_L \rightarrow \nu_R) = \left(\frac{m_D}{2 E_+}\right)^2 \sin^2 E_+ t \tag{20}
\]

\[
P(\nu_L \rightarrow \nu^c_R) = P(\nu_L \rightarrow \nu^c_L) = 0 \tag{21}
\]

while for pure Majorana neutrinos \((m_D = 0)\) we recover \[10\]

\[
P(\nu_L \rightarrow \nu^c_R) = \left(\frac{m_M}{2 E_+}\right)^2 \sin^2 E_+ t \tag{22}
\]

\[
P(\nu_L \rightarrow \nu_R) = P(\nu_L \rightarrow \nu^c_L) = 0 \tag{23}
\]

Note that for the chirality-changing processes, the transition probabilities \([10], [17]\) contain suppression factors, as stated in the previous section. In the ultrarelativistic limit the expressions for the transition probabilities simplify, and we obtain (for the leading terms)

\[
P(\nu_L \rightarrow \nu_R) = \frac{m_D^2}{4 k^2} \sin^2 k t \tag{24}
\]

\[
P(\nu_L \rightarrow \nu^c_R) = \frac{m_M^2}{4 k^2} \sin^2 k t \tag{25}
\]

\[
P(\nu_L \rightarrow \nu^c_L) = \sin^2 \frac{m_D m_M}{4 k} t \tag{26}
\]

At the leading order we thus recover the Pontecorvo formula for the survival probability \[11], [4]:

\[
P(\nu_L \rightarrow \nu_L) = 1 - \sin^2 \frac{m_D m_M}{4 k} t \tag{27}
\]

Note that, apart from the suppression factors, the chirality-changing and the chirality-preserving transitions have very different periods, the oscillation length for the former being very much shorter than the one for the latter.

We also observe that for ultrarelativistic propagation the expression for the probability of \(\nu_L \rightarrow \nu_R\) Dirac-Majorana neutrino oscillations coincides with that for Dirac neutrinos, while the probability for \(\nu_L \rightarrow \nu^c_R\) transitions for Dirac-Majorana neutrinos and pure Majorana neutrinos are also the same (cfr eqs. \(20\),\(24\) and eqs. \(22\),\(25\)). This is not a real surprising feature, given that in the ultrarelativistic limit the differences between Dirac, Majorana and Dirac-Majorana neutrinos tend to disappear. Nevertheless, we stress that both processes can take place for Dirac-Majorana neutrinos and the fact that \(\nu_L \rightarrow \nu_R\) transition is quite exclusively governed by the Dirac mass term, while \(\nu_L \rightarrow \nu^c_R\) by the Majorana mass term. Instead, both mass term must be non vanishing for \(\nu_L \rightarrow \nu^c_L\) oscillation to occur.

From the experimental study on disappearance experiments for neutrino oscillations \[4\], one can obtain the following limits for \(\nu_e\) and \(\nu_\mu\) Dirac and Majorana masses using \(27\) \[1\]:

\[
m_D^{\nu_e} m_M^{\nu_e} \leq 7.5 \times 10^{-3} \text{ eV}^2 \tag{28}
\]
and

\[ m_{D}^{\nu_e} m_{M}^{\nu_{\mu}} \leq 0.23 \text{ eV}^2 \]  \hspace{1cm} (29)

or

\[ m_{D}^{\nu_e} m_{M}^{\nu_{\mu}} \geq 1500 \text{ eV}^2 \]  \hspace{1cm} (30)

3 Matter Oscillations

As recognized many years ago [12], when neutrinos propagate in a medium the influence of matter on neutrino oscillations can be very important. For example, for \( \nu_e \rightarrow \nu_\mu \) oscillations (for Dirac as well as for Majorana neutrinos) this is due to the fact that mass eigenstates (which propagate as free plane waves in vacuum) do not coincide with flavour eigenstates (which weak-interact with the matter), and this difference leads to a “resonance” in neutrino oscillations, that can be reached for definite values of the squared masses difference between the mass eigenstates, the mixing angle and the matter density of the medium (for given energy of the neutrino beam). The presence of a resonance, obviously, completely alters the features of neutrino oscillations with respect to the vacuum case.

This is the reason of why we now study matter oscillations of Dirac-Majorana neutrinos; however, as we will show, for flavour-conserving oscillations (considered in this paper) no resonance occurs for ultrarelativistic neutrinos in usual conditions.

Let us start by noting that the effective potential experienced by neutrinos in a medium with \( N_e \) electrons per unit volume and \( N_n \) neutrons per unit volume is given by (see for example [12], [13], [14])

\[ V_{\nu_L} = -b_L \]  \hspace{1cm} (31)
\[ V_{\nu_R} = +b_L \]  \hspace{1cm} (32)
\[ V_{\nu_R} = V_{\nu_L} = 0 \]  \hspace{1cm} (33)

where

\[ -b_L = \sqrt{2} G_F \left( N_e - \frac{1}{2} N_n \right) \]  \hspace{1cm} (34)

for the electron flavour, while

\[ -b_L = -\frac{G_F}{\sqrt{2}} N_n \]  \hspace{1cm} (35)

for the \( \mu \) and \( \tau \) flavours (obviously, sterile neutrinos experience no effective potential). Here \( G_F \) is the Fermi coupling constant, and we consider non magnetized matter (the same is valid for strongly magnetized matter using for \( V \) the expressions given in [13]).

Since the neutrino states interacting with the matter are the flavour eigenstates, which are linear combinations of the Majorana states in (6), (7), now the complete hamiltonian describing neutrino propagation in the medium cannot be block-diagonalized as in the
vacuum case. The eigenvalue equation is then

\[
H \begin{pmatrix}
  n_1^+ \\
  n_2^+ \\
  n_1^- \\
  n_2^-
\end{pmatrix} = \omega \begin{pmatrix}
  n_1^+ \\
  n_2^+ \\
  n_1^- \\
  n_2^-
\end{pmatrix}
\]

(36)

with

\[
H = H_{\text{cin}} + H_b
\]

(37)

where

\[
H_{\text{cin}} = \begin{pmatrix}
  k & m_+ & 0 & 0 \\
  m_+ & -k & 0 & 0 \\
  0 & 0 & k & m_- \\
  0 & 0 & m_- & -k
\end{pmatrix}
\]

(38)

\[
H_b = \frac{1}{2} \begin{pmatrix}
  -b_L & 0 & b_L & 0 \\
  0 & b_L & 0 & b_L \\
  b_L & 0 & -b_L & 0 \\
  0 & b_L & 0 & b_L
\end{pmatrix}
\]

(39)

Now, to find the transition probabilities one firstly has to diagonalize the complete hamiltonian in (37). This can be done analytically, because the eigenvalue equation corresponds to a fourth degree algebraic equation whose solutions in terms of radicals are known. However, if we do so, we will obtain ugly formulae for the probabilities, and the physical content of them cannot be extracted in an easy way. So we will now proceed to consider some physical approximations in the framework of which we can easily get the transition probabilities and discuss their implications.

First of all we want to analyze the problem of occurrence of resonances in the transition processes (for relativistic neutrinos), looking for possible level crossings in the flavour eigenstate basis [12]. The hamiltonian, and then the eigenvalue equation, in this basis can be obtained from (38), (39) remembering the relations in (6), (7):

\[
\begin{pmatrix}
  k & 0 & m_D/2 & m_M/2 \\
  0 & k - b_L & m_M/2 & m_D/2 \\
  m_D/2 & m_M/2 & -k + b_L & 0 \\
  m_M/2 & m_D/2 & 0 & -k
\end{pmatrix}
\begin{pmatrix}
  \nu_L^c \\
  \nu_L \\
  \nu_R^c \\
  \nu_R
\end{pmatrix} = \omega
\begin{pmatrix}
  \nu_L^c \\
  \nu_L \\
  \nu_R^c \\
  \nu_R
\end{pmatrix}
\]

(40)

From this we immediately recognize that the level crossing for \( \nu_L \rightarrow \nu_L^c \) transition is given by the condition

\[
b_L = 0
\]

(41)

while those for \( \nu_L \rightarrow \nu_R^c \) and \( \nu_L \rightarrow \nu_R \) transitions are given, respectively, by

\[
\begin{aligned}
b_L &= k \\
b_L &= 2k
\end{aligned}
\]

(42) (43)
The last two conditions are never verified by relativistic neutrinos because the effective potential for media encountered in Nature never exceeds some eV (this “huge” value holding for the very dense matter of a neutron star), due to the Fermi coupling constant in (34), (35).

The condition for $\nu_L \to \nu_{\ell}^c$ transition in (41) is not properly a “resonance condition”, because it does not involve neutrino energy and mass parameters; it tells us, however, that the amplitude of oscillations is maximum in vacuum, so that the interaction of $\nu_L$ with the medium can only suppress neutrino oscillation (as already discussed in [1], this is obvious looking at eq. (26), where we see that in vacuum the amplitude of oscillation is already 1). Nevertheless, the implications of (41) can be important in the study of propagation of $\nu_{\ell}L$ during the neutronization phase of a supernova, where the condition $N_\ell = N_n/2$ is fulfilled.

Then we can conclude by saying that in “normal” media, at least for relativistic neutrinos, no resonance occurs for (flavour-conserving) oscillations of Dirac-Majorana neutrinos. With this result in mind, we now proceed to evaluate the transition probabilities in two unusual limits, namely for ultrarelativistic neutrinos satisfying the conditions

$$\text{case A: } \quad b_L \ll m_{D,M}, k$$

or

$$\text{case B: } \quad m_{D,M} \ll b_L, k$$

Since neutrinos will meet no resonance during their propagation in matter (we do not discuss here the situation envisaged above, for which $N_\ell = N_n/2$) these two cases can be analyzed by using perturbation theory.

### 3.1 Case A

In the eigenvalue problem (36), (37) we consider $H_0 = H_{cin}$ as the unperturbed hamiltonian and $H_b$ as a perturbation. The unperturbed energy levels and states are given by (12), (13) and (14), respectively. By means of standard techniques, we find that, at first order in $b_L$, the hamiltonian eigenvalues are approximatively given by

$$E_1 = -\left(E_+ - \frac{b_L}{2} \cos 2\theta_+\right) = -E_{+m}$$ (46)

$$E_2 = E_{+m}$$ (47)

$$E_3 = -\left(E_- - \frac{b_L}{2} \cos 2\theta_-\right) = -E_{-m}$$ (48)

$$E_2 = E_{-m}$$ (49)

\[\text{However, there is an exception offered by the cosmological relic neutrinos in the Universe, whose momentum is } 10^{-2} \div 10^{-4} \text{ eV}\]
while the (approximated) matter eigenstates and flavour eigenstates are related in the following way:

\[
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-C_+ + D_- & -C_+ - D_- & A_+ - B_- & A_+ + B_- \\
A_+ + B_- & A_+ - B_- & C_+ + D_- & C_+ - D_- \\
-C_- + D_+ & C_- + D_+ & A_- - B_+ & A_- + B_+ \\
A_- + B_+ & -A_- + B_+ & C_- + D_- & -C_- + D_-
\end{pmatrix}
\begin{pmatrix}
\nu^c_L \\
\nu_L \\
\nu_R \\
\nu_R
\end{pmatrix}
\] (50)

with

\[
A_+ = \cos \theta_+ - \frac{b_L}{4} \sin \theta_+ \sin 2\theta_+ \\
B_+ = \frac{b_L}{2} \left( \frac{\sin \theta_+ \sin (\theta_+ - \theta_-)}{E_+ + E_-} - \frac{\cos \theta_+ \cos (\theta_+ - \theta_-)}{E_+ - E_-} \right) \\
C_+ = \sin \theta_+ + \frac{b_L}{4} \cos \theta_+ \sin 2\theta_+ \\
D_+ = -\frac{b_L}{2} \left( \frac{\cos \theta_+ \sin (\theta_+ - \theta_-)}{E_+ + E_-} + \frac{\sin \theta_+ \cos (\theta_+ - \theta_-)}{E_+ - E_-} \right)
\] (51) (52) (53) (54) (55)

and similarly for \( A_-, B_-, C_-, D_- \) by substituting "+" \( \rightarrow \) "-". With some calculations, one can check that the matter mixing matrix in (50) is unitary (orthogonal) at first order in \( b_L \). Given this matrix, we are now able to calculate the transition probabilities for matter oscillations in the present limit. For simplicity, we report here only the formulae which are appropriate for ultrarelativistic neutrinos:

\[
P_m(\nu_L \rightarrow \nu_R) \approx \frac{m_L^2}{4k^2} \left( 1 + \frac{b_L}{k} \right) \sin^2 kt
\] (56)

\[
P_m(\nu_L \rightarrow \nu_R^c) \approx \frac{m_L^2}{4k^2} \left( 1 + 2 \frac{b_L}{k} \right) \sin^2 kt
\] (57)

\[
P_m(\nu_L \rightarrow \nu_L^c) \approx \sin^2 \frac{m_D m_M}{4k} \left( 1 + \frac{b_L}{2k} \right) t
\] (58)

\[
P_m(\nu_L \rightarrow \nu_L) \approx 1 - P_m(\nu_L \rightarrow \nu_L^c) - P_m(\nu_L \rightarrow \nu_R) - P_m(\nu_L \rightarrow \nu_R^c)
\] (59)

The last relation corrects eq. (55) of ref. [1].

Confronting the obtained results with the expressions (24)-(27) holding for the vacuum case, we can observe that for the chirality-changing transitions the period of matter and vacuum oscillations are the same, while their amplitude change (because of the sign of \( b_L \), the amplitude of matter oscillations are always lower than those of vacuum oscillations for \( \nu_\mu \) and \( \nu_\tau \), while for \( \nu_e \) all possibilities can occur). The opposite happens for \( \nu_L \rightarrow \nu_L^c \) oscillations. In any case, the corrections due to matter interaction are quite insignificant, because of the very smallness of the effective potential with respect to neutrino momentum in normal situations, except for relic neutrinos propagating in very dense stars
as in the study developed in \[14\]. However, for the last case, relations (58)-(59) would not apply because of the non relativistic propagation; the right expressions obtained relaxing the assumption of ultrarelativistic propagation can be found in the appendix.

### 3.2 Case B

Let us now consider the situation in which the mass parameters are the smallest ones, namely \([13]\) holds. In this case, it is more convenient to work in the flavour basis and then solve the eigenvalue equation \((10)\) by considering \(H = \text{diag}\{k, k-b_L, -k+b_L, -k\}\) as the unperturbed hamiltonian and the remaining hamiltonian mass term in \((10)\) as a perturbation. At second order in the perturbation parameters we thus obtain the following energy levels:

\[
E_1 \simeq k + \frac{m_D^2}{4(2k-b_L)} + \frac{m_M^2}{8k} \tag{60}
\]

\[
E_2 \simeq k - b_L + \frac{m_D^2}{4(2k-b_L)} + \frac{m_M^2}{8(k-b_L)} \tag{61}
\]

\[
E_3 \simeq -E_2 \tag{62}
\]

\[
E_4 \simeq -E_1 \tag{63}
\]

Instead, at the same order, the energy eigenstates are given by

\[
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4
\end{pmatrix} =
\begin{pmatrix}
1 - \frac{1}{2}(A^2 + C^2) & AC \delta_1 & A & C \\
-AB \delta_2 & 1 - \frac{1}{2}(A^2 + B^2) & B & A \\
-A & -B & 1 - \frac{1}{2}(A^2 + B^2) & -AB \delta_2 \\
-C & -A & AC \delta_1 & 1 - \frac{1}{2}(A^2 + C^2)
\end{pmatrix}
\begin{pmatrix}
\nu_L^c \\
\nu_L \\
\nu_R^c \\
\nu_R
\end{pmatrix} \tag{64}
\]

where

\[
A = \frac{m_D}{2(2k-b_L)} \tag{65}
\]

\[
B = \frac{m_M}{4(k-b_L)} \tag{66}
\]

\[
C = \frac{m_M}{4k} \tag{67}
\]

\[
\delta_1 = \frac{4k-b_L}{b_L}, \quad \delta_2 = \frac{4k-3b_L}{b_L} \tag{68}
\]

With the mixing matrix defined in \((14)\) we can now obtain the expressions for the transition probabilities

\[
P_m(\nu_L \rightarrow \nu_R) \simeq 4 A^2 \sin^2 \frac{E_1 + E_2}{2} t \tag{69}
\]

\[
P_m(\nu_L \rightarrow \nu_L^c) \simeq 4 B^2 \sin^2 E_2 t \tag{70}
\]

\[
P_m(\nu_L \rightarrow \nu_L^c) \simeq 0 \tag{71}
\]
More interestingly, for $b_L \ll k$ we have

\begin{align*}
P_m(\nu_L \rightarrow \nu_R) &\simeq \frac{m_D^2}{4k^2} \left(1 + \frac{b_L}{k}\right) \sin^2 k t \quad (72) \\
P_m(\nu_L \rightarrow \nu_R^c) &\simeq \frac{m_M^2}{4k^2} \left(1 + 2\frac{b_L}{k}\right) \sin^2 k t \quad (73) \\
P_m(\nu_L \rightarrow \nu_L^c) &\simeq 0 \quad (74)
\end{align*}

and we immediately recognize that the transition probabilities for $\nu_L \rightarrow \nu_R$ and $\nu_L \rightarrow \nu_R^c$ are coincident with those calculated for the case A, this showing that the presence of a medium has practically in any regime no influence on these processes. Instead, the very intriguing fact is that the probability for $\nu_L \rightarrow \nu_L^c$ substantially vanishes in the limit (45): one can check that $P_m(\nu_L \rightarrow \nu_L^c)$ has in fact an oscillatory behaviour with an amplitude of the order $\frac{m_D^2}{4k^2} \frac{m_M^2}{4k^2}$. This feature is a general one of all the chirality-preserving transitions, and is a consequence of the fact that, in the present limit, the physical propagating neutrino fields are not predominantly Majorana states. In fact let us consider, for example, the state $\nu_2$ in (64). Switching off the perturbation, this correspond to the flavour state $\nu_L$. Because of the mass interaction, this state acquires, at first order, a small $\nu_R$ and/or $\nu_R^c$ component letting non zero $m_D$ and/or $m_M$ but no fraction of $\nu_L^c$ enters in $\nu_2$ at this order. Only at the second order, for Dirac and Majorana masses both non vanishing, $\nu_2$ acquires a $\nu_L$ component; so the predominant flavour content of $\nu_2$ is only due to $\nu_L$ but not to $\nu_L$ and $\nu_L^c$ in almost equal parts as happens in vacuum (or in the case A), and this explains the result (74). Analogous considerations hold for the other chirality-preserving transitions.

4 Discussion and conclusions

In this paper we have studied flavour-conserving oscillations of Dirac-Majorana neutrinos when these propagate in vacuum as well as in a dense medium.

In vacuum, since the physical neutrino field of definite helicity is predominantly composed of states with a given related chirality and has only a small component (for non zero masses) of states with the other chirality, both chirality-changing and chirality-preserving transitions are possible, but the first ones are suppressed with respect to the others [6].

In the ultrarelativistic limit, $\nu_L \rightarrow \nu_R$ oscillations are ruled exclusively by the Dirac mass term, and $\nu_L \rightarrow \nu_R^c$ ones exclusively by the Majorana mass term, while $\nu_L \rightarrow \nu_L^c$ transitions can take place only if both $m_D$ and $m_M$ are non zero (more precisely, the Dirac mass term is essential for the existence of the sterile $\nu_L^c$ state, while the total lepton number violating Majorana mass term has to be present for the $\Delta L = 2$ process $\nu_L \rightarrow \nu_L^c$ to occur).

When energetic neutrinos propagate in a medium, the oscillations considered here undergo no resonant enhancement since the effective potential experienced by neutrinos is too low with respect to neutrino momentum, and this justifies a perturbative analysis of the phenomena, as made in the present paper. The chirality-changing transitions proceed in
matter practically as in vacuum, apart from an insignificant modification of the oscillation amplitude. For the $\nu_L \to \nu^c_L$ transitions, if the mass interaction is more effective than the matter interaction (case A, (44)), the period of oscillation slightly changes with respect to the vacuum case, but even here the correction is unimportant because of the smallness of the effective potential. Instead the situation dramatically change if the interaction of neutrinos with matter is predominant on mass interaction. Due to the fact that, in this case, the physical propagating neutrino field is no longer dominated by the vacuum Majorana mass states (as in the previous case), the probability for $\nu_L \to \nu^c_L$ transitions is practically zero. This result would then suggest a peculiar situation in which one can study neutrino oscillation phenomena by using the fact that $P(\nu_L \to \nu^c_L)$ in (26) has an oscillatory behaviour in time (or distance) while $P_m(\nu_L \to \nu^c_L)$ in (74) is de facto time independent. Indeed, let us suppose that at $x = 0$ a pure $\nu_L$ beam is produced and at $x = L$ a neutrino detector is placed. If between the source and the detector there is the vacuum, then at $x = L$ we have a probability to detect $\nu^c_L$ given by (26) with $t \simeq x = L$. The situation change if between the source and the detector we place a dense medium from $x_1$ to $x_2$ (with $x_2 - x_1 = L_m$) with a density satisfying (45): in this case, because of (74), the probability of finding a $\nu^c_L$ at $x_1$ and at $x_2$ is the same, and the effective length for oscillation is no longer $L$ but $L - L_m$. So, we can detect at $x = L$ a $\nu^c_L$ with a probability again given by (26) but with $t \simeq x = L - L_m$. This method would be useful to study oscillations of neutrinos with a very small mass (the relation (45) has to be satisfied); however, we think that it is unfair to apply this method in a very near future. Nevertheless, the results contained in this paper could be interesting for astrophysics and especially cosmology, where very dense matter indeed exists.

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Appendix

From (50), assuming no hierarchy of \( m_D, m_M \) with respect to neutrino momentum \( k \), we obtain the following expressions for the transition probabilities:

\[
P_m(\nu_L \to \nu_R) \simeq \frac{1}{4} \left( \frac{m_+}{E_+} \sin E_+ t + \frac{m_-}{E_-} \sin E_- t \right)^2 + \frac{b_L}{4} \left( \frac{m_+}{E_+} \sin E_+ t + \right.
\left. \frac{m_-}{E_-} \sin E_- t \right) \left( \frac{m_+ k}{E_+^3} \sin E_+ t + \frac{m_- k}{E_-^3} \sin E_- t \right) \tag{75}
\]

\[
P_m(\nu_L \to \nu^c_R) \simeq \frac{1}{4} \left( \frac{m_+}{E_+} \sin E_+ t - \frac{m_-}{E_-} \sin E_- t \right)^2 + \frac{b_L}{2} \left( \frac{m_+}{E_+} \sin E_+ t + \right.
\left. \frac{m_-}{E_-} \sin E_- t \right) \left( \frac{m_+ k}{2 E_+^3} \sin E_+ t - \frac{m_- k}{2 E_-^3} \sin E_- t + \right.
\left. \frac{m_+ - m_-}{E_+^2 - E_-^2} \frac{k}{E_+ E_-} (E_+ \sin E_- t - E_- \sin E_+ t) \right) \tag{76}
\]

\[
P(\nu_L \to \nu^c_L) \simeq \sin^2 \frac{E_+ m + E_- m}{2} t \sin^2 \frac{E_+ m - E_- m}{2} t + \frac{1}{4} \left( \frac{k}{E_+} \sin E_+ t + \right.
\left. \frac{k}{E_-} \sin E_- t \right)^2 + \frac{b_L}{2} \left( \frac{k}{E_+} \sin E_+ t - \frac{k}{E_-} \sin E_- t \right) \times
\left. \left( \frac{m_+^2}{E_+^2} \sin E_- t - \frac{m_-^2}{E_-^2} \sin E_+ t \right) \right) \tag{77}
\]

In the limit \( m_D, m_M \ll k \) these expressions reduce to those reported in eqs. (56)-(58).

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