EXISTENCE OF THE LATTICE ON GENERAL $H$-TYPE GROUPS

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Abstract. Let $\mathcal{N}$ be a two step nilpotent Lie algebra endowed with non-degenerate scalar product $\langle \cdot, \cdot \rangle$ and let $\mathcal{N} = V \oplus Z$, where $Z$ is the center of the Lie algebra and $V$ its orthogonal complement with respect to the scalar product. We prove that if $(V, \langle \cdot, \cdot \rangle_V)$ is the Clifford module for the Clifford algebra $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ such that the homomorphism $J: \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \to \text{End}(V)$ is skew symmetric with respect to the scalar product $\langle \cdot, \cdot \rangle_V$, or in other words the Lie algebra $\mathcal{N}$ satisfies conditions of general $H$-type Lie algebras [7, 14], then there is a basis with respect to which the structural constants of the Lie algebra $\mathcal{N}$ are all $\pm 1$ or 0.

1. Introduction and definitions

We denote by $\langle \cdot, \cdot \rangle_V$ a real valued symmetric non-degenerate bi-linear form defined on a real vector space $V$ and call it a scalar product. If the form is positive definite, we denote it by $(\cdot, \cdot)$ and call it an inner product. We use the notation $[\cdot, \cdot]$ for commutator, or in other words for a skew symmetric bi-linear vector valued form. The $H$-type Lie algebras were introduced by A. Kaplan in [19] and were widely studied, see, for instance [6, 10, 12, 20, 21]. In works [7, 9, 14] the analogous of classical $H$-type Lie algebras were introduced and studied.

Definition 1. A 2-step nilpotent Lie algebra $\mathcal{N}$ endowed with a scalar product $\langle \cdot, \cdot \rangle$ is called a general $H$-type algebra, if

1. $\mathcal{N} = V \oplus Z$, where $Z$ is the center of the Lie algebra $\mathcal{N}$, which is non-degenerate subspace of the scalar product vector space $(\mathcal{N}, \langle \cdot, \cdot \rangle)$, and $V$ its orthogonal complement (we call it the horizontal space),

2. the skew symmetric map $J: Z \to \text{End}(V)$ defined by

$$\langle J_z u, v \rangle = \langle z, [u, v] \rangle$$

satisfies the orthogonality condition

$$\langle J_z u, J_z v \rangle = \langle z, z \rangle \langle u, v \rangle.$$

Conditions 1 and 2 imply

$$J_z^2 = -\langle z, z \rangle \text{Id}_V$$

see, for example, [7, 14, 23].

Due to (3) the horizontal space $V$ becomes a $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$-module, where $\langle \cdot, \cdot \rangle_Z$ is the restriction of the scalar product $\langle \cdot, \cdot \rangle$ onto the center $Z$. So, from the definition we see that any general $H$-type algebra $\mathcal{N}$ defines a $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$-module $V$. Moreover, the module $V$ is endowed with the scalar product $\langle \cdot, \cdot \rangle_V$, obtained by the restriction of $\langle \cdot, \cdot \rangle$ on $V$, such that the representations $J_z$ are skew symmetric with respect to the scalar product $\langle \cdot, \cdot \rangle_V$ for any $z \in Z$. 

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1
From the other side, if we assume that $V$ is a Clifford module for some Clifford algebra $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$, and $V$ carried a scalar product $\langle \cdot, \cdot \rangle_V$ such that (2) holds, then $J$ is skew symmetric with respect to $\langle \cdot, \cdot \rangle_V$:

$$
\langle J_z u, v \rangle_V = -\langle u, J_z v \rangle_V.
$$

(4)

Therefore, one can define the Lie bracket $[\cdot, \cdot] : V \times V \to Z$ by using (1) with $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_V + \langle \cdot, \cdot \rangle_Z$; and show that the Lie algebra $\mathcal{N} = (V \oplus_Z \mathbb{R} \cdot 1, [\cdot, \cdot])$ is a general $H$-type algebra, see [7, 10, 14, 20].

In general, among the conditions (2), (3), and (4) any two of them imply the third one.

We say that a $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$-module $V$ is an admissible module, if there is a scalar product $\langle \cdot, \cdot \rangle_V$ defined on $V$ such that the representations $J_z : \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \to \text{End}(V)$, satisfy the skew symmetry condition (4) for any $z \in Z$. The scalar product $\langle \cdot, \cdot \rangle_V$ will be called an admissible scalar product. The following is known about admissible modules. Let $V$ be a given $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$-module and $\{z_1, \ldots, z_n\}$ an orthonormal basis of $Z$ with respect to the scalar product $\langle \cdot, \cdot \rangle_Z$. Let us denote by $J_{z_j} \in \text{End}(V)$ the representations of the generators $z_1, \ldots, z_n$ of the Clifford algebra $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$. Then $J_{z_j}$ satisfy

$$
J_{z_j}^2 = -\langle z_j, z_j \rangle_Z \text{Id}_V, \quad J_{z_j} J_{z_j} = -J_{z_j} J_{z_j}, \quad j = 1, \ldots, n, \quad i \neq j.
$$

It is always possible to find an inner product $\langle \cdot, \cdot \rangle_V$ on $V$ such that, the representations $J_{z_j}$ satisfy the orthogonality condition (2), since the group generated by the operators $\{J_{z_j}\}_{j=1}^n$ is a finite group in $\text{End}(V)$ (see [17]). In the special case of Clifford algebra $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$, generated by an inner product space $(Z, \langle \cdot, \cdot \rangle_Z)$ the chosen inner product $\langle \cdot, \cdot \rangle_V$ on $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$-module $V$ satisfies the orthogonality condition (2) for any $z_j, z \in Z$ and not only for representations $J_{z_j}$ of generators $z_j, j = 1, \ldots, n$, see [24]. As a consequence, in this case we obtain the skew symmetry property (4) and $(V, \langle \cdot, \cdot \rangle_V)$ became an admissible module. The Lie algebra $\mathcal{N} = V \oplus_Z \mathbb{R}$, where the Lie bracket is defined in (1) by making use of skew symmetric maps $J_z$ and the inner product on $\mathcal{N}$ is the sum of inner products on $V$ and $Z$, is the $H$-type algebra introduced by A. Kaplan in [19]. We call such an algebra a classical $H$-type Lie algebra.

Let $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ be a Clifford algebra generated by an arbitrary scalar product $\langle \cdot, \cdot \rangle_Z$. It was shown in [7] that given a $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$-module $V$ there always exists a scalar product $\langle \cdot, \cdot \rangle_V$ on $V$ (or on $V \oplus V$), such that the map $J_z : Z \to \text{End}(V)$ (or a modified map $\overline{J}_z : Z \to \text{End}(V \oplus V)$) satisfies (2), or equivalently (4), for an arbitrary $z \in Z$. As a consequence, we obtain that for any Clifford algebra $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ there exists an admissible module $(V, \langle \cdot, \cdot \rangle_V)$ (or $(V \oplus V, \langle \cdot, \cdot \rangle_{V \oplus V})$). Moreover, the admissible module $(V, \langle \cdot, \cdot \rangle_V)$ (or $(V \oplus V, \langle \cdot, \cdot \rangle_{V \oplus V})$) will be necessarily a neutral space, that is the dimensions of the maximal positive and negative definite subspace of $V$ (or $V \oplus V$) coincide, see Proposition 1 and [7]. The corresponding 2-step nilpotent Lie algebra satisfies Definition 1 and is called general $H$-type algebra.

Now, let $\mathcal{N}$ be a simply connected, nilpotent Lie group and $\Gamma$ its discrete subgroup such that the quotient space $\Gamma \backslash \mathcal{N}$ is compact. Then the group $\Gamma$ is called lattice and quotient $\Gamma \backslash \mathcal{N}$ is called a compact nilmanifold, see, for instance, [13]. Nilmanifolds, as a generalization of higher dimensional tori, play important role in study of the sub-Riemannian geometry, the Riemannian geometry with singularities, hypoelliptic operators, and spectral properties of differential operators of the Grushin type, see for example [2, 3, 4, 5, 16, 18]. Also see [1, 8] for such study of the compact sub-Riemannian manifolds coming from simple Lie groups. According to the Mal’cev criterium [25] a nilpotent Lie group $\mathcal{N}$ admits a lattice if and only if the corresponding Lie algebra of $\mathcal{N}$ has a basis with rational structure constants. Not all, even 2-step, nilpotent Lie algebras admit such a basis. In the work [11] it was shown that classical $H$-type Lie algebras $\mathcal{N}$ have integer structure constants, or more precisely, there is a basis $\{v_1, \ldots, v_m\}$ of $V$ and a basis $\{z_1, \ldots, z_n\}$ of $Z$, such that $[v_\alpha, v_\beta] = \sum_{k=1}^n A^k_{\alpha \beta} z_k$, where the numbers $A^k_{\alpha \beta}$ equal 0, 1, or $-1$. So it will be natural to ask whether general H-type algebras have such a basis too.

In the present work we show the following statement.
Theorem 1. Let \( \mathcal{N} = V \oplus \perp Z, [\cdot, \cdot] \) be a general \( H \)-type algebra. Then there is an orthonormal basis \( \{v_1, \ldots, v_m, z_1, \ldots, z_n\} \) of \( V \oplus \perp Z \) such that \( [v_\alpha, v_\beta] = \sum_{k=1}^{n} A_{\alpha\beta}^k z_k \), where the coefficients \( A_{\alpha\beta}^k \) are equal to \( \pm 1 \), or 0.

Denote by \( \langle \cdot, \cdot \rangle_V \) and \( \langle \cdot, \cdot \rangle_Z \) the restrictions of the scalar product to the subspaces \( V \) and \( Z \), and assume that the scalar product spaces \( (V, \langle \cdot, \cdot \rangle_V) \) and \( (Z, \langle \cdot, \cdot \rangle_Z) \) are nondegenerate. Let

\[
\nu_\alpha^V = \langle v_\alpha, v_\alpha \rangle_V, \quad \alpha = 1, \ldots, m, \quad \nu_\alpha^Z = \langle z_\alpha, z_\alpha \rangle_Z, \quad k = 1, \ldots, n
\]

be corresponding indices. Let \( J_z : \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \to \text{End}(V) \) be representations of orthonormal generators \( z_1, \ldots, z_n \) of \( \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \). We write

\[
(v_\alpha, v_\beta) = \sum_{k=1}^{n} A_{\alpha\beta}^k z_k \quad \text{and} \quad J_z v_\alpha = \sum_{\beta=1}^{m} B_{\alpha\beta}^k v_\beta.
\]

Then, as a consequence of (1) and (5), we obtain

\[
(J_z v_\alpha, v_\beta)_V = \langle z_\alpha, [v_\alpha, v_\beta] \rangle_Z \implies \nu_\beta^V B_{\alpha\beta}^k = \nu_\alpha^Z A_{\alpha\beta}^k.
\]

Therefore the result of Theorem 1 can be reformulated as follows.

Theorem 2. Given a scalar product space \( (Z, \langle \cdot, \cdot \rangle_Z) \) with an orthonormal basis \( z_1, \ldots, z_n \) there is an admissible Clifford \( \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \)-module \( (V, \langle \cdot, \cdot \rangle_V) \) of minimal dimension with representations

\[
J : \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \to \text{End}(V)
\]

and an orthonormal basis \( v_1, \ldots, v_m \) on \( (V, \langle \cdot, \cdot \rangle_V) \), such that \( J_z \) satisfies (2), (3) and moreover,

\[
(J_z v_\alpha, v_\beta)_V = \pm 1, \quad \text{or} \quad 0, \quad \text{for all} \quad k = 1, \ldots, n, \quad \alpha, \beta = 1, \ldots, m.
\]

In the following we always use the identification \( \text{Cl}_{r,s} \cong \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \), arising from the isomorphism \( (Z, \langle \cdot, \cdot \rangle_Z) \cong \mathbb{R}^{r,s} \). Here \( \mathbb{R}^{r,s} \) is the space \( \mathbb{R}^{r+s} \) with the quadratic form \( Q_{r,s}(x) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2 \).

We call \( \text{Cl}_{r,s} \)-modules, satisfying Theorem 2 admissible integral modules and the corresponding orthonormal basis \( \{v_\alpha\} \) integral basis. The existence of admissible integral \( \text{Cl}_{r,0} \)-modules was shown in [11] and here we reconstruct \( \text{Cl}_{r,0} \)-modules with an integral basis by making use of a different method, see also an observation in Section 9. The admissible integral \( \text{Cl}_{r,0} \)-modules lead to the presents of a lattice on classical \( H \)-type groups. Notice also, that in the work [13], the existence of a rational structure constants on classical \( H \)-type algebras was shown by realizing its Lie algebra as a direct sum of the space \( \mathbb{R}^m \) and the center \( Z \), given as a Lie triple system embedded in a subspace of \( \mathfrak{so}(m) \).

In the present work we construct explicitly an orthonormal basis of any minimal dimensional admissible \( \text{Cl}_{r,s} \)-module with respect to which the structure constants \( A_{\alpha\beta}^k \) defined in (5), or equivalently in (6), equal to \( \pm 1 \) or 0.

There are several methods of construction of such an integral basis. To be able to use the Bott 8-periodicity the number of necessary admissible integral modules is 64. We use the isomorphism \( \text{Cl}_{r,s+1} \cong \text{Cl}_{r,s+1} \) that preserves the integral basis, we take the tensor products with \( \text{Cl}_{4,4} \)-module or with \( \text{Cl}_{1,1} \)-module, the construction of \( \text{Cl}_{r,1} \)-module from \( \text{Cl}_{r,0} \)-module and reduce the number of required modules to 28. To explain the main idea for the construction of the integral basis in the remaining cases we recall the terminology. A vector \( v \in V \) is called

- \textbf{spacelike} if \( \langle v, v \rangle_V > 0 \) or \( v = 0 \);
- \textbf{timelike} if \( \langle v, v \rangle_V < 0 \);
- \textbf{null} if \( \langle v, v \rangle_V = 0 \).

A linear map \( P : V \to V \) is called \textit{involution} if \( P^2 = \text{Id}_V \) and \textit{anti-involution} if \( P^2 = -\text{Id}_V \). We say that a bijective linear map \( T : V \to V \) is an \textit{isometry} if

\[
\langle T v, T v \rangle_V = \langle v, v \rangle_V \quad \text{for all vectors} \quad v \in V.
\]
and it is an anti-isometry if $\langle Tv, Tv \rangle_V = -\langle v, v \rangle_V$ for all vectors. The principal method for the construction of integral bases starts by picking up a maximal number of mutually commuting isometric involutions together with “complementary” isometries or anti-isometries satisfying some commutation relations with the original involutions. These choice of involutions and complementary operators give an orthogonal decomposition of the representation space for the Clifford algebra. Choosing a common spacelike eigenvector of the original involutions we construct an integral basis by means of action on it of representations of the orthonormal generators for the corresponding Clifford algebra.

There are several differences in the construction of orthogonal decompositions of the representation spaces by those involutions and complementary operators. The purpose of the present work is, not only to show the existence of an integral structure for all $\text{Cl}_{r,s}$-modules, but also to present several possible methods for such kind of constructions, especially for the cases of low dimensions.

The structure of this paper is as follows: Section 2 is an auxiliary section where we collected the information about properties of admissible modules and auxiliary technical lemmas. In Section 3 we prove that the isomorphism between Clifford algebras $\text{Cl}_{r+1}$ and $\text{Cl}_{r+1}$ preserves the admissible integral modules. This isomorphism reduces significantly the number of the Clifford modules where we need to construct integral basis before we are able to apply the Bott periodicity. In Section 4 we show the existence of admissible integral $\text{Cl}_{0,s}$-modules of minimal dimensions for $s \leq 8$. We set apart this section to emphasise the difference between the admissible modules for Clifford algebras $\text{Cl}_{r,0}$ and $\text{Cl}_{0,s}$. In Section 5 we construct a minimal admissible module of $\text{Cl}_{1,1}$ with an integral basis basing on the existence of integral basis for the algebra $\text{Cl}_{r,0}$. Section 6 devoted to the construction of integral structures on admissible $\text{Cl}_{r,s}$-modules for $0 \leq r, s \leq 8$ with $r + s \leq 8$. In the section we actively develop a method of the simultaneous orthogonal decomposition of eigenspaces for a collection of mutually commuting isometric involutions. We also exploit results of Sections 3 and 5. In Section 7 we prove some theorems that allow to use the Bott periodicity of Clifford algebras for construction of integral structures. We also construct admissible modules with integral basis for $\text{Cl}_{r+1, s+1}$ and $\text{Cl}_{0, n+2}$ based on the admissible module of $\text{Cl}_{1,1}$ and $\text{Cl}_{2,0}$. This method shows that the tensor product representation with some modification gives us an admissible module with integral basis, but it need not be minimal. It remains to decompose this admissible module into minimal one’s together with an integral basis. Section 8 deals with integral structures on admissible $\text{Cl}_{r,s}$-modules for $r + s \geq 9$. In the last Section 9 we make some observations about the presented constructions. Appendix contains the table of Clifford algebras, where circled Clifford algebras have the admissible modules of double dimension compare with the irreducible modules. It is also easy to see the symmetry with respect to the axis $r - s = -1$ that allows to use the construction of Section 3.

2. Properties of admissible $\text{Cl}_{r,s}$-modules

We recall the basic properties of admissible $\text{Cl}_{r,s}$-modules when $s > 0$, see also [7]. We say that $W$ is an admissible sub-module of an admissible module $(V, \langle \cdot, \cdot \rangle_V)$ if $W$ is a Clifford sub-module of $V$ and the restriction of $\langle \cdot, \cdot \rangle_V$ on $W$, denoted by $\langle \cdot, \cdot \rangle_W$, is an admissible scalar product. There are decompositions of a given admissible $\text{Cl}_{r,s}$-module, $s > 0$, into non-admissible sub-modules, see Remarks 2 and 4 for examples. In the following proposition we give conditions that ensure a decomposition of an admissible Clifford module into admissible sub-modules.

**Proposition 1.** Let $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible $\text{Cl}_{r,s}$-module with $s > 0$ and $J_{z_k}$, $k = 1, \ldots, r + s$, representations of the orthonormal generators $z_1, \ldots, z_{r+s}$ of the Clifford algebra $\text{Cl}_{r,s}$.

1. Then the scalar product space $(V, \langle \cdot, \cdot \rangle_V)$ is neutral, i.e. the maximal dimension of subspaces where the restriction of $\langle \cdot, \cdot \rangle_V$ is positive or negative definite coincide and, particularly, the dimension of $V$ can be only even.

2. If $W$ is an admissible sub-module of a Clifford $\text{Cl}_{r,s}$-module, then $W^\perp$ is also an admissible sub-module. Hence, we have the decomposition of an admissible $\text{Cl}_{r,s}$-module $(V, \langle \cdot, \cdot \rangle_V)$ into admissible sub-modules.
Applying the same arguments to \( W \) the orthogonal complement, that shows the first statement.

The scalar product restricted to the orthogonal complement to \( \mathbb{R}^{r,s} \) and invariant the orthogonal complement \( W \) and particularly, \( \langle v, z \rangle_W = \langle z, v \rangle_W \) for all \( z \in \mathbb{R}^{r,s} \). The scalar product restricted to the orthogonal complement to \( W \) is non-degenerate and neutral. Applying the same arguments to \( W^\perp \), we decompose it into two dimensional neutral subspace and the orthogonal complement, that shows the first statement.

If \( W \) is an admissible sub-module of a Clifford \( \text{Cl}_{r,s} \)-module, then the action of each \( J_z \) leaves invariant the orthogonal complement \( W^\perp = \{ v \in V \mid \langle w, v \rangle_W = 0 \text{ for any } w \in W \} \). Indeed, if \( \tilde{w} \in W^\perp \) and \( w \in W \), then

\[
\langle J_z \tilde{w}, w \rangle_V = -\langle \tilde{w}, J_z w \rangle_V = 0.
\]

Since the scalar product restricted to \( W^\perp \) is non-degenerate, \( W^\perp \) is also an admissible sub-module.

Further we show the existence of non-trivial scalar product satisfying two conditions (consequently all three) among \([2], [3], \) and \([4]\) for Clifford \( \text{Cl}_{r,s} \)-modules with \( s > 0 \).

**Lemma 1.** Let \( V \) be an irreducible module of a Clifford algebra \( \text{Cl}_{r,s} \) :

\[
J : \text{Cl}_{r,s} \to \text{End}(V), \quad J^2 = -\langle z, z \rangle_{\mathbb{R}^{r,s}} I_d, \quad z \in \mathbb{R}^{r,s},
\]

with a symmetric bilinear form \( \langle \cdot, \cdot \rangle_V : V \times V \to \mathbb{R} \) which satisfies

\[
\langle J_z v, w \rangle_V + \langle v, J_z w \rangle_V = 0 \quad \text{for any } z \in \mathbb{R}^{r,s} \text{ and any } v, w \in V.
\]

Then the scalar product \( \langle \cdot, \cdot \rangle_V \) is non-degenerate or identically vanishes.

**Proof.** Let \( N = \{ v \in V \mid \langle v, w \rangle_V = 0 \text{ for any } w \in V \} \). Then for any \( z \in \mathbb{R}^{r,s} \) and \( v \in N \)

\[
\langle J_z v, w \rangle_V + \langle v, J_z w \rangle_V = 0, \quad \text{and} \quad \langle v, J_z w \rangle_V = 0, \quad \text{for all } w \in V.
\]

Hence \( \langle J_z v, w \rangle_V = 0 \), which shows that the subspace \( N \) is invariant under the action of the Clifford algebra. So once we have an element \( v \in V \) with \( \langle v, v \rangle_V \neq 0 \) then \( N \) must be the trivial space \( \{0\} \) or entire \( V \) due to the irreducibility of the module \( V \).

As was mentioned above, in the case of classical \( H \)-type algebras, there is an admissible inner product for any \( \text{Cl}_{r,0} \)-module, particularly, any irreducible module can be an admissible module with an inner product. However, for \( s > 0 \) not all irreducible \( \text{Cl}_{r,s} \)-modules can be admissible modules. For instance, the Clifford algebra \( \text{Cl}_{0,2} \) is isomorphic to the algebra \( \mathbb{R}(2) \) of \((2 \times 2)\) real matrices, and the irreducible module is 2 dimensional whereas the admissible \( \text{Cl}_{0,2} \)-module of minimal dimension is isomorphic to \( \mathbb{R}^{2,2} \), see Section [4]. The table presented in Appendix shows the Clifford algebras, where the circled Clifford algebras has admissible modules of double dimension comparing with irreducible modules. However the following properties are still hold.

**Proposition 2.** ([4]) Let \( V \) be a \( \text{Cl}_{r,s} \)-module, then there is a scalar product on \( V \) or on \( V \oplus V \) with respect to which the resulting \( \text{Cl}_{r,s} \)-module is an admissible module. The representation on \( V \oplus V \) should be redefined in an obvious way.

Particularly for irreducible modules \( V \) the following corollary holds.

**Corollary 1.** An irreducible \( \text{Cl}_{r,s} \)-module can be either an admissible module of minimal dimension or the double of the irreducible module is the admissible module of minimal dimension. As a result, a generalized \( H \)-type algebra constructed from the minimal dimensional admissible module is unique up to isomorphism.
Any irreducible Cl_{r,s}-module \( V \) can be generated by a non-zero vector \( v \in V \) and the subsequent actions by \( J_z, z \in \mathbb{R}^{r,s} \) (or Clifford multiplication by \( z \)). Similarly, we have the following statement.

**Proposition 3.** Any minimal dimensional admissible module is generated by a non-null vector.

**Lemma 2.** Let \( Cl_{r,s} \) be a Clifford algebra with orthonormal generators \( \{ z_i \}, i = 1, \ldots, r + s \) and \( (V, \langle \cdot, \cdot \rangle_V) \) an admissible \( Cl_{r,s}\)-module with an orthonormal basis \( \{ v_\alpha \}_{\alpha=1}^{\dim V} \). Then \( J_z, v_\alpha \not= \pm J_z, v_\alpha \) for \( i \not= j \).

**Proof.** Let us assume that \( J_z, v_\alpha = \pm J_z, v_\alpha \) for some \( i \not= j \) and an element \( v_\alpha \) of the orthonormal basis. From the assumption we have \( J_z, v_\alpha = 0 \) and hence \( J_z, v_\alpha = - \langle z_i \pm z_j, z_i \pm z_j \rangle_{\mathbb{R}^{r,s}} v_\alpha = 0 \), which implies that \( \langle z_i \pm z_j, z_i \pm z_j \rangle_{\mathbb{R}^{r,s}} = 0 = \langle z_i, z_i \rangle_{\mathbb{R}^{r,s}} + \langle z_j, z_j \rangle_{\mathbb{R}^{r,s}} \). So, one of \( z_i \) or \( z_j \) is spacelike, and the other is timelike. The assumption also implies that \( J_z, v_\alpha = \pm v_\alpha \), which leads to
\[
\langle \pm v_\alpha, \pm v_\alpha \rangle_V = \langle J_z, J_z, v_\alpha, J_z, J_z, v_\alpha \rangle_V = \langle z_i, z_i \rangle_{\mathbb{R}^{r,s}} \langle z_j, z_j \rangle_{\mathbb{R}^{r,s}} \langle v_\alpha, v_\alpha \rangle_V = - \langle v_\alpha, v_\alpha \rangle_V.
\]
This is a contradiction since \( v_\alpha \) is a non-null vector.

**Corollary 2.** Under assumptions of Lemma 2 if we additionally assume that each operator \( J_{z_i} \) permutes the basis \( \{ v_\alpha \} \) up to sign, that is \( J_{z_i}, v_\alpha = \pm v_\beta \) for any \( i, \alpha \) and some \( \beta \), then with the given orthonormal generators \( \{ z_i \} \) of the Clifford algebra the basis \( \{ v_\alpha, z_i \} \) of the general Lie algebra \( V \oplus_{\perp} \mathbb{R}^{r,s} \) has structure constants \( A_{\alpha,\beta} \) equal \( \pm 1 \) or zero.

In fact, in the present work we show that for any minimal dimensional admissible module \( (V, \langle \cdot, \cdot \rangle_V) \), one can find a vector \( w \in V \) with \( \langle w, w \rangle_V = 1 \) such that the \( 2^{r+s} \)-vectors
\[
w, J_1 w, \ldots, J_{r+s} w, \quad J_1 J_2 w, \ldots, J_{r+s-1} J_{r+s} w, \quad \ldots, J_1 J_2 \cdots J_{r+s} w,
\]
satisfy the property that any two vectors \( J_i, J_2 \cdots J_{r+s} w \) and \( J_{r+s-1} J_{r+s} \cdots J_{r+i} \) coincide up to sign or orthogonal and therefore we can select an orthonormal basis which is produced from \( w \in V \) by action of generators \( J_i = J_{z_i} \). Moreover \( J_{z_i} \) act on the obtained basis as permutations up to sign. Hence we prove that any general H-type algebra have an integral structure according Lemma 2 and Corollary 2.

Further we collect some auxiliary technical lemmas, that will be used later. We also say that a linear operator \( \Omega : V \to V \) on a scalar product space \( (V, \langle \cdot, \cdot \rangle_V) \) is symmetric if \( \langle \Omega v, w \rangle_V = \langle v, \Omega w \rangle_V \) for all \( v, w \in V \).

**Lemma 3.** Let \( (V, \langle \cdot, \cdot \rangle_V) \) be an admissible module, \( \Omega : V \to V \) a symmetric linear operator such that \( \Omega^2 = -\text{Id}_V \). Then for any \( w \in V \) with \( \langle w, w \rangle_V = 1 \) there is \( \lambda \in \mathbb{R} \) such that \( \bar{w} = w + \lambda \Omega(w) \) satisfies:
\[
\langle \bar{w}, \Omega \bar{w} \rangle_V = 0, \quad \text{and} \quad \langle \bar{w}, \bar{w} \rangle_V = 1.
\]

**Proof.** First we claim that \( \langle \Omega w, \Omega w \rangle_V = -\langle w, w \rangle_V \). Indeed
\[
\langle \Omega w, \Omega w \rangle_V = \langle \Omega^2 w, w \rangle_V = -\langle w, w \rangle_V.
\]

Let \( w \in V \) be a vector such that \( \langle w, w \rangle_V = 1 \) and assume \( \langle w, \Omega w \rangle_V = a \). If \( a = 0 \), then we choose \( \lambda = 0 \). Thus, we can assume that \( \langle w, \Omega w \rangle_V = a \not= 0 \). Then, by solving the equation
\[
\langle \bar{w}, \Omega \bar{w} \rangle_V = \langle w + \lambda \Omega w, -\lambda w + \Omega w \rangle_V = -\lambda + a - \lambda^2 a - \lambda = - (a \lambda^2 + 2 \lambda - a) = 0,
\]
we find the solutions of this equation as
\[
\lambda = -\frac{1}{a} \pm \sqrt{1 + \frac{1}{a^2}} \quad \text{for} \quad a \not= 0.
\]

For this \( \lambda \) we get
\[
\langle \bar{w}, \bar{w} \rangle_V = \langle w + \lambda \Omega w, w + \lambda \Omega w \rangle_V = 1 - \lambda^2 + 2 \lambda a = 2 \left( a + \frac{1}{a} \right) \left( -\frac{1}{a} \pm \sqrt{1 + \frac{1}{a^2}} \right) = 2 \lambda \frac{a^2 + 1}{a} \not= 0.
\]
If $a > 0$ then we choose $\lambda = -\frac{1}{a} + \sqrt{1 + \frac{1}{a^2}} > 0$ and if $a < 0$ then we choose $\lambda = -\frac{1}{a} - \sqrt{1 + \frac{1}{a^2}} < 0$. This choice makes the product $\langle \hat{w}, \hat{w} \rangle_V$ strictly positive. Normalizing $\hat{w}$ we get $\langle \hat{w}, \hat{w} \rangle = 1$. □

The next lemma is a generalization of the previous one.

**Lemma 4.** Let $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible module, $\Omega_1, \ldots, \Omega_l$ symmetric linear operators on $V$ such that

1) $\Omega_k^2 = -\text{Id}_V$, $k = 1, \ldots, l$;
2) $\Omega_k \Omega_j = -\Omega_j \Omega_k$ for all $k, j = 1, \ldots, l$.

Then for any $w \in V$ with $\langle w, w \rangle_V = 1$ there is a vector $\tilde{w}$ satisfying:

$$\langle \tilde{w}, \Omega_k \tilde{w} \rangle_V = 0, \quad \text{and} \quad \langle \tilde{w}, \tilde{w} \rangle_V = 1, \quad k = 1, \ldots, l.$$  

**Proof.** Notice that symmetry of operators and property 2) imply

$$\langle \Omega_k v, \Omega_j v \rangle_V = 0$$

for any $v \in V$. Indeed $\langle \Omega_k v, \Omega_j v \rangle_V = \langle \Omega_j v, \Omega_k v \rangle_V = -\langle \Omega_k v, \Omega_j v \rangle_V = -\langle \Omega_j v, \Omega_k v \rangle_V$, that shows (8).

To prove Lemma 4, we choose $w \in V$ with $\langle w, w \rangle_V = 1$. Apply Lemma 3 to $w$ and $\Omega_1$ and construct $w_1 \in V$ such that

$$\langle w_1, w_1 \rangle_V = 1, \quad \langle w_1, \Omega_1 w_1 \rangle_V = 0.$$  

Then we define $w_2 = w_1 + \lambda_2 \Omega_2 w_1$ and find that for suitable $\lambda_2 \in \mathbb{R}$ we have

$$\langle w_2, w_2 \rangle_V = 1, \quad \langle w_2, \Omega_2 w_2 \rangle_V = 0.$$  

Moreover,

$$\langle w_2, \Omega_1 w_2 \rangle_V = \langle w_1, \Omega_1 w_1 \rangle_V + \lambda_2 \langle w_1, \Omega_1 \Omega_2 w_1 \rangle_V + \lambda_2 \langle \Omega_2 w_1, \Omega_1 w_1 \rangle_V + \lambda_2^2 \langle \Omega_2 w_1, \Omega_1 \Omega_2 w_1 \rangle_V = 0$$

by assumptions of Lemma 4 and properties (9) and (10). Now, applying Lemma 3 we assume that the vector $w_k = w_{k-1} + \lambda_k \Omega_k w_{k-1}$, $2 < k < l$, is chosen and satisfies

$$\langle w_k, w_k \rangle_V = 1, \quad \langle \Omega_k w_k, w_k \rangle_V = 0,$$

where it was shown that

$$\langle w_{k-1}, w_{k-1} \rangle_V = 1, \quad \langle w_{k-1}, \Omega_j w_{k-1} \rangle_V = 0, \quad j = 1, \ldots, k - 1.$$  

Then

$$\langle w_k, \Omega_j w_k \rangle_V = \langle w_{k-1}, \Omega_j w_{k-1} \rangle_V + \lambda_k \langle w_{k-1}, \Omega_j \Omega_k w_{k-1} \rangle_V + \lambda_k \langle \Omega_k w_{k-1}, \Omega_j w_{k-1} \rangle_V + \lambda_k^2 \langle \Omega_k w_{k-1}, \Omega_j \Omega_k w_{k-1} \rangle_V = 0,$$

for any $j = 1, \ldots, k - 1$.

by (9) and assumption on operators $\Omega_j$. Denoting the last vector $w_l$ by $\tilde{w}$, we finish the proof of Lemma 4. □

**Corollary 3.** Let $(V, \langle \cdot, \cdot \rangle_V)$ and $\Omega_1, \ldots, \Omega_l$ satisfies the conditions of Lemma 4. Let $P$ be a linear operator on $V$ such that $P^2 = \text{Id}_V$, and $P\Omega_k = \Omega_k P$, $k = 1, \ldots, l$. If $w \in V$ satisfies $Pw = w$ and $\langle w, w \rangle_V = 1$, then the vector $\tilde{w}$ constructed in Lemma 4 is also eigenvector of $P$: $P\tilde{w} = \tilde{w}$.

**Proof.** Let $w \in V$ satisfies $Pw = w$ and $\langle w, w \rangle_V = 1$. Then for the vector $w_1 = w + \lambda_1 \Omega_1 w$ we calculate $Pw_1 = Pw + \lambda_1 P\Omega_1 w = w + \lambda_1 \Omega_1 Pw = w_1$. Thus, we proceed further by induction and prove the Corollary.

□

One of the principal parts in our construction is a presence of an operator having orthogonal eigenspaces. The following lemma describes some of them.
Lemma 5. Let \((V, \langle \cdot, \cdot \rangle_V)\) be a neutral scalar product space. Assume that an involution \(P : V \to V\) is either symmetric or isometric operator. We denote its eigenspaces \(E_+ = \{ v \in V \mid Pv = v \}, \ E_- = \{ v \in V \mid Pv = -v \}\). Then the decomposition \(P = E_+ \oplus E_-\) is orthogonal with respect to \(\langle \cdot, \cdot \rangle_V\).

Proof. Let \(w \in E_+\) and \(v \in E_-\) be arbitrary vectors. If involution \(P\) is symmetric then we argue as follows \(\langle w, v \rangle_V = \langle Pw, v \rangle_V = \langle w, Pv \rangle_V = -\langle w, v \rangle_V\).

Let \(P\) be an isometry. Then \(\langle w, v \rangle_V = \langle Pw, Pv \rangle_V = -\langle w, v \rangle_V\), where in the first equality we used the isometry property of \(P\) and in the second the definition of eigenvectors. \(\square\)

From now on we fix the notation \(E_\pm\) for eigenspaces of an involution \(P\) corresponding to eigenvalues \pm 1.

Lemma 6. Let \((V, \langle \cdot, \cdot \rangle_V)\) be a neutral scalar product space and \(P : V \to V\) be an isometric involution. Then we have the following cases.

1) If a linear map \(T : V \to V\) is an isometry such that \(PT = -TP\), then each of eigenspaces \(E_\pm\) of \(P\) is a neutral scalar product space with respect to the restriction of the scalar product \(\langle \cdot, \cdot \rangle_V\) on each of \(E_\pm\).

2) If a linear map \(T : V \to V\) is an anti-isometry such that \(PT = -TP\), then the restriction of \(\langle \cdot, \cdot \rangle_V\) on each of \(E_\pm\) is non-degenerate neutral or sign definite.

3) If a linear map \(T : V \to V\) is an anti-isometry such that \(PT = TP\), then the restriction of \(\langle \cdot, \cdot \rangle_V\) on each of \(E_\pm\) is non-degenerate neutral.

Proof. To show the first statement we observe that the isometry \(T\) acts as an isometry from \(E_+\) to \(E_-\). Since the eigenspaces \(E_\pm\) are orthogonal, the scalar product \(\langle \cdot, \cdot \rangle_{E_+}\) restricted to each of \(E_\pm\) is non-degenerate. If the scalar product restricted to \(E_+\) would be positive definite, then the scalar product restricted to \(E_-\) would be also positive definite, since the map \(T\) is an isometry that contradicts the assumption that space \((V, \langle \cdot, \cdot \rangle_V)\) is neutral. The same arguments shows that the restriction to \(E_+\) could not be negative definite. So the scalar product restricted to \(E_+\) and therefore to \(E_-\) should be neutral.

In order to prove the second statement, we observe that since \(T : E_+ \to E_-\) is an anti-isometry, the restriction of \(\langle \cdot, \cdot \rangle_V\) to \(E_+\) can be sign definite and the restriction of \(\langle \cdot, \cdot \rangle_V\) to \(E_-\) will have opposite sign due to neutral nature of \((V, \langle \cdot, \cdot \rangle_V)\).

In the third case since the eigenspaces \(E_\pm\) are invariant under \(T\) but contains spacelike and timelike vectors, they decompose into subspaces of equal dimensions where the restriction of \(\langle \cdot, \cdot \rangle_V\) sign definite but of opposite signs. \(\square\)

Note that assumptions made in the case 2 of Lemma 6 does not guarantee the existence of \(w \in V\) such that

\[
Pw = w, \quad \langle w, w \rangle_V = 1.
\]

The following lemma contains a benchmark example for our work, describing one of possible solutions of this problem.

Lemma 7. Let \((V, \langle \cdot, \cdot \rangle_V)\) be an admissible \(\text{Cl}_{r,s}\)-module, \(z_1, \ldots, z_{r+s}\) orthonormal generators of the Clifford algebra \(\text{Cl}_{r,s}\), and \(J_{z_i} \in \text{End}(V)\), \(i = 1, \ldots, r + s\) are representations for the Clifford algebra. Assume that \(P = J_{z_{i_1}}J_{z_{i_2}}J_{z_{i_3}}J_{z_{i_4}}, i_1 \neq i_2 \neq i_3 \neq i_4\), is an isometric involution and \(T : V \to V\) is an anti-isometry such that \(PT = -TP\). Then there is a vector \(w \in V\) satisfying (10) or \(P\) can be modified to other isometric involution \(P\) such that (10) holds for \(P\).

Proof. First we notice that operator \(P = J_{z_{i_1}}J_{z_{i_2}}J_{z_{i_3}}J_{z_{i_4}}, i_1 \neq i_2 \neq i_3 \neq i_4\) is also symmetric.

We apply Lemma 6 item 2). If the restriction of \(\langle \cdot, \cdot \rangle_V\) on \(E_+\) is positive definite or neutral, then we are done. If the restriction is negative definite, then we define the operator \(\hat{P} = J_{z_{i_2}}J_{z_{i_3}}J_{z_{i_4}}\) and denote by \(\hat{E}_\pm\) its eigenspaces corresponding to eigenvalues \pm 1. Thus if \(w \in E_+\), then \(Pw =\)
−Pw = −w and therefore w ∈ E_. Continuing to argue in the same way, we conclude that E_ = E_ and E_ = E_. So, we change the operator P and its eigenspaces E_ to the operator P and the corresponding eigenvectors E_ to satisfy \(10\).

To ensure existence of w ∈ V satisfying \(10\) for a general isometric involution P we need to have one more operator commuting with P. More precisely we state the following generalisation of Lemma 6.

**Lemma 8.** Let \((V, \langle \cdot, \cdot \rangle_V)\) be a neutral scalar product space. Let P be an isometric involution and assume that there are two anti-isometric operators \(T_i: V \to V\), \(i = 1, 2\), such that

\[T_1 P = -PT_1, \quad \text{and} \quad T_2 P = PT_2.\]

Then the eigenspaces \(E_\pm\) of P are non-trivial and the scalar product \(\langle \cdot, \cdot \rangle_V\) restricted to each of \(E_\pm\) is non-degenerate and neutral.

**Proof.** We only need to explain why the scalar product \(\langle \cdot, \cdot \rangle_V\) restricted to each of \(E_\pm\) is neutral. The non-degeneracy of the restriction of \(\langle \cdot, \cdot \rangle_V\) to \(E_\pm\) is shown as in Lemma 6.

The presence of the operator \(T_1\) ensures that the restriction of \(\langle \cdot, \cdot \rangle_V\) to \(E_\pm\) is neutral or sign definite. Actually the restriction of \(\langle \cdot, \cdot \rangle_V\) to \(E_\pm\) can not be sign definite. Indeed, since \(T_2\) preserves \(E_+\) and it is an anti-isometry, the space \(E_+\) contains both spacelike and timelike vectors forming subspaces of an equal dimension as was shown in the proof of Proposition 1. The same arguments, applied to \(E_-,\) finish the proof. □

**Definition 2.** Let \(P_1, \ldots, P_m\) be isometric mutually commuting involutions defined on a neutral scalar product space \((V, \langle \cdot, \cdot \rangle_V)\). Then the collection \(T_{m+1}\) of linear operators on \(V\) is called complementary operators to the family \(P_1, \ldots, P_m\) if

\[
P_1 T_1 = -T_1 P_1, \quad P_1 T_2 = T_2 P_1, \quad \ldots \quad P_1 T_m = T_m P_1, \quad P_1 T_{m+1} = T_{m+1} P_1,
\]

\[
P_2 T_2 = -T_2 P_2, \quad \ldots \quad P_2 T_m = T_m P_2, \quad P_2 T_{m+1} = T_{m+1} P_2,
\]

\[\vdots\]

\[
P_m T_m = -T_m P_m, \quad P_m T_{m+1} = T_{m+1} P_m.
\]

**Remark 1.** In some situations the operator \(T_{m+1}\) can be omitted, but we still call the system of operators \(T_1, \ldots, T_m\) complementary to \(P_1, \ldots, P_m\).

Based on Lemmas 6 and 8 we construct an integral structure by giving an explicit simultaneous eigenspace decomposition of a given admissible module by a family of isometric involutions and their complementary operators. Simultaneously, we calculate the dimension of the minimal admissible modules for all cases.

## 3. Isomorphism Preserving Admissibility

There are several types of isomorphisms between Clifford algebras. Among them the periodicity with the period 8

\[
\text{Cl}_{r+8, s} \cong \text{Cl}_{r+4, s+4} \cong \text{Cl}_{r, s} \otimes \mathbb{R}(16), \quad \text{Cl}_{r, s+8} \cong \text{Cl}_{r+4, s+4} \cong \text{Cl}_{r, s} \otimes \mathbb{R}(16)
\]

are basic and used to construct an integral basis for all the cases \(\text{Cl}_{r, s}\) after we prove the existence of the integral basis for \(\text{Cl}_{r, s}\) of \(0 \leq r, s \leq 7\) and \(\text{Cl}_{8, 0} \cong \mathbb{R}(8), \text{Cl}_{0, 8} \cong \mathbb{R}(8),\) see Theorems 5, 6 and 7.

Not all isomorphisms of the Clifford algebras lead to the isometric admissible modules, for instance, the isomorphism \(\text{Cl}_{r,s+4} \cong \text{Cl}_{8,r+4}\) does not preserve the admissibility in general, since in particular, the isomorphism \(\text{Cl}_{0,4} \cong \text{Cl}_{4,0}\) does not directly give us a required scalar product from the positive one for \(\text{Cl}_{4,0}\)-module to a neutral one for \(\text{Cl}_{0,4}\)-module. In this section we show that the isomorphism \(\text{Cl}_{r,s+1} \cong \text{Cl}_{s,r+1}\) preserves the admissibility.
We recall an isomorphism between $\text{Cl}_{r,s+1}$ and $\text{Cl}_{s,r+1}$. It is given as follows: let $z_1, \ldots, z_r$, $\zeta_1, \ldots, \zeta_{s+1}$ be the orthonormal generators of $\text{Cl}_{r,s+1}$ with the property
$$\langle z_i, z_i \rangle_{\mathbb{R}^{r,s+1}} = 1, \quad \langle \zeta_j, \zeta_j \rangle_{\mathbb{R}^{r,s+1}} = -1.$$ Likewise let $a_1, \ldots, a_s, b_1, \ldots, b_{r+1}$ be orthonormal generators of $\text{Cl}_{s,r+1}$ with
$$\langle a_i, a_i \rangle_{\mathbb{R}^{s,r+1}} = 1, \quad \langle b_j, b_j \rangle_{\mathbb{R}^{s,r+1}} = -1.$$ We define a correspondence $\Phi: \mathbb{R}^{r,s+1} \to \text{Cl}_{s,r+1}$ by
$$\begin{align*}
&z_1 \mapsto b_1 b_{r+1}, \quad \zeta_1 \mapsto a_1 b_{r+1}, \\
&z_2 \mapsto b_2 b_{r+1}, \quad \ldots \quad \ldots, \\
&z_r \mapsto b_r b_{r+1}, \quad \zeta_{s+1} \mapsto b_{r+1}.
\end{align*}$$
Then we have $\Phi(z_i)^2 = -1$, $\Phi(\zeta_j)^2 = 1$ and
$$\Phi(z_i)\Phi(\zeta_j) + \Phi(\zeta_j)\Phi(z_i) = 0 \quad \text{for any} \quad i = 1, \ldots, r, \quad j = 1, \ldots, s,$$ and
$$\Phi(z_{i_1})\Phi(z_{i_2}) + \Phi(z_{i_2})\Phi(z_{i_1}) = 0, \quad \Phi(\zeta_{j_1})\Phi(\zeta_{j_2}) + \Phi(\zeta_{j_2})\Phi(\zeta_{j_1}) = 0 \quad \text{for any} \quad i_k \neq i_l, \quad j_k \neq j_l.$$ Since the vectors $\{\Phi(z_i), \Phi(\zeta_j)\}_{i,j}$ are linearly independent, one can extend the isomorphism to an isomorphism of the Clifford algebras $\text{Cl}_{r,s+1}$ and $\text{Cl}_{s,r+1}$.

**Theorem 3.** Let $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible module of the Clifford algebra $\text{Cl}_{s,r+1}$ and $J: \text{Cl}_{s,r+1} \to \text{End}(V)$ its representation. Then the Clifford module $(V, \langle \cdot, \cdot \rangle_V)$ with representation
$$J \circ \Phi: \text{Cl}_{r,s+1} \to \text{End}(V)$$
is admissible.

**Proof.** The skew symmetry condition
$$\langle J_\Phi(z_i), v \rangle_V + \langle v, J_\Phi(z_i) \rangle_V = 0 \quad \text{for any} \quad v, w \in V$$holds by the following
$$\langle J_\Phi(z_i), v \rangle_V = \langle J_b, J_{b_{r+1}} v, w \rangle_V = -\langle J_{b_{r+1}}, J_b w \rangle_V = \langle v, J_{b_{r+1}} J_b w \rangle_V = -\langle v, J_b J_{b_{r+1}} w \rangle_V = -\langle v, J_\Phi(z_i) w \rangle_V \quad \text{for} \quad i \leq r.$$ The rest of the cases can be shown in a similar way. \Box

**Corollary 4.** If an integral basis for an admissible $\text{Cl}_{s,r+1}$-module satisfies the assumptions of Lemma 2 and Corollary 2, then it is also an integral basis of the admissible $\text{Cl}_{r,s+1}$-module.

**Proof.** Indeed, let $\{v_\alpha\}$ be an integral basis for $\text{Cl}_{s,r+1}$-module, then
$$\langle J_\Phi(z_i) v_\alpha, v_\beta \rangle_V = \langle J_b, J_{b_{r+1}} v_\alpha, v_\beta \rangle_V = -\langle J_{b_{r+1}}, J_b v_\alpha, v_\beta \rangle_V = \pm 1 \text{ or } 0,$$since the vectors $J_{b_{r+1}} v_\alpha$ and $J_b v_\beta$ are also ones of the basis vectors up to sign by the assumption. \Box

This Lemma 2 and Corollary 4 allows to reduce the construction of integral bases from 64 to 42 cases, and we shall construct such bases for the Clifford algebras
$$\text{Cl}_{r,s}, \quad \text{for} \quad 0 \leq r \leq s \leq 7 \quad \text{and} \quad \text{Cl}_{r,r+1} \quad \text{for} \quad r = 0, \ldots, 6.$$

4. **Integral structure on admissible $\text{Cl}_{0,s}$-modules**

In this section we present admissible integral $\text{Cl}_{0,s}$-modules for $s = 1, \ldots, 8$, constructing them directly.
4.1. Integral structure on admissible $\text{Cl}_{0,1}$-module. In this case the Clifford algebra is isomorphic to the space $\mathbb{R} \oplus \mathbb{R}$, where the isomorphism is given by

$$1_{\text{Cl}_{0,1}} \mapsto (1, 1), \quad z_1 \mapsto (1, -1)$$

and then it is continued to the algebra isomorphism. The generator $z_1$ of $\text{Cl}_{0,1}$ satisfies $\langle z_1, z_1 \rangle_{\mathbb{R}^0,1} = -1$. Let $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible module and the endomorphism $J_{z_1}$ be such that $J_{z_1}^2 = \text{Id}_V$. We pick up $w \in V$ with $\langle w, w \rangle_V = 1$. Then $\langle J_{z_1}w, w \rangle_V = 0$, that gives us two linearly independent vectors. We can choose the basis

$$v_1 = w, \quad v_2 = J_{z_1}w.$$ 

Then, since

$$\langle v_1, v_1 \rangle_V = 1, \quad \langle v_2, v_2 \rangle_V = \langle z_1, z_1 \rangle_{\mathbb{R}^0,1} \langle w, w \rangle_V = -1$$

the basis is orthonormal and we obtain one non-vanishing relation $\langle J_{z_1}v_1, v_2 \rangle_V = -1$. It gives the following commutation relation

$$\langle J_{z_1}v_1, v_2 \rangle_V = \langle z_1, [v_1, v_2] \rangle_{\mathbb{R}^0,1} \implies [v_1, v_2] = z_1.$$

This Lie algebra $\langle \mathbb{R}^{1,1} \oplus \mathbb{R}^{0,1}, [\cdot, \cdot] \rangle$ is the Heisenberg algebra. The natural choice of the metric of index $(1,1)$ on the space $V \cong \mathbb{R}^{1,1}$ leads to studying of the Lorentzian Heisenberg groups, see [15, 22]. The constructed admissible module is of minimal dimension, but it is not irreducible, since the irreducible module has dimension 1.

Remark 2. Let $u = w + J_{z_1}w$, then $u \neq 0$ and $\langle u, u \rangle_V = 0$, also $J_{z_1}u = u$. Hence the subspace $W = \text{span} \{u\}$ of $V$ is a $\text{Cl}_{0,1}$-module with the trivial scalar product. So, in general a sub-module of an admissible $\text{Cl}_{r,s}$-module need not be an admissible module, as was observed at the beginning of Section 3.

Remark 3. The Clifford algebra $\text{Cl}_{1,0}$ is isomorphic to the field $\mathbb{C}$ of complex numbers and the corresponding $H$-type Lie algebra $(\mathbb{R}^{2,0} \oplus \mathbb{R}^{1,0}, [\cdot, \cdot])$ is also three dimensional Heisenberg algebra with the Euclidean metric as a natural choice on the horizontal space. We see that different Clifford algebras $\text{Cl}_{1,0}$ and $\text{Cl}_{0,1}$ lead to different general $H$-type Lie algebras, having isomorphic underlying Lie algebras, if we discard the presence of the metric.

4.2. Integral structure on admissible $\text{Cl}_{0,2}$-module. The Clifford algebra $\text{Cl}_{0,2}$ is isomorphic to the space $\mathbb{R}(2)$ of $(2 \times 2)$-matrices with real entries. Let $z_1, z_2$ be generators of $\text{Cl}_{0,2}$ with $\langle z_1, z_1 \rangle_{\mathbb{R}^0,2} = -1$ and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{0,2}$-module. Then $J_{z_1}^2 := J_{z_2}^2 = \text{Id}_V$. Choose $w \in V$ such that $\langle w, w \rangle_V = 1$. The basis

$$v_1 = w, \quad v_2 = J_{z_1}J_{z_1}w, \quad v_3 = J_{z_1}w, \quad v_4 = J_{z_2}w$$

is orthonormal and satisfies $\langle v_1, v_1 \rangle_V = \langle v_2, v_2 \rangle_V = -\langle v_3, v_3 \rangle_V = -\langle v_4, v_4 \rangle_V = 1$. This implies that the scalar product restricted to the subspace spanned by the four vectors $\{v_1, v_2, v_3, v_4\}$ is non-degenerate. Moreover, since the action of $J_j$, $j = 1, 2$, on the basis $\{v_i\}_{i=1}^4$ permutes this basis up to sign, the basis is integral that gives the following non-vanishing commutation relations

$$[v_1, v_3] = [v_2, v_4] = z_1, \quad [v_1, v_4] = -[v_2, v_3] = z_2.$$

Remark 4. Let $u_1 = v_1 + v_3$ and $u_2 = v_2 - v_4$, then

$$J_{z_1}u_1 = u_1, \quad J_{z_1}u_2 = -u_2, \quad J_{z_2}u_1 = u_2, \quad J_{z_2}u_2 = -u_1.$$ 

The scalar product on the subspace $W = \text{span} \{u_1, u_2\}$ vanishes, so the irreducible sub-module $W$ is not admissible.

We emphasise that the construction of admissible integral modules sometimes gives the irreducible admissible integral module, but in other cases the resulting module exceeds the dimension of the irreducible module twice.
4.3. Integral structure on admissible $\Cl_{0,3}$-module. The Clifford algebra $\Cl_{0,3}$ is isomorphic to the space $\mathbb{C}(2)$ of $(2 \times 2)$-matrices with complex entries. Let $z_1, z_2, z_3$ be generators of $\Cl_{0,3}$ with $\langle z_i, z_i \rangle_{\mathbb{R}^{0,3}} = -1, i = 1, 2, 3,$ and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\Cl_{0,3}$-module. Then $J_i^2 := J_i^2 = \text{Id}_V, i = 1, 2, 3$. Choose $w \in V$ such that $\langle w, w \rangle_V = 1$. In general, the scalar product $\langle w, J_1J_2J_3w \rangle_V$ does not vanish. Nevertheless, since the operator $\Omega = J_1J_2J_3$ satisfies the conditions of Lemma 3 we can choose the initial vector making the product $\langle w, J_1J_2J_3w \rangle_V$ equal to 0. Fix such a vector $w \in V$ and pick up the vectors

$$v_1 = w, \quad v_2 = J_1J_2w, \quad v_3 = J_1J_3w, \quad v_4 = J_2J_3w,$$

$$v_5 = J_1w, \quad v_6 = J_2w, \quad v_7 = J_3w, \quad v_8 = J_1J_2J_3w,$$

which satisfy $\langle v_\alpha, v_\alpha \rangle_V = 1, \alpha = 1, 2, 3, 4$ and $\langle v_\alpha, v_\alpha \rangle_V = -1$ for $\alpha = 5, 6, 7, 8$. Moreover, since we have

$$0 = \langle v_1, v_3 \rangle_V = \langle v_2, v_7 \rangle_V = \langle v_3, v_6 \rangle_V = -\langle v_4, v_5 \rangle_V$$

the basis $\{v_1, \ldots, v_8\}$ is orthonormal with respect to the scalar product $\langle \cdot, \cdot \rangle_V$ by the choice of initial vector $w$ and admissibility of the scalar product.

It is clear that the action of $J_j, j = 1, 2, 3$, on the basis $\{v_\alpha\}_{\alpha = 1}^8$ permutes it up to the sign. Thus we conclude $\langle J_jv_\alpha, v_\beta \rangle_V = \pm 1$ or $\langle J_jv_\alpha, v_\beta \rangle_V = 0$ for any $j = 1, 2, 3$ and any $\alpha, \beta = 1, \ldots, 8$. Precisely, we obtain the following non-vanishing commutators

$$[v_1, v_5] = [v_2, v_6] = [v_3, v_7] = [v_4, v_8] = z_1, \quad [v_1, v_6] = [v_2, v_5] = -[v_3, v_8] = [v_4, v_7] = z_2$$

We apply Proposition 1 part (2) and note, that $W = \text{span} \{v_1, \ldots, v_8\}$ is invariant under the action of the algebra $\Cl_{0,3}$ and the scalar product $\langle \cdot, \cdot \rangle_V$ restricted to $W$ (we denote it by $\langle \cdot, \cdot \rangle_W$) is non-degenerate. So we constructed an admissible sub-module $(W, \langle \cdot, \cdot \rangle_W)$. The orthogonal complement $W^\perp = \{x \in V \mid \langle x, v \rangle_V = 0 \text{ for any } v \in W\}$ is also invariant under the action of the Clifford algebra $\Cl_{0,3}$ and the scalar product restricted to $W^\perp$ is non-degenerate. This procedure implies that the given admissible $\Cl_{0,3}$-module $(V, \langle \cdot, \cdot \rangle_V)$ is decomposed into a finite sum of the minimal 8-dimensional admissible modules $(W, \langle \cdot, \cdot \rangle_W)$ of $\Cl_{0,3}$.

4.4. Integral structure on admissible $\Cl_{0,4}$-module. The Clifford algebra $\Cl_{0,4}$ is isomorphic to the space $\mathbb{H}(2)$. Let $z_j, j = 1, 2, 3, 4$ be generators of $\Cl_{0,4}$ with $\langle z_j, z_j \rangle_{\mathbb{R}^{0,4}} = -1$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\Cl_{0,4}$-module. Then $J_j^2 := J_j^2 = \text{Id}_V, j = 1, 2, 3, 4$. The operator $P = J_1J_2J_3J_4$ is isometric involution and $T = J_1$ is an anti-isometry such that $PT = -TP$. Applying Lemma 7 we find $w \in V$ such that $Pw = w$ and $\langle w, w \rangle_V = 1$. Then we get relations

$$J_1w = J_2J_3J_4w, \quad J_2w = -J_1J_3J_4w, \quad J_3w = J_1J_2J_4w, \quad J_4w = -J_1J_2J_3w,$$

$$J_1J_2w = -J_3J_4w, \quad J_1J_3w = J_2J_4w, \quad J_1J_4w = -J_2J_3w,$$

that allows us to choose 8 orthogonal vectors

$$v_1 = w, \quad v_2 = J_1J_2w, \quad v_3 = J_1J_3w, \quad v_4 = J_1J_4w,$$

$$v_5 = J_1w, \quad v_6 = J_2w, \quad v_7 = J_3w, \quad v_8 = J_4w,$$

satisfying $\langle v_\alpha, v_\alpha \rangle_V = 1, \alpha = 1, 2, 3, 4$ and $\langle v_\alpha, v_\alpha \rangle_V = -1$ for $\alpha = 5, 6, 7, 8$.

The relations 111 shows that the action of $J_j, j = 1, 2, 3, 4$, on the basis $\{v_\alpha\}_{\alpha = 1}^8$ permutes elements of the basis up to sign. We conclude $\langle J_jv_\alpha, v_\beta \rangle_V = \pm 1$ or 0 for $\alpha, \beta = 1, \ldots, 8$.

The subspace spanned by vectors 12 is invariant under the action of the Clifford algebra $\Cl_{0,4}$. Using Proposition 11 (2) we conclude that constructed an admissible sub-module $W = \text{span} \{v_1, \ldots, v_8\}$, is of minimal dimension admissible integral module. This module is irreducible.
4.5. Integral structure on admissible $\text{Cl}_{0,5}$-module. The Clifford algebra $\text{Cl}_{0,5}$ is isomorphic to the direct sum $\mathbb{H}(2) \oplus \mathbb{H}(2)$ of spaces of $(2 \times 2)$-matrices with quaternion entries. Let $z_j$, $j = 1, \ldots, 5$, be generators of $\text{Cl}_{0,5}$ with $\langle z_j, z_j \rangle_{\mathbb{R}^{0,5}} = -1$ and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{0,5}$-module. Then $J_j^2 := J_{z_j}^2 = \text{Id}_V$, $j = 1, \ldots, 5$. Consider the isometric involution $P_1 = J_1 J_2 J_3 J_4$ and operator $T = J_5$. Then the restriction of $\langle \cdot, \cdot \rangle_V$ to the eigenspace $E_+$ of the involution $P$ is neutral by Lemma 6. Thus we can find $w \in E_+$ such that $(w, w)_V = 1$.

Recalling the relations (11) and taking into account the invariance of $E_+$ under $J_5$, we present an eigenspace decomposition of the involution $P_1$.

| Involution | Eigenvector | Eigenvalues |
|------------|-------------|-------------|
| $P_1$      | $w, J_1 J_2 w, J_1 J_3 w, J_1 J_4 w$ | $+1$ |
|            | $J_5 w, J_1 J_2 J_5 w, J_1 J_3 J_5 w, J_1 J_4 J_5 w$ | $-1$ |

It allows to choose the linear independent elements

$$v_1 = w, \quad v_2 = J_1 J_2 w, \quad v_3 = J_1 J_3 w, \quad v_4 = J_1 J_4 w,$$
$$v_5 = J_5, \quad v_6 = J_2 J_5 w, \quad v_7 = J_3 J_5 w, \quad v_8 = J_4 J_5 w,$$
$$v_9 = J_1 w, \quad v_{10} = J_2 w, \quad v_{11} = J_3 w, \quad v_{12} = J_4 w,$$
$$v_{13} = J_5 w, \quad v_{14} = J_1 J_2 J_5 w, \quad v_{15} = J_1 J_3 J_5 w, \quad v_{16} = J_1 J_4 J_5 w.$$  

(13)

satisfying $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 8$ and $\langle v_\alpha, v_\alpha \rangle_V = -1$, $\alpha = 9, \ldots, 16$. Unfortunately not all the vectors in (13) are orthogonal for an arbitrary choice of $w \in V$. The possible non-vanishing relations are

$$\langle v_1, v_{14} \rangle_V = \langle v_2, v_{15} \rangle_V = \langle v_3, v_{16} \rangle_V = \langle v_4, v_{10} \rangle_V = \langle v_5, v_{11} \rangle_V = \langle v_6, v_{12} \rangle_V = a,$$
$$\langle v_7, v_{12} \rangle_V = -a,$$
$$\langle v_1, v_{15} \rangle_V = \langle v_2, v_{16} \rangle_V = \langle v_3, v_{11} \rangle_V = \langle v_4, v_{12} \rangle_V = b,$$
$$\langle v_5, v_{12} \rangle_V = -b,$$
$$\langle v_2, v_{14} \rangle_V = \langle v_3, v_{15} \rangle_V = \langle v_4, v_{11} \rangle_V = \langle v_5, v_{10} \rangle_V = c,$$
$$\langle v_6, v_{11} \rangle_V = -c.$$

If we denote $\Omega_1 = J_1 J_2 J_5$, $\Omega_2 = J_1 J_3 J_5$, and $\Omega_3 = J_1 J_4 J_5$, then we apply Lemma 4 and Corollary 4 and find $\tilde{w}$ making all the vectors in (13) orthogonal.

Moreover the relations (11) and Table 1 shows that the action of $J_j$, $j = 1, \ldots, 5$ permutes the basis up to sign. The constructed admissible integral sub-module $(W, \langle \cdot, \cdot \rangle_W)$, $W = \text{span} \{v_1, \ldots, v_{16}\}$, is of minimal dimension, but is not irreducible, since the irreducible module of the Clifford algebra $\text{Cl}_{0,5} \cong \mathbb{H}(2) \oplus \mathbb{H}(2)$ is of dimension 8.

4.6. Integral structure on admissible $\text{Cl}_{0,6}$-module. The Clifford algebra $\text{Cl}_{0,6}$ is isomorphic to the space $\mathbb{H}(4)$. Let $z_j$, $j = 1, \ldots, 6$ be orthogonal generators of $\text{Cl}_{0,6}$ with $\langle z_j, z_j \rangle_{\mathbb{R}^{0,6}} = -1$ and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{0,6}$-module. Then $J_j^2 := J_{z_j}^2 = \text{Id}_V$, $j = 1, \ldots, 6$. Consider two isometric and mutually commuting involutions $P_1 = J_1 J_2 J_3 J_4$ and $P_2 = J_1 J_2 J_5 J_6$. In order to construct a collection of complementary operators we present the table of commuting relations with generators $J_j$, $j = 1, \ldots, 6$.

| Commutation relations: $\text{Cl}_{0,6}$ case |
|---------------------------------------------|
| Involution | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ | $P_1 = J_1 J_2 J_3 J_4$ | $P_2 = J_1 J_2 J_5 J_6$ |
|------------|-----|-----|-----|-----|-----|-----|----------------------------|----------------------------|
| $P_1$      | a   | a   | a   | a   | c   | c   | a                          | a                          |
| $P_2$      | a   | a   | c   | c   | a   | a   | a                          | a                          |
Here “a” denotes that the corresponding operators anti-commute and “c” means that they are commuting. Then we give the table of complementary operators, where we use the sign (+ → −) to emphasise that the operator is an isometry and (+ → +) to point out the anti-isometric operator.

Complementary operator: $\text{Cl}_{0,6}$ case

| Involution \ Comp. op. | $J_1 (+ \to -)$ | $J_5 (+ \to -)$ | $J_2 J_3 J_5 (+ \to -)$ |
|------------------------|-----------------|-----------------|--------------------------|
| $P_1 = J_1 J_2 J_3 J_4$ | $a$             | $c$             | $c$                      |
| $P_2 = J_1 J_2 J_5 J_6$ | $a$             | $c$             |                          |

From these two tables we deduce, that since $P_1 J_1 = - J_1 P_1$ and $P_1 J_5 = J_5 P_1$ the eigenspaces $E_{1\pm}$ of the operator $P_1$ are neutral spaces by Lemma [8]. Then we apply Lemma [8] to the neutral space $E_{1+}$, the operator $P_2$ and anti-isometries $T_1 = J_5$, $T_2 = J_2 J_3 J_5$ and conclude that $E_{1+}$ is decomposed into two eigenspaces $E_{2\pm}$ of the operator $P_2$ which both are neutral. The same arguments are applied to the neutral space $E_{1-}$ and operators $P_2$ and $T_1 = J_5$, $T_2 = J_2 J_3 J_5$. Thus we can find $w \in E_{2+} \cap E_{1+} \subset V$ such that

$$P_1 w = J_1 J_2 J_3 J_4 w = w, \quad P_2 w = J_1 J_2 J_5 J_6 w = w, \quad \text{and} \quad \langle w, w \rangle_V = 1.$$  

Then we fix linear dependent vectors

$$
\begin{align*}
J_1 w &= J_2 J_3 J_4 w = J_2 J_5 J_6 w, & J_1 J_2 w &= J_3 J_4 w = J_5 J_6 w, \\
J_2 w &= - J_1 J_3 J_4 w = - J_1 J_5 J_6 w, & J_1 J_3 w &= J_2 J_4 w, \\
J_3 w &= J_1 J_2 J_4 w = - J_4 J_5 J_6 w, & J_1 J_4 w &= - J_2 J_3 w, \\
J_4 w &= - J_1 J_2 J_3 w = J_3 J_5 J_6 w, & J_1 J_5 w &= J_2 J_6 w, \\
J_5 w &= J_1 J_2 J_6 w = - J_4 J_5 J_6 w, & J_3 J_5 w &= - J_1 J_6 w, \\
J_6 w &= - J_1 J_2 J_5 w = J_3 J_4 J_5 w, & J_3 J_6 w &= J_4 J_5 w.
\end{align*}

(15)

$$
\begin{align*}
J_1 J_3 J_5 w &= - J_1 J_4 J_6 w = J_2 J_3 J_5 J_6 w = J_2 J_4 J_5 w, \\
J_1 J_3 J_6 w &= J_1 J_4 J_5 w = - J_2 J_3 J_5 w = J_2 J_4 J_6 w.
\end{align*}

We present the simultaneous eigenspace decomposition by the involutions $P_1$ and $P_2$ taking into account (14) and (15):

**Table 2. Eigenspace decomposition: $\text{Cl}_{0,6}$ case**

| Involution | Eigenvalues | $\pm 1$ | $\mp 1$ |
|------------|-------------|---------|---------|
| $P_1 = J_1 J_2 J_3 J_4$ | $+$ | $-1$ | $+1$ |
| $P_2 = J_1 J_2 J_5 J_6$ | $+$ | $-1$ | $+1$ |
| Eigenvector | $w$, $J_1 J_2 w$, $J_1 J_3 w$, $J_1 J_4 w$, $J_3 J_5 w$, $J_3 J_6 w$, $J_5 w$, $J_6 w$ |

We choose the vectors

$$
\begin{align*}
v_1 &= w, & v_2 &= J_1 J_2 w, & v_3 &= J_1 J_3 w, & v_4 &= J_1 J_4 w, \\
v_5 &= J_1 J_5 w, & v_6 &= J_1 J_6 w, & v_7 &= J_3 J_5 w, & v_8 &= J_3 J_6 w, \\
v_9 &= J_1 w, & v_{10} &= J_2 w, & v_{11} &= J_3 w, & v_{12} &= J_4 w, \\
v_{13} &= J_5 w, & v_{14} &= J_6 w, & v_{15} &= J_1 J_3 J_5 w, & v_{16} &= J_1 J_3 J_6 w,
\end{align*}

(16)

satisfying $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 8$ and $\langle v_\alpha, v_\alpha \rangle_V = -1$, $\alpha = 9, \ldots, 16$. Since not all of them can be orthogonal, we apply Lemma [9] to operators $\Omega_1 = J_1 J_3 J_5$, and $\Omega_2 = J_1 J_3 J_6$ in order to choose correct vector $w \in E_{1+} \cap E_{2+} \subset V$. We need to use the orthogonalisation only for these operators, since other triplets are linear dependent with $\Omega_1$ and $\Omega_2$. Thus the basis (16) is orthonormal and the action of any $J_j$, $j = 1, \ldots, 6$ permutes it up to sign, by (15). We conclude that the constructed 16 dimensional sub-module $(W, \langle \cdot, \cdot \rangle_W)$ is of minimal dimension. Moreover it is irreducible.
4.7. Integral structure on admissible $\text{Cl}_{0,7}$-module. The Clifford algebra $\text{Cl}_{0,7}$ is isomorphic to the space $\mathbb{C}(8)$. Let $z_j$, $j = 1, \ldots, 7$, be generators of $\text{Cl}_{0,7}$ with $\langle z_j, z_j \rangle_{\text{Cl}_{0,7}} = -1$ and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{0,7}$-module. Then $J_j^2 := J_j^2 = \text{Id}_V$, $j = 1, \ldots, 7$. In this case we fix three mutually commuting symmetric isometric involutions $P_1 = J_1 J_2 J_3 J_4$, $P_2 = J_1 J_2 J_5 J_6$ and $P_3 = J_2 J_3 J_6 J_7$. This gives the table of commuting relations with generators and the collection of complementary operators.

Commutation relations: $\text{Cl}_{0,7}$ case

| Involution \ Generator | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ | $J_7$ |
|------------------------|-------|-------|-------|-------|-------|-------|-------|
| $P_1 = J_1 J_2 J_3 J_4$ | $a$   | $a$   | $a$   | $a$   | $c$   | $c$   | $c$   |
| $P_2 = J_1 J_2 J_5 J_6$ | $a$   | $a$   | $c$   | $c$   | $a$   | $a$   | $c$   |
| $P_3 = J_2 J_3 J_6 J_7$ | $c$   | $a$   | $a$   | $c$   | $a$   | $a$   | $c$   |

Complementary operator: $\text{Cl}_{0,7}$ case

| Involution \ Comp. op. \ $J_1(+ \to -)$ | $J_5(+ \to -)$ | $J_7(+ \to -)$ | $J_5 J_6 J_7(+ \to -)$ |
|-----------------------------------------|----------------|----------------|-------------------------|
| $P_1 = J_1 J_2 J_3 J_4$                | $a$            | $c$            | $c$                     |
| $P_2 = J_1 J_2 J_5 J_6$                | $a$            | $c$            | $c$                     |
| $P_3 = J_2 J_3 J_6 J_7$                | $a$            | $c$            | $c$                     |

From these relations, applying Lemma 3, we can choose a vector $w \in V$ such that

$$ P_1 w = w, \quad P_2 w = w, \quad P_3 w = w, \quad \text{and } \langle w, w \rangle_V = 1, $$

since the common eigenspaces of all three involutions are all neutral spaces. Then we have a simultaneous eigenspace decomposition of a subspace $W \subset V$ spanned by the 16 eigenvectors

$$ v_1 = w, \quad v_2 = J_1 J_3 w, \quad v_3 = J_1 J_3 w, \quad v_4 = J_1 J_4 w, $$

$$ v_5 = J_1 J_5 w, \quad v_6 = J_1 J_6 w, \quad v_7 = J_1 J_7 w, \quad v_8 = J_3 J_6 w, $$

$$ v_9 = J_1 w, \quad v_{10} = J_2 w, \quad v_{11} = J_3 w, \quad v_{12} = J_4 w, $$

$$ v_{13} = J_5 w, \quad v_{14} = J_6 w, \quad v_{15} = J_7 w, \quad v_{16} = J_1 J_3 J_6 w, $$

satisfying $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 8$ and $\langle v_\alpha, v_\alpha \rangle_V = -1$, $\alpha = 9, \ldots, 16$. Where we used (17) and Table 3. Eigenspace decomposition: $\text{Cl}_{0,7}$ case

| Involutions | $P_1$ | $P_2$ | $P_3$ | $J_1 J_6 w$ | $J_1 J_7 w$ | $J_2 J_6 w$ | $J_2 J_7 w$ | $J_3 J_6 w$ | $J_3 J_7 w$ | $J_4 J_6 w$ | $J_4 J_7 w$ | $J_5 J_6 w$ | $J_5 J_7 w$ | $J_6 J_7 w$ |
|-------------|-------|-------|-------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $+$         | $1$   | $+1$  | $-1$  | $+1$        | $-1$        | $+1$        | $-1$        | $+1$        | $-1$        | $+1$        | $-1$        | $+1$        | $-1$        | $+1$        |

the following linear dependence relations.

$$ J_1 J_2 = J_3 J_4 = J_2 J_5 J_6 = J_3 J_5 J_7 w = -J_4 J_6 J_7 w, $$

$$ J_2 J_2 = -J_1 J_3 J_4 = -J_1 J_5 J_6 = J_3 J_5 J_7 w = -J_4 J_6 J_7 w, $$

$$ J_3 J_2 = J_1 J_3 J_4 = -J_1 J_5 J_6 = -J_3 J_5 J_7 w = -J_4 J_6 J_7 w, $$

(18)
\[ J_1 J_3 J_6 w = -J_1 J_2 J_7 w = J_1 J_4 J_5 w = -J_2 J_3 J_5 w = J_2 J_4 J_6 w = J_3 J_4 J_7 w = J_5 J_6 J_7 w. \]

At the final step, we apply Lemma 3 for the operator \( \Omega = J_1 J_3 J_6 w \) to make the basis orthogonal. The relations (18) shows that the action of \( J_i, i = 1, \ldots, 7 \) permutes the orthonormal basis \( \{ v_\alpha \}_{\alpha=1}^{16} \) up to sign. We conclude that the constructed 16 dimensional sub-module \( (W, \langle \cdot, \cdot \rangle_W) \) is of minimal dimension. Moreover it is irreducible.

4.8. Integral structure on admissible \( \text{Cl}_{0,8} \)-module. The Clifford algebra \( \text{Cl}_{0,8} \) is isomorphic to the space \( \mathbb{R}(16) \). Let \( z_j, j = 1, \ldots, 8 \), be orthonormal generators of \( \text{Cl}_{0,8} \) with \( \langle z_j, z_j \rangle_{\mathbb{R}^{0,8}} = -1 \) and \( (V, \langle \cdot, \cdot \rangle_V) \) an admissible \( \text{Cl}_{0,8} \)-module. Then \( J_j^2 := J_j^2 \) is \( \text{Id}_V, j = 1, \ldots, 8 \). We fix four isometric mutually commuting involutions \( P_i \):

\[
P_1 = J_1 J_2 J_3 J_4, \quad P_2 = J_1 J_2 J_5 J_6, \quad P_3 = J_2 J_3 J_5 J_7, \quad \text{and} \quad P_4 = J_1 J_2 J_7 J_8.
\]

In this case to find 5 anti-isometric complementary operators is impossible and we chose the different strategy. The tables of commutation relations of the involutions with the generators and the family of complementary operators are the following:

| Commutation relations: \( \text{Cl}_{0,8} \) case |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Involution | \( J_1 J_5 (+ \rightarrow +) \) | \( J_1 J_3 (+ \rightarrow +) \) | \( J_1 J_2 (+ \rightarrow +) \) | \( J_8 (+ \rightarrow -) \) |
| \( P_1 = J_1 J_2 J_3 J_4 \) | \( a \) | \( c \) | \( c \) | \( c \) |
| \( P_2 = J_1 J_2 J_5 J_6 \) | \( a \) | \( a \) | \( c \) | \( c \) |
| \( P_3 = J_2 J_3 J_5 J_7 \) | \( a \) | \( a \) | \( a \) | \( c \) |
| \( P_4 = J_1 J_2 J_7 J_8 \) | \( a \) | \( a \) | \( a \) | \( a \) |

| Complementary operator: \( \text{Cl}_{0,8} \) case |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Involution | \( J_1 J_5 (+ \rightarrow +) \) | \( J_1 J_3 (+ \rightarrow +) \) | \( J_1 J_2 (+ \rightarrow +) \) | \( J_8 (+ \rightarrow -) \) |
| \( P_1 = J_1 J_2 J_3 J_4 \) | \( a \) | \( c \) | \( c \) | \( c \) |
| \( P_2 = J_1 J_2 J_5 J_6 \) | \( a \) | \( a \) | \( c \) | \( c \) |
| \( P_3 = J_2 J_3 J_5 J_7 \) | \( a \) | \( a \) | \( a \) | \( c \) |
| \( P_4 = J_1 J_2 J_7 J_8 \) | \( a \) | \( a \) | \( a \) | \( a \) |

We apply Lemma 6 item 1) three times to pairs \( (P_1, T_1 = J_1 J_5), (P_2, T_2 = J_1 J_3), \) and \( (P_3, T_3 = J_1 J_2) \) and conclude that the common 1-eigenspace of \( P_1, P_2, P_3 \) is neutral. We decompose it into two orthogonal eigenspaces \( E_{4 \pm} \) of \( P_4 \). If the space \( E_{4+} \) is negative definite, we use Lemma 7 and modify the operator \( P \). From these relations we can choose a vector \( w \in V \) such that

\[
P_1 w = P_2 w = P_3 w = P_4 w = w \quad \text{and} \quad \langle w, w \rangle_V = 1.
\]

We act successively by \( J_j \) on \( w \) and get a simultaneous eigenspace decomposition spanned by 16 orthogonal vectors

\[
\begin{align*}
v_1 &= w, & v_2 &= J_1 J_2 w, & v_3 &= J_1 J_3 w, & v_4 &= J_1 J_4 w, \\
v_5 &= J_1 J_5, & v_6 &= J_1 J_6, & v_7 &= J_1 J_7, & v_8 &= J_1 J_8 w, \\
v_9 &= J_1 w, & v_{10} &= J_2 w, & v_{11} &= J_3 w, & v_{12} &= J_4 w, \\
v_{13} &= J_5 w, & v_{14} &= J_6 w, & v_{15} &= J_7 w, & v_{16} &= J_8 w,
\end{align*}
\]

with \( \langle v_\alpha, v_\alpha \rangle_V = 1, \alpha = 1, \ldots, 8 \) and \( \langle v_\alpha, v_\alpha \rangle_V = -1, \alpha = 9, \ldots, 16 \).

| Table 4. Eigenspace decomposition: \( \text{Cl}_{0,8} \) case |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Involution | \( J_1 J_5 \) | \( J_1 J_3 \) | \( J_1 J_2 \) | \( J_7 w \) | \( J_6 w \) | \( J_1 J_4 w \) | \( J_5 w \) | \( J_1 J_3 w \) |
| \( P_1 \) | \( +1 \) | \( +1 \) | \( +1 \) | \( +1 \) | \( +1 \) | \( +1 \) | \( +1 \) | \( +1 \) |
| \( P_2 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) |
| \( P_3 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) | \( -1 \) |
The presence of new involution $P_4$ eliminate additional linear dependent vectors and as in the previous cases it can be shown that generators acts by permutation on the basis. Therefore, we constructed an integral structure of a minimal admissible and irreducible $\text{Cl}_{0,8}$-module. For the completeness we present all linear relations up to sign.

$$
J_1J_2 = \pm J_3J_4 = \pm J_5J_6 = \pm J_7J_8,
$$

$$
J_1J_3 = \pm J_2J_4 = \pm J_5J_7 = \pm J_6J_8,
$$

$$
J_1J_4 = \pm J_2J_3 = \pm J_5J_7 = \pm J_6J_8,
$$

$$
J_1J_5 = \pm J_2J_6 = \pm J_3J_8 = \pm J_4J_7,
$$

$$
J_1J_6 = \pm J_2J_5 = \pm J_3J_7 = \pm J_4J_8,
$$

$$
J_1J_7 = \pm J_2J_8 = \pm J_3J_6 = \pm J_4J_5,
$$

$$
J_1J_8 = \pm J_2J_7 = \pm J_3J_5 = \pm J_4J_6.
$$

(19)

5. Integral structure on admissible $\text{Cl}_{r,1}$-modules

In this section we show the existence of the integral structure for $\text{Cl}_{r,1}$-modules based on the existence of those for $\text{Cl}_{r,0}$-modules. In the next section we will deal with the classical cases $\text{Cl}_{r,0}$ directly.

Let $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible $\text{Cl}_{r,1}$-module. Denote by $J_i = J_{z_i}$, $i = 1, \ldots, r + 1$ the representations of orthonormal generators of the algebra $\text{Cl}_{r,1}$ such that

$$
J_i^2 = J_{z_i}^2 = -\text{Id}_V, \quad i = 1, \ldots, r,
$$

$$
J_{r+1}^2 = J_{z_{r+1}}^2 = \text{Id}_V.
$$

Let $U$ be a subspace of $V$ invariant under the action of $J_i = J_{z_i}$, $i = 1, \ldots, r$, and let $U = \text{span} \{u_1, \ldots, u_l\}$, be a basis such that

- $\{u_\alpha\}_{\alpha=1}^l$ is an orthonormal basis of $U$;
- maps $J_i = J_{z_i}$, $i = 1, \ldots, r$, permute the basis $\{u_\alpha\}_{\alpha=1}^l$ up to sign;
- $\langle J_iu_\alpha, u_\beta \rangle_V = \pm 1$ or 0.

Denote $\tilde{u}_\alpha = J_{r+1}u_\alpha$, $\alpha = 1, \ldots, l$, $\tilde{U} = \text{span} \{\tilde{u}_1, \ldots, \tilde{u}_l\}$, and set the space

$$
W = \text{span} \{u_1, \ldots, u_l, \tilde{u}_1, \ldots, \tilde{u}_l\}.
$$

Then

Theorem 4. In the notations above if the decomposition $W = U \oplus \tilde{U}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_V$, then $(W, \langle \cdot, \cdot \rangle_W)$, where $\langle \cdot, \cdot \rangle_W$ is the restriction of $\langle \cdot, \cdot \rangle_V$ onto $W$ is an admissible integral $\text{Cl}_{r,1}$-module. If $U$ has minimal dimension, then $W$ is an admissible integral $\text{Cl}_{r,1}$-module of minimal dimension.
We conclude that restriction of \(\langle \cdot, \cdot \rangle_V\) to \(U\) will be positive definite or negative definite since
\[\langle J_iu_\alpha, J_iu_\alpha \rangle_V = \langle z_i, z_i \rangle_{\mathbb{R}^r} : \langle u_\alpha, u_\alpha \rangle_V = \langle u_\alpha, u_\alpha \rangle_V, \quad i = 1, \ldots, r.\]

Let us assume that it is positive definite. Then \(\tilde{U} = \text{span}\{\tilde{u}_1, \ldots, \tilde{u}_l\}\) is isomorphic to \(U\) and the restriction of \(\langle \cdot, \cdot \rangle_V\) on \(\tilde{U}\) is negative definite since
\[\langle \tilde{u}_\alpha, \tilde{u}_\alpha \rangle_V = \langle J_{r+1}u_\alpha, J_{r+1}u_\alpha \rangle_V = \langle z_{r+1}, z_{r+1} \rangle_{\mathbb{R}^r} : \langle u_\alpha, u_\alpha \rangle_V = -\langle u_\alpha, u_\alpha \rangle_V.\]

We conclude that \(J_{r+1}: U \to \tilde{U}\) defines an anti-isometry with respect to restrictions of the scalar product \(\langle \cdot, \cdot \rangle_V\) onto spaces \(U\) and \(\tilde{U}\) and the space \((W, \langle \cdot, \cdot \rangle_W)\), where \(W = U \oplus \tilde{U}\) and \(\langle \cdot, \cdot \rangle_W\) is the restriction of \(\langle \cdot, \cdot \rangle_V\) onto \(W\) is neutral.

The space \(W\) is invariant under the action of \(J_i, i = 1, \ldots, r + 1\), and all maps \(J_i, i = 1, \ldots, r + 1\) permute the basis \(\{v_\alpha, \tilde{v}_\alpha\}_{\alpha = 1}^1\) up to sign. We conclude that the sub-module \((W, \langle \cdot, \cdot \rangle_W)\) is admissible integral and has a minimal possible dimension if the space \(U\) has minimal dimension.

\[\square\]

**Corollary 5.** The dimension of the minimal admissible module for \(\text{Cl}_{r,1}\) is always twice of the \(\text{Cl}_{r,0}\)-module.

### 6. Integral Structure on Admissible \(\text{Cl}_{r,s}\)-Modules with \(r + s \leq 8\)

We show the existence of an admissible integral \(\text{Cl}_{r,s}\)-module for \(r + s = 2, \ldots, 8\) by direct construction. In these constructions we start from an admissible module \((V, \langle \cdot, \cdot \rangle_V)\) and find an orthonormal set \(\{v_\alpha, 1 \leq \alpha \leq l\}\) such that the sub-module \((W, \langle \cdot, \cdot \rangle_W)\), where \(W = \text{span}\{v_1, \ldots, v_l\}\) and the scalar product \(\langle \cdot, \cdot \rangle_W\) is the restriction of \(\langle \cdot, \cdot \rangle_V\) onto \(W\), is an admissible integral and has the minimal possible dimension. In some cases \(W\) is not an irreducible sub-module.

#### 6.1. Integral Structure on \(\text{Cl}_{r,s}\)-Admissible Modules with \(r + s = 2\)

##### 6.1.1. Integral Structure on Admissible \(\text{Cl}_{2,0}\)-module.

The Clifford algebra \(\text{Cl}_{2,0}\) is isomorphic to the space \(\mathbb{H}\). Let \(z_1, z_2\) be orthonormal generators of \(\text{Cl}_{2,0}\) with \(\langle z_i, z_i \rangle_{\mathbb{R}^2} = 1, i = 1, 2\). Let \((V, \langle \cdot, \cdot \rangle_V)\) be an admissible \(\text{Cl}_{2,0}\)-module with positive definite inner product, then \(J_1^2 := J_2^2 = -(z_1, z_1)_{\mathbb{R}^2} : \text{Id}_V = -\text{Id}_V\), for \(i = 1, 2\). Choose \(w \in V\) such that \(\langle w, w \rangle_V = 1\). Then the basis
\[v_1 = w, \quad v_2 = J_1w, \quad v_3 = J_2w, \quad v_4 = J_2J_1w\]
is orthogonal and satisfies \(\langle v_\alpha, v_\alpha \rangle_V = 1\) for \(\alpha = 1, 2, 3, 4\). It is easy to see that the admissible module is integral. The admissible module is irreducible as in all case corresponding to \(\text{Cl}_{r,0}\) algebras.

##### 6.1.2. Integral Structure on Admissible \(\text{Cl}_{1,1}\)-module.

The Clifford algebra \(\text{Cl}_{1,1}\) is isomorphic to the space \(\mathbb{R}(2)\) of \((2 \times 2)\)-matrices with real entries. Let \(z_1, z_2\) be orthonormal generators of \(\text{Cl}_{1,1}\) with \(\langle z_1, z_1 \rangle_{\mathbb{R}^2} = 1\) and \(\langle z_2, z_2 \rangle_{\mathbb{R}^2} = -1\). Let \((V, \langle \cdot, \cdot \rangle_V)\) be an admissible \(\text{Cl}_{1,1}\)-module, then
\[J_1^2 := J_2^2 = -(z_1, z_1)_{\mathbb{R}^2} : \text{Id}_V = -\text{Id}_V, \quad J_2^2 := J_2^2 = -(z_2, z_2)_{\mathbb{R}^2} : \text{Id}_V = \text{Id}_V.\]

Choose \(w \in V\) such that \(\langle w, w \rangle_V = 1\). Then the basis \((v_1, v_1) = \langle v_2, v_2 \rangle_V = -\langle v_3, v_3 \rangle_V = -\langle v_4, v_4 \rangle_V = 1\). It is easy to see that the admissible module is integral. The admissible module is not irreducible in this case since the dimension of an irreducible module is 2.

Notice that we could show the existence of the integral structure by making use of the isomorphism \(\text{Cl}_{r,s+1} \cong \text{Cl}_{0,2} \cong \text{Cl}_{1,1} \cong \text{Cl}_{s,r+1}\), see Theorem \[\text{[3]}\] but we prefer the direct construction, since we will use this construction further.

#### 6.1.3. Integral Structure on Admissible \(\text{Cl}_{0,2}\)-module.

The admissible integral \(\text{Cl}_{0,2}\)-module was constructed in Section \[\text{[4]}\]

#### 6.2. Integral Structure on Admissible \(\text{Cl}_{r,s}\)-Modules with \(r + s = 3\).
6.2.1. Integral structure on admissible $\Cl_{3,0}$-module. The Clifford algebra $\Cl_{3,0}$ is isomorphic to the space $\mathbb{H} \oplus \mathbb{H}$. Let $z_1, z_2, z_3$ be orthogonal generators of $\Cl_{3,0}$ with $(z_i, z_i)_{\mathbb{H}^2} = 1$, $i = 1, 2, 3$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\Cl_{3,0}$-module with positive definite metric. Then

$$J_i^2 := J_{z_i}^2 = -(z_i, z_i)_{\mathbb{H}^2} \Id_V = -\Id_V.$$}

Consider the isometric involution $P = J_1J_2J_3$ and pick up a vector $w \in V$ such that $J_1J_2J_3w = w$ and $(w, w)_V = 1$. It is possible, since the 1-eigenspace is an inner product space. We get the following linear dependent vectors

$$J_1w = -J_2J_3w, \quad J_2w = J_1J_3w, \quad J_3w = J_1J_2w.$$

In this case we choose the basis $v_1 = w$, $v_2 = J_1w$, $v_3 = J_2w$, $v_4 = J_3w$. It is orthogonal due to the skew symmetry of $J_i$, $i = 1, 2, 3$. Moreover $(v_\alpha, v_\alpha)_V = 1$, $\alpha = 1, 2, 3, 4$. It is easy to see that the sub-module $W = \text{span} \{v_1, \ldots, v_4\}$, is integral due to the orthogonality of generators, skew symmetry of $J_k$ and the condition $J_1J_2J_3w = w$. The admissible integral sub-module is irreducible with an inner product.

6.2.2. Integral structure on admissible $\Cl_{2,1}$-module. The Clifford algebra $\Cl_{2,1}$ is isomorphic to the space $\mathbb{C}(2)$ of $2 \times 2$ matrices with complex entries. Let $z_1, z_2, z_3$ be generators of $\Cl_{2,1}$ with $(z_i, z_i)_{\mathbb{C}(2)} = 1$, $i = 1, 2$ and $(z_3, z_3)_{\mathbb{C}(2)} = -1$. Let $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible $\Cl_{2,1}$-module. Then $J_i^2 := J_{z_i}^2 = -\Id_V$, $i = 1, 2$. We apply Theorem 4 and chose the following basis

$$v_1 = w, \quad v_2 = J_1w, \quad v_3 = J_2w, \quad v_4 = J_1J_2w,$$

$$v_5 = J_3w, \quad v_6 = J_1J_3w, \quad v_7 = J_2J_3w, \quad v_8 = J_1J_2J_3w.$$}

If the vectors $w$ and $J_1J_2J_3w$ are not orthogonal, we apply Lemma 3 and find the correct vector $\tilde{w}$ making basis orthogonal. Moreover $(v_\alpha, v_\alpha)_V = 1$, $\alpha = 1, \ldots, 4$, $(v_\alpha, v_\alpha)_V = -1$, $\alpha = 5, \ldots, 8$. It is easy to see that the module is integral since $J_i$, $i = 1, 2, 3$, permute the basis up to sign. The admissible integral sub-module is not irreducible since the dimension of the irreducible module is equal to 4.

**Remark 5.** Let $v \in V$ be a non-null vector and assume that one of $J_iJ_jv$ with $i < j$ is independent from $v$, $J_1v$, $J_2v$, $J_3v$. Then the dimension of the admissible subspace is at least 8. Together with the construction above the irreducible module can not be admissible with any choice of non-null vector.

We also can use isomorphism $\Cl_{0,3} \cong \Cl_{2,1}$ and Theorem 3 to prove the existence of integral structure on admissible $\Cl_{2,1}$-module.

6.2.3. Integral structure on admissible $\Cl_{1,2}$-module. The Clifford algebra $\Cl_{1,2}$ is isomorphic to the space $\mathbb{R}(2) \times \mathbb{R}(2)$. Let $z_1, z_2, z_3$ be generators of $\Cl_{1,2}$ with $(z_1, z_1)_{\mathbb{R}(2)} = 1$, $(z_3, z_3)_{\mathbb{R}(2)} = -1$, $i = 2, 3$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\Cl_{1,2}$-module. Then

$$J_i^2 := J_{z_i}^2 = -\Id_V, \quad J_i^2 := J_{z_i}^2 = \Id_V \quad i = 2, 3.$$}

Consider the isometric involution $P = J_1J_2J_3$. The operator $P$ commutes with the anti-isometry $T = J_2$. Thus the eigenspace $E_+$ of $P$ is a neutral space by Lemma 6, case 3). We pick up a vector $w \in V$ such that $Pw = w$ and $(w, w)_V = 1$. It gives the linear dependent vectors

$$(21) \quad J_1w = -J_2J_3w, \quad J_2w = J_1J_3w, \quad J_3w = J_1J_2w.$$}

In this case we choose the basis $v_1 = w$, $v_2 = J_1w$, $v_3 = J_2w$, $v_4 = J_3w$, that is orthogonal as it was shown in the case of $\Cl_{3,0}$-module. Moreover $(v_1, v_1)_V = (v_2, v_2)_V = -\langle v_3, v_3 \rangle_V = -\langle v_4, v_4 \rangle_V = 1$. It is easy to see that the module is integral due to the orthogonality of generators, skew symmetry of $J_k$ and the condition $J_1J_2J_3w = w$. The admissible integral module is not irreducible since the dimension of the irreducible module is equal to 2.

6.2.4. Integral structure on admissible $\Cl_{0,3}$-module. The admissible integral $\Cl_{0,3}$-module was constructed in Section 4.

6.3. Integral structure on admissible $\Cl_{r,s}$-modules with $r + s = 4$. 

LATTICE ON GENERAL $H$-TYPE GROUPS
6.3.1. Integral structure on admissible $\text{Cl}_{4,0}$-module. The Clifford algebra $\text{Cl}_{4,0}$ is isomorphic to the space $\mathbb{H}(2)$. Let $z_1, z_2, z_3, z_4$ be orthogonal generators of $\text{Cl}_{4,0}$ with $(z_i, z_j)_{\mathbb{H}(2)} = 1$, $i = 1, 2, 3, 4$, and let $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible $\text{Cl}_{4,0}$-module with positive definite metric. Then $J_i^2 := J_i^2 = -\text{Id}_V$, $i = 1, 2, 3, 4$. As in the case of $\text{Cl}_{0,4}$-module we consider the isometric involution $P = J_1J_2J_3J_4$ and pick up $w \in V$ such that $Pw = w$ and $(w, w)_V = 1$. It leads to the linear dependence relations similar to (11), where some of them can change sign. The basis (12) is orthonormal with respect to $\langle \cdot, \cdot \rangle_V$. The maps $J_i$, $i = 1, 2, 3, 4$, permute basis vectors up to sign the sub-module $W$ is admissible and integral, but is not irreducible because the dimension of an irreducible module is 4 and with $w = 1$, they commute with $J_1J_2J_3J_4$. Moreover $J_i^2 := J_i^2 = -\text{Id}_V$, $i = 1, 2, 3, 4$.

6.3.2. Integral structure on admissible $\text{Cl}_{3,1}$-module. The Clifford algebra $\text{Cl}_{3,1}$ is isomorphic to the space $\mathbb{R}(4)$. The integral structure on admissible $\text{Cl}_{3,1}$-modules exists according to Theorem 3 and Corollary 4 or Theorem 11.

6.3.3. Integral structure on admissible $\text{Cl}_{2,2}$-module. The Clifford algebra $\text{Cl}_{2,2}$ is isomorphic to the space $\mathbb{R}(4)$. Let $z_1, z_2, z_3, z_4$ be orthogonal generators of $\text{Cl}_{2,2}$ with $(z_i, z_i)_{\mathbb{R}(4)} = 1$, $i = 1, 2, 3, 4$, $(z_j, z_j)_{\mathbb{R}(4)} = -1$, $j = 3, 4$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{2,2}$-module. Then

$$J_i^2 := J_i^2 = -\text{Id}_V, \quad i = 1, 2, \quad J_j^2 := J_j^2 = \text{Id}_V, \quad j = 3, 4.$$  

We consider the isometric involution $P = J_1J_2J_3J_4$ and the isometry $T = J_1$ such that $PT = -TP$. We choose a vector $w \in V$ such that $Pw = w$ and $(w, w)_V = 1$ by Lemma 6, part 1). The linear dependence relations (11) still hold up to sign. We write the basis (12) in the form

$$v_1 = w, \quad v_2 = J_1w, \quad v_3 = J_2w, \quad v_4 = J_1J_2w, \quad v_5 = J_3w, \quad v_6 = J_4w, \quad v_7 = J_1J_3w, \quad v_8 = J_1J_4w.$$  

It is orthonormal and $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, 2, 3, 4$. Since $J_i, i = 1, 2, 3, 4$, permute basis vectors up to sign the sub-module $W$ is span $\{v_1, \ldots, v_8\}$ is integral. The constructed module is not irreducible because the dimension of an irreducible module is 4 and with any choice of a non-null vector $v \in V$ the five vectors $v, J_1v, J_2v, J_3v, J_4v$ are already linear independent.

We give an alternative construction of an admissible $\text{Cl}_{2,2}$-module. As it was mentioned the $\pm 1$-eigenspaces $E_{\pm}$ of $P$ have equal dimension. Consider two operators $\hat{J}_1 = J_1J_2$ and $\hat{J}_2 = J_1J_3$. Since they commute with $P$ they leave invariant $E_{\pm}$. Moreover

$$\hat{J}_i^2 = -\text{Id}_{E_{\pm}}, \quad \hat{J}_i^2 = \text{Id}_{E_{\pm}}, \quad \hat{J}_1\hat{J}_2 = -\hat{J}_2\hat{J}_1.$$  

Thus, the algebra generated by $\hat{J}_1$ and $\hat{J}_2$ in End($E_{+}$) is isomorphic to the Clifford algebra $\text{Cl}_{1,1}$ and the representation is admissible, since, for example,

$$\langle \hat{J}_1u, v \rangle_{E_{+}} = \langle J_1J_2u, v \rangle_{E_{+}} = -\langle J_2u, J_1v \rangle_{E_{+}} = \langle u, J_2J_1v \rangle_{E_{+}} = -\langle u, J_1J_2v \rangle_{E_{+}} = -\langle u, \hat{J}_1\hat{J}_2v \rangle_{E_{+}}.$$  

The same arguments valid for $E_{-}$. Because dim$(E_{+}) = 4$ and $E_{+} \perp E_{-}$, we have an integral structure on $V = E_{+} \oplus \perp E_{-}$ inherited from that of $\text{Cl}_{1,1}$.

6.3.4. Integral structure on admissible $\text{Cl}_{1,3}$-module. The Clifford algebra $\text{Cl}_{1,3}$ is isomorphic to the space $\mathbb{R}(4)$. In this case we can use Theorem 3 since the Clifford algebra $\text{Cl}_{r,s+1} = \text{Cl}_{2,2}$ is isomorphic to $\text{Cl}_{s,r+1} = \text{Cl}_{1,3}$. The orthogonal basis changed to

$$v_1 = w, \quad v_2 = J_3J_4w, \quad v_3 = J_3J_4w, \quad v_4 = -J_3J_4w, \quad v_5 = J_1J_4w, \quad v_6 = J_4w, \quad v_7 = J_1J_2w, \quad v_8 = J_2w,$$

with $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, 2, 3, 4$. Then the constructed sub-module $W = \text{span} \{v_1, \ldots, v_8\}$ is admissible and integral, but is not irreducible because the dimension of an irreducible module is 4.

6.3.5. Integral structure on admissible $\text{Cl}_{0,4}$-module. The admissible integral $\text{Cl}_{0,4}$-module was constructed in Section 11.
6.4. Integral structure on admissible $\text{Cl}_{r,s}$-modules with $r + s = 5$.

6.4.1. Integral structure on admissible $\text{Cl}_{5,0}$-module. The Clifford algebra $\text{Cl}_{5,0}$ is isomorphic to the space $\mathbb{C}(4)$. Let $z_1, \ldots, z_5$ be orthonormal generators of $\text{Cl}_{5,0}$ with $(z_i, z_i)_{\mathbb{C}(5)} = 1$, $i = 1, \ldots, 5$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{5,0}$-module with a neutral product. Then $J_i^2 := J_i^2 = -\text{Id}_V$ for $i = 1, \ldots, 5$. In this case we fix two mutually commuting isometric involutions $P_1 = J_1 J_2 J_3 J_4$ and $P_2 = J_1 J_2 J_5$. Since we can find complementary isometries $T_1 = J_1$ and $T_2 = J_2 J_3$ which satisfy the relations

\begin{equation}
P_1 T_1 = -T_1 P_1, \quad P_1 T_2 = T_2 P_1 \quad P_2 T_2 = -T_2 P_2,
\end{equation}

we may pick up a vector $w \in V$ such that $P_1 w = P_2 w = w$ and $\langle w, w \rangle_V = 1$ by applying Lemma 6 part 1) twice. Then the simultaneous eigenspace decomposition, presented in Table 5, allows to chose the orthonormal basis

$$
\begin{align*}
v_1 &= w, & v_2 &= J_1 w, & v_3 &= J_2 w, & v_4 &= J_3 w, \\
v_5 &= J_4 w, & v_6 &= J_5 w, & v_7 &= J_1 J_3 w, & v_8 &= J_1 J_4 w,
\end{align*}
$$

with $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 8$. Relations (11) and those coming from $J_1 J_2 J_5 w = w$ shows that the action of $J_i$, $i = 1, \ldots, 5$, permutes basis up to sign. Hence we have an integral structure of a minimal dimensional admissible sub-module of $\text{Cl}_{5,0}$-module with an inner product, see Remark 6 and an observation in Section 9. The admissible module is irreducible.

Table 5. Eigenspace decomposition: $\text{Cl}_{5,0}$ case

| Involution | Eigenvalue |
|------------|------------|
| $P_1$      | +1         |
| $P_2$      | −1         |

| Eigenvectors | $w, J_5 w$ | $J_1 J_3 w, J_1 J_4 w$ | $J_1 w, J_2 w$ | $J_3 w, J_4 w$ |

6.4.2. Integral structure on admissible $\text{Cl}_{4,1}$-module. The admissible $\text{Cl}_{4,1}$-module is integral by Theorem 4 If the decomposition on $U$ spanned by (12) and the image $\tilde{J}_5(U)$ is not orthogonal, then we change the vector $w$ to $\tilde{w}$ by Lemma 4 and Corollary 5. We also can use Theorem 3 and Corollary 4 to show the existence of an integral structure.

6.4.3. Integral structure on admissible $\text{Cl}_{3,2}$-module. The Clifford algebra $\text{Cl}_{3,2}$ is isomorphic to the space $\mathbb{C}(4)$. Let $z_1, \ldots, z_5$ be orthogonal generators of $\text{Cl}_{3,2}$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{3,2}$-module. Then $J_i^2 := J_i^2 = -\text{Id}_V$, $i = 1, 2, 3$, $J_i^2 := J_i^2 = \text{Id}_V$, for $j = 4, 5$. The construction is essentially the same as for $\text{Cl}_{5,0}$-module. In this case we choose the mutually commuting isometric involutions $P_1 = J_2 J_3 J_4 J_5$ and $P_2 = J_1 J_2 J_3$ and complementary isometries $T_1 = J_2$ and $T_2 = J_2 J_4$, that satisfy relations (22). We pick up a vector $w \in V$ such that $P_1 w = P_2 w = w$ and $\langle w, w \rangle_V = 1$, which existence is guaranteed by Lemma 6 part 1). The orthonormal basis is

$$
\begin{align*}
v_1 &= w, & v_2 &= J_1 w, & v_3 &= J_2 w, & v_4 &= J_3 w, \\
v_5 &= J_4 w, & v_6 &= J_5 w, & v_7 &= J_2 J_4 w, & v_8 &= J_2 J_5 w,
\end{align*}
$$

with $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 4$ and $\langle v_\alpha, v_\alpha \rangle_V = -1$, $\alpha = 5, \ldots, 8$. In this case we will have enough linear dependent relations showing that action of $J_i$, $i = 1, \ldots, 5$, permutes basis vectors up to sign. These relations are analogous to (11) and those arising from $P_2 w = w$. The constructed module is admissible, integral, and irreducible.
6.4.4. **Integral structure on admissible Cl_{2,3}-module.** The Clifford algebra Cl_{2,3} is isomorphic to the space \( \mathbb{R}(4) \oplus \mathbb{R}(4) \). Let \( z_1, \ldots, z_5 \) be generators of Cl_{2,3}, and \( (V, \langle \cdot, \cdot \rangle_V) \) an admissible Cl_{2,3}-module. Then \( J_1^2 := J_2^2 := -\text{Id}_V, i = 1, 2, J_j^2 := \text{Id}_V, j = 3, 4, 5 \).

We fix two mutually commuting isometric involutions \( P_1 = J_1 J_2 J_3 J_4 \) and \( P_2 = J_1 J_4 J_5 \) and two complementary isometries \( T_1 = J_1 \) and \( T_2 = J_1 J_2 \) which satisfy (22). The common eigenspaces of \( P_1 \) and \( P_2 \) are neutral spaces by Lemma 6, part 1. So we find a common eigenvector \( w \in E_{1+} \cap E_{2+} \subset V \) such that \( P_1 w = P_2 w = w \) with \( \langle w, w \rangle_V = 1 \). We have the following simultaneous eigenspace decomposition of the representation space \( V \) by \( P_1 \) and \( P_2 \) with mutually orthogonal eigenvectors:

**Table 6. Eigenspace decomposition: Cl_{2,3} case**

| Involution | Eigenvalue | Eigenvectors |
|------------|------------|--------------|
| \( P_1 \)  | +1         | \( w, J_1 J_4 w \) |
| \( P_2 \)  | +1         | \( J_1 w, J_4 w \) |

It gives the orthonormal basis

\[
\begin{align*}
v_1 &= w, \\
v_2 &= J_1 w, \\
v_3 &= J_2 w, \\
v_4 &= J_1 J_2 w, \\
v_5 &= J_3 w, \\
v_6 &= J_4 w, \\
v_7 &= J_5 w, \\
v_8 &= J_1 J_4 w,
\end{align*}
\]

with \( \langle v_\alpha, v_\alpha \rangle_V = 1, \alpha = 1, \ldots, 4 \) and \( \langle v_\alpha, v_\alpha \rangle_V = -1, \alpha = 5, \ldots, 8 \). As in previous cases we can show that \( J_i, i = 1, \ldots, 5 \) permute basis vectors up to sign. The constructed module is admissible integral, but not irreducible because we know that with any choice of non-null vector \( v \in V \) the 6 vectors \( v, J_1 v, J_2 v, J_3 v, J_4 v, J_5 v \) are linearly independent but the dimension of the irreducible module is 4.

6.4.5. **Integral structure on admissible Cl_{1,4}-module.** The Clifford algebra Cl_{1,4} is isomorphic to the space \( \mathbb{C}(4) \). We can apply Theorem 3 and Corollary 4 since \( \text{Cl}_{1,4} \cong \text{Cl}_{3,2} \).

6.4.6. **Integral structure on admissible Cl_{0,5}-module.** The admissible integral Cl_{0,5}-module was constructed in Section 4.

6.5. **Integral structure on admissible Cl_{r,s}-modules with \( r + s = 6 \).**

6.5.1. **Integral structure on admissible Cl_{6,0}-module.** The Clifford algebra Cl_{6,0} is isomorphic to the space \( \mathbb{R}(8) \). Let \( z_1, \ldots, z_6 \) be orthonormal generators of Cl_{6,0}, and \( (V, \langle \cdot, \cdot \rangle_V) \) an admissible Cl_{6,0}-module. Then \( J_1^2 := J_2^2 := -\text{Id}_V, i = 1, \ldots, 6 \). We consider three mutually commuting isometric involutions

\[
\begin{align*}
P_1 &= J_1 J_2 J_3 J_4, \\
P_2 &= J_1 J_2 J_5 J_6, \\
P_3 &= J_1 J_4 J_5.
\end{align*}
\]

We start from the common eigenvector \( w \in V \):

\[
(23) \quad P_1 w = P_2 w = P_3 w = w, \quad \langle w, w \rangle_V = 1.
\]

The existence of such \( w \) is guaranteed by the complementary isometries \( T_1 = J_1, T_2 = J_5 \) and \( T_3 = J_5 J_6 \) and Lemma 6, part 1. We sum up the relations between these two families of operators in the following tables:

**Table 7. Complementary operators**

| Involution | \( J_1 (+ \rightarrow +) \) | \( J_5 (+ \rightarrow +) \) | \( J_5 J_6 (+ \rightarrow +) \) |
|------------|----------------|----------------|----------------|
| \( P_1 \)  | \( a \)     | \( c \)     | \( c \)     |
| \( P_2 \)  | \( a \)     | \( a \)     | \( c \)     |
| \( P_3 \)  | \( a \)     | \( a \)     | \( a \)     |
Table 7. Eigenspace decomposition: $\text{Cl}_{6,0}$ case

| Involution | Eigenvalues |
|------------|-------------|
| $P_1$      | $+1$        |
| $P_2$      | $+1$        |
| $P_3$      | $+1$        |

We get $8$ orthonormal vectors

$$v_1 = w, \quad v_2 = J_1 w, \quad v_3 = J_2 w, \quad v_4 = J_3 w,$$
$$v_5 = J_4 w, \quad v_6 = J_5 w, \quad v_7 = J_6 w, \quad v_8 = J_1 J_2 w,$$

with $\langle v_\alpha, v_\alpha \rangle_V = 1$. Any two of these $8$ vectors do not belong to a common eigenspace of $P_1$, $P_2$ and $P_3$, which says that they are orthogonal. Relations (23) lead to relations (15), that still hold up to sign, and additional ones, arising from $P_3 w = w$. It shows that each operator $J_i$ $i = 1, \ldots, 6$ permutes these basis up to the sign. The sub-module spanned by theses $8$ vectors is admissible integral irreducible sub-module with positive definite inner product, see Remark 6 and observation in Section 6.5.2. Integral structure on admissible $\text{Cl}_{5,1}$-module. The Clifford algebra $\text{Cl}_{5,1}$ is isomorphic to the space $\mathbb{H}(4)$. We can apply Theorem 3 or Theorem 3 and Corollary 4 to show the existence of an integral structure.

6.5.3. Integral structure on admissible $\text{Cl}_{4,2}$-module. The Clifford algebra $\text{Cl}_{4,2}$ is isomorphic to the space $\mathbb{H}(4)$. Let $z_1, \ldots, z_6$ be generators of $\text{Cl}_{4,2}$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{4,2}$-module. Then $J_1^2 := J_{z_1}^2 = -\text{Id}_V$, $i = 1, \ldots, 4$, $J_5^2 := J_{z_5}^2 = \text{Id}_V$ for $j = 5, 6$. We fix two mutually commuting isometric involutions $P_1 = J_1 J_2 J_3 J_4$ and $P_2 = J_1 J_2 J_5 J_6$. Then we have two complementary isometries $T_1 = J_1$, $T_2 = J_2 J_3$ which satisfy relations (22). So, the simultaneous eigenspace $E_{1+} \cap E_{2+}$ of $P_1$ and $P_2$ is neutral by Lemma 6, part 1). Hence we find a common eigenvector $w \in V$ satisfying $P_1 w = P_2 w = w$ with $\langle w, w \rangle_V = 1$. It gives a simultaneous eigenspace decomposition of the representation space $V$, generated by Clifford multiplication of $w$, and which is presented in Table 8.

Table 8. Eigenspace decomposition: $\text{Cl}_{4,2}$ case

| Involution | Eigenvalues |
|------------|-------------|
| $P_1$      | $+1$        |
| $P_2$      | $+1$        |

We chose the basis

$$v_1 = w, \quad v_2 = J_1 w, \quad v_3 = J_2 w, \quad v_4 = J_3 w,$$
$$v_5 = J_4 w, \quad v_6 = J_1 J_2 w, \quad v_7 = J_1 J_3 w, \quad v_8 = J_1 J_4 w,$$
$$v_9 = J_5 w, \quad v_{10} = J_6 w, \quad v_{11} = J_1 J_5 w, \quad v_{12} = J_1 J_6 w,$$
$$v_{13} = J_3 J_5 w, \quad v_{14} = J_4 J_6 w, \quad v_{15} = J_1 J_3 J_5 w, \quad v_{16} = J_2 J_3 J_5 w,$$

with $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 8$ and $\langle v_\alpha, v_\alpha \rangle_V = -1$ for $\alpha = 9, \ldots, 16$. The vectors can be made orthogonal if we apply Lemma 3 to operators $J_1 J_3 J_5$ and $J_2 J_3 J_5$. The relations (15) show that all other relations will be made also orthogonal. They also proves that the operators $J_j$, $j = 1, \ldots, 6$, permute the basis up to sign. Hence we constructed a minimal admissible integral sub-module of $\text{Cl}_{4,2}$-module of the dimension $16$, which is irreducible.
6.5.4. **Integral structure on admissible Cl\(_{3,3}\)-module.** The Clifford algebra Cl\(_{3,3}\) is isomorphic to the space \(\mathbb{R}(8)\). Let \(z_1, \ldots, z_6\) be generators of Cl\(_{3,3}\), and \((V, \langle \cdot, \cdot \rangle_V)\) an admissible Cl\(_{3,3}\)-module. Then \(J_i^2 := J_i J_i = -\text{Id}_V\), \(i = 1, 2, 3, J_j^2 := J_j J_j = \text{Id}_V\) for \(j = 4, 5, 6\). We argue similar to the case Cl\(_{6,0}\) but consider different isometric mutually commuting involutions

\[
P_1 = J_1J_2J_4J_5, \quad P_2 = J_2J_3J_5J_6, \quad P_3 = J_1J_2J_3.
\]

Then we have two tables of commutation relations with the generators \(J_i\) and complementary operators.

| Involution \(\setminus\) Generators | \(J_1\) | \(J_2\) | \(J_3\) | \(J_4\) | \(J_5\) | \(J_6\) |
|-------------------------------------|--------|--------|--------|--------|--------|--------|
| \(P_1 = J_1J_2J_4J_5\)            | a      | a      | c      | a      | a      | c      |
| \(P_2 = J_2J_3J_5J_6\)            | c      | a      | a      | c      | a      | a      |
| \(P_3 = J_1J_2J_3\)               | c      | c      | a      | a      | a      | a      |

| Involution \(\setminus\) Comp. op. | \(J_1(\rightarrow +)\) | \(J_3(\rightarrow +)\) | \(J_1J_4(\rightarrow -)\) |
|-------------------------------------|------------------------|------------------------|------------------------|
| \(P_1 = J_1J_2J_4J_5\)            | a                      | c                      | c                      |
| \(P_2 = J_2J_3J_5J_6\)            | a                      | c                      | a                      |
| \(P_3 = J_1J_2J_3\)               | a                      | a                      | a                      |

The tables show that the common eigenspace \(E_{1^+} \cap E_{2^+}\) of operators \(P_1\) and \(P_2\) is neutral scalar product space by Lemma 3 part 1). If the restriction of \(\langle \cdot, \cdot \rangle_V\) on \(E_{1^+} \cap E_{2^+} \cap E_{3^+}\) is negative definite, we apply procedure of Lemma 7 and change the operator \(P_3 = J_1J_2J_3\) to the operator \(\hat{P}_3 = J_2J_1J_3\).

So we can find a vector \(w \in E_{1^+} \cap E_{2^+} \cap E_{3^+}\) \(\subset V\) with properties \(P_1w = P_2w = P_3w = w\) and \(\langle w, w \rangle_V = 1\). Then we obtain Table 9, expressing the simultaneous eigenspace decomposition by the involutions \(P_1, P_2,\) and \(P_3\). It allows us to choose the orthonormal basis (24).

| Involution | Eigenvalues |
|------------|-------------|
| \(P_1\)    | +1          |
| \(P_2\)    | -1          |
| \(P_3\)    | +1          |

| Eigenvector | \(J_1J_4w\) | \(J_3w\) | \(J_6w\) | \(J_1w\) | \(J_4w\) | \(J_2w\) | \(J_5w\) |
|-------------|-------------|--------|--------|--------|--------|--------|--------|
| \(v_1\)     | \(w\)      | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  |
| \(v_2\)     | \(J_1w\)   | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  |
| \(v_3\)     | \(J_2w\)   | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  |
| \(v_4\)     | \(w\)      | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  |
| \(v_5\)     | \(J_4w\)   | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  |
| \(v_6\)     | \(J_5w\)   | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  |
| \(v_7\)     | \(J_6w\)   | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  |
| \(v_8\)     | \(J_1J_4w\)| \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  | \(w\)  |

with \(\langle v_\alpha, v_\alpha \rangle_V = 1, \alpha = 1, \ldots, 4\) and \(\langle v_\alpha, v_\alpha \rangle_V = -1\) for \(\alpha = 5, \ldots, 8\). Moreover, as in the previous case of Cl\(_{0,6}\)-module, relations \(P_1w = P_2w = P_3w = w\) leave only 8 linear independent vectors and shows that operators \(J_j, j = 1, \ldots, 6\) permute the basis up to sign. Finally we conclude that the minimal admissible integral sub-module of Cl\(_{3,3}\)-module is of dimension 8 and it is irreducible.

6.5.5. **Integral structure on admissible Cl\(_{2,4}\)-module.** The Clifford algebra Cl\(_{2,4}\) is isomorphic to the space \(\mathbb{R}(8)\). The integral structure exists according to Theorem 3 and Corollar 4 since Cl\(_{3,3} \cong\) Cl\(_{2,4}\).

6.5.6. **Integral structure on admissible Cl\(_{1,5}\)-module.** The Clifford algebra Cl\(_{1,5}\) is isomorphic to the space \(\mathbb{H}(4)\) and the integral structure exists according to Theorem 3 and Corollar 4 by the isomorphism Cl\(_{4,2} \cong\) Cl\(_{1,5}\).

6.5.7. **Integral structure on admissible Cl\(_{0,6}\)-module.** The integral admissible Cl\(_{0,6}\)-module was constructed in Section 4.

6.6. **Integral structure on admissible Cl\(_{r,s}\)-modules with \(r + s = 7\).**
6.6.1. Integral structure on admissible $\text{Cl}_{7,0}$-module. The Clifford algebra $\text{Cl}_{7,0}$ is isomorphic to the space $\mathbb{R}(8) \oplus \mathbb{R}(8)$. Let $z_1, \ldots, z_7$ be a set of orthonormal generators of $\text{Cl}_{7,0}$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{7,0}$-module. Then $J_i^2 := J_{z_i}^2 = -\text{Id}_V$, $i = 1, \ldots, 7$. We consider four isometric involutions commuting with each other:

$$P_1 = J_1J_2J_3J_4, \quad P_2 = J_1J_2J_3J_6, \quad P_3 = J_2J_3J_6J_7, \quad \text{and} \quad P_4 = J_1J_4J_5.$$  

We start from the common eigenvector $w \in V$ of $P_i$, $i = 1, 2, 3$: $P_1w = P_2w = P_3w = w$. The existence of three isometric complementary to $P_1, P_2, P_3$ isometries

$$T_1 = J_1, \quad T_2 = J_5, \quad \text{and} \quad T_3 = J_7$$

guarantees that the space $E := E_{1+} \cap E_{2+} \cap E_{3+} \subset V$ is neutral by Lemma 6 part 1). Since the isometric involution $P_4$ commutes with $P_1, P_2$ and $P_3$, their common eigenspace $E$ is $P_4$-invariant. We write $E = E_{4+} \oplus E_{4-}$. If the restriction of $\langle \cdot, \cdot \rangle_V$ to $E_{4+}$ is not negative definite we does not need to do anything and we find $w \in E \cap E_{4+}$ such that

$$P_1w = P_2w = P_3w = P_4w = w \quad \text{and} \quad \langle w, w \rangle_V = 1. \quad (25)$$

If the restriction of $\langle \cdot, \cdot \rangle_V$ to $E_{4+}$ is negative definite, then the restriction of $\langle \cdot, \cdot \rangle_V$ to $E_{4-}$ is positive definite. We change the operator $P_4 = J_1J_4J_5$ to the operator $\hat{P}_4 = J_4J_1J_5$. Then for $w \in E_{4-}$ we have $\hat{P}_4w = -P_4w = w$. Thus we can find $w \in E \cap E_{4+}$, where $E_{4+} = E_{4-}$ satisfying (25), where the operator $P$ is changed to $\hat{P}$. To the linear relations (18), that still hold up to sign, we add new ones coming from $P_3w = w$ and choose the orthonormal basis

$$v_1 = w, \quad v_2 = J_1w, \quad v_3 = J_2w, \quad v_4 = J_3w, \quad v_5 = J_4w, \quad v_6 = J_5w, \quad v_7 = J_6w, \quad v_8 = J_7w. \quad (26)$$

with $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 8$. These relations also shows that operators $J_j$, $j = 1, \ldots, 7$ permutes this basis up to sign.

**Remark 6.** Note that we constructed from the given admissible module $V$, where we assumed that the scalar product is neutral a minimal admissible sub-module of the Clifford algebra $\text{Cl}_{7,0}$ whose dimension is 8 and the restriction of the given neutral scalar product to this subspace is positive definite. The sub-module is irreducible. The same constructions were done for the $\text{Cl}_{5,0}$- and $\text{Cl}_{6,0}$-modules.

### Table 10. Eigenspace decomposition: $\text{Cl}_{7,0}$ case

| Involution | $+1$ | $-1$ | $+1$ | $-1$ |
|------------|------|------|------|------|
| $P_1$      |      |      |      |      |
| $P_2$      |      | $+1$ |      | $-1$ |
| $P_3$      |      | $+1$ |      |      |
| $P_4$      |      | $+1$ | $+1$ |      |
| Eigenvectors | $w$ | $J_7w$ | $J_3w$ | $J_6w$ |

6.6.2. Integral structure on admissible $\text{Cl}_{6,1}$-module. The Clifford algebra $\text{Cl}_{6,1}$ is isomorphic to the space $\mathbb{C}(8)$. We can apply Theorem 4 or Theorem 3 and Corollary 4 to show the existence of an integral structure.
6.6.3. **Integral structure on admissible \( \text{Cl}_{5,2} \)-module.** The Clifford algebra \( \text{Cl}_{5,2} \) is isomorphic to the space \( \mathbb{H}(4) \oplus \mathbb{H}(4) \). Let \( z_1, \ldots, z_7 \) be orthonormal generators of \( \text{Cl}_{5,2} \), and \( (V, \langle \cdot, \cdot \rangle_V) \) an admissible \( \text{Cl}_{5,2} \)-module. Then \( J_i^2 := J_i^2 = -\text{Id}_V \), \( i = 1, \ldots, 5 \), \( J_j^2 := J_j^2 = \text{Id}_V \) for \( j = 6, 7 \). The isometric involutions

\[
P_1 = J_1 J_2 J_3 J_4, \quad P_2 = J_1 J_2 J_6 J_7, \quad P_3 = J_5 J_6 J_7.
\]

mutually commute. In this case we have two complementary isometric operators \( T_1 = J_1, T_2 = J_2 J_3 \). Here we present the tables of their commutation relations

\[
\begin{array}{c|cccccc}
\text{Involutions} & \text{Generators} & J_1 & J_2 & J_3 & J_4 & J_5 & J_6 & J_7 \\
\hline
P_1 = J_1 J_2 J_3 J_4 & a & a & a & c & c & c \\
P_2 = J_1 J_2 J_6 J_7 & a & a & c & c & a & a \\
P_3 = J_5 J_6 J_7 & a & a & a & c & c & c \\
\end{array}
\]

\[
\begin{array}{c|cccccc}
\text{Involutions} \setminus \text{Comp. op.} & J_1 (+ \rightarrow +) & J_2 J_3 (+ \rightarrow +) \\
\hline
P_1 = J_1 J_2 J_3 J_4 & a & c \\
P_2 = J_1 J_2 J_6 J_7 & a & a \\
P_3 = J_5 J_6 J_7 & \\
\end{array}
\]

Since there is no a complementary isometric operator with the property that it commutes with \( P_1 \) and \( P_2 \) and anti-commutes with \( P_3 \) we argue as in the case of \( \text{Cl}_{7,0} \) and use the property that isometry \( P_3 \) commutes with \( P_1 \) and \( P_2 \) and if it is necessary we change \( P_3 \) to \( P_3 \). We find a vector \( w \in V \) such that \( P_1 w = P_2 w = P_3 w = w \) and \( \langle w, w \rangle_V = 1 \). This relations will give the orthonormal basis and show that \( J_1 \) acts by permutation on this basis. The eigenspace decomposition is given by Table 11. The basis is given in (27).

**Table 11. Eigenspace decomposition: \( \text{Cl}_{5,2} \) case**

| Involutions | Eigenvalues |
|-------------|-------------|
| \( P_1 \)  | +1          |
| \( P_2 \)  | -1          |
| \( P_3 \)  | +1          |
| Eigenvector |             |
| \( w, J_5 w \) | \( J_6 w, J_7 w \) |
| \( J_1 J_3 J_6 w, J_1 J_3 J_7 w \) | \( J_1 J_4 w, J_1 J_5 w \) |
| \( J_1 J_6 w, J_1 J_7 w \) | \( J_3 J_6 w, J_3 J_7 w \) |

(27)

\[
\begin{align*}
v_1 &= w, & v_2 &= J_1 w, & v_3 &= J_2 w, & v_4 &= J_3 w, \\
v_5 &= J_4 w, & v_6 &= J_5 w, & v_7 &= J_1 J_3 w, & v_8 &= J_1 J_4 w, \\
v_9 &= J_5 w, & v_{10} &= J_6 w, & v_{11} &= J_1 J_6 w, & v_{12} &= J_1 J_7 w, \\
v_{13} &= J_3 J_6 w, & v_{14} &= J_3 J_7 w, & v_{15} &= J_1 J_3 J_6 w, & v_{16} &= J_1 J_3 J_7 w
\end{align*}
\]

with \( \langle v_\alpha, v_\alpha \rangle_V = 1, \alpha = 1, \ldots, 8 \) and \( \langle v_\alpha, v_\alpha \rangle_V = -1 \) for \( \alpha = 9, \ldots, 16 \). Finally, since the vectors \( w \) and \( J_1 J_3 J_6 w \) and \( J_1 J_3 J_7 w \) need not be orthogonal, we apply Lemma 4 to change the vector \( w \) to \( \tilde{w} \).

We constructed an integral structure in the admissible sub-module, which is irreducible.

6.6.4. **Integral structure on admissible \( \text{Cl}_{4,3} \)-module.** The Clifford algebra \( \text{Cl}_{4,3} \) is isomorphic to the space \( \mathbb{C}(8) \). Let \( z_1, \ldots, z_7 \) be generators of \( \text{Cl}_{4,3} \), and \( (V, \langle \cdot, \cdot \rangle_V) \) an admissible \( \text{Cl}_{4,3} \)-module. Then \( J_i^2 := J_i^2 = -\text{Id}_V \), \( i = 1, \ldots, 4 \), \( J_j^2 := J_j^2 = \text{Id}_V \) for \( j = 5, 6, 7 \). We use the same mutually commuting isometries as in the case of \( \text{Cl}_{0,7} \)-module

\[
P_1 = J_1 J_2 J_3 J_4, \quad P_2 = J_1 J_2 J_5 J_6, \quad P_3 = J_2 J_3 J_5 J_7.
\]

We start from the common eigenvector \( w \in V: P_1 w = P_2 w = P_3 w = w, \langle w, w \rangle_V = 1 \). The existence of such a common eigenvector \( w \) is guaranteed by Lemma 6 part 1) and presence of three
complementary isometries $T_1 = J_1$, $T_2 = J_2J_3$ and $T_3 = J_1J_2$ with the commutation relations

| Involutions\Comp. op. | $J_1(\rightarrow +)$ | $J_2J_3(\rightarrow +)$ | $J_1J_2(\rightarrow +)$ |
|-----------------------|-----------------------|------------------------|------------------------|
| $P_1 = J_1J_2J_3J_4$ | $a$                   | $c$                    | $c$                    |
| $P_2 = J_1J_2J_5J_6$ | $a$                   | $c$                    |                        |
| $P_3 = J_2J_3J_4J_7$ |                       |                        | $a$                    |

Then we have simultaneous eigenspace decomposition of $P_i$ showed in Table 12. The eigenvectors listed in Table 12 form a basis. Since the elements $w$ and $J_1J_3J_6w$ are not necessarily orthogonal, we apply Lemma 8 to make them orthogonal. The relations (18) show that $J_i$, $i = 1, \ldots, 7$, permute basis vectors up to sign. The constructed sub-module of admissible $\text{Cl}_{3,4}(w)$-module is admissible, integral, and irreducible.

6.6.5. **Integral structure on admissible $\text{Cl}_{3,4}$-module.** The Clifford algebra $\text{Cl}_{3,4}$ is isomorphic to the space $\mathbb{R}(8) \oplus \mathbb{R}(8)$. Let $z_1, \ldots, z_7$ be orthonormal generators of $\text{Cl}_{3,4}$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible $\text{Cl}_{3,4}$-module. Then $J_i^2 := J_i^2 = -\text{Id}_V$, $i = 1, \ldots, 3$, $J_j^2 := J_j^2 = \text{Id}_V$ for $j = 4, \ldots, 7$. We fix mutually commuting isometric involutions

$$\begin{align*}
P_1 &= J_1J_2J_4J_5, & P_2 &= J_2J_3J_5J_6, & P_3 &= J_1J_2J_6J_7, & P_4 &= J_3J_4J_5.
\end{align*}$$

They have the following commutation relations with representations of generators.

| Involutions\Generators | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ | $J_7$ |
|------------------------|-------|-------|-------|-------|-------|-------|-------|
| $P_1 = J_1J_2J_4J_5$  | $a$   | $a$   | $c$   | $a$   | $c$   | $a$   | $c$   |
| $P_2 = J_2J_3J_5J_6$  | $a$   | $a$   | $c$   | $a$   | $a$   | $c$   | $a$   |
| $P_3 = J_1J_2J_6J_7$  | $a$   | $c$   | $c$   | $c$   | $a$   | $a$   |       |
| $P_4 = J_3J_4J_5$     | $a$   | $a$   | $c$   | $c$   | $c$   | $a$   | $a$   |

Define the complementary operators $T_1 = J_1$, $T_2 = J_3$ and $T_3 = J_7$.

| Involutions\Comp. op. | $J_1(\rightarrow +)$ | $J_3(\rightarrow +)$ | $J_7(\rightarrow +)$ |
|-----------------------|-----------------------|-----------------------|-----------------------|
| $P_1 = J_1J_2J_4J_5$  | $a$                   | $c$                   | $c$                   |
| $P_2 = J_2J_3J_5J_6$  | $a$                   | $c$                   |                        |
| $P_3 = J_1J_2J_6J_7$  | $a$                   | $c$                   |                        |
| $P_4 = J_3J_4J_5$     | $a$                   | $c$                   |                        |

Since there are no complementary isometric operator with the property that it commutes with $P_1$ and $P_2$ and anti-commutes with $P_3$, we only know that the simultaneous eigenspaces of $P_1$ and $P_2$ are neutral spaces by Lemma 6 (part 1). The complementary anti-isometry $T_3 = J_7$ and Lemma 6 (part 2) guaranties that the space $E = E_{1+} \cap E_{2+} \cap E_{3+}$ either neutral or sign definite. We consider both possibilities.

Let the restriction of $\langle \cdot, \cdot \rangle_V$ on $E$ be neutral, then we argue as in the case of $\text{Cl}_{0,7}$ algebra and find (28)

$$P_1w = P_2w = P_3w = P_4w = w \quad \text{and} \quad \langle w, w \rangle_V = 1.$$  

In this case we can directly proceed further and construct an othonormal basis.

Let the restriction of $\langle \cdot, \cdot \rangle_V$ on $E$ be negative definite, then we change $P_3$ to $P_3 \Rightarrow J_2J_1J_6J_7$, that will make the space $E = E_{1+} \cap E_{2+} \cap E_{3+}$ positive definite space.
If the restriction of $\langle \cdot, \cdot \rangle_V$ on $E$ is positive definite, we do nothing. Thus, from now on we can assume that the restriction of $\langle \cdot, \cdot \rangle_V$ on $E = E_{1+} \cap E_{2+} \cap E_{3+}$ is positive definite. Hence it can happen only two cases for $w \in E$:

1. $\langle w + P_4 w, w + P_3 w \rangle_V = 0$, that is $P_4 w = -w$. We change the operator $P_1 = J_3 J_4 J_5$ to the operator $P_1 = J_4 J_3 J_5$, then the vector $w$ is a common eigenvector of all four involutions $P_i$ with the eigenvalue 1, i.e. $w$ satisfies (25).

2. $\langle w + P_4 w, w + P_3 w \rangle_V > 0$. In this case we get the eigenvector $\hat{w} = w + P_4 w$ of $P_4$ with the eigenvalue 1. Normalising the vector $\hat{w}$ we obtain that $\hat{w}$ satisfies (25).

So in the both cases the admissible sub-module of the Clifford algebra $\text{Cl}_{3,4}$ is 8 dimensional and decomposed into 8 common eigenspaces as shown in Table 13. The orthogonal basis is given in (25) with $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 4$ and $\langle v_\alpha, v_\alpha \rangle_V = -1$ for $\alpha = 5, \ldots, 8$. The relations (28) show also that operators $J_j$, $j = 1, \ldots, 7$, permute the basis up to sign. The constructed module is admissible integral, and irreducible.

| Involution | Eigenvectors |
|------------|--------------|
| $P_1$      | 1            |
| $P_2$      | 1            |
| $P_3$      | 1            |
| $P_4$      | 1            |
| $\hat{w}$  | 1            |

**Table 13. Eigenspace decomposition: Cl$_{3,4}$ case**

6.6.6. **Integral structure on admissible Cl$_{2,5}$-module.** The integral structure on the admissible Cl$_{2,5}$-module exists according to Theorem 3 and Corollar 4 since Cl$_{2,5} \cong$ Cl$_{4,3}$.

6.6.7. **Integral structure on admissible Cl$_{1,6}$-module.** The integral structure on the admissible Cl$_{1,6}$-module exists according to Theorem 3 and Corollar 4 since Cl$_{1,6} \cong$ Cl$_{5,2}$.

6.6.8. **Integral structure on admissible Cl$_{0,7}$-module.** The admissible integral Cl$_{0,7}$-module was constructed in Section 4.

6.7. **Integral structure on admissible Cl$_{r,s}$-modules with $r + s = 8$.**

6.7.1. **Integral structure on admissible Cl$_{8,0}$-module.** The Clifford algebra Cl$_{8,0}$ is isomorphic to the space $\mathbb{R}(16)$. Let $z_1, \ldots, z_8$ be generators of Cl$_{8,0}$, and $(V, \langle \cdot, \cdot \rangle_V)$ an admissible Cl$_{8,0}$-module with the positive definite product $\langle \cdot, \cdot \rangle_V$. Note that here we use the known fact that Cl$_{r,0}$-modules are admissible with a positive definite product. Then $J_i^2 := J_i^2 = -1_{V}, i = 1, \ldots, 8$. In this case we argue as in the case of Cl$_{0,8}$-modules and consider the mutually commuting isometric involution

$$P_1 = J_1 J_2 J_3 J_4, \quad P_2 = J_1 J_2 J_5 J_6, \quad P_3 = J_2 J_3 J_5 J_7, \quad P_4 = J_1 J_2 J_7 J_8.$$ 

We start from the common eigenvector $w \in V$: $P_1 w = P_2 w = P_3 w = P_4 w = w$ with $\langle w, w \rangle_V = 1$. The relations (19) are still true and they left 16 elements of orthonormal basis:

$$v_1 = w, \quad v_2 = J_1 J_2 w, \quad v_3 = J_1 J_3 w, \quad v_4 = J_1 J_4 w,$$

$$v_5 = J_1 J_5 w, \quad v_6 = J_1 J_6 w, \quad v_7 = J_1 J_7 w, \quad v_8 = J_1 J_8 w,$$

$$v_9 = J_1 w, \quad v_{10} = J_2 w, \quad v_{11} = J_3 w, \quad v_{12} = J_4 w,$$

$$v_{13} = J_5 w, \quad v_{14} = J_6 w, \quad v_{15} = J_7 w, \quad v_{16} = J_8 w,$$

with $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 16$. We see from (19) that $J_i$, $i = 1, \ldots, 8$, permute basis vectors up to sign. The constructed 16 dimensional sub-module is admissible, integral, and irreducible.
6.7.2. Integral structure on admissible $\text{Cl}_{7,1}$-module. The Clifford algebra $\text{Cl}_{7,1}$ is isomorphic to the space $\mathbb{R}(16)$. We can apply Theorem 6 or Theorem 8 and Corollary 7 to show the existence of an integral structure. The admissible module is irreducible.

6.7.3. Integral structure on $\text{Cl}_{6,2}$-admissible module. The Clifford algebra $\text{Cl}_{6,2}$ is isomorphic to the space $\mathbb{H}(8)$. Let $z_1, \ldots, z_8$ be orthonormal generators of $\text{Cl}_{6,2}$ and $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible $\text{Cl}_{6,2}$-module. Then $J^2_j := J^2_{z_j} = -\text{Id}_V$, $i = 1, \ldots, 6$, $J^2_j := J^2_{z_j} = \text{Id}_V$ for $j = 7, 8$. We consider the three mutually commuting isometric involutions

$$P_1 = J_1 J_2 J_3 J_4, \quad P_2 = J_1 J_2 J_5 J_6, \quad \text{and} \quad P_3 = J_1 J_2 J_7 J_8,$$

and four complementary operators: $T_1 = J_1$, $T_2 = J_5$, $T_3 = J_7$, and $T_4 = J_1 J_3 J_5 J_7$. Then we have commutation relations between involutions $P_i$ and complementary operators $T_j$.

| Involutions | Comp. op. $\langle \cdot \rangle_V$ | $J_1(\to +)$ | $J_5(\to +)$ | $J_7(\to -)$ | $J_1 J_3 J_5 J_7(\to -)$ |
|-------------|-------------------------------|---------------|---------------|---------------|----------------------------|
| $P_1$       | $J_1 J_2 J_3 J_4$             | $a$           | $c$           | $c$           | $c$                        |
| $P_2$       | $J_1 J_2 J_5 J_6$             | $a$           | $c$           | $c$           | $c$                        |
| $P_3$       | $J_1 J_2 J_7 J_8$             | $a$           | $c$           | $c$           | $c$                        |

From these relations the common eigenspace $E_{1+} \cap E_{2+}$ of the first two involutions is neutral space by Lemma 3 part 1). Then we use Lemma 8 and conclude that the common eigenspace $E_{1+} \cap E_{2+} \cap E_{3+}$ of all three involutions $P_i$ is a neutral space. So, we may find an element $w$ such that $P_1 w = P_2 w = P_3 w = w$, and $\langle w, w \rangle_V = 1$. The eigenspace decomposition presented in Table 14. By a direct calculations, we know that each 4 vectors belong to a common eigenspace, especially the vectors $w, J_1 J_2 w, J_1 J_3 J_5 J_7 w, J_1 J_3 J_5 J_8 w$ are in the common eigenspace with eigenvalue of 1. First two are spacelike and orthonormal and also last two are timelike and orthonormal. Unfortunately, first two and last two need not be orthogonal, so we need to apply Lemma 3. Since the operators $J_1 J_3 J_5 J_7 w$ and $J_1 J_3 J_5 J_8 w$ are anti-involutions and anti-commute each other, they satisfy the conditions of Lemma 3. Hence we can obtain a new common eigenvector $\tilde{w}$: $P_1 \tilde{w} = P_2 \tilde{w} = P_3 \tilde{w} = \tilde{w}$, such that $\langle \tilde{w}, J_1 J_3 J_5 J_7 \tilde{w} \rangle_V = \langle \tilde{w}, J_1 J_3 J_5 J_8 \tilde{w} \rangle_V = 0$. The linear dependence relations arising from $P_1 \tilde{w} = P_2 \tilde{w} = P_3 \tilde{w} = \tilde{w}$ shows that all other vectors presented in Table 14 become orthogonal and moreover $J_j, j = 1, \ldots, 8$, permute them up to sign. The vectors listed in Table 14 form an integral basis of 32 dimensional admissible sub-module of $\text{Cl}_{6,2}$-module. Moreover this sub-module is irreducible.

**Table 14. Eigenspace decomposition: $\text{Cl}_{6,2}$ case**

| Involutions | Eigenvalues |
|-------------|-------------|
| $P_1$       | $+1$        |
| $P_2$       | $+1$        |
| $P_3$       | $+1$        |
| Eigenvectors|             |
| $w$         | $J_7 w$     |
| $J_1 J_2 w$ | $J_5 w$     |
| $J_1 J_3 J_5 J_7 w$ | $J_1 J_3 w$ |
| $J_1 J_3 J_5 J_8 w$ | $J_3 J_7 w$ |
| $J_2 J_3 J_5 J_7 w$ | $J_3 J_5 w$ |
| $J_2 J_3 J_5 J_8 w$ | $J_5 J_7 w$ |
| $J_3 J_7 J_8 J_7 w$ | $J_5 J_5 w$ |

6.7.4. Integral structure on admissible $\text{Cl}_{5,3}$-module. The Clifford algebra $\text{Cl}_{5,3}$ is isomorphic to the space $\mathbb{H}(8)$. Let $z_1, \ldots, z_8$ be orthonormal generators of $\text{Cl}_{5,3}$ and $(V, \langle \cdot, \cdot \rangle_V)$ be an admissible $\text{Cl}_{5,3}$-module. Then $J^2_i := J^2_{z_i} = -\text{Id}_V$, $i = 1, \ldots, 5$, $J^2_j := J^2_{z_j} = \text{Id}_V$ for $j = 6, 7, 8$. We consider the three mutually commuting isometric involutions

$$P_1 = J_1 J_2 J_3 J_4, \quad P_2 = J_1 J_2 J_6 J_7, \quad \text{and} \quad P_3 = J_2 J_3 J_7 J_8.$$
In this case, we choose four complementary operators:

\[ T_1 = J_1, \quad T_2 = J_1J_3, \quad T_3 = J_8, \quad \text{and} \quad T_4 = J_1J_3J_5J_7. \]

Then we have commutation relations between involutions \( P_i \) and complementary operators \( T_j \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Involutions} & \text{Comp. op.} & J_1(+ \rightarrow +) & J_1J_3(+ \rightarrow +) & J_8(+ \rightarrow -) & J_1J_3J_5J_7(+ \rightarrow -) \\
\hline
P_1 & J_1J_2J_3J_4 & a & c & c & c \\
\hline
P_2 & J_1J_2J_5J_7 & a & c & c & c \\
\hline
P_3 & J_2J_3J_7J_8 & a & c & c & c \\
\hline
\end{array}
\]

From these relations the common eigenspace \( E_{1+} \cap E_{2+} \cap E_{3+} \) of the first two involutions is neutral space by Lemma 6 part 1). Then we use Lemma 8 and conclude that the common eigenspace \( E_{1+} \cap E_{2+} \cap E_{3+} \) of all three involutions \( P_i \) is a neutral space. So, we may find an element \( w \) such that \( P_1w = P_2w = P_3w = w \) and \( \langle w, w \rangle_V = 1 \). The eigenspace decomposition presented in Table 15. We need to apply Lemma 9 to operators \( \Omega_1 = J_1J_2J_8 \) and \( \Omega_2 = J_1J_2J_5J_8 \) to make the vectors in \( E_{1+} \cap E_{2+} \cap E_{3+} \) orthogonal. It also makes all other vectors orthogonal by relations \( P_1w = P_2w = P_3w = w \). The same relations show that \( J_j, j = 1, \ldots, 8 \) permute the basis up to sign. It proves that basis listed in Table 15 is integral. Finally we notice that the constructed sub-module is irreducible, since its dimension is 32.

6.7.5. Integral structure on admissible \( \text{Cl}_{4,4} \)-module. The Clifford algebra \( \text{Cl}_{4,4} \) is isomorphic to the space \( \mathbb{R}(16) \). Let \( z_1, \ldots, z_8 \) be generators of \( \text{Cl}_{4,4} \), and \( (V, \langle \cdot, \cdot \rangle_V) \) an admissible \( \text{Cl}_{4,4} \)-module. Then \( J_i^2 := J_{z_i}^2 = -\text{Id}_V, i = 1, \ldots, 4, \) \( J_j^2 := J_{z_j}^2 = \text{Id}_V \) for \( j = 5, \ldots, 8 \). Choose mutually commuting isometric involutions

\[ P_1 = J_1J_2J_3J_4, \quad P_2 = J_1J_2J_5J_6, \quad P_3 = J_2J_3J_5J_7, \quad \text{and} \quad P_4 = J_1J_2J_7J_8, \]

and four complementary operators \( T_1 = J_1, \quad T_2 = J_1J_3, \quad T_3 = J_8, \quad \text{and} \quad T_4 = J_1J_3J_5J_7, \) and four complementary operators \( T_5 = J_1J_2J_3J_4, \quad T_6 = J_1J_2J_5J_6, \quad T_7 = J_7J_8, \quad \text{and} \quad T_8 = J_1J_2J_7J_8, \)

Here are the tables of the commutation relations with generators and complementary operators:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Involutions} & \text{Generator} & J_1 & J_2 & J_3 & J_4 & J_5 & J_6 & J_7 & J_8 \\
\hline
P_1 & J_1J_2J_3J_4 & a & a & a & a & c & c & c & c \\
\hline
P_2 & J_1J_2J_5J_6 & a & a & c & c & c & a & c & c \\
\hline
P_3 & J_2J_3J_5J_7 & c & a & a & c & a & a & c & c \\
\hline
P_4 & J_1J_2J_7J_8 & a & a & c & c & c & a & c & a \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Involutions} & \text{Comp. op.} & J_1(+ \rightarrow +) & J_1J_3(+ \rightarrow +) & J_1J_2(+ \rightarrow +) & J_8(+ \rightarrow -) \\
\hline
P_1 & J_1J_2J_3J_4 & a & c & c & c \\
\hline
P_2 & J_1J_2J_5J_6 & a & c & c & c \\
\hline
P_3 & J_2J_3J_5J_7 & a & c & c & c \\
\hline
P_4 & J_1J_2J_7J_8 & a & c & c & c \\
\hline
\end{array}
\]

From these relations we can choose a vector \( w \in V \) such that

\[ P_1w = P_2w = P_3w = P_4w = w, \quad \text{and} \quad \langle w, w \rangle_V = 1, \]
by Lemma 6 part 1) and Lemma 7. Hence by the relations (19) we have a simultaneous eigenspace decomposition of a subspace in $V$ spanned by the 16 common eigenvectors that form the basis

$$
v_1 = w, \quad v_2 = J_1 w, \quad v_3 = J_2 w, \quad v_4 = J_3 w,$$
$$v_5 = J_4 w, \quad v_6 = J_1 J_2 w, \quad v_7 = J_1 J_3 w, \quad v_8 = J_4 w,$$
$$v_9 = J_5 w, \quad v_{10} = J_6 w, \quad v_{11} = J_7 w, \quad v_{12} = J_8 w,$$
$$v_{13} = J_1 J_5 w, \quad v_{14} = J_1 J_6 w, \quad v_{15} = J_1 J_7 w, \quad v_{16} = J_1 J_8 w,$$

with $\langle v_\alpha, v_\alpha \rangle_V = 1$, $\alpha = 1, \ldots, 8$ and $\langle v_\alpha, v_\alpha \rangle_V = -1$ for $\alpha = 9, \ldots, 16$. The relations (19) also shows that $J_j$, $j = 1, \ldots, 8$ permutes the basis up to sign. So the minimal admissible module of $\mathrm{Cl}_{4,4}$ is 16 dimensional, irreducible and integral.

### Table 16. Eigenspace decomposition: $\mathrm{Cl}_{4,4}$ case

| Involution $P_j$ | Eigenvalues $\pm 1$ |
|------------------|----------------------|
| $P_1$            | $+1$                |
| $P_2$            | $+1$                |
| $P_3$            | $+1$                |
| $P_4$            | $+1$                |
| Eigenvectors $v$ | $J_1 w$, $J_2 w$, $J_3 w$, $J_4 w$, $J_5 w$, $J_6 w$, $J_7 w$, $J_8 w$, $J_9 w$, $J_{10} w$, $J_{11} w$, $J_{12} w$, $J_{13} w$, $J_{14} w$, $J_{15} w$, $J_{16} w$ |

shows that $J_j$, $j = 1, \ldots, 8$ permutes the basis up to sign. So the minimal admissible module of $\mathrm{Cl}_{4,4}$ is 16 dimensional, irreducible and integral.

#### 6.7.6. Integral structure on admissible $\mathrm{Cl}_{3,5}$-module.

The integral structure on the admissible $\mathrm{Cl}_{3,5}$-module exists according to Theorem 3 and Corollary 4 since $\mathrm{Cl}_{3,5} \cong \mathrm{Cl}_{4,4}$.

#### 6.7.7. Integral structure on admissible $\mathrm{Cl}_{2,6}$-module.

The integral structure on the admissible $\mathrm{Cl}_{2,6}$-module exists according to Theorem 3 and Corollary 4 since $\mathrm{Cl}_{2,6} \cong \mathrm{Cl}_{5,3}$.

#### 6.7.8. Integral structure on admissible $\mathrm{Cl}_{1,7}$-module.

The integral structure on the admissible $\mathrm{Cl}_{1,7}$-module exists according to Theorem 3 and Corollary 4 since $\mathrm{Cl}_{1,7} \cong \mathrm{Cl}_{6,2}$.

#### 6.7.9. Integral structure on admissible $\mathrm{Cl}_{0,8}$-module.

Admissible integral $\mathrm{Cl}_{0,8}$-module was constructed in Section 4.

#### 7. Admissible modules obtained by tensor product

**7.1. Bott periodicity and admissible modules of dimensions** $r + s > 8$. In this section we present some theorems that allow to use the Bott periodicity

$$\mathrm{Cl}_{r+8,s} \cong \mathrm{Cl}_{r+4,s+4} \cong \mathrm{Cl}_{r,s} \otimes \mathbb{R}(16), \quad \mathrm{Cl}_{r,s+8} \cong \mathrm{Cl}_{r+4,s+4} \cong \mathrm{Cl}_{r,s} \otimes \mathbb{R}(16)$$

of Clifford algebras in order to prove that $\mathrm{Cl}_{r,s}$-modules are integer for $r + s > 8$.

**Theorem 5.** Let us assume that $(V, \langle \cdot, \cdot \rangle_V)$ is an admissible integral $\mathrm{Cl}_{r,s}$-module and $(U, \langle \cdot, \cdot \rangle_U)$ is an admissible integral $\mathrm{Cl}_{0,8}$-module, where the representations $J_{y_j} \in \text{End}(U)$ permute the integral basis of $U$ up to sign for all orthonormal generators $y_j$ of the Clifford algebra $\mathrm{Cl}_{0,8}$. Then the tensor product vector space $(V \otimes U, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_U)$ is an admissible integral $\mathrm{Cl}_{r+s+8}$-module.
Proof. Let \((z_1, \ldots, z_r, \zeta_1, \ldots, \zeta_s)\) be orthonormal generators of the Clifford algebra \(\text{Cl}_{r,s}\) with the quadratic form \(Q_{r,s}(a) = \sum_{i=1}^{r} a_i^2 - \sum_{k=1}^{s} a_k^2\) for \(a = \sum_{i=1}^{r} a_i z_i + \sum_{k=1}^{s} \alpha_k \zeta_k\). Let \(\{y_1, \ldots, y_8\}\) be orthonormal generators for \(\text{Cl}_{0,8}\) with quadratic form \(Q_{0,8}(b) = -\sum_{j=1}^{8} b_j^2\) for \(b = \sum_{j=1}^{8} b_j y_j\) and, finally let \(\tilde{z}_1, \ldots, \tilde{z}_r, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_s + 8\) be orthonormal generators for the Clifford algebra \(\text{Cl}_{r,s+8}\) with quadratic form \(Q_{r,s+8}(c) = \sum_{i=1}^{r} c_i^2 - \sum_{k=1}^{s+8} c_k^2\) for \(c = \sum_{i=1}^{r} c_i \tilde{z}_i + \sum_{k=1}^{s+8} \zeta_k \tilde{\zeta}_k\).

We know that the minimal admissible \(\text{Cl}_{0,8}\)-module \((U, \langle \cdot, \cdot \rangle_U)\) is isomorphic to \(\mathbb{R}^{8,8}\) with quadratic form \(Q_{8,8}(u) = \sum_{i=1}^{8} u_i^2 - \sum_{j=9}^{16} u_j^2\) for \(u = \sum_{i=1}^{16} u_i e_i\), where \(e_i, i = 1, \ldots, 16\) is the standard basis in \(\mathbb{R}^{8,8}\). This module is also irreducible. Then the endomorphisms \(J_{y_j} \in \text{End}(\mathbb{R}^{8,8})\) are such that \(J_{y_j}^2 = \text{Id}_{\mathbb{R}^{8,8}}, j = 1, \ldots, 8, J_{y_j} J_{y_i} = -J_{y_i} J_{y_j}\) for \(i \neq j\).

Now one needs to find \(E \in \text{End}(\mathbb{R}^{8,8})\) satisfying conditions
\[
(29) \quad E J_{y_j} = -J_{y_j} E, \quad j = 1, \ldots, 8, \quad E^2 = \text{Id}_{\mathbb{R}^{8,8}},
\]
\[
(30) \quad \langle E u, u' \rangle_{\mathbb{R}^{8,8}} = \langle u, E u' \rangle_{\mathbb{R}^{8,8}} \quad \text{for} \quad u, u' \in \mathbb{R}^{8,8},
\]
where the scalar product in (30) is the scalar product defined by the quadratic form \(Q_{8,8}\). Define \(E = \prod_{j=1}^{8} J_{y_j}\) to be a volume form for the Clifford algebra \(\text{Cl}_{0,8}\). Then it is easy to check that \(E\) satisfies conditions (29) and (30).

Denote \(\bar{V} = V \otimes \mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_{\bar{V}} = \langle \cdot, \cdot \rangle_{V} \otimes \mathbb{R}^{8,8}\) and notice that the scalar product \(\langle \cdot, \cdot \rangle_{\bar{V}}\) is non-degenerate. Set
\[
\mathbb{R}^{r,s+8} \ni z_i \mapsto \tilde{J}_{z_i} = J_{z_i} \otimes E \in \text{End}(V \otimes \mathbb{R}^{8,8}), \quad i = 1, \ldots, r,
\]
\[
\mathbb{R}^{r,s+8} \ni \zeta_k \mapsto \tilde{J}_{\zeta_k} = J_{\zeta_k} \otimes E \in \text{End}(V \otimes \mathbb{R}^{8,8}), \quad k = 1, \ldots, s,
\]
\[
\mathbb{R}^{r,s+8} \ni \zeta_{s+j} \mapsto \tilde{J}_{\zeta_{s+j}} = \text{Id}_V \otimes J_{y_j} \in \text{End}(V \otimes \mathbb{R}^{8,8}), \quad j = 1, \ldots, 8,
\]
where \(J_{z_i}, J_{\zeta_k} \in \text{End}(V), i = 1, \ldots, r, k = 1, \ldots, s, s + 1, \ldots, 8\), such that \(J_{z_i}^2 = -\text{Id}_V, J_{\zeta_k}^2 = \text{Id}_V, \text{and } J_{y_j} \in \text{End}(\mathbb{R}^{8,8}), j = 1, \ldots, 8\), such that \(J_{y_j}^2 = \text{Id}_{\mathbb{R}^{8,8}}\). Then, it is easy to see that
\[
\tilde{J}_{z_i}^2 = -\text{Id}_{\bar{V}} \quad \text{for} \quad i = 1, \ldots, r, \quad \tilde{J}_{\zeta_k}^2 = \text{Id}_{\bar{V}} \quad \text{for} \quad k = 1, \ldots, s + 8.
\]
Moreover, we have for \(v \otimes e \in V \otimes \mathbb{R}^{8,8}\),
\[
\tilde{J}_{\zeta_{s+j}} \tilde{J}_{z_i} v \otimes e = v \otimes J_{y_j} J_{y_j} e = -v \otimes J_{y_j} J_{y_j} e = -\tilde{J}_{\zeta_{s+j}} \tilde{J}_{z_i} v \otimes e
\]
for \(j_1, j_2 = 1, \ldots, 8, j_1 \neq j_2\).
\[
\tilde{J}_{\zeta_k} \tilde{J}_{\zeta_{s+j}} v \otimes e = J_{\zeta_k} v \otimes E J_{y_j} e = -J_{\zeta_k} v \otimes J_{y_j} E e = -\tilde{J}_{\zeta_{s+j}} \tilde{J}_{\zeta_k} v \otimes e
\]
for \(k = 1, \ldots, s, j = 1, \ldots, 8\).
\[
\tilde{J}_{z_i} \tilde{J}_{\zeta_{s+j}} v \otimes e = J_{z_i} v \otimes E J_{y_j} e = -J_{z_i} v \otimes J_{y_j} E e = -\tilde{J}_{z_i} \tilde{J}_{\zeta_{s+j}} v \otimes e
\]
for \(i = 1, \ldots, r, j = 1, \ldots, 8\).
\[
\tilde{J}_{z_i} \tilde{J}_{z_i} v \otimes e = J_{z_i} J_{z_i} v \otimes E^2 e = -J_{z_i} J_{z_i} v \otimes E^2 e = -\tilde{J}_{z_i} \tilde{J}_{z_i} v \otimes e
\]
for \(i_1, i_2 = 1, \ldots, r, i_1 \neq i_2\).
\[
\tilde{J}_{\zeta_k} \tilde{J}_{\zeta_k} v \otimes e = J_{\zeta_k} J_{\zeta_k} v \otimes E^2 e = -J_{\zeta_k} J_{\zeta_k} v \otimes E^2 e = -\tilde{J}_{\zeta_k} \tilde{J}_{\zeta_k} v \otimes e
\]
for \(k_1, k_2 = 1, \ldots, s, k_1 \neq k_2\).
\[
\tilde{J}_{z_i} \tilde{J}_{\zeta_k} v \otimes e = J_{z_i} J_{\zeta_k} v \otimes E^2 e = -J_{\zeta_k} J_{z_i} v \otimes E^2 e = -\tilde{J}_{z_i} \tilde{J}_{\zeta_k} v \otimes e
\]
for \(i = 1, \ldots, r, k = 1, \ldots, s\).
The next step is to verify that the scalar product $\langle \cdot, \cdot \rangle_{\hat{V}} = \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}}$ satisfies
\[
\langle \hat{J}_z \hat{v}, \hat{v}' \rangle_{\hat{V}} + \langle \hat{v}, \hat{J}_z \hat{v}' \rangle_{\hat{V}} = 0.
\]
We write $\hat{z} = a + b$, where $a \in \mathbb{R}^{r,s}$, $b \in \mathbb{R}^{0,8}$, and $\hat{v} = v \otimes u$, $\hat{v}' = v' \otimes u'$ for $v, v' \in V$, $u, u' \in \mathbb{R}^{8,8}$. Then
\[
\langle \hat{J}_z \hat{v}, \hat{v}' \rangle_{\hat{V}} + \langle \hat{v}, \hat{J}_z \hat{v}' \rangle_{\hat{V}} = \langle (\hat{J}_a + \hat{J}_b) \hat{v}, \hat{v}' \rangle_{\hat{V}} + \langle \hat{v}, (\hat{J}_a + \hat{J}_b) \hat{v}' \rangle_{\hat{V}}
\]
\[
= \langle \hat{J}_a \hat{v} \otimes E_u, v' \otimes u' \rangle_{\hat{V}} + \langle \hat{v}, \hat{J}_a \hat{v}' \otimes E_u \rangle_{\hat{V}} + \langle v \otimes J_b u, v' \otimes u' \rangle_{\hat{V}} + \langle v \otimes u, J_b v' \otimes E_u \rangle_{\hat{V}}
\]
\[
= \langle J_a v \otimes E_u, v' \otimes u' \rangle_{\hat{V}} + \langle v \otimes u, J_a v' \otimes E_u \rangle_{\hat{V}} + \langle v \otimes J_b u, v' \otimes u' \rangle_{\hat{V}} + \langle v \otimes u, J_b v' \otimes E_u \rangle_{\hat{V}}
\]
\[
= \langle J_a v, v' \rangle_{V} \langle E_u, u' \rangle_{\mathbb{R}^{8,8}} + \langle v, J_a v' \rangle_{V} \langle E_u, u' \rangle_{\mathbb{R}^{8,8}}
\]
\[
+ \langle v, v' \rangle_{\mathbb{R}^{8,8}} \langle J_b u, J_b u' \rangle_{\mathbb{R}^{8,8}} + \langle v, J_b u \rangle_{\mathbb{R}^{8,8}} \langle u, J_a v' \rangle_{\mathbb{R}^{8,8}}
\]
\[
= 0.
\]
\[
\langle \hat{J}_z \hat{v}, \hat{v}' \rangle_{\hat{V}} + \langle \hat{v}, \hat{J}_z \hat{v}' \rangle_{\hat{V}} = 0.
\]

To show that the resulting $\text{Cl}_{r,s+8}$-module is integral we assume that both modules $(V, \langle \cdot, \cdot \rangle_V)$, $(\mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}})$ are integral. Then, if $\{v_0\}_{0}^{\dim V}$ and $\{e_p\}_{p=1}^{16}$ are integral bases for $V$ and $\mathbb{R}^{8,8}$ respectively, we denote by $\{\hat{v}_n = v_\alpha \otimes e_p\}_{n=1}^{\dim V}$ the basis of $\hat{V}$. We assumed that the maps $J_{y_j}$, $j = 1, \ldots, 8$, permute the basis $\{e_p\}_{p=1}^{16}$ up to sign. Then the map $E$ also permutes the basis $\{e_p\}_{p=1}^{16}$. We have
\[
\langle \hat{J}_{\hat{z}} \hat{v}, \hat{v}_m \rangle_{\hat{V}} = \langle J_z v_\alpha \otimes E e_p, v_\beta \otimes e_q \rangle_{\hat{V}} = \langle J_z v_\alpha, v_\beta \rangle_V \cdot \langle E e_p, e_q \rangle_{\mathbb{R}^{8,8}} = \pm 1 or 0,
\]
\[
\langle \hat{J}_{\hat{c}} \hat{v}, \hat{v}_m \rangle_{\hat{V}} = \langle J_c v_\alpha \otimes E e_p, v_\beta \otimes e_q \rangle_{\hat{V}} = \langle J_c v_\alpha, v_\beta \rangle_V \cdot \langle E e_p, e_q \rangle_{\mathbb{R}^{8,8}} = \pm 1 or 0,
\]
for all $i = 1, \ldots, r$, $k = 1, \ldots, s$ and $n, m = 1, \ldots, 16 \dim V$. Analogously
\[
\langle \hat{J}_{\hat{z}} \hat{v}, \hat{v}_m \rangle_{\hat{V}} = \langle v_\alpha \otimes J_y e_p, v_\beta \otimes e_q \rangle_{\hat{V}} = \langle v_\alpha, v_\beta \rangle_V \cdot \langle J_y e_p, e_q \rangle_{\mathbb{R}^{8,8}} = \pm 1 or 0,
\]
for all $j = 1, \ldots, 8$ and $n, m = 1, \ldots, 16 \dim V$.

\[\square\]

**Theorem 6.** Let us assume that $(V, \langle \cdot, \cdot \rangle_V)$ is an admissible integral $\text{Cl}_{r,s}$-module and $(U, \langle \cdot, \cdot \rangle_U)$ is an admissible integral $\text{Cl}_{0,8}$-module, where the representations $J_{y_j} \in \text{End}(U)$ permute the integral basis of $U$ up to sign for all orthonormal generators $y_j$ of the Clifford algebra $\text{Cl}_{8,0}$. Then the scalar product vector space $(V \otimes U, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_U)$ is an admissible integral $\text{Cl}_{r+8,s}$-module.

**Proof.** Let $(z_1, \ldots, z_r, \zeta_1, \ldots, \zeta_s)$ be orthonormal generators of the Clifford algebra $\text{Cl}_{r,s}$ with the quadratic form $Q_{r,s}(a) = \sum_{i=1}^{r} a_i^2 - \sum_{k=1}^{s} \alpha_k^2$ for $a = \sum_{i=1}^{r} a_i z_i + \sum_{k=1}^{s} \alpha_k \zeta_k$. Let $\{y_1, \ldots, y_8\}$ be orthonormal generators for $\text{Cl}_{8,0}$ with quadratic form $Q_{8,0}(b) = \sum_{j=1}^{8} b_j^2$ for $b = \sum_{j=1}^{8} b_j y_j$ and, finally let $\tilde{z}_1, \ldots, \tilde{z}_{r+8}, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_s$ be orthonormal generators for the Clifford algebra $\text{Cl}_{r+8,s}$ with quadratic form $Q_{r+8,s}(c) = \sum_{i=1}^{r+8} c_i^2 - \sum_{k=1}^{s} \tilde{\zeta}_k^2$ for $c = \sum_{i=1}^{r+8} c_i \tilde{z}_i + \sum_{k=1}^{s} \tilde{\zeta}_k \tilde{\zeta}_k$.

We know that the minimal admissible $\text{Cl}_{8,0}$-module $(U, \langle \cdot, \cdot \rangle_U)$ is isomorphic to $\mathbb{R}^{16,0} = \mathbb{R}^{16}$ with quadratic form $Q_{16}(u) = \sum_{i=1}^{16} u_i^2$ for $u = \sum_{i=1}^{16} u_i e_i$, where $e_i$, $i = 1, \ldots, 16$, is the standard basis in $\mathbb{R}^{16}$. This module is also irreducible. Then for the endomorphisms $J_{x_j} \in \text{End}(\mathbb{R}^{16})$ we have $J_{x_j}^2 = -\text{Id}_{\mathbb{R}^{16}}$, $j = 1, \ldots, 8$, $J_{x_j} J_{x_j} = -J_{x_j} J_{x_j}$ for $i \neq j$.

Now we want to find $\mathcal{E} \in \text{End}(\mathbb{R}^{16})$ satisfying conditions
\[
\mathcal{E} J_{x_j} = -J_{x_j} \mathcal{E}, \quad j = 1, \ldots, 8, \quad \mathcal{E}^2 = \text{Id},
\]
\[
(\mathcal{E} u, u')_{\mathbb{R}^{16}} = (u, \mathcal{E} u')_{\mathbb{R}^{16}} \quad \text{for} \quad u, u' \in \mathbb{R}^{10}
\]
where the inner product in (33) is the standard Euclidean product. Define \( \mathcal{E} = \prod_{j=1}^{8} J_{y_j} \) to be a volume form for \( \text{Cl}_{8,0} \). Then it is easy to check that \( \mathcal{E} \) satisfies conditions (32) and (33).

Denote \( \tilde{V} = V \otimes \mathbb{R}^{16}, \langle \cdot, \cdot \rangle_{\tilde{V}} = \langle \cdot, \cdot \rangle_{V} \langle \cdot, \cdot \rangle_{\mathbb{R}^{16}} \) and notice that the scalar product \( \langle \cdot, \cdot \rangle_{\tilde{V}} \) is non-degenerate. Set

\[
\mathbb{R}^{r+8,s} \ni \tilde{z}_i \mapsto J_{\tilde{z}_i} = J_{z_i} \otimes \mathcal{E} \in \text{End}(V \otimes \mathbb{R}^{16}), \quad i = 1, \ldots, r,
\]

\[
\mathbb{R}^{r+8,s} \ni \tilde{\zeta}_k \mapsto J_{\tilde{\zeta}_k} = J_{\zeta_k} \otimes \mathcal{E} \in \text{End}(V \otimes \mathbb{R}^{16}), \quad k = 1, \ldots, s,
\]

\[
\mathbb{R}^{r+8,s} \ni \tilde{z}_{r+j} \mapsto J_{\tilde{z}_{r+j}} = \text{Id}_V \otimes J_{x_j} \in \text{End}(V \otimes \mathbb{R}^{16}), \quad j = 1, \ldots, 8,
\]

where \( J_{z_i}, J_{\zeta_k} \in \text{End}(V), \) \( i = 1, \ldots, r, k = 1, \ldots, s, \) such that \( J_{z_i}^2 = -\text{Id}_V, J_{\zeta_k}^2 = \text{Id}_V, \) and \( J_{x_j} \in \text{End}(\mathbb{R}^{16}), \) \( j = 1, \ldots, 8, \) such that \( J_{x_j}^2 = -\text{Id}_{\mathbb{R}^{16}}. \) Then, we finish the proof as in Theorem 5.

**Theorem 7.** Let us assume that \((V, \langle \cdot, \cdot \rangle_{V})\) is an admissible integral \( \text{Cl}_{r,s} \)-module and \((U, \langle \cdot, \cdot \rangle_{U})\) is an admissible integral \( \text{Cl}_{4,4} \)-module, where the representations \( J_{y_j} \in \text{End}(U) \) permute the integral basis of \( U \) up to sign for all orthonormal generators \( y_j \) of the Clifford algebra \( \text{Cl}_{4,4} \). Then the tensor product vector space \( (V \otimes U, \langle \cdot, \cdot \rangle_{V} \langle \cdot, \cdot \rangle_{U}) \) is an admissible integral \( \text{Cl}_{r+s+4} \)-module.

**Proof.** Let \((z_1, \ldots, z_r, \xi_1, \ldots, \xi_s)\) be orthonormal generators of the Clifford algebra \( \text{Cl}_{r,s} \) with the quadratic form \( Q_{r,s}(a) = \sum_{i=1}^{r} a_i^2 - \sum_{k=1}^{s} a_k^2 \). Let the collection \( \{z_1, z_2, z_3, x_1, y_1, y_2, y_3, y_4\} \) be orthonormal generators for \( \text{Cl}_{4,4} \) with quadratic form \( Q_{4,4}(b) = \sum_{i=1}^{4} b_i^2 - \sum_{j=1}^{4} b_j^2 \). Denote \( \tilde{z}_1, \ldots, \tilde{z}_{r+4}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{s+4} \) orthonormal generators for the Clifford algebra \( \text{Cl}_{r+s+4} \) with quadratic form \( Q_{r+s+4}(c) = \sum_{i=1}^{r+s+4} c_i^2 - \sum_{k=1}^{s+4} c_k^2 \).

We know that the minimal admissible \( \text{Cl}_{4,4} \)-module \((U, \langle \cdot, \cdot \rangle_{U})\) is isomorphic to \( \mathbb{R}^{8,8} \) with quadratic form \( Q_{8,8} \). This module is also irreducible. Then for the endomorphisms \( J_{z_i}, J_{y_j} \in \text{End}(\mathbb{R}^{8,8}) \) we have \( J_{z_i}^2 = -\text{Id}_{\mathbb{R}^{8,8}}, J_{y_j}^2 = \text{Id}_{\mathbb{R}^{8,8}}, j = 1, \ldots, 4, \) and moreover all of them mutually anti-commute.

We define the endomorphism \( \mathcal{E} : \mathbb{R}^{8,8} \rightarrow \mathbb{R}^{8,8} \) by \( \mathcal{E} = \prod_{i=1}^{4} J_{z_i} \prod_{j=1}^{4} J_{y_j} \) or in other words \( \mathcal{E} \) is the volume form for \( \text{Cl}_{4,4} \). Then

\[
\mathcal{E}^2 = \text{Id}_{\mathbb{R}^{8,8}}, \quad \mathcal{E} J_{z_i} = -J_{z_i} \mathcal{E}, \quad \mathcal{E} J_{y_i} = -J_{y_i} \mathcal{E}, \quad i = 1, 2, 3, 4,
\]

and

\[
\langle \mathcal{E} u, u' \rangle_{\mathbb{R}^{8,8}} = \langle u, \mathcal{E} u' \rangle_{\mathbb{R}^{8,8}} \quad \text{for all } u, u' \in \mathbb{R}^{8,8}.
\]

Denote \( \tilde{V} = V \otimes U \cong V \otimes \mathbb{R}^{8,8} \) and non-degenerate scalar product \( \langle \cdot, \cdot \rangle_{\tilde{V}} = \langle \cdot, \cdot \rangle_{V} \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}} \). The space \( \tilde{V} \) has dimension 16 dim \( V \). Set also

\[
\tilde{J}_{\tilde{z}_k} = J_{z_k} \otimes \mathcal{E}, \quad k = 1, \ldots, r,
\]

\[
\tilde{J}_{\tilde{\xi}_l} = J_{\xi_l} \otimes \mathcal{E}, \quad l = 1, \ldots, s,
\]

\[
\tilde{J}_{\tilde{z}_{r+i}} = \text{Id}_V \otimes J_{x_i}, \quad i = 1, 2, 3, 4
\]

\[
\tilde{J}_{\tilde{\xi}_{s+j}} = \text{Id}_V \otimes J_{y_j}, \quad j = 1, 2, 3, 4.
\]

Then it is easy to see that \( \tilde{J}_{\tilde{z}_i} = -\text{Id}_{\tilde{V}}, i = 1, \ldots, r + 4, \tilde{J}_{\tilde{\xi}_i} = \text{Id}_{\tilde{V}} \) for \( j = 1, \ldots, s + 4 \). It can be shown as in Theorem 6 that all \( \tilde{J}_{\tilde{z}_i}, \tilde{J}_{\tilde{\xi}_i} \) mutually anti-commute and the module \( \tilde{V} = V \otimes U \cong V \otimes \mathbb{R}^{8,8} \) is admissible integral \( \text{Cl}_{r+s+4} \)-module.

**Proposition 4.** If the admissible integral \( \text{Cl}_{r,s} \)-module \((V, \langle \cdot, \cdot \rangle_{V})\) in Theorems 3, 4 and 7 is of minimal dimension then the resulting \( \text{Cl}_{r+s+8} \)-module \((V \otimes \mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_{V} \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}})\), \( \text{Cl}_{r+8,s} \)-module \((V \otimes \mathbb{R}^{16}, \langle \cdot, \cdot \rangle_{V} \langle \cdot, \cdot \rangle_{\mathbb{R}^{16}})\), and \( \text{Cl}_{r+s+4} \)-module \((V \otimes \mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_{V} \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}})\) are minimal admissible integral modules.

**Proof.** The admissible integral \( \text{Cl}_{0,8} \)-module \((\mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}})\) is of minimal dimension equals 16. Since admissible and irreducible modules has periodicity 8 the resulting \( \text{Cl}_{r,s+8} \)-module \( \tilde{V} \) will have minimal dimension equals \( \text{dim} \tilde{V} = 16 \text{ dim} V \).
Similar arguments used in the cases of $\mathrm{Cl}_{r+8,s}$ and $\mathrm{Cl}_{r+4,s+4}$-modules.

7.2. Twisted tensor product. In this subsection, we give two methods of construction of an admissible module from a given admissible module of lower dimensions by making use of tensor product. We show two cases, that is a construction of an admissible module for $\mathrm{Cl}_{0,n+2}$ from that of $\mathrm{Cl}_{n,0}$ and $\mathrm{Cl}_{1,2}$. Another one is the construction of an admissible module of $\mathrm{Cl}_{r+1,s+1}$ from that of $\mathrm{Cl}_{r,s}$ and $\mathrm{Cl}_{1,1}$. Both methods also give us the integral structure from those of the lower dimensions. Note that these constructions not always give the minimal dimensional resulting module, even if the initial admissible modules $\mathrm{Cl}_{n,0}$ (or $\mathrm{Cl}_{r,s}$) and $\mathrm{Cl}_{1,2}$ (and $\mathrm{Cl}_{1,1}$) are of minimal dimensions. The resulting module can exceed the minimal dimension two or four times.

First, basing on the isomorphism $\mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{1,2} \cong \mathrm{Cl}_{0,n+2}$ we prove the following theorem.

Theorem 8. Let us assume that $(V, \langle \cdot, \cdot \rangle_V)$ is an admissible integral $\mathrm{Cl}_{n,0}$-module and $(U, \langle \cdot, \cdot \rangle_U)$ is an admissible integral $\mathrm{Cl}_{0,2}$-module. Then the scalar product vector space $(V \otimes U, \langle \cdot, \cdot \rangle_V \otimes \langle \cdot, \cdot \rangle_U)$ is an admissible integral $\mathrm{Cl}_{0,n+2}$-module.

Proof. Let $(z_1, \ldots, z_n)$ be orthonormal generators of the Clifford algebra $\mathrm{Cl}_{n,0}$ with the quadratic form $Q_{n,0}(a) = \sum_{i=1}^{n} a_i^2$ for $a = \sum_{i=1}^{n} a_iz_i$. Let $\{y_1, y_2\}$ be orthonormal generators for $\mathrm{Cl}_{0,2}$ with quadratic form $Q_{0,2}(b) = -b_1^2 - b_2^2$ for $b = b_1y_1 + b_2y_2$ and, finally let $\tilde{z}_1, \ldots, \tilde{z}_{n+2}$ be orthonormal generators for the Clifford algebra $\mathrm{Cl}_{0,n+2}$ with quadratic form $Q_{0,n+2}(c) = -\sum_{i=1}^{n+2} c_i^2$ for $c = \sum_{i=1}^{n+2} c_i\tilde{z}_i$. The map

$$
\begin{align*}
\tilde{z}_i &\mapsto z_i \otimes y_1 y_2, & \text{if } i = 1, \ldots, n, \\
\tilde{z}_{n+1} &\mapsto 1 \otimes y_1, \\
\tilde{z}_{n+2} &\mapsto 1 \otimes y_2,
\end{align*}
$$

defines the isomorphism between the Clifford algebras $\mathrm{Cl}_{0,n+2}$ and $\mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}$.

We saw in the previous section that the minimal admissible $\mathrm{Cl}_{0,2}$-module $(U, \langle \cdot, \cdot \rangle_U)$ is isomorphic to $\mathbb{R}^{2,2}$ with quadratic form $Q_{2,2}(u) = \sum_{i=1}^{2} u_i^2 - \sum_{j=3,4} u_j^2$ for $u = \sum_{i=1}^{4} u_i e_i$, where $e_i$, $i = 1, 2, 3, 4$, is the standard basis in $\mathbb{R}^{2,2}$. Then the endomorphisms $J_{y_1}$ and $J_{y_2}$ from $\text{End}(\mathbb{R}^{2,2})$ are written in the basis $\{e_i\}_{i=1}^{4}$ as follows

$$
J_{y_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_{y_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

We have $J_{y_1}^2 = \text{Id}$, $J_{y_2}^2 = \text{Id}$, $J_{y_1} J_{y_2} = -J_{y_2} J_{y_1}$, and

$$
J_{y_1} J_{y_2} e_1 = e_2, \quad J_{y_1} e_2 = e_3, \quad J_{y_2} e_1 = e_4.
$$

Now we want to find $\mathcal{F} \in \text{End}(\mathbb{R}^{2,2})$ satisfying conditions

$$
(36) \quad \mathcal{F} J_{y_1} = -J_{y_1} \mathcal{F}, \quad \mathcal{F} J_{y_2} = -J_{y_2} \mathcal{F}, \quad \mathcal{F}^2 = -\text{Id},
$$

$$
(37) \quad \langle \mathcal{F}u, u' \rangle_{\mathbb{R}^{2,2}} = \langle u, \mathcal{F}u' \rangle_{\mathbb{R}^{2,2}} \quad \text{for } u, u' \in \mathbb{R}^{2,2}
$$

where the scalar product in (37) is the scalar product defined by the quadratic form $Q_{2,2}$. Conditions (36) imply that the matrix for $\mathcal{F}$ has the form

$$
(38) \quad \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & -a & b \\ -d & c & b & -a \end{pmatrix} \quad \text{with} \quad a^2 - b^2 - c^2 - d^2 = -1, \quad ab = 0, \quad bc = 0, \quad bd = 0.
$$

Checking the condition (37) we find that $b = 0$. 

Denote $\bar{V} = V \otimes \mathbb{R}^{2,2}$, $\langle \cdot, \cdot \rangle_{\bar{V}} = \langle \cdot, \cdot \rangle_{V} \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}}$ and notice that the scalar product $\langle \cdot, \cdot \rangle_{\bar{V}}$ is non-degenerate. Set
\[
\tilde{J}_z = J_z \otimes F, \quad i = 1, \ldots, n, \\
\tilde{J}_{z_{n+1}} = \text{Id}_V \otimes J_{y_1}, \quad \text{and} \quad \tilde{J}_{z_{n+2}} = \text{Id}_V \otimes J_{y_2},
\]
where $J_{z_i} \in \text{End}(V)$, $i = 1, \ldots, n$, such that $J_{z_i}^2 = -\text{Id}$. Then, it is easy to see that $\tilde{J}_{z_i}^2 = \text{Id}_{\bar{V}}$ for $i = 1, \ldots, n + 2$. Moreover, similar to discussions in Theorem 5 we have
\[
\tilde{J}_{z_{n+1}} \tilde{J}_{z_{n+2}} = -\tilde{J}_{z_{n+2}} \tilde{J}_{z_{n+1}}, \quad \tilde{J}_z \tilde{J}_{z_{n+1}} = -\tilde{J}_{z_{n+1}} \tilde{J}_z, \quad \tilde{J}_z \tilde{J}_{z_{n+2}} = -\tilde{J}_{z_{n+2}} \tilde{J}_z, \quad \tilde{J}_z \tilde{J}_{z_j} = -\tilde{J}_{z_j} \tilde{J}_z,
\]
for $i, j = 1, \ldots, n$, $i \neq j$.

The next step is to verify that the scalar product $\langle \cdot, \cdot \rangle_{\bar{V}} = \langle \cdot, \cdot \rangle_{V} \cdot \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}}$ satisfies
\[
\langle \tilde{J}_z \tilde{v}, \tilde{v} \rangle_{\bar{V}} = \langle \tilde{v}, \tilde{J}_z \tilde{v} \rangle_{\bar{V}} = 0.
\]
We write $\tilde{z} = a + b$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^2$, and $\tilde{v} = v \otimes u$, $\tilde{v}' = v' \otimes u'$ for $v, v' \in V$, $u, u' \in \mathbb{R}^{2,2}$. Then we argue as in [11].

To show that the resulting $\text{Cl}_{0,n+2}$-module is integral we assume that both modules $(V, \langle \cdot, \cdot \rangle_V)$, $(\mathbb{R}^{2,2}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}})$ are integral and choose special form of the map $F$, for instance we set $a = d = 0$ and $c = 1$. Then, if $\{v_\alpha\}$ and $\{e_p\}$ are integral bases for $V$ and $\mathbb{R}^{2,2}$ respectively, then for $\tilde{v}_n = v_\alpha \otimes e_p$
\[
\langle \tilde{J}_z \tilde{v}_n, \tilde{v}_m \rangle_{\bar{V}} = \langle J_z v_\alpha \otimes F e_p, v_\beta \otimes e_q \rangle_{\bar{V}} = \langle J_z v_\alpha, v_\beta \rangle_V \cdot \langle F e_p, e_q \rangle_{\mathbb{R}^{2,2}} = \pm 1 \text{ or } 0.
\]

\begin{remark}
Note that $F \neq \pm J_{y_1} J_{y_2}$.
\end{remark}

\begin{theorem}
Let us assume that $(V, \langle \cdot, \cdot \rangle_V)$ is an admissible integral $\text{Cl}_{r,s}$-module and $(U, \langle \cdot, \cdot \rangle_U)$ is the minimal dimensional admissible integral $\text{Cl}_{1,1}$-module, then $(V \otimes U, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_U)$ is an admissible integral $\text{Cl}_{r+1,s+1}$-module.
\end{theorem}

\begin{proof}
Let $(z_1, \ldots, z_r, \zeta_1, \ldots, \zeta_s)$ be orthonormal generators of the Clifford algebra $\text{Cl}_{r,s}$ with the quadratic form $\mathbb{Q}_{r,s}(a) = \sum_{i=1}^r a_i^2 - \sum_{j=1}^s a_j^2$ for $a = \sum_{i=1}^r a_i z_i + \sum_{j=1}^s a_j \zeta_j$ and let $\{x, y\}$ be orthonormal generators for $\text{Cl}_{1,1}$ with quadratic form $\mathbb{Q}_{2,2}(b) = b_1^2 - b_2^2$ for $b = b_1 x + b_2 y$. Denote by $(\tilde{z}_1, \ldots, \tilde{z}_{r+1}, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_{s+1})$ orthonormal generators for the Clifford algebra $\text{Cl}_{r+1,s+1}$ with the quadratic form $\mathbb{Q}_{r+1,s+1}(c) = \sum_{i=1}^{r+1} c_i^2 - \sum_{j=1}^{s+1} c_j^2$ for $c = \sum_{i=1}^{r+1} c_i \tilde{z}_i + \sum_{j=1}^{s+1} c_j \tilde{\zeta}_j$. It is known that there is the isomorphism between $\text{Cl}_{r+1,s+1}$ and $\text{Cl}_{r,s} \otimes \text{Cl}_{1,1}$ given by the following relation between the generators $\tilde{z}_1, \ldots, \tilde{z}_{r+1}, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_{s+1}$ of $\text{Cl}_{r+1,s+1}$ and generators of $\text{Cl}_{r,s} \otimes \text{Cl}_{1,1}$:
\[
\begin{align*}
\tilde{z}_i &\cong z_i \otimes xy, \quad \text{if} \quad i = 1, \ldots, r, \\
\tilde{\zeta}_j &\cong \zeta_j \otimes xy, \quad \text{if} \quad j = 1, \ldots, s, \\
\tilde{z}_{r+1} &\cong x, \\
\tilde{\zeta}_{s+1} &\cong y.
\end{align*}
\]

We saw in the previous section that the admissible $\text{Cl}_{1,1}$-module $(U, \langle \cdot, \cdot \rangle_U)$ is isomorphic to $\mathbb{R}^{2,2}$ with quadratic form $\mathbb{Q}_{2,2}(u) = \sum_{i=1}^{2} u_i^2 - \sum_{j=3}^{4} u_j^2$ for $u = \sum_{i=1}^{4} u_i e_i$, where $e_i$, $i = 1, 2, 3, 4$ is the standard basis in $\mathbb{R}^{2,2}$. Then the endomorphisms $J_{y_1}$ and $J_{y_2}$ from $\text{End}(\mathbb{R}^{2,2})$ are written in the basis $\{e_i\}_{i=1}^{4}$ as follows
\[
J_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_y J_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Then $J_x^2 = -\text{Id}$, $J_y^2 = \text{Id}$, $J_x J_y = -J_y J_x$, and $J_x e_1 = e_2$, $J_y e_1 = e_3$, $J_y J_x e_1 = e_4$. 


We need to find the endomorphism $F: \mathbb{R}^{2,2} \to \mathbb{R}^{2,2}$ such that
\begin{equation}
F^2 = \text{Id}, \quad FJ_x = -J_x F, \quad FJ_y = -J_y F,
\end{equation}
\begin{equation}
\langle Fu, u' \rangle_{\mathbb{R}^{2,2}} = \langle u, Fu' \rangle_{\mathbb{R}^{2,2}} \quad \text{for all} \quad u, u' \in \mathbb{R}^{2,2}.
\end{equation}
Checking conditions (39) and (40) we find that matrix for $F$ has the form
\[
\begin{pmatrix}
a & b & c & d \\
b & -a & d & -c \\
-c & -d & -a & -b \\
-d & c & -b & a
\end{pmatrix}
\]
with $a^2 + b^2 - c^2 - d^2 = 1$ and $bc = ad$.

Denote $\tilde{V} = V \otimes \mathbb{R}^{2,2}$ and non-degenerate scalar product $\langle \cdot, \cdot \rangle_{\tilde{V}} = \langle \cdot, \cdot \rangle_V \cdot \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}}$. Set also
\[
\tilde{J}_{z_i} = J_{z_i} \otimes F, \quad i = 1, \ldots, r, \quad \tilde{J}_{\xi_j} = J_{\xi_j} \otimes F, \quad j = 1, \ldots, s,
\]
\[
\tilde{J}_{x_{i+1}} = \text{Id}_V \otimes J_x, \quad \tilde{J}_{\zeta_{i+1}} = \text{Id}_V \otimes J_y.
\]
Here $J_{z_i}, J_{\xi_j} \in \text{End}(V)$, $i = 1, \ldots, r$, $j = 1, \ldots, s$ are such that $J^2_{z_i} = -\text{Id}, J^2_{\xi_j} = \text{Id}$. Then it is easy to see that $\tilde{J}^2_{z_i} = -\text{Id}_{\tilde{V}}, \tilde{J}^2_{\xi_j} = \text{Id}_{\tilde{V}}$, $\forall i = 1, \ldots, r + 1, j = 1, \ldots, s + 1$ due to (39) and $J^2_{x_i} = -\text{Id}_{\mathbb{R}^{2,2}}, J^2_{\eta} = \text{Id}_{\mathbb{R}^{2,2}}$. We also obtain that $\tilde{J}_{z_i}$ and $\tilde{J}_{\xi_j}$ mutually anti-commute for all $i = 1, \ldots, r + 1$ and for $j = 1, \ldots, s + 1$. Now we verify as in (31) that the scalar product $\langle \cdot, \cdot \rangle_{\tilde{V}} = \langle \cdot, \cdot \rangle_V \cdot \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}}$ satisfies the skew symmetry property. As in the previous case we show that if both modules $(V, \langle \cdot, \cdot \rangle_V), (\mathbb{R}^{2,2}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}})$ are integral, then the resulting module is integral. For this we can choose the map $F$ with the matrix having the entries $a = 1$ and $b = c = d = 0$.

**Remark 8.** Observe that $J_xJ_y \neq \pm F$ since if we choose $d = \pm 1$, then $a$ or $b$ must be different from zero.

8. Integral $Cl_{r,s}$-modules with $r + s \geq 9$

8.1. Integral structure on $Cl_{r,s}$-modules of dimension $r + s = 9$. Since
\[
Cl_{9,0} \cong Cl_{1,0} \otimes Cl_{8,0}, \quad Cl_{8,1} \cong Cl_{0,1} \otimes Cl_{8,0}
\]
we apply Theorem [1] For the cases
\[
Cl_{0,9} \cong Cl_{0,1} \otimes Cl_{9,0}, \quad Cl_{1,8} \cong Cl_{1,0} \otimes Cl_{0,8}
\]
we apply Theorem [5] We use Theorem [7] and get the integral structure due to the isomorphisms
\[
Cl_{5,4} \cong Cl_{1,0} \otimes Cl_{4,4}, \quad Cl_{4,5} \cong Cl_{0,1} \otimes Cl_{4,4}.
\]
For $Cl_{6,3}$- and $Cl_{2,7}$-modules we exploit the isomorphisms
\[
Cl_{6,3} \cong Cl_{5,2} \otimes Cl_{1,1}, \quad Cl_{2,7} \cong Cl_{1,6} \otimes Cl_{1,1}
\]
and Theorem [9] In this case, counting dimensions, we conclude that $Cl_{6,3}$ and $Cl_{2,7}$ are of minimal dimensions. Note that we could only construct the admissible integral module $Cl_{2,7} \cong Cl_{1,6} \otimes Cl_{1,1}$ and then use Theorem [3] and Corollary [4] to justify the existence of an integral structure on $Cl_{6,3}$ module, by the isomorphism $Cl_{6,3} \cong Cl_{2,7}$. We also apply Theorem [3] and Corollary [4] to the isomorphic Clifford algebras
\[
Cl_{8,r+1} = Cl_{5,4} \cong Cl_{3,6} = Cl_{r,s+1} \quad \text{and} \quad Cl_{s,r+1} = Cl_{1,8} \cong Cl_{7,2} = Cl_{r,s+1}
\]
and pullback the integral admissible structure of $Cl_{s,r+1}$-module to the module of $Cl_{r,s+1}$.
8.2. Integral structure on $\text{Cl}_{r,s}$-modules of dimension $r + s = 10$. The integral admissible module for Clifford algebras $\text{Cl}_{10,0}$, $\text{Cl}_{9,1}$, and $\text{Cl}_{8,2}$ we construct by applying Theorem 6. For the cases of $\text{Cl}_{2,8}$, $\text{Cl}_{1,9}$, and $\text{Cl}_{0,10}$ we use Theorem 5. We use Theorem 7 and get the integral structure for modules of $\text{Cl}_{4,6}$, $\text{Cl}_{5,5}$, and $\text{Cl}_{6,4}$. We also apply Theorem 3 to the isomorphic Clifford algebras $\text{Cl}_{8,2}$, $\text{Cl}_{9,1}$, and $\text{Cl}_{10,0}$. We use Theorem 7 and get the integral structure for modules of $\text{Cl}_{4,6}$, $\text{Cl}_{5,5}$, and $\text{Cl}_{6,4}$. We also apply Theorem 3 to the isomorphic Clifford algebras $\text{Cl}_{s,r+1} \cong \text{Cl}_{6,4}$, $\text{Cl}_{3,7} = \text{Cl}_{r,s+1}$ and $\text{Cl}_{s,r+1} = \text{Cl}_{2,8} \cong \text{Cl}_{7,3} = \text{Cl}_{r,s+1}$ and pullback the integral admissible structure of $\text{Cl}_{s,r+1}$-module to the module of $\text{Cl}_{r,s+1}$.

8.3. Integral structure on $\text{Cl}_{r,s}$-modules of dimension $r + s > 10$. For the rest of cases we use Theorems 5, 6, and 7.

9. Final remarks

(1) The constructed admissible integral modules show that the corresponding general $H$-type Lie algebras admit integer structural constants. The natural question arises: how many different Lie algebras are behind the general $H$-type Lie algebras if we discard the presence of the scalar product? Having in hand the integral basis, it is easier to answer this question. As we noticed in Remark 3 both of the Lie algebras based on spaces $\mathbb{R}^{1,1} \oplus \mathbb{R}^{0,1}$ and $\mathbb{R}^{2,0} \oplus \mathbb{R}^{1,0}$, corresponding to $\text{Cl}_{1,0}$- and $\text{Cl}_{0,1}$-modules, are isomorphic to the three dimensional Heisenberg algebra, although the metrics are different. The 6-dimensional Lie algebras based on vector spaces $\mathbb{R}^{4,0} \oplus \mathbb{R}^{2,0}$, $\mathbb{R}^{2,2} \oplus \mathbb{R}^{0,2}$, related to $\text{Cl}_{2,0}$- and $\text{Cl}_{0,2}$-modules, are also isomorphic, but not isomorphic to the Lie algebra based on the space $\mathbb{R}^{2,2} \oplus \mathbb{R}^{1,1}$, arising from $\text{Cl}_{1,1}$-module. This can be proved by making use of the above constructed integral basis. The details including these and more general relations between $\mathbb{R}^{k,k} \oplus \mathbb{R}^{r,s}$ is treated in a forthcoming paper, see also [26] and [27] for the classification of low dimensional nilpotent Lie algebras.

(2) Note that for construction of integral structures on admissible $\text{Cl}_{r,1}$-modules, we could used the isomorphism $\text{Cl}_{0,r+1} \cong \text{Cl}_{r,1}$, but not Theorem 4 and get the integral basis from ones constructed in Section 4. Or vice versa, we could get all the integral modules $\text{Cl}_{0,r+1}$ from Section 4 by making use Theorem 3 and Corollary 4 from modules $\text{Cl}_{r,1}$, where we first construct $\text{Cl}_{r,1}$-module by making use of Theorem 4.

(3) In Section 6 we constructed admissible integral $\text{Cl}_{r,0}$-modules. It is known classical case that admissible modules for the Clifford algebras $\text{Cl}_{r,0}$ have a positive definite product. We based on this knowledge when we found integral structures for $\text{Cl}_{r,0}$-modules with $r = 1, 2, 3, 4, 8$. For the rest of the cases we started the construction from the assumption that the module is admissible with a neutral scalar product and then the construction of integral bases leads to the sub-module with positive definite product. This shows that our method not only gives the integral basis but also detects the possible signature of the scalar product.

(4) The integral admissible $\text{Cl}_{0,s}$-modules, constructed in Section 4 could be also found by using Theorem 8. In this case the resulting $\text{Cl}_{0,3}$- and $\text{Cl}_{0,5}$-modules would be of minimal dimensions, but $\text{Cl}_{0,s}$-modules for $r = 2, 4, 6, 7, 8$ would exceed the minimal dimension twice. That was also a reason why we constructed all $\text{Cl}_{0,s}$-modules directly.

10. Appendix
| r + s | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-------|----|----|----|----|----|----|----|----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0     |    |    |    |    |    |    |    |    |  R  |     |     |     |     |     |     |     |
| 1     |    |    |    |    |    |    |    |    |  R  |  R  |  C  |     |     |     |     |     |
| 2     |    |    |    |    |    |    |    |    |  R  |  R  |     |     |     |     |     |     |
| 3     |    |    |    |    |    |    |    |    |     |     |     |     |     |     |     |     |
| 4     |    |    |    |    |    |    |    |    |  R  |  R  |     |     |     |     |     |     |
| 5     |    |    |    |    |    |    |    |    |     |     |     |     |     |     |     |     |
| 6     |    |    |    |    |    |    |    |    |  R  |  R  |     |     |     |     |     |     |
| 7     |    |    |    |    |    |    |    |    |     |     |     |     |     |     |     |     |
| 8     |    |    |    |    |    |    |    |    |  R  |  R  |     |     |     |     |     |     |
| 9     |    |    |    |    |    |    |    |    |     |     |     |     |     |     |     |     |
| 10    |    |    |    |    |    |    |    |    |  R  |  R  |     |     |     |     |     |     |
| 11    |    |    |    |    |    |    |    |    |     |     |     |     |     |     |     |     |


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