On the Boundary Value Problems of \( \Psi \)-Hilfer Fractional Differential Equations

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Abstract

In the current paper, we derive the comparison results for the homogeneous and non-homogeneous linear initial value problem (IVP) for \( \Psi \)-Hilfer fractional differential equations. In the presence of upper and lower solutions, the obtained comparison results and the location of roots theorem utilized to prove the existence and uniqueness of the solution for the linear \( \Psi \)-Hilfer boundary value problem (BVP) through the linear non-homogeneous \( \Psi \)-Hilfer IVP. Assuming the existence of lower solution \( w_0 \) and upper solution \( z_0 \), we establish the existence of minimal and maximal solutions for the nonlinear \( \Psi \)-Hilfer BVP in the line segment \([w_0, z_0]\) of the weighted space \( C_{\gamma; \Psi}(J, \mathbb{R}) \).

Further, it demonstrated that the iterative Picard type sequences that began with lower and upper solutions respectively converges to a minimal and maximal solutions, and that started with any point on a line segment converge to the exact solution of nonlinear \( \Psi \)-Hilfer BVP. Finally, an example is provided in support of the main results we acquired.

Key words: \( \Psi \)-Hilfer fractional derivative; Boundary value problems, Existence and uniqueness; Upper and lower solutions; Extremal solutions, Monotone iterative technique.

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1 Introduction

The boundary value problems (BVPs) of fractional differential equations (FDEs) have significant applications [1, 2, 3, 4] in mathematical physics, financial mathematics, mathematical biology, biochemical system, biomedical engineering etc. Because of widespread applications, fractional-order BVPs were analyzed by numerous researchers for the existence of solutions by utilizing various procedures, for example, fixed point theorems [5, 6, 7, 8], fixed point index theory [9, 10, 11, 12], measure of non-compactness method [13, 14, 15], upper and lower solution method [16, 17, 18, 19], and so forth.

The technique of upper and lower solutions combined with the monotone iterative approach utilized effectively to derive sufficient conditions about the existence and uniqueness of solutions for the differential equations of integer order [20, 21, 22] and fractional order [23, 24, 25, 26, 27, 28] subject to initial or boundary condition. The primary thought behind this method is to construct the monotonic sequences for the corresponding differential equations, where the initial approximations taken are upper and lower solutions. At that point, it will be demonstrated that the sequences converge monotonically to the corresponding maximal and minimal solutions.
The fractional derivative with respect to another function is presented by Kilbas et al. [29] and Almeida [30] respectively in the sense of Riemann-Liouville and Caputo fractional derivative. Stimulated by the concept of fractional derivative with respect to another function, in [31] Sousa and Oliveira introduced Ψ-Hilfer fractional derivative operator \( H D_{0+}^{\alpha,\beta;\Psi}(\cdot) \) and investigated its important properties. Authors have proved that the Ψ-Hilfer derivative operator is more generalized and incorporates numerous fractional derivatives as its special cases. For more details, we refer the reader to the paper [31]. For the analysis of nonlinear Ψ-Hilfer FDEs about existence and uniqueness of the solution, and its qualitative properties, such as Ulam-Hyers stability and various types of data dependence results, we refer the reader to the recent papers [32, 33, 34, 35, 36, 37, 38, 39, 40].

Motivated by the work mentioned above and the results obtained by Lin et al. [28], in the present paper, we discuss the existence and uniqueness results for linear and nonlinear BVPs of Ψ-Hilfer FDEs by the method of upper and lower solutions combined with monotone iterative technique.

To discuss the existence and uniqueness of solutions for BVPs of Ψ-Hilfer FDEs via method of upper and lower solutions, we need to derive the proper fractional differential inequalities in the setting of Ψ-Hilfer derivative as comparison results. Therefore, initially we obtain the comparison results for the homogeneous linear initial value problem (IVP) for Ψ-Hilfer FDEs of the form

\[
\begin{align*}
H D_{0+}^{\alpha,\beta;\Psi} y(t) - M y(t) &= 0, \quad t \in (0, T], \\
I_{0+}^{1-\gamma;\Psi} y(0) &= 0,
\end{align*}
\]

(1.1)

and non-homogeneous linear IVP for Ψ-Hilfer FDEs. Here \( M > 0 \) (\( M \in \mathbb{R} \), \( \Psi \in C^1([0,T],\mathbb{R}) \)) is an increasing function such that \( \Psi'(t) \neq 0 \), \( t \in [0,T] \), \( H D_{0+}^{\alpha,\beta;\Psi}(\cdot) \) is the Ψ-Hilfer fractional derivative of order \( \alpha \) (\( 0 < \alpha < 1 \)) and type \( \beta \) (\( 0 \leq \beta \leq 1 \)), \( \gamma = \alpha + \beta (1 - \alpha) \) and \( I_{0+}^{1-\gamma;\Psi}(\cdot) \) is the Ψ-Riemann-Liouville fractional integral of order \( 1 - \gamma \).

With the help of acquired comparison results for the linear Ψ-Hilfer IVP and the location of roots theorem [31], assuming existence of lower and upper solution, we investigated the existence and uniqueness of the solution for the following linear Ψ-Hilfer BVP

\[
\begin{align*}
H D_{0+}^{\alpha,\beta;\Psi} y(t) - M y(t) &= g(t), \quad t \in (0, T], \\
I_{0+}^{1-\gamma;\Psi} y(0) &= r I_{0+}^{1-\gamma;\Psi} y(T),
\end{align*}
\]

(1.2)

(1.3)

through the linear non-homogeneous Ψ-Hilfer IVP, where \( g \in C_{1-\gamma;\Psi} (J, \mathbb{R}) \), \( J = [0, T] \) and \( 0 < r < \frac{1}{E_{\alpha,1}(M(\Psi(T) - \Psi(0))^{\alpha})} \). The two parameter Mittag-Leffler function \( E_{\alpha,1}(\cdot) \) and the weighted space \( C_{1-\gamma;\Psi} (J, \mathbb{R}) \) will be defined later in the preliminary section.

Next, we consider the nonlinear Ψ-Hilfer BVP of the form

\[
\begin{align*}
H D_{0+}^{\alpha,\beta;\Psi} y(t) - M y(t) &= f(t, y(t)), \quad t \in (0, T], \\
I_{0+}^{1-\gamma;\Psi} y(0) &= r I_{0+}^{1-\gamma;\Psi} y(T),
\end{align*}
\]

(1.4)

(1.5)

where \( f(\cdot, y(\cdot)) \in C_{1-\gamma;\Psi} (J, \mathbb{R}) \) for each \( y \in C_{1-\gamma;\Psi} (J, \mathbb{R}) \). Utilizing the existence and uniqueness results that are obtained for the linear non-homogeneous BVP of Ψ-Hilfer FDEs, and assuming that the nonlinear Ψ-Hilfer FDEs (1.4)-(1.5) has lower solution \( w_0 \) and upper solution \( z_0 \), it is proved that there exist minimal and maximal solutions on the line.
segment \([w_0, z_0]\) of the ordered Banach space \((\text{weighted space}) C_{1-\gamma, \Psi}(J, \mathbb{R})\). Further, it is proved that the Picard iterative sequences \(\{w_n\}_{n=1}^{\infty}\) and \(\{z_n\}_{n=1}^{\infty}\) starting respectively with \(w_0\) (lower solution) and \(z_0\) (upper solution) are monotonic in the ordered Banach space \(C_{1-\gamma, \Psi}(J, \mathbb{R})\) and converges respectively to minimal and maximal solutions of the nonlinear BVP of \(\Psi\)-Hilfer FDEs (1.4)-(1.5).

Assuming that the function \(f\) satisfies the one-sided Lipschitz condition, we have shown that the Picard iterative sequences beginning with any arbitrary point on the line segment \([w_0, z_0]\) converges to the unique solution of the nonlinear \(\Psi\)-Hilfer BVP (1.4)-(1.5). Further, the error bound between the \(n^{th}\) approximation \(y_n\) and the exact solution \(y^*\) of the nonlinear \(\Psi\)-Hilfer BVP (1.4)-(1.5) is obtained with respect to the norm on the weighted space \(C_{1-\gamma, \Psi}(J, \mathbb{R})\).

Finally, an example is provided to illustrate the existence and uniqueness results that we acquired through the method of lower and upper solution.

The outcomes acquired in the present are the generalization of the results derived in [28] and can be achieved by taking \(\beta = 1\) and \(\Psi(t) = t\). The \(\Psi\)-Hilfer derivative \(H^{D_{0+}^{\alpha, \beta}}\Psi(\cdot)\) is generalized derivative operator that incorporates many notable fractional derivatives recorded in [31] as its special cases including most widely used derivative operators, such as, Riemann-Liouville derivative [29], Caputo derivative [29], Hadmard derivative [29], Erdély-Kober derivative [29], Hilfer derivative [42], Katugampola derivative [43] etc. Along these lines, the outcomes acquired in the current paper are likewise valid for the fractional derivatives listed in [31] as particular cases of the \(\Psi\)-Hilfer derivative.

The paper is composed as follows: In section 2, we collect a few definitions and essential outcomes about \(\Psi\)-Hilfer fractional derivative. Further, we have provided some results which assume a significant role in our analysis. In Section 3, we prove the comparison results for \(\Psi\)-Hilfer FDEs. Section 4 deals with the existence and uniqueness of the linear \(\Psi\)-Hilfer BVPs. In Section 5, the existence and uniqueness results are proved for the nonlinear \(\Psi\)-Hilfer BVPs (1.4)-(1.5). An example is provided in section 6 to verify the assurance of our primary outcomes.

2 Preliminaries

Let \([a, b]\) \((0 < a < b < \infty)\) be a finite interval and \(\Psi \in C^1([a, b], \mathbb{R})\) be an increasing function such that \(\Psi' (t) \neq 0, t \in [a, b]\). Consider the weighted space [31]

\[
\mathcal{X} := C_{1-\gamma, \Psi}[a, b] = \left\{ h : [a, b] \to \mathbb{R} \mid (\Psi(t) - \Psi(a))^{1-\gamma} h(t) \in C[a, b] \right\}, \quad 0 < \gamma \leq 1,
\]

endowed with the norm

\[
\|h\|_{C_{1-\gamma, \Psi}[a, b]} = \max_{t \in [a, b]} \left| (\Psi(t) - \Psi(a))^{1-\gamma} h(t) \right|, \tag{2.1}
\]

where \(\gamma = \alpha + \beta(1 - \alpha)\). Then, \((\mathcal{X}, \|\cdot\|_{C_{1-\gamma, \Psi}(J, \mathbb{R})})\) is a partially ordered Banach space with the partial order relation \(\preceq\) defined by

\[
x, y \in \mathcal{X}, \ x \preceq y \text{ if and only if } x(t) \leq y(t), \ t \in (0, T).
\]
Lemma 2.1 ([29]) Let $\mu > 0$ ($\mu \in \mathbb{R}$), $h$ be an integrable function defined on $[a, b]$. Then, the $\Psi$-Riemann–Liouville fractional integral of a function $h$ of order $\mu$ with respect to $\Psi$ is given by

$$I_{a+}^\mu;\Psi h(t) = \frac{1}{\Gamma(\mu)} \int_a^t \Psi'(s) (\Psi(t) - \Psi(s))^{\mu-1} h(s) ds.$$ 

Definition 2.2 ([31]) Let $0 < \alpha < 1$ and $h \in C^1([a, b], \mathbb{R})$. Then, the $\Psi$-Hilfer fractional derivative of a function $h$ of order $\alpha$ and type $\beta$ ($0 \leq \beta \leq 1$), is defined by

$$H D_{a+}^{\alpha;\beta;\Psi} h(t) = I_{a+}^{\beta(1-\alpha);\Psi} \left( \frac{1}{\Psi'(t)} \frac{d}{dt} I_{a+}^{(1-\beta)(1-\alpha);\Psi} h(t) \right).$$ 

Lemma 2.1 ([29, 31]) Let $\mu_i > 0$ ($i = 1, 2$) and $\delta > 0$. Then,

(a) $I_{a+}^{\mu_1;\Psi} I_{a+}^{\mu_2;\Psi} h(t) = I_{a+}^{\mu_1+\mu_2;\Psi} h(t)$.

(b) $I_{a+}^{\mu;\Psi} (\Psi(t) - \Psi(a))^{\delta-1} = \frac{1}{\Gamma(\delta)} (\Psi(t) - \Psi(a))^{\mu+\delta-1}$.

Lemma 2.2 ([31]) If $h \in C^1[a, b]$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then

(a) $I_{a+}^{\alpha;\Psi} H D_{a+}^{\alpha;\beta;\Psi} h(t) = h(t) - \frac{(\Psi(t) - \Psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_{a+}^{1-\gamma;\Psi} h(a)$.

(b) $H D_{a+}^{\alpha;\beta;\Psi} I_{a+}^{\alpha;\Psi} h(t) = h(t)$.

Lemma 2.3 ([35]) If $\mu > 0$ and $0 \leq \omega < 1$, then $I_{a+}^{\mu;\Psi} (\cdot)$ is bounded from $C_\omega;\Psi [a, b]$ to $C_\omega;\Psi [a, b]$. In addition, if $\omega \leq \mu$, then $I_{a+}^{\mu;\Psi} (\cdot)$ is bounded from $C_\omega;\Psi [a, b]$ to $C [a, b]$.

Lemma 2.4 ([36]) Let $g \in C_{1-\gamma;\Psi} (J, \mathbb{R})$ and $\eta \in \mathbb{R}$. Then, the solution of the Cauchy problem for FDEs with constant coefficient involving $\Psi$-Hilfer fractional derivative,

$$H D_{0+}^{\alpha;\beta;\Psi} y(t) - \eta y(t) = g(t), \quad t \in (0, T],$$

is given by

$$y(t) = y_0 (\Psi(t) - \Psi(0))^{\gamma-1} E_{\alpha, \gamma} (\eta (\Psi(t) - \Psi(0))^\alpha)$$

$$+ \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} (\eta (\Psi(t) - \Psi(s))^\alpha) g(s) ds, \quad t \in (0, T].$$

Lemma 2.5 ([44]) Let $n_1, n_2 > 0$, ($n_i \in \mathbb{R}$), $i = 1, 2$. Consider the two parameter Mittag–Leffler function defined by $E_{n_1, n_2} (z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(n_1 k + n_2)}, z \in \mathbb{C}$. Then, the power series defining $E_{n_1, n_2} (z)$ is convergent for all $z \in \mathbb{C}$.

Lemma 2.6 (Location of roots theorem ([41])) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on $I$. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.
Lemma 2.7 ([21]) Let $X$ be a partially ordered Banach space, $\{x_n\} \subset X$ a monotone sequence and relatively compact set, then $\{x_n\}$ is convergent.

Lemma 2.8 ([21]) Let $X$ be a partially ordered Banach space, $x_n \leq y_n (n = 1, 2, 3, \cdots)$ if $x_n \to x^*$, $y_n \to y^*$ we have $x^* \leq y^*$.

Definition 2.3 Let $u \in C_{1-\gamma; \Psi}(J, \mathbb{R})$. We say that $u$ is a lower solution of the linear $\Psi$-Hilfer BVP (1.2)–(1.3) if

$$H^D_{0+}^{\alpha,\beta; \Psi} u(t) - Mu(t) \leq g(t) - a_u(t), \ t \in (0, T],$$

where

$$a_u(t) = \begin{cases} 0, & \text{if } r I_{0+}^{1-\gamma; \Psi} u(T) \geq I_{0+}^{1-\gamma; \Psi} u(0) \\ \frac{1}{\Gamma(\delta+1)} \left[ (\Psi(t) - \Psi(0))^{\delta} - (\Psi(T) - \Psi(0))^{1-\gamma+\delta} \right], & \text{if } r I_{0+}^{1-\gamma; \Psi} u(T) < I_{0+}^{1-\gamma; \Psi} u(0), \end{cases}$$

and

$$\xi(t) = \frac{\Gamma(2+\delta-\gamma)}{\Gamma(\delta+1)} \left[ (\Psi(t) - \Psi(0))^{\delta} - (\Psi(T) - \Psi(0))^{1-\gamma+\delta} \right], \ \delta > 0, \ t \in J.$$ 

Clearly, $\xi \in C_{1-\gamma; \Psi}(J, \mathbb{R})$.

Definition 2.4 Let $v \in C_{1-\gamma; \Psi}(J, \mathbb{R})$. We say that $v$ is an upper solution of the linear $\Psi$-Hilfer BVP (1.2)–(1.3) if

$$H^D_{0+}^{\alpha,\beta; \Psi} v(t) - Mv(t) \geq g(t) + b_v(t), \ t \in (0, T],$$

where

$$b_v(t) = \begin{cases} 0, & \text{if } r I_{0+}^{1-\gamma; \Psi} v(T) \leq I_{0+}^{1-\gamma; \Psi} v(0) \\ \frac{1}{\Gamma(\delta+1)} \left[ (\Psi(t) - \Psi(0))^{\delta} - (\Psi(T) - \Psi(0))^{1-\gamma+\delta} \right], & \text{if } r I_{0+}^{1-\gamma; \Psi} v(T) > I_{0+}^{1-\gamma; \Psi} v(0), \end{cases}$$

where $\xi(t)$ is defined as in the equation (2.3).

3 Comparison Theorems

Theorem 3.1 Assume that $y \in C_{1-\gamma; \Psi}(J, \mathbb{R})$ and satisfies

$$H^D_{0+}^{\alpha,\beta; \Psi} y(t) - My(t) \leq 0, \ t \in (0, T],$$

$$I_{0+}^{1-\gamma; \Psi} y(0) \leq 0. \quad (3.2)$$

Then, $y(t) \leq 0, \ t \in (0, T]$. 

Proof: Consider the following linear IVP

\[ H_{D_{0+}^\alpha,\beta; \Psi} y(t) - M y(t) = \sigma(t), \quad t \in (0, T], \quad (3.3) \]

\[ \mathcal{I}_{0+}^{1-\gamma; \Psi} y(0) = y_0, \quad (3.4) \]

where \( \sigma \in C_{1-\gamma; \Psi} (J, \mathbb{R}) \). By Lemma 2.4, the linear IVP (3.3)-(3.4) has a unique solution given by

\[ y(t) = y_0 (\Psi(t) - \Psi(0))^{-1} E_{\alpha, \gamma} \left( M (\Psi(t) - \Psi(0))^\alpha \right) \]

\[ + \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} (M (\Psi(t) - \Psi(s))^\alpha) \sigma(s) ds, \quad t \in (0, T]. \quad (3.5) \]

From equations (3.2) and (3.4), we have \( y_0 \leq 0 \). Since \( \Psi \) is an increasing function, we have \( (\Psi(t) - \Psi(0))^{\gamma-1} \geq 0 \), \( t \in J \) and \( (\Psi(t) - \Psi(0))^{\alpha} \geq 0 \), \( t \in J \). This gives, \( E_{\alpha, \gamma} (M (\Psi(t) - \Psi(0))^\alpha) > 0 \), \( t \in J \). Therefore, from (3.5), we have

\[ y(t) \leq \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} (M (\Psi(t) - \Psi(s))^\alpha) \sigma(s) ds, \quad t \in (0, T]. \quad (3.6) \]

From equations (3.1) and (3.3), it follows that \( \sigma(t) \leq 0 \), \( t \in J \). Since \( \Psi : J \to \mathbb{R} \) is an increasing continuous function, we have \( \Psi'(t) > 0 \), \( t \in J \), \( (\Psi(t) - \Psi(s))^{\alpha-1} \geq 0 \), \( t \geq s \geq 0 \) and \( E_{\alpha, \gamma} (M (\Psi(t) - \Psi(s))^\alpha) > 0 \), \( t \geq s \geq 0 \). Therefore,

\[ \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} (M (\Psi(t) - \Psi(s))^\alpha) \sigma(s) \leq 0, \quad 0 \leq s \leq t \leq T. \quad (3.7) \]

From (3.6) and (3.7), we obtain

\[ y(t) \leq 0, \quad t \in (0, T]. \]

\[ \square \]

**Theorem 3.2** Let \( y \in C_{1-\gamma; \Psi} (J, \mathbb{R}) \) satisfies

\[ H_{D_{0+}^\alpha,\beta; \Psi} y(t) - M y(t) \leq -a_u(t), \quad t \in (0, T], \quad (3.8) \]

\[ \mathcal{I}_{0+}^{1-\gamma; \Psi} y(0) \leq 0. \quad (3.9) \]

Then, \( y(t) \leq 0, \quad t \in (0, T]. \)

**Proof:** In the view of definition of \( a_u \) given in (2.2), we give the proof in following two cases.

**Case 1:** If \( r \mathcal{I}_{0+}^{1-\gamma; \Psi} u(T) \geq \mathcal{I}_{0+}^{1-\gamma; \Psi} u(0) \) then \( a_u(t) = 0, \quad t \in (0, T] \). Thus, equations (3.8)-(3.9) reduces to equations (3.1)-(3.2). Applying Theorem 3.1 we obtain \( y(t) \leq 0, \quad t \in (0, T] \).

**Case 2:** If \( r \mathcal{I}_{0+}^{1-\gamma; \Psi} u(T) < \mathcal{I}_{0+}^{1-\gamma; \Psi} u(0) \) then

\[ a_u(t) = \frac{1}{r} \left( H_{D_{0+}^\alpha,\beta; \Psi} \xi(t) - M \xi(t) \right) \left( \mathcal{I}_{0+}^{1-\gamma; \Psi} u(0) - r \mathcal{I}_{0+}^{1-\gamma; \Psi} u(T) \right), \quad t \in (0, T]. \]

Define

\[ \rho(t) = y(t) + \frac{1}{r} \xi(t) \left( \mathcal{I}_{0+}^{1-\gamma; \Psi} u(0) - r \mathcal{I}_{0+}^{1-\gamma; \Psi} u(T) \right), \quad t \in (0, T]. \quad (3.10) \]
By assumption
\[ I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) > 0. \]

Further, from equation (2.3), we have \( \xi(t) \geq 0, \ t \in J. \) Hence, we have
\[ \frac{1}{r} \xi(t) \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right) \geq 0, \ t \in J. \] (3.11)

From (3.10) and (3.11), it follows that
\[ y(t) \leq \rho(t), \ t \in (0, T]. \] (3.12)

Next, using (3.10), we have
\[
H D_{0+}^{\alpha,\beta;\Psi} \rho(t) - M \rho(t)
= H D_{0+}^{\alpha,\beta;\Psi} \left[ y(t) + \frac{1}{r} \xi(t) \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right) \right]
- M \left[ y(t) + \frac{1}{r} \xi(t) \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right) \right]
= H D_{0+}^{\alpha,\beta;\Psi} y(t) - M y(t) + \frac{1}{r} H D_{0+}^{\alpha,\beta;\Psi} \xi(t) \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right)
= H D_{0+}^{\alpha,\beta;\Psi} y(t) - M y(t) + a_u(t), \ t \in (0, T].
\]

Using the inequality (3.8), above equation reduces to the following inequality
\[ H D_{0+}^{\alpha,\beta;\Psi} \rho(t) - M \rho(t) \leq 0, \ t \in (0, T]. \] (3.13)

Further, using (3.10) and Lemma 2.1(b), we have
\[
I_{0+}^{1-\gamma;\Psi} \rho(t)
= I_{0+}^{1-\gamma;\Psi} y(t) + \frac{1}{r} \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right) I_{0+}^{1-\gamma;\Psi} \xi(t)
= I_{0+}^{1-\gamma;\Psi} y(t)
+ \frac{1}{r} \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right) I_{0+}^{1-\gamma;\Psi} \left[ \frac{\Gamma(2 + \delta - \gamma)}{\Gamma(\delta + 1)} \frac{(\Psi(t) - \Psi(0))^{1-\gamma+\delta}}{(\Psi(T) - \Psi(0))^{1-\gamma+\delta}} \right]
= I_{0+}^{1-\gamma;\Psi} y(t) + \frac{1}{r} \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right) \frac{(\Psi(t) - \Psi(0))^{1-\gamma+\delta}}{(\Psi(T) - \Psi(0))^{1-\gamma+\delta}}. \] (3.14)

From (3.9) and (3.14), it follows that
\[ I_{0+}^{1-\gamma;\Psi} \rho(0) \leq 0. \] (3.15)

By applying Theorem 3.1 to the inequalities (3.13) and (3.15), we have
\[ \rho(t) \leq 0, \ t \in (0, T]. \] (3.16)

Further, from inequalities (3.12) and (3.16), we obtain \( y(t) \leq 0, \ t \in (0, T]. \)

Following similar type of steps as in the proof of Theorem 3.2, one can easily prove the following theorem.
Theorem 3.3 Let $y \in C_{1-\gamma;\psi}(J, \mathbb{R})$ satisfies

$$
\begin{align*}
&{^H_{0+}}D_{\gamma}^{\alpha_1, \beta_1} y(t) - Mq(t) \leq -b_v(t), \quad t \in (0, T], \\
&{^\eta_{I_0+}}^{1-\gamma;\psi} y(0) \leq 0.
\end{align*}
$$

(3.17) (3.18)

Then, $y(t) \leq 0$, $t \in (0, T]$.

4 Existence and uniqueness for the linear $\Psi$-Hilfer BVP

In this section, using the method of upper and lower solutions, we derive the existence and uniqueness results for the linear $\Psi$-Hilfer BVP (1.2)-(1.3).

Theorem 4.1 Assume that there exist upper and lower solutions $v, u \in C_{1-\gamma;\psi}(J, \mathbb{R})$ respectively of the linear $\Psi$-Hilfer BVP (1.2)-(1.3) such that $u \leq v$. Then, the linear $\Psi$-Hilfer BVP (1.2)-(1.3) has a unique solution $y \in C_{1-\gamma;\psi}(J, \mathbb{R})$ that satisfy $u \leq y \leq v$.

Proof: We give the proof in following two parts.

Part I: In this part we prove that the linear $\Psi$-Hilfer BVP (1.2)-(1.3) has a unique solution. Consider the functions $p, q \in C_{1-\gamma;\psi}(J, \mathbb{R})$ defined by

$$
p(t) = \begin{cases} 
ru(t), & \text{if } r {^\eta_{I_0+}}^{1-\gamma;\psi} u(T) \geq {^\eta_{I_0+}}^{1-\gamma;\psi} u(0) \\
ru(t) + \xi(t) \left( {^\eta_{I_0+}}^{1-\gamma;\psi} u(0) - r {^\eta_{I_0+}}^{1-\gamma;\psi} u(T) \right), & \text{if } r {^\eta_{I_0+}}^{1-\gamma;\psi} u(T) < {^\eta_{I_0+}}^{1-\gamma;\psi} u(0),
\end{cases}
$$

and

$$
q(t) = \begin{cases} 
rv(t), & \text{if } r {^\eta_{I_0+}}^{1-\gamma;\psi} v(T) \leq {^\eta_{I_0+}}^{1-\gamma;\psi} v(0) \\
rv(t) - \xi(t) \left( r {^\eta_{I_0+}}^{1-\gamma;\psi} v(T) - {^\eta_{I_0+}}^{1-\gamma;\psi} v(0) \right), & \text{if } r {^\eta_{I_0+}}^{1-\gamma;\psi} v(T) > {^\eta_{I_0+}}^{1-\gamma;\psi} v(0),
\end{cases}
$$

(4.1) (4.2)

where $\xi(t)$ is defined as in the equation (2.3).

Case 1: If $r {^\eta_{I_0+}}^{1-\gamma;\psi} u(T) \geq {^\eta_{I_0+}}^{1-\gamma;\psi} u(0)$ then $p(t) = ru(t)$, $t \in (0, T]$. Therefore,

$$
{^\eta_{I_0+}}^{1-\gamma;\psi} p(0) = r {^\eta_{I_0+}}^{1-\gamma;\psi} u(0).
$$

Further,

$$
{^\eta_{I_0+}}^{1-\gamma;\psi} p(T) = r {^\eta_{I_0+}}^{1-\gamma;\psi} u(T) \geq {^\eta_{I_0+}}^{1-\gamma;\psi} u(0) = \frac{{^\eta_{I_0+}}^{1-\gamma;\psi} p(0)}{r}.
$$

Thus,

$$
r {^\eta_{I_0+}}^{1-\gamma;\psi} p(T) \geq {^\eta_{I_0+}}^{1-\gamma;\psi} p(0).
$$

Case 2: If $r {^\eta_{I_0+}}^{1-\gamma;\psi} u(T) < {^\eta_{I_0+}}^{1-\gamma;\psi} u(0)$ then $p(t) = ru(t) + \xi(t) \left( {^\eta_{I_0+}}^{1-\gamma;\psi} u(0) - r {^\eta_{I_0+}}^{1-\gamma;\psi} u(T) \right)$, $t \in (0, T]$. Therefore,

$$
{^\eta_{I_0+}}^{1-\gamma;\psi} p(0) = r {^\eta_{I_0+}}^{1-\gamma;\psi} u(0) + {^\eta_{I_0+}}^{1-\gamma;\psi} \xi(0) \left( {^\eta_{I_0+}}^{1-\gamma;\psi} u(0) - r {^\eta_{I_0+}}^{1-\gamma;\psi} u(T) \right). 
$$

(4.3)

But from (2.3) and Lemma 2.1(b), we have

$$
{^\eta_{I_0+}}^{1-\gamma;\psi} \xi(t) = \frac{(\Psi(t) - \Psi(0))^{1-\gamma+\delta}}{(\Psi(T) - \Psi(0))^{1-\gamma+\delta}}, \quad t \in J.
$$

(4.4)
From (4.3) and (4.4), it follows that
\[
I_{0+}^{1-\gamma;\Psi} p(0) = r I_{0+}^{1-\gamma;\Psi} u(0).
\]
Further,
\[
I_{0+}^{1-\gamma;\Psi} p(T) = r I_{0+}^{1-\gamma;\Psi} u(T) + I_{0+}^{1-\gamma;\Psi} \xi(T) \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right).
\]
(4.5)
Again from (4.4) and (4.5), it follows that
\[
I_{0+}^{1-\gamma;\Psi} p(T) = I_{0+}^{1-\gamma;\Psi} u(0) = \frac{I_{0+}^{1-\gamma;\Psi} p(0)}{r}.
\]
Thus,
\[
r I_{0+}^{1-\gamma;\Psi} p(T) = I_{0+}^{1-\gamma;\Psi} p(0).
\]
From Case 1 and Case 2, we have
\[
I_{0+}^{1-\gamma;\Psi} p(0) = r I_{0+}^{1-\gamma;\Psi} u(0),
\]
(4.6)
\[
r I_{0+}^{1-\gamma;\Psi} p(T) \geq I_{0+}^{1-\gamma;\Psi} p(0).
\]
(4.7)
On the similar line for the function \( q \) one can obtain the following relations
\[
I_{0+}^{1-\gamma;\Psi} q(0) = r I_{0+}^{1-\gamma;\Psi} v(0),
\]
(4.8)
\[
r I_{0+}^{1-\gamma;\Psi} q(T) \leq I_{0+}^{1-\gamma;\Psi} q(0).
\]
(4.9)
Next, our aim is to prove that \( p \in C_{1-\gamma;\Psi}(J, \mathbb{R}) \) satisfies the following fractional differential inequality
\[
D_{0+}^{\alpha,\beta;\Psi} p(t) - M p(t) \leq r g(t), \; t \in (0, T].
\]
(4.10)
If \( r I_{0+}^{1-\gamma;\Psi} u(T) \geq I_{0+}^{1-\gamma;\Psi} u(0) \), then \( a_u(t) = 0, \; t \in (0, T] \). Further, \( u \) is a lower solution of linear \( \Psi \)-Hilfer BVP (1.2)-(1.3). Therefore, we have
\[
D_{0+}^{\alpha,\beta;\Psi} p(t) - M p(t) = r \left[ D_{0+}^{\alpha,\beta;\Psi} u(t) - M u(t) \right] \\
\leq r [g(t) - a_u(t)] \\
= r g(t), \; t \in (0, T].
\]
If \( r I_{0+}^{1-\gamma;\Psi} u(T) < I_{0+}^{1-\gamma;\Psi} u(0) \), then
\[
D_{0+}^{\alpha,\beta;\Psi} p(t) - M p(t) \\
= D_{0+}^{\alpha,\beta;\Psi} \left[ r u(t) + \xi(t) \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right) \right] \\
- M \left[ r u(t) + \xi(t) \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right) \right] \\
= r \left[ D_{0+}^{\alpha,\beta;\Psi} u(t) - M u(t) \right] \\
+ \left[ D_{0+}^{\alpha,\beta;\Psi} \xi(t) - M \xi(t) \right] \left( I_{0+}^{1-\gamma;\Psi} u(0) - r I_{0+}^{1-\gamma;\Psi} u(T) \right), \; t \in (0, T].
\]
Using the definition of $a_u(t)$ and the fact that $u$ is a lower solution of linear $\Psi$-Hilfer BVP \eqref{1.2}--\eqref{1.3} from above equation, we obtain
\[
H^\alpha_0,\beta_0^\tau \mathcal{D} p(t) - M p(t) \leq r [g(t) - a_u(t)] + r a_u(t) = r g(t), \ t \in (0, T].
\]
In both cases $r \mathcal{I}^{1-\gamma}_0 \Psi u(T) \geq \mathcal{I}^{1-\gamma}_0 \Psi u(0)$ and $r \mathcal{I}^{1-\gamma}_0 \Psi u(T) < \mathcal{I}^{1-\gamma}_0 \Psi u(0)$, we have proved that the function $p \in C_{1-\gamma} \Psi (J, \mathbb{R})$ defined in equation \eqref{4.1} satisfies the fractional differential inequality \eqref{4.10}. On the similar line, one can prove that the function $q \in C_{1-\gamma} \Psi (J, \mathbb{R})$ defined in equation \eqref{4.2} satisfies the following fractional differential inequality
\[
H^\alpha_0,\beta_0^\tau q(t) - M q(t) \geq r g(t), \ t \in (0, T]. \tag{4.11}
\]
Define $\sigma(t) = p(t) - q(t), \ t \in (0, T]$. Then, $\sigma \in C_{1-\gamma} \Psi (J, \mathbb{R})$. By using fractional differential inequalities \eqref{4.10} and \eqref{4.11}, we obtain
\[
H^\alpha_0,\beta_0^\tau \mathcal{D} \sigma(t) = H^\alpha_0,\beta_0^\tau \mathcal{D} p(t) - H^\alpha_0,\beta_0^\tau \mathcal{D} q(t)
\leq [r g(t) + M p(t)] - [r g(t) + M q(t)]
= M \sigma(t), \ t \in (0, T].
\]
Further, by using equations \eqref{4.6} and \eqref{4.8} and hypothesis, we obtain
\[
\mathcal{I}^{1-\gamma}_0 \Psi \sigma(0) = \mathcal{I}^{1-\gamma}_0 \Psi p(0) - \mathcal{I}^{1-\gamma}_0 \Psi q(0)
= r \left[ \mathcal{I}^{1-\gamma}_0 \Psi u(0) - \mathcal{I}^{1-\gamma}_0 \Psi v(0) \right]
\leq 0.
\]
We have proved that $\sigma \in C_{1-\gamma} \Psi (J, \mathbb{R})$ satisfies
\[
\begin{align*}
\mathcal{D} & \mathcal{I}^{1-\gamma}_0 \Psi \sigma(t) - M \sigma(t) \leq 0, \ t \in (0, T], \\
\mathcal{I}^{1-\gamma}_0 \Psi \sigma(0) & \leq 0.
\end{align*}
\]
By applying Theorem \ref{3.1} we obtain $\sigma(t) \leq 0, \ t \in (0, T]$. This gives
\[
p(t) \leq q(t), \ t \in (0, T].
\]
Next, for any $\lambda \in \mathbb{R}$, consider the following linear $\Psi$-Hilfer FDEs subject to initial condition
\[
\begin{align*}
\mathcal{D} & \mathcal{I}^{1-\gamma}_0 \Psi y(t) - My(t) = g(t), \ t \in (0, T], \\
\mathcal{I}^{1-\gamma}_0 \Psi y(t)|_{t=0} & = \lambda.
\end{align*}
\tag{4.12}
\]
Then, by Lemma \ref{2.4} it has a unique solution in $C_{1-\gamma} \Psi (J, \mathbb{R})$ given by
\[
y(t, \lambda) = \lambda (\Psi(t) - \Psi(0))^{\gamma-1} E_{\alpha, \gamma} \left( M(\Psi(t) - \Psi(0))^\alpha \right)
+ \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} \left( M(\Psi(t) - \Psi(s))^\alpha \right) g(s) ds, \ t \in (0, T]. \tag{4.13}
\]
Since $g \in C_{1-\gamma} \Psi (J, \mathbb{R})$, the function $(\Psi(\cdot) - \Psi(0))^{1-\gamma} y(\cdot, \lambda)$ is continuous on $J$ for each $\lambda \in \mathbb{R}$. 
Define $\chi(t) = p(t) - r y(t, \lambda)$, $t \in (0, T]$, where $y(t, \lambda)$ is a solution of (4.12). Then, $\chi \in C_{1-\gamma; \Psi}(J, \mathbb{R})$. Take any $\lambda \in \mathbb{R}$ such that

$$I_{0+}^{1-\gamma; \Psi} p(T) \leq \lambda \leq I_{0+}^{1-\gamma; \Psi} q(T).$$

(4.14)

From equations (4.10) and (4.12), we have

$$H D_{0+}^{\alpha, \beta; \Psi} \chi(t) - M \chi(t) = H D_{0+}^{\alpha, \beta; \Psi} [p(t) - r y(t, \lambda)] - M [p(t) - r y(t, \lambda)]$$

$$= \left[ H D_{0+}^{\alpha, \beta; \Psi} p(t) - M p(t) \right] - r \left[ H D_{0+}^{\alpha, \beta; \Psi} y(t, \lambda) - M y(t, \lambda) \right]$$

$$\leq r g(t) - r g(t) = 0, t \in (0, T].$$

Further, by using inequalities (4.7) and (4.11) and initial condition in (4.12), we obtain

$$I_{0+}^{1-\gamma; \Psi} \chi(0) = I_{0+}^{1-\gamma; \Psi} p(0) - r I_{0+}^{1-\gamma; \Psi} y(t, \lambda)|_{t=0}$$

$$\leq r \left[ I_{0+}^{1-\gamma; \Psi} p(T) - \lambda \right]$$

$$\leq 0.$$

Therefore, $\chi \in C_{1-\gamma; \Psi}(J, \mathbb{R})$ satisfies

$$\begin{cases}
H D_{0+}^{\alpha, \beta; \Psi} \chi(t) - M \chi(t) \leq 0, t \in (0, T], \\
I_{0+}^{1-\gamma; \Psi} \chi(0) \leq 0.
\end{cases}$$

By applying Theorem 3.1 we obtain $\chi(t) \leq 0, t \in (0, T]$. This implies $p(t) \leq r y(t, \lambda), t \in (0, T]$. On the similar line one can prove that $r y(t, \lambda) \leq q(t), t \in (0, T]$. Therefore,

$$p(t) \leq r y(t, \lambda) \leq q(t), t \in (0, T].$$

Since $\Psi$-Riemann-Liouville fractional integral operator $I_{0+}^{1-\gamma; \Psi}$ is monotonic, from above inequalities, we obtain

$$I_{0+}^{1-\gamma; \Psi} p(t) \leq r I_{0+}^{1-\gamma; \Psi} y(t, \lambda) \leq I_{0+}^{1-\gamma; \Psi} q(t), t \in (0, T].$$

Therefore, we can write

$$I_{0+}^{1-\gamma; \Psi} p(T) \leq r I_{0+}^{1-\gamma; \Psi} y(T, \lambda) \leq I_{0+}^{1-\gamma; \Psi} q(T),$$

(4.15)

for any $\lambda \in \left[ I_{0+}^{1-\gamma; \Psi} p(T), I_{0+}^{1-\gamma; \Psi} q(T) \right]$.

Define

$$g(\lambda) = r I_{0+}^{1-\gamma; \Psi} y(T, \lambda) - \lambda, \lambda \in \left[ I_{0+}^{1-\gamma; \Psi} p(T), I_{0+}^{1-\gamma; \Psi} q(T) \right].$$

(4.16)

Since two parameter Mittag-Leffler function is analytic, using it’s series representation the equation (4.13) can be written as

$$y(t, \lambda) = \lambda (\Psi(t) - \Psi(0))^{\gamma-1} \sum_{k=0}^{\infty} \frac{(M(\Psi(t) - \Psi(0))^\alpha)^k}{\Gamma(k\alpha + \gamma)}$$

$$+ \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \sum_{k=0}^{\infty} \frac{(M(\Psi(t) - \Psi(s))^\alpha)^k}{\Gamma(k\alpha + \alpha)} g(s)ds$$

where $M=\max_{\lambda \in [0,1]} |\Psi(\lambda)|$ and $\Psi(\lambda)$ is a real valued function.
Therefore, we have

\[\lambda \sum_{k=0}^{\infty} \frac{M^k}{\Gamma(k\alpha + \gamma)} (\Psi(t) - \Psi(0))^{k\alpha + \gamma - 1} + \sum_{k=0}^{\infty} \frac{M^k}{\Gamma(\alpha(k + 1))} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^\alpha (k + 1) - 1 g(s) ds\]

\[= \lambda \sum_{k=0}^{\infty} \frac{M^k}{\Gamma(k\alpha + \gamma)} (\Psi(t) - \Psi(0))^{k\alpha + \gamma - 1} + \sum_{k=0}^{\infty} M^k \mathcal{I}_{0+}^{1-\gamma; \Psi} g(t), \ t \in (0, T].\]

Using Lemma 2.1 from above equation, we obtain

\[\mathcal{I}_{0+}^{1-\gamma; \Psi} g(t, \lambda)\]

\[= \lambda \sum_{k=0}^{\infty} \frac{M^k}{\Gamma(k\alpha + \gamma)} \mathcal{I}_{0+}^{1-\gamma; \Psi} (\Psi(t) - \Psi(0))^{k\alpha + \gamma - 1} + \sum_{k=0}^{\infty} M^k \mathcal{I}_{0+}^{1-\gamma; \Psi} \mathcal{I}_{0+}^{\alpha(k+1); \Psi} g(t)\]

\[= \lambda \sum_{k=0}^{\infty} \frac{M^k}{\Gamma(k\alpha + \gamma + \gamma - 1)} (\Psi(t) - \Psi(0))^{k\alpha + \gamma - 1} + \sum_{k=0}^{\infty} M^k \mathcal{I}_{0+}^{1-\gamma+\alpha(k+1); \Psi} g(t)\]

\[= \lambda \sum_{k=0}^{\infty} \frac{M^k}{\Gamma(k\alpha + 1)} (\Psi(t) - \Psi(0))^{k\alpha} + \sum_{k=0}^{\infty} M^k \mathcal{I}_{0+}^{1-\gamma+\alpha(k+1); \Psi} g(t)\]

\[= \lambda E_{\alpha, 1} \left( M (\Psi(t) - \Psi(0))^\alpha \right)

+ \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-\gamma} \sum_{k=0}^{\infty} \frac{(M(\Psi(t) - \Psi(s))^\alpha)^k}{\Gamma(k\alpha + \alpha + 1 - \gamma)} g(s) ds\]

\[= \lambda E_{\alpha, 1} \left( M (\Psi(t) - \Psi(0))^\alpha \right)

+ \int_0^t \Psi'(s) (\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha, \alpha+1-\gamma} (M (\Psi(T) - \Psi(s))^\alpha) g(s) ds, \ t \in (0, T].\]

Therefore,

\[\mathcal{I}_{0+}^{1-\gamma; \Psi} g(t, \lambda)|_{t=T}

= \lambda E_{\alpha, 1} \left( M (\Psi(T) - \Psi(0))^\alpha \right)

+ \int_0^T \Psi'(s) (\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha, \alpha+1-\gamma} (M (\Psi(T) - \Psi(s))^\alpha) g(s) ds. \quad (4.17)\]

Using equation (4.17) in equation (4.16), we get

\[g(\lambda) = r \left\{ \lambda E_{\alpha, 1} \left( M (\Psi(T) - \Psi(0))^\alpha \right) \right.\]

\[+ \int_0^T \Psi'(s) (\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha, \alpha+1-\gamma} (M (\Psi(T) - \Psi(s))^\alpha) g(s) ds \}

\[- \lambda.\]

Differentiating above equation with respect to \(\lambda\) and using the condition on \(r\), we obtain

\[g'(\lambda) = r E_{\alpha, 1} \left( M (\Psi(T) - \Psi(0))^\alpha \right) - 1 < 0.\]

This implies \(g\) is strictly decreasing function on the closed interval \([\mathcal{I}_{0+}^{1-\gamma; \Psi} p(T), \mathcal{I}_{0+}^{1-\gamma; \Psi} q(T)]\).

Therefore, we have

\[g \left( \mathcal{I}_{0+}^{1-\gamma; \Psi} p(T) \right) > g \left( \mathcal{I}_{0+}^{1-\gamma; \Psi} q(T) \right).\]
Next, we show that the equation $g(\lambda) = 0$ has atmost one solution on $\mathbb{R}$. Using equation (4.15), we obtain
\[
g\left(\mathcal{I}_{0+}^{1-\gamma; \Psi} q(T)\right) = \left[r \mathcal{I}_{0+}^{1-\gamma; \Psi} y(T, \mathcal{I}_{0+}^{1-\gamma; \Psi} q(T)) - \mathcal{I}_{0+}^{1-\gamma; \Psi} q(T)\right] \leq 0,
\]
and
\[
g\left(\mathcal{I}_{0+}^{1-\gamma; \Psi} p(T)\right) = \left[r \mathcal{I}_{0+}^{1-\gamma; \Psi} y(T, \mathcal{I}_{0+}^{1-\gamma; \Psi} p(T)) - \mathcal{I}_{0+}^{1-\gamma; \Psi} p(T)\right] \geq 0.
\]
Since $g$ is continuous on the closed interval $\left[\mathcal{I}_{0+}^{1-\gamma; \Psi} p(T), \mathcal{I}_{0+}^{1-\gamma; \Psi} q(T)\right]$ and satisfies the conditions (4.18) and (4.19), using the location of root theorem given in Lemma 2.6 coupled with strictly decreasing nature of $g$, there exist at most one $\lambda_0 \in \left[\mathcal{I}_{0+}^{1-\gamma; \Psi} p(T), \mathcal{I}_{0+}^{1-\gamma; \Psi} q(T)\right]$ such that
\[
g(\lambda_0) = 0.
\]
From equations (4.12), (4.16) and (4.20), we obtain
\[
r \mathcal{I}_{0+}^{1-\gamma; \Psi} y(t, \lambda_0)|_{t=T} = \lambda_0 = \mathcal{I}_{0+}^{1-\gamma; \Psi} y(t)|_{t=0}.
\]
From equations (4.12) and (4.13), it follows that
\[
y(t, \lambda_0) = \lambda_0 (\Psi(t) - \Psi(0))^{\gamma-1} E_{\alpha, \gamma} (M(\Psi(t) - \Psi(0))^{\alpha})
+ \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} (M(\Psi(t) - \Psi(s))^{\alpha}) g(s)ds
\]
is the unique solution of linear $\Psi$-Hilfer FDEs with initial condition
\[
\left\{
\begin{array}{l}
H^{\alpha, \beta; \Psi} y(t) - My(t) = g(t), \quad t \in (0, T], \\
\mathcal{I}_{0+}^{1-\gamma; \Psi} y(t)|_{t=0} = \lambda_0.
\end{array}
\right.
\]
Further, from (4.21) and (4.23) it follows that, $y(t, \lambda_0)$ given in (4.22) is the unique solution of linear $\Psi$-Hilfer BVP (1.2)-(1.3).

**Part II:** In this part we prove that the unique solution $y(t, \lambda_0)$ obtained in the first part satisfies
\[
u \leq y(\cdot, \lambda_0) \leq u \quad \text{in} \quad C_{1-\gamma; \Psi}(J, \mathbb{R}).
\]
Define $h(t) = u(t) - y(t, \lambda_0)$, $t \in (0, T]$. Clearly $h \in C_{1-\gamma; \Psi}(J, \mathbb{R})$.

Case 1: If $r \mathcal{I}_{0+}^{1-\gamma; \Psi} u(T) \geq \mathcal{I}_{0+}^{1-\gamma; \Psi} u(0)$, then $a_u(t) = 0$, $t \in (0, T]$. Since $u$ is a lower solution of the linear $\Psi$-Hilfer BVP (1.2)-(1.3) and $y(t, \lambda_0)$ is the solution of linear $\Psi$-Hilfer FDEs (4.23), we obtain
\[
H^{\alpha, \beta; \Psi} h(t) - M h(t) = H^{\alpha, \beta; \Psi} [u(t) - y(t, \lambda_0)] - M [u(t) - y(t, \lambda_0)]
= \left[H^{\alpha, \beta; \Psi} u(t) - M u(t)\right] - \left[H^{\alpha, \beta; \Psi} y(t, \lambda_0) - M y(t, \lambda_0)\right]
\leq g(t) - a_u(t) - g(t)
= 0, \quad t \in (0, T].
\]

Further, using the equations (4.6) and (4.7) and initial condition in (4.23), we have
\[
\mathcal{I}_{0+}^{1-\gamma; \Psi} h(0) = \mathcal{I}_{0+}^{1-\gamma; \Psi} u(0) - \mathcal{I}_{0+}^{1-\gamma; \Psi} y(t, \lambda_0)|_{t=0}
\]
\[ I_{0^+}^{1-\gamma;\Psi}p(t) - I_{0^+}^{1-\gamma;\Psi}y(t, \lambda_0) |_{t=0} \leq I_{0^+}^{1-\gamma;\Psi}p(T) - \lambda_0 \leq 0. \]

Applying Theorem 3.1 to the fractional inequalities

\[
\begin{align*}
\left\{ \begin{array}{l}
H^{\alpha,\beta;\Psi}D_{0^+} h(t) - M h(t) \leq 0, \quad t \in (0, T], \\
I_{0^+}^{1-\gamma;\Psi}h(0) \leq 0,
\end{array} \right.
\end{align*}
\]

we obtain \( h(t) \leq 0, \quad t \in (0, T]. \) This gives, \( u(t) \leq y(t, \lambda_0), \quad t \in (0, T]. \)

Case 2: If \( r I_{0^+}^{1-\gamma;\Psi}u(T) < I_{0^+}^{1-\gamma;\Psi}u(0) \) then \( a_u(t) \neq 0, \quad t \in (0, T] \) as defined in (2.2). Since \( u \) is a lower solution of the linear \( \Psi \)-Hilfer BVP (1.2)-(1.3) and \( y(t, \lambda_0) \) is the solution of linear \( \Psi \)-Hilfer FDEs (4.23), we obtain

\[
\begin{align*}
H^{\alpha,\beta;\Psi}D_{0^+} h(t) - M h(t) &= H^{\alpha,\beta;\Psi}D_{0^+} [u(t) - y(t, \lambda_0)] - M [u(t) - y(t, \lambda_0)] \\
&= \left[ H^{\alpha,\beta;\Psi}D_{0^+} u(t) - M u(t) \right] - \left[ H^{\alpha,\beta;\Psi}D_{0^+} y(t, \lambda_0) - M y(t, \lambda_0) \right] \\
&\leq g(t) - a_u(t) - g(t) \\
&= -a_u(t), \quad t \in (0, T].
\end{align*}
\]

Further, we have \( I_{0^+}^{1-\gamma;\Psi}h(0) \leq 0. \) By applying Theorem 3.2 to the fractional inequalities

\[
\begin{align*}
\left\{ \begin{array}{l}
H^{\alpha,\beta;\Psi}D_{0^+} h(t) - M h(t) \leq g(t) - a_u(t) - g(t) = -a_u(t), \quad t \in (0, T], \\
I_{0^+}^{1-\gamma;\Psi}h(0) \leq 0,
\end{array} \right.
\end{align*}
\]

we have \( h(t) \leq 0, \quad t \in (0, T]. \) This implies, \( u(t) \leq y(t, \lambda_0), \quad t \in (0, T]. \) From Case 1 and Case 2, it follows that

\[
u(t) \geq y(t, \lambda_0), \quad t \in (0, T].
\]

Following the similar approach, one can show that

\[
v(t) \geq y(t, \lambda_0), \quad t \in (0, T].
\]

Therefore, we have

\[
u(t) \leq y(t, \lambda_0) \leq v(t), \quad t \in (0, T].
\]

By Part I and Part II, it follows that \( y(\cdot, \lambda_0) \in C_{1-\gamma;\Psi} (J, \mathbb{R}) \) is the unique solution of the linear \( \Psi \)-Hilfer BVP (1.2)-(1.3) that satisfies the condition

\[
u \leq y(\cdot, \lambda_0) \leq v \quad \text{in} \quad C_{1-\gamma;\Psi} (J, \mathbb{R}).
\]

This completes the proof. \( \square \)
5 Existence and uniqueness for the nonlinear $\Psi$-Hilfer BVP

**Definition 5.1** We say that $w_0 \in C_{1-\gamma;\Psi}(J, \mathbb{R})$ and $z_0 \in C_{1-\gamma;\Psi}(J, \mathbb{R})$ are the lower and upper solutions respectively of the nonlinear $\Psi$-Hilfer BVP [1.4]-[1.5] if

$$H^D_{0+}^{\alpha, \beta; \Psi} w_0(t) - Mw_0(t) \leq f(t, w_0(t)) - a_{w_0}(t), \quad t \in (0, T],$$

and

$$H^D_{0+}^{\alpha, \beta; \Psi} z_0(t) - Mz_0(t) \geq f(t, z_0(t)) + b_{z_0}(t), \quad t \in (0, T],$$

where

$$a_{w_0}(t) = \begin{cases} 0, & \text{if } r \mathcal{T}^{1-\gamma;\Psi}_{0+} w_0(T) \geq \mathcal{T}^{1-\gamma;\Psi}_{0+} w_0(0) \\ \frac{1}{r} \left( H^D_{0+}^{\alpha, \beta; \Psi} \xi(t) - r \mathcal{T}^{1-\gamma;\Psi}_{0+} w_0(0) \right), & \text{if } r \mathcal{T}^{1-\gamma;\Psi}_{0+} w_0(T) < \mathcal{T}^{1-\gamma;\Psi}_{0+} w_0(0), \end{cases}$$

and

$$b_{z_0}(t) = \begin{cases} 0, & \text{if } r \mathcal{T}^{1-\gamma;\Psi}_{0+} z_0(T) \leq \mathcal{T}^{1-\gamma;\Psi}_{0+} z_0(0) \\ \frac{1}{r} \left( H^D_{0+}^{\alpha, \beta; \Psi} \xi(t) - r \mathcal{T}^{1-\gamma;\Psi}_{0+} z_0(T) - \mathcal{T}^{1-\gamma;\Psi}_{0+} z_0(0) \right), & \text{if } r \mathcal{T}^{1-\gamma;\Psi}_{0+} z_0(T) > \mathcal{T}^{1-\gamma;\Psi}_{0+} z_0(0), \end{cases}$$

and $\xi$ is the function as defined in [2.3].

**Theorem 5.1** Assume $z_0 \in C_{1-\gamma;\Psi}(J, \mathbb{R})$ and $w_0 \in C_{1-\gamma;\Psi}(J, \mathbb{R})$ are the upper and lower solutions respectively of the nonlinear $\Psi$-Hilfer BVP [1.4]-[1.5] such that $w_0 \leq z_0$. Further, assume that:

(H1) the function $f$ satisfies

(i) $f(\cdot, y(\cdot)) \in C_{1-\gamma;\Psi}(J, \mathbb{R})$ for each $y \in C_{1-\gamma;\Psi}(J, \mathbb{R})$,

(ii) $f(t, y_1) \leq f(t, y_2)$, for any $y_1, y_2 \in \mathbb{R}$ with $y_1 \leq y_2$ and $t \in (0, T]$.

Then, the nonlinear $\Psi$-Hilfer BVP [1.4]-[1.5] has a minimal solution $w^* \in C_{1-\gamma;\Psi}(J, \mathbb{R})$ and a maximal solution $z^* \in C_{1-\gamma;\Psi}(J, \mathbb{R})$ in the line segment

$$[w_0, z_0] = \{ y \in C_{1-\gamma;\Psi}(J, \mathbb{R}) : w_0 \leq y \leq z_0 \}.$$ 

Further, if $\{w_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are the iterative sequences defined by

$$w_n(t) = \frac{r (\Psi(t) - \Psi(0))^{\gamma-1} E_{\alpha, \gamma} (M(\Psi(t) - \Psi(0))^\alpha)}{1 - r E_{\alpha, 1} (M(\Psi(T) - \Psi(0))^\alpha)}$$

$$\times \int_0^T \Psi(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha, \alpha+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) f(s, w_{n-1}(s))ds$$

$$+ \int_0^t \Psi(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} (M(\Psi(t) - \Psi(s))^\alpha) f(s, w_{n-1}(s))ds$$

and

$$z_n(t) = \frac{r (\Psi(t) - \Psi(0))^{\gamma-1} E_{\alpha, \gamma} (M(\Psi(t) - \Psi(0))^\alpha)}{1 - r E_{\alpha, 1} (M(\Psi(T) - \Psi(0))^\alpha)}$$

$$\times \int_0^T \Psi(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha, \alpha+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) f(s, z_{n-1}(s))ds$$

$$+ \int_0^t \Psi(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} (M(\Psi(t) - \Psi(s))^\alpha) f(s, z_{n-1}(s))ds$$

and
\[
\times \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha, \alpha+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) f(s, z_{n-1}(s))ds
\]
\[
+ \int_0^T \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} (M(\Psi(t) - \Psi(s))^\alpha) f(s, z_{n-1}(s))ds
\]

then \(\{w_n\}_{n=1}^\infty\) and \(\{z_n\}_{n=1}^\infty\) are the monotonic sequences in \(C_{1-\gamma}; \psi(J, \mathbb{R})\) such that

\[
\lim_{n \to \infty} \|w_n - w^*\|_{C_{1-\gamma}; \psi(J, \mathbb{R})} = 0
\]

and

\[
\lim_{n \to \infty} \|z_n - z^*\|_{C_{1-\gamma}; \psi(J, \mathbb{R})} = 0.
\]

**Proof:** We give the proof in five parts.

**Part 1:** We denote \(\mathcal{D} = [w_0, z_0]\). For any \(\varphi \in \mathcal{D}\), we consider the following linear \(\psi\)-Hilfer BVP

\[
\begin{aligned}
&\mathcal{D}_{0+}^{\alpha, \beta; \Psi} y(t) - My(t) = f(t, \varphi(t)), \quad t \in (0, T], \\
&\mathcal{I}_{0+}^{1-\gamma; \Psi} y(0) = r \mathcal{I}_{0+}^{1-\gamma; \Psi} y(T).
\end{aligned}
\]

(5.1)

Since \(w_0, z_0\) are lower and upper solutions respectively of the nonlinear \(\psi\)-Hilfer BVP (1.4)-(1.5), using hypothesis (H1)(ii), we have

\[
\mathcal{D}_{0+}^{\alpha, \beta; \Psi} w_0(t) - Mw_0(t) \leq f(t, w_0(t)) - aw_0(t) \leq f(t, \varphi(t)) - aw_0(t), \quad t \in (0, T]
\]

and

\[
\mathcal{D}_{0+}^{\alpha, \beta; \Psi} z_0(t) - Mz_0(t) \geq f(t, z_0(t)) + bz_0(t) \geq f(t, \varphi(t)) + bz_0(t), \quad t \in (0, T].
\]

This implies \(w_0\) and \(z_0\) are the lower and upper solutions respectively of the linear \(\psi\)-Hilfer BVP (5.1). In the view of Theorem 4.1, the linear \(\psi\)-Hilfer BVP (5.1) has a unique solution \(y \in \mathcal{D}\), given by

\[
y(t) = \mathcal{I}_{0+}^{1-\gamma; \Psi} y(0) \left(\Psi(t) - \Psi(0))^{\gamma-1} E_{\alpha, \gamma} (M(\Psi(t) - \Psi(0))^\alpha)\right)
\]
\[
+ \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha} (M(\Psi(t) - \Psi(s))^\alpha) f(s, \varphi(s))ds, \quad t \in (0, T].
\]

(5.2)

Following the similar steps as in the proof of the Theorem 4.1 from equation (5.2), we have

\[
\mathcal{I}_{0+}^{1-\gamma; \Psi} y(T) = \mathcal{I}_{0+}^{1-\gamma; \Psi} y(0) E_{\alpha, 1} (M(\Psi(T) - \Psi(0))^\alpha)
\]
\[
+ \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha, \alpha+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) f(s, \varphi(s))ds.
\]

Since \(\mathcal{I}_{0+}^{1-\gamma; \Psi} y(0) = r \mathcal{I}_{0+}^{1-\gamma; \Psi} y(T),\) we have

\[
\mathcal{I}_{0+}^{1-\gamma; \Psi} y(0) = r \mathcal{I}_{0+}^{1-\gamma; \Psi} y(0) E_{\alpha, 1} (M(\Psi(T) - \Psi(0))^\alpha)
\]
\[
+ r \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha, \alpha+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) f(s, \varphi(s))ds.
\]
Hence,
\[ I_{0+}^{1-\gamma; \Psi} y(0) = \frac{r}{[1 - r E_{\alpha,1} (M (\Psi(T) - \Psi(0))^\alpha)]} \]
\[ \times \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha,\alpha+1-\gamma} (M (\Psi(T) - \Psi(s))^\alpha) f(s, \varphi(s))ds. \]  
(5.3)

Using equation (5.3) in (5.2), we get
\[ y(t) = \frac{r (\Psi(t) - \Psi(0))^{\gamma-1} E_{\alpha,\gamma} (M (\Psi(t) - \Psi(0))^\alpha)}{[1 - r E_{\alpha,1} (M (\Psi(T) - \Psi(0))^\alpha)]} \]
\[ \times \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha,\alpha+1-\gamma} (M (\Psi(T) - \Psi(s))^\alpha) f(s, \varphi(s))ds \]
\[ + \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha,\alpha} (M (\Psi(t) - \Psi(s))^\alpha) f(s, \varphi(s))ds, \ t \in (0, T]. \]  
(5.4)

Consider the operator \( A : \mathcal{D} \to \mathcal{X} \) defined by
\[ A\varphi(t) = \frac{r (\Psi(t) - \Psi(0))^{\gamma-1} E_{\alpha,\gamma} (M (\Psi(t) - \Psi(0))^\alpha)}{[1 - r E_{\alpha,1} (M (\Psi(T) - \Psi(0))^\alpha)]} \]
\[ \times \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha,\alpha+1-\gamma} (M (\Psi(T) - \Psi(s))^\alpha) f(s, \varphi(s))ds \]
\[ + \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha,\alpha} (M (\Psi(t) - \Psi(s))^\alpha) f(s, \varphi(s))ds, \ t \in (0, T]. \]  
(5.4)

It follows that, \( A\varphi(t) \) is the solution of the linear BVP (5.1). Using the Theorem 4.1 we have
\[ w_0(t) \leq A\varphi(t) \leq z_0(t), \ \text{for} \ \varphi \in \mathcal{D} = [w_0, z_0] \text{ and } t \in (0, T]. \]

In particular, we have \( w_0 \leq A w_0 \) and \( A z_0 \leq z_0 \) in \( C_{1-\gamma; \Psi} (J, \mathbb{R}) \).

**Part 2:** In this part, we prove that \( A : \mathcal{D} \to \mathcal{X} \) is completely continuous operator.

Firstly, we prove that, \( A(\mathcal{D}) \) is uniformly bounded. Since \( (\Psi(\cdot) - \Psi(0))^{1-\gamma} f(\cdot, y(\cdot)) \) is continuous on compact interval \( J \), it is bounded. Hence, there exists constant \( \mathfrak{K} > 0 \) such that
\[ \max_{t \in J} |(\Psi(t) - \Psi(0))^{1-\gamma} f(t, y(t))| \leq \mathfrak{K}. \]  
(5.5)

Using increasing nature of \( \Psi \), condition [5.5] and the Lemma [2.1]b), for any \( t \in J \), we obtain
\[ |(\Psi(t) - \Psi(0))^{1-\gamma} A\varphi(t)| \]
\[ = \frac{r E_{\alpha,\gamma} (M (\Psi(t) - \Psi(0))^\alpha)}{[1 - r E_{\alpha,1} (M (\Psi(T) - \Psi(0))^\alpha)]} \]
\[ \times \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha,\alpha+1-\gamma} (M (\Psi(T) - \Psi(s))^\alpha) f(s, \varphi(s))ds \]
\[ + (\Psi(t) - \Psi(0))^{1-\gamma} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha,\alpha} (M (\Psi(t) - \Psi(s))^\alpha) f(s, \varphi(s))ds \]
This implies, 

\[
\begin{aligned}
(r E_{a, \gamma} (M(\Psi(T) - \Psi(0))^\alpha) E_{a, a+1-\gamma} (M(\Psi(T) - \Psi(0))^\alpha) & \leq \\
\frac{1}{1 - r E_{a, 1} (M(\Psi(T) - \Psi(0))^\alpha)} & \times \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma}(\Psi(s) - \Psi(0))^{\gamma-1} \left| (\Psi(s) - \Psi(0))^{1-\gamma} f(s, \varphi(s)) \right| ds \\
+ (\Psi(T) - \Psi(0))^{1-\gamma} E_{a,a} (M(\Psi(T) - \Psi(0))^\alpha) & \times \int_0^T \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1}(\Psi(s) - \Psi(0))^{\gamma-1} \left| (\Psi(s) - \Psi(0))^{1-\gamma} f(s, \varphi(s)) \right| ds \\
& \leq \frac{1}{1 - r E_{a, 1} (M(\Psi(T) - \Psi(0))^\alpha)} & \times \mathfrak{R} \Gamma(\alpha - \gamma + 1) \left[ \mathfrak{T}_{0+}^{\alpha-\gamma+1; \Psi(T) - \Psi(0))^{\gamma-1}} \right]_{t=T} \\
+ (\Psi(T) - \Psi(0))^{1-\gamma} E_{a,a} (M(\Psi(T) - \Psi(0))^\alpha) & \mathfrak{R} \Gamma(\alpha) \mathfrak{T}_{0+}^{\alpha; \Psi(T) - \Psi(0))^{\gamma-1}} \\
& \leq \frac{1}{1 - r E_{a, 1} (M(\Psi(T) - \Psi(0))^\alpha)} & \times \mathfrak{R} \Gamma(\alpha - \gamma + 1) \left[ \mathfrak{R} \Gamma(\alpha) \frac{\Gamma(\gamma)}{\Gamma(\gamma + \alpha)} (\Psi(T) - \Psi(0))^{\alpha+\gamma-1} \right] \\
& \mathfrak{R} (\Psi(T) - \Psi(0))^\alpha & \times \left\{ B(\alpha - \gamma + 1, \gamma) \frac{r E_{a, \gamma} (M(\Psi(T) - \Psi(0))^\alpha) E_{a, a+1-\gamma} (M(\Psi(T) - \Psi(0))^\alpha)}{1 - r E_{a, 1} (M(\Psi(T) - \Psi(0))^\alpha)} \\
+ B(\alpha, \gamma) E_{a,a} (M(\Psi(T) - \Psi(0))^\alpha) \right\} \\
& := \omega \\
\end{aligned}
\]

Then, for any \( \varphi \in \mathfrak{D} \), we obtain

\[
\|A \varphi\|_{C_{1-\gamma; \Psi}} (J, \mathbb{R}) \leq \omega, \ \varphi \in \mathfrak{D}.
\]

This implies, \( A(\mathfrak{D}) \) is uniformly bounded. Next, we prove that \( A(\mathfrak{D}) \) is equicontinuous. Let any \( t_1, t_2 \in J \) such that \( t_2 > t_1 \). Then, for any \( \varphi \in \mathfrak{D} \), we obtain

\[
\left| (\Psi(t_2) - \Psi(0))^{1-\gamma} A \varphi(t_2) - (\Psi(t_1) - \Psi(0))^{1-\gamma} A \varphi(t_1) \right| = \left\{ \frac{r E_{a, \gamma} (M(\Psi(t_2) - \Psi(0))^\alpha)}{1 - r E_{a, 1} (M(\Psi(T) - \Psi(0))^\alpha)} \\
\times \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{a,a+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) f(s, \varphi(s)) ds \\
+ (\Psi(t_2) - \Psi(0))^{1-\gamma} \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\alpha-1} E_{a,a} (M(\Psi(t_2) - \Psi(s))^\alpha) f(s, \varphi(s)) ds \\
- \frac{r E_{a, \gamma} (M(\Psi(t_1) - \Psi(0))^\alpha)}{1 - r E_{a, 1} (M(\Psi(T) - \Psi(0))^\alpha)} \\
\times \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{a,a+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) f(s, \varphi(s)) ds \right\}
\]
\[
\left. \frac{r E_{\alpha,\alpha+1-\gamma} (M (\Psi(T) - \Psi(0))^\alpha)}{[1 - r E_{\alpha,1} (M (\Psi(T) - \Psi(0))^\alpha)]} \left| E_{\alpha,\gamma} (M (\Psi(t_2) - \Psi(0))^\alpha) - E_{\alpha,\gamma} (M (\Psi(t_1) - \Psi(0))^\alpha) \right| \right. \\
+ \left. \frac{\mathfrak{r} \Gamma(\alpha - \gamma + 1) \left[ I_{0+}^{\alpha-\gamma+1} \Psi(t - \Psi(0))^{-1} \right]_{t=T}}{1 - r E_{\alpha,1} (M (\Psi(T) - \Psi(0))^\alpha)} \right. \\
+ \left. \frac{\mathfrak{r} r E_{\alpha,\alpha+1-\gamma} (M (\Psi(T) - \Psi(0))^\alpha) B(\alpha - \gamma + 1, \gamma) (\Psi(T) - \Psi(0))^\alpha}{[1 - r E_{\alpha,1} (M (\Psi(T) - \Psi(0))^\alpha)]} \right. \\
\times \left. \left| E_{\alpha,\gamma} (M (\Psi(t_2) - \Psi(0))^\alpha) - E_{\alpha,\gamma} (M (\Psi(t_1) - \Psi(0))^\alpha) \right| \right. \\
+ \left. \frac{\mathfrak{r} E_{\alpha,\alpha} (M (\Psi(T) - \Psi(0))^\alpha) B(\alpha, \gamma) \left| (\Psi(t_2) - \Psi(0))^\alpha - (\Psi(t_1) - \Psi(0))^\alpha \right|}{1 - r E_{\alpha,1} (M (\Psi(T) - \Psi(0))^\alpha)} \right. \\
\left. \times \left| E_{\alpha,\gamma} (M (\Psi(t_2) - \Psi(0))^\alpha) - E_{\alpha,\gamma} (M (\Psi(t_1) - \Psi(0))^\alpha) \right| \right. \\
\left. \times \left| (\Psi(t_2) - \Psi(0))^{1-\gamma} A_{\varphi} (t_2) - (\Psi(t_1) - \Psi(0))^{1-\gamma} A_{\varphi} (t_1) \right| \rightarrow 0 \text{ as } |t_2 - t_1| \rightarrow 0. \tag{5.6} \right]
\]

By Lemma 2.5, two parameter Mittag-Leffler function is uniformly continuous. Therefore, we have

\[
\left| E_{\alpha,\gamma} (M (\Psi(t_2) - \Psi(0))^\alpha) - E_{\alpha,\gamma} (M (\Psi(t_1) - \Psi(0))^\alpha) \right| \rightarrow 0 \text{ as } |t_2 - t_1| \rightarrow 0. \tag{5.7}
\]

Further, using the continuity of \(\Psi\), we have

\[
\left| (\Psi(t_2) - \Psi(0))^\alpha - (\Psi(t_1) - \Psi(0))^\alpha \right| \rightarrow 0 \text{ as } |t_2 - t_1| \rightarrow 0. \tag{5.8}
\]

Using the conditions (5.7) and (5.8) in the inequality (5.6), we obtain

\[
\left| (\Psi(t_2) - \Psi(0))^{1-\gamma} A_{\varphi} (t_2) - (\Psi(t_1) - \Psi(0))^{1-\gamma} A_{\varphi} (t_1) \right| \rightarrow 0 \text{ as } |t_2 - t_1| \rightarrow 0.
\]

This proves \(A(\mathfrak{D})\) is equicontinuous set of family of functions. Therefore, by Arzelà-Ascoli theorem, \(A(\mathfrak{D})\) is relatively compact. Note that, the continuity of operator \(A\) follows from hypothesis (H1)(i). We have proved \(A : \mathfrak{D} \rightarrow \mathcal{X}\) is completely continuous operator.

**Part 3:** In this part, it is proved that \(A : \mathfrak{D} \rightarrow \mathcal{X}\) is monotonically increasing operator.

Let any \(\delta_1, \delta_2 \in \mathfrak{D}\) with \(w_0 \leq \delta_1 \leq \delta_2 \leq z_0\). Define \(B(t) = f(t, \delta_2(t)) - f(t, \delta_1(t)), t \in (0, T]\). Then from hypothesis (H1)(ii), we have \(B(t) \geq 0, t \in (0, T]\). For any \(t \in (0, T]\),

\[
A\delta_2(t) - A\delta_1(t)
\]
This implies \( w \) tively compact also. Therefore, by applying Lemma 2.7, there exists the fixed point of an operator \( A \) C w the limits (5.9) and (5.10), we obtain such that

For each \( n \), we have

Part 4: For each \( n(n=1,2,3,\cdots) \) define \( w_n = A w_{n-1} \) and \( z_n = A z_{n-1} \). By Part 1, it follows that

\[ w_0 \leq A w_0 = w_1 \text{ and } z_1 = A z_0 \leq z_0 \text{ in } C_{1-\gamma;\Psi}(J,\mathbb{R}). \]

Therefore, by using increasing nature of an operator \( A \), we have

\[ w_1 \leq w_2 \leq \cdots \leq w_n \leq z_n \leq \cdots \leq z_2 \leq z_1 \text{ in } C_{1-\gamma;\Psi}(J,\mathbb{R}). \]

This implies \( \{w_n\}_{n=1}^{\infty} \) and \( \{z_n\}_{n=1}^{\infty} \) are the monotonic sequences in \( A(D) \subseteq X \) which are relatively compact also. Therefore, by applying Lemma 2.7, there exists \( w^*,z^* \in C_{1-\gamma;\Psi}(J,\mathbb{R}) \) such that

\[ \lim_{n \to \infty} \|w_n - w^*\|_{C_{1-\gamma;\Psi}(J,\mathbb{R})} = 0 \quad (5.9) \]

and

\[ \lim_{n \to \infty} \|z_n - z^*\|_{C_{1-\gamma;\Psi}(J,\mathbb{R})} = 0. \quad (5.10) \]

Since for each \( n \), \( w_n = A w_{n-1} \) and \( z_n = A z_{n-1} \), by continuity of the operator \( A \) and using the limits (5.9) and (5.10), we obtain

\[ w^* = A w^* \text{ and } z^* = A z^*. \]

Therefore, \( w^* \) and \( z^* \) are the fixed points of an operator \( A \). Further, we know that, \( y \in C_{1-\gamma;\Psi}(J,\mathbb{R}) \) is the solution of the nonlinear \( \Psi \)-Hilfer BVP (1.4), (1.5) if and only if it is the fixed point of an operator \( A \). Thus, \( w^* \) and \( z^* \) are the solutions of the nonlinear \( \Psi \)-Hilfer
BVP (1.4)-(1.5).

**Part 5:** Finally, we prove that \( w^* \) and \( z^* \) are the minimal and the maximal solutions respectively of the nonlinear \( \Psi \)-Hilfer BVP (1.4)-(1.5). Let \( y \in [w_0, z_0] \) be any solution of the \( \Psi \)-Hilfer BVP (1.4)-(1.5). Then,

\[
y = Ay \quad \text{and} \quad w_0 \preceq y \preceq z_0.
\]

Since \( A \) is an increasing operator, we have

\[
w_1 = Ay_0 \preceq y \preceq z_0 = z_1.
\]

Again, using the increasing nature of an operator \( A \), from above inequality, we obtain

\[
w_2 \preceq y \preceq z_2.
\]

Continuing in this way, we obtain

\[
w_n \preceq y \preceq z_n, \quad n = 1, 2, 3, \ldots.
\]

Taking the limit as \( n \to \infty \) in the above inequality with respect to the norm \( \| \cdot \|_{C_{1-\gamma,\Psi}(J, \mathbb{R})} \), we obtain

\[
w^* \preceq y \preceq z^* \quad \text{in} \quad C_{1-\gamma,\Psi}(J, \mathbb{R}).
\]

Therefore, \( w^* \) and \( z^* \) are the minimal solution and the maximal solution respectively in \( C_{1-\gamma,\Psi}(J, \mathbb{R}) \) of the nonlinear \( \Psi \)-Hilfer BVP (1.4)-(1.5). \( \Box \)

**Theorem 5.2** Suppose that the conditions of Theorem 5.1 hold, and let there exists a constant \( \tilde{L} \in \left[ 0, \frac{1}{(\Psi(T) - \Psi(0))^\alpha} \Omega^{-1} \right) \), where

\[
\Omega = B(\alpha - \gamma + 1, \gamma)^r E_{\alpha, \gamma}(M(\Psi(T) - \Psi(0))^\alpha) E_{\alpha, \alpha+1-\gamma}(M(\Psi(T) - \Psi(0))^\alpha)
\]

\[
+ B(\alpha, \gamma) E_{\alpha, \alpha}(M(\Psi(T) - \Psi(0))^\alpha) \]  (5.11)

Further, assume that \( f \) satisfies

\[
f(t, x_2) - f(t, x_1) \leq \tilde{L}(x_2 - x_1), \quad \text{for any} \quad x_1, x_2 \in \mathbb{R} \quad \text{and} \quad x_1 \leq x_2. \]  (5.12)

Then, the \( \Psi \)-Hilfer BVP (1.4)-(1.5) has a unique solution \( y^* \) in \( [w_0, z_0] \). Moreover, for each \( y_0 \in [w_0, z_0] \) the iterative sequence defined by

\[
y_n(t) = \frac{r(\Psi(t) - \Psi(0))^{\gamma-1} E_{\alpha, \gamma}(M(\Psi(t) - \Psi(0))^\alpha)}{[1 - r E_{\alpha, 1}(M(\Psi(T) - \Psi(0))^\alpha)]}
\]

\[
\times \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha, \alpha+1-\gamma}(M(\Psi(T) - \Psi(s))^\alpha) f(s, y_{n-1}(s)) ds
\]

\[
+ \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha, \alpha}(M(\Psi(t) - \Psi(s))^\alpha) f(s, y_{n-1}(s)) ds, \quad n = 1, 2, 3, \ldots,
\]

is such that

\[
\lim_{n \to \infty} \| y_n - y^* \|_{C_{1-\gamma,\Psi}(J, \mathbb{R})} = 0
\]
and
\[
\|y_n - y^*\|_{C_{1-\gamma,\Psi}(J, \mathbb{R})} \leq \rho^n \|z_0 - w_0\|_{C_{1-\gamma,\Psi}(J, \mathbb{R})},
\]
where
\[
\rho = \Omega ((\Psi(T) - \Psi(0))^\alpha \tilde{L}). \tag{5.13}
\]

**Proof:** Let any \(w_1, z_1 \in [w_0, z_0]\) with \(w_1 \leq z_1\). Then using the definition of operator \(A\) defined in \((5.4)\) and the condition \((5.12)\), we obtain
\[
\left((\Psi(t) - \Psi(0))^{1-\gamma} (Az_1(t) - Aw_1(t))\right)
= \left\{\begin{array}{l}
\frac{r E_{\alpha,\gamma} (M(\Psi(t) - \Psi(0))^\alpha)}{[1 - r E_{\alpha,1} (M(\Psi(T) - \Psi(0))^\alpha)]} \\
\times \int_0^T \Psi(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha,\alpha+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) f(s, z_1(s))ds \\
+ (\Psi(t) - \Psi(0))^{1-\gamma} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha,\alpha} (M(\Psi(t) - \Psi(s))^\alpha) f(s, z_1(s))ds
\end{array}\right\}
- \left\{\begin{array}{l}
\frac{r E_{\alpha,\gamma} (M(\Psi(t) - \Psi(0))^\alpha)}{[1 - r E_{\alpha,1} (M(\Psi(T) - \Psi(0))^\alpha)]} \\
\times \int_0^T \Psi(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha,\alpha+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) f(s, w_1(s))ds \\
+ (\Psi(t) - \Psi(0))^{1-\gamma} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha,\alpha} (M(\Psi(t) - \Psi(s))^\alpha) f(s, w_1(s))ds
\end{array}\right\}
\leq \tilde{L} \frac{r E_{\alpha,\gamma} (M(\Psi(t) - \Psi(0))^\alpha)}{[1 - r E_{\alpha,1} (M(\Psi(T) - \Psi(0))^\alpha)]} \int_0^T \Psi'(s)(\Psi(T) - \Psi(s))^{\alpha-\gamma} E_{\alpha,\alpha+1-\gamma} (M(\Psi(T) - \Psi(s))^\alpha) \\
\times (\Psi(s) - \Psi(0))^{\gamma-1} (\Psi(s) - \Psi(0))^{1-\gamma} [z_1(s) - w_1(s)] ds \\
+ \tilde{L} (\Psi(t) - \Psi(0))^{1-\gamma} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} E_{\alpha,\alpha} (M(\Psi(t) - \Psi(s))^\alpha) \\
\times (\Psi(s) - \Psi(0))^{\gamma-1} (\Psi(s) - \Psi(0))^{1-\gamma} [z_1(s) - w_1(s)] ds \\
\leq \tilde{L} \frac{r E_{\alpha,\gamma} (M(\Psi(T) - \Psi(0))^\alpha)}{[1 - r E_{\alpha,1} (M(\Psi(T) - \Psi(0))^\alpha)]} E_{\alpha,\alpha+1-\gamma} (M(\Psi(T) - \Psi(0))^\alpha) \|z_1 - w_1\|_{C_{1-\gamma,\Psi}(J, \mathbb{R})} \\
\times \Gamma(\alpha - \gamma + 1) \left[\int_0^\infty \frac{\Gamma(\alpha - \gamma + 1)}{\Gamma(\alpha + 1)} (\Psi(T) - \Psi(0))^{\gamma-1} \right]_{t=T} \\
+ \tilde{L} (\Psi(T) - \Psi(0))^{1-\gamma} E_{\alpha,\alpha} (M(\Psi(T) - \Psi(0))^\alpha) \|z_1 - w_1\|_{C_{1-\gamma,\Psi}(J, \mathbb{R})} \\
\times \Gamma(\alpha - \gamma + 1) \frac{\Gamma(\gamma)}{\Gamma(\alpha + 1)} (\Psi(T) - \Psi(0))^\alpha + \tilde{L} (\Psi(T) - \Psi(0))^{1-\gamma} E_{\alpha,\alpha} (M(\Psi(T) - \Psi(0))^\alpha) \\
\times \|z_1 - w_1\|_{C_{1-\gamma,\Psi}(J, \mathbb{R})} \Gamma(\alpha) \frac{\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} (\Psi(T) - \Psi(0))^\alpha + \tilde{L} (\Psi(T) - \Psi(0))^\alpha \|z_1 - w_1\|_{C_{1-\gamma,\Psi}(J, \mathbb{R})}
Consider the sequences

\[ \times \left\{ B(\alpha - \gamma + 1, \gamma) \frac{r E_{\alpha, \gamma} (M(\Psi(T) - \Psi(0))^{\alpha})}{1 - r E_{\alpha, 1} (M(\Psi(T) - \Psi(0))^{\alpha})} E_{\alpha, \alpha+1-\gamma} (M(\Psi(T) - \Psi(0))^{\alpha}) + B(\alpha, \gamma) E_{\alpha, \alpha} (M(\Psi(T) - \Psi(0))^{\alpha}) \right\}. \]

Therefore,

\[ \|Az_1 - Aw_1\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) \leq \tilde{L} (\Psi(T) - \Psi(0))^{\alpha} \|z_1 - w_1\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) \Omega. \]

Using (5.13), above inequality reduces to

\[ \|Az_1 - Aw_1\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) \leq \varrho \|z_1 - w_1\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}), \quad w_0 \leq w_1 \leq z_1 \leq z_0. \]

Consider the sequences \( \{z_n\} \) and \( \{w_n\} \) defined in Theorem 5.1. Then \( z_n = A z_{n-1} \) and \( w_n = A w_{n-1} (n = 1, 2, 3, \cdots). \) By repeated application of the above inequality, we obtain

\[ \|z_n - w_n\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) = \|Az_{n-1} - Aw_{n-1}\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) \leq \varrho \|z_{n-1} - w_{n-1}\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) = \varrho \|Az_{n-2} - Aw_{n-2}\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) \leq \varrho \|Az_{n-3} - Aw_{n-3}\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) \leq \cdots \leq \varrho^n \|z_0 - w_0\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}). \] (5.14)

Using the condition on \( \tilde{L} \), we obtain \( 0 \leq \varrho < 1 \). This implies that \( \varrho^n \to 0 \) as \( n \to \infty \). Therefore, from the inequality (5.14), it follows that

\[ \lim_{n \to \infty} \|z_n - w_n\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) \to 0. \]

By applying Theorem 5.1, there exists minimal solution \( w^* \) and maximal solution \( z^* \) in \( [w_0, z_0] \) such that

\[ Aw^* = w^* \quad \text{and} \quad Az^* = z^*. \] (5.15)

Further,

\[ \lim_{n \to \infty} \|w_n - w^*\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) = 0 \quad \text{and} \quad \lim_{n \to \infty} \|z_n - z^*\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) = 0. \] (5.16)

Using the equations in (5.16) and the continuity of norm, we have

\[ 0 = \lim_{n \to \infty} \|z_n - w_n\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}) = \|z^* - w^*\|_{C_{1-\gamma, \psi}}(J, \mathbb{R}). \]

This gives

\[ z^* = w^* := y^* \quad \text{in} \quad C_{1-\gamma, \psi}(J, \mathbb{R}). \] (5.17)

From equations (5.15) and (5.17), we have

\[ Ay^* = y^*. \]

Thus, we have a unique \( y^* \in [w_0, z_0] \) such that

\[ \lim_{n \to \infty} \|w_n - y^*\|_{C_{1-\gamma, \psi}(J, \mathbb{R})} = 0 \quad \text{and} \quad \lim_{n \to \infty} \|z_n - y^*\|_{C_{1-\gamma, \psi}(J, \mathbb{R})} = 0. \] (5.18)
Since \( \{w_n\}_{n=1}^\infty \subseteq C_{1-\gamma;\Psi}(J, \mathbb{R}) \) is increasing bounded sequence and \( \{z_n\}_{n=1}^\infty \subseteq C_{1-\gamma;\Psi}(J, \mathbb{R}) \) is decreasing bounded sequence, from (5.18) it follows that
\[
  w_n \preceq y^* \preceq z_n \quad \text{in} \quad C_{1-\gamma;\Psi}(J, \mathbb{R}).
\] 
(5.19)

For each \( y_0 \in [w_0, z_0] \), consider the iterative sequence \( y_n = A y_{n-1} \). Then using the increasing nature of an operator \( A \) and the definitions of \( w_n \) and \( z_n \), we obtain
\[
  w_n \preceq y_n \preceq z_n \quad \text{in} \quad C_{1-\gamma;\Psi}(J, \mathbb{R}).
\] 
(5.20)

Using the inequalities (5.14), (5.19) and (5.20), for each \( t \in J \), we have
\[
  \left| (\Psi(t) - \Psi(0))^{1-\gamma} (y_n(t) - y^*(t)) \right| \leq \left| (\Psi(t) - \Psi(0))^{1-\gamma} (z_n(t) - w_n(t)) \right| \leq \|z_n - w_n\|_{C_{1-\gamma;\Psi}(J, \mathbb{R})} \leq \varrho^n \|z_0 - w_0\|_{C_{1-\gamma;\Psi}(J, \mathbb{R})}.
\]
Therefore,
\[
  \|y_n - y^*\|_{C_{1-\gamma;\Psi}(J, \mathbb{R})} \leq \varrho^n \|z_0 - w_0\|_{C_{1-\gamma;\Psi}(J, \mathbb{R})}.
\] 
(5.21)

From the above inequality it follows that
\[
  \lim_{n \to \infty} \|y_n - y^*\|_{C_{1-\gamma;\Psi}(J, \mathbb{R})} = 0.
\]

Observe that the inequality (5.21) gives the error bound with respect to the \( \| \cdot \|_{C_{1-\gamma;\Psi}(J, \mathbb{R})} \) between approximation \( y_n \) and the exact solution \( y^* \) of the nonlinear \( \Psi \)-Hilfer BVP \((1.4)-(1.5)\).

\section{Example}

We consider a specific case of the problem \((1.4)-(1.5)\) to illustrate the main results that we acquired.

\textbf{Example 6.1} Consider the following BVP for the nonlinear Caputo FDEs
\[
  C^\alpha D^\beta_{0+} y(t) = \frac{\sqrt{\pi}}{10} - \frac{\sqrt{t+1}}{25} + \frac{1}{25} \sin \left( \frac{\sqrt{t+1}}{5} \right) + \frac{1}{25} (5y(t) - \sin (y(t))), \quad t \in (0, 1], 
\]
\[
  y(0) = \frac{1}{2} y(1).
\] 
(6.1)
(6.2)

One can verify that \( y^*(t) = \frac{\sqrt{t+1}}{5} \), \( t \in [0, 1] \) is an exact solution of the problem \((6.1)-(6.2)\). Comparing the above problem with the nonlinear \( \Psi \)-Hilfer BVP \((1.4)-(1.5)\), we obtain
\[
  \alpha = \frac{1}{2}, \beta = 1, \gamma = \alpha + \beta(1 - \alpha) = 1, M = 0, r = \frac{1}{2}, T = 1 \quad \text{and} \quad \Psi(t) = t, t \in [0, 1].
\] 
(6.3)
In this case the weighted space $C_{1-\gamma; \Psi}([0, 1], \mathbb{R})$ reduces to the space of continuous functions $C([0, 1], \mathbb{R})$ endowed with the supremum norm.

Define the function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t, y) = \frac{\sqrt{\pi}}{10} - \frac{\sqrt{t} + 1}{25} + \frac{1}{25} \sin\left(\frac{\sqrt{t} + 1}{5}\right) + \frac{1}{25} (5y - \sin y). \quad (6.4)$$

(1) For any $k \in \mathbb{R}$ consider the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{f}(y) = k + \frac{1}{25} (5y - \sin y), \quad y \in \mathbb{R}.$$ Then, $\tilde{f}'(y) = \frac{1}{25} (5 - \cos y) \geq 0$, for all $y \in \mathbb{R}. \quad$ Therefore the function $\tilde{f}$ is increasing on $\mathbb{R}$ for any real $k$. This implies the function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ defined in (6.4) is increasing in $y \in \mathbb{R}$ for each $t \in [0, 1]$.

(2) Let any $y_1, y_2 \in \mathbb{R}$ with $y_1 \leq y_2$. Then

$$f(t, y_2) - f(t, y_1) = \frac{1}{25} \{5(y_2 - y_1) - (\sin y_2 - \sin y_1)\} \leq \frac{1}{25} \{5|y_2 - y_1| + |\sin y_2 - \sin y_1|\}. \quad (6.5)$$

Since $\sin y$ is continuous and differentiable on the interval $[y_1, y_2]$, applying mean value theorem, there exists $\tilde{y} \in [y_1, y_2]$ such that

$$\frac{\sin y_2 - \sin y_1}{y_2 - y_1} = \cos \tilde{y}.$$ This implies $|\sin y_2 - \sin y_1| \leq |y_2 - y_1|$. Therefore, the inequality (6.5) reduces to

$$f(t, y_2) - f(t, y_1) \leq \frac{6}{25} (y_2 - y_1), \quad \text{for any } y_1, y_2 \in \mathbb{R} \text{ with } y_1 \leq y_2. \quad (6.6)$$

Comparing the above inequality with (5.12), we have $\tilde{L} = \frac{6}{25}$.

(3) Next, we prove that $\tilde{L} \in [0, \Omega^{-1})$ where $\Omega$ is defined in (5.11). Using the values given in (6.3), the equation (5.11) reduces to

$$\Omega = B\left(\frac{1}{2} - 1 + 1, 1\right) \frac{1}{2} E_{\frac{1}{2}+1}(0) E_{\frac{1}{2}+\frac{1}{2}+1}(0) + B\left(\frac{1}{2}, 1\right) E_{\frac{1}{2}, \frac{1}{2}}(0).$$ Using Lemma 2.5, we obtain

$$E_{n_1, n_2}(0) = \frac{1}{\Gamma(n_2)}, \quad n_1, \ n_2 > 0. \quad (6.7)$$

Therefore

$$\Omega = \Gamma\left(\frac{1}{2}\right) \Gamma\left(1\right) \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \frac{1}{2} \Gamma\left(\frac{1}{2}\right) + \Gamma\left(\frac{1}{2}\right) \Gamma\left(1\right) \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{\Gamma\left(\frac{1}{2}\right)} = \frac{4}{\sqrt{\pi}}.$$
Note that, for $\tilde{L} = \frac{6}{25} \sqrt{\pi}$ and $\Omega = \frac{1}{\sqrt{\pi}}$, we have $\tilde{L} \in \left[0, \Omega^{-1}\right)$.

(4) Define $z_0(t) = \sqrt{t} + 1$, $t \in [0, 1]$ and $w_0(t) = \frac{-(\sqrt{t} + 1)}{6}$, $t \in [0, 1]$. Then,

$$h_1(t) := C D_{0+}^{\frac{1}{2}} z_0(t) = \frac{\sqrt{\pi}}{2},$$

$$h_2(t) := f(t, z_0(t)) = \frac{\sqrt{\pi}}{10} - \frac{\sqrt{t} + 1}{25} + \frac{1}{25} \sin \left(\frac{\sqrt{t} + 1}{5}\right) + \frac{1}{25} \left(5 \left(\sqrt{t} + 1\right) - \sin \left(\sqrt{t} + 1\right)\right),$$

$$h_3(t) := C D_{0+}^{\frac{1}{2}} w_0(t) = -\frac{\sqrt{\pi}}{12},$$

$$h_4(t) := f(t, w_0(t)) = \frac{\sqrt{\pi}}{10} - \frac{\sqrt{t} + 1}{25} + \frac{1}{25} \sin \left(\frac{\sqrt{t} + 1}{5}\right) + \frac{1}{25} \left(5 \left(-\frac{(\sqrt{t} + 1)}{6}\right) - \sin \left(-\frac{(\sqrt{t} + 1)}{6}\right)\right).$$

From Figure 1, it follows that

$$C D_{0+}^{\frac{1}{2}} z_0(t) \geq f(t, z_0(t)), \; t \in [0, 1].$$

Further, using the definition of $z_0$, we have $z_0(0) = \frac{1}{2} z_0(1)$ and hence $b_{z_0}(t) = 0$. This proves that $z_0(t) = \sqrt{t} + 1$, $t \in [0, 1]$ is an upper solution of the problem (6.1)-(6.2). Again, from Figure 2, it follows that

$$C D_{0+}^{\frac{1}{2}} w_0(t) \leq f(t, w_0(t)), \; t \in [0, 1].$$

Further, by definition of $w_0$, we have $w_0(0) = \frac{1}{2} w_0(1)$ and $a_{w_0}(t) = 0$. This implies $w_0(t) = -\frac{\sqrt{t} + 1}{6}$, $t \in [0, 1]$ is a lower solution of the problem (6.1)-(6.2). Since all the assumptions of Theorem 5.2 are satisfied, it guarantee the existence of a unique solution $y^*$ in $[w_0, z_0]$ of the problem (6.1)-(6.2). Indeed,

$$-\frac{\sqrt{t} + 1}{6} \leq \frac{\sqrt{t} + 1}{5} \leq \sqrt{t} + 1, \text{ for all } t \in [0, 1],$$

implies

$$w_0(t) \leq y^*(t) \leq z_0(t), \; t \in [0, 1].$$
This implies \( y^* \in [w_0, z_0] \), where \( y^* \), \( w_0 \) and \( z_0 \) defined above, respectively are the exact, lower and upper solutions of the problem (6.1)-(6.2).

(5) Using (6.3) and (6.7), the sequence \( \{y_n\}_{n=1}^{\infty} \) defined in the Theorem 5.2 reduces to

\[
y_n(t) = \frac{1}{\sqrt{\pi}} \int_0^1 (1-s)^{-\frac{1}{2}} f(s,y_{n-1}(s))ds + \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-\frac{1}{2}} f(s,y_{n-1}(s))ds
\]

where \( y_0 \in [w_0, z_0] \). By Theorem 5.2

\[
\|y_n - y^*\|_{C([0,1],\mathbb{R})} \leq \varrho^n \|z_0 - w_0\|_{C([0,1],\mathbb{R})}. \tag{6.8}
\]

Using (6.3) and the values of \( \Omega \) and \( \tilde{L} \) determined above, from (5.13), we obtain

\[
\rho = \Omega \tilde{L} = \frac{4}{\sqrt{\pi}} \frac{6}{25} = \frac{24}{25 \sqrt{\pi}}.
\]

Thus, from the inequality (6.8), we have

\[
\|y_n - y^*\|_{C([0,1],\mathbb{R})} \leq \left( \frac{24}{25 \sqrt{\pi}} \right)^n \sup_{t \in [0,1]} |z_0(t) - w_0(t)|
\]

\[
= \left( \frac{24}{25 \sqrt{\pi}} \right)^n \sup_{t \in [0,1]} \sqrt{t} + 1 - \left( -\sqrt{t} + \frac{1}{6} \right)
\]

\[
= \left( \frac{24}{25 \sqrt{\pi}} \right)^n \frac{7}{6} \sup_{t \in [0,1]} \sqrt{t} + 1.
\]

Therefore,

\[
\|y_n - y^*\|_{C([0,1],\mathbb{R})} \leq \frac{7}{3} \left( \frac{24}{25 \sqrt{\pi}} \right)^n. \tag{6.9}
\]

The inequality (6.9) gives the error between \( n^{th} \) approximation \( y_n \) and exact solution \( y^* \) of the problem (6.1)-(6.2). Since \( \frac{24}{25 \sqrt{\pi}} < 1 \), it follows that \( y_n \to y^* \) in \( C([0,1],\mathbb{R}) \) as \( n \to \infty \).

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