A Novel Variational Principle in Electrostatics and its Consequences

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Abstract. We propose a novel variational principle in electrostatics and show that one can derive mirror equation in the context of image problem starting from this principle. The corresponding Euler-Lagrange equation is seen to lead to Green’s differential equation (also known as Thomson’s equation).
1. Introduction

The method of images is a well-known powerful tool for solving boundary value problems in electromagnetism. In 1848, Sir W Thomson (also known as Lord Kelvin) introduced this method in a paper published in Cambridge and Dublin Mathematical Journal [1]. He showed that when a charge is placed outside a grounded conducting sphere, the electric potential outside equals the potential of the given charge plus that of another charge imagined inside the sphere with its surface removed. Thus, outside the sphere, the charge induced on the original spherical surface has the same effect as that of the image charge conceived inside the sphere. The image charge is similar to the virtual image, formed in a mirror that would seem to emit the rays of light (to an external observer) which are originally reflected by the mirror. Physically, The electric field lines do not enter inside the conducting sphere (the electric field inside a conductor is zero [15]) like rays of light that do not enter inside a spherical mirror.

After Lord Kelvin, J C Maxwell applied the method to solve various electrostatic problems [2] involving conducting spheres and planes. His work was followed by Jeans [3] who found that the method was applicable to problems with dielectric boundaries as well. Due course of time, this method was applied to solve many complicated problems of diverse categories. Whereas the simplest applications are found in standard university texts (see [4] or [5]), some of the rigorous results can be found in [7]-[11]. The method has also been applied to the problems in magnetostatics [12], [13] and fluid dynamics [14] also.

However, the earlier authors in this field do not seem to have contributed on the analogy between the image charge and the image of a body (formed in a mirror). Probably, they did not take the analogy seriously. Rather, there are articles where it is indicated that no such analogy exists. For instance, the authors of [11] mention ‘in the problem of a point charge in the presence of a grounded perfectly conducting sphere, also a well-known example of an electrostatic problem, the latter does not work as a mirror’. This is not unlikely; because, only if a charge is placed near an infinite grounded conducting plane, the image charge and its position mimic the mirror image of a body formed in a plane mirror. In all other cases, the analogy is not at all apparent. For a charge \( q \) placed at a distance \( y \) from the center of a sphere of radius \( a \), the image charge \( -\frac{aq}{y} \) is conceived at a distance \( \frac{a^2}{y} \) from the center. This result does not seem to be related to the virtual image formation in a spherical mirror where the position and magnification of the image is given by a mirror equation and magnification relation [16]. Hence, it is not surprising that Maxwell, while defining the image charge, says ‘They do not correspond to them in actual position, or merely approximate character of optical foci’ [2].

Recently, the current author has observed [18] that actually an analogy exists between the electrostatic image and the virtual image formed in a mirror. For the above problem, it is seen that all the required information about the image charge can be deduced from a mirror equation and a magnification formula. Instead of calculating
the distances from the center of the sphere, if the distances are calculated from the point of intersection of the line joining the two charges and the sphere, the object charge distance becomes \( u = (y - a) \) and the image charge distance becomes \( v = (a - \frac{a^2}{y}) \). Then, with the usual sign convention that for reflection in a spherical mirror, \( u \) is positive and both \( v \) and focal length \( f \) are negative, the image charge distance can be deduced from:

\[
\frac{1}{u} + \frac{1}{v} = \frac{1}{f} \tag{1.1}
\]

with focal length \( f = -a \). Again, the magnitude of the image charge \( q' \) can be obtained from the following magnification formula:

\[
\frac{q'}{q} = \frac{v}{u} = -\frac{a}{y} \tag{1.2}
\]

These well-known relations describing the reflection of light from a spherical mirror are taught in all the high school or undergraduate geometrical optics courses (see [16],[17]). Whereas this observation justifies the intuitive basis and the naming of image problems, its theoretical basis is not very clear. It is natural to ask why the equations (1.1) and (1.2) are valid in the image problems in electrostatics. Speculation on the reflection of field lines by conducting surface seems unaccountable as we do not have law of reflection in electrostatics as we have in optics. We find ourselves in a paradoxical situation where we see that our results are correct, but we cannot explain them. The present article is a sequel of [18] and here we shall try to give a plausible answer to this question.

The image formation in geometrical optics by refraction through (or reflection in) lenses (or mirrors) is adequately described by ‘Fermat’s principle’ ([19], [20]). One can derive the mirror equation (1.1) for reflection in a spherical mirror, by minimizing the optical path length. With this in mind, we ask if an equivalent form of Fermat’s Principle can be fitted in the existing framework of electrostatics. We propose the principle to be of the following form (resembling closely to Fermat’s principle):

\[
\delta \int_{a}^{b} E \cdot dr = 0 \tag{1.3}
\]

between two fixed points \( a \) and \( b \); here \( E \) denotes the electrostatic field. In this paper, we shall show that the mirror equation (1.1) in the grounded conducting sphere image problem can be reached starting from \( \delta \int_{a}^{b} E \cdot dr = 0 \) in the same manner as the usual mirror equation in optics is derived from Fermat’s Principle.

The proposed principle is relevant also in the context of Green’s differential equation [22] (known as Thomson’s equation as well). This relation between the normal derivative of the electric field across a conducting surface and the local mean surface curvature is given as:

\[
\frac{dE}{dn} = -E \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \tag{1.4}
\]

-where \( R_1 \) and \( R_2 \) are the principal radii of curvature of the surface at a given location. There are plenty of proofs of the relation in the literature (see [23]-[26]). We shall see
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that the Euler-Lagrange equation that follow from \( \delta \int_{a}^{b} E \cdot dr = 0 \) can be used with Gauss’s theorem to prove (1.4).

2. Meaning of \( \delta \int_{a}^{b} E \cdot dr = 0 \)

As the electrostatic potential difference between two fixed points along any contour is the same, the proposition \( \delta \Phi = \delta \int E \cdot dr = 0 \) appears to be redundant. So, what is the meaning of this statement? Although \( \Phi(b) - \Phi(a) \) along any path is the same, not all of these paths are allowed by \( \nabla \times E = 0 \) and Laplace’s equation. The proposed principle is supposed to pick up those paths that satisfy all these constraints. To demonstrate this claim, we derive the Euler-Lagrange equation in the next section.

3. Derivation of Euler-Lagrange Equation from \( \delta \int_{1}^{2} E \cdot dr = 0 \)

3.1. Derivation

We proposed the novel variational principle in electrostatics of the following form:

\[
\delta \int_{1}^{2} E \cdot dr = 0 \quad (3.1)
\]

We may expand the left hand side of (3.1):

\[
\delta \int_{1}^{2} E \cdot dr = \delta \int_{1}^{2} [E \, dr] = \int [\delta E \, dr + E \delta (dr)] = \int \left[ \left( \frac{\partial E}{\partial r} \right) \cdot \delta r + E \delta (dr) \right]
\]

The second term contains \( \delta (dr) \) which can be found from Figure 1:

Figure 1. Variation of differential path length
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\[ \delta(dr) = |dr + d(\delta r)| - |dr| \]
\[ = \sqrt{dr \cdot dr + 2dr \cdot d(\delta r) + d(\delta r) \cdot d(\delta r)} - \sqrt{dr \cdot dr} \]
\[ \sim \sqrt{dr \cdot dr + 2dr \cdot d(\delta r)} - \sqrt{dr \cdot dr} \]
\[ = dr \left[ \sqrt{1 + 2 \frac{dr \cdot d(\delta r)}{dr^2}} \right] - dr \]

If we take the first order term in the binomial expansion of the above square root, we get:

\[ \delta(dr) = \frac{dr}{dr} \cdot d(\delta r) \quad (3.2) \]

Thus, the expression for \( \delta \int_1^2 E \cdot dr \) becomes:

\[ \delta \int_1^2 E \cdot dr = \int_1^2 \left[ \left( \frac{\partial E}{\partial r} \cdot \delta r \right) dr + E \delta(dr) \right] \]
\[ = \int_1^2 \left[ \left( \frac{\partial E}{\partial r} \cdot \delta r \right) dr + \left( \frac{dr}{dr} \cdot d(\delta r) \right) \right] \]
\[ = \int_1^2 \left[ \left( \frac{\partial E}{\partial r} \cdot \delta r \right) dr - d \left( E \frac{dr}{dr} \right) \cdot \delta r \right] + \left[ \frac{dr}{dr} \delta r \right] \]
- using integration by parts. The last term vanishes as \( \delta r = 0 \) at the end points. If we insist that \( \delta \int_1^2 E \cdot dr = 0 \), then the above equations imply:

\[ \delta \int_1^2 E \cdot dr = \int_1^2 \left[ \left( \frac{\partial E}{\partial r} \right) dr - d \left( E \frac{dr}{dr} \right) \right] \cdot \delta r = 0 \]

For arbitrary \( \delta r \), the above leads to:

\[ \frac{\partial E}{\partial r} = \frac{d}{dr} \left( E \frac{dr}{ds} \right) = \frac{d}{ds} \left( E \frac{dr}{ds} \right) \quad (3.3) \]

### 3.2. A Digression to Validate \( \delta \int_1^2 E \cdot dr = 0 \)

The principle may be validated further by showing that equation (3.3) is compatible with \( \nabla \times E = 0 \). Consider the x component:

\[ \frac{\partial E}{\partial x} = \frac{dE_x}{ds} \quad (3.4) \]

and similar equations for y and z components. If we expand the L.H.S. of (3.4), we get:

\[ \frac{\partial}{\partial x} \sqrt{E_x^2 + E_y^2 + E_z^2} = \frac{E_x \partial E_x}{E} + \frac{E_y \partial E_y}{E} + \frac{E_z \partial E_z}{E} \]

Similarly, expanding the total derivative in the R.H.S. we get:

\[ \frac{dE_x}{ds} = \frac{\partial E_x}{\partial x} \frac{dx}{ds} + \frac{\partial E_x}{\partial y} \frac{dy}{ds} + \frac{\partial E_x}{\partial z} \frac{dz}{ds} \]
\[ = \frac{E_x \partial E_x}{E} + \frac{E_y \partial E_x}{E} + \frac{E_z \partial E_x}{E} \]
where we have used $\frac{dx}{ds} = \frac{E_y}{E}$ etc. Thus, for (3.4) to hold, we must have

$$\frac{E_y \partial E_y}{E \partial x} + \frac{E_z \partial E_z}{E \partial x} = \frac{E_y \partial E_x}{E \partial y} + \frac{E_z \partial E_x}{E \partial z}$$

Or, we must have (after rearranging the terms),

$$\frac{E_y}{E} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = \frac{E_z}{E} \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right)$$

-which is vacuously satisfied as the factors in the parentheses in both sides of the equation are zero, from $\nabla \times E = 0$. Similar results follow for $y$ and $z$ counterparts of (3.4) also. Thus, $\delta \int_1^2 E \cdot dr = 0$ is validated. We interpret the result as: although any path between two fixed points gives the same value of the integral $\int E \, dr$, the variational principle proposed above allows only those paths (among all) who satisfy $\nabla \times E = 0$. Only this $E$ derives from the unique solution to Poisson’s or Laplace’s equation.

4. Mirror Equation from Variational Principle

Before attacking the problem in electrostatics, let us review the corresponding problem in optics. The “Fermat’s principle” is the variational principle used in optics. It is given as $[20] \delta \int_1^2 k \cdot dl = 0$ (here $k$ is the wave vector).

4.1. In Optics

In Figure 2, a ray of light ($k_o$) from source $S$ gets reflected from a point $R$ on the mirror surface. To an observer, light ray ($k_i$) seems to come from the image $I$. In this system, phase $\psi$ is given as:

$$\psi = \int_S^R k_o \cdot dl_o + \int_R^I k_i \cdot dl_i$$

$$= \int_S^R k_o \, dl_o - \int_R^I k_i \, dl_i$$

$$= \text{constant} \times \left[ \int_S^R dl_o - \int_R^I dl_i \right]$$
The subscripts \( o \) and \( i \) denote the outside and inside of the spherical mirror, respectively. The negative sign comes before \( k_i \) because its actual direction is opposite to that of \( dl_i \). Thus, \( \text{optical path length} \ L_{op} \) is given as (in the small angle limit):

\[
L_{op} = SR - RI = u - v + \frac{1}{2} a^2 \left( \frac{1}{u} - \frac{1}{v} + \frac{2}{a} \right) \theta^2
\]  

(4.1)

The symbols are explained in Figure 2. We know that the variation of the optical path with respect to the angle \( \theta \) gives the desired mirror formula (1.1) with focal length \( |f| = \frac{a}{2} \). The minimum \( L_{op} \) is guaranteed by the condition \( \frac{\partial L_{op}}{\partial \theta} \big|_{\theta \to 0} = 0 \).

\subsection*{4.2. In Electrostatics}

It has been found [18] that the location of the image charge in the grounded conducting sphere image problem can be extracted from a mirror equation (1.1). Whereas the distance of the image charge from the center of the sphere can also be found from standard texts ([4]-[5]), we shall make use of the principle \( \delta \int_{a}^{b} \mathbf{E} \cdot dl = 0 \) to see if we can derived (1.1). Instead of a point charge, we assume that the real charge is distributed uniformly over a very small sphere \( S \) which is finite nevertheless. We call its center as 1. If we do not assume this, \( \int_{1}^{R} E_o \cdot dl_o \) will diverge from the lower limit. Since inversion of a sphere in a bigger sphere is another sphere [21], the image charge \( I \) is also spherical. We denote its center by 2. Evidently, \( \int_{2}^{R} E_i \cdot dl_i \) does not diverge from the upper limit (potential at a point inside a continuous charge distribution is finite).

Let us assume that we have no prior information about the position or the value of image charge. We define the \textit{electric path potential} (refer to Discussions) between the real charge and the image charge as (Figure 3):

\[
\Phi^* = - \int_{1}^{R} E_o \cdot dl_o - \int_{2}^{R} E_i \cdot dl_i = - \int_{1}^{R} E_o \cdot dl_o + \int_{2}^{R} E_i \cdot dl_i
\]  

(4.2)

The symbols are explained in figure 3. Notice that the \( * \) sign is applied to make it explicit that \( \Phi^* \) between 1 and 2 is different from the potential \( \Phi(r) \) which is the solution of Laplace’s equation outside the sphere. Thus,

\[
\Phi^* = [\Phi_S(R) - \Phi_S(1)] - [\Phi_I(R) - \Phi_I(2)]
\]  

(4.3)
or, more explicitly,

$$\Phi^* = \frac{q}{\sqrt{y^2 + a^2 - 2y\cos\theta}} - \frac{q'}{\sqrt{a^2 + y^2 - 2ay\cos\theta}} - C$$  \hspace{1cm} (4.4)$$

-where $C = (\Phi_S(1) - \Phi_I(2))$ is finite and independent of $\theta$. At this point we do not specify the potential on the sphere. This boundary condition will be invoked later.

Writing $-2ya \cos\theta = -2ya + 2ya(1 - \cos\theta)$, we get:

$$\Phi^* = \frac{q}{\sqrt{y^2 + a^2 - 2ya + 2ya(1 - \cos\theta)}} - \frac{q'}{\sqrt{a^2 + y^2 - 2ay' + 2ay'(1 - \cos\theta)}} - C$$  \hspace{1cm} (4.5)$$

Using real charge distance $u = (y - a)$ and image charge distance $v = (a - y')$, we find

$$\Phi^* = \frac{q}{\sqrt{u^2 + 2a(a + u)(1 - \cos\theta)}} - \frac{q'}{\sqrt{v^2 + 2a(a - v)(1 - \cos\theta)}} - C$$  \hspace{1cm} (4.6)$$

Now, we set the derivative of $\Phi^*$ with respect to the angle $\theta$ to zero:

$$\frac{\partial \Phi^*}{\partial \theta} = -a \sin\theta \left[ \frac{q(a + u)}{(u^2 + 2a(a + u)(1 - \cos\theta))^{3/2}} - \frac{q'(a - v)}{(v^2 + 2a(a - v)(1 - \cos\theta))^{3/2}} \right] = 0$$  \hspace{1cm} (4.7)$$

We are following the notion that electric path potential $\Phi^*$ must be stationary along all the electric field paths. This must be the case because field lines from all possible angles are responsible for image formation. Thus, $\Phi^*$ is independent of the parameter $\theta$. No extreme condition like $\frac{\partial \Phi^*}{\partial \theta} |_{\theta = 0}$ is needed. Let us now use the boundary condition (1.2) that at the pole $P$, the potential $\Phi(P) = 0$:

$$\frac{q}{u} = -\frac{q'}{v}$$  \hspace{1cm} (4.8)$$

which can be used to assert that $q = uk$ and $q' = -vk$ where $k$ is a non-zero constant. Using (4.8), (4.7) reduces to

$$-ka \sin\theta \left[ \frac{u(a + u)}{(u^2 + 2a(a + u)(1 - \cos\theta))^{3/2}} + \frac{v(a - v)}{(v^2 + 2a(a - v)(1 - \cos\theta))^{3/2}} \right] = 0$$  \hspace{1cm} (4.9)$$

In general $\theta \neq 0$; therefore, we have,

$$\left[ \frac{u(a + u)}{(u^2 + 2a(a + u)(1 - \cos\theta))^{3/2}} + \frac{v(a - v)}{(v^2 + 2a(a - v)(1 - \cos\theta))^{3/2}} \right] = 0$$  \hspace{1cm} (4.10)$$

From (4.10), we get the following:

$$u^2(a + u)^2[v^2 + 2a(a - v)(1 - \cos\theta)]^3 - v^2(a - v)^2[u^2 + 2a(a + u)(1 - \cos\theta)]^3 = 0$$  \hspace{1cm} (4.11)$$

From the problem, it is clear that (4.11) must hold for all values of $\theta$. It is an identity rather than an equation. So, each coefficient of $(1 - \cos\theta)$ or its higher powers must be equated to zero. The term independent of $\theta$ yields:

$$u^2(a + u)^2v^6 = v^2(a - v)^2u^6$$  \hspace{1cm} (4.12)$$

Canceling common non-zero factors and taking positive square root, we get

$$u^2(a - v) = v^2(a + u)$$  \hspace{1cm} (4.13)$$
This can be written as
\[ a(v + u)(v - u) = -uv(u + v) \]  (4.14)

Canceling the common factor \((u + v)\), we get
\[ a(v - u) = -uv \]  (4.15)

Or, in more familiar form,
\[ \frac{1}{u} - \frac{1}{v} = -\frac{1}{a} \]  (4.16)

The reader is strongly encouraged to check that the same or trivial result follows by equating higher powers of \((1 - \cos \theta)\). Now, we know that real charge distance is \(u = y - a\). Then, (4.16) gives the image charge distance \(v = \frac{a}{y}(y - a)\) behind the mirror. The boundary condition (4.8) gives the magnitude of image charge \(q' = -\frac{a}{y}\). After we get information about both distance and value of the image charge, we can construct the Green’s function for the problem and the problem is solved.

Taking the sign convention, real charge distance \(u\) to be positive when image charge distance \(v\) is negative and the focal length \(a\) is also negative, (4.16) can be put in the familiar form:
\[ \frac{1}{u} + \frac{1}{v} = \frac{1}{f} \]  (4.17)

5. Proof of Green’s Differential Equation

In vector notation, the “Euler-Lagrange” equation of \(\delta \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} = 0\) is given by:
\[ \nabla |\mathbf{E}| = \frac{d}{ds}(|\mathbf{E}| \frac{d\mathbf{r}}{ds}) = \frac{d\mathbf{E}}{ds} \]

Let us write the equation for an electric field near a conducting surface. The electric field is perpendicular to the conducting surface [15]. That is, \(\mathbf{E} = \mathbf{E}^\parallel + \mathbf{E}^\perp = \mathbf{E}^\perp\) where \(\mathbf{E}^\perp\) and \(\mathbf{E}^\parallel\) denote normal and tangential components of the electric field. Denoting infinitesimal displacement \(ds\) along direction of \(\mathbf{E}^\perp\) by \(dn\) and the unit vector normal to the conducting surface by \(\mathbf{n}\), we have:
\[ \nabla \cdot \mathbf{E} = \nabla \cdot (|\mathbf{E}|\mathbf{n}) \\
= \nabla |\mathbf{E}| \cdot \mathbf{n} + |\mathbf{E}| \nabla \cdot \mathbf{n} \\
= \frac{d\mathbf{E}}{dn} \cdot \mathbf{n} + 2\kappa |\mathbf{E}| \]

-where we have used mean curvature, \(\kappa = \frac{1}{2} \nabla \cdot \mathbf{n}\) [27]. If we work just outside the conductor, Laplace’s equation holds and \(\nabla \cdot \mathbf{E} = 0\). Thus, in this region,
\[ \frac{d|\mathbf{E}|}{dn} = \frac{d\mathbf{E}}{dn} \cdot \mathbf{n} \]
\[ = -2\kappa |\mathbf{E}| \]
\[ = -|\mathbf{E}| \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]
Thus, with little effort we have proved Thomson’s equation:

\[
\frac{dE}{dn} = -2\kappa E = -E \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]

(5.1)

In this proof, we see the value of the vector relation \( \nabla E \cdot dn = dE \), which we identified with the “Euler-Lagrange” equation of \( \delta \int_a^b E \cdot dl = 0 \).

6. Discussions

In this article, we saw that a novel variational principle can be conceived in electrostatics. We used this principle to justify the observations made in [18] and to prove Green’s differential equation [22]. The ‘Euler-Lagrange’ equation obtained from this principle was identified to be a well-known vector relation.

In the proof of the mirror equation, the ‘electric path potential’ was constructed in a way so that the contour of the integral \( \int E \cdot dl \) stays always superimposed with the local \( E \) field direction. Exactly the same is done in optics (subsection 4.1). The contributions to the phase from wave vectors \( k_o \) and \( k_i \) are treated separately. So, electrostatic path potential is just the electrostatic twin of the optical path length. In this proof, we did not use any ‘law of reflection’ which, in optics, is actually a manifestation of the boundary conditions of the wave vector \( k \) at the interface between two media. Our proof was facilitated by the use of (4.8), which is the boundary condition in the context of this problem. Thus, the boundary conditions play a very crucial role in the formation of all virtual images. An ideal point charge could not be taken for this work, to avoid divergence. That is not a big problem, as no charge is ideal in reality. Also, the concise proof of Green’s differential equation shows the relevance of the ‘Euler-Lagrange’ equation.

Image method is also applicable to magnetostatic boundary value problems (for example, one can solve the problem of a magnetic dipole placed in front of an infinite superconducting plane by image method). Other problems may be found in various literatures (Hague [12] or Q.G.Lin [13]). They can be seen in the light of the present article in the following way. In the current free region, magnetic field may be expressed as the gradient of a scalar potential \( U(r) \), and both \( \nabla \times B = 0 \) and Laplace’s equation \( \nabla^2 U(r) = 0 \) apply. Thus, we see that a calculation parallel to the one described in section 3 (with \( U(r) \leftrightarrow \Phi(r) \)) is possible if \( ds \) is taken parallel to \( B \) field lines. Then, we can speak of \( \delta \int_a^b B \cdot dl = 0 \) in magnetostatics also. Physically, magnetic field cannot penetrate inside a superconductor and the interface again behaves like a mirror that effectively reflects magnetic field lines. The argument is strengthened by the observation that Green’s differential equation (section 5) is valid for magnetostatic field as well [28].

Clearly, the work presented in this paper reveals an unfamiliar face of the image problems. We feel that the proposed principle may be explored even further to understand the physical world. Hopefully, the interdisciplinary nature of the article will attract the general audience.
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