Closed-loop Quantum Parameter Estimation: Spins in a Magnetic Field

JM Geremia, John K. Stockton, and Hideo Mabuchi
Norman Bridge Laboratory of Physics, California Institute of Technology, Pasadena CA 91125

We present an experimental demonstration of closed-loop quantum parameter estimation in which real-time feedback is used to achieve robustness to modeling uncertainty. By performing broadband estimation of a magnetic field acting on hyperfine spins in a cold atom ensemble, we show that accuracy is not compromised by fluctuations in total atom number even though the measured signal in our canonical configuration depends only on the product of the field and atom number. This methodology could be essential for efforts to utilize conditional squeezing in spin-resonance measurements.

PACS numbers: 06.20.-i, 32.80.Pj, 32.80.Qk

Optimal design of experimental procedures and data analysis strategies can often be accomplished using techniques from parameter estimation theory. Such an approach can be essential for obtaining acceptable performance in scenarios where modeling is subject to some degree of uncertainty, and has seen widespread use in fields that lie near the interface of physics and information science. Prominent current examples of such fields include metrology, optical communication, and computation with novel substrates. As micro- and nano-scale systems with manifestly non-classical behavior have gained importance in these areas, researchers have devoted increasing attention to the extension of parameter estimation methodologies to problems involving quantum mechanical inference and dynamics. Our aim in this article is to establish that real-time feedback plays a central role in the quantum regime— as it does in classical scenarios— enabling robust parameter estimation in the presence of significant modeling uncertainty.

In quantum parameter estimation the central objective is to extract information about a static or time-dependent parameter in a Hamiltonian, via direct or indirect measurements performed on a system that evolves according to this Hamiltonian. An elementary example of such a process is estimating the amplitude of a magnetic field, \( b(t) \), by observing short-time Larmor precession of a spin ensemble \( \hat{F} \). (here we will consider an ensemble consisting of hyperfine spins in a cloud of laser-cooled atoms). The field-spin interaction is described by a magnetic dipole Hamiltonian,

\[
\hat{H}[t; b(t)] = \hbar \gamma b(t) \cdot \hat{F}
\]

where \( \gamma \) is the gyromagnetic ratio and \( \hat{F} \) is the total angular momentum operator for the ensemble.

A canonical estimation procedure is depicted in Fig. 1(A). An atomic spin ensemble is prepared such that its net magnetization, or Bloch vector, \( \langle \hat{F} \rangle \), achieves a coherent spin state along the \( x \)-axis. In the presence of an external magnetic field, \( b(t) \), the bulk magnetization precesses around \( b \) with instantaneous frequency, \( \omega_L(t) = \gamma |b(t)| \). Generally, it is arranged such that \( b(t) = b(t)\hat{y} \) lies along the \( y \)-axis so that the spin state can be resolved from a continuous measurement of \( \langle \hat{F}_z \rangle \). An estimate, \( \hat{b} \), of the external magnetic field is then obtained (by regression or filtering) from the small-angle relation

\[
\hat{b} = \frac{\langle \hat{F}_z \rangle}{\gamma |\langle \hat{F} \rangle|} \hat{y}, \quad t \ll \omega_L^{-1}.
\]

This procedure can achieve Heisenberg limited sensitivity by exploiting conditional spin-squeezing. The problem with this approach is that it requires accurate knowledge of the net magnetization, \( |\langle \hat{F} \rangle| \), a quantity that unfortunately varies due to decoherence and shot-to-shot fluctuations in the atom number. Uncertainty in \( |\langle \hat{F} \rangle| \) directly translates into uncertainty in \( \hat{b} \).

Fig. 1(B) depicts an alternative estimation procedure that is robust to fluctuations in \( |\langle \hat{F} \rangle| \). The spin ensemble and \( \langle \hat{F}_z \rangle \) measurement are situated within a feedback control loop that attempts to stabilize \( \langle \hat{F}_x \rangle \) to a reference value, \( r(t) = 0 \). In the presence of a time-varying magnetic field signal, \( b(t) = b(t)\hat{y} \), the controller imposes a compensating field, \( \hat{b}_c(t) \approx -b(t)\hat{y} \), to try to suppress the atomic Larmor precession. The closed-loop estimate is then given by \( \hat{b}(t) = -\hat{b}_c \), and its accuracy is determined by the controller’s ability to respond promptly and
accurately to changes in \( b(t) \). In this work we demonstrate that standard design techniques enable the implementation of feedback controllers with excellent tracking and high robustness to fluctuations in \( |{\bf F}(t)| \). This illustrates new utility for real-time feedback in cold atom physics, as previous investigations have focused on active control of atomic motion \([11, 12]\).

Fig. 2 provides a schematic of our experimental apparatus implemented according to the design in Fig. 1B. It consists of a cold atom ensemble, a Faraday polarimeter for continuously probing the atomic Bloch vector, and a high-speed Helmholtz coil along the \( y \)-axis for applying feedback. The spin system is provided by the \( 6^3S_{1/2}(F=4) \) ground state hyperfine manifold in Cs, which contains \((2F+1) = 9\) energetically degenerate Zeeman sublevels in zero field. Therefore, the net angular momentum of the polarized spin state is given by, \( Nh \sqrt{F(F+1)} \approx 4Nh \), for an ensemble of \( N \) atoms.

Cold atom samples were obtained by loading \( N \approx 10^9 \) neutral \(^{133}\)Cs atoms into a magneto-optical trap (MOT) from a \( \sim 1 \times 10^{-8} \) Torr background vapor, using optical trapping beams (30 mW each, 2.5 cm diameter) derived from an injection-locked diode laser. The atoms were cooled to \( 1 \) \( \mu \)K via a 5 ms \( \sigma_+ / \sigma_- \) polarization gradient cooling phase, and a coherent spin state was produced by optical pumping with \( \sigma_+ \) polarized light (100 \( \mu \)W with a 6.2 mm Gaussian waist) tuned to the \( 6^3S_{1/2}(F=4) \rightarrow 6^3P_{3/2}(F'=5) \) hyperfine transition. A 45 mW, 2.5 cm diameter re-pumping beam was used.

Continuous weak measurement of \( \hat{F}_y \) was implemented using a free-running diode laser, blue-detuned from the \((F=4) \rightarrow (F'=5)\) transition by \( \Delta = 2 \) GHz. The 65 \( \mu \)W probe beam was linearly polarized by a high extinction \((> 10^8)\) Glan-Thompson prism polarizer and mode-matched to the Gaussian waist of the atomic spatial distribution. Relaxation of the coherent spin state due to the probe light was measured to be \( T_2 = 11.2 \) ms.

Faraday rotation of the probe light was detected using a balanced polarimeter (1 MHz detector bandwidth) constructed from a high extinction Glan-Thompson polarizing beam splitter. This configuration produces a photocurrent, \( y(t) \), proportional to the \( z \)-component of the spin angular momentum \( \hat{F}_z \).

\[
y(t) = 2\sqrt{M(\hat{F}_z(t)) + \zeta(t)}
\]

where \( M \) is the measurement strength and \( \zeta(t) \) reflects measurement noise. Background magnetic fields were nulled to \( < 100 \) \( \mu \)G with large \( (d=1 \text{ m}) \) external three-axis Helmholtz coils by Larmor precession measurements.

The feedback system in Fig. 2 utilizes the photocurrent, \( y(t) \), to control the strength of a Hamiltonian,

\[
\hat{H}_c(t) = \hbar \gamma b_c[y(t)] \hat{F}_y
\]

that rotates the Bloch vector around the \( y \)-axis. It is the job of the controller to determine the appropriate feedback strength, \( b_c(t) = \beta u(t) \), based on the observation \( y(t) \). For a linear controller,

\[
u(t) = \int_0^t C(t - \tau)y(\tau) \, d\tau,
\]

where \( C(t) \) must be designed to satisfy tracking and robustness objectives. The controller output, \( u(t) \), is used to program a current supply that drives a \( y \)-axis Helmholtz coil surrounding the atomic sample. This requires accurate calibration (obtained in our case from Larmor frequency measurements) of the gain, \( \beta \), from current supply programming voltage, \( u(t) \), to the feedback field, \( b_c(t) \).

In control theory, it is customary to design \( C(t) \) in the frequency rather than time domain by taking Laplace transforms of the functions, \( C(t) \rightarrow C(s) \), \( y(t) \rightarrow y(s) \), etc., where \( s = j\omega \) and \( j = \sqrt{-1} \). This results in the closed-loop frequency response of the control system,

\[
T(s) = \frac{y(s)}{r(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)}
\]

where \( P(s) = y(s)/u(s) \) is the transfer function from the feedback coil programming voltage to the photocurrent. The tracking objective is to adjust the gain and phase of \( C(s) \) such that \( T(s) \approx 1 \) over as large a frequency range as possible. Standard results from robust loop-shaping theory \([13, 14]\) lead to a controller design,

\[
C(s) = \frac{Q(s)}{1 + P(s)Q(s)}, \quad Q = \frac{W(s)}{P_{mp}(s)}
\]

where \( W(s) \) is a stable, strictly proper \( [W(s \rightarrow \infty) = 0] \) function that adjusts the bandwidth of \( C(s) \) and \( P_{mp}(s) \) is the minimum phase contribution to \( P(s) \).
Ideally, the Helmholtz coil current supply would introduce no additional frequency dependence to \( P(s) \), so that \( P(s) \) would be determined solely by atomic dynamics. In the limit, \( \langle \tilde{F}_z \rangle \ll |\tilde{F}| \), which is maintained in closed-loop, this would be [using \( \langle \tilde{F}_z(t) \rangle = \exp(-t/T_2) \sin(\omega_L t) \)],

\[
P(s) \approx 2\gamma \sqrt{|\tilde{M}|} \frac{1}{s}
\]

and proportional feedback [constant \( C(s) \)] would provide good tracking. However, it is not always possible to construct a current supply with sufficient bandwidth. Finite available supply power places an upper bound on the closed-loop bandwidth for driving an inductive load, and a more intelligent \( C(s) \) design is necessary. Our Helmholtz coil supplies were constructed using high-power (500 W) operational amplifiers and displayed 73 degrees of phase delay at 100 kHz. With proportional control, this would be insufficient phase margin.

Fig. 3 shows the measured transfer function, \( P(s) \), for the combined atomic ensemble, Faraday polarimeter and feedback coil system. It was generated by utilizing a network analyzer to perform a swept sine analysis of \( P(s) \). For each data point, the analyzer was triggered following preparation of a coherent spin state according to the cooling and optical pumping procedure described above. Although \( P(s) \) is approximately an integrator, it displays substantially larger phase delay at higher frequency; a fit of the transfer function yields the model,

\[
P(s) = \frac{1.6 \times 10^4 (8.0 \times 10^5 - s)}{s^2 + 4.1 \times 10^8 s + 4.0 \times 10^9}.
\]

By factoring this transfer function into its minimum-phase and all-pass components, \( P(s) = P_{\text{mp}}(s)P_{\text{ap}}(s) \), a stabilizing feedback controller was obtained using Eq. 4. \( W(s) \) was chosen to be a single-pole (Butterworth) low-pass filter with \( f_c = 1 \) MHz which yielded the transfer function, \( C(s) \), in Fig. 3(A). The feedback controller was implemented using high bandwidth analog electronics and resulted in the closed-loop transfer function, \( T(s) \), depicted in Fig. 3(B). The significance of \( T(s) \) is that \( ||1 - T(s)|| \) provides a measure of the tracking error, and thus the error in the parameter estimation.

Fig. 4 shows a demonstration of the closed-loop parameter estimation procedure for both stationary and time-dependent magnetic fields generated by an auxiliary \( y \)-axis Helmholtz coil (refer to Fig. 2). Plot (A) indicates the effect of real-time feedback on the polarimetry photocurrent, \( y(t) \). At time, \( t = 0.5 \) ms, a field of \( B_y = 50 \) mG was turned on, with feedback disabled, resulting in atomic Larmor precession. However, when the feedback loop was closed at \( t = 1 \) ms, it acted to null out the applied field and lock the Bloch vector into the \( xy \)-plane. This simultaneously provided an estimation of the applied field, \( \tilde{\mathbf{b}} = -\mathbf{b}_{x} = -\beta u(t)\dot{y} \). Plot (B) demonstrates our capability to track time-varying fields within the feedback bandwidth of 100 kHz; here, feedback was enabled the entire time during a 5 kHz bandwidth applied field.

To demonstrate robustness, the parameter estimation error was measured as a function of atom number, \( N \), by varying the MOT loading time, \( t_L \), according to a calibration of \( N \) versus \( t_L \) (accurate to 40%) obtained from resonant fluorescence imaging. The estimation error, \( \Delta \mathbf{b} \), was computed from ensemble averages, \( E[||\mathbf{b}(t) - \tilde{\mathbf{b}}(t)||] \), over 100 replicate measurements for each sampled value of \( N \) according to the semi-norm,

\[
||\mathbf{b}(t)||_2 = \frac{1}{T} \int_0^T |\mathbf{b}(t)|^2 dt.
\]

As seen in Fig. 4, \( \Delta \mathbf{b} \) was essentially unaffected by fluctuations in \( N \) spanning three orders of magnitude. The residual estimation uncertainty of approximately 713 \( \mu \text{G} \)
FIG. 4: Single-shot, closed-loop parameter estimation of a stationary (A) and time-varying (B) applied magnetic field $b(t)$. In (A) feedback is enabled at $t = 1$ ms (dotted line). The labels identify several artifacts in the data: (1) cross-talk between the driving and feedback coils, (2) background field fluctuations due to power supply noise at 51.3 kHz, and (3) noise in the driving field that is also revealed by the estimator (4).

is the result of tracking error due to finite controller gain which provides motivation for higher bandwidth feedback systems. A closed-loop estimator with a 1 MHz unity-gain point is expected to provide $\sim 10$ nG field resolution similar to current magnetometers [15].

Although the current experiment did not produce an appreciable degree of spin-squeezing, it has demonstrated a connection between feedback and robustness (to inevitable atom-number fluctuations) that will apply equally well in future experiments with improved sensitivity [10]. Hence the closed-loop methodology we advocate should enable—and may even be essential for—the implementation of proposals to utilize conditional spin-squeezing for sensitivity beyond the Standard Quantum Limit in various applications of spin resonance with cold atoms, such as magnetometry [14], atomic frequency standards, and matter-wave gravimetry.

This work was supported by the NSF (PHY-9987541, EIA-0086038), the ONR (N00014-00-1-0479), and the Caltech MURI Center for Quantum Networks (DAAD19-00-1-0374). JKS acknowledges a Hertz fellowship. We thank Andrew Berglund and Michael Armen for experimental assistance and Andrew Doherty and Poul Jessen for helpful discussions. Additional information is available at [4](http://munity.caltech.edu/Ensemble)

* Electronic address: [Jgeremia@Caltech.EDU](mailto:Jgeremia@Caltech.EDU)
[1] F. Verstraete, A. C. Doherty, and H. Mabuchi, Phys. Rev. A 64, 032111 (2001).
[2] A. S. Holevo, *Statistical Structure of Quantum Theory* (Springer-Verlag, Berlin, 2001).
[3] J. Gambetta and H. M. Wiseman, Phys. Rev. A 64, 042105 (2001).
[4] J. Dupont-Roc, S. Haroche, and C. Cohen-Tannoudji, Phys. Lett. A 28, 638 (1969).
[5] G. A. Smith, S. Chaudhury, and P. S. Jessen, J. Opt. B: Quant. Semiclass. Opt. 5, 323 (2003).
[6] A. Silberfarb and I. Deutsch, Phys. Rev. A 68, 013817 (2003).
[7] J. Geremia, *et al.*, quant-ph/0306192 (2003).
[8] M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).
[9] A. Kuzmich, L. Mandel, and N. P. Bigelow, Phys. Rev. Lett. 85, 1594 (2000).
[10] L. K. Thomsen, S. Mancini, and H. M. Wiseman, Phys. Rev. A 65, 061801 (2002).
[11] N. Morrow, S. K. Dutta, and G. Raithel, Phys. Rev. Lett. 88, 093003 (2002).
[12] T. Fischer, *et al.*, Phys. Rev. Lett 88, 163002 (2002).
[13] J. Doyle, B. Francis, and A. Tannenbaum, *Feedback Control Theory* (Macmillan Publishing Co., New York, 1990).
[14] K. Zhou and J. Doyle, *Essentials of Robust Control* (Prentice-Hall, Inc., New Jersey, 1997), 1st ed.
[15] I. K. Kominis, *et al.*, Nature 422, 596 (2003).
[16] J. K. Stockton, *et al.*, in preparation (2003).

FIG. 5: Robustness of the single-shot, closed-loop estimation to atom number fluctuations (refer to text).