A variational approach to $S^1$-harmonic maps and applications

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Abstract: We present a renormalization procedure for the Dirichlet Lagrangian for maps from surfaces with or without boundary into $S^1$, whose finite energy critical points are the $S^1$-harmonic maps with isolated singularities. We give some applications of this renormalization scheme in two different frameworks. The first application has to do with the renormalization of the Willmore energy for Lagrangian singular immersions into Kähler-Einstein surfaces while the second application is dealing with frame energies for surfaces immersions into Euclidian spaces.

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I Introduction

I.1 $S^1$–harmonic maps with point singularities

A map $u \in C^\infty(B^n,S^{m-1})$ is by definition a smooth harmonic map from an $n$–dimensional Euclidian ball into a $m-1$ dimensional sphere in $\mathbb{R}^m$ if

$$\Delta u \wedge u = 0 \quad \text{in } B^n. \tag{1.1}$$

where $\Delta$ is denoting the standard negative Laplacian on $B^n$. Equation (1.1) can be interpreted as follows: at any point $x \in B^n$ the Laplacian of the map $u$ is orthogonal to the tangent space of the sphere at $u(x)$:

$$\forall \ x \in B^n \quad \Delta u \perp T_{u(x)}S^{m-1}. \tag{1.2}$$

This condition generates a non-linear equation known as the harmonic map equation

$$-\Delta u = u|\nabla u|^2 \quad \text{in } B^n. \tag{1.3}$$

This equation is in fact variational in the sense that it is the Euler-Lagrange Equation of the Dirichlet Energy

$$E(u) := \frac{1}{2} \int_{B^n} |\nabla u|^2 \, dx^n. \tag{1.4}$$

More precisely smooth solutions to (1.2) are smooth critical points of $E$ among maps taking values into $S^{m-1}$ for the following variations

$$\forall \varphi \in C^\infty_0(B^n,\mathbb{R}^m) \quad \frac{d}{dt} \bigg|_{t=0} E\left(\frac{u + t\varphi}{|u + t\varphi|}\right) = 0. \tag{1.5}$$

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Considering harmonic maps which are smooth exclusively is bringing to numerous limitations. For instance, given a smooth map \( g_0 : \partial B^3 \to S^2 \) of topological degree 0 it is still unknown if the following problem has a smooth solution

\[
\begin{aligned}
-\Delta u &= u|\nabla u|^2 \quad \text{in } B^3 \\
|u &= g_0 \quad \text{on } \partial B^3 ,
\end{aligned}
\]  

(I.6)

while it has been proved (see [19]) by a pure minimization procedure in the Sobolev space \( W^{1,2}_{g_0}(B^3, S^2) \) of maps with finite Dirichlet energy and trace equal to \( g_0 \) that there exists a solution to (I.6) with isolated singularities. In fact, there exist boundary data \( g_0 \) for which any minimizer must have point singularities (see [10]). These singularities have a unique tangent cone of the form

\[
u_0 : x \in B^3 \mapsto \frac{x}{|x|} \in S^2 ,
\]  

(I.7)

modulo the composition by an isometry \( R \in O(3) \). Observe that in one dimension lower the map

\[
u_0 : z \in D^2 \mapsto \frac{z}{|z|} \in S^1 .
\]  

(I.8)

is still harmonic away from the origin (in the sense above) but is missing to have finite Dirichlet energy by very little since \( |\nabla \nu_0| \simeq |x|^{-1} \) and only belongs to the Weak Marcinkiewicz Space or Lorentz Space \( L^{2,\infty}(D^2) \) (see the beginning of section II) but does not belong to \( L^2(D^2) \). Nevertheless \( \nu_0 \) satisfies a weak version of (I.1) in the form

\[
\text{div}(u \wedge \nabla u) = 0 \quad \text{in } D'(D^2) .
\]  

(I.9)

The weak solutions to (I.9) with point singularities were until now considered as “semi variational” in the sense that they solve a weak version of an Euler-Lagrange equation for a Lagrangian, the Dirichlet energy of maps into the circle, which is infinite for these solutions. In the pioneer work on the subject by F.Bethuel, H.Brezis and F.Hélein [2] these \( S^1 \)-harmonic maps are obtained as weak limit of critical points of the Ginzburg-Landau energy

\[
E_\varepsilon(u) := \frac{1}{2} \int_{D^2} |\nabla u|^2 + \frac{1}{4 \varepsilon^2} (1 - |u|^2)^2 \, dx^2
\]  

(I.10)

One of the drawbacks of these variational formulations is that they are requiring a renormalization procedure due to the asymptotic production of infinite energy and this renormalization can generate tedious and lengthy analysis. Another drawback from these approaches comes from the difficulty to deal with “natural” boundary conditions.

The main purpose of the present work is to remedy to these difficulties and to present a direct variational formulation of singular solutions to (I.9). This formulation in particular enable to treat more general boundary conditions than the ones considered in the Ginzburg-Landau theory. The original motivation for our work is related to the theory of Hamiltonian stationary Lagrangian surfaces and is explained in section V.1.1.

In order to explain our approach we take the simplest framework of maps from \( C \) into \( S^1 \). Let \( g : C \to S^1 \subset S^1 \) such that \( \nabla g \in L^{2,\infty}(C) \). We proceed to the Hodge decomposition\(^1\) in \( L^{2,\infty}(C) \) of \( g^{-1}\nabla g : \) there exist two real valued functions \( a_g \) and \( b_g \) with \( \nabla a_g \) and \( \nabla b_g \) in \( L^{2,\infty}(C) \) such that

\[
g^{-1}\nabla g = i \nabla^+ a_g + i \nabla b_g,
\]  

(I.11)

\(^1\)By \( g^{-1}\nabla g \) we denote the complex multiplication of \( g^{-1} \) with \( \partial_{z_1} g \) and \( \partial_{z_2} g \) respectively.
where $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. Using complex coordinates, (I.11) becomes also
\[ g^{-1} \frac{\partial g}{\partial \bar{z}} = - \frac{\partial a_g}{\partial \bar{z}} + i \frac{\partial b_g}{\partial \bar{z}} \]  
(I.12)
which is equivalent to
\[ \frac{\partial}{\partial \bar{z}} (e^{a_g - ib_g}) = 0 \]  
(I.13)
For instance, let’s assume that $g$ is an $S^1$–harmonic map satisfying (I.9) equal to the following product of elementary maps of the form (I.8)
\[ g(z) := \prod_{j=1}^{Q} \frac{z - p_j}{|z - p_j|} \prod_{j=1}^{Q} \frac{|z - q_j|}{z - q_j} \]  
(I.14)
on $\mathbb{C}$. This gives
\[ g^{-1} \nabla g = i \sum_{j=1}^{Q} \frac{\nabla^\perp |z - p_j|}{|z - p_j|} - i \sum_{j=1}^{Q} \frac{\nabla^\perp |z - q_j|}{|z - q_j|}. \]
Thus we can choose $b_g = 0$ and
\[ a_g := \sum_{j=1}^{Q} \log \frac{|z - p_j|}{|z - q_j|}, \]
and
\[ e^{a_g} g = \prod_{j=1}^{Q} \frac{z - p_j}{z - q_j}. \]  
(I.15)
is meromorphic. It is then natural to introduce a “regularization” of this map by taking its inverse stereographic projection into $\mathbb{C}P^1$:
\[ u_g := \pi^{-1} (e^{a_g} g) \]  
(I.16)
The map we have generated through this procedure is a conformal harmonic map into $S^2$. We will call it the “$S^2$ lift” of the $S^1$ valued map $g$.

We then naturally introduce the following definitions.

**Definition I.1.** Let $g \in W^{1,p}(D^2, S^1)$ for some $p > 1$. Let $b_g$ be the unique solution to
\[ \begin{cases} 
\Delta b_g = \text{div} (-ig^{-1} \nabla g) & \text{in } D^2 \\
 b_g = 0 & \text{on } \partial D^2.
\end{cases} \]  
(I.17)

Let $a_g \in W^{1,p}(D^2, \mathbb{R})$ with average 0 on $D^2$ and $u_g$ such that
\[ \nabla^\perp a_g = -ig^{-1} \nabla g - \nabla b_g \quad \text{and} \quad u_g := \pi^{-1} (e^{a_g} g), \]  
(I.18)
where $\pi$ is the stereographic projection from $S^2$ into $\mathbb{C}$ sending the north pole to zero. We call $u_g$ the “$S^2$ lift of $g$”. We introduce the “renormalized Dirichlet Energy” of $g$ the be the following energy
\[ \mathcal{E}(g) := \frac{1}{4} \int_{D^2} (|\nabla u_g|^2 + |\nabla b_g|^2) \, dx^2. \]
Remark I.1. Observe that the previous definition given for the flat disc extends word by word to the simply connected Riemann surfaces \( \mathbb{C} \) and \( \mathbb{C}P^1 \). This is also the case for a general closed Riemann surface \( \Sigma \) with or without boundary, modulo the addition of the \( L^2 \) norms of the harmonic forms involved in the Hodge decomposition of \( g^{-1}dg \).

The motivation for the denomination “renormalized Dirichlet Energy” is justified by the following fact: assume that \( g \in W^{1,2}(D^2, S^1) \) is equal to 1 on the boundary, then \( g \) admits a lift \( \phi \in W^{1,2}_0(D^2, \mathbb{R}) \) such that \( g = e^{i\phi} \). Thus \( b_g = \phi \) and

\[
E(g) = E(g) = \frac{1}{2} \int_{D^2} |\nabla g|^2 \, dx^2.
\]

The advantage of \( E \) over \( E \) is that it can be finite still allowing \( g \) to have point singularities of the form (I.8). In particular, if \( g \) is defined as in (I.14) then the renormalized energy \( E \) (computed as an integral over the whole \( \mathbb{C} \)) is equal to

\[
E(g) = 2\pi Q
\]

for any choice of points \( p_1, ..., p_Q, q_1, ..., q_Q \) in \( \mathbb{C} \).

To see this observe that since \( u_g \) is holomorphic\(^4\)

\[
\frac{1}{4} \int_{\mathbb{C}} |\nabla u_g|^2 \, dx^2 = \frac{1}{2} \int_{\mathbb{C}} u_g^* d\text{vol}_{S^2} = \frac{1}{2} \frac{4\pi}{2} \deg(u_g)
\]

and by considering the preimages of points in a neighbourhood of the north or the south pole we see that

\[
\deg(u_g) = Q.
\]

We recall from [6], [4] and that the obstruction for approximating strongly an arbitrary map \( g \in W^{1,1}(D^2, S^1) \) by smooth maps into \( S^1 \) is given by the distribution

\[
\text{div}(ig^{-1}\nabla^\perp g)
\]

More precisely, it is proven in [4] (Theorem 3') that for such a map \( g \in W^{1,1}(D^2, S^1) \) there exists an at most countable family of pairs of points \( p_i \in \overline{D^2} \) and integers \( d_i \) such that

\[
\text{div}(ig^{-1}\nabla^\perp g) = 2\pi \sum_{i \in I} d_i \delta_{p_i},
\]

and the convergence has to be understood in the sense that there exists an at most countable family of segments with integer multiplicity such that, denoting \( J \) the associated integer rectifiable current,

\[
\partial J = \sum_{i \in I} d_i \delta_{p_i} \quad \text{and} \quad \inf \left\{ M(J) : \partial J = \sum_{i \in I} d_i \delta_{p_i} \right\} \leq \int_{D^2} |\nabla g| \, dx^2.
\]

Here \( M(J) \) denotes the mass of the 1–current \( J \).

We have moreover for any sequence \( g_k \in C^\infty(D^2, S^1) \) satisfying

\[
g_k \rightarrow g \text{ a.e.}
\]

\[
\int_{D^2} |\nabla g| \, dx^2 + 2\pi \inf \left\{ M(J) : \partial J = \sum_{i \in I} d_i \delta_{p_i} \right\} \leq \liminf_{k \to +\infty} \int_{D^2} |\nabla g_k| \, dx^2
\]

Notice that the map \( u_g \) can be extended to an holomorphic map on \( \mathbb{C}P^1 \), therefore its degree is well defined.
and the inequality is optimal for any $g$ (see Theorem 1’ in [4]). For a detailed description of maps in $W^{1,1}((\partial D^2, S^1)$ we refer to [5] (especially Chapters 1 and 2).

The mass distribution
$$
\sum_{i \in I} d_i \delta_{p_i}
$$
is also called **topological singular set of $g$** (see [11]).

In the present work we are interested in the subspace of maps in $W^{1,p}(D^2, S^1)$ such that this mass distribution is discrete.

**Definition I.2** (Isolated/finite topological singularities). Let $g \in W^{1,p}(D^2, S^1)$ for some $p > 1$. Assume that
$$
\text{div} \left( i g^{-1} \nabla g \right) = 2 \pi \sum_{i \in I} d_i \delta_{p_i},
$$
(I.19)

where $I$ is an at most countable index set and for any $i \in I$ $p_i \in D^2$ and $d_i \in \mathbb{Z}$. Assume also that the points $p_i$ in $D^2$ are isolated. Any such map will be referred to as a $W^{1,p}$ $S^1$-valued map with isolated topological singularities or $S^1$-valued map with discrete topological singular set.

Now assume that $g_0 := g|_{\partial D^2} \in W^{1,1}(\partial D^2, S^1)$. Assume that there exist $Q \in \mathbb{N}$, $p_i \in \overline{D^2}$, $d_i \in \mathbb{Z}$ for any $i \in \{1, \ldots, Q\}$ so that
$$
i \int_{\partial D^2} g_0^{-1} \partial_\theta g_0 + 2 \pi \sum_{i=1}^{Q} d_i + \pi \sum_{p_i \in \partial D^2} d_i = 0
$$
and assume that for any $p_i \in \partial D^2$ $d_i$ is even.

Assume that for any $\phi \in C^\infty(D^2)$
$$
\int_{D^2} ig^{-1} \nabla g \nabla \phi = -i \int_{\partial D^2} \phi g_0^{-1} \partial_\theta g_0 - 2 \pi \sum_{i=1}^{Q} d_i \phi(p_i) - \pi \sum_{p_i \in \partial D^2} d_i \phi(p_i).
$$

In this case we refer to the points $p_i \in \overline{D^2}$ as **topological singularities** of $g$ and we say that $g$ has finitely many topological singularities in $\overline{D^2}$. □

The following theorem, which is one of the main results of the present work, gives the sequential weak completeness of $S^1$-valued maps with discrete topological singular set under controlled “renormalized Dirichlet Energy” and the sequential weak completeness of $S^1$-valued maps with finitely many topological singularities in $D^2$ under controlled “renormalized Dirichlet Energy” and controlled $W^{1,1}$-norm at the boundary.

**Theorem I.1.** a) Let $(g_k)_{k \in \mathbb{N}}$ be a sequence of maps in $W^{1,1,2,\infty}(D^2, S^1)$ uniformly bounded in $W^{1,p}(D^2, S^1)$ for some $p > 1$ and such that for any $k \in \mathbb{N}$ $g_k$ has isolated topological singularities in $D^2$.

Assume that
$$
\limsup_{k \to +\infty} \mathcal{E}(g_k) < +\infty.
$$
(I.20)

\[^4\]This assumption will be clarified in Remark 2.
Then there exists a subsequence \( g_{k'} \) and a map \( g_\infty \in W^{1,\infty,1}_{\text{loc}}(\partial D^2, S^1) \) with isolated topological singularities such that
\[
\mathcal{E}(g_\infty) \leq \liminf_{k' \to +\infty} \mathcal{E}(g_{k'}) \quad \text{and} \quad \nabla g_{k'} \rightharpoonup \nabla g_\infty \quad \text{weakly in } L^{1,\infty}_{\text{loc}}(\partial D^2). \tag{I.21}
\]

\[\square\]

b) Let \( g_0 \in W^{1,1}(\partial D^2, S^1) \). Let \( (g_k)_{k \in \mathbb{N}} \) be a sequence of \( S^1 \)-valued map in \( W^{1,p}(D^2, S^1) \) for some \( p > 1 \) (where \( p \) might depend on \( k \)) with finitely many topological singularities in \( \overline{D^2} \) and with trace equal to \( g_0 \) on \( \partial D^2 \). Assume that
\[
\limsup_{k \to +\infty} \mathcal{E}(g_k) < \infty. \tag{I.22}
\]

Then there exists a subsequence \( g_{k'} \) and a \( W^{1,\infty,1} \) \( S^1 \)-valued map \( g_\infty \) with finitely many topological singularities in \( D^2 \) such that
\[
\mathcal{E}(g_\infty) \leq \liminf_{k' \to +\infty} \mathcal{E}(g_{k'}) \quad \text{and} \quad \nabla g_{k'} \rightharpoonup \nabla g_\infty \quad \text{weakly in } L^{1,\infty}_{\text{loc}}(\partial D^2). \tag{I.23}
\]

One important step in the proof of Theorem I.1 consists in the following a-priori estimate on the number of the topological singularities of a function \( g \in W^{1,p}(D^2, S^1) \).

**Theorem I.2.** Let \( g_0 \in W^{1,1}(\partial D^2, S^1) \). Assume that \( g \in W^{1,p}(D^2, S^1) \) (for some \( p > 1 \)) is an \( S^1 \)-valued map with finitely many topological singularities in \( \overline{D^2} \). Assume that either no topological singularities lie on \( \partial D^2 \) or \( g_0 \in W^{1,1}(\partial D^2) \). Then
\[
\sum_{i=1}^{Q} |d_i| \leq C \left[ \mathcal{E}(g) + \|\partial_0 g_0\|_{L^1(\partial D^2)} \right]. \tag{I.24}
\]

for some universal constant \( C > 0 \).

\[\square\]

**Remark I.2.** We believe that Theorem I.2 remains true even if we do not assume a priori that \( g \) has finitely many topological singularities in \( D^2 \): the finiteness of the number of singularities should be a consequence of the finiteness of \( \mathcal{E}(g) \) combined with the \( W^{1,1} \)-bound at the boundary. The \( W^{1,1} \)-bound at the boundary seems necessary and it could be that another bound with the same scaling property such as \( g \in H^{1/2}(\partial D^2, S^1) \) does not imply the finiteness of the number of topological singularities.\(^5\)

The next Theorem is the third main result of the present paper, it says that the critical points of \( \mathcal{E} \) are \( S^1 \)-harmonic maps and vice versa.

**Theorem I.3.** Let \( g \in W^{1,p}(D^2, S^1) \) be as in Definition I.1. Assume that
\[
\mathcal{E}(g) < \infty.
\]

Then \( g \) solves the weak \( S^1 \) harmonic map equation (I.4) if and only if \( g \) is a critical point of the "renormalized Dirichlet Energy" for smooth variations in the target, that is
\[
\forall \psi \in C_0^\infty(D^2, \mathbb{R}) \quad \frac{d}{dt} \bigg|_{t=0} \mathcal{E}(ge^{it\psi}) = 0.
\]
Moreover, if \( g \) has isolated topological singularities \( g \) solves the weak \( S^1 \)-harmonic map equation (I.4) if and only if its lift \( u_g \) is a conformal harmonic map into \( S^2 \).

\(^5\)While the slightly stronger assumption \( \Delta^{1/4} g \in L^{2,1}(\partial D^2, S^1) \) (where \( L^{2,1}(\partial D^2) \) is the Lorentz space predual of the weak \( L^2 \) space \( L^{2,\infty}(\partial D^2) \)) should imply that \( g \) has finitely many topological singularities in \( \overline{D^2} \).
The behaviour of $E$ under variations of different type is addressed in Remark 3. Combining the results above and the fact that any map in $W^{1,1}(\partial D^2, S^1)$ admits a finite “renormalized Dirichlet Energy” extension (see Lemma 4), we obtain the following result.

**Corollary I.1.** Let $g_0 \in W^{1,1}(\partial D^2, S^1)$, then there exists an $S^1$-harmonic map $g_{\text{min}}$ minimizing $E$ among the functions $g \in W^{1,2,\infty}(D^2, S^1)$ with finitely many singularities in $D^2$ and satisfying $g|_{\partial D^2} = g_0$.

**Remark I.3.** It is still an open question to know whether or not in Corollary I.1 the degrees are all equal to $+1$ and whether $Q$ is equal to the topological degree of $g_0$. One could also wonder if one should expect singularities to be located at the boundary or not. While these questions are settled in [2] thanks to the careful analysis of the diverging part of the Ginzburg-Landau energy (i.e. the coefficient in front of $\log \varepsilon^{-1}$), in the present situation there is no such leading diverging term imposing restrictions on the configuration $(d_i, p_i)$ and these questions are left open at this stage.

We conclude this introduction with the following open problem.

**Open problem I.1.** A more natural trace space than $W^{1,1}(\partial D^2, S^1)$ to consider for $S^1$-harmonic map is the trace space $H^{1/2}(\partial D^2, S^1)$. In particular it would be interesting to investigate whether any trace in $H^{1/2}(\partial D^2, S^1)$ admits a finite “renormalized Dirichlet Energy” extension - we believe this is the case - and if there exist finite energy extensions (minimal or not) with infinitely many singular points accumulating at the boundary. This last fact cannot be excluded a priori.

The paper is organized as follows. In chapter II we recall the definition of some of the functions spaces we will be using throughout the paper and we fix some notations. We then present some preliminary results about the energy and the functions introduced in Definition I.1. In section III we give a proof of Theorem I.1 and Theorem I.2. In section IV we give a proof of Theorem I.3. In section V we present two applications of the ideas introduced in this work: the first has to do with the renormalization of the Willmore energy for Lagrangian singular immersions into Kähler-Einstein surfaces while the second is dealing with frame energies for surfaces immersions into Euclidian spaces.

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**II Notation and preliminary results**

**II.1 Notation**

Let $\Omega \subset \mathbb{R}^n$ be a domain. Recall that a function $f : \Omega \to \mathbb{R}$ is said to belong to the weak $L^2$ space $L^{2,\infty}(\Omega, \mathbb{R})$ if $f$ is measurable and

$$[f]_{L^{2,\infty}} := \sup \{ \gamma d_f(\gamma)^{\frac{1}{2}}, \gamma > 0 \}$$

is finite, where

$$d_f(\alpha) = \mathcal{L}^n(\{x \in \Omega : |f(x)| > \alpha\}).$$
$[\cdot]_{L^2,\infty}$ is a quasi-norm on $L^2,\infty(\Omega, \mathbb{R})$ and $L^2,\infty(\Omega, \mathbb{R})$ can be made into a Banach space by introducing a norm $\|\cdot\|_{L^2,\infty}$ equivalent to $[\cdot]_{L^2,\infty}$ (as quasi-norm) (see Exercise 1.1.12 in [7]). We also recall the definition of the following space:

$$W^{1,(2,\infty)}(\Omega, \mathbb{R}) = \left\{ u \in D'(\Omega), \nabla u \in L^2,\infty \right\}.$$ 

$W^{1,(2,\infty)}(\Omega, \mathbb{R})$ is a Banach space with norm

$$\|f\|_{W^{1,(2,\infty)}} = \|f\|_{L^2,\infty} + \|\nabla f\|_{L^2,\infty}.$$ 

Observe that

$$[f]_{W^{1,(2,\infty)}} := \|\nabla f\|_{L^2,\infty}$$

defines a semi-norm on $W^{1,(2,\infty)}(\Omega, \mathbb{R})$. At times it will be useful to consider the space $\dot{W}^{1,(2,\infty)}(\Omega)$ obtained as the quotient of $W^{1,(2,\infty)}(\Omega)$ by the constant functions. $\dot{W}^{1,(2,\infty)}(\Omega)$ is again a Banach space and the seminorm $[\cdot]_{W^{1,(2,\infty)}}$ induces a norm on $\dot{W}^{1,(2,\infty)}(\Omega)$.

In the following we will often consider functions with values in $\mathbb{C} \simeq \mathbb{R}^2$. Sometimes it will be convenient to look at this space as $\mathbb{C}$, while in other occasions as $\mathbb{R}^2$. To avoid confusion, we will denote the complex multiplication of two elements $\alpha, \beta \in \mathbb{C}$ as

$$\alpha \beta,$$

while we will denote their $\mathbb{R}^2$-scalar product as

$$\alpha \cdot \beta.$$ 

Moreover, when considering the product of gradients we will use the following notation: if $f, g : \mathbb{R}^2 \to \mathbb{R}^n$,

$$< \nabla f, \nabla g > = \sum_{i=1}^{2} \sum_{j=1}^{n} \partial_x f^i \partial_x g^j.$$ 

### II.2 Degree of a map between manifolds

We briefly recall here the notion of degree of a map between smooth manifolds, as we will make large use of it in the present article. For more details see Chapter 7 in [1].

Let $M$ and $N$ be two oriented, compact, connected smooth $n$–manifolds without boundary. Let $f : M \to N$ be a smooth map. For any regular value $y \in N$ of $f$ let

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sgn } df_x,$$

where $\text{sgn } df_x = 1$ if $df_x$ is orientation preserving and $\text{sgn } df_x = -1$ if $df_x$ is orientation reversing. One can show that $\deg(f, y)$ does not depend on the choice of $y$, therefore we can define the degree of $f$ as

$$\deg(f) := \deg(f, y)$$

for any regular value $y \in N$ of the map $f$.

The degree of $f$ can also be characterized as follows: $\deg(f)$ is the only integer such that for any smooth $n$–form $\omega$ on $N$

$$\int_M f^* \omega = \deg(f) \int_N \omega.$$
When $M = N = S^1$, the notion of degree of a map from $M$ to $N$ can be extended to continuous maps. In fact given a continuous map $f : S^1 \to S^1$ and a continuous parametrization $\phi : [0, 1] \to S^1$ of $S^1$ as a closed curve (with $\phi(0) = \phi(1)$) one can show that there exists a continuous lift $\tilde{f} : [0, 1] \to \mathbb{R}$ such that

$$f \circ \phi(x) = e^{i\tilde{f}(x)} \quad \forall x \in S^1.$$ 

Then the degree of $f$ is defined as

$$\text{deg}(f) = \frac{1}{2\pi} \left( \tilde{f}(1) - \tilde{f}(0) \right).$$

**II.3 Preliminary results for general $g$**

In this subsection and in the next we collect some preliminary results for functions $g$ as in Definition I.1. Here we do not make further assumptions on $g$ (in particular we do not assume that $g$ has isolated or finitely many topological singularities). In the next subsection we will focus on functions with finitely many topological singularities.

**Lemma 1.** Let $g \in W^{1,p}(D^2, S^1)$ (for some $p > 1$) be as in Definition I.1 and assume that

$$\mathcal{E}(g) < \infty.$$ 

Let

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{e^{2x}}{(1 + e^{2x})^2}.$$ 

Then

$$\mathcal{E}(g) = \int_{D^2} f(a_g) \left( |\nabla g|^2 + |\nabla a_g|^2 \right) + \frac{1}{4} \int_{D^2} |\nabla b_g|^2. \quad \text{(II.24)}$$

Moreover

$$\int_{D^2} f(a_g) |\nabla a_g|^2 = \int_{D^2} |\nabla \arctan e^{a_g}|^2$$

and if $b_g = 0$ in $D^2$

$$\mathcal{E}(g) = 2 \int_{D^2} f(a_g) |\nabla a_g|^2.$$ 

**Proof.** We compute

$$\mathcal{E}(g) = \frac{1}{4} \int_{D^2} |D\pi^{-1}(e^{a_g}) D(e^{a_g})|^2 + \frac{1}{4} \int_{D^2} |\nabla b_g|^2.$$ 

Now

$$D\pi^{-1}(e^{a_g}) D(e^{a_g}) = \frac{2}{1 + |e^{a_g}|^2} D(e^{a_g}) = 2 \frac{e^{a_g} Dg + e^{a_g} gDa_g}{1 + e^{2a_g}}.$$ 

Therefore

$$\mathcal{E}(g) = \int_{D^2} f(a_g) \left( |\nabla g|^2 + |\nabla a_g|^2 \right) + \frac{1}{4} \int_{D^2} |\nabla b_g|^2 \quad \text{(II.25)}$$

(here we used the fact that since $g$ takes values in $S^1$, $g \cdot Dg = 0$).

Moreover let

$$H : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \arctan e^x,$$
then \( H' = f^\frac{1}{2} \) and \( H \circ a_g \in W^{1,1}(D^2) \) with
\[
\nabla (H \circ a_g) = (f(a_g))^\frac{1}{2} \nabla a_g.
\]

Therefore
\[
\int_{D^2} f(a_g) \left| \nabla a_g \right|^2 = \int_{D^2} \left| \nabla (H \circ a_g) \right|^2 = \int_{D^2} |\nabla \arctan e^{a_g}|^2.
\]

Finally, if \( b_g = 0 \) in \( D^2 \) then
\[
-ig^{-1} \nabla g = \nabla^\perp a_g,
\]
therefore
\[
|\nabla g| = |\nabla a_g|
\]
and so it follows from (II.25) that
\[
\mathcal{E}(g) = 2 \int_{D^2} f(a_g) \left| \nabla a_g \right|^2.
\]

Lemma 2. Let \( g \in W^{1,p}(D^2, S^1) \) (for some \( p > 1 \)) be as in Definition I.1 and assume that \( \mathcal{E}(g) < \infty \).

Let
\[
\dot{g} = ge^{-ib_g}.
\]

Then \( \dot{g} \in W^{1,p'}(D^2, S^1) \) with \( p' = \min\{p, 2\} \), \( a_{\dot{g}} = a_g \), \( b_{\dot{g}} = 0 \) and
\[
\mathcal{E}(g) = 2 \int_{D^2} f(a_g) \left| \nabla a_g \right|^2 + \int_{D^2} \left( f(a_g) + \frac{1}{4} \right) |\nabla b_g|^2
\]

\[
= \mathcal{E}(\dot{g}) + \int_{D^2} \left( f(a_g) + \frac{1}{4} \right) |\nabla b_g|^2,
\]
where \( f \) is as in Lemma I.

In particular
\[
\mathcal{E}(\dot{g}) \leq \mathcal{E}(g).
\]

Proof. We compute
\[
\nabla \dot{g} = e^{-ib_g} \nabla g - ig e^{-ib_g} \nabla b_g.
\]

Therefore \( \dot{g} \in W^{1,p'}(D^2, S^1) \), where \( p' = \min\{p, 2\} \), and
\[
\dot{g}^{-1} \nabla \dot{g} = g^{-1} \nabla g - i \nabla b_g = i \nabla^\perp a_g.
\]

Thus
\[
a_{\dot{g}} = a_g \text{ and } b_{\dot{g}} = 0.
\]

By Lemma I there holds
\[
\mathcal{E}(\dot{g}) = 2 \int_{D^2} f(a_g) \left| \nabla a_g \right|^2 \tag{II.26}
\]
and
\[
\mathcal{E}(g) = \int_{D^2} f(a_g) \left( |\nabla^\perp a_g + \nabla b_g|^2 + |\nabla a_g|^2 \right) + \frac{1}{4} |\nabla b_g|^2. \tag{II.27}
\]
Now we claim that
\[ \int_{D^2} f(a_g) \nabla^\perp a_g \nabla b_g = 0. \] (II.28)

In fact let
\[ F : \mathbb{R} \to \mathbb{R}, \quad x \mapsto -\frac{1}{2(e^{2x} + 1)}, \]
then \( F' = f \), therefore
\[ f(a_g) \nabla^\perp a_g = \nabla^\perp (F \circ a_g). \]

Now for any \( \phi \in C^\infty_c(D^2, \mathbb{R}) \)
\[ \hat{D^2} f(a_g) \nabla^\perp a_g \nabla \phi = -\lim_{n \to \infty} \int_{D^2} f(a_g) \nabla^\perp \phi = \int_{D^2} F \circ a_g \operatorname{div}(\nabla^\perp \phi) = 0. \] (II.29)

As \( b_g \in W^{1,2}(D^2) \) there exists a sequence \( (\phi_n)_{n \in \mathbb{N}} \) in \( C^\infty_c(D^2) \) such that
\[ \phi_n \to b_g \text{ in } W^{1,2}(D^2, \mathbb{R}). \]

Now notice that by Lemma 1
\[ \int_{D^2} |f(a_g) \nabla^\perp a_g|^2 \le \int_{D^2} f(a_g) |\nabla a_g|^2 < \infty, \]
therefore
\[ \int_{D^2} f(a_g) \nabla^\perp a_g \nabla b_g = \lim_{n \to \infty} \int_{D^2} f(a_g) \nabla^\perp a_g \nabla \phi_n = 0. \]

This concludes the proof of (II.28).

Now by (II.27)
\[ \mathcal{E}(g) = \int_{D^2} f(a_g) \left(2|\nabla a_g|^2 + |\nabla b_g|^2\right) + \frac{1}{4}|\nabla b_g|^2. \]
Comparing with (II.26) we obtain
\[ \mathcal{E}(g) = \mathcal{E}(\tilde{g}) + \int_{D^2} \left( f(a_g) + \frac{1}{4} \right) |\nabla b_g|^2. \]

Then in particular
\[ \mathcal{E}(\tilde{g}) \le \mathcal{E}(g). \]

\[ \quad \] \[ \quad \]

II.4 Functions with finitely many topological singularities

In this subsection we first give a more explicit expression for maps \( g \) as in Definition I.1 (and their corresponding \( a_g \)) when \( g \) has finitely many topological singularities in \( \overline{D^2} \). We will then show that for any boundary datum \( g_0 \in W^{1,1}(\partial D^2, \mathbb{R}) \) it is possible to find an extension in \( D^2 \) with finite renormalized Dirichlet Energy.

Lemma 3 (A more explicit form for \( a \)). Let \( g_0 \in W^{1,1}(\partial D^2, S^1) \). Let \( Q \in \mathbb{N} \) and for any \( i \in \{1, ..., Q\} \) let \( p_i \in \overline{D^2} \) and \( d_i \in \mathbb{Z} \). Assume that
\[ i \int_{\partial D^2} g_0^{-1} \partial g g_0 + 2\pi \sum_{i=1}^{Q} d_i + \pi \sum_{i=1}^{Q} d_i = 0. \]
For any $i \in \{1, \ldots, Q\}$ assume that whenever $p_i \in \partial D^2$, $d_i$ is even.
Let $a \in W^{1,1}(D^2, \mathbb{R})$ and assume that for any $\phi \in C^\infty(D^2)$

$$
\int_{D^2} \nabla a \nabla \phi = -i \int_{\partial D^2} \phi g_0^{-1} \partial_\theta g_0 - 2\pi \sum_{i=1}^{Q} d_i \phi(p_i) - \pi \sum_{i=1}^{Q} d_i \phi(p_i). \tag{II.30}
$$

Let

$$
\Phi(x) := \sum_{i=1}^{Q} d_i \log|x - p_i| \quad \text{and} \quad \tilde{a}(x) := a(x) - \Phi(x) \tag{II.31}
$$

for any $x \in D^2$. Then

$$
a = \tilde{a} + \Phi \tag{II.32}
$$

and for any $\phi \in C^\infty(D^2)$

Let

$$
\beta(x) = -i g_0^{-1} \partial_\theta g_0 - \sum_{i=1}^{Q} d_i \partial_\nu \log|x - p_i| - \frac{1}{2} \sum_{i=1}^{Q} d_i.
$$

for any $x \in \partial D^2$. In particular $\tilde{a}$ is harmonic in $D^2$.

Proof. Observe that if $p \in D^2$

$$
\int_{D^2} \nabla \phi \nabla \log|x - p| = \lim_{\varepsilon \to 0} \int_{\partial(D^2 \setminus B_\varepsilon(p))} \phi \partial_\nu \log|x - p| \tag{II.33}
$$

$$
= \int_{\partial D^2} \phi \partial_\nu \log|x - p| - \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(p)} \phi \frac{1}{\varepsilon}
$$

$$
= \int_{\partial D^2} \phi \partial_\nu \log|x - p| - 2\pi \phi(p).
$$

On the other hand, if $p \in \partial D^2$

$$
\int_{D^2} \nabla \phi \nabla \log|x - p| = \lim_{\varepsilon \to 0} \int_{\partial(D^2 \setminus B_\varepsilon(p))} \phi \partial_\nu \log|x - p| \tag{II.34}
$$

$$
= \lim_{\varepsilon \to 0} \int_{\partial D^2 \setminus B_\varepsilon(p)} \phi \frac{(x - p) \cdot x}{|x - p|^2} - \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(p) \cap D^2} \phi \frac{1}{\varepsilon}
$$

$$
= \frac{1}{2} \int_{\partial D^2} \phi - \pi \phi(p)
$$

where we used the fact that for any $x, p \in \partial D^2$

$$
\frac{(x - p) \cdot x}{|x - p|^2} = \frac{1 - x \cdot p}{|x - p|^2} = \frac{p \cdot (p - x)}{|x - p|^2}
$$

and therefore

$$
\frac{(x - p) \cdot x}{|x - p|^2} = \frac{1}{2} \frac{(x - p) \cdot (x - p)}{|x - p|^2} = \frac{1}{2}.
$$

Thus the function $\tilde{a}$ defined in (II.31) satisfies (II.32). \hfill \square
Remark 1. Whenever \(a \in W^{1,p}(D^2)\) satisfies (II.30) we will say that \(a\) is a solution of
\[
\begin{cases}
\Delta a = 2\pi \sum_{i=1}^{Q} d_i \delta_{p_i} & \text{in } D^2 \\
\partial_n a = -i g_0^{-1} \partial_\nu g_0 & \text{on } \partial D^2.
\end{cases}
\]

Corollary 1 (A more explicit form for \(g\)). Let \(g\) be a \(S^1\)-valued map in \(W^{1,p}\) (for some \(p > 1\)) with finitely many topological singularities in \(D^2\). Assume that \(a_g\) in the Hodge decomposition (I.18) satisfies the assumptions of Lemma 3. Then \(g\) has the following form:
\[
ge(z) = \prod_{i=1}^{Q} \left( \frac{z - p_i}{|z - p_i|} \right)^{-d_i} e^{i\varphi},
\]
where
\[
\varphi = H(\tilde{a}_g) + b_g
\]
up to an additive constant. Here \(\tilde{a}\) is the function introduced in (II.31) and \(H(\tilde{a})\) denotes the harmonic conjugate of \(\tilde{a}\) in \(D^2\), i.e.
\[
\nabla H(\tilde{a}_g) = \nabla^\perp \tilde{a}_g \text{ and } H(\tilde{a}_g)(0) = 0.
\]

Proof. The result follows from Lemma 3 and the fact that for any \(p \in \overline{D^2}\)
\[
\nabla \left( \frac{z-p}{|z-p|} \right) = i \frac{\nabla^\perp |z-p|}{|z-p|}.
\]

The following Lemma shows that for any given \(g_0 \in W^{1,1}(\partial D^2, S^1)\) the class of functions \(g \in W^{1,2}(D^2, S^1)\) with \(g|_{\partial D^2} = g_0\), with finitely many topological singularities and with
\[
\mathcal{E}(g) < \infty
\]
is not empty.

Lemma 4. Let \(d \in \mathbb{Z}\). Let \(g_0 \in W^{1,1}(\partial D^2, S^1)\) with \(\deg(g_0) = d\). Then there exist \(g \in W^{1,2}(D^2, S^1)\) as in Definition I.1 with finitely many topological singularities such that \(g|_{\partial D^2} = g_0\) and
\[
\mathcal{E}(u_g) < \infty.
\]

More precisely, there exists a map
\[
\text{Ext}_d : W^{1,1}_{\text{deg}=d}(\partial D^2, S^1) \to W^{1,2}(D^2, S^2)
\]
sending a boundary datum \(g_0\) to a function \(u_g\) as in (I.16) corresponding to a function \(g \in W^{1,2}(D^2, S^1)\) with finitely many topological singularities, with \(g|_{\partial D^2} = g_0\), and so that
\[
\|\nabla g\|_{L^2} \leq C (\|g_0\|_{W^{1,1}} + |d|)
\]
for some constant \(C\) and
\[
\frac{1}{2} \int_{D^2} |\nabla u_g|^2 \leq \frac{\pi^2}{2} \|\partial_\nu g_0\|_{L^1(\partial D^2)} + 4\pi|d|.
\]
Proof. Let \( g_0 \in W^{1,1}(\partial D^2, S^1) \) with \( \deg(g_0) = 0 \). Then there exists a lift \( \phi_0 \in C^\infty(S^1, \mathbb{R}) \) such that
\[
g_0 = e^{i\phi_0} \text{ on } \partial D^2.
\]
Let \( \phi \) be the solution of the Cauchy problem
\[
\begin{cases}
\Delta \phi = 0 & \text{in } D^2 \\
\phi = \phi_0 & \text{on } \partial D^2.
\end{cases}
\]
and let \( g = e^{i\phi} \) in \( D^2 \). Then by Lemma 12 \( g \in W^{1,2}(D^2, S^1) \) with
\[
\|\nabla g\|_{L^2, \infty} \leq C\|g_0\|_{W^{1,1}}.
\]
ang \( g \) has no topological singularities in \( D^2 \). Let \( a : \overline{D^2} \to \mathbb{R} \) so that
\[
\hat{S}^2 F^*dvol_{S^2} = 0
\]
and so
\[
\int_{S^2_+} (u_g \circ \mu_+)^*dvol_{S^2} = \int_{S^2_+} (A \circ \mu_-)^*dvol_{S^2}.
\]
Now since \( u_g \) is holomorphic and \( \mu_+ \) is orientation preserving,
\[
\int_{S^2} \#(u_g \circ \mu_+)^{-1}(y) d\text{vol}_{S^2} = \int_{S^2} (u_g \circ \mu_+)^* d\text{vol}_{S^2} = - \int_{S^2} (A \circ \mu_-)^* d\text{vol}_{S^2}.
\]
Here and in the following the symbol \( \# \) denotes the cardinality of a set.
We also have
\[
- \int_{S^2} (A \circ \mu_-)^* d\text{vol}_{S^2} = - \text{deg}(\mu_-) \int_{D^2} A^* d\text{vol}_{S^2} \leq \int_{D^2} |JA|.
\]
To estimate the last term, let’s introduce the following function:
\[
\overline{A} : D^2 \to S^2, \quad (r, \varphi) \mapsto (g_0(e^{i\varphi}), r\pi).
\] (II.37)
(here we are using the same coordinates as in \( \text{(II.36)} \)). Then for any \( y \in S^2 \)
\[
\#A^{-1}(y) \leq \#\overline{A}^{-1}(y).
\]
Therefore, by the area formula,
\[
\int_{D^2} |JA| d\sigma^2 = \int_{S^2} \#A^{-1}(y) d\text{vol}_{S^2} \leq \int_{S^2} \#\overline{A}^{-1}(y) d\text{vol}_{S^2} = \int_{D^2} |J\overline{A}| d\sigma^2. \tag{II.38}
\]
One computes that
\[
|J\overline{A}(r, \theta)| \leq \pi^2 |\partial_\theta g_0(e^{i\theta})|,
\]
therefore
\[
\int_{D^2} |J\overline{A}| d\sigma^2 \leq \pi^2 \int_0^1 r \int_{\partial D^2} |\partial_\theta g_0| d\theta dr = \frac{\pi^2}{2} \|\partial_\theta g_0\|_{L^1(\partial D^2)}
\]
Since \( u_g \) is holomorphic, we conclude that
\[
\frac{1}{2} \int_{D^2} |\nabla u_g|^2 d\sigma^2 = \int_{S^2} \#u_g^{-1}(y) d\text{vol}_{S^2} = \int_{S^2} \#(u_g \circ \mu_+)^{-1}(y) d\text{vol}_{S^2} \leq \frac{\pi^2}{2} \|\partial_\theta g_0\|_{L^1(\partial D^2)}. \tag{II.39}
\]
Therefore the procedure described above induces a bounded continuous map
\[
\text{Ext}_0 : W^{1,1}_{\text{deg}=0}(\partial D^2, S^1) \to W^{1,2}(D^2, S^2), \quad g_0 \mapsto u_g.
\]
In fact, given a generic map \( g_0 \in W^{1,1}_{\text{deg}=0}(\partial D^2, S^1) \) let \( (g^n_0)_{n\in\mathbb{N}} \) be a sequence of degree zero maps in \( C^\infty(\partial D^2, S^1) \) such that
\[
g^n_0 \to g_0 \text{ in } W^{1,1}(\partial D^2).
\]
Then by Lemma \([12]\)
\[
g_n \to g \text{ and } a_n \to a \text{ in } W^{1,(2,\infty)}(D^2).
\]
In particular, up to a subsequence,
\[
u_{g_n} \to u_g \text{ a.e.}
\]
Upon considering a further subsequence, the weak lower semi-continuity of the norm implies that estimate \( \text{(II.39)} \) passes to the limit and thus holds for \( u_g \).
Next let’s consider the case where \( g_0 \in W^{1,1}(\partial D^2, S^1) \) and \( \text{deg}(g_0) = d \) for some \( d \in \mathbb{Z} \). Let
\[
\tilde{g}_0 = \left( \frac{z}{|z|} \right)^{-d} g_0 \text{ on } \partial D^2.
\]
Let \( \tilde{\phi} \) be its harmonic extension in \( D^2 \) and set
\[
g := \left( \frac{z}{|z|} \right)^d e^{i\tilde{\phi}}.
\]
Then \( g|_{\partial D^2} = g_0 \) and by Lemma 12
\[
\| \nabla g \|_{L^{2,\infty}} \leq C \left( |d| + \| \partial_\theta g_0 \|_{L^1(\partial D^2)} \right).
\]
(II.40)

Notice that the corresponding function \( a_g \) in the decomposition (I.18) is given by
\[
a_g = d \log |z| - \mathcal{H}(\tilde{\phi}),
\]
where \( \mathcal{H}(\tilde{\phi}) \) is the harmonic conjugate of \( \tilde{\phi} \), therefore
\[
\| a_g \|_{L^{2,\infty}} \leq C \left( |d| + \| \partial_\theta g_0 \|_{L^1(\partial D^2)} \right).
\]
(II.41)

As above let
\[
\pi^{-1}(e^{a_g} g) \text{ in } D^2.
\]

Let’s assume now that \( g_0 \in C^\infty(\partial D^2, S^1) \) and let \( A \) be the map introduced in (II.36).

Again the maps \( u_g \circ \mu_+ \) and \( A \circ \mu_- \) can be glued along \( \partial D^2 \times \{0\} \) to obtain a Lipschitz continuous map
\[
F : S^2 \to S^2.
\]

Now \( \text{deg}(F) = d \), as one can see considering the preimages of point around the south pole (if \( d \) is negative) or the north pole (if \( d \) is positive).

Therefore
\[
\int_{S^2} F^* d\text{vol}_{S^2} = d4\pi
\]

Thus, arguing as above, we obtain
\[
\int_{S^2} \#(u_g \circ \mu_+)^{-1}(y) d\text{vol}_{S^2} \leq \int_{D^2} |J A| dx^2 + 4\pi|d|.
\]

As estimate (II.38) remains true for the function \( \overline{A} \) introduced in (II.37), we conclude that
\[
\frac{1}{2} \int_{D^2} |\nabla u_g|^2 dx^2 = \int_{S^2} \#u_g^{-1}(y) d\text{vol}_{S^2} = \int_{S^2} \#(u_g \circ \mu_+)^{-1}(y) d\text{vol}_{S^2} \leq \frac{\pi^2}{2} \| \partial_\theta g_0 \|_{L^1(\partial D^2)} + 4\pi|d|.
\]
(II.42)

Just as above one can verify that the prescription
\[
g_0 \mapsto u_g
\]
induces a bounded map
\[
\text{Ext}_d : W^{1,1}_{\text{deg}=d}(\partial D^2, S^1) \to \hat{W}^{1,2}(D^2, S^2)
\]
such that for any \( g_0 \in W^{1,1}(\partial D^2, S^1) \) the corresponding \( u_g \) is the “\( S^2 \) lift” of a map \( g \in W^{1,(2,\infty)}(\partial D^2, S^1) \) with finitely many topological singularities in \( D^2 \) such that estimates (II.40) and (II.42) hold true.
II.5 Stability of the renormalized Dirichlet Energy

Next we show that when the boundary datum \( g_0 \) lies in \( H^\frac{1}{2}(\partial D^2) \), the renormalized energy is stable under displacements of the topological singularities, even if a topological singularity is pushed to the boundary.

**Lemma 5.** Let \( g_0 \in H^\frac{1}{2}(\partial D^2, S^1) \), let \( Q \in \mathbb{N} \). Let \( p_i \in \overline{D^2} \) and \( d_i \in \mathbb{Z} \setminus \{0\} \) for any \( i \in \{1, ..., Q\} \). For any \( i \in \{1, ..., Q\} \) if \( p_i \in \partial D^2 \) assume that \( d_i \) is even.

Let \((p^k_i)_{k \in \mathbb{N}}\) be a sequence of points in \( \overline{D^2} \) such that \( p^k_i \to p_1 \). For any \( k \in \mathbb{N} \) let \( a_k \) denote the zero-average solution of

\[
\begin{align*}
\Delta a_k &= 2\pi \left( d_1 \delta_{p_1} + \sum_{i=2}^{Q} d_i \delta_{p_i} \right) \quad \text{in } D^2 \\
\partial_\nu a_k &= -ig_0^{-1}\partial_\theta g_0 \quad \text{on } \partial D^2
\end{align*}
\]

and let \( a \) denote the zero-average solution of

\[
\begin{align*}
\Delta a &= 2\pi \left( d_1 \delta_{p_1} + \sum_{i=2}^{Q} d_i \delta_{p_i} \right) \quad \text{in } D^2 \\
\partial_\nu a &= -ig_0^{-1}\partial_\theta g_0 \quad \text{on } \partial D^2.
\end{align*}
\]

Then

\[
\lim_{k \to \infty} \int_{D^2} f(a_k)|\nabla a_k|^2 = \int_{D^2} f(a)|\nabla a|^2.
\]

**Proof.**

**Claim 1:**

\[-ig_0^{-1}\partial_\theta g_0 \in H^{-\frac{1}{2}}(\partial D^2)\].

**Proof of Claim 1.** Let

\[d := \text{deg}(g_0)\]

and let

\[\tilde{g}_0(e^{i\theta}) := g_0(e^{i\theta})e^{-id\theta} \quad \forall \theta \in [0, 2\pi)\].

Then \( \tilde{g}_0 \in H^\frac{1}{2} \) and

\[\text{deg}(\tilde{g}_0) = 0\].

Therefore by Theorem 1 in [3] there exists a function \( \varphi_0 \in H^\frac{1}{2}(\partial D^2) \) such that

\[\tilde{g}_0 = e^{i\varphi_0}\].

Now we claim that

\[\partial_\theta \varphi_0 = -ie^{-i\varphi_0} \partial_\theta e^{i\varphi_0}. \quad \text{(II.43)}\]

To see this let \((\varphi_n)_{n \in \mathbb{N}}\) be a sequence in \( C^\infty(\partial D^2) \) such that

\[\varphi_n \to \varphi_0 \text{ in } H^\frac{1}{2}(\partial D^2)\].

Then for any \( n \in \mathbb{N} \)

\[\partial_\theta \varphi_n = -ie^{-i\varphi_n} \partial_\theta e^{i\varphi_n} \].
By Lemma 14 there holds

\[ e^{i\varphi_n} \to e^{i\varphi_0} \text{ in } H^{\frac{1}{2}}(\partial D^2). \]

Therefore

\[ \partial_\theta e^{i\varphi_n} \to \partial_\theta e^{i\varphi_0} \text{ in } H^{-\frac{1}{2}}(\partial D^2) \]

and so

\[ e^{i\varphi_n} \partial_\theta e^{i\varphi_n} \to e^{-i\varphi_0} \partial_\theta e^{i\varphi_0} \text{ in } D'(\partial D^2). \]

On the other hand

\[ \partial_\varphi_n \to \partial_\varphi_0 \text{ in } H^{-\frac{1}{2}}(\partial D^2), \]

therefore (II.43) follows.

Now we compute

\[ e^{-i\varphi_0} \partial_\theta e^{i\varphi_0} = \tilde{g}_0^{-1} \partial_\theta \tilde{g}_0 = g_0^{-1} e^{-i\theta \partial_\theta} \partial_\theta (g_0 e^{i\theta \partial_\theta}) = g_0^{-1} \partial_\theta g_0 + id. \]

As

\[ e^{i\varphi_0} \partial_\theta e^{i\varphi_0} \in H^{-\frac{1}{2}}(\partial D^2) \]

by (II.43) and clearly \( id \in H^{-\frac{1}{2}}(\partial D^2) \) we conclude that

\[ -g_0^{-1} \partial_\theta g_0 \in H^{-\frac{1}{2}}(\partial D^2). \]

Let \( \delta > 0 \) (to be determined later) and let \( h \in C^\infty(\partial D^2, S^1) \) such that

\[ \int_{\partial D^2} h = -i \int_{\partial D^2} g_0^{-1} \partial_\theta g_0 \]

and

\[ \| -ig_0^{-1} \partial_\theta g_0 - h \|_{H^{-\frac{1}{2}}(\partial D^2)} < \delta. \]

Let \( a_1, a_2, a_3 \) be zero-mean solutions of

\[ \begin{cases} \Delta a_1 = 2\pi \sum_{i=1}^Q d_i \delta_{p_i} - \sum_{i=1}^Q d_i & \text{in } \overline{D^2} \\ \partial_{\nu} a_1 = 0 & \text{on } \partial D^2 \end{cases} \]  

(II.44)

\[ \begin{cases} \Delta a_2 = \sum_{i=1}^Q d_i & \text{in } D^2 \\ \partial_{\nu} a_2 = h & \text{on } \partial D^2 \end{cases} \]

\[ \begin{cases} \Delta a_3 = 0 & \text{in } D^2 \\ \partial_{\nu} a_3 = -ig_0^{-1} \partial_\theta g_0 - h & \text{on } \partial D^2. \end{cases} \]
Then $a = a_1 + a_2 + a_3$.

Notice that $a_2$ is smooth, $a_3$ lies in $H^1(D^2)$ with

$$\|a_3\|_{H^1} \leq \delta$$

and only $a_1$ depends on the positions and the degrees of the topological singularities.

For any $k \in \mathbb{N}$ let $a_2^k, a_3^k$ be defined analogously and observe that for any $k \in \mathbb{N}$ $a_2^k = a_2$ and $a_3^k = a_3$.

**Claim 2:**

$$a_1(x) = \sum_{i=1}^{Q} d_i \left( \log|x - p_i| + \log|x - \overline{p_i}| - \frac{1}{2}|x|^2 \right) \quad (\text{II.45})$$

up to an additive constant, with the convention that if $p = 0$

$$\log|x - \overline{p}| \equiv 0.$$

**Proof of Claim 2.** First we observe that by linearity it is enough to check the Claim for $Q = 1$ and $d_1 = 1$. Let $p \in \overline{D^2}$ denote the only singularity of $a_1$. Let $\tilde{a}_1$ denote the function defined by the right hand side of (II.45).

If $p = 0$ the Claim is clear. If $p \in \partial D^2$ then $\overline{p} = p$ and

$$\tilde{a}_1(x) = 2 \log|x - p| - \frac{1}{2}.$$

By Computation (II.34) for any $\phi \in C^\infty(D^2)$ there holds

$$\int_{D^2} \nabla \phi \cdot \nabla \tilde{a}_1 = \int_{D^2} \phi - 2\pi \phi(p),$$

then $\tilde{a}_1$ is a solution of (II.44) and thus it differs from $a_1$ at most by an additive constant.

Let’s consider the case where $p \in D^2$ and $p \neq 0$. It is clear that $\tilde{a}_1$ satisfies

$$\Delta \tilde{a}_1 = 2\pi \delta_p - 1 \text{ in } D^2.$$  

We still need to check that $\tilde{a}_1$ also satisfies the Neumann boundary condition satisfied by $a_1$.

Let

$$\tau : \mathbb{C} \setminus \{1\} \to \mathbb{C}, \quad z \mapsto \frac{1 + z}{1 - z}.$$  

Observe that $\tau$ restricts to a biholomorphic map from a neighbourhood of $\overline{D^2} \setminus \{1\}$ to a neighbourhood of $\overline{\mathbb{H}}$, whose inverse is given by

$$\tau^{-1} : \mathbb{C} \setminus \{-1\} \to \mathbb{C}, \quad w \mapsto \frac{w - 1}{w + 1}.$$  

Notice that for any $p \in \mathbb{C} \setminus \{-1\}$

$$\overline{p}^{-1} = \tau^{-1}(\overline{\tau(p)}).$$

Now set

$$F_p : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \log|z - \tau(p)| + \log|z + \overline{\tau(p)}|.$$
For any $z \in \mathbb{C} \setminus \{1\}$

$$F_p \circ \tau(z) = \log|1 + z - \tau(p)(1 - z)| + \log|1 + z + \tau(p)(1 - z)| - 2 \log|1 - z|$$

$$= \log \left| z + \frac{1 - \tau(p)}{1 + \tau(p)} \right| + \log|1 + \tau(p)| + \log \left| z + \frac{1 + \tau(p)}{1 - \tau(p)} \right| + \log|1 - \tau(p)| - 2 \log|1 - z|$$

$$= \tilde{a}_1(z) - \frac{1}{2}|z|^2 + \log|1 + \tau(p)| + \log|1 - \tau(p)| - 2 \log|1 - z|.$$  

Recall that

$$\partial_{\nu} \log|1 - z| = \frac{1}{2} \text{ on } \partial D^2 \setminus \{1\},$$

as shown in (II.34). Moreover

$$\partial_{\nu}(F_p \circ \tau)(z) = DF_p(\tau(z))\partial_{\nu}\tau(z) = 0 \text{ on } \partial D^2 \setminus \{1\}$$

since $\partial_{\nu}\tau(z)$ is orthogonal to the imaginary axis, and thus $DF_p$ vanishes in that direction. Therefore

$$\partial_{\nu} \left( \tilde{a}_1(z) - \frac{1}{2}|z|^2 \right) = \partial_{\nu}(F_p \circ \tau)(z) + 2\partial_{\nu}\log|1 - x| = 1$$

on $\partial D^2 \setminus \{1\}$.

Now

$$\partial_{\nu} \frac{1}{2}|x|^2 = 1 \text{ on } \partial D^2,$$

therefore

$$\partial_{\nu}\tilde{a}_1 = 0 \text{ on } \partial D^2.$$  

\[\square\]

Claim 2 implies that

$$a_1(x) = \sum_{i=1}^{Q} d_i \left( \log|x - p_i| - \int_{D^2} \log|y - p_i| dy \right)$$

$$+ \sum_{i=1}^{Q} d_i \left( \log|x - \bar{p}_i| - \int_{D^2} \log|y - \bar{p}_i| dy \right) + \left( \frac{1}{2}|x|^2 - \frac{\pi}{4} \right) \sum_{i=1}^{Q} d_i.$$ 

Observe that the analogous result holds for $a_1^k$ for any $k \in \mathbb{N}$.

**Claim 3**: There exists a constant $C$ such that for any $k \in \mathbb{N}$

$$f(a_k)|\nabla a_k|^2 \leq C \left( e^{2a_3} + |\nabla(a_2 + a_3)|^2 \right)$$  

(II.46)

**Proof of Claim 3.** Let $k \in \mathbb{N}$. To simplify the notation, in the proof of Claim 3 we will set $p_1 = p_k^i$. Notice first that

$$f(a_k)|\nabla a_k|^2 \leq 2 \left( f(a_k)|\nabla a_k^1|^2 + |\nabla(a_k^2 + a_k^3)|^2 \right),$$
therefore it is enough to show that there exists a constant $C$, independent from $k$, such that
\[ f(a_k) |\nabla a_k|^2 \leq Ce^{2a_3}. \]

The key step will consist in proving the following estimate: for any $x \in D^2$
\[ \prod_{j=1}^{Q} \left( |x - p_j^{-1}|^{2d_j} \exp \left( - \int_{\partial D^2} \log |y - p_j^{-1}|^{2d_j} dy \right) \right) \leq C \min \left( 1, |x - p_l^{-1}|^{2d_l} \right) \]  
(II.47)
for some constant $C$ depending only on $Q$ and the degrees $d_1, \ldots, d_Q$.

If $p_l^{-1} \in B_2(0)$,
\[ \left| \int_{\partial D^2} \log |y - p_l^{-1}| dy \right| \leq \frac{1}{2\pi} \int_{B_l(0)} |\log |y|| dy \]
and therefore
\[ |x - p_l^{-1}|^{2d_l} \exp \left( - \int_{\partial D^2} \log |y - p_l^{-1}|^{2d_l} dy \right) \leq C \]  
(II.48)
for some constant $C$ independent from $p_l$, for any $x \in D^2$.

On the other hand if $p_l^{-1} \notin B_2(0)$
\[ |x - p_l^{-1}|^{2d_l} \geq 1 \]
and
\[ \left| \log |x - p_l^{-1}|^{2d_l} - \int_{\partial D^2} \log |y - p_l^{-1}|^{2d_l} dy \right| \leq 4 |d_l| \sup_{y \in D^2} \frac{1}{|y - p_l^{-1}|} \leq 4 |d_l| \]
for any $x \in D^2$. Thus in this case
\[ \exp \left( \log |x - p_l^{-1}|^{2d_l} - \int_{\partial D^2} \log |y - p_l^{-1}|^{2d_l} dy \right) \leq e^{4|d_l|} \]  
(II.49)
for any $x \in D^2$.

Combining (II.48) and (II.49) we obtain (II.47).

We also have for any $i \in \{1, \ldots, Q\}$, for any $x \in D^2$
\[ \prod_{j=1}^{Q} \left( |x - p_j|^{2d_j} \exp \left( - \int_{\partial D^2} \log |y - p_j|^{2d_j} dy \right) \right) \leq C \min \left( 1, |x - p_i|^{2d_i} \right) \]
for some constant $C$ depending only on $Q$ and the degrees $d_1, \ldots, d_Q$.

Now for any $x \in D^2$
\[ \nabla a_k^1(x) = \sum_{i=1}^{Q} d_i \left( \frac{x - p_i}{|x - p_i|^2} + \frac{x - p_l^{-1}}{|x - p_l^{-1}|^2} - x \right), \]
therefore we conclude that for any $x \in D^2$
\[ e^{2a_1} |\nabla a_k|^2 = \prod_{i=1}^{Q} e^{d_i|x|^2} |x - p_i|^{2d_i} |x - p_l^{-1}|^{2d_l} |\nabla a_k|^2 \leq C \]
for some constant $C$ depending only on $Q$ and the degrees $d_1, \ldots, d_Q$.

Then
\[ f(a_k) |\nabla a_k|^2 \leq e^{2a_k} |\nabla a_k|^2 \leq C \|e^{2a_2}\|_{L^\infty} e^{2a_3}. \]
We notice that the right hand side of \((\text{II.46})\) is integrable if \(\delta\) is chosen to be sufficiently small. In fact \(a_2\) and \(a_3\) lie in \(H^1(D^2)\), moreover since \(a_3 \in H^1(D^2) \subset BMO(D^2)\), by the John-Nirenberg Theorem (see Corollary 3.1.7 in [8])
\[
\int_{D^2} e^{2a_3(x)} \, dx < \infty
\]
if \(\delta\) is chosen sufficiently small (and thus the \(BMO\)-norm of \(a_3\) is sufficiently small).

**Claim 4:**
\[
f(a_k)|\nabla a_k|^2 \to f(a)|\nabla a|^2 \text{ a.e.}
\]

**Proof of Claim 4.** It is enough to check that
\[
\log |x - p_1^k| + \log |x - \overline{p}_1^{k-1}| - \int_{D^2} \left( \log |y - p_1^k| + \log |y - \overline{p}_1^{k-1}| \right) \, dy \to \log |x - \overline{p}| - \log |x - \overline{p}_1^{k-1}| \text{ a.e.}
\]
\((\text{II.50})\)

and
\[
\frac{x - p_1^k}{|x - p_1^k|^2} + \frac{x - \overline{p}_1^{k-1}}{|x - \overline{p}_1^{k-1}|^2} \to \frac{x - p_1}{|x - p_1|^2} + \frac{x - \overline{p}_1}{|x - \overline{p}_1|^2} \text{ a.e.}
\]
\((\text{II.51})\)

with the convention that if \(p = 0\) then
\[
\log |x - \overline{p}| = 0 \quad \text{and} \quad \frac{x - \overline{p}_1^{k-1}}{|x - \overline{p}_1^{k-1}|^2} \equiv 0.
\]

When \(p_1 \neq 0\) both \((\text{II.50})\) and \((\text{II.51})\) are clear.
When \(p_1 = 0\) then we can assume without loss of generality that \(p_1^k \neq 0\) for any \(k \in \mathbb{N}\). In order to show \((\text{II.50})\) in this case we have to check that
\[
\log |x - \overline{p}_1^{k-1}| - \int_{D^2} \log |y - \overline{p}_1^{k-1}| \, dy \to 0 \text{ a.e.}
\]
In fact
\[
\int_{D^2} \left( \log |x - \overline{p}_1^{k-1}| - \log |y - \overline{p}_1^{k-1}| \right) \, dy \leq 2 \sup_{z \in D^2} \frac{1}{|z - \overline{p}_1^{k-1}|}
\]
and since \(p_1^k \to 0\), the right hand side tends to zero.
In order to show \((\text{II.51})\) when \(p_1 = 0\) we have to check that
\[
\frac{x - \overline{p}_1^{k-1}}{|x - \overline{p}_1^{k-1}|^2} \to 0 \text{ a.e.,}
\]
but this is clear, since \(p_1^k \to 0\).

\[
\lim_{k \to \infty} \int_{D^2} f(a_k)|\nabla a_k|^2 = \int_{D^2} f(a)|\nabla a|^2.
\]
Remark 2. The proof of Lemma 5 (and in particular Claim 2) shows that when a topological singularity of degree $d$ approaches the boundary, in the limit it becomes a singularity of degree $2d$. This justifies the fact that throughout this paper we require the degree of topological singularities lying on $\partial D^2$ to be even.

Open problem II.2. Does the result of Lemma 5 remains true if we assume that $g_0$ lies in $W^{1,1}(\partial D^2)$? And is the renormalized Dirichlet Energy stable under perturbation of the boundary datum $g_0$ (in $W^{1,1}(\partial D^2)$ or $H^2(\partial D^2)$)?

III Proof of Theorem I.1 and Theorem I.2

In this section we provide a proof of Theorem I.1 and Theorem I.2. Theorem I.2 follows directly from the a priori estimate on the number of topological singularities given by Lemma 6 and Corollary 2. The result of Theorem I.2 is then applied to prove Theorem I.1.

III.1 Proof of Theorem I.2

Lemma 6 (A priori estimate on the number of singularities). Let $g_0 \in W^{1,1}(\partial D^2, S^1)$. Let $Q \in \mathbb{N}$ and for any $i \in \{1, \ldots, Q\}$ let $p_i \in D^2$ and $d_i \in \mathbb{Z}$. Assume that

$$i \int_{\partial D^2} g_0^{-1} \partial g_0 + 2\pi \sum_{i=1}^{Q} d_i = 0. \quad (III.52)$$

Let $a \in W^{1,1}(D^2, \mathbb{R})$ and assume that for any $\phi \in C^\infty(D^2)$

$$\int_{D^2} \nabla a \nabla \phi = -i \int_{\partial D^2} \phi g_0^{-1} \partial g_0 - 2\pi \sum_{i=1}^{Q} d_i \phi(p_i). \quad (III.53)$$

Let

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{e^{2x}}{(1 + e^{2x})^2}. \quad f$$

Then

$$\pi \sum_{i=1}^{Q} d_i \leq \frac{1}{2} \|\partial g_0\|_{L^1(\partial D^2)} + \int_{D^2} f(a)|\nabla a|^2. \quad (III.54)$$

and thus

$$\sum_{i=1}^{Q} |d_i| \leq \frac{3}{2\pi} \|\partial g_0\|_{L^1} + \frac{2}{\pi} \int_{D^2} f(a)|\nabla a|^2. \quad (III.55)$$

Proof. Assume first that $g_0 \in C^\infty(\partial D^2, S^1)$.

For any $t \in \mathbb{R}$ let

$$A_t := \{x \in D^2, a(x) < t\}.$$ 

Claim 1: for a.e. $t \in \mathbb{R}$

$$\int_{a^{-1}(t)} \partial_n a = i \int_{\partial D^2 \cap A_t} g_0^{-1} \partial g_0 + 2\pi \sum_{i=1}^{Q} d_i, \quad (III.56)$$

where $\nu$ is the outer normal vector of the set $A_t$. 

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Proof of Claim 1. Since \( a \in C^\infty(\overline{D^2} \setminus \{p_1, ..., p_Q\}) \), for a.e. \( t \in \mathbb{R} \) the set \( A_t \) is an open subset of \( D^2 \) such that \( \partial A_t \) is piecewise smooth and does not contain any topological singularity.

As \( a \) solves
\[
\Delta a = 2\pi \sum_{i=1}^{Q} d_i \text{ in } D^2,
\]
for any such \( t \) there holds
\[
\int_{\partial A_t} \partial_r a = 2\pi \sum_{i=1 \atop p_i \in A_t}^{Q} d_i. \tag{III.54}
\]

Now
\[
\partial A_t = a^{-1}(t) \cup (\partial D^2 \cap A_t)
\]
and
\[
\partial_r a = -ig_0^{-1} \partial_\theta g_0 \text{ on } \partial D^2.
\]
Moreover for any \( t \in \mathbb{R} \) a point \( p_i \) lies in \( A_t \) if and only if \( d_i > 0 \), since
\[
a(x) = d_i \log|x - p_i| + O(1)
\]
in a neighbourhood of \( p_i \) (see Lemma 3). Thus \((III.54)\) implies
\[
\int_{a^{-1}(t)} \partial_r a = i \int_{\partial D^2 \cap A_t} g_0^{-1} \partial_\theta g_0 + 2\pi \sum_{i=1 \atop d_i > 0}^{Q} d_i.
\]

Since the derivative of \( a \) vanishes along \( a^{-1}(t) \), on \( a^{-1}(t) \) there holds
\[
\partial_r a = |\nabla a|.
\]

Therefore Claim 1 implies that for a.e. \( t \in \mathbb{R} \)
\[
2\pi \sum_{i=1 \atop d_i > 0}^{Q} d_i \leq \| \partial_\theta g_0 \|_{L^1(\partial D^2)} + \int_{a^{-1}(t)} \partial_r a. \tag{III.55}
\]

Now since \( a \in W^{1,1} \), by Theorem 11 in [9] there exists a representative of \( a \) for which the co-area formula holds. For such a representative we have
\[
\int_{D^2} f(a)|\nabla a|^2 dx = \int_{\mathbb{R}} \left( \int_{a^{-1}(t)} f(a(x))|\nabla a(x)| d\mathcal{H}^1(x) \right) dt \tag{III.56}
\]
\[
= \int_{\mathbb{R}} f(t) \left( \int_{a^{-1}(t)} \partial_r a(x) d\mathcal{H}^1(x) \right) dt.
\]

Observe that
\[
\int_{\mathbb{R}} f(t) dt = \int_{\mathbb{R}} \frac{e^{2t}}{(1 + e^{2t})^2} dt = \frac{1}{2} \int_{0}^{\infty} \frac{1}{(1 + x)^2} dx = \frac{1}{2}
\]
Therefore multiplying both sides of (III.55) by \( f(t) \) and integrating on \( \mathbb{R} \) we obtain
\[
\pi \sum_{i=1}^{Q} d_i \leq \frac{1}{2} \| \partial_\theta g_0 \|_{L^1(\partial D^2)} + \int_{\mathbb{R}} f(t) \left( \int_{a^{-1}(t)} \partial_\nu a \right) dt.
\]
Thus by (III.56)
\[
\pi \sum_{i=1}^{Q} d_i \leq \frac{1}{2} \| \partial_\theta g_0 \|_{L^1(\partial D^2)} + \int_{D^2} f(a)|\nabla a|^2 \, dx
\]
and combining (III.52) and (III.56) we obtain
\[
\sum_{i=1}^{Q} |d_i| = 2 \sum_{i=1}^{Q} d_i - \sum_{i=1}^{Q} d_i \leq \frac{2}{\pi} \left( \frac{1}{2} \| \partial_\theta g_0 \|_{L^1(\partial D^2)} + \int_{D^2} f(a)|\nabla a|^2 \, dx \right) + \frac{1}{2\pi} \| \partial_\theta g_0 \|_{L^1}
= \frac{3}{2\pi} \| \partial_\theta g_0 \|_{L^1} + \frac{2}{\pi} \int_{D^2} f(a)|\nabla a|^2.
\]
This concludes the proof under the assumption that \( g_0 \in C^\infty(\partial D^2, S^1) \).

Next consider the case where \( g_0 \) is a generic element of \( W^{1,1}(\partial D^2, S^1) \). Let
\[
R := \sup_{i \in \{1, \ldots, Q\}} |p_i| < 1.
\]
Since \( a \) is smooth in \( D^2 \setminus \{p_1, \ldots, p_Q\} \), arguing as above we see that for any \( r \in (R, 1) \), for any \( t \in \mathbb{R} \)
\[
\int_{a^{-1}(t) \cap D^2} |\nabla a| = -\int_{a^{-1}(t) \cap D^2} \partial_\nu a = \int_{\partial D^2 \cap A_t} \partial_\nu a + 2\pi \sum_{i=1}^{Q} d_i.
\] (III.57)
The following Claim implies that taking the limit \( r \to 1^- \) in (III.57) we recover estimate (III.55), therefore we can conclude as in the previous case.

Claim 2:
\[
\lim_{r \to 1^-} \left| \int_{\partial D^2 \cap A_t} \partial_\nu a \right| \leq \| \partial_\theta g \|_{L^1(\partial D^2)}.
\]
Proof of Claim 2. For any \( r \in (R, 1) \) let
\[
t_r : \partial D^2 \to \partial D^2_r, \quad x \mapsto rx.
\]
To prove the Claim it is enough to show that
\[
\frac{1}{r} \partial_\nu a|_{\partial D^2_r} \circ t_r \to -ig_0^{-1}\partial_\theta g_0 \text{ in } L^1(\partial D^2)
\]
as \( r \to 1^- \) (where \( \partial_\nu \) denotes the outer normal derivative on \( \partial D^2_r \)). To see this let’s write
\[
a = \bar{a} + \Phi
\]
as in Lemma 3 Then
\[
\partial_\nu \Phi|_{\partial D^2_r} \circ t_r \to \partial_\nu \Phi|_{\partial D^2} \text{ in } L^1(\partial D^2)
\]
as \( r \to 1^- \). Moreover since \( \tilde{a} \) is harmonic we have, using polar coordinates,

\[
\partial_r \tilde{a}(r, \theta) = \tilde{a}|_{\partial D^2} \ast \partial_r P_r(\theta) = \frac{1}{r} \tilde{a}|_{\partial D^2} \ast H \partial_\theta P_r(\theta) = \frac{1}{r} H (\partial_\theta \tilde{a}|_{\partial D^2}) \ast P_r(\theta).
\]

where \( P_r \) denotes the Poisson kernel and \( H \) the Hilbert transform, and we made use of the identity

\[
H \partial_\theta P_r(\theta) = r \partial_r P_r(\theta).
\]

Next we claim that

\[
H (\partial_\theta \tilde{a}|_{\partial D^2}) = -i g_0^{-1} \partial_\theta g_0 - \partial_r \Phi|_{\partial D^2}. \tag{III.58}
\]

To see this let \( \phi \in C^\infty(\partial D^2, \mathbb{R}) \) and denote by \( \tilde{\phi} \) its harmonic extension in \( D^2 \). Then

\[
\langle H \partial_\theta \tilde{a}, \phi \rangle = \int_{\partial D^2} \tilde{a} H \partial_\theta \phi = \int_{\partial D^2} \tilde{a} \partial_r \phi = \int_{D^2} \nabla \tilde{a} \nabla \tilde{\phi} = -\int_{\partial D^2} \phi (i g_0^{-1} \partial_\theta g_0 + \partial_r \Phi|_{\partial D^2}).
\]

In the last step we made use of assumption (III.53) and Lemma 3. As \( \phi \) was arbitrary we conclude that (III.58) holds true.

Then in particular

\[
\partial_\nu \tilde{a}|_{\partial D^2} \circ \iota_r = -\frac{1}{r} (i g_0 \partial_\theta g_0 + \partial_r \Phi) \ast P_r \to -i g_0 \partial_\theta g_0 - \partial_r \Phi \text{ in } L^1(\partial D^2)
\]

as \( r \to 1^- \), since \( (P_r)_{r \in (0,1)} \) is a family of approximated identities.

From the previous result and Lemma 5 we deduce that the same estimate holds if we allow singular points to lie on \( \partial D^2 \) provided that \( g_0 \in W^{1,1} \cap H^{1/2}(\partial D^2) \):

**Corollary 2.** Let \( g_0 \in W^{1,1} \cap H^{1/2}(\partial D^2) \). Let \( Q \in \mathbb{N} \) and for any \( i \in \{1, \ldots, Q\} \) let \( p_i \in \overline{D^2} \) and \( d_i \in \mathbb{Z} \). Assume that

\[
i \int_{\partial D^2} g_0^{-1} \partial_\theta g_0 + 2 \pi \sum_{p_i \in D^2} d_i + \pi \sum_{p_i \in \partial D^2} d_i = 0.
\]

For any \( i \in \{1, \ldots, Q\} \) assume that whenever \( p_i \in \partial D^2 \) \( d_i \) is even.

Let \( a \in W^{1,1}(D^2, \mathbb{R}) \) and assume that for any \( \phi \in C^\infty(D^2) \)

\[
\int_{D^2} \nabla a \nabla \phi = -i \int_{S^1} \phi g_0^{-1} \partial_\theta g_0 - 2 \pi \sum_{p_i \in D^2} d_i \phi(p_i) - \pi \sum_{p_i \in \partial D^2} d_i \phi(p_i).
\]

Then

\[
\pi \sum_{\{d_i \neq 0\} p_i \in D^2} d_i + \frac{1}{2} \pi \sum_{\{d_i \neq 0\} p_i \in \partial D^2} d_i \leq \frac{1}{2} \| \partial_\theta g_0 \|_{L^1(\partial D^2)} + \int_{D^2} f(a) |\nabla a|^2 \, dx \tag{III.59}
\]

and thus

\[
\sum_{i=1}^Q |d_i| \leq \frac{3}{\pi} \| \partial_\theta g_0 \|_{L^1} + \frac{4}{\pi} \int_{D^2} f(a) |\nabla a|^2 \tag{III.60}
\]
Proof. Assume that $p_1, ..., p_Q$ lie on $\partial D^2$ while $p_{Q+1}, ..., p_Q$ lie on $D^2$. For any $i \in \{1, ..., Q\}$ let $\hat{d}_i = \frac{d_i}{2}$. Let $\varepsilon > 0$. According to Lemma 5 for any $i \in \{1, ..., Q\}$ we can choose a point $p_i^\ast \in D^2$ such that the corresponding function $a_i$ (with degrees $\hat{d}_1, ..., \hat{d}_{Q'}, \hat{d}_{Q'+1}, ..., \hat{d}_Q$ and same outer normal derivative as $a$) satisfies

$$
\left| \int_{D^2} f(a_i) |\nabla a_i|^2 - \int_{D^2} f(a) |\nabla a|^2 \right| < \varepsilon.
$$

Now since all all the topological singularities of $a_i$ lie in $D^2$, Lemma 6 implies

$$
\pi \sum_{d_i > 0 \atop p_i \in D^2} d_i + \frac{1}{2} \pi \sum_{d_i > 0 \atop p_i \in \partial D^2} d_i = \pi \sum_{i=1}^Q d_i + \pi \sum_{i=Q'+1}^Q \hat{d}_i
$$

$$
\leq \frac{1}{2} ||\partial_\theta g_0||_{L^1(\partial D^2)} + \int_{D^2} f(a) |\nabla a_i|^2 \, dx
$$

$$
\leq \frac{1}{2} ||\partial_\theta g_0||_{L^1(\partial D^2)} + \int_{D^2} f(a) |\nabla a|^2 \, dx + \varepsilon.
$$

Letting $\varepsilon$ tend to 0 we obtain \text{(III.59)}, from which \text{(III.60)} can be deduced as in the proof of Lemma 6. \hfill \Box

**Open problem III.3.** Although we assume $g_0$ to lie in $H^\frac{1}{2}(\partial D^2)$ in Corollary 3, the $H^\frac{1}{2}$-norm of $g_0$ does not appear on the right hand side of estimate \text{(III.59)}. It is natural to wonder if the result remains true if we only assume $g_0$ to lie in $W^{1,1}(\partial D^2)$, or if we could substitute the $W^{1,1}$-norm with the $H^\frac{1}{2}$-norm of $g_0$ in \text{(III.59)} (assuming only $g_0 \in H^\frac{1}{2}(\partial D^2)$).

From Lemma 6 we deduce also the following estimate.

**Lemma 7.** Let $a$ be as in Lemma 6 or as in Corollary 3. There exists a constant $C$ independent from such a that

$$
||\nabla a||_{L^2(\partial D^2)} \leq C \left( ||g_0||_{W^{1,1}(\partial D^2)} + ||\nabla u_\delta||_{L^2(D^2)} \right)
$$

\text{(III.61)}

**Proof.** Let $X \in L^{2,1}(D^2)$ be a vector field. Then by Lemma 11 there exist functions $\eta \in W^{1,(2,1)}(D^2)$, $\xi \in W_0^{1,(2,1)}(D^2)$ such that

$$
||\eta||_{W^{1,(2,1)}(D^2)} \leq C ||X||_{L^{2,1}(D^2)}
$$

and

$$
X = \nabla \eta + \nabla ^\perp \xi.
$$

Recall that $W^{1,(2,1)}(D^2) \subset C^0(\overline{D^2})$ and the embedding is continuous. Then there holds

$$
\int_{D^2} X \nabla a = \int_{D^2} \nabla \eta \nabla a + \int_{D^2} \nabla ^\perp \xi \nabla a = \int_{D^2} \nabla \eta \nabla a
$$

$$
= -i \int_{\partial D^2} \eta g_0^{-1} \partial_\theta g_0 - 2\pi \sum_{i=1}^Q d_i \eta(p_i) - \pi \sum_{i=1}^Q d_i \eta(p_i).
$$

Here we used the fact that since $\xi$ has vanishing trace on $\partial D^2$, integrating by parts we get

$$
\int_{D^2} \nabla ^\perp \xi \nabla a = 0.
$$
Moreover we know from Lemma 5 (or Corollary 2) that
\[
\sum_{i=1}^{Q} |d_i| \leq C \left( \int_{D^2} f(a) |\nabla a|^2 + \|g_0\|_{W^{1,1}(\partial D^2)} \right)
\]
Therefore there holds
\[
\int_{D^2} X \nabla a \leq C \left( \|g_0^{-1} \partial_0 g_0\|_{L^1(\partial D^2)} + \int_{D^2} f(a) |\nabla a|^2 dx + \|g_0\|_{W^{1,1}(\partial D^2)} \right) \|\eta\|_{L^\infty(D^2)} \\
\leq C \left( \mathcal{E}(u_\eta) + \|g_0\|_{W^{1,1}(\partial D^2)} \right) \|X\|_{L^{2,1}(D^2)}.
\]
As the above estimate holds true for any $X \in L^{2,1}(D^2)$ the statement follows. \hfill \Box

### III.2 Proof of Theorem I.1

We are now ready to prove Theorem I.1.

**Proof of Theorem I.1.** a) First we observe that by condition (I.20) the sequence $(b_{g_k})_{k \in \mathbb{N}}$ is bounded in $W^{1,2}(D^2)$. Therefore there exists a function $b \in W^{1,2}(D^2)$ such that
\[
b_{g_k} \rightharpoonup b \quad \text{weakly in} \quad W^{1,2}(D^2) \quad \text{and} \quad \text{a.e.}
\]
up to a subsequence.

**Claim 1:** There is a subsequence of $(g_k)_{k \in \mathbb{N}}$, say indexed by $\Lambda \subset \mathbb{N}$, a function $g \in W^{1,1(\mathbb{R}, 1)} \cap W^{1,\varrho}(D^2, \mathbb{S}^1)$, a zero average function $a \in W^{1,1(\mathbb{R}, 1)} \cap W^{1,1}(D^2, \mathbb{R})$ and $b \in W^{2,1}(D^2, \mathbb{R})$ such that
\[
a_{g_k} \rightarrow a \quad \text{a.e. and} \quad \nabla a_{g_k} \rightharpoonup \nabla a \quad \text{weakly-* in} \quad L^{2,\infty}_{\text{loc}}(D^2),
\]
\[
g_k \rightarrow g \quad \text{a.e. and} \quad \nabla g_k \rightharpoonup \nabla g \quad \text{weakly-* in} \quad L^{2,\infty}_{\text{loc}}(D^2)
\]
along $\Lambda$,
\[
-ig^{-1} \nabla g = \nabla^2 a + \nabla b
\]
and $g$ has isolated topological singularities in $D^2$.

**Proof of Claim 1.** First observe that by Lemma 13 for any $n \in \mathbb{N}$ there exists $r_n \in (1 - \frac{1}{n}, 1)$ and a subsequence indexed by $\Lambda_n \subset \mathbb{N}$ such that for any $k \in \Lambda_n$ $g_k$ has no topological singularities on $\partial D^2_{r_n}$ and
\[
\sup_{k \in \Lambda_n} \int_{\partial D^2_{r_n}} |ig^{-1} \nabla g_k + \nabla b_{g_k}| < \infty.
\]
Thus by Lemma 4 (applied to the sequence of functions $(e^{ib_{g_k}} g_k)_{k \in \Lambda_n}$) the number of topological singularities of $a_k$ in $D_{r_n}$ (counted with multiplicities) is uniformly bounded for all $k \in \Lambda_n$. Now let $n \in \mathbb{N}$. By Lemma 1 (applied to $D_{r_n}$) the sequence $(\nabla a_{g_k})_{k \in \Lambda_n}$ is bounded in $L^{2,\infty}(D^2_{r_n})$. By Poincaré Lemma and Banach-Alaoglu Theorem 4 the sequence
\[
\left(a_{g_k} - \int_{D_{r_n}} a_{g_k}\right)_{k \in \Lambda_n}
\]
6Since $L^{2,\infty}(D^2_{r_n}) = (L^{2,1}(D^2_{r_n}))^*$ and $L^{2,1}(D^2_{r_n})$ is separable, Banach-Alaoglu holds for $L^{2,\infty}(D^2_{r_n})$. 28
has a subsequence converging weakly in $W^{1,(2,\infty)}(D_{r_n})$ to a function $a_n \in W^{1,(2,\infty)}(D_{r_n})$.
Iterating this argument for any $n \in \mathbb{N}$ and extracting a diagonal subsequence we find for any $n \in \mathbb{N}$ a function $a_n \in W^{1,(2,\infty)}(D_{r_n}^2)$ such that for $n, m \in \mathbb{N}$, $n \leq m$ the function $a_n - a_m$ is constant in $D_{r_n} \cap D_{r_m}$.
All these functions can be glued together (subtracting a constant whenever necessary) to obtain a function $a \in W^{1,(2,\infty)}_{\text{loc}}(D^2)$ (uniquely defined up to an additive constant) such that
\[
\nabla a_{g_k} \to \nabla a \text{ weakly-}^* \text{ in } L^{2,\infty}_{\text{loc}}(D^2)
\]
up to a subsequence. Moreover since the sequence $(a_{g_k})_{k \in \mathbb{N}}$ is bounded in $W^{1,p'}(D^2)$ (with $p' = \min\{2, p\}$), taking a further subsequence if necessary we may assume that the convergence also takes place weakly in $W^{1,p'}(D^2)$ a.e.. Then $a \in W^{1,1}(D^2)$ and the weak $W^{1,p'}$-convergence determines the additive constant in the definition of $a$. In particular $a$ has average zero.
For any $n \in \mathbb{N}$ Corollary 1 (applied to $D_{r_n}^2$) implies that the sequence $(g_k)_{k \in \Lambda_n}$ is also bounded in $W^{1,(2,\infty)}(D_{r_n}^2)$, since for any $k \in \Lambda_n$
\[
|\nabla g_k| \leq |\nabla \varphi_k| + \sum_{i=1}^{Q_k} \frac{|d_i^k|}{|z - p_i^k|} \leq |\nabla^a a_{g_k}| + |\nabla b_{g_k}| + \sum_{i=1}^{Q_k} \frac{|d_i^k|}{|z - p_i^k|},
\]
where $\varphi_k$ was defined in Corollary 1 and $\sum_{i=1}^{Q_k} |d_i^k|$ is uniformly bounded for any $k \in \Lambda_n$.
Therefore applying Banach-Alaoglu Theorem and Rellich-Kondrachov Theorem to the sequence $(g_k|_{D_{r_n}^2})_{k \in \Lambda_n}$ for any $n \in \mathbb{N}$ and extracting a diagonal subsequence we find a function $g \in W^{1,(2,\infty)}_{\text{loc}}(D^2, S^1)$ such that
\[
g_k \to g \text{ a.e. and } \nabla g_k \to \nabla g \text{ weakly-}^* \text{ in } L^{2,\infty}_{\text{loc}}(D^2)
\]
up to a subsequence. Moreover since the sequence $(g_k)_{k \in \mathbb{N}}$ is bounded in $W^{1,p}(D^2)$, taking a further subsequence if necessary we may assume that the convergence also takes place weakly in $W^{1,p}(D^2)$. In particular $g \in W^{1,p}(D^2)$.
It follows that
\[
g_k^{-1} \nabla g_k \to g^{-1} \nabla g \text{ in } D'(D^2)
\]
up to a subsequence. Therefore, since for any $k \in \mathbb{N}$
\[
-g_k^{-1} \nabla g_k = \nabla^a a_{g_k} + \nabla b_{g_k},
\]
we conclude that
\[
-g^{-1} \nabla g = \nabla^a a + \nabla b.
\]
We still need to check that $g$ has isolated singularities in $D^2$. To this end let’s fix $n \in \mathbb{N}$. Observe that for any $k \in \Lambda_n$, $a_{g_k}$ satisfies
\[
\Delta a_{g_k} = 2\pi \sum_{i=1}^{Q_k} d_i^k \delta_{p_i^k} \text{ in } D_{r_n}^2
\]
for some integers $Q_k, d^k_1, \ldots, d^k_{Q_k}$ and points $p_1, \ldots, p_{Q_k}$ in $D^2_{r_n}$, where $\sum_{i=1}^{Q_k} |d^k_i|$ is uniformly bounded for any $k \in \Lambda_n$.

Upon passing to a subsequence we may assume that the integers $Q_k$ and $d^k_i$ do not depend on $k$ (therefore we will drop the $k$ in the notation) and that for any $i \in \{1, \ldots, Q\}$

$$p^k_i \to p_i$$

for some points $p_1, \ldots, p_Q \in \overline{D^2_{r_n}}$. Then for any $\varphi \in C^\infty_c(D^2_{r_n})$

$$\int_{D^2_{r_n}} \nabla \varphi \nabla a = \lim_{k \to \infty} \int_{D^2_{r_n}} \nabla \varphi \nabla a_g = 2\pi \sum_{i=1}^{Q} d_i \varphi(p_i),$$

therefore

$$\text{div}(ig^{-1}\nabla g) = \Delta a = 2\pi \sum_{i=1}^{Q} d_i \delta_{p_i} \text{ in } D^2_{r_n}.$$ 

Thus $g$ has finitely many topological singularities in $D^2_{r_n}$ and since this is true for any $n \in \mathbb{N}$ we conclude that $g$ has isolated topological singularities in $D^2$.

\textbf{Claim 2:} There exists a subsequence of $(g_k)_{k \in \Lambda}$, say indexed by $\tilde{\Lambda}$, such that

$$\mathcal{E}(g) \leq \liminf_{k \to \infty} \mathcal{E}(g_k).$$

\textit{Proof of Claim 2.} Observe that by condition (I.1) there is a subsequence, say indexed by $\tilde{\Lambda} \subset \Lambda$, and a map $u \in W^{1,2}(D^2, S^2)$ such that

$$u_{g_k} \rightharpoonup u \text{ weakly in } W^{1,2}(D^2, S^2) \text{ and a.e. along } \tilde{\Lambda}. \text{ Then}

$$\int_{D^2} |\nabla u|^2 \leq \liminf_{k \to \infty} \int_{D^2} |\nabla u_{g_k}|^2.$$

Since for any $k \in \mathbb{N}$

$$u_{g_k} = \pi^{-1}(g_k e^{a_k}),$$

the pointwise convergence of $(g_k)_{k \in \tilde{\Lambda}}, (a_k)_{k \in \tilde{\Lambda}}$ and $(u_{g_k})_{k \in \tilde{\Lambda}}$ implies that

$$u = \pi^{-1}(g e^{a}).$$

Moreover replacing $\tilde{\Lambda}$ with a subsequence if necessary we have

$$\int_{D^2} |\nabla b|^2 \leq \liminf_{k \to \infty} \int_{D^2} |\nabla b_{g_k}|^2.$$

Therefore

$$\mathcal{E}(g) = \frac{1}{4} \int_{D^2} |\nabla u|^2 + \frac{1}{4} \int_{D^2} |\nabla b|^2 \leq \liminf_{k \to \infty} \mathcal{E}(g_k).$$
b) By Lemma 7 (applied to the case $g_0 \in W^{1,1} \cap H^1(\partial D^2)$) the sequence $(\nabla a_{g_k})_{g_k}$ is bounded in $L^{2,\infty}(D^2)$. By Corollary 1 the sequence $(g_k)_{k \in \mathbb{N}}$ is bounded in $W^{1,2}(D^2)$. Therefore by Banach-Alaoglu Theorem and Rellich-Kondrachov Theorem we can find a zero average function $a \in W^{1,2}(D^2)$, $b \in W^{1,2}(D^2)$, $g \in W^{1,2}(D^2, S^1)$ such that
\[a_k \rightarrow a \text{ a.e. and } \nabla a_k \rightharpoonup \nabla a \text{ weakly-}^* \text{ in } L^{2,\infty}(D^2),\]
\[b_k \rightarrow b \text{ weakly in } W^{1,2}(D^2)\]
along a subsequence and
\[-ig^{-1}\nabla g = \nabla a + \nabla b \text{ in } D^2. \quad (\text{III.62})\]
Moreover following the argument of Claim 2 in part a) we see that
\[\mathcal{E}(u_g) \leq \liminf_{k \rightarrow \infty, k \in \Lambda} \mathcal{E}(u_{g_k})\]
where $\Lambda \subset \mathbb{N}$ is the index set of a further subsequence.

**Claim 1:** $g$ has finitely many topological singularities in $\overline{D^2}$.

**Proof of Claim 1.** For any $k \in \mathbb{N}$ denote $p_{1,k}, ..., p_{Q_k}$, $d_{1,k}, ..., d_{Q_k}$ the topological singularities of $g_k$ and their degrees. By Corollary 1 the number of topological singularities of $g_k$ and their degrees are uniformly bounded for all $k \in \mathbb{N}$. Therefore there exists a subsequence, say indexed by $\Lambda' \subset \Lambda$, $Q \in \mathbb{N}$ points $p_1, ..., p_Q \in \overline{D^2}$ and degrees $d_1, ..., d_Q \in \mathbb{Z}$ such that for any $i \in \{1, ..., Q\}$
\[d^k_i = d_i \quad \forall k \in \Lambda'\]
and
\[\lim_{k \rightarrow \infty, k \in \Lambda'} p^k_i = p_i.\]
Thus for any $\phi \in C^{\infty}(\overline{D^2})$ there holds
\[\int_{D^2} \nabla \phi \nabla a = \lim_{k \rightarrow \infty, k \in \Lambda'} \int_{D^2} \nabla \phi \nabla a_k = -i \int_{\partial D^2} \phi g_0^{-1} \partial_\theta g_0 - 2\pi \sum_{i=1}^{Q} d_i \phi(p_i) - \pi \sum_{i=1}^{Q} d_i \phi(p_i),\]
so $g$ has finitely many topological singularities in $\overline{D^2}$. \qed
IV Proof of Theorem I.3

In this section we prove Theorem I.3. We divide the proof in two steps, corresponding to the two following Lemmas.

Lemma 8. Let $g \in W^{1,1}(D^2, S^1)$ be as in Definition I.1. Assume that $\mathcal{E}(g) < \infty$.

Then the following conditions are equivalent:

1. $g$ is an $S^1$-harmonic map,
2. $b_g \equiv 0$ in $D^2$.
3. $u_g$ is conformal.

Proof. Assume first that $g$ is $S^1$-harmonic. We claim that $b_g \equiv 0$. By assumption

$$\text{div}(g^{-1}\nabla g) = 0.$$ 

Plugging in the decomposition (I.18) we obtain

$$\text{div}(\nabla^\perp a_g + \nabla b_g) = 0.$$ 

Now since

$$\text{div}(\nabla^\perp a_g) = 0,$$

$b_g$ solves

$$\begin{cases} 
\Delta b_g = 0 & \text{in } D^2 \\
b_g = 0 & \text{on } \partial D^2.
\end{cases}$$

Since $b_g \in H^1(D^2)$ we conclude that $b_g \equiv 0$ in $D^2$.

We now show the converse: assume that $b_g \equiv 0$. Then

$$g^{-1}\nabla g = i \nabla^\perp a_g.$$ 

Therefore

$$-i \text{div}(g^{-1}\nabla g) = \text{div}(\nabla^\perp a_g) = 0.$$ 

We conclude that $g$ is an $S^1$-harmonic map.

Next assume that $g$ has isolated topological singularities. We will show that $b_g \equiv 0$ if and only if $u_g$ is conformal.

Assume first that $b_g \equiv 0$. Then

$$g^{-1}\nabla g = i \nabla^\perp a_g.$$ 

Observe that

$$\nabla u_g = D\pi^{-1}(e^{a_g}g)e^{a_g}(\nabla g + g\nabla a_g) \tag{IV.63}$$

Then

$$\nabla u_g = D\pi^{-1}(e^{a_g}g)e^{a_g}(i \nabla^\perp a_g + \nabla a_g).$$
As \( \pi \) is conformal and
\[
\nabla a_g(x) + i \nabla^\perp a_g(x)
\]
defines a conformal map at any point \( x \in D^2 \) away from the singular points of \( a_g \) we conclude that \( u_g \) is conformal away from the topological singularities of \( g \).

Now if we consider separately the preimages of \( S^2 \setminus \{S\} \) and \( S^2 \setminus \{N\} \) (where \( S \) and \( N \) denote the south and the north pole of \( S^2 \) respectively) and look at the composition \( \pi_i \circ u_g \) for the corresponding stereographic projection \( \pi_i \) we notice that the singularities of \( u_g \) in \( D^2 \), corresponding to the topological singularities of \( g \) are removable. We conclude that \( u_g \) is conformal.

Conversely, if we assume that \( u_g \) is conformal computation (IV.63) implies that
\[
-ig^{-1}\nabla g = \nabla^\perp a_g
\]
is conformal a.e., therefore
\[
-ig^{-1}\nabla g = \nabla^\perp a_g
\]
and thus \( b_g \equiv 0 \).

**Lemma 9.** Let \( g \in W^{1,p}(D^2; S^1) \) for some \( p > 1 \) with finite renormalized Dirichlet Energy. Let
\[
f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{e^{2x}}{(1 + e^{2x})^2}.
\]

Then \( g \) is a critical point of the renormalized Dirichlet Energy for smooth variations in the target, that is
\[
\forall \psi \in C_c^\infty(D^2, \mathbb{R}) \quad \frac{d}{dt} \bigg|_{t=0} \mathcal{E}(ge^{it\psi}) = 0 \tag{IV.64}
\]
if and only if \( b_g \equiv 0 \) in \( D^2 \).

**Proof.** Let \( \psi \in C_c^\infty(D^2, \mathbb{R}) \). For any \( t > 0 \) let
\[
g_t = ge^{it\psi}.
\]

Then
\[
\nabla g_t = \nabla g e^{it\psi} + itg e^{it\psi} \nabla \psi.
\]

Therefore
\[
g_t^{-1}\nabla g_t = g^{-1}\nabla g + it\nabla \psi = i\nabla^\perp a_g + i\nabla b_g + it\nabla \psi.
\]

Thus it follows from (II.24) that
\[
\mathcal{E}(g_t) - \mathcal{E}(g) = \int_{D^2} f(a_g) \left( |\nabla g_t|^2 - |\nabla g|^2 \right) + \frac{1}{4} \int_{D^2} \left( |\nabla b_g + t\nabla \psi|^2 - |\nabla b_g|^2 \right) \tag{IV.65}
\]
\[
= 2t \int_{D^2} f(a_g) \left( < -ig^{-1}\nabla g, \nabla \psi > + \frac{1}{4} < \nabla b_g, \nabla \psi > \right)
\]
\[
+ t^2 \int_{D^2} \left( f(a_g)|\nabla \psi|^2 + \frac{1}{4} |\nabla \psi|^2 \right)
\]
and we get
\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}(g_t) = 2 \int_{D^2} f(a_g) < -ig^{-1}\nabla g, \nabla \psi > + \frac{1}{4} < \nabla b_g, \nabla \psi > .
\]
Therefore \( g \) is a critical point of \( \mathcal{E} \) (with respect to the variations introduced in (IV.64)) if and only if
\[
\text{div} \left( -i f(a_g) g^{-1} \nabla g + \frac{1}{4} \nabla b_g \right) = 0.
\]
(IV.66)

Plugging in the Hodge decomposition (I.18) in (IV.66) we obtain
\[
\text{div} \left( f(a_g)(\nabla^\perp a_g + \nabla b_g) + \frac{1}{4} \nabla b_g \right) = 0.
\]

Recall that
\[
\text{div} \left( f(a_g) \nabla^\perp a_g \right) = 0
\]
as we saw in (II.29). Therefore we can rewrite (IV.66) as
\[
\text{div} \left( f(a_g) \nabla b_g + \frac{1}{4} \nabla b_g \right) = 0.
\]
(IV.67)

We claim that the only solution in \( W^{1,2}_0(D^2, \mathbb{R}) \) of (IV.67) is \( b_g \equiv 0 \).

As the energy \( \mathcal{E} \) is strictly convex, it has a single critical point, which has to be \( b_g \equiv 0 \).
We conclude that \( b_g \) is a solution of (IV.66) if and only if \( b_g \equiv 0 \).

\[\square\]

**Remark 3.** One could also consider variations of the type

\[
g_t = \frac{g + t \phi}{|g + t \phi|}
\]

where \( \phi \in C^\infty_c(D^2, \mathbb{R}^2) \). Nevertheless there are \( S^1 \)-harmonic maps \( g \) with finite energy \( \mathcal{E} \) and maps \( \phi \in C^\infty_c(D^2, \mathbb{R}^2) \), for which the energy \( \mathcal{E}(g_t) \) is not finite for any \( t \neq 0 \).

For instance let \( g(re^{i\theta}) = e^{i\theta} \), \( e_1 \) the first basis vector of \( \mathbb{R}^2 \) and \( \eta \in C^\infty_c((-1, 1)) \) such that \( \eta \equiv 1 \) on \((-1/2, 1/2)\). Set
\[
g_t(re^{i\theta}) = \frac{e^{i\theta} + t \eta(r)e_1}{|e^{i\theta} + t \eta(r)e_1|}.
\]

For \( t \) sufficiently small there holds \( g_t = ge^{i\psi_t} \), where
\[
\psi_t(re^{i\theta}) = -\arctan \left( \frac{t \eta(r) \sin \theta}{1 + t \eta(r) \cos \theta} \right).
\]

Therefore
\[
g_t^{-1} \nabla g_t = g^{-1} \nabla g + i \nabla \psi_t.
\]

Since for our choice of \( g \) we have \( b_g \equiv 0 \), there holds \( b_{g_t} = \psi_t \). But \( \psi_t \notin L^2(D^2) \) for \( t \neq 0 \) (as \( \psi_t \) only depends on \( \theta \) in a neighbourhood of zero), therefore \( \mathcal{E}(g_t) = \infty \) whenever \( t \neq 0 \).

By requiring \( g \) to have isolated topological singularities in \( D^2 \) and considering variations as above with \( \phi \) supported away from the topological singularities of \( g \) one can obtain a result analogous to Lemma [4].
V Applications

V.1 The Lagrangian-Willmore Energy

V.1.1 The Hamiltonian stationary condition and Schoen-Wolfson isolated singularities

Let \( \omega := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \) be the standard symplectic form in \( \mathbb{R}^4 \) compatible with the standard complex structure \( J_0 \) such that \( J_0 \partial x_k = \partial y_k \). We shall denote \( dz_k = dx_k + i dy_k \). An immersion \( G \) from a surface \( \Sigma \) into \( (\mathbb{R}^4, \omega) \) is called Lagrangian if

\[ G^* \omega = 0. \]

This condition is equivalent to the fact that \( J_0 \) realizes an isometry between the tangent plane to the immersion and the normal plane. A short computation gives the existence of a map \( g \) from \( \Sigma \) into \( S^1 \) such that

\[ G^* dz_1 \wedge dz_2 = g \, dvol_G \]

where \( dvol_G \) denotes the volume form associated to the induced metric \( G^* g_{\mathbb{R}^4} \). The map \( g \) is called the Lagrangian angle function.

Consider conformal coordinates for \( G \) with respect to \( G^* g_{\mathbb{R}^4} \), with \( G^* g_{\mathbb{R}^4} = e^{2\lambda} dx^2 \); then a direct computation gives that

\[ i \Delta G = i^{-1} g^{-1} \nabla G \cdot \nabla G \iff \text{div} (g \nabla G) = 0. \]

We also deduce the following expression for the mean curvature vector

\[ \vec{H}_G := 2^{-1} e^{-2\lambda} \Delta G = -2^{-1} e^{-2\lambda} g^{-1} \nabla G \cdot \nabla G. \] (V.1)

The vector-fields preserving the Lagrangian condition infinitesimally in the ambient space \( \mathbb{R}^4 \) are called Hamiltonian vector fields and are of the form

\[ X = J_0 \nabla f = (-\partial y_1 f, \partial x_1 f, -\partial y_2 f, \partial x_2 f), \]

where \( f \) is an arbitrary function. Thus the condition for being a critical point of the area under local perturbations preserving the Lagrangian condition (the so called Hamiltonian stationary condition introduced originally by Oh [16]) is given by

\[ \forall f \in C_0^\infty (\mathbb{R}^4) \quad 0 = \int_\Sigma J_0 \nabla f (G) \cdot \vec{H}_G \, dvol_G = \int_\Sigma <G^* df, i^{-1} g^{-1} d\varphi>_g \, dvol_G. \] (V.2)

Assume that \( G \) is a smooth local immersion. Then, locally, every differential \( d\varphi \) can be written in the form \( d\varphi = G^* df \) for some Hamiltonian \( f \). Therefore the Hamiltonian Stationary Equation is equivalent to the \( S^1 \)-harmonic map equation with respect to the \( G^* g_{\mathbb{R}^4} \) metric on \( \Sigma \):

\[ d^* (g^{-1} d\varphi) = 0, \] (V.3)

where again the Hodge operation \( * \) is the one given by the complex structure on \( \Sigma \) induced by \( G^* g_{\mathbb{R}^4} \). Observe that \( g = e^{i\theta_0} \) is a constant \( S^1 \)-harmonic map if and only if \( G \) is minimal and Lagrangian (i.e. \( \vec{H}_G = 0 \)), which is also equivalent to the fact that \( G \) is calibrated by the form

\[ \Omega := e^{-i\theta_0} dz_1 \wedge dz_2 \] (V.4)

and \( G \) realizes a so called special Lagrangian immersion.
All the above up to (V.3) extends to the general case of a Kähler-Einstein Surface and up to (V.4) to the general case of a Kähler-Einstein Surface with trivial canonical bundle: for which there exists a global nowhere vanishing holomorphic $(2,0)$ form (Calabi-Yau surfaces).

In their pioneering work on the analysis of Hamiltonian Stationary Maps R. Schoen and J. Wolfson aimed at constructing in any integral Lagrangian homology class (or also in the slightly more restrictive constraint of any Hamiltonian isotopy class [15]) of a closed Kähler-Einstein manifold a Lagrangian surface minimizing the area in this class. The main contribution of [20] is the successful implementation of the minimization procedure. They proved that any such a class is realized by a smooth minimal immersion away from isolated singular points and that (V.3) holds. The singularities however can be “worse” than classical branched points which are common for minimal surfaces and which are also present of course. For instance, at such singularities, the Gauss map of the immersion cannot be extended smoothly and it is proved in [20] that these singularities (called “Schoen-Wolfson cones”) correspond to singularities around which the $S^1$-harmonic map has non-zero degree (in fact $+1$ or $-1$ degree) and coincide exactly with the singularities of $\mathfrak{g}$. In [23], Wolfson is giving examples of Lagrangian spherical integer homology classes whose $S^1$-Lagrangian area minimizers are not minimal and hence the associated $S^1$-harmonic maps $\mathfrak{g}$ must have singularities and are only in $W^{1, (2,\infty)}(S^2, S^1)$ (and not in $W^{1,2}(S^2, S^1)$).

V.1.2 The Renormalized Lagrangian Willmore Energy

From (V.1) we deduce that the Willmore Energy for a Lagrangian immersion $G$ into a Kähler-Einstein surface is given by

$$W(G) := \frac{1}{4} \int_{\Sigma} |d\mathfrak{g}|_{g}^{2} \text{dvol}_{G}.$$  \hspace{1cm} (V.5)

This is just $1/2$ of the Dirichlet energy of the Lagrangian angle function $\mathfrak{g}$.

From the previous subsection it is clear that for any Hamiltonian stationary surface in a Calabi-Yau 2-fold which is not special Lagrangian the Willmore energy is infinite

$$W(G) = +\infty$$

and there is an obvious need to renormalize it.

Following the first part of the paper we introduce the Renormalized Lagrangian-Willmore Energy. Let $G$ be a Lagrangian map from a closed surface $\Sigma$ into a Kähler-Einstein surface $N$ realizing a Lipschitz weak immersion (in the sense introduced in [17]) away from point singularities with an underlying smooth conformal structure $h$ and in such a way that $\mathfrak{g}$ is in $W^{1,p}(\Sigma, S^1)$ for some $p > 1$ and has finitely many topological singularities. We call such an immersion a singular immersion. Let $a_{\mathfrak{g}}, b_{\mathfrak{g}}$ and $h_{\mathfrak{g}}$ be such that

$$-i\mathfrak{g}^{-1}d\mathfrak{g} = *da_{\mathfrak{g}} + db_{\mathfrak{g}} + h_{\mathfrak{g}},$$  \hspace{1cm} (V.6)

where $h_{\mathfrak{g}}$ realizes an harmonic one form. Following the first part of paper we introduce the Renormalized Lagrangian-Willmore Energy.

$$\mathcal{W}(G) := \frac{1}{4} \int_{\Sigma} |da_{\mathfrak{g}}|_{h}^{2} + |db_{\mathfrak{g}}|_{h}^{2} + |h_{\mathfrak{g}}|_{h}^{2} \text{dvol}_{h},$$  \hspace{1cm} (V.7)

where $u_{\mathfrak{g}}$ is the “$S^2$ lift” of $\mathfrak{g}$ introduced in Definition [11]. Such a singular Lagrangian immersion being given, we define the singular Lagrangian degree of $G$ to be

$$\text{deg}_{\text{Lag}}(G) = \text{deg}(u_{\mathfrak{g}}).$$

From the previous we deduce the following Proposition.
Proposition V.1. Let $G$ be a singular Lagrangian immersion of $S^2$ into a Kähler-Einstein manifold. The map $G$ is Hamiltonian stationary if and only if $b_g \equiv 0$ and we have then

$$W(G) = 2\pi \deg_{\text{Lag}}(G).$$

We propose the following open problems.

Open problem V.4. Study the sequential weak closure of singular weak Lagrangian immersions under Renormalized Lagrangian-Willmore Energy control in the spirit of [17].

Observe that thanks to inequality (I.23) (or more exactly its counterpart in the Lagrangian immersion framework) the control of the point singularities is guaranteed by the control of the Renormalized Lagrangian-Willmore Energy.

Open problem V.5. Study the minimization of the Renormalized Lagrangian-Willmore Energy among singular weak Lagrangian immersions in Hamiltonian isotopy classes.

V.2 Frame Energy

Let $\Phi$ be an immersion of a torus $\Sigma = T^2$ in $\mathbb{R}^n$ and let $\varepsilon := (\vec{e}_1, \vec{e}_2)$ be an associated tangent frame, that is $\varepsilon(x)$ forms an orthonormal basis of $\Phi^* T_x \Sigma$. The associated frame energy is just the Dirichlet energy of $\varepsilon$:

$$F(\Phi, \varepsilon) := \frac{1}{2} \int_{T^2} |d\vec{e}_1|^2_{g_\Phi} + |d\vec{e}_2|^2_{g_\Phi} \, d\text{vol}_{g_\Phi}. \quad (V.8)$$

It has been originally introduced in [22]. Its minimization in isotopy classes of immersions has been performed in [14] while its link with the Alvarez-Polyakov anomaly has been established in [18]. For a fixed immersion the optimal frame satisfies the Coulomb condition

$$d^*(\vec{e}_1 \cdot d\vec{e}_2) = 0. \quad (V.9)$$

Observe that the passage from one such a frame to another Coulomb frame $(\vec{f}_1, \vec{f}_2)$ is achieved through an $S^1$-harmonic map $g$ given by

$$\vec{e}_1 \cdot d\vec{e}_2 = \vec{f}_1 \cdot df_2 - ig^{-1} \, dg. \quad (V.10)$$

It is clear that for a closed oriented surface $\Sigma$ of genus different from 1 any such a frame has singularities and its Dirichlet energy is infinite

$$F(\Phi, \varepsilon) = +\infty.$$ 

Thus there is an obvious need to renormalize the frame energy.

The renormalization we are proposing corresponds to a “bundle version” of the passage from $E$ to $\mathcal{E}$ in the Introduction. Let $\vec{e}_1'$ be a (possibly singular at isolated points) section of the unit tangent bundle to the immersion $\Phi$ of a simply connected closed oriented smooth surface $\Sigma$. Assume that

$$\vec{e}_2 \cdot d\vec{e}_1' \in L^p(\Sigma)$$

\footnote{We are avoiding harmonic forms in this first presentation.}
for some $p > 1$. Let $a_\varepsilon, b_\varepsilon \in W^{1,p}(\Sigma)$ such that
\[
\begin{aligned}
\vec{e}_2 \cdot d\vec{e}_1 &= *a_\varepsilon + db_\varepsilon \text{ in } \Sigma \\
\int_\Sigma a_\varepsilon &= 0 \\
\int_\Sigma b_\varepsilon &= 0.
\end{aligned}
\tag{V.11}
\]

Let
\[
v_\varepsilon := e^{a_\varepsilon} \vec{e}_1 \text{ on } D^2.
\]

Then $v_\varepsilon$ defines a section of the tangent bundle of $\Sigma$. We remark that the section $v_\varepsilon$ has the following representation in charts: let
\[
\phi : \Omega \to U
\]
be a conformal diffeomorphism from a domain $\Omega \subset \mathbb{C}$ to an open subset $U \subset \Sigma$. For $i \in \{1, 2\}$ let
\[
e_i := \phi^* \vec{e}_i.
\]

Then, in the standard metric of $\Omega$,
\[
|e_1| = e^{-\lambda},
\]
where $\lambda$ denotes the conformal factor of the map $\phi$. Therefore the Hodge decomposition in (V.11) takes the form
\[
e^\lambda e_2 \cdot d(e^\lambda e_1) = e^{2\lambda} e_2 \cdot de_1 = *d(a_\varepsilon - \lambda) + db_\varepsilon.
\]

Here we can see how $e^\lambda e_1$ corresponds heuristically to the function $g$ studied above.

We define the renormalized frame energy of the frame $\varepsilon$ as follows:
\[
\mathcal{F}(\varepsilon) := \int_\Sigma \frac{e^{2a_\varepsilon}}{(1 + e^{2a_\varepsilon})^2} |Dv_\varepsilon|_{g_\varepsilon}^2 + \frac{1}{4} \int_\Sigma |db_\varepsilon|^2_{g_\varepsilon}. \tag{V.12}
\]

Here $D$ denotes the covariant derivative of $\Sigma$.

Notice that the weight in the first integral of (V.12) corresponds to the one generated by the differential of an inverse stereographic projection on the tangent planes.

Notice also that the energy $\mathcal{F}$ depends on the immersion $\tilde{\Phi}$ only through the metric on $\Sigma$ induced by $\tilde{\Phi}$. In particular the energy $\mathcal{F}$ is an intrinsic object.

In the following we assume that the frame $\varepsilon$ satisfies the equation:
\[
\Delta_g a_\varepsilon = 2\pi \sum_{i=1}^Q d_i \delta_{p_i},
\]
where $Q \in \mathbb{N}$ and for any $i \in \{1, \ldots, Q\}$ $d_i \in \mathbb{Z} \setminus \{0\}$, $p_i \in \Sigma$. We call the points $p_1, \ldots, p_Q$ topological singularities of $\varepsilon$.

With the methods presented above one proves the following Lemma.

**Lemma 10.** Let $\Sigma$ and $\varepsilon$ be as above, assume that
\[
\mathcal{F}(\varepsilon) < \infty.
\]

Then the following conditions are equivalent:
1. \( e \) is a critical point of the energy \( \mathcal{E} \), in the sense that
\[
\forall \psi \in C^\infty(\Sigma, \mathbb{R}) \quad \frac{d}{dt} \bigg|_{t=0} \mathcal{E}(e^{it\psi}e) = 0,
\]
2. \( e \) satisfies the Coulomb condition, i.e. \( b_e \equiv 0 \),
3. the section \( v_e \) is holomorphic outside of the topological singularities of \( e \).

If \( \Sigma \) is not simply connected there is a unique harmonic 1-form \( h_e \) such that
\[
\vec{e}_2 \cdot d\vec{e}_1 = *da_e + db_e + h_e
\]
and the renormalized frame energy becomes
\[
\mathcal{F}(e) := \int_\Sigma \frac{e^{2a_e}}{(1 + e^{2a_e})^2} |Dv_e|_{g_\Sigma}^2 + \frac{1}{4} \int_\Sigma \left( |db_e|_{g_\Sigma}^2 + |h_e|_{g_\Sigma}^2 \right). \tag{V.13}
\]
If moreover \( \Sigma \) has a boundary, one could define a renormalized frame energy in the same spirit as (V.13) by means of an \( L^p \)-Hodge decomposition for manifolds with boundary (see Corollary 10.5.1 in [12]).

It would be interesting to study the following question.

**Open problem V.6.** Consider an immersion \( \tilde{\Phi} \) of a an oriented surface with boundary and study the link between the minimal renormalized frame energy among all tangent frames such that the first vector is tangent to the boundary and the Alvarez-Polyakov anomaly associated to this immersion. 

### VI Appendix: some auxiliary results

**Lemma 11.** Let \( p \in (1, \infty) \), \( q \in [1, \infty] \), let \( X \in L^{p,q}(D^2) \). Then there exists \( \eta \in W^{1,(p,q)}(D^2) \), \( \xi \in W_0^{1,(p,q)}(D^2) \) so that
\[
X = \nabla \eta + \nabla^\perp \xi \text{ in } D^2
\]
and
\[
\|\eta\|_{W^{1,(p,q)}} \leq C\|X\|_{L^{p,q}}, \quad \|\xi\|_{W^{1,(p,q)}} \leq C\|X\|_{L^{p,q}}
\]
for some constant \( C \) independent from \( X \).

**Proof.** The result is well known for \( L^p \) spaces. In particular, for fixed \( p \in (1, \infty) \) there exists a continuous linear map from \( L^p(D^2) \) to \( W^{1,1}(D^2) \) sending a vector space \( X \) to the solution \( \eta \) of
\[
\begin{align*}
\Delta \eta &= \text{div}X \quad \text{in } D^2 \\
\eta &= 0 \quad \text{on } \partial D^2.
\end{align*}
\]
By interpolation, one can define an analogous map from \( L^{p,q}(D^2) \) to \( W^{1,(p,q)}(D^2) \). Now given \( X \in L^{p,q}(D^2) \) and \( \eta \) as above, there holds
\[
\text{div}(X - \nabla \eta) = 0.
\]
Therefore there exists \( \xi \in W^{1,(p,q)}(D^2) \) such that \( \nabla^\perp \xi = X - \nabla \eta \). \( \square \)
Lemma 12. Let \( g_0 \in W^{1,1}(\partial D^2, S^1) \). Then there exists an harmonic map \( g \in W^{1,2}(D^2) \) with trace \( g_0 \) on \( \partial D^2 \), and the map sending \( g_0 \in W^{1,1}(\partial D^2, S^1) \) to \( g \in W^{1,2}(D^2, S^1) \) is a bounded continuous map.

Proof. Let \( g_0 \in W^{1,1}(\partial D^2, S^1) \). First we show that \( g_0 \) has a lift in \(BV(\partial D^2, \mathbb{R})\) (up to an additive constant). In fact, for any \( \theta \in [0,2\pi] \) let

\[
\phi(\theta) = -i \int_0^\theta g_0^{-1}(e^{i\alpha}) \partial_\theta g_0(e^{i\alpha}) d\alpha.
\]

Set

\[
\lambda(e^{i\theta}) = \phi(\theta)
\]

for any \( \theta \in [0,2\pi] \). Then \( \lambda \in BV(\partial D^2, \mathbb{R}) \) with

\[
\|\lambda\|_{BV} \leq C \|g_0\|_{W^{1,1}} \tag{VI.1}
\]

and

\[
g_0 = g_0(1)e^{i\lambda} \text{ on } \partial D^2.
\]

We claim that the map

\[
W^{1,1}(\partial D^2, S^1) \to BV(\partial D^2, \mathbb{R}), \quad g_0 \mapsto \lambda
\]

is continuous. In fact let \( g_0 \in W^{1,1}(\partial D^2, S^1) \) and let \( (g_0^n)_{n \in \mathbb{N}} \) be a sequence in \( W^{1,1}(\partial D^2, S^1) \) such that

\[
g_0^n \to g_0 \text{ in } W^{1,1}(\partial D^2).
\]

Observe that by estimate \(\text{(VI.1)}\) for any \( n \in \mathbb{N} \)

\[
\|\lambda g_0^n - \lambda g_0\|_{BV} \leq C \|g_0^n g_0^{-1}\|_{W^{1,1}}.
\]

Now let \( \Lambda \subset \mathbb{N} \) the index set of a subsequence of \( (g_0^n)_{n \in \mathbb{N}} \) such that

\[
g_0^n \to g_0 \text{ a.e. along } \Lambda.
\]

Then by Dominated Convergence

\[
\lim_{n \to \infty} \|g_0^n g_0^{-1}\|_{L^1} = 0.
\]

Moreover for any \( n \in \mathbb{N} \)

\[
\partial_\theta (g_0^n g_0^{-1}) = \partial_\theta (g_0^n - g_0) g_0^{-1} + (g_0^n - g_0) \partial_\theta g_0^{-1},
\]

therefore

\[
\|\partial_\theta (g_0^n g_0^{-1})\|_{L^1} \leq \|\partial_\theta (g_0^n - g_0)\|_{L^1} + \|\partial_\theta g_0\|_{L^1} \|g_0^n - g_0\|_{L^1} \tag{VI.2}
\]

and by Dominated Convergence the right hand side of \(\text{(VI.2)}\) converges to zero along \( \Lambda \). Thus for any subsequence of \( (\lambda g_0^n)_{n \in \mathbb{N}} \) there exists a further subsequence along which

\[
\lambda g_0^n \to \lambda g_0 \text{ in } BV(\partial D^2).
\]

This concludes the proof of the Claim.

Next we consider the following Cauchy problem:

\[
\begin{cases}
\Delta \psi = 0 & \text{in } D^2 \\
\psi = \lambda & \text{on } \partial D^2.
\end{cases} \tag{VI.3}
\]
We claim that the solution $\psi$ to (VI.3) lies in $W^{1,(2,\infty)}(D^2)$. To see this, assume first that $\lambda$ is a smooth function. Now let $X$ be a vector field on $\overline{D^2}$ smooth up to the boundary, let $\eta \in W^{1,(2,1)}_0(D^2)$ and $\xi \in W^{1,(2,1)}(D^2)$ as in Lemma 11 so that
\[ X = \nabla \eta + \nabla^\perp \xi \]
and
\[ \|\xi\|_{W^{1,(2,1)}} \leq C \|X\|_{L^{2,1}}. \]
Then
\[ \int_{D^2} X \nabla \psi = \int_{D^2} \nabla \eta \nabla \psi + \int_{D^2} \nabla^\perp \xi \nabla \psi = \int_{D^2} \nabla^\perp \xi \nabla \psi = \int_{\partial D^2} \xi \partial_\theta \psi = \int_{\partial D^2} \xi \partial_\theta \lambda. \tag{VI.4} \]
Here we used the fact that since $\eta$ has vanishing trace on $\partial D^2$ and $\psi$ is harmonic,
\[ \int_{D^2} \nabla \eta \nabla \psi = 0. \]
We also used the fact that, by the divergence Theorem,
\[ \int_{D^2} \nabla^\perp \xi \nabla \psi = -\int_{D^2} \text{div}(\xi \nabla^\perp \psi) = -\int_{\partial D^2} \xi \nabla^\perp \psi \cdot \nu = \int_{\partial D^2} \xi \partial_\theta \psi. \]
Since $W^{1,(2,1)}(D^2)$ embeds continuously in $L^\infty(D^2)$, it follows from (VI.4) that
\[ \|\nabla \psi\|_{L^{2,\infty}} \leq C \|\lambda\|_{BV} \]
and therefore\(^8\)
\[ \|\psi\|_{W^{1,(2,\infty)}} \leq C \|\lambda\|_{BV}. \tag{VI.5} \]
By density of the smooth functions in $BV(\partial D^2)$ the estimate extends by continuity to any $\lambda \in BV(\partial D^2)$.
Now given $g_0 \in W^{1}(\partial D^2, S^1)$ let $\lambda_{g_0}$ and $\psi_{\lambda_{g_0}}$ as above. Set
\[ g := g_0(1)e^{i\psi_{\lambda_{g_0}}} \text{ on } \partial D^2. \]
Then by construction $g|_{\partial D^2} = g_0$,
\[ \|\nabla g\|_{L^{2,\infty}} \leq C \|g_0\|_{W^{1,1}} \]
and the prescription
\[ g_0 \mapsto g \]
defines a continuous map
\[ W^{1,1}(\partial D^2, S^1) \to W^{1,(2,\infty)}(D^2, S^1). \]
\[ \square \]
\(^8\)This can be shown by contradiction, just as in the classical proof of Poincaré inequality.
Lemma 13. Let $R \in (0, 1)$, let $(u_n)_{n \in \mathbb{N}}$ be a sequence of non-negative functions in $W^{1,1}(D_2^2 \setminus D_R^2)$ defined pointwise. Assume that
\[
\sup_{n \in \mathbb{N}} \int_{D_2^2 \setminus D_R^2} u_n \leq C
\]
for some constant $C$.
Let $\varepsilon > 0$. Then there exists a measurable subset $A \subset (R, 1)$ with positive Lebesgue measure and a subsequence, say indexed by $\Lambda \subset \mathbb{N}$, such that
\[
\int_{\partial D_R^2} u_n \leq C_\varepsilon := \frac{C}{1 - R} + \varepsilon
\]
for any $R \in A$, for any $n \in \Lambda$.

Proof. Assume by contradiction that the statement is false. Then for a.e. $r \in (R, 1)$ there exists $N(r)$ such that for any $n \geq N(r)$
\[
\int_{\partial D_r^2} u_n > C_\varepsilon
\]
For any $n \in \mathbb{N}$ set
\[
A(n) = \{ r \in (R, 1) \text{ s.t. } N(r) \leq n \}.
\]
Then $A(n) \subset A(m)$ for any $n, m \in \mathbb{N}$ with $n \leq m$ and
\[
\left| \bigcup_{n \in \mathbb{N}} A(n) \right| = 1 - R.
\]
Observe that for any $n \in \mathbb{N}$
\[
C \geq \int_{A(n)} \left( \int_{\partial D_r^2} u_n \right) dr \geq |A(n)| C_\varepsilon.
\]
If we let $n$ tend to infinity we obtain
\[
C \geq C_\varepsilon (1 - R),
\]
a contradiction. \hfill \Box

Lemma 14. Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence in $H^{1/2}(\partial D^2)$ converging in $H^{1/2}(\partial D^2)$ to a function $\varphi$, i.e.
\[
\varphi_n \to \varphi \text{ in } H^{1/2}(\partial D^2).
\]
Let
\[
F : \mathbb{R} \to \mathbb{R}^n
\]
be a Lipschitz-continuous function. Then
\[
F \circ \varphi_n \to F \circ \varphi \text{ in } H^{1/2}(\partial D^2, \mathbb{R}^n).
\]
Proof. It is clear that
\[ F \circ \varphi_n \to F \circ \varphi \text{ in } L^2(\partial D^2, \mathbb{R}^n). \]
We would like to show that
\[ \lim_{n \to \infty} \int_{\partial D^2} \int_{\partial D^2} \frac{|(F(\varphi_n(x)) - F(\varphi(x))) - (F(\varphi_n(y)) - F(\varphi(y)))|^2}{|x - y|^2} \, dx \, dy. \quad \text{(VI.6)} \]
Observe that for a.e. \((x, y) \in \partial D^2 \times \partial D^2\),
\[ |(F(\varphi_n(x)) - F(\varphi(x))) - (F(\varphi_n(y)) - F(\varphi(y)))| \leq L \left( |\varphi_n(x) - \varphi_n(y)| + |\varphi(x) - \varphi(y)| \right), \]
where \(L\) denotes the Lipschitz constant of the function \(F\).
Now since
\[ \varphi_n \to \varphi \text{ in } H^\frac{1}{2}(\partial D^2), \]
there holds
\[ \left| \frac{\varphi_n(x) - \varphi_n(y)}{|x - y|} \right|^2 \to \left| \frac{\varphi(x) - \varphi(y)}{|x - y|} \right|^2 \text{ in } L^1(\partial D^2 \times \partial D^2). \]
Therefore there exists a function
\[ B \in L^1(\partial D^2 \times \partial D^2) \]
and a subsequence of \((\varphi_n)_{n \in \mathbb{N}}\), say indexed by \(\Lambda \subset \mathbb{N}\), so that for any \(n \in \Lambda\)
\[ \left| \frac{\varphi_n(x) - \varphi_n(y)}{|x - y|} \right|^2 \leq B(x, y) \text{ a.e. in } \partial D^2 \times \partial D^2. \]
Thus by Dominated Convergence \((\text{VI.6})\) holds true for a subsequence. Since this argument can be repeated for any subsequence of \((\varphi_n)_{n \in \mathbb{N}}\), the statement holds true. \qed

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