The state complexity of random DFAs

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Abstract

The state complexity of a Deterministic Finite-state automaton (DFA) is the number of states in its minimal equivalent DFA. We study the state complexity of random \(n\)-state DFAs over a \(k\)-symbol alphabet, drawn uniformly from the set \([n]^{n} \times [k] \times 2^{[n]}\) of all such automata. We show that, with high probability, the latter is \(\alpha_k n + O(\sqrt{n \log n})\) for a certain explicit constant \(\alpha_k\).

1 Introduction

A randomly generated deterministic finite automaton (DFA) on \(n\) states and \(k\) symbols is drawn as follows: for each state and each of the \(k\) symbols in the alphabet, the transition arrow’s destination is chosen uniformly at random among the \(n\) states; the \(nk\) random choices are independent\(^1\). Then each state is chosen to be accepting (or not) independently with probability \(1/2\). This natural model for a “typical” DFA goes back to [13] and was considered in [1, 10] in the context of learning theory. In particular, in [1] it is shown (perhaps surprisingly) that random DFAs possess sufficient complexity to embed nontrivial parity problems.

Let us define the state complexity of a DFA \(M\) as the number of states in the canonical (minimal) DFA equivalent to \(M\), and denote it by \(\|M\|\). In this paper, we study the state complexity of random DFAs in the model defined above.

Related work We are not aware of previous literature dealing with the specific problem we have posed. The somewhat related problem of enumerating finite automata according to various criteria has been extensively studied; see [6] and the references therein. Some recent results include enumeration of minimal automata [2], generation of random complete DFAs [4], and enumeration and generation of accessible DFAs [3]. In a different line of enquiry, Pittel investigated the distributions induced by transitive closures [11] and rumor spreading [12].

2 Background and notation

We use standard automata-theoretic notation throughout; the reader is referred to [9, 14] for background. We put \([n] = \{0, \ldots, n - 1\}\). Thus, \([k]\) is a \(k\)-ary alphabet and \([k]^*\) is the set of all finite words (strings) over this alphabet. The notation \(|\cdot|\) is used for both word length and

\(^1\)By symmetry, we may always take the state \(q = 1\) to be the starting state.
set cardinality. Standard order-of-magnitude notation $o(\cdot)$ and $O(\cdot)$ is used, as well as their “with high probability” variants $o_P(\cdot)$ and $O_P(\cdot)$. The $\tilde{O}(\cdot)$ notation ignores polylog factors.

An $n$-state $k$-ary Deterministic Finite-state Automaton is a tuple $M = (Q, q_0, A, \delta)$ where

- $Q = [n]$ is the set of states
- $q_0 = 1$ is the starting state;
- $A \subseteq [n]$ is the set of accepting states;
- $\delta : [n] \times [k] \to [n]$ is the transition function.

The transition function $\delta$ may be extended to $[n] \times [k]^*$ via the recursion

$$\delta(q, u_1 u_2 \cdots u_n) = \delta(\delta(q, u_1), u_2 \cdots u_n).$$

If the accepting states are unspecified, the transition function $\delta$ induces a directed multi-graph on $n$ nodes with regular outdegree $k$, called a $k$-ary semiautomaton.

We recall the standard equivalence relation over the states of a DFA: a word $x \in [k]^*$ distinguishes between the states $p, q \in [n]$ if exactly one of the states $\delta(p, x), \delta(q, x)$ is accepting.

If no $x \in [k]^*$ distinguishes between $p$ and $q$, these states are equivalent, denoted by $p \equiv q$.

A standard high-level algorithm\(^2\) for minimizing a DFA proceeds in two stages:

- **REMOVE-UNREACHABLE**: Remove all states $q$ such that there is no directed path from the starting state $q_0$ to $q$.
- **COLLAPSE-EQUIVALENT**: Collapse each set of mutually equivalent states into a single state.

### 3 Main results

Our main result is the following estimate on the state complexity of random DFAs:

**Theorem 1.** Let $M_n^{(k)}$ be a random DFA on $n$ states and $k$ symbols drawn uniformly from $[n][n] \times [k] \times 2^n$. Then, for any fixed $k \geq 2$ and sufficiently large $n$,

$$P\left(\|M_n^{(k)}\| - \alpha_k n > \sqrt{n} \log n\right) = \Theta\left(\frac{1}{n^k}\right),$$

where $\alpha_k$ is unique positive root\(^3\) of $x = 1 - e^{-kx}$. In particular,

$$E\|M_n^{(k)}\| = \alpha_k n + O(\sqrt{n} \log n).$$

**Remark 2.** Observe that $0.7968 \approx \alpha_2 < \alpha_3 < \ldots < \alpha_\infty = 1$. For $k = 1$, the behavior of $\|M_n^{(1)}\|$ is qualitatively different than described in Theorem 1. The equation $x = 1 - e^{-x}$ has no positive solution and $E\|M_n^{(1)}\| = \Theta(\sqrt{n})$, which follows from the analysis in [11].

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\(^2\)Hopcroft’s celebrated algorithm [8] for minimizing a DFA has runtime complexity $O(n \log n)$.

\(^3\)A closed-form expression for $\alpha_k$ is possible via the Lambert $W$ function [5]: $\alpha_k = 1 + W(-ke^{-k})/k$. This constant also appears (as $\omega_k$) in [2] and seems to be intimately related to average reachability properties of semiautomata in $[n][n] \times [k]$ under the uniform measure.
Remark 3. The lower bound of $\Omega(1/n^k)$ in (2) is trivial, since with probability $1/n^k$, all of state 1’s arrows point back to itself and $\left\| M_n^{(k)} \right\| = 1$.

Our proof of Theorem 1 proceeds in two principal stages. First we show that in our model, a random semiautomaton has roughly $\alpha_k n$ reachable states with high probability. As in [13], we refer to the number of reachable states as the accessibility spectrum of the automaton.

**Theorem 4.** Let $R_n^{(k)}$ be the accessibility spectrum of a random semiautomaton on $n$ states and $k$ symbols drawn uniformly from $[n]^{[n] \times [k]}$. Then, for every fixed $k \geq 2$ and as $n \to \infty$,

$\mathbb{P}\left( \left| R_n^{(k)} - \alpha_k n \right| > \sqrt{n \log n} \right) = O\left( \frac{1}{n^k} \right).$  \hspace{1cm} (3)

Second, we show that with high probability, very few states are lost when equivalent ones are merged. Define $E_n^{(k)}$ to be the number of “excess” reachable states:

$$E_n^{(k)} = R_n^{(k)} - \left\| M_n^{(k)} \right\|. $$

Note that in principle we need only show that the number of states lost due to merging is small after the unreachables are removed, but we will actually show that this is true even without removing them.

**Theorem 5.** For every fixed $k \geq 2$,

$$\mathbb{P}\left( E_n^{(k)} > C_k \frac{\log n}{\log \log n} \right) = O\left( \frac{1}{n^k} \right)$$

for an appropriate constant $C_k$.

Remark 6. Theorem 5 continues to hold when each state is accepting with probability $0 < p < 1$ instead of $1/2$; only the constants $C_k$ and those implicit in $O(\cdot)$ will change.

4 Proofs

**Lemma 7.** Define the function

$$F(t) = n(1 - (1 - 1/n)^t) - (t - 1)/2, \hspace{1cm} 1 \leq t \leq n. $$

Then, for sufficiently large $n$,

$$\frac{F^2(t)}{t} \geq \begin{cases} 
0.01t, & t \leq n/2, \\
\Omega(\log^2 n), & t \in [n/2, \alpha_2 n - \sqrt{n \log n}] \cup [\alpha_2 n + \sqrt{n \log n}, \infty). 
\end{cases}$$

**Proof.** We have $F(0) = 1/2$, and for $t \leq n/2$

$$F'(t) = -\frac{1}{2} + n \log \left(1 + \frac{1}{n-1}\right) \cdot \left(1 - \frac{1}{n}\right)^t \geq -\frac{1}{2} + n \left(\frac{1}{n-1} - \frac{1}{2(n-1)^2}\right) \cdot \left(1 - \frac{1}{n}\right)^{n/2} \geq -\frac{1}{2} + \frac{n}{n-1} \cdot \frac{2n-3}{2n-2} \cdot \left(\frac{1}{\sqrt{e}} - o(1)\right) \geq \frac{1}{\sqrt{e}} - \frac{1}{2} - o(1) > 0.1.$$
This proves the estimate on $F^2(t)/t$ in the range $[1, n/2]$.

Now consider $t \in [n/2, \alpha_2 n - \sqrt{n} \log n]$, and observe that $F(t) = H(t) + O(1)$, where

$$H(t) = n - t/2 - n \exp(-t/n).$$

By the definition of $\alpha_2$, we have $H(\alpha_2 n) = H(0) = 0$. Furthermore, $H''(t) = -\exp(-t)/n < 0$, and so $H$ is concave with $H'(n \log 2) = 0$, and therefore increasing on $[0, n \log 2]$ and decreasing on $[n \log 2, \infty)$. Hence, to lower-bound $H^2(t)/t$ in the given range, it suffices to estimate $H$ at its right endpoint:

$$H(\alpha n - \sqrt{n} \log n) = \frac{1}{2} \sqrt{n} \log n - e^{-\alpha} \sqrt{n} \log n + O(\log^2 n) = \Omega(\sqrt{n} \log n).$$

Since $H'(t) < 1/2 - e^{-\alpha}$ for $t > \alpha n$, we have $H(\alpha_2 n + x) = \Omega(x)$ for $x > 0$, which completes the proof. \(\square\)

Proof of Theorem 4. We will prove the theorem for $k = 2$; the general case is completely analogous — only the constants implicit in $O(1/n^k)$ will vary with $k$. For readability, we will write $\alpha = \alpha_2$ and $R_n = R_{n}^{(2)}$. It will be convenient to embed $R_n$ in a slightly more general random process. Fix $n \geq 1$, and define the sequence of random variables $(\nu_t)_{t=1}^{\infty}$, as follows:

$$\nu_1 = 1,$$

$$\nu_{t+1} = \begin{cases} \nu_t, & \text{with probability } \nu_t/n, \\ \nu_t + 1, & \text{with probability } 1 - \nu_t/n. \end{cases}$$

Clearly, $\nu_t$ is with probability $1$ nondecreasing, upper-bounded by $n$, and reaches $n$ after a finite number of steps. Let us also define

$$\omega_t = 2\nu_t + 1 - t, \quad t \geq 1.$$  \hspace{1cm} (4)

Now consider the following process for generating random directed multigraphs with regular outdegree 2. For time steps $t = 1, 2, \ldots$, we will maintain the set of nodes $N_t$, reached from $q_0 = 1$ by time $t$, and two sets of edges: open edges $O_t$ and closed edges $C_t$. A closed edge $c$ is an ordinary directed arrow from a source node $p$ to a destination node $q$ marked with a $\sigma \in [k]$ and denoted by $c = (p \xrightarrow{\sigma} q)$. An open edge $o$ has a specified source $p$ but an as yet unspecified destination; such an edge will be denoted by $o = (p \xrightarrow{} \star)$. We initialize $N_1 = \{1\}$, $C_1 = \emptyset$ and $O_1 = \{(1 \xrightarrow{0} \star), (1 \xrightarrow{1} \star)\}$. At time $t + 1$, some (arbitrarily chosen\footnote{It is easy to see that the distribution of $N_t, C_t, O_t$ is unaffected by the order in which the open edges are selected.}) open edge in $o \in O_t$ (if one exists) chooses a destination node $q$ as follows:

(i) $q \in N_t$ with probability $|N_t|/n$ (that is, $o$ will point to a previously reached node);

(ii) $q \in [n] \setminus N_t$ with probability $1 - |N_t|/n$.

In event (i), $O_{t+1} = O_t \setminus \{o\}$, while in event (ii), $O_{t+1} = (O_t \setminus \{o\}) \cup \{(q \xrightarrow{0} \star), (q \xrightarrow{1} \star)\}$; in both cases, $N_{t+1} = N_t \cup \{q\}$ and $C_{t+1} = C_t \cup \{o\}$.

The random semiautomaton embeds into the process $(\nu_t, \omega_t)$ via the following natural correspondence: $|N_t| = \nu_t$ and $|O_t| = \omega_t$ as long as the latter is nonnegative (in particular,
the correspondence breaks down for \( t > 2n + 1 \), since \( \omega_t \) becomes negative). Let \( \tau \) be the smallest \( t \) for which \( \omega_t = 0 \) — i.e., the first time there are no longer any open edges to choose from. Then the pair \((N_\tau, C_\tau)\) defines\(^5\) a semiautomaton with accessibility spectrum \( R_n = \nu_\tau \), drawn uniformly from \([n]^{[n]} \times \{0,1\}\). Hence, proving (3) amounts to showing that

\[
P( |\tau - 2\alpha n| > \sqrt{n} \log n) = O \left( \frac{1}{n^2} \right).
\]  

(5)

Indeed, (5) implies that \( \tau = (2 + o_P(1)) \alpha n \), and \( \nu_\tau = (1 + o_P(1)) \nu_{\alpha n} \). Since, by definition, \( \tau \) is the smallest \( t \) for which \( \nu_t = (t - 1)/2 \), we have

\[
P(\tau \in [a, b]) \leq \mathbb{P}(\exists t \in [a, b] : \nu_t = (t - 1)/2)
\]

\[
\leq \mathbb{P}(\exists t \in [a, b] : \nu_t \leq (t - 1)/2).
\]

(6)

We estimate the left tail of \( \tau \) as follows:

\[
P(\tau \leq \alpha n - \sqrt{n} \log n) \leq P_0 + P_1 + P_2,
\]

where

\[
P_0 = \mathbb{P}(\tau \in [1, 150 \log n]),
\]

\[
P_1 = \mathbb{P}(\tau \in [150 \log n, n/2]),
\]

\[
P_2 = \mathbb{P}(\tau \in [n/2, \alpha n - \sqrt{n} \log n]).
\]

To bound \( P_0 \), we first argue, by elementary combinatorics, that \( \mathbb{P}(\omega_3 < 3) = O(1/n^2) \). Now we condition on the high-probability event that there are at least 3 open arrows available after 3 steps. If all of the open arrows have been exhausted between time \( t = 4 \) and \( t = T \), then certainly at least three of these arrows must point back to the \( O(T) \) previous states. Thus, for \( T = 150 \log n \),

\[
P_0 \in O \left( \frac{1}{n^2} + \binom{T}{3} \left( \frac{T}{n} \right)^3 \right) \subset O \left( \frac{1}{n^2} \right).
\]

To bound \( P_1 \) and \( P_2 \), we observe that an alternate interpretation is possible for \( \nu_t \). Namely, when \( t \) balls are thrown into \( n \) bins uniformly at random, the number of non-empty bins is distributed as \( \nu_t \). We also observe that

\[
\mathbb{E}\nu_t = n(1 - (1 - 1/n)^t), \quad t \geq 1.
\]  

(7)

Now by the Chernoff bound for negatively associated random variables [7, Prop. 5, Thm. 13],

\[
P(\nu_t - \mathbb{E}\nu_t \leq -\Delta) \leq \exp(-2\Delta^2/t), \quad \Delta > 0.
\]  

(8)

Hence,

\[
P_1 \leq \sum_{t=150 \log n}^{n/2} \mathbb{P}(\nu_t - \mathbb{E}\nu_t \leq -F(t)),
\]

\[
P_2 \leq \sum_{t=n/2}^{\alpha n - \sqrt{n} \log n} \mathbb{P}(\nu_t - \mathbb{E}\nu_t \leq -F(t)),
\]

\(^5\)Since the quantity of interest is the accessibility spectrum, it is unnecessary to define transitions out of unreachable states.
where $F(t)$ is as in Lemma 7. The estimates in the lemma and (8) yield

$$ P_1 \leq \frac{n}{2} \exp(-3 \log n) \in O\left(\frac{1}{n^2}\right) $$

and

$$ P_2 \in O(n) \exp(-\Omega(\log^2 n)) \subset O\left(\frac{1}{n^2}\right). $$

This proves the left-tail estimate in (5). To prove the corresponding right-tail estimate, we observe that, analogously to (6),

$$ P(\tau \in [a, b]) \leq P(\exists t \in [a, b] : \nu_t \geq (t-1)/2). $$

The deviation probability is bounded as in (8):

$$ P(\nu_t - E\nu_t \geq \Delta) \leq \exp(-2\Delta^2/t), \quad \Delta > 0. $$

Hence

$$ P(\tau > \alpha n + \sqrt{n \log n}) \leq \sum_{t=\alpha n + \sqrt{n \log n}}^{n} P(\nu_t - E\nu_t \geq G(t)), $$

where $G(t) = -F(t)$. Invoking again Lemma 7, we have

$$ P(\tau > \alpha n + \sqrt{n \log n}) \in O(n) \exp(-\Omega(\log^2 n)) \subset O\left(\frac{1}{n^2}\right). $$

Proof of Theorem 5. Again, for ease of exposition, we only prove the claim for $k = 2$. We start by explaining the idea of the proof. We need to show that there are usually “few” pairs of equivalent states. Let us start by describing two “typical” situations in which equivalent states emerge. The first is where a state is mapped into itself by every member of $\{0, 1\}$. Two such states are equivalent if and only if both are accepting or both are rejecting, which happens with a probability of 1/2. More generally, if from each of the two states one can reach very few states, then there is a non-negligible probability that the states are equivalent. Thus, we will show that there are few states with small accessibility spectra. In the preceding sentence, “few” means (with high probability) “at most 2”, while “small” means “less than $4 \log_2 n$”.

The second principal reason for two states $q, q'$ to be equivalent is that $\delta(q, 0) = \delta(q', 0)$ and $\delta(q, 1) = \delta(q', 1)$. Again, $q$ and $q'$ are equivalent in this case with probability 1/2. Thus, we will need to show that there are few pairs of states $q, q'$ for which there are few words in $\{0, 1\}^*$ taking $q$ and $q'$ to distinct states. Here, the first “few” means “at most $C \log n/ \log \log n$” and the second means “up to $4 \log_2 n$”.

Let us now consider the above scenarios in more detail. The (random) set of states reachable from $q$ is given by $\{q, \delta(q, 0), \delta(q, 1), \delta(q, 00), \delta(q, 01), \ldots\}$. Thus, the states reachable from $q$ reside on a binary tree whose edges are marked by letters in $\{0, 1\}$. Each time the
random DFA selects a state \( p = \delta(q, w) \), if \( p \) is already in the tree, the edge that would create a directed cycle is not drawn. We refer to the resulting tree as the tree growing from \( q \). Its size is the accessibility spectrum of \( q \), denoted by \( S(q) \).

Let \( C > 0 \) be a constant to be determined later. A state’s accessibility spectrum is said to be small if it is below \( C \log_2 n \). As in the proof of Theorem 4, the probability of a given state having a small accessibility spectrum is \( O(1/n^2) \). A similar argument shows that the joint probability of any pair of states \( q, q' \) having small accessibility spectra is \( \tilde{O}(1/n^4) \). Indeed, consider the event of \( S(q') \) being small, conditioned on \( S(q) \) being such. Draw the states \( \delta(q', 0), \delta(q', 1), \delta(q', 00), \ldots \) similarly to the proof of Theorem 4. The event in question is contained in the event whereby, in the course of the first \( C \log_2 n \) steps of the process of “closing” the open edges, we encounter at least twice either a state visited already or a state contained in the event whereby, in the course of the first \( C \log_2 n \) steps of the process of “closing” the open edges, we encounter at least twice either a state visited already or a state belonging to the tree growing from \( q \). The probability of the latter event is clearly \( O(\log^4/n^2) \).

Hence,

\[
\mathbb{P}(S(q), S(q') \text{ are both small}) = O\left(\frac{\log^4 n}{n^4}\right).
\]

Carrying this line of reasoning over to triples, we have that the probability of any three states having small accessibility spectra is \( \tilde{O}(1/n^6) \) — and therefore,

\[
\mathbb{P}(\text{there are 3 distinct states with small accessibility spectra}) \leq \tilde{O}\left(\frac{1}{n^6}\frac{\binom{n}{3}}{}\right) = \tilde{O}\left(\frac{1}{n^3}\right) \subset O\left(\frac{1}{n^2}\right).
\]

In view of the discussion above, we may assume (after removing up to \( 2 \) states) that all states have large accessibility spectra. Consider two states \( q, q' \). Let \( T \) be a tree of size \( m = C \log_2 n \) growing from \( q \) (this will typically be a subtree of a larger tree of size \( O(\alpha n) \)). The nodes of \( T \) are \( \delta(q, w_1), \delta(q, w_2), \ldots, \delta(q, w_m) \) for certain words \( w_1, w_2, \ldots, w_m \in \{0, 1\}^* \). If \( \delta(q, w_i) \neq \delta(q', w_i), i = 1, 2, \ldots, m \), then the probability of \( q, q' \) being equivalent is at most \( 1/2^m = 1/n^C \). (Note that this holds even if the states \( \delta(q', w_i), 1 \leq i \leq m \), are not mutually distinct, in fact even if they all coincide. Similarly, it does not matter if some of the states \( \delta(q', w_i) \) coincide with some of the \( \delta(q, w_j) \), as long as \( i \neq j \).) The probability that both \( \delta(q, 0) = \delta(q', 0) \) and \( \delta(q, 1) = \delta(q', 1) \) is \( 1/n^2 \). Call a state pair satisfying these equalities a dud. The union bound does not yield a non-trivial upper bound on the number of duds, and a more refined analysis will be needed. Clearly, the probability that \( d \) specific pairwise disjoint pairs are duds is \( 1/n^{2d} \). Now the probability that there exist \( d \) disjoint duds is at most

\[
\frac{1}{n^{2d}} \binom{n}{2d} (2d - 1) (2d - 3) \cdots 1 \leq \frac{1}{n^{2d}} \cdot \frac{n^{2d}}{(2d)!} \cdot \frac{(2d)!}{2^{2d}d!} = \frac{1}{2^{2d}d!}.
\]

Choosing \( d = 3 \log n / \log \log n \) and applying Stirling’s formula, we see that the probability of there being \( d \) disjoint duds is \( O(1/n^2) \).

Other than duds — pairs “dying” right away after 2 steps — we must consider pairs dying after 4, 6, \ldots, \( C \log_2 n \) steps. However, the probability that a pair will die after 4 steps is \( O(1/n^3) \), after 6 steps — \( O(1/n^4) \), and so forth. Hence, the probability that there will be \( 2 \) pairs for which the process dies after 4 steps is

\[
\binom{n}{4} \cdot O\left(\frac{1}{n^6}\right) = O\left(\frac{1}{n^2}\right).
\]
that there will be a pair that dies after 6 steps,
\[ \binom{n}{2} \cdot O \left( \frac{1}{n^4} \right) = O \left( \frac{1}{n^2} \right) , \]
and that some pair will die after \( t \in [8, 4 \log_2 n] \) steps,
\[ \binom{n}{2} \cdot \tilde{O} \left( \frac{1}{n^5} \right) \subset O \left( \frac{1}{n^7} \right) . \]

Now for two states \( q, q' \) reaching distinct states for many words \( w_i \), the probability of being equivalent is at most \( 1/n^C \). Thus, it suffices to take \( C = 4 \) to bound the probability of any pair of states growing large trees yet being equivalent by
\[ \binom{n}{2} \cdot O \left( \frac{1}{n^4} \right) = O \left( \frac{1}{n^2} \right) . \]

**Proof of Theorem 1.** Follows immediately from Theorems 4 and 5 since the former estimates the number of states remaining after \textsc{remove-unreachable} and the latter bounds the number of states lost after \textsc{collapse-equivalent}. \qed

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