Covariate-Assisted Community Detection on Sparse Networks

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Abstract

Community detection is an important problem when processing network data. Traditionally, this is done by exploiting the connections between nodes, but connections can be too sparse to detect communities in many real datasets. Node covariates can be used to assist community detection; see Binkiewicz et al. (2017); Weng and Feng (2022); Yan and Sarkar (2021); Yang et al. (2013). However, how to combine covariates with network connections is challenging, because covariates may be high-dimensional and inconsistent with community labels.

To study the relationship between covariates and communities, we propose the degree corrected stochastic block model with node covariates (DCSBM-NC). It allows degree heterogeneity among communities and inconsistent labels between communities and covariates. Based on DCSBM-NC, we design the adjusted neighbor-covariate (ANC) data matrix, which leverages covariate information to assist community detection. We then propose the covariate-assisted spectral clustering on ratios of singular vectors (CA-SCORE) method on the ANC matrix. We prove that CA-SCORE successfully recovers community labels when 1) the network is relatively dense; 2) the covariate class labels match the community labels; 3) the data is a mixture of 1) and 2). CA-SCORE has good performance on synthetic and real datasets. The algorithm is implemented in the R (R Core Team (2021)) package CASCORE.

Keywords: Spectral Clustering; Asymptotic Analysis; Degree-Corrected Stochastic Block Model; Node Attribute; Random Matrix Theory.

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\section{Introduction}

Network data can be commonly found in various directions of science, such as biology (Binkiewicz et al., 2017; Chen and Yuan, 2006; Sporns and Betzel, 2016; Deco and Corbetta, 2011), ecology (Jacob et al., 2011), and sociology (Gil-Mendieta and Schmidt, 1996; Leskovec and McAuley, 2012; Ying et al., 2018). Even the literature of statistics itself can be viewed as a network: each paper can be seen as a node, and a citation is an edge from one node to another. Mathematically speaking, a network can be presented by an adjacency matrix \( A \in \{0,1\}^{n \times n} \), where \( n \) is the number of nodes, and

\begin{equation}
A(i, j) = I\{\text{node } i \text{ and node } j \text{ is connected}\}, 1 \leq i, j \leq n.
\end{equation}

Among the many insights that network data can provide, community detection is one of the most important. With community detection, we can label, or cluster, nodes of the network into several communities. Members of each community may hold similar political views, research interest, or biological functions. In other words, the community label provides us a degree of interpretability over the nodes and greatly simplifies their inter-relationship.

Due to its importance, community detection for networks has been studied with great interest in the last decade. The seminal work (Holland et al. (1983)) proposed a simple stochastic block model (SBM) to model network data and it has lead to many promising community detection results. Later works (Karrer and Newman (2011); Bickel and Chen (2009); Yan et al. (2014); Zhao et al. (2012)) generalized SBM to degree-corrected SBM (DCSBM) which allows degree heterogeneity. Existing community detection methods include the maximum likelihood methods (Bickel et al. (2013); Mariadassou et al. (2010); Wang and Bickel (2017)) and spectral clustering methods (Amini et al. (2013); Chaudhuri et al. (2012a); Rohe et al. (2011); Jin (2015)). Among these methods, the spectral clustering methods are usually computationally cheaper and hence preferred for large networks;
see Amini et al. (2013), Binkiewicz et al. (2017), Joseph and Yu (2016) and Rohe et al. (2011).

For some network data in practice, communities may not be detectable using the adjacency information only. One possibility is that two communities may have identical connection pattern. Another possibility is that a real network often has many sparsely connected nodes, which form small spectral components that cannot help on community detection, or are simply isolated from others. As a rather extreme example, in the citation network dataset discussed in Section 5.2 (published in Ji and Jin (2016)), more than 29% of the nodes are isolated nodes. The lack of connections with others makes the labelling task impossible. In fact, many existing community detection methods can only be applied to a connected network.

For networks with sparse connections, we may seek assistance from the node covariate information, which can be collected from data features associated with each node. The meaning of covariate information depends on the dataset in discussion. For example, in citation networks, the covariates can be obtained as abstracts. In the network data provided by an online radio app LastFM (Rozemberczki and Sarkar, 2020), two users can be seen as connected nodes if they are friends on the app, while the covariates are given by the list of artists they follow. Let \( x_i \in \mathbb{R}^p \) denote the covariate vector of node \( i \). The covariate matrix is defined as

\[
X = \begin{pmatrix} x_1, x_2, \cdots, x_n \end{pmatrix} \in \mathbb{R}^{n \times p}
\]  

We expect these covariate features partially reflect the community structure, so using them will improve the community detection results. In particular, even if a node is isolated from others, we may still be able to learn its community, since its covariate should be similar to the ones from its community members.
The idea of using covariate to assist community detection has been studied in recent years. Newman and Clauset (2016) and Weng and Feng (2022) use the covariates to build a prior for the community labels, where the covariates are required to be low-dimensional. For high-dimensional covariates, Binkiewicz et al. (2017), Yang et al. (2013) and Yan and Sarkar (2021) provide some solutions, but they all require the covariates and the network share the same community structure. This assumption may not hold in practice. As pointed out in Zhang et al. (2016) and Huang and Feng (2018), it is possible that the covariates have a class structure that is different from network community structure. To solve this problem, both papers use the covariates to build an edge weightage. Zhang et al. (2016) then optimizes a joint community detection criteria analogous to the modularity. Huang and Feng (2018) proposes the pairwise covariates-adjusted stochastic block model, and finds the maximum likelihood estimate of the community labels. However, computation and storage of $n \times n \times p$ matrix is required in both approaches, which is a huge numerical cost even for moderately large $p$ and $n$. In summary, existing methods often impose strong requirements on the covariate and their dimensions, which can be unrealistic in some applications.

In this article, we propose a general model for the covariates, where the covariate class can be different from the community structure of the adjacency matrix $A$, and the covariate dimensions $p$ can be either a constant or increasing with $n$. (Note that we use “community” when referring to the structure in $A$, and “covariate class” or “class” when referring the structure in $X$.) To combine the information from $A$ and $X$, we consider an adjusted neighbor covariate (ANC) matrix:

$$Y(\alpha) = AX + \Lambda_\alpha X,$$  \hfill (1.3)

where $\Lambda_\alpha$ is a diagonal matrix. The diagonal component $\alpha_i$ depends on an overall tuning parameter $\alpha$ and the degree of node $i$. By selecting a proper $\alpha$, the rows of $Y(\alpha)$ correspond
to popular nodes are mainly decided by rows of $AX$, and the rows correspond to unpopular nodes are mainly decided by rows of $X$.

To recover the community labels from $Y(\alpha)$, let $K$ be the number of communities and $\hat{\Xi} \in \mathcal{R}^{n \times K}$ be the matrix containing the top $K$ left eigenvectors of $Y(\alpha)$. Consider normalizing rows in $\hat{\Xi}$ so that each row has norm 1. We will show that, with high probability, all normalized rows corresponding to nodes in community $k$ are close to a center $m_k \in \mathcal{R}^K$, \( k = 1, 2, \ldots, K \). Hence, we can detect the community label by applying $k$-means on the normalized rows of $\hat{\Xi}$. This approach is named as Covariate Assisted Spectral Clustering on Ratios of Eigenvectors (CA-SCORE).

Since CA-SCORE only requires the first few singular vectors of $Y(\alpha)$, it is computationally very efficient. Furthermore, we prove it can work well on the networks where the node degrees have severe heterogeneity among communities. For \textit{relatively dense} communities, CA-SCORE inverts the covariate class label information in $X$ to be the community information. For \textit{extremely sparse} communities, CA-SCORE relies on the covariates. Putting all communities together, we prove that the overall error rate of CA-SCORE is proportional to square root of the mismatching rate between extremely sparse communities and their covariate class labels. When this mismatching rate goes to 0, CA-SCORE successfully recovers the community labels.

In Section 2, we introduce the problem formulation, set up the ANC matrix $Y(\alpha)$ and propose the algorithm CA-SCORE. In Section 3, we present our main theoretical results about CA-SCORE, followed by numerical results on synthetic data in Section 4 and real datasets in Section 5. In Section 6, we discuss possible extensions. The proofs are deferred to Appendix in 8 and the supplementary materials.
1.1 Notations

Throughout this paper, we add a subscript $X$ to the symbols denoting quantities about covariates, to differentiate them from those about the network. For a matrix $A$, we use $\lambda_k(A)$ to denote the $k$-th largest singular value of $A$. Let $\|A\|$ denote the spectral norm of $A$. For a vector $a$, we use $\|a\|_q$ to denote the $\ell_q$ norm of $a$ and ignores $q$ when $q = 2$. If $A$ is a matrix, $\text{diag}(A)$ denotes the vector formed by its diagonal entries. If $a$ is a vector, $\text{diag}(a)$ denotes a diagonal matrix that the diagonals are the entries of $a$. For two series $a_n$ and $b_n$, we say $a_n \asymp b_n$ if there is a constant $C$, such that $a_n \leq C b_n$ and $b_n \leq C a_n$ when $n$ is large enough. We say $a_n \precsim b_n$ if $\lim_{n \to \infty} a_n/b_n \leq 1$.

2 Model Setup and Methodology

A network with covariates can be represented as a duplex $(A, X)$, where $A \in \mathbb{R}^{n \times n}$ is the adjacency matrix and $X \in \mathbb{R}^{n \times p}$ is the covariate matrix. Suppose $\ell \in \mathbb{R}^n$ is the community label vector, of which each entry $\ell(i) \in \{1, 2, \ldots, K\}$ is the community label of node $i$. Our objective is to estimate the unknown $\ell$.

To solve the problem, we first establish the model for $(A, X)$ in Sections 2.1 and 2.2, called the DCSBM with Node Covariates. Then we introduce the intuition of ANC matrix $Y(\alpha)$ in 2.3, followed by the algorithm in Section 2.4.

2.1 Sparse Network Model

Following Holland et al. (1983) and Karrer and Newman (2011), we use the degree-corrected stochastic block model (DCSBM) for the network $A$. It is a generalization of stochastic block model, which is commonly used in network analysis; see Bickel and Chen (2009), Jin
(2015), Yan et al. (2014), and Zhao et al. (2012). Consider $A$ to be symmetric with 0 on the diagonals and independent Bernoulli random variables on the upper off-diagonals. Let $\Pi \in \{0, 1\}^{n \times K}$ be the community label matrix, where $\Pi(i, j) = I\{\ell(i) = j\}$. The connection probability between communities is parameterized as $P \in \mathcal{R}^{K \times K}$, where $P(k, l)$ denotes the probability of connections between a community $k$ node and a community $l$ node. To model degree heterogeneity, let $\theta_i$ denote the popularity of node $i$ and $\Theta = \text{diag}(\theta)$. With DCSBM, the expectation matrix $\mathbb{E}[A]$ is parameterized as

\begin{equation}
\mathbb{E}[A] = \Omega_A - \text{diag}(\Omega_A), \quad \Omega_A = \Theta \Pi P \Pi' \Theta. \tag{2.4}
\end{equation}

Element-wisely, for two nodes $i$ and $j$, $P(A(i, j) = 1) = \Omega_A(i, j) = \theta_i \theta_j P(\ell(i), \ell(j))$. It depends on the popularity of node $i$ and node $j$, and the connection probability between the communities $\ell(i)$ and $\ell(j)$.

When $\theta_i$’s are $O(1)$ constants, the connections are dense and the community detection problem is easy to solve. It’s often more interesting to consider the case where $\theta_i \to 0$ when $n \to \infty$, for any $i = 1, 2, \ldots, n$. It’s common to assume all $\theta_i$’s are at the same convergence rate; see Amini et al. (2013), Jin (2015) and Krzakala et al. (2013). However, some communities in practice tend to make more connections while other communities have more isolated nodes. In Joseph and Yu (2016), they are named as dense and weak clusters. Here we define an intensity parameter $\rho_k$ for community $k$.

**Definition 2.1.** Suppose for each community $k$, there is a constant $C > 0$, s.t. $\theta_i \leq C \theta_j$ if $\ell(i) = \ell(j) = k$. We call $\rho_k = \rho_k(n) = \min_{i: \ell(i) = k} \theta_i^2$ the intensity of community $k$.

We assume the convergence rate of $\theta_i$ to 0 is the same within one community; but these convergence rates can vary among communities. With this setup, we allow severe heterogeneity between communities in terms of their convergence rates.
Naturally, the adjacency matrix will play a more important role in recovery of communities with larger $\rho_k$. Hence, we define two types of communities.

**Definition 2.2.** For community $k$ with intensity $\rho_k$, we call it a

1. **Key Community**, if $\frac{n}{\log n} \rho_k \to \infty$ and $\rho_k \gtrsim \rho_j$, $j \neq k$;

2. **Secondary community**, if it is not a key community.

Note $\rho_k$’s for different communities may have the same order. So there can be multiple key communities. Denote $\mathcal{D}$ as the set of key communities. According to elements in $\mathcal{D}$, there are three kinds of networks.

- Relatively dense network: $\mathcal{D} = \{1, 2, \cdots, K\}$, i.e. all the communities share the same intensity and the connections are relatively dense.

- Extremely sparse network: $\mathcal{D}$ is empty, i.e. nodes in the most dense community still have a bounded degree when $n \to \infty$.

- Mixed network: $\mathcal{D}$ is neither empty nor the whole set, i.e. there are some communities with dense connections and some communities with sparse connections.

When $\mathcal{D}$ is non-empty, it can be shown that the orders of average degree, signal matrix norm $\|\Omega_A\|$ and noise matrix norm $\|A - \Omega_A\|$ are all decided by $\rho_k$ where $k \in \mathcal{D}$. That’s why they are called the key communities.

Based on the key communities, we define

\[
\rho_A = \begin{cases} 
\max_k \rho_k, & \text{if } \mathcal{D} \text{ is not empty;} \\
1, & \text{if } \mathcal{D} \text{ is empty.}
\end{cases}
\]
When $\mathcal{D}$ is non-empty, $\rho_A \propto \rho_k$ for $k \in \mathcal{D}$ and it represents the overall network intensity.

When there are no key communities, $\rho_A$ is taken to be an upper bound of the network intensity. By the definition of key communities, for any type of network, $n\rho_A \to \infty$.

When $\mathcal{D}$ is non-empty, we further check the gap between $\rho_A$ and the intensity of secondary communities. Denote $\mathcal{D}^c$ as the set of secondary communities. Define $\beta$ as

\[
\beta = \lim_{n \to \infty} \frac{1}{2} \log_n \frac{\rho_A}{\max_{k \in \mathcal{D}^c} \rho_k}.
\]

The definition of $\beta$ means $\rho_l \leq n^{-2\beta} \rho_A \propto n^{-2\beta} \rho_k$ for any $l \in \mathcal{D}^c$ and $k \in \mathcal{D}$. Hence, $\beta$ is the intensity gap between the key communities and the most dense community in $\mathcal{D}^c$. When $\mathcal{D}^c$ is empty, then $\beta$ can be any small non-zero constant.

\subsection{DCSBM with Node Covariates}

For covariates, we use the standard cluster model (Jin and Wang (2016); Jin et al. (2017b)), where $x_i$'s independently follow the distribution

\[
x_i \sim F_{\ell_X(i)}, \quad i = 1, 2, \ldots, n.
\]

Here $\ell_X(i) \in \{1, 2, \ldots, K_X\}$ denotes the covariate class label of node $i$, which can be different from the community label $\ell(i)$. $F_k$ is some general distribution in this section, and we will discuss cases where $F_k$ is a Gaussian or a multinomial in Section 3. Let $Q \in \mathcal{R}^{K_X \times p}$ be the matrix consisting of the mean vectors $\mu_1, \mu_2, \ldots, \mu_{K_X}$, where $\mu_k = \mathbb{E}[F_k] \in \mathcal{R}^p$. Let $\Pi_X$ be the label matrix corresponding to $\ell_X$, then there is $\mathbb{E}[X|\Pi_X] = \Pi_X Q$.

The remaining part is how to model the relationship between $\Pi$ and $\Pi_X$. Consider a node $i$ in community $k$. Let $t_k \in \mathcal{R}^{K_X}$ be the conditional probability that it has covariate class label $k_x$, which means $t_k(k_x) = P(\ell_X(i) = k_x|\ell(i) = k), k_x = 1, 2, \ldots, K_X$. Let $T^{(A,X)}$

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1We take $\rho_A = 1$ here, but any quantity between 1 and $\max_k \sqrt{\rho_k}$ will work.
be the matrix consisting of row vectors \( t_k \)'s, that is
\[
T^{(A,X)}(k,k_x) = P(\ell_X(i) = k_x | \ell(i) = k), \quad k = 1, 2, \cdots, K, \quad k_x = 1, 2, \cdots, K_X. \tag{2.8}
\]

Therefore, \( T^{(A,X)} \) describes the relationship between covariate classes and communities.

The expectation matrix \( \mathbb{E}[X] \) follows,
\[
\Omega_X = \mathbb{E}[X] = \Pi T^{(A,X)} Q. \tag{2.9}
\]

We call the model for \((A, X)\) as DCSBM with Node Covariates (DCSBM-NC), if (2.4) and (2.7)–(2.9) are satisfied.

### 2.3 Adjusted Neighbor Covariate Matrix

Our goal is to retrieve the community labels \( \ell \) based on \((A, X)\). Under DCSBM-NC, the covariate \( x_i \) follows \( F_{\ell_X(i)} \), which may not depend on \( \ell(i) \). This is a major challenge of using \( X \) to improve the recovery of \( \ell \). To solve this problem, we define
\[
y_i = \sum_{j:A(i,j)=1} x_j = \sum_j A(i,j) x_j, \quad \text{i.e. the sum of the neighbours' covariates.}
\]

The expectation of \( y_i \) is \( \mathbb{E}[y_i] = \theta_i \sum_{j=1,j\neq i}^n P(\ell(i), \ell(j)) \theta_j \mu_{\ell_X(j)} \), which now depends on \( \theta_i \) and \( \ell(i) \) only. Such property is inherited by the left singular vectors of \( AX \); see Lemma 2.1 below.

The usage of \( y_i \) is helpful when all the node have many neighbors. Now we consider the mixed network case that the community intensity \( \rho_k \)'s have different orders. Nodes in secondary communities may have too few neighbors to reveal the community labels. In the extreme case that node \( i \) is isolated, \( y_i = 0 \). For such nodes, we have to adjust \( y_i \) to include the covariate vector of the node itself, \( x_i \). Let \( d(i) \) be the degree of node \( i \). Let \( \bar{d} \) be the average of \( d(i), 1 \leq i \leq n \). Define the adjusted neighbour covariate (ANC) vector
\[
y_i(\alpha) = \sum_j A(i,j) x_j + \alpha_i x_i, \quad \text{where } \alpha_i = \alpha \min\{\frac{n^{-\beta}(\bar{d} + 1)}{d(i) + 1}, 1\}. \tag{2.10}
\]
Here $\beta$ is the intensity gap defined in (2.6) and $\alpha = n\rho_{A}$ in theoretical analysis. For practical data, the selection of $\alpha$ and $\beta$ is discussed in Sections 2.4 and 4.2.

The corrected vector $y_i(\alpha)$ combines both the neighbors’ covariate and the covariate of the node itself. For high-degree nodes in key communities, $d_i \approx \bar{d}$, so that $\alpha_i \approx n\rho_{A}n^{-\beta}$, which is at a lower order than $\|A\|$. Hence, $x_i$ will play a secondary role in $y_i(\alpha)$. For low-degree nodes in secondary communities, $\alpha_i \geq n\rho_{A} \gg \|A\|$ holds with large probability. Thus, $y_i(\alpha)$ will be dominated by $x_i$.

Putting all $y_i$’s together, we define the ANC matrix

$$Y(\alpha) = (A + \Lambda_{\alpha})X.$$ 

Denote $\Omega(\alpha) = \Omega_{A}\Omega_{X} + \Lambda_{\alpha}\Omega_{X}$ as the population version of $Y(\alpha)$. Consider the singular value decomposition of $\Omega(\alpha)$. Let $\Omega(\alpha) = \Xi \Lambda U'$ where $\Xi \in \mathcal{R}^{n \times K}$ consisting of top $K$ left singular vectors. Lemma 2.1 below shows that each row of $\Xi$ can be written as a weighted summation of $v_k$ and $w_k$, where $v_k$ and $w_k$ depends on the community label $k$ only. Consider all nodes in community $k$. According to the definition of $\alpha_i$ in (2.10), if community $k$ has a large intensity $\rho_k$, then these nodes have rows $\Xi_i \approx \theta_i v_k$; if community $k$ has a small intensity $\rho_k$, then these nodes have rows $\Xi_i \approx \alpha_i w_k$. In short, each community has a common center, up to a constant.

**Lemma 2.1.** Consider $\Omega(\alpha) = \Omega_{A}\Omega_{X} + \Lambda_{\alpha}\Omega_{X}$. Denote the singular value decomposition as $\Omega(\alpha) = \Xi \Lambda U'$. Under DCSBM-NC, there is

$$\Xi = \Theta \Pi V + \Lambda_{\alpha} \Pi W = \begin{bmatrix} \quad \quad \theta_1 v_{\ell(1)} + \alpha_1 w_{\ell(1)} \\
\quad \theta_2 v_{\ell(2)} + \alpha_2 w_{\ell(2)} \\
\quad \vdots \\
\quad \theta_n v_{\ell(n)} + \alpha_n w_{\ell(n)} \end{bmatrix},$$
where $w_k$’s are the row vectors of $W = (A, X)^TQUA^{-1} \in \mathcal{R}^{K \times K}$ and $v_k$’s are the row vectors of $V = PP'\Theta\Pi W \in \mathcal{R}^{K \times K}$.

To see how the ANC matrix $Y(\alpha)$ works, we examine its effects on a LastFM app user data; see Rozemberczki and Sarkar (2020). This network records app users from several countries and takes their mutual friendship as the connections. The covariate $x_i$ is the list of artists user $i$ follows. The true labels are the (unspecified) countries. Here we only consider the users in Country 5, 8 and 14 as an example since they have comparable sizes; more discussions of this issue can be found in Section 5. In Figure 1, we show the top two singular vectors (normalized by row norms) of $X$, $AX$ and $Y(\alpha)$ with $\alpha = 12$ in left panel, middle panel and right panel, respectively. Different types of point shapes represent different country labels, which is the underlying truth. It can be seen that nodes from different communities are better separated in the right panel. Actually, if we apply $k$-means to the normalized singular vectors, then the community detection error rates are 0.21 for the left panel and 0.16 for the right panel. In this example, $Y(\alpha)$ shows its improvement on community detection. More analysis of this dataset is allocated in Section 5.

2.4 Covariate Assisted SCORE (CA-SCORE)

By above discussions, we can see the normalized left singular vectors of $Y(\alpha)$ are well separated by communities. To understand it, recall that $\Xi_i \approx \theta_i v_{\ell(i)}$ for key communities and $\Xi_i \approx \alpha_i w_{\ell(i)}$ for secondary communities. We normalize the row $\Xi_i$ so that it has norm 1 and denote the normalized row as $R_i$. Then $R_i \approx v_{\ell(i)}/\|v_{\ell(i)}\|$ for communities with large $\rho_k$ and $R_i \approx w_{\ell(i)}/\|w_{\ell(i)}\|$ for the other communities. Therefore, $R_i$’s share the same center as long as these nodes are in the same community, no matter $\rho_k$ is large or small.

This paves the way to utilise ANC matrix $Y(\alpha)$ under DCSBM-NC. Let $\hat{\Xi} \in \mathcal{R}^{n \times K}$ be...
Figure 1: The top 2 normalized left singular vectors of $X$ (left panel), $AX$ (middle panel) and $Y(\alpha)$ (right panel). The nodes have true labels as Country 5 (circles), Country 8 (triangles) and Country 14 (pluses).

the matrix consisting of the top $K$ left singular vectors of $Y(\alpha)$. Define the normalized matrix $\hat{R} \in \mathbb{R}^{n \times K}$, that is

\[
\hat{R}(i, k) = \frac{\hat{\xi}_i(k)}{\|\hat{\xi}_i\|}, \quad \hat{\xi}_i \text{ is the } i\text{-th row of } \hat{\Xi}, \ k = 1, 2, \cdots, K.
\] (2.11)

Rows of $\hat{R}$ that corresponding to nodes in the same community are close to the same center. Apply $k$-means on $\hat{R}$ and we get the community detection results. We call this procedure the Covariate Assisted SCORE (CA-SCORE) method. Details are summarized in Table 1.

In Jin (2015), a similar mechanism is leveraged for the community detection problem using only the adjacency matrix $A$ under DCSBM. Let $\Xi(A)$ be the matrix of top $K$ eigenvectors of $A$, then the rows $\Xi_i(A) \approx \theta_i u_{\ell(i)}$. Jin (2015) proposes the Spectral Clustering On Ratios-of-Eigenvectors (SCORE) method, where the entry-wise ratio is taken between eigenvectors to get rid of the constants $\theta_i$. Our method is similar with SCORE in the way of normalizing the rows $\Xi_i$, but the problem and model of interest are very different. We
are interested in \((A, X)\) instead of \(A\) only. Under DCSBM-NC, \(\hat{R}_i\) is close to the center formed by \(A\) or formed by \(X\), depending on the community intensity \(\rho_k\). The technical proof is more complicated.

| Input: | the adjacency matrix \(A\), the covariate matrix \(X\), the number of classes \(K\); the tuning parameter \(\alpha\) (optional). |
|-------|---------------------------------------------------------------|
| Output: | Predicted community label vector \(\hat{\ell}\) |
| Step 1 | Find degree \(d_i = \sum_{j=1}^{n} A(i, j)\). Let \(\alpha_i = \alpha \ast \min\{\frac{\text{median}(d)}{d_i + 1}, 1\}\). Let \(\Lambda_{\alpha}\) be the diagonal matrix with diagonals as \(\alpha_i\). |
| Step 2 | Let \(Y(\alpha) = (A + \Lambda_{\alpha})X\) be the ANC matrix. |
| Step 3 | Obtain the first \(K\) leading left singular vectors of \(Y(\alpha)\), denoted as \(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_K\). |
| Step 4 | Denote \(\hat{\Xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_K)\). Let \(\hat{\Xi}_i\) be the \(i\)-th row vector of \(\hat{\Xi}\). |
| Step 5 | Construct the ratio matrix \(\hat{R}\) by \(\hat{R}(i, k) = \frac{\hat{\Xi}_i(k)}{\|\hat{\Xi}_i\|}\), \(i = 1, \ldots, n\), \(k = 1, \ldots, K\). |
| Step 6 | Estimate \(\hat{\ell}\) by applying \(k\)-means to \(\hat{R}\) with number of clusters as \(K\), i.e. \(\hat{\ell} = \arg\min_l L(\hat{R}, l)\), where \(L(\hat{R}, l) = \sum_{k=1}^{K} \sum_{i:l(i)=k} (\hat{R}_i - \bar{R}_k)^2\). Here, \(\hat{R}_i\) is \(i\)-th row of \(\hat{R}\) and \(\bar{R}_k\) is the average of \(\hat{R}_i\)'s with \(l(i) = k\). |

Table 1: Algorithm: Covariate Assisted SCORE

**Remark 1.** For practical data, we use \(\text{median}(d)/\bar{d}\) in place of \(n^{-\beta}\) in (2.10). It works well in synthetic and real datasets; see Sections 4 and 5.

**Remark 2.** We have a tuning parameter \(\alpha\) to decide whether the network is overall dense enough. When \(\alpha\) is unknown, we choose \(\alpha\) by a grid search on \((\alpha_{\text{min}}, \alpha_{\text{max}})\), where \(\alpha_{\text{min}} = \lambda_K(A)/4\) and \(\alpha_{\text{max}} = \lambda_1(A) \log(n)/\bar{d}\). The minimum value \(\alpha_{\text{min}} = \|A\|/4\) is comparatively small so that the information mainly depends on \(A\), but it is not exactly zero so that it can be applied for unconnected networks. The maximum value \(\alpha_{\text{max}} = \lambda_1(A) \log(n)/\bar{d}\), which is bigger for sparse network and smaller for dense network. The optimal \(\alpha\) is chosen to be the one with smallest \(L(\hat{R}, \hat{\ell})\) defined in Step 6; as Binkiewicz et al. (2017). More
details of this procedure are located in Section 4.2.

**Remark 3.** For Step 6, we apply built-in kmeans function in R, which finds a local optima by the algorithm in Hartigan and Wong (1979).

### 3 Convergence of CA-SCORE

In this section, we show that CA-SCORE yields satisfactory community detection results under DCSBM-NC and regular conditions. To facilitate the discussion, we first introduce several cases and then establish the results in Section 3.1 and 3.2. The proof is deferred to Appendix 8, where we further introduce a more general theorem on DCSBM-NC.

We propose some regularity conditions on the network and covariates. These conditions are to assure the network will not degenerate.

**Assumption 3.1.** For a DCSBM with $\Omega_A = \Theta \Pi \Pi' \Theta$, when $n \to \infty$, assume that

1. the number of communities $K > 0$ is fixed;
2. there is a constant $r_0 > 0$, so that $n_k/n \geq r_0$ for each community $k = 1, \ldots, K$;
3. $P \in \mathcal{R}^{K \times K}$ is fixed, with $P(k,k) > 0$ for all $k$;
4. Let $\rho_k = \min_{i: l(i) = k} \theta_i^2$. There exists a constant $C > 0$, so that $\rho_l(i) \leq \theta_i^2 \leq C \rho_l(i)$ holds for any $i = 1, 2, \ldots, n$.

**Assumption 3.2.** For a DCSBM-NC with $\Omega_X = \Pi T^{(A,X)} Q$, when $n \to \infty$, assume that

1. $T^{(A,X)}$ is fixed with $\text{rank}(T^{(A,X)}) = K$;
2. All the mean covariate vectors $\mu_{k_x}$ have norms at the same order, denoted as $L_{\mu}$. It means there exists $c, C > 0$ so that $c L_{\mu} \leq \|\mu_k\| \leq C L_{\mu}$, $k = 1, 2, \ldots, K_X$. 

3. rank\( (Q) = K_X \) and \( \lambda_{K_X}(Q) \gtrsim L_{\mu} \).

Condition 1) is to assure the relationship between \( \Pi \) and \( \Pi_X \) is fixed when \( n \to \infty \). Further, each community has a special pattern in \( T^{(A,X)} \) and \( K_X \geq K \). Condition 2) requires all the mean vectors are at the same order \( L_{\mu} \), which can be regarded as covariate information strength. Here, \( L_{\mu} \) may increase when \( n \to \infty \). Condition 3) requires the mean vectors to be linearly independent, and a control over the smallest non-zero singular value.

Finally we set up \( \Lambda_\alpha \). Recall \( d = \sum_{i=1}^{n} d_i \) is the average degree. Define \( \Lambda_\alpha \) as a diagonal matrix with diagonals \( \alpha_i \), where

\[
\alpha_i = \begin{cases} 
n\rho_A \min \left\{ \frac{n^{-\beta E[d_i]}}{d_i+1}, 1 \right\}, & \text{if } D \text{ is not empty}; 
n\rho_A, & \text{if } D \text{ is empty}.
\end{cases}
\] (3.12)

### 3.1 Gaussian Covariates

Recall that \( D \) is the set of key communities. Let \( K_D = |D| \) be the number of key communities and \( P_D \in \mathbb{R}^{K_D \times K_D} \) be the sub-matrix of \( P \) related to \( D \), the set of key communities. Define the mismatching rate between \( \ell \) and \( \ell_X \) for the secondary communities, as

\[
q_0 = \max_{k \in D^c}(1 - T^{(A,X)}(k,k)).
\] (3.13)

We have the following theorem when \( X \) follow Gaussian distributions.

**Theorem 3.3.** Suppose \( X_i \sim N(\mu_{\ell_X(i)}, \Sigma_{\ell_X(i)}) \), where \( \mu_k \in \mathbb{R}^p \) is the mean vector and \( \Sigma_k \) is the corresponding covariance matrix, \( j = 1, \ldots, K_X \).

Under the conditions of Assumptions 3.1 and 3.2, further suppose \( P_D \) has full rank and \( \sigma^2_{\Sigma} = \max \{ \|\Sigma_1\|, \|\Sigma_2\|, \ldots, \|\Sigma_{K_X}\| \} \) is bounded. Let \( \hat{\ell} \) be the estimated label by CA-SCORE with \( \Lambda_\alpha \) defined in (3.12). When \( n \to \infty \) and \( p = o(n) \), with probability \( 1 - O(1/n) \), there
exists a permutation $\pi$ on \{1,2,\ldots,K\}, so that

$$Err_n = \frac{1}{n} \sum_{i=1}^{n} I\{\pi(\hat{\ell}_i) \neq \ell_i\} \leq C((\sqrt{\frac{p \log n}{n}} + \frac{1}{\sqrt{n \rho_A}}) \frac{1 + L_{\mu}}{L_{\mu}} + \frac{\sqrt{1 + q_0 L_{\mu}^2}}{L_{\mu}/\sigma_\Sigma} + n^{-\beta}).$$

In particular, when the covariate information strength $L_{\mu}$ defined in Assumption 3.1 goes to infinity, CA-SCORE will recover the community labels with error rate $C\sigma_\Sigma \sqrt{q_0}$.

Theorem 3.3 explains the role of key and secondary communities in community detection results. For nodes in the key communities, $\ell(i)$’s are recovered by $AX$. $P_D$ must have full rank, which means the communities can be identified in $\Omega_A$. The corresponding error rate can achieve 0. For nodes in the secondary communities, $\ell(i)$’s are recovered by $X$. The error rate of this part comes from the mismatch between $\ell$ and $\ell_X$, summarized by $q_0$, which is decided by $D^c$.

When the covariates are Gaussian distributed, the dimension of covariates $p$ appears in the error term as $\sqrt{p \log n/n}$, which goes to 0 for $p = O(1)$ or $p = o(n)$. We further need the covariate information $L_{\mu} \to \infty$ to assure successful community detection. The results can also be extended to sub-Gaussian distributions.

To better understand Theorem 3.3, we consider two special cases in the following two corollaries, which focus on the relatively dense network and the extremely sparse network.

**Corollary 3.1.** Suppose $X_i \sim N(\mu_{\ell_X(i)}, \Sigma_{\ell_X(i)})$, where $\mu_k \in \mathcal{R}^p$ is the mean vector and $\Sigma_k$ is the corresponding covariance matrix, $j = 1,\ldots,K_X$. Denote $\sigma_\Sigma^2 = \max_k \|\Sigma_k\|$.

Suppose Assumptions 3.1 and 3.2 hold, $\mathcal{D} = \{1,2,\cdots,K\}$, $\text{rank}(P) = K$, $\sigma_\Sigma$ is bounded and $L_{\mu} \to \infty$. Let $\hat{\ell}$ be the estimated label of CA-SCORE with $\Lambda_\alpha$ defined in (3.12). When $n \to \infty$ and $p = o(n)$, with probability $1 - O(1/n)$, there exists a permutation $\pi$ on \{1,2,\ldots,K\}, so that

$$Err_n = \frac{1}{n} \sum_{i=1}^{n} I\{\pi(\hat{\ell}_i) \neq \ell_i\} = o(1).$$
In Corollary 3.1, we consider the relatively dense network, where all communities are key communities. For this case, \( \rho_k \)'s are all at the same order with \( n\rho_k \to \infty \). According to literature, the community labels can be recovered by \( A \); see Amini et al. (2013); Jin (2015); Krzakala et al. (2013). For this case, \( \Lambda_\alpha \) will end up with \( \alpha_i = o(n\rho_k) \), which is a small term compared to \( ||A|| \simeq n\rho_k \). The community detection results mainly depend on \( AX \) and achieves error rate 0.

**Corollary 3.2.** Suppose \( X_i \sim N(\mu_{\ell X(i)}, \Sigma_{\ell X(i)}) \), where \( \mu_k \in \mathbb{R}^p \) is the mean vector and \( \Sigma_k \) is the corresponding covariance matrix, \( j = 1, \ldots, K_X \). Denote \( \sigma^2_\Sigma = \max_k ||\Sigma_k|| \).

Suppose Assumptions 3.1 and 3.2 hold, \( \mathcal{D} \) is empty, \( \sigma_\Sigma \) is bounded and \( L_\mu \to \infty \). Let \( \hat{\ell} \) be the estimated label of CA-Score with \( \Lambda_\alpha \) defined in (3.12). When \( n \to \infty \) and \( p = o(n) \), with probability \( 1 - O(1/n) \), there exists a permutation \( \pi \) on \( \{1, 2, \cdots, K\} \), so that

\[
Err_n = \frac{1}{n} \sum_{i=1}^{n} I\{\pi(\hat{\ell}_i) \neq \ell_i\} \leq C\sigma_\Sigma \sqrt{\bar{q}_0} + o(1).
\]

In Corollary 3.2, we consider the extremely sparse network, where there are no key communities. For this case, \( n\rho_k \) is bounded for any \( k \), \( \rho_A = 1 \) and \( \alpha_i \asymp n \). Therefore, \( \Lambda_\alpha X \) dominates \( Y(\alpha) \). The community detection results mainly depend on \( X \) and the error rate \( \sqrt{\bar{q}_0} \) is not avoidable due to the random design.

### 3.2 Multinomial Covariates

For LastFM app user data and the citation network data, the covariates are of discrete values. While the covariate dimension \( p \) is relatively large, more than two thirds of \( X \) entries are 0. The non-zero entries are all integers. The multinomial covariates model is a better model for such datasets, we have a separate theorem as follows.
Theorem 3.4. Suppose $X_i \sim Multinomial(\psi_{\ell(i)}, m_{\ell(i)})$, where $\psi_k \in \mathbb{R}^p$, $k = 1, \ldots, K_x$ are probability vectors and $m_k$ is the multiplicity.

Suppose Assumptions 3.1 and 3.2 hold, $\mathcal{D} = \{1, 2, \cdots, K\}$, $\text{rank}(P) = K$ and $m_k \leq m$ for a universal $m > 0$. Let $\hat{\ell}$ be the estimated label of CA-SCORE with $\Lambda_{\alpha}$ defined in (3.12). There is a constant $C > 0$, When $n \to \infty$ and $p = O(1)$, with probability $1 - O(1/n)$, there exists a permutation $\pi$ on $\{1, 2, \cdots, K\}$, so that

$$Err_n = \frac{1}{n} \sum_{i=1}^{n} I\{\pi(\hat{\ell}_i) \neq \ell_i\} \leq C(\rho_A n)^{-1/2}.$$

Compared with Gaussian covariates, the noise is strong for multivariate covariates. Hence we require $p = O(1)$ to control noise. However, in Section 4, we can see CA-SCORE still works when $p > n$.

4 Experiments

In this section, we applied our method on large scale simulated networks with high dimensional covariates. The results of the newly proposed method are denoted as CA-SCORE. We compare them with the methods that use only the adjacency matrix and methods that use both the adjacency matrix and the covariates. The former set of methods contains (i) spectral clustering on ratio of eigenvectors (SCORE) (Jin (2015)), (ii) normalized PCA (nPCA) that applies spectral clustering on penalized Laplacian matrix (Chung and Graham (1997)), and (iii) ordinary PCA (oPCA) that applies spectral clustering on $A$ directly (Chaudhuri et al. (2012b)). The latter set of methods contains (i) covariate-assisted spectral clustering (CASC) in Binkiewicz et al. (2017) and (ii) semidefinite programming on the summation of the adjacency matrix and the covariates kernel matrix (SDP) in Yan and Sarkar (2021). For each method, we take the number of community $K$ as known, and
match the estimated label with the true label in the way that the mis-classification nodes are the least.

Note that SCORE requires the network to be connected. So we apply SCORE on the giant component only and the error rate for SCORE is calculated based on the giant component. In our simulation settings, the giant component occupies more than 99% of all nodes in Experiment 1 and 88% to 100% of all nodes in Experiment 2. For all the other methods including CA-SCORE, the results are based on the whole network.

4.1 Simulation Studies

We conduct two experiments to explore the performance of CA-SCORE when the community labels and covariate class labels mismatch and the network intensity changes. We consider Gaussian covariates in Experiment 1 and multinomial covariates in Experiment 2. For both, we consider a moderately large \( p \) and a large \( p \).

**Experiment 1.** To investigate the performance when there is mismatching between community labels and covariate class labels, we let \( K = 4 \) and \( K_X = 8 \). For Node \( i \), we assign its covariate class label in the way that \( \ell_X(i) = \ell(i) \) with probability \( a \) and \( \ell_X(i) \) be one of the other 7 classes with probability \( (1 - a)/7 \). In other words, \( T^{(A,X)}(k, k) = a \) and \( T^{(A,X)}(k, j) = (1 - a)/7 \) for \( j \neq k \).

To set up network \( A \), we take \( n = 1000 \), which produces a big network. The nodes are randomly assigned to each of \( K = 4 \) communities with equal probability. The connection matrix \( P \) has 1 on the diagonals and 0.3 on the off-diagonals. The degree heterogeneity is defined as \( \theta_i = u_i \sqrt{\rho_k} \), where \( u_i \sim \text{Uniform}(0.1, 0.4) \) and \( \rho_k = 1/2 \) for \( k = 1, 2 \) and \( \rho_k = 1 \) for \( k = 3, 4 \). Hence, communities 1 and 2 have sparser connections than communities 3 and 4. To construct \( X \), we first set up \( \Omega_X = \Pi_X Q \). The label matrix \( \Pi_X \) is generated from \( \ell \).
Figure 2: The community detection error rate among 50 repetitions versus the matching rate $a$ between community labels and covariate class labels.

Let $a = 0.4 + 0.05i$, $i = 0, 1, \cdots, 12$, where a larger $a$ indicates more similarity between the community labels and covariate class labels. We run 50 repetitions. The error rate for $p = 800$ and $p = 2000$ are presented in Figure 2.

In Figure 2, the error rates of SCORE, nPCA and oPCA are stable around 0.5, which is reasonable since they are based on the network only. Note that there are 4 communities, an error rate 0.5 means there is some information in the estimated labels, but not much. Incorporating the covariates, CA-SCORE, CASC and SDP all show improvements, especially when $a > 0.7$ and $p = 2000$. When the information in covariates are also weak, say $a < 0.7$ or $p = 800$, then only CA-SCORE can combine the two resources of information.
and improve the community detection results.

Experiment 2. To explore the algorithms’ performance when the network intensity changes, we set up network $A$ in the following way. Take $n = 600$ and $K = 4$. Let $P$ be 1 on the diagonals and 0.3 on the off-diagonals, which is the same with Experiment 1. We define the node popularity parameter $\theta_i = \theta_{\text{min}} + (0.5 - \theta_{\text{min}}) \times (i/n)^2$, $i = 1, 2, \cdots, n$. We will test the methods with $\theta_{\text{min}} = 0.01k$, where $k = 1, 2, \cdots, 25$. The overall network intensity increases when $\theta_{\text{min}}$ gets bigger.

Now we consider $X$. Let $p = 400$ be a moderately large covariate dimension and $p = 800$ be a large covariate dimension. Let the number of covariate classes be $K_X = 10$. For node $i$, we take $\ell_X(i) = \ell(i)$ with probability 0.7 and $\ell_X(i) = k$ with probability $0.3/9$ when $k \neq \ell(i)$. For each node, the covariate follows a multinomial distribution that

$$x_i \sim \text{Multinomial}(m_i, \psi_{\ell_X(i)}).$$

We take $m_i \sim \text{Uniform}(70, 120)$, $i = 1, 2, \cdots, n$. For the probability vector $\psi_k$, we first generate $\tilde{\psi}_k$ with independent Uniform$(0, 1)$ entries and then let $\psi_k = \tilde{\psi}_k/\|\tilde{\psi}_k\|_1$.

For each choice of $\theta_{\text{min}}$, the error rates over 50 repetitions are summarized in Figure 3. As $\theta_{\text{min}}$ increases, the network density increases, hence the error rates of network-based methods decrease fast. For sparse network with small $\theta_{\text{min}}$, combining the covariate information largely improves results, especially for CA-SCORE. When $\theta_{\text{min}}$ increases to moderately dense, CASC and SDP methods still rely mainly on the covariates and their performance remain constant, while the newly proposed CA-SCORE improves with the network density. For dense network with a large $\theta_{\text{min}}$, CASC switches to the information in $A$ suddenly and all methods excluding SDP enjoy similar small error rates.
Figure 3: The community detection error rate among 50 repetitions versus the minimal node popularity $\theta_{\min}$. Left panel: results when $p = 400$. Right panel: results when $p = 800$.

4.2 Choice of Tuning Parameter

There is a tuning parameter $\alpha$ in CA-SCORE. We propose to select $\alpha$ in a data adaptive range that minimizes the within-cluster sum of squares in $k$-means.

To explore the robustness of CA-SCORE versus $\alpha$, we generated $A$ and $X$ the same as Experiment 2 with $\theta_{\min} = 0.10$ and $p = 800$. Consider a broad range $\alpha \in (0.05, 5 \times \|A\|)$. We take 40 values of $\alpha$ in this range and check the corresponding mis-classification error rate. In Figure 4, we show how the error rate changes versus $\alpha$, where we can see an optimal value of $\alpha$ that the error rate is small. The dashed line represents the adaptive $\alpha$ chosen in the algorithm, which is close to the optimal value.
Figure 4: The error rate versus $\alpha$. Dashed line: Adaptive $\alpha$ chosen in the algorithm.

5 Real World Networks

5.1 LastFM Network

LastFM asian dataset is a social network of LastFM app users. This dataset was collected and cleaned by Rozemberczki and Sarkar (2020) from the public API in March 2020. The nodes are users of LastFM from 18 unspecified Asian countries. Users on this app can follow each other, which can be seen as the edges of the network. We removed the smallest country with only 16 users, because it is too small to provide label information. Each user has a list of liked artists, which can be regarded as the node covariates. The goal is to estimate the country of each node by the network with covariates.

The sizes of users in countries vary significantly from 54 to 1572. Most existing community detection methods require the community sizes to be similar and non-degenerate.

Datasets are available at https://github.com/benedekrozemberczki/FEATHER/tree/master/new_node_level_dataset.
Therefore, we decompose them into 4 groups: small-sized countries (less than 100 nodes), medium-sized countries (between 100 and 300 nodes), large-sized countries (between 300 and 1000) and huge-sized countries (> 1000). For each group, we first screen the artists by selecting the regional popular artists who enjoys the largest ratio between the number of fans in the corresponding group and the overall number of fans, and then applied CA-SCORE, CASC, SDP, and nPCA on the dataset. We do not include SCORE since the giant component is only half of the whole network for the small-sized countries group, which will lead to unfair comparison. The processing and comparison code is available at https://tinyurl.com/CASCORE-Network.

| Country Group  | n   | p   | K | CA-SCORE | CASC | SDP | nPCA |
|---------------|-----|-----|---|----------|------|-----|------|
| Small size    | 343 | 194 | 6 | .227     | .178 | .545| .510 |
| Medium size   | 612 | 324 | 3 | .041     | .044 | .361| .444 |
| Large size    | 2488| 600 | 5 | .274     | .371 | .513| .643 |
| Huge size     | 3691| 600 | 3 | .018     | .019 | .416| .512 |

Table 2: Community detection error rates and NMI for LastFM app users. Columns 1-4 are the basic information of the dataset, including the group of countries, the network size n, the number of covariates p and the number of countries K. The community detection error rates and NMI of all methods are in Columns 6-9.

The error rates and Normalized Mutual Information (NMI) of all methods are presented in Table 2. A lower error rate, or a higher NMI, means better recovery results. According to Table 2, if we rely on the network to recover the community labels, then nPCA shows
an error rate around 0.5 and NMI between 0.1 to 0.4. Combining with the covariates, then the results are significantly improved, especially by CA-SCORE and CASC. CA-SCORE slightly improved the result over CASC on the medium-sized and huge-sized countries. It largely outperforms CASC on the large-sized countries but worse than CASC on the small-sized countries.

5.2 Citation Network

The citation network was published in Ji and Jin (2016). It contains 3232 papers published in Annals of Statistics (AoS), Journal of American Statistical Association (JASA), Journal of Royal Statistical Society (Series B) (JRSS-B) and Biometrika from 2003 to the first half of 2012. Each paper is treated as a node and the edges show whether they have cited the same paper or are cited by one paper simultaneously. For each paper, the abstract is record, for which the text is analyzed as the covariates. The relevant data, code and estimated labels can be found at https://tinyurl.com/CASCORE-Network.

The citation network has 3232 nodes as papers and the covariate vector has length $p = 4095$. Compared to the large number of $n$ and $p$, only 0.5% of the adjacency matrix entries and 1.4% of the covariate matrix entries are non-zeros. It contains 995 connected components, where the giant component has 2179 nodes and all other components containing no more than 9 nodes. In particular, 957 of this network are isolated nodes. If we apply SCORE, we have to remove about one third of nodes to consider the giant component only.

We apply CA-SCORE to the data with $K = 5$. For each community, we check the most popular papers and corpus. The results can be interpreted as: Variable selection (regression) community; Variable selection (semi-parametric) community; Large-scale multiple testing community; Biostatistics community; and Bayesian community. Compare to the
results on the authors in Ji and Jin (2016), both methods generate the variable selection community, large-scale multiple testing community, biostatistics community, and Bayesian community (non-parametric community in Ji and Jin (2016)). One thing to note is that the variable selection community is decomposed into two communities, one about regression and one about the semi-parametric models. According to the network on nodes with at least 50 connections in Figure 5, the biostatistics community and variable selection regression community is well separated while the semi-parametric community has connections with both.

In Table 3, we record the community sizes which are comparatively balanced, and also number of nodes in the giant component in each community. CA-SCORE presents good community detection results even for the isolated nodes. We randomly check two papers with no connections at all. Node 2893 has title “Bayesian pseudo-empirical-likelihood
intervals for complex surveys (Rao and Wu, 2010)” and is classified into the Bayesian community. Node 2481 that titled “Testing dependence among serially correlated multi-category variables (Pesaran and Timmermann, 2009)” is classified into the Multiple Testing community.

| Nodes             | Regression | Multiple Testing | Biostatistics | Bayesian | Semi-parametric |
|-------------------|------------|------------------|---------------|----------|-----------------|
| Whole Network     | 250        | 324              | 499           | 1050     | 1109            |
| Giant Component   | 235        | 205              | 308           | 670      | 761             |
| Other Components  | 15         | 119              | 191           | 380      | 348             |

Table 3: The community sizes of the citation network by CA-SCORE.

6 Discussion

In this article, we model the relationship between the adjacency matrix and the node covariates by DCSBM-NC. Under DCSBM-NC, we propose the CA-SCORE method which combines the information from $A$ and $X$, in ways of the ANC matrix. ANC matrix converts the covariate labels into community labels by the node neighbors. Such usage of neighbors can be found in other network problems; see Hu et al. (2022). The information from covariates support a satisfactory community detection result, even when the expected degree of $A$ is bounded.

The idea of CA-SCORE is inspired by the SCORE method for DCSBM; see Jin (2015). In recent years, there are plenty of works following SCORE, such as mixed-SCORE on mixed membership stochastic blockmodels, D-SCORE on directed networks, Hier-SCORE on hierarchical community detection (Ji and Jin (2016); Jin et al. (2017a); Ji et al. (2022)), etc. How to adapt and interpret such community structure when there are node covariates will be an exciting problem that requires a lot of work.
In this article, we mainly discuss the high-dimensional covariates. For such a case, a proper feature screening step should help. Hence, it is of interest to design a network-assisted feature screening step that works for random variables with excessive zeros, with which features related to the community labels will be kept. In Zhao et al. (2022), a network-assisted dimension reduction method for covaiartes has been proposed, which yields good theoretical results when \( p \) is small (\( p = 5 \) in the simulation studies). The screening method for large \( p \) still needs exploration.

7 Supplementary Materials

Title: Additional proofs and experimental results (.pdf file).

8 Appendix

In this Appendix, we present a general theorem about CA-SCORE. We set up new assumptions that targets on \( \Omega_A \) and \( \Omega_X \) directly. It can be viewed as the results of CA-SCORE on the oracle case. The assumptions on \( \Omega \) are as follows.

Assumption 8.1. The ideal model \( \Omega = \Omega(\alpha) := (\Omega_A + \Lambda_\alpha)\Omega_X \) is identifiable using CA-SCORE if the follows are satisfied when \( n \to \infty \):

1. The \( K \)-th spectral gap of \( \Omega \) is proportional to its spectral norm, i.e., there exists \( c_{\text{eig}} > 0 \), \( \Delta_K(\Omega) := \lambda_K(\Omega) - \lambda_{K+1}(\Omega) \geq c_{\text{eig}} \|\Omega\| \).

2. Let \( \Xi \) be the matrix consisting of the top \( K \) leading left singular vectors of \( \Omega \). The row norms of \( \Xi \) are of order \( n^{-1/2} \), i.e., for some \( c_\xi > 0 \),

\[
\|\Xi_i\| \geq c_\xi \sqrt{K} n^{-1/2}, \quad i = 1, \ldots, n. \tag{8.14}
\]
3. There exists $K$ vectors $m_k(\Omega) \in \mathcal{R}^K$ and $c_m > 0$, so that $\|m_k(\Omega) - m_l(\Omega)\| \geq c_m$, $1 \leq k, l \leq K$, and $\delta > 0$ so that $\|R_i - m_{\ell(i)}\|^2 \leq \delta_1$ for all $i$.

4. There is a constant $r_0 > 0$, so that $n_k/n \geq \rho$ for each community $k = 1, \ldots, K$.

Condition 1) is a regular condition so that the spectral gap is non-degenerate; see Jin (2015); Lei and Rinaldo (2015). Condition 2) is to guarantee that the rows in the ratio matrix $R$ are meaningful. Actually, since that the column norm of $\Xi$ is always 1, so in general case $|\Xi(i,j)| = O(n^{-1/2})$ and $\|\Xi\| = \sqrt{\sum_{j=1}^{K} |\Xi(i,j)|^2} = O(n^{-1/2})$. Condition 3) is to guarantee that the communities are distinct. Condition 4) is a regular condition to ensure the communities are reasonably large; see Jin (2015); Yan and Sarkar (2021); Binkiewicz et al. (2017); Yang et al. (2013); Lei and Rinaldo (2015); Zhang et al. (2016).

We are now ready to state the oracle property of CA-SCORE as the following theorem. Under these conditions, CA-SCORE always achieves successful community detection results on the ideal matrix $\Omega$.

**Theorem 8.2.** Suppose Assumption 8.1 holds with some $\delta_1 < c_2^2/4$ and $\|Y(\alpha) - \Omega\| \leq \delta_2\|\Omega\|$ with probability $1 - O(1/n)$. Let $\hat{\ell}$ be the estimated labels by CA-SCORE. Denote $C_1 = \frac{32}{c_{\text{eig}}} + \frac{4}{c_\xi}$. Then the follows hold with probability $1 - O(1/n)$,

1. For estimated community $k$, define the corresponding community center as the average of corresponding rows in $\hat{R}$, i.e. $\hat{m}_k = \frac{1}{|\{i: \hat{\ell}(i) = k\}|} \sum_{i: \hat{\ell}(i) = k} \hat{R}_i$. Up to an orthogonal matrix $O \in \mathcal{R}^{K \times K}$ and a permutation $\pi$ on $\{1, 2, \cdots, K\}$, there is

   $$\|\hat{m}_{\pi(k)}O' - m_k\| \leq \frac{\sqrt{(C_1\delta_2 + \delta_1)/\rho}}{\rho}, \quad k = 1, 2, \cdots, K;$$

2. With the matching $\pi$ in 1), the mis-classification error rate by CA-SCORE is bounded,

   $$\text{Err}_n = \frac{1}{n} \sum_{i=1}^{n} I\{\pi(\hat{\ell}_i) \neq \ell_i\} \leq \frac{16(C_1\delta_2 + \delta_1)}{c_m^2}.$$
Therefore, the community detection error rate converges to 0 when $\delta_1 \to 0$ and $\delta_2 \to 0$. The proof of this theorem can be found in Supplementary Materials.

Based on the main theorem, we can prove Theorems 3.1 and 3.4. We first show that as long as Assumptions 3.1 and 3.2 hold, then Assumption 8.1 here hold. Then we show how the mis-classification error is related to the noise term. Detailed proofs in supplementary materials due to space limit.

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