Density Perturbations in String Cosmology

Michele Maggiore\textsuperscript{a,b} and Riccardo Sturani\textsuperscript{b,c}

\textsuperscript{(a)} INFN, sezione di Pisa, Pisa, Italy
\textsuperscript{(b)} Dipartimento di Fisica dell’Università, piazza Torricelli 2, I-56100 Pisa, Italy
\textsuperscript{(c)} Scuola Normale Superiore, piazza dei Cavalieri 7, I-56125 Pisa, Italy.

We discuss the generation and evolution of density perturbations during the large curvature phase of string cosmology. We find that perturbations in the scalar components of the metric evolve with cosmic time as \(\exp(\gamma H_s t)\) where \(H_s\) is the Hubble constant during the string phase and \(\gamma\) is a constant which is determined by an algebraic equation involving all orders in the string coupling \(\alpha'\). The seed for the perturbations can be provided by massive string modes.

I. INTRODUCTION

A very important issue in any cosmological model is the generation of density perturbations which evolved into the presently observed large scale structures of the Universe. At present, two main mechanisms have been studied. One is the amplification of zero point fluctuations at the transition between two cosmological phases, as in the transition from an inflationary phase to a standard radiation dominated phase; and the second possibility relies on topological defects produced at a GUT phase transition (see e.g. \cite{1,2}).

In the last few years, a cosmological model based on the low energy effective action of string theory has been proposed by Gasperini and Veneziano \cite{3–5}. The model addresses the kinematical problems of standard cosmology thanks to a superinflationary 'pre-big-bang' phase (see however ref. \cite{6} for critical remarks) and to a subsequent De-Sitter like inflationary stage, and has a number of interesting phenomenological consequences \cite{5,7–11}. The purpose of this paper is to examine the issue of the generation and evolution of density perturbations in this model.

The analysis of refs. \cite{4,12} indicates that the mechanism of amplification of vacuum fluctuations is not effective in string cosmology at very large wavelengths and it is probably unable to produce density fluctuations at the level \(\delta\rho/\rho \sim 10^{-5}\) as required for COBE anisotropy and for galaxies formation. Of course, it is still possible that density perturbations are generated by cosmic strings or other topological defects at the GUT phase transition, or in general after the end of the high curvature phase of the model. In this paper, however, we explore the possibility that a genuinely 'stringy' effect in the high curvature regime is responsible for the generation of density fluctuations.

The paper is organized as follows. In sect. 2 we briefly recall the main features of the cosmological model. In sect. 3 we compute the evolution of scalar perturbations during the high curvature regime in a simplified setting. (The details of the full analysis are relegated in the Appendix.) In sect. 4 we discuss a possible mechanism for the generation of the seed for scalar perturbation. Sect. 5 contains the conclusions.

II. A SHORT SUMMARY OF STRING COSMOLOGY

In this section we present the main features of the model and we establish the notations. For a more detailed presentation we refer the reader to refs. \cite{3–5}. The model is obtained by considering the Neveu-Schwarz bosonic sector of string theory. The low-energy effective action is built from the massless states of the string: the graviton \(g_{\mu\nu}\), the dilaton \(\phi\) and the antisymmetric field \(B_{\mu\nu}\) and, in the string frame, it reads

\[
S^{(0)} = -\frac{1}{2\lambda^2} \int d^4x \sqrt{-g} e^{-\phi} \left[ R + (\nabla\phi)^2 - \frac{1}{12} H^2 \right], \tag{1}
\]
where $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$, $R$ is the Ricci scalar and $\lambda_s$ is the string length. This action receives two kinds of corrections \cite{14}. The analogue of quantum field theory loop corrections, which are governed by $e^\phi$, and genuinely stringy corrections which are controlled by the string constant $\alpha'$ (related to the string length $\lambda_s$ and to the string tension $\mu$ by $\sqrt{2\alpha'} = \lambda_s = 1/\sqrt{\pi\mu}$).

In the pre-big-bang scenario the evolution starts from the string perturbative vacuum at low curvature and $\phi \to -\infty$, where both kind of corrections are negligible and physics is well described by the effective action \cite{14}, and it follows a phase of superinflationary evolution, which ends in a singularity at a finite value of cosmic time. However, when the curvature reaches a value of order one in string units, $\alpha'$ corrections to eq. (1), e.g. terms $\alpha'R_{\mu\nu\rho\sigma}^2$, become important. As shown in \cite{13}, at first order in $\alpha'$ they do indeed stop the growth of the curvature: the effective action at order $\alpha'$ receives a correction which depends on the Gauss-Bonnet term $R_{\text{GB}}^2$ \cite{14} and after a field redefinition at order $\alpha'$ it can be written in the form

$$S^{(1)} = -\frac{1}{2\lambda_s^2} \int d^4x \sqrt{-g} e^{-\phi} \left[ R + (\nabla\phi)^2 - \frac{\alpha'}{4} (R_{\text{GB}}^2 - (\nabla\phi)^4) \right],$$

where we have reabsorbed into $\alpha'$ a factor $k = 1, 1/2$ for bosonic and heterotic string respectively and we have assumed that the torsion background $B_{\mu\nu}$ is trivial.

The action \cite{14} has a solution of the equations of motion of the form $H = \text{const.} = H_0$, $\phi = \text{const.} = c$, where $H$ is the Hubble constant, $H_0, c$ are numerical constants of order $1/\lambda_s$, and the dot is the derivative with respect to cosmic time. This solution acts as a late time attractor of the lowest order pre-big-bang solution obtained from eq. (1). Of course, when terms $O(\alpha')$ are crucial, also terms $O(\alpha'^2)$, etc. become crucial. Therefore we are in a truly stringy regime. The general structure of the equations of motions at all orders in $\alpha'$ is discussed in ref. \cite{13}, where it is found that a solution of the form $H = \text{const.} = H_0$, $\phi = \text{const.} = c$, persists at all orders in $\alpha'$ if a set of two algebraic equations in the two unknown $c, H_0$, involving all orders in $\alpha'$, has real solutions. In the following, we assume that this is indeed the case.

Finally, this 'string phase' should be matched to the standard radiation dominated phase. The problems connected with implementing this transition ('graceful exit') have been investigated recently by various authors \cite{13}.

### III. EVOLUTION OF DENSITY PERTURBATIONS DURING THE STRING PHASE

Metric perturbations around a FRW background can be classified according to their properties under spatial rotations. Scalar perturbations, which couple to density fluctuations, can be written in the longitudinal gauge in terms of two scalar functions $\psi_1, \psi_2$ as \cite{14}

$$ds^2 = a^2(\eta) \left\{ (1 + 2\psi_1)d\eta^2 - (1 - 2\psi_2)dx^2 \right\},$$

where $a$ is the scale factor, $\eta$ is conformal time, and we are considering a spatially flat 4-dimensional FRW background. In this section, to illustrate our results, we limit ourselves to fluctuations with $\psi_1 = -\psi_2$, i.e. to metrics which can be written in terms of a single function $h$ as

$$g_{\mu\nu}(\eta, x) = [1 + h(\eta, x)] g_{\mu\nu}(\eta)$$

where $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$ is the background metric. The fact that these fluctuations are simply fluctuations in the scale factor allows a substantial simplifications of the relevant equations and allows us to present equations which are not too lengthy. As it will be clear below, we are only interested in general properties of the solutions, and we will show in the Appendix that these properties still hold if we consider the general case with $\psi_1$ and $\psi_2$ independent.

We also expand the dilaton as $\phi(\eta, x) = \phi_0(\eta) + \varphi(\eta, x)$. We make these substitutions into eq. (1) and expand to second order in $h, \varphi$. The condition that the linear terms vanish give the dynamical equations of motion for the background,

$$\ddot{R} - (\nabla \phi_0)^2 + 2\nabla^2 \phi_0 - \alpha' \left[ \frac{1}{4} (R_{\text{GB}}^2 + 3(\nabla \phi_0)^4) - (\nabla \phi_0)^2 \nabla^2 \phi_0 - 2\nabla^\mu \phi_0 \nabla^\nu \phi_0 \nabla_\mu \nabla_\nu \phi_0 \right] = 0$$

*We use the sign conventions $\eta_{\mu\nu} = (+, -, -, -)$ and $R^\nu_{\nu\mu\rho} = \partial_\mu \Gamma^\nu_{\rho\sigma} - \ldots$. 

2
\[ \tilde{R} - 2(\nabla \phi_0)^2 + 3 \nabla^2 \phi_0 - \alpha' G^{\mu \nu} (\nabla_\mu \phi_0 \nabla_\nu \phi_0 - \nabla_\mu \nabla_\nu \phi_0) = 0. \]  

(6)

(The overbar denotes quantities constructed with \( \bar{g}_{\mu \nu} \), and \( G^{\mu \nu} \) is the Einstein tensor. Indices are raised and lowered with the background metric, and covariant derivatives \( \nabla_\mu \) are with respect to the background metric. The first equation is obtained with a variation with respect to \( \phi \) and the second with respect to \( h \).

Furthermore, the variation with respect to the lapse function gives a constraint on the initial values.

For our homogeneous background, \( \bar{g}_{\mu \nu} = a^2(\eta) \eta_{\mu \nu}, \phi_0 = \phi(\eta) \) the dynamical equations of motion, eqs. (11,12), become

\[
2\ddot{\phi}_0 - \dot{a}^2 \phi - \dot{a} \dot{\phi}_0 - \alpha' \left\{ \frac{\dot{a}^2}{a^2} - \frac{6 \ddot{a}}{a} - \frac{\ddot{\phi}_0}{\phi} - \frac{3 \dot{\phi}_0}{\phi} - \frac{3 \dot{\phi}_0^2}{a^2} \right\} = 0, \tag{7}
\]

\[
3\ddot{\phi}_0 - 2\dot{a}^2 + 9\ddot{\phi}_0 - \frac{6 \dot{a}^2}{a^2} - \frac{6 \ddot{a}}{a} - \alpha' \left\{ -6 \ddot{\phi}_0 - \frac{6 \dot{\phi}_0}{a} + \frac{3 \ddot{\phi}_0^2 + \ddot{\phi}_0^2}{a^2} - \frac{3 \dot{\phi}_0}{a^3} \right\} = 0, \tag{8}
\]

while the constraint equation is

\[
- \frac{\dot{a}^2}{a^2} - \dot{\phi}_0^2 + \frac{6 \dot{a} \dot{\phi}_0}{a} + \frac{3 \dot{\phi}_0^2}{4} = 0. \tag{9}
\]

Eqs. (7,8,9) have the solution \( H = \dot{H}_s, \phi = c \) which, in terms of conformal time, reads

\[
a(\eta) = -\frac{1}{\dot{H}_s \eta}, \quad \phi(\eta) = -\frac{c}{\dot{H}_s} \log(-\dot{H}_s \eta) \quad (\eta < 0). \tag{10}
\]

Notice that the ansatz (10) reduces eqs. (7,8,9) to 3 equations in 2 unknowns \( \dot{H}_s, c \); remarkably, on this ansatz the 3 equations are not independent and admit a common solution. This goes through at all orders in \( \alpha' \) due to general properties of the beta functions of the underlying sigma model (13).

Expanding the action at second order in \( \varphi, h \) and then varying it, or linearizing eqs. (6,8), we get the equations of motion for the fluctuations, which read

\[
2 \nabla^\mu \phi_0 \nabla_\mu \varphi - 2 \nabla^2 \varphi + 3 \nabla^2 h - 2 \nabla^\mu \phi_0 \nabla_\mu h - h (R - (\nabla \phi_0)^2 + 2 \nabla^2 \phi_0) + \alpha' \left[ -G^{\mu \nu} \nabla_\mu \nabla_\nu \varphi + \nabla_\mu \varphi \left( 4 \nabla_\nu \phi_0 \nabla_\nu \nabla^\mu \phi_0 - 3 \nabla_\mu \phi_0 (\nabla \phi_0)^2 + 2 \nabla_\mu \phi_0 \nabla^2 \phi_0 \right) + \nabla_\mu \nabla_\nu \varphi \left( 2 \nabla_\mu \phi_0 \nabla_\nu \phi_0 + G^{\mu \nu} (\nabla \phi_0)^2 \right) \right] = 0, \tag{11}
\]

and

\[
-4 \nabla^\mu \phi_0 \nabla_\mu \varphi + 3 \nabla^2 \varphi - 3 \nabla^2 h + 3 \nabla_\mu \phi_0 \nabla_\mu h + \alpha' \left[ (\nabla_\mu \nabla_\nu h) \tilde{R}^{\mu \nu} + \nabla_\mu \nabla_\nu \phi_0 - \nabla_\mu \phi_0 \nabla_\nu \phi_0 - G^{\mu \nu} (\nabla^2 \phi_0 - (\nabla \phi_0)^2) \right] + \nabla_\mu h \left( \frac{1}{2} \tilde{R} - R^{\mu \nu} \nabla_\mu \nabla_\nu \phi_0 + h \tilde{G}^{\mu \nu} \left( \nabla_\mu \nabla_\nu \phi_0 - \nabla_\mu \phi_0 \nabla_\nu \phi_0 \right) + 2 \nabla_\mu \nabla_\nu \tilde{G}^{\mu \nu} \nabla_\mu \nabla_\nu \varphi \right] = 0. \tag{12}
\]

(12)

(In the above expressions we have not yet used the specific form of the background.) The linearization of the constraint equation is of course satisfied with the initial conditions \( h = \varphi = 0 \).

Although formally correct, eqs. (11,12,13) suffer from a serious ambiguity. Consider for instance the coefficient of the term \( \alpha' h \) in eq. (12), i.e., \( \tilde{G}^{\mu \nu} (\nabla_\mu \phi_0 \nabla_\nu \phi_0 - \nabla_\mu \phi_0 \nabla_\nu \phi_0) \). We are free to add to \( \tilde{G}^{\mu \nu} \) a term which vanishes because of the equations of motions of the background, e.g., any arbitrary constant times the left-hand side of eq. (11). The latter, however, is made of two separately non zero terms, one of order zero in \( \alpha' \) and one of order \( \alpha' \). In a formal expansion in powers of \( \alpha' \) of the coefficient of \( h \) in eq. (12), part of the term that we have added would contribute to order \( \alpha' \) and part to order \( \alpha'^2 \); at the order at which we are working, this second part would be discarded. Thus, the coefficient of \( h \) in eq. (12), at order \( \alpha' \), could be equally well written as

\[
- \alpha' \left[ \tilde{G}^{\mu \nu} (\nabla_\mu \phi_0 \nabla_\nu \phi_0 - \nabla_\mu \phi_0 \nabla_\nu \phi_0) + c_1 (\tilde{R} - (\nabla \phi_0)^2 + 2 \nabla^2 \phi_0) \right] , \tag{13}
\]

with \( c_1 \) an arbitrary constant. The same holds for all other operators in eqs. (11,12,13); for instance the coefficient of the term \( \nabla_\mu \nabla_\nu h \) can be modified adding a term proportional to \( \tilde{G}^{\mu \nu} \) times the left-hand side of eq. (11). Therefore, a truncation of the coefficients at a finite order in \( \alpha' \) gives ambiguous results.

One might try to argue that there is at least a 'natural' prescription for the coefficients at any finite order which consists of taking eqs. (11,12) as they are, without adding terms proportional to the equations of motion. However,
even this is not true. Consider for instance the term \((\nabla^2 \phi_0 - (\nabla \phi_0)^2)\) which appears at order \(\alpha'\) in eq. (12). If we insert the explicit expression for \(\phi_0\), eq. (10), this is equal to \(3cH_0 - c^2\). However, if we combine eqs. (5) and (12) we find that \((\nabla^2 \phi_0 - (\nabla \phi_0)^2)\) is \(O(\alpha')\) and, since this term appears in eq. (12) already multiplied by \(\alpha'\), it gives a contribution \(O(\alpha'^2)\). Clearly, there is no natural choice between these two ways of treating this term.

What we learn, therefore, is that a truncation at any finite order in \(\alpha'\) gives completely arbitrary results. Only the full answer, including all orders in \(\alpha'\), is meaningful. This is a consequence of the fact that the solution around which we are expanding emerges only after \(\alpha'\) corrections are taken into account, and it is not a solution of the lowest order effective action, eq. (1). So, the background solution mixes terms of different order in \(\alpha'\).

We can however still use eqs. (13,12) in order to learn something on the structure of the equations at all orders. Using the explicit form of the background, eq. (10), we find that eqs. (13,12) can be recast in the general form

\[
\eta^2 \left( a_1^{(i)} \frac{d^2 h}{d\eta^2} - a_2^{(i)} \nabla^2 h \right) + a_3^{(i)} h + a_4^{(i)} \eta \frac{dh}{d\eta} + \eta^2 \left( b_1^{(i)} \frac{d^2 \varphi}{d\eta^2} - b_2^{(i)} \nabla^2 \varphi \right) + b_3^{(i)} \eta \frac{d\varphi}{d\eta} = 0,
\]

where \(\nabla\) is the flat space spatial gradient and the index \(i = 1, 2\) enumerates the two equations obtained from eqs. (11,12), respectively. The coefficients \(a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, a_4^{(i)}, b_1^{(i)}, b_2^{(i)}, b_3^{(i)}\) are independent of \(\eta\) and are formally given as a power expansion in \(\alpha'\). From eqs. (11,12) we formally get for instance \(a_1^{(2)} = -3H_0^2 + \alpha'(3H_0^2 - cH_0^2)\), etc., but as explained before, a truncation at finite order in \(\alpha'\) is actually meaningless. Note that this would be true even if, for some numerical accident, \(H_0, c\) were much smaller than one in units \(\alpha' = 1\).

Let us introduce the modes \(h_k(\eta)\) performing the Fourier transform of \(h(\eta, x)\) with respect to \(x\), and similarly for \(\varphi\). Consider in eq. (14) the term \(a_2^{(i)} \frac{d^2 h}{d\eta^2} - a_2^{(i)} \nabla^2 h\), which can be rewritten as \(a_2^{(i)} [(1/c_0^2) \frac{d^2 h_k}{d\eta^2} + k^2 h_k]\) where \(c_0 = (a_2/a_1)^{1/2}\) represents, physically the speed of the perturbation and, on general grounds, \(c_0 < 1\). For wavelengths with \(\eta k c_0 < 1\) the gradient term \(k^2 h_k\) is small compared to \((1/c_0^2) \frac{d^2 h_k}{d\eta^2} \sim k^2 h_k/(c_0^2 \eta^2)\). Of course, this is just the condition that the physical wavelength of the perturbation is greater than the Jeans length. On the other hand, if \(c_0 < 1\), there is a window of values of the wavelength for which at the same time the perturbation is still within the horizon, \(k\eta > 1\). The latter condition is important because for super-horizon-sized modes it becomes relevant the fact that the scalar perturbations \(\psi_1, \psi_2\) are not exactly gauge-invariant. Indeed, \(\psi_1, \psi_2\) are the Bardeen variables written in the longitudinal gauge (1,1,1), and are gauge-invariant only at first order for small coordinate transformations. The effect of non gauge invariance becomes important for physical wavelength larger than the horizon, \(\lambda_{\text{phys}} > H^{-1}\), where \(\lambda_{\text{phys}} = a(\eta)\lambda\). In a DeSitter background this means \(k \eta < 1\) where \(k = 1/\lambda\) is the comoving wavenumber. For \(k \eta < 1\) a growing perturbation can be a gauge artefact with no physical meaning.

If instead \(1 < k \eta < 1/c_0\) there is a range of wavelengths for which all gradient terms can be neglected but the perturbations are still within the horizon; then we see that eq. (14) for \(h_k, \varphi_k\) depends on \(\eta\) and \(d/\eta\) only through the combination \(\eta d/\eta\). This was not obvious a priori. On dimensional grounds, \(\eta c_0 \ll 1\), \(\eta c_0 \ll 1\), \(\eta c_0 \ll 1\), and therefore the coefficients \(a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, a_4^{(i)}, b_1^{(i)}, b_2^{(i)}, b_3^{(i)}\), etc., could have been functions of \(\eta\), while instead they only depend on the constants \(H_0, c, \alpha'\). This result can be traced to the specific form of the De Sitter background and of the dilaton, eq. (1).

To understand this point, observe for instance that in this background, using conformal time, the Ricci tensor is given by \(\tilde{R}^{\mu
u} = -3H_0^2 \eta^2 \eta^{\mu\nu}\), while \(\tilde{\varphi}^{\mu
u} = H_0^2 \eta^2 \eta^{\mu\nu}\), \(\partial^{\mu} \phi_0 = \tilde{\varphi}^{\mu\nu} \partial_{\nu} \phi_0 = -cH_0 \eta \delta^{0}_{\nu}\) and similarly \(\nabla^{\mu} \nabla^{\nu} \phi_0 \sim \eta^2\). So, in a term like \(\nabla^{\mu} \nabla^{\nu} h\) in eqs. (13,12), independently of whether we use \(\tilde{R}^{\mu\nu}, \tilde{\varphi}^{\mu\nu}\) or derivatives with respect to \(\phi_0\) to saturate the indices \(\mu, \nu\), we get a factor of \(\eta\) for each index to be saturated. One can easily check that the argument goes through for all operators one can construct, and therefore it holds at all orders in \(\alpha'\). For instance, at higher orders we will have terms with more than two derivatives, e.g. \(\nabla_\mu \nabla_\nu \nabla_\rho \phi h\). Whatever operator we use to saturate the four indices, we get a term \(\sim \eta^4 d^4 h/d\eta^4\).

The existence of a range of values of \(k\) for which the perturbation is within the horizon, \(k \eta > 1\), but still the gradient term can be neglected, is a general properties due basically to the fact that the speed of the scalar perturbation (‘speed of sound’) is smaller than the speed of light, and we expect that it goes through at all orders in \(\alpha'\), when higher order spatial and temporal derivatives come into play. If this is the case then, for such wavelengths, the full equations governing the perturbations, at all orders in \(\alpha'\), have the general form

\[
F_i(\eta) \frac{d}{d\eta} h_k(\eta) + G_i(\eta) \frac{d}{d\eta} \varphi_k(\eta) = 0,
\]

where \(i = 1, 2\) enumerate the equation and \(F_i, G_i\) are unknown functions. The non-trivial content of eq. (15) is that they depend on \(\eta\) only through the combination \(\eta d/\eta\). Looking for solutions of the form \(h_k \sim \eta^\gamma, \varphi_k \sim \eta^\gamma\), reduces eqs. (15) to a set of two algebraic equations for the two unknown \(\beta, \gamma\):

\[
F_1(\gamma) + G_1(\beta) = 0, \quad F_2(\gamma) + G_2(\beta) = 0.
\]
The situation is particularly interesting if the above equations have a real and negative solution for $\gamma$. In fact, since in DeSitter space the relation between conformal and cosmic time is $-H_s \eta = \exp(-H_s t)$, we get an exponentially growing mode

$$h_k \sim \exp\{|\gamma|H_s t\}.$$ \hfill (17)

The maximum amplification which can be obtained is related to the duration of the string phase. Denoting with $\eta_s$ the value of conformal time at which the string phase begins and with $\eta_f$ the value at which it ends, \(\eta_s, \eta_f < 0, |\eta_s| > |\eta_f|\) the maximum amplification factor is \((\eta_s/\eta_f)^{\gamma^*}\). The parameter \((\eta_s/\eta_f)\) (equal to $f_1/f_s$ in the notation of ref. [1]) is a free parameter of the model, and can be very large. In fact, the most interesting phenomenological situation from the point of view of the production of relic gravitational waves is realized when this number is at least of order $10^8$ [8,11].

We see therefore that string cosmology has a possible built-in mechanism which allows to amplify in a substantial way very tiny initial inhomogeneities.

Of course, the above result only holds in the linear regime. However, this is sufficient for our purpose, which is to find a mechanism which generates density fluctuations at the level \((\delta \rho/\rho)_k \sim h_k \sim 10^{-5}\) at wavelength of the order of the present Hubble radius or at the scale of galaxies formation.

IV. THE SEED FOR PERTURBATIONS

In the previous section we found that, depending on the form of some unknown algebraic functions $F_i, G_i$, which involve a full (all-orders in $\alpha'$) computation, we can have a very large enhancement of initial density fluctuations. The next step is to identify a possible candidate mechanism which generates a seed which is then amplified.

In the dilaton dominated regime where the action (1) gives a good description of physics, there is no mechanism which can generate inhomogeneities. In fact, as recently shown by Veneziano [18], even starting with generic inhomogeneous initial conditions the model evolves into a highly homogeneous spacetime. Therefore, we have to look again into the string phase in order to find a seed for density perturbations.

Until now we have limited our attention to the massless modes of the string. However, massive string modes are associated with $\alpha'$ corrections to the action [18,20], and must be included for consistency once we include corrections of the type $\alpha'R^2_{\mu\nu\rho\sigma}$. In fact the complete effective action of the string has the general form [20]

$$S = \int d^2 \sigma \sum_{n=0}^{\infty} (\alpha')^n \sum_{i_1=1}^{N_1} \mathcal{O}^{(n)}_i (\partial X)^{(n)} B^{(n)}_{i_n}(X),$$ \hfill (18)

where the string coordinates have been rescaled, $X^\mu \rightarrow \sqrt{\alpha'}X^\mu$, $B_{i_n}$ are the background field at a given massive level (rescaled so that they are dimensionless) and $\mathcal{O}^{(n)}_i$ the corresponding operators; $N_1$ is the number of irreducible operators for given $n$. The power of $\alpha'$ is uniquely connected with the mass level. At least working order by order in the mass level [20], the condition of quantum Weyl invariance of the action induces $\alpha'$ corrections to the equations of motion derived from eq. (1). Since we have seen that $\alpha'$ corrections have a crucial effect in the string phase, it is clear that in this phase it is not appropriate to restrict the attention to the graviton-dilaton-antisymmetric tensor field sector.

Of course, a description of the full stringy regime is very difficult to obtain. However, a number of general properties are well understood from the study of strings at very large temperature and densities [21]. The main results are as follows. At low temperatures and densities the canonical ensemble provides a good description of a gas of strings. As $T$ approaches the Hagedorn temperature $T_H$, the canonical ensemble breaks down and one has to resort to the more fundamental microcanonical ensemble. As $T \rightarrow T_H$ the canonical energy density tends to a finite value $\rho_c$ (in dimensions $d \geq 4$). The most intriguing result is that if we increase the energy density beyond this critical value, all the excess energy goes into a single highly excited string, rather than raising the temperature of the string ensemble. In the high curvature phase of string cosmology the parameter which controls the value of the curvature is $H_s$. If $H_s$ exceeds a critical value $H_c$, the energy density of the gravitational field exceeds the critical energy density $\rho_c$ and the excess energy goes into a very excited string. More precisely, we should expect the formation of one very excited string in each horizon volume, since regions separated by more than a horizon distance behave independently.\footnote{Here we are using the common abuse of language of calling $H^{-1}$ the horizon distance. Actually, in string cosmology the... This...}
means that, if $H_0 > H_c$, the favored thermodynamical configuration is highly inhomogeneous, and the massive string modes can act as the seed for density perturbations.

To understand more in detail the effect of the highly excited strings, let us denote by $\Theta_{\mu \nu}$ their energy momentum tensor. It acts effectively as an external source, and the equation of motion of the perturbations obtained with a variation with respect to $g_{\mu \nu}$ or, in our case, with respect to $h$ (which corresponds to $i = 2$ in eq. (15)) becomes, at all orders in $\alpha'$,

$$F_2(\eta \frac{d}{d\eta}) h_{kk}(\eta) + G_2(\eta \frac{d}{d\eta}) \varphi_k(\eta) = \lambda^2 e^{\gamma \Theta},$$

(19)

where $\Theta = \bar{g}^{\mu \nu} \Theta_{\mu \nu}$, while the equation with $i = 1$ remains homogeneous.

As already mentioned, eqs. (14), and therefore their linearization, eqs. (18,19), do not exhaust the content of the classical theory. Another equation is obtained varying the action with respect to the lapse function $N$, which is then set equal to one with a choice of gauge. This equation, of course, is a constraint on the initial data, which is conserved by the dynamical equations of motions. The linearization of the constraint equations gives as the initial conditions for the perturbations $h = \varphi = 0$ (and similarly all the time derivatives of $h, \varphi$ can be set to zero up to one order less than the higher order derivative appearing in the dynamical equations). Since the energy momentum tensor is coupled to the lapse function, the constraint equation in the presence of massive string modes has the general form $C(h, \varphi) = \Theta$ where $C$ is a function of $h, \varphi$ and of their derivatives which vanishes when all its arguments are zero.

Therefore, we see that the ‘external source’ $\Theta$ has a two-fold effect. First, it forces $h$ to have a non-zero initial value, through the constraint equation. And second, the evolution of the perturbation will be the sum of a particular solution of the inhomogeneous equation (19) and of the general solution of the associated homogeneous equation, which includes (if $\gamma < 0$) the exponentially growing mode (17). These two effects, which are quite general, correspond to what have been termed as ‘initial compensation’ and ‘subsequent compensation’ in ref. 22.

It is quite difficult to estimate the form of the spectrum of density perturbations, since it depends on the details of the string phase, and we leave the problem open for further work. Even more difficult is to estimate the numerical value of $(\delta \rho/\rho)_k$, since it involves in an essential way the numerical values of $\eta_s, \eta_1, \gamma$ on which we have no clue.

V. CONCLUSIONS

The problem of the generation and evolution of density perturbations in string cosmology appears to be a very difficult one, since it fully involves the details of the high curvature stringy regime, for which we do not have an adequate theoretical understanding.

Still, a number of general considerations appear to be possible. It turns out that in a number of situations some dynamical questions concerning the string phase and which therefore in principle are very complicated, can actually be reduced to the issue of the existence of a certain algebraic equation involving all orders in $\alpha'$.

An example of this phenomenon has been found in ref. 13, where the issue of the existence of a solution of the equations of motion which forbids the lowest order pre-big bang solutions from running into a singularity has been shown to reduce, to all orders in $\alpha'$, to the existence of a real solution of an algebraic equation. In this paper we have found a similar characterization for the existence of exponentially growing scalar perturbations during the string phase. The basic mechanism is that, for wavelengths larger than the Jeans length, in a DeSitter background with a dilaton growing linearly with cosmic time, the equations of motion of the perturbation have solutions of the form $\eta^{-1}$ and, in DeSitter space, $\eta \sim \exp(-H_s t)$; this results, for $\gamma < 0$, in an exponentially growing mode.

From the physical point of view, both the exponential growth of the perturbation, and the generation of seeds from highly excited strings appear under rather general circumstances and it is therefore possible that string cosmology has its own, genuinely stringy mechanism for generating large scale density perturbations.

As already stressed in 13, an all order solution should correspond to an exact conformal field theory (see 23 for work along these lines) and our algebraic conditions could have a natural counterpart in the conformal field theory approach.

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particle horizon is infinite 4, but still $H^{-1}$ gives the scale of the distance between points which formally recede from each other at the speed of light.
APPENDIX A:

In this Appendix we examine the equations governing the evolution of the perturbations in the general case with \( \psi_1 \) and \( \psi_2 \) independent. Technically, we lose the advantage of having a metric conformal to Minkowski metric, and as a consequence most of our equations become very lengthy. However we are only interested in the general properties of these equations, and in particular in the fact that they only depend on \( \eta d/d\eta \) once spatial gradients have been neglected.

The equations of motion for the background are of course unchanged. We now have three dynamical equations of motion for the fluctuations, which can be obtained by expanding the action at second order and performing the variations with respect to \( \varphi, \psi_1, \psi_2 \). The variation w.r.t. \( \varphi \) gives

\[
2\nabla_\mu \varphi \nabla^\mu \phi_0 - 2\nabla^2 \varphi + 2\nabla_\mu (\psi_1 + \psi_2) \nabla^\mu \phi_0 + 4 \nabla_0 (\psi_1 + \psi_2) \nabla^0 \phi_0 - 4 \nabla_\mu \psi_1 \nabla^\mu \phi_0 - 6\nabla^2 \psi_2 + 2\nabla_k \nabla^k (\psi_1 + \psi_2) + \\
+ (\psi_1 + 3\psi_2) \bar{R}^0_0 - (\psi_1 - \psi_2) \bar{R}^i_i + (\psi_2 - \psi_1) (2\nabla^2 \phi_0 - \nabla_\mu \nabla^\mu \phi_0) + 2(\psi_1 + \psi_2)(2\nabla_0 \nabla^0 \phi_0 - \nabla_0 \phi_0 \nabla^0 \phi_0) + \\
- \frac{\alpha'}{4} \{ 8\nabla_\mu \nabla^\mu \bar{G}_\mu^\nu - R^2_{GB} \} + 3\bar{R}^0_{GB} (\psi_1 + \psi_2) + 4(\nabla \phi_0)^2 (\nabla^2 \varphi - 3\nabla \phi_0 \nabla \varphi) + 16\nabla_\mu \nabla^\mu \phi_0 \nabla^\nu \phi_0 \nabla \varphi + \\
+ 8\nabla^2 \phi_0 \nabla_\mu \phi_0 \nabla^\mu \phi_0 + 8\nabla_\mu \nabla^\nu \phi_0 \nabla^\mu \phi_0 \nabla^\nu \phi_0 + (\psi_1 + \psi_2) \left[ -36 \nabla_0 \nabla^0 \phi_0 \nabla_0 \nabla^0 \phi_0 + 9(\nabla \phi_0)^4 + \\
- 4\nabla_i \nabla^i \phi_0 \nabla_0 \phi_0 \nabla^0 \phi_0 \right] - 12 \nabla_0 (\psi_1 + \psi_2)(\nabla_0 \phi_0 \nabla_0 \phi_0 \nabla^0 \phi_0) \right] = 0.
\]

where repeated latin indices like \( i, j, k \ldots \) mean summation over the three spatial coordinates and the index 0 denotes the time coordinate. The variation w.r.t. \( \psi_1 \) gives

\[
2 \nabla_i \nabla^i \varphi - 2\nabla_\mu \varphi \nabla^\mu \phi_0 + 3(\psi_1 + \psi_2) \bar{R}^0_0 - (\psi_1 + \psi_2) \bar{R}^i_i + \\
+ (\psi_1 + \psi_2) \left[ (\nabla \phi_0)^2 + 2\nabla_0 \phi_0 \nabla^0 \phi_0 - 2\nabla_\mu \phi_0 \nabla^\mu \phi_0 + 4\nabla_k \nabla^k \phi_0 - 2\nabla_\mu \phi_0 \nabla^\mu \phi_0 - 4\nabla_0 \phi_0 \nabla^0 \phi_0 + \\
- \frac{\alpha'}{4} \{ 3\nabla_0 \bar{R}^0_0 + 9\bar{R}^0_{GB}\psi_1 - \psi_1) + 40 \nabla_\mu \nabla^\mu \phi_0 \nabla^2 \phi_0 - 2\nabla_0 \phi_0 \nabla^0 \phi_0 \nabla_\mu \phi_0 \nabla^\mu \phi_0 + (\psi_1 + \psi_2) \left[ 36 \nabla_0 \nabla^0 \phi_0 + 9(\nabla \phi_0)^4 + \\
- 4\nabla_i \nabla^i \phi_0 \nabla_0 \phi_0 \nabla^0 \phi_0 \right] - 12 \nabla_0 (\psi_1 + \psi_2)(\nabla_0 \phi_0 \nabla_0 \phi_0 \nabla^0 \phi_0) \right] = 0.
\]

To simplify the equations, in the terms between curly brackets we have used the fact that \( \phi_0 \) does not depend on spatial coordinates, and that during the stringy phase every component of Riemann and Ricci tensor with an equal

\[
A_i (\eta) \frac{d}{d\eta} \psi_1 (\eta) + B_i (\eta) \frac{d}{d\eta} \psi_2 (\eta) + C_i (\eta) \frac{d}{d\eta} \varphi_k (\eta) = 0,
\]

where now \( i = 1, 2, 3 \), and therefore we can look for solutions \( \psi_1 \sim \eta^{\gamma_1}, \psi_2 \sim \eta^{\gamma_2}, \varphi \sim \eta^\beta \), which reduce eqs. \( \text{(A2)} \) to a set of 3 algebraic equations for 3 unknowns \( \gamma_1, \gamma_2, \beta \). The general situation is therefore completely analogous to that discussed in the text.
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