From Schrödinger spectra
to orthogonal polynomials,
via a functional equation

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Abstract The main difference between certain spectral problems for linear Schrödinger operators, e.g. the almost Mathieu equation, and three-term recurrence relations for orthogonal polynomials is that in the former the index ranges across $\mathbb{Z}$ and in the latter only across $\mathbb{Z}^+$. We present a technique that, by a mixture of Dirichlet and Taylor expansions, translates the almost Mathieu equation and its generalizations to three term recurrence relations. This opens up the possibility of exploiting the full power of the theory of orthogonal polynomials in the analysis of Schrödinger spectra. Aforementioned three-term recurrence relations share the property that their coefficients are almost periodic. We generalize a method of proof, due originally to Jeff Geronimo and Walter van Assche, to investigate essential support of the Borel measure of associated orthogonal polynomials, thereby deriving information on the underlying absolutely continuous spectra of Schrödinger operators.

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1 The almost Mathieu equation and orthogonal polynomials

The point of departure of our analysis is the almost Mathieu equation (also known as the Harper equation). We seek \( \lambda \in \mathbb{R} \) and \( \{a_n\}_{n \in \mathbb{Z}} \in \ell_1[\mathbb{Z}] \) that satisfy

\[
a_{n-1} - 2\kappa \cos(\alpha n + \beta) a_n + a_{n+1} = \lambda a_n, \quad n \in \mathbb{Z}.
\]

(1.1)

\( \alpha, \beta \) and \( \kappa \neq 0 \) being given real constants.

The almost Mathieu equation features in a number of applications \([9]\) and has been already extensively studied \([1, 3, 14, 15]\). The purpose of our analysis is not to reveal new features of the spectrum of (1.1) per se, since the latter is quite comprehensively known. Instead, we intend to demonstrate that the almost Mathieu equation exhibits an intriguing connection with orthogonal polynomials, a connection that lends itself to far reaching generalizations.

Let \( \omega = e^{i\alpha} \), \( b_1 = \kappa e^{i\beta} \) and \( b_2 = \kappa e^{-i\beta} = \bar{b}_1 \). We rewrite (1.1) as

\[
(b_1 \omega^{2n} + b_2) a_n = \omega^n (a_{n-1} - \lambda a_n + a_{n+1}), \quad n \in \mathbb{Z}
\]

(1.2)

and consider the Dirichlet expansion

\[
y(z) = \sum_{n=-\infty}^{\infty} a_n \exp \left\{ b_1^{\frac{1}{2}} \omega^n z \right\}, \quad z \in \mathbb{C}.
\]

The choice of a specific branch of the square root of \( b_1 \) is arbitrary. Note that, since \( |\omega| = 1 \), it is easy to demonstrate that \( \{a_n\}_{n \in \mathbb{Z}} \in \ell_1[\mathbb{Z}] \) implies convergence of the series for all \( z \in \mathbb{C} \) \([7, 10]\).

We multiply (1.2) by \( \exp \left\{ b_1^{\frac{1}{2}} \omega^n z \right\} \) and sum for \( n \in \mathbb{Z} \). Since

\[
y'(z) = b_1^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} a_n \omega^n e^{b_1^{\frac{1}{2}} \omega^n z}, \quad y''(z) = b_1 \sum_{n=-\infty}^{\infty} a_n \omega^{2n} e^{b_1^{\frac{1}{2}} \omega^n z},
\]

we readily deduce that \( y \) obeys the functional differential equation

\[
y''(z) + b_2 y(z) = b_1^{\frac{1}{2}} \left\{ \omega^{-1} y'(\omega^{-1} z) - \lambda y'(z) + \omega y'(\omega z) \right\}.
\]

(1.3)

The solution of (1.3) is determined uniquely by the values of \( y(0) \) and \( y'(0) \).

Dirichlet expansions have been employed by Gregorí Defel and Stanislav Molchanov \([4]\) to investigate the simplified spectral problem

\[
a_{n-1} - 2\kappa q^n a_n + a_{n-1} = \lambda a_n, \quad n \in \mathbb{Z},
\]

where \( \kappa \in \mathbb{R} \setminus \{0\} \) and \( q > 0 \), and they have demonstrated that it obeys another functional differential equation. Both the Derfel–Molchanov equation and (1.3) are a generalization of the pantograph equation, that has been extensively analysed in \([13]\) and \([10]\). However, it is important to emphasize that (1.3) has an important feature that sets it apart from other functional equations of the pantograph type, namely that, unless \( \omega \in \mathbb{R} \), its evolution
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makes sense only for complex \(z\) and proceeds along circles of constant \(|z|\), emanating from the origin.

Inasmuch as the equation (1.3) can be analysed directly, our next step entails expanding it in Taylor series. Thus, letting

\[
y(z) = \sum_{m=0}^{\infty} \frac{y_m}{m!} z^m,
\]

substitution in (1.3) readily yields

\[
y_{m+1} = \sqrt{b_1 - \frac{1}{2}} (2 \cos \alpha m - \lambda) y_m - b_2 y_{m-1}, \quad m = 1, 2, \ldots. \tag{1.4}
\]

It is beneficial to treat \(y_m\) as a function of the spectral parameter \(\lambda\) and to define

\[
\tilde{y}_m(t) := b_2^{-m/2} y_m\left(-\left(b_1 b_2\right)^{1/2} t\right), \quad m = 0, 1, \ldots.
\]

Brief manipulation affirms that (1.4) is equivalent to

\[
\tilde{y}_{m+1}(t) = \left(t + \frac{2}{\kappa} \cos \alpha m\right) \tilde{y}_m(t) - \tilde{y}_{m-1}(t), \quad m = 0, 1, \ldots. \tag{1.5}
\]

To specify the solution of (1.5) in a unique fashion we need to choose \(\tilde{y}_0\) and \(\tilde{y}_1\), which, of course, corresponds to equipping (1.3) with requisite initial conditions. We note in passing that \(t = -\kappa \lambda\), hence real initial values in (1.3) correspond to \(\tilde{y}_0, \tilde{y}_1 \in \mathbb{R}\).

Each solution of (1.5) is a linear combination of two linearly independent solutions. Setting \(\sigma = 2/\kappa\), we let

\[
r_{-1}(t) \equiv 0,
\]

\[
r_0(t) \equiv 1,
\]

\[
r_{m+1}(t) = (t + \sigma \cos \alpha m) r_m(t) - r_{m-1}(t), \quad m = 0, 1, \ldots, \tag{1.6}
\]

and

\[
s_{-1}(t) \equiv 0,
\]

\[
s_0(t) \equiv 1,
\]

\[
s_{m+1}(t) = (t + \sigma \cos \alpha (m + 1)) s_m(t) - s_{m-1}(t), \quad m = 0, 1, \ldots. \tag{1.7}
\]

It is trivial to verify that \(\{r_m\}_{m \in \mathbb{Z}_+}\) and \(\{s_{m+1}\}_{m \in \mathbb{Z}_+}\) are linearly independent, hence they span all solutions of (1.5). Moreover, we observe that each \(r_m(t)\) and \(s_m(t)\) is an \(m\)th degree monic polynomial in \(t\). This is a crucial observation, by virtue of the Favard theorem \[2\]. Given any three-term recurrence relation of the form

\[
p_{-1}(t) \equiv 0,
\]

\[
p_0(t) \equiv 1,
\]

\[
p_{m+1}(t) = (t + c_m) p_m(t) - d_m p_{m-1}(t), \quad m = 0, 1, \ldots,
\]
the monic polynomial sequence \( \{p_m\}_{m \in \mathbb{Z}^+} \) is orthogonal with respect to some Borel measure \( d \varphi \), i.e.

\[
\int_{\mathbb{R}} p_m(t)p_n(t) \, d \varphi(t) = 0, \quad m \neq n.
\]

The essential support \( \Xi \) of \( d \varphi \) is of central importance to the matter in hand, since it is an easy consequence of, for example, the \( n \)th root asymptotics of orthogonal polynomials \[16\] that, as long as \( \Xi \) is bounded, the sum \( \sum_{m=0}^{\infty} p_m(t)z^m/m! \) converges for \( z \in \mathbb{C} \), whereas convergence fails for \( z \neq 0 \) for \( t \notin \Xi \). Thus, travelling all the way back from orthogonal polynomials to functional equations, and hence to the almost Mathieu equation, we deduce that, subject to the linear transformation \( \lambda = -t/\kappa \), essential supports of the Borel measures corresponding to (1.6) and (1.7) result in the essential spectrum of (1.1).

We note that, in principle, the Favard theorem falls short of producing a unique measure and it is entirely possible that there exist many Borel measures that produce an identical set of monic orthogonal polynomials. This, however, is ruled out by the determinacy of the underlying Hamburger moment problem and the latter can be affirmed for both (1.6) and (1.7) by the Carleman criterion \[2\].

The remainder of this paper is devoted toward the determination of \( \Xi \). In Section 2 we demonstrate, by extending a technique due to Jeff Geronimo and Walter van Assche, that the essential support – and, indeed, the underlying Borel measure – can be specified explicitly when \( \alpha/\pi \) is rational. We prove in essence that \( d \varphi \) is a linear combination of Chebyshev measures of the second kind, supported on a set of disjoint intervals.

Various results on irrational \( \alpha/\pi \) are reported in Section 3. Inasmuch as the general form of \( d \varphi \) is currently a matter for conjecture, we derive a number of results that, beside being of interest on their own merit, are fully consistent with known results about the almost Mathieu operator, in particular with the theorem that the absolutely continuous spectrum of (1.1) is (for irrational \( \alpha/\pi \)) a Cantor set.

The equation (1.1) has been extensively studied in the past and, inasmuch as there are few outstanding conjectures, our knowledge of the spectrum of the almost Mathieu operator is quite comprehensive. Although the approach of this paper introduces a new perspective, there is no claim that Sections 2–3 add to the current state of knowledge of Schrödinger spectra. This state of affairs is remedied in Section 4, where the framework of our discussion undergoes a far reaching generalization. Firstly, we demonstrate that general periodic potentials with a finite number of Fourier harmonics lend themselves to similar analysis, except that, instead of orthogonal polynomials, the outcome is a generalized eigenvalue problem for a certain matrix pencil. Secondly, we prove that a multivariate extension of the almost Mathieu equation can be ‘transformed’ by our techniques to a problem in (univariate) orthogonal polynomials.

A possible application of our analysis, and in particular of Section 4, is numerical computation of the essential spectrum of (1.1) and of its generalizations. We do not pursue this further in the present paper.

The observation that (1.1) is ‘almost’ a three-term recurrence relation – only the index range is wrong – hence that the almost Mathieu equation might be connected with orthogonal polynomials, is not new. Most prominently, it had been made in \[12\], where it motivated a very interesting generalization of Chebyshev polynomials. The most important innovation in the present paper is in a technique that reduces the index range from \( \mathbb{Z} \)
to $\mathbb{Z}^+$ by passing from Dirichlet to Taylor expansions and which can be extended to cater for a substantially more general problem.

## 2 Orthogonal polynomials with periodic recurrence coefficients

The focus of our attention in this section is the three-term recurrence

\[ p_{-1}(t) \equiv 0, \]
\[ p_0(t) \equiv 1, \]
\[ p_m(t) = (t - \alpha_m)p_m(t) - p_{m-1}(t), \quad m = 0, 1, \ldots, \]  

(2.1)

where the sequence \( \{\alpha_m\}_{m \in \mathbb{Z}^+} \) is \( K \)-periodic,

\[ \alpha_{m+K} = \alpha_m, \quad m = 0, 1, \ldots \]  

(2.2)

Note that both (1.7) and (1.7) assume this form when \( \alpha / \pi \) is rational. Our objective is to determine the Borel measure that renders \( \{p_m\}_{m \in \mathbb{Z}^+} \) into an orthogonal polynomial system (OPS).

In [6] the authors consider the following problem. Let \( \{Q_m\}_{m \in \mathbb{Z}^+} \) be an OPS whose measure has an essential support \( \Xi_0 \subseteq [-1, 1] \) and let \( T \) be a given \( N \)-th degree polynomial. Setting \( \Xi = T^{-1}(\Xi_0) \) (the latter set is, generically, a union of \( \leq N \) disjoint intervals), they derive a new OPS, whose Borel measure is supported in \( \Xi \), explicitly in terms of \( \{Q_m\}_{m \in \mathbb{Z}^+} \). This construction is intimately related to the discussion of this section, except that we need, in a manner of speech, to travel in the opposite direction. As it turns out, the Borel measure associated with (2.7) inhabits a sets of disjoint intervals and we identify it by choosing an appropriate polynomial transformation \( T \).

Let

\[ q_n(t) := p_{(n+1)K-1}(t), \quad n = 0, 1, \ldots. \]

Note that \( q_{-1} \equiv 0 \) and, moreover, (2.3) and (2.2) imply

\[ p_{nK}(t) = (t - \alpha_0)q_{n-1} - p_{nK-2}(t), \quad n = 1, 2, \ldots. \]  

(2.3)

We seek polynomials \( \alpha_{\ell}, \beta_{\ell}, \ell = 0, 1, \ldots, K - 1 \), such that

\[ p_{nK+\ell}(t) = a_{\ell}(t)q_{n-1}(t) - b_{\ell}(t)p_{nK-2}(t), \quad \ell = 0, 1, \ldots, K - 1, \quad n = 1, 2, \ldots. \]

Because of (2.3) and the definition of \( q_n \), we have

\[ a_{-1}(t) \equiv 1, \quad b_{-1}(t) \equiv 0, \]
\[ a_0(t) = t - \alpha_0, \quad b_0(t) \equiv 1. \]  

(2.4)

We next substitute in the recurrence relation (2.4) and, by virtue of (2.2), obtain

\[ p_{nK+\ell+1}(t) = (t - \alpha_{\ell+1})p_{nK+\ell}(t) - p_{nK+\ell-1}(t) \]
\[ = (t - \alpha_{\ell+1})\{a_{\ell}(t)q_{n-1}(t) - b_{\ell}(t)p_{nK-2}(t)\} \]
\[ - \{a_{\ell-1}(t)q_{n-1}(t) - b_{\ell-1}(t)p_{nK-2}(t)\}. \]
Thus, comparing coefficients, we derive the recurrences

\[ a_{\ell+1}(t) = (t - \alpha_{\ell+1})a_{\ell}(t) - a_{\ell-1}(t), \quad \ell = 0, 1, \ldots, K - 2, \tag{2.5} \]

\[ b_{\ell+1}(t) = (t - \alpha_{\ell+1})b_{\ell}(t) - b_{\ell-1}(t), \quad \ell = 0, 1, \ldots, K - 2, \tag{2.6} \]

which, in tandem with (2.3), determine \( \{a_{\ell}, b_{\ell}\}_{\ell=0}^{K-1} \).

Let \( \ell = K - 1 \), then

\[ p_{nK-2}(t) = \frac{a_{K-1}(t)q_{n-1}(t) - q_{n}(t)}{b_{K-1}(t)} \]

and, shifting the index,

\[ p_{(n+1)K-2}(t) = \frac{a_{K-1}(t)q_{n}(t) - q_{n+1}(t)}{b_{K-1}(t)} \]

Substituting both expressions into

\[ p_{(n+1)K-2}(t) = a_{K-2}(t)q_{n-1}(t) - b_{K-2}(t)p_{nK-2}(t) \]

yields the recurrence relation

\[ q_{n+1}(t) = (a_{K-1}(t) - b_{K-2}(t))q_{n}(t) - \Delta_{n-2}(t)q_{n-1}(t), \tag{2.7} \]

where

\[ \Delta_{\ell}(t) = \det \begin{bmatrix} a_{\ell}(t) & a_{\ell+1}(t) \\ b_{\ell}(t) & b_{\ell+1}(t) \end{bmatrix}, \quad \ell = 0, 1, \ldots, K - 1. \]

We multiply (2.6) by \( a_{\ell}(t) \), (2.3) by \( b_{\ell}(t) \) and subtract from each other. This readily affirms by induction that

\[ \Delta_{\ell}(t) = \Delta_{\ell-1}(t) = \cdots = 1 \]

and (2.7) simplifies into

\[ q_{n+1}(t) = (a_{K-1}(t) - b_{K-2}(t))q_{n}(t) - q_{n-1}(t), \quad n = 0, 1, \ldots. \tag{2.8} \]

Note that (2.8) is consistent with \( n = 0 \), since \( q_{-1} \equiv 0 \). To further simplify the recurrence, we observe that \( q_{0}(t) = a_{K-1}(t) - b_{K-2}(t) \), hence, letting

\[ \tilde{q}_{n}(x) := \frac{q_{n}(t)}{q_{0}(t)}, \quad n = -1, 0, \ldots, \]

where \( x = q_{0}(t) \), we obtain the three-term recurrence

\[ \tilde{q}_{-1}(x) \equiv 0, \]

\[ \tilde{q}_{0}(x) \equiv 1, \]

\[ \tilde{q}_{n+1}(x) = x\tilde{q}_{n}(x) - \tilde{q}_{n-1}(x), \quad n = 0, 1, \ldots. \]

Thus, each \( \tilde{q}_{n} \) is an \( n \)th degree monic polynomial and, by virtue of the Favard theorem, \( \{\tilde{q}_{n}\}_{n \in \mathbb{Z}^+} \) is an OPS. It can be easily identified as a shifted and scaled Chebyshev polynomial of the second kind,

\[ \tilde{q}_{n}(x) = 2^{n}U_{n}\left(\frac{x}{2}\right), \quad n = 0, 1, \ldots. \]
We thus deduce that
\[ q_n(t) = 2^n q_0(t) U_n \left( \frac{1}{2} q_0(t) \right), \quad n = 0, 1, \ldots \] (2.9)

Before we identify the underlying Borel measure, let us ‘fill in’ the remaining values of \( p_m \). By definition, \( p_{n-1} = q_{n-1} \), \( p_{n+1} = q_n \), hence the recurrence (2.1) gives
\[ (x - \alpha_1)p_{nK}(t) - p_{nK+1}(t) = q_{n-1}(t), \]
\[ -p_{nK+\ell-1}(t) + (t - \alpha_{\ell+1})p_{nK+\ell}(t) - p_{nK+\ell+1}(t) = 0, \quad \ell = 1, 2, \ldots, K - 3, \]
\[ -p_{(n+1)K-3}(t) + (t - \alpha_{K-1})p_{(n+1)K-2}(t) = q_n(t). \]

This is a linear system of equations, which we write as
\[ A_{K-1} p_n = q_n, \] (2.10)
where
\[
A_m = \begin{bmatrix}
  t - \alpha_1 & -1 & & & \\
  -1 & t - \alpha_2 & -1 & & \\
  & -1 & t - \alpha_3 & -1 & \\
  & & \ddots & \ddots & \ddots \\
  & & & -1 & t - \alpha_{m-1} & -1 \\
  & & & & -1 & t - \alpha_m \\
\end{bmatrix}, \quad m = 1, 2, \ldots, K - 1,
\]
\[ p_n = \begin{bmatrix} p_{nK}(t) \\ p_{nK+1}(t) \\ \vdots \\ p_{(n+1)K-3}(t) \\ p_{(n+1)K-2}(t) \end{bmatrix} \quad \text{and} \quad q_n = \begin{bmatrix} q_{n-1}(t) \\ 0 \\ \vdots \\ 0 \\ q_n(t) \end{bmatrix}. \]

We expand the determinant of \( A_m \) in its bottom row and rightmost column. This results in a three-term recurrence relation and comparison with (2.4) and (2.6) affirms that \( \det A_m = b_m(t) \). Hence, solving (2.1) with Cramer’s rule, we deduce that there exist \((K-2)\)-degree polynomials \( \bar{a}_\ell \) and \( \bar{b}_\ell \), \( \ell = 0, 1, \ldots, K - 2 \), such that
\[ p_{nK+\ell}(t) = \frac{\bar{a}_\ell(t)q_{n-1}(t) + \bar{b}_\ell(t)q_n(t)}{b_{K-1}(t)}, \quad \ell = 0, 1, \ldots, K - 2. \] (2.11)

Bearing in mind the definition of \( a_\ell \) and \( b_\ell \),
\[ p_{nK+\ell}(t) = a_\ell(t)q_{n-1}(t) - b_\ell(t)p_{nK-2}(t), \]
we obtain from (2.11) the identity
\[ \frac{\bar{a}_\ell(t)q_{n-1}(t) + \bar{b}_\ell(t)q_n(t)}{b_{K-1}(t)} = a_\ell(t)q_{n-1}(t) - \frac{b_\ell(t)(\bar{a}_{K-2}(t)q_{n-2}(t) + \bar{b}_{K-2}(t)q_{n-1}(t))}{b_{K-1}(t)}. \]
We next substitute
\[ q_{n-2}(t) = (a_{K-1}(t) - b_{K-2}(t))q_{n-1}(t) - q_n(t) \]  
(pace (2.8)) and rearrange terms, whereby
\[
\{\hat{b}_\ell(t) - b_\ell(t)\tilde{a}_{K-2}(t)\}q_n(t)
= \{-\hat{a}_\ell + a_\ell(t)b_{K-1}(t) - (a_{K-1}(t)b_\ell(t) - b_\ell(t)b_{K-2}(t))\tilde{a}_{K-2}(t) - b_\ell(t)\hat{b}_{K-2}(t)\}\tilde{a}_{K-2}(t).
\]
However, consecutive orthogonal polynomials \(q_{n-1}\) and \(q_n\) cannot share zeros [3], therefore both sides of the last equality identically vanish and we derive the explicit expressions
\[
\begin{align*}
\hat{a}_\ell(t) &= a_\ell(t)b_{K-1}(t) - a_{K-1}(t)b_\ell(t) - b_\ell(t)b_{K-2}(t)\tilde{a}_{K-2}(t) - b_\ell(t)\hat{b}_{K-2}(t), \quad (2.12) \\
\hat{b}_\ell(t) &= b_\ell(t)\tilde{a}_{K-2}(t).
\end{align*}
\]
Letting \(\ell = K - 2\) in (2.13) gives \(\hat{b}_{K-2}(t) = b_{K-2}(t)\tilde{a}_{K-2}(t)\) and we substitute this into (2.12). The outcome is
\[ \hat{a}_\ell(t) = a_\ell(t)b_{K-1}(t) - a_{K-1}(t)b_\ell(t)\tilde{a}_{K-2}(t). \]  
In particular, \(\ell = K - 2\) and the definition of \(\Delta_m\) result in
\[
(1 + a_{K-1}(t)b_{K-2}(t))\tilde{a}_{K-2}(t) = a_{K-2}(t)b_{K-1}(t) = a_{K-1}(t)b_{K-2}(t) + \Delta_K(t) = 1 + a_{K-1}(t)b_{K-2}(t).
\]
Since \(a_{K-1}b_{K-2} \neq -1\), we conclude that \(\tilde{a}_{K-2} \equiv 1\) and substitution in (2.13) and (2.14) yields the explicit formulæ
\[ \hat{a}_\ell = \det \begin{bmatrix} a_\ell(t) & a_{K-1}(t) \\ b_\ell(t) & b_{K-1}(t) \end{bmatrix}, \quad \hat{b}_\ell(t) = b_\ell(t), \quad \ell = 0, 1, \ldots, K - 2. \]

**Theorem 1** The OPS \(\{p_m\}_{m \in \mathbb{Z}^+}\) has an explicit representation in the form
\[ p_{nK+\ell}(t) = \frac{1}{b_{K-1}(t)} \left\{ \det \begin{bmatrix} a_\ell(t) & a_{K-1}(t) \\ b_\ell(t) & b_{K-1}(t) \end{bmatrix} q_{n-1}(t) + b_\ell(t)q_n(t) \right\}, \]  
where \(n = 0, 1, \ldots, \ell = 0, 1, \ldots, K - 1\) and the OPS \(\{q_m\}_{m \in \mathbb{Z}^+}\) satisfies the three-term recurrence (2.8). \(\square\)

Note that letting \(\ell = K - 1\) or \(\ell = -1\) in (2.15), in tandem with (2.4), results in \(p_{(n+1)K-1} = q_n\) and \(p_{nK-1} = q_{n-1}\) respectively, as required.

Let \(t \in \Xi\). Then, by the discussion preceding the representation (2.9), we know that \(\frac{1}{2}q_0(t) \in [-1, 1]\), and there exists \(\theta \in [-\pi, \pi]\) such that \(\frac{1}{2}q_0(t) = \cos \theta\). Since, by the definition of Chebyshev polynomials of the second kind,
\[ U_n(\cos \theta) = \frac{\sin(n + 1)\theta}{\sin \theta}, \]
We stipulate that $d\psi \ll \xi$

Case 1: the construction of Geronimo and van Assche in [6], with

We commence by observing that, by virtue of Theorem 1, everything depends on the support to that of Geronimo and van Assche – and we present here a complete derivation of $\Xi$. We

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Similarly to [3], we seek the inverse function to $x = \frac{1}{2} q_0(t)$. Let $\xi_1 < \xi_2 < \cdots < \xi_s$ be all the minima and maxima of $q_0$ in $\mathbb{R}$ (of course, $s \leq K - 2$) and define $\xi_0 = -\infty$, $\xi_{s+1} = \infty$. In each interval $[\xi_j, \xi_{j+1}]$, $j = 0, 1, \ldots, s$, the function $\frac{1}{2} q_0(t)$ is monotone, hence it possesses there a well-defined inverse. We denote it by $X_j(x)$, hence $\frac{1}{2} q_0(X_j(x)) = x$. Changing the integration variable, we have

We recall that $\{U_n\}_{n \in \mathbb{Z}^+}$ is an OPS with respect to the Borel measure $(1 - x^2)^{\frac{1}{2}} \, dx$, supported by $x \in [-1, 1]$. Thus, for every $j = 0, 1, \ldots, s$ we distinguish among the following cases:

Case 1: $q_0(\xi_j) \leq -2$ and $2 \geq q_0(\xi_{j+1})$. We stipulate that $d\psi(X_j(x))$ vanishes for all

Since $X_j$ increases monotonically in $[\xi_j, \xi_{j+1}]$, the contribution of this interval to (2.16) is

$$2^{n+m} \int_{-1}^1 U_n(x) U_m(x) \, d\psi(|X_j(x)|). \tag{2.17}$$
Case 2: \( q_0(\xi_{j+1}) \leq -2 \) and \( 2 \leq q_0(\xi_j) \).
Likewise, we require that the support of \( d\psi(X_j) \) is restricted to \([-1, 1]\). \( q_0 \) decreases monotonically within \([\xi_j, \xi_{j+1}]\) and straightforward manipulation affirms that (2.17) represents the contribution of this interval to (2.16).

Case 3: \( \min\{q_0(\xi_j), q_0(\xi_{j+1})\} > -2 \) or \( \max\{q_0(\xi_j), q_0(\xi_{j+1})\} < 2 \).
In that case we cannot fit \([-1, 1]\) into \([\xi_j, \xi_{j+1}]\), hence we stipulate that \( d\psi(X_j) \) is not supported in \([\xi_j, \xi_{j+1}]\).

Let \( \nu_1 < \nu_2 < \ldots < \nu_r \) be all the indices in \( \{0, 1, \ldots, s\} \) such that either Case 1 or Case 2 holds. We require that \( r \geq 1 \). Then (2.16) reduces to

\[
I_{n,m} = \int_{-1}^{1} U_n(x) U_m(x) \sum_{\ell=1}^{r} d\psi(|X_{\nu_\ell}(x)|).
\]

Since the Hamburger moment problem for the Chebyshev measure of the second kind is determinate, it follows that necessarily

\[
\sum_{\ell=1}^{r} d\psi(|X_{\nu_\ell}(x)|) = (1 - x^2)^{1/2} \, dx, \quad x \in [-1, 1].
\]

Theorem 2 The orthogonality measure corresponding to the OPS \( \{p_m\}_{m \in \mathbb{Z}^+} \) is supported by

\[
\Xi = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_r,
\]

where for each \( \ell = 1, 2, \ldots, r \) \( \mathcal{I}_\ell \subseteq [\xi_{\nu_\ell}, \xi_{\nu_{\ell+1}}] \) is the unique interval such that \( |q_0(t)| = 2 \) at its endpoints.

Proof. Follows at once from our construction. \( \square \)

Figure 1 displays two examples of the present construction, for different cases of \( q_0 \). In each case \( \Xi \) is the union of the ‘thick’ intervals.

Harking back to (1.6) and (1.7), we let \( \alpha_m = -\sigma \cos \alpha m \) and \( \alpha_m = -\sigma \cos (m+1) \alpha \), \( m = 0, 1, \ldots \), respectively, where \( \alpha = 2 \pi L/K \). Thus, \( \{\alpha_m\}_{m \in \mathbb{Z}^+} \) is indeed \( K \)-periodic. Unsurprisingly, the outcome of our analysis are the familiar spectral bounds \([1, 3]\). The merits of our approach are, however, not just in providing an alternative proof of known results but also in extending the framework to the multivariate case in Section 4.

We mention in passing that the analysis of this section can be easily extended to recurrences of the form

\[
\begin{align*}
p_{-1}(t) & \equiv 0, \\
p_0(t) & \equiv 1, \\
p_{m+1}(t) & = (t - \alpha_m)p_m(t) - \beta_m p_{m-1}(t), \quad m = 0, 1, \ldots,
\end{align*}
\]

where both \( \{\alpha_m\}_{m \in \mathbb{Z}^+} \) and \( \{\beta_m\}_{m \in \mathbb{Z}^+} \) are \( K \)-periodic. This, however, is of little relevance to the theme of this paper.
Figure 1: The sets $\Xi$ for two different polynomials $q_0$. 
3 Orthogonal polynomials with almost periodic recurrence coefficients

The three-term recurrence relations (1.4) and (1.7) assume, \( \alpha/\pi \) being irrational, almost-periodic recurrence coefficients and this state of affairs is even more important in a multivariate generalization of (1.1) in Section 4. Unfortunately, no general theory exists to cater for orthogonal polynomials with almost periodic recurrence coefficients. The theme of the present section is a preliminary and – in the nature of things – incomplete investigation of the case when an irrational \( \alpha/\pi \) is approximated by rationals. In other words, we commence with the \( K \)-periodic recurrence (2.1), except that we will allow the period \( K \) to become unbounded.

The motivation for our analysis is an observation which is interesting on its own merit. Denote by \( \sigma_K \) the value of \( q_{K-1}(0) \) for \( \alpha \ell = - \cos \frac{2\pi \ell}{K}, \ell = 0, 1, \ldots, K - 1, \) i.e.

\[
\sigma_K = \det \begin{bmatrix}
\cos \frac{2\pi}{K} & 1 & \cos \frac{4\pi}{K} & 1 \\
1 & \cos \frac{4\pi}{K} & 1 & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \cos \frac{2(K-1)\pi}{K} & \ddots & \ddots & \cos \frac{2(K-2)\pi}{K} \\
1 & \cos \frac{2(K-1)\pi}{K} & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}, \quad K = 1, 2, \ldots
\]

Computation indicates that

\[
\sigma_{4L} \equiv 0, \quad \lim_{L \to \infty} \sigma_{4L+2} = 0, \quad \lim_{L \to \infty} \sigma_{4L+1} = - \lim_{L \to \infty} \sigma_{4L+3} = \frac{2\sqrt{3}}{3}. \tag{3.1}
\]

It is easy to prove that \( \sigma_{4L} = 0 \) for all \( L \geq 1 \). Thus, let \( K = 4L \) and set \( \ell = L \) in

\[
\nu_{\ell} = \cos \frac{2\pi \ell}{K} \nu_{\ell-1} - \nu_{\ell-2}, \quad \ell = 1, 2, \ldots, 4L - 1 \tag{3.2}
\]

(note that (3.2), in tandem with \( \nu_{-1} = 0, \nu_0 = 1 \), yields \( \sigma_K = \nu_{K-1} \)). This yields \( \nu_L + \nu_{L-2} = 0 \) and we claim that, in general,

\[
\nu_{L+k} + (-1)^k \nu_{L-k} = 0, \quad k = -1, 0, \ldots, L - 1. \tag{3.3}
\]

We have already proved (3.3) for \( k = 0 \) and it is trivially true for \( k = -1 \). We continue by induction and assume that (3.3) is true for \( k = -1, 0, \ldots, s - 1 \). Letting \( \ell = L \pm s \) in (3.2), we have

\[
\begin{align*}
\nu_{L+s} &= - \sin \frac{\pi s}{2L} \nu_{L+s-1} - \nu_{L+s-2}; \\
\nu_{L-s} &= + \sin \frac{\pi s}{2L} \nu_{L-s-1} - \nu_{L-s-2}.
\end{align*}
\]

We multiply the second equation by \( (-1)^s \) and add to the first, thus

\[
(\nu_{L+s} + (-1)^s \nu_{L-s}) = - \sin \frac{\pi s}{2L} (\nu_{L+s-1} + (-1)^{s-1} \nu_{L-s+1}) - (\nu_{L+s-2} + (-1)^{s-2} \nu_{L-s+2})
\]

and (3.3) follows at once.
Hence, letting \( k = L - 1 \) in (3.3) and recalling that \( \nu_{-1} = 0 \), we obtain \( \nu_{2L-1} = 0 \). Moreover, similarly to (3.3), we can prove that
\[
\nu_{3L+k} + (-1)^k \nu_{3L-k} = 0, \quad k = -1, 0, \ldots, L - 1
\]
and \( k = L - 1 \) gives
\[
\sigma_{4L} = \nu_{4L-1} = (-1)^L \nu_{2L-1} = 0.
\]
This completes the proof of the lemma.

Other observations in (3.1) are also true and in the sequel we prove them in a generalized setting.

Given \( \sigma \in (-1, 1) \), we define
\[
A_{-1}(t) \equiv 0, \quad A_0(t) \equiv 1,
A_n(t) = t \xi_n A_{n-1}(t) - A_{n-2}(t), \quad n = 1, 2, \ldots, (3.4)
\]
where \( \{\xi_n\}_{n=1}^\infty \) is a given real sequence. To emphasize the dependence on parameters, we write, as and when necessary, \( A_n(\cdot) = A_n(\cdot; \xi_1, \xi_2, \ldots, \xi_n) \).

An alternative representation of \( A_n \) is
\[
A_n(t) = \det \begin{bmatrix}
    t \xi_1 & 1 \\
    1 & t \xi_2 & \ddots \\
    \vdots & \ddots & \ddots & 1 \\
    1 & t \xi_n
\end{bmatrix}, \quad n = 1, 2, \ldots,
\]
hence \( A_n \) is an \( n \)th degree polynomial. We observe that
\[
A_n(0) = \begin{cases}
    (-1)^s & : n = 2s, \\
    0 & : n = 2s + 1.
\end{cases}
\]
Moreover, differentiating with respect to \( t \), we obtain
\[
A'_n(t) = \sum_{k=1}^n \det \begin{bmatrix}
    t \xi_1 & 1 & & & \\
    1 & t \xi_2 & \ddots & & \\
    \vdots & \ddots & \ddots & 1 & \\
    1 & t \xi_k & 0 & \xi_k & 0 \\
    \vdots & \ddots & \ddots & \ddots & 1 \\
    1 & t \xi_{k+1} & & & 1 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\
    1 & t \xi_n & & & & 1
\end{bmatrix},
\]
therefore, expanding in the \( k \)th row, we derive the identity
\[
A'_n(t; \xi_1, \ldots, \xi_n) = \sum_{k=1}^n \xi_k A_{k-1}(t; \xi_1, \ldots, \xi_{k-1}) A_{n-k}(t; \xi_{k+1}, \ldots, x_n). \quad (3.5)
\]
Proposition 3 $A_n^{(r)}(0) = 0$ whenever $n + r$ is odd, hence the polynomial $A_n$ has the same parity as $n$.

Proof. By induction on $r$. The assertion is true for $r = 0$. Moreover, repeatedly differentiating (3.3) with the Leibnitz rule and letting $t = 0$, we obtain

$$A_{2n}^{(2r+1)}(0; \xi_1, \ldots, \xi_{2n}) = \sum_{\ell=0}^{2r} \binom{2r}{\ell} 2n \ell A_{k-1}^{(\ell)}(0; \xi_1, \ldots, \xi_{k-1}) A_{2n-k}^{(2r-\ell)}(0; \xi_{k+1}, \ldots, \xi_{2n}).$$

But

$$(k - 1) + \ell \quad \text{is even} \quad \iff \quad (2n - k) + (2r - \ell) \quad \text{is odd},$$

therefore for all $\ell = 0, 1, \ldots, 2r$ and $k = 1, 2, \ldots, 2n$ the induction hypothesis affirms that at least one of the terms in the product vanishes. Similar argument demonstrates that $A_{2n+1}^{(r)}(0) = 0$. □

We therefore let for all $n = 0, 1, \ldots, s = 1, 2, \ldots,$

$$A_{2n}(t; \xi, \ldots, \xi_{2n+s-1}) = \sum_{r=0}^{n} B_{2n,s}^{(2r)} t^{2r},$$

$$A_{2n+1}(t; \xi, \ldots, \xi_{2n+s}) = \sum_{r=0}^{n} B_{2n+1,s}^{(2r+1)} t^{2r+1}.$$

Substitution into (3.4) (where we replace $\xi$ by $\xi_{n+s-1}$) results in the recurrences

$$B_{2n,s}^{(2r)} = \xi_{s+2n-1} B_{2n-1,s}^{(2r-1)} - B_{2n-1,s}^{(2r)}, \quad (3.6)$$

$$B_{2n+1,s}^{(2r+1)} = \xi_{s+2n} B_{2n,s}^{(2r)} - B_{2n-1,s}^{(2r+1)}, \quad (3.7)$$

Given $q \in \mathbb{C}$, we recall that the $q$-factorial symbol is defined as

$$(z; q)_n = \prod_{k=0}^{n-1} (1 - q^k z), \quad z \in \mathbb{C}, \quad n \in \mathbb{Z} \cup \{\infty\},$$

whereas the $q$-binomial reads

$$\left[ \begin{array}{c} n \\ m \end{array} \right] := \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}, \quad 0 \leq m \leq n.$$
Proof. By induction on \( n \), using (3.8) and (3.9). Obviously, the assertion of the lemma is true for \( n = 0 \). Otherwise, for even values,

\[
\left(q^{n+\frac{q-1}{2}} + q^{-n-\frac{q-1}{2}}\right)B_{2n-1}^{(2r)-1} - B_{2n-2,s}^{(2r)} = (-1)^{n+r}q^{-(r-\frac{q-1}{2})(2n+s-r-1)}\left(q^{n+\frac{q-1}{2}} + q^{-n-\frac{q-1}{2}}\right)^{n+r-1}
\]

This accomplishes a single inductive step for (3.8). We prove (3.9) in an identical manner, by considering odd values of \( n \).

Recall that our interest in the polynomials \( A_n \) has been sparked by the observation (3.7). Thus, we require to recover cosine terms, and to this end we choose \( q \) of unit modulus.

**Proposition 5** Suppose that \( q^{n+1} = 1 \) and \( q^m \neq 1 \) for \( m = 1, 2, \ldots, n \). Then \( B_{2n+1,1}^{(2r+1)} = 0 \) for all \( 0 \leq r \leq \frac{n}{2} \).

**Proof.** Let \( 2r \leq n \). We have from (3.3) that

\[
B_{2n+1,1}^{(2r+1)} = (-1)^{n+r}q^{\left(r+\frac{1}{2}\right)(r+1)}\frac{(q; q)_{n+r+1}}{(q; q)_{2r+1}(q; q)_{n-r}}(-q^{-r}; q)_{2r+1}.
\]

However,

\[
\frac{(q; q)_{n+r+1}}{(q; q)_{n-r}} = \prod_{\ell=-r}^{r} (1 - q^\ell) = 0,
\]

whereas, because of our restriction on \( r \),

\[
(q; q)_{2r+1} = \prod_{\ell=1}^{2r} (1 - q^\ell) \neq 0,
\]

since \( q \) is a root of unity of minimal degree \( n + 1 \). The proposition follows.

**Corollary** Let \( q = \exp\frac{2\pi im}{n+1} \), where \( m \) and \( n \) are relatively prime. Then it is true that

\[
\lim_{n \to \infty} A_{2n+1}(t) = 0
\]

for every \( t \in (-1, 1) \).

**Proof.** Straightforward, since \( A_{2n+1}(t) = O\left(t^{\frac{1}{2}n}\right) \).
Proposition 6 Suppose that \( \omega = q^{\frac{1}{2n}} \) is a root of unity of minimal degree \( 2n + 1 \). Then, for all \( r = 0, 1, \ldots, n \) it is true that

\[
(-1)^n B_{2n,1}^{(2r)} = \prod_{\ell=1}^{r} \frac{\sin(2\ell \phi - 1)}{\sin 2\ell \phi},
\]  

(3.11)

where \( \phi = \arg \omega \).

Proof. Since \( q^{n + \frac{1}{2}} = 1 \), it follows from (3.3) that

\[
(-1)^n B_{2n,1}^{(2r)} = (-1)^r q^{r(r+\frac{1}{2})} \frac{(q;q)_{n+r}}{(q;q)_{2r}(q;q)_{n-r}}(-q^{-r+\frac{1}{2}};q)_{2r}.
\]

But

\[
\frac{(q;q)_{n+r}}{(q;q)_{n-r}} = \prod_{\ell=-r}^{r-1} (1 - q^{2\ell+1}).
\]

Moreover, \((q;q)_{2r} \neq 0\) for \( r = 0, 1, \ldots, n \), since \( 2n + 1 \) is the least nontrivial degree of the root of unity \( q \), and we deduce that

\[
(-1)^n B_{2n,1}^{(2r)} = (-1)^r q^{r(r+\frac{1}{2})} \prod_{\ell=-r}^{r-1} \frac{(1 - q^{2\ell+1})}{(1 - q^{\ell})}.
\]

But

\[
\prod_{\ell=1}^{2r} (1 - q^{\ell}) = \prod_{\ell=1}^{2r} \omega^{\ell}(\omega^\ell - \omega^{-\ell}) = q^{r(r+\frac{1}{2})} \prod_{\ell=1}^{2r} (\omega^\ell - \omega^{-\ell})
\]

and, likewise,

\[
\prod_{\ell=-r}^{r-1} (1 - q^{2\ell+1}) = \prod_{\ell=-r}^{r-1} \omega^{2\ell+1}(\omega^{2\ell+1} - \omega^{-2\ell-1}) = (-1)^{r-1} \prod_{\ell=0}^{r-1} (\omega^{2\ell+1} - \omega^{-2\ell-1})^2.
\]

Consequently,

\[
(-1)^n B_{2n,1}^{(2r)} = \frac{\prod_{\ell=0}^{r-1} (\omega^{2\ell+1} - \omega^{-2\ell-1})^2}{\prod_{\ell=1}^{r} (\omega^\ell - \omega^{-\ell})}.
\]

This is precisely the identity (3.11). \( \square \)

Next, we consider progression to a limit as \( n \to \infty \) — (3.1) is a special case. Thus, suppose that we have a sequence \( \Phi = \{ \phi_n \}_{n \in \mathbb{I}} \), where \( \phi_n = 2\pi m_n/(n + 1) \), \( m_n \in \mathbb{Z}^+ \), \( \mathbb{I} \subseteq \mathbb{Z}^+ \) is a set of infinite cardinality and

\[
\lim_{n \to \infty} \phi_n = \phi \in [0, 2\pi).
\]

Set

\[
C(t, \Phi) = \lim_{n \to \infty} (-1)^{[n/2]} A_n(t; \xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_n^{(n)}),
\]
where \( \xi^{(n)}_\ell = 2 \cos \ell \phi_n, \ell = 1, 2, \ldots, n, n = 1, 2, \ldots. \) Thus, \( \xi^{(n)}_\ell = q^n_\ell + q^{-n}_\ell, \) where \( q_n = \exp \frac{4\pi i}{n+1}. \) Consequently, according to Proposition 5, if \( \mathcal{I} \) consists of only odd indices, necessarily \( C(t, \Phi) \equiv 0. \) This proves, incidentally, that \( \sigma_{2L} \to 0 \) in (3.1).

**Lemma 7** Suppose that \( \mathcal{I} \subseteq 2\mathbb{Z}^+ \) and that \( \phi = 0. \) Then, provided that \( m_n = o(n^{3/4}) \), it is true that \( C(t, \Phi) = (1 - t^2)^{-1/4}. \)

**Proof.** Since
\[
\frac{\sin(2\ell - 1)\phi_n}{\sin 2\ell \phi_n} = \frac{2\ell - 1}{2\ell} + \mathcal{O}\left(\phi_n^3\right),
\]
we deduce from (3.11) that
\[
(-1)^n B_{2n,1}^{(2r)} = r^{-r} \binom{2r}{r} + \mathcal{O}\left(n\phi_n^3\right).
\]
According to the assumption, \( \mathcal{O}(n\phi_n^3) = o(1) \) and the lemma follows from
\[
\sum_{r=0}^{\infty} \binom{2r}{r} \frac{t^r}{4^r} = \frac{1}{\sqrt{1 - t^2}}.
\]

Letting \( t = \frac{1}{2} \) affirms the remaining part of (3.1).

Similarly to the last proposition, it is possible to derive an explicit expression for \( C(t, \Phi) \), provided that \( \phi/\pi \) is rational and that \( n(\phi - \phi_n)^3 = o(1) \) as \( n \to \infty. \) The derivation is long and it will be published elsewhere. It suffices to mention here the remarkable sensitivity of \( C(t, \Phi) \) to both the choice of \( \mathcal{I} \) and to the specific nature of \( \phi. \)

What happens when \( \phi/\pi \) is irrational? This is, as things stand, an open problem. It is possible to show that, formally,
\[
C(t, \Phi) = \sum_{\ell=0}^{\infty} t^{2\ell} \prod_{k=1}^\ell \frac{\sin(2k - 1)\phi}{2k\phi} = \sum_{\ell=0}^{\infty} t^{2\ell} q^\frac{1}{4} \left(\frac{q^{1/2}; q}{q; q}\right)_\ell \frac{(q^{1/2}; q)_\ell}{(q; q)_\ell}.
\]
where \( q = e^{4i\phi}. \) The latter series can be summed up by means of the Gauß–Heine theorem for \( |q| < 1 \) and, after simple manipulation, for \( |q| > 1. \) Unfortunately, because of a breakdown in Hölder-continuity across \( |q| = 1, \) it is impossible to deduce its value on the unit circle from the values within and without by means, for example, of the Sokhotsky formula (8).

Clearly, there is much to be done to understand better the behaviour of the \( p_n \)'s when the period \( K \) becomes infinite. In this section we have established few results with regard to the values at the origin. They should be regarded as a preliminary foray into an interesting problem in orthogonal polynomial theory cum linear algebra and we hope to return to this theme in the future.

### 4 Generalizations

There are two natural ways of generalizing an almost Mathieu equation (1.1), by either specifying a more general periodic potential or replacing the index by a multi-index. Remarkably, the basic framework of this paper – replacing a doubly-infinite recurrence by a
functional equation which, in turn, is replaced by a singly-infinite recurrence – survives both generalizations! In the present section we describe briefly this state of affairs.

Firstly, suppose that the cosine term in (1.1) is replaced by a more general harmonic term and we consider the spectral problem

\[ a_{n-1} - 2 \left\{ \sum_{\ell=1}^{m} \kappa_{\ell} \cos(n \ell \theta + \psi_{\ell}) \right\} a_n + a_{n+1} = \lambda a_n, \quad n \in \mathbb{Z}. \]  

(4.1)

We assume that \( \kappa_1, \kappa_2, \ldots, \kappa_m \in \mathbb{R} \) and, without loss of generality, that \( \kappa_m \neq 0 \). Letting \( q = e^{i\theta} \), we set

\[ \kappa_{\ell}^* = e^{i\psi_{\ell}} \kappa_{\ell}, \quad \kappa_{-\ell}^* = e^{-i\psi_{\ell}} \kappa_{\ell}, \quad \ell = 1, 2, \ldots, m, \]

and \( \kappa_0^* = 0 \). Therefore (4.1) assumes the form

\[ q^{mn} a_{n-1} - \left\{ \sum_{\ell=0}^{2m} \kappa_{-\ell-m}^{*} q^{n \ell} \right\} a_n + q^{mn} a_{n+1} = \lambda q^{mn} a_n, \quad n \in \mathbb{Z}. \]  

(4.2)

Let \( c \in \mathbb{C} \setminus \{0\} \) and consider the Dirichlet series \( y(t) = \sum_{n=-\infty}^{\infty} a_n \exp\{cq^n t\} \). Since, formally,

\[ y^{(\ell)}(t) = c^\ell \sum_{n=-\infty}^{\infty} q^n a_n e^{cq^n t}, \quad \ell = 0, 1, \ldots, \]

we obtain from (4.2) the functional differential equation

\[ \sum_{\ell=0}^{2m} \kappa_{-\ell-m}^{*} c^{-\ell} y^{(\ell)}(t) = q^m y^{(m)}(qt) - \lambda y^{(m)}(t) + q^{-m} y^{(m)}(q^{-1} t). \]  

(4.3)

The derivation is identical to that of (1.3) and is left to the reader.

In line with Section 1, we next expand the solution of (4.3) in Taylor series, \( y(t) = \sum_{n=0}^{\infty} p_n t^n / n! \). This readily yields

\[ \sum_{\ell=0}^{2m} \kappa_{-\ell-m}^{*} c^{-\ell} p_{n+\ell} = (q^{n+m} + q^{-n-m} - \lambda) p_{n+m}, \quad n = 0, 1, \ldots, \]

hence, replacing \( n + m \) by \( n \),

\[ \sum_{\ell=-m}^{m} \kappa_{\ell}^{*} c^{-\ell} p_{n+\ell} = (2 \cos n \theta - \lambda) p_n, \quad \ell = m, m + 1, \ldots. \]

Finally, we choose \( c = \exp\{i\psi_{m}/m\} \), hence \( \kappa_{m}^{*} c^{-m} = \kappa_{m}^{*} = \kappa_{m} \in \mathbb{R} \setminus \{0\} \). We thus define \( \alpha_{\ell} = c^{-\ell} \kappa_{\ell}^{*} / \kappa_{m} \), \( |\ell| \leq m \) and replace \( \lambda \) by \( -\lambda \kappa_{m} \). This results in the recurrence

\[ \sum_{\ell=-m}^{m} \alpha_{\ell} p_{n+\ell} = (\lambda - \beta_n) p_n, \quad n = m, m + 1, \ldots, \]  

(4.4)

where

\[ \beta_n = - \frac{\cos n \theta}{\kappa_{m}}, \quad n = m, m + 1, \ldots. \]
Note that, inasmuch as the \( \alpha_\ell \)s may be complex, we have \( \alpha_{-\ell} = \bar{\alpha}_\ell \), \( \ell = 1, 2, \ldots, m \), \( \alpha_0 = 0 \).

The recurrence (4.4) is spanned by \( 2m \) linearly independent solutions. However, unless \( m = 1 \), it is no longer true that, for appropriate choice of \( p_0, p_1, \ldots, p_{2m-1} \), each \( p_n \) is a polynomial of degree \( n + k \) for some \( k \), independent of \( n \). Indeed, it is easy to verify that the degree of \( p_n \) increases roughly as \( [n/m] \). Hence, orthogonality is lost. Fortunately, an important feature of orthogonal polynomials, namely that their zeros are eigenvalues of a truncated Jacobi matrix \([2]\), can be generalized to the present framework. It is possible to show that the zeros of \( p_n \) are generalized eigenvalues of a specific pencil of 'truncated' matrices and this provides a handle on their location. We expect to address ourselves to this issue in a future publication.

Another generalization of (1.1) allows the index \( n \) to be replaced by a multi-index \( n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d \). Thus, let \( e_\ell \in \mathbb{Z}^d \) be the \( \ell \)th unit vector, \( \ell = 1, 2, \ldots, d \), and consider the spectral problem

\[
\sum_{\ell=1}^{d} (a_{n+e_\ell} + a_{n-e_\ell}) - 2\kappa \cos \left( \sum_{\ell=1}^{d} \alpha_\ell n_\ell + \beta \right) a_n = \lambda a_n, \quad n \in \mathbb{Z}^d. \tag{4.5}
\]

In line with Section 1, we let

\[
b_1 = be^{i\beta}, \quad b_2 = e^{-i\beta}, \quad q_\ell = e^{i\alpha_\ell}, \quad \ell = 1, 2, \ldots, d,
\]

whereupon (4.5) becomes

\[
\sum_{\ell=1}^{d} q^n(a_{n+e_\ell} + a_{n-e_\ell}) - (b_1 q^{2n} + b_2) a_n = \lambda q^n a_n, \quad n \in \mathbb{Z}^d. \tag{4.6}
\]

The last formula employs standard multi-index notation, e.g. \( q^n = q_1^{n_1} q_2^{n_2} \cdots q_d^{n_d} \).

We let formally

\[
y(t) = \sum_{n \in \mathbb{Z}^d} a_n \exp \left\{ b_1^{\frac{1}{2}} q^n t \right\}
\]

and note that

\[
y'(t) = b_1^{\frac{1}{2}} \sum_{n \in \mathbb{Z}^d} a_n q^n \exp \left\{ b_1^{\frac{1}{2}} q^n t \right\},
\]

\[
y''(t) = b_1^{\frac{1}{2}} \sum_{n \in \mathbb{Z}^d} a_n q^{2n} \exp \left\{ b_1^{\frac{1}{2}} q^n t \right\}.
\]

Therefore, multiplying (4.6) by \( \exp \left\{ b_1^{\frac{1}{2}} q^n t \right\} \) and summing up for \( n \in \mathbb{Z}^d \) yields, after brief manipulation, the complex functional differential equation

\[
y''(t) + b_2 y(t) = b_1^{\frac{1}{2}} \left\{ \sum_{\ell=1}^{d} (q_\ell^{-1} y'(q_\ell^{-1} t) + q_\ell y'(q_\ell t)) - \lambda y'(t) \right\}. \tag{4.7}
\]
Equation (4.7) is of independent interest, being a special case of the equation
\[ y''(t) + c_1 y'(t) + c_2 y(t) = \int_0^{2\pi} y(e^{i\theta} t) \, d\mu(\theta), \]
where \(d\mu\) is a complex-valued Borel measure. This, in turn, is similar to the functional integro-differential equations of the form
\[ y''(t) + c_1 y'(t) + c_2 y(t) = \int_0^1 y(q t) \, d\eta(q), \]
say, where \(d\eta\) is, again, a complex-valued Borel measure. Equations of this kind have been considered by the present author, jointly with Yunkang Liu [11], with an emphasis on their dynamics and asymptotic behaviour. In the present paper, however, we are interested in the spectral problem for (4.7), and to this end we again expand \(y\) in Taylor series,
\[ y(t) = \sum_{m=0}^{\infty} y_m t^m / m! \]
It is easy to affirm by substitution into (4.7) the three-term recurrence relation
\[ y_{m+1} = b_1^{1/2} \left( 2 \sum_{\ell=1}^d \cos \alpha_{\ell} m - \lambda \right) y_m - b_2 y_{m-1}, \quad m = 1, 2, \ldots \quad (4.8) \]
Note a most remarkable phenomenon – although (4.5) is \(d\)-dimensional, the index in (4.8) lives in \(\mathbb{Z}^+!\) In other words, the dimensionality of the resultant three-term recurrence is independent of \(d\) – it is, instead, expressed as the number of harmonics in the recurrence coefficient. Moreover, inasmuch as (4.8) is more complicated for \(d \geq 2\) than its one-dimensional counterpart (1.4), both recurrences display similar qualitative characteristics. In particular, we can use the theory of Section 2 to cater for the case of \(\alpha_1 / \pi, \alpha_2 / \pi, \ldots, \alpha_d / \pi\) being all rational.

In line with the analysis of Section 2, we let \(\tilde{y}_m(t) = b_1^{1/2} y_m(-(b_1 b_2)^2 t), m \in \mathbb{Z}^+\), whereupon (4.8) becomes
\[ \tilde{y}_{m+1}(t) = \left( t + \frac{2}{\kappa} \sum_{\ell=1}^d \cos \alpha_{\ell} m \right) \tilde{y}_m(t) - \tilde{y}_{m-1}(t), \quad m = 1, 2, \ldots \quad (4.9) \]
To recover all solutions of (4.9) we need to consider a linearly independent two-dimensional set of solutions. Letting \(r_{-1} = 0, r_0 = 1\) and \(s_0 = 0, s_1 = 1\), we recover, similarly to (1.6–7), two sequences \(\{r_m\}_{m \in \mathbb{Z}^+}\) and \(\{s_m\}_{m \in \mathbb{Z}^+}\) that span all solutions of (1.9) and such that \(\deg r_m = m, \deg s_m = m - 1\). In other words, by the Favard theorem both \(\{r_m\}_{m \in \mathbb{Z}^+}\) and \(\{s_m\}_{m \in \mathbb{Z}^+}\) are OPS and, in line with our analysis of the one-dimensional almost Mathieu equation (1.4), we are in position to exploit the theory of orthogonal polynomials.

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