SO(10) and SU(6) Unified Theories on an Elongated Rectangle

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Abstract

Maximally supersymmetric SO(10) and SU(6) unified theories are constructed on the orbifold $T^2/(Z_2 \times Z'_2)$, with one length scale $R_5$ taken much larger than the other, $R_6$. The effective theory below $1/R_6$ is found to be the highly successful $SU(5)$ theory in 5D with natural doublet-triplet splitting, no proton decay from operators of dimension four or five, unified mass relations for heavier generations only, and a precise prediction for gauge coupling unification. A more unified gauge symmetry, and the possibility of Higgs doublets being components of the higher dimensional gauge multiplet, are therefore compatible with a large energy interval where physics is described by $SU(5)$ gauge symmetry in 5D. This leads to the distinctive branching ratios for proton decay from $SU(5)$ gauge boson exchange, $p \to l^+\pi^0, l^+K^0, \nu\pi^+, \nu K^+ \ (l = e, \mu)$, for well-motivated locations for matter. Several phenomenological features of the higher unified gauge symmetry are discussed, including the role of an extra $U(1)$ gauge symmetry, which survives compactification, in the generation of neutrino masses.
1 Introduction

The unification of the three standard model gauge couplings with weak scale supersymmetry suggests a threshold for some unified physics at very high energies. In previous papers we have shown that gauge coupling unification may occur in higher dimensional unified theories when the gauge symmetry is broken by boundary conditions [1, 2, 3]. The resulting explicit local breaking of the gauge symmetry at boundaries of the space does not destroy gauge coupling unification providing the volume of the bulk is large [1]. In particular, we have found that the simplest such theory — SU(5) in 5D — possesses a set of remarkable features, making it extremely attractive as the effective field theory description of nature above the compactification scale, $M_5 = 1/R_5 \approx 10^{15}$ GeV, right up to the scale of strong coupling, $M_s \approx 10^{17}$ GeV [2, 3]. In this paper we go beyond this effective field theory, taking a closer look at the energy interval just below strong coupling. In particular we find that the positive features of the 5D effective theory are maintained even if a sixth dimension opens up just before strong coupling. These features are then seen to arise from a more symmetrical field theory, with gauge group $SO(10)$ [4] or $SU(6)$ and $N = 4$ supersymmetry from the 4D viewpoint.

The gauge symmetry of the SU(5) effective theory is illustrated in Fig. 1: it results from imposing a translation boundary condition under $x^5 \to x^5 + 2\pi R_5$ of $(+,+,+,−,−)$ in the SU(5) space. Higgs hypermultiplets in the $5 + \overline{5}$ ($H + \bar{H}$) representation are located in the bulk, while matter in $\overline{5}$ ($F$) and $10$ ($T$) representations can reside either in the bulk or on the SU(5) invariant fixed point. No larger multiplets are needed.

Above the compactification scale the gauge couplings receive power law corrections, but these corrections are universal because of the bulk SU(5) gauge symmetry. However, the SU(5) breaking defect at $x^5 = \pi R_5$ induces a relative logarithmic running of the gauge couplings above $M_5$, which follows from the pattern of SU(5) breaking in the Kaluza-Klein towers of the gauge and Higgs supermultiplets. The resulting correction to gauge coupling unification

$$\delta \alpha_s \simeq -\frac{3}{7\pi} \alpha_s^2 \ln \frac{\pi M_s}{M_5},$$

precisely corrects the central value for the prediction for gauge coupling unification from $\alpha_s(M_Z) \simeq 0.130$ to $\simeq 0.118$ [2], which should be compared with the experimental value of $\alpha_s^{\exp}(M_Z) = 0.117 \pm 0.002$ [3]. If the third generation $T_3 + F_3$ is located at $x^5 = 0$, a unified mass relation occurs for $m_b / m_t$. This unified mass relation holds at the compactification scale $M_5$, which is smaller than the 4D unification scale, so that the 4D prediction for the bottom quark mass is corrected by

$$\frac{\delta m_b}{m_b} \simeq -\frac{20g^2 - 5y_t^2}{112\pi^2} \ln \frac{\pi M_s}{M_5},$$

1
In the fifth dimension, space is a line segment bounded by branes at $x^5 = 0$ and at $x^5 = \pi R_5$. Here, solid and dotted lines represent the profiles of gauge transformation parameters for $SU(3)_C \times SU(2)_L \times U(1)_Y$, $\xi_{321}$, and $SU(5)/(SU(3)_C \times SU(2)_L \times U(1)_Y)$, $\xi_X$, respectively. Because $\xi_X(x^5 = \pi R_5) = 0$, explicit point defect symmetry breaking occurs at the $x^5 = \pi R_5$ brane, which only respects $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry.

serving to slightly improve the agreement with data. Breaking $SU(5)$ by the translation boundary condition is not just a simple way to break the gauge symmetry: it also leads to a very elegant understanding of why there is a light Higgs doublet but not a light colored triplet. Furthermore the theory possesses a $U(1)_R$ symmetry which leads to the removal of all baryon number violating operators at dimension four and five, solving yet another long standing difficulty of 4D supersymmetric grand unified theories. Finally, matter which resides on the $SU(5)$ fixed point is expected to be heavy, since its mass is not suppressed by a volume dilution factor, and to exhibit $SU(5)$ mass relations, reflecting the symmetry at that point. On the other hand, matter in the bulk will be light, and will not respect $SU(5)$ mass relations because the zero mode structure is greatly affected by the $SU(5)$ breaking defect at $x^5 = \pi R_5$ — hence there is a successful correlation: only the heavier fermions are expected to exhibit unified mass relations. A simple, realistic grand unified construction is complete.

While there are certainly other issues that can be addressed within the context of the 5D effective theory, such as the location of each matter field and the breaking of supersymmetry, in this paper we study how this effective theory can emerge from a more unified theory at higher energies. The 5D theory may emerge directly from string theory at $M_s$, or there may be some strongly coupled field theory at $M_s$. In this paper we explore the possibility that the minimal
5D $SU(5)$ theory of Ref. [2] is the valid effective field theory over a large energy interval from $M_5$ to $M_6 = 1/R_6$, but is incorporated into a 6D field theory at $M_6$ which is still perturbative. We will show that the successes of the 5D theory are maintained as long as $M_6/M_5$ is large enough, but this still allows a sufficiently large energy interval $M_6/M_5$ to study the 6D field theory. In constructing a complete $SU(5)$ theory in 5D, an additional $U(1)$ gauge interaction was needed for a variety of reasons — in particular to understand the see-saw mechanism for neutrino masses [10]. In the 6D context this $U(1)$ arises naturally if the rank of the bulk gauge group $G$ is one larger than that of $SU(5)$. We are led to study two cases for $G$: $SO(10)$ and $SU(6)$. In both cases we take the bulk to contain a maximal amount of supersymmetry, $N = 2$ in 6D and therefore $N = 4$ in 4D, which guarantees that the theory is free from all anomalies providing the 4D anomalies vanish. In the case of $G = SU(6)$, gauge and Higgs fields may be unified into a single 6D gauge supermultiplet.

2 Theories with $N = 2$ Supersymmetry in 6D

In this section we construct a set of 6D supersymmetric unified theories, which provide effective descriptions of nature just below the cutoff scale $M_s$ where the theories become strongly coupled and embedded into some more fundamental theory. Below the scale of $M_6 \approx 10^{16} - 10^{17}$ GeV, which is taken to be a factor of a few smaller than $M_s$, these theories are reduced to the 5D $SU(5)$ theory of Ref. [2] (with an extra $U(1)$ factor) which is an appropriate effective field theory describing the physics over a wide energy interval from $M_6$ down to the scale of the fifth dimension, $M_5 \approx 10^{15}$ GeV.

In general, 6D supersymmetric gauge theories compactified on orbifolds are subject to stringent constraints from anomaly cancellation: not only low energy 4D anomalies arising on fixed points but also anomalies in the 6D bulk must be canceled [11]. In theories with 6D $N = 1$ supersymmetry, these constraints are extremely restrictive, making it difficult to find completely realistic anomaly-free theories. Although it is possible to construct such theories, in most cases we have to rely on the Green-Schwarz mechanism to cancel the bulk anomalies, requiring extra axion-like states in the low-energy theory. Therefore, in this paper we consider 6D $N = 2$ theories in which the cancellation of the bulk anomalies is automatic due to the vector-like nature of these theories in 6D. We require that our theories have two separate mass scales $M_5$ and $M_6 (\gg M_5)$ and that they reduce to 5D $N = 1$ theories between $M_5$ and $M_6$ and 4D $N = 1$ theories below $M_5$. Then we find that $T^2/(Z_2 \times Z_2')$ is the unique simple orbifold on which our 6D theories are compactified — we are led to consider 6D $N = 2$ supersymmetric gauge theories with gauge group $G$, compactified on the $T^2/(Z_2 \times Z_2')$ orbifold having two different radii $R_5 \gg R_6$. The structure of these theories are very rich, having four 5D fixed lines and four 4D
fixed points with differing gauge and supersymmetries. While 6D $N = 2$ supersymmetry allows only the gauge multiplet to be located in the bulk, we can introduce a variety of matter and Higgs fields on 4D or 5D fixed sub-spaces. In sub-section 2.1 we construct completely realistic theories based on $G = SO(10)$ and $G = SU(6)$, in which the Higgs fields are located on a 5D fixed line. In sub-section 2.2 we consider the theories where the Higgs fields are unified with the gauge fields into a single 6D gauge supermultiplet. This gauge-Higgs unification selects the gauge group $G = SU(6)$.

### 2.1 Gauge unification on asymmetric $T^2/(Z_2 \times Z'_2)$

The compactification on the $T^2/(Z_2 \times Z'_2)$ orbifold is obtained by identifying points of the infinite plane $R^2$ under four operations, $Z_5 : (x^5, x^6) \to (-x^5, x^6)$, $Z_6 : (x^5, x^6) \to (x^5, -x^6)$, $T_5 : (x^5, x^6) \to (x^5 + 2\pi R_5, x^6)$ and $T_6 : (x^5, x^6) \to (x^5, x^6 + 2\pi R_6)$. Here, for simplicity, we have taken the two translations $T_5$ and $T_6$ to be in orthogonal directions. We take two radii to be highly asymmetric, $R_5 \gg R_6$, as discussed before. In 6D $N = 2$ theories, the only field which can be introduced in the 6D bulk is a gauge supermultiplet. Under the 4D $N = 1$ superfield language, this multiplet is decomposed into a vector superfield $V$ and three chiral superfields $\Sigma_5$, $\Sigma_6$ and $\Phi$, where $\Sigma_5$ ($\Sigma_6$) contains the fifth (sixth) component of the gauge field, $A_5$ ($A_6$), in its lowest component; all these superfields transform as adjoint under the gauge group $G$.

The boundary conditions for the gauge multiplet are given by

\[
\begin{align*}
V(x^5, x^6) &= V(-x^5, x^6) = V(x^5, -x^6), \\
\Sigma_5(x^5, x^6) &= -\Sigma_5(-x^5, x^6) = -\Sigma_5(x^5, -x^6), \\
\Sigma_6(x^5, x^6) &= \Sigma_6(-x^5, x^6) = -\Sigma_6(x^5, -x^6), \\
\Phi(x^5, x^6) &= -\Phi(-x^5, x^6) = -\Phi(x^5, -x^6),
\end{align*}
\]

(3)

and

\[
\begin{align*}
V(x^5, x^6) &= P_5 V(x^5 + 2\pi R_5, x^6) P_5^{-1} = P_6 V(x^5, x^6 + 2\pi R_6) P_6^{-1}, \\
\Sigma_5(x^5, x^6) &= P_5 \Sigma_5(x^5 + 2\pi R_5, x^6) P_5^{-1} = P_6 \Sigma_5(x^5, x^6 + 2\pi R_6) P_6^{-1}, \\
\Sigma_6(x^5, x^6) &= P_5 \Sigma_6(x^5 + 2\pi R_5, x^6) P_5^{-1} = P_6 \Sigma_6(x^5, x^6 + 2\pi R_6) P_6^{-1}, \\
\Phi(x^5, x^6) &= P_5 \Phi(x^5 + 2\pi R_5, x^6) P_5^{-1} = P_6 \Phi(x^5, x^6 + 2\pi R_6) P_6^{-1},
\end{align*}
\]

(4)

where $P_5$ and $P_6$ are matrices acting on the gauge space, which in general do not commute with the gauge generators. By choosing these matrices, we can have a variety of patterns for the gauge breaking structure in the extra dimensions. The resulting gauge and supersymmetry structure in the 2D extra dimensions is summarized in Fig. 2. The gauge and supersymmetries at each fixed point are given by the intersection of those on the adjacent fixed lines; for example, $G_3 = G_1 \cap G_2$.

To construct theories which reduce below $M_6$ to the 5D $SU(5)$ theory with an extra $U(1)$ gauge interaction, we take either $G = SO(10)$ or $G = SU(6)$. We first consider the case.
of $G = SO(10)$. The $SO(10)$ unified theories in 6D were first considered in Refs. \cite{12, 13}. In particular, 6D $SO(10)$ theories on $T^2/(Z_2 \times Z_2')$ have been constructed in Ref. \cite{13}, and here we follow the notation used there. For alternative implementations of $SO(10)$ in higher dimensions, see Ref. \cite{14}. The generators $T^a$ of $SO(10)$ are imaginary and antisymmetric 10×10 matrices, which are conveniently written as tensor products of 2×2 and 5×5 matrices, giving $\sigma_0 \otimes A_5$, $\sigma_1 \otimes A_5$, $\sigma_2 \otimes S_5$ and $\sigma_3 \otimes A_5$ as a complete set. Here $\sigma_0$ is the 2×2 unit matrix and $\sigma_{1,2,3}$ are the Pauli spin matrices; $S_5$ and $A_5$ are 5×5 matrices that are real and symmetric, and imaginary and antisymmetric, respectively. The $\sigma_0 \otimes A_5$ and $\sigma_2 \otimes S_5$ generators form an $SU(5) \otimes U(1)_X$ subgroup of $SO(10)$, with $U(1)_X$ given by $\sigma_2 \otimes I_5$. We choose our basis so that the standard model gauge group is contained in this $SU(5)$ (Georgi-Glashow $SU(5)$ \cite{15}), with $SU(3)_C$ contained in $\sigma_0 \otimes A_3$ and $\sigma_2 \otimes S_3$ and $SU(2)_L$ contained in $\sigma_0 \otimes A_2$ and $\sigma_2 \otimes S_2$, where $A_3$ and $S_3$ have indices 1,2,3 and $A_2$ and $S_2$ have indices 4,5.

In order to obtain an effective 5D $SU(5)$ theory below $M_6 = 1/R_6$, we have to choose $P_6 = \sigma_2 \otimes I_5$, giving $G_2 = SU(5) \times U(1)_X$. For $P_5$, we have two choices $P_5 = \sigma_0 \otimes \text{diag}(1,1,1,-1,-1)$ and $P_5 = \sigma_2 \otimes \text{diag}(1,1,1,-1,-1)$, giving $G_1 = SU(4)_C \times SU(2)_L \times SU(2)_R$ (Pati-Salam group \cite{16}) and $G_1 = SU(5)' \times U(1)'_X$ (flipped $SU(5)$ \cite{17}), respectively. In either case, $G_3 = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$, and the unbroken gauge symmetries below $M_5 = 1/R_5$ is the standard model gauge group with an extra $U(1)_X$ \cite{18, 12}. The massless fields arising from the gauge multiplet are only vector superfields, $V$, of the $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$ gauge group; all the other fields are heavy with masses larger than $\sim M_5$. Below, we construct
theories concentrating on the case with \( G_1 = SU(4)_C \times SU(2)_L \times SU(2)_R \), but completely realistic theories are also obtained in the other choice of \( G_1 = SU(5)' \times U(1)'_X \).

Having fixed the gauge symmetry structure, we now consider the Higgs fields. In the effective 5D \( SU(5) \) theory below \( M_6 \), the standard model Higgs doublets arise from two hypermultiplets of the \( 5 + \bar{5} \) representation, \( \{ H, H^c \} + \{ \bar{H}, \bar{H}^c \} \), located in the bulk. Here, we have used 4D \( N = 1 \) superfield language: \( H \) and \( \bar{H}^c \) (\( H^c \) and \( \bar{H} \)) are 4D chiral superfields transforming as \( 5 \) (\( \bar{5} \)) under \( SU(5) \). This implies that we have to introduce Higgs hypermultiplets on the 5D fixed line, either \( x^6 = 0 \) or \( x^6 = \pi R_6 \), in our 6D theory. Although we can construct realistic theories in both cases, here we choose to put them on the \( x^6 = \pi R_6 \) fixed line with quantum numbers given by \( \{ H, H^c \}(5, -2) \) and \( \{ \bar{H}, \bar{H}^c \}(5, 2) \), where the numbers in parentheses represent gauge quantum numbers for unconjugated chiral superfields under the \( SU(5) \times U(1)_X \) gauge group, which is unbroken on the \( x^6 = \pi R_6 \) fixed line.\(^1\) (In the case of the Higgs on the \( x^6 = 0 \) fixed line, it arises from a single hypermultiplet \( \{ H, H^c \} \), transforming as \( 10 \) under \( SO(10) \).) In general, the boundary conditions for a hypermultiplet \( \{ \Phi, \Phi^c \} \) located on a fixed line with a constant \( x^6 \) are given by

\[
\Phi(x^5) = \Phi(-x^5) = \eta_\Phi P_5 \cdot \Phi(x^5 + 2\pi R_5), \\
\Phi^c(x^5) = -\Phi^c(-x^5) = \eta_\Phi P_5 \cdot \Phi^c(x^5 + 2\pi R_5),
\]

where \( \eta_\Phi = \pm 1 \), and the matrix \( P_5 \) acts on the gauge space. If \( x^6 = \pi R_6 \), we have to use \( \hat{P}_5 \), instead of \( P_5 \), which is obtained by projecting \( P_5 \) on the \( SU(5) \times U(1)_X \) gauge space. Choosing \( \eta_H = \eta_{\bar{H}} = -1 \), we obtain only the two Higgs doublets of the minimal supersymmetric standard model (MSSM) at low energies. Thus, at this stage, the low energy matter content below \( \sim M_5 \) is the vector multiplets of \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \) and the two Higgs doublets of the MSSM.

How about quarks and leptons? Since our theory has \( M_5 \approx 10^{15} \) GeV, which we will see in more detail later, the first generation matter coming from a \( 10 \) representation of \( SU(5) \) must propagate in the fifth dimension to avoid too rapid proton decay caused by an exchange of the broken gauge bosons. This implies that we have to put two hypermultiplets \( \{ T_{1} + T_{1}^c \}(10, 1) + \{ T_{1}^c + T_{1}^c \}(10, 1) \) with \( \eta_{T_{1}} = -\eta_{T_{1}^c} = 1 \) on the \( x^6 = \pi R_6 \) fixed line. We then obtain MSSM quark and lepton superfields \( Q_1, U_1 \) and \( E_1 \) at low energies as zero modes of these multiplets. For the other matter fields, we have three options: introducing on a 4D fixed point, on a short fixed line with a constant \( x^5 \), or on a long 5D fixed line with a constant \( x^6 \). We can make a choice for each matter field from these options. While there are many possible matter configurations leading to realistic fermion mass matrices, here we focus on the case where the bottom and

\(^1\) We have normalized \( U(1)_X \) charges such that \( 10 \) of \( SO(10) \) decomposes into \( (5, -2) + (\bar{5}, 2) \) under the \( SU(5) \times U(1)_X \) subgroup.
tau Yukawa couplings are unified around the unified mass scale reflecting the underlying \(SU(5)\) gauge structure [18]. This forces us to introduce the third generation matter either on the \((x^5, x^6) = (0, \pi R_6)\) fixed point or on the \(x^5 = 0\) fixed line, for the present choice of the Higgs location. As an example, here we choose to put \(T_3\) and \(F_3\) on the fixed point and the fixed line, respectively: we introduce a chiral superfield \(T_3(10, 1)\) on the \(SU(5) \times U(1)_X\) fixed point at \((x^5, x^6) = (0, \pi R_6)\), and a hypermultiplet \(\{\Psi_3 + \Psi_5^6\}(16)\) on the \(SO(10)\) fixed line of \(x^5 = 0\).

The general boundary conditions for a hypermultiplet \(\{\Phi, \Phi^c\}\) located on a fixed line with a constant \(x^5\) are given by

\[
\begin{align*}
\Phi(x^6) &= \Phi(-x^6) = \zeta_\Phi \hat{P}_6 \cdot \Phi(x^6 + 2\pi R_6), \\
\Phi^c(x^6) &= -\Phi^c(-x^6) = \zeta_\Phi \hat{P}_6 \cdot \Phi^c(x^6 + 2\pi R_6),
\end{align*}
\]  

(6)

where \(\zeta_\Phi = \pm 1\), and \(\hat{P}_6\) is a matrix obtained by projecting \(P_6\) on the corresponding gauge space unbroken on the fixed line. Choosing \(\zeta_{\Psi_3}\) appropriately, we find that \(5 + 1\) components of \(SU(5)\) remain as zero modes from \(\{\Psi_3, \Psi_5^6\}\): in the effective 5D \(SU(5) \times U(1)_X\) theory below \(M_6\), the hypermultiplet \(\{\Psi_3, \Psi_5^6\}\) reproduces brane fields \(F_3(5, -3) + N_3(1, 5)\) localized on the \(SU(5) \times U(1)_X\) invariant fixed point at \(x^5 = 0\). Thus, together with \(T_3(10, 1)\) located on the \((x^5, x^6) = (0, \pi R_6)\) fixed point, we recover a complete set of the third generation matter \(T_3 + F_3 + N_3\) on the \(x^5 = 0\) brane in the effective 5D \(SU(5) \times U(1)_X\) theory below \(M_6\).

The configuration for the other matter fields are less restrictive. The only significant constraint is that \(T_2\) and \(F_2\) cannot both be confined to the \(x^5 = 0\) plane, to avoid the unwanted \(SU(5)\) mass relation for \(m_s/m_\mu\): at least one of \(T_2\) and \(F_2\) must propagate in the fifth dimension or be located on subspaces with \(x^5 = \pi R_5\). There are many possibilities which satisfy this criteria and lead to realistic fermion mass matrices, but here we do not exhaust all of these possibilities; rather we present some new mechanisms for understanding matter quantum numbers which can be implemented but cannot be fully understood in the 5D \(SU(5)\) theory context. First, in the present 6D theory, we can introduce matter fields on the \(x^5 = \pi R_5\) fixed line, forming representations under \(SU(4)_C \times SU(2)_L \times SU(2)_R\) such as \((4, 2, 1)\) or \((4, 1, 2)\). In the effective 5D \(SU(5)\) theory below \(M_6\), these fields are reduced to brane fields localized on the \(SU(5)\) breaking fixed point at \(x^5 = \pi R_5\) and thus do not necessarily represent properties for \(SU(5)\) matter; for instance, these fields are not subject to gauge boson mediated proton decay or \(SU(5)\) Yukawa relations. Nevertheless, the hypercharges for these fields are appropriately quantized, since they come from a multiplet of non-Abelian gauge group, \(SU(4)_C \times SU(2)_L \times SU(2)_R\). Therefore, although there is no reason in the effective 5D theory for why hypercharges for these fields are appropriately quantized (since the unbroken gauge symmetry on the \(x^5 = \pi R_5\) brane is \(SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X\)), we can understand the quantization in the more fundamental (higher dimensional) theory. This provides a general
way of understanding the quantization of $U(1)$ charges which are arbitrary in the effective field theory of interest. For instance, we can imagine that $U(1)_X$ charges for matter fields introduced on the $SU(5) \times U(1)_X$ fixed line will be quantized in a similar way in some higher dimensional theory above $M_s$.

We also find an interesting mechanism of realizing textures for fermion mass matrices. Suppose we introduce some of the matter fields on the $x^5 = 0$ plane (either the fixed line or one of two fixed points) and some on the $x^5 = \pi R_5$ plane. In this case, the fields on $x^5 = 0$ cannot couple to those on $x^5 = \pi R_5$ due to locality in the extra dimensions. This leads to texture zeros in Yukawa matrices, which are not guaranteed by any symmetry of the low energy effective field theory. Clearly, this mechanism can also be used in the 5D $SU(5)$ theory, although in this case we have to assume an appropriate $U(1)_Y$ charge quantization for matter on the $x^5 = \pi R_5$ brane.

While it is quite interesting to pursue completely realistic and predictive theories of flavor within the present framework using the above mechanisms, here we just present a simple example of matter configuration which reduces to that of Ref. [3] in the effective 5D $SU(5) \times U(1)_X$ theory below $M_6$. We introduce two hypermultiplets $\{\Psi_1 + \Psi_1^c\}(16)$ and $\{\Psi_2 + \Psi_2^c\}(16)$ on the $x^5 = 0$ fixed line. Choosing $\zeta_{\Psi_1} = \zeta_{\Psi_2} = \zeta_{\Psi_3}$, these hypermultiplets give $F_{1,2}(5, -3) + N_{1,2}(1, 5)$ on the $x^5 = 0$ brane in the effective 5D theory. The only remaining field is the second generation matter coming from $10$ of $SU(5)$, which we introduce on the $x^6 = \pi R_6$ brane as two hypermultiplets $\{T_2 + T_2^c\}(10, 1) + \{T_2' + T_2'^c\}(10, 1)$ with $\eta_{T_2} = -\eta_{T_2'} = 1$. This completes three generations of matter, $T_{1,2,3}, F_{1,2,3}$ and $N_{1,2,3}$, for low energy fields below $M_5$. This example of matter configuration is summarized in Fig. [3].

The Yukawa couplings are introduced on the $(x^5, x^6) = (0, \pi R_6)$ fixed point. Since the gauge symmetry on this fixed point is $SU(5) \times U(1)_X$, they take the form

$$W = \hat{T}\hat{T}H + \hat{F}\hat{F}\bar{H} + \hat{F}\hat{N}H,$$

(7)

where $\hat{T}$, $\hat{F}$ and $\hat{N}$ run for all the components of matter fields in $10$, $5$ and $1$ representations of $SU(5)$, respectively. For instance, in the example of matter configuration in Fig. [3], $\hat{T}$ runs for $T_3, T_{1,2}$ and $T_{1,2}';$ $\hat{F}$ for $F_{1,2,3} \subset \Psi_{1,2,3};$ and $\hat{N}$ for $N_{1,2,3} \subset \Psi_{1,2,3}$. Here, we have omitted coefficients for the operators suppressed by appropriate powers of $M_s$. In the low energy 4D

\footnote{Another intriguing mechanism for charge quantization is to use an anomaly inflow in the bulk through the Chern-Simons term [19]. Suppose we introduce $T(10)$ on the $SU(5)$ brane at $x^5 = 0$ and $D(3, 1, \alpha/3) + L(1, 2, -\alpha/2)$ on the $SU(3)_C \times SU(2)_L \times U(1)_Y$ brane at $x^5 = \pi R_5$, in the effective 5D $SU(5)$ theory. This theory is consistent only if $\alpha = 1$, in which case we can cancel all gauge anomalies by introducing the Chern-Simons term in the bulk with an appropriate coefficient. This mechanism even allows a fractional quantization of $U(1)$ charges: in a way which does not arise from embedding the $U(1)$ factor together with the other gauge factors in a larger non-Abelian gauge group [14].}

\footnote{We could also introduce Yukawa couplings for $T_{1,2}$ and $T_{1,2}'$ on the $(x^5, x^6) = (\pi R_5, \pi R_6)$ fixed point, which do not respect the $SU(5)$ symmetry.}
Figure 3: An example of the matter configuration in the 6D $SO(10)$ theory on asymmetric $T^2/(Z_2 \times Z'_2)$. The numbers in the square brackets represent unbroken gauge symmetries on the corresponding (sub-)spaces: 10, 5-1, 4-2-2 and 3-2-1-1 denote $SO(10)$, $SU(5) \times U(1)_X$, $SU(4)_C \times SU(2)_L \times SU(2)_R$ and $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$, respectively.

theory, these couplings reproduce usual Yukawa couplings of the MSSM (with neutrino Yukawa couplings). Since various matter fields propagate in differing dimensions, various 4D Yukawa couplings have suppressions by powers of different volume factors. In particular, if a matter field propagates in the fifth (sixth) dimension, it carries a suppression factor $\epsilon_5 = (M'_5/M_5)^{1/2}$ ($\epsilon_6 = (M'_6/M_6)^{1/2}$), where $M'_5 \equiv M_5/\pi$ ($M'_6 \equiv M_6/\pi$). In the example of Fig. 3, this leads to

$$W_{4D} \approx (T_1 \ T_2 \ T_3) \begin{pmatrix} \epsilon_5^2 & \epsilon_5^2 & \epsilon_5 \\ \epsilon_5^2 & \epsilon_5^2 & \epsilon_5 \\ \epsilon_5 & \epsilon_5 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} H + \epsilon_6 (T_1 \ T_2 \ T_3) \begin{pmatrix} \epsilon_5 & \epsilon_5 & \epsilon_5 \\ \epsilon_5 & \epsilon_5 & \epsilon_5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} H.$$ (8)

Here we have displayed only the gross structure that follows from the volume suppression factors, and omitted the coupling parameters of the brane-localized Yukawa interactions. Only underlined entries respect $SU(5)$, since the other entries involve $T_{1,2}$ which actually represent quarks and leptons from differing $SU(5)$ bulk multiplets. The only unified mass relation is for $b/\tau$. This mass matrix structure is the same as that of Ref. 3, except that now there is an extra suppression factor $\epsilon_6$ in the $T F \bar{H}$ Yukawa couplings relative to the $T T H$ ones. As we will see later in section 3, we can imagine $\epsilon_5 \simeq 0.1$ and $\epsilon_6 \simeq 0.3$ as a realistic parameter region. Thus this extra suppression factor provides part of the suppression for $b/t$, and allows tan $\beta$ to be moderate ($\approx 15$) rather than large ($\approx 50$).

Here we make one brief comment. The 6D theory has a very rich structure with several different ways of assigning Higgs and matter multiplets leading to the same 5D $SU(5)$ theory. These different assignments can lead to a differing pattern of Yukawa couplings, reflecting the higher symmetries of the 6D theory. For example, the Higgs doublets may be components of

```
Table 1: $U(1)_R$ charges for 4D vector and chiral superfields, normalized such that the superspace coordinates $\theta^a$ of the 4D $N=1$ supersymmetry carry a unit charge. Here, $M$ represents all the matter fields; for the example of the matter configuration in Fig. 3, $M$ stands for $T_3$, $T_{1,2}$, $T'_{1,2}$ and $\Psi_{1,2,3}$, and $M^c$ for $T'^c_{1,2}$, $T''^c_{1,2}$ and $\Psi^c_{1,2,3}$.

| $U(1)_R$ | $\theta^a$ | $V$ | $\Sigma_5$ | $\Sigma_6$ | $\Phi$ | $H$ | $H^c$ | $H'$ | $H'^c$ | $M$ | $M^c$ |
|----------|-----------|-----|------------|------------|-------|-----|------|------|-------|-----|-----|
| $1$      | $0$       | $0$ | $0$        | $2$        | $0$   | $2$ | $0$  | $2$  | $1$   | $1$ |     |

Below the scale $M_6$, our theory is reduced to the 5D $SU(5)$ theory of Ref. [2], with a gauge interaction $U(1)_X \subset SO(10)/SU(5)$. The breaking of $U(1)_X$ and the generation of small neutrino masses and a weak scale mass term ($\mu$ term) for the Higgs doublets can be accomplished along the lines of Ref. [3]: neutrino masses are generated by the see-saw mechanism [10], and the $\mu$ term is generated by the vacuum readjustment mechanism [20]. All the positive features of the 5D $SU(5)$ theory are maintained; in particular, the successful predictions from gauge and Yukawa coupling unification are preserved, which we will see in more detail in section 3.

We finally discuss the other possibilities. We can also construct realistic theories based on $G = SU(6)$, instead of $G = SO(10)$. The boundary conditions for the bulk gauge supermultiplet are given by Eqs. (3, 4), but now $P_5$ and $P_6$ are acting on the $SU(6)$ space. To obtain the theory which reduces to 5D $SU(5)$ below $M_6$, we take $P_6 = \text{diag}(1,1,1,1,1,-1)$, giving $G_2 = SU(5) \times U(1)_X$. For $P_5$, we have two choices $P_5 = \text{diag}(1,1,1,1,1,-1)$ and $P_5 = \text{diag}(-1,-1,1,1,1,1)$, giving $G_1 = SU(3) \times SU(3) \times U(1)$ and $G_1 = SU(4) \times SU(2) \times U(1)$, respectively. In either case, $G_3 = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$, and the unbroken gauge symmetries below $M_5 = 1/R_5$ are the standard model gauge group with an extra $U(1)_X$. The two Higgs hypermultiplets are introduced on the $x^6 = \pi R_6$ fixed line with quantum numbers given by $\{H, H^c\}(5,-2)$ and $\{\bar{H}, \bar{H}^c\}(5,2)$. The boundary conditions are given by Eq. (5).
with $\eta_H = -1$, giving two MSSM Higgs doublets at low energies. The quarks and leptons are introduced either on the $SU(5) \times U(1)_X$ fixed point, $(x^5, x^6) = (0, \pi R_6)$, or fixed line, $x^6 = \pi R_6$, with Yukawa couplings located on the $(x^5, x^6) = (0, \pi R_6)$ fixed point. Again, the $U(1)_R$ symmetry can be introduced with the charge assignments given in Table 1.

2.2 Gauge-Higgs unification on asymmetric $T^2/(Z_2 \times Z_2')$

In this sub-section we construct theories where the Higgs and gauge fields are unified into a single 6D gauge supermultiplet. This type of theories has been considered in Ref. 8, but here we impose slightly different boundary conditions so that the theory is reduced to the 5D $SU(5)$ theory below the scale of the sixth dimension, $M_6$.

We again consider 6D $N = 2$ supersymmetric gauge theories with gauge group $G$, compactified on the asymmetric $T^2/(Z_2 \times Z_2')$ orbifold with $R_5 \gg R_6$. The boundary conditions for the gauge multiplet are given by

\[
\begin{align*}
V(x^5, x^6) &= PV(-x^5, x^6) P^{-1} = PV(x^5, -x^6) P^{-1}, \\
\Sigma_5(x^5, x^6) &= -P \Sigma_5(-x^5, x^6) P^{-1} = P \Sigma_5(x^5, -x^6) P^{-1}, \\
\Sigma_6(x^5, x^6) &= P \Sigma_6(-x^5, x^6) P^{-1} = -P \Sigma_6(x^5, -x^6) P^{-1}, \\
\Phi(x^5, x^6) &= -P \Phi(-x^5, x^6) P^{-1} = -P \Phi(x^5, -x^6) P^{-1},
\end{align*}
\] (9)

and

\[
\begin{align*}
V(x^5, x^6) &= P_5 V(x^5 + 2\pi R_5, x^6) P_5^{-1} = V(x^5, x^6 + 2\pi R_6), \\
\Sigma_5(x^5, x^6) &= P_5 \Sigma_5(x^5 + 2\pi R_5, x^6) P_5^{-1} = \Sigma_5(x^5, x^6 + 2\pi R_6), \\
\Sigma_6(x^5, x^6) &= P_5 \Sigma_6(x^5 + 2\pi R_5, x^6) P_5^{-1} = \Sigma_6(x^5, x^6 + 2\pi R_6), \\
\Phi(x^5, x^6) &= P_5 \Phi(x^5 + 2\pi R_5, x^6) P_5^{-1} = \Phi(x^5, x^6 + 2\pi R_6),
\end{align*}
\] (10)

where $P$ and $P_5$ are matrices acting on the gauge space. The resulting gauge and supersymmetry structure in the 2D extra dimensions is summarized in Fig. 1. Unlike the case of the previous section, the gauge symmetry structure is symmetric about $x^6 = \pi R_6/2$ in the sixth dimension while it is still asymmetric about $x^5 = \pi R_5/2$ in the fifth dimension.

An important point for the boundary conditions in Eqs. (9, 10) is that they can leave low energy fields other than those from the 4D vector superfield $V$. Suppose we take $G = SU(6)$ and $P = \text{diag}(1, 1, 1, 1, 1, -1)$. Then, in the effective 5D theory below $M_6$, we have components of $\{\Phi, \Sigma_6\}$ propagating in the 5D bulk, in addition to the $SU(5) \times U(1)_X$ gauge multiplet coming from $\{V, \Sigma_5\}$. Since the components of $\{\Phi, \Sigma_6\}$ remaining below $M_6$ appear as 5D hypermultiplets transforming as $(5, -2) + (\bar{5}, 2)$ under $SU(5) \times U(1)_X$, we can identify these fields to be the two Higgs hypermultiplets located in the bulk of the effective 5D $SU(5) \times U(1)_X$ theory. This allows us to construct theories where the gauge and bulk Higgs multiplets in 5D are unified into a single gauge multiplet in 6D.
Figure 4: The structure of gauge and supersymmetries in the 2D extra dimensions with boundary conditions Eqs. (9, 10).

Now, we explicitly construct a completely realistic theory. We take $G = SU(6)$ and $P = \text{diag}(1, 1, 1, 1, 1, -1)$ to reproduce two Higgs hypermultiplets, $\{H, H^c\}$ and $\{\bar{H}, \bar{H}^c\}$, in the effective 5D theory below $M_6$. They arise from the $SU(6)/(SU(5) \times U(1)_X)$ components of the $\Phi$ and $\Sigma_6$ fields as $H + \bar{H} \subset \Phi$ and $H^c + \bar{H}^c \subset \Sigma_6$. The matrix $P_5$ must be chosen as $P_5 = \text{diag}(1, 1, 1, 1, 1, -1)$ to obtain the two MSSM Higgs doublets at low energies. This fixes the gauge structure to be $G_1 = SU(3) \times SU(3) \times U(1)_Y$ and $G_2 = SU(5) \times U(1)_X$ and $G_3 = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$. The quark and lepton fields are introduced either on the $SU(5) \times U(1)_X$ fixed points, $(x^5, x^6) = (0, 0), (0, \pi R_6)$, or fixed lines, $x^6 = 0, \pi R_6$. If we locate matter fields in both $x^6 = 0$ and $x^6 = \pi R_6$ planes, the fields on the $x^6 = 0$ plane and on the $x^6 = \pi R_6$ plane do not have Yukawa couplings, giving texture zeros in the low energy 4D theory. While it is interesting to work out more complicated cases, here we just present an example of realistic matter configuration which reduces to that of Ref. [3] in the 5D effective theory below $M_6$. We introduce four chiral superfields $T_3(10, 1) + F_{1,2,3}(\bar{5}, -3)$ on the $(x^5, x^6) = (0, \pi R_6)$ fixed point, and four hypermultiplets $\{T_{1,2} + T_{1,2}^c\}(10, 1) + \{T_{1,2} + T_{1,2}^c\}(10, 1)$ with $\eta_{T_{1,2}} = -\eta_{T_{1,2}^c} = 1$ on the $x^6 = \pi R_6$ fixed line. We also introduce three right-handed neutrino fields to cancel $U(1)_X$ anomalies, which can be located either on the fixed point as $N(1, 5)$ or on the fixed line as $\{N + N^c\}(1, 5)$ with $\eta_N = 1$. This completes the three generations of matter (including right-handed neutrinos) at low energies.

The Yukawa couplings of the form in Eq. (7) are introduced on the $(x^5, x^6) = (0, \pi R_6)$ fixed point, but now $H$ and $\bar{H}$ fields arise from the component, $\Phi$, of the 6D gauge multiplet. It is important to notice that $\Phi$ transforms linearly under the gauge group so that it can have texture zeros in the low energy 4D theory. While it is interesting to work out more complicated cases, here we just present an example of realistic matter configuration which reduces to that of Ref. [3] in the 5D effective theory below $M_6$. We introduce four chiral superfields $T_3(10, 1) + F_{1,2,3}(\bar{5}, -3)$ on the $(x^5, x^6) = (0, \pi R_6)$ fixed point, and four hypermultiplets $\{T_{1,2} + T_{1,2}^c\}(10, 1) + \{T_{1,2} + T_{1,2}^c\}(10, 1)$ with $\eta_{T_{1,2}} = -\eta_{T_{1,2}^c} = 1$ on the $x^6 = \pi R_6$ fixed line. We also introduce three right-handed neutrino fields to cancel $U(1)_X$ anomalies, which can be located either on the fixed point as $N(1, 5)$ or on the fixed line as $\{N + N^c\}(1, 5)$ with $\eta_N = 1$. This completes the three generations of matter (including right-handed neutrinos) at low energies.

As in the previous sub-section, we have made choices for the $U(1)_X$ quantum numbers of matter so that Yukawa couplings with the Higgs in $\Phi$ are allowed.
Yukawa couplings to quarks and leptons without contradicting to the higher dimensional gauge invariance \[8\]. Below \(M_5\), we obtain the 4D Yukawa couplings of the form Eq. (8) with \(\epsilon_6 = 1\), which explains a part of the observed structure of fermion mass matrices, including the unified relation for \(m_b/m_\tau\), the absence of the corresponding relation for \(m_s/m_\mu\), large neutrino mixing angles, and a hierarchy of masses between the third and the first two generations.

In the theories with gauge-Higgs unification, the \(R\) charges of the Higgs fields are determined because they are part of the gauge multiplet. In particular the Higgs fields, \(H\) and \(\bar{H}\), must transform non-trivially under the \(R\) symmetry so that the 6D supersymmetric gauge kinetic term is invariant. This forces us to use a discrete \(R\) symmetry to forbid unwanted brane operators while keeping the Yukawa couplings. We thus consider the \(Z_{4,R}\) symmetry with charge assignments given in Table 2. We require that all the operators in the Lagrangian must be invariant under \(Z_{4,R}\): the terms in the superpotential (Kähler potential) must carry charges of +2 (0) mod 4. This forbids all unwanted brane operators, such as \([H\bar{H}]_{\theta^2}\) (i.e. \([\Phi^2]_{\theta^2}\) and \([\hat{T}\hat{T}\hat{T}\hat{F}]_{\theta^2}\), completing the solutions to the doublet-triplet and proton decay problems.

The \(U(1)_X\) breaking and the generation of small neutrino masses can be achieved preserving \(Z_{4,R}\) symmetry. Specifically, we introduce \([X(B\bar{B} - \Lambda^2) + \hat{B}NN]_{\theta^2}\) on the \((x^5, x^6) = (0, \pi R_6)\) brane, where \(X(1, 0), B(1, 10)\) and \(\hat{B}(1, -10)\) are chiral superfields with the \(SU(5) \times U(1)_X\) quantum numbers given in the parentheses; the \(Z_{4,R}\) charges for the \(X, B\) and \(\hat{B}\) fields are given by 2, -2 and 2, respectively. This brane superpotential gives vacuum expectation values \(\langle B \rangle = \langle \bar{B} \rangle = \Lambda\) and consequently the right-handed neutrino masses of order \(\Lambda\). Taking \(\Lambda \approx 10^{14}\) GeV, we obtain small neutrino masses of desirable sizes through the see-saw mechanism.

The expectation values \(\langle B \rangle = \langle \bar{B} \rangle \neq 0\) break both \(Z_{4,R}\) and \(U(1)_X\) symmetries, but it leaves another unbroken discrete \(Z'_{4,R}\) symmetry that is a linear combination of \(Z_{4,R}\) and \(U(1)_X\): \(Z'_{4,R} = Z_{4,R} + (1/5)U(1)_X\). (To make all charges integer, we have to take a linear combination, \(Z_{4,R} + (1/5)U(1)_X + (24/5)U(1)_Y\).) This \(Z'_{4,R}\) symmetry is sufficient to forbid all the unwanted operators, and thus no large \(\mu\) term or dimension four/five proton decay operators are generated by this symmetry breaking. A \(\mu\) term of the order of the weak scale can be generated though

| \(Z_{4,R}\) | \(\theta^a\) | \(V\) | \(\Sigma_5\) | \(\Sigma_6\) | \(\Phi\) | \(M\) | \(M^c\) |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 2 | 0 | 2 |
the $Z_{4,R}'$ breaking encoded in supersymmetry breaking parameters, through the mechanism of Ref. [20]. The usual MSSM $R$ parity, which is the subgroup of $Z_{4,R}'$, could remain unbroken.

3 Gauge and Yukawa Coupling Unification

In this section we show that our asymmetric 6D theories preserve successful predictions from gauge and Yukawa couplings obtained in Refs. [3, 3]. We present an explicit analysis in the present theories, following the general analysis of Ref. [2], which elucidates some of the general features for the behavior of gauge couplings in higher dimensional unified field theories.

We first consider the effective (Wilsonian) action at the cutoff scale $M_s$. No matter what the physics above $M_s$ is, the general form for the gauge kinetic terms are given by

$$ S = \int \! d^6x \left[ \frac{1}{g_6^2} F_{\mu\nu} F_{\mu\nu} + \sum_{\hat{x}_5=0,\pi R_5} \delta(x_5 - \hat{x}_5) \frac{1}{g_5^2} F_{\mu\nu} F_{\mu\nu} + \sum_{\hat{x}_6=0,\pi R_6} \delta(x_6 - \hat{x}_6) \frac{1}{g_4^2} F_{\mu\nu} F_{\mu\nu} + \sum_{\hat{x}_5=0,\pi R_5, \hat{x}_6=0,\pi R_6} \frac{1}{g_4^2} (\hat{x}_5, \hat{x}_6) \right]. $$

Here, a term located on a subspace only respects the gauge symmetry operative on that subspace. Note that this form is guaranteed by the restricted unified gauge symmetry (position dependent gauge symmetry) of the effective higher dimensional field theory below $M_s$. The 4D gauge couplings for the zero modes are obtained by integrating over the extra dimensions:

$$ \frac{1}{g_{4D,i}^2} = \frac{\pi^2 R_5 R_6}{g_6^2} + \sum_{\hat{x}_5=0,\pi R_5} \frac{\pi R_6}{g_5^2(\hat{x}_5,*)} + \sum_{\hat{x}_6=0,\pi R_6} \frac{\pi R_5}{g_4^2(\hat{x}_6,*)} + \sum_{\hat{x}_5=0,\pi R_5, \hat{x}_6=0,\pi R_6} \frac{1}{g_4^2(\hat{x}_5,\hat{x}_6)}, $$

where $i$ runs for $SU(3)_C$, $SU(2)_L$, $U(1)_Y$ and $U(1)_X$. Since the theory is strongly coupled at the scale $M_s$, we can estimate the coefficients for various operators by requiring that all loop contributions become comparable at this scale. By carefully evaluating loop expansion parameters, we find that $1/g_6^2 \simeq C M_s^2/16\pi^4$, $1/g_5^2(\hat{x}_5,*) \simeq 1/g_5^2(\hat{x}_6,*) \simeq C M_s/16\pi^3$ and $1/g_4^2(\hat{x}_5,\hat{x}_6) \simeq C/16\pi^2$, giving

$$ \frac{1}{g_{4D,i}^2} \simeq \frac{C(M_s R_5)(M_s R_6)}{16\pi^2} + \frac{C(M_s R_6)}{16\pi^2} + \frac{C(M_s R_5)}{16\pi^2} + \frac{C}{16\pi^2} = \frac{C(M_s R_5)(M_s R_6)}{16\pi^2} \left(1 + \frac{1}{M_s R_5} + \frac{1}{M_s R_6} + \frac{1}{(M_s R_5)(M_s R_6)}\right), $$

where $C$ is a group theoretical factor, and the first, second, third and fourth terms represent contributions from the 6D bulk, 5D short fixed lines, 5D long fixed lines and 4D fixed points,
respectively; here we have omitted unknown order-one coefficients for each term. We thus find that the contributions from short fixed lines and fixed points are suppressed by the large volume factor $M_s R_5$, while those from long fixed lines have only small suppression by $M_s R_6$. (Recall, $M_s R_5 \gg M_s R_6 \sim 1$.) However, since the gauge symmetries on the long fixed lines (and in the bulk) contain $SU(5)$, this is sufficient for guaranteeing successful gauge coupling unification. In other words, although there are unknown $SU(5)$-violating contributions to the 4D (zero-mode) gauge couplings at $M_s$ coming from the gauge kinetic terms localized on 5D short fixed lines and 4D fixed points, they are suppressed by the large radius of the fifth dimension. Here we take $M_s R_5 \simeq 30$ to suppress these unknown contributions to a negligible level. This implies that $M_s R_6$ must be a factor of a few to obtain order-one 4D gauge coupling constants, $g_{4D,i} = O(1)$.

Having obtained gauge coupling unification at the scale $M_s$, we next consider radiative corrections coming from an energy interval between $M_s$ and $M_6$. In this energy interval, 4D gauge couplings receive power corrections. From the higher dimensional point of view, these corrections arise from radiative corrections to the gauge kinetic terms localized on 5D fixed lines. (Since we have 6D $N = 2$ supersymmetry in the bulk, the bulk gauge kinetic term does not receive any quantum correction.) However, we can show that the size of the power corrections is always smaller than the tree-level estimates given in Eq. (13). Specifically, the $SU(5)$-violating power-law corrections to $1/g_{4D,i}^2$ are given by $\simeq (b/16\pi^2)(M_s/M_6)$, where $b$ is an appropriate beta-function coefficients, so that these corrections are at most the same size with the tree-level values given at $M_s$.\footnote{In addition, there are $SU(5)$-symmetric power-law corrections of size $\simeq (b'/16\pi^2)(M_s/M_5)$, which also do not exceed the tree-level values at $M_s$.} Note that this is a general consequence of the effective field theory framework. In an effective field theory, power corrections are scheme dependent and can always be absorbed into the definitions of the tree-level parameters at the cutoff scale. Therefore, by appropriately estimating the size of the tree-level terms at the cutoff scale, we can always forget about the presence of power corrections; these are contributions coming from the physics at or above $M_s$ and cannot be computed in the effective field theory framework. Unlike power-law corrections, logarithmically divergent contributions and finite contributions are calculable and meaningful quantities in the effective field theory. In our case, there are logarithmic contributions to $1/g_{4D,i}^2$ coming from the running of the gauge kinetic operators localized on the 4D fixed points, whose sizes are given by $\simeq (b'/16\pi^2)\ln(\pi M_s/M_6)$. However, since $\ln(\pi M_s/M_6)$ is not a large quantity, they are not much larger than unknown tree-level contributions of order $C/16\pi^2$. Therefore, here we do not include this contribution to the calculation of $\alpha_s(M_Z)$. We also neglect the finite correction at the scale $M_6$, since it is also similar in size to unknown tree-level contributions. However, it is important to notice that these are calculable contributions and, if one wants, can be included for the prediction of $\alpha_s(M_Z)$.\footnote{In addition, there are $SU(5)$-symmetric power-law corrections of size $\simeq (b'/16\pi^2)(M_s/M_5)$, which also do not exceed the tree-level values at $M_s$.}
Now, we match our 6D theory to the effective 5D theory. Since we have found that quantum corrections between $M_s$ and $M_6$ do not change the tree-level coefficients much, the gauge kinetic terms at the scale $M_6$ are still given by Eq. (11) with various coefficients taking the sizes given just above Eq. (13). Integrating over the sixth dimension, we obtain the effective action at $M_6$ in the effective 5D theory:

$$S = \int d^5 x \left[ \frac{1}{g_5^2} F_{\mu\nu} F_{\mu\nu} + \sum_{\hat{x}_5 = 0, \pi R_5} \delta(x_5 - \hat{x}_5) \frac{1}{g_{4,\hat{x}_5}^2} F_{\mu\nu} F_{\mu\nu} \right],$$

(14)

where various coefficients are given by

$$\frac{1}{g_5^2} = \frac{\pi R_6}{g_6^2} + \sum_{\hat{x}_6 = 0, \pi R_6} \frac{1}{g_{5,(*,\hat{x}_6)}^2} \\approx \frac{C M_s^2 R_6}{16 \pi^3} \left( 1 + \frac{1}{M_s R_6} \right),$$

(15)

$$\frac{1}{g_{4,\hat{x}_5}^2} = \frac{\pi R_6}{g_{5,\hat{x}_5}^2} + \sum_{\hat{x}_6 = 0, \pi R_6} \frac{1}{g_{4,(\hat{x}_5,\hat{x}_6)}^2} \\approx \frac{C M_s R_6}{16 \pi^2} \left( 1 + \frac{1}{M_s R_6} \right).$$

(16)

Thus the gauge structure of the bulk (brane-localized) kinetic energy, $g_5$ ($g_{4,\hat{x}_5}$), arises from those of the bulk and long fixed lines (short fixed lines and fixed points) in the original 6D theory.

We first consider the bulk gauge coupling, $g_5$. It comes from a linear combination of the couplings of the 6D bulk and 5D long fixed lines in the original theory. In any of the models discussed in the previous section, there is a 5D long fixed line on which the gauge symmetry is only $SU(5) \times U(1)_X$. Since the effect from 5D long fixed lines on $g_5$ is unsuppressed (suppressed only by a small volume factor of $M_s R_6$), we do not find any particular relation between the bulk gauge couplings for $SU(5)$ and $U(1)_X$ in the effective 5D theory. On the other hand, $g_5$ clearly respects $SU(5)$, since the $SU(5)$ gauge symmetry remains unbroken in the bulk of the 5D effective theory; in fact, the bulk and long fixed lines of the original theory always respect $SU(5)$ by construction. Similar considerations show that the brane-localized gauge couplings $g_{4,\hat{x}_5=0}$ and $g_{4,\hat{x}_5=\pi R}$ respect (only) $SU(5) \times U(1)_X$ and $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$, respectively.

Summarizing so far, we have obtained the effective 5D $SU(5) \times U(1)_X$ theory at $M_6$, where the sizes of the bulk and brane gauge couplings are given by $1/g_5^2 \simeq C M_s^2 R_6/16 \pi^3$ and $1/g_{4,\hat{x}_5}^2 \simeq C M_s R_6/16 \pi^2$, respectively. The bulk gauge coupling and the brane gauge coupling at $x^5 = 0$ respect $SU(5) \times U(1)_X$ symmetry, while the brane coupling at $x^5 = \pi R_5$ respects only $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$. Since $M_s R_6$ is not much larger than unity, we find that the situation is almost the same with the minimal 5D theory in Ref. [2] with the cutoff scale replaced by $M_6$. Of course, now the bulk and brane gauge couplings are slightly smaller than the case of a single extra dimension, $1/g_5^2 \simeq C M_s/16 \pi^3$ and $1/g_{4,\hat{x}_5}^2 \simeq C/16 \pi^2$, due to the volume
suppression from the sixth dimension, leading to a somewhat smaller size for the fifth dimension. However, this effect is numerically not so large that the correction to gauge coupling unification is still dominated by a logarithmic contribution coming from the energy interval between $M_6$ and $M_5$. Below we explicitly show that this contribution corrects the prediction for $\alpha_s(M_Z)$ to the values which agree well with experiment, by looking at the behavior of the gauge couplings at the energy scale below $M_6$.

The bulk and brane gauge couplings receive power-law and logarithmic corrections, respectively, in the energy interval between $M_6$ and $M_5$. The correction to the bulk coupling, $1/g_5^2$, is $SU(5)$ symmetric and has a size $\simeq (b/16\pi^2)(M_6/M_5)$. It is dominated at the scale $M_6$ and, in fact, can also be interpreted as the finite correction at $M_6$. This correction does not contribute to the prediction of $\alpha_s(M_Z)$ because it is $SU(5)$ symmetric. On the other hand, the brane couplings receive logarithmic contributions, which intrinsically arise from the physics between $M_6$ and $M_5$ and cannot be attributed to any other corrections. Furthermore, since the fixed point at $x^5 = \pi R_5$ does not respect $SU(5)$, they are not $SU(5)$ symmetric and contribute to the prediction of $\alpha_s(M_Z)$. The beta-function coefficients for this logarithmic evolution are given by $(b_1, b_2, b_3) = (0, -4, -6)$ plus an $SU(5)$ symmetric piece coming from matter [1], so that we obtain

$$\delta \alpha_s \simeq -\frac{3}{7\pi} \alpha_s^2 \left( \ln \frac{\pi M_6}{M_5} + \Delta \right),$$

(17)

where $\Delta = O(1)$ represents effects from unknown brane-localized operators at $M_s$, logarithmic corrections between $M_s$ and $M_6$, and finite corrections at $M_6$ and $M_5$ [2]. Substituting $M_6/M_5 \simeq 20$ as an example, we obtain $\delta \alpha_s \simeq -0.01$. Although the error is larger than that in the case where the theory is five dimensional up to the cutoff scale, the correction in Eq. (17) still significantly improves the agreement with data. Therefore, we find that our theories retain the successful feature of the two-stage gauge coupling unification of the minimal 5D $SU(5)$ theory, with $M_5 \approx 10^{15}$ GeV and with $M_6$ and $M_5$ in the region of $10^{16} - 10^{17}$ GeV.

We next consider Yukawa coupling unification. In our theories, whether the Yukawa couplings are unified or not depends on the location of matter. Here we consider the case where the third generation matter is located on the $x^5 = 0$ subspace (either on the fixed line or on a fixed point) so that we have $b/\tau$ Yukawa unification. The behavior of Yukawa couplings is quite different from that of gauge couplings. In fact, by integrating out the physics above $M_5$, we find that no violation of $SU(5)$ is felt by bottom and tau Yukawa couplings above $M_5$. Thus, they start to deviate at $M_5$, giving the correction to the 4D prediction for the bottom quark mass

$$\frac{\delta m_b}{m_b} \simeq -\frac{20 g^2 - 5 y_t^2}{112 \pi^2} \ln \frac{\pi M_6}{M_5},$$

(18)

where $g$ and $y_t$ are the 4D gauge and Yukawa couplings around the scale $M_5$ [3]. Again, although
this expression is not as precise as the case of the exact single extra dimension, we find that it improves the agreement with data.

So far, we have considered the case where there is a non-negligible energy interval in which the physics is described by perturbative 6D theories. Here we comment on the possibility of taking the limit \( M_s R_6 \rightarrow 1 \). In this limit, the predictions for gauge and Yukawa coupling unification are reduced to those of Refs. 4, 5, where the theory is five dimensional up to the cutoff scale. Although it does not make much sense to talk about the field theoretic 6D theories in this case, we expect that some aspects of our constructions, such as the patterns of gauge symmetry breaking and/or the unification of gauge and Higgs fields, persist even in this limit, presumably as intermediate steps for string compactification. It would be interesting to further pursue the present line of constructions to more symmetrical theories having a larger gauge group and/or number of dimensions, such as \( E_8 \) in 10D.

4 Conclusions

We have constructed maximally supersymmetric \( SO(10) \) and \( SU(6) \) models in 6D, on the orbifold \( T^2/(Z_2 \times Z'_2) \), with one dimension, \( R_5 \), much larger than the other, \( R_6 \). A set of boundary conditions leads to a pattern of local breaking of the gauge symmetry such that, at distance scales larger than \( R_6 \), the theory reduces to \( SU(5) \times U(1)_X \) in 5D. The \( SU(5) \) sector corresponds to the highly predictive and successful minimal unified theory in 5D, while the \( U(1)_X \) gauge symmetry allows an understanding of the scales of both neutrino masses and the \( \mu \) parameter. Thus these highly symmetric \( SO(10) \) and \( SU(6) \) theories reproduce the predictions, for example for \( \alpha_s(M_Z) \), \( m_b/m_\tau \) and proton decay, of the minimal 5D \( SU(5) \) theory 4, 5, although with a slightly reduced precision. In the case that the theory is five dimensional up to the scale of strong coupling, we have previously argued that, with \( F_i \) all on the \( SU(5) \) invariant brane as expected from \( b/\tau \) Yukawa unification and the observed large neutrino mixing angles, exchange of the broken \( SU(5) \) gauge boson leads to \( p \rightarrow l^+\pi^0, l^+K^0, \bar{\nu}\pi^+, \bar{\nu}K^+ \) (\( l = e, \mu \)) with a lifetime of order \( 10^{34} \) years 5. In the case of 6D, with \( M_s/M_6 \lesssim 2 \), the broken gauge boson mass is increased only by a factor \( \lesssim 3 \), so that the proton decay lifetime is still expected to be in the range \( 10^{34} - 10^{36} \) years. Finally, the lightness of the Higgs doublets can be understood whether they originate as fields in 5D or, for the case of \( SU(6) \), in 6D, where they are identified as components of the gauge supermultiplet.

These more unified theories in 6D offer new possibilities for phenomenology beyond those of the 5D theory. The mass ratio \( m_t/m_b \) may arise partially from \( M_s R_6 \), allowing moderate values for \( \tan \beta \); or alternatively the \( t, b \) and \( \tau \) Yukawa couplings may all be unified in an \( SO(10) \) theory, with \( \tan \beta \approx 50 \). In the 5D theory, fields located at the fixed point with
$SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry are not forced to have their hypercharge quantized. However, if the field propagates in a sixth dimension, even if it is very small, the hypercharge will be quantized as long as $U(1)_Y$ is embedded into a non-Abelian gauge group on this fixed line. Finally, a larger variety of locations for matter increases the number of ways in which texture zeros may occur in the Yukawa matrices.

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