ROOK THEORETIC PROOFS OF SOME IDENTITIES RELATED TO SPIVEY’S BELL NUMBER FORMULA

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ABSTRACT. We use rook placements to prove Spivey’s Bell number formula and other identities related to it, in particular, some convolution identities involving Stirling numbers and relations involving Bell numbers. To cover as many special cases as possible, we work on the generalized Stirling numbers that arise from the rook model of Goldman and Haglund. An alternative combinatorial interpretation for the Type II generalized $q$-Stirling numbers of Remmel and Wachs is also introduced in which the method used to obtain the earlier identities can be adapted easily.

1. INTRODUCTION

Let $B(n)$ denote the $n$-th Bell number and $S(n,k)$ denote the Stirling number of the second kind. Spivey [15] established combinatorially the following identity for Bell numbers

$$B(n + m) = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} \binom{n}{k} S(m,j) B(k).$$

Various alternative proofs and extensions of this identity have appeared in the literature. For instance, Gould and Quaintance [6] provided a generating function proof, which, in turn, was extended by Xu [17] in the case of Hsu and Shuie’s [7] generalized Stirling numbers. Katriel [8] proved a $q$-analogue using certain $q$-differential operators. Still, Belbachir and Mihoubi [2] proved (1) using a decomposition of the Bell polynomial into a certain polynomial basis. See also [9, Theorem 10], [10, Theorem 5.3] and [13] for other generalizations and methods. The present authors also derived a generalization [4, Theorem 4.5] of Spivey’s identity in the case of the generalized $q$-Stirling numbers that arise from normal ordering.

In this short paper, we derive variations of (1) and other related identities using the aforementioned rook model which we describe in Section 2. The results are presented in Section 3. In Section 4, we introduce a new rook model for the Type II generalized $q$-Stirling numbers of Remmel and Wachs [14] which is a variation of Goldman and Haglund’s. We show how the method used in deriving the earlier identities can be easily adapted under this model.

2. GOLDMAN AND HAGLUND’S ROOK MODEL

Our presentation of the rook model is slightly different from the description given by Goldman and Haglund. The reader may verify that our presentation is essentially equivalent to the original definition (see [5, Section 4]).

![Figure 1. The board outlined by VUVVVUUV.](image)

Let $w$ be a word consisting of the letters $U$ and $V$. If we let $U$ correspond to a unit horizontal step and $V$ a unit vertical step, then $w$ outlines a Ferrers boards (or simply, board), which we denote by $B(w)$. For example, Figure 1 shows the board outlined by $VUVVVUUV$. Note that we allow the rightmost columns and the bottommost rows to be empty, i.e., not containing any cell. The initial $U$’s and final $V$’s, if any,
are extraneous when we consider rook placements, but they are natural when we relate rook placements with normal ordering (see [4, 16]). Alternatively, we can also describe a board by the length of its columns.

We say that each cell in $B$ has default pre-weight $1$. Let $s \in \mathbb{R}$. A placement of $k$ rooks on a board $B$ is a marking of $k$ cells of $B$ with "$\bullet$" subject to the following:

1. First, $k$ columns in $B$ are chosen where rooks will be placed. The rooks are then placed from right to left such that at most one rook is placed in each column.
2. A placement of a rook adds $s - 1$ to the pre-weight of every cell to its left in the same row.
3. A rook is said to “cancel” all cells lying above it. A canceled cell has pre-weight $0$.

The rows and columns of a board $B$ are numbered from top to bottom and from right to left, respectively. We denote by $C_k(B; s)$ the set of all placement of $k$ rooks on $B$. A cell with pre-weight $t$ is assigned the weight $q^t$ if it does not contain a rook, and $[t]_q = q^t - 1$ if it contains a rook. The weight of a rook placement $\phi$, denoted by $\text{wt}(\phi)$, is the product of the weight of the cells.

Denote by $J_n$ the board outlined by $(V U)^n$. Figure 2 shows a rook placement on $J_5$ where the values in the cells indicate the pre-weights. This particular rook placement has weight $q^{2s+2}[s]_q$.

![Figure 2. A rook placement on $J_5$.](image_url)

We will denote by $S_{s,q}[n,k]$ the sum of the weights of all placements of $n - k$ rooks on $J_n$, i.e.,

$$S_{s,q}[n,k] = \sum_{\phi \in C_{n-k}(J_n; s)} \text{wt}(\phi).$$

**Remark** 1. If, for instance, a column has two cells where the bottom cell has pre-weight $c_1$ and the top cell has pre-weight $c_2$, then the placement of a rook on the bottom cell has weight $[c_1]_q$ while the placement of a rook on the top cell has weight $q[c_2]_q$. Hence, the total weight of all rook placements on this column is $[c_1] + q[c_2]_q = [c_1 + c_2]_q$. In general, if $c$ is the sum of the pre-weights of the cells in a column, then the total weight of all rook placements on the column is $[c]_q$.

**Proposition 2.** The number $S_{s,q}[n,k]$ satisfies the recurrence

$$S_{s,q}[n,k] = q^{s(n-1)-(s-1)(k-1)}S_{s,q}[n-1,k-1] + [s(n-1) - (s-1)k]_q S_{s,q}[n-1,k],$$

with initial conditions $S_{s,q}[n,0] = S_{s,q}[0,n] = 0$, where $0 \neq 0$ if $n = 0$ and $0$ if $n \neq 0$.

**Proof.** If there is a rook on the $n$-th column, then the other columns form a rook placement from $C_{n-1-k}(J_{n-1}; s)$. Due to the placement of $n - 1 - k$ rooks on the first $n - 1$ columns, the $n$-th column has pre-weight $(n - 1) + (s - 1)(n - 1 - k) = s(n - 1) - (s - 1)k$. Hence, the total weight contributed by all rook placements on the $n$-th column is $[s(n-1) - (s-1)k]_q$.

If there is no rook on the $n$-th column, then the remaining columns form a rook placement from $C_{n-k}(J_{n-1}; s)$. Because of the placement of $n - k$ rooks on the first $n - 1$ columns, the $n$-th column has pre-weight $(n - 1) + (s - 1)(n - 1 - k) = s(n - 1) - (s - 1)(k - 1)$, which implies that the $n$-th column contributes a weight of $q^{s(n-1)-(s-1)(k-1)}$.

By comparing recurrences, we see that $h^{n-k}S_{s,q}[n,k]$ equals the number $S_{h,s,q}[n,k]$ in [4] and $S_{s,h}(n,k;q)$ in [11]. These numbers are coefficients of the string $(V U)^n$ in its normally ordered form (an equivalent expression where the $V$’s are to the right of $U$’s) where $V, U$ satisfy $UV = qVU + hV^s$. When $h = 1, q = 1$, the cases $s = 1$ and $s = 0$ produce the usual Stirling number of the first kind and Stirling number of the second kind, respectively. (See [11] for other special cases and their relationship with normal ordering.)

The concept of pre-weights provides us with some degree of flexibility in further generalizing the earlier model. For instance, instead of adding $s - 1$ to the pre-weight of each cell to the left of a rook, we can fix a rule $R$ which specifies, given a placement of rook in a cell, the cell (in each column to the right of a rook) that
receives the additional pre-weight. Denote by $C^R_k(B; s)$ the set of placement of $k$ rooks on the board $B$ under this rule. In particular, we shall find useful the rule $\mathcal{R}$, which specifies that the $i$-th rook (from the right) adds a pre-weight of $s - 1$ to the $i$-th cell above the bottom cell in every column to its left. Of course, when the default rule is used, we simply use the notation $C_k(B; s)$.

In addition to specifying a rule $R$, we can also assign other values for the default pre-weights. For $\alpha \in \mathbb{R}$, denote by $J^\prime_{n, \alpha}$ the board outlined by $(VU)^nV$ where the bottom cell in each column each has default pre-weight $\alpha$. Figure 3 shows one rook placement in $C^3_{s}(J^\prime_{4, \alpha}; s)$, which has weight $q^{2s+2\alpha}[\alpha^3_3][s]_q$.

Figure 3. A rook placement on $J^\prime_{4, \alpha}$ using rule $\mathcal{R}$.

**Proposition 3.** Given any rule $R$ and two boards $B_1$ and $B_2$ with the same number of non-empty columns such that the corresponding columns have identical total column pre-weights, we have $\sum_{\phi \in C^R_k(B_1; s)} wt(\phi) = \sum_{\phi \in C^R_k(B_2; s)} wt(\phi)$. Furthermore, given a board $B$ and two rules $R_1$ and $R_2$, then $\sum_{\phi \in C^R_k(B; s)} wt(\phi) = \sum_{\phi \in C^R_k(B; s)} wt(\phi)$.

**Proof.** The first statement follows from the fact that the weight of a rook placement is completely determined by the total pre-weight of each column, regardless of the distribution of the pre-weights of the individual cells. The second statement holds since different rules applied on the same board result to boards with identical total column pre-weights.

3. Results

Define the $n$-th generalized Bell polynomial as $B_{s,q}[n; x] = \sum_{k=0}^{n} S_{s,q}[n, k]x^k$. The $n$-th generalized Bell number, denoted by $B_{s,q}[n]$, is defined as $B_{s,q}[n] = \left( [n]_q! \right)$.

The $q$-binomial coefficients are given by $[n]_{k,q} = \binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$. Clearly, $[n]_{k,q} = \frac{[n]_q!}{[k]_q! [n-k]_q!}$. They also satisfy the relations (see [12, Table 1 and Identity (2.2)])

$$\binom{n}{k}_q = \sum_{0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n-k} \leq k} q^{t_1+t_2+\cdots+t_{n-k}}$$

$$\binom{n}{k}_q = \sum_{t_0+t_1+t_2+\cdots+t_k = n-k} q^{t_0+t_1+2t_2+\cdots+kt_k}.$$

In this section, we show how Spivey’s identity, specifically, its generalization involving $S_{s,q}[n, k]$, is a consequence of a convolution identity. We then proceed by deriving other convolution identities and the identities for Bell numbers that arise from them.

A version of Lemma 4, Theorem 5 and Corollary 6 that follow already appeared in [4], but only the case where $s \in \mathbb{N}$ and with pre-weights interpreted as subdivisions in cells was proved in detail. We present here a complete proof to make the paper self-contained.

**Lemma 4.** Let $\phi$ be a placement of $k$ rooks on the board $J_n$. Then, there exists a unique (possibly empty) collection $\mathcal{C}$ of columns in $\phi$, such that if $|\mathcal{C}| = \mu + 1$, then (a) these columns have a rook in the bottom $1, 2, \ldots, \mu + 1$ cells and (b) every column not in $\mathcal{C}$ contains at least $1 + t$ uncanceled cells not containing a rook, where $t$ is the number of columns in $\mathcal{C}$ to the right of that column.

**Proof.** Let $\phi \in C^R_k(J^\prime_{n, \alpha}; s)$. The elements of $\mathcal{C}$ can be obtained iteratively as follows. Let $c_1$ be the first column from the right containing a rook on the bottom cell. If there exists no such $c_1$, then $\mathcal{C} = \varnothing$. Let $c_2$ be the next column containing a rook in one of the bottom two cells, i.e., either on the bottom cell or on the cell above the bottom cell. If there exists no such $c_2$, then $\mathcal{C} = \{c_1\}$. We can continue this process as long as needed until the elements of $\mathcal{C}$ are all determined. Note that this process also shows the uniqueness of $\mathcal{C}$. \qed
\textbf{Theorem 5.} Let \( n, m, k \in \mathbb{N} \). Then,
\[
S_{s,q}[n+m,k] = \sum_{r=0}^{n} \sum_{j=0}^{m} S_{s,q}[m,j] q^{r(j(1-s) + sm)} \binom{n}{r} S_{s,q}[r,k-j] \prod_{i=0}^{n-r-1} [j(1-s) + sm + si]_{q}.
\] (2)

\textit{Proof.} The number \( S_{s,q}[n+m,k] \) equals the sum of the weights of all rook placements in the set \( C_{n+m-k}(J_{n+m};s) \).

The rooks may be placed as follows. First, place \( m-j \) rooks in columns \( 2, \ldots, m \). The sum of the weights of all such placements is \( S_{s,q}[m,j] \). Next, place the remaining \( n+j-k \) rooks in columns \( m+1, \ldots, n \). Due to the placement of the first \( m-j \) rooks, columns \( m+1, \ldots, n \) form a board whose column pre-weights are identical to those of \( J_{n;\alpha} \), with \( \alpha = j(1-s) + sm \). By Proposition 3, we can obtain the sum of the weights of all placements of \( n+j-k \) rooks in columns \( m+1, \ldots, n \) of \( J_{n+m} \) by considering, instead, the board \( J_{n;\alpha} \) under some convenient rule, which we choose to be rule \( R \) defined in Section 2, i.e., by computing the sum of the weights of the rook placements in \( C_{n+j-k}(J_{n;\alpha};s) \).

Let \( \phi \in C_{n+j-k}(J_{n;\alpha};s) \). By Lemma 4, we can write \( \phi \) uniquely as a pair \( (C_{\phi}, \phi - C_{\phi}) \), where \( C_{\phi} \) are the cells of \( \phi \) that either lie in the columns that satisfy the properties in the lemma with \( \mu = n-r-1 \) or on the bottom \( 1+t \) cells of the other columns, and \( \phi - C_{\phi} \) are the cells in \( \phi \) not in \( C_{\phi} \). One sees that \( \phi - C_{\phi} \) forms some rook placement in \( C_{r-k+j}(J_{r};s) \). This accounts for the factor \( S_{s,q}[r,k-j] \).

Let \( L_{\mu} \) be the set of all such possible \( C_{\phi} \), with \( \phi \in C_{n+j-k}(J_{n;\alpha};s) \) with \( \mu = n-r-1 \) as in Lemma 4. It suffices to compute the sum of the weights of the elements of \( L_{\mu} \). Clearly, the columns containing rooks contribute a weight of \( \prod_{i=0}^{n-r-1} [j(1-s) + sm + si]_{q} \) and the bottom \( r \) cells of the columns not containing rooks contribute a weight of \( q^{r(0(1-s) + sm)} \) regardless of where the rooks are. On the other hand, the weight contributed by the remaining cells depends on the placement of the rooks, and varies as \( q^{s_{t_1} s_{t_2} \cdots s_{t_{n-r}}} \) for some \( 0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n-r} \leq r \). Hence, these cells contribute
\[
\sum_{0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n-r} \leq r} q^{s_{t_1} s_{t_2} \cdots s_{t_{n-r}}} = \binom{n}{r} r_{q}.
\]
\( \square \)

The previous theorem then allows us to obtain, not just a generalization of (1), but also the corresponding recursion for the Bell polynomials.

\textbf{Corollary 6.} Let \( n, m \in \mathbb{N} \). Then,
\[
B_{s,q}[n+m;x] = \sum_{r=0}^{n} \sum_{j=0}^{m} S_{s,q}[m,j] q^{r(j(1-s) + sm)} \binom{n}{r} B_{s,q}[r;x] x^{j} \prod_{i=0}^{n-r-1} [j(1-s) + sm + si]_{q} .
\] (3)

In particular,
\[
B_{s,q}[n+m] = \sum_{r=0}^{n} \sum_{j=0}^{m} S_{s,q}[m,j] q^{r(j(1-s) + sm)} \binom{n}{r} B_{s,q}[r] \prod_{i=0}^{n-r-1} [j(1-s) + sm + si]_{q} .
\] (4)

\textit{Proof.} Multiply both sides of (2) by \( x^{k} \) and take the sum over all \( 0 \leq k \leq n + m \). \( \square \)

The next result is a variation of Theorem 5.

\textbf{Theorem 7.} Let \( n, m \in \mathbb{N} \). Then,
\[
S_{s,q}[n+m,k] = \sum_{r=0}^{n} \sum_{j=0}^{m} S_{s,q}[m,j] q^{(j(1-s) + sm)(k-j)} \binom{r}{k-j} S_{s,q}[n,r] \prod_{i=0}^{r-k+j-1} [j(1-s) + sm + i(s-1)]_{q}.
\]

\textit{Proof.} Again, we shall consider the rook placements in \( C_{n+m-k}(J_{n+m};s) \). Place \( m-j \) rooks in columns \( 2, \ldots, m \). The sum of the weights of all such placements is \( S_{s,q}[m,j] \). The difference with Theorem 5 is in the manner in which we place the rooks in the remaining columns and our use of the default rule instead of \( R \).

We now count the sum of the weights of the rook placements in \( C_{n+j-k}(J_{n;\alpha};s) \) by dividing columns \( m+1, \ldots, n \) into two. First, we place \( n-r \) rooks in rows \( m+1, \ldots, n+m-1 \) of columns \( m+1, \ldots, n+m \). The sum of the weights of all such placements is \( S_{s,q}[n,r] \). Then, place the remaining \( l = r + j - k \) rooks in rows \( 1, 2, \ldots, m \) of columns \( m+1, \ldots, n+m \) which have not been canceled by the \( n-r \) rooks. These
columns form an \( m \) by \( r \) rectangular board we denote by \( B \). Due to the placement of \( m - j \) rooks in columns \( 1, \ldots, m \), the total pre-weight of each column in \( B \) is \( j(1-s) + sm \). To count the total weight contributed by the placement of \( r \) rooks on \( B \), we will use, instead, the \( m \) by \( r \) board \( B' \) where each cell has default pre-weight \( (j(1-s) + sm)/m \) using the default rule.

Suppose that \( l \) columns of \( B' \) were rooks will be placed has been chosen. The weight contributed by all possible placements of rooks in the first of the \( l \) columns is \( [j(1-s)+sm]q \), the second column \( [j(1-s)+sm+s-1]q \), etc. Hence the weight contributed by the columns of \( B' \) containing rooks is \( \prod_{i=0}^{l-1}[j(1-s)+sm+i(s-1)]q \), and this is independent of the choice of columns. On the other hand, the weight contributed by the remaining columns varies, depending on the placement of the rooks, as \( q^{j(1-s)+sm}t_0 q^{j(1-s)+sm+(s-1)} t_1 \ldots q^{j(1-s)+sm+l(s-1)} t_l \) for some \( t_0 + t_1 + \ldots + t_l = r - l \). Finally,

\[
\sum_{t_0+t_1+\ldots+t_l=r-l} q^{j(1-s)+sm}t_0 q^{j(1-s)+sm+(s-1)} t_1 \ldots q^{j(1-s)+sm+l(s-1)} t_l = q^{j(1-s)+sm}(r-l) \left[ \begin{array}{c} r \\ l \end{array} \right]_{q^{s-1}}.
\]

**Corollary 8.** Let \( n, m \in \mathbb{N} \). Then,

\[
B_{s,q}[n + m; x] = \sum_{r,j,k} S_{s,q}[m, j]q^{j(1-s) + sm}(k-j) \left[ \begin{array}{c} r \\ k-j \end{array} \right]_{q^{s-1}} S_{s,q}[n, r] x^k \prod_{i=0}^{r-k+j-1} [j(1-s) + sm + i(s-1)]q.
\]

In particular,

\[
B_{s,q}[n + m] = \sum_{r,j,k} S_{s,q}[m, j]q^{j(1-s) + sm}(k-j) \left[ \begin{array}{c} r \\ k-j \end{array} \right]_{q^{s-1}} S_{s,q}[n, r] \prod_{i=0}^{r-k+j-1} [j(1-s) + sm + i(s-1)]q.
\]

The proofs of the Theorems 5 and 7 also give us the following result which will be used in deriving other convolution identities. The number \( C^{R}_{n-k}(J_{n; n}; s) \) itself can be considered as a kind of generalized Stirling number, noting that \( C^{R}_{n-k}(J_{n; 0}; s) = S_{s,q}[n, k] \). Interestingly, the individual terms on the \( RHS \) of (5) and (6) are not equal for fixed \( r \).

**Proposition 9.** Let \( \alpha \in \mathbb{R}, n, k \in \mathbb{N} \). Then, for any rule \( R \),

\[
C^{R}_{n-k}(J_{n; \alpha}; s) = \sum_{r=0}^{n} q^{ar} \left[ \begin{array}{c} n \\ r \end{array} \right]_{q^{s}} S_{s,q}[r, k] \prod_{i=0}^{n-r-1} [\alpha + si]q \tag{5}
\]

\[
C^{R}_{n-k}(J'_{n; \alpha}; s) = \sum_{r=0}^{n} q^{ak} S_{s,q}[n, r] \left[ \begin{array}{c} r \\ k \end{array} \right]_{q^{s-1}} \prod_{i=0}^{r-k-1} [\alpha + i(s-1)]q. \tag{6}
\]

As we have seen, the generalizations of (1) given in Corollaries 6 and 8 where obtained using convolution identities for \( S_{s,q}[n, k] \). Taking cue from the identities in [12, Theorem 2.6], we derive another set of convolution identities.

**Theorem 10.** For \( n, m, j \in \mathbb{N} \), we have

\[
S_{s,q}[n + 1, m + j + 1] = \sum_{k=m}^{n} \sum_{r=0}^{n-k} S_{s,q}[k, m] q^{r((k-m)(s-1)+k+1)+(k-m)(s-1)} \left[ \begin{array}{c} n-k \\ r \end{array} \right]_{q^{s}} S_{s,q}[r, j] \prod_{i=0}^{n-k-r-1} [(k-m)(s-1) + k + 1 + si]q, \tag{7}
\]

\[
S_{s,q}[n + 1, m + j + 1] = \sum_{k=m}^{n} \sum_{r=0}^{n-k} S_{s,q}[k, m] q^{(j+1)(k+(k-m)(s-1)) + j} \left[ \begin{array}{c} r \\ j \end{array} \right]_{q^{s-1}} \prod_{t=0}^{r-j-1} [(k-m)(s-1) + k + 1 + t(s-1)]q. \tag{8}
\]
Proof. For a given rook placement in $C_{n-m-j}(J_{n+1};s)$, there exists a unique $k$ such that there are $k - m$ rooks in columns $2, \ldots, k$, $0$ rooks in column $k + 1$ and $n - j - k$ rooks in columns $k + 2, k + 3, \ldots, n + 1$. Indeed, if there exists $k_1 < k_2$ satisfying these properties, then there are $k_2 - k_1$ rooks in columns $k_1 + 1, k_1 + 2, \ldots, k_2$. This is impossible since there is no rook in column $k_1 + 1$.

Suppose that $k - m$ rooks have been placed in columns $1, 2, \ldots, k$. The weight contributed by the $k + 1$-th column is $q^{k+(k-m)(s-1)}$. Using Proposition 9 with the substitutions $\alpha \rightarrow k + 1 + (k-m)(s-1), n \rightarrow n-k, k \rightarrow j, i \rightarrow r$, we obtain two different expressions for the sum of the weights of the placements of $n - j - k$ rooks in columns $k + 2, k + 3, \ldots, n + 1$.

Remark 11. By taking the sum of both sides of (7) and (8) over all $0 \leq m, j \leq n + 1$, we obtain an expression for the sum $\sum_{k=0}^{n+1} kS_{s,q}[n + 1, k]$.

Remark 12. We briefly discuss some special cases.

By letting $s = 0$ in (4), we get the following $q$-analogue of (1) which was previously derived by Katriel [8]

$$B_q[n + m] = \sum_{r=0}^{n} \sum_{j=0}^{m} S_q[m, j] q^{rj} \left( \begin{array}{c} n \\ r \end{array} \right) B_q[r][j]_{q}^{n-r},$$

where, $S_{0,q}[n, k] = S_q[n, k]$ and $B_{0,q}[n + m] = B_q[n + m]$ are the $q$-Stirling number of the second kind and the $q$-Bell number, respectively.

A “dual” of Spivey’s identity was derived by Mező [13] and is given by

$$(n + m)! = \sum_{r=0}^{n} \sum_{j=0}^{m} c(m, j) \left( \begin{array}{c} m \\ j \end{array} \right) r^k q^{r-r-k+j}.$$

(9)

where $a^r = a(a + 1) \cdots (a + b - 1)$ and $c(n, k)$ is the Stirling number of the first kind. One can check that Identity (4) reduces to (9) when $s = 1, q = 1$. Using Corollary 8, we deduce another expression for $(n + m)!$, namely

$$(n + m)! = \sum_{r, j, k} c(m, j) \left( \begin{array}{c} r \\ k-j \end{array} \right) c(n, r)m^{r-k+j}.$$

(Is there a simple combinatorial proof of this, possibly similar to Mező’s?)

Other identities for the the classical Stirling numbers and Bell numbers and their $q$-analogues may be obtained as straight-forward corollaries of the results in this section. For instance, Theorem 7 gives

$$S_q[n + m, k] = \sum_{r=0}^{n} \sum_{j=0}^{m} S_q[m, j] q^{r(k-j)} \left( \begin{array}{c} r \\ k-j \end{array} \right) S_q[n, r][j][j-1]q \cdots [k - r + 1]q,$$

which appears to be new.

The proposition that follows generalizes Theorems 5, 7 and 10. As before, two explicit forms, which we no longer state, can be obtained through an application of Proposition 9. The proofs only describe the needed partition of the boards.

Proposition 13. Let $m_1, \ldots, m_n, k \in \mathbb{N}$. Then,

$$S_{s,q}[m_1 + \cdots + m_n, k] = \sum_{j_1 + \cdots + j_n = k} \prod_{i=1}^{n} C_{m_i-j_i} \left( J_{m_i} \left( J_{m_i} \sum_{i=1}^{j_i} i (i + 1) ! ; s \right) \right).$$

In addition, for $n, m_1, \ldots, m_j \in \mathbb{N}$, we have

$$S_{s,q}[n+j-1, m_1 + \cdots + m_j + j-1] = \sum_{k_1 + \cdots + k_j = n \atop k_1, \ldots, k_n, k \in \mathbb{N}} q^{\sum_{i=1}^{j} k_i + (k_i - m_i)(s-1)} \prod_{i=1}^{j} C_{k_i-m_i} \left( J_{k_i} \left( J_{k_i} \sum_{i=1}^{k_i} i (i + 1) ! ; s \right) \right).$$

Proof. For the first identity: starting from the right, divide the the board $J_{m_1 + \cdots + m_n}$ into $n$ groups $G_1, \ldots, G_n$ consisting of $m_1, \ldots, m_n$ adjacent columns, respectively. Then, for $1 \leq i \leq n$, place $m_i - j_i$ rooks on the columns in $G_i$. 

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4. Extension to Type II $p,q$-analogues

Hsu and Shiu [7] introduced generalized Stirling numbers via the relations

$$(x|\alpha)_n = \sum_{k=0}^{n} S_{n,k}^{1}(\alpha,\beta,p)(x-\rho|\beta)_k$$

$$(x|\beta)_n = \sum_{k=0}^{n} S_{n,k}^{2}(\alpha,\beta,p)(x+\rho|\alpha)_k$$

where $(z|\alpha)_0 = 1$ and $(z|\gamma)_n = z(z-\gamma)\cdots(z-(n-1)\gamma)$ for every positive integer $n$. When $(\alpha,\beta,\gamma) = (0,1,0)$, $S_{n,k}^{1}(\alpha,\beta,p)$ and $S_{n,k}^{2}(\alpha,\beta,p)$ become the usual Stirling number of the first kind and Stirling number of the second kind, respectively. Since $S_{n,k}^{1}(\alpha,\beta,p) = S_{n,k}^{2}(\alpha,\beta,0)$, it suffices to consider just one of these numbers, namely $S_{n,k}^{1}(\alpha,\beta,p)$.

Let $[\gamma]_{p,q} = \frac{p^\gamma - q^\gamma}{p-q}$ for any $\gamma \in \mathbb{R}$. The Type II $p,q$-analogues of these numbers were introduced by Remmel and Wachs [14] by the following recursions ($\rho$ appears as $-r$ in [14])

$$S_{n,k}^{1,p,q}(\alpha,\beta,p) = q^{(k-1)\beta - (n-1)\alpha + \rho} p^{k\beta - (n-1)\alpha + \rho} S_{n-1,k}^{1,p,q}(\alpha,\beta,p) + \frac{p^{n-k}}{\beta - (n-1)\alpha + \rho} S_{n-1,k}^{1,p,q}(\alpha,\beta,p)$$

$$S_{n,k}^{2,p,q}(\alpha,\beta,p) = q^{-(k-1)\alpha - (n-1)\beta} p^{-(n-k)(\alpha - \rho) - (n-1)(\beta - \rho)} S_{n-1,k-1}^{2,p,q}(\alpha,\beta,p) + \frac{p^{n-k}}{\alpha - (n-1)(\beta - \rho)} S_{n-1,k-1}^{2,p,q}(\alpha,\beta,p)$$

where in addition $S_{n,k}^{1,p,q}(\alpha,\beta,p) = S_{n,k}^{2,p,q}(\alpha,\beta,p) = 1$ and $S_{n,k}^{1,p,q}(\alpha,\beta,p) = S_{n,k}^{2,p,q}(\alpha,\beta,p) = 0$ if $k = 0$ or $k > n$.

Since $S_{n,k}^{1,p,q}(\alpha,\beta,p) = S_{n,k}^{2,p,q}(\alpha,\beta,-\rho)$, we also consider just one of these numbers, which we choose to be $S_{n,k}^{1,p,q}(\alpha,\beta,p)$. When $(\alpha,\beta,\rho) = (0,1,0)$ and $p = q = 1$, $S_{n,k}^{1,p,q}(\alpha,\beta,p) = S_{n,k}(n,k)$.

Remmel and Wachs also gave combinatorial interpretations for these numbers for a certain choice of parameters. Our goal in this section is to introduce an alternative rook interpretation for $S_{n,k}^{1,1,q}(\alpha,\beta,p)$ which adapts the method used in Section 3. To do this, we will derive the analogue of Theorem 5 and Theorem 6. We leave the corresponding versions of the other identities in Section 3 to the interested reader.

Let $c,d \in \mathbb{R}$ and denote by $J_{n}^{c,d}$ the board outlined by $(VU)^n$ such that each bottom cell has default pre-weight $d$ and all other cells have default pre-weight $c + d$. Let $S_{s,q}^{c,d}[n,k]$ denote the sum of the weights of all placements of $n - k$ rooks on $J_{n}^{c,d}$.

**Remark 14.** Since $J_{n}^{0,0} = J_{n}$, it follows that $S_{s,q}^{1,0}[n,k] = S_{s,q}[n,k]$.

**Remark 15.** A “special case” of $J_{n}^{c,d}$ with $c = m, d = 0$ are the $m$-jump boards $Jb_{n,m}$ [5, Example 3] which have column heights $0, m, 2m, \ldots, (n-1)m$. One sees that these two boards have identical column pre-weights.

**Proposition 16.** The number $S_{s,q}^{c,d}[n,k]$ satisfies the recurrence

$$S_{s,q}^{c,d}[n,k] = q^{c(n-1)+d+(n-k)(s-1)} S_{s,q}^{c,d}[n-1,k-1] + [c(n-1) + d + (n-k-1)(s-1)] q S_{s,q}^{c,d}[n-1,k]$$

Hence,

$$S_{s,q}^{1,1,q}[n,k] = S_{s,q}^{1,1}[n,k].$$

**Proof.** The proof is similar to that of Proposition 2. \hfill \Box

**Theorem 17.** For $n, m, k \in \mathbb{N}$, we have

$$S_{s,q}^{c,d}[n + m, k] = \sum_{r=0}^{n} \sum_{j=0}^{m} S_{s,q}^{c,d}[m,j] q^{r(d+mc+(m-j)(s-1))}$$

$$= \begin{cases} n \\ r \end{cases} q^{d+mc+(m-j)(s-1)} \prod_{i=0}^{n-r-1} \left[ d+mc+(m-j)(s-1)+(c+s-1) \right]_q.$$
This implies that
\[ S_{n,k}^{1,1,q}(\alpha, \beta, \rho) = \sum_{r=0}^{n} \sum_{j=0}^{m} S_{m,j}^{1,1,q}(\alpha, \beta, \rho) q^{(\beta j - \alpha m - \rho)} \left[ \begin{array}{c} n \\ r \end{array} \right] B_{r-k-j}^{1,1,q}(\alpha, \beta, 0) \prod_{i=0}^{n-r-1} [\beta j - \rho - \alpha(m + i)]_q. \]

**Proof.** We proceed as in Theorem 5. The \( m - j \) rooks in columns 2, \ldots, \( m \) explains the factor \( S_{s,q}^{c,d}[m, j] \). We then determine the sum of the weights of all placements of \( n + j - k \) rooks in the remaining columns.

The placement of \( m - j \) rooks adds \( (m - j)(s - 1) \) to the total pre-weights of columns \( m + 1, \ldots, n \). These columns form a board whose column pre-weights are identical to the board outlined by \((VU)^n\) such that the bottom cells have pre-weight \( d + mc + (m - j)(s - 1) \) and the other cells have pre-weight \( c \). For convenience, let us denote such board by \( B \).

Note that Lemma 4 is a statement regarding only the placement of rooks and thus, it holds true regardless of the default pre-weights of the cells and the rule used. Hence, we can use this lemma on \( B \) with final \( \mathcal{R} \), reasoning as in Theorem 5.

Let \( \phi \in C_{n+j-k}^{\mathcal{R}}(B; s) \). Again, by Lemma 4, we can write \( \phi \) uniquely as \( (C_\phi, \phi - C_\phi) \). Note that \( \phi - C_\phi \) forms a rook placement in \( C_{n-k+j}(J_{r}^{c,d}; s) \) and not in \( C_{n-k+j}(J_{r}^{c,d}; s) \). This is because \( \phi - C_\phi \) does not include any bottom cell of \( \phi \).

The board \( B \) is similar to \( J_{n,s}^{r} \) with \( \alpha = d + mc + (m - j)(s - 1) \), the only difference is that the non-bottom cells of \( B \) have pre-weight \( c \). Hence, a cell with pre-weight \( c \) which receives an additional pre-weight due to the placement of a rook following rule \( \mathcal{R} \) will have pre-weight \( c + s - 1 \).

Define \( B_{n+k}^{1,1,q}(\alpha, \beta, \rho) = \sum_{k=0}^{n} S_{n,k}^{1,1,q}(\alpha, \beta, \rho) \).

**Corollary 18.** For \( n, m, \in \mathbb{N} \), we have
\[ B_{n+m,k}^{1,1,q}(\alpha, \beta, \rho) = \sum_{r=0}^{n} \sum_{j=0}^{m} S_{m,j}^{1,1,q}(\alpha, \beta, \rho) q^{(\beta j - \alpha m - \rho)} \left[ \begin{array}{c} n \\ r \end{array} \right] B_{r-k-j}^{1,1,q}(\alpha, \beta, 0) \prod_{i=0}^{n-r-1} [\beta j - \rho - \alpha(m + i)]_q. \]  

Letting \( q = 1 \) produces a generalization of Spivey’s identity, which is a variant of Xu’s result [17, Corollary 8]. Also, Mezö [13, Theorem 2] obtained a special case of (10) for \( (\alpha, \beta, \rho) = (0, 1, \rho) \).

**Theorem 19.** Let \( n, m, \in \mathbb{N} \). Then,
\[ S_{s,q}^{c,d}[n + m, k] = \sum_{r=0}^{n} \sum_{j=0}^{m} S_{s,q}^{c,d}[m, j] q^{(d + mc + (m - j)(s - 1))(k - j)} \left[ \begin{array}{c} r \\ k-j \end{array} \right] S_{s,q}^{0,0}[n, r] \prod_{i=0}^{r-k+j-1} [d + mc + (m - j)(s - 1) + i(s - 1)]_q. \]

This gives
\[ S_{n+m,k}^{1,1,q}(\alpha, \beta, \rho) = \sum_{r=0}^{n} \sum_{j=0}^{m} S_{m,j}^{1,1,q}(\alpha, \beta, \rho) q^{(\rho + \alpha m - \beta j)(j - k)} \left[ \begin{array}{c} r \\ k-j \end{array} \right] S_{n,r}^{1,1,q}(\alpha, \beta, 0) \prod_{i=0}^{r-k+j-1} [\rho + \alpha m - \beta(j + i)]_q. \]  

**Proof.** Instead of using the board \( J_{n}^{c,d} \), we use the board outlined by \((VU)^n\) such that the bottom cells of columns \( 2, \ldots, m \) have pre-weight \( c + d \), the cells in row \( m \) have pre-weight \( c + d \), and all other cells have pre-weight \( c \). We then proceed as in the proof of Theorem 7.

5. Some Remarks

Mezö’s dual of Spivey’s identity is based on the relation \( n! = \sum_{k=0}^{n} c(n, k) \). However, it is not true that \([n]_q! = \sum_{k=0}^{n} c_q[n, k] \) where \( c_q[n, k] = S_{1,q}[n, k] \). It would be interesting to derive an expression for \([n + m]_q! \) similar to Mezö’s for \( q \)-Stirling numbers of the first kind.
We also ask if a modification of the rook model for the Type II generalized \( q \)-Stirling numbers can be made to give another combinatorial interpretation for both the Type I and Type II \( p, q \)-analogues. It is also possible that the techniques outlined in this paper may be modified to obtain the corresponding identities for the generalized Stirling numbers that arise from other rook models, such as those in [14] and [3].

Lastly, we note that other forms of convolution identities are given in [1, Theorem 1] and [12, page 5]. Can these identities and their \( S_{s,q}[n,k] \) versions be proved using partitions on rook boards?

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