The generating function for the Bessel point process and a system of coupled Painlevé V equations

Christophe Charlier,∗ Antoine Doeraene†

August 13, 2018

Abstract

We study the joint probability generating function for \( k \) occupancy numbers on disjoint intervals in the Bessel point process. This generating function can be expressed as a Fredholm determinant. We obtain an expression for it in terms of a system of coupled Painlevé V equations, which are derived from a Lax pair of a Riemann-Hilbert problem. This generalizes a result of Tracy and Widom [24], which corresponds to the case \( k = 1 \). We also provide some examples and applications. In particular, several relevant quantities can be expressed in terms of the generating function, like the gap probability on a union of disjoint bounded intervals, the gap between the two smallest particles, and large \( n \) asymptotics for \( n \times n \) Hankel determinants with a Laguerre weight possessing several jumps discontinuities near the hard edge.

1 Introduction

The Bessel point process is a determinantal point process on \( \mathbb{R}^+ \) arising as a limit point process of a wide range of mathematical models in random matrix theory [14, 15]. A celebrated toy example is the behaviour near 0 of the squared singular values of Ginibre matrices, also known as the Laguerre Unitary Ensemble [25]. Other examples include non-intersecting squared Bessel paths [17], and the conditional Circular Unitary Ensemble near the edges [5].

The main feature of determinantal point processes on a set \( A \subseteq \mathbb{R} \) is that for all \( n \in \mathbb{N}_{>0} \), the \( n \)-point correlation function \( \rho_n : A^n \to \mathbb{R} \) is expressed in terms of a correlation kernel \( K : A \times A \to \mathbb{R} \) as follows

\[
\rho_n(x_1, \ldots, x_n) = \det (K(x_j, x_\ell))_{j, \ell=1}^n.
\]

In the Bessel point process, \( A = \mathbb{R}^+ \) and the kernel is given by

\[
K_{\text{Be}}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x - y)}, \quad \alpha > -1,
\]

(1.1)

where \( J_\alpha \) stands for the Bessel function of the first kind of order \( \alpha \) (see [22] formula 10.2.2] for a definition of \( J_\alpha \)).

∗Department of Mathematics, KTH Royal Institute of Technology, Lindstedtsvägen 25, SE-114 28 Stockholm, Sweden. e-mail: cchar@kth.se
†Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Chemin du Cyclotron 2, B-1348 Louvain-La-Neuve, Belgium. e-mail: antoine.doeraene@uclouvain.be
Important quantities related to point processes are occupancy numbers. Given a Borel set $B \subseteq \mathbb{R}^+$, the occupancy number $n_B$ is the random variable defined as the number of particles that fall into $B$. Determinantal point processes are always locally finite, i.e. $n_B$ is finite with probability 1 for $B$ bounded. Moreover, all particles are distinct with probability 1. In particular, it allows us to enumerate particles for the Bessel point process in the following way,

$$0 < \zeta_1 < \zeta_2 < \zeta_3 < ...$$

In this paper, we focus on the joint behaviour of a finite number of particles, which can be completely understood via the joint probability generating function of the occupancy numbers of some particular sets. Let $k \in \mathbb{N}_{>0}$, $\vec{s} = (s_1, ..., s_k) \in \mathbb{C}^k$ and $\vec{x} = (x_1, ..., x_k) \in (\mathbb{R}^+)^k$ be such that $0 = x_0 < x_1 < x_2 < ... < x_k < +\infty$. We will be interested in the function

$$F(\vec{x}, \vec{s}) = \mathbb{E} \left( \prod_{j=1}^{k} s_j^{n(x_{j-1}, x_j)} \right) = \sum_{m_1, ..., m_k \geq 0} \mathbb{P} \left( \bigcap_{j=1}^{k} n(x_{j-1}, x_j) = m_j \right) \prod_{j=1}^{k} s_j^{m_j}. \quad (1.2)$$

It is known [23, Theorem 2] that $F(\vec{x}, \vec{s})$ is an entire function in $s_1, ..., s_k$ and can be expressed as a Fredholm determinant as follows

$$F(\vec{x}, \vec{s}) = \det \left( 1 - \chi_{(0,x_1)}(s_1) \prod_{j=1}^{k} (1 - s_j) K_{Be}^{n(x_{j-1}, x_j)} \right), \quad (1.3)$$

where $K_{Be}$ denotes the integral operator acting on $L^2(\mathbb{R}^+)$ whose kernel is the Bessel kernel $K_{Be}$, and where $\chi_A$ is the projection operator onto $L^2(A)$.

The goal of this paper is to express $F(\vec{x}, \vec{s})$ explicitly in terms of $k$ functions which satisfy a system of $k$ coupled Painlevé V equations. Analogous generating functions for the Airy point process have been recently studied in [11] (for a general $k \in \mathbb{N}_{>0}$), and in [26] (for the case $k = 2$ with an extra root-type singularity). In both cases, the authors expressed it in terms of a system of coupled Painlevé II equations.

**Tracy-Widom formula for $k = 1$**

In [24], Tracy and Widom have studied $F(x_1, s_1)$, i.e. the case $k = 1$. This is the probability generating function of $n_{(0,x_1)}$. In particular, we can deduce from $F(x_1, s_1)$ the probability distribution of the $\ell$-th smallest particle $\zeta_\ell$ as follows

$$\mathbb{P}(\zeta_\ell > x_1) = \mathbb{P}(n_{(0,x_1)} < \ell) = \sum_{j=0}^{\ell-1} \frac{1}{j!} \partial_{s_1}^{j} F(x_1, s_1) \bigg|_{s_1=0}. \quad (1.4)$$

Their theorem states that for $0 \leq s_1 < 1$ and $x_1 > 0$,

$$F(x_1, s_1) = \exp \left( -\frac{1}{4} \int_0^{x_1} \log \left( \frac{x_1}{\xi} \right) q^2(\xi; s_1) d\xi \right), \quad (1.5)$$

where $q(\xi; s_1)$ satisfies the Painlevé V equation given by

$$\xi q(1 - q^2)(\xi q q')' + \xi (1 - q^2)^2 \left( (\xi q')' + \frac{q}{4} \right) + \xi^2 q(q')^2 = \alpha^2 \frac{q}{4}, \quad (1.6)$$

with boundary condition $q(\xi; s_1) \sim \sqrt{1 - s_1 J_0(\sqrt{\xi})}$ as $\xi \to 0$, and where primes denote derivatives with respect to $\xi$. 

2
Joint distribution for \( k \) particles

Let us start with the case \( k = 2 \) for simplicity. Let \( m_1, m_2 \in \mathbb{N}_0 \) be such that \( m_1 < m_2 \). If \( 0 < x_1 < x_2 < +\infty \), the joint distribution of the \( m_1 \)-th and \( m_2 \)-th smallest particles in the Bessel point process is given in terms of \( F((x_1, x_2), (s_1, s_2)) \) by

\[
P(\zeta_{m_1} > x_1, \zeta_{m_2} > x_2) = \sum_{j_1 < m_1 \atop j_1 + j_2 < m_2} \mathbb{P}(n(0, x_1) = j_1, n(x_1, x_2) = j_2) = \sum_{j_1 < m_1 \atop j_1 + j_2 < m_2} \frac{1}{j_1! j_2!} \partial_{s_1}^{j_1} \partial_{s_2}^{j_2} F((x_1, x_2), (s_1, s_2)) \big|_{s_1 = s_2 = 0} .
\]

(1.7)

More generally, for \( k \in \mathbb{N}_0 \), the function \( F \) can be used to express the joint probability distribution of any \( k \) distinct particles. The general formula for \( k > 2 \) can be easily generalized from (1.7). Let \( m_1, \ldots, m_k \in \mathbb{N}_0 \) and \( \bar{x} = (x_1, \ldots, x_k) \in (\mathbb{R}^+)^k \) be such that \( m_1 < m_2 < \cdots < m_k \) and \( x_1 < \cdots < x_k \). We have

\[
P(\cap_{j=1}^k (\zeta_{m_j} > x_j)) = \sum_{j_1 < m_1 \atop \cdots \atop j_k < m_k} \mathbb{P}\left( \bigcap_{j=1}^k (n(x_{j-1}, x_j) = m_j) \right) = \sum_{j_1 < m_1 \atop \cdots \atop j_k < m_k} \frac{1}{j_1! j_2! \cdots j_k!} \partial_{s_1}^{j_1} \partial_{s_2}^{j_2} \cdots \partial_{s_k}^{j_k} F(\bar{x}, \bar{s}) \big|_{\bar{x} = 0},
\]

where the sum is taken over all indices \( j_1, \ldots, j_k \) such that

\[
j_1 < m_1, \quad j_1 + j_2 < m_2, \quad \ldots \quad \sum_{i=1}^k j_i < m_k.
\]

We give other quantities of interest which can be expressed in terms of \( F \) in Section 2.

Tracy-Widom type formula for \( F \)

The Tracy-Widom formula (1.5) characterized \( F \) in the case \( k = 1 \) in terms of a function \( q \) which satisfies the Painlevé V equation (1.6). The main result of this paper gives a generalisation of that for a general \( k \in \mathbb{N}_0 \). We find that \( F \) can be expressed in terms of \( k \) functions \( q_1, \ldots, q_k \), which satisfy a system of \( k \) coupled Painlevé V equations with Bessel boundary conditions at 0. The theorem reads as follows.

**Theorem 1.1** Let \( \bar{r} = (r_1, \ldots, r_k) \in (\mathbb{R}^+)^k \) and \( \bar{s} = (s_1, \ldots, s_k) \in [0, 1]^k \) be such that

\[
r_j > r_{j-1}, \text{ for } j = 1, \ldots, k, \text{ where } r_0 := 0,
\]

\[
s_j \neq s_{j+1}, \text{ for } j = 1, \ldots, k, \text{ where } s_{k+1} := 1.
\]

(1.9)

(1.10)

For \( x > 0 \), the joint probability generating function \( F(\bar{r} x, \bar{s}) \) is given by

\[
F(\bar{r} x, \bar{s}) = \prod_{j=1}^k \exp \left( -\frac{r_j}{4} \int_0^x \log \left( \frac{x}{\xi} \right) q_j^2(\xi; \bar{r}, \bar{s}) d\xi \right),
\]

(1.11)

where the functions \( q_1(\xi; \bar{r}, \bar{s}), \ldots, q_k(\xi; \bar{r}, \bar{s}) \) satisfy the system of \( k \) equations given by

\[
\xi q_j \left( 1 - \sum_{\ell=1}^k q_\ell^2 \right) \sum_{\ell=1}^k (\xi q_\ell q_\ell')' + \xi \left( 1 - \sum_{\ell=1}^k q_\ell^2 \right)^2 \left( (\xi q_j)' + \frac{r_j q_j}{4} \right) + \xi^2 q_j \left( \sum_{\ell=1}^k q_\ell q_\ell' \right)^2 = \alpha^2 q_j \left( \frac{4}{4} \right).
\]
where \( j = 1, 2, \ldots, k \), and where primes denote derivatives with respect to \( \xi \). Furthermore, for every \( j \in \{1, \ldots, k\} \), \( q_j^2(\xi; \vec{r}, \vec{s}) \) is real for \( \xi > 0 \) and satisfies the boundary condition

\[
q_j(\xi; \vec{r}, \vec{s}) = \sqrt{s_{j+1} - s_j} J_\alpha(\sqrt{r_j} \xi)(1 + \mathcal{O}(\xi)), \quad \text{as } \xi \to 0.
\]  

(1.13)

Remark 1.2 Theorem 1.1 is a generalization for \( k \in \mathbb{N}_{>0} \) of the Tracy-Widom formula. Indeed, if \( k = 1 \), \( x = x_1 \) and \( r_1 = 1 \), the above formulas (1.11) and (1.12) are reduced to (1.5) and (1.6), and \( q_1 \) given in Theorem 1.1 and \( q \) given by (1.6) satisfy the same boundary condition at 0.

Remark 1.3 The system (1.12) with boundary conditions (1.13) given in Theorem 1.1 has at least one solution \( (q_1, \ldots, q_k) \), but there is no guarantee that this solution is unique. Therefore, the functions \( q_1, \ldots, q_k \) that appear in (1.11) are not defined through the system (1.12), but they are explicitly constructed from the solution \( \Phi \) of a Riemann-Hilbert (RH) problem. This RH problem is presented in Section 3.

The asymptotic behaviour (1.13) allows to compute directly the small \( x \) asymptotics of \( F(\vec{r}x, \vec{s}) \), and is given in the following corollary.

Corollary 1.4 Let \( x > 0 \), fix \( \vec{r} = (r_1, \ldots, r_k) \in (\mathbb{R}^+)^k \) independent of \( x \) such that \( r_1 < \ldots < r_k \), and fix \( \vec{s} = (s_1, \ldots, s_k) \in [0, 1]^k \) independent of \( x \) such that \( s_j \neq s_{j+1} \) for \( j = 1, \ldots, k \) with \( s_{k+1} = 1 \). We have

\[
F(\vec{r}x, \vec{s}) = 1 - \sum_{j=1}^{k} (s_{j+1} - s_j) J_{\alpha+1}(\sqrt{r_j} x)^2 + \mathcal{O}(x^{2+\alpha}), \quad \text{as } x \to 0.
\]  

(1.14)

Proof. This a direct consequence of Theorem 1.1 together with the formula \( z \Gamma(z) = \Gamma(z+1) \) and the limiting behaviour \( J_\alpha(x) = \left( \frac{x}{2} \right)^\alpha \frac{1}{\Gamma(\alpha+1)} (1 + \mathcal{O}(x^2)) \) as \( x \to 0 \) (see [22, formula 10.7.3]). \( \square \)

Asymptotics for \( q_1, \ldots, q_k \) as \( s_j \to s_{j+1} \) or as \( r_j \to r_{j-1} \)

In Theorem 1.1 it is essential that the conditions (1.9) and (1.10) hold. Suppose that one of these conditions is not satisfied, i.e. suppose we have \( s_j = s_{j+1} \) or \( r_j = r_{j-1} \) for a certain \( j \in \{1, \ldots, k\} \). Then, from (1.12), we have

\[
F(\vec{r}x, \vec{s}) = F(\vec{r}^{[j]}x, \vec{s}^{[j]}),
\]  

(1.15)

where for a given vector \( \vec{w} = (w_1, \ldots, w_k) \), we use the notation \( \vec{w}^{[j]} \) for the vector \( \vec{w} \) with its \( j \)-th component removed, i.e. \( \vec{w}^{[j]} = (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_k) \). Theorem 1.1 applied to the right-hand side of (1.15) allows to rewrite \( F(\vec{r}x, \vec{s}) \) in terms of a solution of a system of \( k-1 \) coupled Painlevé equations. Thus, if \( \vec{r} \) and \( \vec{s} \) satisfy (1.9) and (1.10), as \( s_j \to s_{j+1} \) or \( r_j \to r_{j-1} \) and \( \xi \) fixed, we should observe \( ||q(\xi; \vec{r}, \vec{s}) - q^{[j]}(\xi; \vec{r}^{[j]}, \vec{s}^{[j]})|| \to 0 \), where \( q = (q_1, \ldots, q_k) \). Theorem 1.5 below gives such asymptotics in the above degenerate cases.
Theorem 1.5 Fix $x > 0$ and let $\bar{r} = (r_1, ..., r_k) \in (\mathbb{R}^+)^k$ and $\bar{s} = (s_1, ..., s_k) \in [0, 1)^k$ be such that (1.9) and (1.10) are satisfied.

1. Let $j \in \{1, ..., k\}$. As $s_j \to s_{j+1}$, we have

\begin{align}
q_j^2(x; \bar{r}, \bar{s}) &= \mathcal{O}(|s_j - s_{j+1}|), \\
|q_j^2(x; \bar{r}, \bar{s}) - q_j^2(x; \bar{r}^{[j]}, \bar{s}^{[j]})| &= \mathcal{O}(|s_j - s_{j+1}|),
\end{align}

and (1.17) holds for any $\ell \neq j$, and where $\tilde{\ell} = \ell$ if $\ell < j - 1$ and $\tilde{\ell} = \ell - 1$ if $\ell > j$.

2. Let $j \in \{2, ..., k\}$. As $r_j \to r_{j-1}$ and if $s_{j+1} \neq s_{j-1}$, we have

\begin{align}
q_{j-1}^2(x; \bar{r}, \bar{s}) &= \frac{s_j - s_{j-1}}{s_{j+1} - s_{j-1}} q_{j-1}^2(x; \bar{r}^{[j]}, \bar{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}), \\
q_j^2(x; \bar{r}, \bar{s}) &= \frac{s_j - s_{j+1}}{s_{j+1} - s_{j-1}} q_{j-1}^2(x; \bar{r}^{[j]}, \bar{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}), \\
|q_j^2(x; \bar{r}, \bar{s}) - q_j^2(x; \bar{r}^{[j]}, \bar{s}^{[j]})| &= \mathcal{O}(r_j - r_{j-1}),
\end{align}

and (1.20) holds for any $\ell \neq j - 1, \ell \neq j$, and where $\tilde{\ell} = \ell$ if $\ell < j - 2$ and $\tilde{\ell} = \ell - 1$ if $\ell > j$.

3. As $r_1 \to 0$, we have

\begin{align}
q_1^2(x; \bar{r}, \bar{s}) &= \mathcal{O}(r_1^{\alpha}), \\
|q_1^2(x; \bar{r}, \bar{s}) - q_1^2(x; \bar{r}^{[1]}, \bar{s}^{[1]})| &= \mathcal{O}(r_1),
\end{align}

and (1.22) holds for any $\ell \geq 2$, and where $\tilde{\ell} = \ell - 1$.

Outline

We provide some examples and applications of our results in Section 2. The system of $k$ coupled Painlevé V equations for $q_1, ..., q_k$, given by (1.12), is obtained from a Lax pair of a model Riemann-Hilbert (RH) problem which is introduced in Section 3 and whose solution is denoted $\Phi$. In Section 4 using the procedure introduced by Its, Izergin, Korepin and Slavnov [19] for integrable operators, we relate $F$ with $\Phi$ through a differential identity, which we integrate to prove Theorem 1.1. In Section 5 we perform the Deift/Zhou [11] [12] steepest descent method on the model RH problem to obtain the asymptotic behaviour of $q_j(x)$ as $x \to 0$. The first and second part of Theorem 1.5 are obtained in Section 6 and Section 7 respectively, via a more direct steepest descent method on the RH problem.

2 Examples and applications

The applications presented in this section are similar to those presented in [6] (for the Airy point process), and are adapted here for the Bessel point process.
2.1 Gap probability on a union of disjoint bounded intervals

A gap in a point process is the event of finding no particle in a certain set. The Tracy-Widom distribution given by (1.5) corresponds to the gap probability for an interval of the form \((0, a]\), where \(0 < a < +\infty\), and can be rewritten as

\[
P(n_{(0, a)} = 0) = F(a, 0) = \exp \left( \frac{a}{4} \int_0^1 \log(\xi) q_1^2(\xi) d\xi \right),
\]

where in the above expression we have used the definition of \(F\) given by (1.2) for the first equality, and where we have applied Theorem 1.1 (with \(k = 1, x = 1, r_1 = a, s_1 = 0\)) for the second equality. The gap probability in the Bessel point process for a single interval of the form \((a, b]\), with \(0 < a < b < +\infty\), is given by

\[
P(n_{(a, b)} = 0) = F((a, b), (1, 0)) = \exp \left( \frac{a}{4} \int_0^1 \log(\xi) q_1^2(\xi) d\xi \right) \exp \left( \frac{b}{4} \int_0^1 \log(\xi) q_2^2(\xi) d\xi \right),
\]

where we have used Theorem 1.1 (with \(k = 2, x = 1, r = (a, b), s = (1, 0)\)) for the second equality.

This computation can be generalized for the gap probability of any finite union of disjoint bounded intervals. Let \(\ell \in \mathbb{N} > 0\) be the number of intervals and \(0 < a_1 < b_1 < a_2 < ... < b_\ell < +\infty\), we have

\[
P\left( \bigcap_{j=1}^{\ell} n_{(a_j, b_j)} = 0 \right) = F((a_1, b_1, ..., a_\ell, b_\ell), (1, 0, ..., 1, 0)) = \exp \left( \frac{1}{4} \int_0^1 \log(\xi) \sum_{j=1}^{\ell} (a_j q_{2j-1}^2(\xi) + b_j q_{2j}^2(\xi)) d\xi \right),
\]

where we have applied Theorem 1.1 with \(k = 2\ell, x = 1, r = (a_1, b_1, ..., a_\ell, b_\ell)\) and \(s = (1, 0, ..., 1, 0)\), and where the \(2\ell\) functions \(q_1, ..., q_{2\ell}\) satisfy the system (1.12).

2.2 Distribution of the smallest particle in the thinned and conditional Bessel point process

The generating function \(F\) is also useful in the context of thinning. The thinning of a determinantal point process is a procedure introduced by Bohigas and Pato [3, 4] that consists in building a new point process by removing each particle independently with a certain probability.

We consider a constant and independent thinning of the Bessel point process. Given a realization \(0 < \zeta_1 < \zeta_2 < ...\), it consists of removing each of these particles independently with the same probability \(s \in (0, 1)\). The thinned point process is composed of the remaining particles \(0 < \xi_1 < \xi_2 < ...\), and is again a determinantal point process, whose correlation kernel is given by \((1 - s)K^\text{Be}\) (see [20]). For a given Borel set \(B \subset \mathbb{R}^+\), we denote \(\tilde{n}_B\) for the occupancy number of \(B\) in the thinned point process. The probability distribution of \(\xi_1\) (smallest particle of the thinned point process) can be deduced from \(F\) with \(k = 1\), since by (1.2) we have

\[
P(\xi_1 > x) = \sum_{j=0}^{+\infty} P(n_{(0, x)} = j \cap \tilde{n}_{(0, x)} = 0) = \sum_{j=0}^{+\infty} P(n_{(0, x)} = j) s^j = F(x, s),
\]
and where $F(x, s)$ admits the Tracy-Widom formula (1.5). We can also consider another situation, where we have information about the thinned point process. Suppose that we observe the event $\tilde{n}_{(0,x_2)} = 0$ for a certain $x_2 > 0$ (we condition on this event), and from there, we want to retrieve information on $\zeta_1$. The distribution of $\zeta_1|\tilde{n}_{(0,x_2)} = 0$ (the smallest particle in the conditional point process) is given by

$$P\left(\zeta_1|\tilde{n}_{(0,x_2)} = 0 > x_1\right) = P(\zeta_1 > x_1|\zeta_1 > x_2) = \frac{P(\zeta_1 > x_1 \cap \zeta_1 > x_2)}{P(\zeta_1 > x_2)},$$

(2.3)

where $0 < x_1 < x_2$. The denominator in the above expression is just given by $F(x_2, s)$, as shown in (2.2). The numerator is slightly more involved, and can be expressed in terms of $F$ with $k = 2$ as follows

$$P(\zeta_1 > x_1 \cap \zeta_1 > x_2) = \sum_{j=0}^{+\infty} s^j P\left(n_{(0,x_1)} = 0 \cap n_{(x_1,x_2)} = j\right) = F((x_1, x_2), (0, s)).$$

(2.4)

Therefore, Theorem 1.1 allows us to express the distribution of the smallest particle in the conditional point process as

$$P\left(\zeta_1|\tilde{n}_{(0,x_2)} = 0 > x_1\right) = \exp\left(\frac{1}{4} \int_0^1 \left[ x_1 q_1(\xi) + x_2 (q_2(\xi) - \tilde{q}(\xi)) \right] \log \xi d\xi \right),$$

(2.5)

where $q_1, q_2$ satisfy the system (1.12) with $k = 2$, $x = 1$, $\vec{r} = (x_1, x_2)$, $\vec{s} = (0, s)$ and $\tilde{q}$ satisfies (1.12) with $k = 1$, $x = 1$, $r_1 = x_2$ and $s_1 = s$.

2.3 Smallest LUE eigenvalues

The Bessel point process appears as a limiting point process for eigenvalues of random matrices whose spectrum possesses a hard edge. The most well-known example is the Laguerre Unitary Ensemble (LUE), which is the set of $n \times n$ positive definite Hermitian matrices $M$ endowed with the probability measure

$$\frac{1}{Z_{n,\alpha}} (\det M)^\alpha e^{-\text{Tr} M} dM, \quad dM = \prod_{j=1}^n dM_{ii} \prod_{1 \leq i < j \leq n} d\text{Re} M_{ij} d\text{Im} M_{ij},$$

(2.6)

where $Z_{n,\alpha}$ is the normalization constant. Since the matrix $M$ is positive definite, its eigenvalues $\lambda_1, ..., \lambda_n$ are positive and $0$ is a hard edge of the spectrum. By integrating over the unitary group the probability measure (2.6), it reduces to the probability measure on $(\mathbb{R}^+)^n$ given by

$$\frac{1}{n! Z_{n,\alpha}} \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)^2 \prod_{j=1}^n \lambda_j^\alpha e^{-\lambda_j} d\lambda_j,$$

(2.7)

where $Z_{n,\alpha}$ is the partition function. It is well-known [7] that (2.7) is a determinantal point process whose correlation kernel is

$$K^{\text{LUE}}_n(\lambda, \nu) = \sqrt{w(\lambda)w(\nu)} \sum_{j=0}^{n-1} p_j(\lambda)p_j(\nu), \quad \lambda, \nu > 0,$$

(2.8)

where $w(x) = x^\alpha e^{-x}$ and $p_j(x)$ is the Laguerre orthonormal polynomial of degree $j$, i.e. it satisfies

$$\int_0^\infty p_j(x)p_\ell(x)w(x)dx = \delta_{j\ell}, \quad \text{for } \ell = 0, 1, 2, ..., j.$$  

(2.9)
Near the hard edge, the LUE kernel converges to the Bessel kernel as $n \to \infty$. More precisely, after the rescaling

$$x_j = 4n\lambda_j$$

for $j = 1, ..., n$, (2.10)

the following limit holds

$$\lim_{n \to \infty} \frac{1}{4n} K_n^{\text{LUE}} \left( \frac{x}{4n}, \frac{y}{4n} \right) = K^{\text{Be}}(x, y).$$

(2.11)

This limit implies also trace-norm convergence of the associated operator when acting on bounded intervals. Therefore, after the proper rescaling between $\vec{\lambda}$ and $\vec{x}$ given by (2.10), we have

$$F_{n}^{\text{LUE}}(\vec{x}, \vec{s}) := \det \left( 1 - \chi(0, \lambda_j) \sum_{j=1}^{k} (1-s_j) K_n^{\text{LUE}} \chi(\lambda_{j-1}, \lambda_j) \right) = F(\vec{x}, \vec{s}) + o(1) \quad (2.12)$$

as $n \to \infty$, and where $\lambda_0 := 0$ and $K_n^{\text{LUE}}$ is the integral operator whose kernel is $K_n^{\text{LUE}}$. On the other hand, $F_{n}^{\text{LUE}}(\vec{x}, \vec{s})$ can also be written as the following ratio of Hankel determinants

$$F_{n}^{\text{LUE}}(\vec{x}, \vec{s}) = \frac{\det \left( \int_{0}^{\infty} w(x) \left( 1 - \sum_{j=1}^{k} (1-s_j) \chi(\lambda_{j-1}, \lambda_j) x^{i+j-2} dx \right)^n \right)_{i,j=1}}{\det \left( \int_{0}^{\infty} w(x) x^{i+j-2} dx \right)_{i,j=1}}, \quad (2.13)$$

where the denominator of the above expression is the partition function of the LUE and is well-known (see [21, formula 17.6.5]). In particular, Theorem 1.1 together with (2.12) implies large $n$ asymptotics for the ratio (2.13) up to constant term, but this does not provide an estimate for error term $o(1)$ in (2.12).

### 2.4 Ratio probability between the two smallest particles

Two quantities of interest are the ratio and the gap probabilities between the two smallest particles in the Bessel point process, namely $Q_{\alpha}(r) = \mathbb{P} \left( \frac{\zeta_2}{\zeta_1} > r \right)$ and $G_{\alpha}(d) = \mathbb{P} \left( \zeta_2 - \zeta_1 > d \right)$, where $r > 1$ is the size of the ratio and $d > 0$ is the size of the gap. The ratio probability was obtained in [1] and the gap probability in [16]. Note that Theorem 1.1 expresses quantities related to ratios of particles more naturally than quantities related to differences of particles. Indeed, if we choose $\vec{r} = (1, r_2, ..., r_k)$ (i.e. $r_1 = 1$) in [1.11], the numbers $r_2, ..., r_k$ are related to the ratios $\frac{\zeta_2}{\zeta_1}, ..., \frac{\zeta_k}{\zeta_1}$. In this section, we start by expressing $Q_{\alpha}(r)$ in terms of $F$. By definition, we have

$$Q_{\alpha}(r) = \int_{0}^{\infty} \partial_{x} \mathbb{P} \left( \zeta_1 \leq \xi \land \zeta_2 > r x \right)_{\xi=x} dx,$$

(2.14)

$$= \int_{0}^{\infty} \partial_{x} \mathbb{P} \left( n(0, \xi) = 1 \land n(\xi, rx) = 0 \right)_{\xi=x} dx.$$

The probability in the integrand can be obtained from the generating function (1.2) as follows

$$\partial_{s} F((\xi, rx), (s, 0))|_{s=0} = \mathbb{P} \left( n(0, \xi) = 1 \land n(\xi, rx) = 0 \right),$$

(2.15)
and thus
\[
Q_\alpha(r) = \left. \int_0^\infty \partial_\xi \partial_s F((\xi, rx), (s, 0)) \right|_{s=0}^{\xi=x} dx \\
= \left. \int_0^\infty \partial_{r_1} \partial_s F((r_1 x, rx), (s, 0)) \right|_{s=0, r_1=1} \frac{dx}{x}, \\
= \left. \int_0^\infty \partial_{r_1} \partial_s \exp \left( -\frac{1}{4} \int_0^x (r_1 q_1^2(\xi) + r q_2^2(\xi)) \log \left( \frac{x}{\xi} \right) d\xi \right) \right|_{s=0, r_1=1} \frac{dx}{x},
\]
where we have applied Theorem 1.1 with \( k = 2, \bar{r} = (r_1, r), \bar{s} = (s, 0) \). It is worth comparing this formula with the result obtained in [1, Theorem 1.7], which is given by
\[
Q_\alpha(r) = \frac{1}{4^{\alpha+1} \Gamma(1+\alpha) \Gamma(2+\alpha)} \int_0^\infty x^\alpha e^{I(x;r)} dx,
\]
where
\[
I(x; r) = -\frac{1}{4} \int_0^x (\bar{q}_1^2(\xi; r) + r \bar{q}_2^2(\xi; r)) \log \left( \frac{x}{\xi} \right) d\xi.
\]

The functions \( \bar{q}_1^2(\xi; r) \) and \( \bar{q}_2^2(\xi; r) \) are real and analytic for \( \xi \in (0, \infty) \) and \( r \in (1, \infty) \), and they satisfy the following system of two coupled Painlevé V equations:
\[
\begin{align*}
\xi \bar{q}_1 \left( 1 - 2 \sum_{j=1}^{2} \bar{q}_j^2 \right) \sum_{j=1}^{2} (\xi \bar{q}_j \bar{q}_j')' &+ \left[ \xi \left( (\xi \bar{q}_1')' + \frac{\bar{q}_1}{4} \right) + \frac{1}{\bar{q}_1} \right] \left( 1 - 2 \sum_{j=1}^{2} \bar{q}_j^2 \right) + \xi^2 \bar{q}_1 \left( \sum_{j=1}^{2} \bar{q}_j \bar{q}_j' \right)^2 = \frac{\alpha^2 - \bar{q}_1}{4}, \\
\xi \bar{q}_2 \left( 1 - 2 \sum_{j=1}^{2} \bar{q}_j^2 \right) \sum_{j=1}^{2} (\xi \bar{q}_j \bar{q}_j')' &+ \xi \left( (\xi \bar{q}_2')' + \frac{\bar{q}_2}{4} \right) \left( 1 - 2 \sum_{j=1}^{2} \bar{q}_j^2 \right) + \xi^2 \bar{q}_2 \left( \sum_{j=1}^{2} \bar{q}_j \bar{q}_j' \right)^2 = \frac{\alpha^2 - \bar{q}_2}{4},
\end{align*}
\]
where primes denote derivatives with respect to \( \xi \). Furthermore, the functions \( \bar{q}_1 \) and \( \bar{q}_2 \) satisfy the following boundary conditions: as \( \xi \to 0 \), we have
\[
\begin{align*}
\bar{q}_1(\xi) &= \sqrt{\frac{2}{\alpha + 2}} (1 + \mathcal{O}(\xi)), \\
\bar{q}_2(\xi) &= (1 - r^{-1}) J_{\alpha+2}(\sqrt{\xi})(1 + \mathcal{O}(\xi)) = \frac{(1 - r^{-1})(r \xi)^{\alpha+2}}{2^\alpha \Gamma(\alpha + 3)} (1 + \mathcal{O}(\xi)).
\end{align*}
\]
The authors obtained also other asymptotics for \( \bar{q}_1(\xi; r) \) and \( \bar{q}_2(\xi; r) \) in various regimes of \( r \) and \( x \) (see [1, Theorem 1.1] for more details). The main differences between the system (2.18) for \( \bar{q}_1 \) and \( \bar{q}_2 \) with the system for \( q_1 \) and \( q_2 \) lie in the extra term \( \frac{1}{\bar{q}_1} \) in the first equation of (2.18), as well as the small \( \xi \) asymptotics of \( \bar{q}_1(\xi) \), see (2.19).

### 3 Model RH problem

In order to have compact notations in the coming sections, we define
\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.
\]

\[\text{9}\]
Figure 1: The jump contour for $\Phi$ with $k = 3$, and the four sectors $\mathcal{I}_i$, $i = 1, 2, 3, 4$.

We also define for $y \in \mathbb{R}$ the following piecewise constant matrix:

$$H_y(z) = \begin{cases} 
I, & \text{for } -\frac{2\pi}{3} < \arg(z - y) < \frac{2\pi}{3}, \\
\left( \begin{array}{cc} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{array} \right), & \text{for } \frac{2\pi}{3} < \arg(z - y) < \pi, \\
\left( \begin{array}{cc} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{array} \right), & \text{for } -\pi < \arg(z - y) < -\frac{2\pi}{3},
\end{cases}$$

where the principal branch is chosen for the argument, such that $\arg(z - y) = 0$ for $z > y$.

Let $0 = x_0 < x_1 < \ldots < x_k < +\infty$ and $s_1, \ldots, s_k \in [0, 1]$, $s_{k+1} = 1$ be such that $s_{j+1} \neq s_j$ for $j \in \{1, \ldots, k\}$. The solution of our model RH problem will be denoted by $\Phi(z; \vec{x}, \vec{s})$, where $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{s} = (s_1, \ldots, s_k)$. When there is no confusion, we will just denote it by $\Phi(z)$ where the dependence in $\vec{x}$ and $\vec{s}$ is omitted.

**RH problem for $\Phi$**

(a) $\Phi : \mathbb{C} \setminus \Sigma_\Phi \to \mathbb{C}^{2 \times 2}$ is analytic, where the contour $\Sigma_\Phi = ((-\infty, 0] \cup \Sigma_1 \cup \Sigma_2)$ is oriented as shown in Figure 1 with

$$\Sigma_1 = -x_k + e^{\frac{2\pi i j}{3}} \mathbb{R}^+, \quad \Sigma_2 = -x_k + e^{-\frac{2\pi i j}{3}} \mathbb{R}^+.$$ 

(b) The limits of $\Phi(z)$ as $z$ approaches $\Sigma_\Phi \setminus \{0, -x_1, \ldots, -x_k\}$ from the left (+ side) and from the right (− side) exist, are continuous on $\Sigma_\Phi \setminus \{0, -x_1, \ldots, -x_k\}$ and are denoted by $\Phi_+$ and $\Phi_+$ respectively. Furthermore they are related by:

\[
\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \quad z \in \Sigma_1, \tag{3.3}
\]

\[
\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, -x_k), \tag{3.4}
\]

\[
\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \quad z \in \Sigma_2, \tag{3.5}
\]

\[
\Phi_+(z) = \Phi_-(z) \begin{pmatrix} e^{\pi i \alpha} & s_j \\ 0 & e^{-\pi i \alpha} \end{pmatrix}, \quad z \in (-x_j, -x_{j-1}), \tag{3.6}
\]
where \( j = 1, \ldots, k \).

(c) As \( z \to \infty \), we have

\[
\Phi(z) = \left( I + \Phi_1(\vec{x}, \vec{s})z^{-1} + \mathcal{O}(z^{-2}) \right) z^{-\frac{a}{2}} Ne^{\frac{1}{2} i \sigma_3},
\]  

(3.7)

where the principal branch is chosen for each root, and \( \Phi_1 \) is given by

\[
\Phi_1(\vec{x}, \vec{s}) = \begin{pmatrix}
v(\vec{x}, \vec{s}) & -it(\vec{x}, \vec{s}) \\
-ip(\vec{x}, \vec{s}) & -v(\vec{x}, \vec{s})
\end{pmatrix}.
\]

(3.8)

The fact that \( \Phi_1 \) is traceless follows directly from the relation \( \det \Phi \equiv 1 \).

As \( z \) tends to \(-x_j, j \in \{1, \ldots, k\}\), \( \Phi \) takes the form

\[
\Phi(z) = \Phi_{0,j}(z) \begin{pmatrix}
1 & \frac{s_{j+1} - s_j}{2\pi i} \log(z + x_j) \\
0 & 1
\end{pmatrix} V_j(z) e^{\frac{2i\alpha}{2} \theta(z) \sigma_3 H_{-x_k}(z)},
\]

(3.9)

where \( \Phi_{0,j}(z) = \Phi_{0,j}(z; \vec{r}, \vec{s}) \) is analytic in a neighbourhood of \((-x_{j+1}, -x_{j-1})\), satisfies \( \det \Phi_{0,j} \equiv 1 \), and \( \theta(z), V_j(z) \) are piecewise constant and defined by

\[
\theta(z) = \begin{cases}
+1, & \text{Im } z > 0, \\
-1, & \text{Im } z < 0,
\end{cases}
\]

\[
V_j(z) = \begin{cases}
I, & \text{Im } z > 0, \\
\begin{pmatrix}
1 & -s_j \\
0 & 1
\end{pmatrix}, & \text{Im } z < 0.
\end{cases}
\]

(3.10)

As \( z \) tends to 0, the behaviour of \( \Phi \) is

\[
\Phi(z) = \Phi_{0,0}(z) z^\frac{\alpha}{2} \sigma_3 \begin{pmatrix}
1 & s_1 h(z) \\
0 & 1
\end{pmatrix}, \quad \alpha > -1,
\]

(3.11)

where \( \Phi_{0,0}(z) \) is analytic in a neighbourhood of \((-x_1, \infty)\), satisfies \( \det \Phi_{0,0} \equiv 1 \) and

\[
h(z) = \begin{cases}
1 & \alpha \not\in \mathbb{N}, \\
\frac{2i \sin(\pi \alpha)}{2\pi i} \log z, & \alpha \in \mathbb{N}.
\end{cases}
\]

(3.12)

**Remark 3.1** The solution of the RH problem for \( \Phi \) is unique. This follows by standard arguments, based on the fact that \( \det \Phi(z) \equiv 1 \), see e.g. [7, Theorem 7.18]. We will prove the existence of the solution in Section 4, see in particular (4.17) and comments below.

**Remark 3.2** We can verify that \( \sigma_3 \Phi(\overline{z}) \sigma_3 \) is also a solution of the RH problem for \( \Phi \). Thus, by uniqueness of the solution (see Remark 3.1), we have

\[
\Phi(z) = \sigma_3 \Phi(\overline{z}) \sigma_3.
\]

(3.13)

This means that there is some symmetry in the problem. In particular, this relation implies that the functions \( v, t \) and \( p \) that appear in (3.8) are real.
Lax pair

In this subsection, we obtain a system of $k$ ordinary differential equations for $k$ functions associated to $\Phi$. We derive these equations using Lax pair techniques. The following computations are similar to those done in [1] for the distribution of the ratio between the two smallest eigenvalues in the Laguerre Unitary Ensemble. We introduce a new parameter $x > 0$, and we begin with the following transformation on $\Phi$:

$$
\tilde{\Phi}(z; x) = \tilde{E}(x)\Phi(x^2 z; \bar{r}x^2, \bar{s}), \quad \tilde{E}(x) = \begin{pmatrix} 1 & 0 \\ \frac{1}{t(\bar{r}x^2, \bar{s})} & 1 \end{pmatrix} e^{\frac{\pi}{4}x^2 x^2},
$$

(3.14)

where we have omitted the dependence of $\tilde{\Phi}$ in $\bar{r}$ and $\bar{s}$. Note that with this transformation, $\tilde{\Phi}$ satisfies an RH problem whose contour does not depend on $x$. By standard arguments, $\tilde{\Phi}(z; x)$ is analytic in $x$ for $x$ in a compact subset of $(0, \infty)$. By differentiating $\tilde{\Phi}$ with respect to $z$ and $x$, we obtain a Lax pair of the form

$$
\begin{cases}
\partial_z \tilde{\Phi}(z; x) = A(z; x)\tilde{\Phi}(z; x), \\
\partial_x \tilde{\Phi}(z; x) = B(z; x)\tilde{\Phi}(z; x),
\end{cases}
$$

(3.15)

where we have also omitted the dependence of $A$ and $B$ in $\bar{r}$ and $\bar{s}$. Since $\tilde{\Phi}$, $\partial_z \tilde{\Phi}(z; x)$ and $\partial_x \tilde{\Phi}(z; x)$ have the same jumps, $A$ and $B$ are meromorphic in $z \in \mathbb{C}$. From (3.9) and (3.11), $B$ is an entire function in $z$ and $A$ has simple poles in $z$ at $0, -r_1, \ldots, -r_k$. We can use (3.7) to obtain an explicit expression for $B$:

$$
B(z; x) = B_0(x) + zB_1, \quad B_0(x) = \begin{pmatrix} 0 \\ u(x) \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
$$

(3.16)

where $u(x) = \frac{2\bar{t}(\bar{r}x^2, \bar{s})x^2 + t(\bar{r}x^2, \bar{s})^2 - 2\nu(\bar{r}x^2, \bar{s}) - t(\bar{r}x^2, \bar{s})}{x^2}$, and $t'(\bar{r}x^2) = \partial_y t(y)|_{y=x^2}$. On the other hand, $A$ can be written as

$$
A(z; x) = A_\infty(x) + \sum_{j=0}^k \frac{A_j(x)}{z + r_j}.
$$

(3.17)

The matrix $A_\infty$ can also be explicitly evaluated by using (3.7), we have

$$
A_\infty(x) = \begin{pmatrix} 0 & 0 \\ \frac{1}{x} & 0 \end{pmatrix}.
$$

(3.18)

Since $\det \tilde{\Phi}(z)$ is constant, $A$ is traceless and we can also write

$$
A(z; x) = \begin{pmatrix} a(z; x) & b(z; x) \\ c(z; x) & -a(z; x) \end{pmatrix}, \quad b(z; x) = \sum_{j=0}^k \frac{b_j(x)}{z + r_j}.
$$

(3.19)

We will derive a system of ordinary differential equations for $b_0(x), b_1(x), \ldots, b_k(x)$ and $u(x)$ from the compatibility condition

$$
\partial_x \partial_z \tilde{\Phi}(z; x) = \partial_z \partial_x \tilde{\Phi}(z; x),
$$

(3.20)

which by using (3.15) is equivalent to

$$
\partial_x A - \partial_z B + AB - BA = 0.
$$

(3.21)
This condition gives rise to the three following equations for \( a, b, c, \text{ and } u \):

\[
\begin{align*}
0 &= c - b(z + u) - a', \\
0 &= 2a + b', \\
0 &= 2a(z + u) - c' + 1,
\end{align*}
\]

where primes denote derivatives with respect to \( x \). In particular \( a \) and \( c \) can be expressed in terms of \( b \). Thus we can write the determinant of \( A \) as

\[
\det A = -b^2(z + u) + \frac{(b^2)''}{4} - \frac{3}{4}(b')^2.
\]

Expanding \( \det A(z) \) around \( z = 0, -r_1, ..., -r_k \) and \( \infty \) using on one hand \( \Phi(3.19) \) and \( (3.25) \), and on the other hand \( (3.7), (3.9) \) and \( (3.11) \), and by expanding \( A_{12}(z) = b(z) \) around \( z = \infty \), we obtain

\[
\sum_{j=0}^{k} b_j(x) = \frac{x}{2},
\]

\[
(u(x) - r_j)b_j(x)^2 + \frac{1}{4}b_j'(x)^2 - \frac{1}{2}b_j(x)b_j''(x) = 0, \quad j = 1, ..., k
\]

\[
u(x)b_0(x)^2 + \frac{1}{4}b_0'(x)^2 - \frac{1}{2}b_0(x)b_0''(x) = \frac{\alpha^2}{4}.
\]

**Definition 3.3** We define \( q_j \) in terms of \( b_j \) as follows:

\[
q_j^2(x) = \frac{2b_j(\sqrt{x})}{\sqrt{x}}, \quad j = 1, ..., k.
\]

We can use \( (3.26) \) and \( (3.28) \) to express \( u \) and \( b_0 \) in terms of \( b_1, ..., b_k \), and therefore in terms of \( q_1, ..., q_k \). By substituting these expressions for \( u \) and \( b_0 \) in \( (3.27) \), we obtain the system of \( k \) coupled Painlevé V equations given by \( (1.12) \). Also, from \( (3.13) \), if \( z \in \mathbb{R} \setminus \{-r_k, ..., -r_1, 0\} \), we have \( b(z; x) = b(z; x) \). This implies that \( b_0, ..., b_k \), and therefore \( q_1^2, ..., q_k^2 \), are all real functions of \( x \in \mathbb{R}^+ \).

**Proposition 3.4** below will be useful in Section 4 to integrate the identity \( (4.21) \).

**Proposition 3.4** For each \( j = 1, 2, ..., k \), there holds the relation

\[
\partial_x \left( x \lim_{z \to -r_j x} [\Phi^{-1}(z; \vec{r} x, \vec{s})\Phi'(z; \vec{r} x, \vec{s})]_{21} \right) = \frac{2\pi i e^{-\pi i \alpha} q_j^2(x)}{s_{j+1} - s_j} \frac{1}{4},
\]

where the limit is taken from \( z \in \mathbb{I}_4 \), with \( \mathbb{I}_4 \) as shown in Figure 1 and where \( \Phi' = \partial_z \Phi \).

**Proof.** We recall that \( \Phi_{0,j}(z; \vec{r} x, \vec{s}) \) defined in \( (3.9) \) is invertible and analytic in \( z \) in a neighbourhood of \( -r_j x \). By expanding it around \( -r_j x \), we can write

\[
\Phi_{0,j}(z; \vec{r} x, \vec{s}) = E_j(x)(I + F_j(x)(z + r_j x) + O((z + r_j x)^2)), \quad \text{as } z \to -r_j x,
\]

for certain matrices \( E_j \) and \( F_j \) (they depend also on \( \vec{r} \) and \( \vec{s} \)). Therefore, we have

\[
\lim_{z \to -r_j x} [\Phi^{-1}(z; \vec{r} x, \vec{s})\Phi'(z; \vec{r} x, \vec{s})]_{21} = e^{-\pi i \alpha} [\Phi^{-1}_{0,j}(-r_j x)\Phi'_{0,j}(-r_j x)]_{21},
\]

\[
= e^{-\pi i \alpha} F_{j,21}(x),
\]

where \( F_{j,21} \) is the \( 2 \times 1 \) block of \( F_j \) and \( \Phi^{-1}_{0,j}(-r_j x) \) is the inverse of \( \Phi_{0,j}(-r_j x) \) at \( -r_j x \).
where the limit is taken from \( z \in \mathcal{I}_4 \). On the other hand, taking the limit \( z \to -r_j \) in the \( B \)-equation in the Lax pair (3.15) leads to

\[
\partial_x (\bar{E}(x)E_j(x^2)) = (B_0(x) - r_jB_1)\bar{E}(x)E_j(x^2) - \bar{E}(x)E_j(x^2)K(x), \\
\partial_x (x^2 F_j(x^2)) = (\bar{E}(x)E_j(x^2))^{-1}B_1\bar{E}(x)E_j(x^2) + x^2[K(x), F_j(x^2)],
\]

where \( K(x) = \begin{pmatrix} 0 & \frac{s_{j+1} - s_j}{\pi x} \\ 0 & 0 \end{pmatrix} \). In particular, taking the (2,1) entry in (3.34) and using the fact that \( \det E_j(x) = 1 \), it leads to

\[
\partial_x (x^2 F_{j,21}(x^2)) = ix E_{j,11}(x^2)^2.
\]

By the change of variables \( x^2 \to x \), this can be rewritten as

\[
\partial_x (xF_{j,21}(x)) = \frac{i}{2} E_{j,11}(x^2)^2.
\]

We also have, by the \( A \)-equation in the Lax pair (3.15), as \( z \to -r_j \)

\[
A(z; x) = \frac{s_{j+1} - s_j}{2\pi i(z + r_j)} \bar{E}(x) \left( \begin{array}{cc} -E_{j,11}(x^2)E_{j,21}(x^2) & E_{j,11}(x^2)^2 \\ -E_{j,21}(x^2)^2 & E_{j,11}(x^2) \end{array} \right) \bar{E}(x)^{-1} + O(1).
\]

Equation (3.37) implies then

\[
b_j(x) = \frac{s_{j+1} - s_j}{2\pi} x E_{j,11}(x^2)^2.
\]

Thus by (3.32), (3.36) and (3.29), we obtain the claim.

\[\square\]

4 Proof of Theorem 1.1

We start the proof of Theorem 1.1 by following a theory developed by Its, Izergin, Korepin and Slavnov [19], which was also developed by Bertola and Cafasso in [2], to express the quantities \( \partial_x \log F(x, t) \), \( j = 1, \ldots, k \) in terms of a RH problem related to an integrable kernel \( R \) (the solution of this RH problem will be denoted \( Y \)). Then, we will relate \( Y \) to \( \Phi \) and finally integrate these identities. Let \( K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) be given by

\[
K(u, v) = \chi_{(0,x_k)}(u) \sum_{j=1}^{k} (1 - s_j)K^{Be}(u,v)\chi_{(x_{j-1},x_j)}(v), \quad u, v > 0.
\]

This is the kernel of a trace class integral operator \( K \) acting on \( L^2(\mathbb{R}^+) \). The kernel \( K \) is integrable in the sense of Its, Izergin, Korepin and Slavnov, i.e. it can be written in the form

\[
K(u, v) = \frac{f^T(u)g(v)}{u - v}, \quad f^T(u)g(u) = 0, \quad u, v > 0,
\]

where \( f(u) \) and \( g(v) \) are given by

\[
f(u) = \frac{1}{2} \left( \frac{\chi_{(0,x_k)}(u)J_\alpha(\sqrt{u})}{\chi_{(0,x_k)}(u)\sqrt{u}J'_\alpha(\sqrt{u})} \right), \quad g(v) = \left( \sum_{j=1}^{k} (1 - s_j)\sqrt{u}J'_\alpha(\sqrt{u})\chi_{(x_{j-1},x_j)}(v) \right).
\]
Also, by using the connection formula $I_\alpha(e^{\frac{\pi i}{2}} \sqrt{u}) = e^{\frac{\alpha \pi i}{2}} I_\alpha(\sqrt{u})$ for $u > 0$ (see formula 10.27.6), $f(u)$ and $g(v)$ can be rewritten in terms of $P_{\text{Be}}$ (this is the solution of a modified Bessel model RH problem, and is defined in the Appendix, see (A.7)) as follows:

$$
\begin{align*}
  f(u) &= e^{-\alpha \pi i} \frac{\alpha i}{2\sqrt{\pi}} \chi_{(0,x_k)}(u) \sigma_3 P_{\text{Be}+}(-u) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } u > 0, \\
g(v) &= e^{-\alpha \pi i} \frac{\alpha i}{2\sqrt{\pi}} \sum_{j=1}^{k} (1 - s_j) \chi(x_{j-1},x_j)(v) \sigma_3 P_{\text{Be}+}^{-1}(-v)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{for } v > 0.
\end{align*}
$$

In the Bessel point process, for all bounded Borel set $B$ with non zero Lebesgue measure, we have $P(n_B = 0) > 0$. Therefore, from (1.2) and (1.3) we have $\det(1 - K) > 0$ if $s_1, ..., s_k \in [0,1]$. By standard properties of trace class operators (see e.g. [13, page 1029]), we have

$$\partial_{x_j} \log \det(1 - K) = -\text{Tr} \left( (1 - K)^{-1} \partial_{x_j} K \right), \quad j = 1, ..., k. \tag{4.5}$$

In our case, it can be rewritten more explicitly as

$$
\begin{align*}
  \text{Tr} \left( (1 - K)^{-1} \partial_{x_j} K \right) &= (s_{j+1} - s_j) \text{Tr} \left( (1 - K)^{-1} K^{\text{Be}} \delta_{x_j} \right) \\
  &= \frac{s_{j+1} - s_j}{1 - s_j} \lim_{u \to x_j} [(1 - K)^{-1} K](u, u) \\
  &= \frac{s_{j+1} - s_j}{1 - s_j} \lim_{u \to x_j} R(u, u)
\end{align*}
$$

where $R$ is the kernel for the resolvent operator $\mathcal{R}$ defined by

$$1 + \mathcal{R} = (1 - K)^{-1}. \tag{4.7}$$

If $s_j = 1$, then we take the limit $z \searrow x_j$ instead, and the above formula is replaced by

$$
\text{Tr} \left( (1 - K)^{-1} \partial_{x_j} K \right) = \frac{s_{j+1} - s_j}{1 - s_{j+1}} \lim_{z \searrow x_j} R(z, z), \tag{4.8}
$$

which is well defined since $s_{j+1} \neq s_j$. Let us now define the matrix $Y$ by

$$Y(z) = I - \int_{0}^{x_k} F(\mu) g^T(\mu) \frac{d\mu}{z - \mu}, \quad F(\mu) = \begin{pmatrix} (1 - K)^{-1} f_1(\mu) \\ (1 - K)^{-1} f_2(\mu) \end{pmatrix}. \tag{4.9}$$

The function $Y$ satisfies the following RH problem [3].

**RH problem for Y**

(a) $Y : \mathbb{C} \setminus [0, x_k] \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) For $u \in (0, x_k) \setminus \{x_1, ..., x_k\}$, the limits $\lim_{\epsilon \to 0^+} Y(u \pm i\epsilon)$ exist, are denoted $Y_+(u)$ and $Y_-(u)$ respectively, are continuous as functions of $u \in (0, x_k)$, and satisfy furthermore the jump relation

$$Y_+(u) = Y_-(u) J_Y(u), \quad J_Y(u) = I - 2\pi i f(u) g^T(u). \tag{4.10}$$

(c) $Y(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$. 

15
(d) \( Y(z) = \mathcal{O}(\log(z-x_j)) \) as \( z \to x_j \), for each \( j = 0, \ldots, k \) (with \( x_0 = 0 \)).

For \( u, v \in (0, x_k) \), the resolvent can now be written as

\[
R(u, v) = \frac{F_T(u)G(v)}{u-v}, \quad \text{where} \quad F(u) = Y_+(u)f(u) \quad \text{and} \quad G(v) = (Y_+^{-1}(v))^T g(v). \tag{4.11}
\]

Now we want to relate \( Y \) with \( \Phi \). Let us consider \( X(z) = \tilde{Y}(z)\tilde{P}_{Be}(z) \), where \( \tilde{Y}(z) = \sigma_3 Y(-z)\sigma_3 \) and \( \tilde{P}_{Be} \) is the solution of a modified Bessel model RH problem, defined in \( (A.7) \). Since \( \tilde{Y} \) is analytic on \( \Sigma_1 \cup \Sigma_2 \cup (-\infty, -x_k) \), from the jumps of \( \tilde{P}_{Be} \), it is direct that \( X \) has exactly the same jumps as \( \Phi \) on \( \Sigma_1 \cup \Sigma_2 \cup (-\infty, -x_k) \). The jumps \( J_X \) of \( X \) are \emph{a priori} more involved on \( (-x_k, 0) \). They are given by

\[
J_X(-u) = \begin{pmatrix} e^{-\pi i\alpha} & 0 \\ 0 & e^{-\pi i\alpha} \end{pmatrix} \tilde{P}_{Be,+}^{-1}(-u) J_{\tilde{Y}}(-u) \tilde{P}_{Be,+}^{-1}(-u), \quad u \in (0, x_k),
\]

where \( J_{\tilde{Y}} \) is the jump of \( \tilde{Y} \), given by

\[
J_{\tilde{Y}}(-u) = \sigma_3 J_Y(u)^{-1} \sigma_3, \quad u \in (0, x_k). \tag{4.13}
\]

For \( u \in (0, x_k) \), by \( (4.13) \) and \( (4.14) \), we have

\[
J_{\tilde{Y}}(-u) = \tilde{P}_{Be,+}^{-1}(-u) \begin{pmatrix} 1 & -e^{-\pi i\alpha} \sum_{j=1}^{k} (1-s_j) \chi(x_{j-1}, x_j)(u) \\ 0 & 1 \end{pmatrix} \tilde{P}_{Be,+}^{-1}(-u). \tag{4.14}
\]

By plugging it into \( (4.12) \), \( J_X \) is simply reduced to

\[
J_X(-u) = \begin{pmatrix} e^{-\pi i\alpha} & 0 \\ 0 & e^{-\pi i\alpha} \end{pmatrix} \sum_{j=1}^{k} s_j \chi(x_{j-1}, x_j)(u), \quad u \in (0, x_k),
\]

which is precisely the same jump as \( \Phi(z; \vec{x}, \vec{s}) \) for \( z \in (-x_k, 0) \). On the other hand, from \( (A.9) \), as \( z \to \infty \) we have

\[
X(z) = e^{-\frac{\pi i\alpha}{4}} \left( \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{8}(4\alpha^2 + 3) \end{pmatrix} \right) \left( I + \mathcal{O}(z^{-1}) \right) I + \mathcal{O}(z^{-1}) \right) \left( z^{-\frac{\alpha}{4}} Ne^{\frac{3}{4} z^2\sigma_3} \right). \tag{4.16}
\]

Thus by uniqueness of the solution of the RH problem for \( \Phi \), see Remark \( 3.1 \) we have

\[
\Phi(z; \vec{x}, \vec{s}) = \left( \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{8}(4\alpha^2 + 3) \end{pmatrix} \right) e^{-\frac{\pi i\alpha}{4}} \tilde{Y}(z) \tilde{P}_{Be}(z). \tag{4.17}
\]

Since from our proof, the matrix \( \tilde{Y} \) on the right hand side exists and is constructed explicitly in terms of \( (1-K)^{-1} \) (see \( (4.9) \)), it also proves the existence of a solution for the RH problem for \( \Phi \). Note that \( (4.3) \) and \( (4.4) \) can equivalently be written as

\[
\tilde{P}_{Be,-}^{-1}(-u)\sigma_3 f(u) = \frac{c}{2} \chi(0, x_k)(u) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{P}_{Be,-}^{-1}(-v)^T \sigma_3 g(v) = c \sum_{j=1}^{k} (1-s_j) \chi(x_{j-1}, x_j)(v) \begin{pmatrix} 0 & 1 \end{pmatrix},
\]

where \( u, v \in \mathbb{R}^+ \) and \( c = \frac{e^{\frac{\pi i\alpha}{4}} - \frac{\alpha i}{\sqrt{\pi}}}{\sqrt{\pi}} \). Thus for \( u, v \in \mathbb{R}^+ \), we have

\[
R(u, v) = \frac{c^2}{2} \left[ \Phi_{-}(\vec{v}; \vec{x}, \vec{s}) \Phi_{-}(u; \vec{x}, \vec{s}) \right] \chi(0, x_k)(u) \sum_{j=1}^{k} (1-s_j) \chi(x_{j-1}, x_j)(v) \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{4.18}
\]
By taking the limit \( v \to u \) for a certain \( u \in (x_{j-1}, x_j) \) with \( j \in \{1, ..., k\} \) in the above expression, we obtain

\[
R(u, u) = -\frac{e^2}{2} (1 - s_j) \left[ \Phi_-( -u; \bar{x}, \bar{s})^{-1} \Phi'_-( -u; \bar{x}, \bar{s}) \right]_{21}.
\] (4.19)

Taking now the limit \( u \not\to x_j \) in (4.19) and substituting the result in (4.5) and (4.6), we obtain an explicit differential identity in terms of \( \Phi \) for each \( j \in \{1, ..., k\} \):

\[
\partial_{x_j} \log F(\bar{x}, \bar{s}) = -(s_{j+1} - s_j) \frac{e^{\pi i \alpha}}{2\pi i} \lim_{z \to x_j} \left[ \Phi^{-1} (z; \bar{x}, \bar{s}) \Phi'(z; \bar{x}, \bar{s}) \right]_{21},
\] (4.20)

where the limit is taken from \( z \in I_4 \), with \( I_4 \) as shown in Figure 1. By simple compositions, we can use the above identities to get

\[
\partial_x \log F(\bar{r}x, \bar{s}) = \sum_{j=1}^k r_j \partial_{x_j} \log F(\bar{x}, \bar{s})|_{\bar{x} = \bar{r}x} = -\sum_{j=1}^k r_j (s_{j+1} - s_j) \frac{e^{\pi i \alpha}}{2\pi i} \lim_{z \to r_j x} \left[ \Phi^{-1} (z; \bar{r}x, \bar{s}) \Phi'(z; \bar{r}x, \bar{s}) \right]_{21}.
\] (4.21)

Let \( \epsilon \) and \( x \) be such that \( 0 < \epsilon < x \). By integrating the above expression from \( \epsilon \) to \( x \), this gives

\[
\log \frac{F(\bar{r}x, \bar{s})}{F(\bar{r}\epsilon, \bar{s})} = -\sum_{j=1}^k r_j (s_{j+1} - s_j) \frac{e^{\pi i \alpha}}{2\pi i} \int_{\epsilon}^x \lim_{z \to r_j \xi} \left[ \Phi^{-1} (z; \bar{r}\xi, \bar{s}) \Phi'(z; \bar{r}\xi, \bar{s}) \right]_{21} d\xi.
\] (4.22)

Integrating it by parts and using Proposition 3.4, one has

\[
\int_{\epsilon}^x \lim_{z \to r_j \xi} \left[ \Phi^{-1} (z; \bar{r}\xi, \bar{s}) \Phi'(z; \bar{r}\xi, \bar{s}) \right]_{21} d\xi = \log \left( \frac{x}{\epsilon} \right) \epsilon \int_{\epsilon}^x \lim_{z \to r_j \xi} \left[ \Phi^{-1} (z; \bar{r}\xi, \bar{s}) \Phi'(z; \bar{r}\xi, \bar{s}) \right]_{21}
\]

\[
+ 2\pi i e^{-\pi i \alpha} \int_{\epsilon}^x \log \left( \frac{x}{\xi} \right) \frac{q_j^2(\xi)}{4} d\xi.
\] (4.23)

We will prove in the next section that

\[
\lim_{\epsilon \to 0} \left( \frac{x}{\epsilon} \right) \epsilon \int_{\epsilon}^x \lim_{z \to r_j \xi} \left[ \Phi^{-1} (z; \bar{r}\xi, \bar{s}) \Phi'(z; \bar{r}\xi, \bar{s}) \right]_{21} = 0,
\] (4.24)

and that \( \int_{0}^{x} \log \left( \frac{\xi}{x} \right) q_j^2(\xi) d\xi \in \mathbb{R} \) for every \( j \in \{1, ..., k\} \). (4.25)

Thus, taking the limit \( \epsilon \to 0 \) in (4.23) and in (4.22) gives, using \( F(0, s) = 1 \) (see (4.3)), the following identity

\[
\log F(\bar{r}x, \bar{s}) = -\sum_{k=1}^k \frac{r_j}{4} \int_{0}^{x} \log \left( \frac{x}{\xi} \right) q_j^2(\xi) d\xi.
\] (4.26)

Apart from (4.24) and (4.25), this finishes the proof of Theorem 1.1.

5 Small \( x \) asymptotics

In this section, we perform a Deift/Zhou steepest descent [11, 12, 9, 10] to obtain small \( x \) asymptotics for \( \Phi(z; \bar{r}x, \bar{s}) \) uniformly in \( z \), and where \( \bar{r} \) and \( \bar{s} \) are independent of \( x \) and satisfy conditions (1.9) and (1.10).
5.1 First transformation $\Phi \mapsto W$

The first transformation consists of making the rays $\Sigma_1$ and $\Sigma_2$ ending at 0 instead of $-r_kx$, we define

$$W(z) = \Phi(z; \vec{r}x, \vec{s})H_{-r_kx}(z)^{-1}H_0(z). \quad (5.1)$$

It satisfies the following RH problem.

**RH problem for $W$**

(a) $W : \mathbb{C} \setminus \left(\left(-\infty, 0\right) \cup e^{\pm \frac{2\pi i}{3}}\mathbb{R}^+\right) \to \mathbb{C}^{2 \times 2}$ is analytic, where the rays $e^{\pm \frac{2\pi i}{3}}\mathbb{R}^+$ are oriented from $e^{\pm \frac{2\pi i}{3}}\infty$ to 0.

(b) The jumps for $W$ are given by

$$W_+(z) = W_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pm \pi i\alpha} & 1 \end{pmatrix}, \quad z \in e^{\pm \frac{2\pi i}{3}}\mathbb{R}^+, \quad (5.2)$$

$$W_+(z) = W_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in \left(-\infty, -r_kx\right), \quad (5.3)$$

$$W_+(z) = W_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \quad z \in e^{-\frac{2\pi i}{3}}\mathbb{R}^+, \quad (5.4)$$

$$W_+(z) = W_-(z) \begin{pmatrix} e^{\pi i\alpha} & s_j \\ s_j & 2 \end{pmatrix} e^{-\pi i\alpha} \begin{pmatrix} 1 & s_j \\ s_j & 1 \end{pmatrix}, \quad z \in \left(-r_jx, -r_j-1x\right), \quad (5.5)$$

where $j = 1, ..., k$.

(c) As $z \to \infty$, we have

$$W(z) = (I + O(z^{-1}))z^{-\frac{\sigma_3}{2}}Ne^{z^\frac{\sigma_3}{2}}. \quad (5.6)$$

As $z$ tends to $-r_jx$, $j \in \{1, ..., k\}$, the behaviour of $W$ is

$$W(z) = \Phi_{0,j}(z) \begin{pmatrix} 1 & \frac{s_{j+1} - s_j}{2\pi i} \log(z + r_jx) \\ 0 & 1 \end{pmatrix} V_j(z)e^{\frac{\pi i\alpha}{2} \theta(z)\sigma_3}H_0(z). \quad (5.7)$$

As $z$ tends to 0, the behaviour of $W$ is

$$W(z) = \Phi_{0,0}(z)z^{\frac{\sigma_3}{2}} \begin{pmatrix} 1 & s_1h(z) \\ 0 & 1 \end{pmatrix} H_0(z). \quad (5.8)$$

5.2 Global parametrix

Ignoring a small neighbourhood of 0, we are left with a Riemann-Hilbert problem which is independent of $x$. We denote the solution of this RH problem $P^{(\infty)}$. The jumps of $P^{(\infty)}$, as well as its asymptotic behaviour at $\infty$ (5.6), are the same of those of the Bessel model RH problem of order $\alpha$ presented in Appendix A (the solution of the Bessel model RH problem is denoted $P_{Be}(z; \alpha)$). If we don’t specify the behaviour of the global parametrix near $z = 0$, the solution is not unique and for example $P^{(\infty)}(z) = P_{Be}(z; \alpha + 2n)$ for any $n \in \mathbb{N}$ is a solution. In order to have later the matching condition with the local parametrix, see (5.10), we choose the global parametrix to be

$$P^{(\infty)}(z) = P_{Be}(z; \alpha). \quad (5.9)$$
5.3 Local parametrix

Inside a fixed disk $D_0$ around 0, we want the local parametrix $P$ to satisfy the following RH problem:

**RH problem for $P$**

(a) $P : D_0 \setminus \left( (-\infty, 0] \cup e^{\pm \frac{2\pi i}{3} \mathbb{R}}^+ \right) \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) For $z \in D_0 \cap \left( (-\infty, 0] \cup e^{\pm \frac{2\pi i}{3} \mathbb{R}}^+ \right)$, $P(z)$ has the same jumps as $W(z)$, i.e. we have $P^{-1}(z)P_+(z) = W^{-1}(z)W_+(z)$.

(c) As $x \to 0$, we have

$$P(z) = \left( I + \mathcal{O}(x) \right) P^{(\infty)}(z),$$

uniformly for $z \in \partial D_0$.

(d) As $z$ tends to $-r_jx$, $j \in \{0, 1, ..., k\}$, we have

$$W(z)P^{-1}(z) = \mathcal{O}(1).$$

It can be directly verified that the following matrix satisfies conditions (a), (b) and (d) of the above RH problem:

$$P(z) = P_{Be,0}(z; \alpha) \begin{pmatrix} 1 & f(z; x) \\ 0 & 1 \end{pmatrix} z^\alpha \sigma_3 \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z),$$

(5.12)

where $P_{Be,0}(z; \alpha)$ is analytic in a neighbourhood of 0 and defined in (A.4), and where $f$ is given by

$$f(z; x) = \frac{-1}{2\pi i} \sum_{j=1}^{k} (1 - s_j) \int_{-r_jx}^{-r_j-1x} \frac{|s|^\alpha}{s-z} ds.$$ (5.13)

From (A.4) and (5.9), we have

$$P(z)P^{(\infty)}(z)^{-1} = P_{Be,0}(z; \alpha) \begin{pmatrix} 1 & f(z; x) \\ 0 & 1 \end{pmatrix} P_{Be,0}(z; \alpha)^{-1} = I + \mathcal{O}(x),$$

(5.14)

as $x \to 0$ uniformly for $z \in \partial D_0$, and the matching condition (5.10) holds.

5.4 Small norm RH problem

We define

$$R(z) = \begin{cases} W(z)P^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus D_0, \\ W(z)P(z)^{-1}, & z \in D_0. \end{cases}$$

(5.15)

Since $P^{(\infty)}$ (resp. $P$) has the same jumps as $W$ on $\mathbb{C} \setminus D_0$ (resp. on $D_0$), $R$ is analytic on $\mathbb{C} \setminus (\partial D_0 \cup \{0, -r_1x, ..., -r_kx\})$. Furthermore, from (5.11), $R$ is bounded near $0, -r_1x, ..., -r_kx$ and thus $0, -r_1x, ..., -r_kx$ are removable singularities. It follows that $R$ is analytic on $\mathbb{C} \setminus \partial D_0$. Also, from (5.6), (5.9) and (A.2), we have that $R(z) =$
\( I + \mathcal{O}(z^{-1}) \) as \( z \to \infty \). Let us put the clockwise orientation on \( \partial D_0 \). The jumps \( J_R(z) = R^{-1}(z)R_+(z) \) satisfy

\[
J_R(z) = P(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}(x), \quad \text{as } x \to 0 \text{ uniformly for } z \in \partial D_0, \quad (5.16)
\]

where we have used \((5.10)\). It follows from standard theory for small norm RH problems that \( R \) exists for sufficiently small \( x \) and satisfies

\[
R(z) = I + \mathcal{O}(x), \quad R'(z) = \mathcal{O}(x), \quad (5.17)
\]

uniformly for \( z \in \mathbb{C} \setminus \partial D_0 \). We are now in a position to compute the small \( x \) asymptotics of \( b_0(x), \ldots, b_k(x) \). Inverting the transformations \( R \mapsto W \mapsto \Phi \), we obtain for \( z \in D_0 \) that

\[
\Phi(z; r \vec{x}, s) = R(z)P_{Be,0}(z; \alpha) \left( \begin{array}{c} f(z; x) \\ 1 \end{array} \right) z^{\frac{\alpha}{2}} \left( \begin{array}{c} h(z) \\ 1 \end{array} \right) H_{-r \vec{x}}(z). \quad (5.18)
\]

By \((3.14)\), \((3.15)\) and \((3.19)\), for any \( j \in \{1, \ldots, k\} \) we have

\[
b_j(\sqrt{x}) = i \sqrt{x} \lim_{z \to r_j} (z + r_j) \left[ \partial_z (\Phi(xz; r \vec{x}, s)) \Phi^{-1}(xz; r \vec{x}, s) \right]_{12}. \quad (5.19)
\]

Using \((5.18)\) and the small \( x \) asymptotics for \( R \) given by \((5.17)\), after some calculations we obtain for \( j \in \{1, \ldots, k\} \) that

\[
b_j(\sqrt{x}) = i \sqrt{x}(1 + \mathcal{O}(x))P_{Be,0,11}^2(0; \alpha) \lim_{z \to r_j} (z + r_j) \partial_z \left( f(xz; x) \right). \quad (5.20)
\]

For \( j \in \{1, \ldots, k - 1\} \), only two terms in the sum \((5.13)\) contribute to this limit. After a straightforward calculation we have that

\[
\lim_{z \to r_j} (z + r_j) \int_{-r_{j-1}x}^{-r_jx} \frac{x|s|^\alpha}{(s - xz)^2} ds = - (r_jx)^\alpha,
\]

\[
\lim_{z \to r_j} (z + r_j) \int_{-r_{j+1}x}^{-r_jx} \frac{x|s|^\alpha}{(s - xz)^2} ds = (r_jx)^\alpha.
\]

For \( j = k \), the analysis is slightly simpler, because only one term in the sum \((5.13)\) contributes to the limit \((5.19)\). Thus, we obtain as \( x \to 0 \)

\[
b_j(\sqrt{x}) = \frac{\sqrt{x}}{2\pi} P_{Be,0,11}^2(0; \alpha)(r_jx)^\alpha(s_{j+1} - s_j)(1 + \mathcal{O}(x)), \quad j \in \{1, \ldots, k\}. \quad (5.21)
\]

We will now use the explicit form of \( P_{Be} \) given in the appendix, see \((A.5)\). Since \( P_{Be,0,11}(z; \alpha) \) is an entire function in \( z \), we can obtain \( P_{Be,0,11}(0; \alpha) \) by taking the limit \( z \to 0 \) from any region. In particular, for \( z \in \{ z \in \mathbb{C} : |\arg(z)| < \frac{2\pi}{3} \} \), we have

\[
P_{Be,0,11}(z; \alpha) = \sqrt{\pi} z^{-\frac{\alpha}{2}} I_\alpha(z^\frac{3}{2}). \quad (5.22)
\]

By using the behaviour of \( I_\alpha(z) \) as \( z \to 0 \) (see \cite{22}, formula 10.30.1) we obtain \( P_{Be,0,11}(0; \alpha) = \frac{\sqrt{\pi}}{2^{1/(\alpha+1)}} \). The equation \((5.21)\) can now be rewritten as

\[
b_j(\sqrt{x}) = \frac{\sqrt{x}(s_{j+1} - s_j)}{2} J_\alpha(\sqrt{r_jx})^2(1 + \mathcal{O}(x)), \quad \text{as } x \to 0, \quad (5.23)
\]
for any \( j \in \{1, \ldots, k\} \). By the definition of \( q_j \), see (3.29), we have

\[
q_j(x) = \sqrt{s_{j+1} - s_j} J_\alpha(\sqrt{x}), \quad \text{as } x \to 0,
\]

which is precisely the boundary conditions of the system (1.12). In particular, the functions \( q_1^2(x), \ldots, q_k^2(x) \) are integrable on \((0, \epsilon)\) for any \( \epsilon > 0 \), and this proves (4.23).

Also, (5.18) implies that as \( x \to 0 \) we have

\[
\lim_{z \to -r_jx} [\Phi^{-1}(z; \bar{r}x, \bar{s})\Phi(z; \bar{r}x, \bar{s})]_{21} = e^{-\pi i \alpha(r_jx)\alpha} \left( [P_{Be,0}^{-1}(0; \alpha)P_{Be,0}^\ell(0; \alpha)]_{21} + O(x) \right) = O(x^\alpha),
\]

for every \( j \in \{1, \ldots, k\} \), and where the limit in taken from \( z \in \mathcal{I}_4 \). This proves (4.24).

6 Asymptotics for \( s_j \to s_{j+1}, \ j \in \{1, \ldots, k\} \)

In this section, we perform a Deift/Zhou steepest descent \([11, 12, 9, 10]\) to obtain asymptotics as \( s_j \to s_{j+1} \) for \( \Phi(z; \bar{r}x, \bar{s}) \) uniformly in \( z \), and where \( \bar{r} \) and \( \bar{s} \) satisfy conditions (1.9) and (1.10). Let us fixed \( j \in \{1, \ldots, k\} \) in this section. If \( j \neq 1 \), we assume furthermore that \( s_{j+1} \neq s_j \). When \( s_j \to s_{j+1} \), the jumps of \( \Phi \) on \((-r_jx, -r_{j-1}x)\) tend to be the same as those on \((-r_jx, -r_{j-1}x)\) and therefore the logarithmic singularity at \( z = -r_jx \) for \( \Phi(z; \bar{r}x, \bar{s}) \) tends to disappear. Consider \( U_j \), a fixed open neighbourhood of \([-r_jx, -r_{j-1}x]\) with smooth boundaries, sufficiently small such that it does not include any \(-r_\ell x, \ell \neq j, \ell \neq j-1 \). Outside \( U_j \), the model RH problem \( \Phi(z; \bar{r}^j x, \bar{s}^j) \) possesses exactly the same jumps and the same large \( z \) asymptotics than \( \Phi(z; \bar{r}x, \bar{s}) \), and thus heuristically it is a good approximation of \( \Phi(z; \bar{r}x, \bar{s}) \) for \( z \in \mathbb{C} \setminus U_j \). Furthermore, for \( z \in U_j \), from (3.9) and (3.11), \( \Phi(z; \bar{r}^j x, \bar{s}^j) \) can be written as

\[
\Phi(z; \bar{r}^j x, \bar{s}^j) = \Phi^{*}_{0,j-1}(z) \begin{pmatrix} 1 & \frac{s_{j+1} - s_{j-1}}{2\pi i} \log(z + r_{j-1}x) \\ 0 & 1 \end{pmatrix} V_{j-1}(z)e^{\frac{\pi i}{2} \theta(z) \sigma_3} H_{\bar{r}r_jx}(z) \tag{6.1}
\]

if \( j \in \{2, \ldots, k\} \), and as

\[
\Phi(z; \bar{r}^j x, \bar{s}^j) = \Phi^{*}_{0,0}(z) z^{\frac{\sigma_3}{2}} \begin{pmatrix} 1 & s_2 h(z) \\ 0 & 1 \end{pmatrix}, \quad \text{if } j = 1. \tag{6.2}
\]

Therefore, we define the local parametrix inside \( U_j \) by

\[
P(z) = \Phi^{*}_{0,j-1}(z) \begin{pmatrix} 1 & \frac{s_{j+1} - s_{j-1}}{2\pi i} \log(z + r_{j-1}x) + \frac{s_{j-1} - s_{j-2}}{2\pi i} \log(z + r_{j-2}x) \\ 0 & 1 \end{pmatrix} V_{j-1}(z)e^{\frac{\pi i}{2} \theta(z) \sigma_3} H_{\bar{r}r_jx}(z) \times V_{j-1}(z) e^{\frac{\pi i}{2} \theta(z) \sigma_3} H_{\bar{r}r_jx}(z), \tag{6.3}
\]

if \( j \in \{2, \ldots, k\} \), and by

\[
P(z) = \Phi^{*}_{0,0}(z) \begin{pmatrix} 1 & \tilde{F}(z;x) \\ 0 & 1 \end{pmatrix} z^{\frac{\sigma_3}{2}} \begin{pmatrix} 1 & s_2 h(z) \\ 0 & 1 \end{pmatrix}, \quad \text{if } j = 1, \tag{6.4}
\]

where \( \tilde{F}(z;x) = -\frac{s_2 - s_1}{2\pi i} \int_{-r_1x}^{r_jx} \frac{\sigma_3}{s-z} ds \). It is direct to check that \( P(z) \) has exactly the same jumps as \( \Phi(z; \bar{r}x, \bar{s}) \) inside \( U_j \). We define

\[
R(z) = \begin{cases} \Phi(z; \bar{r}x, \bar{s}) \Phi(z; \bar{r}^j x, \bar{s}^j)^{-1}, & \text{for } z \in \mathbb{C} \setminus U_j, \\ \Phi(z; \bar{r}x, \bar{s}) P(z)^{-1}, & \text{for } z \in U_j. \end{cases} \tag{6.5}
\]
From the above remarks, it follows that $R(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$ and $R(z)$ has no jump at all inside and outside $U_j$, and has removable singularities at $0, -x_1, \ldots, -x_k$. Let us denote the boundaries of $U_j$ by $\partial U_j$, whose orientation is chosen to be clockwise. For $z \in \partial U_j$, the jumps $J_R$ of $R$ satisfy $J_R(z) = P(z)\Phi(z; x, r^{[j]}, s^{[j]})^{-1}$, or more explicitly

$$J_R(z) = \begin{cases} \Phi^*_{0, j-1}(z) \left( 1 \over s_j - s - \frac{s_{j+1} - s_j}{2\pi i} \log \left( \frac{z + r^{[j]}x}{z + r_{j-1}x} \right) \right) \Phi^*_{0, j-1}(z)^{-1}, & \text{if } j \in \{2, \ldots, k\}, \\ \Phi^*_{0, 0}(z) \begin{pmatrix} 1 & \bar{f}(z; x) \\ 0 & 1 \end{pmatrix} \Phi^*_{0, 0}(z)^{-1}, & \text{if } j = 1. \end{cases} \quad (6.6)$$

In all the cases, we thus have $J_R(z) = I + \mathcal{O}(s_{j+1} - s_j)$ as $s_{j+1} - s_j \to 0$ uniformly for $z \in \partial U_j$. It follows from standard analysis for small norm RH problems that $R$ exists for sufficiently small $s_{j+1} - s_j$ and satisfies

$$R(z) = I + \mathcal{O}(s_{j+1} - s_j), \quad R'(z) = \mathcal{O}(s_{j+1} - s_j), \quad (6.7)$$

uniformly for $z \in \mathbb{C} \setminus \partial U_j$. We now turn to the small $s_{j+1} - s_j$ asymptotics for $q_1, \ldots, q_k$. For convenience, we rewrite (5.19), but we explicit the dependence in $\bar{r}$ and in $\bar{s}$:

$$b_{\ell}(\sqrt{z}; \bar{r}, \bar{s}) = i\sqrt{z} \lim_{z \to -r_{\ell}} (z + r_{\ell}) \left[ \partial_z(\Phi(xz; \bar{r}x, \bar{s})) \Phi^{-1}(xz; \bar{r}x, \bar{s}) \right]_{12}, \quad (6.8)$$

for any $\ell \in \{1, \ldots, k\}$.

If $\ell \neq j$ and $\ell \neq j - 1$, then $-r_{\ell}x \notin U_j$, and in the above limit, from (6.5) we have to use $\Phi(\ell, j, \bar{r}x, \bar{s}) = R(xz)\Phi(xz; \bar{r}^{[j]}x, \bar{s}^{[j]})$, and thus

$$\partial_z(\Phi(xz; \bar{r}x, \bar{s})) \Phi^{-1}(xz; \bar{r}x, \bar{s}) = \partial_z(R(xz))R(xz)^{-1} + R(xz)\partial_z(\Phi(xz; \bar{r}^{[j]}x, \bar{s}^{[j]}))\Phi^{-1}(xz; \bar{r}^{[j]}x, \bar{s}^{[j]})R(xz)^{-1}. \quad (6.9)$$

By (6.7) and (6.8), this implies

$$b_{\ell}(\sqrt{z}; \bar{r}, \bar{s}) = b_{\ell}(\sqrt{z}; \bar{r}^{[j]}x, \bar{s}^{[j]}) + \mathcal{O}(s_{j+1} - s_j), \quad \ell \notin \{j, j - 1\}, \quad (6.10)$$

where $\bar{\ell} = \ell$ if $\ell < j - 1$ and $\bar{\ell} = \ell - 1$ if $\ell > j$. If $\ell \in \{j, j - 1\}$, then $-r_{\ell}x \in U_j$ and we have to use the local parametrix:

$$b_{\ell}(\sqrt{z}; \bar{r}, \bar{s}) = i\sqrt{z} \lim_{z \to -r_{\ell}} (z + r_{\ell}) \left[ \partial_z(R(xz))R(xz)^{-1} \right.$$

$$\left. + R(xz)\partial_z(\Phi(xz))\Phi^{-1}(xz)R^{-1}(xz) \right]_{12}. \quad (6.11)$$

Note from (6.1) and (6.3) that for $j \in \{2, \ldots, k\}$ we have as $z \to -r_{j-1}$ that

$$\frac{[\partial_z(\Phi(xz))\Phi^{-1}(xz)]_{12}}{[\partial_z(\Phi(xz; \bar{r}^{[j]}x, \bar{s}^{[j]}))\Phi^{-1}(xz; \bar{r}^{[j]}x, \bar{s}^{[j]})]_{12}} \sim \left( 1 - \frac{s_{j+1} - s_j}{s_{j+1} - s_{j-1}} \right),$$

and for $j = 1$, from (6.2) and (6.4), we have as $z \to 0$ that

$$\frac{[\partial_z(\Phi(xz))\Phi^{-1}(xz)]_{12}}{[\partial_z(\Phi(xz; \bar{r}^{[j]}x, \bar{s}^{[j]}))\Phi^{-1}(xz; \bar{r}^{[j]}x, \bar{s}^{[j]})]_{12}} \sim \frac{[\Phi^*_{0,0}(0) \begin{pmatrix} 1 & \bar{f}(0) \\ 0 & 1 \end{pmatrix} \sigma_3 \begin{pmatrix} 1 & -\bar{f}(0) \\ 0 & 1 \end{pmatrix} \Phi^*_{0,0}(0)^{-1}]_{12}}{[\Phi^*_{0,0}(0)\sigma_3\Phi^*_{0,0}(0)^{-1}]_{12}}.$$

22
Thus, we also obtain \( b_{j-1}(\sqrt{x}; \bar{r}, \bar{s}) = b_{j-1}(\sqrt{x}; \bar{r}^{[j]}, \bar{s}^{[j]}) + \mathcal{O}(s_{j+1} - s_j) \) as \( s_j \to s_{j+1} \). When \( \ell = j \in \{2, ..., k\} \), from (6.3) and (6.8), we have as \( s_j \to s_{j+1} \)

\[
b_j(\sqrt{x}; \bar{r}, \bar{s}) = i\sqrt{x} \frac{s_j - s_{j-1}}{2\pi i} (\Phi^*_{0,j-1}(-r_j x) )_{11}^2 (1 + \mathcal{O}(s_{j+1} - s_j)) = \mathcal{O}(s_j - s_{j+1}). \tag{6.12}
\]

For \( \ell = j = 1 \), from (6.4), we obtain similarly that

\[
b_1(\sqrt{x}; \bar{r}, \bar{s}) = i\sqrt{x} \frac{s_2 - s_1}{2\pi i} (r_1 x)^\alpha (\Phi^*_{0,0}(-r_1 x) )_{11}^2 (1 + \mathcal{O}(s_2 - s_1)) = \mathcal{O}(s_2 - s_1). \tag{6.13}
\]

This finishes the proof of part 1 of Theorem 1.5.

### 7 Asymptotics for \( r_j \to r_{j-1}, \ j \in \{1, ..., k\} \)

As \( r_j \to r_{j-1} \), the jumps along \((-r_j x, -r_{j-1} x)\) tends to disappear. Thus, we do the exactly the same steepest descent as in the previous section. The computations are very similar and we will give less details. We define \( R \) exactly as in (6.5). By (6.6), we have \( J_R(z) = \mathcal{O}(r_j - r_{j-1}) \) as \( r_j \to r_{j-1} \) uniformly for \( z \in \partial U_j \). Thus, by standard theory for small norm RH problems, \( R \) exists for sufficiently small \( r_j - r_{j-1} \) and satisfies

\[
R(z) = I + \mathcal{O}(r_j - r_{j-1}), \quad R'(z) = \mathcal{O}(r_j - r_{j-1}), \tag{7.1}
\]

uniformly for \( z \in \mathbb{C} \setminus \partial U_j \). From (6.8), (6.9) together with (7.1), we have

\[
b_\ell(\sqrt{x}; \bar{r}, \bar{s}) = b_\ell(\sqrt{x}; \bar{r}^{[j]}, \bar{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}), \quad \text{for any } \ell \in \{1, ..., k\} \setminus \{j, j-1\}. \tag{7.2}
\]

Let \( j \in \{2, ..., k\} \). From (6.3), (6.11) and (7.1), we obtain

\[
b_{j-1}(\sqrt{x}; \bar{r}, \bar{s}) = \frac{s_j - s_{j-1}}{s_{j+1} - s_j} b_{j-1}(\sqrt{x}; \bar{r}^{[j]}, \bar{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}), \tag{7.3}
\]

\[
b_j(\sqrt{x}; \bar{r}, \bar{s}) = \frac{s_j - s_{j-1}}{s_{j+1} - s_j} b_{j-1}(\sqrt{x}; \bar{r}^{[j]}, \bar{s}^{[j]}) + \mathcal{O}(r_j - r_{j-1}). \tag{7.4}
\]

This proves part 2 of Theorem 1.5. If \( j = 1 \), the computations are very similar to (6.13). From (6.4), (6.11) and (7.1), we obtain

\[
b_1(\sqrt{x}; \bar{r}, \bar{s}) = i\sqrt{x} \frac{s_2 - s_1}{2\pi i} (r_1 x)^\alpha (\Phi^*_{0,0}(-r_1 x) )_{11}^2 (1 + \mathcal{O}(r_1)) = \mathcal{O}(r_1^\alpha), \tag{7.5}
\]

which is the part 3 of Theorem 1.5.

### A Bessel model RH problem

In this appendix, we present the well-known Bessel model RH problem, whose solution is denoted \( P_{Be} \) and depends on a parameter \( \alpha > -1 \). At the end of the appendix, we also define \( \widetilde{P}_{Be} \), which a obtained from \( P_{Be} \) by a simple transformation and satisfied a modified version of the Bessel model RH problem.
RH problem for $P_{Be}(z) = P_{Be}(z; \alpha)$

(a) $P_{Be} : \mathbb{C} \setminus \Sigma_B \to \mathbb{C}^{2 \times 2}$ is analytic, where $\Sigma_B$ is shown in Figure 2.

(b) $P_{Be}$ satisfies the jump conditions

\[
P_{Be,+}(z) = P_{Be,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in \mathbb{R}^-,
\]

\[
P_{Be,+}(z) = P_{Be,-}(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \quad z \in e^{\frac{2\pi}{3}} \mathbb{R}^+,
\]

\[
P_{Be,+}(z) = P_{Be,-}(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \quad z \in e^{-\frac{2\pi}{3}} \mathbb{R}^+.
\]

(c) As $z \to \infty$, $z \notin \Sigma_B$, we have

\[
P_{Be}(z) = (I + \mathcal{O}(z^{-1})) \, z^{-\frac{e^{\pi i \alpha}}{3}} \, Ne^{\frac{1}{2} \pi i \sigma_3}.
\]

(d) As $z$ tends to 0, the behaviour of $P_{Be}(z)$ is

\[
P_{Be}(z) = \begin{cases} 
\mathcal{O}(1) z^{\frac{\pi}{3} \sigma_3}, & \text{if } \alpha > 0, \\
\mathcal{O}(z^{-\frac{\pi}{3}}), & \frac{2\pi}{3} < \left| \arg z \right| < \pi,
\end{cases}
\]

\[
P_{Be}(z) = \mathcal{O}(\log z), \quad \text{if } \alpha = 0,
\]

\[
P_{Be}(z) = \mathcal{O}(z^{-\frac{\pi}{3}}), \quad \text{if } \alpha < 0.
\]

Note that by deleting the jumps of $P_{Be}$ around the origin, we obtain the relation

\[
P_{Be}(z) = P_{Be,0}(z) z^{\frac{\pi}{3} \sigma_3} \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z), \quad z \in \mathbb{C} \setminus \Sigma_B,
\]

where $P_{Be,0}$ is an entire function, $h$ is defined in (3.12) and $H_0$ is defined in (3.2). It was shown in [18] that the unique solution to this RH problem is given by

\[
P_{Be}(z) = \begin{pmatrix} \frac{1}{8} & 0 \\ \frac{1}{8} (4\alpha^2 + 3) & 1 \end{pmatrix} \begin{pmatrix} I_\alpha(z^{\frac{1}{2}}) \\ \pi z^{\frac{1}{2}} I_\alpha(z^{\frac{1}{2}}) \end{pmatrix} \begin{pmatrix} I_\alpha(z^{\frac{1}{2}}) \\ \pi z^{\frac{1}{2}} I_\alpha'(z^{\frac{1}{2}}) \end{pmatrix} H_0(z).
\]

Figure 2: The jump contour $\Sigma_B$ for $P_{Be}(z)$.
where $I_\alpha$ and $K_\alpha$ are the modified Bessel functions of the first and second kind.

Note that
\[
\begin{pmatrix}
I_\alpha(z^{\frac{1}{2}}) & \frac{i}{\pi}K_\alpha(z^{\frac{1}{2}}) \\
\pi iz^{\frac{1}{2}}I'_\alpha(z^{\frac{1}{2}}) & -z^{\frac{1}{2}}K'_\alpha(z^{\frac{1}{2}})
\end{pmatrix} \begin{align*}
H_0(z) & \text{ can be rewritten as}
\end{align*}
\]
\[
\begin{cases}
I_\alpha(z^{\frac{1}{2}}) & \frac{i}{\pi}K_\alpha(z^{\frac{1}{2}}) \\
\pi iz^{\frac{1}{2}}I'_\alpha(z^{\frac{1}{2}}) & -z^{\frac{1}{2}}K'_\alpha(z^{\frac{1}{2}})
\end{cases}, & \text{if } |\arg\ z| < \frac{2\pi}{3},
\]
\[
\begin{cases}
\frac{1}{2}H_\alpha^{(1)}((z)^{\frac{1}{2}}) & \frac{1}{2}H_\alpha^{(2)}((z)^{\frac{1}{2}}) \\
\frac{1}{2}\pi z^{\frac{1}{2}}(H_\alpha^{(1)})'((z)^{\frac{1}{2}}) & \frac{1}{2}\pi z^{\frac{1}{2}}(H_\alpha^{(2)})'((z)^{\frac{1}{2}})
\end{cases}, & \text{if } \frac{2\pi}{3} < \arg\ z < \pi,
\]
\[
\begin{cases}
\frac{1}{2}H_\alpha^{(2)}((z)^{\frac{1}{2}}) & -\frac{1}{2}H_\alpha^{(1)}((z)^{\frac{1}{2}}) \\
-\frac{1}{2}\pi z^{\frac{1}{2}}(H_\alpha^{(2)})'((z)^{\frac{1}{2}}) & \frac{1}{2}\pi z^{\frac{1}{2}}(H_\alpha^{(1)})'((z)^{\frac{1}{2}})
\end{cases}, & \text{if } -\pi < \arg\ z < -\frac{2\pi}{3},
\]
\[
(A.6)
\]

where $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are the Hankel functions of the first and second kind respectively.

We will also use a modified version of the above RH problem. We define

\[
\tilde{P}_{\text{Be}}(z) = e^{-\frac{\pi i}{4}\sigma_3}z^{-\frac{\alpha}{2}} \begin{pmatrix}
I_\alpha(z^{\frac{1}{2}}) & \frac{i}{\pi}K_\alpha(z^{\frac{1}{2}}) \\
\pi iz^{\frac{1}{2}}I'_\alpha(z^{\frac{1}{2}}) & -z^{\frac{1}{2}}K'_\alpha(z^{\frac{1}{2}})
\end{pmatrix} H_{-x_k}(z).
\]

From (A.5), we have that $\tilde{P}_{\text{Be}}$ has exactly the same jumps as $P_{\text{Be}}$ on $\Sigma_1 \cup \Sigma_2 \cup (-\infty,-x_k)$. We can compute the jumps of $\tilde{P}_{\text{Be}}$ on $(-x_k,0)$ either from the properties of the Bessel functions, or from the jumps of $P_{\text{Be}}$. We obtain

\[
\tilde{P}_{\text{Be},+}(z) = \tilde{P}_{\text{Be},-}(z) \begin{pmatrix}
ed^{\pi i\alpha} & 0 \\
0 & e^{-\pi i\alpha}
\end{pmatrix}, & z \in (-x_k,0).
\]

(A.8)

Also, from (A.2), as $z \to \infty$, $z \notin \Sigma_\Phi$, we have

\[
\tilde{P}_{\text{Be}}(z) = e^{-\frac{\pi i}{4}}z^{-\frac{\alpha}{2}} \begin{pmatrix}
\frac{1}{8}(4\alpha^2 + 3) & 0 \\
0 & 1
\end{pmatrix} (I + O(z^{-1})) z^{-\frac{\alpha}{2}} N e^{\frac{i}{4}z^2}. \tag{A.9}
\]

\begin{section}{Acknowledgements}

C. Charlier was supported by the European Research Council under the European Union’s Seventh Framework Programme (FP/2007/2013)/ ERC Grant Agreement n. 307074. Both authors also acknowledge support by the Belgian Interuniversity Attraction Pole P07/18. We acknowledge the anonymous referee for a careful reading and for useful remarks.

\begin{section}{References}

[1] M. Atkin, C. Charlier and S. Zohren, On the ratio probability of the smallest eigenvalues in the Laguerre Unitary Ensemble, \textcolor{blue}{arXiv:1611.00631}.

[2] M. Bertola and M. Cafasso, Riemann-Hilbert approach to multi-time processes: the Airy and the Pearcey cases, \textit{Phys. D} \textbf{241} (2012), no. 23–24, 2237–2245.

[3] O. Bohigas and M.P. Pato, Missing levels in correlated spectra, \textit{Phys. Lett. B595} (2004), 171–176.

\end{section}

\end{document}
[4] O. Bohigas and M.P. Pato, Randomly incomplete spectra and intermediate statistics, *Phys. Rev. E* (3) 74 (2006).

[5] C. Charlier and T. Claeys, Thinning and conditioning of the Circular Unitary Ensemble, *Random Matrices Theory Appl.* 6 (2017), 51 pp.

[6] T. Claeys and A. Doeraene, The generating function for the Airy point process and a system of coupled Painlevé II equations, arXiv:1708.03481.

[7] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Amer. Math. Soc. 3 (2000).

[8] P. Deift, A. Its, and X. Zhou, A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, *Ann. Math.* 278 (1997), 149–235.

[9] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* 52 (1999), 1335–1425.

[10] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* 52 (1999), 1491–1552.

[11] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems, *Bull. Amer. Math. Soc. (N.S.)* 26 (1992), 119–123.

[12] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. Math.* 137 (1993), 295–368.

[13] N. Dunford and J. Schwartz, *Linear operators, part II: spectral theory*, Interscience, New York (1963).

[14] P.J. Forrester, The spectrum edge of random matrix ensembles, *Nuclear Phys. B* 402 (1993), 709–728.

[15] P.J. Forrester and T. Nagao, Asymptotic correlations at the spectrum edge of random matrices, *Nuclear Phys. B* 435 (1995), 401–420.

[16] P.J. Forrester and N.S. Witte, The distribution of the first eigenvalue spacing at the hard edge of the Laguerre unitary ensemble, *Kyushu J. Math.* 61 (2007), no. 2, 457–526.

[17] A.B.J. Kuijlaars, A. Martínez-Finkelshtein and F. Wielonsky, Non-intersecting squared Bessel paths and multiple orthogonal polynomials for modified Bessel weights. *Comm. Math. Phys.* 286 (2009), 217–275.

[18] A.B.J. Kuijlaars, K.T.–R. McLaughlin, W. Van Assche and M. Vanlessen, The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1,1]$, *Adv. Math.* 188 (2004), 337–398.
[19] A. Its, A.G. Izergin, V.E. Korepin and N.A. Slavnov, Differential equations for quantum correlation functions, In proceedings of the Conference on Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory, Volume 4, (1990) 1003–1037.

[20] F. Lavancier, J. Moller and E. Rubak, Determinantal point process models and statistical inference: Extended version, J. Royal Stat. Soc.: Series B 77 (2015), no. 4, 853–877.

[21] M.L. Mehta, Random matrices, Third Edition, Pure and Applied Mathematics Series 142, Elsevier Academic Press, 2004.

[22] F.W.J. Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller and B.V. Saunders, NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.13 of 2016-09-16.

[23] A. Soshnikov, Determinantal random point fields, Russian Math. Surveys 55 (2000), no. 5, 923–975.

[24] C.A. Tracy and H. Widom, Level spacing distributions and the Bessel kernel. Comm. Math. Phys. 161 (1994), no. 2, 289–309.

[25] M. Vanlessen, Strong asymptotics of Laguerre-type orthogonal polynomials and applications in random matrix theory, Constr. Approx. 25 (2007), 125–175.

[26] S.–X. Xu and D. Dai, Tracy-Widom distributions in critical unitary random matrix ensembles and the coupled Painlevé II system, arXiv:1708.06113.