Integrating Schur polynomials using iterated residues at infinity

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Abstract

This is a note in which I show examples of computations done using the formulas obtained in [MZ], which express push-forwards in equivariant cohomology as iterated residues at infinity. In particular, I show how to obtain some well-known results ([P-R]) on integrals of Schur polynomials over the Lagrangian Grassmannian $LG(n)$ and the orthogonal Grassmannian $OG(n)$.

1 Introduction

Let $LG(n)$ be the Lagrangian grassmannian, parametrizing the maximal isotropic subspaces of $\mathbb{C}^{2n}$ equipped with the standard symplectic form.

The fixed points of the torus action can be parametrized using the subsets $I \subseteq \{1, \ldots, n\}$:

$$p_I = \text{Span}\{q_i, p_j : i \in I, j \notin I\},$$

where $q_1, \ldots, q_n, p_n, \ldots, p_1$ are the coordinates on $\mathbb{C}^{2n}$.

Weights of the torus action on the tangent space are equal to $\{\pm t_i \pm t_j : 1 \leq i < j \leq n\} \cup \{\pm 2t_i : i = 1, \ldots, n\}$, where the $+$ sign appears whenever $i, j \in I$.

In this case, the Atiyah-Bott-Berline-Vergne formula [A-B] gives:

$$\int_{LG(n)} \phi(\mathcal{R}) = \sum_I V(t_i, -t_j : i \in I, j \notin I) \prod_{i,j=1}^n (\pm 2t_i)(\pm t_i \pm t_j),$$

where $\phi(\mathcal{R})$ is a characteristic class of the tautological bundle, which at the fixed points of the action is given by a polynomial $V$. The right-hand side can be expressed as a residue at infinity as follows:
Formula 1.
\[ \int_{LG(n)} \phi(\mathcal{R}) = Res_{z=\infty} V(z_1, ..., z_n) \prod_{i,j}(z_j - z_i) \prod_{i=1}^n (t_i - z_i) (t_i + z_i) \prod_{i,j}(t_i + t_j)(t_j - t_i). \]

The proof can be found in [MZ]. I will use this formula to prove the following result [P-R]: Let \( \lambda \) be a partition of length \( < n + 1 \) and let \( \rho(n) \) be the partition \( (n, n-1, \ldots, 1) \). Let \( s_\lambda \) denote the Schur polynomial corresponding to the partition \( \lambda \). Furthermore, let \( \omega : LG(n) \to pt \) and let \( \omega^*: H^*_T(LG(n)) \to H^*_T(pt) \) be the integration over the fiber of \( \omega \).

Let \( Q \) be the tautological quotient rank \( n \) bundle on \( LG(n) \). Then, the Schur polynomial \( s_\lambda(Q) \) has a nonzero image under \( \omega^* \) only if \( \lambda = 2\mu + \rho(n) \) for some partition \( \mu \). In terms of Young diagrams, this means that the diagram corresponding to \( \lambda \) contains the diagram of the standard partition \( \rho(n) \), and has additionally an even number of boxes added in each row. In this case, the image is:
\[ \omega^* s_\lambda(Q) = s_\lambda^{[2]}(\mathbb{C}^{2n}), \]
where \( s_\lambda^{[2]} \) is obtained from \( s_\lambda \) by replacing each \( e_i \) in the presentation of \( s_\lambda \) as a polynomial in elementary symmetric functions \( e_i \), by \( (-1)^i c_2(\mathbb{C}^{2n}) \). In other words, \( s_\lambda^{[2]}(t_1, \ldots, t_n) = s_\lambda(-t_1^2, \ldots, -t_n^2) \).

2 Push-forward of Schur polynomials

2.1 An example

Before we prove it, let us look at an example for \( n = 2 \). Then, the formula 1 has the form:
\[ \int_{LG(2)} s_{(\lambda_1, \lambda_2)}(\mathcal{R}) = Res_{z_1=z_2=\infty} \frac{s_{(\lambda_1, \lambda_2)}(z_1, z_2)(z_2 - z_1)}{(t_1 - z_1)(t_1 + z_1)(t_2 - z_2)(t_2 + z_2)(t_1 + t_2)(t_2 - t_1)} = \]
\[ = \frac{1}{t_2^2 - t_1^2} Res_{z_1=z_2=\infty} \det \begin{bmatrix} z_1^{\lambda_1+1} & z_2^{\lambda_1+1} \\ z_1^{\lambda_2} & z_2^{\lambda_2} \end{bmatrix} (z_2 - z_1) = \frac{1}{t_2^2 - t_1^2}. \]

To compute the residue, we use the fact, that \( Res_{z=\infty} f(z) = -\frac{1}{z} Res_{z=0} f(z) \), so

2
\[
\star = \frac{1}{z_1 z_2} Res_{z_1 = z_2 = 0} \frac{\det \begin{bmatrix}
    z_1^{-(\lambda_1+1)} & z_2^{-(\lambda_1+1)} \\
    z_1^{-\lambda_2} & z_2^{-\lambda_2}
\end{bmatrix}}{(t_1 - z_1^{-1})(t_1 + z_1^{-1})(t_2 - z_1^{-1})(t_2 + z_1^{-1})} = Res_{z_1 = z_2 = 0} \frac{\det \begin{bmatrix}
    z_1^{-(\lambda_1+1)} & z_2^{-(\lambda_1+1)} \\
    z_1^{-\lambda_2} & z_2^{-\lambda_2}
\end{bmatrix}}{(t_1 z_1 - 1)(t_1 z_1 + 1)(t_2 z_2 - 1)(t_2 z_2 + 1)}
\]

Note that computing a residue at zero of a function is taking the coefficient corresponding to \(z^{-1}\) in the Laurent series expansion. Thus, in order to compute \(\star\) we need to expand in power series the function

\[
\frac{1}{(t_1 z_1 - 1)(t_1 z_1 + 1)(t_2 z_2 - 1)(t_2 z_2 + 1)} = \frac{1}{((t_1 z_1)^2 - 1)((t_2 z_2)^2 - 1)}.
\]

multiply by the determinant coming from the Schur polynomial, and take the coefficient corresponding to \(z_1^{-1} z_2^{-1}\).

Proceeding in the steps described above, we have:

\[
\frac{1}{((t_1 z_1)^2 - 1)((t_2 z_2)^2 - 1)} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_1^{2i} t_2^{2j} z_1^i z_2^j = ***.
\]

Now we expand the determinant:

\[
\det \begin{bmatrix}
    z_1^{-(\lambda_1+1)} & z_2^{-(\lambda_1+1)} \\
    z_1^{-\lambda_2} & z_2^{-\lambda_2}
\end{bmatrix} = z_1^{-(\lambda_1+1)} z_2^{-\lambda_2} - z_2^{-(\lambda_1+1)} z_1^{-\lambda_2} = A + B
\]

From this we can easily see the first claim of the theorem:

The coefficients in the series *** are even, so if \(\lambda_1 + 1\) is odd or \(\lambda_2\) is even, then the coefficient at \(z_1^{-1} z_2^{-1}\) in the result is zero, because it is zero both in \(A \cdot ***\) and in \(B \cdot ***\). Also, if \(\lambda_1 < 2\) or \(\lambda_2 < 1\) then the result is zero, since \((A + B) \cdot ***\) is nonsingular at 0. This shows that the expression \(\star\) can only be nonzero if \(\lambda_1\) is even and greater than 2, \(\lambda_2\) is odd and greater than 1, so \(\lambda = (2, 1) + 2\mu\).

In this case, the \(\star\) splits into a sum of two contributions coming from \(A\) and \(B\):

**Contribution from** \(A\):

\[
X_A = A \cdot *** = z_1^{-(\lambda_1+1)} z_2^{-\lambda_2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_1^{2i} t_2^{2j} z_1^i z_2^j = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_1^{2i} t_2^{2j} z_1^{2i-(\lambda_1+1)} z_2^{-\lambda_2}
\]
The coefficient at $z_1^{-1}z_2^{-1}$ is equal to $t_1^{\lambda_1}t_2^{\lambda_2-1}$.

Similarly, the contribution from $B$ is:

$$X_B = B \ast \ast = z_2^{-(\lambda_1+1)}z_1^{-\lambda_2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_1^{2i}t_2^{2j}z_1^{-2i}z_2^{-2j} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_1^{2i}t_2^{2j}z_1^{-2i-\lambda_2}z_2^{2j-(\lambda_1+1)}$$

The coefficient at $z_1^{-1}z_2^{-1}$ is equal to $t_1^{\lambda_2-1}t_2^{\lambda_1}$.

Finally,

$$\ast = X_A - X_B = t_1^{\lambda_1}t_2^{\lambda_2-1} - t_1^{\lambda_2-1}t_2^{\lambda_1} = \det \begin{bmatrix} t_1^{\lambda_1} & t_2^{\lambda_1} \\ t_1^{\lambda_2-1} & t_2^{\lambda_2-1} \end{bmatrix},$$

and so the push-forward is

$$\omega_s \lambda(\mathcal{R}) = \frac{1}{t_2^2 - t_1^2} \ast = \frac{\det \begin{bmatrix} t_1^{\lambda_1} & t_2^{\lambda_1} \\ t_1^{\lambda_2-1} & t_2^{\lambda_2-1} \end{bmatrix}}{t_2^2 - t_1^2} = \frac{\det \begin{bmatrix} (t_2^2)^{\lambda_1/2} & (t_2^2)^{\lambda_1/2} \\ (t_2^2)^{(\lambda_2-1)/2} & (t_2^2)^{(\lambda_2-1)/2} \end{bmatrix}}{t_2^2 - t_1^2} = \frac{\det \begin{bmatrix} (t_1^2)^{\mu_1+1} & (t_2^2)^{\mu_1+1} \\ (t_1^2)^{\mu_2} & (t_2^2)^{\mu_2} \end{bmatrix}}{t_2^2 - t_1^2} = s_\mu(t_1^2, t_2^2),$$

which is exactly what we wanted.

### 2.2 A general procedure

The case of $n = 2$ can be easily generalized to work for any $n$. We need to proceed in exactly the same steps:

- Use the iterated residue formula.
- Change variables to compute the residue at 0 and simplify the expression.
- Expand the denominator into a power series.
- Expand the determinant coming from the Schur polynomial, compute the contributions from the summands, and add them up.
Let’s follow this procedure.

\[
\int_{L^2(n)} s(\lambda_1, \ldots, \lambda_n)(\mathcal{R}) = Res_{z_1 = \ldots = z_n = \infty} \frac{s_\lambda(z_1, \ldots, z_n) \prod_{i<j} (z_j - z_i)}{\prod_{i=1}^n (t_i - z_i)(t_i + z_i) \prod_{i<j} (t_i + t_j)(t_j - t_i)} = \frac{1}{\prod_{i<j}(t_j^2 - t_i^2)} \cdot \star
\]

Now we change variables, to compute the residue at 0:

\[
\star = (-1)^n \frac{1}{z_1^2 \cdots z_n^2} Res_{z_1 = \ldots = z_n = 0} \frac{1}{z_1 \cdots z_n \prod_{i=1}^n (t_i z_i - 1)(t_i z_i + 1)}
\]

The series expansion of the denominator is:

\[
\frac{1}{((t_1 z_1)^2 - 1) \cdots ((t_n z_n)^2 - 1)} = \sum_{i_1 = 0}^{\infty} \sum_{i_2 = 0}^{\infty} \cdots \frac{t_1^{2i_1} \cdots t_n^{2i_n}}{z_1^{2i_1} \cdots z_n^{2i_n}} = \star \star.
\]

Now we expand the determinant as the sum over all permutations:

\[
\det \begin{bmatrix}
  z_1^{-(\lambda_1 + n - 1)} & z_2^{-(\lambda_1 + n - 1)} & \cdots & z_n^{-(\lambda_1 + n - 1)} \\
  z_1^{-(\lambda_2 + n - 2)} & z_2^{-(\lambda_2 + n - 2)} & \cdots & z_n^{-(\lambda_2 + n - 2)} \\
  \vdots & \vdots & \ddots & \vdots \\
  z_1^{-\lambda_n} & z_2^{-\lambda_n} & \cdots & z_n^{-\lambda_n}
\end{bmatrix} = \sum_{\sigma \in \Sigma_n} (-1)^{sgn(\sigma)} z_{\sigma(1)}^{-(\lambda_1 + 1)} z_{\sigma(2)}^{-(\lambda_2 + 2)} \cdots z_{\sigma(n)}^{-(\lambda_n + n)}
\]

and compute the contribution coming from one summand:

\[
(-1)^{sgn(\sigma)} z_{\sigma(1)}^{-(\lambda_1 + n - 1)} z_{\sigma(2)}^{-(\lambda_2 + n - 2)} \cdots z_{\sigma(n)}^{-(\lambda_n)} \cdot \star \star = (-1)^{sgn(\sigma)} z_{\sigma(1)}^{-(\lambda_1 + n - 1)} z_{\sigma(2)}^{-(\lambda_2 + n - 2)} \cdots z_{\sigma(n)}^{-(\lambda_n)} \cdot \sum_{i_1, \ldots, i_n = 0}^{\infty} t_1^{2i_1} \cdots t_n^{2i_n} z_1^{2i_1} \cdots z_n^{2i_n}
\]
\[ = (-1)^{\text{sgn}(\sigma)} z_{\sigma(1)}^{-\lambda_1} z_{\sigma(2)}^{-\lambda_2} \cdots z_{\sigma(n)}^{-\lambda_n} \sum_{i_{\sigma(1)}, \ldots, i_{\sigma(n)} = 0}^\infty t_{\sigma(1)}^{2i_{\sigma(1)}} \cdots t_{\sigma(n)}^{2i_{\sigma(n)}} z_{\sigma(1)}^{2i_{\sigma(1)}} \cdots z_{\sigma(n)}^{2i_{\sigma(n)}} \]

The coefficient at \( z_1^{-1} \cdots z_n^{-1} \) is \( t_{\sigma(1)}^{2i_{\sigma(1)}} \cdots t_{\sigma(n)}^{2i_{\sigma(n)}} \), where \( i_{\sigma(j)} \) must satisfy:

\[ 2i_{\sigma(j)} - (\lambda_j + n - j) = -1 \text{ for } j = 1, \ldots, n. \]

The solutions are \( 2i_{\sigma(j)} - (\lambda_j + n - j) = -1 \) for \( j = 1, \ldots, n \). The coefficient we are looking for is always zero, unless \( \lambda_j + n - j - 1 \) is even, i.e., \( \lambda_j = 2k_j - n + j + 1 = 2k_j - n + j + 1 + (n + j - 1) - (n + j - 1) = 2k_j - 2n + 2 + \rho(n)j \), so \( \lambda \) is of the form \( \rho(n) + 2\mu \) for some partition \( \mu \).

Finally, if \( \lambda = \rho(n) + 2\mu \), the sum of all contributions is

\[ * = \sum_{\sigma \in \Sigma_n} (-1)^{\text{sgn}(\sigma)} t_{\sigma(1)}^{2i_{\sigma(1)}} \cdots t_{\sigma(n)}^{2i_{\sigma(n)}} = \sum_{\sigma \in \Sigma_n} (-1)^{\text{sgn}(\sigma)} t_{\sigma(1)}^{\lambda_1 - n - 2} t_{\sigma(2)}^{\lambda_2 - n - 3} \cdots t_{\sigma(n)}^{\lambda_n - 1} = \]

\[ = \det \begin{bmatrix} t_1^{\lambda_1 - n - 2} & t_2^{\lambda_1 - n - 2} & \cdots & t_n^{\lambda_1 - n - 2} \\ t_1^{\lambda_2 - n - 3} & t_2^{\lambda_2 - n - 3} & \cdots & t_n^{\lambda_2 - n - 3} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\lambda_n - 1} & t_2^{\lambda_n - 1} & \cdots & t_n^{\lambda_n - 1} \end{bmatrix} \]

The resulting expression for the push-forward is then:

\[ \frac{1}{\prod_{i<j}(t_j^2 - t_i^2)} \det \begin{bmatrix} (t_1^2)^{\lambda_1 - n - 2}/2 & (t_2^2)^{\lambda_1 - n - 2}/2 & \cdots & (t_n^2)^{\lambda_1 - n - 2}/2 \\ (t_1^2)^{\lambda_2 - n - 3}/2 & (t_2^2)^{\lambda_2 - n - 3}/2 & \cdots & (t_n^2)^{\lambda_2 - n - 3}/2 \\ \vdots & \vdots & \ddots & \vdots \\ (t_1^2)^{\lambda_n - 1}/2 & (t_2^2)^{\lambda_n - 1}/2 & \cdots & (t_n^2)^{\lambda_n - 1}/2 \end{bmatrix} \]

\[ = \det \begin{bmatrix} (t_1^2)^{\mu_1 + n - 1} & (t_2^2)^{\mu_1 + n - 1} & \cdots & (t_n^2)^{\mu_1 + n - 1} \\ (t_1^2)^{\mu_2 + n - 2} & (t_2^2)^{\mu_2 + n - 2} & \cdots & (t_n^2)^{\mu_2 + n - 2} \\ \vdots & \vdots & \ddots & \vdots \\ (t_1^2)^{\mu_n} & (t_2^2)^{\mu_n} & \cdots & (t_n^2)^{\mu_n} \end{bmatrix} \]

\[ = \frac{1}{\prod_{i<j}(t_j^2 - t_i^2)} = s_\mu(t_1^2, \ldots, t_n^2), \]

which is the expression we were supposed to prove.

Analogous results can be obtained in the same manner for the orthogonal grassmannians, for example for \( OG(n, 2n + 1) \) one can prove the following formula [P-R] (using the notation as for the Lagrangian case):

\[ \omega_* s_\lambda(Q) = 2^n s_\mu^{[2]}(\mathbb{C}^{2n}). \]
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