Equality Is Not Equity: Proportional Fairness in Federated Learning

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Abstract
Ensuring fairness of machine learning (ML) algorithms is becoming an increasingly important mission for ML service providers. This is even more critical and challenging in the federated learning (FL) scenario, given a large number of diverse participating clients. Simply mandating equality across clients could lead to many undesirable consequences, potentially discouraging high-performing clients and resulting in sub-optimal overall performance. In order to achieve better equity rather than equality, in this work, we introduce and study proportional fairness (PF) in FL, which has a deep connection with game theory. By viewing FL from a cooperative game perspective, where the players (clients) collaboratively learn a good model, we formulate PF as Nash bargaining solutions. Based on this concept, we propose PropFair, a novel and easy-to-implement algorithm for effectively finding PF solutions, and we prove its convergence properties. We illustrate through experiments that PropFair consistently improves the worst-case and the overall performances simultaneously over state-of-the-art fair FL algorithms for a wide array of vision and language datasets, thus achieving better equity.

1 Introduction
Federated learning (FL) has attracted an intensive amount of attention since its invention (McMahan et al., 2017). It lies in the domain of cooperative AI: different clients collaborate to learn some shared knowledge. At the same time, it addresses the concern of privacy: users’ data will not be sent to a central server and the training process is done locally on each client. In the classic federated average (FedAvg) algorithm (McMahan et al., 2017), the role of the server is simply to aggregate the local parameters of different clients, and redistribute their average. Variants of FedAvg have been proposed since then, such as FedProx (Li et al., 2020a) and FedOpt (Reddi et al., 2020). For a comprehensive survey on federated optimization, see Wang et al. (2021) and Kairouz et al. (2021).

Despite recent fast development, fairness still remains an important challenge for FL. Due to model / data / hardware heterogeneity, different clients typically exhibit quite diverse performances. Some efforts have been made to address this challenge. For instance, Mohri et al. (2019) developed a framework named Agnostic Federated Learning (AFL), by optimizing the worst-case client loss function through minimax optimization. Inspired by \(\alpha\)-fairness (Lan et al., 2010) that is widely used in wireless networks, Li et al. (2020b) designed \(q\)-Fair Federated Learning (\(q\)-FFL). Li et al. (2021) further proposed to achieve fairness through personalizing the FL model for each client.

In the literature of FL, fairness is mostly evaluated through the standard deviation of performances among clients (e.g. Li et al., 2020b, 2021). According to this practice, an algorithm is deemed fairer if it exhibits smaller standard deviation among clients, which is known as equal treatment in social welfare. However, it is well-known that equality and equity can be vastly different (Bronfenbrenner, 1973). It is the latter that tends
Table 1: Comparison between the uniformity of absolute performance and relative performance.

| value | $f_1$ | $f_2$ | $1 - \frac{f_1^*}{f_1}$ | $1 - \frac{f_2^*}{f_2}$ |
|-------|-------|-------|--------------------------|--------------------------|
| $\bar{x}$ | $-7/16$ | 1.10 | 94.3% | 8.7% |
| $\bar{x}$ | $4/5$ | 0.08 | 24.2% | 24.2% |

to reflect different needs from different clients, and hence is often preferred in many practical scenarios. Let us illustrate this point with a simple example:

**Example 1.1 (Equality is not equity, Table 1).** Suppose we have two clients with the loss functions

$$f_1(x) = \frac{1}{2} (x - 1)^2 + \frac{1}{16} \quad \text{and} \quad f_2(x) = \frac{1}{2} x^2 + 1,$$

respectively. Demanding equal treatment would require $f_1(\bar{x}) = f_2(\bar{x})$, leading to $\bar{x} = -\frac{7}{16}$ for both clients. However, from Table 1, the optimal loss of client 1 (i.e., $f_1^* = \frac{1}{16}$) is 94.3% smaller than the equal solution $\bar{x}$. On the other hand, for client 2, the optimal loss (i.e., $f_2^* = 1$) is only 8.7% smaller than the equal solution $\bar{x}$. In other words, to achieve equality, client 1 has to sacrifice its performance significantly. It might be fairer to consider the alternative:

$$1 - \frac{f_1^*}{f_1(\bar{x})} = 1 - \frac{f_2^*}{f_2(\bar{x})},$$

where one of the solutions is $\bar{x} = 0.8$. In this case, both clients sacrifice equally ($\approx 24.2\%$), indicating a more equitable treatment toward both of them.

The above example reveals that, if different clients have different loss (utility) functions and optimal values, simply mandating equality can actually harm the better-performing clients disproportionately. Thus, we argue that a more suitable criterion would be the relative loss (utility), rather than the absolute one. This motivates us to apply the concept of proportional fairness (PF) in FL.

Yet, analyzing and achieving PF in FL is not an easy task. To accomplish this mission, we make the following contributions:

- We unify existing fair FL algorithms with Kolmogorov’s generalized mean and study these algorithms (including our newly proposed one) with a dual view. This allows us to interpret the fairness requirement of each algorithm in the same framework.
- Within this unified framework, we introduce proportional fairness for FL as a new fairness rule, which also has deep roots in cooperative game theory.
- Based on the Nash bargaining solutions for cooperative games, we propose a new algorithm called *PropFair* to achieve PF in FL, and we prove its convergence properties under standard assumptions.
- Experiments on vision and language datasets confirm that PropFair achieves better worst-case client performance than existing fair FL algorithms, with no degradations and often improvements in the overall performance. Thus, it achieves enhanced equity.

**Notations.** We use $f_i := \mathbb{E}_{(x,y) \sim D_i} [\ell(\cdot, (x,y))]$ for the loss function of client $i$ with distribution $D_i$, with $\ell$ the cross entropy loss. We denote $F = \frac{1}{n} \sum_i f_i$ with $n$ the number of clients. We use $u_i$ for the (positive) utility of client $i$. The notation $\| \cdot \|_p$ means the $\ell_p$ norm.
2 Unifying Fair FL Algorithms

Fairness has been a perennial and venerable topic for social welfare (Fleurbaey & Maniquet, 2011). Since its inception, federated learning incorporates fairness as an integral part (Kairouz et al., 2021). This is because in FL a number of clients work collaboratively, despite of different objectives and resources. How to encourage the participation of various clients, without projecting an impression of benefiting only a (selected) few, is utterly important for the success of FL and remains a significant challenge in practice.

Formally, we have \( n \) clients in FL, indexed by \( i = 1, \ldots, n \), each aiming to minimize a loss function \( f_i = f_i(\theta) \), based on one’s proprietary data. A centralized server coordinates the joint effort by performing some aggregation to update the common \( \theta \) parameter. This can be treated as multi-objective optimization (MOO) (Hu et al., 2020), which has been intensively studied in the field of operation research (Jahn et al., 2009). Our goal is to minimize a series of (non-negative) loss functions \( f_1, f_2, \ldots, f_n \) based on their best trade-offs. In the following, we show that many existing algorithms in FL can be treated in a surprisingly unified way, and it also sets up the stage for our main development.

2.1 Unifying through the generalized mean

One can solve the trade-off among clients through scalarizing the nonnegative vector losses \( f = (f_1, \ldots, f_n) \) with the generalized mean (Kolmogorov, 1930):

\[
A_s(f) := s^{-1} \left( \frac{1}{n} \sum_{i=1}^n s(f_i) \right),
\]

where \( s : I \rightarrow I \) is a (strictly) monotonic and continuous function on an interval \( I \). Kolmogorov (1930) also provided a complete characterization of such generalized means. For instance, the generalized mean is clearly permutation invariant and increasing in each \( f_i \). The generalized mean provides a convenient aggregation strategy for the different goals of each client in FL. Indeed, many existing FL algorithms can be cast as special cases of this perspective.

**FedAvg (McMahan et al. 2017).** Simply setting \( s(t) = t \) results in the usual arithmetic mean:

\[
\min_{\theta} \frac{1}{n} \|f(\theta)\|_1 \equiv \min_{\theta} \sum_{i=1}^n f_i(\theta).
\]

This corresponds to the utilitarian (Maskin 1978) point of view, where we maximize the total performance, without much regard to the individual wellness of each client.

**q-FFL (Li et al. 2020b).** Setting \( s(t) = t^{q+1} \) for some \( q \geq 0 \) yields the usual power mean:

\[
\min_{\theta} n^{-\frac{1}{q+1}} \|f(\theta)\|_{q+1} \equiv \min_{\theta} \sum_{i=1}^n f_i^{q+1}(\theta).
\]

This corresponds to \( \alpha \)-fairness (Lan et al. 2010). Obviously, with \( q = 0 \), q-FFL reduces to FedAvg, but with a larger \( q \), q-FFL emphasizes more on clients who suffer a larger loss.

**AFL (Mohri et al. 2019).** Letting \( q \rightarrow \infty \) in q-FFL, we recover agnostic FL:

\[
\min_{\theta} \|f(\theta)\|_{\infty} \equiv \min_{\theta} \max_{i=1,\ldots,n} f_i(\theta),
\]

This corresponds to the egalitarian (Rawls 1974) point of view and aims at maximizing the performance of the most disadvantaged client, while potentially putting all other clients on hold.

\(^1\)It is certainly possible to allow each client to have its own version of parameters. For simplicity, we do not consider personalization in this work.
2.2 Dual view of fair FL

We can also supply a dual view of existing FL algorithms that is perhaps more revealing. Concretely, let \( s \) be (strictly) increasing, convex and thrice differentiable. Then, the generalized mean function \( A_s \) is convex iff \(-s' \) is convex (Theorem 1, Ben-Tal & Teboulle 1986). Applying conjugation we obtain the equivalent problem:

\[
\min_\theta A_s(f(\theta)) \equiv \min_\theta \max_{\lambda \geq 0} \sum_i \lambda_i f_i(\theta) - A_s^*(\lambda),
\]

where \( A_s^*(\lambda) := \sup_f \lambda^T f - A_s(f) \) is the Fenchel conjugate of \( A_s \). We recall that \( f_i \geq 0 \) is assumed throughout and the non-negative constraint \( \lambda \geq 0 \) is due to the monotonicity of \( A_s \). In particular, for \( s(t) = t^{q+1} \), we obtain the conjugate function corresponding to \( q \)-FFL:

\[
A_s^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \geq 0 \text{ and } \|\lambda\|_{(q+1)/q} \leq n^{-\frac{1}{q+1}} \\ \infty, & \text{otherwise} \end{cases}
\]

For convex losses, strong duality allows us to swap the minimization and maximization:

\[
\max_\lambda \min_\theta \sum_i \lambda_i f_i(\theta) - A_s^*(\lambda).
\]

Thus, if we are given the dual variable \( \lambda \), FL methods based on a generalized mean with increasing and convex \( s \) (e.g. \( q \)-FFL and AFL) are essentially FedAvg with a fine-tuned weighting \( \lambda \). Of course, in practice, obtaining \( \lambda \) is by no means easy and specialized algorithms need to be developed (e.g., Mohri et al. 2019).

Interestingly, equal treatment (a.k.a. demographic parity) can also be reformulated in a similar way:

\[
\min_{(\theta, t)} f_i(\theta) = t \equiv \min_\theta \max_{\lambda} \sum_i \lambda_i f_i(\theta).
\]

Unlike AFL (or \( q \)-FFL), here \( \lambda \) needs not be non-negative. For instance, the \( \lambda \) for the equal treatment solution \( \pi \) in Example 1.1 is given as \( \lambda = (-\frac{7}{10}, \frac{2}{10}) \). There is also a potential downside: if the ranges of individual losses do not overlap, then the minimum will be undefined (i.e., \(-\infty\)). In fact, this is (one of) the reason(s) why egalitarian systems usually constrain \( \lambda \) to be nonnegative.

So far, we have showed that many popular fair FL algorithms, including equal treatment, can be rewritten as the following minimax problem:

\[
\min_\theta \max_{\lambda \in \Lambda} \sum_i \lambda_i f_i(\theta),
\]

where the dual set \( \Lambda \) can vary. We summarize the above discussion in Table 2 along with the new result (see Definition 3.1 and (4.2)) that we develop in the following sections.

3 Proportional Fairness (PF)

In this subsection we propose the notion of PF, which optimizes the total relative utilities. For example, in FL, the utility could be the test accuracy of each client model, or the negated training loss (plus a constant). As we will see, for convex utility sets, PF is equivalent to the Nash bargaining solution in game theory (Nash 1950; Harsanyi, 1956).

Although maximizing utilities is equivalent to minimizing losses in §2, the utility point of view brings us new insights, and draws connections to the game theory. We will come back to the loss function view in §4.1.

Let us first introduce the notion of proportional fairness:
| Fairness | FL algorithm | Objective | Requirement of $\lambda$ | Reference |
|----------|--------------|-----------|--------------------------|-----------|
| Utilitarian | FedAvg | $\sum_i f_i$ | $\lambda_1 = \cdots = \lambda_n = 1/n$ | McMahan et al. (2017) |
| Egalitarian | Agnostic FL | $\max_i f_i$ | $\lambda \geq 0, \frac{1^\top \lambda}{\|\lambda\|_p} = \frac{1}{n}$ | Mohri et al. (2019) |
| $\alpha$-fairness | $q$-FedAvg | $\sum_i f_i^{q+1}$ | $\lambda \geq 0, \|\lambda\|_{(q+1)/q} \leq \frac{1}{n^{1/q}}$ | Li et al. (2020b) |

**Table 2:** Different fairness concepts and their corresponding FL algorithms. $f_i$ is the loss function for the $i^{th}$ client. See also §2 and (4.2).

Definition 3.1 (proportional fairness, PF, e.g., Kelly et al. 1998). Suppose $U \subseteq \mathbb{R}^n_{++}$. An $n$-dimensional utility vector $u^* \in U$ is called proportionally fair if for all $u \in U$:

$$\sum_{i=1}^n \frac{u_i - u_i^*}{u_i^*} \leq 0.$$  \hspace{1cm} (3.1)

Intuitively, $\frac{u_i - u_i^*}{u_i^*}$ is the relative utility gain for player $i$ given its utility switched from $u_i^*$ to $u_i$. PF simply states that at the solution $u^*$, the total relative utility cannot be improved. To further motivate why relativity is essential, let us mention the celebrated St. Petersburg paradox:

Example 3.2 (importance of relative scale). Suppose we flip a fair coin and get paid $2^k$ dollars if $k$ consecutive heads show up before the first tail. What is a fair ticket price $p$ for one to play this game? If we compute the expected monetary value of the game, we will arrive at obvious absurdity:

$$p = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \infty.$$  \hspace{1cm} (3.2)

The great insight that Daniel Bernoulli realized in 1738 is that monetary value only matters in relative scale, i.e., increasing one’s wealth in fixed amount matters less and less as one’s wealth accumulates, while doubling one’s wealth perhaps matters more equally irrespective of one’s absolute wealth. Thus, instead of the expected utility, Bernoulli proposed a logarithmic transformation of the monetary value:

$$p = \sum_{k=1}^{\infty} \log(2^k) \cdot \frac{1}{2^k},$$  \hspace{1cm} (3.3)

which is indeed finite. We note that Bernoulli’s insight is also instrumental in von Neumann and Morgenstern’s later development of their expected utility theory.

3.1 Nash bargaining solution and proportional fairness

An $n$-player bargaining game is a cooperative game where all players can improve their utilities (Morgenstern & von Neumann 1953) simultaneously through cooperation. Since the seminal work of Nash (1950), the bargaining problem has been an important topic in game theory (Kalai 1977; Roth 1979; Muthoo 1999). Recall that $u_i$ is the utility for player $i$. The Nash bargaining solution is defined as:

Definition 3.3 (Nash bargaining solution as fairness). The Nash bargaining (NB) solution for $n$ players solves the following optimization problem:

$$\max_{u \in \mathcal{U}} \Pi(u), \text{ where } \Pi(u) := \prod_{i=1}^n u_i,$$  \hspace{1cm} (3.4)

and $\mathcal{U} \subseteq \mathbb{R}^n_{++}$ is a non-empty convex compact set.
Figure 1: Figure inspired by Nash (1950). $\mathcal{U}$: the feasible set of utilities. Blue line: maximizers of the total utility, on which the Nash bargaining solution $u^*$ stands out as the most equitable.

In cooperative games, we assume that there is a choice such that all players can benefit from cooperation, and thus the set $\mathcal{U}$ is non-empty. The objective in Def. 3.3 is also known as the Nash product. The compactness assumption on $\mathcal{U}$ guarantees the existence of a Nash bargaining solution while convexity of $\mathcal{U}$ also ensures its uniqueness. Axiomatic characterizations of the Nash bargaining solution are well-known, for instance by the following four axioms: Pareto optimality, symmetry, scale equivariance and monotonicity (e.g., Maschler et al., 2020, Theorem 16.35).

Applying concavity we can upper bound the Nash product:

\[
\frac{1}{n} \log \Pi(u) \leq \log \sum_i u_i, \tag{3.5}
\]

with equality holds iff $u_1 = \cdots = u_n$. This assures us that if the Nash product is large then the total utility should also be large. On the other hand, if any of the utilities is close to 0, then the Nash product would also be negligible. Therefore, maximizing the Nash product achieves equity and efficiency simultaneously. Figure 1 gives an illustration of the Nash bargaining solution. Among all the solutions that maximize the total utility, the Nash bargaining solution achieves equal utility for the two players, and thus achieving equity (and equality in this case). In fact, the Nash bargaining solution is equivalent to PF for convex utility sets:

**Proposition 3.4 (equivalence, e.g. [Kelly et al. 1998; Boche & Schubert 2009]).** For any convex compact set $\mathcal{U} \subseteq \mathbb{R}^n_{++}$, a point $u \in \mathcal{U}$ is the Nash bargaining solution iff it is proportionally fair. If $\mathcal{U}$ is non-convex, then a PF solution, when exists, necessarily maximizes the Nash product.

This equivalence follows directly from the fact that (3.1) is the optimality condition (a.k.a. variational inequality) for (3.4). With this equivalence, we show that PF (and the Nash bargaining solution) differs from equal treatment:

**Proposition 3.5 (PF $\neq$ equal treatment).** If an equal treatment solution does not maximize total utility $\sum_i u_i$, then it cannot be proportionally fair.

We remark that the condition is very mild: if an equal treatment solution maximizes total utility, then it simply means there is no need to perform FL in the first place. In the wide presence of heterogeneity (of different clients) in FL, this condition almost always hold in practice.

## 4 Algorithm

Let us propose PropFair based on the notion of Nash bargaining solutions as discussed in the last section. We also prove its convergence under standard assumptions in FL.
Algorithm 1: PropFair

1. **Input:** global epoch $T$, client number $n$, loss function $f_i$ for client $i$, number of samples $n_i$ for client $i$, initial global model $\theta_0$, local step number $K_i$, baseline $M$, threshold $\epsilon$

2. for $t$ in $0, 1, \ldots, T - 1$ do

3. randomly select $S_t \subseteq [n]$

4. $\theta_{t,0}^{(i)} = \theta_t$ for $i \in S_t$, $N = \sum_{i \in S_t} n_i$

5. for $i$ in $S_t$ do // in parallel

6. draw $K_i$ batches from client $i$

7. for $S$ in the $K_i$ batches do

8. $\ell_S^i(\theta) = \frac{1}{|S|} \sum_{(x,y) \in S} \ell(\theta, (x,y))$

9. if $M - \ell_S^i(\theta) \geq \epsilon$ then

10. $f_i^\log(\theta) = -\log(M - \ell_S^i(\theta))$

11. else

12. $f_i^\log(\theta) = \ell_S^i(\theta)/M$

13. $\theta_{t,j}^{(i)} - \theta_{t,j-1}^{(i)} - \eta \nabla f_i^\log(\theta_{t,j-1}^{(i)})$

14. $\theta_{t+1} = \sum_{i \in S_t} \frac{n_i}{N} \theta_{t,K_i}$

15. **Output:** global model $\theta_T$

### 4.1 PropFair

In classical federated learning, a global model $\theta$, often a neural network, is shared across clients, and all utilities are parameterized by $\theta$. Therefore, our goal is to find a global model to maximize the Nash product introduced in Def. 3.3:

$$\max_{\theta} \sum_i \log u_i(\theta). \quad (4.1)$$

For example, the function $u_i$ can be defined as the test accuracy for client $i$ with model $\theta$, or simply the negated training loss (plus some constant) as a viable proxy. Therefore we take $u_i(\theta) = M - f_i(\theta)$, where $M$ is some utility baseline and $f_i$ is the loss function. The resulting objective becomes:

$$\max_{\theta} \sum_i \log(M - f_i(\theta)). \quad (4.2)$$

Based on (4.2) we propose Algorithm 1 called PropFair.

One might think an easier way to define the objective is to simply maximizing $\prod_i f_i$. However, this will encourage the $f_i$’s to be even more disparate. For instance, $(f_1, f_2) = (0, 1)$ has smaller product than $(f_1, f_2) = (\frac{1}{2}, \frac{1}{2})$.

**Relation with FedAvg.** When $f_i(\theta)$ is small (compared to $M$), the loss function for client $i$ becomes:

$$f_i^\log(\theta) = -\log(M - f_i(\theta)) \approx -\log M + \frac{f_i(\theta)}{M}.$$

Thus, FedAvg can be regarded as a first-order approximation of PropFair.

**Implementation tricks.** To avoid negative numbers in the logarithm, we replace $-\log(M - \ell_S^i(\theta))$ with $\ell_S^i(\theta)/M$ if $M - \ell_S^i(\theta) < \epsilon$ (here $\epsilon > 0$). An added benefit is that it also stabilizes the optimization procedure, since if $M - \ell_S^i(\theta)$ is small for some batch, taking the gradient of $\log(M - \ell_S^i(\theta))$ would lead to a large effective learning rate (see (4.7)).
Convexity of the utility set. In Proposition 3.4 we showed the equivalence between PF and Nash bargaining when the utility set is convex. We note that $U$ can be convex even if $f_i$ is not. For instance, in the extreme case with one client, $f_1(\theta) = \log \theta$ and $\theta \in [1, 2]$, the utility set $U = [M - \log 2, M]$ is clearly convex. This is specifically suitable for deep learning where $f_i$’s are usually non-convex.

4.2 PropFair as a generalized mean

We can interpret PropFair from the perspective of generalized mean. If we define $s(t) = -\log(M - t)$, then the generalized mean becomes:

$$A_s(f) = M - \frac{n}{\prod_{i=1}^n (M - f_i)^{1/n}}.$$  

(4.3)

This is a totally different type of generalized mean from the $\ell_p$-norm (including $\ell_\infty$-norm) discussed in §2.1.

We can derive $A^*_s(\lambda)$ as ( Appendix D):

$$A^*_s(\lambda) = M(\lambda^\top 1 - 1), \text{ if } \lambda \geq 0, \prod_i (n\lambda_i) \geq 1,$$

(4.4)

and infinity otherwise. With this dual view, we can also treat PropFair as minimizing a weighted combination of loss functions (plus constants), in a similar way as other algorithms in §2.2 (see Appendix D for details):

**Proposition 4.1 (dual view of PropFair).** The generalized mean (4.3) can be written as:

$$A_s(f) = \max_{\lambda \geq 0, \prod_i (n\lambda_i) \geq 1} \lambda^\top f - M(\lambda^\top 1 - 1),$$

(4.5)

where the maximizer is achieved at:

$$\prod_i (n\lambda_i) = 1 \text{ and } \lambda_i \propto \frac{1}{M - f_i}.$$  

(4.6)

We can revisit Table 2 for the dual view of different fair FL algorithms. Intuitively, since $\lambda_i$ increases with $f_i$, we see that PropFair puts a proportionally larger weight on clients with a larger loss. Therefore, it achieves enhanced equity.

4.3 Convergence guarantee

From the composition rule, if $f_i$ is convex for each client $i$, then $M - f_i(\theta)$ is concave, and thus $\sum_i \log(M - f_i(\theta))$ is concave as well (e.g., Boyd et al., 2004). In other words, for convex losses, our optimization problem (4.2) is still convex as we are maximizing over concave functions. Moreover, our PropFair objective is convex even when $f_i$’s are not (e.g. $f_1(\theta) = M - e^{-\|\theta\|^2}$ and we have only one client). This shows the wider applicability of our method in non-convex settings. In fact, most deep learning problems are non-convex.

4.3.1 Adaptive learning rate and curvature

For simplicity we take $\varphi(t) = -\log(M - t)$. We compute the 1st- and 2nd-order derivatives of $\varphi \circ f_i$:

$$\nabla(\varphi \circ f_i) = \frac{\nabla f_i}{M - f_i},$$

(4.7)

$$\nabla^2(\varphi \circ f_i) = \frac{(M - f_i)\nabla^2 f_i + (\nabla f_i)(\nabla f_i)^\top}{(M - f_i)^2}.$$  

(4.8)

From the above equations we observe that:
• At each local gradient step, the gradient $\nabla(\phi \circ f_i)$ has the same direction as $\nabla f_i$, and the only difference is the step size. Compared to FedAvg, PropFair automatically has an adaptive learning rate for each client. When the local client loss function $f_i$ is small, the learning rate is smaller; when $f_i$ is large, the learning rate is bigger. This agrees with our intuition that to achieve fairness, a worse-off client should be allowed to take a more aggressive learning rate, while a better-off client moves more slowly to “wait for” other clients.

• In the Hessian $\nabla^2(\phi \circ f_i)$, an additional positive semi-definite (p.s.d.) term $(\nabla f_i)(\nabla f_i)^\top$ is added. Thus, $\nabla^2(\phi \circ f_i)$ can be p.s.d. even if the original Hessian $\nabla^2 f_i$ is not. Moreover, the denominator $(M - f_i)^2$ has a similar effect of coordinating the curvatures of various clients as in the gradients.

4.3.2 Convergence results

Let us now formally prove the convergence of PropFair by bounding its progress, using standard assumptions [Li et al., 2019; Reddi et al., 2020] such as Lipschitz smoothness and bounded variance. Every norm discussed in this subsection is Euclidean (including the proofs in App. A). Recall that:

$$f_i(\theta) := \mathbb{E}_{(x,y) \sim D_i}[\ell(\theta, (x,y))], F(\theta) = \frac{1}{n} \sum_{i=1}^n f_i(\theta).$$

We first assume Lipschitz smoothness for each function $f_i$:

**Assumption 4.2 (Lipschitz smoothness).** Each function $f_i$ is $L$-Lipschitz smooth, i.e., for any $\theta, \theta' \in \mathbb{R}^d$ and any $i \in [n]$, we have $\|\nabla f_i(\theta) - \nabla f_i(\theta')\| \leq L \|\theta - \theta'\|$.

Since in practice we use SGD for training, we consider the effect of mini-batches. We assume that the variance of mini-batches (local variance) and the variance among clients (global variance) are bounded.

**Assumption 4.3 (bounded variance).** For any $\theta \in \mathbb{R}^d$ and any $i, j \in [n]$, we have $\|\nabla f_i(\theta) - \nabla f_j(\theta)\|^2 \leq \sigma_g^2$ the following inequality:

$$\mathbb{E}_{S \sim D_i^n} \left\| \frac{1}{|S|} \sum_{(x,y) \in S} \nabla \ell(\theta, (x,y)) \right\|^2 \leq \sigma_g^2.\]$$

Following the notation of [Reddi et al., 2020], we use $\sigma_g^2$ and $\sigma_{l,i}^2$ to denote the global and local variances for client $i$. With these two assumptions we obtain the convergence result for FedAvg (see details in Appendix C). Note that for client $i$, the number of local steps is $K_i$ with learning rate $\eta$.

**Theorem 4.4 (FedAvg).** Denote $p_i = \frac{n_i}{n}$. Given Assumptions 4.2 and 4.3 assume that the local learning rate satisfies $\eta \leq \frac{1}{6LK_i}$ for any $i \in [n]$ and

$$\eta \leq \frac{1}{L} \sqrt{\frac{1}{24(e - 2)(\sum_i p_i^2)(\sum K_i^3)}}, \quad (4.9)$$

Running FedAvg for $T$ global epochs we have:

$$\min_{0 \leq t \leq T-1} \mathbb{E}\|\nabla F(\theta_t)\|^2 \leq \frac{12}{(11\mu - 9)\eta} \left( \frac{F_0 - F^*}{T} + \Psi_\sigma \right),$$

with $\mu = \sum_i p_i K_i$ for full participation and $\mu = \min_i K_i$ for partial participation, $F_0 = F(\theta_0)$, $\ell^* = \min_{\theta} \ell(\theta)$ the optimal value, and

$$\Psi_\sigma = \frac{\eta}{2} \left( \sum_{i=1}^n p_i^2 \sum_{i=1}^n K_i^3 (\sigma_{l,i}^2 + 2\sigma_g^2) + 2(e - 2)\eta^2 L^2 \sum_{i=1}^n K_i^3 (\sigma_{l,i}^2 + 6K_i\sigma_g^2) \right).$$
The detailed proof can be found in Appendix A. Let us discuss a few highlights of Theorem 4.4:

- Our result is quite general: we allow for both full and partial participation; multiple and heterogeneous local steps; and non-uniform aggregation with weight $p_i$.
- Compared to our result, Reddi et al. [2020] only considers homogeneous number of steps for each client, and uniform aggregation for Adam-type methods. Li et al. [2019] also considers homogeneous number of steps and adds strong convexity for their proof. These assumptions may not reflect the practical use of FedAvg. For example, usually each client has a different number of samples and they may take different numbers of local updates. Moreover, for neural networks, (global) strong convexity is usually not present.
- When the variance terms $\sigma_{l,i} = \sigma_g = 0$, and the local step number $K_i = 1$, our result reduces to that of gradient descent under the usual Lipschitz smooth assumption, where $\min_{0 \leq t \leq T-1} E \| \nabla \ell(\theta_t) \|^2 = O\left(\frac{1}{T}\right)$.
- The variance term $\Psi$ decreases with smaller local steps $K_i$, which agrees with our intuition that each $K_i$ should be as small as possible given the communication constraint. Moreover, to minimize $\sum_i p_i^2$, we should take $p_i = 1/n$ for each client $i$, which means if the samples are more evenly distributed across clients, the error is smaller.

We can also apply Theorem 4.4 to our PropFair objective as a corollary. Recall that the objective of PropFair is:

$$!min_\theta \pi(\theta) := \frac{1}{n} \sum_{i=1}^{n} \log(M - f_i(\theta)).$$

In order to apply Theorem 4.4, we need an additional assumption about the client function $f_i$, due to (1.7).

**Assumption 4.5 (boundedness, Lipschitz continuity and bounded variance for client losses).** For any $i \in [n]$, $\theta \in \mathbb{R}^d$ and any batch $S \sim \mathcal{D}_i^m$ of $m$ i.i.d. samples, we have:

$$0 \leq \ell_S(\theta) := \frac{1}{|S|} \sum_{(x,y) \in S} \ell(\theta, (x,y)) \leq \frac{M}{2},$$

and $\|f_i(\theta) - f_i(\theta')\| \leq L_0 \|\theta - \theta'\|$. We also assume that for any $i,j \in [n]$ and $\theta \in \mathbb{R}^d$, $\|f_i(\theta) - f_j(\theta)\| \leq \sigma_{0,i}$ and $E_{S \sim \mathcal{D}_i^m} \left\| \frac{1}{|S|} \sum_{(x,y) \in S} \ell(\theta, (x,y)) - f_i(\theta) \right\|^2 \leq \sigma_{0,i,j}^2$.

From the Lipschitzness we can obtain an upper bound for the gradient: $\|\nabla f_i(\theta)\| \leq L_0$ for any $\theta \in \mathbb{R}^d$. With this additional assumption and Theorem 4.4, we find:

**Theorem 4.6 (PropFair).** Denote $\tilde{L} = \frac{1}{M^2} (\frac{3}{2} M L + L_0^2)$. Given Assumptions 4.2, 4.3 and 4.5, assume that the local learning rate satisfies $\eta \leq \frac{1}{6 L K L}$, for any $i \in [n]$ and (4.9) holds. By running Algorithm 1 for $T$ global epochs we have:

$$\min_{0 \leq t \leq T-1} E \| \nabla \pi(\theta_t) \|^2 \leq \frac{12}{(11 \mu - 9) \eta} \left( \frac{\pi_0 - \pi^*}{T} + \tilde{\Psi}_{\sigma}\right),$$

with $\mu$ the same meaning as in Theorem 4.4, $\pi_0 = \pi(\theta_0)$, $\pi^* = \min_{\theta} \pi(\theta)$ the optimal value, and

$$\tilde{\Psi}_{\sigma} = \frac{\eta}{2} \left( \sum_{i=1}^{n} p_i^2 \left[ \sum_{i=1}^{n} K_i^2 (\tilde{\sigma}_{l,i}^2 + 2\tilde{\sigma}_g^2) \right. \right.$$

$$\left. + 2(e - 2)\eta^2 L^2 \sum_{i=1}^{n} K_i^3 \left( \tilde{\sigma}_{l,i}^2 + 6 K_i \tilde{\sigma}_g^2 \right) \right],$$

where $\tilde{\sigma}_{l,i}^2 = \frac{\eta}{\mu} (9 M^2 \sigma_{l,i}^2 + 4 L_0^2 \sigma_{0,l,i}^2)$ and $\tilde{\sigma}_g = \frac{\eta}{\mu} \left( \frac{3}{2} \sigma_g + \frac{2}{\mu} \sigma_{0,g} \right)$.

The same comments after Theorem 4.4 apply equally well here. Moreover, in presence of convexity, it follows easily from Theorem 4.6 that PropFair converges to a neighborhood of the optimal solution, whose size is controlled by the heterogeneity of clients and the variance of mini-batches.
Figure 2: Mean and worst-case test accuracies for different algorithms. The accuracies are in percentage. (top left): CIFAR-10; (top middle): CIFAR-100; (top right): CINIC-10; (bottom left): Shakespeare. (bottom right): StackOverflow.

Table 3: Comparison between the solutions found by different algorithms for Example 1.1. $x_{alg}$ denotes the solution found by optimizing the corresponding objective for each algorithm. For $q$-FFL we take $q = 1$ and for PropFair we take $M = 2$.

|       | $f_1$ | $f_2$ | $\sum_i f_i$ | $\sum_i f_i^2$ | $\prod_i (2 - f_i)$ |
|-------|-------|-------|--------------|-----------------|-------------------|
| $x_{FedAvg}$ | 0.19  | 1.13  | 1.31         | 1.30            | 1.59              |
| $x_{q-FFL}$  | 0.34  | 1.03  | 1.38         | **1.18**        | 1.60              |
| $x_{AFL}$    | 0.56  | 1.00  | 1.56         | 1.32            | 1.44              |
| $x_{PropFair}$ | 0.27  | 1.06  | 1.33         | 1.20            | **1.62**          |

5 Experiments

In this section, we test the ability of PropFair (Algorithm 1) for training proportionally fair models. Our experiments confirm that PropFair achieves better worst-case as well as overall performances at the same time. See Appendix B for more details.

5.1 Toy example

We revisit Example 1.1 and study the solutions obtained by different generalized means in §2.1. From Table 3 we find that different algorithms indeed successfully optimize their respective generalized mean: for FedAvg the total loss $f_1 + f_2$ is minimal; for $q$-FFL the $\ell_2$-norm of $(f_1, f_2)$ is minimal; for AFL the worst-case loss $\max_i f_i$ is minimal; and for PropFair the Nash product $\prod_i (2 - f_i)$ is maximal.

5.2 Experimental setup

We first give details on our datasets, models and hyperparameters, which are in accordance with existing works. For a comprehensive survey of benchmarking FL algorithms, see e.g. Caldas et al. (2018); He et al. (2020). More detailed experimental setup can be found in Appendix B.1.

Datasets. We follow standard benchmark datasets as in the existing literature, including CIFAR-10/100 (Krizhevsky et al., 2009), CINIC-10 (Darlow et al., 2018), Shakespeare (McMahan et al., 2017) and Stack-
Figure 3: Mean and worst-case test accuracies of PropFair for different choices of $M$ on the CIFAR-100 dataset. The dotted lines are the mean/worst-case test accuracies of FedAvg.

Overflow. For vision datasets (CIFAR-10/100 and CINIC-10), the task is image classification, and following Wang et al. (2019b) we use Dirichlet allocation to split the dataset into different clients. For language datasets, the task is next-word prediction. We use the default realistic partition based on different users. For more details of the datasets and our splitting methods, see Appendix B.1.

Train-test split. We first partition the dataset into different clients, and further split each client dataset into its own training and test sets. This reflects the real scenario, where each client evaluates the performance by himself/herself.

Models, optimizer and loss function. For CIFAR-10/100 we use ResNet-18 (He et al., 2016) with group normalization (Wu & He, 2018). As discussed by Hsieh et al. (2020), group normalization (with num_groups=2) works better than batch normalization, especially in the federated settings. For Shakespeare and StackOverflow datasets we use LSTMs (Hochreiter & Schmidhuber, 1997). We implement stochastic gradient descent (SGD) and cross entropy loss functions throughout our experiments. We find the best learning rate through grid search (see Appendix B).

Baseline algorithms. We compare our PropFair algorithm with common FL baselines for fairness (see Table 2), including FedAvg (McMahan et al., 2017), $q$-FFL (Li et al., 2020b) and AFL (Mohri et al., 2019). For $q$-FFL we find the best hyperparameter $q$ from \{0.1, 1, 5\}.

Other hyperparameters. We implement full participation and one local epoch throughout (with many local steps for each client). Due to data heterogeneity, the number of local steps $K_i$ for each client $i$ varies. For CIFAR-10/100 we partition the data into 10 clients; for CINIC-10 we choose 50 clients; for language datasets we choose 20 clients.

5.3 Comparison between PropFair and baselines

In Figure 2 we compare different algorithms with the average and the worst-case test accuracies across clients. In order to rule out the factor of optimization, we train all algorithms till convergence. We can see that across various vision and language datasets, PropFair has the best mean and worst-case performance over baselines. Specifically, for the Shakespeare dataset, PropFair is better than the baselines by nearly 2% w.r.t. the worst-case test accuracy, while also having the best average performance. For other datasets, we can also see that PropFair remains consistently competitive against the baselines (see Appendix B.3 for more details).

As we explained in §4.3, PropFair automatically enjoys customized learning rates, and encourages the worse-end clients to converge faster, which might explain the reason why it is better in mean and worst-case performances, compared to FedAvg. As mentioned in Pathak & Wainwright (2020), FedAvg might not always
converge to the optimal solution that maximizes the total utility, and our experiments show that PropFair can find a better solution.

It is also worth mentioning that although the goal of AFL is to maximize the worst-case performance, we find that PropFair consistently outperforms AFL, possibly because the minimax optimization of AFL is much more challenging.

Similarly, since the goal of $q$-FFL is to encourage equality, it may (disproportionately) suppress the high-performing clients and hurt the overall performance. Even though it is possible to tune the hyperparameter $q$ for better tradeoff, we find that the final performance is quite sensitive to this value. Comparably, PropFair is more robust to the variation of hyperparameters. We will demonstrate this point in the next subsection.

5.4 Ablation study of PropFair

Let us study the effect of the utility baseline $M$ in PropFair (Algorithm 1). From Figure 3, we can see that PropFair is not very sensitive to the choice of $M$. Within a large range of $M$, PropFair can achieve better worst-case accuracy and mean accuracy than FedAvg. This confirms that we do not need to tune the hyperparameter $M$ excessively for obtaining reasonably equitable performance. Also, we can see that as $M$ becomes larger, the performance of PropFair gradually reduces to that of FedAvg, as we predicted in § 4.1.

6 Conclusions

In this work, we introduce the concept of proportional fairness (PF) into the field of federated learning (FL), which is deeply rooted in cooperative game theory. By showing the equivalence between PF and the Nash bargaining solution in the context of FL, we propose PropFair that maximizes the product of client utilities. We prove the convergence of PropFair in the heterogeneous setting, and verify its empirical competitiveness. Compared to other fair FL algorithms, PropFair achieves better worst-case performance with no sacrifice (in fact often improvement) on the overall performance, and thus yields better equity. We believe that the proportional fairness guarantee of PropFair is a welcome addition to existing FL systems, and we plan to further study its applicability in training generative models.
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A Proofs

**Proposition 3.4 (equivalence, e.g. [Kelly et al., 1998; Boche & Schubert, 2009])**. For any convex compact set \( U \subseteq \mathbb{R}^n_{++} \), a point \( u \in U \) is the Nash bargaining solution iff it is proportionally fair. If \( U \) is non-convex, then a PF solution, when exists, necessarily maximizes the Nash product.

**Proof.** The Nash bargaining solution \( u^* \) is equivalent to the maximum of the following:

\[
\max_{u \in U} \sum_{i=1}^{n} \log u_i. \tag{A.1}
\]

Since \( U \) is convex and \( \sum_{i=1}^{n} \log u_i \) is concave in \( u \), the necessary and sufficient optimality condition (e.g., Bertsekas, 1997) is:

\[
\langle u - u^*, \nabla \sum_{i=1}^{n} \log u_i^* \rangle \leq 0, \text{ for any } u \in U, \tag{A.2}
\]

or equivalently, (3.1). If \( U \) is non-convex, then the optimality condition (A.2) also holds for the convex hull of \( U \). Therefore, \( u^* \) is a maximizer of \( \sum_{i=1}^{n} \log u_i \) in the convex hull of \( U \) and thus \( U \).

**Proposition 3.5 (PF ≠ equal treatment)**. If an equal treatment solution does not maximize total utility \( \sum_i u_i \), then it cannot be proportionally fair.

**Proof.** Define \( \theta_{\text{equal}} \) to be the equal treatment solution where all \( u_i(\theta_{\text{equal}}) \) are equal to the same value. We can prove this proposition by showing that:

\[
\sum_{i=1}^{n} \frac{u_i(\theta_{\text{max}}) - u_i(\theta_{\text{equal}})}{u_i(\theta_{\text{equal}})} = \frac{1}{u_i(\theta_{\text{equal}})} \sum_{i=1}^{n} u_i(\theta_{\text{max}}) - n > 0,
\]

where \( \theta_{\text{max}} \) is the maximizer of the total utility \( \sum_{i=1}^{n} u_i \).

**Theorem 4.4 (FedAvg).** Denote \( p_i = \frac{n_i}{N} \). Given Assumptions 4.2 and 4.3 assume that the local learning rate satisfies \( \eta \leq \frac{1}{6L K_i} \) for any \( i \in [n] \) and

\[
\eta \leq \frac{1}{L} \sqrt{\frac{1}{24(e-2)(\sum_i p_i^2)(\sum_i K_i^2)}}. \tag{4.9}
\]

Running FedAvg for \( T \) global epochs we have:

\[
\min_{0 \leq t \leq T-1} \mathbb{E}\|\nabla F(\theta_t)\|^2 \leq \frac{12}{(11\mu - 9)\eta} \left( \frac{F_0 - F^*}{T} + \Psi_\sigma \right),
\]

with \( \mu = \sum_i p_i K_i \) for full participation and \( \mu = \min_i K_i \) for partial participation, \( F_0 = F(\theta_0) \), \( \ell^* = \min_\theta \ell(\theta) \) the optimal value, and

\[
\Psi_\sigma = \frac{\eta}{2} \left( \sum_{i=1}^{n} p_i^2 \right) \left[ \sum_{i=1}^{n} K_i^2 (\sigma_{l,i}^2 + 2\sigma_g^2) + 2(e-2)\eta^2 L^2 \sum_{i=1}^{n} K_i^3 (\sigma_{l,i}^2 + 6K_i \sigma_g^2) \right].
\]
Proof. We first assume full participation in the following theorem. The partial participation version is an easy extension and we discuss it in the end. We use \( \theta_{t,i}^{(i)} \) to denote the model parameters of client \( i \) at global epoch \( t \) and local step \( j \). Due to the synchronization step, we have \( \theta_{t,0}^{(i)} = \theta_t \), the global model at step \( t \), and

\[
\theta_{t+1} = \sum_{i=1}^{n} p_i \theta_{t,i}^{(i)}, \quad p_i = \frac{n_i}{N}, \quad (A.3)
\]

where \( K_i \) is the local number of steps of client \( i \). We also have:

\[
\theta_{t,j}^{(i)} = \theta_{t,j-1}^{(i)} - \eta g_{t,j}^{(i)}, \text{ for all } j \in [K_i]. \quad (A.4)
\]

where \( g_{t,j}^{(i)} \) is an unbiased estimator of \( \nabla f_i(\theta_{t,j-1}^{(i)}) \) for \( j \in [K_i] \). Combining (A.3) and (A.4) we have:

\[
\theta_{t+1} = \theta_t - \eta \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} g_{t,j}^{(i)}. \quad (A.5)
\]

Part I  Since each \( f_i \) is \( L \)-Lipschitz smooth, so is their average \( f_t \), from which we obtain that:

\[
F(\theta_{t+1}) \leq F(\theta_t) + \langle \nabla F(\theta_t), \theta_{t+1} - \theta_t \rangle + \frac{L}{2} \|\theta_{t+1} - \theta_t\|^2. \quad (A.6)
\]

Plugging in (A.5) yields:

\[
F(\theta_{t+1}) \leq F(\theta_t) - \eta \left( \nabla F(\theta_t) \cdot \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} g_{t,j}^{(i)} \right) + \frac{L\eta^2}{2} \left( \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} g_{t,j}^{(i)} \right) + \frac{L\eta^2}{2} \left( \sum_{i=1}^{n} p_i K_i \|\nabla F(\theta_t)\| \right)^2. \quad (A.7)
\]

From the following identity:

\[
g_{t,j}^{(i)} = g_{t,j}^{(i)} - \nabla F(\theta_t) + \nabla F(\theta_t), \quad (A.8)
\]

we write (A.7) as:

\[
F(\theta_{t+1}) \leq F(\theta_t) - \eta \sum_{i=1}^{n} p_i K_i \|\nabla F(\theta_t)\|^2 - \eta \left( \nabla F(\theta_t) \cdot \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{t,j}^{(i)} - \nabla F(\theta_t)) \right) + \frac{L\eta^2}{2} \left( \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{t,j}^{(i)} - \nabla F(\theta_t)) \right) + \frac{L\eta^2}{2} \left( \sum_{i=1}^{n} p_i K_i \|\nabla F(\theta_t)\| \right)^2. \quad (A.9)
\]

By further expanding the last term we have:

\[
F(\theta_{t+1}) \leq F(\theta_t) - \eta \sum_{i=1}^{n} p_i K_i \|\nabla F(\theta_t)\|^2 - \eta \left( \nabla F(\theta_t) \cdot \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{t,j}^{(i)} - \nabla F(\theta_t)) \right) + \frac{L\eta^2}{2} \left( \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{t,j}^{(i)} - \nabla F(\theta_t)) \right)^2 + \frac{L\eta^2}{2} \left( \sum_{i=1}^{n} p_i K_i \right)^2 \|\nabla F(\theta_t)\|^2 + \frac{L\eta^2}{2} \left( \sum_{i=1}^{n} p_i K_i \|\nabla F(\theta_t)\| \right)^2 \quad (A.10)
\]

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For simplicity we use $\mu$ as a shorthand for $\sum_{i=1}^{n} p_i K_i$. Grouping similar terms together gives:

$$F(\theta_{t+1}) \leq F(\theta_t) - \eta \mu \left(1 - \frac{L\eta}{2}\right) \left\| \nabla F(\theta_t) \right\|^2 - \eta (1 - L\eta\mu) \left( \nabla F(\theta_t), \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{i,j}^{(i)} - \nabla F(\theta_t)) \right) +$$

$$+ \frac{L\eta^2}{2} \left\| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{i,j}^{(i)} - \nabla F(\theta_t)) \right\|^2 .$$

(A.11)

Taking the expectation on both sides and with Cauchy–Schwarz inequality, we have:

$$EF(\theta_{t+1}) \leq EF(\theta_t) - \eta \mu \left(1 - \frac{L\eta}{2}\right) E\|\nabla F(\theta_t)\|^2 +$$

$$+ \eta (1 - L\eta\mu) E \left\| \nabla F(\theta_t) \right\| \cdot \left\| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (\nabla f_i(\theta_{t,j-1}^{(i)}) - \nabla F(\theta_t)) \right\| + \frac{L\eta^2}{2} E \left\| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{i,j}^{(i)} - \nabla F(\theta_t)) \right\|^2$$

$$\leq EF(\theta_t) + \left( -\eta \mu \left(1 - \frac{L\eta}{2}\right) + \frac{1}{2} \eta (1 - L\eta\mu) \right) E\|\nabla F(\theta_t)\|^2 +$$

$$+ \left( \frac{L\eta^2}{2} + \frac{1}{2} \eta (1 - L\eta\mu) \right) E \left\| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{i,j}^{(i)} - \nabla F(\theta_t)) \right\|^2 ,$$

(A.12)

where we used the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ in the second line. Let us now study the two coefficients separately. From the assumption, $6\eta L K_i \leq 1$ for any $i \in [n]$, and thus $6L\mu \eta \leq 1$. Hence we have:

$$-\eta \mu \left(1 - \frac{L\eta}{2}\right) + \frac{1}{2} \eta (1 - L\eta\mu) \leq -\eta \left( \mu - \frac{1}{2} \frac{L\eta}{2} \mu^2 + \frac{1}{2} L\eta\mu \right)$$

$$\leq -\eta \left( \frac{11\mu - 6}{12} + \frac{L\eta\mu}{2} \right)$$

$$\leq -\eta \left( \frac{11\mu - 6}{12} \right) .$$

(A.13)

Moreover, the coefficient of the third term can be upper bounded as:

$$\frac{L\eta^2}{2} + \frac{1}{2} \eta (1 - L\eta\mu) = \frac{\eta}{2} + \frac{L}{2} \eta^2 (1 - \mu) \leq \frac{\eta}{2} .$$

(A.14)

Therefore, (A.12) becomes:

$$EF(\theta_{t+1}) \leq EF(\theta_t) - \eta \frac{11\mu - 6}{12} E\|\nabla F(\theta_t)\|^2 + \frac{\eta}{2} E \left\| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{i,j}^{(i)} - \nabla F(\theta_t)) \right\|^2 .$$

(A.15)

**Part II** With the following identity:

$$g_{i,j}^{(i)} - \nabla F(\theta_t) = g_{i,j}^{(i)} - \nabla f_i(\theta_{t,j-1}^{(i)}) + \nabla f_i(\theta_{t,j-1}^{(i)}) - \nabla F(\theta_t) ,$$

(A.16)

the last term of (A.12) can be simplified as:

$$E \left\| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{i,j}^{(i)} - \nabla f_i(\theta_{t,j-1}^{(i)})) \right\|^2 + E \left\| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (\nabla f_i(\theta_{t,j-1}^{(i)}) - \nabla F(\theta_t)) \right\|^2 ,$$

(A.17)
where we note that \( g_{t,j}^{(i)} \) is an unbiased estimator of \( \nabla f_i(\theta_{t,j-1}^{(i)}) \). We first bound the first term of (A.17):

\[
E \left| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{t,j}^{(i)} - \nabla f_i(\theta_{t,j-1}^{(i)})) \right|^2 \leq E \left( \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} \left| g_{t,j}^{(i)} - \nabla f_i(\theta_{t,j-1}^{(i)}) \right| \right)^2 
\]

\[
\leq E \left( \sum_{i=1}^{n} p_i^2 \sum_{j=1}^{K_i} \left| g_{t,j}^{(i)} - \nabla f_i(\theta_{t,j-1}^{(i)}) \right| \right)^2 
\]

\[
\leq E \left( \sum_{i=1}^{n} p_i^2 \right) \sum_{i=1}^{n} K_i \sum_{j=1}^{K_i} \left| g_{t,j}^{(i)} - \nabla f_i(\theta_{t,j-1}^{(i)}) \right|^2 
\]

\[
= \left( \sum_{i=1}^{n} p_i^2 \right) \sum_{i=1}^{n} K_i \sum_{j=1}^{K_i} E \left| g_{t,j}^{(i)} - \nabla f_i(\theta_{t,j-1}^{(i)}) \right|^2 
\]

\[
\leq \left( \sum_{i=1}^{n} p_i^2 \right) \sum_{i=1}^{n} K_i^2 \sigma_{t,i}^2, \quad (A.18)
\]

where in the first line, we used triangle inequality; in the second and third lines, we used the Cauchy–Schwarz inequality; in the fourth line we used the linearity of expectation; in the final line we used Assumption 4.3.

Similarly we bound the second term of (A.17):

\[
E \left| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (\nabla f_i(\theta_{t,j-1}^{(i)}) - \nabla F(\theta_t)) \right|^2 \leq \left( \sum_{i=1}^{n} p_i^2 \right) \sum_{i=1}^{n} K_i \sum_{j=1}^{K_i} E \left| \nabla f_i(\theta_{t,j-1}^{(i)}) - \nabla F(\theta_t) \right|^2. \quad (A.19)
\]

With the following identity:

\[
\nabla f_i(\theta_{t,j-1}^{(i)}) - \nabla F(\theta_t) = \nabla f_i(\theta_{t,j-1}^{(i)}) - \nabla f_i(\theta_t) + \nabla f_i(\theta_t) - \nabla F(\theta_t), \quad (A.20)
\]

and taking the squared norm on both sides, we have:

\[
\left| \nabla f_i(\theta_{t,j-1}^{(i)}) - \nabla F(\theta_t) \right|^2 \leq 2 \left| \nabla f_i(\theta_{t,j-1}^{(i)}) - \nabla f_i(\theta_t) \right|^2 + 2 \left| \nabla f_i(\theta_t) - \nabla F(\theta_t) \right|^2 
\]

\[
\leq 2L^2 \left| \theta_{t,j-1}^{(i)} - \theta_t \right|^2 + 2\sigma_g^2, \quad (A.21)
\]

where we note that:

\[
\left| \nabla f_i(\theta_t) - \nabla F(\theta_t) \right|^2 = \left| \nabla f_i(\theta_t) - \frac{1}{n} \sum_{i=1}^{n} \nabla f_j(\theta_t) \right|^2 
\]

\[
= \frac{1}{n^2} \left| \sum_{i=1}^{n} (\nabla f_i(\theta_t) - \nabla f_j(\theta_t)) \right|^2 
\]

\[
\leq \frac{1}{n^2} \left( \sum_{i=1}^{n} \left| \nabla f_i(\theta_t) - \nabla f_j(\theta_t) \right| \right)^2 
\]

\[
\leq \frac{1}{n^2} \cdot n \cdot \sum_{i=1}^{n} \left| \nabla f_i(\theta_t) - \nabla f_j(\theta_t) \right|^2 
\]

\[
\leq \sigma_g^2, \quad (A.22)
\]
where in the third line we used the triangle inequality and in the last line we used Assumption 4.3. Plugging (A.21) into (A.19) yields:

\[ E \left\| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (\nabla f_i(\theta_i^{(i)})) - \nabla F(\theta_1) \right\|^2 \leq 2 \left( \sum_{i=1}^{n} p_i^2 \right) \left( \sum_{i=1}^{n} K_i^2 \sigma_i^2 + 2L^2 \left( \sum_{i=1}^{n} p_i^2 \right) \sum_{j=1}^{K_i} E\|\theta_i^{(i)} - \theta_1\|^2. \right) \]

(A.23)

Bringing (A.18) and (A.23) into (A.17) we write:

\[ E \left\| \sum_{i=1}^{n} p_i \sum_{j=1}^{K_i} (g_{t,j}^{(i)} - \nabla F(\theta_t)) \right\|^2 \leq \left( \sum_{i=1}^{n} p_i^2 \right) \left( \sum_{i=1}^{n} K_i^2 (\sigma_{t,i}^2 + 2\sigma_g^2) + 2L^2 \left( \sum_{i=1}^{n} p_i^2 \right) \sum_{j=1}^{K_i} E\|\theta_i^{(i)} - \theta_t\|^2 \right) \]

\[ \leq \left( \sum_{i=1}^{n} p_i^2 \right) \left( \sum_{i=1}^{n} K_i^2 (\sigma_{t,j}^2 + 2\sigma_g^2) + 2L^2 \left( \sum_{i=1}^{n} p_i^2 \right) \sum_{j=1}^{K_i} E\|\theta_i^{(i)} - \theta_t\|^2. \right) \]

(A.24)

**Part III**  Now let us give an upper bound for \( E\|\theta_t^{(i)} - \theta_t\|^2 \). From (A.24), we only need to focus on \( K_i \geq 2 \) and \( j = 1, \ldots, K_i - 1 \) since \( \theta_i^{(i)} = \theta_t \). For \( j \in [K_i - 1] \), we have from (A.4):

\[ E\|\theta_t^{(i)} - \theta_t\|^2 = E\|\theta_i^{(i)} - \theta_t - \eta h_{i,j}\|^2 \]

\[ = E\|\theta_i^{(i)} - \theta_t - \eta \nabla f_i(\theta_i^{(i)}) + \eta \nabla f_i(\theta_i^{(i)}) - h_{i,j}\|^2 \]

\[ = E\|\theta_i^{(i)} - \theta_t - \eta \nabla f_i(\theta_i^{(i)})\|^2 + E\eta^2\|\nabla f_i(\theta_i^{(i)}) - h_{i,j}\|^2 \]

\[ = E\|\theta_i^{(i)} - \theta_t - \eta \nabla f_i(\theta_i^{(i)})\|^2 + \eta^2 \sigma_i^2, \]

(A.25)

where in the third line we note that \( g_{i,j}^{(i)} \) is an unbiased estimator of \( \nabla f_i(\theta_i^{(i)}) \) and in the last line we used Assumption 4.3. The first term of (A.25) can be bounded as:

\[ E\|\theta_t^{(i)} - \theta_t - \eta \nabla f_i(\theta_i^{(i)})\|^2 \leq \left( 1 + \frac{1}{2K_i - 1} \right) E\|\theta_i^{(i)} - \theta_t\|^2 + 2K_i \eta^2 \|\nabla f_i(\theta_i^{(i)})\|^2, \]

(A.26)

where we used \( \|a + b\|^2 = (1 + \frac{1}{2})\|a\|^2 + (1 + \alpha)\|b\|^2 \) for any vectors \( a, b \) with the same dimension and \( \alpha > 0 \). Since

\[ \nabla f_i(\theta_i^{(i)}) = (\nabla f_i(\theta_i^{(i)})) - (\nabla f_i(\theta_t)) + (\nabla f_i(\theta_t) - \nabla F(\theta_t)) + \nabla F(\theta_t), \]

(A.27)

taking the squared norm on both sides we have (note that \( (a + b + c)^2 \leq 3(a^2 + b^2 + c^2) \)):

\[ \|\nabla f_i(\theta_i^{(i)})\|^2 \leq 3\|\nabla f_i(\theta_i^{(i)}) - \nabla f_i(\theta_t)\|^2 + 3\|\nabla f_i(\theta_t) - \nabla F(\theta_t)\|^2 \]

\[ \leq 3L^2\|\theta_i^{(i)} - \theta_t\|^2 + \eta^2 \sigma_i^2 + 3\|\nabla F(\theta_t)\|^2, \]

(A.28)

where in the second line we used (A.22) and Assumption 4.2. Plugging (A.28) into (A.26) we find:

\[ E\|\theta_t^{(i)} - \theta_t - \eta \nabla f_i(\theta_i^{(i)})\|^2 \leq \left( 1 + \frac{1}{2K_i - 1} + 6K_i \eta^2 L^2 \right) E\|\theta_t^{(i)} - \theta_t\|^2 + 6K_i \eta^2 \sigma_i^2 + 6K_i \eta^2 \|\nabla F(\theta_t)\|^2. \]

(A.29)

Combined with (A.25), we obtain:

\[ E\|\theta_t^{(i)} - \theta_t\|^2 \leq \left( 1 + \frac{1}{2K_i - 1} + 6K_i \eta^2 L^2 \right) E\|\theta_t^{(i)} - \theta_t\|^2 + \eta^2 (\sigma_i^2 + 6K_i \sigma_g^2) + 6K_i \eta^2 \|\nabla F(\theta_t)\|^2. \]

(A.30)
Recall that we assumed $\eta \leq \min\{\frac{1}{6K_iL}\}$. For $K_i \geq 2$ (note the assumption at the beginning of Part III), we have:

$$1 + \frac{1}{2K_i - 1} + 6K_i\eta^2L^2 \leq 1 + \frac{1}{2K_i - 1} + \frac{1}{6K_i} \leq 1 + \frac{1}{K_i}.$$ (A.31)

Therefore, [A.30] becomes:

$$\mathbb{E}\|\theta_{t,j}^{(1)} - \theta_t\|^2 \leq \left(1 + \frac{1}{K_i}\right) \mathbb{E}\|\theta_{t,j-1}^{(1)} - \theta_t\|^2 + \eta^2(\sigma_{t,i}^2 + 6K_i\sigma_g^2) + 6K_i\eta^2\|\nabla F(\theta_t)\|^2.$$ (A.32)

We can treat $\{a_j = \mathbb{E}\|\theta_{t,j}^{(i)} - \theta_t\|^2\}_{j=1}^{K_i-1}$ as a sequence. Unrolling this sequence and with $a_0 = 0$, we have:

$$\mathbb{E}\|\theta_{t,j}^{(i)} - \theta_t\|^2 \leq \left(1 + \frac{1}{K_i}\right)^j - 1 \left(\frac{\eta^2(\sigma_{t,i}^2 + 6K_i\sigma_g^2) + 6K_i\eta^2\|\nabla F(\theta_t)\|^2}{1 + \frac{1}{K_i}}\right)$$

$$= K_i \left(\left(1 + \frac{1}{K_i}\right)^j - 1\right) \left(\eta^2(\sigma_{t,i}^2 + 6K_i\sigma_g^2) + 6K_i\eta^2\|\nabla F(\theta_t)\|^2\right).$$ (A.33)

Summing over $j = 0, 1, \ldots, K_i - 1$ gives:

$$\sum_{j=0}^{K_i-1} \mathbb{E}\|\theta_{t,j}^{(i)} - \theta_t\|^2 \leq K_i^2 \left(1 + \frac{1}{K_i}\right)^{K_i} - 2 \left(\eta^2(\sigma_{t,i}^2 + 6K_i\sigma_g^2) + 6K_i\eta^2\|\nabla F(\theta_t)\|^2\right)$$

$$\leq (e - 2)K_i^2 \left(\eta^2(\sigma_{t,i}^2 + 6K_i\sigma_g^2) + 6K_i\eta^2\|\nabla F(\theta_t)\|^2\right)$$

$$= (e - 2)K_i^2\eta^2(\sigma_{t,i}^2 + 6K_i\sigma_g^2 + 6K_i\|\nabla F(\theta_t)\|^2)$$ (A.34)

where in the first line we used the geometric series formula $1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q-1}$; in the second line we used the fact that $\left(1 + \frac{1}{K_i}\right)^{K_i} \leq e$ for $K_i \geq 1$, with $e$ the natural logarithm.

**Part IV** We finally put things together and finish our proof. Combining (A.15), (A.24) and (A.34) we have:

$$EF(\theta_{t+1}) \leq EF(\theta_t) - \eta\frac{11\mu - 6}{12} - \mathbb{E}\|\nabla F(\theta_t)\|^2 + 6(e - 2)\eta^3L^2 \left(\sum_{i=1}^{n} p_i^2\right) \sum_{i=1}^{n} K_i^4\|\nabla F(\theta_t)\|^2 + \Psi_\sigma,$$ (A.35)

where we denote

$$\Psi_\sigma = \frac{\eta}{2} \left(\sum_{i=1}^{n} p_i^2\right) \left[\sum_{i=1}^{n} K_i^2(\sigma_{t,i}^2 + 2\sigma_g^2) + 2(e - 2)\eta^2L^2 \sum_{i=1}^{n} K_i^3\left(\sigma_{t,i}^2 + 6K_i\sigma_g^2\right)\right].$$ (A.36)

Since we assumed:

$$\eta \leq \frac{1}{L} \sqrt{\frac{1}{24(e - 2)(\sum_{i=1}^{n} p_i^2)(\sum_{i=1}^{n} K_i^4)},}$$ (A.37)

we have:

$$\frac{11\mu - 6}{12} - 6(e - 2)L^2 \left(\sum_{i=1}^{n} p_i^2\right) \sum_{i=1}^{n} K_i^4\eta^2 \geq \frac{11\mu - 6}{12} - \frac{1}{4} \geq \frac{11\mu - 9}{12}. $$ (A.38)
Therefore, (A.35) becomes:

$$EF(\theta_{t+1}) \leq EF(\theta_t) - \eta \frac{11\mu - 9}{12} E\|\nabla F(\theta_t)\|^2 + \Psi_\sigma.$$  \hspace{1cm} (A.39)

With some algebra we obtain:

$$\eta \frac{11\mu - 9}{12} E\|\nabla F(\theta_t)\|^2 \leq E[F(\theta_t) - F(\theta_{t+1})] + \Psi_\sigma.$$  \hspace{1cm} (A.40)

Summing both sides over \( t = 0, \ldots, T - 1 \) and dividing by \( T \), we have:

$$\eta \frac{11\mu - 9}{12} \frac{1}{T} \sum_{t=0}^{T-1} E\|\nabla F(\theta_t)\|^2 \leq \frac{F(\theta_0) - F^*}{T} + \Psi_\sigma,$$  \hspace{1cm} (A.41)

which gives:

$$\min_{0 \leq t \leq T-1} E\|\nabla F(\theta_t)\|^2 \leq \frac{12}{(11\mu - 9)\eta} \left( \frac{F(\theta_0) - F^*}{T} + \Psi_\sigma \right),$$  \hspace{1cm} (A.42)

with \( F^* = \min_\theta F(\theta) \) the optimal value.

Finally, for the partial participation, it suffices to replace the client set \( \{1, \ldots, n\} \) with its subset. Note that after this substitution, the new variance term satisfies \( \Psi'_\sigma \leq \Psi_\sigma \) since this term increases with more participants, and (A.38) still holds since we subtract a smaller term with partial participation. We also need to modify (A.38) so we further lower bound (A.38) with \( \mu \geq \min_i K_i \).

**Theorem 4.6 (PropFair).** Denote \( \bar{L} = \frac{4}{\pi^2} (3ML + L_0^2) \). Given Assumptions 4.2, 4.3 and 4.5, assume that the local learning rate satisfies \( \eta \leq \frac{1}{6\bar{L}K} \) for any \( i \in [n] \) and (4.9) holds. By running Algorithm 1 for \( T \) global epochs we have:

$$\min_{0 \leq t \leq T-1} E\|\nabla \pi(\theta_t)\|^2 \leq \frac{12}{(11\mu - 9)\eta} \left( \frac{\pi_0 - \pi^*}{T} + \bar{\Psi}_\sigma \right),$$

with \( \mu \) the same meaning as in Theorem 4.4. \( \pi_0 = \pi(\theta_0) \), \( \pi^* = \min_\theta \pi(\theta) \) the optimal value, and

$$\bar{\Psi}_\sigma = \eta \left( \sum_{i=1}^{n} \frac{p_i^2}{\pi^2} \left[ \sum_{i=1}^{n} K_i^2 (\bar{\sigma}_{l,i}^2 + 2\bar{\sigma}_{g}^2) \right] + 2(e - 2)\eta^2 L^2 \sum_{i=1}^{n} K_i^2 (\bar{\sigma}_{l,i}^2 + 6K_i \bar{\sigma}_{g}^2) \right),$$

where \( \bar{\sigma}_{l,i}^2 = \frac{8}{\pi^2} (9M^2 \sigma_{l,i}^2 + 4L_0^2 \sigma_{0,l,i}^2) \) and \( \bar{\sigma}_{g} = \frac{1}{M^2} \left( \frac{2}{\pi^2} \sigma_g + \frac{24}{\pi^2} \sigma_{0,g} \right) \).

**Proof.** It suffices to determine the constants in Assumptions 4.2 and 4.3 after adding the composition \( \varphi \circ f_i \). For Assumption 4.2 we write:

$$\|\nabla(\varphi \circ f_i)(\theta) - \nabla(\varphi \circ f_i)(\theta')\| = \left\| \frac{\nabla f_i(\theta)}{M - f_i(\theta)} - \frac{\nabla f_i(\theta')}{M - f_i(\theta')} \right\|$$

$$= \left\| \frac{M(\nabla f_i(\theta) - \nabla f_i(\theta')) - \nabla f_i(\theta)f_i(\theta') + \nabla f_i(\theta')f_i(\theta)}{(M - f_i(\theta))(M - f_i(\theta'))} \right\|$$

$$\leq \frac{4}{M^2} (M\|\nabla f_i(\theta) - \nabla f_i(\theta')\| + \|\nabla f_i(\theta)f_i(\theta') - \nabla f_i(\theta')f_i(\theta)\|)$$

$$\leq \frac{4}{M^2} (ML\|\theta - \theta'\| + \|\nabla f_i(\theta)f_i(\theta') - \nabla f_i(\theta')f_i(\theta)\|).$$  \hspace{1cm} (A.43)
The second term in the parenthesis above can be computed as:

\[
\|\nabla f_i(\theta) f_i(\theta') - \nabla f_i(\theta') f_i(\theta)\| \leq \frac{4}{M^2} (M\|\nabla f_i - \nabla f_j\| + \|\nabla f_i - \nabla f_j\| \theta - \theta') \leq L_0\|\theta' - \theta\| + L_0 \frac{M^2}{2} \|\theta' - \theta\|.
\]

where in the second line we used triangle inequality; in the fourth line we used Assumptions 4.2 and 4.5.

Plugging back to (A.43) we have:

\[
\|\nabla (\varphi \circ f_i)(\theta) - \nabla (\varphi \circ f_i)(\theta')\| \leq \frac{4}{M^2} \left(3\frac{M}{L} + L_0^2\right) \|\theta - \theta'\|.
\]

Let us now figure out the variance terms. For the global variance term, we similarly write:

\[
\|\nabla (\varphi \circ f_i)(\theta) - \nabla (\varphi \circ f_j)(\theta)\| = \left\| \frac{\nabla f_i(\theta)}{M - f_i(\theta)} - \frac{\nabla f_j(\theta)}{M - f_j(\theta)} \right\|
\]

\[
= \left\| \frac{M(\nabla f_i(\theta) - \nabla f_j(\theta))}{(M - f_i(\theta))(M - f_j(\theta))} \right\|
\]

\[
\leq \frac{4}{M^2} (M\|\nabla f_i(\theta) - \nabla f_j(\theta)\| + \|\nabla f_i(\theta) f_j(\theta) - \nabla f_j(\theta) f_i(\theta)\|)
\]

\[
\leq \frac{4}{M^2}(M\sigma_g + \|\nabla f_i(\theta) f_j(\theta) - \nabla f_j(\theta) f_i(\theta)\|). \tag{A.46}
\]

The second term in the parenthesis above can be computed as:

\[
\|\nabla f_i(\theta) f_j(\theta) - \nabla f_j(\theta) f_i(\theta)\| = \|\nabla f_i(\theta) f_j(\theta) - \nabla f_i(\theta) f_i(\theta) + \nabla f_i(\theta) f_i(\theta) - \nabla f_j(\theta) f_i(\theta)\|
\]

\[
\leq \|\nabla f_i(\theta) f_j(\theta) - \nabla f_i(\theta) f_i(\theta)\| + \|\nabla f_i(\theta) f_i(\theta) - \nabla f_j(\theta) f_i(\theta)\|
\]

\[
= \|\nabla f_i(\theta)\| \cdot \|f_i(\theta) - f_i(\theta)\| + \|\nabla f_i(\theta) - \nabla f_j(\theta)\| \cdot \|f_i(\theta)\|
\]

\[
\leq L_0\sigma_0,g + \frac{M}{2}\sigma_g. \tag{A.47}
\]

where in the second and fourth lines we used triangle inequality; in the last line we used Assumptions 4.2 and 4.5.

Plugging (A.47) into (A.46) we find:

\[
\|\nabla (\varphi \circ f_i)(\theta) - \nabla (\varphi \circ f_j)(\theta)\|^2 \leq \left(\frac{4}{M} \left(3\frac{M}{L} + L_0\sigma_0,g\right)\right)^2.
\]

Let us finally compute the new local variance term. Denote

\[
\ell_S(\theta) := \frac{1}{|S|} \sum_{(x,y) \in S} \nabla \ell(\theta, (x,y)),
\]

where \(S\) is a batch with \(m\) samples i.i.d. drawn from \(\mathcal{D}_i\). We have:

\[
\|\nabla (\varphi \circ f_i)(\theta) - \nabla (\varphi \circ \ell_S)(\theta)\| = \left\| \frac{\nabla f_i(\theta)}{M - f_i(\theta)} - \frac{\nabla \ell_S(\theta)}{M - \ell_S(\theta)} \right\|
\]

\[
\leq \frac{4}{M^2} \left(3\frac{M}{L} \|\nabla f_i(\theta) - \nabla \ell_S(\theta)\| + L_0\|f_i(\theta) - \ell_S(\theta)\|\right), \tag{A.50}
\]

25
and the derivation follows similarly as (A.46). Taking the square on both sides and taking the expectation over \( S \sim D^m \), we obtain:

\[
E_{S \sim D^m} \| \nabla (\varphi \circ f_i)(\theta) - \nabla (\varphi \circ \ell_S)(\theta) \|^2 \leq \frac{16}{M^4} \left( \frac{9M^2}{4} E_{S \sim D^m} \| \nabla f_i(\theta) - \nabla \ell_S(\theta) \|^2 + \right.
\]
\[
+ 2L_0^2 E_{S \sim D^m} \| f_i(\theta) - \ell_S(\theta) \|^2 \right)
\]
\[
\leq \frac{8}{M^4} \left( 9M^2 \sigma_{i,i}^2 + 4L_0^2 \sigma_{0,i,i}^2 \right), \quad (A.51)
\]

where in the first line we used \((a+b)^2 \leq 2(a^2 + b^2)\) and in the second line we used Assumptions 4.3 and 4.5.

### B Additional Experiments

In this section, we provide more details about our experimental results.

#### B.1 Datasets and models

We describe the benchmark datasets in this subsection. For all datasets we fix the batch size to be 64.

**CIFAR-10/100** [Krizhevsky et al., 2009] are standard image classification datasets. There are 50000 samples with 10/100 balanced classes for CIFAR-10/100. By doing Dirichlet allocation [Wang et al., 2019a] we achieve the heterogeneity of label distributions. For all samples in each class \( k \), denoted as the set \( S_k \), we split \( S_k = S_{k,1} \cup S_{k,2} \ldots S_{k,n} \) into \( n \) clients according a symmetric Dirichlet distribution \( \text{Dir}(0.5) \). Then we gather the samples for client \( j \) as \( S_{1,j} \cup S_{2,j} \ldots S_{C,j} \) if we have \( C \) classes in total. We note that some of the clients might have too few samples (a few hundred). In this case the FL algorithm might overfit for such client and we regenerate the data split. We choose the number of clients to be 10 for both CIFAR-10/100. For each of the client dataset, we split it further into 60% training data and 40% test data for CIFAR-10. For CIFAR-100 we split the client dataset into 50% training data and 50% test data.

**CINIC-10** [Darlow et al., 2018] is an extension of CIFAR-10, which contains the images from CIFAR-10 and a selection of ImageNet database images (210,000 images downscaled to 32×32). We split the dataset into 50 clients due to the increased sample size. We use the symmetric Dirichlet distribution \( \text{Dir}(0.5) \) for partitioning the total 270,000 samples into 50 clients. For each client, we randomly select 50% of the client dataset as the training set and the rest as the test set.

**Shakespeare** [Shakespeare, 1614; McMahan et al., 2017] is a text dataset of Shakespeare dialogues, and we use it for the task of next character prediction. We treat each speaking role as a client resulting in a natural heterogeneous partition. We first filter out the clients with less than 10,000 samples and sample 20 clients from the remaining. Also, each client’s dataset is split into 50% for training and 50% for test.

**StackOverflow** [The Tensorflow Federated Authors, 2019] is another popular text dataset for benchmarking FL algorithms [Reddi et al., 2020; He et al., 2020]. Our task is next word prediction and our model is LSTM [Hochreiter & Schmidhuber, 1997]. We use the natural non-iid partition of StackOverflow and randomly select 20 users with between 10000 and 12500 samples. Afterwards, we split their data into training and test sets with an equal ratio.

In Table 4, we summarize these datasets, our partition methods, as well as the models we implement.
Table 4: Details of the experiments and the used datasets. ResNet-18 is the residual neural network defined in He et al. (2016). GN: group normalization (Wu & He, 2018); FC: fully connected layer; CNN: convolutional neural network; Conv: convolution layer; RNN: recurrent neural network; LSTM: long-term short memory layer. The plus sign means composition.

| Datasets   | Training set size | Test set size | Partition method          | # of clients | Model            |
|------------|-------------------|---------------|----------------------------|--------------|------------------|
| CIFAR-10   | 24948             | 25052         | Dirichlet partition ($\beta = 0.5$) | 10           | ResNet-18 + GN  |
| CIFAR-100  | 29636             | 20364         | Dirichlet partition ($\beta = 0.5$) | 10           | ResNet-18 + GN  |
| CINIC-10   | 134709            | 134970        | Dirichlet partition ($\beta = 0.5$) | 50           | CNN (2 Conv + 3 FC) |
| Shakespeare| 178796            | 177231        | realistic partition        | 20           | RNN (1 LSTM + 1 FC) |
| StackOverflow | 109671          | 109621        | realistic partition        | 20           | RNN (1 LSTM + 2 FC) |

B.2 Algorithms to compare and tuning hyperparameters

We compare our PropFair algorithm with common FL baselines, including FedAvg (McMahan et al., 2017), q-FFL (Li et al., 2020b) and AFL (Mohri et al., 2019). For each dataset and each algorithm (algorithms with different hyperparameters are counted as different), we find the best learning rate from a grid. Here are the grids we used for each dataset:

- CIFAR-10: \{1e-4, 1e-3, 2e-3, 5e-3, 1e-2, 2e-2, 5e-2, 1e-1\};
- CIFAR-100: \{1e-4, 1e-3, 2e-3, 5e-3, 1e-2, 2e-2, 5e-2, 1e-1\};
- CINIC-10: \{1e-4, 1e-3, 2e-3, 5e-3, 1e-2, 2e-2, 5e-2, 1e-1\};
- Shakespeare: \{1e-3, 1e-2, 5e-2, 1e-1, 5e-1, 1, 2, 5, 10\};
- StackOverflow: \{1e-3, 1e-2, 5e-2, 1e-1, 5e-1, 1, 2, 5, 10\}.

For AFL we tune the learning rate $\gamma_w$ from the corresponding grid and choose the default hyperparameter $\gamma_{\lambda} = 0.1$. We also find that $\gamma_{\lambda} = 0.01$ has worse performance than $\gamma_{\lambda} = 0.1$ for all datasets, and thus we stick to $\gamma_{\lambda} = 0.1$ throughout. For q-FFL, we run the $q$-FedAvg algorithm from Li et al. (2020b) with the default Lipschitz constant $L = 1/\eta$ from where $\eta$ is the learning rate. For each dataset we tune $q$ from \{0.1, 1, 5\}. For vision datasets we find $q = 0.1$ has the best performance and for language datasets we find $q = 5$ often leads to divergence during training. For PropFair we tune $M$ (Algorithm 1) from $M = 2, 3, 4, 5$. Table 5 shows the optimal values of $M$ used for different datasets:

| Datasets               | CIFAR-10 | CIFAR-100 | CINIC-10 | Shakespeare | StackOverflow |
|------------------------|----------|-----------|----------|------------|---------------|
| $M$                    | 5        | 2         | 5        | 2          | 4             |

After finding the best hyperparameters for each algorithm, we record the best learning rates in Table 6.

B.3 Detailed results

In Table 7 we report different statistics across clients, for all the algorithms and datasets we study in this work. These statistical quantities include:

- The mean of test accuracies of all clients;

https://github.com/litian96/fair_flearn/tree/master/flearn/trainers
Table 6: The best learning rates used for different datasets and algorithms, based on grid search.

| Datasets   | FedAvg | q-FFL | AFL | PropFair |
|------------|--------|-------|-----|----------|
| CIFAR-10   | 5e-3   | 2e-3  | 2e-3| 1e-2     |
| CIFAR-100  | 1e-2   | 1e-2  | 1e-2| 1e-2     |
| CINIC-10   | 1e-2   | 5e-3  | 1e-2| 2e-2     |
| Shakespeare| 1      | 5e-1  | 1   | 2        |
| StackOverflow | 5e-1 | 1e-1  | 5e-2| 5e-1     |

- The standard deviation of client accuracies;
- The worst test accuracy among the clients;
- The best test accuracy among the clients.

For each algorithm we take three different runs and report the mean and standard deviation of different statistical indices. Table 7 shows that PropFair is consistently advantageous across various datasets over the state-of-the-art algorithms.

Table 7: Comparison among federated learning algorithms on CIFAR-10, CIFAR-100, CINIC-10, Shakespeare and StackOverflow datasets with test accuracies (%) from clients. All algorithms are fine-tuned. **Mean**: the averaged performance across all clients; **Std**: standard deviation of client test accuracies; **Worst/Best**: the worst/best test accuracy from clients.

| Dataset     | Algorithm | Mean     | Std     | Worst    | Best     |
|-------------|-----------|----------|---------|----------|----------|
| CIFAR-10    | FedAvg    | 78.52±0.32| 15.27±0.19| 57.16±1.61| 99.92±0.07|
|             | q-FFL     | 77.35±0.17| 15.15±0.23| 56.51±0.28| 98.01±0.45|
|             | AFL       | 78.01±0.24| 15.34±0.10| 57.43±0.26| 100.0±0.00|
|             | PropFair  | 78.47±0.09| 14.79±0.07| 58.14±0.29| 98.55±0.12|
| CIFAR-100   | FedAvg    | 60.80±0.16| 3.39±0.08 | 53.46±0.27| 65.47±0.44|
|             | q-FFL     | 60.51±0.07| 3.19±0.03 | 53.57±0.14| 65.47±0.45|
|             | AFL       | 60.42±0.11| 3.37±0.11 | 53.51±0.31| 65.28±0.28|
|             | PropFair  | 61.36±0.24| 3.02±0.17 | 54.80±0.44| 65.73±0.16|
| CINIC-10    | FedAvg    | 81.29±0.09| 14.42±0.05| 49.93±0.43| 100.00±0.00|
|             | q-FFL     | 81.90±0.08| 14.22±0.05| 50.91±0.60| 100.00±0.00|
|             | AFL       | 80.17±0.34| 14.04±0.17| 49.02±1.78| 99.54±0.05|
|             | PropFair  | 81.96±0.23| 14.11±0.03| 51.04±0.69| 100.00±0.00|
| Shakespeare | FedAvg    | 47.52±0.28| 1.22±0.05 | 44.64±0.41| 49.47±0.07|
|             | q-FFL     | 47.14±0.38| 1.31±0.07 | 43.81±0.48| 49.33±0.45|
|             | AFL       | 47.45±0.17| 1.28±0.12 | 44.73±0.33| 49.81±0.12|
|             | PropFair  | 48.90±0.17| 1.26±0.01 | 46.69±0.35| 51.17±0.28|
| StackOverflow | FedAvg | 39.88±0.11| 7.11±0.06 | 25.05±0.31| 49.02±0.05|
|             | q-FFL     | 39.13±0.06| 7.25±0.01 | 24.03±0.15| 48.63±0.05|
|             | AFL       | 36.73±0.23| 7.17±0.09 | 22.24±0.13| 46.04±0.37|
|             | PropFair  | 41.15±0.08| 6.98±0.02 | 26.35±0.18| 49.92±0.03|
Algorithm 2: FedAvg

Input: global epoch $T$, client number $n$, loss function $f_i$, number of samples $n_i$ for client $i$, initial global model $\theta_0$, local step number $K_i$ for client $i$, learning rate $\eta$

for $t$ in 0, 1, ..., $T - 1$

randomly select $S_t \subseteq [n]$

$\theta^{(i)}_t = \theta_t$ for $i \in S_t$, $N = \sum_{i \in S_t} n_i$

for $i$ in $S_t$ do // in parallel

starting from $\theta^{(i)}_t$, take $K_i$ local SGD steps on $f_i$ to find $\theta^{(i)}_{t+1}$

$\theta_{t+1} = \frac{\sum_{i \in S_t} \frac{n_i}{N} \theta^{(i)}_{t+1}}{\sum_{i \in S_t} \frac{n_i}{N}}$

Output: global model $\theta_T$

C FedAvg Algorithm

For easy reference, we include FedAvg \citep{McMahan2017} in Algorithm 2 whose goal is to optimize the overall performance. At each round, each client takes local SGD steps to minimize the loss function based on the client data. Afterwards, the server computes a weighted average of the parameters of these participating clients, and shares this average among them.

D Dual View of PropFair

Let us derive the dual of the generalized mean for PropFair in the same framework as in Section 2.2. Note that

$$s(t) = -\log(M - t), \quad (D.1)$$

and therefore the generalized mean is:

$$A_s(f) = s^{-1} \left( \frac{1}{n} \sum_{i} s(f_i) \right) = M - \left( \prod_{i=1}^{n} (M - f_i) \right)^{1/n}. \quad (D.2)$$

We observe that $A_s$ is a convex function, since it is composition of geometric mean (which concave from Mahler’s inequality) and affine transformation. Now we compute the dual function

$$A^*_s(\lambda) = \sup_{f \leq M1} \lambda^\top f - A_s(f)$$

$$= \sup_{f \leq M1} \lambda^\top f + \left( \prod_{i=1}^{n} (M - f_i) \right)^{1/n} - M \quad (D.3)$$

If any entry $\lambda_i$ is non-positive, clearly we can let $f_i \to -\infty$ so that $A^*_s(\lambda) \to \infty$. For positive $\lambda$, and $\prod_{i=1}^{n} (n\lambda_i) < 1$, we can take $f_i = M - \frac{c}{n\lambda_i}$ and get:

$$\lambda^\top f + \left( \prod_{i=1}^{n} (M - f_i) \right)^{1/n} - M = \sum_{i=1}^{n} \left( M\lambda_i - \frac{c}{n} \right) + \left( \prod_{i=1}^{n} \frac{c}{n\lambda_i} \right)^{1/n} - M$$

$$= M(\lambda^\top 1 - 1) + \left( \frac{1}{\prod (n\lambda_i)} - 1 \right) c \quad (D.4)$$
We remark that

We focus on the inner maximization so that we know the weights we put on each client: we have

To further demonstrate the dual view of different algorithms. We extend Table 3 and study the linear weight

where in the second line we used the AM-GM inequality and in the last line we used

Thus, we verify again that the optimal value of (D.9) is:

where the equality is attained iff

(D.5)

where in the second line we used the AM-GM inequality and in the last line we used \( \prod_{i=1}^{n} (n \lambda_i) \geq 1 \). This equality can always be achieved by taking \( f = M \mathbf{1} \). In summary, we have:

We remark that \( A^*_s(\lambda) \) is closed (since its domain is closed). If we want to enforce \( f \geq \mathbf{0} \) when computing the dual function, we simply apply the convolution formula:

However, the formula for \( A^*_s \) suffices for our purpose so we need not compute the above explicitly.

Applying the above conjugation result we can rewrite PropFair as:

We focus on the inner maximization so that we know the weights we put on each client:

Using the AM-GM inequality we have:

where the equality is attained iff \( \prod_{i=1}^{n} (n \lambda_i) = 1 \) and

(D.11)

Thus, we verify again that the optimal value of (D.9) is:

and we retrieve our original objective. (D.11) tells us that we are essentially solving a linearly weighted combination of \( f_1, \ldots, f_n \), but with more weights on the worse-off clients, since \( \frac{1}{M-f_i} \) is larger for larger \( f_i \).

D.1 More details about Example 1.1

To further demonstrate the dual view of different algorithms. We extend Table 3 and study the linear weight \( \lambda \) for these algorithms in Table 8. One can verify that for \( q \)-FFL \( ( q = 1 ) \) we have \( \| \lambda \|_2^2 = \frac{1}{2} \) and for PropFair we have \( \sqrt{\lambda_1 \lambda_2} = \frac{1}{2} \) and \( \lambda_i \propto \frac{1}{M-f_i} \) (we round the exact values so the equations may become inexact).
Table 8: Comparison between the solutions found by different algorithms for Example 1.1. x_{alg} denotes the solution found by optimizing the corresponding objective for each algorithm. For \textit{q-FFL} we take \( q = 1 \) and for PropFair we take \( M = 2 \).

|          | value | \( f_1 \) | \( f_2 \) | \( \sum_i f_i \) | \( \sum_i f_i^2 \) | \( \prod_i (2 - f_i) \) | \( \lambda_1 \) | \( \lambda_2 \) |
|----------|-------|-----------|-----------|-----------------|-----------------|-----------------|-----------|-----------|
| \textit{xFedAvg} | 1/2   | 0.19      | 1.13      | 1.31            | 1.30            | 1.59            | 1/2       | 1/2       |
| \textit{x_q-FFL} | 1/4   | 0.34      | 1.03      | 1.38            | 1.18            | 1.60            | \( \frac{1}{2\sqrt{5}} \) | \( \frac{3}{2\sqrt{5}} \) |
| \textit{xAFI}     | 0     | 0.56      | 1.00      | 1.32            | 1.44            | 0               | 1         |
| \textit{xPropFair} | 0.35  | 0.27      | 1.06      | 1.33            | 1.20            | 1.62            | 0.37      | 0.68      |

E More Related Work

In this appendix we introduce more related work, including multi-objective optimization, fairness in FL, as well as various definitions of fairness from multiple fields.

E.1 Multi-objective optimization

Multi-objective optimization (MOO) has been intensively studied in the field of operation research (Geoffrion, 1968; Yu & Zeleny, 1975; Jahn et al., 2009). The goal of MOO is to minimize a series of objectives \( f_1, f_2, \ldots, f_n \) based on their best trade-offs. This is directly related to federated learning (Hu et al., 2020) because one can treat the loss function of each client as an objective.

In MOO, Pareto optimality is often desired. To find a Pareto optimum, one way is to use an aggregating objective (a.k.a. scalarizing function [Lootsma et al., 1995]). We list some common choices of this aggregating objective:

- **Linear weighting method** [Geoffrion, 1968]: this method converts MOO into the problem of minimizing the convex combination of client objectives:

  \[
  \min_{x \in X} \sum_{i=1}^{n} \lambda_i f_i(x),
  \]  
  \( \text{E.1} \)

  with \( \lambda \in \Delta_{n-1} \) in the \((n-1)\)-simplex, and \( X \) a set of \( x \). Such solution is always Pareto optimal and the method has been used in FedAvg (McMahan et al., 2017). A well-known difficulty is that it cannot generate point in the nonconvex part of the Pareto front (Audet et al., 2008).

- **Reference point** [Audet et al., 2008]: This method requires proximity to the \textit{ideal point}: \( r = (\min_{x \in X} f_1(x), \ldots, \min_{x \in X} f_n(x)) \), measured by the \( \ell_q \)-norm:

  \[
  \min_{x \in X} \| f(x) - r \|_q := \sum_{i=1}^{n} (f_i(x) - r_i)^q,
  \]  
  \( \text{E.2} \)

  with \( f(x) := (f_1(x), \ldots, f_n(x)) \) and \( \| \cdot \|_q \) the \( \ell_q \)-norm \( (q \geq 1) \). This method has been applied to federated learning as \textit{q-FFL} (Li et al., 2020b) (by assuming \( r = 0 \)).

- **Weighted geometric mean** [Lootsma et al., 1995]: this method converts MOO to a single-objective formulation by maximizing the weighted geometric mean between elements of the \textit{nadir point} and the client objectives:

  \[
  \max_{x \in X} \prod_{i=1}^{n} (p_i - f_i(x))^{\lambda_i}, \text{ such that } f_i(x) \leq p_i \text{ for any } i \text{ and } x \in X,
  \]  
  \( \text{E.3} \)
where \( p \) is called a nadir point, defined as (Lootsma et al., 1995):

\[
p_i = \max_{j=1,2,...,n} f_i(x^*_j),
\]

(E.4)

with \( x^*_j = \arg \min_{x \in X} f_j(x) \) the optimizer of function \( f_j \). The \( \lambda_i \)'s are the weights for each client and they are positive. If we take \( \lambda = (\lambda_1, \ldots, \lambda_n) = 1 \), then it resembles our objective in (4.2).

### E.2 Fairness in federated learning

As FL has been deployed to more and more real-world applications, it has become a major challenge to guarantee that FL models have no discrimination against certain clients and/or sensitive attributes. Since different participants may contribute differently to the final model’s quality, it is necessary to provide a fair mechanism to encourage user participation.

Besides the related work we mentioned in the main paper (McMahan et al., 2017; Mohri et al., 2019; Li et al., 2020), another direction of research tries to directly encourage the involvement of user participation, by providing some rewards to fairly recognize the contributions of clients. For example, Lyu et al. (2020) designed a local credibility mutual evaluation mechanism to enforce good contributors get more credits. Concretely, each client computes the contribution of every other client by investigating the label similarities of the synthetic samples generated by the clients’ differential private GANs (Goodfellow et al., 2014). Kang et al. (2020) proposed a pairwise measurement of contribution. Reputation scores are kept at each client for all other clients, and are updated by a multi-weight subjective logic model. Yu et al. (2020) proposed a Federated Learning Incentivizer (FLI) payoff-sharing scheme, which dynamically divides a given budget among clients by optimizing their joint utility while minimizing their discrepancy. The objective function takes into account the amount of payoff and the waiting time to receive the payoff. Wang et al. (2020) analyzed the contribution from the data side, and proposed the federated Shapley Value (SV) for data valuation. While preserving the desirable properties of the canonical SV, this federated SV can be calculated with no extra communication overhead, making it suitable for the FL scenarios.

The above methods already applied some objective functions that reflect the concept of proportional fairness, e.g., payoff proportional to the contribution. However, they mostly apply fixed contribution-reward assignment rules, without explicit definitions of proportional fairness or theoretical guarantee.

### E.3 Definitions of fairness

Fairness has been a perennial topic in social choice (Sen, 1986), communication (Jain et al., 1984), law (Rawls, 1999) and machine learning (Barocas et al., 2017). Whenever we have multiple agents and limited resources, we need fairness to allocate the resources. There have been many definitions of fairness, such as individual fairness (Dwork et al., 2012), demographic fairness, counterfactual fairness and proportional fairness.

In this section, we introduce definitions of fairness from various perspectives including social choice, communication and machine learning, and study the implications in the setting of FL.

#### E.3.1 Social Choice and Law

We review some principles for fairness and justice in social choice (Sen, 1986) and law (Rawls, 1999), which resembles FL: we can treat the shared global model as a public policy and clients as social agents.

- **Utilitarian rule (Maskin, 1978):** suppose we have \( n \) clients and their loss functions are \( f_i \), the utilitarian rule aims to minimize the sum of the loss functions, e.g.,

\[
\min_{\theta} \sum_i f_i(\theta),
\]

(E.5)

with \( \theta \) the global model parameter. This utilitarian rule represents the utilitarian philosophy: as long as the overall performance of the whole society is optimal, we call the society to be fair. A utilitarian
policy is Pareto-optimal but not vice versa. With model homogeneity, equation (E.5) is nothing but the objective for FedAvg (McMahan et al., 2017), although the FedAvg algorithm may not always converge to the global optimum even in linear regression (Pathak & Wainwright, 2020).

- **Egalitarian rule (Rawls, 1974, 1999):** The egalitarian rule, also known as the maximin criterion represents egalitarianism in political philosophy. Instead of maximizing the overall performance as in (E.5), an egalitarian wants to maximize the performance of the worst-case client, i.e., we solve the following optimization problem:

  $$\min_{\theta} \max_i f_i(\theta). \quad (E.6)$$

  This accords with Agnostic FL (Mohri et al., 2019). The egalitarian problem (E.6) may not always be Pareto optimal, e.g., \((f_1, f_2, f_3) = (1, 1, 1)\) and \((f_1, f_2, f_3) = (1, 0.9, 0.8)\) can both be the optimal solution of (E.6), but the former is not Pareto optimal.

### E.3.2 Fairness in wireless communications

Since resource allocation is common in communication, different notions of fairness have also been proposed and studied. We review some common fairness definitions in communication:

- **Max-min fairness / Pareto optimal (Bertsekas & Gallager, 1987):** this definition says at the fair solution, one cannot simultaneously improve the performance of all clients, which is equivalent to the definition of Pareto optimal. The corresponding algorithm in FL for finding a Pareto optimum is FedMGDA+ (Hu et al., 2020).

- **α-fairness (Lan et al., 2010):** the optimization problem is:

  $$\max_{\theta} \sum_i u_i(\theta)^\alpha, \ \alpha > 0. \quad (E.7)$$

  The optimal solution is always Pareto optimal. When \(\alpha \to \infty\), the optimal solution corresponds to the egalitarian solution (a.k.a. Agnostic FL); when \(\alpha = 1\), the optimization problem corresponds to the utilitarian rule, i.e., FedAvg.

- **Proportional-fair rule (Kelly et al., 1998; Bertsimas et al., 2011):** proportional fairness aims to find a solution \(\theta^*\) such that for all \(\theta\) in the domain:

  $$\sum_i \frac{u_i(\theta) - u_i(\theta^*)}{u_i(\theta^*)} \leq 0, \quad (E.8)$$

  with \(u_i\) the utility function of client \(i\), e.g., the test accuracy. This problem aims to find a policy such that the total relative utility cannot be improved. Proportional-fairness has been studied in communication (e.g. Seo & Lee, 2006) for scheduling but the application in FL has not been seen.

- **Harmonic mean (Dashti et al., 2013):** the method maximizes the harmonic mean of the utility functions of each client, that is, we solve the following optimization problem:

  $$\max_{\theta} \frac{n}{\sum_i u_i(\theta)^{-1}} \quad (E.9)$$

  In a similar vein we can find its optimality condition, assuming the utility set \(\mathcal{U}\) is convex:

  $$\sum_{i=1}^n \frac{u_i - u_i^*}{(u_i^*)^2} \leq 0, \text{ for all } u \in \mathcal{U}. \quad (E.10)$$

  Compared to proportional fairness, it simply amounts to squaring the denominator.
E.3.3 Fairness in machine learning

Fairness has been studied in machine learning for almost a decade (Barocas et al., 2017). A large body of work focuses on proposing machine learning algorithms for achieving different definitions of fairness. These definitions are often incompatible with each other, i.e., one cannot achieve two definitions of fairness simultaneously. Let us review some common definitions, using classification as an illustrating example:

- **Group fairness / statistical parity / demographic parity** (DP, Dwork et al., 2012; Zemel et al., 2013): this definition requires that the prediction is independent of the subgroup (e.g., race, gender). Denote $Y$ as the prediction and $S$ as the sensitive attribute, this definition requires $Y \perp S$, where the symbol $\perp$ denotes statistical independence. This is the simplest definition of fairness, and probably what people think of at a first thought. However, this definition can be problematic. For instance, suppose in an FL system a subgroup of clients have poor performance (e.g., due to communication, memory), and then to achieve better group fairness one can deliberately lower the performance of high-performing clients, and thus the overall performance is lower. Moreover, DP would forbid us to achieve the optimal performance if the true labels are not independent of the sensitive attribute (Hardt et al., 2016; Zhao & Gordon, 2019).

- **Equalized odds (EO)** (Hardt et al., 2016): this definition requires demographic parity given each true label class. Define $T$ as the random variable for the true label. Equalized odds requires that $Y \perp S | T$ for any $T$ and equal opportunity requires that $Y \perp S | T$ for some $T$. Different from DP, this conditioning allows the prediction to align with the true label. In the binary setting, EO and DP cannot be simultaneously achieved (Barocas et al., 2017).

- **Calibration / Predictive Rate Parity** (Gebel, 2009): this definition requires that among the samples having a prediction score $Y$, the expectation of the true label $T$ should match the prediction score, i.e., $E[T|Y] = Y$. In the context of fairness, calibration says that $T \perp S | Y$. Under mild assumptions, calibration and EO cannot be simultaneously achieved (Pleiss et al., 2017). Similarly, calibration and DP cannot be simultaneously achieved.

- **Individual fairness** (Dwork et al., 2012): this concept requires that similar samples, as measured by some metric, should have similar predictions.

- **Counterfactual fairness** (Kusner et al., 2017): this definition requires that from any sample, the prediction should be the same had the sensitive attribute taken different values. It follows the notion of counterfactual from casual inference (Pearl et al., 2000).

- **Accuracy parity** (Zafar et al., 2017): the accuracy for each group remains the same.

Since many concepts conflict with each other (Barocas et al., 2017), there is no unified definition of fairness. In light of this, a dynamical definition of fairness has been proposed (Awasthi et al., 2020). Algorithms for achieving different definitions of fairness include mutual information (Zemel et al., 2013), representation learning (Zemel et al., 2013; Zhao & Gordon, 2019) and Rényi correlation (Baharlouei et al., 2019).