The Dynamical and Static Casimir Effects
and
the Thermodynamic Instability

E. Sassaroli* [1, 2], Y. N. Srivastava [3,2], J. Swain [2] and A. Widom [2]

1. Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics
Massachusetts Institute of Technology, Cambridge, MA 02139, USA
2. Physics Department, Northeastern University Boston, Massachusetts, 02115, USA
3. Dipartimento di Fisica and I.N.F.N, Universita’ degli Studi di Perugia, 06100 Perugia, Italy

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ABSTRACT

The dynamical Casimir effect for the ideal case of two perfectly conducting non-charged parallel plates, is discussed using the zero-point energy summation method to the first order in perturbation theory. We show that it is possible to create photon radiation when the two plates are modulated rapidly in time. Moreover we point out that the static Casimir energy between two conducting non-charged parallel plates violates the thermodynamic stability condition normally associated with the second law of thermodynamics.

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1. Introduction

It has become common practice to use the label Casimir effect [1] whenever one wishes to describe a non-classical electromagnetic force of attraction between two perfectly conducting non-charged parallel plates. The central idea behind this effect is that the vacuum can be “perturbed” by inserting objects to produce measurable effects. The electromagnetic vacuum near a conducting plate is different from the one in the empty space. The perturbed vacuum gives rise to an attractive long range force between the two plates.

Over the past fifty years, Casimir forces have been calculated for a variety of different geometries. Examples include plane dielectric surfaces, cavities, polarizable particles, etc. Different methods have been developed to evaluate these forces: the mode summation method, the local Green function method, source theory, etc. [2-5].

Regardless of the way the Casimir forces are calculated, they are obtained by holding the geometry either fixed or by varying it only quasi-statically. We use the term dynamical Casimir effect [6] when the geometry of the system is varied more quickly in time. In this case photon production becomes possible simply by accelerating the vacuum.

In this paper we discuss the dynamical Casimir effect for the ideal case of two perfectly conducting non-charged parallel plates [7]. The frequencies of the zero-point electromagnetic energy between the plates depend on the distance between them. If we modulate the distance in time the zero-point electromagnetic energy becomes also a function of time. The result is the possibility of photon production, if the modulation occurs quickly enough in time.

There is much recent discussion in the literature about the possibility of photon production by the dynamical Casimir effect, see for example [8-24] and references there.

Moreover, we point out here that the Casimir vacuum between two perfectly conducting plates is thermodynamically instable having a zero temperature negative compressibility [25].
2. The dynamical Casimir effect

The static Casimir energy for two perfectly conducting plates separated by a distance \( d \) (\( L_x = L_y = L \) and \( L_z = d \) with \( L \gg d \)) is given by the difference between the zero-point energy when the plates are placed at a distance \( d \) and when the distance between them is infinite

\[
U(d) = E_o(d) - E_o(\infty) = -\frac{\pi^2 \hbar c L^2}{720d^3}. \quad (1)
\]

Now suppose that the distance \( d \) varies as a function of time \( d(t) \). We assume a constant initial value \( d_i = a \) and after a time \( \Delta t \) the modulation stops and the distance \( d \) reaches a final value, which for simplicity we assume equal to the initial value \( d_f = a \).

The possibility of photon production due to the effect of the plate modulation can be described as follows.

The Hamiltonian for a single electromagnetic mode before the modulation is

\[
H_{\text{in}} = \hbar \omega_{\text{in}} (a_{\text{in}}^\dagger a_{\text{in}} + \frac{1}{2}), \quad (2)
\]

with

\[
\omega_{\text{in}} = \omega = c \sqrt{\left(\frac{\pi l}{L}\right)^2 + \left(\frac{\pi m}{L}\right)^2 + \left(\frac{\pi n}{a}\right)^2}, \quad (3)
\]

where \( l, m, n = 0, 1, 2, \ldots \), with the restriction that only one integer at a time can be zero. After the modulation the Hamiltonian is

\[
H_{\text{out}} = \hbar \omega_{\text{out}} (a_{\text{out}}^\dagger a_{\text{out}} + \frac{1}{2}), \quad (4)
\]

with \( \omega_{\text{out}} = \omega_{\text{in}} \).

We can relate the ladder operators \( a_{\text{in}} \) and \( a_{\text{out}} \) through a Bogolubov transformation

\[
a_{\text{out}} = U a_{\text{in}} + V a_{\text{in}}^\dagger, \quad a_{\text{out}}^\dagger = U^* a_{\text{out}}^\dagger + V^* a_{\text{out}}, \quad (5)
\]

where the relation

\[
|U|^2 - |V|^2 = 1, \quad (6)
\]
is required by the equal time commutation relation.

The mean number of photons produced through the modulation of the single electromagnetic mode [26]

\[ N = \langle 0 | a_{\text{out}}^\dagger a_{\text{out}} | 0 \rangle, \]  

obeys the relation

\[ N = |V|^2, \]  

through Eqs. (5) and (6), with \( a_{\text{in}} | 0 \rangle = 0. \)

To the first order in perturbation theory \( N \) can be determined in terms of the scattering solutions of the equation [27]

\[ \frac{d^2}{dt^2} q(t) + \omega^2(t) q(t) = 0, \]  

with

\[ \omega(t) = c \sqrt{\left( \frac{\pi l}{L} \right)^2 + \left( \frac{\pi m}{L} \right)^2 + \left( \frac{\pi n}{d(t)} \right)^2}, \]  

subject to the boundary condition that for large \( t \) the time dependent frequency \( \omega(t) \) goes to the unperturbed values

\[ \omega(t \to -\infty) = \omega_{\text{in}} = \omega, \quad \omega(t \to \infty) = \omega_{\text{out}} = \omega. \]

Eq. (9) is formally identical to a one-dimensional Schrödinger equation, where \( t \to x. \)

If the potential \( \omega(t) \) is symmetrical respect to the origin and goes to zero for large values of \( |t| \), then the asymptotic solutions of the Eq. (9) are

\[ q(t) = \begin{cases} 
A e^{i \omega t} + B e^{-i \omega t}, & t \to -\infty, \\
F e^{i \omega t} + G e^{-i \omega t}, & t \to +\infty.
\end{cases} \]

We can relate the coefficients \( A, B, F, G \) through the matrix \( M \) defined by

\[ \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \alpha_1 + i \beta_1 & i \beta_2 \\ -i \beta_2 & \alpha_1 - i \beta_2 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \]

where \( \alpha_1, \beta_{1,2} \) are real numbers satisfying the relation

\[ \alpha_1^2 + \beta_1^2 - \beta_2^2 = 1. \]
It is possible to express $N$ in terms of the reflection coefficient $R$

$$R = \frac{|B|^2}{|A|^2}$$  \hspace{1cm} (15)

as

$$N = |V|^2 = \frac{R}{1 - R}.$$  \hspace{1cm} (16)

Eq. (16) can be easily obtained by noticing that $[1/(1 - R) - R/(1 - R)]$ satisfies Eq. (6) with $|U|^2 = 1/(1 - R)$.

Suppose now that the distance between the two plates in modulated periodically in time during some time interval $\Delta t$ by a succession of pulses. Each pulse satisfying Eqs. (9) and (10). The number of photons produced in a given electromagnetic mode is [7]

$$N = \beta^2 \left(\frac{\sin(r\gamma)}{\sin(\gamma)}\right)^2,$$  \hspace{1cm} (17)

where $r$ is the number of pulses and $\gamma$ satisfies the condition

$$\cos(\gamma T) = \alpha_1 \cos(\omega T) + \beta_1 \sin(\omega T),$$  \hspace{1cm} (18)

with $T$ modulation period.

The total number of photons produced is obtained by taking the sum over all the electromagnetic modes up to a cut-off frequency $\nu_c \sim 10^{15}$ Hz.

$$N_{tot} = \sum_{k\lambda} N_{k\lambda}(\omega),$$  \hspace{1cm} (19)

where $\lambda = 1, 2$ is the polarization index.

We can evaluate $N_{tot}$ for the following interesting case. For $L \gg a$ the frequency $\omega$ in Eq. (10) can be written as

$$\omega = 2\pi \nu = \frac{\pi c}{a} \sqrt{x^2 + n^2}, \quad x = ak_||/\pi,$$  \hspace{1cm} (20)

with $k_||^2 = k_x^2 + k_y^2$. If $a \sim 10^{-4}$ cm the frequency $\omega \sim 10^{15}$ Hz i.e. in the visible region of the electromagnetic spectrum.
It is possible to show that for the value of \( a \sim 10^{-4} \) cm and for a modulation period \( T \sim 10^{-3} \) s the total number of photons produced per unit area and per pulse in the limit \( r \to \infty \) is [7]

\[
\mathcal{N} \sim \frac{1}{a^2} \sum_{n=1}^{n_c} \int dx x \beta_2^2(x, \ldots).
\]  

(21)

The dots in Eq. (20) means that the function \( \beta_2 \) is not only a function of \( x \) but also of the parameters which characterize each pulse, such as for example the duration and the modulation strength. If for those values the term \( 1/a^2 \) gives the dominant contribution in Eq. (21) then the photon production can be appreciable, as it can be seen through the following examples.

### 3. Examples

As a first example, we consider a succession of time pulses such that for each pulse the distance between the two plates varies as

\[
d^2(t) = \frac{a^2}{1 + \frac{b^2}{\cosh^2(\pi t/\tau)}},
\]  

(22)

where \( \tau \) describes the pulse duration and \( b \) determines the modulation strength. We assume \( b \sim 5 \), \( a \) and \( T \) given above.

Therefore the modulated single mode frequency is

\[
\omega^2(t) = \omega^2 + \left( \frac{c \pi n b}{a \cosh(\pi t/\tau)} \right)^2,
\]  

(23)

where \( \omega \) is given by Eq. (20). The function \( \beta_2 \) for this case is

\[
\beta_2 \sim \frac{1}{\sinh^2(\omega \tau)}.
\]

The function \( \beta_2 \) goes to zero very quickly and therefore also the integral in Eq. (21) except when \( \tau \sim 1/\omega \). Therefore the photon production can be appreciable only for very sharp pulses.
As a second example we consider a succession of rectangular potential barriers. In the valleys of the potential $\omega(t)$ the distance between the two plates is

$$d^2(t) = a^2,$$  \hspace{1cm} (24)

and in the hills of the potential $\omega(t)$

$$d^2(t) = \frac{a^2}{1 + b^2},$$  \hspace{1cm} (25)

where $b$ is the modulation strength.

The function $\beta_2$ is given by

$$\beta_2 = \frac{1}{2} \left( \frac{\omega}{\omega_2} - \frac{\omega_2}{\omega} \right) \sin(2\omega_2 T),$$  \hspace{1cm} (26)

where $\omega_2$ is

$$\omega_2 = \frac{c \pi}{a} \sqrt{x^2 + n^2(1 + b^2)}.$$  \hspace{1cm} (27)

For the values of $a$, $T$ and $b$ chosen above, the estimated number of produced photons is

$$\sim 10^{11}/\text{area} - \text{pulse},$$  \hspace{1cm} (28)

mainly in the visible region.

Therefore if there is a rapid modulation of the plates photon production becomes possible.

The dynamical Casimir effect has been considered as possible explanation of the phenomenon of sonoluminescence [6]. In the sonoluminescence experiments air bubbles in water, under the effect of sound waves, expand up to the dimension $\sim 50 \mu m$ and then collapse very quickly to a dimension $\sim 0.5 \mu m$. During the collapse a flash of light of duration of femtoseconds is emitted [28]. Although there is not yet a clear explanation of the sonoluminescence phenomenon and different theories have been proposed, the model presented here can give a simple and qualitative explanation of the phenomenon in terms of the dynamical Casimir effect. A pulse varying very quickly in time can describe very approximately what happens when the bubble collapses and the periodicity can give an idea how the bubble can emit light periodically.
4. The static Casimir effect and thermodynamic instability

A property shared by many one loop quantum statistical thermodynamic computations is that a thermodynamic second law instability appears in the final answer. Perhaps the most commonly discussed example of this phenomenon occurs in black hole statistical thermodynamics [29,30]. The entropy and the energy of a stationary non-charged, non-rotating black hole having mass $M$ are given by

$$S = 4\pi k_B \left( \frac{GM^2}{\hbar c} \right), \quad E = mc^2$$  \hspace{1cm} (29)

where $G$ is Newton’s gravitational coupling strength. The black hole temperature, defined as $T = c^2(\partial M/\partial S)$, is then determined by

$$M = \left( \frac{\hbar c^3}{8\pi G k_B T} \right).$$  \hspace{1cm} (30)

The black hole heat capacity $C = c^2(\partial M/\partial T)$ is thereby negative,

$$C = c^2 \frac{\partial M}{\partial T} = -\frac{K_B \hbar c^5}{8\pi GT^2} < 0,$$  \hspace{1cm} (31)

therefore the black hole is thermodynamically instable. An increase in its energy decreases its temperature. This fact is considered by some authors as a possible violation of the second law of thermodynamics.

That a one loop quantum gravity calculation, i.e. the gravitational Casimir effect, produces thermodynamical instability is (perhaps) not very surprising. Even at the Newtonian theoretical level, the long range gravitational attraction upsets the usual convexity conditions otherwise present in the thermodynamic limit of infinite size.

Our purpose is to show that the electrodynamic Casimir effect can also produce thermodynamic instability. To see what is involved, suppose that a material is located inside a box of volume $V = Az$ with a movable piston. The free energy per unit area obeys

$$df = -sdT - Pdz,$$  \hspace{1cm} (32)
where \( S = A s \) and \( F = Af \). If the system is thermodynamically stable then the isothermal compressibility must be non-negative

\[
K_T = -\left( \frac{1}{V} \right) \left( \frac{\partial V}{\partial P} \right)_T = -\left( \frac{1}{z} \right) \left( \frac{\partial z}{\partial P} \right)_T \geq 0. \tag{33}
\]

However the condition given by Eq. (33) is not satisfied for the case of the electromagnetic vacuum, which is therefore thermodynamically unstable.

Let \( f(z, T) \) be the free energy per unit area of the vacuum inside two perfectly conducting non-charged parallel plate The ground state energy per unit area of this vacuum is given by

\[
\epsilon(z) = \lim_{T \to 0} f(z, T) = -\left( \frac{\pi^2}{720} \right) \left( \frac{\hbar c}{z^3} \right), \tag{34}
\]

yielding the pressure

\[
P_0 = \lim_{T \to 0} P(z, T) = -\left( \frac{\pi^2}{240} \right) \left( \frac{\hbar c}{z^4} \right). \tag{35}
\]

The zero temperature compressibility then reads

\[
K_0 = \lim_{T \to 0} K_T = -\left( \frac{60}{\pi^2} \right) \left( \frac{z^4}{\hbar c} \right) < 0. \tag{36}
\]

The negative compressibility \((K_0 < 0)\) in Eq.(36) violates the stability condition, as written in Eq.(33).

The notion of negative compressibility matter is as old as the van der Waals approximation to the equations of state of a material [31]. It has always been stated that such equations of state require supplementary conditions such as equal area constructions and so forth (see for example Ref. [31] pp 257-62). Furthermore, the second thermodynamic law has been thought to put a complete and total veto on observing the totally unstable part of the van der Waals curve; i.e. \( K_T < 0 \) exists formally in the approximation, but is strictly forbidden from observation.

So now we have a paradox, and perhaps an interesting energy source. For the Casimir force, and even for Coulomb’s law, the regime \( K_T < 0 \) may be asserted to be real.
To end in order to see under which condition there is stability, consider the vacuum fluid in a cylinder separated by an internal movable piston from a given material. Then the total free energy of the system at zero temperature is

\[ f_{tot} = f(z, 0) + f'(z - L, 0), \]  

(37)

where \( f(z - L)' \) is the free energy of the material and \( L \) is the total length of the cylinder with \( V = AL \).

The stability condition dictates that

\[ \frac{\partial^2 f'}{\partial z^2} \geq 0, \]  

(38)

and therefore

\[ \frac{1}{K_o} + \frac{1}{K_m} \geq 0, \]  

(39)

where \( K_m \) is the compressibility of the material. Hence thermodynamic stability is reached when

\[ K_m \geq |K_o|, \]  

(40)

5. Summary

We have shown that it is possible to create photon radiation by modulating the vacuum between two perfectly conducting plates when the distance between them is sufficiently small and the modulation is very rapid in time.

Moreover we have considered the stability condition associated with the electromagnetic vacuum between two perfectly conducting non-charged parallel plates, which can be thermodynamically unstable.
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