Critical Velocity of Vortex Nucleation in Rotating Superfluid $^3$He-A

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(March 24, 2022)

We have measured the critical velocity $v_c$ at which $^3$He-A in a rotating cylinder becomes unstable against the formation of quantized vortex lines with continuous (singularity-free) core structure. We find that $v_c$ is distributed between a maximum and minimum limit, which we ascribe to a dependence on the texture of the orbital angular momentum $\hat{l}(\mathbf{r})$ in the cylinder. Slow cool down through $T_c$ in rotation yields $I(\mathbf{r})$ textures for which the measured $v_c$’s are in good agreement with the calculated instability of the expected $I$ texture.

PACS numbers: 67.57.Fg, 47.32.-y, 05.70Fh

A first order transition from one phase to another is associated with hysteresis because of the difficulty of nucleating the new phase. Two effects generally reduce the hysteresis. Firstly, thermal or quantum fluctuations cause the new phase to appear before the energy barrier separating the two energy minima vanishes. Secondly, surfaces, impurities, or other external agents reduce the energy barrier from its intrinsic value. Both phenomena are of crucial importance for the long standing problem of critical velocities and vortex nucleation in superfluids [1], but occur also in more usual phenomena like formation of water droplets or gas bubbles [3]. The purpose of the present work is to study an exceptional case of vortex nucleation where neither fluctuations nor external surfaces should play a role: superfluid $^3$He-A.

In usual superfluids and superconductors the phase slip takes place by creation and motion of zeros in the order parameter [1]. The A phase of $^3$He is exceptional because the phase slip arises from the motion of the local angular momentum axis $I(\mathbf{r})$. The characteristic length scale of the $I(\mathbf{r})$ texture is macroscopic $\sim 10 \mu m$. Therefore all thermal and quantum fluctuations are negligible. Moreover, a rigid boundary condition fixes $I$ perpendicular to the wall of the experimental container. Thus the processes responsible for the critical velocity take place further than $10 \mu m$ from the wall, beyond the reach of surface roughness. Instead, the critical velocity $v_c$ for the phase slip depends on the initial $I$ texture. We measure the critical velocity in a rotating cylinder, and find that it may vary within a factor of 6. However, by cooling slowly through the superfluid transition temperature $T_c$, the equilibrium texture is created and the measured $v_c$ is in agreement with theoretical calculations.

Anisotropic superflow.—In ordinary superconductors and superfluids, the order parameter has a phase factor $\exp[i\phi(\mathbf{r})]$, and the superfluid velocity is defined as the gradient of the phase, $\mathbf{v}_s \propto \nabla \phi$. In $^3$He-A there is an additional phase factor $\exp[i\phi_I(\hat{p})]$, which depends on the azimuthal angle $\phi_I$ of the quasiparticle momentum $\hat{p}$ with respect to the angular momentum axis $\hat{l}$. Instead of resolving the two phases separately, one may only define the total phase factor, which can be expressed as $(\hat{m} + i\hat{n}) \cdot \hat{p}$. Here $\hat{l}$, $\hat{m}$, and $\hat{n}$ form an orthonormal triad, which generally depends on the location $\mathbf{r}$. The superfluid velocity is defined as $\mathbf{v}_s = \frac{\hbar}{2m} \sum \hat{m}_k \nabla \hat{n}_k$, where the prefactor $\frac{\hbar}{2m}$ equals Planck’s constant divided by twice the mass of a $^3$He atom. This leads to several unusual features. For example, let us take an initially uniform $\hat{l} \equiv \hat{z}$, and then tilt the triads so that $I(z)$ forms a helix with an opening angle $\beta$ and a wave vector $q$. This leads to a change in $v_{az}$ by $\frac{\hbar}{2m} (1 - \cos \beta) q$ without a change in the externally applied phase difference $\phi$. Thus $^3$He-A can respond to flow by forming an $I$ texture.

The energetics of the current carrying states is contained in the energy functional [5]

$$E = \frac{1}{2} \rho w^2 + \frac{1}{2}(\rho || - \rho \perp) (\hat{l} \cdot \hat{w})^2$$

$$- C \mathbf{w} \cdot \nabla \times \hat{l} + C_0 (\hat{l} \cdot \hat{w})(\hat{l} \cdot \nabla \times \hat{l})$$

$$+ \frac{1}{2} K_s (\nabla \times \hat{l})^2 + \frac{1}{2} K_0 (\hat{l} \cdot \nabla \times \hat{l})^2 + \frac{1}{2} K_{ss} (\hat{l} \times (\nabla \times \hat{l}))^2$$

$$+ \frac{1}{2} K_{sl}(\hat{l} \cdot \nabla) \hat{d}^2 + \frac{1}{2} K_{l \ddot{l}} (\hat{l} \times \nabla \times \hat{l})_d) \hat{d}^2$$

$$- \frac{1}{2} g_d (\hat{d} \cdot \hat{l})^2 + \frac{1}{2} g_d (\hat{d} \cdot \mathbf{H})^2.$$  

Here the first two terms describe the anisotropic kinetic energy arising from the counterflow $\mathbf{w} = \mathbf{v}_n - \mathbf{v}_s$. The terms with coefficients $C$ and $C_0$ contribute to the coupling between $\mathbf{w}$ and inhomogeneous $I(\mathbf{r})$. The five terms with $K_i$ coefficients are gradient energies for $\hat{l}(\mathbf{r})$ and the spin anisotropy vector $\hat{d}(\mathbf{r})$. The last two terms arise from the magnetic dipole-dipole interaction, and from the external magnetic field $\mathbf{H}$. In the absence of the dipole coupling, all uniform current carrying states would be unstable [1]. Thus the dipole interaction determines the scale of the critical velocity $v_d = \sqrt{g_d/\rho_l} \sim 1$ mm/s and the length scale $\xi_d = \frac{2\pi}{\sqrt{g_d/\rho_l}} \sim 10 \mu m$. Except for a tiny region near $T_c$ (Fig. 2), $v_d$ is much smaller than is needed to nucleate usual “singular vortices”, such as one encounters in other superfluids [1].

Let us drive the current by applying a normal fluid velocity $\mathbf{v}_n$. (This is equivalent to applying a phase dif-
ference $\Delta \phi = \frac{2\pi}{N} v_n L$ between two points a distance $L$ apart.) The flow properties depend on the magnitude and orientation of the magnetic field $\mathbf{H}$ [10]. In general, a uniform state of $\mathbf{H}$ is stable for velocities smaller than a first critical velocity $v_{c1}$. At larger velocities there often is a stable helical texture. The opening angle $\beta$ of the helix grows continuously from zero with increasing $v_n$ until a second critical velocity $v_{c2}$. There the helix becomes unstable, and the resulting state depends on how the flow is applied. A continuously sustained dissipative $\mathbf{H}$ texture is found when a constant current is driven in a channel. This case has been studied by several groups, most extensively by Bozler and collaborators [11] [12]. In our rotating cylinder, a stationary state is restored after one or more vortex lines are formed in a phase slip.

Experiment.—Our sample container is a cylinder which is rotated around its axis with angular velocity $\Omega$. The radius of the cylinder $R \sim 2$ mm is large compared to $\xi$, which means that the normal velocity $v_n = \Omega \times \mathbf{r}$ is rectilinear to a good approximation near the cylindrical walls. Therefore, one should see transitions at $\Omega_{c1} = v_{c1}/R$ and $\Omega_{c2} = v_{c2}/R$ corresponding to the critical values of one-dimensional flow. At $\Omega_{c2}$ dissipation sets in only temporarily when a vortex line is created. It is driven by the Magnus force into a vortex bundle in the center of the cylinder. As a consequence, the effective driving velocity $v = [\Omega - \Omega_0(N)] R$ is reduced to a subcritical value. Here $N$ is the number of vortices in the bundle, $\Omega_0 = \kappa N / 2 \pi R^2$, and $\kappa$ the circulation of one vortex line. Compared to channel flow, our experiment has high resolution in the measurement of $v_{c2}$, since the vortices can be counted with a precision of $\pm 5$ lines from the continuous-wave NMR spectrum. Our sensitivity to a helical texture is poor and $v_{c1}$ has not been observed.

Three different sample cylinders have been used, which were fabricated from epoxy or fused quartz with radii $R = 2 - 2.5$ mm, heights $L = 6 - 7$ mm, and different surface roughness [3]. No systematic dependence of $v_n$ on the container was found. A small orifice in the center of the bottom plate of the cylinder provides thermal contact via a liquid $^4$He column to the refrigerator. The NMR field $\mathbf{H}$, which is large compared to the dipole field $H_d = \sqrt{4 \pi / 9n} \approx 3$ mT, is either axial ($\parallel \Omega$) or transverse ($\perp \Omega$). The NMR absorption spectrum has two peaks. The frequency shift of the main peak from the Larmor value is used for thermometry. The satellite peak arises from vortex lines [4] and its intensity is proportional to the number of lines $N$.

Results.—Fig. 1 shows two measurements of the amplitude of the satellite peak as a function of $\Omega$. The acceleration is started from rest ($\Omega = N = 0$) at a slow rate $|d\Omega/dt| = 10^{-3} - 10^{-5}$ rad/s$^2$, and the satellite intensity remains zero until a critical velocity $v_c = \Omega_c R$. In the top frame the amplitude starts to increase linearly when $\Omega_c$ is exceeded. This means that vortices are nucleated regularly at a $v_c$ that is approximately independent of the number of vortices $N$ in the center of the cylinder.

The acceleration in Fig. 1a is stopped at a velocity $\Omega_{max}$. In order to determine $N$ and $v_c$ in this state, the rotation is decelerated until at $\Omega_{min}(N)$ vortices start to annihilate [6]. The regular parallelogram-like shape of the acceleration – deceleration loop in Fig. 1a shows that $v_c$ is approximately constant.

On repeating the measurement, we generally find a considerable spread in $v_c$. The distribution is evident in Fig. 2, which shows $v_c$ measured for different histories of sample preparation. Because this variation is much larger than seen in one measuring run (Fig. 1a), it has to arise from different metastable states of the system. The only source of metastability in our system are different superfluid states, i.e. different textures of $\mathbf{H}$, $\mathbf{m}$, $\mathbf{n}$, and $\mathbf{d}$. Additional evidence for the textural origin of the spread is discussed below.

In Fig. 1b the response to acceleration is a sudden jump in the signal, which corresponds to a burst of $\Delta N \approx 90$ vortex lines. The critical velocity $v_c$ after the jump is changed and is generally smaller than before. The subsequent deceleration to the annihilation threshold in Fig. 1b shows that $v_c$ is reduced from 1.2 mm/s to 0.26 mm/s. Obviously, this behavior is caused by a transition from one texture to another. The vortex bursts (o in Fig. 2) take place only at high temperatures $T \geq 0.7 T_c$. This is consistent with the fact that the energy barriers between different textures are smaller at higher temperatures.

Curve $v_{c2}$ in Fig. 2 is a theoretical result for the vor-
tex instability of an initially uniform texture. It roughly agrees with the largest measured \( v_c \)'s. Some measured values are larger than the theoretical upper limit, which may arise from inaccurate parameter values in the calculation (see below). In order to justify theoretically the texture dependence of \( v_c \), we have calculated two simple cases of initially inhomogeneous texture. A locked soliton (LS) is a planar object where both \( \mathbf{d} \) and \( \mathbf{l} \) turn from parallel to \( \mathbf{v} \) on one side to antiparallel to \( \mathbf{v} \) on the other side of the wall, while they are locked to each other (\( \mathbf{d} = \mathbf{l} \)). Such a texture becomes unstable against vortex formation at the velocity \( v_{c,LS} \). Another type of domain wall is an unlocked soliton (US), where \( \mathbf{d} = \mathbf{l} \) on one side and \( \mathbf{d} = -\mathbf{l} \) on the other. When its plane is perpendicular to \( \mathbf{v} \), vortices are created at the velocity \( v_{c,US} \). These simple cases span the spread of the measured \( v_c \) in Fig. 2. This makes it plausible that the three-dimensional texture and its defects in the cylinder are responsible for all of the variation in \( v_c \).

The data in Fig. 3 has been collected under a variety of prehistories of sample preparation. We now demonstrate that more reproducible results are obtained if the initial state is prepared using a procedure which favors an equilibrium texture. Experimentally textural metastability is best avoided by cooling slowly (\( dT/dt \approx -1 \) \( \mu \)K/min) through \( T_c \) at nonzero \( \Omega \). For consistent \( v_c \) values, \( \Omega \) must not be reduced to zero at any point during the \( v_c \) measurements. The critical velocities measured under these conditions are shown in Fig. 3 in axial (\( \bullet \), \( \bullet \)) and in transverse field (\( \circ \), \( \bullet \)).

Insets (a) and (b) in Fig. 3 display two candidates for the \( \mathbf{I} \) texture in a rotating cylinder in axial field. The “circular” texture (a) is symmetric in rotations around the cylinder axis but the “double-half” (b) has only reflection symmetry. In both textures the boundary condition fixes \( \mathbf{I} \) perpendicular to the wall but the counterflow bends it azimuthal already at distances \( \sim \xi_d \). Concerning the flow properties, the circular texture is equivalent to the uniform texture, and thus vortices are here expected to be created at \( v_{c,2} \). In contrast, the double-half texture contains two locked solitons, which means that vortices are created already at \( v_{c,LS} \). The fact that the measured \( v_c \) data in Fig. 3 lies consistently above \( v_{c,LS} \) suggests that the circular texture is created in cooling through \( T_c \) in rotation. Thus the measurement of \( v_c \) gives information about textures which is difficult to get by other techniques.

In transverse field the texture depicted in inset (c) is expected. The flow is here parallel to the field in two sectors of the cylinder. There it becomes unstable toward vortex formation at \( v_{c,II} = v_{c,\rho_l} / \sqrt{\rho_l \rho_l} \), as calculated by Fetter. The measured data (\( \circ \), \( \bullet \), \( \triangle \)) agrees with this prediction.
In contrast, a field to resist the precession. Such precessing states of both field, both $x$ soliton texture around end plates of the cylinder. $H$ arises because the texture at small $v_a$ finite $v_a$ Vollhardt and Maki [22,5]. We find a much lower $v_a$ helixes and eventually sweeps through 180 by Lin-Liu $v_a$ have to be positive.) At temperatures the eigenvalues of the second derivative matrix of $F$ increase with decreasing $v_a$ical flow rate at temperatures $T/T_c > 0.9$ [13].

Calculations.—The critical velocities were calculated assuming that the order parameter depends only on one coordinate $x$. The energy functional $I_0$ was minimized numerically for different values of the drive velocity $v_n$. The values of the parameters are from Ref. [19], except that we use $g_0$ determined from the NMR shift in the B phase, corrected by trivial strong-coupling effects. No adjustable parameters are contained in the calculations.

For helical textures the minimization gives the energy $F(v_n, q)$, where $q$ is the fundamental wave vector of the helix. Although the periodic boundary conditions in the rotating container tend to fix $q$, our simulation of the dynamics gives transitions from one value of $q$ to another. As a consequence, $q$ approaches $q_0(v_n)$, which corresponds to the minimum of $F(v_n, q)$. (For stability both the eigenvalues of the second derivative matrix of $F(v_n, q)$ have to be positive.) At $v_{c2}$ the opening angle $\beta(x)$ locally starts to grow larger than $\beta \lesssim 60^\circ$ found for stable helices and eventually sweeps through 180°. Previously $v_{c1}$ and an estimate of $v_{c2}$ have been calculated near $T_c$ by Lin-Liu et al. [4]. We find that in high field (Fig. 2) the helical texture is not stable above $0.8 T_c$, but at lower temperatures $v_{c2}$ grows considerably above $v_{c1}$.

In low fields ($H \lesssim H_d$) $v_{c1}$ drops and vanishes at $H = 0$ when $T < 0.85 T_c$, while $v_{c2}$ is nearly independent of $H$. The observed reduction of the measured $v_c$ probably arises because the texture at small $H$ becomes more susceptible to different perturbations such as heat flows or end plates of the cylinder.

In the presence of solitons, the precession of the whole soliton texture around $x$ leads to phase slippage. In zero field, both $v_{cLS}$ and $v_{cUS}$ vanish because there is nothing to resist the precession. Such precessing states of both the LS and the US have been studied in Refs. [20,21]. In contrast, a field $H$ perpendicular to $x$ gives rise to a finite $v_c$, as calculated approximately for the US by Vollhardt and Maki [22,5]. We find a much lower $v_{cUS}$ than reported previously.

Conclusions.—Our measurements of vortex formation in $^3$He-A are the first to allow detailed comparison with theoretical calculations. The quantitative agreement is much better than found for $v_c$ in other superfluids ($^4$He-II and $^3$He-B). We find that the critical velocity depends on the bulk $\bar{I}(r)$ texture. The maximal critical velocity we associate with the equilibrium texture in axial field, while the minimal velocity is a characteristic of textures incorporating unlocked solitons.

We thank R. Hänninen and J. Ruohio for valuable help. This work is funded by the EU Human Capital and Mobility Program (no. CHGECT94-0069).

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