A NATURAL GEOMETRIC CONSTRUCTION UNDERLYING A CLASS OF LAX PAIRS

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Abstract. In the framework of the theory of differential coverings [2], we discuss a general geometric construction that serves the base for the so-called Lax pairs containing differentiation with respect to the spectral parameter [4]. Such kind of objects arise, for example, when studying integrability properties of equations like the Gibbons-Tsarev one [1].

Introduction

The Gibbons-Tsarev equation

\[
z_{yy} + z_x z_{xy} - z_y z_{xx} + 1 = 0,
\]

(1)

see, e.g., [1], arises as a reduction of the Benney chain and possesses many properties of integrable systems. On the other hand, it is not integrable in the exact solitonic sense (for example, the equation does not admit Hamiltonian operators). In particular, equation (1) is the compatibility condition for the system

\[
\begin{align*}
\varphi_x &= \frac{1}{z_y + z_x \varphi - \varphi^2}, & \varphi_y &= -\frac{z_x - \varphi}{z_y + z_x \varphi - \varphi^2}.
\end{align*}
\]

(2)

Introducing a new function \( \psi = \psi(\varphi) \), one can consider the system (‘a Lax pair’)

\[
\begin{align*}
\psi_x &= \frac{1}{z_y + z_x \psi - \psi^2} \psi\varphi, & \psi_y &= \frac{z_x - \varphi}{z_y + z_x \psi - \psi^2} \psi\varphi
\end{align*}
\]

(3)

(see, for example, [4]), whose compatibility condition is the Gibbons-Tsarev equation as well. Taking now \( \varphi \) for a parameter in (3) and expanding the resulting system in Laurent series, one can describe an infinite family of nonlocal conservation laws and the corresponding infinite algebra of nonlocal symmetries.

Of course, these computations can be done in a more general situation. Let \( \mathcal{E} \) be a PDE imposed on an unknown function \( z(x, y) \) and \( \varphi_x = X(x, y, [z], \varphi), \varphi_y = Y(x, y, [z], \varphi) \) be a system compatible by virtue of \( \mathcal{E} \), where \([z]\) denotes a collection consisting of \( z \) itself and its derivatives up to a certain order. Then one can pass to the system \( \psi_x = -X(x, y, [z], \varphi)\psi\varphi, \psi_y = -Y(x, y, [z], \varphi)\psi\varphi \) also compatible over \( \mathcal{E} \). As I see it myself, this seemingly simple construction invokes two questions at least: (a) why do
the signs change? (b) how does the unknown function \( \varphi \) in the initial system transform to an independent variable in the resulting one?

Below we shall discuss a general geometric construction that (hopefully) answers these questions and explains the trick of passing from System (2) to System (3). In Section 1 we briefly recall the necessary definitions and results from the geometric theory of differential equations (see [3], for example), including the nonlocal theory, [2]. Section 2 concerns with the definition of jet spaces associated to integrable distributions. Finally, in Section 3 the main construction is described.

1. The basic notions and notation

Consider a smooth manifold \( M, \text{dim } M = n \), and a locally trivial vector fiber bundle \( \pi: E \to M, \text{rank } \pi = m \). Denote by \( \Gamma(\pi) \) the \( \mathcal{C}^\infty(M) \)-module of sections of the bundle \( \pi \). Let \( \pi_k: J_k(\pi) \to M \) be the corresponding bundle of \( k \)-jets, \( k = 1, \ldots, \infty \), and let \( \pi_{k,l}: J_k(\pi) \to J_l(\pi), \ k > l \), denote the natural projections. The bundle \( \pi_\infty \) possesses a natural flat connection \( \mathcal{C}_X \), which is called the Cartan connection and is defined by the condition

\[
\mathcal{C}_X(\varphi) = X(\varphi),
\]

where \( X \) is a vector field on \( M \), \( \varphi \) is a smooth function on \( J_\infty(\pi) \), \( f \) is a section of \( \pi \), and \( j_\infty(f) \in \Gamma(\pi_\infty) \) denotes the infinite jet of this section.

The corresponding horizontal distribution \( z \mapsto \mathcal{C}_z \), \( z \in J_\infty(\pi) \), on \( J_\infty(\pi) \) is integrable and is called the Cartan distribution.

An (infinitely prolonged) differential equation will be understood as a submanifold \( E \subset J_\infty(\pi) \) such that all the fields of the form \( \mathcal{C}_X \) are tangent to this submanifold. Consequently, the Cartan connection can be restricted to the projection \( \pi_\infty|_E \), while the Cartan distribution is restricted to \( E \). We shall keep the same notation for these restrictions.

Let \( \mathcal{E} \subset J_\infty(\tilde{\pi}) \), \( \tilde{\pi}: \tilde{E} \to M \), be another equation. A smooth vector bundle \( \tau: \tilde{E} \to E \) with coordinates \( w^1, \ldots, w^r \) along the fibers (the functions \( w^\gamma \) are called nonlocal variables) is given by the extensions of the total derivatives

\[
\tilde{D}_{x^i} = D_{x^i} + X_i,
\]

where \( \sigma \) is the multi-index corresponding to the variables \( x^i \). A covering structure in the bundle \( \tau: \mathcal{E} \to \mathcal{E} \) with coordinates \( w^1, \ldots, w^r \) along the fibers (the functions \( w^\gamma \) are called nonlocal variables) is given by the extensions of the total derivatives.
where $X_i$ are $\tau$-vertical vector fields (i.e., the fields of the form $\sum_{\gamma} X_i^\gamma \partial/\partial w^\gamma$) that satisfy the relations

$$D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0$$

for all $i < j$.

A covering $\tau$ is called trivial if $\mathcal{E}_X(C^\infty(\tilde{E})) \subset C^\infty(\tilde{E})$ for all vector fields $X \in D(M)$. A morphism of two coverings $\tau_1: \tilde{E}_\alpha \rightarrow \tilde{E}$, $\alpha = 1, 2$, of ranks $r_1$ and $r_2$, respectively, is a smooth map $\varphi: \tilde{E}_1 \rightarrow \tilde{E}_2$ such that

1. $\tau_1 = \tau_2 \circ \varphi$ and
2. $\varphi^*(\tilde{E}_1) \subset \tilde{E}_2(z)$, $z \in \tilde{E}$.

Coordinates. A covering $\tau$ is trivial if and only if the vertical fields in

Equalities (4) vanish.

If $w^\alpha_i$ are the nonlocal variables in the covering $\tau_\alpha$, $\alpha = 1, 2$, while the covering structures are given by the vertical fields

$$\sum_{\gamma} X_i^{\alpha,\gamma} \frac{\partial}{\partial w^\gamma}$$

then a morphism $\varphi$ of $\tau_1$ to $\tau_2$ is determined by a system of functions $w_1^\gamma = \varphi^*(w_1^\gamma) = \varphi^*(w_1^\gamma)$, $\gamma = 1, \ldots, r_2$. In particular, $\varphi$ is a morphism to the trivial covering if

$$\tilde{D}_{x^i}(\varphi^*) = 0.$$  \hspace{1cm} (5)

Such morphisms will be called trivializing ones.

2. Jets associated to distributions

Consider a smooth manifold $N$, $\dim N \leq \infty$. Let $\mathcal{D}$ be an integrable distribution on this manifold, rank $\mathcal{D} = s < \infty$, i.e., a locally projective finite-rank submodule $\mathcal{D} \subset D(N)$ such that $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$. Let also $\xi: F \rightarrow N$ be a locally trivial vector bundle over $N$. We say that two sections $f$ and $f' \in \Gamma(\xi)$ are $k$-equivalent over $\mathcal{D}$ at a point $z \in N$ if

$$(X_1 \ldots X_l(f))|_z = (X_1 \ldots X_l(f'))|_z, \quad l \leq k,$$

for any collection of vector fields $X_1, \ldots, X_l \in \mathcal{D}$. Let $[f]_z^k$ denote the equivalence class of $f$. The set

$$J_\mathcal{D}^k(\xi) = \{ [f]_z^k \mid z \in N, f \in \Gamma(\xi) \}, \quad k = 0, 1, \ldots, \infty,$$

is endowed with a natural smooth manifold structure (cf. [3]) and is called the the manifold of $k$-jets associated to the distribution $\mathcal{D}$. One has the natural projections $\xi_k: J_\mathcal{D}^k(\xi) \rightarrow M$ and $\xi_{k,l}: J_\mathcal{D}^k(\xi) \rightarrow J_\mathcal{D}^l(\xi)$, $k > l$, and all these maps are locally trivial fiber bundles while the functions $\xi_k(f): M \rightarrow J_\mathcal{D}^k(\xi)$, $z \mapsto [f]_z^k$, (jets of sections) are smooth sections of the bundles $\xi_k$.

Example. If $\mathcal{D} = D(N)$ we obtain the classical definition of jet spaces.

Example. If $N = \mathcal{E}$ is an equation and $\mathcal{D} = \mathcal{C}$ coincides with the Cartan distribution we arrive to the definition of horizontal jets, see [3].
The bundle of infinite jets $\xi_\infty$ admits a flat $\mathcal{D}$-connection denoted by
\[ \nabla^\mathcal{D}: \mathcal{D} \to D(J^\infty_\mathcal{D}(\xi)) \]
and which is also called the Cartan connection: for any point $\theta = [f]_z^\infty$, a field $X \in \mathcal{D}$ and a function $\varphi \in C^\infty(J^\infty_\mathcal{D}(\xi))$, we set
\[ \nabla^\mathcal{D}_X(\varphi)|_\theta = X(j_\infty(f)^*\varphi)|_z. \]
The corresponding horizontal distribution on $J^\infty_\mathcal{D}(\xi)$ will be denoted by $\Delta^\mathcal{D}$; it is integrable, i.e., $[\Delta^\mathcal{D}, \Delta^\mathcal{D}] \subset \Delta^\mathcal{D}$.

A submanifold $E_k \subset J^k_\mathcal{D}(\xi)$ is called a $\mathcal{D}$-equation of order $k$. As in the case of ‘usual’ equations, one can define the $\mathcal{D}$-prolongations $E^{(l)}_k \subset J^{k+l}_\mathcal{D}(\xi)$ of finite and infinite orders. The latter, $E \subset J^\infty_\mathcal{D}(\xi)$, will be shortly called just a $\mathcal{D}$-equation. The map $\xi_\infty: E \to M$ inherits the Cartan connection, while the submanifolds $E$ carries the corresponding horizontal distribution.

Let $S$ be a $\xi_\infty$-vertical vector field on $E$; it is called a (higher infinitesimal) $\mathcal{D}$-symmetry of the equation $E$ if $[S, \Delta^\mathcal{D}] \subset \Delta^\mathcal{D}$.

Coordinates. Let $y^1, \ldots, y^k, \ldots$ be local coordinates in $N$, $v^1, \ldots, v^r$ be coordinates if fibers of a trivialization of the bundle $\xi$. Finally, let $Y_1, \ldots, Y_s$ be a local basis of the distribution $\mathcal{D}$,
\[ [Y_i, Y_j] = \sum_k c^k_{ij} Y_k, \quad c^k_{ij} \in C^\infty(J^\infty_\mathcal{D}(\xi)). \]
Let $\sigma = i_1 \ldots i_{|\sigma|}$ be a multi-index of arbitrary but finite length $|\sigma|$, $1 \leq i_\alpha \leq s$. Let us define the coordinate functions $v^j_\sigma$ on $J^\infty_\mathcal{D}(\xi)$ by setting
\[ v^j_\sigma(\theta) = Y_{i_1} \ldots Y_{i_{|\sigma|}}(f^j)|_z, \quad \theta = [f]_z^\infty, \quad f = (f^1, \ldots, f^l). \]
Then the lifts $\nabla^\mathcal{D}_Y$ of vector fields $Y \in \mathcal{D}$ to $J^\infty_\mathcal{D}(\xi)$, i.e., the connection $\nabla^\mathcal{D}$, are defined by the equalities
\[ \nabla^\mathcal{D}_Y(v^j_\sigma) = v^j_{\alpha \sigma}. \]
One has the following relations
\[ v^j_{\alpha \sigma} = v^j_{i_1 \alpha \sigma} + \sum_k c^k_{\alpha i_1} v^j_{k \sigma}, \quad (6) \]
where $\bar{\sigma} = i_2 \ldots i_{|\sigma|}$. The functions
\[ v^j_K = \underbrace{\underbrace{v^1_{\underbrace{k_1 \text{times}}}} \ldots \underbrace{v^s_{\underbrace{k_s \text{times}}}}}_{k_1 \ldots k_s \text{times}} \]
may be taken for independent coordinates.

Obviously, $\mathcal{D}$-coverings of $\mathcal{D}$-equations are also defined in a straightforward way.
3. Jets over differential coverings

Let us now consider the constructions of Section 2 when \( N = \tilde{E} \), where \( \tilde{E} \) is the covering equation in some covering \( \tau: \tilde{E} \to E \). Then the manifold \( \tilde{E} \) carries three integrable distributions:

- the horizontal distribution denoted by \( \mathcal{H} \) which coincides with the Cartan distribution \( \tilde{C} \) on \( \tilde{E} \);
- the vertical distribution \( \mathcal{V} \) generated by \( \tau \)-vertical vector fields;
- the total distribution \( \mathcal{T} \) which is the sum of the previous two.

Respectively, one can define the jet spaces associated to these distributions and the diagram

![Diagram](image)

is commutative (the projections \( \xi^\mathcal{H} \) and \( \xi^\mathcal{V} \) are defined in an obvious way).

Note that the distribution \( \Delta^\mathcal{T} \), just like the original total distribution \( \mathcal{T} \), is the sum of the distributions \( \Delta^\mathcal{H} \) and \( \Delta^\mathcal{V} \), where \( \Delta^\mathcal{H} \) and \( \Delta^\mathcal{V} \) are obtained by lifting \( \mathcal{H} \) and \( \mathcal{V} \) using the Cartan connection associated to the distribution \( \mathcal{T} \). Thus, the jet manifold \( J^\infty_\mathcal{T}(\xi) \) possesses three different geometries. Note also that the projection \( \xi^\mathcal{H} \) is a covering with respect to the \( \mathcal{T} \)-geometry while the \( \xi^\mathcal{V} \) is a covering in the \( \mathcal{H} \)-geometry of the space \( J^\infty_\mathcal{T}(\xi) \).

Note also that for any \( \mathcal{H} \)-equation \( \mathcal{W} \subset J^\infty_\mathcal{H}(\xi) \) one can consider its pre-image \( (\xi^\mathcal{H})^{-1}(\mathcal{W}) \) in \( J^\infty_\mathcal{T}(\xi) \) and the infinite \( \mathcal{T} \)-prolongation of the latter. The resulting \( \mathcal{T} \)-equation is denoted by \( \tilde{\mathcal{W}} \) also possesses three different geometries, while the map \( \xi^\mathcal{H} \big|_{\tilde{\mathcal{W}}} \) is a covering in the geometry associated to the horizontal distribution. Exactly the same situation arises when one considers \( \mathcal{V} \)-equations.

**Coordinates.** Let the covering structure in the bundle \( \tau: \tilde{E} \to E \) be defined by vector fields \( \partial/\partial w^\alpha \) and choose

\[
\frac{\partial}{\partial w^1}, \cdots, \frac{\partial}{\partial w^r}
\]

for a basis in the space of \( \tau \)-vertical fields. Then

\[
[D_{x^i}, D_{x^j}] = 0, \quad \left[ \frac{\partial}{\partial w^\alpha}, \frac{\partial}{\partial w^\beta} \right] = 0, \quad \left[ \frac{\partial}{\partial w^\alpha}, D_{x^i} \right] = \sum_\beta \frac{\partial X_\beta}{\partial w^\alpha} \frac{\partial}{\partial w^\beta}.
\]

(7)

Denote by \( \sigma \) the multi-index that corresponds to the variables \( x^i \). Let also the multi-index \( \rho \) correspond to the variables \( w^\alpha \). Then we obtain the following coordinate functions in the jet spaces:

- \( v^j_\rho \) in \( J^\infty_\mathcal{V}(\xi) \):
\[ \Delta_{\xi} = \tilde{D}_{x^i} + \sum v^j_{i\sigma\rho} \frac{\partial}{\partial v^j_{i\sigma\rho}}, \quad \Delta_{\omega} = \frac{\partial}{\partial \omega^{\alpha}} + \sum v^j_{\alpha\sigma\rho} \frac{\partial}{\partial \omega^{\alpha}}, \]

while the functions \( v^j_{i\sigma\rho}, v^j_{\alpha\sigma\rho} \) are computed accordingly to Equations (6) and (7).

Existence of different geometries on the manifold \( J_{\infty}^\xi(\xi) \) leads to different notions of a symmetry. Below, we shall need the following one:

A vertical field \( S \) is called a gauge \( \mathcal{V} \)-symmetry if
\[
[ S, \Delta_{\mathcal{V}} ] \subset \Delta_{\mathcal{V}}.
\]

Similar to the classical case, one has the following description of the symmetries:

**Proposition 1.** Gauge \( \mathcal{V} \)-symmetries are in one-to-one correspondence with sections of the pull-back \((\xi_{\infty}^\xi)^*(\xi_{\infty}^\xi)\). In local coordinates, to any such a section \( h \) of the form \( v^j_{\alpha} = h^j_{\alpha}(\theta), \theta \in J_{\infty}^\xi(\xi) \), there corresponds the symmetry
\[
E_h = \sum_{\rho,\gamma,j} \Delta_{\rho} (h^j_{\alpha}) \frac{\partial}{\partial v^j_{\rho\sigma}};
\]

where \( \Delta_{\rho} \) is the composition of the total derivatives \( \Delta_{\omega^{\alpha}} \) corresponding to the multi-index \( \rho \).

If \( \mathcal{W} \subset J_{\infty}^\xi(\xi) \) is a \( \mathcal{T} \)-equation then the Lie algebra of its gauge \( \mathcal{V} \)-symmetries will be denoted by \( \text{gs}(\mathcal{W}) \).

4. **Jets of Automorphisms**

Consider now a particular case of the above described constructions: let \( \xi = \tau^*(\tau) \). Sections of this bundle are naturally identified with morphisms of the bundle \( \tau \), i.e., with smooth maps \( \varphi: \tilde{E} \to \tilde{E} \), such that \( \tau \circ \varphi = \tau \).

Consider the submanifold \( \sum_{\gamma}^\infty (\xi_{\infty}^\xi)(\xi_{\infty}^\xi) \in J_{\infty}^\xi(\xi) \) in \( J_{\infty}^\xi(\xi) \) and the \( \mathcal{H} \)-equation
\[
\mathcal{W}(\tau) = \{ \theta \in J_{\infty}^\xi(\xi) \mid \theta = \varphi_{\infty}^{\gamma}, \varphi(z) = z, z \in \tilde{E} \} \subset J_{\infty}^\xi(\xi).
\]

Let \( \mathcal{W}(\tau) \subset J_{\infty}^\xi(\xi) \) be the \( \mathcal{T} \)-prolongation of this equation endowed with the \( \mathcal{H} \)-geometry, i.e., equipped with the horizontal distribution, as it was described above. Then we obtain the infinite-dimensional covering
\[
\tilde{\tau}: \mathcal{W}(\tau) \to \tilde{E},
\]

which is associated with the initial covering \( \tau \) in a canonical way.

**Coordinates.** Let \( \varphi_{\sigma}^{\alpha} \) be local coordinates in \( \sum_{\gamma}^\infty (\xi_{\infty}^\xi)(\xi_{\infty}^\xi) \), where \( \sigma \) is the multi-index corresponding to the variables \( x^1, \ldots, x^n \). Then the equation \( \mathcal{W}(\tau) \) is determined by the relations
\[
\varphi_{\sigma}^{\gamma} \equiv \tilde{D}_{x^i}(\varphi_{\sigma}^{\gamma}) = 0, \quad i = 1, \ldots, \dim M, \quad \gamma = 1, \ldots, \text{rank } \tau,
\]
or
\[
\frac{\partial \varphi^\gamma}{\partial x^i} + \sum_{\alpha} X_i^\alpha \frac{\partial \varphi^\gamma}{\partial w^\alpha} = 0.
\] (9)

The nonlocal variables in the covering ˜τ are the functions ˆ\varphi^\gamma_\rho = Δ^\gamma_\rho(\varphi^\gamma), where ρ is the multi-index that corresponds to the variables w^1, ..., w^r. Then the equation ˜W(τ) is given by the infinite system
\[
\frac{\partial \varphi^\gamma_\rho}{\partial x^i} + Δ^\gamma_\rho\left(\sum_{\alpha} X_i^\alpha \varphi^\gamma_\alpha\right) = 0.
\]

Thus, the covering structure in the bundle ˜τ is described by the vector fields
\[
\Delta^\gamma_\rho x^i = D_{x^i} + \sum_{\rho, \gamma} \Delta^\gamma_\rho\left(\sum_{\alpha} X_i^\alpha \varphi^\gamma_\alpha\right) \frac{\partial}{\partial \varphi^\gamma_\rho},
\]
or
\[
\Delta^\gamma_\rho x^i = D_{x^i} + E_{h_i},
\] (10)

where
\[
h_i = \left(\sum_{\alpha} X_i^\alpha \varphi^1_\alpha, \ldots, \sum_{\alpha} X_i^\alpha \varphi^r_\alpha\right), \quad i = 1, \ldots, n.
\]

Summarizing the above discussion, we arrive to the following

**Proposition 2.** The covering ˜τ: ˜W(τ) → E defined by equalities (10) is a zero-curvature representation with values in the Lie algebra of gauge V-symmetries gs(W).

Obviously, the construction introduced here is a geometrical generalization of the ‘Lax pairs’ mentioned in Introduction.

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