ONLINE ORDERING STRATEGY FOR THE DISCRETE NEWSVENDOR PROBLEM WITH ORDER VALUE-BASED FREE-SHIPPING

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ABSTRACT. Suppliers always provide free-shipping for retailers whose total order value exceeds or equals an explicit promotion threshold. This paper incorporates a shipping fee in the discrete multi-period newsvendor problem and applies Weak Aggregating Algorithm (WAA) to offer explicit online ordering strategy. It further considers an extended case with salvage value and shortage cost. In particular, online ordering strategies are derived based on return loss function. Numerical examples serve to illustrate the competitive performance of the proposed ordering strategies. Results show that newsvendors’ cumulative return losses are affected by the threshold of the order value-based free-shipping. Moreover, the introduction of salvage value and shortage cost greatly improves the competitive performance of online ordering strategies.

1. Introduction. In the traditional newsvendor problem, the decision-maker must decide the order quantity of a product at the beginning of a future sale during which the demand is usually uncertain [1, 2]. The newsvendor problem addresses how to make a perfect ordering decision for perishable products, such as newspapers, bread, and seasonal products. Indeed, no replenishment is possible during the selling period. Thus, the decision-maker must try his or her best to achieve maximum returns by balancing the risk of lost sales due to understocking with that of inventory spoilage due to overstocking. Due to the practical application of this problem, it has been widely studied since it was put forward. At present, the latest paper concentrated on the case which supply capacity is random and the newsvendor is loss-averse [3].

The classical newsvendor model assumes the knowledge of statistical demand information. It aims for a perfect ordering strategy that can achieve maximum single-period returns. Many scholars have extended the classical newsvendor problem from different perspectives. Khouja (1999)[4] and Qin et al. (2011)[5] have summarized all of the extensions of the single-period problem. These studies have

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relied on the statistical hypothesis of the future demand. However, statistical demand information is often unavailable or difficult to know. The only information we can obtain is the historical demand. Thus, some scholars have paid attention to the newsvendor problem knowing only part of the demand information. For instance, they have used the Bayesian decision method to strengthen the authenticity of parameters step by step when only the type of demand distribution is known [6, 7]. Hence, these works remain within the statistical hypothesis.

The development of online algorithm provides scholars a chance to avoid the statistical hypothesis of uncertain demand when studying the newsvendor problem. WAA is an online algorithm that combines predictions of different experts with dynamically updated weights [8]. Levina et al. (2010)[9] first applied WAA to the multi-period newsvendor problem based on stationary expert advice, and illustrated that the cumulative gain obtained with their given strategy was as high as that of the best expert. The stationary expert advice means that each expert respectively keeps his or her order quantity at the same value throughout trading periods. Recently, Zhang et al. (2016)[10] continued to provide competitive strategies for both real- and integer-valued order quantities and extended their work to case that considered both salvage value and shortage cost. Meanwhile, Zhang and Yang (2016)[11] formulated a two-product, multi-period newsvendor problem in which the total demand of the two products was fixed.

With the rapid development of electronic commerce and logistics industry, more and more retailers order on the internet. In this case, the shipping fee affects the retailers ordering decision, because it is an important factor in ordering cost. Lewis et al. (2006)[12] found that the free-shipping policy could greatly increase order incidence and basket size. In fact, many suppliers offer free-shipping to retailers whose total order value exceeds or equals a threshold, which is referred to in this paper as order value-based free-shipping. Several studies have addressed the newsvendor problem with free-shipping [13, 14, 15]. However, the free-shipping factor is rarely considered in the multi-period newsvendor model. In the paper, we introduce the order value-based free-shipping factor in the discrete multi-period newsvendor problem and apply WAA to offer explicit online ordering strategy following the work of Zhang et al. (2016)[10]. Compared with previous related work, our contributions are as follows. First, we use the free-shipping factor in the discrete multi-period newsvendor problem, which is helpful for retailers who need to determine integral order quantities over a finite horizon. Second, we study the newsvendor problem with consideration of the return loss aspect. We define the return loss function to evaluate the experts’ performance and provide newsvendor online ordering strategy, which is helpful for the retailers who care about loss more than return. Finally, we expand our study to a case in which salvage value and shortage cost are taken into account.

The rest of this paper is organized as follows. In Section 2, we provide a detailed explanation of WAA and its application to the multi-period newsvendor problem. Section 3 presents the explicit online ordering strategy for the discrete multi-period newsvendor problem with order value-based free-shipping. It also demonstrates the generalized result of the extended newsvendor problem in which salvage value and shortage cost are considered. Section 4 uses numerical analysis to illustrate the competitive performance of our two proposed strategies. Finally, Section 5 concludes the paper.
2. WAA and its application to the multi-period newsvendor problem.

2.1. The newsvendor protocol based on return loss function. In our newsvendor problem, order quantities are decided sequentially over a finite or infinite horizon rather than all at once. The decision-making process can be considered as a perfect-information game played between two players, newsvendor and nature, according to the following newsvending protocol.

- $L_0 := 0$.
- In each period $n = 1, 2, \ldots$:
  - The newsvendor announces $y_n \in [0, B]$.
  - Nature announces $d_n \in [0, \infty)$.
  - $L_n := L_{n-1} + \lambda(d_n, y_n)$.

The protocol involves the return loss function $\lambda : [0, \infty) \times [0, B] \to \mathbb{R}$, which is defined as the difference between the optimal offline gain that can be achieved in introspection and the newsvendor’s gain. The intuition behind the protocol is as follows. Every morning, a newsvendor orders a number $y_n$ of copies of a newspaper. The day’s demand is $d_n$, which the newsvendor learns at the end of the day. In fact, the demand for newspapers or something similar is usually an integer. Hence, in the paper we consider the discrete newsvendor problem in which both demand and order quantity are integers. Each copy is bought for $c$ and sold for $p$. In the basic newsvendor problem [9], the newsvendor’s gain represents the total revenue from selling the min($d_n, y_n$) copies that he manages to sell minus the total cost $cy_n$ of all copies. It is assumed that the price $p$ exceeds the cost $c$, $c < p$. We are given an upper bound $B > 0$ on the number of newspapers that can be purchased by the newsvendor. $L_n$ is the newsvendor’s cumulative return losses over $n$ periods. In the paper, we introduce the conditional free-shipping in the basic newsvendor problem and use WAA to offer an explicit online ordering strategy to newsvendor from the perspective of return loss.

2.2. WAA. WAA is an online learning approach that makes decisions sequentially rather than all at once. For the newsvendor problem proposed here, there is a match between the decision-making process of WAA and of how to decide order quantities. The online ordering strategy for the discrete newsvendor problem with conditional free-shipping is constructed using WAA. The idea of WAA is as follows. Given a set of experts who provide decisions in each period, the decision-maker uses WAA to combine these decisions in a certain way. In the paper, we refer to the decision-maker as the newsvendor and the decision problem as the newsvendor problem.

An initial weight distribution on a set of experts is defined when WAA begins. In each period, the weights assigned to the experts are recomputed after the actual demand becomes known, to reflect the change in the newsvendor’s level of trust in each of the experts’ decision. The set of experts is denoted by $\Theta$, which is assumed to be a measurable space. Both experts and newsvendor choose their decisions from a decision set $T$. The actual demand set is denoted by $\Omega$. A return loss function $\lambda$ is defined on $\Omega \times T$. In period $n$, given the newsvendor’s decision $\gamma_n \in T$ and nature’s outcome $\omega_n \in \Omega$, the newsvendor’s return loss is $l_n := \lambda(\omega_n, \gamma_n)$; and given an expert $\theta$’s decision $\gamma^\theta_n$, his or her return loss is $l^\theta_n := \lambda(\omega_n, \gamma^\theta_n)$. Therefore, the cumulative return losses for newsvendor and expert $\theta$ during the first $n$ periods are $L_n := \sum_{i=1}^n l_i$ and $L^\theta_n := \sum_{i=1}^n l^\theta_i$. The main parameter of the WAA is the initial probability measure $q(d\theta)$ on $\Theta$, interpreted as the prior weights assigned to the
experts. Weights are continuously recomputed and are represented by a probability measure $p_n(d\theta)$ in period $n$. To be specific, $p_n(d\theta), \theta \in \Theta$ are experts’ weights in period $n$. They denote the weight of each expert when combining their opinions in period $n$. Intuitively, the less cumulative return loss an expert obtains, the greater weight he has. The pseudo code of WAA is as follows:

- Initial cumulative return losses for newsvendor and all experts are 0: $L_0 := 0; L_0^\theta := 0, \theta \in \Theta$.
- In each period $n = 1, 2, \cdots$:
  - The experts’ weights are recomputed:
    $$p_n(d\theta) := \frac{\beta_n^{L_{n-1}^\theta} q(d\theta)}{\int_\Theta \beta_n^{L_{n-1}^\theta} q(d\theta)},$$
    where $\theta \in \Theta$ and $\beta_n := e^{-1/\sqrt{n}}$, which guarantees that a greater weight can be assigned to the expert whose cumulative return loss is relatively low.
  - Experts give their decisions $\gamma_n^\theta, \theta \in \Theta$.
  - The newsvendor announces his or her decision $\gamma_n := \int_\Theta \gamma_n^\theta p_n(d\theta)$.
  - Nature announces the outcome $\omega_n$.
  - Cumulative return losses are then updated: $L_n := L_{n-1} + \lambda(\omega_n, \gamma_n)$ and $L_n^\theta := L_{n-1}^\theta + \lambda(\omega_n, \gamma_n^\theta), \theta \in \Theta$.

Levina et al. (2010) [9] first applied WAA to the basic newsvendor problem in which the order quantity was real-valued. Based on the gain function, they provided the newsvendor an explicit online ordering strategy and proved that WAA’s cumulative gains are competitive with the weighted average of all experts’ cumulative gains in the case of the bounded gain function.

3. Discrete newsvendor problem with order value-based free-shipping.

With the emergence of online ordering, the shipping fee gradually attracts retailers’ attention. In this section, we study the multi-period newsvendor problem with conditional free-shipping and discrete order quantity. We consider order value-based free-shipping, which means that the shipping fee is 0 if the total order value exceeds or equals a certain threshold. We assume that only one type of product can be bought. To clearly present an ordering strategy for the decision-maker who cares about loss, we define the return loss function as the difference between the optimal offline gain and the newsvendor’s gain. The optimal offline gain is achieved in introspection, as it is computed when the demand sequence is available, and the newsvendor’s gain is the gain achieved by our proposed ordering strategy. We continue to apply WAA to stationary expert advice to offer online ordering strategy.

3.1. The case with order value-based free-shipping factor. We introduce the order value-based free-shipping factor in the discrete multi-period newsvendor problem. We assume that the threshold for the order value-based free-shipping is $V_0$ and that $V_n$ is the newsvendor’s total order value in period $n$. The shipping fee is $v$ if it is required in a period. We then have the following:

$$\text{shipping fee} = \begin{cases} v, & 0 < V_n < V_0; \\ 0, & V_n \geq V_0. \end{cases}$$

Let $d_n$ denote the actual demand in period $n$, and its boundary is $B$ (both $d_n$ and $B$ are integers). Then, under the order value-based free-shipping condition, the
newsvendor who orders quantity $\theta := y$ in period $n$ obtains the following return loss:

$$l_n := (p - c)d_n - vI_{d_n < c} - [p \min(y, d_n) - cy - vI_{0 < cy < V}],$$  \hspace{1cm} (1)

where $I_{\{\cdots\}}$ the indicator function that equals 1 if it satisfies the condition, and 0 otherwise.

To obtain the explicit online ordering strategy, we follow the method presented by [9]. At the beginning of day $n$, we assume that $d_1, d_2, \ldots, d_{n-1}$, which are integers between 0 to $B$, are the historical demand sequence. We consider that $d^{(1)}, \ldots, d^{(n-1)}, d^{(n)}$ are the order statistics of the historical demand sequence. Here we assume that there exists $k_1 \in \{1, 2, \ldots, n\}$, that satisfies $d^{(k_1)} = [V_0/c]$. Similarly, we assume that $d^{(0)} := 0, d^{(n+1)} := B$ and $m_k := d^{(k+1)} - d^{(k)}$, $k = 0, 1, \ldots, n$.

To facilitate the expression of Theorem 3.1, we define the following:

$$a := [p(n-1-k) - c(n-1)]n^{-1/2}, k = 0, 1, \ldots, n-1;$$

$$\sigma_1 := e^{\left|\sum_{i=1}^{n-1} d^{(i)} - p \frac{n-1}{n} \sum_{i=1}^{n-1} v/\sqrt{n}\right|}, k = 0, 1, \ldots, k_1 - 1;$$

$$\sigma_2 := e^{\left|\sum_{i=1}^{n-1} d^{(i)} - p \frac{n-1}{n} \sum_{i=1}^{k_1} d^{(i)} + \frac{k_1}{\sqrt{n}}\right|}, k = k_1, k_1 + 1, \ldots, n - 1.$$

The definitions $a, \sigma_1$ and $\sigma_2$ mentioned above are symbols. We can compute their exact values using known parameters $p, c, v, k_1$ and historical demand sequence $d_1, \ldots, d_{n-1}$ in each period $n$.

By applying WAA to stationary expert advice (also called fixed-stock strategy) that respectively keep the order quantity $y \in \{0, 1, ..., B\}$ at the same value throughout all decision periods, we can obtain the explicit online ordering strategy of the discrete multi-period newsvendor problem with order value-based free-shipping. The conclusion is given in Theorem 3.1.

**Theorem 3.1.** With order value-based free-shipping factor, the order quantity for the discrete multi-period newsvendor problem in period $n$ is as follows:

$$y_n = \begin{cases} y_n^\circ, & \text{if } y_n^\circ \text{ is integer;} \\ \lfloor y_n^\circ \rfloor, & \text{with probability of } [y_n^\circ] - y_n^\circ; \\ \lceil y_n^\circ \rceil, & \text{with probability of } y_n^\circ - [y_n^\circ]. \end{cases}$$

where

$$y_n^\circ = \sum_{k=0}^{k_1-1} \sigma_1 (d^{(k)}e^{a(d^{(k)} - d^{(k+1)})}e^{a(d^{(k+1)} + 1)} + e^{a(d^{(k+1)} + 1)}(1 - e^{am_k}))$$

$$+ \sum_{k=k_1}^{n-1} \sigma_2 (e^{a(d^{(k)}) - [1 - e^{a(m_k + 1)}]} + \sum_{k=k_1}^{n-1} \sigma_2 e^{a(d^{(k)}) - [1 - e^{a(m_k + 1)}]}1 - e^{a(m_k + 1)} \right) \right) \right|, k = k_1, k_1 + 1, \ldots, n - 1.$$

**Proof.** To set WAA, we firstly consider $q(d\theta)$ as uniform, i.e., $q(d\theta) := \frac{1}{B+1}$ for the sake of simplicity. In Section 4, we illustrate that the effect of different $q(d\theta)$ on cumulative return losses can be neglected by numerical analysis. Based on the
return loss function (1) and WAA’s procedure, the order quantity in period \( n \) can be written as follows:

\[
y^0_n = \sum_{y=0}^{B} y p_n(dy) = \frac{\sum_{y=0}^{B} ye^{-L^y_{n-1}/\sqrt{n}}}{B} \frac{(B+1)}{y} (B+1) = \frac{\sum_{y=0}^{B} ye^{-L^y_{n-1}/\sqrt{n}}}{B} \frac{(B+1)}{y} = \frac{\sum_{y=0}^{B} ye^{-L^y_{n-1}/\sqrt{n}}}{B} = \frac{a_n}{b_n} \quad (3)
\]

According to the threshold, we can decompose the online ordering strategy (3) into two parts, then we further obtain:

\[
y^0_n = \frac{\sum_{k=0}^{k_1-1} \sum_{y=d(k)}^{d(k+1)-1} ye^{-L^y_{n-1}/\sqrt{n}} + \sum_{k=k_1}^{n-1} \sum_{y=d(k)}^{d(k+1)-1} ye^{-L^y_{n-1}/\sqrt{n}}}{\sum_{k=0}^{k_1-1} \sum_{y=d(k)}^{d(k+1)-1} e^{-L^y_{n-1}/\sqrt{n}} + \sum_{k=k_1}^{n-1} \sum_{y=d(k)}^{d(k+1)-1} e^{-L^y_{n-1}/\sqrt{n}}}
\]

With a fixed period \( k \), when \( d(k) \leq y \leq d(k+1) \),

\[
L^y_{n-1} = p \sum_{i=k+1}^{n-1} d(i) - c \sum_{i=1}^{n-1} d(i) - v \sum_{i=1}^{n-1} I_{0<y<d(i)<V_0} + v \sum_{i=1}^{n-1} I_{0<y<V_0} + [c(n-1) - p(n-1-k)]y
\]

Then,

\[
a_n = \frac{\sum_{k=0}^{k_1-1} \sum_{y=d(k)}^{d(k+1)-1} ye^{-L^y_{n-1}/\sqrt{n}} + \sum_{k=k_1}^{n-1} \sum_{y=d(k)}^{d(k+1)-1} ye^{-L^y_{n-1}/\sqrt{n}}}{\sum_{k=0}^{k_1-1} \sum_{y=d(k)}^{d(k+1)-1} e^{-L^y_{n-1}/\sqrt{n}} + \sum_{k=k_1}^{n-1} \sum_{y=d(k)}^{d(k+1)-1} e^{-L^y_{n-1}/\sqrt{n}}}
\]

\[
= \frac{\sum_{k=0}^{k_1-1} \sum_{y=d(k)}^{d(k+1)-1} ye^{-L^y_{n-1}/\sqrt{n}} + \sum_{k=k_1}^{n-1} \sum_{y=d(k)}^{d(k+1)-1} ye^{-L^y_{n-1}/\sqrt{n}}}{\sum_{k=0}^{k_1-1} \sum_{y=d(k)}^{d(k+1)-1} e^{-L^y_{n-1}/\sqrt{n}} + \sum_{k=k_1}^{n-1} \sum_{y=d(k)}^{d(k+1)-1} e^{-L^y_{n-1}/\sqrt{n}}}
\]

\[
= \sum_{k=0}^{k_1-1} \sigma_1 \left[ \frac{d(k)e^{a\sigma(k)} - d(k+1)e^{a\sigma(k+1)}}{1 - e^a} \right] + \frac{e^{a\sigma(k+1)}(1 - e^{am_k})}{(1 - e^a)^2}
\]

\[
+ \sum_{k=k_1}^{n-1} \sigma_2 \left[ \frac{d(k)e^{a\sigma(k)} - d(k+1)e^{a\sigma(k+1)}}{1 - e^a} \right] + \frac{e^{a\sigma(k+1)}(1 - e^{am_k})}{(1 - e^a)^2}
\]

Similarly, we have the following:

\[
b_n = \sum_{y=0}^{B} e^{-L^y_{n-1}/\sqrt{n}} = \sum_{k=0}^{k_1-1} \sigma_1 \sum_{y=d(k)}^{d(k+1)-1} e^{ay} + \sum_{k=k_1}^{n-1} \sigma_2 \sum_{y=d(k)}^{d(k+1)-1} e^{ay}
\]

\[
= \sum_{k=0}^{k_1-1} \sigma_1 \left[ \frac{1 - e^{a(m_k+1)}}{1 - e^a} \right] + \sum_{k=k_1}^{n-1} \sigma_2 \left[ \frac{1 - e^{a(m_k+1)}}{1 - e^a} \right]
\]

Hence, we obtain the final explicit online ordering strategy \( y^0_n \). After taking integer, (2) can be obtained. □
Theorem 3.1 presents an explicit online ordering strategy for the discrete multi-period newsvendor problem with order value-based free-shipping, which we call swaa.

3.2. Extension to the case where salvage value and shortage cost are considered. In real life, it is difficult for retailers to make perfect decisions for unknown demand. It is often possible to order too much or too little. The first will cause inventory cost if the order is greater than the actual demand, and the second results in the risk of losing opportunities and business credibility if the order is inferior to the demand. Both cases will affect retailers’ management efficiency. Thus, we extend the discrete multi-period newsvendor model with order value-based free-shipping to the case in which salvage value and shortage cost are considered. Symbols added in this section are as follows. \( s \) represents the unit salvage value and \( h \) represents the unit shortage cost. We assumed that \( s > h \). To facilitate the presentation of results in Theorem 3.2, we define the following:

\[
\begin{align*}
 b := & \left[p(n - 1 - k) + k s - c(n - 1) + h(n - 1 - k)\right]n^{-1/2}; \\
\sigma_3 & := e \left[ c \sum_{i=1}^{n-1} d_{(i)} - \sum_{i=1}^{n-1} v^\ast(p+h) \sum_{i=k+1}^{n-1} d_{(i)} - s \sum_{i=1}^{n-1} d_{(i)} / \sqrt{n} \right]; \\
\sigma_4 & := e \left[ c \sum_{i=1}^{n-1} d_{(i)} + b \sum_{i=1}^{n-1} v^\ast(p+h) \sum_{i=k+1}^{n-1} d_{(i)} - s \sum_{i=1}^{n-1} d_{(i)} / \sqrt{n} \right].
\end{align*}
\]

Similarly, the values of \( b, \sigma_3 \) and \( \sigma_4 \) can be computed in period \( n \).

In this case, the return loss function becomes the following:

\[
\begin{align*}
l_n^0 := & (p - c)d_n - vI_{\{0 < d_n < V_0\}} - \left[p \min(y, d_n) - cy - vI_{\{0 < y < V_0\}}ight. \\
& \left. + s(\max(d_n, y) - d_n) - h(d_n - \min(d_n, y))\right]
\end{align*}
\]

Based on the return loss function above, the corresponding online ordering strategy can be obtained using the same method.

**Theorem 3.2.** For the discrete newsvendor problem with salvage value and shortage cost and order value-based free-shipping, the order quantity in period \( n \) is as follows:

\[
\hat{y}_n = \begin{cases} 
\hat{y}_n^c, & \text{if } \hat{y}_n^c \text{ is integer}; \\
\left\lfloor \hat{y}_n^c \right\rfloor, & \text{with probability of } \left\lfloor \hat{y}_n^c \right\rfloor - \hat{y}_n^c; \\
\left\lceil \hat{y}_n^c \right\rceil, & \text{with probability of } \hat{y}_n^c - \left\lfloor \hat{y}_n^c \right\rfloor.
\end{cases}
\]

where

\[
\begin{align*}
\hat{y}_n^c = & \sum_{k=0}^{k_1-1} \sigma_3 \frac{d_{(k)} e^{b(d_{(k)} - d_{(k+1)})} e^{b(d_{(k+1)} + 1)}}{1 - e^b} + \frac{b e^{b(d_{(k+1)} + 1)}}{1 - e^b}; \\
+ & \sum_{k=0}^{k_1-1} \sigma_3 \frac{d_{(k)} e^{b(d_{(k)} - 1 - e^b(m_{k+1}) + 1)}}{1 - e^b} + \frac{e^{b(d_{(k+1)} + 1)}}{1 - e^b}; \\
+ & \sum_{k=k_1}^{n-1} \sigma_4 \frac{d_{(k)} e^{b(d_{(k)} - d_{(k+1)})} e^{b(d_{(k+1)} + 1)}}{1 - e^b} + \frac{b e^{b(d_{(k+1)} + 1)}}{1 - e^b}; \\
+ & \sum_{k=0}^{k_1-1} \sigma_3 \frac{d_{(k)} e^{b(d_{(k)} - 1 - e^b(m_{k+1}) + 1)}}{1 - e^b} + \frac{e^{b(d_{(k+1)} + 1)}}{1 - e^b};
\end{align*}
\]

The proof for Theorem 3.2 is the same as Theorem 3.1. We call the online ordering strategy proposed in Theorem 3.2 twaa.

**Corollary 1.** When \( s = 0 \) and \( h = 0 \), we get \( a = b, \sigma_1 = \sigma_3 \) and \( \sigma_2 = \sigma_4 \). It means that (5) degrades into (2), thereby demonstrating that the introductions of salvage
value and shortage cost generalize the discrete newsvendor problem presented in section 3.1.

Remark 1. The return loss function defined by (1) is not convex in \( y \) when studying the newsvendor problem with the return loss function. Thus, theoretical guarantees for online ordering strategies \( \text{swaa} \) and \( \text{twaa} \) cannot be obtained by using the results of WAA. In Section 4, we provide some numerical examples to illustrate the competitive performance of our two proposed online ordering strategies.

4. Numerical analysis. In this section, we use numerical examples to illustrate the competitive performance of our proposed online ordering strategies (\( \text{swaa} \) and \( \text{twaa} \)). We take the data from JD.COM, which is the biggest proprietary-type electric commercial enterprise in China. We can easily obtain information about the shipping fee a retailer should pay when buying products using the following function:

\[
v = \begin{cases} 
6, & 0 < V_n < 99; \\
0, & V_n \geq 99.
\end{cases}
\]

We assume that uncertain demands belong to \( \{0,1,2,3,4,5\} \), i.e., the upper boundary of the actual demand \( B = 5 \). Following Alfares and Elmorra(2005)[16], we set the unit selling price at \( \hat{p} = 50 \), unit cost at \( c = 35 \) in the original newsvendor model proposed in section 3.1. In the extended newsvendor model proposed in section 3.2, we set the unit salvage value at \( s = 15 \) and unit shortage cost at \( h = 8 \), considering that salvage value and shortage cost are much smaller than the unit selling price. We also assume that stationary ordering strategies provided by experts are \( \theta = 0,1,2,3,4,5 \). We use MATLAB to solve this problem and use the random integers it produces as the actual demand sequence.

To show the relationship between free-shipping threshold \( V_0 \) and cumulative return losses, we randomly generate the stochastic demand sequence for 30 times when \( V_0 = 99 \) and \( V_0 = 110 \), respectively. In each trial, we compute strategies’ cumulative return losses and their corresponding best experts’ cumulative return losses (denoted \( \text{best}_i \), with \( i = 1,2 \)). We also calculate the ratio of the former to the latter and call it \( \text{ratio}_i \), with \( i = 1,2 \). Computed results are presented in Table 1 and Table 2 for strategies \( \text{swaa} \) and \( \text{twaa} \), respectively. For strategy \( \text{swaa} \), we discover that cumulative return losses are greater when \( V_0 \) is greater, by comparing the left and right sides in Table 1. Indeed, when \( V_0 \) is greater, customers will order more to enjoy the free-shipping policy. Moreover, the averages of \( \text{ratio}_1 \)s under \( V_0 = 99 \) and \( V_0 = 110 \) are 1.0764 and 1.0916, respectively, which highlight the good competitive performance of strategy \( \text{swaa} \), as its cumulative return losses are almost as small as those of \( \text{best}_1 \). In addition, the standard deviations of \( \text{ratio}_1 \)s under \( V_0 = 99 \) and \( V_0 = 110 \) are 0.0016 and 0.0035, respectively, which shows the robustness of the online ordering strategy \( \text{swaa} \). Similarly, Table 2 presents the relationship between \( V_0 \) and the cumulative return losses of strategy \( \text{twaa} \). A conclusion similar to Table 1 can be drawn from Table 2. Furthermore, comparing \( \text{ratio}_1 \) presented in Table 1 and \( \text{ratio}_2 \) presented in Table 2 under the same \( V_0 \) (\( V_0 = 99 \)), we can conclude that strategy \( \text{twaa} \) improves competitive performance because the average of \( \text{ratio}_2 \)s (1.0613) is smaller than the average of \( \text{ratio}_1 \)s (1.0764). Figure 1 and 2 show the cumulative return losses \( \text{swaa} \) and \( \text{best}_1 \) achieved when \( V_0 = 99 \) and \( V_0 = 110 \), respectively. These figures illustrate our conclusion clearly.

To set WAA, prior weights \( q(d\theta) \) for each expert should be decided first. We first assume that \( q(d\theta) \) is uniform in Theorem 3.1 and 3.2 to give the explicit online
Table 1. Cumulative return losses of swaa and best1 under different $V_0$

| Trials | $V_0 = 99$ | $V_0 = 110$ |
|--------|------------|-------------|
|        | swaa       | best1       | ratio1 | swaa      | best1       | ratio1 |
| 1      | 1298.8     | 1107.4      | 1.1728 | 1406.8    | 1216.4      | 1.1565 |
| 2      | 1089.8     | 964.60      | 1.1298 | 1119.8    | 999.30      | 1.1206 |
| 3      | 1161.9     | 1049.2      | 1.1074 | 1209.9    | 1019.2      | 1.1871 |
| 4      | 991.80     | 962.50      | 1.0304 | 1081.8    | 1100.5      | 0.9830 |
| 5      | 1080.6     | 999.70      | 1.0622 | 1182.6    | 1040.4      | 1.1367 |
| 6      | 1104.2     | 1000.8      | 1.1033 | 1224.2    | 1059.5      | 1.1555 |
| 7      | 1093.5     | 1026.3      | 1.0655 | 1099.5    | 1008.3      | 1.0904 |
| 8      | 953.60     | 924.40      | 1.0316 | 983.60    | 900.40      | 1.0924 |
| 9      | 1130.6     | 990.30      | 1.1417 | 1124.6    | 1037.0      | 1.0845 |
| 10     | 888.60     | 867.30      | 1.0246 | 900.60    | 837.30      | 1.0756 |
| 11     | 1114.2     | 1005.2      | 1.1084 | 1126.2    | 975.20      | 1.1548 |
| 12     | 922.70     | 922.70      | 1.0000 | 1072.7    | 1072.7      | 1.0000 |
| 13     | 916.90     | 857.20      | 1.0696 | 1030.9    | 1007.2      | 1.0235 |
| 14     | 1064.1     | 1005.5      | 1.0583 | 1118.1    | 1064.2      | 1.0506 |
| 15     | 983.90     | 928.20      | 1.0600 | 1079.9    | 906.60      | 1.1912 |
| 16     | 1254.0     | 1155.3      | 1.0854 | 1368.0    | 1202.0      | 1.1381 |
| 17     | 764.10     | 742.90      | 1.0285 | 860.10    | 844.90      | 1.0180 |
| 18     | 1129.6     | 1035.0      | 1.0914 | 1141.6    | 1049.0      | 1.0883 |
| 19     | 1209.5     | 1113.3      | 1.0864 | 1215.5    | 1107.3      | 1.0977 |
| 20     | 890.10     | 879.70      | 1.0118 | 890.10    | 843.70      | 1.0763 |
| 21     | 1145.3     | 1053.3      | 1.0873 | 1157.3    | 1047.3      | 1.1050 |
| 22     | 1177.9     | 1061.9      | 1.1092 | 1213.9    | 1037.9      | 1.1696 |
| 23     | 832.90     | 801.10      | 1.0397 | 898.90    | 927.10      | 0.9696 |
| 24     | 1086.8     | 989.70      | 1.0981 | 1098.8    | 953.70      | 1.1521 |
| 25     | 1065.8     | 1010.2      | 1.0550 | 1125.8    | 1050.9      | 1.0713 |
| 26     | 960.60     | 861.90      | 1.1145 | 1092.6    | 1017.9      | 1.0734 |
| 27     | 1138.0     | 1057.2      | 1.0764 | 1162.0    | 1033.2      | 1.1247 |
| 28     | 895.80     | 832.80      | 1.0756 | 985.80    | 964.80      | 1.0218 |
| 29     | 1023.7     | 981.30      | 1.0432 | 1131.7    | 1101.3      | 1.0276 |
| 30     | 1244.6     | 1107.4      | 1.1239 | 1352.6    | 1216.4      | 1.1120 |

ordering strategy without information about the market. We then compare the effect of different prior weights on cumulative return losses. We randomly generate integer demands 30 times as the demand sequence for 30 days and compute the cumulative return losses achieved everyday by the two proposed strategies under either uniform or norm distribution. Results of swaa and twaa are presented in Figure 3 and 4, respectively. In Figure 3, the difference of cumulative return losses
Table 2. Cumulative return losses of twaa and best2 under different $V_0$

| Trials | $V_0 = 99$ | $V_0 = 110$ |
|--------|------------|-------------|
|        | twaa | best2 | ratio2 | twaa | best2 | ratio2 |
| 1      | 1016.4 | 935.60 | 1.0864 | 1058.4 | 935.60 | 1.0827 |
| 2      | 1430.7 | 1360.8 | 1.0514 | 1454.7 | 1384.8 | 1.0505 |
| 3      | 877.00 | 832.00 | 1.0541 | 895.00 | 850.00 | 1.0529 |
| 4      | 980.00 | 943.60 | 1.0386 | 1010.0 | 973.60 | 1.0374 |
| 5      | 888.80 | 814.80 | 1.0908 | 924.80 | 850.80 | 1.0870 |
| 6      | 1030.6 | 1081.6 | 0.9528 | 1048.6 | 1117.6 | 0.9383 |
| 7      | 959.30 | 929.60 | 1.0319 | 977.30 | 947.60 | 1.0313 |
| 8      | 1017.7 | 969.2 | 1.0500 | 1041.7 | 1005.2 | 1.0363 |
| 9      | 784.80 | 722.40 | 1.0864 | 808.80 | 746.40 | 1.0836 |
| 10     | 1270.0 | 1157.6 | 1.0971 | 1312.0 | 1199.6 | 1.0937 |
| 11     | 1131.5 | 1127.2 | 1.0038 | 1143.5 | 1163.2 | 0.9831 |
| 12     | 964.60 | 924.80 | 1.0430 | 1000.6 | 966.80 | 1.0350 |
| 13     | 1265.9 | 1140.8 | 1.1097 | 1295.9 | 1182.8 | 1.0956 |
| 14     | 722.50 | 663.60 | 1.0888 | 746.50 | 687.60 | 1.0857 |
| 15     | 1366.4 | 1272.8 | 1.0735 | 1390.4 | 1296.8 | 1.0722 |
| 16     | 1061.2 | 1012.0 | 1.0486 | 1121.2 | 1072.0 | 1.0459 |
| 17     | 1212.7 | 1146.8 | 1.0575 | 1230.7 | 1170.8 | 1.0512 |
| 18     | 844.40 | 810.40 | 1.0420 | 856.40 | 822.40 | 1.0413 |
| 19     | 1238.4 | 1163.6 | 1.0643 | 1262.4 | 1187.6 | 1.0630 |
| 20     | 1260.7 | 1230.4 | 1.0246 | 1278.7 | 1272.4 | 1.0050 |
| 21     | 1300.2 | 1200.0 | 1.0835 | 1330.2 | 1230.0 | 1.0815 |
| 22     | 1195.0 | 1102.8 | 1.0836 | 1219.0 | 1132.8 | 1.0761 |
| 23     | 1008.9 | 947.10 | 1.0653 | 1044.9 | 1001.1 | 1.0438 |
| 24     | 1187.1 | 1096.8 | 1.0823 | 1211.1 | 1132.8 | 1.0691 |
| 25     | 1155.4 | 1087.9 | 1.0620 | 1173.4 | 1135.9 | 1.0330 |
| 26     | 832.90 | 792.60 | 1.0508 | 844.90 | 804.60 | 1.0501 |
| 27     | 1035.3 | 942.00 | 1.0990 | 1035.3 | 942.00 | 1.0990 |
| 28     | 936.20 | 861.70 | 1.0865 | 942.20 | 873.70 | 1.0784 |
| 29     | 974.80 | 907.50 | 1.0742 | 1016.8 | 949.50 | 1.0709 |
| 30     | 872.50 | 825.00 | 1.0576 | 896.50 | 873.00 | 1.0269 |

achieved under two distributions is not obvious, as the red snowflake representing the achieved cumulative return losses under uniform distribution is close to the green dot that presents the achieved cumulative return losses under norm distribution for each day. We can draw a similar conclusion from Figure 4. Therefore, the effect of choosing different distributions as prior weights can be neglected in the analysis of strategy’s competitive performance.
Figure 1. Daily cumulative return losses of \textit{swaa} and \textit{best1} when $V_0 = 99$

Figure 2. Daily cumulative return losses of \textit{swaa} and \textit{best1} when $V_0 = 110$

Figure 3. Cumulative return losses \textit{swaa} achieved under uniform and norm distribution
Third, Figure 1 and Figure 5 intuitively present the daily cumulative return losses of strategies $swaa$, $twaa$, and their corresponding best experts when $V_0 = 99$. In each figure, the proposed strategy traces the best expert’s decision, as the red dotted line representing the proposed strategy follows a similar trend as the one standing for the corresponding best expert. Furthermore, by comparing Figure 1 and Figure 5, we discover that the cumulative return losses of proposed strategy and best expert greatly decrease when both salvage value and shortage cost are considered. It means that the introductions of salvage value and shortage cost greatly improve the competitive performance of the online ordering strategy. A convincing explanation for this phenomenon is that salvage value is bigger than shortage cost. As a result, the decision-maker may order more to avoid more loss, which is consistent with our expectation.

Finally, we analyze the robustness of the proposed online ordering strategies. We first generate 100 random integers as the demand sequence in 100 days. For $n = 20, 40, 60, 80, 100$, we compute the average($Avg1$) and standard deviation($SD1$) of ratios in the last two lines in Table 3 for $swaa$. From the table, we can conclude that the average and standard deviation of strategy $swaa$ gradually decrease when
It means that the competitive performance of the strategy *swaa* is stabilized gradually as the decision period increases. A similar conclusion for the strategy *twaa* can be obtained by analyzing the following Table 4.

**Table 3.** *swaa*’s robustness in different computational days

| Trials | Days 20 | Days 40 | Days 60 | Days 80 | Days 100 |
|--------|---------|---------|---------|---------|---------|
| 1      | 1.2102  | 1.1728  | 1.1041  | 1.0327  | 1.0385  |
| 2      | 1.1738  | 1.0852  | 1.0933  | 1.0680  | 1.0718  |
| 3      | 1.1524  | 1.1362  | 1.1578  | 1.0160  | 1.0789  |
| 4      | 1.2868  | 1.1640  | 1.1158  | 1.0695  | 1.0328  |
| 5      | 1.1268  | 1.2441  | 1.1610  | 1.1056  | 1.0996  |
| 6      | 1.1801  | 1.0671  | 1.0995  | 1.0356  | 1.0640  |
| 7      | 1.3219  | 1.0925  | 1.1321  | 1.0791  | 1.0466  |
| 8      | 1.2262  | 1.1151  | 1.0661  | 1.0559  | 1.0424  |
| 9      | 1.1983  | 1.0736  | 1.0796  | 1.0563  | 1.0899  |
| 10     | 1.1783  | 1.0686  | 1.1073  | 1.0661  | 1.0602  |
| **Avg1** | 1.2055 | 1.1222  | 1.1117  | 1.0585  | 1.0445  |
| **SD1** | 0.00321 | 0.00302 | 0.00087 | 0.00059 | 0.00046 |

**Table 4.** *twaa*’s robustness in different computational days

| Trials | Days 20 | Days 40 | Days 60 | Days 80 | Days 100 |
|--------|---------|---------|---------|---------|---------|
| 1      | 1.1845  | 1.1008  | 1.0993  | 1.0353  | 1.0300  |
| 2      | 1.2484  | 1.1490  | 1.0404  | 1.0976  | 1.0805  |
| 3      | 1.2752  | 1.1033  | 1.0457  | 1.0805  | 1.0503  |
| 4      | 1.2479  | 1.0672  | 1.0644  | 1.0674  | 1.0556  |
| 5      | 1.1546  | 1.1039  | 1.0984  | 1.0617  | 1.0298  |
| 6      | 1.2251  | 1.0257  | 1.0684  | 1.0938  | 1.0303  |
| 7      | 1.1617  | 1.1161  | 1.1026  | 1.0269  | 1.0380  |
| 8      | 1.1078  | 1.1347  | 1.0910  | 1.0505  | 1.0264  |
| 9      | 1.1772  | 1.0503  | 1.0776  | 1.0630  | 1.0696  |
| 10     | 1.2711  | 1.0892  | 1.0463  | 1.0652  | 1.0343  |
| **Avg2** | 1.2053 | 1.0940  | 1.0734  | 1.0642  | 1.0445  |
| **SD2** | 0.00285 | 0.00127 | 0.00052 | 0.00047 | 0.00032 |

5. **Conclusion.** The paper incorporates the shipping fee factor in the discrete multi-period newsvendor problem and uses WAA to offer the explicit online ordering strategy. Because the return loss function for our proposed newsvendor
model is not convex, our proposed online ordering strategies cannot offer theoretical guarantees by using the results of WAA. Thus, our next work will focus on identifying the technique to deal with theoretical results. This paper simply considers the order value-based free-shipping condition and assumes that only one type of product can be bought. For future research, we may extend our strategy to the case considering both piece-based free-shipping and multi-product. Future research may also consider the dynamic expert advice.

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