REPRESENTATIONS OF THE DOUBLE BURNSIDE ALGEBRA AND
COHOMOLOGY OF THE EXTRASPECIAL $p$-GROUP II

AKIHIKO HIDA AND NOBUAKI YAGITA

Abstract. Let $E$ be the extraspecial $p$-group of order $p^3$ and exponent $p$ where $p$ is an odd prime. We determine the mod $p$ cohomology $H^*(X,\mathbb{F}_p)$ of a summand $X$ in the stable splitting of $p$-completed classifying space $BE$. In the previous paper [Representations of the double Burnside algebra and cohomology of the extraspecial $p$-group, J. Algebra 409 (2014) 265-319], we determined these cohomology modulo nilpotence. In this paper, we consider the whole part of the cohomology. Moreover, we consider the stable splittings of $BG$ for some finite groups with Sylow $p$-subgroup $E$ related with the three dimensional linear group $L_3(p)$.

1. Introduction

Let $p$ be an odd prime and $E = p_+^{1+2}$ the extraspecial $p$-group of order $p^3$ and exponent $p$. In the previous paper [7], we determined the composition factor of $H^*(E) = (\mathbb{F}_p \otimes H^*(E,\mathbb{Z}))/\sqrt{(0)}$ as a right $A_p(E, E)$-module, where $A_p(E, E)$ is a double Burnside algebra of $E$ over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. In this paper, we consider the whole part of the cohomology $H^*(E,\mathbb{F}_p)$ and determine the composition factor of $H^*(E,\mathbb{F}_p)$ as an $A_p(E, E)$-module.

The mod $p$ cohomology ring $H^*(E,\mathbb{F}_p)$ of $E$ is completely known by [9], but the structure is very complicated. We shall study $H^*(E,\mathbb{F}_p)$ through the integral cohomology ring $H^*(E, \mathbb{Z})$ as in [13] and [14]. Let $H^{even}(E,\mathbb{Z})$ (resp. $H^{odd}(E,\mathbb{Z})$) be the even (resp. odd) degree part of $H^*(E,\mathbb{Z})$. Let $N = \sqrt{(0)}$ in $\mathbb{F}_p \otimes H^{even}(E,\mathbb{Z})$. Then we have that $\mathbb{F}_p \otimes H^{even}(E,\mathbb{Z})/N \cong H^*(E)$. On the other hand, the Milnor operator $Q_1$ induces an isomorphism $H^{odd}(E,\mathbb{Z}) \cong (y_1v, y_2v)H^*(E)$ where $(y_1v, y_2v)H^*(E)$ is the ideal of $H^*(E)$ generated by $y_1v, y_2v \in H^{2p+2}(E)$ (see the first part of section 2).

Let $M = \oplus M^n$ and $L = \oplus L^n$ be graded $A_p(E, E)$-modules such that $M^n$ and $L^n$ are finite dimensional for every $n$. We write as $M \leftrightarrow L$ if $M^n$ and $L^n$ have some composition factors (with same multiplicity), that is, $[M^n] = [L^n]$ in the Grothendieck group $K_0(A_p(E, E))$.

Using this notation, the structure of $H^*(E,\mathbb{F}_p)$ can be stated as follows.

Theorem 1.1. (1) As $A_p(E, E)$-modules,

$$H^{even}(E,\mathbb{F}_p) \leftrightarrow H^*(E) \oplus N \oplus (y_1v, y_2v)H^*(E)[-2p].$$

(2) As $A_p(E, E)$-modules,

$$H^{odd}(E,\mathbb{F}_p) \leftrightarrow (y_1v, y_2v)H^*(E)[-2p + 1] \oplus (N \oplus H^+(E))[-1].$$

Here, for a graded $\mathbb{F}_p$-subspace $M$ of $H^*(E)$, we denote by $M[i]$ the graded vector space with $M[i]^n = M^{n-i}$. Since the composition factors of $N$ and $(y_1v, y_2v)H^*(E)$ are determined in Proposition 3.1 and Theorem 3.5, we can get the composition factors of $H^*(E,\mathbb{F}_p)$ completely.
The indecomposable summands in the complete stable splitting of the $p$-completed classifying space $BE_p^\wedge$ correspond to primitive idempotents in $A_p(E, E)$. Moreover, they correspond to simple $A_p(E, E)$-modules. We simply write $BE$ for $BE_p^\wedge$. Let $X$ be a summand in $BE$ which corresponds to a simple $A_p(E, E)$-module $S$. Then the multiplicity of $X$ in $BE$ is equal to the dimension of $S$ as an $\mathbb{F}_p$-vector space since $\mathbb{F}_p$ is a splitting field for $A_p(E, E)$. By results above, we can get the cohomology $H^*(X, \mathbb{F}_p)$ (See Remark 3.6).

Let $G$ be a finite group with Sylow $p$-subgroup $E$. Then the multiplicity of $X$ in $BG$ is equal to the dimension of $S[G]$ where $[G]$ is an element of $A_p(E, E)$ corresponds to the $(E, E)$-biset $G$. See [1], [2], [11] for details.

In [15], the second author studied the splitting of $BG$ for various finite groups $G$ whose Sylow $p$-subgroup is $E$ and $p$-local finite groups on $E$. In this paper, we consider the stable splitting for groups related with the linear group $L_3(p)$ which were not treated in [15] in general. We use some simple $A_p(E, E)$-submodules of $H^*(E)$ and determine the multiplicity of summands in $BG$ for $G = L_3(p), L_3(p) : 2, L_3(p).3, L_3(p).S_3$ (Theorem 4.17 4.18 4.19 4.20).

Combining these results and results in [15], we have the complete information on the stable splitting of finite groups or $p$-local finite groups which have at least two $F$-radical maximal elementary abelian $p$-subgroups in $E$, by the classification in [12], where $F$ is a fusion system of $G$.

In particular, for $p = 7$, we obtain a diagram which describes inclusions of some fusion systems and stable splitting (Theorem 4.23). This result supplements the results of [15, section 9], in which the splitting of sporadic simple groups are mainly studied.

In section 2, we review the main results of [7] which will be used in section 3. In section 4, we prove Theorem 4.11 and determine the structures of ideals $N$ and $(y_1^2, y_2^2)H^*(E)$. In section 4, we consider $H^*(G)$ and the stable splitting for finite group $G$ which has a Sylow $p$-subgroup $E$. Finally, in section 5, we consider the case $p = 3$ and state some remarks.

2. Preliminary results on $H^*(E)$

In this section, we quote some results from [7]. Let $p$ be an odd prime. Let

$$E = \langle a, b, c \mid [a, b] = c, \, a^p = b^p = c^p = [a, c] = [b, c] = 1 \rangle$$

be the extraspecial $p$-group of order $p^3$ and exponent $p$. Let $A_i = \langle c, ab^i \rangle$ for $0 \leq i \leq p - 1$ and $A_\infty = \langle c, b \rangle$. Then

$$A(E) = \{A_0, A_1, \ldots, A_{p-1}, A_\infty\}$$

is the set of all maximal elementary abelian $p$-subgroups of $E$.

The cohomology of $E$ is known by [8, 9, 15]. In particular, $H^*(E) = (\mathbb{F}_p \otimes H^*(E, \mathbb{F}_p))/\sqrt{(0)}$ is generated by

$$y_1, y_2, C, v$$

with

$$\deg y_1 = 2, \, \deg C = 2p - 2, \, \deg v = 2p$$

subject to the following relations:

$$y_1^p y_2 - y_1 y_2^p = 0, \, Cy_i = y_i^p, \, C^2 = y_1^{2p-2} + y_2^{2p-2} - y_1^{p-1} y_2^{p-1},$$

We set $V = v^{p-1}$ and $Y_i = y_i^{p-1}$. 
Let $R$ be a subalgebra of $H^*(E)$ and $x_1, \ldots, x_r$ elements of $H^*(E)$. We set

$$R\{x_1, \ldots, x_r\} = \sum_{i=1}^r Rx_i$$

if $x_1, \ldots, x_r$ are linearly independent over $R$. Moreover, if $W = \sum_{i=1}^r \mathbb{F}_p x_i$ is a $\mathbb{F}_p$-vector space spanned by $x_1, \ldots, x_r$, then we set

$$R\{W\} = R\{x_1, \ldots, x_r\}.$$

We consider the action of $\text{Out}(E) = \text{GL}_2(\mathbb{F}_p)$ on $H^*(E)$. Let $S^i$ be the homogeneous part of degree $2i$ in $\mathbb{F}_p[y_1, y_2]$. Then $p(p-1)$ simple $\mathbb{F}_p \text{Out}(E)$-modules

$$S^i \otimes (\text{det})^q \quad (0 \leq i \leq p-1, 0 \leq q \leq p-2)$$

give the complete set of representatives of nonisomorphic simple $\mathbb{F}_p \text{Out}(E)$-modules. Let us write

$$\mathcal{CA} = \mathbb{F}_p[C, V]$$

and

$$\mathcal{DA} = \mathbb{F}_p[D_1, D_2]$$

where $D_1 = C^p + V$, $D_2 = CV$. Then $\mathcal{CA} = H^*(E)^{\text{Out}(E)}$, the $\text{Out}(E)$-invariants, and the restriction map induces an isomorphism

$$\mathcal{DA} \xrightarrow{\sim} H^*(A)^{\text{Out}(A)}$$

for all $A \in \mathcal{A}(E)$.

Let

$$T^i = \mathbb{F}_p\{y_1^{p-1}y_2, y_1^{p-2}y_2^{i+1}, \ldots, y_1^iy_2^{p-1}\}$$

for $1 \leq i \leq p-2$. Then $S^{p-1+i} = CS^i + T^i$. The $\mathbb{F}_p$-subspace $CS^i$ is a $\text{GL}_2(\mathbb{F}_p)$-submodule of $CS^i + T^i$ and

$$(CS^i + T^i)/CS^i \cong (S^{p-1-i} \otimes \text{det}^i).$$

Moreover we have the following expression [7, Theorem 4.4]:

$$H^*(E) = \mathbb{F}_p[C, v]\left\{ (\bigoplus_{i=0}^{p-1} S^i) \oplus (\bigoplus_{i=1}^{p-2} T^i) \right\} = \mathcal{CA}\left\{ (\bigoplus_{i=0}^{p-2} S^i v^q) \oplus T^i v^q) \right\}$$

where $S^0 = \mathbb{F}_p$ and $T^0 = S^{p-1}$.

Let $C_p$ be a cyclic group of order $p$ and let $U_i = H^{2i}(C_p, \mathbb{F}_p)$ ($0 \leq i \leq p-2$). Then $U_i$ are simple $\mathbb{F}_p \text{Out}(C_p)$-modules. Let $A \in \mathcal{A}(E)$ be a maximal elementary abelian $p$-subgroup of $E$. Let $S(A)^i = H^{2i}(A)$. Then $S(A)^i \otimes \text{det}^q$ ($0 \leq i \leq p-1, 0 \leq q \leq p-2$) are simple modules for $\text{Out}(A) = \text{GL}_2(\mathbb{F}_p)$.

Let $P$ be a general finite $p$-group and $A_p(P, P)$ the double Burnside algebra of $P$ over $\mathbb{F}_p$. The simple $A_p(P, P)$-modules corresponds to some pairs $(Q, V)$ where $Q \leq P$ and $V$ is a simple $\mathbb{F}_p \text{Out}(Q)$-module, see [2], [4], [11]. In this paper, we denote the simple $A_p(P, P)$-module corresponds to the pair $(Q, V)$ by $S(P, Q, V)$.

On the other hand, Dietz and Pridy [5] studied the stable splitting of $BE$ and determined the multiplicity of each summand. In particular, their result implies the classification of simple $A_p(E, E)$-modules.
Proposition 2.1 ([5, 7 Proposition 10.1]). The simple $A_p(E, E)$-modules are given as follows:

1. $S(E, E, S^i \otimes \det^q)$ for $0 \leq i \leq p - 1$, $0 \leq q \leq p - 2$,
   \[
   \dim S(E, E, S^i \otimes \det^q) = i + 1.
   \]

2. $S(E, A, S(A)^{p-1} \otimes \det^q)$ for $0 \leq q \leq p - 2$,
   \[
   \dim S(E, A, S(A)^{p-1} \otimes \det^q) = p + 1.
   \]

3. $S(E, C_p, U_i)$ for $0 \leq i \leq p - 2$,
   \[
   \dim S(E, C_p, U_i) = \begin{cases} 
   p + 1 & (i = 0) \\
   i + 1 & (1 \leq i \leq p - 2).
   \end{cases}
   \]

4. $S(E, 1, \mathbb{F}_p)$, $\dim S(E, 1, \mathbb{F}_p) = 1$.

To describe the composition factor of $H^*(E)$ as an $A_p(E, E)$-module, we need the following $\mathbb{F}_p$-subspace of $H^*(E)$.

Definition 2.2. Let $S$ be a simple $A_p(E, E)$-module. Let $\Gamma_S$ be the following $\mathbb{F}_p$-subspace of $H^*(E)$:

1. If $S = S(E, C_p, U_i)$, then
   \[
   \Gamma_S = \left\{ \begin{array}{ll}
   \mathbb{F}_p[C]\{\mathbb{F}_p C + S^{p-1}\} & (i = 0) \\
   \mathbb{F}_p[C]\{S^i\} & (1 \leq i \leq p - 2).
   \end{array} \right.
   \]

2. If $S = S(E, A, S(A)^{p-1} \otimes \det^q)$, then
   \[
   \Gamma_S = \left\{ \begin{array}{ll}
   DA_{\oplus 0 \leq j \leq p-1}D_2C^j(\mathbb{F}_p C + S^{p-1})) & (q = 0) \\
   DA_{\oplus 0 \leq j \leq p-1}v^qC^j(CS^q + T^q) & (1 \leq q \leq p - 2).
   \end{array} \right.
   \]

3. \[
   \Gamma_S = \left\{ \begin{array}{ll}
   DA^+ & (S = S(E, E, S^0)) \\
   CA\{v^q\} & (S = S(E, E, \det^q), 1 \leq q \leq p - 2) \\
   DA\{VS^{p-1}\} & (S = S(E, E, S^{p-1}) \\
   CA\{v^qS^{p-1}\} & (S = S(E, E, S^{p-1} \otimes \det^q), 1 \leq q \leq p - 2)
   \end{array} \right.
   \]

4. Let
   \[
   S = S^i v^q, \quad T = T^{p-i-1} v^s
   \]
   for $1 \leq i \leq p - 2$, $0 \leq q \leq p - 2$, where $s \equiv i + q \pmod{p - 1}$, $0 \leq s \leq p - 2$. Let $\Gamma_{S(E,E,S^i \otimes \det^q)}$ be the following $\mathbb{F}_p$-subspace:

   \[
   \begin{array}{llll}
   CA\{VS\} \oplus DA\{VT\} & (q \equiv 2i \equiv 0) \\
   CA\{VS\} \oplus CA\{T\} & (q \equiv 0, \ 2i \not\equiv 0) \\
   DA\{S\} \oplus DA\{VT\} & (i = q, \ 3i \equiv 0) \\
   DA\{S\} \oplus CA\{T\} & (i = q, \ 3i \not\equiv 0) \\
   CA\{S\} \oplus DA\{VT\} & (q \not\equiv 0, \ i \not\equiv q, \ q + 2i \equiv 0) \\
   CA\{S\} \oplus CA\{T\} & (q \not\equiv 0, \ i \not\equiv q, \ q + 2i \not\equiv 0).
   \end{array}
   \]

5. \[
   \Gamma_{S(E,1,\mathbb{F}_p)} = \mathbb{F}_p = H^0(E).
   \]
The following theorem is the main result of [7]. If $S$ is a simple $A_p(E, E)$-module, then there exists an idempotent $e_S$ such that $Se_S = S$ and $S'e_S = 0$ for any simple module $S' \not\cong S$. We call $e_S$ an idempotent which corresponds to $S$.

**Theorem 2.3 ([7 Theorem 10.2, 10.3, 10.4, 10.5]).** Let $S$ be a simple $A_p(E, E)$-module. Then there exists an idempotent $e_S$ which corresponds to $S$ such that

$$H^*(E)e_S = \Gamma_se_S \cong \Gamma_S.$$ 

If we see the minimal degree of non zero part of $\Gamma_S$, we have the following corollary.

**Corollary 2.4.** Every simple $A_p(E, E)$-module appears as a composition factor in $H^{2n}(E)$ for some $n \leq (p + 2)(p - 1)$.

Let $\Gamma^n_S = \Gamma_S \cap H^n(E)$ be the degree $n$ part of $\Gamma_S$. Then by Theorem 2.3

$$\sum_S \dim \Gamma^n_S = \dim H^n(E)$$

for any $n \geq 0$. In fact we have the following.

**Proposition 2.5.** $H^*(E)$ is a direct sum of $\mathbb{F}_p$-subspaces $\Gamma_S$ where $S$ runs over the representatives of the isomorphism classes of simple $A_p(E, E)$-modules.

**Proof.** By Theorem 2.3 it suffices to show that $H^*(E) = \sum \Gamma_S$. We shall show that $\mathbb{C}A\{S^i v^q\}$ ($0 \leq i \leq p - 1$, $0 \leq q \leq p - 2$) and $\mathbb{C}A\{T^i v^q\}$ ($1 \leq i \leq p - 2$, $0 \leq q \leq p - 2$) are contained in $\sum \Gamma_S$.

First consider $\mathbb{C}A\{S^i v^q\}$. If $(i, q) = (0, 0)$,

$$\mathbb{C}A = \mathbb{F}[C, D_1] = \mathbb{F}[D_1] \oplus \mathbb{C}A\{C\} = \mathbb{F}[D_1] \oplus \mathbb{F}[C]\{C\} \oplus \mathbb{C}A\{D_2\} = \mathbb{F}[D_1] \oplus \mathbb{F}[C]\{C\} \oplus \sum_{j=0}^{p-1} \mathbb{D}A\{D_2 C^{j+1}\} \oplus \mathbb{D}A\{D_2\}$$

then this is contained in the sum of the subspaces of Definition 2.2 (1)(2)(3)(5). If $(i, q) = (0, q)$, $1 \leq q \leq p - 2$, or $(i, q) = (p - 1, q)$, $1 \leq q \leq p - 2$, then $\mathbb{C}A v^q$ and $\mathbb{C}A S^{p-1} v^q$ are contained in the subspace of Definition 2.2 (3). If $(i, q) = (p - 1, 0)$, then

$$\mathbb{C}A = \mathbb{F}[C] \oplus \mathbb{C}A\{V\} = \mathbb{F}[C] \oplus \mathbb{D}A\{V\} \oplus \mathbb{D}A\{D_2, D_2 C, \ldots, D_2 C^{p-1}\}$$

and we have

$$\mathbb{C}A\{S^{p-1}\} = \mathbb{F}_p[C]\{S^{p-1}\} \oplus \mathbb{D}A\{V S^{p-1}\} \oplus (\oplus_{j=0}^{p-1} \mathbb{D}A\{D_2 C^j S^{p-1}\}).$$

This is contained in the sum of subspaces of Definition 2.2 (1)(2)(3).

Consider $(i, q)$ ($1 \leq i \leq p - 2$, $0 \leq q \leq p - 2$). If $q = 0$, then

$$\mathbb{C}A\{S^i\} = \mathbb{F}_p[C]\{S^i\} \oplus \mathbb{C}A\{V S^i\}$$

and this is contained in the sum of subspaces of Definition 2.2 (1)(4). If $i = q$, then

$$\mathbb{C}A\{S^i v^q\} = \mathbb{D}A\{S^i v^q\} \oplus (\oplus_{j=0}^{p-1} \mathbb{D}A\{C^{j+1} S^i v^q\})$$
and this is contained in the sum of the subspaces of Definition 2.2 (2)(4). If \( q \neq 0 \) and \( i \neq q \), then \( \mathbb{C} \mathbb{A} \{ S^i v^q \} \) is contained in the subspace of Definition 2.2 (4).

Next, consider \( T^k v^m \) (\( 1 \leq k \leq p - 2 \), \( 0 \leq m \leq p - 2 \)). Let \( i = p - k - 1 \), \( q \equiv m + k \mod (p - 1) \), \( 0 \leq q \leq p - 2 \), \( s = m \). Then

\[
T^k v^m = T^{p-i-1} v^s
\]

where \( 1 \leq i \leq p - 2 \), \( 0 \leq q \leq p - 2 \), \( s \equiv i + q \mod (p - 1) \) and

\[
q + 2i \equiv m + k + 2(p - k - 1) \equiv m - k \mod (p - 1).
\]

If \( k \neq m \), then \( q + 2i \neq 0 \) and \( \mathbb{C} \mathbb{A} \{ T^k v^m \} \) is contained in the subspace of Definition 2.2 (4). If \( k = m \), then \( q + 2i \equiv 0 \) and

\[
\mathbb{C} \mathbb{A} \{ T^k v^m \} = \mathbb{D} \mathbb{A} \{ \bigoplus_{j=0}^{p-1} T^k C^j v^m \} \oplus \mathbb{D} \mathbb{A} \{ T^k V v^m \}
\]

since \( \mathbb{C} \mathbb{A} = \mathbb{D} \mathbb{A} \{ 1, C, \ldots, C^{p-1}, V \} \). This is contained in the sum of the subspaces of Definition 2.2 (2)(4).

\[
\square
\]

3. Composition factors of \( H^*(E, \mathbb{F}_p) \)

In this section, we study the \( A_p(E, E) \)-module structure of \( H^*(E, \mathbb{F}_p) \). First, we shall prove Theorem 1.1. The even degree part \( H^{even}(E, \mathbb{Z}) \) of integral cohomology ring is generated by

\[
y_1, y_2, b_2, \ldots, b_{p-2}, C, v
\]

with

\[
\text{deg } y_i = 2, \text{ deg } b_i = 2i
\]

subject to the following relations:

\[
py_i = pb_j = pC = 0, \ p^2 v = 0,
\]

\[
y_1 y_2^p - y_2^p y_1 = 0,
\]

\[
y_i b_k = b_k b_i = C b_j = 0,
\]

\[
y_i C = y_i^p, \ C^2 = y_1^{2p-2} + y_2^{2p-2} - x_1^{p-1} y_2^{p-1}
\]

by [10] or [8, Theorem 3] (see [13] also). In particular, \( p^2(H^{2n}(E, \mathbb{Z})) = 0 \) for any \( n > 0 \).

On the other hand, the odd degree part of integral cohomology ring \( H^{odd}(E, \mathbb{Z}) \) is annihilate by \( p \) and so it is considered as an \( \mathbb{F}_p[y_1, y_2, v] \)-module. As an \( \mathbb{F}_p[y_1, y_2, v] \)-module, \( H^{odd}(E, \mathbb{Z}) \) is generated by two elements \( a_1 \) and \( a_2 \) with deg \( a_i = 3 \) subject to the following relations:

\[
y_1 a_2 - y_2 a_1 = 0, \ y_1^p a_2 - y_2^p a_1 = 0.
\]

Let \( H^*(E, \mathbb{Z}) \rightarrow H^*(E, \mathbb{F}_p) \) be the natural map induced by \( \mathbb{Z} \rightarrow \mathbb{F}_p \). We use the same letters for the images of \( y_i, b_j, C, v \) in \( H^*(E, \mathbb{F}_p) \). Then

\[
N = \mathbb{F}_p[v] \{ b_2, \ldots, b_{p-2} \} = \sqrt{0}
\]

in

\[
H^{even}(E, \mathbb{Z})/pH^{even}(E, \mathbb{Z}) = \mathbb{F}_p \otimes_{\mathbb{Z}} H^{even}(E, \mathbb{Z}).
\]

Since

\[
H^*(E) = (\mathbb{F}_p \otimes_{\mathbb{Z}} H^*(E, \mathbb{Z}))/\sqrt{0} = (\mathbb{F}_p \otimes_{\mathbb{Z}} H^{even}(E, \mathbb{Z}))/N,
\]

there is a short exact sequence of \( A_p(E, E) \)-modules,

\[
0 \rightarrow N \rightarrow H^{even}(E, \mathbb{Z})/pH^{even}(E, \mathbb{Z}) \rightarrow H^*(E) \rightarrow 0.
\]
On the other hand, since $pH^{\text{odd}}(E, \mathbb{Z}) = 0$, there is a short exact sequence of $A_p(E, E)$-modules,

$$\begin{array}{c}
0 \rightarrow H^{\text{even}}(E, \mathbb{Z})/pH^{\text{even}}(E, \mathbb{Z}) \rightarrow H^{\text{even}}(E, \mathbb{F}_p) \rightarrow H^{\text{odd}}(E, \mathbb{Z})[-1] \rightarrow 0.
\end{array}$$

Let $(y_1v, y_2v)H^{\text{even}}(E, \mathbb{Z})$ (resp. $(y_1v, y_2v)H^*(E)$) be the ideal of $H^{\text{even}}(E, \mathbb{Z})$ (resp. $H^*(E)$) generated by $y_1v$ and $y_2v$. Since $py_i = 0$ and $y_iN = 0$, it follows that

$$(y_1v, y_2v)H^{\text{even}}(E, \mathbb{Z}) \cong (y_1v, y_2v)H^*(E).$$

Here we use Milnor’s primitive operator $Q_1 = P^1\beta - \beta P^1$ on $H^*(-, \mathbb{F}_p)$. This operator induces a map $Q_1$ on $H^*(-, \mathbb{Z})$ such that the following diagram commutes:

$$\begin{array}{ccc}
H^{\text{odd}}(E, \mathbb{Z}) & \xrightarrow{Q_1} & H^{\text{even}}(E, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^{\text{odd}}(E, \mathbb{F}_p) & \xrightarrow{Q_1} & H^{\text{even}}(E, \mathbb{F}_p).
\end{array}$$

Moreover, $Q_1$ induces an isomorphism of $A_p(E, E)$-modules,

$$Q_1 : H^{\text{odd}}(E, \mathbb{Z}) \xrightarrow{\sim} (y_1v, y_2v)H^{\text{even}}(E, \mathbb{Z}) \cong (y_1v, y_2v)H^*(E)$$

(see [13, section 1]). Then we have

$$H^{\text{odd}}(E, \mathbb{Z}) \cong (y_1v, y_2v)H^*(E)[-2p + 1]$$

and the proof of the first part of Theorem [11] is completed by the exact sequences (3.1) and (3.2).

Next we consider the odd degree part $H^{\text{odd}}(E, \mathbb{F}_p)$. Let $K = \{x \in H^{\text{even}}(E, \mathbb{Z}) \mid px = 0\}$. Then there exists a short exact sequence of $A_p(E, E)$-modules,

$$0 \rightarrow H^{\text{odd}}(E, \mathbb{Z}) \rightarrow H^{\text{odd}}(E, \mathbb{F}_p) \rightarrow K[-1] \rightarrow 0.$$  

Let $H = H^{\text{even}}(E, \mathbb{Z}) \cap H^+(E, \mathbb{Z})$. Since $p^2H = 0$, $pH \subset K$. Moreover, $K$, $pH$ and $H/K$ are $A_p(E, E)$-modules.

Since the map $p : H \rightarrow H$ is a homomorphism of $A_\mathbb{Z}(E, E)$-modules,

$$H/K \cong pH$$

as $A_\mathbb{Z}(E, E)$-modules. Since these are modules for $A_p(E, E) = A_\mathbb{Z}(E, E)/pA_\mathbb{Z}(E, E)$, these are isomorphic as $A_p(E, E)$-modules. Hence we have

$$K \leftrightarrow pH \oplus K/pH \leftrightarrow H/K \oplus K/pH \leftrightarrow H/pH.$$  

Moreover, since

$$H/pH = \mathbb{F}_p \otimes H \leftrightarrow N \oplus H^+(E),$$

the proof of the second part of Theorem [11] is completed by (3.3) and (3.4).

Next we shall see the structure of $N$. Note that $\text{res}_A^{\mathbb{F}}(b_i) = 0$ for any $i$ and any maximal elementary abelian $p$-subgroup $A$ of $E$. Since the action of $g \in \text{GL}_2(\mathbb{F}_p)$ is given by

$$g^*(b_i) = \det(g)^i b_i, \quad g^*(v) = \det(g)v$$

(see [8, Theorem 3]), $N$ is a direct sum of simple $A_p(E, E)$-modules isomorphic to $S(E, E, \det^i)$ for $0 \leq i \leq p - 2$. Hence we have the following:
Proposition 3.1. Let

$$N_q = \mathbb{F}_p \langle v^kb_i^j \mid k \geq 0, k + i \equiv q \mod (p-1) \rangle$$

for $0 \leq q \leq p-2$. Then

$$N = \bigoplus_{0 \leq q \leq p-2} N_q$$

and

$$N_q \cong \bigoplus S(E, E, \det^q)$$

as $A_p(E, E)$-modules.

Next, we shall consider the structure of the ideal $(y_1v, y_2v)H^*(E)$. Let $I = (y_1v, y_2v)H^*(E)$. Let $\Gamma_S$ be the $\mathbb{F}_p$-subspace defined in Definition 2.2 for each simple $A_p(E, E)$-module $S$. We shall show that

$$Ie_S \cong I \cap \Gamma_S$$

for an idempotent $e_S$ which corresponds to $S$ and determine the $\mathbb{F}_p$-subspace $I \cap \Gamma_S$ explicitly.

Lemma 3.2. Let

$$L = \mathbb{F}_p[y_1, y_2, C] \oplus \mathbb{F}_p[V] \langle v, \ldots, v^{p-2}, Cv, \ldots, Cv^{p-2} \rangle \oplus \mathbb{F}_p[D_1] \{D_1\} \oplus \mathbb{F}_p[D_1] \{D_2\}$$

where $\mathbb{F}_p[y_1, y_2, C]$ is the subalgebra of $H^*(E)$ generated by $y_1, y_2$ and $C$. Then

$$H^*(E) = I \oplus L.$$

Proof. Let $(y_1, y_2, C)$ be the ideal of $H^*(E)$ generated by $y_1, y_2$ and $C$. Since

$$D_1 = C^p + V \equiv V \mod (y_1, y_2, C),$$

we have

$$H^*(E) = \mathbb{F}_p[v] \oplus (y_1, y_2, C)$$

$$= \mathbb{F}_p \oplus \mathbb{F}_p[V] \{v, \ldots, v^{p-2}\} \oplus \mathbb{F}[V] \{V\} \oplus (y_1, y_2, C)$$

$$= \mathbb{F}_p \oplus \mathbb{F}_p[V] \{v, \ldots, v^{p-2}\} \oplus \mathbb{F}[D_1] \{D_1\} \oplus (y_1, y_2, C).$$

On the other hand, since

$$D_2D_1 = CV(C^p + V) \equiv D_2V \mod I,$$

it follows that

$$\mathbb{F}_p \oplus (y_1, y_2, C)$$

$$= \mathbb{F}_p[y_1, y_2, C] \oplus \mathbb{F}_p[v] \{Cv\} \oplus I$$

$$= \mathbb{F}_p[y_1, y_2, C] \oplus \mathbb{F}_p[V] \{Cv, \ldots, Cv^{p-2}\} \oplus \mathbb{F}_p[V] \{CV\} \oplus I$$

$$= \mathbb{F}_p[y_1, y_2, C] \oplus \mathbb{F}_p[V] \{Cv, \ldots, Cv^{p-2}\} \oplus \mathbb{F}_p[D_1] \{D_2\} \oplus I$$

and we have $H^*(E) = I \oplus L$. \qed

Lemma 3.3. For each simple $A_p(E, E)$-module $S$, we have

$$\Gamma_S = (I \cap \Gamma_S) \oplus (L \cap \Gamma_S)$$
where $I = (y_1 v, y_2 v) H^*(E)$ and $L$ is an $\mathbb{F}_p$-subspace defined in Lemma 3.2. Moreover,

1. If $S = S(E, C_p, U_i)$, then $I \cap \Gamma_S = 0$.
2. If $S = S(E, A, S^{p-1} \otimes \det^q)$, then
   \[ I \cap \Gamma_S = \Gamma_S = \left\{ \begin{array}{ll}
   \text{DA}\{\bigoplus_{0 \leq j \leq p-1} D_2 C_j(\mathbb{F}_p C + S^{p-1})\} & (q = 0) \\
   \text{DA}\{\bigoplus_{0 \leq j \leq p-1} v^q C_j(C S^q + T^q)\} & (1 \leq q \leq p - 2).
   \end{array} \right. \]

3. \[ I \cap \Gamma_S = \left\{ \begin{array}{ll}
   \text{DA}\{D_2^2\} & (S = S(E, E, S^0)) \\
   \text{CA}\{C^2 v^q\} & (S = S(E, E, \det^q), 1 \leq q \leq p - 2) \\
   \text{DA}\{V S^{p-1}\} & (S = S(E, E, S^p)) \\
   \text{CA}\{v^q S^{p-1}\} & (S = S(E, E, S^{p-1} \otimes \det^q), 1 \leq q \leq p - 2)
   \end{array} \right. \]

4. Let
   \[ S = S^i v^q, \quad T = T^{p-i-1} v^s \]

for $1 \leq i \leq p - 2, 0 \leq q \leq p - 2$, where $s \equiv i + q \pmod{p-1}, 0 \leq s \leq p - 2$. Then

$I \cap \Gamma_{S(E, E, S^p \otimes \det^q)}$ is the following $\mathbb{F}_p$-subspace:

\[ \begin{array}{ll}
   \text{CA}\{V S\} \oplus \text{DA}\{V T\} & (q \equiv 2i \equiv 0) \\
   \text{CA}\{V S\} \oplus \text{CA}\{T\} & (q \equiv 0, 2i \not\equiv 0) \\
   \text{DA}\{S\} \oplus \text{DA}\{VT\} & (i = q, 3i \not\equiv 0) \\
   \text{DA}\{S\} \oplus \text{CA}\{T\} & (i = q, 3i \not\equiv 0, 2i \not\equiv 0) \\
   \text{DA}\{S\} \oplus \text{CA}\{VT\} & (i = q, 3i \not\equiv 0, 2i \equiv 0) \\
   \text{CA}\{S\} \oplus \text{DA}\{VT\} & (q \not= 0, i \not= q, q + 2i \equiv 0) \\
   \text{CA}\{S\} \oplus \text{CA}\{T\} & (q \not= 0, i \not= q, q + 2i \not\equiv 0, i + q \not\equiv 0) \\
   \text{CA}\{S\} \oplus \text{CA}\{VT\} & (q \not= 0, i \not= q, q + 2i \not\equiv 0, i + q \equiv 0).
   \end{array} \]

5. $I \cap \Gamma_{S(1, 1, p^i)} = 0$.

Proof. (1) If $S = S(E, C_p, U_i) (0 \leq i \leq p - 2)$, then $I \cap \Gamma_S = 0$ and $\Gamma_S \subset L$ by Definition 2.2.

(2) If $S = S(E, A, S^{p-1} \otimes \det^q) (0 \leq q \leq p - 2)$, then $\Gamma_S \subset I$ since $D_2 C = C^2 V \in I$.

(3) If $S = S(E, E, S^0)$, then $\Gamma_S = \text{DA}^+$,

\[ \text{DA}^+ = \text{DA}\{D_2^2\} \oplus \mathbb{F}_p[D_1]\{D_1, D_2\}. \]

Since $\text{DA}\{D_2^2\} \subset I$ and $\mathbb{F}_p[D_1]\{D_1, D_2\} \subset L$, we have

\[ \text{DA}^+ = (I \cap \Gamma_S) \oplus (L \cap \Gamma_S) \]

and $I \cap \Gamma_S = \text{DA}\{D_2^2\}$.

If $S = S(E, E, \det^q) (1 \leq q \leq p - 2)$, then $\Gamma_S = \text{CA}\{v^q\}$,

\[ \text{CA}\{v^q\} = \text{CA}\{C^2 v^q\} \oplus \mathbb{F}_p[V]\{v^q, Cv^q\}. \]

Since $C^2 v^q \in I$ and $\mathbb{F}_p[V]\{v^q, Cv^q\} \subset L$, it follows that

\[ \text{CA}\{v^q\} = (I \cap \Gamma_S) \oplus (L \cap \Gamma_S) \]

and $I \cap \Gamma_S = \text{CA}\{C^2 v^q\}$.

If $S = S(E, E, S^{p-1})$, then $\Gamma_S = \text{DA}\{V S^{p-1}\}$. If $S = S(E, E, S^{p-1} \otimes \det^q) (1 \leq q \leq p - 2)$, then $\Gamma_S = \text{CA}\{v^q S^{p-1}\}$. In these cases, $\Gamma_S \subset I$.

(4) Let $S = S^i v^q (1 \leq i \leq p - 2, 0 \leq q \leq p - 2)$. If $q = 0$ then $SV \subset I$. If $q \not= 0$ then $S \subset I$. Hence the first term of $\Gamma_S$ is contained in $I$. 
Let $T = T^{p-i-1}v^{s}$, $s \equiv i + q \mod (p - 1)$, $0 \leq s \leq p - 2$. If $s \equiv 0$, then $VT \subset I$. If $i + q \neq 0$, then $T \subset I$. Hence the second term of $\Gamma_S$ is contained in $I$ unless $i = q$, $3i \neq 0$, $2i \equiv 0$, or $q \neq 0$. If $i \neq q$, $q + 2i \neq 0$ in these cases,

$$\text{CA} \{T\} = \mathbb{F}_p[C]\{T\} \oplus \text{CA} \{VT\}$$

where $\text{CA} \{VT\} \subset I$, $\mathbb{F}_p[C]\{T\} \subset L$. Hence

$$\text{CA} \{T\} = \text{CA} \{T\} \cap I \oplus \mathbb{F}_p[C]\{T\} \cap I$$

and $\text{CA} \{T\} \cap I = \text{CA} \{VT\}$. \hfill \square

**Lemma 3.4.** Let $I = (y_1v, y_2v)H^*(E)$. Then

$$I = \bigoplus_S (I \cap \Gamma_S).$$

**Proof.** First, $H^*(E) = \bigoplus_S \Gamma_S = I \oplus L$ by Proposition 2.5 and Lemma 3.2. On the other hand,

$$\Gamma_S = (I \cap \Gamma_S) \oplus (L \cap \Gamma_S)$$

by Lemma 3.3. Hence

$$H^*(E) = \bigoplus \Gamma_S = \bigoplus ((I \cap \Gamma_S) \oplus (L \cap \Gamma_S))$$

$$= (\bigoplus (I \cap \Gamma_S)) \oplus (\bigoplus (L \cap \Gamma_S)) \subset I \oplus L = H^*(E).$$

Hence we have

$$I = \bigoplus_S (I \cap \Gamma_S).$$

\hfill \square

Now, we determine the $\mathbb{F}_p$-vector space $Ie_S$ for any simple $A_p(E, E)$-module $S$.

**Theorem 3.5.** Let $S$ be a simple $A_p(E, E)$-module. Then there exists an idempotent $e_S$ corresponding to $S$ such that

$$Ie_S = (I \cap \Gamma_S)e_S \cong I \cap \Gamma_S.$$ 

**Proof.** By Theorem 2.3, $\Gamma_S \cong \Gamma_S e_S$ for some idempotent corresponding to $S$. Hence $e_S$ induces an isomorphism

$$I \cap \Gamma_S \cong (I \cap \Gamma_S)e_S.$$

For a graded $\mathbb{F}_p$-subspace $M \subset H^*(E)$, let $M^n = H^n(E) \cap M$. Then

$$\dim I^n = \sum_S \dim (I^n)e_S \geq \sum_S \dim (I^n \cap \Gamma_S)e_S$$

$$= \sum_S \dim I^n \cap \Gamma_S = \dim I^n.$$

The last equality follows from Lemma 3.4. Hence we have

$$Ie_S = (I \cap \Gamma_S)e_S.$$ 

\hfill \square
Remark 3.6. Let $X$ be the indecomposable summand in the complete stable splitting of $BE$ which corresponds to a simple $A_p(E, E)$-modules $S$. Let $e_S$ be an idempotent which correspond to $S$ as above. Then we can get the $\mathbb{F}_p$-vector space

$$H^*(E, \mathbb{F}_p) e_S \cong H^*(\bigvee X, \mathbb{F}_p) \cong \bigoplus d H^*(X, \mathbb{F}_p)$$

where $\bigvee X$ is a wedge sum of $d = \dim S$ copies of $X$, from Theorem 1.1, Proposition 3.1 and Theorem 3.3.

Corollary 3.7. Every simple $A_p(E, E)$-module appears as a composition factor in $H^{2n}(E, \mathbb{F}_p)$ for some $n \leq p^2 - 2$.

Proof. Let $H(p^2 - 2) = \oplus_{n=0}^{p^2-2} H^{2n}(E, \mathbb{F}_p)$. For a simple $A_p(E, E)$-module $S$, let $2\gamma(S)$ be the lowest degree such that $(\Gamma_S \cap I)^{2\gamma(S)} \neq 0$. If $2\gamma(S) \leq 2(p + 2)(p - 1)$, then $2(\gamma(S) - p) \leq 2(p^2 - 2)$. Since $S$ appears in the degree $2(\gamma(S) - p)$ part of $I[-2p]$, it follows that $S$ appears in $H^{2(\gamma(S) - p)}(E, \mathbb{F}_p)$ by Theorem 1.1. Hence $S$ appears in $H(p^2 - 2)$. In particular, if $\Gamma_S \leq I$, then $S$ appears in $H(p^2 - 2)$ by Corollary 2.4. This implies that simple modules

$$S(E, A, S(A)^{p-1} \otimes \det q)(0 \leq q \leq p - 2), \ S(E, E, S^{p-1} \otimes \det q)(0 \leq q \leq p - 2)$$

appear in $H(p^2 - 2)$.

On the other hand, since the degrees of $C^2v^q$, $(1 \leq q \leq p - 2)$ and $VS^i$, $S^iv^q$, $(1 \leq i \leq p - 2, 0 \leq q \leq p - 2)$ are all smaller than $\deg D_2S^{p-1} = 2(p + 2)(p - 1)$, we have that $S(E, E, \det q)$, $(1 \leq q \leq p - 2)$ and $S(E, E, S^i \otimes \det q)$, $(1 \leq i \leq p - 2, 0 \leq q \leq p - 2)$ appear in $H(p^2 - 2)$.

Moreover, since $S(E, C_p, U_i)$ $(0 \leq i \leq p - 2)$ appears in $H^2(E) \oplus \cdots \oplus H^{2(p-1)}(E)$, it appears in $H(p^2 - 2)$.

Finally, we consider $S(E, E, S^0)$. Since it appears in $H^{2p(p-1)}(E)$ and $2p(p - 1) = \deg D_1 \leq 2(p^2 - 2)$, $S(E, E, S^0)$ appears in $H(p^2 - 2)$. This completes the proof. \qed

4. Stable splitting of groups related to $L_3(p)$

In this section, we consider the stable splitting of $BG$ for groups $G$ having $E$ as a Sylow $p$-subgroup, in particular the linear group $L_3(p)$ and its extensions.

Benson and Fehbach [2], Martino and Priddy [11] prove the following theorem on complete stable splitting. Let $P$ be a finite $p$-group. If $G$ is a finite group which contains $E$, then $G$ is considered as an $(E, E)$-biset. We denote by $[G]$ the element of $A_p(E, E)$ corresponding to $G$. Let $S(P, Q, V)$ be the simple $A_p(P, P)$-module which corresponds to $(Q, V)$ where $Q$ is a subgroup of $P$ and $V$ is a simple $\mathbb{F}_p\text{Out}(P)$-module.

Theorem 4.1 (2, [11]). Let $G$ be a finite group with Sylow $p$-subgroup $P$. The complete stable splitting of $BG$ is given by

$$BG \sim \bigvee_{(Q, V)} \dim(S(G, Q, V))X_{S(P, Q, V)}$$

where $S(G, Q, V) = S(P, Q, V)[G]$.

Let

$$H^*(G) = (\mathbb{F}_p \otimes H^*(G, \mathbb{Z}))/\sqrt{(0)}$$

for a finite group $G$. From Corollary 2.4 and Corollary 3.7 we have the following.
Lemma 4.4 denotes the subspace consists of
responds to the simple
A
M
p
way to compute it from the information on the cohomology
H
M
Out
([13, Theorem 4.3],[15, Theorem 3.1])
Theorem 4.3
If
Corollary 4.2.
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direct summands in the stable splitting of
for all
(2)
If
for all
0 ≤ n ≤ p2 − 2, then BG1 ∼ BG2.
In general, the computation of these dim S(G, Q, V) is not so easy. Hence we study the way to compute it from the information on the cohomology H*(G). (In fact, in [15], most direct summands in the stable splitting of BG are computed from H*(G).)
Let \( \mathcal{F}_G \) be the fusion system on \( E \) determined by \( G \). Let \( \mathcal{F}_G^{ec-rad} \) be the set of \( \mathcal{F}_G^{ec} \)-radical maximal elementary abelian \( p \)-subgroups of \( E \). If \( A \) is a maximal elementary abelian \( p \)-subgroup of \( E \), then \( A \in \mathcal{F}_G^{ec-rad} \) if and only if \( W_G(A) = N_G(A)/C_G(A) = \text{Out}_{\mathcal{F}_G}(A) \geq SL_2(\mathbb{F}_p) \) by [12] Lemma 4.1. Let \( W_G(E) = N_G(E)/EC_G(E) = \text{Out}_{\mathcal{F}_G}(E) \).

**Theorem 4.3** ([13] Theorem 4.3,[15] Theorem 3.1). Let \( G \) have the Sylow \( p \)-subgroup \( E \), then
\[
H^*(G) \cong H^*(E)[G] = H^*(E)^{W_G(E)} \cap (\cap_{A \in \mathcal{F}_G^{ec-rad}} (\text{res}_A^E)^{-1}(H^*(A)^{W_G(A)})).
\]
Moreover, if \( M \) is an \( A_p(E,E) \)-submodule of \( H^*(E) \), then
\[
M[G] = M^{W_G(E)} \cap (\cap_{A \in \mathcal{F}_G^{ec-rad}} (\text{res}_A^E)^{-1}(H^*(A)^{W_G(A)})).
\]

**Proof.** The first part follows from Alperin’s fusion theorem ([3] Theorem A.10]). Let \( M \) be an \( A_p(E,E) \)-submodule of \( H^*(E) \). Since \( [G][G] \in \mathbb{F}_p[G] \), it follows that \( M[G] = M \cap H^*(E)[G] \). Hence the result follows from the first part. 

Let \( X_{i,q} \) be the indecomposable summand in the stable splitting of \( BE \) which corresponds to the simple \( A_p(E,E) \)-module \( S(E,E,S^i \otimes \text{det}^q) \). For \( 0 \leq q \leq p - 2 \), let \( L(2,q) \) (resp. \( L(1,q) \)) be the summand which corresponds to the simple \( A_p(E,E) \)-module \( S(E,A,S(A)^{p-1} \otimes \text{det}^q) \) (resp. \( S(E,C_p,U_q) \)). We set \( M(2) = L(1,0) \cup L(2,0) \).

Suppose that the stable splitting of \( BG \) is written as
\[
BG \sim (\vee_{i,q} m(G)_{i,q} X_{i,q}) \vee (\vee_q m(G,2,q)L(2,q)) \vee (\vee_q m(G,1,q)L(1,q)).
\]
Recall that
\[
H^{2q}(E) \cong \begin{cases} S(E,C_p,U_i) & (1 \leq q \leq p-2) \\ S(E,C_p,U_0) & (q = p-1) \end{cases}
\]
by Theorem 2.3. Hence,

**Lemma 4.4** ([15] Corollary 4.6)). The multiplicity \( m(G,1,q) \) for \( L(1,q) \) is given by
\[
m(G,1,q) = \begin{cases} \dim H^{2q}(G) & (1 \leq q \leq p-2) \\ \dim H^{2(p-1)}(G) & (q = 0). \end{cases}
\]
The multiplicity \( n(G)_{i,q} \) of \( X_{i,q} \) depends only on \( W_G(E) = N_G(E)/EC_G(E) \). For \( H \leq \text{GL}_2(\mathbb{F}_p) \) and \( \text{GL}_2(\mathbb{F}_p) \)-submodule \( M \) of \( H^*(E) \), let
\[
M^H = \{ m \in M \mid mh = m \text{ for any } h \in H \}
\]denotes the subspace consists of \( H \)-invariant elements. Then we have the following lemma.
Lemma 4.5 ([15] Lemma 4.7). The multiplicity $n(G)_{i,q}$ of $X_{i,q}$ in $BG$ is given by

$$n(G)_{i,q} = \dim(S^i v^q)^{W_G(E)}.$$ 

Next problem is to seek the multiplicity $m(G, 2)_q$ for the summand $L(2, q)$ in $BG$. We can prove,

Lemma 4.6 ([15] Proposition 4.9]). The multiplicity of $L(2, 0)$ in $BG$ is given by

$$m(G, 2)_0 = \sharp_G(A) - \sharp_G(F^{ec}A)$$

where $\sharp_G(A)$ (resp. $\sharp_G(F^{ec}A)$) is the number of $G$-conjugacy classes of rank two elementary abelian $p$-subgroups in $E$ (resp. subgroups in $F_G^{ec}$-rad).

Lemma 4.7 ([15] Corollary 4.10]). The multiplicity of $L(1, 0)$ in $BG$ is given by

$$m(G, 1)_0 = \dim H^{2(p-1)}(G) = \sharp_G(A) - \sharp_G(F^{ec}A).$$

Remark 4.8. By Lemma 4.6 and 4.7, $m(G, 1)_0 = m(G, 2)_0$, namely, $L(1, 0)$ and $L(2, 0)$ always appear in $BG$ as $M(2) = L(1, 0) \lor L(2, 0)$. On the other hand, in Corollary 2.4 all simple modules except for $S(E, A, S^{p-1})$ appear in $H^{2n}(E)$ for $n \leq p^2 - 1$. Note that the minimal $n$ such that $S(E, E, S^{p-1})$ appears in $H^{2n}(E)$ is $p^2 - 1 = \frac{1}{2} \deg(VS^{p-1})$. Hence we may replace the bound $(p + 2)(p - 1)$ by $p^2 - 1$ in Corollary 2.4 (1).

For the number $m(G, 2)_q$ for $q \neq 0$, it seems that there is not a good way to find it. However we give some condition such that $m(G, 2)_q = 0$.

Lemma 4.9 ([15] Lemma 4.11]). Let $\xi \in \mathbb{F}_p^*$ be a primitive $(p - 1)$-th root of 1. Suppose that $G \supset E$: $\langle \text{diag}(\xi, \xi) \rangle$. If $\xi^{3k} \neq 1$, then $BG$ does not contain the summand $L(2, k)$, i.e., $m(G, 2)_k = 0$.

Let $\xi$ be the multiplicative generator of $\mathbb{F}_p^*$ as above. Let

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2(\mathbb{F}_p).$$

Let

$$T = \langle \text{diag}(\xi, \xi), \text{diag}(\xi, 1) \rangle$$

be a torus in $\text{GL}_2(\mathbb{F}_p)$. Then $w$ normalizes $T$. Let $T \langle w \rangle$ be the semidirect product of $T$ by $\langle w \rangle$.

Lemma 4.10. Assume that $1 \leq l \leq p - 1$, $0 \leq k \leq p - 2$. Then

$$(S^l v^k)^T = \begin{cases} 
\mathbb{F}_p y_1^l y_2^p v^{p-1-i} & (l = 2i, \ k = p - 1 - i, \ 1 \leq i \leq \frac{p-1}{2}) \\
\mathbb{F}_p y_1^p v^{p-1} \oplus \mathbb{F}_p y_2^{p-1} & (l = p - 1, \ k = 0) \\
0 & (\text{otherwise})
\end{cases}$$

and

$$(S^l v^k)^T \langle w \rangle = \begin{cases} 
\mathbb{F}_p y_1^{2j} y_2^{2j} v^{p-1-2j} & (l = 4j, \ k = p - 1 - 2j, \ 1 \leq j \leq \frac{p-1}{4}) \\
\mathbb{F}_p (y_1^{p-1} + y_2^{p-1}) & (l = p - 1, \ k = 0) \\
0 & (\text{otherwise}).
\end{cases}$$
Hence $y_1^j y_2^{l-j} v^k$ is $T$-invariant if and only if $i + k \equiv 0 \mod p - 1$, namely, $l \equiv 2i \mod p - 1$ and $i + k \equiv 0 \mod p - 1$. If $k = 0$, then $i = 0, p - 1, l = p - 1$. If $k > 0$, then $i = p - 1 - k, l = 2i$. Since $1 \leq l \leq p - 1$, it follows that $i \leq \frac{p - 1}{2}$.

Next consider the action of $w$. Since $w$ interchanges $y_1$ and $y_2$, $wv = -v, y_1^j y_2^{p - 1 - i}$ is $w$-invariant if and only if $i$ is even. □

Now assume that $p - 1 = 3m$. Note that $m$ is even. We set

$$H = \langle \text{diag}(\xi, \xi), \text{diag}(\xi^3, 1) \rangle.$$  

Then $w$ normalizes $H$. Let $H \langle w \rangle$ be the semidirect product of $H$ by $\langle w \rangle$.

**Lemma 4.11.** Assume that $p - 1 = 3m$. Let $m = 2n$. Assume that $1 \leq l \leq p - 1, 0 \leq k \leq p - 2$. Then $(S^l v^k)H$ is equal to the following vector space.

$$\begin{align*}
\{ & \mathbb{F}_p \{ y_1^{p-1}, y_1^2, y_2^m, y_1^2, y_2^m, y_2^{p-1} \} & (l = p - 1, k = 0) \\
& \mathbb{F}_p \{ y_1^{i-n} y_2^{i+n v^3 m-i}, y_1^{i+n} y_2^{i+n v^3 m-i} \} & (l = 2i, k = 3n - i, 1 \leq i < 3n) \\
& \mathbb{F}_p \{ y_1^{i-m} y_2^{i+m v^3 m-i}, y_1^{i-m} y_2^{i+m v^3 m-i} \} & (l = 2i, k = (p - 1) - i, m \leq i < 3n) \\
& 0 & (\text{otherwise}).
\end{align*}$$

**Proof.** We consider the action of $\text{diag}(\xi, \xi)$ and $\text{diag}(\xi^3, 1)$ on $y_1^j y_2^{l-j} v^k$ for $0 \leq j \leq l$. Since

$$\text{diag}(\xi, \xi) y_1^j y_2^{l-j} v^k = \xi^{l+2k} y_1^j y_2^{l-j} v^k$$

and

$$\text{diag}(\xi^3, 1) y_1^j y_2^{l-j} v^k = \xi^{3i+3k} y_1^j y_2^{l-j} v^k,$$

$y_1^j y_2^{l-j} v^k$ is $H$-invariant if and only if

$$\begin{cases}
 l + 2k \equiv 0 & \mod p - 1 \\
 j + k \equiv 0 & \mod m.
\end{cases}$$

Note that the condition $l + 2k \equiv 0 \mod p - 1$ implies that $l$ is even, so we set $l = 2i, 1 \leq i \leq 3n$. Then $y_1^j y_2^{l-j} v^k \in (S^l v^k)H$ if and only if

$$\begin{cases}
 i + k \equiv 0 & \mod 3n \\
 j + k \equiv 0 & \mod m.
\end{cases}$$

Since $0 \leq k \leq p - 2, k \equiv -i \mod 3n$ if and only if

$$k = 3n - i \text{ or } (p - 1) - i.$$  

First, assume $k = 3n - i$. Then $j \equiv -k = i - 3n \mod m$ if and only if $j = i + sn$ for some integer $s \in \mathbb{Z}$ such that $s \equiv 1 \mod 2$. Then

$$0 \leq j \leq l = 2i \iff 0 \leq i + sn \leq 2i$$

$$\iff -i \leq sn \leq i$$

$$\iff |s| \leq \frac{i}{n}.$$
If $1 \leq i < n$, namely, $\frac{i}{n} < 1$, then there is no $s \in \mathbb{Z}$ such that $|s| \leq \frac{i}{n}$ with $s \equiv 1 \mod 2$.

Assume that $n \leq i \leq 3n$. If $i = 3n$, namely, $l = p - 1$ and $k = 0$, then, since $\frac{i}{n} = 3$, $|s| \leq \frac{i}{n}$ if and only if $s = -3, -1, 1, 3$. Then

$$j = i + sn = (3 + s)n = 0, m, 2m, 3m.$$ If $n \leq i < 3n$, namely, $1 \leq \frac{i}{n} < 3$, then $|s| \leq \frac{i}{n}$ if and only if $s = -1, 1$ since $s \equiv 1 \mod 2$. So we have

$$j = i + sn = i \pm n.$$ Next we consider the case $k = (p - 1) - i$. Then,

$$j \equiv -k = i - (p - 1) \mod m$$ if and only if $j = i + sm$ for some $s \in \mathbb{Z}$. Then,

$$0 \leq j = i + sm \leq l = 2i \iff -i \leq sm \leq i$$

$$\iff |s| \leq \frac{i}{m}.$$ If $1 \leq i < m$, then this implies $s = 0$ and $j = i$. If $m \leq i \leq 3n$, namely, $1 \leq \frac{i}{m} \leq \frac{3}{2} < 2$, then this implies $s = 0, \pm 1$. Hence we have $j = i, i \pm m$. This completes the proof. \hfill \Box

Lemma 4.12. Assume that $p - 1 = 3m$. Let $m = 2n$. Assume that $1 \leq l \leq p - 1$, $0 \leq k \leq p - 2$. Then $(S^l v^k)^{H(w)}$ is equal to the following vector space.

$$\begin{cases}
F_p \{ y_1^{l-1} + y_2^{l-1}, y_1^{2m} y_2 + y_1 y_2^{2m} \} & (l = p - 1, \ k = 0) \\
F_p \{(y_1^{i-n} y_2^{i+n} + (-1)^{3n-i} y_1^{i-n} y_2^{i+n}) v^{3n-i}\} & (l = 2i, \ k = 3n - i, \ n \leq i < 3n) \\
F_p \{y_1 y_2 v^{3m-i}\} & (l = 2i, \ k = (p - 1) - i, \ 1 \leq i < m, \ i: \text{even}) \\
F_p \{(y_1^{i-m} y_2^{i+m} + y_1^{i+m} y_2^{i-m}) v^{3m-i}, y_1 y_2 v^{3m-i}\} & (l = 2i, \ k = (p - 1) - i, \ m \leq i \leq 3n, \ i: \text{even}) \\
F_p \{(y_1^{i-m} y_2^{i+m} - y_1^{i+m} y_2^{i-m}) v^{3m-i}\} & (l = 2i, \ k = (p - 1) - i, \ m \leq i \leq 3n, \ i: \text{odd}) \\
0 & (\text{otherwise})
\end{cases}$$

Proof. Since $w$ interchanges $y_1$ and $y_2$, $vw = (\det w) v = -v$, this lemma follows from the previous lemma. \hfill \Box

Let $p - 1 = 3m$. We consider the multiplicity $m(G, 2)_m$ and $m(G, 2)_{2m}$ in some cases. Recall that $(CS^q + T^q)v^q \cong S(E, A, S(A)^{p-1} \otimes \det v)$ for $1 \leq q \leq p - 2$ by [7, Corollary 9.3]. This module has a basis

$$(y_1^iy_2^j)^v, \ (i = 0, \ q \leq i \leq p - 1 + q, \ j = p - 1 + q - i).$$

Note that

$$y_1^iy_2^j, \ (i = 0, \ p \leq i \leq p - 1 + q, \ j = p - 1 + q - i)$$

is a basis of $CS^q$. On the other hand,

$$y_1^iy_2^j, \ (q \leq i \leq p - 1, \ j = p - 1 + q - i)$$

is a basis of $T^q$. Moreover, if $q = m$ or $q = 2m$, then all elements in $(CS^q + T^q)v^q$ are diag($\xi, \xi$)-invariant since

$$\text{diag}(\xi, \xi)(y_1^iy_2^j)^v = \xi^{i+j+2q}(y_1^iy_2^j)^v = \xi^{p-1+2q}(y_1^iy_2^j)^v = (y_1^iy_2^j)^v.$$
Lemma 4.13. Let $M = (CS^m + T^m)v^m$.
(1) $M^T$ has a basis
\[ y_1^{2m}y_2^{2m}v^m. \]
(2) $M^H$ has a basis
\[ y_1^{4m}v^m, \ y_1^3y_2^mv^m, \ y_1^{2m}y_2^mv^m, \ y_1y_2^{3m}v^m, \ y_2^{4m}v^m. \]
(3) $M^{T(w)}$ has a basis
\[ y_1^{2m}y_2^{2m}v^m. \]
(4) $M^{H(w)}$ has a basis
\[ (y_1^{4m} + y_2^{4m})v^m = C(y_1^m + y_2^m)v^m, \ y_1^my_2^m(y_1^{2m} + y_2^{2m})v^m, \ y_1^{2m}y_2^{2m}v^m. \]

Proof. Since
\[ \text{diag}(\xi, 1)(y_1^iy_2^jv^m) = \xi^{i+m}(y_1^iy_2^jv^m) \]
and
\[ \text{diag}(\xi^3, 1)(y_1^iy_2^jv^m) = \xi^{3(i+m)}(y_1^iy_2^jv^m) = \xi^{3i}(y_1^iy_2^jv^m), \]
y_1^iy_2^jv^m is $T$-invariant if and only if $i + m \equiv 0 \pmod{p - 1}$. Moreover, $y_1^iy_2^jv^m$ is $H$-invariant if and only if $i \equiv 0 \pmod{m}$.

(1) Since $i = 0$ or $m \leq i \leq p - 1 + m = 4m$, $i + m \equiv 0 \pmod{p - 1}$ if and only if $i = 2m$.
(2) Since $i = 0$ or $m \leq i \leq p - 1 + m = 4m$, $i \equiv 0 \pmod{m}$ if and only if $i = 0, m, 2m, 3m, 4m$.
(3) (4) Since $m$ is even, $w$ acts on $v^m$ trivially. On the other hand $w$ interchanges $y_1$ and $y_2$. Hence the results follow from (1) and (2). \hfill \square

Similarly, we have the following.

Lemma 4.14. Let $M = (CS^{2m} + T^{2m})v^{2m}$.
(1) $M^T$ has a basis
\[ C'y_1^my_2^mv^{2m} = y_1^{4m}y_2^{2m}v^{2m}. \]
(2) $M^H$ has a basis
\[ y_1^{5m}v^{2m}, \ C'y_1^my_2^mv^{2m} = y_1^{4m}y_2^{2m}v^{2m}, \ y_1^{3m}y_2^{3m}v^{2m}, \ y_1^{2m}y_2^{3m}v^{2m}, \ y_1^{5m}y_2^{5m}v^{2m}. \]
(3) $M^{T(w)}$ has a basis
\[ C'y_1^my_2^mv^{2m} = y_1^{4m}y_2^{2m}v^{2m}. \]
(4) $M^{H(w)}$ has a basis
\[ (y_1^{5m} + y_2^{5m})v^{2m} = C(y_1^{2m} + y_2^{2m})v^{2m}, \ y_1^{2m}y_2^{2m}(y_1^m + y_2^m)v^{2m}, \ C'y_1^my_2^mv^{2m}. \]

If $A \in \mathcal{A}(E)$ is a maximal elementary abelian $p$-subgroup of $E$, then
\[ H^*(A) = \mathbb{F}_p[y_A, u_A], \ \deg y_A = \deg u_A = 2. \]

We may assume that
\[ \text{res}^E_{A_i}(y_1) = y_{A_i}, \ \text{res}^E_{A_i}(y_2) = iy_{A_i} \text{ for } i \in \mathbb{F}_p \]
and
\[ \text{res}^E_{A_\infty}(y_1) = 0, \ \text{res}^E_{A_\infty}(y_2) = y_{A_\infty}. \]

Moreover,
\[ \text{res}^E_{A}(C) = y_A^{p-1}, \ \text{res}^E_{A}(v) = u_A^p - y_A^{p-1}u \]
for any $A \in \mathcal{A}(E)$ (see [1], section 4).
Lemma 4.15. Let $1 \leq q \leq p - 2$. Then
\[(CS^q + T^q)v^q[G] = ((CS^q + T^q)v^q)^{W_G(E)} \cap (\cap_{A \in \mathcal{F}_p^{\text{ec}} \text{-rad}} \ker \text{res}_E^A)\]

Proof. Let $g = y_A$ and $u = u_A$ for $A \in \mathcal{A}(E)$. Then
\[\text{res}_E^A((CS^q + T^q)v^q) = \mathbb{F}_p y^{p-1+q} \text{res}_A^E(v^q) = \mathbb{F}_p y^{p-1}(yu^p - y^p u)^q.\]
If $g \in \text{Aut}(A) = \text{GL}_2(\mathbb{F}_p)$, then
\[g(yu^p - y^p u) = (\det g)(yu^p - y^p u)\]
and $y^{p-1}(yu^p - y^p u)^q$ is not $\text{SL}_2(\mathbb{F}_p)$-invariant, hence the result follows from Theorem 4.3.

\[\square\]

Proposition 4.16. Suppose that $\mathcal{F}_p^{\text{ec}} \text{-rad} = \{A_0, A_\infty\}$. Then $m(G, 2)_m = m(G, 2)_{2m}$ and we have the following values:

| \(W_G(E)\) | \(m(G, 2)_m = m(G, 2)_{2m}\) | \(H\) | \(H\langle w \rangle\) | \(T\) | \(T\langle w \rangle\) |
|-------------|----------------|------|--------------|------|----------------|
| \(H\langle w \rangle\) | 2 | \(L_3(p) : 2\) | 3 | \((p - 1)\) |
| \(T\langle w \rangle\) | 2 | \(L_3(p) : S_3\) | 3 | \((p - 1)\) |
| \(T\langle w \rangle\) | 2 | \(L_3(p) : 2\) | 3 | \((p - 1)\) |

Proof. Since
\[\text{res}_{A_0}^E(y_1) \neq 0, \quad \text{res}_{A_0}^E(y_2) = 0\]
and
\[\text{res}_{A_\infty}^E(y_1) = 0, \quad \text{res}_{A_\infty}^E(y_2) \neq 0,\]
the results follows from Lemma 4.13, Lemma 4.14 and 4.15.

Next we study the stable splitting of $BG$ for some $G$ related to the linear group $L_3(p)$. There are 6 saturated fusion systems related to $L_3(p)$ [12, p. 46, Table 1.1].

| \(W_G(E)\) | $|\mathcal{F}_p^{\text{ec}}\text{-rad}|$ | Group | $p$ |
|-------------|----------------|------|-----|
| $H$ | 1 + 1 | $L_3(p)$ | 3 | \((p - 1)\) |
| $H\langle w \rangle$ | 2 | $L_3(p) : 2$ | 3 | \((p - 1)\) |
| $T$ | 1 + 1 | $L_3(p) : 3$ | 3 | \((p - 1)\) |
| $T\langle w \rangle$ | 2 | $L_3(p) : S_3$ | 3 | \((p - 1)\) |
| $T\langle w \rangle$ | 2 | $L_3(p) : 2$ | 3 | \((p - 1)\) |

We determine the stable splittings of these 6 groups. Note that the these results, with the results in [15], give a complete information on the splitting for fusion systems on $E$ with $|\mathcal{F}_p^{\text{ec}}\text{-rad}| \geq 2$ by the classification in [12].

Let
\[X = X_{0,0} \lor 2X_{p-1,0} \lor (\lor_{1 \leq i \leq (p-1)/2} X_{2i, p-1-i}) \lor M(2)\]
and
\[X' = X_{0,0} \lor X_{p-1,0} \lor (\lor_{1 \leq j \leq (p-1)/4} X_{4j, p-1-2j}) \lor M(2).\]

Theorem 4.17. Suppose that $W_G(E) = T$ and $\mathcal{F}_p^{\text{ec}} \text{-rad} = \{A_0, A_\infty\}$. If $3 \nmid p - 1$ then $BG$ is stably homotopic to $X$. If $p - 1 = 3m$, then $BG$ is stably homotopic to $X \lor L(2, m) \lor L(2, 2m)$. 
Proof. By Lemma 4.4 and Lemma 4.10, \( L(1, q) \) (1 \( \leq q \leq p - 2 \)) is not contained in \( BG \). Since \( A_i \) (1 \( \leq i \leq p - 1 \)) are \( T \)-conjugate, there are three conjugacy classes of maximal elementary abelian \( p \)-subgroups and two of them consist of \( \mathcal{F}^{ec}_G \)-radical subgroups. From Lemma 4.6 and Lemma 4.7, just one \( L(2, 0) \) (and one \( L(1, 0) \)) is contained in \( BG \). Moreover if 3 does not divide \( p - 1 \), then \( L(2, q) \) is not contained in \( BG \) for each 1 \( \leq q \leq p - 2 \) from Lemma 4.9.

Moreover, by Lemma 4.5 and Lemma 4.10

\[
n(G)_{l,k} = \begin{cases} 
1 & (l = 2i, \ k = p - 1 - i, \ 1 \leq i \leq \frac{p-1}{2}) \\
2 & (l = p - 1, \ k = 0) \\
0 & \text{(otherwise)}.
\end{cases}
\]

On the other hand, if \( p - 1 = 3m \), then \( m(G, 2)_m = m(G, 2)_{2m} = 1 \) by Proposition 4.16. This completes the proof.

Theorem 4.18. Suppose that \( W_G(E) = T(w) \) and \( \mathcal{F}^{ec}_G \)-rad = \( \{A_0, A_\infty\} \). If 3 \( \nmid \ p - 1 \) then \( BG \) is stably homotopic to \( X' \). If \( p - 1 = 3m \), then \( BG \) is stably homotopic to \( X' \lor L(2, m) \lor L(2, 2m) \).

Proof. The proof is similar to that of previous Theorem. By Lemma 4.4 and Lemma 4.10, \( L(1, q) \) (1 \( \leq q \leq p - 2 \)) is not contained in \( BG \). Since \( A_i \) (1 \( \leq i \leq p - 1 \)) are \( T \)-conjugate, there are two conjugacy classes of maximal elementary abelian \( p \)-subgroups and one of them consists of \( \mathcal{F}^{ec}_G \)-radical subgroups. From Lemma 4.6 and Lemma 4.7, just one \( L(2, p - 1) \) (and one \( L(1, p - 1) \)) is contained in \( BG \). Moreover if 3 does not divide \( p - 1 \), then \( L(2, q) \) is not contained in \( BG \) for each 1 \( \leq q \leq p - 2 \) from Lemma 4.9.

Moreover, by Lemma 4.5 and Lemma 4.10

\[
n(G)_{l,k} = \begin{cases} 
1 & (l = 4j, \ k = p - 1 - 2j, \ 1 \leq j \leq \frac{p-1}{4}) \\
1 & (l = p - 1, \ k = 0) \\
0 & \text{(otherwise)}.
\end{cases}
\]

On the other hand, if \( p - 1 = 3m \), then \( m(G, 2)_m = m(G, 2)_{2m} = 1 \) by Proposition 4.16. This completes the proof.

Next assume that \( p - 1 = 3m \). Let \( m = 2n \).

Theorem 4.19. Suppose that \( W_G(E) = H \) and \( \mathcal{F}^{ec}_G \)-rad = \( \{A_0, A_\infty\} \). Then \( BG \) is stably homotopic to

\[
X_{0,0} \lor 4X_{p-1,0} \lor 2(\lor_{n \leq i < 3n} X_{2i,3m-i}) \lor (\lor_{1 \leq i < m} X_{2i,3m-i}) \lor 3(\lor_{m \leq i < 3n} X_{2i,3m-i}) \lor 3(M(2) \lor L(2, m) \lor L(2, 2m)).
\]

Proof. By Lemma 4.4 and Lemma 4.11, \( m(G, 1)_q = 0 \) for 1 \( \leq q \leq p - 2 \). On the other hand, by Lemma 4.6 and Corollary 4.7, \( m(G, 1)_0 = m(G, 2)_0 = 5 - 2 = 3 \). By Lemma 4.9, \( m(G, 2)_k = 0 \) for 1 \( \leq k \leq p - 2 \), \( k \neq m, 2m \). By Proposition 4.16, \( m(G, 2)_m = \)
The multiplicity \( n(G)_{i,q} \) is obtained by Lemma 4.11. By Lemma 4.11

\[
n(G)_{i,q} = \begin{cases} 
4 & (l = p - 1, q = 0) \\
2 & (l = 2i, q = 3n - i, n \leq i < 3n) \\
1 & (l = 2i, q = (p - 1) - i, 1 \leq i < m) \\
3 & (l = 2i, q = (p - 1) - i, m \leq i \leq 3n) \\
0 & \text{(otherwise)}. 
\end{cases}
\]

\[\Box\]

**Theorem 4.20.** Suppose that \( W_G(E) = H(w) \) and \( F_G^{\text{ef}} \)-rad = \( \{A_0, A_\infty\} \). Then \( BG \) is stably homotopic to

\[
X_{0,0} \lor 2X_{p-1,0} \lor (\lor_{n \leq i < 3n} X_{2i,3n-i}) \lor (\lor_{1 \leq j \leq 3n/2} X_{4j,3m-2j}) \lor (\lor_{m \leq i \leq 3n} X_{2i,(p-1)-i}) \\
\lor 2(M(2) \lor L(2, m) \lor L(2, 2m)).
\]

**Proof.** By Lemma 4.14 and Lemma 4.12, \( m(G, 1)_q = 0 \) for \( 1 \leq q \leq p - 2 \). By Lemma 4.6 and Corollary 4.7, \( m(G, 1)_0 = m(G, 2)_0 = 3 - 1 = 2 \). By Lemma 4.9, \( m(G, 2)_k = 0 \) for \( 1 \leq k \leq p - 2, k \neq m, 2m \). By Proposition 4.16, \( m(G, 2)_m = m(G, 2)_{2m} = 2 \). The multiplicity \( n(G)_{i,q} \) is obtained by Lemma 4.12. By Lemma 4.12

\[
n(G)_{i,q} = \begin{cases} 
2 & (l = p - 1, q = 0) \\
1 & (l = 2i, q = 3n - i, n \leq i < 3n) \\
1 & (l = 2i, q = (p - 1) - i, 1 \leq i < m, \; i: \; \text{even}) \\
2 & (l = 2i, q = (p - 1) - i, m \leq i \leq 3n, \; i: \; \text{even}) \\
1 & (l = 2i, q = (p - 1) - i, m \leq i \leq 3n, \; i: \; \text{odd}) \\
0 & \text{(otherwise)}. 
\end{cases}
\]

Moreover, consider the 3rd, 4th and 5th cases. We have

\[
(\lor_{1 \leq i < m, i: \; \text{even}} X_{2i,(p-1)-i}) \lor 2(\lor_{m \leq i \leq 3n, i: \; \text{even}} X_{2i,(p-1)-i}) \lor (\lor_{m \leq i \leq 3n, i: \; \text{odd}} X_{2i,(p-1)-i})
\]

\[
= (\lor_{1 \leq i \leq 3n, i: \; \text{even}} X_{2i,(p-1)-i}) \lor (\lor_{m \leq i \leq 3n} X_{2i,(p-1)-i})
\]

\[
= (\lor_{1 \leq j \leq 3n/2} X_{4j,(p-1)-2j}) \lor (\lor_{m \leq i \leq 3n} X_{2i,(p-1)-i}).
\]

This completes the proof. \( \Box \)

Next we consider the specific case, that is, \( p = 7 \). We give a result which supplements the result on splitting for \( p = 7 \) in [15].

**Example 4.21.** Let \( p = 7, \; p - 1 = 6, \; m = 2, \; n = 1 \). Suppose that \( F_G^{\text{ef}} \)-rad = \( \{A_0, A_\infty\} \).

(1) If \( W_G(E) = T \), then

\[
BG \sim X_{0,0} \lor X_{2,5} \lor X_{4,4} \lor 2X_{6,0} \lor X_{6,3} \lor M(2) \lor L(2, 2) \lor L(2, 4).
\]

(2) If \( W_G(E) = T \langle w \rangle \), then

\[
BG \sim X_{0,0} \lor X_{4,4} \lor X_{6,0} \lor M(2) \lor L(2, 2) \lor L(2, 4).
\]

(3) If \( W_G(E) = H \), then

\[
BG \sim X_{0,0} \lor 2X_{2,2} \lor X_{2,5} \lor 2X_{4,1} \lor 3X_{4,4} \lor 4X_{6,0} \lor 3X_{6,3} \\
\lor 3(M(2) \lor L(2, 2) \lor L(2, 4)).
\]
(4) If $W_G(E) = H\langle w \rangle$, then

$$BG \sim X_{0,0} \lor X_{2,2} \lor X_{4,1} \lor 2X_{4,4} \lor 2X_{6,0} \lor X_{6,3}$$

$$\lor 2(M(2) \lor L(2,2) \lor L(2,4)).$$

Let $G_1$ and $G_2$ be finite groups with Sylow $p$-subgroup $E$. If $F_{G_1}$ is (isomorphic to) a subfusion system of $F_{G_2}$, then $BG_1 \sim BG_2 \lor X$ for some summand $X$ of $BG_1$. In this case, we write

$$G_2 \leftarrow X \rightarrow G_1$$

We use same notation for fusion systems.

In [15], the second author considered the graphs related to the splitting of sporadic simple groups and some exotic fusion system for $p = 7$ and obtained the following.

**Theorem 4.22** ([15] Theorem 9.4). Let $p = 7$. We have the following two sequences:

$$X_{0,0} \lor X_{4,4} \sim RV_3 \xleftarrow{X_{2,2} \lor X_{6,0}} RV_2 \xleftarrow{M(2) \lor L(2,2) \lor L(2,4)} O'N : 2 \xleftarrow{X_{4,1} \lor X_{4,4} \lor X_{6,0} \lor X_{6,3}} O'N$$

$$X_{0,0} \lor X_{4,4} \lor X_{6,0} \sim RV_1 \xleftarrow{M(2)} Fi_{24} \xleftarrow{X_{2,2} \lor X_{6,0} \lor X_{6,3}} Fi'_{24} \xleftarrow{M(2) \lor L(2,2) \lor L(2,4)} He : 2 \xleftarrow{X_{3,0} \lor X_{3,3} \lor X_{5,2} \lor X_{5,5} \lor L(1,3) \lor L(2,3)} He.$$

where $RV_1, RV_2, RV_3$ are the exotic fusion systems of Ruiz and Viruel [12].

Now we add more information on the splittings for $p = 7$.

**Theorem 4.23.** Let $p = 7$. We have the following diagram:

$$RV_1 \xleftarrow{M(2) \lor \tilde{L}} L_3(7).S_3 \xleftarrow{Y'} Y \xleftarrow{L_3(7).3} L_3(7)$$

$$Y \lor Z \lor M(2) \lor \tilde{L} \quad Y \lor Z \lor M(2) \lor \tilde{L} \quad Y \lor Y' \lor Z \lor M(2) \lor \tilde{L} \quad Y \lor Y' \lor Z \lor M(2) \lor \tilde{L} \quad \quad Y \lor Y' \lor Z \lor M(2) \lor \tilde{L}$$

$$O'N \xleftarrow{M(2) \lor \tilde{L}} L_3(7) : 2 \xleftarrow{Y} Y \lor Z \lor M(2) \lor 2\tilde{L} \xleftarrow{Y \lor Y' \lor Z \lor M(2) \lor \tilde{L}} L_3(7)$$

$$RV_1 \xleftarrow{M(2)} Fi_{24} \xleftarrow{Y} Y \xleftarrow{Fi'_{24}}$$

where

$$Y = X_{2,2} \lor X_{6,0} \lor X_{6,3}, \quad Y' = X_{2,5} \lor X_{6,0} \lor X_{6,3}, \quad Z = X_{4,1} \lor X_{4,4}$$

$$\tilde{L} = L(2,2) \lor L(2,4).$$
Proof. We have the following table by \cite{12} Lemma 4.9, Lemma 4.16.

| group (fusion system) | $\text{Out}_F(E)$ | $\mathcal{F}^{ec}\text{-rad}$ | $\text{Out}_F(A)$ |
|-----------------------|------------------|-----------------|----------------|
| $L_3(7)$              | $6 \times 2 = \langle 3I, u \rangle$ | $\{A_0\} \{A_\infty\}$ | $\text{SL}_2(7) : 2$ |
| $L_3(7)$              | $6 \times 2 = \langle 3I, w \rangle$ | $\{A_4\} \{A_6\}$ | $\text{SL}_2(7) : 2$ |
| $L_3(7) : 2$          | $(6 \times 2) : 2 = \langle 3I, u, w \rangle$ | $\{A_0, A_\infty\}$ | $\text{SL}_2(7) : 2$ |
| $L_3(7) : 2$          | $(6 \times 2) : 2 = \langle 3I, u, w \rangle$ | $\{A_1, A_6\}$ | $\text{SL}_2(7) : 2$ |
| $O'N$                 | $(6 \times 2) : 2 = \langle 3I, u, w \rangle$ | $\{A_0, A_\infty\} \{A_1, A_6\}$ | $\text{SL}_2(7) : 2$ |
| $L_3(7).3$            | $6^2 = T$ | $\{A_0\} \{A_\infty\}$ | $\text{GL}_2(7)$ |
| $L_3(7).S_3$          | $6^2 : 2 = T \langle w \rangle$ | $\{A_0, A_\infty\}$ | $\text{GL}_2(7)$ |
| $Fi_{24}$             | $6 \times S_3 = \langle 3I, s, w \rangle$ | $\{A_1, A_2, A_4\} \{A_3, A_5, A_6\}$ | $\text{SL}_2(7) : 2$ |
| $Fi_{24}$             | $6^2 : 2 = T \langle w \rangle$ | $\{A_1, \ldots, A_6\}$ | $\text{SL}_2(7) : 2$ |
| $RV_1$                | $6^2 : 2 = T \langle w \rangle$ | $\{A_0, A_\infty\} \{A_1, \ldots, A_6\}$ | $\text{GL}_2(7), \text{SL}_2(7) : 2$ |

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

and $T = \{\text{diag}(\alpha, \beta) \mid \alpha, \beta \in \mathbb{F}_7^\times\}$ is the subgroup of all invertible diagonal matrices. The set $\mathcal{F}^{ec}\text{-rad}$ is separated by conjugacy classes and $\text{Out}_F(A)$ is described for each representative $A$ of conjugacy classes in $\mathcal{F}^{ec}\text{-rad}$ if they are different. Note that if we take the generators $a$ and $b$ of $E$ suitably, we can obtained the two rows in the case of $L_3(7)$ and $L_3(7) : 2$. For example, consider $G = L_3(7)$. Let $E$ be the group of all upper triangular matrices with diagonal entry 1. The subgroups in $\mathcal{F}^{ec}\text{-rad}$ are

$$\left\{ \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid \alpha, \beta \in \mathbb{F}_p \right\}, \quad \left\{ \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \mid \alpha, \beta \in \mathbb{F}_p \right\}.$$

If we take

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

then $\mathcal{F}^{ec}_G\text{-rad} = \{A_0, A_\infty\}$ and $\text{Out}_{\mathcal{F}_G}(E) = \langle 3I, u \rangle = H$. On the other hand, if we take

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

then $\mathcal{F}^{ec}_G\text{-rad} = \{A_1, A_6\}$ and $\text{Out}_{\mathcal{F}_G}(E) = \langle 3I, w \rangle$.

The inclusions of fusion systems are obtained by the table above. For $Fi_{24}' \leftarrow L_3(7)$ and $Fi_{24} \leftarrow L_3(7) : 2$, we use the second rows of $L_3(7)$ and $L_3(7) : 2$. The information on the summands are obtained by Example 4.21 and Theorem 4.22. \qed

Remark 4.24. As we can see from Theorem 4.22 or 4.23 above, $B(Fi_{24})$ is a stable summand of $B(O'N)$, $B(O'N) \sim B(Fi_{24}) \vee Y \vee Z \vee L$, but the fusion system of $O'N$ is not isomorphic to a subfusution system of fusion system of $Fi_{24}$, namely,

$$Fi_{24} \xrightarrow{Y \vee Z \vee L} O'N$$

does not hold.
Let $\mathcal{F}_0 = \mathcal{F}_{0^*/N}$, $\mathcal{F}_1 = \mathcal{F}_{0^{ec}}$. By [12] Lemma 4.3, for each $A_i \in \mathcal{F}_0^{ec}$-rad, there exists an element of order 6 in $Out(\mathcal{F}_0(E)) \leq GL_2(F_p)$ which has an eigenvalue 3 with eigenvector $(1 \ i)$ (0 \ 1) if $i = \infty$ and determinant 5. Hence there exists an involution in $Out(\mathcal{F}_0(E))$ which has an eigenvalue $-1$ with eigenvector $(1 \ i)$ (0 \ 1) if $i = \infty$ and determinant $-1$.

We may assume that $Out(\mathcal{F}_1(E)) = 6^2 : 2 = T(w)$ and $\mathcal{F}_1^{ec}$-rad = $\{A_1, \ldots, A_6\}$ as above. Suppose that $K = Out(\mathcal{F}_0(E)) \cong (6 \times 2) : 2 \leq Out(\mathcal{F}_1(E))$. Then $K$ contains exactly 4 involutions with determinant $-1$. Moreover $K \supset \langle \text{diag}(-1, 1), \text{diag}(1, -1) \rangle$ since $\langle \text{diag}(-1, 1), \text{diag}(1, -1) \rangle \triangleleft (6^2 : 2)$. Note that $|\mathcal{F}_0^{ec}\text{-rad}| = 4$. Since $\text{diag}(-1, 1)$ (resp. $\text{diag}(1, -1)$) has an eigenvalue $-1$ with eigenvector $(1 \ 0)$ (resp. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$) and determinant $-1$, it follows that $A_0, A_\infty \in \mathcal{F}_0^{ec}$-rad for any choice of $K \leq 6^2 : 2 = T(w)$. Hence $\mathcal{F}_0$ is not isomorphic to a subfusional system of $\mathcal{F}_1$.

5. Some remarks on the case $p = 3$

Recall that $H^3(E, \mathbb{Z}) = F_p\{a_1, a_2\}$. The short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{i} F_p \rightarrow 0$$

induces the following short exact sequence

$$(5.1) \quad 0 \rightarrow H^2(E, \mathbb{Z}) \xrightarrow{j} H^2(F, F_p) \xrightarrow{\hat{\beta}} H^3(E, \mathbb{Z}) \rightarrow 0.$$ 

Hence there exist elements $a'_1, a'_2 \in H^2(E, F_p)$ such that $\hat{\beta}(a'_i) = a_i$ and $H^2(E, F_p) = F_p\{y_1, y_2, a'_1, a'_2\}$.

We consider the action of $Out(E) = GL_2(F_p)$. The sequence (5.1) is a sequence of $F_pGL_2(F_p)$-modules and the map $Q_1 \hat{\beta}$ induces an isomorphism of $F_pGL_2(F_p)$-modules,

$$H^2(E, F_p)/H^2(E, \mathbb{Z}) \rightarrow H^3(E, \mathbb{Z}) \rightarrow F_p\{y_1 v, y_2 v\} \cong S^1 \otimes \text{det}.$$ 

Now, consider the sequence

$$H^{2p}(E, \mathbb{Z}) \xrightarrow{j_*} H^{2p}(E, F_p) \xrightarrow{\hat{\beta}} H^{2p+1}(E, \mathbb{Z}) \xrightarrow{p} H^{2p+1}(E, \mathbb{Z}) \xrightarrow{j_*} H^{2p+1}(E, F_p).$$

By taking the $p$-th power, we have an $F_pGL_2(F_p)$-morphism,

$$H^2(E, F_p) \rightarrow H^{2p}(E, F_p).$$

Since $\beta = j_* \hat{\beta}$ is the Bockstein homomorphism, we have $\beta((a'_i)^p) = 0$. Moreover, since $j_* : H^{2p+1}(E, \mathbb{Z}) \rightarrow H^{2p+1}(E, F_p)$ is injective, $\hat{\beta}((a'_i)^p) = 0$. Hence, it follows that $(a'_i)^p \in j_*(H^{2p}(E, \mathbb{Z})) \cong H^{2p}(E)$.

On the other hand, since

$$H^{2p}(E) \cong CS^1 + T^1 + F_p\{v\} \leftrightarrow S^1 \oplus (S^{p-2} \otimes \text{det}) \oplus \text{det},$$

we see that if $p > 3$ then $(a'_i)^p = 0$. Since $H^{even}(E, F_p)$ is generated by $1, a'_1, a'_2$ as a module over $F_p \otimes H^{even}(E, \mathbb{Z})$, it follows that

$$H^{even}(E, F_p)/\sqrt{(0)} = (F_p \otimes H^{even}(E, \mathbb{Z}))/\sqrt{(0)}$$
and in particular,
\[ H^*(E, \mathbb{F}_p) / \sqrt{0} = (\mathbb{F}_p \otimes H^*(E, \mathbb{Z})) / \sqrt{0} = H^*(E) \]
for \( p > 3 \).

On the other hand, if \( p = 3 \), then the sequence (5.1) does not split and \( \mathbb{F}_3 \{y_1, y_2\} \) is the unique nontrivial \( \mathbb{F}_3 \text{GL}_2(\mathbb{F}_3) \)-submodule of \( H^2(E, \mathbb{F}_3) \) (cf. [9] p.74). This implies that \( (a'_i)^p \neq 0 \) for any \( n > 0 \) and in particular, we see that \( a'_i \) is not nilpotent.

In fact, the structure of \( H^*(E, \mathbb{F}_3) / \sqrt{0} \) is known by the result of Leary [9 Theorem 7] and we have the following:

**Proposition 5.1.** Assume that \( p = 3 \). Then \( H^*(E, \mathbb{F}_3) / \sqrt{0} \) is generated by 
\[ y_1, y_2, a'_1, a'_2, v \]
with 
\[ \deg y_i = \deg a'_i = 2, \ \deg v = 6 \]
subject to the following relations:
\[ y_1^3y_2 - y_1y_2^3 = 0 \]
\[ a'_1a'_2 = a'_1 y_1 = a'_2 y_2 = y_1 y_2, \ (a'_i)^2 = (a'_i)^2 = a'_1 y_2 = a'_2 y_1. \]

Since \( H^4(E, \mathbb{F}_3) / (H^4(E, \mathbb{F}_3) \cap \sqrt{0}) \) is spanned by \( y_1^2, y_1 y_2, y_2^2, (a'_i)^2 \) and \( \dim_{\mathbb{F}_3} H^4(E) = 4 \),
\[ H^4(E, \mathbb{F}_3) / (H^4(E, \mathbb{F}_3) \cap \sqrt{0}) = H^4(E). \]
In particular, \( (a'_i)^2 \in H^4(E) \) and hence we have
\[ H^*(E, \mathbb{F}_3) / \sqrt{0} = H^*(E) \oplus \mathbb{F}_3[v] \{F_3a'_1 + F_3a'_2\}. \]
Since \( H^2(E, \mathbb{F}_3) / H^2(E) \cong S^1 \otimes \det \) as \( \mathbb{F}_3 \text{GL}_2(\mathbb{F}_3) \)-modules,
\[ (H^*(E, \mathbb{F}_3) / \sqrt{0}) / H^*(E) \cong \mathbb{F}_3[v] \otimes (S^1 \otimes \det) \]
as \( \mathbb{F}_3 \text{GL}_2(\mathbb{F}_3) \)-modules. If \( Q \) is a proper subgroup of \( E \), then \( H^*(Q, \mathbb{F}_3) / \sqrt{0} = H^*(Q) \).

Hence
\[ (H^*(E, \mathbb{F}_3) / \sqrt{0}) A_3(Q, E) A_3(E, Q) \subset (H^*(Q, \mathbb{F}_3) / \sqrt{0}) A_3(E, Q) \]
\[ = H^*(Q) A_3(E, Q) \subset H^*(E). \]
In particular, \( (H^*(E, \mathbb{F}_3) / \sqrt{0}) / H^*(E) \) is annihilated by \( A_3(Q, E) A_3(E, Q) \) for any \( Q < E \).

Hence, every composition factors of \( (H^*(E, \mathbb{F}_3) / \sqrt{0}) / H^*(E) \) as an \( A_3(E, E) \)-module is isomorphic to \( S(E, E, S^i \otimes \det^i) \) for some \( i, q \) and we have the following:

**Proposition 5.2.**
\[ (H^n(E, \mathbb{F}_3) / \sqrt{0}) / H^n(E) \cong \begin{cases} S(E, E, S^i \otimes \det) & (n \equiv 2 \mod 12) \\ S(E, E, S^i) & (n \equiv 8 \mod 12) \\ 0 & \text{(otherwise)} \end{cases} \]

**Corollary 5.3.** Let \( X_{0,0} \) be the summand which corresponding to the simple module \( S(E, E, \mathbb{F}_p) \) and \( e \) be the corresponding idempotent in \( A_p(E, E) \). Then
\[ (H^*(E, \mathbb{F}_3) / \sqrt{0}) e \cong H^*(E) e \cong \mathbb{D}A^+. \]
At last of this paper, we see more closely the cohomology $H^*(X)$ of a summand $X$ in the stable splitting of $BG$ with $E \in \text{Syl}_3(G)$ in the case $p = 3$. The lowest degree and some of the second lowest degree $* > 0$ with $H^{2*}(X) \neq 0$ are given as follows:

\[
\begin{align*}
L(1, 1) & : |S^1| = 1 & L(1, 0) & : |y^2| = 2, \\
L(2, 1) & : |CS^1v| = 6 & L(2, 0) & : |S^2D_2| = 10, \\
X_{0,0} & : |V| = 6 & X_{0,1} & : |v| = 3 \ (|Cv| = 5) \\
X_{1,0} & : |S^1V| = 7 & X_{1,1} & : |T^1| = 3 \ (|S^1v| = 4) \\
X_{2,0} & : |S^2V| = 8 & X_{2,1} & : |S^2v| = 5
\end{align*}
\]

where $|x| = \frac{1}{2} \deg x$ for an element or a subspace of $H^*(E)$. First note that $BG$ always contains $X_{0,0}$. The lowest degree of nonzero elements in $H^{2*}(L(i, j))$ or $H^{2*}(X_{i,q}), (i, q) \neq (0, 0)$ are all different except for $X_{0,1}$ and $X_{1,1}$. On the other hand we see $H^4(X_{0,1}) = 0$ but $H^4(X_{1,1}) \cong \mathbb{F}_3$. Moreover $L(1, 0)$ and $L(2, 0)$ have same multiplicity by Lemma 4.6 and 4.7. Hence we can count the numbers of

\[
L(1, 1), M(2) = L(1, 0) \lor L(2, 0), X_{1,1}, X_{0,1}, X_{2,1}, L(2, 1), X_{1,0}, X_{2,0}
\]

from $H^{2*}(G)$ for $* = 1, 2, 4, 3, 5, 6, 7, 8$. Thus we have the following result which is similar to Corollary 4.2 (1) (See Remark 4.8).

**Theorem 5.4.** Let $G_1$ and $G_2$ be finite groups with same Sylow 3-subgroup $E$. If

\[
\dim H^{2n}(G_1) = \dim H^{2n}(G_2)
\]

for $n \leq 8$, then $BG_1 \sim BG_2$.

For example, let $G_1 = 2F_4(2)'$ and $G_2 = J_4$. Then by [15, Theorem 6.2],

\[
B(2F_4(2)') \sim BJ_4 \lor X_{2,0}.
\]

Hence $H^{2n}(2F_4(2)') \cong H^{2n}(J_4)$ for $n < 8$ and $\dim H^{16}(2F_4(2)') > \dim H^{16}(J_4)$. See [15, section 6] for details.

**Remark 5.5.** If $G$ has a Sylow 3-subgroup $E$, then $BG$ is homotopic to the classifying space of one of the groups listed in [15, Theorem 6.2]. Moreover the cohomology of each dominant summand $X_{i,j}$ of $BE$, expect for $X_{1,0}$ and $X_{1,1}$, is deduced from the cohomology of those finite groups.

On the other hand, as we can see from the graph in [15, Theorem 6.2], $X_{1,0}$ and $X_{1,1}$ always appear as $X_{1,0} \lor X_{1,1}$. We shall give an brief explanation of this fact. Let

\[
H = W_G(E) = N_G(E) / E\text{C}_G(E) \leq \text{Out}(E) = \text{GL}_2(\mathbb{F}_3).
\]

Note that $H$ is a $3'$-group, in fact, 2-group. The multiplicity of $X_{1,j}$ in the stable splitting of $BG$ is equal to $\dim(S^1 \otimes \text{det}^j)^H$. We have to show that

\[
\dim(S^1)^H = \dim(S^1 \otimes \text{det})^H.
\]

We may assume that $H \neq 1$. Then $\dim(S^1)^H = 1$ if and only if $H$ is conjugate to the subgroup $\langle \text{diag}(1, -1) \rangle$ in $\text{GL}_2(\mathbb{F}_3)$. Similarly $\dim(S^1 \otimes \text{det})^H = 1$ if and only if $H$ is conjugate to the subgroup $\langle \text{diag}(1, -1) \rangle$ in $\text{GL}_2(\mathbb{F}_3)$. Hence we have

\[
\dim(S^1)^H = \dim(S^1 \otimes \text{det})^H
\]

and this implies that $X_{1,0}$ and $X_{1,1}$ appear in $BG$ with same multiplicity.
References

[1] D. J. Benson, Stably splitting $BG$, Bull. Amer. Math. Soc. 33 (1996) 189-198.
[2] D. J. Benson, M. Feshbach, Stable splittings of classifying spaces of finite groups, Topology 31 (1992) 157-176.
[3] C. Broto, R. Levi, B. Oliver, The homotopy theory of fusion systems, J. Amer. Math. Soc. 16 (2003) 779-856.
[4] S. Bouc, Biset functor for finite groups, Lecture Notes in Mathematics 1990, Springer (2010).
[5] J. Dietz, S. Priddy, The stable homotopy type of rank two $p$-groups, Homotopy theory and its applications, Contemp. Math. 188, Amer. Math. Soc., Providence, RI, (1995) 93-103.
[6] D. J. Green, On the cohomology of the sporadic simple group $J_4$, Math. Proc. Cambridge Philos. Soc. 113 (1993) 253-266.
[7] A. Hida, N. Yagita, Representations of the double Burnside algebra and cohomology of the extraspecial $p$-group, J. Algebra 409 (2014) 265-319.
[8] I. J. Leary, The integral cohomology rings of some $p$-groups, Math. Proc. Cambridge Philos. Soc. 110 (1991) 25-32.
[9] I. J. Leary, The mod-$p$ cohomology rings of some $p$-groups, Math. Proc. Cambridge Philos. Soc. 112 (1992) 63-75.
[10] G. Lewis, The integral cohomology rings of groups of order $p^3$, Trans. Amer. Math. Soc. 132 (1968) 501-529.
[11] J. Martino, S. Priddy, The complete stable splitting for the classifying space of a finite group, Topology 31 (1992) 143-156.
[12] A. Ruiz, A. Viruel, The classification of $p$-local finite groups over the extraspecial group of order $p^3$ and exponent $p$, Math. Z. 248 (2004) 45-65.
[13] M. Tezuka, N. Yagita, On odd prime components of cohomologies of sporadic simple groups and the rings of universal stable elements, J. Algebra 183 (1996) 483-513.
[14] N. Yagita, On odd degree parts of cohomology of sporadic simple groups whose Sylow $p$-subgroup is the extra-special $p$-group of order $p^3$, J. Algebra 201 (1998) 373-391.
[15] N. Yagita, Stable splitting and cohomology of $p$-local finite groups over the extraspecial $p$-group of order $p^3$ and exponent $p$, Geom. Topol. Monogr. 11 (2007) 399-434.

Akihiko Hida, Faculty of Education, Saitama University, Shimo-okubo 255, Sakura-ku, Saitama-city, Saitama, Japan
E-mail address: ahida@mail.saitama-u.ac.jp

Nobuaki Yagita, Department of Mathematics, Faculty of Education, Ibaraki University, Mito, Ibaraki, Japan
E-mail address: yagita@mx.ibaraki.ac.jp