Euclidean spinor Green’s functions
in the spacetime of a straight cosmic string

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Abstract

The spinor Green’s function and the twisted spinor Green’s function in an Euclidean space with a conical-type line singularity are generally determined. In particular, in the neighbourhood of the point source, they are expressed as a sum of the usual Euclidean spinor Green’s function and a regular term. In four dimension, these determinations can be used to calculate the vacuum energy density and the twisted one for a massless spinor field in the spacetime of a straight cosmic string.
cosmic string. In the Minkowski spacetime, the vacuum energy density for a massive twisted spinor field is explicitly determined.

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I Introduction

We consider an Euclidean space with a conical-type line singularity which is described by the metric

\[ ds^2 = d\rho^2 + B^2 \rho^2 d\varphi^2 + (dx^3)^2 + \cdots + (dx^n)^2 \]  

in a coordinate system \((\rho, \varphi, x^i), i = 3, \ldots, n\) such that \(\rho \geq 0\) and \(0 \leq \varphi < 2\pi\). Metric (1) is characterized by an arbitrary positive constant \(B\), different from zero. The line of singularity is located at \(\rho = 0\) but it disappears for \(B = 1\).

The aim of this paper is to determine in metric (1) the spinor Green’s function \(S^{(n)}(x, x_0; m)\) and the twisted spinor Green’s function \(S^{(n)(T)}(x, x_0; m)\).

We are motivated by the fact that metric (1) can result from the complexification of the time coordinate in the metric

\[ ds^2 = d\rho^2 + B^2 \rho^2 d\varphi^2 + dz^2 - dt^2 \]  

which represents a straight cosmic string [1], \(B\) being related to the linear mass density \(\mu\) by \(B = 1 - 4G\mu\) (units are chosen such that \(c = \hbar = 1\)). In this case, the spacetime is the product of a cone by \(\mathbb{R}^2\) and we have \(0 < B \leq 1\). Now, to study the quantum field theory of a spinor field of mass \(m\), it is possible to work within the Euclidean approach by performing a Wick rotation \(t = -i\tau\), therefore the determination of the Euclidean spinor Green’s function \(S^E_E(x, x_0; m)\) and the twisted one \(S^{(T)}_E(x, x_0; m)\) for the Dirac operator in metric (2) must enable one to evaluate the vacuum expectation values of the energy-momentum operator either for a free spinor field or for a free twisted spinor field [2, 3].

Dowker [4] for a massless spinor field and recently Bezerra and Bezerra de Mello [5] in three dimensions have written this spinor Green’s function
as contour integrals in the complex plan. In the present work, we give an integral expression of $S^{(n)}(x, x_0; m)$ and $S^{(n)(T)}(x, x_0; m)$. We give also a convenient form in which each spinor Green’s function is a sum of the usual spinor Green’s function and a regular term, assuming that point $x$ is near $x_0$. Then the coincidence limit $x = x_0$ of this term and its derivatives may enable us to calculate straightforwardly the vacuum energy-momentum tensor.

It should be emphasized that our method of determining $S^{(n)}(x, x_0; m)$ and $S^{(n)(T)}(x, x_0; m)$ will be based on the fact that they can be derived from the scalar Green’s function $G^{(n)}(x, x_0; m)$ satisfying the following condition of periodicity

$$G^{(n)}(\rho, \varphi + 2\pi, x^i; m) = \exp(2\pi i\gamma)G^{(n)}(\rho, \varphi, x^i; m)$$

(3)

for a certain constant $\gamma$ ($0 \leq \gamma < 1$). Such Green’s functions have been previously derived by Guimarães and Linet [6].

The plan of the present work is as follows. In section 2, we recall some properties about the spinor fields. We determine explicitly the spinor Green’s function in section 3 and the twisted one in section 4. For massless spinor fields in the spacetime of a straight cosmic string, the vacuum energy density and the twisted one are calculated in section 5. The case of a massive twisted spinor field in the Minkowski spacetime is treated in section 6. In section 7, we add some concluding remarks.

### II Preliminaries

We introduce the vierbein $e^\mu_a$ and the Riemannian connection $\omega_{\mu a b c} = e^\nu_b \nabla_\mu e_{c\nu}$ where $\nabla_\mu$ is the covariante derivative ($\mu = 1, \ldots, n$ and $a = 1, \ldots, n$).
We also introduce the elements $\gamma^a$ which satisfy the Clifford algebra

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2I \delta^{ab}$$  \hspace{1cm} (4)$$

where $I$ is the unit element. In the dimensions $n = 2p$ and $n = 2p + 1$, the $\gamma^a$ are represented by complex matrices $2^p \times 2^p$. In the present paper, we choose these matrices so that $\gamma^a \dagger = -\gamma^a$ where $\dagger$ means Hermitian conjugate. The gauge-covariant derivative $D_\mu$ of a spinor field is equal to $\partial_\mu + \Gamma_\mu$ where

$$\Gamma_\mu = \frac{1}{2} \omega_{\mu ab} \sigma^{ab}$$  \hspace{1cm} (5)$$

with $\sigma^{ab} = (\gamma^a \gamma^b - \gamma^b \gamma^a)/4$.

In metric (1), we can choose the vierbein having the components

$$e^\mu_a = (0, \frac{1}{B \rho^2}, 0, \cdots, 0) \quad e^\mu_a = \delta^\mu_a \quad a \neq 2$$  \hspace{1cm} (6)$$

and then spinor connection (3) has the components

$$\Gamma_2 = -\frac{B}{4} (\gamma^2 \gamma^2 - \gamma^2 \gamma^2) \quad \Gamma_i = 0 \quad i \neq 2$$  \hspace{1cm} (7)$$

However, choice (6) of the vierbein yields some subtleties and for instance De Sousa Gerbert and Deser (7) use another choice. The coordinates $(\rho, \varphi)$ in metric (1) can be related to Cartesian-like coordinates $(x^1, x^2)$ as explained in appendix A. In the associated vierbein to coordinates $(x^1, x^2)$, the spinorial components of a spinor field are well defined. In coordinates $(\rho, \varphi)$, we must identify the hypersurfaces $\varphi = 0$ and $\varphi = 2\pi$ in the space described by metric (1) but the spinorial components of this spinor field, expressed now in vierbein (6), cannot identify. We have to take into account the transformation law of the spinorial components resulting from the change of vierbein. When we use vierbein (6), we have to demand that the spinorial components $\Phi$ satisfy

$$\Phi(\rho, \varphi = 2\pi, x^i) = -\Phi(\rho, \varphi = 0, x^i)$$  \hspace{1cm} (8)$$
as well as its successive derivatives with respect to $\varphi$.

We now write the Dirac operator for a spinor field of mass $m$ in metric (1) with choice (6) of vierbein. The spinor Green’s function $S^{(n)}(x, x_0; m)$ must obey the Dirac equation

$$
(e_\mu^a \gamma_a \partial_\mu + \frac{\gamma_1}{2\rho} + mI)S^{(n)} = -I \frac{1}{B \rho} \delta(\rho - \rho_0) \delta(\varphi - \varphi_0) \delta^{(n-2)}(x^i - x_0^i)
$$

and, according to (8), it must satisfy the requirement

$$
S^{(n)}(\rho, \varphi + 2\pi, x^i; m) = -S^{(n)}(\rho, \varphi, x^i; m)
$$

as well as its successive derivatives with respect to $\varphi$. Moreover, we must impose that $S^{(n)}(x, x_0; m)$ vanishes when the points $x$ and $x_0$ are infinitely separated.

III Determination of $S^{(n)}(x, x_0; m)$

We seek $S^{(n)}(x, x_0; m)$ in the following form

$$
S^{(n)}(x, x_0; m) = (e_\mu^a \gamma_a \partial_\mu + \frac{\gamma_1}{2\rho} - mI)G^{(n)}(x, x_0; m)
$$

where $G^{(n)}(x, x_0; m)$ obey the square of the Dirac operator but this equation does not coincide with the Laplacian in metric (1). However in the present situation, we can put

$$
G^{(n)}(x, x_0; m) = I \Re H^{(n)}(x, x_0; m) + \frac{\gamma_1}{2\rho} \Im H^{(n)}(x, x_0; m)
$$

with

$$
H^{(n)}(x, x_0; m) = \exp(iB \varphi - \varphi_0)G^{(n)}_\gamma(x, x_0; m)
$$

where $\Re$ and $\Im$ denote respectively the real and the imaginary part. Then we verify that $G^{(n)}_\gamma(x, x_0; m)$ satisfies the Laplacian in metric (1)

$$
\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{B^2 \rho^2} \frac{\partial^2}{\partial \varphi^2} + \cdots + \frac{\partial^2}{\partial (x^n)^2} - m^2\right)G^{(n)}_\gamma = 0
$$
\[- \frac{1}{B\rho} \delta(\rho - \rho_0) \delta(\varphi - \varphi_0) \delta^{(n-2)}(x^i - x^i_0) \]  

(13)

The sign of \(m\) is not fixed but really we put \(|m|^2\) in (13).

With a function \(G_\gamma^{(n)}(x, x_0; m)\) obeying (13), we obtain a solution to equation (9) but we must choose this function so that \(S^{(n)}(x, x_0; m)\) verifies requirement (10). To obtain this, we see that \(G_\gamma^{(n)}(x, x_0; m)\) must satisfy

\[G_\gamma^{(n)}(\rho, \varphi + 2\pi, x^i; m) = \exp(2\pi i \gamma) G_\gamma^{(n)}(\rho, \varphi, x^i; m)\]  

(14)

as well as its derivatives with respect to \(\varphi\), where the constant \(\gamma\) is the fractional part of the number \(1/2 - B/2\) (\(0 \leq \gamma < 1\)). Such scalar Green’s functions have been previously derived by Guimarães and Linet [6]. In the usual case where metric (1) describes a cone \((B \leq 1)\), we have the following value of \(\gamma\)

\[\gamma = \frac{1}{2} - \frac{B}{2} \quad \text{with} \quad 0 \leq \gamma < \frac{1}{2}\]  

(15)

We had firstly derived in [6] an integral expression of \(G_\gamma^{(n)}(x, x_0; m)\). As a consequence, we might obtain an integral expression of \(S^{(n)}(x, x_0; m)\) by formulas (11) and (12) that we will not reproduce here. In four dimension, when the mass \(m\) vanishes, we have \(D_\gamma^{(4)}(x, x_0)\) in closed form

\[D_\gamma^{(4)} = \frac{e^{i(\varphi - \varphi_0)\gamma} \sinh[\xi_4(1 - \gamma)/B] + e^{-i(\varphi - \varphi_0)(1 - \gamma)} \sinh(\xi_4\gamma/B)}{8\pi^2 B \rho \rho_0 \sinh(\xi_4) \sinh(\xi_4/B) - \cos(\varphi - \varphi_0)]}\]  

(16)

where \(\xi_4\) is defined by

\[\cosh \xi_4 = \frac{\rho^2 + \rho_0^2 + (x^3 - x_0^3)^2 + (x^4 - x_0^4)^2}{2\rho \rho_0}\]

Consequently, the Euclidean spinor Green’s function \(S_E(x, x_0)\) for the Dirac operator without mass can be obtained in closed form by applying formula
With the restriction $B \leq 1$, (15) holds and we get

$$S_E(x, x_0) = (e^{\mu} \gamma \partial_\mu + \frac{\gamma_0}{2\rho}) \{ I \cos \varphi - \varphi_0 \sinh[\xi_4 (1/B + 1)/2] + \sinh[\xi_4 (1/B - 1)/2] \}
$$

$$\cdot 2 \cdot \frac{8\pi^2 B \rho_0 \sinh \xi_4 [\cosh \xi_4 / B - \cos(\varphi - \varphi_0)]}{8 \pi^2 B \rho_0 \sinh \xi_4 [\cosh \xi_4 / B - \cos(\varphi - \varphi_0)]}$$

We had secondly derived in [6] a local form of $G^{(n)}_\gamma(x, x_0; m)$ in which it is a sum of the usual Euclidean scalar Green’s function and a regular term $G^{* (n)}_\gamma(x, x_0; m)$ valid when the points $x$ and $x_0$ belong to the subset of the space defined by

$$\pi B - 2\pi < \varphi - \varphi_0 < 2\pi - \pi B$$

in the case $B > 1/2$ in which we restrict ourselves. We have

$$G^{(n)}_\gamma = \frac{m^{n/2-1}}{(2\pi)^{n/2} r_n^{n/2-1}} K_{n/2-1}(m r_n) + G^{* (n)}_\gamma(x, x_0; m)$$

(19)

with $r_n = \sqrt{\rho^2 + \rho_0^2 + 2\rho \rho_0 \cos B(\varphi - \varphi_0) + \cdots + (x^n - x_0^n)^2}$, the $K_\mu$ being the modified Bessel function, and the regular term $G^{* (n)}_\gamma(x, x_0; m)$ is given by

$$G^{* (n)}_\gamma = \frac{m^{n/2-1}}{(2\pi)^{n/2+1} B} \int_0^\infty K_{n/2-1}[m R_n(u)] [R_n(u)]^{n/2-1} F^{(\gamma)}_B(u, \varphi - \varphi_0) du$$

(20)

with $R_n(u) = \sqrt{\rho^2 + \rho_0^2 + 2\rho \rho_0 \cosh u + \cdots + (x^n - x_0^n)^2}$ and the function $F^{(\gamma)}_B(u, \psi)$ has the expression

$$F^{(\gamma)}_B(u, \psi) = \frac{e^{i(\psi + \pi/B)\gamma} \cosh[\frac{u(1-\gamma)}{B}] - e^{-i(\psi + \pi/B)(1-\gamma)} \cosh \frac{u\gamma}{B}}{\cosh \frac{\psi}{B} - \cos(\psi + \frac{\pi}{B})}
$$

$$- \frac{e^{i(\psi - \pi/B)\gamma} \cosh[\frac{u(1-\gamma)}{B}] - e^{-i(\psi - \pi/B)(1-\gamma)} \cosh \frac{u\gamma}{B}}{\cosh \frac{\psi}{B} - \cos(\psi - \frac{\pi}{B})}$$

(21)
From formulas (11) and (12), we can determine the spinor Green’s function

\[ S^{(n)}(x, x_0; m) \]

under a local form valid in the subset (18) of the space. We obtain

\[
S^{(n)} = \left( e^{\mu} a \gamma^a \partial_{\mu} + \frac{\gamma^1}{2\rho} - mI \right)
\]

\[
\left\{ \left( I \cos B \frac{\varphi - \varphi_0}{2} + \gamma^1 \gamma_2 \sin B \frac{\varphi - \varphi_0}{2} \right) K_{n/2-1}(mr_n) \right\}
\]

\[ + S^{*^{(n)}} \]

(22)

where the regular term \( S^{*^{(n)}}(x, x_0; m) \) can be expressed by formula (11)

\[
S^{*^{(n)}}(x, x_0; m) = \left( e^{\mu} a \gamma^a \partial_{\mu} + \frac{\gamma^1}{2\rho} - mI \right)
\]

\[
\left\{ I \Re H^{*^{(n)}}(x, x_0; m) + \gamma^1 \gamma_2 \Im H^{*^{(n)}}(x, x_0; m) \right\} \]

(23)

The regular term \( H^{*^{(n)}}(x, x_0; m) \) appearing in (23) is given by

\[
H^{*^{(n)}} = \frac{m^{n/2-1}}{(2\pi)^{n/2+1}B} \int_0^\infty \frac{K_{n/2-1}[mR_n(u)]}{[R_n(u)]^{n/2-1}} H_B^{(\gamma)}(u, \varphi - \varphi_0) du \]

(24)

where the function \( H_B^{(\gamma)}(u, \psi) \) has the expression

\[
H_B^{(\gamma)}(u, \psi) = \exp(iB \frac{\psi}{2}) F_B^{(\gamma)}(u, \psi) \]

(25)

Local form (22) is valid for \( B > 1/2 \).

Assuming moreover \( B \leq 1 \), we have \( \gamma \) given by (13). We specialize expression (25) for this value of \( \gamma \); we find the real part

\[
\Re H_B^{\left( \frac{1}{2} - \frac{\psi}{2} \right)}(u, \psi) = \cos(\psi \frac{\pi}{2} + B) \cosh(\frac{u}{2} (\frac{1}{B} + 1)) - \cosh(\frac{u}{2} (\frac{1}{B} - 1)) \cosh(\frac{u}{2} - \cos(\psi - \frac{\pi}{B}))
\]

(26)
and the imaginary part

\[ \Im H_B^{(\frac{1}{2} - \frac{B}{2})}(u, \psi) = \sin\left(\frac{\psi}{2} + \frac{\pi}{2B}\right) \cosh\left[\frac{\psi}{2}(\frac{1}{B} + 1)\right] \cosh\left[\frac{\psi}{2}(\frac{1}{B} - 1)\right] \]

\[ + \sin\left(\frac{\psi}{2} - \frac{\pi}{2B}\right) \cosh\left[\frac{\psi}{2}(\frac{1}{B} + 1)\right] \cosh\left[\frac{\psi}{2}(\frac{1}{B} - 1)\right] \cosh\left[\frac{\psi}{2} - \cos(\psi + \frac{\pi}{B})\right] \]  

(27)

By setting \( n = 4 \) in formula (22), we obtain the expression of the Euclidean spinor Green’s function \( S_E(x, x_0; m) \) for point \( x \) near \( x_0 \) in the case \( B > 1/2 \). Since it will be needed in a next section, we write down it when the mass \( m \) vanishes; we have

\[ S_E = (e^{\frac{\mu}{2}}\gamma^a\partial_\mu + \gamma^1) \]

\[ \{ (I \cos B\varphi - \varphi_0 + \gamma^1\gamma^2\sin B(\varphi - \varphi_0)) \frac{1}{4\pi^2r_4^4} \]

\[ + I\Re H^{(4)} + \gamma^1\gamma^2\Im H^{(4)} \} \]  

(28)

where the regular term \( H^{(4)}(x, x_0; m) \) has now the integral expression

\[ H^{(4)} = \frac{1}{8\pi^3B} \int_0^\infty \frac{1}{[R_4(u)]^2} H_B^{(\gamma)}(u, \varphi - \varphi_0)du \]  

(29)

**IV Determination of \( S^{(n)}(T)(x, x_0; m) \)**

Isham [8] has shown that twisted fields can be defined in a spacetime which is not simply connected. In space described by metric (1), the axis \( \rho = 0 \) can be removed from the space. So, the twisted fields are obtained by requiring that they are antiperiodic about this axis. Twisted scalar fields have already considered in the spacetime of a straight cosmic string [12] and in the particular case of the Minkowski spacetime [13, 14].
In metric (1), the property of antiperiodicity for a twisted spinor field is required in the Cartesian-like coordinates \((x^1, x^2)\) with the associated vierbein. Now we work with choice (3) of vierbein, therefore we demand that the spinorial components of the twisted spinor field \(\Phi\) satisfy

\[
\Phi(\rho, \varphi = 2\pi, x^i; m) = \Phi(\rho, \varphi = 0, x^i; m)
\]  

(30)

instead of requirement (8) for a spinor field. The twisted spinor Green’s function \(S^{(n)(T)}(x, x_0; m)\) for the Dirac operator must satisfy the condition

\[
S^{(n)(T)}(\rho, \varphi + 2\pi, x^i; m) = S^{(n)(T)}(\rho, \varphi, x^i; m)
\]

(31)
as well as its successive derivatives with respect to \(\varphi\). It is obvious that \(S^{(n)(T)}(x, x_0; m)\) can be determined by formulas (11) and (12) following the method developed in section 3 but the scalar Green’s function \(G^{(n)}_{\gamma}(x, x_0; m)\) is now characterized by a constant \(\gamma^{(T)}\) which is the fractional part of the number \(1 - B/2\). In the case where \(B \leq 1\), we have the following value of \(\gamma^{(T)}\)

\[
\gamma^{(T)} = 1 - \frac{B}{2} \quad \text{with} \quad \frac{1}{2} \leq \gamma^{(T)} < 1
\]

(32)

In metric (2) of the spacetime of a straight cosmic string, the quantum field theory for a twisted spinor field can be done from the Euclidean twisted spinor Green’s function \(S_{E}^{(T)}(x, x_0; m)\) obtained by setting \(n = 4\). By this way, we determine an explicit expression of the Euclidean twisted spinor Green’s function in the massless case. In the massive case, we can obtain \(S_{E}^{(T)}(x, x_0; m)\) in a local form valid in subset (18) in which we take \(\gamma^{(T)}\) instead of \(\gamma\) in formula (24).
V  Vacuum energy density (massless spinor)

Within the Euclidean quantum field theory of a spinor field in the space-time of a straight cosmic string, the fundamental quantity in the free case is the Euclidean spinor Green’s function $S_E(x, x_0; m)$ which coincides with $S^{(4)}(x, x_0; m)$. To renormalize, it is convenient to use local form (22) of the spinor Green’s function valid for $B > 1/2$. The expectation values of the energy-momentum operator is performed by removing the usual spinor Euclidean Green’s function in expression (22). Since metric (4) is locally flat we obtain merely the vacuum energy-momentum tensor in the Euclidean approach by the usual formula [9]

$$< T_{\mu\nu} >= \frac{1}{4} tr [\gamma^\mu (e_{\mu\nu}(\partial_{\nu} - \partial_{\nu_0}) + e_{\nu\mu}(\partial_{\mu} - \partial_{\mu_0}))S^{(4)}(x, x_0; m)] |_{x=x_0} \tag{33}$$

where $\partial_{\nu_0}$ denotes the derivative with respect to $x_0$. We point out that only the antisymmetrical part of $S^{(4)}(x, x_0; m)$ with respect to $x$ and $x_0$ occurs in formula (33). The vacuum energy-momentum tensor is conserved.

The vacuum energy density $< T_{tt} >$ is given by $- < T_{\tau\tau} >$ since $t = -i\tau$ in a Wick rotation. From expression (33), we obtain the general form

$$< T_{tt}(x) >= -4\partial_{\tau}\Re H^{(4)}(x, x_0; m) |_{x=x_0} \tag{34}$$

where we have used the fact that expression (24) of $H^{(4)}(x, x_0; m)$ depends on $\tau$ and $\tau_0$ by $(\tau - \tau_0)^2$.

We now confine ourselves to massless spinors. According to expression (24) of $H^{(4)}(x, x_0)$, we obtain immediately

$$< T_{tt}(x) >= \frac{1}{\pi^2 B} \int_{0}^{\infty} \frac{1}{[R_4(u)]^2} H_B^{(\gamma)}(u, 0) du \tag{35}$$
We rewrite (35) in the form

\[ \langle T_{tt} (x) \rangle = \frac{2}{\rho^2} w_4 (\gamma) \]  

(36)

where \( w_4 (\gamma) \) is the following integral

\[ w_4 (\gamma) = -\frac{1}{4\pi^3 B} \int_0^\infty \frac{\sin \frac{\pi u}{B} \cosh \left[ \frac{\pi (1-\gamma)}{B} \right]}{(1 + \cosh u)^2 (\cosh \frac{u}{B} - \cos \frac{\pi}{B})} \]  

(37)

Integrals of type (37) have been studied by Dowker [4, 10]. For \( B > 1/2 \), we have the explicit expression

\[ w_4 (\gamma) = -\frac{1}{720\pi^2} \left\{ 11 - \frac{15}{B^2} \left[ 4 (\gamma - \frac{1}{2})^2 - \frac{1}{3} \right] + \frac{15}{8B^4} \left[ 16 (\gamma - \frac{1}{2})^4 - 8 (\gamma - \frac{1}{2})^2 + \frac{7}{15} \right] \right\} \]  

(38)

So, for \( B > 1/2 \), we have explicitly determined in general the vacuum energy density which is given by by formula (38). Its value depends on the choice of \( \gamma \). With restriction \( B \leq 1 \), \( \gamma \) is given by (13) and then we find the result of Frolov and Serebriany [11]

\[ \langle T_{tt} (x) \rangle = -\frac{1}{2880\pi^2 \rho^4} \left[ \frac{1}{B^2} - 1 \right \left( \frac{7}{B^2} + 17 \right] \]  

(39)

It vanishes for \( B = 1 \).

We now turn to the twisted case where \( \gamma \) is given by (32) when \( B \leq 1 \). From formula (38) we find the twisted vacuum energy density

\[ \langle T_{tt} (x) \rangle^{(T)} = -\frac{1}{360\pi^2 \rho^4} \left[ -\frac{17}{8} + \frac{45}{2B} - \frac{5}{2B^2} - \frac{1}{B^4} \right] \]  

(40)

As noticed by DeWitt et al [15], the twisted vacuum has the lower vacuum energy density in the case of spinor fields. Actually, we see that

\[ \langle T_{tt} (x) \rangle^{(T)} < \langle T_{tt} (x) \rangle \quad \text{for} \quad \frac{1}{2} < B \leq 1 \]  

(41)

in our situation. They coincide in the limit where the constant \( B = 1/2 \).

More generally for a massive spinor field, our method should be enabled us to express the vacuum energy density under an explicit integral form.
VI Massive twisted spinor field in Minkowski spacetime

In the Minkowski spacetime, characterized by $B = 1$, a twisted spinor field can be defined. The massless case has been already considered by Ford [13]. The method of the present paper can be applied to evaluate the twisted vacuum energy density. Since $\gamma^{(T)} = 1/2$ in this case, the twisted spinor Green’s function $S_E^{(T)}(x, x_0; m)$ is derived from the twisted scalar Green’s function $G_{1/2}^{(4)}(x, x_0; m)$.

For $B > 1/2$, we write down $S_E^{(T)}(x, x_0; m)$ in a local form valid in subset (18) of the Euclidean space

$$S_E^{(T)} = (e^\mu \gamma^a \partial_\mu + \gamma^{1/2} - mI)$$

$$\{\cos \frac{\varphi - \varphi_0}{2} + \gamma^{1/2} \sin \frac{\varphi - \varphi_0}{2} \frac{m}{4\pi^2 r_4} K_1(m r_4)$$

$$I\Re H^{(4)} + \gamma^{1/2} \Im H^{(4)}\}$$

where the regular term $H^{(4)}(x, x_0; m)$ has now the integral expression

$$H^{(4)} = \frac{m}{8\pi^3} \int_0^\infty \frac{K_1[m R_4(u)]}{R_4(u)} H_1^{(1/2)}(u, \varphi - \varphi_0) du$$

in which $H_1^{(1/2)}(u, \psi)$ is obtained from (23) under the form

$$H_1^{(1/2)}(u, \psi) = -2(1 + \cos \psi + i \sin \psi) \frac{\cosh u/2}{\cosh u + \cos \psi}$$

By using formula (34), we obtain the twisted vacuum energy density as a definite integral

$$< T_{tt}(x) >^{(T)} = -\frac{m^2}{2\pi^3 \rho^2} \int_0^\infty \frac{K_0(2m \rho \cosh v)}{(\cosh v)^3} dv$$

$$-\frac{m}{2\pi^3 \rho^3} \int_0^\infty \frac{K_1(2m \rho \cosh v)}{(\cosh v)^4} dv$$

(45)
In the appendix B, we give some definite integrals of this type which are useful for our problem. We can thereby obtain the expression of the twisted vacuum energy density in closed form

\[
<T_{tt}(x)>^{(T)} = -\frac{m^2}{32\pi^2\rho^2}[4m^2\rho^2E_1(2m\rho) \\
+(1-2m\rho + \frac{3}{m\rho} + \frac{3}{2m^2\rho^2})\exp(-2m\rho)]
\]

(46)

where \(E_1\) is the exponential integral function. The twisted vacuum energy density is always negative and its range is \(1/2m\).

In the limit where \(m\rho \ll 1\), expression (46) becomes asymptotically

\[
<T_{tt}(x)>^{(T)} \sim -\frac{3}{64\pi^2\rho^4}
\]

(47)

The twisted vacuum energy density (47) coincides with expression (40), directly established for a massless twisted spinor field, in which we set \(B = 1\).

VII Conclusion

We have explicitly determined the spinor Green’s function and the twisted spinor Green’s function in an Euclidean space with a conical-type line singularity. It should be emphasized that their local expression when point \(x\) is near \(x_0\) is always convenient to evaluate straightforwardly the vacuum energy-momentum tensor within the framework of quantum field theory of free spinor fields. In the massless case, we have calculated the vacuum energy density and the twisted vacuum energy density for an arbitrary constant \(B\) (\(B > 1/2\)) in the four dimensional spacetime. In the massive case, we might obtain it under the form of definite integrals. For a massive twisted spinor field in the Minkowski spacetime, we have performed the integration in terms of elementary functions.
Our method can also give the spinor Green’s functions, in the vierbein used in this work, which satisfy the condition

\[ S_{\Phi/\Phi_0}^{(n)}(\rho, \varphi + 2\pi, x^i) = -\exp(2\pi i \Phi/\Phi_0)S_{\Phi/\Phi_0}^{(n)}(\rho, \varphi, x^i) \]

for constants \( \Phi/\Phi_0 \). We must simply introduce in our formulas the scalar Green’s function \( G^{(n)}(x, x_0; m) \) where \( \gamma \) is the fractional part of the number \( 1/2 - B/2 - \Phi/\Phi_0 \). In contrast to the case of a scalar charged field, the quantum field theory of a charged spinor field in a magnetic flux \( \Phi, \Phi_0 \) being the quantum flux, cannot describe by the \( S_{\Phi/\Phi_0}^{(d)}(x, x_0; m) \) determined in this manner. Indeed, the axis \( \rho = 0 \) is excluded in this kind of method \cite{16}. The boundary conditions on the axis \( \rho = 0 \) must be carefully examined in this situation \cite{17, 18}.

A Appendix: transformation law of the spinorial components

By making the coordinate transformation \( \rho = \rho^1/B \), metric (1) takes the form

\[ ds^2 = B^2\rho^{2B-2}(d\rho^2 + \rho^2 d\varphi^2) + (dx^3)^2 + \cdots (dx^n)^2 \]

In the coordinates \( (\rho, \varphi, x^i) \), vierbein (3) has now the components

\[ e^\mu_1 = \left( \frac{1}{B^{B-1}}, 0, 0, \ldots, 0 \right) \]

\[ e^\mu_2 = \left( 0, \frac{1}{B^{B-1}}, 0, \ldots, 0 \right) \]

\[ e^\mu_3 = \delta^\mu_3, \quad 3, \ldots, n \]
We now introduce the Cartesian coordinates \((x^1, x^2)\) related to \((\rho, \varphi)\). The vierbein \(e^\mu_a\) associated to the Cartesian coordinates \((x^\mu)\) have the components

\[
e^\mu_1 = \left(\frac{1}{B\rho^{B-1}}, 0, 0, \ldots, 0\right)
\]

\[
e^\mu_2 = \left(0, \frac{1}{B\rho^{B-1}}, 0, \ldots, 0\right)
\]

\[
e^\mu_a = \delta^\mu_a \quad a = 3, \ldots, n
\]

Let a spinor well defined in the coordinates \((x^\mu)\); its spinorial components \(\Phi\), defined relatively to the associated vierbein \(e^\mu_a\), are regular. But in vierbein \(e^\mu_a\), this spinor has the following spinorial components \(\Phi\)

\[
\Phi(\rho, \varphi, x^i) = (I \cos \frac{\varphi}{2} + \gamma^1 \gamma^2 \sin \frac{\varphi}{2} ) \Phi(\rho, \varphi, x^i)
\]

that we have expressed in coordinates \((\rho, \varphi, x^i)\). We can immediately return to usual coordinates \((\rho, \varphi, x^i)\) and we see that we obtain requirement (8) for the spinorial components \(\Phi\).

The spinor Green’s function is a bi-spinor whose the spinorial components \(S^{(n)}(x, x_0; m)\) defined relatively to vierbein \(e^\mu_a\) are well defined. The spinorial components in vierbein \(e^\mu_a\) are obtained by a transformation law which can be recast in the form

\[
S^{(n)}(\rho, \varphi, x^i) = (I \cos \frac{\varphi - \varphi_0}{2} + \gamma^1 \gamma^2 \sin \frac{\varphi - \varphi_0}{2} ) S^{(n)}(\rho, \varphi, x^i)
\]

It yields also condition (10).

**B Appendix: a list of some definite integrals**

To our knowledge, these following definite integrals are not in the mathematical tables under this form. The two first integrals have been derived
in the context of a twisted scalar field in the spacetime of a straight cosmic string [19]. For a positive real number \( b \), we have the formulas

\[
\int_0^\infty \frac{K_0(b \cosh v)}{\cosh v} dv = \frac{\pi}{2} E_1(b)
\]

\[
\int_0^\infty \frac{K_1(b \cosh v)}{(\cosh v)^2} dv = \frac{\pi}{4} \left[ -b E_1(b) + \left( \frac{1}{b} + 1 \right) \exp(-b) \right]
\]

in which \( K_\mu \) denotes the modified Bessel function and \( E_1 \) the exponential integral function.

The two other integrals can be obtained by performing some integrations by parts and by using the above mentioned integrals. Furthermore, we recall the identities

\[
K_0'(x) = -K_1(x) \quad \text{and} \quad K_1'(x) = -K_0 - \frac{1}{x} K_1(x)
\]

We give the final results without proof

\[
\int_0^\infty \frac{K_0(b \cosh v)}{(\cosh v)^3} dv = \frac{\pi}{8} \left[ (2 + b^2) E_1(b) + (1 - b) \exp(-b) \right]
\]

\[
\int_0^\infty \frac{K_1(b \cosh v)}{(\cosh v)^4} dv = \frac{\pi}{32} \left[ (-4b - b^3) E_1(b) + \left( \frac{6}{b} + 6 - b + b^2 \right) \exp(-b) \right]
\]

We have checked these definite integrals by a numerical analysis.
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