Autocorrelation functions for quantum particles in supersymmetric Pöschl-Teller potentials

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Abstract. We consider autocorrelation functions for supersymmetric quantum mechanical systems (consisting of a fermion and a boson) confined in trigonometric Pöschl–Teller partner potentials. We study the limit of rescaled autocorrelation functions (at random time) as the localization of the initial state goes to infinity. The limiting distribution can be described using pairs of Jacobi theta functions on a suitably defined homogeneous space, as a corollary of the work of Cellarosi and Marklof. A construction by Contreras-Astorga and Fernández provides large classes of Pöschl-Teller partner potentials to which our analysis applies.

1 Introduction

1.1 Supersymmetric Quantum Mechanics

Supersymmetric (SUSY) quantum mechanics is the study of a pair of Hamiltonians (in units where \( \hbar = \text{mass} = 1 \))

\[
H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + V_0(x), \quad H_1 = -\frac{1}{2} \frac{d^2}{dx^2} + V_1(x)
\]

that are intertwined by a differential operator \( A \) and its adjoint \( A^\dagger \) as

\[
H_0 A = A H_1, \quad H_1 A^\dagger = A^\dagger H_0.
\]

The term “supersymmetric” is motivated as follows. If we define the operator matrices

\[
Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad H_{ss} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix},
\]

\[
Q_1 = \frac{Q^\dagger + Q}{\sqrt{2}}, \quad Q_2 = \frac{Q^\dagger - Q}{i\sqrt{2}}.
\]
then we have the commutator and anticommutator relations of the supersymmetry algebra with \( N = 2 \) generators

\[
\begin{align*}
\{Q_1, H_{ss}\} &= 0, \\
\{Q_2, H_{ss}\} &= 0, \\
\{Q_1, Q_1\} &= Q_2, \\
\{Q_1, Q_2\} &= 0,
\end{align*}
\]

where \( [X, Y] = XY - YX \) and \( \{X, Y\} = XY + YX \) denote the commutator and the anticommutator of \( X \) and \( Y \), respectively. The algebra (1.5) corresponds to the simplest supersymmetric quantum system (see [9], as well as [2, §5]). The operators \( Q, Q^\dagger \) are called supercharges or supersymmetry generators, and the Hamiltonians \( H_0, H_1 \) (as well as the potentials \( V_0, V_1 \)) are called supersymmetric partners.

Given a general 1-dimensional Hamiltonian \( H_0 \) whose eigenfunctions and eigenvalues are known, there is an intertwining method due to Sukumar (see [5] for an historical account, tracing back to the work of Dirac) to construct various partners \( H_1 \) using various differential operators \( A, A^\dagger \). The advantage of the intertwining relations (1.2) is to generate eigenvalues and eigenfunctions of \( H_1 \) from those of \( H_0 \). In general, if \( A \) is a differential operator of order \( k \), then the spectra of \( H_0 \) and \( H_1 \) differ by at most \( k \) values. Moreover, if we denote by

\[
H = \begin{pmatrix} H_1 & 0 \\ 0 & H_0 \end{pmatrix}
\]

the physical Hamiltonian, then the supersymmetric Hamiltonian \( H_{ss} \) in (1.3) and (1.5) can be expressed as a polynomial of degree \( k \) in \( H \). It is therefore enough to study the time-independent Schrödinger equation

\[
H\psi = E\psi,
\]

where \( \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_0(x) \end{pmatrix} \). From the physical point of view, SUSY predicts that each particle of the Standard Model has a partner particle with a spin that differs by a half unit. Therefore, we can think of \( H \) as describing the joint action of \( H_0 \) on fermions (particles with half-integer spin) and of \( H_1 \) on bosons (particles with integer spin).

### 1.2 Autocorrelation Functions

The time-dependent Schrödinger equation corresponding to (1.7) is

\[
i \frac{\partial}{\partial t} \Psi = H\Psi,
\]

where \( \Psi(x, t) = \begin{pmatrix} \Psi_1(x, t) \\ \Psi_0(x, t) \end{pmatrix} \). We consider the 1-dimensional case in which \( x \in I \), where \( I \subseteq \mathbb{R} \) denotes an interval. Moreover, we restrict our analysis to the class of physical Hamiltonians \( H \) acting on \( L^2(I) \oplus L^2(I) \) with purely discrete spectrum. This means that there exists an orthonormal basis of \( L^2(I) \oplus L^2(I) \) consisting of eigenfunctions of \( H \), say \( \psi_n(x) = \begin{pmatrix} \psi_{1,n}(x) \\ \psi_{0,n}(x) \end{pmatrix} \) with eigenvalues \( E_n, n \geq 0 \). In this case, the solution of (1.8) with initial condition \( \Psi(x, 0) \) is given in terms of the evolution
The operator $U(t) = e^{-iHt}$ satisfies

$$\Psi(x, t) = U(t)\Psi(x, 0) = \sum_{n=0}^{\infty} c_n e^{-iE_n t} \psi_n(x),$$

where

$$c_n = \left( \begin{array}{c} c_{1,n} \\ c_{0,n} \end{array} \right), \quad c_{\ell,n} = \int_I \psi_\ell(x,0) \overline{\psi_{\ell,n}(x)} \, dx, \quad \ell = 0, 1.$$ 

The autocorrelation function for the initial state $\Psi(x, 0)$ is the function $A : \mathbb{R} \to \mathbb{C}^2$,

$$A(t) = \int_I \Psi(x,0) \overline{\Psi(x,t)} \, dx = \left( \sum_{n \geq 0} |c_{1,n}|^2 e^{iE_n t} \right) / \left( \sum_{n \geq 0} |c_{0,n}|^2 e^{iE_n t} \right).$$

We further restrict our attention to the case of autocorrelation functions $A(t)$ as in (1.11) where the coefficients $c_{\ell,n}$ have a special form, namely,

$$|c_{\ell,n}|^2 = \frac{1}{N} f_\ell \left( \frac{n}{N} \right),$$

for some nonnegative functions $f_1, f_0$ supported on $\mathbb{R}_{\geq 0}$ and some $N \geq 1$. This is a natural choice, which appears in several interesting cases. Define

$$A_N(f_1, f_0; t) = \left( \frac{1}{N} \sum_{n \geq 0} f_1 \left( \frac{n}{N} \right) e^{iE_n t} \right) / \left( \frac{1}{N} \sum_{n \geq 0} f_0 \left( \frac{n}{N} \right) e^{iE_n t} \right).$$

We will consider the large $N$ regime, which physically represents the case of initial states $\Psi(x, 0)$ that are highly localized in space. We will focus on the autocorrelation functions for a class of Hamiltonians $H$ as in (1.6), where the spectra of $H_0$ and of $H_1$ differ by at most finitely many values. For random $t$, we will treat $A_N(f_1, f_0; t)$ as a random variable at the scale $N^{-1/2}$. In this case, as $N \to \infty$, the contribution of finitely many eigenvalues to $A_N(f_1, f_0, t)$ is $O(N^{-1})$ and hence negligible in our analysis. Therefore, without loss of generality, we can assume that $H_0$ and $H_1$ have the same spectrum, and thus, $\text{sp}(H) = \text{sp}(H_0) = \text{sp}(H_1)$.

### 1.3 Trigonometric Pöschl–Teller Potentials

Let us discuss the case when $H_0$ is the Hamiltonian corresponding to the so-called trigonometric Pöschl–Teller potential with parameters $(\alpha, \beta)$, i.e.,

$$V_0(x) = \frac{(\alpha - 1) \alpha}{2 \sin^2(x)} + \frac{(\beta - 1) \beta}{2 \cos^2(x)}, \quad \alpha, \beta > 1, \quad 0 \leq x \leq \frac{\pi}{2}.$$ 

The potential $V_0$ can be interpreted as an infinite well, confining the particle to the 1-dimensional box $[0, \pi]$, with “soft walls”, where the the parameters $\alpha$ and $\beta$ represent the strength of the reflection of the particle off the two walls. Denote $\gamma = \alpha + \beta$. The
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The potential $V_0(x)$ for $\alpha = \sqrt{2}$ and $\beta = 4$ and the probability densities $x \mapsto |\psi_{0,n}(x)|^2$ on $[0, \frac{\pi}{2}]$ for $0 \leq n \leq 4$.

The eigenvalues of $H_0$ on $L^2([0, \frac{\pi}{2}])$ are

$$E_n = \frac{1}{2} (2n + \gamma)^2, \quad n \geq 0.$$  

The corresponding normalized eigenfunctions are

$$\psi_{0,n}(x) = \sqrt{\frac{2(2n + \gamma)\Gamma(n + \gamma)(\alpha + \frac{1}{2})_n}{n! \Gamma(\alpha + \frac{1}{2})\Gamma(\beta + \frac{1}{2})(\beta + \frac{1}{2})_n}} \sin^\alpha(x) \cos^\beta(x)$$

$$\times \, _2F_1(-n, n + \gamma; \alpha + \frac{1}{2}; \sin^2(x)),$$

where $\, _2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}$ is the hypergeometric function, and $(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}$ is the Pochhammer symbol; see [3, 4].

Using various differential operators $A, A^\dagger$ as in (1.2), Contreras-Astorga and Fernández [3] were able to construct various families of partner potentials $V_1$ such that $H_0$ and $H_1$ are isospectral. We discuss two such families in Section 4.

1.4 The Main Theorem

Let $H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + V_0$ be the Hamiltonian with with Pöschl-Teller potential (1.14) with parameters $(\alpha, \beta)$. Let $\gamma = \alpha + \beta$. Let $H_1$ be any supersymmetric partner of $H_0$ such that $\text{sp}(H_1)$ and $\text{sp}(H_0)$ differ by at most finitely many eigenvalues. Fix two compactly supported, Riemann integrable functions $f_0, f_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Let $\lambda$ be a probability measure on $\mathbb{R}$, absolutely continuous with respect to the Lebesgue measure. Denote by $\rho$ the density of $\lambda$, i.e., $d\lambda(t) = \rho(t)dt$. Rescaled autocorrelation functions are viewed as random variables, i.e., as $C^2$-valued measurable functions of $t \in (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$. 
More precisely, consider the random variables

\[(1.17) \quad X_N : (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \rightarrow (\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2)), \quad X_N(t) = \sqrt{N}A_N(f_0, f_1; t), \quad N \geq 1.\]

Throughout the paper, \(\mathcal{B}(\cdot)\) denotes the Borel \(\sigma\)-algebra. Our main result is the following theorem.

**Theorem 1.1** Assume \(\gamma \notin \mathbb{Q}\). Then the random variables \(X_N\) have a limiting distribution as \(N \to \infty\). That is, there exists a (non-degenerate) random variable \(X\) on \(\mathbb{C}^2\) such that \(X_N\) converge in law to \(X\) as \(N \to \infty\). Moreover, the law of \(X\) (which does not depend on \(\rho\) nor \(\gamma\)) can be realized as the push forward onto \(\mathbb{C}^2\) of the Haar measure on a homogeneous space \(\Gamma \backslash G\) via an explicit function \(\Gamma \backslash G \rightarrow \mathbb{C}^2\).

**Remark 1.2** In this case, \(\Gamma < G\) is a lattice in the Lie group \(G\), and \(\Gamma \backslash G\) has finite volume. We describe \(\Gamma \backslash G\), its normalized Haar measure, and the function \(X\) on \(\Gamma \backslash G\) explicitly in Section 2. The proof of Theorem 1.1 is provided in Section 3.

In particular, we show that the law of \(X\) is not the product of two measures on \(\mathbb{C}\). This means that the two random variables

\[(1.18) \quad \frac{1}{\sqrt{N}} \sum_{n \geq 0} f_0 \left( \frac{n}{N} \right) e^{iE_n t} \quad \text{and} \quad \frac{1}{\sqrt{N}} \sum_{n \geq 0} f_1 \left( \frac{n}{N} \right) e^{iE_n t}\]

do not become independent as \(N \to \infty\). This can be interpreted physically as follows. The probability distribution of the autocorrelation function (at a random time) of an individual particle is altered if we condition on the event that the autocorrelation function of the partner particle lies in a certain set in \(\mathbb{C}\), at least for highly localized particles. This means, in particular, that if we observe—at a random time \(t\)—a localized quantum particle undergoing a quantum revival (i.e., we observe a large value of its correlation function), then the probability that the partner particle (which we may have no access to) is also undergoing a quantum revival at time \(t\) is not the same as the unconditional probability of the same event; see Remark 3.2.

As an illustration, in Figure 2, we plot the real and imaginary parts of both components of \(X_N(t)\) when \(f_1\) and \(f_0\) are indicator functions for different values of \(N\).

### 2 The Limiting Random Variable \(X\)

In this section, we describe explicitly the random variable \(X\) featured in Theorem 1.1. We refer the reader to [1, §2] for more details.

#### 2.1 The Universal Jacobi Group \(G\)

Let \(\mathcal{H} := \{w \in \mathbb{C} : \mathcal{I}(w) > 0\}\) denote the upper half plane. The group \(\text{SL}(2, \mathbb{R})\) acts on \(\mathcal{H}\) by Möbius transformations \(z \mapsto gz := \frac{az + b}{cz + d}\), where \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})\). Every
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Figure 2: The real and imaginary parts of the two components of the rescaled autocorrelation function \( t \mapsto X_N(t) = (X_N(f_1, t), X_N(f_0, t)) \) for \( f_1 = I_{[0,1]} \) (left panels) and \( f_0 = I_{[1/3,4/3]} \) (right panels) and \( N = 10 \) (top four panels) and \( N = 40 \) (bottom four panels). In all the panels, \( \alpha = \sqrt{2} \) and \( \beta = 3 \).

\( g \in \text{SL}(2, \mathbb{R}) \) can be written uniquely via Iwasawa decomposition as

\[
(2.1) \quad g = n_x a_y k_\phi,
\]

where

\[
(2.2) \quad n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a_y = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}, \quad k_\phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix},
\]

and \( z = x + iy \in \mathcal{H}, \phi \in [0, 2\pi) \). Set \( e(t) := e^{2\pi it} \) and \( \varepsilon_g(z) = (cz + d)/|cz + d| \). The universal cover of \( \text{SL}(2, \mathbb{R}) \) is defined as

\[
(2.3) \quad \tilde{\text{SL}}(2, \mathbb{R}) := \{ [g, \beta_g] : g \in \text{SL}(2, \mathbb{R}), \beta_g \text{ a continuous function on } \mathcal{H} \} 
\]

such that \( e(i\beta_g(z)) = \varepsilon_g(z) \).
and has the group structure given by
\begin{equation}
[g, \beta_g^2][h, \beta_h^3] = [gh, \beta_{gh}^3], \quad \beta_{gh}^3(z) = \beta_g^1(hz) + \beta_h^3(z),
\end{equation}

\begin{equation}
[g, \beta_g^{-1}] = [g^{-1}, \beta_g']^{-1}, \quad \beta_g'(z) = -\beta_g(g^{-1}z).
\end{equation}

The group $\tilde{\text{SL}}(2, \mathbb{R})$ is identified with $\mathfrak{h} \times \mathbb{R}$ via $[g, \beta_g] \mapsto (z, \phi) = (gi, \beta_g(i))$ and acts on $\mathfrak{h}$ by
\begin{equation}
[g, \beta_g](z, \phi) = (gz, \phi + \beta_g(z)).
\end{equation}

The Iwasawa decomposition (2.1) of $\text{SL}(2, \mathbb{R})$ extends to a decomposition of $\tilde{\text{SL}}(2, \mathbb{R})$: for every $\tilde{g} = [g, \beta_g] \in \tilde{\text{SL}}(2, \mathbb{R})$, we have
\begin{equation}
\tilde{g} = [g, \beta_g] = n_x a_y k_\phi = [n_x, 0][a_y, 0][k_\phi, \beta_{k_\phi}].
\end{equation}

Let $\omega$ be the standard symplectic form on $\mathbb{R}^2$, $\omega(\xi, \xi') = xy' - yx'$, where $\xi = \begin{pmatrix} x \\ y \end{pmatrix}$, $\xi' = \begin{pmatrix} x' \\ y' \end{pmatrix}$. The Heisenberg group $\mathbb{H}(\mathbb{R})$ is defined as $\mathbb{R}^2 \times \mathbb{R}$ with the multiplication law
\begin{equation}
(\xi, t)(\xi', t') = (\xi + \xi', t + t' + \frac{1}{2} \omega(\xi, \xi')).
\end{equation}

We consider universal Jacobi group
\begin{equation}
G = \tilde{\text{SL}}(2, \mathbb{R}) \ltimes \mathbb{H}(\mathbb{R}),
\end{equation}

having the multiplication law
\begin{equation}
([g, \beta_g]; \xi, \zeta)([g', \beta_{g'}]; \xi', \zeta') = ([gg', \beta_{gg'}]; \xi + g\xi', \zeta + \xi' + \frac{1}{2} \omega(\xi, g\xi')),
\end{equation}

where $\beta_{gg'}(z) = \beta_g(g'z) + \beta_{g'}(z)$. The Haar measure on $G$ is given in coordinates $(x + iy, \phi; \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \zeta)$ by
\begin{equation}
d\mu(g) = \frac{dx
dy
d\phi
d\xi_1
d\xi_2
d\zeta}{y^2}.
\end{equation}

### 2.2 Hermite Expansion

Let $H_k$ be the $k$-th Hermite polynomial
\begin{equation}
H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} e^{-t^2} = k! \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^m (2t)^{k-2m}}{m!(k-2m)!}.
\end{equation}

We define the Hermite functions
\begin{equation}
\psi_k(t) = (2\pi)^{\frac{1}{4}} h_k(\sqrt{2\pi}t) = (2^{k-\frac{1}{2}} k!)^{-1/2} H_k(\sqrt{2\pi} t) e^{-t^2},
\end{equation}
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which form an orthonormal basis for $L^2(\mathbb{R}, dx)$. Let $f \in L^2(\mathbb{R})$. Following [6], we set

\begin{equation}
(2.14) \quad f_\phi(t) = \sum_{k=0}^{\infty} \hat{f}(k) e^{-i(2k+1)\phi/2} \psi_k(t),
\end{equation}

where $\hat{f}(k) = \langle f, \psi_k \rangle_{L^2(\mathbb{R})}$. For an equivalent definition of $f_\phi$, see [1, §2.3]. The space of functions $f : \mathbb{R} \to \mathbb{R}$ for which $f_\phi$ has a prescribed decay at infinity, uniformly in $\phi$, is denoted by $S_\eta(\mathbb{R})$. More precisely,

\begin{equation}
(2.15) \quad S_\eta(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} : \sup_{t, \phi} |f_\phi(t)| (1 + |t|)^\eta < \infty \};
\end{equation}

see e.g., [8].

2.3 Jacobi Theta Functions on $G$

For $g = (z, \phi; \xi, \zeta) \in G$ and $f \in S_\eta(\mathbb{R})$, $\eta > 1$, define the Jacobi theta function as

\begin{equation}
(2.16) \quad \Theta_f(z, \phi; \xi, \zeta) = y^{1/4} e(\zeta - \frac{1}{2} \xi_1 \xi_2) \sum_{n \in \mathbb{Z}} f_\phi((n - \xi_2)y^{1/2}) e\left( \frac{1}{2}(n-\xi_2)^2 x + n\xi_1 \right),
\end{equation}

where $z = x + iy$, $\xi = (\xi_1, \xi_2)$ and $f_\phi$ is as in (2.14).

2.4 A Lattice $\Gamma < G$ such that $\Theta_f$ is $\Gamma$-invariant

Consider the following elements of $G$, each written in two ways using the identification described in Section 2.1:

\begin{equation}
(2.17) \quad y_1 = \left( \left( \begin{array}{c} 0 & -1 \\ 1 & 0 \end{array} \right), \arg \right) ; \left( 0, \frac{1}{8} \right) = \left( i, \frac{\pi}{2}; 0, \frac{1}{8} \right),
\end{equation}

\begin{equation}
(2.18) \quad y_2 = \left( \left( \begin{array}{c} 1 & 1 \\ 0 & 1 \end{array} \right); \left( 1/2, 0 \right), 0 \right) = \left( 1 + i; 0; \left( 1/2, 0 \right), 0 \right),
\end{equation}

\begin{equation}
(2.19) \quad y_3 = \left( \left( \begin{array}{c} 1 & 0 \\ 0 & 1 \end{array} \right); \left( 0, 1 \right), 0 \right) = \left( i, 0; \left( 0, 1 \right), 0 \right),
\end{equation}

\begin{equation}
(2.20) \quad y_4 = \left( \left( \begin{array}{c} 1 & 0 \\ 0 & 1 \end{array} \right); \left( 0, 0 \right), 1 \right) = \left( i, 0; \left( 0, 0 \right), 1 \right),
\end{equation}

\begin{equation}
(2.21) \quad y_5 = \left( \left( \begin{array}{c} 1 & 0 \\ 0 & 1 \end{array} \right); \left( 0, 0 \right), 1 \right) = (i, 0; 0, 1).
It was shown in [7] that for \( i = 1, \ldots, 5 \), we have \( \Theta_f(y_i; g) = \Theta_f(g) \) for every \( g \in G \).

The Jacobi theta function \( \Theta_f \) is therefore invariant under the left action by the group

\[
\Gamma = \langle y_1, y_2, y_3, y_4, y_5 \rangle < G.
\]

This means that \( \Theta_f \) is well defined on the quotient \( \Gamma \backslash G \). The group \( \Gamma \) is a lattice in \( G \), and \( \Gamma \backslash G \) is a 4-torus bundle over the modular surface \( SL(2, \mathbb{Z}) \backslash \mathfrak{H} \). In particular, \( \Gamma \backslash G \) is non-compact. A fundamental domain for the action of \( \Gamma \) on \( G \) is

\[
\mathcal{F}_\Gamma = \left\{ (z, \phi; \xi, \zeta) \in \mathcal{F}_{SL(2, \mathbb{Z})} \times [0, \pi) \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \right\},
\]

where \( \mathcal{F}_{SL(2, \mathbb{Z})} \) is a fundamental domain of the modular group in \( \mathfrak{H} \). The hyperbolic area of \( \mathcal{F}_{SL(2, \mathbb{Z})} \) is \( \pi/3 \), and hence, by (2.11), we have that \( \mu(\Gamma \backslash G) = \mu(\mathcal{F}_\Gamma) = \pi^2/3 \).

**Remark 2.1** Although we defined the Jacobi theta function in (2.16) assuming that \( f \) is regular enough (\( \eta > 1 \)), it can be shown \( \Theta_f \) is a well-defined element of \( L^2(\Gamma \backslash G, \mu) \) (in fact, of \( L^4(\Gamma \backslash G, \mu) \)) provided \( f \in L^2(\mathbb{R}) \); see [1, §2.9].

We are finally ready to define \( X \). Given \( f_0, f_1 \) as in the Section 1.4, set \( X : \Gamma \backslash G \to \mathbb{C}^2 \) as

\[
X(\Gamma g) = \left( \Theta_{f_1}(\Gamma g), \Theta_{f_2}(\Gamma g) \right).
\]

Observe that \( X : (\Gamma \backslash G, \mathcal{B}(\Gamma \backslash G), \frac{1}{\pi \times \pi} \mu) \rightarrow (\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2)) \) is a random variable whose law is simply the push forward of the normalized Haar measure \( \frac{1}{\pi \times \pi} \mu \) onto \( \mathbb{C}^2 \) via \( X \). The properties of the law of each component \( \Theta_f(\Gamma g) \) when \( \Gamma g \) is Haar-random on \( \Gamma \backslash G \) were studied in [1]. In particular, we have the following lemma.

**Lemma 2.2** [1] Let \( \eta > 1 \) and \( f \in S_\eta \). Then for all sufficiently large \( R > 0 \), we have

\[
\mu(\{ \Gamma g \in \Gamma \backslash G : |\Theta_f(\Gamma g)| > R \}) = \frac{2}{3} D(f) R^{-\eta} \left( 1 + O_\eta(R^{-\eta}) \right),
\]

where \( D(f) := \int_{-\infty}^{\infty} \int_0^\pi |f_\phi(w)|^2 d\phi dw \) and \( \kappa_\eta(f) := \sup_{w, \phi} |f_\phi(w)|(1 + |w|)^\eta \).

If \( f \) is less regular (e.g., when \( f \) is the indicator of an interval, in which case \( f \in S_1 \) and Lemma 2.2 does not apply) a delicate analysis can be performed to study the tail asymptotics (see [1, §3.6]). In any case, the complex-valued random variable \( \Theta_f \) has heavy tails and all moments of order \( p \geq 6 \) are infinite.

### 3 Proof of Theorem 1.1

Recall (2.7). The following theorem describes how certain “irrational” horocycle lifts of the form \( u \mapsto M(u) \Psi^u \), with \( M(u) \in G \) and \( \Psi^u = (\tilde{n}_u, (0), 0) \), become equidistributed in \( \Gamma \backslash G \) under the action of the geodesic flow \( \Phi^t = (\tilde{a}_e, (0), 0) \). It follows immediately from [1, Corollary 4.3].
**Theorem 3.1** Let $\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a probability density, and let $F : \Gamma \setminus G \to \mathbb{C}^2$ be a bounded continuous function. For any $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and any $\varrho \in \mathbb{R}$, we have

\[
\lim_{s \to \infty} \int_{\mathbb{R}} F(\Gamma \setminus G; \left( \frac{\partial u}{0}, \varrho u \right)) \sigma(u) du = \frac{1}{\mu(\Gamma \setminus G)} \int_{\Gamma \setminus G} F(\Gamma g) d\mu(\Gamma g).
\]

Note that the limit depends neither on $\sigma$, nor on $\theta$ (provided it is irrational), nor on $\varrho$. Our assumptions on the Pöschl–Teller potential $V_0$ and on the functions $f_0, f_1$, along with (1.15), allow us to write

\[
\frac{1}{2\pi} E_n t = \frac{1}{2\pi} \frac{(2n + \gamma)^2}{2} t = \frac{1}{2} (n - \xi_2)^2 u + n \xi_1 + \zeta - \frac{1}{2} \xi_1 \xi_2,
\]

where $\xi_1 = \frac{\gamma}{2} u$, $\xi_2 = 0$, $\zeta = \frac{\gamma^2}{8} u$, and $u = \frac{2n}{\pi}$. Therefore, using (2.16) and setting $s = 2 \log N$, we have

\[
\frac{1}{\sqrt{N}} \sum_{n \geq 0} f_\ell \left( \frac{n}{N} \right) e^{iE_n t} = \Theta f_\ell \left( u + i e^{-s}, 0; \left( \frac{\gamma}{2} u, \frac{\gamma^2}{8} u \right) \right), \quad \ell = 0, 1.
\]

Recall that $\rho$ denotes the density of the probability measure $\lambda$ on $\mathbb{R}$, and that $t$ in (1.17) is distributed according to the law $\lambda$. Therefore, $u = \frac{2n}{\pi}$ has density $\sigma = \frac{n}{2} \rho \left( \frac{2n}{\pi} \right)$.

We can write

\[
X_N(t) = X \left( u + i e^{-s}, 0; \left( \frac{\gamma}{2} u, \frac{\gamma^2}{8} u \right) \right),
\]

and proving Theorem 1.1 is equivalent to showing that for every bounded, continuous function $h : \mathbb{C}^2 \to \mathbb{R}$, we have that

\[
\lim_{s \to \infty} \int_{\mathbb{R}} h \left( X \left( u + i e^{-s}, 0; \left( \frac{\gamma}{2} u, \frac{\gamma^2}{8} u \right) \right) \right) \sigma(u) du
\]

\[
= \frac{1}{\mu(\Gamma \setminus G)} \int_{\Gamma \setminus G} h(X(\Gamma g)) d\mu(\Gamma g).
\]

Observe that if $f_0, f_1 \in \mathcal{S}_\eta$ with $\eta > 1$, then $h \circ X$ is bounded and continuous, and, since we are assuming that $\gamma$ is irrational, we can apply Theorem 3.1 to achieve (3.3).

Finally, if $f_0$ and $f_1$ are bounded, Riemann-integrable functions, then we can use a standard approximation argument, analogous to the one used in [1, Lemmata 4.5–4.9].

**Remark 3.2** Since the two components of $X$ are both functions of the same random variable ($\Gamma g$ distributed in $\Gamma \setminus G$ according to the normalized Haar measure), it is clear that the law on $\mathbb{C}^2$ of $X$ is not the product measure of its two marginals on $\mathbb{C}$. The analogue of Theorem 1.1 for rational $\gamma$ can also be considered, but the statement of Theorem 3.1 needs to be modified. In fact, the horocycle lifts do not equidistribute on the homogeneous space $\Gamma \setminus G$; instead, they equidistribute on a submanifold of positive codimension of the form $\Gamma_{\gamma} \setminus G$. In this case, the random variable $\Gamma g$ on the right-hand side of (3.1) would be distributed according the normalized Haar measure on $\Gamma_{\gamma} \setminus G$. 

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The extension of Theorem 3.1 for rational $\gamma$ is the subject of a joint work in progress with Tariq Osman.

4 A Class of SUSY Partner Pöschl–Teller Potentials

In this section, we explore two constructions that, given a Pöschl–Teller potential, allow us to construct infinitely many isospectral supersymmetric partners, thus providing us with a plethora of cases in which our Theorem 1.1 can be applied. Let $H_0$ be the Hamiltonian with potential $V_0$ as in (1.14). Recall that $\gamma = \alpha + \beta$. A general solution to the equation $H_0 u = \varepsilon u$ (regardless of boundary conditions) for any $\varepsilon > 0$ is

$$u_{\varepsilon,a,b}(x) = \sin^a(x) \cos^b(x) \left( a \, _2F_1\left( \frac{\gamma}{2} + \sqrt{\frac{\varepsilon}{2}}, \frac{\gamma}{2} - \sqrt{\frac{\varepsilon}{2}}; \alpha + \frac{1}{2}; \sin^2(x) \right) + b \, \sin^{1-2a}(x) \, _2F_1\left( \frac{1+\beta-\alpha}{2} + \sqrt{\frac{\varepsilon}{2}}, \frac{1+\beta-\alpha}{2} - \sqrt{\frac{\varepsilon}{2}}; \alpha; \sin^2(x) \right) \right).$$

Using first-order intertwining operators of the form

$$A = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + \kappa(x) \right), \quad A^\dagger = \frac{1}{\sqrt{2}} \left( - \frac{d}{dx} + \kappa(x) \right),$$

Contreras–Astorga and Fernández [3] were able to construct a 1-parameter family of partner potentials $V_1$ such that $H_0$ and $H_1$ are isospectral; see also [5]. Specifically,

$$V_1(x) = \frac{a(\alpha + 1)}{2 \sin^2(x)} + \frac{(\beta - 2)(\beta - 1)}{2 \cos^2(x)} - (\log v_\varepsilon(x))'', \quad \alpha > 1, \beta > 2,$$

where

$$v_\varepsilon(x) = \frac{u_{\varepsilon,1,0}(x)}{\sin^a(x) \cos^{1-\beta}(x)} = \cos^{2\beta-1}(x) \, _2F_1\left( \frac{\gamma}{2} + \sqrt{\frac{\varepsilon}{2}}, \frac{\gamma}{2} - \sqrt{\frac{\varepsilon}{2}}; \alpha + \frac{1}{2}; \sin^2(x) \right),$$

and $\varepsilon < E_0 = \frac{\gamma^2}{2}$ is an arbitrary real parameter. The corresponding eigenfunctions for $H_1$ are

$$\psi_{1,n} = \frac{A^\dagger \psi_{0,n}}{\sqrt{E_n - \varepsilon}},$$

where $\kappa(x) = (\log u_{\varepsilon,1,0}(x))'$ in (4.2). Such a potential $V_1$ for $\alpha = \sqrt{2}, \beta = 3$, and $\varepsilon = 9$ is shown in Figure 3, along with the supersymmetric partner $V_0$.

In the same paper [3], the authors also used second-order intertwining operators $B, B^\dagger$ of the form so that $H_1 B^\dagger = B^\dagger H_0$, where

$$B^\dagger = \frac{1}{2} \left( \frac{d^2}{dx^2} - \eta(x) \frac{d}{dx} + \theta(x) \right),$$
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\[ \alpha = \sqrt{2}, \beta = 3, \epsilon = 9 \]

**Figure 3:** The potentials \( V_0 \) and \( V_1 \) of two isospectral supersymmetric partner Hamiltonians, given by (1.14) and (4.3), respectively.

...to construct a 2-parameters family of partner potentials \( V_1 \) such that \( H_0 \) and \( H_1 \) are isospectral. Specifically,

\[ V_1(x) = \frac{(\alpha + 1)(\alpha + 2)}{2 \sin^2(x)} + \frac{(\beta - 3)(\beta - 2)}{2 \cos^2(x)} - (\log W(x))'', \quad \alpha > 1, \beta > 3, \]  

where

\[ W(x) = \frac{W(u_{\epsilon_1,1,0}, u_{\epsilon_2,1,0})}{\sin^{2\alpha+1}(x) \cos^{3-2\beta}(x)}, \]  

\( W(f,g) = f'g - fg' \) denotes the Wronskian of \( f \) and \( g \), \( u_{\epsilon_i,1,0} \) is as in (4.1) for \( i = 1,2 \), and \( \epsilon_1, \epsilon_2 \) are real parameters such that \( E_l < \epsilon_2 < \epsilon_1 < E_{l+1} \) for some \( l \geq 0 \). In this case, the normalized eigenfunctions of \( H_1 \) are

\[ \psi_{1,n} = \frac{B^\dagger \psi_{0,n}}{\sqrt{(E_n - \epsilon_1)(E_n - \epsilon_2)}}, \]

where in (4.6), we have \( \eta = (\log( W(u_{\epsilon_1,1,0}, u_{\epsilon_2,1,0})))', \theta = \frac{\eta'}{2} + \frac{\eta^2}{2} - 2 V_0 + d, \) and \( d = \epsilon_1 + \epsilon_2. \)

In all these classes of supersymmetric partner potentials, our Theorem 1.1 applies, provided \( \gamma \) is irrational.

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