Complexity of commuting Hamiltonians on a square lattice of qubits

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Abstract

We consider the computational complexity of Hamiltonians which are sums of commuting terms acting on plaquettes in a square lattice of qubits, and we show that deciding whether the ground state minimizes the energy of each local term individually is in the complexity class \( \mathbf{NP} \). That is, if the ground states has this property, this can be proven using a classical certificate which can be efficiently verified on a classical computer. Different to previous results on commuting Hamiltonians, our certificate proves the existence of such a state without giving instructions on how to prepare it.

1 Introduction

Understanding the ground state properties of spin systems on a lattice is of central importance in many-body physics, but at the same time, it is a highly challenging problem in many scenarios. An important step in understanding its difficulty has been the insight that computing e.g. the ground state energy of a classical spin system is, in general, an \( \mathbf{NP} \)-complete problem [1]: While the energy of any given spin configuration can be easily computed, finding the configuration with minimal energy is in general a difficult task – it can be as hard as any problem in \( \mathbf{NP} \), i.e., any problem whose solution can be efficiently verified. For quantum spin systems, an additional difficulty arises: Generally, we cannot even expect to have an efficient description of the ground state. Thus, it seems that the only statement we can make about the difficulty of the problem is that given a quantum register with
the ground state, we will be able to efficiently estimate its energy using a quantum computer. Indeed, it has been shown that this is the best we can say, as the problem of estimating the ground state energy of a quantum system is a complete problem for the class \( \text{QMA} \) \(^2\), the quantum analogue of \( \text{NP} \): It contains all problems which have a quantum solution which can be efficiently checked on a quantum computer, and thus, determining the ground state energy of a quantum spin system is as hard as any of these problems; in fact, the problem retains its hardness even when restricted to two-dimensional lattices of qubits with nearest-neighbor interactions \(^3\) or one-dimensional chains \(^5\).

It is an interesting question to understand the reasons underlying the additional complexity of quantum spin systems as compared to classical systems. To this end, restricted version of the problem which lie between classical and general quantum spin Hamiltonians have been studied: For instance, it has been shown that so-called stoquastic Hamiltonians, where all off-diagonal elements are negative, have a complexity which lies in between \( \text{NP} \) and \( \text{QMA} \), as those systems can be related to classical random processes \( ^6\), \(^7\); in fact, these are exactly the Hamiltonians which allow for Quantum Monte Carlo simulations as they do not exhibit a sign problem.

Another restricted class of Hamiltonians are commuting Hamiltonians, that is, Hamiltonians which can be written as a sum of mutually commuting few-body terms. For those systems, all terms can be simultaneously diagonalized, just as for classical systems; however, the corresponding eigenbasis can be highly entangled, making it unclear whether a useful classical description of the ground state can be provided. In fact, commuting Hamiltonians encompass systems which exhibit rich non-classical physics, in particular models with topological order and even anyonic excitations, such as Kitaev’s toric code and quantum double models \(^8\), or Levin and Wen’s string net models \(^9\). Commuting Hamiltonians are also of interest since the fixed points of renormalization flows in gapped phases are expected to be commuting Hamiltonians, and thus understanding their structure might give insight into the structure of gapped quantum phases. Finally, understanding the complexity of commuting Hamiltonians is of interest in quantum complexity, as it might be a step towards a quantum PCP theorem, which would assess how the difficulty of estimating the ground state energy is related to the desired accuracy which is integer for commuting projectors.

What is know about the complexity of finding the ground state energy of commuting Hamiltonians, or rather, of determining whether the ground state minimizes each term in the Hamiltonian individually – the commuting Hamiltonian problem? For lattices in two and higher dimensions,
COMMUTING HAMILTONIAN is an NP-hard problem, as it e.g. encompasses the Ising model on a planar graph \cite{1}. On the other hand, it is not clear whether the general COMMUTING HAMILTONIAN problem is inside \textit{NP}, since it is not clear in general how to provide an efficiently checkable description of the ground state. For two-local (i.e., two-body) Hamiltonians, Bravyi and Vyalyi \cite{10} have shown that the problem is in \textit{NP} by using \textit{C*-algebraic} techniques (their result also implies that one-dimensional commuting Hamiltonians are efficiently solvable); subsequently, Aharonov and Eldar \cite{11} have proven containment in \textit{NP} for Hamiltonians with three-body interactions both for qubits on arbitrary graphs, and qutrits on nearly-Euclidean interaction graphs. In both of these works, the classical certificates do not only prove the problem to be in \textit{NP}, but can in fact be used to construct constant depth quantum circuits which \textit{prepare} the ground state. This, in particular, implies that the corresponding Hamiltonians – including qutrits with three-body interactions – cannot exhibit topological order \cite{11,12}. On the other hand, Kitaev’s toric code, which is the ground state of a commuting Hamiltonian with four-body interactions of qubits, does have topological order, and thus, we cannot expect any approach which yields a low-depth circuit to work beyond three qutrits.

In this paper, we study the COMMUTING HAMILTONIAN problem on a square lattice of qubits with plaquette-wise interactions, and prove that it is in \textit{NP}. That is, we consider a square lattice of qubits, with a Hamiltonian with mutually commuting terms acting on the four qubits adjacent to each plaquette, and show that the problem of deciding whether its ground state minimizes the energy of each local term in the Hamiltonian is in \textit{NP}: i.e., in case the ground state has this property, a classical certificate exists which can be checked efficiently by a quantum computer. Our approach differs considerably from the aforementioned approaches in that the certificate cannot be used to devise a quantum circuit for preparing the ground state, and is thus also applicable to systems with topological order; it should be noted that the same holds true for the proof in Ref. \cite{10} that COMMUTING HAMILTONIAN with factorizing projectors is in \textit{NP}.

2 The setup

We will consider a 2D square lattice with spins on the vertices, and either open or periodic boundary conditions. The Hamiltonian

$$H = \sum_p h_p$$
consists of plaquette terms \( h_p \) which act on the four spins adjacent to the plaquette \( p \), and we impose that all the terms in the Hamiltonian commute, 
\[
[ h_p, h_q ] = 0 \quad \forall \ p, q.
\]

As the Hamiltonian terms commute, there is a basis of eigenstates of \( H \) which are simultaneously eigenstates of all the \( h_p \). We would like to know the computational difficulty of the following problem, called \textsc{commuting hamiltonian}: Is there an eigenstate \( |\psi\rangle \) of \( H \) which minimizes the energies for all \( h_p \) individually, i.e., are the ground states of \( H \) also ground states of each \( h_p \)? In the following, we will show that in the case of qubits, the existence of such a state can be proven within \textsc{np}, i.e., there is a classical certificate which proves the existence of such a \( |\psi\rangle \), and which can be checked efficiently classically. Note that on the other hand, it is clear that the problem is \textsc{np}–hard, as it e.g. encompasses classical Ising spin glasses in a field which are known to be \textsc{np}–hard even for two-level systems \( \Pi \).

For the following, it will be useful to reformulate \textsc{commuting hamiltonian} as follows: Define the local ground state projectors \( \Pi_p \) as the projectors onto the ground state subspace of \( h_p \); the \( \Pi_p \) commute again, \( [\Pi_p, \Pi_q] = 0 \). Then,
\[
\Pi_{GS} = \prod_{p \in P} \Pi_p
\]
is the projector onto the subspace spanned by the states which are ground states of all \( h_p \). Since \textsc{commuting hamiltonian} asks whether such states exist, it is equivalent to asking whether \( \Pi_{GS} \neq 0 \).

3 \textsc{commuting hamiltonian in np}

3.1 Two layers

We start by coloring the plaquettes of the square lattice black and white in a checkerboard pattern, and denote the set of black and white plaquettes by \( P_B \) and \( P_W \), respectively. Let
\[
\Pi_B = \prod_{p \in P_B} \Pi_p \quad \text{and} \quad \Pi_W = \prod_{p \in P_W} \Pi_p
\]
be the projectors onto the joint ground state space of the black and white \( h_p \), respectively; then, \textsc{commuting hamiltonian} corresponds to determining whether \( \Pi_B \Pi_W \neq 0 \). This is equivalent to asking whether

\[
\text{tr}[\Pi_B \Pi_W] \neq 0 \quad \text{(1)}
\]
(this can be seen using eigendecompositions of $\Pi_W$ and $\Pi_B$), and we will consider this formulation of the problem from now on; to prove COMMUTING HAMILTONIAN is contained in $\text{NP}$, we therefore need to show that a classical certificate for the validity of (11) can be provided.

A helpful example to keep in mind in the first part of our discussion is Kitaev’s toric code [8]: There, $\Pi_{p} = \frac{1}{2}(I + Z^\otimes 4)$ for $p \in \mathcal{P}_B$, and $\Pi_{p} = \frac{1}{2}(I + X^\otimes 4)$ for $p \in \mathcal{P}_W$, with $X$ and $Z$ the Pauli matrices.

3.2 The structure of one layer

In the following, let us study the structure of each layer individually (we will w.l.o.g. choose black). To this end, we will use a result of Bravyi and Vyalyi based on $C^*$-algebraic techniques [10]; a detailed explanation of those techniques can also be found in [11]. The basic insight from Ref. [10] is the structure of two commuting projectors. Consider two projectors $L \equiv L_{AB} \otimes 1_C$ and $R \equiv 1_A \otimes R_{BC}$ acting on a space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$; the two operators overlap on $\mathcal{H}_B$. Now consider the Schmidt decompositions

\[
L = \sum A^i_L \otimes B^i_L \quad \text{and} \quad R = \sum B^i_R \otimes C^i_R ,
\]

i.e. $\text{tr}[A^i_L (A^j_L)^\dagger] = 0$ for $i \neq j$, and similarly for $B^i_L$, $B^i_R$, and $C^i_R$. Then, $[L, R] = 0$ implies that $[B^i_L, B^j_R] = 0$ for all $i, j$, and thus, $B^i_L$ and $B^i_R$ span commuting $C^*$-algebras, cf. [10]. Using the standard form of finite $C^*$-algebras, it follows that the space $\mathcal{H}_B$ has a canonical decomposition

\[
\mathcal{H}_B = \bigoplus_\alpha \mathcal{H}^\alpha_L \otimes \mathcal{H}^\alpha_R \otimes \mathcal{H}^\alpha_Z =: \mathcal{H}^\alpha_B
\]

where the $B^i_L$ span the full matrix algebra on $\mathcal{H}^\alpha_L$ while acting trivially on the rest, and correspondingly for $B^i_R$ and $\mathcal{H}^\alpha_R$.

This shows that the space $\mathcal{H}_B$ can be cut into direct sum “slices” (the $\alpha$) such that in each slice, $L$ and $R$ act on independent subsystems. More formally, there exists a decomposition $1 = \sum_\alpha \pi_\alpha$ of $\mathcal{H}_B$, with $\pi_\alpha$ the projectors onto $\mathcal{H}^\alpha_B$, such that

$[\pi_\alpha, L] = 0$ and $[\pi_\alpha, R] = 0$

and thus

\[
L = \sum_{\alpha, \beta} \pi_\alpha L \pi_\beta = \sum_\alpha \pi_\alpha L \pi_\alpha , \quad R = \sum_\alpha \pi_\alpha R \pi_\alpha ,
\]
where $L^\alpha$ and $R^\alpha$ act on different subsystems $H_L^\alpha$ and $H_R^\alpha$, i.e., factorize. Note that the above decomposition allows to compute the $\pi_\alpha$ and thus the $L_\alpha$ and $R_\alpha$ efficiently.

Each vertex in the black sublattice is acted upon by exactly two commuting projectors $\Pi_p$; thus, we can apply the preceding argument to all vertices to find decompositions $\pi^{x}_\alpha$, $\sum_{\alpha_v} \pi^{x}_\alpha = 1$, of the Hilbert space at each vertex $v$, such that $\Pi_B$ projected onto the slice $\vec{\alpha} = (\alpha_v)_{v \in V}$ factorizes,

$$\bigotimes_{p \in P_B} \Pi_{p}^{\vec{\alpha}} = \prod_{v \in V} \pi^{x}_\alpha \Pi_B \pi^{x}_\alpha$$

(this implicitly defines the $\Pi_{p}^{\vec{\alpha}}$), and $\Pi_B$ can be written as

$$\Pi_B = \bigoplus_{\vec{\alpha}} \bigotimes_{p \in P_B} \Pi_{p}^{\vec{\alpha}} \equiv \sum_{\vec{\alpha}} \bigotimes_{p \in P_B} \Pi_{p}^{\vec{\alpha}} . \quad (3)$$

Note that in the sum on the right hand side, we implicitly regard the tensor products as being canonically embedded into the full Hilbert space, and we will use this convention in the following.

The analogous decomposition can be performed for the white sublattice, yielding a (in general different!) decomposition

$$\Pi_W = \bigoplus_{\vec{\beta}} \bigotimes_{p \in P_W} \Pi_{p}^{\vec{\beta}} \equiv \sum_{\vec{\beta}} \bigotimes_{p \in P_W} \Pi_{p}^{\vec{\beta}} . \quad (4)$$

Note that in order to distinguish the decomposition for the the black and the white layer, we strictly use labels $\vec{\alpha}$ and $\vec{\beta}$, respectively; moreover, we denote the projectors decomposing the white layer by $\tilde{\pi}^{x}_{\beta_v}$.

E.g., for Kitaev’s toric code the $\pi^{x}_\alpha$ are projectors onto the $Z$ eigenstates, and the $\tilde{\pi}^{x}_{\beta_v}$ onto the $X$ eigenstates.

### 3.3 Combining the layers

Using the structure of $\Pi_B$ and $\Pi_W$, Eqs. (3) and (4), we can rewrite the commuting Hamiltonian problem, Eq. (1), as

$$0 \neq \sum_{\vec{\alpha}, \vec{\beta}} \text{tr} \left[ \left( \bigotimes_{p \in P_B} \Pi_{p}^{\vec{\alpha}} \right) \left( \bigotimes_{p \in P_W} \Pi_{p}^{\vec{\beta}} \right) \right] ; \quad (5)$$

recall that we regard the tensor products as being canonically embedded into the full Hilbert space. Since each of the traces is positive (as it is the
trace of the product of two positive operators), the above is equivalent to the existence of $\vec{\alpha}$ and $\vec{\beta}$ such that $\Omega(\vec{\alpha}, \vec{\beta}) \neq 0$, where

$$\Omega(\vec{\alpha}, \vec{\beta}) := \text{tr} \left[ \left( \bigotimes_{p \in \mathcal{P}_B} \Pi_p^{\vec{\alpha}} \right) \left( \bigotimes_{p \in \mathcal{P}_W} \Pi_p^{\vec{\beta}} \right) \right]. \quad (6)$$

Thus, we can ask the prover to provide us as a certificate with some $\vec{\alpha}$ and $\vec{\beta}$ for which $\Omega(\vec{\alpha}, \vec{\beta}) \neq 0$; if we can moreover show that $\Omega(\vec{\alpha}, \vec{\beta})$ can be computed efficiently (or rather in NP), this will prove that commuting Hamiltonian is in NP.

Note that $\Omega(\vec{\alpha}, \vec{\beta})$ can be interpreted as the overlap of the (unnormalized) states $\bigotimes \Pi_p^{\vec{\alpha}}$ and $\bigotimes \Pi_p^{\vec{\beta}}$, both of which are tensor products of states supported on individual plaquettes, but with different partitions in the two layers. Computing such an overlap can in general be as hard as contracting Projected Entangled Pair States (PEPS) \[13\], i.e., PP-hard \[14\]: any PEPS can be expressed this way by using one layer for the bonds and the other for the PEPS projections. Of course, the fact that these states arise from two commuting layers $\Pi_B$ and $\Pi_W$ yields additional constraints, and we will show in the following that those constraints allow for the efficient evaluation of $\Omega(\vec{\alpha}, \vec{\beta})$ in the case of qubits.

For Kitaev’s toric code, e.g., we could choose $\vec{\alpha} = \vec{\beta} = (0, \ldots, 0)$: This yields

$$\Omega(\vec{\alpha}, \vec{\beta}) = \text{tr}\left[ (|0\rangle\langle 0|)^\otimes N (|+\rangle\langle +|)^\otimes N \right] = 2^{-N} \neq 0,$$

proving the existence of a zero-energy ground state; note that this certificate does not carry any information on how to prepare the state.

### 3.4 Computing the overlap

Let us now show that for a lattice of qubits, the overlap $\mathcal{O}(\vec{\alpha}, \vec{\beta})$ can be computed efficiently. To this end, we will show that the computation of the overlap can be decomposed into a product of overlaps of one-dimensional structures which can be computed efficiently.

Let us first consider the black layer. For each vertex $v$, the decomposition \[2\] of the local Hilbert space can either be trivial (no direct sum) or non-trivial, $1 = \sum \pi_{\alpha_v}$. In the former case, this implies that at most one of the adjacent plaquette terms $\Pi_p$ acts non-trivially of vertex $v$; in the latter case, the sum consists of exactly two one-dimensional projectors $\pi_{\alpha_v}$, making use of the fact that the Hilbert space at each site is a qubit, i.e., two-dimensional. We will denote the set of vertices with a non-trivial
decomposition in the black layer by $F_B$, and in the white layer by $F_W$ (in which the one-dimensional projectors are labelled $\bar{\pi}_v^u$).

In $\Omega(\vec{\alpha}, \vec{\beta})$, all vertices in $F_B \cup F_W$ contribute only a one-dimensional subspace and thus can be traced out: That is, all vertices in $F_B \cap F_W$ can be removed (taking care whether the overlap of the one-dimensional projectors is non-vanishing), while for vertices where only one layer has a one-dimensional decomposition, this yields new effective plaquette terms $\rho_p$ in the other layer by projecting the original plaquette terms $\Pi_p$ onto that one-dimensional subspace; thus, the problem of checking whether $\Omega(\vec{\alpha}, \vec{\beta})$ is non-zero reduces to computing the overlap of the new effective plaquette terms $\rho_p$. Formally, this reads

$$\Omega(\vec{\alpha}, \vec{\beta}) = \text{tr} \left[ \bigotimes_{p \in P_B} \Pi^\vec{\alpha}_p \left( \bigotimes_{p \in P_W} \Pi^\vec{\beta}_p \right) \right]$$

$\hspace{1cm} = \text{tr} \left[ \left( \prod_{w \in F_W} \bar{\pi}_w^u \bigotimes_{p \in P_B} \Pi^\vec{\alpha}_p \prod_{w \in F_W} \bar{\pi}_w^v \right) \left( \prod_{v \in F_B \setminus F_W} \pi_v^v \bigotimes_{p \in P_W} \Pi^\vec{\beta}_p \prod_{v \in F_B \setminus F_W} \pi_v^v \right) \right] \hspace{1cm} \quad (\ast)$$

$\hspace{1cm} = \prod_{v \in F_B \cap F_W} \text{tr} \left[ \pi_v^v \bar{\pi}_v^u \right] \times \text{tr} \left[ \left( \bigotimes_{p \in P_B} \rho_p \right) \left( \bigotimes_{p \in P_W} \rho_p \right) \right]. 

(7)$

Here, we have used in step (A) that for all $v \in F_W$,

$$\bigotimes_{p \in P_W} \Pi^\vec{\beta}_p = \bar{\pi}_v^u \left( \bigotimes_{p \in P_W} \Pi^\vec{\beta}_p \right) \bar{\pi}_v^v$$

and analogously for the black layer; in step (B), we have defined new effective plaquette terms $\rho_p$ by virtue of

$$\bigotimes_{p \in P_B} \rho_p = \text{tr}_{F_B \cup F_W} \left[ \left( \prod_{w \in F_W \setminus F_B} \bar{\pi}_w^u \bigotimes_{p \in P_B} \Pi^\vec{\alpha}_p \prod_{w \in F_W \setminus F_B} \bar{\pi}_w^v \right) \right],$$

and correspondingly for the white plaquettes; the first factor in (7) takes care of the terms in $F_B \cap F_W$ in (\ast). Note that the $\rho_p$ are now only supported on those vertices not in $F_B \cup F_W$, as those have been traced out.

The task of checking whether $\Omega(\vec{\alpha}, \vec{\beta}) \neq 0$ has thus been reduced to checking this for (7): For the first term, this can be clearly done efficiently,
and so, it remains to prove that the overlap

$$\Theta := \text{tr} \left[ \left( \bigotimes_{p \in \mathcal{P}_B} \rho_p \right) \left( \bigotimes_{p \in \mathcal{P}_W} \rho_p \right) \right]$$

(8)

can be computed efficiently. Note that the $\rho_p$ are now supported on plaquettes of a square lattice with vertices missing. Moreover, while the $\rho_p$ do no longer commute, in each layer at most one $\rho_p$ acts non-trivially on each vertex; we will make use of this fact repeatedly in the following.

The situation encountered in computing the overlap $\Theta$ is depicted in Fig. 1. Here, the dots in each plaquette denote the vertices on which $\rho_p$ acts non-trivially (the lines just connect the vertices involved in $\rho_p$). If the $\rho_p$ on adjacent plaquettes act non-trivially on the same qubit (we will say they “overlap”), they form connected structures which we need to contract in order to evaluate $\Theta$. For one-dimensional structures as the one on in Fig. 1a, this contraction can be carried out efficiently: One starts from one pla-
quette and proceeds along one direction of the one-dimensional chain, always tracing out the degrees of freedom on the inside. This way, at every point in the computation only the state at the boundary (which has fixed size) needs to be stored, and thus, the contraction can be carried out efficiently. On the other hand, branching structures like the one in Fig. 1b can in general not be contracted efficiently, since the size of the boundary is a priori not bounded; in fact, e.g. quantum circuits, and even postselected quantum circuits, can be encoded this way, making such contractions in general a computationally hard task.

However, as we will show in the following, the structures formed by the $\rho_p$ in $\Theta$, Eq. (8), will always be one-dimensional, and thus $\Theta$ can be computed efficiently. To this end, we will consider the state $\rho_C$ on a plaquette $C$ (the “central” plaquette), and show that it can overlap non-trivially with the states $\rho_p$ of at most two of the adjacent plaquettes, thus ruling out branching structures as the one on the right of Fig. 1. We will make intensive use of the fact that in each layer, at most one plaquette term $\rho_p$ can act non-trivially on any given vertex; in the graphical notation of Fig. 1, we will highlight this fact by placing a cross opposite of any dot:

![Diagram](image)

(9)

Here, the dot indicates that the corresponding $\rho_p$ acts non-trivially on a vertex qubit (the diamond in the center), while the cross indicates that the corresponding $\rho_p$ does act trivially. Note that this in particular implies that $\rho_C$ can at most overlap non-trivially with the four horizontally and vertically adjacent plaquettes from the other layer, but not with diagonally adjacent plaquettes.

The simplest case is when the state $\rho_C$ on the central plaquette $C$ involves only two vertices non-trivially, for instance

![Diagram](image)

Now, both qubits 1 and 3 can be acted upon non-trivially by at most one white plaquette – the other has to be empty, following the rule (9) that opposite of any dot there has to be a cross; this way, only one-dimensional
structures can arise:

This clearly holds for any possible $\rho_C$ which acts non-trivially on only two qubits, and for any configuration of the adjacent plaquettes; it follows that only one-dimensional structures can emerge this way.

In order to understand the cases where $\rho_C$ acts non-trivially on three or four qubits, let us first analyze the following situation:

Here, $\rho_C$ acts non-trivially at least on qubits 1 and 2, and $\rho_L$ acts non-trivially on qubit 1 which implies that $\rho_T$ acts trivially on it. In the following, we will show that this implies that $\rho_T$ also has to act trivially on qubit 2. We will prove this by contradiction, so assume that $\rho_T$ acts non-trivially on qubit 2. Since $\rho_T$ is obtained from the original projector $\Pi_T$ on that plaquette by a partial projection on some of the other vertices, this implies that $\Pi_T$ acts non-trivially on qubit 2 (where it spans the whole $\mathbb{C}^*$–algebra, since we have traced out all vertices where this was not the case). On the other hand, $\rho_L$ and thus $\Pi_L$ acts non-trivially on qubit 1, and thus spans the whole $\mathbb{C}^*$–algebra on it. Since $\Pi_L$ and $\Pi_T$ commute, this means that $\Pi_T$ acts trivially on qubit 1; that is, $\Pi_T$ and $\Pi_C$ need to commute on qubit 2 alone: However, since $\Pi_T$ spans the whole $\mathbb{C}^*$–algebra on qubit 2, this would imply that $\Pi_C$ and thus $\rho_C$ has to act trivially on qubit 2, giving a contradiction. Thus, we have the following “Lemma”:

Let us now consider the case where the state on the central plaquette involves all four qubits non-trivially, and let us start by assuming w.l.o.g.
that $\rho_L$ acts non-trivially on qubit 1:

![Diagram](image)

This implies that qubit 1 is not acted upon by $\rho_T$. Using Eq. (10), we infer that $\rho_T$ cannot involve qubit 2 either,

![Diagram](image)

Eq. (10) also shows that $\rho_B$ has to act trivially on qubit 4 – otherwise, $\rho_L$ would act trivially on qubit 4 and thus qubit 1, which it doesn’t:

![Diagram](image)

In order to obtain a branching structure, both $\rho_B$ and $\rho_R$ need to act non-trivially on some of the qubits. However, if $\rho_B$ acts non-trivially on qubit 3, Eq. (10) implies that $\rho_R$ has to act trivially on qubits 2 and 3:

![Diagram](image)

This shows that for $\rho_C$ acting non-trivially on all four qubits, the central plaquette can only couple to at most two adjacent plaquettes, forming one-dimensional structures.
It remains to study what happens in the case of tripartite entanglement on the central plaquette. We start from the following configuration:

(Note that we don’t make any assumptions on how the Hilbert space of qubit 2 decomposes.) Clearly, in order to obtain a branching structure, either $\rho_T$ has to act non-trivially on qubit 1, or $\rho_R$ has to act non-trivially on qubit 3. We consider w.l.o.g. the first possibility and infer from Eq. (10) that $\rho_L$ has to act trivially on both qubits 1 and 4:

In order to obtain a branching structure, we now have to get both $\rho_R$ and $\rho_B$ involved. However, if $\rho_R$ acts non-trivially on qubit 3, Eq. (11) implies that $\rho_B$ has to act trivially on both qubits 3 and 4, and thus, the structure formed around $\rho_C$ will again be one-dimensional.

Together, this shows that the overlap $\Theta$, Eq. (8), decays into one-dimensional structures for which the overlap can be computed efficiently. In turn, this implies that for given $\vec{\alpha}$ and $\vec{\beta}$, $\Omega(\vec{\alpha}, \vec{\beta})$ can be computed efficiently, and thus, the commuting Hamiltonian problem on a square lattice of qubits with plaquette interactions is in $\mathbf{NP}$.

### 3.5 Finite accuracy

In the preceding proof, we have assumed infinite accuracy, but as we will now show, our argument still applies if we compute with finite accuracy. To this end, let $N$ denote the number of qubits in the system; we will need to show that the computation time scales as $\text{poly}(N)$. We assume that the Hamiltonian terms are given exactly and can be represented with $\text{poly}(N)$ digits. First, note that the trace in Eq. (11), which equals the sum in Eq. (5), evaluates to an integer, and thus, there exists at least one pair $(\vec{\alpha}, \vec{\beta})$ such
that $\Omega(\vec{\alpha}, \vec{\beta}) \geq 2^{-2N}$. If we request this particular $(\vec{\alpha}, \vec{\beta})$ as a certificate, it is sufficient if we can determine $\Omega(\vec{\alpha}, \vec{\beta})$ to $2N + 1 = \text{poly}(N)$ digits. (This is crucial, since there can be $(\vec{\alpha}, \vec{\beta})$ for which $\Omega(\vec{\alpha}, \vec{\beta})$ is arbitrarily small.) $\Omega(\vec{\alpha}, \vec{\beta})$ is obtained from contracting a polynomial number of terms which are either $\Pi_p$ (which are known exactly) or $\pi_{\vec{\alpha}_e}^v$ and $\bar{\pi}_{\vec{\beta}_e}^v$, and the latter can be determined to $\text{poly}(N)$ accuracy from the C*-decomposition (2), which is the solution to a (fixed-size) eigenvalue problem. It follows that $\Omega(\vec{\alpha}, \vec{\beta})$ can be computed to the required $\text{poly}(N)$ accuracy in $\text{poly}(N)$ time, and our proof still applies.

4 Summary

We have studied the commuting Hamiltonian problem on a square lattice of qubits with plaquette-wise interaction and shown that the problem is $\text{NP}$–complete. Differently speaking, we have shown that there exists a classical certificate for the fact that the ground state of the system minimizes each term locally which can be checked efficiently on a classical computer. The central idea for our proof has been to split the system into two layers in each of which the commuting terms overlap on individual sites, and to argue that the existence of a state minimizing all local terms is equivalent to the existence of a pair of ground states for the two layers with non-zero overlap. Each layer could be decomposed using the C*-algebraic techniques introduced to the problem in [10], allowing to find an efficient description of its ground state subspace. Finally, we showed that the overlap of ground states of two layers can be computed efficiently, by showing that it gives rise to of one-dimensional structures only. A somewhat surprising feature of our approach is that while it certifies the existence of a ground state, it cannot (to our knowledge) be used to devise a way how to prepare the ground state; in fact, due to the possibility of having topological order in such systems, any circuit preparing their ground states would need to have at least logarithmic depth, or linear depth if it was local [12].

Our method does, in principle, also apply beyond qubits: We can still split the system into two layers, decompose both of them into direct sum slices $\vec{\alpha}$ and $\vec{\beta}$, and ask the prover to provide labels $\vec{\alpha}$ and $\vec{\beta}$ with non-zero overlap $\Omega(\vec{\alpha}, \vec{\beta})$. While we cannot make sure any more that $\Omega(\vec{\alpha}, \vec{\beta})$ can be computed efficiently, we can always ask the prover to also provide us with an instruction on how to efficiently contract the states, in case there is a way to do so, e.g. by providing the optimal contraction order. In particular, this applies to the case where the decomposition in the direct sum gives
one-dimensional spaces, such as in Kitaev’s toric code or quantum double models; as well as to cases where the $\rho_p$ are separable states. Our idea also applies to any other graph which can be split into two layers in such a way that the C$^*$-technique of [10] can be applied to each of them, and in fact to any type of Hamiltonian which is composed of two layers with eigenbases for the zero-energy subspace whose overlap can be computed efficiently, such as for product bases. Note that on the other hand, a decomposition into three layers cannot be used for our purposes, since for three positive operators $A$, $B$, and $C$, $\text{tr}[ABC]$ can have both real and imaginary parts of either sign, so that Eq. (5) is no longer equivalent to Eq. (6) being non-zero (as the $\Pi_{p}^{\alpha}$ and $\Pi_{p}^{\beta}$ do not commute any more).

An interesting open question relating to the present approach to the problem is whether it can be generalized beyond qubits. For four-level systems and beyond, this is likely not the case, since the local Hilbert space can decompose into two qubits, and thus operators commuting on a single spin can both act non-trivially on it, i.e., Eq. (9) does not hold any more. On the other hand, for qutrits this is not the case once we have fixed a slice in the direct sum; yet, it is not clear how to establish a version of Eq. (10). In particular, the non-trivial projections $\pi_{\alpha}^{v}$ can now have both rank 1 and 2, and in the latter case we cannot simply trace out the corresponding degree of freedom; it is however not clear that this does rule out an analogue to Eq. (10).

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