Noncommutative Complex Structures on Quantum Homogeneous Spaces

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Abstract

A new framework for noncommutative complex geometry on quantum homogeneous spaces is introduced. The main ingredients used are covariant differential calculi and Takeuchi’s categorical equivalence for faithfully flat quantum homogeneous spaces. A number of basic results are established, producing a simple set of necessary and sufficient conditions for noncommutative complex structures to exist. Throughout, the framework is applied to the quantum projective spaces endowed with the Heckenberger–Kolb calculus.

1 Introduction

Classical complex geometry is a subject of remarkable richness and beauty with deep connections to modern physics. Yet despite over twenty-five years of noncommutative geometry, the development of noncommutative complex geometry is still in its infancy. What we do have is a large number of examples which demand consideration as noncommutative complex spaces. We cite, among others, noncommutative tori [8, 39], noncommutative projective algebraic varieties [36], fuzzy flag manifolds [28], and (most importantly from our point of view) examples arising from the theory of quantum groups [11, 26]. These objects are of central importance to areas such as the construction of spectral triples [9, 7, 35], noncommutative mirror symmetry [11, 34, 37], and localisation for quantum groups [2, 3, 22].

Thus far, there have been two attempts to formulate a general framework for noncommutative complex geometry. The first, due to Khalkhali, Landi, and van Suijlekom [16], was introduced to provide a context for their work on the noncommutative complex geometry of the Podleś sphere. This followed on from earlier work of Majid [26], Schwartz and Polishchuk [34], and Connes [5, 6]. Khalkhali and Moatadelfro [17, 18] would go on to apply this framework to D’Andrea and Dąbrowski’s work [7] on the higher order quantum projective spaces.

Subsequently, Beggs and Smith introduced a second more comprehensive approach to noncommutative complex geometry in [4]. Their motive was to provide a framework for quantising the intimate relationship between complex differential geometry and complex projective geometry. They foresee that the rich interaction between algebraic and
analytic techniques occurring in the classical setting will carry over to the noncommu-
tative world.

The more modest aim of this paper is to begin the development of a theory of noncommu-
tative complex geometry for quantum group homogeneous spaces. This will be done very
much in the style of Majid’s noncommutative Riemannian geometry [24, 26, 25], with
the only significant difference being that here we will not need to assume that our quan-
tum homogeneous spaces are Hopf–Galois extensions, while we will assume that they
are faithfully flat. We first introduce the notion of a covariant noncommutative complex
structure for a total differential calculus. Then, by calling on our assumption of faithful
flatness, we use Takeuchi’s categorical equivalence to establish a simple set of neces-
sary and sufficient conditions for such noncommutative complex structures to exist. In
subsequent work, it is intended to build upon these results and formulate noncommu-
tative generalisations of Hodge theory and Kähler geometry for quantum homogeneous
spaces [31]. Indeed, the first steps in this direction have already been taken [32].

For this undertaking to be worthwhile, however, it will need to be applicable to a good
many interesting examples. Recall that classically one of the most important classes
of homogeneous complex manifolds is the family of generalised flag manifolds. As has
been known for a long time, these spaces admit a direct $q$-deformation in terms of the
Drinfeld–Jimbo quantum groups [21, 38, 41]. Somewhat more recently, it was shown
by Heckenberger and Kolb [11] that the Dolbeault double complex of the irreducible
flag manifolds survives this $q$-deformation intact. This result gives us one of the most
important families of noncommutative complex structures that we have, and as such,
provides an invaluable testing ground for any newly proposed theory of noncommu-
tative complex geometry.

Heckenberger and Kolb undertook their work in the absence of a general framework for
noncommutative complex geometry. While they produced an accomplished and compre-
hensive treatment of the $q$-deformed Dolbeault complexes, the fundamental processes at
work were obscured by the complexity of the calculations. Moreover, their technical style
of presentation is quite difficult to use as a basis for future work. Subsequent papers on
the geometry of the quantum flag manifolds would choose to follow a different approach
[20, 7].

In this paper we show that, for the special case of the quantum projective spaces, the
work of Heckenberger and Kolb can be understood in terms of our general framework
for noncommutative complex geometry. This allows for a significant simplification of
the required calculations, and helps identify some of the underlying general processes
at work. It is foreseen that this work will prove easily extendable to all the irreducible
quantum flag manifolds. Moreover, it is hoped to extend it even further to include all
the quantum flag manifolds, and in so doing, produce new examples of noncommuta-
tive complex structures. As mentioned above, it is also hoped to use this new simplified
presentation to identify noncommutative Hodge and Kähler structures hidden in the
Dolbeault complexes of Heckenberger and Kolb.
The paper is organised as follows: In section 2 we introduce some well-known material about quantum homogeneous spaces, Takeuchi’s categorical equivalence, covariant differential calculi, the framing result of Majid, and the classification result of Hermisson. The presentation will differ somewhat from standard in the presentation of Majid and Hermisson’s work.

In Section 3 we discuss the quantum special unitary group, its coquasi-triangular structure, and the quantum projective spaces. Moreover, we give an explicit presentation of the ideal corresponding to the Heckenberger–Kolb calculus for these spaces.

In Section 4 we introduce one of the fundamental results of the paper. We show that we can restrict Takeuchi’s equivalence to a monoidal equivalence between two subcategories of $\mathcal{M}_M$ and $\mathcal{M}_H^G$. Crucially, this allows us to take tensor products of framings.

In Section 5 we build upon this work and show how to frame the maximal prolongation of a covariant first-order differential calculus. We then show how our method can be greatly simplified by making a suitable choice of calculus on the total space.

In Section 6 we introduce a new variation on the existing definitions of noncommutative complex structure, and provide a simple set of necessary and sufficient conditions for such structures to exist.

Finally, in Section 7 we follow [4] and introduce a notion of integrability modeled directly on the classical case. We then construct a simple method for verifying integrability.

Throughout, the family of quantum projective spaces, endowed with the Heckenberger–Kolb calculus, is taken as the motivating set of examples. In each section, the newly constructed general theory is applied to these examples in detail, building up step by step, to a final explicit presentation of the $\varphi$-deformed Dolbeault double complexes.

\section{Preliminaries}

In this section we recall Takeuchi’s categorical equivalence for faithfully flat quantum homogeneous spaces, and some of the consequences of this result for the theory of covariant differential calculi. With the exception of the somewhat novel presentations of Majid’s framing theorem and Hermisson’s classification, all of the material found here is well-known.

\subsection{Quantum Homogeneous Spaces and Takeuchi’s Categorical Equivalence}

Let $G$ be a Hopf algebra with comultiplication $\Delta_G$, counit $\varepsilon_G$, antipode $S_G$, unit $1_G$, and multiplication $m_G$ (where no confusion arises, we will drop explicit reference to $G$ when denoting these operators). Throughout, we use Sweedler notation, as well as denoting $g^+ := g - \varepsilon(g)1$, for $g \in G$, and $V^+ = V \cap \ker(\varepsilon)$, for $V$ a subspace of $G$. For a right $G$-comodule $V$ with coaction $\Delta_R$, we say that an element $v \in V$ is coinvariant if $\Delta_R(v) = v \otimes 1$. We denote the subspace of all coinvariant elements by $V^G$, and call it
the coinvariant subspace of the coaction. More generally, a covariant subspace $W \subseteq V$ is defined to be a subspace that is also a sub-comodule of $V$.

For $H$ also a Hopf algebra, a homogeneous right $H$-coaction on $G$ is a coaction of the form $(\text{id} \otimes \pi) \circ \Delta$, where $\pi : G \to H$ is a Hopf algebra map. We call the coinvariant subspace $M := GH$ of such a coaction a quantum homogeneous space. As is easy to see, $M$ will always be a subalgebra of $G$. Moreover, it can be shown without difficulty that the coaction of $G$ restricts to a right $G$-coaction on $M$, and that

$$\pi(m) = \varepsilon(m)1_H,$$

(for all $m \in M$). (1)

In this paper we will always use the symbols $G, H, \pi$ and $M$ in this sense. We also note that $G$ is itself a trivial example of a quantum homogeneous space, where $\pi = \varepsilon$.

Let us now introduce $\mathcal{G}_M$, the category whose objects are the $M$-bimodules $E$ endowed with a left $G$-coaction $\Delta_L$, satisfying the compatibility condition

$$\Delta_L(me') = m(1)e(-1)m'(1) \otimes m(2)e(0)m'(2),$$

(for all $m, m' \in M, e \in E$),

and whose morphisms are both $M$-bimodule and left $G$-comodule maps. Moreover, let $\mathcal{M}_M^H$ denote the category whose objects $V$ are the right $M$-modules endowed with a right $H$-coaction satisfying the compatibility condition

$$\Delta_R(vm) = v(0)m(2) \otimes S(\pi(m(1)))v(1),$$

(for all $v \in V, m \in M$),

and whose morphisms are both left $M$-module and right $H$-comodule maps. In what follows, for sake of clarity, we will denote the right $M$-action on an object in $\mathcal{G}_M$ by juxtaposition, while we will denote the right $M$-action on an object in $\mathcal{M}_M^H$ by $\lhd$.

For any object $V$ in $\mathcal{M}_M^H$, we can associate to it a corresponding object in $\mathcal{G}_M$ as follows: Consider the coinvariant subspace $(G \otimes V)^H$, where $G \otimes V$ is endowed with the usual tensor product coaction. We can give $(G \otimes V)^H$ the structure of an object in $\mathcal{G}_M$ by defining right and left $M$-actions according to

$$m(\sum_i g^i \otimes v^i) = \sum_i mg^i \otimes v^i, \quad (\sum_i g^i \otimes v^i)m = \sum_i g^i m(1) \otimes (v^i \triangleleft m(2)),$$

and defining a left $G$-coaction according to

$$\Delta_L(\sum_i g^i \otimes v^i) = \sum_i g^i_{(1)} \otimes g^i_{(2)} \otimes v^i.$$

A framing, for an object $E$ in $\mathcal{G}_M$, is a pair $(V, t)$ where $V$ is an object in $\mathcal{M}_M^H$, and $t$ is an isomorphism from $E$ to $(G \otimes V)^H$. A natural question to ask is whether a framing exists for every object in $\mathcal{G}_M$, and when it does how many different choices of framing there are. In order to address this question, we will need to introduce some additional structures.
The right $M$-module structure of $E$ clearly restricts to a right $M$-module structure on $E/(M^+E)$. Moreover, it can be shown using (1) that the left $G$-module structure of $E$ induces a right $H$-comodule structure on $E/(M^+E)$ defined by

$$\Delta_R(e) = e(0) \otimes S(\pi(e_{-1})), \quad (e \in E),$$

(3)

where $\pi$ denotes the coset of $e$ in $E/(M^+E)$. To show that these two structures are compatible in the sense of (2) is routine. Thus, we have given $E/(M^+E)$ the structure of an object in $\mathcal{M}^H_M$. Consider now the functors

$$\Phi_M : \mathcal{G}M_M \to \mathcal{M}^H_M, \quad \Phi_M(E) = E/(M^+E),$$

$$\Psi_M : \mathcal{M}^H_M \to \mathcal{G}M_M, \quad \Psi_M(V) = (G \otimes V)^H.$$

Where for $f : E \to F$ a morphism in $\mathcal{G}M_M$, the morphism $\Phi_M(f) : \Phi_M(E) \to \Phi_M(F)$ is the function to which $f$ descends on $\Phi_M(E)$. While for $\varphi : V \to W$ a morphism in $\mathcal{M}^H_M$, we define $\Psi_M(\varphi) := 1 \otimes \varphi$. To show that both morphisms are well-defined is routine.

Moreover, using some basic linear algebra arguments, it can also be shown that, for $E, F$ two objects in $\mathcal{G}M_M$, and $V, W$ two objects in $\mathcal{M}^H_M$, we have

$$\Phi(E \oplus F) = \Phi(E) \oplus \Phi(F), \quad \Psi(V \oplus W) = \Psi(V) \oplus \Psi(W),$$

(4)

and if we further assume that $E \subseteq F$, and $V \subseteq W$, then

$$\Phi(E/F) = \Phi(E)/\Phi(F), \quad \Psi(V/W) = \Psi(V)/\Phi(W).$$

(5)

A natural question to ask is when this induces an equivalence of categories. This leads us to the notion of faithful flatness: We say that $G$ is a faithfully flat module over $M$ if the tensor product functor $G \otimes_M - : \mathcal{M} \to \mathcal{C}M$, from the category of left $M$-modules to the category of complex vector spaces, preserves and reflects exact sequences.

**Theorem 2.1 (Takeuchi [40])** Let $\pi : G \to H$ be a quantum homogeneous space for which $G$ is a faithfully flat right module over $M = G^H$. A natural isomorphism between $\Psi_M \circ \Phi_M$ and the identity is determined by

$$\text{frame}_M : E \to \Psi_M \circ \Phi_M(E), \quad e \mapsto e(0) \otimes \pi(1),$$

$$\text{frame}_M^\perp : \Phi_M \circ \Psi_M(V) \to V, \quad \sum_i g^i \otimes v^i \mapsto \sum_i \varepsilon(g^i)v^i,$$

(6)

(7)

giving an equivalence of categories between $\mathcal{G}M_M$ and $\mathcal{M}^H_M$.

Thus we see that we have a framing $(\Phi(E), \text{frame}_M)$ for every object $E$ in $\mathcal{G}M_M$. Now for any other framing $s : E \to (G \otimes V)^H$, for $V$ some object in $\mathcal{M}^H_M$, we have the isomorphism

$$\sigma := \text{frame}_M^\perp \circ \Phi_M(s) : \Phi_M(E) \to V,$$

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which gives us the re-expression \( s = \Psi_M(\sigma) \circ \text{frame}_M \). It follows that every framing of \( \mathcal{E} \) is of the form \((V, \Psi_M(\sigma) \circ \text{frame}_M)\), where \( V \) is some object in \( \mathcal{M}_M^H \), and \( \sigma : \Phi_M(\mathcal{E}) \to V \) is an isomorphism in \( \mathcal{M}_M^H \).

We should note that the original presentation of this work by Takeuchi uses somewhat different conventions. Most noticeably, the notion of cotens or product \( G \square_H \Phi_M(\mathcal{E}) \) is used instead of coinvariant subspace \((G \otimes \Phi_M(\mathcal{E}))^H \). However, as is easily seen, the two notions are equivalent. Another important point to note is that the existence of the isomorphism from \( \mathcal{E} \) to \( \Psi_M \circ \Phi_M(\mathcal{E}) \) does not depend on the assumption of faithful flatness, as the following lemma shows:

**Lemma 2.2** For a (not necessarily faithfully flat) quantum homogeneous space \( M \), the map \( \text{frame}_M \) is an isomorphism, with its inverse being given by

\[
\text{frame}_M^{-1} : \Psi_M \circ \Phi_M(\mathcal{E}) \to \mathcal{E}, \quad \sum_i g_i \otimes e_i \mapsto \sum_i g_i S(e_{(-1)})e_{(0)}.
\]

**Proof.** Let us begin by showing that \( \text{frame}_M^{-1} \) is well-defined: For \( \sum_i g_i \otimes m^i e^i \) an element in \((G \otimes \mathcal{E})^H\), with \( m^i \in M^+ \), for all \( i \), we have

\[
\sum_i f^i S((m^i e^i)_{(-1)})(m^i e^i)_{(0)} = \sum_i f^i S(m^i_{(1)} e^i_{(-1)}) m^i_{(2)} e^i_{(0)} = \sum_i f^i S(e^i_{(-1)}) S(m^i_{(1)}) m^i_{(2)} e^i_{(0)} = \sum_i \varepsilon(m^i) f^i S(e^i_{(-1)}) e^i_{(0)} = 0.
\]

Thus, \( \text{frame}_M^{-1} \) descends to a well-defined map on \((G \otimes \Phi_M(\mathcal{E}))^H\). That \( \text{frame}_M^{-1} \) is indeed the inverse of \( \text{frame}_M \) follows from

\[
\text{frame}_M^{-1} \circ \text{frame}_M(e) = \text{frame}_M^{-1}(e_{(-1)} \otimes \overline{e(0)}) = e_{-2} S(e_{(-1)}) e_{(0)} = \varepsilon(e_{(-1)}) e_{(0)} = e.
\]

Thus, we see that even in the absence of faithful flatness, a framing will exist for any \( \mathcal{E} \in \mathcal{M}_M \). However, without faithful flatness we are not guaranteed the existence of an inverse for \( \text{frame}_M^{-1} \).

### 2.2 Covariant First Order Differential Calculi

Let \( A \) be an algebra. (In what follows all algebras are assumed to be unital.) A first-order differential calculus over \( A \) is a pair \((\Omega^1, d)\), where \( \Omega^1 \) is an \( A-A \)-bimodule and \( d : A \to \Omega^1 \) is a linear map for which the Leibniz rule holds

\[
d(ab) = a(db) + (da)b, \quad (a, b, \in A),
\]
and for which $\Omega^1 = \text{span}_C\{adb | a, b \in A\}$. (Where no confusion arises we will drop explicit reference to $d$ and denote a calculus by its bimodule $\Omega^1$ alone.) We call an element of $\Omega^1$ a one-form. The universal first-order differential calculus over $A$ is the pair $(\Omega^1_u(A), d_u)$, where $\Omega^1_u(A)$ is the kernel of the product map $m : A \otimes A \rightarrow A$ endowed with the obvious bimodule structure, and $d_u$ is defined by

$$d_u : A \rightarrow \Omega^1_u(A), \quad a \mapsto 1 \otimes a - a \otimes 1.$$  

It is not difficult to show that every calculus over $A$ is of the form $(\Omega^1_u(A)/N, \text{proj} \circ d_u)$, where $N$ is a $A$-sub-bimodule of $\Omega^1_u(A)$, and $\text{proj} : \Omega^1_u(A) \rightarrow \Omega^1_u(A)/N$ is the canonical projection. Moreover, this association between calculi and sub-bimodules is bijective.

A differential calculus $\Omega^1(A)$ over a left $G$-comodule $A$ is said to be left-covariant if there exists a (necessarily unique) left-coaction $\Delta_L : \Omega^1(A) \rightarrow G \otimes \Omega^1(A)$ such that

$$\Delta_L(ab) = \Delta(a)(\text{id} \otimes d)\Delta(b), \quad (a, b \in A).$$

Clearly this can happen if, and only if, the corresponding sub-bimodule $N \subseteq \Omega^1_u(A)$ is left-covariant, giving us a correspondence between left-covariant calculi and left-covariant sub-bimodules of $\Omega^1_u(A)$. Furthermore, for $M$ the base of a quantum homogeneous space, any left-covariant calculus has the structure of an object in $G_M^H M_M$. Thus, Takeuchi’s theorem induces a correspondence between left-covariant calculi $\Omega^1(M)$ and sub-objects of $\Phi_M(\Omega^1_u(M))$ in $M_M^H$.

A problem with this last classification is that our generator and relation presentation of $\Phi_M(\Omega^1_u(M))$ is not particularly easy to work with. However, the following very useful result tells us that there is an isomorphism between $\Phi_M(\Omega^1_u(M))$ and $M^+$, where we consider $M^+$ as an object in $M_M^H$ according to the obvious right $M$-module structure, and the right $H$-comodule structure defined by $\Delta_{M,R}(m) = m_{(2)} \otimes S(\pi(m_{(1)}))$, for $m \in M^+$. (Note that the proof given here does not assume that $G$ is a Hopf–Galois extension of $M$ as is done in [24]. However, this more general result is implicit in the original proof.)

**Theorem 2.3** [Majid [24]] For a (not necessarily faithfully flat or Hopf–Galois) quantum homogeneous space $M$, we have an isomorphism

$$\sigma : \Phi_M(\Omega^1_u(M)) \rightarrow M^+, \quad \sum_i m^i dn^i \mapsto \sum_i \varepsilon(m^i)(n^i)^+, \quad (9)$$

and a corresponding framing $(M^+, s := \Psi(\sigma) \circ \text{frame}_M)$, which we call the canonical framing. Explicitly $s$ acts according to

$$s : \Omega^1_u(M) \rightarrow (G \otimes M^+)^H, \quad mdn \mapsto mn_{(1)} \otimes (n_{(2)})^+. \quad (10)$$

**Proof.** We begin by showing that the map $\sigma$ is well-defined as a right $M$-module map: Consider the right $M$-module map

$$\varepsilon \otimes \text{id} : M \otimes M \rightarrow M, \quad m \otimes n \mapsto \varepsilon(m)n.$$  

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Now that we have shown that it is clear that this restriction is exactly the map \( \sigma \) defined in (2). That \( \sigma \) is also a right \( H \)-comodule map is clear from

\[
(\sigma \otimes \text{id}) \circ \Delta_R(mdn) = (\sigma \otimes \text{id})(m \otimes n - mn \otimes 1) = \varepsilon(m)(n - \varepsilon(n)1) = \varepsilon(m)n^+,
\]

it is clear that this restriction is exactly the map \( \sigma \) defined in (2). That \( \sigma \) is also a right \( H \)-comodule map is clear from

\[
(\sigma \otimes \text{id}) \circ \Delta_R(mdn) = (\sigma \otimes \text{id})(m(2)dn(2) \otimes S(\pi(m(1)n(1))))
\]

For a faithfully flat quantum homogeneous space \( \text{Corollary 2.4 (Hermisson [12])} \)

Combining this result with the classification of covariant calculi on quantum homogeneous spaces discussed earlier, gives us the classification result of Hermisson:

**Corollary 2.4 (Hermisson [12])** For a faithfully flat quantum homogeneous space \( M \), there is a bijective correspondence between left-covariant first-order differential calculi over \( M \), and the sub-objects of \( M^+ \) in \( \mathcal{M}_M^H \).

Now for such a calculus \( \Omega^1(M) \simeq \Omega^1_u(M)/N \), with its corresponding ideal \( \sigma(\Phi_M(N)) \), the canonical framing clearly descends to a framing

\[
s : \Omega^1(M) \to (G \otimes V_M)^H, \quad ab \mapsto ab(1) \otimes (b(2))^+,
\]
where we have denoted $V_M := M^+/\sigma(\Phi_M(N))$. We will call $V_M$ the cotangent space of the calculus, and we again call $s$ the canonical framing of the calculus. We define the dimension of the calculus to be the dimension of $V_M$. It is easy to see from \([8]\) that an explicit presentation of the inverse of the canonical framing is given

$$s^{-1} : (G \otimes V_M)^H \to \Omega^1(M), \quad \sum_i f^i \otimes v^i \mapsto \sum_i f^i S(v^i_1)dv^i_2. \quad (11)$$

If we drop the assumption of faithful flatness, then because of Lemma 2.2 we will still have a corresponding framing for every covariant calculus. However, we are not guaranteed an equivalence between calculi and ideals. This is essentially what is established in Majid’s second framing theorem in \([20]\). For the special case of the trivial quantum homogeneous space $G$ (where the faithful flatness condition is trivial), the results of Majid and Hermisson reduce to Woronowicz’s celebrated classification of left-covariant calculi over a Hopf algebra $G$. For such a calculus $\Omega^1(G)$, we will denote its cotangent space by $\Lambda^1_G$, and call it the space of left-invariant one forms.

If $(\Omega^1, d)$ is a differential calculus over a $*$-algebra $A$ such that the involution of $A$ extends to an involutive conjugate-linear map $*$ on $\Omega^1$, for which $(a b)^* = (b^*) a^*$, for all $a, b \in A$, then we say that $(\Omega^1, d)$ is a first-order differential $*$-calculus. It is easy to see that the universal calculus $\Omega^1_u(A)$ over any $*$-algebra $A$ always has a unique $*$-calculus structure. Moreover, any non-universal calculus of the form $\Omega^1(A) = \Omega^1(A)/N$ is a $*$-calculus if, and only if, $N^* = N$.

Let us now assume that both $G$ and $H$ are Hopf $*$-algebras, and that $\pi$ is a Hopf $*$-algebra map. It is easy to see that in this case $M$ is a $*$-subalgebra of $G$. In general it is not known how to tell that a calculus $\Omega^1(M)$ over $M$ is a $*$-calculus, directly from the corresponding sub-object of $M^+$. However, we can show without too difficulty, that for the universal $*$-calculus $\Omega^1_u(G)$, the corresponding $*$-map on $G \otimes G^+$ acts according to

$$* : G \otimes G^+ \to G \otimes G^+, \quad g \otimes v \mapsto g^* \otimes S(v^*) g^*.(12)$$

Thus, for $\Omega^1(G)$ a non-universal calculus over $G$, with corresponding sub-object $I_G \subseteq G^+$, we have that $\Omega^1(G)$ is a $*$-calculus if, and only if,

$$\{S(v^*) | v \in I_G\} = I_G.$$

Now if $\Omega^1(G)$ restricts to $\Omega^1(M)$ on $M$, then since $(m n)^* = d(n^*) m^* \in \Omega^1(M)$, for any $m, n \in M$, the $*$-structure on $\Omega^1(G)$ must induce a $*$-structure on $\Omega^1(M)$. This provides us with a crude method for establishing that $\Omega^1(M)$ has a $*$-structure.

2.3 Total Differential Calculi

We now come to noncommutative higher differential forms: For $(Y, +)$ a group, a $Y$- graded algebra is an algebra of the form $A = \bigoplus_{y \in Y} A_y$, where each $A_y$ is a linear subspace of $A$, and $A_y A_z \subseteq A_{y+z}$, for all $y, z \in Y$. If $a \in A_y$, then we say that $a$ is
a homogeneous element of degree $y$. A homogenous mapping of degree $d$ on $A$ is a linear mapping $L : A \to A$ such that if $a \in A^y$, then $L(a) \in A^{y+d}$. We say that a subspace $B$ of $A$ is homogeneous if it admits a decomposition $B = \oplus_{y \in Y} B^y$, with $B^y \subseteq A^y$, for all $y \in Y$.

A triple $(A, \partial, \overline{\partial})$ is called a double complex if $A$ is an $\mathbb{N}_0^2$-graded algebra, $\partial$ is homogeneous mapping of degree $(1, 0)$, $\overline{\partial}$ is homogeneous mapping of degree $(0, 1)$, and

$$\partial^2 = \overline{\partial}^2 = 0, \quad \partial \circ \overline{\partial} = -\overline{\partial} \circ \partial.$$ 

A graded derivation $d$ on an $\mathbb{N}_0^0$-graded algebra $A$ is a homogenous mapping of degree 1 that satisfies the graded Liebniz rule

$$d(ab) = d(a)b + (-1)^n a dB,$$

(for all $a \in A^n$, $b \in A$).

A pair $(A, d)$ is called a differential algebra if $A$ is an $\mathbb{N}_0^0$-graded algebra and $d$ is a graded derivation on $A$ such that $d^2 = 0$. The operator $d$ is called the differential of the algebra.

**Definition 2.5.** A total differential calculus over an algebra $A$ is a differential algebra $(\Omega(A), d)$, such that $\Omega^0 = A$, and

$$\Omega^k = \text{span}_{\mathbb{C}} \{a_0 da_1 \wedge \cdots \wedge da_k \mid a_0, \ldots, a_k \in A\}. \quad (13)$$

Following the classical example of the de Rham complex, we will always use $\wedge$ to denote the multiplication between total calculus elements, both of order greater than or equal to 1.

In commutative geometry the higher forms are constructed as exterior powers of the one-forms. In the noncommutative setting such a construction is not in general well-defined. However, there exists an alternative formulation of the higher forms which is well-defined for noncommutative algebras: For $(\Omega^1(A), d)$ a first-order differential calculus with corresponding sub-bimodule $N \subseteq \Omega^1(A)$, denote by $\Omega^\bullet(A)$ the quotient of the tensor algebra $\bigoplus_{k=0}^{\infty} (\Omega^1(A))^{\otimes A^k}$ by $\langle d(N) \rangle$, where $\langle d(N) \rangle$ is the subalgebra of the tensor algebra generated by $d(N)$. As a little thought will confirm, the exterior derivative $d$ has a unique extension to a map $d : \Omega^\bullet(A) \to \Omega^\bullet(A)$, such that $(\Omega^\bullet(A), d)$ has the structure of a total differential calculus. We call this total differential calculus the maximal prolongation of $(\Omega^1(A), d)$. The maximal prolongation is easily seen to be unique, in the sense that any other calculus extending $(\Omega^1(A), d)$ can be obtained as a quotient of the maximal prolongation by an ideal of $\ker(d)$. It is clear that $\langle d(N) \rangle$ is homogeneous with respect to the $\mathbb{N}_0$-grading of the tensor algebra. We will denote the corresponding decomposition by

$$\langle d(N) \rangle = \bigoplus_{n \in \mathbb{N}_{\geq 2}} \langle d(N)_k \rangle. \quad (14)$$

As is well known and easily seen, each $\langle d(N)_k \rangle$ is an object in $\mathcal{G}_M \mathcal{M}_M$. This means that the natural comodule structure of the tensor algebra descends to a comodule structure on
\(\Omega^\bullet(A)\), giving it the structure of an object in \(G_M\). For the special case of the universal calculus, its maximal prolongation is just its tensor algebra. An important point to note is that the maximal prolongation of \(\Omega^1(A)\) can also be constructed as the quotient of the tensor algebra of \(\Omega^1_u(A)\) by the subalgebra \(\langle N + dN \rangle\), with the total derivative being obtained by restriction.

If \((\Omega^\bullet, d)\) is a differential calculus over a \(*\)-algebra \(A\) such that the involution of \(A\) extends to an involutive conjugate-linear map \(*\) on \(\Omega^\bullet\), for which \((d\omega)^* = d\omega^*, \text{ for all } \omega \in \Omega\), and

\[
(\omega_p \omega_q)^* = (-1)^{pq} \omega_q^* \omega_p^*, \quad \text{for all } \omega_p \in \Omega^p, \omega_q \in \Omega^q,
\]

then we say that \((\Omega, d)\) is a \(\text{total } *\)-differential calculus. It is easy to see that if \(\Omega^1\) is a first order \(*\)-calculus, then its maximal prolongation is canonically a total \(*\)-calculus.

### 3 The Quantum Projective Spaces

In this section we introduce the Heckenberger–Kolb first-order differential calculus for \(\mathbb{C}_q[\mathbb{C}P^{N-1}]\). With the exception of the generating set for the calculus ideal given in Subsection 3.3, this material is all quite well-known. We begin by describing the quantum unitary group \(\mathbb{C}_q[U_N]\) and the quantum special unitary group \(\mathbb{C}_q[SU_N]\), then we define the quantum projective spaces, and finally we introduce the calculus itself \(\Omega^1(\mathbb{C}P^{N-1})\).

#### 3.1 The Quantum Special Unitary Group \(\mathbb{C}_q[SU_N]\)

We begin by fixing notation and recalling the various definitions and constructions needed to introduce the quantum unitary group and the quantum special unitary group. (Where proofs or basic details are omitted we refer the reader to [19, 33].)

For \(q \in (0, 1]\) and \(\nu := q - q^{-1}\), let \(\mathbb{C}_q[M_N]\) be the quotient of the free noncommutative algebra \(\mathbb{C}\left\langle u^i_j, \mid i, j = 1, \ldots, N \right\rangle\) by the ideal generated by the elements

\[
\begin{align*}
 u_k^i u_k^j &- q u_k^j u_k^i, & u_k^i u_k^j - q u_k^j u_k^i, & \quad (1 \leq i < j \leq N); \\
 u_l^i u_l^j &- u_l^j u_l^i, & u_l^i u_l^j - u_l^j u_l^i - \nu u_l^j u_l^i, & \quad (1 \leq i < j \leq N, 1 \leq k < l \leq N).
\end{align*}
\]

These generators can be more compactly presented as

\[
\sum_{w,x=1}^{N} R_{wx}^{ac} u^w_b u^x_d - \sum_{y,z=1}^{N} R_{bd}^{yz} u^y_a u^z_c, \quad (1 \leq a, b, c, d \leq N), \quad (15)
\]

where, for \(H\) the Heaviside step function with \(H(0) = 0\), we have denoted

\[
R_{ijl} = q^\delta_{ij} \delta_{il} \delta_{kj} + \nu H(k - i) \delta_{ij} \delta_{kl}. \quad (16)
\]
We can put a bialgebra structure on $C_q[M_N]$ by introducing a coproduct $\Delta$, and counit $\varepsilon$, uniquely defined by $\Delta(u^i_j) := \sum_{k=1}^N u^i_k \otimes u^k_j$, and $\varepsilon(u^i_j) := \delta_{ij}$. The quantum determinant of $C_q[M_N]$ is the element

$$\det_N := \sum_{\pi \in S_N} (-q)^{\ell(\pi)} u^1_{\pi(1)} u^{2}_{\pi(2)} \cdots u^N_{\pi(N)},$$

where summation is taken over all permutations $\pi$ of the set of $N$ elements, and $\ell(\pi)$ is the length of $\pi$. As is well-known, $\det_N$ is a central and grouplike element of the bialgebra. The centrality of $\det_N$ makes it easy to adjoin an inverse $\det^{-1}_N$. Both $\Delta$ and $\varepsilon$ have extensions to this larger algebra, which are uniquely determined by $\Delta(\det^{-1}_N) = \det^{-1}_N \otimes \det^{-1}_N$, and $\varepsilon(\det^{-1}_N) = 1$. The result is a new bialgebra which we denote by $C_q[GL_N]$. We can endow $C_q[GL_N]$ with a Hopf algebra structure by defining

$$S(\det^{-1}_N) = \det_N, \quad S(u^i_j) = (-q)^{i-j} \sum_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} u^{i_1}_{\pi(1)} u^{i_2}_{\pi(2)} \cdots u^{i_{N-1}}_{\pi(l_{N-1})} \det^{-1}_N,$$

where $\{k_1, \ldots, k_{N-1}\} = \{1, \ldots, N\} \setminus \{j\}$, and $\{l_1, \ldots, l_{N-1}\} = \{1, \ldots, N\} \setminus \{i\}$ as ordered sets. Moreover, we can give $C_q[GL_N]$ a Hopf *-algebra structure by setting $\det^{-*}_N = \det_N$, and $(u^i_j)^* = S(u^j_i)$. We denote this Hopf *-algebra by $C_q[U_N]$, and call it the quantum unitary group of order $N$. If we quotient $C_q[U_N]$ by the ideal $\langle \det^{-1}_N \rangle$, then the resulting algebra is again a Hopf *-algebra. We denote it by $C_q[SU_N]$, and call it the quantum special unitary group of order $N$.

As is well-known [33], for each $N^{th}$-root $q^{1/N}$ of $q$, we have a map

$$r : C_q[SU_N] \otimes C_q[SU_N] \to C, \quad u^i_j \otimes u^k_l \to q^{-\frac{1}{N}} R^{kl}_{ij},$$

which we call the coquasi-triangular structure map of $C_q[SU_N]$, for $q^{1/N}$. We can use $r$ to define a family of maps $\{Q_{kl} | k, l = 1, \ldots, N\}$ by setting

$$Q_{kl} : C_q[SU_N] \to C, \quad f \mapsto \sum_{a=1}^N r(u^a_k \otimes f(1)) r(f(2) \otimes u^a_l).$$

Using this family of maps, an $N^2$-dimensional representation $Q$ can be defined by

$$Q : C_q[SU_N] \to M_N(C) \quad h \mapsto [Q_{kl}(h)]_{kl}.$$

We call $Q$ the quantum Killing representation of $C_q[SU_N]$. Explicit formulae for the action of $Q$ on some distinguished elements of $C_q[SU_N]$ can be found in [33].

### 3.2 The Quantum Projective Spaces $C_q[C^P^{N-1}]$

We are now ready to introduce the quantum projective spaces. As mentioned earlier, they form a subfamily of the quantum flag manifolds, and will serve as our motivating set of examples. We use a description, introduced in [27], that presents quantum $(N-1)$-projective space as the coinvariant subalgebra of a $C_q[U_{N-1}]$-coaction on $C_q[SU_N]$. This subalgebra is a $q$-deformation of the coordinate algebra of the complex manifold $SU_N/U_{N-1}$. (Recall that classically $C^P^{N-1}$ is isomorphic to $SU_N/U_{N-1}$.)
Definition 3.1. Let $\alpha_N : C_q[SU_N] \to C_q[U_{N-1}]$ be the surjective Hopf algebra map defined by setting $\alpha_N(u_i^1) = \text{det}_{N-1}^{-1}$; $\alpha_N(u_i^1) = 0$, for $i = 2, \ldots, N$; and $\alpha_N(u_i^j) = u_j^{-1}$, for $i, j = 2, \ldots, N$. Quantum projective $(N-1)$-space $C_q[CP^{N-1}]$ is defined to be the coinvariant subspace of the corresponding homogeneous coaction $\Delta_{SU_N, \alpha_N} = (\text{id} \otimes \alpha_N) \circ \Delta$, that is,

$$C_q[CP^{N-1}] := \{ f \in C_q[SU_N] \mid \Delta_{SU_N, \alpha_N}(f) = f \otimes 1 \}.$$ 

Now let us consider the element $z_{ij} := u_i^1 S(u_j^1)$. From the following calculation, we can see that $z_{ij}$ is contained in $C_q[CP^{N-1}]$, for all $i, j = 1, \ldots, N$:

$$\Delta_{SU_N}(z_{ij}) = \Delta_{SU_N}(u_i^1 S(u_j^1)) = (\text{id} \otimes \alpha_N)(\sum_{a,b=1}^N u_a^i S(u_b^j) \otimes u_a^i S(u_b^j))$$

$$= \sum_{a,b=1}^N u_a^i S(u_b^j) \otimes \alpha_N(u_a^i S(u_b^j)) = u_a^i S(u_b^j) \otimes \alpha_N(u_a^i S(u_b^j))$$

$$= u_a^i S(u_b^j) \otimes \text{det}_{N-1}^{-1} \text{det}_N = z_{ij} \otimes 1.$$ 

Moreover, it can be shown that $C_q[CP^{N-1}]$ is generated as a $C$-algebra by the set \{$z_{ij} \mid i, j = 1, \ldots, N$\}. (See [30, 19] for more details.)

As one would hope, $C_q[SU_N]$ is a faithfully flat module over $C_q[CP^{N-1}]$. This was originally established in [29].

An important family of examples of objects in $SU_{CP^{N-1}}$, $M_{CP^{N-1}}$, is the quantum line bundles $\mathcal{E}_p$, for $p \in \mathbb{Z}$: The module $\mathcal{E}_p$ is defined to be $\Psi_{CP^{N-1}}(C)$, where $C$ considered as an object in $M_{CP^{N-1}}$, according to the unique $C[U_{N-1}]$-coaction for which $\lambda \mapsto \lambda \otimes \text{det}_{N-1}^{-p}$, for $\lambda \in C$. Clearly, we have that $\mathcal{E}_0 = C_q[CP^{N-1}]$.

### 3.3 The Heckenberger–Kolb Calculus $\Omega_q^1(CP^{N-1})$

Let us now introduce an ideal of $C_q[CP^{N-1}]$ that will play a central role in the rest of this paper:

Lemma 3.2 The ideal

$$I_{CP^{N-1}} := \langle z_{ij}, z_{il}z_{kl}, z_{lk}z_{kl} \mid i, j = 2, \ldots, N; k, l = 1, \ldots, N, (k, l) \neq (1, 1) \rangle$$

is covariant with respect to the $C_q[U_{N-1}]$-coaction $\Delta_{CP^{N-1}}$. 

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We call the corresponding calculus the Heckenberger–Kolb calculus and denote it by $\Omega^1_q(\mathbb{C}P^{N-1})$. (For a proof that this is indeed the calculus identified in the classification result [10] of Heckenberger and Kolb see [30] [32].)

We would now like to find a basis for $\ker(Q)$. A convenient way to do this is by embedding $\Omega^1_q(\mathbb{C}P^{N-1})$ into a calculus on $\mathbb{C}_q[SU_N]$: We recall that $\ker(Q)$ is a right ideal of $\mathbb{C}_q[\mathbb{C}P^{N-1}]^+$, and that the corresponding calculus is the well-known bicovariant calculus on $\mathbb{C}_q[SU_N]$ first introduced in [15] (see [23] for details). Moreover, it was shown in [30] [32] that

$$I_{SU_N} := \{ \ker(Q) + u^i_j \mid i, j \geq 2 \}$$

is also a right ideal of $\mathbb{C}_q[SU_N]^+$. Hence we have a corresponding calculus on $\mathbb{C}_q[SU_N]$ which we denote by $\Omega^1_q(SU_N)$, and whose space of left-covariant one-forms we denote by $\Lambda^1_{SU_N} := \mathbb{C}_q[SU_N]^+ / I_{SU_N}$. It was also shown in [30] that a basis of $\Lambda^1_{SU_N}$ is given by

$$e_i^+ := u_i^{1+1}, \quad e_i^0 := u_i^1 - 1, \quad e_i^- := u_i^{i+1}, \quad (i = 1, \ldots, N - 1).$$

For $i = 2, \ldots, N; j = 3, \ldots, N; i < j$, and $k = 1, \ldots, N$, all the non-zero actions of the generators $e_i^\pm$ are given by

$$e_i^+ \triangleleft u_{j+1}^{j+1} = q^{\alpha_{ij} + \delta_{ij} - \frac{2}{q}} e_i^\pm, \quad e_i^+ \triangleleft u_{i+1}^{j+1} = q^{-\frac{2}{q}} \nu e_j^+ \nu_i, \quad e_i^- \triangleleft u_{j+1}^{j+1} = q^{\frac{2}{q}} \nu e_j^- \nu_i, \quad (18)$$

while, the non-zero actions of the antipodes of the generators are given by

$$e_i^+ \triangleleft S(u_{i+1}^j) = -q^{\frac{2}{q}} \nu e_j^+ \nu_i, \quad e_i^- \triangleleft S(u_{i+1}^j) = -q^{2(i-j+1+\frac{1}{q})} \nu e_j^- \nu_i \quad (19)$$

Moreover, we have the relations

$$S(u_i^1) = -q^{\frac{2}{q} - 1} e_i^+ = -q^{\frac{2}{q} - 1} e_i^-.$$ (21)

and, that $u_i^1 \triangleright f = q^{2\delta_{ki} - \frac{2}{q}} f$, and $S(u_i^1) \triangleright f = q^{\frac{2}{q} - 2\delta_{ki}} f$, for all $f \in \mathbb{C}_q[SU_N]$. We are now ready to find a basis for $V_{\mathbb{C}P^{N-1}}$.
Lemma 3.3 The canonical map \( \hat{\iota} : V_{\mathbb{C}P^{N-1}} \rightarrow \Lambda^{1}_{SU_{N}} \) is an embedding, with respect to which a \( 2(N-1) \)-dimensional basis of \( V_{\mathbb{C}P^{N-1}} \) is given by
\[
\{ e_{i}^{\pm} | i = 1, \ldots, N-1 \}. \tag{22}\]

Proof. It follows easily from (18) and (19) that \( I_{\mathbb{C}P^{N-1}} \) is contained in \( I_{SU_{N}} \), and so, \( \hat{\iota} \) is well-defined. Next we note that a spanning set for \( V_{\mathbb{C}P^{N-1}} \) is given by \( \{ z_{i1}, z_{1i} | i = 2, \ldots, N \} \). Using (18) and (19) again, it is trivial to show that
\[
z_{i1} = q_{2i-1} e_{i-1}^{+}, \quad z_{1i} = q_{1-2i} e_{i-1}^{-}, \quad (i = 2, \ldots, N). \tag{23}\]
This means that \( \{ z_{i1}, z_{1i} | i = 2, \ldots, N \} \) is a linearly independent set; that we have an embedding of \( V_{\mathbb{C}P^{N-1}} \) in \( \Lambda^{1}_{SU_{N}} \); and that the set given in (22) is a basis of \( \Omega_{q}^{1}(\mathbb{C}P^{N-1}) \).
\[\square\]

4 Tensor Products and Framings

A natural question to ask is whether or not one can extend the canonical framing of a left-covariant calculus \( \Omega^{1}(M) \) to a framing for its tensor powers \( (\Omega^{1}(M))^\otimes M^{k} \), for any \( k \in \mathbb{N} \). In this section we will use Takeuchi’s categorical equivalence to show that, for a distinguished class of calculi, this can indeed be done.

4.1 A Monoidal Equivalence of Categories

The category \( \mathcal{G}_{M} \) has a natural monoidal structure \( \otimes_{M} \), where for \( E, F \) two objects in \( \mathcal{G}_{M} \), we define \( E \otimes_{M} F \) to be the usual bimodule tensor product endowed with the obvious left \( G \)-comodule structure
\[
\Delta_{L} : E \otimes_{M} F \rightarrow G \otimes E \otimes_{M} F, \quad e \otimes_{M} f \mapsto e(-1) f(-1) \otimes e(0) \otimes_{M} f(0). \tag{24}\]
However, for \( \mathcal{M}_{H} \) no such obvious monoidal structure exists. This leads us to consider a particular subcategory of \( \mathcal{M}_{H} \) defined as follows: Let \( \mathcal{M}_{0}^{H} \) be the strictly full monoidal subcategory of \( \mathcal{M}_{H}^{\otimes M} \) whose objects \( V \) are those endowed with the trivial right action
\[
v \triangleright m = \varepsilon(m) v, \quad (v \in V, m \in M). \]
This category has a natural monoidal structure \( \otimes \), where for \( V, W \) two objects in \( \mathcal{M}_{0}^{H} \), we define \( V \otimes W \) to be the usual vector space tensor product, endowed with the trivial right \( H \)-comodule structure given by
\[
\Delta_{R} : V \otimes W \rightarrow V \otimes W \otimes H, \quad v \otimes w \mapsto v(0) \otimes w(0) \otimes w(1) v(1). \tag{25}\]
That these two structures are compatible in the sense of (2) follows easily from (1).

One should now of course ask what the corresponding subcategory of \( \mathcal{G}_{H} \mathcal{M}_{M} \) is. As a candidate we propose the strictly full subcategory whose objects \( E \) are those satisfying
\( E M^+ \subseteq M^+ E \). As a moment’s thought will confirm, for \( E, F \) two objects in \( G_M \), their tensor product \( E \otimes_M F \) is still an object in \( G_M \). Thus it is clear that \( G_M \) is a monoidal subcategory of \( G_M \). Moreover, as the following theorem demonstrates, it is monoidally equivalent to \( \mathcal{M}_0^H \).

**Theorem 4.1** The functor \( \Phi_M \) restricts to an equivalence of categories between \( G_M \) and \( \mathcal{M}_0^H \). Moreover, for any two objects \( E, F \in G_M \), the natural transformation

\[
\mu_{E,F} : \Phi_M(E \otimes_M F) \to \Phi_M(E) \otimes \Phi_M(F), \quad \overline{v \otimes_M w} \mapsto \overline{v} \otimes \overline{w},
\]

(26) gives an equivalence of monoidal categories between \( G_M \) and \( \mathcal{M}_0^H \).

**Proof.** Let us first show that \( \Phi \) restricts to an equivalence of categories between \( G_M \) and \( \mathcal{M}_0^H \): If \( E \) is an object in \( G_M \), then for any \( e \in E \), and \( m \in M^+ \), we must have, from the definitions of \( G_M \) and \( \Phi_M(E) \), that \( \mathcal{E} \triangleleft m = 0 \). Hence, for any \( n \in M \), we have

\[
\mathcal{E} \triangleleft n = \mathcal{E} \triangleleft (n^+ + \mathcal{E}(n)1) = \mathcal{E} \triangleleft n^+ + \mathcal{E} \triangleleft (\mathcal{E}(n)1) = \mathcal{E}(n)\mathcal{E}.
\]

Thus, \( \Phi_M(E) \) is well-defined as an object in \( \mathcal{M}_0^H \). Conversely, if \( V \) is an object in \( \mathcal{M}_0^H \), then for any element \( \sum_i f^i \otimes v^i \) in \( \Psi(V) \), the right action of \( M \) on \( \Psi_M(V) \) must operate according to

\[
(\sum_i f^i \otimes v^i)m = \sum_i f^i m(1) \otimes (v^i \triangleleft m(2)) = \sum_i f^i m(1) \mathcal{E}(m(2)) \otimes v^i = \sum_i f^i m \otimes v^i.
\]

Now if \( m \in M^+ \), then \( \sum_i f^i m \otimes v^i \) must be an element of \( \text{ker}(\text{frame}_M^+ \mathcal{F}) \). But since \( \text{ker}(\text{frame}_M^+ \mathcal{F}) \) is equal to \( M^+ \Psi_M(V) \), we must have that \( (\sum_i f^i \otimes v^i)m \) is contained in \( M^+ \Psi_M(V) \). Hence \( \Psi_M(V) \) is well-defined as an object in \( G_M \). That gives an equivalence of categories now follows from the fact that \( \Phi_M : G_M \to \mathcal{M}_0^H \) is an equivalence of categories, and that \( G_M \), and \( \mathcal{M}_0^H \), are both full subcategories of \( G_M \), and \( \mathcal{M}_0^H \), respectively.

We now turn to showing that \( \mu_{E,F} \) is a natural isomorphism: It is trivial that \( \mu_{E,F} \) is well-defined as a right \( M \)-module map. To see that it is also a \( H \)-comodule map, note first that the right comodule structure on \( \Phi_M(E \otimes_M F) \) acts according to

\[
\Delta_R : e \otimes_M F \mapsto e(0) \otimes_M f(0) \otimes S(e(-1)f(-1)), \quad (e \in E, f \in F).
\]

By (26), the right comodule structure on \( \Phi_M(E) \otimes \Phi_M(F) \) acts according to

\[
\Delta_R : \overline{e} \otimes \overline{f} \mapsto \overline{e(0)} \otimes \overline{f(0)} \otimes S(e(-1))S(f(-1)), \quad (e \in E, f \in F).
\]

Hence, \( \mu_{E,F} \) is indeed a morphism in \( \mathcal{M}_0^H \). It remains to show that the inverse morphism, which would send \( \overline{v} \otimes \overline{w} \) to \( v \otimes w \), is well-defined. But this follows directly from the fact that

\[
(M^+ E) \otimes_M F + E \otimes_M (M^+ F) = M^+(E \otimes_M F).
\]

This result allows us to identify \( \Phi_M(\Omega^1(M)^{\otimes_M k}) \) and \( \Phi_M(\Omega^1(M))^{\otimes k} \), and gives us the following corollary.
Corollary 4.2 Let $\Omega^1(M)$ be a left-covariant first-order differential calculus with canonical framing $(V_M, \sigma)$. If $\Omega^1(M)$ is contained in the subcategory $\mathcal{M}_0^H$, then we have a framing $(V_M^{\otimes k}, \sigma^k)$, where
\[ \sigma^k : \Phi_M(\Omega^{\otimes M^k}(M)) \to V_M^{\otimes k}, \quad \omega_1 \otimes \cdots \otimes \omega_k \mapsto \sigma(\omega_1) \otimes \cdots \otimes \sigma(\omega_k). \]

4.2 Framing Calculi

For $\Omega^1(G)$ a left-covariant differential calculus over a Hopf algebra $G$, it can quite often happen that $\Omega^1(G)$ is not an object in $\mathcal{G}_G M_0$, meaning we cannot frame its tensor powers using the above approach. An obvious example is the calculus $\Omega^1_q(SU_N)$ introduced in Section 3. For such calculi consider the framing $((\Lambda^1_G)^{\otimes k}, t^k)$, where
\[ t^k : (\Lambda^1_G)^{\otimes G} : (\Omega^1(G)^{\otimes G}) \to (\Lambda^1_G)^{\otimes k}, \quad (k \geq 2), \]
with $c^k : (G \otimes \Lambda^1_G)^{\otimes G} \to G \otimes (\Lambda^1_G)^{\otimes k}$ the obvious identification. We denote the corresponding isomorphism in $\mathcal{M}_G G$ by
\[ \tau^k : \Phi_G((\Omega^1(G)^{\otimes G}) \to (\Lambda^1_G)^{\otimes k}. \]
Explicitly, $\tau^k$ acts on $g^1 dg^2 \otimes_G dg^3 \otimes_G \cdots \otimes_G dg^k$ to give
\[ \varepsilon(g^0)(g^1)^+g^0(1) \cdots g^0(k-1)^+g^0(1)^+g^0(2)^+ \cdots g^0(k-1)^+ \otimes (g^1)^+ \cdots (g^k)^+). \]
As we shall now show, for certain distinguished calculi on $G$, we can use $\tau^k$ to give a new framing for tensor powers of $\Omega^1(M)$:

Definition 4.3. For any first-order differential calculus $\Omega^1(M)$ over $M$, a framing calculus $\Omega^1(G)$ is a first-order differential calculus for $G$ such that

1. $\Omega^1(G)$ restricts to $\Omega^1(M)$ on $M$, by which we mean
\[ \Omega^1(M) = \{ \sum_i m^i dn^i \in \Omega^1(G) \mid m^i, n^i \in M, \text{ for all } i \}; \]

2. $\Omega^1(M)G \subseteq G \Omega^1(M)$.

Now $\Omega^1(M)$ and $\Omega^1(G)$ live in two ostensibly different categories. For sake of clarity, we should spend a little time exploring the relationship between $\mathcal{G}_G M_G$ and $G_M M_M$; as well as the relationship between $\mathcal{G}_G M_G$ and $\mathcal{M}_M^H$. First we note that, since every $G$-$G$-bimodule is obviously an $M$-$M$-bimodule, we have the forgetful inclusion of $\mathcal{G}_G M_G$ in $G_M M_M$, which remembers only the $M$-$M$-bimodule structure of the objects of $G_M M_G$. On the other side of Takeuchi’s equivalence, it is easy to see that the only coaction on a right $G$-module that is compatible in the sense of $[2]$, is the trivial coaction. Thus, $\mathcal{G}_G M_G$ is equivalent to $\mathcal{M}_G G$, giving us a forgetful inclusion of $\mathcal{G}_G M_G$ in $\mathcal{M}_M^H$. 

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Let us denote by $i : \Omega^1(M) \to \Omega^1(G)$ the embedding of $\Omega^1(M)$ into $\Omega^1(G)$. With respect to the inclusion of $G_M$ in $G_M^G$, it is clear that $i$ is a morphism in $G_M^G$, as are its tensor powers $i^\otimes k : (\Omega^1(M))^\otimes M^k \to (\Omega^1(G))^\otimes G^k$, for $k \in \mathbb{N}$. An important question to ask is when $i^\otimes k$ is an embedding, for $k \geq 2$. To address this question we will need to introduce two important commutative diagrams: First consider the maps

$$i^k := \text{proj} \circ \Phi_M(i^\otimes k) : \Phi_M((\Omega^1(M))^\otimes M^k) \to \Phi_G((\Omega^1(G))^\otimes G^k), \quad (k \geq 2)$$

where $\text{proj} : \Phi \to (\Omega^1(M))^\otimes M$ is the canonical projection. Since

$$i^\otimes k = \text{frame}^G_1 \circ \Psi_M(i^k) \circ \text{frame}_M,$$

it is clear that $i^\otimes k$ is an embedding if, and only if, $i^k$ is an embedding. We are now ready to introduce our first commutative diagram:

$$
\begin{array}{ccc}
\Phi_G(\Omega^1(G)) & \xrightarrow{\sigma} & \Lambda_1^G \\
\downarrow{i} & & \downarrow{i} \\
\Phi_M(\Omega^1(M)) & \xrightarrow{\sigma} & V_M,
\end{array}
$$

where $i$ is the descent of the embedding $M^+ \hookrightarrow G^+$. It is clear that $i$ is a morphism in $G_M^G$, as are its tensors powers $i^\otimes k : V^\otimes k \hookrightarrow (\Lambda_1^G)^\otimes k$. For higher powers of $k$, we have the analogous diagram

$$
\begin{array}{ccc}
\Phi_G((\Omega^1(G))^\otimes G^k) & \xrightarrow{\tau^k} & (\Lambda_1^G)^\otimes k \\
\downarrow{i^k} & & \downarrow{\gamma^k} \\
\Phi_M((\Omega^1(M))^\otimes M^k) & \xrightarrow{\sigma^k} & V_M^\otimes k,
\end{array}
$$

where $\gamma^k$ is the unique map for which the diagram is commutative. Explicitly, the action of $\gamma^k$ is given by

$$\gamma^k(m^1 \otimes \cdots \otimes m^k) = \tau^k \circ i^k \circ (\sigma^k)^{-1}(m^1 \otimes \cdots \otimes m^k) = \tau^k(dm^1 \otimes_G \cdots \otimes_G dm^k),$$

$$= m^1 m^2_{(1)} \cdots m^k_{(1)} \otimes_G \cdots \otimes_G (m^1_{(k-2)})^+ m^k_{(k-2)} \otimes_G (m^k_{(k-1)})^+.$$  

Thus, we should note that unless $\Omega^1(G)$ is an object in $G_M^G$, we have no guarantee that $\gamma^k$ is equal to $i^\otimes k$. With these maps and diagrams in hand we are now ready to give a sufficient criteria for $i^\otimes k$ to be an embedding:

**Lemma 4.4** If $\Omega^1(M)$ is a finite dimensional calculus, then $\gamma^k$ is an embedding, and hence $i^k$ and $i^\otimes k$ are embeddings.

**Proof.** If the image of $\gamma^k$ could be shown equal to $i^\otimes k(V_M^\otimes k)$, then, since we are assuming $\Omega^1(M)$ to be finite dimensional, it would follow that $\gamma^k$ was an isomorphism. As a first
step towards establishing this, we note that \( i(\Omega^1(M)) \) is well-defined as an object in \( G_M^* \mathcal{M}_0 \), and so, we can identify \( \Phi_M(i(\Omega^1(M)) \otimes M^k) \) and \( \Phi_M(i(\Omega^1(M))) \otimes M^k \), giving us the isomorphism

\[
\sigma^k : \Phi_M(i(\Omega^1(M)) \otimes M^k) \to \hat{\gamma}^k(V_M^\otimes k).
\]

Combining this fact with the commutative diagram in (27), gives us the new diagram

\[
\begin{array}{ccc}
\Phi_G(\Omega^1(G)) \otimes k & \xrightarrow{\tau^k} & (\Lambda^1_G) \otimes k \\
\text{proj} & & \text{proj} \\
\Phi_M(i(\Omega^1(M)) \otimes M^k) & \xrightarrow{\sigma^k} & \hat{\gamma}^k(V_M^\otimes k),
\end{array}
\]

where proj is the canonical projection, and \( \gamma^k \) is defined so as to make the diagram commutative. Now for an arbitrary element \( m_1 \otimes \cdots \otimes m^k \) in \( \hat{\gamma}^k(V_M^\otimes k) \), it follows from condition 2 of the framing calculus definition that

\[
m_1^1 S(m_{(1)}^1) \otimes m_{(2)}^2 S(m_{(2)}^2) \otimes \cdots \otimes m_{(k)}^k S(m_{(k)}^k) \otimes m_{(2)}^1 \in \hat{\gamma}^k(V_M^\otimes k).
\]

Let us look at the image of this element under \( \gamma^k \), for the first few values of \( k \): For \( k = 2 \), we have

\[
\gamma^2 m_1^1 S(m_{(1)}^1) \otimes m_{(2)}^2 = m_1^1 S(m_{(1)}^1) m_{(2)}^2 \otimes (m_{(3)}^3) = m_1 \otimes m_2.
\]

For \( k = 3 \), we have

\[
\gamma^3 m_1^1 S(m_{(1)}^1) \otimes m_{(2)}^2 S(m_{(2)}^2) \otimes m_{(3)}^3 = m_1^1 S(m_{(1)}^1) m_{(2)}^2 \otimes (m_{(3)}^3) = m_1 \otimes m_2 \otimes m_3.
\]

Continuing in this manner for subsequent values of \( k \), it becomes easy to see that in general

\[
\gamma^k m_1^1 S(m_{(1)}^1) \otimes m_{(2)}^2 S(m_{(2)}^2) \otimes \cdots \otimes m_{(k)}^k = m_1 \otimes m_2 \otimes \cdots \otimes m_k.
\]

Hence, \( \hat{\gamma}^k(V_M^\otimes k) \) is mapped surjectively onto itself by \( \gamma^k \), immediately implying that \( \gamma^k \) and \( i^k \) are embeddings.

As a direct consequence we get the following corollary:

**Corollary 4.5** When \( \Omega^1(M) \) is finite dimensional, the pair \( (V_M^\otimes k, \tau^k \circ i^k) \) (or equivalently the pair \( (V_M^\otimes k, \gamma^k \circ \sigma^k) \)) is a framing for \( (\Omega^1(M)) \otimes M^k \).

We note that, if \( \Omega^1(G) \) is an object in \( G_M^* \mathcal{M}_0 \), then \( \gamma^k = \hat{\gamma}^k \) and the two framings \( (V_M^\otimes k, \tau^k \circ i^k) \) and \( (V_M, \sigma^k) \) are equal.
4.3 The Heckenberger–Kolb Calculus

From the relations given in (18), (19), and (20), it is clear that $C_q[CP^{N-1}]$ acts on $V_{CP^{N-1}}$ according to

$$e^\pm_1 \triangleleft z_{11} = e^\pm_1, \quad e^\pm_i \triangleleft z_{ij} = 0, \quad ((i, j) \neq (1, 1)). \quad (29)$$

Thus, $\Omega^1(CP^{N-1})$ is an object in $G_M^0$, and we have a well-defined framing $(V_{CP^{N-1}}^{\otimes k}, \sigma^k)$ for $(\Omega^1(CP^{N-1}))^{\otimes CP^{N-1}}_k$.

We would now like to find a framing calculus for $\Omega^1_q(CP^{N-1})$. As one might guess, the calculus $\Omega^1_q(SU_N)$ introduced in Section 2 fits the role. To see this we first recall that $\Omega^1_q(SU_N)$ restricts to $\Omega^1_q(CP^{N-1})$ on $C_q[CP^{N-1}]$. Moreover, the right actions given in (29) show that $V_{CP^{N-1}}$ is a right submodule of $\Lambda^1_{SU_N}$. Hence, $\Omega^1_q(SU_N)$ is indeed a framing calculus for $\Omega^1_q(CP^{N-1})$.

5 Framing the Maximal Prolongation

Building on the work of the previous section, we will now show how to find an analogue of the canonical framing for the maximal prolongation of a quantum homogeneous base space calculus. Moreover, we show how a suitable choice of calculus on the total space, extending the base calculus, can be used to express this framing in a very explicit form.

5.1 A Direct Approach

Let $\Omega^1(M)$ be a left-covariant first-order differential calculus over a quantum homogeneous base space $M$, and let $N_M$ be the corresponding sub-bimodule of the universal calculus over $M$. If we denote $I^k_M := \sigma^k(\Phi_M((dN_M)_k))$, for $k \geq 2$, then it is clear from (5) that $\sigma^k$ descends to an isomorphism

$$\sigma^k : \Phi_M(\Omega^k(M)) \to \sigma^k(\Phi_M((\Omega^1(M))^{\otimes M^k})/\sigma^k(\Phi_M((dN_M)_k)) = V^\otimes_k / I^k_M = : V^k_M.$$  

In order for this isomorphism to be of use to us, we will need to find a convenient description of $I^k_M$. The following lemma brings us some way towards this goal.

**Lemma 5.1** For a left-covariant first-order differential calculus $\Omega^1(M)$, which is an object in $G_M^0$, we have

$$I^2_M = \{ \sum_i \overline{m_i} \otimes \overline{n_i} \mid \sum_i m^i dt^i \in N_M \}, \quad (30)$$

or equivalently that

$$I^2_M = \{ \sum_i (f^i S(v^{(1)}))^+ \otimes (v^{(2)})^+ \mid \sum_i f^i \otimes v^i \in (G \otimes I_M)^H \}. \quad (31)$$

Moreover, for $k \geq 3$, we have $I^k_M = \bigoplus_{a+b=k-2} V^a_M \otimes I^2_M \otimes V^b_M$.
Proof. It follows immediately from the properties of the total derivative $d$, and the construction of the maximal prolongation, that

$$\Phi_M(d(N_M)) = \left\{ \sum_i dm_i \otimes_M dn_i \mid \sum_i m_i n_i \in N_M \right\}.$$

(32)

Operating on (32) by $\sigma^2$ then gives us (30). One derives (31) from (11) in the same way.

For $k \geq 3$, the construction of the maximal prolongation tells us that

$$\langle d(N_M) \rangle_k = \bigoplus_{a+b=k-2} (\Omega^1(M))^\otimes_M a \otimes_M d(N_M) \otimes_M (\Omega^1(M))^\otimes_M b.$$

The fact that $\Omega^1(M)$ is an object in $G_M^M 0$, and that $d(N_M)$ is a sub-object of $\Omega^1(M)$ in $G_M^M M$, easily implies that $\Phi_M(d(N_M))$ is an object in $M_H^0$. This in turn tells us that $d(N_M)$ is an object in $G_M^M 0$. Thus, since $\Phi_M$ restricts to a monoidal functor on $G_M^M 0$, we have

$$\Phi_M(\langle dN_M \rangle_k) = \bigoplus_{a+b=k-2} (\Phi_M(\Omega^1(M)))^\otimes_M a \otimes_M (\Phi_M(dN_M)) \otimes_M (\Phi_M(\Omega^1(M)))^\otimes_M b.$$

Operating on this by $\sigma^k$ gives us the required expression for $I^k_M$. □

5.2 Framing Calculi and the Maximal Prolongation

While Lemma 5.1 gives an explicit description of the ideal $I^k_M$, it requires a complete description of the generating relations of the calculus $\Omega^1(M)$ before one can begin calculating. This is more or less the approach followed in [11], and it leads to the type of heavily technical calculations that we are trying to avoid. Instead, in this section we will show that one can use a framing calculus to find a simple description of $I^k_M$ in terms of any generating set of $I_M^k$.

Theorem 5.2 Let $\Omega^1(G)$ be a framing calculus for $\Omega^1(M)$, with $\Lambda^1_G$ its space of left-invariant one forms. We have the equality

$$i^\otimes_2(I^2_M) = \text{span}_C \{ S(z(1)) \otimes_G (z(2))^+ \mid z \in \text{Gen}(I_M) \} \subseteq (\Lambda^1_G)^\otimes_2,$$

where Gen$(I_M)$ is any subset of $I_M$ that generates it as a right $M$-module.

Proof. In the first part of the proof we establish the identity

$$i^2(\Phi(dN_M)) = \{ d(S(z(1))) \otimes_G d(z(2)) \mid z \in I_M \}.$$

We begin with the inclusion $i^2(\Phi(dN_M)) \subseteq \{ d(S(z(1))) \otimes_G d(z(2)) \mid z \in I_M \}$: It is clear from (11) that we have

$$i^\otimes_2(dN_M) = \left\{ \sum_i d(g^i S(v^i(1))) \otimes_G d(v^i(2)) \mid \sum g^i \otimes v^i \in (G \otimes I_M)^H \right\}.$$
For each $i$, since $S(v_i) d(v_i) = 0$ in $Ω^1(G)$, it holds in $(Ω^1(G)) ⊗ G^2$ that
\[
d(g' S(v_i)) ⊗ G dv_i = d(g' S(v_i)) ⊗ G dv_i + g' dS(v_i) ⊗ G dv_i
\]
\[
= d(g' S(v_i)) dv_i + g' dS(v_i) dv_i
\]
\[
= g' dS(v_i) dv_i.
\]
Thus, we have that
\[
i^2(Ω^2 G_N) = \{ \sum_i g' dS(v_i) ⊗ G dv_i \mid \sum_i g' ⊗ v ∈ (G ⊗ I_M)^H \}.
\]
This in turn implies that
\[
i^2(Ω^2 G_N) = \{ \sum_i ι(d(S(v_i))) ⊗ G dv_i \mid \sum_i g' ⊗ v ∈ (G ⊗ I_M)^H \}.
\]
From which it is clear that
\[
i^2(Ω^2 G_N) = \{ d(S(v_i)) ⊗ G dv_i \mid v ∈ I_M \},
\]
giving us the required inclusion.

We now turn to the opposite inclusion of $\{ d(S(v_i)) ⊗ G dv_i \mid v ∈ I_M \}$ in $i^2(Ω^2 G_N)$.
From Takeuchi’s theorem we have that the image of $(G ⊗ I_M)^H$ under frame $H_M$ is equal to $I_M$. In other words, for any $z ∈ I_M$, we have an element $\{ g' ⊗ v \}$ contained in $(G ⊗ I_M)^H$ such that $z = \sum_i ε(g') v$. This gives us that
\[
dS(z) ⊗ G dz = \sum_i ε(g') dS(v_i) ⊗ G dv_i.
\]
Since $z$ tells us that $\sum_i ε(g') dS(v_i) ⊗ dv_i$ is an element of $i^2(Φ_M(d(N_M)))$, we must have $dS(z) ⊗ dz$ contained in $i^2(Φ(d(N_M)))$. This gives us the required opposite inclusion, and hence the required equality.

Let us now move onto the second part of the proof where we find the image of $i^2(Ω^2 G_N)$ under $τ^2$:
\[
τ^2(Ω^2 G_N) = \{ τ^2 dS(z) ⊗ G dz \mid z ∈ I_M \}
\]
\[
= \{ (S(z))^+ ⊗ (z^+) \mid z ∈ I_M \}
\]
\[
= \{ (z - z^+) ⊗ z^+ \mid z ∈ I_M \}
\]
\[
= \{ z(1) ⊗ z(2) - T ⊗ z - z ⊗ T \mid z ∈ I_M \}
\]
\[
= \{ z(1) ⊗ z(2) \mid z ∈ I_M \}
\]
For any $m ∈ M^+$, the fact that $V_M$ is an object in $M^H_0$ means that
\[
(zm)(1) ⊗ (zm)(2) = z(1)m(1) ⊗ z(2)m(2) = z(1)m(1) ⊗ z(2)ε(m(2))
\]
\[
= z(1) ⊗ z(2) = ε(m)z(1) ⊗ z(2).
\]
Hence, for any generating subset \( \text{Gen}(I_M) \) of \( I_M \), we have

\[
\tau^2 \circ \iota^2 (\Phi_M(dN_M)) = \text{span}_C \{ z_i(1) \otimes z_j(2) \mid z \in \text{Gen}(I_M) \}.
\]

We begin the final part of the proof by noting that, for any \( z \in I_M \),

\[
S(z(1))^+ \otimes (z(2))^+ = S(z(1)) \otimes (z(2))^+ - 1 \otimes z = S(z(1)) \otimes (z(2))^+.
\]

Since \( \gamma^2 = \gamma^2 \circ \iota^2 \) (where \( \gamma^2 \) is defined in the commutative diagram (28)), the theorem would follow if we could show that \( \gamma^2 \) acted on \( \text{span}_C \{ (S(z(1)))^+ \otimes (z(2))^+ \mid z \in \text{Gen}(I_M) \} \) to give \( \text{span}_C \{ z_i(1) \otimes z_j(2) \mid z \in \text{Gen}(I_M) \} \). But this follows directly from the calculation

\[
\gamma^2 (S(z(1))^+ \otimes (z(2))^+) = \tau^2 (d(S(z(1))) \otimes G d(z(2))) = S(z(1))^+ z(2) \otimes z(3) = z(1) \otimes z(2) + 1 \otimes z.
\]

\[
= -z(1) \otimes z(2).
\]

\[
\Box
\]

### 5.3 The Heckenberger–Kolb Calculus

We will now present two applications of the general theory developed in this section. First, we take the calculus \( \Omega_q^1(SU_N) \) as a framing calculus for \( \Omega_q^1(\mathbb{C}P^{N-1}) \), and use it to explicitly describe the maximal prolongation of \( \Omega_q^1(\mathbb{C}P^{N-1}) \). Secondly, we take the famous three-dimensional Woronowicz calculus \( \Gamma_q^1(SU_2) \) as a framing calculus for \( \Omega_q^1(SU_2) \), and use it to describe the maximal prolongation of \( \Omega_q^1(\mathbb{C}P^1) \). We see that these two descriptions for the maximal prolongation of \( \Omega_q^1(SU_2) \) agree, as of course they should.

#### 5.3.1 The Calculus \( \Omega_q^1(SU_N) \) as a Framing Calculus for the Heckenberger–Kolb Calculus \( \Omega_q^1(\mathbb{C}P^{N-1}) \)

We recall from Section 4 that \( \Omega_q^1(SU_N) \) satisfies all the requirements to be a framing for \( \Omega_q^1(\mathbb{C}P^{N-1}) \). Hence, following Theorem 5.2, we can use it to calculate the maximal prolongation of \( \Omega_q^1(\mathbb{C}P^{N-1}) \).

**Proposition 5.3** The subspace \( I_{\mathbb{C}P^{N-1}}^2 \) is spanned by the elements

\[
e^i_1 \otimes e_j^+ + q e^i_1 \otimes e_j^-, \quad e^i_1 \otimes e_i^- + q^{-2} e^i_1 \otimes e_i^+ - q^{2i-1} \nu \sum_{a=i+1}^{N-1} q^{-2a} e^a_1 \otimes e^a_1, \quad (34)
\]

\[
e^i_1 \otimes e_h^- + q e^i_1 \otimes e_i^-, \quad e^i_1 \otimes e_h^+ + q e^i_1 \otimes e_i^+, \quad e^i_1 \otimes e_i^+, \quad e_i^- \otimes e_i^-,
\]

\[
e_i^- \otimes e_j^- + q e_i^- \otimes e_j^+, \quad e_i^+ \otimes e_i^+ + q e_i^+ \otimes e_i^-, \quad e_i^+ \otimes e_i^+, \quad e_i^- \otimes e_i^-,
\]

(35)
for $h, i, j = 1, \ldots, N - 1, i \neq j,$ and $h < i$. Hence, $V_{C_{P}}^{k}$ is a $(2(N-1))$-dimensional vector space, with a basis given by

$$\{e_{i_{1}}^{+} \land \cdots \land e_{i_{m}}^{+} \land e_{j_{1}}^{-} \land \cdots \land e_{j_{m}}^{-} \mid i_{1} < \cdots < i_{m}; j_{1} < \cdots < j_{m}\}.$$ 

**Proof.** Consider the generating set of $I_{C_{P}}^{N-1}$, as given in Section 3.3, 

$$\{z_{ij}, z_{i1}z_{kl}, z_{i1}z_{k1} \mid i, j = 2, \ldots, N; i \neq j; (k, l) \neq (1, 1)\}$$

For $z_{ij}$, we have

$$S((z_{ij})_{(1)}) \otimes (z_{ij})_{(2)} = \sum_{a, b=1}^{N} S(u_{a}^{i}S(u_{b}^{j})) \otimes u_{a}^{i}S(u_{b}^{j}) = \sum_{a, b=1}^{N} S^{2}(u_{a}^{i})S(u_{b}^{j}) \otimes u_{a}^{i}S(u_{b}^{j})$$

$$= \sum_{a, b=1}^{N} q^{2(b-j)}u_{a}^{i}S(u_{b}^{j}) \otimes u_{a}^{i}S(u_{b}^{j})$$

From the formulae given in Section 3.3, for the actions of the elements of $C_{P}[SU_{N}]$ on $V_{C_{P}}^{N-1}$, it is easy to conclude that the summand $u_{a}^{i}S(u_{b}^{j}) \otimes u_{a}^{i}S(u_{b}^{j})$ is non-zero only if $a = i, b = 1$, or $a = 1, b = j$. Thus, we must have

$$S((z_{ij})_{(1)}) \otimes (z_{ij})_{(2)} = q^{2(1-j)}u_{1}^{i}S(u_{1}^{j}) \otimes u_{1}^{i}S(u_{1}^{j}) + u_{1}^{i}S(u_{j}^{1}) \otimes u_{1}^{i}S(u_{j}^{1})$$

$$= -q^{4i+1-2j}(q^{2}e_{i-1}^{+} \otimes e_{j-1}^{-} + e_{j-1}^{-} \otimes e_{i-1}^{+}).$$

A similar analysis will show that $S((z_{i1}z_{1j})_{(1)}) \otimes (z_{i1}z_{1j})_{(2)}$, and $S((z_{i1}z_{1i})_{(1)}) \otimes (z_{i1}z_{1i})_{(2)}$, are also equal to scalar multiples of

$$q^{3}e_{i-1}^{+} \otimes e_{j-1}^{-} + e_{j-1}^{-} \otimes e_{i-1}^{+}.$$ 

Let us now assume from now on that $i < j$. Then for $z_{i1}z_{j1}$, and $z_{j1}z_{i1}$, that $S((z_{i1}z_{j1})_{(1)}) \otimes (z_{i1}z_{j1})_{(2)}$, and $S((z_{j1}z_{i1})_{(1)}) \otimes (z_{j1}z_{i1})_{(2)}$, are both equal to linear multiples of the element

$$e_{i-1}^{+} \otimes e_{j-1}^{+} + qe_{j-1}^{+} \otimes e_{i-1}^{+}.$$ 

While for $z_{i1}z_{j1}$, and $z_{i1}z_{j1}$, we have that $S((z_{i1}z_{j1})_{(1)}) \otimes (z_{i1}z_{j1})_{(2)}$, and $S((z_{j1}z_{i1})_{(1)}) \otimes (z_{j1}z_{i1})_{(2)}$, are both equal to linear multiples of the element

$$e_{j-1}^{+} \otimes e_{i-1}^{+} + q^{-1}e_{j-1}^{-} \otimes e_{i-1}^{+}.$$ 

The generators $z_{ii}$, $z_{i1}z_{1i}$, and $z_{i1}z_{1i}$, give in all three cases a linear multiple of

$$e_{i-1}^{+} \otimes e_{i-1}^{-} + q^{-1}e_{i-1}^{-} \otimes e_{i-1}^{+} - q^{2i-1} \sum_{a=1+1}^{N-1} q^{-2a}e_{a-1}^{-} \otimes e_{a-1}^{+}.$$ 

Similarly, the generators $z_{i1}z_{1i}$, and $z_{i1}z_{1i}$ give scalar multiples of $e_{i-1}^{+} \otimes e_{i-1}^{-}$, and $e_{i-1}^{-} \otimes e_{i-1}^{-}$ respectively. Finally, for $k, l \neq 1$, we get that

$$S((z_{i1}z_{kl})_{(1)}) \otimes (z_{i1}z_{kl})_{(2)} = S((z_{i1}z_{kl})_{(1)}) \otimes (z_{i1}z_{kl})_{(2)} = 0.$$ 

$\square$
5.3.2 The Woronowicz Calculus $\Gamma_q^1(SU_2)$ as a Framing Calculus for $\Omega_q^1(CP^1)$

In this subsection we specialise to the case of $C_q[CP^1]$, and use the three-dimensional Woronowicz calculus $\Omega_q^1(SU_2)$ on $C_q[SU_2]$ as a framing calculus for the Heckenberger–Kolb calculus. (See [30] for a detailed presentation of the Woronowicz calculus in the conventions of this paper). We do this firstly to demonstrate that there can exist more than one framing calculus for any given base space calculus, and secondly to highlight the fact that the description produced is independent of the choice of framing calculus.

Denote by $I_{SU_2}$ the ideal corresponding to the Woronowicz calculus. We will recall that the cotangent space $V_{C_{P1}} := C_q[CP^1]^+/I_{SU_2}$ has a basis given by

$$
e^+ := \overline{c}, \quad e^0 := \overline{a - 1}, \quad e^+ := \overline{b}.$$  

Moreover, from the description of $I_{SU_2}$ given in (??), it is easy to see that the non-zero actions of the generators of $C_2[SU_2]$ on $e^+$ and $e^-$ are given by

$$e^\pm \triangleleft a = q^{-\pm}e^\pm, \quad e^\pm \triangleleft d = qe^\pm. \quad (36)$$

It is also clear that $I_{C_{P1}} = \langle b^2, bc, c^2 \rangle$, the ideal corresponding to the Heckenberger–Kolb calculus, is contained in $I_{SU_2}$, giving us a well-defined map $V_{C_{P1}} \rightarrow \Lambda^1_{SU_2}$. With respect to this map, we have that $\overline{ab} = e^-$, and $\overline{cd} = qe^+$, showing that the map is in fact an inclusion. Since it is clear from (36) that $V_{C_{P1}}$ is a right $C_q[SU_2]$-submodule of $\Lambda^1_{SU_2}$, we have that $C_q[SU_2]$ is a framing calculus for $\Omega_q^1(CP^1)$. We can now use Theorem 5.2 to find a framing for the maximal prolongation of $\Omega_q^1(CP^1)$:

**Lemma 5.4** It holds that

$$I_{C_{P1}, N-1}^2 = \text{span}_{C_q} \{ e^+ \otimes e^+, e^- \otimes e^-, e^+ \otimes e^- + q^{-2}e^- \otimes e^+ \}, \quad (37)$$

and hence that $V_{C_{P1}}^2 = C_qe^+ \otimes e^-$, while $V_{C_{P1}}^k = \{0\}$, for all $k \geq 3$.

**Proof.** Take the generating set $\{b^2, bc, c^2\}$ for $I_{C_{P1}}$. For $b^2$ we have that

$$S((b^2)_{(1)}) \otimes ((b^2)_{(2)})^T \equiv S(a^2) \otimes \overline{b^2} + (1 + q^{-2})S(ab) \otimes \overline{bd} + S(b^2) \otimes (a^2)^T = (1 + q^{-2})(-q^{-1}\overline{bd} \otimes \overline{bd}) = -(1 + q^{-2})qe^- \otimes e^-.$$  

For $c^2$, we have that

$$S((c^2)_{(1)}) \otimes ((c^2)_{(2)})^T \equiv S(c^2) \otimes (a^2)^T + (1 + q^{-2})S(cd) \otimes \overline{ac} + S(d^2) \otimes c^T = (1 + q^2)(-q^{-1}\overline{ac} \otimes \overline{ac}) = -q^{-5}(1 + q^2)e^+ \otimes e^+.$$  

Finally, for $bc$, we have that

$$S((bc)_{(1)}) \otimes ((bc)_{(2)})^T \equiv S(ac) \otimes \overline{ba} + S(ad) \otimes \overline{bc} + S(bc) \otimes (da)^T + S(bd) \otimes dc$$

$$= -qcd \otimes \overline{ba} - q^{-1}ab \otimes dc = -q\overline{c} \otimes \overline{a} - q^{-1}\overline{b} \otimes \overline{c}$$

$$= -q(e^- \otimes e^+ + q^2e^+ \otimes e^-).$$

This gives the three elements in (37), along with the implied descriptions of the higher forms. \[\square\]
6 Covariant Almost Complex Structures

We begin this section by introducing our definition of an almost complex structure over a general algebra. We then specialise to the case where this algebra is a quantum homogeneous space, and give a simple set of necessary and sufficient conditions for such an almost complex structure to exist. Finally, we apply this general theory to the Heckenberger–Kolb calculus for the quantum projective spaces.

6.1 Almost Complex Structures

Let us first introduce the wedge map \( \wedge \) for a total differential calculus \( \Omega^\bullet(A) \), by defining

\[
\wedge: \Omega^k(A) \otimes_A \Omega^l(A) \rightarrow \Omega^{k+l}(A), \quad \omega \otimes \omega' \mapsto \omega \wedge \omega'.
\]

Next, we introduce the central definition of the paper:

**Definition 6.1.** An almost complex structure for a total \( \ast \)-differential calculus \( \Omega^\bullet(A) \) over a \( \ast \)-algebra \( A \), is an \( \mathbb{N}^2 \)-algebra grading \( \bigoplus_{(p,q) \in \mathbb{N}^2} \Omega^{(p,q)} \) for \( \Omega^\bullet(A) \) such that, for all \( (p,q) \in \mathbb{N}^2 \):

1. \( \Omega^k(A) = \bigoplus_{p+q=k} \Omega^{(p,q)} \);

2. the wedge map restricts to isomorphisms

\[
\wedge: \Omega^{(p,0)} \otimes_A \Omega^{(0,q)} \rightarrow \Omega^{(p,q)}, \quad \wedge: \Omega^{(0,q)} \otimes_A \Omega^{(p,0)} \rightarrow \Omega^{(p,q)}; \quad (38)
\]

3. \( \ast(\Omega^{(p,q)}) = \Omega^{(q,p)} \).

We call an element of \( \Omega^{(p,q)} \) a \( (p,q) \)-form.

Classically every decomposition of the cotangent bundle into two sub-bimodules extends to an almost complex structure. As the following proposition shows, things are more complicated in the noncommutative setting. The proof requires us to consider the unique \( \mathbb{N}^2 \)-grading of the tensor algebra \( \bigoplus_{k=0}^{\infty} \Omega^k(A) \otimes_A \) of \( \Omega^1(A) \) extending a bimodule decomposition \( \Omega^1(A) = \Omega^{(1,0)} \oplus \Omega^{(0,1)} \). Explicitly, the decomposition \( \Omega^{\otimes(p,q)} := \bigoplus_{(p,q) \in \mathbb{N}^2} \Omega^{(p,q)} \) is defined by

\[
\Omega^{\otimes(p,q)} := \{ w \in \Omega^{p+q}(A) \mid \pi(\omega) \in (\Omega^{(1,0)})^{\otimes p} \otimes_A (\Omega^{(0,1)})^{\otimes q}, \text{ for some } \pi \in S_{p+q} \},
\]

where \( S_{p+q} \) is the permutation group on \( p+q \) objects, acting \( \mathbb{C} \)-linearly on \( \Omega^{p+q}(A) \) in the obvious way.

**Theorem 6.2** For \( \Omega^1(A) \) a first-order differential calculus over an algebra \( A \), and \( \Omega^1(A) = \Omega^{(1,0)} \oplus \Omega^{(0,1)} \) a decomposition of \( \Omega^1(A) \) into sub-bimodules, we have that:
1. The decomposition has at most one extension, satisfying condition (1), to an $\mathbb{N}_0^2$-
grading of the maximal prolongation of $\Omega^1(A)$;

2. Such an extension exists if, and only if, $d(N)$ is homogeneous with respect to the
decomposition

\[
(\Omega^1)^{\otimes A^2} = \Omega^{\otimes (2,0)} \oplus \Omega^{\otimes (1,1)} \oplus \Omega^{(0,2)}. \tag{39}
\]

3. When this decomposition exists, the maps in condition 2 of the almost complex
structure definition are isomorphisms if, and only if, $\wedge$ restricts to isomorphisms
\[
\wedge : \Omega^{(1,0)} \otimes_A \Omega^{(0,1)} \to \Omega^{(1,1)}, \quad \wedge : \Omega^{(0,1)} \otimes_A \Omega^{(1,0)} \to \Omega^{(1,1)}; \tag{40}
\]

4. Moreover, condition 3 holds if, and only if, $*(\Omega^{(1,0)}) = \Omega^{(0,1)}$, or equivalently if,
and only if, $*(\Omega^{(0,1)}) = \Omega^{(1,0)}$.

**Proof.** We begin by giving a sufficient condition for an $\mathbb{N}_0^2$-grading, extending the
decomposition of $\Omega^1(A)$, to exist: For some $\omega \in d(N)$, we denote the decomposition of $\omega$
with respect to (39) by $\omega := \omega_1 + \omega_2 + \omega_3$. By definition $d(N)$ is homogeneous
with respect to (39) if, for each $\omega$, we have $\omega_1, \omega_2, \omega_3 \in d(N)$. In this case, for any
homogeneous elements $\nu, \nu'$ in the tensor algebra of $\Omega^1(A)$, the decomposition of the
element $\nu \otimes_A \omega \otimes_A \nu'$, with respect to $\Omega^{\otimes (\bullet, \bullet)}$, is given by
\[
\nu \otimes_A \omega \otimes_A \nu' = \nu \otimes_A \omega_1 \otimes_A \nu' + \nu \otimes_A \omega_2 \otimes_A \nu' + \nu \otimes_A \omega_3 \otimes_A \nu'.
\]
It is clear that $\nu \otimes_A \omega_i \otimes_A \nu' \in \langle d(N) \rangle$, for $i = 1, 2, 3$. Now since every element of
$\langle d(N) \rangle$ is a sum of elements of the form $\nu \otimes \omega \otimes \nu'$, we see that homogeneity of $d(N)$
with respect to (39), implies homogeneity of $\langle d(N) \rangle$ with respect to $\Omega^{\otimes (\bullet, \bullet)}$. In this case,
$\Omega^{\otimes (\bullet, \bullet)}$ clearly descends to a grading $\Omega^{(\bullet, \bullet)}$ on the maximal prolongation. Finally, we
note that if $d(N)$ is not homogeneous with respect to (39), then clearly $\Omega^{\otimes (\bullet, \bullet)}$ cannot
descend to a grading on the maximal prolongation.

We will now show that this grading is the only possible $\mathbb{N}_0^2$-grading on the maximal
prolongation extending the decomposition of $\Omega^1(A)$: For another such distinct grading
$\Gamma^{(\bullet, \bullet)}$ to exist, there would have to be an element $\omega \in \Omega^{\otimes (p,q)}$, for some $(p, q) \in \mathbb{N}_0^2$,
such that the image of $\omega$ in $\Omega^*(A)$ was not contained in $\Gamma^{(p,q)}$. Now it is clear from the
definition of $\Omega^{\otimes (p,q)}$ that every element of $\Omega^{\otimes (p,q)}$ is of the form
\[
\omega := \sum_{i=1}^{\infty} \omega_i^1 \otimes \cdots \otimes \omega_i^{p+q}, \tag{41}
\]
where each $\omega_i^1 \otimes \cdots \otimes \omega_i^{p+q}$ has exactly $p$ of its factors contained in $\Omega^{(1,0)}$, and $q$ of its
factors contained in $\Omega^{(0,1)}$. However, the general properties of a graded algebra imply
that the image of such an element in $\Omega^*(A)$ must be contained in $\Gamma^{(p,q)}$. Thus, we can
conclude that there exists no other grading on the maximal prolongation extending the
decomposition $\Omega^1(A)$. This gives us the first and second parts of the theorem.
Now we come to showing that when this \( \mathbb{N}_{\geq 0}^2 \)-grading exists, condition 2 of the definition of an almost complex structure holds if, and only if, the maps in (40) are isomorphisms. Let us begin by establishing that surjectivity of the first map in (38) follows from surjectivity of the second map in (40): Let

\[
\omega := \sum_i \omega_i^j \wedge \cdots \wedge \omega_{p+q}^j,
\]

be a general element of \( \Omega^{(p,q)} \), where, just as in (41), each \( \omega_i^j \wedge \cdots \wedge \omega_{p+q}^j \) has exactly \( p \) of its factors contained in \( \Omega^{(1,0)} \), and \( q \) of its factors contained in \( \Omega^{(0,1)} \). If for each of these summands, there exists no pair of adjacent factors \( \omega_i^j \wedge \omega_{k+1}^j \), for some \( 1 \leq k < p + q \), such that \( \omega_k \in \Omega^{(0,1)} \), and \( \omega_{k+1} \in \Omega^{(1,0)} \), then it is clear that \( \omega \) is contained in the image of \( \Omega^{(p,0)} \otimes_A \Omega^{(0,q)} \) under \( \wedge \). If such an adjacent pair does exist, then since we are assuming the first map in (40) to be surjective, there must exist an element \( \sum_j \nu_j \otimes_A \nu_j' \) in \( \Omega^{(1,0)} \otimes_A \Omega^{(0,1)} \), such that

\[
\sum_j \nu_j \wedge \nu_j' = \omega_i \wedge \omega_{i+1}.
\]

If upon inserting this relation into \( \omega \) we obtain a presentation of \( \omega \) whose summands contain no other such pairs of adjacent factors, then it is clear that \( \omega \) is contained in the image of \( \Omega^{(p,0)} \otimes_A \Omega^{(0,q)} \) under \( \wedge \). If such adjacent pairs do exist, then it is easy to see that by successive applications of this process, one will eventually arrive at a presentation of \( \omega \) containing none. Thus, it is clear that \( \omega \) is contained in the image of \( \Omega^{(p,0)} \otimes_A \Omega^{(0,q)} \) under \( \wedge \), which is to say that surjectivity of the first map in (38) follows from surjectivity of the first map in (40). That surjectivity of the second map in (38) follows from surjectivity of the second map in (40) is established in an exactly analogous manner.

We now move on to establishing injectivity. As a little thought will confirm, the first map of (38) would be seen to be injective if it could be shown that, for all \( (p, q) \in \mathbb{N}_{\geq 0}^2 \),

\[
\langle d(N) \rangle \cap (\Omega \otimes (p,0) \otimes_A \Omega \otimes (0,q)) = \langle dN \rangle_{(p,0)} \otimes_A \Omega \otimes (0,q) + \Omega \otimes (p,0) \otimes_A \langle dN \rangle_{(0,q)},
\]

where \( \langle dN \rangle_{(p,0)} \) and \( \langle dN \rangle_{(0,q)} \) are the \( \otimes (p,0) \) and \( \otimes (0,q) \), homogeneous components of \( \langle dN \rangle \) respectively. To see that this is so, consider the general element \( \sum_i \nu_i \otimes \omega_i \otimes \nu'_i \) of \( \langle d(N) \rangle \), with each \( \nu_i, \nu'_i \) contained in the tensor algebra of \( \Omega^1(A) \), and each \( \omega_i \) a homogeneous element of \( d(N) \). Since the first mapping in (40) is an isomorphism, it must hold that

\[
d(N) \cap (\Omega^{(1,0)} \otimes_A \Omega^{(0,1)}) = \{ 0 \}.
\]

This implies that \( \sum_i \nu_i \otimes \omega_i \otimes \nu'_i \) is contained in \( \Omega \otimes (p,0) \otimes_A \Omega \otimes (0,q) \) only if \( \omega^j \in \Omega \otimes (2,0) \), or \( \omega^j \in \Omega \otimes (0,2) \). It now follows that (40) holds, and hence that the first map of condition 3 is injective. That the second map of (38) is injective is established analogously. Thus, we have established the third part of the theorem.
We now come to the fourth and final part of the theorem. Note first that since the $\ast$-map is involutive, assuming $\ast(\Omega^{(1,0)}) = \Omega^{(0,1)}$ is clearly equivalent to assuming $\ast(\Omega^{(0,1)}) = \Omega^{(1,0)}$. Next we note that, for a general element $\omega$ in $\Omega^{(p,q)}$ as given in (42), the properties of a graded $\ast$-algebra imply that

$$\omega^\ast := \sum_{i=1}^{(p+q)} (-1)^{\binom{p+q}{2}} (\omega_p^\ast \wedge \cdots \wedge (\omega_q^\ast) = \sum_{i=1}^{(p+q)} \omega_i^\ast \wedge \cdots \wedge (\omega_i^\ast).$$

(44)

Our two equivalent assumptions, and the properties of a graded algebra, now imply that $\omega^\ast$ must be contained in $\Omega^{(q,p)}$, giving us that $\ast(\Omega^{(p,q)}) \subseteq \Omega^{(q,p)}$. The opposite inclusion is established analogously, giving us the desired equality. □

We now recall that every total calculus extending $(\Omega^1(A), d)$ can be obtained as a quotient of the maximal prolongation by an ideal $I \subseteq \ker(d)$. It is not difficult to see that a decomposition of $\Omega^1(A) = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ is extendable to an almost complex structure on such a total calculus if, and only if, it is extendable to an almost complex structure on the maximal prolongation with respect to which $I$ is homogeneous. This gives us a classification of all almost complex structures over the algebra $A$. However, since at present we have no interesting examples of such structures, we will not pursue this observation here.

### 6.2 Covariant Almost Complex Structures

We say that an almost complex structure $\Omega(\bullet, \bullet)$ for a quantum homogeneous space $M = G^H$ is **left-covariant** if we have

$$\Delta_L(\Omega^{(p,q)}) \subseteq G \otimes \Omega^{(p,q)}, \quad \text{(for all } (p, q) \in \mathbb{N}^2).$$

As a little thought will confirm, an almost complex structure will be covariant if, and only if,

$$\Delta_L(\Omega^{(1,0)}) \subseteq G \otimes \Omega^{(1,0)}, \quad \Delta_L(\Omega^{(0,1)}) \subseteq G \otimes \Omega^{(0,1)}.$$

For covariant almost complex structures we will of course have each $\Omega^{(p,q)}$ contained as an object in $G^H_M$. For the special case that $\Omega^1(M)$ is an object in $G^H_M$, we denote

$$V^{\otimes(p,q)} := \sigma^{p+q}(\Phi_M(\Omega^{\otimes(p,q)})).$$

Clearly, it follows from the definition of an almost complex structure that we have $V^k_M = \bigoplus_{p+q=k} V^{(p,q)}_M$. Another important fact is that since $\wedge$ is clearly a morphism in $G^H_M$, we have a corresponding morphism $\Phi_M(\wedge)$ in $M^H$. Moreover, since we have given $\Phi_M$ the structure of a monoidal functor, we can consider $\Phi_M(\wedge)$ as a morphism

$$\Phi_M(\wedge) : (\Phi_M(\Omega^1(M)))^\otimes \rightarrow \Phi_M(\Omega^2(M)).$$

We use this to define a new morphism

$$\wedge_\sigma := \sigma^2 \circ \Phi_M(\wedge) \circ (\sigma^2)^{-1} : V^\otimes \rightarrow V^2.$$

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The following corollary shows that for covariant complex structures, we have a convenient reformulation of Theorem 6.2.

**Corollary 6.3** For a left-covariant first-order differential calculus $\Omega^1(M)$, with canonical framing $(V_M, s)$, we have that:

1. Decompositions of $\Omega^1(M) = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ into left-covariant bimodules correspond to decompositions of $V_M = V_M^{(1,0)} \oplus V_M^{(0,1)}$ into right-covariant right comodules.

2. Such a decomposition extends to an $\mathbb{N}^2_0$-grading of the maximal prolongation of $\Omega^1(M)$ if, and only if, $I^2_M$ is homogeneous with respect to the decomposition

$$V_M^{\otimes 2} = V_M^{\otimes (2,0)} \oplus V_M^{\otimes (1,1)} \oplus V_M^{\otimes (0,2)}. \quad (45)$$

3. If $\Omega^1(M)$ is contained as an object in the category $G_M^M_0$, then condition 3 of the almost complex structure definition is satisfied if, and only if, we have isomorphisms

$$\wedge \sigma : V_M^{(1,0)} \otimes V_M^{(0,1)} \to V_M^{(1,1)}, \quad \wedge \sigma : V_M^{(0,1)} \otimes V_M^{(1,0)} \to V_M^{(1,1)}. \quad (46)$$

**Proof.** Since $\Phi$ obeys (4), every covariant bimodule decompositions of $\Omega^1(M)$ induces a covariant right module decomposition of $\Phi(\Omega^1(M))$. Conversely, since $\Psi$ obeys (4), every covariant right module decomposition of $\Phi(\Omega^1(M))$ induces a covariant bimodule decomposition of $\Omega^1(M)$. This gives an equivalence between decompositions of $\Omega^1(M)$ and decompositions of $\Phi(\Omega^1(M))$. The first part of the proof now follows from the fact that $s$ is an isomorphism in $G_M^M_0$, and $\sigma$ is an isomorphism in $M^H_M$.

Turning now to the second part of the proof, we see that, using an analogous argument to the one above, one can establish an equivalence between decompositions of $(\Omega^1(M))^{\otimes M^2}$ and decompositions of $V_M^{\otimes 2}$. Moreover, the decomposition of $(\Omega^1(M))$ given in (39), corresponds to the decomposition of $V_M^{\otimes 2}$ given in (45). Properties (4) and (5) of the functors $\Psi$ and $\Phi$ now imply that $dN$ is homogeneous with respect to (39) if, and only if, $I^2_M$ is homogeneous with respect to (39). Part 2 of the corollary now follows from part 2 of Theorem 6.2.

For the last part of the proof, we note that since $\sigma^2$ is an isomorphism, the functorial properties of $\Phi_M$ imply that the maps in (40) are isomorphisms if, and only if, the maps in (46) are isomorphisms. Part 3 of the corollary now follows from part 3 of Theorem 6.2.

Finally, we come to finding an easily verifiable reformulation of the $*$-condition. As for first order differential $*$-calculi, the fact that the $*$-map is not a bimodule map means that Takeuchi’s equivalence will be of no use here. However, just as for first order differential $*$-calculi, there exists a convenient direct reformulation.
Proposition 6.4 Let $\Omega^1(M)$ be a first order differential $*$-calculus in $G^*_M$, and let $\Omega^{••}$ be an $\mathbb{N}^2_0$-grading for its maximal prolongation $\Omega^{•}(M)$ satisfying the first condition of an almost complex structure. If $\Omega^1(G)$ is a framing $*$-calculus for $\Omega^1(M)$, whose $*$-map restricts to the $*$-map of $\Omega^1(M)$, and with respect to which

$$
\Omega^{(1,0)}G \subseteq G\Omega^{(1,0)}, \quad \Omega^{(0,1)}G \subseteq G\Omega^{(0,1)},
$$

then we have $*(\Omega^{(p,q)}) = \Omega^{(q,p)}$ if, and only if,

$$
\{S(m)^* | m \in V^{(1,0)}\} = V^{(0,1)}.
$$

Proof. From (12), we see that if (47) and (48) holds, then

$$(G \otimes V^{(1,0)})^* = G \otimes V^{(0,1)}.$$ 

From this it easily follows that $*(\Omega^{(1,0)}) = \Omega^{(0,1)}$. Part 4 of Theorem 6.2 now implies that $*(\Omega^{(p,q)}) = \Omega^{(q,p)}$.

Conversely, let us assume that there exists a $\tau \in V^{(1,0)}$ such that $S(\tau)^* \notin V^{(0,1)}$. With respect to the choice of framing calculus, we have

$$
(s^{-1}(1 \otimes \tau))^* = s^{-1}(1 \otimes \tau^*) = s^{-1}(1 \otimes S(\tau)^*) \notin s^{-1}(G \otimes V^{(0,1)}) = G\Omega^{(0,1)}.
$$

However, since $v \in V^{(1,0)}$, we must also have $s^{-1}(1 \otimes v) = \sum a_i \omega_i$, for some $a_i \in G, \omega_i \in \Omega^{(1,0)}$. If we had an almost complex structure, then $\omega_i^*$ would be contained in $\Omega^{(0,1)}$, for all $i$, giving us that

$$
\sum (a_i \omega_i)^* = \sum (\omega_i)^* a_i^* \in \Omega^{(1,0)}G \subseteq G\Omega^{(0,1)}.
$$

Since this contradicts (49), we are forced to conclude that, for some $\omega_i$, we have $\omega_i^* \notin \Omega^{(0,1)}$, and consequently that we do not have an almost complex structure.

6.3 An Almost Complex Structure for the Maximal Prolongation of the Heckenberger–Kolb Calculus

We will now use Corollary 6.3 to investigate the maximal prolongation of the Heckenberger–Kolb calculus. Consider first the canonical decomposition $V_{CPN-1} = V^{(1,0)} \oplus V^{(0,1)}$, where

$$
V^{(1,0)} := \text{span}_C\{e_i^+ | i = 2, \ldots, N\}, \quad V^{(0,1)} := \text{span}_C\{e_i^- | i = 2, \ldots, N\}.
$$

This gives the corresponding decomposition

$$
\Omega^1_q(CP^{N-1}) = \Omega^{(1,0)} \oplus \Omega^{(0,1)} := \Psi_{CP^{N-1}}(V^{(1,0)}) \oplus \Psi_{CP^{N-1}}(V^{(0,1)}).
$$

As we will see below, this decomposition extends to an almost complex structure for the maximal prolongation of $\Omega^1_q(CP^{N-1})$. 

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Classically, it holds that $\Omega^1_q(N-1,0)$ is isomorphic to $E_N$, and that $\Omega^0_q(N-1)$ is isomorphic to $E_{-N}$. Moreover, it holds that $CP^{N-1}$ is orientable, which is to say that $\Omega^0_q(N-1,0) \simeq C_q[CP^{N-1}]$. These are important properties and one would naturally hope that they generalise to the quantum setting. The following proposition tells us that this is also the case.

**Proposition 6.5** The decomposition $\Omega^1_q(CP^{N-1}) = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ extends to an almost complex structure on the maximal prolongation of $\Omega^1(CP^{N-1})$. Moreover, it holds that:

$\Omega^{(N-1,0)}_q \simeq E_{-N}$, $\Omega^{(0,N-1)}_q \simeq E_N$, and, as a direct consequence

$$\Omega^{(N-1,N-1)}_q \simeq C_q[CP^{N-1}].$$

**Proof.** That $I^2_{CP^{N-1}}$ is homogeneous with respect to the decomposition from part 2 of the corollary follows directly from Proposition 5.3, as does the fact the maps in (46) are isomorphisms. We now turn to Proposition 6.4, noting first that (47) follows directly from the module relations given in (18). Moreover, (48) follows from the fact that for $i = 2, \ldots, N$, we have

$$S(u^i_1)^* = S^{-1}((u^i_1)^*) = S^{-1} \circ S(u^i_1) = u^-_i \in V^{(0,1)},$$

where we have used the standard Hopf $*$-algebra identity $* \circ S = S^{-1} \circ *$. Thus, the decomposition $\Omega^1_q(CP^{N-1}) = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ extends to an almost complex structure for the maximal prolongation of $\Omega^1_q(CP^{N-1})$.

We now come to the second part of the proposition, beginning with the action of $\Delta_M^{N-1}$ on $V^{(0,N-1)} \simeq C e^+_1 \wedge \cdots e^+_N$:

$$\Delta_M^{N-1}(e^+_1 \wedge \cdots \wedge e^+_N) = \sum_{l=1}^{N-1} \sum_{k=1}^{N-1} e^+_k \wedge \cdots \wedge e^+_{k_{N-l}} \otimes S(u^1_k) \cdots S(u^{N-1}_{k_{N-l}}) \det^{1-N}_{N-1} \quad \text{for } 1 \leq l \leq N-1,$$

$$= \sum_{l=1}^{N-1} e^+_l \wedge \cdots \wedge e^+_N \otimes S(u^1_{N-l} \cdots u^{N-1}_{k_{N-l}}) \det^{1-N}_{N-1}.$$

Now since any summand with a repeated basis element in the first tensor factor will be zero, we must have

$$\Delta_M^{N-1}(e^+_1 \wedge \cdots e^+_N) = \sum_{\pi \in S_N} e^+_{\pi(1)} \wedge \cdots e^+_{\pi(N-1)} \otimes S(u^1_{\pi(1)} \cdots u^{N-1}_{\pi(N-1)}) \det^{N-1}_{N-1}.$$

As a little thought will confirm $e^+_{\pi(1)} \wedge \cdots e^+_{\pi(N-1)} = (-q)^{-\text{sgn}(\pi)} e^+_{\pi(N-1)} \cdots e^+_{\pi(1)}$ for any $\pi \in S_N$. Thus, since it is clear that

$$\sum_{\pi \in S_N} (-q)^{-\text{sgn}(\pi)} u^1_{\pi(N-1)} \cdots u^1_{\pi(1)} = \det_{N-1},$$

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we must have
\[
\Delta_M^{N-1}(e_1^+ \wedge \cdots \wedge e_{N-1}^+) = e_1^+ \wedge \cdots \wedge e_{N-1}^+ \otimes S(\det) \det^{-N}_{N-1}
\]
\[
= e_1^+ \wedge \cdots \wedge e_{N-1}^+ \otimes \det^{-N}_{N-1}.
\]
Thus, as we would have hoped, it holds that \(\Omega^{(0,N-1)}(CP^{N-1}) \cong \mathcal{E}_{-N}\).

An exactly analogous argument will establish that \(\Omega^{(N-1,0)}(CP^{N-1})\) is isomorphic to \(\mathcal{E}_{N}\). While the fact that \(\Omega^{(N-1,N-1)}(CP^{N-1})\) is isomorphic to \(\mathbb{C}_q[CP^{N-1}]\), follows as a direct consequence of these two results.

\[\Box\]

7 Integrability and Complex Structures

In this section we will show how the classical notion of integrability transfers directly to the noncommutative setting. Mirroring the classical picture, we demonstrate how integrability of an almost complex structure implies the existence of a quantum Dolbeault double complex. Moreover, with respect to a choice of framing calculus, we give a simple set of sufficient criteria for a complex structure to be integrable.

We begin with two lemmas whose proofs carry over directly from the classical case. (It should be noted that these results have already appeared in [4], where one can find a more comprehensive treatment of integrability in the noncommutative setting.)

Lemma 7.1

If \(\bigoplus_{(p,q)\in\mathbb{N}_2} \Omega^{(p,q)}\) is an almost-complex structure for a total calculus \(\Omega^*(A)\) over an algebra \(A\), then the following two conditions are equivalent:

1. \(d(\Omega^{(1,0)}) \subseteq \Omega^{(2,0)} \oplus \Omega^{(1,1)}\),
2. \(d(\Omega^{(0,1)}) \subseteq \Omega^{(1,1)} \oplus \Omega^{(0,2)}\).

Proof. For any \(\omega \in \Omega^{(0,1)}\), the properties of an almost complex structure imply that \(\omega^* \in \Omega^{(1,0)}\). Thus if we assume \(1\) it must hold that \(d\omega^* \in \Omega^{(2,0)} \oplus \Omega^{(1,1)}\). This in turn implies that \(d\omega = (d\omega^*)^* \in \Omega^{(1,1)} \oplus \Omega^{(0,2)}\), showing us that \(2\) holds. The proof in other other direction is entirely analogous. \(\Box\)

If these conditions hold for an almost-complex structure, then we say that it is integrable. We will usually call an integrable almost-complex structure a complex structure. (To see how the formulation of integrability that we have generalised is equivalent to the more standard formulation, see [13].)

With a view to exploring some of the consequences of integrability, we now introduce two new operators: For \(\bigoplus_{(p,q)\in\mathbb{N}_2} \Omega^{(p,q)}\) an almost complex structure, we define \(\partial\), and \(\bar{\partial}\), to be the unique order (1, 0), and (0, 1) respectively, homogeneous operators for which

\[
\partial|_{\Omega^{(p,q)}} = \text{proj}_{\Omega^{(p+1,q)}} \circ d,
\]

\[
\bar{\partial}|_{\Omega^{(p,q)}} = \text{proj}_{\Omega^{(p,q+1)}} \circ d.
\]

where \(\text{proj}_{\Omega^{(p+1,q)}}\), and \(\text{proj}_{\Omega^{(p,q+1)}}\), are the projections onto \(\Omega^{(p+1,q)}\), and \(\Omega^{(p,q+1)}\) respectively.

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Lemma 7.2 If an almost complex structure $\bigoplus_{(p,q) \in \mathbb{N}^2} \Omega^{(p,q)}$ is integrable, then

1. $d = \partial + \overline{\partial}$;
2. $(\bigoplus_{(p,q) \in \mathbb{N}^2} \Omega^{(p,q)}, \partial, \overline{\partial})$ is a double complex;
3. $\partial(a^*) = (\overline{\partial}a)^*$, and $\overline{\partial}(a^*) = (\partial a)^*$, for all $a \in A$;
4. both $\partial$ and $\overline{\partial}$ satisfy the graded Liebniz rule.

Proof. We begin by proving that $d = \partial + \overline{\partial}$: Since $\Omega^{(p,q)}$ is spanned by products of $p$ elements of $\Omega^{(1,0)}$, and $q$ elements of $\Omega^{(0,1)}$, it follows from the Liebniz rule and the assumption of integrability that $d \omega \in \Omega^{(p+1,q)} \oplus \Omega^{(p,q+1)}$, (for all $\omega \in \Omega^{(p,q)}$).

Thus, we must have that $d = \overline{\partial} + \partial$.

Let us now move on to the second part of the proof: Since $d^2 = 0$, we have

$$0 = d^2 = (\partial + \overline{\partial}) \circ (\partial + \overline{\partial}) = \partial^2 + (\overline{\partial} \circ \partial + \partial \circ \overline{\partial}) + \overline{\partial}^2.$$

For any $\omega \in \Omega^k(M)$, it is easy to see that any non-zero images of $\omega$ under $\partial^2$, $\overline{\partial} \partial + \partial \overline{\partial}$, and $\overline{\partial}^2$, would lie in complementary subspaces of $\Omega^{k+2}(M)$. Thus, it must hold that

$$\partial^2 = 0, \quad \partial \circ \overline{\partial} = -\overline{\partial} \circ \partial, \quad \overline{\partial}^2 = 0,$$

showing that we have a double complex.

For the third part of the proof, we first note that since $d(a^*) = (\partial a)^*$, we have

$$\partial(a^*) + \overline{\partial}(a^*) = (\partial a)^* + (\overline{\partial}a)^*.$$

Now $\partial(a^*)$ and $(\overline{\partial}a)^*$ both lie in $\Omega^{(1,0)}$, while $\overline{\partial}(a^*)$ and $(\partial a)^*$ both lie in $\Omega^{(0,1)}$. Since these are again complementary subspaces of $\Omega^1(M)$, we must have $\partial(a^*) = (\overline{\partial}a)^*$, and $\overline{\partial}(a^*) = (\partial a)^*$.

The fourth part of the lemma is an analogously consequence of the Liebniz rule of d. □

Thus we see that integrability in the noncommutative setting has many of the same properties as classical integrability. Inspired by the classical case we call the double complex $(\bigoplus_{(p,q) \in \mathbb{N}^2} \Omega^{(p,q)}, \partial, \overline{\partial})$ the quantum Dolbeault double complex of an integrable complex structure.
7.1 Integrability for a Covariant Complex Structure

Directly verifying that an almost complex structure is integrable can lead to quite involved calculations. So we would like to use the assumption of covariance to find a simple set of sufficient criteria (analogous to our method for verifying the existence of an almost-complex structure given in the previous section). This will require us to make a choice of linear complement $V_M^+$ to $\tilde{\mathcal{H}}(V_M)$ in $\Lambda^1_G$. With respect to this choice of complement, we will write

$$(V_M^\otimes 2)^\perp := (\tilde{\mathcal{H}}(V_M) \otimes V_M^+ + (V_M^+ \otimes \tilde{\mathcal{H}}(V_M)) \oplus (V_M^+)\otimes 2,$$

for the corresponding linear complement to $\tilde{\mathcal{H}}(V_M)^\otimes 2$ in $(\Lambda^1_G)\otimes 2$. Moreover, we will say that a subset $\{m_j^1\} \subseteq M^+$ descends to a spanning set of $V^{(1,0)}$, if we have that $\text{span}_G\{m_j^1\} = V^{(1,0)}$. We state the result in terms of the holomorphic cotangent space, however, as is clear from the proof, exactly analogous result holds for the anti-holomorphic cotangent space $V^{(0,1)}$.

**Proposition 7.3** Let $\Omega^{(\bullet, \bullet)}$ be a covariant almost-complex structure over a quantum homogeneous space $M = G^H$, such that $\Omega^1(M)$ is a finite dimensional as a first-order differential calculus. Moreover, let $V_M^+$ be a choice of linear complement to $\tilde{\mathcal{H}}(V_M)$ in $\Lambda^1_G$, and $\{m_j^1\}$ a subset of $M^+$ that descends to a spanning set of $V^{(1,0)}$. It holds that $\Omega^{(\bullet, \bullet)}$ is integrable if, for all $m_j^1 \in \{m_j^1\}$, and $v \in \Lambda^1_G$, we have that

$$(v \circ S(m_j^1)) \otimes (m_j^1)^+ \in \tilde{\mathcal{H}}(V_M^{\otimes 2}) \oplus \tilde{\mathcal{H}}(V_M^{\otimes (1,1)}) \oplus (V_M^+)\otimes 2.$$

(50)

**Proof.** It is clear that $d(\Omega^{(1,0)})$ is contained in $\Omega^{(2,0)} \oplus \Omega^{(1,1)}$ if we have

$$\Phi_M(d(\Omega^{(1,0)})) \subseteq \Phi_M(\Omega^{(2,0)}) \oplus \Phi_M(\Omega^{(1,1)}).$$

(51)

We will establish the proposition by demonstrating that this happens when (50) holds: From (11) it is clear that

$$\Omega^{(1,0)} = s^{-1}((G \otimes V^{(1,0)})^H) = \{ \sum_j f_j S(m_j^1) d(m_j^2) | \sum_j f_j \otimes m_j^1 \in (G \otimes V^{(1,0)})^H \}.$$

This in turn implies that

$$\Phi_M(d(\Omega^{(1,0)})) = \{ \sum_j d(f_j S(m_j^1)) \otimes d(m_j^2) | \sum_j f_j \otimes m_j^1 \in (G \otimes V^{(1,0)})^H \},$$

giving us the equality

$$\sigma^2(\Phi_M(d(\Omega^{(1,0)})) = \{ \sum_j (f_j S(m_j^1))^+ \otimes (m_j^2)^+ | \sum_j f_j \otimes m_j^1 \in (G \otimes V^{(1,0)})^H \}. $$

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Now as a little thought will confirm, this means that (51) holds if, for each such
\[ \sum_j (f^j S(m^j_{(1)}))^+ \otimes (m^j_{(2)})^+ \text{ we have that its image in } (\Lambda^1_L)^{\otimes 2} \text{ under } \hat{\gamma}^{\otimes 2} \text{satisfies} \]
\[ \sum_j (f^j S(m^j_{(1)}))^+ \otimes (m^j_{(2)})^+ \in \hat{\gamma}^{\otimes 2}(V_M^{\otimes (2,0)}) \oplus \hat{\gamma}^{\otimes 2}(V_M^{\otimes (1,1)}). \]

As a little more thought will confirm, this will hold if, for each \( j \), we have
\[ (f^j S(m^j_{(1)}))^+ \otimes (m^j_{(2)})^+ \in \hat{\gamma}^{\otimes 2}(V_M^{\otimes (2,0)}) \oplus \hat{\gamma}^{\otimes 2}(V_M^{\otimes (1,1)}) \oplus (V_M^{\otimes 2})^\perp. \]

That (51) is implied by the requirements of the proposition now follows from the equality
\[ (f^j S(m^j_{(1)}))^+ \otimes (m^j_{(2)})^+ = (f^j S(m^j_{(1)})) \otimes (m^j_{(2)})^+. \]

\[ \square \]

7.2 Integrability and an Alternative Construction of the Maximal Prolongation

For an almost complex structure \( \Omega^{(\bullet, \bullet)} \), the pairs \( (\Omega^{(1,0)}, \partial) \) and \( (\Omega^{(0,1)}, \overline{\partial}) \) are each first order differential calculi. Thus, one can consider their maximal prolongations. Let us denote the \( k \)-forms of the maximal prolongation of \( \Omega^{(1,0)} \) by \( (\Omega^{(1,0)})^k \), and the \( k \)-forms of the maximal prolongation of \( \Omega^{(0,1)} \) by \( (\Omega^{(0,1)})^k \). It is natural to ask when we have
\[ (\Omega^{(1,0)})^k = \Omega^{(k,0)}, \quad (\Omega^{(0,1)})^k = \Omega^{(0,k)}. \]

The following result tells us that this condition is in fact equivalent to integrability.

**Lemma 7.4** For an almost complex structure \( \Omega^{(\bullet, \bullet)} \), the equalities in (52) are equivalent to each other, and to integrability.

**Proof.** Let \( \{\omega_i^-\}_i \) be a subset of \( \Omega^1_u(M) \), such that \( \text{span}_C\{\omega_i^-\} = \Omega^{(0,1)} \), where by abuse of notation we have used the same symbol for \( \omega_i^- \), as for its coset in \( \Omega^1(M) \). If \( N_M \) is the sub-bimodule of \( \Omega^1_u(M) \) corresponding to \( \Omega^1(M) \), then it is clear that the sub-bimodule of \( \Omega^1_u(M) \) corresponding to \( (\Omega^{(1,0)}, \partial) \) is given by
\[ N_M^+ := N_M + \text{span}_C\{\omega_i^-\}_i. \]

Now from the definition of the maximal prolongation, we have that
\[ (\Omega^{(1,0)})^k = (\Omega^{(1,0)})^{\otimes k}/\langle \partial N_M^+ \rangle_k, \]
while Theorem 6.2 tells us that
\[ \Omega^{(k,0)} = \Omega^{\otimes (k,0)}/\langle dN_M \rangle_{(k,0)} = (\Omega^{(1,0)})^{\otimes k}/\langle dN_M \rangle_{(k,0)}. \]
It is easy to see that
\[
\langle \partial N^+_M \rangle_k = \bigoplus_{a+b=k-2} (\Omega^{(1,0)} \otimes_M^a \otimes_M (\partial N^+_M) \otimes_M (\Omega^{(1,0)}) \otimes_M^b
\]
and
\[
\langle dN_M \rangle_{(0,k)} = \bigoplus_{a+b=k-2} (\Omega^{(1,0)} \otimes_M^a \otimes_M \langle dN_M \rangle_{(2,0)} \otimes_M (\Omega^{(1,0)}) \otimes_M^b.
\]
Thus, the first equality in (52) is equivalent to \( \partial N^+_M = \langle dN_M \rangle_{(2,0)} \).

As a little careful thought will confirm, we have
\[
\langle dN_M \rangle_{(2,0)} = \partial N_M,
\]
Thus, the first equality in (52) amounts to having \( \partial \omega_i^- = 0 \), for all \( i \). But this holds if, and only if, our almost complex structure is integrable.

That the second equality in (52) is equivalent to integrability is proved in exactly the same way. \( \square \)

7.3 The Heckenberger–Kolb Calculus

We will now show that the almost-complex structure on \( \Omega^*_q(\mathbb{C}P^{N-1}) \), introduced in the previous section, is integrable: 7.3

Proposition 7.5 The almost-complex structure \( \Omega^{(\bullet, \bullet)}(\mathbb{C}P^{N-1}) \) is integrable.

**Proof.** We will establish the proposition by showing that (50) holds for the calculus, where as a choice of complement to \( V \otimes C^{N-1} \) in \( \Lambda^1_{SU_N} \), we take \( V_{\mathbb{C}P^{N-1}} = \mathbb{C}e^0 \).

For \( z_{1I} = u_1^1S(u_1^1) \in V^{(1,0)} \), with \( i = 2, \ldots, N \), we have, for \( k = 1, \ldots, N - 1 \), that
\[
(e_k^+ \triangleright S((z_{1I})_{(1)})) \otimes ((z_{1I})_{(2)})^+ = \sum_{a,b=1}^N (e_k^+ \triangleright S(u_a^1S(u_b^1))) \otimes (u_a^qS(u_b^1))^+
\]
\[
= \sum_{a,b=1}^{N} q^{2(b-1)}(e_k^+ \triangleright (u_a^bS(u_b^a))) \otimes (u_a^qS(u_b^1))^+
\]
\[
= \sum_{a=1}^{N} (e_k^+ \triangleright (u_1^1S(u_a^1))) \otimes (u_1^qS(u_1^1))^+
\]
\[
= \sum_{a=2}^{N} (e_k^+ \triangleright S(u_a^1)) \otimes u_1^q
\]
\[
= (e_k^+ \triangleright S(u_1^1)) \otimes u_1^q + (e_k^+ \triangleright S(u_{k+1}^1)) \otimes u_1^{k+1}.
\]
From the relations given in Lemma ??, it is clear that \((e_k^+ \triangleleft S(u_i^j)) \otimes u_1^j\) is equal to a linear multiple of \(e_k^+ \otimes e_i^+\), while \((e_k^+ \triangleleft S((u_{-1}^{k+1})) \otimes u_1^{k+1}\) is equal to a linear multiple of \(e_{k-1}^+ \otimes e_k^+\). Thus, we have that \((e_k^+ \triangleleft S((z_{i1}^{(1)})) \otimes ((z_{i1}^{(2)}))^\dagger\) is contained in \(\mathcal{N}(V^{(2,0)})\).

The corresponding calculations for \(e^-\), and \(e^0\), follow similarly. Hence, the requirements of (60) are satisfied, and our almost-complex structure is in fact a complex structure. □

We will finish by explicitly demonstrating how the \(q\)-deformed de Rham complex we have constructed for the quantum projective spaces relates to the \(q\)-deformed de Rham show constructed by Heckenberger and Kolb in [10, 11]. We begin by recalling the celebrated classification result for the special case of the quantum projective spaces. Just before, however, we will need to recall a simple definition: A left-covariant first-order calculus over an algebra \(A\) is called irreducible if it does not possess any non-trivial quotients by a left-covariant \(A\)-bimodule. We now state the result:

**Theorem 7.6** [10] There exist exactly two non-isomorphic finite-dimensional irreducible left-covariant first-order differential calculi over quantum projective \((N - 1)\)-space. Each has dimension \(N - 1\).

Since both \(\Omega_q^{(1,0)}\) and \(\Omega_q^{(0,1)}\) have dimension \(N - 1\), they must both be irreducible (since otherwise there would exist an irreducible left-covariant calculus of dimension strictly less than \(N - 1\) in contradiction of the theorem). Moreover, it is easy to see that \(\Omega_q^{(1,0)}\) and \(\Omega_q^{(0,1)}\) correspond to different ideals of \(C_q[CP^{N-1}]^+\), and consequently are non-isomorphic. This gives us the following corollary:

**Corollary 7.7** The two calculi identified in Theorem 7.6 are \(\Omega_q^{(1,0)}\) and \(\Omega_q^{(0,1)}\).

Heckenberger and Kolb constructed a total differential calculus extending the direct sum calculus \(\Omega_q^{(1,0)} \oplus \Omega_q^{(0,1)}\) as follows: They took the maximal prolongations of \(\Omega_q^{(1,0)}\), and \(\Omega_q^{(0,1)}\), and defined

\[
\Omega_q^k(CP^{N-1}) := \bigoplus_{a+b=k} (\Omega_q^{(1,0)})^a \otimes C_q[CP^{N-1}] (\Omega_q^{(0,1)})^b,
\]

where \((\Omega_q^{(1,0)})^a\) is the space of \(a\)-forms of the maximal prolongation of \(\Omega_q^{(1,0)}\), and \((\Omega_q^{(0,1)})^b\) is the space of \(b\)-forms of the maximal prolongation of \(\Omega_q^{(0,1)}\). They then showed that the partial derivatives \(\partial\) and \(\mathcal{D}\) could be extended to operators on the direct sum \(\bigoplus_{k=1}^{2(N-1)} \Omega_q^k(CP^{N-1})\) giving it the structure of a double complex.

That Heckenberger and Kolb’s construction of \(\Omega_q^k(CP^{N-1})\) is isomorphic to our construction follows from Lemma 7.4 and the integrability of our calculus. That the two constructions of the exterior derivative agree follows from the fact that there exists only one exterior derivative on the maximal prolongation of any first-order differential calculus.
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