Infinite products with strongly $B$-multiplicative exponents

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To Professor Kátai on the occasion of his 70th birthday

Abstract

Let $N_{1,B}(n)$ denote the number of ones in the $B$-ary expansion of an integer $n$. Woods introduced the infinite product $P := \prod_{n \geq 0} \left( \frac{2n+1}{2n+2} \right)^{(-1)^{N_{1,2}(n)}}$ and Robbins proved that $P = 1/\sqrt{2}$. Related products were studied by several authors. We show that a trick for proving that $P^2 = 1/2$ (knowing that $P$ converges) can be extended to evaluating new products with (generalized) strongly $B$-multiplicative exponents. A simple example is

$$\prod_{n \geq 0} \left( \frac{Bn+1}{Bn+2} \right)^{(-1)^{N_{1,B}(n)}} = \frac{1}{\sqrt{B}}$$

MSC: 11A63, 11Y60.

1 Introduction

In 1985 the following infinite product, for which no closed expression is known, appeared in [8, p. 193 and p. 209]:

$$R := \prod_{n \geq 1} \left( \frac{(4n+1)(4n+2)}{4n(4n+3)} \right)^{\varepsilon(n)}$$

where $(\varepsilon(n))_{n \geq 0}$ is the $\pm 1$ Prouhet-Thue-Morse sequence, defined by

$$\varepsilon(n) = (-1)^{N_{1,2}(n)}$$

with $N_{1,2}(n)$ being the number of ones in the binary expansion of $n$. (For more on the Prouhet-Thue-Morse sequence, see for example [5].)
On the one hand, it is not difficult to see that $R = \frac{3}{2^{\sqrt{2}}}$, where

$$Q := \prod_{n \geq 1} \left( \frac{2n}{2n+1} \right)^{\varepsilon(n)} .$$

Namely, splitting the simpler product into even and odd indices and using the relations $\varepsilon(2n) = \varepsilon(n)$ and $\varepsilon(2n+1) = -\varepsilon(n)$, we get

$$Q = \left( \prod_{n \geq 1} \left( \frac{4n}{4n+1} \right)^{\varepsilon(n)} \right) \left( \prod_{n \geq 0} \left( \frac{4n+2}{4n+3} \right)^{-\varepsilon(n)} \right) = \frac{3}{2} \prod_{n \geq 1} \left( \frac{4n(4n+3)}{(4n+1)(4n+2)} \right)^{\varepsilon(n)} = \frac{3}{2R} .$$

(Note that, whereas the logarithm of $R$ is an absolutely convergent series, the logarithm of $Q$ – and similarly the logarithm of the product $P$ below – is a conditionally convergent series, as can be seen by partial summation, using the fact that the sums $\sum_{0 \leq k \leq n} \varepsilon(k)$ only take the values $+1$, $0$ and $-1$, hence are bounded.)

On the other hand, the product $Q$ reminds us of the Woods-Robbins product \[18, 12\]

$$P := \prod_{n \geq 0} \left( \frac{2n+1}{2n+2} \right)^{\varepsilon(n)} = \frac{1}{\sqrt{2}}$$

(generalized for example in \[13, 1, 2, 3, 4, 14\]).

In 1987 during a stay at the University of Chicago, the first author, convinced that the computation of the infinite product $Q$ should not resist the even-odd splitting techniques he was using with J. Shallit, discovered the following trick. First write $QP$ as

$$QP = \left( \frac{1}{2} \right)^{\varepsilon(0)} \prod_{n \geq 1} \left( \frac{2n}{2n+1} \cdot \frac{2n+1}{2n+2} \right)^{\varepsilon(n)} = \frac{1}{2} \prod_{n \geq 1} \left( \frac{n}{n+1} \right)^{\varepsilon(n)} .$$

Now split the indices as we did above, obtaining

$$\prod_{n \geq 1} \left( \frac{n}{n+1} \right)^{\varepsilon(n)} = \left( \prod_{n \geq 1} \left( \frac{2n}{2n+1} \right)^{\varepsilon(n)} \right) \left( \prod_{n \geq 0} \left( \frac{2n+1}{2n+2} \right)^{-\varepsilon(n)} \right) = QP^{-1} .$$

This gives $QP = \frac{1}{2} QP^{-1}$: as the hope of computing $Q$ fades, the trick at least yields an easy way to compute $P = 1/\sqrt{2}$. By extending this trick to $B$-ary expansions, the second author \[14\] found the generalization of $P = 1/\sqrt{2}$ given in Corollary \[5\] of Section \[4.2\].

It happens that the sequence $(\varepsilon(n))_{n \geq 0}$ is strongly $2$-multiplicative (see Definition \[1\] in the next section). The purpose of this paper is to extend the trick to products with more general exponents. For example, we prove the following.

Let $B > 1$ be an integer. For $k = 1, \ldots, B - 1$ define $N_{k,B}(n)$ to be the number of occurrences of the digit $k$ in the $B$-ary expansion of the integer $n$. Also, let

$$s_B(n) := \sum_{0 < k < B} kN_{k,B}(n)$$
be the sum of the $B$-ary digits of $n$, and let $q > 1$ be an integer. Then

$$
\prod_{n \geq 0} \left( \frac{Bn + k}{Bn + k + 1} \right)^{(-1)^{N_k,B(n)}} = \frac{1}{\sqrt{B}},
$$

$$
\prod_{n \geq 0} \prod_{\substack{0 < k < B \\text{mod } q}} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\sin \frac{\pi k}{q} \sin \frac{\pi (2s_B(n) + k)}{q}} = \frac{1}{\sqrt{B}},
$$

and

$$
\prod_{n \geq 0} \prod_{\substack{0 < k < B \\text{mod } q}} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\sin \frac{\pi k}{q} \cos \frac{\pi (2s_B(n) + k)}{q}} = 1.
$$

Note that the use of the trick is not necessarily the only way to compute products of this type: real analysis is used for computing $P$ in [12] and to compute products more general than $P$ in [13]; the core of [1] is the use of Dirichlet series, while [2] deals with complex power series and the second part of [3] with real integrals. It may even happen that, in some cases, the use of the trick gives less general results than other methods. For example, in Remark 3 we show that Corollary 5 of [14] can also be obtained as an easy consequence of [2, Theorem 1].

2 Strongly $B$-multiplicative sequences

We recall the classical definition of a strongly $B$-multiplicative sequence. (For this and for the definitions of $B$-multiplicative, $B$-additive, and strongly $B$-additive, see [6, 9, 7, 11, 10].)

**Definition 1.** Let $B \geq 2$ be an integer. A sequence of complex numbers $(u(n))_{n \geq 0}$ is strongly $B$-multiplicative if $u(0) = 1$ and, for all $n \geq 0$ and all $k \in \{0, 1, \ldots, B-1\}$,

$$u(Bn + k) = u(n)u(k).$$

**Example 1.** If $z$ is any complex number, then the sequence $u$ defined by $u(0) := 1$ and $u(n) := z^{s_B(n)}$ for $n \geq 1$ is strongly $B$-multiplicative.

**Remark 1.** If we do not impose the condition $u(0) = 1$ in Definition 1, then either $u(0) = 0$ holds, or the sequence $(u(n))_{n \geq 0}$ must be identically 0. To see this, note that the relation $u(Bn + k) = u(n)u(k)$ implies, with $n = k = 0$, that $u(0) = u(0)^2$. Hence $u(0) = 1$ or $u(0) = 0$. If $u(0) = 0$, then taking $n = 0$ in the relation gives $u(k) = 0$ for all $k \in \{0, 1, \ldots, B-1\}$, which by (1) implies $u(n) = 0$ for all $n \geq 0$.

**Proposition 1.** If the sequence $(u(n))_{n \geq 0}$ is strongly $B$-multiplicative, and if the $B$-ary expansion of $n \geq 1$ is $n = \sum_j e_j(n)B^j$, then $u(n) = \prod_j u(e_j(n))$. In particular, the only strongly $B$-multiplicative sequence with $u(1) = u(2) = \cdots = u(B-1) = \theta$, where $\theta = 0$ or 1, is the sequence $1, \theta, \theta, \theta, \ldots$.  

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Proof. Use induction on the number of base $B$ digits of $n$. ■

We now generalize the notion of a strongly $B$-multiplicative sequence different from $1, 0, 0, 0, \ldots$

**Definition 2.** Let $B \geq 2$ be an integer. A sequence of complex numbers $(u(n))_{n \geq 0}$ satisfies Hypothesis $H_B$ if there exist an integer $n_0 \geq B$ and complex numbers $v(0), v(1), \ldots, v(B-1)$ such that $u(n_0) \neq 0$ and, for all $n \geq 1$ and all $k = 0, 1, \ldots, B - 1$,  
\[ u(Bn + k) = u(n)v(k). \]

**Proposition 2.**

1. If a sequence $(u(n))_{n \geq 0}$ satisfies Hypothesis $H_B$, then the values $v(0), v(1), \ldots, v(B-1)$ are uniquely determined.

2. A sequence $(u(n))_{n \geq 0}$ has $u(0) = 1$ and satisfies Hypothesis $H_B$ with $u(Bn + k) = u(n)v(k)$ not only for $n \geq 1$ but also for $n = 0$, if and only if the sequence is strongly $B$-multiplicative and not equal to 1, 0, 0, 0, \ldots. In that case, $v(k) = u(k)$ for $k = 0, 1, \ldots, B - 1$.

Proof. If the sequence $(u(n))_{n \geq 0}$ satisfies Hypothesis $H_B$, then $v(k) = u(Bn_0 + k)/u(n_0)$ for $k = 0, 1, \ldots, B - 1$. This implies (1).

To prove the “only if” part of (2), take $n = 0$ in the relation $u(Bn + k) = u(n)v(k)$, yielding $u(k) = u(0)v(k) = v(k)$ for $k = 0, 1, \ldots, B - 1$. Hence $u(Bn + k) = u(n)u(k)$ for all $n \geq 0$ and $k = 0, 1, \ldots, B - 1$. Thus $(u(n))_{n \geq 0}$ is strongly $B$-multiplicative. Since $u(n_0) \neq 0$ for some $n_0 \geq B$, the sequence is not 1, 0, 0, 0, \ldots.

Conversely, suppose that $(u(n))_{n \geq 0}$ is strongly $B$-multiplicative and is not 1, 0, 0, 0, \ldots. Then there exists an integer $\ell_0 \geq 1$ such that $u(\ell_0) \neq 0$. Hence $n_0 := B\ell_0 \geq B$ and $u(n_0) = u(B\ell_0) = u(\ell_0)u(0) = u(\ell_0) \neq 0$. Defining $v(k) := u(k)$ for $k = 0, 1, \ldots, B - 1$, we see that $(u(n))_{n \geq 0}$ satisfies Hypothesis $H_B$, and the proposition follows. ■

**Example 2.** We construct a sequence which satisfies Hypothesis $H_B$ but is not strongly $B$-multiplicative. Let $z$ be a complex number, with $z \notin \{0, 1\}$, and define $u(n) := z^{N_0,B(n)}$, where $N_0,B(n)$ counts the number of zeros in the $B$-ary expansion of $n$ for $n > 0$, and $N_0,B(0) := 0$ (which corresponds to representing zero by the empty sum, that is, the empty word). Note that for all $n \geq 1$ the relation $N_0,B(Bn) = N_0,B(n) + 1$ holds, and for all $k \in \{1, 2, \ldots, B - 1\}$ and all $n \geq 0$ the relation $N_0,B(Bn + k) = N_0,B(n) + N_0,B(k)$ holds. Hence the nonzero sequence $(u(n))_{n \geq 0}$ satisfies Hypothesis $H_B$, with $v(0) := z$ and $v(k) := 1 = u(k)$ for $k = 1, 2, \ldots, B - 1$. But the sequence is not strongly $B$-multiplicative: $u(B \times 1 + 0) = z \neq 1 = u(1)u(0)$.

**Remark 2.** The alternative definition $N_0,B(0) := 1$ (which would correspond to representing zero by the single digit 0 instead of by the empty word) would also not lead to a strongly $B$-multiplicative sequence $u$, since then $u(0) = z \neq 1$, which does not agree with Definition \[.\] (see also Remark \[.\]). On the other hand, the new sequence would still satisfy Hypothesis $H_B$, with the same values $v(k)$, as the same proof shows, since $u(0)$ does not appear in it.
3 Convergence of infinite products

Inspired by the Woods-Robbins product $P$, we want to study products given in the following lemma.

**Lemma 1.** Let $B > 1$ be an integer. Let $(u(n))_{n \geq 0}$ be a sequence of complex numbers with $|u(n)| \leq 1$ for all $n \geq 0$. Suppose that it satisfies Hypothesis $\mathcal{H}_B$ with $|v(k)| \leq 1$ for all $k \in \{0, 1, \ldots, B-1\}$, and that $\left| \sum_{0 \leq k < B} v(k) \right| < B$. Then for each $k \in \{0, 1, \ldots, B-1\}$, the infinite product

$$ \prod_{n \geq \delta_k} \left( \frac{Bn + k}{Bn + k + 1} \right)^{u(n)} $$

converges, where $\delta_k$ — a special case of the Kronecker delta — is defined by

$$ \delta_k := \begin{cases} 
0 & \text{if } k \neq 0 \\
1 & \text{if } k = 0. 
\end{cases} $$

**Proof.** For $N = 1, 2, \ldots$, let

$$ F(N) := \sum_{0 \leq n < N} u(n). $$

Also define for $j = 1, 2, \ldots, B-1$

$$ G(j) := \sum_{0 \leq n < j} v(n) $$

and set $G(0) := 0$. Then, for each $b \in \{0, 1, \ldots, B-1\}$, and for every $N \geq 1$,

$$ F(BN + b) = \sum_{0 \leq n < BN} u(n) + \sum_{BN \leq n < BN+b} u(n) $$

$$ = \sum_{0 \leq n < N} \sum_{0 \leq \ell < B} u(Bn + \ell) + \sum_{0 \leq \ell < b} u(BN + \ell) $$

$$ = \sum_{0 \leq \ell < B} u(\ell) + \sum_{1 \leq n < N} \sum_{0 \leq \ell < B} u(n)v(\ell) + u(N) \sum_{0 \leq \ell < b} v(\ell). $$

Hence, using $|u(n)| \leq 1$ and $|G(b)| \leq B - 1 < B$,

$$ |F(BN + b)| = |F(B) + (F(N) - u(0))G(B) + u(N)G(b)| $$

$$ < |F(B) - u(0)G(B)| + |F(N)||G(B)| + B. $$

This gives the case $d = 1$ of the following inequality, which holds for $d \geq 1$ and $e_\ell \in \{0, 1, \ldots, B-1\}$, and which is proved by induction on $d$ using $|F(e_\ell)| \leq B$:

$$ \left| F \left( \sum_{0 \leq \ell \leq d} e_\ell B^\ell \right) \right| < |F(B) - u(0)G(B)| \left( 1 + \sum_{1 \leq \ell \leq d-1} |G(B)|^\ell \right) + B \left( 1 + \sum_{1 \leq \ell \leq d} |G(B)|^\ell \right). $$

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Hence

\[ F\left( \sum_{0 \leq t \leq d} e_t B^t \right) < \begin{cases} B(3d + 1) & \text{if } |G(B)| \leq 1, \\ 3B |G(B)|^{d+1} - 1 & \text{if } |G(B)| > 1. \end{cases} \]

This implies that for some constant \( C = C(B, v) \), and for every \( N \) large enough,

\[ |F(N)| < \begin{cases} C \log N & \text{if } |G(B)| \leq 1, \\ \frac{C|G(B)|}{\log B} & \text{if } |G(B)| > 1. \end{cases} \]

Since \( |G(B)| < B \) by hypothesis, we can define \( \alpha \in (0, 1) \) by

\[ \alpha := \begin{cases} \frac{1}{2} \frac{\log |G(B)|}{\log B} & \text{if } |G(B)| \leq 1, \\ \frac{1}{2} \frac{\log |G(B)|}{\log B} & \text{if } |G(B)| > 1. \end{cases} \]

Hence for every \( N \) large enough \( |F(N)| < CN^\alpha \). It follows, using summation by parts, that the series \( \sum_n u(n) \log \frac{Bn + k}{Bn + k + 1} \) converges, hence the lemma. \( \blacksquare \)

**Remark 3.**

(1) Here and in what follows, expressions of the form \( a^z \), where \( a \) is a positive real number and \( z \) a complex number, are defined by \( a^z := e^{z \log a} \), and \( \log a \) is real.

(2) For more precise estimates of summatory functions of (strongly) \( B \)-multiplicative sequences, see for example [7, 10]. (In [10] strongly \( B \)-multiplicative sequences are called completely \( B \)-multiplicative.)

## 4 Evaluation of infinite products

This section is devoted to computing some infinite products with exponents that satisfy Hypothesis \( \mathcal{H}_B \), including some whose exponents are strongly \( B \)-multiplicative.

### 4.1 General results

**Theorem 1.** Let \( B > 1 \) be an integer. Let \( (u(n))_{n \geq 0} \) be a sequence of complex numbers with \( |u(n)| \leq 1 \) for all \( n \geq 0 \). Suppose that \( u \) satisfies Hypothesis \( \mathcal{H}_B \), with complex numbers \( v(0), v(1), \ldots, v(B-1) \) such that \( |v(k)| \leq 1 \) for \( k \in \{0, 1, \ldots, B-1\} \) and \( |\sum_{0 \leq k < B} v(k)| < B \). Then the following relation between nonempty products holds:

\[
\prod_{0 \leq k < B} \prod_{n \geq \delta_k} \left( \frac{Bn + k}{Bn + k + 1} \right)^{u(n)(1-v(k))} = \frac{1}{B^{u(0)}} \prod_{0 < k < B} \left( \frac{k}{k + 1} \right)^{u(k)-u(0)v(k)}.
\]

**Proof.** The condition \( |\sum_{0 \leq k < B} v(k)| < B \) prevents \( v \) from being identically equal to 1 on \( \{0, 1, \ldots, B-1\} \), so the left side of the equation is not empty. Since \( B > 1 \), so is the right.

We first show that

\[
\prod_{0 \leq k < B} \prod_{n \geq \delta_k} \left( \frac{Bn + k}{Bn + k + 1} \right)^{u(n)} = \frac{1}{B^{u(0)}} \prod_{n \geq 1} \left( \frac{n}{n + 1} \right)^{u(n)} \quad (*)
\]
Example 3. As in Example 2, the sequence \( u \) defined by \( u(n) = z^{N_0, B(n)} \), with \( z \notin \{0, 1\} \), satisfies Hypothesis \( \mathcal{H}_B \), and \( \sum_{0 \leq k < B} v(k) = z + B - 1 \). If furthermore \( |z| \leq 1 \), then

\[
\prod_{n \geq 1} \left( \frac{Bn}{Bn + 1} \right)^{u(n)(1-z)^{N_0, B(n)}} = B.
\]

Corollary 1. Fix an integer \( B > 1 \). If \((u(n))_{n \geq 0}\) is strongly \( B \)-multiplicative, satisfies \( |u(n)| \leq 1 \) for all \( n \geq 0 \), and is not equal to either of the sequences 1, 0, 0, 0, . . . or 1, 1, 1, . . . , then

\[
\prod_{n \geq 0} \prod_{0 < k < B \atop u(k) \neq 1} \left( \frac{Bn + k}{Bn + k + 1} \right)^{u(n)(1-u(k))} = \frac{1}{B}.
\]

Proof. Using Theorem 1 and Proposition 2, part (2) it suffices to prove that \( | \sum_{0 \leq k < B} u_k | < B \). Since \( |u_n| \leq 1 \) for all \( n \geq 0 \), we have \( | \sum_{0 \leq k < B} u_k | \leq B \). From the equality case of the triangle inequality, it thus suffices to prove that the numbers \( u_0, u_1, \ldots, u_{B-1} \) are not all equal to a same complex number \( z \) with \( |z| = 1 \). If they were, then, since \( u_0 = 1 \), we would have \( u_0 = u_1 = \ldots = u_{B-1} = 1 \). Hence \((u(n))_{n \geq 0} = 1, 1, 1, \ldots \) from Proposition 1, a contradiction.

Addendum. Theorem 1 and Corollary 7 can be strengthened, as follows.

(1) If \( B, u, \) and \( v \) satisfy the hypotheses of Theorem 1, then

\[
\sum_{0 \leq k < B \atop u(k) \neq 1} (1 - v(k)) \sum_{n \geq \delta_k} u(n) \log \frac{Bn + k}{Bn + k + 1} = -u(0) \log B + \sum_{0 < k < B} (u(k) - u(0)v(k)) \log \frac{k}{k + 1}.
\]
(2) If $B$ and $u$ satisfy the hypotheses of Corollary 1, then
\[
\sum_{n \geq 0} \sum_{\substack{0 < k < B \\ u(k) \neq 1}} u(n)(1 - u(k)) \log \frac{Bn + k}{Bn + k + 1} = - \log B.
\]

Proof. Write the proofs of Theorem 1 and Corollary 1 additively instead of multiplicatively.

Remark 4. The Addendum cannot be proved by just taking logarithms in the formulas in Theorem 1 and Corollary 1. To illustrate the problem, note that while
\[
\prod_{n \geq 0} e^{(-1)^n 8i} = 1
\]
(because the product converges to $e^{2\pi i}$), the log equation is false:
\[
\sum_{n \geq 0} \frac{(-1)^n 8i}{2n + 1} = 2\pi i \neq 0 = \log 1.
\]

Example 4. With the same $u$ and $z$ as in Example 3, Addendum (1) yields
\[
\sum_{n \geq 0} z^{N_{0,B}(n)} \log \frac{Bn}{Bn + 1} = \log B + \frac{1}{z - 1}.
\]
Hence
\[
\prod_{n \geq 1} \left( \frac{Bn}{Bn + 1} \right)^{z^{N_{0,B}(n)}} = B^{\frac{1}{z - 1}}.
\]
(Note the similarity between this product and the one in Example 3. Neither implies the other, but of course the preceding log equation implies both.)

If we modify the sequence $u$ as in Remark 2, we get the same two formulas, because the value $N_{0,B}(0)$ does not appear in them.

Corollary 2. Fix integers $B, q, p$ with $B > 1$, $q > p > 0$, and $B \equiv 1 \mod q$. Then
\[
\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \not\equiv 0 \mod q}} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\sin \frac{\pi kp}{q} \sin \frac{\pi (2n+k)p}{q}} = \frac{1}{\sqrt{B}}
\]
and
\[
\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \not\equiv 0 \mod q}} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\sin \frac{\pi kp}{q} \cos \frac{\pi (2n+k)p}{q}} = 1.
\]
Proof. Let \( \omega := e^{2\pi ip/q} \). Since \( B \equiv 1 \mod q \), we may take \( u(n) := \omega^n \) in Addendum (2), yielding the formula

\[
\sum_{n \geq 0} \sum_{0 < k < B, \ k \not\equiv 0 \mod q} \omega^n(1 - \omega^k) \log \frac{Bn + k}{Bn + k + 1} = -\log B.
\]

Writing \( \omega^n(1 - \omega^k) = -2i\omega^{n+k} \sin \frac{\pi kp}{q} \), and multiplying the real and imaginary parts of the formula by \( 1/2 \), the result follows. \( \blacksquare \)

Example 5. Take \( B = 5 \), \( p = 1 \), and \( q = 4 \). Squaring the products, we get

Define \( \sigma(n) \) to be \(+1\) if \( n \) is a square modulo \( 4 \), and \(-1\) otherwise, that is,

\[
\sigma(n) := \begin{cases} +1 & \text{if } n \equiv 0 \text{ or } 1 \mod 4, \\ -1 & \text{if } n \equiv 2 \text{ or } 3 \mod 4. \end{cases}
\]

Then

\[
\prod_{n \geq 0} \left( \frac{5n + 1}{5n + 2} \right)^{\sigma(n)} \left( \frac{5n + 2}{5n + 3} \right)^{\sigma(n) + \sigma(n + 1)} \left( \frac{5n + 3}{5n + 4} \right)^{\sigma(n + 1)} = \frac{1}{5}
\]

and

\[
\prod_{n \geq 0} \left( \frac{5n + 1}{5n + 2} \right)^{\sigma(n - 1)} \left( \frac{5n + 2}{5n + 3} \right)^{\sigma(n - 1) + \sigma(n)} \left( \frac{5n + 3}{5n + 4} \right)^{\sigma(n)} = 1.
\]

4.2 The sum-of-digits function \( s_B(n) \)

Other products can also be obtained from Corollary 1. We give three corollaries, each of which generalizes the Woods-Robbins formula \( P = 1/\sqrt{2} \) in the Introduction. Recall that \( s_B(n) \) denotes the sum of the \( B \)-ary digits of the integer \( n \).

Corollary 3. Fix an integer \( B > 1 \) and a complex number \( z \) with \( |z| \leq 1 \). If \( z \notin \{0, 1\} \), then

\[
\prod_{n \geq 0} \prod_{0 < k < B, \ z^k \neq 1} \left( \frac{Bn + k}{Bn + k + 1} \right)^{s_B(n)(1 - z^k)} = \frac{1}{B}.
\]

Proof. Take \( u(n) := z^{s_B(n)} \) in Corollary 1 and note that \( s_B(k) = k \) when \( 0 < k < B \). \( \blacksquare \)

Example 6. Take \( B = 2 \) and \( z = 1/2 \). Squaring the product, we obtain

\[
\prod_{n \geq 0} \left( \frac{2n + 1}{2n + 2} \right)^{(1/2)^{s_2(n)}} = \frac{1}{4}.
\]

Corollary 4. Let \( B, p, q \) be integers with \( B > 1 \) and \( q > p > 0 \). Then

\[
\prod_{n \geq 0} \prod_{0 < k < B, \ k \not\equiv 0 \mod q} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\sin \frac{\pi kp}{q}} \sin \frac{\pi (2s_B(n) + k)p}{q} = \frac{1}{\sqrt{B}}
\]
and
\[
\prod_{n \geq 0} \prod_{0 < k < B \mod q} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\sin \frac{\pi kp}{q} \cos \frac{\pi (2s_B(n) + k)p}{q}} = 1.
\]

**Proof.** Use the proof of Corollary \[2\] but replace \( B \equiv 1 \mod q \) with \( s_B(Bn + k) = s_B(n) + k \) when \( 0 \leq k < B \), and replace \( \omega^n \) with \( \omega^{s_B(n)} \).

**Example 7.** Take \( B = 2, q = 4, \) and \( p = 1 \). Squaring the products and defining \( \sigma(n) \) as in Example \[5\], we get
\[
\prod_{n \geq 0} \left( \frac{2n + 1}{2n + 2} \right)^{\sigma(s_2(n))} = \frac{1}{2} \quad \text{and} \quad \prod_{n \geq 0} \left( \frac{2n + 1}{2n + 2} \right)^{\sigma(s_2(n) + 1)} = 1.
\]

In the same spirit, we recover a result from \[3\] p. 369-370).

**Example 8.** Taking \( B = q = 3 \) and \( p = 1 \) in Corollary \[4\] we obtain two infinite products. Raising the second to the power \(-2/\sqrt{3}\) and multiplying by the square of the first, we get

*Define \( \theta(n) \) by*

\[
\theta(n) := \begin{cases} 
1 & \text{if } n \equiv 0 \text{ or } 1 \mod 3, \\
-2 & \text{if } n \equiv 2 \mod 3.
\end{cases}
\]

Then
\[
\prod_{n \geq 0} (3n + 1)^{\theta(s_3(n))}(3n + 2)^{\theta(s_3(n) + 1)}(3n + 3)^{\theta(s_3(n) + 2)} = \frac{1}{3}.
\]

**Corollary 5 (\[14\]).** Let \( B > 1 \) be an integer. Then
\[
\prod_{n \geq 0} \prod_{0 < k < B \mod q} \left( \frac{Bn + k}{Bn + k + 1} \right)^{(-1)^{s_B(n)}} = \frac{1}{\sqrt{B}}.
\]

**Proof.** Take \( z = -1 \) in Corollary \[3\] (or take \( q = 2 \) and \( p = 1 \) in Corollary \[4\]).

**Example 9.** With \( B = 2 \), since \( s_2(n) = N_{1,2}(n) \), we recover the Woods-Robbins formula \( P = 1/\sqrt{2} \). Taking \( B = 6 \) gives
\[
\prod_{n \geq 0} \left( \frac{(6n + 1)(6n + 3)(6n + 5)}{(6n + 2)(6n + 4)(6n + 6)} \right)^{(-1)^{s_6(n)}} = \frac{1}{\sqrt{6}}.
\]

**Remark 5.** Corollary \[\text{[5]}\] can also be obtained from \[\text{[2]}\] Theorem 1], as follows. Taking \( x \) equal to \(-1\) and \( j \) equal to 0 in that theorem gives
\[
\sum_{n \geq 0} (-1)^{s_B(n)} \log \frac{n + 1}{B[n/B] + B} = -\frac{1}{2} \log B
\]
where \([x]\) is the integer part of \(x\). But the series is equal to
\[
\sum_{m \geq 0} \sum_{0 \leq k < B} (-1)^{s_B(Bm+k)} \log \frac{Bm + k + 1}{Bm + B} = \sum_{m \geq 0} (-1)^{s_B(m)} \sum_{0 \leq k < B} (-1)^k \log \frac{Bm + k + 1}{Bm + B}
\]
\[
= \sum_{m \geq 0} (-1)^{s_B(m)} \left( \sum_{k \text{ odd}} \log \frac{Bm + k}{Bm + k + 1} \right)
\]
where the last equality follows by looking separately at the cases \(B\) even and \(B\) odd.

### 4.3 The counting function \(N_{j,B}(n)\)

We can also compute some infinite products associated with counting the number of occurrences of one or several given digits in the base \(B\) expansion of an integer.

**Definition 3.** If \(B\) is an integer \(\geq 2\) and if \(j\) is in \(\{0, 1, \ldots, B-1\}\), let \(N_{j,B}(n)\) be the number of occurrences of the digit \(j\) in the \(B\)-ary expansion of \(n\) when \(n > 0\), and set \(N_{j,B}(0) := 0\).

**Corollary 6.** Let \(B, q, p\) be integers with \(B > 1\) and \(q > p > 0\). Let \(J\) be a nonempty, proper subset of \(\{0, 1, \ldots, B-1\}\). Define \(N_{j,B}(n) := \sum_{j \in J} N_{j,B}(n)\). Then the following equalities hold:
\[
\prod_{k \in J} \prod_{n \geq \delta_k} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\sin \left( \frac{2\pi N_{j,B}(n)+1}{q} \right)} = B^{\frac{-1}{2} \sin \frac{\pi p}{q}}
\]
and
\[
\prod_{k \in J} \prod_{n \geq \delta_k} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\cos \left( \frac{2\pi N_{j,B}(n)+1}{q} \right)} = 1.
\]

**Proof.** Let \(\omega := e^{2\pi i/p/q}\). We denote \(u_{q,j,B}(n) := \omega^{N_{j,B}(n)}\) and \(u_{q,j,B}(n) := \prod_{j \in J} u_{q,j,B}(n) = \omega^{N_{j,B}(n)}\). Note that, for every \(j\) in \(\{1, 2, \ldots, B-1\}\), the sequence \((u_{q,j,B}(n))_{n \geq 0}\) is strongly \(B\)-multiplicative and nonzero, hence satisfies Hypothesis \(\mathcal{H}_B\). The sequence \((u_{q,0,B}(n))_{n \geq 0}\) also satisfies Hypothesis \(\mathcal{H}_B\), as is seen by taking \(z = \omega\) in Example 2. Therefore the sequence \((u_{q,j,B}(n))_{n \geq 0}\) satisfies Hypothesis \(\mathcal{H}_B\), with, for \(k = 0, 1, \ldots, B-1\), the value \(v(k) := \omega\) if \(k \in J\) and \(v(k) := 1\) otherwise.

Now \(|u_{q,j,B}(n)| = 1\) for \(n \geq 0\), and \(|v(k)| = 1\) for \(k = 0, 1, \ldots, B-1\). Furthermore, \(|\sum_{0 \leq k < B} v(k)| < B\), since \(v\) is not constant on \(\{0, 1, \ldots, B-1\}\). Thus we may apply Addendum (1) with \(u(n) := u_{q,j,B}(n)\), obtaining
\[
(1 - \omega) \sum_{k \in J} \sum_{n \geq \delta_k} \omega^{N_{j,B}(n)} \log \frac{Bn + k}{Bn + k + 1} = - \log B.
\]
Writing \((1 - \omega)\omega^{N_{j,B}(n)} = -2i\omega^{N_{j,B}(n)+1/2} \sin \frac{\pi p}{q}\), and taking the real and imaginary parts of the summation, the result follows. ■
Example 10. Taking $q = 2$ and $p = 1$ in the first formula gives
\[
\prod_{k \in J} \prod_{n \geq \delta_k} \left( \frac{Bn + k}{Bn + k + 1} \right)^{(-1)^{N_{j,B}(n)}} = \frac{1}{\sqrt{B}}.
\]

An application is an alternate proof of Corollary 5: take $J$ to be the set of odd numbers in \{1, 2, \ldots, B - 1\}; since $s_B(n) = \sum_{0 < k < B} k N_{k,B}(n)$, it follows that $(-1) \sum_{j \in J} N_{j,B}(n) = (-1)^{s_B(n)}$.

Remark 6. Corollary 6 requires that $J$ be a proper subset of \{0, 1, \ldots, B - 1\}. Suppose instead that $J = \{0, 1, \ldots, B - 1\}$. Then $N_{j,B}(n)$ is the number of $B$-ary digits of $n$ if $n > 0$ (that is, $N_{j,B}(n) = \left\lfloor \frac{\log n}{\log B} \right\rfloor + 1$), and $N_{j,B}(0) = 0$. In that case, Corollary 6 does not apply, and the products may diverge. For example, when $B = q = 2$ and $p = 1$ the logarithm of the first product is equal to the series
\[
- \log 2 + \sum_{n \geq 1} (-1)^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \log \frac{n + 1}{n},
\]
which does not converge. However, note its resemblance with Vacca’s (convergent) series for Euler’s constant \cite{16}
\[
\gamma = \sum_{n \geq 1} \left\lfloor \frac{\log n}{\log 2} \right\rfloor \frac{(-1)^n}{n}.
\]

Corollary 7. Let $B, q, p$ be integers with $B > 1$ and $q > p > 0$. Then for $k = 0, 1, \ldots, B - 1$ the following equalities hold:
\[
\prod_{n \geq \delta_k} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\sin \left(\frac{\pi(2N_{k,B}(n)+1)p}{q}\right)} = B^{-2 \sin \frac{\pi p}{q}}
\]
and
\[
\prod_{n \geq \delta_k} \left( \frac{Bn + k}{Bn + k + 1} \right)^{\cos \left(\frac{\pi(2N_{k,B}(n)+1)p}{q}\right)} = 1.
\]

Proof. Take $J := \{k\}$ in Corollary 6 (The case $k = 0$ and $p = 1$ is Example 4 with $z = e^{2\pi i/q}$.)

Example 11. Taking $q = 2$ and $p = 1$ in the first formula (or taking $J = \{k\}$ in Example 10) yields
\[
\prod_{n \geq \delta_k} \left( \frac{Bn + k}{Bn + k + 1} \right)^{(-1)^{N_{k,B}(n)}} = \frac{1}{\sqrt{B}}.
\]

In particular, if $B = 2$ the choice $k = 1$ gives the Woods-Robbins formula $P = 1/\sqrt{2}$, and $k = 0$ gives
\[
\prod_{n \geq 1} \left( \frac{2n}{2n + 1} \right)^{(-1)^{N_{0,2}(n)}} = \frac{1}{\sqrt{2}}.
\]
Remark 7. For base $B = 2$, the formulas in Example 11 are special cases of results in [4], where $N_{j,2}(n)$ is generalized to counting the number of occurrences of a given word in the binary expansion of $n$. On the other hand, the value of the product $Q$ in the Introduction,

$$Q = \prod_{n \geq 1} \left( \frac{2n}{2n+1} \right)^{(-1)^{N_{j,2}(n)}}$$

remains a mystery.

Example 12. Take $B = q = 3$ and $p = 1$. Raising the first product to the power $2/\sqrt{3}$ and squaring the second, we obtain

Define $\eta(n)$ by

$$\eta(n) := \begin{cases} +1 & \text{if } n \equiv 0 \mod 3, \\ 0 & \text{if } n \equiv 1 \mod 3, \\ -1 & \text{if } n \equiv 2 \mod 3, \end{cases}$$

and define $\theta(n)$ as in Example 8. Then for $k = 0, 1, 2$

$$\prod_{n \geq \delta_k} \left( \frac{3n + k}{3n + k + 1} \right)^{\eta(N_{k,3}(n))} = \frac{1}{3^{2/3}} \quad \text{and} \quad \prod_{n \geq \delta_k} \left( \frac{3n + k}{3n + k + 1} \right)^{\theta(N_{k,3}(n) + 1)} = 1.$$ 

4.4 The Gamma function

It can happen that the exponent in some of our products is a periodic function of $n$. For example, this is obviously the case in Corollary 2. To take another example, it is not hard to see that if $B$ odd, then $(-1)^{s_B(n)} = (-1)^n$. Hence Corollary 5 gives

$$\prod_{n \geq 0} \prod_{0 < k < B \text{ odd}} \left( \frac{Bn + k}{Bn + k + 1} \right)^{(-1)^n} = \frac{1}{\sqrt{B}} \quad (B \text{ odd}). \quad (**)$$

(This formula can also be obtained from Corollary 2 with $q = 2$ and $p = 1$.) For instance

$$P_{1,3} := \prod_{n \geq 0} \left( \frac{3n + 1}{3n + 2} \right)^{(-1)^n} = \frac{1}{\sqrt{3}}.$$ 

The product $P_{1,3}$ can also be computed using the following corollary of the Weierstrass product for the Gamma function [17, Section 12.13].

If $d$ is a positive integer and $a_1 + a_2 + \cdots + a_d = b_1 + b_2 + \cdots + b_d$, where the $a_j$ and $b_j$ are complex numbers and no $b_j$ is zero or a negative integer, then

$$\prod_{n \geq 0} \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_d)} = \frac{\Gamma(b_1) \cdots \Gamma(b_d)}{\Gamma(a_1) \cdots \Gamma(a_d)}.$$ 

Combining this with the relation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ gives $P_{1,3} = 1/\sqrt{3}$. 

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The computation can be generalized, using Gauss’ multiplication theorem for the Gamma function, to give another proof of Corollary 5 for $B$ odd. Likewise, an analog of the odd-$B$ case of Corollary 5 can be proved for even $k$:

$$\prod_{n \geq 1} \prod_{0 \leq k < B, k \text{ even}} \left( \frac{Bn + k}{Bn + k + 1} \right)^{(-1)^n} = \frac{\pi \sqrt{B}}{2^B} \left( \frac{B - 1}{(B - 1)/2} \right) (B \text{ odd}).$$

Multiplying this by the formula

$$\prod_{n \geq 1} \prod_{0 < k < B, k \text{ odd}} \left( \frac{Bn + k}{Bn + k + 1} \right)^{(-1)^n} = \frac{2^{B-1}}{\sqrt{B}} \left( \frac{B - 1}{(B - 1)/2} \right)^{-1} (B \text{ odd}),$$

which is (**) rewritten, yields Wallis’ product for $\pi$. (For an evaluation of the preceding two products when $B = 2$, see [15, Example 7].)

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