The spherical symmetry Black hole collapse in expanding universe

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The spherical symmetry Black holes are considered in expanding background. The singularity line and the marginally trapped tube surface behavior are discussed. In particular, we address the conditions of whether a dynamical horizon forms for these cosmological black holes. We also discuss the cosmological constant effect on these black holes and the redshift of the light which comes from the marginally trapped tube surface.

I. INTRODUCTION

The study of black holes in stationary and asymptotically flat spacetimes has led to many remarkable insights. But, as we know, our universe is not stationary and is in fact undergoing cosmological expansion. Let us use the term cosmological black hole for any solution of Einstein equations representing a collapsing overdense region in a cosmological background, leading to an infinite density at its center [1]. The first attempt in this direction is due to McVittie [2] who introduced a spacetime metric that represents a point mass embedded in a Friedmann-Robertson-Walker (FRW) universe. There have been several other attempts to construct solutions of Einstein equations representing such a collapsing central mass. Gluing of a Schwarzschild manifold to an expanding FRW manifold is one of such attempts, made first by Einstein and Straus [3].

Now, a widely used metric to describe the gravitational collapse of a spherically symmetric dust cloud is the so-called Tolman-Bondi-Lemaitre (LTB) metric [4]. These models have been extensively studied for the validity of the cosmic censorship conjecture [5, 6]. It was pointed out in [7] that the model admits cosmological black holes.

Much of the literature on BH’s focuses on stationary and asymptotically flat situations [9]. It is therefore desirable to have black hole models embedded in a cosmological environment to see if there may be considerable differences from the familiar Schwarzschild black

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hole and if there are quasi-local characteristics of it in place of the global and teleological event horizon. The need for a local definition of black holes and their horizons have led to concepts such as Hayward’s trapping horizons [10], Ashtekar’s isolated horizons [12], Ashtekar and Krishnan’s dynamical horizon (DH) [13], and Booth and Fairhurst’s slowly evolving horizon [14].

There have been some previous studies of BH’s in cosmological situations, for example, to glue two different LTB metrics to study the structure formation out of an initial mass condensation or the formation of a galaxy with a central black hole [15] and [16]. However, since the structure of the metric outside these mass condensations are fixed by hand to match to a specific galaxy or cluster feature, we are faced with the shortcomings of the cut and paste models. Harada et. al was also interested in the behavior of primordial black holes within cosmological models to probe the gravitational memory and back hole radiation in expanding universe [17].

Our main interest is to consider the black hole (BH) properties within the expanding universe model. We consider different properties of the LTB metric as a cosmological BH in section II. These properties are generalized to perfect fluid in section III. Section IV is devoted to the cosmological constant effect on spherical cosmological BH.

II. LTB METRIC

The LTB [4] metric is a spherically symmetric non-static solution of the Einstein equations with a dust source.

The LTB metric may be written in synchronous coordinates as

\[ ds^2 = -dt^2 + \frac{R'^2}{1 + f(r)} dr^2 + R(t, r)^2 d\Omega^2. \]  

and represents a pressure-less perfect fluid satisfying

\[ \rho(r, t) = \frac{2M'(r)}{R^2 R'}, \quad \dot{R}^2 = f + \frac{2M}{R}. \]

Here dot and prime denote partial derivatives with respect to the parameters \( t \) and \( r \) respectively. The angular distance \( R \), depending on the value of \( f \), is given by

\[ R = -\frac{M}{f} (1 - \cos(\eta(r, t))), \]

\[ \eta - \sin(\eta) = \frac{(-f)^{3/2}}{M} (t - t_n(r)), \]

for \( f < 0 \), and

\[ R = \left(\frac{9}{2}M\right)^{\frac{1}{3}} (t - t_n)^{\frac{2}{3}}, \]

(4)
for \( f = 0 \), and

\[
R = \frac{M}{f} (\cosh(\eta(r,t)) - 1),
\]

\[
\sinh(\eta) - \eta = \frac{f^{3/2}}{M} (t - t_n(r)), \quad (5)
\]

for \( f > 0 \).

The metric is covariant under the rescaling \( r \to \tilde{r}(r) \). Therefore, one can fix one of the three free functions of the metric, i.e. \( t_n(r), f(r), \) and \( M(r) \). One can shows that \( M(r) \) represents the mass accumulation in a 2-sphere with radius \( r \), more precisely, the Misner-Sharp mass \([8]\).

There are two generic singularities of this metric: the big bang and big crunch singularity (shell focusing singularity) at \( R(t,r) = 0 \), and the shell crossing one at \( R'(t,r) = 0 \). However, if \( \frac{M'}{Rt'} \) and \( \frac{M}{R} \) is finite at \( R = 0 \) then there is no shell focusing singularity. Similarity, if \( \frac{M'}{R'} \) is finite at \( R' = 0 \) then there is no shell crossing singularity. In addition, to get rid of the unnecessary complications of the shell focusing singularity, corresponding to a non-simultaneous big bang singularity, we will assume \( t_n(r) = 0 \). This will enable us to concentrate on the behavior of the collapse of an overdense region in an expanding universe without interfering with the complexity of the inherent bang singularity of the metric \([7]\).

**Proposition II.1** There exists a marginally trapped tube for dust cosmological BH.

As we know any dynamical BH has collapsing region (\( \dot{R} < 0 \)). If we calculate the expansion for ingoing and outgoing null geodesic then we get, \( \theta_\ell \propto (1 - \frac{\sqrt{2M + f}}{\sqrt{1+f}}) \), \( \theta_n \propto (-1 - \frac{\sqrt{2M + f}}{\sqrt{1+f}}) < 0 \). Hence, we see in collapsing region, the expansion for null outgoing geodesic changes its sign from negative to positive at \( R = 2M \) as we go to larger \( R \) and ingoing null geodesics expansion is negative everywhere. Therefore, the 3-manifold \( R = 2M \) is a marginally trapped tube (MTT).

Now we prove that, there is MTT between singularity line (\( R = 0, \eta = 2\pi \)) and \( \dot{R} = 0, \eta = \pi \) (boundary between collapsing and expanding region). It can be seen from \([3]\) that we have \( R > 2M \) on \( \dot{R} = 0 \) (it is sufficient to put \( f > -1 \) and \( \eta = \pi \)). We know that at the singularity (\( R=0 \)) the Misner-Sharp mass in non zero \( M(r = r_s) = M_s > R_s = 0 \), so if we look at the \( R \) and 2M values at \( R \) plane, there is a \( R_0 \) between \( R = 0 \) and \( \dot{R} = 0 \) that \( R_0 = 2M \) so we have the apparent horizon. This MTT can be spacelike, timelike or null surface.
A. LTB solutions as a cosmological BH

For LTB solutions to be asymptotically FRW certain conditions have to be fulfilled. We first note that FRW spaces are special cases of LTB metrics: if \( R(r, t) = ra(t) \) we obviously get the homogeneous FRW solutions. For the vanishing bang time this corresponds to \( M(r) = cr^3, f(r) = -r^2, 0, r^2 \). Therefore, to have an asymptotically open FRW LTB solution must have \( M(r) = cr^3, f(r) = r^2 \).

Assumptions: The first assumption is that the Misner-Sharp mass of round spheres increases monotonically concerning comoving coordinate \( r \); \( M'(r) > 0 \). Our second assumption is that we have no shell crossing and \( R' > 0 \) everywhere. The third assumption expresses the idea that wants to look for conditions leading to an overdense region near the center \( r = 0 \), within an expanding universe with \( \dot{R} > 0 \) (at least) far from the center. However, overdensities in a region around the center require \( \dot{R} < 0 \), corresponding to the collapse phase of the overdense region, which we may assume to start at a time \( t_c > 0 \). We will say that a LTB solution represents a LTB cosmological BH if it satisfied these conditions. Finally, equations (3), (4), and (5) it is easily seen that for the collapsing region one has to have \( f(r) < 0 \). In contrast, for the universe outside there must be \( f(r) > 0 \) having expansion for late time (it can be \( f(r) < 0 \) and we have expanding phase, but after some time this region changes the phase from expanding to collapsing phase). Hence there must be at least one root for \( f(r) \) (see Fig.1).

The collapse of the overdense region leads to two new conditions on the metric coefficients. First, we see from [7], that at any constant time shells corresponding to \( 0 < \eta(r) < \pi \) are in an expanding phase and those corresponding to \( \pi < \eta(r) < 2\pi \) are in the collapsing phase.

We will now look for LTB solutions fulfilling these assumptions.

Proposition II.2 There is a finite time for LTB cosmological BH at which \( \frac{dr}{dt} \mid_{MTT} \) becomes positive.

Proof: For LTB metric we have,

\[
R' = \left( \frac{M'}{M} - \frac{f'}{f} \right)R + \left( -\dot{R} \right) \left( \frac{M'}{M} - 3 \frac{f'}{f} \right). \tag{6}
\]

In apparent horizon, \( \frac{dr}{dt} \mid_{AH} = \frac{-R}{R^2 - 2M^2} \). According to before proposition. [II.1], we have \( \dot{R} \mid_{AH} = -\sqrt{1 + \frac{f}{f'}} \neq 0 \). We know form the above equation that

\[
R' - 2M' \mid_{MTT} = -\frac{f'}{f} R + \left( -\dot{R} \right) \left( \frac{M'}{M} - 3 \frac{f'}{f} \right). \tag{7}
\]

The first term on right hand side can be negative or positive and the second term is +. If the first term be negative, it is easy to see from above equation that there will be a finite \( t_0 \) that \( \frac{f'}{f} R < \left( -t_0 \dot{R} \right) \left( \frac{M'}{M} - 3 \frac{f'}{f} \right) \) on MTT (because the second term is increasing according
to $t$ in collapsing region while the first term remain finite collapsing region). Hence, $\frac{dr}{dt}|_{AH} > 0$.

As we see there is a finite time $t_0$ at which $R' - 2M' > 0$ and so $R' - M' > 0$; therefor apparent horizon become space like [7]. This is a dynamical horizon which is good candidate for black hole boundary. We assume that the collapsing region is limited a finite region $r_0$ in the expanding region then apparent horizon is limited too. Consider following proposition from analysis,
Proposition II.3 If \( \frac{dr(t)}{dt} > 0 \) and \( r < r_0 \), for any \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \) there will be \( t_0 \) such that \( \frac{dr}{dt} < \epsilon_1 \) or \( \frac{dr}{dt} < \frac{1}{r^2} \).

As \( t \) becomes larger and larger, the \( \frac{dr}{dt} |_{\text{AH}} \) tends to zero. One can see that the \( r \) coordinate at event horizon has the same behavior of singularity line at late time \([18]\).

Proposition II.4 The singularity line slope in the \( t - r \) plane is increasing for cosmological dust collapse.

We know that \( r \) is the comoving radius. We want to construct a collapsing spherical matter in expanding universe, so we expect that the singularity forms at the center, \( r = 0 \) firstly and then any other shells fall into the singularity. Each shell which is more nearer to the center will fall into the singularity earlier. Therefore, the above expression in \( t - r \) plane can be mathematically express as \( \frac{dr}{dt} |_{\text{singularity}} = t'_s(r) > 0 \). We should note that \( t_s(r) \) and \( \dot{R} = 0 \) have the same behavior in \( t - r \) plane because they correspond to the \( \eta = 2\pi \) and \( \eta = \pi \) respectively and their difference is only 2 factor.

Now consider the first root of the \( f(r = r_0) = 0 \) which is outside collapsing region and is in the expanding phase (at any time). It has finite mass \( M \) which corresponds to \( \eta = 0 \). The singularity line \( t_s(r) \) and \( \dot{R} = 0 \) cannot reach to these surface in any time at \( t - r \) plan. Hence, there is a \( r = r_0 \) (such that \( f(r = r_0) = 0 \)) that the \( t_s(r) \) and \( \dot{R} = 0 \) curves become asymptote to \( r = r_0 \) in \( t - r \) plan at late time.

Although the space time is not asymptotically flat, for the class of LTB BHs under consideration we can still introduce the notion at an event horizon as ”the very last ray to reach future singularity” or ”the light ray that divides those observers who cannot escape the future singularity from thus who do” then we have the following result.

Proposition II.5 Every trapped surface \( (T) \) for cosmological BH is located inside the event horizon.

Our matter is dust and satisfies energy conditions. Suppose that there is a point \( p \) that is located outside the event horizon. Therefore, it can send a ray to point \( q \) infinity (not singularity). Hence, according to theorem 9.3.11 at \([9]\) there is null geodesic \( \gamma \) from \( p \) to \( q \) is orthogonal to \( T \) has no conjugate point between \( T \) and \( q \). However, this is impossible, because according to theorem 9.3.6, \( \gamma \) must have a conjugate point within affine parameter \( 2/|\theta_0| \) from \( p \), where \( \theta_0 < 0 \) is the expansion at \( p \) of the orthogonal null geodesic congruence from \( T \) to which \( \gamma \) belongs.

Geometry of the dynamical horizon \( H \) is represented by The unit normal to \( H \) by \( \hat{\tau}^a \); \( g_{ab}\hat{\tau}^a\hat{\tau}^b = -1 \). The unit space-like vector orthogonal to \( S \) and tangent to \( H \) is denoted by \( \hat{r}^a \). Finally, we will fix the rescaling freedom in the choice of null normals via \( \ell^a = \hat{\tau}^a + \hat{r}^a \) and \( n^a = \hat{\tau}^a - \hat{r}^a \). As is usual in general relativity, the notion of energy is tied to a choice of a vector field. The definition of a dynamical horizon provides a preferred direction field; that is along \( \ell^a \). To fix the proportionality factor, or the lapse \( N \), let us first introduce the
area radius $R$, a function which is constant on each $S$ and satisfies $a_S = 4\pi R^2$. Since we already know that area is monotonically increasing, $R$ is a good coordinate on $H$. Now, the 3-volume $d^3V$ on $H$ can be decomposed as $d^3V = |\partial R|^{-1}dRd^2V$ where $\partial$ denotes the gradient on $H$. Therefore, as we will see, our calculations will simplify if we choose $N_R = |\partial R|$

Fix two cross sections $S_1$ and $S_2$ of $H$ and denote by $\Delta H$ the portion of $H$ they bound. We are interested in calculating the flux of energy associated with $\xi^a_{(R)} = N_R \ell^a$ across $\Delta H$. Denote the flux of matter energy across $\Delta H$ by $\mathcal{F}_{\text{matter}}^{(R)}$:

$$
\mathcal{F}_{\text{matter}}^{(R)} := \int_{\Delta H} T_{ab} \hat{\nabla}^a \xi_b^{(R)} d^3V = \frac{1}{G} \left(M(r_2) - M(r_1)\right).
$$

(8)

LTB BHs, dynamical horizon behaves like (2) at late time, matter flux become zero and the horizon radius become fixed at radius $R = 2M(r_0)$ where $M(r_0)$ is Misner-Sharp mass in the horizon. For more practical case we can calculate the matter accretion according time along the dynamical horizon by

$$
\left. \frac{dM(r)}{dt} \right|_{DH} = \left. \frac{dM(r)}{dr} \right|_{DH} \left. \frac{dr}{dt} \right|_{DH}.
$$

(9)

To show that the dynamical horizon becomes isolated horizon, we have to show that the dynamical horizon become Slowly evolving horizon at late time. It can be seen from [7] that

$$
\left. \frac{dt}{dr} \right|_{AH} = \frac{R' - 2M'}{-R} = \frac{R' - 2M'}{1 + f}.
$$

(10)

We know $R' > 0$ has regular behavior (no shell crossing) and $f(r)$ is finite along the MTT. From Proposition II.3 we $R' - 2M'|_{AH}$ becomes large and $R' - M'|_{AH}$ do too. We follow the Booth [19] calculations to show that our dynamical horizon becomes a slowly evolving horizon at a late time. In defining a slowly evolving horizon, it is convenient to further restrict the scaling of the null vectors. To do this we label the foliating MTSs with a parameter $\nu$ and choose the scaling and an evolution parameter $C$ so that

$$
V^\mu = \ell^\mu - cn^\mu
$$

is tangent to DH and

$$\mathcal{L}_V V = 1.
$$

(12)

According to [19], if we calculate $c$ and $\epsilon$ along the DH for LTB metric, we have

$$
c = 2\sqrt{1 + f} \left. \frac{M'}{R' - M'} \right|_{AH}
$$

(13)

$$
\epsilon^2 = 8\sqrt{1 + f} \left. \frac{M'}{R' - M'} \right|_{AH}
$$

(14)
if $c, \epsilon << 1$, then slowly evolving horizons conditions become satisfied. As we see in above, $R' - M'|_{AH}$ becomes large at large $t$, therefore our DH becomes slowly evolving horizons at a late time. We can see from the Einstein equation that the density become small around the apparent horizon ($R$ and $M'$ are finite because we have no singularity at there) and we will have a void around the apparent horizon at a late time.

### B. Redshift

If an emitter sends a light ray to an observer with a null vector $k^\mu$, the relative light redshift that is calculated by an observer with 4-velocity $u^\mu$ is,

$$1 + z = \frac{(k_\mu u^\mu)_e}{(k_\mu u^\mu)_o}. \tag{15}$$

From [20], we can can calculate the redshift for observer who sit at $r = \text{const}$ with $u^\mu = (1, 0, 0, 0)$ as below,

$$k^t = c_0 \exp \left( -\int \frac{\dot{R}'}{\sqrt{1 + f}} dr \right) = c_0 \exp \left( \int \frac{-1}{\sqrt{1 + f}} \left( \frac{M'}{RR} - \frac{MR'}{RR^2} + \frac{f'}{2R} \right) dr \right). \tag{16}$$

As we saw in the last section, $R'$ becomes large for a late time in the apparent horizon and $(k_\mu u^\mu)_e$ will have a larger value as $t$ become larger. Therefore, the light that comes out from this surface to the external observer will be fainted and has an infinite redshift relative to the distant observer at large time.

### III. COSMOLOGICAL BH WITH PERFECT FLUID

In the last section, we limited ourselves to a dust source and studied it at BH formation in expanding background. In this section, we want to probe if the same properties of the dust cosmological BH collapse are held for general perfect fluid collapse. We assume perfect fluid with an equation of state of the form $p = w\rho$. If $w = 0$ the fluid becomes dust. We now assume that $w \neq 0$ and has a finite value. Take a collapsing ideal fluid within a compact spherically symmetric spacetime region described by the following metric in the comoving coordinates $(t, r, \theta, \phi)$:

$$ds^2 = -e^{2\nu(t,r)} dt^2 + e^{2\psi(t,r)} dr^2 + R(t,r)^2 d\Omega^2. \tag{17}$$

Assuming the energy-momentum tensor for the perfect fluid in the form

$$T^t_t = -\rho(t,r), \quad T^r_r = p_r(t,r), \quad T^\theta_\theta = p_\theta(t,r) = w\rho(t,r),$$  

$$T^\phi_\phi = p_\phi(t,r) = w\rho(t,r). \tag{18}$$
with the week energy condition
\[ \rho \geq 0 \quad \rho + p_r \geq 0 \quad \rho + p_\theta \geq 0, \]
where \( w \) is constant. Einstein equations give,
\[ \rho = \frac{2M'}{R^2 R'}, \quad p_r = -\frac{2\dot{M}}{R^2 R'}, \]
\[ \nu' = \frac{2(p_\theta - p_r)}{\rho + p_r} \frac{R'}{R} - \frac{\rho'}{\rho + p_r}, \]
\[ -2\dot{R}' + R' \frac{\dot{G}}{G} + \frac{\dot{H}'}{H} = 0, \]
where
\[ G = e^{-2\psi}(R')^2, \quad H = e^{-2\nu}(\dot{R})^2, \]
and \( M \) is defined by
\[ G - H = 1 - \frac{2M}{R}. \]
The function \( M \) can also be written as
\[ M = \frac{1}{2} \int_0^R \rho R^2 dR, \]
or
\[ M = \frac{1}{8\pi} \int_0^r \rho \sqrt{(1 + \left(\frac{dR}{d\tau}\right)^2 - \frac{2M}{R})} d^3V, \]
where
\[ d^3V = 4\pi e^\psi R' dr, \]
and
\[ \frac{d}{d\tau} = e^{-\nu} \frac{d}{dt}. \]
The last form of the function \( M \) indicates that when considered as energy, it includes a contribution from the kinetic energy and the gravitational potential energy. \( M \) is called the Misner-Sharp energy.
Hayward [11] showed that in the Newtonian limit of a perfect fluid, \( M \) yields the Newtonian mass to leading order and the Newtonian kinetic and potential energy to the next order. In a vacuum, \( M \) reduces to the Schwarzschild energy. At null and spatial infinity, \( M \) reduces to the Bondi-Sachs and Arnowitt-Deser-Misner energies respectively [11]. Similar to dust case the flux of matter energy across \( \Delta H \) by \( F_{\text{matter}}^{(R)} \):
\[ F_{\text{matter}}^{(R)} := \int_{\Delta H} T_{ab} \dot{\tau}^a \epsilon^b \xi^{(R)} d^3V = \]
\[ \frac{1}{G}(M(t_2, r_2) - M(t_1, r_1)). \]
We assume $M'(t,r) > 0$ and $\dot{M}(t,r) > 0$ in collapsing region which came form positive sign of density and pressure. 

**Existence of marginally trapped tube for cosmological BH:** As we know any dynamical BH has collapsing region ($\dot{R} < 0$). If we calculate the expansion for ingoing and outgoing null geodesic then we get, $\theta_\ell \propto (1 - \sqrt{1 + \frac{2M}{R} - \frac{1}{G}})$, $\theta_\nu \propto (-1 - \sqrt{1 + \frac{2M}{R} - \frac{1}{G}}) < 0$. Hence, we see in collapsing region, the expansion for null outgoing geodesic changes its sign from negative to positive at $R = 2M$ as we go to larger $R$ and ingoing null geodesics expansion is negative everywhere. Therefore, 3-manifold $R = 2M$ is *marginally trapped tube*.

For proof that there exist MTT between the singularity line and the the boundary between expansion region and collapsing region, we know that at $\dot{R} = 0$ so $H = 0$ and from (24) we have $G > 0$ then $R > 2m|_{R=0}$. On the other hand in singularity $2M(t,r_s) > R = 0$. If we look at the $R$ and $2M$ values at the $R$ plane, there is a $R_0$ between $R = 0$ and $\dot{R} = 0$ that $R_0 = 2M$ so we have the MTT.

Similarly, we assume that $M' > 0$ (which means that Misner-Sharp mass is increasing), we have no shell crossing $e^{2\psi(t,r)} > 0$(or equivalently for finite $G R' > 0$), $\dot{M} > 0$ in collapsing region and $\dot{M} < 0$ in expanding region. In the region between collapsing and expanding matter; $\dot{R} = 0$ and then $\dot{M} = 0$ because from (20) we have no singularity in pressure. These condition are compatible with $\rho$ and $p$ positive sign.

**Proposition III.1** the Singularity line slope in $t\rightarrow r$ plane increases for general spherical fluid collapse.

We want to show if we assume that singularity line has another shape for example part $A$ and part $B$ (see Fig.3), then we get contradictions with our assumptions. Consider part $A$, if these case exist then we have $R(r_1) > R(r_2) = 0$ which is in contradiction with $R' > 0$ and Consider part $B$ if these cases exist then we have $R(t_1) > R(t_2) = 0$ which is in contradiction with $\dot{R} < 0$. Therefore, $t_s' > 0$ comes from the above assumptions for cosmological BH collapse.

Cosmological expansion assumption forces singularity to be limited in the part of the space time around the center, and the other part expands with cosmological expansion. Hence, there is an event horizon that separates causal geodesics that fall into the singularity from other geodesics. This event horizon is asymptotic to the singularity at a late time. Similar to the dust case, the trapped region is inside the event horizon see Proposition II.5.

**Proposition III.2** The apparent horizon slope in $t\rightarrow r$ plane becomes infinite at late time.

We know that the singularity slop is strictly positive and singularity is limited in a finite $r$, therefor its slop becomes infinite (see Proposition II.3). The apparent horizon is among
the singularity line and the event horizon, so as even horizon tends to the singularity then apparent horizon tends to singularity and it becomes asymptotic to the singularity at late time and its slop becomes infinite.

We want to show that MTT becomes slowly evolving horizons large time. We know the expression

$$\frac{dt}{dr}|_{AH} = \frac{R' - 2M'}{2\dot{M} - \dot{R}}.$$  \hspace{1cm} (30)

is large at from above proposition late time. We know $R' > 0$ has regular behavior (no shell crossing for finite $G$), $\dot{R} < 0$ and $\dot{M} \geq 0$ are finite along the apparent horizon. Therefore, we can infer $2\dot{M} - \dot{R}$ is finite. Since the density is finite around the horizon, so $R' - 2M' = R' - 2R'\rho\dot{R}^2$ should be large and then $R'|_{AH} > 0$ becomes large.

If we want than MTT becomes space like, we should have

$$\frac{dt}{dr}|_{AH} = -1 < \frac{1}{1 - \frac{2M'}{R'}} < 1. \hspace{1cm} (31)$$

Because of $\dot{M}|_{AH} > 0$ and $\dot{R}|_{AH} < 0$, the above condition will be satisfied and MTT becomes dynamical horizon at large time.

Similar to the LTB case, it can be shown that the dynamical horizon satisfying \textit{slowly evolving horizons} at large time. According to \cite{19}, if we calculate $c$ and $\epsilon$ along the DH for perfect fluid metric, we have

$$c = 2\frac{M'\dot{R} + wM'\dot{R}}{M'\dot{R} - wM'\dot{R} - \dot{R}\rho}\bigg{|}_{AH}$$  \hspace{1cm} (32)
\[
\epsilon^2 = 8 \frac{M' \dot{R} + wM' \dot{R}}{M'\dot{R} - wM'\dot{R} - RR'}|_{AH},
\]
which we put \( \dot{M} = -wM' \frac{\dot{R}}{R} \) from the perfect fluid assumption. If \( c, \epsilon \ll 1 \), then slowly evolving horizons conditions become satisfied. As we calculate above, \( R'\big|_{AH} \) become large at large \( t \), therefore the dynamical horizon becomes slowly evolving horizons at large time. Similar to LTB, the density becomes small around the \( (R \text{ and } M' \text{ are finite around the MTT}) \) and void will be formed around the dynamical horizon at a late time.

IV. COSMOLOGICAL BH WITH COSMOLOGICAL CONSTANT

One of the characteristic properties of a black hole is its horizon. One can show that cosmological constant can affect the nature of the singularity and final fate of the black hole \[21\]. But we want to show that the local behavior of the gravitational collapse is not affected by the cosmological constant as a dark energy candidate. If we calculate the expansion for ingoing and outgoing null geodesic then we get,

\[
\theta^{(\ell)} \propto (1 - \sqrt{1 + \frac{\Lambda}{3} R^2 + \frac{2M}{G} - \frac{1}{3}}),
\]

\[
\theta^{(n)} \propto (1 - \sqrt{1 + \frac{\Lambda}{3} R^2 + \frac{2M}{G} - \frac{1}{3}}) < 0.
\]

So the apparent horizon surface change from \( 2M = R \) to \( 2M = R - \frac{\Lambda R^3}{3} \) surface with the same properties.

For general perfect fluid we have the same change in apparent horizon surface from expansion of light ray, \( \theta^{(\ell)} \propto (1 - \sqrt{1 + \frac{\Lambda}{3} R^2 + \frac{2M}{G} - \frac{1}{3}}), \theta^{(n)} \propto (1 - \sqrt{1 + \frac{\Lambda}{3} R^2 + \frac{2M}{G} - \frac{1}{3}}) < 0 \) (The apparent horizon surface \( 2M = R - \frac{\Lambda R^3}{3} \)).

\[
F_{\text{matter}}^{(R)} := \int_{\Delta H} (T_{ab} - \frac{\Lambda}{8\pi G} g_{ab}) \hat{\tau}^{a} \xi^{b} (R) d^3V.
\]

It can be seen from \[13\] that the area law becomes,

\[
dE' = \frac{\tilde{\kappa}_R}{8\pi G} (1 - \Lambda R^2) d(4\pi R^2)
\]

which \( \kappa_R = \frac{1}{2\pi R} \) is effective surface gravity. We can integrate from the above equation

\[
E'\bigg|_{AH} = \frac{1}{2G} (R - \frac{\Lambda R^3}{3}) = \frac{M(r)}{G}.
\]

If we compare a cosmological BH with a radius \( R_0 \) in de Sitter background \( (\Lambda > 0) \) with non de Sitter background, we will get that de Sitter background black hole will have less Misner-Sharp mass \( M \) in black hole with radius \( R_0 \).

The local property of the black hole evolution will not change with adding the cosmological constant term because if we put cosmological constant realistic value in the Einstein equation with \( \Lambda \ll 10^{-35} \text{s}^{-2} \) and mass \( M = 10^{12} M_\odot \), we get

\[
G - H = 1 - \frac{2M}{R} + \frac{\Lambda}{3} R^2, \quad \frac{\Lambda}{3} R^2 \ll \frac{2M}{R},
\]

(37)
Hence, cosmological constant don’t affect on local black hole physics.

V. DISCUSSION

Considering the black hole as a dynamical object in expanding background shows new properties that a Schwarzschild black hole is not able to show. In particular, the mass and the flux of matter can have different properties in expanding background. Recent analysis has shown that these dynamical BHs have the flexibility to explore some basic questions such as cosmic censorship conjecture, quasi-local definition masses and BH thermodynamic in non-asymptotic flat background [5, 13]. Thus, the first attempt is making a good model for these BH (cosmological BH) with a reasonable kind of matter [7]. In this paper, we assumed some acceptable properties for these BH and consider its singularity and its horizons behavior along the time. We found that the singularity line shows a special behavior and MTT becomes dynamical horizon as a candidate for BH boundary in non-asymptotic flat background. Furthermore, in terms of global behavior in time, Misner-Sharp mass has this ability to describe the mass of the cosmological BH and its time evolution across the dynamical horizon becomes the fluid accretion. In contrast to the Schwarzschild BH, the light with infinite redshift cannot come from the apparent horizon to distance observer. We showed that for the dust cosmological collapse, the apparent horizon becomes infinite redshift surface only at the late time for distance observer. Finally, we found that we can neglect the cosmological constant (as a dark energy candidate) for the local behavior of the BH in time.

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