ADM mass and quasilocal energy of black hole in the deformed Hořava-Lifshitz gravity

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Abstract

Inspired by the Einstein-Born-Infeld black hole, we introduce the isolated horizon to study the Kehagias-Sfetsos (KS) black hole in the deformed Hořava-Lifshitz gravity. This is because the KS black hole is more close to the Einstein-Born-Infeld black hole than the Reissner-Nordström black hole. We find the horizon and ADM masses by using the first law of thermodynamics and the area-law entropy. The mass parameter $m$ is identified with the quasilocal energy at infinity. Accordingly, we discuss the phase transition between the KS and Schwarzschild black holes by considering the heat capacity and free energy.

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1 Introduction

Recently Hořava has proposed a renormalizable theory of gravity at a Lifshitz point \[1, 2\], which may be regarded as a UV complete candidate for general relativity. At short distances the theory of Hořava-Lifshitz (HL) gravity describes interacting non-relativistic gravitons and is supposed to be power counting renormalizable in (1+3) dimensions. Recently, its black hole solutions have been intensively investigated \[3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16\].

Concerning the static spherically symmetric (SSS) solutions, Lü-Mei-Pope (LMP) have obtained the black hole solution with dynamical parameter $\lambda$ \[3\] and topological black holes were found in \[4\]. Its thermodynamics were studied in \[7, 8\] but there remain unclear issues in defining the ADM mass and entropy because its asymptotes are Lifshitz.

On the other hand, Kehagias and Sfetsos (KS) have found the “$\lambda = 1$” black hole solution in asymptotically flat spacetimes using the deformed HL gravity with parameter $\omega$ \[9\]. Its thermodynamics seemed to be nicely defined when using the first law of thermodynamics in Ref.\[10\]. However, the entropy takes a very unusual form as $S = A/4 + (\pi/\omega) \ln[A/4]$ \[14\]. Thus, one has to explain why a logarithmic term $(\pi/\omega) \ln[A/4]$ appears for the entropy of black hole in the deformed HL gravity \[17, 18\]. This term arises because one has used the first law of $dS = dm/T_H$ to derive the entropy, provided that the Hawking temperature $T_H$ and the mass $m$ are known. Indeed, the mass $m$ was defined naively by the condition of the zero metric function $f_{KS} = 0$. Actually, $m$ is not the Arnowitt-Deser-Misner (ADM) mass $M_{ADM}$ defined at infinity because the metric function $f_{KS}$ is different from the Reissner-Norström (RN) black hole, but it is similar to $f_{BI}$ of the Einstein-Born-Infeld (EBI) black hole. Here we will identify the mass parameter $m$ with the quasilocal energy $E(\infty)$ at infinity. However, for the Schwarzschild and RN black holes, their ADM masses are just quasilocal energies at infinity.

Introducing the isolated horizon formalism \[19\], one may resolve the unsatisfactory and uncomplete description of the KS black hole given by concepts such as ADM mass and event horizon. This formalism provides a more complete description of what happens in the neighborhood of the horizon. In this formalism, one considers spacetimes with an interior boundary, which satisfy quasilocal boundary condition, insuring that the horizon remains isolated. The boundary condition means that quasilocal charges could be defined at horizon, which remain constant in time. These charges include horizon mass, horizon electric charge, and horizon magnetic charge. Importantly, the first law of black hole thermodynamics for quantities defined only at horizon arises naturally, as part of the requirements of a consistent
Hamiltonian formulation. In addition, Ashteker-Corichi-Sudarsky (ACS) conjecture on the relation between the colored black holes and their solitonic analogs implies that the ADM mass consists of two contributions: black hole horizon and solitonic residue. Hence, the colored black holes with index $n$ (EBI black hole with index $b^2$) can be regarded as bound states of ordinary black holes and their solitons. We insist that the isolated horizon formalism is also applicable to the KS black hole.

Comparing the KS black hole $(m, \omega)$ with the EBI black hole $(M, Q, b)$, one observes an apparent correspondence such that $m \leftrightarrow Q^2$ (magnetic charge) and $\omega \leftrightarrow b^2$ (non-linear coupling parameter). This implies that the EBI black hole may play a role in understanding the KS black hole from the deformed HL gravity. At infinity, the EBI black hole $(M, Q, b^2)$ is indistinguishable from the RN black hole $(M, Q)$, which implies that $b^2$ is considered as a free parameter like color index $n$. Similarly, at infinity, the KS black hole $(m, \omega)$ is indistinguishable form the Schwarzschild black hole $(m)$. Hence, we have an index relation of $n \sim b^2 \sim \omega$.

Furthermore, it was well known that many different kinds of black holes from string theories have the Bekenstein-Hawking entropy of $S_{BH} = A/4$ [21]. Hence, it would be better to use the Bekenstein-Hawking entropy to derive the horizon mass and ADM mass via the first law of $dM_h = T_H dS_{BH}$.

In this work, we obtain the horizon and ADM mass of KS black hole in the deformed HL gravity. In deriving these masses, we use the first law of thermodynamics and the Bekenstein-Hawking entropy. Also, we confirmed that the horizon mass satisfies the modified Smarr formula.

## 2 HL gravity

Introducing the ADM formalism where the metric is parameterized

$$ds^2_{\text{ADM}} = -N^2 dt^2 + g_{ij} \left( dx^i - N^i dt \right) \left( dx^j - N^j dt \right),$$

the Einstein-Hilbert action can be expressed as

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} N \left[ K_{ij} K^{ij} - K^2 + R - 2\Lambda \right],$$

where $G$ is Newton’s constant and extrinsic curvature $K_{ij}$ takes the form

$$K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right).$$
Here, a dot denotes a derivative with respect to \( t \). An action of the non-relativistic renormalizable gravitational theory is given by

\[
S_{HL} = \int dt d^3 x \left[ \mathcal{L}_K + \mathcal{L}_V \right],
\]

where the kinetic Lagrangian is given by

\[
\mathcal{L}_K = \frac{2}{\kappa^2} \sqrt{g} N K_{ij} G^{ijkl} K_{kl} = \frac{2}{\kappa^2} \sqrt{g} N \left( K_{ij} K^{ij} - \lambda K^2 \right),
\]

with the DeWitt metric

\[
G^{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} - g^{il} g^{jk} \right) - \frac{\lambda}{3 \lambda - 1} g_{ijkl},
\]

and its inverse metric

\[
G_{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} - g^{il} g^{jk} \right) - \frac{\lambda}{3 \lambda - 1} g^{ijkl}.
\]

The potential Lagrangian is determined by the detailed balance condition as

\[
\mathcal{L}_V = -\frac{\kappa^2}{2} \sqrt{g} N E^{ij} G_{ijkl} E^{kl} = \sqrt{g} N \left\{ \frac{\kappa^2 \mu^2}{8(1 - 3 \lambda)} \left( \frac{1 - 4 \lambda}{4} R^2 + \Lambda_W R - 3 \Lambda_W^2 \right) \right. \\
\left. - \frac{\kappa^2}{2 w^4} \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu w^2}{2} R^{ij} \right) \right\}.
\]

Here the \( E \) tensor is defined by

\[
E^{ij} = \frac{1}{w^2} C^{ij} - \frac{\mu}{2} \left( R^{ij} - \frac{R}{2} g^{ij} + \Lambda_W g^{ij} \right)
\]

with the Cotton tensor \( C_{ij} \)

\[
C^{ij} = \frac{\epsilon^{ik\ell}}{\sqrt{g}} \nabla_k \left( R^{ij} - \frac{1}{4} R \delta^{ij} \right).
\]

Explicitly, \( E_{ij} \) could be derived from the Euclidean topologically massive gravity

\[
E^{ij} = \frac{1}{\sqrt{g}} \frac{\delta W_{TMG}}{\delta g_{ij}}
\]

with

\[
W_{TMG} = \frac{1}{w^2} \int d^3 x \epsilon^{ik\ell} \left( \Gamma^m_{il} \partial_j \Gamma^l_{km} + \frac{2}{3} \Gamma^m_{ij} \Gamma^{lm} \Gamma^{kn} \right) - \mu \int d^3 x \sqrt{g} (R - 2 \Lambda_W),
\]

where \( \epsilon^{123} = 1 \).

In the IR limit, comparing \( \mathcal{L}_0 \) with Eq. (2) of general relativity, the speed of light, Newton’s constant and the cosmological constant are given by

\[
c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda_W}{1 - 3 \lambda}}, \quad G = \frac{\kappa^2}{32\pi c}, \quad \Lambda_{cc} = \frac{3}{2} \Lambda_W.
\]
The equations of motion were derived in [22] and [3]. We would like to mention that the IR vacuum of this theory is anti-de Sitter (AdS$_4$) spacetimes. Hence, it is interesting to take a limit of the theory, which may lead to a Minkowski vacuum in the IR sector. To this end, one may deform the theory by introducing a soft violation term of “$\mu^4 R$” ($\tilde{L}_V = L_V + \sqrt{g} N \mu^4 R$) and then, take the $\Lambda_W \to 0$ limit [9]. We call this as the “deformed HL gravity”. This theory does not alter the UV property of the HL gravity, while it changes the IR property. That is, there exists a Minkowski vacuum, instead of an AdS vacuum. In the IR limit, the speed of light and Newton’s constant are given by
\[ c^2 = \frac{\kappa^2 \mu^4}{2}, \quad G = \frac{\kappa^2}{32\pi}, \quad \lambda = 1. \] (14)

3 KS black hole and old thermodynamics

A static spherically symmetric (SSS) solution to the deformed HL gravity was obtained by considering the line element
\[ ds^2_{SSS} = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \] (15)

For this purpose, we choose the case of $K_{ij} = 0$ and $C_{ij} = 0$. Actually, the above SSS solution could be derived from the deformed potential Lagrangian given by
\[ \tilde{L}_V = \mu^4 \sqrt{g} N \left[ R + \frac{1}{2\omega} \left( \frac{4\lambda - 1}{3\lambda - 1} R^2 - \frac{2}{\omega} R_{ij} R_{ij} \right) \right], \] (16)

where an important parameter,
\[ \omega = \frac{16\mu^2}{\kappa^2} \] (17)

specifies the deformed HL gravity. Hence, it is emphasized that we have relaxed both the projectability restriction and detailed balance condition [1, 23], since the lapse function $N(r)$ depends on the spatial coordinate $r$ as well as a soft violation term of $\mu^4 R$ is included.

Substituting the metric ansatz (15) into $\tilde{L}_V$, one has the reduced Lagrangian
\[ \tilde{L}_V = \mu^4 \sqrt{f} N \left[ \frac{\lambda - 1}{2\omega} f'^2 - \frac{2\lambda (f - 1)}{\omega r} f' + \frac{(2\lambda - 1)(f - 1)^2}{\omega r^2} - 2(1 - f - rf') \right]. \] (18)

For $\lambda = 1$, the KS solution is given by [9]
\[ f_{KS} = N_{KS}^2 = 1 + \omega r^2 \left( 1 - \sqrt{1 + \frac{4m}{\omega r^3}} \right), \] (19)

where $m$ is a mass parameter. In the limit of $\omega \to \infty$ ($\kappa^2 \to 0$ and thus, $\mu^4 R$ dominates), it reduces to the Schwarzschild metric function
\[ f_s(r) = 1 - \frac{2m}{r}. \] (20)
From the condition of \( f_{KS}(r_{\pm}) = 0 \), the outer (inner) horizons are given by
\[
r_{\pm} = m \pm \sqrt{m^2 - \frac{1}{2\omega}}
\]
which takes the same form as in the RN hole obtained from Einstein-Maxwell action (linear electrodynamics)
\[
r_{\pm}^{RN} = M \pm \sqrt{M^2 - Q^2}
\]
when considering a naive correspondence of
\[
m \leftrightarrow M, \quad \frac{1}{2\omega} \leftrightarrow Q^2.
\]
In order to have a black hole solution, it requires that the mass parameter satisfies the following bound,
\[
m^2 \geq \frac{1}{2\omega}.
\]

Based on the assumption that the mass parameter \( m \) from the condition of \( f_{KS} = 0 \) could represent the ADM mass, thermodynamic quantities of Hawking temperature and heat capacity for the KS black hole were derived as
\[
m(r_{\pm}) = \frac{1 + 2\omega r^2_{\pm}}{4\omega r_{\pm}}, \quad T_H = \frac{2\omega r^2_{\pm} - 1}{8\pi r_{\pm}(\omega r^2_{\pm} + 1)}, \quad C_\omega = -\frac{2\pi}{\omega} \left[ \frac{(\omega r^2_{\pm} + 1)^2(2\omega r^2_{\pm} - 1)}{2\omega^2 r^4_{\pm} - 5\omega r^2_{\pm} - 1} \right].
\]
Using the first law of thermodynamics, the entropy was calculated as
\[
S = \int dr_+ \left[ \frac{1}{T_H} \frac{\partial m}{\partial r_+} \right] + S_0,
\]
which leads to
\[
S = \frac{A}{4} + \frac{\pi}{\omega} \ln \left[ \frac{A}{4} \right]
\]
with \( A/4 = \pi r^2_+ \) and \( S_0 = \pi \ln \pi/\omega \). We note that in the limit of \( \omega \to \infty \), Eq. (27) reduces to the Bekenstein-Hawking entropy of Schwarzschild black hole as
\[
S_{BH} = \frac{A}{4}.
\]
If the entropy (27) is correct, the logarithmic term should represent a feature of KS black hole in the deformed HL gravity. However, there was no way to explain the appearance of logarithmic term unless either quantum correction or thermodynamic correction is considered [17].
4 Einstein-Born-Infeld black holes

First of all, we expand the metric function for large \( r \) as

\[
\begin{align*}
f_{KS} & \simeq 1 + \left( -\frac{2m}{r} + \frac{2m^2}{\omega r^4} - \frac{4m^3}{\omega^2 r^7} + \frac{10m^4}{\omega^3 r^{10}} - \frac{28m^5}{\omega^4 r^{13}} + \cdots \right) \\
& \equiv 1 - \frac{2\tilde{m}(r)}{r},
\end{align*}
\]

(29)

where the mass function \( \tilde{m}(r) \) is introduced to take into account the whole \( m \)-dependent terms. In the limit of \( \omega \to \infty \), it is obvious that \( f_{KS} \to f_s \). We note that the absence of \( 1/r^2 \)-term implies that the deformed HL gravity is a purely gravity theory. Also, Eq. (29) shows clearly a different behavior from the RN metric function

\[
f_{RN} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.
\]

(31)

Hence, the naive correspondence (23) is questionable. In order to find a proper case, one introduces a (3+1)-gravity coupled with nonlinear electrodynamics known as the Einstein-Born-Infeld (EBI) action [20]

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R_4}{16\pi G} + L(P, \tilde{Q}) \right],
\]

(32)

where the Born-Infeld Lagrangian is given by

\[
L(P, \tilde{Q}) = -\frac{P_{\mu\nu}F_{\mu\nu}}{2} + K(P, \tilde{Q}),
\]

(33)

with the structural function \( K(P, \tilde{Q}) \)

\[
K(P, \tilde{Q}) = b^2 \left( 1 - \sqrt{1 - \frac{2P}{b^2} + \frac{Q^2}{b^4}} \right)
\]

(34)

with \( P \) and \( \tilde{Q} \) the invariants of \( P_{\mu\nu} \). Here, the constant \( b^2 \) is the Born-Infeld parameter. We introduce the line element with the metric function \( f_{BI}(r) \) as follows:

\[
ds_{BI}^2 = -f_{BI}(r)dt^2 + f_{BI}(r)^{-1}dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).
\]

(35)

Choosing the SSS background (35), the electrically (magnetically) charged solution is obtained by taking

\[
F_{01} = \frac{Q e}{\sqrt{r^4 + \frac{Q^2}{b^4}}}, \quad P_{01} = \frac{Q}{r^2}.
\]

(36)
In this work, we consider the magnetically charged case only. The EBI black hole solution can be written as

\[ f_{BI}(r) = 1 - \frac{2M}{r} + 2b^2 r^2 \left( 1 - \sqrt{1 + \frac{Q^2}{b^2 r^4}} \right) + \frac{4Q^2}{3r} G(r), \quad (37) \]

where \( G'(r) \) denotes the derivative of \( G(r) \) with respect to its argument. Importantly, comparing (19) with third term of (37) leads to other correspondence

\[ m \leftrightarrow Q^2, \quad \omega \leftrightarrow b^2. \quad (39) \]

For the EBI black hole, \( G \) takes the form

\[ G(r) = \int_0^\infty \frac{ds}{\sqrt{s^4 + \frac{Q^2}{b^2}}} = \frac{1}{r} F\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; -\frac{Q^2}{b^2 r^4}\right], \quad (40) \]

where \( F \) is the hypergeometric function. In the presence of negative cosmological constant and electric charge, its solution and thermodynamics were given in Refs. [24, 25, 26]. On the other hand, for the soliton-like solution, it takes the form

\[ G(r) = -\int_0^r \frac{ds}{\sqrt{s^4 + \frac{Q^2}{b^2}}}. \quad (41) \]

For large \( r \), the expansion of \( f_{BI} \) is given by

\[ f_{BI} \simeq 1 - \frac{2M}{r} + \left( \frac{Q^2}{r^2} - \frac{Q^4}{20b^2 r^6} + \frac{Q^6}{75b^4 r^{10}} - \frac{5Q^8}{832b^6 r^{14}} + \cdots \right). \quad (42) \]

At this stage, we note two limiting cases as guided black holes to study the EBI black hole. In the limit of \( Q \to 0 \), this metric function reduces to the Schwarzschild case (20), while in the limit of \( b \to \infty \) and \( Q \neq 0 \), this metric function reduces to the RN black hole (31). Comparing (29) with (42), one notes that the correspondence (39) holds roughly.

At infinity, the ADM mass \( M_{ADM} \) is obtained from the condition of \( f_{BI}(r_+) = 0 \) [20]

\[ M_{ADM}(r_+, Q, b) = \frac{r_+}{2} + \frac{b^2 r_+^3}{3} \left( 1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}} \right) + \frac{2Q^2}{3r_+} F\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; -\frac{Q^2}{b^2 r_+^4}\right]. \quad (43) \]

This is possible because \( M \)-term is a single one in the metric function \( f_{BI} \). On the other hand, the horizon mass \( M_h \) is defined to be

\[ M_h(r_+, Q, b) = \frac{r_+}{2} + \frac{b^2 r_+^3}{3} \left( 1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}} \right) - \frac{2Q^2}{3} \int_0^{r_+} \frac{ds}{\sqrt{s^4 + \frac{Q^2}{b^2}}}. \quad (44) \]
In addition, we note that the soliton mass is obtained as

\[ M_{\text{sol}} = M_{\text{ADM}} - M_h = \frac{2Q\sqrt{QbK[1/2]}}{3}, \]

where \( K[1/2] \) is the complete elliptical integral of the first kind given by

\[ K[\frac{1}{2}] = \frac{\Gamma[\frac{1}{4}]\Gamma[\frac{5}{4}]}{\Gamma[\frac{1}{2}]} . \]

Finally, we emphasize that the horizon mass (44) is also derived from the first law of the thermodynamics

\[ dM_h = T_H dS_{BH} \rightarrow M_h = \int_{0}^{r_+} T_H dS_{BH}, \]

where the Hawking temperature is defined by

\[ T_H(\tilde{r}_+, Q, b) = \frac{f'_{BI}(\tilde{r}_+)}{4\pi} = \frac{1}{4\pi} \left[ \frac{1}{\tilde{r}_+} + 2b^2\tilde{r}_+ \left( 1 - \sqrt{1 + \frac{Q^2}{b^2\tilde{r}_+^2}} \right) \right] . \]

In deriving the horizon mass \( M_h \), we use the integration formula

\[ \int_{0}^{r_+} \sqrt{r^4 + \frac{Q^2}{b^2}} dr = \frac{r_+^3}{3} \sqrt{1 + \frac{Q^2}{b^2r_+^4}} + \frac{2}{3} \frac{Q^2}{b^2} \int_{0}^{r_+} \frac{dr}{\sqrt{r^4 + \frac{Q^2}{b^2}}} . \]

However, we note that this is possible only for a magnetically charged EBI black hole, but not for an electrically charged EBI black hole [27]. The reason is that if the variation of electric charge \( Q_e \) is taken into account, the first law (47) is changed into \( dM_h = T_H dS_{BH} + \Phi dQ_e \) at horizon.

### 5 Horizon and ADM masses, and quasilocal energy

It was shown that the horizon and ADM masses of EBI [27], colored [19], and Bardeen black holes [28] are also derived from the first law and the area-law entropy if one uses magnetic charges. Considering two correspondences (23) and (39), we may consider \( \omega \) as “a pseudo magnetic charge”. We remind that the KS metric function (29) contains an infinite \( m \)-dependent terms and thus, one could not use \( f_{KS} = 0 \) to obtain the horizon and ADM masses as in the EBI black hole. It suggests that one way to derive these masses is to use the first law because \( \omega \) belongs to a pseudo magnetic charge.

Now we are in a position to derive the horizon mass for the KS black hole. Using the Bekenstein-Hawking entropy \( S_{BH} = \pi r_+^2 \) and the Hawking temperature in (25), the horizon mass is obtained from the first law

\[ M_h(r_+, \omega) = \int T_H dS_{BH} = \frac{r_+}{2} - \frac{3}{4} \int_{0}^{r_+} \frac{dr}{\omega r^2 + 1} = \frac{r_+}{2} - \frac{3\tan^{-1}(\sqrt{\omega}r_+)}{4\sqrt{\omega}} . \]
On the other hand, the ADM mass is calculated to be

$$M_{ADM}(r_+, \omega) = \frac{r_+}{2} + \frac{3}{4} \int_{r_+}^{\infty} \frac{dr}{\omega r^2 + 1} dr = \frac{r_+}{2} - \frac{3 \tan^{-1}(\sqrt{\omega} r_+)}{4 \sqrt{\omega}} + \frac{3\pi}{8\sqrt{\omega}}. \tag{51}$$

Using the relation for large $x$

$$\tan^{-1} x = \frac{x}{1 + x^2} F\left[1, 1; \frac{3}{2}, \frac{x^2}{1 + x^2}\right] = \sum_{n=0}^{\infty} \frac{2^{2n} (n!)^2}{(2n + 1)!} \frac{x^{2n+1}}{(1 + x^2)^{n+1}}, \tag{52}$$

one finds that the horizon mass takes a series form

$$M_h \simeq M_H - \frac{\omega r_+^3}{2(1 + \omega r_+^2)^2} \left[1 + \frac{4\omega r_+}{5(1 + \omega r_+^2)} + \cdots \right], \tag{53}$$

where the first term represents the Komar charge \(^2\) at horizon

$$M_H = \tilde{m}(r_+) - r_+ \tilde{m}'(r_+) = \frac{r_+}{2} - \frac{3r_+}{4(1 + \omega r_+^2)} = 2T_H S_{BH}, \tag{55}$$

and remaining terms denote the potential $V(r_+)$ in the modified Smarr formula.

Other important quantity of quasilocal energy is defined by \([31, 32]\)

$$E(r) = \frac{1}{8\pi} \int_B d^2x \sqrt{\sigma} (k - k_0), \tag{56}$$

where $B$ is the two dimensional spherical surface $S^2$ with surface area $A = 4\pi r^2$, $k$ is the trace of the extrinsic curvature of $B$, $\sigma_{ij}$ is the induced metric of $B$, and $k_0$ is the reference term of the Minkowski spacetimes. Interpreting the Komar charge and quasilocal energy \([30]\), the former (gravitational charge) measures the strength of the gravitational pull exerted by a body, while the gravitational field energy (quasilocal energy difference between horizon and infinity) is related to the amount of curvature of space. For the RN black hole, both quantities are equals at horizon. Especially, this is a nonvariational identity which relates quantities at horizon and at infinity, in a different way to the first law of black hole thermodynamics, where variations of certain quantities at horizon and at infinity are related.

For the SSS metric \([15]\), it turns out that the boundary condition of $\tilde{m}(r \to \infty) = m$ satisfies asymptotic flatness. Then, the quasilocal energy inside a spherical surface of radius $r \geq r_+$ is given by

$$E(r) = r - r \sqrt{1 - \frac{2\tilde{m}(r)}{r}} \tag{57}$$

\(^2\)The Komar charge is originally defined by

$$M_c = \frac{1}{4\pi} \int_B g \cdot ds, \tag{54}$$

where $g = -n \nabla (\ln n)$ and $n = \sqrt{-t^a t_a}$ with $t^a$ a timelike Killing vector \([29, 30]\). In this case, the KS system is a spatial three-surface $\Sigma$ bounded by a two-surface $B = S^2$. 

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whose expansion is given for large \( r \)

\[
E(r) \simeq m + \frac{m^2}{2r} + \frac{m^3}{2r^2} + \cdots.
\] (58)

On the other hand, the RN metric function provides its quasilocal energy \[33\]

\[
E_{RN}(r) \simeq M + \frac{M^2 - Q^2}{2r} + \cdots
\] (59)

The quasilocal energy provides an interesting difference between the horizon and infinity

\[
E(r_+) - E(\infty) = \sqrt{m^2(r_+) - \frac{1}{2\omega}} = \frac{r_+}{2} - \frac{1}{4\omega r_+}
\] (60)

where

\[
E(\infty) = m.
\] (61)

For the extremal black hole at \( r_+ = r_e = 1/\sqrt{2\omega} \), this quantity vanishes. In addition, the mass parameter at horizon takes the form

\[
m(r_+) = \frac{r_+}{2} + \frac{1}{4\omega r_+}
\] (62)

which is obviously different from the horizon mass \( M_h \). Hence, we identify the mass parameter \( m \) with the quasilocal energy at infinity.

Furthermore, we show that the inequality

\[
E(r_+) - E(\infty) > M_H(r_+)
\] (63)

is satisfied \[33\] and thus, the equality achieves when adding a new term \( \Delta(r_+) \) as

\[
E(r_+) - E(\infty) = M_H(r_+) + \Delta(r_+) = 2T_H \left( S_{BH} + \frac{\pi}{\omega} \right)
\] (64)

with

\[
\Delta(r_+) = \frac{2\omega r_+^2 - 1}{4\omega r_+ (\omega r_+^2 + 1)} = \frac{2\pi}{\omega} T_H.
\] (65)

However, the RN black hole satisfies the equality as

\[
E_{RN}(r_+) - E_{RN}(\infty) = \sqrt{M^2 - Q^2} = M^{RN}_H(r_+).
\] (66)

At this stage, we show that ACS conjecture \[19\] is satisfied by the KS black hole. From Fig. 1, the KS horizon mass \( M_h(r_+, \omega) \) is always less than Schwarzschild mass \( M_s = r_+/2 \) for any value \( \omega \), while the KS ADM mass \( M_{ADM} \) is always greater than the Schwarzschild mass \( M_s \). We note that \( M_h \) becomes negative for small black holes, while \( M_{ADM} \) is always positive.
Let us calculate the difference between ADM and horizon masses

\[ M_{\text{ADM}} - M_h = \frac{3\pi}{8\sqrt{\omega}}, \]  
\[ (67) \]

which may be interpreted as a solitonic mass. Comparing (45) with (67) leads to a relation

\[ \omega \sim \frac{1}{Q^3 b}. \]  
\[ (68) \]

On the other hand, the positivity of mass difference may imply a potential instability \[ [19, 27] \]. That is, a perturbation in the initial data will trigger the KS black hole to decay to a Schwarzschild black hole. Furthermore, the difference between the KS ADM mass and the Schwarzschild mass turns out to be positive as

\[ M_{\text{ADM}} - M_s = \frac{3\pi}{8\sqrt{\omega}} - \frac{3\tan^{-1}(\sqrt{\omega r^2})}{4\sqrt{\omega}} > 0. \]  
\[ (69) \]

On this basis, one might conjecture that the KS black hole is unstable. In order to study a phase transition between two black holes, however, we use the heat capacity and free energy.

### 6 Phase transitions

The two important quantities for determining the black hole phase transition are heat capacity and free energy \[ [26] \]. The heat capacity is defined by

\[ C_{\omega} = \left( \frac{dM_h}{dT_H} \right)_\omega = \left( \frac{dM_{\text{ADM}}}{dT_H} \right)_\omega = -\frac{2\pi r^2_+ (\omega r^2_+ + 1)(2\omega r^2_+ - 1)}{2\omega^2 r^4_+ - 5\omega r^2_+ - 1} \]  
\[ (70) \]
Figure 2: Graphs of temperature, heat capacity, and free energy with $\omega = 1$. Left: Hawking temperature is zero ($T_H(r_e, \omega) = 0$) at the extremal point $r_e = 0.71$, while it is maximum ($T_H = T_m$) at $r_m = 1.64$. A dashed curve denotes the temperature $T_s = \frac{1}{4\pi r_+}$ of the Schwarzschild black hole. Two are quite different for small black holes. Center: $C_\omega(r_+, \omega)$ shows a blow-up point at $r_m = 1.64$, dividing it into $C_\omega > 0$ and $C_\omega < 0$. Note that $C_\omega(r_e, \omega) = 0$. A dashed curve denotes the heat capacity $C_s = -2\pi r_+^2$ of the Schwarzschild black hole. Right: upper dashed, solid, and lower dashed curves represent $F_{ADM}(r_+, \omega)$, $F_s = r_+ / 4$, and $F_h(r_+, \omega)$, respectively.

which seems to be different from the old heat capacity in (25), but its characteristic is not changed. This quantity is crucial for testing local thermodynamic stability. As is depicted in Fig. 2, the heat capacity blows up at the maximum temperature point $r_+ = r_m$, dividing it into $C_\omega > 0$ and $C_\omega < 0$. The former case of small black hole is thermodynamically stable, while the latter of large black hole is thermodynamically unstable, like the Schwarzschild black hole. This is clear because the KS black hole has an extremal black hole which is considered to be a stable remnant as a final stage of black hole evaporation via Hawking radiation [34]. This feature contrasts sharply to that of Schwarzschild black hole, showing that a negative heat capacity makes it hotter and causes the horizon area to shrink. This process escalates until the horizon collapses rapidly onto the singularity amid an explosive radiation of quanta.

The free energy usually determines a global thermodynamic stability when combining with the heat capacity. We have two kinds of free energies. The free energy at horizon is defined by

$$F_h = M_h - T_H S_{BH} = \frac{r_+}{2} - 3 \tan^{-1}(\sqrt{\omega r_+}) - \frac{(2\omega r_+^2 - 1)}{8(\omega r_+^2 + 1)},$$

(71)

because we consider $\omega$ as a pseudo magnetic charge. On the other hand, the free energy based on the ADM mass takes a different form as

$$F_{ADM} = M_{ADM} - T_H S_{BH} = \frac{r_+}{2} - 3 \frac{\tan^{-1}(\sqrt{\omega r_+})}{4\sqrt{\omega}} + \frac{3\pi}{8\sqrt{\omega}} - \frac{(2\omega r_+^2 - 1)}{8(\omega r_+^2 + 1)}.$$

(72)
As is shown in Fig. 2, we find an important sequence

\[ F_{\text{ADM}} > F_s > F_h. \]  

(73)

This implies that if one uses \( F_{\text{ADM}} \) instead of \( F_h \), one could not explain the local stability of small black hole, arriving at the extremal black hole as a stable remnant. In this case, the horizon free energy \( F_h \) is more appropriate for understanding the feature of KS black hole than the ADM free energy. If one uses the ADM free energy to describe the phase transition, the Schwarzschild black hole is more stable than the KS black hole. This implies that a perturbation on the KS black hole may induce the KS black hole to decay to the Schwarzschild black hole. This is consistent with the previous argument based on the mass difference.

### 7 Discussions

Applying the isolated horizon formalism to the KS black hole, we have found the horizon and ADM mass by using the first law and the Bekenstein-Hawking entropy. These masses take obviously different from the mass parameter \( m \) in Eq. (25) obtained from \( f_{KS} = 0 \). Importantly, we have identified the mass parameter \( m \) with the quasilocal energy at infinity.

This implies that the colored black hole with color index \( n \), the EBI black hole with coupling parameter \( b^2 \), the Bardeen black hole with magnetic charge \( g \), and the KS black hole with parameter \( \omega \) belong to the same category of black holes which need a careful study to find the correct thermodynamics. In this sense, the deformed potential Lagrangian (16) may be regarded as a non-linear gravity theory of \( R \) with coupling parameter \( \omega \).

In order to study a phase transition between the KS black hole and Schwarzschild black hole, we introduce the heat capacity and the free energy. We did not discuss the black hole phase transition only by mentioning the mass difference. The heat capacity shows the local thermodynamic stability, while the free energy describes the global thermodynamic stability. One basic difference between two black hole is that the KS black hole has an extremal black hole with \( C_\omega = 0 \) and \( F_h < 0 \), while the Schwarzschild black hole has not. In order to explain the stable remnant at extremal point, we need to use the horizon free energy but not the ADM free energy.

In conclusion, we have obtained the horizon and ADM masses for the KS black hole in the deformed HL gravity. In deriving these masses, we use the first law of thermodynamics and the area-law entropy. Hence, all thermodynamic quantities are well defined without any pathology. A remaining issue is that the ADM mass is always greater than the horizon mass,
implying that the KS black hole likely decays to the Schwarzschild black hole. However, this issue is hard to be accepted because it unlikely occurs when considering the extremal black hole as a stable remnant.

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