Non-relativistic scalar field on the quantum plane

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Abstract

We apply the coherent state approach to the non-commutative plane to check the one-loop finiteness of the two-point and four-point functions of a non-relativistic scalar field theory in 2 + 1 dimensions. We show that the two-point and four-point functions of the model are finite at one-loop level and one recovers the divergent behavior of the model in the limit \( \theta \to 0^+ \) by appropriate redefinition of the non-commutativity parameter.

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1. Introduction

The origin of the recent interests in non-commutative field theories refers to the fact that they may appear naturally in string/M-theories [1,2]. Historically it was a hope that the idea of non-commutative space–time may provide a mechanism, which can cure the UV divergent behavior of the quantum field theory [3]. However, Filk pointed out that the non-commutative field theories exhibit the same divergent behavior of the usual commutative field theories [4]. In non-commutative field theories one replaces the space–time coordinates \( x^\mu \) with operators \( \hat{x}^\mu \), which do not commute with each other, i.e.,

\[
[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu \nu},
\]

with \( \theta^{\mu \nu} \) as a real antisymmetric matrix. Thus because of the uncertainty relation

\[
\Delta x^\mu \Delta x^\nu \geq \frac{|\theta^{\mu \nu}|}{2}
\]

induced by the relation (1), short distance scales in the \( x^\mu \) direction correspond to the large distance scales in the \( x^\nu \) direction and vice versa. So there will be a mixing between the ultraviolet and infrared behaviors of the field theories in non-commutative space–times. Such a problem, which is called the “UV/IR mixing problem”, is characteristic of the models based on the \( \star \)-product approach [5].

In a non-commutative space–time the usual product between the fields must be replaced by the Weyl–
Moyal bracket or $\star$-product defined as
\[
 f(x) \star g(x) = \lim_{\gamma \to y} \exp \left( \frac{i}{2} \gamma^\mu \partial^\nu \right) f(x) g(y). \tag{3}
\]
The $\star$-product is the usual starting point in the most studies about the non-commutative field theories since it encodes the non-commutativity of space–time for the product of the several fields defined in the same point in the following sense
\[
 f_1(\hat{x}) f_2(\hat{x}) \cdots f_n(\hat{x}) \to f_1(x) \star f_2(x) \cdots \star f_n(x). \tag{4}
\]
Recently, authors in [6,7] have developed a new approach to study the non-commutative space–time. Their main idea is to use the expectation values of the operators between the coherent states of the model. This approach to study the non-commutative space–time was replaced by its mean value as
\[
 A(\alpha) = \alpha |A\rangle \langle \alpha|, \quad \langle \alpha | \alpha \rangle^* = \langle \alpha | A^\dagger, \tag{7}
\]
then the mean position of the particle on the quantum plane is defined as
\[
 x^1 = \langle \alpha | \hat{y}^1 | \alpha \rangle, \quad x^2 = \langle \alpha | \hat{y}^2 | \alpha \rangle. \tag{8}
\]
More generally any function $F(\hat{y})$ on the non-commutative plane will be replaced by its mean value as
\[
 F(x) = \langle \alpha | F(\hat{y}) | \alpha \rangle = \int \frac{d^2p}{(2\pi)^2} f(p) e^{\hat{y}\cdot p} \phi^p, \quad p^2 = |p|^2. \tag{9}
\]
So in coherent state formalism the fuzziness of space manifests itself by smearing the fields over space by modification of the kernel of integral via a Gaussian damping factor as in (9). Thus by means of the general recipe
\[
 f(k) \xrightarrow{\text{non-commutativity}} f(k) e^{-\frac{\theta}{2}k^2}, \tag{10}
\]
for any auxiliary kernel $f(k)$, one immediately finds the non-relativistic free particle momentum space propagator as (setting $\hbar = m = 1$)
\[
 g_0(k, \omega; \theta) = \frac{e^{-\frac{\theta}{2}k^2}}{\omega - \frac{k^2}{2} + i\epsilon}, \quad k = |k|, \tag{11}
\]
bearing in mind that
\[
 G_0(x, t) = \int \frac{d^2k d\omega}{(2\pi)^3} g_0(k, \omega) e^{i(x \cdot k - \omega t)} = \int \frac{d^2k d\omega}{(2\pi)^3} e^{i(x \cdot k - \omega t)} \frac{e^{-\frac{\theta}{2}k^2}}{\omega - \frac{k^2}{2} + i\epsilon}. \tag{12}
\]
3. Self-interacting scalar field on the quantum plane

We consider a self-interacting model of the non-relativistic boson field characterized by the Lagrangian

$$\mathcal{L} = \phi^\dagger(x)\left(i\partial_t + \nabla^2\right)\phi(x) - \frac{\lambda}{4}\phi^\dagger(x)\phi(x)\phi(x)\phi(x).$$  \hspace{1cm} (13)

One-loop correction to the two-point function comes from the term (Fig. 1)

$$-i\Sigma^{(1)}(\Lambda) = \frac{\lambda}{2} \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} g_0(k,\omega) = \frac{\lambda}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{\omega - k^2/2 + i\epsilon} = -\frac{i\lambda}{16\pi} \Lambda^2,$$  \hspace{1cm} (14)

where the UV cut-off parameter $\Lambda$ is introduced. This divergent term can be removed by adding the appropriate counter term to the Lagrangian (13). However, as we shall see later, this expression is no longer divergent on the quantum plane.

The one-loop quantum correction associated with the four-point correction is given by (Fig. 2) [9]

$$\Gamma(k_0, \omega_0; \Lambda) = i\frac{\lambda^2}{2} \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} g_0(k,\omega) g_0(k_0 - k, \omega_0 - \omega)$$

$$= -\frac{\lambda^2}{8\pi} \left(\log \frac{\Lambda^2}{\omega_0 - k^2/4} + i\pi\right),$$  \hspace{1cm} (15)

with

$$k_0 = k_1 + k_2 = k'_1 + k'_2, \quad \omega_0 = \omega_1 + \omega_2 = \omega'_1 + \omega'_2,$$  \hspace{1cm} (16)

where the unprimed and primed quantities refer respectively to the incoming and outgoing particles. So once again we are left with a divergent term (in this case logarithmically divergent) but as the case of two-point function, the expression (15) is no longer UV divergent on the quantum plane and one obtains a finite result for the integral (15), when the non-commutativity of plane is taken into account.

Now by employing the general recipe (10), we proceed further to evaluate the one-loop correction associated with two-point and four-point functions (14) and (15) on the quantum plane.

First let us focus on the two-point function. From (11) we have

$$-i\Sigma^{(1)}(\theta) = \frac{\lambda}{2} \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} e^{-\theta k^2/2} g_0(k,\omega; \theta)$$

$$= \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} e^{-\frac{\theta}{2}k^2} = -\frac{i\lambda}{4\pi \theta},$$  \hspace{1cm} (17)

which, in contrast to (14), is obviously a finite result. By redefinition of the non-commutativity parameter as $\theta = 4/\Lambda^2$ the expression (17) in the limit $\theta \to 0^+$ coincides with Eq. (14) in the limit $\Lambda \to \infty$.

$$\lim_{\theta \to 0^+} (-i\Sigma^{(1)}(\theta)) = \lim_{\theta \to 0^+} -\frac{i\lambda}{4\pi \theta} = \lim_{\Lambda \to \infty} -\frac{i\lambda}{16\pi} \Lambda^2 = \lim_{\Lambda \to \infty} (-i\Sigma^{(1)}(\Lambda)).$$  \hspace{1cm} (18)

So one recovers the one-loop correction to the two-point function on the usual commutative plane from the corresponding term on the quantum plane in the limit $\theta \to 0^+$. 
Next we have the one-loop correction to the four-point function on the quantum plane as

\[ \Gamma(k_0, \omega_0; \theta) = \frac{i\lambda^2}{2} \int \frac{d^2 k d\omega}{(2\pi)^3} \times g_0(k, \omega; \theta) g_0(k_0 - k, \omega_0 - \omega; \theta) \]

\[ = \frac{i\lambda^2}{2} \int \frac{d^2 k d\omega}{(2\pi)^3} \frac{e^{-\frac{\eta}{2}k^2}}{k^2 - \omega_0 - \omega + \frac{k_0^2}{4} - 2i\epsilon}, \]  

(19)

with \( k_0 = |k_0| \). After integration over \( \omega \) and shifting \( k \to k + k_0/2 \), we arrive at

\[ \Gamma(k_0, \omega_0; \theta) = -\frac{\lambda^2}{2} e^{-\frac{\eta}{2}k_0^2} \]

\[ \times \int \frac{d^2 k}{(2\pi)^2} \frac{e^{-\frac{\eta}{2}k^2}}{k^2 - \omega_0 + \frac{k_0^2}{4} - 2i\epsilon}, \]  

(20)

or

\[ \Gamma(k_0, \omega_0; \theta) \]

\[ = -\frac{\lambda^2}{8\pi} \left[ e^{-\frac{\eta}{2}k_0^2} P \int_0^\infty \frac{dx}{x - (\omega_0 - \frac{k_0^2}{4})/2} \right] + i\pi e^{-\frac{\eta}{2}k_0^2}, \]

\[ x = k^2/2, \]  

(21)

where we have invoked the well-known formula

\[ \frac{1}{x \pm i\epsilon} = P \frac{1}{x} \mp i\pi \delta(x). \]  

(22)

Integral (21) is singular at \( 0 \leq (\omega_0 - \frac{k_0^2}{4})/2 \) (see Appendix A). Hence the Cauchy principal value of the integral in (21) must be evaluated carefully. We proceed further by noting that

\[ P \int_0^\infty \frac{dx}{x - \Omega^1} \]

\[ = \lim_{\delta \to 0} \int_0^{\Omega - \delta} dx \frac{e^{-\theta x}}{x - \Omega} + \lim_{\delta \to 0} \int_{\Omega + \delta}^\infty dx \frac{e^{-\theta x}}{x - \Omega}, \]

\[ \Omega = \left( \omega_0 - \frac{k_0^2}{4} \right)/2. \]  

(23)

For the first integral of right-hand side of Eq. (23) we have

\[ \int_0^{\Omega - \delta} dx \frac{e^{-\theta x}}{x - \Omega} = e^{-\theta \Omega} \left[ \ln(\delta) - \ln(-\Omega + \delta) - \sum_{n=1}^{\infty} \frac{\theta^n}{nn!} (\delta^n) \right], \]  

(24)

while the second integral reads

\[ \int_{\Omega + \delta}^\infty dx \frac{e^{-\theta x}}{x - \Omega} = e^{-\theta \Omega} \left[ -\gamma - \ln(\Omega) - \sum_{n=1}^{\infty} \frac{(-1)^n}{nn!} (\Omega^n) \right]. \]  

(25)

where the exponential function \( E_i(\alpha x) \) is defined as

\[ E_i(\alpha x) = \int_x^\infty dt \frac{e^{-\alpha t}}{t} \]

\[ = -\gamma - \ln(\alpha x) - \sum_{n=1}^{\infty} \frac{(-1)^n}{nn!} (\alpha x)^n. \]  

(26)

The symbol \( \gamma \) stands for the Euler–Mascheroni constant. By substituting the results obtained above in (21) we are finally left with

\[ \Gamma(k_0, \omega_0; \theta) = -\frac{\lambda^2}{8\pi} e^{-\frac{\eta}{2}k_0^2} \left[ -\gamma - \ln\left( \frac{\theta}{2} \left( \omega_0 - \frac{k_0^2}{4} \right) \right) \right] \]

\[ - \sum_{n=1}^{\infty} \frac{(\theta \Omega)^n}{nn!} + i\pi \].  

(27)

Now let us explore the \( \theta \to 0^+ \) limit of the one-loop contribution to the four-point function on the quantum plane. To this end we expand the factor \( \exp(-\theta \omega_0/2) \) in a power series as

\[ e^{-\frac{\theta}{2} \omega_0} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \omega_0^n}{2^n \theta^n}, \]  

(28)

then by noting that

\[ \lim_{x \to 0^+} x^m \ln x = 0, \quad m \in \mathbb{N}, \]  

(29)
we are left with the following result for the one-loop correction to the four-point function as

\[
\lim_{\theta \to 0^+} \Gamma(k_0, \omega_0; \theta) = -\frac{\lambda^2}{8\pi} \left( -\gamma - \ln \left[ \frac{\theta}{2} \left( \omega_0 - \frac{k_0^2}{4} \right) \right] + i\pi \right)
\]

\[
= \lim_{\Lambda \to \infty} -\frac{\lambda^2}{8\pi} \left( -\gamma - \ln 2 + \ln \left[ \frac{\Lambda^2}{\omega_0 - \frac{k_0^2}{4}} \right] + i\pi \right)
\]

\[
= \lim_{\Lambda \to \infty} \Gamma(k_0, \omega_0; \Lambda), \quad (30)
\]

where we have set \(\theta = \frac{4}{\Lambda^2}\) once again and ignored the constant \(-\gamma - \ln 2\) since \(\Lambda^2\) is a very large quantity.

Hence as the case of two-point function the UV divergent one-loop four-point function on the usual commutative plane can be recovered from the corresponding (finite) term on the quantum plane in the limit \(\theta \to 0^+\). So as the results (17) and (27) imply, we get finite result for the two-point and four-point functions on the quantum plane. In the limit \(\theta \to 0^+\), i.e., when the non-commutativity between the coordinates disappears, both of the two-point and four-point functions diverge and one recovers the divergent behavior of the two-point and four-point functions on the usual commutative plane.

### Appendix A

We prove that \(0 \leq \Omega\). From (16) we have

\[
k_0^2 = \omega_0 + \frac{k_1 \cdot k_2}{2},
\]

(A.1)

since \(\omega_i = k_i^2 / 2, (i = 1, 2)\). Thus \(\Omega\) reads

\[
4\Omega = \omega_0 - k_1 \cdot k_2 = (k_1^2 + k_2^2) / 2 - k_1 \cdot k_2, \quad (A.2)
\]

but \(k_1 \cdot k_2 \leq k_1^2 \leq (k_1^2 + k_2^2) / 2\), which implies \(0 \leq \Omega\).

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