Hack’s law in a drainage network model: a Brownian web approach

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Abstract

Hack (1957), while studying the drainage system in the Shenandoah valley and
the adjacent mountains of Virginia, observed a power law relation \( l \sim a^{0.6} \) between
the length \( l \) of a stream from its source to a divide and the area \( a \) of the basin
that collects the precipitation contributing to the stream as tributaries. We study the
tributary structure of Howard’s drainage network model of headward growth
and branching studied by Gangopadhyay et al. (2004). We show that the exponent
of Hack’s law is \( 2/3 \) for Howard’s model. Our study is based on a scaling of the
process whereby the limit of the watershed area of a stream is area of a Brownian
excursion process. To obtain this we define a dual of the model and show that
under diffusive scaling, both the original network and its dual converge jointly to
the standard Brownian web and its dual.

Key words: Brownian excursion, Brownian meander, Brownian web, Hack’s law.

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1 Introduction

River basin geomorphology is a very old subject of study initiated by Horton (1945).
Hack (1957), studying the drainage system in the Shenandoah valley and the adjacent
mountains of Virginia, observed a power law relation

\[ l \sim a^{0.6} \]

between the length \( l \) of a stream from its source to a divide and the area of the basin
\( a \) that collects the precipitation contributing to the stream as tributaries. Langbein
(1947) corroborated this power law through studies of nearly 400 different streams in
northeastern United States. This empirical relation is widely accepted nowadays
albeit with a different exponent (see Gray (1961), Muller (1973)) and is called Hack’s

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law. Mandelbrot (1983) mentions Hack’s law to strengthen his contention that “if all rivers as well as their basins are mutually similar, the fractal length-area argument predicts (river’s length)$^{1/D}$ is proportional to (basin’s area)$^{1/2}$” where $D > 1$ is the fractal dimension of the river. In this connection it is worth remarking that the Hurst exponent in fractional Brownian motion and in time series analysis arose from the study of the Nile basin by Hurst (1927) where he proposed the relation $l_\perp = l_\|^{0.9}$ as that governing the width, $l_\perp$, and the length, $l_\|$, of the smallest rectangular region containing the drainage system.

Various statistical models of drainage networks have been proposed (see Rodriguez-Iturbe et al. (1997) for a detail survey). In this paper we study the tributary structure of a 2-dimensional drainage network called the Howard’s model of headward growth and branching (see Rodriguez-Iturbe et al. (1997)). Our study is based on a scaling of the process and we obtain the watershed area of a stream as the area of a Brownian excursion process. This gives a statistical explanation of Hack’s law and justifies the remark of Giacometti et al. (1996): “From the results we suggest that a statistical framework referring to the scaling invariance of the entire basin structure should be used in the interpretation of Hack’s law.”

We first present an informal description of the model: suppose that the vertices of the $d$-dimensional lattice $\mathbb{Z}^d$ are open or closed with probability $p (0 < p < 1)$ and $1 - p$ respectively, independently of all other vertices. Each open vertex $u \in \mathbb{Z}^d$ represents a water source and connects to a unique open vertex $v \in \mathbb{Z}^d$. These edges represent the channels through which water can flow. The connecting vertex $v$ is chosen so that the $d$-th co-ordinate of $v$ is one more than that of $u$ and $v$ has the minimum $L_1$ distance from $u$. In case of non-uniqueness of such a vertex, we choose one of the closest open vertices with equal probability, independently of everything else.

Let $V$ denote the set of open vertices and $h(u)$ denote the uniquely chosen vertex to which $u$ connects, as described above. Set $\langle u, h(u) \rangle$ as the edge (channel) connecting $u$ and $h(u)$. From the construction it follows that the random graph, $G = (V, E)$ with edge set $E := \{\langle u, h(u) \rangle : u \in V\}$, does not contain any circuit. This model has been studied by Gangopadhyay et al. (2004) and the following results were obtained:

**Theorem 1.1.** Let $0 < p < 1$.

(i) For $d = 2$ and $d = 3$, $G$ consists of one single tree almost surely, and, for $d \geq 4$, $G$ is a forest consisting of infinitely many disjoint trees almost surely.

(ii) For any $d \geq 2$, the graph $G$ contains no bi-infinite path almost surely.
\((x, t) \in \mathbb{Z}^2\), we consider \(k_0 = \min\{|k| : k \in \mathbb{Z}, B(x+k,t+1) = 1\}\). Clearly, \(k_0\) is almost surely finite. Now, we define,

\[
h(x, t) := \begin{cases} 
(x + k_0, t + 1) & \text{if } (x + k_0, t + 1) \in V, (x - k_0, t + 1) \not\in V \\
(x - k_0, t + 1) & \text{if } (x - k_0, t + 1) \in V, (x + k_0, t + 1) \not\in V 
\end{cases}
\]

or \((x + k_0, t + 1), (x - k_0, t + 1) \in V, U(x,t) = 1\) and \((x + k_0, t + 1), (x - k_0, t + 1) \in V, U(x,t) = -1\).

For any \(k \geq 0\), let

\[
h^{k+1}(x, t) := h(h^k(x, t)) \quad \text{with} \quad h^0(x, t) := (x, t),
\]

\[
C_k(x, t) := \begin{cases} 
\{(y, t-k) \in V : h^k(y, t-k) = (x, t)\} & \text{if } (x, t) \in V, 0 \leq k < L(x, t) \\
\emptyset & \text{otherwise,}
\end{cases}
\]

\[
C(x, t) := \bigcup_{k \geq 0} C_k(x, t).
\]

Here \(h^k(x, t)\) represents the 'k-th generation progeny' of \((x, t)\), the sets \(C_k(x, t)\) and \(C(x, t)\) denote, respectively, the set of \(k\)-th generation ancestors and the set of all ancestors of \((x, t)\); \(C(x, t) = \emptyset\) if \((x, t) \not\in V\). In the terminology of drainage network, \(C(x, t)\) represents the region of precipitation, the water from which is channelled through the open point \((x, t)\). From Theorem 1.1 (ii), we have that \(C(x, t)\) is finite almost surely.

Now, we define

\[
L(x, t) := \inf\{k \geq 0 : C_k(x, t) = \emptyset\},
\]

as the 'length of the channel', which as earlier is finite almost surely. Our first result is about the length of the channel.

**Theorem 1.2.** For \((x, t) \in \mathbb{Z} \times \mathbb{Z}\), we have

\[
\lim_{n \to \infty} \sqrt{n} \mathbb{P}(L(x, t) > n) = \frac{1}{\gamma_0 \sqrt{\pi}},
\]

where \(\gamma_0^2 := \gamma_0^2(p) = \frac{(1-p)(2-2p+p^2)}{p^2(2-p)^2}\).

Next we define

\[
r_k(x, t) := \begin{cases} 
\max\{u : (u, t-k) \in C_k(x, t)\} & \text{if } (x, t) \in V \text{ and } 0 \leq k < L(x, t), \\
0 & \text{otherwise,}
\end{cases}
\]

\[
l_k(x, t) := \begin{cases} 
\min\{u : (u, t-k) \in C_k(x, t)\} & \text{if } (x, t) \in V \text{ and } 0 \leq k < L(x, t), \\
0 & \text{otherwise,}
\end{cases}
\]

\[
D_k(x, t) := r_k(x, t) - l_k(x, t).
\]

The quantity \(D_k(x, t)\) denotes the width of the set of all \(k\)-th generation ancestors of \((x, t)\). We define the width process \(D_n^{(x,t)}(s)\) and the cluster process \(K_n^{(x,t)}(s)\) for \(s \geq 0\).
as follows: for \( k = 0, 1, \ldots \) and \( k/n \leq s \leq (k + 1)/n \),

\[
D_n^{(x,t)}(s) := \frac{D_k(x,t)}{\gamma_0 \sqrt{n}} + \frac{(ns - [ns])}{\gamma_0 \sqrt{n}} (D_{k+1}(x,t) - D_k(x,t))
\]

\[
K_n^{(x,t)}(s) := \frac{\# C_k(x,t)}{\gamma_0 \sqrt{n}} + \frac{(ns - [ns])}{\gamma_0 \sqrt{n}} (\# C_{k+1}(x,t) - \# C_k(x,t))
\]

(2)

where \( \gamma_0 > 0 \) is as in the statement of Theorem 1.2. In other words, \( D_n^{(x,t)}(s) \) (respectively \( K_n^{(x,t)}(s) \)) is defined \( D_k(x,t)/\gamma_0 \sqrt{n} \) (respectively \( \# C_k(x,t)/\gamma_0 \sqrt{n} \)) at time points \( s = k/n \) and, at other time points defined by linear interpolation.

To describe our results we need to introduce two processes, Brownian meander and Brownian excursion, studied by Durrett et al. (1972) and references therein. Further, both of these processes are continuous non-homogeneous Markov process (see Belkin (1972) and references therein). Further, \( W^+(0) = 0 \) and, for \( x \geq 0 \), \( \mathbb{P}(W^+(1) \leq x) = 1 - \exp(-x^2/2) \), i.e. \( W^+(1) \) follows a Rayleigh distribution.

We also need some random variables obtained as functionals of these two processes. In particular, let

\[
I_0^+ := \int_0^1 W_0^+(t) dt \quad \text{and} \quad M_0^+ := \max\{W_0^+(t) : t \in [0, 1]\}.
\]

Janson et al. (2007) showed that, as \( x \to \infty \),

\[
\mathbb{P}(I_0^+ > x) \sim \frac{6 \sqrt{6}}{\sqrt{\pi}} x \exp(-6x^2) \quad \text{and, the density,} \quad f_{I_0^+}(x) \sim \frac{72 \sqrt{6}}{\sqrt{\pi}} x^2 \exp(-6x^2).
\]

The random variable \( M_0^+ \) is continuous, having a strictly positive density on \((0, \infty)\) (see Durrett et al. (1977)) and for \( x > 0 \),

\[
\mathbb{P}(M_0^+ \leq x) = 1 + 2 \sum_{k=1}^{\infty} \exp(-(2kx)^2/2)[1 - (2kx)^2] \quad \text{with} \quad \mathbb{E}(M_0^+) = \sqrt{\pi}/2.
\]

For \( f \in C[0, \infty] \) let \( f|_{[0,1]} \) denotes the restriction of \( f \) over \([0, 1]\). Our next result is about the weak convergence of the width process \( D_n|_{[0,1]} \) and the cluster process \( K_n|_{[0,1]} \) under diffusive scaling:

**Theorem 1.3.** As \( n \to \infty \), we have
(i) \( D_n^{(x,t)}|_{[0,1]}(L(x,t) > n) \Rightarrow \sqrt{2}W^+ \),

(ii) \( \sup\{|pD_n^{(x,t)}(s) - K_n^{(x,t)}(s)| : s \in [0,1]\} \{L(x,t) > n\} \xrightarrow{p} 0 \).

The following corollary is an immediate consequence of Theorem 1.3:

**Corollary 1.3.1.** For \( u > 0 \), as \( n \to \infty \) we have

(i) \( \sqrt{n}\mathbb{P}\left[1/n(L(x,t), (#C(x,t))^{2/3}) \in B\right] \to \mu(B) \) \hspace{1cm} (6)

(ii) \( \sqrt{n}\mathbb{P}\left[1/n(L(x,t), (D_{max}(x,t))^{1/2}) \in B\right] \to \nu(B) \) \hspace{1cm} (7)

with \( \mu \) and \( \nu \) being given by

\[
\mu(B) = \int \int_B \frac{3\sqrt{v}}{4\sqrt{2\pi}\gamma_0^2pt^3} f_{I_0^+}(\frac{v^2}{\gamma_0pt\sqrt{2t^3}})dvdt,
\]

\[
\nu(B) = \int \int_B \frac{v}{2\sqrt{2\pi}\gamma_0^2pt^2} f_{M_0^+}(\frac{v^2}{\gamma_0pt\sqrt{2t}})dvdt,
\]

where \( f_{I_0^+} \) and \( f_{M_0^+} \) denote the density functions of \( I_0^+ \) and \( M_0^+ \) respectively. Moreover we have

\[
\sqrt{n}\mathbb{P}\left[1/n(L(x,t), (#C(x,t))^\alpha) \in (\tau, \infty) \times (\lambda, \infty)\right] = \begin{cases} 0 & \text{if } \alpha < 2/3 \\ (\pi \tau \gamma_0^2)^{-1/2} & \text{if } \alpha > 2/3 \end{cases} \hspace{1cm} (8)
\]

and

\[
\sqrt{n}\mathbb{P}\left[1/n(L(x,t), (D_{max}(x,t))^\alpha) \in (\tau, \infty) \times (\lambda, \infty)\right] = \begin{cases} 0 & \text{if } \alpha < 1/2 \\ (\pi \tau \gamma_0^2)^{-1/2} & \text{if } \alpha > 1/2. \end{cases} \hspace{1cm} (9)
\]

Our next result states that the exponent of Hack’s law is 2/3 for Howard’s model. In addition we obtain a scaling law for the length of the stream \( \text{vis-à-vis} \) the maximum width of the region of precipitation, i.e.,

\[
D_{max}(x,t) := \max\{D_k(x,t) : 0 \leq k < L(x,t)\}. \hspace{1cm} (5)
\]

It should be noted that Leopold et al. (1962) obtained an exponent of 0.64 through computer simulations.

**Theorem 1.4.** Let \( E := (0, \infty) \times (0, \infty) \). There exist measures \( \mu \) and \( \nu \) on the Borel \( \sigma \)-algebra on \( E \) such that for any \( (x,t) \in \mathbb{Z}^2 \) and any Borel set \( B \subseteq E \) we have

\[
\sqrt{n}\mathbb{P}\left[1/n(L(x,t), (#C(x,t))^\alpha) \in (\tau, \infty) \times (\lambda, \infty)\right] = \begin{cases} 0 & \text{if } \alpha < 2/3 \\ (\pi \tau \gamma_0^2)^{-1/2} & \text{if } \alpha > 2/3 \end{cases} \hspace{1cm} (8)
\]

and

\[
\sqrt{n}\mathbb{P}\left[1/n(L(x,t), (D_{max}(x,t))^\alpha) \in (\tau, \infty) \times (\lambda, \infty)\right] = \begin{cases} 0 & \text{if } \alpha < 1/2 \\ (\pi \tau \gamma_0^2)^{-1/2} & \text{if } \alpha > 1/2. \end{cases} \hspace{1cm} (9)
\]
The estimates of the densities \( f_{I_0^+} \) and \( f_{M_0^+} \) imply that \( \mu \) and \( \nu \) are finite measures on \( E \). Note that the above theorem asserts that the distribution of \((L(x,t), (\#C(x,t))^{2/3}) \) (as well as that of \((L(x,t), (D_{\max}(x,t))^{1/2}) \) has a regularly varying tail (see Resnick (2007) page 172).

An immediate consequence of the above theorem is the following:

**Corollary 1.4.1.** As \( n \to \infty \) for \( u > 0 \), we have

\[
(i) \quad \sqrt{n} \Pr(\#C(x,t) > \sqrt{2n^3 \gamma_0 pu}) \to \frac{1}{2\sqrt{\pi \gamma_0}} \int_0^{\infty} t^{-\frac{3}{2}} \bar{F}_{I_0^+}(ut^{-\frac{3}{2}}) \, dt,
\]

\[
(ii) \quad \sqrt{n} \Pr(D_{\max}(x,t) > \sqrt{2n^3 \gamma_0 pu}) \to \frac{1}{2\sqrt{\pi \gamma_0}} \int_0^{\infty} t^{-\frac{3}{2}} \bar{F}_{M_0^+}(ut^{-\frac{3}{2}}) \, dt
\]

where \( F_{I_0^+} \) and \( F_{M_0^+} \) are the distribution functions of \( I_0^+ \) and \( M_0^+ \) respectively and \( \bar{F}_{I_0^+} := 1 - F_{I_0^+}, \bar{F}_{M_0^+} := 1 - F_{M_0^+} \).

The proofs of the above theorems are based on a scaling of the process. In the next section we define a dual graph and show that as processes, under a suitable scaling, the original and the dual processes converge jointly to the Brownian web and its dual in distribution (the double Brownian web). This invariance principle is used in Sections 3 and 4 to prove the theorems. In this connection it is worth noting that in Proposition 2.8, we have provided an alternate characterization of the dual of Brownian web which is of independent interest. This characterization is suitable for proving the joint convergence of coalescing non-crossing path family and its dual to the double Brownian web and has been used in Theorem 2.10 to achieve the required convergence.

We should mention here that the Brownian web appears as a universal scaling limit for various network models (see Fontes et al. (2004), Ferrari et al. (2005), Coletti et al. (2009)). It is reasonable to expect that with suitable modifications our method will give similar results in other network models. Our results will hold for any network model which admits a dual and satisfies (i) conditions listed in Remark 2.1, (ii) the scaled model and its dual converges weakly to the double Brownian web (see Section 2) and (iii) a certain sequence of counting random variables are uniformly integrable (see Lemma 3.3).

2 Dual process and the double Brownian web

2.1 Dual process

For the graph \( G \) we now describe a dual process such that the set of ancestors \( C(x,t) \) (defined in the previous section) of a vertex \((x,t) \in V \) is bounded by two dual paths. The dependency inherent in the graph \( G \) implies that, although the cluster is bounded by two dual paths, these paths are not given by independent random walks. The dual vertices are precisely the mid-points between any two consecutive open vertices on each horizontal line \( \{y = n\}, n \in \mathbb{Z} \) with each dual vertex having a unique offspring dual vertex in the negative direction of the y-axis. Before giving a formal definition, we direct the attention of the reader to Figure 1.
Figure 1: The black points are open points, the gray points are the points of the dual process and the gray (dashed) paths are the dual paths.

For \( u \in \mathbb{Z}^2 \), we define,

\[
J_u^+ := \inf\{ k : k \geq 1, (u(1) + k, u(2)) \in V \} \\
J_u^- := \inf\{ k : k \geq 1, (u(1) - k, u(2)) \in V \}.
\] (10)

Next, we define \( r(u) := (u(1) + J_u^+, u(2)) \) and \( l(u) := (u(1) - J_u^-, u(2)) \), as the first open point to the right (open right neighbour) and the first open point to the left (open left neighbour) of \( u \) respectively. For \( (x,t) \in V \), let \( \tilde{r}(x,t) := (x + J_{(x,t)}^+/2, t) \) and \( \tilde{l}(x,t) := (x - J_{(x,t)}^-/2, t) \) denote respectively the right dual neighbour and the left dual neighbour of \( (x,t) \) in the dual vertex set. Finally, the dual vertex set is given by

\[
\hat{V} := \{ \tilde{r}(x,t), \tilde{l}(x,t) : (x,t) \in V \}.
\]

Remark 2.1: Note that the vertex set of the dual graph is a subset of \( \frac{1}{2}\mathbb{Z} \times \mathbb{Z} \). Before we proceed, we list some properties of the graph \( G \) and its dual \( \hat{G} \).
(1) $\mathcal{G}$ uniquely specifies the dual graph $\hat{\mathcal{G}}$ and the dual edges do not intersect the original edges. The construction ensures that $\hat{\mathcal{G}}$ does not contain any circuit.

(2) For $(x,t) \in V$, the cluster $C(x,t)$ is enclosed within the dual paths starting from $\hat{\rho}(x,t)$ and $\hat{\eta}(x,t)$. The boundedness of $C(x,t)$ for every $(x,t) \in V$ implies that these two dual paths coalesce, thus $\hat{\mathcal{G}}$ is a single tree.

(3) Since paths starting from any two open vertices in the original graph coalesce and the dual edges do not cross the original edges, there is no bi-infinite path in $\hat{\mathcal{G}}$. □

Fix $(u,s) \in \hat{V}$ and for $k \geq 1$, set $\hat{h}_k(u,s) := \hat{h}(\hat{h}^{k-1}(u,s))$ where $\hat{h}^0(u,s) := (u,s)$. Let $\hat{h}_k(u,s) := (X_{k}^{(u,s)}, s - k)$ for $k \geq 0$. Given $\hat{X}_k^{(u,s)} = v$, we have $\hat{X}_{k+1}^{(u,s)} = \hat{X}_{1}^{(v,s-k)} = (a^*(v,s-k) + a'(v,s-k))/2$. To find the distribution of $\hat{X}_1^{(v,s-k)}$ we note that

(a) if $v \notin \mathbb{Z}$, then $v - 1/2 \in \mathbb{Z}$ and

$$a^*(v,s-k) = v - 1/2 + J_{(v-1/2,s-k-1)}^+$$

and

$$a'(v,s-k) = v + 1/2 - J_{(v+1/2,s-k-1)}^-;$$

(b) if $v \in \mathbb{Z}$ and $(v,s-k-1) \notin V$ then

$$a^*(v,s-k) = v + J_{(v,s-k-1)}^+$$

and

$$a'(v,s-k) = v - J_{(v,s-k-1)}^-;$$

(c) if $v \in \mathbb{Z}$ and $(v,s-k-1) \in V$, then note that the open right neighbour $r(v,s-k)$ and open left neighbour $l(v,s-k)$, flanking the dual vertex $(v,s-k)$ from both sides, are equidistant from $(v,s-k-1)$. Thus, either $U_{(v,s-k-1)} = 1$, in which case

$$a^*(v,s-k) = v$$

and

$$a'(v,s-k) = v - J_{(v,s-k-1)}^-;$$

or $U_{(v,s-k-1)} = -1$, in which case

$$a^*(v,s-k) = v$$

and

$$a'(v,s-k) = v + J_{(v,s-k-1)}^+.$$

We note that in all the three cases above, $\hat{X}_{k+1}^{(u,s)}$ is a function of $\hat{X}_{k}^{(u,s)}$ and the collection of random variables $\{(B_u, U_u) : u(2) = s - k - 1 \in \mathbb{Z}\}$. Thus by the random mapping representation (see, for example, Levin et al. 2008) we have

**Proposition 2.2.** For $(u,s) \in \hat{V}$ the process $\{\hat{X}_k^{(u,s)} : k \geq 0\}$ is a time homogeneous Markov process.

Before we proceed, we make the following observations about the transition probabilities of the Markov process. Let $G$ be a geometric random variable taking values in $\{1,2,\ldots\}$, i.e., $\mathbb{P}(G = l) = p(1-p)^{l-1}$ for $l \geq 1$. For any $u \in \mathbb{Z} \times \mathbb{Z}$, the random variables $J_{u}^+$ and $J_{u}^-$ are i.i.d. copies of the geometric random variable $G$ independent of $B_u$. Further, if $u_1, u_2 \in \mathbb{Z}^2$ are such that $u_1(1) \geq u_2(1) - 1$ and $u_1(2) = u_2(2)$, the random variables $J_{u_1}^+$ and $J_{u_2}^-$ are also independent. Now, for $u \notin \mathbb{Z}$ and $v \in \mathbb{Z}/2$, we have

$$\mathbb{P}(\hat{X}_1^{(u,s)} - \hat{X}_0^{(u,s)} = v | \hat{X}_0^{(u,s)} = u) = \mathbb{P}(J_{(u-1/2,s-1)}^+ - J_{(u+1/2,s-1)}^- = 2v) = \mathbb{P}(G_1 - G_2 = 2v)$$

(12)
where $G_1$ and $G_2$ are i.i.d. copies of $G$, defined above. If $u \in \mathbb{Z}$ and $v \in \mathbb{Z}/2$, we have, 
using notation from (c) above
\[
\mathbb{P}(\hat{X}_1^{(u,s)} - \hat{X}_0^{(u,s)} = v | \hat{X}_0^{(u,s)} = u) = (1 - p)^{\mathbb{P}(G_1 - G_2 = 2v)} + p^{\mathbb{P}(G = 2v)/2} + p^{\mathbb{P}(G = -2v)/2}
\]
where $G_1$ and $G_2$ are as above. It is therefore obvious that the transition probabilities of $\hat{X}_k^{(u,s)}$ depend on whether the present state is an integer or not.

From equations (12) and (13), it immediately follows that

**Proposition 2.3.** For any $(u, s) \in \hat{V}$, \{\hat{X}_k^{(u,s)} : k \geq 0\} is an $L^2$-martingale with respect to the filtration $\mathcal{F}_k := \sigma(\{B_u, U_u : u \in \mathbb{Z}^2, u(2) \geq s - k\})$.

### 2.2 Dual Brownian web

In this section we briefly describe the dual Brownian web $\hat{W}$ associated with $W$ and present an easily verifiable alternate characterization of the dual Brownian web $\hat{W}$.

The Brownian web originated as the diffusive scaling limit of the coalescing simple random walk paths starting from every point on the space-time lattice (see Arratia (1979), Arratia (1981)). Thus we can think of the Brownian web as a collection of one-dimensional coalescing Brownian motions starting from every point on the space-time lattice (see Arratia).

We recall relevant details from Fontes et al. (2004).

Let $\mathbb{R}^2_\pi$ denote the completion of the space time plane $\mathbb{R}^2$ with respect to the metric
\[
\rho((x_1, t_1), (x_2, t_2)) := |\tanh(t_1) - \tanh(t_2)| \lor \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right|.
\]

As a topological space $\mathbb{R}^2_\pi$ can be identified with the continuous image of $[-\infty, \infty]^2$ under a map that identifies the line $[-\infty, \infty] \times \{\infty\}$ with the point $(\ast, \infty)$, and the line $[-\infty, \infty] \times \{-\infty\}$ with the point $(\ast, -\infty)$. A path $\pi$ in $\mathbb{R}^2_\pi$ with starting time $\sigma_x \in [-\infty, \infty]$ is a mapping $\pi : [\sigma_x, \infty) \to [-\infty, \infty] \cup \{\ast\}$ such that $\pi(\infty) = \ast$ and, when $\sigma_x = -\infty$, $\pi(-\infty) = \ast$. Also $t \mapsto (\pi(t), t)$ is a continuous map from $[\sigma_x, \infty]$ to $(\mathbb{R}^2_\pi, \rho)$.

We then define $\Pi$ to be the space of all paths in $\mathbb{R}^2_\pi$ with all possible starting times in $[-\infty, \infty]$. The following metric, for $\pi_1, \pi_2 \in \Pi$
\[
d_\Pi(\pi_1, \pi_2) := |\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})| \lor \sup_{t \geq \sigma_{\pi_1} \land \sigma_{\pi_2}} \left| \frac{\tanh(\pi_1(t \lor \sigma_{\pi_1}))}{1 + |t|} - \frac{\tanh(\pi_2(t \lor \sigma_{\pi_2}))}{1 + |t|} \right|
\]

makes $\Pi$ a complete, separable metric space.

**Remark 2.4:** Convergence in this metric can be described as locally uniform convergence of paths as well as convergence of starting times. Therefore, for any $\epsilon > 0$ and $m > 0$, we can choose $\epsilon_1(= f(\epsilon, m)) > 0$ such that for $\pi_1, \pi_2 \in \Pi$ with \{$(\pi_i(t), t) : t \in [\sigma_{\pi_i}, m]$\} $\subseteq [-m, m] \times [-m, m]$ for $i = 1, 2$, $d_\Pi(\pi_1, \pi_2) < \epsilon_1$ implies that $|||\pi_1(\sigma_{\pi_1}), \sigma_{\pi_1}) - (\pi_2(\sigma_{\pi_2}), \sigma_{\pi_2})||_2 < \epsilon$ and $sup\{|\pi_1(t) - \pi_2(t)| : t \in [max\{\sigma_{\pi_1}, \sigma_{\pi_2}\}, m]\} < \epsilon$. We will use this later several times.\[\square\]
Let $\mathcal{H}$ be the space of compact subsets of $(\Pi, d_\Pi)$ equipped with the Hausdorff metric $d_\mathcal{H}$ given by,

$$d_\mathcal{H}(K_1, K_2) := \sup_{\pi_1 \in K_1} \inf_{\pi_2 \in K_2} d_\Pi(\pi_1, \pi_2) \lor \sup_{\pi_2 \in K_2} \inf_{\pi_1 \in K_1} d_\Pi(\pi_1, \pi_2).$$

The space $(\mathcal{H}, d_\mathcal{H})$ is a complete separable metric space. Let $\mathcal{B}_\mathcal{H}$ be the Borel $\sigma-$algebra on the metric space $(\mathcal{H}, d_\mathcal{H})$. The Brownian web $\mathcal{W}$ is an $(\mathcal{H}, \mathcal{B}_\mathcal{H})$ valued random variable.

We now recall from [Fontes et al. 2004] Theorem 2.1, the following characterization of the Brownian web $\mathcal{W}$:

**Theorem 2.5.** There exists an $(\mathcal{H}, \mathcal{B}_\mathcal{H})$ valued random variable $\mathcal{W}$ such that whose distribution uniquely determined by the following properties:

(a) for each deterministic point $z \in \mathbb{R}^2$ there is a unique path $\pi^z \in \mathcal{W}$ almost surely;

(b) for finite set of deterministic points $z^1, \ldots, z^k \in \mathbb{R}^2$, the collection $(\pi^{z^1}, \ldots, \pi^{z^k})$ is distributed as coalescing Brownian motions;

(c) for any deterministic dense set $D \in \mathbb{R}^2$, $\mathcal{W}$ is the closure of $\{\pi^z : z \in D\}$ in $(\Pi, d_\Pi)$ almost surely.

Before introducing the dual Brownian web we require a similar metric space on the collection of backward paths. As in the definition of $\Pi$, let $\overset{\sim}{\Pi}$ be the collection of all paths $\overset{\sim}{\pi}$ with starting time $\sigma_\overset{\sim}{\pi} \in [-\infty, \infty]$ such that $\overset{\sim}{\pi} : [\sigma_\overset{\sim}{\pi}], [\sigma_\overset{\sim}{\pi}] \cup \{\ast\}$ with $\overset{\sim}{\pi}(-\infty) = \ast$ and, when $\sigma_\overset{\sim}{\pi} = +\infty$, $\overset{\sim}{\pi}(\infty) = \ast$. As earlier $t \mapsto (\overset{\sim}{\pi}(t), t)$ is a continuous map from $[\sigma_\overset{\sim}{\pi}], (\mathbb{R}^2, \rho)$. We equip $\overset{\sim}{\Pi}$ with the metric

$$d_{\overset{\sim}{\Pi}}(\overset{\sim}{\pi}_1, \overset{\sim}{\pi}_2) = |\tanh(\sigma_{\overset{\sim}{\pi}_1}) - \tanh(\sigma_{\overset{\sim}{\pi}_2})| \lor \sup_{t \leq \sigma_{\overset{\sim}{\pi}_1} \lor \sigma_{\overset{\sim}{\pi}_2}} \left| \frac{\tanh(\overset{\sim}{\pi}_1(t \land \sigma_{\overset{\sim}{\pi}_1})) - \tanh(\overset{\sim}{\pi}_2(t \land \sigma_{\overset{\sim}{\pi}_2}))}{1 + |t|} \right|$$

making $(\overset{\sim}{\Pi}, d_{\overset{\sim}{\Pi}})$ a complete, separable metric space. The metric space of compact sets of paths of $\overset{\sim}{\Pi}$ is denoted by $(\overset{\sim}{\mathcal{H}}, d_{\overset{\sim}{\mathcal{H}}})$, where $d_{\overset{\sim}{\mathcal{H}}}$ is the Hausdorff metric on $\overset{\sim}{\mathcal{H}}$, and let $\overset{\sim}{\mathcal{B}}_{\overset{\sim}{\mathcal{H}}}$ be the corresponding Borel $\sigma$ field.

**2.3 Properties of $(\mathcal{W}, \overset{\sim}{\mathcal{W}})$**

The Brownian web and its dual $(\mathcal{W}, \overset{\sim}{\mathcal{W}})$ is a $(\mathcal{H} \times \overset{\sim}{\mathcal{H}}, \mathcal{B}_{\mathcal{H}} \times \mathcal{B}_{\overset{\sim}{\mathcal{H}}})$ valued random variable such that $\mathcal{W}$ and $\overset{\sim}{\mathcal{W}}$ uniquely determine each other almost surely with $\overset{\sim}{\mathcal{W}}$ being equally distributed as $-\mathcal{W}$, the Brownian web rotated 180° about the origin. The interaction between the paths in $\mathcal{W}$ and $\overset{\sim}{\mathcal{W}}$ is that of Skorohod reflection (see [Soucaliuc et al. 2000]). We list some properties which hold almost surely.

(a) Let $D, \overset{\sim}{D} \subseteq \mathbb{R}^2$ be two deterministic dense sets. There exist unique paths $\pi^{(x,t)} \in \mathcal{W}$ and $\overset{\sim}{\pi}^{(y,s)} \in \overset{\sim}{\mathcal{W}}$ starting from any $(x, t) \in D$ and $(y, s) \in \overset{\sim}{D}$ respectively.
(b) As in Fontes et al. (2003), for \((W, \hat{W})\) and \((x, t) \in \mathbb{R}^2\), we define

\[
m_{\text{in}}(x, t) := \lim_{\epsilon \downarrow 0} \{ \text{number of paths in } W \text{ starting at some } t - \epsilon \text{ that pass through } (x, t) \text{ and are disjoint in } (t - \epsilon, t) \};
\]

\[
m_{\text{out}}(x, t) := \lim_{\epsilon \downarrow 0} \{ \text{number of paths in } W \text{ starting at } (x, t) \text{ that are disjoint in } (t, t + \epsilon) \};
\]

The type of a point \((x, t)\) is given by \((m_{\text{in}}(x, t), m_{\text{out}}(x, t))\). Similarly we define \(\hat{m}_{\text{in}}(x, t)\) and \(\hat{m}_{\text{out}}(x, t)\) for the dual paths. It is known that (see Proposition 5.12, Theorem 5.16 of Fontes et al. (2003))

(i) \(m_{\text{in}}(x, t) + 1 = \hat{m}_{\text{out}}(x, t)\) and \(m_{\text{out}}(x, t) - 1 = \hat{m}_{\text{in}}(x, t)\).

(ii) Every deterministic point \((x, t) \in \mathbb{R}^2\) is of type \((0, 1)\).

(iii) For any deterministic time \(t\), each point on \(\mathbb{R} \times \{t\}\) is of either type \((0, 1)\), \((0, 2)\) or \((1, 1)\) in \(W\).

(c) For \(\pi^1, \pi^2 \in W\), let \(t^{\pi^1, \pi^2} := \inf\{ t : t > \max\{\sigma_{\pi^1}, \sigma_{\pi^2}\}, \pi^1(t) = \pi^2(t) \}\). Then, \(t^{\pi^1, \pi^2} < \infty\) and for all \(s > t^{\pi^1, \pi^2}\), \(\pi^1(s) = \pi^2(s)\), i.e., the paths coalesce at the time of intersection. For \(\{\pi_n : n \geq 1\} \subseteq W\) with \(d_H(\pi_n, \pi) \to 0\), we have that \(t^{\pi_n, \pi} \to \sigma\) as \(n \to \infty\) (see Sun et al. (2008)).

(d) For \(\pi^1 \in W\) with \((\pi^1(\sigma_{\pi^1}), \sigma_{\pi^1})\) of type \((0, 1)\), \((0, 2)\), \((1, 1)\), \((2, 1)\) or \((1, 2)\) and for any \(\epsilon > 0\), there exists a path \(\pi^2 \in W\) such that \(\sigma_{\pi^2} < \sigma_{\pi^1}\) and \(\pi^2(t) = \pi^1(t)\) for all \(t \geq \sigma_{\pi^1} + \epsilon\) (follows from the proof of Lemma 3.4 of Sun et al. (2008)).

(e) For \(\pi \in W\), \(\hat{\pi} \in \hat{W}\) and

(i) for no \(s, t \in [\pi, \sigma_{\pi}]\), we have \((\pi(s) - \hat{\pi}(s))(\pi(t) - \hat{\pi}(t)) < 0\), i.e., no forward path of \(W\) crosses a dual path of \(\hat{W}\);

(ii) \(\int_{\sigma} \{\sigma(s) = \hat{\pi}(s)\}ds = 0\), i.e., forward paths of \(W\) and dual paths of \(\hat{W}\) “spend zero Lebesgue time together” (see Sun et al. (2008)).

(f) For any \(s > 0\), the sets \(\{\pi(t + s) : \pi \in W, \sigma_{\pi} \leq t\}\) and \(\{\pi(t - s) : \pi \in \hat{W}, \sigma_{\pi} \geq t\}\) are locally finite.

We introduce some notation to study the sets \(\{\pi(t + s) : \pi \in W, \sigma_{\pi} \leq t\}\) and \(\{\hat{\pi}(t - s) : \hat{\pi} \in \hat{W}, \sigma_{\hat{\pi}} \geq t\}\). For a \((H, B_H)\) valued random variable \(K\) and \(t \in \mathbb{R}\) let \(K_{t^-} := \{\pi : \pi \in K\text{ and } \sigma_{\pi} \leq t\}\). Similarly for a \((\hat{H}, B_{\hat{H}})\) valued random variable \(\hat{K}\) and \(t \in \mathbb{R}\) let \(\hat{K}_{t^+} := \{\hat{\pi} : \hat{\pi} \in \hat{K}\text{ and } \sigma_{\hat{\pi}} \geq t\}\). For \(t_1, t_2 \in \mathbb{R}, t_2 > t_1\) and a \((H, B_H)\) valued random variable \(K\), define

\[
\mathcal{M}_K(t_1, t_2) := \{\pi(t_2) : \pi \in K_{t_1^-}, \pi(t_2) \in [0, 1]\};
\]

\[
\xi_K(t_1, t_2) := \#\mathcal{M}_K(t_1, t_2),
\]

(14)
i.e., $\xi_K(t_1, t_2)$ denotes the number of distinct points in $[0, 1] \times t_2$ which are on some path in $K^{1_{t’}}$. We note that for $t > 0$, $M_{W}(t_0, t_0 + t) = N_{W}(t_0, t; 0, 1)$ as defined in Sun et al. (2008). It is known that for all $t > 0$ the random variable $\xi_W(t_0, t_0 + t)$ is finite almost surely (see ($E_1$) in Theorem 1.3 in Sun et al. (2008)) with

$$
E(\xi_W(t_0, t_0 + t)) = \frac{1}{\sqrt{\pi t}}.
$$

Moreover, from the properties listed above we obtain the following Proposition.

**Proposition 2.6.** For any $t_0 < t_1$ almost surely we have

(i) $M_{W}(t_0, t_1) \cap \mathbb{Q} = \emptyset$;

(ii) each point in $M_{W}(t_0, t_1)$ is of type $(1, 1)$;

(iii) for each $x \in M_{W}(t_0, t_1)$ there exists $\pi \in W$ with $\sigma_\pi < t_0$ and $\pi(t_1) = x$;

(iv) $\lim_{s \downarrow t_0} \xi_W(s, t_1) = \xi_W(t_0, t_1)$;

(v) for each $x \in M_{W}(t_0, t_1)$ there exist exactly two paths $\hat{\pi}_x^{(x, t_1)}$ and $\hat{\pi}_x^{(x, t_1)}$ in $\hat{W}$ starting from $(x, t_1)$ with $\hat{\pi}_x^{(x, t_1)}(t) > \hat{\pi}_x^{(x, t_1)}(t)$ for all $[t_0, t_1)$.

**Proof:** (i), (ii), (iii) and (v) follow directly from the properties of $(W, \hat{W})$ listed above. For (iv) we consider $t_0 = 0$ and $t_1 = 1$. For other choices of $t_0$ and $t_1$, the argument is similar. Clearly $M_{W}(0, 1) \subseteq M_{W}(s, 1)$ for all $0 < s < 1$. Therefore, $M_{W}(0, 1) \subseteq \cap_{s \in (0, 1)} M_{W}(s, 1)$ and hence $\lim_{s \downarrow 0} \xi_W(s, 1) \geq \xi_W(0, 1)$. Let $m = m(\omega) > 0$ be such that

$$
\{(y, t) : \hat{\pi}_y^{(0, 1)}(t) \leq y \leq \hat{\pi}_y^{(1, 1)}(t), 0 \leq t \leq 1, \hat{\pi}_y^{(0, 1)}, \hat{\pi}_y^{(1, 1)} \in \hat{W}^{1+} \} \subset [-m, m]^2.
$$

(16)

If $\lim_{s \downarrow 0} \xi_W(\omega)(s, 1) > \xi_W(\omega)(0, 1)$, then there exist $x \in M_{W}(\omega)(1/2, 1) \setminus M_{W}(\omega)(0, 1)$ and a sequence of paths $\{\pi^n(1) = x\}$ for all $n$, such that $0 < \sigma_\pi^n$ and $\sigma_\pi^n \downarrow 0$. By compactness of $W(\omega)$ it follows that there exists a convergent subsequence $\{\pi^{n_k} : k \in \mathbb{N}\}$ and $x \in W(\omega)$ such that $\pi^{n_k} \rightharpoonup \pi$ in $(\Pi, d_H)$ as $k \to \infty$. Hence we must have $\sigma_\pi = 0$ and $\pi(1) = x$ which contradicts the choice of $x$ and the proof follows.

There are several ways to construct $\hat{W}$ from $W$. In this paper we follow the wedge characterization provided by Sun et al. (2008). For $\pi^r, \pi^l \in \hat{W}$ with coalescing time $t^{\pi^r, \pi^l}$ and $\pi^\dagger(\max\{\sigma_{\pi^r}, \sigma_{\pi^l}\}) > \pi^\dagger(\max\{\sigma_{\pi^r}, \sigma_{\pi^l}\})$, the wedge with right boundary $\pi^r$ and left boundary $\pi^l$, is an open set in $\mathbb{R}^2$ given by

$$
A = A(\pi^r, \pi^l) := \{(y, s) : \max\{\sigma_{\pi^l}, \sigma_{\pi^r}\} < s < t^{\pi^r, \pi^l}, \pi^l(s) < y < \pi^r(s)\}.
$$

(17)

A path $\hat{\pi} \in \hat{\Pi}$, is said to enter the wedge $A$ from outside if there exist $t_1$ and $t_2$ with $\sigma_{\hat{\pi}} > t_1 > t_2$ such that $(\hat{\pi}(t_1), t_1) \notin A$ and $(\hat{\pi}(t_2), t_2) \in A$.

From Theorem 1.9 in Sun et al. (2008) it follows that the dual Brownian web $\hat{W}$ associated with the Brownian web $W$ satisfies the following wedge characterization.
Theorem 2.7. Let \((W, \tilde{W})\) be a Brownian web and its dual. Then almost surely

\[ \tilde{W} = \{ \hat{\pi} : \hat{\pi} \in \tilde{\Pi} \text{ and does not enter any wedge in } W \text{ from outside} \}. \]

Because of Theorem 2.7 for a \((\mathcal{H} \times \hat{\mathcal{H}}, B_\mathcal{H} \times B_{\hat{\mathcal{H}}} )\) valued random variable \((W, Z)\) to show that \(Z = \tilde{W}\), it suffices to check that \(Z\) satisfies the wedge condition. Here we present an alternate condition which is easier to check.

Proposition 2.8. Let \((W, Z)\) be a \((\mathcal{H} \times \hat{\mathcal{H}}, B_\mathcal{H} \times B_{\hat{\mathcal{H}}} )\) valued random variable such that

1. for any deterministic \((x, t) \in \mathbb{R}^2\), there exists a path \(\hat{\pi}^{(x,t)} \in Z\) starting at \((x, t)\) and going backward in time almost surely;

2. paths in \(Z\) do not cross paths in \(W\) almost surely, i.e., there does not exist any \(\pi \in W, \hat{\pi} \in Z\) and \(t_1, t_2 \in (\sigma_\pi, \sigma_{\hat{\pi}})\) such that \((\hat{\pi}(t_1) - \pi(t_1))(\hat{\pi}(t_2) - \pi(t_2)) < 0\) almost surely;

3. paths in \(Z\) and paths in \(W\) do not coincide over any time interval almost surely, i.e., for any \(\pi \in W\) and \(\hat{\pi} \in Z\) and for no pair of points \(t_1 < t_2\) with \(\sigma_\pi \leq t_1 < t_2 \leq \sigma_{\hat{\pi}}\) we have \(\hat{\pi}(t) = \pi(t)\) for all \(t \in [t_1, t_2]\) almost surely.

Then \(Z = \tilde{W}\) almost surely.

Proof: We show that \(\hat{\pi} \in Z\) does not enter any wedge in \(W\) from outside. Suppose, on the contrary, \(\hat{\pi} \in Z\) enters the wedge \(A(\pi^*, \pi^l)\) from outside. Set \(T_A := \{ (\pi^*(t), t) : t \geq t^{\pi^*, \pi^l} \} = \{ (\pi^l(t), t) : t \geq t^{\pi^*, \pi^l} \}\). The dual path \(\hat{\pi}\) must satisfy that there exists a pair of time points \(t_1, t_2\) with \(\sigma_{\hat{\pi}} > t_1 > t_2\) such that either (a) \((\hat{\pi}(t_1), t_1) \notin \tilde{A} \cup T_A\) and \((\hat{\pi}(t_2), t_2) \in A\) or (b) \((\hat{\pi}(t_1), t_1) \notin \tilde{A}\) and \((\hat{\pi}(t_2), t_2) \in A\) but for all \(t_2 \leq t < \sigma_{\hat{\pi}}, (\hat{\pi}(t), t) \in \tilde{A} \cup T_A\).

In case (a), either \(\hat{\pi}(t_1) < \pi^l(t_1)\), in which case \((\hat{\pi}(t_1) - \pi^l(t_1))(\hat{\pi}(t_2) - \pi^l(t_2)) < 0\) or \(\hat{\pi}(t_1) > \pi^l(t_1)\), in which case \((\hat{\pi}(t_1) - \pi^l(t_1))(\hat{\pi}(t_2) - \pi^l(t_2)) < 0\). By condition (2), this is not possible almost surely.

In case (b), we have that \(\hat{\pi}(t) = \pi^l(t)\) for all \(t^{\pi^*, \pi^l} \leq t < \sigma_{\hat{\pi}}\) which, by condition (3), is not possible almost surely. This proves that no path in \(Z\) enters any wedge from outside almost surely and hence \(\tilde{Z} \subseteq \tilde{W}\) almost surely.

To show \(\tilde{W} \subseteq \tilde{Z}\), we first observe that since \(Z\) is compact, it is enough to show that for any \(\hat{\pi} \in \tilde{W}\) and \(\epsilon > 0\) and finitely many time points \(t_k < t_{k-1} < \cdots < t_1 < \sigma_{\hat{\pi}}\) with \(t_i \in \mathbb{Q}\), there exists \(\tilde{\pi} \in Z\) such that \(|\tilde{\pi}(t_i) - \hat{\pi}(t_i)| < \epsilon\) for all \(i = 1, \ldots, k\).

We recall here that for any \((x, t) \in \mathbb{Q} \times \mathbb{Q}\) there exists almost surely a unique path \(\pi^{(x,t)} \in W\) such that the finite dimensional distributions of \(\{\pi^{(x,t)} : (x, t) \in \mathbb{Q} \times \mathbb{Q}\}\) are given by that of coalescing Brownian motions. Furthermore, by assumption (1), for every \((x, t) \in \mathbb{Q} \times \mathbb{Q}\), there is a path \(\hat{\pi}^{(x,t)} \in Z\) almost surely.

We use ideas introduced in Sun et al. (2008) to create a fish-trap using paths of \(W\), which will ensure that a path of \(Z\) lies close to the given path. In other words, we construct two collections of paths \(f_{\text{left}}\) and \(f_{\text{right}}\) in \(W\), each member of \(f_{\text{left}}\) lying to the
left of the dual path \( \hat{\pi} \) in \( \hat{W} \) and each member of \( f_{\text{right}} \) lying to the right of \( \hat{\pi} \). Since paths in \( Z \) cannot cross paths in \( W \), the construction also ensures that any path of \( Z \), starting at the right of the top-most member of \( f_{\text{left}} \) cannot weave through the paths of \( f_{\text{left}} \) and will always remain to the right of all the paths in \( f_{\text{left}} \). Using the fact that there is a path in \( Z \), starting from every point having both co-ordinates rational, we will conclude the result.

We pick a rational \( x_k^{(\text{left})} \in (\hat{\pi}(t_k) - \epsilon/2, \hat{\pi}(t_k)) \) and start a path \( \pi_k^{(\text{left})} \in W \) from \( (x_k^{(\text{left})}, t_k) \). We have \( \pi_k^{(\text{left})}(t_k) < \hat{\pi}(t_k) \) almost surely. Now, we choose a rational \( x_{k-1}^{(\text{left})} \) such that \( \max\{\pi_k^{(\text{left})}(t_k), \hat{\pi}(t_k) - \epsilon/2\} < x_{k-1} < \min\{\pi_k^{(\text{right})}(t_k), \hat{\pi}(t_k) + \epsilon/2\} \) and start a path \( \pi_{k-1}^{(\text{left})} \in W \) from \( (x_{k-1}^{(\text{left})}, t_{k-1}) \). We continue this way and construct the family of paths \( \{\pi_j^{(\text{left})} : j = 2, 3, \ldots, k\} \) with starting point of the \( j \) th path being \( (x_j^{(\text{left})}, t_j) \) for \( j = 2, 3, \ldots, k \). Clearly each of these paths stays to the left of \( \hat{\pi} \). We construct similarly another collection of paths \( \{\pi_j^{(\text{right})} : j = 2, 3, \ldots, k\} \) with starting point of the \( j \) th path being \( (x_j^{(\text{right})}, t_j) \) for \( j = 2, 3, \ldots, k \), whose paths stay to the right of \( \hat{\pi} \). This collection of paths constitutes the fish-trap. Now, consider \( x_1 \in \mathbb{Q} \) such that \( \max\{\pi_2^{(\text{left})}(t_1), \hat{\pi}(t_1) - \epsilon/2\} < x_1 < \min\{\pi_2^{(\text{right})}(t_1), \hat{\pi}(t_1) + \epsilon/2\} \). and start a path \( \hat{\pi}_2^{(x_1,t_1)} \in Z \) from the point \( (x_1, t_1) \). Since no paths of \( Z \) and \( W \) cross each other, on \( [t_k, t_1] \) the backward path \( \hat{\pi}_2^{(x_1,t_1)} \) must stay in between \( \{\pi_j^{(\text{left})} : j = 2, 3, \ldots, k\} \) and \( \{\pi_j^{(\text{right})} : j = 2, 3, \ldots, k\} \). Therefore, we have \( |\hat{\pi}_2^{(x_1,t_1)}(t_i) - \pi(t_i)| < \epsilon \) for \( i = 1, \ldots, k \). This completes the proof.

\[ \square \]

2.4 Convergence to the double Brownian web

For any \((x, t) \in V\) the path \( \pi^{(x,t)} \) in the random graph \( G \) is obtained as the piecewise linear function \( \pi^{(x,t)} : [t, \infty) \rightarrow \mathbb{R} \) with \( \pi^{(x,t)}(t+k) = h^k(x, t)(1) \) for every \( k \geq 0 \) and \( \pi^{(x,t)} \) being linear in the interval \([t+k, t+k+1]\). Similarly for \((x, t) \in \hat{V}\), the dual path \( \hat{\pi}^{(x,t)} \) is the piecewise linear function \( \hat{\pi}^{(x,t)} : (-\infty, t] \rightarrow \mathbb{R} \) with \( \hat{\pi}^{(x,t)}(t-k) = \hat{h}^k(x, t)(1) \) for every \( k \geq 0 \) and \( \hat{\pi}^{(x,t)} \) being linear in the interval \([t-k-1, t-k]\). Let \( \mathcal{X} := \{\pi^{(x,t)} : (x, t) \in V\} \) and \( \hat{\mathcal{X}} := \{\hat{\pi}^{(x,t)} : (x, t) \in \hat{V}\} \) be the collection of all possible paths and dual paths admitted by \( G \) and \( \hat{G} \).

For a given \( \gamma > 0 \) and a path \( \pi \) with starting time \( \sigma_\pi \), the scaled path \( \pi_n(\gamma) : [\sigma_\pi/n, \infty) \rightarrow [-\infty, \infty] \) is given by \( \pi_n(\gamma)(t) = \pi(nt)/(\sqrt{n}\gamma) \) for each \( n \geq 1 \). Thus, the starting time of the scaled path \( \pi_n(\gamma) \) is \( \sigma_{\pi_n(\gamma)} = \sigma_\pi/n \). Similarly for the backward path \( \hat{\pi} \), the scaled version is \( \hat{\pi}_n(\gamma) : [-\infty, \sigma_\pi/n] \rightarrow [-\infty, \infty] \) given by \( \hat{\pi}_n(\gamma)(t) = \hat{\pi}(nt)/(\sqrt{n}\gamma) \) for each \( n \geq 1 \). For each \( n \geq 1 \), let \( \mathcal{X}_n = \mathcal{X}_n(\gamma) := \{\pi_n^{(x,t)}(\gamma) : (x, t) \in V\} \) and \( \hat{\mathcal{X}}_n = \hat{\mathcal{X}}_n(\gamma) := \{\hat{\pi}_n^{(x,t)}(\gamma) : (x, t) \in \hat{V}\} \) be the collections of all the \( n \) th order diffusively scaled paths and dual paths respectively.

The closure \( \overline{\mathcal{X}}_n(\gamma) \) of \( \mathcal{X}_n(\gamma) \) in \((\Pi, d_\Pi)\) and the closure \( \overline{\hat{\mathcal{X}}}_n(\gamma) \) of \( \hat{\mathcal{X}}_n(\gamma) \) in \((\hat{\Pi}, d_{\hat{\Pi}})\) are \((H, B_H)\) and \((\hat{H}, B_{\hat{H}})\) valued random variables respectively. Coletti et al. (2009) showed that
Theorem 2.9. For $\gamma_0 := \gamma_0(p)$ as in Theorem 1.2 as $n \to \infty$, $\bar{X}_n(\gamma_0)$ converges weakly to the standard Brownian Web $\mathcal{W}$.

Our main result of this section is the joint invariance principle for $\{(\bar{X}_n(\gamma_0), \bar{X}_n(\gamma_0)) : n \geq 1\}$ considered as $(\mathcal{H} \times \hat{\mathcal{H}}, \mathcal{B}_\mathcal{H} \times \mathcal{B}_{\hat{\mathcal{H}}})$ valued random variables.

Theorem 2.10. $\{(\bar{X}_n(\gamma_0), \hat{X}_n(\gamma_0)) : n \geq 0\}$ converges weakly to $(\mathcal{W}, \hat{\mathcal{W}})$ as $n \to \infty$.

We require the following propositions to prove Theorem 2.10. We say that $\mathcal{H}$ is the standard Brownian motion. Since $\mathcal{H}$ is a Brownian motion going back in time.

Proposition 2.11. For any deterministic point $(x, t) \in \mathbb{R}^2$, there exists a sequence of paths $\hat{\gamma}_n(x, t) \in \hat{X}_n(\gamma_0)$ which converges in distribution to $\hat{X}(x, t)$.

Proof: For any $(x, t) \in \mathbb{R}^2$ fix $t_n = \lfloor nt \rfloor$ and $x_n = \max\{(\lfloor \sqrt{n}x \rfloor + j : j \leq 0, (\lfloor \sqrt{n}x \rfloor + j, t_n) \in \hat{V}\}$. Let $\hat{\gamma}_n(x, t) \in \hat{X}_n(\gamma_0)$ be the scaling of the path $\hat{\gamma}(x, t_n) \in \hat{X}$.

Since $\mathcal{G}$ is invariant under translation by lattice points and $\mathcal{G}$ is uniquely determined by $\mathcal{G}$, the conditional distribution of $\{x_n, t_n + \hat{\gamma}^j(0, 0) : j \geq 0\}$ given $(0, 0) \in \mathcal{V}$ is the same as that of $\{\hat{\gamma}^j(x_n, t_n) : j \geq 0\}$. We observe that $(x_n/(\sqrt{n}x_0), t_n/n) \to (x, t)$ as $n \to \infty$ almost surely. Hence, it suffices to prove that the scaled dual path starting from $(0, 0)$ given $(0, 0) \in \mathcal{V}$ converges in distribution to $\hat{X}(0, 0)$.

From Proposition 2.3 we see that $\hat{X}^{(0,0)} = \hat{\gamma}(0, 0)(1)$ is an $L^2$ martingale with respect to the filtration $\sigma(B(x, s), U(x, s) : x \in \mathbb{Z}, s \geq -k)$. Let

$$\eta_n(u) := s_n^{-1}[\hat{X}^{(0,0)}_j + (\hat{X}^{(0,0)}_{j+1} - \hat{X}^{(0,0)}_j)(us_n^2 - s_j^2)/(s_{j+1}^2 - s_j^2)]$$

for $u \in [0, \infty)$ and $s_n^2 \leq u s_n^2 < s_{n+1}^2$, where $s_n^2 = \sum_{j=1}^n \mathbb{E}((\hat{X}^{(0,0)}_j - \hat{X}^{(0,0)}_{j-1})^2)$. We know $\eta_n$ converges in distribution to a standard Brownian motion (see Theorem 3, Brown (1971)). Since $s_n^2/(\sqrt{n}) \to 1$, it can be seen that $u \in [0, \infty], \eta_n(u) - \hat{\gamma}^0(0)(u) \to 0$ in probability for any $M > 0$. So by Slutsky’s theorem, we conclude that $\hat{\gamma}^0(0)$ converges in distribution to a standard Brownian motion going backward in time.

The next result helps in estimating the probability that a direct path and a dual path stay close to each other for some time period. Given $m \in \mathbb{N}$ and $\varepsilon, \delta > 0$ we define the event

$$B_n = B_n(\delta, m) := \{\text{there exist } \pi_1^m, \pi_2^m, \pi_3^m \in X_n \text{ such that } \sigma_{\pi_1^m}, \sigma_{\pi_2^m} \leq 0, \sigma_{\pi_3^m} \leq |n\delta|/n, \\
\pi_1^m(0) \in [-m, m], |\pi_2^m(0) - \pi_3^m(0)| < \varepsilon, \text{ with } \pi_1^m(|n\delta|/n) = \pi_2^m(|n\delta|/n), \\
|\pi_1^m(|n\delta|/n) - \pi_3^m(|n\delta|/n)| < \varepsilon, \text{ with } \pi_1^m(2|n\delta|/n) 
eq \pi_3^m(2|n\delta|/n)\}.$$

Lemma 2.12. For any $m \in \mathbb{N}$ and $\varepsilon, \delta > 0$, we have

$$\mathbb{P}(B_n(\delta, m)) \leq C_1(\delta, m)\varepsilon$$

where $C_1(\delta, m)$ is a positive constant, depending only on $\delta$ and $m$. 

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Proof: Let $\mathcal{D}_n^e$ be the unscaled version of the event $\mathcal{B}_n$, i.e.,

$$
\mathcal{D}_n^e := \{ \text{there exist } (x, 0), (y, 0), (z, [n\delta]) \in V \text{ such that } x \in [-m\sqrt{n}\gamma_0, m\sqrt{n}\gamma_0], \\
\quad |x - y| < \sqrt{n}\epsilon_0 \text{ and } h^{[n\delta]}(x, 0) \neq h^{[n\delta]}(y, 0), \\
\quad |h^{[n\delta]}(x, 0)(1) - z| < \sqrt{n}\epsilon_0, h^{2[n\delta]}(x, 0) \neq h^{[n\delta]}(z, [n\delta]) \}.
$$

For $\omega \in \mathcal{D}_n^e$, suppose $x, y$ are as in the definition above and assume that $x < y$. Set $l = \max\{x + j : h^{[n\delta]}(x, 0) = h^{[n\delta]}(x + j, 0)\}$. Clearly, $-m\sqrt{n}\gamma_0 \leq x \leq l < y \leq (m + \epsilon)\sqrt{n}\gamma_0$ and $h^{[n\delta]}(x, 0)(1) = h^{[n\delta]}(l, 0)(1) < h^{[n\delta]}(l + 1, 0)(1) \leq h^{[n\delta]}(y, 0)(1)$. Assume that $h^{[n\delta]}(x, 0)(1) = k$ for some $k \in \mathbb{Z}$. Then, $z$ in the definition above satisfies

$$
z \in (k - \sqrt{n}\epsilon\gamma_0, k + \sqrt{n}\epsilon\gamma_0) \text{ and } h^{[n\delta]}(k, [n\delta]) \neq h^{[n\delta]}(z, [n\delta]).
$$

So, by non-crossing property of paths, it must be the case that $h^{[n\delta]}(k - [\sqrt{n}\epsilon\gamma_0] - 1, [n\delta]) \neq h^{[n\delta]}(k + [\sqrt{n}\epsilon\gamma_0] + 1, [n\delta])$. Thus, we must have $\omega \in H^{(L)}(n, \delta, \epsilon)$ where for $l \in \mathbb{Z}$,

$$
H^{(L)}_{l,k}(n, \delta, \epsilon) := \bigl(h^{[n\delta]}(l, 0)(1) = k \neq h^{[n\delta]}(l + 1, 0)(1) \text{ and } h^{[n\delta]}(k - [\sqrt{n}\epsilon\gamma_0] - 1, [n\delta]) \neq h^{[n\delta]}(k + [\sqrt{n}\epsilon\gamma_0] + 1, [n\delta]) \bigr);
$$

Similarly for $\omega \in \mathcal{D}_n^e$ such that $x > y$, set $r = \min\{x - j : h^{[n\delta]}(x, 0) = h^{[n\delta]}(x - j, 0)\}$. As earlier, $\omega \in H^{(R)}(n, \delta, \epsilon)$ where for $r \in \mathbb{Z}$,

$$
H^{(R)}_{r,k}(n, \delta, \epsilon) := \bigl(h^{[n\delta]}(r, 0)(1) = k \neq h^{[n\delta]}(r - 1, 0)(1) \text{ and } h^{[n\delta]}(k - [\sqrt{n}\epsilon\gamma_0] - 1, [n\delta]) \neq h^{[n\delta]}(k + [\sqrt{n}\epsilon\gamma_0] + 1, [n\delta]) \bigr);
$$

Thus, $\mathcal{D}_n^e \subseteq H^{(L)}(n, \delta, \epsilon) \cup H^{(R)}(n, \delta, \epsilon)$. We note that the events $\{h^{[n\delta]}(l, 0)(1) = k \neq h^{[n\delta]}(l + 1, 0)(1)\}$ and $\{h^{[n\delta]}(k - [\sqrt{n}\epsilon\gamma_0] - 1, [n\delta]) \neq h^{[n\delta]}(k + [\sqrt{n}\epsilon\gamma_0] + 1, [n\delta])\}$ are independent as the first event depends only on the realizations $\{(B_u, U_u) : u \in \mathbb{Z}^2, 1 \leq u(2) \leq [n\delta]\}$ while the second event depends on the realizations $\{(B_u, U_u) : u \in \mathbb{Z}^2, u(2) > [n\delta]\}$.

We have from Theorem 4 of Coletti et al. (2009),

$$
P\{h^{[n\delta]}(k - [\sqrt{n}\epsilon\gamma_0] - 1, [n\delta]) \neq h^{[n\delta]}(k + [\sqrt{n}\epsilon\gamma_0] + 1, [n\delta])\} \leq C_2(2\sqrt{n}\gamma_0\epsilon + 3) \leq \sqrt{\frac{1}{n\delta}} \leq C_3(\delta)\epsilon
$$

where $C_2, C_3(\delta) > 0$ are constants. Hence,

$$
P(H^{(L)}_{l,k}(n, \delta, \epsilon)) \leq C_3(\delta)\epsilon P\{h^{[n\delta]}(l, 0)(1) = k \neq h^{[n\delta]}(l + 1, 0)(1)\}.
$$
Now, the events \( \{ h^{[n\delta]}(l,0)(1) = k \neq h^{[n\delta]}(l+1,0)(1) \} \) are disjoint for distinct values of \( k \). Hence,

\[
\mathbb{P}(\cup_{k \in \mathbb{Z}} H_{t,k}^{(L)}(n,\delta,\epsilon)) \leq C_3(\delta)\epsilon \sum_{k \in \mathbb{Z}} \mathbb{P}\{ h^{[n\delta]}(l,0)(1) = k \neq h^{[n\delta]}(l+1,0)(1) \}
\]

\[
= C_3(\delta)\epsilon \mathbb{P}\{ h^{[n\delta]}(l,0) \neq h^{[n\delta]}(l+1,0) \}
\]

\[
= C_3(\delta)\epsilon C_2/\sqrt{n\delta}
\]

where the last step again follows from Theorem 4 of [Coletti et al. (2009)]#ref. The above argument also holds for \( \cup_{k \in \mathbb{Z}} H_{t,k}^{(R)}(n,\delta,\epsilon) \) to yield \( \mathbb{P}(D_n^\epsilon) \leq C_1(\delta,m)\epsilon \) for a proper choice of \( C_1(\delta,m) \). This completes the proof of Lemma 2.12.

**Proof of Theorem 2.10**: Since \( \mathcal{X} \) consists of non-crossing paths only, Proposition 2.11 implies the tightness of the family \( \{ \hat{X}_n : n \geq 1 \} \) (see Proposition B.2 in the Appendix of Fontes et al. (2004)). The joint family \( \{(\hat{X}_n,\hat{X}_n) : n \geq 1 \} \) is tight since each of the two marginals is tight. To prove Theorem 2.10 it suffices to show that for any subsequential limit \((\mathcal{W},\mathcal{Z})\) of \( \{(\hat{X}_n,\hat{X}_n) : n \geq 1 \} \), the random variable \( \mathcal{Z} \) satisfies the conditions given in Proposition 2.8.

Consider a convergent subsequence of \( \{(\hat{X}_n,\hat{X}_n) : n \geq 1 \} \) such that \((\mathcal{W},\mathcal{Z})\) is its weak limit and by Skorohod’s representation theorem, we may assume that the convergence happens almost surely. For ease of notation, we denote the convergent subsequence by itself.

From Proposition 2.11 it follows that for any deterministic \((x,t) \in \mathbb{R}^2\) there exists a path \( \hat{\pi} \in \mathcal{Z} \) starting at \((x,t)\) going backward in time almost surely.

Next we need to show that paths in \( \mathcal{Z} \) do not cross paths in \( \mathcal{W} \) almost surely. It is enough to consider paths in a compact set \([-m,m] \times [-m,m] \) for \( m \in \mathbb{N} \). Now, suppose that a backward path \( \hat{\pi} \in \mathcal{Z} \) crosses a forward path \( \pi \in \mathcal{W} \) in \([-m,m] \times [-m,m] \). More precisely there exist \( \hat{\pi} \in \mathcal{Z} \) and \( \pi \in \mathcal{W} \) such that we have the following:

(a) \( m > \sigma_\pi > \sigma_\pi > -m, -m \leq \pi(t), \hat{\pi}(t) \leq m \) for all \( t \in [\sigma_\pi, \sigma_\pi] \),

(b) there exist \( \sigma_\pi < t_1 < t_2 < \sigma_\pi \) such that \( (\pi(t_1) - \hat{\pi}(t_1))(\pi(t_2) - \hat{\pi}(t_2)) < 0 \).

By continuity, we can choose \( \epsilon' > 0 \) so that \( \left[ (\pi(t_1) + u_1) - (\hat{\pi}(t_1) + u_2) \right] \left[ (\pi(t_2) + u_3) - (\hat{\pi}(t_2) + u_4) \right] < 0 \) for all \( -\epsilon' < u_1, u_2, u_3, u_4 < \epsilon' \). Choose \( \epsilon = \min\{ (\sigma_\pi - t_2)/3, (t_1 - \sigma_\pi)/3, \epsilon' \} \) and set \( \epsilon_1 = f(\epsilon, m) \), as described in Remark 2.4.

From the almost sure convergence of \( (\hat{X}_n,\hat{X}_n) \) to \((\mathcal{W},\mathcal{Z})\), for any realization \( \omega \) of the above event, we may choose \( n_0(=n_0(\omega)) \) so that there exists \( (\pi^{n_0}, \hat{\pi}^{n_0}) \in \hat{X}_{n_0} \times \hat{X}_{n_0} \) with \( d\|\pi(t), \pi^{n_0}\) < \( \hat{\pi}(t), \hat{\pi}^{n_0}\) < \( \pi(t) - \hat{\pi}(t) \) for all \( t \in [t_1, t_2] \), \( \sup\{ |\pi(t) - \hat{\pi}(t) : t \in [t_1, t_2] \} < \epsilon' \). Thus, by our choice of \( \epsilon' \), we obtain that \( (\pi^{n_0}(t_1) - \hat{\pi}^{n_0}(t_1))(\pi^{n_0}(t_2) - \hat{\pi}^{n_0}(t_2)) < 0 \), i.e., the paths \( (\pi^{n_0}, \hat{\pi}^{n_0}) \in \hat{X}_{n_0} \times \hat{X}_{n_0} \) cross each other, yielding a contradiction.
Now, to prove that condition (3) in Proposition 2.8 is satisfied, we define the following event: for \( \delta > 0 \) and positive integer \( m \geq 1 \), let

\[
A(\delta, m) := \{ \text{there exist paths } \pi \in \mathcal{W} \text{ and } \hat{\pi} \in \mathcal{Z} \text{ with } \sigma_\pi, \sigma_{\hat{\pi}} \in (-m, m), \\
\text{and there exists } t_0 \text{ such that } \sigma_\pi < t_0 < t_0 + \delta < \sigma_{\hat{\pi}}, \\
\text{and } -m < \pi(t) = \hat{\pi}(t) < m \text{ for all } t \in [t_0, t_0 + \delta] \}.
\]

It is enough to show that for any fixed \( \delta > 0 \) and for \( m \geq 1 \), we have \( \mathbb{P}(A(\delta, m)) = 0 \). For any \( 0 < \epsilon < 1 \), we also define

\[
A'(\delta, m) := \{ \text{there exist paths } \pi \in \mathcal{W} \text{ and } \hat{\pi} \in \mathcal{Z} \text{ with } \sigma_\pi, \sigma_{\hat{\pi}} \in (-m, m), \\
\text{and there exists } t_0 \text{ such that } \sigma_\pi < t_0 < t_0 + \delta < \sigma_{\hat{\pi}} \text{ and } \pi(t), \hat{\pi}(t) \in (-m, m), \\
\text{for } t \in [t_0, t_0 + \delta] \text{ and } \sup \{|\pi(t) - \hat{\pi}(t)| : t \in [t_0, t_0 + \delta]\} < \epsilon \}.
\]

Clearly, we have \( A(\delta, m) \subseteq \cap_{\epsilon>0} A'(\delta, m) \). Further, we have \( A'(\delta, m) \) is decreasing in \( \epsilon \), so that \( \mathbb{P}(A(\delta, m)) \leq \lim_{\epsilon \to 0} \mathbb{P}(A'(\delta, m)) \).

Now, for every \( n \geq 1 \) and \( j \geq 1 \), set \( h_n = |n\delta/3|/n \) and \( t_j = -m + jh_n \). Let

\[
B'_n(\delta, m; j) := \{ \text{there exist } \pi^n_1, \pi^n_2, \pi^n_3 \in \mathcal{X}_n \text{ such that } \sigma_{\pi^n_1}, \sigma_{\pi^n_2} \leq t^n_j, \sigma_{\pi^n_3} \leq t^n_{j+1}, \\
\pi^n_1(t^n_j) \in [-2m, 2m], |\pi^n_1(t^n_j) - \pi^n_2(t^n_j)| < 4\epsilon, \text{ with } \pi^n_1(t^n_{j+1}) \neq \pi^n_2(t^n_{j+1}) \text{ and } |\pi^n_1(t^n_{j+1}) - \pi^n_3(t^n_{j+1})| < 4\epsilon, \text{ with } \pi^n_1(t^n_{j+2}) \neq \pi^n_3(t^n_{j+2}) \}.
\]

We observe that the event \( B'_n(\delta, m; j) \) is a translation of the event \( B'_{n'}(\delta, 2m) \), considered in Lemma 2.12 where the starting points of the paths are shifted up by \( t^n_j \), with \( \delta \) and \( \epsilon \) replaced by \( \delta/3 \) and \( 4\epsilon \) respectively. Hence, by translation invariance of our model and Lemma 2.12 we have \( \mathbb{P}(B'_n(\delta, m; j)) = \mathbb{P}(B'_{n'}(\delta, 2m)) \leq 4C_1(\delta/3, 2m) \epsilon \) for all \( n \geq 1 \).

We show that \( A'(\delta, m) \subseteq \liminf_{n \to \infty} \bigcup_{j=1}^{[\delta/m]} B'_n(\delta, m; j) \), which implies that

\[
\mathbb{P}(A(\delta, m)) \leq \liminf_{\epsilon \to 0} \mathbb{P}(A'(\delta, m)) \leq \limsup_{\epsilon \to 0} \mathbb{P}\left( \liminf_{n \to \infty} \bigcup_{j=1}^{[\delta/m]} B'_n(\delta, m; j) \right) \\
\leq \limsup_{\epsilon \to 0} \sum_{j=1}^{[\delta/m]} \mathbb{P}(B'_n(\delta, m; j)) \leq \limsup_{\epsilon \to 0} \frac{6m}{\delta} C_1(\delta/3, m) \epsilon = 0.
\]

For any realization \( \omega \in A'(\delta, m) \), we have \( \pi \in \mathcal{W} \) and \( \hat{\pi} \in \mathcal{Z} \), with their starting times \( \sigma_\pi, \sigma_{\hat{\pi}} \in (-m, m) \) such that \( \sigma_\pi < t_0 < t_0 + \delta < \sigma_{\hat{\pi}} \), sup\{\( |\pi(t) - \hat{\pi}(t)| : t \in [t_0, t_0 + \delta] \}\} < \epsilon, \text{ and } -m < \pi(t), \hat{\pi}(t) < m \text{ for } t \in [t_0, t_0 + \delta]. \) We choose \( \epsilon' = \min\{\epsilon/2, (\sigma_{\hat{\pi}} - t_0 - \delta)/3, (t_0 - \sigma_{\hat{\pi}})/3\} \) and set \( \epsilon_1 = f(\epsilon', m) \), as in Remark 2.4. Using the almost sure convergence of \( (\chi_n, \hat{\chi}_n) \) to \( (W, Z) \), we choose \( n_0(= n_0(\omega)) \geq n_0 \) such that for all \( n \geq n_0 \), there exist \( \pi^n_1 \in \chi_n \) and \( \hat{\pi}^n \in \hat{\chi}_n \) with max\{\( d_\Pi(\pi, \pi^n_1), d_\Pi(\hat{\pi}, \hat{\pi}^n) \}\} < \epsilon_1. \text{ From the choice of } \epsilon_1 \text{ (see Remark 2.4), it is ensured that } \sigma_{\pi^n_1} > t_0 + \delta \text{ and } \sigma_{\hat{\pi}^n} < t_0 \text{ and } \sup\{\{\pi^n_1(t) - \pi(t), |\hat{\pi}^n(t) - \hat{\pi}(t)| : t \in [t_0, t_0 + \delta] \} < \epsilon/2. \text{ Therefore, we have that } \sup\{\{\pi^n_1(t) - \pi(t), |\hat{\pi}^n(t) - \hat{\pi}(t)| : t \in [t_0, t_0 + \delta] \} < 2\epsilon \text{ for all } n \geq n_0. \)

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Now we divide the time interval \([-m, m]\) with parallel horizontal lines at \(y = t_n^j\) for \(j = 1, 2, \ldots\). Now, from the choice of \(n_0\), there must exist \(1 \leq j \leq \left\lfloor \frac{6m}{\sqrt{n}} \right\rfloor\) such that the path \(\pi^n(t)\) starts below \(t_n^j\) and \(\tilde{\pi}^n(t)\) starts above \(t_n^{j+2}\) and the paths stay within \(2\epsilon\) distance of each other in the time interval \([t_n^j, t_n^{j+2}]\). Now \((\pi^n_1(t_n^j), t_n^j)\) and \((\tilde{\pi}^n(t_n^j), t_n^j)\) are scaled open vertex and scaled dual vertex respectively. If \(\pi^n_1(t_n^j) < \tilde{\pi}^n(t_n^j)\), by our definition of the dual vertex, we must have at least one more scaled open vertex, say \((\alpha^1_n, t_n^j)\), such that \(\pi^n(t_n^j) < \alpha^1_n < \tilde{\pi}^n(t_n^j) + 2\epsilon\), so that \(\pi^n_2(t_n^j) < \alpha^1_n < \pi^n(t_n^j) + 4\epsilon\). In such a case, the scaled path \(\pi^n_2\), starting from the scaled open point \((\alpha^1_n, t_n^j)\), will not meet the path \(\pi^n_1\), at least till time point \(t_n^{j+1}\), i.e., \(\pi^n_2(t_n^{j+1}) \neq \pi^n_1(t_n^{j+1})\), since the scaled dual path \(\tilde{\pi}^n\) staying in between \(\pi^n_1\) and \(\pi^n_2\) starts before \(t_n^{j+2}\). If \(\pi^n_2(t_n^{j+1}) < \pi^n_1(t_n^{j+1}) + 4\epsilon\), we take \(\pi^n_3\) as the continuation of \(\pi^n_2\) from \(t_n^{j+1}\). Again the paths \(\pi^n_1\) and \(\pi^n_3\) will not meet before time point \(t_n^{j+2}\). If \(\pi^n_2(t_n^{j+1}) \geq \pi^n_1(t_n^{j+1}) + 4\epsilon\), using the same logic as above, there exists another scaled open vertex \((\alpha^2_n, t_n^{j+1})\) such that \(\pi^n_1(t_n^{j+1}) < \alpha^2_n < \pi^n_1(t_n^{j+1}) + 4\epsilon\) and the scaled path \(\pi^n_3\) starting from the scaled open point \((\alpha^2_n, t_n^{j+1})\) does not meet \(\pi^n_1\) until \(t_n^{j+2}\). In the case \(\pi^n(t_n^j) < \pi^n_1(t_n^j)\), similar argument holds. Therefore, the event \(\bigcup_{j=1}^{2n} B_n(\delta, m; j)\) occurs.

\(\square\)

**Remark 2.13:** Modifying the proof of Lemma \(2.12\) suitably, it can be shown that the probability of the event \(P(A^*(\delta, m))\) decays faster than any power of \(\epsilon\).

\(\square\)

### 3 Proof of Theorem 1.2

Let \(\xi := \xi_W(0, 1)\) and \(\xi_n := \xi_{X_n}(0, 1)\) as defined in \((14)\). The proof of Theorem 1.2 follows from the following Proposition.

**Proposition 3.1.** \(E[\xi_n] \to E[\xi]\) as \(n \to \infty\).

We first complete the proof of Theorem 1.2 assuming Proposition 3.1.

**Proof of Theorem 1.2.** Using the translation invariance of our model, we have,

\[
\sqrt{n}\gamma_0 P(L(0, 0) > n) = \sum_{k=0}^{\lfloor \sqrt{n} \gamma_0 \rfloor} E(1 \{L(k; n) > n\}) \times \frac{\sqrt{n}\gamma_0}{\lfloor \sqrt{n} \gamma_0 \rfloor + 1} = E(\xi_n) \times \frac{\sqrt{n}\gamma_0}{\lfloor \sqrt{n} \gamma_0 \rfloor + 1} \to E(\xi) = \frac{1}{\sqrt{\pi}} \text{ as } n \to \infty.
\]

This proves Theorem 1.2.

\(\square\)

Proposition 3.1 will be proved through a sequence of lemmas.

To state the next lemma we need to recall from Theorem 2.10 that \((\hat{X}_n, \tilde{X}_n) \Rightarrow (\mathcal{W}, \tilde{\mathcal{W}})\) as \(n \to \infty\). Using Skorohod’s representation theorem we assume that we are working on a probability space where \(d_{\mathcal{H}}((\hat{X}_n, \tilde{X}_n), (\mathcal{W}, \tilde{\mathcal{W}})) \to 0\) almost surely as \(n \to \infty\).

**Lemma 3.2.** For \(t_1 > t_0\) we have

\[
P(\xi_{\hat{X}_n}(t_0, t_1) \neq \xi_W(t_0, t_1) \text{ for infinitely many } n) = 0.
\]

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**Proof:** We prove the lemma for \( t_0 = 0 \) and \( t_1 = 1 \), i.e., for \( \xi_n = \xi_{\mathcal{X}}(0, 1) \) and \( \xi_{\mathcal{W}}(0, 1) \), the proof for general \( t_0, t_1 \) being similar. First we show that, for all \( k \geq 0 \),

\[
\liminf_{n \to \infty} 1\{\xi_n \geq k\} \geq 1\{\xi \geq k\} \text{ almost surely.} \tag{18}
\]

Indeed, for \( k = 0 \), both \( 1\{\xi_n \geq k\} \) and \( 1\{\xi \geq k\} \) equal 1. For \( k \geq 1 \), (18) follows from almost sure convergence of \((\hat{X}_n, \hat{X}_n)\) to \((\mathcal{W}, \hat{\mathcal{W}})\). Indeed, from Proposition 2.6 (i) we have that \( \mathcal{M}_{\mathcal{W}}(0, 1) \subseteq (0, 1) \). Since \( \mathcal{M}_{\mathcal{W}}(0, 1) \) is finite, from Proposition 2.6 (iii), we can obtain \( n \) and paths in \( \hat{X}_n \) starting strictly below the x-axis which approximate the paths contributing to \( \mathcal{M}_{\mathcal{W}}(0, 1) \).

To complete the proof, we need to show that \( \mathbb{P}(\limsup_{n \to \infty} \{\xi_n > \xi\}) = 0 \). This is equivalent to showing that \( \mathbb{P}(\Omega_k^\xi) = 0 \) for all \( k \geq 0 \), where

\[
\Omega_k^\xi := \{\omega : \xi_n(\omega) > \xi(\omega) = k \text{ for infinitely many } n\}.
\]

Consider \( k = 0 \) first. From Proposition 2.6 it follows that on the event \( \xi = 0 \), almost surely we can obtain \( \gamma := \gamma(\omega) > 0 \) such that \( \mathcal{M}_{\mathcal{W}}(0, 1) \cap (\mathcal{N}, 1 + \gamma) = \emptyset \). From the almost sure convergence of \((\hat{X}_n, \hat{X}_n)\) to \((\mathcal{W}, \hat{\mathcal{W}})\), we have \( \mathbb{P}(\Omega_0^\xi) = 0 \).

Let \( \omega \in \Omega_k^\xi \). From Proposition 2.6 it follows that there exists \( s_1 = s_1(\omega) \in (0, 1) \) such that \( \xi_{\mathcal{W}}(s_1, 1) = \xi = k \). For \( i = 1, \ldots, k \) choose \( \pi_i \in \mathcal{W}^{s_1}(\omega) \) such that \( \{\pi_i(1) : 1 \leq i \leq k\} = \mathcal{M}_{\mathcal{W}}(0, 1) \) and there exists \( s_2 = s_2(\omega) \in (s_1, 1) \) such that \( \{\pi(s_2) : \pi \in \mathcal{W}^{s_1}(\omega), \pi(1) \in [0, 1]\} = \{\pi_i(s_2) : 1 \leq i \leq k\} \), i.e., the paths leading to any single point considered in \( \mathcal{M}_{\mathcal{W}}(0, 1) = \mathcal{M}_{\mathcal{W}}(s_1, 1) \) have coalesced before time \( s_2 \). Choose \( m = m(\omega) > 0 \) as in (16) and we observe that

\[
\Omega_k^\xi \subseteq \{\text{there exist } \pi_i \in \mathcal{W}^{s_1} \text{ for some } 1 \leq i \leq k, \{\pi^n_1, \pi^n_2 \in \hat{X}_n^{s_1} : n \in \mathbb{N}\} \text{ with}
\]

\[
\pi^n_1(1) \neq \pi^n_2(1), \max\{|\pi^n_1(t) - \pi^n_2(t)| : t \in [s_2, 1], j = 1, 2\} \to 0 \text{ as } n \to \infty
\]

\[
\subseteq \{\text{there exist } \pi_i \in \mathcal{W}^{s_1} \text{ for some } 1 \leq i \leq k, \{\hat{\pi}^n \in \hat{X}_n^{s_1} : n \in \mathbb{N}\} \text{ with}
\]

\[
\max\{|\hat{\pi}^n(t) - \pi_i(t)| : t \in [s_2, 1]\} \to 0 \text{ as } n \to \infty
\]

\[
\subseteq \{\text{there exist } \pi_i \in \mathcal{W}^{s_1} \text{ for some } 1 \leq i \leq k \text{ and } \hat{\pi} \in \mathcal{W}^{s_1} \text{ with}
\]

\[
\hat{\pi}(t) = \pi_i(t) \text{ for all } t \in [s_2, 1].
\]

This violates property (d) of \((\mathcal{W}, \hat{\mathcal{W}})\) listed in the earlier subsection and proves that \( \mathbb{P}(\cup_{k \geq 0} \Omega_k^\xi) = 0 \).

Lemma 3.2 immediately gives the following corollary.

**Corollary 3.2.1.** As \( n \to \infty \), \( \xi_n \) converges in distribution to \( \xi \).

Corollary 3.2.1 along with the following lemma completes the proof of Proposition 3.1.

**Lemma 3.3.** The family \( \{\xi_n : n \in \mathbb{N}\} \) is uniformly integrable.
Proof: For $m \in \mathbb{N}$, let $K_m = [-m, m]^2 \cap \mathbb{Z}^2$ and $\Omega_m := \{(0, 1), (0, -1), (1, 1), (1, -1)\}^{K_m}$. We assign the product probability measure $\mathbb{P}'$ whose marginals for $u \in K_m$ are given by

$$
\mathbb{P}'\{\zeta : \zeta(u) = (a, b)\} = \begin{cases} 
\frac{2^a}{(1 + p)^2} & \text{for } a = 1 \text{ and } b \in \{1, -1\} \\
\frac{2^b}{(1 + p)^2} & \text{for } a = 0 \text{ and } b \in \{1, -1\}.
\end{cases}
$$

$\mathbb{P}'$ is the measure induced by the random variables $\{(B_u, U_u) : u \in K_m\}$.

For $\zeta \in \Omega_m$ and for $K \subseteq K_m$, the $K$ cylinder of $\zeta$ is given by $C(\zeta, K) := \{\zeta' : \zeta'(u) = \zeta(u) \text{ for all } u \in K\}$. For any two events $A, B \subseteq \Omega_m$, let

$$A \circ B := \{\zeta : \text{there exists } K = K(\zeta) \subseteq K_m \text{ such that } C(\zeta, K) \subseteq A, \text{ and } C(\zeta, K') \subseteq B \text{ for } K' = K_m \setminus K\}
$$

denote the disjoint occurrence of $A$ and $B$. Note that this definition is associative, i.e., for any $A, B, C \subseteq \Omega_m$ we have $(A \circ B) \circ C = A \circ (B \circ C)$.

Let

$$F^m_n := \{\text{there exists } (u_1, n), (u_2, n) \in \hat{V} \text{ with } 0 \leq u_1 < u_2 \leq \sqrt{n} \gamma_0 \text{ and } (v^1, l), (v^2, l) \in V \text{ for all } 0 \leq l \leq n \text{ such that } -m \leq v^1 < \hat{h}^l(u_1, n)(1) < \hat{h}^l(u_2, n)(1) < v^2 \leq m\},
$$

$$E^m_n(k) := \{\text{for } 1 \leq i \neq j \leq k, \text{ there exists } (x_i, 0) \in V \text{ with } h^n(x_i, 0)(1) \in [0, \sqrt{n} \gamma_0] \text{ and } h^n(x_i, 0) \neq h^n(x_j, 0), \text{ and } -m \leq h^n(x_i, 0)(1) \leq m \text{ for all } 0 \leq l \leq n\}.$$

We claim that for all $k \geq 2$,

$$E^m_n(3k) \subseteq \underbrace{F^m_n \circ F^m_n \circ \cdots \circ F^m_n}_{k \text{ times}}. \quad (19)$$

We prove it for $k = 2$. For general $k$, the proof is similar. Let $(u_1, n) \in \hat{V}, 1 \leq i \leq 5$ and $(x_i, 0) \in V, 1 \leq i \leq 6$ be as in Figure 2. The region explored to obtain $\hat{h}^l(u_1, n), 1 \leq j \leq n$ is contained within $\cup_{l=0}^{n-1}[h^l(x_i, 0)(1), h^l(x_{i+1}, 0)(1)] \times \{l\}$. Thus the regions explored to obtain the dual paths starting from $(u_1, n), (u_2, n)$ and the dual paths starting from $(u_4, n), (u_5, n)$ are disjoint (see Figure 2). Hence it follows that $E^m_n(6) \subseteq F^m_n \circ F^m_n$.

Since the event $E^m_n(k)$ is monotonic in $m$, from (19) we get

$$\mathbb{P}(\xi_n \geq 3k) = \mathbb{P}(\lim_{m \to \infty} E^m_n(3k)) = \lim_{m \to \infty} \mathbb{P}(E^m_n(3k)) \leq \lim_{m \to \infty} \mathbb{P}(F^m_n \circ \cdots \circ F^m_n) = \lim_{m \to \infty} \mathbb{P}'(F^m_n \circ \cdots \circ F^m_n).
$$

Applying BKR inequality (see [Reimer 2000]) we get

$$\mathbb{P}(\xi_n \geq 3k) \leq \lim_{m \to \infty} (\mathbb{P}(F^m_n))^k = (\mathbb{P}(\lim_{m \to \infty} F^m_n))^k = (\mathbb{P}(F_n))^k \quad (20)$$

where $F_n := \{\text{there exist } (u_1, n), (u_2, n) \in \hat{V} \text{ with } 0 \leq u_1 < u_2 \leq \sqrt{n} \gamma_0 \text{ such that } h^n(u_1, n) \neq h^n(u_2, n)\}$.
For any \((x, t) \in \mathbb{R}^2\) fix \(t_n = \lfloor nt \rfloor\) and \(x_n = \max \{ \lfloor \sqrt{n} \gamma_0 x \rfloor + j : j \leq 0, (\lfloor \sqrt{n} \gamma_0 x \rfloor + j, t_n) \in \hat{V} \}.\) Let \(\hat{\theta}_n(x, t) = \hat{X}_n(\gamma_0)\) be the scaling of the path \(\hat{\pi}(x_n, t_n) \in \hat{X}.\) Define
\[
F_n' := \{ \hat{\theta}_n(0, 1) \text{ and } \hat{\theta}_n(1, 1) \text{ do not coalesce in time 1} \}.
\]
We observe that \(F_n \subseteq F_n'.\) Now \(\mathbb{P}(F_n')\) converges to the probability that two independent Brownian motions starting at a distance 1 from each other do not meet by time 1. Since \(\lim_{n \to \infty} \mathbb{P}(F_n') < 1,\) the family \(\{ \xi_n : n \in \mathbb{N} \}\) is uniformly integrable.

4 Proofs of Theorem 1.3 and 1.4

Recall that \(\hat{r}(x, t)\) and \(\hat{l}(x, t)\) denote the right and left dual neighbours, respectively, of \((x, t) \in V.\) Let \(\hat{D}_k(x, t) := \hat{h}_k(\hat{r}(x, t))(1) - \hat{h}_k(\hat{l}(x, t))(1)\) and consider the continuous function \(\hat{D}_n(x, t) \in C[0, \infty)\) given by
\[
\hat{D}_n(x, t)(s) := \frac{\hat{D}_k(x, t)}{\gamma_0 \sqrt{n}} + \frac{(ns - \lfloor ns \rfloor)}{\gamma_0 \sqrt{n}}(\hat{D}_{k+1}(x, t) - \hat{D}_k(x, t)) \quad \text{for } \frac{k}{n} \leq s \leq \frac{k+1}{n}.
\]
For \(\tau > 0\) and \(T \geq 0, \hat{\pi} \in \hat{\Pi}\) with \(\sigma_{\hat{\pi}} \geq \tau,\) let \(g^{\tau, T}_{\hat{\pi}} \in C[0, \tau + T]\) be given by
\[
g^{\tau, T}_{\hat{\pi}}(t) := \hat{\pi}(\tau) - t \quad \text{for } 0 \leq t \leq \tau + T.
\]
For \((K, \hat{K}) \in \mathcal{H} \times \hat{\mathcal{H}}\) and for \(O \subseteq C[0, \tau + T]\) we define
\[
\mathcal{M}_{(K, \hat{K})}(\tau, T, O) := \{ x : x = \pi(\tau) \text{ for some } \pi \in K^{0-} \text{ and there exist } \hat{\pi}^1, \hat{\pi}^2 \in \hat{K}\}^{\tau+}
\]
with \(0 \leq \hat{\pi}^1(\tau) \leq x \leq \hat{\pi}^2(\tau) \leq 1,\) such that \(g^{\tau, T}_{\hat{\pi}^1} - g^{\tau, T}_{\hat{\pi}^2} \in O\) and there does not exist \(\hat{\pi} \in \hat{K}\) with \(\hat{\pi}^1(\tau) < \hat{\pi}(\tau) < \hat{\pi}^2(\tau)\).
Let $\kappa(\tau, T, O) := \# M(\mathcal{W}, \mathcal{W})(\tau, T, O)$, and $\kappa_n(\tau, T, O) := \# M_{(\lambda_n, \tilde{\lambda}_n)}(\tau, T, O)$. Comparing with the definitions introduced in [14], we have

$$\kappa(\tau, T, O) \leq \xi(0, \tau), \quad \kappa_n(\tau, T, O) \leq \xi_n(0, \tau) \quad \text{for all } n \geq 1. \quad (23)$$

Let

$$R_n(\tau, T, O) := \#\{i : 0 \leq i \leq n - 1, g_{\tilde{h}_n(i/n, r)}^r(\tau) \neq g_{\tilde{h}_n(i+1/n, r)}^r(\tau) \text{ and } g_{\tilde{h}_n(i/n, r)}^r - g_{\tilde{h}_n(i+1/n, r)}^r \in E\}.$$  

From Proposition 2.6, we know that for each $x \in \mathcal{M}(0, \tau)$, starting from $(x, \tau)$ there exist exactly two dual paths $\tilde{\pi}^r(x, \tau), \tilde{\pi}^l(x, \tau) \in \tilde{\mathcal{W}}$ (say) such that $\tilde{\pi}^r(x, \tau)(0) > \tilde{\pi}^l(x, \tau)(0)$. Let $g_r(\cdot, \tau, T), g_l(\cdot, \tau, T) \in C[0, \tau + T]$ given by $g_r(x, \tau, T)(t) := \tilde{\pi}^r(x, \tau)(\tau - t)$ and $g_l(x, \tau, T)(t) := \tilde{\pi}^l(x, \tau)(\tau - t), 0 \leq t \leq \tau + T$. Hence $\kappa(\tau, T, O) = \#\{x \in \mathcal{M}(0, \tau) \text{ with } g_r(x, \tau, T) - g_l(x, \tau, T) \in E\}$. In the following we consider $O$ such that

$$\mathbb{P}(g_r(x, \tau, T) - g_l(x, \tau, T) \in \delta(O) \text{ for some } x \in \mathcal{M}(0, \tau)) = 0 \quad (25)$$

where $\delta(O)$ denotes the boundary of $O$. The next lemma allows us to compute $\mathbb{E}[\kappa(\tau, T, O)]$.

**Lemma 4.1.** For $\tau > 0$ and $T \geq 0$ and $O$ satisfying (25), we have

$$\lim_{n \to \infty} \mathbb{E}[R_n(\tau, T, O)] = \mathbb{E}[\kappa(\tau, T, O)]. \quad (26)$$

**Proof:** Choose $\omega$ so that $(\mathcal{W}(\omega), \tilde{\mathcal{W}}(\omega))$ satisfies the properties listed in Subsection 2.2. Hence, we have $\mathcal{M}(\mathcal{W}(\omega))(0, \tau) \cap \mathcal{Q} = \emptyset$. For each $x \in \mathcal{M}(\mathcal{W}(\omega))(0, \tau)$, set $l_n^x = l_n^\tau(\omega) = \lfloor nx \rfloor/n$ and $r_n^x = l_n^\tau(1 + (1/n))$. Since there are exactly two dual paths $\tilde{\pi}_r^a(x, \tau)$ and $\tilde{\pi}_l^a(x, \tau)$ starting from $(x, \tau)$ with $\tilde{\pi}_r^a(x, \tau)(t) > \tilde{\pi}_l^a(x, \tau)(t)$ for all $t \in [0, \tau)$, from Proposition 3.2 (e) of Sun et al. (2008) it follows that $\tilde{\pi}_r^a(l_n^\tau, r)$ and $\tilde{\pi}_l^a(l_n^\tau, r)$ converge to $\tilde{\pi}_r^a(x, \tau)$ and $\tilde{\pi}_l^a(x, \tau)$ respectively in $(\tilde{\Pi}, d_{\tilde{\Pi}})$ as $n \to \infty$. Hence $g_r(x, \tau, T)$ and $g_l(x, \tau, T)$ converge to $g_l(x, \tau, T)$, $g_r(x, \tau, T)$ respectively in $(C[0, \tau + T], || \cdot ||_{\infty})$ as $n \to \infty$. Since $g_l(x, \tau, T) - g_l(x, \tau, T) \notin \delta(O)$ almost surely, we have $\mathbb{P}(g_r(x, \tau, T) - g_l(x, \tau, T) \in \text{int}(O) \cup \text{int}(\bar{O})) = 1$. Hence it follows that $\lim_{n \to \infty} R_n(\tau, T, O) = \kappa(\tau, T, O)$ almost surely.

To show the family $\{R_n(\tau, T, O) : n \in \mathbb{N}\}$ is uniformly integrable, we observe that $R_n(\tau, T, O) \leq \xi(0, \tau)$ for all $n$ and $\mathbb{E}[\xi(0, \tau)] < \infty$.

The following lemma is the main result in this section and is the main tool for establishing Theorem 1.5 and Theorem 1.4.

**Lemma 4.2.** For $\tau > 0$ and $T \geq 0$ and $O$ satisfying (25), we have

$$\lim_{n \to \infty} \mathbb{E}[\kappa_n(\tau, T, O)] = \mathbb{E}[\kappa(\tau, T, O)]. \quad (27)$$
Proof: We assume that we are working on a probability space such that \( (\hat{X}_n, \hat{\mathcal{H}}_n) \) converges to \( (\mathcal{W}, \hat{\mathcal{W}}) \) almost surely in \( (\mathcal{H} \times \hat{\mathcal{H}}, d_{\mathcal{H} \times \hat{\mathcal{H}}}) \) as \( n \to \infty \).

We prove it for \( \tau = 1 \). The argument for general \( \tau > 0 \) is similar. Choose \( \omega \) so that
\[
\omega \in \cap_{\nu \in (0,1]} \{ \lim_{n \to \infty} \xi_{\hat{\mathcal{H}}_n} (1 - \nu, 1) = \xi_{\mathcal{W}} (1 - \nu, 1) \}
\]
as well as \( (\mathcal{W}(\omega), \hat{\mathcal{W}}(\omega)) \) satisfies the properties listed in Subsection 2.2. From Lemma 3.2 we have that such a choice is possible for almost all \( \omega \). Let \( \xi_{\mathcal{W}}(0,1)(\omega) = k \). Since we have \( \lim_{n \to \infty} \xi_{\hat{\mathcal{H}}_n}(0,1)(\omega) = \xi(0,1)(\omega) \), from [22] for \( k = 0 \), we have \( \kappa_n(1, T, \mathcal{O}) = \kappa(1, T, \mathcal{O}) = 0 \) for all \( n \) large.

Suppose \( \xi(0,1) = k \) for some \( k \geq 1 \). Fix \( \epsilon > 0 \) and for \( m = m(\omega) > 0 \) as in [10] choose \( \delta = \delta(\omega) \in \mathbb{Q} \cap (0, 1) \) such that \( \sup \{ \hat{\theta}(1 - s) - \hat{\theta}(1 - t) : s, t \in [0, 2\delta], \hat{\theta}(1) \in [0, 1] \} < \epsilon/8 \). Assume that \( \mathcal{M}_{\mathcal{W}}(0,1) = \{ x_1, \ldots, x_k \} \) and \( \mathcal{M}_{\mathcal{W}}(1 - \delta, 1) = \{ x_1, \ldots, x_{k+1} \} \) for some \( l \geq 0 \). We choose \( \gamma_\delta > 0 \) such that for \( x, y \in \mathcal{M}_{\mathcal{W}}(1 - \delta, 1) \), \( x \neq y \), we have \( |x - y| > 2\gamma_\delta \) and \( (x - \gamma_\delta, x + \gamma_\delta) \subset (0, 1) \).

From Proposition 2.6 it also follows that for each \( x_i \) there exist dual paths \( \hat{\pi}^{(x_i, 1)}_r, \hat{\pi}^{(x_i, 1)}_l \) both starting at \( (x_i, 1) \) such that (a) \( \hat{\pi}^{(x_i, 1)}_r(1 - \delta) > \hat{\pi}^{(x_i, 1)}_l(1 - \delta) \) for all \( 1 \leq i \leq k + l \) and (b) \( \hat{\pi}^{(x_i, 1)}_r(0) > \hat{\pi}^{(x_i, 1)}_l(0) \) for all \( 1 \leq i \leq k \). Since \( \hat{\pi}^{(x_i, 1)}_r \) and \( \hat{\pi}^{(x_i, 1)}_l \) are both continuous functions, there exists \( \nu_\delta > 0 \) such that \( \hat{\pi}^{(x_i, 1)}_r(1 - \delta - \nu_\delta) > \hat{\pi}^{(x_i, 1)}_l(1 - \delta - \nu_\delta) \) for all \( 1 \leq i \leq k + l \). Set \( \zeta_\delta := \min \{ \min \{ \hat{\pi}^{(x_i, 1)}_l - \hat{\pi}^{(x_i, 1)}_r(1 - \delta - \nu_\delta) : 1 \leq i \leq k + l \}, \min \{ \hat{\pi}^{(x_i, 1)}_l - \hat{\pi}^{(x_i, 1)}_r(0) : 1 \leq i \leq k \} \} > 0 \).

Let \( n_0 = n_0(\omega) > 4/\nu_\delta \) be such that, for all \( n \geq n_0 \),

(i) \( \xi_{\mathcal{H}_n}(1 - \delta, 1) = \xi_{\mathcal{W}}(1 - \delta, 1), \xi_{\mathcal{H}_n}(0,1) = \xi_{\mathcal{W}}(0,1) \) and

(ii) for all \( 1 \leq i \leq k + l \), there exist \( \hat{\pi}^{(x_i, 1)}_{i, r}, \hat{\pi}^{(x_i, 1)}_{i, l} \in \hat{\mathcal{X}}^{1+} \) with

\[
\max \left\{ \sup_{s \in [0,1+T]} \left| \hat{\pi}^{(x_i, 1)}_{i, r} - \hat{\pi}^{(x_i, 1)}_{i, l} \right|(1 - s) \right\} \leq \min \{ \gamma_\delta, \zeta_\delta/4, \epsilon/4 \}. \tag{28}
\]

The choice of \( n_0 \) and \( \xi_\delta \) ensures that \( \hat{\pi}^{(x_i, 1)}_{i, r}(0) > \hat{\pi}^{(x_i, 1)}_{i, l}(0) \) for all \( 1 \leq i \leq k \) and \( \hat{\pi}^{(x_i, 1)}_{i, r}(1 - \delta - \nu_\delta) < \hat{\pi}^{(x_i, 1)}_{i, l}(1 - \delta - \nu_\delta) \) for all \( 1 \leq i \leq k + l \). Therefore, there exist forward paths \( \theta^{(n)} \in \hat{\mathcal{X}}_n^{1-} \) such that \( x_i - \gamma_\delta < \hat{\pi}^{(x_i, 1)}_{i, r}(1) < \theta^{(n)}(1) < x_i + \gamma_\delta \) for all \( 1 \leq i \leq k \). Further since \( n_0 > 4/\nu_\delta \), there exist forward paths \( \theta^{(n)} \in \hat{\mathcal{X}}_n^{1-} \) such that \( x_i - \gamma_\delta < \hat{\pi}^{(x_i, 1)}_{i, r}(1) < \theta^{(n)}(1) < \hat{\pi}^{(x_i, 1)}_{i, l}(1) < x_i + \gamma_\delta \) for all \( k + 1 \leq i \leq k + l \). For all \( n \geq n_0 \) we have \( \# \mathcal{M}_{\mathcal{X}_n}(1 - \delta, 1) = k + \ell \) and hence \( \{ \theta^{(n)}(1) : 1 \leq i \leq k + \ell \} = \mathcal{M}_{\mathcal{X}_n}(1 - \delta, 1) \). Thus for each \( i \),

there exists a unique \( x_i^n := \theta^{(n)}(1) \in \mathcal{M}_{\mathcal{X}_n}(1 - \delta, 1) \) with \( |x_i^n - x_i| < \gamma_\delta \). \tag{29}

We observe that \( (y^n_-, n) := (\sqrt{n} \gamma_0 \theta^{(n)}(1), n) \in V \) for all \( 1 \leq i \leq k \) and \( \hat{\pi}^{(y^n_-, n)}(s), \hat{\mathcal{I}}^{(y^n_-, n)} \) are the right and left dual neighbours of \( (y^n_-, n) \) respectively. Also

\[
\hat{\pi}^{(y^n_-, n)}_{i, l}(s) \leq \hat{\pi}^{(y^n_-, n)}(s) < \theta^{(n)}(s) < \hat{\pi}^{(y^n_-, n)}(s) \leq \hat{\pi}^{(y^n_-, n)}_{i, r}(s) \quad \text{for} \quad 0 \leq s \leq 1.
\]
If \( \hat{\pi}_n^r(y_{i,n}) (1 - 2\delta) \neq \hat{\pi}_n^{l,r}(1 - 2\delta) \) then, as in the choice of \( \theta_i^n \in \mathcal{K}_n^{(1 - \delta)} \), there exists a forward path \( \psi_i^n \in \mathcal{K}_n^{(1 - \delta)} \) such that \( z_i^n = \psi_i^n(1) \in \mathcal{M}_n^{(1 - \delta, 1)} \) with

\[
x_i - \gamma_\delta < x_i^n < \hat{\pi}_n^r(y_{i,n}) (1) < z_i^n < \hat{\pi}_n^{l,r}(1) < x_i + \gamma_\delta
\]

which contradicts the uniqueness of \( x_i^n \) as given in (29). Hence for all \( n \geq n_0 \) we must have

\[
\hat{\pi}_n^r(y_{i,n}) (1 - 2\delta) = \hat{\pi}_n^{l,r}(1 - 2\delta) \quad \text{and} \quad \hat{\pi}_n^{l,r}(1 - 2\delta) = \hat{\pi}_n^{l,r}(1 - 2\delta)
\]

for all \( 1 \leq i \leq k \). For each \( 1 \leq i \leq k \) as \( \hat{\pi}_r^{(x_i,1)}(1) - \hat{\pi}_r^{(x_i,1)}(1) = 0 \), we have \( \sup_{s \in [0,2\delta]} (\hat{\pi}_r^{(x_i,1)}(1) - \hat{\pi}_r^{(x_i,1)}(1)) < \epsilon/4 \). Clearly \( (\hat{\pi}_r^{y_{i,n}}(1) - \hat{\pi}_r^{y_{i,n}}(1)))(s) \leq \hat{\pi}_n^{l,r}(1 - s) \) for all \( s \leq 1 \) and from (28) we obtain \( \sup_{s \in [0,2\delta]} (\hat{\pi}_n^{l,r}(1 - s) - \hat{\pi}_n^{l,r}(1 - s)) < \epsilon \) for all \( n \geq n_0 \). Hence from (30) we have \( \sup_{s \in [0,1]} \{(\hat{\pi}_r^{y_{i,n}}(1) - \hat{\pi}_r^{y_{i,n}}(1)))(s) - (\hat{\pi}_r^{y_{i,n}}(1) - \hat{\pi}_r^{y_{i,n}}(1)))(s)\} < \epsilon \) for all \( n \geq n_0 \). Hence we have

\[
\left(g_{1,T}^{(x_i, y_{i,n})} - g_{1,T}^{(x_i, y_{i,n})}\right) \to \left(g_{1,T}^{(x_i,1,T)} - g_{1,T}^{(x_i,1,T)}\right)
\]

as \( n \to \infty \) in \( (C[0,1 + T], \| \cdot \|_\infty) \).

Since \( g_{r}^{(x_i,1,T)} - g_{l}^{(x_i,1,T)} \notin \delta(O) \) almost surely, we have \( \lim_{n \to \infty} \kappa_n(1, T, O) = \kappa(1, T, O) \) almost surely.

Since \( \kappa_n(1, T, O) \leq \kappa_n(0,1) \) for all \( n \geq 1 \), from Lemma 3.3 it follows that the family \( \{\kappa_n(1, T, O) : n \in \mathbb{N}\} \) is uniformly integrable and hence

\[
\mathbb{E}[\kappa_n(1, T, O)] \to \mathbb{E}[\kappa(1, T, O)] \quad \text{as} \quad n \to \infty.
\]

This completes the proof.

\[
\square
\]

4.1 Theorem 1.3

For \( \tau > 0 \) and \( T \geq 0 \) and a Brownian motion \( W \) with \( W(0) = 0 \), let \( S^\tau := \inf\{t \geq 0 : W(t + s) > W(t) \text{ for all } 0 \leq s \leq \tau\} \). Following the same argument as in Lemma 2.2 Bolthausen (1976), we have \( S^\tau \) is finite almost surely. We define \( W^\tau \) on \( C[0, \infty) \) and \( W^\tau,T \) on \( C[0, \tau + T] \) as

\[
W^\tau(t) := W(S^\tau + t) - W(S^\tau) \text{ for } t \geq 0;
\]

\[
W^\tau,T := W^\tau|_{[0,\tau+T]},
\]

where for \( f \in C[0, \infty) \), \( f|_{[0,\tau+T]} \) denotes the restriction of \( f \) on \( [0, \tau + T] \). Bolthausen (1976) showed that \( W^{\tau,0} \overset{d}{=} W^{\tau} \) and from scaling invariance property of Brownian motion it follows that \( \{W^{\tau,0}(s) : s \in [0, \tau]\} \overset{d}{=\sqrt{\tau}}W^{\tau}(s/\tau) : s \in [0, \tau]\}. Durrett et al. (1977) (Theorem 2.1) proved that \( W|_{\{\min_{s \in [0, \tau]} W(s) \geq -\epsilon\}} \Rightarrow W^{\tau} \text{ as } \epsilon \downarrow 0 \). Using Theorem 2.1 of Durrett et al. (1977) and scaling property of \( W^\tau \), given above we obtain the following:
Lemma 4.3. For $\tau > 0$ and $T \geq 0$ considering $B$ as a standard Brownian motion on $[0, \tau + T]$ starting from 0, we have $B\{\min_{t \in [0, \tau]} B(t) \geq -1/n\} \Rightarrow W^{\tau, T}$ as $n \to \infty$.

Proof: For $c < d$, let $C_b(C[c, d], \mathbb{R})$ be the space of all real valued bounded continuous functions on $C[c, d]$. For $h_1 \in C_b(C[0, \tau], \mathbb{R})$ and $h_2 \in C_b(C[\tau, \tau + T], \mathbb{R})$ we define $h_1 \otimes h_2 \in C_b(C[0, \tau + T], \mathbb{R})$ given by $h_1 \otimes h_2(f) := h_1(f_{[0, \tau]} h_2(f_{[\tau, \tau + T]}))$. Define $A := \{\sum_{i=1}^{k} h_1_i \otimes h_2_i : k \geq 1, h_1_i \in C_b(C[0, \tau], \mathbb{R})$ and $h_2_i \in C_b(C[\tau, \tau + T], \mathbb{R})$ for all $1 \leq i \leq k\}$. Take any $h = h_1 \otimes h_2$. By Theorem 4.5 of Ethier et al. (1986), it suffices to show that $\mathbb{E}(h(W)|\{\min_{t \in [0, \tau]} W(t) \geq -1/n\}) \to \mathbb{E}(h(W^{\tau, T}))$ as $n \to \infty$. For $x \in \mathbb{R}$ let $\mathbb{P}_x$ denote the probability measure of a Brownian motion on $[\tau, \tau + T]$ taking value $x$ at time $\tau$ and $\mathbb{E}_x$ the expectation with respect to measure $\mathbb{P}_x$. For $\mathbb{E}(h(W^{\tau, T}))$ we obtain

\[
\mathbb{E}[h(W^{\tau, T})] = \mathbb{E}[h_1(W^{\tau, T}[0, \tau])h_2(W^{\tau, T}[\tau, \tau + T])]
\]

where $\tilde{W}$ is a Brownian motion on $[\tau, \tau + T]$ independent of $W^T$ with $\tilde{W}(\tau) = W^{\tau, 0}(\tau)$. Since $S^T + \tau$ is a stopping time, the last equality follows from strong Markov property of Brownian motion. We now observe that

\[
\mathbb{E}[h(B)|\min_{t \in [0, \tau]} B(t) \geq -1/n] = \mathbb{E}[h_1(B[0, \tau])h_2(B[\tau, \tau + T])|\min_{t \in [0, \tau]} B(t) \geq -1/n]
\]

where $\tilde{B}$ is a Brownian motion on $[\tau, \tau + T]$ independent of $B$ with $\tilde{B}(\tau) = B(\tau)$ and the penultimate equality follows from Markov property. For $f \in C[0, \tau]$ the map $f \to h_1(f)\mathbb{E}_f(\tau)(h_2(\tilde{B}))$ is continuous. From Theorem 2.1 of Durrett et al. (1977) and from property (g) of $W^T$ described in Section it follows that $B[0, \tau]|\{\min_{t \in [0, \tau]} B(t) \geq -1/n\} \Rightarrow W^{\tau, 0}$. Hence we have $\mathbb{E}(h_1(B[0, \tau])\mathbb{E}_B(\tau)(h_2(\tilde{B})))|\{\min_{t \in [0, \tau]} B(t) \geq -1/n\} \to \mathbb{E}(h_1(W^{\tau, 0})\mathbb{E}_{W^{\tau, 0}(\tau)}(h_2(W)))$, which completes the proof. □
Let $\tilde{W}^\tau$ be a process on $C[0, \infty)$ given by

$$
\tilde{W}^\tau(t) := \begin{cases} 
W^\tau(t) & \text{if } 0 \leq t \leq \tau \\
W^\tau(\tau) + \tilde{W}(t) & \text{otherwise}
\end{cases}
$$

where $\tilde{W}$ is a Brownian motion on $[\tau, \infty)$ with $\tilde{W}(\tau) = 0$ and independent of $W^\tau$. Similar argument as that of Lemma 4.3 gives us the following corollary.

**Corollary 4.3.1.** For $\tau > 0$ and $T \geq 0$ we have, $W^\tau \overset{d}{=} \tilde{W}^\tau$ and $W^{\tau,T} \overset{d}{=} \tilde{W}^{\tau,T} := \tilde{W}^\tau|_{[0,\tau+T]}$.

For $f \in C[0, \tau+T]$, let $t_f := \min\{\inf\{s : s \in (0, \tau+T), f(s) = 0\}, \tau+T\}$. By definition $t_f = \tau + T$ if $f(s) \neq 0$ for all $s \in (0, \tau+T]$. Consider the mapping $H : C[0, \tau+T] \to C[0, \tau+T]$ given by $H(f)(t) := \{t \leq t_f\} f(t)$. We define

$$
W^{+,\tau,T} := H(W^{\tau,T}).
$$

Let $O$ be the collection of open sets in $(C[0, \tau+T], || \cdot ||)$ such that for any $O \in O$

(a) $O = \bigcap_{i=1}^k B(f_i, r_i)$ for some $k \geq 1$, $f_1, \ldots, f_k \in C[0,1]$ and $r_1, \ldots, r_k > 0$ where $|f_i(0)| \neq r_i$ for all $1 \leq i \leq k$;

(b) $\mathbb{P}(\sqrt{2}W^{+,\tau,T} \in \delta(O)) = 0$.

**Lemma 4.4.** For $O = \bigcap_{i=1}^k B(f_i, r_i) \in O$, we have

(a) $O$ satisfies [24], i.e., $\mathbb{P}(g_r^{(x),\tau,T} - g_l^{(y),\tau,T} \in \delta(O)) = 0$ for some $x \in \mathcal{M}_C(0, \tau)) = 0$.

(b) $\lim_{n \to \infty} \mathbb{E}[R_n(\tau, T, O)] = \mathbb{E}[\kappa(\tau, T, O)] = \mathbb{P}(\sqrt{2}W^{+,\tau,T} \in O)/\sqrt{\pi \tau}$.

**Proof:** To prove (a), we claim

$$
\{\text{there exists } x \in \mathcal{M}_W(0, \tau) \text{ with } g_r^{(x),\tau,T} - g_l^{(y),\tau,T} \in \delta(O)\} \subseteq \liminf_{n \to \infty} G_n
$$

where

$$
G_n := \{\text{there exists } 0 \leq i < n \text{ with } g_r^{\tau,T}_{[\tau^{(i+1)/n}, \tau]}(\tau) \neq g_l^{\tau,T}_{[\tau^{(i)/n}, \tau]}(\tau) \text{ and } g_r^{\tau,T}_{[\tau^{(i+1)/n}, \tau]} - g_l^{\tau,T}_{[\tau^{(i)/n}, \tau]} \in \delta(O)\}.
$$

Fix $x = x(\omega) \in \mathcal{M}_W(0, \tau)$. We observe that $g_r^{(x),\tau,T}(0) - g_l^{(x),\tau,T}(0) = 0$. If $0 \notin (f_i(0) - r_i, f_i(0) + r_i)$ for some $1 \leq i \leq k$, then $g_r^{(x),\tau,T} - g_l^{(x),\tau,T} \notin \delta(O)$, i.e., we must have $g_r^{(x),\tau,T}(0) - g_l^{(x),\tau,T}(0) = 0 \in (f_i(0) - r_i, f_i(0) + r_i)$ for all $1 \leq i \leq k$. Hence there exists $s(x) > 0$ such that for all $1 \leq i \leq k$, $sup\{|g_r^{(x),\tau,T}(t) - g_l^{(x),\tau,T}(t) - f_i(t)| : 0 \leq t \leq s(x)| < r_i$. Now, set $s_0 = \min\{s(x) : x \in \mathcal{M}_W(0, \tau)\}$. Since $\mathcal{M}_W(0, \tau)$ is finite, $s_0 > 0$. Let $n_1 = n_1(\omega)$ be such that for all $n \geq n_1$ and $1 \leq i \leq k$

$$
\sup\{|g_r^{\tau,T}_{[\tau^{(i+1)/n}, \tau]}(t) - g_l^{\tau,T}_{[\tau^{(i)/n}, \tau]}(t) - f_i(t)| : 0 \leq t \leq s_0| < r_i.
$$

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From property (c) of the Brownian web and its dual, we have \( \hat{t}^{(\sigma,\tau),\pi}_{\tau} \rightarrow \tau \) and \( \hat{t}^{(\rho,\tau),\pi}_{\tau} \rightarrow \tau \). Hence, let \( n_0 = n_0(\omega) \geq n_1 \) be such that for all \( n \geq n_0 \)
\[
g_{\hat{t}^{(\sigma,\tau),\pi}_{\tau}}(t) = g_{\hat{t}^{(\rho,\tau),\pi}_{\tau}}(t) \text{ for } t \geq s_0,
\]
i.e., if \( g_{\hat{t}^{(\sigma,\tau),\pi}_{\tau}}(x) - g_{\hat{t}^{(\rho,\tau),\pi}_{\tau}}(x) \in \delta(O) \), then \( g_{\hat{t}^{(\sigma,\tau),\pi}_{\tau}}(x) - g_{\hat{t}^{(\rho,\tau),\pi}_{\tau}}(x) \in \delta(O) \) which justifies (44).

From (44) and the fact that \( g_{\hat{t}^{(\sigma,\tau),\pi}_{\tau}}(x) - g_{\hat{t}^{(\rho,\tau),\pi}_{\tau}}(x) \) is a convergence determining class in \( (0, \tau + T) \), we have
\[
\mathbb{P}(\text{there exists } x \in \mathcal{M}_{W}(0, \tau) \text{ with } g_{\hat{t}^{(\sigma,\tau),\pi}_{\tau}}(x) - g_{\hat{t}^{(\rho,\tau),\pi}_{\tau}}(x) \in \delta(O))
\leq \mathbb{P}(\lim_{n \rightarrow \infty} G_n)
\leq \lim_{n \rightarrow \infty} \mathbb{P}(H(1/n + \sqrt{2}W) \in \delta(O), 1/n + \min_{t \in [0,\tau]} \sqrt{2}W(t) > 0).
\]

Let \( A \subset C[0, \tau + T] \) be such that
\[
A := \{ f \in C[0, \tau + T] : t_f = \tau + T \text{ or } t_f < \tau + T \text{ and for every } \epsilon > 0 \text{ there exists } s \in (t_f, \min\{t_f + \epsilon, \tau + T\}) \text{ with } f(s) < 0 \}.
\]

From Corollary 4.3.1 it follows that \( \mathbb{P}(W^{\tau,T} \in A) = 1 \). For \( f \in A \) and a sequence \( \{ f_n : n \in \mathbb{N} \} \) with \( \lim_{n \rightarrow \infty} f_n = f \) in \( (C[0, \tau + T], || \cdot ||_{\infty}) \), we have \( H(f_n) \rightarrow H(f) \) also. Hence using the continuous mapping theorem, we have
\[
\lim_{n \rightarrow \infty} \mathbb{P}(H(1/n + \sqrt{2}W) \in \delta(O) | \min_{t \in [0,\tau]} \sqrt{2}W(t) > -1/n)n \mathbb{P}(\min_{t \in [0,\tau]} \sqrt{2}W(t) > -1/n)
= \lim_{n \rightarrow \infty} \mathbb{P}(H(1/n + \sqrt{2}W) \in \delta(O) | \min_{t \in [0,\tau]} \sqrt{2}W(t) > -1/n)n(2\Phi(1/\sqrt{2\tau n}) - 1)
\leq \mathbb{P}(\sqrt{2}W^{\tau,T} \in \delta(O)) / \sqrt{\pi \tau} = 0
\]
where the penultimate step follows from Lemma 4.3 and Slutsky’s theorem and the last step follows from conditions on \( O \).

To show (b) observe that from Lemma 4.1 and Lemma 4.3(a) for all \( O \in \mathcal{O} \) we have \( \lim_{n \rightarrow \infty} \mathbb{E}[\mathcal{R}_n(\tau, T, O)] = \mathbb{E}[\kappa(\tau, T, O)] \). This completes the proof.

Now, to complete the proof of Theorem 4.4 we need the following lemmas.

**Lemma 4.5.** For \( \tau > 0 \) and \( T \geq 0 \) we have \( \hat{D}_n^{(0,0)}|_{[0,\tau + T]}(L(0,0) > n\tau) \Rightarrow \sqrt{2}W^{+,\tau,T} \) as \( n \rightarrow \infty \).

**Proof:** For any two sets \( O_1, O_2 \in \mathcal{O} \) we have \( \delta(O_1 \cap O_2) \subseteq \delta(O_1) \cup \delta(O_2) \) and hence the collection \( \mathcal{O} \) is closed under finite intersection. Fix \( f \in C[0, \tau + T] \) and \( \epsilon > 0 \) and choose \( 0 < \nu < \epsilon \) such that \( \mathbb{P}(\sqrt{2}W^{+,\tau,T} \in \delta(B(\nu, f))) = 0 \) and \( |f(0)| \neq \nu \). Clearly \( f \in B(f, \nu) \subseteq B(f, \epsilon) \subseteq B(f, \nu) \). By Corollary 1 in page 14 of [Billingsley (1968)](https://example.com), the collection \( \mathcal{O} \) forms a convergence determining class in \( (C[0, \tau + T], || \cdot ||_{\infty}) \). Hence to prove (a), it is enough to show that \( \mathbb{P}(\hat{D}_n^{(0,0)}|_{[0,\tau + T]} \in O|L(0,0) > n\tau) \rightarrow \mathbb{P}(\sqrt{2}W^{+,\tau,T} \in O) \) for all \( O \in \mathcal{O} \).
Using translation invariance of our model, we have
\[ \mathbb{P}(\hat{D}_n(0,0)|_{[0,\tau+T]} \in O|L(0,0) > n\tau) \frac{\mathbb{E}[k_n(\tau,T,O)]}{\mathbb{E}[\xi_{x_n}(0,\tau)]} \rightarrow \frac{\mathbb{E}[k(\tau,T,O)]}{\mathbb{E}[\xi_{W}(0,\tau)]} = \mathbb{P}(\sqrt{2}W^{+},\tau,T \in O). \]
This completes the proof. \qed

**Lemma 4.6.** For \( \tau > 0 \) and \( T \geq 0 \) we have

(a) \( \sup\{\hat{D}_n(0,0)(s) - D_k(0,0)(s) : s \in [0, \tau + T]\}|\{L(0,0) > n\tau\} \xrightarrow{P} 0 \) as \( n \to \infty \).

(b) \( \sup\{K_n(0,0)(s) - pD_k(0,0)(s) : s \in [0, \tau + T]\}|\{L(0,0) > n\tau\} \xrightarrow{P} 0 \) as \( n \to \infty \).

**Proof:** For part (a) fix \( 0 < \alpha < 1/2 \) and it is enough to show that \( \mathbb{P}(\max\{|\hat{D}_k(0,0) - D_k(0,0)| : 0 \leq k \leq n(\tau + T) + 1\} \geq n^{\alpha}|L(0,0) > n\tau) \to 0 \) as \( n \to \infty \). By Theorem 1.2 we have \( \sqrt{n}\mathbb{P}(L(0,0) > n\tau) \to \frac{1}{\sqrt{\pi}}. \) Hence it suffices to show that \( \sqrt{n}\mathbb{P}(\max\{|\hat{D}_k(0,0) - D_k(0,0)| : 0 \leq k \leq n(\tau + T) + 1\} \geq n^{\alpha}, L(0,0) > n\tau) \to 0 \) as \( n \to \infty \). Define \( H^{(r)}_k = a^r(\hat{h}^{k-1}(\hat{r}(0,0))) - a^l(\hat{h}^{k-1}(\hat{r}(0,0))) \) and \( H^{(l)}_k = a^r(\hat{h}^{k-1}(\hat{t}(0,0))) - a^l(\hat{h}^{k-1}(\hat{t}(0,0))) \) for \( 1 \leq k \leq n(\tau + T) + 1 \) where for \( (x,t) \in \hat{V} \), \( a^r(x,t) \) and \( a^l(x,t) \) are defined as in (11). From the construction of the dual process, we observe that \( 2|\hat{D}_k(0,0) - D_k(0,0)| = H^{(r)}_k + H^{(l)}_k \) for \( 1 \leq k \leq n(\tau + T) + 1 \). Thus, we have \( \{\max\{|\hat{D}_k(0,0) - D_k(0,0)| : 0 \leq k \leq n(\tau + T) + 1\} \geq n^{\alpha}, L(0,0) > n\tau\} \subseteq E_n \cup \{\hat{r}(0,0) - \hat{t}(0,0) \geq n^{\alpha}, (0,0) \in V\} \) where the event \( E_n \) is defined by

\[
E_n := \bigcup_{k=1}^{\lfloor n(\tau+T) \rfloor +1} \{H^{(r)}_k \geq n^{\alpha}, (0,0) \in V\} \cup \bigcup_{k=1}^{\lfloor n(\tau+T) \rfloor +1} \{H^{(l)}_k \geq n^{\alpha}, (0,0) \in V\}. \tag{36}
\]

For \( \mathbb{P}\{H^{(r)}_k \geq n^{\alpha}, (0,0) \in V\} \), we have

\[
\mathbb{P}\{H^{(r)}_k \geq n^{\alpha}, (0,0) \in V\} = \sum_{u_r \in \mathbb{Z}/2} \mathbb{P}(\hat{h}^{k-1}(\hat{r}(0,0)) = (u_r, -k+1), (0,0) \in V) \times \mathbb{P}\{a^r(u_r, -k+1) - a^l(u_r, -k+1) \geq n^{\alpha} | \hat{h}^{k-1}(\hat{r}(0,0)) = (u_r, -k+1), (0,0) \in V\}.
\]

The event \( \{\hat{h}^{k-1}(\hat{r}(0,0)) = (u_r, -k+1), (0,0) \in V\} \) depends on \( \{(B_u, U_u) : -k+1 \leq u(2) < 0\} \) while, from the definition of \( a^r(u_r, -k+1) \) and \( a^l(u_r, -k+1) \) for \( u_r \in \mathbb{Z}/2 \) (see (11)), the event \( \{a^r(u_r, -k+1) - a^l(u_r, -k+1) \geq n^{\alpha}\} \) depends only on \( \{(B_u, U_u) : u(2) = -k\} \) and hence is independent of the conditioning event. Further, we have \( \mathbb{P}\{a^r(u_r, -k+1) - a^l(u_r, -k+1) \geq n^{\alpha}\} \leq (1 - p)^{n^{\alpha}-1} \). Similar argument holds for \( \mathbb{P}\{H^{(l)}_k \geq n^{\alpha}, (0,0) \in V\} \). Therefore we have

\[
\sqrt{n}\mathbb{P}(\max\{|\hat{D}_k(0,0) - D_k(0,0)| : 0 \leq k \leq n(\tau + T) + 1\} \geq n^{\alpha}, L(0,0) > n)
\leq \sqrt{n}\left[\mathbb{P}(E_n) + \mathbb{P}(\hat{r}(0,0) - \hat{t}(0,0) \geq n^{\alpha}, (0,0) \in V]\right]
\leq \sqrt{n}\left[2(n(\tau + T) + 1)(1 - p)^{n^{\alpha}-1} + 2(1 - p)^{n^{\alpha}-1}\right] \to 0 \text{ as } n \to \infty.
\]
This completes the proof of part (a).

To show (b), we fix $\epsilon, \delta > 0$ and choose $m_0 > \epsilon/p$ so that $\mathbb{P}(\sup\{\sqrt{2}W^+, T(\tau) : t \in [0, \tau + T]\} \geq m_0) < \delta/8$. From Lemma 4.5 we have $\limsup_{n \to \infty} \mathbb{P}(\sup\{\hat{D}_n^{(0,0)}(t) : t \in [0, \tau + T]\} \geq m_0) < \delta/2$. Hence from the choice of $m_0$ we have that there exists $n_1$ such that for all $n \geq n_1$ we have $\mathbb{P}(\sup\{\hat{D}_n^{(0,0)}(t) : t \in [0, \tau + T]\} > m_0) < \delta/4$. Hence, we can show (b) for all $n \geq n_1$.

Since $D_k(0,0) \leq \hat{D}_k(0,0)$ for all $k \geq 0$, we also have $\mathbb{P}(\sup\{\hat{D}_n^{(0,0)}(t) : t \in [0, \tau + T]\} > m_0) < \delta$ for all $n \geq n_1$. Let $A_n := \{\sup\{\hat{D}_n^{(0,0)}(t) : t \in [0, \tau + T]\} > m_0\}$.

We have

$$\{\{\max\{|pD_k(0,0) - \#C_k(0,0)| : 0 \leq k \leq |n(\tau + T) + 1\} \geq \epsilon \gamma_0 \sqrt{n}\} \cap A_n^c\} \subseteq A_n$$

Further, we can write

$$\left\{\max\{|pD_k(0,0) - \#C_k(0,0)| : 0 \leq k \leq |n(\tau + T) + 1\} \geq \epsilon \gamma_0 \sqrt{n}\right\} \cap A_n^c \cap \{L(0,0) > n\tau\} \subseteq \bigcup_{k=1}^{n(\tau + T) + 1} F_k$$

where

$$F_k := \{|pD_k(0,0) - \#C_k(0,0)| \geq \epsilon \gamma_0 \sqrt{n}, 0 \leq D_k(0,0) \leq m_0 \gamma_0 \sqrt{n}, (0,0) \in V\}.$$ 

To compute $\mathbb{P}(F_k)$ we obtain

$$\mathbb{P}(F_k) \leq \mathbb{P}(\{pD_k(0,0) - \#C_k(0,0)| \geq \epsilon \gamma_0 \sqrt{n}, \frac{\epsilon \gamma_0 \sqrt{n}}{4} \leq D_k(0,0) \leq m_0 \gamma_0 \sqrt{n}, (0,0) \in V\})$$

The inequality follows from the fact that $\max\{pD_k(0,0), \#C_k(0,0)\} \leq D_k(0,0) + 1$ and hence $|pD_k(0,0) - \#C_k(0,0)| \leq 2(D_k(0,0) + 1)$. Now, we condition on the possible positions of the left dual and the right dual paths at the $(k - 1)$ th step. Set $A_{u_t, u_r}^{(k-1)} = \{\hat{h}^{k-1}(\vec{r}(0,0)) = (u_r, -k + 1), \hat{h}^{k-1}(\vec{i}(0,0)) = (u_t, -k + 1), (0,0) \in V\}$ for $u_t, u_r \in \mathbb{Z}/2$ with $u_t \leq u_r$. Then we have

$$\mathbb{P}(F_k) = \sum_{u_t \leq u_r \in \mathbb{Z}/2} \mathbb{P}(A_{u_t, u_r}^{(k-1)}) \mathbb{P}\left\{|pD_k(0,0) - \#C_k(0,0)| \geq \epsilon \gamma_0 \sqrt{n}, \frac{\epsilon \gamma_0 \sqrt{n}}{4} \leq D_k(0,0) \leq m_0 \gamma_0 \sqrt{n}, A_{u_t, u_r}^{(k-1)}\right\}.$$ 

To compute the conditional probability we split the event by specifying the values $a^l(u_r, -k + 1)$ and $a^r(u_t, -k + 1)$ for $i_1, i_2 \in \mathbb{Z}$. On this event $G_{i_1, i_2}$ we have $D_k(0,0) = 1\{i_2 > i_1\}(i_2 - i_1)$. Further, we observe that $\#C_k(0,0) = 2 + Z_k$ where $Z_k := \#\{j : i_1 < j < i_2, (j, -k) \in V\}$. Let us denote $\Sigma = \{(i_1, i_2) \in \mathbb{Z}: i_2 > i_1, \frac{\epsilon \gamma_0 \sqrt{n}}{4} \leq (i_2 - i_1) \leq m_0 \gamma_0 \sqrt{n}\}$. Hence, we can write the conditional probability above as

$$\mathbb{P}\left\{|pD_k(0,0) - \#C_k(0,0)| \geq \epsilon \gamma_0 \sqrt{n}, \frac{\epsilon \gamma_0 \sqrt{n}}{4} \leq D_k(0,0) \leq m_0 \gamma_0 \sqrt{n}, A_{u_t, u_r}^{(k-1)}\right\}$$

$$\leq \sum_{(i_1, i_2) \in \Sigma} \mathbb{P}(G_{i_1, i_2} \cap \{|Z_k - p(i_2 - i_1 - 1)| \geq \frac{\epsilon \gamma_0 \sqrt{n}}{2}\} A_{u_t, u_r}^{(k-1)})$$
Also we observe that $A_{u_1,u_2}^{(k_1-1)}$ depends on $\{(B_u,U_u) : -k + 1 \leq u(2) \leq 0\}$ while the event $G_{i_1,i_2}$ depends on $\{(B_u,U_u) : u(1) \geq i_2$ or $u(1) \leq i_1, u(2) = -k\}$ and the event $\{|Z_k - p(i_2 - i_1 - 1)| \geq \frac{c n \sqrt{p}}{2}\}$ depends on $\{(B_u,U_u) : i_1 < u(1) < i_2, u(2) = -k\}$ and $Z_k$ follows binomial distribution with parameter $(i_2 - i_1 - 1, p)$. Hence, the events are independent. Thus, using Chernoff bound (see Theorem 4.3 Motwani et al. (1995)) we conclude that

$$\mathbb{P}(F_k) \leq \sum_{u \leq u_r \in \mathbb{Z}/2} \mathbb{P}(A_{u_1,u_2}^{(k_1-1)}) \sum_{(i_1,i_2) \in \Sigma} \mathbb{P}(G_{i_1,i_2}) 2 \exp \left( -\frac{(\epsilon/40p)^2 m \gamma_0 \sqrt{n} p}{16} \right) \leq 2 \exp \left( -\frac{\epsilon^2 \gamma_0 \sqrt{n}}{16 m \gamma_0 p} \right),$$

and we have

$$\mathbb{P}(\left\{ \max_{0 \leq k \leq [n(\tau + T)] + 1} \{|pD_{k}(0,0) - \#C_{k}(0,0)| \geq \epsilon \gamma_0 \sqrt{n} \} \cap A_{n}^c \cap \{L(0,0) > n \tau\} \right\} \leq \sum_{k=0}^{[n(\tau + T)] + 1} \mathbb{P}(F_k) \leq 2(n(\tau + T) + 1) \exp \left( -\frac{\epsilon^2 \gamma_0 \sqrt{n}}{16 m \gamma_0 p} \right).$$

Because of Theorem 1.2 we have

$$\mathbb{P}(\left\{ \max_{0 \leq j \leq [n(\tau + T)] + 1} \{|pD_{j}(0,0) - \#C_{j}(0,0)| \geq \epsilon \gamma_0 \sqrt{n} \} \cap A_{n}^c \cap \{L(0,0) > n \tau\} \right\} \rightarrow 0$$

as $n \rightarrow \infty$. This completes the proof.

**Proof of Theorem 1.3**: We remarked that $W^{1,0} \overset{d}{=} W^+$. As $W^{+,1,0} = W^{1,0}$, the proof of Theorem 1.3 follows from Lemmas 4.3 and 1.6 and Slutsky’s Theorem with the choice of $\tau = 1$ and $T = 0$.

### 4.2 Proof of Theorem 1.4

Set $\tau > 0$ and for any $T \geq 0$ and $\delta > 0$ following the same argument as in Lemma 1.3 we have for any $0 < \epsilon < \delta$

$$\mathbb{P}\left( \int_{0}^{\tau+T} (g_{\epsilon}(x,\tau,t) - g_{\eta}(x,\tau,t))(t) dt = \delta \text{ for some } x \in \mathcal{M}_{W}(0,\tau) \right) \leq \frac{1}{\sqrt{\pi \tau}} \mathbb{P}\left( \sqrt{2} \int_{0}^{\tau+T} W^{\tau,T}(t) dt \in [\delta - \epsilon, \delta + \epsilon] \right)$$

where $W^{\tau,T}$ is defined as in (33). As $\epsilon \rightarrow 0$, we have

$$\mathbb{P}\left( \int_{0}^{\tau+T} (g_{\epsilon}(x,\tau,t) - g_{\eta}(x,\tau,t))(t) dt = \delta \text{ for some } x \in \mathcal{M}_{W}(0,\tau) \right) \leq \frac{1}{\sqrt{\pi \tau}} \mathbb{P}\left( \sqrt{2} \int_{0}^{\tau+T} W^{\tau,T}(t) dt = \delta \right).$$
The set \( \mathbb{R}^+ := \{ \delta : \mathbb{P}(\int_0^{\tau+T} W^+,\tau,T(t)dt = \delta) > 0 \text{ for some } T \in \mathbb{Q}^+ \} \) is at most countable. Hence \( \mathbb{R} \setminus \mathbb{R}^+ \) is dense in \( \mathbb{R} \).

Recall that \( W^\tau \) is a process on \([0, \infty)\) defined as in (32). Let \( t^\tau := \inf\{ t > 0 : W^\tau(t) = 0 \} \). From definition of \( W^\tau \) and from Corollary 4.3 it follows that \( t^\tau \in [\tau, \infty) \) almost surely. Let \( W^{+,t}(t) := 1\{ t \leq t^\tau \} W^\tau(t) \) be another process defined on \([0, \infty)\). Hence \( \int_0^{\infty} W^{+,t}(t)dt < \infty \) almost surely. We observe that \( W^{+,t}|[0,\tau] = W^\tau|[0,\tau] \) and for \( T \geq 0, W^{+,t,T} = W^{+,t}|[0,\tau+T] \). For \( \lambda \in \mathbb{R} \setminus \mathbb{R}^+ \) and \( \lambda > 0 \), let \( \bar{\lambda} := \lambda^{\delta/2}(\gamma_0 \rho)^{-1} \). We show that

**Lemma 4.7.** For \( \tau, \lambda > 0 \), \( T \geq 0 \) and \( \lambda \in \mathbb{R} \setminus \mathbb{R}^+ \),

\[
\lim_{n \to \infty} \sqrt{n} \mathbb{P}(L(0,0) > n\tau, \sum_{k=0}^{[n(\tau+T)]} \#C_k(0,0) > (\lambda n)^{3/2}) = \frac{1}{\gamma_0 \sqrt{\pi \tau}} \mathbb{P}\left( \sqrt{2} \int_0^{\tau+T} W^{+,\tau,T}(t)dt > \bar{\lambda} \right) = \frac{1}{\gamma_0 \sqrt{\pi \tau}} \mathbb{P}\left( \sqrt{2} \int_0^{\tau+T} W^{+,\tau}(t)dt > \bar{\lambda} \right).
\]

**Proof:** Since \( \lambda \in \mathbb{R} \setminus \mathbb{R}^+ \), the proof follows from Lemmas 4.2 and 4.3. \( \square \)

Lemma 4.7 implies the following

**Lemma 4.8.** For \( \tau, \lambda > 0 \) and \( \lambda \in \mathbb{R} \setminus \mathbb{R}^+ \),

\[
\lim_{n \to \infty} \sqrt{n} \mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (\lambda n)^{3/2}) = \frac{1}{\gamma_0 \sqrt{\pi \tau}} \mathbb{P}\left( \sqrt{2} \int_0^{\infty} W^{+,\tau}(t)dt > \bar{\lambda} \right).
\]

**Proof:** Fix \( \epsilon > 0 \) and choose \( T_0 \geq 0 \) such that \( (\sqrt{\pi \tau \gamma_0})^{-1} < \epsilon \). For any \( T > 0 \) clearly \( \mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (\lambda n)^{3/2}) \geq \mathbb{P}(L(0,0) > n\tau, \sum_{k=0}^{[n(\tau+T)]} \#C_k(0,0) > (\lambda n)^{3/2}) \), and hence from Lemma 4.7 we have for all \( T \geq T_0 \)

\[
\lim_{n \to \infty} \sqrt{n} \mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (\lambda n)^{3/2}) \geq \frac{1}{\gamma_0 \sqrt{\pi \tau}} \mathbb{P}\left( \int_0^{\tau+T} W^{+,\tau,T}(t)dt > \bar{\lambda} \right) = \frac{1}{\gamma_0 \sqrt{\pi \tau}} \mathbb{P}\left( \int_0^{\tau+T} W^{+,\tau}(t)dt > \bar{\lambda} \right).
\]

On the other hand we observe that

\[
\mathbb{P}(L(0,0) > n\tau, \sum_{k=0}^{[n(\tau+T)]} \#C_k(0,0) > (\lambda n)^{3/2}) \leq \mathbb{P}(L(0,0) > n\tau, \sum_{k=0}^{[n(\tau+T)]} \#C_k(0,0) > (\lambda n)^{3/2}) + \mathbb{P}(L(0,0) > n(\tau + T)).
\]

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Using Lemma 4.7 and Theorem 1.2 from the choice of $T_0$, we have for all $T \geq T_0$

$$\limsup_{n \to \infty} \sqrt{n} \mathbb{P}(L(0,0) > n\tau, \#C(0,0) > (\lambda n)^{3/2}) \leq \frac{1}{\gamma_0 \sqrt{\pi \tau}} \mathbb{P}(\int_0^{\tau+T} W^{+}(t) dt > \lambda \bar{\beta}) + \epsilon.$$ 

This completes the proof. \qed

**Remark 4.9:** Later we show that Lemma 4.7 and 4.8 hold for all $\lambda > 0$. \qed

To calculate the distribution of $\int_0^\infty W^{+}(t) dt$ we need the following lemmas. The next lemma gives the distribution of $t^\tau$.

**Lemma 4.10.** For $\tau > 0$ and $T \geq 0$ we have,

$$1 - \mathbb{P}(t^\tau \leq \tau + T) = \mathbb{P}(W^{+}(\tau + T) > 0) = \frac{\sqrt{\tau}}{\sqrt{\tau + T}}.$$ 

**Proof:** From the scaling properties described as in Section 1, it suffices to show the result for $\tau = 1$. We know that $W^{+}(1)$ follows Rayleigh distribution. From Corollary 4.3.1 integrating by parts we obtain

$$\mathbb{P}(W^{+}(1 + t) > 0) = \mathbb{P}(\min\{W^{+}(t) : t \in [1, 1 + T]\} > 0)$$
$$= \int_0^\infty (2\Phi(x) - 1) x \exp(-\frac{x^2}{2}) dx$$
$$= 2 \int_0^\infty (2\pi T)^{-\frac{1}{2}} \exp(-\frac{x^2}{2}(1 + T^{-1})) dx$$
$$= \frac{1 + T}{1 + T}.$$

This completes the proof. \qed

Using Lemma 4.10 we obtain the following.

**Lemma 4.11.** For $\tau > 0, \lambda > 0, T \geq 0, t \in [\tau, \tau + T]$ and a sequence $\{\tau + j_n h_n : n \geq 1\}$ with $h_n \to 0$ and $(\tau + j_n h_n) \to t$ as $n \to \infty$, we have

$$\lim_{n \to \infty} \mathbb{P}(\int_0^{\tau+j_n h_n} W^{+}(s) ds > \lambda W^{+}(\tau + (j_n + 1) h_n) = 0) = \bar{F}_{I_0^+} (\lambda t^{-\frac{3}{2}})$$

where $\bar{F}_{I_0^+}$ is the distribution function of $I_0^+$. 

**Proof:** For simplicity of notation we take $\tau_n := \tau + j_n h_n$ and $\gamma_n := \tau + (j_n + 1) h_n$.

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\[ \mathbb{P}(\int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda | W^{+,-\tau_n}(\gamma_n) = 0) \]

\[ = \mathbb{E}(1\{ \int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda, W^{+,-\tau_n}(\gamma_n) = 0 \}) \mathbb{P}(W^{+,-\tau_n}(\gamma_n) = 0)^{-1} \]

\[ = \sum_{k=0}^{\infty} \mathbb{E}(1\{ \int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda, W^{+,-\tau_n}(\gamma_n) = 0, W^{\tau_n}(\tau_n) \in (k\nu, (k+1)\nu] \}) \]

\[ \mathbb{P}(W^{+,-\tau_n}(\gamma_n) = 0)^{-1} \]

\[ = \sum_{k=0}^{\infty} \mathbb{E}(1\{ \int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda, W^{+,-\tau_n}(\tau_n) \in (k\nu, (k+1)\nu], \min_{s \in [\tau_n, \gamma_n]} (W^{\tau_n}(\tau_n) + \tilde{W}(s)) \leq 0 \}) \]

\[ \mathbb{P}(W^{+,-\tau_n}(\gamma_n) = 0)^{-1} \]

The last equality follows from Corollary 4.3.1, where \( \tilde{W} \) is a Brownian motion on \([\tau_n, \infty)\)
starting from origin and independent of $W^{\tau_n}$. Hence we have

$$
P(\int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda | W^{+\tau_n}(\gamma_n) = 0)
$$

$$
= \sum_{k=0}^{\infty} \mathbb{E}(\frac{1}{\tau_n} \int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda, W^{\tau_n}(\tau_n) \in (k
$$

$$
= \sum_{k=0}^{\infty} \mathbb{P}(\int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda, W^{\tau_n}(\tau_n) \in (k
$$

$$
\geq \sum_{k=0}^{\infty} \mathbb{P}(\int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda, W^{\tau_n}(\tau_n) \in (k
$$

$$
= 2 \sum_{k=0}^{\infty} \mathbb{P}(\int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda, W^{\tau_n}(\tau_n) \in (k
$$

$$
= 2 \sum_{k=0}^{\infty} \mathbb{P}(\int_0^{\tau_n} W^{\tau_n}(s) ds > \lambda, W^{\tau_n}(\tau_n) \leq (k + 1) \nu)(\Phi(\frac{(k + 2) \nu}{\sqrt{h_n}}) - \Phi(\frac{(k + 1) \nu}{\sqrt{h_n}}))
$$

$$
= 2 \sum_{k=0}^{\infty} \mathbb{P}(\int_0^{1} W^{1}(s) ds > \lambda \tau_n^{\frac{3}{2}} | W^{1}(1) \leq (k + 1) \nu \tau_n^{\frac{3}{2}}) \mathbb{P}(W^{1}(1) \leq (k + 1) \nu \tau_n^{-\frac{3}{2}})
$$

$$
= 2 \sum_{k=0}^{\infty} \mathbb{P}(\int_0^{1} W^{1}(s) ds > \lambda \tau_n^{\frac{3}{2}} | W^{1}(1) \leq (k + 1) \nu \tau_n^{-\frac{3}{2}}) \mathbb{P}(W^{1}(1) \leq (k + 1) \nu \tau_n^{-\frac{3}{2}})
$$

$$
\phi(s^k_{\nu}^{\frac{\nu}{\sqrt{h_n}}}) \nu h_n^{-\frac{3}{2}} (1 - \tau_n^{\frac{3}{2}} \gamma_n^{-\frac{1}{2}})^{-1}.
$$

The last step follows from Lemma 4.10 and Lagrange’s mean value theorem where $s^k_{\nu} \in ((k + 1) \nu h_n^{-\frac{3}{2}}, (k + 2) \nu h_n^{-\frac{1}{2}})$ is such that $\nu h_n^{-\frac{3}{2}} \phi(s^k_{\nu}) = \Phi(\frac{(k + 2) \nu}{\sqrt{h_n}}) - \Phi(\frac{(k + 1) \nu}{\sqrt{h_n}})$. Letting
\( \nu \to 0 \) we get
\[
P(\int_0^{\tau_n} W^+(s)ds > \lambda |W^{+}\cdot \tau_n(\gamma_n) = 0) \\
\geq \int_0^{\infty} 2(1 - \frac{\frac{1}{\gamma_n}}{\gamma_n} - \frac{1}{\tau_n})^{-1} P(\int_0^{1} W^+(s)ds > \lambda \tau_n^{-\frac{1}{2}} | W^+(1) \leq u h_n^{\frac{1}{2}} \tau_n^{-\frac{1}{2}}) \\
P(W^+(1) \leq u h_n^{\frac{1}{2}} \tau_n^{-\frac{1}{2}} | u) du \\
= \int_0^{\infty} 2 \gamma_n h_n^{-1}(1 + \frac{1}{\tau_n} \gamma_n^{-\frac{1}{2}}) \Phi(\int_0^{1} W^+(s)ds > \lambda \tau_n^{-\frac{1}{2}} | W^+(1) \leq u h_n^{\frac{1}{2}} \tau_n^{-\frac{1}{2}}) \\
(1 - \exp(-\frac{u^2 h_n}{2 \tau_n})) \phi(u) du.
\]
The last equality follows from property (a) of \( W^+ \) described in Section [1]. Let \( f_n : (0, \infty) \to \mathbb{R} \) be defined as
\[
f_n(u) := 2 \gamma_n h_n^{-1}(1 + \frac{1}{\gamma_n} \gamma_n^{-\frac{1}{2}}) \Phi(\int_0^{1} W^+(s)ds > \lambda \tau_n^{-\frac{1}{2}} | W^+(1) \leq u h_n^{\frac{1}{2}} \tau_n^{-\frac{1}{2}})(1 - \exp(-\frac{u^2 h_n}{2 \tau_n})).
\]
We observe that for \( u > 0 \), as \( n \to \infty \),

(i) \( u h_n^{\frac{1}{2}} \tau_n^{-\frac{1}{2}} \to 0 \);

(ii) \( \lambda \tau_n^{-\frac{1}{2}} \to \lambda t^{-\frac{1}{2}} \);

(iii) \( (1 - \exp(-\frac{u^2 h_n}{2 \tau_n}))(1 + \frac{1}{\gamma_n} \gamma_n^{-\frac{1}{2}}) \frac{2 h_n}{\tau_n} \to 2 u^2 \).

Since the map \( \Gamma : C[0,1] \to \mathbb{R} \) given by \( \Gamma(g) = \int_0^1 g(s)ds \) is continuous and \( F^+_0(\lambda) \) is a continuous random variable (see [Janson et al. (2007)]), by Theorem 2.1 of [Durrett et al. (1977)] and from continuous mapping theorem it follows that the distribution function of \( \int_0^1 W^+(s)ds | \{W^+(1) \leq u h_n^{\frac{1}{2}} \tau_n^{-\frac{1}{2}} \} \) converges uniformly to \( F^+_0(\lambda) \). Hence we have \( f_n(u) \to f(u) \) where \( f(u) := 2 u^2 \Phi(\int_0^1 W^0_\tau(s)ds > \lambda t^{-\frac{1}{2}}) \). This allows us to obtain
\[
\liminf_{n \to \infty} P(\int_0^{\tau_n} W^+\cdot \tau_n(s)ds > \lambda |W^{+}\cdot \tau_n(\gamma_n) = 0) \geq F^+_0(\lambda t^{-\frac{1}{2}}).
\]

Similar argument gives us that
\[
\limsup_{n \to \infty} P(\int_0^{\tau_n} W^+\cdot \tau_n(s)ds > \lambda |W^{+}\cdot \tau_n(\gamma_n) = 0) \leq F^+_0(\lambda t^{-\frac{1}{2}})
\]
where in order to obtain an upper bound we use the fact for \( u \in (k\nu,(k + 1)\nu] \), \( P(\min_{t \in [\tau_n,\tau_n]} (u + W(t)) \leq 0) \leq 2(1 - \Phi(\frac{ku}{\sqrt{\lambda}})) \). This completes the proof. \( \square \)

Next we obtain the distribution of \( \int_0^{\infty} W^+(t)dt \).
Lemma 4.12. For $\tau, \lambda > 0$, we have

$$\mathbb{P}(\int_{0}^{T} W^{+\tau}(t) dt > \lambda) = \frac{\sqrt{T}}{2} \int_{\tau}^{T} t^{-\frac{3}{4}} \tilde{F}_{I_{0}}(\lambda t^{-\frac{3}{4}}) dt.$$  

**Proof:** For any $T \geq 0$,

$$\mathbb{P}(\int_{0}^{T} \tau W^{+\tau}(t) dt > \lambda, W^{+\tau}(\tau + T) = 0) \leq \mathbb{P}(\int_{0}^{\infty} \tau W^{+\tau}(t) dt > \lambda) \leq \mathbb{P}(\int_{0}^{T} \tau W^{+\tau}(t) dt > \lambda, W^{+\tau}(\tau + T) = 0) + \mathbb{P}(W^{+\tau}(\tau + T) > 0).$$

From Lemma 4.10 $\mathbb{P}(W^{+\tau}(\tau + T) > 0) \to 0$ as $T \to \infty$; hence it suffices to show

$$\mathbb{P}(\int_{0}^{T} \tau W^{+\tau}(t) dt > \lambda, W^{+\tau}(\tau + T) = 0) = \frac{\sqrt{T}}{2} \int_{\tau}^{T} t^{-\frac{3}{4}} \tilde{F}_{I_{0}}(\lambda t^{-\frac{3}{4}}) dt.$$  

We proceed as follows

$$\mathbb{P}(\int_{0}^{T} \tau W^{+\tau}(t) dt > \lambda, W^{+\tau}(\tau + T) = 0) = \sum_{j=0}^{[T/h]} \mathbb{P}(\int_{0}^{\tau+jh} W^{+\tau}(t) dt > \lambda, W^{+\tau}(\tau + jh) > 0, W^{+\tau}(\tau + (j + 1)h) = 0)$$

$$= \sum_{j=0}^{[T/h]} \mathbb{P}(\int_{0}^{\tau+jh} W^{+\tau}(t) dt > \lambda, W^{+\tau}(\tau + jh) > 0, W^{+\tau}(\tau + (j + 1)h) = 0)$$

$$= \sum_{j=0}^{[T/h]} \mathbb{P}(W^{+\tau}(\tau + jh) > 0, W^{+\tau}(\tau + (j + 1)h) = 0)$$

$$= \sum_{j=0}^{[T/h]} \mathbb{P}(W^{+\tau}(\tau + (j + 1)h) > 0, W^{+\tau}(\tau + (j + 1)h) = 0)$$

$$= \sum_{j=0}^{[T/h]} \mathbb{P}(\int_{0}^{\tau+jh} W^{+\tau+jh}(t) dt > \lambda, W^{+\tau+jh}(\tau + (j + 1)h) = 0)$$

$$\tau \frac{3}{2}(\tau + jh)^{-\frac{1}{2}} - (\tau + (j + 1)h)^{-\frac{1}{2}},$$

where we have used Lemma 4.10 and the fact that on the event $\{W^{\tau+jh}(\tau + jh) > 0\}, S^{\tau+jh} = S^{\tau+jh}$ and $W^{+\tau+jh}[0,\tau+jh] = W^{+\tau+jh}[0,\tau+jh]$. Letting $h \to 0$ using Lemma 4.11 we have

$$\mathbb{P}(\int_{0}^{T} \tau W^{+\tau}(t) dt > \lambda, W^{+\tau}(\tau + T) = 0) = \frac{\sqrt{T}}{2} \int_{\tau}^{T} t^{-\frac{3}{4}} \tilde{F}_{I_{0}}(\lambda t^{-\frac{3}{4}}) dt.$$  

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This completes the proof. □

Since \( J_0^+ \) is a continuous random variable (see [Janson et al. (2007)]) for any \( \tau > 0 \) the map \( \lambda \mapsto \int_0^\infty \left( 4\pi \tau \gamma_0^2 t^3 \right)^{-\frac{3}{2}} \bar{F}_{I_0^+} \left( \lambda t^{-\frac{3}{2}} \right) dt \) is also continuous. This justifies Remark 4.9.

From Remark 4.9 and Lemma 4.12 we derive the following.

Corollary 4.12.1. For \( \lambda > 0 \), we have

\[
\lim_{n \to \infty} \sqrt{n} \mathbb{P}(\#C(0, 0) > (\lambda n)^{3/2}) = \frac{1}{2\gamma_0 \sqrt{\pi \tau}} \int_0^\infty \bar{F}_{I_0^+} \left( \lambda t^{-\frac{3}{2}} \right) dt.
\]

Proof: For any \( \tau > 0 \) we have, \( \mathbb{P}(\#C(0, 0) > (n\lambda)^{3/2}) \geq \mathbb{P}(L(0, 0) > n\tau, \#C(0, 0) > (n\lambda)^{3/2}) \) and hence \( \liminf_{n \to \infty} \sqrt{n} \mathbb{P}(\#C(0, 0) > (n\lambda)^{3/2}) \geq \int_0^\infty \left( 4\pi \tau \gamma_0^2 t^3 \right)^{-\frac{3}{2}} \bar{F}_{I_0^+} \left( \lambda t^{-\frac{3}{2}} \right) dt. \)

Fix \( \varepsilon > 0 \). We observe that

\[
\sqrt{n} \mathbb{P}(L(0, 0) \leq n\tau, \#C(0, 0) > (n\lambda)^{3/2}) \leq \sqrt{n} \mathbb{P}(L(0, 0) \leq n\tau, \sum_{k=0}^{\lfloor n\tau \rfloor} \hat{D}_k(0, 0) > (n\lambda)^{3/2}) \leq \sqrt{n} \mathbb{P}(\sum_{k=0}^{\lfloor n\tau \rfloor} \hat{D}_k(0, 0), n\lambda)^{3/2}) \leq \sqrt{n} \mathbb{E}[\sum_{k=0}^{\lfloor n\tau \rfloor} \hat{D}_k(0, 0)](n\lambda)^{-3/2} = \sqrt{n}(\lfloor n\tau \rfloor + 1)\mathbb{E}(\bar{D}_0(0, 0))(n\lambda)^{-3/2},
\]

where we have used the fact that \( \{\hat{D}_k(0, 0) = \hat{h}^k(\hat{r}(0, 0))(1) - \hat{h}^k(\hat{r}(0, 0))(1) : k \geq 0\} \) is a martingale (see Proposition 2.3). From the earlier discussions it also follows that \( \mathbb{E}(\bar{D}_0(0, 0)) \leq 2\mathbb{E}(G) = 2(1 - p) p^{-1} \) where \( G \) is a geometric random variable. Choose \( \tau_0 > 0 \) such that \( \tau_0 < \frac{\lambda p}{2(1 - p)} \). Hence for \( \tau \leq \tau_0 \) we have \( \limsup_{n \to \infty} \sqrt{n} \mathbb{P}(L(0, 0) \leq n\tau, \#C(0, 0) > (n\lambda)^{3/2}) \leq \varepsilon. \) This allows us to obtain

\[
\limsup_{n \to \infty} \sqrt{n} \mathbb{P}(\#C(0, 0) > (n\lambda)^{3/2}) = \limsup_{n \to \infty} \sqrt{n} \mathbb{P}(L(0, 0) > n\tau, \#C(0, 0) > (n\lambda)^{3/2}) + \mathbb{P}(L(0, 0) \leq n\tau, \#C(0, 0) > (n\lambda)^{3/2}) \leq \frac{1}{2\gamma_0 \sqrt{\pi \tau}} \int_0^\infty \bar{F}_{I_0^+} \left( \lambda t^{-\frac{3}{2}} \right) dt + \varepsilon.
\]

This completes the proof. □

Proof of Theorem 1.4 From Lemma 6.1 of [Resnick (2007)] page 174, it follows that Lemma 4.7 together with Corollary 4.12.1 and Theorem 1.2 prove (i).

Fix \( \tau > 0, \lambda > 0 \). For \( \alpha < 2/3, \delta > 0 \) and for all large \( n \), we have \( \mathbb{P}(L(0, 0) > n\tau, \#C(0, 0) > (n\lambda)^{1/\alpha}) \leq \mathbb{P}(L(0, 0) > n\tau, \#C(0, 0) > (n\delta)^{3/2}) \). Fix any \( \varepsilon > 0 \) and choose \( \delta = \delta(\varepsilon) > 0 \) so that \( \mathbb{P}(\sqrt{2} \int_0^\infty W^+(t) dt > \delta)(\sqrt{\pi \tau \gamma_0})^{-1} < \varepsilon, \) where
\( \delta = \delta^{3/2}(\gamma_0 p)^{-1} \). From Lemma 4.8, we have \( \limsup_{n \to \infty} \sqrt{n} P(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) < \epsilon \).

On the other hand, from property (a) of \( W^+ \) and property (g) of \( W^\tau \) it follows that \( P(\int_0^\infty W^+(\tau(t)) dt > 0) = 1 \) for \( \tau > 0 \). Now for \( \alpha > 2/3 \) and \( \delta > 0 \) we have \( P(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) = P(L(0,0) > n\tau, \#C(0,0) > (n\delta)^{3/2}) \) for all large \( n \). Again from Lemma 4.8, we have \( \liminf_{n \to \infty} \sqrt{n} P(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) \geq \mathbb{P}(\sqrt{2} \int_0^\infty W^+(\tau(t)) dt > \delta)(\sqrt{\pi\tau\gamma_0})^{-1} \). Since \( \limsup_{n \to \infty} \sqrt{n} P(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) \leq \lim_{n \to \infty} \sqrt{n} P(L(0,0) > n\tau) = (\sqrt{\pi\tau\gamma_0})^{-1} \), letting \( \delta \to 0 \), it follows that \( \lim_{n \to \infty} \sqrt{n} P(L(0,0) > n\tau, \#C(0,0) > (n\lambda)^{1/\alpha}) = (\sqrt{\pi\tau\gamma_0})^{-1} \) for \( \alpha > 2/3 \). This completes the proof of (8).

The argument for \( (L(0,0), (D_{\max}(0,0))^{1/2}) \) being similar is omitted. \( \square \)

**Mathematical Notation**

- \( V \): the set of all open vertices–on page 2;
- \( G \): the random graph associated with Howard’s model–on page 2;
- \( B(x,t) \): Bernoulli random variable associated with the vertex \( (x,t) \)–on page 2;
- \( U(x,t) \): the random variable taking values \(+1, -1\) with probability \(1/2\) and \(1/2\) associated with the vertex \( (x,t) \)–on page 2;
- \( h(x,t) \): the progeny of \( (x,t) \) in the Howard’s model–on page 3;
- \( h^k(x,t) \): the \( k \)-th generation progeny of \( (x,t) \)–on page 3;
- \( C_k(x,t), C(x,t) \): the set of the \( k \)-th generation ancestors and the set of all ancestors of \( (x,t) \) respectively–on page 3;
- \( L(x,t) \): the height of the ancestry line of \( (x,t) \)–on page 3;
- \( \gamma_0 = \sqrt{(1-p)(2-2p+p^2)}/p^2(2-p)^2 \): a constant depending upon \( p \)–on page 3;
- \( D_k(x,t) \): the width of the set of the \( k \)-th generation ancestors of \( (x,t) \)–on page 3;
- \( D_n(x,t), K_n(x,t) \): diffusively scaled width process and cluster process respectively–on page 4;
- \( W^+, W^0_+ \): the standard Brownian meander and Brownian excursion respectively–on page 4;
- \( I_0^+, M_0^+ = \max\{W_0^+(s) : s \in [0,1]\} \): on page 4;
- \( f|_{[a,b]} \): restriction of \( f \) over \([a,b]\)–on page 4;
- \( f_{1+}, F_{1+} \): density function and distribution function associated with \( I_0^+ \) respectively–on page 5;
• $f_{M_0^+}, F_{M_0^+}$: density function and distribution function associated with $M_0^+$ respectively–on page 6;

• $\bar{F}_{I_0^+} = 1 - F_{I_0^-}, \bar{F}_{M_0^+} = 1 - F_{M_0^-}$ on page 5;

• $J^+_{(x,t)}, J^-_{(x,t)}$: (10) on page 7;

• $\hat{r}(x,t), \hat{l}(x,t)$: right and left dual neighbour of open $(x,t)$ respectively–on page 7;

• $\hat{V}$: the set of dual vertices–on page 7;

• $a^r(u,s), a^l(u,s)$: (11) on page 7;

• $\hat{h}_k(u,s)$: $k$-th progeny of dual vertex $(u,s)$ in the dual graph–on page 8;

• $\hat{X}_k(u,s) = \hat{h}_k(u,s)(1)$–on page 8;

• $\hat{\mathcal{W}}, \hat{\mathcal{W}}$: the standard Brownian web and its dual–on page 9.

• $(R^2, \rho), (\mathcal{H}, d_{\mathcal{H}})$: complete separable metric spaces–on pages 9, 10;

• $(\hat{\mathcal{H}}, d_{\hat{\mathcal{H}}})$: complete separable metric spaces–on page 10;

• $(m_{in}(x,t), m_{out}(x,t))$: type of $(x,t)$ in the Brownian web $\mathcal{W}$–on page 11;

• $\sigma_\pi, \sigma_{\hat{\pi}}$: the starting time of a forward path $\pi$ and a backward path $\hat{\pi}$ respectively on pages 9 and 10.

• $t_{\pi_1,\pi_2}$: the intersection time of the two paths $\pi_1$ and $\pi_2$–on page 11;

• $K^t$: paths in $K$ which start before time $t$–on page 11;

• $\hat{K}^t$: paths in $\hat{K}$ which start after time $t$–on page 11.

• $\mathcal{M}_K(t_1, t_2), \xi_K(t_1, t_2)$: (14) on page 11;

• $\hat{\mathcal{N}}^{(x,t)}, \hat{\mathcal{N}}^{(x,t)}_1$: there exist exactly two dual paths in $\hat{\mathcal{W}}$ starting at $(x,t)$ for $(x,t)$ of type $(1,1)$ in $\hat{\mathcal{W}}$–on page 12;

• $A(\pi^r, \pi^l)$: wedge with right boundary $\pi^r$ and left boundary $\pi^l$–on (17) page 12;

• $\hat{\pi}^{(x,t)}, \hat{\pi}^{(x,t)}$: forward path starting from $(x,t)$;

• $\hat{\pi}^{(u,s)}$: backward path starting from $(u,s)$;

• $\pi^{(x,t)}_n, \pi^{(x,t)}$: $n$-th order diffusively scaled version of $\pi^{(x,t)}$ and $\hat{\pi}^{(x,t)}$ respectively–on page 13;

• $\mathcal{X}$, $\hat{\mathcal{X}}$: collection of all paths in the graph and dual paths in the dual graph respectively–on page 13;
• $\mathcal{X}_n, \tilde{\mathcal{X}}_n$: $n$-th order diffusively scaled versions of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ respectively–on page 13;

• $\bar{\mathcal{X}}_n, \tilde{\bar{\mathcal{X}}}_n$: closure of $\mathcal{X}_n$ and $\tilde{\mathcal{X}}_n$ in $(\Pi, d_\Pi)$ and $(\tilde{\Pi}, d_{\tilde{\Pi}})$ respectively–on page 14;

• $\hat{D}_{n}(x,t)$: (21) on Page 22;

• $g^{\tau,T}_{\pi}$: (22) on Page 22;

• $\mathcal{M}_{(K,\tilde{K})}(\tau,T,O)$: on Page 22;

• $\kappa(\tau,T,O) = \#\mathcal{M}_{(\bar{\mathcal{W}},\tilde{\mathcal{W}})}(\tau,T,O), \kappa_n(\tau,T,O) = \#\mathcal{M}_{(\bar{\mathcal{X}}_n,\tilde{\mathcal{X}}_n)}(\tau,T,O)$: on Page 22;

• $\mathcal{R}_n(\tau,T,O)$: (24) on Page 22;

• $g^{(x),\tau,T}_{\nu,\pi} = g^{\tau,T}_{\nu(x,\tau)}$, $g^{(x),\tau,T}_{\ell,\pi} = g^{\tau,T}_{\ell(x,\tau)}$ on Page 23;

• $S^\tau$, $W^\tau$, $W^{\tau,T}$: (32) on page 25;

• $C_b(C[c,d], \mathbb{R})$: space of all real valued bounded continuous functions on $C[c,d]$–on page 26;

• $H(f)$: on page 27 before (33);

• $W^{\tau,+}$: (33) on page 27;

• $t^\tau$: on page 31;

• $W^\tau$: on page 32.

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