Violation of the viscosity/entropy bound in translationally invariant non-Fermi liquids

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Shear viscosity is an important characterization of how a many-body system behaves like a fluid. Here we study the shear viscosity of a strongly-interacting solvable model in two spatial dimensions, consisting of coupled Sachdev-Ye-Kitaev (SYK) islands. As temperature is lowered, the model exhibits a crossover from an incoherent metal with local criticality to a marginal fermi liquid. We find that while the shear viscosity to entropy density ratio satisfies the Kovtun-Son-Starinets (KSS) bound in the marginal Fermi liquid regime, it can strongly violate the KSS bound within a finite and robust temperature range in the incoherent metal regime, implying nearly perfect fluidity of the incoherent metal with local criticality. To the best of our knowledge, it provides the first translationally invariant example violating the KSS bound with known gauge-gravity correspondence.

Introduction.—Fluid mechanics is among the oldest and the most fundamental subjects in physics. A generic many-body system with globally conserved quantities, such as mass, energy, and momentum, will exhibit fluidity if the local thermalization time scale is much less than the relaxation time scale of the conserved quantities. As a result, universal properties of a fluid can provide extremely useful insights in understanding the correlated many-body systems with complicated interactions between their constituents, like the ultra-cold Fermi gases in the unitary regime and quark-gluon plasma (QGP) produced in relativistic heavy-ion collisions, where no control parameter exists [1]. More recently, owing to the advances of experimental techniques, quantum fluid behaviors are also witnessed in correlated electrons in lattice systems [2,3]. Interestingly, the theory of fluids also receives a boost from the development of holographic principles [4,5]. A fundamental characterization of fluids is the shear viscosity that measures the resistance of a fluid to the shear stress. Since viscosity generates entropy and causes dissipation, a good fluid should have small shear viscosity. However, the viscosity cannot be arbitrarily small. Namely, like the uncertainty principle, the fundamental laws of nature put a lower bound on the ratio of shear viscosity to entropy. Based on the AdS/CFT correspondence, Kovtun, Son and Starinets conjectured a lower bound (KSS bound) on the ratio of shear viscosity to entropy in the strongly coupled, non-quasiparticle systems [7], i.e., $\eta/\mathcal{S} \geq 1/4\pi$, where $\eta$ and $\mathcal{S}$ refer to shear viscosity and entropy density, respectively.

The closer the ratio, $\eta/\mathcal{S}$, of a many-body system is to the KSS bound, the better the system behaves as a perfect fluid. Thus, it is of great interest and importance to explore the scarce examples that saturate, or even violate the KSS bound. Among holographic systems, the KSS bound is obeyed in Einstein gravity with both rotational and translational symmetries, while a weaker bound [8–12] is obeyed in higher-derivative gravity theory. When rotational symmetry is broken, like in anisotropic black branes [13,15], the Goldstone vector bosons are generated, and the shear viscosity of the spin-1 component violates the KSS bound in a parametric manner. Additionally, the black brane solution for Gauss-Bonnet massive gravity and Rastall AdS Massive gravity both show the violation of KSS bound [16]. For isotropic black branes with linear axion fields, the KSS bound can also be violated, but shear viscosity does not have a hydrodynamic interpretation since momentum is no longer a conserved charge [17,23].

On the other hand, among many-body systems, as expected, the minimal of the ratio, $\eta/\mathcal{S}$, occurs at the fixed point exhibiting emergent conformal symmetry, where the quasiparticle description often invalidates. When the fixed point locates at zero temperature, the ratio should be a universal number associated with the universality class that the system falls into. The examples include the electron fluid in graphene [24], the Luttinger-Abrikosov-Beneslavskii phase in three dimensional quadratic band touching semimetal [25], and Ising nematic quantum critical point in two dimensional metals [26]. However, if the fixed point locates at finite temperature, the ratio shows a non-universal behavior as a function of temperature. The well-studied unitary quantum gases and the QGP fall into this class [27,28]. In unitary quantum gases, the minimal of the ratio occurs at an intermediate temperature range associated with the superfluid transition, providing possible examples violating the KSS bound [29], while at the zero-temperature limit the gapless Goldstone modes lead to a divergent ratio.

Recently, Patel et al. [34] as well as Chowdhury et al. [35] constructed a two-dimensional strongly correlated solvable model, consisting of coupled Sachdev-Ye-Kitaev (SYK) islands as shown in Fig. 1(a). This model is extremely interesting due to the facts that the SYK model...
is believed to have a gravity dual \[33, 36-41\] with maximal chaos \[42\], and that though the model exhibits marginal Fermi liquid (MFL) with well-defined quasiparticle at low temperature, it exhibits an intermediate-temperature incoherent metal (IM) regime where the quasiparticle description invalidates, much similar to the case of unitary chaos \[42\], and that though the model exhibits marginal Fermi liquid (MFL), IM (incoherent) and semi-classical regime, exhibiting different behaviors. The ratio violates the KSS bound indicated by the dashed line in the IM regime.

The model.—As shown in Fig. 1(a), we consider a two-dimensional lattice model, with \(M\) flavors of conduction electrons, \(c_{ri}, i = 1, \ldots, M,\) and \(N\) flavors of valence electrons, \(f_{ri}, i = 1, \ldots, N,\) at site \(r:\)

\[
H = -\sum_{r,r'} \sum_{i=1}^{M} \left( t_{rr'} c_{ri}^{\dagger} c_{r'i} + h.c. \right) + \sum_{r} \left[ -\mu_c \sum_{i=1}^{M} c_{ri}^{\dagger} c_{ri} \right] -\mu_f \sum_{i=1}^{N} f_{ri}^{\dagger} f_{ri} + \sum_{i,j=1}^{M} \sum_{k,l=1}^{N} \frac{g_{ijkl}}{N M^{1/2}} f_{ri}^{\dagger} f_{rj}^{\dagger} c_{rk} c_{rl}
+ \sum_{i,j,k,l=1}^{N} \frac{J_{ijkl}}{N^{3/2}} f_{ri}^{\dagger} f_{rj}^{\dagger} f_{rk} f_{rl},
\]  

(1)

where \(t_{rr'}\) is the hopping amplitude of \(c\) fermions between sites \(r\) and \(r',\) and \(\mu_c, \mu_f\) denote the chemical potential of \(c\) and \(f\) fermions, respectively. The local interaction strength \(g_{ijkl}\) and \(J_{ijkl}\) are random numbers which satisfies \(\langle J_{ijkl} J_{klij} \rangle = \frac{g^2}{2}\) and \(\langle g_{ijkl} g_{klij} \rangle = g^2\) and all other \(\langle \ldots \rangle\) are vanishing. Here \(\langle \ldots \rangle\) means disorder average. Note that the coupling constants \(g_{ijkl}\) and \(J_{ijkl}\) on different sites not only have the same distribution, but are identical in each realization. In the following, we choose the hopping amplitude to be a function depending on \(|r - r'|\), for instance, \(t_{rr'} = t_0 \delta_{r'-r+\epsilon_i}\), where \(\epsilon_i\) is the primitive lattice vector. As a result, the Hamiltonian is translationally invariant. If \(g = 0\), the model can be viewed as two independent subsystems: the conducting \(c\) fermions with a hopping \(t_{rr'}\), and the local \(f\) fermions with SYK interaction at each site. Finite \(g > 0\) will couple the two subsystems, as illustrated in Fig. 1(a).

They interact through a random exchange with effective strength \(g\), similar to the Kondo lattice model \[43, 46\].

We will consider large \(N\) and \(M\) limit, while keep their ratio, \(M/N\), fixed. The Green’s functions are given by \[35\]

\[
G^c(k, i\omega_n) = \frac{1}{i\omega_n - \epsilon_k + \mu_c - \Sigma_{c_f}(k, i\omega_n)},
\]

(2)

\[
G^f(k, i\omega_n) = \frac{1}{i\omega_n + \mu_f - \Sigma_{f_f}(k, i\omega_n) - \Sigma_f(k, i\omega_n)}
\]

(3)

where \(k\) and \(\omega_n\) denote momentum and Matsubara frequency, \(\Sigma_{c_f}, \Sigma_{f_f}\) and \(\Sigma_f\) refer to the self-energy resulted from the coupling between \(c\) and \(f\) fermions and self interaction of \(f\) fermions, respectively. Local critical \(f\) fermion propagator, i.e., \(G^f(k, i\omega_n) = G^f(i\omega_n)\), is always a consistent solution to the saddle point equations \[47\].

Especially, in the case \(M/N = 0\), the saddle point equations of \(f\) fermions are identical to the zero dimensional complex SYK model and the solutions in the conformal limit are given by \[48\]

\[
G^f(r) = -\frac{\pi^{\frac{1}{2}} \frac{4}{3} \cosh \frac{1}{2} (2\pi \xi) \left( \frac{T}{\sin (\pi T)} \right) \left( \frac{T}{\sin (\pi T)} \right)^{\frac{1}{2}} e^{-2\pi \xi T},
\]

where \(\xi\) is a parameter controlling the particle-hole asymmetry, and \(r \in [0, \beta]\) is the imaginary time.

Now, moving to the propagator of \(c\) fermion, we will follow Ref. \[31\] closely. Though the model in Ref. \[31\] breaks translational symmetry by the locally independent disorder, we show in Supplemental Materials \[47\] that at \(\frac{M}{N} \ll 1\), both models have the same saddle point solutions. In the limit \(g^2 \gg tJ\), there exists a crossover temperature, \(T_{inc} \sim \frac{tJ^{3/2}}{g^2}\), between the MFL regime in the lower temperature and the IM regime in higher temperature. When \(T \ll T_{inc}\), the hopping term between conduction electrons dominates, and the self-energy of
the c fermion yields \[^{33}\text{[17]}\]

\[ \Sigma_{c}^{\text{MFL}}(i\omega_n) = \frac{ig^2 T}{2Jt \cosh^{1/2}(2\pi\epsilon)\pi^{3/2}} \left( \frac{\omega_n}{T} \ln \left( \frac{2\pi Te^{\epsilon}e^{-1}}{J} \right) \right. \]

\[ \left. + \frac{\omega_n}{T} \psi \left( \frac{-i\omega_n}{2\pi T} + \pi \right) \right), \tag{4} \]

where \( \psi \) is the digamma function, and \( \gamma_E = 0.577 \) is the Euler-Mascheroni constant. The self-energy shows that the c fermions exhibit a MFL behavior. Indeed, in this regime, the model a linear-in-T resistivity as well as a \( T \ln T \) entropy density \[^{33}\text{[19]}\], i.e., \( S_{\text{MFL}} \sim \frac{g^2}{2\pi c} \sqrt{T + T \ln \frac{T}{\mu}} \).

On the other hand, when \( T > T_{\text{inc}} \), the interacting term between the conduction and the valence band electrons dominates. Since the interacting term is local, the c fermion propagator will also exhibit local critical behavior \[^{33}\text{[17]}\]. The c fermion self-energy reads \[^{33}\text{[17]}\]

\[ \Sigma_{c}^{\text{IM}}(i\omega_n) = \frac{iT^{1/2}g\Lambda^{1/2}2^{1/4}(0)(-1)^{1/2}(1 + e^{2\pi\epsilon}c)\pi^{1/4}e^{2\pi\epsilon}}{\pi^{1/4}J^{1/2}2^{1/2}(i + e^{2\pi\epsilon}c) \cosh^{1/2}(2\pi\epsilon)} \]

\[ \times \frac{\Gamma(\frac{1}{4} + i\epsilon + c)}{\Gamma(\frac{1}{4} + i\epsilon + \frac{\pi}{2\pi T})}, \tag{5} \]

where \( \Gamma \) denotes Gamma function, and \( \epsilon \) is a parameter related to the conduction band filling. At small \( \epsilon_{c} \) limit, \( \epsilon \approx -\mu_{f}/(\pi^{1/4}/\sqrt{2}) \) and \( \epsilon_{c} \approx -\pi^{1/4} \cosh^{1/4}(2\pi\epsilon) \mu_{c}/g \). The form of self-energy indicates the quasiparticle does not exist, and the conduction electrons enter the IM regime. As the Green’s functions of both c and f fermions are local SYK-type \[^{34}\], the entropy density scales as \( S_{\text{IM}} \sim \frac{M}{4\pi T} \), where the first and second term come from c fermions and f fermions, respectively \[^{34}\].

Shear viscosity. — The shear viscosity is usually evaluated via the Kubo formula \( \eta = \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im}G_{R}^{xy,xy}(\omega,0) \), where \( G_{R}^{xy,xy} \) is the retarded Green’s function of xy component of the energy-momentum tensor, i.e.,

\[ iG_{R}^{xy,xy}(\omega,\mathbf{p}) = \int dt d\mathbf{x} e^{i(\omega t - \mathbf{p} \cdot \mathbf{x})} \theta(t) \langle [T_{xy}(t,\mathbf{x}), T_{xy}(0,0)] \rangle. \]

where \( \theta(t) \) denotes the step function such that \( \theta(t) = 1 \) for \( t \geq 0 \) and zero otherwise, and […] is commutator.

In the following, we consider the isotropic dispersion \( \epsilon_{k} = \frac{k^{2}}{2m} - \frac{n^{2}}{2} \) with \( -\Lambda/2 \leq \epsilon \leq \Lambda/2 \). Generalization to other dispersions is straightforward, and won’t change our results qualitatively. Note that the lattice constant has been taken to be 1, so momentum \( k \) becomes dimensionless, and we have the relations \( m \sim \frac{1}{t} \sim \frac{1}{\Lambda} \sim \nu(0) \), where \( \nu(0) \) denotes the density of states at Fermi level. For the isotropic dispersion, the density of state is a constant, \( \nu(0) \equiv \int k^{2} \delta(\epsilon - \epsilon_{k}) = \nu(0) \), \( \int k^{2} \equiv \int \frac{d^{2}k}{(2\pi)^{2}} \), irrespective of the energy. The tensor \( T_{xy} \) of c fermions is given by \( T_{xy}(p) = \sum k \Gamma_{0}(p,k) k \chi = H.c., \) where \( \chi_{k} = \int dx \chi_{k}^{*}(x) \chi_{k} \), and \( \Gamma_{0}(p,k) = (k_{x} + i)k_{y} + p_{z}^{2} \) for the isotropic dispersion.

As shown in Fig. 2 to the leading nontrivial order in large-N limit, the self-consistent equation for the full vertex \( \Gamma \) is

\[ \Gamma(p,q) = \Gamma_{0}(p,q) + \frac{1}{N} \sum_{i} \int q' F^{(i)}(p,q,q') \Gamma(p,q')(6) \]

where \( \int k = \int_{k_{y}} \int_{k_{x}} \int_{k_{z}} \equiv T_{\text{inc}} \sum_{\omega_{c}} \) and \( F^{(i)} \) is represented in the second and third diagram in Fig. 2 i.e.,

\[ F^{(1)} = -g^{2} \int G^{f}(q - q' + k)G^{f}(q' - q - k)G^{c}(q')G^{c}(p + q'). \]

Because we are interested in the uniform case, i.e., \( p = 0 \),

\[ \int q' F^{(1)}(0,p_{0};q,q') \Gamma(0,p_{0};q') = -g^{2} \int_{k_{0}} \int_{q_{0}} G^{f}(q - q' + k)G^{f}(q' - q - k) \]

\[ \times \int_{q'_{0}} G^{c}(q',q_{0})G^{c}(q',p_{0} + q_{0}) \Gamma(0,p_{0};q'), \tag{7} \]

Eq. 7 vanishes since it is odd in \( q'_{0} \) (or \( q'_{0} \)). Owing to the same reason, we find that \( F^{(2)} \) on the right-hand side in Fig. 2 also vanishes. Therefore, the vertex corrections vanish, \( \Gamma(0,p_{0};q) = \Gamma_{0}(0;p_{0};q) = 0 \). Thus, to leading order in \( 1/N \), the shear viscosity is given by the sum over the set of ladder diagrams shown in Fig. 3 and the spectral representation of shear viscosity is \[^{34}\text{[17]}\]

\[ \eta = \frac{M}{4\pi} \int_{-\infty}^{\infty} d\omega \left( -\frac{dF(\omega)}{d\omega} \right) \int_{-\infty}^{\infty} d\epsilon \Theta_{xy}(\epsilon) A^{c}(\omega,\epsilon)^{2}, \tag{8} \]

where \( n_{F}(\omega) = 1/(e^{\omega/M} + 1) \) is the Fermi-Dirac distribution, \( A^{c}(\omega,\epsilon) = -2\text{Im}G^{c}(i\omega_{n} \rightarrow \omega + i0^{+},\epsilon) \) denotes the spectral function, and \( \Theta_{xy}(\epsilon) = \int \frac{d^{2}k}{(2\pi)^{2}} \epsilon(k_{x}^{2} + k_{y}^{2})^{2} \delta(\epsilon - \epsilon_{k}) \) is the transport density of states for shear viscosity.

Shear viscosity in MFL regime. — In the MFL regime, the Fermi surface is well defined and the leading temperature-dependence contribution to viscosity comes from the states near Fermi surface, \( \epsilon = 0 \). This allows us to approximate \( \Theta_{xy}(\epsilon) \) by the value at Fermi surface,
The spectral function is independent of $\epsilon$, $A^{\text{IM}}_{\text{c}}(\omega, \epsilon) = A^{\text{IM}}_{\text{c}}(\omega)$. As a result, the shear viscosity splits into two independent integrations,

$$\eta^{\text{IM}} = \frac{M}{16\pi T} \int d\epsilon \Theta_{xy}(\epsilon) \int d\omega \text{sech}^2\left(\frac{\omega}{2T}\right) A^{\text{IM}}_{\text{c}}(\omega)^2,$$

both of which can be evaluated directly [27], and the final result is

$$\eta^{\text{IM}}(T) = \frac{M \pi^2}{24} \frac{\Lambda^2}{g^2 T} \cosh^2(2\pi \epsilon).$$

In the IM regime, the entropy density corresponds to $c$ fermions is given by $S^{\text{IM}}_{\text{c}} \sim M JT/\pi$, so the ratio between shear viscosity and entropy density is given by

$$\frac{\eta^{\text{IM}}}{S^{\text{IM}}_{\text{c}}} \sim \frac{\cosh^2(2\pi \epsilon) \Lambda^2}{\cos(2\pi \epsilon) T^2}.$$  

If $\Lambda \ll J$, there exists a robust temperature window in the IM regime, i.e., $\Lambda \ll T \ll \min(J, g^2/J)$, such that the KSS bound is strongly violated!

In fact, the scaling form of the shear viscosity obtained in the IM regime, $\eta \propto T^{-1}$, is a universal property for local critical systems. In local critical regime, the local interaction dominates over hoppings between different sites, and in turn dictates the scaling dimension of fermions. The most generic local interaction allowed by $U(1)$ symmetry is of quartic order. Thus, the local critical freedoms, i.e., the $c$ fermions in our case, have scaling dimension 1/4, and consequently the spectral weight $\Lambda \propto T^{-1/2}$. Furthermore, the local criticality also renders the vertex correction vanishing, and leads to the spectral representation of shear viscosity, as shown in Eq. (8). These reasons lead to the scaling form of shear viscosity $\eta \propto T^{-1}$. Note that though the scaling form is the same in the MFL regime, the origins behind them are different, i.e., the shear viscosity is determined by quasiparticle lifetime in the MFL as discussed before. The essential point for the violation of the KSS bound is that the scaling form in the IM regime can survive in an intermediate-temperature range, which lead to a robust energy window violating the bound, as indicated in Fig. 1(b).

**Conclusions and discussions.**—In this paper, we study the shear viscosity in a translationally invariant, strongly correlated solvable model [34][35]. By using Kubo formula, we get the interesting behaviors of shear viscosity as a function of temperature in the MFL regimes, which is related to the quasiparticle lifetime, and in the IM regimes, which is a general result from local criticality. As shown in Fig. 1(b), we further find a robust temperature range in the IM regime where the ratio of shear viscosity to entropy density, $\eta/S_c$, can strongly violate the KSS bound. To the best of our knowledge, it is for the first time that the perfect fluidity behaviors are discovered in the coupled local critical SYK models in an intermediate-temperature range.
Though a similar violation of the KSS bound is also reported in unitary quantum gases by dynamic mean field theory calculation [32], the SYK model has a better holographic interpretation [36, 39] and analytical controllability than the model used in Ref. [32]. Thus our calculations provide the first translationally invariant example violating the KSS bound with known gauge-gravity correspondence. Moreover, as indicated in Ref. [35, 50], we also expect that the model in this paper has a description of semi-holography: $f$ fermions form the bulk geometry while $c$ fermions live on the boundary. From this point of view, the $\eta/S_c$ we calculate here is different from the one calculated in those full-holographic models, where the entropy is black hole entropy. To compare our result with those full-holographic results, one should replace the $S_c$ in $\eta/S_c$ by the entropy density of the whole system consisted of both $f$ fermions and $c$ fermions. Since $S_f \propto N \gg S_c \propto M$, we have $\eta/S_f \propto M/N \rightarrow 0$, at the $M/N \ll 1$ limit. Here, the KSS bound is violated trivially, since the entropy density comes from an immobile contribution, $S_f$, with $U(1)$ symmetry at each site.

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Summing the relevant Feynman diagrams in the large-N limit, the saddle-point equations are given by

$$G_c(k, i\omega) = \frac{1}{i\omega_n - \epsilon_k + \mu_c - \Sigma_{cf}(k, i\omega_n)},$$  \hspace{1cm} (S1)

$$G_f(k, i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma_{cf}^f(k, i\omega_n) - \Sigma_f(k, i\omega_n)},$$  \hspace{1cm} (S2)

$$\Sigma_{cf}(k, i\omega_n) = -g^2 \int_{k'} G_c(k', i\omega_{n'}) \Pi_f(k + k', i\omega_n + i\omega_{n'}),$$  \hspace{1cm} (S3)

$$\Sigma_{cf}^f(k, i\omega_n) = -\frac{M}{N} g^2 \int_{k'} G_f(k', i\omega_{n'}) \Pi_c(k + k', i\omega_n + i\omega_{n'}),$$  \hspace{1cm} (S4)

$$\Sigma_f(k, i\omega_n) = -J^2 \int_{k'} G_f(k', i\omega_{n'}) \Pi_f(k + k', i\omega_n + i\omega_{n'}),$$  \hspace{1cm} (S5)

$$\Pi_f(q, i\Omega_n) = \int_k G^f(k, i\omega_n) G^f(q + k, i\Omega_n + i\omega_n),$$  \hspace{1cm} (S6)

$$\Pi_c(q, i\Omega_n) = \int_k G^c(k, i\omega_n) G^c(q + k, i\Omega_n + i\omega_n),$$  \hspace{1cm} (S7)

where $k$ and $\omega_n$ denote momentum and Matsubara frequency, $G^i$, $i = c, f$ refers to the Green’s function of $c$ and $f$ fermion, respectively, and $\int_k = \int_{k_0} \int_{k'} \int_{k_0} = T \sum_{\omega_n} \int_{k} = \int \frac{d^3k}{(2\pi)^3}$. It is easy to check from the saddle point equations that local critical $f$ fermion propagator, i.e., $G^f(k, i\omega_n) = G^f(i\omega_n)$, is always a consistent solution to the saddle point equations. Indeed, at the $M/N \to 0$ limit, the $f$ fermion propagator is

$$G^f(\tau) = -\frac{\pi^2}{J^2 \sqrt{1 + e^{-2\pi\xi}} \left( \frac{T}{\sin(\pi T\tau)} \right)^2} e^{-2\pi\xi T\tau}.$$

SUPPLEMENTAL MATERIAL

A. Saddle point solutions



where $E$ is a parameter controlling the particle-hole asymmetry, and $\tau \in [0, \beta]$ is the imaginary time. For finite $M/N$, a local critical form of $f$ fermion propagator is still consistent with the full saddle point equations. Moreover, according to Ref. [1, 3], finite $M/N$ correction is subleading. Thus, we assume the local critical solution holds at a small but finite $M/N$, and focus on the case $M/N \ll 0$.

Moving to the $c$ fermion propagators, we will follow Ref. [3] closely. The self-energy of $c$ fermion is given by Eq. (S3). Since $G^f$ is local critical, we can see from Eqs. (S3) and (S6) that $\Sigma_{cf}$ is also independent of momentum, i.e., $\Sigma_{cf}(k, \omega_n) = \Sigma_{cf}(\omega_n)$, and consequently $\Sigma_{cf}(\tau) = -g^2 G^c(\tau) G^f(-\tau)$, with $G^c(\tau) \equiv T \sum_{\omega_n} G^c(\omega_n)$ and $G^c(\omega_n) \equiv \int_k G^c(k, \omega_n)$. Then with the assumption $\text{sgn}(\text{Im}[\Sigma_{cf}(\omega_n)]) = -\text{sgn}(\omega_n)$, and in the limit of infinite bandwidth $\Lambda \to \infty$ (i.e., bandwidth is the largest energy scale), $G^c(\omega_n) \approx \nu(0) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega_n - \epsilon - \Sigma_{cf}(\omega_n)} = -\frac{1}{4} \nu(0) \text{sgn}(\omega_n)$, and $G^c(\tau) = -\frac{\nu(0) T}{2 \sin(\pi T \tau)}$, where $\nu(0)$ is the density of state at fermi level. The self-energy of the $c$ fermion yields [3]

$$\Sigma_{c}^{\text{MFL}}(\omega_n) = \frac{i g^2 T}{2 J \cosh^{1/2}(2 \pi E) \pi^{3/2}} \left( \frac{\omega_n}{T} \ln \left( \frac{2 \pi T \gamma_E - 1}{J} \right) + \frac{\omega_n}{T} \psi \left( -\frac{i \omega_n}{2 \pi T} + \pi \right) \right),$$

where $\psi$ is the digamma function, and $\gamma_E = 0.577$ is the Euler-Mascheroni constant. The self-energy indicate that in the large bandwidth limit, the $c$ fermions exhibit a MFL behavior.

On the other hand, in the limit where $|\omega_n + \mu_c - \Sigma^c(\omega_n)| \gg \Lambda$, one can find local critical solutions of SYK type for both $c$ and $f$ fermions [3] at conformal limit. Namely, the $f$ fermion propagator is still given by Eq. (S3), while the $c$ fermion will enter the IM regime, whose propagator reads [3]

$$G^c(\omega_n) \approx \frac{1}{2\pi(\mu_c - \Sigma_{cf}(\omega_n))},$$

where the self-energy is given by

$$\Sigma_{cf}(\omega_n) = \frac{i T \frac{g^2}{\pi^{3/2}} \nu_c^\frac{1}{4}(0) \left( -1 \right)^{\frac{1}{4}} (1 + e^{4 \pi E c}) \frac{e^{2 \pi E}}{2 \pi i J \frac{1}{4} \left( 2 \pi \right) \Gamma \left( \frac{3}{4} + i E_c + \frac{\omega_n}{2 \pi T} \right) \Gamma \left( \frac{3}{4} + i E_c + \frac{\omega_n}{2 \pi T} \right)},$$

where $E \simeq -\frac{\mu_f}{\pi^{3/4} \sqrt{2}}$ and $E_c \simeq -\frac{\pi^3}{8} \cosh^{1/4}(2 \pi E) \mu_c / g$ at small $\mu_f / J, \mu_c / g$ limit. Note Eq. (S10) is only valid provided $T \gg T_{\text{inc}}$ and $g^2 \gg \Lambda J$.

**B. The derivation of shear viscosity in terms of spectral function**

We prove that the shear viscosity defined via the Kubo formula

$$\eta = \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G_{xy,xy}^R(\omega, 0),$$

$$G_{xy,xy}^R(\omega, 0) = -i \int dt d\delta e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle,$$

is equivalent to [3] in terms of spectral functions.

The $xy$-component of the uniform energy-momentum tensor for $c$-fermions is given by

$$T_{xy} = \int \frac{d^2 k}{(2\pi)^2} \frac{k_x k_y}{m} c_k c_{-k}.$$

To obtain the retarded Green function, we first use the imaginary time formula. In the tree level, we have

$$G_{xy,xy}(i\Omega, 0) = -MT \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \left( \frac{k_x k_y}{m} \right)^2 G_c(i\omega_n, k) G_c(i\omega_n + i\Omega, k).$$

Using the spectral representation, $G(z) = \int \frac{d\omega}{2\pi} A^c(\omega) A^c(\omega + \Omega)$, one is able to sum over Matsubara frequencies and continue to real frequency

$$\text{Im} T \sum_{\omega_n} G(i\omega_n) G(i\omega_n + \Omega + i\delta) = -\frac{1}{2} \int \frac{d\omega'}{2\pi} A^c(\omega') A^c(\omega' + \Omega) [n_F(\omega') - n_F(\omega' + \Omega)].$$
We obtain the imaginary part of the retarded Green’s function
\[ \text{Im}G_{xy,xy}^{R}(\Omega,0) = \frac{M}{2} \int \frac{d^{2}k}{(2\pi)^{2}} \left( \frac{k_{x}k_{y}}{m} \right)^{2} \int \frac{d\omega}{2\pi} A^{c}(\omega,k)A^{c}(\omega+\Omega,n_{F}(\omega) - n_{F}(\omega + i\Omega)). \] (S16)

The shear viscosity is then given by
\[ \eta = \frac{M}{2} \int_{-\infty}^{\infty} d\omega \left( - \frac{\partial n_{F}}{\partial \omega} \right) \int_{-\infty}^{\infty} d\epsilon \Theta_{xy}(\epsilon)A^{c}(\omega,\epsilon)^{2}, \] (S17)
where \( \Theta_{xy}(\epsilon) \equiv \int \frac{d^{2}k}{(2\pi)^{2}} \left( \frac{k_{x}k_{y}}{m} \right)^{2} \delta(\epsilon - \epsilon_{k}) \).

C. Shear viscosity in marginal fermi liquid

In MFL regime, the well-defined fermi surface allows us to approximate the density of states \( \nu(\epsilon) \) at energy \( \epsilon \) by density of states at fermi surface \( \nu(0) \). Then we have
\[ \Theta_{xy}(\epsilon) = m^{2}v_{F}^{4} \int \frac{d^{2}k}{(2\pi)^{2}} \cos^{2} \theta \sin^{2} \theta \delta(\epsilon - \epsilon_{k}) \approx \frac{m^{2}v_{F}^{4}}{16\pi} \nu(0) \approx \frac{\nu(0)}{16\pi m^{2}}, \] (S18)
where in the last step, we use the relation \( v_{F} \sim \frac{1}{m} \) in the isotropic dispersion. The shear viscosity is given by
\[ \eta_{\text{MFL}} = \frac{M}{16\pi T} \int d\omega \text{sech}^{2}\left( \frac{\omega}{2T} \right) \int d\epsilon \Theta_{xy}(\epsilon)A^{c}_{\text{MFL}}(\omega,\epsilon)^{2} \] (S19)
\[ = \frac{M}{16\pi T} \frac{m^{2}v_{F}^{4}}{16\pi} \nu(0) \int d\omega \text{sech}^{2}\left( \frac{\omega}{2T} \right) \int d\epsilon A^{c}_{\text{MFL}}(\omega,\epsilon)^{2} \] (S20)
\[ = \frac{Mm^{2}v_{F}^{4}\nu(0)}{128\pi T} \int d\omega \frac{\text{sech}^{2}\left( \frac{\omega}{2T} \right)}{[2m\Sigma_{c}^{\text{MFL}}(\omega)]} \] (S21)
\[ \approx 0.030062gT \frac{M^{2}J}{g^{2}T} \frac{\cosh^{2}(2\pi \epsilon)}{2}, \] (S22)
where in the last line, we have used the relation \( v_{F} \sim \frac{1}{m} \sim \frac{1}{\nu(0)} \sim t \) in the isotropic dispersion.

D. Shear viscosity in incoherent metal

For the dispersion relation \( \epsilon_{k} = \frac{k^{2}}{2m} - \frac{A}{2} \) with bandwidth \( \epsilon_{k} \in [-\frac{A}{2}, \frac{A}{2}] \), we have
\[ \Theta_{xy}(\epsilon) = \int \frac{d^{2}k}{(2\pi)^{2}} \left( \frac{k_{x}k_{y}}{m} \right)^{2} \delta(\epsilon - \epsilon_{k}) \] (S23)
\[ = \frac{1}{(2\pi m)^{2}} \int d\theta \cos^{2} \theta \sin^{2} \theta \int dk \epsilon_{k} \delta(\epsilon - \epsilon_{k}) \] (S24)
\[ = \frac{m}{4\pi} \left( \epsilon + \frac{A}{2} \right)^{2} \theta \left( \frac{A}{2} - |\epsilon| \right), \] (S25)
where \( \theta(x) \) is the unit step function. One can also find \( \Theta_{xy} \) using Fourier transform \[4, 5\], which exactly gives the same result. The spectral function of \( c \) fermion in IM region is given by \[3\]
\[ A^{c}(\omega,\epsilon) = A^{c}(\omega) = -29\Re \left[ \frac{e^{i \frac{\pi}{4} J^{1/4} 1/2 \cosh^{1/4}(2\pi \epsilon)(i + e^{2\pi \epsilon}) \Gamma(1/2 - i \frac{\omega - 2\pi \epsilon}{4\pi})}{g^{T} 1/2 \sqrt{1 + e^{4\pi \epsilon}}} \right] \] (S26)
which is independent of \( \epsilon \) as a result of local criticality. Then the shear viscosity is given by
\[ \eta = \frac{M}{16\pi T} \int d\epsilon \Theta_{xy}(\epsilon) \int d\omega \text{sech}^{2}\left( \frac{\omega}{2T} \right) A^{c}(\omega)^{2} = \frac{M^{2}J}{16\pi T} \frac{8\pi^{5/2} J \cosh^{1/2}(2\pi \epsilon)}{g^{2} \cosh(2\pi \epsilon)} = \frac{M^{2}J}{24} \frac{\Lambda^{2} J \cosh^{1/2}(2\pi \epsilon)}{g^{2}T \cosh(2\pi \epsilon)} \] (S27)
where we have used \( \int d\Theta_{xy}(\epsilon) = \frac{\Lambda^2}{12\pi} \), and

\[
\int d\omega \text{sech}^2\left(\frac{\beta \omega}{2}\right)A^r(\omega)^2 = \frac{16\pi^{5/2}J \cosh^{1/2}(2\pi \mathcal{E})}{g^2 T} \int d\omega \frac{1}{2 \cosh(2\pi \mathcal{E}_c)} \frac{\text{sech}(\beta \omega - 2\pi \mathcal{E}_c)}{\Gamma \left( \frac{3}{4} + i \frac{\beta \omega - 2\pi \mathcal{E}_c}{2\pi} \right) \Gamma \left( \frac{3}{4} - i \frac{\beta \omega - 2\pi \mathcal{E}_c}{2\pi} \right)}^2 \tag{S28}
\]

\[
= \frac{8\pi^{5/2}J \cosh^{1/2}(2\pi \mathcal{E})}{g^2 \cosh(2\pi \mathcal{E}_c)} \int dx \frac{\text{sech}(x)}{\Gamma \left( \frac{3}{4} + i \frac{x}{2\pi} \right) \Gamma \left( \frac{3}{4} - i \frac{x}{2\pi} \right)}^2 = \frac{8\pi^{5/2}J \cosh^{1/2}(2\pi \mathcal{E})}{g^2 \cosh(2\pi \mathcal{E}_c)}. \tag{S29}
\]

### E. Thermal diffusion constant

We calculate the thermal diffusion coefficient in both regimes by using the results given in Ref. [3]. The thermal diffusivity can be given by Einstein’s relation

\[ D = \frac{\kappa_0}{c_V}, \tag{S30} \]

where \( \kappa_0 \) is the ‘closed-circuit’ thermal conductivity and \( c_V \) is the specific heat.

In MFL regime, from Ref. [3], we have \( \kappa_0^{MFL} \sim MJ^2/g^2 \) and \( c_V^{MFL} \sim M(g^2/\ell^2)(T/J) \ln(J/T) \), where we have set \( \mathcal{E} = 0 \) in the following calculations. The thermal diffusion constant scales as

\[ D^{MFL} \sim \frac{J^{2+4}}{g^4 T \ln(\frac{J}{T})}. \tag{S31} \]

Note that as \( T \to 0 \), the thermal diffusion constant becomes divergent same as the shear viscosity. Since \( T \ll T_{inc} \), we conclude that \( D^{MFL} \gg \frac{L^2}{g^2} \frac{1}{\ln(g/\ell)} \).

Similarly, in the IM regime, one has \( \kappa_0^{IM} \sim MJ^2/g^2 \) and \( c_V^{IM} \sim MJT/g^2 \) Ref. [3]. The thermal diffusion constant scales as

\[ D^{IM} \sim \frac{\pi^{5/2}A^2}{64 T}. \tag{S32} \]

Due to the IM existing only at temperature above \( T_{inc} \), we always have \( D^{IM} \ll \frac{\pi^{5/2}g^2}{64 J^{2+4}} \). In the MFL regime, the thermal diffusion has a \( 1/T \) dependence due to local criticality. It was argued that the fast ‘Planckian’ dissipation together with the causality of diffusion results in an upper bound of diffusivity [6]. The results found in this work strongly implies that the shear viscosity and the upper bound of diffusivity maybe deeply connected.

### F. Relation to the DC conductivity

In MFL regime, similar to case of shear viscosity, the inverse lifetime Eq. (11) also gives rise to the \( T^{-1} \) dependence of DC conductivity. From [3]

\[ \sigma_{DC}^{MFL} \sim \frac{M}{m \gamma} \sim \frac{MJ^2}{T g^2} \tag{S33} \]

From uncertainty principle, the metallic conductivity in 2D is bounded below by the Mott-Ioffe-Regel (MIR) limit, \( \sigma = \frac{e^2}{m \ell} \geq \frac{1}{\ell} \), where \( \ell \) is the electronic mean free path and the charge unit is omitted. The conductivity obtained here can be lower than the MIR limit \( 1/h \) numerically by tuning parameters, although the MFL is not rigorously a bad metal.

In the IM regime, the DC conductivity

\[ \sigma_{DC}^{IM} \sim \frac{MA^2 J \cosh^{1/2}(2\pi \mathcal{E})}{g^4 T \cosh(2\pi \mathcal{E}_c)} \tag{S34} \]

shares the same scaling form with the shear viscosity in Eq. (15). It is not surprising. Firstly, because of local criticality, the spectral density is independent of momentum. Secondly, the vertex of shear viscosity and conductivity has the same scaling, which is \( 1/m \sim t \). The combination of above two features completely determine the scaling form.
Both of shear viscosity and DC conductivity vanish when $T \gg T_{inc}$ due to the same scaling forms in Eqs. [15] and [S34]. To reach $T \gg T_{inc}$, one can consider the decouple limit $t \to 0$ while keeping other couplings and temperature fixed, which agrees with the fact that transport coefficients die out. Furthermore, the entropy $S_c$ contributed by $c$-fermion keep fixed under the decouple limit, which is equal to the entropy of the SYK model with $J_{IM} = g^2/J$. From this point of view, the violation of the KSS bound of $\eta/S_c$ here shares the same reason with the deviation from the MIR limit of $\sigma_{DC}$ in the incoherent metal regime.

These two bounds can be understood from the inverse lifetime for the $c$ fermions Eq. (11). In the MFL regime with temperature $T \ll T_{inc}$ and $t \gg g$, $J \gg T$, the $c$ fermions’ lifetime behaves as $\tau_h \sim \frac{T_{inc} t T}{\g^2} \gg \frac{1}{t}$. However, in the IM regime, due to local criticality, the universal ‘Planckian’ time $\tau_h \sim 1/T$ give the temperature dependence of transport coefficients.

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