Ashtekar variables, self-dual metrics, and \( w_\infty \)

Viqar Husain*
International Center for Theoretical Physics
P. O. Box 586, 34100 Trieste, Italy

Abstract

The self-duality equations for the Riemann tensor are studied using the Ashtekar Hamiltonian formulation for general relativity. These equations may be written as dynamical equations for three divergence free vector fields on a three dimensional surface in the spacetime.

A simplified form of these equations, describing metrics with a one Killing field symmetry are written down, and it shown that a particular sector of these equations has a Hamiltonian form where the Hamiltonian is an arbitrary function on a two-surface. In particular, any element of the \( w_\infty \) algebra may be chosen as a the Hamiltonian.

For a special choice of this Hamiltonian, an infinite set of solutions of the self-duality equations are given. These solutions are parametrized by elements of the \( w_\infty \) algebra, which in turn leads to an explicit form of four dimensional complex self-dual metrics that are in one to one correspondence with elements of this algebra.

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* email: viqar@itsictp.bitnet. Address after 30th September 1992: Dept. of Physics, University of Alberta, Edmonton, Canada T6G 2J1.
Introduction

Self-duality plays an important role in four dimensional field theories that involve two-forms. In Yang-Mills theories the solutions to the Euclidean space field equations with self-dual curvature, the ‘instantons’, are solutions that give a finite action. Because of this, these field configurations play an important role in the evaluation of the path integrals that involve Yang-Mills fields. In the past few years these solutions have played an essential role in mathematics as well, in the question of constructing invariants of four manifolds. It has been shown that certain invariants may be calculated from path integrals of ‘topological’ field theories, theories that are diffeomorphism invariant and have no local degrees of freedom [1].

In general relativity, one role played by self duality is that it provides a way of investigating a subset of the space of solutions to the Einstein equations. A substantial amount of work has been done in this area and there are several interesting results. It has been shown that for asymptotically flat spacetimes, the scattering problem for self-dual initial data is trivial [2], that is, the evolved data at future null infinity is the same as the ingoing selfdual data at past null infinity. Penrose has proposed that the ‘non-linear gravitons’ [3], metrics with self-dual and anti-self-dual Weyl curvature, should play the essential role in quantum gravity as the fundamental quanta, as opposed to the linear spin two particles. A purely classical result of this work is that the general solution of the self-dual equations may be written down. This solution is however in twistor space and the general solutions, as spacetime metrics, are not available. Neither is there a Hamiltonian formulation for this sector of the Einstein equations, and the integrability according to Liouville, which involves constructing an infinite set of commuting conserved quantities, hasn’t been demonstrated.

In attempts at quantization of general relativity, it now appears that self-duality may again play an essential role [4]. The Hamiltonian formulation of the Einstein equations due to Ashtekar makes use of the self-dual part of the spin-connection. Its projection onto spacelike surfaces provides a coordinate on the phase space of general relativity, and a spatial (densitized) triad is the conjugate momentum. These variables have proven to be quite useful in pursuing the Dirac quantization procedure [5,6].

Some connections between two dimensional conformal field theories and the self-dual Einstein equations have been discussed recently [7,8]. An interesting result of this work is that the self-duality equations for a class of metrics are the same as the field equations for a particular two dimensional conformal field theory, the continuum limit of the Toda theory. Using this connection it was demonstrated in [7] that the self-dual Einstein equations for this class of metrics have an associated infinite dimensional symmetry algebra, the algebra of area preserving diffeomorphisms of a two dimensional surface [9], a subset of which is the $w_\infty$ algebra. This algebra is the generalization of the Virasoro algebra that includes all higher spins $2,3,...\infty$. Below, this result appears in a direct way where it is shown that a sector of the self-duality equations has a Hamiltonian form where the Hamiltonian is an arbitrary function on a two dimensional surface, and furthermore, that metrics parametrized by elements of the $w_\infty$ algebra may be explicitly written down.

For the purposes of this paper, the Ashtekar hamiltonian variables, being connected with self-duality, provide a simple way to write down a first order form of the (anti)self-dual Einstein equations [10]. It has been shown that this form of the self-dual Einstein equations may be derived from the self-dual Yang-Mills equations [11], and they also appear as half of the Hamilton equations in the weak coupling ($G \to 0$) limit of the full Einstein equations (written in the Ashtekar formulation) [12].

The contents of this paper are as follows: in the next section, the derivation of the self-dual Einstein equations from the Ashtekar variables is reviewed. In section 3 these equations are simplified via a one Killing field reduction, and it is shown that a sector of the resulting equations has a natural Hamiltonian formulation where the Hamiltonian is an arbitrary function on a two-surface. The Hamiltonians may be chosen as elements of the $w_\infty$ algebra, and for a particular such choice, the Hamilton equations are integrated explicitly. The initial data used for this integration has a $w_\infty$ algebra associated with it, and the resulting metrics are therefore in one to one correspondence with elements of this algebra. The metrics are given explicitly. The last section is a discussion of the results regarding connections with two-dimesional physics and possibilities for constructing more general solutions, as well as quantization.

The self-dual equations
This section is a review of the derivation of the self-dual Einstein equations from the Ashtekar Hamiltonian variables \([10]\).

The Einstein equations for real metrics on a real Euclidean manifold may be derived from a Palatini type action where the field variables are vierbeins \(e^a_i\), and the spin-connection \(\omega^a_{ij}\). The \(\mu, \nu = 0, ..., 3\) are spacetime indices and \(i, j, ... = 0, ..., 3\) are internal \(so(4)\) indices. The lagrangian density is the 4-form \(\dot{e}^{ijkl}(e^i \wedge e^j \wedge R^{kl}[\omega])\), where \(R^{kl}\) is the curvature 2-form of \(\omega^{ij}\). The Hamiltonian 3+1 decomposition of this action leads essentially to the usual form of the phase space and constraints \([13]\).

An alternative lagrangian density \([14]\) that also leads to the Einstein equations is formed by replacing the curvature of \(\omega^{ij}\) by the curvature of its (anti)self-dual part with respect to the internal \(so(4)\) indices. The fundamental Poisson bracket is

\[
S = \int_M \epsilon_{ijkl} \, e^i \wedge e^j \wedge - F^{kl}[-A]
\]

where \(-F^{ij}\) is the curvature of \(-A^{ij}\). This replacement does not affect the Einstein equations because \(-F^{ij} = \frac{1}{2}(R^{ij} - \frac{1}{2} \epsilon^{ij}_{\ k}\ R^{kl}[\omega])\), and the variation of the second term as a function of the vierbein is zero. (Recall that \(R^{ij}[\omega] = R^{ij}[{-A} + \frac{\partial}{\partial A}] \equiv - F^{ij}[{-A}] + \frac{\partial F^{ij}}{\partial A}\)). It is the Hamiltonian decomposition of the action (0) that leads to the Ashtekar variables. (Note that one can use either the self-dual or the anti-self-dual part of the curvature, (or equivalently, the connection), to construct the Lagrangian. We are using the anti-self-dual part in order to get equations for self-dual metrics, as will become clear below).

To find the Hamiltonian versions of such actions on (Euclidean) 4-manifolds \(M\), one assumes the spacetime is of topology \(\Sigma \times \mathbb{R}\). \(\mathbb{R}\) is the ‘time’ direction, which is fixed by introducing a vector field \(t^\mu\) on \(M\) such that \(t^\mu = N n^\mu\), where \(n^\mu\) are unit normals to the ‘spacelike’ surfaces \(\Sigma\) and \(N\) is the lapse function. Each surface carries a ‘time’ label \(t\). The function \(t\) is determined by \(\mathcal{L}_t = 1\), where \(\mathcal{L}_t\) denotes the Lie derivative with respect to \(t^\mu\). The \(n^\mu\) may be used determine the metric on \(\Sigma\), \(g\), from the spacetime metric \(g : g_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu\) (with \(g_{\mu\nu}\) determined by the vierbeins). This ‘spatial’ metric may be used to project tensors on \(M\) to tensors on \(\Sigma\).

For the action (0), the phase space coordinate is the spatial projection of \(-A^{ij}_\mu\), \(A^i_a\), which is now valued in \(so(3)\) \((i = 1, ..., 3)\), and the conjugate momentum is the densitized dreibein \(E^a_i\). \((a, b, ... = 1, ..., 3\) denote spatial indices). The fundamental Poisson bracket is

\[
\{A^i_a(x), E^{bj}_a(y)\} = \delta^i_j \delta^b_a \delta^3(x - y)
\]

(For details of the Hamiltonian decomposition see \([5,6,13,14]\)). The first class constraints on the phase space corresponding to the invariances of the action are

\[
\partial_\lambda E^{ai} + \epsilon^{ijk} A^a_j E^{ak} = 0 \tag{1}
\]

\[
F_{ab} E^{bi} = 0 \tag{2}
\]

\[
H \equiv \epsilon^{ijk} E^{ai} E^{bj} F_{ab} = 0 \tag{3}
\]

where \(F_{ab} = \partial_{(a} A^i_{b)(j} + \epsilon^{ijk} A^i_a A^k_b\). (1) is the Gauss law constraint generating triad rotations and gauge tranformations, (2) generates spatial diffeomorphisms, and (3) generates time reparametrizations or ‘evolution’ of the initial data on the spacelike surfaces. The evolution equations are the Hamiltonian equations for the phase space variables with respect to an arbitrary ‘lapse’ density \(\mathcal{N}\), which is a density of weight \(-1\):

\[
\mathcal{H}(\mathcal{N}) = \int_\Sigma N \mathcal{H}
\]

and

\[
\dot{E}^{ai} = \{E^{ai}, H(\mathcal{N})\}_{PB} = \epsilon^{ijk} D_b(\mathcal{N} E^{aj} E^{bk}) \tag{4}
\]

and

\[
\dot{A}^i_a = \{A^i_a, H(\mathcal{N})\}_{PB} = \mathcal{N} \epsilon^{ijk} E^{bj} F_{ab} \tag{5}
\]
where $D_a$ is the covariant derivative for the connection $A^i_a$. Note that $\mathcal{N}$ is related to the lapse function $N$ by $N = (\text{det} q)^{1/2} \mathcal{N}$. The equations (4-5), together with the constraints (1-3), are equivalent to the full empty space Einstein equations without a cosmological constant.

We now consider the restrictions of these that will yield (Euclidean) spacetime metrics $g_{\mu\nu}$ with self-dual Riemann tensor: $R_{\mu\nu\rho\sigma} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} R_{\rho\sigma}$. That this is equivalent to the condition $R_{\mu\nu}^i = \frac{1}{2} \varepsilon_{ijkl} R^{ijl}$ is easily verified (using $R_{\mu\nu\rho\sigma} = R_{\mu\nu}^{ij} e_{i\alpha} e_{j\beta}$). The self duality condition is (by definition) equivalent to the condition $-F_{\mu\nu} [-\mathcal{A}] = 0$. Now, since the spatial projection of this $-F$ is determined by the spatial metrics $q_{\mu\nu} = q^a_{\mu} q^b_{\nu} ( - F_{\alpha\beta} )$, this implies that the curvature of the phase space coordinate $A^i_a$, $F_{ab}^i [\mathcal{A}]$, is zero if $- F_{\alpha\beta} = 0$. Thus, $F_{ab}^i = 0$ is the phase space condition corresponding to the self-duality of the spacetime Riemann curvature. One can now work in a gauge where $A^i_a$ itself is zero. Either of these conditions immediately solve the constraints (2) and (3). Also, if $A^i_a$ is chosen to be zero on the initial data surface, it remains zero under the evolution equations (5). The remaining equations, with this condition imposed are

$$\partial_a E^{a\alpha} = 0 \quad (6)$$

$$\dot{E}^{ai} = \varepsilon^{ijk} \partial_k (\mathcal{N} E^{a\alpha} E^{bj}) \quad (7)$$

These equations may be further simplified my making a specific choice for the arbitrary lapse density $\mathcal{N}$. This corresponds to fixing ‘time’ gauge. Choosing $\mathcal{N}$ to be a constant ($\dot{\mathcal{N}} = \mathcal{L}_t \mathcal{N} = 0; \partial_a \mathcal{N} = 0$) and defining the vector fields $V_i^a = \mathcal{N} E_i^a$ (recall that $\mathcal{N}$ and $E_i^a$ are densities of weight $-1$ and $+1$ respectively), we can rewrite these remaining equations as equations for the three vector fields $V_i^a$

$$\partial_a V_i^a = 0 \quad (8)$$

$$\dot{V}_i^a = \varepsilon^{ijk} [V_j, V_k]^a \quad (9)$$

where $[\ ,\ ]$ denotes the Lie bracket of the vector fields. Equations (8-9) are the new form of the self-dual Einstein equations. The spacetime metric is constructed from a solution of these equations via

$$g^{ab} = D^{-1} \mathcal{N} (V_i^a V^b_i + t^a t^b) \quad (10)$$

where $D = \text{det} V = \varepsilon^{ijk} \epsilon_{abc} V_i^a V_j^b V_k^c$.

Although real so(4) connections and vierbeins were used in the steps leading to equations (8-9), which give real Euclidean metrics, the complex metric case may be discussed similarly. The same final equations result. Therefore, complex metrics with self-dual Riemann curvature may be constructed by starting with a triad of complex vector fields $V_i^a$ satisfying (8-9).

**The self-dual metrics**

In this section, a one Killing field reduction of the self-duality equations (8) and (9) is described. It is shown that a sector of these equations has a Hamiltonian form where the Hamiltonians are arbitrary functions on a two-surface. For a specific Hamiltonian, an infinite class of solutions to the Hamilton equations are given. These solutions may be associated with elements of the $\mathfrak{w}_\infty$ algebra and the infinite set of metrics determined by them are presented.

The spatial surface $\Sigma$ has a fixed volume element $\tilde{\Omega}_{abc}$. This is because the divergence free condition (8) is with respect to a fixed non dynamical volume form. We fix a flat coordinate system $(x, y, \theta)$ on some neighbourhood of $\Sigma$. We would like to construct a self-dual metric that has a one Killing vector field symmetry with respect to the vector field $\omega^a = (\partial/\partial \theta)^a$. $\Sigma$ will then have the topology $R \times \Sigma^2$ for arbitrary two surfaces $\Sigma^2$, on which the coordinates are $(x, y)$. (It may be possible to compactify the orbits of the Killing field to $S^1$ after specific metrics are determined). A two dimensional volume form $\Omega$ on $\Sigma^2$ can be fixed by

$$\Omega_{ab} = \tilde{\Omega}_{abc} \frac{\partial}{\partial \theta} \epsilon^c \quad (11)$$
Locally, the coordinates \((x, y)\) may be chosen such that \(\Omega_{xy} = -\Omega_{yx} = 1\). This volume form can be used to define Poisson brackets for functions on (a local neighbourhood) of \(\Sigma^2\) via \(\{f, g\} = \Omega^{ab} \partial_a f \partial_b g\). These brackets will be used below.

In order to satisfy the Killing symmetry condition \(L_u g_{ab} = 0\), we must have by (10) that \([u, V^a_i] = 0 = [u, t]^a\) for all \(i\). An ansatz for \(V^a_i\) that satisfies this condition and solves the divergence free conditions (8) is

\[
V^a_i = \Omega^{ab} \partial_b \Lambda_i + \Pi_i (\frac{\partial}{\partial \theta})^a
\]  

where \(\Lambda_i\) and \(\Pi_i\) are at this stage six arbitrary real functions of \((x, y, t)\). Substituting these into the dynamical equations (9) gives in a straightforward way the equations for the \(s\) functions.

\[
\dot{\Lambda}_i = \varepsilon_{ijk} \{\Lambda_k, \Lambda_j\}
\]

and

\[
\dot{\Pi}_i = 2\varepsilon_{ijk} \{\Pi_k, \Lambda_j\}
\]

We now consider a particular set of solutions to these equations. This may be viewed as an ansatz to solve (13-14). Set \(\Lambda_1 = \Lambda(x, y, t)\), \(\Lambda_2 = -i\Lambda(x, y, t)\), and \(\alpha = \Pi_1 + i\Pi_2\). The first two of these imply that \(\Lambda_3\) is arbitrary and time independent, and equations (13-14) become

\[
\dot{\Lambda} = \{\Lambda, H\}
\]

\[
\dot{\alpha} = \{\alpha, H\}
\]

\[
\dot{\Pi}_3 = 2\varepsilon_{ijk} \{\alpha^* \Lambda, \Lambda\}
\]

\[
\{\Pi_3, \Lambda\} = 0
\]

where * denotes complex conjugation. (Note that the choice of \(\Lambda_2\) makes the vector fields (12) complex and so the metrics (10) will also be complex). (15a) and (15b) are Hamiltonian evolution equations for \(\Lambda\) and \(\alpha\) with arbitrary Hamiltonian \(H \equiv 2t\Lambda_3\), and (15c)-(15d) determine \(\Pi_3\) via an evolution and a ‘constraint’ equation from solutions of the first two. The constraint equation results from the consistency condition that the evolution equations for \(\alpha\) and \(\alpha^*\) be complex conjugates of each other.

There is a natural way to write down a series of Hamiltonians \(H\) (or choices of the functions \(\Lambda_3\)), that form a \(w_\infty\) algebra. The functions \(w^s_n(x, y) = x^{n+s-1} y^{s-1}\) form this algebra via Poisson brackets with respect to (11):

\[
\{w^s_m, w^t_n\} = ((t-1)m - (s-1)n)w^{s+t-2}_{m+n}
\]

where \(m, n \in \mathbb{Z}\) and \(s, t\) are integers \(\geq 2\). This algebra contains the Virasoro algebra which arises for \(s = t = 2\)

\[
\{w^2_m, w^2_n\} = (m-n)w^2_{m+n}.
\]

The \(w^s_n\) also satisfy the relation

\[
\{w^2_m, w^s_n\} = ((s-1)m-n)w^s_{m+n}
\]

In order to find some explicit solutions to (15), we consider the specific Hamiltonian \(H = iw_0^2 = ixy\) (\(\Lambda_3 = u_0^2/2\)) and the initial condition \(\Lambda(x, y, t = 0) = w^0_m = x^{m+s-1} y^{s-1}\). Then it is easy to verify using (18) that some solutions to (15a) are

\[
\Lambda^s_m(x, y, t) = \Lambda_0 e^{int} w^s_m(x, y)
\]

where \(\Lambda_0\) is an arbitrary constant. These solutions have a natural associated \(w_\infty\) symmetry namely

\[
\{\Lambda^s_m, \Lambda^t_n\} = ((t-1)m - (s-1)n)\Lambda^{s+t-2}_{m+n}
\]
which is just the (built in) symmetry associated with the initial data \( \Lambda(x, y, t = 0) \). Note also that the Hamiltonian used to obtain these solutions is in fact just \( L_0(\equiv w_0^2) \) of the Virasoro algebra (17) (where \( L_n \equiv w_n^2 \)).

Given these solutions to (15a), one still needs to solve (15b)-(15d) to determine metrics explicitly. One easy solution is to set the \( \Pi_i \) equal to three constants \( C_i \). With this the vector fields (12) become

\[
V_1^a = \Lambda_0 e^{int} \Omega^{ab} \partial_b w_m^s + C_1 \left( \frac{\partial}{\partial \theta} \right)^a
\]

\[
V_2^a = -i \Lambda_0 e^{int} \Omega^{ab} \partial_b w_m^s + C_2 \left( \frac{\partial}{\partial \theta} \right)^a
\]

\[
V_3^a = \frac{1}{2} \Omega^{ab} \partial_b w_0^2 + C_3 \left( \frac{\partial}{\partial \theta} \right)^a
\]

(20)

The metric corresponding to this solution obtained from (10) is

\[
ds^2 = -2i(Ax + By)^{-1}\left[ \frac{1}{4}(Ax + By)^2 dt^2 + \left( \frac{Cy^2}{4} - A^2 \right) dx^2 + \left( \frac{Cx^2}{4} - B^2 \right) dy^2 - \frac{1}{2}(Ax + By) dxd \theta - \frac{1}{2}(Ax^2 + Bxy) dyd \theta \right]\]

(21)

where

\[
A = -\Lambda_0 e^{int}(C_1 - iC_2) \partial_x w_m^s - \frac{C_3 y}{2} \quad B = \Lambda_0 e^{int}(C_1 - iC_2) \partial_y w_m^s + \frac{C_3 x}{2} \quad C = C_1^2 + C_2^2 + C_3^2
\]

We note that by letting \( s = 2 \) in (21) we get a smaller set of metrics parametrized by elements of the Virasoro sub-algebra (17) of \( w_\infty \).

To find a more general solution to (15b)-(15d), we can write a a solution similar to (19) for \( \alpha \): \( \alpha^s_n = \alpha_0 e^{int} w_n^{s'} \), where at this stage there is no relation between \( n, s' \) here and the \( m, s \) of (19). With this \( \alpha \) we have from (15c) that \( \Pi_3 = 2i \{ w_n^s, w_m^s \} e^{i(m-n)t} \), and from (15d) that \( \{ \Pi_3, \Lambda \} = \Lambda_0 e^{int} \{ \Pi_3, w_m^s \} = 0 \). The first of these gives the \( (x, y) \) dependence of \( \Pi_3 \) to be proportional to \( w_m^{s+s'-2} \), and the second gives a relation between \( s, s' \) and \( m \) given by

\[
\{ w_m^{s+s'-2}, w_m^s \} = [(s - 1)(m + n) - (s' + s - 3)m] w_{2m+n}^{2s+s'-4} = 0
\]

This is satisfied by \( s' = s + 1 \) and \( m = n \). Thus, the solutions to (15b)-(15d) consistent with (19) are

\[
\alpha = \alpha_0 e^{int} w_m^{s+1} \quad \Pi_3 = -2i \alpha_0 \Lambda_0 w_m^{2s-1} t + k.
\]

(22)

where \( k \) is an arbitrary constant. The more general metrics obtained from this solution are obtained from (21) by replacing the constants \( C_i \) by the functions \( C_1 = \alpha_0 \cos nt w_m^{s+1}, C_2 = \alpha_0 \sin nt w_m^{s+1} \), and \( C_3 = \Pi_3 \).

To summarize, the main result of this section are the metrics (21) labelled by elements \( w_n^s \) of the algebra (16), which are obtained by using \( L_0 \equiv w_0^2 \) as the Hamiltonian in equations (15). This result displays rather explicitly the previously discussed [7] \( w_\infty \) infinite spin symmetries associated with the self-duality condition of the Riemann curvature.

**Discussion**

Using the Ashtekar variables, we have given an infinite set of metrics with self-dual Riemann curvature, where the metrics are parametrized by elements of the \( w_\infty \) algebra. This is however not an exhaustive set of solutions for the one-Killing field reduction that is used. This is because we have considered only one class of initial data for the evolution equations and only one specific ‘Hamiltonian’, \( w_0^2 \), in equations (15).
may choose other Hamiltonians and study these equations further and it would be of interest to see what other choices give a rich class of solutions.

There are a number of known metrics with self-dual Riemann curvature [15]. If the metrics are real and geodesically complete without singularities, they are called gravitational instantons. Examples are the Eguchi-Hanson and Taub-NUT metrics [15]. Such metrics are determined by starting with an appropriate ansatz in some local coordinates, (such as spherical coordinates on $R^4$ for the Eguchi-Hanson case), and then determining the manifold globally by suitably eliminating any curvature singularities, or by determining maximal extensions in appropriate coordinates, or both. In this regard it would be interesting to study the global structure of the metrics given here to see if they have real sections that are free of singularities.

There is an ansatz more general than (12) which involves adding terms corresponding to elements of the cohomology group, $H^1(\Sigma^2)$, if for example, one starts with $\Sigma^2 = R^2$ with some punctures. The terms to be added to (12) would be of the form $g_r(t)\Omega^{ab}\omega_b^{(r)}$ for elements $\omega^{(r)} (r = 1, \ldots, n)$ of $H^1(\Sigma^2)$ and arbitrary functions $g_r(t)$. More generally, one may try an ansatz using triads with specific symmetries and try to fix the spatial topologies from the start. In such cases also it would be relevant to consider the contributions discussed in this paragraph. Fixing the spatial topology by a suitable choice of triads may, however, lead to a specific metric rather than a large class of metrics (as happens for example, with the Eguchi-Hanson ansatz). It may still be useful to pursue this case since the $w_{\infty}$ generators may be expanded in a basis geared to the appropriate topology of the 2-surfaces, and one may get a similar structure associated with solutions of (15) for more general topologies.

The reduction of the Einstein equations discussed here, involving self-duality, a Killing symmetry and the ansatz that gives the Hamiltonian form (15) may be viewed as a particular midi-superspace model. Since there is an infinite dimensional symmetry algebra on the solution space, it may be possible to describe a quantization in terms of representations of $w_{\infty}$.

This work was motivated by the desire to find a general solution to the unreduced self-dual equations (8)-(9), and to construct explicitly the infinite number of associated conserved quantities. This would, if it can be done, perhaps provide a connection in terms of metrics, with the general twistor space solution to the self-dual equations [3].

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