Sharp Hardy inequalities in an exterior of a ball

Nikolai Kutev\textsuperscript{*} Tsviatko Rangelov \textsuperscript{*}

Abstract

New Hardy type inequalities in sectorial area and as a limit in an exterior of a ball are proved. Sharpness of the inequalities is shown as well.

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1 Introduction

The classical Hardy inequality, proved in Hardy [7, 8] states

\[
\int_{0}^{\infty} |u'(x)|^p x^\alpha dx \geq \left(\frac{p - 1 - \alpha}{p}\right)^p \int_{0}^{\infty} x^{-p+\alpha} |u(x)|^p dx
\]

where \(1 < p < \infty\), \(\alpha < p - 1\) and \(u(x)\) is absolutely continuous on \([0, \infty)\), \(u(0) = 0\).

There are several generalizations of (1) in multidimensional case mostly in bounded domains, see Davies [4], Ghoussoub and Moradifam [6], Balinsky et al. [3], Kutev and Rangelov [9] and the literature therein. For unbounded domains, in the exterior of a ball, only few generalizations of (1) are reported in Wang and Zhu [10], Adimurthi et al. [1], Avkhadiev and Laptev [2].

For example, in Theorem 1.1 in Wang and Zhu [10] for all function \(u \in D^{1,2}_a(R^n)\), where \(D^{1,2}_a(R^n)\) is the weighted Sobolev space - the completion of \(C^\infty_0(R^n)\) with the norm \(\int_{\mathbb{R}^n} |x|^{-2a} |u|^2 dx\), the following inequality for \(a < \frac{n-2}{2}\) is proved

\[
\int_{B^c_1} |x|^{-2a} |\nabla u|^2 dx \geq \left(\frac{n-2-2a}{2}\right)^2 \int_{B^c_1} \frac{|u|^2}{|x|^{2(a+1)}} dx + \frac{n-2-2a}{2} \int_{\partial B^c_1} u^2,
\]

where \(B^c_1 = \{|x| > 1\}\).

Let us mention that in Wang and Zhu [10] Hardy inequalities with weights in unbounded domains \(\Omega \subset \mathbb{R}^n, 0 \notin \partial\Omega\) are also considered, see Theorem 1.3 and Remark 1.5.

In Adimurthi et al. [1], Corollary, for \(n \geq 3\) and \(u \in C^\infty_0(B^c_1)\) the limiting case of the Caffarelli–Kohn–Nirenberg inequality

\[
\int_{B^c_1} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{B^c_1} \frac{u^2}{|x|^{n}} dx + C_n(a) \left[ \int_{B^c_1} X_1^{2(n-1)} \left( a, \frac{1}{|x|} \right) u^{\frac{n}{n-2}} dx \right] \frac{n-2}{n},
\]

\textsuperscript{*}Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113, Sofia, Bulgaria

Corresponding author: T. Rangelov, rangelov@math.bas.bg
is proved, where \( X_1(a, s) = a - \ln s \)^{-1}, \( a > 0, 0 < s \leq 1 \) and the constant \( C_n(a) \) is given explicitly.

Finally, in Corollary 1 and Remark 1 in Avkhadiev and Laptev [2] the following Hardy inequality is proved for \( n \geq 3 \) and \( u \in W_{1,2}^0(B^c_r) \)

\[
\int_{B^c_r} |\nabla u|^2 dx \geq \left( \frac{n-2}{2} \right)^2 \int_{B^c_r} \frac{u^2}{|x|^2} dx + \frac{1}{4} \int_{B^c_r} \left( \frac{1}{|x-r|^2} - \frac{1}{|x|^2} \right) u^2 dx.
\]  \( \text{(4)} \)

At the end of the paper we compare the inequalities (2), (3), (4) with our results.

The aim of the present work is to derive new Hardy inequalities in the exterior of a ball. There are listed also functions for which these inequalities are sharp, i.e., inequalities with an optimal constant of the leading term become equations.

\section{Inequalities in sectorial area}

We start with Hardy inequalities in sectorial area \( B_R \setminus B_r \) where \( B_R, B_r \) are balls centered at zero, \( 0 < r < R \). Let \( 1 < p, p' = \frac{p}{p-1}, 2 \leq n \) and denote \( m = \frac{p-n}{p-1} = \frac{p-n}{p} p' \).

For functions \( u \) such that \( \int_{B_R \setminus B_r} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p dx < \infty \) let us define two sets

\[
M_1(r, R) = \begin{cases} \frac{R^m - \hat{R}^m}{m} & \frac{1}{p-1} \int_{\partial B_{\hat{R}}} |u|^p d\sigma \to 0, \ \hat{R} \to R - 0, \ m \neq 0, \\
\ln \frac{R}{\hat{R}} & \frac{1}{1-n} \int_{\partial B_{\hat{R}}} |u|^n d\sigma \to 0, \ \hat{R} \to R - 0, \ m = 0 \end{cases}
\]

\[
M_2(r, R) = \begin{cases} \frac{\hat{r}^m - r^m}{m} & \frac{1}{p-1} \int_{\partial B_r} |u|^p d\sigma \to 0, \ \hat{r} \to r + 0, \ m \neq 0, \\
\ln \frac{\hat{r}}{r} & \frac{1}{1-n} \int_{\partial B_r} |u|^n d\sigma \to 0, \ \hat{r} \to r + 0, \ m = 0 \end{cases}
\]

where \( \langle , \rangle \) is a scalar product in \( \mathbb{R}^n \).

Let functions \( \psi_j \) be solutions of the problems:

\[
-\Delta_p \psi_1 = 0, \ \text{in} \ B_R \setminus B_r, \ \psi_1|_{\partial B_R} = 0, \ \psi_1|_{\partial B_r} = 1,
\]

\[
-\Delta_p \psi_2 = 0, \ \text{in} \ B_R \setminus B_r, \ \psi_2|_{\partial B_R} = 1, \ \psi_2|_{\partial B_r} = 0.
\]

Their explicit form is:

\[
\psi_1(x) = \begin{cases} \frac{R^m - |x|^m}{R^m - r^m}, & m \neq 0, \\
\ln \frac{R}{|x|} & m = 0 \end{cases}, \ \psi_2(x) = \begin{cases} \frac{|x|^m - r^m}{R^m - r^m}, & m \neq 0, \\
\ln \frac{|x|}{r} & m = 0 \end{cases}.
\]
We can define vector functions \( f_i \) as \( f_i = \frac{\nabla \psi_i}{\psi_i} \) in \( B_R \setminus \hat{B}_r \) and

\[
L^i(u) = \int_{B_R \setminus B_r} |\frac{\nabla \psi_i}{\psi_i} \nabla u|_1^p dx = \int_{B_R \setminus B_r} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p dx,
\]

\[
K^i(u) = \int_{B_R \setminus B_r} \left| \frac{\nabla \psi_i}{\psi_i} \right|^p dx = \int_{B_R \setminus B_r} |f_i|^p |u|^p dx,
\]

\[
K^i_0(u) = \int_{\partial(B_R \setminus B_r)} \left| \frac{\nabla \psi_i}{\psi_i} \right|^{p-2} \langle \frac{\nabla \psi_i}{\psi_i}, \nu \rangle |u|^p ds = \int_{\partial(B_R \setminus B_r)} \langle f_i, \nu \rangle |u|^p ds.
\]

Here \( \nu \) is the outward normal to \( B_R \setminus B_r \).

The following Theorem takes place.

**Theorem 2.1.** For every \( u \in M_1(r, R) \) we have

\[
L^i(u) \geq \left( \frac{1}{p} \right)^p \frac{K^i_0(u) + (p-1)K^i_{\hat{b}}(u)^p}{(K^i_{\hat{b}}(u))^{p-1}} = K^i(u).
\]

where \( \nu \) is the outward normal to \( B_R \setminus B_r \).

**Proof.** We follow the proof of Proposition 1 in Fabricant et al. [5]. Since

\[
\int_{B_R \setminus B_r} \langle f_i, \nabla |u|^p \rangle dx = p \int_{B_R \setminus B_r} |u|^{p-2} u \langle f_i, \nabla u \rangle dx,
\]

where \( r < \hat{r} < \hat{R} < R \). Then applying Hölder inequality on the rhs of (7) with \( \frac{\langle x, \nabla u \rangle}{|x|} \) and \( |f_i||u|^{p-2}u \) as factors of the integrand we get

\[
\int_{B_R \setminus B_r} \langle f_i, \nabla |u|^p \rangle dx \leq p \left( \int_{B_R \setminus B_r} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p dx \right)^{1/p} \times \left( \int_{B_R \setminus B_r} |f_i|^p |u|^p dx \right)^{p-1}.
\]

Rising to \( p \) power both sides of (8) it follows that

\[
\int_{B_R \setminus B_r} \left| \frac{\langle x, \nabla u \rangle}{|x|} \right|^p dx \geq \frac{1}{p} \left( \int_{B_R \setminus B_r} \langle f_i, \nabla |u|^p \rangle dx \right)^p \left( \int_{B_R \setminus B_r} |f_i|^p |u|^p dx \right)^{p-1}.
\]

Integrating by parts the numerator of the right hand side of (9) we get

\[
\frac{1}{p} \int_{B_R \setminus B_r} \langle f_i, \nabla |u|^p \rangle dx = \frac{1}{p} \int_{\partial B_R \cup \partial B_r} \langle f_i, \nu \rangle |u|^p dS - \frac{1}{p} \int_{B_R \setminus B_r} \text{div} f_i |u|^p dx
\]

\[
= \frac{1}{p} \int_{\partial B_R \cup \partial B_r} \langle f_i, \nu \rangle |u|^p dS + \left( \frac{p-1}{p} \right) \int_{B_R \setminus B_r} |f|^p |u|^p dx
\]

\[
\rightarrow \frac{1}{p} ((p-1)K^i + K^i_0), \text{ when } \hat{R} \to R - 0, \hat{r} \to r + 0.
\]
Note that \( \int_{\partial B_R \setminus \partial B_r} \langle f_i, \nu \rangle |u|^p dS \geq 0 \) for \( u \in M_i(r, R) \), since \( \nu |_{\partial B_R} = \frac{x}{|x|} |_{\partial B_r} \), \( \nu |_{\partial B_r} = -\frac{x}{|x|} |_{\partial B_r} \). From (9) and (10) we obtain (6) since

\(- \text{div} f_i = (p-1)|f_i|^p'.\)

\[ \square \]

3 Inequalities in an exterior of a ball \( B_r^c = R^n \setminus \overline{B_r} \)

Let us introduce functions \( u \) such that

\[ \int_{B^c_r} \left| \frac{< x, \nabla u >}{|x|} \right|^p dx < \infty, \quad \int_{B^c_r} \frac{|u|^p}{|x|^{(n-1)p'}} dx < \infty \]

and define two sets

\[ M_1(r, \infty) = \begin{cases} R^{1-n} \int_{\partial B_r} |u|^p d\sigma \to 0, & R \to \infty, \quad m \geq 0, \\ R^{1-p} \int_{\partial B_r} |u|^p d\sigma \to 0, & R \to \infty, \quad m < 0. \end{cases} \]

\[ M_2(r, \infty) = \begin{cases} \frac{r^m - r^n}{m}^{1-p} \int_{\partial B_r} |u|^p d\sigma \to 0, & r \to r + 0, \quad m \neq 0, \\ \left( \ln \frac{r}{r'} \right)^{1-p} \int_{\partial B_r} |u|^p d\sigma \to 0, & r \to r + 0, \quad m = 0. \end{cases} \]

In a similar way we can prove Theorem 2.1, replacing \( B_R \setminus \overline{B_r} \) with \( B^c_r = R^n \setminus \overline{B_r} \) and \( \partial (B_R \setminus \overline{B_r}) \) with \( \partial B^c_r = \partial B_r \) and \( L^1(u), K^i_0(u), K^i_1(u) \) define in (5) for \( R \to \infty \). The inequalities below for functions of \( M_i(r, \infty), i = 1, 2 \) can be obtained with a limit \( R \to \infty \).

**Proposition 3.1.** For every \( u \in M_1(r, \infty) \) the following inequalities hold:

(i) \( \left( \int_{B^c_r} \frac{|u|^p}{|x|^{(n-1)p'}} dx \right)^\frac{1}{p'} \left( \int_{B^c_r} \frac{|< x, \nabla u >|^p}{|x|} dx \right)^\frac{1}{p} \geq \frac{1}{p} r^{1-n} \int_{\partial B_r} |u|^p dS, \quad \text{for} \quad m > 0. \)

With function \( u_\alpha(x) = e^{-\alpha|x|^m}, \alpha > 0 \) inequality (11) becomes equality.

(ii) For \( m < 0 \) we get:

\[ \left( \int_{B^c_r} \frac{|< x, \nabla u >|^p}{|x|} dx \right)^\frac{1}{p} \geq \left( \int_{B^c_r} \frac{|u|^p}{|x|^p} dx \right)^\frac{1}{p} \geq \frac{|m|}{p'} \left( \int_{B^c_r} \frac{|u|^p}{|x|^p} dx \right)^\frac{1}{p'}, \quad \text{for} \quad m < 0. \]

With function \( u_k(x) = |x|^{km}, \quad k > p' \) inequality (12) becomes an equality.
(iii) 
\[ \left( \int_{B_r} |u|^n dx \right)^{\frac{1}{n}} \leq \left( \int_{B_r} \frac{\langle x, \nabla u \rangle}{|x|} \right)^{\frac{1}{n}} \] 
\[ \geq \frac{1}{n} r^{1-n} \int_{\partial B_r} |u|^n dS, \text{ for } m = 0. \]

With function \( u_q(x) = |x|^q, q < 0 \) inequality (13) becomes equality.

Proof. For \( m \neq 0 \) the inequality (6) has the form
\[ \left( \int_{B_R \setminus B_r} \frac{\langle x, \nabla u \rangle^p}{|x|^p} dx \right)^{\frac{1}{p}} \geq \frac{1}{p'} \left( \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} R^{m-|x|^m}} dx \right)^{\frac{1}{p}}. \]
\[ + \frac{1}{p} r^{1-n} \int_{\partial B_r} |u|^p dS \left( \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} R^{m-|x|^m}} dx \right)^{-\frac{1}{p'}}. \]

Analogously, for \( m = 0 \), i.e. \( p = n \) the inequality (6) becomes
\[ \left( \int_{B_R \setminus B_r} \frac{\langle x, \nabla u \rangle^n}{|x|^n} dx \right) \geq \frac{n-1}{n} \left( \int_{B_R \setminus B_r} \frac{|u|^n}{|x|^n \ln \frac{R}{|x|}} dx \right)^{\frac{1}{n}}. \]
\[ + \frac{1}{n} \left( r \ln \frac{R}{r} \right)^{1-n} \int_{\partial B_r} |u|^p dS \left( \int_{B_R \setminus B_r} \frac{|u|^n}{|x|^n \ln \frac{R}{|x|}} dx \right)^{\frac{1-n}{n}}. \]

(i) For \( m > 0 \), after the limit \( R \to \infty \) in (14) we obtain
\[ \frac{1}{p'} \left( \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} R^{m-|x|^m}} dx \right)^{\frac{1}{p'}} \to R \to \infty 0 \text{ and} \]
\[ \frac{1}{p} r^{1-n} \int_{\partial B_r} |u|^p dS \left( \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} R^{m-|x|^m}} dx \right)^{-\frac{1}{p'}} \]
\[ \to R \to \infty \frac{1}{p} r^{1-n} \int_{\partial B_r} |u|^p dS \left( \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'}} dx \right)^{-\frac{1}{p'}}. \]

and hence (11) holds.

Let us check that with the function \( u_\alpha(x) = e^{-\alpha |x|^n}, \alpha > 0 \) inequality (11) becomes an
equality. Simply computation gives us
\[
I_{1\alpha} = \left( \int_{B_r^c} e^{-\alpha|x|^m} \frac{dx}{|x|} \right)^{\frac{1}{p}} = \left( \int_{S_1} \int_r^\infty e^{-\alpha\rho^{m-1}} \rho^{-(n-1)p'} \rho^{-1} \rho d\rho dS \right)^{\frac{1}{p}}
\]
\[
= |S_1|^{\frac{1}{p'}} \left( \frac{1}{m} \int_r^\infty e^{-\alpha\rho^{m-1}} \rho^m \right)^{\frac{1}{p'}} = |S_1|^{\frac{1}{p'}} \frac{1}{(\alpha m)^{\frac{1}{p}}} e^{-\alpha(p-1)m},
\]
\[
I_{2\alpha} = \left( \int_{B_r^c} \left| \frac{x}{|x|} \right|^p dx \right)^{\frac{1}{p}} = \left( \int_{S_1} \int_r^\infty e^{-\alpha\rho^{m-1}} (\alpha m)^{\frac{1}{p}} \rho^{-(n-1)p'} \rho^{-1} \rho d\rho dS \right)^{\frac{1}{p}}
\]
\[
= |S_1|^{\frac{1}{p}} \left( \frac{(\alpha m)^{\frac{1}{p}}}{m \alpha p} \int_r^\infty e^{-\alpha\rho^{m-1}} \rho^m \right)^{\frac{1}{p}} = |S_1|^{\frac{1}{p}} \frac{\alpha m}{(\alpha m)^{\frac{1}{p}}} e^{-\alpha \rho^{-m}},
\]
\[
I_{3\alpha} = \frac{1}{p} \int_{\partial B_r} |u_{\alpha}|^p dS = \frac{1}{p} \int_{S_1} e^{-\alpha \rho^{-m}} \rho^{-n} dS = \left( \frac{1}{p} \right) e^{-\alpha \rho^{-m}},
\]
and we get the equality \( I_{1\alpha} I_{2\alpha} = I_{3\alpha} \).

(ii) For \( m < 0 \), after the limit \( R \to \infty \) in (14) we obtain (12). Moreover, inequality (12) becomes equality. Indeed,
\[
I_{1k} = \left( \int_{B_r^c} \left| \frac{x}{|x|} \right|^p dx \right)^{\frac{1}{p}} = \left( \int_{B_r^c} km|x|^{km-1} |x|^{p} dx \right)^{\frac{1}{p}}
\]
\[
= |km| \left( |S_1| \int_r^\infty \rho^{(km-1)p} \rho^{n-1} d\rho \right)^{\frac{1}{p}} = |km| \left( |S_1| \frac{\rho^{(km-1)p+n}}{|(km-1)p+n|} \right)^{\frac{1}{p}},
\]
\[
I_{2k} = \frac{m}{p'} \left( \int_{B_r^c} |u_k|^p \frac{dx}{|x|} \right)^{\frac{1}{p}} = \frac{m}{p'} \left( |S_1| \int_r^\infty \rho^{(km-1)p+n-1} d\rho \right)^{\frac{1}{p}}
\]
\[
= \frac{m}{p'} \left( |S_1| \frac{\rho^{(km-1)p+n}}{|(km-1)p+n|} \right)^{\frac{1}{p}},
\]
\[
I_{3k} = \frac{1}{p} \int_{\partial B_r} |u_k|^p dS \left( \int_{B_r^c} \frac{|u_k|^p}{|x|^p} dx \right)^{-\frac{1}{p'}}
\]
\[
= \frac{1}{p} \int_{\partial B_r} |u_k|^p dS \left( \frac{1}{|S_1| \rho^{(km-1)p+n-1}} \right)^{-\frac{1}{p'}} \left( |S_1| \frac{\rho^{(km-1)p+n}}{|(km-1)p+n|} \right)^{-\frac{1}{p'}}
\]
\[
= \frac{1}{p} \left( |S_1| \frac{\rho^{(km-1)p+n}}{|(km-1)p+n|} \right)^{\frac{1}{p'}} \left( |(km-1)p+n| \right)^{\frac{1}{p'}}.
\]
Since
\[
\frac{|km|}{|(km-1)p+n|} = \frac{m}{p'} \frac{1}{|km-1)p+n|} + \frac{1}{p}
\]
we get the equality $I_{1k} = I_{2k} + I_{3k}$.

(iii) For $m = 0$, after the limit $R \to \infty$ in (15), since

\[
\frac{n-1}{n} \left( \int_{B_R \setminus B_r} \frac{|u|^n}{|x^n| \ln R/|x|} \, dx \right)^{\frac{1}{n}} \to_{R \to \infty} 0 \quad \text{and} \quad \frac{1}{n} \left( r \ln \frac{R}{r} \right)^{1-n} \int_{\partial B_r} |u|^n dS \left( \int_{B_R \setminus B_r} \frac{|u|^n}{|x^n| \ln R/|x|} \, dx \right)^{\frac{1}{n}}
\]

\[
= \frac{1}{n} r^{1-n} \ln^{1-n} R \left( 1 - \frac{\ln r}{\ln R} \right)^{1-n} \int_{\partial B_r} |u|^n dS \frac{1}{\ln^{1-n} R} \left( \int_{B_R \setminus B_r} \frac{|u|^n}{|x^n|} \left[ 1 - \frac{\ln |x|}{mR} \right] \, dx \right)^{\frac{1}{n}}
\]

\[
\to_{R \to \infty} \frac{1}{n} r^{1-n} \int_{\partial B_r} |u|^n dS \left( \int_{B_r^c} \frac{|u|^n}{|x^n|} \, dx \right)^{\frac{1}{n}}
\]

we obtain (13). Let us check that with function $u_q(x) = |x|^q, q < 0$ inequality (13) becomes equality. Indeed,

\[
I_{1q} = \left( \int_{B_r^c} \frac{|u_q|^n}{|x^n|} \, dx \right)^{\frac{n-1}{n}} = \left( |S_1| \int_{r}^{\infty} \rho^{q-1} \rho^{n-1} \, d\rho \right)^{\frac{n-1}{n}} = |S_1|^{\frac{n-1}{n}} (nq)^{-\frac{n-1}{n}} r^{(n-1)q},
\]

\[
I_{2q} = \left( \int_{B_r^c} \frac{<x, \nabla u_q>}'n}{|x|} \, dx \right)^{\frac{1}{n}} = \left( |S_1| \int_{r}^{\infty} q^n \rho^{q-1} \rho^{n-1} \, d\rho \right)^{\frac{1}{n}} = |S_1|^{\frac{1}{n}} |q|^{\frac{n-1}{n}} n^{-\frac{1}{n}} r^q,
\]

\[
I_{3q} = \frac{1}{n} r^{1-n} \int_{\partial B_r} |u_q|^n dS = \frac{1}{n} r^{1-n} |S_1| r^{nq} r^{n-1} = |S_1| \frac{1}{n} r^m,
\]

and we get the equality $I_{1q} I_{2q} = I_{3q}$.

**Proposition 3.2.** For every $u \in M_2(r, \infty)$ the following inequalities hold:

(i) 

\[
\left( \int_{B_r^c} \frac{<x, \nabla u>^p}{|x|} \, dx \right)^{\frac{1}{p}} \geq \frac{m}{p'} \left( \int_{B_r^c} \frac{|u|^p}{|x|^{(n-1)p'} |r^m - |x|^m|^p} \, dx \right)^{\frac{1}{p'}}
\]

\[
+ \frac{1}{p} \limsup_{R \to \infty} \left[ \int_{\partial B_R} |u|^p dS \left( \int_{B_r^c} \frac{|u|^p}{|x|^{(n-1)p'} |x^m - r^m|^p} \, dx \right)^{-\frac{1}{p'}} \right]^{\frac{1}{p}}
\]

for $m > 0$.

(16)
With function $u_\varepsilon(x) = |x|^{-\frac{m(1-\varepsilon)}{p'}} (|x|^m - r^m)^{\frac{(1+\varepsilon)}{p'}}$, $0 < \varepsilon < 1$ inequality (16) is $\varepsilon$-sharp, i.e.

$$(m/p')^p \leq \frac{L^1(u_\varepsilon)}{[\left(K^1(u_\varepsilon)\right)^{\frac{1}{p'}} + K^1_0(u_\varepsilon) (K^1(u_\varepsilon))^{-\frac{1}{p'}}]^p} \leq \frac{L^1(u_\varepsilon)}{K^1(u_\varepsilon)} \leq \left(\frac{m}{p'}\right)^p (1+\varepsilon)^p. \quad (17)$$

(ii)

$$\left(\int_{B^\varepsilon} \frac{|x, \nabla u|^p}{|x|} \, dx\right)^{\frac{1}{p}} \geq \frac{|m|}{p'} \left(\int_{B^\varepsilon} \frac{|u|^p}{|x|^{(n-1)p'} |r^m - |x|^m|^p} \, dx\right)^{\frac{1}{p'}}$$

$$+ \frac{r^{n-p}}{p} \limsup_{R \to \infty} \left[ K^{1-n} \int_{\partial B^R} |u|^p \, dS \right] \left(\int_{B^\varepsilon} \frac{|u|^p}{|x|^{(n-1)p'} |r^m - |x|^m|^p} \, dx\right)^{\frac{1}{p'}}$$

for $m < 0$.

With function $u_s(x) = (r^m - |x|^m)^s$, $s > \frac{1}{p'}$ inequality (18) becomes equality.

(iii)

$$\left(\int_{B^\varepsilon} \frac{|x, \nabla u|^n}{|x|^n} \, dx\right)^{\frac{1}{n}} \geq \frac{n-1}{n} \left(\int_{B^\varepsilon} \frac{|u|^n}{|x|^n \ln \frac{|x|}{r}} \, dx\right)^{\frac{1}{n}}$$

$$+ \frac{1}{n} \limsup_{R \to \infty} \left[ (R \ln \frac{R}{r})^{1-n} \int_{\partial B^R} |u|^n \, dS \right] \left(\int_{B^\varepsilon} \frac{|u|^n}{|x|^n \ln \frac{|x|}{r}} \, dx\right)^{\frac{1-n}{n}}$$

for $m = 0$.

For function

$$u_\eta(x) = \begin{cases} 
\left(\ln \frac{|x|}{r}\right)^{\frac{n-1}{n}(1+\frac{\eta}{2})} M^{\frac{n-1}{n}(1-\frac{\eta}{2})}, & r < |x| < M \\
\left(\ln \frac{|x|}{r}\right)^{\frac{n-1}{n}(1+\frac{\eta}{2})} |x|^{\frac{n-1}{n}(1-\frac{\eta}{2})}, & M < |x|
\end{cases},$$

where $0 < \eta < 1$, inequality (19) is $\eta$-sharp, i.e.

$$\left(\frac{n-1}{n}\right)^n \leq \frac{L^1(u_\eta)}{K^1(u_\eta)} \leq \left(\frac{n-1}{n}\right)^n (1+\eta)^n. \quad (20)$$
Proof. For \( m \neq 0 \) inequality (6) has the form

\[
\left( \int_{B_R \setminus B_r} \frac{|x, \nabla u>|^p}{|x|^p} \right)^{\frac{1}{p}} \geq \frac{|m|}{p'} \left( \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} |r^m - |x|^m|^p} \right)^{\frac{1}{p'}}
\]

\[ + \frac{1}{p} R^{1-n} |R^m - r^m|^{1-p} \int_{\partial B_R} |u|^p dS \left( \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} |r^m - |x|^m|^p} \right)^{-\frac{1}{p'}}. \tag{21} \]

while for \( m = 0 \) (6) becomes

\[
\left( \int_{B_R \setminus B_r} \frac{|x, \nabla u>|^n}{|x|^n} \right)^{\frac{1}{n}} \geq \frac{n-1}{n} \left( \int_{B_R \setminus B_r} \frac{|u|^n}{|x|^n |\ln \frac{|x|}{r}|} \right)^{\frac{1}{n}}
\]

\[ + \frac{1}{n} \left( R \ln R \right)^{1-n} \int_{\partial B_r} |u|^n dS \left( \int_{B_R \setminus B_r} \frac{|u|^n}{|x|^n |\ln \frac{|x|}{r}|} \right)^{-\frac{1}{n}}. \tag{22} \]

(i) If \( m > 0 \) after the limit \( R \to \infty \) in (21) we obtain

\[
\frac{1}{p} \limsup_{R \to \infty} \left[ R^{1-n} |R^m - r^m|^{1-p} \int_{\partial B_R} |u|^p dS \right]
\]

\[
= \frac{1}{p} \limsup_{R \to \infty} \left[ R^{1-n} R^{m-p} \left| 1 - \frac{r^m}{R^m} \right|^{1-p} \int_{\partial B_R} |u|^p dS \right]
\]

\[
= \frac{1}{p} \limsup_{R \to \infty} R^{1-p} \int_{\partial B_R} |u|^p dS
\]

which proves (16).

For the function \( u_\varepsilon(x) = |x|^{-\frac{m(1-\varepsilon)}{p'}} (|x|^m - r^m)^{\frac{(1+\varepsilon)}{p'}} \), \( 0 < \varepsilon < 1 \), \( u_\varepsilon(x) \in M_2(r, \infty) \)
simple computation give us

\[ I_1^\varepsilon = L^1(u_\varepsilon) = \int_{B_\varepsilon^c} \left| \frac{\nabla u_\varepsilon}{|x|} \right|^p dx \]

\[ = \left( \frac{m}{p'} \right)^p \int_{B_\varepsilon^c} |x|^{-m(1-\varepsilon)(p-1)-p} \left( |x|^m - r_m^{1+\varepsilon(p-1)-p} \right) \]

\[ \times [ (1+\varepsilon)|x|^m - (1-\varepsilon) \left( |x|^m - r_m^p \right) ]^p dx \]

\[ = \left( \frac{m}{p'} \right)^p |S_1| \int_r^\infty (\rho^m - r_m^{1+\varepsilon(p-1)-p}) \rho^{-m(1-\varepsilon)(p-1)-p+n-1} d\rho \]

\[ \leq \left( \frac{m}{p'} \right)^p |S_1| (1+\varepsilon)^p \int_r^\infty (\rho^m - r_m^{1+\varepsilon(p-1)-p}) \rho^{-m(1-\varepsilon)(p-1)-p+n-1+mp} d\rho \]

\[ = \left( \frac{m}{p'} \right)^p |S_1| (1-\varepsilon)^p \int_r^\infty (\rho^m - r_m^{1+\varepsilon(p-1)-p}) \rho^{-m(1-\varepsilon)(p-1)-\frac{n-1}{p-1}} d\rho, \]

because \( n-1-p+mp = -\frac{n-1}{p-1} \).

\[ I_2^\varepsilon = K^1(u_\varepsilon) = \int_{B_\varepsilon^c} |x|^{-m(1-\varepsilon)(p-1)-(n-1)p'} \left( |x|^m - r_m^{1+\varepsilon(p-1)-p} \right) dx \]

\[ = |S_1| \int_r^\infty (\rho^m - r_m^{1+\varepsilon(p-1)-p}) \rho^{-m(1-\varepsilon)(p-1)-\frac{n-1}{p-1}} d\rho. \]

Thus (17) follows immediately from expressions of \( I_1^\varepsilon \) and \( I_2^\varepsilon \).

(ii) When \( m < 0 \), after the limit \( R \to \infty \) in (21) we get

\[ \frac{1}{p} \limsup_{R \to \infty} \left[ R^{1-n} |R^m - r_m|^{1-p} \int_{\partial B_R} |u|^{p} dS \right] \]

\[ = \frac{1}{p} \limsup_{R \to \infty} \left[ \rho^{n-p} R^{1-n} \left( \frac{R^m}{r^m} - 1 \right)^{1-p} \int_{\partial B_R} |u|^{p} dS \right] \]

\[ = \frac{r^{n-p}}{p} \limsup_{R \to \infty} R^{1-n} \int_{\partial B_R} |u|^p dS \]

which proves (18).
For the function \( u_s(x) = (r^m - |x|^m)^s \), \( s > \frac{1}{p'} \) we have the identities

\[
I_{1s} = \left( \int_{B_r} \left| \frac{x \cdot \nabla u_s}{|x|} \right|^p dx \right)^{\frac{1}{p}} = (s^p |m| |S_1| \int_r^\infty \rho^{p(m-1)n-1} (r^m - \rho^m)^{(s-1)p} d\rho)^{\frac{1}{p}}
\]

\[
= s|m||S_1|^{\frac{1}{p}} \left( -\frac{1}{m} \int_r^\infty (r^m - \rho^m)^{(s-1)p} d(r^m - \rho^m) \right)^{\frac{1}{p}}
\]

\[
= s|m|^{\frac{1}{p'}} |S_1|^{\frac{1}{p'}} \frac{\omega_{(s-1)p+1}}{((s-1)p+1)^{\frac{1}{p'}}} |(s-1)p + 1|^{-\frac{1}{p'}}
\]

because \( p(m-1) + n-1 = m-1 \).

\[
I_{2s} = \int_{B_r} \frac{|u_s|^p}{|x|^{(n-1)p'}} |r^m - |x|^m|^p dx = |S_1| \int_r^\infty \rho^{(n-1)(1-p')} |r^m - \rho^m|^{(s-1)p} d\rho
\]

\[
= |S_1| r^{m(s-1)p+1} / |m| (s-1)p + 1.
\]

\[
I_{3s} = \frac{r^{n-p}}{p} \lim \sup_{R \to \infty} \left[ R^{1-n} \int_{\partial B_R} |r^m - R^m|^p dS \right]
\]

\[
= \frac{r^{n-p}}{p} \lim \sup_{R \to \infty} |R^m - r^m|^p |S_1| = \frac{|S_1|}{p^{n-p+m}}.
\]

Simple computation gives us the equality

\[
\frac{m}{p'} (I_{2s})^{\frac{1}{p}} + I_{3s} (I_{2s})^{-\frac{1}{p'}}
\]

\[
= \frac{|S_1| |m|^{\frac{1}{p'}} \frac{\omega_{(s-1)p+1}}{((s-1)p+1)^{\frac{1}{p'}}}}{p'(s-1)p + 1} + \frac{|S_1| r^{n-p+m} |S_1|}{p^{\frac{1}{p'}}} \left( \frac{|S_1|}{|m|} \right)^{-\frac{1}{p'}} \left[ \frac{r^{m(s-1)p+1}}{(s-1)p + 1} \right]^{-\frac{1}{p'}}
\]

\[
= \frac{1}{p'} |m|^{\frac{1}{p'}} |S_1|^{\frac{1}{p'}} \frac{\omega_{(s-1)p+1}}{((s-1)p+1)^{\frac{1}{p'}}} \left( \frac{p-1}{((s-1)p+1)^{\frac{1}{p'}}} + [(s-1)p + 1]^{-\frac{1}{p'}} \right)
\]

\[
= s |m|^{\frac{1}{p'}} |S_1|^{\frac{1}{p'}} \frac{\omega_{(s-1)p+1}}{((s-1)p+1)^{\frac{1}{p'}}} |(s-1)p + 1|^{-\frac{1}{p'}} = I_{1s}
\]

because

\[
n - p + m - \frac{m}{p'} [(s-1)p + 1] = \frac{m}{p'} [(s-1)p + 1]
\]

(iii) After the limit \( R \to \infty \) in (22) we get (19).

The function \( u_\eta(x) \) belongs to \( M_2(r, \infty) \) for \( m = 0 \). Moreover, for \( r < |x| < M \) we have the equalities

\[
I_{1\eta} = \int M^1(u_\eta) = M^{(n-1)(1-\eta)} \left( 1 + \frac{\eta}{2} \right) |S_1| \int_r^M \left( \ln \frac{\rho}{r} \right)^{(n-1)(1+\frac{\eta}{2})-n} \rho^{-1} d\rho,
\]

\[
I_{2\eta} = K^1(u_\eta) = M^{(n-1)(1+\frac{\eta}{2})} |S_1| \int_r^M \left( \ln \frac{\rho}{r} \right)^{(n-1)(1+\frac{\eta}{2})-n} \rho^{-1} d\rho.
\]
and hence
\[
\frac{L^1(u_\eta)}{K^1(u_\eta)} = \left(\frac{n-1}{n}\right)^n \left(1 + \frac{\eta}{2}\right)^n \leq \left(\frac{n-1}{n}\right)^n (1 + \eta)^n.
\]

Tedious calculations give us for \(|x| \geq M\) the identities
\[
I_{\eta_1} = L^1(u_\eta) = \left(\frac{n-1}{n}\right)^n |S_1| \int_{M} \rho^{(n-1)(1-\frac{\eta}{2})-1} \left(\ln \frac{\rho}{r}\right)^{(n-1)(1+\frac{\eta}{2})-n} \times \left[\left(1 - \frac{\eta}{2}\right) \frac{1}{\rho} \ln \frac{\rho}{r} + \left(1 + \frac{\eta}{2}\right)\right]^n d\rho
\]
\[
I_{\eta_2} = K^1(u_\eta) = |S_1| \int_{M} \rho^{(n-1)(1-\frac{\eta}{2})-1} \left(\ln \frac{\rho}{r}\right)^{(n-1)(1+\frac{\eta}{2})-n},
\]
If \(M\) is sufficiently large, i.e.,
\[
\frac{1}{M} \ln \frac{M}{r} < \frac{\eta}{2 - \eta} \quad \text{and} \quad M > er,
\]
then
\[
\left(1 - \frac{\eta}{2}\right) \frac{1}{\rho} \ln \frac{\rho}{r} + 1 + \frac{\eta}{2} \leq \left(1 - \frac{\eta}{2}\right) \frac{\eta}{2 - \eta} + 1 + \frac{\eta}{2} = 1 + \eta,
\]
because the function \(h(\rho) = \frac{1}{\rho} \ln \frac{\rho}{r}\) is monotone decreasing for \(\rho > er\). Thus (20) follows from expressions of \(I_{\eta_1}\) and \(I_{\eta_2}\) above for \(|x| \geq r\). \(\square\)

4 Sharp inequalities in an exterior of a ball \(B^c_r\) for \(u \in W^{1,p}_0(B^c_r)\)

For \(m < 0\), i.e. \(p < n\), we can combine inequalities (14) and (12) for functions \(u \in M(r, \infty)\), where
\[
M(r, \infty) = \begin{cases} 
  u \in W^{1,p}_0(B^c_r), \\
  \left(\frac{\tilde{r}^m - r^m}{m}\right)^{1-p} \int_{\partial B^c_r} |u|^p d\sigma \to 0, \quad \tilde{r} \to r + 0, \\
  R^{1-p} \int_{\partial B_R} |u|^p d\sigma \to 0, \quad \text{for} \ R \to \infty.
\end{cases}
\]

For \(r < \gamma < \infty\), \(\gamma = 2^{\frac{1}{1-p}}r\), we define
\[
L_1(u) = \int_{B^c_r \setminus B^c_{\gamma}} \left|\frac{x}{|x|}\right|^p \frac{u^p}{|x|^p} dx, \quad L_2(u) = \int_{B^c_{\gamma}} \left|\frac{x}{|x|}\right|^p \frac{|u|^p}{|x|^p} dx,
\]
\[
K_{11}(u) = |m|^p \int_{B^c_r \setminus B^c_{\gamma}} \frac{|u|^p}{|x|^{(n-1)p} (|x|^m - r^m)^p} dx, \quad K_{12}(u) = |m|^p \int_{B^c_{\gamma}} \frac{|u|^p}{|x|^p} dx,
\]
\[
K_{01}(u) = |m|^{p-1} \gamma^{1-p} \int_{\partial B^c_r} |u|^p d\sigma = K_{02}(u) = |m|^{p-1} \gamma^{1-p} \int_{\partial B^c_{\gamma}} |u|^p d\sigma = K_0.
\]
Proposition 4.1. If \( \gamma = 2^{1/|m|_r} \) then for every \( u \in M(r, \infty) \) the inequality

\[
L(u) = \sum_{j=1}^{2} L_j(u) = \int_{B_r^c} \left( \frac{|x, \nabla u|}{|x|} \right)^p \, dx \geq \left( \frac{1}{p} \right)^p \sum_{j=1}^{2} \frac{[K_{0j}(u) + (p-1)K_{1j}(u)]^p}{(K_{1j}(u))^{p-1}} = K(u),
\]

holds in \( B_r^c \).

The inequality (23) becomes an equality for functions \( u_\beta(x) \in M(r, \infty) \), \( \beta > 1/p' \) where

\[
u_\beta(x) = \begin{cases} (r^m - |x|^m)^\beta & x \in B_\gamma \setminus B_r, \\ |x|^{m\beta} & x \in B_r^c. \end{cases}
\]

Proof. As in Theorem 2.1 and Proposition 3.1 ii) we get

\[
L_1(u) \geq K_1(u) \quad \text{in} \quad B_\gamma \setminus B_r,
\]

\[
L_2(u) \geq K_2(u) \quad \text{in} \quad B_r^c
\]

where

\[
K_j(u) = \frac{[K_{0j}(u) + (p-1)K_{1j}(u)]^p}{(K_{1j}(u))^{p-1}}.
\]

With the choice of \( \gamma = 2^{1/|m|_r} \), so \( \gamma^p = \gamma^{(n-1)p'}(r^m - \gamma^m)^p \), we have continuous kernel for \( K_{1j} \) on \( \partial B_\gamma \).

Adding inequalities in (24) we get for \( u \in M(r, \infty) \) \( L(u) = \sum_{j=1}^{2} L_j(u) \geq \sum_{j=1}^{2} K_j(u) \).

Hence from the Jounel inequality we obtain (23).

Using (12) in \( B_r^c \) and (15) in \( B_\gamma \setminus B_r \) it follows that the inequality (23) becomes equality for functions \( u_\beta(x), \beta > 1/p' \).

Proposition 4.1 gives sharp Hardy inequality in an exterior of a ball \( B_r^c \) for functions \( u \in W^{1,p}_0(B_r^c) \), \( p < n \).

We will illustrate Proposition 3.2 ii) and Proposition 4.1 in the following examples.

Example 4.1. For \( p = 2, n \geq 3, r = 1, a = 0 \) and \( m = 2 - n < 0 \) from Proposition 3.1 ii) it follows that the Hardy inequality

\[
\int_{B_1^c} |\nabla u|^2 \, dx \geq \left( \frac{n-2}{2} \right)^2 \int_{B_1^c} \frac{|u|^2}{|x|^2} \, dx + \frac{n-2}{2} \int_{\partial B_1^c} u^2 + \frac{1}{4} \int_{\partial B_1^c} u^2 \left( \int_{B_1^c} \frac{|u|^2}{|x|^2} \right)^{-1}, \tag{25}
\]

holds. Note that (25) has an additional term in the right hand side in comparison with (2). Moreover, for the functions \( u_k(x) = |x|^{k(2-n)}, k > 2 \) inequality (25) becomes an equality, i.e., inequality (25) is sharp.
Example 4.2. For $p = 2, n \geq 3, m = 2 - n < 0, \gamma = 2 \frac{1}{n - 2} r$, and every function $u \in W^{1,2}_0(B^r_c)$ the Hardy inequality (24) becomes

$$
\int_{B^r_c} \left| \frac{x, \nabla u}{|x|} \right|^p \, dx \geq \left( \frac{n - 2}{2} \right)^2 \int_{B^r_c} \frac{u^2}{|x|^2} \, dx + 2 \left( \frac{n - 2}{2} \right)^2 \int_{B^r_c \setminus B^r_r} \frac{|x|^{n-2}}{\left( |x|^{n-2} - 1 \right)^2} u^2 \, dx + 2 \frac{1}{r} (n - 2) \int_{\partial B^r_c} u^2 \, dS
$$

$$
+ 2 \frac{n - 2}{r^2} \left( \frac{4}{n - 2} \right)^2 \left( \int_{\partial B^r_r} u^2 \, dS \right)^2
$$

$$
\times \left[ \left( \int_{B^r_c \setminus B^r_r} \frac{u^2}{|x|^2} \left( \frac{|x|^{n-2}}{\left( |x|^{n-2} - 1 \right)^2} \right) \, dx \right)^{-2} + \left( \int_{B^r_r} \frac{u^2}{|x|^2} \, dx \right)^{-2} \right] \tag{26}
$$

Moreover, for function $u_\beta(x)$ defined in Proposition 4.1 for $m = 2 - n$, inequality (26) becomes equality.

Let us mention that Hardy inequality (26) has the same leading term in the right hand side as in inequality (3), but inequality (26) is sharp one.

Finally, it is difficult to compare inequality (4) with (26), but (26) is sharp one, i.e., for the functions $u_\beta(x)$ defined in Proposition 4.1, inequality (26) becomes an equality.

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