Remote operations and interactions for systems of arbitrary dimensional Hilbert space: a state-operator approach

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Abstract

We present a systematic simple method for constructing deterministic remote operations on single and multiple systems of arbitrary discrete dimensionality. These operations include remote rotations, remote interactions and measurements. The resources needed for an operation on a two-level system are one ebit and a bidirectional communication of two cbits, and for an n-level system, a pair of entangled n-level particles and two classical “nits”. In the latter case, there are $n - 1$ possible distinct operations per one n-level entangled pair. Similar results apply for generating interaction between a pair of remote systems and for remote measurements. We further consider remote operations on $N$ spatially distributed systems, and show that the number of possible distinct operations increases here exponentially, with the available number of entangled pairs that are initially distributed between the systems. Our results follow from the properties of a hybrid state-operator object (“stator”), which describes quantum correlations between states and operations.

1 Introduction

Over the recent years entanglement has been examined as a resource which allows new types of communication tasks such as teleportation, dense coding, and other local manipulations of entanglement\textsuperscript{[1]}. These studies exploit the
relation between quantum non-locality and the structure of the Hilbert space. A more recent avenue of research examines the relation between entanglement and the dynamical evolution of several systems. Here, two basic questions have been examined: First, what is the entanglement creation capability of a given Hamiltonian that acts on a pair of systems \[^{2}\]. The second question deals with the reverse problem: what types of non-local operations on two or more remote systems can be generated, using a given resource of entangled states, by applying local operations and performing classical communication (LOCC).

In this article we will be interested in the second question. Previous work has demonstrated that certain operations like a remote controlled-not (CNOT), may consume less entanglement than what is needed when applying teleportation techniques\[^{3}\]. For probabilistic non-local operations, an isomorphism between the physical operations and the required entanglement has been discovered, which for certain operations necessitates less than one ebit per operation\[^{4}\]. A closely related question, raised by Huelga et. al.\[^{5}\], concerns the possibility of implementing a unitary transformation on a remote system.

The purpose of this article is to present a systematic approach for constructing a class of deterministic remote unitary transformation, and remote interactions between several distributed systems. We assume that the parties share entangled states and are allowed to perform only local operations and (bidirectional) classical communication.

A special characteristic of our method is that the generators which give rise to the transformation, are controlled locally by the two parties. The structure of the complete operation is in a sense “split” and determined by the local observers that posses the distributed parts of the system. Therefore, in the special case that the generators are known only locally, one cannot perform the operation using ordinary teleportation techniques.

To clarify this, consider the remote unitary operation

$$U_B = \exp\left[i\alpha\sigma_{n_B}\right]$$

that Alice and Bob wish to apply on a state $|\Psi_B\rangle$ of Bob. The axis $n_B$, which defines $\sigma_{n_B} = \vec{n}_B \cdot \vec{\sigma}_B$, is determined by Bob, while the angle of rotation, $\alpha$, by Alice.
Similarly, if Alice and Bob wish to apply a remote interaction

\[ U_{AB} = \exp\left[i\alpha \sigma_{n_A} \sigma_{n_B}\right] \]  \hspace{1cm} (2)

on a pair of spins in some arbitrary state |\Psi_{AB}\rangle, with one spin at the hands of Alice and the other with Bob, then, the axes \( n_A \) and \( n_B \), which fix the local generators \( \sigma_{n_A} \) and \( \sigma_{n_B} \) are controlled locally by Alice and Bob respectively.

Our approach relies on the properties of a new hybrid object which we introduce in section 2. This object describes quantum correlations between states of one party, say Alice, and operations acting on an arbitrary state of Bob. It turns out that certain remote operations can be translated to certain properties of this hybrid state-operator object, which we will refer to as a “stator”. The possible remote operations are hence associated with properties of the stator alone and are independent of the nature of the state(s) upon we intend to act remotely. By identifying the appropriate stator we are able to apply a remote operation on an arbitrary state(s).

In section 3, we describe the physical context in which stators can be prepared by applying LOCC on shared entanglement and the system. In section 4, we show how to use stators to construct remote rotations for a 2-level (spin-half) system. Then, in section 5, we consider the general problem of operating on an n-level system. In section 6, we study the case of N multiple systems, and in section 7, we show how to promote remote unitary operations into remote interactions and measurements.

2 The Stator

We begin by introducing a new object, which we shall refer to as a “stator”. A stator is a hybrid linear construction of states in Alice’s Hilbert space and operators acting on Bob’s system. The purpose of introducing this object is twofold: First, stators simplify considerably the construction of remote unitary operations and interactions via entanglement by providing us with a systematic general approach which can be easily generalized to an arbitrary number of n-level system. Second, we found that these objects, which describe quantum correlations between states on one side and operators on the other side, assist us to develop an intuition regarding remote operations which may turn out helpful in other problems.

Let us then begin by defining what is a stator. We denote the Hilbert spaces of two remote observers, Alice and Bob, by \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively.
Instead of describing quantum correlations between states of Alice and Bob, we wish now to describe quantum correlations between states in $\mathcal{H}_A$ and resulting actions described by operators, in $O(\mathcal{H}_B)$, acting on an arbitrary state in $\mathcal{H}_B$. Hence we now construct a hybrid state-operator or shortly a “stator”, $S$, that lives in the space

$$S \in \{\mathcal{H}_A \times O(\mathcal{H}_B)\}$$

In close analogy to an entangled state, a stator has the general form

$$S = \sum_{ij} c_{ij} |i_A⟩ ⊗ O_{Bj}$$

with $|i_A⟩ \in \mathcal{H}_A$, $O_{Bj} \in O(\mathcal{H}_B)$, and $c_{ij}$ as c-numbers.

Although this structure resembles the form of an entangled state it does not describe a fixed amount of entanglement because it applies to any general state of Bob. When we act with a stator on a general state $|\Psi_B⟩ \in \mathcal{H}_B$ we get

$$S|\Psi_B⟩ \in \mathcal{H}_A ⊗ \mathcal{H}_B$$

Therefore, even if the stator has a maximal entanglement-like structure (as in eq. (7) below), the measure of entanglement, $E(S|\Psi_B⟩)$, depends on the nature of $|\Psi_B⟩$. For a general state of Bob, we may get any value between of $E(S|\Psi_B⟩)$, from zero to one even for a “maximal” stator.

Most important to us will be the following property. For every stator we can construct an eigenoperator equation

$$O_A S = \lambda_B S$$

Thus, by operating on the stator with an operator $O_A \in \mathcal{H}_A$ in Alice’s Hilbert space, we get back the same stator multiplied by an eigenoperator now acting in Bob’s Hilbert space $\mathcal{H}_B$.

In general, the operators $O_A$ and eigenoperators $\lambda_B$ need not be hermitian. However, as we shall see, for certain classes of stators, relevant to the present problem, the operators and eigenoperators are both hermitian.

As a first example, let dim$\mathcal{H}_A = 2$ be spanned by the eigenstates $|0_A⟩$ and $|1_A⟩$ of $σ_z_A$, and consider the operator $σ_{n_B} ∈ O(\mathcal{H}_B)$ such that $σ_{n_B}^2 = I_B$. Consider now the stator

$$S = |0_A⟩ ⊗ I_B + |1_A⟩ ⊗ σ_{n_B}$$

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which we shall refer to in the sequel as a 2-level stator. S satisfies the eigenoperator equation:

$$\sigma_x S = \sigma_n B S$$

(8)

As straightforward, but useful consequence, any analytic function f also satisfies

$$f(\sigma_x) S = f(\sigma_n) S$$

(9)

and particularly

$$e^{i\alpha \sigma_x} S = e^{i\alpha \sigma_n} S$$

(10)

where $\alpha$ is any real number or hermitian operator in $H_A$. The relation above already indicates why stators can be useful for generating remote operations. We note that a unitary operation of Alice gives rise to a similar unitary operation acting on Bob’s side.

The above construction can be generalized to the case $\dim H_A = n$, which becomes relevant if Bob owns an n-level system. Let $|i_B\rangle$, $i = 0, 1 \cdots n - 1$, be an orthogonal basis of $H_A$, and choose $U_B \in O(H_B)$ be the n'th root of the unity: $U_B^n = I_B$. We can than construct the n-level stator

$$S = |0_A\rangle \otimes I_B + |1_A\rangle \otimes U_B + \cdots |n - 1_A\rangle \otimes U_B^{n-1}$$

(11)

The relevant eigenoperator equation than becomes

$$V_A S = U_B S$$

(12)

where $V_A$ is a shift operator defined by: $V_A|m_A\rangle = |(m-1)_A\rangle$, $m = 1 \cdots n - 1$, and $V_A|0_A\rangle = |(n-1)_A\rangle$. By operating with $V_A + V_A^\dagger$ we then obtain the hermitian eigenoperator $U_B + U_B^\dagger$, and acting with $i(V_A - V_A^\dagger)$ yields the eigenoperator $i(U_B - U_B^\dagger)$. Similarly we can construct any powers of $U_B + U_B^\dagger$ and $i(U_B - U_B^\dagger)$.

We can further generalize our construction to the case that Bob has at hand several systems (which may be removed from each other) of arbitrary dimension. We discuss this case in section 6.

### 3 Preparation of Stators

We have seen that stators allow us to obtain state-independent relations between Alice’s actions and their result on Bob’s state. We proceed then to describe the process that will be referred to as “preparation” of a stator.
Figure 1: Preparation of a stator acting on Bob’s state.

Hence, given by an unknown state, \( |\Psi_B\rangle \in \mathcal{H}_B \), and some shared entangled state \( |\text{ent}\rangle \), our aim is transform this initial state by performing some LOCC operation into

\[
|\text{ent}\rangle \otimes |\Psi_B\rangle \rightarrow S|\Psi_B\rangle
\]

We first describe in details the simplest case in which Alice and Bob use one ebit of shared entanglement to prepare a 2-level stator as depicted in Figure 1. The initial state at the hands of Alice and Bob is in this case

\[
\frac{1}{\sqrt{2}} \left( |0_a0_b\rangle + |1_a1_b\rangle \right) \otimes |\Psi_B\rangle
\]

For practical purposes, in the following we will denote by the small letters, \( a \) and \( b \), the shared ancillary entangled systems of Alice and Bob respectively.

Bob starts by performing a CNOT interaction (with respect to \( \sigma_{n_B} \)) between the qubit (b) and his state \( |\Psi_B\rangle \), described by the unitary transformation

\[
U_{bB} = |0_b\rangle \langle 0_b| \otimes I_B + |1_b\rangle \langle 1_b| \otimes \sigma_{n_B}
\]

Here \( \sigma_{n_B} \) is an operator acting in \( \mathcal{H}_B \) satisfying \( \sigma_{n_B}^2 = I_B \). ( \( \mathcal{H}_B \) need not be 2-dimensional; for instance Bob’s system may contain several spins, in which case \( \sigma_{n_B} = \sigma_{n_B}^1 \sigma_{n_B}^2 \cdots \)).

This yields the state

\[
\frac{1}{\sqrt{2}} \left( |0_a0_b\rangle \otimes I_B + |1_a1_b\rangle \otimes \sigma_{n_B} \right) |\Psi_B\rangle
\]
Next he performs a measurement of $\sigma_x$ of the entangled qubit to project out a certain value. The resulting state is now

$$\frac{1}{2}(|0_a\rangle \pm |1_b\rangle) \otimes \left(|0\rangle_a \otimes I_B \pm |1\rangle_a \otimes \sigma_{n_B}\right)|\Psi_B\rangle$$ \hspace{1cm} (17)

Finally Bob informs Alice what was the result of his measurement by sending Alice one classical bit of information. For the case that $\sigma_x = -1$ Alice performs a trivial $\pi$ rotation around the $\hat{z}$ axis and flips the $-$ sign to a $+$ sign. The resulting state of the system is now given by

$$\frac{1}{2}(|0_a\rangle \pm |1_b\rangle) \otimes \left(|0\rangle_a \otimes I_B \mp |1\rangle_a \otimes \sigma_{n_B}\right)|\Psi_B\rangle$$ \hspace{1cm} (18)

Since Bob’s previously entangled qubit factors out, the final state of Alice’s qubit and Bob’s system can be obtained by letting the stator

$$\mathcal{S} = |0_a\rangle \otimes I_B + |1_a\rangle \otimes \sigma_{n_B}$$ \hspace{1cm} (19)

act on $|\Psi_B\rangle$. This completes the preparation of a 2-level stator $\mathcal{S}$ which now operates on Bob’s system.

We further discuss preparation of n-level stators in connection to remote operations on an n-level system in section 5.

## 4 Remote Unitary transformations

Suppose that Bob has a system in the unknown state $|\Psi_B\rangle$ on which Alice and Bob wish to act on with a unitary transformation described by a rotation

$$U_B = e^{i\alpha\sigma_{n_B}}$$ \hspace{1cm} (20)

with $\sigma_{n_B}^2 = I_B$ and $\alpha$ a real arbitrary number.

We will now show that the transformation (20) can be performed, provided that the generator $\sigma_{n_B}$ is known to Bob, and the parameter $\alpha$ of rotation is know to Alice. To this end, they start by using a shared ebit to prepare, as described in the previous section, the stator

$$\mathcal{S} = |0_a\rangle \otimes I_B + |1_a\rangle \otimes \sigma_{n_B}$$ \hspace{1cm} (21)

which operates on Bob’s state. $\sigma_{n_B}$ enters here as a result Bob’s choice to perform a CNOT with respect to $\sigma_{n_B}$ as in eq. (13).
Next, Alice performs on her qubit a unitary transformation
\[ U_a = e^{i\alpha \sigma_x a} \]  \hspace{1cm} (22)
where \( \sigma_{xa} |0_a\rangle = |1_a\rangle \) and \( \sigma_{xa} |1_a\rangle = |0_a\rangle \). Using the fact that when acted with \( \sigma_{xa} \) the stator satisfies an eigenoperator equation with an eigenoperator \( \sigma_{nB} \) we have
\[ e^{i\alpha \sigma_x a} S = e^{i\alpha \sigma_{nB}} S \]  \hspace{1cm} (23)
Hence after the rotation the state is
\[ \left( |0_a\rangle \otimes I_B + |1_a\rangle \otimes \sigma_{nB} \right) e^{i\alpha \sigma_{nB}} |\psi_B\rangle \]  \hspace{1cm} (24)
Depending upon the final state of Alice’s qubit, they managed to produced the required rotation, modulo possible extra trivial rotations. To eliminate these rotations, Alice measures the state of her qubit. If it is \( |0_a\rangle \), we have produced the required transformation. If it turns out to be in the state \( |1_a\rangle \) she needs to inform Bob to perform a trivial \( \pi \) rotation, \( U_\pi = \exp(i\pi \sigma_{nB}/2) \), which corrects for the extra \( \sigma_{nB} \) above. This completes the process.

The resources that Alice and Bob require for remote rotation applied on a 2-level system are hence, one e-bit of shared entanglement and two cbits. They communicate one cbit first from Bob to Alice to prepare the stator, and one cbit from Alice back to Bob to complete the required rotation with probability 1. For both cbits we have that \( p(1) = p(0) = 1/2 \), i.e. they are unbiased. Therefore the exchanged classical communication contains no information on the state of Bob or the angle of rotation.
The role of the exchanged cbits is as follows: the first cbit is needed in order to obtain the correct stator (fix the sign in eq. (17)). Without this one would have obtained with probability 1/2 the correct rotation $U$ and with probability 1/2 the rotation $U^\dagger$. (For the case of remote measurements discussed in section 7, this uncertainty in the sign may be irrelevant initially and may be corrected at later stage of the process.) The second cbit sent from Alice to Bob is clearly needed from causality requirement. A process that uses less than one cbit of communication from Alice to Bob clearly violates causality.

5 Remote operations on n-level systems

We now apply our method for the case of an n-level system. First we identify the n-level stator with the appropriate generator of rotations as an eigenoperator. To prepare this stator, Alice and Bob apply LOCC on their shared entangled state and Bob’s n-level system. Next Alice performs a unitary transformation on half of the entangled pair on her side, followed by a measurement, and informs Bob via a classical channel how to correct his system to complete the rotation.

As we shall shortly see the required resources in this case are two maximally entangled n-level systems, and two classical “nits” (each containing $n$ possible values), one sent from Bob to Alice to complete the preparation, and the second from Alice back to Bob to complete the remote operation. However, unlike the 2-level case, the number of possible unitary operation per given entangled n-level pair is here larger and given by any general linear combinations of $n-1$ generators. The rotation around a given axis is one of the possible operations.

To illustrate this let us first demonstrate the process for the case $n = 3$ of a spin one particle. For a rotation around the $z$-axis (with the axis of rotation been chosen as before by Bob) we need to identify a stator that satisfies the eigenoperator equation

$$AS = L_Z S$$

where $L_Z$ is the appropriate generator of rotation. When applying $A^2$ on the stator we get $A^2 S = L_x^2 S$. Therefore $L_x^2$ is another eigenoperator of $S$. Since $L_x^2 = L_Z$ these are the only eigenoperators.
Since for \( n = 3 \) we have two distinct eigenoperators, the most general remote transformation which we are able to construct, using two maximally entangled 3-level systems (qutrit), has the form

\[
U_B = e^{i(\alpha L_z + \beta L_z^2)}
\]  

(26)

where \( \alpha \) and \( \beta \) are chosen by Alice.

Recalling the discussion in section 2, the appropriate \( S \) for this case is a 3-level stator of the form

\[
S = |0_a \rangle \otimes I_{\Psi_B} + |1_a \rangle \otimes U_{\Psi_B} + |2_a \rangle \otimes U_{\Psi_B}^2
\]  

(27)

where the requirement \( U_{\Psi_B}^3 = I_{\Psi_B} \) dictates the form

\[
U_{\Psi_B} = e^{\frac{2\pi i}{3} L_z}
\]  

(28)

(Here we used the subscript \( \Psi_B \) in \( U_{\Psi_B} \) in order to distinguish between the full remote operation \( U_B \) applied by Alice and Bob and local transformations \( U_{\Psi_B} \) applied by Bob). Since for a spin one particle we have \( e^{i\theta L_z} = 1 + i \sin \theta L_z + L_z^2(cos \theta - 1) \), we identify the operator \( A \) in eq. (25) as

\[
A = \frac{1}{2i \sin \frac{4\pi}{3}}(V - V^\dagger)
\]  

(29)

where \( V \) and \( V^\dagger \) are the raising and lowering operators defined in section 2.

Having identified the required stator and the operators \( A \), we next describe the preparation and rotation process. We begin with a shared pair of maximally entangled qutrits and Bob’s state \( |\Psi_B \rangle \):

\[
\left( |0_a 0_b \rangle + |1_a 1_b \rangle + |2_a 2_b \rangle \right)|\Psi_B \rangle
\]  

(30)

Bob applies the unitary operation \( U_{bB} \) on his state and his half (b) of the entangled pair:

\[
U_{bB} = |0_b \rangle \langle 0_b | \otimes I_{\Psi_B} + |1_b \rangle \langle 1_b | \otimes U_{\Psi_B} + |2_b \rangle \langle 2_b | \otimes U_{\Psi_B}^2
\]  

(31)

This results with the state

\[
|\Psi_{tot} \rangle = \left[ |0_a 0_b \rangle \otimes I_{\Psi_B} + |1_a 1_b \rangle \otimes U_{\Psi_B} + |2_a 2_b \rangle \otimes U_{\Psi_B}^2 \right]|\Psi_B \rangle
\]  

(32)
Now to generate the stator (27), we need to eliminate Bob’s entangled particle. Hence Bob measures his particle \( b \) in the following basis:

\[
|0'_b\rangle = \frac{1}{\sqrt{3}}(|0_b\rangle + |1_b\rangle + |2_b\rangle) \\
|1'_b\rangle = \frac{1}{\sqrt{3}}(|0_b\rangle + e^{\frac{2\pi i}{3}}|1_b\rangle + e^{\frac{4\pi i}{3}}|2_b\rangle) \\
|2'_b\rangle = \frac{1}{\sqrt{3}}(|0_b\rangle + e^{\frac{4\pi i}{3}}|1_b\rangle + e^{\frac{2\pi i}{3}}|2_b\rangle)
\] (33)

Let’s rewrite the state (32) in the terms of the new basis vectors:

\[
|\Psi_{tot}\rangle = \{|0_a\rangle \otimes I_{\Psi_B} + |1_a\rangle \otimes U_{\Psi_B} + |2_a\rangle \otimes U_{\Psi_B}^2\}|0'_b\rangle \\
+ \{|0_a\rangle \otimes I_{\Psi_B} + e^{\frac{2\pi i}{3}}|1_a\rangle \otimes U_{\Psi_B} + e^{\frac{4\pi i}{3}}|2_a\rangle \otimes U_{\Psi_B}^2\}|1'_b\rangle \\
+ \{|0_a\rangle \otimes I_{\Psi_B} + e^{\frac{4\pi i}{3}}|1_a\rangle \otimes U_{\Psi_B} + e^{\frac{2\pi i}{3}}|2_a\rangle \otimes U_{\Psi_B}^2\}|2'_b\rangle \} |\Psi_B\rangle
\] (34)

According to one of the three particular outcomes of Bob’s measurement of the particle \( b \) the state of Alice’s particle \( a \) and Bob’s particle \( \Psi_B \) evolves to

\[
|0_a\rangle \otimes I_{\Psi_B} + |1_a\rangle \otimes U_{\Psi_B} + |2_a\rangle \otimes U_{\Psi_B}^2 |\Psi_B\rangle
\] (35)

or to the states

\[
|0_a\rangle \otimes I_{\Psi_B} + e^{\frac{2\pi i}{3}}|1_a\rangle \otimes U_{\Psi_B} + e^{\frac{4\pi i}{3}}|2_a\rangle \otimes U_{\Psi_B}^2 |\Psi_B\rangle
\]
and

\[
|0_a\rangle \otimes I_{\Psi_B} + e^{\frac{4\pi i}{3}}|1_a\rangle \otimes U_{\Psi_B} + e^{\frac{2\pi i}{3}}|2_a\rangle \otimes U_{\Psi_B}^2 |\Psi_B\rangle.
\]

Bob transmits this classical outcome (classical ”trit”) to Alice. Notice that the three results appear with equal probability of 1/3 hence the classical trit is unbiased. In the last two cases Alice performs the following transformations on her particle \( a \) in order to correct the state to the form (35):

\[
C_1 = |0_a\rangle \langle 0_a| + e^{\frac{4\pi i}{3}}|1_a\rangle \langle 1_a| + e^{\frac{2\pi i}{3}}|2_a\rangle \langle 2_a| \quad \text{and} \quad C_2 = |0_a\rangle \langle 0_a| + e^{\frac{2\pi i}{3}}|1_a\rangle \langle 1_a| + e^{\frac{4\pi i}{3}}|2_a\rangle \langle 2_a|.
\]

We can interpret (35) as the stator (27) operating on the state \( |\Psi_B\rangle \). This completes the preparation process.

In order to generate a general rotation, Alice acts on her particle with the unitary operator

\[
U_a = e^{i(\alpha A + \beta A^2)}
\] (36)
and performs a measurement to collapses the state into one of the states $|n_a\rangle$. Notice that as in the preparation process the results are again unbiased. She then sends a classical trit to and informs Bob the result of her measurement. For the cases that Alice obtained $|1_a\rangle$ or $|2_a\rangle$, Bob then performs the rotations $U^2_B$ and $U_B$, respectively. This completes the procedure or generating a remote rotation.

The above procedure can be applied for an arbitrary $n$-level system. The maximally entangled state of two qutrits is then replaced by a maximally entangled pair of $n$-level systems. After applying the interaction $U_{bb}$ the total state becomes

$$\left|\Psi_{tot}\right\rangle = \left[\sum_{m=0}^{n-1} |n_a n_b\rangle \otimes U^m_{\Psi B}\right]|\Psi_B\rangle$$

with

$$U_{\Psi B} = e^{2\pi i n L_Z}$$

and $L_Z$ the appropriate rotation generator for the $n$-level system.

Bob then performs a measurement of his half of the entangled pair $b$ in the following basis:

$$|m'_b\rangle = \frac{1}{\sqrt{n}} \sum_{m_b=0}^{n-1} e^{2\pi i m'_b m_b}|m_b\rangle$$

where $m'_b = 0...n-1$. He sends to Alice one “nit” to inform her which of the $n$-possible outcomes was obtained. Alice on her side operates the relevant unitary operation. It can be shown that for $n > 3$ the relevant operator $A$ in equation (25) becomes a linear combinations of powers of $V - V^\dagger$ for $n$ odd, and of $V + V^\dagger$ for even $n$. The total number of independent combinations is $n - 1$. To complete the process Alice then performs a measurement and send Bob one classical nit. This enables him to perform one of the operations $U^m_{\Psi B}$, $m = 0...n-1$ which complete the process.

To summarize: for an $n$-level system, we use the resources of one pair of maximally entangled $n$-level system and a two way classical communication of one nit in each direction. This enables to apply a general remote transformation of the form

$$U_B = e^{i(\alpha_1 L_Z + \alpha_2 L_Z^2 + ... + \alpha_{n-1} L_Z^{n-1})}$$

where Bob determines the axis $Z$, and Alice determines the $n - 1$ angles $\alpha_i$. 
Figure 3: Remote operation on \( N \) distributed systems.

6 Operations on multiple systems

Consider next the case that on Bob’s side we have \( N \) distinguishable separate systems in some arbitrary state \( |\Psi_{B_1\ldots N}\rangle\):

\[
|\Psi_{B_1\ldots N}\rangle \in \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_N}
\]  

(41)

with \( \dim \mathcal{H}_{B_i} = n_i \). The \( N \) systems may be distributed to \( N \) different remote spatially separated locations denoted by \( B_i \).

To examine the operations possible by our method we further assume that we distribute between Alice and \( B_i \) \( N \) maximally entangled pairs as depicted in figure 3. For a given system of dimensionality \( n_i \) we match a maximally entangled \( n_i \)-level pair shared between Alice and \( B_i \).

Clearly we now can repeat our method and generate \( n_i - 1 \) operations on the \( i \)'th system by using the shared entangled pairs to prepare \( N \) stators, each one connecting between Alice and the system \( B_i \). However it now turns out that with \( N \) stators at hand we can generate an exponentially larger class
of operations, most of them corresponding to interactions between several remote subsystems.

To exemplify this, consider first the simplest case of \( N \) 2-level (spin-half) systems. In this case the resources needed are \( N \) shared ebits between Alice and \( B_i \) and classical bidirectional communication of \( 2N \) classical bits: two cbits between Alice and a given \( B_i \). As before each \( B_i \) has the choice of fixing the local axis of rotation which fixes \( N \) generators \( \sigma_{n_{B_i}} \), \( i = 1...N \).

We can repeat the preparation of a stator \( S_i \) for each spin separately as described in section 3. The total stator is then

\[
S_{tot} = \bigotimes_{i=1}^{N} \left( |0_{a_i}\rangle \otimes I_{B_i} + |1_{a_i}\rangle \otimes \sigma_{n_{B_i}} \right) \tag{42}
\]

The above stator satisfies an eigenoperator equations

\[
\sigma_{x_i} S_{tot} = \sigma_{B_i} S_{tot} \tag{43}
\]

However since the different \( N \) generators commute, we also have that any product of separate eigenoperators is also an eigenoperator. The total number of eigenoperators is then

\[
\sum_{m=1}^{N} C_N^m = 2^N - 1 \tag{44}
\]

It follows then that Alice has the freedom of selecting the \( 2^N - 1 \) angles that generate rotations and interactions between the spins.

For example, the most general remote operation for the case \( N = 3 \) becomes

\[
U_B = \exp \left[ i \sum_{m=1}^{3} \alpha_m \sigma_{B_m} + i \frac{1}{2} \sum_{m \neq n} \beta_{mn} \sigma_{B_m} \sigma_{B_n} + i \gamma \sigma_{B_1} \sigma_{B_2} \sigma_{B_3} \right] \tag{45}
\]

We can easily apply our method for any configuration of \( N \) separated \( n_i \)-levels systems. (In general \( n_i \) may not be equal.) Let us consider the case with \( n_i = n \) for all \( i \). Then the total number of operators is easily computed to be

\[
\sum_{m=1}^{N} (n-1)^m C_N^m = n^N - 1 \tag{46}
\]

Therefore with the aid of \( N \) pairs of \( n \)-level maximally entangled pairs and bidirectional classical communication of \( 2N \) nits we can apply \( n^N - 1 \) remote operations.
Finally, we note that the $N$ separated subsystems can be viewed as a single system of dimensionality $D = \otimes_{i=1}^{N} n_i$. Hence by the results of the previous section, we can use one D-level stator to act on the system as a whole. The number of distinct operation will then given by $D - 1$, in agreement with the results obtained in eqs. (14,46).

7 Generating remote interactions and measurements

In the last section we have already seen examples where Alice can act remotely on several spatially separated systems and effectively generate an interaction between remote subsystems. For instance for two remote spins systems Alice can use two ebits and four cbits to generate the interaction

$$U_{B_1,B_2} = e^{i\alpha B_1 \sigma B_2}$$

(47)

Here the local axes of rotation, $\bar{n}_i$ ($\sigma_{B_i} = \bar{n}_i \cdot \sigma_{B_i}$), are determined locally by the local observers $B_i$, and the coupling strength $\alpha$ is controlled by Alice.

There is yet another simple method to generate remote interaction between Bob’s system and a system $A$ located with Alice. Inspecting eq. (10), we note that in fact the angle $\alpha$ can be promoted to an operator acting on a system $A$ of Alice. Hence in the case of a 2-level stator, with an eigenoperator $\sigma_{n_B}$ we have also the relation

$$e^{i\lambda O_{A} \sigma x_A} S = e^{i\lambda O_{A} \sigma x_B} S$$

(48)

where the stator $S$ is defined as in eq. (7), and $O_A$ is an hermitian operator acting on an arbitrary dimensional system $A$ of Alice.

A simple generalization of the procedures in sections 3. and 4. now allows performing remote interaction between separate systems. The only modification needed is to replace the unitary rotation performed by Alice to her half of the entangled pair, $(a)$, with the unitary operation

$$U_{Aa} = e^{i\lambda O_A \sigma x_a}$$

(49)

acting on $(a)$ and on her system $A$. (For the n-level case $\sigma_{x_a}$ needs to be replaced by an appropriate operator, e.g. the operator $A$ defined in eq. (29)).
For example suppose Alice’s system is another spin-half particle, and we wish to apply remotely a CNOT operation between Alice’s and Bob’s spins. To this end we need to apply the transformation

$$U_{AB} = e^{i\frac{\lambda}{4} \sigma_{xA}(1-\sigma_{xB})}$$

But this is the special case of applying the transformation (48) while taking $\lambda = \pi/4$, $O_A = \sigma_{xA}$ and $n_B = x_B$, followed with a simple local rotation $e^{i\frac{\pi}{4} \sigma_{xA}}$.

As a special case of remote interactions we can further consider remote measurements. Hence Alice’s system A will be considered as a measuring device. We can use another spin as a measuring device (pointer) or let us introduce a continuous measuring device with conjugate coordinates $P$ and $Q$, where $P$ plays the role of the “pointer”.

Let us describe a remote “Stern-Gerlach” measurement of Bob’s spin system along a certain direction. Alice informs Bob to fix the axis $n_B$ according to the direction she wishes to perform the measurement. After completing the preparation of the stator she applies the unitary operation

$$U_{MD,a} = e^{iQ\sigma_{xa}}$$

that yields the state

$$S e^{iQ\sigma_{nB}} |\Psi_B\rangle |MD\rangle$$

She can now observe the variable $P$ of the measuring device and read the outcome of the measurement. The final state of the system $B$ still does not correspond to the outcome of the measurement. For this she needs to measure the entangled particle ($a$) and communicate to Bob the result (not the result of the measurement!). Bob uses this random information to corrects the state to match the result of the measurement.

We conclude with several comments. The remote measurement process can be in fact completed instantaneously on a space-like surface; Alice does not need to wait to obtain a classical bit to perform her measurement. In this case Alice generates the operation $\exp(\pm iQ\sigma_{nB})$ with probability 1/2 for each $\pm$ possibility. Hence, in accordance with causality, the result of the measurement can be interpreted only after she obtains the classical bit from Bob. This approach can be easily generalized for general systems as well as for performing measurements non-local observables.

Finally, it is interesting to note that the present method consumes less entanglement resources (one ebit instead of two, and two cbits instead of
four) compared with methods using teleportations. It can also be shown that some non-local measurements can be performed using the present method but cannot by using teleportation.

8 Conclusion

We presented a systematic method for constructing deterministic remote operations on single and multiple systems of arbitrary dimensions. Our approach requires bidirectional classical communication of unbiased bits between the parties and leaves the control over the generators that act on each system at the hands of the local observers. In this way the control over the full structure of the unitary operation is split between several remote observers. It is also worth to mention that when the local information is kept secret, the operations cannot be achieved using teleportation-like schemes. These properties may be helpful for constructing new cryptographic tools.

To facilitate the construction of remote operations we have introduced a new object – the stator – which describes correlations between states of one system and operations acting on an arbitrary state of another remote system. We hope that stators may turn out useful for other problems regarding the relation between entanglement and remote interactions.

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