Multiple scales and singular limits of perfect fluids

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Abstract

In this article our goal is to study the singular limits for a scaled barotropic Euler system modelling a rotating, compressible and inviscid fluid, where Mach number $\epsilon\mu$, Rossby number $\epsilon$ and Froude number $\epsilon^n$ are proportional to a small parameter $\epsilon \to 0$. The fluid is confined to an infinite slab, the limit behaviour is identified as the incompressible Euler system. For well-prepared initial data, the convergence is shown on the life span time interval of the strong solutions of the target system, whereas a class of generalized dissipative solutions is considered for the primitive system. The technique can be adapted to the compressible Navier–Stokes system in the subcritical range of the adiabatic exponent $\gamma$ with $1 < \gamma \leq \frac{3}{2}$, where the weak solutions are not known to exist.

Keywords: Compressible Euler system, rotating fluids, dissipative solution, low Mach and Rossby number limit, multiple scales, compressible Navier–Stokes system.

AMS classification: Primary: 76U05; Secondary: 35Q35, 35D99, 76N10

1 Introduction

We study models of rotating fluids as described in Chemin et.al. [11]. Let $T > 0$ and $\Omega(\subset \mathbb{R}^3) = \mathbb{R}^2 \times (0, 1)$ be an infinite slab. We consider the scaled compressible Euler equation in time-space cylinder $Q_T = (0, T) \times \Omega$ describing the time evolution of the mass density $q = q(t, x)$ and the momentum field $m = m(t, x)$ of a rotating inviscid fluid with axis of rotation $b = (0, 0, 1)$:

- Conservation of Mass:

\begin{equation}
\partial_t q + \text{div}_x m = 0.
\end{equation}

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• Conservation of Momentum:
\[
\partial_t \mathbf{m} + \text{div}_x \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \frac{1}{\text{Ma}^2} \nabla_x p(\rho) + \frac{1}{\text{Ro}} \mathbf{b} \times \mathbf{m} = \frac{1}{\text{Fr}^2} \rho \nabla_x G. \tag{1.2}
\]

• The scaled system contains characteristic numbers:
  
  \begin{align*}
  \text{Ma} & \sim \varepsilon^m, \\
  \text{Ro} & \sim \varepsilon, \\
  \text{Fr} & \sim \varepsilon^n \quad \text{for} \quad \varepsilon > 0, \ m, n > 0.
\end{align*} \tag{1.3}

• Pressure Law: The pressure \( p \) and the density \( \rho \) of the fluid are interrelated by the standard isentropic law
\[
p(\rho) = a\rho^\gamma, \quad a > 0, \quad \gamma > 1. \tag{1.4}
\]

• Boundary condition: Here we consider slip condition on the horizontal boundary, i.e.
\[
\mathbf{m} \cdot \mathbf{n} = 0, \quad \mathbf{n} = (0, 0, \pm 1). \tag{1.5}
\]

• Far field condition: Let us introduce the notation \( x = (x_h, x_3) \) and \( P_h(x) = x_h \). For each \( \varepsilon > 0 \), we identify a static solution that satisfies (1.1)-(1.2) with (1.3). More specifically, a static solutions is a pair \( (\tilde{\rho}_\varepsilon, 0) \), where the density profile \( \tilde{\rho}_\varepsilon \) satisfies
\[
\nabla_x p(\tilde{\rho}_\varepsilon) = \varepsilon^{2(m-n)} \tilde{\rho}_\varepsilon \nabla_x G. \tag{1.6}
\]

In general, there are infinitely many static solutions for a given potential \( G \). We assume the far field condition as,
\[
|\rho - \tilde{\rho}_\varepsilon| \to 0, \quad \mathbf{m} \to 0 \quad \text{as} \quad |x_h| \to \infty. \tag{1.7}
\]

• Initial data: For each \( \varepsilon > 0 \), we supplement the initial data as
\[
\rho(0, \cdot) = \rho_{\varepsilon,0}, \quad \mathbf{m}(0, \cdot) = \mathbf{m}_{\varepsilon,0}. \tag{1.8}
\]

• Choice of \( G \): As a matter of fact, the driving potential \( G \) can be seen as a sum of the centrifugal force proportional to the norm of the horizontal component of the spatial variable i.e. \( (x_1^2 + x_2^2) \) and the gravitational force acting in the vertical direction \( x_3 \). We omit the effect of the centrifugal force in the present paper motivated by certain meteorological models. Instead we consider
\[
G(x) = -x_3 \quad \text{in} \ \Omega \tag{1.9}
\]
corresponding to the gravitational force acting in the vertical direction.
We consider singular limit problem for $\epsilon \to 0$ in the multiscale regime:

$$\frac{m}{2} > n \geq 1.$$  \hspace{1cm} (1.10)

Thus we study the effect of low Mach number limit (also called incompressible limit), low Rossby number limit and low Froude number limit acting simultaneously on the system (1.1)-(1.2).

Formally, we observe that low Mach number limit regime indicates the fluid becomes incompressible and low Rossby number limit indicates fast rotation of the fluid and as a consequence of that fluid becomes planner (two-dimensional).

As solutions of the (primitive) compressible Euler systems are expected to develop singularities (shock waves) in a finite time, there are two approaches to deal with the singular limit problem.

I. The first approach consists of considering classical(strong) solutions of the primitive system and expecting it converges to the classical solutions of the target system. Here, the main and highly non-trivial issue is to ensure that the lifespan of the strong solutions is bounded below away from zero uniformly with respect to the singular parameter.

II. The second approach is based on the concept of weak, measure–valued or dissipative solutions of the primitive system. Under proper choice of initial data one can show convergence provided the target system admits smooth solution.

For the first approach in the low Mach number limits we have results by Ebin [12], Kleinermann and Majda [25], Schochet [30], and many others. For rotating fluids there are results by Babin, Mahalov and Nicolaenko [2, 3] and Chemin et. al. [11].

In the case of second approach, most of the results dealing with weak solutions have been studied for compressible Navier–Stokes system with additional consideration of high Reynolds number limit. For rotating fluids there are several results, see Feireisl, Gallagher and Novotný [13], Feireisl et. al. [14], Feireisl and Novotný [21, 22] and Li [26].

Since existence of global-in-time weak solution of compressible Euler equation satisfying energy inequality is still open for general initial data. Hence it is important to consider measure–valued solution or newly developed dissipative solution for this system. The concept of measure–valued solutions has been studied in various context, like, analysis of numerical schemes etc. In the following articles by Alibert and Bouchitté [1], Gwiazda, Świerczewska-Gwiazda and Wiedemann [23], Březina and Feireisl [8], Březina [7], Basarić [4], Feireisl and Lukáčová-Medvidová [18] we observe the development of theory on measure valued solution for different models describing compressible fluids mainly with the help of Young measures.

Recently, Feireisl, Lukáčová-Medvidová and Mizerová in [19] and Breit, Feireisl and Hofmanová [5] give a new definition for compressible Euler system, termed as dissipative solution without involving Young measures.

The advantages to consider the second approach are,

- Weak or measure valued solutions to the primitive system exist globally in time. Hence the result depends only on the life span of the target problem that may be finite.

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The convergence holds for a large class of generalized solutions which indicates certain stability of the limit solution of the target system.

In particular, the results involving generalized solutions are better in the sense that convergence holds for a larger class of solutions and on the life span of the limit system.

There is a series of works dealing with the low Mach number limit in the framework of measure–valued solutions. In Feireisl, Klingenberg and Markfelder [17], Bruell and Feireisl [9], Březina and Mácha [10], it is shown that measure–valued solution of primitive system which describes some compressible inviscid fluid converges to strong solution of incompressible target system under consideration of suitable initial data. The ‘single–scale’ limit of our system i.e. \( m = 1 \) and \( G = 0 \), has been studied by Nečasová and Tong in [28], again with the help of measure–valued solution.

The framework of measure–valued solutions can be applied also in the context of the Navier–Stokes system. Although weak solutions are available here, their existence is constrained by the technical condition for the adiabatic exponent \( \gamma > \frac{3}{2} \). To handle this technical restriction, Feireisl et. al. [16] introduced the concept of dissipative measure-valued solution in terms of the Young measure. Here we use a slightly different approach introducing dissipative solution for Navier–Stokes system without an explicit presence of the Young measure. In such a way, we extend the convergence result to the Navier–Stokes system with high Reynolds number limit in the regime where the existence of weak solutions is not known.

In our approach, it is very important to consider proper initial data mainly termed as well-prepared and ill-prepared initial data. Feireisl and Novotný in [20], explain that for ill-prepared data the presence of Rossby-acoustic waves play an important role in analysis of singular limits. Meanwhile this effect was absent in well-prepared data. Here we deal with the well-prepared initial data.

Our main goal is to prove that under suitable choice of initial data a dissipative solution of compressible rotating Euler system in low Mach and low Rossby regime converges to strong solution of incompressible Euler system in 2D. Hence our plan for the article is,

1. Derivation of limit system.
2. Definition of dissipative solution.
3. Singular limit for ‘well-prepared’ data.
4. Extension to Navier–Stokes system.

1.1 Notation:

- To begin, we introduce a function \( \chi = \chi(\rho) \) such that

\[
\chi(\cdot) \in C^\infty_c(0, \infty), \quad 0 \leq \chi \leq 1, \quad \chi(\rho) = 1 \text{ if } \frac{1}{2} \leq \rho \leq 2. \quad (1.11)
\]

For a function, \( H = H(\rho, \mathbf{u}) \) we set

\[
[H]_{\text{ess}} = \chi(\rho)H(\rho, \mathbf{u}), \quad [H]_{\text{res}} = (1 - \chi(\rho))H(\rho, \mathbf{u}). \quad (1.12)
\]
Without loss of generality, we assume the ‘normalized’ setting for $p$ as
\[ p'(1) = 1. \tag{1.13} \]

Let us define pressure potential as,
\[ P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, dz. \tag{1.14} \]

As a consequence of that we have,
\[ \varrho P'(\varrho) - P(\varrho) = p(\varrho) \text{ and } \varrho P''(\varrho) = p'(\varrho) \text{ for } \varrho > 0. \tag{1.15} \]

## 2 Derivation of Limit systems

Here is an informal justification how we obtain the target system. First we note that $(\tilde{\varrho}_e, 0)$ is a steady state solution for (1.1)-(1.2).

Let us consider
\[ \begin{align*}
\varrho_e &= \tilde{\varrho}_e + \varepsilon^m \varrho_e^{(1)} + \varepsilon^{2m} \varrho_e^{(2)} + \cdots, \\
m_e &= \tilde{\varrho}_e v + \varepsilon^m m_e^{(1)} + \varepsilon^{2m} m_e^{(2)} + \cdots.
\end{align*} \]

As a consequence of the above we obtain,
\[ p(\varrho_e) = p(\tilde{\varrho}_e) + \varepsilon^m p'(\tilde{\varrho}_e) \varrho_e^{(1)} + \varepsilon^{2m} p'(\tilde{\varrho}_e) \varrho_e^{(2)} + \frac{1}{2} \varepsilon p''(\tilde{\varrho}_e) (\varrho_e^{(1)})^2 + o(\varepsilon^3). \]

We have static solution $(\tilde{\varrho}_e, 0)$
\[ \nabla \times p'(\tilde{\varrho}_e) = \varepsilon^{2(m-n)} \tilde{\varrho}_e \nabla \times G. \tag{2.1} \]

Clearly condition on $m$ and $n$ in (1.10) indicate $\lim_{\varepsilon \to 0} \nabla \times P'(\tilde{\varrho}_e) = 0$.

Without loss of generality we assume,
\[ \tilde{\varrho}_e \approx \varrho + \varepsilon^{2(m-n)}. \]

So we obtain,
\[ \tilde{\varrho} \text{div}_x v + \varepsilon^m (\partial_t \varrho_e^{(1)} + \text{div}_x (m_e^{(1)})) + o(\varepsilon^{2m}) = 0 \]
and
\[ \begin{align*}
\tilde{\varrho} \text{div}_x v + (v \cdot \nabla_x) v + \nabla_x (p'(\tilde{\varrho}) \varrho_e^{(1)}) &+ \frac{1}{2} \varepsilon p''(\tilde{\varrho})(\varrho_e^{(1)})^2 + \varepsilon^{m-1} b \times m_e^{(1)} \\
+ \frac{1}{\varepsilon^m} p'(\tilde{\varrho}) \nabla \times \varrho_e^{(1)} + \frac{1}{\varepsilon} (\tilde{\varrho} b \times v) + o(\varepsilon) &= 0. \tag{2.2}
\end{align*} \]

Let $\mathbb{H}$ be the Helmontz projection, then we have,
\[ \mathbb{H} \left( \partial_t (\varrho_e u_e) + \text{div}_x \left( \frac{m_e \otimes m_e}{\varrho_e^{(1)}} \right) + \frac{1}{\varepsilon} b \times \varrho_e u_e \right) = \mathbb{H} \left( \frac{1}{\varepsilon^{2m}} \varrho_e \nabla \times G \right). \tag{2.3} \]
Assuming \( m_{\epsilon} \to \tilde{\eta}v \) in some strong sense, multiplying the above equation by \( \epsilon \) and using our standard expansion technique, we obtain,

\[
\mathcal{H}[b \times \tilde{\eta}v] = 0
\]

From above relations we get,

\[
b \times v = \nabla_x \psi,
\]

\[
\psi(x_3) = 0, \quad \psi(x) = q(x_h), \quad \nabla \perp x_h \psi = v_h, \quad v_h = (v_1, v_2),
\]

\[
\text{div}_x v_h = 0, \quad \nabla_{x_3} v_h = 0, \quad \text{with \( \nabla \perp x_h \equiv (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}) \).}
\]

Thus we have,

\[
v_{1x_3} = 0, \quad v_{2x_3} = 0, \quad v(x) = v(x_h).
\]

Also boundary condition will lead us to conclude \( v_3(x_h, x_3) = 0 \). Thus we have \( v = (v_h(x_h), 0) \). Finally we obtain,

\[
\text{div}_x v_h = 0,
\]

\[
\partial_t v_h + (v_h \cdot \nabla x_h)v_h + \nabla x_h \Pi = 0. \tag{2.4}
\]

The above system is 2D Euler equation.

### 3 Definition of dissipative solution

#### 3.1 Choice of Static Solution:

As we have noticed during the informal discussion a static solution \( (\tilde{\eta}_{\epsilon}, 0) \) satisfies,

\[
\nabla_x p(\tilde{\eta}_{\epsilon}) = \epsilon^{2(m-n)} \tilde{\eta}_{\epsilon} \nabla_x G
\]

\[
\Rightarrow \nabla_x P'(\tilde{\eta}_{\epsilon}) = \epsilon^{2(m-n)} \nabla_x G.
\]

As a consequence of \( G = (0, 0, -x_3) \), we have \( \tilde{\eta}_{\epsilon}(x) = \tilde{\eta}_{\epsilon}(x_3) \). With an extra assumption \( \tilde{\eta}_{\epsilon}(0) = 1 \) we obtain

\[
P'(\tilde{\eta}_{\epsilon}) = -\epsilon^{2(m-n)} x_3 + P'(1). \tag{3.1}
\]

Finally we choose static solution \( \tilde{\eta}_{\epsilon} \) with the property,

\[
\sup_{x_3 \in [0,1]} |\tilde{\eta}_{\epsilon}(x_3) - 1| \leq \epsilon^{2(m-n)}, \quad \sup_{x_3 \in [0,1]} |\nabla_x \tilde{\eta}_{\epsilon}(x_3)| \leq \epsilon^{2(m-n)}. \tag{3.2}
\]

As \( m > n \), asymptotically, the static solution approaches the constant state \( \tilde{\eta} = 1 \) as \( \epsilon \to 0 \).

Now we will give the definition of dissipative solution for the system of our consideration.
Definition 3.1. Let $\epsilon > 0$ and $\tilde{\epsilon}_e > 0$. We say functions $\varrho_e, u_e$ with,

$$q_e - \tilde{\epsilon}_e \in C_{\text{weak}}([0, T]; L^2 + L^2(\Omega)), \quad \varrho_e \geq 0, \quad m_e \in C_{\text{weak}}([0, T]; L^2 + L^{\frac{2n}{m}}(\Omega)), $$

are a dissipative solution to the compressible Euler equation (1.1)-(1.7) with initial data $q_{0,e}, (q u)_{0,e}$ satisfying,

$$q_{0,e} \geq 0, \quad E_{0,e} = \int_{\Omega} \left( \frac{1}{2} \frac{|m_{0,e}|^2}{\varrho_{0,e}} + P(q_{0,e}) - (q_{0,e} - \tilde{\epsilon}_e)P'(\tilde{\epsilon}_e) - P(\tilde{\epsilon}_e) \right) \, dx < \infty, $$

if there exist the turbulent defect measures

$$\mathcal{R}_{m_e} \in L^\infty(0, T; \mathcal{M}^+(\Omega; \mathbb{R}^{d \times d}_\text{sym})), \quad \mathcal{R}_{e} \in L^\infty(0, T; \mathcal{M}^+(\Omega)), $$

satisfying compatibility condition

$$\lambda_1 \text{trace} \mathcal{R}_{m_e} \leq \mathcal{R}_{e} \leq \lambda_2 \text{trace} \mathcal{R}_{m_e}, \quad \lambda_1, \lambda_2 > 0,$$

such that the following holds,

- **Equation of Continuity:** For any $\tau \in (0, T)$ and any $\varphi \in C^1_c([0, T] \times \Omega)$ it holds

  $$[ \int_{\Omega} q_e \varphi \, dx ]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \left[ q_e \partial_t \varphi + m_e \cdot \nabla \varphi \right] \, dx \, dt. $$

- **Momentum equation:** For any $\tau \in (0, T)$ and any $\varphi \in C^1_c([0, T] \times \Omega; \mathbb{R}^d)$ with $\varphi \cdot n|_{\partial \Omega} = 0$, it holds

  $$\left[ \int_{\Omega} m_e (\tau, \cdot) \cdot \varphi (\tau, \cdot) \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \left[ m_e \cdot \partial_t \varphi + \frac{m_e \otimes m_e}{\varrho_e} : \nabla \varphi + \frac{1}{\epsilon^{2m}} P(q_e) \text{div} \varphi + \frac{1}{\epsilon} b \times m_e \cdot \varphi \right] \, dx \, dt + \int_0^\tau \int_{\Omega} \left[ \frac{1}{\epsilon^{2n}} q_e \nabla x G \cdot \varphi \right] \, dx \, dt + \int_0^\tau \int_{\Omega} \nabla x \varphi \cdot d\mathcal{R}_{m_e} \, dt. $$

- **Energy inequality:** The total energy $E$ is defined in $[0, T]$ as,

  $$E_e(\tau) = \int_{\Omega} \left( \frac{1}{2} \frac{|m_e|^2}{\varrho_e} + \frac{1}{\epsilon^{2m}} (P(q_e) - (q_e - \tilde{\epsilon}_e)P'(\tilde{\epsilon}_e)) - P(\tilde{\epsilon}_e) \right) (\tau, \cdot) \, dx. $$

It satisfies,

$$E_e(\tau) + \int_{\Omega} d\mathcal{R}_{e}(\tau, \cdot) \leq E_{0,e}$$

for a.a. $\tau > 0$. 

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Remark 3.2. It is important to define the function
\[(q_\epsilon, m_\epsilon) \mapsto \frac{1}{2} \frac{|m_\epsilon|^2}{q_\epsilon}\]
on the vacuum set as
\[(q_\epsilon, m_\epsilon) \mapsto \frac{1}{2} \frac{|m_\epsilon|^2}{q_\epsilon} = \begin{cases} 
\frac{1}{2} \frac{|m_\epsilon|^2}{q_\epsilon} & \text{if } q_\epsilon \neq 0, m_\epsilon \neq 0, \\
0 & \text{if } q_\epsilon = 0, m_\epsilon = 0, \\
\infty & \text{if } q_\epsilon = 0, m_\epsilon \neq 0.
\end{cases}\]
It follows from the energy inequality (3.9) that for each \(\epsilon > 0\),
\[L^4 \left( \left\{ (t, x) \in (0, T) \times \Omega \left| \frac{1}{2} \frac{|m_\epsilon|^2}{q_\epsilon} = \infty \right. \right\} \right) = 0,\]
where \(L^4\) is the Lebesgue measure in \(\mathbb{R}^4\).

Theorem 3.3. Suppose \(\Omega\) be the domain specified above and pressure follows (1.4). If \((q_{0,\epsilon}, m_{0,\epsilon})\) satisfies (3.4), then there exists dissipative solution as defined in definition (3.1).

The proof this theorem follows in similar lines of Breit, Feireisl and Hofmanová as in [5, 6]. We have to adopt it for unbounded domain as suggested in Basarič [4].

4 Singular limit for "Well-prepared" initial Data

4.1 Target System

Taking motivation from [21] we expect the target system as,
\[
\begin{align*}
\text{div}_{x_h} v_h &= 0, \text{ in } \mathbb{R}^2, \\
\partial_t v_h + (v_h \cdot \nabla_{x_h}) v_h + \nabla_{x_h} \Pi &= 0, \text{ in } \mathbb{R}^2.
\end{align*}
\] (4.1)

The result stated below by Kato and Lai in [24] ensures the existence and uniqueness of Euler system in 2D:

Proposition 4.1. Let
\[v_0 \in W^{k,2}(\mathbb{R}^2; \mathbb{R}^2), \ k \geq 3, \ \text{div}_{x_h} v_0 = 0 \]
be given. Then the system (4.1) supplemented with initial data \(v_h(0) = v_0\) admits regular solution \((v_h, \Pi)\), unique in the class
\[
\begin{align*}
v_h &\in C([0, T]; W^{k,2}(\mathbb{R}^2; \mathbb{R}^2)), \ \partial_t v_h, v_h &\in C([0, T]; W^{k-1,2}(\mathbb{R}^2; \mathbb{R}^2)), \\
\Pi &\in C([0, T]; W^{k,2}(\mathbb{R}^2)),
\end{align*}
\] (4.2)
with \(\text{div}_{x_h} v_h = 0\).

Taking motivation from (2.2) we consider another equation that describes some non-oscillatory part,
\[\nabla_{x} q_\epsilon + \epsilon^{m-1} b \times v = 0.\] (4.3)

Also we choose \(q_\epsilon(0, \cdot) = q_{0,\epsilon}\) such that \(-\Delta_{x_h} q_{0,\epsilon} = \epsilon^{m-1} \tilde{q} \text{curl}_{x_h} P_h(v_0)\).
4.2 Definition of "Well-prepared Data"

Definition 4.2. We say that the set of initial data \( \{(\varrho_{0,e}, m_{0,e})\}_{e>0} \) is "well-prepared" if,

\[
\begin{align*}
\varrho_e(0, \cdot) &= \varrho_{0,e} = \bar{\varrho}_e + e^m \varrho_{0,e}^{(1)} \quad \text{with} \quad \varrho_{0,e}^{(1)} \rightarrow 0 \text{ in } L^2(\Omega), \\
u_{0,e} &= \frac{m_{0,e}}{\varrho_{0,e}} \rightarrow v_0 = (v_0^{(1)}, v_0^{(2)}, 0) \text{ in } L^2(\Omega; \mathbb{R}^3) \text{ with } \text{div}_x h v_0 = 0.
\end{align*}
\]

(4.4)

4.3 Main Theorem:

Theorem 4.3. Let pressure \( p \) follows (1.4). We assume that the initial data is well-prepared, i.e. it follows (4.4). Let \( v_0 \in W^{k,2}(\Omega) \) with \( k \geq 3 \). Let \( (\varrho_e, u_e) \) be a dissipative solution as in definition (3.1) in \((0, T) \times \Omega\). Then,

\[
\text{ess sup}_{t \in (0,T)} \| \varrho_e - \bar{\varrho}_e \|_{(L^2+L^\gamma)(\Omega)} \leq e^m c
\]

(4.5)

where, \( v = (v_h, 0) \) is the unique solution of Euler system with initial data \( v_0 \).

In the following subsections we give the proof.

4.4 Relative energy inequality

In our approach, relative energy functional plays an important role. We consider,

\[
\mathcal{E}_e(t) = \mathcal{E}(\varrho_e, m_e|\bar{\varrho}, \bar{u})(t) \quad := \int_{\Omega} \left[ \frac{1}{2} \frac{m_e}{\varrho_e} - \bar{u} \right]^2 + \frac{1}{e^{2m}} [P(\varrho_e) - P(\bar{\varrho}) - P'(\bar{\varrho})(\varrho_e - \bar{\varrho})] (t, \cdot) \, dx,
\]

(4.6)

where \( \bar{\varrho}, \bar{u} \) are arbitrary smooth test functions with \( \bar{\varrho} - \bar{\varrho}_e \) have compact support and \( \bar{u} \cdot n = 0 \) on \( \partial \Omega \).

Remark 4.4. The relative energy is a coercive functional (see. Bruell et. al. [9]) satisfying the estimate,

\[
\mathcal{E}(\varrho_e, u_e|\bar{\varrho}, \bar{u})(t) \geq \int_{\Omega} \left[ \frac{m_e}{\varrho_e} - \bar{m} \right]^2 \, dx + \int_{\Omega} \left[ \frac{|m_e|^2}{\varrho_e} \right] \, dx + \frac{1}{e^{2m}} \int_{\Omega} \left[ (\varrho_e - \bar{\varrho})^2 \right] \, dx + \frac{1}{e^{2m}} \int_{\Omega} \left[ 1 \right] \, dx + |\bar{\varrho}_e| \, dx.
\]

(4.7)
Using definition (3.1) we obtain relative energy inequality,

\[ \mathcal{E}_c(\tau) + \int_{\Omega} d\mathcal{R}_c(\tau, \cdot) \]

\[ \leq \mathcal{E}_c(0) - \int_0^\tau \int_{\Omega} (m_e - \varrho_e \bar{u}) \cdot \partial_t \bar{u} \, dx \, dt - \int_0^\tau \int_{\Omega} \left( \frac{(m_e - \varrho_e \bar{u}) \otimes m_e}{\varrho_e} \right) : \nabla \bar{u} \, dx \, dt \]

\[ - \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_{\Omega} (p(\varrho_e) - p(\bar{\varrho})) \text{div}_\tau \bar{u} \, dx \, dt + \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_{\Omega} (\bar{\varrho} - \varrho_e) \partial_t P'(\bar{\varrho}) \, dx \, dt \]

\[ + \frac{1}{\varepsilon} \int_0^\tau \int_{\Omega} b \times m_e \cdot \left( \bar{u} - \frac{m_e}{\varrho_e} \right) \, dx \, dt \]

\[ + \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_{\Omega} (\bar{\varrho} \bar{u} - m_e) \cdot (\nabla \times P'(\bar{\varrho}) - \nabla \times P'(\varrho_e)) \, dx \, dt \]

\[ - \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_{\Omega} (\varrho_e - \bar{\varrho}) \nabla G \cdot \bar{u} \, dx \, dt - \int_0^\tau \int_{\Omega} \nabla \bar{u} : d\mathcal{R}_{mc}(t, \cdot) \, dt. \]

(4.8)

Following [4] we can extend the above inequality for functions \((\bar{\varrho}, \bar{u})\) having Sobolev regularities, i.e. \((\bar{\varrho} - \varrho_e, \bar{u}) \in C^1([0, T]; W^{k,2}_0(\Omega)) \times C^1([0, T]; W^{k,2}(\Omega; \mathbb{R}^3))\) with \(k \geq 3\).

We rewrite the above inequality as,

\[ \left[ \mathcal{E}_c(\varrho_e, m_e|\bar{\varrho}; \bar{u}) \right]_0^\tau + \int_{\Omega} d\mathcal{R}_c(\tau, \cdot) + \left[ \mathcal{R}_c(\varrho_e, m_e|\bar{\varrho}; \bar{u}) \right] \leq 0. \]

(4.9)

Suppose, \((\bar{\varrho} - \varrho_e, \bar{u}) \in C^1([0, T]; W^{k,2}_0(\Omega)) \times C^1([0, T]; W^{k,2}(\Omega; \mathbb{R}^3))\) with \(\bar{u} \cdot n|_{\partial \Omega} = 0\) and \(k \geq 3\). We know \(C^1([0, T]; C^\infty(\Omega))\) is dense in Sobolev space \(C^1([0, T]; W^{k,2}(\Omega))\) and \(\{ \bar{u} \in C^1([0, T]; C^\infty(\Omega; \mathbb{R}^3)) | \bar{u} \cdot n|_{\partial \Omega} = 0 \} \) is dense in \(\{ \bar{u} \in C^1([0, T]; W^{k,2}(\Omega; \mathbb{R}^3)) | \bar{u} \cdot n|_{\partial \Omega} = 0 \} \).

For \(\delta > 0\) we have, \(\bar{r} \in C^1([0, T]; C^\infty(\Omega))\) and \(\bar{v} \in C^1([0, T]; C^\infty(\Omega))\) with \(\bar{v} \cdot n|_{\partial \Omega} = 0\).

\[ \| \bar{r} - \varrho \|_{C^1([0, T]; W^{k,2}(\Omega))} + \| \bar{v} - \bar{u} \|_{C^1([0, T]; W^{k,2}(\Omega))} < \delta. \]

Following Theorem 2.3 of [4] we can show that,

\[ \left[ \mathcal{E}_c(\varrho_e, m_e|\bar{\varrho}; \bar{u}) \right]_0^\tau + \int_{\Omega} d\mathcal{R}_c(\tau, \cdot) + \left[ \mathcal{R}_c(\varrho_e, m_e|\bar{\varrho}; \bar{u}) \right] \]

\[ \leq \left[ \mathcal{E}_c(\varrho_e, m_e|\bar{r}; \bar{v}) \right]_0^\tau + \int_{\Omega} d\mathcal{R}_c(\tau, \cdot) + \left[ \mathcal{R}_c(\varrho_e, m_e|\bar{r}; \bar{v}) \right] + C\delta \]

(4.10)

Thus for Sobolev functions, relative energy inequality (4.8) is true.
4.5 Convergence: Part 1

First with $\bar{u} = 0, \bar{\rho} = \bar{\rho}_e$ as test functions we have the following bounds,

\[
\text{ess sup}_{t \in (0,T)} \left\| \frac{m_e}{\sqrt{\rho_e}} \right\|_{L^2(\Omega; \mathbb{R}^3)} \leq C,
\]

\[
\text{ess sup}_{t \in (0,T)} \left\| \left[ \frac{\rho_e - \bar{\rho}_e}{\epsilon^m} \right] \right\|_{L^2(\Omega)} \leq C, \tag{4.11}
\]

\[
\text{ess sup}_{t \in (0,T)} \| [\rho_e]_{\text{res}} \|^2 \gamma (\Omega) + \text{ess sup}_{t \in (0,T)} \| [1]_{\text{res}} \|^2 \gamma (\Omega) \leq \epsilon^{2m} C.
\]

Now we want to calculate $\| \rho_e - \bar{\rho}_e \|_{(L^2 + L^\gamma)(\Omega)}$. We rewrite,

\[
\| \rho_e - \bar{\rho}_e \|_{(L^2 + L^\gamma)(\Omega)} \leq \| [\rho_e - \bar{\rho}_e]_{\text{ess}} \|_{L^2(\Omega)} + \| [\rho_e - \bar{\rho}_e]_{\text{res}} \|_{L^\gamma(\Omega)}.
\]

From the above estimates and using fact $m > \gamma$ we have,

\[
\text{ess sup}_{t \in (0,T)} \| \rho_e - \bar{\rho}_e \|_{(L^2 + L^\gamma)(\Omega)} \leq \epsilon^m C.
\]

Now above estimates conclude that,

\[
\rho_e \to 1 \text{ in } L^\infty(0,T; L^q_{\text{loc}}(\Omega)) \text{ for any } 1 \leq q < \gamma. \tag{4.12}
\]

Combining above estimates we obtain,

\[
\frac{m_e}{\sqrt{\rho_e}} \to u \text{ in } L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)),
\]

and

\[
m_e \to u \text{ weakly-(*) in } L^\infty(0,T; L^2 + L^{2\gamma/\gamma+1}(\Omega; \mathbb{R}^3)),
\]

passing to suitable subsequences.

Finally we may let $\epsilon \to 0$ in the continuity equation to deduce that,

\[
\int_0^T \int_\Omega u \cdot \nabla \phi \, dx \, dt = 0, \quad \forall \phi \in C^\infty_c(\Omega).
\]

4.6 Convergence: Part 2

Here we choose proper test functions and will show that $\lim_{\epsilon \to 0} E_\epsilon(t) = 0$.

We consider,

\[
\bar{u} = v, \quad \bar{\rho} = \bar{\rho}_e + \epsilon^m q_e, \tag{4.13}
\]
where, \( \mathbf{v} = (\mathbf{v}_h, 0) \), \( \mathbf{v}_h \) as a solution of (4.1), \( q_e \) solves (4.3) and \( \tilde{q}_e \) satisfies (1.6). Using relation of \( q_e \) and \( \mathbf{v} \) we obtain,

\[
E_e(\tau) + \int_0^\tau \int_{\Omega} | \partial_\tau \mathbf{v} + (\mathbf{v} \cdot \nabla \mathbf{v}) | \ dx \ dt \\
\leq E_e(0) - \int_0^\tau \int_{\Omega} (m_e - q_e \mathbf{v}) \cdot (\partial_\tau \mathbf{v} + (\mathbf{v} \cdot \nabla \mathbf{v}) \mathbf{v}) \ dx \ dt \\
- \int_0^\tau \int_{\Omega} (m_e - q_e \mathbf{v}) \Theta (m_e - q_e \mathbf{v}) : \nabla \mathbf{u} \ dx \ dt \\
+ \frac{1}{\epsilon^{2m}} \int_0^\tau \int_{\Omega} (\tilde{q}_e - \tilde{q}_e) \Theta \mathbf{P}'(\tilde{q}_e) \ dx \ dt \\
+ \frac{1}{\epsilon^{2m}} \int_0^\tau \int_{\Omega} m_e \cdot \nabla \mathbf{v} (P''(\tilde{q}_e) - P''(1)) \ dx \ dt \\
+ \frac{1}{\epsilon^{2m}} \int_0^\tau \int_{\Omega} m_e \cdot (P''(\tilde{q}_e) - P''(\tilde{q}_e)) \nabla \tilde{q}_e \ dx \ dt \\
- \frac{1}{\epsilon^{2m}} \int_0^\tau \int_{\Omega} (q_e - \tilde{q}) \nabla \mathbf{G} \cdot \mathbf{u} \ dx \ dt - \int_0^\tau \int_{\Omega} \nabla \mathbf{u} : \text{d} \mathbf{R}_m(t, \cdot) \ dx \ dt \ = \Sigma_{i=1}^8 \mathcal{L}_i. 
\]

(4.14)

From the relation \( \nabla \mathbf{v} q_e + \epsilon^{m-1} \mathbf{b} \times (\mathbf{v}_h, 0) = 0 \), and (4.2) we can conclude that

\[
\| q_e \|_{L^\infty(0,T;L^2(\Omega))} + \| \eta q_e \|_{L^\infty(0,T;L^2(\Omega))} + \| \nabla \mathbf{v} q_e \|_{L^\infty(0,T;L^2(\Omega))} \leq \epsilon^{m-1} c, \ \forall q \geq 2. 
\]

(4.15)

Also we have \( \| q_{0,e} \|_{L^\infty(0,T;L^2(\Omega))} \leq \epsilon^{m-1} c. \)

Let us calculate each term \( \mathcal{L}_i, i = 1(1)8 \) of (4.14). For term \( \mathcal{L}_1 \) we have,

\[
E_e(q_{0,e}, (\mathbf{u}q)_{0,e} | \tilde{q}_e + \epsilon q_{0,e}, \mathbf{v}_0) \leq \| (\mathbf{u}q)_{0,e} - \mathbf{v}_0 \|_{L^2(\Omega)}^2 + \| q_{0,e}^1 - q_{0,e} \|_{L^2(\Omega)}^2.
\]

Consideration of well prepared data yields,

\[
| \mathcal{L}_1 | \leq \xi(\epsilon). 
\]

(4.16)

From now on we use this generic function \( \xi(\epsilon) \), such that \( \lim_{\epsilon \to 0} \xi(\epsilon) = 0. \)

Now using our convergence in earlier part and \( \mathbf{v}_h \) solves (4.1) we obtain,

\[
\mathcal{L}_2 = -\int_0^\tau \int_{\Omega} (m_e - q_e \mathbf{v}) \cdot (\partial_\tau \mathbf{v} + (\mathbf{v} \cdot \nabla \mathbf{v}) \mathbf{v}) \ dx \ dt \\
\quad = -\int_0^\tau \int_{\Omega} (m_e - \mathbf{v}) \cdot \nabla \mathbf{v} \Pi \ dx \ dt + \int_0^\tau \int_{\Omega} (1 - q_e) \mathbf{v} \cdot \nabla \mathbf{v} \Pi \ dx \ dt \\
\quad = \mathcal{L}_{2/r} + \mathcal{L}_{2/r}.
\]

It is easy to verify that,

\[
\mathcal{L}_{2/r} \xrightarrow{\epsilon \to 0} \int_0^\tau \int_{\Omega} (\mathbf{u} - \mathbf{v}) \nabla \mathbf{v} \Pi \ dx \ dt = 0.
\]
We observe that, 

\[ |\mathcal{L}_{2n}| \leq \tilde{\zeta}(e). \] 

Thus we can conclude, 

\[ |\mathcal{L}_2| \leq \tilde{\zeta}(e). \] 

(4.17)

Now we also obtain,

\[ |\mathcal{L}_3| \leq \|\nabla x_q v_h\|_{L^\infty(0,T; L^2(\Omega))} \int_0^T \mathcal{E}_e(t) \, dt. \] 

(4.18)

We want to estimate the term \( \mathcal{L}_4 \). First we rewrite it as,

\[ \mathcal{L}_4 = \frac{1}{e^m} \int_0^T \int_{\Omega} (\bar{q} - q_e)^{P''}(\bar{q}) \partial_t q_e \, dx \, dt \]

\[ = \int_0^T \int_{\Omega} \frac{\bar{q}_e - q_e}{e^m} \partial_t q_e P''(\bar{q}) \, dx \, dt + \int_0^T \int_{\Omega} q_e \partial_t q_e P''(\bar{q}) \, dx \, dt. \]

We observe that,

\[ |\mathcal{L}_4| \leq \sup_{t \in (0,T)} \left\| \frac{\bar{q}_e - \bar{q}}{e^m} \right\|_{L^2(\Omega)} \left\| \partial_t q_e \right\|_{L^\infty(0,T; L^2(\Omega))} \]

\[ + \sup_{t \in (0,T)} \left\| \frac{\bar{q}_e - \bar{q}}{e^m} \right\|_{L^7(\Omega)} \left\| \partial_t q_e \right\|_{L^\infty(0,T; L^7(\Omega))} \]

\[ + \left\| \partial_t q_e \right\|_{L^\infty(0,T; L^2(\Omega))} \left\| q_e \right\|_{L^\infty(0,T; L^2(\Omega))}. \]

Using bound of \( \bar{q}_e \) as in (4.15) and (4.11), we obtain,

\[ |\mathcal{L}_4| \leq \tilde{\zeta}(e). \] 

(4.19)

We write \( \mathcal{L}_5 \) as,

\[ \mathcal{L}_5 = \frac{1}{e^m} \int_0^T \int_{\Omega} m_e \cdot \nabla x \{ q_e (\bar{q} - 1) P''(\eta(x)) \} \, dx \, dt \]

\[ = \frac{1}{e^m} \int_0^T \int_{\Omega} m_e \cdot \nabla x q_e (\bar{q}_e - 1) P''(\eta(x)) \, dx \, dt \]

\[ + \int_0^T \int_{\Omega} m_e \cdot \nabla x q_e q_e P''(\eta(x)) \, dx \, dt. \]

By using (4.11) and (4.15) we observe,

\[ |\mathcal{L}_5| \leq e^{m-2m} \| m_e \|_{L^\infty(0,T; L^2+L^{2\gamma+1}(\Omega; \mathbb{R}^3))} \| \nabla x q_e \|_{L^\infty(0,T; L^2+L^{2\gamma+1}(\Omega; \mathbb{R}^3))} \]

\[ + \| m_e \|_{L^\infty(0,T; L^2+L^{2\gamma+1}(\Omega; \mathbb{R}^3))} \| q_e \nabla x q_e \|_{L^\infty(0,T; L^2+L^{2\gamma+1}(\Omega; \mathbb{R}^3))} \]

\[ \leq \tilde{\zeta}(e). \] 

(4.20)

Similarly for \( \mathcal{L}_6 \) we rewrite as,

\[ \mathcal{L}_6 = \frac{1}{e^{2m}} \int_0^T \int_{\Omega} m_e \cdot (\bar{q} - \bar{q}_e) P''(\tilde{\zeta}(x)) \nabla x \bar{q}_e \, dx \, dt \]

\[ = \frac{1}{e^m} \int_0^T \int_{\Omega} m_e \cdot q_e P''(\tilde{\zeta}(x)) \nabla x \bar{q}_e \, dx \, dt. \]
Arguing in the same line as before we have,
\[ |\mathcal{L}_6| \leq e^{m-2n} \| m_\varepsilon \|_{L^\infty(0,T;L^2+L^{2n/\gamma+3}(\Omega;\mathbb{R}^3))} \| q_\varepsilon \|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \]
\[ \leq \xi(\varepsilon). \]
Now choice of \( G \) implies
\[ \mathcal{L}_7 = 0. \]
We also have,
\[ |\mathcal{L}_8| \leq \int_0^T \int_\Omega \mathcal{R}_{\varepsilon} \, dt. \]
Thus combining all estimates \((4.16)-(4.23)\) we have,
\[ E_\varepsilon(\tau) + \int_\Omega d \mathcal{R}_{\varepsilon}(\tau, \cdot) \leq \xi(\varepsilon) + c \int_0^\tau E_\varepsilon(t) \, dt + c \int_0^\tau \int_\Omega d \mathcal{R}_{\varepsilon} \, dt. \]
Using Grönwall lemma we have,
\[ E_\varepsilon(\tau) + \int_\Omega d \mathcal{R}_{\varepsilon}(\tau, \cdot) \leq \xi(\varepsilon)C(T), \]
where \( \xi(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Using coercivity of relative energy functional we obtain,
\[ \limsup_{\varepsilon \to 0} \int_K \left| \frac{m_\varepsilon}{\sqrt{\rho_\varepsilon}} - v \right|^2 \, dx \leq C(T) \limsup_{\varepsilon \to 0} \xi(\varepsilon), \]
where, \( K \subset \Omega \) is a compact set. Hence we can conclude that \( v = u = v_h \). Also we have,
\[ \frac{m_\varepsilon}{\sqrt{\rho_\varepsilon}} \to v \strongly \text{ in } L^1_{\text{loc}}((0, T) \times \Omega; \mathbb{R}^3). \]

It ends proof of the theorem.

5 Extension to the Navier–Stokes System

Here our goal is to give a proper definition of dissipative solution for Navier–Stokes equation. We consider another characteristic number i.e. Reynolds number. In high Reynolds number limit, we will obtain the same target system.

5.1 Definition of dissipative Solution for Navier-Stokes system:

Let \( \rho \) be the density and \( u \) be the velocity. In time-space cylinder \( Q_T = (0, T) \times \Omega \), we consider:

- Conservation of Mass:
  \[ \partial_t \rho + \text{div}_x (\rho u) = 0. \]
Conservation of Momentum:
\[
\partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \frac{1}{Ma^2} \nabla_x p(\varrho) + \frac{1}{Ro} \mathbf{b} \times \varrho u = \frac{1}{Re} \text{div}_x S(\nabla_x u) + \frac{1}{Fr^2} \varrho \nabla_x G.
\] (5.2)

Constitutive Relation: Here \(S(\nabla_x u)\) is Newtonian stress tensor defined by
\[
S(\nabla_x u) = \mu \left( \frac{\nabla_x u + \nabla^T_x u}{2} - \frac{1}{d}(\text{div}_x u)I \right) + \lambda (\text{div}_x u)I,
\] (5.3)
where \(\mu > 0\) and \(\lambda > 0\) are the shear and bulk viscosity coefficients, respectively.

The scaled system contains all specified characteristic numbers as in (1.3) along with,
\[
\text{Re} = \text{Reynolds number}.
\]
Here we consider,
\[
\text{Ma} \approx \epsilon^m, \text{Ro} \approx \epsilon, \text{Re} \approx \epsilon^{-\alpha}, \text{Fr} \approx \epsilon^n \text{ for } \epsilon > 0, m, n, \alpha > 0 \text{ and } \frac{m}{2} > n \geq 1.
\] (5.4)

Pressure Law: In an isentropic setting, the pressure \(p\) and the density \(\varrho\) of the fluid are interrelated by
\[
p(\varrho) = a\varrho^\gamma, \, a > 0, \, \gamma > 1.
\] (5.5)

Boundary condition: Here we consider complete slip condition for velocity on the horizontal boundary i.e.
\[
\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = [S(\nabla_x u) \cdot \mathbf{n}]_{\text{tan}}|_{\partial \Omega} = 0, \, \mathbf{n} = (0, 0, \pm 1).
\] (5.6)

Far field condition: Let \((\bar{\varrho}_e, 0)\) be a static solution. we assume the condition as,
\[
|\rho - \bar{\rho}_e| \to 0, \, \mathbf{u} \to 0 \text{ as } |x_h| \to \infty.
\] (5.7)

Initial data: For each \(\epsilon > 0\), we supplement the initial data as
\[
\varrho(0, \cdot) = \varrho_{0e,0} (\varrho u)(0, \cdot) = (\varrho u)_{\epsilon,0}.
\] (5.8)

Here we provide the definition of dissipative solution for the Navier–Stokes system.

**Definition 5.1.** Let \(\epsilon > 0\) and \(\bar{\varrho}_e > 0\). We say functions \(\varrho_e, \mathbf{u}_e\) with,
\[
\varrho_e - \bar{\rho}_e \in C_{\text{weak}}([0, T]; L^2 + L^\gamma(\Omega)), \, \varrho_e \geq 0, \, \varrho_e \mathbf{u}_e \in C_{\text{weak}}([0, T]; L^2 + L^{\gamma 2/\gamma}(\Omega)),
\] (5.9)
\[
\mathbf{u}_e \in L^2(0, T; W^{1,2}(\Omega)), \, \mathbf{u}_e \cdot \mathbf{n}|_{\partial \Omega} = 0,
\] (5.10)
are a **dissipative solution** to (5.1)-(5.7) with initial data \( q_{0,e}, (q_{0}u)_{0,e} \) satisfying,
\[
q_{0,e} \geq 0, \quad E_{0,e} = \int_{\Omega} \left( \frac{1}{2} \frac{|(q_{0}u)_{0,e}|^2}{q_{0,e}} + P(q_{0,e}) - (q_{0,e} - \bar{q}_e)P'(\bar{q}_e) - P(\bar{q}_e) \right) \, dx < \infty,
\]

if there exist the **turbulent defect measures**
\[
\mathcal{R}_{m_e} \in L^\infty(0,T; \mathcal{M}^+(\underline{\Omega}; \mathbb{R}^{d \times d}_{\text{sym}})), \quad \mathcal{R}_{e} \in L^\infty(0,T; \mathcal{M}^+(\Omega)),
\]
satisfying compatibility condition
\[
\lambda_1 \text{trace} \mathcal{R}_{m_e} \leq \mathcal{R}_{e} \leq \lambda_2 \text{trace} \mathcal{R}_{m_e}, \quad \lambda_1, \lambda_2 > 0,
\]
such that the following holds,

**Equation of Continuity:** For any \( \tau \in (0,T) \) and any \( \varphi \in C^1_c([0,T] \times \Omega) \) it holds
\[
\left[ \int_{\Omega} q_{e} \varphi \, dx \right]_{t=0}^{t=\tau} = \int_{0}^{\tau} \int_{\Omega} \left[ q_{e} \varphi_{t} + q_{e} u_{e} \cdot \nabla \varphi \right] \, dx \, dt.
\]

**Momentum equation:** For any \( \tau \in (0,T) \) and any \( \varphi \in C^1_c([0,T] \times \Omega; \mathbb{R}^d) \) with \( \varphi \cdot n \mid_{\partial \Omega} = 0 \), it holds
\[
\left[ \int_{\Omega} q_{e} u_{e}(\tau, \cdot) \cdot \varphi(\tau, \cdot) \, dx \right]_{t=0}^{t=\tau} = \int_{0}^{\tau} \int_{\Omega} \left[ q_{e} u_{e} \cdot \varphi_{t} + q_{e} u_{e} \otimes u_{e} : \nabla \varphi + \frac{1}{\varepsilon} \nu_{e} \partial_{\nu} \varphi + \frac{1}{\varepsilon} b \times q_{e} u_{e} \cdot \varphi \right] \, dx \, dt
\]
\[
- \int_{0}^{\tau} \int_{\Omega} [\varepsilon \partial_{\nu}(\nabla_{x} u) : \nabla_{x} \varphi - \frac{1}{\varepsilon^{2m}} q_{e} \nabla_{x} G : \varphi] \, dx \, dt + \int_{0}^{\tau} \int_{\Omega} \nabla_{x} \varphi : d\mathcal{R}_{m_e} \, dt.
\]

**Energy inequality:** The total energy \( E \) is defined in \( [0,T] \) as,
\[
E_{e}(\tau) = \int_{\Omega} \left( \frac{1}{2} q_{e} |u_{e}|^2 + \frac{1}{\varepsilon^{2m}} (P(q_{e}) - (q_{e} - \bar{q}_e)P'(\bar{q}_e)) - P(\bar{q}_e) \right)(\tau, \cdot) \, dx.
\]

It satisfies,
\[
E_{e}(\tau) + e^{\alpha} \int_{0}^{\tau} \int_{\Omega} S(\nabla_{x} u) : \nabla_{x} u \, dx \, dt + \int_{\Omega} d \mathcal{R}_{e} (\tau, \cdot) \leq E_{0,e}
\]
for a.a. \( \tau > 0 \).

**Remark 5.2.** The class of test functions in the momentum equations correspond to the complete slip (Navier slip) boundary conditions. These are necessary to avoid problems with boundary layer.
Theorem 5.3. Suppose $\Omega$ be the domain specified above and pressure follows (5.5). If $(q_{0,\epsilon}, (\eta u)_{0,\epsilon})$ satisfies (5.11), then there exists dissipative solution as defined above.

We prove the existence theorem for $\epsilon = 1$.

Proof. Here we give an extended outline of proof. We know that the existence theory in the class of finite energy weak solutions was developed by Lions [27] and later extended by Feireisl [13] to the so far subcritical exponent $\gamma > \frac{3}{2}$. For unbounded domain similar result has been proposed by Novotný and Pokorný in [29].

Here our goal is to add $\delta \nabla x q^T$ in the momentum equation with $\Gamma \geq \frac{3}{2}$ and then the system admits a finite energy weak solutions $(q_{\delta}, u_{\delta})$. Then we will show that this approximate solution converges to a dissipative solution of above described system. This motivates the following approximate problem,

\[ \partial_t q_{\delta} + \text{div}_x (q_{\delta} u_{\delta}) = 0, \]

\[ \partial_t (q_{\delta} u_{\delta}) + \text{div}_x (q_{\delta} u_{\delta} \otimes u_{\delta}) + \nabla x p(q_{\delta}) + \delta \nabla x q_{\delta}^T + b \times q_{\delta} u_{\delta} = \text{div}_x S(\nabla x u_{\delta}) + q_{\delta} \nabla x G, \] (5.17)

\[ u_{\delta} \cdot n|_{\partial \Omega} = [S(\nabla x u_{\delta}) \cdot n]|_{\partial \Omega} = 0. \] (5.18)

\[ (\Omega) = \int_\Omega \left( \frac{1}{2} \left| \frac{(\eta u)_{0,\delta}}{q_{0,\delta}} \right|^2 + H(q_{0,\delta}) - (q_{0,\delta} - \tilde{q}_{\delta}) H'(\tilde{q}_{\delta}) - H(\tilde{q}_{\delta}) \right) \, dx \]

\[ \rightarrow E_0 = \int_\Omega \left( \frac{1}{2} \left| \frac{(\eta u)_{0}}{q_{0}} \right|^2 + P(q_0) - (q_0 - \tilde{q}) P'(\tilde{q}) - P(\tilde{q}) \right) \, dx \text{ in } L^1(\Omega), \]

where $H(s) = \frac{1}{1 - \gamma} p(s) + \delta \frac{1}{1 - 3\gamma} s^\gamma$. Clearly from existence of weak solution we have several apriori bounds, i.e.

\[ \|q_{\delta} - \tilde{q}_{\delta}\|_{L^\infty(0,T;L^2+L^7(\Omega))} \leq C, \]

\[ \|\sqrt{q_{\delta} u_{\delta}}\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \]

\[ \|u_{\delta}\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)} \leq C, \]

\[ \|H(q_{\delta}) - (q_{\delta} - \tilde{q}_{\delta}) H'(\tilde{q}_{\delta}) - H(\tilde{q}_{\delta})\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \]

As a consequence of that we obtain,

\[ \|q_{\delta} u_{\delta}\|_{L^\infty(0,T;L^2+L^{5\gamma+1}(\Omega;\mathbb{R}^3))} \leq C, \]

\[ \text{ess sup}_{t \in [0,T]} \int_\Omega \delta q_{\delta}^T \, dx \leq C. \] (5.21)
From the above bounds we get,

$$
\delta u \rightarrow u \text{ weakly in } L^2(0, T; W^{1,2}(\Omega : \mathbb{R}^3)).
$$

Following the above consequence we also conclude,

$$
\delta u \rightarrow u \text{ weakly in } L^2(0, T; W^{1,2}(\Omega : \mathbb{R}^3)).
$$

Let us introduce the conservative variable \( m_\delta = \delta u \). In terms of momentum we rewrite kinetic energy as,

$$
\left( \delta \rho, m_\delta \right) \mapsto \frac{1}{2} \left( \frac{|m_\delta|^2}{\rho_\delta} \right) = \begin{cases} 
\frac{1}{2} \left( \frac{|(\delta \rho \delta \cdot u_\delta)|^2}{\delta \rho} \right) & \text{if } \delta \rho \neq 0, m_\delta \neq 0, \\
0 & \text{if } \delta \rho = 0, m_\delta = 0, \\
\infty & \text{if } \delta \rho = 0, m_\delta \neq 0.
\end{cases}
$$

As an observation we have the above map is convex l.s.c. From energy inequality, it is worth to notice that it is \( \infty \) only on a measure zero set in (0, T) \times \Omega. Using convexity of \( p(\cdot) \) and \([\rho, m] \mapsto \frac{m \times m}{\rho} \), and also using fact \( L^1(\Omega) \) continuously embedded in \( M(\bar{\Omega}) \), we conclude,

$$
\begin{align*}
\frac{m_\delta \times m_\delta}{\rho_\delta} &\rightarrow \frac{m \times m}{\rho} \text{ weakly-(*'} in \ L^\infty(0, T; M(\bar{\Omega}; \mathbb{R}^{d \times d})), \\
p(\rho_\delta) &\rightarrow p(\rho) \text{ weakly-(*'} in \ L^\infty(0, T; M(\bar{\Omega})), \\
\delta \rho_\delta (t, \cdot) &\rightarrow \zeta \text{ weakly-(*'} in \ L^\infty(0, T; M(\bar{\Omega})).
\end{align*}
$$

We choose,

$$
\begin{align*}
\mathcal{R}_m &= \left[ \frac{m \times m}{\rho} - \frac{m \times m}{\rho} \right] + \left[ p(\rho) - p(\rho) + \zeta \right], \tag{5.23}
\end{align*}
$$

and

$$
\begin{align*}
\mathcal{R}_e &= \frac{1}{2} \frac{|m|^2}{\rho} - \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho) - P(\rho) + \frac{1}{1 - 1}, \tag{5.24}
\end{align*}
$$

Clearly the compatibility of two turbulent defect measure is clear from above equations. Arguing similarly as in Breit et. al. [6] we obtain,

$$
\mathcal{R}_m \in L^\infty(0, T; M^+(\bar{\Omega}; \mathbb{R}^{d \times d})), \mathcal{R}_e \in L^\infty(0, T; M^+(\bar{\Omega})).
$$

Now we are in a position to conclude that \( \rho, u, \mathcal{R}_m \) and \( \mathcal{R}_e \) is a dissipative solution for the Navier–Stokes equation.

Finally we state the theorem,
Theorem 5.4. Let pressure $p$ follows $(5.5)$. We assume that the initial data is well-prepared, i.e. it follows $(4.4)$. We also consider $\mathbf{v}_0 \in W^{k,2}$ with $k \geq 3$. Let $(\rho_c, \mathbf{u}_c)$ be a dissipative solution as in definition $(5.1)$ in $(0,T) \times \Omega$. Then,

$$\text{ess sup}_{t \in (0,T)} \| \rho_c - \bar{\rho}_c \|_{(L^2+L^\gamma)(\Omega)} \leq c^m, \quad (5.25)$$

$$\sqrt{\rho_c} \mathbf{u}_c \to \mathbf{v} \quad \left\{ \begin{array}{l}
\text{weakly}^{(*)} \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \\
\text{strongly in } L^1_{\text{loc}}((0,T) \times \Omega;\mathbb{R}^3),
\end{array} \right.$$

where, $\mathbf{v} = (\mathbf{v}_h,0)$ and $\mathbf{v}_h$ is the unique solution of Euler system $(4.1)$ with initial data $\mathbf{v}_0$.

Proof. The proof is similar as before only we have to consider few extra terms, see [21].

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References

[1] J. J. Alibert and G. Bouchitté, Non-uniform integrability and generalized Young measures, J. Convex Anal. 4 (1997), no. 1, 129–147. MR1459885
[2] A. Babin, A. Mahalov, and B. Nicolaenko, Global regularity of 3D rotating Navier-Stokes equations for resonant domains, Indiana Univ. Math. J. 48 (1999), no. 3, 1133–1176. MR1736966
[3] , 3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity, Indiana Univ. Math. J. 50 (2001), no. Special Issue, 1–35. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000). MR1855663
[4] D. Basarić, Vanishing viscosity limit for the compressible navier-stokes system via measure-valued solutions, arXiv e-prints (2019Mar), arXiv:1903.05886, available at 1903.05886.
[5] D. Breit, E. Feireisl, and M. Hofmanová, Dissipative solutions and semiflow selection for the complete euler system, arXiv e-prints (2019Apr), arXiv:1904.00622, available at 1904.00622.
[6] , Generalized solutions to models of inviscid fluids, arXiv e-prints (2019Jul), arXiv:1904.00622, available at 1904.00622.
[7] J. Březina, Existence of measure-valued solutions to a complete Euler system for a perfect gas, arXiv e-prints (2018May), arXiv:1805.05570, available at 1805.05570.
[8] J. Březina and E. Feireisl, Measure-valued solutions to the complete Euler system, J. Math. Soc. Japan 70 (2018), no. 4, 1227–1245. MR3868717
[9] G. Bruell and E. Feireisl, On a singular limit for stratified compressible fluids, arXiv e-prints (2018Feb), arXiv:1802.10340, available at 1802.10340.
[10] J. Březina and V. Mácha, Low stratification of the complete euler system, arXiv e-prints (2018Dec), arXiv:1812.08465, available at 1812.08465.
[11] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier, Mathematical geophysics, Oxford Lecture Series in Mathematics and its Applications, vol. 32, The Clarendon Press, Oxford University Press, Oxford, 2006. An introduction to rotating fluids and the Navier-Stokes equations. MR2228849
[12] D. G. Ebin, The motion of slightly compressible fluids viewed as a motion with strong constraining force, Ann. of Math. (2) **105** (1977), no. 1, 141–200. MR0431261

[13] E. Feireisl, Dynamics of viscous compressible fluids, Oxford Lecture Series in Mathematics and its Applications, vol. 26, Oxford University Press, Oxford, 2004. MR2040667

[14] E. Feireisl, I. Gallagher, D. Gerard-Varet, and A. Novotný, Multi-scale analysis of compressible viscous and rotating fluids, Comm. Math. Phys. **314** (2012), no. 3, 641–670. MR2964771

[15] E. Feireisl, I. Gallagher, and A. Novotný, A singular limit for compressible rotating fluids, SIAM J. Math. Anal. **44** (2012), no. 1, 192–205. MR2888285

[16] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann, Dissipative measure-valued solutions to the compressible Navier-Stokes system, Calc. Var. Partial Differential Equations **55** (2016), no. 6, Art. 141, 20. MR3567640

[17] E. Feireisl, C. Klingenberg, and S. Markfelder, On the low mach number limit for the compressible Euler system, SIAM J. Math. Anal. **51** (2019), no. 2, 1496–1513. MR3942857

[18] E. Feireisl and M. Lukáčová-Medvidová, Convergence of a mixed finite element–finite volume scheme for the isentropic Navier-Stokes system via dissipative measure-valued solutions, Found. Comput. Math. **18** (2018), no. 3, 703–730. MR3807359

[19] E. Feireisl, M. Lukáčová-Medvidová, and H. Mizerová, $K-$convergence as a new tool in numerical analysis, arXiv e-prints (2019Mar), arXiv:1904.00297, available at [1904.00297](https://arxiv.org/abs/1904.00297).

[20] E. Feireisl and A. Novotný, Singular limits in thermodynamics of viscous fluids, Advances in Mathematical Fluid Mechanics, Birkhäuser Verlag, Basel, 2009. MR2499296

[21] E. Feireisl and A. Novotný, Multiple scales and singular limits for compressible rotating fluids with general initial data, Comm. Partial Differential Equations **39** (2014), no. 6, 1104–1127. MR3200090

[22] _____, Scale interactions in compressible rotating fluids, Ann. Mat. Pura Appl. (4) **193** (2014), no. 6, 1703–1725. MR3275259

[23] P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann, Weak-strong uniqueness for measure-valued solutions of some compressible fluid models, Nonlinearity **28** (2015), no. 11, 3873–3890. MR3424896

[24] T. Kato and C. Y. Lai, Nonlinear evolution equations and the Euler flow, J. Funct. Anal. **56** (1984), no. 1, 15–28. MR735703

[25] S. Klainerman and A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, Comm. Pure Appl. Math. **34** (1981), no. 4, 481–524. MR615627

[26] Y. Li, Singular limit for rotating compressible fluids with centrifugal force in a finite cylinder (2019).

[27] P.-L. Lions, Mathematical topics in fluid mechanics. Vol. 2, Oxford Lecture Series in Mathematics and its Applications, vol. 10, The Clarendon Press, Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications. MR1637634

[28] S. Nečasová and T. Tang, On a singular limit for the compressible rotating Euler system (2018).

[29] A. Novotný and M. Pokorný, Stabilization to equilibria of compressible Navier-Stokes equations with infinite mass, Comput. Math. Appl. **53** (2007), no. 3–4, 437–451. MR2323702

[30] S. Schochet, The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit, Comm. Math. Phys. **104** (1986), no. 1, 49–75. MR834481