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1. Introduction

As is well known, there is a congruence between the Fourier coefficients of Eisenstein series and those of cuspidal Hecke eigenforms for $SL_2(\mathbb{Z})$. This type of congruence is not only interesting in its own right but also has an important application to number theory as shown by Ribet [Ri1]. Since then, there have been so many important results about the congruence of elliptic modular forms (cf. [Hi1],[Hi2],[Hi3].) In the case of Hilbert modular forms or Siegel modular forms, it is sometimes more natural and important to consider the congruence between the Hecke eigenvalues of Hecke eigenforms modulo a prime ideal $\mathfrak{p}$. We call such a $\mathfrak{p}$ a prime ideal giving the congruence or a congruence prime. For a cuspidal Hecke eigenform $f$ for $SL_2(\mathbb{Z})$, let $\hat{f}$ be a lift of $f$ to the space $\mathfrak{M}_1(\Gamma')$ of modular forms of weight $l$ for a modular group $\Gamma'$. Here we mean by the lift of $f$ a cuspidal Hecke eigenform whose certain L-function can be expressed in terms of certain L-functions of $f$. As typical examples of the lift we can consider the Doi-Naganuma lift, the Saito-Kurokawa lift, and the Duke-Imamoglu-Ikeda lift. We then consider the following problem:

**Problem.** Characterize the prime ideals giving the congruence between $\hat{f}$ and a cuspidal Hecke eigenform in $\mathfrak{M}_1(\Gamma')$ not coming from the lift. In particular characterize them in terms of special values of certain L-functions of $f$.

This type of problem was first investigated in the Doi-Naganuma lift case by Doi, Hida, and Ishii [D-H-I]. In our previous paper [Ka2], we considered the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$ and the special values of their standard zeta functions. In particular, we proposed a conjecture concerning the congruence between Saito-Kurokawa lifts and non-Saito-Kurokawa
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lifts, and proved it under certain condition. In this paper, we con-
sider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-
Imamoglu-Ikeda lifts, which is a generalization of our previous conjec-
ture.

In Section 3, we review a result concerning the relationship between
the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbb{Z})$
and the special values of their standard zeta functions. In Section
4, we propose a conjecture concerning the congruence between Duke-
Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, and prove it
under a certain condition.

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Notation. For a commutative ring $R$, we denote by $M_{mn}(R)$ the
set of $(m, n)$-matrices with entries in $R$. In particular put $M_n(R) =
M_{nn}(R)$. Here we understand $M_{mn}(R)$ the set of the empty matrix
if $m = 0$ or $n = 0$. For an $(m, n)$-matrix $X$ and an $(m, m)$-matrix
$A$, we write $A[X] = {}^tXAX$, where $^tX$ denotes the transpose of $X$. Let
$a$ be an element of $R$. Then for an element $X$ of $M_{mn}(R)$ we often
use the same symbol $X$ to denote the coset $X \mod aM_{mn}(R)$. Put
$GL_n(R) = \{ A \in M_n(R) \mid \det A \in R^* \}$, where $\det A$ denotes the
determinant of a square matrix $A$, and $R^*$ denotes the unit group of
$R$. Let $S_n(R)$ denote the set of symmetric matrices of degree $n$ with
entries in $R$. Furthermore, for an integral domain $R$ of characteristic
different from 2, let $H_n(R)$ denote the set of half-integral matrices of
degree $n$ over $R$, that is, $H_n(R)$ is the set of symmetric matrices of
degree $n$ whose $(i, j)$-component belongs to $R$ or $\frac{1}{2}R$ according as
$i = j$ or not. For a subset $S$ of $M_n(R)$ we denote by $S^\times$ the subset of $S$
consisting of non-degenerate matrices. In particular, if $S$ is a subset
of $S_n(R)$ with $R$ the field of real numbers, we denote by $S_{>0}$ (resp.
$S_{\geq 0}$) the subset of $S$ consisting of positive definite (resp. semi-positive
definite) matrices. Let $R'$ be a subring of $R$. Two symmetric matrices
$A$ and $A'$ with entries in $R$ are called equivalent over $R'$ with each
other and write $A \sim_{R'} A'$ if there is an element $X$ of $GL_n(R')$ such that
$A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For
square matrices $X$ and $Y$ we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2. Standard zeta functions of Siegel modular forms

For a complex number $x$ put $e(x) = \exp(2\pi \sqrt{-1}x)$. Furthermore put
$J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where $1_n$ denotes the unit matrix of degree $n$. For
a subring $K$ of $R$, put

$$GSp_n(K)^+ = \{ M \in GL_{2n}(K) \mid J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0 \},$$

and

$$Sp_n(K) = \{ M \in GSp_n(K)^+ \mid J_n[M] = J_n \}.$$  

Furthermore, put

$$\Gamma^{(n)} = Sp_n(Z) = \{ M \in GL_{2n}(Z) \mid J_n[M] = J_n \}.$$  

We sometimes write an element $M$ of $GSp_n(K)$ as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D \in M_2(K)$. We define a subgroup $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$ as

$$\Gamma_0^{(n)}(N) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid C \equiv O_n \mod N \}.$$  

Let $H_n$ be Siegel’s upper half-space. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(R)^+$ and $Z \in H_n$ put

$$M(Z) = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = \det(CZ + D).$$

Furthermore, for a function $f$ on $H_n$ and an integer $k$ we define $f|_k M$ as

$$(f|_k M)(Z) = \det(M)^{k/2}j(M, Z)^{-k}f(M(Z)).$$

For an integer or half integral $l$ and the subgroup $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$, we denote by $\mathcal{M}_k(\Gamma_0^{(n)}(N))$ (resp. $\mathcal{M}_k^\infty(\Gamma_0^{(n)}(N))$) the space of holomorphic (resp. $C^\infty$-) modular forms of weight $k$ with respect to $\Gamma_0^{(n)}(N)$. We denote by $\mathcal{M}_k(\Gamma_0^{(n)}(N))$ the sub-space of $\mathcal{M}_k(\Gamma_0^{(n)}(N))$ consisting of cusp forms. Let $f$ be a holomorphic modular form of weight $k$ with respect to $\Gamma_0^{(n)}(N)$. Then $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in H(Z) \geq 0} a_f(A)e(\text{tr}(AZ));$$

and in particular if $f$ is a cusp form, $f$ has the following Fourier expansion:

$$f(Z) = \sum_{A \in H(Z) \geq 0} a_f(A)e(\text{tr}(AZ)),$$

where tr denotes the trace of a matrix. Let $dv$ denote the invariant volume element on $H_n$ defined by $dv = \det(\text{Im}(Z))^{n-1}\wedge_{1 \leq j \leq l \leq n}(dx_{j1} \wedge dy_{jl})$. Here for $Z \in H_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices
(x_{ij}) and (y_{ij}). For two $C^{\infty}$-modular forms $f$ and $g$ of weight $l$ with respect to $\Gamma_0^{(n)}(N)$ we define the Petersson scalar product $\langle f, g \rangle$ by

$$\langle f, g \rangle = [\Gamma^{(n)} : \Gamma_0^{(n)}(N)]^{-1} \int_{\Gamma_0^{(n)}(N) \backslash \mathbb{H}_n} f(Z)g(Z) \det(\text{Im}(Z))^{(l)}dv,$$

provided the integral converges.

Let $L_n = \mathbb{L}_Q(GSp_n(\mathbb{Q})^+, \Gamma^{(n)})$ denote the Hecke algebra over $\mathbb{Q}$ associated with the Hecke pair $(GSp_n(\mathbb{Q})^+, \Gamma^{(n)})$. Furthermore, let $L'_n = \mathbb{L}_Q(Sp_n(\mathbb{Q}), \Gamma^{(n)})$ denote the Hecke algebra over $\mathbb{Q}$ associated with the Hecke pair $(Sp_n(\mathbb{Q}), \Gamma^{(n)})$. For each integer $m$ define an element $T(m)$ of $L_n$ by

$$T(m) = \sum_{d_1, \ldots, d_n, e_1, \ldots, e_n} \Gamma^{(n)}(d_1 \perp \ldots \perp d_n \perp e_1 \perp \ldots \perp e_n)\Gamma^{(n)},$$

where $d_1, \ldots, d_n, e_1, \ldots, e_n$ run over all positive integer satisfying

$$d_i|d_{i+1}, e_{i+1}|e_i \ (i = 1, \ldots, n-1), d_n|e_n, d_ie_i = m \ (i = 1, \ldots, n).$$

Furthermore, for $i = 1, \ldots, n$ and a prime number $p$ put

$$T_i(p^2) = \Gamma^{(n)}(1_{n-i} \perp p1_i \perp p21_{n-i} \perp p1_i)\Gamma^{(n)},$$

and $(p^{\pm 1}) = \Gamma^{(n)}(p^{\pm 1}1_n)\Gamma^{(n)}$. As is well known, $L_n$ is generated over $\mathbb{Q}$ by all $T(p), T_i(p^2) \ (i = 1, \ldots, n)$, and $(p^{\pm 1})$. We denote by $L'_n$ the subalgebra of $L_n$ generated over $\mathbb{Z}$ by all $T(p)$ and $T_i(p^2) \ (i = 1, \ldots, n)$. Let $T = \Gamma^{(n)}M\Gamma^{(n)}$ be an element of $L_n \otimes \mathbb{C}$. Write $T$ as $T = \bigcup \Gamma^{(n)}_{\gamma}$ and for $f \in \mathcal{M}_k(\Gamma^{(n)})$ define the Hecke operator $|kT$ associated to $T$ as

$$f|kT = \det(M)^{k/2-(n+1)/2} \sum_{\gamma} f|k\gamma.$$

We call this action the Hecke operator as usual (cf. [A].) If $f$ is an eigenfunction of a Hecke operator $T \in L_n \otimes \mathbb{C}$, we denote by $\lambda_f(T)$ its eigenvalue. Let $L$ be a subalgebra of $L_n$. We call $f \in \mathcal{M}_k(\Gamma^{(n)})$ a Hecke eigenform for $L$ if it is a common eigenfunction of all Hecke operators in $L$. In particular if $L = L_n$ we simply call $f$ a Hecke eigenform. Furthermore, we denote by $\mathcal{M}(f)$ the field generated over $\mathbb{Q}$ by eigenvalues of all $T \in L_n$ as in Section 1. As is well known, $\mathcal{M}(f)$ is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field $K$, let $\mathcal{O}_K$ denote the ring of integers in $K$. The following assertion has been proved in [Mi2] (see also [Ka2].)

**Theorem 2.1** Let $k \geq n + 1$. Let $f \in \mathcal{M}_k(\Gamma^{(n)})$ be a common eigenform in $L'_n$. Then $\lambda_f(T)$ belongs to $\mathcal{O}_K(f)$ for any $T \in L'_n$. 

Let $L_{np} = L(GSp_n(Q)^+ \cap GL_{2n}(Z[p^{-1}]), \Gamma(n))$ be the Hecke algebra associated with the pair $(GSp_n(Q)^+ \cap GL_{2n}(Z[p^{-1}]), \Gamma(n))$. $L_{np}$ can be considered as a subalgebra of $L_n$, and is generated over $Q$ by $T(p)$ and $T_i(p^2) (i = 1, 2, \ldots, n)$. We now review the Satake $p$-parameters of $L_{np}$; let $P_n = Q[X_0^\pm, X_1^\pm, \ldots, X_n^\pm]$ be the ring of Laurent polynomials in $X_0, X_1, \ldots, X_n$ over $Q$. Let $W_n$ be the group of $Q$-automorphisms of $P_n$ generated by all permutations in variables $X_1, \ldots, X_n$ and by the automorphisms $\tau_1, \ldots, \tau_n$ defined by

$$\tau_i(X_0) = X_0 X_i, \tau_i(X_i) = X_i^{-1}, \tau_i(X_j) = X_j (j \neq i).$$

Furthermore, a group $\tilde{W}_n$ isomorphic to $W_n$ acts on the set $T_n = (C^\times)^{n+1}$ in a way similarly to above. Then there exists a $Q$-algebra isomorphism $\Phi_{np}$, called the Satake isomorphism, from $L_{np}$ to the $W_n$-invariant subring $P_n^{W_n}$ of $P_n$. Then for a $Q$-algebra homomorphism $\lambda$ from $L_{np}$ to $C$, there exists an element $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$ of $T_n$ satisfying

$$\lambda(\Phi_{np}^{-1}(F(X_0, X_1, \ldots, X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$$

for $F \in P_n^{W_n}$. The equivalence class of $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda))$ under the action of $W_n$ is uniquely determined by $\lambda$. We call this the Satake parameters of $L_{np}$ determined by $\lambda$.

Now assume that an element $f$ of $M_k(Sp_n(Z))$ is a Hecke eigenform. Then for each prime number $p$, $f$ defines a $Q$-algebra homomorphism $\lambda_{f,p}$ from $L_{np}$ to $C$ in a usual way, and we denote by $\alpha_0(p, \lambda), \alpha_1(p, \lambda), \ldots, \alpha_n(p, \lambda)$ the Satake parameters of $L_{np}$ determined by $f$. We then define the standard zeta function $L(f, s, \St)$ by

$$L(s, f, \St) = \prod_p \prod_{i=1}^{n} \{(1 - p^{-s})(1 - \alpha_i(p)p^{-s})(1 - \alpha_i(p)^{-1}p^{-s})\}^{-1}.$$

Let $f(z) = \sum_{A \in \Gamma(n)(Z)} a(A)e(\text{tr}(Az))$ be a Hecke eigenform in $\S_k(\Gamma(n))$.

For a positive integer $m \leq k - n$ such that $m \equiv n \mod 2$ put

$$\Lambda(f, m, \St) = (-1)^{(m+1)/2+nk/2} \frac{L(f, m, \St)}{\Gamma(m+1) \prod_{i=1}^{n} \Gamma(2k-n-i)}.$$

Then the following theorem is due to Böcherer [B2] and Mizumoto [Mi].

**Theorem 2.2.** Let $l, k$ and $n$ be positive integers such that $\rho(n) \leq l \leq k-n$, where $\rho(n) = 3$, or 1 according as $n \equiv 1 \mod 4$ and $n \geq 5$, or not. Let $f \in \S_k(\Gamma(n))$ be a Hecke eigenform. Then $\Lambda(f, m, \St)$ belongs to $Q(f)$. 
For later purpose, we consider a special element in $L_{np}$; the polynomial $X_0^2 X_1 X_2 \cdots X_n \sum_{i=1}^{n} (X_i + X_i^{-1})$ is an element of $P_n$, and thus we can define an element $\Phi_{np}^{-1}(X_0^2 X_1 X_2 \cdots X_n \sum_{i=1}^{n} (X_i + X_i^{-1}))$ of $L_{np}$, which is denoted by $r_1$.

**Proposition 2.3.** Under the above notation the element $r_1$ belongs to $L'_n$, and we have

$$\lambda_f(r_1) = p^{n(k-(n+1)/2)} \sum_{i=1}^{n} (\alpha_i(p) + \alpha_i(p)^{-1}).$$

**Proof.** By a careful analysis of the computation in page 159-160 of [A], we see that $r_1$ is a $\mathbb{Z}$-linear combination of $T_i(p^2) (i = 1, \ldots, n)$, and therefore we can prove the first assertion. Furthermore, by Lemma 3.3.34 of [A], we can prove the second assertion.

3. Congruence of modular forms and special values of the standard zeta functions

In this section we review a result concerning the congruence between the Hecke eigenvalues of modular forms of the same weight following [Ka2]. Let $K$ be an algebraic number field, and $\mathcal{O} = \mathcal{O}_K$ the ring of integers in $K$. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}$, we denote by $\mathcal{O}_{\mathfrak{p}}$ the localization of $\mathcal{O}$ at $\mathfrak{p}$ in $K$. Let $\mathfrak{m}$ be a fractional ideal in $K$. If $\mathfrak{m} = \mathfrak{p}\mathfrak{A}$ with $\mathfrak{A}\mathfrak{O}_{\mathfrak{p}} = \mathfrak{O}_{\mathfrak{p}}$ we write $\text{ord}_{\mathfrak{p}} = e$. We simply write $\text{ord}_{\mathfrak{p}}((c)) = \text{ord}_{\mathfrak{p}}(c)$ for $c \in K$. Now let $f$ be a Hecke eigenform in $\mathfrak{S}_k(\Gamma(n))$ and $M$ be a subspace of $\mathfrak{S}_k(\Gamma(n))$ stable under Hecke operators $T \in \mathfrak{L}_n$. Assume that $M$ is contained in $(Cf)^\perp$, where $(Cf)^\perp$ is the orthogonal complement of $Cf$ in $\mathfrak{S}_k(\Gamma(n))$ with respect to the Petersson product. Let $K$ be an algebraic number field containing $\mathbb{Q}(f)$. A prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ is called a congruence prime of $f$ with respect to $M$ if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \mod \tilde{\mathfrak{p}}$$

for any $T \in \mathfrak{L}_n'$, where $\tilde{\mathfrak{p}}$ is the prime ideal of $\mathcal{O}_{K\mathbb{Q}(f)}$ lying above $\mathfrak{p}$. If $M = (Cf)^\perp$, we simply call $\mathfrak{p}$ a congruence prime of $f$.

Now we consider the relation between the congruence primes and the standard zeta values. To consider this, we have to normalize the standard zeta value $\Lambda(f,l,\text{St})$ for a Hecke eigenform $f$ because it is not uniquely determined by the system of Hecke eigenvalues of $f$. We note that there is no reasonable normalization of cuspidal Hecke eigenform in the higher degree case unlike the elliptic modular case.
Thus we define the following quantities: for a Hecke eigenform $f(z) = \sum_A a_f(A)e(\text{tr}(Az))$ in $\mathcal{S}_k(\Gamma(n))$, let $\mathfrak{Z}_f$ be the $\mathcal{O}_{\mathbb{Q}(f)}$-module generated by all $a_f(A)$'s. Assume that there exists a complex number $c$ such that all the Fourier coefficients $cf$ belong to $\mathbb{Q}(f)$. Then $\mathfrak{Z}_f$ is a fractional ideal in $\mathbb{Q}(f)$, and therefore, so is $\Lambda(f, l, S_l)\mathfrak{Z}_f^2$ if $l$ satisfies the condition in Theorem 2.2. We note that this fractional ideal does not depend on the choice of $c$. We also note that the value $N_{\mathbb{Q}(f)}(\Lambda(f, l, S_l))N(\mathfrak{Z}_f)^2$ does not depend on the choice of $c$, where $N(\mathfrak{Z}_f)$ is the norm of the ideal $\mathfrak{Z}_f$. Then, we have

**Theorem 3.1.** Let $f$ be a Hecke eigenform in $\mathcal{S}_k(\Gamma(n))$. Assume that there exists a complex number $c$ such that all the Fourier coefficients $cf$ belong to $\mathbb{Q}(f)$. Let $l$ be a positive integer satisfying the condition in Theorem 2.2. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$. Assume that $\text{ord}_{\mathfrak{p}}(\Lambda(f, l, S_l)\mathfrak{Z}_f^2) < 0$ and that it does not divide $(2l - 1)!$. Then $\mathfrak{p}$ is a congruence prime of $f$. In particular, if a rational prime number $p$ divides the denominator of $N_{\mathbb{Q}(f)}(\Lambda(f, l, S_l))N(\mathfrak{Z}_f)^2$, then $p$ is divisible by some congruence prime of $f$.

Now for a Hecke eigenform $f$ in $\mathcal{S}_k(\Gamma(n))$, let $\mathfrak{Z}_f$ denote the subspace of $\mathcal{S}_k(\Gamma(n))$ spanned by all Hecke eigenforms with the same system of the Hecke eigenvalues as $f$.

**Corollary.** In addition to the above notation and the assumption, assume that $\mathcal{S}_k(\Gamma(n))$ has the multiplicity one property. Then $\mathfrak{p}$ is a congruence prime of $f$ with respect to $\mathfrak{Z}_f^\perp$. In particular, if a rational prime number $p$ divides the denominator of $N_{\mathbb{Q}(f)}(\Lambda(f, l, S_l))N(\mathfrak{Z}_f)^2$, then $p$ is divisible by some congruence prime of $f$ with respect to $\mathfrak{Z}_f^\perp$.

4. **Congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts**

In this section, we consider the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts. Throughout this section and the next, we assume that $n$ and $k$ are even positive integers. Let

$$f(z) = \sum_{m=1}^{\infty} a(m)e(mz)$$

be a normalized Hecke eigenform of weight $2k - n$ with respect to $SL_2(\mathbb{Z})$. For a Dirichlet character $\chi$, we then define the L-function
$L(s, f)$ of $f$ twisted by $\chi$ by

$$L(s, f) = \prod_p \{(1 - \chi(p)\beta_p p^{k-n/2-1/2-s})(1 - \chi(p)\beta_p^{-1} p^{k-n/2-1/2-s})\}^{-1},$$

where $\beta_p$ is a non-zero complex number such that $\beta_p + \beta_p^{-1} = p^{k-n/2+1/2}a(p)$.

We simply write $L(s, f)$ as $L(s, f, \chi)$ if $\chi$ is the principal character. Furthermore, let $\tilde{f}$ be the cusp form of weight $k-n/2+1/2$ belonging to the Kohnen plus space corresponding to $f$ via the Shimura correspondence (cf. [Ko1]). Then $\tilde{f}$ has the following Fourier expansion:

$$\tilde{f}(z) = \sum c(e) e(ez),$$

where $e$ runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \mod 4$.

We then put

$$a_{I_n(f)}(T) = c(\|T\|) \prod_p (p^{k-n/2-1/2}\beta_p)_{\nu_p(T)} F_p(T, p^{-(n+1)/2}\beta_p^{-1}).$$

We note that $a_{I_n(f)}(T)$ does not depend on the choice of $\beta_p$. Define a Fourier series $I_n(f)(Z)$ by

$$I_n(f)(Z) = \sum_{T \in \mathcal{H}_n(Z), \gamma > 0} a_{I_n(f)}(T) e(\text{tr}(TZ)).$$

In [Ik1] Ikeda showed that $I_n(f)(Z)$ is a cusp form of weight $k$ with respect to $\Gamma^{(a)}$ and a Hecke eigenform for $L_n^\phi$ such that

$$L(s, I_n(f), St) = \zeta(s) \prod_{i=1}^n L(s + k - i, f).$$

This was first conjecture by Duke and Imamoglu. Thus we call $I_n(f)$ the Duke-Imamoglu-Ikeda lift of $f$. We note that we have $\mathcal{Q}(\tilde{f}) = \mathcal{Q}(I_n(f)) = \mathcal{Q}(f)$. Furthermore, we have $\mathfrak{F}_f = \mathfrak{F}_{I_n(f)}$, where $\mathfrak{F}_f$ is the $\mathcal{Q}(f)$-module generated by all the Fourier coefficients of $\tilde{f}$.

Now to consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, first we prove the following:

**Proposition 4.1** $I_n(f)$ is a Hecke eigenform.

We note that Ikeda proved in [Ik1] that $I_n(f)$ is a Hecke eigenform for $L_n^\phi$ but has not proved that it is a Hecke eigenform for $L_n$. This was pointed to us by B. Heim (see [He].) We thank him for his comment. We also note that an explicit form of the spinor L-function of $I_n(f)$ was obtained by Murakawa [Mu] and Schmidt [Sch] assuming that $I_n(f)$ is a Hecke eigenform.
Proof of Proposition 4.1. We have only to prove that $I_n(f)$ is an eigenfunction of $T(p)$ for any prime $p$. The proof may be more or less well known, but for the convenience of the readers we here give the proof. For a modular form

$$F(Z) = \sum_B c_F(B)e(\text{tr}(BZ)),$$

let $c_F^{(p)}(B)$ be the $B$-th Fourier coefficient of $F|T(p)$. Then for any positive definite matrix $B$ we have

$$c_F^{(p)}(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\cdots|d_n|p} d_1^m d_2^{m-1} \cdots d_n \times \sum_{D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n} \det D^{-k} c_F(p^{-1}A[t^t D]),$$

where $\Lambda_n = GL_n(\mathbb{Z})$.

Now let $E_{n,k}(Z)$ be the Siegel Eisenstein series of degree $n$ and of weight $k$ defined by

$$E_{n,k}(Z) = \sum_{\gamma \in \Gamma_{n,\infty} \setminus \Gamma_n} j(\gamma, Z)^{-k}.$$

For $k \geq n + 1$, the Siegel Eisenstein series $E_{n,k}(Z)$ is a holomorphic modular form of weight $k$ with respect to $\Gamma_n$. Furthermore, $E_{n,k}(Z)$ is a Hecke eigenform and in particular we have

$$E_{n,k}|T(p)(Z) = h_{n,p}(p^k) E_{n,k}(Z),$$

where

$$h_{n,p}(X) = 1 + \sum_{r=1}^n \sum_{1 \leq i_1 < \cdots < i_r \leq n} p^{-\sum_{j=1}^r i_j} X^{r}.$$

Let $c_{n,k}(B)$ be the $B$-th Fourier coefficient of $E_{n,k}(Z)$. Then we have

$$h_{n,p}(p^k)c_{n,k}(B) = p^{nk-n(n+1)/2} \sum_{d_1|d_2|\cdots|d_n|p} d_1^m d_2^{m-1} \cdots d_n \times \sum_{D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n} \det D^{-k} c_{n,k}(p^{-1}B[t^t D]).$$

Let $B$ be positive definite. Then we have

$$c_{n,k}(B) = a_{n,k}(\det 2B)^{k-(n+1)/2} L(k - n/2, \chi_B) \prod_q F_q(B, p^{-k}),$$

where $a_{n,k}$ is a non-zero constant depending only on $n$ and $k$. We note that we have

$$F_q(p^{-1}B[t^t D], X) = F_q(B, X)$$
for any $D \in \Lambda_n(d_1 \perp \cdots \perp d_n)\Lambda_n$ with $d_1 | \cdots | d_n | p$ if $q \neq p$. Thus we have

$$h_{n,p}(p^k)F_p(B, p^{-k}) = \sum_{\epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_n \leq 1} p^{n\epsilon_1+(n-1)\epsilon_2+\cdots+n\epsilon_n}(k-n-1)$$

$$\times \sum_{D \in \Lambda_n \lambda_{n,p}(p^1 \perp \cdots \perp p^n)\Lambda_n} F_p(p^{-1}B\lvert^D, p^{-k}).$$

The both-hand sides of the above are polynomials in $p^k$ and the equality holds for infinitely many $k$. Thus we have

$$h_{n,p}(X^{-1})F_p(B, X) = \sum_{\epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_n \leq 1} p^{n\epsilon_1+(n-1)\epsilon_2+\cdots+n\epsilon_n}(X^{-1}p^{-n-1}(\epsilon_1+\cdots+\epsilon_n)$$

$$\times \sum_{D \in \Lambda_n \lambda_{n,p}(p^1 \perp \cdots \perp p^n)\Lambda_n} F_p(p^{-1}B\lvert^D, X)$$

as polynomials in $X$ and $X^{-1}$. Thus we have

$$(p^{k-(n+1)/2}X)^{n/2}h_{n,p}(p^{(n+1)/2}X^{-1})(p^{k-(n+1)/2}X^{-1})^{\nu_p(\lvert^D)}F_p(B, p^{-(n+1)/2}X)$$

$$= p^{nk-n(n+1)/2} \sum_{\epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_n \leq 1} p^{n\epsilon_1+(n-1)\epsilon_2+\cdots+n\epsilon_n}$$

$$\times \sum_{D \in \Lambda_n \lambda_{n,p}(p^1 \perp \cdots \perp p^n)\Lambda_n} \det D^{-k}(p^{k-(n+1)/2}X^{-1})^{\nu_p(\lvert^D)}F_p(p^{-1}B\lvert^D, p^{-(n+1)/2}X).$$

We recall that we have

$$c_{f_n}(\lvert_B) = c_f(\lvert_B^{p^{-1}B\lvert^D})^{k-(n+1)/2} \prod_q (\beta_q)^{\nu_q(\lvert_B)} F_q(B, q^{-(n+1)/2}B^{\beta_q}),$$

where $\beta_q$ is the Satake $q$-parameter of $f$. We also note that $c_f(\lvert_B^{p^{-1}B\lvert^D}) = c_f(\lvert_B)$ for any $D$. Thus we have

$$(p^{k-(n+1)/2}A_p^{-1})^{n/2} h_{n,p}(p^{(n+1)/2}A_p)c_{f_n}(\lvert_B)$$

$$= p^{nk-n(n+1)/2} \sum_{d_1 | d_2 | \cdots | d_n | p} d_1^p d_2^p \cdots d_n \sum_{D \in \Lambda_n (d_1 \perp \cdots \perp d_n)\Lambda_n} \det D^{-k} c_{f_n}(\lvert_B^{p^{-1}B\lvert^D}).$$

This proves the assertion.

Let $f$ be a primitive form in $\mathfrak{S}_{2k-n}(\Gamma(1))$. Let $\{f_1, \ldots, f_d\}$ be a basis of $\mathfrak{S}_{2k-n}(\Gamma(1))$ consisting of primitive forms. Let $K$ be an algebraic number field containing $\mathbb{Q}(f_1) \cdots \mathbb{Q}(f_d)$, and $A = \mathfrak{O}_K$. To formulate our conjecture exactly, we introduce the Eichler-Shimura periods as follows (cf. Hida [Hi3]). Let $\mathfrak{P}$ be a prime ideal in $K$. Let $A_{\mathfrak{P}}$ be a valuation ring in $K$ corresponding to $\mathfrak{P}$. Assume that the residual characteristic of $A_{\mathfrak{P}}$ is greater than or equal to 5. Let $L(2k-n-2, A_{\mathfrak{P}})$ be the module of homogeneous polynomials of degree $2k-n-2$ in the variables $X, Y$.
with coefficients in $A_{\mathfrak{p}}$. We define the action of $M_2(\mathbb{Z}) \cap \text{GL}_2(\mathbb{Q})$ on $L(2k - n - 2, A_{\mathfrak{p}})$ via
\[
\gamma \cdot P(X, Y) = P((X, Y)(\gamma)^t),
\]
where $\gamma' = (\det \gamma)^{-1}$. Let $H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ be the parabolic cohomology group of $\Gamma^{(1)}$ with values in $L(2k - n - 2, A_{\mathfrak{p}})$. Fix a point $z_0 \in \mathcal{H}_1$. Let $g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$ or $g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$. We then define the differential $\omega(g)$ as
\[
\omega(g)(z) = \begin{cases} 2\pi ig(z)(X - zY)^n dz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \\ 2\pi \sqrt{-1}g(z)(X - \bar{z}Y)^n dz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}), \end{cases}
\]
and define the cohomology class $\delta(g)$ of the 1-cocycle of $\Gamma^{(1)}$, as
\[
\gamma \in \Gamma^{(1)} \rightarrow \int_{z_0}^{\gamma(z_0)} \omega(g).
\]
The mapping $g \rightarrow \delta(g)$ induces the isomorphism
\[
\delta : \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \oplus \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \rightarrow H^1_p(\Gamma^{(1)}, L(2k - n - 2, \mathbb{C})),
\]
which is called the Eichler-Shimura isomorphism. We can define the action of Hecke algebra $\mathcal{L}_1$ on $H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ in a natural manner. Furthermore, we can define the action $F_\infty$ on $H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ as
\[
F_\infty(\delta(g)(z)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta(g)(-\bar{z}),
\]
and this action commutes with the Hecke action. For a primitive form $f$ and $j = \pm 1$, we define the subspace $H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$ of $H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ as
\[
H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j] = \{ x \in H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}})) : x|T = \lambda_f(T)x \text{ for } T \in \mathcal{L}_1, \text{ and } F_\infty(x) = jx \}.
\]
Since $A_{\mathfrak{p}}$ is a principal ideal domain, $H^1_p(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$ is a free module of rank one over $A_{\mathfrak{p}}$. For each $j = \pm 1$ take a basis $\eta(f, j, A_{\mathfrak{p}})$ of $H^1_p(\Gamma^{(1)}, (2k - n - 2, A_{\mathfrak{p}}))[f, j]$ and define a complex number $\Omega(f, j; A_{\mathfrak{p}})$ by
\[
(\delta(f) + jF_\infty(\delta(f)))/2 = \Omega(f, j; A_{\mathfrak{p}})\eta(f, j; A_{\mathfrak{p}}).
\]
This $\Omega(f, j; A_{\mathfrak{p}})$ is uniquely determined up to constant multiple of units in $A_{\mathfrak{p}}$. We call $\Omega(f, +; A_{\mathfrak{p}})$ and $\Omega(f, -; A_{\mathfrak{p}})$ the Eichler-Shimura periods. For $j = \pm, 1 \leq l \leq 2k - n - 1$, and a Dirichlet character $\chi$ such
that $\chi(-1) = j(-1)^{l-1}$, put

$$L(l, f, \chi) = L(l, f; A_\mathfrak{p}^l) = \frac{\Gamma(l)L(l, f, \chi)}{\tau(\chi)(2\pi \sqrt{-1})^l\Omega(f, j; A_\mathfrak{p}^l)},$$

where $\tau(\chi)$ is the Gauss sum of $\chi$. In particular, put $L(l, f; A_\mathfrak{p}^l) = L(l, f, \chi; \Psi)$ if $\chi$ is the principal character. Furthermore, put

$$L(s, f, S_l) = 4(2\pi)^{-s-2k+n+1}\Gamma(s)\Gamma(s+2k-n-1)L(s, f, S_l).$$

It is well-known that $L(l, f, \chi)$ belongs to the field $K(\chi)$ generated over $K$ by all the values of $\chi$, and $L(l, f, S_l)$ belongs to $\mathbb{Q}(f)$ (cf. [Bo].)

**Conjecture A.** Let $K$ and $f$ be as above. Assume that $k > n$. Let $\mathfrak{p}$ be a prime ideal of $K$ not dividing $(2k-1)!$. Then $\mathfrak{p}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{E}_k(\Gamma^{(n)}))$ if $\mathfrak{p}$ divides $L(k, f)\prod_{i=1}^{n/2-1} L(2i+1, f, S_l)$.

**Remark.** This is an analogue of the Doi-Hida-Ishii conjecture concerning the congruence primes of the Doi-Naganuma lifting [D-H-I]. (See also [Ka1].) We also note that this type of conjecture has been proposed by Harder [Ha] in the case of vector valued Siegel modular forms.

Now to explain why our conjecture is reasonable, we refer to Ikeda’s conjecture on the Petersson inner product of the Duke-Imamoglu-Ikeda lifting. Let $f$ and $\tilde{f}$ be as above. Put

$$\tilde{\xi}(s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s),$$

and

$$\Lambda(s, f) = 2(2\pi)^{-s}\Gamma(s)L(s, f).$$

**Theorem 4.2.** (Katsurada and Kawamura [K-K]) In addition to the above notation and the assumption, assume $k > n$. Then we have

$$\tilde{\xi}(n)\Lambda(k, f)\prod_{i=1}^{n/2-1} L(2i-1, f, S_l)\tilde{\xi}(2i) = 2^n \frac{\langle I_n(f), I_n(f) \rangle}{\langle f, f \rangle^{n/2-1}\langle \tilde{f}, \tilde{f} \rangle},$$

where $\alpha$ is an integer depending only on $n$ and $k$. 
We note that the above theorem was conjectured by Ikeda [Ik2] under more general setting. We note that the theorem has been proved by Kohnen and Skoruppa [K-S] in case $n = 2$.

**Proposition 4.3** Under the above notation and the assumption we have for any fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$ and $L(k - n/2, f, \chi_D) \neq 0$ we have

$$\frac{c(|D|)^2}{\langle I_n(f), I_n(f) \rangle} = a_{n,k} \frac{\langle f, f \rangle^n/2 |D|^{k-n/2} L(k - n/2, f, \chi_D)}{n/2-1} \prod_{i=1}^{n/2-1} L(2i+1, f, St) \tilde{\xi}(2i)$$

with some algebraic number $a_{n,k}$ depending only on $n, k$.

**Proof.** By the result in Kohnen-Zagier [K-Z], for any fundamental discriminant $D$ such that $(-1)^{n/2}D > 0$ we have

$$\frac{c(|D|)^2}{\langle f, f \rangle} = \frac{2^{k-n/2-1} |D|^{k-n/2-1/2} \Lambda(k - n/2, f, \chi_D)}{\langle f, f \rangle}.$$ 

Thus the assertion holds.

**Lemma 4.4.** Let $f$ be as above.

1. Let $r_1$ be an element of $L'_n$ in Proposition 2.3. Then we have

$$\lambda_{I_n(f)}(r_1) = p^{(n-1)k-n(n+1)/2} a_f(p) \sum_{i=1}^{n} p^i.$$

2. Let $n = 2$. Then we have

$$\lambda_{I_2(f)}(T(p)) = a_f(p) + p^{2k-n-1} + p^{2k-n-2}.$$

**Lemma 4.5.** Let $d$ be a fundamental discriminant such that $(-1)^{n/2}d > 0$.

1. Assume that $d \neq 1$. Then there exists a positive definite half integral matrix $A$ of degree $n$ such that $(-1)^{n/2} \det(2A) = d$.
2. Assume $n \equiv 0 \mod 8$. Then there exists a positive definite half integral matrix $A$ of degree $n$ such that $\det(2A) = 1$.
3. Assume that $n \equiv 4 \mod 8$. Then for any prime number $q$ there exists a positive definite half integral matrix $A$ of degree $n$ such that $\det(2A) = q^2$.

**Proof.** (1) For a non-degenerate symmetric matrix $A$ with entries in $Q_p$ let $h_p(A)$ be the Hasse invariant of $A$. First let $n \equiv 2 \mod 4$ and $d = -4$. Take a family $\{A_p\}_p$ of half integral matrices over $Z_p$ of
degree $n$ such that $A_p = 1_n$ if $p \neq 2$, and $A_2 = (-1)^{(n-2)/4}1_2 \perp H_{n/2-1}$, where $H_n = H \perp \ldots \perp H$ with $H = \left( \begin{array}{cc} 0 & 1/2 \\ 1/2 & 0 \end{array} \right)$. Then we have $\det A = 2^{2-n} \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ for any $p$, and $h_p(A) = 1$ for any $p$. Thus by [I-S, Proposition 2.1], we prove the assertion for this case. Finally assume that $d$ contains an odd prime factor $q$. For $p \neq p_0$ we take a matrix $A_p$ so that $\det A_p = 2^{-n}d \in \mathbb{Q}_q^\times/(\mathbb{Q}_q^\times)^2$. Then for almost all $p$ we have $h_p(A_p) = 1$. We take $\xi \in \mathbb{Z}_p^*$ such that $(q, -\xi) = \prod_{p \neq q} h_p(A_p)$, and put $A_q = \xi d \perp (-\xi) \perp 1_{n-2}$. Then we have $\det A_0 = 2^{-n}d \in \mathbb{Q}_q^\times/(\mathbb{Q}_q^\times)^2$, and $h_p(A_q) \prod_{p \neq q} = 1$. Thus again by [I-S, Proposition 2.1], we prove the assertion for this case.

(2) It is well known that there exists a positive definite half-integral matrix $E_8$ of degree 8 such that $\det(2E_8) = 1$. Thus $A = E_8 \perp \ldots \perp E_8$ satisfies the required condition.

(3) Let $q \neq 2$. Then, take a family $\{A_p\}$ of half-integral matrices over $\mathbb{Z}_p$ of degree $n$ such that $A_q \sim_{\mathbb{Z}_p} q \perp (-q\xi) \perp (-\xi) \perp 1_{n-3}$ with $(\xi^q_q) = -1, A_2 = H_{n/2}$, and $A_p = 1_n$ for $p \neq 2$. Then by the same argument as in (1) we can show that there exists a positive definite half integral matrix $A$ of degree $n$ such that $\det(2A) = q^2$ such that $A \sim_{\mathbb{Z}_n} A_p$ for any $p$. Let $q = 2$. Then the matrix $A' = \left( \begin{array}{cc} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{array} \right)^{(n-4)/8}$ is a positive definite and $\det(2A') = 4$. Thus the matrix $A' \perp E_8 \perp \ldots \perp E_8$ satisfies the required condition.

**Proposition 4.6.** Let $k$ and $n$ be positive even integers. Let $d$ be a fundamental discriminant. Let $f$ be a primitive form in $S_{2k-n}(\Gamma_1)$. Let $\mathfrak{p}$ be a prime ideal in $K$. Then there exists a positive definite half integral matrices $A$ of degree $n$ such that $c_{I_{n,f}}(A) = c_f(|d|)q$ with $q$ not divisible by $\mathfrak{p}$.
Proof. First assume that \( d \neq 1 \), or \( n \not\equiv 4 \mod 8 \). (1) By (1) and (2) of Lemma 4.5, there exists a matrix \( A \) such that \( a_A = d \). Thus we have \( c_{I_0} (f) (A) = c_f (|d|) \). This proves the assertion.

Next assume that \( n \equiv 4 \mod 8 \) and that \( d = 1 \). Assume that \( c_f(q) + q^{k-n/2-1}(-q - 1) \) is divisible by \( \mathfrak{P} \) for any prime number \( q \). Let \( p \) be a prime number divisible by \( \mathfrak{P} \). Fix an imbedding \( \iota_p : \overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_p \), and let \( \rho_{f,p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{Q}}_p) \) be the Galois representation attached to \( f \). Then by Chebotarev density theorem, the semi-simplification \( \overline{\rho}_{f,p}^{ss} \) of \( \overline{\rho}_{f,p} \) can be expressed as

\[
\overline{\rho}_{f,p}^{ss} = \overline{\chi}_p^{k-n/2} \oplus \overline{\chi}_p^{k-n/2-1}
\]

with \( \overline{\chi}_p \) the \( p \)-adic mod \( p \) cyclotomic character. On the other hand, by the Fontaine-Messing [Fo-Me] and Fontaine-Laffaille [Fo-La], \( \overline{\rho}_{f,p}^{ss}/I_p \) should be \( \overline{\chi}_p^{2k-n-1} \oplus 1 \) or \( \omega_2^{2k-n-1} \oplus \omega_2^{p(2k-n-1)} \) with \( \omega_2 \) the fundamental character of level 2, where \( I_p \) denotes the inertia group of \( p \) in \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). This is impossible because \( k > 2 \). Thus there exists a prime number \( q \) such that \( c_f(q) + q^{k-n/2-1}(-q - 1) \) is not divisible by \( \mathfrak{P} \). For such a \( q \), take a positive definite matrix \( A \) in (3) of Lemma 4.5. Then

\[
c_{I_0} (f) (A) = c(1)q^{k-(n+1)/2} \beta_q F_q (A, q^{-(n+1)/2} \beta_q^{-1}).
\]

By [Ka1], we have

\[F_q (B, X) = 1 - Xq^{(n-2)/2}(q^2 + q) + q^3(Xq^{(n-2)/2}).\]

Thus we have

\[c_{I_0} (f) (A) = c(1)(c_f(q) + q^{k-n/2-1}(-q - 1)).\]

Thus the assertion holds.

**Theorem 4.7.** Let \( k \geq 2n+4 \). Let \( K \) and \( f \) be as above. Assume that the Conjecture B holds for \( f \). Let \( \mathfrak{P} \) be a prime ideal of \( K \). Furthermore assume that

1. \( \mathfrak{P} \) divides \( L(k, f) \prod_{i=1}^{n/2-1} L(2i + 1, f, \text{St}) \).
2. \( \mathfrak{P} \) does not divide \( \tilde{\xi}(2m) \prod_{i=1}^{n} L(2m + k - i, f) L(k - n/2, f, \chi_D) D(2k - 1)! \) for some integer \( n/2 + 1 \leq m \leq k/2 - n/2 - 1 \), and for some fundamental discriminant \( D \) such that \( (-1)^{n/2} D > 0 \).

Then \( \mathfrak{P} \) is a congruence prime of \( I_0 (f) \) with respect to \( CI_0 (f) \). Furthermore assume that the following condition hold:
(3) \( \Psi \) does not divide

\[
\langle f, f \rangle
\]

\[
\frac{C_{k,n}}{\Omega(f, +, A\Psi)\Omega(f, -, A\Psi)},
\]

where \( C_{k,n} = 1 \) or \( \prod_{q \leq (2k-n)/12} (1 + q + \cdots + q^{n-1}) \) according as \( n = 2 \) or not.

Then \( \Psi \) is a congruence prime of \( I_n(f) \) with respect to \( (\mathfrak{E}_k(I_n)^* )^- \).

**Proof.** Let \( \Psi \) be a prime ideal satisfying the condition (1) and (2). For the \( D \) above, take a matrix \( A \in \mathcal{H}_n(\mathbb{Z})_>0 \) so that \( c_{I_n(f)}(A) = c_f(\lvert D \rvert)q \) with \( q \) not divisible by \( \Psi \). Then by Proposition 4.3, we have

\[
\Lambda(2m, I_n(f), St)c_{I_n(f)}(A)^2 = \Lambda(2m, I_n(f), St)c_f(\lvert D \rvert)^2 q^2
\]

\[
= \epsilon_{k,m} \prod_{i=1}^n L(2m + k - i, f)|D|^{k-n/2} L(k - n/2, f, \chi_D)
\]

\[
\times \left( \frac{\Omega(f, +; \Psi)\Omega(f, -; A\Psi)}{\langle f, f \rangle} \right)^{n/2},
\]

where \( \epsilon_{k,m} \) is a rational number whose numerator is not divided by \( \Psi \).

We note that \( \frac{\langle f, f \rangle}{\Omega(f, +; A\Psi)\Omega(f, -; A\Psi)} \) is \( \Psi \)-integral. Thus by assumptions (1) and (2), \( \Psi \) divides \( (\Lambda(2m, I_n(f), St)c_{I_n(f)}(A)^2)^{-1} \), and thus it divides \( (\Lambda(2m, I_n(f), St)c_{I_n(f)}(A)^2)^{-1} \). We note that \( I_n(f) \) satisfies the assumption in Theorem 3.1. Thus by Theorem 3.1, there exits a Hecke eigenform \( G \in \mathcal{C}(I_n(f))^\perp \) such that

\[
\lambda_G(T) \equiv \lambda_{I_n(f)}(T) \mod \Psi
\]

for any \( T \in \mathbb{L}_n' \). Assume that we have \( G = I_n(g) \) with some primitive form \( g(z) = \sum_{m=1}^\infty a_g(m)e(mz) \in \mathcal{E}_{2k-n}(\Gamma(1)) \). Let \( n = 2 \). Then by (1) of Proposition 4.2, \( \Psi \) is also a congruence prime of \( f \). Let \( n \geq 4 \). Then by (1) of Proposition 4.4, we have

\[
(p^{n-1} + \cdots + p + 1)a_f(p) \equiv (p^{n-1} + \cdots + p + 1)a_g(p) \mod \Psi
\]

for any prime number \( p \) not divisible by \( \Psi \). By assumption (3), in particular, for any \( p \leq (2k - n)/12 \), we have

\[
a_f(p) \equiv a_g(p) \mod \Psi.
\]

Thus by Sturm [Stur], \( \Psi \) is also a congruence prime of \( f \). Thus by [Hi2] and [Ri2], \( \Psi \) divides \( \frac{\langle f, f \rangle}{\Omega(f, +; A\Psi)\Omega(f, -; A\Psi)} \), which contradicts the assumption (3). Thus \( \Psi \) is a congruence prime of \( I_n(f) \) with respect to \( (\mathfrak{E}_k(\Gamma(n))^* )^- \).
Example Let \( n = 4 \) and \( k = 18 \). Then we have \( \dim S_{18}(\Gamma_4) \approx 16 \) (cf. Poor and Yuen\[P-Y\]) and \( \dim S_{18}(\Gamma_4)^* = \dim S_{32}(\Gamma_1) = 2 \). Take a primitive form \( f \in S_{32}(\Gamma_1) \). Then we have \([\mathbb{Q}(f) : \mathbb{Q}] = 2\), and \(211 = \mathfrak{p}\mathfrak{q}'\) in \( \mathbb{Q}(f) \). Then we have

\[
N_{\mathbb{Q}(f)/\mathbb{Q}}(L(18, f)) = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 211,
\]

\[
N_{\mathbb{Q}(f)/\mathbb{Q}}(\prod_{i=1}^4 L(24-i, f)) = 2^{19} \cdot 3^{13} \cdot 5^5 \cdot 7^8 \cdot 11^2 \cdot 13^5 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 503 \cdot 1307 \cdot 14243,
\]

and

\[
\tilde{\xi}(6) = 2^{-2} \cdot 3^{-2} \cdot 7^{-1}
\]

(cf. Stein \[Ste\].) Furthermore, by a direct computation we see neither \( \mathfrak{p} \) nor \( \mathfrak{p}' \) is a congruence prime of \( \tilde{f} \) with respect to \( C\tilde{g} \) for another primitive form \( g \in S_{32}(\Gamma_1) \). Thus by Theorem 4.7, \( \mathfrak{p} \) or \( \mathfrak{p}' \) is a congruence prime of \( \tilde{f} \) with respect to \( S_{18}(\Gamma_4)^{\perp} \).

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