A Problem on Edge-magic Labelings of Cycles

S. C. López, F. A. Muntaner-Batle, and M. Rius-Font

Abstract. In 1970, Kotzig and Rosa defined the concept of edge-magic labelings as follows. Let $G$ be a simple $(p, q)$-graph (that is, a graph of order $p$ and size $q$ without loops or multiple edges). A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ is an edge-magic labeling of $G$ if $f(u) + f(uv) + f(v) = k$, for all $uv \in E(G)$. A graph that admits an edge-magic labeling is called an edge-magic graph, and $k$ is called the magic sum of the labeling. An old conjecture of Godbold and Slater states that all possible theoretical magic sums are attained for each cycle of order $n \geq 7$. Motivated by this conjecture, we prove that for all $n_0 \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that the cycle $C_n$ admits at least $n_0$ edge-magic labelings with at least $n_0$ mutually distinct magic sums. We do this by providing a lower bound for the number of magic sums of the cycle $C_n$, depending on the sum of the exponents of the odd primes appearing in the prime factorization of $n$.

1 Introduction

For the graph theory terminology and notation not defined in this paper we refer the reader to any one of the following sources [3, 5, 8, 15]. In 1970, Kotzig and Rosa [10] defined the concept of edge-magic labelings as follows. Let $G$ be a simple $(p, q)$-graph (that is, a graph of order $p$ and size $q$ without loops or multiple edges). A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ is an edge-magic labeling of $G$ if $f(u) + f(uv) + f(v) = k$, for all $uv \in E(G)$. A graph that admits an edge-magic labeling is called an edge-magic graph, and $k$ is called the valence, the magic sum [15], or the magic weight [3] of the labeling.

Godbold and Slater introduced the following conjecture in [9].

Conjecture 1.1 ([9]) For $n = 2t + 1 \geq 7$ and $5t + 4 \leq j \leq 7t + 5$ there is an edge-magic labeling of $C_n$ with magic sum $k = j$. For $n = 2t \geq 4$ and $5t + 2 \leq j \leq 7t + 1$ there is an edge-magic labeling of $C_n$ with magic sum $k = j$.

We mention that the lower bound (resp. the upper bound) on the magic sum comes from assigning the lowest (resp. the highest) numbers to the vertices of the cycle. Motivated by this conjecture we introduce the following theorem. The goal of this paper is to prove it.

Theorem 1.2 For all $n_0 \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that the cycle $C_n$ admits at least $n_0$ edge-magic labelings with at least $n_0$ mutually distinct magic sums.
2 The Tools

Figueroa-Centeno et al. introduced the following definition in [7]. Let $D$ be a digraph and let $\Gamma = \{F_i\}_{i=1}^m$ be a family of digraphs such that $V(F_i) = V$ for every $i \in \{1, 2, \ldots, m\}$. Consider a function $h : E(D) \to \Gamma$. Then the product $D \otimes_h \Gamma$ is the digraph with vertex set $V(D) \times V$ and $((a, b), (c, d)) \in E(D \otimes_h \Gamma)$ if and only if $(a, c) \in E(D)$ and $(b, d) \in E(h(a, c))$. The adjacency matrix of $D \otimes_h \Gamma$, namely $A(D \otimes_h \Gamma)$, is obtained by replacing every 0 entry of $A(D)$, the adjacency matrix of $D$, by the $|V| \times |V|$ null matrix and every 1 entry of $A(D)$ by $A(h(a, c))$.

The following restriction of edge-magic labelings introduced independently by Acharya and Hegde [1] and by Enomoto et al. [6] will prove to be of great help in the rest of this document. Let $G$ be a $(p, q)$-graph. Then $G$ is a super edge-magic graph [1, 6] if there is an edge-magic labeling of $G$, namely $\sigma : V(G) \cup E(G) \to \{i\}_{i=1}^{p+q}$, with the extra property that $f(V(G)) = \{i\}_{i=1}^{p}$. The labeling $f$ is called a super edge-magic labeling of $G$. All cycles are edge-magic [9]. However, a cycle $C_p$ is super edge-magic if and only if $p$ is odd [6]. As in [7], a digraph $D$ is said to admit a labeling $f$ if its underlying graph, $\text{und}(D)$, admits $l$. From now on, let $D_p$ be the set of all 1-regular super edge-magic labeled digraphs of odd order $p$, $p \geq 3$, where each vertex takes the name of the label assigned to it. Then we have the following theorem.

**Theorem 2.1** ([7]) Let $D$ be a (super) edge-magic digraph, and let $h : E(D) \to S_p$ be any function. Then $\text{und}(D \otimes_h S_p)$ is (super) edge-magic.

The key point in the proof (see also [11]) is to rename the vertices of $D$ and each element of $D_p$ after the labels of their corresponding (super) edge-magic labeling $f$ and their super edge-magic labelings respectively and to define the labels of the product as follows: (i) the vertex $(i, j) \in V(D \otimes_h S_p)$ receives the label: $p(i-1) + j$ and (ii) the arc $((i, j), (i', j')) \in E(D \otimes_h S_p)$ receives the label: $p(e-1) + (3p+3)/2 - (j + j')$, where $e$ is the label of $(i, i')$ in $D$. Thus, for each arc $((i, j), (i', j')) \in E(D \otimes_h S_p)$, coming from an arc $e = (i, i') \in E(D)$ and an arc $(j, j') \in E(h(i, i'))$, the sum of labels is constant and equal to $p(i + i' + e - 3) + (3p + 3)/2$. That is, $p(\sigma_f - 3) + (3p + 3)/2$, where $\sigma_f$ denotes the magic sum of the labeling $f$ of $D$. Therefore, we obtain the following proposition.

**Proposition 2.2** Let $\bar{f}$ be the edge-magic labeling of the graph $\text{und}(D \otimes_h S_p)$ obtained in Theorem 2.1 from a labeling $f$ of $D$. Then the magic sum of $\bar{f}$, $\sigma_{\bar{f}}$, is given by the formula

$$\sigma_{\bar{f}} = p(\sigma_f - 3) + \frac{3p + 3}{2},$$

where $\sigma_f$ is the magic sum of $f$.

**Corollary 2.3** Let $D$ be an edge-magic digraph and assume that there exist two edge-magic labelings of $D$, $f$ and $g$, such that $\sigma_f \neq \sigma_g$. If we denote by $\bar{f}$ and $\bar{g}$ the edge-magic labelings of the graph $\text{und}(D \otimes_h S_p)$ when using the edge-magic labelings $f$ and $g$ of $D$ respectively, then we get $|\sigma_{\bar{f}} - \sigma_{\bar{g}}| \geq 3$.
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Proof Since $\sigma_f \neq \sigma_g$, we get the inequality $|\sigma_f - \sigma_g| \geq 1$. Thus, by using (2.1) we obtain that $|\sigma_f - \sigma_g| = |p(\sigma_f - \sigma_g)| \geq 3$.

The following two results appear in [15].

Theorem 2.4 ([15]) Every odd cycle $C_n$ has an edge-magic labeling with magic sum $3n + 1$ and an edge-magic labeling with magic sum $3n + 2$.

Theorem 2.5 ([15]) Every even cycle $C_n$ has an edge-magic labeling with magic sum $(5n + 4)/2$.

Next, we state the following two structural results. We denote by $\overrightarrow{C_n}$ and $\overleftarrow{C_n}$ the two possible strong orientations of the cycle $C_n$, where the vertices of $C_n$ are the elements of the set $\{i\}_{i=1}^n$.

Theorem 2.6 ([7]) Let $h: E(\overrightarrow{C_m}) \to \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ be any constant function. Then $\text{und}(\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}) = \gcd(m, n) C_{\text{lcm}(m, n)}$.

Theorem 2.7 ([2]) Let $m, n \in \mathbb{N}$ and consider the product $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ where $h: E(\overrightarrow{C_m}) \to \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$. Let $g$ be a generator of a cyclic subgroup of $\mathbb{Z}_n$, namely $\langle g \rangle$, such that $|\langle g \rangle| = k$. Also let $N_g(h^-) < m$ be a natural number that satisfies the congruence relation $m - 2N_g(h^-) \equiv g \pmod{n}$.

If the function $h$ assigns $\overleftarrow{C_n}$ to exactly $N_g(h^-)$ arcs of $\overrightarrow{C_m}$, then the product $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ consists of exactly $n/k$ disjoint copies of a strongly oriented cycle $\overrightarrow{C_{mk}}$. In particular, if $\gcd(g, n) = 1$, then $\langle g \rangle = \mathbb{Z}_n$, and if the function $h$ assigns $\overleftarrow{C_n}$ to exactly $N_g(h^-)$ arcs of $\overrightarrow{C_m}$, then $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\} \cong \overrightarrow{C_{mn}}$.

Corollary 2.8 Let $n \geq 3$ be an odd integer and suppose that $m \geq 3$ is an integer such that either $m$ is odd or $m \geq n$. Then there exists a function $h: E(\overrightarrow{C_m}) \to \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ such that $\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\} \cong \overrightarrow{C_{mn}}$.

Proof We have that $\langle 1 \rangle = \mathbb{Z}_n$, and since $n$ is odd, the congruence relation $m - 2r \equiv 1 \pmod{n}$ can be solved, with $0 < r < m$. Therefore, inheriting the notation of Theorem 2.7, by considering any function $h$ with $N_1(h^-) = r$, we get the desired result.

3 Proof of the Main Result

We start this section by showing four edge-magic labelings of $C_3$ with consecutive magic sums in Figure 1.

We are now ready to prove Theorem 1.2.
Proof of Theorem 1.2 We already know that $C_3$ admits 4 edge-magic labelings with 4 consecutive edge-magic magic sums (notice that the labeling corresponding to magic sum 9 is super edge-magic). Call these labelings $l_1, l_2, l_3, l_4$, where the magic sum of $l_i$ is less than the magic sum of $l_j$ if and only if $i < j$ ($i, j \in \{1, 2, 3, 4\}$), and denote by $C_3^l$ the copy of $C_3$, where each vertex takes the name of the label that $l_i$ has assigned to it. Also let $\vec{C}_3^l$ be the digraph obtained from $C_3^l$ with the edges oriented cyclically. Recall that, we denote by $\vec{C}_3$ and $\vec{C}_3^\gamma$ the two possible strong orientations of $C_3$, where the vertices of $C_3$ are labeled in a super edge-magic way. Let $\gamma = \{\vec{C}_3, \vec{C}_3^\gamma\}$.

By Corollary 2.8, for all $i \in \{1, 2, 3, 4\}$ there exists a function $h_i : E(\vec{C}_3^l) \to \Gamma$ such that $\text{und}(\vec{C}_3^l \otimes h_i \Gamma) \cong C_9$. Also, any two magic sums of the labelings obtained for $\vec{C}_3^l \otimes h_i \Gamma$ differ, by Corollary 2.3, by at least three units. But we know by Theorem 2.4 that magic sums 28 and 29 appear for different edge-magic labelings of $C_9$. Hence, the cycle $C_9$ admits at least 5 edge-magic labelings with 5 different magic sums. Let the labelings that provide these magic sums be $l_1^i$, where the magic sum of $l_1^i$ is less than the magic sum of $l_1^j$ if and only if $i < j$ ($i, j \in \{1, 2, \ldots, 5\}$).

If we repeat the process with $\vec{C}_3^l \otimes h_i \Gamma$, where $h_i : E(\vec{C}_3^l) \to \Gamma$ is a function as in Corollary 2.8, we obtain 5 edge-magic labelings of $C_{27}$ with 5 different magic sums. But, again by Corollary 2.3, either magic sum 82 or magic sum 83, does not appear among these 5 magic sums, since among these 5 magic sums no two magic sums are consecutive. But we know by Theorem 2.4 that these two magic sums, 82 and 83, appear for an edge-magic labeling of $C_{27}$. Hence, there are at least 6 magic sums for edge-magic labelings of $C_{27}$.

Repeating this process inductively, we obtain that each cycle of order $3^\alpha$ admits at least $3 + \alpha$ edge-magic labelings with at least $3 + \alpha$ mutually different magic sums. Therefore, we get the desired result.

Notice that, using a similar idea to the one in the proof of Theorem 1.2, we can obtain the following theorem.

**Theorem 3.1** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique prime factorization (up to ordering) of an odd number $n$. Then $C_n$ admits at least $1 + \sum_{i=1}^k \alpha_i$ edge-magic labelings with at least $1 + \sum_{i=1}^k \alpha_i$ mutually different magic sums.

Using Theorem 2.5 and the previous construction, we can prove the next theorem.
Theorem 3.2 Let \( n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) be the unique prime factorization of an even number \( n \), with \( p_1 > p_2 > \cdots > p_k \). Then \( C_n \) admits at least \( \sum_{i=1}^k \alpha_i \) edge-magic labelings with at least \( \sum_{i=1}^k \alpha_i \) mutually different magic sums. If \( \alpha \geq 2 \), this lower bound can be improved to \( 1 + \sum_{i=1}^k \alpha_i \).

Proof Assume first that \( \alpha \geq 2 \). By Theorem 2.5, the cycle of order \( 2^\alpha \) has an edge-magic labeling \( l \) with magic sum \( 5 \cdot 2^{\alpha-1} + 2 \). Let \( C^l_{2^\alpha} \) be the copy of \( C_{2^\alpha} \), where each vertex takes the name of the label that \( l \) has assigned to it, and for each \( i = 1, 2, \ldots, k \) let

\[
\Gamma_i = \{ \frac{\circlearrowright}{\circlearrowleft} C_{p_i}, C_{p_i} \},
\]

where the vertices of \( C_{p_i} \) are labeled in a super edge-magic way. Also let \( \hat{C}^l_{2^\alpha} \) be the digraph obtained from \( C^l_{2^\alpha} \), such that the edges have been oriented cyclically.

By Theorem 2.6, any constant function \( h: E(\hat{C}^l_{2^\alpha}) \rightarrow \Gamma_1 \) gives

\[
C^l_{2^\alpha - p_i} \cong \text{und}(\hat{C}^l_{2^\alpha} \otimes_h \Gamma_1).
\]

Notice that, by Proposition 2.2, the induced edge-magic labeling on \( C^l_{2^\alpha - p_i} \) has magic sum

\[
p_1 (\sigma_1 - 3) + \frac{3p_1 + 3}{2} = 5p_1 \cdot 2^{\alpha-1} + \frac{p_1 + 3}{2}.
\]

Since by Theorem 2.5, the cycle \( C^l_{2^\alpha - p_i} \) has an edge-magic labeling with magic sum \( 5p_1 \cdot 2^{\alpha-1} + 2 \), we get that \( C^l_{2^\alpha - p_i} \) admits two edge-magic labelings with two different magic sums. Assume that \( \sum_{i=1}^k \alpha_i \geq 2 \) (otherwise the result is proved) and call these labelings \( l_1, l_2 \), where the magic sum of \( l_1 \) is less than the magic sum of \( l_2 \). Denote by \( C^l_{2^\alpha - p_i} \) the copy of \( C^l_{2^\alpha - p_i} \), where each vertex takes the name of the label that \( l_i \) has assigned to it. Also let \( \hat{C}^l_{2^\alpha - p_i} \) be the digraph obtained from \( C^l_{2^\alpha - p_i} \), such that the edges have been oriented cyclically.

By Corollary 2.8, for each \( i \in \{1, 2\} \) and for some fixed \( j \in \{1, 2, \ldots, k\} \), there exists a function

\[
h_j: E(\hat{C}^l_{2^\alpha - p_i}) \rightarrow \Gamma_j
\]

such that \( \text{und}(\hat{C}^l_{2^\alpha - p_i} \otimes_h \Gamma_j) \cong C^l_{2^\alpha - p_i,j} \). We take \( j = 1 \) when \( \alpha_1 > 2 \), and \( j = 2 \) when \( \alpha_1 = 1 \). Also, by Corollary 2.3, the two magic sums of the labelings obtained from \( \hat{C}^l_{2^\alpha - p_i} \otimes_h \Gamma_j \) differ, by at least three units. Moreover, the minimum of them, that is \( p_1 (\sigma_0 - 3) + (3p_1 + 3) / 2 = 5p_1 p_j \cdot 2^{\alpha-1} + (p_1 + 3) / 2 \), is bigger than the magic sum guaranteed by Theorem 2.5. Hence, the cycle \( C^l_{2^\alpha - p_i,j} \) has at least three edge-magic labelings with at least three mutually different magic sums.

Repeating this process inductively, following the order of primes, we obtain that each cycle of order \( 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) admits at least \( 1 + \sum_{i=1}^k \alpha_i \) edge-magic labelings with at least \( 1 + \sum_{i=1}^k \alpha_i \) mutually different magic sums.

Assume now that \( \alpha = 1 \). In this case, we proceed as in the case \( \alpha \geq 2 \), but starting with the cycle of length \( 2^\alpha p_1 \). Therefore, we get the desired result. \( \square \)
3.1 Conclusions

Interest seems to be growing on the study of the magic sums of edge-magic and super edge-magic labelings (see [12, 13] for instance). In this paper we have concentrated our efforts in the study of the set of edge-magic magic sums for cycles. This is an old problem that appeared in [9] and that has remained unsolved for 15 years. Very little progress has been made towards a solution of it since then. In fact, for many years only four magic sums have been known for $C_n$, except for small values of $n$ where the problem has been treated using computers (see [4]). It was not until 2009 that a paper appeared [14], in which the author proved a result similar to the one introduced in this paper. However, the method used in [14] and the method introduced in this paper are absolutely different. We feel that to try to combine both methods could be a very interesting line of research in the future. So far, we remark that concerning this problem about the valences of $C_n$ we have only two different methods that allow us to show that the number of magic sums of the cycle $C_n$ grows unbounded for the values of $n$. However the original question found in [9] remains unsolved, and we feel that, at this point, we are very far away from a final solution.

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