A partial element equivalent circuit-metamodel combination for fast tolerance analysis of electromagnetic systems

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Abstract
In this letter, we propose a combination of the partial element equivalent circuit method with metamodelling in order to achieve fast tolerance analysis of electromagnetic systems. The proposed model combination can be interpreted as a multifidelity modeling approach. This technique is inspired by the multilevel Monte Carlo method and provides great benefits in terms of computational resources.

KEYWORDS
multilevel Monte Carlo, partial element equivalent circuit method, tolerance analysis

1 | INTRODUCTION

Electromagnetic (EM) solvers are fundamental for a variety of important design problems, such as microwave devices, signal and power integrity. The Partial Element Equivalent Circuit (PEEC) method has gained popularity thanks to its ability to represent the electromagnetic behavior of physical structures by means of circuit elements. As in general for EM solvers, the discretization and subsequent solution of Maxwell’s equations can result in large-scale systems of equations demanding high computational resources. Therefore, the brute force use of EM solvers in design tasks requiring a large number of simulations, such as optimization and tolerance analysis, becomes infeasible.

Among all the techniques in the literature to speed-up the design flow of complex systems, we focus on multifidelity modeling in this letter. The main idea behind multifidelity modeling is to combine models with a different level of accuracy and computational complexity. High accuracy requires high computational resources and vice versa. Several flavors of multifidelity models can be found in the literature with applications in several domains, such as optimization and tolerance analysis.

In the tolerance analysis area, the Monte Carlo (MC) technique is very easy to implement and well-established. One drawback of this technique is the slow convergence that might potentially require a high number of simulations to accurately estimate stochastic information (e.g., mean and variance). Several techniques have been proposed to accelerate the MC analysis, such as Quasi Monte Carlo and Stochastic Collocation techniques. Multifidelity modeling to accelerate MC has been proposed as a multi-level Monte Carlo (MLMC) method. MLMC has been recently applied to low-frequency EM problems. The EM applications are examples with a single frequency and the mean value of some quantities of interest is estimated. For an analysis of a dynamical system over a frequency range, assuring that the same number and definition of levels is effective for each frequency might be a difficult task. The set of models needs to satisfy some specific conditions about computational complexity and variance in order to achieve an optimal convergence rate. This affects choices such as the number of models (levels) and how to choose the levels.

In EM problems, a natural way to define levels is by changing the discretization mesh size. One additional attractive feature of MLMC is that an error control on the moments estimation is available, as it is the case for MC.

In this letter, we propose a simple but very effective multilevel choice for systems that span a broadband frequency range. We select only two models, one high-fidelity model and one low-fidelity model. The high-fidelity model is defined by a PEEC model with a mesh discretization that
ensures high accuracy. This also brings a high computational cost. The low-fidelity model is built based on metamodelling techniques. Metamodels are very efficient and provide a functional relationship between input and output variables. In our case, these variables correspond to physical design parameters and EM quantities of interest (e.g., scattering parameters), respectively. Several basis functions can be used to build this mathematical link, such as radial basis functions, polynomials, and Gaussian processes. The high-fidelity model is kept as a reference for accuracy and used a few times during the tolerance analysis. Instead, the low-fidelity model is built based on metamodelling. Several flavors of the PEEC method exist depending on how the exponential function in the integrals above is treated. In this letter, we used the so-called quasi-static (qs) case where the exponential terms are kept into the integrals when computing the partial inductances and potential coefficients terms. This formulation is also more accurate than the so-called quasi-static (qs) case where the exponential terms are just fixed equal to one.

2 | PEEC METHOD

The PEEC method\(^1\) applies the Galerkin procedure to Maxwell’s equations in Lorentz gauge condition. After the discretization of materials volumes and surfaces into elementary regions, the Electric Field Integral Equation and continuity equations are used to obtain an EM model that has a circuit description. Considering a transfer function representation, the PEEC method provides a descriptor state-space description. For example, we show below an impedance representation

\[
(sC(s) + G(s))X(s) = BL_p(s), \quad V_p(s) = B^T X(s),
\]

(1)

where \(V_p(s)\) and \(L_p(s)\) are the port voltages and currents, and \(s\) is the Laplace variable. Admittance and scattering parameters representation are also possible. In the PEEC matrices above, the EM phenomena are represented in terms of resistances, partial inductances and partial capacitances (circuit interpretation of the EM phenomena).

The entries of the partial inductance sub-matrix \(L_p(s)\) and potential coefficient sub-matrix \(P(s)\) in the block-structure of the matrix \(C(s)\) are defined as follows

\[
L_{p,n,m}(s) = \iiint_{V_n \times V_m} g(r, r', s) dV_m dV_n, \quad P_{n,m}(s) = \tfrac{1}{s} \iint_{S_m \times S_n} g(r, r', s) dS_m dS_n, \quad g(r, r', s) = \frac{1}{4\pi} \frac{e^{-\tau r}}{|r-r'|},
\]

(2)

using the scalar Green’s functions with retardation of the type

\[
g(r, r', s) = \frac{1}{4\pi} \frac{e^{-\tau r}}{|r-r'|},
\]

(3)

with the delay equal to \(\tau = \frac{|r-r'|}{c_0}\) and \(c_0\) equal to the speed of light in vacuum. The reader can refer to [1] for more details about the PEEC method. Several flavors of the PEEC method exist depending on how the exponential function in the integrals above is treated. In this letter, we used the so-called center-to-center approximation:

\[
L_{p,n,m}^{(cc)}(s) = e^{-s R_{n,m}} L_{p,n,m}(s), \quad P_{n,m}^{(cc)}(s) = e^{-s R_{n,m}} P_{n,m},
\]

(4)

where \(R_{n,m}\) is the Euclidean distance between the centers of the mesh cells \(n\) and \(m\). The delay center-to-center between each couple of mesh cells is used to approximate the delay of the interaction of each mesh cell couple and the exponential term is taken out of the integrals. This PEEC formulation has been proven accurate for several application cases and less computationally expensive than the PEEC solution where the exponential terms are kept into the integrals when computing the partial inductances and potential coefficients terms. This formulation is also more accurate than the so-called quasi-static (qs) case where the exponential terms are just fixed equal to one.

3 | MONTE CARLO AND MULTIFIDELITY MODELING

In the design of a system of interest, design parameters such as layout features influence the system behavior and therefore the design performances. Design tasks, such as optimization and tolerance analysis, require multiple simulations where the design parameters are varied. Let us consider \(d\) design parameters as \(\xi = [\xi_1, \ldots, \xi_d] \in \mathbb{R}^d\), then we can introduce the influence of these parameters into the state-space PEEC Equation (1) as

\[
(sC(s, \xi) + G(s, \xi))X(s, \xi) = BL_p(s), \quad V_p(s, \xi) = B^T X(s, \xi),
\]

(5)

Our contribution targets tolerance analysis and the estimation of stochastic quantities, such as mean and variance, of some EM quantities of interest that depend on random variations of the \(\xi\) parameters. We estimate stochastic moments of the scattering parameters \(S(s, \xi)\) in this letter. The formulas in what follows are considered for a scalar function for ease of notation. The extension towards vectors and matrices is straightforward considering element-wise operations. The \(m\)-th raw and central stochastic moment of \(S(s, \xi)\) (without loss of generality, we assume here continuous random variables) can be expressed as

\[
\mathbb{E}[Q^m(s, \xi)] = \int \mathcal{Q}^m(s, \xi) W(\xi) d\xi,
\]

(6)

with \(m \in \mathbb{N}\), where \(\mathbb{E}[-]\) represents the expectation operator, the \(Q(s, \xi)\) quantity can denote \(S(s, \xi)\) or \(S(s, \xi) - \mathbb{E}[S(s, \xi)]\), for raw and central moments, respectively. We
call here $\mu_m(Q)$ and $\mu_m(Q)$ the $m$-th raw and central moments, respectively. $W(\xi)$ is a joint probability density function.

In this letter, we use the following assumption: although the exact EM behavior is not perfectly known due to Maxwell’s equations discretization in EM solvers, the common way to use EM solvers in engineering to get accurate results is by choosing settings (e.g., a fine mesh discretization) that can provide a high accuracy. Therefore, we assume that the response $S(s, \xi)$ provided by the PEEC solver is the correct EM response. An estimation of the moments above described can be obtained by the MC approach, where a set of random samples based on the joint probability density function of $\xi$ is used. After the function $S(s, \xi)$ is sampled in the $\xi$ space, the collected data can be used to estimate stochastic moments by sample-based formulas. It is important to use estimators that are unbiased and have minimal variance compared with other unbiased estimators. In this perspective, for example the mean $E[S(s, \xi)]$ and variance $E[(S(s, \xi) - E[S(s, \xi)])^2]$ can be estimated using $R$ independent and identically distributed (i.i.d.) samples $S_R := (S(s, \xi),)_{r=1,\ldots,R}$ as

$$E[S(s, \xi)] \approx m_1(S_R(s, \xi)) := \frac{1}{R} \sum_{r=1}^{R} S(s, \xi, r), \tag{7a}$$

$$E[(S(s, \xi) - E[S(s, \xi)])^2] \approx h_2(S_R(s, \xi)) = \frac{R T_2 - T_1}{(R-1)R}, \tag{7b}$$

where $T_a$ is defined based on the samples $S(s, \xi, r), r = 1,\ldots,R$ as a power sum $T_a = \sum_{r=1}^{R} S^a(s, \xi, r)$ with $a \in \mathbb{N}$, using the so-called $h$-statistics. Both (7a)-(7b) represent an unbiased estimator of the corresponding moments. For the estimation of the second central moment (the variance), we use the $h$-statistics that provides an unbiased estimator with minimal variance with respect to other unbiased estimators. Generalization of (7a)-(7b) for higher order moments can be found in \cite{10,11}.

In this letter, we consider mean and variance as the stochastic moments of interest. The described estimators are based on random sampling and their mean squared error (MSE) can be defined as a bias term squared and a variance term. We omit $(s, \xi)$ into $S(s, \xi)$ and $S_R(s, \xi)$ for ease of notation in what follows. In the case of an unbiased estimator, such as $m_1$ and $h_2$, only the variance term will appear.

$$\text{MSE}(m_1(S_R)) = E[(m_1(S_R) - \mu_1(S))^2] = \text{Var}(m_1(S_R)), \tag{8a}$$

$$\text{MSE}(h_2(S_R)) = E[(h_2(S_R) - \mu_2(S))^2] = \text{Var}(h_2(S_R)), \tag{8b}$$

Using $h_2(S_R)$ to estimate $\mu_2(S)$ provides an unbiased estimator of $\text{Var}(m_1(S_R))$. For this formula, estimating the needed samples $R$ to achieve a certain target MSE is straightforward. An unbiased estimator for $\text{Var}(h_2(S_R))$ is discussed in \cite{11}.

Both variance terms in (8c) are of order $O(R^{-1})$, which allows reformulating $\text{Var}(h_2(S_R))$ in a more suitable form in order to estimate the needed samples $R$ to achieve a certain target MSE. The maximum number of samples between the two estimations is then chosen. We note that MC and the proposed multifidelity approach are initialized with an initial number of samples, then the needed number of samples to achieve the target MSE value is estimated, new simulations are performed and the loop continues till the target MSE value is achieved.

The MLMC technique generalizes the MC technique. A set of models is combined in a specific form. Let us consider $L+1$ levels and therefore $L+1$ models that provide $\{S_l(s, \xi)\}_{l=0}^{L}$ as corresponding output. We assume the $L$-th model is the most accurate and computationally expensive, while the 0-th model is the least accurate and computationally expensive in the set of $L+1$ levels. We assume the $L$-th model gives what we consider the exact EM behavior $S_L = S$ and this model is also used by the MC analysis. The estimation of mean and variance of $S(s, \xi)$ can be obtained as described in \cite{4,11}, respectively

$$E[S(s, \xi)] \approx m_{L,\text{MLMC}}(\{S_l\}_{l=0}^{L}) = m_1(S_{0,R_0}) + \sum_{l=1}^{L} m_1(S_{l,R_l} - S_{l-1,R_l}), \tag{9}$$

$$E[(S(s, \xi) - E[S(s, \xi)])^2] \approx h_{2,\text{MLMC}}(\{S_l\}_{l=0}^{L}) = h_2(S_{0,R_0}) + \sum_{l=1}^{L} \left( h_2(S_{l,R_l}) - h_2(S_{l-1,R_l}) \right). \tag{10}$$

Considering the assumption $S_L = S$, both estimators $m_{L,\text{MLMC}}$ and $h_{2,\text{MLMC}}$ are unbiased. Concerning their MSE (thus their variance), we assume to use i.i.d. samples to sample and evaluate $S_0(s, \xi)$ and each pair $[S_l(s, \xi), S_{l-1}(s, \xi)]$. The following variance formula for the $m_{L,\text{MLMC}}$ estimator can be obtained \cite{4}.

$$\text{Var}(m_{L,\text{MLMC}}) = \text{Var}(m_1(S_{0,R_0})) + \sum_{l=1}^{L} \text{Var}(m_1(S_{l,R_l} - S_{l-1,R_l})), \tag{11}$$

A similar formula can be obtained for $h_{2,\text{MLMC}}$ as a sum of variance contributions. However, the formula is more involved and we refer to \cite{11} for the details.

For a chosen MSE value related to the computation of mean and variance of $S(s, \xi)$, the number of samples
The metamodel is then built

\[
\{ R_i \}_{i=0}^L
\]

needed to achieve this error level and minimize the total computational cost of the tolerance analysis can be found by simple analytical formulas obtained after a constrained minimization problem.\(^4,11\) Therefore, as in (8c), the needed number of samples can be easily computed.\(^4,11\)

Choosing the number and definition of levels might not be a simple choice, especially for a broadband design of interest. These choices strongly influence the speed-up that MLMC can achieve over MC. What we propose is to reduce the number of levels to only two levels. This removes in principle some flexibility, but it provides a very simple and effective choice. Therefore, we are in the scenario of one high-fidelity model (HFM) and one low-fidelity model (LFM). The LFM is generated by using metamodelling techniques and it accurately approximates the HFM. Therefore, we consider a metamodel as

\[
S_{\text{metamodel}}(s, \xi) \approx S_L(s, \xi).\]

In this letter, the LFM is created by using a Latin hypercube sampling (LHS) approach and radial basis functions (RBF) and polynomial functions for the metamodel construction. The LHS is used to choose the samples \( \xi_q, q = 1, \ldots, Q \), the data \( S_L(s, \xi_q) \) is collected from \( Q \) HFM simulations and the metamodel is then built

\[
S_{\text{metamodel}}(s, \xi) = \sum_{q=1}^{Q} w_q(s) \phi(\xi - \xi_q) + \sum_{b=1}^{B} w_b(s) p_b(\xi).
\]

More details about the linear system to solve in order to compute the coefficients \( w_q, w_b \) can be found in.\(^12\) Different radial basis functions \( \phi(\xi - \xi_q) \) exist in the literature. We selected as an example the multiquadrics basis functions

\[
\phi(\xi - \xi_q) = r_q + c^2
\]

where \( r_q \) represents the euclidean distance between \( \xi \) and \( \xi_q \) and \( c \) is a positive parameter. Other choices are possible. The \( B \) polynomial functions were chosen to be \( [1, \xi(1), \xi(2), \ldots, \xi(d)] \), where \( d \) is the number of design parameters.

After the LFM generation, the HFM (level 1) is kept as a reference for accuracy and called a few times (this means a few samples) during the tolerance analysis. Instead, the LFM (level 0) takes care of a very high percentage of the needed simulations and accurately approximates the behavior of the HFM. In this way, their difference (level 1 - level 0) will be low and a few samples will be called by this difference term in the related MLMC formulas. We note that it might be tempting to use only the LFM. However, this will introduce a bias term into the \( MSE \) formulas, which might be difficult to compute accurately. We note that having a more or less accurate LFM will bring less or more samples for the difference term (level 1 - level 0) into the MLMC formulas. The generation of a metamodel can be automated by adaptive sampling and modeling techniques. However, this is beyond the scope of this letter. Figure 1 shows the flowchart of the proposed technique.

## 4 NUMERICAL RESULTS

A finger capacitor was used as a validation example to illustrate the advantages of the proposed technique. Figure 2 shows the layout of the structure. The thickness of conductors and dielectrics is equal to 100 \( \mu m \) and 840 \( \mu m \), respectively; \( WP = 300 \mu m, WC = 400 \mu m, WR = 700 \mu m \). The stochastic parameters are uniform random variables varying in the \[10 \pm 10\% \] range with central values \( L = 1400 \mu m, W = 600 \mu m, G = 200 \mu m, LP = 950 \mu m \) and \( GR = 650 \mu m \). The electrical port of interest is defined by the blue dots. The metal is copper, while the substrate is made of a dielectric with relative permittivity of 6.5. The frequency range of interest is \([1 - 7] GHz\) and it was sampled by 51 samples. Each frequency sample is considered separately for the tolerance analysis. The 5D space of the stochastic parameters is composed of \( \xi = [L, W, G, LP, GR] \). This space was sampled by a MC technique and the proposed technique. The HFM is a PEEC model with a mesh resolution \( \lambda_{\text{min}}/40 \) (\( \lambda \) is the wavelength). A higher mesh resolution leads to the same accuracy of the EM results (and therefore to an unnecessary higher computational complexity), while a lower mesh resolution leads to accuracy losses.

![FIGURE 1 Flowchart of the proposed technique](https://wileyonlinelibrary.com)

![FIGURE 2 Layout of the finger capacitor](https://wileyonlinelibrary.com)
This is the only model used by the MC approach. The EM quantity of interest is the scattering parameter $S_{11}(s=j\omega f, \xi)$, where $f$ denotes the frequency variable.

The average CPU time needed on an average performance laptop by the HFM model for one frequency sample simulation is 178 s, while the same CPU time quantity for the metamodel is equal to 1 ms. The LHS sampling used 50 samples in the $\xi$ space as samples dedicated to build the metamodel $S_{\text{metamodel}}(j\omega f, \xi)$. We consider an $MSE$ equal to $10^{-6}$ as accuracy objective for the estimators of mean and variance of the real and imaginary part of $S_{11}$. The needed number of samples to run these two techniques is chosen by (8c) for MC and by the formulas in $^4, ^{11}$ for the proposed two-level multifidelity technique. Both techniques were initialized with 50 $\xi$ random samples for each level involved (the HFM level only in MC, the LFM and HFM levels in the multifidelity scheme). Then, the needed number of samples to achieve the desired $MSE$ value was estimated, new simulations were performed and the loop continued till the desired $MSE$ value was achieved. Figure 3 compares the mean and variance estimation of the real and imaginary part of $S_{11}$ for the MC analysis and proposed technique. Figure 4 shows the speed-up of the proposed technique with respect to a MC analysis, when both techniques target the same $MSE$ value. This speed-up is calculated considering for each frequency sample: a) for the MC, the CPU time needed

**FIGURE 3** Mean and variance of the real and imaginary part of $S_{11}$ [Color figure can be viewed at wileyonlinelibrary.com]

**FIGURE 4** Speed-up of the proposed technique with respect to a MC analysis [Color figure can be viewed at wileyonlinelibrary.com]

**FIGURE 5** Speed-up of a five levels multilevel Monte Carlo (MLMC) based on different partial element equivalent circuit (PEEC) mesh resolutions with respect to a MC analysis [Color figure can be viewed at wileyonlinelibrary.com]
to run the MC using the HFM; b) for the proposed technique, the CPU time needed to collect the $S_{11}$ data samples related to the LHS sampling, to build the metamodel and to run the MLMC analysis with the two levels (LFM and HFM). As demonstrated by the results, the proposed multifidelity scheme achieves a significant speed-up ensuring the same $MSE$ of a MC analysis.

We also performed a comparison with a standard way of choosing levels based on multiple PEEC mesh resolutions. Five levels were used corresponding to the mesh resolutions $\lambda_{\text{min}}/20, \lambda_{\text{min}}/25, \lambda_{\text{min}}/30, \lambda_{\text{min}}/35, \lambda_{\text{min}}/40$. The fifth level is equal to the HFM of our proposed technique. Targeting the same accuracy $MSE = 10^{-6}$, the speed-up obtained by the five levels MLMC with respect to the MC analysis is shown in Figure 5. These results show that this standard approach to define levels in MLMC provides a significantly reduced speed-up with respect to our proposed technique.

5 | CONCLUSION

A two-level multifidelity method has been proposed for fast tolerance analysis of EM systems that span a broadband frequency range. The combination of the PEEC method with metamodelling results to be very effective in accurately estimating stochastic quantities of interest, while achieving high speed-ups with respect to a MC analysis. Pertinent numerical results have validated the proposed technique.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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