Domain Wall and Periodic Solutions of Coupled $\phi^4$ Models in an External Field

Avinash Khare
Institute of Physics, Bhubaneswar, Orissa 751005, India

Avadh Saxena
Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Abstract:
Coupled double well ($\phi^4$) one-dimensional potentials abound in both condensed matter physics and field theory. Here we provide an exhaustive set of exact periodic solutions of a coupled $\phi^4$ model in an external field in terms of elliptic functions (domain wall arrays) and obtain single domain wall solutions in specific limits. We also calculate the energy and interaction between solitons for various solutions. Both topological and nontopological (e.g. some pulse-like solutions in the presence of a conjugate field) domain walls are obtained. We relate some of these solutions to the recently observed magnetic domain walls in certain multiferroic materials and also in the field theory context wherever possible. Discrete analogs of these coupled models, relevant for structural transitions on a lattice, are also considered.
1 Introduction

There are many physical situations, both in condensed matter and field theory, where two double well potentials model the phenomena of interest with a specific coupling between the two fields. One such phenomenon of current intense interest is the coexistence of magnetism and ferroelectricity (i.e. magnetoelectricity) in a given material. This is a highly desired functionality in technological applications involving cross-field response, switching and actuation. In general, this phenomenon is referred to as multiferroic behavior[1]. Recently, two different classes of (single phase) multiferroics, namely the orthorhombically distorted perovskites[2] and rare earth hexagonal structures[3], have emerged. The latter show magnetic domain walls in the basal planes which can be modeled by a coupled $\phi^4$ model[4] in the presence of a magnetic field. Coupled $\phi^4$ models[5,6,7] also arise in the context of many ferroelectric and other second order phase transitions. The coupled $\phi^4$ model for multiferroics[4] has a biquadratic coupling whereas the coupled $\phi^4$ model for a surface phase transition with hydration forces[7], relevant in biophysics context, has a bilinear coupling. Other types of couplings are also known for structural phase transitions with strain[8].

Similarly, there are analogous coupled models in field theory[9,10]. Several related models have been discussed in the literature and their soliton solutions have been found[11,12,13,14,15,16,17,18] including periodic ones[19,20,21]. Here our motivation is to obtain various possible domain wall solutions of these models with either a bilinear or a biquadratic coupling and then connect to experimental observations wherever possible.

The paper is organized as follows. In Sec. II we provide the solutions for the coupled $\phi^4$ model with an explicit biquadratic coupling in the presence of an external field (with an additional linear-quadratic coupling) and calculate their energy as well as interaction between the solitons. We also obtain solutions in the limit of no field. In Sec. III we consider a coupled $\phi^4$ model with a bilinear coupling. In Sec. IV we obtain several solutions of a coupled discrete $\phi^4$ model with biquadratic coupling (but without an external field). In Sec. V we obtain a solution of the coupled discrete $\phi^4$ model with bilinear coupling. To the best of our knowledge, no solution of the coupled $\phi^4$ model in an external field is known. Similarly, no solutions of either the coupled continuum $\phi^4$ model with a bilinear coupling or the discrete case are known. Even
for the continuum coupled case with a biquadratic coupling and zero external field, out of the six possible solutions only three were known previously [20] but the other three are new. Finally, we conclude in Sec. VI with remarks on related models.

2 Coupled $\phi^4$ solutions in an external field

We consider several exact solutions of the coupled $\phi^4$ system in a magnetic field ($H_z$) as given in Ref. [4] for hexagonal multiferroics. In particular, there are nine periodic solutions (valid for arbitrary $m$, the modulus of elliptic functions), which at $m = 1$ reduce to just four solutions. In particular, there is one “bright-bright”, one “dark-dark” and one each of bright-dark and dark-bright solutions. Notice that the equations of motion are asymmetric in the two scalar fields ($\phi$ and $\psi$) due to different coupling of the scalar fields to the magnetic field. Thus, the dark-bright and the bright-dark solutions are distinct.

The potential, with a biquadratic coupling between the two fields and in an external magnetic field ($H_z$), is given by

$$V = \alpha_1 \phi^2 + \beta_1 \phi^4 + \alpha_2 \psi^2 + \beta_2 \psi^4 + \gamma \phi^2 \psi^2 - H_z [\rho_1 \phi + \rho_2 \phi^3 + \rho_3 \phi \psi^2],$$  

(1)

where $\alpha_i$, $\beta_i$, $\gamma$ and $\rho_i$ are material (or system) dependent parameters. Hence the (static) equations of motion are

$$\frac{d^2 \phi}{dx^2} = 2\alpha_1 \phi + 4\beta_1 \phi^3 + 2\gamma \phi^2 \psi^2 - H_z [\rho_1 + 3\rho_2 \phi^2 + \rho_3 \phi \psi^2],$$  

(2)

$$\frac{d^2 \psi}{dx^2} = 2\alpha_2 \psi + 4\beta_2 \psi^3 + 2\gamma \phi^2 \psi^2 - 2H_z \rho_3 \phi \psi.$$  

(3)

These coupled set of equations admit several periodic solutions which we now discuss one by one systematically.

For static solutions the energy is given by

$$E = \int \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{1}{2} \left( \frac{d\psi}{dx} \right)^2 + V(\phi, \psi) \right] dx,$$  

(4)

where the limits of integration are from $-\infty$ to $\infty$ in the case of hyperbolic solutions (i.e., single solitons) on the full line. On the other hand, in the case of periodic solutions (i.e. soliton lattices), the limits are
from $-K(m)$ to $+K(m)$. Here $K(m)$ [and $E(m)$ below] denote the complete elliptic integral of the first (and second) kind \[22\]. Using equations of motion, one can show that for all of our solutions

$$V(\phi, \psi) = \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + \frac{1}{2} \left( \frac{d\psi}{dx} \right)^2 \right] + C,$$

(5)

where the constant $C$ in general varies from solution to solution. Hence the energy $\hat{E} = E - \int C \, dx$ is given by

$$\hat{E} \equiv E - \int C \, dx = \int \left[ \left( \frac{d\phi}{dx} \right)^2 + \left( \frac{d\psi}{dx} \right)^2 \right] \, dx.$$

(6)

Below we will give explicit expressions for energy in the case of all nine periodic solutions (and hence the corresponding four hyperbolic solutions). In each case we also provide an expression for the constant $C$.

### 2.1 Solution I

We look for the most general periodic solutions in terms of the Jacobi elliptic functions $\text{sn}(x, m)$, $\text{cn}(x, m)$ and $\text{dn}(x, m) \[22\]. It is not difficult to show that

$$\phi = F + A\text{sn}[D(x + x_0), m], \quad \psi = G + B\text{sn}[D(x + x_0), m],$$

(7)

is an exact solution provided the following eight coupled equations are satisfied

$$2\alpha_1 F + 4\beta_1 F^3 + 2\gamma FG^2 - H_z\rho_1 - 3H_z\rho_2 F^2 - H_z\rho_3 G^2 = 0,$$

(8)

$$2\alpha_1 A + 12\beta_1 F^2 A + 4\gamma BFG + 2\gamma AG^2 - 6H_z\rho_2 AF - 2H_z\rho_3 BG = -(1 + m)AD^2,$$

(9)

$$12\beta_1 F A^2 + 2\gamma FB^2 + 4\gamma ABG - 3H_z\rho_2 A^2 - H_z\rho_3 B^2 = 0,$$

(10)

$$2\beta_1 A^2 + \gamma B^2 = mD^2,$$

(11)

$$2\alpha_2 G + 4\beta_2 G^3 + 2\gamma GF^2 - 2H_z\rho_3 GF = 0,$$

(12)

$$2\alpha_2 B + 12\beta_2 G^2 B + 4\gamma AFG + 2\gamma BF^2 - 2H_z\rho_3 (BF + AG) = -(1 + m)BD^2,$$

(13)

$$12\beta_2 GB^2 + 2\gamma GA^2 + 4\gamma ABF - 2H_z\rho_3 AB = 0,$$

(14)

$$2\beta_2 B^2 + \gamma A^2 = mD^2.$$ 

(15)
Here \( A \) and \( B \) denote the amplitudes of the “kink lattice”, \( F \) and \( G \) are constants, \( D \) is an inverse characteristic length and \( x_0 \) is the (arbitrary) location of the kink. Five of these equations determine the five unknowns \( A, B, D, F, G \) while the other three equations, give three constraints between the nine parameters \( \alpha_{1,2}, \beta_{1,2}, \gamma, H_z, \rho_1, \rho_2, \rho_3 \). In particular, \( A \) and \( B \) are given by

\[
A^2 = \frac{mD^2[2\beta_2 - \gamma]}{[4\beta_1\beta_2 - \gamma^2]}, \quad B^2 = \frac{mD^2[2\beta_1 - \gamma]}{[4\beta_1\beta_2 - \gamma^2]}. \tag{16}
\]

Thus this solution exists provided \( 2\beta_1 > \gamma \) and \( 2\beta_2 > \gamma \).

It may be noted here that in case both \( F, G = 0 \) then no solution exists so long as \( H_z \neq 0 \). In fact this is true for all the nine solutions that we discuss below. There are two special cases (i.e., when either \( F = 0 \) or \( G = 0 \)) when the analysis becomes somewhat simpler.

**G=0, F\( \neq \)0:**

In this case \( A, B \) are again given by Eq. (16) while

\[
(1 + m)D^2 = \frac{\rho_2\rho_3H_z^2}{2\gamma} - \frac{2\rho_1\gamma}{\rho_3}, \quad F = \frac{\rho_3H_z}{2\gamma}. \tag{17}
\]

The three (and not four, since one of the equations is identically satisfied) constraints are

\[
\beta_1 = \frac{\rho_2\gamma}{2\rho_3}, \quad \alpha_1 = \frac{\rho_2\rho_3H_z^2}{2\gamma} + \frac{\rho_1\gamma}{\rho_3}, \quad \alpha_2 = \frac{\rho_3H_z^2}{4\gamma} - \frac{\rho_2\rho_3H_z^2}{4\gamma} + \frac{\rho_1\gamma}{\rho_3}. \tag{18}
\]

**m=1:**

In this limiting case we have a bright-bright soliton solution given by

\[
\phi = F + A \tanh[D(x + x_0)], \quad \psi = B \tanh[D(x + x_0)], \quad (19)
\]

with \( A, B \) and \( D \) determined by

\[
A^2 = \frac{D^2[2\beta_2 - \gamma]}{[4\beta_1\beta_2 - \gamma^2]}, \quad B^2 = \frac{D^2[2\beta_1 - \gamma]}{[4\beta_1\beta_2 - \gamma^2]}, \quad 2D^2 = \frac{\rho_2\rho_3H_z^2}{2\gamma} - \frac{2\rho_1\gamma}{\rho_3}, \tag{20}
\]

while the other relations remain unchanged and are again given by Eqs. (17) and (18).

**F=0, G \( \neq \)0:**

In this case \( A, B \) are again given by Eq. (16) with

\[
(1 + m)D^2 = 4\alpha_2 + \frac{2\rho_3H_zAG}{B}, \quad G^2 = -\frac{\alpha_2}{2\beta_2}, \tag{21}
\]

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and the corresponding four constraints are

\[ \beta_2 = \frac{\rho_3 \alpha_2}{2\rho_1}, \quad (1 + m)D^2 = -2\alpha_1 - 2\gamma G^2 + \frac{2\rho_3 H_zBG}{A}, \]

\[ 4\gamma ABG = 3H_z\rho_2 A^2 + H_z\rho_3 B^2, \quad \rho_3 H_z AB = \gamma G A^2 + 6\beta_2 GB^2. \] (22)

The solution at \( m = 1 \) can be easily written down as above.

**Special case of \( H_z = 0 \):**

In this case the field equations are completely symmetric in the two fields \( \phi, \psi \) and solution exists even if both \( F, G = 0 \). In particular, with \( H_z = 0 \), the solution is as given by Eq. (7) but with \( F = 0 = G \) where \( A, B \) are again as given by Eq. (16) and further both \( \alpha_1, \alpha_2 \) turn out to be negative, i.e.

\[ \alpha_1 = \alpha_2 = -\frac{(1 + m)D^2}{2}. \] (23)

**Special case of \( \gamma^2 = 4\beta_1\beta_2 \)**

One can show that the solution (7) exists even in case \( \gamma^2 = 4\beta_1\beta_2 \). It turns out such a solution exists only if

\[ 2\beta_1 = 2\beta_2 = \gamma, \] (24)

and that in this case one cannot determine \( A, B \). However, they must satisfy the constraint

\[ A^2 + B^2 = \frac{mD^2}{\gamma}. \] (25)

Other relations can be easily worked out depending on if \( F \) or \( G \) (or neither) is zero. For example, in case \( G = 0, F \neq 0 \), one has

\[ \rho_2 = \rho_3, \quad \alpha_2 = \frac{\rho_1 \gamma}{\rho_3}, \quad \alpha_1 - \alpha_2 = \frac{\rho_3^2 H_z^2}{2\gamma}. \] (26)

**Energy:** Corresponding to the periodic solution [Eq. (7) with \( G = 0 \)] the energy \( \hat{E} \) and the constant \( C \) are given by

\[ \hat{E} = \frac{2(A^2 + B^2)D}{3m}[(1 + m)E(m) - (1 - m)K(m)], \]

\[ C = -\frac{1}{2}(A^2 + B^2)D^2 - F^2[\alpha_1 + 3\beta_1 F^2 - 2H_z\rho_2 F]. \] (27)

It is worth pointing out that even in the case of the solution (7) with either \( F = 0, G \neq 0 \) or both \( F, G \) nonzero, the energy \( \hat{E} \) is the same. Only the value of \( C \) is different. For example, in the case of \( F = 0, \)
\[ G \neq 0, \text{ } C \text{ is given by} \]
\[ C = -\frac{1}{2}(A^2 + B^2)D^2 - \frac{\alpha_2^2}{2\beta_2}, \quad (28) \]

while in the case of both \( F, G \) being nonzero, \( C \) is
\[ C = -\frac{1}{2}(A^2 + B^2)D^2 - F^2[\alpha_1 + 3\beta_1 F^2 - 2H_z \rho_2 F] - G^2[\alpha_2 + 3\beta_2 G^2 + 3\gamma F^2 - 2H_z \rho_3 F]. \quad (29) \]

On using the expansion formulas for \( E(m) \) and \( K(m) \) around \( m = 1 \) as given in [22]
\[ K(m) = \ln \left( \frac{4}{\sqrt{1-m}} \right) + \frac{(1-m)}{4} \left( \ln \left( \frac{4}{\sqrt{1-m}} \right) - 1 \right) + \ldots \quad (30) \]
\[ E(m) = 1 + \frac{(1-m)}{2} \left( \ln \left( \frac{4}{\sqrt{1-m}} \right) - \frac{1}{2} \right) + \ldots, \quad (31) \]

for \( m \) near one, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-bright) soliton solution [Eq. (19)] plus the interaction energy. We find
\[ \hat{E} = E_{kink} + E_{int} = (A^2 + B^2)D \left[ \frac{4}{3} + \frac{(1-m)}{3} \right]. \quad (32) \]

Note that this solution exists only when \( 2\beta_1 \geq \gamma, 2\beta_2 \geq \gamma \) and \( 4\beta_1 \beta_2 \geq \gamma^2 \). The interaction energy vanishes at exactly \( m = 1 \), as it should.

### 2.2 Solution II

A different type of solution ("pulse lattice") is given by
\[ \phi = F + A \cn[D(x + x_0), m], \quad \psi = G + B \cn[D(x + x_0), m], \quad (33) \]

provided the following eight coupled equations are satisfied
\[ 2\alpha_1 F + 4\beta_1 F^3 + 2\gamma FG^2 - H_z \rho_1 - 3H_z \rho_2 F^2 - H_z \rho_3 G^2 = 0, \quad (34) \]
\[ 2\alpha_1 A + 12\beta_1 F^2 A + 4\gamma BFG + 2\gamma AG^2 - 6H_z \rho_2 AF - 2H_z \rho_3 BG = (2m - 1)AD^2, \quad (35) \]
\[ 12\beta_1 F^2 A + 2\gamma FB^2 + 4\gamma ABG - 3H_z \rho_2 A^2 - H_z \rho_3 B^2 = 0, \quad (36) \]
\[ 2\beta_1 A^2 + \gamma B^2 = -mD^2, \quad (37) \]
\[ 2\alpha_2 G + 4\beta_2 G^3 + 2\gamma GF^2 - 2H_z \rho_3 GF = 0, \quad (38) \]
\[2\alpha_2 B + 12\beta_2 G^2 B + 4\gamma AFG + 2\gamma BF^2 - 2H_z\rho_3(BF + AG) = (2m - 1)BD^2, \quad (39)\]
\[12\beta_2 GB^2 + 2\gamma GA^2 + 4\gamma ABF - 2H_z\rho_3 AB = 0, \quad (40)\]
\[2\beta_2 B^2 + \gamma A^2 = -mD^2. \quad (41)\]

Notice that two of these equations are meaningful only if \(\gamma < 0\) since \(\beta_1, \beta_2 > 0\) from stability considerations. Thus, we write \(\gamma = -|\gamma|\). Five of these equations determine the five unknowns \(A, B, D, F, G\) while other three equations, give three constraints between the nine parameters \(\alpha_{1,2}, \beta_{1,2}, \gamma, H_z, \rho_1, \rho_2, \rho_3\). In particular, \(A\) and \(B\) are given by
\[A^2 = \frac{mD^2[2\beta_2 + |\gamma|]}{|\gamma^2 - 4\beta_1\beta_2|}, \quad B^2 = \frac{mD^2[2\beta_1 + |\gamma|]}{|\gamma^2 - 4\beta_1\beta_2|}. \quad (42)\]
Thus this solution exists provided \(\gamma^2 > 4\beta_1\beta_2\).

There are two special cases when the analysis becomes somewhat simpler and we consider both the cases one by one.

**G=0, F\neq 0:**

In this case \(A, B\) are again given by Eq. (42) while
\[(2m - 1)D^2 = \frac{\rho_2\rho_3 H_z^2}{2|\gamma|} - \frac{2\rho_1|\gamma|}{\rho_3}, \quad F = -\frac{\rho_3 H_z}{2|\gamma|}, \quad (43)\]
and the corresponding three constraints are
\[\beta_1 = -\frac{\rho_2|\gamma|}{2\rho_3}, \quad \alpha_1 = -\frac{\rho_2\rho_3 H_z^2}{2|\gamma|}, \quad \frac{\rho_1|\gamma|}{\rho_3}, \quad \alpha_2 = \frac{\rho_2 H_z^2}{4|\gamma|} + \frac{\rho_2\rho_3 H_z^2}{4|\gamma|} - \frac{\rho_1|\gamma|}{\rho_3}. \quad (44)\]

**F=0, G\neq 0:**

In this case \(A, B\) are again given by Eq. (42) with
\[(2m - 1)D^2 = 2\alpha_1 - \frac{2\rho_3 H_z BG}{A} - 2|\gamma|G^2, \quad G^2 = -\frac{\alpha_2}{2\beta_2}, \quad (45)\]
and the corresponding four constraints are
\[\beta_1 = \frac{\rho_3\alpha_2}{2\rho_1}, \quad \alpha_1 = -\frac{2\rho_3 H_z B G}{A} - 4\alpha_2 - \frac{2\rho_3 H_z AG}{B}, \quad (2m - 1)D^2 = -4\alpha_2 - \frac{2\rho_3 H_z AG}{B}, \quad (46)\]
\[-4|\gamma|ABG = 3H_z\rho_2 A^2 + H_z\rho_3 B^2, \quad \rho_3 H_z AB = -|\gamma|GA^2 + 6\beta_2 GB^2. \quad (46)\]

**Special case of \(H_z=0\):**
In this case the field equations are completely symmetric in the two fields $\phi, \psi$ and a solution exists even when both $F, G = 0$. In particular, with $H_z = 0$, the solution is as given by Eq. [33] but with $F = 0 = G$ [20] where $A, B$ are again as given by Eq. [12] and furthermore, $\alpha_1, \alpha_2$ are positive (negative) so long as $m > (<) 1/2$, i.e.

$$\alpha_1 = \alpha_2 = \frac{(2m - 1) D^2}{2}.$$  \hspace{1cm} (47)

**Energy:** Corresponding to the “pulse lattice” solution [Eq. (33) with $G = 0$] the energy is given by

$$\hat{E} = \frac{2(A^2 + B^2)D}{3m}[(2m - 1)E(m) + (1 - m)K(m)],$$

$$C = -\frac{1}{2}(1 - m)(A^2 + B^2)D^2 - F^2[\alpha_1 + 3\beta_1 F^2 - 2H_z \rho_2 F].$$  \hspace{1cm} (48)

It is worth pointing out that even in the case of the solution [33] with either $F = 0, G \neq 0$ or both $F, G$ nonzero, the energy $\hat{E}$ is the same. Only the value of $C$ is different. For example, in the case of $F = 0$, $G \neq 0$, $C$ is given by

$$C = -\frac{1}{2}(1 - m)(A^2 + B^2)D^2 - \frac{\alpha_2^2}{4\beta_2},$$  \hspace{1cm} (49)

while in the case of both $F, G$ being nonzero, $C$ is

$$C = -\frac{1}{2}(1 - m)(A^2 + B^2)D^2 - F^2[\alpha_1 + 3\beta_1 F^2 - 2H_z \rho_2 F] - G^2[\alpha_2 + 3\beta_2 G^2 + 3\gamma F^2 - 2H_z \rho_3 F].$$  \hspace{1cm} (50)

For $m$ near one, the energy of this periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution

$$\phi = F + A \text{sech}[D(x + x_0)], \quad \psi = B \text{sech}[D(x + x_0)],$$  \hspace{1cm} (51)

plus the interaction energy. We find

$$\hat{E} = E_{\text{pulse}} + E_{\text{int}} = (A^2 + B^2)D \left[\frac{2}{3} - \frac{5(1 - m)}{6} + (1 - m) \ln \left(\frac{4}{\sqrt{1 - m}}\right)\right].$$  \hspace{1cm} (52)

Note that this solution exists only when $\gamma < 0, 4\beta_1 \beta_2 < \gamma^2$. Again, the interaction energy vanishes at $m = 1$. 

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2.3 Solution III

In this case, there is another “pulse lattice” solution which is given by

\[ \phi = F + Adn[D(x + x_0), m], \quad \psi = G + Bdn[D(x + x_0), m], \]  

provided the following eight coupled equations are satisfied

\[ 2\alpha_1 F + 4\beta_1 F^3 + 2\gamma FG^2 - H_z \rho_1 - 3H_z \rho_2 F^2 - H_z \rho_3 G^2 = 0, \]  
\[ 2\alpha_1 A + 12\beta_1 F^2 A + 4\gamma BFG + 2\gamma AG^2 - 6H_z \rho_2 AF - 2H_z \rho_3 BG = (2 - m) AD^2, \]  
\[ 12\beta_1 FA^2 + 2\gamma FB^2 + 4\gamma ABG - 3H_z \rho_2 A^2 - H_z \rho_3 B^2 = 0, \]  
\[ 2\beta_1 A^2 + \gamma B^2 = -D^2, \]  
\[ 2\alpha_2 G + 4\beta_2 G^3 + 2\gamma GF^2 - 2H_z \rho_3 GF = 0, \]  
\[ 2\alpha_2 B + 12\beta_2 G^2 B + 4\gamma AFG + 2\gamma BF^2 - 2H_z \rho_3 (BF + AG) = (2 - m) BD^2, \]  
\[ 12\beta_2 GB^2 + 2\gamma GA^2 + 4\gamma ABF - 2H_z \rho_3 AB = 0, \]  
\[ 2\beta_2 B^2 + \gamma A^2 = -D^2. \]  

Notice that (as in the cn – cn case) two of these equations are meaningful only if \( \gamma < 0 \) since \( \beta_1, \beta_2 > 0 \) from stability considerations. We therefore write \( \gamma = -|\gamma| \). Five of these equations determine the five unknowns \( A, B, D, F, G \) while the other three equations give three constraints between the nine parameters \( \alpha_{1,2}, \beta_{1,2}, |\gamma|, H_z, \rho_1, \rho_2, \rho_3 \). In particular, \( A \) and \( B \) are given by

\[ A^2 = \frac{D^2[2\beta_2 + |\gamma|]}{|\gamma|^2 - 4\beta_1 \beta_2}, \quad B^2 = \frac{D^2[2\beta_1 + |\gamma|]}{||\gamma|^2 - 4\beta_1 \beta_2|}. \]  

There are two special cases when the analysis becomes somewhat simpler and we consider both the cases one by one.

**G=0, F\neq0:**

In this case \( A, B \) are again given by Eq. (62) while

\[ (2 - m) D^2 = \frac{\rho_2 \rho_3 H_z^2}{2|\gamma|} - \frac{2\rho_1 |\gamma|}{\rho_3}, \quad F = -\frac{\rho_3 H_z}{2|\gamma|}, \]  

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and the three constraints are
\[
\begin{align*}
\beta_1 &= -\frac{\rho_2|\gamma|}{2\rho_3}, \\
\alpha_1 &= -\frac{\rho_2\rho_3^2 H_z^2 - \rho_1|\gamma|}{4|\gamma|}, \\
\alpha_2 &= -\frac{\rho_2\rho_3^2 H_z^2 + \rho_2\rho_3^2 H_z^2 - \rho_1|\gamma|}{4|\gamma|}.
\end{align*}
\] (64)

**F=0, G\neq0:**

In this case \( A, B \) are again given by Eq. (62) with
\[
(2 - m)D^2 = 2\alpha_1 - \frac{2\rho_3 H_z BG}{A} - 2|\gamma|G^2, \quad G^2 = -\frac{\alpha_2}{2\beta_2},
\] (65)

and the corresponding four constraints are
\[
\beta_2 = \frac{\rho_3\alpha_2}{2\rho_1}, \quad (2 - m)D^2 = -4\alpha_2 - \frac{2\rho_3 H_z AG}{B},
\]
\[
-4|\gamma|ABG = 3H_z\rho_2 A^2 + H_z\rho_3 B^2, \quad \rho_3 H_z AB = -|\gamma|GA^2 + 6\beta_2 GB^2.
\] (66)

**Special case of \( H_z=0 \):**

With \( H_z = 0 \), the solution is as given by Eq. (63) but with \( F = 0 = G \) where \( A, B \) are again as given by Eq. (62) and furthermore, both \( \alpha_1, \alpha_2 \) are positive definite.
\[
\alpha_1 = \alpha_2 = \frac{(2 - m)D^2}{2}.
\] (67)

**Energy:** Corresponding to the “pulse lattice” solution [Eq. (53) with \( G = 0 \)] the energy is given by
\[
\tilde{E} = \frac{2(A^2 + B^2)D}{3}[(2 - m)E(m) - (1 - m)K(m)],
\]
\[
C = \frac{1}{2}(1 - m)(A^2 + B^2)D^2 - F^2[\alpha_1 + 3\beta_1 F^2 - 2H_z\rho_2 F].
\] (68)

It is worth pointing out that even in the case of the solution (63) with either \( F = 0, G \neq 0 \) or both \( F, G \) nonzero, the energy \( \tilde{E} \) is the same. Only the value of \( C \) is different. For example, in the case of \( F = 0, G \neq 0 \), \( C \) is given by
\[
C = \frac{1}{2}(1 - m)(A^2 + B^2)D^2 - \frac{\alpha_3}{4\beta_2},
\] (69)

while in the case of both \( F, G \) being nonzero, \( C \) is
\[
C = \frac{1}{2}(1 - m)(A^2 + B^2)D^2 - F^2[\alpha_1 + 3\beta_1 F^2 - 2H_z\rho_2 F] - G^2[\alpha_2 + 3\beta_2 G^2 + 3\gamma F^2 - 2H_z\rho_3 F].
\] (70)
For \( m \) near one, the energy of this periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution as given by Eq. (51) plus the interaction energy. We find

\[
\hat{E} = E_{\text{pulse}} + E_{\text{int}} = (A^2 + B^2)D \left[ \frac{2}{3} - \frac{(1 - m)}{2} - (1 - m) \ln \left( \frac{4}{\sqrt{1 - m}} \right) \right].
\]  

(71)

Note that this solution also exists only when \( \gamma < 0, 4\beta_1\beta_2 < \gamma^2 \). Again, the interaction energy vanishes at \( m = 1 \).

### 2.4 Solution IV

In addition to the \( cn - cn \) and \( dn - dn \) solutions discussed above, there are two novel (mixed) soliton solutions of \( dn - cn \) and \( cn - dn \) type. Let us discuss them one by one. We shall see that for these two solutions (in fact it is true for all the six solutions that we discuss below) \( G \) is necessarily zero while \( F \) is necessarily nonzero (unless \( H_z = 0 \)), otherwise the solution does not exist. In particular, it is easily shown that

\[
\phi = F + A\text{cn}[D(x + x_0), m], \quad \psi = G + B\text{cn}[D(x + x_0), m],
\]

(72)

is a solution provided \( G = 0 \) and further, the following seven coupled equations are satisfied

\[
2\alpha_1 F + 4\beta_1 F^3 + (2/m)\gamma(m - 1)FB^2 - H_z\rho_1 - 3H_z\rho_2 F^2 + (1/m)(1 - m)H_z\rho_3 B^2 = 0,
\]

(73)

\[
2\alpha_1 A + 12\beta_1 F^2 A + (2/m)(m - 1)\gamma AB^2 - 6H_z\rho_2 AF = (2 - m)AD^2,
\]

(74)

\[
12\beta_1 FA^2 + (2/m)\gamma FB^2 - 3H_z\rho_2 A^2 - (1/m)H_z\rho_3 B^2 = 0,
\]

(75)

\[
\gamma B^2 + 2m\beta_1 A^2 = -mD^2,
\]

(76)

\[
2\alpha_2 B + 2(1 - m)\gamma A^2 B + 2\gamma BF^2 - 2H_z\rho_3 BF = (2m - 1)BD^2,
\]

(77)

\[
2\beta_2 B^2 + m\gamma A^2 = -mD^2,
\]

(78)

\[
4\gamma FAB - 2H_z\rho_3 AB = 0.
\]

(79)

Again a solution exists only if \( \gamma < 0 \) and thus we write \( \gamma = -|\gamma| \). The solution (72) exists provided

\[
A^2 = \frac{D^2[|\gamma| + 2\beta_2]}{[\gamma^2 - 4\beta_1\beta_2]}, \quad B^2 = \frac{mD^2[2\beta_1 + |\gamma|]}{[\gamma^2 - 4\beta_1\beta_2]},
\]

(80)
where
\[
(2 - m)D^2 = \frac{2}{m}(1 - m)|\gamma|B^2 - \frac{2\rho_1|\gamma|}{\rho_3} + \frac{\rho_2\rho_3 H_z^2}{2|\gamma|}, \quad F = -\frac{\rho_3 H_z}{2|\gamma|},
\] (81)
and the three constraints are
\[
\beta_1 = -\frac{\rho_2|\gamma|}{2\rho_3}, \quad \alpha_2 = -\frac{\rho_2\rho_3 H_z^2}{2|\gamma|} - \frac{\rho_1|\gamma|}{\rho_3},
\]
\[
(2m - 1)D^2 = 2\alpha_2 - 2(1 - m)|\gamma|A^2 + \frac{H_z^2\rho_3^2}{2|\gamma|}.
\] (82)

**Special case of** $H_z = 0$:

With $H_z = 0$, the solution is as given by Eq. (72) but with $F = 0 = G$ where $A$ and $B$ are again as given by Eq. (80) and furthermore, $\alpha_2$ is positive, i.e.
\[
\alpha_1 = \frac{mD^2}{2} - 2(1 - m)\beta_1 A^2, \quad m\alpha_2 = m\frac{D^2}{2} + 2(1 - m)\beta_2 B^2 > 0.
\] (83)

**Energy**: Corresponding to the “pulse lattice” solution [Eq. (72) with $G = 0$] the energy $\hat{E}$ and $C$ are given by
\[
\hat{E} = \frac{2D}{3m} \left( [(2 - m)mA^2 + (2m - 1)B^2]E(m) + (1 - m)(B^2 - 2mA^2)K(m) \right),
\]
\[
C = \frac{1}{2}A^2D^2(1 - m) - F^2[\alpha_1 + 3\beta_1 F^2 - 2H_z\rho_2 F] + \frac{(1 - m)}{m^2}[m\gamma F^2 - \alpha_2 m + \beta_2(1 - m)B^2]B^2.
\] (84)

For $m$ near one, the energy of this periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution (51) plus the interaction energy. We find
\[
\hat{E} = E_{\text{pulse}} + E_{\text{int}} = D \left[ \frac{2}{3}(A^2 + B^2) + \frac{(1 - m)}{6}(3A^2 - 5B^2) + (1 - m)(B^2 - A^2) \ln \left( \frac{4}{\sqrt{1 - m}} \right) \right].
\] (85)

Note that this solution also exists only when $\gamma < 0, 4\beta_1\beta_2 < \gamma^2$. The interaction energy vanishes for $m = 1$, as it should.

**2.5 Solution V**

It is easily shown that there is also a cn – dn solution which is distinct from the above dn – cn solution.

This solution is given by
\[
\phi = F + Acn[D(x + x_0), m], \quad \psi = G + Bdn[D(x + x_0), m],
\] (86)
provided \( G = 0 \) and the following seven coupled equations are satisfied

\[
2\alpha_1 F + 4\beta_1 F^3 + 2\gamma(1 - m)FB^2 - Hz\rho_1 - 3Hz\rho_2 F^2 - (1 - m)Hz\rho_3 B^2 = 0, \tag{87}
\]

\[
2\alpha_1 A + 12\beta_1 F^2 A + 2(1 - m)\gamma AB^2 - 6Hz\rho_2 AF = -(1 - 2m)AD^2, \tag{88}
\]

\[
12\beta_1 FA^2 + 2m\gamma FB^2 - 3Hz\rho_2 A^2 - mHz\rho_3 B^2 = 0, \tag{89}
\]

\[
m\gamma B^2 + 2\beta_1 A^2 = -mD^2, \tag{90}
\]

\[
2\alpha_2 B - (2/m)(1 - m)\gamma A^2 B + 2\gamma BF^2 - 2Hz\rho_3 BF = (2 - m)BD^2, \tag{91}
\]

\[
2m\beta_2 B^2 + \gamma A^2 = -mD^2, \tag{92}
\]

\[
4\gamma FAB - 2Hz\rho_3 AB = 0. \tag{93}
\]

Again a solution exists only if \( \gamma < 0 \) and hence we put \( \gamma = -|\gamma| \). The solution turns out to be

\[
A^2 = \frac{mD^2[|\gamma| + 2\beta_2]}{[|\gamma|^2 - 4\beta_1\beta_2]}, \quad B^2 = \frac{D^2[2\beta_1 + |\gamma|]}{[|\gamma|^2 - 4\beta_1\beta_2]}, \tag{94}
\]

where

\[
(2m - 1)D^2 = -2(1 - m)|\gamma|B^2 - \frac{2\rho_1 |\gamma|}{\rho_3} + \frac{\rho_2 \rho_3 H^2}{2|\gamma|}, \quad F = -\frac{\rho_3 Hz}{2|\gamma|}, \tag{95}
\]

and the three constraints are

\[
\beta_1 = -\frac{\rho_2 |\gamma|}{2\rho_3}, \quad \alpha_1 = -\frac{\rho_2 \rho_3 H^2}{2|\gamma|} - \frac{\rho_1 |\gamma|}{\rho_3},
\]

\[
(2 - m)D^2 = 2\alpha_2 + (2/m)(1 - m)|\gamma|A^2 + \frac{H^2 \rho_3}{2|\gamma|}. \tag{96}
\]

**Special case of \( Hz = 0 \):**

With \( Hz = 0 \), the solution is as given by Eq. (86) but with \( F = 0 = G \) where \( A \) and \( B \) are again as given by Eq. (94) and furthermore, \( \alpha_1 \) is positive, i.e.

\[
m\alpha_1 = m \frac{D^2}{2} + 2(1 - m)\beta_1 A^2 > 0, \quad \alpha_2 = \frac{mD^2}{2} - 2(1 - m)\beta_2 B^2. \tag{97}
\]

It is worth pointing out that with \( Hz = 0 \), the field equations are completely symmetric between the two fields \( \phi \) and \( \psi \) and hence the solutions IV and V are identical in the limit \( Hz = 0 \).
In the special case of \( m = 1 \) and \( G = 0, F \neq 0 \), all the four solutions II to V reduce to a dark-dark type soliton solution \((51)\), i.e.

\[
\phi = F + \text{Asech}[D(x + x_0)], \quad \psi = B\text{sech}[D(x + x_0)],
\]

with \( A, B \), and \( D \) given by

\[
A^2 = \frac{D^2[|\gamma| + 2\beta_2]}{|\gamma|^2 - 4\beta_1\beta_2}, \quad B^2 = \frac{D^2[2\beta_1 + |\gamma|]}{|\gamma|^2 - 4\beta_1\beta_2}, \quad D^2 = -\frac{2\rho_1|\gamma|}{\rho_3} + \frac{\rho_2\rho_3H_z^2}{2|\gamma|},
\]

while the other relations remain unchanged and are again given by Eqs. \((98)\) and \((99)\).

Similarly, in the special case of \( m = 1 \) and \( F = 0, G \neq 0 \), both the solutions II and III reduce to a dark-dark type soliton solution

\[
\phi = \text{Asech}[D(x + x_0)], \quad \psi = G + B\text{sech}[D(x + x_0)],
\]

with \( A \) and \( B \) again given by Eq. \((99)\) while

\[
D^2 = 2\alpha_1 - \frac{2\rho_3H_zBG}{A} - 2|\gamma|G^2, \quad D^2 = -4\alpha_2 - \frac{2\rho_3H_zAG}{B}.
\]

Other relations remain unchanged and are again given by Eqs. \((103)\) and \((104)\).

**Energy:** Corresponding to the “pulse lattice” solution [Eq. \((86)\) with \( G = 0 \)] the energy is given by

\[
\hat{E} = \frac{2D}{3m}\left[(2 - m)kB^2 + (2m - 1)A^2E(m) + (1 - m)(A^2 - 2mB^2)K(m)\right],
\]

\[
C = \frac{1}{2}A^2D^2(1 - m) - F^2[\alpha_1 + 3\beta_1B^2 - 2H_z\rho_2F] + [\gamma F^2 + \alpha_2 + \beta_2(1 - m)B^2]B^2.
\]

For \( m \) near one, the energy of this periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution [Eq. \((51)\)] plus the interaction energy. We find

\[
\hat{E} = E_{\text{pulse}} + E_{\text{int}} = D\left[\frac{2}{3}(A^2 + B^2) + \frac{(1 - m)}{6}(3B^2 - 5A^2) + (1 - m)(A^2 - B^2)\ln\left(\frac{4}{\sqrt{1 - m}}\right)\right].
\]

Note that this solution also exists only when \( \gamma < 0, 4\beta_1\beta_2 < \gamma^2 \). Again, the interaction energy vanishes at \( m = 1 \). It is amusing to note that the energy of the solution V is easily obtained from that of solution IV by simply interchanging \( A \) and \( B \).
2.6 Solution VI

Apart from the solutions which at $m = 1$ reduce to the bright-bright and dark-dark solutions, there are four solutions which at $m = 1$ go over to either bright-dark or dark-bright solutions, which we now discuss one by one.

One such solution, kink-like in $\phi$ and pulse-like in $\psi$, is

$$
\phi = F + A \text{sn}[D(x + x_0), m], \quad \psi = G + B \text{cn}[D(x + x_0), m],
$$

provided $G = 0$ and the following seven coupled equations are satisfied

$$
2\alpha_1 F + 4\beta_1 F^3 + 2\gamma FB - H_z \rho_1 - 3H_z \rho_2 F^2 - H_z \rho_3 B^2 = 0,
$$

$$
2\alpha_1 A + 12\beta_1 F^2 A + 2\gamma AB^2 - 6H_z \rho_2 AF = -(1 + m)AD^2,
$$

$$
12\beta_1 FA^2 - 2\gamma FB^2 - 3H_z \rho_2 A^2 + H_z \rho_3 B^2 = 0,
$$

$$
2\beta_1 A^2 - \gamma B^2 = mD^2,
$$

$$
2\alpha_2 B + 2\gamma A^2 B + 2\gamma BF^2 - 2H_z \rho_3 BF = (2m - 1)BD^2,
$$

$$
2\beta_2 B^2 - \gamma A^2 = -mD^2,
$$

$$
(4\gamma F - 2H_z \rho_3)AB = 0.
$$

In this case it turns out that the solution exists only if either $2\beta_1 > \gamma > 2\beta_2$, $\gamma^2 > 4\beta_1 \beta_2$ or $2\beta_2 > \gamma > 2\beta_1$, $\gamma^2 < 4\beta_1 \beta_2$. We find that

$$
A^2 = \frac{mD^2[\gamma - 2\beta_2]}{[\gamma^2 - 4\beta_1 \beta_2]}, \quad B^2 = \frac{mD^2[2\beta_1 - \gamma]}{[\gamma^2 - 4\beta_1 \beta_2]},
$$

where

$$
-(1 + m)D^2 = 2\gamma B^2 + \frac{2\rho_1 \gamma}{\rho_3} - \frac{\rho_2 \rho_3 H_z^2}{2\gamma}, \quad F = \frac{\rho_3 H_z}{2\gamma},
$$

and the three constraints are

$$
\beta_1 = \frac{\rho_2 \gamma}{2\rho_3}, \quad \alpha_1 = \frac{\rho_2 \rho_3 H_z^2}{2\gamma} + \frac{\rho_1 \gamma}{\rho_3},
$$

$$
(2m - 1)D^2 = 2\alpha_2 + 2\gamma A^2 - \frac{H_z^2 \rho_3^2}{2\gamma}.
$$
Special case of $H_z = 0$:

With $H_z = 0$, the solution is again given by Eq. (104) but with $F = 0 = G$ where $A$ and $B$ are again as given by Eq. (112) and furthermore, $\alpha_1, \alpha_2$ turn out to be negative, i.e.

$$\alpha_1 = -\frac{(1 + m)D^2}{2} - \gamma B^2, \quad \alpha_2 = -\frac{D^2}{2} - 2\beta_2B^2.$$  \hfill (115)

Special case of $\gamma^2 = 4\beta_1\beta_2$

One can show that the solution (104) exists even in case $\gamma^2 = 4\beta_1\beta_2$. It turns out such a solution exists only if

$$2\beta_1 = 2\beta_2 = \gamma,$$ \hfill (116)

and that in this case one cannot determine $A, B$. However, they must satisfy the constraint

$$A^2 - B^2 = \frac{mD^2}{\gamma}.$$ \hfill (117)

Further, one has

$$\rho_2 = \rho_3, \quad \alpha_1 = \frac{\rho_3^2H_z^2}{2\gamma}, \quad \alpha_2 - \alpha_1 = \frac{mD^2}{2} - \frac{\rho_3^2H_z^2}{2\gamma}. \hfill (118)$$

Energy: Corresponding to the mixed “kink-pulse lattice” solution [Eq. (104)] the energy is given by

$$\hat{E} = \frac{2D}{3m}\left( [(2 - m)mb^2 + (1 + m)A^2]E(m) - (1 - m)(A^2 + 2B^2)K(m) \right),$$

$$C = -\frac{1}{2}A^2D^2 - F^2[\alpha_1 + 3\beta_1F^2 - 2H_z\rho_2F] + [-\gamma F^2 + \alpha_2 + \beta_2B^2]B^2.$$ \hfill (119)

For $m$ near one, the energy of this periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-dark) soliton solution

$$\phi = F + A\tanh[D(x + x_0)], \quad \psi = B\text{sech}[D(x + x_0)],$$ \hfill (120)

plus the interaction energy. We find

$$\hat{E} = E_{\text{soliton}} + E_{\text{int}} = D \left[ \frac{2}{3}(2A^2 + B^2) + \frac{(1 - m)}{6}(2A^2 + 3B^2) - (1 - m)B^2 \ln\left( \frac{4}{1 - m} \right) \right].$$ \hfill (121)

Note that this solution also exists only when either $2\beta_1 \geq \gamma \geq 2\beta_2$ and $\gamma^2 \geq 4\beta_1\beta_2$ or $2\beta_2 \geq \gamma \geq 2\beta_1$ and $\gamma^2 \leq 4\beta_1\beta_2$. The interaction energy vanishes at $m = 1$. 

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It is easy to show that another such (kink- and pulse-like) solution is

$$\phi = F + A \sin[D(x + x_0), m], \quad \psi = G + B \sin[D(x + x_0), m],$$

(122)

provided $G = 0$ and the following seven coupled equations are satisfied

\begin{align*}
2\alpha_1 F + 4\beta_1 F^3 + 2\gamma F B^2 - H_z \rho_1 - 3H_z \rho_2 F^2 - H_z \rho_3 B^2 &= 0, \quad (123) \\
2\alpha_1 A + 12\beta_1 F^2 A + 2\gamma AB^2 - 6H_z \rho_2 AF &= -(1 + m) AD^2, \quad (124) \\
12\beta_1 F A^2 - 2m\gamma F B^2 - 3H_z \rho_2 A^2 + mH_z \rho_3 B^2 &= 0, \quad (125) \\
2\beta_1 A^2 - m\gamma B^2 &= m D^2, \quad (126) \\
2\alpha_2 B + (2/m)\gamma A^2 B + 2\gamma BF^2 - 2H_z \rho_3 BF &= (2 - m) BD^2, \quad (127) \\
2m\beta_2 B^2 - \gamma A^2 &= -m D^2, \quad (128) \\
4\gamma F AB - 2H_z \rho_3 AB &= 0. \quad (129)
\end{align*}

In this case also it turns out that the solution exists only if either $2\beta_1 > \gamma > 2\beta_2$, $\gamma^2 > 4\beta_1 \beta_2$ or $2\beta_2 > \gamma > 2\beta_1$, $\gamma^2 < 4\beta_1 \beta_2$. We find that

$$A^2 = \frac{m D^2 [\gamma - 2\beta_2]}{[\gamma^2 - 4\beta_1 \beta_2]}, \quad B^2 = \frac{D^2 [2\beta_1 - \gamma]}{[\gamma^2 - 4\beta_1 \beta_2]},$$

(130)

where

$$-(1 + m) D^2 = 2\gamma B^2 + \frac{2\rho_1 \gamma}{\rho_3} - \frac{\rho_2 \rho_3 H_z^2}{2\gamma}, \quad F = \frac{\rho_3 H_z}{2\gamma},$$

(131)

and the three constraints are

$$\beta_1 = \frac{\rho_2 \gamma}{2\rho_3}, \quad \alpha_1 = \frac{\rho_2 \rho_3 H_z^2}{2\gamma} + \frac{\rho_1 \gamma}{\rho_3},$$

$$2 - m) D^2 = 2\alpha_2 + (2/m) \gamma A^2 - \frac{H_z^2 \rho_3^2}{2\gamma}.$$ (132)

Special case of $H_z = 0$: 

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With $H_z = 0$, the solution is again given by Eq. (122) but with $F = 0 = G$ where $A$ and $B$ are again as given by Eq. (130) and furthermore, $\alpha_1, \alpha_2$ turn out to be negative, i.e.

$$\alpha_1 = -\frac{(1 + m)D^2}{2} - \gamma B^2, \quad \alpha_2 = -\frac{mD^2}{2} - 2\beta_2 B^2.$$  \hfill (133)

**m=1:**

In the special case of $m = 1$ and $G = 0, F \neq 0$, both solutions VI and VII reduce to a bright-dark type of solution as given by Eq. (120), i.e.

$$\phi = F + A \tanh[D(x + x_0)], \quad \psi = B \sech[D(x + x_0)],$$  \hfill (134)

with $A, B$ and $D$ given by

$$A^2 = \frac{D^2[\gamma - 2\beta_2]}{[\gamma^2 - 4\beta_1\beta_2]}, \quad B^2 = \frac{D^2[2\beta_1 - \gamma]}{[\gamma^2 - 4\beta_1\beta_2]}, \quad D^2 = -\gamma B^2 - \frac{\rho_1\gamma}{\rho_3} + \frac{\rho_2\rho_3 H_z^2}{4\gamma},$$  \hfill (135)

while the other relations remain unchanged and are again given by Eqs. (131) and (132).

**Special case of $\gamma^2 = 4\beta_1\beta_2$**

One can show that the solution (122) exists even in case $\gamma^2 = 4\beta_1\beta_2$. It turns out such a solution exists only if

$$2\beta_1 = 2\beta_2 = \gamma,$$  \hfill (136)

and that in this case one cannot determine $A, B$. However, they must satisfy the constraint

$$A^2 - mB^2 = \frac{mD^2}{\gamma}.$$  \hfill (137)

Further, one has

$$\rho_2 = \rho_3, \quad \alpha_1 = \frac{\rho_2^2 H_z^2}{2\gamma} + \frac{\rho_1\gamma}{\rho_3}, \quad \alpha_2 - \alpha_1 = \frac{D^2}{2} - \frac{\rho_2 H_z^2}{2\gamma}.$$  \hfill (138)

**Energy:** Corresponding to the mixed “kink-pulse lattice” solution [Eq. (122)], the energy and the constant $C$ are given by

$$\hat{E} = \frac{2D}{3m}\left( (2m - 1)B^2 + (1 + m)A^2 \right)E(m) - (1 - m)(A^2 - B^2)K(m),$$  \hfill (139)

$$C = -\frac{1}{2}A^2D^2 - F^2[\alpha_1 + 3\beta_1 F^2 - 2H_z \rho_2 F] + [-\gamma F^2 + \alpha_2 + \beta_2 B^2]B^2.$$  \hfill (140)
For \( m \) near one, the energy of this periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-dark) soliton solution [Eq. (120)] plus the interaction energy. We find

\[
E = E_{\text{soliton}} + E_{\text{int}} = D \left[ \frac{2}{3} (2A^2 + B^2) + \frac{(1 - m)}{6} (2A^2 - 5B^2) + (1 - m)B^2 \ln \left( \frac{4}{\sqrt{1 - m}} \right) \right].
\] (141)

Note that this solution also exists only when either \( 2\beta_1 \geq \gamma \geq 2\beta_2 \) and \( \gamma^2 \geq 4\beta_1\beta_2 \) or \( 2\beta_2 \geq \gamma \geq 2\beta_1 \) and \( \gamma^2 \leq 4\beta_1\beta_2 \). The interaction energy vanishes for \( m = 1 \).

2.8 Solution VIII

Finally, we discuss two periodic solutions both of which at \( m = 1 \) reduce to a dark-bright type of solution. The first such (pulse-like in \( \phi \) and kink-like in \( \psi \)) solution is given by

\[
\phi = F + A\text{cn}[D(x + x_0), m], \quad \psi = G + B\text{sn}[D(x + x_0), m],
\] (142)

provided \( G = 0 \) and the following seven coupled equations are satisfied

\[
2\alpha_1 F + 4\beta_1 F^3 + 2\gamma FB^2 - H_z\rho_1 - 3H_z\rho_2 F^2 - H_z\rho_3 B^2 = 0, \quad (143)
\]
\[
2\alpha_1 A + 12\beta_1 F^2 A + 2\gamma AB^2 - 6H_z\rho_2 AF = (2m - 1)AD^2, \quad (144)
\]
\[
12\beta_1 FA^2 - 2\gamma FB^2 - 3H_z\rho_2 A^2 + H_z\rho_3 B^2 = 0, \quad (145)
\]
\[
2\beta_1 A^2 - \gamma B^2 = -mD^2, \quad (146)
\]
\[
2\beta_2 B^2 - \gamma A^2 = mD^2, \quad (147)
\]
\[
4\gamma FAB - 2H_z\rho_3 AB = 0. \quad (148)
\]

In this case it turns out that the solution exists only if either \( 2\beta_1 > \gamma > 2\beta_2 \), \( \gamma^2 < 4\beta_1\beta_2 \) or \( 2\beta_2 > \gamma > 2\beta_1 \), \( \gamma^2 > 4\beta_1\beta_2 \). We find that

\[
A^2 = \frac{mD^2[\gamma - 2\beta_2]}{[4\beta_1\beta_2 - \gamma^2]}, \quad B^2 = \frac{mD^2[2\beta_1 - \gamma]}{[4\beta_1\beta_2 - \gamma^2]}, \quad (150)
\]

where

\[
(2m - 1)D^2 = 2\gamma B^2 + \frac{2\rho_1\gamma}{\rho_3} - \frac{\rho_2\rho_3H_z^2}{2\gamma}, \quad F = \frac{\rho_3H_z}{2\gamma}, \quad (151)
\]
and the three constraints are

\[
\beta_1 = \frac{\rho_2 \gamma}{2 \rho_3}, \quad \alpha_1 = \frac{\rho_2 \rho_3 H_z^2}{2 \gamma} + \frac{\rho_1 \gamma}{\rho_3}, \\
-(1 + m)D^2 = 2\alpha_2 + 2\gamma A^2 - \frac{H_z^2 \rho_3^2}{2\gamma}. \tag{152}
\]

**Special case of** \(H_z = 0\):

When \(H_z = 0\), the solution is again given by Eq. (142) but with \(F = 0 = G\) where \(A\) and \(B\) are again as given by Eq. (150) and furthermore, \(\alpha_1, \alpha_2\) turn out to be negative, i.e.

\[
\alpha_1 = -\frac{D^2}{2} - 2\beta_1 A^2, \quad \alpha_2 = -\frac{(1 + m)D^2}{2} - \gamma A^2. \tag{153}
\]

It is worth pointing out that with \(H_z = 0\), the field equations are completely symmetric between the two fields \(\phi\) and \(\psi\) and hence the solutions VI and VIII are identical in the limit \(H_z = 0\).

**Special case of** \(\gamma^2 = 4\beta_1 \beta_2\)

One can show that the solution (142) exists even in case \(\gamma^2 = 4\beta_1 \beta_2\). It turns out such a solution exists only if

\[
2\beta_1 = 2\beta_2 = \gamma, \tag{154}
\]

and that in this case one cannot determine \(A, B\). However, they must satisfy the constraint

\[
B^2 - A^2 = \frac{mD^2}{\gamma}. \tag{155}
\]

Further, one has

\[
\rho_2 = \rho_3, \quad \alpha_1 = \frac{\rho_3^2 H_z^2}{2\gamma} + \frac{\rho_1 \gamma}{\rho_3}, \quad \alpha_1 - \alpha_2 = \frac{mD^2}{2} + \frac{\rho_3^2 H_z^2}{2\gamma}. \tag{156}
\]

**Energy**: Corresponding to the mixed “kink-pulse lattice” solution [Eq. (12)] the energy and the constant \(C\) are given by

\[
E - V_{\text{min}} = \frac{2D}{3m}\left(\frac{1}{2}(1 - m)A^2 + (1 + m)B^2\right)E(m) - (1 - m)(B^2 - A^2)K(m), \tag{157}
\]

\[
C = -\frac{1}{2}(1 - m)A^2 D^2 - F^2[\alpha_1 + 3\beta_1 F^2 - 2H_z \rho_2 F] + [-\gamma F^2 + \alpha_1 + \beta_2 B^2]B^2. \tag{158}
\]

For \(m\) near one, the energy of this periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-bright) soliton solution

\[
\phi = F + A \text{sech}[D(x + x_0)], \quad \psi = B \tanh[D(x + x_0)], \tag{159}
\]
plus the interaction energy. We find

$$E = E_{\text{soliton}} + E_{\text{int}} = D \left[ \frac{2}{3} (2B^2 + A^2) + \frac{(1 - m)}{6} (2B^2 - 5A^2) + (1 - m)A^2 \ln \left( \frac{4}{\sqrt{1 - m}} \right) \right].$$  \hspace{1cm} (160)

Note that this solution also exists only when either $2\beta_1 \geq \gamma \geq 2\beta_2$ and $\gamma^2 \leq 4\beta_1\beta_2$ or $2\beta_2 \geq \gamma \geq 2\beta_1$ and $\gamma^2 \geq 4\beta_1\beta_2$. Again, the interaction energy vanishes at $m = 1$. It is amusing to note that the energy of the solution VIII is easily obtained from that of solution VII by simply interchanging $A$ and $B$.

### 2.9 Solution IX

Another solution which at $m = 1$ reduces to a dark-bright type of soliton solution is given by

$$\phi = F + A\text{dn}[D(x + x_0), m], \quad \psi = G + B\text{sn}[D(x + x_0), m],$$  \hspace{1cm} (161)

provided $G = 0$ and the following seven coupled equations are satisfied

$$2\alpha_1 F + 4\beta_1 F^3 + (2/m)\gamma FB^2 - H_z\rho_1 - 3H_z\rho_2 F^2 - \frac{H_z\rho_3 B^2}{m} = 0,$$  \hspace{1cm} (162)

$$2\alpha_1 A + 12\beta_1 F^2 A + (2/m)\gamma AB^2 - 6H_z\rho_2 AF = (2 - m)AD^2,$$  \hspace{1cm} (163)

$$12\beta_1 FA^2 - (2/m)\gamma FB^2 - 3H_z\rho_2 A^2 + \frac{H_z\rho_3 B^2}{m} = 0,$$  \hspace{1cm} (164)

$$2m\beta_1 A^2 - \gamma B^2 = -mD^2,$$  \hspace{1cm} (165)

$$2\alpha_2 B + 2\gamma A^2 B + 2\gamma BF^2 - 2H_z\rho_3 BF = -(1 + m)BD^2,$$  \hspace{1cm} (166)

$$2\beta_2 B^2 - m\gamma A^2 = mD^2,$$  \hspace{1cm} (167)

$$4\gamma FAB - 2H_z\rho_3 AB = 0.$$  \hspace{1cm} (168)

In this case it turns out that the solution exists only if either $2\beta_1 > \gamma > 2\beta_2$, $\gamma^2 < 4\beta_1\beta_2$ or $2\beta_2 > \gamma > 2\beta_1$, $\gamma^2 > 4\beta_1\beta_2$. We find that

$$A^2 = \frac{D^2[\gamma - 2\beta_2]}{[4\beta_1\beta_2 - \gamma^2]}, \quad B^2 = \frac{mD^2[2\beta_1 - \gamma]}{[4\beta_1\beta_2 - \gamma^2]},$$  \hspace{1cm} (169)

where

$$(2 - m)D^2 = (2/m)\gamma B^2 + \frac{2\rho_1\gamma}{\rho_3} - \frac{\rho_2\rho_3 H_z^2}{2\gamma}, \quad F = \frac{\rho_3 H_z}{2\gamma},$$  \hspace{1cm} (170)
and the three constraints are

\[\beta_1 = \frac{\rho_2 \gamma}{2 \rho_3}, \quad \alpha_1 = \frac{\rho_2 \rho_3 H^2 z^2}{2 \gamma} + \frac{\rho_1 \gamma}{\rho_3}, \]

\[-(1 + m)D^2 = 2 \alpha_2 + 2 \gamma A^2 - \frac{H^2 \rho_3^2}{2 \gamma}. \quad (171)\]

**Special case of** \(H_z = 0: \)**

When \(H_z = 0\), the solution is again given by Eq. (161) but with \(F = 0 = G\) where \(A\) and \(B\) are again as given by Eq. (169) and furthermore, \(\alpha_1, \alpha_2\) turn out to be negative, i.e.

\[\alpha_1 = -\frac{mD^2}{2} - 2 \beta_1 A^2, \quad \alpha_2 = -\frac{(1 + m)D^2}{2} - \gamma A^2. \quad (172)\]

It is worth pointing out that with \(H_z = 0\), the field equations are completely symmetric between the two fields \(\phi\) and \(\psi\) and hence the solutions VII and IX are identical in the limit \(H_z = 0\).

**m=1:**

In the special case of \(m = 1\) and \(G = 0, F \neq 0\), both solutions VIII and IX reduce to a dark-bright type of solution (159), i.e.

\[\phi = F + A \text{sech}[D(x + x_0)], \quad \psi = B \tanh[D(x + x_0)], \quad (173)\]

with \(A, B\) and \(D\) given by

\[A^2 = \frac{D^2 \gamma - 2 \beta_2}{4 \beta_1 \beta_2 - \gamma^2}, \quad B^2 = \frac{D^2 [2 \beta_1 - \gamma]}{4 \beta_1 \beta_2 - \gamma^2}, \quad D^2 = 2 \gamma B^2 + \frac{2 \rho_1 \gamma}{\rho_3} - \frac{\rho_2 \rho_3 H^2 z^2}{2 \gamma}. \quad (174)\]

while the other relations remain unchanged and are given by Eqs. (151) and (152).

**Special case of** \(\gamma^2 = 4 \beta_1 \beta_2\)

One can show that the solution (161) exists even in case \(\gamma^2 = 4 \beta_1 \beta_2\). It turns out such a solution exists only if

\[2 \beta_1 = 2 \beta_2 = \gamma, \quad (175)\]

and that in this case one cannot determine \(A, B\). However, they must satisfy the constraint

\[B^2 - mA^2 = \frac{mD^2}{\gamma}. \quad (176)\]
Further, one has
\[ \rho_2 = \rho_3, \quad \alpha_1 = \frac{\rho_2^2 H_z^2}{2\gamma} + \frac{\rho_1 \gamma}{\rho_3}, \quad \alpha_1 - \alpha_2 = \frac{D^2}{2} + \frac{\rho_2^2 H_z^2}{2\gamma}. \] (177)

**Energy:** Corresponding to the mixed “pulse-kink lattice” solution [Eq. (161)] the energy and the constant \( C \) are given by
\[ E - V_{\text{min}} = \frac{2D}{3m} \left( [(2 - m) m A^2 + (1 + m) B^2] E(m) - (1 - m)(B^2 + 2A^2) K(m) \right), \] (178)
\[ C = \frac{1}{2}(1 - m) A^2 D^2 - F^2 [\alpha_1 + 3\beta_1 F^2 - 2H_z \rho_2 F] + [-m\gamma F^2 + m\alpha_2 + \beta_2 B^2] \frac{B^2}{m^2}. \] (179)

For \( m \) near one, the energy of this periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-bright) soliton solution [Eq. (159)] plus the interaction energy. We find
\[ E = E_{\text{soliton}} + E_{\text{int}} = D \left[ \frac{2}{3} (2B^2 + A^2) + \left( \frac{1 - m}{6} \right) (2B^2 + 3A^2) - (1 - m) A^2 \ln \left( \frac{4}{\sqrt{1 - m}} \right) \right]. \] (180)

Note that this solution also exists only when either \( 2\beta_1 \geq \gamma \geq 2\beta_2 \) and \( \gamma^2 \leq 4\beta_1 \beta_2 \) or \( 2\beta_2 \geq \gamma \geq 2\beta_1 \) and \( \gamma^2 \geq 4\beta_1 \beta_2 \). The interaction energy vanishes at \( m = 1 \). It is amusing to note that the energy of the solution IX is easily obtained from that of solution VI by simply interchanging \( A \) and \( B \).

Summarizing, we have obtained nine periodic solutions in terms of Jacobi elliptic functions, in the case of coupled \( \phi^4 \) field theory with biquadratic coupling and an external magnetic field. This was possible because the magnetic field interaction is not symmetric between the two fields \( \phi \) and \( \psi \). In the special case when the modulus parameter \( m \) of the Jacobi elliptic function is one, these nine solutions reduce to four different soliton solutions valid on the full line and expressed in terms of hyperbolic functions. Note, however, that in case the external field \( H_z = 0 \), instead of nine, we only obtain six distinct periodic solutions (of which three are previously known \([20]\)), which in the limit \( m = 1 \), give three distinct soliton solutions.

It is worth emphasizing the restrictions on the various parameters in the case of the nine solutions. For example, in the case of sn – sn solution (with \( G = 0 \), \( 2\beta_1 \geq \gamma \geq 2\beta_2 \)). Further, in the special case of \( H_z = 0 = F = G \), one can show that \( \alpha_1 < 0, \alpha_2 < 0 \). On the other hand, in the case of cn – cn, dn – dn, cn – dn and dn – cn solutions (with \( G = 0 \), \( \gamma < 0 \) and further \( \gamma^2 > 4\beta_1 \beta_2 \). In the special case of \( H_z = 0 = F = G \), in addition one finds that (i) for cn – cn case \( \alpha_1, \alpha_2 > (\gamma) 0 \) provided \( m > (\gamma) 1/2 \)
(ii) $\alpha_1 > 0$, $\alpha_2 > 0$ for $dn - dn$ solution (iii) for $dn - cn$ solution, $\alpha_2 > 0$ (iv) $\alpha_1 < 0$ for $cn - dn$ solution. Instead, for $sn - cn$ as well as $sn - dn$ solutions, either $2\beta_1 \geq \gamma \geq 2\beta_2$ and $\gamma^2 \leq 4\beta_1\beta_2$ or $2\beta_2 \geq \gamma \geq 2\beta_1$ and $\gamma^2 \leq 4\beta_2\beta_4$. Finally, in the $cn - sn$ and $dn - sn$ cases, either $2\beta_1 \geq \gamma \geq 2\beta_2$ and $\gamma^2 \leq 4\beta_1\beta_2$ or $2\beta_2 \geq \gamma \geq 2\beta_1$ and $\gamma^2 \geq 4\beta_1\beta_2$. If in addition $H_z = 0 = F = G$, then for all the four solutions (i.e. $sn - cn$, $cn - sn$, $sn - dn$, $dn - sn$), $\alpha_1 < 0$ and $\alpha_2 < 0$.

3 Solutions with bilinear coupling

Several years ago, a coupled $\phi^4$ model was considered in the context of a surface phase transition with hydration forces \cite{7} which is similar to the one considered in the last section except that there was no external magnetic field $H_z$ and instead of a biquadratic coupling there was a bilinear coupling between the two fields. The purpose of this section is to obtain a bright-bright soliton solution of that model.

The potential (i.e. free energy) is given by

$$V = \alpha_1 \phi^2 + \beta_1 \phi^4 + \alpha_2 \psi^2 + \beta_2 \psi^4 + \delta_1 (\phi - \psi)^2 + \delta_2 (\phi + \psi)^2,$$

with $\delta_1$, $\delta_2$ being the coupling parameters between the two fields. The (static) equations of motion which follow from here are

$$\frac{d^2 \phi}{dx^2} = 2\alpha_1 \phi + 4\beta_1 \phi^3 + 2\delta_1 (\phi - \psi) + 2\delta_2 (\phi + \psi),$$

$$\frac{d^2 \psi}{dx^2} = 2\alpha_2 \psi + 4\beta_2 \psi^3 - 2\delta_1 (\phi - \psi) + 2\delta_2 (\phi + \psi).$$

It is not difficult to show that this pair of coupled field equations admits the “kink-kink” type periodic solution

$$\phi = A \text{sn}[D(x + x_0), m], \quad \psi = B \text{sn}[D(x + x_0), m],$$

provided

$$mD^2 = 2\beta_1 A^2 = 2\beta_2 B^2,$$

$$A^2 = \frac{-m}{(1 + m)\beta_1} \left[ \alpha_1 + \delta_1 + \delta_2 + (\delta_2 - \delta_1) \sqrt{\frac{\beta_1}{\beta_2}} \right],$$
and furthermore the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2$ satisfy the constraint

$$\alpha_2 - \alpha_1 = (\delta_1 - \delta_2) \left[ \sqrt{\frac{\beta_2}{\beta_1}} - \sqrt{\frac{\beta_1}{\beta_2}} \right]. \quad (186)$$

Since $A^2 > 0$, the relation (185) gives us a strong constraint on some of the parameters. The energy $\hat{E}$ and the constant $C$ corresponding to the periodic solution [Eq. (184)] are

$$\hat{E} = \frac{2(A^2 + B^2)D}{3m} \left[ (1 + m)E(m) - (1 - m)K(m) \right],$$

$$C = -\frac{1}{2}(A^2 + B^2)D^2. \quad (187)$$

In the limit $m = 1$, this solution reduces to the bright-bright soliton solution

$$\phi = A \tanh[D(x + x_0), m], \quad \psi = B \tanh[D(x + x_0)], \quad (188)$$

provided

$$D^2 = 2\beta_1 A^2 = 2\beta_2 B^2,$$

$$A^2 = -\frac{1}{2\beta_1} \left[ \alpha_1 + \delta_1 + \delta_2 + (\delta_2 - \delta_1) \sqrt{\frac{\beta_1}{\beta_2}} \right], \quad (189)$$

while the relation (186) remains unchanged. For $m$ near one, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-bright) soliton solution [Eq. (188)] plus the interaction energy. We find

$$\hat{E} = E_{\text{kink}} + E_{\text{int}} = (A^2 + B^2)D \left[ \frac{4}{3} + \frac{(1 - m)}{3} \right]. \quad (190)$$

The interaction energy vanishes at $m = 1$. In view of the requirement $\beta_1, \beta_2 > 0$ arising from stability, we are unable to find any other solution to this coupled set of equations with a bilinear coupling.

4 Solutions of discrete coupled $\phi^4$-type equations with biquadratic coupling

Discrete coupled $\phi^4$ models arise in the context of structural transitions on a lattice, collective proton dynamics in ice [23], etc. The purpose of this section is to give an exhaustive list of solutions to the
discrete coupled $\phi^4$-type equations with biquadratic coupling (but in the absence of an external magnetic field $H_z$). In the next section, we shall obtain a solution of the discrete coupled $\phi^4$-type equations with bilinear coupling.

We start from the coupled static field equations (2) and (3). The discrete analog of these field equations, for $H_z = 0$ has the form

\begin{align}
\frac{1}{h^2} \left( \phi_{n+1} + \phi_{n-1} - 2\phi_n \right) - 2\alpha_1 \phi_n - 2[2\beta_1 \phi_n^2 + \gamma \psi_n^2] \phi_n &= 0, \\
\frac{1}{h^2} \left( \psi_{n+1} + \psi_{n-1} - 2\psi_n \right) - 2\alpha_2 \psi_n - 2[2\beta_2 \psi_n^2 + \gamma \phi_n^2] \psi_n &= 0.
\end{align}

We are unable to find any solution to this coupled set of field equations. However, as in the Ablowitz-Ladik discretization of the discrete nonlinear Schrödinger equation [24], if we replace $\phi_n$ and $\psi_n$ in the last term in Eqs. (191) and (192) by their average, then we can find exact solutions to this coupled system. In particular, instead of Eqs. (191) and (192), we consider the discretized equations

\begin{align}
\frac{1}{h^2} \left( \phi_{n+1} + \phi_{n-1} - 2\phi_n \right) - 2\alpha_1 \phi_n - [2\beta_1 \phi_n^2 + \gamma \psi_n^2] \phi_n + \phi_{n+1} + \phi_{n-1} &= 0, \\
\frac{1}{h^2} \left( \psi_{n+1} + \psi_{n-1} - 2\psi_n \right) - 2\alpha_2 \psi_n - [2\beta_2 \psi_n^2 + \gamma \phi_n^2] \psi_n + \psi_{n+1} + \psi_{n-1} &= 0.
\end{align}

Note that single solitons and their stability in coupled Ablowitz-Ladik chains have been studied previously [25]. We now show that this modified set of coupled discrete equations has six different periodic solutions which in the limit $m = 1$ reduce to the bright-bright, bright-dark and dark-dark soliton solutions. In all the solutions, we shall see that the static kink can be placed anywhere with respect to the lattice. Hence we suspect that in all these cases, there may be an absence of the Peierls-Nabarro barrier [26, 27, 28], which is the energy cost associated with moving a localized solution such as a soliton by a half lattice constant on a discrete lattice. It would be nice if one can demonstrate this explicitly.

4.1 Solution I

It is easy to show that the field Eqs. (193) and (194) admit the kink-kink type solution

\begin{equation}
\phi_n = A \text{ sn}[hD(n + x_0), m], \quad \psi_n = B \text{ sn}[hD(n + x_0), m],
\end{equation}
provided

\[
A^2 = \frac{m(2\beta_2 - \gamma)\text{sn}^2(hD, m)}{h^2(4\beta_1\beta_2 - \gamma^2)}, \quad B^2 = \frac{m(2\beta_1 - \gamma)\text{sn}^2(hD, m)}{h^2(4\beta_1\beta_2 - \gamma^2)},
\]

\[
\alpha_1 = \alpha_2 = -\frac{1}{h^2}[1 - \text{cn}(hD, m)\text{dn}(hD, m)],
\]

(196)

where \(h\) is the lattice spacing. Thus, note that as in the continuum case, this solution exists provided

\(2\beta_1 > \gamma, 2\beta_2 > \gamma, \alpha_1 < 0, \alpha_2 < 0\). It is interesting to note that the solutions to both the discrete and the continuum model exist under the same set of conditions.

**Continuum Limit:** It is instructive to consider the continuum limit \(h \to 0\) and show that the above solution smoothly goes over to the corresponding continuum solution. In particular, on using the fact that as \(h \to 0\)

\[
\text{sn}(hD, m) \to hD, \quad \text{cn}^2(hD, m) \to 1 - h^2D^2, \quad \text{dn}^2(hD, m) \to 1 - mh^2D^2,
\]

(197)

it readily follows that the above solution indeed reduces to the corresponding continuum solution [Eq. (7)] obtained in Sec. II (when \(H_z = F = G = 0\), i.e.

\[
A^2 = \frac{m(2\beta_2 - \gamma)D^2}{(4\beta_1\beta_2 - \gamma^2)}, \quad B^2 = \frac{m(2\beta_1 - \gamma)D^2}{(4\beta_1\beta_2 - \gamma^2)}, \quad \alpha_1 = \alpha_2 = -\frac{(1 + m)D^2}{2}.
\]

(198)

In fact we shall see that all the six solutions of this coupled discrete model smoothly go over to the corresponding continuum solutions obtained in Sec. II in the limit \(h \to 0\).

In the limit \(m = 1\), the periodic solution (195) reduces to the bright-bright soliton solution

\[
\phi_n = A \tanh[hD(n + x_0)], \quad \psi_n = B \tanh[hD(n + x_0)].
\]

(199)

**Special case of \(\gamma^2 = 4\beta_1\beta_2\)**

One can show that the solution (195) exists even in case \(\gamma^2 = 4\beta_1\beta_2\). It turns out that such a solution exists only if

\[
2\beta_1 = 2\beta_2 = \gamma,
\]

(200)

and that in this case one cannot determine \(A, B\). However, they must satisfy the constraint

\[
A^2 + B^2 = \frac{m\text{sn}^2(hD, m)}{h^2\gamma}.
\]

(201)

In the continuum limit \(h \to 0\), as expected, this reduces to the constraint equation (25) obtained in Sec. II.
4.2 Solution II

It is easily shown that a kink-pulse type solution

\[ \phi_n = A \text{sn}[hD(n+x_0),m], \quad \psi_n = B \text{cn}[hD(n+x_0),m], \tag{202} \]

is an exact solution to the field Eqs. (193) and (194) provided

\[ A^2 = \frac{m(\gamma - 2\beta_2)\text{sn}^2(hD,m)}{h^2(\gamma^2 - 4\beta_1\beta_2)}, \quad B^2 = \frac{m(2\beta_1 - \gamma)\text{sn}^2(hD,m)}{h^2(\gamma^2 - 4\beta_1\beta_2)}. \tag{203} \]

Furthermore,

\[ -\alpha_1 = \frac{1}{h^2} + \frac{\text{cn}(hD,m)[2m\beta_1(\gamma - 2\beta_2)\text{sn}^2(hD,m) - (\gamma^2 - 4\beta_1\beta_2)]}{h^2(\gamma^2 - 4\beta_1\beta_2)\text{dn}(hD,m)}, \tag{204} \]

\[ -\alpha_2 = \frac{1}{h^2} + \frac{\text{cn}(hD,m)[m\gamma(\gamma - 2\beta_2)\text{sn}^2(hD,m) - (\gamma^2 - 4\beta_1\beta_2)]}{h^2(\gamma^2 - 4\beta_1\beta_2)\text{dn}^2(hD,m)}. \tag{205} \]

Again, as in the continuum case either \( 2\beta_1 > \gamma > 2\beta_2, \gamma^2 > 4\beta_1\beta_2 \) or \( 2\beta_2 > \gamma > 2\beta_1, 4\beta_1\beta_2 > \gamma^2 \). It is, however, not clear here if \( \alpha_1 \) and \( \alpha_2 \) have a definite sign. However, in the limit \( m = 1 \), as in the continuum case, one finds that \( \alpha_1 < 0, \alpha_2 < 0 \).

It is readily checked that in the continuum limit this solution smoothly goes over to the corresponding continuum solution, Eqs. (104) and (112). Furthermore, the corresponding bright-dark solution is easily obtained in the limit \( m = 1 \).

**Special case of \( \gamma^2 = 4\beta_1\beta_2 \)**

One can show that the solution (202) exists even in case \( \gamma^2 = 4\beta_1\beta_2 \). It turns out such a solution exists only if

\[ 2\beta_1 = 2\beta_2 = \gamma, \tag{206} \]

and that in this case one cannot determine \( A, B \). However, they must satisfy the constraint

\[ A^2 - B^2\text{dn}^2(hD,m) = \frac{m\text{sn}^2(hD,m)}{h^2\gamma}. \tag{207} \]

In the continuum limit \( h \to 0 \), as expected, this reduces to the constraint equation (117) obtained in Sec. II.
4.3 Solution III

Yet another kink-pulse type solution is given by

\[ \phi_n = A \text{sn}[hD(n + x_0), m], \quad \psi_n = B \text{dn}[hD(n + x_0), m], \]

(208)

provided

\[ A^2 = \frac{m(\gamma - 2\beta_2)\text{sn}^2(hD, m)}{h^2(\gamma^2 - 4\beta_1\beta_2)}, \quad B^2 = \frac{(2\beta_1 - \gamma)\text{sn}^2(hD, m)}{h^2(\gamma^2 - 4\beta_1\beta_2)}. \]

(209)

Furthermore,

\[ -\alpha_1 = \frac{1}{h^2} + \frac{\text{dn}(hD, m)[2\beta_1(\gamma - 2\beta_2)\text{sn}^2(hD, m) - (\gamma^2 - 4\beta_1\beta_2)]}{h^2(\gamma^2 - 4\beta_1\beta_2)\text{cn}(hD, m)}, \]

(210)

\[ -\alpha_2 = \frac{1}{h^2} + \frac{\text{dn}(hD, m)[\gamma(\gamma - 2\beta_2)\text{sn}^2(hD, m) - (\gamma^2 - 4\beta_1\beta_2)]}{h^2(\gamma^2 - 4\beta_1\beta_2)\text{cn}^2(hD, m)}. \]

(211)

Special case of \( \gamma^2 = 4\beta_1\beta_2 \)

One can show that the solution (208) exists even in case \( \gamma^2 = 4\beta_1\beta_2 \). It turns out such a solution exists only if

\[ 2\beta_1 = 2\beta_2 = \gamma, \]

(212)

and that in this case one cannot determine \( A, B \). However, they must satisfy the constraint

\[ A^2 - mB^2\text{cn}^2(hD, m) = \frac{m\text{sn}^2(hD, m)}{h^2\gamma}. \]

(213)

In the continuum limit \( h \to 0 \), as expected, this reduces to the constraint equation (137) obtained in Sec. II.

Note that in the \( m = 1 \) limit the solutions II and III reduce to the same bright-dark soliton solution. In addition, as in the continuum case, this solution exists if either \( 2\beta_1 \geq \gamma \geq 2\beta_2, \gamma^2 \geq 4\beta_1\beta_2 \) or \( 2\beta_2 \geq \gamma \geq 2\beta_1, 4\beta_1\beta_2 \geq \gamma^2 \). It is, however, not clear here if \( \alpha_1 \) and \( \alpha_2 \) have a definite sign. However, in the limit \( m = 1 \), as in the continuum case, one finds that \( \alpha_1 < 0 \) and \( \alpha_2 < 0 \).

4.4 Solution IV

Finally, we present three periodic solutions, all of which in the limit \( m = 1 \) reduce to the dark-dark soliton solution. One of the pulse-pulse type periodic solution is given by

\[ \phi_n = A \text{cn}[hD(n + x_0), m], \quad \psi_n = B \text{cn}[hD(n + x_0), m], \]

(214)
provided as in the continuum case, $\gamma < 0$ and $\gamma^2 > 4\beta_1\beta_2$. We find

$$A^2 = \frac{2m(\beta_2 + |\gamma|)sn^2(hD, m)}{h^2(\gamma^2 - 4\beta_1\beta_2)dn^2(hD, m)}, \quad B^2 = \frac{2m(\beta_1 + |\gamma|)sn^2(hD, m)}{h^2(\gamma^2 - 4\beta_1\beta_2)dn^2(hD, m)},$$

$$\alpha_1 = \alpha_2 = -\frac{1}{h^2} \left[ 1 - \frac{\text{cn}(hD, m)}{\text{dn}(hD, m)} \right].$$

(215)

Using Eq. (197) it is easily shown that as in the continuum case, $\alpha_1, \alpha_2 > (\cdot) 0$ provided $m > (\cdot) 1/2$. This solution is equivalent to the continuum solution, Eqs. (33) and (42).

### 4.5 Solution V

Another pulse-pulse type solution is given by

$$\phi_n = A \text{dn}[hD(n + x_0), m], \quad \psi_n = B \text{dn}[hD(n + x_0), m],$$

(216)

provided as in the continuum case, $\gamma < 0$ and $\gamma^2 > 4\beta_1\beta_2$. We find

$$A^2 = \frac{2(\beta_2 + |\gamma|)sn^2(hD, m)}{h^2(\gamma^2 - 4\beta_1\beta_2)cn^2(hD, m)}, \quad B^2 = \frac{2(\beta_1 + |\gamma|)sn^2(hD, m)}{h^2(\gamma^2 - 4\beta_1\beta_2)cn^2(hD, m)},$$

$$\alpha_1 = \alpha_2 = -\frac{1}{h^2} \left[ 1 - \frac{\text{cn}(hD, m)}{\text{dn}(hD, m)} \right].$$

(217)

Using Eq. (197) it is easily shown that as in the continuum case, $\alpha_1, \alpha_2 > 0$. This solution is equivalent to the continuum solution, Eqs. (33) and (62).

### 4.6 Solution VI

Yet another pulse-pulse type solution is

$$\phi_n = A \text{dn}[hD(n + x_0), m], \quad \psi_n = B \text{cn}[hD(n + x_0), m],$$

(218)

provided as in the continuum case, $\gamma < 0$ and $\gamma^2 > 4\beta_1\beta_2$. We find

$$A^2 = \frac{m(2\beta_2 + |\gamma|)sn^2(hD, m)}{h^2(\gamma^2 - 4\beta_1\beta_2)cn^2(hD, m)}, \quad B^2 = \frac{m(2\beta_1 + |\gamma|)sn^2(hD, m)}{h^2(\gamma^2 - 4\beta_1\beta_2)cn^2(hD, m)},$$

(219)

Furthermore,

$$-\alpha_1 = \frac{1}{h^2} + \frac{[2\beta_1(2\beta_2 + |\gamma|)dn^2(hD, m) - \gamma(2\beta_1 + |\gamma|)cn^2(hD, m)]}{h^2(\gamma^2 - 4\beta_1\beta_2)hn(hD, m)cn^2(hD, m)}.$$

(220)
\[- \alpha_2 = \frac{1}{h^2} + \frac{[2\beta_2(2\beta_1 + |\gamma|)\text{cn}^2(hD, m) - |\gamma||(\alpha_1 + 2\beta_2)\text{dn}^2(hD, m))]}{h^2(\gamma^2 - 4\beta_1 \beta_2)\text{dtn}^2(hD, m)\text{cn}(hD, m)}. \] (221)

It is not clear if in general \( \alpha_1 \) and \( \alpha_2 \) have a definite sign. However, it is easily checked that at \( m = 1 \), \( \alpha_1, \alpha_2 > 0 \). This solution is equivalent to the continuum solution, Eqs. (72) and (80).

Note that the last three (i.e., IV, V, VI) solutions reduce to the (same) dark-dark soliton solution in the limit \( m = 1 \). Since we do not know the Hamiltonian corresponding to Eqs. (193) and (194), we are unable to find the energy and soliton interaction explicitly for any of the above discrete solutions. Similarly, for \( H_z \neq 0 \) we have not succeeded in finding exact solutions.

5 Solutions of discrete coupled \( \phi^4 \)-type equations with bilinear coupling

Coupled lattice chains, with a bilinear coupling, undergoing a second order structural phase transition can represent this case. We start from the coupled static field Eqs. (182) and (183). The discrete analog of these field equations has the form

\[
\frac{1}{h^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) - 2[\alpha_1 + \delta_1 + \delta_2]\phi_n + 2[\delta_1 - \delta_2]\psi_n - 4\beta_1 \phi_n^3 = 0, \tag{222}
\]

\[
\frac{1}{h^2}(\psi_{n+1} + \psi_{n-1} - 2\psi_n) - 2[\alpha_2 + \delta_1 + \delta_2]\psi_n + 2[\delta_1 - \delta_2]\phi_n - 4\beta_2 \psi_n^3 = 0. \tag{223}
\]

We are unable to find any solution to this coupled set of field equations. However, as in the Ablowitz-Ladik discretization of the discrete nonlinear Schrödinger equation [24], if we replace \( \phi_n \) and \( \psi_n \) in the last term in Eqs. (222) and (223) by their average, then we can find exact solutions to this coupled system. In particular, instead of Eqs. (222) and (223), we consider the discretized equations

\[
\frac{1}{h^2}(\phi_{n+1} + \phi_{n-1} - 2\phi_n) - 2[\alpha_1 + \delta_1 + \delta_2]\phi_n + 2[\delta_1 - \delta_2]\psi_n - 2\beta_1 \phi_n^2[\phi_{n+1} + \phi_{n-1}] = 0, \tag{224}
\]

\[
\frac{1}{h^2}(\psi_{n+1} + \psi_{n-1} - 2\psi_n) - 2[\alpha_2 + \delta_1 + \delta_2]\psi_n + 2[\delta_1 - \delta_2]\phi_n - 2\beta_2 \psi_n^2[\psi_{n+1} + \psi_{n-1}] = 0. \tag{225}
\]

It is easy to show that the field Eqs. (224) and (225) admit the kink-kink type solution

\[
\phi_n = A \text{sn}[hD(n + x_0), m], \quad \psi_n = B \text{sn}[hD(n + x_0), m], \tag{226}
\]
provided

\[
\frac{\text{msn}^2(hD, m)}{h^2} = 2\beta_1 A^2 = 2\beta_2 B^2, \\
A^2 = \frac{-\text{msn}^2(hD, m)}{2\beta_1 [1 - \text{cn}(hD, m)\text{dn}(hD, m)]} \left[ \alpha_1 + \delta_1 + \delta_2 + (\delta_2 - \delta_1) \sqrt{\frac{\beta_1}{\beta_2}} \right],
\]

(227)

and furthermore the parameters \( \alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2 \) satisfy the constraint

\[
\alpha_2 - \alpha_1 = (\delta_1 - \delta_2) \left[ \sqrt{\frac{\beta_2}{\beta_1}} - \sqrt{\frac{\beta_1}{\beta_2}} \right].
\]

(228)

As expected, in the continuum limit of \( h \to 0 \), this solution smoothly goes over to the corresponding continuum solution, Eqs. (184), (185) and (186), obtained in Sec. III. Since we do not know the Hamiltonian corresponding to Eqs. (224) and (225), we are unable to find the energy and soliton interaction explicitly for this discrete solution. Similarly, for \( H_z \neq 0 \) we have not succeeded in finding an exact solution.

6 Conclusion

We have systematically provided an exhaustive set of exact periodic domain wall solutions for a coupled \( \phi^4 \) model with and without an external field, and for both bilinear and biquadratic couplings. Only a bright-bright solution could be obtained for the bilinear case. For both the biquadratic and bilinear couplings the corresponding discrete case was also considered–with an Ablowitz-Ladik like modification of the coupled discrete equations– and we obtained several exact solutions. For the solutions of the discrete model, the calculation of the Peierls-Nabarro barrier \([26, 27, 28]\) and soliton scattering \([29, 30]\) remain topics of further study. Similarly, scattering of solitons in the coupled \( \phi^4 \) continuum and discrete models with either the biquadratic or bilinear coupling is an interesting open issue. To this end, the static solutions presented here need to be boosted with a certain velocity.

It would be instructive to explore whether the nine different solutions reported in Sec. 2 (or the six solutions in Sec. 4) are completely disjoint or if there are any possible bifurcations linking them via, for instance, analytical continuation. We have not tried to carry out an explicit stability analysis of various periodic solutions. However, the energy calculations and interaction energy between solitons (for \( m \sim 1 \)) in the case of both the biquadratic and bilinear couplings could provide useful insight in this direction.
Our results are relevant for spin configurations, domain walls and magnetic phase transitions in multiferroic materials \[3, 4\]; periodic domain walls are yet to be observed in the hexagonal multiferroics \[3\]. Similarly, our solutions are important for understanding structural phase transitions in ferroelectrics \[5, 6\] and elastic materials \[8\], biophysics problems such as multilamellar lipid systems \[7\] as well as field theoretic contexts \[9, 10\]. These ideas and exact solutions can be generalized to other coupled models such as $\phi^6$ (for first order phase transitions) and will be discussed elsewhere \[31\].

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