Regular prism tilings in $\widetilde{\text{SL}_2\mathbb{R}}$ space

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Abstract. $\widetilde{\text{SL}_2\mathbb{R}}$ geometry is one of the eight 3-dimensional Thurston geometries, it can be derived from the 3-dimensional Lie group of all $2 \times 2$ real matrices with determinant one. Our aim is to describe and visualize the regular infinite or bounded $p$-gonal prism tilings in $\widetilde{\text{SL}_2\mathbb{R}}$. For this purpose we introduce the notion of infinite and bounded prisms, prove that there exist infinitely many regular infinite $p$-gonal face-to-face prism tilings $T_p^i(q)$ and infinitely many regular bounded $p$-gonal non-face-to-face $\widetilde{\text{SL}_2\mathbb{R}}$ prism tilings $T_p(q)$ for integer parameters $p, q; \, 3 \leq p, \, \frac{2p^2}{p^2 - 2} < q$. Moreover, we describe the symmetry group of $T_p(q)$ via its index 2 rotational subgroup, denoted by $pq2_1$. Surprisingly this group already occurred in our former work (Molnár et al., J Geometry, 95:91–133, 2009) in another context. We also develop a method to determine the data of the space filling regular infinite and bounded prism tilings. We apply the above procedure to $T_3^i(q)$ and $T_3(q)$ where $6 < q$ and visualize them and the corresponding tilings. E. Molnár showed, that homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathbb{P}^3$ and 3-space $\mathbb{P}^3(V^4, V_1, \mathbb{R})$. In our work we will use this projective model of $\widetilde{\text{SL}_2\mathbb{R}}$ and in this manner the prisms and prism tilings can be visualized on the Euclidean screen of a computer.

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1. On $\widetilde{\text{SL}_2\mathbb{R}}$ geometry

The real $2 \times 2$ matrices $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ with unit determinant $ad - bc = 1$ constitute a Lie transformation group by the usual product operation, taken to act on row matrices as on point coordinates on the right as follows

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\[(z^0, z^1) \begin{pmatrix} d & b \\ c & a \end{pmatrix} = (z^0 d + z^1 c, z^0 b + z^1 a) = (w^0, w^1)\] 

with \(w = \frac{w^1}{w^0} = \frac{b + \frac{z^1}{z^0} a}{d + \frac{z^1}{z^0} c} = \frac{b + za}{d + zc}\),

as action on the complex projective line \(\mathbb{C}^\infty\) (see [2,3]).

This group is a 3-dimensional manifold, because of its 3 independent real coordinates and with its usual neighbourhood topology [6,7,15]. In order to model the above structure in the projective sphere \(\mathcal{P}S^3\) and in the projective space \(\mathcal{P}^3\) (see [2]), we introduce the new projective coordinates \((x^0, x^1, x^2, x^3)\) where

\[
a := x^0 + x^3, \quad b := x^1 + x^2, \quad c := -x^1 + x^2, \quad d := x^0 - x^3,
\]

with the positive, then the non-zero multiplicative equivalence as projective freedom in \(\mathcal{P}S^3\) and in \(\mathcal{P}^3\), respectively. Then it follows that

\[
0 > bc - ad = -x^0 x^0 - x^1 x^1 + x^2 x^2 + x^3 x^3
\]

(1.2)

describes the interior of the above one-sheeted hyperboloid solid \(H\) in the usual Euclidean coordinate simplex with the origin \(E_0(1; 0; 0; 0)\) and the ideal points of the axes \(E_1^\infty(0; 1; 0; 0), E_2^\infty(0; 0; 1; 0), E_3^\infty(0; 0; 0; 1)\). We consider the collineation group \(G_*\) that acts on the projective sphere \(\mathcal{P}^3\) and preserves a polarity i.e. a scalar product of signature \((-+++)\), this group leaves the one sheeted hyperboloid solid \(H\) invariant. We have to choose an appropriate subgroup \(G\) of \(G_*\) as isometry group, then the universal covering group and space \(\tilde{H}\) of \(H\) will be the hyperboloid model of \(\tilde{\text{SL}}_2\mathbb{R}\) (see Fig. 1 and [2]).

**Figure 1.** The hyperboloid model of the \(\tilde{\text{SL}}_2\mathbb{R}\) geometry
The specific isometries \( \mathbf{S}(\phi)(\phi \in \mathbb{R}) \) constitute a one parameter group given by the matrices:

\[
\mathbf{S}(\phi) : (s_i^j(\phi)) = \begin{pmatrix}
\cos \phi & \sin \phi & 0 & 0 \\
-\sin \phi & \cos \phi & 0 & 0 \\
0 & 0 & \cos \phi & -\sin \phi \\
0 & 0 & \sin \phi & \cos \phi 
\end{pmatrix}
\]  \hspace{1cm} (1.3)

The elements of \( \mathbf{S}(\phi) \) are the so-called \textit{fibre translations}. We obtain a unique fibre line to each \( X(x^0; x^1; x^2; x^3) \in \tilde{\mathcal{H}} \) as the orbit by right action of \( \mathbf{S}(\phi) \) on \( X \). The coordinates of points lying on the fibre line through \( X \) can be expressed as the images of \( X \) by \( \mathbf{S}(\phi) \):

\[
(x^0; x^1; x^2; x^3) \xrightarrow{\mathbf{S}(\phi)} (x^0 \cos \phi - x^1 \sin \phi; x^0 \sin \phi + x^1 \cos \phi; x^2 \cos \phi + x^3 \sin \phi; -x^2 \sin \phi + x^3 \cos \phi).
\]  \hspace{1cm} (1.4)

The points of a fibre line through \( X \) by usual inhomogeneous Euclidean coordinates \( x = \frac{x^1}{x^0}, y = \frac{x^2}{x^0}, z = \frac{x^3}{x^0}, x^0 \neq 0 \) are given by

\[
(1; x; y; z) \xrightarrow{\mathbf{S}(\phi)} \left( 1; \frac{x + \tan \phi}{1 - x \tan \phi}; \frac{y + z \tan \phi}{1 - x \tan \phi}; \frac{z - y \tan \phi}{1 - x \tan \phi} \right)
\]  \hspace{1cm} (1.5)

for the projective space \( \mathcal{P}^3 \), where ideal points (at infinity) conventionally occur. The \( \pi \) periodicity of the above maps can be seen from formula \( (1.5) \).

In \( (1.3) \) and \( (1.4) \) we can see the \( 2\pi \) periodicity of \( \phi \), moreover the (logical) extension to \( \phi \in \mathbb{R} \), as real parameter, to have the universal covers \( \tilde{\mathcal{H}} \) and \( \mathbf{SL}_2 \mathbb{R} \), respectively, through the projective sphere \( \mathcal{P}S^3 \). The elements of the isometry group of \( \mathbf{SL}_2 \mathbb{R} \) (and so by the above extension the isometries of \( \mathbf{SL}_2 \mathbb{R} \)) can be described by the matrix \( (a_i^j) \) (see [2] and [3])

\[
(a_i^j) = \begin{pmatrix}
a_0^0 & a_0^1 & a_0^2 & a_0^3 \\
\mp a_0^1 & a_0^0 & a_0^3 & \mp a_0^2 \\
a_0^2 & a_0^1 & a_0^2 & a_0^3 \\
\pm a_0^1 & \mp a_0^2 & a_0^3 & \pm a_0^2 
\end{pmatrix}
\]

where

\[-(a_0^0)^2 - (a_0^1)^2 + (a_0^2)^2 + (a_0^3)^2 = -1, \quad -(a_0^2)^2 - (a_0^1)^2 + (a_0^2)^2 + (a_0^3)^2 = 1, \quad -(a_0^0)^2 - a_0^1 a_0^2 + a_0^2 a_0^3 + a_0^3 a_0^2 = 0 = -a_0^0 a_0^1 + a_0^1 a_0^2 - a_0^2 a_0^3 + a_0^3 a_0^2 \] (1.6)

can be assumed. Moreover, we have the projective proportionality, of course. We define the translation group \( G_T \), as a subgroup of the isometry group of \( \mathbf{SL}_2 \mathbb{R} \), the isometries acting transitively on the points of \( \mathcal{H} \) and by the above extension on the points of \( \mathbf{SL}_2 \mathbb{R} \) and \( \tilde{\mathcal{H}} \). \( G_T \) maps the origin \( E_0(1; 0; 0; 0) \) onto \( X(x^0; x^1; x^2; x^3) \). These isometries and their inverses (up to a positive
determinant factor) can be given by the following matrices:

\[
T : (t^j_i) = \begin{pmatrix}
  x^0 & x^1 & x^2 & x^3 \\
-x^1 & x^0 & x^3 & -x^2 \\
x^2 & x^3 & x^0 & x^1 \\
x^3 & -x^2 & -x^1 & x^0
\end{pmatrix},
\]

\[
T^{-1} : (T^k_j) = \begin{pmatrix}
  x^0 & -x^1 & -x^2 & -x^3 \\
x^1 & x^0 & -x^3 & x^2 \\
x^2 & -x^3 & x^0 & -x^1 \\
x^3 & x^2 & x^1 & x^0
\end{pmatrix}.
\] (1.7)

The rotation about the fibre line through the origin \(E_0(1; 0; 0; 0)\) by angle \(\omega(-\pi < \omega \leq \pi)\) can be expressed by the following matrix (see [2])

\[
R_{E_0}(\omega) : (r^j_i(E_0, \omega)) = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \omega & \sin \omega \\
0 & 0 & -\sin \omega & \cos \omega
\end{pmatrix},
\] (1.8)

and the rotation \(R_X(\omega)\) about the fibre line through \(X(x^0; x^1; x^2; x^3)\) by angle \(\omega\) can be derived by formulas (1.7) and (1.8):

\[
R_X(\omega) = T^{-1}R_{E_0}(\omega)T : (r^j_i(X, \omega))
\] (1.9)

in the sense of (1.6). Especially if \(X \sim (\cosh r, 0, \sinh r, 0) \sim (1, 0, \tanh r, 0)\), we get an important simplification in a short form:

\[
\begin{align*}
  r^0_0 &= 1 + \sinh^2 r (1 - \cos \omega) = 1 + 2 \sinh^2 r \sin^2 \omega/2, \\
  r^1_0 &= \sinh^2 r \sin \omega = 2 \sinh^2 r \sin \omega/2 \cos \omega/2, \\
  r^2_0 &= \cosh r \sinh r (1 - \cos \omega) = 2 \cosh r \sinh r \sin^2 \omega/2, \\
  r^3_0 &= -\cosh r \sinh r \sin \omega = -2 \cosh r \sinh r \sin \omega/2 \cos \omega/2, \\
  r^0_1 &= -\cosh r \sinh r \sin \omega = -2 \cosh r \sinh r \sin \omega/2 \cos \omega/2, \\
  r^1_2 &= -\cosh r \sinh r \sin \omega = -2 \cosh r \sinh r \sin \omega/2 \cos \omega/2, \\
  r^2_2 &= 1 - \cosh^2 r (1 - \cos \omega) = 1 - 2 \cosh^2 r \sin^2 \omega/2, \\
  r^3_2 &= \cosh^2 r \sin \omega = 2 \cosh^2 r \sin \omega/2 \cos \omega/2.
\end{align*}
\] (1.9')

Horizontal intersection of the hyperboloid solid \(H\) with the plane \(E_0E_\infty E_\infty\) provides the hyperbolic \(H^2\) base plane of the model \(\tilde{H} = \tilde{\text{SL}}_2\mathbb{R}\). The fibre through \(X\) intersects the base plane \(z^1 = x = 0\) in the foot point

\[
Z(z^0 = x^0 x^0 + x^1 x^1; z^1 = 0; z^2 = x^0 x^2 - x^1 x^3; z^3 = x^0 x^3 + x^1 x^2).
\] (1.10)
We introduce a so-called hyperboloid parametrization by [2] as follows

\[
\begin{align*}
    x^0 &= \cosh r \cos \phi, \\
    x^1 &= \cosh r \sin \phi, \\
    x^2 &= \sinh r \cos (\theta - \phi), \\
    x^3 &= \sinh r \sin (\theta - \phi),
\end{align*}
\]

(1.11)

where \((r, \theta)\) are the polar coordinates of the base plane and \(\phi\) is just the fibre coordinate. We note that

\[-x^0 x^0 - x^1 x^1 + x^2 x^2 + x^3 x^3 = -\cosh^2 r + \sinh^2 r = -1 < 0.\]

The inhomogeneous coordinates corresponding to (1.11), that play an important role in the later visualization of prism tilings in \(E^3\), are given by

\[
\begin{align*}
    x &= \frac{x^1}{x^0} = \tan \phi, \\
    y &= \frac{x^2}{x^0} = \tanh r \frac{\cos (\theta - \phi)}{\cos \phi}, \\
    z &= \frac{x^3}{x^0} = \tanh r \frac{\sin (\theta - \phi)}{\cos \phi}.
\end{align*}
\]

(1.12)

2. Prisms and prism tilings in \(\widetilde{SL}_2\mathbb{R}\)

After having investigated the prisms and prism-like tilings in \(S^2 \times \mathbb{R}\) and \(H^2 \times \mathbb{R}\) spaces (see [10] and [11]) we consider the analogous problem in the \(\widetilde{SL}_2\mathbb{R}\) space.

**Definition 2.1.** Let \(P^i\) be an infinite solid that is bounded by certain surfaces. These will be determined (later on) by certain “side fibre lines” passing through the vertices of a \(p\)-gon \((P^b)\) in the base plane. The images of \(P^i\) by \(\widetilde{SL}_2\mathbb{R}\) isometries are called infinite \(p\)-sided prisms.

The common part of \(P^i\) with the base plane is the base figure of \(P^i\) that is denoted by \(P\) and its vertices coincide with the vertices of \(P^b\), but \(P\) is not assumed to be a polygon.

**Definition 2.2.** A \(p\)-sided bounded prism is a solid where a \(p\)-sided infinite prism \(P^i\) is cut by its base figure \(P\) and the translated copy \(P^t\) of \(P\) by a fibre translation, given by (1.3) and so (1.5). The faces \(P\) and \(P^t\) are called cover faces.

**Definition 2.3.** The previous prisms are said to be regular if the side surfaces are congruent to each other under \(\widetilde{SL}_2\mathbb{R}\) rotations about the central fiber line (e.g. through the origin, Fig. 2).
Remark 2.1. 1. It is a natural assumption that the cover faces are derived as the images of the base plane under translation of the $\tilde{SL}_2\mathbb{R}$ space i.e. the cover faces lie in parallel Euclidean planes of the model.

2. It is clear that there exist regular $p$-gonal $\tilde{SL}_2\mathbb{R}$ prisms for $3 \leq p \in \mathbb{N}$.

A family of closed sets called tiles forms a tessellation or tiling of a space if their union is the whole space and every two distinct tiles in the family have disjoint interiors. A tiling is said to be monohedral if all of the tiles are congruent to each other. At present the space is $\tilde{SL}_2\mathbb{R}$ and the tiles are congruent regular infinite or bounded prisms. A tiling is called face-to-face if the intersection of any two tiles is either empty or a common vertex, edge or face of both tiles, respectively. Otherwise it is non-face-to-face.

2.1. Regular bounded prism tilings

In this work we examine the regular bounded prism tilings $T_p(q)$. Then each vertex of a tiling is a proper point of $\tilde{SL}_2\mathbb{R}$, and the prism is a topological polyhedron having at each vertex one $p$-gonal cover face (it is not a polygon at all in the usual sense) and two “skew” quadrangles. Moreover, at each side edge $q$ prisms meet regularly, i.e. by $q$-rotations, $3 \leq q$. We shall see in Theorem 2.1 that the regular prism tiling $T_p(q)$ exists and let $P_p(q)$ be one of its tiles whose $P$ (and so $P^b$ as well) is centered in the origin with vertices $A_1A_2\ldots A_p$ in the base plane. It is clear that the side curves $c_{A_iA_{i+1}} (i = 1\ldots p, A_{p+1} \equiv A_1)$ of the base figure are derived from each other.
by $\frac{2\pi}{p}$ rotation about the $x$ axis. The corresponding vertices $B_1B_2\ldots B_p$ are generated by a fibre translation $\tau$ given by (1.3) with a positive real parameter $\Phi$. The cover faces $A_1,\ldots ,A_p, B_1,\ldots ,B_p$ and the side surfaces form a $p$-sided regular prism $\mathcal{P}_p(q)$ in $\widetilde{\text{SL}_2\mathbb{R}}$. $\mathcal{T}_p(q)$ will be generated by its rotational isometry group $\Gamma_{p(q)} = pq^2 \subset \text{Isom}(\widetilde{\text{SL}_2\mathbb{R}})$ which is given by its fundamental domain $\mathcal{F} = A_1 A_2 O A_1^s A_2^s O^s , A_1^r = B_p, A_2^r = B_1, O^s = O^r$, a piece-wise linear topological polyhedron and the group presentation (see Fig. 3 for $p = 3$ ) by a standard procedure [1], called Poincaré algorithm as follows. The generators will pair the bent (piecewise linear) faces of $\mathcal{F}$:

\[ a : O A B_p O^s (O) \rightarrow O A^2 B_1 O^s (O), \]
\[ b : A_1 A_2 B_1 (A_1) \rightarrow A_1 B_p B_1 (A_1), \]
\[ s : O A_1 A_2 (O) \rightarrow O^s B_p B_1 (O^s) \]

mapping $\mathcal{F}$ onto its neighbours $\mathcal{F}^a, \mathcal{F}^b, \mathcal{F}^s$, respectively. E.g. for the face $a^{-1}$ a point $A$ (relatively freely, e.g. in the segment $OB_p$) is taken. Then the union of triangles $AO^a, AO^s B_p , AB_p A_1, O A_1 O$ will be the face $a^{-1}$. Then the $a$-image $A^a$ is taken in $OB_1$ for the face $a = A^a O O^a \cup A^a O^s B_1 \cup A^a B_1 A_2 \cup A^a A_2 O$, as usual. The relations are induced by the edge equivalence classes \{ $\{ 0 O' \}; \{ A_1 B_1 \}; \{ O A_1, O A_2, O'B_1, O'B_p \}; \{ A_1 A_2, A_1 B_p, A_2 B_1, B_p B_1 \}$ \}. So we get the group

\[ pq2_1 = \{ a, b, s : a^p = b^q = asa^{-1} s^{-1} = b a b^{-1} s^{-1} = 1 \} \]
\[ = \{ a, b : a^p = b^q = ababa^{-1}b^{-1} a^{-1} b^{-1} = 1 \}. \quad (2.1) \]

Here $a$ is a $p$-rotation by angle $\frac{2\pi}{p}$ about the fibre line through the origin ($x$ axis), $b$ is a $q$-rotation about the fibre line $A_1 B_1$ and $s = bab$ is a screw
motion. Then we get the second presentation in (2.1), i.e. \( abab = baba =: \tau \) is a fibre translation. Then \( ab \) is a \( 2_1 \) screw motion about a fibre line that determines the fibre translation \( \tau \). This group \( pq2_1 \) in (2.1) surprisingly occurred in §6 of our paper [5] at double link \( K_{p,q} \). The fibre lines through the vertices \( A_iB_i \) are denoted by \( f_i \), \( (i = 1, \ldots, p) \) and the fibre line through the “midpoint” \( H \) of the curve \( c_{\bar{A}_i\bar{A}_p} \) is denoted by \( f_0 \). This \( f_0 \) will be a half screw axis of \( ab \) (see later on).

The metric data of the fundamental domain \( F \) and the group \( pq2_1 \) will be determined in the following.

**Theorem 2.1.** In \( \widetilde{SL}_2\mathbb{R} \) there exist infinitely many regular \( p \)-gonal non-face-to-face prism tilings \( T_p(q) \) with group \( \Gamma_p(q) = pq2_1 \) for the integer parameters \( p, q \) where \( 3 \leq p, \frac{2p}{p-2} < q \). Namely, we have the characteristics as follows:

1. There are appropriate vertices \( A_1, A_2, \ldots, A_p \) of the base figure of prism \( \mathcal{P}_p(q) \), i.e. there is a parameter \( b = \tanh r \) in formulas (1.9′) so that the image \( A_b^2 \) at the \( q \)-gonal rotation \( b \) about fibre \( A_1B_1 \) will be \( B_p \) lying on the fibre line through \( A_p \).
2. There are appropriate side surfaces of prism \( \mathcal{P}_p(q) \) containing the corresponding side fibre lines, i.e. there is a side curve \( c_{A_1A_2} \) of the base figure between \( A_1 \) and \( A_2 \) whose image \( c_{A_1A_2}^b \) under rotation \( b \) lies on the side surface generated by the base side curve \( c_{A_1A_p} \).
3. The prism tilings \( T_p(q) \) are not face-to-face.

**Proof.** We have to prove the above three statements in i–iii, respectively:

(i.) We shall give the metric presentation of the group \( pq2_1 \) (see formulas in (2.1)). The matrix \( h_i^j \) of \( h = ab \) by (1.8) and (1.9′) will be in the sense of (1.6) as follows:

\[
\begin{align*}
    h_0^0 &= 1 + 2 \sinh^2 r \sin^2 \frac{\pi}{q}, \quad h_1^0 = 2 \sinh^2 r \sin \frac{\pi}{q} \cos \frac{\pi}{q}, \\
    h_2^0 &= 2 \cosh r \sinh r \sin^2 \frac{\pi}{q}, \quad h_3^0 = -2 \cosh r \sinh^2 r \sin \frac{\pi}{q} \cos \frac{\pi}{q}, \\
    h_0^1 &= -2 \cosh r \sinh r \sin \frac{\pi}{q} \sin \left( \frac{2\pi}{p} + \frac{\pi}{q} \right), \\
    h_0^2 &= -2 \cosh r \sinh r \sin \frac{\pi}{q} \sin \left( \frac{2\pi}{p} + \frac{\pi}{q} \right), \\
    h_0^3 &= \cos \frac{2\pi}{p} - 2 \cosh^2 r \sin \frac{\pi}{q} \sin \left( \frac{2\pi}{p} + \frac{\pi}{q} \right), \\
    h_2^1 &= \sin \frac{2\pi}{p} + 2 \cosh^2 r \sin \frac{\pi}{q} \cos \left( \frac{2\pi}{p} + \frac{\pi}{q} \right), \\
    h_2^2 &= \sin \frac{2\pi}{p} + 2 \cosh^2 r \sin \frac{\pi}{q} \cos \left( \frac{2\pi}{p} + \frac{\pi}{q} \right).
\end{align*}
\]
\[ h^2 = abab = baba = \tau \] above has to be of form (1.3). This yields

\[ \tanh^2 r = \frac{\cos \left( \frac{\pi}{p} + \frac{\pi}{q} \right)}{\cos \left( \frac{\pi}{p} - \frac{\pi}{q} \right)} \]

and the metric parameter \( b \) in the convenient form

\[ b = \tanh(OA_1) = \sqrt{\frac{1 - \tan \frac{\pi}{p} \tan \frac{\pi}{q}}{1 + \tan \frac{\pi}{p} \tan \frac{\pi}{q}}} = \sqrt{\frac{\cos \left( \frac{\pi}{p} + \frac{\pi}{q} \right) \cos \left( \frac{\pi}{p} - \frac{\pi}{q} \right)}{\cos \left( \frac{\pi}{p} - \frac{\pi}{q} \right)}}. \tag{2.2} \]

The above matrix of \( h = ab \) will be

\[
(h_i^j) = \begin{pmatrix}
C & S & Q \tan \frac{\pi}{q} & -Q \\
-S & -Q & -Q & -Q \tan \frac{\pi}{q} \\
-Q \cos \left( \frac{\pi}{p} + \frac{\pi}{q} \right) & -Q \cos \left( \frac{\pi}{p} + \frac{\pi}{q} \right) & \cos \frac{\pi}{q} & S \\
-Q \cos \left( \frac{\pi}{p} + \frac{\pi}{q} \right) & Q \sin \left( \frac{\pi}{p} + \frac{\pi}{q} \right) & \cos \frac{\pi}{q} & -\tan \frac{\pi}{q} C
\end{pmatrix}
\]

where \( Q := \sqrt{\cos \left( \frac{\pi}{p} + \frac{\pi}{q} \right) \cos \left( \frac{\pi}{p} - \frac{\pi}{q} \right)} \), \( C := \cos \left( \frac{\pi}{2} - \frac{\pi}{p} - \frac{\pi}{q} \right) \) and \( S := \sin \left( \frac{\pi}{2} - \frac{\pi}{p} - \frac{\pi}{q} \right) \).

Hence the translation \( \tau \) has the parameter by (1.3)

\[ 2\Phi = \pi - \frac{2\pi}{p} - \frac{2\pi}{q}. \]

(ii.) The invariant fibre line through \( H \) (Fig. 3.) of the half-screw \( h = ab \) is \( f_0 \), then the points \( C(\Theta) \) of the side curve \( c_{A_1 A_p} \) in the base figure \( P \) can be determined exactly. Namely, \( H^{abf} = H \) then

\[ C(\Theta)^{abf_0} = C(\Theta) \tag{2.3} \]

provides the polar coordinate equation for the radius \( c(\Theta) = OC(\Theta) \) and thus the complicated equation of \( c_{A_1 A_p} \) can be determined exactly for all possible parameters \( p, q \) (see Figs. 4, 5). For this the above matrix of \( h = ab \) can be used by (2.2) and (1.10). Moreover, \( \Theta \in \left[ -\frac{2\pi}{p}, 0 \right] \) can be assumed, \( f_0 \) is the half-turn about \( f_0 \) above (by (1.9)) and here \( f \) simply denotes the fibre projection into the base plan (by (1.10)).

(iii.) The image of the plane with equation \( x = k \) (for any fixed \( k \in \mathbb{R} \)) is invariant under rotations about the fibre line through the origin. Therefore, its image, at an arbitrary translation given by parameters \( (t^0, t^1, t^2, t^3) \) (see (1.7)), is an invariant plane under rotation \( \mathbb{R}_T(\omega) \) about
Figure 4. The simplified construction of the side surface $S_{A_1A_4}$ for the regular trigonal prism $P_3(7)$ by fibre projection of the line $A_1F$ through $f_0$ in (ii). $F$ is the midpoint of $A_3B_3$.

Figure 5. The construction of the side surface $S_{A_1A_4}$ for the regular 4-gonal prism $P_4(6)$.

the fibre line through the point $T(t^0; t^1; t^2; t^3)$ (see (1.9)). We get the next Lemma by (1.7).

Lemma 2.2. The rotation $R_T(\omega)$ leaves the plane $\mathcal{L}$ of equation (with inhomogeneous variables $x, y, z$) invariant

$$x(kt^1 - t^0) + y(t^3 - kt^2) - z(kt^3 + t^2) + t^0k + t^1 = 0 \quad (2.4)$$

for any fixed $k \in \mathbb{R}$.

It is clear, that the base plane and $\mathcal{L}$ (see (2.4)) are different planes if $T \neq O$, hence the third statement is proved.
Figure 6. Regular trigonal prism $P_3(7)$ ($A_1A_2A_3B_1B_2B_3$) with the base figure $A_1B_3A_3^p = A_1^pA_2^pA_3^p$ of its neighbouring prism.

Table 1. The values of the parameter $b$ for some $p = 3; 6 < q \in \mathbb{N}$ (see (2.2))

| (p, q) | b                |
|-------|------------------|
| (3,7) | $\approx 0.30007426$ |
| (3,8) | $\approx 0.40561640$ |
| (3,9) | $\approx 0.47611091$ |
| (3,10) | $\approx 0.52893551$ |
| (3,50) | $\approx 0.89636657$ |
| (3,1000) | $\approx 0.99457331$ |

Therefore, the above prism tilings $T_p(q)$ are not face-to-face (see Fig. 6). Theorem 2.1 has been proved.

For example, Figs. 3, 4 show the simplified construction of the side surfaces of $P_3(7)$ with its base polygon. The equation of the curve $c_{A_1A_3}$ of $P_3(7)$ can be determined as the arguments in (ii) show it. The half-screw $ab = h$ has the invariant fibre line $f_0$ with footpoint $H$. Then the midpoint $F$ of the fibre $A_3B_3$ determines the straight line $A_1F$ through $f_0$ on the side surface of $P_3(7)$. See Fig. 4 where $A_p = A_3 \sim (1; 0; b \cos \frac{-7\pi}{3}; b \sin \frac{-7\pi}{3})$ with $b \approx 0.30007426$ (from Table 1), $B_p = B_3 \sim (1; 0.150725; -0.189628; -0.237786)$ and $F \sim (1; 0.074940; -0.169890; -0.249182)$. (arctan(0.074940) $\approx \frac{\pi}{3} - \frac{\pi}{7}$). The equations of the other side curves $c_{A_iA_{i+1}}$ ($i = 2, 3$, $A_4 \equiv A_1$) of the base figure are derived from the equation of $c_{A_1A_3}$ by $\frac{2\pi}{3}$ rotation about the $x$ axis (see Figs. 3, 6). The data of $P_3(q)$ for some $6 < q \in \mathbb{N}$ are collected in Table 1. From the above arguments we can formulate the following

Lemma 2.3. Any side surface of the above regular prism $P_p(q)$ is a part of a one sheeted hyperboloid.

Our computer figures and animations nicely show these (e.g. Figs. 2, 7).
2.1.1. Regular infinite prism tilings. In this section we study the regular infinite prism tilings $\mathcal{T}_i^p(q)$. We consider the regular bounded prism tiling $\mathcal{T}_p(q)$ which exists for parameters $p, q$ above. Let $\mathcal{P}_p(q)$ be one of its tiles whose $\mathcal{P}$ (and so $\mathcal{P}^b$ as well) is centered at the origin with vertices $A_1A_2A_3\ldots A_p$ in the base plane. The corresponding vertices $B_1B_2B_3\ldots B_p$ are generated by a fibre translation $\tau$ given by (1.3) with parameter $2\Phi = \pi - \frac{2\pi}{p} - \frac{2\pi}{q}$. The images of the topological polyhedron $\mathcal{P}_p(q)$ by the translations $\langle \tau \rangle$ form an infinite prism $\mathcal{P}_i^p(q)$ (see Definitions 2.1, 2.2, 2.3).

From the construction of the bounded prism tilings it follows that rotations through $\omega = \frac{2\pi}{q}$ about the fibre lines $f_i$ maps the corresponding side face onto the neighbouring one.

Therefore, we get the following:

**Theorem 2.4.** There exist regular infinite face-to-face prism tilings $\mathcal{T}_i^p(q)$ for integers $p, q$ where $3 \leq p, \frac{2p}{p-2} < q$.

For example, we have described $\mathcal{P}_i^4(6)$ with its base polygon in Fig. 7, where the parameter $b = \frac{\sqrt{6} - \sqrt{2}}{2}$.

It is interesting to consider further tilings in 3-dimensional Thurston geometries, because important information (can be relevant to the Euclidean crystallography as well) of the “crystal structures” is included by the “space filling polyhedra” (see e.g. [4, 8, 9, 12, 13]).
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