Genuine multipartite correlations distribution in the criticality of the Lipkin-Meshkov-Glick model

Antônio C. Lourenço, Susane Calegari, Thiago O. Maciel, Tiago Debarba, Gabriel T. Landi, and Eduardo I. Duzzioni

1Departamento de Física, Universidade Federal de Santa Catarina, CEP 88040-900, Florianópolis, Santa Catarina, Brazil
2Departamento Acadêmico de Ciências da Natureza, Universidade Tecnológica Federal do Paraná (UTFPR), Campus Cornélio Procópio, Avenida Alberto Carazzai 1640, Cornélio Procópio, Paraná 86300-000, Brazil.
3Instituto de Física, Universidade de São Paulo, CEP 05314-970, São Paulo, São Paulo, Brazil

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I. INTRODUCTION

Phase transitions emerge from the complex correlations developed between the microscopic constituents. Understanding and characterizing these correlations has, therefore, always been a central problem in statistical physics. This is particularly more so for quantum phase transitions, for which one may employ concepts from quantum information theory. For example, the divergence of the entanglement entropy as one crosses the critical point is related to the underlying conformal theory that dictates the universal properties of a quantum phase transition [1]. This has led to several studies aimed at characterizing entanglement in a variety of different critical systems [2–6], including the first direct experimental measurement of the entanglement entropy in a superfluid [7].

Most of our present knowledge on this subject, however, is restricted to bipartite correlations. The extension to a multipartite scenario is highly nontrivial, for two main reasons. The first is related to the factorial large number of partitions that one can divide a system comprised of \( N \) parties, making the problem difficult to analyze. The second is related to the difficulties in constructing measures of genuine multipartite correlations (GMC) [8]. In a system with \( N \) parts, a genuine correlation of order \( k \leq N \) represents the total amount of correlations that cannot be obtained from clusters of size smaller than \( k \).

Hence, GMC should be able to quantify what part of the total correlations is distributed between clusters of different sizes. This has applications in, e.g., dimerization in aperiodic spin chains [9] or the formation of strings in the Fermi-Hubbard model [10]. GMC can therefore be a valuable tool in our understanding of quantum criticality.

The characterization of multipartite correlations in quantum critical systems has thus far focused almost exclusively in multipartite entanglement, which has been explored in a variety of models [1,11–14]. The current available measures of multipartite entanglement, however, are either ill defined or too complex to be computed [15] (for a review on approaches to characterize multipartite entanglement in many-body systems, see Ref. [16]). For this reason, such studies still remain scarce.

More recently, Girolami et al. have put forth a formalism for computing GMC which relies only on knowledge of the quantum relative entropy (Kullback-Leibler divergence) [17]. The formalism accounts for both quantum and classical correlations and is based on general distance-based concepts, formalized in Ref. [18], thus making it much more tractable. This framework has since been applied to GHZ [19] as well as Dicke states [20].

In this work, we calculate genuine \( k \)-partite correlations in the ground state of the Lipkin-Meshkov-Glick (LMG) model by the framework presented in Ref. [17]. Also, we show how is the distribution of correlations for a system of many particles and that these correlations signal the already known second order quantum phase transition (QPT). In addition, we use a method of finite-size scaling (FSS) to find the critical exponent of some orders \( k \) of correlations with emphasis on the total correlation, the bipartite correlation, and the tripartite correlation.

This work is organized as follows: In Sec. II is presented the materials and methods, where we introduce the measures used to calculate the genuine \( k \)-partite correlations, the LMG model, its QPT, and the FSS theory. The analysis of the genuine \( k \)-partite correlations, the verification that all orders of \( k \) signal the second order QPT, and the achievement of the critical exponents via FSS for total correlation, bipartite correlation, and tripartite correlation, are presented in
Sec. III. Finally, the conclusions and perspectives are left for Sec. IV.

II. MATERIALS AND METHODS

A. Measures of genuine k-partite correlations

Consider an N-partite system described by the density matrix \( \rho_N \in D({\mathcal H}_N = \mathcal{H}_{1} \otimes \ldots \otimes \mathcal{H}_{N}) \), where each partition \( \rho_j = tr_{N\setminus j}(\rho_N) \) is the state of the subsystem \( j \), where \( tr_{N\setminus j} \) indicates the trace over all partitions except \( j \), such that \( \rho_j \in D(\mathcal{H}_j) \). It is important to emphasize that each partitioning \( \mathcal{H}_{ij} \) can also be a multipartite system, indeed the number of subpartitions in each subsystem will be useful to define the genuine correlations. Now, let us consider that the system has \( m \) partitions \( \{ \mathcal{H}_j \}_{j=1}^{m} \) and \( k \) of them number the partitions in each subsystem

\[
S_k = \{ S_{[1]}, \ldots, S_{[k]} \}, \quad k \leq k,
\]

such that \( \sum_{j=1}^{m} k_j = N \). In our case each \( S_{ij} \) is a qubit system.

One can define the set of genuine uncorrelated states for a given order higher than \( k \). In this way, it is possible to define a specific partitioning considering an integer number \( 2 \leq k \leq N \) and the coarse grained partitioning \( \{ \mathcal{H}_k \otimes \ldots \otimes \mathcal{H}_k \} \), where each cluster \( \mathcal{H}_k \) includes at most \( k \) subsystems [17].

Definition 1 (k-partite genuine product states). It is defined a set of states that has up to \( k \) subsystems as

\[
P_k := \{ \sigma_N = \bigotimes_{i=1}^{m} \otimes_{j=1}^{k} \rho_{kj} \}, \quad k \leq k,
\]

where \( \sigma_k \) is a subsystem of \( k \) particles, this set contains all the sets \( P_k \) with \( k' \leq k \), such that \( P_1 \subseteq P_2 \cdots \subseteq P_{N-1} \subseteq P_N \).

In order to calculate the GMC of order higher than \( k \), it is used the relative entropy as a pseudodistance

\[
S_k^{k\rightarrow N}(\rho_N) = \min_{\sigma_N} S(\rho_N || \sigma),
\]

where the minimization is taken over all product states \( \sigma = \bigotimes_{i=1}^{m} \otimes_{j=1}^{k} \rho_{kj} \in P_k \). The state \( \sigma \) that minimizes \( S_k^{k\rightarrow N}(\rho_N) \) will be the product of the reduced states of \( \rho_N \) [17,18,21]. Therefore

\[
S_k^{k\rightarrow N}(\rho_N) = S(\rho_N || \bigotimes_{i=1}^{m} \otimes_{j=1}^{k} \rho_{kj}) = \sum_{i=1}^{m} S(\rho_{ki}) - S(\rho_N).
\]

For states with permutation symmetry Eq. (4) can be simply written as

\[
S_k^{k\rightarrow N}(\rho_N) = \lfloor N/k \rfloor S(\rho_k) + (1 - \delta_N \mod k,0)S(\rho_N \mod k) - S(\rho_N),
\]

where \( \lfloor N/k \rfloor \) is the floor function, which is the greatest integer less than or equal to \( N/k \). The \( \rho_N \mod k \) describe the subsystem \( N \mod k \). If we choose \( k = 1 \),

\[
S_1^{1\rightarrow N}(\rho_N) = N S(\rho_1) - S(\rho_N)
\]

describes the total correlations presented in the system.

The genuine k-partite correlations can be defined as the difference between the correlations of order higher than \( k - 1 \rightarrow N \) and those of order higher than \( k \rightarrow N \)

\[
S_k^{k\rightarrow N}(\rho_N) = S^{k-1\rightarrow N}(\rho_N) - S^{k\rightarrow N}(\rho_N).
\]

Once the correlations of order higher than \( k - 1 \) encapsulates those ones of order higher than \( k \), the difference between them returns only the genuine k-partite correlations. A nice interpretation of the GMC of order \( k \) introduced above is that the sum of all GMC gives the total correlation in the system, \( S^{1\rightarrow N}(\rho_N) = \sum_{k=2}^{N} S_k^{k\rightarrow N}(\rho_N) \), as can be verified from Eq. (8).

B. Lipkin-Meshkov-Glick model

The LMG model, as studied here, is a system composed of \( N \) spins 1/2 fully connected with anisotropy controllable by the parameter \( \gamma \) and an external transversal magnetic field \( h \) acting on it. This model has first and second order QPTs depending on the values of the control parameters \( h \) and \( \gamma \). Here, we limit our analysis to the cases in which \( \gamma = 0.5 \) and the external field varies in the range \( 0 \leq h \leq 2 \). The critical point at \( h = 1 \) signals a second order QPT. Such phase transition has already been studied according to the quantum information theory in Refs. [4–6,22–24]. The Hamiltonian of the LMG is described by [4–6]

\[
H = -\frac{\lambda}{N} \sum_{i<j} (\sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y) - h \sum_{i=1}^{N} \sigma_i^z,
\]

where \( \lambda \) is the ferromagnetic coupling factor, \( \lambda = 1 \) was chosen for the sake of simplicity, and \( \sigma_i^x, \sigma_i^y, \sigma_i^z \) are Pauli matrices with \( a = x, y, z \). Employing \( J_0 = \sum_{i=1}^{N} \sigma_i^a \), the Hamiltonian can be represented by collective spin operators as

\[
H = -\frac{\lambda}{N} (1 + \gamma) (J_x^2 + J_z^2) - 2hJ_z - \frac{\lambda}{2N} ((1 - \gamma) (J_x^2 + J_z^2)),
\]

with \( J \) being the total collective angular momentum, \( J_z \) is its projection along the \( z \) direction and \( J_0 \) and \( J_x \) are collective ladder operators of lowering and raising, respectively. The ground state of the LMG model is a linear combination of Dicke states, which are eigenstates of \( J^2 \) and \( J_z \),

\[
J^2 |J, M \rangle = J(J+1) |J, M \rangle, \quad J_z |J, M \rangle = M |J, M \rangle,
\]

where \( J = N/2 \) and \( M = -N/2, -N/2 + 1, \ldots, N/2 - 1, N/2 \). Instead of \( |J, M \rangle \), it will be used the following representation of Dicke states \( |N, n_e \rangle \), in which \( n_e \) is the number of excited spins. The Dicke states are totally symmetric by permutation of particles and can be represented by

\[
|N, n_e \rangle = \frac{1}{\sqrt{\binom{N}{n_e}}} \sum_{i} \mathcal{P}_i |0 \rangle^{N-n_e} \otimes |1 \rangle^{n_e}.
\]

The sum is taken over all possible permutations of \( n_e \) described by the permutation operator \( \mathcal{P}_i \) and \( \binom{N}{n_e} \) is the binomial coefficient required to normalize the Dicke state. Therefore, the ground state of the LMG model is

\[
|\Psi \rangle = \sum_{n_e=0}^{N} P_{n_e} |N, n_e \rangle,
\]
with $P_{n_e}$ being the amplitudes of probability of occurrence of a Dicke state with $n_e$ excitations, which ones are obtained from numerical diagonalization of the Hamiltonian in Eq. (10).

Quantum phase transitions of the LMG model

In Ref. [5], where by mean field approximation was determined the phase diagram of the LMG model, the authors show that for the region of $0 \leq h \leq 1$ (broken phase), the ground state of the system is double degenerated for $\gamma \neq 1$, for $\gamma = 1.0$ the ground state is infinitely degenerated, and for $1 < h \leq 2$ symmetric phase the ground state is unique for all $\gamma$. The second order QPT in the LMG model occurs due to the competition between the spins interaction and the effect of the external field $h$ applied over the spin chain. When the external field is strong enough ($h \geq 1$), all the spins begin to align with it so that the state of the system in this phase has no correlations between the spins. For $h = 0$ and $\gamma = 0$ the ground state of the system is described by a GHZ-like state [6]. Even though we are using the anisotropy parameter $\gamma = 0.5$ for all of our calculations, the ground state is still an approximation of the GHZ-like state, so the spins are correlated. Some previous works in quantum information theory used entanglement [4–6,22–27] and others correlations [28–34] as order parameter to detect the second order phase transition of this model. Furthermore, there are also some previous works in quantum information that make use of FSS [4,5,27,29–31,33,34] for the calculus of exponents in some previous works in quantum information that make use of FSS [4,5,27,29–31,33,34] for the calculus of exponents in the LMG model.

Since the LMG model is an infinity coordinated system, because all particles interact with all others equally, the system does not have the concepts of length and dimensionality defined [35], such as made to study finite size scaling in others systems. Then, the number of particles $N$ is the only one variable in the analysis of FSS exponents. Here, we analyze the behavior of the genuine $k$-partite correlations near the second order QPT.

The method to extract the exponent that we apply is the following: we take the minimum derivative of $k$-partite correlation and calculate the $k$-partite correlation at this point as function of $N$, thus assuming that relation of correlation and $N$ obeys a power law, we take the logarithm of both variables and we are able to get the exponent relate to the genuine $k$-partite correlation.

III. RESULTS

In this section we calculate numerically the genuine $k$-partite correlations across the QPT for some values of $k$. By induction, we conclude that all orders of the genuine multipartite correlations signal the second order QPT at $h = 1$. Following the same reasoning, we calculate critical exponents for some genuine $k$-partite correlations and evidence that all exponents are in the range $[-1/2, 1/2]$.

From Eq. (6), we notice that to compute the multipartite correlations of order higher than $k$, $S^k \rightarrow N(\rho_k)$, it is necessary to calculate the reduced density matrix $\rho_k = \text{Tr}_{N/k} \rho_N$ from $\rho_N$. For the ground state, which one is a superposition of Dicke states, as in Eq. (13), we can do the Schmidt decomposition

of the Dicke states [36,37] as

$$|\Psi\rangle = \sum_{n_e=0}^{N} P_{n_e} \sum_{L=0}^{n_e} \lambda_{L} |L, L_e \rangle \otimes |N-L, n_e - L_e \rangle,$$  

where $\lambda_{L} = \sqrt{\binom{N-L}{L} \binom{N}{n_e-L_e}}$, getting the reduced density matrix from this state and tracing out $|L_e, L \rangle$ we obtain the reduced density matrix of $k$ spins

$$\rho_k = \sum_{n'_e, n_e=0}^{N} \sum_{L=0}^{n_e} \left[ \frac{\binom{L}{L_e} \binom{N-L}{n'_e-L_e} \binom{N}{n'_e} \binom{N}{n_e}}{\sqrt{\binom{N}{n'_e-L_e} \binom{N}{n'_e}} \sqrt{\binom{N-L}{n_e-L_e} \binom{N}{n_e-L_e}}} \right] P_{n'_e} P_{n_e}^{*} |k, n'_e - L_e \rangle \langle k, n'_e - L_e|,$$

where $0 \leq n'_e - L_e \leq N - L = k$ and $0 \leq n'_e - L_e \leq N - L = k$. The coefficients $P_{n_e}$ depend on the ground state of the LMG Hamiltonian in Eq. (10).

A. Genuine $k$-partite correlations in the ground state of the LMG model

We recall that the anisotropy parameter is fixed as $\gamma = 0.5$ in all numerical analysis of the GMC of order $k$ presented in the ground state of the LMG model. In Fig. 1 and inset it is possible to observe the behavior of the GMC of orders $k = 2, 3, 4, 13$ for $N = 156$ spins. For $h = 0$ the ground state of the LMG model is approximately a GHZ state [6], which implies that the GMC can be described approximately by the expression $S^k = \lfloor N/(k-1) \rfloor - \lceil N/k \rceil$. Also, as $k$ increases the genuine $k$-partite correlations decrease and become more resistant to variations of the magnetic field. Such a tendency also appears in Ref. [6] for the calculus of entanglement.

FIG. 1. Genuine $k$-partite correlations presented in the ground state of the LMG model for $N = 156$, $\gamma = 0.5$, and $k = 2, 3, 4, 13$ as function of $h$. In the phase in which $h > 1$ the spins are aligned to the external field so that they are not correlated. The inset is a zoom of the figure to a better visualization of the behavior of the GMC of order $k = 13$ near the phase transition.

|\Psi\rangle = \sum_{n_e=0}^{N} P_{n_e} \sum_{L=0}^{n_e} \lambda_{L} |L, L_e \rangle \otimes |N-L, n_e - L_e \rangle,$$

where $\lambda_{L} = \sqrt{\binom{N-L}{L} \binom{N}{n_e-L_e}}$, getting the reduced density matrix from this state and tracing out $|L_e, L \rangle$ we obtain the reduced density matrix of $k$ spins

$$\rho_k = \sum_{n'_e, n_e=0}^{N} \sum_{L=0}^{n_e} \left[ \frac{\binom{L}{L_e} \binom{N-L}{n'_e-L_e} \binom{N}{n'_e} \binom{N}{n_e}}{\sqrt{\binom{N}{n'_e-L_e} \binom{N}{n'_e}} \sqrt{\binom{N-L}{n_e-L_e} \binom{N}{n_e-L_e}}} \right] P_{n'_e} P_{n_e}^{*} |k, n'_e - L_e \rangle \langle k, n'_e - L_e|,$$

where $0 \leq n'_e - L_e \leq N - L = k$ and $0 \leq n'_e - L_e \leq N - L = k$. The coefficients $P_{n_e}$ depend on the ground state of the LMG Hamiltonian in Eq. (10).
to the behavior of the distance measure second order QPT in the LMG model, we call the attention Figure 2(a) shows the distance in Eq. (6), or equivalently, there is an unpaired block of for \( N \) and its connection to the total number of spins \( N \). Figure 2(a) shows the distance \( S^{k \rightarrow N} \) for \( N = 200 \) and \( h = 1 \) as function of \( k \). The distance is monotonically decreasing with the block size \( k \). For small values of \( k \) the decreasing is smooth, but as \( k \) becomes comparable to \( N \) abrupt changes occur in form of a ladder. Such an effect was already verified for Dicke states in Ref. [20] and comes from the floor function \( \mod N = k \). As can be seen, this is enough to remove the role played by the floor function and consequently removing the ladder behavior. This kind of imposition allow us to get the critical exponents straightforwardly, since the unpaired blocks cause abrupt changes in the GMC of order \( k \).

B. Quantum phase transition and genuine \( k \)-partite correlations

Our first task is to show that all GMC of order \( k \) \( (1 \leq k \leq N) \) are able to signal the second order QPT. For the sake of simplicity we included the value \( k = 1 \), which means the total correlations, see Eq. (7). In order to verify that the GMC signal the QPT we calculate the first derivative of \( S^k \) with respect to the parameter \( h \) and show that in the thermodynamic limit it is nonanalytic for some value of \( h \), named critical parameter \( h_c \). This behavior is shown in Fig. 3, where we conclude that the minimum value of \( dS^k/dh \) (for \( k = 1, 2, 3, N \)) occurs for some value of the control parameter \( h_{\text{min}} \) which tends to \( h_c = 1 \) for \( N \rightarrow \infty \).

This result is in agreement with Fig. 1 inset, where the GMC of higher orders disappear faster after the phase transition \( h_c = 1 \) and with Fig. 4 which explores the thermodynamic limit. We observe that the same procedure has been performed for other values of \( k \) not reported here, but which ones corroborate the conclusion presented above. Therefore, the GMC of order \( k \) are able to signal the already known second order QPT in the LMG model [5].

FIG. 2. The GMC of order higher than \( k \), \( S^{k \rightarrow N} \), as function of \( k \) for \( N = 200, \gamma = 0.5, \) and \( h = 1 \). In (a) all integers values of \( k \) from \( 1 \leq k \leq N \) are considered, while in (b) only the values of \( k \) which satisfy the constraint \( N \mod k = 0 \) are taken into account.

FIG. 3. The first order derivative of the total correlations \((k = 1)\) and GMC of orders \( k = 2, 3, N \) as function of the external control parameter \( h \). The minimum value of the derivatives occurs for values of \( h \) closer to \( h_c = 1 \) as \( N \) increase.

FIG. 4. Values of the control parameter \( h \) in which the first order derivative of the total correlations and GMC of orders \( k = 2, 3, N \) are minimum as function of the number of spins in the system \( N \). The solid lines are logarithm fit of the points to show that in the thermodynamic limit \((N \rightarrow \infty h_{\text{min}}) \rightarrow h_c = 1 \). The constraint \( N \mod 6 = 0 \) has been imposed for the curves with \( k = 2, 3 \).
CONCLUSIONS AND PERSPECTIVES

We analysed the genuine multipartite correlations in the Lipkin-Meshkov-Glick model according to the measure introduced by Girolami and coworkers [17]. Within this framework we were able to calculate the genuine $k$-partite correlations and show they behave for different partition sizes. Also, we verify that the genuine $k$-partite correlations signal the second order quantum phase transition in the LMG model. Furthermore, we obtained the critical exponents through finite-size scaling analysis for the total correlations and genuine multipartite correlations of order $k = 2, 4, 5, N/4, N/2, N$. From that we observed that the genuine multipartite correlations go to zero across the second order quantum phase transition in the thermodynamic limit when the partition size $k$ increases with the number of particles of the system.

As perspective for future works, it would be interesting to certify if the critical exponents of the genuine multipartite correlations of order $k$ are confined in the interval $[−1/2, 1/2]$ for all $k$. Likewise, the analysis of weaving, a measure of correlation also proposed by Girolami et al. [17] which is the sum of all genuine multipartite correlations, to see if it is possible to gain additional information on the scalability of the genuine multipartite correlations across the quantum phase transition.

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| Table I. Critical exponents for different values of $k$ of the GMC $S^k$ vs $\ell$ across the second order QPT in the LMG model. |
| $k$ | Critical exponent |
|-----|------------------|
| 1   | $0.508 \pm 0.001$ |
| 2   | $0.313 \pm 0.001$ |
| 3   | $0.317 \pm 0.002$ |
| 4   | $0.333 \pm 0.004$ |
| 5   | $0.350 \pm 0.003$ |
| $N/4$ | $-0.377 \pm 0.002$ |
| $N/2$ | $-0.4540 \pm 0.0007$ |
| $N$ | $-0.492 \pm 0.003$ |

critical exponent becomes negative. Also, the GMC across the quantum phase transition diminish faster for higher orders of $k$ when it depends on $N$. In the particular case of the GMC of order $N$ it goes to zero at a rate greater than for the other exponents. If we analyze the GMC of order $k$ per particle in the thermodynamic limit across the second order QPT, $\lim_{N \to \infty} S^k(\rho_N)/N = 0$, it becomes null, as expected for the classical world.
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