HEAT KERNEL COEFFICIENTS FOR TWO-DIMENSIONAL SCHröDINGER OPERATORS

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ABSTRACT. In this note, we compute the Hadamard coefficients of (algebraically) integrable Schrödinger operators in two dimensions. These operators first appeared in [BL] and [B] in connection with Huygens’ principle, and our result completes, in a sense, the investigation initiated in those papers.

Let $L = -\Delta_n + V$ be a Schrödinger operator on $\mathbb{R}^n$ ($n \geq 1$) with $C^\infty$-smooth potential $V = V(x)$ defined in some open domain $\Omega \subseteq \mathbb{R}^n$. Recall that the heat kernel of $L$ is the solution $\Phi_+(x, \xi, t) \in C^\infty(\Omega \times \Omega \times \mathbb{R}_+^1)$ of the initial value problem

$$\frac{\partial}{\partial t} \Phi_+(x, \xi, t) + L \Phi_+(x, \xi, t) = 0, \quad \lim_{t \to 0^+} \Phi_+(x, \xi, t) = \delta(x - \xi),$$

where $\delta(x - \xi)$ is the Dirac delta-function on $\Omega$ with support at $\xi$. It is well known (see, e.g., [R], Sect. 3.2.1) that $\Phi_+(x, \xi, t)$ has an asymptotic expansion of the form

$$\Phi_+(x, \xi, t) \sim e^{-|x-\xi|^2/4t} \left(1 + \sum_{\nu=1}^{\infty} U_\nu(x, \xi) t^\nu\right)$$

as $t \to 0^+$, with coefficients $U_\nu(x, \xi) \in C^\infty(\Omega \times \Omega)$ determined by the following transport equations

$$(x - \xi, \partial_x) U_\nu(x, \xi) + \nu U_\nu(x, \xi) = -L[U_{\nu-1}(\cdot, \xi)](x), \quad \nu = 1, 2, \ldots$$

The system (3) has a unique solution $\{U_\nu(x, \xi)\}_{\nu=0}^\infty$ if one sets $U_0(x, \xi) \equiv 1$ and requires each $U_\nu(x, \xi)$ to be bounded in a neighborhood of the diagonal $x = \xi$. Following [G], we will refer to $\{U_\nu(x, \xi)\}$ as the Hadamard coefficients of the operator $L$.

In general, calculating the Hadamard coefficients for a given potential $V$ is a difficult problem. Of special interest are potentials, for which the heat kernel expansion (2) is finite, i.e. the sum in the right-hand side of (2) has only finitely many nonzero terms. In this case, formula (2) yields not only a “short-time” asymptotics, but an exact analytic representation for $\Phi_+(x, \xi, t)$ valid for all $t \in \mathbb{R}_+^1$.

Such potentials are usually called Huygens’ potentials in view of an important role they play in the theory of Huygens’ principle (see, e.g., [BV], Ch. 1).

The problem of describing all Huygens potentials goes back to Hadamard’s classical treatise [H] and still remains open in all dimensions, except for $n = 1$ (see [L]) and $n = 2$ (see [I]). In dimension one, these potentials coincide with the well-known Adler-Moser potentials [AM], which are the rational solutions of the Korteweg-de Vries hierarchy of nonlinear integrable PDE’s (see [S] or [BV], Ch. 3, Sect. 3.2); apart from the original works [LS] and [L], their Hadamard coefficients have been studied recently in [Gr], [I], [I2] and [Ha].

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In the present paper, we will deal with the two-dimensional Huygens potentials, which have received so far much less attention (see, however, [CFV]). Our main result (Theorem 1) provides simple and explicit formulas for the Hadamard coefficients of the corresponding Schrödinger operators. It is surprising that these formulas do not seem to have analogues in dimension one: in a sense, the two-dimensional case is simpler than the one-dimensional one!

We begin by recalling the main result of [BL] and [B]. Let \( k = (k_0, k_1, \ldots, k_m) \) be a finite, strictly increasing sequence of integers, with \( k_0 = 0 \), and let \( \varphi_i \) be real numbers given one for each integer \( k_i \), with \( \varphi_0 = 0 \). Passing to the polar coordinates \( (x_1, x_2) = (r \cos \varphi, r \sin \varphi) \) in \( \mathbb{R}^2 \), we associate to these data the following potential

\[
V_k(x_1, x_2) = -\frac{2}{r^2} \frac{\partial^2}{\partial \varphi^2} \log \text{Wr}[\chi_0, \chi_1, \ldots, \chi_m],
\]

where \( \chi_i := \cos(k_i \varphi + \varphi_i), \ i = 0, 1, \ldots, m, \) and \( \text{Wr}[\chi_0, \chi_1, \ldots, \chi_m] \) is the Wronskian of the set \( \{\chi_i\} \) taken with respect to the variable \( \varphi \). As all \( k_i \)'s are integers, \( V_k \) is a single-valued rational function on \( \mathbb{R}^2 \), homogeneous of degree \(-2\), whose analytic continuation to \( \mathbb{C}^2 \) has singularities along certain lines passing through the origin. For example, for \( k = (0, 1, 3, 4) \), with all \( \varphi_i \)'s being 0, we have

\[
V_k(x_1, x_2) = \frac{12 (49 x_1^4 + 28 x_1^2 x_2^2 - x_2^4)}{x_2^4 (7 x_1^2 + x_2^2)^2}.
\]

In general, \( V_k \) depends on both the choice of \( k_i \)'s and the choice of \( \varphi_i \)'s, though we suppressed the latter from our notation.

**Theorem 1** ([BL] [B]). A (locally) smooth function \( V \) on \( \mathbb{R}^2 \), which is homogeneous of degree \(-2\), is a Huygens potential if and only if \( V = V_k(x_1, x_2) \) for some integer sequence \( k = (k_i) \) and real numbers \( \varphi_i \).

**Remark.** The “if” part of Theorem 1 was first proven in [BL] and then reproven by a different method, together with the “only if” part, in [B] (see loc. cit., Theorem 1.1). The assumption that \( V \) is homogeneous can be relaxed and replaced by a weaker condition that \( V \) and all the Hadamard coefficients of \( V \) are algebraic functions (see [CFV]).

The main result of this paper can now be encapsulated in

**Theorem 2.** The Hadamard coefficients of \( \mathcal{L} = -\Delta_2 + V_k \) with potential \( \Psi \) are given, in terms of polar coordinates \( x = (r \cos \varphi, r \sin \varphi) \) and \( \xi = (q \cos \phi, q \sin \phi) \), by

\[
U_\nu(x, \xi) = \frac{(-2)^\nu}{(r q)^\nu} \sum_{i=0}^m c_i \Psi_i(\varphi) \Psi_i(\phi) T^{(\nu)}_{k_i}(\cos(\varphi - \phi)), \quad \forall \nu \geq 0,
\]

where

\[
\Psi_i := \frac{\text{Wr}[\chi_0, \chi_1, \ldots, \chi_{i-1}, \chi_{i+1}, \ldots, \chi_m]}{\text{Wr}[\chi_0, \chi_1, \ldots, \chi_m]}, \quad c_i := \prod_{j=0 \atop j \neq i}^m (k_i^2 - k_j^2),
\]

and \( T_N(z) := \cos(N \arccos z) \) is the \( N \)-th Chebyshev polynomial, with \( T^{(\nu)}_N(z) \) being its derivative of order \( \nu \) with respect to \( z \).
Remark. It follows immediately from (3) that \( U_{\nu}(x, \xi) \equiv 0 \) for \( \nu > k_m \), implying that \( V_k \) is a Huygens potential. For \( k \) of length one, i.e. \( k = (0, N) \), formula (4) yields the Calogero-Moser potential of dihedral type \( I_2(N) \), and in this special case the coefficients \( \delta \) have already been found in [1] (see loc. cit., Sect. IV, Example 1).

There are several ways to prove Theorem 2. Perhaps, the most straightforward one is to use a differential recurrence relation between the Hadamard coefficients of operators \( \mathcal{L}_k \) and \( \mathcal{L}_{k'} \), where \( k = (k, k_{m+1}) \) is obtained by adding one integer on top of \( k \) (see [1], (8)). This method requires double induction (in \( m \) and \( \nu \)) and leads to rather unwieldy calculations.

Here, we will offer a more illuminating argument based on the remarkable fact that the coefficients \( U_{\nu}(x, \xi) \) appear not only in fundamental solutions of the heat equation but also in its elliptic and hyperbolic counterparts. Instead of the Cauchy problem (1), we will consider

\[
\mathcal{L}'(G(x, \xi))(x) = \delta(x - \xi) \quad \text{with} \quad G(x, \xi) \in \mathcal{D}'(\Omega) ,
\]

where \( \mathcal{D}'(\Omega) \) denotes the space of distributions on \( C^\infty \)-functions with compact support in \( \Omega \). Of course, unlike the heat kernel, \( G(x, \xi) \) is not uniquely determined by (7), but only up to adding smooth functions from \( \ker(\mathcal{L}) \). The problem is now to describe the singularities of \( G(x, \xi) \). In modern language, the solution to this classical problem is given in terms of Riesz distributions, and as in the hyperbolic case (see, e.g., [1]), it depends on whether \( n \) is even or odd. Specifically, if \( n \geq 2 \) is even, \( G(x, \xi) \) has the following asymptotics “in smoothness” (cf. [1], Sect. 3, (1.4)):

\[
G(x, \xi) \sim \sum_{\nu = 0}^{\infty} \tilde{U}_{\nu}(x, \xi) S_{\nu}' \frac{\gamma^n}{n!} \quad \text{for all} \quad \nu \geq 0 ,
\]

where \( \gamma = |x - \xi|^2 \) is the square of the Euclidean distance between \( x \) and \( \xi \), \( S_{\lambda}(t) := t_{+}^{\lambda+1}/\Gamma(\lambda+1) \) is the family of Riemann-Liouville distributions on \( \mathbb{R}^1 \), depending analytically on the parameter \( \lambda \in \mathbb{C} \), and \( S_{\lambda}' := dS_{\lambda}/d\lambda \) are the adjoint distributions of \( S_{\lambda} \) with respect to \( \lambda \) (see [2], Ch. VI, Sect. 2). The coefficients \( \tilde{U}_{\nu}(x, \xi) \) in (8) are smooth functions, which satisfy, up to rescaling factor \( -1/4 \), the same transport equations as the heat kernel coefficients \( U_{\nu}(x, \xi) \). Like in the heat kernel case, this can be verified by direct calculation, substituting (8) into (7). By uniqueness of the regular solution of (6), we thus have

\[
\tilde{U}_{\nu}(x, \xi) = \left( -\frac{1}{4} \right)^{\nu} U_{\nu}(x, \xi) \quad \text{for all} \quad \nu \geq 0 .
\]

Now, let \( n = 2 \) and assume that \( V = V(x_1, x_2) \) is (locally) analytic, as are our potentials (4). In this case, the distributions \( S_{\lambda}' \) appear in (8) only with non-negative integer \( \lambda \)'s, and for such \( \lambda \)'s they can be easily calculated

\[
S_{\lambda}'(t) := \frac{dS_{\lambda}(t)}{d\lambda} \bigg|_{\lambda = \nu} = \frac{1}{\nu!} t_{+}^{\nu+1} \log t + C_{\nu} t_{+}^{\nu} , \quad \nu = 0, 1, 2, \ldots ,
\]

where \( C_{\nu} \in \mathbb{R} \) are some constants, with \( C_0 = 0 \). Substituting (8) and (10) in (8), we get

\[
G(x, \xi) \sim W(x, \xi) \log \gamma + \ldots ,
\]

\footnote{For an excellent survey on this classical subject we refer the reader to [3]}. 

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where “…” denote smooth functions, which do not contribute to singularities of $G(x, \xi)$, and

$$W(x, \xi) := \sum_{\nu=0}^{\infty} \frac{1}{(-4)^{\nu}\nu!} U_{\nu}(x, \xi) \gamma^{\nu}.$$  

Since $V$ is analytic, the series (11) uniformly converges in a neighborhood of the diagonal $x = \xi$. This was already observed by Hadamard, who called $W(x, \xi)$ the logarithmic term of the “elementary solution” $G(x, \xi)$. He also showed that $W(x, \xi)$ is the (unique) analytic solution of the Goursat problem

$$\mathcal{L}[W(\cdot, \xi)@(x) = 0, \quad W|_{\gamma = 0} = 1,$$

with the boundary condition given on the (complex) characteristics of $\mathcal{L}$ (see [H], Ch. III, Sect. 46).

Now, to prove Theorem 2 we will simply solve (12) for the operator $\mathcal{L}$ with potential (4), using the method of separation of variables, and then recover the coefficients $U_{\nu}(x, \xi)$ by expanding the solution in the vicinity of $\gamma = 0$ as in (11).

First, we introduce a one-dimensional Schrödinger operator $L$ by writing $\mathcal{L}$ in terms of polar coordinates (cf. [B], Sect. 3):

$$L = \frac{-\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} L.$$  

Explicitly, $L = -\partial^2/\partial \varphi^2 + V_k(\cos \varphi, \sin \varphi)$, where $V_k$ is given by (4). Next, we prove

**Lemma 1.** If $\Psi = \Psi(\varphi)$ satisfies $L[\Psi] = k^2 \Psi$ with some integer $k \geq 0$, then

$$\Psi(\varphi) T_k[(r/\varrho + \varrho/r)/2] \in \text{Ker}(\mathcal{L}) \text{ for any } \varrho \neq 0.$$  

**Proof.** Changing the variables $(r, \varphi) \mapsto (z, \varphi)$ in (13), with $z := (r/\varrho + \varrho/r)/2$, yields

$$L = \frac{1}{r^2} \left[ (1 - z^2) \frac{\partial^2}{\partial z^2} - z \frac{\partial}{\partial z} + L \right].$$  

Since $T_k(z)$ is an eigenfunction of $(1 - z^2)d^2/dz^2 - zd/dz$ with eigenvalue $-k^2$, the claim is obvious. □

**Lemma 2.** The functions $\Psi_i$ defined by (6) satisfy

$$L[\Psi_i] = k_i^2 \Psi_i, \quad i = 0, 1, \ldots, m,$$

$$\sum_{i=0}^{m} c_i \Psi_i(\varphi) \Psi_i(\phi) \cos(k_i(\varphi - \phi)) = 1.$$  

**Proof.** The equations (14) simply say that $\Psi_i$’s are eigenfunctions of $L$ with eigenvalues $k_i^2$. This follows directly from Crum’s classical theorem (see [C], p. 124).

The identity (14) seems more interesting: we could not find it in the literature, so we will prove it in detail by induction on $m$.

The case $m = 1$ is straightforward. Writing $S_m(\varphi, \phi)$ for the left-hand side of (15), we now fix $m \geq 1$ and assume that $S_m(\varphi, \phi) \equiv 1$ for all sequences of integers $(k_0 = 0, k_1, \ldots, k_m)$ and reals $(\varphi_0 = 0, \varphi_1, \ldots, \varphi_m)$ of length $m$. To make the
induction step we add \( k_{m+1} \in \mathbb{Z}, \ k_{m+1} > k_m \), and \( \varphi_{m+1} \in \mathbb{R} \) to these sequences and consider

\[
S_{m+1}(\varphi, \phi) := \sum_{i=0}^{m+1} \tilde{c}_i \tilde{\Psi}_i(\varphi) \tilde{\Psi}_i(\phi) \cos(k_i(\varphi - \phi)),
\]

where \( \tilde{\Psi}_i \) and \( \tilde{c}_i \) are defined by formulas (6) with \( m \) replaced by \( m+1 \). By Crum’s Theorem, \( \tilde{\Psi}_i \)'s are eigenfunctions of an operator \( \tilde{L} \) obtained from \( L \) by applying the Darboux transformation (cf. [B], (3.29)):

\[
L = A_m^* \circ A_m + k_{m+1}^2 \mapsto \tilde{L} = A_m^* \circ A_m + k_{m+1}^2,
\]

where

\[
A_m := \tilde{\Psi}_m^{-1} \circ \left( \frac{\partial}{\partial \varphi} \right) \circ \tilde{\Psi}_m, \quad A_m^* := -\tilde{\Psi}_m^{-1} \circ \left( \frac{\partial}{\partial \varphi} \right) \circ \tilde{\Psi}_m^{-1}.
\]

It is immediate from (18) that \( A_m^*[\tilde{\Psi}_{m+1}] = 0 \). On the other hand, we have

\[
A_m^*[\tilde{\Psi}_i] = -\tilde{\Psi}_i \quad \text{for all} \ i = 0, \ldots, m.
\]

In fact, a trivial calculation shows that (19) is equivalent to

\[
\frac{\partial}{\partial \varphi} \left( \frac{W_i}{W_{m+1}} \right) = \frac{W W_i_{m+1}}{W_{m+1}^2},
\]

where \( W \) denotes the Wronskian of the set \( \{\chi_0, \chi_1, \ldots, \chi_{m+1}\} \), and \( W_i, W_{m+1} \) are the Wronskians of this set with functions \( \chi_i \) and \( \{\chi_i, \chi_{m+1}\} \) being omitted. Now, to prove (20) consider the system of linear equations

\[
\sum_{i=0}^m \chi_i^{(k)} y_i = \chi_{m+1}^{(k)}, \quad k = 0, 1, \ldots, m.
\]

By Cramer’s Rule, the solution to this system is given by \( y_i = (-1)^{m+i} W_i/W_{m+1} \). On the other hand, differentiating both sides of (21) with respect to \( \varphi \) yields

\[
\sum_{i=0}^m \chi_i^{(k)} y_i' = 0 \quad \text{for} \ k = 0, 1, \ldots, m-1 \quad \text{and} \ \sum_{i=0}^m \chi_i^{(m)} y_i' = W_{m+1} \quad \text{for} \ k = m.
\]

Solving these equations for \( y_i' \), we get

\[
y_i' = (-1)^{m+i} W W_i_{m+1} W_{m+1}^2,
\]

which is equivalent to (20).

It follows from (19) that \( A_m[\tilde{\Psi}_i] = -A_m A_m^*[\tilde{\Psi}_i] = -(\tilde{L} - k_{m+1}^2) \tilde{\Psi}_i \). Hence

\[
A_m[\tilde{\Psi}_i] = (k_{m+1}^2 - k_i^2) \tilde{\Psi}_i \quad \text{for all} \ i = 0, \ldots, m.
\]

Now, applying \( A_m \) and \( A_m^* \) to (16), we can formally relate \( S_{m+1}(\varphi, \phi) \) to \( S_m(\varphi, \phi) \).

In fact, a straightforward calculation using (10) and (22) shows

\[
A_m^*[S_{m+1}(\cdot, \phi)] + A_m^*[S_{m+1}(\varphi, \cdot)] = A_m[S_m(\cdot, \phi)] + A_m[S_m(\varphi, \cdot)],
\]

where omitted are the variables on which the differential operators act. Since \( S_m(\varphi, \phi) \equiv 1 \) by induction assumption, we can regard (23) as a first order PDE for the function \( S_{m+1}(\varphi, \phi) \). Specifically, substituting \( S_1 = 1 \) into (23) yields

\[
- \left( \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \phi} \right) S_{m+1} + \left( \frac{\partial B}{\partial \varphi} + \frac{\partial B}{\partial \phi} \right) S_{m+1} = \frac{\partial B}{\partial \varphi} + \frac{\partial B}{\partial \phi},
\]

where \( B := \log[\tilde{\Psi}_{m+1}(\varphi) \tilde{\Psi}_{m+1}(\phi)] \). This PDE can be easily integrated: changing the variables \( x := (\varphi + \phi)/2, t := (\varphi - \phi)/2 \) and \( F := (S_{m+1} - 1) e^{-B} \) transforms
to the equation $\partial F/\partial x = 0$, which means that $F$ depends only on $t$. It follows that
\begin{equation}
S_{m+1}(\varphi, \phi) = 1 + F(\varphi - \phi) \tilde{\Psi}_{m+1}(\varphi) \tilde{\Psi}_{m+1}(\phi),
\end{equation}
where $F$ is a differentiable function of one variable defined on $(0, 2\pi)$.

Now, if we replace $k_m$ with $k_{m+1}$ in the sum $S_m$ and repeat the above argument, adding $k_m$ to the partition $(k_0 = 0, k_1, \ldots, k_{m-1}, k_{m+1})$, then, instead of (25), we get
\begin{equation}
S_{m+1}(\varphi, \phi) = 1 + G(\varphi - \phi) \tilde{\Psi}_m(\varphi) \tilde{\Psi}_m(\phi),
\end{equation}
where $G$ is another differentiable function on $(0, 2\pi)$. Comparing (24) and (26) shows that $F = G \equiv 0$. Indeed, if one of these functions ($F$ say) is nonzero, then by continuity $F(\varphi - \phi) \neq 0$ in an open subset of $(0, 2\pi) \times (0, 2\pi)$, and in that subset we have
\begin{equation}
G(\varphi - \phi)/F(\varphi - \phi) = h(\varphi) h(\phi),
\end{equation}
where $h := \tilde{\Psi}_{m+1}/\tilde{\Psi}_m$. Differentiating both sides of (27) with respect to $\varphi$ and $\phi$ and adding the results yields $h'(\varphi) h(\phi) + h(\varphi) h'(\phi) = 0$. Whence, letting $\varphi = \phi$, we see that $h$ must be constant. This means that $\tilde{\Psi}_{m+1}$ is a multiple of $\tilde{\Psi}_m$, which is impossible, since $\tilde{\Psi}_m$ and $\tilde{\Psi}_{m+1}$ are nonzero eigenfunctions of $\tilde{L}$ corresponding to different eigenvalues ($k_m^2$ and $k_{m+1}^2$ respectively). Thus $F \equiv 0$, and therefore $S_{m+1}(\varphi, \phi) \equiv 1$, finishing the induction. □

Now, combining the results of Lemmas 1 and 2, we see at once that
\begin{equation}
W := \sum_{i=0}^m c_i \Psi_i(\varphi) \tilde{\Psi}_i(\phi) T_{k_i}[(r/\rho + \rho/r)/2]
\end{equation}
is a solution to (14). Indeed, by (14) and Lemma 1, each summand of (28) lies in the kernel of $L$, and hence $L[W] = 0$ by linearity of $L$. On the other hand, in polar coordinates $\gamma = r^2 + \rho^2 - 2 r \rho \cos(\varphi - \phi)$, so
\begin{equation}
\frac{1}{2} \left( \frac{\rho}{\rho} + \frac{\rho}{r} \right) = \frac{\gamma}{2 r \rho} + \cos(\varphi - \phi),
\end{equation}
and therefore
\begin{equation}
T_{k_i}[(r/\rho + \rho/r)/2] = \cos(k_i(\varphi - \phi)) \quad \text{on} \quad \gamma = 0
\end{equation}
for all $i = 0, 1, \ldots, m$. The boundary condition for $W$ follows then from (15).

Now, by (13), Lemma 3.1, $U_\nu(x, \xi)$ are homogeneous functions of $x$ and $\xi$, which can be written in terms of the polar coordinates as
\begin{equation}
U_\nu(x, \xi) = \frac{1}{(r \rho)^\nu} \sigma_\nu(\varphi, \phi), \quad \nu \geq 0.
\end{equation}
Clearly, there is at most one expansion of $W(x, \xi)$ of the form (11) with coefficients (30). To find this expansion, we take the obvious Taylor formulas (see (29))
\begin{equation}
T_{k_i}[(r/\rho + \rho/r)/2] = \sum_{\nu = 0}^{\infty} \frac{1}{\nu!} \left( \frac{\gamma}{2 r \rho} \right)^\nu T_{k_i}^{(\nu)}(\cos(\varphi - \phi)),
\end{equation}
substitute (31) into (28) and reorder summations. Collecting coefficients under the different powers of $\gamma$ gives then the desired formulas (5).
In the end, we note that our operators $\mathcal{L}$ are examples of algebraically integrable Schrödinger operators (see [CV]). Such Schrödinger operators possess special eigenfunctions called the Baker-Akhiezer functions. Like fundamental solutions above, the Baker-Akhiezer functions have asymptotic expansions, with coefficients satisfying (up to rescaling factor $1/2$) the same transport equations [3]. This was originally discovered in the special case of Calogero-Moser potentials in [BV], but the argument of [BV] applies to any homogeneous operator (see, e.g., [CFV]). In combination with Theorem 2, this yields

Theorem 3. The Baker-Akhiezer function of $\mathcal{L} = -\Delta_2 + V_k(x_1, x_2)$ is given by

$$\Psi_{BA}(x, \xi) = \left( \sum_{\nu=0}^{k_m} \frac{1}{2^\nu} U_\nu(x, \xi) \right) e^{(x, \xi)},$$

where $U_\nu(x, \xi)$ are the same coefficients as in [5].

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