On optimal mean-field type control problems of stochastic systems with jump processes under partial information

Yaozhong Hu\textsuperscript{1} \footnote{Y. Hu is partially supported by a grant from the Simons Foundation No.209206. D. Nualart is supported by the NSF grant DMS1208625. Q. Zhou is supported by the National Natural Science Foundation of China (No 11001029 and 11371362) and the Fundamental Research Funds for the Central Universities (No BUPT2012RC0709).} \footnote{E-mail: hu@math.ku.edu} \ David Nualart\textsuperscript{2} \footnote{E-mail: nualart@math.ku.edu} \ Qing Zhou\textsuperscript{3} \footnote{Corresponding author. E-mail: zqleii@bupt.edu.cn}

1. 2. Department of Mathematics, University of Kansas, Lawrence, Kansas, 66045 USA
3. School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China

Abstract

This paper considers the problem of partially observed optimal control for forward stochastic systems which are driven by Brownian motions and an independent Poisson random measure with a feature that the cost functional is of mean-field type. When all the system coefficients and the objective performance functionals are allowed to be random, possibly non-Markovian, Malliavin calculus is employed to derive a maximum principle for the optimal control of such a system where the adjoint process is explicitly expressed. We also investigate the mean-field type optimal control problems for systems driven by mean-field type stochastic differential equations (SDEs in short) with jump processes, in which the coefficients contain not only the state process but also its marginal distribution under partially observed information. The maximum principle is established using convex variational technique with an illustrating example about linear-quadratic optimal control.

Keywords: maximum principle; mean-field type; partial information; Girsanov’s theorem; forward stochastic differential equations; Malliavin calculus; jump diffusion

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1 Introduction

Let $T > 0$ be a fixed time horizon and let $(\Omega, \mathcal{F}, \mathbb{P}_0)$ be a probability space equipped with a right continuous filtration $(\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions (we use the notation $\mathbb{P}_0$ here to reserve $\mathbb{P}$ for a future use). Let $((W_1(t), W_2(t)), 0 \leq t \leq T)$ be a two dimensional $\mathcal{F}_t$-Brownian motion and let $N(dt, dz)$ be a $\mathcal{F}_t$-Poisson random measure on $[0,T] \times \mathbb{R}_0$ with intensity $\mu(dz)$, independent of the Brownian motion $W_1$ and $W_2$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. We denote the compensated Poisson measure by $\tilde{N}(dt, dz) := N(dt, dz) - \mu(dz)dt$. 
In this paper we shall study the mean field optimal control problems which have the following characteristics. The state equation is given by the following mean field stochastic differential equation with jumps:

\[
\begin{align*}
\dot{x}^v(t) &= b(t, x^v(t), v(t)) dt + \sigma(t, x^v(t), v(t)) dW_1(t) \\
&\quad + \int_{\mathbb{R}_0} \gamma(t, x^v(t-), v(t), z) \tilde{N}(dt, dz), \\
x^v(0) &= x_0, \quad t \in [0, T],
\end{align*}
\]  

(1.1)

where \( v(\cdot) \) is a control process taking values in a nonempty, closed convex subset \( U \subseteq \mathbb{R} \) and the expectation \( \mathbb{E}_0 \) is related to the probability measure \( \mathbb{P}_0 \). The conditions on the coefficients will be made specific in Section 4. To describe the conditions on the control \( v \), we assume the state process \( x^v(t) \) is not completely observable. Instead, it is partially observed and the observation is corrupted with noise. The observation equation is given by the following equation:

\[
\begin{align*}
\begin{cases}
    dY(t) &= h(t, x^v(t)) dt + dW_2(t), \\
    Y(0) &= 0,
\end{cases}
\end{align*}
\]  

(1.2)

Thus the control process \( v(t) \) will be an \( \mathcal{F}_Y^t \)-adapted processes. More precisely, we give the following definition of admissible controls.

**Definition 1.1** Let \( U \subseteq \mathbb{R} \) be a nonempty, closed and convex subset which will be the range of control \( v \). Let \( \mathcal{F}_Y^t = \sigma(Y_s, 0 \leq s \leq t) \) be the \( \sigma \)-algebra generated by \( Y \). A control process \( v : [0, T] \times \Omega \to U \) is called admissible if \( v(t) \) is \( \mathcal{F}_Y^t \)-adapted and \( \sup_{0 \leq t \leq T} \mathbb{E}_0 |v(t)|^2 < \infty \). The set of all admissible controls is denoted by \( \mathcal{U}_{ad} \).

We introduce the following cost functional

\[
J(v(\cdot)) = \mathbb{E}_0 \left[ \int_0^T l(t, x^v(t), \mathbb{E}_0[f(x^v(t))], v(t)) dt + \phi(x^v(T), \mathbb{E}_0[g(x^v(T))]) \right],
\]  

(1.3)

where \( l : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R} \) and \( \phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are given mappings. \( \mathbb{E}_0 \) is the expectation with respect to the probability measure \( \mathbb{P}_0 \). \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are given functions such that \( \mathbb{E}_0[|f(x^v(t))|] < \infty \), for all \( t \) and \( \mathbb{E}_0[|g(x^v(T))|] < \infty \).

Now we can state our mean-field type control (MFC) problem as follows.

**Problem (MFC):** Find \( u(\cdot) \in \mathcal{U}_{ad} \) (if it exists) such that

\[
J(u(\cdot)) = \min_{v(\cdot) \in \mathcal{U}_{ad}} J(v(\cdot)).
\]

Our objective in this paper is to establish a maximum principle for the optimal control to satisfy. This will be given in Section 4. However, we shall pay a particular attention to the case when the state equation does not contain the mean field, namely, when the state equation is given by the following equation

\[
\begin{align*}
\dot{x}^v(t) &= b(t, x^v(t), v(t)) dt + \sigma(t, x^v(t), v(t)) dW_1(t) \\
&\quad + \int_{\mathbb{R}_0} \gamma(t, x^v(t-), v(t), z) \tilde{N}(dt, dz), \\
x^v(0) &= x_0, \quad t \in [0, T].
\end{align*}
\]  

(1.4)

In this case we can use Malliavin calculus to obtain more explicit form of the maximum principle. This is done in Section 3.
It seems that this paper is the first to study the problem of minimizing \( (1.3) \) subject to state constraint \((1.1)\) and observation constraint \((1.2)\). When full information is available, the maximum principle is obtained in \([9]\), \([16]\) and references therein.

The work closely related to ours is the work \([8]\), where the authors have already used Malliavin calculus to obtain the maximum principle. The difference is that their partial information flow is related to this paper, for reader’s convenience.

\[ E \left[ \sum_{i=1}^{n} I_n(f_n) \right] \]

where \( E \) is the expectation with respect to \( \mathbb{P} \).

A general reference for this presentation is \([3]\), \([4]\) and \([11]\). See also the book \([5]\).

### 2.1 Malliavin calculus for \( W(\cdot) \)

A natural starting point is the Wiener-Itô chaos expansion theorem, which states that any \( F \in L^2(\mathcal{F}_t, \mathbb{P}) \) (where in this case \( \mathcal{F}_t = \mathcal{F}_t^W \) is the \( \sigma \)-algebra generated by \( W(s); 0 \leq s \leq t \)) can be written as

\[ F = \sum_{n=0}^{\infty} I_n(f_n), \tag{2.1} \]

for a unique sequence of symmetric deterministic functions \( f_n \in L^2(\lambda^n) \), where \( \lambda \) is a Lebesgue measure on \([0,T]\) and

\[ I_n(f_n) = n! \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, \cdots, t_n) dW(t_1) \cdots dW(t_n) \]

(the \( n \)-times iterated integral of \( f_n \) with respect to \( W(\cdot) \) for \( n = 1, 2, \cdots \) and \( I_0(f_0) = f_0 \) when \( f_0 \) is a constant.

### 2. A brief review of Malliavin calculus for Lévy processes

In this section, we recall the basic definitions and properties of Malliavin calculus for Brownian motion \( W(\cdot) \) and \( N(ds, dz) \) related to this paper, for reader’s convenience.

Let \( L^2(\lambda^n) \) be the space of deterministic real functions \( f \) such that

\[ \| f \|_{L^2(\lambda^n)} = \left( \int_{[0,T]^n} f^2(t_1, t_2, \cdots, t_n) dt_1 dt_2 \cdots dt_n \right)^{1/2} < \infty, \]

where \( \lambda(dt) \) denotes the Lebesgue measure on \([0,T]\).

Let \( L^2((\lambda \times \mu)^n) \) be the space of deterministic real functions \( f \) such that

\[ \| f \|_{L^2((\lambda \times \mu)^n)} = \left( \int_{[0,T]^n \times \mathbb{R}^n} f^2(t_1, z_1, t_2, z_2, \cdots, t_n, z_n) dt_1 d\mu(dz_1) dt_2 d\mu(dz_2) \cdots dt_n d\mu(dz_n) \right)^{1/2} < \infty. \]

\( L^2(\lambda \times \mathbb{P}) \) can be similarly denoted.

A general reference for this presentation is \([3]\), \([4]\) and \([11]\). See also the book \([5]\).
Moreover, we have the isometry
\[
\mathbb{E}[F^2] = \|F\|_{L^2(\mathbb{P})}^2 = \sum_{n=0}^{\infty} n!\|f_n\|_{L^2(\mathbb{R}^n)}^2.
\]

**Definition 2.1 (Malliavin derivative \(D_t\)).** Let \(\mathcal{D}_{1,2}^{(W)}\) be the space of all \(F \in L^2(\mathcal{F}_T, \mathbb{P})\) such that its chaos expansion (2.1) satisfies
\[
\|F\|_{\mathcal{D}_{1,2}^{(W)}}^2 := \sum_{n=1}^{\infty} n!\|f_n\|_{L^2(\mathbb{R}^n)}^2 < \infty.
\]

For \(F \in \mathcal{D}_{1,2}^{(W)}\) and \(t \in [0,T]\), we define the Malliavin derivative of \(F\) at \(t\) (with respect to \(W(\cdot)\)), \(D_tF\), by
\[
D_tF = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t)),
\]
where the notation \(I_{n-1}(f_n(\cdot,t))\) means that we apply the \((n-1)\)-times iterated integral to the first \(n-1\) variables \(t_1, \ldots, t_{n-1}\) of \(f_n(t_1, t_2, \ldots, t_n)\) and keep the last variable \(t_n = t\) as a parameter.

Some basic properties of the Malliavin derivative \(D_t\) are the following:
(i) Chain rule ([III], page 29)
Suppose \(F_1, \ldots, F_m \in \mathcal{D}_{1,2}^{(W)}\) and that \(\psi : \mathbb{R}^m \to \mathbb{R}\) is \(C^1\) with bounded partial derivatives. Then
\[
\psi(F_1, \ldots, F_m) \in \mathcal{D}_{1,2}^{(W)}
\]
and
\[
D_t\psi(F_1, \ldots, F_m) = \sum_{i=1}^{m} \frac{\partial \psi}{\partial x_i}(F_1, \ldots, F_m)D_tF_i.
\]
(ii) Integration by parts/duality formula ([III], page 35)
Suppose \(h(t)\) is \(\mathcal{F}_t\)-adapted with \(\mathbb{E}[\int_0^T h^2(t)dt] < \infty\) and let \(F \in \mathcal{D}_{1,2}^{(W)}\). Then
\[
\mathbb{E}\left[ F \int_0^T h(t)dW(t) \right] = \mathbb{E}\left[ \int_0^T h(t)D_tF dt \right].
\]

### 2.2 Malliavin calculus for \(N(\cdot)\)

The construction of a stochastic derivative/Malliavin derivative in the pure jump martingale case follows the same lines as in the Brownian motion case. In this case, the corresponding Wiener-Itô chaos expansion theorem states that any \(F \in L^2(\mathcal{F}_T, \mathbb{P})\) (where in this case \(\mathcal{F}_t = \mathcal{F}_t^N\) is the \(\sigma\)-algebra generated by \(\int_0^t \int_A N(dr, dz)\); \(0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}_0)\)) can be written as
\[
F = \sum_{n=0}^{\infty} I_n(f_n); \ f_n \in \hat{L}^2((\lambda \times \mu)^n),
\]
where \(\mathcal{B}(\mathbb{R}_0)\) is the Borel \(\sigma\)-field generated by the open subset \(O\) of \(\mathbb{R}_0\), whose closure \(\bar{O}\) does not contain the point 0, and \(\hat{L}^2((\lambda \times \mu)^n)\) is the space of functions \(f_n(t_1, z_1, \ldots, t_n, z_n); t_i \in [0,T], z_i \in \mathbb{R}_0\) such that \(f_n \in L^2((\lambda \times \mu)^n)\) and \(f_n\) is symmetric with respect to the pairs of variables \((t_1, z_1), \ldots, (t_n, z_n)\).
It is important to note that in this case the \( n \)-times iterated integral \( I_n(f_n) \) is taken with respect to \( \tilde{N}(dt, dz) \). Thus, we define
\[
I_n(f_n) = n! \int_0^T \int_{\mathbb{R}_0} \cdots \int_{\mathbb{R}_0} f_n(t_1, z_1, \cdots, t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n),
\]
for \( f_n \in L^2((\lambda \times \mu)^n) \).

Then Itô isometry for stochastic integrals with respect to \( \tilde{N}(dt, dz) \) gives the following isometry for the chaos expansion:
\[
\|F\|_{L^2(\mathbb{P})}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2((\lambda \times \mu)^n)}^2.
\]

As in the Brownian motion case, we use the chaos expansion to define the Malliavin derivative. Note that in this case there are two parameters \( t, z \), where \( t \) represents time and \( z \neq 0 \) represents a generic jump size.

**Definition 2.2 (Malliavin derivative \( D_{t,z} \))** ([3], [4]) Let \( \mathcal{D}_{1,2}^{(N)} \) be the space of all \( F \in L^2(\mathcal{F}_T, \mathbb{P}) \) such that its chaos expansion (2.4) satisfies
\[
\|F\|^2_{\mathcal{D}_{1,2}^{(N)}} = \sum_{n=1}^{\infty} n! \|f_n\|^2_{L^2((\lambda \times \mu)^n)} < \infty.
\]

For \( F \in \mathcal{D}_{1,2}^{(N)} \), we define the Malliavin derivative of \( F \) at \( (t, z) \) (with respect to \( N(\cdot) \)), \( D_{t,z} F \), by
\[
D_{t,z} F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t, z)),
\]
where \( I_{n-1}(f_n(\cdot, t, z)) \) means that we perform the \((n-1)\)-times iterated integral with respect to \( \tilde{N} \) to the first \( n \) \(-\)times iterated integral with respect to \( \tilde{N} \), where \( I_{n-1}(f_n(\cdot, t, z)) \) is \( \tilde{N} \)-measurable and \( f_n(\cdot, t, z) \) is \( \tilde{N} \)-measurable with respect to \( \tilde{N} \).

The properties of \( D_{t,z} \) corresponding to the properties (2.2) and (2.3) of \( D_t \) are the following:
(i) Chain rule ([4], [7])
Suppose \( F_1, \cdots, F_m \in \mathcal{D}_{1,2}^{(N)} \) and that \( \varphi : \mathbb{R}^m \to \mathbb{R} \) is continuous and bounded. Then
\[
D_{t,z} \varphi(F_1, \cdots, F_m) = \varphi(F_1 + D_{t,z} F_1, \cdots, F_m + D_{t,z} F_m) - \varphi(F_1, \cdots, F_m).
\]
(ii) Integration by parts/duality formula ([4])
Suppose \( \Psi(t, z) \) is \( \mathcal{F}_t \)-adapted and \( \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} \Psi^2(t, z) \mu(dz) dt] < \infty \) and let \( F \in \mathcal{D}_{1,2}^{(N)} \). Then
\[
\mathbb{E} \left[ F \int_0^T \int_{\mathbb{R}_0} \Psi(t, z) \tilde{N}(dt, dz) \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}_0} \Psi(t, z) D_{t,z} F \mu(dz) dt \right].
\]

### 3 Stochastic maximum principle for mean-field type optimal control- Malliavin calculus approach

In this section, we derive the maximum principle for the mean field optimal control problem of minimizing (1.3) over \( v(\cdot) \in \mathcal{U}_{ad} \) subject to (1.4) and (1.2).

We make some assumptions on the coefficients \( b, \sigma : [0, T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R} \) and \( \gamma : [0, T] \times \mathbb{R} \times U \times \mathbb{R}_0 \times \Omega \to \mathbb{R} \):
(A1) The functions $b$, $\sigma$ and $\gamma$ are almost surely continuous with respect to their variables $t$, $x$, $v$. For any $t \in [0, T]$, the functions $b$ and $\sigma$ are continuously differentiable with respect to $x$ and $v$ with uniformly bounded derivatives $b_x$, $b_v$, $\sigma_x$ and $\sigma_v$.

\[
\sup_{0 \leq t \leq T, x \in \mathbb{R}, v \in U, \omega \in \Omega} \left( |b_x(t, x, v, \omega)| + |b_v(t, x, v, \omega)| + |\sigma_x(t, x, v, \omega)| + |\sigma_v(t, x, v, \omega)| \right) < \infty.
\]

(3.1)

The function $\gamma$ is continuously differentiable in $(x, v)$ and there is a constant $C$ such that

\[
\sup_{0 \leq t \leq T, \omega \in \Omega} \left( \int_{\mathbb{R}_0} |\gamma(t, x, v, z, \omega)|^2 \mu(dz) \right)^{\frac{1}{2}} \leq C(1 + |x| + |v|).
\]

Moreover, we assume that $\int_{\mathbb{R}_0} |\gamma_x(t, x, v, z)|^2 \mu(dz)$ and $\int_{\mathbb{R}_0} |\gamma_v(t, x, v, z)|^2 \mu(dz)$ are continuous with respect to $(x, v)$ and uniformly bounded for $0 \leq t \leq T, x \in \mathbb{R}, v \in U$.

(A2) The function $h_t(x)$ is almost surely continuous on $t \in [0, T]$ and $x \in \mathbb{R}$. For any $t \in [0, T]$, the function $h : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is continuously differentiable with respect to $x$ and

\[
\sup_{t \in [0, T], x \in \mathbb{R}, \omega \in \Omega} |h_t(x)| + h_x(t, x)| < \infty.
\]

For any $x$, $h_t(x, \omega)$ is an $\mathcal{F}_t$-adapted process.

The state process $(x^y(t), 0 \leq t \leq T)$ is not observable itself, but is observed partially and corrupted with noise so that we have $(Y(t), 0 \leq t \leq T)$ defined by (1.2) available. Our control will be based on the observation of the process $Y$ up time instant $t$.

The intrinsic difficulty arising from the fact that the control $v$ depends on the observation $Y$, which itself is dependent on the control $v$. The approach via Girsanov transformation offers a way to overcome this difficulty. Let

\[
\rho_0^y(t) = \exp \left\{ - \int_0^t h(s, x(s))dW_2(s) - \frac{1}{2} \int_0^t h^2(s, x(s))ds \right\}.
\]

(3.2)

Define

\[
\frac{d\mathbb{P}^y}{d\mathbb{P}^0} = \rho_0^y(T) = \exp \left\{ - \int_0^T h(s, x(s))dW_2(s) - \frac{1}{2} \int_0^T h^2(s, x(s))ds \right\}.
\]

(3.3)

Then from the Girsanov theorem and the Kallianpur-Striebel formula, we know that under the probability measure $\mathbb{P}^y$, $(Y(t), 0 \leq t \leq T)$ is a Brownian motion, independent of $W_1$ and $N$. From now on we shall use this probability measure $\mathbb{P}^y$. Now we denote by $\mathbb{P}$ the probability measure, under which $(W_1(t), Y(t), 0 \leq t \leq T)$ is a two dimensional Brownian motion and $N$ is a Poisson random measure independent of $W_1$ and $Y$. The original probability measure can be represented as

\[
\frac{d\mathbb{P}^0}{d\mathbb{P}} = \frac{1}{\rho_0^y(T)} = \exp \left\{ \int_0^T h(s, x(s))dW_2(s) + \frac{1}{2} \int_0^T h^2(s, x(s))ds \right\} = \rho^y(T),
\]

(3.4)

where

\[
\rho^y(t) = \exp \left\{ \int_0^t h(s, x(s))dY(s) - \frac{1}{2} \int_0^t h^2(s, x(s))ds \right\}.
\]

(3.5)
It is easy to see that

$$\begin{align*}
\begin{cases}
    d\rho^v(t) &= h(t, x^v(t))\rho^v(t) dY(s), \\
    \rho^v(0) &= 1.
\end{cases}
\end{align*}$$

(3.6)

With the new probability measure $P$ and denoting the expectation with respect to $P$ by $\mathbb{E}$, the performance functional $J^v$ can be rewritten as

$$J(v(\cdot)) = \mathbb{E} \left[ \rho^v(T) \int_0^T l(t, x^v(t), \mathbb{E}[\rho^v(T) f(x^v(t))], v(t)) dt \\
+ \rho^v(T) \phi(x^v(T), \mathbb{E}[\rho^v(T) g(x^v(T))]) \right]$$

$$= \mathbb{E} \left[ \int_0^T \rho^v(t) l(t, x^v(t), \mathbb{E}[\rho^v(t) f(x^v(t))], v(t)) dt \\
+ \rho^v(T) \phi(x^v(T), \mathbb{E}[\rho^v(T) g(x^v(T))]) \right]$$

(3.7)

by the martingale property of $\rho^v(t)$. Therefore, the problem (MFC) is equivalent to minimizing (3.7) over $\mathcal{U}_{ad}$ subject to (1.4), where in the definition of $\mathcal{U}_{ad}$ the observation $(Y(t), 0 \leq t \leq T)$ is a Brownian motion, independent of $W_t$ and $N$ and $\rho^v(t)$ is given by (3.6).

Let $D_{1,2}$ denote the set of all random variables which are Malliavin differentiable with respect to all of $W_t(\cdot)$, $Y(\cdot)$, and $\mathcal{N}(\cdot, \cdot)$.

Furthermore, let us introduce some notations.

$$\mathcal{L}^2_{\mathcal{F}}(0, T) = \left\{ \phi(t, \omega) \text{ is an } \mathbb{R}\text{-valued progressively measurable process such that} \right\}$$

$$\mathbb{E} \left( \int_0^T |\phi(t)|^2 dt \right) < \infty \right\}.$$ 

$$\mathcal{M}^2_{\mathcal{F}}(0, T; \mathbb{R}) = \left\{ \phi(t, z, \omega) \text{ is an } \mathbb{R}\text{-valued progressively measurable process such that} \right\}$$

$$\mathbb{E} \left( \int_0^T \int_{\mathbb{R}_0} |\phi(t, z)|^2 \mu(dz) dt \right) < \infty \right\}.$$ 

$$\mathcal{L}_{12}(\mathbb{R}) = \left\{ F(t, \omega) \text{ is an } \mathbb{R}\text{-valued progressively measurable process such that} \right\}$$

for almost everywhere $0 \leq t \leq T$, $F(t, \cdot) \in D_{1,2}$ and

$$\|F\|_{1,2}^2 := \mathbb{E} \left( \int_0^T |F(t, \omega)|^2 dt + \int_0^T \int_{\mathbb{R}_0} |D_s^{(v)} F(t, \omega)|^2 ds dt \\
+ \int_0^T \int_{\mathbb{R}_0} |D_s F(t, \omega)|^2 \mu(dz) ds dt \right) < \infty \right\}.$$ 

(3.8)

Let $u(\cdot), v(\cdot) \in \mathcal{U}_{ad}$ be given and fixed such that $u(\cdot) + v(\cdot) \in \mathcal{U}_{ad}$. For any $0 \leq \varepsilon \leq 1$, we take the variational control $u^\varepsilon(\cdot) = u(\cdot) + \varepsilon v(\cdot)$. Because $\mathcal{U}_{ad}$ is convex, $u^\varepsilon(\cdot)$ belongs to $\mathcal{U}_{ad}$. Denote by $x^\varepsilon(\cdot)$ and $\rho^\varepsilon(\cdot)$ the solutions of (1.4) and (3.6) corresponding to the control $u^\varepsilon(\cdot)$. When $\varepsilon = 0$, denote $x = x^0(\cdot)$ and $\rho = \rho^0(\cdot)$. To obtain the maximum principle for the optimal control problem of minimizing (3.7) over $v(\cdot) \in \mathcal{U}_{ad}$, we use $\frac{d}{d\varepsilon} \big|_{\varepsilon=0} J^\varepsilon = 0$. To compute $\frac{d}{d\varepsilon} \big|_{\varepsilon=0} J^\varepsilon$ we need to compute $\frac{d}{d\varepsilon} \big|_{\varepsilon=0} J^\varepsilon$ and $\frac{d}{d\varepsilon} \big|_{\varepsilon=0} \rho^\varepsilon$, which are given by the following lemma 3.2. First, we need
Lemma 3.1 Under assumptions (A1) and (A2), for any \(v(\cdot) \in \mathcal{U}_ad\), there is a constant \(C\) such that

\[
\sup_{0 \leq t \leq T} \mathbb{E}[x^v(t)]^2 \leq C(1 + \sup_{0 \leq t \leq T} \mathbb{E}v^2(t)), \quad \sup_{0 \leq t \leq T} \mathbb{E}[x^v(t) - x(t)]^2 \leq Ce^2, \quad \sup_{0 \leq t \leq T} \mathbb{E}[p^v(t)]^2 \leq Ce^2.
\]

Proof. This is a directly application of Burkholder-Davis-Gundy inequality. ■

Consider the following linear stochastic differential equations (which will be the equations satisfied by \(\frac{d}{d\varepsilon}|_{\varepsilon=0}x^\varepsilon\) and \(\frac{d}{d\varepsilon}|_{\varepsilon=0}p^\varepsilon\)).

\[
\begin{align*}
    dx_1(t) &= \left( b_x(t)x_1(t) + b_y(t)v(t) \right)dt + \left( \sigma_x(t)x_1(t) + \sigma_y(t)v(t) \right)dW_1(t) \\
    &\quad + \int_{\mathbb{R}_0} \left[ \gamma_x(t,x(t-),u,z)x_1(t-) + \gamma_y(t,x(t-),u,z)v(t) \right] \mathcal{N}(dt,dz), \\
    x_1(0) &= 0,
\end{align*}
\]

and

\[
\begin{align*}
    dp_1(t) &= \left( p_1(t)h(t,x(t)) + p(t)h_x(t)x_1(t) \right)dy(t), \\
    p_1(0) &= 0,
\end{align*}
\]

where the notations \(b_x(t)\) and so on are defined in (3.8) of this section. In view of the boundedness of \(b_x, b_y, \sigma_x, \sigma_y, \gamma_x, \gamma_y, h\) and \(h_x\), (3.9) and (3.10) admit unique solutions \(x_1(\cdot), p_1(\cdot) \in \mathcal{L}^2_{\mathbb{P}}(0,T;\mathbb{R})\). (See also [1] and [13]). Obviously,

\[
p_1(t) = p(t) \left( \int_0^t h_x(s)x_1(s)dy(s) - \int_0^t h_x(s)h(s,x(s))x_1(s)ds \right), \quad 0 \leq t \leq T.
\]

By Lemma 1 and Lemma 2 in [14], we have the following lemma, which states that \(\frac{d}{d\varepsilon}|_{\varepsilon=0}x^\varepsilon = x_1(t)\) and \(\frac{d}{d\varepsilon}|_{\varepsilon=0}p^\varepsilon = p_1(t)\).

Lemma 3.2 Let the assumptions (A1) and (A2) hold. Then

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{x^\varepsilon(t) - x(t)}{\varepsilon} - x_1(t) \right|^2 = 0, \quad \lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{p^\varepsilon(t) - p(t)}{\varepsilon} - p_1(t) \right|^2 = 0.
\]

The following assumptions are needed to obtain the maximum principle.

(A3) The functions \(l : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \times \Omega \to \mathbb{R}\) and \(\phi : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}\) are almost surely continuously differentiable with respect to \((t,x,y,v) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times U\) and \((x,y) \in \mathbb{R} \times U\), respectively, and satisfying

\[
\mathbb{E} \left[ \int_0^T \rho^y(t)|l(t,x^v(t),\mathbb{E}[\rho^y(t)f(x^v(t))],v(t))|dt + \rho^y(T)|\phi(x^v(T),\mathbb{E}[\rho^y(T)g(x^v(T))])| \right] < \infty.
\]

\(\phi\) is almost surely twice continuously differentiable with respect to \(x\) with first and second order bounded derivatives. \(f: \mathbb{R} \to \mathbb{R}\) and \(g: \mathbb{R} \to \mathbb{R}\) are both twice continuously differentiable with first and second order bounded derivatives.
(A4) (i) For any \( t, \tau \), such that \( t + \tau \in [0, T] \), and bounded \( \mathcal{F}_t^Y \)-measurable random variable \( \beta \), we formulate the control process \( v(s) \in U \), with

\[
v(s) = \beta I_{[t,t+\tau]}(s), \quad s \in [0, T],
\]

where \( I_{[t,t+\tau]}(s) \) is the indicator function on the set \([t, t+\tau]\).

(ii) For any \( v(s) \in \mathcal{F}_s^Y \) with \( v(s) \) bounded, \( s \in [0, T] \), there is an \( \delta > 0 \) such that \( u(\cdot) + \varepsilon v(\cdot) \in \mathcal{U}_{ad} \) for \( \varepsilon \in (\delta, \delta) \).

To describe the maximum principle we define the following adjoint processes \( q(\cdot), k(\cdot) \) and \( r(\cdot, \cdot) \) as follows. Let

\[
G(t, s) = \exp\left( \int_t^s \left[ b_x(r) - \frac{1}{2} \sigma_x^2(r) \right] dr + \int_t^s \sigma_x(r) dW_1(r) \\
+ \int_t^s \int_{\mathbb{R}^3} \ln(1 + \gamma_x(r, x(r), u, z)) \tilde{N}(dr, dz) \\
+ \int_t^s \int_{\mathbb{R}^3} \left[ \ln(1 + \gamma_x(r, x(r), u, z)) - \gamma_x(r, x(r), u, z) \right] \mu(dz) dr \right); \quad s > t;
\]

\[
\Sigma(t) = \rho(T) \left( \Phi_x + g'(x(T)) \mathbb{E}_0[\Phi_y] \right) \\
+ \int_t^T \rho(s) \left( I_x(s) + f'(x(s)) \mathbb{E}_0[I_y(s)] \right) ds;
\]

\[
\Pi(t) = \rho(T) \left( \Phi + g(x(T)) \mathbb{E}_0[\Phi_y] \right) \\
+ \int_t^T \rho(s) \left( I(u(s)) + f(x(s)) \mathbb{E}_0[I_y(s)] \right) ds; \quad (3.11)
\]

\[
\Lambda(t) = \rho(T) \left( \Phi + g(x(T)) \mathbb{E}_0[\Phi_y] \right) h(t, x(t)) \\
+ \int_t^T \rho(s) \left( I(u(s)) + f(x(s)) \mathbb{E}_0[I_y(s)] \right) h(t, x(t)) ds; \quad (3.12)
\]

\[
H_x(t) = \Sigma(t) b_x(t) + \sigma_x(t) D^{(W_1)}_t \Sigma(t) + \alpha_x(t) \left( \mathcal{D}^{(Y)}_t \Pi(t) - \Lambda(t) \right) \\
+ \int_{\mathbb{R}^3} \gamma_x(t, x(t), u, z) D_t z \Sigma(t) \mu(dz);
\]

\[
\Theta(t, s) = H_x(s) G(t, s).
\]

Finally we denote

\[
q(t) := \Sigma(t) + \int_t^T \Theta(t, s) ds,
\]

\[
k(t) := D^{(W_1)}_t q(t),
\]

\[
r(t, z) := D_{t, z} q(t). \quad (3.13)
\]

Now we state our main theorem of this section.

**Theorem 3.1** Let the assumptions (A1), (A2), (A3) and (A4) hold. Assume that \( u(\cdot) \) is a local minimum for \( J(v(\cdot)) \), in the sense that for all bounded \( v(\cdot) \in \mathcal{U}_{ad} \), there exists a \( \delta > 0 \) such that \( u(\cdot) + \varepsilon v(\cdot) \in \mathcal{U}_{ad} \) for any \( \varepsilon \in (-\delta, \delta) \) and

\[
f(\varepsilon) = J(u(\cdot) + \varepsilon v(\cdot)), \quad \varepsilon \in (-\delta, \delta),
\]
attains its minimum at $\varepsilon = 0$. Assume that $\rho(t)$, $l(u(t))$, $l_1(t)$ and $\Theta(t,s)$ are in $L_{1,2}(\mathbb{R})$ for all $0 \leq t \leq s \leq T$. Then we have

$$
\mathbb{E} \left[ H_v(t, x(t), \mathbb{E}_0[f(x(t))], u(t); q(t), k(t), r(t, \cdot)) \mid \mathcal{F}_t^Y \right] = 0,
$$

where $H_v$ is defined by

$$
H_v(t, x, y, v; q, k, r) = b_v(t, x, v)q + \sigma_v(t, x, v)k
+ \int_{\mathbb{R}_0} r(t, z)\gamma_v(t, x, v, z)\mu(dz) + \rho(t)l_v(t, x, y, v).
$$

**Proof.** If $u(\cdot)$ is a local minimum for $J(v(\cdot))$, then $\frac{d}{d\varepsilon} J(\varepsilon) |_{\varepsilon=0} = 0$. Since

$$
\frac{d}{d\varepsilon} J(\varepsilon) |_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{J(u(\cdot) + \varepsilon v(\cdot)) - J(u(\cdot))}{\varepsilon}
= \lim_{\varepsilon \to 0} \mathbb{E} \left\{ \int_0^T \left[ (\rho^{u+E\varepsilon}(t) - \rho(t)) l(t, x(t), \mathbb{E}[\rho(t)f(x(t))], u)
+ \rho^{u+E\varepsilon}(t) \left( l(t, x^{u+E\varepsilon}(t), \mathbb{E}[\rho^{u+E\varepsilon}(t)f(x^{u+E\varepsilon}(t))], u + \varepsilon v)
- l(t, x(t), \mathbb{E}[\rho(t)f(x(t))], u) \right) \right] dt
+ (\rho^{u+E\varepsilon}(T) - \rho(T)) \phi(x(T), \mathbb{E}[\rho(T)g(x(T))])
+ \rho^{u+E\varepsilon}(T) \left( \phi(x^{u+E\varepsilon}(T), \mathbb{E}[\rho^{u+E\varepsilon}(T)g(x^{u+E\varepsilon}(T))])
- \phi(x(T), \mathbb{E}[\rho(T)g(x(T))]) \right) \right\},
$$

it follows from Lemma 3.2 that

$$
\frac{d}{d\varepsilon} J(\varepsilon) |_{\varepsilon=0} = \mathbb{E} \left[ \int_0^T \rho_1(t)l(t, x(t), \mathbb{E}[f(x(t))], u) dt \right]
+ \mathbb{E} \int_0^T \left[ l_1(t, x(t), \mathbb{E}[f(x(t))], u)\rho(t)x_1(t)
+ \rho(t)l_1(t, x(t), \mathbb{E}[f(x(t))], u)\mathbb{E}\left( f'(x(t))\rho(t)x_1(t) + \rho_1(t)f(x(t)) \right) \right] dt
+ \mathbb{E} \int_0^T l_v(t, x(t), \mathbb{E}[f(x(t))], u)\rho(t)v(t) dt
+ \mathbb{E} \left[ \phi(x(T), \mathbb{E}_0[g(x(T))])\rho_1(T) + \phi(x(T), \mathbb{E}_0[g(x(T))])\rho(T)x_1(T)
+ \rho(T)\phi_1(x(T), \mathbb{E}_0[g(x(T))])\mathbb{E}\left( g'(x(T))\rho(T)x_1(T) + \rho_1(T)g(x(T)) \right) \right].
$$

For the convenience of computation, we may adjust the order of the terms in the right side of the
above equation such that
\[
\frac{d}{d\varepsilon} \mathcal{J}(\varepsilon) \big|_{\varepsilon=0} = \mathbb{E} \left[ \phi(x(T), \mathbb{E}_0[g(x(T))]) \mathbf{p}_1(T) + \int_0^T \mathbf{p}_1(t) l(t, x(t), \mathbb{E}_0[f(x(t))], u) dt \right] + \mathbb{E} \left[ \mathbb{E}_0 [\Phi_y(x(T), \mathbb{E}_0[g(x(T))])] \mathbf{p}_1(T) g(x(T)) \right] + \mathbb{E} \int_0^T \mathbb{E}_0 [I_y(t, x(t), \mathbb{E}_0[f(x(t))], u)] \mathbf{p}_1(t) f(x(t)) dt + \mathbb{E} \left[ \Phi_v(x(T), \mathbb{E}_0[g(x(T))]) \mathbf{p}(T)x_1(T) \right] + \mathbb{E}_0 [\Phi_y(x(T), \mathbb{E}_0[g(x(T))]) g'(x(T)) \mathbf{p}(T)x_1(T)] + \mathbb{E} \left[ \int_0^T l_x(t, x(t), \mathbb{E}_0[f(x(t))], u) \mathbf{p}(t)x_1(t) dt \right]
\]
where in the forth identity we have used the Fubini theorem. Similarly,
\[
\int_0^T \mathbb{E}_0 [I_y(t, x(t), \mathbb{E}_0[f(x(t))], u)] \mathbf{p}(t)f(t)x_1(t)dt = \mathbb{E} \left[ \int_0^T l_t(t, x(t), \mathbb{E}_0[f(x(t))], u) \mathbf{p}(t)v(t)dt \right]
\]
(3.14)
where in the forth identity we have used \( \mathbb{E}(A\mathbb{E}(B)) = \mathbb{E}(B\mathbb{E}(A)) \). Since \( \Phi, \Phi_x, \mathbb{E}_0[\Phi_y(g(x(T)) \mathbb{E}_0[\Phi_y(g'(x(T))) \in \mathbb{L}_{1,2}(\mathbb{R}) \right), l(u(t)), l(t), \mathbb{E}_0[I_y(t)]f(x(t)), \mathbb{E}_0[I_y(t)]f'(x(t)) \) and \( \Theta(t,s) \) are in \( \mathbb{L}_{1,2}(\mathbb{R}) \) for all \( 0 \leq t \leq s \leq T \), according to (2.3) and (2.6), we have
\[
I_1 = \mathbb{E} \left( \phi(x(T), \mathbb{E}_0[g(x(T))]) \mathbf{p}(T) \left[ \int_0^T h_x(t)x_1(t)dY(t) - \int_0^T h_x(t)h(t, x(t))x_1(t)dt \right] \right)
\]
(3.15)
\[
I_2 \begin{aligned}
&= \mathbb{E} \int_0^T \mathbb{E}_0[f(x(t))], u) \mathbf{p}(t) \left( \int_0^t h_x(s)x_1(s)dY(s) - \int_0^t h_x(s)h(s, x(s))x_1(s)ds \right) \right) \\
&= \mathbb{E} \int_0^T h_x(t)x_1(t) \left\{ \int_t^T D_t^Y \left( \mathbf{p}(s)l(u(s)) \right) - \mathbb{P}(s,l(u(s))h(t,x(t)) \right\} dt.
\] (3.16)

Note that, in deriving the last identity in (3.16), we have used the Fubini theorem. Similarly,
\[
I_3 \begin{aligned}
&= \mathbb{E} \left\{ g(x(T))\mathbb{E}_0(\phi_y(x(T), \mathbb{E}_0[g(x(T))]) \mathbf{p}(T) \left[ \int_0^T h_x(t)x_1(t)dY(t) \\
&- \int_0^T h_x(t)h(t, x(t))x_1(t)dt \right] \right\} \\
&= \mathbb{E} \int_0^T h_x(t)x_1(t) \left\{ D_t^Y \left( \mathbf{p}(T)g(x(T))\mathbb{E}_0[\phi_y] \right) - \mathbb{P}(T)g(x(T))\mathbb{E}_0[\phi_y]h(t, x(t)) \right\} dt
\] (3.17)

and
\[
I_4 \begin{aligned}
&= \mathbb{E} \int_0^T f(x(t))\mathbb{E}_0[I_y(t)] \mathbf{p}(t) \left[ \int_0^t h_x(s)x_1(s)dY(s) - \int_0^t h_x(s)h(s, x(s))x_1(s)ds \right] dt \\
&= \mathbb{E} \int_0^T h_x(t)x_1(t) \left( \int_t^T D_t^Y \left( \mathbf{p}(s)f(x(s))\mathbb{E}_0[I_y(s)] \right) - \mathbb{P}(s,f(x(s))\mathbb{E}_0[I_y(s)] \right) ds \right. dt.
\] (3.18)
Then from (3.15), (3.16), (3.17), (3.18), (3.11) (the definition of $I$) and (3.12) (the definition of $\Lambda$) it follows that

$$I_1 + I_2 + I_3 + I_4 = \mathbb{E} \int_0^T \left\{ D^Y_t \left( \rho(T) \phi + D^Y_t (\rho(T)g(x(T))\mathbb{E}_0[\phi]) \right) \\
+ \int_t^T D^Y_s \left( \rho(s)l(u(s)) \right) ds + \int_t^T D^Y_s \left( \rho(s)f(x(s))\mathbb{E}_0[l_y(s)] \right) ds \\
- \left( \phi + g(x(T))\mathbb{E}_0[\phi] \right) \rho(T)h(t,x(t)) \\
- \int_t^T \left( l(u(s)) + f(x(s))\mathbb{E}_0[l_y(s)] \right) \rho(s)h(t,x(t)) ds \right\} h_x(t)x_1(t) dt$$

$$= \mathbb{E} \int_0^T h_x(t)x_1(t) \left( D^Y_t \Pi(t) - \Lambda(t) \right) dt. \quad (3.19)$$

Similarly, according to (2.3) and (2.6), we have

$$I_5 + I_6 = \mathbb{E} \left\{ \rho(T) \left( \phi_x + g' (x(T))\mathbb{E}_0[\phi] \right) \right\} \left[ \int_0^T \left( b_x(t)x_1(t) + b_v(t)v(t) \right) dt \\
+ \int_0^T \left( \sigma_x(t)x_1(t) + \sigma_v(t)v(t) \right) dW_1(t) \\
+ \int_0^T \int_{\mathbb{R}_0} \left[ \gamma_x(t,x(t),u,z)x_1(t) + \gamma_v(t,x(t),u,z)v(t) \right] \tilde{N}(dt, dz) \right\}$$

$$= \mathbb{E} \int_0^T \left\{ \rho(T) \left( \phi_x + g' (x(T))\mathbb{E}_0[\phi] \right) \right\} \left( b_x(t)x_1(t) + b_v(t)v(t) \right)$$

$$+ \left( \sigma_x(t)x_1(t) + \sigma_v(t)v(t) \right) D^W_1 \rho(T) \left( \phi_x + g' (x(T))\mathbb{E}_0[\phi] \right)$$

$$+ \int_{\mathbb{R}_0} \left[ \gamma_x(t,x(t),u,z)x_1(t) + \gamma_v(t,x(t),u,z)v(t) \right] D_{x,z} \rho(T) \left( \phi_x + g' (x(T))\mathbb{E}_0[\phi] \right) \mu(dz) \} dt. \quad (3.20)$$

and

$$I_7 + I_8 = \mathbb{E} \int_0^T \rho(t) \left( I_x(t) + f'(x(t))\mathbb{E}_0[l_y(t)] \right) \left\{ \int_0^t \left( b_x(s)x_1(s) + b_v(s)v(s) \right) ds \\
+ \int_0^t \left( \sigma_x(s)x_1(s) + \sigma_v(s)v(s) \right) dW_1(s) \\
+ \int_0^t \int_{\mathbb{R}_0} \left[ \gamma_x(s,x(s),u,z)x_1(s) + \gamma_v(s,x(s),u,z)v(s) \right] \tilde{N}(ds, dz) \right\}$$

$$= \mathbb{E} \int_0^T \int_0^t \left\{ \rho(t) \left( I_x(t) + f'(x(t))\mathbb{E}_0[l_y(t)] \right) \right\} \left( b_x(s)x_1(s) + b_v(s)v(s) \right)$$

$$+ \left( \sigma_x(s)x_1(s) + \sigma_v(s)v(s) \right) D^W_1 \rho(t) \left( I_x(t) + f'(x(t))\mathbb{E}_0[l_y(t)] \right)$$

$$+ \int_{\mathbb{R}_0} \left[ \gamma_x(s,x(s),u,z)x_1(s) + \gamma_v(s,x(s),u,z)v(s) \right] D_{x,z} \rho(t) \left( I_x(t) + f'(x(t))\mathbb{E}_0[l_y(t)] \right) \mu(dz) \} ds dt.$$
By the Fubini theorem, we have

\[ I_7 + I_8 = \mathbb{E} \int_0^T \int_s^T \left\{ \rho(t) \left( l_x(t) + f'(x(t)) \mathbb{E}_0[l_y(t)] \right) (b_x(s)x_1(s) + b_v(s)v(s)) \\
+ (\sigma_x(s)x_1(s) + \sigma_v(s)v(s)) D_{t}^{[W_t]} \rho(t) \left( l_x(t) + f'(x(t)) \mathbb{E}_0[l_y(t)] \right) \\
+ \int_{R_0} [\gamma_x(s,x(s-),u,z)x_1(s-) + \gamma_v(s,x(s-),u,z)v(s)] \\
D_{s,z} \rho(t) \left( l_x(t) + f'(x(t)) \mathbb{E}_0[l_y(t)] \right) \mu(dz) \right\} dt ds \]

\[ = \mathbb{E} \int_0^T \left\{ \int_t^T \rho(s) \left( l_x(s) + f'(x(s)) \mathbb{E}_0[l_y(s)] \right) ds \left( b_x(s)x_1(s) + b_v(s)v(t) \right) \\
+ (\sigma_x(s)x_1(t) + \sigma_v(t)v(t)) D_{t}^{[W_t]} \rho(s) \left( l_x(s) + f'(x(s)) \mathbb{E}_0[l_y(s)] \right) ds \\
+ \int_{t}^T \int_{R_0} [\gamma_x(t,x(t-),u,z)x_1(t-) + \gamma_v(t,x(t-),u,z)v(t)] \\
D_{t,z} \rho(s) \left( l_x(s) + f'(x(s)) \mathbb{E}_0[l_y(s)] \right) \mu(dz) ds \right\} dt. \tag{3.21} \]

Then it follows from (3.20) and (3.21) that

\[ I_5 + I_6 + I_7 + I_8 \]

\[ = \mathbb{E} \int_0^T \left\{ \Sigma(t) \left( b_x(t)x_1(t) + b_v(t)v(t) \right) + \left( \sigma_x(t)x_1(t) + \sigma_v(t)v(t) \right) D_{t}^{[W_t]} \Sigma(t) \\
+ \int_{R_0} [\gamma_x(t,x(t-),u,z)x_1(t-) + \gamma_v(t,x(t-),u,z)v(t)] D_{t,z} \Sigma(t) \mu(dz) \right\} dt. \tag{3.22} \]

We insert (3.19) and (3.22) into (3.14) to transform the equation \( \frac{d}{d\varepsilon} \mathcal{I}(\varepsilon) |_{\varepsilon=0} = 0 \) to

\[ \mathbb{E} \int_0^T \left[ \Sigma(t) b_x(t) + \sigma_x(t) D_{t}^{[W_t]} \Sigma(t) + h_x(t) \left( D_{t}^{[Y]} \Pi(t) - \Lambda(t) \right) \\
+ \int_{R_0} [\gamma_x(t,x(t-),u,z) D_{t,z} \Sigma(t) \mu(dz)] x_1(t) dt \\
+ \mathbb{E} \int_0^T \left[ \Sigma(t) b_v(t) + \sigma_v(t) D_{t}^{[W_t]} \Sigma(t) + \rho(t) l_v(t,x(t), \mathbb{E}_0[f(x(t))], u) \right. \\
+ \left. \int_{R_0} [\gamma_v(t,x(t-),u,z) D_{t,z} \Sigma(t) \mu(dz)] v(t) dt \right] = 0. \tag{3.23} \]

To simplify the equation (3.23), we take

\[ v(s) = \bar{\beta} I_{t,t+c}(s), \]

where \( \bar{\beta} = \beta(\omega) \) is a bounded \( \mathcal{F}_{t}^{Y} \)-measurable random variables, \( 0 \leq t \leq t+\tau \leq T. \) It is easy to see from (3.9) that

\[ x_1(s) = 0 \quad \text{for} \quad 0 \leq s \leq t. \tag{3.24} \]

Then (3.23) can be written as

\[ \mathcal{I}_1(\tau) + \mathcal{I}_2(\tau) = 0 \tag{3.25} \]
with

\[ f_1(\tau) = \mathbb{E} \int_t^T \left[ \Sigma(s) b_x(s) + \sigma_x(s) \sigma(\mathcal{W}_t) \Sigma(s) + h_x(s) \left( D_t^Y(u) \Pi(s) - \Lambda(s) \right) \right. \\
\left. + \int_{\mathbb{R}_0} \gamma_x(s, x(s-), u, z) D_x \Sigma(s) \mu(dz) \right] x_1(s) ds \]

and

\[ f_2(\tau) = \mathbb{E} \int_t^{t+\tau} \beta \left[ \Sigma(s) b_v(s) + \sigma_v(s) \sigma(\mathcal{W}_t) \Sigma(s) + \rho(s) I_v(s, x(s), \mathbb{E}_0[f(x(s))], u) \right. \\
\left. + \int_{\mathbb{R}_0} \gamma_v(s, x(s-), u, z) D_v \Sigma(s) \mu(dz) \right] ds. \]

Since (3.25) holds for all \( \tau \in [0, T-t] \) we differentiate it to obtain

\[ \frac{d}{d\tau} \bigg|_{\tau=0} f_1(\tau) + \frac{d}{d\tau} \bigg|_{\tau=0} f_2(\tau) = 0. \] (3.26)

First we compute \( \frac{d}{d\tau} \bigg|_{\tau=0} f_1(\tau) \). Note that with the special control \( v(s) = \beta I_{(t, t+\tau)}(s) \), we derive for \( s \geq t + \tau \)

\[ dx_1(s) = x_1(s) \left( b_x(s) ds + \sigma_x(s) d\mathcal{W}_1(s) + \int_{\mathbb{R}_0} \gamma_x(s, x(s-), u, z) \tilde{N}(ds, dz) \right), \]

Solving the above equation, we get

\[ x_1(s) = x_1(t+\tau) G(t+\tau, s), \quad s \geq t + \tau, \]

where

\[ x_1(t+\tau) = \beta \int_t^{t+\tau} \left( b_v(r) dr + \sigma_v(r) d\mathcal{W}_1(r) + \int_{\mathbb{R}_0} \gamma_v(r, x(r-), u, z) \tilde{N}(dr, dz) \right) \\
+ \int_t^{t+\tau} x_1(r) \left( b_x(r) dr + \sigma_x(r) d\mathcal{W}_1(r) + \int_{\mathbb{R}_0} \gamma_x(r, x(r-), u, z) \tilde{N}(dr, dz) \right). \]

Then

\[ \frac{d}{d\tau} f_1(\tau) \bigg|_{\tau=0} = \frac{d}{d\tau} \mathbb{E} \left[ \int_t^{t+\tau} H_x(s) x_1(1+\tau) G(t+\tau, s) ds \right] \bigg|_{\tau=0} \]

\[ = \int_t^{t+\tau} \frac{d}{d\tau} \mathbb{E} \left[ H_x(s) x_1(1+\tau) G(t+\tau, s) ds \right] \bigg|_{\tau=0} ds \]

\[ = \int_t^{t+\tau} \frac{d}{d\tau} \mathbb{E} \left[ x_1(1+\tau) \Theta(t, s) ds \right] \bigg|_{\tau=0} ds \]

\[ = f_{11} + f_{12}, \]

where

\[ f_{11} = \int_t^{t+\tau} \frac{d}{d\tau} \mathbb{E} \left\{ \Theta(t, s) \int_t^{t+\tau} x_1(r) \left( b_x(r) dr + \sigma_x(r) d\mathcal{W}_1(r) \right) + \int_{\mathbb{R}_0} \gamma_x(r, x(r-), u, z) \tilde{N}(dr, dz) \right\} \bigg|_{\tau=0} ds \]
and

\[ J_{12} = \int_t^T \frac{d}{d\tau} \mathbb{E} \left\{ \beta \Theta(t,s) \int_t^{\tau+\tau} \left[ b_v(r) dr + \sigma_v(r) dW_1(r) \right] + \int_{\mathbb{R}_0} \gamma_v(r,x(r-),u,z) \tilde{N}(dr,dz) \right\} ds. \]

According to (3.24), (3.25), (2.6) and the fact that \( x(t) = 0 \), it is not difficult to derive that

\[ J_{11} = 0 \]

and

\[ J_{12} = \mathbb{E} \int_t^T \beta \left( \Theta(t,s) b_v(t) + \sigma_v(t) D_{\tau}^{(W)} \Theta(t,s) + \int_{\mathbb{R}_0} \gamma_v(s,x(s-),u,z) D_{(\tau,s)} \Theta(t,s) \mu(dz) \right) ds. \quad (3.27) \]

Now we proceed to calculate the value of \( \frac{d}{d\tau} J_2(\tau)|_{\tau=0} \). As in the computation for \( \frac{d}{d\tau} J_2(\tau)|_{\tau=0} \) we have

\[ \frac{d}{d\tau} J_2(\tau)|_{\tau=0} = \mathbb{E} \left\{ \beta \left[ \Sigma(t) b_v(t) + \sigma_v(t) D_{\tau}^{(W)} \Sigma(t) + \rho(t) l_v(t) + \int_{\mathbb{R}_0} \gamma_v(t,x(t-),u,z) D_{(\tau,s)} \Sigma(t) \mu(dz) \right] \right\}. \quad (3.28) \]

From (3.13), (3.14), (3.27) and (3.28), the equation (3.26) becomes

\[ \mathbb{E} \left\{ \beta \left[ b_v(t) q(t) + \sigma_v(t) k(t) + \rho(t) l_v(t) + \int_{\mathbb{R}_0} \gamma_v(t,x(t-),u,z) r(t,z) \mu(dz) \right] \right\} = 0. \]

Since the above equality holds for any bounded \( \mathcal{F}_t^Y \)-measurable \( \beta \), we conclude that

\[ 0 = \mathbb{E} \left[ \left\{ H_v(t,x(t),\mathbb{E}_0[f(x(t))]),u(t);q(t),k(t),r(t,\cdot) \right\} | \mathcal{F}_t^Y \right]. \]

The proof of the theorem is then completed.

**An application to linear-quadratic control problem** We consider an economic quantity \( x^v(\cdot) \), which can be interpreted as cash-balance, wealth, and an intrinsic value process in different fields of insurance, mathematical finance, and mathematical economy, respectively. Suppose that \( x^v(\cdot) \) is governed by

\[
\begin{cases}
  dx^v(t) &= \left( A(t)x^v(t) + B(t)v(t) \right) dt + \left( C(t)x^v(t) + D(t)v(t) \right) dW_1(t) \\
  &+ \int_{\mathbb{R}_0} \left( F_i(z)x^v(t-)+G_i(z)v(t) \right) \tilde{N}(dt,dz), \quad t \in [0,T], \\
  x^v(0) &= x_0 \in \mathbb{R},
\end{cases}
\]

where \( v(\cdot) \) is the control strategy of a policymaker, and \( A(t), B(t), C(t), D(t), F_i(z) \) and \( G_i(z) \) are uniformly bounded \( \mathcal{F}_t^Y \)-adapted stochastic processes with value in \( \mathbb{R} \). In fact, it is possible for the policymaker to partially observe \( x(\cdot) \), due to the inaccuracies in measurements, discreteness of account information, or possible delay in the actual payments. See, e.g., Huang, Wang, and Wu.
to see from (3.29) and (3.30) that

subject to (3.29) and (3.30), where model:

\[ x(t) = \frac{1}{\beta} \alpha(t, x^v(t)) - \frac{1}{2} \beta \] \[ Y(0) = 0, \]

(3.30)

where \( x(\cdot) \) is the underlying factor which is partially observed through the observation \( Y(\cdot) \), \( \beta > 0 \) is a constant, and \( \alpha \) satisfies an assumption similar to \( h \) (see, e.g., Assumption (A2)). A typical example of \( Y(\cdot) \) in reality is the logarithm of the stock price \( S(\cdot) \) related to \( x(\cdot) \). Specifically, set \( S(t) = s_0 e^{\beta Y(t)} \) with a constant \( s_0 > 0 \). Obviously, the stock price \( S(\cdot) \) is the information available to the policymaker. Moreover, it follows from Itô’s formula that

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= \left[ \alpha(t, x^v(t)) dt + \beta dW_2(t) \right], \\
S(0) &= s_0,
\end{align*}
\]

Note that the above factor model is inspired by those of Nagai and Peng [10] and Xiong and Zhou [15].

Assume that the objective of the policymaker is to minimize

\[
J(v(\cdot)) = \frac{1}{2} \mathbb{E}_0 \left\{ \int_0^T \left[ L(t) \left( x^v(t) - \mathbb{E}_0[x^v(t)] \right)^2 \right] dt + N \left( x^v(T) - \mathbb{E}_0[x^v(T)] \right)^2 \right\},
\]

subject to (3.29) and (3.30), where \( M(t) \geq 0, L(t) \geq 0 \) are uniformly bounded deterministic functions with value in \( \mathbb{R} \) and \( M(t) \) is referred to as a dynamic benchmark. \( N \geq 0 \) is a constant. Equation (3.31) implies that the policymaker wants to not only prevent the control strategy from large deviation but also minimize the risk of the economic quantity.

In what follows, we solve the linear-quadratic problem with the help of Theorem 2.1. It is easy to see from (3.29) and (3.30) that

\[
\begin{align*}
b(t, x, v) &= A(t)x + B(t)v, \\
\sigma(t, x, v) &= C(t)x + D(t)v, \\
\gamma(t, x, v, z) &= F_t(z)x + G_t(z)v, \\
h(t, x) &= \frac{1}{\beta} \alpha(t, x) - \frac{1}{2} \beta.
\end{align*}
\]

As we know,

\[
\rho^v(t) = \exp \left\{ \int_0^t h(s, x^v(s)) dY(s) - \frac{1}{2} \int_0^t h^2(s, x^v(s)) ds \right\}.
\]

If \( u \) is the optimal control, then we denote \( \rho(t) = \rho^u(t), 0 \leq t \leq T \). The new adjoint processes are written as

\[
\begin{align*}
q(t) &= \Sigma(t) + \int_t^T H(s) G(t, s) ds, \\
k(t) &= D_{t}^{(t, y)} q(t), \\
r(t, z) &= D_{t, z} q(t),
\end{align*}
\]

with

\[
\Sigma(t) = N \rho(T) \left( x(T) - \mathbb{E}_0[x(T)] \right) + \int_t^T L(s) \rho(s) \left( x(s) - \mathbb{E}_0[x(s)] \right) ds.
\]
According to Theorem 2.1 and (3.32), we have the following proposition.

For any \( t \leq s \leq T \), then it is necessary to satisfy

\[
\Pi(t) = \frac{1}{2} \mathcal{N}(T) \left( x(T) - \mathbb{E}_0[x(T)] \right)^2 + \frac{1}{2} \int_t^T \rho(s) \left[ L(s) \left( x(s) - \mathbb{E}_0[x(s)] \right)^2 + \left( u(s) - M(s) \right)^2 \right] ds,
\]

\[
\Lambda(t) = \frac{1}{2} \mathcal{N}(T) \left( x(T) - \mathbb{E}_0[x(T)] \right)^2 h(t,x(t)) + \frac{1}{2} \int_t^T \rho(s) \left[ L(s) \left( x(s) - \mathbb{E}_0[x(s)] \right)^2 + \left( u(s) - M(s) \right)^2 \right] h(t,x(t)) ds.
\]

According to Theorem 2.1 and (3.32), we have the following proposition.

**Proposition 3.2** If \( u(\cdot) \) is an optimal control strategy and \( \rho(t), \frac{1}{\beta} \alpha_x(t,x) G(t,s) \in \mathbb{L}_{1,2}(\mathbb{R}), 0 \leq t \leq s \leq T \), then it is necessary to satisfy

\[
u(t) = M(t) - B(t) \mathbb{E}[q(t)|\mathcal{F}_t] - D(t) \mathbb{E}[D_t^{(W_1)} q(t)|\mathcal{F}_t] - \mathbb{E} \left[ \int_{\mathbb{R}_0} G_t(z) D_{t,z} q(t) \mu(dz) | \mathcal{F}_t \right].
\]

### 4 Maximum principle for jump-diffusion mean-field SDEs

In this section, we study the mean field stochastic optimal control problem to minimize (1.3). However, the system is given by a nonlinear SDE of mean-field type (which is also called McKean-Vlasov equations) with jumps, namely, (1.1). The observation is as (1.2) and we define the admissible control as Definition 1.1.

As for the first problem treated in the previous section, we need to deal with the problem of minimizing the performance functional (3.7) subject to new state equation (1.1) and the observation (1.2). The Radon-Nikodym derivative \( \rho^* \) is still given by (3.6). To obtain the maximum principle for this problem, we make the following assumptions in this section.

**H1** For any \( t \in [0,T] \) and \( z \in \mathbb{R}_0, b(t,x,y,v), \sigma(t,x,y,v) \) and \( \gamma(t,x,y,v,z) \) are continuously differentiable functions of \( x, y \) and \( v \) and their derivatives \( b_x, b_y, b_v, \sigma_x, \sigma_y, \sigma_v, \int_{\mathbb{R}_0} |\gamma_x(t, x, y, v, z)|^2 \mu(dz), \int_{\mathbb{R}_0} |\gamma_y(t, x, y, v, z)|^2 \mu(dz) \) and \( \int_{\mathbb{R}_0} |\gamma_v(t, x, y, v, z)|^2 \mu(dz) \) are uniformly bounded. Suppose also that there is a constant \( C > 0 \) such that

\[
|b(t,x,y,v)|^2 + |\sigma(t,x,y,v)|^2 + \int_{\mathbb{R}_0} |\gamma(t,x,y,v,z)|^2 \mu(dz) \leq C(1 + |x|^2 + |y|^2 + |v|^2).
\]
(H2) For any \( t \in [0, T] \), the function \( h \) is continuously differentiable with respect to \( x \) and its derivative \( h_x \) are uniformly bounded.

(H3) For any \( t \in [0, T] \), the functions \( l \) and \( \phi \) are continuously differentiable with respect to \((x, y, v) \in \mathbb{R} \times \mathbb{R} \times U \) and \((x, y) \in \mathbb{R} \times \mathbb{R} \), respectively. The derivatives of \( l \) and \( \phi \) are uniformly Lipschitz continuous. Moreover, there is a constant \( C > 0 \) such that

\[
|l(t, x, y, v)| + |\phi(x, y)| \leq C(1 + x^2 + y^2 + v^2),
\]

\[
|\phi_x(x, y)| + |\phi_y(x, y)| \leq C(1 + |x| + |y|),
\]

\[
|l_x(t, x, y, v)| + |l_y(t, x, y, v)| + |l_v(t, x, y, v)| \leq C(1 + |x| + |y| + |v|).
\]

\( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are both continuously differentiable with bounded derivatives \( f'(x) \) and \( g'(x) \).

All the above mentioned functions in (H1), (H2) and (H3) are deterministic.

Suppose that \( u(\cdot) \in \mathcal{U}_{ad} \) is an optimal control process and \( x(\cdot) \) is the corresponding state process. We want to obtain the maximum principle for \( u \) and \( x \). Namely, we want to find necessary conditions that \( u \) and \( x \) must satisfy. We shall follow the same argument as in the previous section. But we can no longer use Malliavin calculus because of the mean field's appearance in the state equation (4.1). Let \( v(\cdot) \) be another arbitrary control process in \( \mathcal{U}_{ad} \). Since \( \mathcal{U}_{ad} \) is convex, the following perturbed control process \( u^\varepsilon(\cdot) \) is also an element of \( \mathcal{U}_{ad} \):

\[
u^\varepsilon(t) = u(t) + \varepsilon (v(t) - u(t)), \quad 0 \leq \varepsilon \leq 1.
\]

We follow all the notations used in the previous section. For example, we denote by \( x^\varepsilon(\cdot) \) and \( \rho^\varepsilon(\cdot) \) the states of (4.1) and (3.6) along with the control \( u^\varepsilon(\cdot) \). When \( \varepsilon = 0 \), denote \( x = x(\cdot) \) and \( \rho = \rho(\cdot) \). Furthermore, suppose that \( v(\cdot) \in \mathcal{U}_{ad} \) such that \( v'(\cdot) = v(\cdot) - u(\cdot) \in \mathcal{U}_{ad} \), then \( v'(\cdot) + u(\cdot) \in \mathcal{U}_{ad} \).

The equation for the derivative \( \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} x^\varepsilon(t) \) will be

\[
\begin{cases}
dx(t) = & \left( b_x(t)x(t) + b_y(t)E_0[x(t)] + b_v(t)(v(t) - u(t)) \right)dt \\
& + \left( \sigma_x(t)x(t) + \sigma_y(t)E_0[x(t)] + \sigma_v(t)(v(t) - u(t)) \right)dW(t) \\
& + \int_{\mathbb{R}_0} \left[ \gamma_1(t, z)x(t) + \gamma_y(t, z)E_0[x(t)] + \gamma_v(t, z)(v(t) - u(t)) \right]N(dt, dz), \quad x_1(0) = 0,
\end{cases}
\]

and \( \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \rho^\varepsilon(t) \) will satisfy

\[
\begin{cases}
d\rho_1(t) = & \left( \rho_1(t)h_t(x(t)) + \rho(t)h_x(t)x(t) \right)dY(s), \\
\rho_1(0) = & 0,
\end{cases}
\]

where while the equation is exactly the same as (8.10) but with \( x_1(t) \) being given by (4.2). Obviously,

\[
\rho_1(t) = \rho(t) \left( \int_0^t h_x(s)x_1(s)ds \right) - \int_0^t h_x(s)h(s, x(s))x_1(s)ds, \quad 0 \leq t \leq T.
\]

In fact, we have the following
Lemma 4.1 Let assumptions (H1) and (H2) hold. Then

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{|x^\varepsilon(t) - x(t) - x_1(t)|^2}{\varepsilon} \right] = 0, \quad \lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{|p^\varepsilon(t) - p(t) - \rho_1(t)|^2}{\varepsilon} \right] = 0.
\]

Proof. By Lemma 4.3 in [16], we have \( \lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{|x^\varepsilon(t) - x(t) - x_1(t)|^2}{\varepsilon} \right] = 0 \). In order to prove the second equality, we apply the Itô formula to \( \eta(t) := \frac{p^\varepsilon(t) - p(t)}{\varepsilon} - \rho_1(t) \) to obtain

\[
\left\{ \begin{array}{l}
d\eta(t) = \left( \eta(t) h(t, x^\varepsilon(t)) + p(t) A^\varepsilon(t) \xi(t) + p(t) \left( A^\varepsilon(t) - h_x(t) \right) x_1(t) \\
\quad + \rho_1(t) \left( h(t, x^\varepsilon(t)) - h(t, x(t)) \right) \right) dt \\
\eta(0) = 0,
\end{array} \right.
\]

with \( A^\varepsilon(t) = \int_0^1 h_x(s) x(s) + \theta h_x(s, x_1(s) + \xi(s)) d\theta \), \( \xi(t) = \frac{x^\varepsilon(t) - x(t)}{\varepsilon} - x_1(t) \).

Then we have

\[
\mathbb{E} \eta^2(t) = \mathbb{E} \int_0^T \left( \eta(t) h(t, x^\varepsilon(t)) + p(t) A^\varepsilon(t) \xi(t) + p(t) \left( A^\varepsilon(t) - h_x(t) \right) x_1(t) \\
\quad + \rho_1(t) \left( h(t, x^\varepsilon(t)) - h(t, x(t)) \right) \right)^2 dt \leq K_0 \mathbb{E} \int_0^T \eta^2(t) dt + o(\varepsilon),
\]

where \( K_0 > 0 \) is a constant. Now the Gronwall inequality yields the lemma. \( \blacksquare \)

Since \( u(\cdot) \) is an optimal control, we have

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} J(u_\varepsilon(\cdot)) \geq 0.
\]

Using Lemma 4.1 and almost the same argument as for the equation (3.14), we obtain

Lemma 4.2 Under (H1), (H2) and (H3), if \( u(\cdot) \) is an optimal control and \( v(\cdot) \) is any given control process in \( \mathcal{U}_{ad} \) such that \( v(\cdot) - u(\cdot) \in \mathcal{U}_{ad} \), then we have

\[
\mathbb{E} \left\{ \int_0^T \left[ p(t) \left( l_x(t) x_1(t) + f'(x(t)) x_1(t) \mathbb{E}_0[l_x(t)] + l_v(t) (v(t) - u(t)) \right) \\
\quad + \rho_1(t) \left( f(x(t)) \mathbb{E}_0[l_y(t)] + l(u(t)) \right) \right] dt + \rho(T) \phi(x_1(T)) \\
\quad + \rho(T) x_1(T) g'(x(T)) \mathbb{E}_0[\phi_3] + \rho_1(T) \left( g(x(T)) \mathbb{E}_0[\phi_3] + \phi \right) \right\} \geq 0.
\]

(4.4)

Now we shall write the optimality condition (4.4) by a backward mean field stochastic differential equation. For any \( u(\cdot) \in \mathcal{U}_{ad} \) and the corresponding state trajectory \( x(\cdot) \), we define the first order adjoint process \( (p(\cdot), q(\cdot), R(\cdot, \cdot)) \) as follows:

\[
\left\{ \begin{array}{l}
-dp(t) = \left( b_x(t) p(t) + p(t) \mathbb{E}[b_y(t) p(t)] + \sigma_x(t) q(t) + p(t) \mathbb{E}[\sigma_y(t) q(t)] \\
\quad + p(t) \left( h_x(t) Q(t) + l_x(t) + f'(x(t)) \mathbb{E}_0[l_x(t)] \right) \right) + \int_{\mathbb{R}_0} \gamma_x(t, z) R(t, z) \mu(dz) \\
\quad + \int_{\mathbb{R}_0} p(t) \mathbb{E}[\gamma_y(t, z) R(t, z)] \mu(dz) dt - q(t) dW_1(t) - \int_{\mathbb{R}_0} R(t, z) \tilde{N}(dt, dz),
\end{array} \right.
\]

\[
p(T) = \rho(T) \left( \Phi_x + g'(x(T)) \math!\text{E}_0[\Phi_3] \right).
\]

(4.5)
where \((P(\cdot), Q(\cdot), G(\cdot, \cdot))\) is defined by

\[
\begin{align*}
-dP(t) &= \left(I(u(t)) + f(x(t))E_0[l_y(t)]\right)dt - Q(t)dW_2(t) - \int_{\mathbb{R}_0} G(t,z)\tilde{N}(dt,dz), \\
P(T) &= \phi + g(x(T))E_0[\phi_y],
\end{align*}
\]  

(4.6)

which is a mean-field backward stochastic differential equation (BSDE for short) and from \([16]\) this BSDE admits unique solution triplet \((p, q, R)\). Then we define the usual Hamiltonian associated with the mean-field stochastic control problem as follows

\[
H(t,x,y,v;p,q,R(\cdot),Q,\rho) = b(t,x,y,v)p + \sigma(t,x,y,v)q + \int_{\mathbb{R}_0} R(t,z)\gamma(t,x,y,v,z)\mu(dz)
+ h(t,x)Q + l(t,x,y,v)\rho.
\]

(4.7)

**Theorem 4.1** Under \((H1), (H2)\) and \((H3)\), if \(u(\cdot)\) is an optimal control and \(v(\cdot)\) is any given control process in \(\mathcal{U}_{ad}\) such that \(v(\cdot) - u(\cdot) \in \mathcal{U}_{ad}\), then it is necessary to satisfy that

\[
\mathbb{E}\left[ H_v(t,x(t),E_0[f(x(t))],u(t);p(t),q(t),R(t,\cdot),Q(\cdot),\rho(t))(v(t) - u(t)) \mid \mathcal{F}_t \right] \geq 0,
\]

(4.8)

where \((p(\cdot), q(\cdot), R(\cdot, \cdot))\) and \(Q(\cdot)\) are the solutions of \((4.5)\) and \((4.6)\), respectively.

**Proof.** Applying Itô’s formula to \(\rho_1(\cdot)P(\cdot)\) and \(p(\cdot)x_1(\cdot)\), we obtain

\[
\mathbb{E}\left[ \rho_1(T)\left(\phi + g(x(T))E_0[\phi_y]\right) \right] = \mathbb{E}\int_0^T \left[ \rho(t)Q(t)h_x(t)x_1(t) - \rho_1(t)\left(I(u(t)) + f(x(t))E_0[l_y(t)]\right) \\
+ f(x(t))E_0[l_y(t)] \right]dt
\]

and

\[
\mathbb{E}\left[ x_1(T)\rho(T)\left(\phi + g(x(T))E_0[\phi_y]\right) \right] = \mathbb{E}\int_0^T \left[ \left(b_v(t)p(t) + \sigma_v(t)q(t) \right) \\
+ \int_{\mathbb{R}_0} \gamma_v(t,z)R(t,z)\mu(dz) \right]v(t) - \left(I_x(t) + h_x(t)Q(t) + f'(x(t))E_0[l_y(t)] \right)Q(t)x_1(t) \right]dt.
\]

Inserting the above two equations into the variational inequality \((4.4)\), we have

\[
\mathbb{E}\int_0^T \left( b_v(t)p(t) + \sigma_v(t)q(t) + \int_{\mathbb{R}_0} \gamma_v(t,z)R(t,z)\mu(dz) + I_x(t) \right)(v(t) - u(t))dt \geq 0,
\]

Thus, the proof is completed. \(\blacksquare\)

**Remark 4.2** If \(u(\cdot)\) is a local minimum for the performance functional \(J\) (given by \((3.7)\)), in the sense that for all bounded \(v(\cdot) \in \mathcal{U}_{ad}\), there exists an \(\delta > 0\) such that \(u(\cdot) + \varepsilon v(\cdot) \in \mathcal{U}_{ad}\) for any \(\varepsilon \in (-\delta, \delta)\) and

\[
f(\varepsilon) = J(u(\cdot) + \varepsilon v(\cdot)), \quad \varepsilon \in (-\delta, \delta),
\]

attains its minimum at \(\varepsilon = 0\), then \((4.4)\) and hence \((4.8)\) are identities.

**Applications**
We aim to illustrate Theorem 4.1 by a linear-quadratic (LQ) example as in Section 3. Consider the following LQ optimal control problem with partial information. Namely, we want to minimize $J(v(\cdot))$, where

$$J(v(\cdot)) = \frac{1}{2} \mathbb{E}_0 \left\{ \int_0^T \left[ L(t) \left( x^v(t) - \mathbb{E}_0[x^v(t)] \right)^2 + O(t) \left( v(t) - M(t) \right)^2 \right] dt ight\}$$

subject to

$$\begin{cases}
  d x^v(t) &= \left( A(t)x^v(t) + B(t)\mathbb{E}_0[x^v(t)] + C(t)v(t) \right) dt + \left( D(t)x^v(t) + E(t)\mathbb{E}_0[x^v(t)] \right) dW_1(t) + \int_{\mathbb{R}_0} \left( S(t,z)x^v(t-z) + K(t,z)\mathbb{E}_0[x^v(t-z)] \right) dt \\
  &+ I(t,z)v(t) \hat{N}(dt,dz), \\
  x^v(0) &= x_0 \in \mathbb{R}.
\end{cases}$$

The observation is $dY(t) = h(t,x^v(t))dt + dW_2(t), \quad Y(0) = 0$.

Here $L(\cdot) \geq 0$, $O(\cdot) > 0$, $\frac{1}{2} M(\cdot) A(\cdot) + B(\cdot) C(\cdot) + C(\cdot) B(\cdot)$, $D(\cdot), E(\cdot), F(\cdot), S(\cdot, \cdot), K(\cdot, \cdot, \cdot), I(\cdot, \cdot, \cdot)$ are uniformly bounded and deterministic; $N \geq 0$ is a constant. $h$ satisfies the assumption (H2). Theorem 4.1 is valid. Thus, we define the Hamiltonian as below.

$$H(t,x,y,v;p,q,R(\cdot),Q,R) = \left( A(t)x + B(t)y + C(t)v \right) p + \left( D(t)x + E(t)y + F(t)v \right) q$$

$$+ \int_{\mathbb{R}_0} R(t,z) \left( S(t,z)x + K(t,z)y + I(t,z)v \right) \mu(dz) + h(t,x)Q$$

$$+ \frac{1}{2} pL(t)(x-y)^2 + \frac{1}{2} \rho O(t)(v-M(t))^2. \quad (4.9)$$

The corresponding adjoint process $(p(\cdot), q(\cdot), R(\cdot, \cdot))$ is defined as follows:

$$\begin{cases}
  -dp(t) &= \left[ A(t)p(t) + \rho(t) \mathbb{E}[B(t)p(t)] + D(t)q(t) + \rho(t) \mathbb{E}[E(t)q(t)] \right] \\
  &+ \rho(t) \left( h_x(t)Q(t) + L(t) \left( x(t) - \mathbb{E}_0[x(t)] \right) \right) \\
  &+ \int_{\mathbb{R}_0} S(t,z)R(t,z)\mu(dz) + \int_{\mathbb{R}_0} \rho(t) \mathbb{E}[K(t,z)R(t,z)]\mu(dz) dt \\
  -q(t)dW_1(t) - \int_{\mathbb{R}_0} R(t,z)\hat{N}(dt,dz), \\
  \left( p(T) = Np(T) \left( x(T) - \mathbb{E}_0[x(T)] \right) \right),
\end{cases}$$

where $(P(\cdot), Q(\cdot), G(\cdot, \cdot))$ is defined by

$$\begin{cases}
  -dP(t) &= \frac{1}{2} \left[ L(t) \left( x(t) - \mathbb{E}_0[x(t)] \right)^2 + O(t) \left( u(t) - M(t) \right)^2 \right] dt \\
  -Q(t)dW_1(t) - \int_{\mathbb{R}_0} G(t,z)\hat{N}(dt,dz), \\
  P(T) &= \frac{1}{2} N \left( x(T) - \mathbb{E}_0[x(T)] \right)^2.
\end{cases}$$

By Remark 4.2 if $u(\cdot)$ is a local minimum, then it is necessary to satisfy

$$\mathbb{E} \left[ \rho(t)O(t)(u(t) - M(t)) + C(t)p(t) + F(t)q(t) + \int_{\mathbb{R}_0} R(t,z)I(t,z)\mu(dz) | \mathcal{F}_t^Y \right] = 0,$$
where \((p(\cdot), q(\cdot), R(\cdot, \cdot))\) is the solution to (4.10). Then

\[
u(t) = \frac{1}{\rho(t) O(t)} \mathbb{E} \left[ C(t)p(t) + F(t)q(t) + \int_{\mathbb{R}_0} R(t, z) I(t, z) \mu(dz) \mid \mathcal{F}_t \right] + M(t). \tag{4.11}
\]

**Proposition 4.3** If \(u(\cdot)\) is an optimal control strategy, then it is necessary to satisfy (4.11).

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