QUANTITATIVE RECURRENCE PROPERTIES FOR SELF-CONFORMAL SETS

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Abstract. In this paper we study the quantitative recurrence properties of self-conformal sets $X$ equipped with the map $T : X \rightarrow X$ induced by the left shift. In particular, given a function $\varphi : \mathbb{N} \rightarrow (0, \infty)$, we study the metric properties of the set

$$R(T, \varphi) = \{x \in X : |T^n x - x| < \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Our main result is that under the open set condition, for the natural measure $\mu$ supported on $X$, $\mu$-almost every $x \in X$ is contained in $R(T, \varphi)$ when an appropriate volume sum diverges. We also prove a complementary result which states that for self-similar sets satisfying the open set condition, when the volume sum converges, then $\mu$-almost every $x \in X$ is not contained in $R(T, \varphi)$.

1. Introduction

The notion of recurrence is of central importance within Dynamical Systems and Ergodic Theory. A well known theorem due to Poincaré states that if $(X, \mathcal{B}, \mu)$ is a probability space, and $T : X \rightarrow X$ is a measure preserving transformation, then for any $E \in \mathcal{B}$ we have

$$\mu(\{x \in E : T^n x \in E \text{ for infinitely many } n \in \mathbb{N}\}) = \mu(E).$$

If $X$ is endowed with a metric $d$ so that $(X, d)$ is a separable metric space, and $\mathcal{B}$ is the Borel $\sigma$-algebra, then Poincaré’s theorem allows us to conclude the following topological statement:

$$(1.1) \quad \lim_{n \to \infty} \inf d(T^n x, x) = 0$$

for $\mu$-a.e. $x \in X$. We call $(X, \mathcal{B}, \mu, d, T)$ a metric measure-preserving system or an m.m.p.s. The information provided by $(1.1)$ is qualitative in nature. It tells us nothing about the speed at which an orbit can recur upon its initial point. One of the first general quantitative recurrence results was proved by Boshernitzan in [4].
Theorem 1.1 (\[4\]). Let \((X, \mathcal{B}, \mu, d, T)\) be a m.m.p.s. Assume that for some \(\alpha > 0\) the 
\(\alpha\)-dimensional Hausdorff measure \(\mathcal{H}^{\alpha}\) is \(\sigma\)-finite on \((X, d)\). Then for \(\mu\)-a.e \(x \in X\) we have 
\[
\liminf_{n \to \infty} n^{1/\alpha} d(T^nx, x) < \infty.
\]
Moreover, if \(\mathcal{H}^{\alpha}(X) = 0\) then for \(\mu\)-a.e \(x \in X\) 
\[
\liminf_{n \to \infty} n^{1/\alpha} d(T^nx, x) = 0.
\]

Building upon the work of Boshernitzan, Barreira and Saussol in [2] showed how the 
lower local dimension of a measure can be used to obtain quantitative recurrence results.

Theorem 1.2 ([2]). If \(T : X \to X\) is a Borel measurable map on \(X \subset \mathbb{R}^d\), and \(\mu\) is a 
\(T\)-invariant Borel probability measure on \(X\), then for \(\mu\)-almost every \(x \in X\), we have 
\[
\liminf_{n \to \infty} d(T^nx, x) = 0 \text{ for any } \alpha > \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.
\]

A suitable framework for describing recurrence quantitatively is the following. Given 
\((X, \mathcal{B}, \mu, d, T)\) an m.m.p.s. and \(\varphi : \mathbb{N} \times X \to (0, \infty)\), let 
\[
R(T, \varphi) := \{x \in X : d(T^nx, x) < \varphi(n, x) \text{ for infinitely many } n \in \mathbb{N}\}.
\]
Typically one is interested in determining the metric properties of \(R(T, \varphi)\) and relating 
these properties to \(T\) and \(\varphi\). This was the line of research pursued by Tan and Wang in [17] 
where they calculated the Hausdorff dimension of \(R(T, \varphi)\) when \(T\) is the \(\beta\)-transformation. 
Later Seuret and Wang proved similar results for self-conformal sets in [15]. As remarked 
upon by Chang et al in [5], very few results exist on the Hausdorff measure of \(R(T, \varphi)\). 
In this paper we continue the line of research instigated in [5] and obtain results on the 
Hausdorff measure of \(R(T, \varphi)\) when \(X\) is a self-conformal set and \(T\) is the natural map 
induced by the left shift. Before introducing our problem formally, we would like to mention 
a related topic and include some references.

The shrinking target problem is concerned with determining the speed at which the orbit 
of a \(\mu\)-typical point accumulates on a fixed point \(x_0\). The shrinking target problem and 
the problem of obtaining quantitative recurrence results have many common features. One 
can define a suitable analogue of the set \(R(T, \varphi)\) and ask what are its metric properties. 
For the shrinking target problem much more is known about the Hausdorff measure of this 
set see [6] [9], for results on the Hausdorff dimension of this set see [10] [13] [18].
1.1. Statement of results. Let $V \subset \mathbb{R}^d$ be an open set, a $C^1$ map $\phi : V \to \mathbb{R}^d$ is a conformal mapping if it preserves angles. Equivalently $\phi$ is a conformal mapping if the differential $\phi'$ satisfies $|\phi'(x)y| = |\phi'(x)||y|$ for all $x \in V$ and $y \in \mathbb{R}^d$. Let $\Phi = \{\phi_i\}_{i \in D}$ be a finite set of contractions on a compact set $Y \subset \mathbb{R}^d$, i.e. there exists $r \in (0,1)$ such that $|\phi_i(x) - \phi_i(y)| \leq r|x - y|$ for all $x,y \in Y$. We say that $\Phi$ is a conformal iterated function system if each $\phi_i$ can be extended to an injective conformal contraction on some open connected neighbourhood $V$ that contains $Y$ and $0 < \inf_{x \in V} |\phi_i'(x)| \leq \sup_{x \in V} |\phi_i'(x)| < 1$. Throughout this paper we will assume that the differentials are Hölder continuous. This means there exists $\alpha > 0$ and $c > 0$ such that

$$||\phi_i'(x)| - |\phi_i'(y)|| \leq c|x - y|^{\alpha}$$

for all $x,y \in V$. A well known result due to Hutchinson [11] implies that for any conformal iterated function system there exists a unique non-empty compact set $X \subset \mathbb{R}^d$ such that

$$X = \bigcup_{i \in D} \phi_i(X).$$

We call the set $X$ the self-conformal set of $\Phi$. The simplest self-conformal sets are those that arise from similarities. We say that a contraction $\phi : Y \to Y$ is a similarity if $\phi = r \cdot O + t$ for some $r \in (0,1)$, some $d \times d$ orthogonal matrix $O$, and some $t \in \mathbb{R}^d$. Clearly any similarity is a conformal mapping. When $\Phi$ consists solely of similarities we say that $X$ is a self-similar set.

In what follows, if $I = (i_1, \ldots, i_n)$ then we let $\phi_I = \phi_{i_1} \circ \cdots \circ \phi_{i_n}$, $X_I = \phi_I(X)$, and $|I|$ will denote the length of $I$. We may refer to the set $X_I$ as a cylinder or a cylinder set. For a word $I \in \cup_n D^n$ we let $I^\infty$ denote the element of $D^\infty$ obtained by repeating $I$ indefinitely. Similarly for $k \geq 1$ we let $I^k$ denote the word $I$ repeated $k$ times. When $\Phi$ consists of similarities then we let $O_I = O_{i_1} \circ \cdots \circ O_{i_n}$ and $r_I = \prod_{m=1}^n r_{i_m}$.

We say that a conformal iterated function system $\Phi$ satisfies the open set condition if there exists an open set $O \subset \mathbb{R}^d$ such that $\phi_i(O) \subseteq O$ for all $i \in D$ and $\phi_i(O) \cap \phi_j(O) = \emptyset$ for $i \neq j$. Under the open set condition, the Hausdorff dimension of the self-conformal set $X$ is the unique solution to $P(s) = 0$ where

$$P(s) := \lim_{n \to \infty} \frac{1}{n} \log \inf_{x \in X} \sum_{I \in D^n} |\phi_I(x)|^s = \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} \sum_{I \in D^n} |\phi_I(x)|^s.$$  

When $\Phi$ consists of similarities then the unique solution to $P(s) = 0$ can be expressed more succinctly as the unique solution to

$$(1.2) \quad \sum_{i \in D} r_i^s = 1.$$
In what follows we denote the Hausdorff dimension of a self-conformal set satisfying the open set condition by $\gamma$, i.e. $\gamma$ is the unique solution to $P(s) = 0$. The conformal iterated system we are referring to should be clear from the context. In addition to $\gamma$ being given by the zero of some appropriate pressure equation, under the open set condition it is known that $\mathcal{H}^\gamma(X)$ is positive and finite. Moreover under the open set condition $X$ is Ahlfors regular, this means that there exists $C > 1$ such that
\begin{equation}
\frac{r^\gamma}{C} \leq \mathcal{H}^\gamma(X \cap B(x, r)) \leq Cr^\gamma
\end{equation}
for all $x \in X$ and $0 < r < \text{Diam}(X)$. These results are well known and date back to the work of Ruelle [14]. For a proof see [8].

One can encode elements of a self-conformal set using sequences in $\mathcal{D}^\mathbb{N}$ as follows. Let $\pi: \mathcal{D}^\mathbb{N} \to X$ be given by
\[
\pi((i_m)) = \lim_{n \to \infty} (\phi_{i_1} \circ \cdots \circ \phi_{i_n})(0).
\]
The map $\pi$ is surjective. Moreover, equipping $\mathcal{D}^\mathbb{N}$ with the product topology it can be shown that $\pi$ is continuous. For any $x \in X$, we call any sequence $(i_m) \in \mathcal{D}^\mathbb{N}$ such that $\pi((i_m)) = x$ a coding of $x$. Without any separation hypothesis on the IFS, it is possible that a typical $x \in X$ will have multiple, possibly infinitely many, distinct codings. However, assuming the open set condition, $\mathcal{H}^\gamma$-almost every $x \in X$ has a unique coding. This follows from Theorem 3.7. from [12]. With this fact in mind we now define our map $T : X \to X$ induced by the left shift on $\mathcal{D}^\mathbb{N}$. Let $Tx = \pi((i_{m+1}))$ where $(i_m)$ is an arbitrary choice of coding for $x$. Since $\mathcal{H}^\gamma$-almost every $x \in X$ has a unique coding, it follows from the definition of $T$ that $T^n x = \pi((i_{m+n}))$ for $\mathcal{H}^\gamma$-almost every $x \in X$ for any $n \in \mathbb{N}$. We will only be interested in statements which hold for $\mathcal{H}^\gamma$-almost every $x \in X$. As such we can effectively ignore those points with multiple codings and assume that $T$ maps $x$ to the point whose coding is the unique coding of $x$ with the first digit removed.

Recalling the definition of $R(T, \varphi)$ from our introduction, and taking $d$ to be the usual Euclidean metric, we may now state our main result.

**Theorem 1.3.** Let $\Phi$ be a conformal iterated function system satisfying the open set condition and $\varphi : \mathbb{N} \to (0, \infty)$. Then the following statements are true:

1. If $\Phi$ consists of similarities and $\sum_{n=1}^\infty \varphi(n)^\gamma < \infty$ then $\mathcal{H}^\gamma(R(T, \varphi)) = 0$.
2. If $\sum_{n=1}^\infty \varphi(n)^\gamma = \infty$ then $\mathcal{H}^\gamma(R(T, \varphi)) = \mathcal{H}^\gamma(X)$.

Theorem 1.3 was proved in [5] by Chang et al for homogeneous self-similar sets in $\mathbb{R}$ satisfying the strong separation condition (i.e. $\phi_i(X) \cap \phi_j(X) = \emptyset$ for all $i \neq j$). As such Theorem 1.3 significantly improves upon [5] as it allows for a more general class of iterated
function systems and has weaker separation hypothesis.

**Notation.** Given two positive real valued functions $f$ and $g$ defined on some set $S$, we write $f \preceq g$ if there exists a positive constant $C$ such that $f(x) \leq C g(x)$ for all $x \in S$. Similarly we write $f \succeq g$ if $g \preceq f$. We write $f \asymp g$ if $f \preceq g$ and $f \succeq g$.

2. **Proof of Theorem 1.3**

2.1. **Proof of Theorem 1.3.1. (Convergence part).** Throughout this section we assume that $\Phi$ consists of similarities. The proof of the convergence part of Theorem 1.3 will be a consequence of the following lemma.

**Lemma 2.1.** For $n$ sufficiently large, for each $I \in \mathcal{D}^n$ there exists a point $x_I \in \mathbb{R}^d$ such that

$$\{x \in X : |T^n x - x| < \varphi(n)\} \subseteq \bigcup_{I \in \mathcal{D}^n} B(x_I, 2r_I \varphi(n)).$$

**Proof.** Let $x \in X$ and suppose that $I \in \mathcal{D}^n$ is such that $x = \phi_I(T^n x)$. Then we have

$$x = r_I O_I T^n x + s_I,$$

where $s_I \in \mathbb{R}^d$ depends solely upon $I$. Rearranging (2.1) we have

$$(2.2) \quad T^n x = O_I^{-1} \left(\frac{x - s_I}{r_I}\right).$$

If we assume $x$ satisfies $|T^n x - x| < \varphi(n)$, then (2.2) implies

$$|(O_I^{-1} - r_I I_d)x - O_I^{-1} s_I| < r_I \varphi(n).$$

Here $I_d$ is the $d \times d$ identity matrix. For $n$ sufficiently large the linear map $O_I^{-1} - r_I I_d$ is invertible for any $I \in \mathcal{D}^n$. Equation (2.3) therefore tells us that for $n$ sufficiently large

$$(2.4) \quad \{x \in X : |T^n x - x| < \varphi(n)\} \subseteq \bigcup_{I \in \mathcal{D}^n} (O_I^{-1} - r_I I_d)^{-1} (B(O_I^{-1} s_I, r_I \varphi(n))).$$

The image of a unit ball under the linear map $(O_I^{-1} - r_I I_d)^{-1}$ is an ellipse whose semi-axes have lengths equal to the singular values of $(O_I^{-1} - r_I I_d)^{-1}$. Importantly the singular values of a matrix when interpreted as a function from $\mathbb{R}^d \to \mathbb{R}^d$ is continuous. Moreover, as the group of $d \times d$ orthogonal matrices is compact, the singular value function restricted to some small neighbourhood of the set of $d \times d$ orthogonal matrices is uniformly continuous. Since the singular values of an orthogonal matrix are all equal to 1, it follows from uniform continuity that for $n$ sufficiently large we have

$$(O_I^{-1} - r_I I_d)^{-1} (B(O_I^{-1} s_I, r_I \varphi(n))) \subseteq B(x_I, 2r_I \varphi(n)).$$
for any $I \in \mathcal{D}^n$ where $x_I := (O_I^{-1} - r_I I_d)^{-1}(O_I^{-1} s_I)$. Combining this fact with (2.4) we have shown that
\[
\{ x \in X : |T^n x - x| < \varphi(n) \} \subseteq \bigcup_{I \in \mathcal{D}^n} B(x_I, 2r_I \varphi(n))
\]
for $n$ sufficiently large. \hfill \Box

**Proof of Theorem 1.3.1.** Assume $\varphi$ is such that $\sum_{n=1}^{\infty} \varphi(n)^{\gamma} < \infty$. By Lemma 2.1 and the definition of Hausdorff measure (see [7]) we have the following:

\[
\mathcal{H}^{\gamma}(R(T, \varphi)) \leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} \sum_{I \in \mathcal{D}^n} Diam(B(x_I, 2r_I \varphi(n)))^\gamma \\
= \liminf_{N \to \infty} \sum_{n=N}^{\infty} \sum_{I \in \mathcal{D}^n} (4r_I \varphi(n))^\gamma \\
= \liminf_{N \to \infty} 4^\gamma \sum_{n=N}^{\infty} \varphi(n)^{\gamma} \sum_{I \in \mathcal{D}^n} r_I^{\gamma} \\
\overset{(2.2)}{=} \liminf_{N \to \infty} 4^\gamma \sum_{n=N}^{\infty} \varphi(n)^{\gamma} \\
= 0.
\]

In the last line we used our assumption $\sum_{n=1}^{\infty} \varphi(n)^{\gamma} < \infty$. \hfill \Box

2.2. **Proof of Theorem 1.3.2. (Divergence part).** Our proof of the divergence part of Theorem 1.3 is based upon the proof of Theorem 1.4. from [1]. Before diving into our proof we state some properties of self-conformal sets.

Suppose $\Phi$ is a conformal iterated function system satisfying the open set condition. For any $I \in \bigcup_n \mathcal{D}^n$ we let
\[
\tilde{X}_I := \{ x \in X_I : x \text{ has a unique coding} \}.
\]
Since $\mathcal{H}^{\gamma}$-almost every $x \in X$ has a unique coding, we have
\[
\mathcal{H}^{\gamma}(X_I) = \mathcal{H}^{\gamma}(\tilde{X}_I) \tag{2.5}
\]
for any $I \in \bigcup_n \mathcal{D}^n$. Let $\mu := \mathcal{H}^{\gamma}|_X$ be the $\gamma$-dimensional Hausdorff measure restricted to $X$. The properties stated below are well known for cylinders without the unique coding restriction. These properties still hold for the sets $\tilde{X}_I$ because of (2.5).

We have the following:
• For any \( n \in \mathbb{N} \) and \( I, J \in \mathcal{D}^n \) such that \( I \neq J \), we have

\[
\mu(\tilde{X}_I \cap \tilde{X}_J) = 0.
\]

• For any \( I, J \in \bigcup_n \mathcal{D}^n \)

\[
\mu(\tilde{X}_{IJ}) \asymp \mu(\tilde{X}_I) \mu(\tilde{X}_J).
\]

• For any \( I \in \bigcup_n \mathcal{D}^n \)

\[
\mu(\tilde{X}_I) \asymp \text{Diam}(X_I)^\gamma.
\]

• There exists \( \kappa \in (0, 1) \) such that for any \( I \in \bigcup_n \mathcal{D}^n \)

\[
\mu(\tilde{X}_I) \leq \kappa^{|I|}.
\]

• Let \( x \in X \) and \((i_m) \in \mathcal{D}^\mathbb{N}\) be a coding of \( x \). For any \( 0 < r < \text{Diam}(X) \) there exists \( N \in \mathbb{N} \) such that

\[
X_{i_1, \ldots, i_N} \subset B(x, r) \text{ and } \text{Diam}(X_{i_1, \ldots, i_N}) \leq r.
\]

• For any \( n \in \mathbb{N} \) we have

\[
\sum_{I \in \mathcal{D}^n} \mu(\tilde{X}_I) = \mu(X).
\]

Similarly, for any \( J \in \bigcup_n \mathcal{D}^n \) and \( n \geq |J| \) we have

\[
\sum_{J \text{ is a prefix of } I} \mu(\tilde{X}_I) = \mu(\tilde{X}_J).
\]

In the above we have denoted the concatenation of two words \( I \) and \( J \) by \( IJ \). Property (2.6) follows from Theorem 3.7 from [12]. For a proof of the remaining properties see [8] and [14]. Properties (2.7), (2.8), and (2.9) are essentially a consequence of the fact that \( \mu \) is equivalent to the pushforward of a suitably defined Gibbs measure for a particular H"older continuous potential. The proof of (2.10) is standard. Properties (2.11) and (2.12) are a consequence of (2.6) and the fact \( X = \bigcup_{I \in \mathcal{D}^\mathbb{N}} \phi_I(X) \) for any \( n \in \mathbb{N} \).

We will also need the following two lemmas.

**Lemma 2.2.** Let \( X \) be a compact set in \( \mathbb{R}^d \) and let \( \mu \) be a finite doubling measure on \( X \) such that any open set is \( \mu \)-measurable. Let \( E \) be a Borel subset of \( X \). Assume that there are constants \( r_0, c > 0 \) such that for any ball \( B \) with radius less than \( r_0 \) and centre in \( X \) we have

\[
\mu(E \cap B) > c \mu(B).
\]

Then \( \mu(X \setminus E) = 0 \).
For a proof of Lemma 2.2 see [3, §8]. Note that a measure \( \mu \) supported on a compact set \( X \) is doubling if there exists a constant \( C > 1 \) such that for any \( x \in X \) and \( r > 0 \) we have
\[
\mu(B(x, 2r)) \leq C\mu(B(x, r)).
\]
Since \( X \) is Ahlfors regular it follows from (1.3) that \( \mu \) is automatically a doubling measure.

**Lemma 2.3.** Let \( X \) be a compact set in \( \mathbb{R}^d \) and let \( \mu \) be a finite measure on \( X \). Also, let \( E_n \) be a sequence of \( \mu \)-measurable sets such that \( \sum_{n=1}^{\infty} \mu(E_n) = \infty \). Then
\[
\mu(\limsup_{n \to \infty} E_n) \geq \limsup_{Q \to \infty} \frac{\left( \sum_{n=1}^{Q} \mu(E_n) \right)^2}{\sum_{n,m=1}^{Q} \mu(E_n \cap E_m)}.
\]

For a proof of Lemma 2.3 see [16, Lemma 5].

We may now proceed with our proof of the divergence part of Theorem 1.3. Let \( I = (i_1, \ldots, i_n) \in \mathcal{D}^n \) and consider the ball \( B(\pi(I^\infty), \varphi(n)/2) \). Applying (2.10) we know that there exists \( k_I \geq 0 \) and \( 1 \leq s_I \leq n - 1 \) such that
\[
X_{I^{k_I}(i_1, \ldots, i_{s_I})} \subseteq B(\pi(I^\infty), \varphi(n)/2)
\]
and
\[
\text{Diam}(X_{I^{k_I}(i_1, \ldots, i_{s_I})}) \approx \frac{\varphi(n)}{2}.
\]

Now consider the set \( \tilde{X}_{I^{k_I+1}(i_1, \ldots, i_{s_I})} \). For any \( x \in \tilde{X}_{I^{k_I+1}(i_1, \ldots, i_{s_I})} \) we have \( T^n x \in \tilde{X}_{I^{k_I}(i_1, \ldots, i_{s_I})} \). Moreover, since
\[
\tilde{X}_{I^{k_I+1}(i_1, \ldots, i_{s_I})} \subseteq X_{I^{k_I}(i_1, \ldots, i_{s_I})} \subseteq B(\pi(I^\infty), \varphi(n)/2),
\]
we may conclude by the triangle inequality that if \( x \in \tilde{X}_{I^{k_I+1}(i_1, \ldots, i_{s_I})} \) then \( |T^n x - x| < \varphi(n) \).

So if we let
\[
E'_n = \bigcup_{I \in \mathcal{D}^n} \tilde{X}_{I^{k_I+1}(i_1, \ldots, i_{s_I})}
\]
then
\[
\limsup_{n \to \infty} E'_n \subseteq R(T, \varphi).
\]

To prove the divergence part of Theorem 1.3 it suffices to show that \( \mu(\limsup_{n \to \infty} E'_n) = \mu(X) \). To do this we will apply Lemma 2.2. As such let us fix an arbitrary ball \( B \) with centre in \( X \) and radius less then \( \text{Diam}(X) \). Applying (2.10) we know that there exists \( J \in \cup_n \mathcal{D}^n \) such that \( X_J \subseteq B \) and \( \text{Diam}(X_J) \approx \text{Radius}(B) \). By (2.8) and (1.3) we know that \( \mu(X_J) \approx \mu(B) \). Therefore to prove the divergence part of Theorem 1.3 instead of proving that there exists \( c > 0 \) such that \( \mu(\limsup_{n \to \infty} E'_n \cap B) > c\mu(B) \), it suffices to show that there exists \( c > 0 \) such that \( \mu(\limsup_{n \to \infty} E'_n \cap X_J) > c\mu(X_J) \).
For $n \geq |J|$ if we let

$$E_n = \bigcup_{I \in \mathcal{P}^n} \tilde{X}_{I^{k+1}(i_1, \ldots, i_s)}$$

then $\limsup_{n \to \infty} E_n \subset \limsup_{n \to \infty} E'_n \cap X_J$. Therefore to prove the divergence part of Theorem 1.3 it suffices to show that there exists $c > 0$ such that

$$\limsup_{n \to \infty} E_n > c \mu(X_J).$$

To do this we will use Lemma 2.3. To use this lemma we first have to check $\sum_{n=|J|}^{\infty} \mu(E_n) = \infty$.

**Lemma 2.4.** We have $\sum_{n=|J|}^{Q} \mu(E_n) \sim \mu(X_J) \sum_{n=|J|}^{\infty} \varphi(n)^\gamma$.

**Proof.** The following holds

$$\sum_{n=|J|}^{Q} \mu(E_n) = \sum_{n=|J|}^{Q} \mu \left( \bigcup_{I \in \mathcal{P}^n} \tilde{X}_{I^{k+1}(i_1, \ldots, i_s)} \right)$$

$$\overset{(2.6)}{=} \sum_{n=|J|}^{Q} \sum_{I \in \mathcal{P}^n} \mu(\tilde{X}_{I^{k+1}(i_1, \ldots, i_s)})$$

$$\overset{(2.7)}{=} \sum_{n=|J|}^{Q} \sum_{I \in \mathcal{P}^n} \mu(\tilde{X}_I) \mu(\tilde{X}_{I^{k+1}(i_1, \ldots, i_s)})$$

$$\overset{(2.8)}{=} \sum_{n=|J|}^{Q} \varphi(n)^\gamma \sum_{I \in \mathcal{P}^n} \mu(\tilde{X}_I)$$

$$\overset{(2.14)}{=} \mu(\tilde{X}_J) \sum_{n=|J|}^{Q} \varphi(n)^\gamma$$

$$\overset{(2.15)}{=} \mu(X_J) \sum_{n=|J|}^{Q} \varphi(n)^\gamma.$$

Lemma 2.4 shows that when $\sum_{n=1}^{\infty} \varphi(n)^\gamma = \infty$ the sequence $(E_n)$ satisfies the hypothesis of Lemma 2.3. It will also be used in some of our later calculations.
Lemma 2.5. Let $I \in D^n$ be such that $J$ is a prefix of $I$. Then for $m > n$ we have

$$\mu(\bar{X}_{I^{k+1}(i_1, \ldots, i_s)} \cap E_m) \leq \mu(\bar{X}_J) \mu(\bar{X}_{i_{j+1}, \ldots, i_n}) \kappa^{m-n} \phi(m)^\gamma + \mu(\bar{X}_J) \mu(\bar{X}_{i_{j+1}, \ldots, i_n}) \phi(m)^\gamma \phi(n)^\gamma.$$  

Proof. Fix $I \in D^n$ such that $J$ is a prefix of $I$ and let $m > n$. It is useful to consider two separate cases. We first consider the case where $m \leq (k + 1)n + s_I$.

If $m \leq (k + 1)n + s_I$ then there exists at most one $\tilde{I} \in D^m$ such that

$$\mu(\bar{X}_{\tilde{I}^{k+1}(i_1, \ldots, i_s)} \cap \bar{X}_{I^{k+1}(i_1, \ldots, i_s)}) > 0.$$  

Therefore

$$\mu(\bar{X}_{I^{k+1}(i_1, \ldots, i_s)} \cap E_m) = \mu(\bar{X}_{I^{k+1}(i_1, \ldots, i_s)} \cap \bar{X}_{I^{k+1}(\tilde{i}_1, \ldots, \tilde{i}_s)})$$  

$$\leq \mu(\bar{X}_{I^{k+1}(\tilde{i}_1, \ldots, \tilde{i}_s)})$$  

$$\leq \mu(\bar{X}_J) \mu(\bar{X}_{I^{k+1}(\tilde{i}_1, \ldots, \tilde{i}_s)})$$  

$$\leq \mu(\bar{X}_J) \mu(\bar{X}_{i_{j+1}, \ldots, i_n}) \mu(\bar{X}_{i_{n+1}, \ldots, \tilde{i}_m}) \mu(\bar{X}_{I^{k+1}(\tilde{i}_1, \ldots, \tilde{i}_s)})$$  

$$\leq \mu(\bar{X}_J) \mu(\bar{X}_{i_{j+1}, \ldots, i_n}) \kappa^{m-n} \mu(\bar{X}_{I^{k+1}(\tilde{i}_1, \ldots, \tilde{i}_s)})$$  

$$\leq \mu(\bar{X}_J) \mu(\bar{X}_{i_{j+1}, \ldots, i_n}) \kappa^{m-n} \phi(m)^\gamma.$$  

Summarising the above, we have shown that if $n < m \leq (k + 1)n + s_I$ then

$$(2.15) \quad \mu(\bar{X}_{I^{k+1}(i_1, \ldots, i_s)} \cap E_m) \leq \mu(\bar{X}_J) \mu(\bar{X}_{i_{j+1}, \ldots, i_n}) \kappa^{m-n} \phi(m)^\gamma.$$  

Now suppose $m > (k + 1)n + s_I$. Then

$$\mu(\bar{X}_{I^{k+1}(i_1, \ldots, i_s)} \cap E_m) = \sum_{I \in D^m} \mu(\bar{X}_{I^{k+1}(\tilde{i}_1, \ldots, \tilde{i}_s)})$$  

$$\sum_{I \in D^m} \mu(\bar{X}_I) \mu(\bar{X}_{I^{k+1}(\tilde{i}_1, \ldots, \tilde{i}_s)})$$  

$$= \sum_{I \in D^m} \mu(\bar{X}_I) \phi(m)^\gamma.$$  

Therefore
Adding together the bounds provided by (2.15) and (2.16), we see that for any $\mu$:

\begin{align*}
\mu(\tilde{X}_{I^k+1(i_1,\ldots,i_j)})\varphi(m)^\gamma \\
\lesssim \mu(\tilde{X}_I)\mu(\tilde{X}_{I^k(i_1,\ldots,i_j)})\varphi(m)^\gamma \\
\lesssim \mu(\tilde{X}_I)\mu(\tilde{X}_{i_{|I|+1},\ldots,i_n})\mu(\tilde{X}_{I^k(i_1,\ldots,i_j)})\varphi(m)^\gamma \\
\lesssim \mu(\tilde{X}_I)\mu(\tilde{X}_{i_{|I|+1},\ldots,i_n})\varphi(n)^\gamma \varphi(m)^\gamma.
\end{align*}

Summarising the above, we have shown that if $m > (k_I + 1)n + s_I$ then

\begin{equation}
(2.16)
\mu(\tilde{X}_{I^k+1(i_1,\ldots,i_j)} \cap E_m) \lesssim \mu(\tilde{X}_I)\mu(\tilde{X}_{i_{|I|+1},\ldots,i_n})\varphi(m)^\gamma \varphi(n)^\gamma.
\end{equation}

Adding together the bounds provided by (2.15) and (2.16), we see that for any $m > n$ we have

\begin{align*}
\mu(\tilde{X}_{I^k+1(i_1,\ldots,i_j)} \cap E_m) &\lesssim \mu(\tilde{X}_I)\mu(\tilde{X}_{i_{|I|+1},\ldots,i_n})\kappa^{m-n}\varphi(m)^\gamma \\
&\quad + \mu(\tilde{X}_I)\mu(\tilde{X}_{i_{|I|+1},\ldots,i_n})\varphi(m)^\gamma \varphi(n)^\gamma.
\end{align*}

This completes our proof.

\begin{proof}

Proposition 2.6.

\[
\sum_{n,m=|J|}^Q \mu(E_n \cap E_m) \leq \mu(X_I) \left( \sum_{n=|J|}^Q \varphi(n)^\gamma + \left( \sum_{n=|J|}^Q \varphi(n) \right)^2 \right).
\]

Proof. We have the following

\begin{align*}
\sum_{n,m=|J|}^Q \mu(E_n \cap E_m) &= \sum_{n=|J|}^Q \mu(E_n) + 2 \sum_{n=|J|}^Q \sum_{m=n+1}^{Q-1} \mu(E_n \cap E_m) \\
&= \sum_{n=|J|}^Q \mu(E_n) + 2 \sum_{n=|J|}^Q \sum_{n=|J|}^{Q-1} \sum_{m=n+1}^{Q} \mu(\tilde{X}_{I^k+1(i_1,\ldots,i_j)} \cap E_m).
\end{align*}

Applying Lemma (2.5) to the above we obtain

\begin{equation}
(2.17)
\sum_{n,m=|J|}^Q \mu(E_n \cap E_m) \lesssim \sum_{n=|J|}^Q \mu(E_n) + \sum_{n=|J|}^{Q-1} \sum_{J \text{ is a prefix of } I} \sum_{m=n+1}^{Q} \mu(\tilde{X}_I)\mu(\tilde{X}_{i_{|I|+1},\ldots,i_n})\kappa^{m-n}\varphi(m)^\gamma \\
+ \sum_{n=|J|}^{Q-1} \sum_{J \text{ is a prefix of } I} \sum_{m=n+1}^{Q} \mu(\tilde{X}_I)\mu(\tilde{X}_{i_{|I|+1},\ldots,i_n})\varphi(m)^\gamma \varphi(n)^\gamma.
\end{equation}
\end{proof}
We focus on the three terms on the right hand side of (2.17) individually. By Lemma 2.4 we know that the following holds for the first term

\[ \sum_{n=|J|}^{Q} \mu(E_n) \lesssim \mu(X_J) \sum_{n=|J|}^{Q} \varphi(n)^\gamma. \]

Now let us focus on the second term in (2.17). We have

\[ \sum_{n=|J|}^{Q-1} \sum_{I \in \mathcal{D}^n} \mu(\tilde{X}_J) \mu(\tilde{X}_{I_{|J|+1}, \ldots, I_n}) \kappa^{m-n} \varphi(m)^\gamma \]

\[ = \mu(\tilde{X}_J) \sum_{n=|J|}^{Q-1} \sum_{I \in \mathcal{D}^n} \mu(\tilde{X}_{I_{|J|+1}, \ldots, I_n}) \sum_{m=n+1}^{Q} \kappa^{m-n} \varphi(m)^\gamma \]

\[ \leq \mu(\tilde{X}_J) \sum_{n=|J|}^{Q-1} \sum_{m=n+1}^{Q} \kappa^{m-n} \varphi(m)^\gamma \]

\[ \leq \mu(\tilde{X}_J) \sum_{n=|J|}^{Q-1} \sum_{m=n+1}^{Q} \kappa^{m-n} \varphi(m)^\gamma \]

\[ \leq \mu(\tilde{X}_J) \sum_{m=|J|+1}^{Q} \varphi(m)^\gamma. \]

It remains to bound the third term:

\[ \sum_{n=|J|}^{Q-1} \sum_{I \in \mathcal{D}^n} \sum_{m=n+1}^{Q} \mu(\tilde{X}_J) \mu(\tilde{X}_{I_{|J|+1}, \ldots, I_n}) \kappa^{m-n} \varphi(m)^\gamma \varphi(n)^\gamma \]

\[ = \mu(\tilde{X}_J) \sum_{n=|J|}^{Q-1} \sum_{I \in \mathcal{D}^n} \mu(\tilde{X}_{I_{|J|+1}, \ldots, I_n}) \sum_{m=n+1}^{Q} \varphi(m)^\gamma \]

\[ \leq \mu(\tilde{X}_J) \sum_{n=|J|}^{Q-1} \sum_{I \in \mathcal{D}^n} \mu(\tilde{X}_{I_{|J|+1}, \ldots, I_n}) \sum_{m=n+1}^{Q} \varphi(m)^\gamma \]

\[ \leq \mu(\tilde{X}_J) \sum_{n=|J|}^{Q-1} \sum_{I \in \mathcal{D}^n} \mu(\tilde{X}_{I_{|J|+1}, \ldots, I_n}) \sum_{m=n+1}^{Q} \varphi(m)^\gamma \]

\[ \leq \mu(\tilde{X}_J) \sum_{n=|J|}^{Q-1} \sum_{I \in \mathcal{D}^n} \mu(\tilde{X}_{I_{|J|+1}, \ldots, I_n}) \sum_{m=n+1}^{Q} \varphi(m)^\gamma. \]
Collecting the bounds provided by (2.18), (2.19), and (2.20), and substituting them into (2.17) completes our proof.

With Proposition 2.6 we can now complete our proof of Theorem 1.3.

**Proof of Theorem 1.3.** By (2.14) to prove the divergence part of Theorem 1.3 it suffices to show that there exists $c > 0$ such that

\[
\mu(\limsup_{n \to \infty} E_n) > c \mu(X_J).
\]

By Lemma 2.4 we know that \( \sum_{n=1}^{\infty} \phi(n)^\gamma = \infty \). Therefore Lemma 2.3 and Lemma 2.4 combined tell us that

\[
\mu(\limsup_{n \to \infty} E_n) \geq \limsup_{Q \to \infty} \frac{Q^2 \sum_{n=|J|}^{Q} \mu(E_n \cap E_m)}{\sum_{n,m=|J|}^{Q} \mu(E_n \cap E_m)} \geq \limsup_{Q \to \infty} \frac{\mu(\tilde{X}_J)^2 \left( \sum_{n=|J|}^{Q} \phi(n)^\gamma \right)^2}{\sum_{n,m=|J|}^{Q} \mu(E_n \cap E_m)}.
\]

Since \( \sum_{n=1}^{\infty} \phi(n)^\gamma = \infty \), we know that for any $Q$ sufficiently large we have

\[
\sum_{n=|J|}^{Q} \phi(n)^\gamma < \left( \sum_{n=|J|}^{Q} \phi(n)^\gamma \right)^2.
\]

Combining this observation with (2.22) and Proposition 2.6, it follows that

\[
\mu(\limsup_{n \to \infty} E_n) \geq \mu(X_J),
\]

i.e. there exists $c > 0$ such that $\mu(\limsup_{n \to \infty} E_n) > c \mu(X_J)$. So we have shown that (2.21) holds. This completes our proof.

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