Abstract. Let $R$ be the ring of algebraic integers in a number field $K$ and let $\Lambda$ be a maximal order in a semisimple $K$-algebra $B$. Building on our previous work, we compute the smallest number of algebra generators of $\Lambda$ considered as an $R$-algebra. This reproves and vastly extends the results of P.A.B. Pleasants, who considered the case when $B$ is a number field. In order to achieve our goal, we obtain several results about counting generators of algebras which have finitely many elements. These results should be of independent interest.

Mathematics Subject Classification (2010). Primary 16H10, 16P10, 16S15, 15B33. Secondary 11R45, 11R52.

Keywords: density, smallest number of generators, probability of generating.

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1. Introduction

The investigation of rings of algebraic integers that can be generated by one element as a ring is an old topic in algebraic number theory. Such rings and their field of fractions are often called monogenic. In the early 1970s J. Browkin raised a more general problem of finding the smallest number of ring generators for the ring of integers of a given number field. This question was answered by P. A. B. Pleasants [11]. The
The present work can be considered as a vast generalization of the results of [11], because we address the same question for maximal orders in finite dimensional semisimple \( \mathbb{Q} \)-algebras.

In group theory, the smallest number of generators of a finite group and the structure of generating sets have been studied for a long time, starting with the seminal works by Hall [2], Neumann \& Neumann [8] and Gashütz [11]. The present paper is a continuation of our investigation in [3], where we have developed methods to compute the smallest number of generators of algebras of finite type over certain commutative rings. In particular, if \( R \) is the ring of integers in a number field and \( A \) is an \( R \)-algebra finitely generated as an \( R \)-module, then the computation of the smallest number of algebra generators of \( A \) often reduces to a computation, for each maximal ideal \( m \) of \( R \), of the smallest number of generators of \( A/mA \) as an \( R/m \)-algebra (see the beginning of Section 5 for exact statements). With this in mind, in Section 2 we develop techniques to count the number of generating tuples of \( R \)-algebras which have finitely many elements. This often reduces to the problem of counting generating tuples of matrix algebras over finite fields. This problem is handled in Section 3. In Section 4, we review some background material about maximal orders in simple algebras. Section 5 contains our main results, Theorem 5.8 and Theorem 5.9. These theorems provide exact formulas for the smallest number of generators of a maximal \( R \)-order in a semisimple algebra over a number field with the ring of integers \( R \).

Acknowledgments. This work was partially supported by a grant from the Simons Foundation (#245855 to Marcin Mazur).

2. Generators of some finite dimensional algebras over fields

The Jacobson radical of a ring \( S \) is denoted by \( J(S) \).

Lemma 2.1. Let \( A = \prod_{i=1}^n A_i \) be a product of (left or right) Artinian algebras and let \( B \) be a subalgebra of \( A \) such that

(i) for each \( i \), the projections of \( A \) onto \( A_i \) maps \( B \) onto \( A_i \).

(ii) \( A = B + J(A) \).

Then \( B = A \).
Proof. Fix $i \in \{1, 2, \ldots, m\}$. Let $e_i = (u_1, \ldots, u_m)$ be the element of $A$ such that $u_i = 1$ and $u_j = 0$ for all $j \neq i$. Condition (ii) ensures that $B$ contains an element $a = (a_1, \ldots, a_m)$ such that $a - e_i \in J(A)$. Thus $a_i$ is invertible in $A_i$ and $a_j \in J(A_j)$ for all $j \neq i$. By (i), $B$ contains an element $b = (b_1, \ldots, b_m)$ such that $b_i = a_i^{-1}$. Thus $ab = (c_1, \ldots, c_m) \in B$, where $c_i = 1$ and $c_j \in J(A_j)$ for all $j \neq i$. Since $J(A_j)$ are nilpotent, $(ab)^N = e_i$ for $N$ sufficiently large. Thus $B$ contains the idempotents $e_1, \ldots, e_m$. Condition (i) implies now that $B = A$. \qed

Recall that an Artinian algebra $A$ is called primary if $A/J(A)$ is simple.

Theorem 2.2. Let $R$ be a commutative ring. Let $A = \prod_{i=1}^m A_i$ be a product of Artinian primary $R$-algebras, such that each $A_i$ is finitely generated as an $R$-module and such that the simple quotients of each $A_i$ are isomorphic as $R$-algebras. Let $F = R/I$, where $I$ is the annihilator of the simple quotient of each $A_i$. Let $k$ and $m$ be positive integers. Then elements

$$a_1 = (a_{11}, \ldots, a_{1m}), \ldots, a_k = (a_{k1}, \ldots, a_{km})$$

of $A$ generate $A$ as an $R$-algebra if and only if the following two conditions are satisfied:

(1) For any $i = 1, \ldots, m$, the elements $a_{1i}, \ldots, a_{ki}$ generate $A_i$ as an $R$-algebra.

(2) For any $i \neq j$ there is no isomorphism $\Psi : A_j/J(A_j) \to A_i/J(A_i)$ of $F$-algebras such that

$$a_{1i} + J(A_i) = \Psi(a_{1j} + J(A_j)), \ldots, a_{ki} + J(A_i) = \Psi(a_{kj} + J(A_j)).$$

Note that a special case of this theorem, when $J(A) = 0$, is proved in [3, Theorem 6.1]. Indeed, as we will see in the course of the proof of Theorem 2.2, $F$ is a field and each quotient $A_i/J(A_i)$ is a finite dimensional simple $F$-algebra. If $J(A) = 0$, then $J(A_i) = 0$ for all $i$ and $A$ can be considered as a product of $m$ copies of a finite dimensional simple $F$-algebra as in the Theorem 6.1 of [3].

Proof. Note that the center $Z_i$ of $A_i/J(A_i)$ is a field and $F$ can be identified with a subring of $Z_i$ such that $A_i/J(A_i)$ is a finitely generated $F$-module. It follows that every element of $A_i/J(A_i)$ is integral over $F$. In particular, the field $Z_i$ is integral over $F$ and therefore $F$ is also a field. Thus $A_i/J(A_i)$ is a finite dimensional simple $F$-algebra.

We will use the fact that Theorem 2.2 is true when $J = 0$ (see [3, Theorem 6.1]).
The necessity of condition (1) is clear. If \( a_1, \ldots, a_k \) generate the \( R \)-algebra \( A \), then the images of these elements in \( A/J(A) \) generate \( A/J(A) \) as an \( F \)-algebra. The necessity of condition (2) is now a simple consequence of the fact that Theorem 2.2 is true for the \( F \)-algebra \( A/J = \prod_{i=1}^{m} A_i/J(A_i) \).

Conversely, suppose that conditions (1) and (2) are satisfied. Then the images of \( a_1, \ldots, a_k \) in \( A/J(A) \) generate \( A/J(A) \) as an \( F \)-algebra (we use again the fact that Theorem 2.2 is true for the \( F \)-algebra \( A/J(A) \)). Let \( B \) be the \( R \)-subalgebra of \( A \) generated by \( a_1, \ldots, a_k \). The quotient map \( A \to A/J(A) \) maps \( B \) surjectively onto \( A/J(A) \). Thus \( B \) satisfies condition (ii) of Lemma 2.1. Condition (1) implies that \( B \) also satisfies (i) of Lemma 2.1. Thus \( B = A \). \( \square \)

The following result is an algebra analog of a well known and very useful result of Gaschütz [1] for groups. It is clear from the proof that analogs of this result hold for other finite algebraic structures.

**Theorem 2.3.** Let \( R \) be a commutative ring, let \( A \) be an \( R \)-algebra of finite cardinality and let \( f : A \to B \) be an epimorphism of \( R \)-algebras. For any natural number \( k \) there is a non-negative integer \( g_k(A, B) \) with the following property: any sequence of \( k \) generators of the \( R \)-algebra \( B \) can be lifted to exactly \( g_k(A, B) \) sequences of \( k \) generators of \( A \).

**Proof.** We fix \( B \) and prove the result by induction on the order of \( A \). If \( |A| = |B| \) then \( f \) is an isomorphism and the result is clear with \( g_k(A, B) = 1 \). Consider a surjective homomorphism \( f : A \to B \) of \( R \)-algebras and let \( h \) be the cardinality of the kernel of \( f \). Let \( \Phi \) be the family of all \( R \)-subalgebras of \( A \) which are mapped onto \( B \) by \( f \). Consider any \( k \) elements \( b_1, \ldots, b_k \) of \( B \) which generate \( B \) as an \( R \)-algebra. The number of \( k \)-tuples \( a_1, \ldots, a_k \) of elements of \( A \) such that \( f(a_i) = b_i \) for \( i = 1, \ldots, k \), is equal to \( h^k \). Any such \( k \)-tuple generates an \( R \)-algebra in \( \Phi \). By the inductive assumption, the number of such \( k \)-tuples which generate \( A \) is \( h^k - \sum_{C \in \Phi, C \neq A} g_k(C, B) \). \( \square \)

**Remark 2.4.** Note that if \( B \) can not be generated by \( k \) elements, then the numbers \( g_k(A, B) \) are not well defined. It will however not matter what (finite) value we set for \( g_k(A, B) \) in this case, and we will use any convenient value (to keep our formulas uniform).
Proposition 2.5. Let $A$ be an $R$-algebra with a nilpotent ideal $J$. If $B$ is a subalgebra of $A$ such that $B + J^2 = A$ then $B = A$.

Proof. Let $K = B \cap J$. Then $K$ is a nilpotent ideal of $B$ and $J = K + J^2$. We claim that $J^m = K^m + J^{m+1}$ for all positive integers $m$. Indeed, this holds for $m = 1$ and if it holds for some $m$ then

$$J^{m+1} = J \cdot J^m = (K + J^2)(K^m + J^{m+1}) = K^{m+1} + J^2 \cdot K^m + K \cdot J^m + J^{m+3} \subseteq K^{m+1} + J^{m+2} \subseteq J^{m+1}$$

so the result holds for $m+1$. Thus our claim follows by induction. Since $J$ is nilpotent, let $n$ be largest such that $J^n \neq 0$. Then $J^n = K^n$ and by backward induction we conclude that $J^m = K^m$ for all $m$. Thus $J = K$, so $J \subseteq B$ and $A = B + J^2 \subseteq B$. □

As an immediate corollary we get the following result.

Corollary 2.6. If $A$ is an Artinian $R$-algebra and $b_1, \ldots, b_k$ generate the $R$-algebra $A/J(A)^2$ then any lifts $a_1, \ldots, a_k$ of $b_1, \ldots, b_k$ to $A$ generate $A$ as an $R$-algebra. In particular, if $A$ is finite then $g_k(A, A/J(A)^2) = |J(A)^2|^k$ for every positive integer $k$.

Theorem 2.7. Let $A$ be a finite dimensional algebra over a field $F$ such that $J(A)^2 = 0$ and $J(A)$ is simple as a $A/J(A)$-bimodule. Suppose furthermore that $A/J(A)$ is a separable $F$-algebra. For $k \geq 2$, any sequence of $k$ elements which generate the $F$-algebra $A/J(A)$ can be lifted to a sequence of $k$ elements in $A$ which generate $A$ as an $F$-algebra. If $A$ is commutative, the same holds for $k = 1$.

Suppose furthermore that $F$ is finite and $J(A) \neq 0$, and let $C$ be the intersection of $J(A)$ and the center of $A$. Then $g_k(A, A/J(A)) = |J(A)|^k - |J(A)/C|$.

Proof. The result is obvious when $J(A) = 0$, so we assume that $J(A) \neq 0$. Recall the following fundamental result of Wedderburn and Malcev (see [10], Chapter 11):

Theorem 2.8. Let $F$ be field and $T$ a finite dimensional $F$-algebra with Jacobson radical $J$ such that $T/J$ is a separable $F$-algebra. Then $T$ has a subalgebra $S$ such that $T = S \oplus J$. Moreover, any two such subalgebras are conjugate by a unit of the form $1 - u$ for some $u \in J$. 
Choose a subalgebra $S$ of $A$ such that $A = S \oplus J(A)$. If $B$ is any subalgebra of $A$ such that $A = B + J(A)$ then $J(B) = B \cap J(A)$ and $J(B)$ is a $B/J(B)$-bimodule. Thus $A/J(A) = B/J(B)$ and $J(B)$ is a $A/J(A)$-bi-submodule of $J(A)$. Since $J(A)$ is simple, either $J(B) = 0$ or $J(B) = J(A)$. In the latter case we have $A = B$, and in the former case $B = (1 - u)S(1 + u)$ for some $u \in J(A)$.

Let $x_1, \ldots, x_k$ be elements of $A/J(A)$ which generate $A/J(A)$ as an $F$-algebra. Let $s_i$ be the unique element of $S$ such that $x_i = s_i + J(A)$. Any lift of $x_1, \ldots, x_k$ to $A$ is of the form $s_1 + u_1, \ldots, s_k + u_k$ for some $u_i \in J(A)$. We will show that $s_1 + u_1, \ldots, s_k + u_k$ do not generate $A$ as an $F$-algebra if and only if there is $u \in J(A)$ such that $u_i = [s_i, u]$ for $i = 1, \ldots, k$. Let $B$ be the $F$-subalgebra of $A$ generated by $s_1 + u_1, \ldots, s_k + u_k$. Then $A = B + J(A)$. If $B \neq A$ then $B = (1-u)S(1+u)$ for some $u \in J(A)$. It follows that $s_i + u_i = (1-u)s_i(1+u) = s_i + [s_i, u]$ (as both sides are elements of $B$ which lift $x_i$). Conversely, if $u_i = [s_i, u]$ for $i = 1, \ldots, k$ and some $u \in J(A)$, then $B = (1-u)S(1+u) so B \neq A$.

Consider the map $\phi : J(A) \longrightarrow \prod_{i=1}^{k} J(A)$ given by $\phi(u) = ([s_1, u], \ldots, [s_k, u])$. This is an $F$-linear map so it can not be onto when $k \geq 2$. When $A$ is abelian then this map is trivial, so it is not onto even for $k = 1$. This means that we can always choose $u_i \in J(A)$, $i = 1, \ldots, k$ such that $s_1 + u_1, \ldots, s_k + u_k$ generate $A$ as an $F$-algebra. In other words, $x_1, \ldots, x_k$ can be lifted to a sequence generating $A$ as an $F$-algebra.

Suppose now that $F$ is finite. Then the number of lifts of $x_1, \ldots, x_k$ to $A$ is equal to $|J(A)|^k$. The number of lifts which do not generate $A$ is equal to the cardinality of the image of $\phi$. Note that the kernel of $\phi$ is exactly $C$, so the image of $\phi$ has cardinality $|J(A)/C|$. Thus $g_k(A, A/J(A)) = |J(A)|^k - |J(A)/C|$. \hfill \square

3. Matrix rings over finite fields

As we have seen in the previous section, computations of the number of generating tuples of an algebra $A$ with finitely many elements can often be reduced to same computations for $A/J(A)$. This in turn reduces to computations of generating tuples for finite simple algebras. Any such an algebra is of the form $M_n(F)$, where $F$ is a finite field. In this section we extend some results about generators of matrix rings
over finite fields obtained in \cite{3}. Let us first recall some of the results from \cite{3}. Note that our notation here will differ slightly from the one introduced in \cite{3}.

For a finite $R$-algebra $A$ and a positive integer $k$ we denote by $g_k(A, R)$ the number of $k$-tuples of elements in $A$ which generate $A$ as an $R$-algebra. If there is no ambiguity about $R$ we simply write $g_k(A)$. Recall that for a quotient $B$ of $A$ we defined $g_k(A, B)$ as the number of ways a $k$-tuple generating the $R$-algebra $B$ can be lifted to a $k$-tuple generating the $R$-algebra $A$ (note that we use an italic $g$ here, and not a roman $g$). If there is a need to indicate $R$ explicitly, we will write $g_k(A, B, R)$ for $g_k(A, B)$.

Thus $g_k(A, R) = g_k(B, R) g_k(A, B, R)$. To simplify the notation, we set $g_k(n, q, r) = g_k(M_n(F_q r), F_q)$ and $g_k(n, q) = g_k(n, q, 1)$. We can now state Theorem 6.3. from \cite{3}:

\textbf{Theorem 3.1.} Let $A = M_n(F_q s)$. Then $A^m$ can be generated by $k$ elements as an $F_q$-algebra iff

$$m \leq \frac{g_k(n, q, s)}{s |PGL_n(F_q)|}.$$ 

Furthermore,

$$g_k(A^m, F_q) = \prod_{i=0}^{m-1} (g_k(n, q, s) - i \cdot s \cdot |PGL_n(F_q)|).$$

The following theorem summarizes the results of section 7 of \cite{3}.

\textbf{Theorem 3.2.} (i) $g_k(1, q) = q^k$.

(ii) $g_k(2, q) = q^{2k+1}(q^{k-1} - 1)(q^k - 1)$.

(iii) $g_k(3, q) = q^{3k+4}(q^{k-1} - 1)(q^{k-1} + 1)(q^k - 1)(q^{3k-2} + q^{2k-2} - q^k - 2q^{k-1} - q^{k-2} + q + 1)$.

(iv) $g_k(n, q) \geq q^{kn^2 - 2\frac{n^2 + n}{2}q^{n^2(k-1)(n-1)}}$.

The computation of the numbers $g_k(n, q)$ for $n \geq 4$ remains one of the main outstanding problems in this area. The main result of this section expresses the numbers $g_k(n, q, r)$ in terms of the numbers $g_k(n, q)$. We start with the following proposition.

\textbf{Proposition 3.3.} Let $A$ be a subring of $M_n(F_q)$. Then $F_q A = M_n(F_q)$ if and only if $A$ is conjugate to $M_n(F_\pi)$ for some subfield $F_\pi$ of $F_q$.

\textbf{Proof.} Let $Z$ be the center of $A$. We claim that $Z \subseteq F_q$. Indeed, the assumption that $F_q A = M_n(F_q)$ implies that any element central in $A$ remains central in $M_n(F_q)$. Thus $Z = F_\pi$ is a subfield of $F_q$. Let $J$ be the Jacobson radical of $A$. Then $J$ is nilpotent and
\(\mathbb{F}_q J\) is a nilpotent ideal of \(M_n(\mathbb{F}_q)\). It follows that \(\mathbb{F}_q J = 0\), and consequently \(J = 0\). This proves that \(A\) is semisimple. Since the center of \(A\) is a field, \(A\) is simple. Thus \(A\) is isomorphic as \(\mathbb{F}_r\)-algebra to \(M_r(\mathbb{F}_r)\) for some \(r\). The embedding \(A \hookrightarrow M_n(\mathbb{F}_q)\) extends to an \(\mathbb{F}_q\)-algebra homomorphism \(f : A \otimes_{\mathbb{F}_r} \mathbb{F}_q \to M_n(\mathbb{F}_q)\), which is surjective by the assumption that \(\mathbb{F}_q A = M_n(\mathbb{F}_q)\). On the other hand, \(A \otimes_{\mathbb{F}_r} \mathbb{F}_q \simeq M_r(\mathbb{F}_q)\) is a central simple \(\mathbb{F}_q\)-algebra, so \(f\) is injective. It follows that \(f\) is an isomorphism and \(r = n\). There is an \(\mathbb{F}_r\)-algebra isomorphism \(g : A \to M_n(\mathbb{F}_r)\) which extends to an \(\mathbb{F}_q\)-algebra isomorphism \(g^* : A \otimes_{\mathbb{F}_r} \mathbb{F}_q \to M_n(\mathbb{F}_q)\). By the Noether-Skolem theorem, there is an invertible matrix \(u \in M_n(\mathbb{F}_q)\) such that \(f(x) = u g(x) u^{-1}\) for all \(x \in A \otimes_{\mathbb{F}_r} \mathbb{F}_q\). It follows that \(A = u M_n(\mathbb{F}_r) u^{-1}\).

Conversely, if \(A = u M_n(\mathbb{F}_r) u^{-1}\) then clearly \(\mathbb{F}_q A = M_n(\mathbb{F}_q)\).

\[\square\]

**Theorem 3.4.**

\[
\frac{g_m(n, q, r)}{|\text{PGL}_n(\mathbb{F}_q^r)|} = \sum_{s|r} \frac{g_m(n, q^s)}{|\text{PGL}_n(\mathbb{F}_q^s)|} \mu(r/s).
\]

**Proof.** Note that an \(m\)-tuple of elements in \(M_n(\mathbb{F}_{q^r})\) generates \(M_n(\mathbb{F}_{q^r})\) as an \(\mathbb{F}_{q^r}\)-algebra if and only if the \(\mathbb{F}_q\)-algebra \(A\) it generates satisfies \(\mathbb{F}_{q^r} A = M_n(\mathbb{F}_{q^r})\). Let \(\Phi\) be the collection of all \(\mathbb{F}_q\)-subalgebras \(A\) of \(M_n(\mathbb{F}_{q^r})\) such that \(\mathbb{F}_{q^r} A = M_n(\mathbb{F}_{q^r})\). Thus

\[g_m(n, q^r) = \sum_{A \in \Phi} g_m(A, \mathbb{F}_q).
\]

By Proposition 3.3 \(A \in \Phi\) if and only if \(A\) is conjugate to \(M_n(\mathbb{F}_{q^r})\) for some divisor \(s\) of \(r\). Note that if \(C \in \text{GL}_n(\mathbb{F}_{q^r})\) satisfies \(C M_n(\mathbb{F}_{q^s}) C^{-1} = M_n(\mathbb{F}_{q^r})\), then conjugation by \(C\) induces \(\mathbb{F}_{q^r}\)-algebra automorphism of \(M_n(\mathbb{F}_{q^r})\). Since all such automorphisms are inner by the Noether-Skolem theorem, there is \(D \in \text{GL}_n(\mathbb{F}_{q^r})\) such that \(D^{-1} C\) centralizes \(M_n(\mathbb{F}_{q^r})\). It follows that \(D^{-1} C \in \mathbb{F}_{q^r}\). Consequently, the number of matrices \(C\) which stabilize \(M_n(\mathbb{F}_{q^r})\) is equal to \(|\text{GL}_n(\mathbb{F}_{q^r})|/(q^r - 1)/(q^s - 1)|. The number of elements in \(\Phi\) which are conjugate to \(M_n(\mathbb{F}_{q^r})\) is then equal to \(\frac{|\text{GL}_n(\mathbb{F}_{q^r})|/(q^s - 1)}{|\text{GL}_n(\mathbb{F}_{q^r})|/(q^r - 1)}\). Note that if \(A\) is conjugate to \(M_n(\mathbb{F}_{q^r})\) then \(g_{m}(A, \mathbb{F}_q) = g_m(n, q, s)\). Thus we get the following formula.

\[g_m(n, q^r) = \sum_{s|r} g_m(n, q, s) \frac{|\text{GL}_n(\mathbb{F}_{q^r})|/(q^s - 1)}{|\text{GL}_n(\mathbb{F}_{q^r})|/(q^r - 1)}.
\]
Thus
\[ \frac{g_m(n, q^r)}{|\text{PGL}_n(\mathbb{F}_{q^r})|} = \sum_{s|r} \frac{g_m(n, q^s)}{|\text{PGL}_n(\mathbb{F}_{q^s})|} \]

By Möbius inversion formula, we get
\[ \frac{g_m(n, q, r)}{|\text{PGL}_n(\mathbb{F}_{q^r})|} = \sum_{s|r} \frac{g_m(n, q^s)}{|\text{PGL}_n(\mathbb{F}_{q^s})|} \mu(r/s). \]

We will need the following technical result.

**Lemma 3.5.** For any positive integers \( r, t, n \) such that \( n \neq 2 \) there is a constant \( c(n, r, t) \) such that
\[ g_2(M_n(\mathbb{F}_{q^s}), \mathbb{F}_q) \geq q^{2n^2rt} \left(1 - \frac{25t}{q^2}\right)^t \]
for all \( q \geq c(n, r, t) \).

**Proof.** When \( n = 1 = r \), Theorem 3.1 and Theorem 3.2(i) yield
\[ g_2((\mathbb{F}_q)^t, \mathbb{F}_q) = \prod_{i=0}^{t-1} (q^2 - i) \geq q^{2t} \left(1 - \frac{25t}{q^2}\right)^t. \]
Suppose now that either \( n \geq 3 \) or \( n = 1 \) and \( r \geq 2 \). By Theorem 3.4
\[ g_2(n, q, r) \geq g_2(n, q^r) - \sum_{s|r, s < r} q^{2sn^2} \frac{|\text{PGL}_n(\mathbb{F}_{q^s})|}{|\text{PGL}_n(\mathbb{F}_{q^r})|}. \]
Note that
\[ \frac{a^y - 1}{a^x - 1} \leq \frac{y}{x} a^{y-x} \]
for any \( a > 1 \) and \( 0 < x < y \) (for a simple proof use the Mean Value Theorem). Using this inequality, we get
\[ \frac{|\text{PGL}_n(\mathbb{F}_{q^r})|}{|\text{PGL}_n(\mathbb{F}_{q^s})|} = \frac{q^s - 1}{q^r - 1} \prod_{i=0}^{n-1} q^{r^n - q^i} \leq \frac{q^{(r-s)n} - q^i}{q^{(r-s)i} - q^{i-s}} \prod_{i=0}^{n-2} q^{(r-s)(n-i)} = \left(\frac{r}{s}\right)^{n-1} q^{(r-s)(n^2-1)}. \]
Since \( 1 \leq s \leq r/2 \), we have
\[ g_2(n, q, r) \geq g_2(n, q^r) - q^n q^{\frac{3}{2}r(n^2-1)+r}. \]
When \( n = 1 \), \( r > 1 \), we get by Theorem 3.2(i) that
\[
g_2(1, q, r) \geq q^{2r} - rq^r
\]
and using Theorem 3.1 we conclude that
\[
g_2((\mathbb{F}_{q^r})^t, \mathbb{F}_q) = \prod_{i=0}^{t-1}(g_2(1, q, r) - ir) \geq q^{2rt} \left(1 - \frac{2r}{q^t}\right)^t \geq q^{2rt} \left(1 - \frac{4}{q^2}\right)^t \geq q^{2rt} \left(1 - \frac{25t}{q^2}\right)^t
\]
for \( q > \max(25, t, r) \).

Suppose now that \( n \geq 3 \). By Theorem 3.2(iv) we have
\[
g_2(n, q, r) \geq q^{2rn^2} - \frac{2(n+6)/2}{q^2} q^{(2n^2-n+1)r} - r^n q^{\frac{3r(n^2-1)+r}{2}}.
\]
Note that \( |\text{PGL}_n(\mathbb{F}_{q^r})| \leq q^{r(n^2-1)} \). Hence, when \( q > \max(tr, rn) \) and \( 0 \leq i < t \) we have
\[
g_2(n, q, r) - ir|\text{PGL}_n(\mathbb{F}_{q^r})| \geq q^{2rn^2} - \frac{2(n+6)/2}{q^2} q^{(2n^2-n+1)r} - r^n q^{\frac{3r(n^2-1)+r}{2}} - irq^{r(n^2-1)} \geq q^{2rn^2} \left(1 - \frac{25t}{q^2}\right)^t.
\]
Using Theorem 3.1 we see that
\[
g_2(M_n(\mathbb{F}_{q^r})^t, \mathbb{F}_q) \geq q^{2rn^2r} \left(1 - \frac{25t}{q^2}\right)^t
\]
provided \( q > \max(tr, rn, 25) \).

4. Maximal orders in simple algebras

In this section we review some facts about orders in simple algebras. All the results and their proofs can be found in the excellent book [13].

Let \( R \) be a Dedekind domain with field of fractions \( K \) and let \( H \) be a central simple \( K \)-algebra. Let \( \Lambda \) be a maximal order in \( H \) and let \( P \) be a maximal ideal of \( R \). Our goal is to get some understanding of algebras of the form \( \Lambda/P^t\Lambda \). Let us consider the \( P \)-completions \( \Lambda_P, H_P, K_P, R_P \) of \( \Lambda, H, K \) and \( R \) respectively. Then \( \Lambda_P \) is a maximal order in the central simple \( K_P \)-algebra \( H_P \). \( R_P \) is a complete discrete valuation ring with maximal ideal \( \hat{P} \) and \( \Lambda/P^t\Lambda = \Lambda_P/\hat{P}^t\Lambda_P \).

There are a positive integer \( r_P = r \), a finite dimensional central division algebra \( D \) over \( K_P \) with the unique maximal order \( \Delta \), and an isomorphism \( H_P \cong M_r(D) \) which
identifies $\Lambda_P$ with $M_r(\Delta)$. We call $r_P$ the \textbf{local capacity} of $H$ at $P$. The index of $D$ is denoted by $m_P = m$ and called the \textbf{local index} of $H$ at $P$ (so the dimension of $D$ over $K_P$ is $m_P^2$). We say that $P$ is \textbf{ramified} in $H$ if $m_P > 1$, and \textbf{unramified} if $m_P = 1$. Furthermore, there is an element $\pi \in \Delta$ such that ideals in $\Delta$ are exactly the subsets of the form $\pi^j \Delta$, $j = 1, 2, \ldots$ (so, in particular, each one-sided ideal in $\Delta$ is two-sided). Now $\hat{\mathcal{P}} \Delta = \pi e \Delta$, where $e = e_P$ is called the \textbf{ramification index} of $D$ over $K_P$. The residue ring $\overline{\Delta} = \Delta / \pi \Delta$ is a division algebra over the field $R_P / \hat{\mathcal{P}} \cong R/P$. The dimension of $\overline{\Delta}$ over $R/P$ is denoted by $f = f_P$ and called the \textbf{inertial degree} of $D$ over $K_P$. Note that $e_P f_P = m_P^2$ (also $e_P | m_P$ and $m_P | f_P$).

Suppose in addition that $R/P \cong \mathbb{F}_q$ is a finite field. Then $e_P = f_P = m_P$ and $\overline{\Delta} \cong \mathbb{F}_{q^m}$. Moreover, $\Delta$ contains a primitive $(q^m - 1)$th root of 1, denoted $\omega$. Given any generator of $\hat{\mathcal{P}}$, one can choose $\pi$ so that $\pi^m \in \hat{\mathcal{P}}$ is the given generator and $\pi \omega = \omega^{q^s} \pi$, where $s$ is a positive integer independent of all the choices (of $\omega$, $\pi$, etc.) and such that $1 \leq s \leq m$, $(s, m) = 1$. We have $\Delta = R[\omega, \pi]$. The fraction $s/m$ is called the \textbf{Hasse invariant} of $D$.

**Theorem 4.1.** Let $R$ be a Dedekind domain with field of fractions $K$ and let $H$ be a central simple $K$-algebra. Let $\Lambda$ be a maximal order in $H$ and let $P$ be a maximal ideal of $R$. Let $t$ a positive integer and let $A = \Lambda / P^t \Lambda$. We use the notation introduced above. Then

(i) $A$ is an Artinian primary $R$-algebra, finitely generated as an $R$-module, and $A/J(A) \cong M_r(\overline{\Delta})$, where $r$ is the local capacity of $H$ at $P$.

(ii) $J(A)/J(A)^2$ is simple as $A/J(A)$-bimodule.

(iii) Suppose in addition that $R/P \cong \mathbb{F}_q$ is finite. Then $A/J(A) \cong M_r(\mathbb{F}_{q^m})$, where $m$ is the local index of $H$ at $P$. Moreover, if $C := \{x \in J(A)/J(A)^2 : ax = xa \text{ for all } a \in A/J(A)\}$

then

$$|C| = \begin{cases} 1, & \text{if } m > 1; \\ q, & \text{if } m = 1. \end{cases}$$

**Proof.** We use the notation and results discussed at the beginning of this section. In particular, we identify $\Lambda_P$ with $M_r(\Delta)$. We start with observing that two sided ideals of
\[ \Lambda_P = M_r(\Delta) \] are of the form \((\pi \Lambda_P)^n = \pi^n \Lambda_P = M_r(\pi^n \Delta), n \in \mathbb{N}. \] Also, \(\hat{P} \Lambda_P = \pi^e \Lambda_P\) and \(\Lambda_P / \pi \Lambda_P \cong M_r(\overline{\Delta})\). It is clear now that \(A \cong \Lambda_P / \hat{P} \Lambda_P = \Lambda_P / \pi^e \Lambda_P\) is an Artinian \(R_P\)-algebra with Jacobson radical \(J(A) = \pi \Lambda_P / \pi^e \Lambda_P\) and therefore \(A / J(A) \cong M_r(\overline{\Delta})\).

Note now that conjugation by \(\pi (x \mapsto \pi^{-1} x \pi)\) defines an automorphism of \(\Lambda_P\) (and of \(\Delta\)) which induces an automorphism \(\phi\) of \(M_r(\overline{\Delta})\). We may identify \(J(A) / J(A)^2 = \pi \Lambda_P / \pi^2 \Lambda_P\) with \(M_r(\overline{\Delta})\) by sending the class of an element \(\pi g, g \in \Lambda_P\) to the image of \(g\) in \(M_r(\overline{\Delta})\). This yields an isomorphism of \(A / J(A)\)-bimodules, where the \(M_r(\overline{\Delta})\)-bimodule structure on \(M_r(\overline{\Delta})\) is given by \(u \cdot x \cdot v = \phi(u)xv\) (where on the right hand side we use the ring multiplication in \(M_r(\overline{\Delta})\)). It is clear now that \(J(A) / J(A)^2\) is simple as \(A / J(A)\)-bimodule.

Finally, suppose that \(R / P \cong \mathbb{F}_q\) is finite. Then \(\overline{\Delta} \cong \mathbb{F}_q^m\) and therefore \(A / J(A) \cong M_r(\mathbb{F}_q^m)\). Choosing a primitive \((q^m - 1)\)th root of unity \(\omega\) and \(\pi\) so that \(\pi \omega = \omega q^e \pi\), we see that in this case the automorphism \(\phi\) is induced on \(M_r(\mathbb{F}_q^m)\) by the automorphism \(F^{-s}\) of \(\mathbb{F}_q^m\), where \(F\) is the Frobenius automorphism of \(\mathbb{F}_q^m\) over \(\mathbb{F}_q\). Under our identifications, \(C\) is identified with \(\{x \in M_r(\mathbb{F}_q^m) : \phi(u)x = xu \text{ for all } u \in M_r(\mathbb{F}_q^m)\}\). When \(m > 1\), \(\phi\) is a non-trivial automorphism and it is easy to see that \(C = \{0\}\) in this case. When \(m = 1\) then \(\phi\) is the identity and \(C\) is identified with the center of \(M_r(\mathbb{F}_q)\), which has \(q\) elements. \(\square\)

5. Maximal orders in semisimple algebras

We start this section by recalling a few results from [3]. For a commutative ring \(R\) and an \(R\)-algebra \(A\) which is finitely generated as an \(R\)-module we write \(r(A, R)\) for the smallest number \(k\) such that \(A\) can be generated by \(k\) elements as an \(R\)-algebra. Let \(R\) be a Dedekind domain with the field of fractions \(K\). For a maximal ideal \(\mathfrak{p}\) of \(R\) define \(r_{\mathfrak{p}}(A) = r(A / \mathfrak{p}A, R / \mathfrak{p})\). Let \(r_K(A) = r(A \otimes_R K, K)\). The following result is a special case of Theorem 5.7. in [3].

**Theorem 5.1.** Let \(R\) be a Dedekind domain with the field of fractions \(K\) and let \(A\) be an \(R\)-algebra which is finitely generated as an \(R\)-module. If \(k \geq r_{\mathfrak{p}}(A)\) for all maximal ideals \(\mathfrak{p}\) of \(R\) and \(k \geq r_K(A) + 1\) then \(A\) can be generated by \(k\) elements as an \(R\)-algebra.

Suppose now that \(R\) is the ring of algebraic integers in a number field \(K\). Then, for each maximal ideal \(\mathfrak{p}\) of \(R\), the residue field \(R / \mathfrak{p}\) is finite with \(N(\mathfrak{p})\) elements. For a
positive integer $k$, let $g_k(A, p) = g_k(A/pA, R/p)$. Suppose that $A$ is a free $R$-module of finite rank. In [3] we introduced the notion of density $\text{den}_k(A)$ of the set of all $k$-tuples in $A^k$ which generate $A$ as and $R$-algebra. Roughly speaking, a choice of an integral basis of $R$ and of a basis of $A$ over $R$ allows us to introduce integral coordinates on all cartesian powers $A^k$, $k \in \mathbb{N}$. For any subset $S$ of $A^k$ and any $N$ we consider the finite set $S(N)$ of all points whose coordinates are in the interval $[-N, N]$. We define the density $\text{den}(S)$ of $S$ as the limit $\lim_{N \to \infty} \frac{|S(N)|}{|A^k(N)|}$ (we do not claim that it always exists).

**Theorem 5.2.** Let $R$ be the ring of algebraic integers in a number field $K$ and let $A$ be an $R$-algebra which is free of rank $m$ as an $R$-module. Then

$$\text{den}_k(A) = \prod_{p \in \text{m-Spec } R} \frac{g_k(A, p)}{N(p)^{mk}}$$

for any positive integer $k$, where $\text{m-Spec } R$ is the set of all maximal ideals of $R$.

Note in particular that if $\text{den}_k(A) > 0$ then $A$ can be generated by $k$ elements as an $R$-algebra. The right hand side of (1) makes sense when $A$ is a projective $R$-module. But so far we have no good definition of density in this case. Nevertheless, we propose the following conjecture.

**Conjecture 5.3.** If $A$ in Theorem 5.2 is finitely generated and projective as an $R$-module and if for some $k$ the right hand side of (1) is positive, then $A$ can be generated by $k$ elements as an $R$-algebra.

Finally, we state the following result, which is Proposition 2.12. in [3].

**Proposition 5.4.** Suppose that $A = \prod_{i=1}^s A_i$ is a product of $R$-algebras $A_1, \ldots, A_s$ such that for any maximal ideal $m$ of $R$ and any $i \neq j$ the $R/m$-algebras $A_i \otimes_R R/m$ and $A_j \otimes_R R/m$ do not have isomorphic quotients. Then elements

$$a_1 = (a_{11}, \ldots, a_{1s}), \ldots, a_k = (a_{k1}, \ldots, a_{ks})$$

of $A$ generate $A$ as an $R$-algebra if and only if the elements $a_{i1}, \ldots, a_{ki}$ generate $A_i$ as an $R$-algebra for $i = 1, \ldots, s$.

Let now $R$ be a Dedekind domain with the field of fractions $K$. Let $H$ be a separable $K$-algebra and let $\Lambda$ be a maximal $R$-order in $H$. Our goal is to apply the above results
in order to find the smallest number of generators of $\Lambda$ as an $R$-algebra. To this end we introduce the following notation.

- $R$ is a Dedekind domain with the field of fractions $K$.
- $P$ is a maximal ideal of $R$.
- $H$ is a separable $K$-algebra and $\Lambda$ is a maximal $R$-order in $H$.
- $H = \prod_i H_i$, where $H_i$ are simple separable $K$-algebras.
- $K_i$ is the center of $H_i$ and $R_i$ is the integral closure of $R$ in $K_i$.
- $\Lambda = \prod_i \Lambda_i$, where $\Lambda_i$ is a maximal $R_i$-order in $H_i$.
- $P R_i = \prod_j P_{e_{i,j}}$, where $P_{i,j}$ are maximal ideals of $R_i$ and $e_{i,j}$ are positive integers.
- $A_{i,j} = \Lambda_i/\Lambda_i$.

It follows that

$$\Lambda/\Lambda = \prod_i \Lambda_i/\Lambda_i = \prod_i \Lambda_i/\Lambda_\Lambda_i = \prod_i A_{i,j}.$$  \hspace{1cm} (2)

We now state our first result of this section.

**Theorem 5.5.** Let $R$ be a Dedekind domain whose residue fields are infinite. Let $K$ be the field of fractions of $R$ and let $H$ be a separable $K$-algebra. Let $\Lambda$ be a maximal $R$-order in $H$. Suppose that for each maximal ideal $P$ of $R$ the quotient of $\Lambda/\Lambda$ by its Jacobson radical is a separable $R/P$-algebra (note that this is true when all residue fields of $R$ are perfect). Then $\Lambda$ can be generated by 3 elements as an $R$-algebra.

**Proof.** We proved in [1] that separable algebras over infinite fields are 2-generated. Thus, $r_K(\Lambda) \leq 2$. By Theorem 5.1, it suffices now to show that for every maximal ideal $P$ of $R$ the $R/P$-algebra $\Lambda/\Lambda$ is 2-generated. To this end, let $P$ be a maximal ideal of $R$. We use the notation introduced above.

By Theorem 4.1 each $A_{i,j}$ in (2) is an Artinian primary $R_i/P_{i,j}$-algebra, finitely generated as $R_i/P_{i,j}$-module. Note that $R_i/P_{i,j}$ is a finite dimensional $R/P$-algebra. It follows that $A_{i,j}$ is a finite dimensional primary $R/P$-algebra. Furthermore, $A_{i,j}/J(A_{i,j})$ is a separable $R/P$-algebra, hence it is 2-generated. By Theorem 4.1 $J(A_{i,j})/J(A_{i,j})^2$ is a simple $A_{i,j}/J(A_{i,j})$-bimodule. Thus any pair of $R/P$-algebra generators of $A_{i,j}/J(A_{i,j})^2$ can be lifted to a pair of $R/P$-algebra generators of $A_{i,j}/J(A_{i,j})^2$ by Theorem 2.7. By Corollary 2.6 any pair of $R/P$-algebra generators of $A_{i,j}/J(A_{i,j})^2$ can be lifted to a pair
of \( R/P \)-algebra generators of \( A_{i,j} \). It follows that any pair of \( R/P \)-algebra generators of \( A_{i,j}/J(A_{i,j}) \) can be lifted to a pair of \( R/P \)-algebra generators of \( A_{i,j} \).

Now let \( B \) be one of the simple quotients of \( \Lambda/PA \) and let \( A_B \) be the product of all those \( A_{i,j} \) which have simple quotient isomorphic to \( B \) as an \( R/P \)-algebra. Then \( \Lambda/PA = \prod B A_B \). By Proposition 5.4 it suffices to show that each \( R/P \)-algebra \( A_B \) is 2-generated (recall that \( R/P \) is an infinite field). As \( A_B/J(A_B) \) is a separable \( R/P \)-algebra, it is 2-generated. It is now clear that Theorem 2.2 implies that any pair of generators of \( A_B/J(A_B) \) lifts to a pair of generators of \( A_B \).

\[ \square \]

Question 5.6. In the notation of Theorem 5.5, is \( \Lambda \) generated by two elements as an \( R \)-algebra?

We expect that the answer is negative in general, but at present we do not have a counterexample. Consider however the following related question. Suppose that \( R \) is a Dedekind domain with fraction field \( K \) and with all residue fields infinite and perfect. Let \( L \) be a finite separable extension of \( K \) and let \( S \) be an integral closure of \( R \) in \( L \). Then \( S \) is a Dedekind domain, finitely generated as \( R \)-module. Theorem 5.1 implies that \( S \) can be generated as an \( R \)-algebra by no more than 2 elements. Is \( S \) generated by 1 element? The answer is yes if \( R \) is a discrete valuation ring. The following example shows that the answer is no in general.

Example 5.7. Let \( K = \mathbb{C}(t) \). Take \( L = K(\sqrt{t-1}, \sqrt[3]{t}) \). Set \( y = \sqrt{t-1}, x = \sqrt[3]{t} \), so \( y^2 = x^3 - 1 \) and \( L = \mathbb{C}(x, y) \) is the function field of the elliptic curve \( y^2 = x^3 - 1 \). Then \( S = \mathbb{C}[x, y] \) is the integral closure of \( R = \mathbb{C}[t] \) in \( L \), and \( S \) can not be generated by one element as an \( R \)-algebra.

Proof. Note that \( L \) is a Galois extension of \( K \) of degree 6. The Galois group of \( L/K \) is cyclic generated by an automorphism \( \tau \) such that \( \tau(y) = -y \) and \( \tau(x) = \lambda x \), where \( \lambda \) is a primitive cube root of 1. Note that the fixed field of \( \tau^2 \) is \( K(y) = \mathbb{C}(y), T_y = \mathbb{C}[y] \) is the integral closure of \( R \) in \( K(y) \), and 1, \( y \) is an integral basis of \( K(y)/K \) with discriminant \( 4(t-1) \). The fixed field of \( \tau^3 \) is \( K(x) = \mathbb{C}(x), T_x = \mathbb{C}[x] \) is the integral closure of \( R \) in \( K(x) \), and 1, \( x, x^2 \) is an integral basis of \( K(x)/K \) with discriminant \(-27t^2 \). As the discriminants of \( K(x)/K \) and \( K(y)/K \) are relatively prime in \( R \), we conclude that the integral closure of \( R \) in \( L = K(x, y) \) is \( S = \mathbb{C}[x, y] \), and 1, \( y, x, x^2, xy, x^2y \) is an integral
basis of $L/K$ with discriminant $(4(t - 1))^3(-27t^2)^2 = 6^6t^4(t - 1)^3$ (see [7, Proposition 2.11]).

Suppose now that $S = R[h]$ for some $h \in S$. Then $1, h, h^2, h^3, h^4, h^5$ is an integral basis of $L/K$ with discriminant

$$\prod_{0 \leq i < j \leq 5} (\tau^i(h) - \tau^j(h))^2.$$

Thus

$$(3) \quad \prod_{0 \leq i < j \leq 5} (\tau^i(h) - \tau^j(h))^2 = ct^4(t - 1)^3$$

for some constant $c \in \mathbb{C}$. We may choose $h$ of the form

$$h = a_1y + a_2x + a_3x^2 + a_4yx + a_5yx^2$$

for some $a_1, \ldots, a_5 \in R$. Then

$$(\tau^i(h) - \tau^{i+3}(h))^2 = 4(t - 1)(a_1 + a_4\lambda^i x + a_5\lambda^{2i} x^2)^2$$

for $i = 0, 1, 2$. It follows from (3) (interpreted as equality in $\mathbb{C}[x]$) that $a_1(x^3) + a_4(x^3)x + a_5(x^3)x^2$ divides $x^{12}$. It follows that two of $a_1, a_4, a_5$ are 0 and the third is of the form $dt^i$ for some $i$. If $a_1 = 0$ or $a_1 = dt^i$ with $i > 0$ then we would have $h = xg$ for some $g \in S$ and the discriminant of $1, h, h^2, h^3, h^4, h^5$ would be divisible by $x^{30} = t^{10}$, a contradiction. It follows that $a_1$ is constant and $a_4 = a_5 = 0$. Now, when $0 \leq i < j \leq 5$ have the same parity (there are six such pairs $i, j$), then

$$(\tau^i(h) - \tau^j(h))^2 = (\lambda^i - \lambda^j)^2 x^2(a_2 + (\lambda^i + \lambda^j)a_3x)^2.$$ 

It follows that $x^{12} \prod(a_2(x^3) + (\lambda^i + \lambda^j)x a_3(x^3))^2$ divides $x^{12}$ in $\mathbb{C}[x]$. This can only happen if $a_3 = 0$ and $a_2$ is constant. We may then assume that $h = x + uy$ for some constant $u \neq 0$. Then

$$(h - \tau(h))^2(\tau^3(h) - \tau^4(h))^2 = [(1 - \lambda)^2x^2 - 4u^2(x^3 - 1)]^2$$

divides $x^{12}$ in $\mathbb{C}[x]$, a contradiction. \qed

At this point we add the following to the notation and assumptions made at the beginning of this section:

- $R$ is the ring of integers in a number field $K$.
- $n_{i,j}$ is the local capacity of $H_i$ at $P_{i,j}$.
- $m_{i,j}$ is the local index of $H_i$ at $P_{i,j}$.
- $R/P$ is a finite field of order $q$, i.e. $R/P \cong \mathbb{F}_q$. 

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- \( f_{i,j} \) is the inertia degree of \( P_{i,j} \) over \( P \), so \( R_i/P_{i,j} \cong \mathbb{F}_{q^{i,j}} \).
- \( I(n,r) = \{(i,j) : n_{i,j} = n \text{ and } f_{i,j}m_{i,j} = r\} \).
- \( M(n,r) \) is the number of elements in \( I(n,r) \).
- \( A(n,r) = \prod_{(i,j) \in I(n,r)} A_{i,j} \).

Note that many of the above notions depend on the maximal ideal \( P \) of \( R \) and, if necessary, they will be treated as functions of \( P \). For example, we will write \( M(n,r,P) \), \( f_{i,j}(P) \), \( I(n,r,P) \) for \( M(n,r) \), \( f_{i,j} \), \( I(n,r) \) respectively, if the dependence on \( P \) needs to be indicated.

By Theorem 4.1, \( A_{i,j} = \Lambda_i/P_{i,j}^{e_{i,j}} \Lambda_i \) is a finite dimensional primary \( R/P \)-algebra with simple quotient isomorphic to \( M_{n_{i,j}}(\mathbb{F}_{q^{i,j}m_{i,j}}) \). Thus \( I(n,r) \) is the set of all pairs \( (i,j) \) such that the simple quotient of \( A_{i,j} \) is isomorphic to \( M_n(\mathbb{F}_q^r) \) as \( R/P = \mathbb{F}_q \)-algebras and \( M(n,r) \) is the number of simple factors of the semisimple algebra \( (\Lambda/PA)/J(\Lambda/PA) \) which are isomorphic to \( M_n(\mathbb{F}_q^r) \).

**Theorem 5.8.** Let \( R \) be the ring of integers in a number field \( K \) and let \( \Lambda \) be a maximal \( R \)-order in a finite dimensional semisimple \( K \)-algebra \( H \). For each maximal ideal \( P \) of \( R \) and positive integers \( n, r \) let \( M(n,r,P) \) be the number of simple factors of the semisimple algebra \( (\Lambda/PA)/J(\Lambda/PA) \) which are isomorphic to \( M_n(\mathbb{F}_q^r) \), where \( q = N(P) \). Let \( h \) be the smallest positive integer such that

\[
M(n,r,P) \leq \frac{g_h(n,q,r)}{r|\text{PGL}_n(\mathbb{F}_q^r)|}
\]

for every maximal ideal \( P \) of \( R \) and any positive integers \( n, r \). If \( h \geq 3 \) then \( h \) is the smallest number of generators of \( \Lambda \) as an \( R \)-algebra. If \( h = 2 \) then the smallest number of generators is 2 or 3. If \( h = 1 \) then \( \Lambda \) is commutative and the smallest number of generators is 1 or 2.

**Proof.** As \( H \) is a separable \( K \)-algebra, we have \( r_K(\Lambda) = 1 \) if \( \Lambda \) is commutative and \( r_K(\Lambda) = 2 \) if \( \Lambda \) is not commutative. By Theorem 5.1, it suffices to prove that \( h \) is the smallest number such that \( g_h(\Lambda, P) > 0 \) for all maximal ideals \( P \) of \( R \).

Consider a maximal ideal \( P \) of \( R \). By [2], we have

\[
\Lambda/PA = \prod A(n,r),
\]
where the product is over all pairs \( n, r \) such that \( I(n, r) \) is non-empty. Any simple quotient of \( A(n, r) = \prod_{(i,j) \in I(n,r)} A_{i,j} \) is isomorphic to \( \mathbb{M}_n(\mathbb{F}_q) \). Thus, when \( I(n, r) \) and \( I(n_1, r_1) \) are distinct and non-empty, the algebras \( A(n, r) \) and \( A(n_1, r_1) \) have no isomorphic quotients. By Proposition 5.4, for any integer \( k > 0 \) we have

\[
g_k(A, P) = \prod_{(i,j) \in I(n,r)} g_k(A_{i,j}, A_{i,j}/J(A_{i,j})) ,
\]

where the product is over all pairs \( n, r \) such that \( I(n, r) \) is non-empty. By Theorem 2.2 we have

\[
g_k(A(n, r), R/P) = g_k(\mathbb{M}_n(\mathbb{F}_q)^{M(n,r)}, \mathbb{F}_q) \prod_{(i,j) \in I(n,r)} g_k(A_{i,j}, A_{i,j}/J(A_{i,j})).
\]

Recall now that \( g_k(A_{i,j}, A_{i,j}/J(A_{i,j})) = g_k(A_{i,j}, A_{i,j}/J(A_{i,j})^2) g_k(A_{i,j}/J(A_{i,j})^2, A_{i,j}/J(A_{i,j})) \).

By Corollary 2.6 \( g_k(A_{i,j}, A_{i,j}/J(A_{i,j})^2) = |J(A_{i,j})^2|^k \). Define \( c_{i,j} \) as follows:

\[
c_{i,j} = \begin{cases} 
1, & \text{if } m_{i,j} > 1; \\
q^{-f_{i,j}}, & \text{if } m_{i,j} = 1 \text{ and } e_{i,j} > 1; \\
0, & \text{if } m_{i,j} = 1 \text{ and } e_{i,j} = 1.
\end{cases}
\]

Then, by Theorem 2.7 and Theorem 4.1 we have

\[
g_k(A_{i,j}/J(A_{i,j})^2, A_{i,j}/J(A_{i,j})) = |J(A_{i,j})/J(A_{i,j})^2|^k - c_{i,j} |J(A_{i,j})/J(A_{i,j})^2|.
\]

Finally, \( |J(A_{i,j})|^t = q^{n^2r(e_{i,j}m_{i,j} - t)} \) for all positive integers \( t \leq e_{i,j}m_{i,j} \). Putting all this together we get

\[
g_k(A(n, r), R/P) = g_k(\mathbb{M}_n(\mathbb{F}_q)^{M(n,r)}, \mathbb{F}_q) \prod_{(i,j) \in I(n,r)} q^{kn^2r(e_{i,j}m_{i,j} - 2)}(q^{kn^2r} - c_{i,j}q^{n^2r}).
\]

By Theorem 3.1 \( g_k(\mathbb{M}_n(\mathbb{F}_q)^{M(n,r)}, \mathbb{F}_q) > 0 \) if and only if (11) holds for \( M(n, r, P) = M(n, r) \). Thus, \( h \) is the smallest positive integer such that \( g_k(\mathbb{M}_n(\mathbb{F}_q)^{M(n,r)}, \mathbb{F}_q) > 0 \) for all maximal ideals \( P \) and all \( n, r \) such that \( M(n, r) > 0 \). If \( h > 1 \), then this is the same as \( g_h(A(n, r), R/P) > 0 \) for all maximal ideals \( P \) and all \( n, r \) such that \( M(n, r) > 0 \), so the theorem is true in this case. If \( h = 1 \), then \( n = 1 \) whenever \( M(n, r) > 0 \). It follows that \( \Lambda \) is commutative and \( c_{i,j} \) are always less than 1. Hence \( g_1(A(n, r), R/P) > 0 \) for all maximal ideals \( P \) and all \( n, r \) such that \( M(n, r) > 0 \) and the theorem holds in this case as well. \( \square \)
When \( h = 1 \) in Theorem 5.8, then there are examples when the smallest number of generators is 2. In fact, finding such examples with additional assumption that \( H \) is a field has been an old topic in algebraic number theory. See the last section of [11] for some examples. The next result shows that when \( h = 2 \) the ambiguity about the smallest number of generators can only happen when \( H \) has a simple factor of dimension 4 over its center.

**Theorem 5.9.** If in Theorem 5.8 \( h = 2 \) and no simple factor of \( H \) has dimension 4 over its center and \( \Lambda \) is free as an \( R \)-module, then the smallest number of generators of \( \Lambda \) as an \( R \)-algebra is 2. If Conjecture 5.3 is true then the assumption that \( \Lambda \) is free can be dropped.

**Proof.** According to Theorem 5.2 and Conjecture 5.3 it suffice to prove that

\[
\prod_{P \in \text{m-Spec } R} \frac{g_2(\Lambda, P)}{N(P)^{2d}} > 0,
\]

where \( d \) is the dimension of \( H \) over \( K \). We have seen in the proof of Theorem 5.8 that all the factors in (8) are positive. In addition, for all but a finite number of maximal ideals \( P \), all the numbers \( m_{i,j} \) and \( e_{i,j} \) are equal to 1 and \( n_{i,j}^2 = d_i \) is the dimension of \( H_i \) over \( K_i \). Let \( C > d \) be such that all \( P \) with \( N(P) > C \) have this property and \( C > c(n, r, t) \) for all \( n, r, t \) such that \( a \leq d, r \leq d, r \leq d \) (here \( c(n, r, t) \) are the constants from Lemma 3.5). Thus, when \( N(P) > C \) then

- \( I(n, r, P) = \{(i, j) : d_i = n_i^2 \text{ and } f_{i,j} = r \} \).  
- \( A(n, r, P) = M_n(\mathbb{F}_{q^r})^{M(n, r, P)} \), where \( q = N(P) \).
- \( d = \sum_{(n, r)} M(n, r, P) r n^2. \)

By (5), (7), and Lemma 3.5 when \( q = N(P) > C \) then

\[
\frac{g_2(\Lambda, P)}{N(P)^{2d}} = \prod_{(n, r)} \frac{g_2(M_n(\mathbb{F}_{q^r})^{M(n, r, P)}, \mathbb{F}_q)}{q^{2M(n, r, P) r n^2}} \geq \prod_{(n, r)} \left( 1 - \frac{25M(n, r, P)}{q^2} \right)^{M(n, r, P)} \geq \left( 1 - \frac{25d}{q^2} \right)^{\sum_{(n, r)} M(n, r, P)} \geq \left( 1 - \frac{25d}{q^2} \right)^d.
\]
The result follows now from the well-known fact that the product
\[
\prod_{P \in \text{m-Spec } R, \ N(P) > 25d} \left( 1 - \frac{25d}{N(P)^2} \right)
\]
converges (i.e. it is positive).

\[\square\]

**Remark 5.10.** Using Theorem 3.2(ii) it is not hard to show that if \(H\) has a simple factor of dimension 4 over its center then the product (8) is indeed 0.

**Example 5.11.** Let \(H = B^m\), where \(B\) is a quaternion algebra over \(\mathbb{Q}\) ramified at exactly the primes \(p_1 < \ldots < p_t\) (and at infinity, if \(t\) is odd). Let \(\Lambda\) be a maximal order in \(H\) so \(\Lambda^m\) is a maximal order in \(H\). We have \(M(1,2,p_i) = m\) and \(M(n,r,p_i) = 0\) if \(n \neq 1\) or \(r \neq 2\). For every prime \(p\) different from the ramified primes we have \(M(2,1,p) = m\) and \(M(n,r,p) = 0\) if \(n \neq 2\) or \(r \neq 1\). Thus we are looking for the smallest \(k\) such that \(m \leq g_k(2, p_i)\) for every \(p\) unramified in \(B\) and \(m \leq g_k(1, p_i, 2) / 2\) for \(i = 1, \ldots, t\). By Theorem 3.2, the first condition is equivalent to
\[
m \leq \frac{p^{2k-1}(p^k - 1)(p^k - p)}{p^2 - 1}.
\]

By Theorem 3.4, the second condition can be stated as
\[
2m \leq p_i^{2k} - p_i^k.
\]

If 2 ramifies in \(B\), i.e. if \(p_1 = 2\), then all these inequalities hold if and only if \(m \leq 2^{k-1}(2^k - 1)\). Thus there is \(u \leq 6\) such that the minimal number of generators of \(\Lambda^m\) is 2 if and only if \(m \leq u\), and it is 3 if and only if \(u < m \leq 28\) and it is \(k > 3\) iff \(2^{k-2}(2^k - 1) < m \leq 2^{k-1}(2^k - 1)\). Suppose now that 2 is unramified. Then our conditions will be satisfied iff \(m \leq 2^{2k-1}(2^k - 1)(2^k - 2)/3\) and \(2m \leq p_i^{2k} - p_i^k\). Note that \(2^{2k-1}(2^k - 1)(2^k - 2)/3 \leq 4^k(4^k - 1)/3\). It follows that if \(p_1 > 4\), i.e. if \(3\) is also unramified, then \(m \leq 2^{2k-1}(2^k - 1)(2^k - 2)/3\) implies the other condition. Thus there is \(u \leq 16\) such that the minimal number of generators of \(\Lambda^m\) is 2 if and only if \(m \leq u\) and it is 3 iff \(u < m \leq 448\) and it is \(k > 3\) iff \(2^{2k-3}(2^k - 1)(2^k - 2)/3 < m \leq 2^{2k-1}(2^k - 1)(2^k - 2)/3\). Finally suppose that \(p_1 = 3\). Then the conditions are \(m \leq 2^{2k-1}(2^k - 1)(2^k - 2)/3\) and \(m \leq 3^k(3^k - 1)/2\). For \(k = 2\) this is equivalent to \(m \leq 16\), and for all \(k \geq 3\) it is equivalent to \(m \leq 3^k(3^k - 1)/2\). Thus there is \(u \leq 16\) such that the minimal number of generators of \(\Lambda^m\) is 2 if and only if \(m \leq u\) and it is 3 iff \(u < m \leq 351\) and it is \(k > 3\).
iff $3^{k-1}(3^{k-1} - 1)/2 < m \leq 3^k(3^k - 1)/2$. In the introduction to [3] we expressed a hope that knowing the smallest number of generators of $\Lambda^m$ for every positive integer $m$ may tell us a lot about the structure of $\Lambda$. This example shows that this is not the case.

**Question 5.12.** What is $u$ in the above example?

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