On the Hydrostatic Approximation of Compressible Anisotropic Navier–Stokes Equations–Rigorous Justification

Hongjun Gao, Šárka Nečasová and Tong Tang

Communicated by Tohru Ozawa

Dedicated to Professor Yoshihiro Shibata on the occasion of his 70th birthday

Abstract. In this work, we derive the hydrostatic approximation by taking the small aspect ratio limit to the Navier–Stokes equations. The aspect ratio (the ratio of the depth to horizontal width) is a geometrical constraint in the general large scale geophysical motions meaning that the vertical scale is significantly smaller than horizontal. We derive the versatile relative entropy inequality. Applying the versatile relative entropy inequality we gave the rigorous derivation of the limit from the compressible Navier–Stokes equations to the compressible Primitive Equations. This is the first work where the relative entropy inequality was used for proving hydrostatic approximation - the compressible Primitive Equations.

Mathematics Subject Classification. 35Q30, 35Q86.

Keywords. Anisotropic Navier–Stokes equations, Aspect ratio limit, Hydrostatic approximation, Compressible Primitive Equations.

1. Introduction

The atmosphere and ocean have attracted considerable attention in the scientific research community, especially for the geophysics, as it has so many fluid dynamic properties and mysterious phenomena. One of the most interesting and physically important features of large-scale meteorology and oceanography is that vertical dimension of the domain is much smaller than the horizontal dimension of domain. Therefore, many scientists have suggested the viscosity coefficients must be anisotropic, such as [12, 51, 55]. The anisotropic Navier–Stokes equations are widely used in geophysical fluid dynamics. In this paper, we consider the following compressible anisotropic Navier–Stokes equations

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) &= \mu_x \Delta_x \mathbf{u} + \mu_z \partial_{zz} \mathbf{u},
\end{align*}
\]

in the thin domain \((0, T) \times \Omega_\epsilon\). Here \(\Omega_\epsilon = \{(x, z)| x \in \mathbb{T}^2, -\epsilon < z < \epsilon\}\), \(x\) denotes the horizontal direction and \(z\) denotes the vertical direction, while, \(\mu_x\) and \(\mu_z\) are given constant horizontal viscous coefficient and vertical viscous coefficient. The velocity \(\mathbf{u} = (v, w)\), where \(v(t, x, z) \in \mathbb{R}^2\) and \(w(t, x, z) \in \mathbb{R}\) represent the horizontal velocity and vertical velocity, respectively. Through out this paper, we use \(\text{div} \mathbf{u} = \text{div}_x v + \partial_z w\) and \(\nabla = (\nabla_x, \partial_z)\) to denote the three-dimensional spatial divergence and gradient respectively, and \(\Delta_x\) stands for horizontal Laplacian. As atmosphere and ocean are the thin layers, where the fluid layer depth is small compared to radius of sphere, Pedlosky [51] pointed out that “the pressure difference between any two points on the same vertical line depends only on the weight of the fluid between these points...”. Here we neglect the gravity and suppose the pressure \(p(\rho)\) satisfies the barotropic pressure law where the pressure and the density are related by the formula: \(p(\rho) = \rho^\gamma\) (\(\gamma > 1\)). Therefore we assume the density...
In the three dimension case. The celebrated breakthrough result was made by Cao and Titi [13]. They
Guillén–González, Masmoudi and Rodríguez–Bellido [33] proved the local existence of strong solutions
in this field. Then PE has historically progressed by concentrated the mathematical arguments developed
and Titi–arxiv:1905.09367, Page 4, line 14.

Under this assumption, the system is rewritten as the following
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) + \frac{\partial}{\partial x}(\rho w) &= 0, \\
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{v} - \Delta_x \mathbf{v} - \epsilon^2 \frac{\partial}{\partial z} \mathbf{v} + \nabla_x p(\rho) &= 0, \\
\rho \frac{\partial w}{\partial t} + \rho (\mathbf{u} \cdot \nabla) w - \Delta_x w - \epsilon^2 \frac{\partial}{\partial z} w + \frac{\partial}{\partial z} p(\rho) &= 0.
\end{align*}
\]

Inspired by [2,39], we introduce the following new unknowns,
\[\mathbf{u}_e = (\mathbf{v}_e, w_e), \quad \mathbf{v}_e(x,z,t) = \mathbf{v}(x,\epsilon z,t), \quad w_e = \frac{1}{\epsilon} w(x,\epsilon z,t), \quad \rho_e = \rho(x,t),\]
for any \((x,z) \in \Omega := \mathbb{T}^2 \times (-1,1)\). Then the system (1.3) becomes the following compressible scaled
Navier–Stokes equations (CNS):
\[
\begin{align*}
\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{v}_e) + \frac{\partial}{\partial x}(\rho_e w_e) &= 0, \\
\rho_e \frac{\partial \mathbf{v}_e}{\partial t} + \rho_e (\mathbf{u}_e \cdot \nabla) \mathbf{v}_e - \Delta_x \mathbf{v}_e - \frac{\partial}{\partial z} \mathbf{v}_e + \nabla_x p(\rho_e) &= 0, \\
\epsilon^2 \left( \frac{\partial}{\partial t} w_e + \rho_e \mathbf{u}_e \cdot \nabla w_e - \Delta_x w_e - \frac{\partial}{\partial z} w_e \right) + \frac{\partial}{\partial z} p(\rho_e) &= 0.
\end{align*}
\]

We supplement the CNS with the following boundary and initial conditions:
\[
\rho_e, \quad \mathbf{u}_e \quad \text{are periodic in } x,y,z, \\
(\rho_e, \mathbf{u}_e) |_{t=0} = (\rho_0, \mathbf{u}_0).
\]

The goal of this work is to investigate the limit process \(\epsilon \to 0\) in the system of (1.4) converges in a certain
sense to the following compressiblePrimitive Equations(CPE):
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) + \frac{\partial}{\partial x}(\rho w) &= 0, \\
\frac{\partial \mathbf{v}}{\partial t} + \nabla(\rho \mathbf{v} \otimes \mathbf{v}) + \frac{\partial}{\partial x}(\rho \mathbf{w}) + \nabla_x p(\rho) &= \Delta_x \mathbf{v} + \frac{\partial}{\partial z} \mathbf{v}, \\
\frac{\partial \rho}{\partial z} p(\rho) &= 0.
\end{align*}
\]
were the first who proved the global well-posedness of PE in the three dimensional case. Then, by virtue of semigroup method, Hieber and Kashiwabara [36] extended this result relaxing the smoothness on the initial data. On the other hand, regarding to inviscid PE (hydrostatic incompressible Euler equations), the existence and uniqueness is an outstanding open problem. Brenier [5] proved the existence of smooth solutions in two-dimensions under the convex horizontal velocity assumptions. Moreover, he suggested (see [6] that the existence problem may be ill-posed in Sobolev spaces. Later, Masmoudi and Wong [48] extended Brenier’s result, removing the convex horizontal velocity assumptions. Partly for historical reasons, the research of geophysical fluid concerns on PE model at incompressible case. However, it is well known that atmosphere and ocean have compressible property. Therefore, it is interesting and natural to consider the PE model at compressible case, meaning CPE. With the constant viscosity coefficients, Gatapov and Kazhikhov [31], Ersoy and Ngom [21] proved the global existence of weak solutions in 2D case. Recently, Liu and Titi [44,46] proved the local existence of strong solutions in 3D case and consider the zero Mach number limit of CPE. On the other hand, Ersoy et al. [20] used the dimensionless number and asymptotic analysis, obtaining the CPE in the case where the viscosity coefficients are depending on the density. Ersoy et al. [20], Tang and Gao [52] showed the stability of weak solutions. The stability means that a subsequence of weak solutions will converge to another weak solutions if it satisfies some uniform bounds. Recently, Liu and Titi [45] and independently Wang et al. [54] used the B–D entropy to prove the global existence of weak solutions.

As stressed by [2,39], the hydrostatic approximation is one of the important feature of PE model. A rigorous justification of the limit passage from anisotropic Navier–Stokes equations to its hydrostatic approximation via the small aspect limit seems to be of obvious practical importance. There are numerous studies of the incompressible convergence. For example, Azérad and Guillén [2] proved the weak solutions of anisotropic Navier–Stokes equations converge to weak solutions of PE. Li and Titi [39] used the method of weak-strong uniqueness to prove the aspect ratio limit of incompressible anisotropic Navier–Stokes equations, that is from weak solutions of anisotropic Navier–Stokes equations to strong solutions of incompressible PE model. Then Giga, Hieber and Kashiwabara et al. [27,28] extended the results into maximal regularity spaces. Recently, Donatelli and Juhasz [18] proved the convergence in downwind-matching coordinates. For the stationary case, readers can refer to [4,9]. On the other hand, based on a revised global Cauchy–Kowalewski theorem, Paicu, Zhang and Zhang [50] proved the incompressible anisotropic Navier–Stokes equations converge to the Prandtl equation in Besov spaces for 2D case. However, for the compressible fluids flows, to the best of authors’ knowledge, there are no results concerning the convergence from compressible Navier–Stokes system (CNS) to compressible Primitive Equations (CPE).

Our goal is to rigorously justify the limit in the framework of weak solutions of CNS. Recently, Bella, Feireisl and Novotný [3], Maltese and Novotný [47] proved the limit passage from 3D compressible Navier–Stokes equations to 1D and 2D compressible Navier–Stokes equations in thin domain. See also result by Ducomet et al. [19]. Heuristically, inspired by their works, we develop and adapt the corresponding idea of relative entropy inequality for compressible Navier–Stokes equations. There are significant differences of the mathematical structure between Navier–Stokes equations and CPE model. Due to the hydrostatic approximation, there is no information for the vertical velocity in the momentum equation of CPE model, and the vertical velocity is determined by the horizontal velocity via the continuity equation, so it is very difficult to analyze the CPE model. Therefore, the classical method used in Navier–Stokes system can not be applied straightforwardly to CPE. Luckily, based on our previous work [30] of weak-strong uniqueness to CPE, we prove the aspect ratio limit of compressible anisotropic Navier–Stokes equations.

This is the first work to use the relative entropy inequality for proving the hydrostatic approximation at the compressible case. For the introduction of the versatile relative entropy inequality, see [29]. We will explain the differences between our present work with [30], see Sect. 3, Remark 3.5. Last but not least, let us mention that the corner-stone analysis of our results is based on the relative energy inequality which was invented by Dafermos, see [17] and by Germain, [32], who introduced it into compressible Navier–Stokes equations. After that Feireisl and his co-authors [23,24,26] generalize the relative energy inequality and apply such inequality for solving various compressible fluid model problems.
The paper is organized as follows. In Sect. 2, we recall some useful inequalities. We introduce the definition of weak solutions, strong solution, relative energy and state the main theorem in Sect. 3. Section 4 is devoted to proof of the convergence.

2. Preliminaries

In this section, we first introduce some basic inequalities needed in the later proof. The first inequality is the so called the generalized Poincaré inequality.

**Lemma 2.1.** Let \( 2 \leq p \leq 6 \) and \( \rho \geq 0 \) such that \( 0 < M \leq \int_{\Omega} \rho \, dx, \int_{\Omega} \rho^\gamma \, dx \leq E_0 \) for some \( (\gamma > 1) \) then

\[
\|f\|_{L^p(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)} + \|\rho^{\frac{1}{2}} f\|_{L^2(\Omega)},
\]

where \( C \) depends on \( M \) and \( E_0 \).

The details of proof can be seen at Feireisl’s monograph [22]. The following is the famous Gagliardo–Nirenberg inequality (see [49]):

**Lemma 2.2.** For a function \( u : \Omega \rightarrow \mathbb{R} \) defined on a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \), \( \forall 1 \leq q, r \leq \infty \), and a natural number \( m \). Suppose that a real number \( \theta \) and a natural number \( j \) are such that

\[
\frac{1}{p} = \frac{j}{n} + \left( \frac{1}{r} - \frac{m}{n} \right) \theta + \frac{1 - \theta}{q},
\]

and

\[
\frac{j}{m} \leq \theta \leq 1,
\]

then there exists constant \( C \) independent of \( u \) such that

\[
\|D^j u\|_{L^p(\Omega)} \leq C \|D^m u\|_{L^r(\Omega)} \|u\|_{L^1(\Omega)}^{1 - \theta} + C \|u\|_{L^s},
\]

where \( s > 0 \) is arbitrary naturally.

3. Main Result

Before stating our main result, we give the definition of a weak solution for CNS and a strong solution for CPE. Recently, Bresch and Jabin [10] introduced different compactness method from Lions or Feireisl which can be applied to anisotropical stress tensor.\(^4\) Bresch and Jabin has proved the global existence of weak solutions for non-monotone pressure and for anisotropical stress tensor. Therefore, following their theory, we can obtain the existence of weak solution for monotone pressure and anisotropic viscous stress tensor. Let us recall their definitions here.

3.1. Dissipative Weak Solutions of CNS

**Definition 3.1.** We say that \( [\rho_\epsilon, \mathbf{u}_\epsilon] \) with \( \mathbf{u}_\epsilon = (v_\epsilon, w_\epsilon) \) is a finite energy weak solution to the system of (1.4), supplemented with initial data (1.5) if \( \rho_\epsilon = \rho_\epsilon(x, t) \) and

\[
\mathbf{u}_\epsilon \in L^2(0, T; H^1(\Omega)), \quad \rho_\epsilon |\mathbf{u}_\epsilon|^2 \in L^\infty(0, T; L^1(\Omega)),
\rho_\epsilon \in L^\infty(0, T; L^\gamma(\Omega)) \cap C([0, T], L^1(\Omega)), \quad \rho_\epsilon \geq 0,
\]

\(^4\)Let us emphasize the result of Bresch and Jabin is valid only for small coefficients of viscosities.
• the continuity equation
\[
\left[ \int_\Omega \rho_c \psi \, dx \, dz \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \rho_c \partial_t \psi + \rho_c \mathbf{v}_c \cdot \nabla_x \psi + \rho_c w_c \partial_z \psi \, dx \, dz \, dt,
\]
holds for all \( \psi \in C^\infty_c((0, T) \times \overline{\Omega}) ;
• the momentum equation
\[
\left[ \int_\Omega \rho_c \mathbf{v}_c \varphi_\mathbf{H} \, dx \, dz \right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \rho_c \mathbf{v}_c \partial_t \varphi_\mathbf{H} \, dx \, dz \, dt - \int_0^\tau \int_\Omega \rho_c \mathbf{u}_c \mathbf{v}_c \cdot \nabla \varphi_\mathbf{H} \, dx \, dz \, dt
+ \int_0^\tau \int_\Omega \nabla \mathbf{v}_c : \nabla \varphi_\mathbf{H} \, dx \, dz \, dt - \int_0^\tau \int_\Omega p(\rho_c) \text{div}_x \varphi_\mathbf{H} \, dx \, dz \, dt = 0,
\]
and
\[
e^2 \left[ \int_\Omega \rho_c w_c \varphi_3 \, dx \, dz \right]_{t=0}^{t=\tau} - e^2 \int_0^\tau \int_\Omega \rho_c w_c \partial_t \varphi_3 \, dx \, dz \, dt - e^2 \int_0^\tau \int_\Omega \rho_c \mathbf{u}_c w_c \cdot \nabla \varphi_3 \, dx \, dz \, dt
+ e^2 \int_0^\tau \int_\Omega \nabla w_c \cdot \nabla \varphi_3 \, dx \, dz \, dt - \int_0^\tau \int_\Omega p(\rho_c) \partial_z \varphi_3 \, dx \, dz \, dt = 0,
\]
holds for any spatially periodic function \( \varphi_\mathbf{H}, \varphi_3 \in C^\infty_c((0, T) \times \overline{\Omega}) \). Combining (3.3)–(3.4), we obtain
\[
\left[ \int_\Omega \rho_c \mathbf{v}_c \varphi_\mathbf{H} \, dx \, dz + e^2 \int_\Omega \rho_c w_c \varphi_3 \, dx \, dz \right]_{t=0}^{t=\tau}
- \int_0^\tau \int_\Omega \rho_c \mathbf{v}_c \partial_t \varphi_\mathbf{H} \, dx \, dz \, dt - e^2 \int_0^\tau \int_\Omega \rho_c w_c \partial_t \varphi_3 \, dx \, dz \, dt
- \int_0^\tau \int_\Omega \rho_c \mathbf{v}_c \otimes \mathbf{v}_c : \nabla \varphi_\mathbf{H} \, dx \, dz \, dt - \int_0^\tau \int_\Omega \rho_c \mathbf{v}_c w_c \cdot \partial_z \varphi_\mathbf{H} \, dx \, dz \, dt
- e^2 \int_0^\tau \int_\Omega \rho_c \mathbf{u}_c w_c \cdot \nabla \varphi_3 \, dx \, dz \, dt - e^2 \int_0^\tau \int_\Omega \rho_c w_c^2 \partial_z \varphi_3 \, dx \, dz \, dt
+ \int_0^\tau \int_\Omega \nabla \mathbf{v}_c : \nabla \varphi_\mathbf{H} \, dx \, dz \, dt + e^2 \int_0^\tau \int_\Omega \nabla w_c \cdot \nabla \varphi_3 \, dx \, dz \, dt - \int_0^\tau \int_\Omega p(\rho_c) \text{div}_x \varphi \, dx \, dz \, dt = 0,
\]
where spatially periodic function \( \varphi = (\varphi_\mathbf{H}, \varphi_3) \in C^\infty_c((0, T) \times \overline{\Omega}) \) and \( \text{div} \varphi = \text{div}_x \varphi_\mathbf{H} + \partial_z \varphi_3 ,
• the energy inequality
\[
\left[ \int \frac{1}{2} \rho_c |\mathbf{v}_c|^2 + \frac{e^2}{2} \rho_c |w_c|^2 + P(\rho_c) \right] \, dx \, dz \big|_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega (|\nabla \mathbf{v}_c|^2 + e^2 |\nabla w_c|^2) \, dx \, dz \, dt \leq 0,
\]
holds for a.a \( \tau \in (0, T) \), where \( P(\rho) = \rho \int_1^\rho \frac{p(\xi)}{\xi^2} \, d\xi \).

**Remark 3.1.** Recently, Abbatiello, Feireisl and Novotný [1] propose a new concept-dissipative turbulent solutions, which is so far the largest class of generalized solutions. We will prove the existence of dissipative turbulent solutions for CNS in future work, and it is easy to prove that every dissipative turbulent solution is a bounded energy weak solution.

### 3.2. Strong Solution of CPE

We say that \((r, \mathbf{U}) \), \( \mathbf{U} = (\mathbf{V}, W) \) is a strong solution to the CPE system (1.6) in \((0, T) \times \Omega \), if
\[
r^2 \in L^\infty(0, T; H^2(\Omega)), \quad \partial_t r^2 \in L^\infty(0, T; H^1(\Omega)), \quad r > 0 \text{ for all } (t, x), \quad \mathbf{V} \in L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)), \quad \partial_t \mathbf{V} \in L^2(0, T; H^2(\Omega)),
\]
with initial data \( r_0^2 \in H^2(\Omega), \ r_0 > 0 \) and \( V_0 \in H^3(\Omega) \). Liu and Titi [44] have proved the local existence of strong solution to CPE system (1.6).

**Remark 3.2.** Let us mention that in the definition of the strong solution we consider the regularity \( V \in L^\infty(0,T;H^3) \), which is higher than the regularity considered by Liu and Titi, see [44] which means \( V \in L^\infty(0,T;H^2) \). The explanation how to get higher regularity of solution can be found in Liu and Titi’s original version-arxiv:1806.09868, see page 21.

**Remark 3.3.** As the density is independent of \( z \), we can obtain the following information of vertical velocity for the weak solution of CNS:

\[
\rho w(x,z,t) = -\text{div}_x(\rho \tilde{V}) + z \text{div}_x(\rho \tilde{V}), \quad \text{in the sense of } H^{-1}(\Omega),
\]  

where\(^5\)

\[
\tilde{V}(x,z,t) = \int_0^z v(x,s,t)ds, \quad \tau(x,t) = \int_0^1 v(x,z,t)dz.
\]

Similarly, we can obtain the same equation for the strong solution of CPE in the classical sense. There is no information about \( w \), so we need to derive its information. We should emphasize that (1.7) is the key step to obtain the existence of weak solution for CPE in [45,54], which is inspired by incompressible case.

### 3.3. Versatile Relative Entropy Inequality

Motivated by [23,24], for any finite energy weak solution \((\rho,u)\), where \( u = (v,w) \), to the CNS system, we introduce the relative energy functional

\[
\mathcal{E}(\rho,u|r,U) = \int_\Omega \left[ \frac{1}{2} \rho |v - V|^2 + \frac{\epsilon^2}{2} \rho |w - W|^2 + P(\rho) - P'(\rho)(\rho - r) - P(r) \right] dx \ dz
\]

\[
= \int_\Omega \left( \frac{1}{2} \rho |v|^2 + \frac{\epsilon^2}{2} \rho |w|^2 + P(\rho) \right) dx \ dz - \int_\Omega (\rho v \cdot V + \epsilon^2 \rho w W) \ dx \ dz
\]

\[
+ \int_\Omega \left[ \frac{\rho |V|^2}{2} + \frac{\epsilon^2}{2} \rho |W|^2 - \rho P'(r) \right] dx \ dz + \int_\Omega P(r) \ dx \ dz
\]

\[
= \sum_{i=1}^4 I_i,
\]

where \( r > 0 \), \( U = (V,W) \) are smooth “test” functions, and spatially periodic function \( V, W \in C^\infty_c([0,T) \times \Omega) \). Here we have used \( rP'(r) - P(r) = p'(r) \).

**Lemma 3.1.** Let \((\rho,v,w)\) be a dissipative weak solution introduced in Definition 3.1. Then \((\rho,v,w)\) satisfy the versatile relative entropy inequality

\[
\mathcal{E}(\rho,u|r,U)|_{t=0}^\tau + \int_0^\tau \int_\Omega \left( (\nabla v : (\nabla v - \nabla V) + \epsilon^2 |\nabla w|^2) \right) \ dx \ dz \ dt
\]

\[
\leq \int_0^\tau \int_\Omega \rho(\partial_t V + v \cdot \nabla_x V + w \partial_z V)(V - v) \ dx \ dz \ dt
\]

\[
+ \epsilon^2 \int_0^\tau \int_\Omega \rho(\partial_t W + v \cdot \nabla_x W + w \partial_z W)(W - w) \ dx \ dz \ dt + \epsilon^2 \int_0^\tau \int_\Omega \nabla w \cdot \nabla W \ dx \ dz \ dt
\]

\[
- \int_0^\tau \int_\Omega P'(r)((\rho - r)\partial_t r + \rho v \cdot \nabla_x r) \ dx \ dz \ dt - \int_0^\tau \int_\Omega p(r)\text{div}_x V \ dx \ dz \ dt.
\]

\(^5\)The calculation of \( \rho w \) can be seen in details see Liu and Titi [46] page 1920 and see [54], page 4.
Proof. From the weak formulation and energy inequality (3.3)–(3.6), we deduce

\[ I_1 |_{t=0}^{\tau} + \int_0^\tau \int_\Omega \left( (|\nabla \mathbf{v}|^2 + \epsilon^2 |\nabla w|^2) \right) dx \, dz \, dt \leq 0, \tag{1.10} \]

\[ I_2 |_{t=0}^{\tau} = -\int_0^\tau \int_\Omega (\rho \mathbf{v} \partial_t \mathbf{V} + \rho \mathbf{v} \otimes \mathbf{v} : \nabla_x \mathbf{v} + \rho \mathbf{v} \mathbf{w} \cdot \partial_z \mathbf{V}) \, dx \, dz \, dt \]

\[ -\int_0^\tau \int_\Omega (\epsilon^2 \rho w \partial_t W + \epsilon^2 \rho w (\mathbf{v} \cdot \nabla_x) W + \epsilon^2 \rho w^2 \partial_z W + p(\rho) \text{div}_x \mathbf{V}) \, dx \, dz \, dt \]

\[ + \int_0^\tau \int_\Omega (\nabla \mathbf{v} : \nabla \mathbf{V} + \epsilon^2 \nabla w \cdot \nabla W) \, dx \, dz \, dt, \tag{1.11} \]

\[ I_3 |_{t=0}^{\tau} = \int_0^\tau \int_\Omega (\rho \partial_t |\mathbf{V}|^2/2 + \rho \mathbf{v} \cdot \nabla_x |\mathbf{V}|^2/2 + \rho w \partial_z |\mathbf{V}|^2/2) \, dx \, dz \, dt \]

\[ + \epsilon^2 \int_0^\tau \int_\Omega (\rho \partial_t |\mathbf{W}|^2/2 + \rho \mathbf{v} \cdot \nabla_x |\mathbf{W}|^2/2 + \rho w \partial_z |\mathbf{W}|^2/2) \, dx \, dz \, dt \]

\[ -\int_0^\tau \int_\Omega (\rho \partial_t P'(r) + \rho \mathbf{v} \cdot \nabla_x P'(r) + \rho w \partial_z P'(r)) \, dx \, dz \, dt \]

\[ = \int_0^\tau \int_\Omega (\rho \mathbf{V} \partial_t \mathbf{V} + \rho \mathbf{V} \cdot \nabla_x \mathbf{V} + \rho w \mathbf{V} \partial_z \mathbf{V}) \, dx \, dz \, dt \]

\[ + \epsilon^2 \int_0^\tau \int_\Omega (\rho W \partial_t W + \rho W \mathbf{v} \cdot \nabla_x W + \rho w W \partial_z W) \, dx \, dz \, dt \]

\[ -\int_0^\tau \int_\Omega (\rho P''(r) \partial_t r + P''(r) \rho \mathbf{v} \cdot \nabla_x r) \, dx \, dz \, dt, \tag{1.12} \]

\[ I_4 |_{t=0}^{\tau} = \int_0^\tau \int_\Omega \partial_t p(r) \, dx \, dz \, dt. \tag{1.13} \]

Summing (1.10)–(1.13) together, we obtain Lemma 3.1.

Remark 3.4. In the context of continuum mechanics, the concept of relative entropy has many successful applications, providing weak-strong uniqueness results, as well as justification of singular limits, for both incompressible and compressible fluid models. However, it is important to point out that there are distinguished differences for relative entropy inequality between incompressible and compressible cases. For more details, see [23,24].

Remark 3.5. Compared with the previous results [30], there are some delicate differences in the process of using the relative energy inequality. We should emphasize that we obtain the weak-strong uniqueness which means that we compare weak solutions of CPE and strong solutions of CPE in [30]. Here, our convergence is between two different systems and this convergence gives us system in 3D to “2.5”D. The role of weak solutions is played by the solutions of CNS, and the strong solutions is played by those of CPE. It means that we should deal with the convergence of the vertical velocity of CNS and the absence of the information on the vertical velocity in CPE. Due to the special structure of CPE, the relative

---

The idea of relative entropy inequality was introduced by Dafermos [17] for hyperbolic equations and by Germain ([32]) to fluid dynamics. After that Feireisl and his coauthors proved that the relative entropy inequality is valid for all smooth test functions with appropriate boundary conditions, see [23]. Precisely, the test functions \( r, \mathbf{U} \) in the relative entropy inequality are arbitrary smooth, \( r \) strictly positive, and \( \mathbf{U} \) satisfy the corresponding boundary conditions. Moreover, it is easy to check that the relative entropy inequality is satisfied as an equality as soon as the solution \( \rho, \mathbf{u} \) is smooth enough. Moreover there is shown so-called the weak-strong uniqueness principle, which means that if the initial data of weak solutions and strong solutions coincide, weak solutions coincide with the unique strong solution on the time interval where the strong solution lives. This principle gives us the possibility to use relative entropy inequality and weak-strong principle to show e.g. singular limits from weak solutions to strong solution with corresponding scaling if the corresponding reminder can be estimated in appropriate way. For details see e.g. book of Feireisl, Novotný, [24].
entropy inequality (3.8) is constructed differently from the ones in [30,47] and we call it the versatile relative entropy inequality.

Based on the relative entropy inequality, we can obtain the following lemma from [23]

**Lemma 3.2.** Let $0 < a < b < \infty$. Then there exists $c = c(a,b) > 0$ such that for all $\rho \in [0,\infty)$ and $r \in [a,b]$ there holds

$$P(\rho) - P'(r)(\rho - r) - P(r) \geq \begin{cases} C|\rho - r|^2, & \text{when } \frac{\rho}{2} < \rho < r, \\ C(1 + \rho^\gamma), & \text{otherwise} \end{cases}$$

where $C = C(a,b)$.

Moreover, from [23], we learn that

$$E(\rho, u|r, U)(t) \in L^\infty(0,T), \quad \int_\Omega \chi_{\rho \geq r} \rho^\gamma dx \, dz \leq C E(\rho, u|r, U)(t),$$

$$\int_\Omega \chi_{\rho \leq \frac{\rho}{2}} \rho dx \, dz \leq C E(\rho, u|r, U)(t), \quad \int_\Omega \chi_{\frac{\rho}{2} < \rho < r} \rho^2 dx \, dz \leq C E(\rho, u|r, U)(t).$$

(1.15)

For a rigorous proof of Lemma 3.2 and (1.15), the reader is referred to [23].

### 3.4. Main Result

Now, we are ready to state our main result.

**Theorem 3.1.** Let $\gamma > 6$, $T_{\text{max}} > 0$ be the life time of strong solution to CPE system (1.6) corresponding to initial data $[r_0, V_0]$. Let $(\rho_\epsilon, u_\epsilon)$, $u_\epsilon = (v_\epsilon, w_\epsilon)$ be a sequence of dissipative weak solutions to the CNS system (1.4) from the initial data $(\rho_{0,\epsilon}, u_{0,\epsilon})$. Suppose that $\rho_{0,\epsilon} > 0$, $r_0 > 0$ and $E(\rho_{0,\epsilon}, u_{0,\epsilon}|r_0, U_0) \to 0$,

$$\text{ess} \sup_{t \in (0, T_{\text{max}})} E(\rho_\epsilon, u_\epsilon|r, U) \to 0,$$

where $U_0 = (V_0, W_0)$, then

$$E(\rho_\epsilon, u_\epsilon|r, U) \to 0,$$

where $U = (V, W)$ and the couple $(r, U)$ satisfy the CPE system (1.6) on the time interval $[0, T_{\text{max}}]$.

**Remark 3.6.** Recently, Bresch and Burtea [11] proved existence of weak solutions for anisotropic compressible Stokes system.

Section 4 is devoted to the proof of the above theorem.

### 4. Convergence

In this section, we will prove the Theorem 3.1. First, we will explain the idea of the proof.

#### 4.1. Main Idea of Proof

The proof of Theorem 3.1 relies on the versatile relative energy inequality by considering the strong solution $(r, U)$, where $U = (V, W)$, as test function in the relative energy inequality (1.8). Firstly, let us recall the versatile relative entropy inequality

$$E(\rho_\epsilon, u_\epsilon|r, U)\bigg|_{t=0}^{t=T} + \int_0^T \int_\Omega (\nabla v_\epsilon \cdot (\nabla v_\epsilon - \nabla V) + \epsilon^2 |\nabla w_\epsilon|^2) \, dx \, dz \, dt$$

$$\leq \int_0^T \int_\Omega (\partial_t V + v_\epsilon \nabla V + w_\epsilon \partial_z V)(V - v_\epsilon) \, dx \, dz \, dt$$

\[Birkhäuser\]
\[ + \epsilon^2 \int_0^T \int_\Omega \rho_\epsilon (\partial_t W + \mathbf{v}_\epsilon \nabla_x W + w_\epsilon \partial_z W)(W - w_\epsilon) \, dx \, dz \, dt + \epsilon^2 \int_0^T \int_\Omega \nabla w_\epsilon \cdot \nabla W \, dx \, dz \, dt \]

\[ - \int_0^T \int_\Omega P''(r)(\rho_\epsilon - r) \partial_t r + \rho_\epsilon \mathbf{v}_\epsilon \nabla_x r) \, dx \, dz \, dt - \int_0^T \int_\Omega p(r) \text{div}_x \mathbf{V} \, dx \, dz \, dt. \]  

(4.1)

The goal now is to find a lower bound of (4.1) in the following form

\[ E(\rho_\epsilon, \mathbf{u}_\epsilon | r, \mathbf{U})(t) + C \int_0^t \| \nabla \mathbf{v}_\epsilon - \nabla \mathbf{V} \|_{L^2}^2 \, dt + \epsilon^2 \int_0^t \| \nabla w_\epsilon \|_{L^2}^2 \, dt \]

and an upper bound of (4.1) in the form

\[ C(\delta) \int_0^t h(t) E(\rho_\epsilon, \mathbf{u}_\epsilon | r, \mathbf{U}) \, dt + \delta \int_0^t \| \nabla \mathbf{v}_\epsilon - \nabla \mathbf{V} \|_{W^{1,2}}^2 \, dt + o(\epsilon^2), \]

with any \( \delta > 0 \), where \( C \) is independent of \( \delta \) and \( \epsilon \), \( h \in L^1(0, T) \) and \( o(\epsilon^2) \to 0 \) when \( \epsilon \to 0 \).

Using the above bounds, we can deduce

\[ E(\rho_\epsilon, \mathbf{u}_\epsilon | r, \mathbf{U})(\tau) \leq C \int_0^\tau h(t) E(\rho_\epsilon, \mathbf{u}_\epsilon | r, \mathbf{U})(t) \, dt + o(\epsilon^2), \]

and applying the Gronwall inequality implies the assertion. In the rest of this section, we will follow this way.

### 4.2. Step 1

We write

\[ \int_\Omega \rho_\epsilon \mathbf{v}_\epsilon (\mathbf{V} - \mathbf{v}_\epsilon) \cdot \nabla_x \mathbf{V} \, dx \, dz = \]

\[ \int_\Omega \rho_\epsilon (\mathbf{v}_\epsilon - \mathbf{V}) (\mathbf{V} - \mathbf{v}_\epsilon) \cdot \nabla_x \mathbf{V} \, dx \, dz + \int_\Omega \rho_\epsilon \mathbf{V} (\mathbf{V} - \mathbf{v}_\epsilon) \cdot \nabla_x \mathbf{V} \, dx \, dz. \]

As \([r, \mathbf{V}, W]\) is a strong solution, it is obvious to obtain that

\[ \int_\Omega \rho_\epsilon (\mathbf{v}_\epsilon - \mathbf{V}) (\mathbf{V} - \mathbf{v}_\epsilon) \cdot \nabla_x \mathbf{V} \, dx \, dz \leq C E(\rho_\epsilon, \mathbf{u}_\epsilon | r, \mathbf{U}). \]

(4.2)

Moreover, the momentum equation reads as

\[ (r \mathbf{V})_t + \text{div}_x (r \mathbf{V} \otimes \mathbf{V}) + \partial_z (r \mathbf{V} W) + \nabla_x p(r) = \Delta \mathbf{V} = \Delta_x \mathbf{V} + \partial_{zz} \mathbf{V}, \]

which implies that

\[ \mathbf{V}_t + \mathbf{V} \cdot \nabla_x \mathbf{V} + W \partial_z \mathbf{V} = -\frac{1}{r} \nabla_x p(r) + \frac{1}{r} \Delta_x \mathbf{V} + \frac{1}{r} \partial_{zz} \mathbf{V}. \]

So we rewrite the preceding two items on the right side of (4.1) as

\[ \int_\Omega \rho_\epsilon [\partial_t \mathbf{V} + \mathbf{V} \nabla_x \mathbf{V} + W \partial_z \mathbf{V} + (\mathbf{v}_\epsilon - \mathbf{V}) \nabla_x \mathbf{V} + (w_\epsilon - W) \partial_z \mathbf{V}] (\mathbf{V} - \mathbf{v}_\epsilon) \, dx \, dz \]

\[ = \int_\Omega \rho_\epsilon [\mathbf{V} - \mathbf{v}_\epsilon] (\Delta_x \mathbf{V} + \partial_{zz} \mathbf{V} - \nabla_x p(r)) \, dx \, dz \]

\[ + \int_\Omega \rho_\epsilon (w_\epsilon - W) (\mathbf{V} - \mathbf{v}_\epsilon) \cdot \partial_z \mathbf{V} \, dx \, dz - \int_\Omega \rho_\epsilon (\mathbf{V} - \mathbf{v}_\epsilon)^2 \nabla_x \mathbf{V}, \]

and

\[ \epsilon^2 \int_0^T \int_\Omega \rho_\epsilon (\partial_t W + \mathbf{v}_\epsilon \nabla_x W + w_\epsilon \partial_z W)(W - w_\epsilon) \, dx \, dz \, dt \]

\[ \leq \int_0^T E(\rho_\epsilon, \mathbf{u}_\epsilon | r, \mathbf{U}) + \epsilon^4 \int_0^T \rho_\epsilon (\partial_t W + \mathbf{v}_\epsilon \nabla_x W + w_\epsilon \partial_z W)^2 \, dx \, dz \, dt \]
\begin{align*}
\mathcal{E}(\rho_\epsilon, u_\epsilon| r, U) + \epsilon^4 \int_0^T \int_\Omega \rho_\epsilon (\partial_t W + V \nabla_x W + W \partial_x W)^2 \, dx \, dz \, dt \\
+ \epsilon^4 \int_0^T \int_\Omega \rho_\epsilon ((v_\epsilon - V) \nabla_x W + (w_\epsilon - W) \partial_x W)^2 \, dx \, dz \, dt.
\end{align*}
\tag{4.3}

Noticing Lemma 3.2, we have
\begin{align*}
\int_0^T \int_\Omega \rho_\epsilon (\partial_t W + V \nabla_x W + W \partial_x W)^2 \, dx \, dz \, dt \\
&= \int_0^T \int_\Omega \chi_{\rho_\epsilon < \xi} \rho_\epsilon (\partial_t W + V \nabla_x W + W \partial_x W)^2 \, dx \, dz \\
&+ \int_0^T \int_\Omega \chi_{\xi \leq \rho_\epsilon \leq \tau} \rho_\epsilon (\partial_t W + V \nabla_x W + W \partial_x W)^2 \, dx \, dz \\
&+ \int_0^T \int_\Omega \chi_{\rho_\epsilon > \tau} \rho_\epsilon (\partial_t W + V \nabla_x W + W \partial_x W)^2 \, dx \, dz \, dt \\
&\leq \int_0^T \int_\Omega \chi_{\rho_\epsilon < \xi} \rho_\epsilon (\partial_t W + V \nabla_x W + W \partial_x W)^2 \, dx \, dz \\
&+ \int_0^T \int_\Omega \chi_{\rho_\epsilon > \tau} \rho_\epsilon (\partial_t W + V \nabla_x W + W \partial_x W)^2 \, dx \, dz \, dt \\
&+ C \int_0^T \int_\Omega \chi_{\xi \leq \rho_\epsilon \leq \tau} \rho_\epsilon (\partial_t W + V \nabla_x W + W \partial_x W)^2 \, dx \, dz \, dt + C \\
&\leq C \int_0^T \mathcal{E}(\rho_\epsilon, u_\epsilon| r, U) + C \\
&\leq C \int_0^T \mathcal{E}(\rho_\epsilon, u_\epsilon| r, U) + C. 
\end{align*}
\tag{4.4}

It is easy to use the definition of relative entropy inequality and Cauchy inequality to obtain
\begin{align*}
\epsilon^4 \int_0^T \int_\Omega \rho_\epsilon ((v_\epsilon - V) \nabla_x W + (w_\epsilon - W) \partial_x W)^2 \, dx \, dz \, dt \leq \epsilon^2 \mathcal{E}(\rho_\epsilon, u_\epsilon| r, U) + o(\epsilon^2). \tag{4.5}
\end{align*}

Putting (4.4)–(4.5) into (4.3) yields
\begin{align*}
\epsilon^2 \int_0^T \int_\Omega \rho_\epsilon (\partial_t W + v_\epsilon \nabla_x W + w_\epsilon \partial_x W)(W - w_\epsilon) \, dx \, dz \, dt \leq C \int_0^T \mathcal{E}(\rho_\epsilon, u_\epsilon| r, U) + o(\epsilon^2).
\end{align*}

Moreover, a simple application of the Cauchy inequality leads to the following
\begin{align*}
\epsilon^2 \int_0^T \int_\Omega \nabla w_\epsilon \cdot \nabla W \, dx \, dz \, dt \leq \epsilon^2 \int_0^T \int_\Omega |\nabla w_\epsilon|^2 \, dx \, dz \, dt + o(\epsilon^2).
\end{align*}

Thus, we obtain that
\begin{align*}
\mathcal{E}(\rho_\epsilon, u_\epsilon| r, U)|_{t=0}^{t=T} + \int_0^T \int_\Omega (\nabla v_\epsilon \cdot (\nabla v_\epsilon - \nabla V) + \frac{\epsilon^2}{2} |\nabla w_\epsilon|^2) \, dx \, dz \, dt \\
&\leq C \int_0^T \mathcal{E}(\rho_\epsilon, u_\epsilon| r, U) dt - \int_0^T \int_\Omega \rho_\epsilon \nabla \cdot (\nabla v_\epsilon + \partial_x V) \, dx \, dz \\
&+ \int_0^T \int_\Omega \frac{\rho_\epsilon}{r} (V - v_\epsilon) \Delta_x V + \partial_x \beta \, dx \, dz - \int_0^T \int_\Omega \frac{\rho_\epsilon}{r} (V - v_\epsilon) \partial_x p(r) \, dx \, dz \\
&+ \int_0^T \int_\Omega \rho_\epsilon (w_\epsilon - W) \nabla v_\epsilon \cdot \partial_x V \, dx \, dz \, dt - \int_0^T \int_\Omega \rho_\epsilon \nabla \cdot \nabla V \, dx \, dz \, dt + o(\epsilon^2).
\end{align*}
4.3. Step 2

The major challenges of the analysis is to estimate the complicated nonlinear term \( \int_{\Omega} \rho_e (w_e - W) (V - v_e) \cdot \partial_z V \, dx \, dz \), we rewrite it as

\[
\int_{\Omega} \rho_e (w_e - W) (V - v_e) \cdot \partial_z V \, dx \, dz = \int_{\Omega} \rho_e w_e (V - v_e) \cdot \partial_z V \, dx \, dz - \int_{\Omega} \rho_e W (V - v_e) \cdot \partial_z V \, dx \, dz.
\]  

(4.6)

A similar heuristic argument from [23, 38] shows that the second term on the right side of (4.6) will be split into three parts

\[
\int_{\Omega} \rho_e W (V - v_e) \cdot \partial_z V \, dx \, dz
\]

\[
= \int_{\Omega} \chi_{\rho_e \leq \frac{1}{2}} \rho_e W (V - v_e) \cdot \partial_z V \, dx \, dz + \int_{\Omega} \chi_{\frac{1}{2} < \rho_e < r} \rho_e W (V - v_e) \cdot \partial_z V \, dx \, dz
\]

\[
+ \int_{\Omega} \chi_{\rho_e \geq r} \rho_e W (V - v_e) \cdot \partial_z V \, dx \, dz
\]

\[
\leq \| \chi_{\rho_e \leq \frac{1}{2}} \|_{L^2(\Omega)} \| r \|_{L^\infty} \| W \partial_z V \|_{L^3} \| V - v_e \|_{L^6(\Omega)} + \int_{\Omega} \chi_{\rho_e \geq r} \rho_e^{\frac{1}{2}} \| W \partial_z V \cdot (V - v_e) \| \, dx \, dz
\]

\[
+ C \| \chi_{\frac{1}{2} < \rho_e < r} \|_{L^2(\Omega)} \| W \partial_z V \|_{L^3} \| V - v_e \|_{L^6(\Omega)}
\]

\[
\leq C \int_{\Omega} \chi_{\rho_e \leq \frac{1}{2}} \, dx \, dz + C \int_{\Omega} \chi_{\frac{1}{2} < \rho_e < r} (\rho_e - r)^2 \, dx \, dz
\]

\[
+ C \int_{\Omega} \chi_{\rho_e \geq r} \rho_e^2 \, dx \, dz + \delta \| V - v_e \|_{L^6(\Omega)}^2
\]

\[
\leq C E (\rho_e, u_e, r, U) + \delta \| \nabla_x V - \nabla_x v_e \|_{L^2(\Omega)}^2 + \delta \| \partial_z V - \partial_z v_e \|_{L^2(\Omega)}^2,
\]  

(4.7)

where in the last inequality, we have used Lemma 2.1.

Remark 4.1. Here

\[
\int_{\Omega} \chi_{\rho_e \geq r} \rho_e W (V - v_e) \cdot \partial_z V \, dx \, dz = \int_{\rho_e \geq r} \rho_e W (V - v_e) \cdot \partial_z V \, dx \, dz.
\]

The decomposition of \( \rho_e \) comes from the essential set and residual set. For details, see [22, 23].

We now turn to analyze the first term on the right hand of (4.6), which is the crucial and difficult part in our proof. Taking (1.7) into it, we have

\[
\int_{\Omega} \rho_e w_e (V - v_e) \cdot \partial_z V \, dx \, dz
\]

\[
= \int_{\Omega} [- \text{div}_x (\rho_e \tilde{v}_e) + z \text{div}_x (\rho_e \nabla v_e)] \partial_z V \cdot (V - v_e) \, dx \, dz
\]

\[
= \int_{\Omega} (\rho_e \tilde{v}_e - z \rho_e v_e) \partial_z \nabla_x V \cdot (V - v_e) \, dx \, dz
\]

\[
+ \int_{\Omega} (\rho_e \tilde{v}_e - z \rho_e v_e) \partial_z V \cdot (\nabla_x V - \nabla_x v_e) \, dx \, dz.
\]

(4.8)

In the following, we will estimate the terms on the right hand side of (4.8). We only need to consider the most complicated terms, the remaining terms can be completed by the similar way. Firstly, we deal
with \( \int_{\Omega} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (V - v_e) dx dz \) as the follows,

\[
\int_{\Omega} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (V - v_e) \ dx \ dz \\
= \int_{\Omega} \rho (\tilde{\nu}_e - \tilde{V}) \partial_z \nabla_x V \cdot (V - v_e) \ dx \ dz + \int_{\Omega} \rho \tilde{V} \partial_z \nabla_x V \cdot (V - v_e) \ dx \ dz \\
= J_1 + J_2,
\]

where \( \tilde{V} = \int_{0}^{z} V(x, s, t) ds \).

Similar to the above analysis, we decompose the term \( J_2 \) into three parts

\[
J_2 = \int_{\Omega} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (V - v_e) \ dx \ dz \\
= \int_{\Omega} \chi_{\rho \leq \frac{1}{2}} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (V - v_e) \ dx \ dz + \int_{\Omega} \chi_{\frac{1}{2} < \rho \leq r} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (V - v_e) \ dx \ dz \\
+ \int_{\Omega} \chi_{\rho \geq r} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (V - v_e) \ dx \ dz \\
\leq \| \chi_{\rho \leq \frac{1}{2}}\|_{L^{2}(\Omega)} \| \nabla \tilde{\nu}_e \nabla_x V \|_{L^{2}(\Omega)} \| V - v_e \|_{L^{2}(\Omega)} \\
+ \| \chi_{\frac{1}{2} < \rho \leq r} \|_{L^{2}(\Omega)} \| \tilde{\nu}_e \partial_z \nabla_x V \|_{L^{2}(\Omega)} \| V - v_e \|_{L^{2}(\Omega)} \\
+ \| \chi_{\rho \geq r} \|_{L^{2}(\Omega)} \| \tilde{\nu}_e \partial_z \nabla_x V \|_{L^{2}(\Omega)} \| V - v_e \|_{L^{2}(\Omega)} \\
\leq C \mathcal{E}(\rho, u_e, r, U) + \delta \| \nabla_x V - \nabla_x v_e \|_{L^{2}(\Omega)} + \delta \| \partial_x V - \partial_x v_e \|_{L^{2}(\Omega)}^2.
\]

On the other hand, by virtue of Cauchy inequality, it follows that

\[
J_1 = \int_{\Omega} (\rho (\tilde{\nu}_e - \tilde{V}) \partial_z \nabla_x V) \cdot (V - v_e) \ dx \ dz \\
\leq \| \partial_z \nabla_x V \|_{L^{2}(\Omega)} \int_{\Omega} \rho (\tilde{\nu}_e - \tilde{V}) \ dx dz + \int_{\Omega} \rho |V - v_e| \ dx dz \\
\leq C \int_{\Omega} \rho |v_e(s) - V(s)| ds \ dx dz + \mathcal{E}(\rho, u_e, r, U) \\
\leq C \int_{\Omega} \rho \int_{0}^{1} |V - v_e|^2 ds \ dx dz + \mathcal{E}(\rho, u_e, r, U) \\
\leq C \int_{0}^{1} \int_{\Omega} \rho |V - v_e|^2 dx dz + \mathcal{E}(\rho, u_e, r, U) \\
\leq C \mathcal{E}(\rho, u_e, r, U).
\] (4.9)

Secondly, we will investigate another complicated nonlinear term \( \int_{\Omega} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (\nabla_x V - \nabla_x v_e) dx dz \). It is straightforward to show that

\[
\int_{\Omega} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (\nabla_x V - \nabla_x v_e) \ dx \ dz \\
= \int_{\Omega} \chi_{\rho \leq \frac{1}{2}} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (\nabla_x V - \nabla_x v_e) \ dx \ dz + \int_{\Omega} \chi_{\frac{1}{2} < \rho \leq r} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (\nabla_x V - \nabla_x v_e) \ dx \ dz,
\] (4.10)

where the first term on the right side of (4.10) is split into two parts as

\[
\int_{\Omega} \chi_{\rho \leq \frac{1}{2}} \rho \tilde{\nu}_e \partial_z \nabla_x V \cdot (\nabla_x V - \nabla_x v_e) \ dx \ dz \\
= \int_{\Omega} \chi_{\rho \leq \frac{1}{2}} (\tilde{\nu}_e - \tilde{V}) \partial_z \nabla_x V \cdot (\nabla_x V - \nabla_x v_e) \ dx \ dz
\]
Recalling (1.15) and (4.9), we have

\[ K \leq E \leq C \| \rho \| \| \partial x V - \partial x V \| L^2(\Omega) \]

The decomposition of remainder of (4.10) is identical to the above as:

\[ \int_{\Omega} \chi_{r > r} \rho \chi (\bar{v} - \bar{V}) \partial x V \cdot (\partial x V - \partial x V) \ dx \ dz = K_1 + K_2, \quad (4.11) \]

where

\[ K_2 \leq \int_{\Omega} \chi_{r > r} \rho \chi (\bar{V} - \bar{V}) \partial x V \cdot (\partial x V - \partial x V) \ dx \ dz \]

It remains to estimate \( K_1 \). Due to Hölder inequality, it follows that

\[ K_1 \leq \int_{\Omega} \chi_{r > r} \rho \chi (\bar{v} - \bar{V}) \partial x V \cdot (\partial x V - \partial x V) \ dx \ dz \]

Recalling (1.15) and (4.9), we have

\[ \| \chi_{r > r} \rho \| L^6(\Omega) \leq C \left( \int_{\Omega} \rho^2 \ dx \ dz \right)^{\frac{1}{2}} \leq E(\rho, u, r, U) \]

and

\[ \| \chi_{r > r} (\bar{v} - \bar{V}) \| L^2(\Omega) = \int_{\Omega} |\bar{v} - \bar{V}|^2 \ dx \ dz = \int_{\Omega} \frac{1}{\rho} \rho \ |\bar{v} - \bar{V}|^2 \ dx \ dz \]
where

\[ h \]

We are now in a position to estimate the remaining terms in the versatile relative energy inequality (4.13).

4.4. Step 3

It is clear to check that

\[ -E \]

The estimate of remainder in (4.8) can be completed by the analogous method. Therefore, we can

Combining the above estimates, we arrive at the conclusion that

\[
\int_0^T K_1 dt \leq C \int_0^T h(t) E(\rho_\varepsilon, u_\varepsilon| r, U)(t) dt + \delta \int_0^T \left\| \nabla_x V_\varepsilon - \nabla_x V \right\|^2_{L^2(\Omega)} dt
\]

where \( h(t) \in L^1(0, T) \).

The estimate of remainder in (4.8) can be completed by the analogous method. Therefore, we can

Then we deduce that

\[
E(\rho_\varepsilon, u_\varepsilon| r, U)_{t=0}^{t=T} + \int_0^T \left( (\nabla_x V_\varepsilon - \nabla_x V) : (\nabla_x V_\varepsilon - \nabla_x V) + |\partial_x V_\varepsilon - \partial_x V|^2 + \varepsilon^2 |\nabla w_\varepsilon|^2 \right) dx dz dt
\]

\[ \leq C \int_0^T \left( (\nabla_x V_\varepsilon - \nabla_x V) : (\nabla_x V_\varepsilon - \nabla_x V) + |\partial_x V_\varepsilon - \partial_x V|^2 + \varepsilon^2 |\nabla w_\varepsilon|^2 \right) dx dz dt
\]

\[ = - \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} (V - v_\varepsilon) \nabla_x p(r) \right) dx dz dt + \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} (V - v_\varepsilon) \nabla_x p(r) \right) dx dz dt
\]

4.4. Step 3

We are now in a position to estimate the remaining terms in the versatile relative energy inequality (4.13).

It is clear to check that

\[
- \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} (V - v_\varepsilon) \nabla_x p(r) \right) dx dz dt + \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} (V - v_\varepsilon) \nabla_x p(r) \right) dx dz dt
\]

\[ = - \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} P''(r) |\partial_r V + P''(r)\partial_r V + \rho_\varepsilon |\nabla_x V_\varepsilon| \cdot |\nabla_x V_\varepsilon| + p(\rho_\varepsilon) \div V dx dz dt
\]

\[ = - \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} P''(r) |\partial_r V + V \cdot \nabla_x r - r P''(r) |\partial_r V + p(\rho_\varepsilon) \div V dx dz dt
\]

\[ = - \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} P''(r) |\partial_r V + V \cdot \nabla_x r - r P''(r) |\partial_r V + p(\rho_\varepsilon) \div V dx dz dt
\]

\[ = - \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} P''(r) |\partial_r V + V \cdot \nabla_x r - r P''(r) |\partial_r V + p(\rho_\varepsilon) \div V dx dz dt
\]

\[ = - \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} P''(r) |\partial_r V + V \cdot \nabla_x r - r P''(r) |\partial_r V + p(\rho_\varepsilon) \div V dx dz dt
\]

\[ = - \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} P''(r) |\partial_r V + V \cdot \nabla_x r - r P''(r) |\partial_r V + p(\rho_\varepsilon) \div V dx dz dt
\]

\[ = - \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} P''(r) |\partial_r V + V \cdot \nabla_x r - r P''(r) |\partial_r V + p(\rho_\varepsilon) \div V dx dz dt
\]

\[ = - \int_0^T \int_\Omega \left( \frac{\rho_\varepsilon}{r} P''(r) |\partial_r V + V \cdot \nabla_x r - r P''(r) |\partial_r V + p(\rho_\varepsilon) \div V dx dz dt
\]
where we have used the fact that $\partial_t r + \operatorname{div} x V r + V \cdot \nabla x r + r \partial_r W = 0$.

Using the analogous argument as in [47] Section 2.2.5, we can easily carry out the following estimate:

$$\left| \int_0^T \int_\Omega \operatorname{div} x V (p(\rho_\epsilon) - p'(r)(\rho_\epsilon - r) - p(r)) \, dx \, dz \, dt \right| \leq C \int_0^T h(t) \mathcal{E}(\rho_\epsilon, u_\epsilon | r, U) dt. \quad (4.15)$$

According to the periodic boundary condition, it follows that

$$\int_0^T \int_\Omega p'(r)(\rho_\epsilon - r) \partial_z W \, dx \, dz \, dt = \int_0^T \int_\Omega^2 \left( \int_0^1 \partial_z W dz \right) p'(r)(\rho_\epsilon - r) dx = 0. \quad (4.16)$$

Furthermore, an argument similar to the one used in [38] Section 6.3 shows that

$$\int_\Omega \left( \frac{\partial e}{r} - 1 \right) (V - v_\epsilon) \left( \Delta x V + \partial_{zz} V \right) \, dx \, dz \leq C \mathcal{E}(\rho_\epsilon, u_\epsilon | r, U) + \delta \| \nabla x v_\epsilon - \nabla x V \|^2_{L^2} + \delta \| \partial_z v_\epsilon - \partial_z V \|^2_{L^2}. \quad (4.17)$$

Therefore, putting (4.13)–(4.16) together, we have

$$\mathcal{E}(\rho_\epsilon, u_\epsilon | r, U)(\tau) \leq C \int_0^T h(t) \mathcal{E}(\rho_\epsilon, u_\epsilon | r, U)(t) dt + o(\epsilon^2). \quad (4.18)$$

Then applying the Gronwall’s inequality, we finish the proof of Theorem 3.1.

Acknowledgements. The authors are very much indebted to the anonymous referees for many helpful suggestions. Moreover, we would like to thank to Prof. A. Novotný for his remarks and suggestions during his visit in Prague in the spring 2021. We thank to Prof. Edriss Titi for his wonderful and great talks on the course “compact course Mathematical Analysis of Geophysical Models and Data Assimilation” held from 26 June to 10 July 2020, which offers insightful and constructive suggestions. The research of H. G is partially supported by the NSFC Grant No. 12171084 and the fundamental Research Funds for the Central Universities No. 2242022R10013. The research of Š.N. was supported by the Czech Sciences Foundation (GAČR), GA19-04243S, 22-01591S (final version), Premium Academia of Š. Nečasová and RVO 67985840. The research of T.T. was supported by the NSFC Grant No. 11801138.

Declarations

Conflicts of interest The author(s) declares that they have no competing interests.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Abbatiello, A., Feireisl, E., Novotný, A.: Generalized solutions to models of compressible viscous fluids. Discrete Contin. Dyn. Syst. 41, 1–28 (2021)
[2] Azérad, P., Guillén, F.: Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics. SIAM J. Math. Anal. 33, 847–859 (2001)
[3] Bella, P., Feireisl, E., Novotný, A.: Dimension reduction for compressible viscous fluids. Acta Appl. Math. 134, 111–121 (2014)
[4] Besson, O., Laydi, M.R.: Some estimates for the anisotropic Navier-Stokes equations and for the hydrostatic approximation. ESAIM:M2AN 7, 855–865 (1992)
[5] Brenier, Y.: Homogeneous hydrostatic flows with convex velocity profiles. Nonlinearity 12, 495–512 (1999)
[6] Brenier, Y.: Remarks on the derivation of the hydrostatic Euler equations. Bull. Sci. Math. 127, 585–595 (2003)
[7] Bressan, D., Guillén-González, F., Masmoudi, N., Rodríguez-Bellido, M.A.: On the uniqueness of weak solutions of the two-dimensional primitive equations. Differential Integral Equations 16, 77–94 (2003)
[8] Bressan, D., Kazhikhov, A., Lemoine, J.: On the two-dimensional hydrostatic Navier-Stokes equations, SIAM. J. Math. Anal. 36, 796–814 (2004)
[9] Bressan, D., Lemoine, J., Simon, J.: A vertical diffusion model for lakes. SIAM J. Math. Anal. 30, 603–622 (1999)
[10] Bresch, D., Jabin, P.E.: Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. Ann. Math. 188, 577–684 (2018)
[11] Bresch, D., Burtea, C.: Global existence of weak solutions for the anisotropic compressible Stokes system, accepted by Ann. I. H. Poincaré
[12] Bryan, K.: A numerical method for the study of the circulation of the world ocean. J. Comp. Phys. 4, 347–376 (1969)
[13] Cao, C.S., Titi, E.S.: Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. Ann. Math. 166, 245–267 (2007)
[14] Cao, C.S., Li, J.K., Titi, E.S.: Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity. Arch. Ration. Mech. Anal. 214, 35–76 (2014)
[15] Cao, C.S., Li, J.K., Titi, E.S.: Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion. Comm. Pure Appl. Math. 69, 1492–1531 (2016)
[16] Cao, C.S., Li, J.K., Titi, E.S.: Strong solutions to the 3D primitive equations with only horizontal dissipation: near $H^1$ initial data. J. Funct. Anal. 272, 4606–4641 (2017)
[17] Dafermos, C.M.: The second law of thermodynamics and stability. Arch. Rational Mech. Anal. 70, 167–179 (1979)
[18] Donatelli, D., Juhasz, N.: The primitive equations of the polluted atmosphere as a weak and strong limit of the 3D Navier–Stokes equations in downwind-matching coordinates, arXiv:2001.05387
[19] Ducomet, B., Nečasová, Š, Pokorný, M., Rodríguez-Bellido, M.A.: Derivation of the Navier-Stokes-Poisson system with radiation for an accretion disk. J. Math. Fluid Mech. 20, 697–719 (2018)
[20] Ersoy, M., Ngom, T., Sy, M.: Compressible primitive equations: formal derivation and stability of weak solutions. Nonlinearity 24, 79–96 (2011)
[21] Ersoy, M., Ngom, T.: Existence of a global weak solution to one model of compressible primitive equations. C. R. Math. Acad. Sci. Paris 350, 379–382 (2012)
[22] Feireisl, E.: Dynamics of viscous compressible fluids, Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford (2004)
[23] Feireisl, E., Jin, J.B., Novotný, A.: Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system. J. Math. Fluid Mech. 14, 717–730 (2012)
[24] Feireisl, E., Novotný, A.: Singular limits in thermodynamics of viscous fluids. Advances in Mathematical Fluid Mechanics. Birkhäuser, Basel (2009)
[25] Feireisl, E., Gallagher, I., Novotný, A.: A singular limit for compressible rotating fluids. SIAM J. Math. Anal. 44(1), 192–205 (2012)
[26] Feireisl, E., Jin, J.B., Novotný, A.: Inviscid incompressible limits of strongly stratified fluids. Asymptot. Anal. 89, 307–329 (2014)
[27] Furukawa, K., Giga, Y., Hieber, M., Hussein, A., Kashiwabara, T.: Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier–Stokes equations, arXiv:1808.02410
[28] Furukawa, K., Giga, Y., Kashiwabara, T.: The hydrostatic approximation for the primitive equations by the scaled Navier–Stokes Equations under the no-slip boundary condition, arXiv:2006.02300
[29] Gao, H.J., Nečasová, Š, Tang, T.: On the hydrostatic approximation of compressible anisotropic Navier-Stokes equations. C. R. Math. 6, 639–644 (2021)
[30] Gao, H.J., Nečasová, Š, Tang, T.: On weak-strong uniqueness and singular limit for the compressible Primitive Equations. Discrete Contin. Dyn. Syst. Ser. A 40, 4287–4305 (2020)
[31] Gatapov, B.V., Kazhikhov, A.V.: Existence of a global solution of a model problem of atmospheric dynamics. Siberian Math. J. 46, 805–812 (2005)
[32] Germain, P.: Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. J. Math. Fluid Mech. 13, 137–146 (2011)
[33] Guillén-González, F., Masmoudi, N., Rodríguez-Bellido, M.A.: Anisotropic estimates and strong solutions of the primitive equations. Differ. n.a Equ. 14, 1381–1408 (2001)
[34] Guo, B.L., Huang, D.W.: Existence of the universal attractor for the 3-D viscous primitive equations of large-scale moist atmosphere. J. Differ. Equ. 251, 457–491 (2011)
[35] Guo, B.L., Huang, D.W., Wang, W.: Diffusion limit of 3D primitive equations of the large-scale ocean under fast oscillating random force. J. Differ. Equ. 259, 2388–2407 (2015)
[36] Hieber, M., Kashiwabara, T.: Global strong well-posedness of the three dimensional primitive equations in $L^p$-spaces. Arch. Ration. Mech. Anal. 221, 1077–1115 (2016)
[37] Hu, N.: The global attractor for the solutions to the 3d viscous primitive equations. Discrete Contin. Dyn. Syst. 17, 159–179 (2007)
[38] Kremli, O., Nečasová, Š, Piasecki, T.: Local existence of strong solution and weak-strong uniqueness for the compressible Navier-Stokes system on moving domains. Proc. Roy. Soc. Edinburgh Sect. A 150, 2255–2300 (2020)
[39] Li, J.K., Titi, E.S.: The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation. J. Math. Pures Appl. 124, 30–58 (2019)
[40] Lions, J.L., Temam, R., Wang, S.H.: On the equations of the large-scale ocean. Nonlinearity 5, 1007–1053 (1992)
[41] Lions, J.L., Temam, R., Wang, S.H.: New formulations of the primitive equations of atmosphere and applications. Nonlinearity 5, 237–288 (1992)
[42] Lions, J.L., Temam, R., Wang, S.H.: Mathematical theory for the coupled atmosphere-ocean models, (CAO III). J. Math. Pures Appl. 74, 105–163 (1995)
[43] Lions, J.L., Temam, R., Wang, S.H.: On mathematical problems for the primitive equations of the ocean: the mesoscale midlatitude case. Nonlinear Anal. 40, 439–482 (2000)
[44] Liu, X., Titi, E.S.: Local well-posedness of strong solutions to the three-dimensional compressible Primitive Equations, arXiv:1806.09868v1
[45] Liu, X., Titi, E.S.: Global existence of weak solutions to the compressible Primitive Equations of atmospheric dynamics with degenerate viscosities. SIAM J. Math. Anal. 51, 1913–1964 (2019)
[46] Liu, X., Titi, E.S.: Zero Mach number limit of the compressible Primitive Equations Part I: well-prepared initial data. Arch. Ration. Mech. Anal. 238, 705–747 (2020)
[47] Maltese, D., Novotný, A.: Compressible Navier-Stokes equations on thin domains. J. Math. Fluid Mech. 16, 571–594 (2014)
[48] Masmoudi, N., Wong, T.K.: On the $H^s$ theory of hydrostatic Euler equations. Arch. Ration. Mech. Anal. 204, 231–271 (2012)
[49] Nirenberg, L.: On elliptic partial differential equations. Ann. Scuola Norm. Sup. Pisa 13, 115–162 (1959)
[50] Paicu, M., Zhang, P., Zhang, Z.F.: On the hydrostatic approximation of the Navier-Stokes equations in a thin strip. Adv. Math. 372, 107293, 42 (2020)
[51] Pedlosky, W.M.: Geophysical Fluid Dynamics. Springer-Verlag, Berlin (1979)
[52] Tang, T., Gao, H.J.: On the stability of weak solution for compressible primitive equations. Acta Appl. Math. 140, 133–145 (2015)
[53] Temam, R., Ziane, M.: Some mathematical problems in geophysical fluid dynamics, Handbook of Mathematical Fluid Dynamics, (2004)
[54] Wang, F.C., Dou, C.S., Jiu, Q.S.: Global weak solutions to 3D compressible primitive equations with density-dependent viscosity. J. Math. Phys. 61(2), 021507, 33 (2020)
[55] Washington, W.M., Parkinson, C.L.: An Introduction to Three-Dimensional Climate Modelling. Oxford University Press, Oxford (1986)

Hongjun Gao
School of Mathematical Sciences
Southeast University
Nanjing 211189
People’s Republic of China
e-mail: gaohj@hotmail.com

Tong Tang
School of Mathematical Science
Yangzhou University
Yangzhou 225002
People’s Republic of China
e-mail: tt0507010156@126.com

Šárka Nečasová
Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25
11567 Praha 1
Czech Republic
e-mail: matus@math.cas.cz

(accepted: June 20, 2022; published online: July 20, 2022)