Multivariate Medial Correlation

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Abstract

We define a multivariate medial correlation coefficient that extends the probabilistic interpretation and properties of Blomqvist’s $\beta$ coefficient, incorporates multivariate marginal dependencies and is a measure of multivariate concordance. We determine the maximum and minimum values attainable and illustrate the results in some models.

keywords: Blomqvist $\beta$, multivariate medial correlation, multivariate concordance measure

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1 Introduction

Let us consider that $X = (X_1, X_2)$ is a real random vector, over the probability space $(\Omega, \mathcal{A}, P)$, with continuous marginal distribution functions $F_{X_i}$, $i = 1, 2$, and let $(U_1, U_2)$ represent the corresponding uniformized vector, that is, $U_i = F_{X_i}(X_i)$, $i = 1, 2$. 
The medial correlation coefficient of \((X_1, X_2)\), which we will represent by \(\beta(X_1, X_2)\) or \(\beta(\mathbf{X})\), is defined by

\[
\beta(X_1, X_2) = P\left(\left(U_1 - \frac{1}{2}\right)\left(U_1 - \frac{1}{2}\right) > 0\right) - P\left(\left(U_1 - \frac{1}{2}\right)\left(U_1 - \frac{1}{2}\right) < 0\right). \tag{1}
\]

The \(\beta\) coefficient introduced by Blomqvist ([1]), has its value in \([-1, 1]\) and compares the propensity for the margins of \((X_1, X_2)\) to take both values above or both values below their respective medians, with the propensity for the occurrence of the contrary event.

Since

\[
\beta(X_1, X_2) = 2\left(P\left(U_1 > \frac{1}{2}, U_2 > \frac{1}{2}\right) + P\left(U_1 < \frac{1}{2}, U_2 < \frac{1}{2}\right)\right) - 1, \tag{2}
\]

and

\[
\beta(X_1, X_2) = 4P\left(U_1 < \frac{1}{2}, U_2 < \frac{1}{2}\right) - 1, \tag{3}
\]

if \(C_X(u_1, u_2)\) and \(\hat{C}_X(u_1, u_2)\), \((u_1, u_2) \in [0, 1]^2\), represent the copula and the survival copula of \(\mathbf{X}\) (Nelsen [7]), respectively, we can say that

\[
\beta(X_1, X_2) = 2\left(C_X\left(\frac{1}{2}, \frac{1}{2}\right) + \hat{C}_X\left(\frac{1}{2}, \frac{1}{2}\right)\right) - 1, \tag{4}
\]

and

\[
\beta(X_1, X_2) = 4C_X\left(\frac{1}{2}, \frac{1}{2}\right) - 1. \tag{5}
\]

The bivariate medial correlation coefficient \(\beta(X_1, X_2)\) enables to compare \(C_X(u_1, u_2)\) on \(Q_L \cup Q_U = [0, \frac{1}{2}]^2 \cup [\frac{1}{2}, 1]^2\) with \(C_X(u_1, u_2)\) on \([0, 1]^2 - Q_L \cup Q_U\) or to compare \(C_X(u_1, u_2)\) on \(Q_L = [0, \frac{1}{2}]^2\) with \(C_X(u_1, u_2)\) on \([0, 1]^2 - Q_L\).

The medial correlation coefficient can be related to other summary measures of dependence in \((X_1, X_2)\), or in \(C_X\), such as Spearman’s \(\rho\) or Kendall’s \(\tau\) (Nelsen [7], Joe [2], Lebed [5] and references therein).

Two bivariate vectors \(\mathbf{X}\) and \(\mathbf{Y}\), or their copulas, can be partially ordered by punctually comparing their copulas. We say that \(\mathbf{X}\) is less concordant than \(\mathbf{Y}\), and we write for that \(\mathbf{X} \prec_c \mathbf{Y}\), if \(C_X(u_1, u_2) \leq C_Y(u_1, u_2)\), \((u_1, u_2) \in [0, 1]^2\), or equivalent,
if $\hat{C}_X(u_1, u_2) \leq \hat{C}_Y(u_1, u_2)$, $(u_1, u_2) \in [0, 1]^2$ (Nelsen [7]).

Thus, from the representations (4) or (5), we verify that

$$\beta(X) \leq \beta(Y).$$

In addition to the increasing with concordance ordering, the bivariate medial correlation coefficient $\beta$ satisfies other properties that shape the definition of measure of concordance according to Scarsini (8).

Considering the product and minimum copulas, respectively, $C_\Pi(u_1, u_2) = u_1 u_2$ and $C_W(u_1, u_2) = u_1 \wedge u_2$, $(u_1, u_2) \in [0, 1]^2$, we have $C_\Pi \prec_c C_X \prec_c C_W$, $\beta(C_\Pi) = 0$, $\beta(C_W) = 1$ and we can also represent $\beta(X_1, X_2)$ by

$$\beta(X_1, X_2) = 2 \left( C_X \left( \frac{1}{2}, \frac{1}{2} \right) - C_\Pi \left( \frac{1}{2}, 1 \right) + \hat{C}_X \left( \frac{1}{2}, \frac{1}{2} \right) - \hat{C}_\Pi \left( \frac{1}{2}, \frac{1}{2} \right) \right).$$

(7)

For a random vector $X = (X_1, ..., X_d)$ with dimension $d > 2$, if we think about generalizing (1) to

$$P \left( \prod_{i=1}^d \left( U_i - \frac{1}{2} \right) > 0 \right) - P \left( \prod_{i=1}^d \left( U_i - \frac{1}{2} \right) < 0 \right)$$

we definitely lose:

(i) interpretation as a measure of propensity for all margins to exceed their respective medians or all margins to be below their medians, and

(ii) information about the behaviour of $C_X$ on $Q_k = \prod_{j=1}^d I_{j_k}$, $k = 1, ..., d - 1$, where $I_{j_k} = [0, \frac{1}{2}]$ for $k$ or $d - k$ values of $j$ and $I_{j_k} = [\frac{1}{2}, 1]$ for the others.

On the other hand, any generalization of $\beta$ in the multivariate context must preserve at least the property (i) and also verify

(iii) $\beta(C_\Pi) = 0$ and $\beta(C_W) = 1$.

The proposals of Nelsen (6), Úbeda-Flores (12) and Schmid and Schmidt (9) manage to keep (i) and (iii) above.

Starting from the multivariate version of (5), $4C_X(\frac{1}{2}, ..., \frac{1}{2}) - 1$, rescaled by considering the quotient between its distance to the corresponding value for $C_\Pi$ and
the maximum value of that distance,

\[
\beta'(X_1, \ldots, X_d) = \frac{4C_X \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) - 1 - \left(4 \left(\frac{1}{2}\right)^d - 1\right)}{4W \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) - 1 - \left(4 \left(\frac{1}{2}\right)^d - 1\right)}
\]

we find Nelsen’s generalization (6).

Úbeda-Flores (12) proposes the extension of (4) in

\[
2 \left( C_X \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) + \hat{C}_X \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \right) - 1,
\]

also rescaled by considering the quotient between its distance to the corresponding value for \(C_\Pi\) and the maximum value of that distance. In this way, we obtain the following generalization of \(\beta\), which we will denote by \(\beta^*\) and where \(\frac{1}{2}\) represents the vector of suitable size and coordinates all equal to \(\frac{1}{2}\):

\[
\beta^*(X_1, \ldots, X_d) = \frac{2 \left( C_X \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) + \hat{C}_X \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \right) - 1 - \left(\frac{1}{2^{d-2}} - 1\right)}{2 \left( W \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) + \hat{W} \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \right) - 1 - \left(\frac{1}{2^{d-2}} - 1\right)}
\]

which coincides with (8) when \(C = \hat{C}\).

Reasoning in an equivalent way about (7), Schmid and Schmidt (9) propose

\[
2 \left( C_X \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) - C_\Pi \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) + \hat{C}_X \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) - \hat{C}_\Pi \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \right) - 1
\]

finding again the expression of Úbeda-Flores (12). In addition to this extension, Schmid and Schmidt (9) make a detailed study of a function resulting from a rescaling of \(C_X(u) + \hat{C}_X(v)\), \(u, v \in [0, 1]^d\), putting emphasis on the tail regions of the copula which determine the degree of large co-movements between the marginal random variables.

In order to keep (i), (ii) and (iii), we have Joe’s sophisticated proposal (3) with
an axiomatic on linear combinations of $C_{\sigma_{i_1}\sigma_{i_2}...\sigma_{i_k}}x\left(\frac{1}{2}\right)$ and $\hat{C}_{\sigma_{i_1}\sigma_{i_2}...\sigma_{i_k}}x\left(\frac{1}{2}\right)$, $1 \leq i_1 < ... < i_K \leq d$, $k = \left[\frac{d+1}{2}\right], ..., d$, where $\sigma_j x$ denotes the $j$-th reflection of $x$, that is, the vector $(X_1, ... X_{j-1}, -X_j, X_{j+1}, ..., X_d)$. Joe’s axiomatic definition allows for various extensions of $\beta$, including those mentioned above and the arithmetic mean of $\beta(X_i, X_j)$, $1 \leq i < j \leq d$.

The extensions referred for $\beta$ increase with the multivariate concordance (Joe [4]). We say that $X = (X_1, ..., X_d)$ is less concordant than $Y = (Y_1, ..., Y_d)$, or $C_X$ is less concordant than $C_Y$, and in this case we write $X \prec_c Y$, when we have

$$C_X(u) \leq C_Y(u) \text{ and } \hat{C}_X(u) \leq \hat{C}_Y(u),$$

for $u \in [0,1]^d$. In the case of $d = 2$ the two conditions are equivalent, as we have already mentioned.

The above proposed generalizations start from extensions of the representations of bivariate $\beta$ in terms of copulas, considering the corresponding multivariate copulas.

The proposal that we will make, in the next section, for a multivariate correlation coefficient $\beta(X)$ starts from a generalization of the probabilistic interpretation of the definition [1] and satisfies the desirable properties for a multivariate concordance measure (Taylor [10], [11]). We present several representations for $\beta(X)$, we demonstrate the main properties, relate it to the previously mentioned coefficients and illustrate with examples.

2 Motivation for the multivariate medial correlation coefficient

For $d \geq 2$, $D = \{1, ..., d\}$, $I \subset D$, $X = (X_1, ..., X_d)$ with continuous marginal distributions and $U = (U_1, ..., X_d) = (F_{X_1}(X_1), ..., F_{X_d}(X_d))$, we define

$$M(I) = \bigvee_{i \in I} U_i \text{ and } W(I) = \bigwedge_{i \in I} U_i,$$

(12)
where $\lor$ and $\land$ are the notations for the maximum and minimum operators, respectively.

When further clarification is needed, we write $M_X(I)$ and $W_X(I)$. Inequalities between vectors are understood by corresponding inequalities between homologous coordinates. By $X_I$ we understand the subvector of $X$ with margins in $I$ and $\mathcal{P}(D)$ represents the family of subsets of $D$.

Let’s fix disjoint $I$ and $J$ in $\mathcal{P}(D)$. The propensity for margins of $X_I$ and margins of $X_J$ simultaneously taking values below the respective medians or simultaneously values above the respective medians is evaluated by $C_{X_I\cup J}(\frac{1}{2}) + \hat{C}_{X_I\cup J}(\frac{1}{2})$, that is, the probability of $U_{I\cup J}$ taking values in $[0, \frac{1}{2}]_{I\cup J} \cup [\frac{1}{2}, 1]_{I\cup J}$. If we want to compare this probability with the probability of $U_{I\cup J}$ taking values in $[0, 1]_{I\cup J} - \left([0, \frac{1}{2}]_{I\cup J} \cup [\frac{1}{2}, 1]_{I\cup J}\right)$, we can do it briefly by calculating the coefficients

\[
\beta(M(I), M(J)) := P((M(I) - \frac{1}{2})(M(J) - \frac{1}{2}) > 0) - P((M(I) - \frac{1}{2})(M(J) - \frac{1}{2}) < 0) = 2\left(P(M(I) > \frac{1}{2}, M(J) > \frac{1}{2}) + P(M(I) < \frac{1}{2}, M(J) < \frac{1}{2})\right) - 1
\]

and

\[
\beta(W(I), W(J)) := P((W(I) - \frac{1}{2})(W(J) - \frac{1}{2}) > 0) - P((W(I) - \frac{1}{2})(W(J) - \frac{1}{2}) < 0) = 2\left(P(W(I) > \frac{1}{2}, W(J) > \frac{1}{2}) + P(W(I) < \frac{1}{2}, W(J) < \frac{1}{2})\right) - 1.
\]

Let us make some comments about

\[
\beta_{I,J}(X) := \frac{\beta(M(I), M(J)) + \beta(W(I), W(J))}{2}.
\]

(i) The expressions (13), (14) and (15) have $\beta(X_i, X_j)$ as a particular case, if we take $I = \{i\}$ and $J = \{j\}$. If $I = D$, $J = \emptyset$ and we consider that $M(\emptyset) = -\infty$ and $W(\emptyset) = +\infty$, then (15) is equal to $C_X(\frac{1}{2}) + \hat{C}_X(\frac{1}{2}) - 1$, which can be rescaled in order to obtain the proposal of Úbeda-Flores (12) and Schmid and Schmidt (9).

(ii) Since $\beta_{I,J}(X)$ is defined as an average of bivariate coefficients, it can be
estimated by the methods available for the bivariate context (Blomqvist [1], Schmid and Schmidt [9] and references therein).

(iii) From (6), the value of (15) increases with the concordances of $(M_I, M_J)$ and $(W_I, W_J)$, that is, for two vectors of dimension $d$, $X$ and $Y$, if

$$P (M_X(I) \leq u, M_X(J) \leq v) \leq P (M_Y(I) \leq u, M_Y(J) \leq v),$$
$$P (W_X(I) > u, W_X(J) > v) \leq P (W_Y(I) > u, W_Y(J) > v),$$

$(u, v) \in [0, 1]^2$, then $\beta_{I,J}(X) \leq \beta_{I,J}(Y)$.

Note that the first inequality is equivalent to the similar inequality for the survival functions of $(M,(I), M,(J))$ and the second is equivalent to the similar inequality for the distribution functions of $(W,(I), W,(J))$, since we are in the context of bivariate vectors.

(iv) Since (16) is implied by $X \prec_c Y$, we can say that $\beta_{I,J}(X)$ increases with the concordance of $X$.

(v) If $C_X = C_W$ we have $\beta_{I,J}(X) = 1$ and if $C_X = C_{\Pi}$ then $\beta_{I,J}(X) = (2^{1-|I|})(2^{1-|J|})$. This value becomes null if and only if $|I| = 1$ or $|J| = 1$.

(vi) A linear combination of $\beta_{\{i\},\{j\}}(X), 1 \leq i < j \leq d$, takes into account the bivariate dependencies in $X$, but if we consider some function of the coefficients $\beta_{I,J}(X)$, with $I, J \in F$, for some family $F \subset \mathcal{P}(D)$ containing non-singular sets, then we will be incorporating multivariate marginal dependencies.

The definition we propose, in the next section, for a multivariate medial correlation coefficient, will be based on the bivariate coefficients $\beta_{\{i\},\{d-i\}}(X), 1 \leq i \leq d$, incorporating the dependency between each margin $X_i$ and $X_{D-i}$, $1 \leq i \leq d$.

Our proposal contains, as a particular case, the Blomqvist bivariate coefficient, extends the probabilistic interpretation (1), takes values in $[-1, 1]$, becoming null naturally when $C_X = C_{\Pi}$ and taking the maximum value when $C_X = C_W$. The rest of the properties we proved allow us to consider it a multivariate concordance measure.
3 A multivariate medial correlation coefficient

Definition 3.1. The multivariate medial correlation coefficient of the vector $X$ with dimension $d$, or of its copula $C_X$, is defined as

$$\beta(X) = \frac{1}{d} d \sum_{i=1}^{d} \beta_{(i),D-i}(X),$$  \hspace{1cm} (17)

where

$$\beta_{(i),D-i}(X) = \frac{\beta(U_i, M(D - \{i\})) + \beta(U_i, W(D - \{i\}))}{2}, \quad i = 1, ..., d.$$  \hspace{1cm} (18)

In what follows, for the sake of simplicity of writing, we sometimes write $D - i$ rather than $D - \{i\}$ and $\beta_{i,D-i}(X)$ for $\beta_{(i),D-i}(X)$.

Below we present some representations of $\beta(X)$ that will be useful to clarify their properties and interpretation. The following

$$\beta_{i,D-i}(X) = 2 \left( P(U_i < \frac{1}{2}, M(D - i) < \frac{1}{2}) + P(U_i > \frac{1}{2}, W(D - i) > \frac{1}{2}) \right) - P(M(D - i) < \frac{1}{2}) - P(W(D - i) > \frac{1}{2}),$$  \hspace{1cm} (19)

holds, generalizing (2). We also have

$$\beta_{i,D-i}(X) = 2 \left( C_X \left( \frac{1}{2} \right) + \hat{C}_X \left( \frac{1}{2} \right) \right) - C_{X_{D-i}} \left( \frac{1}{2} \right) - \hat{C}_{X_{D-i}} \left( \frac{1}{2} \right),$$  \hspace{1cm} (20)

generalizing (1). From the previous relation, it follows that

$$\beta_{i,D-i}(X) = C_X \left( \frac{1}{2} \right) + \hat{C}_X \left( \frac{1}{2} \right) - C_{\sigma_iX} \left( \frac{1}{2} \right) - \hat{C}_{\sigma_iX} \left( \frac{1}{2} \right),$$  \hspace{1cm} (21)

where $\sigma_iX$ is the i-th reflection of $X$, that is, $\sigma_iX = (X_1, ..., X_{i-1}, -X_i, X_{i+1}, ..., X_d)$ and therefore $C_{\sigma_iX}(\frac{1}{2}) = C(U_{i1}, ..., U_{i-1}, -U_i, U_{i+1}, ..., U_d)(\frac{1}{2})$. We then obtain the following ways of representing the coefficient $\beta$.

Proposition 3.1. The multivariate medial correlation coefficient of the vector $X$ with dimension $d$, admits the following representations:

$$\beta(X) = 2 \left( P(U \leq \frac{1}{2}) + P(U > \frac{1}{2}) \right) - \frac{1}{d} \sum_{i=1}^{d} \left( P(U_{D-i} \leq \frac{1}{2}) + P(U_{D-i} > \frac{1}{2}) \right),$$  \hspace{1cm} (22)
\[ \beta(X) = 2 \left( C_X \left( \frac{1}{2} \right) + \hat{C}_X \left( \frac{1}{2} \right) \right) - \frac{1}{d} \sum_{i=1}^{d} \left( C_{X_{D_i}} \left( \frac{1}{2} \right) + \hat{C}_{X_{D_i}} \left( \frac{1}{2} \right) \right), \quad (23) \]

\[ \beta(X) = C_X \left( \frac{1}{2} \right) + \hat{C}_X \left( \frac{1}{2} \right) - \frac{1}{d} \sum_{i=1}^{d} \left( C_{\sigma_i X} \left( \frac{1}{2} \right) + \hat{C}_{\sigma_i X} \left( \frac{1}{2} \right) \right). \quad (24) \]

The relation (24) rewritten in the form

\[ \beta(X) = \frac{1}{d} \sum_{i=1}^{d} \left( C_X \left( \frac{1}{2} \right) - C_{\sigma_i X} \left( \frac{1}{2} \right) + \hat{C}_X \left( \frac{1}{2} \right) - \hat{C}_{\sigma_i X} \left( \frac{1}{2} \right) \right), \]

reinforces the idea that \( \beta(X) \) compares the propensity of each margin \( X_i \) to agree with the remaining margins together, \( X_{D_i} \), and the propensity to disagree with them, when they are all above or all below their respective medians.

In the following, we establish relationships between \( \beta(X) \) and the generalizations referred to in the introduction. By applying the definition (10) of \( \beta^* \), we conclude from the representation (24) that

\[ \beta(X) = \frac{(2^d - 1) \beta^*(X) + 1}{2^d - 1} - \frac{1}{d} \sum_{i=1}^{d} \left( \frac{(2^d - 1) \beta^*(\sigma_i X) + 1}{2^d - 1} \right). \]

By defining \( \bar{N} = \sum_{i=1}^{d} 1_{(U_i > \frac{1}{2})} \), the representation (24) of \( \beta \) leads to

\[ \beta(X) = P(\bar{N} = 0) + P(\bar{N} = d) - \frac{1}{d} \left( P(\bar{N} = 1) + P(\bar{N} = d - 1) \right). \quad (25) \]

That fits Joe’s representation (3.1.1) (3) with \( w_d = 1, w_{d-1} = -\frac{1}{d} \) and the remaining weights \( w_i \) equal to zero.

We refer the properties of \( \beta(X) \) in the next section and end this one with three examples.

**Example 3.1.** Consider \( C_X(u_1, ..., u_4) = (u_1^\delta \wedge u_2) u_1^{1-\delta} (u_3^\alpha \wedge u_4) u_3^{1-\alpha} \), with \( 0 \leq \delta, \alpha \leq 1 \), that is, \( C_X \) is the product of two Marshall-Olkin survival copulas (4). It
holds that

\[ C_X \left( \frac{1}{2} \right) = \hat{C}_X \left( \frac{1}{2} \right) = \left( \frac{1}{2} \right)^{4-\delta-\alpha}, \]

\[ C_{X_{D-1}} \left( \frac{1}{2} \right) = \hat{C}_{X_{D-1}} \left( \frac{1}{2} \right) = C_{X_{D-2}} \left( \frac{1}{2} \right) = \hat{C}_{X_{D-2}} \left( \frac{1}{2} \right) = \left( \frac{1}{2} \right)^{3-\alpha}, \]

\[ C_{X_{D-3}} \left( \frac{1}{2} \right) = \hat{C}_{X_{D-3}} \left( \frac{1}{2} \right) = C_{X_{D-4}} \left( \frac{1}{2} \right) = \hat{C}_{X_{D-4}} \left( \frac{1}{2} \right) = \left( \frac{1}{2} \right)^{3-\delta}. \]

Therefore,

\[ \beta(X) = 2^{\delta+\alpha-2} - 2^{\alpha-3} - 2^{\delta-3}. \]

In the case of \( \delta = \alpha = 0 \) the result agrees with what we expect, since in this case the margins of \( X \) are independent. The expression obtained can be related to \( \beta(X_1, X_2) \) e \( \beta(X_3, X_4) \) through

\[ \beta(X) = 2 \times 2^{\delta+\alpha-3} - 2^{\alpha-3} - 2^{\delta-3} = (2^{\delta+\alpha-3} - 2^{\alpha-3}) + (2^{\delta+\alpha-3} - 2^{\delta-3}) \]

\[ = 2^{\alpha-3} (2^{\delta} - 1) + 2^{\delta-3} (2^{\alpha} - 1) \]

\[ = 2^{\alpha-3} \beta(X_1, X_2) + 2^{\delta-3} \beta(X_3, X_4), \]

We verify that \( \beta(X) \) increases with \( \delta \) and \( \alpha \), generalizing what we already knew to \( \beta(X_1, X_2) \) and \( \beta(X_3, X_4) \). Therefore \( \beta(X) \) increases with the concordance of \( X \).

**Example 3.2.** Let us consider that \( X \) has a trivariate Gumbel copula \( C_X(u_1, u_2, u_3) = \exp \left\{ - \left( \sum_{i=1}^{3} \ln u_i \right)^{1/\delta} \right\} \), with \( 0 < \delta \leq 1 \). It holds that

\[ C_X \left( \frac{1}{2} \right) = 2^{-3^{\delta}}, \quad \hat{C}_X \left( \frac{1}{2} \right) = 3 \times 2^{-2^{\delta}} - 2^{-3^\delta} - 2^{-1} \]

and

\[ C_{X_{D-1}} \left( \frac{1}{2} \right) = \hat{C}_{X_{D-1}} \left( \frac{1}{2} \right) = 2^{-2^\delta}, \quad \text{for } i = 1, 2, 3. \]

Therefore, we obtain \( \beta(X) = 2^{2-2^\delta} - 1 \), coincident with \( \beta(X_i, X_j) \), \( 1 \leq i < j \leq 3 \).
With simple calculations we can also conclude that

$$\beta(-X_1, X_2, X_3) = \frac{-2^{2-2^d} + 1}{3}$$

and that

$$\beta(X_1, X_2, X_3) + \beta(-X_1, X_2, X_3) = \frac{2}{2+1} \beta(X_2, X_3),$$

which corresponds to the verification in this example of a transition property that we present in the next section. Before we present the general expression of the multivariate correlation coefficient for a Gumbel distribution of dimension $d \geq 1$, let’s also calculate it specifically for $d = 4$.

We have

$$C_X \left( \frac{1}{2} \right) = 2^{-4^d}, \quad \hat{C}_X \left( \frac{1}{2} \right) = -1 + 6 \times 2^{-2^d} - 4 \times 2^{-3^d} + 2^{-4^d},$$

and

$$C_{X_{D-i}} \left( \frac{1}{2} \right) = 2^{-3^d}, \quad \hat{C}_{X_{D-i}} \left( \frac{1}{2} \right) = 3 \times 2^{-2^d} - 2^{-3^d} - 2^{-1}, \text{ for } i = 1, 2, 3.$$

Then

$$\beta(X_1, X_2, X_3, X_4) = 4 \times 2^{-4^d} - 8 \times 2^{-3^d} + 9 \times 2^{-2^d} - \frac{3}{2}$$

These results for $d = 2, 3, 4$, calculated directly, can also be obtained from the following general result.

If $d$ is even, we have

$$\beta(X) = \frac{1 - d}{2} + \sum_{k=1}^{d-2} \left( \binom{d-1}{k} + \binom{d}{k+1} \right) (-1)^{k+1} 2^{-(k+1)^d} + 4 \times 2^{-d} + (-1)^{d-1} 2^{-(d-1)^d},$$

(considering that a sum with the initial value of the counter greater than the final one is null) and if $d$ is odd, we have

$$\beta(X) = \frac{1 - d}{2} + \sum_{k=1}^{d-2} \left( \binom{d-1}{k} + \binom{d}{k+1} \right) (-1)^{k+1} 2^{-(k+1)^d} - 2^{-(d-1)^d}.$$

The third example also serves as a motivation for one of the properties in the
next section, on the best lower limit of $\beta(X)$.

**Example 3.3.** Consider $X$ of dimension $d$ such that $U = (U, 1 - U, U_3, ..., U_d)$. Then

\[
\beta(X) = 2 \times (0 + 0) - \frac{1}{d} \left( C_{X_{D-1}} \left( \frac{1}{2} \right) + \hat{C}_{X_{D-1}} \left( \frac{1}{2} \right) + C_{X_{D-2}} \left( \frac{1}{2} \right) + \hat{C}_{X_{D-2}} \left( \frac{1}{2} \right) \right).
\]

It follows that $\beta(X) \geq -\frac{1}{d}$ and if, in particular $(U_3, ..., U_d) = (V, ..., V)$, then $\beta(X) = -\frac{1}{d}$.

### 4 Properties of the multivariate medial correlation coefficient

Since the coefficients $\beta_{\{i\},D-\{i\}}(X)$, $i = 1, ..., d$, take values in $[-1, 1]$, the proposed coefficient takes values in the same range and, as already noted in (iii) and (iv) of section 2, the value of $\beta(X)$ increases with the concordance of $X$, being null for $C_X = C_{\Pi}$. The maximum value is attainable when $C_X = C_W = 1$ and the minimum attainable value is equal to $-\frac{1}{d}$. In fact, from the representation (24) we verify that $\beta(X)$ takes the minimum value when $C_X \left( \frac{1}{2} \right) + \hat{C}_X \left( \frac{1}{2} \right) = 0$ and $\sum_{i=1}^{d} \left( C_{\sigma;X} \left( \frac{1}{2} \right) + \hat{C}_{\sigma;X} \left( \frac{1}{2} \right) \right) = 1$, what happens when, for example, $U_j = 1 - U_i$ for some pair $1 \leq i < j \leq d$ and $U_k = V$ for each $k \in D - \{i, j\}$, analogously to what we saw in the example 3.3.

These properties on the values of the multivariate medial correlation coefficient are arranged in the following proposition.

**Proposition 4.1.** The values of the multivariate medial correlation coefficient for vectors of dimension $d$ satisfy the following properties:

(i) If $X \prec_{c} Y$ then $\beta(X) \leq \beta(Y)$.

(ii) If $C_X = C_{\Pi}$ then $\beta(X) = 0$. 

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(iii) If $C_X = C_W$ then $\beta(X) = 1$.

(iv) The minimum attainable value for $\beta(X)$ is $-\frac{1}{d}$.

In the following proposition we present the properties of continuity, permutation invariance, duality, reflection symmetry and transition, which together with (i)-(iii) of the previous proposition justifies calling the proposed coefficient a concordance measure (Taylor [10], [11]).

**Proposition 4.2.** The values of the multivariate medial correlation coefficient for vectors of dimension $d$ satisfy the following properties:

(i) If $\{C_{X_n}\}_{n \geq 1}$ converges uniformly to $C_X$, $n \to +\infty$, then $\lim_{n \to +\infty} \beta(X_n) = \beta(X)$.

(ii) The value of $\beta(X)$ is invariant for permutations of the margins of $X$.

(iii) $\beta(X) = \beta(-X)$.

(iv) $\sum_{(\epsilon_1, \ldots, \epsilon_d) \in \{-1,1\}^d} \beta(\epsilon_1 X_1, \ldots, \epsilon_d X_d) = 0$.

(v) If $Y$ is a $(d+1)$-dimensional random vector such that $C_Y(u_1, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_d) = C_X(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_d)$ then $\frac{d}{d+1} \beta(X) = \beta(Y) + \beta(\sigma_i Y)$.

**Proof.** The statement of (i) can be obtained, for example, from (23). From the representation (25) we can conclude (ii). The representation (24) leads to (iii) and (iv). Finally to obtain (v), let us note that, by (24), we have

$$\beta(Y) + \beta(\sigma_i Y) = C_Y \left( \frac{1}{2} \right) + C_{\sigma_i Y} \left( \frac{1}{2} \right) + \hat{C}_Y \left( \frac{1}{2} \right) + \hat{C}_{\sigma_i Y} \left( \frac{1}{2} \right)$$

$$- \frac{1}{d+1} \left( C_{\sigma_i Y} \left( \frac{1}{2} \right) + \hat{C}_{\sigma_i Y} \left( \frac{1}{2} \right) + C_{\sigma_i \sigma_i Y} \left( \frac{1}{2} \right) + \hat{C}_{\sigma_i \sigma_i Y} \left( \frac{1}{2} \right) \right)$$

$$- \frac{1}{d+1} \sum_{j=1, j \neq i}^{d+1} \left( C_{\sigma_j Y} \left( \frac{1}{2} \right) + C_{\sigma_j \sigma_i Y} \left( \frac{1}{2} \right) + \hat{C}_{\sigma_j Y} \left( \frac{1}{2} \right) + \hat{C}_{\sigma_j \sigma_i Y} \left( \frac{1}{2} \right) \right)$$

$$= C_X \left( \frac{1}{2} \right) + \hat{C}_X \left( \frac{1}{2} \right) - \frac{1}{d+1} \left( C_X \left( \frac{1}{2} \right) + \hat{C}_X \left( \frac{1}{2} \right) \right)$$

$$- \frac{1}{d+1} \sum_{j=1}^{d} \left( C_{\sigma_j X} \left( \frac{1}{2} \right) + \hat{C}_{\sigma_j X} \left( \frac{1}{2} \right) \right)$$

$$= \frac{d}{d+1} \left( C_X \left( \frac{1}{2} \right) + \hat{C}_X \left( \frac{1}{2} \right) \right) - \frac{d}{d+1} \sum_{j=1}^{d} \left( C_{\sigma_j X} \left( \frac{1}{2} \right) + \hat{C}_{\sigma_j X} \left( \frac{1}{2} \right) \right),$$

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that matches \( \frac{d}{d+1} \beta(X) \), applying again [24].

\[ \square \]

5 Conclusion

The multivariate medial correlation coefficient that we propose extends the probabilistic interpretation and properties of the Blomqvist \( \beta \) coefficient, it is calculable from the copula, incorporates the dependence between each margin of the vector and the vector of the remaining margins and is a measure of multivariate concordance.

The adopted approach envisages the possibility of considering other functions of bivariate coefficients involving extremes of subvectors of \( X \), as well as the possibility of adapting the method to generalize other coefficients of bivariate dependence.

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