Verbally closed virtually free subgroups

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A theorem of Myasnikov and Roman’kov says that any verbally closed subgroup of a finitely generated free group is a retract. We prove that all free (and many virtually free) verbally closed subgroups are retracts in any finitely generated group.

1 Introduction

A subgroup $H$ of a group $G$ is called verbally closed [10] (see also [11], [4], [9]) if any equation of the form

$$w(x_1, x_2, \ldots) = h,$$

where $w$ is an element of the free group $F(x_1, x_2, \ldots)$ and $h \in H$,

having a solution in $G$ has a solution in $H$. If each finite system of equations with coefficients from $H$

$$\{w_1(x_1, x_2, \ldots) = 1, \ldots, w_m(x_1, x_2, \ldots) = 1\}, \text{ where } w_i \in H * F(x_1, x_2, \ldots),$$

having a solution in $G$ has a solution in $H$, then the subgroup $H$ is called algebraically closed in $G$.

Clearly, any retract (i.e. the image of an endomorphism $\rho$ such that $\rho \circ \rho = \rho$) is an algebraically closed subgroup. It is easy to show [10] that for finitely presented groups the converse is also true:

A finitely generated subgroup of a finitely presented group is algebraically closed if and only if it is a retract.

For the (wider) class of verbally closed subgroups, no similar structural description is known. However, in free groups, the situation is simple: verbally closed subgroups, algebraically closed subgroups, and retracts are the same things.

Myasnikov–Roman’kov Theorem [10]. Verbally closed subgroups of finitely generated free groups are retracts.

A similar fact is valid for free nilpotent groups [4]. We generalise the Myasnikov–Roman’kov theorem in two directions: first, we study subgroups of arbitrary groups; and secondly, we study not only free subgroups but also virtually free subgroups $H$, i.e. containing free subgroups of finite index (in $H$).

Main Theorem. Let $G$ be any group and let $H$ be its verbally closed virtually free infinite non-dihedral subgroup containing no infinite abelian noncyclic subgroups. Then

1) $H$ is algebraically closed in $G$;

2) if $G$ is finitely generated over $H$ (i.e. $G = \langle H, X \rangle$ for some finite subset $X \subseteq G$), then $H$ is a retract of $G$;

in particular, $H$ is finitely generated if $G$ is finitely generated.

(For an infinite group, non-dihedral means non-isomorphic to the free product of two groups of order two.)

Note that each nontrivial free subgroup $H$ satisfies all conditions of the theorem and, therefore, is a retract of any finitely generated group containing $H$ as a verbally closed subgroup. Even this corollary seems to be nontrivial. The following corollary strengthens Theorem 1(1) of [9].

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Corollary 1. In a free product of finitely many finite groups, any verbally closed infinite non-dihedral subgroup is a retract.

In Section 2, we discuss examples showing that the main theorem cannot be improved in some sense (there are open questions remaining though). Section 3 contains the proof of the theorem. Our argument is slightly trickier than that in [10] but is also based on the use of Lee words [8].

Let us fix the notation. If $k \in \mathbb{Z}$, $x$ and $y$ are elements of a group, then $x^y$, $x^{ky}$, and $x^{-y}$ denote $y^{-1}xy$, $y^{-1}x^ky$, and $y^{-1}x^{-1}y$, respectively. The commutator subgroup of a group $G$ is denoted by $G'$. If $X$ is a subset of a group, then $|X|$, $(X)$, and $C(X)$ mean the cardinality of $X$, the subgroup generated by $X$, and the centraliser of $X$. The index of a subgroup $H$ of a group $G$ is denoted by $[G:H]$. The symbol $A*B$ denotes the free product of groups $A$ and $B$. $F(x_1, \ldots, x_n)$ or $F_n$ is the free group (with a basis $x_1, \ldots, x_n$).

2 Examples

Let us see whether it is possible to omit some conditions of the Main Theorem.

The subgroup $H$ is infinite. This condition cannot be removed. Let $G$ be the central product of two copies of the quaternion group (of order eight). Clearly, the factors of this product are not retracts (and, therefore, they are not algebraically closed because, for finitely generated subgroups of finitely presented groups, this is the same thing). Indeed, the kernel of such hypothetical retraction must be a nontrivial normal subgroup and the group $G$ is nilpotent. Therefore, this nontrivial normal subgroup must nontrivially intersect the centre of $G$ (see, e.g., [2]). This leads immediately to a contradiction because the centre in this case is contained in both factors.

Let us show that (for instance) the second factor $H$ of $G = Q_8 \times Q_8$ (where $C = \{\pm 1\}$) is verbally closed in $G$. Suppose that an equation
\[ w(x_1, \ldots, x_n) = (1, h'), \]
where $(1, h') \in H$ has a solution $x_i = (h_i, h_i')$ in $G$. Let us show that this equation has a solution in $H$ too. Suppose that the sum $k$ of powers of a variable $x_i$ in $w(x_1, \ldots, x_n)$ is odd. In this case, the equation $x^k = h'$ has a solution $q$ in $Q_8$ and the substitution $x_i = (1, q)$ and $x_j = (1, 1)$ for $j \neq i$ is a solution to the initial equation in $H$.

Now, suppose that all variables occur in $w(x_1, \ldots, x_n)$ with even total powers. In this case, $h' = 1$ or $h' = -1$ (otherwise the equation has no solutions in $G$). If $h' = 1$, then the substitution $x_i = (1, 1)$ for all $i$ is a solution belonging to $H$. If $h' = -1$, then either $x_i = (1, h_i)$ or $x_i' = (1, h_i')$ is a solution (lying in $H$).

The subgroup $H$ is non-dihedral. We do not know whether this condition may be omitted and leave it as an open question.

Any infinite abelian subgroup of $H$ is cyclic. A slight modification of the central product considered above shows that this condition may not be omitted. Put $G = Q_8 \times Q_8 \times F$, where $F$ is a nontrivial free group. Consider the product of the second and third factors as the subgroup $H$. The above argument implies that $H$ is verbally closed but not a retract (because a retract of a retract is a retract and the second factor is not a retract of $G$ and even of $Q_8 \times Q_8$ as proved above).

The subgroup $H$ is virtually free. This condition may not be omitted. Indeed, it is known that the free Burnside group $B(m, n)$ of rank $m \geq 2$ and odd exponent $n \geq 665$ is infinite while all its abelian subgroups are finite (see, e.g., Theorems 1.5 and 3.3 in [1]). Put $G = Q_8 \times Q_8 \times B(2, 2017)$ and let $H$ be the product of the second and third factors. Similarly to the above example, we see that $H$ is verbally closed but not a retract.

The group $G$ is finitely generated over $H$ in assertion 2). The following example shows that this condition may not be omitted. Take the subgroup $H = \{1, 1, \ldots\}$ in the Cartesian (=unrestricted) sum $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots$ of the prime-order cyclic groups. Clearly, $H$ is verbally closed (in abelian groups, verbally closed subgroups are the same as algebraically closed subgroup, and the same as serving (pure) subgroups). On the other hand, $H$ is not a direct summand in $G$ (i.e. $H$ is not a retract). Indeed, if $G = H \oplus D$, then the torsion subgroup $T(G)$ of $G$ (the direct (=restricted) sum of $\mathbb{Z}_2$) must lie in $D$; but then $H \simeq \mathbb{Z}$ is a direct summand in $G \simeq H \oplus (D/T(G))$, which is impossible as $G/T(G)$ is a divisible group.1

1The journal version of the paper contains an error in this example. The authors thank Mikhail Mikheenko for pointing this out in 2023.
Take the subgroup $H = \langle 1 \rangle$ of integers in the additive group $G$ of $p$-adic integers. Clearly, $H$ is verbally closed (in abelian groups verbally closed subgroups are the same thing as algebraically closed subgroup, and the same as serving (pure) subgroups). On the other hand, $G$ admits no nontrivial decompositions into a direct sum (see, e.g., [3]), hence, $H$ is not a retract.

### 3 Proof of the Main Theorem

Note that any virtually free group $H$ is linear (even over $\mathbb{Z}$ if $H$ is countable) because the free group (of any cardinality) is linear (over a field) (see, e.g., [2]). and virtually linear group is also linear. Therefore, $H$ is equationally Noetherian [5], i.e. any system of equations with coefficients from $H$ and finitely many unknowns is equivalent to its finite subsystem. This, in turn, means that $H$ is a retract of any finitely generated over $H$ group containing $H$ as an algebraically closed subgroup [10]. Therefore, it suffices to prove assertion 1) of the theorem.

First, suppose that $H$ is cyclic. Recall that any integer matrix can be reduced to a diagonal matrix by integer elementary transformations. This means that any finite system of equations over $H$

$$\{w_1(x_1, x_2, \ldots) = 1, \ldots, w_m(x_1, x_2, \ldots) = 1\}, \text{ where } w_i \in H \ast F(x_1, x_2, \ldots),$$

can be reduced to the form

$$\{x_1^{n_1} u_1(x_1, x_2, \ldots) = h_1, \ldots, x_m^{n_m} u_m(x_1, x_2, \ldots) = h_m\}, \text{ where } u_i \in (H \ast F(x_1, x_2, \ldots))^t, n_i \in \mathbb{Z}, h_i \in H,$

by means of a finite sequence of transformations of the form $w_i \to w_i w_j^{\pm 1}$ and $x_i \to x_i x_j^{\pm 1}$. The obtained system has the same number of solutions (in $G$ or in $H$) as the initial one. Suppose that this system has a solution in $G$ and suppose also that each word $u_i$ is a product of $s$ commutators in $H \ast F(x_1, x_2, \ldots)$. Then each single equation $x_i^{n_i} [y_j, z_j] \ldots [y_s, z_s] = h_i$ (where $y_j$ and $z_j$ are new variables) has a solution in $G$ and, therefore, has a solution $(\tilde{x}_1, \tilde{y}_1^{(i)}, \tilde{z}_1^{(i)}, \ldots)$ in $H$ (because the subgroup $H$ is verbally closed). Then, obviously, $(\tilde{x}_1, \tilde{x}_2, \ldots)$ is a solution to the entire system (since the commutator subgroup of $H$ is trivial).

The case of virtually cyclic subgroup $H$ can be easily reduced to the case of cyclic $H$ by virtue of the following observation:

an infinite virtually cyclic group containing no infinite noncyclic abelian subgroups is either cyclic or dihedral.

Indeed, any virtually cyclic group contains a finite normal subgroup such that the quotient group is either cyclic or dihedral (see, e.g., [13]). This finite normal subgroup must be trivial because otherwise we can find in its centraliser (which is of finite index) an element of infinite order and obtain an infinite noncyclic abelian subgroup.

Now, consider the more difficult case of non-virtually-cyclic group $H$.

**Lemma 1.** In a virtually free group which is not virtually cyclic, any element decomposes into a product of two infinite-order elements.

**Proof.** Clearly, it suffices to prove this assertion for finitely generated groups. Recall, that a finitely generated virtually free group admits an action on a (directed) tree such that the stabilisers of vertices are finite [7]. Recall also that any fixed-point-free automorphism of a tree has a unique invariant line (the axis) [12].

Consider such an action of a group $H$ on a tree $T$. Let $h \in H$ be an element we want to decompose and let $T_h \subseteq T$ be the fixed point set for $h$. If the order of $h$ is infinite, then $h = h^2 h^{-1}$ is the required decomposition; therefore, we assume that the order of $h$ is finite and $T_h$ is nonempty (and, hence, connected).

Take some element $g \in H$ of infinite order with an axis $l_g$. If $g^{-1} h$ has no fixed points, then its order is infinite and $h = g \cdot (g^{-1} h)$ is a required decomposition. Let $a \in T$ be a fixed point for $g^{-1} h$. Then $h(a) = g(a) = b$.

The equality $h(a) = b$ shows that the path joining $a$ and $b$ must intersect the subtree $T_h$ in a unique point $c$, and $c$ is the midpoint of this path. The equality $g(a) = b$ shows that the line $l_g$ passes through $c$ and intersects the segments $[a, c]$ and $[c, b]$ in segments of the same nonzero length $\delta$ (otherwise, the distances from $l_g$ to $a$ and to $b$ are different and, hence, $g(a) \neq b$). The element $g$ must act as a translation by $2\delta$ on its axis $l_g$. However, we only need the equality $T_h \cap l_g = \{c\}$ now.

\footnote{Note also that a group containing an equationally Noetherian subgroup of finite index, is equationally Noetherian [6].}
Take another infinite-order element \( g' \in H \) with another axis \( l_{g'} \neq l_g \) (such an element exists because the group \( H \) is not virtually cyclic). A similar argument shows that either \( h = g' \cdot ((g')^{-1}h) \) is the required decomposition or \( T_h \cap l_{g'} = \{c'\} \) for some point \( c' \). In the latter case, take the element \( g'' = g^k g' g^{-k} \). Its axis is \( g^k(l_{g'}) \) obviously; and we see that this line does not intersect the subtree \( T_h \) if the number \( k \) is chosen large enough (positive or negative). Therefore, in this last case, \( h = g'' \cdot ((g'')^{-1}h) \) is the required decomposition. (More precisely, if \( c' \neq c \), then \( k \) can be chosen equal to 1 or any other nonzero number; if \( c' = c \), then \( k \) should be choose in such a way that \( |2k\delta| \) is larger than the distance from \( c \) to the endpoint of the segment or half-line \( l_g \cap l_{g'} \).) \( \square \)

**Lemma 2.** If \( h_1 \) and \( h_2 \) are infinite-order elements of a virtually free group all whose infinite abelian subgroup are cyclic and \( h_1^k = h_2^k \) for some nonzero integer \( k \), then \( h_1 = h_2 \).

**Proof.** The roots of an element lie in its centraliser, therefore, it suffices to show that the centraliser of an infinite-order element \( h \) is cyclic. This centraliser \( C(h) \) is a virtually free group (as a subgroup of a virtually free group) with infinite centre. This implies immediately that \( C(h) \) is virtually cyclic. It remains to refer to the fact mentioned in the beginning of this section: a virtually cyclic group without infinite abelian noncyclic subgroups is either cyclic or dihedral. In the case under consideration, the group cannot be dihedral because the centre of the centraliser is nontrivial. \( \square \)

The following lemma is well known.

**Lemma 3.** If a subgroup \( H \) of a group \( G \) is such that any finite system of equations of the form
\[
\{w_1(x_1, \ldots, x_n) = h_1, \ldots, w_m(x_1, \ldots, x_n) = h_m\}, \quad \text{where } w_i \in F(x_1, \ldots, x_n) \text{ and } h_i \in H,
\]
has a solution in \( G \) has a solution in \( H \) too, then \( H \) is algebraically closed.

**Proof.** Just denote the coefficients by new letters and interpret them as variables. For example, the solvability of the equation \( x y h_1 [x^{2023}, h_2] y^{-1} = 1 \) is equivalent to the solvability of the system
\[
\{x y z [x^{2023}, t] y^{-1} = 1, z = h_1, t = h_2\}.
\]

Now, we need a useful tool. Recall that a Lee word in \( m \) variables for the free group of rank \( r \) is an element \( L(z_1, \ldots, z_m) \) of the free group of rank \( m \) such that

1) if \( L(v_1, \ldots, v_m) = L(v'_1, \ldots, v'_m) \neq 1 \) in \( F_r \), then \( v'_i \in F_r \) are obtained from \( v_i \in F_r \) by simultaneous conjugation, i.e., there exists \( w \in F_r \) such that \( v'_i = v^w_i \) for all \( i = 1, \ldots, m \);

2) \( L(v_1, \ldots, v_m) = 1 \) if and only if the elements \( v_1, \ldots, v_m \) of \( F_r \) generate a cyclic subgroup.

In [8], such words were constructed for all integers \( r, m \geq 2 \). Actually, it is easy to see that Lee’s result implies the existence of a universal Lee word in \( m \) variables.

**Lemma 4.** For any positive integer \( m \), there exists an element \( L(z_1, \ldots, z_m) \in F_m \) such that properties 1) and 2) hold in all free groups \( F_r \) and even in \( F_\infty \).

**Proof.** This assertion follows immediately from Lee’s result and the following simple fact:

\[ F_\infty \text{ embeds into } F_2 \text{ as a malnormal subgroup,} \]
i.e. a subgroup \( S \subset F_2 \) such that \( S^f \cap S = \{1\} \) for all \( f \in F_2 \setminus S \). This fact follows, e.g., from a result of [14]:

\[ \text{in a free group, any set satisfying small-cancellation condition C(5) freely generates a malnormal subgroup.} \]

Thus, a Lee word for \( F_2 \) is universal, i.e. it is suitable also for \( F_\infty \). \( \square \)
Let us proceed with the proof of the Main Theorem. Thus, we assume that a verbally closed subgroup \( H \) of a group \( G \) is virtually free, does not contain infinite noncyclic abelian subgroups, and contains a normal (in \( H \)) non-abelian free subgroup \( F \) of index \( N \) (in \( H \)). Applying Lemma 3, we can assume that system (1) has a solution in \( G \) and we have to show that this system has a solution in \( H \).

Let \( L(z_1, \ldots, z_{2m+2}) \) be a universal Lee word in \( 2m + 2 \) variables. By Lemma 1, for each element \( h_i \), we find an infinite-order element \( f_i \in H \) such that the order of \( h_i f_i \) is also infinite. Take two noncommuting element \( u_1, u_2 \in F \) and consider the equation

\[
L\left((w_1(x_1, \ldots, x_n)y_1)^N, \ldots, (w_m(x_1, \ldots, x_n)y_m)^N, y_1^N, \ldots, y_m^N, z_1^N, z_2^N\right) = f,
\]

where \( f = L\left((h_1 f_1)^N, \ldots, (h_m f_m)^N, f_1^N, \ldots, f_m^N, u_1^N, u_2^N\right) \in F \). This equation has a solution in \( G \) by construction (just take the following: a solution to system (1) as \( x_i \), elements \( f_i \) as \( y_i \), and \( u_i \) as \( z_i \)).

The subgroup \( H \) is verbally closed in \( G \) and \( f \in H \); thus, the last equation has a solution \( \bar{x}_i, \bar{y}_j, \bar{z}_k \in H \).

The right-hand side of the equation is a value of a Lee word on some elements of the free group \( F \) (because \( h_i^N \in F \) for all \( h \in H \)); the elements \( u_1^N \) and \( u_2^N \) do not commute (because the elements \( u_1 \) and \( u_2 \) of a free group are chosen non-commuting), so they do not lie in the same cyclic subgroup and, therefore, Property 2) of Lee words implies that \( f \neq 1 \). According to Property 1), we have

\[
(w_l(\bar{x}_1, \ldots, \bar{x}_n)\bar{y}_i)^w = (h_l f_l)^N, \quad \bar{y}_i^w = f_i^N \quad \text{and} \quad \bar{z}_i^w = u_i^N \quad \text{for some} \ w \in F.
\]

By Lemma 2, we obtain the equalities

\[
(w_l(\bar{x}_1, \ldots, \bar{x}_n)\bar{y}_i)^w = h_l f_l, \quad \bar{y}_i^w = f_i, \quad \text{and} \quad \bar{z}_i^w = u_i,
\]

i.e. \( \bar{x}_i^w \in H \) is a solution to system (1) in \( H \) as required.

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