Research Announcement: Finite–time Blow Up and Long–wave Unstable Thin Film Equations

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Abstract

We study short–time existence, long–time existence, finite speed of propagation, and finite–time blow–up of nonnegative solutions for long-wave unstable thin film equations

\[ h_t = -a_0(h^n h_{xxx})_x - a_1(h^m h_x)_x \]

with \( n > 0, a_0 > 0, \) and \( a_1 > 0. \) The existence and finite speed of propagation results extend those of [Comm Pure Appl Math 51:625–661, 1998]. For \( 0 < n < 2 \) we prove the existence of a nonnegative, compactly–supported, strong solution on the line that blows up in finite time. The construction requires that the initial data be nonnegative, compactly supported in \( \mathbb{R}^1, \) be in \( H^1(\mathbb{R}^1), \) and have negative energy. The blow-up is proven for a large range of \((n, m)\) exponents and extends the results of [Indiana Univ Math J 49:1323–1366, 2000].

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1 Introduction

Numerous articles have been published since the early sixties concerning the problem of developing finite–time singularities by solutions of nonlinear parabolic equations (see the survey papers of Levine [31] and of Bandle and Brunner [2]). Such problems arise in various applied fields such as combustion theory, the theory of phase separation in binary alloys, population dynamics and incompressible fluid flow.

Whether or not there is a finite–time singularity, such as \( \|u(\cdot, t)\|_\infty \rightarrow \infty \) as \( t \rightarrow T^* < \infty, \) is strongly affected by the nonlinearity in the PDE. For example, consider the semilinear heat equation on the line:

\[ u_t = u_{xx} + u^p, \]  \hspace{1cm} (1.1)

where \( u \) is real–valued. If \( p \leq 1 \) then a solution of an initial value problem exists for all time. If \( 1 < p \leq 3, \) then any non-trivial solution blows up in finite time. If \( p > 3 \) then some initial data yield solutions that exist for all time and other initial data result in solutions that
have finite–time singularities. The manner in which solutions blow up is well understood computationally and analytically. The blow up is of a focussing type: there are isolated points in space around which the graph of the solution narrows and becomes taller as \( t \uparrow T^* \), converging to delta functions centered at the blow–up points. As \( t \uparrow T^* \), the behaviour of the solution near the blow–up point(s) becomes more and more self–similar. Proving this convergence to a self–similar solution uses the maximum principle, which doesn’t hold for fourth–order equations like the one we study in this article.

We study the longwave-unstable generalized thin film equation,

\[
h_t = -a_0 (|h|^n h_{xxx})_x - a_1 (|h|^m h_x)_x, \quad (1.2)
\]

where \( a_0 > 0 \), \( a_1 > 0 \), and \( h \) is real valued. Perturbing a constant steady state slightly, \( h_0(x) = \bar{h} + \epsilon h_1(x,0) = \bar{h} + \epsilon \cos(\xi x + \phi) \), and linearizing the equation about \( \bar{h} \), the small perturbation \( h_1(x,t) \) will (approximately) satisfy \( h_t = -a_0 |\bar{h}|^n h_{xxx} - a_1 |\bar{h}|^m h_{xx} \). Hence the constant steady state is linearly unstable to long wave perturbations:

\[
\xi^2 < |\bar{h}|^{m-n} a_1/a_0 \quad \Rightarrow \quad h_1(x,t) \sim e^{-\alpha_0 \xi^2 |\bar{h}|^{m-n} t} \cos(\xi x + \phi) \quad \text{grows.} \quad (1.3)
\]

Such equations arise in the modelling of fluids and materials. For example, equation \( (1.2) \) with \( n = m = 1 \) describes a thin jet in a Hele-Shaw cell [16] where \( h \) represents the thickness of the jet and \( x \) is the axial direction; if \( n = 3 \) and \( m = -1 \) equation \( (1.2) \) describes Van der Waals driven rupture of thin films [44] where \( h \) represents the thickness of the film; if \( n = m = 3 \) the equation models fluid droplets hanging from a ceiling [21] with \( h \) representing the thickness of the film, and finally if \( n = 0 \) and \( m = 1 \) the equation is a modified Kuramoto-Sivashinsky equation and describes the solidification of a hyper-cooled melt [9] where \( h \) describes the deviation from flatness of a near planar interface. We note that in the first three cases the solution \( h \) must be nonnegative for the model to make physical sense.

Hocherman and Rosenau [28] considered whether or not equation \( (1.2) \) could have solutions that blow up in finite time. They conjectured that if \( n > m \) then solutions might blow up in finite time but if \( n < m \) they would exist for all time. Indeed, this conjecture is natural if one considers the linear stability of a constant steady state \( \bar{h} \): if \( n > m \) the unstable band \( (1.3) \) grows as \( \bar{h} \to \infty \) suggesting that \( n = m \) is critical.

Hocherman and Rosenau considered general, real–valued solutions. However, if \( n > 0 \) equation \( (1.2) \) may have solutions that are nonnegative for all time. Bertozzi and Pugh [12] proposed that\(^1\) if the boundary conditions are such that the mass, \( \int h(x,t) \, dx \), is conserved then mass conservation, combined with the nonnegativity results in a different balance: \( m = n + 2 \) instead of \( m = n \). For such cases they introduced the regimes

\[
\begin{align*}
  m < n + 2 &\quad \Rightarrow \quad \text{subcritical regime} \\
  m = n + 2 &\quad \Rightarrow \quad \text{critical regime} \\
  m > n + 2 &\quad \Rightarrow \quad \text{supercritical regime}
\end{align*}
\]

\(^1\)In fact, the article considers \( (1.2) \) with general coefficients: \( f(h) \) and \( g(h) \) instead of \( |h|^n \) and \( |h|^m \) respectively. In the following, for simplicity, their results are discussed for the power–law case.
In [12], Bertozzi and Pugh considered equation (1.2) on a finite interval with periodic boundary conditions. For a subset of the subcritical ($m < n + 2$) regime they proved some global-in-time results. Specifically, they proved that given positive initial data, $h_0 > 0$, there is a nonnegative weak solution of (1.2) that exists for all time if $0 < n \leq m < n + 2$. By restricting $n$ further to $0 < n < 3$ they can consider nonnegative initial data, $h_0 \geq 0$. They prove that there is a nonnegative weak solution that exists for all time and also prove the local entropy bound needed for the finite speed of propagation proof for $0 < n < 2$. For the critical ($m = n + 2$) regime, they prove that the above results will hold if the mass $\int h_0 \, dx$ is sufficiently small. Also, they provided numerical simulations suggesting that other initial data can result in solutions that blow up in finite time and conjectured that this is also true for the supercritical ($m > n + 2$) regime.

In [13], they considered equation (1.2) on the line and found some analytical results for the critical and supercritical ($m \geq n + 2$) regimes in the special case of $n = 1$. They introduced a large class of “negative energy” initial data and proved that given initial data $h_0$ with compact support and negative energy there is a nonnegative weak solution $h$ that blows up in finite time: there is a time $T^* < \infty$ such that the weak solution $h(\cdot, t)$ exists on $[0, T^*)$ and

$$\limsup_{t \to T^*} \|h(\cdot, t)\|_{L^\infty} = \infty \quad \text{and} \quad \limsup_{t \to T^*} \|h(\cdot, t)\|_{H^1} = \infty.$$  

The blow-up time $T^*$ depends only on $m$ and $H^1$-norm of the initial data. We note that uniqueness of nonnegative weak solutions of (1.2) is an open problem. Indeed, there are simple counterexamples to uniqueness for the simplest equation $h_t = -(h^n h_{xxx})_x$ (see, e.g. [11]) although it is hypothesized that solutions are unique if one considers the question within a sufficiently restrictive class of weak solutions. For this reason, one cannot exclude the possibility that the initial data $h_0$ might also be achieved by a different weak solution, one that exists for all time.

Their proof relied on a second moment argument, found formally by Bernoff [8]: if $h$ is a smooth compactly-supported solution of (1.2) on $[0, T^*)$ then the second moment of $h$ satisfies

$$\int_{-\infty}^{\infty} x^2 h(x, t) \, dx \leq \int_{-\infty}^{\infty} x^2 h_0(x) \, dx + 6 \mathcal{E}(0) t$$  

holds for all $t \in [0, T^*)$. Here, $\mathcal{E}(0)$ is the energy of the initial data:

$$\mathcal{E}(0) := \int_{-\infty}^{\infty} \left\{ \frac{a_m}{2} h_0^2(x) - \frac{a_{n+1}}{m(m+1)} h_0^{m+1}(x) \right\} \, dx. \quad (1.5)$$

As a result, there could never be a global-in-time nonnegative smooth solution with negative-energy initial data: for such a solution the left-hand side of (1.4) would always be nonnegative but the right-hand side would become negative in finite time. (This argument is strictly formal because, to date, no-one has constructed nonnegative, compactly-supported, smooth solutions on the line.) The blow-up result is found by first proving the short-time existence
of a nonnegative, compactly-supported, weak solution: it exists on $[0, T_0]$ where the larger $\|h_0\|_{H^1}$ is, the smaller $T_0$ will be. Also, the constructed solution satisfies the second moment inequality (1.3) at time $t = T_0$. By “time-stepping” the short-time existence result, they construct a solution on $[0, T^*)$ such that the second moment inequality (1.4) holds at a sequence of times $T_i$ with $T_i \to T^*$. It then follows immediately that $T^*$ must be finite and therefore the $\limsup_{t \to T^*} \|h(\cdot, t)\|_{H^1}$ must be infinite.

Outline of results

The main results of this paper are: short-time existence of nonnegative strong solutions on $[-a, a]$ given nonnegative initial data, finite speed of propagation for these solutions if their initial data had compact support within $(-a, a)$, and finite-time blow-up for solutions of the Cauchy problem that have initial data with negative energy.

First, we consider equation (1.2) on a bounded interval $[-a, a]$ with periodic boundary conditions. Given nonnegative initial data that has finite “entropy”, in Theorem 1 we prove the short-time existence of a nonnegative weak solution if $n > 0$ and $m \geq n/2$. The solution is a “generalized weak solution” as described in Section 2 and the entropy is introduced in Section 3. Additional regularity is proven in Theorem 2 there is a second type of entropy such that if this “$\alpha$-entropy” is also finite for the initial data then there is a “strong solution” which satisfies Theorem 1 and also has some additional regularity. We note that the work [12] described above was primarily concerned with long-time existence results: for this reason the authors only addressed the existence theory for the subcritical ($m < n + 2$) case. However, given finite-entropy initial data their methods easily imply a short-time result for general $n > 0$ as long as $m \geq n$. Our advance is prove the results for the larger range of $m \geq n/2$. The left plot in Figure 1 presents the $(n, m)$ parameter range for which our short-time existence results hold. The darker region represents the parameters for which the methods of [12] would have yielded the results. The lighter region represents the extended area where our methods also yield results.

In Theorem 3 we prove that if the initial data $h_0$ has compact support then the strong solution of Theorem 2 will have finite speed of propagation. Specifically, if the support of the initial data satisfies $\text{supp}(h_0) \subseteq [-r_0, r_0] \subset (-a, a)$ then there is a nondecreasing function $\Gamma(t)$ and a time $T_{\text{speed}}$ such that $\text{supp}(h(\cdot, t)) \subseteq [-r_0 - \Gamma(t), r_0 + \Gamma(t)] \subset (-a, a)$ for every time $t \in [0, T_{\text{speed}}]$. For $0 < n < 2$, we further prove that there is a constant $C$ such that $\Gamma(t) \leq Ct^{1/(n+4)}$. This power law behaviour is the same as has been found for $h_t = -(|h|^n h_{xxx})_x$ for $0 < n < 2$ [5] and $2 \leq n < 3$ [29]; it corresponds to the rate of expansion of the self-similar source-type solution.

The middle plot in Figure 1 presents the $(n, m)$ parameter range for which we were able to prove finite speed of propagation. While we were successful in proving finite speed of propagation for the entire expected range of $m \geq n/2$ if $1/2 < n < 3$, if $0 < n \leq 1/2$ we

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2 Throughout this article, we use phrases like “long-time”, “global-in-time” and “exist for all time” as shorthand for the types of large-time results that have been proven in the thin film literature to date: given a time $T < \infty$ there is a solution $h(\cdot, t)$ for $t \in [0, T]$. Specifically, $T$ can be taken arbitrarily large.
could prove it for only a subset of \( m \geq n/2 \). This technical obstruction is discussed further in Section 7. The \( n \) values are restricted to \( n < 3 \) because for \( n \geq 3 \) if initial data has compact support in \((-a, a)\) then it will have infinite entropy and will not be admissible initial data for Theorems 1 and 2.

To prove Theorem 3 we start by proving a local entropy estimate similar to that of [5] for \( 0 < n < 2 \) and a local energy estimate similar to that of [6, 29] for \( 1/2 < n < 3 \). Using these and well-chosen “localization” (or “test”) functions, we find systems of functional inequalities. In [24], Giacomelli and Shishkov proved an extension of Stampacchia’s lemma to systems of inequalities, we use this to then finish the proof.

A Stampacchia-like lemma for a single inequality was used by [29] to prove finite speed of propagation for \( h_t = -(|h|^nh_{xxx})_x \) and [18] proved an extension of Stampacchia’s lemma (also for a single inequality) to study waiting time phenomena. Similar approaches were subsequently used to study finite speed of propagation and waiting time phenomena for related equations, see [1, 19, 25, 27, 42, 23, 39, 40, 41]. Further, there are finite speed of propagation and waiting time results [12, 35] that use the extension of Stampacchia’s lemma to systems of [24] as well as other types of extensions to systems [41].

Having proven finite speed of propagation in Theorem 3 we use this to prove a short-time existence result for the Cauchy problem. Specifically, for the range of exponents \((n, m)\) of Theorem 3 given nonnegative initial data with bounded support in \( \mathbb{R}^1 \) we construct a nonnegative, compactly supported strong solution on \( \mathbb{R}^1 \times [0, T_0] \) that satisfies the bounds and regularity of Theorems 1 and 2 (with those bounds taken over \( \mathbb{R}^1 \) rather than \( \Omega \)). The larger the \( H^1 \) norm of the initial data, the shorter the time \( T_0 \). In Lemma 8.1 we prove that for a subset of these exponents (see Figure 1), the entropy of the solution satisfies a second-moment inequality at time \( T_0 \)

\[
e^{-\tilde{B}(T_0)} \int_{-\infty}^{\infty} x^2 \tilde{G}_0(h(x, T_0)) \, dx \leq \int_{-\infty}^{\infty} x^2 \tilde{G}_0(h_0) \, dx + \int_0^{T_0} e^{-\tilde{B}(t)} \left( k_1 \mathcal{E}_0(0) + k_2 \int_{-\infty}^{\infty} x^2 h_{xx}^2 \, dx \right) \, dt
\]
where \( k_1 = 2(4 - n) \), \( k_2 = 3a_0(n - 1)/2 \), \( E_0(0) \) is the energy of the initial data (1.5),

\[
\tilde{G}_0(z) = \frac{1}{2-n}z^{2-n}, \quad \text{and} \quad \tilde{B}(t) = \frac{a_0^3(1-n)(2-n)}{2a_0(m-n+1)^2} \int_0^t \|h(\cdot, \tau)\|_{L^\infty(\Omega)}^{2n-m} d\tau.
\]

Note that if \( n = 1 \) then the second-moment inequality for the entropy reduces to the second moment inequality (1.4) used by [13].

"Time-stepping" this existence result, we construct a nonnegative, compactly supported, strong solution on \( \mathbb{R}^1 \times [0, T^*) \) such that our second-moment inequality for the entropy holds for a sequence of times \( T_i \to T^* \). The left-hand side of the inequality is nonnegative. If the initial data has negative energy, then the second term on the right-hand side has an integrand which has the possibility of becoming negative. In Theorem 4 we prove that for \( \text{generalized weak solution} \)

\[
\text{Definition 2.1}
\]

However, there has been some detailed study of the critical regime as proving that there’s a focussing singularity, as suggested by numerical simulations [12]. Ideally, we would have proven finite speed of propagation for all \((n, m)\) with \( 0 < n < 3 \) and all \( n/2 \leq m \) and would have proven the finite time blow up result for all \((n, m)\) with \( 0 < n < 3 \) and all \( m \geq n + 2 \). We believe the obstructions are technical ones.

We close by noting that our blow–up result does not give qualitative information such as proving that there’s a focussing singularity, as suggested by numerical simulations [12]. However, there has been some detailed study of the critical regime \( m = n + 2 \). There, a critical mass \( M_c \) has been identified [33], there are self-similar, compactly-supported, source-type solutions with masses in \([0, M_c)\) [3], and there are self-similar, compactly supported, solutions with masses in \((M_c, M_1)\) that blow up in finite time [37]. Further, these self-similar blow-up solutions have been shown to be linearly stable [38].

\section{Generalized weak solution}

We study the existence of a nonnegative weak solution, \( h(x, t) \), of the initial–boundary value problem

\[
\begin{cases}
ht + (f(h)(a_0h_{xxx} + a_1D''(h)h_x)) = 0 \text{ in } Q_T, \\
\frac{\partial h}{\partial x}(-a, t) = \frac{\partial h}{\partial x}(a, t) \text{ for } t > 0, \ i = 0, 3, \\
h(0, x) = h_0(x) \geq 0,
\end{cases}
\]

where \( a_0 > 0, \ a_1 \geq 0, \ f(h) = |h|^n, \ D''(h) = |h|^{m-n}, \ n > 0, \ m > 0, \ \Omega = (-a, a) \), and \( Q_T = \Omega \times (0, T) \). We consider a weak solution like that considered in [41 5]:

\textbf{Definition 2.1} (generalized weak solution). \textit{Let} \( n > 0, \ m > 0, \ a_0 > 0, \) and \( a_1 \geq 0 \). A function \( h \) is a generalized weak solution of the problem (P) if

\[
\begin{align*}
&h \in C^{1/2,1/8}_{x, t}(\overline{Q}_T) \cap L^\infty(0, T; H^1(\Omega)), \\
h_t \in L^2(0, T; (H^1(\Omega))'), \\
h \in C^{4,1}_{x, t}(\mathcal{P}), \quad \sqrt{f(h)}(a_0h_{xxx} + a_1D''(h)h_x) \in L^2(\mathcal{P}),
\end{align*}
\]

6
where $P = Q_T \setminus (\{h = 0\} \cup \{t = 0\})$ and $h$ satisfies (2.1) in the following sense:

$$\int_0^T \langle h_t(\cdot, t), \phi \rangle \, dt - \iint_P f(h)(a_0 h_{xxx} + a_1 D''(h)h_x)\phi_x \, dx \, dt = 0$$

(2.7)

for all $\phi \in C^1(Q_T)$ with $\phi(-a, \cdot) = \phi(a, \cdot)$;

$h(\cdot, t) \to h(\cdot, 0) = h_0$ pointwise & strongly in $L^2(\Omega)$ as $t \to 0$,

$h(-a, t) = h(a, t) \forall t \in [0, T]$ and $\frac{\partial h}{\partial x_i}(-a, t) = \frac{\partial h}{\partial x_i}(a, t)$

(2.8)

for $i = 1, 3$ at all points of the lateral boundary where $\{h \neq 0\}$.

Because the second term of (2.7) has an integral over $P$ rather than over $Q_T$, the generalized weak solution is “weaker” than a standard weak solution. Also note that the first term of (2.7) uses $h_t \in L^2(0, T; (H^1(\Omega))')$; this is different from the definition of weak solution first introduced by Bernis and Friedman [7]; there, the first term was the integral of $h\phi_t$ integrated over $Q_T$. We do not require a test function $\phi$ to be zero at both ends $t = 0$ and $t = T$ that is crucial for our construction of a continuation of the weak solution.

### 3 Main results

Our main results are: the short-time existence of a nonnegative generalized weak solution (Theorem 1), the short-time existence of a nonnegative strong solution (Theorem 2), finite-speed of propagation and finite-time blow-up for some of these strong solutions (Theorems 3 and 4 respectively).

The short-time existence of generalized weak solutions relies on an integral quantity: $\int G_0(h(x, t)) \, dx$. This “entropy” was introduced by Bernis and Friedman [7], where

$$G_0(z) := \begin{cases} 
\frac{z^{2-n} - n}{(2-n)(1-n)} & \text{if } n \in (0, 1) \cup (2, \infty) \\
\frac{z^{2-n} - n}{(2-n)(1-n)} + z + b & \text{if } n \in (1, 2) \\
z \ln z - z + 1 & \text{if } n = 1 \\
- \ln z + z/e & \text{if } n = 2.
\end{cases} \quad \Rightarrow \quad G''_0(z) = \frac{1}{z^n} \quad (3.1)$$

with $b$ chosen to ensure that $G_0(z) \geq 0$ for all $z \geq 0$.

**Theorem 1 (Existence).** Let $n > 0$, $m > \frac{n}{2}$, $a_0 > 0$, and $a_1 \geq 0$ in equation (1.2). Assume that the nonnegative initial data $h_0 \in H^1(\Omega)$ has finite entropy

$$\int_\Omega G_0(h_0(x)) \, dx < \infty,$$  

(3.2)
where $G_0$ is given by (3.1) and either 1) $h_0(-a) = h_0(a) = 0$ or 2) $h_0(-a) = h_0(a) \neq 0$ and
\[ \frac{\partial h}{\partial x^i}(-a) = \frac{\partial h}{\partial x^i}(a) \]
holds for $i = 1, 2, 3$. Then for some time $T_{loc} > 0$ there exists a nonnegative generalized weak solution $h$ on $Q_{T_{loc}}$ in the sense of the definition (2.7). Furthermore, $h \in L^2(0, T_{loc}; H^2(\Omega)).$

If
\[ D_0(z) = \begin{cases} \frac{z^{m-n+2}}{(m-n+1)(m-n+2)} & \text{if } m - n \notin \{-2, -1\} \\ -\log(z) & \text{if } m - n = -2 \\ z \log(z) & \text{if } m - n = -1 \end{cases}, \]
then the weak solution satisfies
\[ h \in L^2(0, T_{loc}; H^2(\Omega)). \] (3.3)

\[ E_0(T) := \int_\Omega \left\{ \frac{a_0}{2} h_2^2(x, T) - a_1 D_0(h(x, T)) \right\} dx, \quad B_1(T) := \frac{a_1^2}{a_0} \int_0^T \|h(\cdot, t)\|_{L^\infty(\Omega)}^{2m-n} dt, \] (3.5)
and
\[ B_2(T) := \frac{a_1^2}{a_0(2m-n+1)^2} \int_0^T \|h(\cdot, t)\|_{L^\infty(\Omega)}^{4m-n} dt + \frac{a_1^2}{a_0(m-n+1)^2} \int_0^T \|h(\cdot, t)\|_{L^\infty(\Omega)}^{2m-n} dt, \]
then the weak solution satisfies
\[ E_0(T_{loc}) + \int \int_{\{h > 0\}} h^n(a_0 h_{xxx} + a_1 h^{m-n} h_x)^2 dxdt \leq E_0(0), \] (3.6)
\[ \int \Omega h_x^2(x, T_{loc}) dx \leq e^{B_1(T_{loc})} \int \Omega h_{0x}^2(x) dx, \] (3.7)
and
\[ \int \Omega \left\{ h_x^2(x, T_{loc}) + G_0(h(x, T_{loc})) \right\} dx \leq e^{B_2(T_{loc})} \int \Omega \left\{ h_{0x}^2(x) + G_0(h_0(x)) \right\} dx. \] (3.8)

The time of existence, $T_{loc}$, is determined by $a_0, a_1, |\Omega|, \int h_0, \|h_{0x}\|_2$, and $\int G_0(h_0)$.

There is nothing special about the time $T_{loc}$ in the bounds (3.6), (3.7), and (3.8); given a countable collection of times in $[0, T_{loc}]$, one can construct a weak solution for which these bounds will hold at those times. Also, we note that the analogue of Theorem 4.2 in [7] also holds: there exists a nonnegative weak solution with the integral formulation
\[ \int_0^T \langle h_1(\cdot, t), \phi \rangle dt + a_0 \int_{Q_T} (n h^{n-1} h_x h_{xx} \phi_x + h^n h_{xx} \phi_{xx}) dxdt - a_1 \int_{Q_T} h^m h_x \phi_x dxdt = 0. \] (3.9)

We note that the existence theory for the long-wave stable case, $a_1 < 0$, has already been considered by [10, 20].
The hypotheses of Theorem 2 and also has compact support in on Then for some time time $T$ $\frac{n}{4}$ where $c(\star)$. Assume the nonnegative initial data (Regularity) Theorem 2 $\alpha$ and also has finite speed propagation, i.e.

$$G_0^{(\alpha)}(z) = \begin{cases} 
\frac{z^{2-n+\alpha}}{(2-n+\alpha)(1-n+\alpha)} + z + c & \text{if } \alpha \in (-\infty, n-2) \cup (n-1, \infty) \\
\frac{z}{(2-n+\alpha)(1-n+\alpha)} + z + 1 & \text{if } \alpha \in (n-2, n-1) \\
\ln z - z + 1 & \text{if } \alpha = n - 1 \\
-\ln z + z/e & \text{if } \alpha = n - 2,
\end{cases}$$

$$G_0^{(\alpha)\prime}(z) = \frac{z^\alpha}{e^\alpha}$$

where $c$ is chosen to ensure that $G_0^{(\alpha)}(z) \geq 0$ for all $z \geq 0$.

**Theorem 2** (Regularity). Assume the initial data $h_0$ satisfies the hypotheses of Theorem 1 and also has finite $\alpha$-entropy for some $\alpha \in (-1/2, 1)$, $\alpha \neq 0$,

$$\int_{\Omega} G_0^{(\alpha)}(h_0(x)) \, dx < \infty.$$  

Then for some time time $T_{loc}^{(\alpha)} \in (0, T_{loc}]$ there exists a nonnegative generalized weak solution on $Q_{T_{loc}^{(\alpha)}}$ that satisfies Theorem 1 and has the additional regularity

$$h^{\frac{\alpha+2}{2}} \in L^2(0, T_{loc}^{(\alpha)}; H^2(\Omega)) \text{ and } h^{\frac{\alpha+2}{4}} \in L^2(0, T_{loc}^{(\alpha)}; W^1_4(\Omega)).$$

**Theorem 3** (Finite Speed of Propagation). Consider the range of exponents $0 < n \leq \frac{1}{2}$ and $n/2 < m < 6 - n$ or $\frac{1}{2} < n < 3$ and $m \geq n/2$ (see Figure 4). Assume the initial data satisfies the hypotheses of Theorem 2 and also has compact support in $(-a, a)$: $\text{supp}(h_0) \subseteq [-r_0, r_0] \subset (-a, a)$. Then there exists a time $0 < T_{\text{speed}} \leq T_{loc}^{(\alpha)}$ and a nondecreasing function $\Gamma(t) \in C([0, T_{\text{speed}}])$ such that the strong solution from Theorem 2 has finite speed propagation, i.e.

$$\text{supp}(h(\cdot, t)) \subseteq [-r_0 - \Gamma(t), r_0 + \Gamma(t)] \subset (-a, a)$$

for all $t \in [0, T_{\text{speed}}]$. Furthermore, if $0 < n < 2$ there exists a constant $C$ which depends on $n$, $m$, $\alpha$, and $\int h_0$ such that $\Gamma(t) \leq C t^{1/(n+4)}$.

**Theorem 4** (Finite Time Blow Up). Consider the range of exponents $0 < n \leq \frac{1}{2}$ and $4 - n \leq m < 6 - n$ or $\frac{1}{2} < n \leq 1$ and $m \geq 4 - n$ or $1 < n < 2$ and $m \geq n + 2$ (see Figure 4). Assume the nonnegative initial data $h_0$ has compact support, negative energy (3.7)

$$E_0(0) = \int_{-\infty}^{\infty} \left\{ \frac{\alpha}{2} h_0^2 - a_1 D_0(h_0) \right\} \, dx < 0,$$

and satisfies the hypotheses of Theorem 2 with $\Omega = \mathbb{R}$. Then there exists a finite time $T^* < \infty$ and a nonnegative, compactly supported, strong solution $h$ on $\mathbb{R} \times [0, T^*)$ such that

$$\limsup_{t \to T^*} \|h(\cdot, t)\|_{H^1(\mathbb{R})} = \limsup_{t \to T^*} \|h(\cdot, t)\|_{L^\infty(\mathbb{R})} = +\infty.$$  

(3.13)

The solution satisfies the bounds of Theorems 1 and 2 with $\Omega = \mathbb{R}$.  

9
4 Proof of Existence of Generalized Solutions

Our proof of existence of generalized weak solutions defined in the section following the main concept of the proof from [7].

4.1 Regularized Problem

Given $\delta, \varepsilon > 0$, a regularized parabolic problem, similar to that of Bernis and Friedman [7], is considered:

\[
\begin{align*}
(P_{\delta, \varepsilon}) & \quad \begin{cases}
  h_t + (f_{\delta \varepsilon}(h)(a_0 h_{xxx} + a_1 D''_{\varepsilon}(h) h_x))_x = 0, \\
  \frac{\partial h}{\partial x^i}(-a, t) = \frac{\partial h}{\partial x^i}(a, t) \text{ for } t > 0, \ i = 0, 3, \\
  h(x, 0) = h_{0, \varepsilon}(x)
\end{cases}
\end{align*}
\]

(4.1)

where

\[
f_{\delta \varepsilon}(z) := f_{\varepsilon}(z) + \delta = \frac{|z|^{4+n}}{|z|^{m+n} + |z|^n} + \delta, \quad D''_{\varepsilon}(z) := \frac{|z|^{m-n}}{1+|z|^{m-n}}, \varepsilon > 0, \delta > 0.
\]

(4.4)

We note that $f_{\delta \varepsilon} \in C^{1+\gamma}(\mathbb{R}^1)$ and $f_{\delta \varepsilon} D''_{\varepsilon} \in C^{1+\gamma}(\mathbb{R}^1)$ for some $\gamma \in (0, 1)$. The $\delta > 0$ in (4.4) makes the problem (4.1) regular (i.e. uniformly parabolic). The parameter $\varepsilon$ is an approximating parameter which has the effect of increasing the degeneracy from $f(h) \sim |h|^n$ to $f_{\varepsilon}(h) \sim h^4$. For $\varepsilon > 0$, the nonnegative initial data, $h_0$, is approximated via

\[
\begin{align*}
  h_0 + \varepsilon^\theta \leq h_{0, \varepsilon} \in C^{4+\gamma}(\Omega) \text{ for some } 0 < \theta < \frac{2}{5}, \\
  \frac{\partial h_{0, \varepsilon}}{\partial x^i}(-a) = \frac{\partial h_0}{\partial x^i}(a) \text{ for } i = 0, 3, \\
  h_{0, \varepsilon} \rightarrow h_0 \text{ strongly in } H^1(\Omega) \text{ as } \varepsilon \rightarrow 0.
\end{align*}
\]

(4.5)

The effect of $\varepsilon > 0$ in (4.5) is to both “lift” the initial data, making it positive, and to smooth the initial data from $H^1(\Omega)$ to $C^{4+\gamma}(\Omega)$.

By Eidelman [22, Theorem 6.3, p.302], the regularized problem has a unique classical solution $h_{\delta \varepsilon} \in C_{x,t}^{4+\gamma, 1+\gamma/4}(\Omega \times [0, \tau_{\delta \varepsilon}])$ for some time $\tau_{\delta \varepsilon} > 0$. For any fixed value of $\delta$ and $\varepsilon$, by Eidelman [22, Theorem 9.3, p.316] if one can prove a uniform in time an a priori bound $|h_{\delta \varepsilon}(x, t)| \leq A_{\delta \varepsilon} < \infty$ for some longer time interval $[0, T_{\text{loc, } \delta \varepsilon}]$ ($T_{\text{loc, } \delta \varepsilon} > \tau_{\delta \varepsilon}$) and for all $x \in \Omega$ then Schauder-type interior estimates [22, Corollary 2, p.213] imply that the solution $h_{\delta \varepsilon}$ can be continued in time to be in $C_{x,t}^{4+\gamma, 1+\gamma/4}(\Omega \times [0, T_{\text{loc, } \delta \varepsilon}])$.

Although the solution $h_{\delta \varepsilon}$ is initially positive, there is no guarantee that it will remain nonnegative. The goal is to take $\delta \rightarrow 0$, $\varepsilon \rightarrow 0$ in such a way that a) $T_{\text{loc, } \delta \varepsilon} \rightarrow T_{\text{loc}} > 0$, b) the solutions $h_{\delta \varepsilon}$ converge to a (nonnegative) limit, $h$, which is a generalized weak solution, and c) $h$ inherits certain a priori bounds. This is done by proving various a priori estimates for $h_{\delta \varepsilon}$ that are uniform in $\delta$ and $\varepsilon$ and hold on a time interval $[0, T_{\text{loc}}]$ that is independent of $\delta$ and $\varepsilon$. As a result, $\{h_{\delta \varepsilon}\}$ will be a uniformly bounded and equicontinuous (in the $C_{x,t}^{1/2, 1/8}$ norm) family of functions in $\bar{\Omega} \times [0, T_{\text{loc}}]$. Taking $\delta \rightarrow 0$ will result in a family of functions $\{h_\varepsilon\}$ that are classical, positive, unique solutions to the regularized problem with $\delta = 0$. Taking a subsequence of $\varepsilon$ going to zero will then result in the desired generalized weak
solution \( h \). This last step is where the possibility of nonunique weak solutions arise; see [4] for simple examples of how such a construction applied to \( h_t = -(|h|^n h_{xxx})_x \) can result in there being two different solutions arising from the same initial data.

4.2 A priori estimates

We start by proving a priori estimates for classical solutions of the regularized problem (4.1)–(4.5). Appendix A contains the proofs of the lemmas in this section.

We introduce function \( G_{\delta \varepsilon} \) chosen such that

\[
G_{\delta \varepsilon}''(z) = \frac{1}{f_{\delta \varepsilon}(z)} \quad \text{and} \quad G_{\delta \varepsilon}(z) \geq 0 \quad \forall z \in \mathbb{R}^1. \tag{4.6}
\]

This is analogous to the “entropy” function (3.1) introduced by Bernis and Friedman [7].

**Lemma 4.1.** Let \( n > 0 \) and \( m \geq n/2 \). There exist \( \delta_0 > 0, \varepsilon_0 > 0, \) and time \( T_{\text{loc}} > 0 \) such that if \( \delta \in [0, \delta_0], \varepsilon \in (0, \varepsilon_0) \) and \( h_{\delta \varepsilon} \) is a classical solution of the problem (4.1)–(4.5) with initial data \( h_{0, \delta \varepsilon} \) that is built from a nonnegative function \( h_0 \) satisfying the hypotheses of Theorem 4 then for any \( T \in [0, T_{\text{loc}}] \) the following inequalities hold true:

\[
\int_{\Omega} \{ h_{\delta \varepsilon, x}^2(x, T) + \frac{a_1}{2a_0} (1 + \delta) G_{\delta \varepsilon}(h_{\delta \varepsilon}(x, T)) \} \, dx \tag{4.7}
\]

\[
+ a_0 \int_{Q_T} f_{\delta \varepsilon}(h_{\delta \varepsilon}) h_{\delta \varepsilon, xxx}^2 \, dx \, dt \leq K_1 < \infty,
\]

\[
\int_{\Omega} G_{\delta \varepsilon}(h_{\delta \varepsilon}(x, T)) \, dx + \frac{a_0}{2} \int_{Q_T} h_{\delta \varepsilon, xx}^2 \, dx \, dt \leq K_2 < \infty. \tag{4.8}
\]

The energy \( E_{\delta \varepsilon}(t) \) (see (3.3)) satisfies:

\[
E_{\delta \varepsilon}(T) + \int_{Q_T} f_{\delta \varepsilon}(h_{\delta \varepsilon})(a_0 h_{\delta \varepsilon, xxx} + a_1 D''_{\varepsilon}(h_{\delta \varepsilon}) h_{\delta \varepsilon, x})^2 \, dx \, dt = E_{\delta \varepsilon}(0). \tag{4.9}
\]

The time \( T_{\text{loc}} \) and the constants \( K_1 \) and \( K_2 \) are independent of \( \delta \) and \( \varepsilon \).

The existence of \( \delta_0, \varepsilon_0, T_{\text{loc}}, K_1, K_2, \) and \( C_0 \) is constructive; how to find them and what quantities determine them is shown in Section A.

Lemma 4.1 yields uniform-in-\( \delta \)-and-\( \varepsilon \) bounds for \( \int h_{\delta \varepsilon, x}^2 \), \( \int G_{\delta \varepsilon}(h_{\delta \varepsilon}) \), \( \int f_{\delta \varepsilon}(h_{\delta \varepsilon}) h_{\delta \varepsilon, xxx}^2 \), and \( \int h_{\delta \varepsilon, xx}^2 \) which will be key in constructing a nonnegative generalized weak solution. However, these bounds are found in a different manner than in earlier work for the equation \( h_t = -(|h|^n h_{xxx})_x \), for example. Although the inequality (4.8) is unchanged, the inequality (4.7) has an extra term involving \( G_{\delta \varepsilon} \). In the proof, this term was introduced to control additional, lower–order terms. This idea of a “blended” \( \|h_x\|_2 \)-entropy bound was first introduced by Shishkov and Taranets especially for long-wave stable thin film equations with convection [34].
Lemma 4.2. Assume \( \varepsilon_0 \) and \( T_{loc} \) are from Lemma 4.1, \( \delta = 0 \), and \( \varepsilon \in (0, \varepsilon_0) \). If \( h_\varepsilon \) is a positive, classical solution of the problem (4.1)–(4.5) with initial data \( h_{0,\varepsilon} \) satisfying Lemma 4.1 then

\[
\int_\Omega h_{\varepsilon,x}^2(x,T) \, dx \leq e^{B_1,\varepsilon(T)} \int_\Omega h_{0,\varepsilon,x}^2 \, dx, \quad (4.10)
\]

\[
\int_\Omega \{h_{\varepsilon,x}^2(x,T) + G_\varepsilon(h_\varepsilon(x,T))\} \, dx \leq e^{B_2,\varepsilon(T)} \int_\Omega \{h_{0,\varepsilon,x}^2 + G_\varepsilon(h_{0,\varepsilon})\} \, dx \quad (4.11)
\]

hold true for all \( T \in [0, T_{loc}] \). Here

\[
B_{1,\varepsilon}(T) := \frac{a_1}{a_0} \int_0^T \|h_\varepsilon(\cdot,t)\|_{2m-n}^{2} dt,
\]

\[
B_{2,\varepsilon}(T) := \frac{a_1^2}{2a_0^2(2m-n+1)^2} \int_0^T \|h_\varepsilon(\cdot,t)\|_{\infty}^{2m-n} dt + \frac{a_1^2}{2a_0(2m-n+1)^2} \int_0^T \|h_\varepsilon(\cdot,t)\|_{\infty}^{2m-n} dt.
\]

The final a priori bound uses the following functions, parametrized by \( \alpha \), chosen such that \( G_\varepsilon^{(\alpha)} \geq 0 \) and \( G_\varepsilon^{(\alpha)''}(z) = z^\alpha/f_\varepsilon(z) \):

\[
G_\varepsilon^{(\alpha)}(z) = \begin{cases} 
\frac{z^{2n+\alpha}}{(2-n+\alpha)(1-n+\alpha)} + \varepsilon z^{\alpha-2} & \text{if } \alpha \in (-\infty, n-2) \cup (n-1, \infty) \\
\frac{z^{2n+\alpha}}{(2-n+\alpha)(1-n+\alpha)} + \varepsilon (\alpha-3)(\alpha-2) + z + c & \text{if } \alpha \in (n-2, n-1) \\
6 \ln z - z + 1 + \varepsilon  & \text{if } \alpha = n-1 \\
6 \ln z + z/\varepsilon + \varepsilon & \text{if } \alpha = n-2,
\end{cases}
\]

\[
(4.12)
\]

Lemma 4.3. Assume \( \varepsilon_0 \) and \( T_{loc} \) are from Lemma 4.1, \( \delta = 0 \), and \( \varepsilon \in (0, \varepsilon_0) \). Assume \( h_\varepsilon \) is a positive, classical solution of the problem (4.1)–(4.5) with initial data \( h_{0,\varepsilon} \) that is built from a nonnegative function \( h_0 \) satisfying the hypotheses of Theorem 2 then there exists \( K_3, \varepsilon_0^{(\alpha)} \) and \( T_{loc}^{(\alpha)} \) with \( 0 < \varepsilon_0^{(\alpha)} \leq \varepsilon_0 \) and \( 0 < T_{loc}^{(\alpha)} \leq T_{loc} \) such that

\[
\int_\Omega \{h_{\varepsilon,x}^2(x,T) + G_\varepsilon^{(\alpha)}(h_\varepsilon(x,T))\} \, dx + \int_{Q_T} \left[ \beta h_\varepsilon^n h_{\varepsilon,x}^2 + \gamma h_\varepsilon^{\alpha-2} h_{\varepsilon,x}^4 \right] \, dx \, dt \leq K_3 < \infty \quad (4.13)
\]

holds for all \( T \in [0, T_{loc}^{(\alpha)}] \). \( K_3 \) is determined by \( \alpha, \varepsilon_0, a_0, a_1, \Omega \) and \( h_0 \). Here,

\[
\beta = \begin{cases} 
\frac{a_0}{(1-\alpha)} & \text{if } 0 < \alpha < 1, \\
\frac{a_0}{4(1-\alpha)} & \text{if } -1/2 < \alpha < 0
\end{cases}
\]

and

\[
\gamma = \begin{cases} 
\frac{a_0}{(1-\alpha)} & \text{if } 0 < \alpha < 1, \\
\frac{a_0(1+2\alpha)}{36} & \text{if } -1/2 < \alpha < 0.
\end{cases}
\]
Furthermore, \( h_\varepsilon^2 \in B_{R_1}(0) \subset L^2(0,T_{loc};H^2(\Omega)) \) and \( h_\varepsilon^4 \in B_{R_2}(0) \subset L^2(0,T_{loc};W_4^1(\Omega)) \) (4.14) where the radii \( R_1 \) and \( R_2 \) are independent of \( \varepsilon \).

The \( \alpha \)-entropy, \( \int G_\alpha(h) \, dx \), was first introduced for \( \alpha = -1/2 \) in [13] and an a priori bound like that of Lemma 4.3 and regularity results like those of Theorem 2 were found simultaneously and independently in [4] and [11].

### 4.3 Proof of existence and regularity of solutions

Bound (4.7) yields uniform \( L^\infty \) control for classical solutions \( h_{\delta \varepsilon} \), allowing the time of existence \( T_{loc, \delta \varepsilon} \) to be taken as \( T_{loc} \) for all \( \delta \in (0,\delta_0) \) and \( \varepsilon \in (0,\varepsilon_0) \). The existence theory starts by constructing a classical solution \( h_{\delta \varepsilon} \) on \([0,T_{loc}]\) that satisfy the hypotheses of Lemma 4.1 if \( \delta \in (0,\delta_0) \) and \( \varepsilon \in (0,\varepsilon_0) \). The a priori bounds of Lemma 4.1 then allow one to take the regularizing parameter, \( \delta \), to zero and prove that there is a limit \( h_\varepsilon \) and that \( h_\varepsilon \) is a generalized weak solution. One then proves additional regularity for \( h_\varepsilon \); specifically that it is strictly positive, classical, and unique. It then follows that the a priori bounds given by Lemmas 4.1, 4.2, and 4.3 apply to \( h_\varepsilon \). This allows one to take the approximating parameter, \( \varepsilon \), to zero and construct the desired generalized weak solution of Theorems 1 and 2.

**Lemma 4.4.** Assume that the initial data \( h_{0,\varepsilon} \) satisfies (4.3) and is built from a nonnegative function \( h_0 \) that satisfies the hypotheses of Theorem 1. Fix \( \delta = 0 \) and \( \varepsilon \in (0,\varepsilon_0) \) where \( \varepsilon_0 \) is from Lemma 4.1. Then there exists a unique, positive, classical solution \( h_\varepsilon \) on \([0,T_{loc}]\) of problem \( (P_{0,\varepsilon}) \), see (4.1)-(4.3), with initial data \( h_{0,\varepsilon} \) where \( T_{loc} \) is the time from Lemma 4.1.

The proof uses a number of arguments like those presented by Bernis & Friedman [7] and we refer to that article as much as possible.

**Proof.** Fix \( \varepsilon \in (0,\varepsilon_0) \) and assume \( \delta \in (0,\delta_0) \). Because \( G_\delta(x) \geq 0 \), the bound (4.7) yields a uniform-in-\( \delta \)-and-\( \varepsilon \) upper bound on \( |h_{\delta \varepsilon}(x,T)| \) for \((x,T) \in \Omega \times [0,T_{loc}] \). As discussed in Subsection 4.1, this allows the classical solution \( h_{\delta \varepsilon} \) to be extended from \([0,\tau_{\delta \varepsilon}] \) to \([0,T_{loc}] \).

By Section 2 of [7], the a priori bound (4.7) on \( \|h_x(\cdot,T_{loc})\|_2 \) implies that \( h_{\delta \varepsilon} \in C_{x,t}^{1/2,1/8}(\Omega_{T_{loc}}) \) and that \( \{h_{\delta \varepsilon}\} \) is a uniformly bounded, equicontinuous family in \( \Omega_{T_{loc}} \). By the Arzela-Ascoli theorem, there is a subsequence \( \{\delta_k\} \), so that \( h_{\delta_k \varepsilon} \) converges uniformly to a limit \( h_\varepsilon \in C_{x,t}^{1/2,1/8}(\Omega_{T_{loc}}) \).

We now argue that \( h_\varepsilon \) is a generalized weak solution, using methods similar to those of [7, Theorem 3.1]. By construction, \( h_\varepsilon \) is in \( C_{x,t}^{1/2,1/8}(\Omega_{T_{loc}}) \), satisfying the first part of (2.4). The strong convergence \( h_{\delta_k \varepsilon}(\cdot,t) \rightarrow h_\varepsilon(\cdot,t) \) in \( L^2(\Omega) \) follows immediately. The uniform convergence of \( h_{\delta_k \varepsilon} \) to \( h_\varepsilon \) implies the pointwise convergence \( h(\cdot,t) \rightarrow h(\cdot,0) = h_0 \), and so \( h_\varepsilon \) satisfies (2.8).
Because $h_{\delta}\varepsilon$ is a classical solution,

$$\int\int_{Q_T} h_{\delta}\varepsilon,t \phi \, dx dt - \int\int_{Q_T} f_{\delta}(h_{\delta}\varepsilon)(a_0 h_{\delta\varepsilon,xxx} + a_1 D''_{\varepsilon}(h_{\delta}\varepsilon) h_{\delta\varepsilon,x}) \phi_x \, dx dt = 0. \quad (4.15)$$

The bound (4.7) yields a uniform bound on

$$\delta \int\int_{Q_T_{loc}} h_{\delta\varepsilon,xxx}^2 \, dx dt$$

for $\delta \in (0, \delta_0)$. It follows that

$$\delta_k \int\int_{Q_T_{loc}} (a_0 h_{\delta\varepsilon,xxx} + a_1 D''_{\varepsilon}(h_{\delta}\varepsilon) h_{\delta\varepsilon,x}) \phi_x \, dx dt \to 0 \quad \text{as } \delta_k \to 0.$$

Introducing the notation

$$H_{\delta}\varepsilon := f_{\delta}(h_{\delta}\varepsilon)(a_0 h_{\delta\varepsilon,xxx} + a_1 D''_{\varepsilon}(h_{\delta}\varepsilon) h_{\delta\varepsilon,x}) \quad (4.16)$$

the integral formulation (4.15) can be written as

$$\int\int_{Q_T} h_{\delta\varepsilon,t} \phi \, dx dt = \int\int_{Q_T_{loc}} H_{\delta\varepsilon}(x,t) \phi_x(x,t) \, dx dt. \quad (4.17)$$

By the $L^\infty$ control of $h_{\delta\varepsilon}$ and the energy bound (4.9), $H_{\delta\varepsilon}$ is uniformly bounded in $L^2(Q_{T_{loc}})$. Taking a further subsequence of $\{\delta_k\}$ yields $H_{\delta\varepsilon}$ converging weakly to a function $H_{\varepsilon}$ in $L^2(Q_{T_{loc}})$. The regularity theory for uniformly parabolic equations implies that $h_{\delta\varepsilon,t}$, $h_{\delta\varepsilon,x}$, $h_{\delta\varepsilon,xx}$, $h_{\delta\varepsilon,xxx}$, and $h_{\delta\varepsilon,xxxx}$ converge uniformly to $h_{\varepsilon,t}$, $h_{\varepsilon,x}$, $h_{\varepsilon,xx}$, $h_{\varepsilon,xxx}$, and $h_{\varepsilon,xxxx}$ on any compact subset of $\{h_{\varepsilon} > 0\}$, implying (2.9) and the first part of (2.6). Note that because the initial data $h_{0,\delta\varepsilon}$ is in $C^4$ the regularity extends all the way to $t = 0$ which is excluded in the definition of $\mathcal{P}$ in (2.6).

The energy $E_{\delta\varepsilon}(T_{loc})$ is not necessarily positive. However, the a priori bound (4.7), combined with the $L^\infty$ control on $h_{\delta\varepsilon}$, ensures that $E_{\delta\varepsilon}(T_{loc})$ has a uniform lower bound. As a result, the bound (4.9) yields a uniform bound on

$$\int\int_{Q_{T_{loc}}} f_{\delta}(h_{\delta\varepsilon})(a_0 h_{\delta\varepsilon,xxx} + a_1 D''_{\varepsilon}(h_{\delta}\varepsilon) h_{\delta\varepsilon,x})^2 \, dx dt.$$

Using this, one can argue that for any $\sigma > 0$

$$\int\int_{\{h_i < \sigma\}} |H_{\delta\varepsilon} \phi_x| \, dx dt \leq C \sigma^{n/2}$$

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for some $C$ independent of $\delta, \varepsilon,$ and $\sigma$. Taking $\delta_k \to 0$ and using that $\sigma$ is arbitrary, we conclude

$$H_{\delta_k \varepsilon} \to H_\varepsilon = f_\varepsilon(h_\varepsilon)(a_0 h_{\varepsilon, xxx} + a_1 D^n_{x}(h_{\delta_k\varepsilon})h_{\varepsilon, x}) \chi(h_{\varepsilon} > 0).$$

As a result, taking $\delta_k \to 0$ in (4.17) implies $h_\varepsilon$ satisfies (2.7) and the second part of (2.6).

The bound (4.7) yields a uniform bound on $\int h^2_{\delta_k\varepsilon}(x, T) \, dx$ for every $T \in [0, T_{\text{loc}}]$. As a result, $\{h_{\delta_k \varepsilon}\}$ is uniformly bounded in

$$L^\infty(0, T_{\text{loc}}; H^1(\Omega)).$$

Therefore, another refinement of the sequence $\{\delta_k\}$ yields $\{h_{\delta_k \varepsilon}\}$ weakly convergent in this space. As a result, $h_\varepsilon \in L^\infty(0, T_{\text{loc}}; H^1(\Omega))$ and the second part of (2.4) holds.

Having proven then $h_\varepsilon$ is a generalized weak solution, we now prove that $h_\varepsilon$ is a strictly positive, classical, unique solution. This uses the entropy $\int G_{\delta_k}(h_\varepsilon)$ and the a priori bound (4.8). This bound is, up to the coefficient $a_0$, identical to the a priori bound (4.17) in [7]. By construction, the initial data $h_{0\varepsilon}$ is positive (see (3.1)), hence $\int G_{\varepsilon}(h_{0 \varepsilon}) \, dx < \infty$. Also, by construction $f_\varepsilon(z) \approx z^4$ for $z \ll 1$. This implies that the generalized weak solution $h_\varepsilon$ is strictly positive [7] Theorem 4.1]. Because the initial data $h_{0 \varepsilon}$ is in $C^4(\bar{\Omega})$, it follows that $h_\varepsilon$ is a classical solution in $C^{4,1}_{x,t}(\bar{Q}_{T_{\text{loc}}})$. This implies that $h_\varepsilon(\cdot, t) \to h_\varepsilon(\cdot, 0)$ strongly\(^3\) in $C^1(\Omega)$. The proof of Theorem 4.1 in [7] then implies that $h_\varepsilon$ is unique.

Proof of Theorem [7]. As in the proof of Lemma 4.4, following [7], there is a subsequence $\{\varepsilon_k\}$ such that $h_{\varepsilon_k}$ converges uniformly to a function $h \in C^{1/2, 1/8}_{x,t}$ which is a generalized weak solution in the sense of Definition 2.1 with $f(h) = |h|^n$.

The initial data is assumed to have finite entropy: $\int G_0(h_0) < \infty$ where $G_0$ is given by (3.1). This, combined with $f(h) = |h|^n$, implies that the generalized weak solution $h$ is nonnegative and, if $n \in [2, 4]$ the set of points $\{h = 0\}$ in $Q_{T_{\text{loc}}}$ has zero measure and $h$ is positive, smooth, and unique if $n \geq 4$ [7] Theorem 4.1].

To prove (3.6), start by taking $T = T_{\text{loc}}$ in the a priori bound (4.9). As $\varepsilon_k \to 0$, the right-hand side of (4.9) is unchanged. Now, consider the $\varepsilon_k \to 0$ limit of

$$\mathcal{E}_{\varepsilon_k}(T_{\text{loc}}) = \int_\Omega \left( \frac{a_0}{2} h_{\varepsilon_k}^2(x, T_{\text{loc}}) - a_1 D_{x} (h_{\varepsilon_k}(x, T_{\text{loc}})) \right) \, dx,$$

where $D_x$ is defined in (4.4). By the uniform convergence of $h_{\varepsilon_k}$ to $h$, the second term in the energy converges strongly as $\varepsilon_k \to 0$. Hence the bound (4.9) yields a uniform bound on $\{\int_\Omega h_{\varepsilon_k}^2(x, T_{\text{loc}}) \, dx\}$. Taking a further refinement of $\{\varepsilon_k\}$, yields $h_{\varepsilon_k}(\cdot, T_{\text{loc}})$ converging weakly in $L^2(\Omega)$. In a Hilbert space, the norm of the weak limit is less than or equal to the lim inf of the norms of the functions in the sequence, hence $\mathcal{E}_0(T_{\text{loc}}) \leq \liminf_{\varepsilon_k \to 0} \mathcal{E}_{\varepsilon_k}(T_{\text{loc}})$. A uniform bound on $\int f_{\varepsilon}(h_{\varepsilon_k}) (a_0 h_{\varepsilon_k, xxx} + \ldots)^2 \, dx$ also follows from (4.9). Hence $\sqrt{f_{\varepsilon_k}(h_{\varepsilon_k})} (a_0 h_{\varepsilon_k, xxx} + \ldots)$ converges weakly in $L^2(Q_{T_{\text{loc}}})$, after taking a further subsequence. We write the weak limit as two integrals: one over $\{h = 0\}$ and one over

\(^3\) Unlike the definition of weak solution given in [7], Definition 2.1 does not include that the solution converges to the initial data strongly in $H^1(\Omega)$. 

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\( \{h > 0\} \). We can determine the weak limit on \( \{h > 0\} \): as in the proof of Lemma 4.4, the regularity theory for uniformly parabolic equations allows one to argue that the weak limit is \( h^{n/2} (a_0 h_{xxx} + \ldots) \) on \( \{h > 0\} \). Using that 1) the norm of the weak limit is less than or equal to the lim inf of the norms of the functions in the sequence and that 2) the lim inf of a sum is greater than or equal to the sum of the lim infs and dropping the nonnegative term arising from the integral over \( \{h = 0\} \) results in the desired bound (3.6).

It follows from (4.8) that \( \{h \varepsilon_k \} \) converges weakly to some \( v \) in \( L^2(Q_{T_{loc}}) \), combining with strong convergence in \( L^2(0,T;H^1(\Omega)) \) of \( h \varepsilon_k \) to \( h \) by Lemma [D.1] and with the definition of weak derivative, we obtain that \( v = h_{xx} \) and \( h \in L^2(0,T_{loc};H^2(\Omega)) \) that implies (3.3). Hence \( h_{\varepsilon,t} \rightarrow h_t \) weakly in \( L^2(0,T; (H^1(\Omega))' ) \) that implies (2.5). By Lemma [D.2] we also have \( h \in C([0,T_{loc}],L^2(\Omega)) \).

**Proof of Theorem 2.** Fix \( \alpha \in (-\frac{1}{2},1) \). The initial data \( h_0 \) is assumed to have finite entropy \( \int G_0^{(\alpha)}(h_0(x)) \, dx < \infty \), hence Lemma 4.3 holds for the approximate solutions \( \{h \varepsilon_k \} \) where this sequence of approximate solutions is assumed to be the one at the end of the proof of Theorem 1. By (4.14),

\[
\left\{ \frac{\alpha+2}{4} h \varepsilon_k^{\frac{1}{2}} \right\} \text{ is uniformly bounded in } \varepsilon_k \text{ in } L^2(0,T_{loc};H^2(\Omega))
\]

and

\[
\left\{ \frac{\alpha+2}{4} h \varepsilon_k^{\frac{1}{4}} \right\} \text{ is uniformly bounded in } \varepsilon_k \text{ in } L^2(0,T_{loc};W_{1,4}^1(\Omega)).
\]

Taking a further subsequence in \( \{\varepsilon_k\} \), it follows from the proof of [17, Lemma 2.5, p.330] that these sequences converge weakly in \( L^2(0,T_{loc};H^2(\Omega)) \) and \( L^2(0,T_{loc};W_{1,4}^1(\Omega)) \), to \( h^{\frac{\alpha+2}{2}} \) and \( h^{\frac{\alpha+2}{4}} \) respectively.

**5 The subcritical regime: long–time existence of solutions**

**Lemma 5.1.** Let \( h \in H^1(\Omega) \) be a nonnegative function and let \( M = \int h(x) \, dx \). Then

\[
\|h\|_{L^p(\Omega)}^p \leq k_1 M^{\frac{p+2}{3}} \left( \int h_2^2 \, dx \right)^{\frac{p-1}{3}} + k_2 M^p, \quad p \geq 1,
\]

where \( k_1 = 2^{\frac{4-p}{3}} \frac{2(p-1)}{3} (1 - \epsilon)^{-1} \), \( k_2 = |\Omega|^{1-p}(1 - (1 - \epsilon)^{-p}) \), and \( \epsilon \in (0,1) \).

Note that by taking \( h \) to be a constant function, one finds that the constant \( k_2 M^p \) in (5.1) is sharp when \( \epsilon \rightarrow 1 \).
Proof. Let \( v = h - M/|\Omega| \). By (A.3),
\[
\| v \|_{L^p(\Omega)}^p \leq \left( \frac{3}{2} \right)^{2(p-1)} \left( \int_{\Omega} v_x^2 \, dx \right)^{\frac{p-1}{3}} \left( \int_{\Omega} |v| \, dx \right)^{\frac{p+2}{3}}.
\]

Hence, due to the inequality
\[
|a - b|^p \geq (1 - \epsilon)a^p - c_0(\epsilon, p)b^p
\]
for any \( a \geq 0, \, b \geq 0, \, p > 1, \, \epsilon \in (0, 1), \, c_0(\epsilon, p) \geq \frac{1}{(1-(1-\epsilon)^{p-1})^{p-1}},
\]
\[
\| h \|_{L^p(\Omega)}^p \leq \frac{1}{1-\epsilon} \left( \frac{3}{2} \right)^{2(p-1)} \left( \int_{\Omega} h_x^2 \, dx \right)^{\frac{p-1}{3}} \left( \int_{\Omega} |h - M/|\Omega|| \, dx \right)^{\frac{p+2}{3}} + \frac{1}{1-\epsilon} \frac{M^p}{|\Omega|^{p-1}} \leq \frac{1}{1-\epsilon} \left( \frac{3}{2} \right)^{2(p-1)} \left( \int_{\Omega} h_x^2 \, dx \right)^{\frac{p-1}{3}} (2M)^{\frac{p+2}{3}} + \frac{1}{1-\epsilon} \frac{M^p}{|\Omega|^{p-1}}.
\]

Lemma 5.2. Let \( 0 < n/2 \leq m < n + 2 \). Let \( h \) be the generalized solution of Theorem 1. Then
\[
\frac{\alpha_n}{4} \| h(., T_{loc}) \|^2_{H^1(\Omega)} \leq \mathcal{E}_0(0) + c_1 M^{\frac{m-n+1}{2-m+n}} + c_2 M^{m-n+2} + c_3 M^2,
\]
where \( \mathcal{E}_0(0) \) is defined in (3.5), and \( M = \int h_0 \). Moreover, if \( m = n + 2 \) and \( 0 < M < M_c \) then
\[
\frac{\alpha_n}{4} \| h(., T_{loc}) \|^2_{H^1(\Omega)} \leq \frac{2}{3(1-c_4 M^4)} \mathcal{E}_0(0) + \frac{c_5}{1-c_4 M^4} M^4 + c_6 M^2.
\]

Proof of Lemma 5.2. We present the proof for the \( m - n \neq -1, -2 \) case in (3.4), leaving the \( m - n = -1, -2 \) cases to the reader. The first step is to find a priori bound (5.2) that is the analogue of Proposition 2.2 in [12]. From (3.6) we deduce
\[
\frac{\alpha_n}{2} \int_{\Omega} h_x^2 \, dx \leq \mathcal{E}_0(0) + \frac{\alpha_n}{(m-n+1)(m-n+2)} \int_{\Omega} h^{m-n+2} \, dx.
\]

Due to (5.1), we have
\[
\int_{\Omega} h^{m-n+2} \, dx \leq k_1 M^{\frac{m-n+4}{3}} \left( \int_{\Omega} h_x^2 \, dx \right)^{\frac{m-n+1}{3}} + k_2 M^{m-n+2}.
\]

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Thus, from (5.4), in view of (5.5), we find that
\[
\frac{a_0}{2} \int_{\Omega} h_x^2 \, dx - \frac{a_1 k_1 M}{3 (m-n+1)(m-n+2)} \left( \int_{\Omega} h_x^2 \, dx \right)^{\frac{m-n+1}{3}} \leq \mathcal{E}_0(0) + \frac{a_1 k_2 M^{m-n+2}}{(m-n+1)(m-n+2)}. \tag{5.6}
\]
Moreover, due to (5.1), in view of the Young inequality (A.6), we have
\[
\int_{\Omega} h_x^2 \, dx \leq \frac{2}{3} M^\frac{4}{3} \left( \int_{\Omega} h_x^2 \, dx \right)^{\frac{1}{3}} + \frac{M^2}{|\Omega|} \leq \frac{1}{4} \int_{\Omega} h_x^2 \, dx + \left( \frac{2\sqrt{3}}{3} + |\Omega|^{-1} \right) M^2. \tag{5.7}
\]
In the subcritical \((m < n+2)\) case, \(\frac{m-n+1}{3} < 1\) and using (5.6) and (5.7) we deduce (5.2) with
\[
\begin{align*}
c_1 &= \left( \frac{a_1 k_1}{(m-n+1)(m-n+2)} \right)^{\frac{3}{2-m+n}} \left( \frac{8(m-n+1)}{3a_0} \right)^{\frac{m-n+1}{2-m+n}} \frac{2-m+n}{3}, \\
c_2 &= \frac{a_1 k_2}{(m-n+1)(m-n+2)}, \\
c_3 &= \frac{a_0}{2} \left( \frac{8\sqrt{3}}{3} + |\Omega|^{-1} \right).
\end{align*}
\tag{5.8}
\]
In the critical \((m = n+2)\) case, \(\frac{m-n+1}{3} = 1\). If \(M < \frac{a_0}{a_1 k_1} \frac{1}{2}\) then using (5.6) we arrive at
\[
\int_{\Omega} h_x^2 \, dx \leq \frac{12}{6a_0 - a_1 k_1 M^2} \left( \mathcal{E}_0(0) + \frac{a_1 k_2}{12} M^4 \right). \tag{5.9}
\]
Using (5.7), from (5.9) we obtain (5.3) with
\[
\begin{align*}
c_4 &= \frac{a_1 k_1}{6a_0}, \\
c_5 &= \frac{a_1 k_2}{18}, \\
c_6 &= \frac{a_0}{3} \left( \frac{8\sqrt{3}}{3} + |\Omega|^{-1} \right).
\end{align*}
\tag{5.10}
\]
Under certain conditions, a bound closely related to (5.2) implies that if the solution of Theorem 4 is initially constant then it will remain constant for all time:

**Theorem 5.** Assume \(m = n\), the coefficient \(a_1 \geq 0\) in (2.1), and \(|\Omega|^2 < a_0 / |a_1|\). If the initial data is constant, \(h_0 \equiv C > 0\), then the solution of Theorem 4 satisfies \(h(x, t) = C\) for all \(x \in \Omega\) and all \(t > 0\).

For the long-wave unstable case \((a_1 > 0)\) the hypotheses correspond to the domain not being “too large”. We note that it is not yet known whether or not solutions from Theorem 4 are unique and so Theorem 5 does have content: it ensures that the approximation method isn’t producing unexpected (nonconstant) solutions from constant initial data.

**Proof.** Consider the approximate solution \(h_\varepsilon\). The definition of \(\mathcal{E}_\varepsilon(T)\) combined with the uniform-in-time bound (4.9) implies
\[
\frac{a_0}{2} \int_{\Omega} h_{x,x}^2(x, T) \, dx \leq \mathcal{E}_\varepsilon(0) + \frac{|a_1|}{2} \int_{\Omega} h_x^2(x, T) \, dx. \tag{5.11}
\]
Letting $M_\varepsilon = \int h_{0,\varepsilon} \, dx$ and applying Poincaré’s inequality (A.2) to $v_\varepsilon = h_\varepsilon - M_\varepsilon/|\Omega|$ and using $\int h_\varepsilon^2 \, dx = \int v_\varepsilon^2 \, dx + M_\varepsilon^2/|\Omega|$ yields

$$(a_0/2 - |a_1||\Omega|^2/2) \int_{\Omega} h_{\varepsilon,x}^2(x,t) \, dx \leq E_\varepsilon(0) + |a_1|M_\varepsilon^2/2|\Omega|.$$ 

If $h_{0,\varepsilon} \equiv C_\varepsilon = C + \varepsilon^\theta$ this becomes

$$(a_0/2 - |a_1||\Omega|^2/2) \int_{\Omega} h_{\varepsilon,x}^2(x,T) \, dx \leq (|a_1| - a_1)C_\varepsilon^2|\Omega|/2.$$ 

If $a_1 > 0$ and $|\Omega|^2 < a_0/|a_1|$ then $\int h_{\varepsilon,x}^2(x,T) \, dx = 0$ for all $T \in [0,T_{\varepsilon,loc}]$ and that this, combined with the continuity in space and time of $h_\varepsilon$, implies that $h_\varepsilon \equiv C_\varepsilon$ on $Q_{T_{\varepsilon,loc}}$.

Taking the sequence $\{\varepsilon_k\}$ that yields convergence to the solution $h$ of Theorem 1, $h \equiv C$ on $Q_{T_{loc}}$.

This $H^1$ control in time of the generalized solution given by Lemma 5.2 is now used to extend the short–time existence result of Theorem 1 to a long–time existence result:

**Theorem 6.** Let $0 < n/2 \leq m < n + 2$. Let $T_g$ be an arbitrary positive finite number. The generalized weak solution $h$ of Theorem 1 can be continued in time from $[0,T_{loc}]$ to $[0,T_g]$ in such a way that $h$ is also a generalized weak solution and satisfies all the bounds of Theorem 1 (with $T_{loc}$ replaced by $T_g$).

Similarly, the short–time existence of strong solutions (see Theorem 2) can be extended to long–time existence.

**Proof.** To construct a weak solution up to time $T_g$, one applies the local existence theory iteratively, taking the solution at the final time of the current time interval as initial data for the next time interval.

Introduce the times

$$0 = T_0 < T_1 < T_2 < \cdots < T_N < \cdots \quad \text{where} \quad T_N := \sum_{n=0}^{N-1} T_{n,loc} \quad (5.12)$$

and $T_{n,loc}$ is the interval of existence (A.21) for a solution with initial data $h(\cdot,T_n)$:

$$T_{n,loc} := \frac{9}{20c_{11}(\gamma_1 - 1)^{1/3}} \min \left\{ 1, \left( \int_{\Omega} h_x^2(x,T_n) + \frac{2a_2}{a_0} G_0(h(x,T_n)) \, dx \right)^{-(\gamma_1 - 1)} \right\} \quad (5.13)$$

where $\gamma_1 = \max\{3,2m - n\}$ and $c_2$ and $c_{11}$ are given in the proof of Lemma 4.1.
The proof proceeds by contradiction. Assume there exists initial data \( h_0 \), satisfying the hypotheses of Theorem 1, that results in a weak solution that cannot be extended arbitrarily in time:

\[
\sum_{k=0}^{\infty} T_{n,loc} = T^* < \infty \implies \lim_{n \to \infty} T_{n,loc} = 0.
\]

From the definition (5.13) of \( T_{n,loc} \), this implies

\[
\lim_{n \to \infty} \int_{\Omega} \left( h_0^2(x, T_n) + \frac{2c_2}{a_0} G_0(h(x, T_n)) \right) \ dx = \infty. \tag{5.14}
\]

By (5.2),

\[
\frac{a_0}{4} \int_{\Omega} h_0^2(x, T_n) \ dx \leq \mathcal{E}_0(T_{n-1}) + K,
\]

where \( K = c_1 M^{\frac{m-n+4}{2-m+n}} + c_2 M^{m-n+2} + c_3 M^2 \). By (3.6),

\[
\mathcal{E}_0(T_{n-1}) \leq \mathcal{E}_0(T_{n-2}) \leq \ldots \leq \mathcal{E}_0(0).
\]

Combining these,

\[
\frac{a_0}{4} \int_{\Omega} h_0^2(x, T_n) \ dx \leq \mathcal{E}_0(0) + K. \tag{5.15}
\]

By assumption, \( T_n \to T^* < \infty \) as \( n \to \infty \) hence \( \int h_0^2(x, T_n) \) remains bounded. Assumption (5.14) then implies that \( \int G_0(h(x, T_n)) \to \infty \) as \( n \to \infty \).

To continue the argument, we step back to the approximate solutions \( h_\varepsilon \). Let \( T_{n,\varepsilon} \) be the analogue of \( T_n \) and \( T_{n,loc,\varepsilon} \), defined by (A.20), be the analogue of \( T_{n,loc} \). By (A.16),

\[
\int_{\Omega} G_\varepsilon(h_\varepsilon(x, T_{n,\varepsilon})) \ dx \leq \int_{\Omega} G_\varepsilon(h_\varepsilon(x, T_{n-1,\varepsilon})) \ dx
\]

\[
+ c_{10} \int_{T_{n-1,\varepsilon}}^{T_{n,\varepsilon}} \max_{T_{n-1,\varepsilon}} \left\{ 1, \left( \int_{\Omega} h_{\varepsilon}^2(x, T) \ dx \right)^{\gamma_2} \right\} dT
\]

with \( \gamma_2 = 3 \). Using the bound (4.9), one can prove the analogue of Lemma 5.2 for the approximate solution \( h_\varepsilon \). However the bound (5.2) would be replaced by a bound on \( \|h_\varepsilon(\cdot, T)\|_{H^1} \) which holds for all \( T \in [0, T_{n,loc}] \). This bound would then be used to prove a bound like (5.15) to prove boundedness of \( \int h_{\varepsilon,\varepsilon}^2(x, T) \) for all \( T \in [0, T_{n,\varepsilon}] \). Using this bound,

\[
\int_{T_{n-1,\varepsilon}}^{T_{n,\varepsilon}} \left( \int_{\Omega} h_{\varepsilon}^2(x, T) \ dx \right)^{\gamma_2} dT \leq \left( \frac{4}{a_0} \right)^{\gamma_2} \int_{T_{n-1,\varepsilon}}^{T_{n,\varepsilon}} (\mathcal{E}_\varepsilon(0) + K)^{\gamma_2} dT
\]

\[
= \left( \frac{4}{a_0} \right)^{\gamma_2} (\mathcal{E}_\varepsilon(0) + K)^{\gamma_2} T_{n-1,loc,\varepsilon}. \tag{5.17}
\]
If the initial data is such that $4/a_0 \ (E_\varepsilon(0) + K) < 1$ then before using (5.17) in (5.16) we replace $K$ by a larger value so that $4/a_0 \ (E_\varepsilon(0) + K) > 1$. Using (5.17) in (5.16), it follows that

$$\int_\Omega G_\varepsilon(h_\varepsilon(x, T_{n,\varepsilon})) \, dx \leq \int_\Omega G_\varepsilon(h_\varepsilon(x, T_{n-1,\varepsilon})) \, dx + \beta T_{n-1,loc,\varepsilon} \quad (5.18)$$

for $\beta = c_{10}(4/a_0)^{\gamma_2} \ (E_\varepsilon(0) + K)^{\gamma_2}$. Here $\beta$ depends on $|\Omega|$, the coefficients of the PDE, and on the initial data $h_{0,\varepsilon}$.

One now takes the sequence $\{\varepsilon_k\}$ that was used to construct the weak solution of Theorem 1 on the interval $[T_{n-1}, T_n]$. Taking $\varepsilon_k \to 0$, (5.18) yields

$$\int_\Omega G_0(h(x, T_n)) \, dx \leq \int_\Omega G_0(h(x, T_{n-1})) \, dx + \beta T_{n-1,loc}. \quad (5.19)$$

Applying (5.19) iteratively,

$$\int_\Omega G_0(h(x, T_n)) \, dx \leq \int_\Omega G_0(h_0(x)) \, dx + \beta \sum_{k=0}^{n-1} T_{k,loc}$$

$$= \int_\Omega G_0(h_0(x)) \, dx + \beta T_n.$$

This upper bound proves that $\int G_0(h(x, T_n))$ cannot diverge to infinity as $n \to \infty$, finishing the proof.

6 Strong positivity

Proposition 6.1. Let $m \geq n/2$. Assume the initial data $h_0$ satisfies $h_0(x) > 0$ for all $x \in \omega \subseteq \Omega$ where $\omega$ is an open interval. Then

1. if $n > 3/2$ and $\alpha \in (-1/2, \min\{1, n - 2\})$ then the strong solution from Theorem 1 satisfies $h(x, T) > 0$ for almost every $x \in \omega$, for all $T \in [0, T_{loc}^{(\alpha)}]$;

2. if $n > 2$ and $\alpha \in (-1/2, \min\{1, 3n/4 - 2\})$ then the strong solution from Theorem 1 satisfies $h(x, T) > 0$ for all $x \in \omega$, for almost every $T \in [0, T_{loc}^{(\alpha)}]$;

3. if $n \geq 7/2$ and $m \geq n - 1/2$ then the strong solution from Theorem 1 satisfies $h(x, T) > 0$ for all $(x, T) \in \overline{Q_{T_{loc}^{(\alpha)}}}$.

The proof of Proposition 6.1 depends on a local version of the a priori bound (4.13) of Lemma 4.3.
Lemma 6.1. Let $\omega \subseteq \Omega$ be an open interval and $\zeta \in C^2(\overline{\Omega})$ such that $\zeta > 0$ on $\omega$, $\text{supp } \zeta = \overline{\omega}$, and $(\zeta')' = 0$ on $\partial \Omega$. If $\omega = \Omega$, choose $\zeta$ such that $\zeta(-a) = \zeta(a) > 0$. Let $\xi := \zeta^4$.

If the initial data $h_0$ and the time $T_{\text{loc}}^{(a)}$ are as in Theorem 2 then for all $T \in [0, T_{\text{loc}}^{(a)}]$ the strong solution $h$ from Theorem 2 satisfies

$$\int_{\Omega} \xi(x) C_0^{(a)}(h(x, T)) \, dx < \infty \quad (6.1)$$

The proof of Lemma 6.1 is given in Appendix A. The proof of Proposition 6.1 is essentially a combination of the proofs of Theorem 6.1 and Corollary 4.5 in [7] and is provided here for the reader’s convenience.

Proof of Proposition 6.1. Choose the test function $\xi(x)$ to satisfy the hypotheses of Lemma 6.1. Hence, (6.1) holds for every $T \in [0, T_{\text{loc}}^{(a)}]$.

Proof of 1): Assume it is not true that $h(x, T) > 0$ for almost every $x \in \omega$, for all $T \in [0, T_{\text{loc}}^{(a)}]$. Then there is a time $T \in [0, T_{\text{loc}}^{(a)}]$ such that the set $\{x : h(x, T) = 0\} \cap \omega$ has positive measure. Then because $\alpha - n + 2 < 0$,

$$\infty > \int_{\Omega} \xi(x) h^{\alpha - n + 2}(x, T) \, dx \geq \int_{\{h(\cdot, T) = 0\} \cap \omega} \xi(x) h^{\alpha - n + 2}(x, T) \, dx = \infty.$$ 

This contradiction implies there can be no time at which $h$ vanishes on a set of positive measure in $\omega$, as desired.

Proof of 2): We start by noting that by (3.12), $(h^{\alpha+2} x x \xi (\cdot, T) \in L^2(\Omega)$ for almost all $T \in [0, T_{\text{loc}}^{(a)}]$ hence $h^{\frac{(\alpha+2)}{2}}(\cdot, T) \in C^{3/2}(\Omega)$ for almost all $T \in [0, T_{\text{loc}}^{(a)}]$. To prove 2), it therefore suffices to show that if $T$ such that $h^{\frac{(\alpha+2)}{2}}(\cdot, T) \in C^{3/2}(\Omega)$ then $h(x, T) > 0$ on $\omega$. Assume this is not true and there is an $x_0 \in \omega$ and $T_0$ such that $h(x_0, T_0) = 0$ and $h^{\frac{(\alpha+2)}{2}}(\cdot, T_0) \in C^{3/2}(\Omega)$. Then there is a $L$ such that

$$h^{\frac{(\alpha+2)}{2}}(x, T_0) = |h^{\frac{(\alpha+2)}{2}}(x, T_0) - h^{\frac{(\alpha+2)}{2}}(x_0, T_0)| \leq L|x - x_0|^{3/2}.$$ 

Hence

$$\infty > \int_{\Omega} \xi(x) h^{\alpha - n + 2}(x, T_0) \, dx \geq L \frac{2(\alpha - n + 2)}{\alpha + 2} \int_{\Omega} \xi(x)|x - x_0|^{\frac{3(\alpha - n + 2)}{\alpha + 2}} \, dx = \infty.$$ 

This contradiction implies there can be no point $x_0$ such that $h(x_0, T_0) = 0$, as desired. Note that we used $\xi > 0$ on $\omega$ and $x_0 \in \omega$ to conclude that the integral diverges.

Proof of 3): Taking $\alpha = -\frac{1}{2}$ in (A.43), the approximate solution $h_\varepsilon$ satisfies

$$\int_{\Omega} G_{\varepsilon^{-1/2}}(h_\varepsilon(x, T)) \, dx \leq \int_{\Omega} G_{\varepsilon^{-1/2}}(h_{0\varepsilon}) \, dx + \int_{Q_T} h_{\varepsilon}^{m-n-\frac{1}{2}} h_{\varepsilon,x}^2 \, dx dt. \quad (6.2)$$
Now we use the estimate
\[
\int_Q h_{\varepsilon}^{m-n-\frac{1}{2}} h_{\varepsilon,x}^2 \, dx \, dt = -\frac{2}{2m-2n+1} \int_Q h_{\varepsilon}^{m-n+\frac{1}{2}} h_{\varepsilon,xx} \, dx \, dt \\
\leq \frac{2}{2m-2n+1} \left( \int_Q h_{\varepsilon,xx}^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_Q h_{\varepsilon}^{2m-2n+1} \, dx \, dt \right)^{\frac{1}{2}}.
\]

Using \( m \geq n - 1/2 \) and the \( L^\infty \) bound on \( h_{\varepsilon} \) from (1.7) as well as (4.8), we obtain
\[
\int_Q h_{\varepsilon}^{m-n-\frac{1}{2}} h_{\varepsilon,x}^2 \, dx \, dt \leq 2^{2m-n} \int_Q h_{\varepsilon}^{2m-2n+1} \, dx \, dt \leq C \sqrt{T}. \tag{6.3}
\]

Returning to (6.2) and taking \( \varepsilon_k \to 0 \) along the subsequence that yielded the strong solution \( h \), we find that
\[
\int_\Omega G_0^{(-1/2)}(h) \, dx = \frac{1}{(n-3/2)(n-1/2)} \int_\Omega h^{3/2-n}(x,T) \, dx \leq K < \infty. \tag{6.4}
\]

We now prove \( h(x,T) > 0 \) for all \( x \in \Omega \), and for every \( T \in [0,T_{\text{loc}}^{(\alpha)}) \). By (2.4), \( h(\cdot,T) \in C^{1/2}(\Omega) \) for all \( T \in [0,T_{\text{loc}}^{(\alpha)}] \). Assume \( T_0 \) is such that \( h(x_0,T_0) = 0 \) at some \( x_0 \in \Omega \). Then there is a \( L \) such that
\[
h(x,T_0) = |h(x,T_0) - h(x_0,T_0)| \leq L|x-x_0|^{1/2}.
\]

Hence since \( n \geq 7/2 \)
\[
\infty > \int_\Omega h^{\frac{3}{2}-n}(x,T_0) \, dx \geq L^{\frac{3-2n}{2}} \int_\Omega |x-x_0|^{-\frac{2n-3}{4}} \, dx = \infty.
\]

This contradiction implies there can be no point \( x_0 \) such that \( h(x_0,T_0) = 0 \), as desired. \( \Box \)

7 Finite speed of propagation

7.1 Local entropy estimate

Lemma 7.1. Let \( \zeta \in C^{1,2}_{t,x}(\bar{Q}_T) \) such that \( \text{supp} \zeta \subset \Omega \), \( (\zeta^4)_x = 0 \) on \( \partial \Omega \), and \( \zeta^4(-a,t) = \zeta^4(a,t) \). Assume that \( -\frac{a}{2} < \alpha < 1 \), and \( \alpha \neq 0 \). Then there exist a weak solution \( h(x,t) \) in the sense of the theorem Theorem 2, constants \( C_i (i = 1,2,3) \) dependent on \( n, m, \alpha, a_0, \) and
$a_1,$ independent of $\Omega,$ such that for all $T \leq T_{loc}^{(a)}$

\[
\begin{align*}
\int_{\Omega} \zeta^4(x,T)G_0^{(a)}(h(x,T)) \, dx - \int_{Q_T} (\zeta^4)_t G_0^{(a)}(h) \, dxdt + C_1 \int_{Q_T} (h^{\frac{\alpha+2}{2}}_{xx})^2 \zeta^4 \, dxdt \leq \\
\int_{\Omega} \zeta^4(x,0)G_0^{(a)}(h_0) \, dx + C_2 \int_{Q_T} h^{\alpha+2}(\zeta_x^4 + \zeta_x^2 \zeta_{xx}^2) \, dxdt + C_3 \int_{Q_T} h^{2(m-n+1)+\alpha} \zeta^4 \, dxdt. \quad (7.1)
\end{align*}
\]

Sketch of Proof of Lemma 7.1. In the following, we denote the positive, classical solution $h_\varepsilon$ constructed in Lemma 4.3 by $h$ (whenever there is no chance of confusion).

Let $\phi(x,t) = \zeta^4(x,t).$ Recall the entropy function $G_\varepsilon^{(a)}(z)$ defined by (4.12). Multiplying (4.1) by $\phi(x,t)G_\varepsilon^{(a)}(h_\varepsilon),$ and integrating over $Q_T$ yields

\[
\begin{align*}
\int_{\Omega} \phi(x,T)G_\varepsilon^{(a)}(h(x,T)) \, dx - \int_{\Omega} \phi(x,0)G_\varepsilon^{(a)}(h_0) \, dx - & \\
\int_{Q_T} \phi \psi \, dxdt = \int_{Q_T} \phi_x f_\varepsilon(h)G_\varepsilon^{(a)}(h)(a_0 h_{xxx} + a_1 D^n_\varepsilon(h)h_x) \, dxdt + \\
\int_{Q_T} \phi \alpha(a_0 h_x h_{xxx} + a_1 D^n_\varepsilon(h)h_x^2) \, dxdt =: I_1 + I_2. \quad (7.2)
\end{align*}
\]

We now bound the terms $I_1$ and $I_2.$ First,

\[
I_1 = -a_0 \int_{Q_T} \phi_{xx} f_\varepsilon(h)G_\varepsilon^{(a)}(h) h_{xxx} \, dxdt - a_0 \int_{Q_T} \phi_x (h^\alpha + f_\varepsilon'(h)G_\varepsilon^{(a)}(h))h_x h_{xx} \, dxdt - \\
- a_1 \int_{Q_T} \phi_{xx} F_\varepsilon^{(a)}(h) \, dxdt - a_0 \int_{Q_T} \phi_{xx} f_\varepsilon(h)G_\varepsilon^{(a)}(h) h_{xxx} \, dxdt + \int_{Q_T} \phi_x h^\alpha h_x^2 \, dxdt + \\
\frac{a_0 \alpha}{2} \int_{Q_T} \phi_x h^{\alpha-1} h_x^3 \, dxdt - a_0 \int_{Q_T} \phi_x f'_\varepsilon(h)G_\varepsilon^{(a)}(h) h_x h_{xx} \, dxdt - a_1 \int_{Q_T} \phi_{xx} F_\varepsilon^{(a)}(h) \, dxdt, \quad (7.3)
\]

where $F_\varepsilon^{(a)}(z) := \int_0^z f_\varepsilon(s)G_\varepsilon^{(a)}(s)g'_\varepsilon(s) \, ds,$

\[
I_2 = -a_0 \int_{Q_T} (\phi_x h^\alpha h_x h_{xx} + \alpha h^{\alpha-1} h_x^2 h_{xx} + \phi h^\alpha h_{xxx}^2) \, dxdt + \\
- a_1 \int_{Q_T} \phi h^\alpha D^n_\varepsilon(h) h_x^2 \, dxdt - \frac{a_0 \alpha}{2} \int_{Q_T} \phi x h^\alpha h_x^2 \, dxdt + \frac{a_0 \alpha}{6} \int_{Q_T} \phi_x h^{\alpha-1} h_x^3 \, dxdt + \\
\frac{a_0 \alpha(a-1)}{3} \int_{Q_T} \phi h^{\alpha-2} h_x^4 \, dxdt - a_0 \int_{Q_T} \phi h^\alpha h_x^2 \, dxdt + a_1 \int_{Q_T} \phi h^\alpha D^n_\varepsilon(h) h_x^2 \, dxdt. \quad (7.4)
\]
One easily finds that for all $\varepsilon > 0$ and all $z \geq 0$
\[ |f_\varepsilon(z)G_\varepsilon^{n(\alpha)}(z)| \leq \frac{\varepsilon^{\alpha+1}}{|\alpha-n+1|}, \quad |f_\varepsilon'(z)G_\varepsilon^{n(\alpha)}(z)| \leq \frac{\varepsilon^2}{|\alpha-n+1|}, \quad |F_\varepsilon(z)| \leq \frac{\varepsilon^3}{|\alpha-n+1|(m-n+\alpha+2)} + o(\varepsilon). \]
Using these bounds, and the Cauchy inequality, we bound $I_1 + I_2$:
\[ I_1 + I_2 \leq \frac{a_0\alpha(1-\alpha)}{3} \int_Q h^{\alpha-2}h_x^4\phi \, dx dt - a_0 \int_Q h^\alpha h_{xx}^2\phi \, dx dt + \]
\[ a_1 \int_Q h^{m-n+\alpha}h_x^2\phi \, dx dt + a_0 \int_Q h^\alpha h_x^2|\phi_{xx}| \, dx dt + \]
\[ \frac{a_0}{|\alpha-n+1|} \int_Q h^{n+1}|h_{xx}| |\phi_{xx}| \, dx dt + \frac{a_0^2}{|\alpha-n+1|^2} \int_Q h^\alpha |h_x||h_{xx}||\phi_x| \, dx dt + \]
\[ + \frac{4a_0\alpha}{3} \int_Q h^{n+1}h_x^3\phi \, dx dt + \frac{a_1}{|\alpha-n+1|(m-n+\alpha+2)} \int_Q h^{m-n+\alpha+2}|\phi_{xx}| \, dx dt. \quad (7.5) \]
Due to (7.5), we deduce from (7.2) that
\[ \int_\Omega \phi(x, T)G_\varepsilon^{(\alpha)}(h(T)) \, dx - \int_\Omega \phi_t(x, t)G_\varepsilon^{(\alpha)}(h) \, dx dt + a_0 \int_Q h^\alpha h_{xx}^2\phi \, dx dt + \]
\[ a_1 \int_Q h^{m-n+\alpha}h_x^2\phi \, dx dt + a_0 \int_Q h^\alpha h_x^2|\phi_{xx}| \, dx dt + \]
\[ \frac{a_0}{|\alpha-n+1|} \int_Q h^{n+1}|h_{xx}| |\phi_{xx}| \, dx dt + \frac{a_0^2}{|\alpha-n+1|^2} \int_Q h^\alpha |h_x||h_{xx}||\phi_x| \, dx dt + \]
\[ + \frac{4a_0\alpha}{3} \int_Q h^{n+1}h_x^3\phi \, dx dt + \frac{a_1}{|\alpha-n+1|(m-n+\alpha+2)} \int_Q h^{m-n+\alpha+2}|\phi_{xx}| \, dx dt, \quad (7.6) \]
where $\alpha \neq n - 1$. Recalling $\phi = \zeta^4$, and using the Young’s inequality (A.6) and simple transformations, from (7.6) we find that
\[ \int_\Omega \zeta^4(x, T)G_\varepsilon^{(\alpha)}(h(x, T)) \, dx - \int_\Omega (\zeta^4)_tG_\varepsilon^{(\alpha)}(h) \, dx dt + \]
\[ C_1 \int_Q (h^{\alpha+2})_x^2\zeta^4 \, dx dt \leq \int_\Omega \zeta^4(x, 0)G_\varepsilon^{(\alpha)}(h_{0, \varepsilon}) \, dx + \]
\[ C_2 \int_Q h^{\alpha+2}(\zeta_x^4 + \zeta_{xx}^2) \, dx dt + C_3 \int_Q h^{2(m-n+\alpha)}\zeta^4 \, dx dt. \quad (7.7) \]
We now argue that the \( \varepsilon \to 0 \) limit of the right-hand side of (7.7) is finite and bounded by \( K \), allowing us to apply Fatou’s lemma to the left-hand side of (7.7), concluding

\[
\int_{\Omega} \zeta^4(x, T) \left| G_0^{(a)}(h(x, T)) \right| dx - \int_{Q_T} \left| \zeta^4 \right| \left| G_0^{(a)}(h) \right| dx dt + C_1 \int_{Q_T} (h \| G \|_2)_{x, t}^2 \zeta^4 dx dt \leq K < \infty
\]

for every \( T \in [0, T_{loc}^{(a)}] \), as desired. (Note that in taking \( \varepsilon \to 0 \) we will choose the exact same sequence \( \varepsilon_k \) that was used to construct the weak solution \( h \) of Theorem 2. Also, in applying Fatou’s lemma we used the fact that \( \{h = 0\} \) having measure zero in \( Q_{T_{loc}}^{(a)} \) implies \( \{h(\cdot, T)\} \) has measure zero in \( \Omega \).)

It suffices to show that \( \int \zeta^4(x, 0) G_0^{(a)}(h_{0, \varepsilon}) \to \int \zeta^4(x, 0) G_0^{(a)}(h_0) < \infty \) as \( \varepsilon \to 0 \) (the rest of items is bonded as \( h \in L^\infty(0, T_{loc}^{(a)}; H^1(\Omega)) \)). This uses the Lebesgue Dominated Convergence Theorem. First, note that

\[
G_0^{(a)}(z) = \frac{z^{\alpha-n+2}}{(\alpha-n+2)(\alpha-1)} + \frac{\varepsilon z^{\alpha-n+2}}{(\alpha-n+2)(\alpha-3)} = G_0^{(a)}(z) + \frac{\varepsilon z^{\alpha-n+2}}{(\alpha-n+2)(\alpha-3)}
\]

hence if \( h_0(x) > 0 \) then

\[
G_0^{(a)}(h_{0, \varepsilon}(x)) = G_0^{(a)}(h_0(x) + \varepsilon \theta) + \frac{\varepsilon h_0(x) + \varepsilon \theta}{(\alpha-2)(\alpha-3)} \leq G_0^{(a)}(h_0(x) + \varepsilon \theta) + \frac{\varepsilon^{1-\theta(2-a)}}{(\alpha-2)(\alpha-3)}.
\]

Because \( h_0 \) has finite entropy (\( \int G_0^{(a)}(h_0) < \infty \)) it is positive almost everywhere in \( \Omega \). Using this and the fact that \( \theta \) was chosen so that \( \theta < 1/(2-a) < 2/5 \), we have

\[
|\phi(x, 0) G_0^{(a)}(h_{0, \varepsilon}(x))| \leq \phi(x, 0) (G_0^{(a)}(h_0(x)) + c) \leq C (G_0^{(a)}(h_0(x)) + c)
\]

almost everywhere in \( x \) and for all \( \varepsilon < \varepsilon_0 \). The dominating function is in \( L^1 \), because \( h_0 \) has finite entropy.

It remains to show pointwise convergence \( \phi(x, 0) G_0^{(a)}(h_{0, \varepsilon}(x)) \to \phi(x, 0) G_0(h_0(x)) \) almost everywhere in \( x \):

\[
\left| G_0^{(a)}(h_{0, \varepsilon}(x)) - G_0^{(a)}(h_0(x)) \right| \leq \left| G_0^{(a)}(h_{0, \varepsilon}(x)) - G_0^{(a)}(h_{0, \varepsilon}(x)) \right| + \left| G_0^{(a)}(h_{0, \varepsilon}(x)) - G_0^{(a)}(h_0(x)) \right| + \left| G_0^{(a)}(h_{0, \varepsilon}(x)) - G_0^{(a)}(h_{0, \varepsilon}(x)) \right| \\
\leq \frac{\varepsilon^{1-\theta(2-a)}}{(\alpha-2)(\alpha-3)} + \left| G_0^{(a)}(h_{0, \varepsilon}(x)) - G_0^{(a)}(h_0(x)) \right|
\]

As before, \( \frac{\varepsilon^{1-\theta(2-a)}}{(\alpha-2)(\alpha-3)} \) goes to zero by the choice of \( \theta \). The term \( \left| G_0^{(a)}(h_{0, \varepsilon}(x)) - G_0^{(a)}(h_0(x)) \right| \) goes to zero for almost every \( x \in \Omega \) because \( G_0^{(a)}(z) \) is continuous everywhere except at \( z = 0 \).

The proof is similar for the case \( \alpha = n - 1 \) and \( \alpha = n - 2 \). \( \Box \)

### 7.2 Proof of Theorem 3 for the case \( 0 < n < 2 \)

Let \( 0 < n < 2 \), and let \( \text{supp} h_0 \subseteq (-r_0, r_0) \subseteq \Omega \). For an arbitrary \( s \in (0, a - r_0) \) and \( \delta > 0 \) we consider the families of sets

\[
\Omega(s) = \Omega \setminus (-r_0 - s, r_0 + s), \quad Q_T(s) = (0, T) \times \Omega(s).
\]

(7.8)
We introduce a nonnegative cutoff function \( \eta(\tau) \) from the space \( C^2(\mathbb{R}^1) \) with the following properties:

\[
\eta(\tau) = \begin{cases} 
0 & \tau \leq 0, \\
\tau^2(3 - 2\tau) & 0 < \tau < 1, \\
1 & \tau \geq 1.
\end{cases} \tag{7.9}
\]

Next we introduce our main cut-off functions \( \eta_{s,\delta}(x) \in C^2(\overline{\Omega}) \) such that \( 0 \leq \eta_{s,\delta}(x) \leq 1 \) for all \( x \in \Omega \) and possess the following properties:

\[
\eta_{s,\delta}(x) = \eta\left(\frac{|x| - (r_0 + s)}{\delta}\right) = \begin{cases} 
1, & x \in \Omega(s + \delta), \\
0, & x \in \Omega(s),
\end{cases} \quad \left| \left( \eta_{s,\delta} \right) x \right| \leq \frac{3}{\delta} \quad \left| \left( \eta_{s,\delta} \right) xx \right| \leq \frac{6}{\delta^2} \tag{7.10}
\]

for all \( s > 0, \delta > 0 : r_0 + s + \delta < a \). Choosing \( \zeta^4(x, t) = \eta_{s,\delta}(x)e^{-\frac{t}{T}} \), from (7.1) we arrive at

\[
\int_{\Omega(s+\delta)} h^{\alpha-n+2}(T) \, dx + \frac{1}{T} \int \int_{Q_T(s+\delta)} h^{\alpha-n+2} \, dx \, dt + C \int \int_{Q_T(s+\delta)} (h^{\frac{\alpha+2}{2}})^2_{xx} \, dx \, dt \leq
\]

\[
\frac{C}{\delta^4} \int \int_{Q_T(s)} h^{\alpha+2} \, dx \, dt + C \int \int_{Q_T(s)} h^{2(m-n+1)+\alpha} \, dx \, dt =: C \sum_{i=1}^{2} \delta^{-\alpha_i} \int \int_{Q_T(s)} h^{\xi_i} \tag{7.11}
\]

for all \( s > 0 \), where we consider \( (n - 1)_+ < \alpha < 1 \). We apply the Gagliardo-Nirenberg interpolation inequality (see Lemma D.4) in the region \( \Omega(s + \delta) \) to a function \( v := h^{\frac{\alpha+2}{2}} \) with \( a = \frac{2\alpha}{\alpha+2}, \quad b = \frac{2(\alpha-n+2)}{\alpha+2}, \quad d = 2, \quad i = 0, \quad j = 2, \) and \( \theta_i = \frac{(\alpha+2)(\xi_i-\alpha-n-2)}{\xi_i(4\alpha-3n+8)} \) under the conditions:

\[
\alpha - n + 2 < \xi_i \quad \text{for} \quad i = 1, 2. \tag{7.12}
\]

Integrating the resulted inequalities with respect to time and taking into account (7.11), we arrive at the following relations:

\[
\int \int_{Q_T(s+\delta)} h^{\xi_i} \leq C T^{1-\frac{\theta_i \xi_i}{\alpha+2}} \left( \sum_{i=1}^{2} \delta^{-\alpha_i} \int \int_{Q_T(s)} h^{\xi_i} \right)^{1+\nu_i} + C T \left( \sum_{i=1}^{2} \delta^{-\alpha_i} \int \int_{Q_T(s)} h^{\xi_i} \right)^{\frac{\xi_i}{\alpha-n+2}}, \tag{7.13}
\]

where \( \nu_i = \frac{4(\xi_i-\alpha-n-2)}{4\alpha-3n+8} \). These inequalities are true provided that

\[
\frac{\theta_i \xi_i}{\alpha+2} < 1 \iff \xi_i < 5\alpha - 4n + 10 \quad \text{for} \quad i = 1, 2. \tag{7.14}
\]

Simple calculations show that inequalities (7.12) and (7.14) hold with some \( (n - 1)_+ < \alpha < 1 \) if and only if

\[
0 < n < 2, \quad \frac{n}{2} < m < 6 - n.
\]

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Since all integrals on the right-hand sides of (7.13) vanish as \( T \to 0 \), the finite speed of propagations follows from (7.13) by applying Lemma D.5 with \( s_1 = 0 \) and sufficiently small \( T \). Hence,

\[
\text{supp} \, h(T, \cdot) \subset (-r_0 - \Gamma_0(T), r_0 + \Gamma_0(T)) \subseteq \Omega \text{ for all } T : 0 \leq T \leq T_{\text{speed}}. \quad (7.15)
\]

Due to (7.15), we can consider the solution \( h(x,t) \) with compact support in the whole space \( \mathbb{R}^1 \) and for \( T \leq T_{\text{speed}} \). In this case, we can repeat the previous procedure for \( \Omega(s) = \mathbb{R}^1 \setminus (-r_0 - s, r_0 + s) \) and we obtain

\[
G_i(s + \delta) := \int \int_{Q_T(s+\delta)} h^{\xi_i} \leq C T^{1-\frac{\theta_1 \xi_i}{\alpha+2}} \left( \sum_{i=1}^{2} \delta^{-\alpha_i} \int \int_{Q_T(s)} h^{\xi_i} \right)^{1+\nu_i}, \quad (7.16)
\]

instead of (7.13), and

\[
\Gamma_0(T) = C \left( T^{(1-\frac{\theta_1 \xi_1}{\alpha+2})(1+\nu_2)} T^{(1-\frac{\theta_2 \xi_2}{\alpha+2})\nu_1(1+\nu_1)} (G(0))^{\nu_1} \right)^{\frac{1}{4(1+\nu_1)(1+\nu_2)}}, \quad (7.17)
\]

\[
H(s) = CT^{(1-\frac{\theta_1 \xi_1}{\alpha+2})(1+\nu_1)} T^{(1-\frac{\theta_2 \xi_2}{\alpha+2})\nu_2(1+\nu_2)} (G_2(0))^{\nu_2},
\]

where

\[
G(0) = C(T^{(1-\frac{\theta_1 \xi_1}{\alpha+2})(1+\nu_2)}(G_2(0))^{1+\nu_1} + T^{(1-\frac{\theta_2 \xi_2}{\alpha+2})(1+\nu_1)}(G_1(0))^{1+\nu_2}). \quad (7.18)
\]

Now we need to estimate \( G(0) \). With that end in view, we obtain the following estimates:

\[
G_i(0) \leq C_1 (C_2 + C_3 T) \frac{\xi_i-1}{\alpha+5} T^{1-\frac{\xi_i-1}{\alpha+5}}, \quad i = 1, 2. \quad (7.19)
\]

where \( 1 < \xi_i < \alpha + 6 \), and \( C_i \) depends on \( h_0(x) \) only. Really, applying the Gagliardo-Nirenberg interpolation inequality (see Lemma D.4) in \( \Omega = \mathbb{R}^1 \) to a function \( v := h \frac{\partial^2}{\partial x^2} \) with

\[
a = \frac{2 \xi_i}{\alpha+2}, \quad b = \frac{2}{\alpha+2}, \quad d = 2, \quad i = 0, \quad j = 2, \quad \text{and} \quad \theta_i = \frac{(\alpha+2)(\xi_i-1)}{\xi_i(\alpha+5)}
\]

under the condition \( \xi_i > 1 \), we deduce that

\[
\int h^{\xi_i} dx \leq c \| h_0 \|^\frac{2(3 \xi_i+\alpha+2)}{(\alpha+2)(\alpha+5)} \left( \int \left( h \frac{\partial^2}{\partial x^2} \right)^2 dx \right)^\frac{\xi_i-1}{\alpha+5}. \quad (7.20)
\]

Integrating (7.20) with respect to time and taking into account the H"older inequality \( (\frac{\xi_i-1}{\alpha+5} \leq 1 \Rightarrow \xi_i < \alpha + 6 \Rightarrow m \leq n + 2) \), we arrive at the following relations:

\[
G_i(0) \leq c \| h_0 \|^\frac{2(3 \xi_i+\alpha+2)}{(\alpha+2)(\alpha+5)} T^{1-\frac{\xi_i-1}{\alpha+5}} \left( \int \int_{Q_T} \left( h \frac{\partial^2}{\partial x^2} \right)^2 dx \right)^\frac{\xi_i-1}{\alpha+5}, \quad m \leq n + 2. \quad (7.21)
\]

From (7.21), due to (A.32) and (5.2), we find (7.19).

Inserting (7.19) into (7.18), we obtain after straightforward computations that

\[
\Gamma_0(T) \leq \Gamma(T) = CT^{\frac{1}{n+4}}. \quad (7.22)
\]
for $T \leq T_{\text{speed}}$, where $\frac{n}{2} < m \leq n + 2$, and $T_{\text{speed}}$ finds from the condition $H(0) < 1$.

In the case of $m > n + 2$, we have (7.21) for $i = 1$ only, i.e.

$$G_1(0) \leq c \|h_0\|_1^{2(3\xi_1 + \alpha + 2)}(\alpha + 5) T^{1-\frac{\xi_1 - 1}{\alpha + 5}} \int_{Q_T} (h^{\alpha + 2} \frac{\partial}{\partial x})^2 dx. \quad (7.23)$$

Hence

$$G_1(0) \leq C(C + C(1 - (1 - k_1(2m-n)||h_0||_{H^1}^{2m-n} 2) \frac{m+2}{2m-n}) T^{1-\frac{\xi_1 - 1}{\alpha + 5}} \leq C_0 T^{1-\frac{\xi_1 - 1}{\alpha + 5}}. \quad (7.24)$$

Next we need estimate the term $G_2(0)$. Really, by using the inequality (A.3), we obtain

$$G_2(0) \leq b_2 \|h_0\|_1 \frac{\xi_2 + 2}{3} \int_0^T \|h_x\|^2_{\frac{3}{2}}(\xi_2 - 1) T \|h_0\|^2_{H^1} \times$$

$$\int_0^T (1 - \frac{k_1(2m-n)||h_0||_{H^1}^{2m-n} 2}{T}) \frac{2(\xi_2 - 1)}{3(2m-n)} dt = b_2 \|h_0\|_1 \frac{\xi_2 + 2}{3} \|h_0\|^2_{H^1} \times$$

$$\frac{6}{k_1(2m+n-2(a+1))} (1 - (1 - \frac{k_1(2m-n)||h_0||_{H^1}^{2m-n} 2}{3(2m-n)} T) \frac{2m+n-2(a+1)}{3(2m-n)}) \leq C_0 \quad (7.25)$$

for all $T \leq T_{T_0c}$, where $C_0$ depends on $h_0(x)$. Inserting (7.19) for $i = 1$ and (7.25) into (7.18), we obtain (7.22).

### 7.3 Local energy estimate

**Lemma 7.2.** Let $n \in (\frac{1}{2}, 3)$, and $m > \frac{2(n-1)}{3}$, and $\beta > \frac{1-n}{3}$. Let $\zeta \in C^2(\Omega)$ such that supp $\zeta$ in $\Omega$, and $(\zeta^6)' = 0$ on $\partial \Omega$, and $\zeta(-a) = \zeta(a)$. Then there exist constants $C_i$ ($i = 1, 2, 3$) dependent on $n, m, a_0$, and $a_1$, independent of $\Omega$ and $\varepsilon$, such that for all $T \leq T_{T_0c}$

$$\begin{align*}
\int_\Omega \zeta^6 h_x^2(x, T) dx + \int_\Omega \zeta^4 h^{\beta+1}(T) dx + C_1 \int_{Q_T} \zeta^6 (h^{\frac{n+2}{2}})_{xx} dx dt &\leq \int_\Omega \zeta^6 h_0^2(x) dx + \\
\int_\Omega \zeta^4 h_0^{\beta+1} dx + C_2 \int_{Q_T} h^{n+2}(\zeta^6 + \zeta^3 |\zeta_{xx}|^3) dx dt + C_3 \int_{Q_T} h^{3m-2n+2} \zeta^6 dx dt + \\
C_4 \int_{Q_T} \{h^{n+2\beta} \zeta_x^2 + \chi_{(\zeta>0)} h^{n+3\beta-1} + h \frac{6n-3n+6\beta+4}{5} \frac{12}{5} \zeta_x^2 + h \frac{3n+3\beta+1}{2} \zeta^3 \} dx dt. \quad (7.26)
\end{align*}$$

**Sketch of Proof of Lemma 7.2.** Let $\phi(x) = \zeta^6(x)$. Multiplying (4.1) by $-(\phi(x)h_x)_x$, and
integrating on $Q_T$, yields
\[
\frac{1}{2} \int_{\Omega} \phi(x)h_x^2(x,T) \, dx - \frac{1}{2} \int_{\Omega} \phi(x)h_x^2(0,T) \, dx = \\
- \iint_{Q_T} f_{\varepsilon}(h)(a_0 h_{xxx} + a_1 D''(h)h_x)(\phi_{xx}h_x + 2\phi_xh_{xx} + \phi h_{xxx}) \, dxdt = \\
- \iint_{Q_T} f_{\varepsilon}(h)(a_0 h_{xxx} + a_1 D''(h)h_x)\phi_{xx}h_x \, dxdt - \\
2 \iint_{Q_T} f_{\varepsilon}(h)(a_0 h_{xxx} + a_1 D''(h)h_x)\phi_xh_{xx} \, dxdt - \\
\iint_{Q_T} f_{\varepsilon}(h)(a_0 h_{xxx} + a_1 D''(h)h_x)\phi h_{xxx} \, dxdt =: I_1 + I_2 + I_3. \tag{7.27}
\]

We now bound the terms $I_1$, $I_2$ and $I_3$. First,
\[
I_1 = -a_0 \iint_{Q_T} \phi_{xx} f_{\varepsilon}(h) h_{xxx} h_x \, dxdt - a_1 \iint_{Q_T} \phi_{xx} f_{\varepsilon}(h) D''(h) h_x^2 \, dxdt =
\]
\[
- 6a_0 \iint_{Q_T} f_{\varepsilon}(h) h_{xxx} h_x \zeta^4(5\zeta_x^2 + \zeta_x x_x) \, dxdt - 6a_1 \iint_{Q_T} f_{\varepsilon}(h) D''(h) h_x^2 \zeta^4(5\zeta_x^2 + \zeta_x x_x) \, dxdt \\
\leq \epsilon_1 \iint_{Q_T} \zeta^6 \left( f_{\varepsilon}(h) h_{xxx}^2 + h^{n-4} h_x^6 \right) \, dxdt + C(\epsilon_1) \iint_{Q_T} h^{n+2} (\zeta_x^6 + \zeta^3 |x_{xx}|^3) \, dxdt + \\
C(\epsilon_1) \iint_{Q_T} h^{3m-2n+2} \zeta^6 \, dxdt, \tag{7.28}
\]

\[
I_2 = -2a_0 \iint_{Q_T} \phi_x f_{\varepsilon}(h) h_{xxx} h_x \, dxdt - 2a_1 \iint_{Q_T} \phi_x f_{\varepsilon}(h) D''(h) h_{xx}h_x \, dxdt =
\]
\[
- 12a_0 \iint_{Q_T} f_{\varepsilon}(h) h_{xxx} h_x \zeta^5 \zeta_x \, dxdt - 12a_1 \iint_{Q_T} f_{\varepsilon}(h) D''(h) h_{xx}h_x \zeta^5 \zeta_x \, dxdt \\
\leq \epsilon_2 \iint_{Q_T} \zeta^6 \left( f_{\varepsilon}(h) h_{xxx}^2 + h^{n-2} h_x^2 h_{xx}^2 + h^{n-1} |h_{xx}|^3 \right) \, dxdt + \\
C(\epsilon_2) \iint_{Q_T} h^{n+2} \zeta_x^6 \, dxdt + C(\epsilon_2) \iint_{Q_T} h^{3m-2n+2} \zeta^6 \, dxdt, \tag{7.29}
\]

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\[ I_3 = -a_0 \int_{Q_T} \phi f_\varepsilon(h) h_{xxx}^2 \, dx \, dt - a_1 \int_{Q_T} \phi f_\varepsilon(h) \Delta_\varepsilon(h) h_{xxx} \, dx \, dt \leq \\
- a_0 \int_{Q_T} \zeta^6 f_\varepsilon(h) h_{xxx}^2 \, dx \, dt + \varepsilon_3 \int_{Q_T} \zeta^6 (f_\varepsilon(h) h_{xxx}^2 + h^{n-4} h_x^6) \, dx \, dt + \\
C(\varepsilon_3) \int_{Q_T} h^{3m-2n+2} \zeta^6 \, dx \, dt. \quad (7.30) \]

Now, multiplying \((7.11)\) by \(\zeta^4 (h + \gamma)^2\), \(\beta > \frac{1-n}{3}\), \(\gamma > 0\) and integrating on \(Q_T\), using the Young’s inequality \((A.6)\), letting \(\gamma \to 0\), we obtain the following estimate

\[ \int \zeta^4 h^{\beta+1}(T) \, dx \leq \int \zeta^4 h_{0e}^{\beta+1} \, dx + \varepsilon_4 \int \int \zeta^6 \{ f_\varepsilon(h) h_{xxx}^2 + h^{n-4} h_x^6 \} \, dx \, dt + \\
C(\varepsilon_4) \int \int \{ h^{n+2} \zeta_x^2 + \chi(\xi > 0) h^{n+3\beta-1} + h^{6m-n+6\beta+4} \zeta^6 + h^{3m-n+3\beta+1} \zeta^3 \} \, dx \, dt, \quad (7.31) \]

where \(\beta > \max\{\frac{1-n}{3}, -m + \frac{n-1}{3}\} = \frac{1-n}{3}\) as \(m > \frac{2(n-1)}{3}\).

If we now add inequalities \((7.27)\) and \((7.31)\), in view of \((7.28)\)–\((7.30)\), then, applying Lemma \([D.3]\) choosing \(\varepsilon_i > 0\), and letting \(\varepsilon \to 0\), we obtain \((7.26)\). \(\square\)

### 7.4 Proof of Theorem 3 for the case \(\frac{1}{2} < n < 3\)

Let \(\eta_{s,\delta}(x)\) denote by \((7.10)\). Setting \(\xi^6(x) = \eta_{s,\delta}(x)\) into \((7.26)\), after simple transformations, we obtain

\[ \int_{\Omega(s+\delta)} h_\varepsilon^2(x, T) \, dx + \int_{\Omega(s+\delta)} h^{\beta+1}(T) \, dx + C \int \int_{Q_T(s+\delta)} (h^{\frac{n+2}{2}} h_{xxx}) \, dx \, dt \leq \\
\frac{C}{\delta^2} \int_{Q_T(s)} h^{n+2} \, dx \, dt + \frac{C}{\delta^2} \int_{Q_T(s)} h^{n+3} \, dx \, dt + \frac{C}{\delta^3} \int_{Q_T(s)} h^{6m-n+6\beta+4} \, dx \, dt + \\
C \int \int_{Q_T(s)} \{ h^{3m-2n+2} + h^{n+3\beta-1} + h^{3m-n+3\beta+1} \} \, dx \, dt =: C \sum_{i=1}^6 \delta^{-\alpha_i} \int_{Q_T(s)} h^{\xi_i}. \quad (7.32) \]

for all for all \(s > 0\), \(\delta > 0\): \(r_0 + s + \delta < a\). We apply the Gagliardo-Nirenberg interpolation inequality (Lemma \([D.4]\)) in the region \(\Omega(s+\delta)\) to a function \(v := h^{\frac{n+2}{2}}\) with \(a = \frac{2\xi_1}{n+2}\), \(b = \frac{2(\beta+1)}{n+2}\), \(d = 2\), \(i = 0\), \(j = 3\), and \(\theta_i = \frac{(n+2)(\xi_i-\beta-1)}{\xi_i^{(n+3\beta+1)}(n+3\beta+1)}\) under the conditions:

\[ \beta < \xi_i - 1 \quad \text{for} \quad i = 1, 6. \quad (7.33) \]
Integrating the resulted inequalities with respect to time and taking into account (7.32), we arrive at the following relations:

\[
\int_0^T \mathcal{Q}_T(s) (s + \delta) \dot{h} \xi_i \leq C T^{1 - \frac{6 \delta}{n+2}} \left( \sum_{i=1}^6 \delta^{-\alpha_1} \int \mathcal{Q}_T(s) \right) \left( \sum_{i=1}^5 \delta^{-\alpha_i} \int \mathcal{Q}_T(s) \right)^{\xi_i \beta + 1},
\]

(7.34)

where \( \nu_i = \frac{6(\xi_i - 3 - 1)}{n+5\beta+i} \). These inequalities are true provided that

\[
\theta_i \xi_i < 1 \Leftrightarrow \beta > \frac{\xi_i - n - 8}{6} \text{ for } i = 1, 6.
\]

(7.35)

Simple calculations show that inequalities (7.33) and (7.35) hold with some \( \beta > \frac{1-n}{3} \) if and only if

\[
\frac{1}{2} < n < 3, \quad m > \frac{n}{2}.
\]

Since all integrals on the right-hand sides of (7.34) vanish as \( T \to 0 \), the finite speed of propagations follows from (7.34) by applying Lemma D.5 with \( s_1 = 0 \) and sufficiently small \( T \). Hence,

\[
\text{supp } h(T, .) \subset (-r_0 - \Gamma(T), r_0 + \Gamma(T)) \subset \Omega \text{ for all } T : 0 \leq T \leq T_{\text{speed}}.
\]

(7.36)

8 Finite time blow up

Lemma 8.1. Let \( 0 < n < 2, m \geq \max\{n + 2, 2 - n\} \). Then the weak solution \( h \) from Theorem 7 satisfies the second-moment inequality:

\[
e^{-\tilde{B}(T)} \int_\Omega x^2 \tilde{G}_0(h(x, T)) \, dx \leq \int_\Omega x^2 \tilde{G}_0(h_0) \, dx + \int_0^T \left( k_1 E_0(0) + W'(t) + k_2 \int_\Omega x^2 h_{xx}^2 \, dx \right) e^{-\tilde{B}(t)} \, dt \quad (8.1)
\]

for all \( T \in [0, T_{\text{loc}}] \), where \( k_1 = 2(2 - n) \), \( k_2 = \frac{3a_0(n-1)}{2} \). Here

\[
\tilde{G}_0(z) = \frac{1}{2-n} z^{2-n}, \quad \tilde{B}(T) := \frac{a_1^2(2-n)(2-n)}{2a_0(m-n+1)} \int_0^T \|h(\cdot, \tau)\|_{L^\infty(\Omega)}^{2m-n} \, d\tau,
\]

\[
W(T) := \left. \int_0^T x(2a_0 hh_{xx} - a_0(2 - n)h_x^2 + 2a_1 m D_0(h)) \, dt \right|_{\partial\Omega}.
\]

Moreover, if \( h(\cdot, t) \) has a compact support in \( \Omega \) then \( W'(t) \equiv 0 \).

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The proof of Lemma 8.1 is given in Appendix A.

Proof of Theorem 4. Assume $0 < n < 2$ and $m$ satisfies the hypotheses of Theorem 3. By Theorems 2 and 3, there is a compactly supported strong solution $h$ for $t \in [0, T_0]$. Hence, we can assume that $W'(t) = 0$ in Lemma 8.1.

First, we construct a sequence of times $T_0 < T_1 < \ldots$ and extend the strong solution $h$ from the time interval $[0, T_i]$ to the time interval $[0, T_i+1]$. Taking it as an initial datum, we obtain a time interval of existence $T_1 - T_0 = \frac{1}{2\sigma(\gamma_3 - 1)} \left( \int_{\Omega} \left\{ \frac{1}{2} h_x^2(x, T_0) + G^{(\alpha)}_0(h(x, T_0)) \right\} dx \right)^{-(\gamma_3 - 1)},$

by (A.39). Applying Theorem 3 to the time interval $[T_0, T_1]$, we have a strong solution with compact support that satisfies all a priori estimates with the time interval $[0, T_0]$ replaced by $[0, T_1]$. In this way, we construct a nonnegative, compactly supported, strong solution on $\mathbb{R}^1 \times [0, T^*)$ where

$T^* = \lim_{i \to \infty} T_i.$

If $T^* < \infty$ then

$\frac{1}{2\sigma(\gamma_3 - 1)} \left( \int_{\Omega} \left\{ \frac{1}{2} h_x^2(x, T_i) + G^{(\alpha)}_0(h(x, T_i)) \right\} dx \right)^{1-\gamma_3} = T_{i+1} - T_i \to 0.$

Hence, the $H^1$ norms at times $T_i$ must blow up. And, due to (3.8), the $L^\infty$-norm of the solution at times $T_i$ must also blow up. Therefore, to finish the proof it suffices to prove that $T^* < \infty$.

Let

$V(0) = \int_{-\infty}^{\infty} x^2 \tilde{G}_0(h_0) \, dx \quad \text{and} \quad V(T_i) := e^{-\tilde{B}(T_i)} \int_{-\infty}^{\infty} x^2 \tilde{G}_0(h(T_i)) \, dx.$

Using that $k_1 > 0$ and $\mathcal{E}_0(T_i) \leq \mathcal{E}_0(0)$, we apply the inequality (8.1) iteratively to find

$V(T_i) \leq V(0) + k_1 \mathcal{E}_0(0) \int_{-\infty}^{T_i} e^{-\tilde{B}(t)} \, dt + k_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 h_{xx}^2(x, t) \, dx \, dxe^{-\tilde{B}(t)} \, dt$ \hspace{1cm} (8.4)

Case $n = 1$: In this case, $\tilde{B}(t) = 0$ and $k_2 = 0$. Hence (8.4) becomes

$V(T_i) \leq V(0) + k_1 \mathcal{E}_0(0) T_i.$

If $T^* = \infty$ then the right-hand side will become negative, contradicting that the left-hand side is nonnegative. Hence $T^* < \infty$. 

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Case 0 < n < 1: In this case, $k_2 < 0$ and (8.4) implies

$$V(T_i) \leq V(0) + k_1 E_0(0) \int_0^{T_i} e^{-\tilde{B}(t)} \, dt \quad (8.5)$$

We wish to argue that if $T^* = \infty$ then the right-hand side of (8.5) must become negative, leading to a contradiction.

Let $g(t) = \int_0^t e^{-\tilde{B}(s)} \, ds$. Using (A.19), we have $\|h(x,t)\|_\infty \leq K_1 (T_i - t)^{-\frac{1}{2(2m-n-1)}}$ for all $t < T_i$, where $K_1$ is a constant.

$$g(T_i) \geq e^{-K_2 T_i} \int_0^{T_i} e^{K_2 (T_i - s)} s^{-\frac{1}{2(2m-n-1)}} \, ds = e^{-K_2 T_i} \int_0^{T_i} e^{K_2 s} s^{-\frac{1}{2(2m-n-1)}} \, ds \sim T_i^{2m-n-1} \text{ as } T_i \to +\infty. \quad (8.6)$$

Since $E_0(0) < 0$, if $T_i \to +\infty$ then for large times the right-hand side of (8.5) would be negative: an impossibility. Therefore $\lim_{i \to +\infty} T_i = T^* < \infty$.

Case 1 < n < 2: In this case, we write inequality (8.4) in the form:

$$V(T_i) \leq V(0) + k_1 E_0(0) g(T_i) + k_2 f(T_i) \quad (8.7)$$

where

$$f(T_i) = \int_0^{T_i} e^{-\tilde{B}(t)} \int_{-\infty}^{\infty} x^2 h^2_{xx}(x,t) \, dx \, dt.$$ 

Assume $T^* = \infty$ and that $f(T_i)$ grows more slowly than $g(T_i)$ as $T_i \to \infty$. Then the right-hand side of (8.7) will become negative in finite time, which is an impossibility. It therefore suffices to show that $f(T_i)$ grows more slowly than $g(T_i)$.

Due to (A.16) and (A.19) as $\varepsilon \to 0$, we have

$$\frac{m}{2} \int_0^{\infty} \int_{-\infty}^{\infty} h_{xx}^2 \, dx \, dt \leq \int_0^{\infty} G_0(h_0) \, dx + C \int_0^{T_i} \|h_x\|_{2(2m-1)}^2 \, dt \leq$$

$$\int_{-\infty}^{\infty} G_0(h_0) \, dx + C \int_0^{T_i} (T_i - t)^{-\frac{m-1}{2m-n-1}} \, dt = \int_{-\infty}^{\infty} G_0(h_0) \, dx + \frac{C(2m-n-1)}{m-2} T_i^{\frac{m-2}{2m-n-1}}. \quad (8.8)$$
In view of (7.22), we know that \( |x| \leq C(h_0) \) for all \( t \leq T_i \). Hence, due to (8.8), we deduce
\[
f(T_i) \leq \int_{-\infty}^{\infty} x^2 h_{xx}^2 \, dx \, dt \leq C \int_{0}^{T_i} \int_{-\infty}^{\infty} h_{xx}^2 \, dx \, dt \leq C(1 + T_i^{2m-2n-1}).
\]
Comparing with the exponent in (8.6), we see that \( f(T_i) \) grows more slowly than \( g(T_i) \), as desired.

\[\square\]

A Proofs of A Priori Estimates

The first observation is that the periodic boundary conditions imply that classical solutions of equation (4.1) conserve mass:
\[
\int_{\Omega} h_{\delta\varepsilon}(x, t) \, dx = \int_{\Omega} h_{0,\delta\varepsilon}(x) \, dx = M_{\delta\varepsilon} < \infty \text{ for all } t > 0. \tag{A.1}
\]
Further, (4.5) implies \( M_{\delta\varepsilon} \to M = \int h_0 \) as \( \varepsilon, \delta \to 0 \). The initial data in this article have \( M > 0 \), hence \( M_{\delta\varepsilon} > 0 \) for \( \delta \) and \( \varepsilon \) sufficiently small.

Also, we will relate the \( L^p \) norm of \( h \) to the \( L^p \) norm of its zero-mean part as follows:
\[
|h(x)| \leq \left| h(x) - \frac{\Omega}{|\Omega|} M \right| + \frac{\Omega}{|\Omega|} \to ||h||_p^p \leq 2^{p-1} ||v||_p^p + \left( \frac{2}{|\Omega|} \right)^{p-1} M^p
\]
where \( v := h - M/|\Omega| \) and we have assumed that \( M \geq 0 \). We will use the Poincaré inequality which holds for any zero-mean function in \( H^1(\Omega) \):
\[
||v||_p^p \leq b_1 ||v_x||_p^p \quad 1 \leq p < \infty \tag{A.2}
\]
with \( b_1 = |\Omega|^p \).

Also used will be an interpolation inequality [30] Theorem 2.2, p. 62] for functions of zero mean in \( H^1(\Omega) \):
\[
||v||_p^p \leq b_2 ||v_x||_2^{ap} ||v||_r^{(1-a)p} \tag{A.3}
\]
where \( r \geq 1, p \geq r, \)
\[
a = \frac{1-r-1/p}{1/r+1/2}, \quad b_2 = (1 + r/2)^{ap}.
\]
It follows that for any zero-mean function \( v \) in \( H^1(\Omega) \)
\[
||v||_p^p \leq b_3 ||v_x||_2^p, \quad \Rightarrow \quad ||h||_p^p \leq b_4 ||h_x||_2^p + b_5 M_{\delta\varepsilon}^p \tag{A.4}
\]
where
\[
b_3 = \begin{cases} b_1 |\Omega|^{(2-p)/p} & \text{if } 1 \leq p \leq 2, \\ b_2^{(p+2)/2} & \text{if } 2 < p < \infty, \end{cases}, \quad b_4 = 2^{p-1} b_3, \quad b_5 = \left( \frac{2}{|\Omega|} \right)^{p-1}
\]

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To see that \((A.4)\) holds, consider two cases. If \(1 \leq p < 2\), then by \((A.2)\), \(\|v\|_p\) is controlled by \(\|v_x\|_p\). By the Hölder inequality, \(\|v_x\|_p\) is then controlled by \(\|v_x\|_2\). If \(p > 2\) then by \((A.3)\), \(\|v\|_p\) is controlled by \(\|v_x\|_2^{1-a}\|v\|^{1/2}_2\) where \(a = 1/2 - 1/p\). By the Poincaré inequality, \(\|v\|^{1/2}_2\) is controlled by \(\|v_x\|^{1/2}_2\).

If \(0 < p < 1\) then, instead of \((A.4)\), we obtain
\[
\|h\|_p^p \leq \tilde{b}_4 \|h_x\|_2^p + \tilde{b}_5 M_{\delta\varepsilon}^p
\]  
where
\[
\tilde{b}_4 = |\Omega|^{1-p/2} b_4^{p/2}, \quad \tilde{b}_5 = |\Omega|^{1-2/2} b_5^{2/2}.
\]

The Cauchy inequality \(ab \leq \varepsilon a^2 + b^2/(4\varepsilon)\) with \(\varepsilon > 0\) will be used often as will Young’s inequality
\[
ab \leq \varepsilon a^p + \frac{b^q}{q(\varepsilon)^{q/p}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \varepsilon > 0.
\]

**Sketch of Proof of Lemma 4.1**. In the following, we denote the classical solution \(h_{\delta\varepsilon}\) by \(h\) whenever there is no chance of confusion.

To prove the bound \((4.7)\) one starts by multiplying \((4.1)\) by \(-h_{xx}\), integrating over \(Q_T\), and using the periodic boundary conditions \((4.2)\) yields
\[
\frac{1}{2} \int_{\Omega} h_x^2(x, T) \, dx + a_0 \int_{Q_T} f_{\delta\varepsilon}(h) h_{xxx}^2 \, dx \, dt = \frac{1}{2} \int_{\Omega} h_{0,\delta\varepsilon,x}^2(x) \, dx - a_1 \int_{Q_T} f_{\delta\varepsilon}(h) D_{\varepsilon}''(h) h_x h_{xxx} \, dx \, dt.
\]  
The Cauchy inequality is used to bound some terms on the right-hand of \((A.7)\):
\[
a_1 \int_{Q_T} f_{\delta\varepsilon}(h) D_{\varepsilon}''(h) h_x h_{xxx} \, dx \, dt \leq \frac{a_0}{2} \int_{Q_T} f_{\delta\varepsilon}(h) h_{xxx}^2 \, dx \, dt + \frac{a_2^2}{2a_0} \int_{Q_T} f_{\delta\varepsilon}(h)(D_{\varepsilon}''(h))^2 h_x^2 \, dx \, dt.
\]  
Using \((A.8)\) in \((A.7)\) yields
\[
\frac{1}{2} \int_{\Omega} h_x^2(x, T) \, dx + \frac{a_0}{2} \int_{Q_T} f_{\delta\varepsilon}(h) h_{xxx}^2 \, dx \, dt \leq \frac{1}{2} \int_{\Omega} h_{0,\delta\varepsilon,x}^2 \, dx + \frac{a_2^2}{2a_0} \int_{Q_T} f_{\delta\varepsilon}(h)(D_{\varepsilon}''(h))^2 h_x^2 \, dx \, dt \leq \frac{1}{2} \int_{\Omega} h_{0,\delta\varepsilon,x}^2 \, dx + \frac{a_2^2}{2a_0} \int_{Q_T} h_x^2 \, dx \, dt.
\]
Above, we used the bounds $f_{\delta}(z) \leq |z|^n + \delta$, $D''_\varepsilon(z) \leq |z|^{m-n}$, and $D'''_\varepsilon(z) \leq \varepsilon^{-1}$. By the Cauchy inequality, bound (A.4), (A.3) and bound (A.5),

$$\int\int_{Q_T} |h|^{2m-n} h_x^2 \, dx \, dt \leq \frac{1}{2} \int\int_{Q_T} h^{2(2m-n)} \, dx \, dt + \frac{1}{2} \int\int_{Q_T} h_x^4 \, dx \, dt$$

$$\leq \frac{b_4}{2} \int_0^T \left( \int_{\Omega} h_x^2 \, dx \right)^{2m-n} \, dt + \frac{b_5}{2} \int_0^T \left( \int_{\Omega} h_x^3 \, dx \right)^{3} \, dt$$

$$\leq \frac{1}{2} \int\int_{Q_T} h_{xx}^2 \, dx \, dt + \frac{b_5}{8} \int_0^T \left( \int_{\Omega} h_x^2 \, dx \right)^{3} \, dt + \frac{b_5}{2} \int_0^T \left( \int_{\Omega} h_x^3 \, dx \right)^{2m-n} \, dt + c_1 T, \quad (A.10)$$

where $c_1 = M^{2(2m-n)}_\delta b_5/2$, $m \geq \frac{n}{2}$. From (A.9), due to (A.10), we arrive at

$$\frac{1}{2} \int_{\Omega} h_x^2(x, T) \, dx + \frac{an}{2} \int\int_{Q_T} f_{\delta}(h) h_{xx}^2 \, dx \, dt$$

$$\leq \frac{1}{2} \int_{\Omega} h_{0,\delta,x,x}^2 \, dx + c_2 \int\int_{Q_T} h_{xx}^2 \, dx \, dt + c_3 \int\int_{Q_T} h_x^2 \, dx \, dt \left( \int_{\Omega} h_x^3 \, dx \right)^{3} \, dt$$

$$+ c_4 \int_0^T \left( \int_{\Omega} h_x^2 \, dx \right)^{2m-n} \, dt + c_5 \int\int_{Q_T} h_x^2 \, dx \, dt + c_6 T$$

$$\leq \frac{1}{2} \int_{\Omega} h_{0,\delta,x,x}^2 \, dx + c_2 \int\int_{Q_T} h_{xx}^2 \, dx \, dt + c_7 \int_0^T \max \left\{ 1, \left( \int_{\Omega} h_x^2 \, dx \right)^{\gamma_1} \right\} \, dt \quad (A.11)$$

where $\gamma_1 = \max\{3,2m-n\}$, $m \geq \frac{n}{2}$,

$$c_2 = \frac{a^2}{4a_0}, \quad c_3 = \frac{a^2 b_4}{16a_0}, \quad c_4 = \frac{a^2 b_4}{4a_0^2}, \quad c_5 = \frac{a^2}{2a_0 \varepsilon} \frac{\delta}{\varepsilon}$$

$$c_6 = \frac{a^2}{2a_0} c_1, \quad c_7 = c_3 + c_4 + c_5 + c_6.$$

Now, multiplying (4.1) by $G'_{\delta}(h)$, integrating over $Q_T$, and using the periodic boundary conditions (4.2), we obtain

$$\int_{\Omega} G_{\delta}(h(x, T)) \, dx + a_0 \int\int_{Q_T} h_{xx}^2 \, dx \, dt = \int_{\Omega} G_{\delta}(h_{0,\delta}) \, dx + a_1 \int\int_{Q_T} D''_\varepsilon(h) h_x^2 \, dx \, dt$$

$$\leq \int_{\Omega} G_{\delta}(h_{0,\delta}) \, dx + \frac{an}{2} \int\int_{Q_T} h_x^2 \, dx \, dt \quad (A.12)$$
for limiting process on $\delta \to 0$, and

$$\int_{\Omega} G_\varepsilon(h(x,T)) \, dx + a_0 \int_{Q_T} h_{xx}^2 \, dx \, dt \leq \int_{\Omega} G_\varepsilon(h_{0,\varepsilon}) \, dx + a_1 \int_{Q_T} h_{m-n} h_{xx}^2 \, dx \, dt, \quad h_\varepsilon > 0 \quad (A.13)$$

for limiting process on $\varepsilon \to 0$. In the case of (A.13), we estimate the right-hand using the following equality

$$\int_{\Omega} v^a v_x^2 \, dx = \frac{1}{a+1} \int_{\Omega} v^{a+1} v_{xx} \, dx, \quad v > 0, \quad a > -1. \quad (A.14)$$

Really, for $h_\varepsilon > 0$ we deduce by the Cauchy inequality and bound (A.4) that

$$a_1 \int_{Q_T} h_{m-n} h_{xx}^2 \, dx \, dt = -\frac{a_1}{m-n+1} \int_{Q_T} h_{m-n} h_{xx} \, dx \, dt$$

$$\leq \frac{a_0}{2} \int_{Q_T} h_{xx}^2 \, dx \, dt + \frac{a_1^2}{2a_0(m-n+1)^2} \int_{Q_T} h_{2(m-n+1)} \, dx \, dt$$

$$\leq \frac{a_0}{2} \int_{Q_T} h_{xx}^2 \, dx \, dt + c_8 \int_0^T \left( \int_{\Omega} h_x^2 \, dx \right)^{m-n+1} \, dt + c_9 T, \quad (A.15)$$

where $c_8 = \frac{a_1^2}{2a_0(m-n+1)^2} b_4$, $c_9 = \frac{a_1^2}{2a_0(m-n+1)^2} b_5 M_\varepsilon^{2(m-n+1)}$, and $m > n - 1$. Thus, from (A.12) due to (A.15), we deduce

$$\int_{\Omega} G_\varepsilon(h(x,T)) \, dx + \frac{a_0}{2} \int_{Q_T} h_{xx}^2 \, dx \, dt \leq \int_{\Omega} G_\varepsilon(h_{0,\varepsilon}) \, dx$$

$$+ c_8 \int_0^T \left( \int_{\Omega} h_x^2 \, dx \right)^{m-n+1} \, dt + c_9 T \leq \int_{\Omega} G_\varepsilon(h_{0,\varepsilon}) \, dx$$

$$+ c_{10} \int_0^T \max \left\{ 1, \left( \int_{\Omega} h_x^2 \, dx \right)^{\gamma_2} \right\} \, dt, \quad (A.16)$$

where $\gamma_2 = \max\{3, m-n+1\}$, $m > n - 1$, $c_{10} = c_8 + c_9$. 38
Further, from (A.11) and (A.12) we find for limiting process on $\delta \to 0$

$$\int \frac{h_x^2}{\Omega} dx + \frac{2c_7}{\Omega} \int G_{\delta \varepsilon}(h(x, T)) \, dx + a_0 \int \int f_{\delta \varepsilon}(h)h_{xxx}^2 \, dx \, dt \leq \int \frac{h_x^2}{\Omega} dx + \frac{2c_7}{\Omega} \left( \int G_{\delta \varepsilon}(h(x, T)) \, dx + \frac{a_1}{\varepsilon} \int \int h_x^2 \, dx \, dt \right)$$

$$+ 2c_7 \int_0^T \max \left\{ 1, \left( \int h_x^2(x, t) \, dx \right)^{\gamma_1} \right\} \, dt \leq \int \frac{h_x^2}{\Omega} dx$$

$$+ \frac{2c_7}{\Omega} \left( \int G_{\delta \varepsilon}(h(0) \varepsilon) \, dx + c_{11} \int \max \left\{ 1, \left( \int h_x^2(x, t) \, dx \right)^{\gamma_1} \right\} \, dt \right)$$

where $c_{11} = \frac{2c_7a_0}{c_{10}} + 2c_7$, $\gamma_1 = \max\{3, 2m - n\}$, $m \geq \frac{n}{2}$. Similarly, from (A.11) and (A.16) we find for limiting process on $\varepsilon \to 0$

$$\int \frac{h_x^2}{\Omega} dx + \frac{2c_7}{\Omega} \int G_{\varepsilon}(h(x, T)) \, dx + a_0 \int \int f_{\varepsilon}(h)h_{xxx}^2 \, dx \, dt \leq \int \frac{h_x^2}{\Omega} dx$$

$$+ \frac{2c_7}{\Omega} \left( \int G_{\varepsilon}(h(0), \varepsilon) \, dx + c_{11} \int \max \left\{ 1, \left( \int h_x^2(x, t) \, dx \right)^{\gamma_1} \right\} \, dt \right)$$

$$+ 2c_7 \int_0^T \max \left\{ 1, \left( \int h_x^2(x, t) \, dx \right)^{\gamma_1} \right\} \, dt \leq \int \frac{h_x^2}{\Omega} dx$$

$$+ \frac{2c_7}{\Omega} \left( \int G_{\varepsilon}(h(0), \varepsilon) \, dx + c_{11} \int \max \left\{ 1, \left( \int h_x^2(x, t) \, dx \right)^{\gamma_1} \right\} \, dt \right), \quad (A.17)$$

where $c_{11} = \frac{2c_7c_{10}}{c_{10}} + 2c_7$, $\gamma_1 = \max\{3, 2m - n, m - n + 1\}$, $m \geq \max\{\frac{n}{2}, n - 1\}$.

Applying the nonlinear Grönwall lemma [15] to

$$v(T) \leq v(0) + c_{11} \int_0^T \max\{1, v^{\gamma_1}(t)\} \, dt$$

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with \( v(t) = \int (h_x^2(x,t) + 2c_2/a_0 G_{\delta \varepsilon}(h(x,t))) \, dx \) yields

\[
v(t) \leq \begin{cases} 
 v(0) + c_{11}t & \text{if } t < t_0 := \frac{1-v(0)}{c_{11}} \\
 (1 - c_{11}(\gamma_1 - 1)(t - t_0))^{-1/(\gamma_1 - 1)} & \text{if } t \geq t_0 \\
 (v(0)^{1-\gamma_1} - c_{11}(\gamma_1 - 1)t)^{-1/(\gamma_1 - 1)} & \text{if } v(0) \geq 1.
\end{cases}
\]

From this,

\[
\int_{\Omega} \{h_x^2(x,t) + \frac{2c}{a_0} G_{\delta \varepsilon}(h(x,t))\} \, dx 
\leq 2^{n-1} \max \left\{ 1, \int_{\Omega} \left(h_{0,\delta \varepsilon,x}^2(x) + \frac{2c}{a_0} G_{\delta \varepsilon}(h_{0,\delta \varepsilon}(x)) \right) dx \right\} = K_{\delta \varepsilon} < \infty
\]

for all \( t \in [0, T_{\delta \varepsilon, loc}] \) where

\[
T_{\delta \varepsilon, loc} := \frac{1}{2c_{11}(\gamma_1 - 1)} \min \left\{ 1, \left( \int_{\Omega} h_{0,\delta \varepsilon,x}^2(x) + \frac{2c}{a_0} G_{\delta \varepsilon}(h_{0,\delta \varepsilon}(x)) dx \right)^{-(\gamma_1 - 1)} \right\}.
\]

Using the \( \delta \rightarrow 0, \varepsilon \rightarrow 0 \) convergence of the initial data and the choice of \( \theta \in (0,2/5) \) (see (4.3)) as well as the assumption that the initial data \( h_0 \) has finite entropy (3.2), the times \( T_{\delta \varepsilon, loc} \) converge to a positive limit and the upper bound \( K \) in (A.19) can be taken finite and independent of \( \delta \) and \( \varepsilon \) for \( \delta \) and \( \varepsilon \) sufficiently small. (We refer the reader to the end of the proof of Lemma 6.1 in this Appendix for a fuller explanation of a similar case.) Therefore there exists \( \delta_0 > 0 \) and \( \varepsilon_0 > 0 \) and \( K \) such that the bound (A.19) holds for all \( 0 \leq \delta < \delta_0 \) and \( 0 < \varepsilon < \varepsilon_0 \) with \( K \) replacing \( K_{\delta \varepsilon} \) and for all

\[
0 \leq t \leq T_{loc} := \frac{9}{10} \lim_{\varepsilon \rightarrow 0, \delta \rightarrow 0} T_{\delta \varepsilon, loc}.
\]

Using the uniform bound on \( \int h_x^2 \) that (A.19) provides, one can find a uniform-in-\( \delta \) and \( \varepsilon \) bound for the right-hand-side of (4.17) yielding the desired a priori bound (4.17). Similarly, one can find a uniform-in-\( \delta \) and \( \varepsilon \) bound for the right-hand-side of (4.16) yielding the desired a priori bound (4.8).

To prove the bound (4.9), multiply (4.1) by \(-a_0 h_{xx} - a_1 D_\varepsilon'(h)\), integrate over \( Q_T \), integrate by parts, use the periodic boundary conditions (4.2), to find

\[
\mathcal{E}_{\delta \varepsilon}(T) + \int_{Q_T} f_{\delta \varepsilon}(h)(a_0 h_{xxx} + a_1 D_\varepsilon''(h)h_x)^2 \, dx \, dt = \mathcal{E}_{\delta \varepsilon}(0).
\]

The parameters \( \delta_0 \) and \( \varepsilon_0 \) are determined by \( a_0, a_1, |\Omega|, \int h_0, \|h_{0x}\|_2, \int h_0^{2-n}, \) by how quickly \( M_{\delta \varepsilon} \) converges to \( M \), and by how quickly the approximate initial data \( (4.5), h_{0,\delta \varepsilon}, \) converge to \( h_0 \) in \( H^1(\Omega) \).

The time \( T_{loc} \) and the constants \( K_1 \) and \( K_2 \) are determined by \( \delta_0, \varepsilon_0, a_0, a_1, |\Omega|, \int h_0, \|h_{0x}\|_2, \) and \( \int h_0^{2-n} \). 

\[\square\]
Sketch of Proof of Lemma 4.2: In the following, we denote the positive, classical solution \( h_\varepsilon \) by \( h \) whenever there is no chance of confusion.

Taking \( \delta \to 0 \) in (A.9) yields

\[
\frac{1}{2} \int_\Omega h_x^2 \, dx + \frac{a_0}{2} \int_{Q_T} f_\varepsilon(h) h_{xxx}^2 \, dx \, dt \\
\leq \frac{1}{2} \int_\Omega h_{0\varepsilon}^2 \, dx + \frac{a_1^2}{2a_0} \int_{Q_T} |h|^{2m-n} h_x^2 \, dx \, dt \\
\leq \frac{1}{2} \int_\Omega h_{0\varepsilon}^2 \, dx + \frac{a_1^2}{2a_0} \int_0^T \|h(\cdot, t)\|_{2m-n}^{2m-n} \int_{\Omega} h_x^2(x, t) \, dx \, dt. \tag{A.23}
\]

Applying the nonlinear Grönwall lemma [15] to

\[
v(T) \leq v(0) + \int_0^T A_1(t) \, v(t) \, dt
\]

with \( v(t) = \int h_x^2(x, t) \, dx \), \( A_1(t) = a_1^2/a_0 \|h(\cdot, t)\|_{2m-n}^{2m-n} \) yields

\[
v(T) \leq v(0) \, e^{B_1(T)}, \text{ where } B_1(t) = \int_0^t A_1(s) \, ds.
\]

Similarly, taking \( \delta \to 0 \) in (A.9) and (A.12), due to (A.14) and (A.6), yield

\[
\int_{\Omega} h_x^2 \, dx + a_0 \int_{Q_T} f_\varepsilon(h) h_{xxx}^2 \, dx \, dt \\
\leq \int_{\Omega} h_{0\varepsilon}^2 \, dx + \frac{a_1^2}{a_0} \int_{Q_T} h_x^{2m-n} h_x^2 \, dx \, dt \\
= \int_{\Omega} h_{0\varepsilon}^2 \, dx - \frac{a_1^2}{a_0(2m-n+1)} \int_{Q_T} h_x^{2m-n+1} h_{xx} \, dx \, dt \\
\leq \int_{\Omega} h_{0\varepsilon}^2 \, dx \\
+ \frac{a_0}{2} \int_{Q_T} h_{xx}^2 \, dx \, dt + \frac{a_1^4}{2a_0^2(2m-n+1)^2} \int_{Q_T} h_x^{2(2m-n+1)} \, dx \, dt \\
\leq \int_{\Omega} h_{0\varepsilon}^2 \, dx \\
\leq \frac{a_0}{2} \int_{Q_T} h_{xx}^2 \, dx \, dt + \frac{a_1^4}{2a_0^2(2m-n+1)^2} \int_0^T \|h(\cdot, t)\|_{4m-n}^{4m-n} \int_{\Omega} G_\varepsilon(h(x, t)) \, dx \, dt, \tag{A.24}
\]

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\[ \int_\Omega G_\varepsilon(h(x,T)) \, dx + a_0 \int_{Q_T} h_{xx}^2 \, dx \, dt \leq \int_\Omega G_\varepsilon(h_{0,\varepsilon}) \, dx + a_1 \int_{Q_T} h^{m-n} h_x^2 \, dx \, dt \]

\[ \leq \int_\Omega G_\varepsilon(h_{0,\varepsilon}) \, dx - \frac{a_1}{m-n+1} \int_{Q_T} h^{m-n+1} h_{xx} \, dx \, dt \leq \int_\Omega G_\varepsilon(h_{0,\varepsilon}) \, dx \]

\[ + \frac{a_0}{2} \int_{Q_T} h_{xx}^2 \, dx \, dt + \frac{a_1^2}{2a_0(m-n+1)^2} \int_{Q_T} h^{2(m-n+1)} \, dx \, dt \leq \int_\Omega G_\varepsilon(h_{0,\varepsilon}) \, dx \]

\[ + \frac{a_0}{2} \int_{Q_T} h_{xx}^2 \, dx \, dt + \frac{a_1^2}{2a_0(m-n+1)^2} \int_0^T \|h(\cdot,t)\|_{2m-n}^2 \int_\Omega G_\varepsilon(h(x,t)) \, dx \, dt. \quad (A.25) \]

Summing (A.24) and (A.25), we find that

\[ \int_\Omega \{h_x^2(x,T) + G_\varepsilon(h(x,T))\} \, dx + a_0 \int_{Q_T} f_\varepsilon(h) h_{xx}^2 \, dx \, dt \]

\[ \leq \int_\Omega \{h_{0x,x}^2 + G_\varepsilon(h_{0,\varepsilon})\} \, dx + \int_0^T A_2(t) \int \{h_x^2(x,t) + G_\varepsilon(h(x,t))\} \, dx \, dt, \quad (A.26) \]

where

\[ A_2(t) = \frac{a_0^2}{2a_0(2m-n+1)^2} \|h_\varepsilon(\cdot,t)\|_{4m-n}^4 + \frac{a_1^2}{2a_0(m-n+1)^2} \|h_\varepsilon(\cdot,t)\|_{2m-n}^2. \]

Applying the nonlinear Grönwall lemma [15] to

\[ v(T) \leq v(0) + \int_0^T A_2(t) v(t) \, dt \]

with \( v(t) = \int \{h_x^2(x,t) + G_\varepsilon(h(x,t))\} \, dx \), yields

\[ v(T) \leq v(0) e^{B_2(T)}, \text{ where } B_2(t) = \int_0^t A_2(s) \, ds. \]

Sketch of Proof of Lemma 4.3 In the following, we denote the positive, classical solution \( h_\varepsilon \) by \( h \) whenever there is no chance of confusion.

Multiplying (4.11) with \( \delta = 0 \) by \( G_\varepsilon^{(\alpha)}(h) \), integrating over \( Q_T \), taking \( \delta \to 0 \), and using the periodic boundary conditions (4.2), yields

\[ \int_\Omega G_\varepsilon^{(\alpha)}(h(x,T)) \, dx + a_0 \int_{Q_T} h^{\alpha} h_{xx}^2 \, dx \, dt + a_0 \frac{\alpha(1-\alpha)}{3} \int_{Q_T} h^{\alpha-2} h_x^4 \, dx \, dt \quad (A.27) \]

\[ = \int_\Omega G_\varepsilon^{(\alpha)}(h_{0,\varepsilon}) \, dx + a_1 \int_{Q_T} h^{\alpha} D_\varepsilon^{(\alpha)}(h) h_x^2 \, dx \, dt \leq \int_\Omega G_\varepsilon^{(\alpha)}(h_{0,\varepsilon}) \, dx + a_1 \int_{Q_T} h^{\alpha+m-n} h_x^2 \, dx \, dt. \]
Case 1: \( 0 < \alpha < 1 \). The coefficient multiplying \( \int \int h^{\alpha-2}h_x^4 \) in (A.27) is positive and can therefore be used to control the term \( \int \int h^{\alpha+m-n}h_x^2 \) on the right-hand side of (A.27). Specifically, using the Cauchy-Schwartz inequality and the Cauchy inequality,

\[
a_1 \int \int h^{\alpha+m-n}h_x^2 \ dx \ dt \leq a_1 \int \int h^{\alpha+m-n}h_x^2 \ dx \ dt \leq \frac{a_0 \alpha(1-\alpha)}{6} \int \int h^{\alpha-2}h_x^4 \ dx \ dt + \frac{3a_0^2}{2\alpha(1-\alpha)} \int \int h^{\alpha+2(m-n+1)} \ dx \ dt. \quad (A.28)
\]

Using the bound (A.28) in (A.27) yields

\[
\int \int G_x^{(\alpha)}(h(x,T)) \ dx + a_0 \int \int h^\alpha h_x^2 \ dx \ dt + a_0 \frac{(1-\alpha)}{6} \int \int h^{\alpha-2}h_x^4 \ dx \ dt \leq \int \int G_x^{(\alpha)}(h_{0\varepsilon}) \ dx \ + \ \frac{3a_0^2}{2\alpha(1-\alpha)} \int \int h^{\alpha+2(m-n+1)} \ dx \ dt. \quad (A.29)
\]

By (A.4),

\[
\int \int h^{\alpha+2(m-n+1)} \ dx \ dt \leq b_4 \int_0^T \left( \int_\Omega h_x^2 \ dx \right)^{\frac{\alpha}{2}+m-n+1} \ dt + b_5 M^{\alpha+2(m-n+1)} \ T. \quad (A.30)
\]

Using (A.31) in (A.29) yields

\[
\int \int G_x^{(\alpha)}(h(x,T)) \ dx + a_0 \int \int h^\alpha h_x^2 \ dx \ dt + a_0 \frac{(1-\alpha)}{6} \int \int h^{\alpha-2}h_x^4 \ dx \ dt \leq \int \int G_x^{(\alpha)}(h_{0\varepsilon}) \ dx \ + \ \frac{d_1}{\alpha} \int_0^T \left( \int_\Omega h_x^2 \ dx \right)^{\frac{\alpha}{2}+m-n+1} \ dt + d_2 \ T
\]

\[
\leq \int \int G_x^{(\alpha)}(h_{0\varepsilon}) \ dx \ + \ d_3 \int_0^T \ max \left\{ 1, \left( \int_\Omega h_x^2 \ dx \right)^{\frac{\alpha}{2}+m-n+1} \right\} \ dt \quad (A.32)
\]

where

\[
d_1 = b_4 \frac{3a_0^2}{2\alpha(1-\alpha)}, \quad d_2 = b_5 \frac{3a_0^2}{2\alpha(1-\alpha)} M^{\alpha+2(m-n+1)}, \quad d_3 = d_1 + d_2.
\]

Taking \( \delta \rightarrow 0 \) in (A.9) yields

\[
\int \int h_x^2 \ dx \ + \ a_0 \int \int f(h) h_{xxx}^2 \ dx \ dt \leq \int \int h_{0\varepsilon,xxx}^2 \ dx \ + \ \frac{a_0^2}{a_0} \int \int h^{2m-n}h_x^2 \ dx \ dt. \quad (A.33)
\]
Applying the Cauchy inequality,
\[ \frac{a^2}{a_0} \int_Q \int h^{2m-n} h_x^2 \, dx \, dt \leq \frac{a_0 \alpha (1-\alpha)}{6} \int_Q \int h^{2m-n} h_x^4 \, dx \, dt + \frac{3a_4^4}{2a_0^3 \alpha (1-\alpha)} \int_Q \int h^{2(2m-n+1)-\alpha} \, dx \, dt. \] (A.34)

By (A.34),
\[ \int_Q \int h^{2(2m-n+1)-\alpha} \, dx \, dt \leq b_4 \int_0^T \left( \int_\Omega h_x^2 \, dx \right)^{2m-n+1-\frac{\alpha}{2}} \, dt + b_5 M_x^{2(2m-n+1)-\alpha} T. \] (A.35)

Using (A.34) and (A.35) in (A.33) yields
\[ \int_\Omega h_x^2 \, dx + a_0 \int_Q \int f_\varepsilon(h) h_{xx}^2 \, dx \, dt \leq \int_\Omega h_{0x,x}^2 \, dx + \frac{a_0 \alpha (1-\alpha)}{6} \int_Q \int h^{2m-n} h_x^4 \, dx \, dt \\
+ d_4 \int_0^T \left( \int_\Omega h_x^2 \, dx \right)^{2m-n+1-\frac{\alpha}{2}} \, dt + d_5 \int_\Omega h_{0x,x}^2 \, dx \\
+ \frac{a_0 \alpha (1-\alpha)}{6} \int_Q \int h^{2m-n} h_x^4 \, dx \, dt + d_6 \int_0^T \max \left\{ 1, \left( \int_\Omega h_x^2 \, dx \right)^{2m-n+1-\frac{\alpha}{2}} \right\} \, dt \] (A.36)

where
\[ d_4 = \frac{3a_4^4}{2a_0^3 \alpha (1-\alpha)} b_4, \quad d_5 = b_5 \frac{3a_4^4}{2a_0^3 \alpha (1-\alpha)} M_x^{2(2m-n+1)-\alpha}, \quad d_6 = d_4 + d_5. \]

Add
\[ \int_\Omega G_x^{(\alpha)}(h(x,T)) \, dx \]

to both sides of (A.36) and add
\[ a_0 \int_Q h^\alpha h_{xx}^2 \, dx \, dt \]

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to the right–hand side of the resulting inequality. Using (A.32) yields

\[
\int_{\Omega} h_{x}^2(x, T) \, dx + \int_{\Omega} G_{\varepsilon}^{(a)}(h(x, T)) \, dx + a_{0} \int_{Q_{T}} f_{\varepsilon}(h) h_{xxx} \, dx \, dt
\]

(A.37)

\[
\leq \int_{\Omega} h_{0x,x}^2 \, dx + \int_{\Omega} G_{\varepsilon}^{(a)}(h_{0x}) \, dx + d_{3} \int_{0}^{T} \max \left\{ 1, \left( \int_{\Omega} h_{x}^2 \, dx \right)^{\frac{\alpha}{2} + m - n + 1} \right\} \, dt
\]

\[
+ d_{6} \int_{0}^{T} \max \left\{ 1, \left( \int_{\Omega} h_{x}^2 \, dx \right)^{2m - n + 1 - \frac{\alpha}{2}} \right\} \, dt
\]

\[
\leq \int_{\Omega} h_{0x,x}^2 \, dx + \int_{\Omega} G_{\varepsilon}^{(a)}(h_{0x}) \, dx + d_{7} \int_{0}^{T} \max \left\{ 1, \left( \int_{\Omega} h_{x}^2 \, dx \right)^{\gamma_{3}} \right\} \, dt
\]

where \( d_{7} = d_{3} + d_{6} \), \( \gamma_{3} = \max \{ \alpha/2 + m - n + 1, 2m - n + 1 - \alpha/2 \} \).

Applying the nonlinear Grönwall lemma [15] to

\[
v(T) \leq v(0) + d_{7} \int_{0}^{T} \max\{1, v^{\gamma_{3}}(t)\} \, dt
\]

with \( v(T) = \int h_{x}^2(x, T) + G_{\varepsilon}^{(a)}(h(x, T)) \, dx \) yields

\[
v(t) \leq \begin{cases} v(0) + d_{7}t & \text{if } t < t_{0} := \frac{1 - v(0)}{d_{9}} \\
(1 - d_{7}(\gamma_{3} - 1)(t - t_{0}))^{-\frac{1}{\gamma_{3} - 1}} & \text{if } t \geq t_{0} \end{cases}
\]

if \( v(0) < 1 \)

\[
(v(0))^{-\gamma_{3} - 1} - d_{7}(\gamma_{3} - 1)t \, \left( \frac{1}{\gamma_{3} - 1} \right)^{-\gamma_{3} - 1} & \text{if } v(0) \geq 1
\]

From this,

\[
\int_{\Omega} (h_{x}^2(x, T) + G_{\varepsilon}^{(a)}(h(x, T))) \, dx
\]

(A.38)

\[
\leq 2 \frac{1}{\gamma_{3} - 1} \max \left\{ 1, \int_{\Omega} (h_{0x,x}^2(x) + G_{\varepsilon}^{(a)}(h_{0x}(x))) \, dx \right\} = K_{\varepsilon} < \infty
\]

for all

\[
0 \leq T \leq T_{\varepsilon,\text{loc}}^{(a)} := \frac{1}{2d_{5}(\gamma_{3} - 1)} \min \left\{ 1, \left( \int_{\Omega} (h_{0x,x}^2(x) + G_{\varepsilon}^{(a)}(h_{0x}(x))) \, dx \right)^{-(\gamma_{3} - 1)} \right\}
\]

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The bound (A.38) holds for all $0 < \varepsilon < \varepsilon_0$ where $\varepsilon_0$ is from Lemma 4.1 and for all $t \leq \min\{T_{\text{loc}}, T^{(\alpha)}_{\varepsilon,\text{loc}}\}$ where $T_{\text{loc}}$ is from Lemma 4.1.

Using the $\delta \to 0, \varepsilon \to 0$ convergence of the initial data and the choice of $\theta \in (0, 2/5)$ (see (4.5)) as well as the assumption that the initial data $h_0$ has finite $\alpha$-entropy (3.11), the times $T^{(\alpha)}_{\varepsilon,\text{loc}}$ converge to a positive limit and the upper bound $K_{\varepsilon}$ in (A.38) can be taken finite and independent of $\varepsilon$. (We refer the reader to the end of the proof of Lemma 6.1 in this Appendix for a fuller explanation of a similar case.) Therefore there exists $\varepsilon_0^{(\alpha)}$ and $K$ such that the bound (A.38) holds for all $0 < \varepsilon < \varepsilon_0^{(\alpha)}$ with $K$ replacing $K_{\varepsilon}$ and for all

$$0 \leq t \leq T^{(\alpha)}_{\text{loc}} := \min\left\{T_{\text{loc}}, \frac{9}{10} \lim_{\varepsilon \to 0} T^{(\alpha)}_{\varepsilon,\text{loc}}\right\}$$

(A.39)

where $T_{\text{loc}}$ is the time from Lemma 4.1. Also, without loss of generality, $\varepsilon_0^{(\alpha)}$ can be taken to be less than or equal to the $\varepsilon_0$ from Lemma 4.1

Using the uniform bound on $\int h^2$ that (A.38) provides, one can find a uniform-in-$\varepsilon$ bound for the right-hand-side of (A.32) yielding the desired bound

$$\int_{\Omega} G^\varepsilon_\alpha(h(x, T)) \, dx + a_0 \int_{Q_T} h^\alpha h_{xx}^2 \, dxdt + a_0 \frac{\alpha(1-\alpha)}{6} \int_{Q_T} h^{\alpha-2} h_x^4 \, dxdt \leq K_1$$

(A.40)

which holds for all $0 < \varepsilon < \varepsilon_0^{(\alpha)}$ and all $0 \leq T \leq T^{(\alpha)}_{\text{loc}}$.

It remains to argue that (A.40) implies that for all $0 < \varepsilon < \varepsilon_0^{(\alpha)}$ that $h_{\varepsilon}^{\alpha/2+1}$ and $h_{\varepsilon}^{\alpha/4+1/2}$ are contained in balls in $L^2(0, T; H^2(\Omega))$ and $L^2(0, T; W^1_4(\Omega))$ respectively. It suffices to show that

$$\int_{Q_T} (h_{\varepsilon}^{\alpha/2+1})^2_{xx} \, dxdt \leq K, \quad \int_{Q_T} (h_{\varepsilon}^{\alpha/4+1/2})^4_{x} \, dxdt \leq K$$

for some $K$ that is independent of $\varepsilon$ and $T$. The integral $\int_{Q_T} (h_{\varepsilon}^{\alpha/2+1})^2_{xx}$ is a linear combination of $\int_{Q_T} h^{\alpha-2} h_{xx}^4$, $\int_{Q_T} h^{\alpha-1} h_x^2 h_{xx}$, and $\int_{Q_T} h^{\alpha} h_{xx}^2$. Integration by parts and the periodic boundary conditions imply

$$\int_{Q_T} h^{\alpha-2} h_x^4 \, dxdt = \int_{Q_T} h^{\alpha-1} h_x^2 h_{xx} \, dxdt$$

(A.41)

Hence $\int_{Q_T} (h_{\varepsilon}^{\alpha/2+1})^2_{xx}$ is a linear combination of $\int_{Q_T} h^{\alpha-2} h_{xx}^4$, and $\int_{Q_T} h^{\alpha} h_{xx}^2$. By (A.40), the two integrals are uniformly bounded independent of $\varepsilon$ and $T$ hence $\int_{Q_T} (h_{\varepsilon}^{\alpha/2+1})^2_{xx}$ is as well, yielding the first part of (4.14).

The uniform bound of $\int_{Q_T} (h_{\varepsilon}^{\alpha/4+1/2})^4_{x}$ follows immediately from the uniform bound of $\int_{Q_T} h^{\alpha-2} h_{xx}^4$, yielding the second part of (4.14).

**Case 2:** $-\frac{1}{2} < \alpha < 0$. For $\alpha < 0$ the coefficient multiplying $\int h^{\alpha-2} h_{xx}^4$ in (4.27) is negative. However, we will show that if $\alpha > -1/2$ then one can replace this coefficient with a positive coefficient while also controlling the term $\int h^{\alpha} h_{xx}^2$ on the right-hand-side of (4.27).
Applying the Cauchy-Schwartz inequality to the right-hand side of (A.41), dividing by $\sqrt{\int \int h^{\alpha-2}h_x^4}$, and squaring both sides of the resulting inequality yields

$$\int \int \frac{h^{\alpha-2}h_x}{\sqrt{\int \int h^{\alpha-2}h_x^4}} dxdt \leq \frac{9}{(1-\alpha)^2} \int \int h^\alpha h_{xx}^2 dxdt \quad \forall \alpha < 1.$$  

(A.42)

Using (A.42) in (A.27) yields

$$\int \Omega G_\varepsilon^{(\alpha)}(h(x, T)) dx + a_0 \frac{1+2\alpha}{1-\alpha} \int \int h^\alpha h_{xx}^2 dxdt$$

$$\leq \int \Omega G_\varepsilon^{(\alpha)}(h_0\varepsilon) dx + a_1 \int \int h^{\alpha+m-n}h_x^2 dxdt.$$  

(A.43)

Note that if $\alpha > -1/2$ then all the terms on the left-hand side of (A.43) are positive. We now control the term $\int \int h^{\alpha}h_x^2$ on the right-hand side of (A.43).

By integration by parts and the periodic boundary conditions

$$\int \int h^{\alpha+m-n}h_x^2 dxdt = -\frac{1}{\alpha+m-n+1} \int \int h^{\alpha+m-n+1}h_{xx} dxdt$$  

(A.44)

Applying the Cauchy-Schwartz inequality and the Cauchy inequality to (A.44) yields

$$a_1 \int \int h^{\alpha+m-n}h_x^2 dxdt \leq \frac{a_0(1+2\alpha)}{2(1-\alpha)} \int \int h^\alpha h_{xx}^2 dxdt + \frac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(\alpha+m-n+1)^2} \int \int h^\alpha h_{xx}^2 dxdt$$  

(A.45)

Using inequality (A.45) in (A.43) yields

$$\int \Omega G_\varepsilon^{(\alpha)}(h(x, T)) dx + a_0 \frac{1+2\alpha}{2(1-\alpha)} \int \int h^\alpha h_{xx}^2 dxdt$$

$$\leq \int \Omega G_\varepsilon^{(\alpha)}(h_0\varepsilon) dx + \frac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(\alpha+m-n+1)^2} \int \int h^\alpha h_{xx}^2 dxdt + \frac{a_1^2(1-\alpha)}{2a_0(1+2\alpha)(\alpha+m-n+1)^2} \int \int h^\alpha h_{xx}^2 dxdt.$$  

(A.46)

Adding

$$\frac{a_0(1+2\alpha)(1-\alpha)}{36} \int \int h^{\alpha-2}h_x^4 dxdt$$

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to both sides of (A.46) and using the inequality (A.42) yields
\[
\int_{\Omega} G_{\varepsilon}^{(\alpha)}(h(x, T)) \, dx + a_0 \frac{(1+2\alpha)}{4(1-\alpha)} \int_{Q_T} h^{\alpha} h_{xx}^2 \, dx dt \tag{A.47}
\]
\[
+ \frac{a_0(1+2\alpha)(1-\alpha)}{36 \quad \int_{Q_T} h^{\alpha-2} h_x^4 \, dx dt \leq \int_{\Omega} G_{\varepsilon}^{(\alpha)}(h_{0\varepsilon}) \, dx \\
+ \frac{a_0^2(1-\alpha)}{2a_0(1+2\alpha)(\alpha+m-n+1)^2} \int_{Q_T} h^{\alpha+2(m-n+1)} \, dx dt.
\]

Using (A.31) in (A.47) yields
\[
\int_{\Omega} G_{\varepsilon}^{(\alpha)}(h(x, T)) \, dx + a_0 \frac{(1+2\alpha)}{4(1-\alpha)} \int_{Q_T} h^{\alpha} h_{xx}^2 \, dx dt \\
+ \frac{a_0(1+2\alpha)(1-\alpha)}{36 \quad \int_{Q_T} h^{\alpha-2} h_x^4 \, dx dt \leq \int_{\Omega} G_{\varepsilon}^{(\alpha)}(h_{0\varepsilon}) \, dx \\
+ e_1 \int_{0}^{T} \left( \int_{\Omega} h_x^2 \, dx \right)^{\alpha} + m-n+1 \quad dt + e_2 \, T \\
\leq \int_{\Omega} G_{\varepsilon}^{(\alpha)}(h_{0\varepsilon}) \, dx + e_3 \int_{0}^{T} \max \left\{ 1, \left( \int_{\Omega} h_x^2 \, dx \right)^{\alpha} \right\} \quad dt \tag{A.48}
\]

where
\[
e_1 = \frac{a_0^2(1-\alpha)}{2a_0(1+2\alpha)(\alpha+m-n+1)^2} \quad b_4, \quad e_2 = b_5 \frac{a_0^2(1-\alpha)}{2a_0(1+2\alpha)(\alpha+m-n+1)} \quad M_{\varepsilon}^{\alpha+2(m-n+1)},
\]
and \( e_3 = e_1 + e_2 \).

Recall the bound (A.33):
\[
\int_{\Omega} h_x^2 \, dx + a_0 \int_{Q_T} f_{\varepsilon}(h) h_{xx}^2 \, dx dt \leq \int_{\Omega} h_{0\varepsilon,x}^2 \, dx + a_0^2 \int_{Q_T} h^{2m-n} h_x^2 \, dx dt. \tag{A.49}
\]

As before, by the Cauchy-Schwartz inequality and the Cauchy inequality,
\[
\frac{a_0^2}{a_0} \int_{Q_T} h^{2m-n} h_x^2 \, dx dt \leq \frac{a_0(1+2\alpha)(1-\alpha)}{36 \quad \int_{Q_T} h^{\alpha-2} h_x^4 \, dx dt \\
+ \frac{9a_0^4}{a_0(1+2\alpha)(1-\alpha)} \int_{Q_T} h^{2(2m-n+1)-\alpha} \, dx dt. \tag{A.50}
\]
Using (A.50), and (A.35) in (A.49) yields

\[
\int_{\Omega} h_x^2 \, dx + a_0 \int_{Q_T} f_\varepsilon(h) h_{xx}^2 \, dx \, dt \leq \int_{\Omega} h_0^2 \, dx + \frac{a_0(1+2\alpha)(1-\alpha)}{36} \int_{Q_T} h^{\alpha - 2} h_x^4 \, dx \, dt
\]

\[
+ e_4 \int_0^T \left( \int_{\Omega} h_x^2 \, dx \right)^{2m-n+1-\frac{\alpha}{2}} \, dt + e_5 \, T \leq \int_{\Omega} h_0^2 \, dx \quad (A.51)
\]

\[
+ \frac{a_0(1+2\alpha)(1-\alpha)}{36} \int_{Q_T} h^{\alpha - 2} h_x^4 \, dx \, dt + e_6 \int_0^T \max \left\{ 1, \left( \int_{\Omega} h_x^2 \, dx \right)^{2m-n+1-\frac{\alpha}{2}} \right\} \, dt
\]

where

\[
e_4 = \frac{9a_1^4}{a_0^4(1+2\alpha)(1-\alpha)} \, b_4, \quad e_5 = b_5 \frac{9a_1^4}{a_0^4(1+2\alpha)(1-\alpha)} \, M_\varepsilon^{2(2m-n+1) - \alpha}, \quad e_6 = e_4 + e_5.
\]

Add

\[
\int_{\Omega} G_\varepsilon^{(\alpha)}(h(x, T)) \, dx
\]

to both sides of (A.51) and add

\[
\frac{a_0(1+2\alpha)}{4(1-\alpha)} \int_{Q_T} h^{\alpha} h_{xx}^2 \, dx \, dt
\]

to the right-hand side of the resulting inequality. Just as (A.32) and (A.33) yielded (A.37), (A.48) combined with the above inequality yields

\[
\int_{\Omega} h_x^2(x, T) \, dx + \int_{\Omega} G_\varepsilon^{(\alpha)}(h(x, T)) \, dx + a_0 \int_{Q_T} f_\varepsilon(h) h_{xx}^2 \, dx \, dt \quad (A.52)
\]

\[
\leq \int_{\Omega} h_0^2 \, dx + \int_{\Omega} G_\varepsilon^{(\alpha)}(h_0) \, dx + e_3 \int_0^T \max \left\{ 1, \left( \int_{\Omega} h_x^2 \, dx \right)^{\frac{\alpha}{2} + m-n+1} \right\} \, dt
\]

\[
+ e_6 \int_0^T \max \left\{ 1, \left( \int_{\Omega} h_x^2 \, dx \right)^{2m-n+1-\frac{\alpha}{2}} \right\} \, dt
\]

\[
\leq \int_{\Omega} h_0^2 \, dx + \int_{\Omega} G_\varepsilon^{(\alpha)}(h_0) \, dx + e_7 \int_0^T \max \left\{ 1, \left( \int_{\Omega} h_x^2 \, dx \right)^{\gamma_3} \right\} \, dt
\]

where \(e_7 = e_3 + e_6, \quad \gamma_3 = \max\{\alpha/2 + m - n + 1, 2m - n + 1 - \alpha/2\}\).
The rest of the proof now continues as in the $0 < \alpha < 1$ case. Specifically, one finds a bound
\[
\int_{\Omega} (h^2_x(x, T) + G^{(\alpha)}_\varepsilon(h(x, T))) \, dx
\leq 2\gamma^{-1} \max \left\{ 1, \frac{1}{2\gamma}; \int_{\Omega} \left( h^{0,\varepsilon,x}(x) + G^{(\alpha)}_\varepsilon(h_{0,\varepsilon}(x)) \right) \, dx \right\} = K_\varepsilon < \infty
\]
for all
\[
0 \leq T \leq T^{(\varepsilon,loc)} := \min \left\{ 1, \frac{1}{2\gamma}; \int_{\Omega} \left( h^{0,\varepsilon,x}(x) + G^{(\alpha)}_\varepsilon(h_{0,\varepsilon}(x)) \right) \, dx \right\}.
\]

The time $T^{(\varepsilon,loc)}$ is defined as in (A.39) and the uniform bound (A.53) used to bound the right hand side of (A.48) yields the desired bound
\[
\int_{\Omega} G^{(\alpha)}_\varepsilon(h(x, T)) \, dx + \frac{\alpha_0(1 + 2\alpha)}{4(1 - \alpha)} \int_{Q_T} h^\alpha h^2_x \, dxdt
\]
\[
+ \frac{\alpha_0(1 + 2\alpha)(1 - \alpha)}{36} \int_{Q_T} h^{\alpha - 2} h^4_x \, dxdt \leq K_2.
\]  
(A.54)

\[\square\]

\textit{Sketch of Proof of Lemma 6.1.} In the following, we denote the positive, classical solution $h_\varepsilon$ constructed in Lemma 4.3 by $h$ (whenever there is no chance of confusion).

Recall the entropy function $G^{(\alpha)}_0(z)$ defined by (4.12). Using the local entropy inequality (7.1) with $\zeta = \zeta(x)$ from Lemma 7.1, we obtain
\[
\int_{\Omega} \zeta^4(x)G^{(\alpha)}_0(h(x, T)) \, dx + C_1 \int_{Q_T} (h^{\alpha 2}_{-\frac{1}{2}})^2 \zeta^4 \, dxdt \leq \int_{\Omega} \zeta^4(x)G^{(\alpha)}_0(h_0) \, dx +
\]
\[
C_2 \int_{Q_T} h^{\alpha + 2}(\zeta^4_x + \zeta^2_{xx}) \, dxdt + C_3 \int_{Q_T} h^{2(m - n + 1) + \alpha} \zeta^4 \, dxdt.
\]  
(A.55)

Due to $h \in L^\infty(0, T^{(\varepsilon,loc)}; H^1(\Omega))$, we deduce from (A.55) that
\[
\int_{\Omega} \zeta^4(x)G^{(\alpha)}_0(h(x, T)) \, dx \leq \int_{\Omega} \zeta^4(x)G_0(h_0(x)) \, dx + C_4 T \leq K < \infty
\]  
(A.56)

for any $T \in [0, T^{(\varepsilon,loc)}].$  
\[\square\]
B Proof of second moment estimate

Sketch of Proof of Lemma 8.1. Let

\[ \tilde{G}_\varepsilon(z) = \frac{z^{2-n}}{2-n} - \frac{\varepsilon z^s}{s-2}, \quad \tilde{G}'_\varepsilon(z) = \frac{z}{f_\varepsilon(z)}, \quad (B.1) \]

Here we use the equality

\[ f'_\varepsilon(z) = n z^{-1} f_\varepsilon(z) + \varepsilon(s-n)z^{-s}. \]

Multiplying (4.1) for \( \delta = 0 \) by \( x^2 \tilde{G}'_\varepsilon(h) \), and integrating on \( Q_T \), yields

\[
\int_{\Omega} x^2 \tilde{G}_\varepsilon(h) \, dx - \int_{\Omega} x^2 \tilde{G}_\varepsilon(h_0) \, dx = \int_{Q_T} f_\varepsilon(h)(a_0 h_{xxx} + a_1 D'''_\varepsilon(h) h_x)(2x \frac{h}{f_\varepsilon(h)} + x^2 \tilde{G}''_\varepsilon(h) h_x) \, dxdt =
\]

\[
2 \int_{Q_T} xh(a_0 h_{xxx} + a_1 D'''_\varepsilon(h) h_x) \, dxdt +
\]

\[
(1 - n) \int_{Q_T} x^2 h_x(a_0 h_{xxx} + a_1 D'''_\varepsilon(h) h_x) \, dxdt -
\]

\[
\varepsilon(s-n) \int_{Q_T} x^2 h^{-s} f_\varepsilon(h) h_x(a_0 h_{xxx} + a_1 D'''_\varepsilon(h) h_x) \, dxdt =: I_1 + I_2 + I_3. \quad (B.2)
\]

We now bound the terms \( I_1, I_2 \) and \( I_3 \). First,

\[
I_1 = 2a_0 \int_0^T \left. xhh_{xx} \right|_{\partial \Omega} \, dx - 2a_0 \int_{Q_T} \{hh_{xx} + x h_x h_{xx} \} \, dxdt +
\]

\[
2a_1 \int_{Q_T} xhD'''_\varepsilon(h) h_x \, dxdt = \int_0^T x(2a_0 hh_{xx} - a_0 h^2_x + 2a_1 \mathcal{L}_\varepsilon(h)) \left. \right|_{\partial \Omega} \, dt +
\]

\[
3a_0 \int_{Q_T} h_x^2 \, dxdt - 2a_1 \int_{Q_T} \mathcal{L}_\varepsilon(h) \, dxdt, \quad \mathcal{L}_\varepsilon(z) := \int_0^z \tau D''_\varepsilon(\tau) \, d\tau. \quad (B.3)
\]
and

\[
I_2 = a_0(1 - n) \int_Q x^2 h_x h_{xxx} \, dx \, dt + a_1(1 - n) \int_Q x^2 D''_\varepsilon(h) h_x^2 \, dx \, dt = \\
-2a_0(1 - n) \int_Q x h_x h_x \, dx \, dt - a_0(1 - n) \int_Q x^2 h_{xx}^2 \, dx \, dt + \\
a_1(1 - n) \int_Q x^2 D''_\varepsilon(h) h_x^2 \, dx \, dt = -a_0(1 - n) \int_T x^2 \left| h_x \right|_{\partial\Omega} \, dt + a_0(1 - n) \int_Q h_x^2 \, dx \, dt - \\
a_0(1 - n) \int_Q x^2 h_{xx}^2 \, dx \, dt + a_1(1 - n) \int_Q x^2 D''_\varepsilon(h) h_x^2 \, dx \, dt. \tag{B.4}
\]

Using the Hölder inequality and (4.9) with \( \delta = 0 \), we find

\[
I_3 \leq \varepsilon (s - n) \left( \int_Q f(x)(a_0 h_{xxx} + a_1 D''_\varepsilon(h) h_x^2) \, dx \, dt \right)^{\frac{1}{2}} \times \\
\left( \int_Q x^4 h^{-2s} f'(x) h_x^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \varepsilon C \left( \int_Q h_x^2 \, dx \, dt \right)^{\frac{1}{2}}.
\]

Using the Young’s inequality \( ab \leq a^p + \frac{b^q}{q} \), \( \frac{1}{p} + \frac{1}{q} = 1 \) \( \Rightarrow p a b \leq a^p + \frac{b^q}{q} b^q = a^p + (p - 1)b^q \) with \( a = z \frac{s-n}{p} \) and \( b = (\frac{s-n}{p})^q \), we deduce

\[
I_3 \leq \frac{q-1}{q} C \left( \int_Q h_x^2 \, dx \, dt \right)^{\frac{1}{2}} = \frac{q-1}{q} \tilde{C} \left( \int_Q \left( h_x^\alpha \right)^2 \, dx \, dt \right)^{\frac{1}{2}},
\]

choosing \( p = -\frac{s-n}{\alpha} > 1 \) and \( q = \frac{2(s-n)}{2(s_n-\alpha)+\alpha} > 1 \) \( \Rightarrow 0 > \alpha > -2(s - n) \), due to (4.14), we find

\[
I_3 \leq \varepsilon \frac{q-1}{q} \tilde{C} \left( \int_Q \left( h_x^{p-s+n} \right)^2 \, dx \, dt \right)^{\frac{1}{2}} = \\
\varepsilon^{-\frac{\alpha}{s-n}} \tilde{C} \left( \int_Q \left( h_x^{\frac{\alpha+2}{2}} \right)^2 \, dx \, dt \right)^{\frac{1}{2}} \leq C_1 \varepsilon^{-\frac{\alpha}{s-n}}, \tag{B.5}
\]

where the constant \( C_1 > 0 \) is independent of \( \varepsilon \). Hence,

\[
\lim_{\varepsilon \to 0} I_3 \leq 0. \tag{B.6}
\]

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From (B.2)–(B.4), we deduce that

$$
\int_\Omega x^2 \tilde{G}_\varepsilon(h) \, dx = \int_\Omega x^2 \tilde{G}_\varepsilon(h_0) \, dx +

\int_0^T x(2a_0 hh_{xx} - a_0(2-n)h_x^2 + 2a_1 L_\varepsilon(h)) \bigg| \partial_\Omega \ dt + a_0(4-n) \int_{Q_T} h_x^2 \, dx \, dt -

a_0(1-n) \int_{Q_T} x^2 h_{xx}^2 \, dx \, dt + a_1(1-n) \int_{Q_T} x^2 D_\varepsilon''(h)h_x^2 \, dx \, dt -

2a_1 \int_{Q_T} L_\varepsilon(h) \, dx \, dt + I_3. \quad (B.7)
$$

Letting $\varepsilon \to 0$ in (B.7), due to (B.6), we find

$$
\int_\Omega x^2 \tilde{G}_0(h) \, dx \leq \int_\Omega x^2 \tilde{G}_0(h_0) \, dx + a_0(4-n) \int_{Q_T} h_x^2 \, dx \, dt -

2a_1(m-n+1) \int_{Q_T} D_0(h) \, dx \, dt + \int_0^T x(2a_0 hh_{xx} - a_0(2-n)h_x^2 + 2a_1 L_0(h)) \bigg| \partial_\Omega \ dt +

a_0(n-1) \int_{Q_T} x^2 h_{xx}^2 \, dx \, dt + a_1(1-n) \int_{Q_T} x^2 h^{m-n} h_x^2 \, dx \, dt =

\int_\Omega x^2 \tilde{G}_0(h_0) \, dx + 2(4-n) \int_0^T E_0(t) \, dt - 2a_1(m-3) \int_{Q_T} D_0(h) \, dx \, dt +

\int_0^T x(2a_0 hh_{xx} - a_0(2-n)h_x^2 + 2a_1 L_0(h)) \bigg| \partial_\Omega \ dt +

a_0(n-1) \int_{Q_T} x^2 h_{xx}^2 \, dx \, dt + a_1(1-n) \int_{Q_T} x^2 h^{m-n} h_x^2 \, dx \, dt. \quad (B.8)
$$

Due to

$$
\int_{Q_T} x^2 h^{m-n} h_x^2 \, dx \, dt = -2 \int_0^T x D_0(h) \bigg| \partial_\Omega \ dt + 2 \int_{Q_T} D_0(h) \, dx \, dt -

\int_{Q_T} x^2 D'_0(h) h_{xx} \, dx \, dt,
$$

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from (B.8), in view of (3.6), we deduce that

$$\int_{\Omega} x^2 \tilde{G}_0(h) \, dx + 2a_1(m + n - 4) \int_{Q_T} D_0(h) \, dx dt \leq \int_{\Omega} x^2 \tilde{G}_0(h_0) \, dx +$$

$$k_1 \mathcal{E}_0(0)T - (1 - n) \int_{Q_T} x^2 h_{xx}(a_0 h_{xx} + a_1 D'_0(h)) \, dx dt + W(T) =$$

$$\int_{\Omega} x^2 \tilde{G}_0(h_0) \, dx + k_1 \mathcal{E}_0(0)T - \frac{1 - n}{a_0} \int_{Q_T} x^2(a_0 h_{xx} + a_1 D'_0(h))^2 \, dx dt +$$

$$\frac{a_1(1 - n)}{a_0} \int_{Q_T} x^2 D'_0(h)(a_0 h_{xx} + a_1 D'_0(h)) \, dx dt + W(T), \quad (B.9)$$

where \( k_1 = 2(4 - n) \), and \( W(T) \) is from (8.2). From (B.9) we find that

$$\int_{\Omega} x^2 \tilde{G}_0(h) \, dx + 2a_1(m + n - 4) \int_{Q_T} D_0(h) \, dx dt +$$

$$\frac{1 - n}{2a_0} \int_{Q_T} x^2(a_0 h_{xx} + a_1 D'_0(h))^2 \, dx dt \leq \int_{\Omega} x^2 \tilde{G}_0(h_0) \, dx + k_1 \mathcal{E}_0(0)T +$$

$$\frac{a_1^2(1 - n)}{2a_0} \int_{Q_T} x^2(D'_0(h))^2 \, dx dt + W(T) \leq \int_{\Omega} x^2 \tilde{G}_0(h_0) \, dx + k_1 \mathcal{E}_0(0)T +$$

$$\int_{0}^{T} \left( \tilde{A}(t) \int_{\Omega} x^2 \tilde{G}_0(h) \, dx \right) dt + W(T), \quad (B.10)$$

where \( m \geq 4 - n \), and

$$\tilde{A}(t) := \frac{a_1^2(1 - n)(2 - n)}{2a_0(m - n + 1)^2} \| h(., t) \|_{L_{\infty}(\Omega)}^{2m - n}.$$ 

Applying the nonlinear Grönwall lemma [15] to

$$v(T) \leq v(0) + k_1 \mathcal{E}_0(0)T + \int_{0}^{T} \tilde{A}(s)v(s) \, dt + W(T)$$

with \( v(T) = \int_{\Omega} x^2 \tilde{G}_0(h(x, T)) \, dx \) yields

$$v(T) \leq e^{\tilde{B}(T)} \left( v(0) + \int_{0}^{T} (k_1 \mathcal{E}_0(0) + W'(t))e^{-\tilde{B}(t)} \, dt \right)$$

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with $\tilde{B}(T) := \int_0^T \tilde{A}(\tau) \, d\tau$. From this we obtain (8.1) for $0 < n \leq 1$.

In the case of $1 < n < 2$, from (B.9) we find that

$$\int_\Omega x^2 \tilde{G}_0(h) \, dx + 2a_1(m + n - 4) \iint_{Q_T} D_0(h) \, dx \, dt \leq \int_\Omega x^2 \tilde{G}_0(h_0) \, dx + k_1 \mathcal{E}_0(0) T +$$

$$\int_\Omega \frac{3ao(n-1)}{2} \iint_{Q_T} x^2 h_{xx}^2 \, dx \, dt + \frac{a^2(n-1)}{2ao} \iint_{Q_T} x^2 (D_0(h))^2 \, dx \, dt + W(T) \leq \int_\Omega x^2 \tilde{G}_0(h_0) \, dx + $$

$$k_1 \mathcal{E}_0(0) T + k_2 \iint_{Q_T} x^2 h_{xx}^2 \, dx \, dt + \int_0^T \left( \tilde{A}(t) \int_\Omega x^2 \tilde{G}_0(h) \, dx \right) dt + W(T), \quad (B.11)$$

where $m \geq 4 - n$, $k_2 = \frac{3ao(n-1)}{2}$, and $W(t)$ is from (8.2). Applying the Grönwall lemma \cite{15} to (B.11), we obtain the estimate (8.1) for $1 < n < 2$.

\[ \square \]

**Appendix C**

**Lemma D.1.** (\cite{32}) Suppose that $X$, $Y$, and $Z$ are Banach spaces, $X \Subset Y \subset Z$, and $X$ and $Z$ are reflexive. Then the imbedding $\{ u \in L^{p_0}(0, T; X) : \partial_t u \in L^{p_1}(0, T; Z), 1 < p_i < \infty, i = 0, 1 \} \subset L^{p_0}(0, T; Y)$ is compact.

**Lemma D.2.** (\cite{32}) Suppose that $X$, $Y$, and $Z$ are Banach spaces and $X \Subset Y \subset Z$. Then the imbedding $\{ u \in L_\infty(0, T; X) : \partial_t u \in L^p(0, T; Z), p > 1 \} \subset C(0, T; Y)$ is compact.

**Lemma D.3.** (\cite{26, 47}) Let $\Omega \subset \mathbb{R}^N$, $n < 6$, be a bounded convex domain with smooth boundary, and let $n \in \left(2 - \sqrt{1 - \frac{N}{N + 3}}, 3\right)$ for $N > 1$, and $\frac{1}{2} < n < 3$ for $N = 1$. Then the following estimates hold for any strictly positive functions $v \in H^2(\Omega)$ such that $\nabla v \cdot \mathbf{n} = 0$ on $\partial \Omega$ and $\int_\Omega v^n |\nabla \Delta v|^2 < \infty$:

$$\int_\Omega \phi^6 \left\{ v^{n-4} |\nabla v|^6 + v^{n-2} |D^2 v|^2 |\nabla v|^2 \right\} \leq c \left\{ \int_\Omega \phi^6 v^n |\nabla \Delta v|^2 + \int_\{\phi > 0\} v^{n+2} |\nabla \phi|^6 \right\},$$

$$\int_\Omega \phi^6 |\nabla \Delta v|^{n+2} \leq c \left\{ \int_\Omega \phi^6 v^n |\nabla \Delta v|^2 +$$

$$+ \int_\{\phi > 0\} v^{n+2} \left( |\nabla \phi|^6 + v^2 |D^2 \phi|^2 |\nabla \phi|^2 + \phi^3 |\Delta \phi|^3 \right) \right\},$$

where $\phi \in C^2(\Omega)$ is an arbitrary nonnegative function such that the tangential component of $\nabla \phi$ is equal to zero on $\partial \Omega$, and the constant $c > 0$ is independent of $v$.  

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Lemma D.4. (33) If $\Omega \subset \mathbb{R}^N$ is a bounded domain with piecewise-smooth boundary, $a > 1$, $b \in (0,a)$, $d > 1$, and $0 \leq i < j$, $i, j \in \mathbb{N}$, then there exist positive constants $d_1$ and $d_2$ ($d_2 = 0$ if $\Omega$ is unbounded) depending only on $\Omega$, $d$, $j$, $b$, and $N$ such that the following inequality is valid for every $v(x) \in W^{j,d}(\Omega) \cap L^b(\Omega)$:

$$
\|D^i v\|_{L^a(\Omega)} \leq d_1 \|D^j v\|_{L^a(\Omega)}^{\theta} \|v\|_{L^b(\Omega)}^{1-\theta} + d_2 \|v\|_{L^b(\Omega)}, \quad \theta = \frac{\frac{i}{j} \frac{N}{d} - \frac{1}{d}}{\frac{i}{j} \frac{N}{d} + 1} \in \left[\frac{i}{j}, 1\right).
$$

Lemma D.5. (24) Let $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$, $m \geq 1$, and let $\beta = \prod_{j=1}^{m} \beta_j$, $\overline{\beta_i} = \frac{\beta}{\beta_i} = \prod_{j=1, j \neq i}^{m} \beta_j$. Assume that $G_i(s)$ are nonnegative nonincreasing functions satisfying the conditions

$$
G_i(s + \delta) \leq c_i \left( \sum_{i=1}^{m} G_i(s) \right)^{\beta_i} \quad \forall s > 0, \ \delta > 0, \ i = 1, \ldots, m
$$

with real constants $c_i > 0$, $\beta_i > 1$, and $\alpha_i > 0$ for $i = 1, \ldots, m$, and $\alpha_i > 0$ for $i = 1, \ldots, \ell$.

Let $G(s) = \sum_{i=1}^{m} (c_i^1) (G_i(s))^{\overline{\beta_i}}$, and let the function $H(s) = m^\beta \sum_{i=1}^{m} c_i^1 (c_i^1)^{1-\beta_i} (G_i(s))^{\beta_i-1}$ be such that $H(s_1) < 1$ at some $s_1 \geq 0$. Then there exists a positive constant $c > 1$ depending on $m$, $\alpha_i$, $\beta_i$, $\ell$, and $H(s_1)$ such that $G_i(s_0) \equiv 0$ for all $i = 1, \ldots, \ell$, where $s_0 = s_1 + c \sum_{i=1}^{\ell} \left( c_i^1 (c_i^1)^{1-\beta_i} (G(s_1))^{\beta_i-1} \right)^{\frac{1}{\alpha_i \beta}}$.

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