Boost Invariant Surfaces with Pointwise 1-Type Gauss Map in Minkowski 4-Space $\mathbb{E}_1^4$

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Abstract

In this paper, we study spacelike rotational surfaces which are called boost invariant surfaces in Minkowski 4-space $\mathbb{E}_1^4$. We give necessary and sufficient condition for flat spacelike rotational surface to have pointwise 1-type Gauss map. Also, we obtain a characterization for boost invariant marginally trapped surface with pointwise 1-type Gauss map.

Key words: Rotation surface, Gauss map, Pointwise 1-type Gauss map, Marginally trapped surface, Minkowski space.

2000 Mathematics Subject Classification: 53B25 ; 53C50 .

1 Introduction

The notion of finite type mapping was introduced by B.Y. Chen in late 1970’s. A pseudo- Riemannian submanifold $M$ of the $m$–dimensional pseudo-Euclidean space $\mathbb{E}_s^m$ is said to be of finite type if its position vector $x$ can be expressed as a finite sum of eigenvectors of the Laplacian $\Delta$ of $M$, that is, $x = x_0 + x_1 + \ldots + x_k$, where $x_0$ is a constant map, $x_1, \ldots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all different, then $M$ is said to be of $k$–type. This notion of finite type immersions is naturally extended to differentiable maps of $M$ in particular, to Gauss maps of submanifolds $[7]$.

If a submanifold $M$ of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map $G$, then $G$ satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface $M$ of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if $M$ is a hypersphere in $\mathbb{E}^{n+1}$ $[7]$.

However the Laplacian of the Gauss map of some typical well-known surfaces such as a helicoid, a catenoid and right cone in 3-dimensional Euclidean space $E^3$ and a helicoids of the 1st,2nd and 3rd kind, conjugate Enneper’s surface of the
second kind and B-scrolls in 3-dimensional Minkowski space $E^3_1$ take a somewhat different form namely,

$$\Delta G = f (G + C)$$  \hfill (1)

for some non-zero smooth function $f$ on $M$ and some constant vector $C$. This equation is similar to an eigenvalue problem but the smooth function $f$ is not always constant. So a submanifold $M$ of a pseudo-Euclidean space $E^m_s$ is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for some smooth function $f$ on $M$ and some constant vector $C$. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind.

Surfaces in Euclidean space and in pseudo-Euclidean space with pointwise 1-type Gauss map were recently studied in [6], [9], [10], [11], [13], [14], [15], [16], [20], [23], [24]. Also Dursun and Turgay in [12] gave all general rotational surfaces in $E^4_s$ with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan et al. in [3] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan at el. in [4] gave necessary and sufficient conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [26] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus and in [25] studied rotation surfaces in the 4-dimensional Euclidean space with finite type Gauss map. Kim and Yoon in [21] obtained the complete classification theorems for the flat rotation surfaces with finite type Gauss map and pointwise 1-type Gauss map. The authors in [1] studied flat general rotational surfaces in the 4-dimensional Euclidean space $E^4$ with pointwise 1-type Gauss map and they showed that a non-planar flat general rotational surfaces with pointwise 1-type Gauss map is a Lie group if and only if it is a Clifford Torus. Also they gave a characterization for flat general rotation surfaces with pointwise 1-type Gauss map in the 4-dimensional pseudo-Euclidean space $E^4_2$ [2].

On the other hand, trapped surfaces, introduced by Penrose in 1965, have a fundamental role in the study of the singularity theorems in General Relativity. If the mean curvature vector of a surface in $E^4_1$ is timelike everywhere, it is called trapped surfaces; if the mean curvature vector is always null (the mean curvature vector is proportional to one of the null normals), the surface is called marginally trapped surface. Since the mean curvature of such spacelike surface $H$ satisfy $\|H\| = 0$, in mathematical literature these surfaces are called quasi-minimal. In general relativity, marginally trapped surfaces are used the study of the surfaces of black hole.

S.Haesen and M. Ortega in [18] and [19] classified marginally trapped surfaces which are invariant under a spacelike rotations and boost transformations in Minkowski 4-space. Also B. Y. Chen classify marginally trapped Lorentzian flat surfaces and biharmonic surfaces in the Pseudo Euclidean space $E^4_2$ [8]. Milou-
sheva in [22] studied marginally trapped surface with pointwise 1-type Gauss map in Minkowski 4-space and proved that marginally trapped surface is of pointwise 1-type Gauss map if and only if it has parallel mean curvature vector field.

In this paper, we study spacelike surfaces which are invariant under boost transformation (hyperbolic rotations) in Minkowski 4-space. We give a characterization of flat spacelike rotational surface with pointwise 1-type Gauss map. Also we obtain a characterization for boost invariant marginally trapped surface with pointwise 1-type Gauss map and give an example of such surfaces.

2 Preliminaries

Let $E^m_s$ be the $m$–dimensional pseudo-Euclidean space with signature $(s, m - s)$. Then the metric tensor $g$ in $E^m_s$ has the form

$$ g = \sum_{i=1}^{m-s} (dx_i)^2 - \sum_{i=m-s+1}^{m} (dx_i)^2 $$

where $(x_1, ..., x_m)$ is a standard rectangular coordinate system in $E^m_s$.

Let $M$ be an $n$–dimensional pseudo-Riemannian submanifold of a $m$–dimensional pseudo-Euclidean space $E^m_s$. We denote Levi-Civita connections of $E^m_s$ and $M$ by $\tilde{\nabla}$ and $\nabla$, respectively. Let $e_1, ..., e_n, e_{n+1}, ..., e_m$ be an adapted orthonormal frame in $E^m_s$ such that $e_1, ..., e_n$ are tangent to $M$ and $e_{n+1}, ..., e_m$ normal to $M$. We use the following convention on the ranges of indices: $1 \leq i, j, k, ... \leq n$, $n + 1 \leq r, s, t, ... \leq m$, $1 \leq A, B, C, ... \leq m$.

Let $\omega_A$ be the dual-1 form of $e_A$ defined by $\omega_A(X) = \langle e_A, X \rangle$ and $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$. Also, the connection forms $\omega_{AB}$ are defined by

$$ de_A = \sum_B \varepsilon_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0 $$

Then we have

$$ \tilde{\nabla}^e e_k = \sum_{j=1}^{n} \varepsilon_j \omega_{ij} (e_k) e_j + \sum_{r=n+1}^{m} \varepsilon_r h_{ikr} e_r $$

and

$$ \tilde{\nabla}^e e_k = - \sum_{j=1}^{n} \varepsilon_j h_{kj} e_j + \sum_{r=n+1}^{m} \varepsilon_r \omega_{sr} (e_k) e_r, \quad D^e e_k = \sum_{r=n+1}^{m} \omega_{sr} (e_k) e_r, \quad (2) $$

where $D$ is the normal connection, $h_{ik}^r$ the coefficients of the second fundamental form $h$. The mean curvature vector $H$ of $M$ in $E^m_s$ is defined by

$$ H = \frac{1}{n} \sum_{s=n+1}^{m} \sum_{i=1}^{n} \varepsilon_i \varepsilon_s h_{it}^s e_s $$
and the Gaussian curvature $K$ of $M$ is given by

$$K = \sum_{s=n+1}^{m} \varepsilon_s (h_{11}^sh_{22}^s - h_{12}^sh_{21}^s)$$

Also normal curvature tensor $R^D$ of $M$ in $E_s^{m=n+2}$ is given by

$$R^D(e_j, e_k; e_r, e_s) = \sum_{i=1}^{n} \varepsilon_i (h_{ik}^r h_{ij}^s - h_{ij}^r h_{ik}^s)$$

(3)

We recall that a surface $M$ in $E_4^4$ is called extremal surface if its mean curvature vector vanishes. If its Gaussian curvature vanishes, the surface $M$ is called flat surface. If its normal curvature tensor $R^D$ vanishes identically then a surface $M$ in $E_4^4$ is said to have flat normal bundle.

For any real function $f$ on $M$ the Laplacian $\Delta f$ of $f$ is given by

$$\Delta f = -\varepsilon_i \sum_i \left( \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f \right)$$

(4)

Let us now define the Gauss map $G$ of a submanifold $M$ into $G(n, m)$ in $\wedge^n E_s^m$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented $n$–planes through the origin of $E_s^m$ and $\wedge^n E_s^m$ is the vector space obtained by the exterior product of $n$ vectors in $E_s^m$. Let $e_i, \wedge ... \wedge e_n$ and $f_j, \wedge ... \wedge f_n$ be two vectors of $\wedge^n E_s^m$, where $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_m\}$ are orthonormal bases of $E_s^m$. Define an indefinite inner product $\langle , \rangle$ on $\wedge^n E_s^m$ by

$$\langle e_1, \wedge ... \wedge e_n, f_j, \wedge ... \wedge f_n \rangle = \det (\langle e_1, f_j \rangle) .$$

Therefore, for some positive integer $t$, we may identify $\wedge^n E_s^m$ with some Euclidean space $E_s^N$ where $N = \binom{m}{n}$. The map $G : M \rightarrow G(n, m) \subset E_t^N$ defined by $G(p) = (e_1 \wedge ... \wedge e_n)(p)$ is called the Gauss map of $M$, that is, a smooth map which carries a point $p$ in $M$ into the oriented $n$–plane in $E_s^m$ obtained from parallel translation of the tangent space of $M$ at $p$ in $E_s^m$.

3 Boost Invariant Surfaces with Pointwise 1-Type Gauss Map in $E_4^1$

In this section, we consider spacelike surfaces in the Minkowski space $E_4^1$ which are invariant under the following subgroup of direct, linear isometries of $E_4^1$.

$$G = \left\{ \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\},$$

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well-known as boost isometries.

\[ \varphi(t, s) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1(s) \\ 0 \\ \alpha_3(s) \\ \alpha_4(s) \end{pmatrix} \]

\[ M : \varphi(t, s) = (\alpha_1(s) \cosh t, \alpha_1(s) \sinh t, \alpha_3(s), \alpha_4(s)) \quad (5) \]

where the profile curve of \( M \) is unit speed spacelike curve, that is,

\[ - (\alpha_1'(s))^2 + (\alpha_3'(s))^2 + (\alpha_4'(s))^2 = 1. \]

We choose a moving frame \( e_1, e_2, e_3, e_4 \) such that \( e_1, e_2 \) are tangent to \( M \) and \( e_3, e_4 \) are normal to \( M \) which are given by the following:

\[ e_1 = (\alpha_1'(s) \cosh t, \alpha_1'(s) \sinh t, \alpha_3'(s), \alpha_4'(s)) \]
\[ e_2 = (\sinh t, \cosh t, 0, 0) \]
\[ e_3 = \frac{1}{\sqrt{1 + (\alpha_1'(s))^2}}((1 + (\alpha_1'(s))^2) \cosh t, (1 + (\alpha_1'(s))^2) \sinh t, \alpha_1'(s) \alpha_3'(s), \alpha_1'(s) \alpha_4'(s)) \]
\[ e_4 = \frac{1}{\sqrt{1 + (\alpha_1'(s))^2}}(0, 0, -\alpha_4'(s), \alpha_3'(s)) \]

Then it is easily seen that

\[ \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_4, e_4 \rangle = 1, \quad \langle e_3, e_3 \rangle = -1 \]

we have the dual 1-forms as:

\[ \omega_1 = ds \quad \text{and} \quad \omega_2 = \alpha_1(s) dt \quad (6) \]

By a direct computation we have components of the second fundamental form and the connection forms as:

\[ h_{11}^3 = -c(s), \quad h_{12}^3 = 0, \quad h_{22}^3 = -b(s) \]
\[ h_{11}^4 = d(s), \quad h_{12}^4 = 0, \quad h_{22}^4 = 0 \quad (7) \]

\[ \omega_{12} = a(s)b(s)\omega_2, \quad \omega_{13} = -c(s)\omega_1, \quad \omega_{14} = d(s)\omega_1 \]
\[ \omega_{23} = -b(s)\omega_2, \quad \omega_{24} = 0, \quad \omega_{34} = a(s)d(s)\omega_1 \quad (8) \]
By covariant differentiation with respect to $e_1$ and $e_2$ a straightforward calculation gives:

\[ \tilde{\nabla}_{e_1}e_1 = c(s)e_3 + d(s)e_4 \]  \hspace{1cm} (9)

\[ \tilde{\nabla}_{e_2}e_1 = a(s)b(s)e_2 \]

\[ \tilde{\nabla}_{e_1}e_2 = 0 \]

\[ \tilde{\nabla}_{e_2}e_2 = -a(s)b(s)e_1 + b(s)e_3 \]

\[ \tilde{\nabla}_{e_1}e_3 = c(s)e_1 + a(s)d(s)e_4 \]

\[ \tilde{\nabla}_{e_2}e_3 = b(s)e_2 \]

\[ \tilde{\nabla}_{e_1}e_4 = -d(s)e_1 + a(s)d(s)e_3 \]

\[ \tilde{\nabla}_{e_2}e_4 = 0 \]

where

\[ a(s) = \frac{\alpha_1'(s)}{\sqrt{1 + (\alpha_1'(s))^2}} \]  \hspace{1cm} (10)

\[ b(s) = \frac{\sqrt{1 + (\alpha_1'(s))^2}}{\alpha_1(s)} \]  \hspace{1cm} (11)

\[ c(s) = \frac{\alpha_2'(s)}{\sqrt{1 + (\alpha_1'(s))^2}} \]  \hspace{1cm} (12)

\[ d(s) = \frac{-\alpha_3''(s)\alpha_3'(s) + \alpha_2''(s)\alpha_3'(s)}{\sqrt{1 + (\alpha_1'(s))^2}} \]  \hspace{1cm} (13)

The Gaussian curvature $K$ of $M$ is given by

\[ K = -b(s)c(s) \]  \hspace{1cm} (14)

The mean curvature $H$ of $M$ is given by

\[ H = \frac{1}{2} (-h_1e_3 + h_2e_4) \quad h_1 = -(b + c) \quad \text{and} \quad h_2 = d \]  \hspace{1cm} (15)

By using (4), (9) and straight-forward computation, the Laplacian $\Delta G$ of the Gauss map $G$ can be expressed as

\[ \Delta G = A(s) (e_1 \wedge e_2) + B(s) (e_2 \wedge e_3) + D(s) (e_2 \wedge e_4) \]  \hspace{1cm} (16)

where

\[ A(s) = d^2(s) - b^2(s) - c^2(s) \]  \hspace{1cm} (17)

\[ B(s) = b'(s) + c'(s) + a(s)d^2(s) \]  \hspace{1cm} (18)

\[ D(s) = d''(s) + a(s)d(s)(b(s) + c(s)) \]  \hspace{1cm} (19)
Theorem 1. Let $M$ be the flat rotation surface given by the parametrization (5). Then $M$ has pointwise 1-type Gauss map if and only if the profile curve of $M$ is parametrized by

$$
\begin{align*}
\alpha_1(s) &= a_1 \\
\alpha_3(s) &= \frac{1}{a_2} \left(1 + a_1^2\right)^\frac{1}{2} \cos (a_2 s + a_3) \\
\alpha_4(s) &= -\frac{1}{a_2} \left(1 + a_1^2\right)^\frac{1}{2} \sin (a_2 s + a_3)
\end{align*}
$$

or

$$
\begin{align*}
\alpha_1(s) &= b_1 s + b_2 \\
\alpha_3(s) &= \int \left(1 + b_1^2\right)^\frac{1}{2} \cos (b \ln|b_1 s + b_2|) \, ds \\
\alpha_4(s) &= \int \left(1 + b_1^2\right)^\frac{1}{2} \sin (b \ln|b_1 s + b_2|) \, ds
\end{align*}
$$

where $a_1, a_2, a_3, b_1 \neq 0, b_2, b_3$ and $b = \frac{b_3}{b_1(1 + b_1^2)^\frac{1}{2}}$ are real constants.

**Proof.** Let $M$ be the flat rotation surface given by the parametrization (5). We suppose that $M$ has pointwise 1-type Gauss map. By using (1) and (16), we have

$$
\begin{align*}
f + f \langle C, e_1 \wedge e_2 \rangle &= A(s) \\
f \langle C, e_2 \wedge e_3 \rangle &= -B(s) \\
f \langle C, e_2 \wedge e_4 \rangle &= D(s)
\end{align*}
$$

and

$$
\langle C, e_1 \wedge e_3 \rangle = \langle C, e_1 \wedge e_4 \rangle = \langle C, e_3 \wedge e_4 \rangle = 0
$$

By differentiating (23) covariantly with respect to $s$, we have

\[-a(s)B(s) + A(s) - f = 0 \]
\[a(s)D(s) = 0 \]
\[D(s) = 0 \]

In this case, firstly, we assume that $a(s) = 0$ and $D(s) = 0$. From (10), we obtain that $\alpha_1(s) = a_1$. Since the profile curve is unit speed spacelike curve, we can write $(\alpha'_3(s))^2 + (\alpha'_4(s))^2 = 1 + a_1^2$. Also we can put

$$
\begin{align*}
\alpha'_3(s) &= \left(1 + a_1^2\right)^\frac{1}{2} \cos \theta(s) \\
\alpha'_4(s) &= \left(1 + a_1^2\right)^\frac{1}{2} \sin \theta(s)
\end{align*}
$$
where $\theta$ is smooth angle function. On the other hand, since $D(s) = 0$, from (19) we obtain as
\[ d(s) = a_2, \quad a_2 \text{ is non zero constant.} \]  \hspace{1cm} (25)

By using (13), (24) and (25) we get
\[ \theta(s) = a_2 s + a_3 \]  \hspace{1cm} (26)

So from (24) and (26) we have
\[
\begin{align*}
\alpha_3(s) &= \frac{1}{a_2} \left(1 + a_1^2\right)^{\frac{1}{2}} \cos(a_2 s + a_3) \\
\alpha_4(s) &= -\frac{1}{a_2} \left(1 + a_1^2\right)^{\frac{1}{2}} \sin(a_2 s + a_3)
\end{align*}
\]

Now we assume that $a(s) \neq 0$ and $D(s) = 0$. Since $M$ is flat, (12) and (14) imply that
\[ \alpha_1(s) = b_1 s + b_2 \]  \hspace{1cm} (27)

for some constants $b_1 \neq 0$ and $b_2 = 0$. Since the profile curve is unit speed spacelike curve, we can write $(\alpha_3'(s))^2 + (\alpha_4'(s))^2 = 1 + b_1^2$. Also we can put
\[
\begin{align*}
\alpha_3'(s) &= \left(1 + b_1^2\right)^{\frac{1}{2}} \cos \theta(s) \\
\alpha_4'(s) &= \left(1 + b_1^2\right)^{\frac{1}{2}} \sin \theta(s)
\end{align*}
\]  \hspace{1cm} (28)

where $\theta$ is smooth angle function. By using (10), (11) and (19), we get
\[ d(s) = \frac{b_3}{b_1 s + b_2} \]  \hspace{1cm} (29)

On the other hand, by using (13), (27) and (28) we have
\[ d(s) = \left(1 + b_1^2\right)^{\frac{1}{2}} \theta'(s) \]  \hspace{1cm} (30)

By combining (29) and (30) we obtain
\[ \theta(s) = b \ln |b_1 s + b_2| \]  \hspace{1cm} (31)

where $b = \frac{b_3}{b_1(1+b_1^2)^{\frac{1}{2}}}$. So by substituting (31) into (28) we can write
\[
\begin{align*}
\alpha_3(s) &= \int \left(1 + b_1^2\right)^{\frac{1}{2}} \cos \left(b \ln |b_1 s + b_2|\right) \, ds \\
\alpha_4(s) &= \int \left(1 + b_1^2\right)^{\frac{1}{2}} \sin \left(b \ln |b_1 s + b_2|\right) \, ds
\end{align*}
\]
Conversely, the surface \( M \) which is parametrized by (20) and (21) is pointwise 1-type Gauss map for
\[
f(s) = -a(s)b'(s) - a^2(s)d^2(s) + d^2(s) - b^2(s)
\]
and
\[
C(s) = \frac{a(s)b'(s) + a^2(s)d^2(s)}{f(s)} (e_1 \wedge e_2) + \frac{b'(s) + a(s)d^2(s)}{f(s)} (e_2 \wedge e_3)
\]
where it can be easily seen that \( e_1 (C(s)) = 0 \) and \( e_2 (C(s)) = 0 \). This completes the proof.

**Corollary 1.** Let \( M \) be the flat rotation surface given by the parametrization (5). If \( M \) has pointwise 1-type Gauss map then the profile curve of \( M \) is a helix curve.

We will also use the following theorems and corollary.

**Theorem 2.** [17] Let \( M \) be an oriented maximal surface in the Minkowski space \( E^4_1 \). Then \( M \) has pointwise 1-type Gauss map of the first kind if and only if \( M \) has flat normal bundle. Hence the Gauss map \( G \) satisfies (1.1) for \( f = \|h\|^2 \) and \( C = 0 \).

**Theorem 3.** [18] Let \( M \) be a spacelike rotational surface in Minkowski 4-space given by the parametrization (5). If \( M \) is marginally trapped surface then
\[
\begin{align*}
\alpha_3(s) &= \int \left(1 + (\alpha_1')^2\right)^{\frac{1}{2}} \cos \theta(s) \, ds \\
\alpha_4(s) &= \int \left(1 + (\alpha_1')^2\right)^{\frac{1}{2}} \sin \theta(s) \, ds
\end{align*}
\]
and
\[
\theta(s) = -\epsilon \int \frac{1 + (\alpha_1')^2 + \alpha_1''}{\alpha_1 \left(1 + (\alpha_1')^2\right)^{\frac{3}{2}}} \, ds
\]
where \( \epsilon = \pm \).

**Corollary 2.** [18] Let \( M \) be a spacelike rotational surface in Minkowski 4-space given by the parametrization (5). If \( M \) is a extremal surface then a unit profile curve is given by
\[
\alpha(s) = \left( f(s), 0, \cos \zeta_0 \sqrt{a_1} \arctan \left( \frac{s + a_2}{f(s)} \right), \sin \zeta_0 \sqrt{a_1} \arctan \left( \frac{s + a_2}{f(s)} \right) \right),
\]
where \( f(s) = \sqrt{a_1 - (s + a_2)^2} \) and \( a_1, a_2, \zeta_0 \in \mathbb{R}, \ a_1 > 0 \), being integration constants. In particular, the surface \( M \) is immersed in a totally geodesic Lorentzian 3-space.
Theorem 4. Let $M$ be the marginally trapped surface given by the parametrization (5) in Minkowski 4-space. Then $M$ has pointwise 1-type Gauss map if and only if the profile curve is given by

\begin{align*}
\alpha_1(s) &= (\lambda_1 - 1)^{\frac{1}{2}} (u^2(s) + \lambda^2)^{\frac{1}{2}} \\
\alpha_3(s) &= \int \left( \frac{\lambda_1 u^2 + \lambda^2}{u^2 + \lambda^2} \right)^{\frac{1}{2}} \cos \theta(s) \, ds \\
\alpha_4(s) &= \int \left( \frac{\lambda_1 u^2 + \lambda^2}{u^2 + \lambda^2} \right)^{\frac{1}{2}} \sin \theta(s) \, ds
\end{align*}

and

\[ \theta(s) = -\epsilon \frac{\lambda_1}{(\lambda_1 - 1)^{\frac{1}{2}}} \int \left( \frac{u^2 + \lambda^2}{\lambda_1 u^2 + \lambda^2} \right)^{\frac{1}{2}} cos \theta(s) \, ds \]

where $u(s) = \delta s + \lambda_3$, $\lambda = \frac{\lambda_2}{\lambda_1 - 1}$, $\lambda_1$, $\lambda_2$, $\lambda_3$, $a_1$ and $a_2$ are real constants.

Proof. Let $M$ be marginally trapped surface. This means $\|H\| = 0$ that is $\langle H, H \rangle = 0$. By using (15), we get

\[ -(b(s) + c(s)) = \epsilon d(s) \]

where $\epsilon = \pm$. In this case, by using (35) we can rewrite the Laplacian $\Delta G$ of the Gauss map $G$ as

\[ \Delta G = A(s) (e_1 \wedge e_2) - \epsilon N(s) (e_2 \wedge e_3) + N(s) (e_2 \wedge e_4) \]

where

\[ N(s) = d'(s) - \epsilon a(s) d^2(s) \]

We assume that $M$ has pointwise 1-type Gauss map. Then we have

\begin{align*}
f + f \langle C, e_1 \wedge e_2 \rangle &= A(s) \\
f \langle C, e_2 \wedge e_3 \rangle &= \epsilon N(s) \\
f \langle C, e_2 \wedge e_4 \rangle &= N(s)
\end{align*}

and

\[ \langle C, e_1 \wedge e_3 \rangle = \langle C, e_1 \wedge e_4 \rangle = \langle C, e_3 \wedge e_4 \rangle = 0 \]

By differentiating (39) covariantly with respect to $s$, we have

\begin{align*}
\epsilon a(s) N(s) + A(s) - f &= 0 \\
\epsilon a(s) N(s) &= 0 \\
N(s) &= 0
\end{align*}
In this case, firstly, we assume that \(a(s) = 0\) and \(N(s) = 0\). From (10) and (12), we obtain that \(\alpha_1(s) = a_1\) and \(c(s) = 0\), respectively. Hence from (35) we get

\[-b(s) = cd(s)\]  \hspace{1cm} (40)

By using (40) and (17) we obtain that \(A(s) = 0\). So we have that \(f = 0\). This is a contradiction.

Now we assume that \(a(s) \neq 0\) and \(N(s) = 0\). By combining (10), (11), (12), (13), (35) and (37), we obtain a differential equation as follows:

\[
\left(1 + (\alpha_1'(s))^2 + \alpha_1'(s)\alpha_1''(s)\right)\alpha_1(s) \left(1 + (\alpha_1'(s))^2\right) = 0
\]

Since \(\alpha_1 > 0\) and \(1 + (\alpha_1'(s))^2 \neq 0\) we have

\[
1 + (\alpha_1'(s))^2 + \alpha_1'(s)\alpha_1''(s) = \lambda_1
\]

whose the solution

\[
\alpha_1(s) = (\lambda_1 - 1)^{\frac{1}{2}} \left(\delta s + \lambda_3\right)^{\frac{1}{2}} + \frac{\lambda_2}{(\lambda_1 - 1)^{\frac{1}{2}}}
\]  \hspace{1cm} (41)

By using (33) and (41) we get

\[
\theta(s) = -\epsilon \mu \int \frac{(u^2 + \lambda)^{\frac{1}{2}}}{\lambda_1 u^2 + \lambda} ds \hspace{1cm} (42)
\]

where \(u(s) = \delta s + \lambda_3\), \(\lambda = \frac{\lambda_2}{\lambda_1 - 1}\) and \(\mu = \frac{\lambda_1}{(\lambda_1 - 1)^{\frac{1}{2}}}\).

Conversely, the surface \(M\) which is parametrized by (34) has pointwise 1-type Gauss map with

\[
f(s) = 2b(s)c(s)
\]

and

\[
C(s) = 0
\]

This completes the proof. \(\square\)

**Corollary 3.** Let \(M\) be marginally trapped surface given by the parametrization (5) in Minkowski 4-space. Then \(M\) has pointwise 1-type Gauss map then \(M\) is pointwise 1-type Gauss map of the first kind.

**Corollary 4.** Let \(M\) be a spacelike rotational surface in Minkowski 4-space given by the parametrization (5). If \(M\) is extremal surface then \(M\) has pointwise 1-type Gauss map of the first kind.

**Proof.** We assume that \(M\) is a spacelike rotational surface given by the parametrization (5). In that case by using (3) and (7) we obtain that \(M\) has flat normal bundle. Hence from Theorem \(\square\) If \(M\) is extremal surface then \(M\) has pointwise 1-type Gauss map of the first kind. \(\square\)
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