Strong Super-additivity of the entanglement of formation for pure stabilizer states

David Fattal
Hewlett-Packard Laboratories, 1501 Page Mill Road, Palo Alto, CA 94304

Keiji Matsumoto
NI, Tokyo...
(Dated: April 1, 2022)

We prove the strong super-additivity of the entanglement of formation for stabilizer pure states, and the set of mixed states which minimize their average entropy of entanglement as a mixture of stabilizer pure states sharing the same stabilizer group up to phases. The implications of the result on the additivity of the Holevo capacity of a quantum channel transmitting stabilizer states with Pauli noise is discussed.

Among the open problems of quantum information theory, the additivity of the Holevo capacity of a quantum channel is an important one. The Holevo capacity is the optimal zero-error information transmission rate of a quantum channel. The implication of the additivity conjecture is that entangled signal states are not more useful than separable signal states for communication through the quantum channel.

A related additivity conjecture is concerns the so-called entanglement of formation $E_f$, which asymptotic behavior, $\lim_{n \to \infty} \frac{1}{n} E_f(\rho^\otimes n)$, named the entanglement cost $E_c(\rho)$, is the number of maximally entangled pairs required to prepare $\rho$ by LOCC in an asymptotic way. The additivity of $E_f(\rho)$ implies that $E_c(\rho) = E_f(\rho)$, simplifying the computation of $E_c(\rho)$ to large extent, and also that $E_c(\rho \otimes \sigma) = E_c(\rho) + E_c(\sigma)$, meaning that making $\rho$ and $\sigma$ altogether requires the same amount of maximally entangled states as they are produced separately (in other words, there is no catalytic effect in entanglement dilution, different from entanglement distillation).

There is yet another stronger conjecture about $E_f$, its strong super-additivity. If $\rho$ is a state on $A \otimes B$, where $A = A_1 \otimes A_2$ and $B = B_1 \otimes B_2$, then the strong super-additivity of $E_f$ is defined as the property:

$$E_f(\rho) \geq E_f(\rho_1) + E_f(\rho_2),$$

where all $E_f$ are taken with respect to the $\{A, B\}$ partition, and $\rho_i$ are reduced density matrix on $A_i \otimes B_i$. In short, the sum of local entanglements are smaller than global entanglement.

As is pointed out in [1], the strong super-additivity conjecture implies both of previously mentioned two additivity conjectures. Peter Shor also proved that these three additivity conjectures are indeed equivalent [3]. Hence, one of the proof of these conjectures will solve all the problems at once. So far, despite many efforts, these additivity conjectures have been proven only for some special instances [4, 3, 6, 7, 8, 10, 11, 12, 13]. In this letter, we prove the strong super-additivity for a discrete but diverse subclass of states, known as stabilizer states.

In a previous paper [3], we have derived a very simple expression for the entropy of entanglement of a stabilizer state of $n$ qubits for a given bi-partition. Here, we use this result to prove the strong super-additivity of the entanglement of formation for stabilizer states.

Let $\mathcal{P}_n$ denote the Pauli group for $n$ qubits. A pure stabilizer states $|\psi\rangle$ of $n$ qubits is a simultaneous eigenvector of $n$ independent pauli operators with eigenvalue $\pm 1$. The $n$ independent pauli operators generate an abelian subgroup of $\mathcal{P}_n$, called the stabilizer group of $|\psi\rangle$ et denoted $S(\psi)$.

Let $(A, B)$ be a partition of the $n$ qubits. Define $S_A$ (resp. $S_B$) to be the subgroup of $S(\psi)$ containing operators that act trivially on $B$ (resp. $A$) :

$$S_A = \{g_A \otimes I_B \in S(\psi)\} \quad (2)$$
$$S_B = \{I_A \otimes g_B \in S(\psi)\} \quad (3)$$

We will refer to $S_{loc} = S_A \cdot S_B$ as the local subgroup of $S$. Then $|\psi\rangle$ is separable for partition $\{A, B\}$ if and only if $S = S_{loc}$. Otherwise we can find a non-trivial subgroup $S_{AB}$ of $S$ such that :

$$S = S_A \cdot S_B \cdot S_{AB} \quad (4)$$

$S_{AB}$, unlike $S_{loc}$, is not uniquely determined. If $S_{AB}$ satisfies (4), and $g_{loc}$ is an element of $S_{loc}$, then $g_{loc} \cdot S_{AB}$ also satisfies (4). The rank $e_{AB}$ of $S_{AB}$ - defined as its minimum number of generators - turns out to be twice the entropy of entanglement of $|\psi\rangle$.

We now introduce a different partition $(1,2)$ of the same set of qubits. We will exhibit a pure state statistical decomposition of $\text{Tr}_2(|\psi\rangle \langle \psi|)$ in terms of stabilizer states sharing a common stabilizer group up to phases. For this we will use $n_{A_2}$ commuting independent pauli operators $M_i^{A_2}$ with support on $A_2$, and $n_{B_2}$ commuting independent pauli operators $M_j^{B_2}$ with support on $B_2$. Together these operators form a complete commuting independent set for partition 2, that is moreover local for partition...
\{A, B\}. Measuring the commuting operators $M_{A2}$'s on the qubits of $A2$ has $2^{n_{A2}}$ possible outcomes denoted by the binary string $k^{A2}$ of length $n_{A2}$, that we can also use to label the mutually orthogonal post-measurement states $|k^{A2}\rangle$. We define a similar basis with similar notations for $B2$. It is then straightforward to see that an unpreferred successive measurement of the $M_{A2}$'s and $M_{B2}$'s on the initial state $|\psi\rangle$ yields the mixed state:

$$\rho^M = \sum_{k^{A2},k^{B2}} \langle k^{A2}, k^{B2}| \rho |k^{A2}, k^{B2}\rangle$$

This state is a statistical mixture of stabilizer pure states on the sets of all qubits, with common stabilizer group $S^M$ - up to phases - given by:

$$S^M = S^1_M \cdot \langle M_1^{A2} \rangle \cdot \langle M_j^{B2} \rangle$$ (6)

and obtained from $S(\psi)$ after the successive measurements by the prescription described e.g. in [2].

Note that $\langle k^{A2}, k^{B2}| \psi\rangle$ is a pure stabilizer state on partition 1, with stabilizer group $S^M$. We will write this state $\psi(k^{A2}, k^{B2})$ and rewrite the decomposition of $\rho^M$ explicitly as:

$$\rho^M = \sum_{k^{A2},k^{B2}} \psi(k^{A2}, k^{B2}) \langle k^{A2}, k^{B2}| \psi(k^{A2}, k^{B2}), k^{A2}, k^{B2}\rangle$$ (7)

The critical observation now is that the entropy of entanglement of $\psi(k^{A2}, k^{B2})$ with respect to partition $\{A, B\}$ is the same as the entropy of entanglement of $\psi(k^{A2}, k^{B2})$.

On the other hand, using the $|k^{A2}, k^{B2}\rangle$ basis to take the partial trace over partition 2, we get:

$$Tr_2(|\psi\rangle \langle \psi|) = \sum_{k^{A2},k^{B2}} |\psi(k^{A2}, k^{B2}) \langle k^{A2}, k^{B2}| \psi(k^{A2}, k^{B2})\rangle|$$ (8)

which is also a decomposition into pure stabilizer states of same stabilizer group $S^M$. Since the entanglement of formation of $Tr_2(|\psi\rangle \langle \psi|)$ is defined as the minimum average entropy of entanglement over all decompositions of $Tr_2(|\psi\rangle \langle \psi|)$ into pure states, we obtain the key result:

$$E_f(Tr_2(|\psi\rangle \langle \psi|)) \leq e_{AB}(S^M)$$ (9)

A similar inequality hold for the partial trace over the partition 1.

We will now see that there is a systematic way to choose the $M^{A2}$ and $M^{B2}$ operators, and similarly the $M^{A1}$ and $M^{B1}$ operators such that:

$$e_{AB}(S^M_1) + e_{AB}(S^M_2) \leq e_{AB}(S)$$ (10)

which will imply the strong super-additivity.

We start from the decomposition [3] and further break $S_A$ and $S_B$ as follows:

$$S_A = S^1_A \cdot S^2_A \cdot S^1_{A2}$$ (11)
$$S_B = S^1_B \cdot S^2_B \cdot S^1_{B2}$$ (12)

where $S^1_A$ is the subgroup of $S_A$ that acts trivially on the qubits of partition 2.

Before we choose our measurement operators, we need a few lemmas to guide our choice.

First we observe that the post-measurement state obtained after measuring a local operator with respect to $\{A, B\}$ cannot have more entanglement than the pre-measurement state. Simply the local subgroup of the post-measurement state will either be of same size or grow after the measurement, which means that the entanglement stays the same or decrease. This is trivial if $M$ commutes with $S_A$ and $S_B$, since the local subgroup after measurement will contain $S_A \cdot S_B$ therefore at least stays of same size. If $M$ does not commute with $S_A \cdot S_B$, then we can write:

$$S_A \cdot S_B = <s_1, \ldots , s_{n-e_{AB}} >$$ (13)

such that $M$ anti-commutes with $s_1$ but commutes with all other $s_j (2 \leq j \leq n - e_{AB})$ [3]. Then recalling that $M$ is a local operator (i.e. acts on A only or B only), the local subgroup after measurement will contain the subgroup $<M, s_2, \ldots , s_{n-e_{AB}}>$, therefore will again be at least of same size as the pre-measurement local subgroup.

We now derive our most useful tool:

**Lemma**: Measuring a local operator $M$ (acting trivially on A or B) that commutes with $S_A$ and $S_B$ but that is not in the stabilizer group $S$ reduces the entanglement $e_{AB}$ of the post-measurement stabilizer $S^M$ by at least 2.

**proof**: $S_A$ and $S_B$ are preserved by the measurement, but we also know that $M$ must lie in the disentangled subspace of $S^M$, therefore the local subgroup must be at least contain $S_A \cdot S_B \cdot (M)$. Therefore $e_{AB} = n - |S_{loc}|$ must decrease by at least one unit. But we also know from [3] that $e_{AB}$ must be an even number, hence we conclude it must have decreased by at least 2. $\square$

Denote by $P_2$ the projection on partition 2, defined by:

$$P_2(g_1 \otimes g_2) \equiv I_2 \otimes g_2$$ (14)

We claim that $P_2(S^1_A)$ is a subgroup of same rank as $S^2_A$. Indeed, if $S^2_A$ is generated by some $\{g_k\}$, then:

$$\prod P_2(g_k) = I \Rightarrow P_2(\prod g_k) = I \Rightarrow \prod g_k \in S_1 \cup S^1_{A2} = \{I\}$$ (15)
and hence the $P_3(g_k)$ are independent. Also, the $P_2(g_k)$ must all commute with $S^2_A$, since the $g_k$ do.

We can organize the generators of $P_2(S^{12}_A)$ as follows:

$$P_2(S^{12}_A) = Z: <g_j, \bar{g}_j >_{j=1..p}$$

where $Z$ is the center of $P_2(S^{12}_A)$ (i.e. the subgroup that commutes with all elements of the group), and the $(g_j, \bar{g}_j)$ form anti-commuting pairs, commuting with all other generators. We can always find such generators up to some multiplications by elements of the local subgroups, as shown in [3].

For our measurement operators on $A_2$, we pick the $\bar{g}_j$ first, that is choose $M_i^{A_2} = \bar{g}_i$ for $1 \leq i \leq p$, $p$ being the number of non-commuting pairs in $S^{12}_A$. Note that:

$$p \leq \frac{1}{2} |S^{12}_A|$$

After the measurement of all $\bar{g}_j$, all the anti-commuting pairs in $P_2(S^{12}_A)$ are "destroyed", to the benefit of $S^1_A$ and $S^2_A$ each which gain a new independent element: $S^1_A$ receives $\bar{g}_j$, and $S^2_A$ receives $\bar{g}_j$ such that $\bar{g}_j \otimes \bar{g}_j$ was in the initial subgroup $S^{12}_A$.

We then choose the remaining $M_i^{A_2}$ ($i = p + 1..n_{A_2}$) to be the generators of $S^2_A$, the generators of $Z$, and possibly complete the list with elements that commute with $S_A$, $Z$, and the $\bar{g}_j$.

Note that all these operators are local with respect to $\{A, B\}$. Moreover, apart from the elements of $S^2_A$ and the $\bar{g}_j$, they all satisfy the assumptions of the lemma, that is commute with the local subgroup but are not element of the stabilizer group itself. Measuring these operators adds one independent element to both $S_A$ and $S_B$ and reduces the overall entanglement by 2 each time. The number $N_{A_2}$ of these operators is:

$$N_{A_2} = n_{A_2} - |S^2_A| - p$$

using the inequality (19). The rest of the operators we measure on $B_2$ are also local with respect to $\{A, B\}$, and therefore do not increase the entanglement.

Now had we chosen to trace over $B_2$ rather than $A_2$ first, a similar inequality would hold on subspace $B_2$. Therefore the total number $N$ of measurement operators that reduce the overall entanglement (each by 2) is at least:

$$N \geq \max(N_{A_2}, N_{B_2})$$

$$\geq \frac{1}{2} (n_2 - |S^2_A| - |S^2_B|)$$

where we used the fact that the $\max(N_{A_2}, N_{B_2}) \geq \frac{N_{A_2} + N_{B_2}}{2}$.

The residual entanglement $e_{AB}(S^M_1)$ common to the pure states on partition 1 described by the stabilizer group $S^M_1$ resulting from the measurement of operators $M_1^{A_2}$ and $M_1^{B_2}$ therefore obeys the inequality:

$$e_{AB}(S^M_1) \leq e_{AB}(S) - 2N$$

$$\leq e_{AB} - n_2 + |S^2_A| + |S^2_B| + \frac{1}{2} |S^{12}_A| + \frac{1}{2} |S^{12}_B|$$

Similarly, after partial trace over 1, we obtain a similar inequality:

$$e_{AB}(S^M_2) \leq e_{AB}(S) - n_1 + |S^1_A| + |S^1_B| + \frac{1}{2} |S^{12}_A| + \frac{1}{2} |S^{12}_B|$$

Finally using the fact that:

$$|S^1_A| + |S^2_A| + |S^{12}_A| = |S_A|$$

and that:

$$|S_A| + |S_B| = n - e_{AB}$$

we obtain the announced result (10).

To summarize, we have proven that the (possibly mixed) state obtained after partial trace over partition 2 (resp 1) had an entanglement of formation that was upper bounded by the entropy of entanglement corresponding to the stabilizer pure states defined on the global set of qubits and obtained after measuring a complete set of independent commuting operators on partition 2 which are local for the $\{A, B\}$ partition. We could explicit a set of such operators for both partitions 1 and 2 so that the sum of the entropy of entanglement of the post-measurement stabilizer groups was smaller than the original entropy of entanglement for the pure state $|\psi\rangle$, which is equal to its entanglement of formation since it is a pure state.

### Mixed states extension

The result generalizes easily to mixed states for which the decomposition into pure state that minimizes the average entropy of entanglement happens to be a mixture of stabilizer states with same stabilizer group - again up to phases.

Indeed, suppose that the $\psi_j$’s are stabilizer pure states described by same stabilizer group $S$, and that:

$$E_f(\rho) = \sum_j p_j |\psi_j\rangle \langle \psi_j| = \sum_j p_j E(\psi_j)$$

We can write the partial trace of $\rho$ over partition 2 as:

$$Tr_2(\rho) = \sum_j p_j Tr_2(|\psi_j\rangle \langle \psi_j|)$$

Now $Tr_2(|\psi_j\rangle \langle \psi_j|)$ is a mixture of pure stabilizer states with common stabilizer group $S^M_1$ and $Tr_1(|\psi_j\rangle \langle \psi_j|)$ is a mixture of pure stabilizer states with common stabilizer
group $S^M_{2j}$ which, according to the previous analysis, verifies:

$$e_{AB}(S^M_{1j}) + e_{AB}(S^M_{2j}) \leq e_{AB}(S) \quad (31)$$

Now since:

$$E_f(Tr_2(\rho)) \leq \sum_j p_j e_{AB}(S^M_{1j}) \quad (32)$$

$$E_f(Tr_1(\rho)) \leq \sum_j p_j e_{AB}(S^M_{2j}) \quad (33)$$

we obtain the announced result:

$$E_f(Tr_2(\rho)) + E_f(Tr_1(\rho)) \leq e_{AB}(S) \quad (34)$$

We finally prove that mixed stabilizer states of the form:

$$\rho = \sum_{g \in H} g \quad (35)$$

where $H$ is a non-maximal abelian subgroup of $\mathcal{P}_n$ have the property mentioned above, namely minimize the average entropy of entanglement when written as a mixture of pure stabilizer states with common stabilizer group $S$ - up to phases.

To prove this result, we use the fact that $H$ - as any stabilizer group - is LU-equivalent to a stabilizer state for which a stabilizer generator list is composed of single $Z$ operators acting on single qubits of partition $A$ or $B$, single $ZZ$ operators acting simultaneously on a qubit of $A$ and a qubit of $B$, and a number $p$ of locally anti-commuting pairs $XX$, $ZZ$ also acting across the $\{A,B\}$ partition but yet on a disjoint set of qubits [ref]. We will denote the corresponding local unitaries $U_A$ and $U_B$.

We will prove that $E_f(\rho)$ is precisely $p$ the number of locally anti-commuting pairs, and will exhibit a pure stabilizer state decomposition of $\rho$ for which the average entropy of entanglement is $p$. We will start by exhibiting such a state. Consider the non-maximal stabilizer group $\tilde{H} = U_A U_B H U_A^\dagger U_B^\dagger$ and its decomposition into single $Z$’s, $ZZ$’s, and pairs $(XX, ZZ)$’s acting on disjoint supports. We complete the stabilizer generator list of $\tilde{H}$ with single $Z$ operators acting on whatever qubit lays out of the support of the generators of $\tilde{H}$, and a single $Z$ operator acting on the first qubit of every single $ZZ$ operator featuring in that generator list. It is easy to check that the stabilizer group $\tilde{S}$ obtained in this way is a maximal stabilizer group having $\tilde{H}$ as a subgroup. Hence $\rho$ can be expressed as a statistical mixture (with equal weights) of pure stabilizer states all having stabilizer group $U_A^\dagger U_B^\dagger \tilde{S} U_A U_B$, and therefore with average entropy of entanglement equal to $p$. This proves that $E_f(\rho) \leq p$.

Now suppose that we can decompose $\rho$ as:

$$\rho = \sum_j p_j |\psi_j\rangle \langle\psi_j| \quad (36)$$

where the $|\psi_j\rangle$’s are arbitrary pure states. From (35), we see that for any $g \in H$, $g \rho = \rho$ which after the trace implies:

$$\sum_j p_j \langle\psi_j| g |\psi_j\rangle = 1 \quad (37)$$

which itself implies that $\langle\psi_j| g |\psi_j\rangle = 1$ for all $j$, and that:

$$g |\psi_j\rangle = |\psi_j\rangle \quad (38)$$

We will consider for $g$ the first pair of locally anti-commuting generators that we write $(g_Z, g_X)$, for which we can find suitable local unitaries $U_A, U_B$ such that:

$$g_X = U_A U_B X_A X_B U_A^\dagger U_B^\dagger \quad (39)$$

$$g_Z = U_A U_B Z_A Z_B U_A^\dagger U_B^\dagger \quad (40)$$

where $A1$ and $B1$ designate the first qubit of partition $A$ and $B$ respectively. It is then not hard to see that due to (38), the pure state $U_A U_B |\psi_j\rangle$ contains at least $p$ Bell pairs on qubits $A1, B1$, and a pure state on the remaining qubits. Iterating this argument, we see that each state $U_A U_B |\psi_j\rangle$ contains at least $p$ Bell pairs on disjoint qubit pairs, and therefore has entropy of entanglement at least $p$. This proves that $E_f(\rho) \geq p$, and therefore that the stabilizer construction above yields the decomposition of $\rho$ into a mixture of pure states that achieves the minimum average entropy of entanglement.

[1] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, W. K. Wootters, Phys. Rev. A, vol. 54, no. 5, pp. 3824–3851, 1996.
[2] I.L. Chuang and M. Nielsen, book ...
[3] D. Fattal, T.S. Cubitt, Y. Yamamoto, S. Bravyi and I.L. Chuang, quant-ph/0406168.
[4] T. Hiroshima, Phys. Rev. A 73, 012330 (2006)
[5] C. King, J. Math. Phys. vol. 43, no. 10, 4641 – 4653 (2002)
[6] C. King, Information Theory, vol. 49, no. 1, 221 – 229, 2003.
[7] K. Matsumoto, T. Shimono, and A. Winter, Comm, Math. Phys., 2003.
[8] K. Matsumoto, F. Yura, J. Phys. A, (2004).
[9] P. Shor, Comm. Math. Phys. 2003.
[10] P. Shor, J. Math. Phys. Vol. 43, 4334–4340 (2002)
[11] G. Vidal, W. Dür, J. I. Cirac, Phys. Rev. Let., vol. 89, no. 2, 027901, 2002.
[12] M. M. Wolf, J. Eisert, New J. Phys. 7, 93 (2005).
[13] KG.H. Vollbrecht and R.F. Werner, Phys. Rev. A64, 062307(2001).