Wigner functions in covariant and single-time formulations

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Abstract

We will establish the connection between the Lorentz covariant and so-called single-time formulation for the quark Wigner operator. To this end we will discuss the initial value problem for the Wigner operator of a field theory and give a discussion of the gauge-covariant formulation for the Wigner operator including some new results concerning the chiral limit. We discuss the gradient or semi-classical expansion and the color and spinor decomposition of the equations of motion for the Wigner operator. The single-time formulation will be derived from the covariant formulation by taking energy moments of the equations for the Wigner operator. For external fields we prove that only the lowest energy moments of the quark Wigner operator contain dynamical information.
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I. INTRODUCTION

Ultra-relativistic heavy ion experiments nowadays reach extreme beam energies of up to 200 GeV/A at CERN. The compressed nuclear matter becomes so hot and dense that a Quark-Gluon-Plasma (QGP) is expected to form [1]. The quest for this new state of matter [2] motivates the following work. It will deal with the description of ultra-relativistic heavy ion collisions in the framework of a transport theory for the underlying degrees of freedom, namely quarks and gluons. So far Quantum Chromodynamics (QCD) as the gauge theory for quarks and gluons has been successful in describing equilibrium properties (including linear response functions to external perturbations) through (i) perturbation theory in the framework of thermal field theory (TFD) [3,4], especially in connection with the Hard Thermal Loop resummation scheme of Braaten and Pisarski [5], and (ii) (non-perturbatively) through numerical Monte Carlo simulations of lattice gauge theory [6,7]. Since, however, the formation of a QGP is, at least in the early dynamical stages, governed by non-equilibrium processes we also need a kinetic theory which can deal with non-equilibrium phase space dynamics. For this the proper framework is relativistic transport theory.

Classical relativistic kinetic theory [8,9] has been used in practice for relativistic heavy ion collisions in the form of numerical cascade codes, including mean field as well as quantum statistical scattering effects for fermions (Pauli suppression) [10,11] and bosons (stimulated scattering) [12,13]. A classical relativistic transport theory for colored degrees of freedom was developed in [14,15]. The basis for quantum transport theory is the Wigner formalism [16–20]; through a systematic gradient expansion it can be reduced to classical kinetic theory [21].
Quantum transport equations for gauge theories require a gauge covariant definition of the Wigner operator [14,19,22,23]. Furthermore, proper inclusion of field theoretic retardation effects is facilitated by a Lorentz covariant formulation [19].

On the other hand, if one wants to solve the kinetic equations as an initial value problem in time, a Lorentz covariant formulation leads to conceptual difficulties. These will be shortly reviewed in Sec. II. This led Bialynicki-Birula, Gornicki, and Rafelski (BGR) [24] to propose a so-called single-time formulation for relativistic quantum transport theory which breaks explicitly the Lorentz covariance of the theory but can be solved as an initial value problem [24–26].

Our goal in this paper is to establish the connection between the Lorentz covariant and single-time formulations of quantum transport theory. The present work extends the previous study by Zhuang and Heinz [27] to the case of non-Abelian gauge theories. We concentrate on the dynamics of fermion fields in the background of a (classical or fluctuating) gauge field. We start from the Lorentz covariant formulation of relativistic quantum transport theory, discussing its equations of motion and their semiclassical expansion as well as their color and spinor decomposition (Sec. III). Most of this reviews (sometimes in more elegant notation) previous work which is relevant for our analysis here. We do, however, point out that quite generally transport and mass shell equations need to be solved only for 8 of the 16 spinor components, either for the scalar, pseudoscalar, and tensor densities or for the vector and axial vector densities. The remaining 8 components can then be obtained from constraints (Sec. III F). This was apparently not realized before. Also the color decomposition in Sec. III D has not been presented before. Some new results
are also contained in Sec. III G on the chiral (massless) limit and in Sec. III H on
the classical limit. In particular we show that in the chiral limit the kinetic equa-
tions for the right-handed and left-handed components decouple which considerably
simplifies the structure of the covariant transport equations.

In Sec. IV we derive the single-time form of the transport equations by tak-
ing energy moments of the covariant equations. This method was established in
Ref. [27] for simpler theories and is here extended to the case of general (abelian
or non-abelian) gauge theories. The covariant transport equations are equivalent
to an infinite hierarchy of coupled single-time transport equations for the energy
moments of the covariant Wigner operator. We construct this hierarchy and discuss
its practical truncation for applications. The general moment hierarchy is written
down for the Wigner operators and their equations of motion which, after taking
ensemble expectation values, would generate the whole coupled BBGKY hierarchy
of kinetic equations for 1-body, 2-body, etc. phase-space distribution functions.

In the present paper the truncation of the energy moment hierarchy is discussed
explicitly only for the simple limit of external gauge fields (Hartree limit). This
corresponds to Vlasov-type transport equations without collision terms. In this limit
we find a simple and exact truncation scheme (“exact” meaning that no additional
approximations like e.g. a semi-classical gradient expansion are neccessary) in which
only one dynamical single-time transport equation for the lowest energy moment
must be solved while all higher order moments can be obtained from the solution
via constraints. This explains the success of the BGR approach [24] (who only
studied the lowest energy moment and never looked at the full moment hierarchy)
for external field problems and at the same time provides a basis for generalizing it to include back-reaction and collision effects. In the last section of the main text, Sec. IVD, we write these equations out explicitly for QED to leading order of a semiclassical gradient expansion to recover the well-known Vlasov equations supplemented by spin precession terms.

A short summary of the achievements and an outlook on the remaining open problems is given in Sec. V. The Appendix contains helpful formulae and other technical details.

II. THE INITIAL VALUE PROBLEM

The Wigner operator for a field theory with second quantized fields $\Psi, \Psi^\dagger$ is given by the Fourier transformation of the density matrix

$$\hat{\rho}(x_+, x_-) = \Psi(x_+)\Psi^\dagger(x_-), \quad x_\pm = x \pm y/2$$

over the relative coordinate $y = x_+ - x_-$

$$\hat{W}(x, p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-iyp} \hat{\rho} \left( x + \frac{y}{2}, x - \frac{y}{2} \right).$$

$x = (x_+ + x_-)/2$ is the center of mass coordinate, and $p$ is the canonically conjugate momentum to $y$.

Using the translation operators $\exp(-y \cdot \partial_x/2)$ (acting to the right) and $\exp(y \cdot \partial^\dagger_x/2)$ (acting to the left) to rewrite the density matrix,

$$\hat{\rho} \left( x + \frac{y}{2}, x - \frac{y}{2} \right) = \Psi \left( x + \frac{y}{2} \right)\Psi^\dagger \left( x - \frac{y}{2} \right) = \Psi(x) e^{y(\partial^\dagger_x - \partial_x)/2} \Psi^\dagger(x)

= \Psi(x) e^{i\hat{\pi} y} \Psi^\dagger(x), \quad \hat{\pi} = \frac{i}{2} (\partial_x - \partial^\dagger_x),$$

(2.3)
the connection with the classical phase space density becomes apparent:

\[
\int \frac{d^4 y}{(2\pi)^4} e^{-i p \cdot y} \tilde{\rho}(x + y/2, x - y/2) = \int \frac{d^4 y}{(2\pi)^4} \Psi(x) e^{-i(p - \hat{\pi}) \cdot y} \Psi^\dagger(x) = \Psi(x) \delta(p - \hat{\pi}) \Psi^\dagger(x),
\]

Equation (2.4)

\[\hat{\pi} = i(\partial_x - \partial^\dagger_x)/2\]

is the momentum operator. The Wigner operator (more exactly: its ensemble expectation value) thus is a quantum mechanical and off-shell generalization of the classical phase space density

\[f(x, p, t) = \sum_i \delta(x - x_i(t)) \delta(p - p_i(t)),\]

Equation (2.5)

which gives the probability density to have a particle with (on-shell) momentum \(p\) at space-time coordinate \((x, t)\) (where \(x = (x_+ + x_-)/2\)). In contrast to the classical phase-space density \(f\), \(W(x, p) \equiv \langle \tilde{W}(x, p) \rangle\) and its energy average \(W(x, p, t) = \int dp_0 W(x, p)\) are in general not positive definite, but can become negative at small phase-space distances. Only if averaged over sufficiently large phase-space volumes \(\Delta V = \Delta^3 x \Delta^3 p \gg (2\pi)^3\) with some positive definite weight function \(g(x, p)\),

\[f(x, p, t) = \int_{\Delta V} \frac{d^3 x'}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} W(x - x', p - p', t) \cdot g(x', p'),\]

Equation (2.6)

the resulting \(f(x, p, t)\) is positive definite and can be interpreted as a classical phase-space distribution \([10]\).

**A. Initial value problem and energy moments**

For a covariant quantum field theory the density operator \(\tilde{\rho}(x_+, x_-)\) and, in general, all 2-point functions contain two time parameters, \(t_1 = x_0 + y_0/2\) and \(t_2 = x_0 - y_0/2\). The occurrence of two time parameters is necessary for a proper
description of signal propagation and retardation effects. It poses, however, conceptual problems for a transport theoretic approach because the corresponding covariant Wigner density cannot be initialized with the help of asymptotic field configurations at \( t = -\infty \). To see this let us rewrite the covariant Wigner function (see (2.2)) as follows:

\[
W_4(x, p) = \int_{-\infty}^{\infty} dy_0 \frac{d^3y}{(2\pi)^3} e^{ip_0y_0} \left\langle \Psi \left( x + \frac{y}{2}, x_0 + \frac{y_0}{2} \right) \Psi^\dagger \left( x - \frac{y}{2}, x_0 - \frac{y_0}{2} \right) \right\rangle .
\]  

(2.7)

Obviously, due to the Fourier transformation over relative times \( y_0 \), the covariant Wigner function \( W_4 \) at any fixed time \( x_0 \) requires knowledge of the fields \( \Psi, \Psi^\dagger \) at \emph{all} times. \( W_4 \) at \( x_0 \to -\infty \) can thus not be initialized by giving the fields only at \( t = -\infty \).

The equal-time (or single-time) Wigner function

\[
W_3(x, p) = \int \frac{d^3y}{(2\pi)^3} e^{ip\cdot y} \left\langle \Psi \left( x + \frac{y}{2}, x_0 \right) \Psi^\dagger \left( x - \frac{y}{2}, x_0 \right) \right\rangle ,
\]  

(2.8)

which uses the fields at equal times and thus depends only on a single time parameter, has clearly no such problem. Initializing \( W_3 \) at \( x_0 = -\infty \) by vacuum solutions for the fields, it has been used in [25] to calculate the pair production rate in QED in a constant external electric field, reproducing the correct Schwinger rate [29]. On the other hand, by comparing (2.7) and (2.8) one sees that the equal-time Wigner function \( W_3 \) is simply the energy integral of the covariant Wigner function \( W_4 \):

\[
W_3(x, p) = \int_{-\infty}^{\infty} dp_0 W_4(x, p, p_0) .
\]  

(2.9)

As such it contains less dynamical information than \( W_4 \) since the complete off-shell structure of \( W_4 \) is averaged over.
While the formulation of the transport theory as an initial value problem thus seems to require a single-time language, it can clearly not be based on $W_3$ alone. Retardation and memory effects in the covariant approach are reflected in the off-shell behaviour of the covariant Wigner function $W_4$ which should not be thrown away by simply averaging it over $p_0$. Instead, one must study, in addition to $W_3$ (which is the zeroth $p_0$-moment of $W_4$) also the higher $p_0$-moments of the covariant Wigner function:

$$\int_{-\infty}^{\infty} dp_0 \, p_0^n \, W_4(x, p) = \left(\frac{i}{2}\right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int \frac{d^3y}{(2\pi)^3} \, e^{ip\cdot y}$$

$$\times \left\langle \left[ \partial^{(k)}_{x_0} \Psi \left( x + \frac{y}{2}, x_0 \right) \right] \left[ \partial^{(n-k)}_{x_0} \Psi^\dagger \left( x - \frac{y}{2}, x_0 \right) \right] \right\rangle . \quad (2.10)$$

The full information on $W_4$ is equivalent to knowing all (and not just the lowest) of its $p_0$ moments, and initializing them at $t = -\infty$ requires the knowledge of all time derivatives of the fields $\Psi$, $\Psi^\dagger$ at $t = -\infty$ (or, equivalently, to knowing the fields for all times).

This is clearly impractical. It follows from these considerations that a formulation of covariant transport theory as an initial value problem for equal-time energy moments of the covariant Wigner function must be accompanied by a truncation prescription which allows to restrict the description to a finite number of such energy moments.

The strategy of the present paper will thus be to first derive from the underlying field theoretic Lagrangian a set of covariant equations of motion for the covariant Wigner operator $\hat{W}_4$ and then, by taking $p_0$-moments of these covariant equations, set up the coupled hierarchy of equal-time equations for the energy moments of $\hat{W}_4$ [27,30]. We will then study the truncation of this hierarchy.
In general, truncating the moment hierarchy will entail nontrivial approximations. However, if the fermion fields evolve under the influence of an external gauge field, i.e. they do not interact with each other, neither directly via scattering nor indirectly via backreaction of their electric current on the external field, their time evolution is uniquely fixed once their values at $t = -\infty$ have been specified. (The Dirac equation is a first order differential equation in time.) It should thus be sufficient to specify initial conditions for $W_3$ only, but not for any higher energy moments. This can only be true if the time evolution of the system in the equal-time framework can be described completely in terms of $W_3$ only. This turns out to be correct in the sense that, in the external field limit, only for $W_3$ a genuine equal-time transport equation must be solved while all higher energy moments of $W_4$ can be obtained from the solution of this equation via constraints. This will be discussed in more detail in Sec. IV.

B. Formulation with orthogonal polynomials

In Ref. [27] it was suggested to construct the equal-time moments of the covariant Wigner density $W_4$ from orthogonal polynomials of $p_0$ rather than from the powers of $p_0$ used in Eq. (2.10). Orthogonal polynomials have the advantage that the complete function $W_4$ is easily reconstructed from the set of moments using the orthogonality relations. Their disadvantage is that, if they are defined over the full $p_0$ interval $(-\infty, \infty)$, they require additional weight functions which fall off sufficiently rapidly at $\pm \infty$. Only if one restricts $p_0$ to a finite interval $[-a, a]$ one can use orthogonal polynomials without weight function, the Legendre polynomials. This was done
in Ref. [27]. The price for doing so was the technical requirement that the Wigner density $W_4$ falls off as a function of $p_0$ sufficiently fast near $\pm \infty$ that its contributions outside a large but finite interval can be neglected.

If one wants to avoid this technicality and rather work with orthogonal polynomials on the infinite $p_0$ interval, one must use additional weight functions $g(p_0)$ in order to obtain suitable orthogonality relations. We will now show that this does not really lead to a proper single-time formulation. Using the reduced energy variable $z = p_0/E$ with $E = \sqrt{p^2 + m^2}$, the general moment expansion then reads

$$\hat{W}(x, p, p_0) = \sum_{k=0}^{\infty} \frac{1}{N_k} \hat{w}_k(x, p) u_k(z), \quad (2.11)$$

$$\hat{w}_k(x, p) = \int_\infty^\infty dz g(z) u_k(z) \hat{W}(x, p, Ez), \quad (2.12)$$

$$\delta_{kl} N_k = \int_\infty^\infty dz g(z) u_k(z) u_l(z). \quad (2.13)$$

Inserting the definition (2.2) of the Wigner operator we get

$$\hat{w}_k(x, p) = \int dz g(z) \int \frac{d^4 y}{(2\pi)^4} \left[ u_k(i\partial_{y_0}/E)e^{-ip \cdot y} \right] \hat{g}(x_+, x_-)$$

$$= \int \frac{d^4 y}{(2\pi)^4} e^{ip \cdot y} \left[ u_k(i\partial_{y_0}/E) \int dz g(z) e^{-iEz y_0} \right] \hat{g}(x_+, x_-)$$

$$= \int \frac{d^4 y}{(2\pi)^4} e^{ip \cdot y} \left[ \int dz g(z) e^{-iEz y_0} \right] u_k(-i\partial_{y_0}/E) \hat{g}(x_+, x_-), \quad (2.14)$$

with partial integration in the last line, assuming that $\hat{g}(x_+, x_-)$ vanishes identically at $|x_0+|, |x_0-| \to \infty$. However, Eq. (2.14) only then strictly corresponds to a formulation in terms of only a single time parameter if

$$\int_\infty^\infty dz g(z) e^{-iEz y_0} \uparrow \frac{1}{\delta(Ez)} = \delta(y_0). \quad (2.15)$$

Since for orthogonal polynomials on the full $p_0$ interval $g(z) \neq 1$ this is not possible, a strict single-time formulation cannot be achieved in this way.
If we insist on a true single-time formulation there are thus only two possibilities:

(1) One tries to keep the computational advantages of orthogonal polynomials. In this case one must restrict \( p_0 \) to a finite interval and assume that outside that interval the Wigner function vanishes \([27]\). (2) One does not want to place such a drastic restriction on \( W_4 \) but allows for, say, exponential tails in the Wigner function at large \( p_0 \). In that case one should use the moments \((2.14)\) with simple powers of \( p_0 \) which are not orthogonal on each other. We will here take the second choice.

Let us remark that the moment expansion of the covariant Wigner operator needs not be restricted to energy moments. In [31] a systematic expansion of the Wigner operator into moments of all four components of the momentum \( p \) was given. One can show that such an expansion corresponds to the semiclassical \( \hbar \) expansion or, equivalently, to a gradient expansion around \( x \). We will explain this in more detail in Sec. [IV] when we discuss the connection between the expansion in energy moments and a temporal gradient expansion around \( x_0 \).

C. Green functions and the Wigner operator

Related to the initial value problem of non-equilibrium dynamics in field theory is the doubling of the Hilbert space between zero and finite temperature field theory [32].

The connection between the non-equilibrium Wigner formalism and the two-point Green functions of a field theory in thermodynamic equilibrium is given via the density operator [33]:

\[
i G(x_+, x_-) = \langle 0 | T \hat{\rho}(x_+, x_-) | 0 \rangle.
\] (2.16)
$T$ is the path ordering operator for a given time contour $C$. The most perfect formal agreement \cite{18,20,32} between vacuum field theory, finite temperature equilibrium thermodynamics and non-equilibrium dynamics is obtained if one chooses as the time contour $C$ the “Closed Time Path” (CTP) \cite{34} which goes from $t = -\infty$ to $t = +\infty$ infinitesimally above the real time axis, then returns to $t = -\infty$ infinitesimally below the real time axis, and (for equilibrium thermodynamics) finally proceeds vertically to $-\infty - i\beta$ (where $\beta = 1/T$) in order to ensure the KMS condition, i.e. the periodicity of the thermal Green functions in imaginary time. The last piece is missing in non-equilibrium situations in which case the Green functions do not satisfy the KMS condition.

In the CTP formalism the 2-point Green functions can be split into four pieces according to the four possibilities to distribute the two time arguments on the upper and lower parts of the real-time contour. In matrix notation one writes \cite{18,20,32}

\[
G(x, y) = \begin{pmatrix}
G^c(x, y) & G^<(x, y) \\
G^>(x, y) & G^a(x, y)
\end{pmatrix}.
\] (2.17)

The covariant Wigner function $W_4$ is the 4-dimensional Wigner transform (Fourier transform with respect to $y$) of $G^<(x + y/2, x - y/2)$.

In thermodynamic equilibrium formal identities, in particular the KMS condition, allow to express all four components of the $2 \times 2$ matrix (2.17) in terms of a single real spectral density \cite{32}. In non-equilibrium situations, the absence of the KMS condition implies that at least two components of (2.17) are independent and probably should be kept for a complete dynamical description. While $G^c$ and $G^a$ contain also information on the virtual excitations of the vacuum, $G^<$ and $G^>$ (which
are again related by the spectral density) describe exclusively the dynamics of the
real excitations in the medium [20,35]. Here we will be mostly interested in the case
of the external fields where knowledge of the dynamics of $G^<$ resp. $W_4$ is sufficient.

III. LORENTZ-COVARIANT FORMULATION

In the following we will discuss the gauge theories QED and QCD [22,23]. We
will consider only the fermion Wigner operator and treat the photon or gluon degrees
of freedom through their classical gauge fields. Problems connected with a suitable
gauge choice will not be discussed here, the reader is referred to [22,36,37].

A. Gauge-covariant Wigner operator

For a gauge covariant theory the Wigner operator must transform covariantly
under gauge transformations. This is achieved through a gauge covariant definition
of the density operator [14,22]

$$ \hat{\rho} \left( x + \frac{y}{2}, x - \frac{y}{2} \right) = \bar{\psi}(x)e^{y \cdot D^\dagger(x)/2} \otimes e^{-y \cdot D(x)/2}\psi(x), $$

(3.1)

with the covariant derivative

$$ D_\mu(x) = \partial_\mu - igA_\mu(x) $$

(3.2)

instead of the partial derivative in the translation operator of Eq. (2.3). The direct
product in Eq. (3.1) is over spinor and (in the case of QCD) color indices. Both
the fields $\bar{\psi}_\beta^a, \psi_\alpha^a$, $a = 1, 2, \ldots, N_c$, and the covariant translation operators are group
elements of $SU(N_c)$ albeit in different representations.

With (3.1) the Wigner operator reads
The density operator can then be rewritten as

\[ \hat{\rho} \left( x + \frac{y}{2}, x - \frac{y}{2} \right) = \overline{\psi} \left( x + \frac{y}{2} \right) U \left( x + \frac{y}{2}, x \right) \otimes \hat{\rho} \left( x, x - \frac{y}{2} \right) \psi \left( x - \frac{y}{2} \right) . \quad (3.7) \]

In the following we will use the conventions from \[22\] for an operator \( \hat{O} \) operating on the color and spinor indices of \( \hat{\rho} \):

\[ \hat{O} \hat{\rho} \left( x + \frac{y}{2}, x - \frac{y}{2} \right) = \overline{\psi} \left( x + \frac{y}{2} \right) U \left( x + \frac{y}{2}, x \right) \otimes \hat{O} \left( x, x - \frac{y}{2} \right) \psi \left( x - \frac{y}{2} \right) \]

\[ \hat{O} \left( x + \frac{y}{2}, x - \frac{y}{2} \right) \hat{\rho} = \overline{\psi} \left( x + \frac{y}{2} \right) U \left( x + \frac{y}{2}, x \right) \hat{O} \otimes U \left( x, x - \frac{y}{2} \right) \psi \left( x - \frac{y}{2} \right) . \quad (3.8) \]
Color octet objects in QCD will be denoted as matrices, $O = O_a t_a$, with $t_a$ being the generators in the fundamental representation of the gauge group. They are given in Appendix A together with useful color trace formulae. Occasionally we will also use their adjoint representation $T_a$ with

$$(T_a)_{bc} = -i h f_{abc}$$

and norm

$$\text{tr}(T_a T_b) = 3 h^2 \delta_{ab}.$$ (3.10)

With our normalization the $t_a$ have eigenvalues $\sim \frac{h}{2}$ proportional to the physical color-spin of quarks, while the $T_a$ have eigenvalues $\sim h$ corresponding to the color spin of gluons.

From the Wigner operator one obtains the operators for the fermionic baryon (charge) and color currents via

$$j^\text{baryon}_\nu(x) = -\text{tr}\left(\gamma_\nu \bar{\psi}(x)\psi(x)\right) = \int d^4 p \text{ tr}\left(1_{3\times3} \gamma_\nu \hat{W}(x,p)\right),$$ (3.11)

$$j^\text{color}_{\nu a}(x) = -\text{tr}\left(\gamma_\nu \frac{\lambda_a}{2} \bar{\psi}(x)\psi(x)\right) = \int d^4 p \text{ tr}\left(\frac{\lambda_a}{2} \gamma_\nu \hat{W}(x,p)\right).$$ (3.12)

The baryon current projects onto the color singlet, the color current onto the color octet contributions of the Wigner operator. In analogy we will later on perform a color decomposition for the equations of motion of the Wigner operator.

As spinor matrices $\hat{\phi}$ and $\hat{W}$ transform according to

$$\hat{\phi}^\dagger \left(x + \frac{y}{2}, x - \frac{y}{2}\right) = \gamma^0 \hat{\phi} \left(x - \frac{y}{2}, x + \frac{y}{2}\right) \gamma^0$$ (3.13)

with exchanged arguments $x_\pm$ and

$$\hat{W}^\dagger(x,p) = \gamma^0 \hat{W}(x,p) \gamma^0.$$ (3.14)
B. Equations of motion for the Wigner operator

The basis for the dynamics of strong interactions is the Lagrange density of QCD (since flavor quantum numbers don’t matter for our purpose we consider only one quark flavor):

\[ \mathcal{L}_{\text{QCD}} = i \bar{\psi}(x) \gamma^\mu D_\mu(x) \psi(x) - mc^2 \bar{\psi} \psi - \frac{1}{4 \hbar c} \hat{F}_{\mu\nu}(x) \hat{F}^{\mu\nu}(x). \] (3.15)

\( \psi(x), \bar{\psi}(x) = \psi^\dagger \gamma^0 \) are the spinor fields,

\[ D_\mu(x) = \partial_\mu - \frac{ig}{\hbar c} \hat{A}_\mu(x), \] (3.16)

is the covariant derivative (see Eq. (3.2)), \( m \) is the bare mass, and \( \hat{F}_{\mu\nu}(x) \) is the field strength tensor. The latter is defined through the commutator of the covariant derivative,

\[ \hat{F}_{\mu\nu}(x) \equiv - \frac{\hbar c}{ig} [D_\mu, D_\nu] = (\partial_\mu \hat{A}_\nu(x)) - (\partial_\nu \hat{A}_\mu(x)) - \frac{ig}{\hbar c} [\hat{A}_\mu(x), \hat{A}_\nu(x)], \] (3.17)

and thus satisfies the Jacobi identity

\[ [D_\alpha(x), \hat{F}_{\mu\nu}(x)] + [D_\nu(x), \hat{F}_{\alpha\mu}(x)] + [D_\mu(x), \hat{F}_{\nu\alpha}(x)] = 0. \] (3.18)

Since for the semiclassical and gradient expansions we will later need to count powers of \( \hbar \) we displayed them here once explicitly. Until further notice we will, however, from now on return again to natural units with \( \hbar = c = 1 \).

Inserting the Dirac equation and its adjoint,

\[ (i \gamma^\mu D_\mu(x) - m) \psi(x) = 0, \] (3.19)

\[ \bar{\psi}(x)(i \gamma^\mu D^\dagger_\mu(x) - m) = 0, \] (3.20)
(with the hermitean adjoint of the partial derivative defined as before as acting to the left) into the definition (3.7) of the density operator and, following [22,23], pulling the derivatives out in front by using the derivative rules for link operators as given in Appendix C, one finds the following equations of motion for the covariant fermion Wigner operator [22]:

$$2m \hat{W}(x,p) = \gamma^\mu \left[ (iD_\mu(x) + p_\mu - g \int_{-1/2}^0 ds \ (1 - 2s) [^x] \hat{F}_{\nu\mu}(x + is\partial_p)i\partial_\nu^p) \hat{W}(x,p) 
+ \hat{W}(x,p)(-iD_\mu(x) + p_\mu + g \int_{-1/2}^0 ds \ (1 + 2s) [^x] \hat{F}_{\nu\mu}(x + is\partial_p)i\partial_\nu^p) \right] ,$$

(3.21)

$$2m \hat{W}(x,p) = \left[ (-iD_\mu(x) + p_\mu + g \int_{-1/2}^0 ds \ (1 + 2s) [^x] \hat{F}_{\nu\mu}(x + is\partial_p)i\partial_\nu^p) \hat{W}(x,p) 
+ \hat{W}(x,p)(iD_\mu(x) + p_\mu - g \int_{-1/2}^0 ds \ (1 - 2s) [^x] \hat{F}_{\nu\mu}(x + is\partial_p)i\partial_\nu^p) \right] \gamma^\mu .$$

(3.22)

Here, for notational convenience, momentum derivatives standing to the right of the Wigner operator are defined in the sense of partial integration as

$$\hat{W}(x,p)\partial_\nu^1 \ldots \partial_\nu^k = (-1)^k \partial_\nu^1 \ldots \partial_\nu^k \hat{W}(x,p) .$$

(3.23)

The operator

$$[^x] \hat{F}_{\nu\mu}(z(s)) \equiv U(x,z(s)) \hat{F}_{\nu\mu}(z(s))U(z(s),x) \quad \text{with } z(s) = x + sy$$

(3.24)

is called Schwinger string and connects the field strength tensor at point $z$ gauge covariantly with the point $x$ along a straight line path. The Schwinger string is the essential quantity to describe the gluonic degrees of freedom as non-local operators.

To get a more compact notation we define generalized non-local momentum and
derivative operators $\Pi_\mu$ and $\Delta_\mu$, respectively (here we display $\hbar$ and $c$ explicitly for later use):

\[
\Pi_\mu = p_\mu + \frac{2g}{c} \int_{-1/2}^{0} ds \, is \hbar \partial_p^\nu \left[ x \right] \tilde{F}_{\nu\mu}(x + is \partial_p),
\]

\[
\Delta_\mu = \hbar D_\mu(x) + \frac{ig}{c} \int_{-1/2}^{0} ds \, i \hbar \partial_p^\nu \left[ x \right] \tilde{F}_{\nu\mu}(x + is \partial_p).
\]

The equations of motion for the Wigner operator are then

\[
2m \hat{W}(x,p) = \gamma^\mu \left( \{ \Pi_\mu, \hat{W}(x,p) \} + i[\Delta_\mu, \hat{W}(x,p)] \right),
\]

\[
2m \hat{W}(x,p) = \left( \{ \Pi_\mu, \hat{W}(x,p) \} - i[\Delta_\mu, \hat{W}(x,p)] \right) \gamma^\mu.
\]

With $\Pi_\mu^\dagger = \Pi_\mu$, $(i\Delta_\mu)^\dagger = i\Delta_\mu$ and $[\Pi_\mu, \gamma^0] = [\Delta_\mu, \gamma^0] = 0$ we find

\[
\gamma^0 (\Pi_\mu \pm i\Delta_\mu)^\dagger \gamma^0 = \Pi_\mu \pm i\Delta_\mu,
\]

which means that Eqs. (3.27) and (3.28) are adjoint to each other: $\gamma^0 (3.27) \gamma^0 = (3.28)^\dagger$. Adding and subtracting the two equations gives

\[
4m \hat{W}(x,p) = \{ \gamma^\mu, \{ \Pi_\mu, \hat{W}(x,p) \} \} + i[\gamma^\mu, [\Delta_\mu, \hat{W}(x,p)]]],
\]

\[
0 = [\gamma^\mu, \{ \Pi_\mu, \hat{W}(x,p) \}] + i\{ \gamma^\mu, [\Delta_\mu, \hat{W}(x,p)] \}.
\]

This corresponds to a separation into real and imaginary parts. The generalized momentum operator is always working as an anti-commutator, the generalized derivative operator always as a commutator on $\hat{W}$. The only difference between the

*Our notation here differs somewhat from that of Ref. 38; it is optimized for a simultaneous description of non-abelian and abelian gauge interactions, while in 38 only the abelian case of QED with classical external fields was studied. In that limit $\Delta$ and $\Pi$ as defined here reduce to the definitions given in 38, see Eqs. (3.65,3.66) below.
two equations is that for the spinor structure commutators are replaced by anti-commutators. Defining therefore

\[ [\hat{A}, \hat{B}]_\pm \equiv [\hat{A}, \hat{B}] , \quad [\hat{A}, \hat{B}]_\mp \equiv \{ \hat{A}, \hat{B} \} , \]

we can combine the two equations as

\[ [2m\mathbf{1}, \hat{W}(x,p)]_\mp = [\gamma^\mu, \{ \Pi_\mu, \hat{W}(x,p) \}]_\pm + i[\gamma^\mu, [\Delta_\mu, \hat{W}(x,p)]]_\mp . \] (3.33)

These are the equations of motion for the Wigner operator for particles with spin 1/2.

The equations are closed by the Yang-Mills-equations for the field strength tensor. These couple to the Wigner operator for the fermion fields via the current \( j_\nu \):

\[ [D^\mu(x), \hat{F}_{\mu\nu}(x)] = j_\mu(x) = t_a \text{tr} t_a \gamma_\nu \hat{W}(x,p). \] (3.34)

The trace sums over spinor and color degrees of freedom and contains an integral over momentum space.

Eqs. (3.33) and (3.34) contain the full dynamical information of the system on the operator level.

C. Gradient expansion

The classical limit of these equations is obtained via a gradient expansion of the non-local operators \( \Pi \) and \( \Delta \) in Eqs. (3.25,3.26). It is generated by a Taylor expansion for the Schwinger string around \( x \) in powers of \( is\hbar\partial_\mu \). As such it is automatically also an expansion in powers of \( \hbar \); since, however, the color decomposition discussed in the following subsection introduces additional powers of \( \hbar \), we will refer to the
semiclassical expansion in the sense of Wigner and Kirkwood only as the “gradient expansion”. As we will see it is actually an expansion in covariant derivatives. It thus preserves gauge covariance.

The Taylor expansion of the Schwinger string is given by

\[
x^\hat{\mu}(x + is\hbar \partial_p) = \sum_{k=0}^{\infty} \frac{1}{k!} (ish)^k \left[ (\partial_{\mu}\partial_y^n) \right]^k [x^\hat{\mu}(x + y)]_{y=0}.
\] (3.35)

The connection between the expansion of the Schwinger string into local operators

\[
[(\partial_p \cdot \partial_y)^n [x\hat{\mu}(x + y)]_{y=0}
\]

and a gradient expansion in covariant derivatives \(D_\mu(x)\) can be shown from the expansion of the Schwinger string in an exponential as derived in [22]:

\[
[x\hat{\mu}(x + is\hbar \partial_p) = \exp[ish \partial^\alpha \tilde{D}_\alpha(x)] \hat{\mu}(x)
\] (3.36)

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} (ish)^k \left( \partial^\alpha \tilde{D}_\alpha(x) \right)^k [x\hat{\mu}(x)]_{y=0}.
\] (3.37)

\(\tilde{D}_\alpha(x)\) is defined as the covariant derivative in the adjoint representation [22]:

\[
\tilde{D}_\alpha(x) \hat{\mu}(x) \equiv [D\alpha(x), \hat{\mu}(x)].
\] (3.38)

Comparison between Eqs. (3.35) and (3.37) gives the following identity (see also Appendix C for a direct calculation from of the l.h.s. of Eq. (3.38) for \(n = 1\):

\[
[(\partial_p \cdot \partial_y)^n [x\hat{\mu}(x + y)]_{y=0} = \left( \partial_p \cdot D(x) \right)^n, \hat{\mu}(x)]
\] (3.39)

Here we defined a generalized commutator through

\[
[A^n, B] = [A, [A^{n-1}, B]],
\]

\[
[A^0, B] = B
\] (3.40)

or explicitly
\[ [A^n, B] = [A, [A, \ldots, [A, B] \ldots]] = \sum_{k=0}^{n} \binom{n}{k} (-1)^k A^{n-k} B A^k. \] (3.41)

Using Eqs. (3.33,3.39) in Eqs. (3.25,3.26) and performing the s-integration we find for the non-local operators

\[ \Delta_\mu = \hbar D_\mu - \hbar \frac{g}{2c} \sum_{k=0}^{\infty} \left( -\frac{i\hbar}{2} \right)^k \frac{1}{(k+1)!} \left[ \left( \partial_\nu \cdot p \right) \cdot D(x) \right]^k \hat{F}_{\nu\mu}(x) \partial_\nu, \] (3.42)

\[ \Pi_\mu = p_\mu - \hbar \frac{ig}{2c} \sum_{k=0}^{\infty} \left( -\frac{i\hbar}{2} \right)^k \frac{k+1}{(k+2)!} \left[ \left( \partial_\nu \cdot p \right) \cdot D(x) \right]^k \hat{F}_{\nu\mu}(x) \partial_\nu. \] (3.43)

Using the definition (3.17) for the field strength tensor and in (3.43) the identity

\[ D_\mu = \left[ \partial_\nu \cdot D(x), p_\mu \right] \] (3.44)

we finally get

\[ \Delta_\mu = D_\mu(x) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[ \left( -\frac{i\hbar}{2} \partial_\nu \cdot D(x) \right)^k, \hbar D_\mu(x) \right] \] (3.45)

\[ \Pi_\mu = p_\mu + 2 \sum_{k=1}^{\infty} \frac{1 - k}{k!} \left[ \left( -\frac{i\hbar}{2} \partial_\nu \cdot D(x) \right)^k, p_\mu \right]. \] (3.46)

Eqs. (3.45,3.46) identify \( \Delta_\mu \) and \( \Pi_\mu \) as non-local generalizations of the covariant derivative \( \hbar D_\mu \) and the momentum \( p_\mu \).

The (hermitean) operator for the expansion about the classical limit is thus given by

\[ \Delta = \partial_\nu^W \cdot i D^F(x), \] (3.47)

with \( \hbar \) as the expansion parameter. The superscript \( W \) means that the momentum derivative acts on the momentum dependence of the Wigner operator. The superscript \( F \) on the covariant derivative reminds us that it acts only on the field strength operator following it, see Eqs. (3.42,3.43). From Eqs. (3.45,3.46) or (3.44) we find that \( \Delta_\mu \) is one order higher in \( \Delta \) and \( \hbar \) than \( \Pi_\mu \).
Since, at least for QED, the measurable classical fields are the field strengths $F_{\mu\nu}(x)$ we will use for the gradient expansion Eqs. (3.42,3.43) rather than Eqs. (3.45,3.46).

The validity of the gradient expansion requires

$$h \Delta = h \partial_p^W \cdot iD^F(x) \ll 1,$$

(3.48)
i.e. that the Wigner function is sufficiently smooth in momentum space and the field strengths vary sufficiently slowly (in a covariant sense) in coordinate space.

The corresponding length scales on which these functions vary must satisfy

$$(\Delta p)_W \cdot (\Delta x)_F \gg h.$$  

(3.49)

Inserting the expansions (3.42,3.43) together with the Ansatz

$$\hat{W}(x,p) = \sum_{k=0}^{\infty} h^k \hat{W}^{(k)}(x,p)$$

(3.50)

for the Wigner operator into the equations of motion (3.33) we obtain

$$[2m \mathbf{1}, \hat{W}^{(n)}]_\pm =$$

$$\bigg[ \gamma^\mu, 2p_\mu \hat{W}^{(n)} - \frac{ig}{2c} \sum_{k=0}^{n-1} \frac{k+1}{(k+2)!} \{ \left[ \left( \frac{-i}{2} \Delta \right)^k, \hat{F}_{\nu\mu}(x) \right] \partial_{p_\nu} \hat{W}^{(n-k-1)} \} \bigg]_\pm$$

$$+ i \bigg[ \gamma^\mu, \left[ D_\mu(x), \hat{W}^{(n-1)} \right] - \frac{g}{2c} \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[ \left[ \left( \frac{-i}{2} \Delta \right)^k, \hat{F}_{\nu\mu}(x) \right] \partial_{p_\nu} \hat{W}^{(n-k-1)} \right] \bigg]_\mp$$

(3.51)

With Table II for the zeroth and first order terms in the non-local operators we get

$$[2m \mathbf{1}, \hat{W}^{(0)}]_\pm = [\gamma^\mu, 2p_\mu \hat{W}^{(0)}]_\pm$$

(3.52)
to zeroth order and

$$[2m \mathbf{1}, \hat{W}^{(1)}]_\pm = \bigg[ \gamma^\mu, 2p_\mu \hat{W}^{(1)} - \frac{ig}{4c} \left\{ \hat{F}_{\nu\mu}(x) \partial_{p_\nu} \hat{W}^{(0)} \right\} \bigg]_\pm$$

$$+ i \bigg[ \gamma^\mu, \left[ D_\mu(x) - \frac{g}{2c} \hat{F}_{\nu\mu}(x) \partial_{p_\nu} \hat{W}^{(0)} \right] \bigg]_\mp$$

(3.53)
to first order in \( \hbar \).

We will examine the resulting classical limit more closely in the context of the spinor decomposition for the Wigner operator since there we will find an easy connection to the classical mass shell condition and transport equation.

D. Color decomposition

For QCD it is useful to decompose the Wigner operator into its (observable) color singlet and its (unobservable) color octet contributions \[14\]:

\[
\hat{W}(x,p) = \hat{W}_s(x,p) + \hat{W}_a(x,p) t^a \tag{3.54}
\]

\[
\hat{W}_s(x,p) = \frac{1}{3} \text{tr} \hat{W}(x,p), \tag{3.55}
\]

\[
\hat{W}_a(x,p) = \frac{2}{\hbar^2} \text{tr} t^a \hat{W}(x,p) \tag{3.56}
\]

The color decomposition of the Schwinger string \[39\] can be calculated to any fixed order of the gradient expansion from the explicit form

\[
[e] \hat{F}_{\mu\nu}(x + i \hbar \partial_\rho) = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} \left[ \left( \frac{1}{2} \partial^\rho \partial_\alpha 1 + \frac{i g}{\hbar c} \partial^\rho \hat{A}_\alpha(x) t^a \right)^n , \hat{F}_{\nu\mu}^{-}(x) t^b \right], \tag{3.57}
\]

Combining the two decompositions we find to zeroth order in the gradient expansion

singlet: \[ m1 - \gamma^\mu p_\mu, \hat{W}_s^{(0)} \] \[= 0, \tag{3.58}\]

octet: \[ m1 - \gamma^\mu p_\mu, \hat{W}_a^{(0)} \] \[= 0, \tag{3.59}\]

and to first order

singlet: \[ m1 - \gamma^\mu p_\mu, \hat{W}_s^{(1)} \] \[= \frac{ig}{4c} [\gamma^\mu, \partial_\nu \hat{W}_s^{(0)}] \pm = -\frac{ig\hbar^2}{24c} \left( \frac{1}{2} [\gamma^\mu, \partial_\nu \hat{F}^a_{\nu\mu}(x), \hat{W}_s^{(0)a}] \right)_\pm + [\gamma^\mu, \partial_\nu \{ \hat{F}^a_{\nu\mu}(x), \hat{W}_s^{(0)a} \}]_\pm \]
\[
-iq / \hbar c \left[ \gamma^\mu, [\hat{A}^a_\mu(x), \hat{W}^{(0)a}] \right]_\mp 
\]

octet: \[ \left[ m1 - \gamma^\mu p^\mu, \hat{W}^{(1)a} \right]_\pm + i q / 4c \left[ \gamma^\mu, \partial^\mu \hat{W}^{(0)a} \right]_\mp = \]

\[
-iq / 4c \left( \frac{1}{2} \left[ \gamma^\mu, \partial^\nu [\hat{F}^a_{\nu\mu}(x), \hat{W}^{(0)}] \right] \right)_\pm + \left[ \gamma^\mu, \partial^\nu \{\hat{F}^a_{\nu\mu}(x), \hat{W}^{(0)} \} \right]_\mp 
\]

\[
-iq / \hbar c \left[ \gamma^\mu, [\hat{A}^a_\mu(x), \hat{W}^{(0)}] \right]_\mp 
\]

\[
-iq / 8c \left( \frac{1}{2} \left[ \gamma^\mu, \partial^\nu [\hat{F}^{b}_{\nu\mu}(x), \hat{W}^{(0)c}] \right] \right)_\pm + \left[ \gamma^\mu, \partial^\nu \{\hat{F}^{b}_{\nu\mu}(x), \hat{W}^{(0)c} \} \right]_\mp 
\]

\[
-ig \left[ \gamma^\mu, \{\hat{A}^b_\mu(x), \hat{W}^{(0)c} \} \right]_\mp 
\]

Here we used the trace formulae given in Appendix A and the relation \([AB, C] = A[B, C] + [A, C]B\).

These equations exemplify a general result which to prove we will leave to the reader: At any given order \(n\) of the gradient expansion, the “leading” color singlet and octet contributions \(\hat{W}^{(n)}_s, \hat{W}^{(n)}_a\) decouple from each other; the color singlet and octet sectors couple only via the lower order components \(\hat{W}^{(n-1)}\).

Note that the color decomposition brings in additional powers of \(\hbar\) which are not connected to the gradient expansion, but are related to the question to what extent color can be treated classically \([15\]. One should carefully differentiate between the limit of classical color and the classical limit in the kinetic sense which is defined through the gradient expansion.

The transition back from QCD to QED is made by formally setting the color matrices equal to 1. Nonetheless the fields \(\hat{A}_\mu(x)\) are still operators in Fock space and neither commute with the Wigner operator nor with each other. The equations
of motion for the Wigner operator are thus formally identical in QED and QCD, only that in QCD the (anti-)commutators refer additionally to the color indices.

E. External field limit and Wigner function

Equations of motion for the Wigner function (rather than the Wigner operator) are obtained by taking ensemble expectation values of the operator equations. Writing for operators \( \hat{O} \) working on \( \hat{W} \)

\[
\langle [\hat{O}, \hat{W}]_{\pm} \rangle = \langle [\hat{O}] \rangle_{\pm}, \langle \hat{W} \rangle_{\pm} \pm \langle [\hat{O}, \hat{W}]_{\pm} \rangle = \langle [\hat{O}] \rangle_{\pm} \pm C(\hat{O}, \hat{W}), \tag{3.62}
\]

we can split the resulting equations into two parts: One part contains only one particle Wigner functions \( \langle \hat{W} \rangle \) and mean (gluon) fields (Vlasov term), while the other part describes the correlations \( C(\hat{O}, \hat{X}) \) (collision term). (For a discussion of the latter see the formulation with Green functions in [40].) In this subsection we will study the structure of the equations for the Wigner function under the assumption that the correlations can be neglected and that the fields \( \hat{F}_{\mu\nu} \) can be approximated by external fields \( \langle \hat{F}_{\mu\nu} \rangle = F_{\mu\nu} \). Then

\[
\langle \hat{\rho} \hat{F}_{\mu\nu} \rangle = \langle \hat{\rho} \rangle F_{\mu\nu}, \tag{3.63}
\]

and similarly for higher order operator products. This mean field approximation will then lead to a generalized Vlasov equation where collision terms, i.e. multi particle correlations, are neglected.

For QCD the equations of motion for the Wigner function in an external gauge field are formally identical to those for the Wigner operator interacting with arbitrary
gauge field operators, due to the non-commuting color structure. For QED the equations simplify somewhat since now the commutators of the gauge field with the Wigner function vanish, and only the non-trivial commutators with the Dirac matrices survive.

In the external field limit the equations of motion for the electron Wigner function in QED read

\[
2m[1, W(x, p)]_\pm = 2\mathcal{P}_\mu[\gamma^\mu, W(x, p)]_\pm + i\mathcal{D}_\mu[\gamma^\mu, W(x, p)]_\mp, \tag{3.64}
\]

with (c.f. [38])

\[
\mathcal{P}_\mu = p_\mu + e \int_{-1/2}^{1/2} ds \ s F_{\mu\nu}(x + is\partial_\nu) \ i\partial_\nu^\nu, \tag{3.65}
\]

\[
\mathcal{D}_\mu = \partial_\mu + ie \int_{-1/2}^{1/2} ds F_{\mu\nu}(x + is\partial_\nu) \ i\partial_\nu^\nu, \tag{3.66}
\]

where \( e = -g \) is the electric coupling constant. Using Table [1], the two leading terms of the gradient expansion yield the semiclassical equations

\[
[m1 - p_\mu \gamma^\mu, W^{(0)}]_\pm = 0 \tag{3.67}
\]

in zeroth order and

\[
[m1 - p_\mu \gamma^\mu, W^{(1)}]_\pm + \frac{i g}{4c} \partial_\mu[\gamma^\mu, W^{(0)}]_\mp = \frac{i g}{2c} F_{\mu\nu}(x) \partial_\nu^\nu[\gamma^\mu, W^{(0)}]_\mp \tag{3.68}
\]

in first order of the expansion.

For QCD the zeroth order equation is again (3.67), while the first order equation becomes after color decomposition

\[
\text{singlet: } [m1 - \gamma^\mu p_\mu, W^{(1)}]_\pm + \frac{i g}{4c} \partial_\mu[\gamma^\mu, W^{(0)}]_\mp = -\frac{i gh^2}{12c} F_{\nu\mu}^a \partial_\nu^\nu[\gamma^\mu, W^{(0)a}]_\mp, \tag{3.69}
\]
The equations for the color singlet distributions become identical to those of QED if we replace $W_s^{(n)} \rightarrow W^{(n)}$ and $W^{(n)\alpha} \rightarrow N_c(N_c - 1) W^{(n)}$ with $N_c(N_c - 1) = 6$ for QCD.

The above mean field equations will be used in Sec. IV.

F. Spinor decomposition

We now return to the full operator equations and perform a spinor decomposition, using the following basis for the Clifford algebra:

$$\Gamma_i = 1, i\gamma^5, \gamma^\mu, \gamma^\mu \gamma^5, \sigma^{\mu\nu}. \quad (3.71)$$

In this basis the Wigner operator $\hat{W}$ is expanded as

$$\hat{W}(x,p) = \hat{F} + i\gamma^5 \hat{P} + \gamma^\mu \hat{V}_\mu + \gamma^\mu \gamma^5 \hat{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \hat{S}_{\mu\nu}, \quad (3.72)$$

with the hermitian spinor components $\hat{F}$ (scalar), $\hat{P}$ (pseudo-scalar), $\hat{V}_\mu$ (vector), $\hat{A}_\mu$ (axial vector) and $\hat{S}_{\mu\nu}$ (tensor). These components are projected out from $\hat{W}$ by taking traces with the corresponding basis elements $\Gamma_i$ [23]. Inserting the expansion (3.72) into Eqs. (3.33) and projecting onto the various spinor channels as described in [23] one finds (after separating hermitean and antihermitean parts)

$$2m\hat{F} = \{\Pi_\mu, \hat{V}^\mu\} \quad (3.73a)$$

$$2m\hat{P} = [\Delta_\mu, \hat{A}^\mu] \quad (3.73b)$$
\[2m \mathcal{S}_{\mu\nu} = [\Delta_\mu, \hat{V}_\nu] - [\Delta_\nu, \hat{V}_\mu] + \varepsilon_{\alpha\beta\mu\nu} \{\Pi^\alpha, \hat{A}^\beta\}\] (3.73c)

\[2m \hat{V}_\mu = \{\Pi_\mu, \hat{F}\} + [\Delta^\nu, \mathcal{S}_{\mu\nu}]\] (3.73d)

\[2m \hat{A}_\mu = -[\Delta_\mu, \hat{P}] + \{\Pi^\nu, \hat{L}_{\mu\nu}\}\] (3.73e)

and

\[0 = [\Delta_\mu, \hat{V}_\mu]\] (3.74a)

\[0 = \{\Pi_\mu, \hat{A}^\nu\}\] (3.74b)

\[0 = \{\Pi_\mu, \hat{V}_\nu\} - \{\Pi_\nu, \hat{V}_\mu\} - \varepsilon_{\alpha\beta\mu\nu} [\Delta^\alpha, \hat{A}^\beta]\] (3.74c)

\[0 = [\Delta_\mu, \hat{F}] - [\Pi^\nu, \hat{S}_{\mu\nu}]\] (3.74d)

\[0 = \{\Pi_\mu, \hat{P}\} + [\Delta^\nu, \hat{L}_{\mu\nu}]\] (3.74e)

Here we defined the adjoint tensor \(\hat{L}_{\mu\nu}\) to \(\hat{S}_{\mu\nu}\) by

\[\hat{L}_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \hat{S}^{\alpha\beta}.\] (3.75)

Up to the (anti-)commutator structure, these equations are identical to those derived in Ref. [23] for QED in the external field limit. Note that by contraction with the Levi-Civita tensor Eqs. (3.73ii, 3.74iii) can also be written as

\[2m \hat{L}_{\mu\nu} = \varepsilon_{\alpha\beta\mu\nu} [\Delta^\alpha, \hat{V}^\beta] - \{\Pi_\nu, \hat{A}_\mu\} + \{\Pi_\mu, \hat{A}_\nu\}\]

\[0 = [\Delta_\mu, \hat{A}_\nu] - [\Delta_\nu, \hat{A}_\mu] + \varepsilon_{\alpha\beta\mu\nu} \{\Pi^\alpha, \hat{V}^\beta\}.\]

Eqs. (3.73, 3.74) can be rewritten in the form of generalized transport equations and generalized mass shell conditions. To this end we introduce [30] a generalization of the Lorentz-covariant total time derivative (also known as the “drift operator”) \(m \, d\hat{X}(x, p)/d\tau = p_\mu \partial_\mu \hat{X}(x, p)\) via
\[ \{ \Pi_\mu, [\Delta^\mu, \hat{X}] \} \] or \[ [\Delta^\mu, \{ \Pi_\mu, \hat{X} \}] \] (3.76)

and the generalized mass shell operator

\[ M^2 \hat{X} \equiv 4m^2 \hat{X} - \{ \Pi_\mu, \{ \Pi^\mu, \hat{X} \} \} + [\Delta_\mu, [\Delta^\mu, \hat{X}]] \] (3.77)

which will be used to generalize the classical mass shell conditions \((m^2 - p^2) \hat{X}(x, p) = 0\). In the semiclassical limit and for Abelian gauge theories the generalized drift operator becomes the well-known Vlasov-operator \(p_\mu \partial^\mu + e p^\mu \partial^\rho \hat{F}_{\mu\nu}(x)\) while the generalized mass shell operator simply turns into \(m^2 - p^2\).

The details of the derivation of the generalized transport and mass shell equations are given in Appendix D. One finds that only 8 of the 16 operators \(\hat{F}, \hat{P}, \hat{\nu}_\mu, \hat{A}_\mu, \hat{S}_{\mu\nu}\) are independent and satisfy generalized transport and mass shell equations while the other 8 functions are obtained from the solutions via simple constraints. For the set of independent functions one can choose either \(\hat{F}, \hat{P}, \hat{\nu}_\mu, \hat{A}_\mu, \hat{S}_{\mu\nu}\) or \(\hat{\nu}_\mu, \hat{A}_\mu\). We will give here the equations corresponding to the first choice; those for the second choice are found in Appendix D3.

The complete set of covariant equations of motion for the spinor components of \(\hat{W}(x, p)\) is:

(a) Transport equations:

\[ [\Delta_\mu, \{ \Pi^\mu, \hat{F} \}] = [\Delta_\nu \Delta_\mu, \hat{S}^{\mu\nu}] \] (3.78a)

\[ [\Delta_\mu, \{ \Pi^\mu, \hat{P} \}] = [\Delta_\nu \Delta_\mu, \hat{L}^{\mu\nu}] \] (3.78b)

\[ [\Delta_\alpha, \{ \Pi^\alpha, \hat{S}_{\mu\nu} \}] = [\{ p_\mu + \Pi_\mu, p_\nu + \Pi_\nu \}, \hat{F}] + \varepsilon_{\alpha\beta\mu\nu} [\Delta^\alpha \Delta^\beta, \hat{P}] 
\quad - \left( \{ \hat{S}_{\mu\alpha}, [\Delta^\alpha, p_\nu + \Pi_\nu] \} + \{ [p_\mu + \Pi_\mu, \Delta^\alpha], \hat{S}_{\alpha\nu} \} \right) \] (3.78c)

(b) Mass shell equations:
\[ M^2 \hat{F} = \{ [p^\mu + \Pi^\mu, \Delta^\nu], \hat{S}_{\mu\nu} \} \]  
\begin{equation}
(3.79a)
\end{equation}

\[ M^2 \hat{P} = \{ [p^\mu + \Pi^\mu, \Delta^\mu], \hat{L}_{\mu\nu} \} \]  
\begin{equation}
(3.79b)
\end{equation}

\[ M^2 \hat{S}_{\mu\nu} = \{ [p_\mu + \Pi_\mu, \Delta_\nu], \hat{F} \} - \{ [p_\nu + \Pi_\nu, \Delta_\mu], \hat{F} \} 
- [\hat{S}_{\mu\alpha}, [\Delta^\alpha, \Delta_\nu]] - [[\Delta_\mu, \Delta^\alpha], \hat{S}_{\alpha\nu}] 
+ [\hat{S}_{\mu\alpha}, [p^\alpha + \Pi^\alpha, p_\nu + \Pi_\nu]] + [[p_\mu + \Pi_\mu, p^\alpha + \Pi^\alpha], \hat{S}_{\alpha\nu}] 
- \varepsilon_{\alpha\beta\mu\nu} \{ [p^\alpha + \Pi^\alpha, \Delta^\beta], \hat{P} \} \]  
\begin{equation}
(3.79c)
\end{equation}

(c) Constraints:

\[ 0 = [\Delta_\mu, \hat{F}] - \{ \Pi^\nu, \hat{S}_{\mu\nu} \} \]  
\begin{equation}
(3.80a)
\end{equation}

\[ 0 = \{ \Pi_\mu, \hat{P} \} + [\Delta^\nu, \hat{L}_{\mu\nu}] \].  
\begin{equation}
(3.80b)
\end{equation}

(d) Defining equations for the dependent spinor components:

\[ 2m \hat{A}_\mu = -[\Delta_\mu, \hat{P}] + \{ \Pi^\nu, \hat{L}_{\mu\nu} \} \]  
\begin{equation}
(3.81a)
\end{equation}

\[ 2m \hat{V}_\mu = \{ \Pi_\mu, \hat{F} \} + [\Delta^\nu, \hat{S}_{\mu\nu}] \].  
\begin{equation}
(3.81b)
\end{equation}

The general structure that transport and mass shell equations must be solved
for only 8 of the 16 spinor components (either for \( \hat{F}, \hat{P}, \hat{S}_{\mu\nu} \) or for \( \hat{V}_\mu, \hat{A}_\mu \)) while the
other 8 components are obtained from constraints has apparently not been noticed
before. It was not made manifest in the equations given in Ref. [23] where transport
and mass shell conditions were derived for the full Wigner operator but were not
given explicitly for each spinor component. Compared to previous work Eqs. (3.78-
3.81) thus provide a considerable structural simplification of the covariant transport
theory for spinor fields.
G. Chiral Limit

In this subsection we study the covariant kinetic equations in the chiral limit of vanishing fermion mass. The resulting equations of motion for the Wigner operator and its spinor components can be easily obtained by setting \( m = 0 \) in Eq. (3.33) and in Eqs. (3.73, 3.79, 3.81). In particular one sees that now in Eqs. (3.73, 3.74) \( \hat{A}_\mu, \hat{V}_\mu \) completely decouple from \( \hat{F}, \hat{P}, \) and \( \hat{S}_{\mu\nu} \). A closer look at the resulting “transport” and “mass shell” equations reveals, however, that they are no longer linearly independent but become linear combinations of each other. For \( m = 0 \) it no longer makes sense to distinguish between transport and mass shell equations (which both contain the non-linear operators \( \Pi \) and \( \Delta \) in second order), and the transport theory is better formulated on the level of Eqs. (3.73, 3.74) which are linear in the non-local operators.

These equations can be further diagonalized by introducing a “helicity basis” via

\[
\hat{a}_\mu^{(\pm)} = \hat{V}_\mu \pm \hat{A}_\mu, \\
\hat{f}^{(\pm)} = \hat{F} \pm \hat{P}, \\
\hat{s}^{(\pm)}_{\mu\nu} = \hat{S}_{\mu\nu} \pm \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\hat{S}^{\alpha\beta},
\]

with \( \frac{1}{2}\varepsilon_{0ijk}\hat{s}^{jk(\pm)} = \hat{s}^{(\pm)}_{0i} \). We find

\[
0 = \{\Pi_{\mu}, \hat{a}^{\mu(\pm)}\}, \\
0 = [\Delta_{\mu}, \hat{a}^{\mu(\pm)}], \\
0 = [\Delta_{\nu}, \hat{a}^{(\pm)}_{\mu}] - [\Delta_{\mu}, \hat{a}^{(\pm)}_{\nu}] \pm \varepsilon_{\alpha\beta\mu\nu}\{\Pi^{\alpha}, \hat{a}^{\beta(\pm)}\},
\]

as well as
This is the full set of covariant transport equations for gauge theories in the chiral limit. $\hat{a}^{(\pm)}_\mu$, $\hat{f}^{(\pm)}$ and $\hat{s}^{(\pm)}_{\mu\nu}$ generate distribution functions for excitations with defined helicity/chirality.

In a chirally symmetric theory both vector and axial vector currents should be conserved. Defining them by

$$
\hat{\mathbf{v}}_\mu(x) = \int \frac{d^4p}{(2\pi)^4} \hat{V}_\mu(x,p),
$$
$$
\hat{\mathbf{a}}_\mu(x) = \int \frac{d^4p}{(2\pi)^4} \hat{A}_\mu(x,p),
$$

one finds easily from Eq. (3.83b) the conservation laws

$$
0 = [D_\mu(x), \hat{\mathbf{v}}^\mu(x)],
$$
$$
0 = [D_\mu(x), \hat{\mathbf{a}}^\mu(x)].
$$

The color singlet components of both currents (for which $D^\mu(x)$ reduces to $\partial^\mu_x$) thus satisfy the usual conservation laws while the color octet vector and axial vector currents are covariantly conserved.

In the recent literature [41] chirally invariant transport equations were also derived in the framework of the Nambu-Jona-Lasinio model, in the external field limit and to leading order of the gradient expansion. The Lagrangian used there,

$$
\mathcal{L}_{\text{NJL}} = i\bar{\psi}\gamma^\mu \partial_\mu \psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2],
$$

Expaning $\Delta_\mu$ in a Taylor series, all terms containing $\partial_p$ vanish after partial integration since $\hat{A}_\mu, \hat{V}_\mu$ vanish at infinite momentum.
does not contain a coupling to gauge fields, but a self-coupling of the spinors which, if sufficiently strong, leads to spontaneous breaking of the chiral symmetry in the massless case. With the help of the auxiliary fields $\hat{\sigma} = -2G\overline{\psi}\psi$ and $\hat{\pi} = -2G\overline{\psi}i\gamma_5\psi$ the Lagrangian is rewritten as

$$L_{NJL} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - \hat{\sigma}\overline{\psi}\psi - \hat{\pi}\overline{\psi}i\gamma_5\psi - \frac{\hat{\sigma}^2 + \hat{\pi}^2}{4G}. \quad (3.88)$$

This Lagrangian is invariant under the chiral transformation

$$\psi \rightarrow \psi' = \exp\left(-i\gamma_5 \frac{\chi}{2}\right)\psi \quad (3.89)$$
$$\hat{\sigma} \rightarrow \hat{\sigma}' = \hat{\sigma}\cos \chi - \hat{\pi}\sin \chi \quad (3.90)$$
$$\hat{\pi} \rightarrow \hat{\pi}' = \hat{\pi}\cos \chi + \hat{\sigma}\sin \chi, \quad (3.91)$$

resulting in a conserved axial vector current.

¿From this Lagrangian the authors of [41] derived chirally symmetric transport equations in a vacuum state with dynamically broken chiral symmetry ($\langle \hat{\sigma}^2 + \hat{\pi}^2 \rangle \neq 0$). In contrast to our transport equations, these equations do not decouple $V_\mu$ and $A_\mu$ from $F$, $P$ and $S_{\mu \nu}$. The coupling between the two sectors occurs via the vacuum expectation values $\sigma$ and $\pi$ of the auxiliary fields. However, according to Eq. (3.89), $\pi$ can always be made to vanish by an appropriate chiral rotation, with a compensating chiral rotation on the spinor components of the Wigner density [41]. If one does this in the equations of Ref. [41], $\sigma$ takes over the same role as the bare mass $m$ in our equations. In other words, the dynamical mass generated by spontaneous chiral symmetry breakdown couples the two sectors in exactly the same way as a non-zero bare mass $m \neq 0$. In the symmetric state, $\sigma = \pi = 0$, their equations decouple like ours.
H. Classical limit

Exact solutions for the kinetic equations in the free-field limit are given in Appendix E. Here we study their classical limit, by expanding the spinor equations Eqs. (3.78-3.81) to leading order in the gradient expansion. We will then consider explicitly the external field case for QED.

With the generalized mass shell operator to zeroth order \( \bar{h} \),

\[
(M^2)^{(0)} \hat{X}^{(0)} = 4(m^2 - p^2) \hat{X}^{(0)}
\]
which becomes just the classical mass shell operator, we find the following equations

\[
M^{(0)}^2 \hat{F}^{(0)} = 0 \quad (3.93a)
\]

\[
M^{(0)}^2 \hat{P}^{(0)} = 0 \quad (3.93b)
\]

\[
M^{(0)}^2 \hat{S}^{(0)}_{\mu\nu} = 0 \quad (3.93c)
\]

\[
0 = p^\nu \hat{S}^{(0)}_{\mu\nu} \quad (3.93d)
\]

\[
0 = p^\mu \hat{P}^{(0)} \quad (3.93e)
\]

\[
m\hat{A}^{(0)}_\mu = p^\nu \hat{L}^{(0)}_{\mu\nu} \quad (3.93f)
\]

\[
m\hat{V}^{(0)}_\mu = p_\mu \hat{F}^{(0)} \quad (3.93g)
\]

The transport equations vanish identically, meaning that the equations to zeroth order do not contain dynamical information. From Eqs. (3.93f,3.93g) the pseudoscalar density \( \hat{P}^{(0)} \) proves to be zero. The equations are still not complete: they only give information on the functional \( p \) dependence of the spinor components. Only in the semi-classical approximation will we get transport equations for the spinor components \( \hat{X}^{(0)} \) and thus information on their \( x \) dependence. From the equations to first
order we find in addition mass shell conditions and constraints for the first order spinor components $\hat{X}^{(1)}$, that is information on their $p$ dependence.

To first order we find for the mass shell operator

$$M^2 \hat{X} = (M^2)^{(0)} \hat{X}^{(1)} + (M^2)^{(1)} \hat{X}^{(0)}$$

with

$$(M^2)^{(1)} \hat{X}^{(0)} = -\frac{ig}{c} p_\mu \partial_{\nu}[\hat{F}^{\nu\mu}(x), \hat{X}^{(0)}]$$ (3.94)

and the generalized derivative

$$[\Delta_\mu, \{\Pi^\mu, \hat{X}\}]_{\text{first order}} = \{\Pi^\mu, [\Delta_\mu, \hat{X}]\}$$

$$\text{first order} = 2[p^\mu D_\mu(x), \hat{X}^{(0)}] - \frac{g}{c} p^\mu \partial_p \{\hat{F}_{\nu\mu}, \hat{X}^{(0)}\}. \quad (3.95)$$

From an explicit calculation we find further

$$[\Delta_\mu, \Delta_\nu, \hat{X}]_{\text{first order}} = 0 \quad (3.96a)$$

$$\{[p_\mu + \Pi_\mu, \Delta_\nu], \hat{X}\}_{\text{first order}} = \frac{ig}{c} \{\hat{F}_{\nu\mu}, \hat{X}^{(0)}\} \quad (3.96b)$$

$$[[p_\mu + \Pi_\mu, p_\nu + \Pi_\nu], \hat{X}]_{\text{first order}} = \frac{ig}{c} [\hat{F}_{\mu\nu}^{(0)}, \hat{X}^{(0)}] \quad (3.96c)$$

This then leads to the following spinor equations in first order of the gradient expansion

$$[p_\mu D^\mu(x), \hat{F}^{(0)}] - \frac{g}{2c} p^\mu \partial_p \{\hat{F}_{\nu\mu}(x), \hat{F}^{(0)}\} = 0 \quad (3.97a)$$

$$[\hat{F}^{\mu\nu}(x), \hat{S}^{(0)}_{\mu\nu}] = 0 \quad (3.97b)$$

$$[p^\alpha D_\alpha(x), \hat{S}^{(0)}_{\mu\nu}] - \frac{g}{2c} p^\alpha \partial_p \{\hat{F}_{\beta\alpha}(x), \hat{S}^{(0)}_{\mu\nu}\} = \frac{ig}{2c} [\hat{F}_{\mu\nu}(x), \hat{F}^{(0)}] + \frac{g}{2c} \left(\{\hat{F}_{\nu\alpha}(x), \hat{S}^{(0)}_{\mu\alpha}\} - \{\hat{F}_{\mu\alpha}(x), \hat{S}^{(0)}_{\nu\alpha}\}\right) \quad (3.97c)$$
\[(m^2 - p^2) \hat{F}^{(1)} = -i g \frac{q}{4c} p^\mu \partial_\nu [\hat{F}_{\mu\nu}(x), \hat{F}^{(0)}] + \frac{g}{4c} \{\hat{F}^{\mu\nu}(x), \hat{S}^{(0)}_{\mu\nu}\} \quad (3.98a)\]

\[(m^2 - p^2) \hat{P}^{(1)} = \frac{g}{4c} \{\hat{F}^{\mu\nu}(x), \hat{L}^{(0)}_{\mu\nu}\} \quad (3.98b)\]

\[(m^2 - p^2) \hat{S}^{(1)}_{\mu\nu} = -i g \frac{q}{4c} p_\alpha \partial_\beta [\hat{F}^{\alpha\beta}(x), \hat{S}^{(0)}_{\mu\nu}] - i g \frac{4}{4c} \left([\hat{F}_{\alpha\nu}(x), \hat{S}^{(0)}_{\mu\alpha}] - [\hat{F}_{\mu\alpha}(x), \hat{S}^{(0)}_{\alpha\nu}]\right)\]

\[+ \frac{g}{2c} \{\hat{F}_{\mu\nu}(x), \hat{F}^{(0)}\}, \quad (3.98c)\]

\[2p^\nu \hat{S}^{(1)}_{\mu\nu} = [D_\mu(x), \hat{F}^{(0)}] - \frac{g}{2c} \partial^\nu \{\hat{F}_{\nu\mu}(x), \hat{F}^{(0)}\} + i g \frac{4}{4c} \partial_\rho \{\hat{F}^{\rho\nu}(x), \hat{S}^{(0)}_{\mu\nu}\} \quad (3.99a)\]

\[-2p_\mu \hat{P}^{(1)} = [D^\nu(x) \hat{L}^{(0)}_{\mu\nu}] - \frac{g}{2c} \partial_\rho \{\hat{F}^{\rho\nu}(x), \hat{L}^{(0)}_{\mu\nu}\} \quad (3.99b)\]

\[2m \hat{A}^{(1)}_\mu = 2p^\nu \hat{L}^{(1)}_{\mu\nu} - i g \frac{4}{4c} \partial_\rho \{\hat{F}^{\rho\nu}(x), \hat{L}^{(0)}_{\mu\nu}\} \quad (3.100a)\]

\[2m \hat{V}^{(1)}_\mu = 2p_\mu \hat{F}^{(1)} - i g \frac{4}{4c} \partial^\nu [\hat{F}_{\nu\mu}(x), \hat{F}^{(0)}] \]

\[+[D^\nu(x) \hat{S}^{(0)}_{\mu\nu}] - \frac{g}{2c} \partial_\rho \{\hat{F}^{\rho\nu}(x), \hat{S}^{(0)}_{\mu\nu}\} \]. \quad (3.100b)\]

Mass shell equations and constraints couple to the spinor components in first order, whereas the transport equations are equations for the zeroth order spinor components. Because of the vanishing pseudoscalar density \(\hat{P}^{(0)}\) the second transport equation Eq. (3.97ii) becomes a constraint for \(\hat{L}^{(0)}_{\mu\nu}\). Contracting Eq. (3.99i) with \(p_\mu\) and Eq. (3.99ii) with \(p_\gamma\hat{\epsilon}^{\mu\gamma\delta\kappa}\) and comparison with the transport equations Eqs. (3.97i,iii) we find as additional constraints

\[[\hat{F}^{\mu\nu}, \hat{S}^{(0)}_{\mu\nu}] = 0 \quad (3.101a)\]

\[[\hat{F}^{\mu\nu}, \hat{F}^{(0)}] = 0 \quad (3.101b)\]

telling us especially that the field strength tensor commutes with the scalar density \(\hat{F}^{(0)}\).
Thus we can conclude that the leading order in the gradient expansion is given through the mass shell and constraint equations Eqs. (3.93) and the transport equations Eq. (3.97). (The additional constraints Eqs. (3.101i,ii) play no role to this order of the approximation.) To leading order the quarks (and electrons) can be interpreted as on-shell particles obeying a transport equation with gauge covariant drift and Vlasov terms, \([p_\mu D^\mu(x), \hat{X}]\) and \(g/(2c)p_\mu \partial_{p\nu}\{F^\nu\mu(x), \hat{X}\}\), respectively. The pseudoscalar density will contribute only to the next higher order.

1. Classical solutions for QED with external fields

For QED with external fields we get for the zeroth order formally the same equations as Eqs. (3.93), but now the spinor components are no longer operators. With \(\mathcal{P}^{(0)} \equiv 0\) this reads

\[
0 = (m^2 - p^2)\mathcal{F}^{(0)} \\
0 = (m^2 - p^2)\mathcal{S}_{\mu\nu}^{(0)} \\
0 = p^\nu \mathcal{S}_{\mu\nu}^{(0)} \\
2m\mathcal{A}_\mu^{(0)} = 2p^\nu \mathcal{L}_{\mu\nu}^{(0)} \\
2m\mathcal{V}_\mu^{(0)} = 2p_\mu \mathcal{F}^{(0)}.
\]

To derive the first order equations we use that the commutators of the gauge fields and field strength tensor with the spinor components vanish, whereas the anticommutators give a factor 2:

\[
0\equiv \left(p_\mu \partial^\mu + \frac{g}{c} F_{\mu\nu} p^\nu \partial_p\right)\mathcal{F}^{(0)} \\
0\equiv \left(p_\alpha \partial^\alpha + \frac{g}{c} F_{\alpha\beta} p^\beta \partial_p\right)\mathcal{S}_{\mu\nu}^{(0)} + \frac{g}{c} \left(F^\alpha_{\mu}(x)\mathcal{S}_{\alpha\nu}^{(0)} + F^{\alpha\nu}(x)\mathcal{S}_{\mu\alpha}^{(0)}\right)
\]
\[(m^2 - p^2) \mathcal{F}^{(1)} = \frac{g}{4c} F^{\mu\nu}(x) S^{(0)}_{\mu\nu} \]  
\[(m^2 - p^2) \mathcal{P}^{(1)} = \frac{g}{4c} F^{\mu\nu}(x) \mathcal{L}^{(0)}_{\mu\nu} \]  
\[(m^2 - p^2) S^{(1)}_{\mu\nu} = \frac{g}{2c} F^{\mu\nu}(x) \mathcal{F}^{(0)} \]  
\[2p^\nu S_{\mu\nu}^{(1)} = \left( \partial_\mu + \frac{g}{c} F_{\mu\nu} \partial_\nu \right) \mathcal{F}^{(0)} \]  
\[2p_\mu \mathcal{P}^{(1)} = - \left( \partial^\nu + \frac{g}{c} F^{\nu\alpha} \partial_\alpha \right) \mathcal{L}^{(0)}_{\mu\nu} \]  
\[2m A^{(1)}_\mu = 2p^\nu \mathcal{L}^{(1)}_{\mu\nu} \]  
\[2m V^{(1)}_\mu = 2p_\nu \mathcal{F}^{(1)} + \left( \partial^\nu + \frac{g}{c} F^{\nu\alpha} \partial_\alpha \right) S^{(0)}_{\mu\nu} \]  

From the systematics of the expansion one finds that the hierarchy of equations can be solved successively for each order of the spinor components. The equations above for instance allow for an unambiguous solution for the classical spinor components \(X^{(0)}\), without having to take into account higher spinor components. Therefore the hierarchy of equations can be truncated at any order for the spinor components. This remains even true for the more general case of QCD with external fields. The reason for this decoupling is simply that the operator \(\Delta_\mu\) to zeroth order \(\hbar\) gives no contribution and therefore the equations to \(n^{th}\) order contain the \(n^{th}\) order spinor components only together with momenta \(p_\mu\). Allowing for correlations this remains true only if we replace multi-particle correlations by products of one-particle Wigner functions and external fields (Boltzmann Ansatz).

To write down the equations for \(\mathcal{F}^{(0)}, S^{(0)}_{\mu\nu}\) in their usual form with electric and magnetic fields we first use the separation into on-shell particle \(\delta(p_0 - E))\) and anti-particle \(\delta(p_0 + E))\) contributions for \(\mathcal{F}^{(0)}\) and \(S^{(0)}_{\mu\nu}\) according to Eqs. (3.102i,ii) using the following identity.
\[ \delta(p^2 - m^2) = \frac{1}{2|p_0|} (\delta(p_0 - E) + \delta(p_0 + E)), \quad E = \sqrt{p^2 + m^2}. \] (3.104)

In the following we will restrict ourselves to particles; the equations for anti-particles are given through \( p_0 \rightarrow -p_0 \). The totally anti-symmetric field strength tensor is given in terms of electric and magnetic fields by

\[
F^{\mu\nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}.
\] (3.105)

Similarly we define the totally anti-symmetric tensor \( S^{\mu\nu} \)

\[
S^{\mu\nu} = \begin{pmatrix}
0 & -S_x & -S_y & -S_z \\
S_x & 0 & -\sigma_z & \sigma_y \\
S_y & \sigma_z & 0 & -\sigma_x \\
S_z & -\sigma_y & \sigma_x & 0
\end{pmatrix}
\] (3.106)

with spin vector \( S \) and helicity \( \sigma \).

The dual tensor \( L^{\mu\nu} \) for \( S^{\mu\nu} \) is then given by \( L^{\mu\nu} = \frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} S_{\alpha\beta} \) in complete analogy to the dual field strength tensor \( F^{\mu\nu} = \frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} \) [12]. \( L^{\mu\nu} \) is obtained from \( S^{\mu\nu} \) by exchanging \( S \rightarrow \sigma \) and \( \sigma \rightarrow -S \).

From \( p_\nu S^{\mu\nu(0)} = 0 \) we find the constraints

\[
S^{(0)} = -v \times \sigma^{(0)}, \quad v = \frac{p}{E},
\] (3.107)

\[
p \cdot S^{(0)} = 0.
\] (3.108)

The first equation connects \( S^{(0)} \) and \( \sigma^{(0)} \), whereas the second equation is automatically fulfilled.
Using 3-vectors we arrive after some algebra at the following equations for the dynamics of the system as governed by the scalar density $\mathcal{F}^{(0)}$ and helicity $\mathbf{\sigma}^{(0)}$ (coupling constant $g = -e$)

\[
\left[ \partial_0 + \mathbf{v} \cdot \nabla + \frac{e}{c} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_p \right] \mathcal{F}^{(0)}(x, p, E) = 0 \quad (3.109a)
\]

\[
E \left[ \partial_0 + \mathbf{v} \cdot \nabla + \frac{e}{c} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_p \right] \mathbf{\sigma}^{(0)}(x, p, E) = \]

\[- \frac{e}{c} [\mathbf{E} \times (\mathbf{v} \times \mathbf{\sigma}^{(0)}(x, p, E)) + \mathbf{B} \times \mathbf{\sigma}^{(0)}(x, p, E)]. \quad (3.109b)
\]

The other spinor components are dependent variables and connected to scalar and spin density through

\[
\mathbf{S}^{(0)}(x, p, E) = -\mathbf{v} \times \mathbf{\sigma}^{(0)}(x, p, E) \quad (3.110a)
\]

\[
m\mathbf{A}^{(0)}_0(x, p) = \mathbf{p} \cdot \mathbf{\sigma}^{(0)}(x, p, E) \quad (3.110b)
\]

\[
m\mathbf{A}^{(0)}(x, p) = E \mathbf{\sigma}^{(0)}(x, p, E) + \mathbf{p} \times (\mathbf{v} \times \mathbf{\sigma}^{(0)}(x, p, E)) \quad (3.110c)
\]

\[
m\mathbf{V}^{(0)}_0(x, p) = E \mathcal{F}^{(0)}(x, p, E) \quad (3.110d)
\]

\[
m\mathbf{V}^{(0)}(x, p) = \mathbf{p} \mathcal{F}^{(0)}(x, p, E). \quad (3.110e)
\]

Eq. (3.109ii) for the helicity or (with Eq. (3.110i)) the spin density, is a generalization of the BMT equation. Eq. (3.109i) for the scalar density corresponds to the Vlasov equation. The equations for the anti-particles are given through $\mathbf{v} \rightarrow -\mathbf{v}$.

Eqs. (3.109,3.110) are still not sufficient to describe the theory, they have to be coupled to the classical Maxwell equations

\[
\partial_\mu F^{\mu\nu} = j^{\nu} + j^{\nu}_{\text{ext}} \quad (3.111)
\]

\[
\partial_\mu \mathcal{F}^{\mu\nu} = 0. \quad (3.112)
\]

$j^{\nu}_{\text{ext}}$ is an external current and
\[ j^\nu(x) = \int d^4p \text{tr} \gamma^\nu W(x,p), \quad j^\nu = j^\nu_{\text{particles}} + j^\nu_{\text{antiparticles}}, \quad (3.113) \]

is the self-consistent current, coupling the field strength tensor in the Maxwell equations to the Wigner function. We will get contributions only from the vector part and with the mass shell condition the current reads

\[ j^\nu(0) = \frac{2}{m} \int \frac{d^3p}{(2\pi)^3} \left( \frac{\mathcal{F}^{(0)}(x,p,E) + \mathcal{F}^{(0)}(x,p,E)}{\nu} \left( \mathcal{F}^{(0)}(x,p,E) - \mathcal{F}^{(0)}(x,p,E) \right) \right). \quad (3.114) \]

Thus we find that \( S_{\mu\nu} \) couples only in next to leading order to the Maxwell equation. Therefore to zeroth order in the gradient expansion the self-consistent dynamics is governed by the scalar part \( \mathcal{F}^{(0)} \) alone!

\textit{2. Classical solutions for chiral QED with external fields}

Finally we study the massless mean field limit. The equations for \( A_{\mu}(0) \) and \( V_{\mu}(0) \) are now

\begin{align*}
0 &= p_\mu A^{\mu(0)} \quad (3.115a) \\
0 &= p_\mu V^{\mu(0)} \quad (3.115b) \\
0 &= p_\mu A^{(0)}_\nu - p_\nu A^{(0)}_\mu \quad (3.115c) \\
0 &= p_\mu V^{(0)}_\nu - p_\nu V^{(0)}_\mu \quad (3.115d) \\
0 &= \left[ \partial_\mu - \frac{e}{c} F^{\nu\rho}_\mu \partial_\rho \right] A^{\mu(0)} \quad (3.115e) \\
0 &= \left[ \partial_\mu - \frac{e}{c} F^{\nu\rho}_\mu \partial_\rho \right] V^{\mu(0)}. \quad (3.115f)
\end{align*}

Multiplying Eqs. (3.115iii,iv) with \( p_\mu \) and using Eqs. (3.115i,ii) we find that \( A^{(0)}_\mu \) and \( V^{(0)}_\mu \) separate into a contribution from particles (\( \delta(p_0 - |p|) \)) and anti-particles
\( \delta(p_0 - |p|) \). It further follows with Eqs. (3.115iii,iv) that \( A_i^{(0)} \) and \( V_i^{(0)} \) can be expressed through \( A_0^{(0)} \) and \( V_0^{(0)} \)

\[
A^{(0)}(x, p, \pm |p|) = \hat{e}_p A_0^{(0)}(x, p, \pm |p|) \quad (3.116a)
\]

\[
V^{(0)}(x, p, \pm |p|) = \hat{e}_p V_0^{(0)}(x, p, \pm |p|). \quad (3.116b)
\]

\( \hat{e}_p \) denotes the unit vector in the direction of \( p \). The transport equations in terms of the fields \( E, B, S, \sigma \) are for particles \( \delta(p_0 - |p|) \)

\[
0 = \left[ \partial^0 + \hat{e}_p \cdot \nabla + \frac{e}{c} (E + \hat{e}_p \times B) \cdot \nabla_p \right] A_0^{(0)}(x, p, |p|) \quad (3.117)
\]

\[
0 = \left[ \partial^0 + \hat{e}_p \cdot \nabla + \frac{e}{c} (E + \hat{e}_p \times B) \cdot \nabla_p \right] V_0^{(0)}(x, p, |p|). \quad (3.118)
\]

For anti-particles one substitutes \( \hat{e}_p \rightarrow -\hat{e}_p \) and changes \( |p| \rightarrow -|p| \) in the arguments of \( A_0^{(0)}, V_0^{(0)} \). The self-consistent current is now given by

\[
j^{\nu(0)}(x) = \int \frac{d^3p}{(2\pi)^3 |p|} \frac{2}{|p|} \left( V_0^{(0)}(x, p, |p|) + V_0^{(0)}(x, p, -|p|) \right) \hat{e}_p \left( V_0^{(0)}(x, p, |p|) - V_0^{(0)}(x, p, -|p|) \right). \quad (3.119)
\]

Here \( A_\mu \) will couple only in next-to-leading order to the self-consistent dynamics.

The equations for \( F, P \) and \( S_{\mu\nu} \) decouple completely from the equations for \( A_\mu \) and \( V_\mu \), see Sec. [11]. Therefore they will not couple to any order in \( \hbar \) to the self-consistent dynamics so we do not have to treat them here. We should remark nevertheless that for \( F, P \) and \( S_{\mu\nu} \) the hierarchy of the gradient expansion cannot be solved successively starting with the zeroth order components, but the equations of motion for the \( n \)th component \( \hbar \) couple to the \( n + 1 \)st component.
IV. SINGLE-TIME FORMULATION

We are now ready to make the transition to the single-time formulation of quantum transport theory, by taking energy moments of the covariant transport equations derived in the preceding Section. As shown in Sec.II, a fully equivalent single-time description requires the calculation of all energy moments.

A. Moment equations

The starting point of our considerations will be again Eqs. (3.27,3.28) for the Wigner operator. By multiplying from the left and right with $\gamma_0$ we can isolate the term $\{\Pi_0, \hat{W}\}$ on the l.h.s. of the equations. Adding and subtracting then leads to

$$2\{\Pi_0, \hat{W}\} = 2m\{\gamma^0, \hat{W}\} - [\gamma^0, \{\Pi_i, \hat{W}\}] - \{\gamma^0, [i\Delta_i, \hat{W}]\}$$

$$0 = 2m\{\gamma^0, \hat{W}\} - 2[i\Delta_0, \hat{W}] - \{\gamma^0, [\Pi_i, \hat{W}]\} - [\gamma^0, [i\Delta_i, \hat{W}]]$$

Through $\Pi_0$ Eq. (4.1) contains an extra factor $p_0$ on the left, but no generalized time derivative $[i\Delta_0, \hat{W}]$. Eq. (4.2) contains the generalized time derivative but no extra factor $p_0$. Note that the generalized derivative $\Delta$ occurs only in commutators while $\Pi$ occurs only in anticommutators with the Wigner operator.

For the spinor component equations (3.73,3.74) the analogous separation into equations with and without explicit $p_0$ dependence yields

$$\{\Pi_0, \hat{\psi}\} = -2m\hat{\mathcal{F}} + \{\Pi_i, \hat{\psi}^i\}$$

$$\{\Pi^0, \hat{\mathcal{A}}^k\} = 2m\hat{\mathcal{L}}^{0k} + \varepsilon^{ijk}[\Delta_i, \hat{\psi}_j] + \{\Pi^k, \hat{\mathcal{A}}^0\}$$

$$\{\Pi_0, \hat{\mathcal{F}}\} = 2m\hat{\mathcal{V}}_0 - [\Delta_i, \hat{\mathcal{S}}_0]$$
\{\Pi^0, \hat S^j\} = \varepsilon_{0ijk}(2m\hat A_i + [\Delta_i, \hat P]) - \{\Pi^k, \hat S^0_j\} + \{\Pi^j, \hat S^0_k\} \quad (4.3d)

\{\Pi_0, \hat A^0\} = -\{\Pi_i, \hat A^i\} \quad (4.3e)

\{\Pi_0, \hat V_i\} = \{\Pi_i, \hat V_0\} + \varepsilon_{0ijk}[\Delta^j, \hat A^k] \quad (4.3f)

\{\Pi^0, \hat S_{0i}\} = [\Delta_i, \hat F] - \{\Pi^j, \hat S_{ij}\} \quad (4.3g)

\{\Pi_0, \hat P\} = -[\Delta^i, \hat L_{0i}] \quad (4.3h)

and

\begin{align*}
2m\hat P &= [\Delta_\mu, \hat A^\mu] \quad (4.4a) \\
2m\hat S_{0i} &= [\Delta_0, \hat V_i] - [\Delta_i, \hat V_0] + \varepsilon_{0ijk}\{\Pi^j, \hat A^k\} \quad (4.4b) \\
2m\hat V_i &= \{\Pi_i, \hat F\} + [\Delta^\nu, \hat S_{i\nu}] \quad (4.4c) \\
2m\hat A_0 &= -[\Delta_0, \hat P] + \{\Pi^i, \hat L_{0i}\} \quad (4.4d) \\
0 &= [\Delta_\mu, \hat V^\mu] \quad (4.4e) \\
0 &= \{\Pi_i, \hat V_j\} - \{\Pi_j, \hat V_i\} - \varepsilon_{0ijk}([\Delta^0, \hat A^k] - [\Delta^k, \hat A^0]) \quad (4.4f) \\
0 &= [\Delta_0, \hat F] - \{\Pi^i, \hat S_{0i}\} \quad (4.4g) \\
0 &= \{\Pi_i, \hat P\} - [\Delta^0, \hat L_{0i}] + [\Delta^j, \hat L_{ij}] \quad (4.4h)
\end{align*}

Before we can take \(p_0\)-moments of these equations we should first analyze the dependence of the non-local operators \(\Pi\) and \(\Delta\) on the partial derivatives \(\partial_{p_0}\). We perform a Taylor expansion for the Schwinger string in \(\partial_{p_0}\):

\begin{align*}
[x] F_{\nu\mu}(x + is\partial_{p}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{y_0}^{(k)} [x] F_{\nu\mu}(x + y) \bigg|_{y = is\partial_{p}, y_0 = 0} (is\partial_{p_0})^k \quad \text{(QCD)} \quad (4.5) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{x_0}^{(k)} F_{\nu\mu}(x_0, x + is\partial_{p})(is\partial_{p_0})^k \quad \text{(QED, external fields)}
\end{align*}

For QED in the external field limit the higher order \(p_0\) derivatives thus couple simply to the higher order time derivatives of the external fields. As discussed
in Sec. III C, for non-abelian gauge fields the above expansion generates a gauge covariant temporal gradient expansion of the transport equations.

For the calculation of the energy moments the following identity is useful:

\[
\int_{-\infty}^{\infty} dp_0 \, p_0^n \, (\partial_{p_0})^k \hat{X}(x, p_0, p) = \begin{cases} 
0 & n < k, \\
(-1)^k \frac{n!}{(n-k)!} [^{n-k}]\hat{X}(x, p) & n \geq k,
\end{cases}
\]  \tag{4.6}

Here \( \hat{X} \) stands for any spinor component of the covariant Wigner operator, and \([n] \hat{X}\) denotes the \(n\)th \(p_0\)-moment of \(\hat{X}\):

\[
[n] \hat{X}(x, p) = \int_{-\infty}^{\infty} dp_0 \, p_0^n \, \hat{X}(x, p).
\]  \tag{4.7}

Eq. (4.6) is proved by partial integration, using that because of energy conservation we expect the Wigner operator to fall off at large \(p_0\) faster than any power.

Whenever in the (anti-)commutators in the equations of motion one of the non-local operators \(\Pi\) or \(\Delta\) appears to the right of the Wigner operator, the momentum derivatives are defined as in (3.23) to act to the left in the sense of partial integration. For \(k\) momentum derivatives this introduces a factor \((-1)^k\). This motivates the definition of an alternating commutator through

\[
[\hat{A}, \hat{B}]_k \equiv \hat{A}\hat{B} - (-1)^k \hat{B}\hat{A} = \begin{cases} 
[\hat{A}, \hat{B}] & (k \text{ even}), \\
\{\hat{A}, \hat{B}\} & (k \text{ odd}).
\end{cases}
\]  \tag{4.8}

With these definitions one can work out the energy moments of the required commutators resp. anticommutators of the spinor components \(\hat{X}\) with \(\Delta\) and \(\Pi\):

\[
\int dp_0 \, p_0^n [\Delta_{\mu}, \hat{X}(x, p)] = [\Delta_{\mu}(x, p), [n] \hat{X}(x, p)] + \sum_{k=1}^{n} \binom{n}{k} [M(k)\mu(x, p), [^{n-k}]\hat{X}(x, p)]_k
\]  \tag{4.9}

\[
= [D_{\mu}(x), [n] \hat{X}(x, p)] + \sum_{k=0}^{n} \binom{n}{k} [M(k)\mu(x, p), [^{n-k}]\hat{X}(x, p)]_k,
\]  \tag{4.10}
\[
\int dp_0 \{\Pi_0, \hat{X}(x,p)\} = 2 \left[ \Phi(x,p) + \sum_{k=0}^{n} \binom{n}{k} \left[ N_{(k)0}(x,p), [n-k] \hat{X}(x,p) \right] \right]_{k+1} ,
\]

(4.11)

\[
\int dp_0 \{\Pi_i, \hat{X}(x,p)\} = \{\Pi_i(x,p), [n] \hat{X}(x,p)\} + \sum_{k=0}^{n} \binom{n}{k} \left[ N_{(k)i}(x,p), [n-k] \hat{X}(x,p) \right]_{k+1} \]

(4.12)

\[
= 2p_i \left[ n \hat{X}(x,p) + \sum_{k=0}^{n} \binom{n}{k} \left[ N_{(k)i}(x,p), [n-k] \hat{X}(x,p) \right] \right]_{k+1} .
\]

(4.13)

Here we defined the following non-local operators in 3-dimensional momentum space:

\[
\Pi_{\mu}(x,p) = p_{\mu} + \frac{2g}{c} \int_{-1/2}^{0} ds \, i\hbar \partial_p \left[ x \hat{F}_{i\mu}(x + i\hbar \partial_p) \right] ,
\]

(4.14)

\[
\Delta_{\mu}(x,p) = \hbar D_{\mu}(x) + \frac{i\gamma}{c} \int_{-1/2}^{0} ds \, i\hbar \partial_p \left[ x \hat{F}_{i\mu}(x + i\hbar \partial_p) \right] ,
\]

(4.15)

\[
M_{(k)\mu}(x,p) = \frac{gh}{c} \int_{-1/2}^{0} ds \, (-i\hbar s)^{k} \left[ \partial_{g0}^{(k)} [x \hat{F}_{\mu i}(x_0 + y_0, x + i\hbar s \partial_p)] \right]_{y_0=0} \partial_{\mu}^{i} - \frac{ik}{\hbar s} \partial_{g0}^{(k-1)} [x \hat{F}_{\mu 0}(x_0 + y_0, x + i\hbar s \partial_p)] \right]_{y_0=0} ,
\]

(4.16)

\[
N_{(k)\mu}(x,p) = \frac{2g}{c} \int_{-1/2}^{0} ds \, (-i\hbar s)^{k+1} \left[ \partial_{g0}^{(k)} [x \hat{F}_{\mu i}(x_0 + y_0, x + i\hbar s \partial_p)] \right]_{y_0=0} \partial_{\mu}^{i} - \frac{ik}{\hbar s} \partial_{g0}^{(k-1)} [x \hat{F}_{\mu 0}(x_0 + y_0, x + i\hbar s \partial_p)] \right]_{y_0=0} .
\]

(4.17)

Note that we have used for the 3-dimensional momentum space versions of the operators \(\Pi\) and \(\Delta\) the same symbols as for the 4-dimensional ones; which one is meant in a particular equation should be obvious from the context.

For a given power \(n\) of \(p_0\), the term \(\Pi_0\) gives rise to moments which are one order higher than those from all other terms. This term thus indeed couples different orders of the moment hierarchy so that it will not trivially truncate.

The moment equations are now given by

\[
4^{[n+1]} \hat{W} = 2m \left\{ \gamma^0, [n] \hat{W} \right\} - 2 \left\{ N_{(0)0}, [n] \hat{W} \right\} - \left\{ \gamma^0 \gamma^i, \{ \Pi_i, [n] \hat{W} \} \right\} - \left\{ \gamma^0 \gamma^i, [i \Delta_i, [n] \hat{W}] \right\}
\]

\[
- \sum_{k=1}^{n} \binom{n}{k} \left( 2 \left\{ N_{(k)0}, [n-k] \hat{W} \right\}_{k+1} + \left\{ \gamma^0 \gamma^i, [N_{(k)i}, [n-k] \hat{W}]_{k+1} \right\} \right)
\]
\[ + \{ \gamma^0 \gamma^i, [iM_{(k)i}, [n-k] \hat{W}]_{k} \} \]

(constraints) \hfill (4.18)

\[ 0=2m[\gamma^0, [n] \hat{W}] - 2[i\Delta_0, [n] \hat{W}] - \{ \gamma^0 \gamma^i, \{ \Pi_{i}, [n] \hat{W} \} \} - [\gamma^0 \gamma^i, [i\Delta_1, [n] \hat{W}]] \]

\[- \sum_{k=1}^{n} \binom{n}{k} \left( 2[iM_{(k)0}, [n-k] \hat{W}]_{k} + \{ \gamma^0 \gamma^i, [N_{(k)i}, [n-k] \hat{W}]_{k+1} \} 
+ [\gamma^0 \gamma^i, [iM_{(k)i}, [n-k] \hat{W}]_{k}]. \right) \]

(dynamics) \hfill (4.19)

The first of the two sets of equations, which couples the \( n \) lowest moments to the \( n+1 \)st one, contains only derivatives with respect to the spatial components of \( x \) and no time derivatives. We can formally write it as

\[ [n+1] \hat{W}(x, p, x_0) = \sum_{k=0}^{n} O^n_{(k)}(x, p, x_0; \partial_x, \partial_p) [k] \hat{W}(x, p, x_0), \]

which shows that it forms a set of constraints.

The second set of equations determines the time derivative of the \( n \)th moment in terms of all lower moments including the \( n \)th one, which we write formally as

\[ \partial_0 [n] \hat{W}(x, p, x_0) + Q^n_{(n)}(x, p, x_0; \partial_x, \partial_p) [n] \hat{W}(x, p, x_0) \]

\[ = - \sum_{k=0}^{n-1} Q^n_{(k)}(x, p, x_0; \partial_x, \partial_p) [k] \hat{W}(x, p, x_0). \]

This is a time-dependent partial differential equation for the \( n \)th moment, with a source term from the lower moments, which we interpret as a single-time transport equation. The equation for \( n = 0 \) is special because in this case the source term on the r.h.s. of (4.21) vanishes. The operators \( O^n_{(k)}, Q^n_{(k)} \) are functionals of the gauge fields and contain arbitrary powers of the spatial coordinate and momentum derivatives.
The spinor decomposition of the moment equations is given in Appendix F.

B. BGR equations

Limiting ourselves to external fields and considering only the lowest (zeroth) moments we can reduce the dynamical equations (F9-F16) for the spinor components to the BGR equations [24,38]. For QED with external fields we find for the non-local operators

\[
\int dp_0 p_0^n \left[ \Delta_\mu, \hat{X}(x,p) \right] \xrightarrow{\text{external fields}} D_\mu [n] X(x,p) + \sum_{k=1}^{n} \binom{n}{k} K_{(k)\mu} [n-k] X(x,p) \quad (4.22)
\]

\[
\int dp_0 p_0^n \left\{ \Pi_0, \hat{X}(x,p) \right\} \xrightarrow{\text{external fields}} 2^{[n+1]} X(x,p) + 2 \sum_{k=0}^{n} \binom{n}{k} L_{(k)0} [n-k] X(x,p) \quad (4.23)
\]

\[
\int dp_0 p_0^n \left\{ \Pi_i, \hat{X}(x,p) \right\} \xrightarrow{\text{external fields}} 2P_i [n] X(x,p) + 2 \sum_{k=1}^{n} \binom{n}{k} L_{(k)i} [n-k] X(x,p) \quad (4.24)
\]

with the definitions (3.65,3.66) for $P_\mu$, $D_\mu$ and the following operators $K_{(k)\mu}$ and $L_{(k)\mu}$:

\[
K_{(k)\mu} = g \int ds (-is)^{k+1} \partial_{x_0}^{(k)} \hat{F}_{\mu j}(x + is \partial_p, x_0) \partial^j_p + \frac{ik}{s} \partial_{x_0}^{(k-1)} \hat{F}_{\mu 0}(x + is \partial_p, x_0), \quad (4.25)
\]

\[
L_{(k)\mu} = g \int ds (-is)^{k+1} \partial_{x_0}^{(k)} \hat{F}_{\mu j}(x + is \partial_p, x_0) \partial^j_p + \frac{ik}{s} \partial_{x_0}^{(k-1)} \hat{F}_{\mu 0}(x + is \partial_p, x_0). \quad (4.26)
\]

The BGR equations are thus given as follows

\[
0 = 2m [\gamma_0, [0] W] - 2i D_0 [0] W - 2\{\gamma_0, \gamma^i, P_i [0] W\} - [\gamma_0, i D_i [0] W]. \quad (4.27)
\]

In spinor decomposition they are

\[
2m [0] P = D_\mu [0] A^\mu \quad (4.28a)
\]

\[
2m [0] S_{0i} = D_0 [0] \nu_i - D_i [0] \nu_0 + 2\varepsilon_{0ijk} P^j [0] A^k \quad (4.28b)
\]
The BGR equations therefore are just the dynamical equations for the zeroth moments for the case of QED with external fields. The connection with the spinor components \( f_i, g_i, i = 0, 1, 2, 3 \), as introduced by BGR in [23] is

\[
f_0 = [0] \mathcal{V}_0, \quad f_1 = -[0] \mathcal{A}_0, \quad f_2 = [0] \mathcal{P}, \quad f_3 = [0] \mathcal{F},
\]

\[
g_0 = -[0] \mathcal{A}, \quad g_1 = -[0] \mathcal{V}, \quad g_2 = [0] \mathcal{S}, \quad g_3 = -[0] \sigma.
\]

The hierarchy of moment equations Eqs. (4.18,4.19) will in general not truncate. The initial problem, namely that the covariant Wigner operator involves the fermion fields at all times and thus cannot be properly initialized at \( t = -\infty \), has now been reformulated into the task of solving simultaneously infinitely many equations as initial value problems. On the other hand the BGR equations (4.27,4.28) can not describe the theory completely since they lack information on all higher moments. How we can truncate the hierarchy naturally and what role the BGR equations play with respect to the infinite hierarchy of moment equations will become clear in the next subsection where we discuss the moment hierarchy in the limit of external fields.
C. Moment hierarchy for external fields

Since the Dirac equation and its adjoint for (time dependent) external fields are inhomogeneous first order differential equations for the wave functions, we need to specify as initial conditions only the fields $\psi, \psi^\dagger$ for all $\mathbf{x}$ at some initial time $x_0 = t_i$. In terms of moments it should be sufficient to specify $^0W$ at some initial time $t_i$. From the structure of Eqs. (4.20) it is obvious that there exists a recursion relation which allows to express all higher order moments $^nW(\mathbf{x}, \mathbf{p}, x_0)$, $n \geq 1$, through the lowest moment $^0W(\mathbf{x}, \mathbf{p}, x_0)$. Let us write it schematically as

$$^nW(\mathbf{x}, \mathbf{p}, x_0) = \mathcal{P}_n(\mathbf{x}, \mathbf{p}, x_0; \partial_x, \partial_p) ^0W(\mathbf{x}, \mathbf{p}, x_0), \quad n \geq 1,$$

(4.31)

where $\mathcal{P}_n$ is a combination of products of the operators $\mathcal{O}_{(k)}^n$ and thus contains derivatives with respect to $\mathbf{x}$ and $\mathbf{p}$ but no time derivatives. With the help of (4.31) one can rewrite (4.21) for $n \geq 1$ as

$$\partial_0 \left( \mathcal{P}_n ^0W \right) = - \sum_{k=0}^n \mathcal{Q}_{(k)}^n \mathcal{P}_k ^0W, \quad n \geq 1,$$

(4.32)

or

$$\mathcal{P}_n \left( \partial_0 ^0W \right) = - \left( \partial_0 \mathcal{P}_n + \sum_{k=0}^n \mathcal{Q}_{(k)}^n \mathcal{P}_k \right) ^0W, \quad n \geq 1.$$

(4.33)

Eliminating on the l.h.s. of (4.33) $\partial_0 ^0W$ with the help of Eq. (4.21) for $n = 0$ we obtain

$$\mathcal{P}_n \mathcal{Q}_{(0)}^0 ^0W = - \left( \partial_0 \mathcal{P}_n + \sum_{k=0}^n \mathcal{Q}_{(k)}^n \mathcal{P}_k \right) ^0W, \quad n \geq 1.$$

(4.34)

This is, for each $n \geq 1$, a new constraint equation for $^0W$.

We have thus replaced the set (4.21) of single-time transport equations by a single transport equation for $^0W$ (the case $n = 0$ in (4.21)) plus an infinite set of
new constraint equations \((4.34)\) for \(n \geq 1\). The single remaining transport equation reads explicitly

\[
2[i\Delta_0, [^0W] = 2m[\gamma^0, [^0W] - \{\gamma^0\gamma^i, \{\Pi_i, [^0W]\}\} - [\gamma^0\gamma^i, [i\Delta_i, [^0W]]]. \tag{4.35}
\]

This is the non-abelian generalization of the BGR equation \((4.27)\) for QED. Eq. \((4.35)\) is a first order differential equation in \(x_0\) for \([^0W]\) which can be solved uniquely by specifying \([^0W](x, p, x_0)\) at \(x_0 = t_i\). But what is the meaning of the additional constraints \((4.34)\)?

We will now show that they are redundant, i.e. turn into identities after using the constraints \((4.31)\). A brute force proof of this statement was given in \cite{27} for the case of a simple scalar field theory. For QED the statement was proven only for the lowest non-trivial values of \(n\). The reason for this was that for higher \(n\) the explicit calculation of the operators \(\mathcal{P}_n\) in \((4.31)\) becomes extremely cumbersome.

We will here give a more elegant general proof which is also directly applicable to the non-Abelian case which exploits the “equivalence” (with respect to taking higher \(p_0\)-moments) of the two covariant equations from which the single-time transport and constraint equations were derived.

The proof is made more presentable by introducing the following notation for multiple commutators:

\[
A_- X = [A, X], \quad A_+ X = \{A, X\}, \quad A_- B_- X = [A, [B, X]] \quad \text{etc.,} \tag{4.36}
\]

where the commutators or anti-commutators extend to everything to the right of the operator \(A_\pm\). For the spinor structure we will use in addition

\[
g^0 \equiv \gamma^0, \quad g^i \equiv \gamma^0\gamma^i \tag{4.37}
\]
Let us now look at the integrands under the moment integral \( \int dp_0 p_0^n \) for the two equations which gave rise to Eqs. (4.20,4.21):

\[
2p_0^+ W = \left[ 2mg_+^0 - 2N_0^0 + g_-^i \Pi_{i+} - g_+^i i \Delta_{i-} \right] W , \quad (4.38)
\]

\[
0 = \left[ 2mg_-^0 - 2i \Delta_{0-} - g_+^i \Pi_{i+} - g_-^i i \Delta_{i-} \right] W . \quad (4.39)
\]

We will now consider in addition Eq. (4.39) multiplied with \( 2p_0^+ \) (which can be written as \( p_0^+ \) [Eq. (4.39)]

\[
p_0^+ \left[ 2mg_-^0 - 2i \Delta_{0-} - g_+^i \Pi_{i+} - i g_-^i \Delta_{i-} \right] W = 0 . \quad (4.40)
\]

After taking moments \( \int dp_0 p_0^n \), Eqs. (4.39,4.40) give dynamical equations for the moments \( n, n+1 \) while Eq. (4.38) gives constraints which tell us how to eliminate the moment \( n+1 \) through the moments \( k = 0, \ldots, n \).

The idea of the proof is to push in Eq. (4.40) the operator \( p_0^+ \) to the right until it stands in front of \( W \), then to use Eq. (4.38) to eliminate \( p_0^+ W \) and afterwards exploit Eq. (4.39) to eliminate the dynamical part \( \Delta_{0-} W \). The result will be an identity. On the level of moments this is equivalent to the elimination procedure discussed at the beginning of this subsection.

We first push \( p_0^+ \) in Eq. (4.40) to the right and insert Eq. (4.38). Noting that \( g_-^i \) commutes with \( p_0^+, \Delta_{\mu-}, \Pi_{\mu+} \) we find

\[
0 = \left[ mg_-^0 - i \Delta_{0-} - \frac{1}{2} g_+^i \Pi_{i+} - \frac{1}{2} g_-^i \Delta_{i-} \left( 2p_0^+ W \right) \right.
\]

\[
+ \left[ 2(i \Delta_{0-} p_0^+ - p_0^+ i \Delta_{0-}) + g_+^i (\Pi_{i+} p_0^+ - p_0^+ \Pi_{i+}) \right.
\]

\[
+ g_+^i (\Delta_{i-} p_0^+ - p_0^+ \Delta_{i-}) \right] W
\]

\[
= \left[ mg_-^0 - i \Delta_{0-} - \frac{1}{2} g_+^i \Pi_{i+} - \frac{1}{2} g_-^i \Delta_{i-} \right] \left[ 2mg_+^0 - 2N_0^0 - g_+^i \Pi_{i+} - g_-^i \Delta_{i-} \right] \left( 2p_0^+ W \right)
\]
Expanding and combining terms with the same spinor structure leads to

\[ 0 = \left[2(i\Delta_{0-}p_{0+} - p_{0+}i\Delta_{0-}) + g^i_+(\Pi_{i+}p_{0+} - p_{0+}\Pi_{i+}) + g^i_-(\Delta_{i-}p_{0+} - p_{0+}\Delta_{i-})\right]W. \] (4.41)

Now (i.e. after having inserted (4.38)) we push \(\Delta_{0-}\) again back to the right and use Eq. (4.39) for \(\Delta_{0-}\): \(\Delta_{0-}\) again back to the right and

\[ 0 = \left[2m^2(g_0^0 g_+^0 - g_+^0 g_0^-) - 2mN_0_0_0^0 (g_0^- - g_0^0) - m\Pi_{i+}(g_0^- g_+^- + g_+^0 g_0^- - g_0^- g_+^0) - mi\Delta_{i-}(g_0^0 g_+^0 + g_+^0 g_0^- - g_0^- g_+^0 - g_+^0 g_0^-) + g_+^i[\Pi_{i+}\Pi_{0+} - \Pi_{0+}\Pi_{i+} + i\Delta_{0-}i\Delta_{i-} - i\Delta_{i-}i\Delta_{0-}] + g_+^i[i\Delta_{i-}\Pi_{0+} - \Pi_{0+}i\Delta_{i-} + i\Delta_{0-}\Pi_{i+} - \Pi_{i+}i\Delta_{0-}] + \frac{1}{2}(g_+^i g_+^0 - g_+^0 g_+^i)[i\Delta_{i-}i\Delta_j - \Pi_{i+}\Pi_{j+}] + \frac{1}{2}(g_+^i g_+^0 - g_+^0 g_+^i)[\Pi_{i+}i\Delta_{j-} - i\Delta_{i-}\Pi_{j+}]\right]W. \] (4.42)

Expanding and combining terms with the same spinor structure leads to

\[ 0 = \left[2m^2(g_0^- g_+^0 - g_+^0 g_0^-) - 2mN_0_0_0^0 (g_0^- - g_0^0) - m\Pi_{i+}(g_0^- g_+^- + g_+^0 g_0^- - g_0^- g_+^0) - mi\Delta_{i-}(g_0^0 g_+^0 + g_+^0 g_0^- - g_0^- g_+^0 - g_+^0 g_0^-) + g_+^i[\Pi_{i+}\Pi_{0+} - \Pi_{0+}\Pi_{i+} + i\Delta_{0-}i\Delta_{i-} - i\Delta_{i-}i\Delta_{0-}] + g_+^i[i\Delta_{i-}\Pi_{0+} - \Pi_{0+}i\Delta_{i-} + i\Delta_{0-}\Pi_{i+} - \Pi_{i+}i\Delta_{0-}] + \frac{1}{2}(g_+^i g_+^0 - g_+^0 g_+^i)[i\Delta_{i-}i\Delta_j - \Pi_{i+}\Pi_{j+}] + \frac{1}{2}(g_+^i g_+^0 - g_+^0 g_+^i)[\Pi_{i+}i\Delta_{j-} - i\Delta_{i-}\Pi_{j+}]\right]W. \] (4.43)

As can be easily checked the following relations for the spinor matrices hold:

\[ (g_0^- g_+^0 - g_+^0 g_0^-)W = 0 \] (4.44)

\[ (g_0^- g_+^i - g_+^0 g_+^i + g_+^i g_0^- - g_+^i g_+^0)W = 0 \] (4.45)
\[(g^{-0}_0 g^i_0 - g^0_+ g^i_- + g^i_+ g^0_- - g^i_- g^0_+) W = 0\] (4.46)

\[(g^i_+ g^j_+ - g^i_- g^j_-) W = 2(g^i W g^j + g^j W g^i)\] (4.47)

\[(g^i_+ g^j_- - g^i_- g^j_+) W = 2(g^i W g^j - g^j W g^i).\] (4.48)

Therefore the terms containing the mass \(m\) will vanish. The rest can be combined, using the explicit form of the \(\gamma\) matrices, to give

\[0 = \left[ (\Pi^{+\mu} + i\Delta^{\mu}_-) (\Pi^{-\nu} - i\Delta^{-\nu}) - (\Pi^{-\nu} - i\Delta^{-\nu}) (\Pi^{+\mu} + i\Delta^{\mu}_-) \right] \gamma^0 \gamma^\mu W \gamma^\nu \gamma^0.\] (4.49)

With the help of Eqs. (D7) we finally arrive at

\[0 = \left[ p^\mu + \Pi^\mu + i\Delta^\mu, p^\nu + \Pi^\nu - i\Delta^\nu \right] \gamma^0 \gamma^\mu W \gamma^\nu \gamma^0 - \gamma^0 \gamma^\nu W \gamma^\mu \gamma^0 \left[ p^\nu + \Pi^\nu - i\Delta^\nu, p^\mu + \Pi^\mu + i\Delta^\mu \right].\] (4.50)

In Appendix G we show that the commutator vanishes:

\[\left[ p^\mu + \Pi^\mu + i\Delta^\mu, p^\nu + \Pi^\nu - i\Delta^\nu \right] \equiv 0.\] (4.51)

Eq. (4.50) is therefore an identity.

In the case of external fields we thus have a natural truncation of the transport hierarchy on the level of \([0] W\). Its time evolution is uniquely determined by Eq. (4.35) once \(W(x, p, t)\)_{t=t_0} has been specified. All higher moments can be found through successive insertion of the solution into Eq. (4.18). These “constraints” are differential equations in \(x\) and \(p\) without time evolution. For the external field problem the “initial value problem” for the quark Wigner operator has thus been solved, and we have also given an explicit prescription how to find the higher moments. The latter was missing in the work of Refs. [24–26], although the BGR equations Eqs. (1.27) properly describe the dynamics of the zeroth moment. Higher moments do, however,
contain important physical information even for external fields. For example, the energy momentum tensor is given by the first moments (for explicit expressions see [27]) and can therefore be calculated only with the help of the constraint Eq. (4.18) once the solution for the zeroth moment has been obtained.

D. QED and external fields

In this subsection we make the results obtained above explicit for the case of an abelian external gauge field, i.e. for QED with external fields. We will show the explicit connection between the covariant theory and the moment equations in leading order of the gradient expansion and demonstrate that the correct Vlasov (including spin precession terms) limit is recovered from the single-time transport equations.

To zeroth order in the gradient expansion the equations for the moments are

\[ [n+1] \gamma_{0}^{(0)} = m \ [n] \mathcal{F}^{(0)} - p_{i} \ [n] \gamma_{i}^{(0)} \]  
\[ (4.52a) \]

\[ [n+1] \mathcal{F}^{(0)} = m \ [n] \gamma_{0}^{(0)} \]  
\[ (4.52b) \]

\[ [n+1] \gamma_{i}^{(0)} = p_{j} \ [n] \gamma_{j}^{(0)} \]  
\[ (4.52c) \]

\[ m [n] \gamma_{i}^{(0)} = p_{j} \ [n] \mathcal{F}^{(0)} \]  
\[ (4.52d) \]

\[ p_{j} \ [n] \gamma_{j}^{(0)} = p_{i} \ [n] \gamma_{i}^{(0)} \]  
\[ (4.52e) \]

\[ [n+1] \mathcal{P}^{(0)} = 0 \]  
\[ (4.52f) \]

\[ [n] \mathcal{P}^{(0)} = 0 \]  
\[ (4.52g) \]

\[ p_{j} \ [n] \mathcal{P}^{(0)} = 0 \]  
\[ (4.52h) \]
\[ [n+1] A^k(0) = -m [n] L^0k(0) + p^k [n] A^0(0) \] (4.53a)

\[ [n+1] S^{jk}(0) = m \varepsilon^{0ijk} [n] A^i(0) - p^k [n] S^{0j}(0) + p^j [n] S^{0k}(0) \] (4.53b)

\[ [n+1] A^0(0) = -p_i [n] A^i(0) \] (4.53c)

\[ [n+1] S^{0i}(0) = p_j [n] S^{ij}(0) \] (4.53d)

\[ m [n] S^{0i}(0) = \varepsilon^{0ijk} p_j [n] A^0(0)_k \] (4.53e)

\[ m [n] A^0(0) = p_i [n] L^{0i}(0) \] (4.53f)

\[ 0 = p_i [n] S^{0i}(0). \] (4.53g)

Eqs. (4.52vi,vii,viii) show that \([n] P^{(0)}\) vanishes for all \(n\) identically. In addition we get that \(V^{(0)}\) can be expressed by \(\mathcal{F}^{(0)}\) and \(A^{(0)}\) by \(S^{(0)}\). Eqs. (4.52) can be solved now for \([n] \mathcal{F}^{(0)}\) and \([n] V^{(0)}\) by specifying the lowest moments \([0] \mathcal{F}^{(0)}\) and \([0] V^{(0)}\) \((E = \sqrt{p^2 + m^2} > 0, m > 0)\):

\[ [2n] V^{0(0)} = E^{2n} [0] V^{0(0)} \quad [2n+1] V^{0(0)} = E^{2n+1} \frac{E}{m} [0] \mathcal{F}^{(0)} \] (4.54)

\[ [2n] V^{(0)} = E^{2n} \frac{p}{m} [0] \mathcal{F}^{(0)} \quad [2n+1] V^{(0)} = E^{2n+1} \frac{p}{E} [0] V^{0(0)} \] (4.54)

\[ [2n] \mathcal{F}^{(0)} = E^{2n} [0] \mathcal{F}^{(0)} \quad [2n+1] \mathcal{F}^{(0)} = E^{2n+1} \frac{m}{E} [0] V^{0(0)} \] (4.54)

Using

\[ m [0] V^{0(0)} = [1] \mathcal{F}^{(0)} \] (4.55)

we can also write
\[ [2n] \mathcal{V}^{0(0)} = E^{2n} \frac{1}{m} [1] \mathcal{F}^{(0)} \quad [2n+1] \mathcal{V}^{0(0)} = E^{2n+1} \frac{E}{m} [0] \mathcal{F}^{(0)} \]

\[ [2n] \mathcal{V}^{(0)} = E^{2n} \frac{P}{m} [0] \mathcal{F}^{(0)} \quad [2n+1] \mathcal{V}^{(0)} = E^{2n+1} \frac{P}{Em} [1] \mathcal{F}^{(0)} \] (4.56)

\[ [2n] \mathcal{F}^{(0)} = E^{2n} [0] \mathcal{F}^{(0)} \quad [2n+1] \mathcal{F}^{(0)} = E^{2n+1} \frac{1}{E} [1] \mathcal{F}^{(0)}, \]

thus specifying rather \([0] \mathcal{F}^{(0)}\) and \([1] \mathcal{F}^{(0)}\) instead of \([0] \mathcal{F}^{(0)}\) and \([0] \mathcal{V}^{0(0)}\). The connection with particle (upper sign) and anti-particle (lower sign) distribution functions from the Lorentz-covariant theory is found to be

\[ \mathcal{F}(x, p, \pm E) = E [0] \mathcal{F}^{(0)} \pm m [0] \mathcal{V}^{0(0)} = E [0] \mathcal{F}^{(0)} \pm [1] \mathcal{F}^{(0)}. \] (4.57)

Thus the particle and anti-particle contributions of the covariant theory \(\mathcal{F}(x, p, \pm E)\) translate into the sum and difference respectively of \([0] \mathcal{F}\) and \([1] \mathcal{F}\) in single-time formulation. This is immediately obvious from the fact that \([0] \mathcal{F}\) is derived from \(\mathcal{F}\) as an integral over all energies, that is particle and anti-particle contributions

\[ [0] \mathcal{F}(x, p) = \int_{-\infty}^{\infty} dp_0 \mathcal{F}(x, p), \] (4.58)

whereas \([1] \mathcal{F}\) has an additional weight \(p_0\), leading to the difference between particles with positive and negative energies.

The transport equations for the zeroth order moments in the semi-classical approximation are now given by (see Appendix [4])

\[ 0 = E^n \left[ E(\partial^0 + \frac{e}{c} \nabla_p \cdot E)[1] \mathcal{F}^{(0)} + (p \cdot \nabla + \frac{e}{c}(p \times B) \cdot \nabla_p)(E[0] \mathcal{F}^{(0)}) \right] \] (4.59)

\[ 0 = E^n \left[ E(\partial^0 + \frac{e}{c} \nabla_p \cdot E)(E[0] \mathcal{F}^{(0)}) + (p \cdot \nabla + \frac{e}{c}(p \times B) \cdot \nabla_p)[1] \mathcal{F}^{(0)} \right]. \] (4.60)
Adding and subtracting gives us back exactly the transport equations we get from putting the distribution function of the covariant theory on the mass shell

\[ \mathcal{F}(x, p) = \mathcal{F}(x, p, \pm E)\delta(p_0 \mp E), \]

inserting into the transport equations of the covariant theory Eqs. (3.103, 3.109) and integrating over \( \int dp_0 \, p_n \). This shows by example the equivalence between covariant and single-time formulation.

Similar we get for \([n] A^{(0)}\) and \([n] S^{(0)}_{\mu\nu}\) the following equations, if we specify the lowest moments of \([n] S^{ij(0)}\), that is \([0] \sigma^{i(0)}\) and \([1] \sigma^{i(0)}\)

\[
[2n] A^{(0)} = E^{2n} \frac{1}{m} p \cdot [0] \sigma^{(0)} \quad (4.61a)
\]

\[
[2n] A^{(0)} = E^{2n} \frac{1}{m} \left( [1] \sigma^{(0)} + p \times (p \times \frac{[1] \sigma^{(0)}}{E^2}) \right) \quad (4.61b)
\]

\[
[2n] S^{(0)} = -E^{2n} p \times \frac{[1] \sigma^{(0)}}{E^2} \quad (4.61c)
\]

\[
[2n] \sigma^{(0)} = E^{2n} [0] \sigma^{(0)} \quad (4.61d)
\]

\[
[2n+1] A^{(0)} = E^{2n+1} \frac{1}{m} p \cdot [0] \sigma^{(0)} \quad (4.61e)
\]

\[
[2n+1] A^{(0)} = E^{2n+1} \frac{1}{m} \left( [0] \sigma^{(0)} + p \times (p \times \frac{[0] \sigma^{(0)}}{E^2}) \right) \quad (4.61f)
\]

\[
[2n+1] S^{(0)} = -E^{2n+1} p \times \frac{[0] \sigma^{(0)}}{E} \quad (4.61g)
\]

\[
[2n+1] \sigma^{(0)} = E^{2n+1} \frac{[1] \sigma^{(0)}}{E}. \quad (4.61h)
\]

As before we have for the connection with the particle and anti-particle distribution of the covariant theory

\[
\sigma(x, p, \pm E) = E \cdot [0] \sigma^{(0)} \pm [1] \sigma^{(0)}. \quad (4.62)
\]
\[ 0 = E(\partial^0 + \frac{e}{c} \nabla_p \cdot E) [\sigma^{(0)}] + (p \cdot \nabla + \frac{e}{c} (p \times B) \cdot \nabla_p) [0] \sigma^{(0)} \]
\[ + \frac{e}{c} \left( p \cdot \left( E \cdot \frac{[0] \sigma^{(0)}}{E^2} \right) + B \times [1] \sigma^{(0)} \right) \]  

\[ 0 = \left[ E(\partial^0 + \frac{e}{c} \nabla_p \cdot E) [0] \sigma^{(0)} + (p \cdot \nabla + \frac{e}{c} (p \times B) \cdot \nabla_p) [1] \sigma^{(0)} \right] \]
\[ + \frac{e}{c} \left( p \cdot \left( E \cdot \frac{[1] \sigma^{(0)}}{E} \right) + B \times \frac{[0] \sigma^{(0)}}{E} \right). \]  

(4.63a)

(4.63b)

Adding and subtracting gives us again the transport equations for the particle and anti-particle distributions of the covariant theory Eqs. (3.103ii, 3.109ii) after integrating over \( \int dp_0 p_0^n \) and employing the mass shell condition \( \sigma = \sigma(x, p, \pm E) \delta(p_0 \mp E) \).

Eqs. (4.63) are therefore again the generalized BMT-equations Eqs. (3.109), but now formulated with moments.

In the case of QED with external fields we thus showed the explicit connection between the formulation with moments and the covariant theory. To this end we had to solve the moment hierarchy exactly in the classical limit. This solution proved then to correspond to the classical mass shell condition of the covariant theory.

V. CONCLUSIONS

In this paper we have established the connection between the covariant and the so-called single-time formulations of the quantum kinetic theory, which is based on the dynamics of the quantum mechanical Wigner operator. We discussed the role of the time parameter in the density matrix, and we could show that a formulation with a single time parameter necessarily leads to a hierarchy of equations for the energy moments for the Wigner operator. For these moments we found a direct connection with the time derivatives of the wave functions.
A covariant field theory is equivalent to the complete set of its $p_0$-moments. The impossibility to formulate the covariant transport theory as an initial value problem, which we discussed in Sec. [1], thus translates into the impossible task of solving an infinite hierarchy of moment equations. The hierarchy structure lends itself, however, to a systematic approximation scheme via successive truncation.

For fermion dynamics in external classical gauge fields the truncation problem can be solved exactly, in the case of Dirac fields at the level of the lowest $p_0$-moment, the single-time Wigner function $W_3(x, p, t)$. This reflects the fact that for external gauge fields the fermionic theory is a single-particle theory whose dynamics can be solved uniquely once initial conditions for the fermion wave functions $\psi(x, x_0)$ at $x_0 = t_i$ have been specified.

To find the equations of motion for the moments we first reviewed the covariant transport theory, extending some of the results obtained previously in [22], [23]. We introduced a compact notation for generalized non-local momentum and derivative operators which allowed us to cast the equations of motion for the Wigner operator into a form which permitted further manipulation. We then performed a gradient expansion followed by a color decomposition and established the connection between the (local) momentum operator and covariant derivative with their non-local generalizations.

To find the proper mass shell and transport equations for the Wigner operator we had also to perform a spinor decomposition. Thereby we could eliminate half of the initial degrees of freedom to end up with a system of equations for only eight independent spinor components, whereas the other eight components could be
expressed as dependent variables. This improves on previous covariant treatments were the system of mass shell and transport equations always contained redundant information. We derived this system of mass shell and transport equations plus constraints on the one hand for scalar, pseudo-scalar and tensor components, which are a suitable basis of spinor components for massive quarks; alternatively we also derived mass shell, transport and constraint equations for vector and axial vector components, which are more suitable as independent components for the case of vanishing bare quark masses (chiral limit).

That our equations have the proper classical limit was proven by an explicit gradient expansion. We could also find the expected vacuum solutions where the pseudo-scalar and spin density did vanish identically. For chiral theory we found the expected conservation of vector and axial vector current. We also compared our results with recent investigations of quantum transport equations for the Nambu-Jona-Lasinio theory. We showed that the failure of those equations to decouple scalar, pseudoscalar and tensor densities from the vector and axial vector ones was related to the spontaneous breaking of chiral symmetry in that theory, with the dynamically generated mass acting like a non-zero bare mass in our case, thereby coupling the two sectors.

After the discussion of the covariant theory we considered the expansion of our equations in energy moments. This moment expansion was performed with the help of a partial gradient expansion in $\partial p_0$ and resulted in an infinite hierarchy of dynamical equations for the moments, which are coupled by constraints. The equivalence of both formulations was given by construction. For external Abelian
fields to leading order of the gradient expansion we showed the explicit connection of the particle and anti-particle distributions of covariant theory with the zeroth and first moments in the equal time formulation. Finally we were able to show that for the external field case we need only the dynamical equations for the zeroth moments. All higher moments can then be successively calculated from the lowest moments via the constraint equations. This leads for the external field case to a natural truncation condition for the moment hierarchy where the minimal truncation, that is ignoring all but the lowest moments, just corresponds to the BGR equations.

The question now arises whether now the equations of motion for the single-time Wigner operators can be finally applied for the (numerical) description of a QGP or electron-ion plasma. For the transport equations with the full operator structure the answer is certainly no in practice. Further work is required to analyze correlations and collision terms in this case which should be obtained by taking an ensemble average and using appropriate factorization of expectation values. For external fields, however, where the equations reduce to transport equations without a collision term, quantum transport theory has now been formulated in the single-time approach in a way which allows for immediate numerical implementation, for both QED and QCD. For each color and spin component one transport equation must be solved, together (but not simultaneously!) with as many time-independent constraints as given by the highest order of $p_0$-moments which one wants to know.

Quantum transport theory beyond the external field limit and the extraction of suitable collision terms remains a great challenge. For gauge theories the Wigner formalism, through the occurrence of Schwinger strings, strongly suggests to use the
radial gauge \[37\] for which perturbation theory has only recently become viable after
a set of consistent Feynman-rules was established in Ref. \[45\]. Infrared divergences
and the need for resummation in non-abelian gauge theories with massless degrees
of freedom have, however, so far hampered practical progress.

**APPENDIX A: COLOR MATRICES**

The generators of $SU(3)$ in the fundamental representation, $t_a = \hbar \lambda_a$ (where $\lambda_a$
are the Gell-Mann matrices), satisfy the following identities

\[(t^a)^\dagger = t^a\]  \hspace{1cm} (A1)

\[[t^a, t^b] = i \hbar f^{abc} t^c\]  \hspace{1cm} (A2)

\[\left\{ t^a, t^b \right\} = \frac{\hbar^2}{3} \delta^{ab} 1 + \hbar d^{abc} t^c\]  \hspace{1cm} (A3)

\[t^a t^b = \frac{1}{2} \left( \frac{\hbar^2}{3} \delta^{ab} 1 + \hbar \left( d^{abc} + i f^{abc} \right) t^c \right)\]
\[\equiv \frac{1}{2} \left( \frac{\hbar^2}{3} \delta^{ab} 1 + \hbar h^{abc} t^c \right).\]  \hspace{1cm} (A4)

$f^{abc}$ and $d^{abc}$ are the totally anti-symmetric and totally symmetric structure con-
stants of $SU(3)$.

The traces of the generators are given by

\[\text{tr } t^a = 0\]  \hspace{1cm} (A5)

\[\text{tr } t^a t^b = \frac{\hbar^2}{2} \delta^{ab}\]  \hspace{1cm} (A6)

\[\text{tr } t^a t^b t^c = \frac{\hbar^3}{4} \left( d^{abc} + i f^{abc} \right) \equiv \frac{\hbar^3}{4} h^{abc}\]  \hspace{1cm} (A7)

\[\text{tr } t^a t^b t^a t^c = -\frac{\hbar^4}{12} \delta^{bc}.\]  \hspace{1cm} (A8)

One also has
\[ f^{abc} = -\frac{i}{\hbar} \text{tr} (\{t^a, t^b\} t^c) \quad (A9) \]
\[ d^{abc} = \frac{2}{\hbar^3} \text{tr} (\{t^a, t^b\} t^c) \quad (A10) \]
\[ f^{abb} = 0 \quad (A11) \]
\[ d^{abb} = 0 \quad (A12) \]
\[ f^{abc} f^{bcd} = 3 \delta^{ab}. \quad (A13) \]

**APPENDIX B: GAUGE TRANSFORMATIONS**

Let \( S(x) \) be a gauge transformation in \( U(1) \) or \( SU(3) \):

\[ S(x) = e^{i \theta_a(x) t_a}, \quad [\theta_a(x) t_a, S(x)] = 0. \quad (B1) \]

(For \( U(1) \) there is only one generator, \( t_0 = 1 \).) Then the following transformation rules apply:

\[ \psi(x) \to S(x) \psi(x) \quad (B2) \]
\[ \psi^\dagger(x) \to \psi^\dagger(x) S^{-1}(x) \quad (B3) \]
\[ \hat{A}_\mu(x) \to S(x) \left[ \hat{A}_\mu(x) + \frac{1}{g} (\partial_\mu \theta_a(x)) t_a \right] S^{-1}(x) \quad (B4) \]
\[ D_\mu(x) \to S(x) D_\mu(x) S^{-1}(x) \quad (B5) \]
\[ U(x,y) \to S(x) U(x,y) S^{-1}(y) \quad (B6) \]
\[ \hat{g}(x + y/2, x - y/2) \to S(x) \hat{g}(x + y/2, x - y/2) S^{-1}(x) \quad (B7) \]
\[ \hat{W}(x,p) \to S(x) \hat{W}(x,p) S^{-1}(x) \quad (B8) \]
\[ [x] \hat{F}_{\mu\nu}(z(s)) \to S(x) [x] \hat{F}_{\mu\nu}(z(s)) S^{-1}(x) \quad (B9) \]
\[ [D_\mu(x), \hat{W}(x,p)] \to S(x) [D_\mu(x), \hat{W}(x,p)] S^{-1}(x) \quad (B10) \]
\[ \Pi_\mu \rightarrow S(x)\Pi_\mu S^{-1}(x) \quad (B11) \]
\[ \{\Pi_\mu, \hat{W}(x,p)\} \rightarrow S(x)\{\Pi_\mu, \hat{W}(x,p)\}S^{-1}(x) \quad (B12) \]
\[ \Delta_\mu \rightarrow S(x)\Delta_\mu S^{-1}(x) \quad (B13) \]
\[ [\Delta_\mu, \hat{W}(x,p)] \rightarrow S(x)[\Delta_\mu, \hat{W}(x,p)]S^{-1}(x). \quad (B14) \]

**APPENDIX C: LINK OPERATORS**

We recall the formula for the variation of link operators with respect to their end points [22]:

\[ \delta U(b,a) = \delta b i g \hat{A}(b)U(b,a) - i g U(b,a)\hat{A}(a) \delta a \quad (C1) \]
\[ - i g \int_0^1 ds U(b,z(s)) F_{\mu\nu}(z(s)) U(z(s),a) (b - a)^\mu (\delta a + s(\delta b - \delta a))^\nu, \]
\[ z(s) = a + (b - a)s, \]

This yields the following explicit expressions:

\[ \partial_+ \mu U(x_+,x) = -\frac{1}{2} i g U(x_+,x) [\hat{\mathcal{A}}_\mu(x) - 2U(x,x_+)\hat{\mathcal{A}}_\mu(x_+)U(x_+,x) \]
\[ + \int_0^{1/2} ds (1 + 2s) [x] \hat{F}_{\nu\mu}(x + sy) y^\nu], \quad (C2) \]
\[ \partial_+ \mu U(x,x_-) = -\frac{1}{2} i g [\hat{\mathcal{A}}_\mu(x) + \int_{-1/2}^0 ds (1 + 2s) [x] \hat{F}_{\nu\mu}(x + sy) y^\nu] U(x,x_-), \quad (C3) \]
\[ \partial_- \mu U(x_+,x) = -\frac{1}{2} i g [\hat{\mathcal{A}}_\mu(x) + \int_0^{1/2} ds (1 - 2s) [x] \hat{F}_{\nu\mu}(x + sy) y^\nu], \quad (C4) \]
\[ \partial_- \mu U(x,x_-) = -\frac{1}{2} i g [\hat{\mathcal{A}}_\mu(x) + 2U(x,x_-)\hat{\mathcal{A}}_\mu(x_-)U(x_-,x) \]
\[ + \int_{-1/2}^0 ds (1 - 2s) [x] \hat{F}_{\nu\mu}(x + sy) y^\nu] U(x,x_-). \quad (C5) \]

For the derivative of the Schwinger string \([x] \hat{F}_{\nu\mu}(x + sy)\) (see Eq. (3.24)) with respect to \(y\) one finds
\[ \partial_{y^a} [x] F_{\mu\nu}(x + y) = U(x, x + y) \left[ \partial_{y^a} - ig A_\alpha(x + y), F_{\mu\nu}(x + y) \right] U(x + y, x) \]
\[ -ig \int_0^1 ds \, s \, y^\beta \left[ [x] F_{\alpha\beta}(x + sy), [x] F_{\mu\nu}(x + y) \right]. \] (C6)

Its covariant derivative with respect to \( x \) is
\[
\tilde{D}_\mu(x)[x] F_{\alpha\beta}(x + y) \equiv [D_\mu(x), [x] F_{\alpha\beta}(x + y)]
\[ = U(x, x + y) \left( \tilde{D}_\mu(x + y) F_{\alpha\beta}(x + y) \right) U(x + y, x) \]
\[ -ig \int_0^1 ds \, y^\nu \left[ [x] F_{\mu\nu}(x + sy), [x] F_{\alpha\beta}(x + y) \right]. \] (C7)

Combining the last two equations one finds
\[
\tilde{D}_\mu(x)[x] F_{\alpha\beta}(x + y) =
\partial_{y^\mu} [x] F_{\alpha\beta}(x + y) - ig \int_0^1 ds \, (1 - s) \, y^\nu \left[ [x] F_{\mu\nu}(x + sy), [x] F_{\alpha\beta}(x + y) \right]. \] (C8)

**APPENDIX D: SPINOR EQUATIONS**

In this Appendix we combine the spinor equations (3.73,3.74) into generalized mass shell and transport equations for the spinor components. Multiplying the first three of equations in (3.73) and (3.74) each by \( 2m \) and inserting the last two equations (3.73) yields
\[
4m^2 \tilde{F} = \{ \Pi_\mu, \{ \Pi^\mu, \tilde{F} \} \} + \{ \Pi_\mu, [\Delta_\nu, \tilde{S}^{\mu\nu}] \} \] (D1)
\[
4m^2 \tilde{P} = -[\Delta_\mu, [\Delta^\mu, \tilde{P}]] + \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} [\Delta_\mu, \{ \Pi_\nu, \tilde{S}_{\alpha\beta} \}] \] (D2)
\[
4m^2 \tilde{S}_{\mu\nu} = [\Delta_\mu, \{ \Pi_\nu, \tilde{F} \}] - [\Delta_\nu, \{ \Pi_\mu, \tilde{F} \}] + [\Delta_\mu, [\Delta^\alpha, \tilde{S}_{\alpha\mu}]] - [\Delta_\nu, [\Delta^\alpha, \tilde{S}_{\alpha\mu}]] - \varepsilon_{\alpha\beta\mu\nu} \{ \Pi^\alpha, [\Delta^\beta, \tilde{P}] \} + \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} \varepsilon^{\beta\gamma\delta\kappa} \{ \Pi^\alpha, \{ \Pi_\gamma, \tilde{S}_{\delta\kappa} \} \} \] (D3)

as well as
\[ 0 = [\Delta_\mu, \{\Pi_\mu, \hat{\mathcal{F}}\}] + [\Delta_\mu, [\Delta_\nu, \hat{\mathcal{S}}^{\mu\nu}]] \]  
\[ 0 = \{\Pi_\mu, [\Delta_\mu, \hat{\mathcal{P}}]\} - \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \{\Pi_\mu, \{\Pi_\nu, \hat{\mathcal{S}}_{\alpha\beta}\}\} \]  
\[ 0 = \{\Pi_\mu, \{\Pi_\nu, \hat{\mathcal{F}}\}\} - \{\Pi_\nu, \{\Pi_\mu, \hat{\mathcal{F}}\}\} + \{\Pi_\mu, [\Delta^\alpha, \hat{\mathcal{S}}_{\nu\alpha}]\} - \{\Pi_\nu, [\Delta^\alpha, \hat{\mathcal{S}}_{\mu\alpha}]\} + \varepsilon_{\alpha\beta\mu\nu} \Delta^\alpha [\Delta^\beta, \hat{\mathcal{P}}] - \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} \varepsilon^{\gamma\delta\kappa} [\Delta^\alpha, \{\Pi_\gamma, \hat{\mathcal{S}}_{\delta\kappa}\}] \]

With some further manipulations the first three equations will give rise to generalized mass shell constraints, the last three to generalized transport equations for \( \hat{\mathcal{F}}, \hat{\mathcal{P}}, \) and \( \hat{\mathcal{S}}_{\mu\nu} \). In terms of the solutions to these equations the remaining spinor components \( \hat{A}_\mu \) and \( \hat{V}_\mu \) are given by the last two equations (3.73).

For the further manipulations the following identities will be useful:

\[ [\Delta_\mu, [\Delta_\nu, \hat{X}]] - [\Delta_\nu, [\Delta_\mu, \hat{X}]] = [[\Delta_\mu, \Delta_\nu], \hat{X}] \]  
\[ \{\Pi_\mu, [\Delta_\nu, \hat{X}]\} - [\Delta_\nu, \{\Pi_\mu, \hat{X}\}] = \{[p_\mu + \Pi_\mu, \Delta_\nu], \hat{X}\} \]  
\[ \{\Pi_\mu, \{\Pi_\nu, \hat{X}\}\} - \{\Pi_\nu, \{\Pi_\mu, \hat{X}\}\} = [[p_\mu + \Pi_\mu, p_\mu + \Pi_\nu], \hat{X}] \].

The additional factors \( \sim p_\mu \) on the r.h.s. arise from the fact that the commutators in these equations refer to the color and Fock space structure of the operators only and do not consistently include also the momentum derivative operators which, according to (3.23), were defined to act only to the left when standing to the right of a Wigner function. Splitting the generalized momentum operator (3.25) according to \( \Pi_\mu = p_\mu + \pi_\mu \) where the second term \( \pi_\mu \) (like \( \Delta_\mu \) in (3.26)) contains only momentum \emph{derivatives}, but no momentum \emph{factors}, Eq. (D7b), for example, therefore has to be evaluated as follows:

\[ \{p_\mu + \pi_\mu, [\Delta_\nu, \hat{X}]\} - [\Delta_\nu, \{p_\mu + \pi_\mu, \hat{X}\}] = +p_\mu (\Delta_\nu \hat{X}) - p_\mu (\hat{X} \Delta_\nu) + (\Delta_\nu \hat{X}) p_\mu - (\hat{X} \Delta_\nu) p_\mu \]
\[-\Delta_\nu(p_\mu \hat{X}) - \Delta_\nu(\hat{X} p_\mu) + (p_\mu \hat{X}) \Delta_\nu + (\hat{X} p_\mu) \Delta_\nu + \{\pi_\mu, [\Delta_\nu, \hat{X}]\} - [\Delta_\nu, \{\pi_\mu, \hat{X}\}]\]
\[= 2[p_\mu, \Delta_\nu] \hat{X} - 2\hat{X}[\Delta_\nu, p_\mu] + [\pi_\mu, \Delta_\nu] \hat{X} - \hat{X}[\Delta_\nu, \pi_\mu]\]
\[= [p_\mu + \Pi_\mu, \Delta_\nu] \hat{X} - \hat{X}[\Delta_\nu, p_\mu + \Pi_\mu]\]
\[= [[p_\mu + \Pi_\mu, \Delta_\nu], \hat{X}], \quad (D8)\]

and similarly for Eq. (D7c). In the intermediate expressions the round brackets indicate, where ambiguous, how the momentum derivatives act via (3.23); since the momentum derivatives in \(\pi_\mu\) and \(\Delta_\nu\) commute the remaining terms are unambiguous.

We will now use the generalized quadratic mass shell operator

\[M^2 \hat{X} = 4m^2 \hat{X} - \{\Pi_\mu, \{\Pi^\mu, \hat{X}\}\} + [\Delta_\mu, [\Delta^\mu, \hat{X}]] \quad (D9)\]

to combine equal spinor components on the left hand sides of the equations.

1. Mass shell equations

We now derive the mass shell constraints (3.79). To get the mass shell condition for \(\hat{\Phi}\) with the operator \(M^2\) from above we bring the term \(\{\Pi_\mu, \{\Pi^\mu, \hat{\Phi}\}\}\) in Eq. (D9) to the left and add on both sides \([\Delta_\mu, [\Delta^\mu, \hat{\Phi}]]\). Furthermore we employ the constraint Eq. (3.74v) together with the anti-symmetry of \(\hat{S}_{\mu\nu}\) in the form

\[[\Delta_\mu, [\Delta^\mu, \hat{\Phi}]] + \{\Pi_\mu, [\Delta_\nu, \hat{S}^{\mu\nu}]\} = [\Delta_\mu, \{\Pi_\nu, \hat{S}^{\mu\nu}\}] - \{\Pi_\nu, [\Delta_\mu, \hat{S}^{\mu\nu}]\}, \quad (D10)\]

and then use (D7f).

The mass shell condition for \(\hat{\Phi}\) we obtain by subtracting \(\{\Pi_\mu, \{\Pi^\mu, P\}\}\) from Eq. (D2) and using
which follows from Eq. (3.74v). We then again use (D7d).

For the mass shell condition for $\hat{S}_{\mu\nu}$, finally, we rewrite

$$\varepsilon_{\alpha\beta\mu\nu}\{\Pi^\alpha, [\Delta^\beta, \hat{P}]\} = \varepsilon_{\alpha\beta\mu\nu}\{\Pi^\alpha, [\Delta^\beta, \hat{P}]\} - [\Delta^\beta, \{\Pi^\alpha, \hat{S}_{\mu\nu}\}].$$

(D12)

where the constraint (3.74iv) was used in the last line. Inserting now Eqs. (D13,D14) into (D3) and using Eqs. (D7) leads to the mass shell condition for $\hat{S}_{\mu\nu}$.

2. Transport equations

In the following we want to express the transport equations with the generalization

$$[\Delta_\mu, \{\Pi^\mu, \hat{X}\}]$$

(D15)

of the Lorentz covariant (proper time) derivative $m \frac{d}{d\tau} \hat{X} = p_\mu \partial^\mu \hat{X}$ ($\tau$: proper time, $x_0$: global time). For $\hat{F}$ Eq. (D4) is already in the desired form. The transport
equation for $\hat{P}$ follows by acting with $\Delta^\mu$ on Eq. (3.73v). For $\hat{S}_{\mu\nu}$ we use the following identity

$$\{\Pi_\mu, [\Delta^\alpha, \hat{S}_{\alpha\nu}]\} + \{\Pi_\nu, [\Delta^\alpha, \hat{S}_{\mu\alpha}]\} = \{\Pi_\mu, [\Delta^\alpha, \hat{S}_{\alpha\nu}]\} - [\Delta^\alpha, \{\Pi_\mu, \hat{S}_{\alpha\nu}\}]$$

$$\{\Pi_\nu, [\Delta^\alpha, \hat{S}_{\mu\alpha}]\} - [\Delta^\alpha, \{\Pi_\nu, \hat{S}_{\mu\alpha}\}] = \{\Pi_\mu, [\Delta^\alpha, \hat{S}_{\alpha\nu}]\} - [\Delta^\alpha, \{\Pi_\mu, \hat{S}_{\alpha\nu}\}] - [\Delta^\alpha, \{\Pi_\nu, \hat{S}_{\mu\alpha}\}]$$

(D16)

3. Equations for the vector and axial vector densities

Alternatively one can derive transport equations, mass shell conditions, and constraints for the eight vector and axial vector components $\hat{V}_\mu$ as independent functions and obtain the eight other components $\hat{F}_\mu$, $\hat{P}_\mu$ and $\hat{S}_{\mu\nu}$ as dependent functions. The results are:

(a) transport equations:

$$[\Delta^\nu, \{\Pi_\nu, \hat{A}_\mu\}] = \{[\Delta_\nu, p_\mu + \Pi_\mu], \hat{A}_\nu\} - \varepsilon_{\alpha\beta\mu\nu}[\Delta^\alpha \Delta^\beta, \hat{V}_\nu]$$

(D17)

$$[\Delta^\nu, \{\Pi_\nu, \hat{V}_\mu\}] = \{[\Delta_\nu, p_\mu + \Pi_\mu], \hat{V}_\nu\} - \varepsilon_{\alpha\beta\mu\nu}[\Delta^\alpha \Delta^\beta, \hat{A}_\nu]$$

(D18)

(b) mass shell equations:

$$M^2 \hat{A}_\mu = \{[p_\mu + \Pi_\mu, p_\nu + \Pi_\nu], \hat{A}_\nu\} - [[\Delta_\mu, \Delta_\nu], \hat{A}_\nu]$$

$$+ \varepsilon_{\alpha\beta\mu\nu}\{[p_\alpha + \Pi_\alpha, \Delta^\beta], \hat{V}_\nu\}$$

(D19)

$$M^2 \hat{V}_\mu = \{[p_\mu + \Pi_\mu, p_\nu + \Pi_\nu], \hat{V}_\nu\} - [[\Delta_\mu, \Delta_\nu], \hat{V}_\nu]$$

$$+ \varepsilon_{\alpha\beta\mu\nu}\{[p_\alpha + \Pi_\alpha, \Delta^\beta], \hat{A}_\nu\}$$

(D20)

(c) constraints:

$$0 = [\Delta_\mu, \hat{\nu}_\mu]$$

(D21)
\[ 0 = \{\Pi_\mu, \hat{A}^\mu\} \] 
\[ 0 = \{\Pi_\mu, \hat{V}_\nu\} - \{\Pi_\nu, \hat{V}_\mu\} - \varepsilon_{\alpha\beta\mu\nu}[\Delta^\alpha, \hat{A}^\beta] \]

(d) equations defining the dependent functions:

\[ 2m\hat{F} = \{\Pi_\mu, \hat{V}_\mu\} \]  
\[ 2m\hat{P} = [\Delta_\mu, \hat{A}^\mu] \]  
\[ 2m\hat{S}_{\mu\nu} = [\Delta_\mu, \hat{V}_\nu] - [\Delta_\nu, \hat{V}_\mu] + \varepsilon_{\alpha\beta\mu\nu}\{\Pi_\alpha, \hat{A}^\beta\}. \]

**APPENDIX E: VACUUM SOLUTIONS**

1. **Massive quarks**

In the field-free limit the spinor equations can be solved exactly. Setting \( \hat{F}_{\mu\nu}(x) \equiv 0 \) and choosing a gauge in which then also \( \hat{A}_\mu(x) \equiv 0 \), the spinor equations reduce to

\[ 0 = p_\mu \partial^\mu \hat{F} \]  
\[ 0 = p_\mu \partial^\mu \hat{P} \]  
\[ 0 = p_\mu \partial^\mu \hat{S}_{\alpha\beta} \]  
\[ 0 = [4(E^2 - p_0^2) + \Box] \hat{F} \]  
\[ 0 = [4(E^2 - p_0^2) + \Box] \hat{P} \]  
\[ 0 = [4(E^2 - p_0^2) + \Box] \hat{S}_{\mu\nu} \]  
\[ 0 = \frac{1}{2} \partial_\mu \hat{F} - p_\nu \hat{S}_{\mu\nu} \]  
\[ 0 = p_\mu \hat{P} + \frac{1}{4} \varepsilon_{\alpha\beta\mu\nu} \partial^\nu \hat{S}^{\alpha\beta} \]
Here $E^2 = p^2 + m^2$, and the operator $\Box$ is defined through $\Box = \partial_\mu \partial^\mu$.

One can see easily that the transport equations (E1i,ii,iii) are already contained in the constraints (E1vii,viii): For $\hat{F}$ and $\hat{P}$ this follows simply by contraction with $p_\mu$ and $\partial_\mu$ respectively and exploiting the anti-symmetry of $\hat{S}_{\mu\nu} = -\hat{S}_{\nu\mu}$. Eq. (E1iii) follows from Eq. (E1viii) by contraction with $\varepsilon^{\alpha\beta\mu\nu'} p_{\nu'}$.

Spinor components which vanish at $t = -\infty$ remain zero for all times: Formal integration of the transport equation $p_\mu \partial^\mu \hat{X}(x,p) = 0$ for a spinor component $\hat{X}$ leads to

$$p_0 \hat{X}(t + \Delta t) = p_0 \hat{X}(t) - p_i \partial^i \hat{X}(t) \Delta t.$$  

(E2)

With the initial condition $\hat{X}(t \to -\infty) = 0$ we find $\hat{X}(t) \equiv 0$ unless $p_0 = 0$. But $p_0 = 0$ inserted into $[4(E^2 - p_0^2) + \Box] \hat{X} = 0$ gives after formal integration

$$\partial_0 \hat{X}(t + \Delta t) = \partial_0 \hat{X}(t) + (\nabla^2 - 4E^2) \Delta t \hat{X}(t),$$  

(E3)

so that with the initial condition $\hat{X}(t \to -\infty) = 0$ we have

$$\partial_0 \hat{X}(t + \Delta t) = \partial_0 \hat{X}(t).$$  

(E4)

Another formal integration then again yields $\hat{X}(t) \equiv 0$. Imposing for the spin and pseudoscalar densities vacuum initial conditions at $t = -\infty$, $\hat{S}_{\mu\nu}(t \to -\infty) = \hat{P}(t \to -\infty) = 0$, we see that they remain zero for all times, and with them the axial vector density $\hat{A}_\mu$ (see Eq. (E1x)).

The scalar density $\hat{F}$ evolves according to
\[ \partial_\mu \hat{F}(x,p) = 0 \quad \text{(E5)} \]

\[ [\Box + 4(E^2 - p_0^2)] \hat{F}(x,p) = 0 \quad \text{(E6)} \]

From the first equation we get \( \hat{F}(x,p) = \hat{F}(p) \) while the second gives \( (E^2 - p_0^2) \hat{F}(p) = 0 \). The solution is stationary and homogeneous and can be written as a sum of particle and antiparticle contributions,

\[ \hat{F}(x,p) = \frac{1}{2E} [\delta(p_0 - E) + \delta(p_0 + E)] \hat{F}(p), \quad \text{(E7)} \]

with an arbitrary momentum spectrum \( \hat{F}(p) \). The vector density is obtained through Eq. (E7) as

\[ \hat{V}_\mu(x,p) = \frac{p_\mu}{m} \hat{F}(x,p) = \frac{p_\mu}{2mE} [\delta(E - p_0) + \delta(E + p_0)] \hat{F}(p). \quad \text{(E8)} \]

### 2. Massless quarks

In the chiral limit the equations now read

\[ 0 = p_\mu \hat{a}^{\mu(\pm)} \quad \text{(E9a)} \]

\[ 0 = \partial_\mu \hat{a}^{\mu(\pm)} \quad \text{(E9b)} \]

\[ 0 = \partial_\mu \hat{a}^{\nu(\pm)} - \partial_\nu \hat{a}^{\mu(\pm)} \pm 2 \varepsilon_{\alpha\beta\mu\nu} p^\alpha \hat{a}^{\beta(\pm)} \quad \text{(E9c)} \]

for \( \hat{a}^{\mu(\pm)} \) and

\[ 0 = 2p_\mu \hat{f}^{(\pm)} - \partial^\nu \hat{s}^{(\pm)}_{\nu\mu} \quad \text{(E10a)} \]

\[ 0 = \partial_\mu \hat{f}^{(\pm)} + 2p^\nu \hat{s}^{(\pm)}_{\nu\mu} \quad \text{(E10b)} \]

for \( \hat{f}^{(\pm)}, \hat{s}^{(\pm)}_{\mu\nu} \).
With the help of the – linearly dependent – “mass shell” equations we know that the solutions have to fulfill

\[(4p^2 - \Box)\hat{a}_\mu^{(\pm)} = 0\]  \hspace{1cm} (E11)
\[(4p^2 - \Box)\hat{f}^{(\pm)} = 0\]  \hspace{1cm} (E12)
\[(4p^2 - \Box)\hat{s}_{\mu\nu}^{(\pm)} = 0.\]  \hspace{1cm} (E13)

The solutions for the \(x\) dependence would therefore have the form \(\exp(\pm p_\mu x^\mu)\), which does not allow for reasonable boundary conditions at \(x_\mu \to \pm \infty\). Therefore we use the following Ansatz

\[\hat{f}^{(\pm)}(x,p) = \hat{f}^{(\pm)}(p)\delta(p^2)\]  \hspace{1cm} (E14)
\[\hat{a}_\mu^{(\pm)}(x,p) = \hat{a}_\mu^{(\pm)}(p)\delta(p^2)\]  \hspace{1cm} (E15)
\[\hat{s}_{\mu\nu}^{(\pm)}(x,p) = \hat{s}_{\mu\nu}^{(\pm)}(p)\delta(p^2).\]  \hspace{1cm} (E16)

Using Eqs. \((E9a, E10a)\) we find the additional restriction

\[\hat{f}^{(\pm)}(x,p) = \hat{f}^{(\pm)}_0 \delta(p_\mu), \quad \hat{f}^{(\pm)}_0 = \text{const.} \]  \hspace{1cm} (E17)
\[\hat{a}_\mu^{(\pm)}(x,p) = p_\mu \hat{a}_\mu^{(\pm)}(p)\delta(p^2)\]
\[= \frac{1}{2}\hat{a}_0^{(\pm)}(p)\left(\begin{array}{c} \delta(p_0 - |p|) - \delta(p_0 + |p|) \\ -\hat{e}_p\delta(p_0 - |p|) - \hat{e}_p\delta(p_0 + |p|) \end{array}\right), \quad \hat{e}_p = \frac{p}{|p|}. \]  \hspace{1cm} (E18)

Assuming now that the spin density vanishes for \(x_0 \to -\infty\) we find that \(\hat{s}_{\mu\nu}^{(\pm)}\) has to vanish identically since \(\hat{s}_{\mu\nu}^{(\pm)}\) is independent of \(x_\mu\). The contribution from \(\hat{f}^{(\pm)}\) is restricted to \(p_\mu = 0\) and constant in space and therefore irrelevant. Thus we are left with \(\hat{a}_\mu^{(\pm)}\), giving contributions to vector and axial vector components as follows

\[\hat{V}_\mu(x,p) = \frac{1}{4}(\hat{a}_0^{(+)}(p) + \hat{a}_0^{(-)}(p))\left(\begin{array}{c} \delta(p_0 - |p|) - \delta(p_0 + |p|) \\ -\hat{e}_p\delta(p_0 - |p|) - \hat{e}_p\delta(p_0 + |p|) \end{array}\right) \]  \hspace{1cm} (E19)
\[\hat{A}_\mu(x,p) = \frac{1}{4}(\hat{a}_0^{(+)}(p) - \hat{a}_0^{(-)}(p))\left(\begin{array}{c} \delta(p_0 - |p|) - \delta(p_0 + |p|) \\ -\hat{e}_p\delta(p_0 - |p|) - \hat{e}_p\delta(p_0 + |p|) \end{array}\right) \]  \hspace{1cm} (E20)
which have no longer a space and time dependence.

**APPENDIX F: MOMENT EQUATIONS IN SPINOR DECOMPOSITION**

The corresponding moment equations to Eqs. (4.3,4.4) in spinor decomposition are for the constraints (Eq. (4.3)), which couple the \( n + 1 \)th to all the lower moments,

\[
2 \left[ n + 1 \right] \hat{V}^0 = 2m \left[ n \right] \hat{F} - \left\{ N(0), [n] \hat{V}^0 \right\} - \left\{ \Pi_i, [n] \hat{V}_i \right\} - \\
\sum_{k=1}^{n} \binom{n}{k} [N(k)_{\mu}, [n-k] \hat{\mu}]_{k+1}
\]

\[
2 \left[ n + 1 \right] \hat{A}^k = -2m \left[ n \right] \hat{L}^{0k} + \varepsilon^{0ijk} [\Delta_i, [n] \hat{V}_j] - \left\{ N(0), [n] \hat{A}^k \right\} + \left\{ \Pi^k, [n] \hat{A}^0 \right\} + \\
\sum_{k=1}^{n} \binom{n}{k} \left[ \varepsilon^{0ijk} [M(k)_{ij}, [n-k] \hat{\phi}]_{k} - [N^0_{(k)}, [n-k] \hat{A}^k]_{k+1} + [N(k), [n-k] \hat{A}^0]_{k+1} \right]
\]

\[
2 \left[ n + 1 \right] \hat{F} = 2m \left[ n \right] \hat{V}_0 - \left\{ N(0), [n] \hat{F} \right\} - [\Delta^i, [n] \hat{S}_{0i}] - \\
\sum_{k=1}^{n} \binom{n}{k} \left[ \left( N(0)_{[k]}, [n-k] \hat{F}_i \right)_{k+1} + [M^i_{(k)}, [n-k] \hat{S}_{0i}]_k \right]
\]

\[
2 \left[ n + 1 \right] \hat{S}^{ij} = \varepsilon^{0ijk} \left( 2m \left[ n \right] \hat{A}_i + [\Delta_2, [n] \hat{\phi}] \right) - \left\{ N(0)_{[0]}, [n] \hat{S}^{ij} \right\} \\
\sum_{k=1}^{n} \binom{n}{k} \left[ \left[ \varepsilon^{0ijk} \left[ M(k)_{ij}, [n-k] \hat{\phi} \right]_k - [N^0_{(k)}, [n-k] \hat{S}^{jk}]_{k+1} \right] \\
\left[ - [N^0_{(k)}, [n-k] \hat{S}^{0j}]_{k+1} + [N^0_{(k)}, [n-k] \hat{S}^{0k}]_{k+1} \right] \right]
\]

\[
2 \left[ n + 1 \right] \hat{A}^0 = - \left\{ N(0), [n] \hat{A}^0 \right\} - \left\{ \Pi_i, [n] \hat{A}^i \right\} - \\
\sum_{k=1}^{n} \binom{n}{k} \left[ N(k)_{\mu}, [n-k] \hat{\mu} \right]_{k+1}
\]

\[
2 \left[ n + 1 \right] \hat{V}_i = - \left\{ N(0), [n] \hat{V}_i \right\} + \left\{ \Pi_i, [n] \hat{V}_0 \right\} + \varepsilon_{0ijk} [\Delta^j, [n] \hat{A}^k] - \\
\sum_{k=1}^{n} \binom{n}{k} \left[ [N(0)_{[0]}, [n-k] \hat{V}_i]_{k+1} - [N(k)_{ij}, [n-k] \hat{V}_i]_{k+1} - \varepsilon_{0ijk} [M^j_{(k)}, [n-k] \hat{A}^k]_k \right]
\]

\[
2 \left[ n + 1 \right] \hat{S}_{0i} = - \left\{ N(0), [n] \hat{S}_{0i} \right\} - \left\{ \Pi^j, [n] \hat{S}_{ij} \right\} + [\Delta_i, [n] \hat{F}] + \\
\sum_{k=1}^{n} \binom{n}{k} \left[ [M(k)_{ij}, [n-k] \hat{F}_i]_k - [N^0_{(k)}, [n-k] \hat{S}_{iv}]_{k+1} \right]
\]
\[2^{[n+1]}\hat{P} = \{ N_{(0)0}, [n] \hat{P} \} - [\Delta^i, [n] \hat{L}_{0i}] - \sum_{k=1}^{n} \binom{n}{k} \left[ [N_{(k)0}, [n-k] \hat{P}]_{k+1} + [M^i_{(k)}, [n-k] \hat{L}_{0i}]_{k+1} \right] \]  

(F8)

and for the dynamical equations (Eq. [4.14]) containing explicit \( \partial_0 \) derivatives

\[2m^{[n]}\hat{P} = [\Delta_{\mu}, [n] \hat{A}^\mu] + \sum_{k=1}^{n} \binom{n}{k} \left[[M_{(k)\mu}, [n-k] \hat{A}^\mu]\right]_k \]  

(F9)

\[2m^{[n]} \hat{S}_{0i} = [\Delta_0, [n] \hat{V}_i] - [\Delta, [n] \hat{V}_0] + \varepsilon_{0ijk} \{ \Pi^j, [n] \hat{A}^k \} + \sum_{k=1}^{n} \binom{n}{k} \left[[M_{(k)0}, [n-k] \hat{V}_i]\right]_k - [M_{(k)i}, [n-k] \hat{V}_0]_k \]  

(F10)

\[2m^{[n]} \hat{V}_i = \{ \Pi^j, [n] \hat{F} \} + [\Delta^\nu, [n] \hat{S}_{\nu i}] + \sum_{k=1}^{n} \binom{n}{k} \left[[N_{(k)i}, [n-k] \hat{F}]\right]_{k+1} + [M^\nu_{(k)}, [n-k] \hat{S}_{\nu i}]_k \]  

(F11)

\[2m^{[n]} \hat{A}_0 = -[\Delta_0, [n] \hat{P}] + \{ \Pi^i, [n] \hat{L}_{0i} \} - \sum_{k=1}^{n} \binom{n}{k} \left[[M_{(k)0}, [n-k] \hat{P}]\right]_k - [N^i_{(k)}, [n-k] \hat{L}_{0i}]_{k+1} \]  

(F12)

\[0 = [\Delta_{\mu}, [n] \hat{V}^\mu] + \sum_{k=1}^{n} \binom{n}{k} \left[[M_{(k)\mu}, [n-k] \hat{V}^\mu]\right]_k \]  

(F13)

\[0 = \{ \Pi^i, [n] \hat{V}_i \} - \{ \Pi^j, [n] \hat{V}_j \} - \varepsilon_{0ijk} ([\Delta^0, [n] \hat{A}^k] - [\Delta^k, [n] \hat{A}^0]) + \sum_{k=1}^{n} \binom{n}{k} \left[[N_{(k)i}, [n-k] \hat{V}_j]\right]_{k+1} - [N_{(k)j}, [n-k] \hat{V}_i]_{k+1} \]  

\[-\varepsilon_{0ijk} ([M^0_{(k)}, [n-k] \hat{A}^k] - [M^k_{(k)}, [n-k] \hat{A}^0]) \]  

(F14)

\[0 = [\Delta_0, [n] \hat{F}] - \{ \Pi^i, [n] \hat{S}_{0i} \} + \sum_{k=1}^{n} \binom{n}{k} \left[[M_{(k)0}, [n-k] \hat{F}]\right]_k - [N^i_{(k)}, [n-k] \hat{S}_{0i}]_{k+1} \]  

(F15)

\[0 = \{ \Pi^i, [n] \hat{P} \} - [\Delta^0, [n] \hat{L}_{0i}] + [\Delta^j, [n] \hat{L}_{ij}] + \sum_{k=1}^{n} \binom{n}{k} \left[[N_{(k)i}, [n-k] \hat{P}]\right]_{k+1} - [M^0_{(k)}, [n-k] \hat{L}_{0i}]_k + [M^j_{(k)}, [n-k] \hat{L}_{ij}]_k \]  

(F16)

We used again the notations Eq. (4.3) for the non-local operators \( M_{k\mu} \) and \( N_{k\mu} \).
APPENDIX G: THE COMMUTATOR EQ. (4.51)

We show that for classical external fields the commutator Eq. (4.51) vanishes identically. Written out explicitly it reads

\[ 0 = \left[ 2p_\mu + iD_\mu(x) - ig \int_{-1/2}^{0} ds(1 - 2s) F_{\alpha\mu}(x + is\partial_p)\partial^\alpha_p, \right. \]
\[ \left. 2p_\nu - iD_\nu(x) + ig \int_{-1/2}^{0} ds(1 + 2s) F_{\alpha\mu}(x + is\partial_p)\partial^\alpha_p \right]. \tag{G1} \]

To evaluate the commutator we first perform a Taylor expansion for the Schwinger string

\[ F_{\mu\nu}(x + is\partial_p) = \sum_{n=0}^{\infty} \frac{(is)^n}{n!} \left[ (\partial_p\cdot D_{\alpha})^n, F_{\mu\nu}(x) \right] \]
\[ = \sum_{n=0}^{\infty} \frac{(is)^n}{n!} (\partial_p\cdot D_{\alpha})^n F_{\mu\nu}(x). \tag{G2} \]

with the definitions Eq. (3.36,3.38). We will then need the two identities

\[ \tilde{D}_\mu \tilde{D}_\nu X = (\partial_p \cdot \tilde{D})^n F_{\beta\nu} \]
\[ = -ig \sum_{m=0}^{n-1} \binom{n}{m+1} \left[ (\partial_p \cdot \tilde{D})^m F_{\mu\alpha}, (\partial_p \cdot \tilde{D})^{n-m-1} F_{\beta\nu} \right] \partial_p^\alpha \]
\[ \left[ p_\mu, (\partial_p \cdot \tilde{D})^n F_{\beta\nu} \right] = -n(\partial_p \cdot \tilde{D})^{n-1} \tilde{D}_\mu F_{\beta\nu} \]
\[ + ig \sum_{m=0}^{n-1} m \binom{n}{m+1} \left[ (\partial_p \cdot \tilde{D})^{m-1} F_{\mu\alpha}, (\partial_p \cdot \tilde{D})^{n-m-1} F_{\beta\nu} \right] \partial_p^\alpha. \tag{G4} \]

The proof goes by induction as follows:

The validity for \( n = 0 \) is obvious. To make in Eq. (G3) the step from \( n \) to \( n + 1 \) we need the relations

\[ \tilde{D}_\mu \tilde{D}_\nu X - \tilde{D}_\nu \tilde{D}_\mu X = [[D_\mu, D_\nu], X] = -ig[F_{\mu\nu}, X] \tag{G5} \]
\[ (\partial_p \cdot \tilde{D})[A(x), B(x)] = [(\partial_p \cdot \tilde{D})A(x), B(x)] + [A(x), (\partial_p \cdot \tilde{D})B(x)] \tag{G6} \]

Thus we get
\[ \tilde{D}_\mu (\partial_p \cdot \tilde{D})^{n+1} F_{\beta\nu} = \]

\[ = -ig [F_{\mu\alpha}, (\partial_p \cdot \tilde{D})^n F_{\beta\nu}] \partial_\alpha^p + (\partial_p \cdot \tilde{D}) \tilde{D}_\mu (\partial_p \cdot \tilde{D})^n F_{\beta\nu} \]

\[ = -ig [F_{\mu\alpha}, (\partial_p \cdot \tilde{D})^n F_{\beta\nu}] \partial_\alpha^p + (\partial_p \cdot \tilde{D})^{n+1} \tilde{D}_\mu F_{\beta\nu} \]

\[ \quad - ig \sum_{m=0}^{n-1} \left( \frac{n}{m+1} \right) [(\partial_p \cdot \tilde{D})^{m+1} F_{\mu\alpha}, (\partial_p \cdot \tilde{D})^{n-m-1} F_{\beta\nu}] \partial_\alpha^p \]

\[ = -ig [F_{\mu\alpha}, (\partial_p \cdot \tilde{D})^n F_{\beta\nu}] \partial_\alpha^p + (\partial_p \cdot \tilde{D})^{n+1} \tilde{D}_\mu F_{\beta\nu} \]

\[ = -ig \sum_{m=1}^{n} \left( \frac{n}{m+1} \right) [(\partial_p \cdot \tilde{D})^m F_{\mu\alpha}, (\partial_p \cdot \tilde{D})^{n-m} F_{\beta\nu}] \partial_\alpha^p \]

\[ = -ig \sum_{m=1}^{n} \left[ \left( \frac{n}{m+1} \right) + \left( \frac{n}{m+1} \right) \right] [(\partial_p \cdot \tilde{D})^m F_{\mu\alpha}, (\partial_p \cdot \tilde{D})^{n-m} F_{\beta\nu}] \]

\[ = (\partial_p \cdot \tilde{D})^{n+1} \tilde{D}_\mu F_{\beta\nu} - ig \sum_{m=0}^{n} \left( \frac{n+1}{m+1} \right) [(\partial_p \cdot \tilde{D})^m F_{\mu\alpha}, (\partial_p \cdot \tilde{D})^{n-m} F_{\beta\nu}] \partial_\alpha^p. \quad (G7) \]

This proves Eq. (G3). For Eq. (G4) we find for \( n = 1 \)

\[ [p_\mu, \partial_\alpha^p \tilde{D}_\alpha F_{\beta\nu}] = -g_\mu^{\alpha} \tilde{D}_\alpha F_{\beta\nu} = -\tilde{D}_\mu F_{\beta\nu}, \quad (G8) \]

i.e. the equation is correct. The step \( n \to n + 1 \) is given by

\[ [p_\mu, (\partial_p \cdot \tilde{D})^{n+1} F_{\beta\nu}] = \]

\[ = [p_\mu, (\partial_p \cdot \tilde{D}) (\partial_p \cdot \tilde{D})^n F_{\beta\nu} + (\partial_p \cdot \tilde{D}) [p_\mu, (\partial_p \cdot \tilde{D})^n F_{\beta\nu}] \]
Eqs. (4.33) \[ -\bar{D}_\mu (\partial_p \cdot \vec{D})^n F_{\beta\nu} - n(\partial_p \cdot \vec{D})^n \bar{D}_\mu F_{\beta\nu} \]

\[ + ig \sum_{m=0}^{n-1} m \left( \begin{array}{c} n \\ m+1 \end{array} \right) [(\partial_p \cdot \vec{D})^m F_{\mu\alpha}, (\partial_p \cdot \vec{D})^{n-m-1} F_{\beta\nu}] \partial_p^\alpha \]

\[ + ig \sum_{m=0}^{n-1} m \left( \begin{array}{c} n \\ m+1 \end{array} \right) [(\partial_p \cdot \vec{D})^m F_{\mu\alpha}, (\partial_p \cdot \vec{D})^{n-m} F_{\beta\nu}] \partial_p^\alpha \]

Eq. (4.34) \[ -(n+1)(\partial_p \cdot \vec{D})^n \bar{D}_\mu F_{\beta\nu} \]

\[ + ig \sum_{m=0}^{n-1} \left( \begin{array}{c} n \\ m \end{array} \right) [(\partial_p \cdot \vec{D})^{m-1} F_{\mu\alpha}, (\partial_p \cdot \vec{D})^{n-m} F_{\beta\nu}] \partial_p^\alpha \]

\[ + ig \sum_{m=1}^{n} (m-1) \left( \begin{array}{c} n \\ m \end{array} \right) [(\partial_p \cdot \vec{D})^{m-1} F_{\mu\alpha}, (\partial_p \cdot \vec{D})^{n-m} F_{\beta\nu}] \partial_p^\alpha \]

\[ + ig \sum_{m=0}^{n-1} m \left( \begin{array}{c} n \\ m+1 \end{array} \right) [(\partial_p \cdot \vec{D})^{m-1} F_{\mu\alpha}, (\partial_p \cdot \vec{D})^{n-m} F_{\beta\nu}] \partial_p^\alpha \]

\[ = -(n+1)(\partial_p \cdot \vec{D})^n \bar{D}_\mu F_{\beta\nu} \]

\[ + ig \sum_{m=1}^{n-1} [(\partial_p \cdot \vec{D})^{m-1} F_{\mu\alpha}, F_{\beta\nu}] \partial_p^\alpha \]

\[ + ig \sum_{n=1}^{n-1} \left[ (\partial_p \cdot \vec{D})^{m-1} F_{\mu\alpha}, (\partial_p \cdot \vec{D})^{n-m} F_{\beta\nu} \right] \partial_p^\alpha \]

\[ = -n(\partial_p \cdot \vec{D})^n \bar{D}_\mu F_{\beta\nu} \]

\[ + ig \sum_{m=0}^{n-1} \left( \begin{array}{c} n+1 \\ m+1 \end{array} \right) [(\partial_p \cdot \vec{D})^{m-1} F_{\mu\alpha}, (\partial_p \cdot \vec{D})^{n-m} F_{\beta\nu}] \partial_p^\alpha \quad \square. \]

For the Schwinger string we get with the definitions of Eq. (G2) the following commutator relations
\[ \tilde{D}_\mu [x] F_{\beta \nu} (x + i s \partial_\mu) \partial_\beta^2 = \]

\[ \sum_{n=0}^{\infty} \frac{(is)^n}{n!} (\partial_\mu \cdot \tilde{D})^n \tilde{D}_\mu F_{\beta \nu} (x) \partial_\beta^3 \]

\[ + g \sum_{n=0}^{\infty} \frac{i^n s^{n+1}}{n!} \sum_{m=0}^{n} \frac{1}{m+1} \binom{n}{m} [ (\partial_\mu \cdot \tilde{D})^m F_{\mu \alpha} , (\partial_\mu \cdot \tilde{D})^{n-m} F_{\beta \nu}] \partial_\alpha^2 \partial_\beta^2 \] (G10)

and

\[ [p_\mu , [x] F_{\beta \nu} (x + i s \partial_\mu) \partial_\beta^2] = \]

\[ - F_{\mu \nu} (x) - i \sum_{n=0}^{\infty} \frac{i^n s^{n+1}}{n!} (\partial_\mu \cdot \tilde{D})^n \left( \frac{1}{n+1} \tilde{D}_\alpha F_{\mu \nu} (x) + \tilde{D}_\mu F_{\alpha \nu} (x) \right) \partial_\alpha^2 \]

\[ - ig \sum_{n=0}^{\infty} \frac{i^n s^{n+2}}{n!} \sum_{m=0}^{n} \frac{1}{m+2} \binom{n}{m} [ (\partial_\mu \cdot \tilde{D})^m F_{\mu \alpha} , (\partial_\mu \cdot \tilde{D})^{n-m} F_{\beta \nu}] \partial_\alpha^2 \partial_\beta^2. \] (G11)

In addition we find for the commutator of the Schwinger string

\[ [x] F_{\alpha \mu} (x + i s \partial_\mu) \partial_\alpha^\nu , [x] F_{\beta \nu} (x + i t \partial_\nu) \partial_\beta^2] = \]

\[ - \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{m=0}^{n} \binom{n}{m} s^m t^{n-m} [ (\partial_\mu \cdot \tilde{D})^m F_{\mu \alpha} , (\partial_\mu \cdot \tilde{D})^{n-m} F_{\beta \nu}] \partial_\alpha^2 \partial_\beta^2. \] (G12)

Plugging in Eqs. (G10, G11, G12) into the commutator Eq. (G1) and performing the \( s \) integration we get the following equation:

\[ [p_\mu + \Pi_\mu + i \Delta_\mu , p_\nu + \Pi_\nu - i \Delta_\nu] = \]

\[ -2ig (F_{\mu \nu} (x) + F_{\nu \mu} (x)) \]

\[ + g \sum_{n=0}^{\infty} \frac{(-i/2)^n}{n!} (\partial_\mu \cdot \tilde{D})^n \frac{1}{(n+1)(n+3)} \left[ \tilde{D}_\alpha F_{\mu \nu} (x) + \tilde{D}_\mu F_{\nu \alpha} (x) + \tilde{D}_\nu F_{\alpha \mu} (x) \right] \partial_\alpha^\nu \]

\[ + g^2 \sum_{n=0}^{\infty} \frac{(-i/2)^n}{4n!} \sum_{m=0}^{n} \binom{n}{m} f(n, m) [(\partial_\mu \cdot \tilde{D})^m F_{\mu \alpha} , (\partial_\mu \cdot \tilde{D})^{n-m} F_{\beta \nu}] \partial_\alpha^2 \partial_\beta^2, \] (G13)

with

\[ f(n, m) = \frac{1}{(n+2)(n+3)} \left( \frac{1}{m+1} + \frac{2n+5}{n-m+1} \right) \]

\[ + \frac{1}{(n+3)(n+4)} \left( \frac{1}{m+2} - \frac{2n+7}{n-m+2} \right) \]

\[ \frac{2m+3}{(m+1)(m+2)(n-m+1)(n-m+2)}. \] (G14)
The first and second line on the r.h.s of Eq. (G13) vanish due to the antisymmetry of the field strength tensor and the Jacobi identity (3.18). In the third line one shows easily that for each combination of the indices $n \geq 0, 0 \leq m \leq n$ the function $f(n, m)$ vanishes identically. Thus the commutator (4.51) vanishes.

APPENDIX H: QED MOMENT EQUATIONS IN THE CLASSICAL LIMIT

To derive the transport equations for the moments in leading order in the gradient expansion we need the moment equations to order $\bar{\hbar}$. These are with coupling constant $g = -e$ inserted:

$$2^{[n+1]}\mathcal{V}^{0(1)} = 2m^{[n]} F^{(1)} - 2p_i^{[n]} \mathcal{V}^i(1)$$

$$2^{[n+1]} \mathcal{A}^{k(1)} = -2m^{[n]} \mathcal{L}^{0k(1)} + 2p^k[n] \mathcal{A}^{0(1)}$$

$$+ \varepsilon^{0ijk} \left( (\partial_i - \frac{e}{c} \partial_p F_{ij}(x))^{[n]} \mathcal{V}^j(0) + \frac{ne}{c} F_{0i}(x)^{[n-1]} \mathcal{V}^j(0) \right)$$

$$2^{[n+1]} F^{(1)} = 2m^{[n]} \mathcal{V}^{0(1)} - (\partial_i - \frac{e}{c} \partial_p F_{ij}(x))^{[n]} S^{0i(0)} - \frac{ne}{c} F_{i0}(x)^{[n-1]} S^{0i(0)}$$

$$2^{[n+1]} S^{jk(1)} = 2m \varepsilon^{0ijk}[n] \mathcal{A}_i^{(1)} - 2p^k[n] S^{0j(1)} + 2p^j[n] S^{0k(1)}$$

$$2^{[n+1]} \mathcal{A}^{0(1)} = -2p_i^{[n]} \mathcal{A}^i(1)$$

$$2^{[n+1]} \mathcal{V}^i(1) = 2p^j[n] \mathcal{V}^0(1)$$

$$+ \varepsilon^{0ijk} \left( (\partial_j - \frac{e}{c} \partial_p F_{ij}(x))^{[n]} \mathcal{A}_i^{(0)} + \frac{ne}{c} F_{j0}(x)^{[n-1]} \mathcal{A}_i^{(0)} \right)$$

$$2^{[n+1]} \mathcal{A}_i^{(1)} = -2p_j^{[n]} \mathcal{S}^{ij(1)}$$

$$+ (\partial_i - \frac{e}{c} \partial_p F_{ij}(x))^{[n]} \mathcal{F}^{(0)} + \frac{ne}{c} F_{i0}(x)^{[n-1]} \mathcal{F}^{(0)}$$

$$2^{[n+1]} \mathcal{P}^{(1)} = -(\partial_i - \frac{e}{c} \partial_p F_{ij}(x))^{[n]} \mathcal{L}^{0i(0)} - \frac{ne}{c} F_{i0}(x)^{[n-1]} \mathcal{L}^{0i(0)}$$

and
2m^{[n]}\mathcal{P}^{(1)}=(\partial_\mu - \frac{e}{c}\partial_\mu F_{\nu\lambda}(x))^{[n]}A^{\mu(0)} + \frac{ne}{c}F_{\mu0}(x)^{[n-1]}A^{\mu(0)} \\
(9)

2m^{[n]}\mathcal{S}_\nu^{0(1)}=2\varepsilon^{0ijk}p_j^{[n]}\mathcal{A}_k^{(1)} + (\partial^\nu - \frac{e}{c}\partial_{\nu j}F_{0j}(x))^{[n]}\mathcal{V}_\nu^{(0)} \\
(10)

-\left(\partial^\nu - \frac{e}{c}\partial_{\nu j}F_{ij}(x)\right)^{[n]}\mathcal{V}_\nu^{(0)} - \frac{ne}{c}F_{\nu0}(x)^{[n-1]}\mathcal{V}_\nu^{(0)} \\
(11)

2m^{[n]}\mathcal{V}_\nu^{(1)}=2p_\nu^{[n]}\mathcal{F}^{(1)} + (\partial_\nu - \frac{e}{c}\partial_\nu F_{\nu j}(x))^{[n]}\mathcal{S}_{\nu j}^{0(0)} + \frac{ne}{c}F_{\nu0}(x)^{[n-1]}\mathcal{S}_{\nu j}^{0(0)} \\
(12)

2m^{[n]}\mathcal{A}^{0(1)}=2p_i^{[n]}\mathcal{L}^{0(i)} \\
(13)

0=-(\partial_\mu - \frac{e}{c}\partial_\mu F_{\nu j}(x))^{[n]}\mathcal{V}_\nu^{(0)} + \frac{ne}{c}F_{\mu0}(x)^{[n-1]}\mathcal{V}_\nu^{(0)} \\
(14)

0=2p_\nu^{[n]}\mathcal{V}_\nu^{(1)} - 2p_\nu^{[n]}\mathcal{V}_\nu^{(1)} - \varepsilon^{0ijk}(\partial_0 - \frac{e}{c}\partial_0 F_{0j}(x))^{[n]}\mathcal{A}_k^{(0)} \\
+ \varepsilon^{0ijk}\left(\partial_k - \frac{e}{c}\partial_k F_{0j}(x)\right)^{[n]}\mathcal{A}_k^{(0)} \\
(15)

0=-2p_\nu^{[n]}\mathcal{S}_\nu^{0(1)} + (\partial_\nu - \frac{e}{c}\partial_\nu F_{0j}(x))^{[n]}\mathcal{F}^{(0)} \\
(16)

0=2p_\nu^{[n]}\mathcal{P}^{(1)} - (\partial_\nu - \frac{e}{c}\partial_\nu F_{0j}(x))^{[n]}\mathcal{L}^{0(i)} \\
+(\partial_j - \frac{e}{c}\partial_j F_{\nu j}(x))^{[n]}\mathcal{L}^{ij(0)} + \frac{ne}{c}F_{\nu0}(x)^{[n-1]}\mathcal{L}^{ij(0)}.

To find the equations for the moments to zeroth order $\hbar$ we eliminate in Eqs. (H9-H16) with $n \rightarrow n + 1$ the components of order $\hbar$ with the help of Eqs. (H1-H3).

Inserting Eqs. (H2-H7) into Eq. (H11) and Eqs. (H7) into Eq. (H13) gives us (with the unchanged Eq. (H13)) the dynamical equations for the components $^{[\nu]}\mathcal{V}_\nu^{(0)}$, $^{[\nu]}\mathcal{F}^{(0)}$

0=(\partial_\mu - \frac{e}{c}\partial_\mu F_{\nu j}(x))^{[n+1]}\mathcal{V}_\nu^{(0)} + p_j^{(\partial_\mu - \frac{e}{c}\partial_\mu F_{\nu j}(x))^{[n]}\mathcal{V}_\nu^{(0)} + \frac{ne}{c}F_{\nu0}(x)^{[n-1]}\mathcal{V}_\nu^{(0)} \\
\quad \quad \quad - p_j^{(\partial_\nu - \frac{e}{c}\partial_\nu F_{\nu j}(x))^{[n]}\mathcal{V}_\nu^{(0)} - \frac{ne}{c}F_{\nu0}(x)^{[n-1]}\mathcal{V}_\nu^{(0)} \\
\quad \quad \quad - (\partial_j - \frac{e}{c}\partial_j F_{\nu j}(x))^{[n+1]}\mathcal{V}_\nu^{(0)} - \frac{(n + 1)e}{c}F_{\nu0}(x)^{[n]}\mathcal{V}_\nu^{(0)} \\
\quad \quad \quad + m(\partial_j - \frac{e}{c}\partial_j F_{\nu j}(x))^{[n]}\mathcal{F}^{(0)} + m\frac{ne}{c}F_{\nu0}(x)^{[n-1]}\mathcal{F}^{(0)} \\
\quad \quad \quad 0=(\partial_\mu - \frac{e}{c}\partial_\mu F_{\nu j}(x))^{[n]}\mathcal{V}_\nu^{(0)} + \frac{ne}{c}F_{\mu0}(x)^{[n-1]}\mathcal{V}_\nu^{(0)}
(17)
(18)
\[0 = (\partial_0 - \frac{e}{c} \partial^i p_0 F_{0j}(x))^{[n+1]} F^{(0)} + p^j (\partial_i - \frac{e}{c} \partial^j p_i F_{ij}(x))^{[n]} F^{(0)} + \frac{ne}{c} F_{i0}(x) p^{[n-1]} F^{(0)}. \tag{H19}\]

To apply now the constraints for the moments to zeroth order \( \bar{h} \)

\[\begin{align*}
[2n] \mathbf{V}^{(0)} &= E^{2n} \frac{1}{m} [1] F^{(0)} & [2n+1] \mathbf{V}^{(0)} &= E^{2n+1} \frac{E [0]}{m} F^{(0)} \\
[2n] \mathbf{V}^{(0)} &= E^{2n} \frac{p [0]}{m} F^{(0)} & [2n+1] \mathbf{V}^{(0)} &= E^{2n+1} \frac{p [1]}{Em} F^{(0)} \tag{H20}\end{align*}\]

\[\begin{align*}
[2n] \mathbf{F}^{(0)} &= E^{2n} [0] F^{(0)} & [2n+1] \mathbf{F}^{(0)} &= E^{2n+1} \frac{1}{E} [1] F^{(0)} \\
\end{align*}\]

we have to distinguish between even and odd \( n \). For \( n \) even we find with

\[\partial^i p^j = p^j \partial^i + g^{ij}, \quad \partial^i p E = E \partial^i p - \frac{p^j}{E} \tag{H21}\]

from Eq. (H18) the following transport equation for the zeroth moment \([0] F^{(0)}\)

\[0 = E^n \left[ E (\partial_0 - \frac{e}{c} \partial^j p_0 F_{0j}(x)) (E^{[0]} F^{(0)}) + p_j (\partial^j - \frac{e}{c} \partial^k p F_{jk}(x))^{[1]} F^{(0)} \right]. \tag{H22}\]

Similar we find for the odd moments with the help of \( E^{[n]} F^{(0)} \leftrightarrow [n] F^{(1)} \) the transport equation for the first moment \([1] F^{(0)}\)

\[0 = E^n \left[ E (\partial_0 - \frac{e}{c} \partial^k p F_{0j}(x))^{[1]} F^{(0)} + p_j (\partial^j - \frac{e}{c} \partial^k p F_{jk}(x)) (E^{[0]} F^{(0)}) \right]. \tag{H23}\]

Expressed with fields \( E, B \) we finally get

\[\begin{align*}
0 &= E^n \left[ E (\partial^0 + \frac{e}{c} \nabla_p E)^{[1]} F^{(0)} + (p \cdot \nabla + \frac{e}{c} (p \times B) \cdot \nabla_p) (E^{[0]} F^{(0)}) \right] \tag{H24}\\
0 &= E^n \left[ E (\partial^0 + \frac{e}{c} \nabla_p E) (E^{[0]} F^{(0)}) + (p \cdot \nabla + \frac{e}{c} (p \times B) \cdot \nabla_p)^{[1]} F^{(0)} \right]. \tag{H25}\end{align*}\]
The remaining dynamical equations Eq. (H17-H19) prove to contain no new information: Eq. (H17) can be derived from \( p_i \cdot \) Eq. (H22) or \( p_i \cdot \) Eq. (H24), Eq. (H19) leads just to Eq. (H22) and Eq. (H23) with \( n \) replaced by \( n - 1 \).

The transport equations for the other spinor components \([n]S_{\mu
u}^{(0)}\), \([n]A_{\mu}^{(0)}\) we find from Eqs. (H13-H16) for the \( n + 1 \)st moments by eliminating the components to first order \( \hbar \) with the help of Eqs. (H1-H8). The transport equations are thus

\[
0 = (\partial_0 - \frac{e}{c} \partial^i F_{0j}(x))[n+1]A^{(0)}
+ (\partial_i - \frac{e}{c} \partial^j F_{ij}(x))[n+1]A^{(0)} + \frac{(n + 1)e}{c} F_{i0}(x)[n]A^{(0)}
+ m(\partial_i - \frac{e}{c} \partial^j F_{ij}(x))[n]L^{0i(0)} + m\frac{ne}{c} F_{i0}(x)[n-1]L^{0i(0)})
\]

(H26)

\[
0 = (\partial_0 - \frac{e}{c} \partial^i F_{0j}(x))[n+1]S^{0(0)}
+ (\partial_j - \frac{e}{c} \partial^k F_{jk}(x))[n+1]S^{ij(0)} + \frac{(n + 1)e}{c} F_{j0}(x)[n]S^{ij(0)}
- p^i(\partial_j - \frac{e}{c} \partial^k F_{jk}(x))[n]S^{0j(0)} - \frac{ne}{c} p^i F_{j0}(x)[n-1]S^{0j(0)}
- m\varepsilon^{0ijk} \left[ (\partial_j - \frac{e}{c} \partial^l F_{lj}(x))[n]A^{(0)} + \frac{ne}{c} F_{j0}(x)[n-1]A^{(0)} \right]
\]

(H27)

\[
0 = (\partial_0 - \frac{e}{c} \partial^i F_{0j}(x))[n+1]A^{(0)} + p_j(\partial^i - \frac{e}{c} \partial^k F_{jk}(x))[n]A^{(0)} + \frac{ne}{c} p_j F_{i0}(x)[n-1]A^{(0)}
- (\partial^i - \frac{e}{c} \partial^j F_{jk}(x))[n]A^{(0)} - \frac{ne}{c} p_j F_{i0}(x)[n-1]A^{(0)}
- (\partial^i - \frac{e}{c} \partial^j F_{jk}(x))[n+1]A^{(0)} - \frac{(n + 1)e}{c} F_{i0}(x)[n]A^{(0)}
\]

(H28)

\[
0 = (\partial_0 - \frac{e}{c} \partial^i F_{0j}(x))[n+1]L^{0i(0)}
+ p^i(\partial_j - \frac{e}{c} \partial^k F_{jk}(x))[n]L^{0j(0)} + \frac{ne}{c} p^i F_{j0}(x)[n-1]L^{0j(0)}
- (\partial_j - \frac{e}{c} \partial^k F_{jk}(x))[n+1]L^{ij(0)} - \frac{(n + 1)e}{c} F_{j0}(x)[n]L^{ij(0)}.
\]

(H29)

Inserting the constraints for the moments to zeroth order \( \hbar \)

\[
[n]A^{(0)} = E^{2n} \frac{1}{m} \cdot [0] \sigma^{(0)}
\]

(H30)
\[ [2n] \mathbf{A}^{(0)} = E^{2n} \frac{1}{m} \left( [1] \sigma^{(0)} + p \times \left( p \times \frac{[1] \sigma^{(0)}}{E^2} \right) \right) \]  

\[ \textup{(H31)} \]

\[ [2n] \mathbf{S}^{(0)} = -E^{2n} p \times \frac{[1] \sigma^{(0)}}{E^2} \]  

\[ \textup{(H32)} \]

\[ [2n] \sigma^{(0)} = E^{2n}[0] \sigma^{(0)} \]  

\[ \textup{(H33)} \]

\[ [2n+1] \mathbf{A}^{(0)} = E^{2n+1} \frac{1}{m} \left( [0] \sigma^{(0)} \right) \]  

\[ \textup{(H34)} \]

\[ [2n+1] \mathbf{S}^{(0)} = -E^{2n+1} p \times \frac{[0] \sigma^{(0)}}{E} \]  

\[ \textup{(H35)} \]

\[ [2n+1] \sigma^{(0)} = E^{2n+1} \frac{[1] \sigma^{(0)}}{E} \]  

\[ \textup{(H36)} \]

\[ [2n+1] \sigma^{(0)} = E^{2n+1} \frac{[1] \sigma^{(0)}}{E} \]  

\[ \textup{(H37)} \]

we get for instance from Eq. (H29) for even and odd moments transport equations for the first two moments of \( \sigma^{(0)} \). Written with fields \( E \) and \( B \) these are

\[ 0 = E^{2n} \left[ E \left( \partial^0 + \frac{e}{c} \nabla_p \cdot E \right) \frac{[1] \sigma^{(0)}}{E} + \left( p \cdot \nabla + \frac{e}{c} (p \times B) \cdot \nabla_p \right) \frac{[0] \sigma^{(0)}}{E} \right] \]  

\[ + \frac{e}{c} \left( p \left( E \cdot \frac{[1] \sigma^{(0)}}{E^2} \right) + B \times \frac{[0] \sigma^{(0)}}{E} \right) \]  

\[ \textup{(H38)} \]

\[ 0 = E^{2n+1} \left[ E \left( \partial^0 + \frac{e}{c} \nabla_p \cdot E \right) \frac{[0] \sigma^{(0)}}{E} + \left( p \cdot \nabla + \frac{e}{c} (p \times B) \cdot \nabla_p \right) \frac{[1] \sigma^{(0)}}{E} \right] \]  

\[ + \frac{e}{c} \left( p \left( E \cdot \frac{[0] \sigma^{(0)}}{E} \right) + B \times \frac{[1] \sigma^{(0)}}{E} \right) \]  

\[ \textup{(H39)} \]

The other dynamical equations do again contain no new information, leading to Eqs. (H38,H39) or \( p \times \) Eqs. (H38,H39).
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TABLES

|                | \( \Pi_\mu \) | \( \Delta_\mu \) |
|----------------|---------------|------------------|
| **0\(^{\text{th}}\) order** | \( p_\mu \)     | 0                |

\[
\frac{-2\Im}{\hbar}{\cal F}_{\nu\mu}(x)\partial_\mu^\nu h
\]

TABLE I. The first two orders of the \( \hbar \)-expansion for the non-local operators \( \Pi_\mu \) and \( \Delta_\mu \).

|                | \( \mathcal{P}_\mu \) | \( \mathcal{D}_\mu \) |
|----------------|-------------------------|------------------------|
| **0\(^{\text{th}}\) order** | \( p_\mu \)         | 0                      |

\[
[D_\mu(x) - \frac{2}{c}{\cal F}_{\nu\mu}(x)\partial_\mu^\nu] h
\]

TABLE II. The first two orders for the gradient expansion for the non-local operators \( \mathcal{P}_\mu \) and \( \mathcal{D}_\mu \) for QED with external fields.