Doping the chiral spin liquid - topological superconductor or chiral metal?

Xue-Yang Song,1 Ashvin Vishwanath,1 and Ya-Hui Zhang1

1Department of Physics, Harvard University, Cambridge, MA 02138, USA

(Dated: November 20, 2020)

We point out that there are two different chiral spin liquid states on the triangular lattice and discuss the conducting states that are expected on doping them. These states labeled CSL1 and CSL2 are associated with two distinct topological orders with different edge states, although they both spontaneously break time reversal symmetry and exhibit the same quantized spin Hall conductance. While CSL1 is related to the Kalmeyer-Laughlin state, CSL2 is the ν = 4 member of Kitaev’s 16 fold way classification. Both states are described within the Abrikosov fermion representation of spins, and the effect of doping can be accessed by introducing charged holons. On doping CSL2, condensation of charged holons leads to a topological d+id superconductor. However on doping CSL1, in sharp contrast, two different scenarios can arise: first, if holons condense, a chiral metal with doubled unit cell and finite Hall conductivity is obtained. However, in a second novel scenario, the internal magnetic flux adjusts with doping and holons form a bosonic integer quantum Hall (BIQH) state. Remarkably, the latter phase is identical to a d+id superconductor. In this case the Mott insulator to superconductor transition is associated with a bosonic variant of the integer quantum Hall plateau transition for the holon. Finally we connect the above two scenarios to two recent numerical studies of doped chiral spin liquids on triangular lattice. Our work clarifies the complex relation between topological superconductors, chiral spin liquids and quantum criticality.

I. INTRODUCTION

Ever since Anderson conceived of the resonating-valence-bond (RVB) liquid as a precursor state to the superconductor in the cuprates [1], the notion of quantum spin liquids (QSLs) has been inextricably tied to mechanisms for high-temperature superconductivity. A considerable body of work has explored doping spin liquid or Mott insulators[2][21]. It should however be emphasized that not every spin liquid leads to superconductivity upon doping. In referring to QSLs, it is customary to describe them in terms of two features - the emergent gauge group, eg. $Z_2$ and U(1) spin liquids and additionally the statistics of “spinon” excitations carrying spin 1/2 and gauge charge. Here we will consider the case of fermionic spinons[22][23]. A Mott insulator to superconductor transition upon doping is quite natural when the parent Mott insulator is a $Z_2$ spin liquid[2]. Note, a $Z_2$ spin liquid can be described by a parton mean field ansatz with fermionic spinons in a BCS state. Doped charges are accounted for in terms of bosonic holon, which can condense. Holon condensation makes the pairing of spinons evolve into true Cooper pair of electron, leading to a superconducting phase. In contrast, doping a U(1) spin liquid with a spinon Fermi surface naturally gives us a Fermi liquid phase, rather than a superconductor, because there is no pairing of spinons in the parent Mott insulator[24].

This brings us to the subject of this paper, the chiral spin liquid(CSL)[25][26], and the conducting states that emerges on doping. Chiral spin liquids spontaneously break time reversal symmetry and exhibit chiral edge modes. In particular, a speculative connection between fractional statistics and two-dimensional superconducting states, so-called anyon superconductivity, has been studied in the earlier days of high Tc superconductivity[20][30]. Theoretically, chiral spin liquids have been found to be the ground state for various spin 1/2 lattice models on the kagome and triangular lattices[31][37] and also in SU(N) model with $N > 2$[38][41]. On the experimental side, recent observation of a quantized thermal Hall effect in a Kitaev material[42] suggests a non-Abelian chiral spin liquid. These examples motivate us to ask - what is the fate of a chiral spin liquid on doping? Surprisingly we find that this question is more complicated and potentially also more interesting than the usual situation of simple $Z_2$ spin liquid, where the conventional RVB theory works well.

The first complexity is that there are actually two different chiral spin liquids: CSL1 and CSL2. CSL1 is an analog of fractional quantum Hall state in spin system, as first proposed by Kalmeyer-Laughlin[25]. Exploiting the relation between spin 1/2 and hard core bosons, one can map the $ν = \frac{1}{2}$ Laughlin state of boson to a spin state, which corresponds to CSL1. This CSL has two anyons: the trivial anyon $I$ and a semion $s$ which carries spin 1/2. It also has edge states characterized by a chiral central charge $c = 1$. In contrast, CSL2 should be thought as a “gauged” chiral superconductor, whose wavefunction is obtained by Gutzwiller projection of a chiral superconductor, or more precisely a spin singlet superconductor with $d_{x^2-y^2} + id_{xy}$ superconducting pairing, sometimes abbreviated as $d+id$ pairing. The topological order depends on the angular momentum of the corresponding pairing, as classified by Kitaev’s sixteenfold way[43]. With SU(2) spin rotation symmetry, the pairing can only be in the even angular momentum channel, making the gauged $d+id$ superconductor as the simplest chiral spin liquid in the second category. In terms of topological order, gauged $d+id$ CSL has four anyons: trivial anyon , a spinless semion, a spin 1/2 semion and a spin 1/2 fermion. Also the chiral central charge is now $c = 2$. 

In the following we use CSL2 to denote specifically the
gauged d + id state. Both CSL1 and CSL2 can be con-
veniently described using mean field ansatz of Abrikosov
fermions. For CSL1, the fermionic spinons are put in a
Chern insulator phase with C = 2. In contrast, the
spinon is in a d + id superconductor ansatz in the CSL2.
These two ansatz are generically not gauge equivalent
to each other. CSL2 is very similar to a Z2 spin liquid
in terms of wavefunction: it is a Gutzwiller projection of
a d + id superconductor instead of a s wave super-
conductor. Hence the conventional RVB theory can be
obviously generalized to this case and a d + id supercon-
ductor naturally emerges from doping. A dual viewpoint
involves beginning in the d+id superconductor, and driv-
ing a superconductor-insulator transition by condensing
pairs of vortices [44]. This will lead directly to CSL2, and
the anyon excitations described above can be identified
with the remnant of vortex of the d+id superconductor
as shown below.

On the other hand, the wavefunction of CSL1 is a
Gutzwiller projection of Chern insulator. Thus a super-
conductor is not expected from the RVB picture. As far
as we know, all of the chiral spin liquids found in numer-
ics for spin rotation invariant spin 1/2 model are CSL1,
which can be confirmed by matching entanglement spec-
trum with that of SU(2) CFTs [31,32]. This strongly
suggests CSL1 as a more likely candidate relevant to re-
alistic spin 1/2 materials. Then a natural question that
arises is: what should we expect on doping this chiral
spin liquid?

The main result of this paper is to propose two differ-
ent scenarios for doping CSL1 on the triangular lattice.
In the first scenario, a simple generalization of RVB the-
ory using slave boson condensation [2] predicts a doped
Chern insulator. On triangular lattice, this is a chiral
metal with doubled unit cell and staggered loop current
order. It also breaks the C6 rotation symmetry com-
pletely. It has a non-zero electrical Hall conductivity
(\(\sigma_{xy} = -2\)) at zero doping. This first scenario is well described by
a mean field theory of electron, which basically inherits
the mean field ansatz of the spinon in the Mott insulator.
A more unconventional second possibility is a d + id su-
perconductor emerges from a completely different mech-
anism, which can not be captured by simple mean field
theory. In this exotic scenario, the doped holon forms a
bosonic state with integer quantum Hall effect (bosonic-
IQHE) [33,34]. The bosonic holons lead to an opposite
Hall conductivity (\(\sigma_{xy} = -2\)) compared to the fermionic
spinons. The final wavefunction of electron is a prod-
uct of a bosonic IQH state and a fermionic IQH state.
This construction, surprisingly, turns out to describe a
superconductor with the same topological property as the
d + id superconductor as we will show below. A sim-
ilar but distinct theory has been proposed for doping a
Dirac spin liquid on Kagome lattice [37].

Our proposed two scenarios for doping CSL1 are in
agreement with with two recent numerical studies of
doped chiral spin liquid on triangular lattice. Ref. [48]
finds a d + id superconductor from doping a CSL in
the strong Mott regime described by a \(J_1 - J_2 - J_3\)
model [39,50]. In contrast, Ref. [51] observes a chiral
metal phase upon doping a CSL in the weak Mott regime
[35]. The CSLs in the parent Mott insulators in the
above two cases are shown to be both the CSL1 because
the measured entanglement spectrum is consistent with
the SU(2) edge theory [35,40]. Lastly we note that
there are other numerical studies of possible supercon-
ductivity on triangular lattice where the parent state
is not a chiral spin liquid [32,54]. Further work will be
needed to synthesize these observations into a coherent
theory. However it is clear that there are at least two
competing phases upon doping the same CSL, which is
consistent with the framework in our paper.

II. TWO CHIRAL SPIN LIQUIDS ON
TRIANGULAR LATTICE

We adopt a parton construction to decompose the
physical electron operator as a boson carrying its electric
charge (called ‘holon’) and a fermionic spinon carrying
spin indices:

\[ c_{i,\sigma} = b_{i,\sigma}^\dagger f_{i,\sigma} \]  

(1)

where \(\sigma, i\) denote spin and site, respectively. This
formalism introduces a \(U(1)\) gauge redundancy between
the holon and spinon, with a \(U(1)\) gauge field \(a\). Hence it
leads to a gauge constraint \(n_i^h + n_i^f = 1\). Note that there is
a \(SU(2)\) version of the slave boson theory with two
slave bosons per site [2] (see appendix A).

At integer filling \(n_i = 1\), the holons are trivially
gapped and the mean-field for spinons is conveniently
written in the basis \(\psi_i = (f_{i,\uparrow} f_{i,\downarrow})^T\), with Pauli matri-
ces \(\tau^x, \psi, \tau^z\) acting in such spinor space.

\[ \mathcal{H}_{\text{spinon}} = \sum_{\langle ij \rangle} \psi_j^\dagger u_{i,j} \psi_i, \]  

(2)

where one considers only nearest neighbor terms. \(u\)’s are
2×2 matrices, satisfy \(u_{i,j} = u_{j,i}^\dagger\) and contain hopping and
pairing as diagonal, off-diagonal elements, respectively.
We use coordinates shown at upper right of fig 2.

There are two kinds of chiral spin liquid variational
states that break time reversal and mirror symmetry on
triangular lattices that have been extensively studied:
the \(U(1)_2\) chiral spin liquid (CSL1) and the projected
d + id superconductor (CSL2). Below we review the
mean field theory for the two states and discuss their
connection and differences.

A. CSL1: \(U(1)_2\) chiral spin liquid

The \(U(1)_2\) chiral spin liquid, which we study in this pa-
per, is described by a Chern insulator of the spinons with
the special point $\theta$ of translational and six-fold rotation symmetries, while $x, y$ momenta ($\epsilon \leq 0$) at the DSL gapless point $\theta = 0$. Other lowest energy points not shown are $(0, \pi), (\pi, \pi)$. (b) The dispersion at the DSL gapless point $\theta = \pi/6$. (c)(d) The Chern number of valence (conduction) bands is $C = \pm 1$.

Typical realizations consist of a $\pi$ hopping flux of each spinon $f_s$ through a rhombus of the triangular lattice, with the flux $\pi/2 \pm 3\theta$ around an upward and downward triangle for the spinon. This staggered-flux ansatz typically gives a gapped dispersion for each spinon species, forming two Chern bands with $C = \pm 1$; yet at the point $\theta = \pm \pi/6$, the dispersion becomes gapless and Dirac fermions emerge at half-filling, i.e. the system becomes a Dirac spin liquid. See Fig. 1 for the dispersion of the spinon bands. One mean-field ansatz of spinon hopping for such construction reads:

\[
\begin{align*}
    u_{r, r+\hat{x}} &= -e^{i\theta r^x}, \\
    u_{r, r+\hat{y}} &= (-1)^r e^{i\theta r^y}, \\
    u_{r, r+\hat{x}+\hat{y}} &= (-1)^r e^{-i\theta r^x}, (\theta \in (-\pi/3, \pi/3))
\end{align*}
\]

where the $x, y$ coordinates are shown in figure 2. In this gauge choice, the spinon valence band dispersion hits the lowest point at zero momentum and the highest point at momenta $(\pi/3, \pi/3), (\pi/3, -2\pi/3)$, independent of $\theta$.

$H_{\text{spinon}}$ as in eq (3) is invariant under a projective translation and six-fold rotation symmetries, while breaking time-reversal and mirror symmetry except at the special point $\theta = \pm \pi/6$. The projective symmetry group for the spinons reads:

\[
\begin{align*}
    T_x : & \psi_r \rightarrow (-1)^r \psi_{r+\hat{x}}, \\
    T_y : & \psi_r \rightarrow \psi_{r+\hat{y}}, \\
    C_6 : & \psi_r \rightarrow \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \psi_{C_6 r} \ (C_6 r)_y \mod 2 = 0 \\
    & \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \psi_{C_6 r} \ (C_6 r)_y \mod 2 = 1
\end{align*}
\]

where $C_6 r$ is the $C_6$ rotated position of the site at $r$.

The low-energy physics of the $U(1)_2$ CSL is captured by the following action:

\[
L = -\frac{1}{4\pi} \alpha_1 d\alpha_1 - \frac{1}{4\pi} \alpha_2 d\alpha_2 + \frac{1}{2\pi} ad(\alpha_1 + \alpha_2) + \frac{1}{2} A_s d(\alpha_1 - \alpha_2)
\]

(5)

where $A_s$ is the external spin field in the convention that its charge $q_s = 1$ for $S_z = \frac{1}{2}$. $\alpha_1, \alpha_2$ are introduced to represent the IQH of $f_s$ and $f_d$, respectively. $a$ is the internal $U(1)$ gauge field shared by the slave boson $b$ and the spinon $f_s$.

We can also integrate $a$ to lock $\alpha_1 = -\alpha_2 = \alpha$, then we get

\[
L = -\frac{2}{4\pi} ad\alpha + \frac{1}{2\pi} A_s d\alpha
\]

(6)

This is a $U(1)_2$ theory which also describes the $\nu = \frac{1}{2}$ Laughlin state. Thus the $U(1)_2$ CSL has the same topological order as the $\nu = \frac{1}{2}$ Laughlin state. Indeed a model wavefunction using the Laughlin state was proposed by Kalmeyer-Laughlin. But we need to emphasize that a general state with any $\theta \neq \frac{\pi}{6}$ is in the same topological phase as the Kalmeyer-Laughlin state. There are two anyons in this phase: $I$ and $s$, with $s$ as a spin $1/2$ semion.

B. CSL2: Projected $d + id$ superconductor

The projected $d + id$ superconductor is given by the $u$ matrices below:

\[
\begin{align*}
    u_{r, r+\hat{x}} &= \frac{1}{\sqrt{\chi^2 + \eta^2}} (\chi^2 + \eta^2 - r^x), \\
    u_{r, r+\hat{y}} &= \frac{1}{\sqrt{\chi^2 + \eta^2}} (\chi^2 + \eta^2 - r^y), \\
    u_{r, r+\hat{x}+\hat{y}} &= \frac{1}{\sqrt{\chi^2 + \eta^2}} (\chi^2 + \eta^2 - r^x r^y)
\end{align*}
\]

(7)

with $\chi, \eta$ as real parameters and explicit translation invariance. The pairings for 3 bonds generated by $C_3$ rotation have phases $0, 2\pi/3, 4\pi/3$ respectively, the same as that of a $d + id$ superconductor.

There are four anyons labeled by $1, e, m, \epsilon$ in CSL2, $e, m$ are semions with $\pi/2$ self-statistics and trivial mutual statistics. $\epsilon$ is a bound state of $e, m$ and has
fermionic statistics. This state corresponds to \( \nu = 4 \) of Kitaev’s sixteenfold classification of \( Z_2 \) topological order \(^{43}\) and its topological order is \( U(1)_2 \times U(1)_2 \).

C. Relation between two CSLs

We note that these two states described above generically are not equivalent, since the invariant gauge group (IGG) for projected \( d + id \) states is \( Z_2 \), while IGG for \( U(1)_2 \) CSL is \( U(1) \). Moreover, the projected \( d + id \) corresponds to Kitaev \( \nu = 4 \) states in his sixteenfold way for topological orders \(^{43}\) with 2 semions and 1 fermion (spinon), while \( U(1)_2 \) CSL has only one semion.

We note however the following collision between these two ansatz and clarify its meaning. For CSL2, at the special point when \( \eta = \sqrt{2} \chi \), the Wilson loop around one triangle for projected \( d + id \) reads \( \Phi = u_{r,r+2\delta}u_{r+2\delta,r}g_{r+2\delta,r}^{-1}g = i\tau^0 \), equivalent to that of CSL2 (the \( U(1)_2 \) CSL) at \( \theta = 0 \). Hence the two states are gauge equivalent at the special point when CSL2 \( \eta = \sqrt{2} \chi \) and CSL1 has \( \pi/2 \) flux through each triangle. Specifically, there is an SU(2) gauge transform \( g_r \) which rotates the mean field ansatz for CSL1 \(^{43}\) at \( \theta = 0 \) to the mean field \( d + id \) ansatz of CSL2 eq \(^{4}\) at \( \eta = \sqrt{2} \chi \):

\[
\psi_r \rightarrow g_r \psi_r,
\]

\[
g_r = \begin{cases} 
1 & \text{\( r_x \) even, \( r_y \) even} \\
-\frac{1}{\sqrt{4}}(\tau^3 + \sqrt{2} \tau^1) & \text{\( r_x \) odd, \( r_y \) even} \\
\frac{1}{\sqrt{4}}\left(-i\sqrt{3} \tau^3 + \tau^1 \cos \frac{\pi}{2} - \tau^2 \sin \frac{\pi}{2}\right) & \text{\( r_x \) even, \( r_y \) odd} \\
\frac{1}{\sqrt{4}}\left(-i\sqrt{3} \tau^3 + \tau^1 \cos \frac{\pi}{2} - \tau^2 \sin \frac{\pi}{2}\right) & \text{\( r_x \) odd, \( r_y \) odd} 
\end{cases}
\]

At the special point \( \theta = 0 \), the IGG is actually enlarged to SU(2) and the low energy theory is SU(2)_1, which is known to be equivalent to the \( U(1)_2 \) theory. Hence the special point with \( \theta = 0 \) also belongs to the CSL1. At the same time, the point \( \eta = \sqrt{2} \chi \) does not belong to CSL2, since the gauge group is enlarged to SU(2). However, on moving away from this special point, a Higgs condensate develops, lowering the gauge group to \( Z_2 \).

One can ask what the gauge transformation above accomplishes when \( \theta \) shifts away from 0. Then the above rotation on the \( U(1)_2 \) CSL1 does not restore explicitly lattice translation as present in the \( d + id \) ansatz (see appendix \[3\]). The transformed state at \( \theta \neq 0 \) has \( d + id \) pairing, but breaks translation symmetry and despite the fact that it appears to be a pairing state, it secretly possesses a \( U(1) \) IGG.

III. CHIRAL METAL

At hole doping level \( \nu \), the density of the holon and the spinon becomes \( n_h = x \) and \( n_f = 1 - x \). One simple possibility is slave boson condenses at lowest energy state. This is obtained self-consistently for a mean field ansatz of holons from spinon expectation values. Assuming an electron hopping term in the parent Hubbard model of the form \( H_{KE} = -t \sum_{(ij)} c_{i\sigma}^{\dag} c_{j\sigma} + \text{h.c.} \), with \( t > 0 \):

\[
\mathcal{H}_{\text{holon}} = t \sum_{(ij)} \langle f_{i,\uparrow} f_{j,\uparrow}^{\dag} \rangle b_{i}^{\dag} b_{j},
\]

where \( \langle f_{i,\uparrow} f_{j,\uparrow}^{\dag} \rangle \) is computed on the ground state of the spinon mean field \( \mathcal{H}_{\text{spinon}} \) and is proportional to spinon hopping terms on corresponding bonds. Note that the Hamiltonian eq \(^{3}\) and hence \( \mathcal{H}_{\text{holon}} \) have doubled unit cell along \( r_x \) direction, and hence the boson condensation value on site \( i \) depends on \( i \)'s belonging to sublattices \( A, B \), denoted as \( i_s = A, B \). One has

\[
\langle b_{i} \rangle = \sqrt{2 e} u_{b}(i_s)
\]

where \( u_{b}(i_s = A, B) \) is the Bloch wavefunction at zero momentum (lowest energy) for \( \mathcal{H}_{\text{holon}} \). lowest energy for two sublattices \( A, B \). This leads to the electron op-
Spinons are away from integer filling upon doping and states at the valence band top are emptied as Fermi pockets. Electrons, identified as spinons by a $C$ number, partially fill states in a band and form a metal with $\pi$ flux and breaks translation, rotation and time reversal, as the spinon Hamiltonian does. Numerically, we found a nonvanishing bond current expectation pattern of the metal $I_{\text{bond}}(ij) \equiv 1m[c_i^\dagger c_j]$ shown in red in figure 2(a) (parameter $\theta = 0$), at infinitesimal doping. For the electron doping scenario, one changes the parrot definition to $c_i^\dagger = b_i^\dagger f_i$, which corresponds to a particle-hole transformation in the spinon hopping mean field, and hence results in a minus sign multiplying the hopping amplitude. One similarly finds a bond current pattern for doping electrons shown in fig 2(b). We show the current patterns at $\theta = \pm 0.2, \pm 0.4$ upon infinitesimal hole doping in supplementary fig 8.

The electrons have a non-vanishing Hall response as the spinons are in a Chern band. From Ioffe-Larkin rule, one has for the resistivity tensors $\rho_\alpha = \rho_h + \rho_f$ where $\rho_h = 0$ identically since holons have condensed. At small doping, $\rho_e = \rho_f$. Note that when holons condense, an extra term appears for the spinon mean field from $tc_i^\dagger c_j^\dagger s f_i^\dagger f_j^\dagger s$ and substituting electron operators from eq (11).

$$\mathcal{H}_{\text{spinon}} = \mathcal{H}_{\text{spinon}} + 2txu_b(i^s)u_b(j^s)^* f_i^\dagger f_j^\dagger s,$$  \hspace{1cm} (12)

which should be small when light doping and we ignored such terms.

For a free fermion system $\sigma_{xy}$ is given by the Kubo formula as

$$\sigma_{xy} = \frac{e^2}{\hbar} \int \frac{d^2k}{4\pi^2} 2Im[(\frac{\partial u}{\partial k_1}) (\frac{\partial u}{\partial k_2})],$$ \hspace{1cm} (13)

where $u(k_1,k_2)$ is the periodic Bloch wavefunction for $\mathcal{H}_{\text{spinon}}$ and the integration is over filled states. See Appendix [D] for numerical details on calculating Berry curvatures. Fig. 3 shows the hall conductivity $\sigma_{xy}$ as one varies hole doping fraction $x$ and hopping phase $\theta$. The hall conductivity upon doping electrons, meanwhile, is identical to that on the hole side of the same doping level $x$.

Near the gapless point $\theta = \pm \pi/6$, one could approximate the part near valence band top as Dirac fermions with a chiral mass, and analytically (see Appendix [D]) express the hall conductivity by integrating Berry curvatures of filled states, i.e. the hall conductivity as a function of $\Delta = |\pi/6 - \theta|$, $x$ reads

$$\sigma_{xy}(x, \Delta) = \frac{4\sqrt{3}\Delta}{\sqrt{12\Delta^2 + 4\sqrt{3}x}}.$$ \hspace{1cm} (14)

The doubled unit cell implies a density wave formation in the chiral metal where states at $k,k + G$ mix and interfere to enhance the electron density $\rho_G$ at a wavevector of the folded Brillouin zone. Fig 4 presents numerical results on the spectral function $A(k,\omega = 0) = 2Im[G_R(k,\omega = i0^+)]$ where $G_R$ is the retarded Green’s function, details shown in appendix [E] as measured by angle-resolved photo emission spectroscopy (ARPES). The signatures of Fermi surfaces separated by $G$ are modulated by the interference of states at $k,k + G$ on the Fermi surface.

The CSL1 with $\theta = 0$ needs special treatment. At this special point, the spinon ansatz is gauge equivalent to a $d+id$ superconductor. Thus a translation invariant $d+id$ superconductor is also possible upon doping. However, there is no reason to expect a CSL1 state is fine tuned to be at $\theta = 0$. For example, CSL1 found in $J_1 - J_2 - J_3$ model [19] is believed to be close to a Dirac spin liquid at $J_3 = 0$ line. Thus the spinon ansatz should be a Dirac spin liquid with small chiral mass, for which $\theta$ should be close to $\pi/2$. For generic $\theta$, a translation invariant $d+id$ superconductor is not possible in the picture of holon condensation (A translation breaking superconductor is possible and is discussed in Appendix [E]).

A comparison with doping square lattice CSL is in order: On the square lattice, there are also two different CSL: CSL1 and CSL2 as on triangular lattice. But for square lattice, the ansatz of spinons for CSL1 can always be written in the form of $d+id$ superconductor (but without hopping on next nearest neighbor) (please see Appendix [C]). In this case, doping can lead to a translation invariant $d+id$ or a chiral metal, depending on which is energetically favorable. This is different from

\[ \text{FIG. 3: The numerical hall conductivity } \sigma_{xy} \text{ of the chiral metal upon doping holes at fractions } x \text{ at selected values } \theta = 0, 0.2, 0.4, \text{ respectively. The results are expected to be accurate for light doping. The red line is the analytical expression for } \sigma_{xy} \text{ for } \theta = 0.4, \text{ near the gapless point } \theta = \pi/6. \text{ It fits well with the numerical results at small doping } x < 0.1. \text{ } \sigma_{xy} \text{ in the electron doping scenario is identical to that of the hole-doping case at same doping level } x. \]
there is an internal flux $\Phi = f$ with the constraint $n = 0$. At a generic point $\theta = 0$. At a generic point $x = 0.5 \%$. We believe that the final electron phase is translation invariant from the usual RVB wavefunction. In our wavefunction $\Psi(x) = \langle da \rangle = \frac{1 - x}{2}$. In the language of PSG this means that boson $b$ and $f$ are obeying a magnetic translation symmetry: $T_1T_2 = T_2T_1e^{\pm \phi(x)}$, where $\phi$ arises because $b$ and $f$ carry opposite gauge charges. The electron $c = bf$, however, still satisfies the usual translation symmetry because the phase factors from $b$ and $f$ cancel each other. We believe that the final electron phase is translation invariant. The lock of the internal flux to the additional doping keeps the spinon $f$ in the $C = 2$ Chern insulator phase and $b$ in the bIQH phase according to the streda formula at arbitrary doping level $x$. The final wavefunction is a product of two quantum Hall wavefunctions:

$$\Psi(x_1, \ldots, x_N) = P_G \Psi_{bIQH}(x_1, \ldots, x_N) \Psi_{fIQH}(x_1, \ldots, x_N)$$

FIG. 4: The numerical results for spectral function for the chiral metal at zero frequency $A(k, \omega = 0) = 2Im[G_R(k, \omega = i0^+)]$, where the retarded Green’s function $G_R(k, \omega) = \int_0^\infty \frac{d\tau}{i2\pi} e^{i\omega \tau} \langle [c_k^\dagger(t), c_k(0)] \rangle_G$, at selected flux and doping level marked at the top. The spectral function could be measured as in angle-resolved photo emission spectroscopy (ARPES), where the absolute momenta in the original Brillouin zone(dotted hexagonal) are resolved. The Fermi surface is duplicated after a translation of $G$ corresponding to the reciprocal vector of the folded BZ.

IV. TOPOLOGICAL SUPERCONDUCTOR FROM DOPING THE GENERIC KALMEYER-LAUGHLIN SPIN LIQUID CSL1

From the $Z_2$ spin liquid, RVB theory predicts a superconducting phase upon doping. One natural question is: can we also get a superconductor from doping a chiral spin liquid. For CSL2, this is obvious as the spinons are already in a BCS state. For the $U(1)_2$ CSL1 however, the focus here, is not obvious how to do this except at the special point $\theta = 0$. At a generic point with $\theta \neq 0$, there is no gauge transformation which can make the spinon ansatz into a translation invariant $d + id$ superconductor. There is a particular gauge in which spinon are in a translation symmetry breaking superconductor, as shown in Appendix. However, in numerics the $d + id$ superconductor that emerges from doping a $U(1)_2$ CSL is found to be translation invariant. This result therefore cannot be explained by the conventional RVB picture and requires a new theory. In the following we propose precisely such an exotic mechanism for obtaining a topological superconductor from doping the Kalmeyer-Laughlin $U(1)_2$ CSL.

We first turn to the slave boson theory $c_{i,\sigma} = b_i f_{i,\sigma}$ with the constraint $n_b = n_f$. The slave boson $b$ and the spinon $f$ now have opposite gauge charges. At zero doping, there is an internal flux $\Phi = \pi$ per unit cell and the effective filling in terms of the magnetic flux for $b$ and $f$ are $n_b = -2$ and $n_f = 2$ respectively. For the CSL, $f$ is in the $C = 2$ Chern insulator, while the slave boson $b$ is in a trivial Mott insulator with one particle per site. When we decrease the Hubbard $U$ or change the doping, the slave boson $b$ can not be in a Mott insulator anymore and will delocalize after a Mott transition. The usual story is that the slave boson will just condense, which in our case leads to a Chern insulator at zero doping and chiral metal at finite doping, as discussed in the last Section. However, as the slave boson is at filling $n_b = -2$ in terms of the magnetic flux, there is another option after the Mott transition: the slave boson can be in a quantum Hall liquid phase instead of a superfluid phase. The most natural choice is the bosonic integer quantum Hall (bIQH) state, which has been found numerically in lattice model with flux $\Phi = \pi$ per unit cell. As we show later, the resulting phases turns out to be a topological superconductor in the same class of $d + id$ superconductor. The easiest way to see the superconductivity is through the Ioffe-Larkin rule: the resistivity tensor of electron is $\rho_e = \rho_0 + \rho_f$, where $\rho_0, \rho_f$ are the resistivity tensor for $b$ and $f$. As $b$ and $f$ are in quantum Hall phase with opposite Hall conductivity, $\rho_e = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which is the same as that of a superconductor.

A superconductor phase obviously should survive when changing the doping. To keep it in a superconductor phase, we need both $b$ and $f$ in the quantum Hall phase even after doping. At finite doping $x$, it turns out that the internal magnetic flux needs to adjust itself to:

$$\Phi(x) = \frac{\langle da \rangle}{2\pi} = \frac{1 - x}{2}.$$
function, none of $b$ and $f$ have pairing structure, but the resulting phase is a superconductor. In the following we study the topological property of this phase more carefully. The low energy theory is

$$L = L_{b, IQH} + L_{f, IQH}$$

We have

$$L_{f, IQH} = -\frac{1}{4\pi} q_1 d\alpha_1 - \frac{1}{4\pi} q_2 d\alpha_2 + \frac{1}{2\pi} a d(\alpha_1 + \alpha_2) + \frac{1}{2\pi} A_s d(\alpha_1 - \alpha_2)$$

and

$$L_{b, IQH} = \frac{1}{4\pi} \beta_1 d\beta_2 + \frac{1}{4\pi} \beta_2 d\beta_1 + \frac{1}{2\pi} (A_c - a) d(\beta_1 + \beta_2)$$

where we have used the fact that the $K$ matrix for bosonic IQH with $\sigma_{xy} = -2$ is $K = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and charge vector is $q = (1, 1)^T$. The IQH with $\sigma_{xy} = 2$ for fermion is described by $K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and charge vector $q = (1, 1)^T$. $\frac{A_s}{2\pi}$ is the spin gauge field. Here we use the convention that $q_s = \pm 1$ for spin up and down. $A_c$ is the external EM field and we assign charge to the slave boson $a$. $\alpha$ is the internal gauge field shared by the slave boson $b$ and spinon $f$.

We can simplify the action by integrating out $a$ first, which locks $\beta_2 = \alpha_1 + \alpha_2 - \beta_1$. After substitution, we get

$$L = -\frac{2}{4\pi} \beta_1 d\beta_1 - \frac{1}{4\pi} \alpha_1 d\alpha_1 - \frac{1}{4\pi} \alpha_2 d\alpha_2$$

$$+ \frac{1}{2\pi} A_c d(\alpha_1 + \alpha_2) + \frac{1}{2\pi} A_s d(\alpha_1 - \alpha_2) + \frac{1}{2\pi} \beta_1 d(\alpha_1 + \alpha_2)$$

Relabeling $\alpha_c = \frac{\alpha_1 + \alpha_2}{2}$, $\alpha_s = \frac{\alpha_1 - \alpha_2}{2}$, $\beta = -\beta_1 + \frac{\alpha_1 + \alpha_2}{2}$, we can rewrite the action as

$$L = -\frac{2}{4\pi} \beta d\beta - \frac{2}{4\pi} \alpha_s d\alpha_s + \frac{1}{2\pi} A_s d\alpha_s + \frac{2}{2\pi} A_c d\alpha_c$$

We need to emphasize that the quantization of the charge is different from the usual Chern-Simons theory. We have the following dictionary: $q_\beta = -q_\beta$, $q_c = q_1 + q_2 + q_\beta_1$ and $q_s = q_1 - q_2$, where $q_\beta$, $q_1$, and $q_2$ are the charges of $\beta_1, \alpha_1, \alpha_2$. We require $q_\beta, q_1, q_2$ to be integers to satisfy the usual quantization rule of the Chern-Simons theory. The excitation is labeled by $l = (q_\beta, q_c, q_s)$.

Given that there is no Chern-Simons term for $\alpha_c$, it represents a gapless charge mode and we have a superconductor. $\alpha_c$ represents the Goldstone mode and its charge $q_c$ labels the vortex. As the smallest $q_c = q_1 + q_2 + q_\beta_1$ is $\pm 1$, the elementary charge of the superconductor is $Q = 2\frac{q_1 + q_2}{2\pi} = 2$ and we conclude that it is a charge $2e$ superconductor.

The other excitations are listed in Table I. There are in total four different anyons 1, e, m, $\epsilon$, where $e$ and $m$ are two semions and $\epsilon = em$ is a fermion. For each excitation, we can define two charges. The first one is the spin $S_z = \frac{1}{2} q_s$. The second one is the vortex charge $V = q_c$. The statistics of the anyon is $\theta = \frac{\pi}{2} (q_1^2 + q_2^2)$.

Find that the two semions $e, m$ always bind with vortex and cost infinite energy. The only finite energy excitation is the fermion $\epsilon$, which can be identified as a Bogoliubov fermion. $\epsilon$ has a mutual statistics $\theta_{ee} = \theta_{em} = \pi$ with $e, m$, consistent with the braiding of bogoliubov fermion around an elementary vortex. The excitations and their topological properties match that of the $d + id$ superconductor. There is also a spin quantum Hall conductivity $\frac{1}{2 \pi} A_s dA_s$, in agreement with the $d + id$ superconductor[18][19][20]. In summary, we believe the topological property and the symmetry quantum number of the state defined in Eq. (16) is the same as the $d + id$ superconductor.

In the last part of this section, we comment on similarity and difference from our construction here and a previous theory on doped Dirac spin liquid on Kagome lattice[17]. Both our proposal and that of Ref. [17] needs internal magnetic flux adjusting to the doping. The superconductor proposed in Ref. [17] is not a BCS superconductor and it needs time reversal to be broken only after doping. In our case, there is already an internal magnetic flux breaking time reversal in the Mott insulator and the proposed superconductor is in the same class as a $d + id$ superconductor from BCS theory. Hence our proposal may be easier to be realized in realistic models. Actually, a recent numerical study confirms the existence of a $d + id$ superconductor[19] from doping a $U(1)/2$ CSL, consistent with our theory.

### V. DECONFINED SUPERCONDUCTOR-INSULATOR TRANSITION

In this section we discuss the critical point between the chiral spin liquid Mott insulator and the chiral metal or
the topological superconductor. The transition can be
tuned by either doping $x$ or via bandwidth control by
changing the Hubbard $U$ as shown schematically in Fig
3. At finite doping, holons may remain in a Mott insulator,
$i.e.$, localized by disorder and hence the CSL phase
extends to $x \neq 0$ in Fig 3. Usually the bandwidth con-
trolled or doping controlled Mott transition is described by
the condensation of the holons[24]. The new feature
in our theory is the possibility of an unconventional
route: the transition into a bIQH insulator for the holons
actually closes the Mott gap and makes the electrons into a
topological superconductor phase.

The different electron phases correspond to different
phases of holons, as shown in Fig[6] Therefore we can
reduce the critical theory to the transition between dif-
ferent bosonic phases for the slave boson, similar to the
theory in Ref[24]. The transitions marked by black ar-
rows in the fig[6(a)] from a trivial gapped state or an IQH
state to a superfluid, are described by base condensation
and are in the XY universality class in $2 + 1D$. The fi-
nal transition for the corresponding electron phases in
Fig. 6(b) need to further include the gauge field. At
zero doping, the chiral metal is really a Chern insula-
tor. Therefore we can have a deconfined critical point
between a translation symmetry breaking Chern insula-
tor and a translation invariant $d + id$ superconductor,
which we leave to future. In this section we focus on the
superconductor insulator transition marked by the red
arrow in Fig 6.

A key component of this superconductor insulator
transition is the plateu transition for the slave boson.
This transition marked by dotted red arrows from a Mott
insulator to bIQH has been studied in Ref. 60–62. Using
a parton construction with 6 fermionic parton fields, this
transition to superfluid or IQH state is described bybose condensation and are in the XY universality class in 2 + 1D. The final
transition for the corresponding electron phases in
Fig. 6(b) need to further include the gauge field. At
zero doping, the chiral metal is really a Chern insulator.
Therefore we can have a deconfined critical point
between a translation symmetry breaking Chern insulator
and a translation invariant $d + id$ superconductor,
which we leave to future. In this section we focus on the
superconductor insulator transition marked by the red
arrow in Fig 6.

A key component of this superconductor insulator
transition is the plateu transition for the slave boson.
This transition marked by dotted red arrows from a Mott
insulator to bIQH has been studied in Ref. 60–62. Using
a parton construction with 6 fermionic parton fields, this
transition to superfluid or IQH state is described bybose condensation and are in the XY universality class in 2 + 1D. The final
transition for the corresponding electron phases in
Fig. 6(b) need to further include the gauge field. At
zero doping, the chiral metal is really a Chern insulator.
Therefore we can have a deconfined critical point
between a translation symmetry breaking Chern insulator
and a translation invariant $d + id$ superconductor,
which we leave to future. In this section we focus on the
superconductor insulator transition marked by the red
arrow in Fig 6.

A key component of this superconductor insulator
transition is the plateu transition for the slave boson.
This transition marked by dotted red arrows from a Mott
insulator to bIQH has been studied in Ref. 60–62. Using
a parton construction with 6 fermionic parton fields, this
transition to superfluid or IQH state is described bybose condensation and are in the XY universality class in 2 + 1D. The final
transition for the corresponding electron phases in
Fig. 6(b) need to further include the gauge field. At
zero doping, the chiral metal is really a Chern insulator.
Therefore we can have a deconfined critical point
between a translation symmetry breaking Chern insulator
and a translation invariant $d + id$ superconductor,
which we leave to future. In this section we focus on the
superconductor insulator transition marked by the red
arrow in Fig 6.

A key component of this superconductor insulator
transition is the plateu transition for the slave boson.
This transition marked by dotted red arrows from a Mott
insulator to bIQH has been studied in Ref. 60–62. Using
a parton construction with 6 fermionic parton fields, this
transition to superfluid or IQH state is described bybose condensation and are in the XY universality class in 2 + 1D. The final
transition for the corresponding electron phases in
Fig. 6(b) need to further include the gauge field. At
zero doping, the chiral metal is really a Chern insulator.
Therefore we can have a deconfined critical point
between a translation symmetry breaking Chern insulator
and a translation invariant $d + id$ superconductor,
which we leave to future. In this section we focus on the
superconductor insulator transition marked by the red
arrow in Fig 6.

A key component of this superconductor insulator
transition is the plateu transition for the slave boson.
This transition marked by dotted red arrows from a Mott
insulator to bIQH has been studied in Ref. 60–62. Using
a parton construction with 6 fermionic parton fields, this
transition to superfluid or IQH state is described bybose condensation and are in the XY universality class in 2 + 1D. The final
transition for the corresponding electron phases in
Fig. 6(b) need to further include the gauge field. At
zero doping, the chiral metal is really a Chern insulator.
Therefore we can have a deconfined critical point
between a translation symmetry breaking Chern insulator
and a translation invariant $d + id$ superconductor,
which we leave to future. In this section we focus on the
superconductor insulator transition marked by the red
arrow in Fig 6.

A key component of this superconductor insulator
transition is the plateu transition for the slave boson.
This transition marked by dotted red arrows from a Mott
insulator to bIQH has been studied in Ref. 60–62. Using
a parton construction with 6 fermionic parton fields, this
transition to superfluid or IQH state is described bybose condensation and are in the XY universality class in 2 + 1D. The final
transition for the corresponding electron phases in
Fig. 6(b) need to further include the gauge field. At
zero doping, the chiral metal is really a Chern insulator.
Therefore we can have a deconfined critical point
between a translation symmetry breaking Chern insulator
and a translation invariant $d + id$ superconductor,
which we leave to future. In this section we focus on the
superconductor insulator transition marked by the red
arrow in Fig 6.

A key component of this superconductor insulator
transition is the plateu transition for the slave boson.
This transition marked by dotted red arrows from a Mott
insulator to bIQH has been studied in Ref. 60–62. Using
a parton construction with 6 fermionic parton fields, this
transition to superfluid or IQH state is described bybose condensation and are in the XY universality class in 2 + 1D. The final
transition for the corresponding electron phases in
Fig. 6(b) need to further include the gauge field. At
zero doping, the chiral metal is really a Chern insulator.
Therefore we can have a deconfined critical point
between a translation symmetry breaking Chern insulator
and a translation invariant $d + id$ superconductor,
which we leave to future. In this section we focus on the
superconductor insulator transition marked by the red
arrow in Fig 6.
we reproduce the same action for the topological superconductor in last section.

B. Universal properties of the critical point

To obtain the transport properties of the critical theory, one integrates out $\alpha$, in the critical action \([23]\) and the critical theory reads,

\[
\mathcal{L}_{cri} = \sum_{\alpha} \bar{\chi}_{\alpha} [i \partial_{\mu} \chi_{\mu} - \beta_{2\mu}] \chi_{\alpha} + \frac{(\partial_{\mu} \beta_{2\nu} - \partial_{\nu} \beta_{2\mu})^2}{2g^2} + \frac{1}{4\pi} \beta_2 d\beta_2 + \frac{4}{4\pi} A_c d\beta_2 + \frac{2}{4\pi} A_c dA_c - \frac{2}{4\pi} \alpha_s d\alpha_s + \frac{1}{2\pi} A_s d\alpha_s.
\]

The physical currents $J = \delta L_{cri}/\delta A_c$ are

\[
2\pi J_x = -2E_y - 2e_y
\]

\[
2\pi J_y = 2E_x - 2e_x.
\]

where we denote $\epsilon^{\mu\nu}\partial_\nu A_\rho = (B, -E_y, E_x)$. In addition, we have $\epsilon^{\mu\nu}\partial_\nu \beta_{2\rho} = (b, -e_y, e_x)$ and $J_\nu$ as the current for $\chi$. The equations of motion for $\beta_2$ are

\[
2\pi J_x - e_y - 2E_y = 0,
\]

\[
2\pi J_y + e_x + 2E_x = 0.
\]

Combining the above relations and the assumed universal conductivity for $\chi$ at the critical point (QED-3), $J_\nu = \sigma_\chi e_\nu$, one gets the conductivity from eq \([25]\) as

\[
2\pi \sigma_x = \begin{pmatrix}
\frac{4\sigma_\chi}{\sigma_\chi + 1} & -\frac{2(1-\sigma_\chi^2)}{\sigma_\chi + 1} \\
\frac{2(1-\sigma_\chi^2)}{\sigma_\chi + 1} & \frac{4\sigma_\chi}{\sigma_\chi + 1}
\end{pmatrix}, \tag{27}
\]

where $\sigma_\chi = 2\pi \sigma_x$.

Alternatively, to simplify the calculation, we can just use the Ioffe-Larkin rule $\rho_c = \rho_0 + \rho_f$. $\rho_0$ can be found in ref \([60]\) as $\sigma^{-1}$. $\rho_f$ is always $\rho_f/(2\pi) = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$. Adding these together, one gets

\[
\rho_c = \begin{pmatrix}
\frac{\sigma_\chi}{\sigma_\chi + 1} & -\frac{\sigma_\chi^2 + 1}{2(\sigma_\chi^2 + 1)} \\
\frac{\sigma_\chi^2 - 1}{2(\sigma_\chi^2 + 1)} & \frac{\sigma_\chi}{\sigma_\chi + 1}
\end{pmatrix}, \tag{28}
\]

and this is the inverse of eq \([27]\).

In summary, the deconfined superconductor-insulator critical point should have a non-zero universal conductivity and Hall conductivity as shown in Eq. \([27]\). Given that both the superconductor and the CSL insulator have zero Hall conductivity, a non-zero Hall effect only at the critical point is quite remarkable. There is also a spin quantum Hall effect $\frac{1}{2} A_s dA_s$, which remains constant across the transition.

VI. CONCLUSIONS AND OUTLOOK

In summary, we propose two different phases from doping a $U(1)_2$ chiral spin liquid on triangular lattice. One possibility is a chiral metal from simple slave boson condensation picture. Another more exotic possibility is a topological superconductor obtained as putting both the slave boson and the spinon in a quantum Hall phase. We argue that these two scenarios are consistent with two recent numerical studies of doped CSL in two different parameter regimes. Our theory may be relevant to TMD heterobilayer where a spin 1/2 Hubbard model can be simulated on a triangular moiré superlattice \([63–66]\). It has also been proposed that a generalized CSL with different topological order is possible in a $SU(4)$ model on moiré bilayer \([11]\). In future it is interesting to extend our current theory of doped CSL to the case with $SU(N)$ spin.

From a quantum criticality perspective, we propose an unconventional superconductor to Mott insulator transition, as described by a plateau transition of the charged holons. If we start from a $d + id$ superconductor, there are two different Mott transitions towards two different chiral spin liquid insulators. We also find a deconfined critical point between a translation symmetry breaking Chern insulator and a translation invariant topological superconductor. We hope to study this kind of deconfined critical points in more detail in the future.

VII. ACKNOWLEDGEMENT

We would like to thank Ehud Altman, Oleksii Motrunich and Zheng Zhu for useful discussions. We acknowledge funding from a Simons Investigator award (AV) and the Simons Collaboration on Ultra-Quantum Matter, which is a grant from the Simons Foundation (651440, AV). This research is funded in part by the Gordon and Betty Moore Foundation’s EPiQS Initiative, Grant GBMF8683 to AV.

Appendix A: $SU(2)$ and $U(1)$ slave boson formalism

We remark that a full treatment of the parton representation for electrons have an $SU(2)$ gauge symmetry, which is given by two species of spin-0 holons $h_i = (b_{1,i}, b_{2,i})^T$ and spinons $\psi_i = (\psi_{i,\uparrow}, \psi_{i,\downarrow})^T$ to write the electron operators.
as
\[
c_{i,\uparrow} = \frac{1}{\sqrt{2}} h_i^\dagger \psi_i = \frac{1}{\sqrt{2}} (b_{1,i,\uparrow} f_{1,i} + b_{2,i,\uparrow} f_{1,i}),
\]
\[
c_{i,\downarrow} = \frac{1}{\sqrt{2}} h_i^\dagger (i\tau^y \psi_i^\dagger) = \frac{1}{\sqrt{2}} (b_{1,i,\downarrow} f_{1,i} - b_{2,i,\downarrow} f_{1,i}),
\]
where both \( \psi_i \) and \( \tilde{\psi}_i = i\tau^y \psi_i^\dagger \) transform as doublets for \( SU(2) \) group: \( \psi_i \rightarrow U \psi_i, (U \in SU(2)) \) and \( \tilde{\psi}_i \rightarrow U \tilde{\psi}_i, (U \in SU(2)) \). The physical electrons stay invariant under such \( SU(2) \) transform given the holon doublet transform accordingly as \( h_i \rightarrow U h_i \). The gauge-invariant condition for physical Hilbert space (excluding double occupancy) constrain each site to contain an \( SU(2) \) singlet, i.e.
\[
h_i^\dagger \tau h_i + \psi_i^\dagger \tau \psi_i |\Psi_{phys}\rangle = 0,
\]
which for the \( \tau^z \) component reads
\[
b_{1,i} b_{1,i} - b_{2,i} b_{2,i} + \sum_s f_{i,s}^\dagger f_{i,s} = 1.
\]

In the main text parton construction, we effectively fix the gauge for the holon sector as \( b_{1,2} = 0 \) identically and leave out one holon \( b_{1,1} \) as \( b \) in the main text. This is justified since given the \( SU(2) \) gauge group, one could always rotate a state to one with \( b_{1,2} = 0 \) without changing the physical state. The price is a reduced gauge group \( U(1) \) away from half-filling, and that upon transforming the mean field for spinons with \( SU(2) \) rotations, the state changes physically upon doping. For instance, in eq (8) one transforms the spinon ansatz from a \( U(1)_2 \) CSL to \( d + id \), equivalent at half-filling, and yet upon doping holes, the state becomes a chiral metal and \( d + id \) superconductor after condensing \( b_1 \), respectively. Conversely, one could fix the gauge for spinons and then different boson condensation leads to different physical states. In the paper we adopt the first approach, allowing only \( b_1 \) to condense, and hence different spinon ansatz, equivalent at half-filling, give different physical states upon doping nevertheless. Therefore at \( \theta = 0 \) on triangular lattice CSLs, upon doping, either a \( d + id \) or a chiral metal could emerge upon condensing different holon states. This is also true on square lattice (see appendix [C]), where a \( d + id \) ansatz with zero diagonal hopping is equivalent to a chiral spin liquid (DSL with a chiral mass) with an \( SU(2) \) transform. Hence on square lattice doping a CSL gives either \( d + id \) with vanishing diagonal hopping or a chiral metal.

### Appendix B: Competing translation breaking pairing state on doping CSL

At noted in the main text, at \( \theta = 0 \), the \( U(1)_2 \) CSL is equivalent to a projected \( d + id \) spin liquid on triangular lattice. Hence upon doping the \( U(1)_2 \) CSL at \( \theta = 0 \), the resulting states are a competition between \( d + id \) and chiral metal in sec [III]. When \( \theta \) shifts from zero, however, one could still perform the gauge transform eq (9) on \( U(1)_2 \) CSL towards a \( d + id \) pairing ansatz, which could be the competing pairing state with chiral metal upon doping the CSL. Next we analyze the ansatz for such pairing state.

The CSL ansatz eq (3) becomes \( \tilde{u}_{ij} = g_j^\dagger u_{ij} g_i \) after the transform,
\[
\tilde{u}_{r,r+\tilde{x}} = \frac{1}{\sqrt{3}} \left( \frac{e^{i\theta}}{\sqrt{2}e^{i\theta}} \sqrt{2e^{-i\theta}} - e^{-i\theta} \right), \quad (r_x \text{ mod } 2 = 0)
\]
\[
\tilde{u}_{r,r+\tilde{y}} = \frac{1}{\sqrt{3}} \left( \frac{e^{i\theta}}{\sqrt{2}e^{-i\theta}} \sqrt{2e^{-i\theta}} - e^{-i\theta} \right), \quad (r_x \text{ mod } 2 = 1)
\]
\[
\tilde{u}_{r,r+\tilde{y}+\tilde{z}} = \frac{1}{\sqrt{3}} \left( \frac{e^{i\theta}}{\sqrt{2}e^{i\theta+i\pi}} \sqrt{2(1-e^{i\theta})} + e^{i\theta} \right), \quad (r_x \text{ mod } 2 = 0)
\]
\[
\tilde{u}_{r,r+\tilde{y}+\tilde{z}} = \frac{1}{\sqrt{3}} \left( \frac{e^{i\theta}}{\sqrt{2}e^{-i\theta-i\pi}} \sqrt{2(1-e^{-i\theta})} - e^{-i\theta} \right), \quad (r_x \text{ mod } 2 = 1)
\]
\[
\tilde{u}_{r,r+\tilde{y}+\tilde{z}} = \frac{1}{\sqrt{3}} \left( \frac{e^{i\theta}}{\sqrt{2}e^{-i\theta+i\pi}} \sqrt{2(1-e^{i\theta})} - e^{i\theta} \right), \quad (r_x \text{ mod } 2 = 0)
\]
\[
\tilde{u}_{r,r+\tilde{y}+\tilde{z}} = \frac{1}{\sqrt{3}} \left( \frac{e^{i\theta}}{\sqrt{2}e^{i\theta+i\pi}} \sqrt{2(1-e^{-i\theta})} - e^{-i\theta} \right), \quad (r_x \text{ mod } 2 = 1)
\]

One sees that at \( \theta = 0 \), the above mean-field ansatz is identical to \( d + id \) states in eq (7) at \( \eta = \sqrt{2} \chi \), while \( \theta \neq 0 \), the pairing has \( d + id \) symmetry yet the state breaks lattice translation, distinct from the conventional translation invariant \( d + id \) states.
Appendix C: Chiral spin liquids on square lattice

For square lattice, under an appropriate gauge choice, CSL mean-field ansatz in the spinor basis $\psi$ reads,

\[
\begin{align*}
    u_{r,r+\hat{x}} &= \cos(\theta)\tau^z + \sin(\theta)\tau^y, \\
    u_{r,r+\hat{y}} &= \cos(\theta)\tau^z - \sin(\theta)\tau^y, \\
    u_{r,r+\hat{x}+\hat{y}} &= \chi \cos(\theta)\tau^z + \eta \sin(\theta)\tau^x, \\
    u_{r,r-\hat{x}+\hat{y}} &= \chi \cos(\theta)\tau^z - \eta \sin(\theta)\tau^x, \\
\end{align*}
\]

where the IGG is reduced to $Z_2$ when $\theta \neq n\pi/4, (n \in Z)$ and $\chi \neq 0$. This can be seen by calculating Wilson loop $\Phi$ around a square or triangular loop on the square lattice,

\[
\Phi_{\square} = u_{r,r+\hat{x}}u_{r,r+\hat{y}}u_{r,r-\hat{x}}u_{r,r-\hat{y}}, \\
= \cos 4\theta - i \sin 4\theta \tau^x,
\]

\[
\Phi_{\Delta} = u_{r,r+\hat{x}}u_{r,r+\hat{y}}u_{r,r-\hat{x}}u_{r,r-\hat{y}} = \chi (\cos 2\theta \cos \theta \tau^z + \sin 2\theta \cos \theta \tau^y) + \eta (\cos 2\theta \sin \theta \tau^x - i \sin 2\theta \sin \theta).
\]

We note that the $d + id$ ansatz contain two different chiral spin liquids: (I) When $\theta = n\pi/2$ or $\chi = 0$ with arbitrary $\theta$, the IGG is $U(1)$ and we get a CSL1 described by the $U(1)_2$ theory; (II) For a generic ansatz with diagonal hopping, the IGG is $Z_2$ and we have a CSL2 phase in the $\nu = 4$ of the Kitaev’s 16 fold way.

We note that $d + id$ ansatz contains all square $U(1)$ DSL with a chiral mass states (CSL1). The ansatz for $U(1)$ CSL1 is a staggered flux one with diagonal hopping,

\[
\begin{align*}
    u_{r,r+\hat{x}} &= \cos(\theta)\tau^z + (-1)^{r_x+r_y} \sin(\theta)\tau^0, \\
    u_{r,r+\hat{y}} &= \cos(\theta)\tau^z - (-1)^{r_x+r_y} \sin(\theta)\tau^0, \\
    u_{r,r+\hat{x}+\hat{y}} &= (-1)^{r_x+r_y} \eta \sin(\theta)\tau^z, \\
    u_{r,r-\hat{x}+\hat{y}} &= (-1)^{r_x+r_y} \eta \sin(\theta)\tau^z, (\eta \in R)
\end{align*}
\]

where IGG is $U(1)$ generated by $\tau^z$. By a gauge transform, such ansatz is equivalent to $d + id$ with $\chi = 0$ in eq (C1). The gauge transform reads,

\[
\psi_r \rightarrow \frac{1}{\sqrt{2}} (1 + (-1)^{r_x+r_y} \tau^y).
\]

Hence a generic CSL1 phase on square lattice can be written in the $d + id$ ansatz. After doping a CSL1, we can get a translation invariant $d + id$ superconductor from the simple holon condensation picture, in contrast to triangular lattice.

Appendix D: Numerical procedure for Berry curvatures and Dirac fermions near $\theta = \pi/6$

We present some numerical details for Berry curvatures and analytical derivation of Dirac Hamiltonian near $\theta = \pi/6$ gapless point.

Berry connections for Bloch wavefunctions $u(\mathbf{k})$ for the electrons can be approximated numerically as $A_{\epsilon}d\epsilon = \text{Arg}(u(\mathbf{k} + \epsilon))u(\mathbf{k}))$ where $\epsilon$ is a small vector increment in momentum space and Arg gives the phase of the complex number. We mesh the $k_x - k_y$ plane with increments $\epsilon_1 = (\delta, 0), \epsilon_2 = (0, \delta)$, where $k_x, k_y$ are measured by the reciprocal vectors for basis vector $\mathbf{x}, \mathbf{y}$ in the main text, shown in fig 7 (b) and $\delta$ at the order of 0.01 radian. The total Berry curvature on one elementary plaquette of the momentum mesh is given by (shown in Fig 3(a))

\[
\mathcal{F}(\mathbf{k})d^2k = \text{Arg}(u(\mathbf{k} + \epsilon_1))u(\mathbf{k})) + \text{Arg}(u(\mathbf{k} + \epsilon_1 + \epsilon_2))u(\mathbf{k} + \epsilon_1)) + \text{Arg}(u(\mathbf{k} + \epsilon_2))u(\mathbf{k} + \epsilon_1 + \epsilon_2)) + \text{Arg}(u(\mathbf{k}))u(\mathbf{k} + \epsilon_2)).
\]

Note that for bands with nonzero Chern number, Berry curvature on some plaquettes in k-space may have discontinuity of $\pm 2\pi$, since total Berry curvature of one band in the above discretized formula is always zero. One manually makes up for the $2\pi$ discontinuities and then sum the Berry curvature totals $\mathcal{F}(\mathbf{k})d^2k$ over filled $\mathbf{k}$ states for hall conductivity of the spinons.
spinon Hamiltonian, Dirac cones with a small mass emerge near half-filling.

where we used the area a state occupied in k-space for a triangular system with size $L_x L_y$.

Dirac cone as an adiabatic process. Pictorially the Berry phase accumulated is given by $\frac{1}{2} \pi \left(\theta - \frac{\pi}{6}\right)$.

FIG. 7: (a) The numerical procedure to calculate Berry curvature of Bloch wavefunction in k-space. One circles around an elementary plaquette (dashed diamond) and adds up the phase of overlap of wavefunctions at the ends of each bond. (b) The reciprocal vectors for $\hat{x}, \hat{y}$ lattice vectors used in main text and the orthogonal coordinates used for Dirac hamiltonian eq. (D3) $\hat{k}_x, \hat{k}_y$.

At $\theta$ close to $\pm \pi/6$, in the spinon Hamiltonian, Dirac cones with a small mass emerge near half-filling.

We derive the Dirac hamiltonian near the gapless point. The spinon hamiltonian eq. (3) in k-space reads

$$H_{\text{spinon}} = \sum_k \psi_k^\dagger (-2 \cos(\theta + k_y) \tau^z - 2 \cos(\theta + k_x) \tau^x + 2 \cos(\theta - k_x - k_y) \tau^y) \psi_k,$$

where $\psi_k = (f_1, k, f_1^\dagger_{1-k})^T$. The k-space hamiltonian can be Taylor expanded near the gapless point, e.g. $\theta = \pi/6, (k_x, k_y) = (\pi/3, \pi/3)$ and we transform to orthogonal momentum coordinates shown in fig 7(b) $\hat{k}_x = \sqrt{3} k_x, \hat{k}_y = k_y + k_x/2$.

$$H_{\text{Dirac}} = 2\left[\left(\frac{2}{\sqrt{3}} \tau^x - \frac{1}{\sqrt{3}} \tau^z - \frac{1}{\sqrt{3}} \tau^y\right) \hat{k}_x + (\tau^z - \tau^y) \hat{k}_y + (\theta - \frac{\pi}{6}) (\tau^x + \tau^y + \tau^y)\right],$$

with the matrices multiplying $\hat{k}_x, \hat{k}_y$, $\sqrt{3/2}(\theta - \pi/6)$ obeying the anti-commutation relations of gamma matrices for Dirac hamiltonian in $(2 + 1)D$ up to an overall factor $2\sqrt{2}$. Hence indeed the band can be approximated by a Dirac fermion with a small mass. Another Dirac cone appears at $(\pi/3, -2\pi/3)$ in k-space with the same chirality. Note the velocity is uniform, $v = 2\sqrt{2}$ and the mass $m = 2\sqrt{3}(\theta - \pi/6)$. The Berry curvature of the two-component spinor system has been well studied: representing the components of the Dirac hamiltonian $H_{\text{Dirac}} = v k_x \gamma^1 + v k_y \gamma^2 + m \gamma^0$ in a unit vector $\hat{n}(k) = \frac{(v k_x, v k_y, m)}{\sqrt{v^2 (k_x^2 + k_y^2) + m^2}}$, the Berry curvature reads

$$F(k) = \frac{\hat{n} \cdot d_k \hat{n} \times d_k \hat{n}}{2},$$

Pictorially the Berry phase accumulated is given by $1/2$ times the solid angle that $\hat{n}(k)$ sweeps through during an adiabatic process.

To express the hall conductivity as one varies electron doping $x$, one first obtains the density of states near the Dirac cone as

$$\rho(E) = \frac{dN}{dE} = \frac{2\pi k dk}{vdk} \frac{4\pi^2 \sin \frac{\pi}{3}}{L_x L_y},$$

where we used the area a state occupied in k-space for a triangular system with size $L_x, L_y$ as $\frac{4\pi^2 \sin \frac{\pi}{3}}{L_x L_y}$ and $k$ as the norm of $k$ since the dispersion is uniform around Dirac cones. At doping level of $x$, the states in a circle around Dirac point with radius $k_c$ in valence band are emptied, $k_c$ given by

$$\int_0^{k_c} dE \rho(E) = L_x L_y \frac{x}{4},$$

$$k_c^2 = \frac{\pi x}{3},$$

where the factor $1/4$ comes from the 4 Dirac cones in 2 spinon conduction bands.
FIG. 8: The relative current strength $I_{\text{bond}(ij)} = Im[c_i^\dagger c_j]$ of the chiral metal upon infinitesimal doping of holes shown as red numbers on the bond ($\theta$ as shown in upper left of each panel) with arrows indicating current directions. The currents for infinitesimal electron doping are negative of those in hole doped case, and with a shift of the $y$ direction bonds.

The Berry curvature accumulated from states in a circle of radius $k_c$ is given by the solid angle swept by $\hat{n}$ from $|k| = 0, \hat{n} = (0, 0, 1)$ to $|k| = k_c, \hat{n}(k) = \frac{(\nu k_x, \nu k_y, m)}{\sqrt{\nu^2 k_x^2 + m^2}}$, hence

$$\int_{|k| < k_c} d^2 k F(k) = \frac{2\pi}{2} (1 - m \sqrt{m^2 + \nu^2 k_c^2}),$$

(D7)

substituting $k_c$ from eq(D6) and $m, \nu$ from previous analysis, one gets the hall conductivity as the total Chern number from two spinon valence bands $C = 2$, minus the contributions from states around Dirac cones as

$$\sigma_{xy} = \frac{2e^2}{h} - \frac{4\pi e^2}{4\pi^2 h} \left(1 - \frac{2\sqrt{3}\Delta}{\sqrt{12\Delta^2 + 4\sqrt{3}\pi x}}\right),$$

(D8)

where $\Delta = |\pi/6 - \theta|$, which is the expression for $\sigma_{xy}$ as in eq(14).

Appendix E: Calculation of spectral function $A(k, \omega = 0^+)$

We define a plane-wave basis for electrons as

$$c_k^\dagger = \frac{1}{L} \sum_{r} e^{i k \cdot r} c_r^\dagger,$$

$$c_{k,A}^\dagger = \sqrt{\frac{2}{L}} \sum_{r \in A} e^{i k \cdot r} c_r^\dagger,$$

$$c_{k,B}^\dagger = \sqrt{\frac{2}{L}} \sum_{r \in B} e^{i k \cdot r} c_r^\dagger.$$
The first one is used in defining the spectral function as one measures in ARPES. The latter two correspond to plane waves on sublattices $A,B$, respectively. Taking the origin at $A$ site, these operators have the relation,

$$
\hat{c}_k^\dagger = \hat{c}_{k,A}^\dagger + \hat{c}_{k,B}^\dagger,
$$
$$
\hat{c}_k = \hat{c}_{k+G,A} - \hat{c}_{k+G,B},
$$
\hspace{1cm} (E2)

where we used $e^{i\mathbf{G}\cdot \mathbf{r}_{AB}} = -1$, where $r_{AB}$ is the vector connecting sites on two sublattices.

The spectral function measured in the ground state at frequency $\omega = 0^+$ is contributed by eigenstates at the Fermi pockets in the double unit cell Hamiltonian $u_k^\dagger = \alpha_k \hat{c}_{k,A}^\dagger + \beta_k \hat{c}_{k,B}^\dagger$. The coefficients are related to those of the spinon eigenstates by holon condensed values, i.e.

$$
\alpha_k = u_b(A)\alpha_{k,f}
$$
$$
\beta_k = u_b(B)\beta_{k,f},
$$
\hspace{1cm} (E3)

where $\alpha_{k,f},\beta_{k,f}$ are the coefficients for sublattice $A,B$ wavefunctions for spinon eigenstates with an normalization factor multiplied afterwards.

This overlap of the plane wave basis for electrons reads

$$
\langle u_k | \hat{c}_k^\dagger | 0 \rangle = \alpha_k + \beta_k,
$$
$$
\langle u_k | \hat{c}_k^\dagger + G | 0 \rangle = \alpha_k - \beta_k.
$$
\hspace{1cm} (E5)

The spectral function is given by

$$
A(k, \omega = 0^+) = 2\pi \int_{\mathbf{k}' \in FS} dk' |\langle u_k' | \hat{c}_k^\dagger | 0 \rangle|^2.
$$
\hspace{1cm} (E6)

Since the original BZ measured in ARPES is double the size of the reduced one, when measured momenta $k$ lie outside of the reduced BZ, one effectively excites states inside the reduced BZ at its equivalent momentum (i.e., obtained by a translation of a reciprocal lattice vector $G$).

Using eq (E5) to express the spectral function, one finds a simple expression

$$
A(k, \omega = 0^+) = \begin{cases} 
2\pi |\alpha_k + \beta_k|^2 & k \in \text{reduced BZ} \\
2\pi |\alpha_k - \beta_k|^2 & k \not\in \text{reduced BZ}
\end{cases}
$$
\hspace{1cm} (E7)

where $k_0$ is the equivalent momenta for $k$ in the reduced BZ.

[1] P. W. Anderson, “The resonating valence bond state in La2CuO4 and superconductivity,” Science 235, 1196–1198 (1987).

[2] Patrick A. Lee, Naoto Nagaosa, and Xiao-Gang Wen, “Doping a Mott insulator: Physics of high-temperature superconductivity,” Reviews of Modern Physics 78, 17–85 (2006).

[3] X. G. Wen, “Symmetries of the superconducting order parameters in the doped spin-liquid state,” Phys. Rev. B 41, 4212–4219 (1990).

[4] Daniel S. Rokhsar, “Pairing in doped spin liquids: Anyon versus d-wave superconductivity,” Phys. Rev. Lett. 70, 493–496 (1993).

[5] M. Sigrist, T. M. Rice, and F. C. Zhang, “Superconductivity in a quasi-one-dimensional spin liquid,” Phys. Rev. B 49, 12058–12061 (1994).

[6] Michele Fabrizio, “Superconductivity from doping a spin-liquid insulator: A simple one-dimensional example,” Phys. Rev. B 54, 10054–10060 (1996) [arXiv:cond-mat/9603112 [cond-mat]].

[7] T. Senthil and Matthew P. A. Fisher, “Z2 gauge theory of electron fractionalization in strongly correlated systems,” Phys. Rev. B 62, 7850–7881 (2000).

[8] T. Senthil and Patrick A. Lee, “Cuprates as doped U (1) spin liquids,” Phys. Rev. B 71, 174515 (2005) [arXiv:cond-mat/0406066 [cond-mat.str-el]].

[9] R. M. Konik, T. M. Rice, and A. M. Tsvelik, “Doped Spin Liquid: Luttinger Sum Rule and Low Temperature Order,” Phys. Rev. Lett. 96, 086407 (2006) [arXiv:cond-mat/0511268 [cond-mat.str-el]].

[10] Leon Balents and Subir Sachdev, “Dual vortex theory of
doped Mott insulators,” Annals of Physics 322, 2635-2664 (2007), arXiv:cond-mat/0612220 [cond-mat.str-el].

[11] Tsutomu Watanabe, Hisatoshi Yokoyama, Yukio Tanaka, Jun-ichiro Inoue, and Masao Ogata, “Variational Monte Carlo Studies of Pairing Symmetry for the t-J Model on a Triangular Lattice,” Journal of the Physical Society of Japan 73, 3404 (2004).

[12] G. Karakonstantakis, L. Liu, R. Thomale, and S. A. Kivelson, “Correlations and renormalization of the electron-phonon coupling in the honeycomb Hubbard ladder and superconductivity in polycrystalline,” Phys. Rev. B 88, 224512 (2013), arXiv:1307.7676 [cond-mat.str-el].

[13] Kuang Shing Chen, Zi Yang Meng, Unjong Yu, Shuxiang Yang, Mark Jarrell, and Juanu Moreno, “Unconventional superconductivity on the triangular lattice Hubbard model,” Phys. Rev. B 88, 041103 (2013), arXiv:1304.7739 [cond-mat.str-el].

[14] S. R. White, D. J. Scalapino, and S. A. Kivelson, “One Hole in the Two-Leg t-J Ladder and Adiabatic Continuity to the Noninteracting Limit,” Phys. Rev. Lett. 115, 056401 (2015), arXiv:1502.04403 [cond-mat.str-el].

[15] Jordan Venderley and E. A. Kim, “A DMRG Study of Superconductivity in the Triangular Hubbard Ladder Model,” arXiv e-prints, arXiv:1901.11034 (2019), arXiv:1901.11034 [cond-mat.supr-con].

[16] Kai Li, Shun-Li Yu, and Jian-Xin Li, “Global phase diagram, possible chiral spin liquid, and topological superconductivity in the triangular Kitaev-Heisenberg model,” New Journal of Physics 17, 043032 (2015), arXiv:1409.7820 [cond-mat.str-el].

[17] Z. A. Kelly, M. J. Gallagher, and T. M. McQueen, “Electron Doping a Kagaome Spin Liquid,” Physical Review X 6, 041007 (2016), arXiv:1610.04632 [cond-mat.str-el].

[18] Bing Ye, Andrej Mesaros, and Ying Ran, (2016), arXiv:1604.08615.

[19] Yi-Fan Jiang, Hong Yao, and Fan Yang, “Possible superconductivity with Bogoliubov Fermi surface in lightly doped Kagome U(1) spin liquid,” arXiv e-prints, arXiv:2003.02850 (2020), arXiv:2003.02850 [cond-mat.supr-con].

[20] Cheng Peng, Yi-Fan Jiang, Thomas P. Devereaux, and Hong-Chen Jiang, “Evidence of pair-density wave in doping Kitaev spin liquid on the honeycomb lattice,” arXiv e-prints, arXiv:2008.03858 (2020), arXiv:2008.03858.

[21] Yuval Gannot, Yi-Fan Jiang, and Steven A. Kivelson, “Hubbard ladders at small U revisited,” Phys. Rev. B 102, 115136 (2020), arXiv:2007.00661 [cond-mat.str-el].

[22] Xiao-Gang Wen, “Quantum order: a quantum entanglement of many particles,” Physics Letters A 300, 175–181 (2002), arXiv:cond-mat/0110397 [cond-mat.str-el].

[23] Xiao-Gang Wen, “Quantum orders and symmetric spin liquids,” Phys. Rev. B 65, 165113 (2002), arXiv:cond-mat/0107071 [cond-mat.str-el].

[24] T. Senthil, “Theory of a continuous mott transition in two dimensions,” Phys. Rev. B 78, 045109 (2008).

[25] V. Kalmeyer and R.B. Laughlin, “Equivalence of the resonating-valence-bond and fractional quantum hall states,” Physical Review Letters 59, 2095–2098 (1987).

[26] X. G. Wen, Frank Wilczek, and A. Zee, “Chiral spin states and superconductivity,” Phys. Rev. B 39, 11413–11423 (1989).

[27] Yi-Hong Chen, Frank Wilczek, Edward Witten, and Bertrand Halperin, “On anyon superconductivity,” International Journal of Modern Physics B 3, 1001–1067 (1989).

[28] Dung-Hai Lee and Matthew P. A. Fisher, “Anyon superconductivity and the fractional quantum Hall effect,” Phys. Rev. Lett. 63, 903–906 (1989).

[29] Dung-Hai Lee and Charles L. Kane, “Boson-vortex-Skrymion duality, spin-singlet fractional quantum Hall effect, and spin-1/2 anyon superconductivity,” Phys. Rev. Lett. 64, 1313–1317 (1990).

[30] R.B. Laughlin, “Current nature of semiion pairing theory of high-tc superconductors,” International Journal of Modern Physics B 5, 1507–1519 (1991).

[31] B. Bauer, L. Cincio, B. P. Keller, M. Dolfi, G. Vidal, S. Trebst, and A. W. W. Ludwig, “Chiral spin liquid and emergent anyons in a Kagome lattice Mott insulator,” Nature Communications 5, 5137 (2014), arXiv:1401.3017 [cond-mat.str-el].

[32] Bing Ye, Andrej Mesaros, and Ying Ran, (2016), arXiv:1604.08615.

[33] Yi-Fan Jiang, Hong Yao, and Fan Yang, “Possible superconductivity with Bogoliubov Fermi surface in lightly doped Kagome U(1) spin liquid,” arXiv e-prints, arXiv:2003.02850 (2020), arXiv:2003.02850 [cond-mat.supr-con].

[34] Cheng Peng, Yi-Fan Jiang, Thomas P. Devereaux, and Hong-Chen Jiang, “Evidence of pair-density wave in doping Kitaev spin liquid on the honeycomb lattice,” arXiv e-prints, arXiv:2008.03858 (2020), arXiv:2008.03858.

[35] Yuval Gannot, Yi-Fan Jiang, and Steven A. Kivelson, “Hubbard ladders at small U revisited,” Phys. Rev. B 102, 115136 (2020), arXiv:2007.00661 [cond-mat.str-el].

[36] Xiao-Gang Wen, “Quantum order: a quantum entanglement of many particles,” Physics Letters A 300, 175–181 (2002), arXiv:cond-mat/0110397 [cond-mat.str-el].

[37] Xiao-Gang Wen, “Quantum orders and symmetric spin liquids,” Phys. Rev. B 65, 165113 (2002), arXiv:cond-mat/0107071 [cond-mat.str-el].

[38] T. Senthil, “Theory of a continuous mott transition in two dimensions,” Phys. Rev. B 78, 045109 (2008).

[39] V. Kalmeyer and R.B. Laughlin, “Equivalence of the resonating-valence-bond and fractional quantum hall states,” Physical Review Letters 59, 2095–2098 (1987).

[40] X. G. Wen, Frank Wilczek, and A. Zee, “Chiral spin states and superconductivity,” Phys. Rev. B 39, 11413–11423 (1989).

[41] Yi-Hong Chen, Frank Wilczek, Edward Witten, and Bertrand Halperin, “On anyon superconductivity,” International Journal of Modern Physics B 3, 1001–1067 (1989).

[42] Dung-Hai Lee and Matthew P. A. Fisher, “Anyon superconductivity and the fractional quantum Hall effect,” Phys. Rev. Lett. 63, 903–906 (1989).

[43] Dung-Hai Lee and Charles L. Kane, “Boson-vortex-Skrymion duality, spin-singlet fractional quantum Hall effect, and spin-1/2 anyon superconductivity,” Phys. Rev. Lett. 64, 1313–1317 (1990).

[44] R.B. Laughlin, “Current nature of semiion pairing theory of high-tc superconductors,” International Journal of Modern Physics B 5, 1507–1519 (1991).

[45] B. Bauer, L. Cincio, B. P. Keller, M. Dolfi, G. Vidal, S. Trebst, and A. W. W. Ludwig, “Chiral spin liquid and emergent anyons in a Kagome lattice Mott insulator,” Nature Communications 5, 5137 (2014), arXiv:1401.3017 [cond-mat.str-el].

[46] Bing Ye, Andrej Mesaros, and Ying Ran, (2016), arXiv:1604.08615.
[44] Leon Balents, Matthew PA Fisher, and Chetan Nayak, “Nodal liquid theory of the pseudo-gap phase of high-temperature superconductors,” International Journal of Modern Physics B 12, 1033–1068 (1998).

[45] Yuan-Ming Lu and Ashvin Vishwanath, “Quantum phase transition between integer quantum hall states of bosons,” Physical Review B 87, 045129 (2013).

[46] T. Senthil and Michael Levin, “Integer quantum Hall effect for bosons,” Physical Review Letters 110, 046801 (2013).

[47] Wing-Ho Ko, Patrick A. Lee, and Xiao-Gang Wen, “Doped kagome system as exotic superconductor,” Physical Review B 79, 214502 (2009).

[48] Yi-Fan Jiang and Hong-Chen Jiang, “Topological Superconductivity in the Doped Chiral Spin Liquid on the Triangular Lattice,” Physical Review B 96, 075116 (2017), arXiv:1705.00510 [cond-mat.str-el].

[49] Shou-Shu Gong, W. Zhu, J. X. Zhu, D. N. Sheng, and Kun Yang, “Global phase diagram and quantum spin liquids in a spin-1/2 triangular antiferromagnet,” Physical Review B 96, 075116 (2017), arXiv:1704.03418 [cond-mat.str-el].

[50] Jordan Venderley and Eun-Ah Kim, “Density matrix renormalization group study of superconductivity in the triangular lattice Hubbard model,” Physical Review B 89, 100516 (2014), arXiv:1404.2351 [cond-mat.str-el].

[51] S. N. Saadatmand and I. P. McCulloch, “Detection and characterization of symmetry-broken long-range orders in the spin-1/2 triangular Heisenberg model,” Physical Review B 96, 075117 (2017), arXiv:1704.03418 [cond-mat.str-el].

[52] Zheng Zhu, D. N. Sheng, and Ashvin Vishwanath, “Doped Mott Insulators in the Triangular Lattice Hubbard Model,” arXiv e-prints , arXiv:2007.11963 [cond-mat.str-el].

[53] Maissam Barkeshli and John McGreevy, “Continuous transition between fractional quantum hall and superfluid states,” Physical Review B 89, 104512 (2014).

[54] Yi-Fan Jiang and Hong-Chen Jiang, “Quantum phase transitions between bosonic symmetry-protected topological phases in two dimensions: Emergent qed and anyon superfluid,” Physical Review B 89, 195143 (2014).

[55] Jordan Venderley and Eun-Ah Kim, “Density matrix renormalization group study of superconductivity in the triangular lattice hubbard model,” Physical Review B 100, 060506 (2019).

[56] Hong-Chen Jiang, “Superconductivity in the doped quantum spin liquid on the triangular lattice,” arXiv preprint arXiv:1912.06624 (2019).

[57] Yueval Gannot, Yi-Fan Jiang, and Steven A. Kivelson, “Hubbard ladders at small u revisited,” Physical Review B 102, 115136 (2020).

[58] Xue-Yang Song, Chong Wang, Ashvin Vishwanath, and Yin-Chen He, “Unifying description of competing orders in two-dimensional quantum magnets,” Nature Communications 10, 4254 (2019).

[59] Victor Gurarie, Leo Radzihovsky, and Abhinav Prem, “Topological order, symmetry, and hall response of two-dimensional spin-singlet superconductors,” Physical Review B 95, 014508 (2017).

[60] Sergey Moroz, Subhro Bhattacharjee, R. Moessner, and Frank Pollmann, “Bosonic Integer Quantum Hall Effect in an Interacting Lattice Model,” Physical Review Letters 115, 116803 (2015), arXiv:1506.01645 [cond-mat.str-el].

[61] Yuan-Ming Lu and Dung-Hai Lee, “Quantum phase transitions in an Interacting Lattice Model,” Physical Review Letters 115, 116803 (2015), arXiv:1506.01645 [cond-mat.str-el].

[62] N. Read and Dmitry Green, “Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect,” Physical Review B 61, 10267–10297 (2000), arXiv:cond-mat/0004053 [cond-mat.mes-hall].

[63] Frank Pollmann, “Bosonic Integer Quantum Hall Effect in WSe2,” arXiv e-prints , arXiv:1910.12147 (2019), arXiv:1910.12147 [cond-mat.mes-hall].