Fully Dynamic Matching in Bipartite Graphs

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Abstract. We present two fully dynamic algorithms for maximum cardinality matching in bipartite graphs. Our main result is a deterministic algorithm that maintains a \((3/2 + \epsilon)\) approximation in worst-case update time \(O(m^{1/4} \epsilon^{-2.5})\), which is polynomially faster than all previous deterministic algorithms for any constant approximation, and faster than all previous algorithms (randomized included) that achieve a better-than-2 approximation. We also give stronger results for bipartite graphs whose arboricity is at most \(\alpha\), achieving a \((1 + \epsilon)\) approximation in worst-case update time \(O(\alpha(\alpha + \log n))\) for constant \(\epsilon\). Previous results for small arboricity graphs had similar update times but could only maintain a maximal matching (2-approximation). All these previous algorithms, however, were not limited to bipartite graphs.

1 Introduction

The problem of finding a maximum cardinality matching in a bipartite graph is a classic problem in computer science and combinatorial optimization. There are efficient polynomial time algorithms (e.g. [12]), and well-known applications, ranging from early algorithms to minimize transportation costs (e.g. [11,14]) and including recent applications in the area of on-line advertising and social media (e.g. [17,8]). We observe that for matching, the restriction to bipartite graphs is natural and still models many real-world applications and also that in many of these applications, the graph is actually changing over time. We study the fully dynamic variant of the maximum cardinality matching problem in which the goal is to maintain a near-maximum matching in a graph subject to a sequence of edge insertions and deletions. When an edge change occurs, the goal is to maintain the matching in time significantly faster than simply recomputing it from scratch.

One of our results is for bipartite small-arboricity graphs, which we define here. The arboricity of an \(n\)-node \(m\)-edge graph, denoted by \(\alpha(G)\) is \(\max \left\{ \frac{|E(J)|}{|J| - 1} \right\}\), where \(J = (V(J), E(J))\) is any subgraph of \(G\) induced by at least two vertices. Many classes of graphs in practice have constant arboricity, including planar graphs, graphs with bounded genus and graphs with bounded tree width. Every graph has arboricity at most \(O(\sqrt{m})\).

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1.1 Previous Work

In addition to exact algorithms on static graphs, there is previous work on approximating matching and on finding online matchings. Duan and Pettie showed how to find a $(1 + \epsilon)$-approximate weighted matching in nearly linear time [7]; their paper also contains an excellent summary of the history of matching algorithms. Motivated partly by online advertising, there has also been significant work on “online matching” (e.g. [17,8]), both exact and approximate. In most online matching work, the graph is dynamic, but with a restricted set of updates. Typically, one side of the bipartite graph is fixed at the beginning of the algorithm. The vertices on the other side arrive, one at a time, and when a vertex arrives, we learn about all of its incident edges. Deletions are not allowed, nor typically are changes to the matching, although some work also studies models that measure the number of changes needed to maintain a matching [6,9,4].

We now turn to fully dynamic matchings. Algorithms can be classified by update time, approximation ratio, whether they are randomized or deterministic and whether they have a worst-case or amortized update time. The distinction between deterministic and randomized is particularly important here as all of the existing randomized algorithms require the assumption of an oblivious adversary that does not see the algorithm’s random bits; thus, in addition to working only with high probability, randomized dynamic algorithms must make an extra assumption on the model which makes them inadequate in certain settings.

For maintaining an exact maximum matching, the best known update time is $O(n^{1.495})$ (Sankowski [21]), which in dense graphs is much faster than reconstructing the matching from scratch. If we restrict the model to bipartite graphs and to the incremental or decremental setting – where we allow only edge insertions or only edge deletions (but not both) – Bosek et al. [4] show that we can achieve total update time (over all insertions or all deletions) $m\sqrt{n}$ for an exact matching and $mc^{-1}$ for a $(1 + \epsilon)$-matching, which is optimal in the sense that it matches the best known bounds for the static case. For the special case of convex bipartite graphs in the fully dynamic setting, Brodal et al. [5], showed how to maintain an implicit (exact) matching with very fast update but slow query time.

Returning to the general problem of maintaining an explicit matching in a fully dynamic setting, we can achieve a much faster update time than $O(n^{1.495})$ if we allow approximation. One can trivially maintain a maximal (and so 2-approximate) matching in $O(n)$ time per update. Ivkovic and Lloyd [13] showed how to improve the update time to $O((m+n)^{\sqrt{2}/2})$. Onak and Rubinfeld [20] were to first to achieve truly fast update times, presenting a randomized algorithm that maintains a $O(1)$-approximate matching in amortized update time $O(\log^2 n)$ time (with high probability). Baswana et al. [2] improved upon this with a randomized algorithm that maintains a maximal matching (2-approximation) in amortized update $O(\log n)$ time per update. These two algorithms are extremely fast, but suffer from being amortized and inherently randomized, and also from the fact their techniques focus on local changes, and so seem unable to break through the barrier of a 2-approximation.
The first result to achieve a better-than-2 approximation was by Neiman and Solomon [19], who presented a deterministic, worst-case algorithm for maintaining a 3/2-approximate matching. However, the price of this improvement was a huge increase in update time: from $O(\log n)$ to $O(\sqrt{m})$. Gupta and Peng [10] later improved upon the approximation, presenting a deterministic algorithm that maintains a $(1 + \epsilon)$-approximate matching in worst-case update time $O(\sqrt{m}\epsilon^{-2})$ (the same paper achieves an analogous result for maintaining a near-maximum weighted matching in weighted graphs).

The two deterministic algorithms are strongly tethered to the $\sqrt{m}$ bound and do not seem to contain any techniques for breaking past it. An important open question was thus: can we achieve $o(\sqrt{m})$ with a deterministic algorithm? (In fact Onak and Rubinfeld [20] presented a deterministic algorithm with amortized update time $O(\log^2 n)$, but it only achieves a log$(n)$-approximation.) Very recently, Bhattacharya, Henzinger, and Italiano [3] presented a deterministic algorithm with worst-case update time $O(m^{1/3}/\epsilon^{-2})$ that maintains a $(4 + \epsilon)$ approximation; this can be improved to $(3 + \epsilon)$ at the cost of introducing amortization. The same paper presents a deterministic algorithm with amortized update time only $O(\epsilon^{-2} \log n)$ that maintains a $(2 + \epsilon)$ fractional matching. Finally, Neiman and Solomon [19] showed that in graphs of constant arboricity we can maintain a maximal (so 2-approximate) matching in amortized time $O(\log(n)/\log\log(n))$; using a recent dynamic orientation algorithm of Kopelowitz et al. [15], this algorithm yields a $O(\log(n))$ worst-case update time.

Very recently there have been some conditional lower bounds for dynamic approximate matching. Kopelowitz et al. [16] show that assuming 3-sum hardness any algorithm that maintains a matching in which all augmenting paths have length at least 6 requires an update time of $\Omega(m^{1/3} - \zeta)$ for any fixed $\zeta > 0$. Henzinger et al. show that such an algorithm in fact requires $\Omega(m^{1/2} - \zeta)$ time if one assumes the Online Matrix-Vector conjecture.

1.2 Results

If we disregard special cases such as small arboricity or fractional matchings, we see that existing algorithms for dynamic matching seem to fall into two groups: there are fast (mostly randomized) algorithms that do not break through the 2-approximation barrier, and there are slow algorithms with $O(\sqrt{m})$ update that achieve a better-than-2 approximation. Thus the obvious question is whether we can design an algorithm – deterministic or randomized – that achieves a tradeoff between these two: a $o(\sqrt{m})$ update time and a better-than-2 approximation.

We answer this question in the affirmative for bipartite graphs.

**Theorem 1.** Let $G$ be a bipartite graph subject to a series of edge insertions and deletions, and let $\epsilon < 2/3$. Then, we can maintain a $(3/2 + \epsilon)$-approximate matching in $G$ in deterministic worst-case update time $O(m^{1/4}\epsilon^{-2.5})$.

This theorem achieves a new trade-off even if one considers existing randomized algorithms. Focusing on only deterministic algorithms the improvement is even more drastic: our algorithm improves upon not just $\sqrt{m}$ but $m^{1/3}$, and
so achieves the fastest known deterministic update time (excluding the log\((n)\)-approximation of \([20]\)), while still maintaining a better-than-2 approximation. Also, since \(m^{1/4} = O(\sqrt{n})\), our algorithm is the first to achieve a better-than-2 approximation in time strictly sublinear in the number of nodes. Of course, our algorithm has the disadvantage of only working on bipartite graphs.

For small arboricity graphs we also show how to break through the maximal matching (2-approximation) barrier and achieve a \((1 + \epsilon)\)-approximation.

**Theorem 2.** Let \(G\) be a bipartite graph subject to a series of edge insertions and deletions, and let \(\epsilon < 1\). Say that at all times \(G\) has arboricity at most \(\alpha\). Then, we can maintain a \((1 + \epsilon)\)-approximate matching in \(G\) in deterministic worst-case update time \(O(\alpha(\alpha + \log(n)) + \epsilon^{-4}(\alpha + \log(n)) + \epsilon^{-6})\). For constant \(\alpha\) and \(\epsilon\) the update time is \(O(\log(n))\), and for \(\alpha\) and \(\epsilon\) polylogarithmic the update time is polylogarithmic.

Note that a \((1 + \epsilon)\)-approximation with polylog update time is pretty much the best we can hope for. The conditional lower bound of Abboud and Williams \([1]\) provides a strong indication that such a result is likely not possible for general graphs, but we have presented the first class of graphs (bipartite, polylog arboricity) for which it is achievable.

### 1.3 Techniques

We can think of the dynamic matching problem as follows: We are given a dynamic graph \(G\) and want to maintain a large subgraph \(M\) of maximum degree 1. This task turns out to be quite hard because, as the graph evolves, \(M\) is unstable and has few appropriate structural properties.

Very recently, Bhattacharya *et al.* \([3]\) presented the idea of using a transition subgraph \(H\), which they refer to as a kernel of \(G\): the idea is to maintain \(H\) as \(G\) changes, and then maintain \(M\) in \(H\). Maintaining an approximate matching \(M\) is significantly easier in a bounded degree graph, so we need a graph \(H\) that has the following properties: it should have bounded degree, it should be easy to maintain in \(G\), and most importantly, a large matching using edges in \(H\) should be a good approximation to the maximum matching in \(G\).

Our algorithm uses the same basic idea of transition subgraph with bounded degree, but the details are entirely different from those in \([3]\). Their subgraph \(H\) is just a maximal \(B\)-matching with \(B\) around \(m^{1/3}\), that allows some slack on the maximality constraint. The use of a maximal matching is a natural choice in a dynamic setting because maximality is a purely local constraint, and so easier to maintain dynamically. The downside is that as long as one relies on maximality, one can never achieve a better-than-2 approximation; due to other difficulties, their paper in fact only achieves a \((3 + \epsilon)\)-approximation.

The main technical contribution of this paper is to present a new type of bounded-degree subgraph, which we call an edge degree constrained subgraph (EDCS). The problem with a simple \(B\)-matching is that the edges are not sufficiently “spread out” to all the vertices: imagine that \(G\) consists of 4 sets \(L_1, L_2, R_1, R_2\), each of size \(n/2\), where the edges form a complete graph except
that there are no edges between \(L_2\) and \(R_2\). One possible maximal B-matching includes many edges between \(L_1\) and \(R_1\) while leaving \(L_2\) and \(R_2\) completely isolated. The resulting matching is only 2-approximate, which is what we are trying to overcome. Our EDCS circumvents this problem by trying to spread out edges. For each edge, instead of separately upper bounding the matching-degree of each endpoint (B-matching) it upper bounds the sum of the matching-degrees of the endpoints, and then captures the notion of maximality by also lower bounding this sum for edges not in the matching. Using an EDCS prevents the above scenario as the sum of the matching-degrees of edges from \(L_1\) to \(R_2\) will be illegally small unless the matching-degree of \(R_2\) is raised by adding some of those edges to the graph, thus ensuring a larger matching in \(H\).

Although the definition is somewhat similar, the structure of an edge degree constrained subgraph is entirely different from that of a maximal B-matching, and for this reason both our analysis of the approximation factor and our algorithm for maintaining this subgraph are entirely different from those in [3]. In particular, while the constraints in an EDCS seem purely local in that they concern only the degrees of the endpoints of an edge, they in fact have a global effect in a way that they do not in a maximal B-matching. In the latter, as long as an edge does not directly violate the degree constraints, it can always be added to the maximal B-matching, without concern for the edges elsewhere in the graph. But as seen from the above example, this is not true in an EDCS: although the edges from \(L_1\) and \(R_1\) do not themselves violate any constraints, they prevent the constraints between \(L_1\) and \(R_2\) or \(L_2\) and \(R_1\) from being satisfied. An analysis of this global structure is what allows us to go beyond the 2-approximation. On the other hand, the same global structure makes the EDCS more difficult to maintain dynamically; we end up showing that an EDCS contains something akin to augmenting paths, although more locally well behaved. We also develop a general new technique for maintaining a transition subgraph based on dynamic graph orientation, which allows us to reduce the update time from \(O(m^{1/3})\) to \(O(m^{1/4})\). That being said, the additional complications inherent in an EDCS have so far prevented us from extending our results to non-bipartite graphs.

2 Preliminaries

Let \(G = (L \cup R, E)\) be an undirected, unweighted bipartite graph where \(|L| = |R| = n\) and \(|E| = m\). Unless otherwise specified, “graph” will always refer to a bipartite graph. In general, we will often be dealing with graphs other than \(G\), so all of our notation will be explicit about the graph in question. We define \(d_G(v)\) to be the degree of a vertex \(v\) in \(G\); if the graph in question is weighted, then \(d_G(v)\) is the sum of the weights of all incident edges. We define edge degree as \(\delta(u, v) = d(u) + d(v)\). If \(H\) is a subgraph of \(G\), we say that an edge in \(G\) is used if it is also in \(H\), and unused if it is not in \(H\). Throughout this paper we will only be dealing with subgraphs \(H\) that contain the full vertex set of \(G\), so we will use the notion of a subgraph and of a subset of edges of \(G\) interchangeably.

A matching in a graph \(G\) is a set of disjoint edges in \(G\). We let \(\mu(G)\) denote the size of the maximum matching in \(G\). A vertex is called matched if it is
Lemma 1 ([10]). If a dynamic graph $G$ has maximum degree $B$ at all times, then we can maintain a $(1+\epsilon)$-approximate matching under insertions and deletions in worst-case update time $O(Be^{-2})$ per update.

Proof. This lemma immediately follows from a simple algorithm presented in Section 3.2 of [10] which shows how to achieve update time $|E(G)|\epsilon^{-2}/\mu(G)$ (for the transition from worst-case to amortized see appendix A.3 of the same paper), as well as the fact that we always have $|E(G)|/\mu(G) \leq 2B$ because all edges must be incident to one of the $2\mu(G)$ matched vertices in the maximum matching, and each of those vertices have degree at most $B$.

Orientations An orientation of an undirected graph $G$ is an assignment of a direction to each edge in $E$. Given an orientation of edge $(u,v)$ from $u$ to $v$, we will say that $u$ owns edge $(u,v)$ and will define the load of a vertex $v$ to be the number of edges owned by $v$. Orientations of small max load are closely linked to arboricity: every graph with arboricity $\alpha$ has an $\alpha$-orientation [18]. Our algorithms will at all times maintain an orientation of the dynamic graph $G$. The details are in Section F but for the sake of intuition, it suffices to say the following: for a graph with small arboricity $\alpha$, an existing result of Kopelowitz et al. [15] dynamically maintains an orientation with small max-load and small worst-case update time (Theorem 7); for arbitrary graphs, we present a new result that maintains a max load of $O(\sqrt{m})$ in $O(1)$ worst-case update time (Theorem 8).

3 The Framework
We now define the transition subgraph $H$ mentioned in Section 1.3.

Definition 1. An unweighted edge degree constrained subgraph (EDCS) $(G, \beta, \beta^-)$ is a subset of the edges $H \subseteq E$ with the following properties:

(P1) if $(u,v)$ is used (in $H$) then $d_H(u) + d_H(v) \leq \beta$,
(P2) if $(u,v)$ is unused (in $G-H$) then $d_H(u) + d_H(v) \geq \beta^-$.

We also define a similar subgraph where edges in $H$ have weights, effectively allowing them to be used more than once. The properties change somewhat as now used edges can always take more weight, so it makes sense to lower bound the degrees of used edges as well. Recall that the degree of a vertex in a weighted graph is the sum of the weights of the incident edges.

Definition 2. A weighted edge degree constrained subgraph (EDCS) $(G, \beta, \beta^-)$ is a subset of the edges $H \subseteq E$ with positive integer weights that has properties:

(P1) if $(u,v)$ is used then $d_H(u) + d_H(v) \leq \beta$
(P2) for all edges $(u,v)$, we have $d_H(u) + d_H(v) \geq \beta^-$.
Algorithm Outline: To process an edge insertion/deletion in $G$, firstly, we update the small-max-load edge orientation (Theorem 7 or 8 in Appendix F).

Secondly, we update the subgraph $H$ so it remains a valid EDCS of the changed graph $G$ (Section 5); this relies on the graph orientation for efficiency.

Thirdly, we update the $(1 + \epsilon)$-approximate matching in $H$ with respect to the changes to $H$ from the previous step (See Lemma 1). The maintained $(1 + \epsilon)$-approximate matching of $H$ is also our final matching in $G$; the central claim of this paper is that because $H$ is an EDCS, $\mu(H)$ is not too far from $\mu(G)$, so a good approximation to $\mu(H)$ is also a decent approximation to $\mu(G)$ (see Section 4).

There is a subtle difficulty that arises from using a transition graph in a dynamic algorithm. We know from Lemma 1 that if the maximum degree in $H$ is guaranteed to always be below $\Delta_H$, then the time to update a $(1 + \epsilon)$-approximate matching in $H$ will be $O(\Delta_H)$ per update in $H$. But a single change in $G$ could in theory cause many changes in $H$, each of which would take $O(\Delta_H)$ time to process. This motivates the following definition:

**Definition 3.** Let $H$ be a subgraph of a dynamic graph $G$, and let $A$ be an algorithm that modifies the edges of $H$ as $G$ changes. Then, we define the update ratio ($ur$) of $A$ to be the maximum number of edge changes (insertions or deletions) that could be made to $H$ given a single edge change in $G$.

We can now state the main theorems of the paper. We present general and small arboricity graphs separately, but the basic framework described above remains the same in both cases. In all the theorems below, the parameter $\epsilon$ corresponds to the desired approximation ratio (either $(1 + \epsilon)$ or $(3/2 + \epsilon)$).

### 3.1 General Bipartite Graphs

For the sake of intuition, think of $\beta$ in the two theorems below as roughly $m^{1/4}$.

**Theorem 3.** Let $G$ be a bipartite graph, and let $\lambda = \epsilon/4$. Let $H$ be an unweighted EDCS with $\beta^- = \beta(1 - \lambda)$, where $\beta$ is a parameter we will choose later. Then $\mu(H) \geq (2/3 - \epsilon)\mu(G)$.

**Theorem 4.** Let $G$ be a bipartite graph. Let $H$ be an unweighted EDCS with $\beta^- = \beta(1 - \lambda)$, where $\lambda$ is a positive constant less than 1. There is an algorithm that maintains $H$ over updates in $G$ (i.e. maintains $H$ as a valid edge degree constrained subgraph) with the following properties:

- The algorithm has worst-case update time $O((1/\lambda)\left(\beta + \frac{\sqrt{m}}{\lambda}\right))$.
- The update ratio of the algorithm is $O(1/\lambda)$ (see Definition 3).

**Proof of Theorem 1** We use the algorithm outline presented near the beginning of Section 3. We let the transition subgraph $H$ be an unweighted EDCS($G$, $\beta$, $\beta(1 - \lambda)$) with $\lambda = 4\epsilon^{-1} = O(\epsilon^{-1})$ and $\beta = m^{1/4}\epsilon^{1/2}$. By Theorem 4, we can maintain $H$ in worst-case update time $O((1/\lambda)\left(\beta + \frac{\sqrt{m}}{\lambda}\right)) = O(m^{1/4}\epsilon^{-2.5} + m^{1/4}\epsilon^{-5})$ =
\(O(m^{1/4} \epsilon^{-2.5})\). The update ratio is \(O(\lambda^{-1}) = O(\epsilon^{-1})\). Since degrees in \(H\) are clearly bounded by \(\beta\), by Lemma 1 we can maintain a \((1 + \epsilon)\)-approximate matching in \(H\) in time \(O(\beta \epsilon^{-2})\); multiplying by the update ratio of maintaining \(H\) in \(G\), we need \(O(\beta \epsilon^{-3}) = O(m^{1/4} \epsilon^{-2.5})\) time to maintain the matching per change in \(G\). By Theorem 5, \(\mu(H)\) is a \((3/2 + \epsilon)\)-approximation to \(\mu(G)\), so our matching is a \((3/2 + \epsilon)(1 + \epsilon) = (3/2 + \epsilon)\)-approximate matching in \(G\).

\[\square\]

### 3.2 Small Arboricity Graphs

**Theorem 5.** Let \(G\) be a bipartite graph, and let \(\beta > 4\epsilon^{-2}\). Let \(H\) be a weighted EDCS with \(\beta^- = \beta - 1\). Then \(\mu(H) \geq \mu(G)(1 - \epsilon)\).

**Theorem 6.** Let \(G\) be a bipartite graph with arboricity \(\alpha\). Let \(H\) be a weighted EDCS with \(\beta^- = \beta - 1\). There is an algorithm that maintains \(H\) over updates in \(G\) with the following properties:

- The algorithm has worse-case update time \(O(\beta^2(\alpha + \log n) + \alpha(\alpha + \log n))\).
- The update ratio of the algorithm is \(O(\beta)\) (see Definition 3).

The proof of Theorem 2 is analogous to that of Theorem 1 with \(\beta\) set to \(\epsilon^{-2}\) (see Appendix A).

### 4 An EDCS Contains an Approximate Matching

In this section we prove Theorems 3 and 5. Both proofs will be by contradiction; for example, for Theorem 3 to be false, there must be an unweighted EDCS\((G, \beta, \beta(1 - \lambda))\) \(H\) such that \(\mu(H) < (2/3 - \epsilon)\mu(G)\). To exhibit the contradiction, we start by establishing a property that must hold of any subgraph \(H\) defined on the full vertex set of \(G\) for which \(\mu(H)\) is smaller than \(\mu(G)\): the smaller \(\mu(H)\), the more constraining the property. Loosely speaking, the property is a generalization of the fact that the maximum matching on \(H\) establishes an \((S, T)\) cut with no edges crossing in \(H\), but at least \(\mu(G) - \mu(H)\) edges crossing in \(G\). We use the convention that the subscript \(L\) or \(R\) refer to the side of the bipartition in which the vertices lie. The proof of the following lemma involves a careful accounting of augmenting paths and appears in Appendix B.

**Lemma 2.** Let \(G = (V, E_G)\) be a bipartite graph, and let \(H = (V, E_H)\) be a subgraph of \(G\). Then, there exist vertex sets \(S^*_L, S_L, S_R, T^*_R, T_R, T_L\) with the following properties (see right side of Figure 7):

1. \(|S_L| + |T_L| = |S_R| + |T_R| = \mu(H)\).
2. In \(E_H\), all edges incident to \(S_L \cup S^*_L\) go to \(S_R\) and all edges incident to \(T^*_R \cup T_R\) go to \(T_L\).
3. \(G\) contains a perfect matching between \(S_L\) and \(S_R\) and between \(T_L\) and \(T_R\) (\(|S_L| = |S_R|, |T_L| = |T_R|\)).
4. \(|S^*_L| = |T^*_R| = \mu(G) - \mu(H)\) and \(G\) contains a perfect matching between these sets.
Let us say, for contradiction, that $\mu(H)$ is much smaller than $\mu(G)$. Then according to Lemma 2 there is a perfect matching between $S_L^*$ and $T_R^*$ in $G$ but not $H$. Thus, by property P2 of an EDCS, for every edge $(v, w)$ on that matching $d_H(v) + d_H(w)$ must be almost $\beta$. This implies that the average degree in $H$ of vertices in $S_L^*$ and $T_R^*$ must be at least around $\beta/2$. But all the edges in $H$ incident to $S_L^*$ and $T_R^*$ can only go to $S_L$ and $T_R$, which are relatively small if $\mu(H)$ is much smaller than $\mu(G)$. To close the contradiction we argue that because of property P1 of an EDCS, we simply won’t be able to fit all those edges from $S_L^*$ to $S_L$ and $T_R^*$ to $T_L$. We argue this by bounding how high degrees can get in an EDCS. Intuitively, if $U$ and $V$ have equal size and all edges are between $U$ and $V$, we expect the average degree on each side to be no more than $\beta/2$, as if each vertex had degree $\beta/2$ then all edge degrees would be $\beta$ — the maximum allowed by property P1. We now state a generalization of this intuition which shows that if one of the sets $U, V$ is larger than the other, it will have average degree below $\beta/2$; see Appendix C for proof.

Lemma 3. Let us say that in some graph we have disjoint sets $(U, V)$ such that $|U| = \epsilon |V|$, and all edges incident to $U$ go to $V$ (but there may be edges incident to $V$ which do not go to $U$). Let $d(v)$ be the degree of vertex $v$ in this graph, and say that for every edge $(u, v)$ in the graph $d(u) + d(v) \leq \beta$ for some parameter $\beta$. Then, the average degree of vertices in $U$ is at most $\frac{\beta}{\epsilon + 1}$.

Proof of Theorem 3. Let us say, for the sake of contradiction, that we had $\mu(H) < (2/3 - \epsilon) \mu(G)$. Then, we have sets $S_L^*, S_L, S_R, T_R^*, T_R, T_L$ as in Lemma 2 (see right side of Figure 1). By property 4 of this lemma, $S_L^*$ and $T_R^*$ have a perfect matching between them consisting of $\mu(G) - \mu(H)$ edges in $E_G - E_H$ — that is, a perfect matching of unused edges. Thus, by the property P2 of an EDCS, for each edge $(u, v)$ in this matching we have $d_H(u) + d_H(v) \geq \beta (1 - \lambda)$, which implies that the total degree of vertices in $S_L^* \cup T_R^*$ is at least $\beta (1 - \lambda) (\mu(G) - \mu(H))$. Now, by property 4 of Lemma 2 we know that $|S_L^*| = |T_R^*| = \mu(G) - \mu(H)$, so $|S_L^* \cup T_R^*| = 2(\mu(G) - \mu(H))$, so we have:

$$\text{average degree of } S_L^* \cup T_R^* \geq \frac{\beta (1 - \lambda) (\mu(G) - \mu(H))}{2(\mu(G) - \mu(H))} = \frac{\beta (1 - \lambda)}{2}. \tag{1}$$

We argue such a high average degree is not possible. Since $\mu(H) < (2/3 - \epsilon) \mu(G)$:

$$|S_L^* \cup T_R^*| = 2(\mu(G) - \mu(H)) > \mu(H)(1 + \epsilon). \tag{2}$$

Observe that we are now in the situation described in Lemma 3, $S_L^* \cup T_R^*$ corresponds to $U$, and $S_R \cup T_L$ corresponds to $V$. Property 2 of Lemma 2 precisely tells us that all edges from $U$ go to $V$, as needed in Lemma 3. We know from properties 3 and 1 of Lemma 2 that $|V| = |S_R \cup T_L| = |S_R| + |T_L| = |S_L| + |T_L| = \mu(H)$ so by Equation 2 we have $|U| = |S_L^* \cup T_R^*| = \epsilon |V|$ for some $\epsilon > (1 + \epsilon)$. Thus Lemma 3 tells us that the average degree of $U$ is at most $\beta/(1 + \epsilon) \leq \beta/(2 + \epsilon)$, which some simple algebra shows is strictly less than $\beta/(1 + \epsilon)$ since we set $\lambda = \epsilon/4$. We have thus arrived at a contradiction with Equation 1, so our original assumption that $\mu(H) < (2/3 - \epsilon) \mu(G)$ must be false. \qed
Small arboricity graphs: We now turn to Theorem 5. The full proof is left for Appendix D, but we give some intuition here. The statement is very similar to Theorem 3, but with two crucial differences: we are now dealing with a weighted EDCS $H$, and the approximation we need to guarantee is $1 - \epsilon$ instead of $2/3 - \epsilon$. (Note that Theorem 5 is true of general graphs as well; we only use it for small arboricity graphs, however, because a weighted EDCS is difficult to maintain in general graphs.) It may seem unintuitive that a weighted EDCS contains a better matching than an unweighted one since it will in fact have fewer total edges to work with. To show why a weighted EDCS is better, see Figure 2 for a simple example where an unweighted EDCS only contains a $(3/2)$-approximate matching, but a weighted one does not suffer the same issues.

In the proof of Theorem 3 we constructed the sets $S^*_L$, $S_L$, $S_R$, $T^*_R$, $T_R$, $T_L$ from Lemma 2 and then argued that $S^*_L$ (and analogously $T^*_R$) must have low average degree because all of its edges go to $S_R$, so we simply cannot fit that many edges before violating property P1 of an EDCS. Now, we could upper bound the average degree of $S^*_L$ even better if we could argue that there also had to be other edges coming into $S_R$, taking up space. The natural candidate would be the edges on the matching from $S_L$ to $S_R$ guaranteed by property 3 of Lemma 2. In Theorem 3 we were unable to take advantage of these edges because we were dealing with an unweighted EDCS, so a single matching worth of edges did not count for much. The properties of a weighted EDCS, however, can force this single matching to be used multiple times, thus leaving even less space for edges leaving $S^*_L$. The proof of Theorem 5 is thus analogous to that of Theorem 3 but requires a stronger version of Lemma 3 (See Lemma 4 in Appendix D).

5 Maintaining an edge degree constrained subgraph

In this section, we outline the proofs of Theorems 4 and 6. Due to space constraints, we leave the formal proof for Section E.

Recall that $\delta(u, v)$ denotes the edge degree of $(u, v)$, $d_H(u) + d_H(v)$. We define an edge to be full if it is in $H$ and has edge degree $\beta$. We define it to be deficient if it is not in $H$ and has the minimum allowable edge degree $\beta^-$, which is $\beta - 1$ for the weighted EDCS in Theorem 6 and $\beta(1 - \lambda)$ for the unweighted EDCS of Theorem 4. We define a vertex to be increase-safe if it has no incident full edges and decrease-safe if it has no incident deficient edges; it is easy to see that increasing (decreasing) the degree of an increase-safe (decrease-safe) vertex by one does not lead to a violation of any EDCS constraints.

Now, let us say that we delete some edge $(u, v)$ from $G$. If $(u, v)$ was not in the EDCS $H$ then all constraints remain satisfied. Otherwise, deleting $(u, v)$ causes the degrees of $u$ and $v$ to decrease by one. Let us focus on fixing up vertex $v$: vertex $u$ can then be handled analogously. If $v$ was decrease-safe, then all constraints relating to $v$ remain satisfied and we are done. Otherwise, it must have had some incident deficient edge $(v, v_2)$. Adding this edge to $H$ rebalances the degree of $v$ to what it was before the deletion, but now the degree of $v_2$ has increased by one. If $v_2$ was increase-safe, the degree increase does not violate any constraints, and we are done. Otherwise, $v_2$ must have an incident full edge $(v_2, v_3)$ which we delete from the graph; this rebalances $v_2$ but decreases the
degree of $v_3$, so we look for an incident deficient edge. We continue in this fashion until we end on an increase/decrease-safe vertex.

We can thus fix up an edge deletion by finding an alternating path of full and deficient edges that ends in an increase/decrease-safe vertex. Insertions are handled analogously. This process is similar to finding an augmenting path in a matching except that finding an augmenting path is much harder because we might hit a dead end and have to back track; but we can fix up an EDCS by following any sequence of full/deficient edges. Moreover, the resulting alternating path is always simple and contains few edges. Figure 3 illustrates this point. For the small arboricity case (Theorem 6) where $\beta^- = \beta - 1$, it is not hard to see that in any such alternating path the vertex degrees $d_H(v)$ on either side of the bipartition are either increasing or decreasing by 1, so since $d_H(v)$ is always between 0 and $\beta$, the path has length $O(\beta)$. In this small arboricity case, $O(\beta)$ is small because $\beta = O(1/\epsilon^2)$ (See Section A). In the general case (Theorem 4), $\beta$ is large but the gap between $\beta$ and $\beta^-$ is $\beta\lambda$, so degrees on either side change by $\beta\lambda$ and the path has length only $O(1/\lambda)$.

To find such an alternating path of full and deficient edges we maintain a data structure that for any vertex $v$ can return an incident full or deficient edge (whichever is asked for), or indicate that none exists. Since the alternating path will always be short, this data structure will only be queried a small number of times per insertion/deletion in $G$. We maintain this data structure using a dynamic orientation, in which each edge is owned by one of its endpoints (see end of Section 2). Let us focus on the small arboricity case, where the dynamic orientation maintains a small max load. Each vertex will maintain fullness/deficiency information about the edges it does not own, storing each category of edge (full/deficient) in its own list. To find a full/deficient edge incident to some vertex $v$, the data structure simply picks an edge from the corresponding list in $O(1)$ time; if the list is empty, the data structure then manually checks all the edges that $v$ does own: since the max load is small, this can be done efficiently. When the status of a vertex $v$ changes, to maintain itself the data structure must transfer this information along all edges $(v,u)$ that are not owned by $u$, but since these are precisely the edges owned by $v$, there can only be a small number of them.

The basic idea is the same for general bipartite graphs (Theorem 4), except that now the max load is $O(\sqrt{m})$, and we cannot afford to spend $O(\sqrt{m})$ per update. Note that in this case, however, there is a gap of $\beta\lambda$ between full and deficient edges, so intuitively, the degree of a vertex has to change $\beta\lambda$ time before it must be updated in the data structure. This leads to an update time of around $\sqrt{m}/(\beta\lambda)$, as needed in Theorem 4. The details, however, are quite involved, especially since we need a worst-case update time.

6 Conclusion

We have presented the first fully dynamic bipartite matching algorithm to achieve a $o(\sqrt{m})$ update time while maintaining a better-than-2-approximate matching. It is also the fastest known deterministic algorithm for achieving any constant
approximation, and certainly any better-than-2 approximation. The main open questions are in how far we can push this tradeoff. Can we achieve a randomized better-than-2 approximation with update time \( \text{polylog}(n) \)? For deterministic algorithms, can we achieve a constant approximation with update time \( \text{polylog}(n) \), or a \((1 + \epsilon)\)-approximation with update time \( o(\sqrt{m}) \)?

The other natural question is whether our results can be extended to general (non-bipartite) graphs and non-bipartite graphs of small arboricity. The definition of an edge degree constrained subgraph does not inherently rely on bipartiteness, and neither do many of the techniques in this paper. The main obstruction to the generalization seems to lie in the structural property exhibited in Lemma 2. Is there an analogue for non-bipartite graphs?

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A  Proof of Theorem 2

The proof follows from Theorems 3 and 6 and is analogous to the proof of Theorem 1 given in Section 3.1.

We use the algorithm outline presented near the beginning of Section 3. For our transition subgraph \( H \), we use a weighted EDCS \( (G, \beta, \beta - 1) \) with \( \beta = 4\epsilon^{-2} \). By Theorem 6, we can maintain \( H \) in worst-case update time \( O(\beta^2(\alpha + \log(n)) + \alpha(\alpha + \log(n))) = O(\epsilon^{-4}(\alpha + \log(n)) + \alpha(\alpha + \log(n))) \). The update ratio of the algorithm is \( O(\beta) \). This EDCS clearly has max degree \( \beta \) so by Lemma 1, we can then maintain a \((1 + \epsilon)\)-approximate matching in \( H \) in time \( O(\beta\epsilon^{-2}) \); multiplying by the update ratio \( O(\beta) \) of maintaining \( H \) in \( G \), we need \( O(\beta^2\epsilon^{-2}) = O(\epsilon^{-6}) \) time to maintain the matching per change in \( G \). Combining the terms above gives precisely the bound of Theorem 2. By Theorem 5, \( \mu(H) \) is a \((1 + \epsilon)\)-approximation to \( \mu(G) \), so our matching is a \((1 + \epsilon)(1 + \epsilon) = (1 + \epsilon)\)-approximate matching in \( G \), as desired.

B  Proof of Lemma 2

In this section we prove Lemma 2 repeated below.

Lemma 2. Let \( G = (V, E_G) \) be a bipartite graph, and let \( H = (V, E_H) \) be a subgraph of \( G \). Then, there exist vertex sets \( S_L^*, S_L, S_R, T_R^*, T_R, T_L \) with the following properties (see right side of Figure 1):

1. \( |S_L| + |T_L| = |S_R| + |T_R| = \mu(H) \).
2. In \( E_H \), all edges incident to \( S_L \cup S_L^* \) go to \( S_R \) and all edges incident to \( T_R \cup T_L \) go to \( T_R \).
3. \( G \) contains a perfect matching between \( S_L \) and \( S_R \) and between \( T_L \) and \( T_R \) (so \( |S_L| = |S_R|, |T_L| = |T_R| \)).
4. \( |S_L^*| = |T_R^*| = \mu(G) - \mu(H) \) and \( G \) contains a perfect matching between these sets.

Proof. Let \( M(H) \) be some maximum matching in \( H \). We know from basic matching facts that since \( H \) is bipartite, \( M(H) \) induces a residual graph on \( E_H \), as well as the following \((P, Q)\) cut (see left side of Figure 1):

1. \( P = P_L^* \cup P_L \cup P_R \) where \( P_L^* \) contains all free vertices in \( L \) (free in \( M(H) \) that is), and \( P_L \) and \( P_R \) are the vertices in \( L \) and \( R \) that are in reachable from \( P_L^* \) in the residual graph induced by \( M(H) \).
2. \( Q = Q_R^* \cup Q_L \cup Q_R \) where \( Q_R^* \) contains all free vertices in \( R \), and \( Q_L \) and \( Q_R \) contains all vertices in \( L \) and \( R \) that are NOT reachable from \( P_L^* \).

We will start with the \((P, Q)\) cut defined by the sets above and show how to convert it into an \((S, T)\) cut satisfying the conditions of the lemma. We define a \( P - Q \) crossing edge to be an edge between \( P_L \cup P_L^* \) and \( Q_R \cup Q_R^* \).

It is easy to see that the cut \((P, Q)\) satisfies the following properties:

- \( |P_L| + |Q_L| = |P_R| + |Q_R| = \mu(H) \) (all matched vertices are in one of these sets).
Hold.

- \( H \) contains no \( P - Q \) crossing edges; if it did, there would be an augmenting path between two free vertices.

- \( H \) (and so certainly \( G \)) contains a perfect matching between \( P_L \) and \( P_R \) and between \( Q_L \) and \( Q_R \).

Note that the three properties of our \( (P,Q) \) cut correspond exactly to properties 1-3 of the desired \((S,T)\) cut. Let us now examine property 4, which involves looking at the \((P,Q)\) cut from the perspective of \( G \). In \( G \), there must be \( \mu(G) - \mu(H) \) disjoint augmenting paths from \( P_L^* \) to \( Q_R^* \), each of which crosses the \((P,Q)\) cut exactly once, as there can be no backwards edges on this cut; so in particular, \( G \) must contain \( \mu(G) - \mu(H) \) disjoint edges crossing the \((P,Q)\) cut. Now, if all of these crossing edges were from \( P_L^* \) to \( Q_R^* \) then our proof would be complete: we would set \( S_L^* \) and \( T_R^* \) to contain the respective endpoints of these \( \mu(G) - \mu(H) \) edges, and then set \( S_L, S_R, T_L, T_R \) to be \( P_L, P_R, Q_L, Q_R \) respectively, and all properties would be satisfied.

But things are not so simple: though the augmenting paths go from \( S_L^* \) to \( T_R^* \), the \( \mu(G) - \mu(H) \) crossing edges in \( G \) can start in \( P_L \) or end in \( Q_R \) (or both). Thus, there is no direct correspondence between the \((P,Q)\) cut and the desired \((S,T)\) split. Instead, we show how to transition from one to the other (see both sides of Figure 1). Start with a preliminary split where \( S_L^* \) and \( T_R^* \) are empty, and \( S_L, S_R, T_L, T_R \) are equal to \( P_L, P_R, Q_L, Q_R \), so all but property 4 are satisfied.

We now show that for each of the \( \mu(G) - \mu(H) \) disjoint augmenting path in \( G \) – paths that all go from \( P_L^* \) to \( Q_R^* \) – we can rearrange the \((S,T)\) split so that properties 1-3 are maintained, and two new vertices are added, one to \( S_L^* \) and one to \( T_R^* \), such that \( G \) has an edge between them. To maintain property 3 we will explicitly maintain a perfect matching in \( G \) between \( S_L \) and \( S_R \) and between \( T_L \) and \( T_R \); initially our matching is the perfect matching in \( H \) between these sets given by the \( P - Q \) cut (see Figure 1). Each augmenting path will only affect the vertices on the path, so since the augmenting paths are disjoint, we can just focus on a single path, without worrying that they will interfere with each other. Clearly, performing this addition of vertices for all \( \mu(G) - \mu(H) \) augmenting paths will yield an \((S,T)\) split where \( S_L^* \) and \( T_R^* \) have size \( \mu(G) - \mu(H) \) and a perfect matching between them (the desired property 4), and properties 1-3 still hold.

We now focus on a single augmenting path in \( G \). Let the path be \( s_0, s_1, \ldots, s_i, s^*, t^*, t_j, t_{j-1}, \ldots, t_0 \), where \( s_0 \in P_L^* \), \( t_0 \in Q_R^* \), and \( (s^*, t^*) \) is the unique \( P - Q \) crossing edge on the path. Note that since the graph is bipartite, \( i \) and \( j \) must be odd unless \( s^* = s_0 \) or \( t^* = t_0 \). Note also that since this is an augmenting path, \( s_0 \) and \( t_0 \) are the only free vertices (i.e. in \( P_L^* \) and \( Q_R^* \) respectively). Now, if \( s_0 = s^* \) (i.e. the \( P - Q \) crossing edge leaves \( s_0 \)), we simply add \( s^* \) to \( S_L^* \) and make no other changes on the \( S \)-side; otherwise, we move \( s^* \) from \( S_L \) to \( S_L^* \) and move \( s_0 \) to \( S_L \). \( S_L \) and \( S_R \) thus maintain the same size \((S_R \) is in fact completely unchanged\), and the perfect matching from \( S_L \) to \( S_R \) is maintained because we can match \((s_0, s_1), (s_2, s_3), \ldots, (s_{i-1}, s_i)\) (recall that \( i \) is odd). Similarly on the \( T \)-side, if \( t_0 = t^* \) we simply move \( t^* \) to \( T_R^* \); otherwise, we move \( t^* \) from \( T_R \) to \( T_R^* \) and move \( t_0 \) to \( T_R \), which as with the \( S \)-side maintains a perfect matching.
from $T_L$ to $T_R$. The end result is that we have added new vertex $s^*$ to $S_L^*$ and new vertex $t^*$ to $T_R^*$, so since edge $(s^*, t^*)$ is in $G$ by construction (it’s on the augmenting path), we are done.

\section{Proof of Lemma \ref{lem:average_deg}}

The lemma at hand, which was used in Section \ref{sec:proof_of_main_result}, is repeated below.

\textbf{Lemma \ref{lem:average_deg}} Let us say that in some graph we have disjoint sets $(U, V)$ such that $|U| = c|V|$, and all edges incident to $U$ go to $V$ (but there may be edges incident to $V$ which do not go to $U$). Let $d(v)$ be the degree of vertex $v$ in this graph, and say that for every edge $(u, v)$ in the graph $d(u) + d(v) \leq \beta$ for some parameter $\beta$. Then, the average degree of vertices in $U$ is at most $\frac{\beta}{c+1}$.

\textbf{Proof.} We want to upper bound the total number of edges from $U$ to $V$. Now, there clearly exists a solution that maximizes this number in which all edges in the graph are between $U$ and $V$ (i.e., no edges from $V$ to elsewhere); just take a maximum solution that has other edges and remove those – the number of edges leaving $U$ remains the same, and all constraints are clearly still satisfied. Thus, we can assume for this proof that all edges in the graph are between $U$ and $V$.

Let $E$ be the set of edges in the graph. Our goal is to upper bound $|E| = \sum_{u \in U} d(u) = \sum_{v \in V} d(v)$. Now for each edge $(u, v) \in E$ we have the constraint $d(u) + d(v) \leq \beta$. Let us sum the inequality constraints for all edges: this yields $\sum_{(u,v) \in E} d(u) + d(v) \leq |E|\beta$. A closer look at the left hand side shows that since each vertex $v$ appears in exactly $d(v)$ edges in $E$, and each of those edges contributes $d(v)$ to the left hand side,

$$\sum_{(u,v) \in E} d(u) + d(v) = \sum_{u \in U} d(u)^2 + \sum_{v \in V} d(v)^2 \leq |E|\beta. \quad (3)$$

Now that we have an upper bound, we also get a lower bound for $\sum_{u \in U} d(u)^2$ and $\sum_{v \in V} d(v)^2$. Since we know that $\sum_{v \in V} d(v)$ is fixed at $|E|$, the sum of squares is minimized when all of the $d(v)$ are equal, i.e., when $d(v) = |E|/|V|$ for every $v$. The same is true for $U$, where recall that $|U| = c|V|$. This yields:

$$\sum_{u \in U} d(u)^2 + \sum_{v \in V} d(v)^2 \geq \sum_{u \in U} \left(\frac{|E|}{|U|}\right)^2 + \sum_{v \in V} \left(\frac{|E|}{|V|}\right)^2$$

$$= \frac{|E|^2}{|U|} + \frac{|E|^2}{|V|} = \frac{|E|^2}{|V|} \left(1 + \frac{1}{c}\right) = \frac{|E|^2}{|V|} \cdot \frac{1+c}{c} \quad (4)$$

Merging the upper bound from Equation \ref{eq:upper_bound} and the lower bound from Equation \ref{eq:lower_bound} we get that

$$\frac{|E|^2}{|V|} \cdot \frac{1+c}{c} \leq |E|\beta \Rightarrow |E| \leq \beta |V| \cdot \frac{c}{1+c}.$$}

Thus, the average degree of $U$ is at most $|E|/|U| = |E|/c|V| \leq \frac{\beta}{1+c}$, as desired.

We note that bipartiteness was actually not required for the proof – we only needed that all edges incident to $U$ go to $V$, which of course disallows edges whose endpoints are both in $U$. 


D Proof of Theorem 5

In this section we give a full proof of Theorem 5. Recall the intuition given at the end of Section 4; in particular, we will rely on the following generalization of Lemma 3.

Lemma 4. Say that in some graph we have two disjoint sets \( U, V \) such that all edges incident to \( U \) go to \( V \). Let \( V = \{v_1, \ldots, v_n\} \), and let \( U = W \cup X \), where \( W = \{w_1, \ldots, w_n\} \) and \( X = \{X_1, \ldots, X_c\} \) for some \( c < 1 \). Note that \( |W| = |V| \), \( |X| = c|V| \) and \( |U| = (1 + c)|V| \). Now, say that all edges have positive integer weights and that the degree of vertex \( v \) (denoted \( d(v) \)) is the sum of its incident edge weights. Say also that the graph obeys the following degree constraints:

- **Constraint 1:** for every edge \((u, v)\) between \( U \) and \( V \) we have \( d(u) + d(v) \leq \beta \),
- **Constraint 2:** for all \( n \) pairs \((v_i, w_i)\), we have \( d(v_i) + d(w_i) \geq \beta - 1 \).

(Compare this with P2 in the weighted EDCS.)

Then, the average degree in \( X \) is at most \( \beta / (2 + c) + 1/c \) (so around \( \beta / (2 + c) \) for large enough \( \beta \)).

Proof. We want to upper bound the number of edges incident to \( X \). We will start by arguing that there is some graph that maximizes this quantity where all edges of the graph are between \( U \) and \( V \). Let us start with some valid graph that maximizes the number of edges leaving \( X \), but might also have other edges. We will show that we can always remove any edge that is not between \( U \) and \( V \) and then fix up the graph in such a way that none of the degree in \( X \) decrease but all the constraints are still satisfied: repeating this multiple times, we will end up with a graph where all edges are between \( U \) and \( V \) but the total degree of \( X \) is still maximized.

The two constraints above only concern degrees in \( U \) and \( V \), so clearly any edge that is incident to neither \( U \) nor \( V \) can be safely removed. Now let us take some edge \((*, v_i)\) that is incident to \( V \) but not \( U \). Removing this edge decreases the degree of \( v_i \), which might violate constraint 1 concerning \((v_i, w_i)\); thus, we might now have some fixing up to do. To do this, let us define a vertex \( v_i \in V \) to be **deficient** if \( d(v_i) + d(w_i) = \beta - 1 \). Let us define edge \((u, v)\) to be **full** if \( d(u) + d(v) = \beta \). Notice that we can safely raise the degree of any vertex that has no incident full edges without violating any of the constraints; similarly, we can decrease the degree of any vertex \( v_i \in V \) that is not deficient.

Now, once we remove edge \((*, v_i)\) (the edge not between \( U \) and \( V \)) the degree of \( v_i \) is about to decrease. If \( v_i \) is not deficient we allow this to happen and we are done. Otherwise, we add a single unit of weight to edge \((v_i, w_i)\); note that if the edge doesn’t exist we can simply add it to the graph, since we only need to prove that there exists some solution that maximizes the total degree of \( X \) while only using \( U-V \) edges. The degree of \( v_i \) thus remains unchanged, but the degree of \( w_i \) is about to increase. If \( w_i \) has no incident full edges we allow this to happen and we are done. Otherwise, let \((w_i, v_{i_2})\) be one of these full edges, and decrease

...
its weight by 1. The degree of \( w_i \) thus remains unchanged but the degree of \( v_{i2} \) is about the decrease. We now repeat: if \( v_{i2} \) is not deficient we allow its degree to decrease and we are done; else, we add one unit of weight to \( (v_{i2}, w_{i2}) \). If \( w_{i2} \) has no incident full edges we allow its degree to increase and we are done; otherwise we remove one unit of weight from \( w_{i2}, v_{i2} \). As we continue in this fashion, we are always ensuring that all constraints are satisfied. It is also easy to see that no degrees in \( U \) decrease: all of them remain the same (every weight-decrease is preceded by a weight-increase), except for the last vertex examined which might increase its degree by 1. Thus, all we have left to show is that this fixing up process terminates. We show this by proving that \( d(w_{i}) \) is always strictly smaller than \( d(w_{i+1}) \). Since the algorithm didn’t stop at \( d(w_{i}) \) it must have found a full edge \( (w_{i}, v_{i+1}) \), so by definition of full \( d(v_{i+1}) = \beta - d(w_{i}) \). But since the algorithm didn’t stop at \( d(v_{i+1}) \) it must have been deficient, so \( d(w_{i+1}) = \beta - 1 - d(v_{i+1}) = \beta - 1 - (\beta - d(w_{i})) = d(w_{i}) - 1 \) (This argument is analogous to one used in Lemma 3 – see Figure 3).

Thus we can assume for the rest of the proof that all edges in the graph are between \( U \) and \( V \). This implies that the total degree of \( U \) is equal to the total degree of \( V \). We now use Lemma 3 to bound this total degree (recall that in our setup for this lemma \( |V| = |W| = n, |X| = cn \), and \( |U| = (1 + c)n \).

\[
\sum_{v \in V} d(v) = \sum_{u \in U} d(u) \leq \frac{\beta |U|}{1 + (1 + c)} = \frac{\beta n}{2 + c} = \frac{\beta n}{2 + c}.
\]

Now, constraint 2 of our lemma clearly implies that

\[
\sum_{w \in W} d(w) \geq n(\beta - 1) - \sum_{v \in V} d(v) = n(\beta - 1) - \sum_{u \in U} d(u).
\]

But note that since \( U = W \cup X \) we have

\[
\sum_{x \in X} d(x) = \sum_{u \in U} d(u) - \sum_{w \in W} d(w) \leq 2 \sum_{u \in U} d(u) - n(\beta - 1)
\]

\[
\leq 2\beta n \frac{1 + c}{2 + c} - n\beta + n = \beta n \frac{c}{2 + c} + n
\]

Dividing this through by \( |X| = cn \) we get that the average degree in \( |X| \) is at most \( \beta / (2 + c) + 1 / c \).

As in Lemma 3 bipartiteness is not required here; we only need that all edges incident to \( U \) go to \( V \).

**Proof of Theorem 5** The proof is very similar to that of Theorem 3. Say for contradiction that \( \mu(H) < (1 - \epsilon)\mu(G) \). There must exist sets \( S_L^*, S_L, S_R, T_R, T_L \) as in Lemma 2. Following the same logic as in the beginning of Theorem 3 we reach Equation 1, which we repeat here \( (\beta(1 - \lambda) \) is replaced by \( \beta - 1 \) in the transition from an unweighted to a weighted EDCS):

\[
\text{average degree of } S_L^* \cup T_R^* \geq \frac{(\beta - 1)(\mu(G) - \mu(H))}{2(\mu(G) - \mu(H))} = \frac{\beta - 1}{2} \quad (5)
\]

\[
X \end{equation}
As in Theorem 3 we now show that such a high average degree is not possible, thus yielding the desired contradiction. We do this by invoking Lemma 4: \( S^*_L \cup T^*_R \) corresponds to \( X \), \( S_L \cup T_R \) corresponds to \( W \), and \( S_R \cup S_L \) corresponds to \( V \) (so \( U = X \cup W = S^*_L \cup S_L \cup T^*_R \cup T_R \)). Property 2 of Lemma 2 precisely tells us that all edges from \( U \) go to \( V \), as needed in Lemma 4. Constraint 1 of Lemma 4 is clearly satisfied by the property P1 of a weighted EDCS. To see that constraint 2 of Lemma 4 is satisfied, note that by property 3 of Lemma 2 there is a perfect matching between \( W = S_L \cup T_R \) and \( V = S_R \cup T_L \), and by property P2 of a weighted EDCS for each edge \((v,w)\) in this matching we have \( d_H(v) + d_H(w) \geq \beta - 1 \). Finally, straightforwardly combining the size bounds in properties 1 and 4 of Lemma 2 with our to-be-contradicted assumption that \( \mu(H) < (1 - \epsilon)\mu(G) \), we have

\[
|X| = |S^*_L \cup T^*_R| = 2(\mu(G) - \mu(H)) > 2\epsilon\mu(G) = 2\epsilon|V|.
\]

Thus, Lemma 3 tells us that the average degree of \( X \) is at most \( \beta/(2+2\epsilon)+1/(2\epsilon) \), which some simple algebra shows is strictly less than \( (\beta - 1)/2 \) because by the assumption of Theorem 5 we have \( \beta > \frac{4\epsilon - 2}{\epsilon} \) (see footnote \[1\]). We have thus arrived at a contradiction with Equation 1, so our original assumption that \( \mu(H) < (1 - \epsilon)\mu(G) \) must be false. \( \square \)

E Dynamically Maintaining an EDCS in a Bipartite Graph

In this section we provide a formal proof of Theorems 4 and 6. We address the low arboricity case first (Theorem 6), as it is a simpler algorithm and analysis, and then explain how to extend our work to general graphs.

In both cases, we will show that when an edge is inserted or deleted in \( G \), we only need to do a small number of updates to maintain an \( H \) with the desired EDCS properties. We will show that we will always be able to find a specific type of alternating path that will allow us to maintain \( H \). We will need to show that the length of the alternating path is bounded, and that we can also find such a path efficiently. Finding a path will involve using the edge orientation to control exactly which neighbors need to be notified of a change in vertex degree. In the general graph case, we will not be able to efficiently maintain accurate degree counts, so we will only maintain approximate counts and use a bucketing scheme to identify edges of appropriate degree.

E.1 Dynamically Maintaining an EDCS in Small Arboricity Graphs

For small arboricity graphs, we will ultimately maintain a weighted EDCS. We first describe how to maintain a unweighted EDCS with \( \beta^- = \beta - 1 \) and then at

1. \( \frac{\beta}{2+2\epsilon} + \frac{1}{2\epsilon} < \frac{\beta - 1}{2} \) if and only if \( \frac{\beta}{2} \cdot (1 - \frac{1}{1+\epsilon}) > \frac{1}{2} + \frac{1}{2} \), which is true when \( \beta > 4/\epsilon^2 \) because \( \frac{\beta}{2} \cdot (1 - \frac{1}{1+\epsilon}) = \frac{\beta}{2} \cdot \frac{\epsilon}{1+\epsilon} > \frac{\epsilon^2}{4} \cdot \frac{\epsilon}{1+\epsilon} = \frac{\epsilon}{4} > \frac{1}{2} + \frac{1}{2} \), as desired.
the end of this section we extend the result to a weighted EDCS with $\beta^- = \beta - 1$. To maintain the unweighted EDCS we define two classes of edges:

- a full edge $(u, v)$ is in $H$ and has $d_H(u) + d_H(v) = \beta$
- a deficient edge $(u, v)$ has $d_H(u) + d_H(v) = \beta - 1$.

Recall that when an edge is inserted or deleted, we first update the orientation, thereby causing some number of edge flips. Recall that a vertex always accurately knows its own degree, but may not accurately know the degree of all its neighbors. From the orientation, each vertex owns some edges. We will maintain the invariant that we always know accurately the degree of our unowned incident edges. Also recall that we use $\delta(u, v)$ to denote the edge degree of $(u, v)$, $d_H(u) + d_H(v)$.

**Insertion** Consider an edge $(u, v)$ that has just been added to $G$. If $\delta(v, w) \geq \beta - 1$, then we do not add the edge to $H$, and the properties P1 and P2 from Definition 1 remain satisfied. On the other hand, if $\delta(v, w) < \beta - 1$, we want to add $(u, v)$ to $H$. Doing so will increase $d(u)$ and $d(v)$ by 1, which may lead to a violation of P1 for other edges in $H$ that are incident to either $u$ or $v$. Thus, we will need to find a type of alternating path that will allow us to add $(u, v)$ and still maintain P1 and P2 for all vertices.

We say that a vertex $x$ is increase-safe if it has no incident full edges, and say that it is decrease-safe if it has no incident deficient edges. Returning to adding $(u, v)$ to $H$, let’s focus on vertex $v$; we will then deal with vertex $u$ analogously. If $v$ is increase safe, then when we add $(u, v)$ we have not violated P1 for any edges incident to $v$, since there are no full edges incident to $v$. If $v$ is not increase safe, then it must have at least one incident full edge, say $(v, p_1)$. We would like to add $(u, v)$ to $H$ and remove $(v, p_1)$ from $H$, thereby leaving $v$’s degree unchanged. Doing so would decrease $d_H(p_1)$, which we can do only if $p_1$ is decrease safe. If $p_1$ is decrease safe, then adding $(u, v)$ to $H$ and removing $(v, p_1)$ leaves $v$’s degree unchanged, decreases $p_1$’s degree and reestablishes P1 and P2 for all vertices (except possibly $u$). However, if $p_1$ is not decrease-safe, it must have an incident deficient edge, say $(p_1, p_2)$. We can add this edge to $P$ and continue from $p_2$ we did from $v$. We can continue in this manner, stopping when we find either an increase-safe vertex or decrease-safe vertex. Assume that the set of edges we find form a simple path. Then we can exchange the role of the matched and unmatched edges on $P$ thereby reestablishing P1 and P2 for all vertices (except possibly $u$). Note that this path may leave the number of edges in $H$ unchanged, or may increase the number of edges in $H$ by one. Either outcome is acceptable.

We now argue that the set of edges we find do form a simple path. In addition, in order to bound the time, we would like to bound the length of $P$. We do so with the following lemma:

**Lemma 5.** Let $P$ be a path of alternating full and deficient edges. Then $P$ is simple and the length of $P$ is at most $2\beta + 1$. 
Proof. Consider first the case that $P = (p_0, p_1, \ldots, p_k)$ starts with a full edge. Let $d = d_H(p_0)$ and clearly $d \leq \beta$. Since $(p_0, p_1)$ is full, $d_H(p_1) = \beta - d$. Since $(p_1, p_2)$ is deficient, $d_H(p_2) = \beta - d_H(p_1) - 1 = \beta - (\beta - d) - 1 = d - 1$. Continuing, we get that $d_H(p_3) = \beta - d + 1$, $d_H(p_4) = d - 2$, and in general $d_H(p_i) = d - 2i$ for even $i$ (See Figure 3). Since each vertex in $P$ has an incident full edge, all vertices have positive degree, and thus $P$ can have at most $2\beta$ edges. Furthermore, if we consider all the vertices on $P$ that are on the same side of the bipartite graph, they all have distinct $d_H$ values, and hence they must be distinct and the path is therefore simple.

If $P$ starts with a deficient edge, we can go through the same argument. Now the degrees of the odd indexed vertices are decreasing, and we have that the length of the path is at most $2\beta + 1$.

Note that these alternating paths are analogous to augmenting paths in a standard B-matching, but are much more locally well behaved: when searching for an ordinary augmenting path we might reach a dead end and have to backtrack, but in an EDCS following any full/deficient edge is guaranteed to eventually lead towards the desired alternating path. After finding an alternating path from $v$, we repeat the same procedure starting at $u$. (Note that it is fine for the path from $v$ to intersect the path from $u$, as we execute the fixing up procedures sequentially). We have thus shown the following:

**Lemma 6.** After inserting an edge into $G$, we can reestablish $P_1$ and $P_2$ using at most $4\beta$ insertions/deletions from $H$.

**Deletions** Deleting an edge $u$ from $G$ is handled in a similar manner to insertions. If $(u, v)$ is not in $H$, then we do not need to change $H$. If $(u, v)$ is in $H$ and both $u$ and $v$ are decrease-safe, we just remove the edge $(u, v)$. Otherwise, we find an alternating path in the same way we did for insertions and observe that Lemma 5 applies for paths beginning with both full and deficient edges. Thus we have:

**Lemma 7.** After deleting an edge from $G$, we can reestablish $P_1$ and $P_2$ using at most $4\beta$ insertions/deletions from $H$.

**Finding alternating paths** In order to find the alternating paths, we need to maintain the necessary data structures to identify full and deficient edges. Each vertex $v$ will maintain the following information: 1) $d_H(v)$, its degree in $H$, 2) a set $O(v)$ consisting of the edges it owns, 3) a set $F(v)$ consisting of its unowned incident full incident edges, and 4), a set $E(v)$ consisting of it unowned incident deficient edges. Each of these sets has no particular order, and can be maintained easily as a doubly linked list.

We now conclude this section and provide a proof of Theorem 6. In the proof, we will also explain how to maintain a weighted edge degree constrained subgraph rather than an unweighted one.

**Proof of Theorem 6** By Lemmas 6 and 7, the update ratio (see Definition 3) is clearly $O(\beta)$. To bound the update time, we first perform $O(\alpha + \log(n))$
reorientations using Theorem 7, which takes $O(\alpha + \log n)$ time. For each flipped edge $(v, w)$ we update the vertices $v$ and $w$, moving the edge in/out of the lists $O()$, $F()$ and $E()$ as appropriate. This takes $O(1)$ per flip, so $O(\alpha + \log(n))$ time in total.

Next we need to implement the search for the alternating path $P$ of full/deficient edges. At vertex $v$, to search for an incident full edge, just check the set $F(v)$. If it is non-empty, a full edge is found. If it is empty, then check the owned edges. There are only $O(\alpha + \log n)$ owned edges, so this operation takes $O(\alpha + \log n)$ time; by Lemmas 6 and 7, this process is repeated $O(\beta)$ times for a total of $O(\beta(\alpha + \log n))$ time.

Once we find an alternating path, we exchange its matched/unmatched edges to preserve properties P1 and P2; this can change the degrees of at most two vertices (the path’s endpoints) and so change the fullness/deficiency of their edges. The owned neighbors of these vertices may thus have to modify their sets $E()$ and $F()$, but each vertex owns at most $O(\alpha + \log(n))$ edges, so this takes $O(\alpha + \log(n))$ time in total.

We next extend our algorithm to a weighted EDCS by paying an extra factor of $O(\beta)$ in the running time, thinking of the an edge of weight $w$ in the weighted EDCS as $w$ parallel edges in the unweighted one. Observe that all weights are bounded by $\beta$. The only change to the algorithm is the implementation of an edge deletion. Now, if an edge is deleted from $G$, it may have weight up to $\beta$ in $H$. However, we can simply delete from $H$ all $\beta$ (unweighted) parallel edges, using the algorithm for an unweighted EDCS. This will increase the running time by a factor of at most $\beta$.

All together the time to process an insertion/deletion in $G$ is $O(\beta^2(\alpha + \log n) + \alpha(\alpha + \log n))$. □

E.2 Dynamically Maintaining an EDCS in General Bipartite Graphs

In this section, we describe how to maintain $H$ in a general bipartite graph (Theorem 4). At a high level, we use similar ideas to the general case – we will use an orientation to describe a data structure consisting of owned and unowned edges and we will, when edges are inserted or deleted, look for alternating paths of full and deficient edges. There will, however, be several technical differences. First, we will maintain an unweighted edge degree constrained subgraph. The biggest difference however, is that, when we orient edges, by Theorem 8, a vertex may own up to $3\sqrt{m}$ edges. Therefore, when a vertex degree changes, we do not have time to update all $3\sqrt{m}$ neighbors, we will only have time to update a small fraction of them. Thus, we will not be able to assume that we accurately know the degrees of our neighbors, and therefore know which edges are full and which are deficient. To compensate for this lack of knowledge, we will introduce a bucketing scheme, where edges are placed in buckets based on our estimate of their distance label. We will then show that our estimates are not too far off, that is, we will only have to search a small number of buckets to find a full
or deficient edge. We will also have to introduce a larger gap between full and deficient, which will also alter the analysis of the length of an alternating path.

We now describe the details of our approach. We assume familiarity with the Section E.1 and only emphasize the differences. Also, rather than separately dealing with edge reorientations, we just process the flip of an edge \((u, v)\) as a deletion of the edge, and then an insertion in the opposite direction. Since by Theorem 8 each change in \(G\) causes at most \(O(1)\) edge flips, this only increases the running time by a constant factor.

Recall the parameter \(\lambda\) from Theorem 4 and assume for simplicity that \(\lambda \beta\) is an integral multiple of 6. We begin by redefining full and deficient.

- A full edge \((u, v) \in H\) has \(d_H(u) + d_H(v) = \beta\),
- A deficient edge \((u, v) \in G - H\) has \(d_H(u) + d_H(v) = \beta(1 - \lambda)\).

We have, in particular, redefined deficient to be not \(\beta - 1\) but rather a constant fraction of \(\beta\). We add this extra space because we will no longer be able to maintain degrees exactly, and thus when we augment, we will no longer be able to alternate between full and deficient edges but rather between full and a relaxed notion of deficient.

In order to define this relaxed notion, we introduce the notion of edge ranges, which capture the various intermediary levels of deficiency and fullness that an edge can have. We will think of our edge degrees as being partitioned into 8 ranges \(F_i\) (\(F\) for different degrees of fullness) defined in terms of a parameter \(\ell = \beta \lambda / 6\). We will then refer to an edge as being in one of the ranges, depending on its edge degree (Note that when we say an edge is in one of these ranges, this always refers to the actual edge degree, not to any incorrect estimates we may have).

- Range \(F_0\) contains edges with edges degree \(< \beta(1 - \lambda)\) (such edges cannot be unused, as they would violate Property P2 of an EDCS).
- Range \(F_7\) contains edges with edge degree \(\beta\). (These are the full edges.)
- Range \(F_i\), for \(1 \leq i \leq 6\) contains edges with edge degree in \([\beta(1 - \lambda) + \ell (i - 1), \beta(1 - \lambda) + \ell i]\).

We call unused edges in \(F_1, \ldots, F_5\) augmentable. We specifically omit \(F_6\) from the definition of augmentable, even though such edges can be used by property P1 of an EDCS, in order to leave a gap between augmentable edges and full edges.

We also recall that edge vertex accurately knows its own degree. The inaccuracy comes from the inability of a vertex to inform all its neighbors, or even all its owned neighbors of its correct degree.

We now describe the insertion and deletion procedures. As in Section E.1 we will first describe them at a high level, ignoring the implementation details and then fill those in later.

**Insertion.** Consider an edge \((u, v)\) that has just been added to \(G\). We know \(d_H(u)\) and \(d_H(v)\) exactly and can therefore compute edge degree \(\delta(u, v)\) If \(\delta(v, w) \geq\)
Let $P = (p_0, p_1, \ldots, p_k)$ be a path of alternating full and augmentable edges. Then $P$ is simple and the length of $P$ is at most $12/\lambda + 1$.

**Proof.** Consider first the case that $P = (p_0, p_1, \ldots, p_k)$ starts with a full edge. Let $d = d_H(p_0)$ and clearly $d \leq \beta$. Since $(p_0, p_1)$ is full, $d_H(p_1) = \beta - d$. Since $(p_1, p_2)$ is augmentable, we have that $d_H(p_1) + d_H(p_2) \leq \beta - \lambda \beta/6$ which implies that $d_H(p_2) \leq d - \lambda \beta/6$. Continuing, we get that $d_H(p_3) \geq \beta - d + \lambda \beta/6$, $d_H(p_4) \leq d - 2\lambda \beta/6$, and in general $d_H(p_i) \leq d - i\lambda \beta/3$ for even $i$ (Figure 3 contains a similar argument, except there we just had $\beta - 1$ instead of $\beta(1 - \lambda/6)$.) Since $d \leq \beta$, we have that after $\beta/(\beta \lambda/6) = 6/\lambda$ even vertices, the degree will be 0 and the path will terminate. The length of the path will therefore be bounded by twice the number of even indexed vertices plus one for $12/\lambda + 1$. Furthermore, if we consider all the vertices on $P$ that are on the same side of the bipartite graph, they all have distinct $d_H$ values, and hence they must be distinct and the path is therefore simple.

Now consider that $P = (p_0, p_1, \ldots, p_k)$ starts with an augmentable edge. Going through the same argument, we now have that the degrees of the odd vertices remain satisfied. (Recall that $\beta^- = \beta(1 - \lambda)$). On the other hand, if $\delta(v, w) < \beta(1 - \lambda)$, we want to add $(u, v)$ to $H$. Doing so will increase $d(u)$ and $d(v)$ by 1, which may lead to a violation of $P1$ for other edges in $H$ that are incident to either $u$ or $v$. Thus, we will need to find a type of alternating path that will allow us to add $(u, v)$ and still maintain $P1$ and $P2$ for all vertices.

We now say that a vertex $x$ is **increase-safe** if it has no incident full edges, and say that it is **decrease-safe** if it has no incident augmentable edges.

Returning to adding $(u, v)$ to $H$, let’s focus on vertex $v$; we will then deal with vertex $u$ analogously. If $v$ is increase safe, then when we add $(u, v)$ we have not violated $P1$ for any edges incident to $v$, since there are no full edges incident to $v$. If $v$ is not increase safe, then it must have at least one incident full edge, say $(v, p_1)$. We would like to add $(u, v)$ to $H$ and remove $(v, p_1)$ to $H$, thereby leaving $v$’s degree unchanged. Doing so would decrease $d_H(p_1)$, which we can do only if $p_1$ is decrease safe. If $p_1$ is decrease safe, then adding $(u, v)$ to $H$ and removing $(v, p_1)$ leaves $v$’s degree unchanged and decreases $p_1$’s degree and reestablishes $P1$ and $P2$ for all vertices (except possibly $u$). However, if $p_1$ is not decrease-safe, it must have an incident deficient edge, say $(p_1, p_2)$. We can add this edge to $P$ and continue from $p_2$ we did from $v$. We can continue in this manner, stopping when we find either an increase-safe vertex or decrease-safe vertex. Assume that the set of edges we find form a simple path. Then we can exchange the role of the matched and unmatched edges on $P$ thereby reestablishing $P1$ and $P2$ for all vertices (except possibly $u$). Note that this path may leave the number of edges in $H$ unchanged, or may increase the number of edges in $H$ by one. Either outcome is acceptable.

We now argue that the set of edges we find do form a simple path. In addition, in order to bound the time, we would like to bound the length of $P$. We do so with the following lemma:

**Lemma 8.** Let $P$ be a path of alternating full and augmentable edges. Then $P$ is simple and the length of $P$ is at most $12/\lambda + 1$. 
indexed vertices are decreasing, and we have that the length of the path is at most \(12/\lambda + 1\).

After finding an alternating path from \(v\), we repeat the same procedure starting at \(u\). (Note that it is fine for the path from \(v\) to intersect the path from \(u\), as we execute the fixing up procedures sequentially). We have thus shown the following:

**Lemma 9.** After inserting an edge into \(G\), we can reestablish \(P1\) and \(P2\) using at most \(24/\lambda + 2\) insertions/deletions from \(H\).

**Deletions** Deleting an edge \(u\) from \(G\) is handled in a similar manner to insertions. If \((u, v)\) is not in \(H\), then we do not need to change \(H\). If \((u, v)\) is in \(H\) and both \(u\) and \(v\) are decrease-safe, we just remove the edge \((u, v)\). Otherwise, we find an alternating path in the same way we did for insertions and observe that Lemma 8 applies for paths beginning with both full and augmentable edges. Thus we have

**Lemma 10.** After deleting an edge from \(G\), we can reestablish \(P1\) and \(P2\) using at most \(24/\lambda + 2\) insertions/deletions from \(H\).

**Finding Alternating Paths** Now, in order to implement the augmenting procedure, we need to be able to accurately identify when a vertex has an incident full edge and when it has an incident augmentable edge. Identifying the incident full edge is straightforward: a full edge is in \(H\), and \(d_H(v) \leq \beta\), so in \(O(\beta)\) time one can scan all the incident edges in \(H\), both owned and unowned. Identifying an incident augmentable edge is more challenging, and we will need to introduce several additional ideas and data structures.

**Buckets** Conceptually, we want each vertex to maintain for each of its incident edges which of the ranges \(F_0, \ldots, F_7\) that edge belongs to, but we will need to do that in an indirect way. The main challenge that arises is that a vertex may maintain inaccurate information about its neighbors. Another challenge is that if a vertex \(v\) tried to bucket its edge degrees, or even its estimates for edge degrees, then an increase in \(d_H(v)\) would cause all of its incident edges to increase in edge degree, and so might require moving many edges to different buckets.

Our solution is to let each vertex \(u\) maintain (possibly inaccurate) information about the degree of all of its unowned neighbors by bucketing each neighbor \(v\) according to \(u\)’s estimate of \(v\)’s degree, denoted \(\tilde{d}_u(v)\). We will use buckets of width \(\ell = \beta \lambda / 6\), so \(u\) has \(\beta/\ell = 6/\lambda\) buckets \(B_1^u, B_2^u, \ldots, B_6^u/\lambda\). That is, \(B_1^u\) contains neighbors \(v\) for which \(\ell(1 - 1) \leq \tilde{d}_u(v) < \ell\). We say that vertex \(v\) properly belongs in bucket \(B_i^u\) if \(\ell(i - 1) \leq d_H(v) < \ell i\), that is if \(B_i^u\) would be \(v\)’s bucket if \(u\) had accurate information about the degree of \(v\).

**Edge Updates** Let \(r = 18\sqrt{m}/(\lambda \beta)\). Each vertex \(u\) will maintain its owned edges in a doubly-linked circular list \(L^u\) with two pointers \(p\) and \(q\). An information update consists of informing the next \(r\) edges \((u, v)\) on the list of the accurate
value of $d_H(u)$, so that they can update their bucket structures $B^v$. The information update will always start at the pointer $p^v$, and $p^v$ will advance as the information update proceeds. New edges will be added to the list just before $p^v$ (i.e. to the “back” of the list). A second pointer $q^v$ will be the repair pointer, and will point to the next edge to consider including in $H$ (more on this later).

The algorithm We will now describe the algorithm to search for an alternating path of full and augmentable edges.

- As observed before, finding a full edge can be done in $O(\beta)$ time.
- To find an augmentable edge incident to $u$, we first want to see if $u$ has any unowned edges in the range $F_1,\ldots,F_5$. To do this, we start checking the buckets of $u$. We start with the bucket whose degree interval contains $\beta(1 - \lambda) - \ell - d_H(u)$, i.e. the bucket $B^u_i$ such that $\beta(1 - \lambda) - \ell - d_H(u) \in [\ell(i - 1), \ell i)$; the reasoning behind these boundaries will become evident later. We then check bucket $B^u_i$ and then buckets $B^u_{i+1}$ then $B^u_{i+2}$ all the way up to $B^u_{i+7}$ until we find some non-empty bucket. If we don’t find a non-empty bucket, we declare that we have failed to find an augmentable edge, and move on the next step. Otherwise, we pick an arbitrary edge $(u,v)$ from the first non-empty bucket that we find (remember, we always from smaller to larger degree buckets) and check if $(u,v)$ is augmentable by checking if $d_H(u) + d_H(v)$ is in $F_5$ or below; if it is augmentable we augment down it, otherwise we do NOT check for more edges but simply declare that we have failed to find an augmenting edge, and move on to the next step. Note that this whole operation thus takes only $O(1)$ time as we only check 8 buckets. Note also, however, that it is quite possible for $u$ to not find an unowned augmenting edge even though it actually has one: the buckets have inaccurate degrees, so $u$ may happen to pick a vertex $v$ from the bucket that has large degree (and so $(u,v)$ is not augmentable), even though there are other vertices in the bucket with small degree.
- If we didn’t find an augmentable unowned edge in the previous step, we look at the next $r$ owned edges of $u$ by starting at the repair pointer $q^v$ and moving forward $r$ steps on the list. For each one of these edges, we can exactly compute $d_H(u) + d_H(v)$ and check if the edge is augmentable. If we find an augmentable edge we stop and take that edge. Otherwise, we move on the the next step.
- If we make it to this step then we were unable to find an augmentable edge (though one may in fact exist), so we allow this vertex to be the end of the augmenting path, we increase/decrease its degree accordingly, and we perform an information update on this vertex (see above).

In order to prove the correctness of the algorithm, we will prove that it maintains the following invariants:

1. Every time vertex $v$’s degree changes, we execute an information update at $v$. 

2. Say that edge \((u, v)\) is owned by \(v\) and recall that \(\hat{d}_u(v)\) is \(u\)'s estimate of the degree of \(v\) and \(d_H(v)\) is the actual degree of \(v\). Then, we always have \(d_H(v) - \ell \leq \hat{d}_u(v) \leq d_H(v) + \ell\). In particular, since \(\ell\) is the range of a bucket \(B^u_i\), if \(v\) properly belongs in bucket \(B^u_i\) then it is in fact contained in one of buckets \(B^u_{i-1}, B^u_i, \text{ or } B^u_{i+1}\).

3. When a vertex \(u\) checks its unowned neighbors for an augmentable edge, every augmentable edge is in one of the 8 buckets that \(u\) is allowed to look at (though \(u\) may not end up checking that particular edge).

4. If \(u\) has an unowned edge to vertex \(v\) of degree \(d(v)\), then when \(u\) looks in its bucket structure for an unowned augmentable edge, if it picks some edge \((u, w)\) then we must have: \(d(w) \leq d(v) + 3\ell\).

5. As long as \(v\) owns \((u, v)\) and \((u, v)\) is in range \(F_2\) or lower, \(d(u)\) cannot decrease.

6. If \(v\) owns \((v, w)\) and \(d(v)\), over some time range, has decreased by \(\delta\), then at some point in that time range the edge \((v, w)\) was not augmentable.

**Proof that the invariants hold**

1. This clearly holds by the design of our algorithm.
2. \(v\) owns \((u, v)\), so consider the last time \(v\) sent its accurate information to \(u\) during an information update of \(v\). Now, every time the degree of \(v\) changes it updates the information of \(r\) owned neighbors, so since by our orientation (Theorem 8), \(v\) owns at most \(3\sqrt{m}\) edges, the degree of \(v\) can change at most \(3\sqrt{m}/r = \beta \lambda / 6 = \ell\) times before it updates \(u\) again, so \(\hat{d}_u(v)\) is off by an additive factor of at most \(\ell\), as desired.

3. This follows from the fact that the 8 buckets \(u\) is allowed to look at span all degrees between \(\beta(1 - \lambda) - \ell - d_H(u)\) and \(\beta + \ell - d_H(u)\), so since by Invariant 2, \(u\)'s degree information about a vertex is at most one bucket off, any vertex with actual degree \(d_H(v)\) between \(\beta(1 - \lambda) - d_H(u)\) and \(\beta - d_H(u)\), so all edges with edge degree between \(\beta(1 - \lambda)\) and \(\beta\) – which includes all augmentable edges – are in one of these 8 buckets.

4. By Invariant 2 if \(d_H(v)\) properly belongs in bucket \(B^u_i\) then in reality \(\hat{d}_u(v)\) will be in bucket at most \(B^u_{i+1}\). Thus, since we always check lower indexed buckets first, the chosen edge \((v, w)\) will be to a vertex \(w\) for which \(\hat{d}_u(w)\) is in bucket at most \(B^u_{i+1}\). Applying Invariant 2 again, \(d_H(w)\) is in bucket at most \(B^u_{i+2}\), so since \(d_H(v)\) was in bucket \(B^u_i\) and the size of each bucket is \(\ell\), the difference between \(d_H(v)\) and \(d_H(w)\) is at most \(3\ell\).

5. The degree of \(u\) can only decrease if when \(u\) searches for an unowned augmentable edge, and it does not find one. By invariant 4 if during its search for an augmentable edge, \(u\) picks \((u, w)\) then \(d_H(w) \leq d_H(v) + 3\ell\). But since by assumption \((u, v)\) is in \(F_2\) or lower, we must have that \((u, w)\) is in \(F_5\) or lower (the size of each range \(F_i\) is precisely \(\ell\)), so \((u, w)\) is augmentable and the degree of \(u\) does not decrease.

6. Every time the degree of \(v\) decreases it scans \(r\) edges in its repair list. Thus, by the time its degree changes by \(\ell\) it has scanned \(r\ell = 3\sqrt{m}\) edges, so since by our orientation algorithm every vertex owns at most \(3\sqrt{m}\) edges
(Theorem 8), and \( v \) must have reached \((v, w)\); at that point, either \((v, w)\) was already not augmentable and we are done, or it was augmentable in which case \( v \) would augment down it and \((v, w)\) would become used and hence not augmentable.

Using the above invariants we the algorithm always maintains the properties of an edge degree constrained subgraph. First, since the algorithm explicitly checks for full edges we never increase the degree of a vertex that has an adjacent full edge, and so property P1 of an EDCS is always maintained (see Definition 1). To verify property P2, let us say for contradiction that there is some unused edge \((u, v)\) such that \( d_H(u) + d_H(v) < \beta(1 - \lambda) \), i.e. such that \((u, v)\) is in \( F_0 \). Let us say, wlog, that \( v \) owns \((u, v)\). Note note that at some point in the sequence \((u, v)\) must have been not augmentable: if it was augmentable when it was inserted, then the algorithm would have augmented down it, and the only way it could become unused is if it was augmented down when full, in which case it would be in \( F_6 \) and hence not augmentable. There must exist a last time that \((u, v)\) dropped from \( F_3 \) to \( F_2 \), i.e. the time after which it was always in \( F_2 \) or below. By Invariant 5, the degree of \( u \) cannot have decrease since that point in time. By Invariant 6, the degree of \( v \) can decrease by at most \( \ell \). Thus, edge \((u, v)\) cannot be below range \( F_1 \), which contradicts our assumption that it was in \( F_0 \).

We conclude with a summary and proof of Theorem 4.

**Proof of Theorem 4.** From Lemma 8 we have that the update ratio is \( O(1/\lambda) \).

To bound the running time, we examine the time to find a path of length \( O(1/\lambda) \). Each search takes either \( O(\beta) \) time for a full edge, or to find an augmentable edge \( O(\sqrt{m}/(\lambda \beta)) \) time. This gives a total time of \( O((1/\lambda) (\beta + \sqrt{m}/\lambda \beta)) \) time.

\(\square\)

### F Dynamic Orientation

In this section we formally state the dynamic orientation results used by our algorithm, and prove Theorem 8 which is new to this paper.

**Theorem 7** ([15]). Let \( G \) be a graph that always has arboricity at most \( \alpha \). We maintain an orientation, under edge insertions and deletions, with the following properties: the maximum load at all times is \( O(\alpha + \log n) \), the worst-case number of flips per insertion/deletion is also \( O((\alpha + \log(n))) \), and the worst-case time to process an insertion/deletion in \( G \) is \( O(\alpha(\alpha + \log(n))) \). (If we do not have an upper bound on \( \alpha(G) \) in advance there is a variant whose bounds are in terms of the exact arboricity of the current graph, at the expense of an extra \( \log(n) \) factor).

**Theorem 8.** In a graph \( G \), we can maintain an orientation, under insertions and deletions, with the following properties: the max load at all times is at most \( 3\sqrt{m} \), the worst-case number of flips per insert/deletion in \( G \) is \( O(1) \), and the worst-case time spent per insertion/deletion in \( G \) is \( O(1) \).
Proof. For simplicity of analysis, we assume that we begin with a graph with no edges, and update from there. Let us start with a few definitions.

**Definition 4.** Define a vertex to be small if it has degree (not load) less than \(2\sqrt{m}\) and large if it has degree greater than or equal to \(2\sqrt{m}\). Given some orientation, define a vertex in the current orientation to be heavy if it has load greater than \(2\sqrt{m}\).

**Observation 9** A graph can contain at most \(\sqrt{m}\) large vertices, and at most \(2\sqrt{m}\) vertices of degree \(\geq \sqrt{m}\). Otherwise, the total degree of these vertices would be greater than \(2\sqrt{m}\sqrt{m} = 2m\), so the number of edges in the graph would be greater than \(m\).

The above observation makes it clear why given any graph we can compute a \(2\sqrt{m}\)-orientation in linear time. Simply let small vertices own all of their edges; if an edge is between two small vertices or two large vertices, it can go either way. Now no vertex can have load greater than \(2\sqrt{m}\), as then all its owned edge would go to large vertices, which would contradict the observation above.

**Observation 10** The natural way for a vertex to transition from small to large, or from heavy to non-heavy, is due to the insertion or deletion of edges incident to this vertex. But because all of these terms are defined in terms of \(\sqrt{m}\), a vertex can also transition simply because the number of edges has changed, and so \(\sqrt{m}\) has changed. This is certainly not a big deal as \(\sqrt{m}\) changes very slowly, but it is inconvenient for our analysis as we would like to treat \(\sqrt{m}\) as a fixed number. We handle this using a standard technique in dynamic algorithms: for \(\sqrt{m}\) to double, \(m\) would have to increase by a factor of 4, and \(4m = O(m)\) time is enough to slowly construct a new orientation from scratch in the background (by staggering the linear-time algorithm above over many updates). For this reason, we can assume that the number of edges is fixed within a factor of 2, and let \(m\) refer to the upper bound of this range, thus allowing us to treat \(\sqrt{m}\) as fixed.

Observation 9 provides a very simple algorithm for maintaining a \(3\sqrt{m}\)-orientation in amortized update time \(O(1)\). The algorithm is as follows: when a vertex reaches load above \(3\sqrt{m}\), simply scan all of its edges and flip edges going to small vertices. By Observation 9, fewer than \(2\sqrt{m}\) of its edges went to large vertices, so we have transformed the vertex into a non-heavy vertex, without adding any heavy vertices. This transformation requires \(3\sqrt{m}\) time to scan all neighbors, but it takes \(\sqrt{m}\) to turn a non-heavy vertex back into one with degree greater than \(3\sqrt{m}\), so we only need 3 credits per update.

Worst-case update time is only slightly more difficult. Whenever an edge is inserted, if exactly one of the endpoints is small we give that endpoint ownership; otherwise, we assign ownership arbitrarily. Every time the load of a heavy vertex increases, we will scan 5 edges that it owns, and flip any edges going to small vertices. From the other direction, every time the degree of a small vertex decreases we will scan 5 of its edges (owned and not owned) and automatically flip them towards the small vertex. Scanning will occur in a round-robin fashion:
each vertex just stores a list of outgoing edges and another list of outgoing owned edges, and moves along this list; if a new edge is inserted into one of the lists, we put it at the “back of the list”, i.e. right behind the current pointer. We will now show that this algorithm maintains a $3\sqrt{m}$-orientation.

**Invariant 11** As $G$ changes, at no point in time can a vertex $u$ of load more than $3\sqrt{m}$ own an edge $(u,v)$ where $v$ has degree (not load) less than $\sqrt{m}$.

**Proof.** For the sake of contradiction, let us consider the first time that $u$ owns such an edge $(u,v)$. Let $t$ be the last time this edge flipped – i.e. the time after which $u$ always owned the edge. Now, it is clear from our algorithm design that when the flip occurred it could not have been the case that $v$ was small and $u$ was heavy and yet ownership was given to $u$. So at time $t$ either $v$ was large or $u$ was not heavy, but by the current (to-be-contradicted) time, $v$ is small and $u$ is heavy. Let $t^*$ be the very last time that $v$ transitioned from large to small, or $u$ transitioned from non-heavy to heavy, whichever came later. If $v$ transitioning from large to small came later, then $v$ must have experienced at least $\sqrt{m}$ degree decreases since time $t^*$ (it dropped from degree $2\sqrt{m}$ to degree $\sqrt{m}$), and over this time it scanned at least $5\sqrt{m}$ edges. Since scanning is done round-robin, it would certainly have examined edge $(u,v)$ during one of its scans, and would have flipped it because small vertices always flip – contradiction. But similarly, if the later event was $u$ becoming heavy, then $u$ must have experienced at least $\sqrt{m}$ degree increases (from $2\sqrt{m}$ to $3\sqrt{m}$), over which time it scanned $5\sqrt{m}$ edges and so would certainly have scanned $(u,v)$ and would have flipped it because after time $t^*$ $v$ was small – again a contradiction.

The above invariant shows that as our algorithm processes updates to $G$, no vertex can ever have load above $3\sqrt{m}$, as then all of its neighbors would have degree more than $\sqrt{m}$, which is impossible by Observation 9.
Fig. 1. The right side of the figure is the desired \((S, T)\) split of Lemma 2. The left side of the figure is the standard \((P, Q)\) cut induced by a matching. The figure illustrates both their similarities and their differences; the key argument of Lemma 2 is that we can fix up the left side to look like the right. On the left side, the starred sets are free vertices, the \(P\) side of the cut is everyone reachable from \(P^*\) in the residual graph \(H\), and the \(Q\) side of the cut is everyone else. The red dashed edges are edges in \(G\) but not \(H\); there must be such edges crossing the cut, but they need not be between \(P^*\) and \(Q^*\); they can go anywhere from one side of the cut to the other. The single green edges represent a matching in \(H\) between \(Q_L\) and \(Q_R\) and between \(P_L\) and \(P_R\). The thick blue lines represent that there could be a bunch of edges, in \(G\) or in \(H\), between the corresponding sets. The right side is different in two crucial ways. The first is that all the red edges in \(G - H\) must go directly between the two starred sets. The second is that the matching edges between \(T_L\) and \(T_R\) and between \(S_L\) and \(S_R\) are now blue to represent that there may no longer be a matching between them in \(H\) (green), but there is still guaranteed to be a matching between them in \(G\) and possibly \(H\) (blue).
Fig. 2. In this example, we see the problem that arises with an unweighted EDCS. Each side of the bipartite graph is split into 3 equal sized pieces. The thick blue arrows represent bipartite graphs of degree $\beta/2 - 1$, while the other blue edges signify a matching. The blue edges are in $H$ whereas the red dashed edges are in $G$ but not $H$. A maximum matching in $H$ matches only $2/3$ of the vertices, whereas a maximum matching in $G$ matches all of them. The values of $d_H$ are written next to the edge blocks. We see that it is legal to omit the dashed red edges from $H$, since their total degree is more than $\beta(1 - \lambda)$. If we had a weighted EDCS, however, we would be forced to increase the degrees of $R_1$ and $L_1$ by adding multiple copies of edges between $L_1$ and $R_1$ and between $L_3$ and $R_3$. Thus the degrees of $L_1$ and $R_3$ would increase, which would force the degrees of $L_2$ and $R_2$ to decrease (property P1 of a weighted EDCS), but then the red edges would defy property P2 of a weighted EDCS so we would have to add some of them to $H$, and hence increase the size of the maximum matching in $H$. 
Fig. 3. An illustration of the alternating path. We see the edge degree in red next to the vertices. Across deficient (D) edges, they sum to $\beta - 1$ and across full (F) edges, they sum to $\beta$. We see that the distances on the right side are decreasing along the path, while those on the left are increasing.