Exponential Stability of the Mono-tubular Heat Exchanger Equation with Time Delay in Boundary Observation

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Abstract In the present paper, the exponential stability of the mono-tubular heat exchanger equation with boundary observation possessing a time delay and inner control is investigated through a simply proportional feedback. Firstly, the close-loop system is translated into an abstract Cauchy problem in the suitable state space. A uniformly bounded \( C_0 \)-semigroup generated by the close-loop system, which means that the unique solution of the system exist, is shown. Secondly, the spectrum configuration of the closed-loop system is analyzed and the eventual differentiability and the eventual compactness of the semigroup are shown by the resolvent estimates on some resolvent set. This implies that the spectrum determined growth assumption hold. Finally, a sufficient condition, which is on the physical parameters in the system and independent of the time delay, of the exponential stability of the closed-loop system is given.

Keywords: mono-tubular heat exchanger system; boundary control; time delay; exponential stability

1 Introduction

In the past decades, the mono-tubular heat exchanger system have been analyzed by several researchers from the viewpoint of system theory. For example, Xu et al. [1] have treated the heat exchanger equation with zero boundary conditions and proved the exponential stability for each set of physical parameters with the finest estimate of the decay rate, by using Huang’s result [2] on the spectrum determined growth assumption. In [3], Kanoh has introduced a controller with two kinds of feedback loops for the mono-tubular heat exchanger equation. However, he does not discuss on the exponential stability of the closed-loop system, which is one of important properties in the field of dynamical system theory. In [4], Sano analyzes the exponential stability of the mono-tubular heat exchanger system with static output feedback. In [5], Guo and Liang show that the \( C_0 \)-semigroup associated with the closed-loop system of the mono-tubular heat exchanger

*This work was supported by the National Natural Science Foundation of China (11201037, 11371071,11371070).
equation is differentiable after some finite time period. Moreover, they verify that the system is not a Riesz spectral system although the root subspace is complete in the energy Hilbert space. In the meanwhile, Yu and Liu give the differentiability of the system by the different method in [6].

However, in a practical control system, there is often a time delay between the controller to be implemented and the information via the observation of the system. These hereditary effects are sometime unavoidable because they might turn a well-behave system into a wild one. A simple example can be found in Gumowski and Mira [7], where they demonstrated that the occurrence of delays could destroy the stability and cause periodic oscillations in a system governed by differential equation. Another examples from Datko [8] illustrated that an arbitrary small time delay in the control could destabilize a boundary feedback hyperbolic control system. On the other side, the inclusion of an appropriate time delay effect can sometime improve the performance of the system (e.g., see [9]-[10]). The stabilization with time delay in observation or control represents difficult mathematical challenges in the control of distributed parameter systems. However this does not mean that there is no stabilizing controller in the presence of time delay. You can refer to [11]-[16] for some successful examples.

Motivated by these works, we shall introduce time delay to the mono-tubular heat exchanger system and we then investigate the effect of the time delay on exponential stability of the system. More precisely, we assume that a time delay occur in the boundary observation. We want to pose a question. Is the stabilization robust to the time delay for the proportional feedback controller? The present paper is devoted to answering this question.

The content of this paper is organized as follows. In Section 2, we shall introduce the mon-tubular heat exchanger system mentioned above and formulate our problem in a suitable Hilbert space. We show that the closed loop system generates a \( C_0 \)-semigroup of bounded linear operators and obtain the wellposedness of the system. In Section 3, we carry out detailed spectral analysis and obtain the spectrum configuration of the closed-loop system. Furthermore, we present that \( C_0 \)-semigroup is differentiable and compact after some finite time period based on the spectral analysis. This means that the spectrum determined growth assumption hold. In section 4, the stable regions of the closed-loop system are given by verifying that the spectral bound is less than or greater than zero. In the last section, a concise conclusion is given.

2 System description and wellposedness of the system

We shall consider the following type of mono-tubular heat exchanger equation in which the time delay occurs in boundary observation:

\[
\begin{cases}
\frac{\partial z}{\partial t}(t, x) = -\frac{\partial z}{\partial x}(t, x) - az(t, x) + \gamma e^{-bx}u(t), & (t, x) \in (0, \infty) \times (0, 1), \\
z(t, 0) = 0, z(0, x) = z_0(x), & (t, x) \in (0, \infty) \times (0, 1), \\
y(t) = z(t - \tau, 1), & t \in (0, \infty),
\end{cases}
\]
where \( z(t, x) \in \mathbb{R} \) is the temperature variation at the time \( t \) and at the point \( x \in [0, 1] \) with respect to an equilibrium point, \( u(t) \in \mathbb{R} \) is the control input, \( y(t) \in \mathbb{R} \) is the measured output, \( a \) is a positive physical parameter, and \( \gamma e^{-bx} \) denotes the spatial distribution of an actuator, \( b \) and \( \gamma \) being positive constants, \( \tau > 0 \) is the length of time delay.

As usual, we adopt the simple feedback control law \( u(t) = -ky(t) \) in system (2.1) and results in the following closed loop system:

\[
\begin{cases}
\frac{\partial z}{\partial t}(t, x) = -\frac{\partial z}{\partial x}(t, x) - az(t, x) - k\gamma e^{-bx}z(t - \tau, 1), & (t, x) \in (0, \infty) \times (0, 1), \\
z(t, 0) = 0, & t \in (0, \infty), \\
z(0, x) = z_0(x), & x \in (0, 1).
\end{cases}
\]

(2.2)

Setting \( w(t, x) = z(t - x\tau, 1) \), the system (2.2) is equivalent to

\[
\begin{cases}
\frac{\partial z}{\partial t}(t, x) = -\frac{\partial z}{\partial x}(t, x) - az(t, x) - k\gamma e^{-bx}w(t, 1), & (t, x) \in (0, \infty) \times (0, 1), \\
\tau \frac{\partial w}{\partial t}(t, x) = -\frac{\partial w}{\partial x}(t, x), & t \in (0, \infty), \\
z(t, 0) = 0, & z(t, 1) = w(t, 0), \\
z(0, x) = z_0(x), & w(0, x) = f(-\tau x), & x \in (0, 1).
\end{cases}
\]

(2.3)

We take the state Hilbert space \( \mathcal{H} = L^2[0, 1] \times L^2[0, 1] \) equipped with natural inner product \( \langle \cdot, \cdot \rangle \) and the induced norm \( \| \cdot \| \). Define the operator \( A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H} \) as

\[
A \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} -z'(x) - az(x) - k\gamma e^{-bx}w(1) \\ \frac{1}{\tau}w'(x) \end{pmatrix}, \quad (z, w)^\perp \in D(A),
\]

(2.4)

\[
D(A) = \{(z, w) \in H^1(0, 1) \times H^1(0, 1) | z(0) = 0, z(1) = w(0)\}.
\]

Thus, the system (2.3) can be written as

\[
\frac{d}{dt}X(t) = AX(t), \quad X(t) = \begin{pmatrix} z(t, x) \\ w(t, x) \end{pmatrix} \in \mathcal{H} \quad \forall t \geq 0, \quad X(0) = \begin{pmatrix} z_0 \\ f \end{pmatrix}.
\]

(2.5)

If the operator \( A \) generates a \( C_0 \)-semigroups \( T(t) \) on \( \mathcal{H} \), then (2.5) has a unique solution, which is equivalent to the unique solution to (2.2) or (2.3) exists. More precisely, we have the following theorem.

**Theorem 2.1** For any \( k \in \mathbb{R} \), the operator \( A \) defined by (2.4) generates a \( C_0 \)-semigroups on \( \mathcal{H} \).

**Proof.** Firstly, in order to prove that \( A \) generates a \( C_0 \)-semigroup, we introduce a new equivalent inner product in \( \mathcal{H} \)

\[
\langle X_1, X_2 \rangle_\tau = \int_0^1 z_1(x)\overline{z_2(x)}dx + \tau \int_0^1 w_1(x)\overline{w_2(x)}dx, \quad X_i = (z_i, w_i)^\perp \in \mathcal{H} \quad (i = 1, 2).
\]

From the definition of the operator \( A \) in (2.4) it follows that the identities

\[
2Re \left\langle A \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle_\tau = -|w(1)|^2 - 2a \int_0^1 |z(x)|^2dx - \int_0^1 2k\gamma e^{-bx}Re[w(1)\overline{z(x)}]dx
\]

(3)
\[ \leq -|w(1)|^2 - 2a \int_0^1 |z(x)|^2 dx - \int_0^1 2k\gamma e^{-bx} Re[w(1)z(x)] dx \]
\[ \leq -|w(1)|^2 - 2a \int_0^1 |z(x)|^2 dx + \int_0^1 |w(1)|^2 + k^2\gamma^2 |z(x)|^2 dx \]
\[ = (k^2\gamma^2 - 2a) \int_0^1 |z(x)|^2 dx \]

hold. These identities imply that
\[ \text{Re} \left\langle A \left( \begin{array}{c} z \\ w \end{array} \right), \left( \begin{array}{c} z \\ w \end{array} \right) \right\rangle_{\tau} \leq (k^2\gamma^2 / 2 - a) \left\| \left( \begin{array}{c} z \\ w \end{array} \right) \right\|_{\tau}, \tag{2.6} \]
in which \( \| \cdot \|_{\tau} \) denotes the norm induced by the inner product of \( \langle \cdot, \cdot \rangle_{\tau} \) in \( \mathcal{H} \).

Secondly, it is easy to verify that the operator \( A \) is densely defined and closed and its adjoint with respect to the new inner product is given as follows.
\[ A^* \left( \begin{array}{c} z \\ w \end{array} \right) = \left( \begin{array}{c} z'(x) - az(x) \\ k\gamma e^{-bx}z(x) \end{array} \right), \quad (z, w)^\perp \in D(A^*), \]
\[ D(A^*) = \left\{ (z, w) \in H^1(0, 1) \times H^1(0, 1) \middle| w(0) = z(1), w(1) = -k\gamma \int_0^1 e^{-bx}z(x) dx \right\}. \]

By the same method as above, we have that, for all \((z, w)^\perp \in D(A^*)\),
\[ \text{Re} \left\langle A^* \left( \begin{array}{c} z \\ w \end{array} \right), \left( \begin{array}{c} z \\ w \end{array} \right) \right\rangle_{\tau} \]
\[ = -(1/2)|z(0)|^2 - a \int_0^1 |z(x)|^2 dx + \frac{1}{2} \left| k\gamma \int_0^1 e^{-bx}z(x) dx \right|^2 \]
\[ \leq \left( k^2\gamma^2 / 2 - a \right) \left\| \left( \begin{array}{c} z \\ w \end{array} \right) \right\|_{\tau}. \tag{2.7} \]

Finally, it follows from the fact together with \( \text{Re} \) and \( \text{Im} \) that the operator \( A \) generates a \( C_0 \)-semigroups \( T(t) \) on \( (\mathcal{H}, \| \cdot \|_{\tau}) \) satisfying \( \| T(t) \|_{\tau} \leq e^{(k^2\gamma^2 / 2 - a)t} \) by using Corollary 2.2.3 of \( [17] \).

The equivalence of two norms \( \| \cdot \|_{\tau} \) and \( \| \cdot \| \) on \( \mathcal{H} \) implies that there exists a positive constant \( M \) such that \( \| T(t) \| \leq Me^{(k^2\gamma^2 / 2 - a)t} \). Thus, the proof of the theorem is complete. \qed

### 3 Spectral analysis and regularity of semigroup

In order to show the exponential stability of the system \( (2.5) \), we shall analyze the spectral configuration of the operator \( A \). To this end, we first show that the operator \( A \) is of compact resolvent (in other words, \( A \) is a discrete operator).

**Theorem 3.1** We have the following statements:
(1) \( \lambda \in \rho(A) \) if and only if \( \lambda \) satisfies
\[
-k\gamma e^{-\lambda \tau} e^{-(\lambda + a)} \int_0^1 e^{(\lambda + a - b)s} ds \neq 1.
\]

(3.1)

(2) If \( \lambda \in \rho(A) \), then \( R(\lambda, A) \) is compact.

**Proof.** (1) For arbitrary \((h, g)^{-1} \in H\), let us consider the resolvent equation
\[
(\lambda I - A) \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} h \\ g \end{pmatrix}.
\]

It is equivalent to
\[
\begin{align*}
z'(x) &= -(\lambda + a)z(x) - k\gamma e^{-bx} w(1) + h(x), \\
w'(x) &= -\lambda \tau w(x) + \tau g(x), \\
z(0) &= 0, \quad z(1) = w(0). 
\end{align*}
\]

(3.2)

(3.3)

(3.4)

Solving the equations of (3.2) and (3.3) with the help of \( z(0) = 0 \), we have
\[
\begin{align*}
z(x) &= \int_0^x e^{-(\lambda + a)(x-s)} \left[-k\gamma e^{-bs} w(1) + h(s)\right] ds, \\
w(x) &= w(0) e^{-\lambda \tau x} + \tau \int_0^x e^{-\lambda \tau (x-s)} g(s) ds.
\end{align*}
\]

(3.5)

It follows from \( z(1) = w(0) \) that
\[
\begin{align*}
w(0) + k\gamma e^{-(\lambda + a)} \int_0^1 e^{(\lambda + a - b)s} dw(1) &= \int_0^1 e^{-(\lambda + a)(1-s)} h(s) ds, \\
-w(0) e^{-\lambda \tau} + w(1) &= \tau \int_0^1 e^{-\lambda \tau (1-s)} g(s) ds.
\end{align*}
\]

(3.6)

It is easy to see from (3.6) that \( z(x) \) and \( w(x) \) are uniquely determined by \( w(0) \) and \( w(1) \). However, the equation (3.6) on \( w(0) \) and \( w(1) \) has unique solution if and only if its coefficient determinant is not zero. That is
\[
\Delta(\lambda) := \left| \begin{array}{ccc}
1 & -k\gamma e^{-(\lambda + a)} \\
e^{-\lambda \tau} & 1
\end{array} \right| \neq 0.
\]

A simple computation shows that the statement (1) is right.

(2) If let \( r(\lambda) = -k\gamma e^{-(\lambda + a)} \int_0^1 e^{(\lambda + a - b)s} ds \) and \( \lambda \in \rho(A) \), then we have
\[
\begin{align*}
w(0) &= \frac{1}{\Delta(\lambda)} \left[ \int_0^1 e^{-(\lambda + a)(1-s)} h(s) ds + \tau r(\lambda) \int_0^1 e^{-\lambda \tau (1-s)} g(s) ds \right], \\
w(1) &= \frac{1}{\Delta(\lambda)} \left[ \tau \int_0^1 e^{-\lambda \tau (1-s)} g(s) ds + e^{\lambda \tau} \int_0^1 e^{-(\lambda + a)(1-s)} h(s) ds \right].
\end{align*}
\]

(3.7)
It follows from (3.5) that

\[
R(\lambda, A) \left( \begin{array}{c} h(x) \\ g(x) \end{array} \right) = \left( \begin{array}{c} k_\gamma w(1) \int_0^x e^{-(\lambda+a)(x-s)} e^{-b s} ds + \int_0^x e^{-(\lambda+a)(x-s)} h(s) ds \\ w(0) e^{-\lambda x} + \tau \int_0^x e^{-\lambda x} g(s) ds \end{array} \right).
\]  

(3.8)

Set

\[
T_1 \left( \begin{array}{c} h(x) \\ g(x) \end{array} \right) = \left( \begin{array}{c} k_\gamma w(1) \int_0^x e^{-(\lambda+a)(x-s)} e^{-b s} ds \\ w(0) e^{-\lambda x} \end{array} \right),
\]

\[
T_2 \left( \begin{array}{c} h(x) \\ g(x) \end{array} \right) = \left( \begin{array}{c} \int_0^x e^{-(\lambda+a)(x-s)} h(s) ds \\ \tau \int_0^x e^{-\lambda x} g(s) ds \end{array} \right).
\]

Obviously, \( R(\lambda, A) = T_1 + T_2 \) and \( T_1 \) and \( T_2 \) are compact operators on \( H \). These facts imply that \( R(\lambda, A) \) is compact. Thus, the second statement holds and the proof the Theorem 3.1 is completed.

\( \square \)

Similarly, we have the following spectral distribution and omit its proof since it involves only simple calculations.

**Lemma 3.1** \( \lambda = 0 \) is an eigenvalue of the operator \( A \) if and only if \( k_\gamma (e^{-a} - e^{-b}) = a - b \). \( \lambda = -a + b \) is an eigenvalue of the operator \( A \) if and only if \( k_\gamma e^{\tau(a-b)} e^{-b} = 1 \).

In the rest of the paper, we shall assume that \( k_\gamma (e^{-a} - e^{-b}) \neq a - b \) and \( k_\gamma e^{\tau(a-b)} e^{-b} \neq 1 \).

According to Theorem 3.1 and Lemma 3.1, we have the following result.

**Theorem 3.2** If \( k_\gamma e^{\tau(a-b)} e^{-b} \neq 1 \), then \( \sigma(A) \) consists of eigenvalues with finite multiplicity and \( \sigma(A) = \{ \lambda \in \mathbb{C} : \Delta(\lambda) = k_\gamma e^{-\lambda x} \frac{e^{\lambda a} - e^{-\lambda b}}{\lambda + a - b} - 1 = 0 \} \).

In the sequel, we will further study the spectral properties of \( A \).

**Lemma 3.2** Let \( k \neq 0 \) and \( \omega_0 = k^2 \gamma^2 / 2 - a \). There exists \( M_1 > 0 \) such that

\[
\Sigma = \{ \lambda \in \mathbb{C} : \text{Re}\lambda \leq \omega_0, \text{Im}\lambda \geq M_1 e^{-2(1+\tau)\text{Re}\lambda} \} \subseteq \rho(A).
\]

Moreover, for \( \lambda \in \Sigma \),

\[
\left| k_\gamma e^{-\lambda x} \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda + a - b} \right| \leq \frac{1}{2}, \quad (3.9)
\]

**Proof.** Let \( \lambda = x + iy \) and \( x \) and \( y \) be real numbers. Straightforward calculations show that

\[
\left| k_\gamma e^{-\lambda x} \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda + a - b} \right| = \left| k_\gamma e^{-\tau x} \frac{e^{-b} - e^{-x} \cos y - e^{-x} \sin y}{x + a - b + iy} \right|,
\]

(3.10)

\[
= \left| k_\gamma e^{-\tau x} \left( e^{-2b} + e^{-2(x+a)} - 2e^{-(x+a+b)} \cos y \right) \right|^{1/2}
\]

\[
= \left| k_\gamma e^{-\tau x} \left( e^{2x-2b} + e^{-2a} - 2e^{(x-a-b)} \cos y \right) \right|^{1/2}
\]

\[
= \left| k_\gamma \frac{e^{2x-2b} + e^{-2a} - 2e^{(x-a-b)} \cos y}{e^{2(\tau+1)x}(x + a - b)^2 + e^{2(\tau+1)x}y^2} \right|^{1/2}
\]

\[6\]
If \(|y| > e^{-2(\tau+1)x}\), we have
\[
|k| \gamma \left| \frac{e^{2x-2b} + e^{-2a} - 2(e^{x-a-b}) \cos y}{e^{2(\tau+1)x} (x + a - b)^2 + e^{2(\tau+1)x} y^2} \right|^{1/2} \leq |k| \gamma \left| \frac{e^{2x-2b} + e^{-2a} + 2e(x-a-b)}{e^{2(\tau+1)x} (x + a - b)^2 + e^{2(\tau+1)x}} \right|^{1/2} \to 0
\]
as \(x \to -\infty\). This means that there exists a constant \(K_1 > 0\) such that
\[
\left| k \gamma e^{-\lambda x} \frac{e^{-b} - e^{-\lambda-a}}{\lambda + a - b} \right| \leq \frac{1}{2}, \quad \text{if Re}\lambda < -K_1 \text{ and Im}\lambda \geq e^{-2(\tau+1)Re}\lambda \tag{3.11}
\]
In the meanwhile, it follows from
\[
\left| k \gamma e^{-\lambda x} \frac{e^{-b} - e^{-\lambda-a}}{\lambda + a - b} \right| \leq \frac{1}{2}, \quad \text{if } |\lambda| > K_2 \text{ and } -K_1 \leq \text{Re}\lambda \leq \omega_0. \tag{3.12}
\]
that \(\left| k \gamma e^{-\lambda x} \frac{e^{-b} - e^{-\lambda-a}}{\lambda + a - b} \right|\) converges to 0 uniformly with respect to \(-K_1 \leq x \leq \omega_0\) as \(|y| \to \infty\).

Thus, there exists a constant \(K_2 > 0\) such that
\[
\left| k \gamma e^{-\lambda x} \frac{e^{-b} - e^{-\lambda-a}}{\lambda + a - b} \right| \leq \frac{1}{2}, \quad \text{if } |\lambda| > K_2 \text{ and } -K_1 \leq \text{Re}\lambda \leq \omega_0.
\]
Take \(M_1 > \max\{1, K_2e^{2(1+\tau)a} \}\). It follows from (3.11) and (3.12) that
\[
\left| k \gamma e^{-\lambda x} \frac{e^{-b} - e^{-\lambda-a}}{\lambda + a - b} \right| \leq \frac{1}{2}, \quad \text{for } \lambda \in \Sigma.
\]
Thus, by Theorem 3.2 we have that \(\Sigma \subset \rho(A)\). 

Now, we give the eventual regularity of the \(C_0\)-semigroup \((T(t))_{t \geq 0}\) generated by the operator \(A\).

**Theorem 3.3** For any \(k \neq 0\), there exists \(t_0 > 0\) such that \(C_0\)-semigroup \((T(t))_{t \geq 0}\) generated by the operator \(A\) is differentiable for \(t > t_0\).

**Proof.** It follows from (3.1) that \(|\Delta(\lambda)| \leq 1/2\) for \(\lambda \in \Sigma\). Here, \(\Delta(\lambda)\) is defined in the proof of Theorem 3.2. Set
\[
\Sigma_1 = \{ \lambda \in \mathbb{C} | \text{Re}\lambda \leq -a, \text{Im}\lambda \geq M_1 e^{-2(1+\tau)\text{Re}\lambda} \},
\Sigma_2 = \{ \lambda \in \mathbb{C} | -a \leq \text{Re}\lambda \leq \omega_0, \text{Im}\lambda \geq M_1 e^{-2(1+\tau)\text{Re}\lambda} \}.
\]
It is easy to see that \(\Sigma = \Sigma_1 \cup \Sigma_2\). Moreover, it follows from (3.8) that
\[
R(\lambda, A) \begin{pmatrix} h(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} k\gamma w(1) \int_0^x e^{-(\lambda+a)(x-s)} e^{-b s} ds + \frac{1}{\Delta(\lambda)} \int_0^1 e^{-(\lambda+a)(1-s)} h(s) ds \\ \frac{1}{\Delta(\lambda)} \int_0^1 e^{-(\lambda+a)(1-s)} g(s) ds \end{pmatrix},
\]
in which
\[
w(0) = \frac{1}{\Delta(\lambda)} \left[ \int_0^1 e^{-(\lambda+a)(1-s)} h(s) ds + \tau \int_0^1 e^{-(\lambda+a)(1-s)} g(s) ds \right],
w(1) = \frac{1}{\Delta(\lambda)} \left[ \tau \int_0^1 e^{-(\lambda+a)(1-s)} g(s) ds + e^{\lambda \tau} \int_0^1 e^{-(\lambda+a)(1-s)} h(s) ds \right].
\]

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are given in (3.17) and \( r(\lambda) = -k\gamma e^{-\lambda t} \int_0^t e^{(\lambda\omega_a - b)s} ds \).
If let \( \| \cdot \|_2 \) be the usual norm of \( L^2(0,1) \), then we have
\[
|r(\lambda)| \leq \begin{cases} 
|k|\gamma e^{-\text{Re}\lambda}, & \lambda \in \Sigma_1; \\
|k|\gamma \int_0^1 e^{(\lambda\omega_a - b)s} ds, & \lambda \in \Sigma_2.
\end{cases} \tag{3.13}
\]
\[
\left| \int_0^x e^{-\lambda t} e^{-bs} ds \right| \leq \begin{cases} 
 e^{-\text{Re}\lambda}, & \lambda \in \Sigma_1; \\
 1, & \lambda \in \Sigma_2.
\end{cases} \tag{3.14}
\]
\[
\left| \int_0^x e^{-\lambda t} e^{-\omega_0 s} h(s) ds \right| \leq \begin{cases} 
 e^{-\text{Re}\lambda} \| h \|_2, & \lambda \in \Sigma_1; \\
 \| h \|_2, & \lambda \in \Sigma_2; \quad \forall x \in [0,1], \ h(x) \in L^2(0,1). \tag{3.15}
\end{cases}
\]
\[
\left| \int_0^x e^{-\lambda t} e^{-\omega_0 s} g(s) ds \right| \leq \begin{cases} 
 e^{-\text{Re}\lambda} \| g \|_2, & \lambda \in \Sigma_1; \\
 e^{\text{Re}\lambda} \| g \|_2, & \lambda \in \Sigma_2; \quad \forall x \in [0,1], \ g(x) \in L^2(0,1). \tag{3.16}
\end{cases}
\]
It follows from (3.13)-(3.16) that
\[
|w(0)| \leq \begin{cases} 
 L_1 e^{-\text{Re}(\lambda) t} \| (h,g)^+ \|, & \lambda \in \Sigma_1; \\
 L_2 \| (h,g)^+ \|, & \lambda \in \Sigma_2; \tag{3.17}
\end{cases}
\]
\[
|w(1)| \leq \begin{cases} 
 L_3 e^{-\text{Re}(\lambda) t} \| (h,g)^+ \|, & \lambda \in \Sigma_1; \\
 L_4 \| (h,g)^+ \|, & \lambda \in \Sigma_2; \tag{3.18}
\end{cases}
\]
in which
\[
 L_1 = 2\sqrt{2} \max \{ \tau, 1, \tau |\gamma| \}, \quad L_2 = 2\sqrt{2} \max \left\{ 1, \tau e^{\rho_0 t}, \tau e^{\omega_0 t}, \tau |\gamma| \int_0^1 e^{(\omega_0\omega + a-b)s} ds \right\}.
\]
In light of the estimates (3.15)-(3.18), we have
\[
\left\| R(\lambda, A) \begin{pmatrix} h(x) \\ g(x) \end{pmatrix} \right\| \leq \begin{cases} 
 L_3 e^{-2\text{Re}\lambda} \| (h,g)^+ \|, & \lambda \in \Sigma_1; \\
 L_4 \| (h,g)^+ \|, & \lambda \in \Sigma_2; \tag{3.19}
\end{cases}
\]
in which
\[
 L_3 = 2 \sqrt{(k\gamma L_1)^2 + 1 + L_1^2 + \tau^2}, \quad L_4 = 2 \sqrt{(k\gamma L_2)^2 + 1 + L_2^2 e^{2\rho_0 t} + \tau^2 e^{2\rho_0 t}}.
\]
Taking \( L = \max \{ L_3, L_4 e^{2(1+\tau)\omega_0} \} \), we have
\[
\left\| R(\lambda, A) \begin{pmatrix} h(x) \\ g(x) \end{pmatrix} \right\| \leq L e^{-2\text{Re}(\lambda) t} \| (h,g)^+ \|, \quad \forall \begin{pmatrix} h(x) \\ g(x) \end{pmatrix} \in \mathcal{H}, \ \lambda \in \Sigma. \tag{3.20}
\]
Finally, by the definition of \( \Sigma \) and (3.20), we have
\[
\| R(\lambda, A) \| \leq \frac{L}{M_1} M_1 e^{-2\text{Re}(\lambda) t} \leq \frac{L}{M_1} \text{Im} \lambda, \quad \lambda \in \Sigma.
\]
It follows from Theorem 2.4.7 of [18] that there exists \( t_0 > 0 \) such that the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) is differentiable for \( t > t_0 \). The proof of the Theorem 3.3 is complete. \( \square \)
Corollary 3.1 For any \( k \neq 0 \), there exists a \( t_0 > 0 \) such that the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) is compact for \( t > t_0 \). Moreover, the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) satisfies the spectrum-determined growth condition.

Proof. By Theorem 3.3, there exists \( t_0 > 0 \) such that the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) is differentiable for \( t > t_0 \). Thus the \( C_0 \) semigroup \( (T(t))_{t \geq 0} \) is norm continuous for \( t > t_0 \). In addition, we see from Theorem 3.1 that the operator \( A \) has compact resolvent. Therefore, by Corollary 2.3.4 of [18], the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) is compact for \( t > t_0 \). □

4 Exponential stability region of the system (2.2)

For convenience, we first state a feasible result of Ruan, Wei as follows (see [19] and [20]).

Theorem 4.1 Consider the exponential polynomial

\[
P(\lambda, e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) = \lambda^n + p_1^{(0)} \lambda^{n-1} + \cdots + p_{n-1}^{(0)} \lambda + p_n^{(0)} + [p_1^{(1)} \lambda^{n-1} + \cdots + p_{n-1}^{(1)} \lambda + p_n^{(1)}]e^{-\lambda \tau_1} \]

\[
+ \cdots + [p_1^{(m)} \lambda^{n-1} + \cdots + p_{n-1}^{(m)} \lambda + p_n^{(m)}]e^{-\lambda \tau_m}
\]

where \( \tau_i \geq 0 (i = 1, 2, \ldots, m) \) and \( p_j^{(i)} (i = 0, 1, \ldots, m-1, j = 1, 2, \ldots, n) \) are constants. As \( (\tau_1, \tau_2, \ldots, \tau_m) \) vary, the sum of the orders of the zeros of \( P(\lambda, e^{-\lambda \tau_1}, \ldots, e^{-\lambda \tau_m}) \) on the open right half-plane can change only if a zero appears on or crosses the imaginary axis.

In order to apply Theorem 4.1 to discuss the exponential stability region of the system (2.2), we introduce the following result.

Theorem 4.2 When \( \tau = 0 \) in (2.2), if \( 1 - a + \ln(k\gamma) - \ln(3\pi/2) < 0 \), then the system (2.2) is exponential stable.

Proof. According to Theorem 3.2 of [4], we know that the system (2.2) with \( \tau = 0 \) is exponential stable if and only if the spectral bound of the system operator is less than zero. However, Lemma 2.1 of [5] shows that the asymptotic expansions of the eigenvalues of the system operator are as follows:

\[
\lambda_n = -a + \ln(k\gamma) - \ln(\omega_n) + i\left(\omega_n - \frac{\ln(\omega_n)}{\omega_n}\right) + O(n^{-1}), \omega_n = \left(2n - \frac{1}{2}\right)\pi, n \geq 1.
\]

It is easy to see that \( \text{Re}(\lambda_n) \leq \text{Re}(\lambda_{n-1}) \) and the spectral bound of the system operator is less than \( 1 - a + \ln(k\gamma) - \ln(3\pi/2) \). Thus, if \( 1 - a + \ln(k\gamma) - \ln(3\pi/2) \) is less than zero, then the system (2.2) with \( \tau = 0 \) is exponential stable. The proof of the Theorem is complete. □

For further research, we introduce the following notations: \( \alpha = k\gamma e^{-a} \), \( \beta = k\gamma e^{-b} \) and \( \eta = a - b \). Thus, \( \lambda \) is a root of the characteristic equation \( \Delta(\lambda) = 0 \) of the system (2.2) if and only if \( \lambda \) is a root of the equation

\[
\lambda - \alpha e^{-\lambda(\tau+1)} - \beta e^{-\lambda \tau} + \eta = 0, \quad (4.2)
\]
since \( \lambda = -a + b \) is not the eigenvalue of the operator \( A \) (see Lemma 3.1). Moreover, we have

**Theorem 4.3** If \( \alpha + \beta < \eta \) holds, then all roots of \( \Delta(\lambda) = 0 \) have negative real parts.

**Proof.** Based on the observation above, it is sufficient to verify that all roots of (4.2) have negative real parts.

Firstly, it follows from Lemma 3.1 that \( \lambda = 0 \) is not a root of (4.2). Moreover, it follows from Theorem 4.2 that all roots of (4.2) have negative real parts when \( \tau = 0 \).

Now, we prove that (4.2) have no imaginary root. In fact, if \( i\omega \) is a root of (4.2), then we have

\[
i\omega - \alpha [\cos((\tau + 1)\omega) - i \sin((\tau + 1)\omega)] - \beta [\cos(\tau \omega) - i \sin(\tau \omega)] + \eta = 0.
\]

Separating real and imaginary parts, we obtain

\[
\eta = \alpha \cos((\tau + 1)\omega) + \beta \cos(\tau \omega), \quad \omega = -\alpha \sin((\tau + 1)\omega) - \beta \sin(\tau \omega).
\]

This implies that

\[
\eta^2 + \omega^2 - \alpha^2 - \beta^2 = 2\alpha\beta \cos \omega
\]

which is equivalent

\[
\frac{\omega^2 + \eta^2 - \alpha^2 - \beta^2}{2\alpha\beta} = \cos \omega
\]

since \( 2\alpha\beta \) is obviously not zero. The assumption \( \alpha + \beta < \eta \) implies that \( \frac{\eta^2 - \alpha^2 - \beta^2}{2\alpha\beta} > 1 \). Hence the equation (4.3) is meaningless. This show that (4.2) have no imaginary root.

Finally, applying Theorem 4.1 we have that all roots of (4.2) have negative real parts for all \( \tau > 0 \). The proof is completed. \( \square \)

**Theorem 4.4** If \( \alpha + \beta > \eta \) holds, then the equation \( \Delta(\lambda) = 0 \) has at least on root with positive real parts for all \( \tau > 0 \).

**Proof.** It is sufficient to verify that (4.2) has at least on root with positive real parts for all \( \tau > 0 \).

Denote

\[
g(\lambda) = \lambda - \alpha e^{-\lambda(\tau + 1)} - \beta e^{-\lambda\tau} + \eta.
\]

Then \( g(0) = -\alpha - \beta + \eta < 0 \). Obviously, \( \lim_{\lambda \to +\infty} g(\lambda) = +\infty \) and hence, there exists a \( \lambda_0 > 0 \) such that \( g(\lambda_0) = 0 \). This implies that \( \lambda_0 \) is a root of (4.2). \( \square \)

We now could give the main result of the paper.

**Theorem 4.5** If \( k\gamma e^{(a-b)}e^{-b} \neq 1 \), \( 1 - a + \ln(k\gamma) - \ln(3\pi/2) < 0 \) and \( \alpha + \beta < \eta \) hold, then the system (2.2) or (2.3) is exponential stable for all \( \tau > 0 \). However, if \( k\gamma e^{(a-b)}e^{-b} \neq 1 \) and \( \alpha + \beta > \eta \) hold, then the system (2.2) or (2.3) is not stable.
Proof. According to Theorem 4.4, the second statement is obvious. Moreover, it follows from Theorem 3.2 and Theorem 4.3 that the spectral bound $s(A)$ of the operator $A$ is less than or equal to zero for all $\tau > 0$. However, it is easy to see that the imaginary axis is not the asymptote of the zeros of the characteristic function $\Delta(\lambda)$ for all $\tau > 0$. This means that the spectral bound $s(A)$ of the operator $A$ is less than zero for all $\tau > 0$. Hence Corollary 3.1 implies that the first statement hold.

5 Conclusions

In this note, we give the stable regions of the mono-tubular heat exchanger equation with delay in boundary observation in light of the distribution of roots of the characteristic equation $\Delta(\lambda) = 0$. More precisely, we give a sufficient condition of the exponential stability of the mono-tubular heat exchanger equation with delay. This condition is applicable for all positive time delay $\tau$. This means that we answer the question in the section 1. We want to point out specially that the characteristic function $\Delta(\lambda)$ is the product of $e^{-\lambda \tau}$ and the characteristic function of the system without delay in the boundary observation. Moreover, The characteristic function $\Delta(\lambda)$, which is associated with one-order (both time and space) PDE with one delay, has the same type with that of the ODE with two delays. For more examples you can refer to [20] and the references therein. Thus, we can apply the result of [19] to discuss the distribution of roots of the characteristic equation $\Delta(\lambda) = 0$, which is an important part of the paper.

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