Riccati equation - based generalization of Dawson’s integral function

R. Messina, M.A. Jivulescu, A. Messina and A. Napoli

MIUR, CNISM and Dipartimento di Scienze Fisiche ed Astronomiche,
Università di Palermo, via Archirafi 36, 90123 Palermo, Italy

SUMMARY

A new generalization of Dawson’s integral function based on the link between a Riccati nonlinear differential equation and a second-order ordinary differential equation is reported. The MacLaurin expansion of this generalized function is built up and to this end an explicit formula for a generic cofactor of a triangular matrix is deduced.

1. INTRODUCTION

The study of homogeneous linear second-order ordinary differential equations (HLIIODE) is strictly related to that of Riccati equations. It is possible to express the general integral of an arbitrary HLIIODE in terms of a particular solution of the associated Riccati equation. In the first part of this paper we recall this procedure in order to write down explicitly the most general solution of a HLIIODE vanishing at a conveniently chosen fixed point. The structure of such a general solution suggests a very simple way to extend Dawson’s integral function (DIF), appearing in various physical contexts such as spectroscopy, electrical oscillations, heat conduction, astrophysics and so on as well as in applied mathematics [1]-[10]. Quite recently the DIF has emerged in the treatment of the dynamics of the so-called generalized spin star system where the reduced dynamics of a system composed of two central qubits is investigated in the limit of an infinite number of environmental spins [11]. The DIF deserves, in addition, attention on the mathematical side since it is related to other special functions and even because it possesses useful formal properties exploitable for its numerical evaluations [12, 13]. The main scope of this paper is to propose a new class of transcendental functions which may be viewed as generalizations of the DIF. We report in detail the construction of the MacLaurin expansion of a generic element of this class by solving a linear system of infinitely many equations in infinitely many unknowns. We reach this goal evaluating all the cofactors of a triangular matrix of arbitrary finite dimension having all its diagonal elements equal to one.
2. RICCATI GENERAL SOLUTION OF A HLIODE

Consider the following homogenous second-order linear differential equation with variable coefficients

$$y''(x) + b_1(x)y'(x) + b_2(x)y(x) = 0$$  (1)

where $b_1(x)$ and $b_2(x)$ are $C^2$ functions on $(-a, a) \in \mathbb{R}$, $a > 0$. Such a linear equation is invariant under dilatations $y \mapsto \lambda y$ the infinitesimal generator of which is

$$X = y \frac{\partial}{\partial y}.$$  (2)

Lie theory of symmetry of differential equations takes into account such symmetry and, as indicated in [14], the recipe is to look for a new coordinate $u$ such that $X$ becomes

$$X = \frac{\partial}{\partial u}.$$  (3)

The coordinate $X$ is such that $Xu = 1$, i.e. $y \frac{\partial u}{\partial y} = 1$, from which we obtain $y(x) = e^{u(x)}$ (up to multiplication by a constant).

Since in terms of the new coordinate we get

$$y'(x) = u'(x)y(x), \quad y''(x) = \left[ u''(x) + \left( u'(x) \right)^2 \right] y(x)$$  (4)

the transformed differential equation may be cast in the following form:

$$u''(x) + \left( u'(x) \right)^2 + b_1(x)u'(x) + b_2(x) = 0$$  (5)

which takes the form of a Riccati equation

$$z'(x) + z^2(x) + b_1(x)z(x) + b_2(x) = 0$$  (6)

if we put

$$z(x) = u'(x).$$  (7)

Let $\zeta(x)$ be a particular solution of eq.(6). Then on integrating eq.(7) one immediately gets the particular solution of eq.(1)

$$y_1(x) = e^{\int_0^x \zeta(t)dt}$$  (8)

such that $y_1(0) = 1$ and $y'_1(0) = \zeta(0)$. 

2
We recall that, once a solution is known, we can introduce the change of variable \( y(x) = y_1(x)v(x) \) and then, taking into account that
\[
y'(x) = y_1'(x)v(x) + y_1(x)v'(x) \quad y''(x) = y_1''(x)v(x) + 2y_1'(x)v'(x) + y_1(x)v''(x)
\]
and the fact that \( y_1(x) \) is a solution of eq. (11), eq. (11) becomes
\[
v''(x) = a(x)v'(x)
\]
where we have defined
\[
a(x) = -(2\pi(x) + b_1(x)).
\]
A solution of eq. (10) is
\[
v(x) = \int_0^x \exp \left( \int_0^t a(t')dt' \right) dt
\]
from which we obtain the following solution of eq. (11):
\[
y_2(x) = y_1(x) \int_0^x \exp \left[ \int_0^t a(t')dt' \right] dt
\]
satisfying the initial conditions \( y_2(0) = 0 \) and \( y_2'(0) = 1 \).
In view of the fact that
\[
W(0) \equiv \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \pi(0) & 1 \end{pmatrix} = 1 \neq 0,
\]
we immediately write down the following general solution of eq. (11):
\[
y(x) = \exp \left[ \int_0^x \pi(t)dt \right] \left[ C_1 + C_2 \int_0^x \exp \left[ - \int_0^t \left( 2\pi(t') + b_1(t') \right) dt' \right] dt \right].
\]
Requiring \( y(0) = 0 \) yields \( C_1 = 0 \) so that the most general solution \( \overline{y}(x) \) of eq. (11) vanishing at \( x = 0 \) may be cast in the form
\[
\overline{y}(x) = C \exp \left[ \int_0^x \pi(t)dt \right] \cdot \int_0^x \exp \left[ - \int_0^t \left( 2\pi(t') + b_1(t') \right) dt' \right] dt
\]
which satisfies the further Cauchy condition \( \overline{y}'(0) = C \).
3. EXTENDING DAWSON’S INTEGRAL FUNCTION

The form of $\overline{y}(x)$ provides a very favorable starting point to construct generalizations of Dawson’s integral odd function

$$F(2, x) = \exp(-x^2) \int_0^x \exp(t^2) dt = -F(2, -x) \quad (17)$$

where $x \in \mathbb{R}$. In the literature there have appeared extensions like

$$F(p, x) = \exp(-x^p) \int_0^x \exp(t^p) dt \quad (18)$$

with $p = 2, 3, 4, \ldots$ and $x \in \mathbb{R}$, in connections with interesting problems both in applied Physics and in Mathematics [13, 16]. Our scope is to exploit the structure of $\overline{y}(x)$ to propose new generalizations of the original Dawson’s integral function. The first natural extension suggested by equations (17) and (18) is to look for $b_1(x)$ in equation (11) such that

$$\overline{z}(x) = 2\overline{z}(x) + b_1(x) \Rightarrow \overline{z}(x) = -b_1(x) \quad (19)$$

This condition establishes the following link between $b_1(x)$ and $b_2(x)$ throughout the associated Riccati equation (6)

$$-b_1'(x) = -b_1^2(x) + b_1^2(x) - b_2(x) \Rightarrow b_2(x) = b_1(x) \quad (20)$$

so that we may claim that the unique solution of the Cauchy problem

$$\begin{aligned}
&y''(x) + b(x)y'(x) + b'(x)y(x) = y''(x) + \left(b(x)y(x)\right)' = 0 \\
y(0) = 0 & \quad y'(0) = 1
\end{aligned} \quad (21)$$

may be written down as

$$y(x) = \exp[-B(x)] \int_0^x \exp[B(t)] dt \quad (22)$$

where

$$B(x) = \int_0^x b(t) dt. \quad (23)$$

When $b(x) = 2x$ or $b(x) = px^{p-1}$ eq.(22) gives back the function $F(2, x)$ and $F(p, x)$ respectively. We denote the function defined by eq.(22), by $D_b(x)$ and call it the generalized Dawson’s integral function associated to the function $b(x)$.
There is another way of extending Dawson’s integral function easily suggested by equation (16). If we in fact stipulate that

\[
\begin{align*}
\bar{z}(x) &= -\lambda px^{p-1} \\
2\bar{z}(x) + b_1(x) &= -\mu sx^{s-1}
\end{align*}
\]

\[p \in \mathbb{N}^+, \lambda \in \mathbb{R} - \{0\} \quad \text{and} \quad s \in \mathbb{N}^+, \mu \in \mathbb{R} - \{0\}
\]

then \(b_1(x) = 2\lambda px^{p-1} - \mu sx^{s-1}\), and consequently

\[
b_2(x) = \lambda^2 p^2 x^{2p-2} - \lambda \mu spx^{p+s-2} + \lambda p(p-1)x^{p-2}.
\]

Thus we may claim that the function

\[
F[(\lambda, p), (\mu, s); x] \equiv \exp(-\lambda x^p) \int_0^x \exp(\mu t^s)dt
\]

with \(p, s \in \mathbb{N}^+, \lambda, \mu \in \mathbb{R} - \{0\}, x \in \mathbb{R}\) is the unique solution of the following Cauchy problem

\[
\begin{align*}
y''(x) + &\left(2\lambda px^{p-1} - \mu sx^{s-1}\right)y'(x) + \\
&\left(\lambda^2 p^2 x^{2p-2} - \lambda \mu spx^{p+s-2} + \lambda p(p-1)x^{p-2}\right)y(x) = 0 \\
y(0) &= 0 \quad y'(0) = 1
\end{align*}
\]

\[F[(1, p), (1, p); x] \text{ is of course coincident with } F(p, x) \text{ and is included in eq. (22)}
\]

in correspondence with \(b(x) = px^{p-1}\).

4. MACLAURIN EXPANSION OF \(D_b(x)\)

We concentrate on \(D_b(x)\) posing the following question: if \(b(x)\) may be expanded in a MacLaurin series with the radius of convergence \(R\), that is

\[
b(x) = \sum_{n=0}^{+\infty} \frac{b^{(n)}(0)}{n!} x^n,
\]

\(b^{(n)}(x)\) being the \(n\)-th derivative of \(b(x)\), is it possible to find the MacLaurin expansion of \(D_b(x)\) in terms of the class of coefficients \(\{b^{(n)}(0), n \in \mathbb{N}\}\)?

The question is well posed, since the assumption on \(b(x)\) guarantees that \(D_b(x)\) too may be expanded in MacLaurin series with the same radius of convergence \(R\). Then our problem is to build up the explicit expression of
\( D_b^{(k)}(0) \) as a function of \( \{b^{(n)}(0), n \in \mathbb{N}\} \). To this end we begin by observing that for any \( b(x) \)

\[
D_b'(x) = -D_b(x)b(x) + 1, \quad D_b''(x) = -[D_b(x)b(x)]'
\]

\( D_b(0) = 0, \quad D_b'(0) = 1 \) \hspace{1cm} (29)

\[
D_b^{(k+2)}(x) = -[D_b(x)b(x)]^{(k+1)} = -\sum_{n=0}^{k+1} \binom{k+1}{n} D_b^{(n)}(x)b^{(k+1-n)}(x)
\]

\( D_b^{(k+1)}(0) = -kb^{(k-1)}(0) - \sum_{n=2}^{k} \binom{k}{n} D_b^{(n)}(0)b^{(k-n)}(0). \) \hspace{1cm} (31)

To find \( D_b^{(k+1)}(0) \), with \( k \in \mathbb{N}^+ \), we must solve the linear system of \( k \) equations in the \( k \) unknowns \( \{D_b^{(2)}(0), D_b^{(3)}(0), \ldots, D_b^{(k+1)}(0)\} \). Its incomplete matrix is

\[
A_k = \begin{pmatrix}
1 \\
\binom{2}{2}b(0) \\
\binom{3}{2}b^{(1)}(0) \\
\binom{4}{2}b^{(2)}(0) \\
\binom{5}{2}b^{(3)}(0) \\
\vdots \\
\binom{k}{2}b^{(k-2)}(0) \\
\binom{k}{3}b^{(k-3)}(0) \\
\binom{k}{4}b^{(k-4)}(0) \\
\vdots \\
\binom{k}{k}b^{(k)}(0) \\
\end{pmatrix}
\]

and has lower triangular form with all its diagonal elements equal to one. The vector \( B_k \) of the constants appearing in the linear system \( \text{(32)} \) is

\[
B_k = \begin{pmatrix}
-b(0) \\
-\binom{2}{1}b^{(1)}(0) \\
-\binom{3}{1}b^{(2)}(0) \\
\vdots \\
-\binom{k}{1}b^{(k-1)}(0)
\end{pmatrix}.
\] \hspace{1cm} (34)

It is convenient to introduce the following notation for the elements of \( a_{ij} \) of \( A_k \) and \( b_i \) of \( B_k \):

\[
\begin{aligned}
a_{i,j} &= \begin{cases}
0 & i < j \\
1 & i = j \\
\binom{i}{i+1}b^{(i-j-1)}(0) & i > j
\end{cases} \\
b_i &= -\binom{i}{1}b^{(i-1)}(0) = -ib^{(i-1)}(0)
\end{aligned}
\hspace{1cm} (35)
\]
provided that $1 \leq i, j \leq k$. Since $\det A_k = 1$ for any $k \in \mathbb{N}^+$, Cramer’s theorem yields

$$D_b^{(k+1)}(0) = \det P_k$$

where

$$P_k = \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots & -b(0) \\
\left(\frac{2}{3}\right)b(0) & 1 & 0 & 0 & \ldots & -2b(1)(0) \\
\left(\frac{3}{5}\right)b(0) & \left(\frac{5}{7}\right)b(0) & 1 & 0 & \ldots & -3b(2)(0) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\left(\frac{k}{k+1}\right)b(k-2)(0) & \left(\frac{k-2}{k-1}\right)b(k-3)(0) & \ldots & \ldots & \left(\frac{k}{k+1}\right)b(0) & -kb(k-1)(0)
\end{pmatrix}. \quad (38)$$

The determinant associated to the matrix $P_k$ borders the determinant of $A_{k-1}$ through the addition of the $k$-th row and the $k$-th column. The value of $\det P_k$ may thus be evaluated by means of the well known Cauchy formula \[17\] according to which

$$\det B = \det \begin{pmatrix}
c_{11} & c_{12} & \ldots & c_{1n} & \alpha_1 \\
c_{21} & \ldots & \ldots & \ldots & \alpha_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c_{n1} & \ldots & \ldots & \alpha_n & a \\
\beta_1 & \beta_2 & \ldots & \beta_n & a
\end{pmatrix} = a \det C - \sum_{r,s=1}^{n} \alpha_r \beta_s \Delta_{r,s} \quad (39)$$

where $\Delta_{r,s}$ is the cofactor of the corresponding element $c_{r,s}$ of the bordered matrix

$$C = \begin{pmatrix}
c_{11} & c_{12} & \ldots & c_{1n} \\
c_{21} & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots \\
c_{n1} & \ldots & \ldots & c_{nn}
\end{pmatrix}. \quad (40)$$

In our case $B = P_k$ and $C = A_{k-1}$, so that this formula assumes the form

$$D^{(k+1)}(0) = \det P_k = -kb^{(k-1)}(0) + \sum_{i,j=1}^{k-1} i \binom{k}{j+1} b^{(k-j-1)}(0)b^{(i-1)}(0)\Delta_{i,j} \quad (41)$$

$\Delta_{i,j}$ being the cofactor of the correspondent element $a_{i,j}$ ($1 \leq i, j \leq k - 1$) of the matrix $A_{k-1}$. It is easy to convince oneself that $\Delta_{i,j} = 0$ for any $i > j$, since the elimination of the $i$-th row and $j$-th column with $i > j$ gives a triangular matrix with one diagonal element equal to 0. It is in addition evident that $\Delta_{i,i} = 1$ for any $i$. Thus we have only to evaluate the cofactors $C(i,n) \equiv \Delta_{i,i+n}$ with $1 \leq i \leq k-2$ and $1 \leq n \leq k-i-1$ finding the explicit
expression of such cofactors for a generic lower triangular matrix \( C = (c_{r,s}) \) of order \( k - 1 \) with all its diagonal elements equal to 1. We indeed claim that

\[
C(i, n) = \sum_{s=1}^{n} (-1)^s \sum_{\{p\}_{n-s}^n} c_{i+n,i+p_1} c_{i+p_1,i+p_2} \cdots c_{i+p_{s-2},i+p_{s-1}} c_{i+p_{s-1},i+p_s}
\]

(42)

where \( \sum_{\{p\}_{2}^n} \) denotes \( \sum_{n>p_1>p_2>\cdots>p_{s-1}>p_s=0} \) that is summation over all the \( \binom{n-1}{n-s} \) possible sets of \( s \) indices \( \{p_1, p_2, \ldots, p_s\} \), taken in strict decreasing order and such that \( 0 < p_i < n, \quad i = 1, 2, \ldots, s - 1 \) and \( p_s = 0 \). We observe that the terms of eq.(42) corresponding to a fixed value of \( s \) are products of \( s \) elements of the matrix \( A \) with a plus (minus) sign if \( s \) is even (odd). Then, the only term corresponding to \( s = 1 \) is \(-c_{i+n,i}\).

Let us prove eq.(42) by induction on \( n \) for an arbitrary \( i = 1, 2, \ldots, k - 2 \). Firstly we show that

\[
C(i, 1) = -c_{i+1,i}.
\]

(43)

To obtain \( C(i, 1) = \Delta_{i,i+1} \) we have to calculate and multiply by \(-1\) the determinant of the matrix obtained from \( A_{k-1} \) eliminating the \( i \)-th row and the \((i + 1)\)-th column. It is easy to observe that this operation leaves a triangular matrix having \( c_{i+1,i} \) as a diagonal element, and all other diagonal elements equal to 1. As a consequence, we have \( \Delta_{i,i+1} = -c_{i+1,i} \), as stated before. Let us now suppose that, for all \( m \leq n \) we have

\[
C(i, m) = \sum_{s=1}^{m} (-1)^s \sum_{\{p\}_{m-s}^m} c_{i+m,i+p_1} c_{i+p_1,i+p_2} \cdots c_{i+p_{s-2},i+p_{s-1}} c_{i+p_{s-1},i+p_s}.
\]

(44)

Using Laplace’s second theorem with the \( i \)-th row for the cofactors and the \((i + n + 1)\)-th row for the elements we obtain

\[
\sum_{j=1}^{k-1} c_{i+n+1,j} \Delta_{i,j} = 0.
\]

(45)

Using the fact that \( C(i, 0) = 1, \quad c_{i,j} = 0 \) for \( j > i \) whereas \( c_{i,i} = 1 \) we obtain

\[
c_{i+n+1,i} + \sum_{j=i+1}^{i+n} c_{i+n+1,j} C(i, j - i) + C(i, n + 1) = 0.
\]

(46)

Using eq.(44) and posing \( j - i = m \) we have

\[
C(i, n + 1) = -c_{i+n+1,i} + \sum_{m=1}^{n} \sum_{s=1}^{m} (-1)^{s+1} \sum_{\{p\}_{s}^s} c_{i+n+1,i+m} c_{i+m,i+p_1} \cdots c_{i+p_{s-1},i+p_s}.
\]

(47)
Observing that for a fixed value of $s$ the second term contains products of $s + 1$ elements $c_{i,j}$ with the correct sign factor, it is easy to convince oneself that we may write

$$C(i, n + 1) = -c_{i+n+1,i} + \sum_{s=2}^{n+1} (-1)^s \sum_{\{p\}^{(n+1)}} c_{i+n+1,i+p_1} c_{i+p_1,i+p_2} \cdots c_{i+p_{s-1},i+p_s}$$

that is

$$C(i, n + 1) = \sum_{s=1}^{n+1} (-1)^s \sum_{\{p\}^{(n+1)}} c_{i+n+1,i+p_1} c_{i+p_1,i+p_2} \cdots c_{i+p_{s-1},i+p_s}$$

which concludes the demonstration eq.(42).

As far as we know the explicit expression of all the cofactors of a triangular determinant of arbitrary finite order having all its diagonal elements equal to one has not previously appeared in literature.

Summing up we have proved that, for any $1 \leq i, j \leq k - 1$,

$$\Delta_{i,j} = \begin{cases} 
C(i, j - i) & i < j \\
1 & i = j \\
0 & i > j 
\end{cases}$$

This concludes the derivation of the MacLaurin expansion of $D_b(x)$ for any $b(x)$ as well expandable in MacLaurin series. We guess that other general properties may be proved for such class of extended Dawson's integral functions also thanks to the explicit knowledge of their MacLaurin expansions. Moreover special cases obtained in correspondence to particular choices of the function $b(x)$ may find applications both in Physics and in Applied Mathematics.

5. CONCLUDING REMARKS

The structure of the general integral of a linear homogeneous second-order ordinary differential equation in the form given by eq.(15) suggests a very simple way to generalize Dawson's integral function. Such a generalization introduces indeed a class of new functions all possessing the same Dawson-like structure.

In order to explore some properties of this class without choosing the form of $b(x)$, we have coped with the interesting question of finding the MacLaurin expansion of $D_b(x)$ in terms of the MacLaurin coefficients of the same $b(x)$. It is worth noting that the resolution of this problem has lead us to derive
the explicit expression of all the cofactors of a lower triangular matrix which diagonal entries are all equal. We underline that such a derivation may be successfully generalized and applied to a generic triangular matrix. In conclusion we feel that the ideas and methods reported in this paper may be of some help to propose new generalizations of other special functions of interest in physics, chemistry and applied mathematics.

References

[1] Garcia T.T. Voigt profile fitting to quasar absorption lines: an analytic approximation to the Voigt-Hjerting function *Monthly Notices of the Royal Astronomical Society* 369, 2025-2035, 2006

[2] Casini R. The Hanle Effect of the Two-Level Atom in the Weak-Field Approximation *The Astrophysical Journal* 568, 1056-1065, 2002

[3] Kaiser A. et al. Microscopic processes in dielectrics under irradiation by sub-picosecond laser pulses *Physical Review B* 61, 11437-11450, 2000

[4] Peng-Sheng Wei et al. Distribution functions of positive ions and electrons in a plasma near a surface *IEEE Transactions on Plasma Science* 28, 1244-1253, 2000

[5] Cody W.J. and Stoltz I. The use of Taylor series to test accuracy of function programs *ACM Transactions on Mathematical Software (TOMS)* 17, 55 - 63, 1991

[6] Di Rocco H.O. and Aguirre Tellez M. Evaluation of the asymmetric Voigt profile and complex error functions in terms of the Kummer functions *Acta Physica Polonica A* 106, 817-827, 2004

[7] Shippony Z. and Read W.G. A very accurate algorithm for the Voigt profile and complex error functions *Journal of Quantitative Spectroscopy and Radiative Transfer* 50, 635-546, 1993

[8] Wendlandt B.C.H. Temperature in an irradiated thermally conducting translucent medium *Journal of Physics D: Applied Physics* 6, 657-660, 1973

[9] Lehle H. and al. Probing electric fields in protein cavities by using the vibrational Stark effect of carbon monoxide *Biophysical Journal* 88, 1978-1990, 2005

[10] Schreier F. Voigt and complex error function: a comparison of computational methods *Journal of Quantitative Spectroscopy and Radiative Transfer* 48, 743-762, 1992
[11] Hamdouni Y., Fannes M. and Petruccione F. Exact dynamics of a two-qubit system in a spin star environment *Physical Review B* **73**, 245323-12, 2006

[12] Cody W.J., Paciorek K.A. and Thacher H.C. Chebyshev approximations for Dawson’s integral *Mathematics of Computation* **24**, 171-178, 1970

[13] McCabe J.H. A continued fraction expansion, with a truncation error estimate, for Dawson’s integral *Mathematics of Computation* **28**, 811-816, 1974

[14] Cariñena J.F. and Ramos A. A new geometric approach to Lie Systems and Physical Applications *Acta Applicandae Mathematicae* **70**, 43-69, 2002

[15] Sajo E. On the recursive properties of Dowson’s integral *Journal of Physics A: Mathematical and General* **26**, 2977-2987, 1993

[16] Dijkstra D. A continued fraction expansion for a generalization of Dawson’s integral *Mathematics of Computation* **31**, 503-510, 1977

[17] Vein R. and Dale P. *Determinants and their Applications in Mathematical Physics*, Springer, New York, 1999