Epimorphisms Between Linear Orders

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Abstract We study the relation on linear orders induced by order preserving surjections. In particular we show that its restriction to countable orders is a bqo.

Keywords Linear orders · Surjective maps · Better quasi-orders

1 Some Generalities and the Questions

Fraïssé [2] conjectured that the class of countable linear orders was a well-quasi-order (wqo for short) under order preserving injections (also called embeddings). Laver [4] proved that this class is in fact a better-quasi-order (bqo for short), which is much stronger.

We are interested in the somehow dual quasi-order (qo for short) induced by order preserving surjections, or epimorphisms, between linear orders, in particular the countable ones. What are the combinatorial properties of this qo?

In Section 2 we present the basic definitions and facts about linear orders, bqos and wqos, and epimorphisms. The main result of Section 3 states that order preserving surjections induce a bqo on countable linear orders; this is a consequence of a theorem of van Engelen,
Miller and Steel [1] stating that the class of countable linear orders with continuous order preserving injections preserves bqos. We also look at the stronger notion of preserving bqos and show how to adapt it to epimorphisms in order to keep its validity in this setting. In Section 4 we apply the tools that we have developed earlier to describe explicitly the relation of epimorphism on some restricted classes of linear orders, such as ordinals.

2 Background

2.1 On Linear Orders

**Definition 1**

- A *linear order* is a reflexive, transitive, antisymmetric and total relation on a non-empty set $K$; we usually denote it with $\leq_K$ (or $\leq$ when no confusion arises) and we say (abusively) that $K$ is a linear order. We also write $<_K$ or $<$ for the strict part of the order.
- Given a linear order $K$, a *suborder* of $K$ is a subset of $K$ along with the induced order on it.
- A linear order $K$ is *dense* when, given any elements $x$, $y$ in $K$, if $x < y$ holds then there is a $z \in K$ so that $x < z < y$ holds. Denote $\eta$ the unique, up to isomorphism, dense countable linear order without end-points, i.e. the order of the rationals.
- A linear order is *scattered* when $\eta$ is not among its suborders.
- A linear order is $\sigma$-scattered if it is a countable union of scattered suborders.
- Denote by $\text{Lin}$ (resp. $\text{LIN}$) the class of all countable (resp. all) linear orders, and by $\text{Scat}$ the class of scattered countable linear orders.
- An order is *complete* when all its non-empty upper bounded subsets have least upper bound.
- An order is *bounded* if it has both a maximum and a minimum.

**Fact 2** Every linear order $K$ can be completed using Dedekind cuts (see [9, Definition 2.22]).

**Notation** If $x$, $y$ are elements of an order $L$ with $x < y$, square bracket notation will be used to denote the interval they determine. For example $[x, y] = \{ z \mid x \leq z \leq y \}$ denotes the closed interval between $x$ and $y$. Other similar notations such as $]x, y[$ or $[x, \rightarrow[$ are self-explaining.

The closed interval notation $[x, y]$ will be sometimes used regardless of how $x$ and $y$ are ordered, meaning in any case the set of all elements between them.

The following is a useful description of order preserving functions that are continuous with respect to the order topologies. Recall that a basis for the order topology consists of the open intervals, including those of the form $]x, x[$ and $]x, y[$.

**Lemma 3** Let $K$, $L \in \text{LIN}$ and let $f : K \rightarrow L$ be order preserving. Then $f$ is continuous if and only if for every non-empty subset $A$ of $K$, if the supremum (or the infimum) of $A$ exists, then the same holds for $f(A)$ and $f(\sup A) = \sup f(A)$ (or $f(\inf A) = \inf f(A)$).