ORBITAL STABILITY OF ELLIPTIC PERIODIC PEAKONS FOR THE MODIFIED CAMASSA-HOLM EQUATION

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Abstract. The orbital stability of peakons and hyperbolic periodic peakons for the Camassa-Holm equation has been established by Constantin and Strauss in [A. Constantin, W. Strauss, Comm. Pure. Appl. Math. 53 (2000) 603-610] and Lenells in [J. Lenells, Int. Math. Res. Not. 10 (2004) 485-499], respectively. In this paper, we prove the orbital stability of the elliptic periodic peakons for the modified Camassa-Holm equation. By using the invariants of the equation and controlling the extrema of the solution, it is demonstrated that the shapes of these elliptic periodic peakons are stable under small perturbations in the energy space. Throughout the paper, the theory of elliptic functions and elliptic integrals is used in the calculation.

1. Introduction. In [18], Holm and Staley studied an one-dimensional version of an active fluid transport that is described by the following nonlinear equation

\[ m_t + um_x + bu_x m = 0, \]  

(1.1)

with \( m = u - u_{xx}, \) \( u(x,t) \) representing the fluid velocity, while the constant \( b \) is a balance or a bifurcation parameter for the solution behavior. If \( b = 2, \) then (1.1) becomes the Camassa-Holm (CH) equation

\[ u_t - u_{xxx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \]  

(1.2)

Equation (1.2) was proposed as a model for the unidirectional propagation of the shallow water waves over a flat bottom [2, 3], with \( u(x,t) \) representing the water’s free surface in nondimensional variables. It can be found using the method of recursion operators as an example of bi-Hamiltonian equation with an infinite number of conserved functionals by Fokas and Fuchssteiner [14]. The CH equation has attracted much attention in the last two decades because of its interesting properties: complete integrability [2, 14], geometric formulations and the presence of breaking waves [7, 8, 9] (i.e. a wave profile remains bounded while its slope becomes unbounded in finite time).

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In [17], Hakkaev, Iliev and Kirchev studied the following equation:
\[ u_t + (a(u))_x - u_{xxt} = \left( b'(u) \frac{u_x^2}{2} + b(u)u_{xx} \right)_x, \tag{1.3} \]
where \( a, b : R \to R \) are smooth functions. It is easy to see that whatever \( a, b \) is, the equation for travelling wave solutions \( u = \varphi(x - ct) \) has no dissipative terms. Hence, any travelling wave solution of (1.3) is determined from Newton’s equation which we can write in the form \((\varphi')^2 = U(\varphi)\) (see section 2). If \( a(u) = u^3 \) and \( b(u) = u \), then (1.3) becomes the modified Camassa-Holm (mCH) equation
\[ u_t - u_{xxt} = uu_{xxx} + 2u_xu_xx - 3u^2u_x, \tag{1.4} \]
The orbital stability of negative solitary waves to the mCH equation was proved by Yin, Tian and Fan [30]. Recently, Darós and Arruda [12] investigated the orbital instability of cnoidal waves of the mCH equation by using the abstract method of Grillakis, Shatah and Strauss [15].

For the CH equation (1.2), a remarkable property is that it admits the peakons in the following forms
\[ u(x, t) = c \varphi(x - ct) = ce^{-|x-ct|}. \tag{1.5} \]
The peakons were proved to be orbital stable by Constantin and Strauss in [10]. A variational approach for proving the orbital stability of the peakons was introduced by Constantin and Moliént [11]. Using variational approach, the stability of the Camassa-Holm peakons in the dynamics of an integrable shallow-water-type system was investigated by Chen et al. in [4]. Orbital stability of multi-peakon solutions was proved by Dika and Moliént in [13]. Liu, Liu and Qu [24] considered the modified Camassa-Holm equation with cubic nonlinearity, which is integrable and admits the single peakons and multi-peakons. Using energy argument and combining the method of the orbital stability of a single peakon with monotonicity of the local energy norm, they proved that the sum of \( N \) sufficiently decoupled peakons is orbitally stable in the energy space. Moreover, the orbital stability of the single peakons for the DP equation was proved by Lin and Liu [23]. They developed the approach due to Constantin and Strauss [10] in a delicate way. The approach in [10] was extended in [25] to prove the orbital stability of the peakons for the Novikov equation. Recently, Guo et al. [16] proved the orbital stability of peakons for the generalized modified Camassa-Holm (gmCH) equation.

In addition, equation (1.2) has also the periodic peakons
\[ u(x, t) = c \varphi(x - ct), \quad c \in \mathbb{R}, \tag{1.6} \]
where
\[ \varphi(x) = \frac{\cosh \left( \frac{1}{2} - x \right)}{\sinh \left( \frac{1}{2} \right)}, \tag{1.7} \]
and \( \varphi(x) \) is defined for \( x \in [0, 1) \) and extends periodically to the whole real line. Orbital stability of the periodic peakons for the CH equation was proved by Lenells in [20, 21]. Wang and Tian [29] extended Lenell’s approach to prove the orbital stability of the periodic peakons for the Novikov equation.

The nonlinear partial differential equation
\[ \mu(u_t) - u_{xxt} = -2\mu(u)u_x + 2u_xu_xx + uu_{xxx}, \quad t > 0, \quad x \in S^1 = \mathbb{R}/\mathbb{Z}, \tag{1.8} \]
where \( u(x, t) \) is a real-valued spatially periodic function and \( \mu(u) = \int_{S^1} u(x, t)dx \) denotes its mean, was introduced in [19] as an integrable equation arising in the
study of the diffeomorphism group of the circle. It describes the propagation of self-interacting, weakly nonlinear orientation waves in a massive nematic liquid crystal under the influence of an external magnetic field. It was noted in [22] that the \( \mu CH \) equation also admits periodic peakons: For any \( c \in \mathbb{R} \), the periodic peaked traveling wave \( u(x,t) = c\varphi(x - ct) \), where
\[
\varphi(x) = \frac{1}{26} \left( 12x^2 + 23 \right), \quad x \in \left[ -\frac{1}{2}, \frac{1}{2} \right],
\]
and \( \varphi \) is extended periodically to the real line, is a solution of (1.8). Chen, Lenells and Liu [5] proved that the periodic peakons of the \( \mu CH \) equation are orbitally stable. Liu, Qu and Zhang [26] further proved that the periodic peakons of the modified \( \mu CH \) equation are orbitally stable. The approach in [5] was further extended in [28] to prove the orbital stability of the periodic peakons for the generalized \( \mu CH \) equation.

This work has the interest in to investigate the existence and orbital stability of elliptic periodic peakon solutions of the modified Camassa-Holm (mCH) equation (1.4). It was noted in [27] that the mCH equation also admits peakons and periodic peakons with similar formulas (1.5) and (1.6). By constructing certain Lyapunov functionals, it is demonstrated that the shapes of the peakons and periodic peakons are stable under small perturbations in the energy space [27]. Equation (1.4) has the elliptic periodic peakon
\[
u(x,t) = \varphi(x - ct),
\]
where \( \varphi(x) \) is given for \( x \in [-T, T] \) by Jacobi elliptic function
\[
\varphi(x) = \frac{\sqrt{5} - 1}{2} \text{cn}^2 \left( \sqrt{\frac{\sqrt{5}}{8}} x, k \right),
\]
where \( T = \sqrt{\frac{8}{\sqrt{5}}} \text{cn}^{-1} \left( \sqrt{\frac{\sqrt{5}}{2}}, k \right) \), \( k = \sqrt{\frac{1 + \sqrt{5}}{2\sqrt{5}}} \) and \( \text{cn}^{-1}(x,k) \) represents the inverse function of Jacobi elliptic function \( \text{cn}(x,k) \). The local well-posedness of the Cauchy problem for equation (1.4) was established by Hakkaev, Iliev and Kirchev in [17].

The periodic peakons of the CH equation and the \( \mu CH \) equation are given by hyperbolic functions and quadratic functions, respectively. According to the previous statement, their orbital stability has been established. Recently, Chen, Deng and Huang [6] proved that the trigonometric periodic peakons for the modified Camassa-Holm equation are orbitally stable. In the present work, we will prove that the elliptic periodic peakons for the modified Camassa-Holm equation are orbitally stable. To the best of our knowledge, this is the first result about the orbital stability of elliptic periodic peakons.

**Theorem 1.1.** Assume that \( u \in C([0,T); H^1(\mathbb{S})) \) is a solution to (1.4) and \( u(\cdot, t) - \varphi(\cdot - \xi(t)) \) preserves sign, then the elliptic periodic peakon \( \varphi \) is orbitally stable in the following sense. For every \( \varepsilon > 0 \), there is a \( \delta > 0 \), if
\[
\| u(\cdot, 0) - \varphi \|_{H^1(\mathbb{S})} < \delta,
\]
then
\[
\| u(\cdot, t) - \varphi(\cdot - \xi(t) + T) \|_{H^1(\mathbb{S})} < \varepsilon \quad \text{for} \ t \in (0, T),
\]
where \( \xi(t) \in \mathbb{R} \) is any point where the function \( u(\cdot, t) \) attains its maximum.
The proof is inspired by [21] where the case of hyperbolic periodic peakons of (1.4) is considered. The approach taken here is similar but there are differences. The main difference is that in [21] the $\|u - \varphi(\cdot - \xi)\|_{H^1(S)}^2$ associated with a solution $u(x, t)$ is represented by the difference between conservation law $H_1(u)$ and $H_1(\varphi)$ and the difference between max $u$ and max $\varphi$, whereas here the $H^1$-norm $\|u - \varphi(\cdot - \xi)\|_{H^1(S)}^2$ is also related to the integrals $\int_S (u(x) - \varphi(x - \xi)) \varphi^i(x - \xi) dx (i = 1, 2)$ depending on time. However, since this integrals can be controlled by the conservation law $H_0$ if $u(x) - \varphi(x - \xi)$ preserves sign, we can ensure that they remain small enough for later times. We conjecture that the Theorem 1.1 still holds without the hypothesis that $u(x) - \varphi(x - \xi)$ preserves sign.

2. Elliptic periodic peakons of the mCH equation. In this section, we convert equation (1.4) into a planar dynamical system. By substituting $u(x, t) = \varphi(\tau)$ with $\tau = x - ct$ into equation (1.4), then it follows that

$$-c\varphi' + c\varphi''' = \varphi\varphi''' + 2\varphi'\varphi'' - 3\varphi^2\varphi', \quad (2.1)$$

where $\varphi'$ is the derivative with respect to $\tau$. Integrating equation (2.1) once we obtain

$$(\varphi - c)\varphi'' + \frac{1}{2}(\varphi')^2 - \varphi^3 + c\varphi = g, \quad (2.2)$$

where $g$ is the integral constant.

Letting $y = \frac{d\varphi}{d\tau}$, then we obtain the following planar dynamic system

$$\begin{cases}
\frac{d\varphi}{d\tau} = y, \\
\frac{dy}{d\tau} = -\frac{1}{2}y^2 + \varphi^3 - c\varphi + g
\end{cases} \quad (2.3)$$

with first integral

$$H(\varphi, y) = (\varphi - c) \left[ y^2 - \frac{1}{2}(\varphi^3 + d_0\varphi^2 + d_1\varphi + d_2) \right] = h, \quad (2.4)$$

where $d_2 = c^3 - 2c^2 + 4g$, $d_1 = c^2 - 2c$ and $d_0 = c$, and $h$ is an integral constant. Here, we assume that $d_2 = c^3 - 2c^2 + 4g = 0$, then we have $g = \frac{2c^2 - c^3}{4}$. For simplicity, henceforth take $c = 1$, $g = \frac{1}{4}$, we have the following planar system

$$\begin{cases}
\frac{d\varphi}{d\tau} = y, \\
\frac{dy}{d\tau} = -\frac{1}{2}y^2 + \varphi^3 - \varphi + \frac{1}{4}
\end{cases} \quad (2.5)$$

with first integral

$$H(\varphi, y) = (\varphi - 1) \left[ y^2 - \frac{1}{2}(\varphi^3 + \varphi^2 - \varphi) \right]. \quad (2.6)$$

By analyzing the system (2.5), we know that it has three equilibrium points at $(\alpha, 0)$, $(\beta, 0)$, and $(\gamma, 0)$, where $(\alpha, 0)$ and $(\gamma, 0)$ are centers, $(\beta, 0)$ is a saddle point. Where $\alpha > \beta > \gamma$ and $\alpha, \beta, \gamma$ satisfying the relations

$$\begin{cases}
\alpha + \beta + \gamma = 0, \\
\alpha\beta + \alpha\gamma + \beta\gamma = -1, \\
\alpha\beta\gamma = -\frac{1}{4}.
\end{cases}$$
Notice that a periodic peakon corresponds to the heteroclinic orbit (arch curve) defined by \( H(\varphi, y) = 0 \) (see Fig. 1). Now, equation (2.6) becomes

\[
y^2 = \frac{1}{2} (\varphi^3 + \varphi^2 - \varphi).
\]

(2.7)

By using the first equation of system (2.5) to do the integration, we have

\[
\int_{\varphi_{-}}^{\varphi_{+}} \frac{d\varphi}{\sqrt{\varphi (\varphi + \frac{1+\sqrt{5}}{2}) (\varphi - \frac{\sqrt{5}-1}{2})}} = \frac{\sqrt{2}}{2} \int_{0}^{T} d\tau.
\]

(2.8)

Therefore, we obtain the exact parameter representation of \( \varphi \) as following

\[
\varphi(\tau) = \frac{\sqrt{5} - 1}{2 \cn^{2} \left( \sqrt{\frac{\sqrt{5}}{8}} \tau, k \right)}, \quad -T \leq \tau \leq T,
\]

(2.9)

where \( T = \sqrt{\frac{8}{\sqrt{5}}} \cn^{-1} \left( \sqrt{\frac{5-1}{2}}, k \right), k = \sqrt{\frac{1+\sqrt{5}}{2}} \sqrt{5} \), here, \( \cn^{-1}(x, k) \) represents the inverse function of Jacobi elliptic function \( \cn(x, k) \). We can get the following elliptic periodic peakon

\[
\varphi(x) = \frac{\sqrt{5} - 1}{2 \cn^{2} \left( \sqrt{\frac{\sqrt{5}}{8}} (x - 2nT), k \right)}, \quad (2n - 1) T \leq x \leq (2n + 1) T,
\]

(2.10)

where \( n = 0, 1, 2, \cdots \). The profile of elliptic periodic peakon is shown in Fig. 2.
Remark 1. Note that the algebraic curves defined by $H(\varphi, y) = 0$ consists of a closed curve and a open curve (see Fig.1(b)), the open curve corresponds periodic peakon solution (2.10) and the close curve corresponds smooth periodic wave solution

$$u(x, t) = -\frac{1 + \sqrt{5}}{2} \cdot \text{cn}^2 \left( \sqrt{\frac{\sqrt{5}}{8}}(x - ct), \sqrt{\frac{1 + \sqrt{5}}{2\sqrt{5}}} \right).$$

(2.11)

Recently, Darós and Arruda [12] investigated the orbital instability of smooth periodic waves (2.11) of the mCH equation by using the abstract method of Grillakis, Shatah and Strauss [15]. The stability of the periodic peakon seems not to enter the general framework developed in [15], especially because of the non-smoothness of the periodic peakon.

3. Proof of stability. Note that a small change in the shape of a peakon can yield another one with a different speed. The appropriate notion of stability is, therefore, that of orbital stability: a periodic wave with an initial profile close to a peakon remains close to some translate of it for all later times. That is, the shape of the wave remains approximately the same for all times.

Equation (1.4) has the conservation laws

$$H_0[u] = \int_S u \, dx, \quad H_1[u] = \frac{1}{2} \int_S (u^2 + u_x^2) \, dx, \quad H_2[u] = \frac{1}{2} \int_S \left( \frac{u^4}{2} + uu_x^2 \right) \, dx. \quad (3.1)$$

We will identify $S$ with $[-T, T]$ and view functions $u$ on $S$ as periodic functions on the real line with period $2T$. For an integer $n \geq 1$, we let $H^n(S)$ be the Sobolev space of all square integrable functions $f \in L^2(S)$ with distributional derivatives $\partial_x^i f \in L^2(S)$ for $i = 1, \ldots, n$. These Hilbert spaces are endowed with the inner products

$$\langle f, g \rangle_{H^n(S)} = \sum_{i=0}^n \int_S (\partial_x^i f(x)) (\partial_x^i g(x)) \, dx. \quad (3.2)$$

By a solution $u$ of (1.4) on $[-T, T]$ with $T > 0$, we mean a function $u \in C([-T, T]; H^1(S))$ such that equation holds in distributional sense and the functionals $H_i[u], i = 0, 1, 2$, defined in (3.1) are independent of $t \in [-T, T]$.

The periodic peakon $\varphi(x)$ is continuous on $S$ with peak at $x = \pm T$. The extreme value of $\varphi$ are

$$M_\varphi = \varphi(T) = 1, \quad m_\varphi = \varphi(0) = \frac{\sqrt{5} - 1}{2}. \quad (3.3)$$

Moreover, $\varphi$ is smooth on $(-T, T)$. This gives, as $\varphi_{xx}(x) = \frac{3}{4} \varphi^2(x) + \frac{1}{4} \varphi(x) - \frac{1}{4}$ on $(-T, T)$, that the integration-by-parts formula $\int \varphi_{xx} f \, dx = -\int \varphi_x f_x \, dx, f \in H^1(S)$, holds with $\varphi_{xx}(x) = \frac{3}{4} \varphi^2(x) + \frac{1}{4} \varphi(x) - \frac{1}{4} - \sqrt{2} \delta(x - T)$. Here, $\delta$ denotes the Dirac delta distribution and, for simplicity, we abuse notation by writing integrals instead of the $H^{-1}(S)/H^1(S)$ duality pairing. According to appendix B, we have

$$H_0[\varphi] = \int_S \varphi \, dx = \int_0^T \frac{\sqrt{5} - 1}{2 \cdot \text{cn}^2 \left( \sqrt{\frac{\sqrt{5}}{8}} x, k \right)} \, dx$$

$$= \frac{2\sqrt{2} \left( \sqrt{5} - 1 \right)}{1 - k^2} \cdot 5 \cdot \frac{1}{4} \left( (1 - k^2) \sqrt{\frac{\sqrt{5}}{8}} \cdot T \right)$$
\[- E\left(\sqrt{\frac{5}{8}} T\right) + \text{tn}\left(\sqrt{\frac{5}{8}} T\right) \text{dn}\left(\sqrt{\frac{5}{8}} T\right), \quad (3.4)\]

where \(\sqrt{\frac{5}{8}} T = \text{cn}^{-1}\left(\sqrt{\frac{5}{8}} - \frac{1}{2}, k\right)\). We can obtain

\[H_1[\varphi] = \frac{1}{2} \int_S (\varphi^2 + \varphi_x^2) \, dx\]

\[= \frac{1}{2} \int_S (\varphi^2 - \varphi \varphi_{xx}) \, dx\]

\[= \frac{1}{2} \int_S \left[ \varphi^2 - \varphi \left( \frac{3}{4} \varphi^2 + \frac{1}{2} \varphi - \frac{1}{4} - \sqrt{2} \delta(x - T) \right) \right] \, dx\]

\[= \frac{1}{8} I_1 + \frac{1}{4} I_2 - \frac{3}{8} I_3 + \frac{\sqrt{2}}{2} \varphi(T), \quad (3.5)\]

where

\[I_1 = H_0[\varphi], \quad (3.6)\]

\[I_2 = \int_S \varphi^2 \, dx = 2 \int_0^T \left( \frac{\sqrt{5} - 1}{2 \text{cn}^2\left(\sqrt{\frac{5}{8}} x, k\right)} \right)^2 \, dx\]

\[= \frac{\sqrt{2}(\sqrt{5} - 1)^2}{3(1 - k^2)^2} \frac{5^{-1}}{4} \left[ (1 - k^2)(2 - 3k^2) \sqrt{\frac{5}{8}} T + 2(2k^2 - 1)E\left(\sqrt{\frac{5}{8}} T\right) + (2 - 4k^2) \text{tn}\left(\sqrt{\frac{5}{8}} T\right) \text{dn}\left(\sqrt{\frac{5}{8}} T\right) + (1 - k^2) \text{tn}\left(\sqrt{\frac{5}{8}} T\right) \text{nc}\left(\sqrt{\frac{5}{8}} T\right) \text{nc}\left(\sqrt{\frac{5}{8}} T\right) \right], \quad (3.7)\]

\[I_3 = \int_S \varphi^3 \, dx = 2 \int_0^T \left( \frac{\sqrt{5} - 1}{2 \text{cn}^2\left(\sqrt{\frac{5}{8}} x, k\right)} \right)^3 \, dx\]

\[= \frac{\sqrt{2}(\sqrt{5} - 1)^3}{2(1 - k^2)} \frac{5^{-\frac{3}{4}}}{4} \left[ 3k^2 D_2 + 4(1 - 2k^2) D_4 \right.\]

\[+ \text{tn}\left(\sqrt{\frac{5}{8}} x\right) \text{dn}\left(\sqrt{\frac{5}{8}} x\right) \text{nc}\left(\sqrt{\frac{5}{8}} x\right) \left\rfloor_0^T. \quad (3.8)\]

Refer to appendix B for details, by calculation, we can get

\[I_3 = \frac{\sqrt{2}(\sqrt{5} - 1)^3}{2(1 - k^2)} \frac{5^{-\frac{3}{4}}}{4} \left[ \left( 3k^2 + \frac{4(1 - 2k^2)(2 - 3k^2)}{3(1 - k^2)} \right) \sqrt{\frac{5}{8}} T \right.\]

\[+ \frac{4(1 - 2k^2)}{3(1 - k^2)} \text{tn}\left(\sqrt{\frac{5}{8}} x\right) \text{dn}\left(\sqrt{\frac{5}{8}} x\right) \text{nc}\left(\sqrt{\frac{5}{8}} x\right) \left\rfloor_0^T. \]
\[- \left( \frac{3k^2}{1-k^2} + \frac{8(1-2k^2)^2}{3(1-k^2)^2} \right) E\left( \sqrt{\frac{\sqrt{5}}{8}} T \right) + \left( \frac{3k^2}{1-k^2} \right)\]

\[+ \frac{4(1-2k^2)(2-4k^2)}{3(1-k^2)^2} \tan \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \tan \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \right]. \quad (3.9)\]

Therefore

\[H_1[\varphi] = \sqrt{\frac{\sqrt{5}}{2}} \varphi(T) + \frac{\sqrt{2}(\sqrt{5} - 1)}{4(1-k^2)} 5^{-\frac{1}{4}} \left[ \left( \frac{(\sqrt{5} - 1)(2 - 3k^2)}{3} \right) + 1 - k^2 \right]\]

\[- \frac{9k^2(\sqrt{5} - 1)^2}{20} - \frac{(\sqrt{5} - 1)^2(1-2k^2)(2-3k^2)}{5(1-k^2)} \right) \sqrt{\frac{\sqrt{5}}{8}} T\]

\[+ \left( \frac{2(\sqrt{5} - 1)^2(2k^2 - 1)}{5(1-k^2)^2} + \frac{2(\sqrt{5} - 1)(2k^2 - 1)}{3(1-k^2)} \right) - 1\]

\[\left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \tan \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \tan \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \right]. \quad (3.10)\]

Using the identity \( \varphi^2 = \frac{1}{2} (\varphi^2 + \varphi^3) \), we also compute

\[H_2[\varphi] = \frac{1}{2} \int_S \left( \frac{1}{2} \varphi^4 + \varphi \varphi^3 \right) dx = \frac{1}{4} \int_S \varphi^4 dx + \frac{1}{4} \int_S \varphi (\varphi^2 + \varphi^3) dx \quad (3.11)\]

\[= \frac{1}{4} I_2 + \frac{1}{4} I_3 + \frac{1}{2} I_4,\]

where

\[I_4 = \int_S \varphi^4 dx = 2 \int_0^T \left( \frac{\sqrt{5} - 1}{2 \cn^2 \left( \sqrt{\frac{\sqrt{5}}{8}} x, k \right) } \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \right]. \quad (3.11)\]

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\[= \frac{1}{4} I_2 + \frac{1}{4} I_3 + \frac{1}{2} I_4,\]

where

\[I_4 = \int_S \varphi^4 dx = 2 \int_0^T \left( \frac{\sqrt{5} - 1}{2 \cn^2 \left( \sqrt{\frac{\sqrt{5}}{8}} x, k \right) } \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \right]. \quad (3.11)\]
\[ \begin{align*}
+ \text{tn}\left( \sqrt{\frac{5}{8}} T \right) \text{dn}\left( \sqrt{\frac{5}{8}} T \right) \text{nc}^6\left( \sqrt{\frac{5}{8}} T \right) \right]_0^T. \tag{3.12}
\end{align*} \]

Refer to appendix B for details, through a series of calculations, we know

\begin{align*}
I_4 &= \frac{\sqrt{2}(\sqrt{5} - 1)^4}{28(1 - k^2)} \ 5^{-\frac{1}{4}} \ \left[ \left( \frac{5k^2(2 - 3k^2)}{3(1 - k^2)} + \frac{24(1 - 2k^2)^2(2 - 3k^2)}{15(1 - k^2)^2} \right) \\
&\quad + \frac{18k^2(1 - 2k^2)}{5(1 - k^2)^3} \sqrt{\frac{5}{8}} T + \left( \frac{10k^2(2k^2 - 1)}{3(1 - k^2)^2} - \frac{18k^2(1 - 2k^2)}{5(1 - k^2)^2} \right) \\
&\quad - \frac{16(1 - 2k^2)^3}{5(1 - k^2)^3} \ E\left( \sqrt{\frac{5}{8}} T \right) + \left( \frac{18k^2(1 - 2k^2)}{5(1 - k^2)^2} + \frac{5k^2(2 - 4k^2)}{3(1 - k^2)^2} \right) \\
&\quad + \frac{8(1 - 2k^2)^2(2 - 4k^2)}{5(1 - k^2)^3} \text{tn}\left( \sqrt{\frac{5}{8}} T \right) \text{dn}\left( \sqrt{\frac{5}{8}} T \right) \text{nc}^2\left( \sqrt{\frac{5}{8}} T \right) + \left( \frac{5k^2}{3(1 - k^2)} \right) \tag{3.13} \\
&\quad + \frac{8(1 - 2k^2)^2}{5(1 - k^2)^2} \text{tn}\left( \sqrt{\frac{5}{8}} T \right) \text{dn}\left( \sqrt{\frac{5}{8}} T \right) \text{nc}^2\left( \sqrt{\frac{5}{8}} T \right) \tag{3.14} \\
&\quad + \frac{6(1 - 2k^2)}{5(1 - k^2)} \text{tn}\left( \sqrt{\frac{5}{8}} T \right) \text{dn}\left( \sqrt{\frac{5}{8}} T \right) \text{nc}^4\left( \sqrt{\frac{5}{8}} T \right) \\
&\quad + \text{tn}\left( \sqrt{\frac{5}{8}} T \right) \text{dn}\left( \sqrt{\frac{5}{8}} T \right) \text{nc}^6\left( \sqrt{\frac{5}{8}} T \right) \right].
\end{align*} \]

Therefore

\begin{align*}
H_2[\varphi] &= \frac{\sqrt{2}(\sqrt{5} - 1)^2}{4(1 - k^2)} \ 5^{-\frac{1}{4}} \ \left[ \left( \frac{5k^2(\sqrt{5} - 1)^2(2 - 3k^2)}{42(1 - k^2)} - \frac{2 - 3k^2}{3} \right) \\
&\quad + \frac{9k^2(\sqrt{5} - 1)^2(1 - 2k^2)}{35(1 - k^2)^2} + \frac{4(\sqrt{5} - 1)^2(1 - 2k^2)^2(2 - 3k^2)}{35(1 - k^2)^2} \\
&\quad + \frac{2(\sqrt{5} - 1)(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)^2} + \frac{3k^2(\sqrt{5} - 1)}{10} \sqrt{\frac{5}{8}} T \\
&\quad + \left( \frac{5k^2(\sqrt{5} - 1)^2(2k^2 - 1)}{21(1 - k^2)^2} - \frac{9k^2(\sqrt{5} - 1)^2(1 - 2k^2)}{35(1 - k^2)^2} \right) \\
&\quad - \frac{2(2k^2 - 1)}{3(1 - k^2)} \frac{3k^2(\sqrt{5} - 1)}{10(1 - k^2)} - \frac{8(\sqrt{5} - 1)^2(1 - 2k^2)^3}{35(1 - k^2)^3} \\
&\quad - \frac{4(\sqrt{5} - 1)(1 - 2k^2)^2}{15(1 - k^2)^2} \ E\left( \sqrt{\frac{5}{8}} T \right) + \left( \frac{3k^2(\sqrt{5} - 1)}{10(1 - k^2)} \right) \\
&\quad + \frac{4(\sqrt{5} - 1)^2(1 - 2k^2)^2(2 - 4k^2)}{35(1 - k^2)^3} + \frac{9k^2(\sqrt{5} - 1)^2(1 - 2k^2)}{35(1 - k^2)^2} \\
&\quad + \frac{5k^2(\sqrt{5} - 1)^2(2 - 4k^2)}{42(1 - k^2)^2} + \frac{2(\sqrt{5} - 1)(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)^2} \
\end{align*}
but also it is also related to the integrals
conservation law
associated with a solution $H$

On account of $\phi$

Proof. For every $\phi$

Recalling that $\int_{S} [u(x) - \varphi(x - \xi)] \varphi'(x - \xi) dx (i = 1, 2)$
depending on time.

Lemma 3.1. For every $u \in H^1(S)$ and $\xi \in \mathbb{R}$,

\[
H_1[u] - H_1[\varphi] = \frac{1}{2} \| u - \varphi(\cdot - \xi) \|^2_{H^1(S)} - \frac{3}{4} \int_{S} [u(x) - \varphi(x - \xi)] \varphi^2(x - \xi) dx
\]

\[
+ \frac{1}{2} \int_{S} [u(x) - \varphi(x - \xi)] \varphi(x - \xi) dx + \frac{1}{4} \int_{S} [u(x) - \varphi(x - \xi)] dx
\]

\[ - \sqrt{2} [M_\varphi - u(\xi + T)] . \] (3.15)

Proof. We have

\[
\frac{1}{2} \| u - \varphi(\cdot - \xi) \|^2_{H^1(S)}
\]

\[
= H_1[u] + H_1[\varphi] - \int_{S} u_x(x) \varphi_x(x - \xi) dx - \int_{S} u(x) \varphi(x - \xi) dx
\]

\[
= H_1[u] + H_1[\varphi] - \int_{S} u(x + \xi) \varphi_x(x) dx - \int_{S} u(x + \xi) \varphi(x) dx . \] (3.16)

Recalling that $\varphi_{xx}(x) = \frac{3}{4} \varphi^2(x) + \frac{1}{2} \varphi(x) - \frac{1}{4} - \sqrt{2} \delta(x + T)$, we obtain

\[
\int_{S} u(x + \xi) \varphi_{xx}(x) dx = \frac{3}{4} \int_{S} \varphi^2(x) u(x + \xi) dx + \frac{1}{2} \int_{S} \varphi(x) u(x + \xi) dx
\]

\[ - \frac{1}{4} \int_{S} u(x + \xi) dx - \sqrt{2} u(x + T) . \] (3.17)

On account of $\varphi(T) = M_\varphi$, we can get

\[
\frac{1}{2} \| u - \varphi(\cdot - \xi) \|^2_{H^1(S)}
\]

\[ = - \frac{1}{4} \int_{S} [u(x) - \varphi(x - \xi)] dx - \frac{1}{2} \int_{S} [u(x) - \varphi(x - \xi)] \varphi(x - \xi) dx \]
Inspired by (3.19), we define the real function $g$ and $m$

Let $H_0[u]$ and extend it periodically to the real line. We compute the elliptic periodic peakon has maximal height. Indeed, if $u$ of $u$ the elliptic periodic peakon’s momentum, energy and height, then the whole shape $H$ that if a wave $H$

As Lenells said in [21], for a wave profile $u \in H^1(S)$, the functional $H_0[u]$ and $H_1[u]$ represents the momentum and kinetic energy, respectively. Lemma 3.1 says that if a wave $u \in H^1(S)$ has momentum $H_0[u]$, energy $H_1[u]$ and height $M_u$ close to the elliptic periodic peakon’s momentum, energy and height, then the whole shape of $u$ is close to that of the elliptic periodic peakon. Another physically relevant consequence of Lemma 3.1 is that among all waves of fixed momentum and energy, the elliptic periodic peakon has maximal height. Indeed, if $u \in H^1(S)$ is such that $H_0[u] = H_0[\varphi]$, $H_1[u] = H_1[\varphi]$ and $u(\xi) = \max_{x \in S} u(x)$, then $u(\xi) \leq M_\varphi$

Note that the periodic peakon $\varphi$ satisfies the differential equation

$$
\varphi_x = \begin{cases} 
-\sqrt{\frac{1}{2} \varphi \left( \varphi + \frac{1+\sqrt{5}}{2} \right) (\varphi - m_\varphi)}, & -T < x \leq 0, \\
\sqrt{\frac{1}{2} \varphi \left( \varphi + \frac{1+\sqrt{5}}{2} \right) (\varphi - m_\varphi)}, & 0 < x < T.
\end{cases}
$$

Let $u \in H^1(S) \subset C(S)$ be a positive function and write $M = M_u = \max_{x \in S} \{ u(x) \}$ and $m = m_u = \min_{x \in S} \{ u(x) \}$. Let $\xi$ and $\eta$ be such that $u(\xi) = M$ and $u(\eta) = m$. Inspired by (3.19), we define the real function $g(x)$ by

$$
g(x) = \begin{cases} 
u_x + \frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m), & \xi < x \leq \eta, \\
u_x - \frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m), & \eta < x < \xi + 2T,
\end{cases}
$$

and extend it periodically to the real line. We compute

$$
\int_S g^2(x) \, dx = \int_\xi^\eta \left[ u_x + \frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m) \right]^2 \, dx \\
+ \int_\eta^{\xi+2T} \left[ u_x - \frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m) \right]^2 \, dx \\
= \int_\xi^\eta u_x^2 \, dx + 2 \int_\xi^\eta u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx \\
+ \frac{1}{2} \int_\xi^\eta u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m) \, dx + \int_\eta^{\xi+2T} u_x^2 \, dx \\
- 2 \int_\eta^{\xi+2T} u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx \\
+ \frac{1}{2} \int_\eta^{\xi+2T} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m) \, dx.
$$

This proves the lemma.

\[\square\]
Notice that,
\[ \int_\eta^\xi u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx = - \int_\eta^{\xi+2T} u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx, \]
we conclude that
\[ \int_S g^2(x) \, dx = \int_S (u^2 + u_x^2) \, dx + 4 \int_\eta^{\xi} u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx 
+ \frac{1}{2} \int_S u^3 \, dx + \frac{1}{2} \left( \frac{\sqrt{5} - 3}{2} - m \right) \int_S u^2 \, dx - \frac{1 + \sqrt{5}}{4} m H_0[u], \]
(3.22)
where the details calculation of
\[ \int_\eta^{\xi} u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx \]
is given in appendix C. In the same way, we compute
\[ \int_S u g^2(x) \, dx = \int_\eta^{\xi} u \left[ u_x + \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \right]^2 \, dx 
+ \int_\eta^{\xi+2T} u \left[ u_x - \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \right]^2 \, dx 
= \int_\xi^{\eta} u u_x^2 \, dx + 2 \int_\xi^{\eta} u u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx 
+ \frac{1}{2} \int_\xi^{\eta} u^2 \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m) \, dx + \int_\eta^{\xi+2T} u u_x^2 \, dx 
- 2 \int_\eta^{\xi+2T} u u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx 
+ \frac{1}{2} \int_\eta^{\xi+2T} u^2 \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m) \, dx. \]
(3.23)
Since
\[ \int_\xi^{\eta} u u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx = - \int_\eta^{\xi+2T} u u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx, \]
therefore,
\[ \int_S u g^2(x) \, dx = \int_S (u u_x^2 + \frac{u^4}{2}) \, dx + 4 \int_\xi^{\eta} u u_x \sqrt{\frac{1}{2} u \left( u + \frac{1+\sqrt{5}}{2} \right) (u - m)} \, dx 
+ \frac{1}{2} \left( \frac{1 + \sqrt{5}}{2} - m \right) \int_S u^3 \, dx - \frac{1 + \sqrt{5}}{4} m \int_S u^2 \, dx. \]
\[ H_2[u] = \frac{1}{2} \int_S u g^2(x) \, dx - 2 \int_\xi^\eta u u_x \sqrt{\frac{1}{2} u (u + \frac{1 + \sqrt{5}}{2}) (u - m)} \, dx + \frac{1}{4} \left( 1 + \frac{\sqrt{5}}{2} - m \right) \int_S u^3 \, dx - \frac{1 + \frac{\sqrt{5}}{2}}{8} m \int_S u^2 \, dx \]

where the details calculation of

\[ \int_\xi^\eta u u_x \sqrt{\frac{1}{2} u (u + \frac{1 + \sqrt{5}}{2}) (u - m)} \, dx \]

is given in appendix C. Combining (3.22) with (3.24) gives

\[ H_2[u] = 2H_2[u] + 4 \int_\xi^\eta u u_x \sqrt{\frac{1}{2} u (u + \frac{1 + \sqrt{5}}{2}) (u - m)} \, dx \]

\[ + \frac{1}{2} \left( 1 + \frac{\sqrt{5}}{2} - m \right) \int_S u^3 \, dx - \frac{1 + \frac{\sqrt{5}}{2}}{4} m \int_S u^2 \, dx \]

\[ \frac{1}{2} \left( 1 + \frac{\sqrt{5}}{2} - m \right) \int_S u^3 \, dx - \frac{1 + \frac{\sqrt{5}}{2}}{4} m \int_S u^2 \, dx \]

\[ \leq M \int_S g^2(x) \, dx - 2 \int_\xi^\eta u u_x \sqrt{\frac{1}{2} u (u + \frac{1 + \sqrt{5}}{2}) (u - m)} \, dx + \frac{1}{4} \int_S u^3 \, dx \]

\[ + \frac{1}{4} \left( 1 + \frac{\sqrt{5}}{2} - m \right) \int_S u^3 \, dx - \frac{1 + \frac{\sqrt{5}}{2}}{8} m \int_S u^2 \, dx \]

\[ + \frac{1}{4} \left( 1 + \frac{\sqrt{5}}{2} - m \right) \int_S u^3 \, dx - \frac{1}{4} \left( 1 + \frac{\sqrt{5}}{2} - m \right) \int_S u^3 \, dx \]

\[ - 2 \int_\xi^\eta u u_x \sqrt{\frac{1}{2} u (u + \frac{1 + \sqrt{5}}{2}) (u - m)} \, dx. \]

We have actually proved the following lemma

**Lemma 3.2.** For any positive \( u \in H^1(S) \), define a function

\[ F_u : \{(M, m) \in \mathbb{R}^2 : M \geq m > 0\} \to \mathbb{R} \]

by

\[ F_u(M, m) = M \left[ H_1[u] + \frac{1}{4} \int_S u^3 \, dx + \frac{1}{4} \left( \frac{\sqrt{5} - 3}{2} - m \right) \int_S u^2 \, dx \right] \]

\[ - \frac{1 + \frac{\sqrt{5}}{2}}{8} m H_0[u] + 2 \int_\xi^\eta u u_x \sqrt{\frac{1}{2} u (u + \frac{1 + \sqrt{5}}{2}) (u - m)} \, dx \]

\[ - \frac{1}{4} \left( 1 + \frac{\sqrt{5}}{2} - m \right) \int_S u^3 \, dx + \frac{1 + \frac{\sqrt{5}}{2}}{8} m \int_S u^2 \, dx - H_2[u] \]

\[ - 2 \int_\xi^\eta u u_x \sqrt{\frac{1}{2} u (u + \frac{1 + \sqrt{5}}{2}) (u - m)} \, dx. \]
Then
\begin{equation}
F_u(M, m) \geq 0
\end{equation}
where $M_u = \max_{x \in \mathbb{S}} \{u(x)\}$ and $m_u = \min_{x \in \mathbb{S}} \{u(x)\}$. \hfill \Box

The next lemma highlights some properties of the function $F_\psi(M, m)$ associated to the peakon.

**Lemma 3.3.** For the peakon $\psi$, it holds that
\begin{align*}
F_\psi(M_\psi, m_\psi) &= 0, & \frac{\partial F_\psi}{\partial M}(M_\psi, m_\psi) &= 0, \\
\frac{\partial^2 F_\psi}{\partial m}(M_\psi, m_\psi) &= 0, & \frac{\partial^2 F_\psi}{\partial M^2}(M_\psi, m_\psi) &= -\sqrt{2}, \\
\frac{\partial^2 F_\psi}{\partial M \partial m}(M_\psi, m_\psi) &= 0, & \frac{\partial^2 F_\psi}{\partial m^2}(M_\psi, m_\psi) &\approx -0.6936.
\end{align*}

**Proof.** Using $M_\psi = 1$ and $m_\psi = \frac{\sqrt{5} - 1}{2}$, we have
\begin{align*}
F_\psi(M_\psi, m_\psi) &= M_\psi \left[ H_1[\psi] + 2 \int^n_\xi \varphi_x \sqrt{\frac{1}{2} \psi \left( \varphi + \frac{1 + \sqrt{5}}{2} \right) (\varphi - m_\psi)} \, dx \\
&\quad - \frac{1 + \sqrt{5}}{8} m_\psi H_0[\psi] + 1 \frac{3}{4} \left( \frac{\sqrt{5} - 3}{2} - m_\psi \right) \int^S \varphi^2 \, dx \\
&\quad + \frac{1}{3} \int^S \varphi^3 \, dx \right] + 1 \frac{1 + \sqrt{5}}{8} m_\psi \int^S \varphi^2 \, dx - H_2[\psi] \\
&\quad - 2 \int^n_\xi \varphi \varphi_x \sqrt{\frac{1}{2} \psi \left( \varphi + \frac{1 + \sqrt{5}}{2} \right) (\varphi - m_\psi)} \, dx \\
&\quad - \frac{1}{3} \left( \frac{1 + \sqrt{5}}{2} - m_\psi \right) \int^S \varphi^3 \, dx \\
&\quad - \frac{1}{4} H_0[\psi] + H_1[\psi] - H_2[\psi] \\
&\quad + 2 \int^n_\xi \varphi_x \sqrt{\frac{1}{2} \psi \left( \varphi + \frac{1 + \sqrt{5}}{2} \right) (\varphi - m_\psi)} \, dx \\
&\quad - 2 \int^n_\xi \varphi \varphi_x \sqrt{\frac{1}{2} \psi \left( \varphi + \frac{1 + \sqrt{5}}{2} \right) (\varphi - m_\psi)} \, dx.
\end{align*}

Similarly, referring to Appendix C for details, we can get
\begin{align*}
\int^n_\xi \varphi_x \sqrt{\frac{1}{2} \psi \left( \varphi + \frac{1 + \sqrt{5}}{2} \right) (\varphi - m_\psi)} \, dx \\
&= -\sqrt{2} \int^{M_\psi}_{m_\psi} \sqrt{\psi \left( \varphi + \frac{1 + \sqrt{5}}{2} \right) (\varphi - m_\psi)} \, d\varphi \\
&= -\sqrt{2} m_\psi \sqrt{m_\psi + \frac{1 + \sqrt{5}}{2}} \left[ \frac{(2k^2 - 1)(2 - 3k^2)}{3(1 - k^2)} + \frac{3k^2}{5} - k^2 \right].
\end{align*}
\[ \begin{align*}
&\frac{4(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)} \phi_1 + \left( \frac{2(2k^2 - 1)^2}{3(1 - k^2)^2} - \frac{8(1 - 2k^2)^2}{15(1 - k^2)^2} \right) \phi_1 + \left( \frac{3k^2}{5(1 - k^2)} + \frac{4(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)^2} \right) \text{tn}\phi_1 \text{dn}\phi_1 + \left( \frac{4(1 - 2k^2)}{15(1 - k^2)} \right) \frac{2}{3(1 - k^2)} \Phi_1 \text{dn}\phi_1 \text{nc}^2\phi_1 + \frac{1}{5} \text{tn}\phi_1 \text{dn}\phi_1 \text{nc}^4\phi_1, \\
\end{align*} \]

where \( \phi_1 = \text{sn}^{-1} \left( \sqrt{\frac{M_x - m_x}{M_x}} \right) = \sqrt{\frac{\sqrt{m_x}}{3}} \). We also obtain

\[ \begin{align*}
\int_{\xi}^{\eta} \phi_1 \sqrt{1 + \frac{1 + \sqrt{m_x}}{2}} (\varphi - m_\varphi) \, dx &= -\sqrt{\frac{2}{m_x + \frac{1 + \sqrt{m_x}}{2}}} \\
&\quad \left[ \frac{24m_x^4(1 - 2k^2)^2(2 - 3k^2)}{105(1 - k^2)^3} + \frac{3m_x^3k^2(1 + \sqrt{m_x} - m_\varphi)}{5(1 - k^2)} \right] \\
&\quad + \frac{4m_x^3(1 + \sqrt{m_x} - m_\varphi)(1 - 2k^2)^2(2 - 3k^2)}{15(1 - k^2)^2} + \frac{18m_x^4k^2(1 - 2k^2)}{35(1 - k^2)^2} \\
&\quad - \frac{1 + \sqrt{m_x}}{2} m_x^3(2 - 3k^2) \right] \left( \frac{5m_x^3k^2(2 - 3k^2)}{21(1 - k^2)^2} \right) \phi_1 + \left( \frac{10m_x^4k^2(2k^2 - 1)}{21(1 - k^2)^3} \right) \\
&\quad - \frac{18m_x^4k^2(1 - 2k^2)^2}{35(1 - k^2)^3} - \frac{48m_x^4(1 - 2k^2)^3}{105(1 - k^2)^4} + \frac{3m_x^3k^2(1 + \sqrt{m_x} - m_\varphi)}{5(1 - k^2)^2} \\
&\quad - \frac{8m_x^3(1 + \sqrt{m_x} - m_\varphi)(1 - 2k^2)^2}{15(1 - k^2)^3} + \frac{1(1 + \sqrt{m_x})m_x^3(2k^2 - 1)}{3(1 - k^2)^2} \right) E(\phi_1) \\
&\quad + \left( \frac{18m_x^4k^2(1 - 2k^2)^2}{35(1 - k^2)^3} + \frac{3m_x^3k^2(1 + \sqrt{m_x} - m_\varphi)}{5(1 - k^2)^2} + \frac{5m_x^4k^2(2 - 4k^2)}{21(1 - k^2)^3} \right) \\
&\quad - \frac{24m_x^4(1 - 2k^2)^2(2 - 4k^2)}{105(1 - k^2)^4} + \frac{4m_x^3(1 + \sqrt{m_x} - m_\varphi)(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)^3} \\
&\quad + \frac{1 + \sqrt{m_x}}{2} m_x^3(2 - 4k^2) \right] \text{tn}\phi_1 \text{dn}\phi_1 + \left( \frac{4m_x^3(1 + \sqrt{m_x} - m_\varphi)(1 - 2k^2)}{15(1 - k^2)^2} \right) \\
&\quad - \frac{1 + \sqrt{m_x}}{2} m_x^3 \right] \text{tn}\phi_1 \text{dn}\phi_1 \text{nc}^2\phi_1 + \frac{24m_x^4(1 - 2k^2)^2}{35(1 - k^2)^3} + \frac{5m_x^4k^2}{21(1 - k^2)^2} \right) \text{tn}\phi_1 \text{dn}\phi_1 \text{nc}^4\phi_1 \\
&\quad + \left( \frac{6m_x^3(1 - 2k^2)}{35(1 - k^2)^2} + \frac{m_x^3(1 + \sqrt{m_x} - m_\varphi)}{5(1 - k^2)} \right) \text{tn}\phi_1 \text{dn}\phi_1 \text{nc}^6\phi_1 + \frac{m_x^3}{7(1 - k^2)} \text{tn}\phi_1 \text{dn}\phi_1 \text{nc}^6\phi_1. \\
\end{align*} \]
According to Equation (3.4), (3.10), (3.14), (3.29) and (3.30), we know

\[
F_{\varphi}(M_{\varphi}, m_{\varphi}) = J_{10}\varphi_1 + J_{11}E(\varphi_1) + J_{12}\text{tn}(\varphi_1)\text{dn}(\varphi_1) + J_{13}\text{tn}(\varphi_1)\text{dn}(\varphi_1)\text{nc}^2(\varphi_1)
\]

\[
+ J_{14}\text{tn}(\varphi_1)\text{dn}(\varphi_1)\text{nc}^4(\varphi_1) + J_{15}\text{tn}(\varphi_1)\text{dn}(\varphi_1)\text{nc}^6(\varphi_1) + \frac{\sqrt{2}}{2}\varphi(T),
\]

(3.31)

where

\[
J_{10} = \frac{\sqrt{2}(1 - \sqrt{5})}{2} 5^{-\frac{1}{4}} + \frac{\sqrt{2}(\sqrt{5} - 1)}{4(1 - k^2)} \frac{5^{-\frac{1}{4}}}{2} \left( \frac{(\sqrt{5} - 1)(2 - 3k^2)}{3} \right)
\]

\[
- \frac{\sqrt{2}(\sqrt{5} - 1)^2(1 - 2k^2)(2 - 3k^2)}{4(1 - k^2)} + \frac{9k^2(\sqrt{5} - 1)^2}{20} + 1 - k^2
\]

\[
- \frac{\sqrt{2}(\sqrt{5} - 1)^2}{5(1 - k^2)} 5^{-\frac{1}{4}} \left( \frac{5k^2(\sqrt{5} - 1)^2(2 - 3k^2)}{42(1 - k^2)} + \frac{3k^2(\sqrt{5} - 1)}{10} \right)
\]

\[
+ \frac{4(\sqrt{5} - 1)^2(1 - 2k^2)^2}{35(1 - k^2)^2} + \frac{2(\sqrt{5} - 1)(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)}
\]

\[
+ \frac{9k^2(\sqrt{5} - 1)^2(1 - 2k^2)}{35(1 - k^2)^2} - \frac{2 - 3k^2}{3} - 2\sqrt{2}m_{\varphi}\sqrt{m_{\varphi} + \frac{1 + \sqrt{5}}{2}} \left( \frac{3k^2}{5} - k^2 + \frac{(2k^2 - 1)(2 - 3k^2)}{3(1 - k^2)} + \frac{4(1 - k^2)(2 - 3k^2)}{15(1 - k^2)} \right)
\]

\[
+ 2\sqrt{\frac{2}{m_{\varphi} + \frac{1 + \sqrt{5}}{2}}} \left( \frac{24m_{\varphi}^3(1 - 2k^2)^2(2 - 3k^2)}{105(1 - k^2)^3} + \frac{18m_{\varphi}^4k^2(1 - 2k^2)}{35(1 - k^2)^2} \right)
\]

\[
+ \frac{4m_{\varphi}^4(1 + \sqrt{5} - m_{\varphi})(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)^2} + \frac{\frac{1 + \sqrt{5}}{2}m_{\varphi}^3(2 - 3k^2)}{3(1 - k^2)}
\]

\[
+ \frac{3m_{\varphi}^4k^2(\frac{1 + \sqrt{5}}{2} - m_{\varphi})}{5(1 - k^2)^2} + \frac{5m_{\varphi}^4k^2(2 - 3k^2)}{21(1 - k^2)^2},
\]

\[
J_{11} = \frac{\sqrt{2}(\sqrt{5} - 1)}{2(1 - k^2)} 5^{-\frac{1}{4}} + \frac{\sqrt{2}(\sqrt{5} - 1)}{4(1 - k^2)} 5^{-\frac{1}{4}} \left( \frac{2(\sqrt{5} - 1)^2(2k^2 - 1)^2}{5(1 - k^2)^2} - 1 \right)
\]

\[
+ \frac{9k^2(\sqrt{5} - 1)^2}{20(1 - k^2)} + \frac{2(\sqrt{5} - 1)(2k^2 - 1)}{3(1 - k^2)} - \frac{\sqrt{2}(\sqrt{5} - 1)^2}{4(1 - k^2)} 5^{-\frac{1}{4}} \left( \frac{5k^2(\sqrt{5} - 1)^2(2k^2 - 1)}{21(1 - k^2)^2} - \frac{9k^2(\sqrt{5} - 1)^2(1 - 2k^2)}{35(1 - k^2)^2} - \frac{3k^2(\sqrt{5} - 1)}{10(1 - k^2)} \right)
\]

\[
- \frac{8(\sqrt{5} - 1)^2(1 - 2k^2)}{35(1 - k^2)^3} - \frac{4(\sqrt{5} - 1)(1 - 2k^2)^2}{15(1 - k^2)^2} + \frac{2(2k^2 - 1)}{3(1 - k^2)^2}
\]

\[
- \frac{2\sqrt{2}m_{\varphi}\sqrt{m_{\varphi} + \frac{1 + \sqrt{5}}{2}}}{2} \left( \frac{2(2k^2 - 1)^2}{3(1 - k^2)^2} - \frac{3k^2}{5(1 - k^2)^2} + \frac{8(1 - 2k^2)^2}{15(1 - k^2)^2} \right)
\]

\[
+ \frac{k^2}{1 - k^2} + \frac{2}{\sqrt{m_{\varphi} + \frac{1 + \sqrt{5}}{2}}} \left( \frac{10m_{\varphi}^4k^2(2k^2 - 1)}{21(1 - k^2)^3} - \frac{48m_{\varphi}^4(1 - 2k^2)^3}{105(1 - k^2)^4} \right)
\]

\[
- \frac{(1 + \sqrt{5})m_{\varphi}^3(2k^2 - 1)}{3(1 - k^2)^2} - \frac{8m_{\varphi}^3(1 + \sqrt{5} - m_{\varphi})(1 - 2k^2)}{15(1 - k^2)^3}.
\]
\[
J_{12} = -\frac{\sqrt{2}(\sqrt{5} - 1)}{2(1 - k^2)} \sqrt{5} - \frac{1}{4} + \frac{\sqrt{2}(\sqrt{5} - 1)}{4(1 - k^2)} \sqrt{5} - \frac{1}{4} \left( - \frac{9k^2(\sqrt{5} - 1)^2}{20(1 - k^2)} \right) \\
+ \frac{(\sqrt{5} - 1)(2 - 4k^2)}{3(1 - k^2)} \left( \frac{3k^2(\sqrt{5} - 1)}{10(1 - k^2)} + \frac{9k^2(\sqrt{5} - 1)^2(1 - 2k^2)}{35(1 - k^2)^2} \right) \\
- \frac{\sqrt{2}(\sqrt{5} - 1)^2}{4(1 - k^2)} \sqrt{5} - \frac{1}{4} \left( \frac{3k^2(\sqrt{5} - 1)}{10(1 - k^2)} + \frac{9k^2(\sqrt{5} - 1)^2(1 - 2k^2)}{35(1 - k^2)^2} \right) \\
+ \frac{2(\sqrt{5} - 1)(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)^2} + \frac{4(\sqrt{5} - 1)^2(1 - 2k^2)^2(2 - 4k^2)}{35(1 - k^2)^3} \\
- \frac{2 - 4k^2}{3(1 - k^2)} + \frac{5k^2(\sqrt{5} - 1)^2(2 - 4k^2)}{42(1 - k^2)^2} \right) \\
- 2\sqrt{2} \frac{m_\varphi}{m_\varphi + 1 + \sqrt{5}} \left( \frac{24m_\varphi^3(1 - 2k^2)^2(2 - 4k^2)}{21(1 - k^2)^2} - \frac{1 + \sqrt{5}}{3} \frac{m_\varphi^3(2 - 4k^2)}{3(1 - k^2)^2} \right) \\
+ \frac{5m_\varphi^4k^2(2 - 4k^2)}{21(1 - k^2)^3} + \frac{18m_\varphi^4k^2(1 - 2k^2)}{35(1 - k^2)^3} + \frac{3m_\varphi^3k^2(1 + \sqrt{5} - m_\varphi)}{5(1 - k^2)^2} \\
+ \frac{4m_\varphi^3(1 + \sqrt{5} - m_\varphi)(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)^3})
\]

\[
J_{13} = \frac{\sqrt{2}(\sqrt{5} - 1)}{4(1 - k^2)} \sqrt{5} - \frac{1}{4} \left( \frac{(\sqrt{5} - 1)^2(2k^2 - 1)}{5(1 - k^2)} + \frac{1}{3} \right) \\
- \frac{\sqrt{2}(\sqrt{5} - 1)^2}{4(1 - k^2)} \sqrt{5} - \frac{1}{4} \left( \frac{5k^2(\sqrt{5} - 1)^2}{42(1 - k^2)^2} + \frac{4(\sqrt{5} - 1)^2(1 - 2k^2)^2}{35(1 - k^2)^2} \right) \\
+ \frac{2(\sqrt{5} - 1)(1 - 2k^2)}{15(1 - k^2)^2} - \frac{1}{3} \right) \\
- 2\sqrt{2} \frac{m_\varphi}{m_\varphi + 1 + \sqrt{5}} \left( \frac{2k^2 - 1}{3(1 - k^2)} \right) \\
+ \frac{4(1 - 2k^2)}{15(1 - k^2)^2} + \frac{2}{m_\varphi + 1 + \sqrt{5}} \left( \frac{5m_\varphi^4k^2}{21(1 - k^2)^2} + \frac{24m_\varphi^4(1 - 2k^2)^2}{105(1 - k^2)^3} \right) \\
- \frac{1 + \sqrt{5}}{3} \frac{m_\varphi^3(1 + \sqrt{5} - m_\varphi)}{3(1 - k^2)^2} + \frac{4m_\varphi^3(1 + \sqrt{5} - m_\varphi)(1 - 2k^2)}{15(1 - k^2)^3})
\]

\[
J_{14} = \frac{3\sqrt{2}(\sqrt{5} - 1)^3}{80(1 - k^2)} \sqrt{5} - \frac{1}{4} \left( \frac{3(\sqrt{5} - 1)^2(1 - 2k^2)}{35(1 - k^2)^2} \right) \\
+ \frac{\sqrt{5} - 1}{10} \right) + \frac{2}{m_\varphi + 1 + \sqrt{5}} \left( \frac{m_\varphi^3(1 + \sqrt{5} - m_\varphi)}{5(1 - k^2)} + \frac{6m_\varphi^4(1 - 2k^2)}{35(1 - k^2)^2} \right) \\
- \frac{2\sqrt{2}m_\varphi^2}{5} \sqrt{m_\varphi + 1 + \sqrt{5}}.
\]
From equation (3.4), (3.7), (3.9), (3.10) and (3.29), we know

On the one hand, we have

\[
J_{15} = -\frac{\sqrt{2(\sqrt{5} - 1)^4}}{56(1 - k^2)} \cdot 5^{-\frac{1}{4}} + \frac{2m_\varphi^4}{7(1 - k^2)} \cdot \sqrt{\frac{2}{m_\varphi + \frac{1 + \sqrt{5}}{2}}}.
\]

Substituting \( k = \frac{1 + \sqrt{5}}{2\sqrt{5}} \) and \( m_\varphi = \frac{\sqrt{5} - 1}{2} \) into the above formula, we can get \( J_{10}, J_{11}, J_{12}, J_{13} \) and \( J_{15} \) are 0, as well as

\[
J_{14}\text{tn}(\varphi_1)\text{dn}(\varphi_1)\text{nc}(\varphi_1) + \frac{\sqrt{2}}{2}\varphi(T) = 0,
\]

therefore

\[
F_\varphi(M_\varphi, m_\varphi) = 0. \tag{3.32}
\]

On the one hand, we have

\[
\frac{\partial F_\varphi}{\partial M} = H_1[u] + 2\int_\xi^n u_x\sqrt{\frac{1}{2}u(u + \frac{1 + \sqrt{5}}{2})} (u - m) \, dx
\]

\[
+ \frac{1}{4}\int_S u^3 \, dx + \frac{1}{4}\left(\frac{\sqrt{5} - 3}{2} - m\right) \int_S u^2 \, dx - \frac{1 + \sqrt{5}}{8}mH_0[u]. \tag{3.33}
\]

From equation (3.4), (3.7), (3.9), (3.10) and (3.29), we know

\[
\frac{\partial F_\varphi}{\partial M}(M_\varphi, m_\varphi) = -\frac{1}{4}H_0[\varphi] + H_1[\varphi] - \frac{1}{4}\int_S \varphi^2 \, dx + \frac{1}{4}\int_S \varphi^3 \, dx
\]

\[
+ 2\int_\xi^n \varphi_x\sqrt{\frac{1}{2}\varphi(\varphi + \frac{1 + \sqrt{5}}{2})} (\varphi - m_\varphi) \, dx
\]

\[
= J_{20}\varphi_1 + J_{21}E(\varphi_1) + J_{22}\text{tn}(\varphi_1)\text{dn}(\varphi_1)
\]

\[
+ J_{23}\text{tn}(\varphi_1)\text{dn}(\varphi_1)\text{nc}(\varphi_1) + \frac{\sqrt{2}}{2}\varphi(T)
\]

\[
+ J_{24}\text{tn}(\varphi_1)\text{dn}(\varphi_1)\text{nc}(\varphi_1), \tag{3.34}
\]

where

\[
J_{20} = \frac{\sqrt{2}(1 - \sqrt{5})}{2} \cdot 5^{-\frac{1}{4}} + \frac{\sqrt{2}(\sqrt{5} - 1)}{4(1 - k^2)} \cdot 5^{-\frac{1}{4}} \left(1 - \frac{9k^2(\sqrt{5} - 1)^2}{20} \cdot \frac{(\sqrt{5} - 1)^2(1 - 2k^2)(2 - 3k^2)}{5(1 - k^2)} + \frac{(\sqrt{5} - 1)(2 - 3k^2)}{3} - k^2\right)
\]

\[
- \frac{\sqrt{2}(\sqrt{5} - 1)^2(2 - 3k^2)}{12(1 - k^2)} \cdot 5^{-\frac{1}{4}} + \frac{\sqrt{2}(\sqrt{5} - 1)^3}{8(1 - k^2)} \cdot 5^{-\frac{5}{4}} \left(3k^2 + \frac{4(1 - 2k^2)(2 - 3k^2)}{3(1 - k^2)}\right) - 2\sqrt{2}m_\varphi^2 \sqrt{m_\varphi + \frac{1 + \sqrt{5}}{2}} \left(\frac{3k^2}{5} - k^2\right)
\]

\[
+ \frac{(2k^2 - 1)(2 - 3k^2)}{3(1 - k^2)} + \frac{4(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)}
\]

\[
J_{21} = \frac{\sqrt{2}(\sqrt{5} - 1)}{2} \cdot 5^{-\frac{1}{4}} + \frac{\sqrt{2}(\sqrt{5} - 1)}{4(1 - k^2)} \cdot 5^{-\frac{1}{4}} \left(\frac{2(\sqrt{5} - 1)^2(2k^2 - 1)^2}{5(1 - k^2)^2} - 1\right)
\]

\[
+ \frac{9k^2(\sqrt{5} - 1)^2}{20(1 - k^2)} + \frac{2(\sqrt{5} - 1)(2k^2 - 1)}{3(1 - k^2)} - \frac{\sqrt{2}(\sqrt{5} - 1)^3}{8(1 - k^2)} \cdot \frac{1}{5} \left(\frac{3k^2}{1 - k^2} + \frac{8(1 - 2k^2)^2}{3(1 - k^2)^2}\right) - 2\sqrt{2}m_\varphi^2 \sqrt{m_\varphi + \frac{1 + \sqrt{5}}{2}} \left(\frac{k^2}{1 - k^2} + \frac{3k^2}{5(1 - k^2)}\right)
\]
Similarly, we have

\[ J_{22} = \frac{\sqrt{2}(1 - \sqrt{5})}{2(1 - k^2)} 5^{-\frac{1}{4}} + \frac{\sqrt{2}(\sqrt{5} - 1)}{4(1 - k^2)} 5^{-\frac{1}{4}} \left( \frac{(\sqrt{5} - 1)^2(2k^2 - 1)(2 - 4k^2)}{5(1 - k^2)^2} + \frac{9k^2(\sqrt{5} - 1)^2}{20(1 - k^2)} + \frac{(\sqrt{5} - 1)(2 - 4k^2)}{3(1 - k^2)} + 1 \right) + \frac{\sqrt{2}(\sqrt{5} - 1)^3}{8(1 - k^2)} 5^{-\frac{5}{4}} \left( \frac{4(1 - 2k^2)(2 - 4k^2)}{3(1 - k^2)^2} + \frac{3k^2}{1 - k^2} - \frac{\sqrt{2}(\sqrt{5} - 1)^2(2 - 4k^2)}{12(1 - k^2)^2} \right) \]

\[ J_{23} = \frac{\sqrt{2}(\sqrt{5} - 1)}{4(1 - k^2)} 5^{-\frac{1}{4}} \left( \frac{(\sqrt{5} - 1)^2(2k^2 - 1)}{5(1 - k^2)^2} + \frac{\sqrt{5} - 1}{3} \right) \]

\[ - \frac{\sqrt{2}(\sqrt{5} - 1)}{12(1 - k^2)} 5^{-\frac{1}{4}} + \frac{\sqrt{2}(\sqrt{5} - 1)^3(1 - 2k^2)^2}{6(1 - k^2)^2} 5^{-\frac{5}{4}} \]

\[ - 2\sqrt{2}m_\varphi \sqrt{m_\varphi} + \frac{1 + \sqrt{5}}{2} \left( \frac{2k^2 - 1}{3(1 - k^2)} + \frac{4(1 - 2k^2)}{15(1 - k^2)} \right) \]

\[ J_{24} = \frac{3\sqrt{2}(1 - \sqrt{5})^3}{16(1 - k^2)} 5^{-\frac{5}{4}} + \frac{\sqrt{2}(\sqrt{5} - 1)^3}{8(1 - k^2)} 5^{-\frac{5}{4}} - \frac{2\sqrt{2}m_\varphi \sqrt{m_\varphi} + \frac{1 + \sqrt{5}}{2}}{5} \]

Similarly, we get \( J_{20}, J_{21}, J_{22} \) and \( J_{23} \) are 0, as well as

\[ J_{24} \theta_n(\varphi_1)dn(\varphi_1)nc^4(\varphi_1) + \frac{\sqrt{2}}{2} \varphi(T) = 0. \]

Hence,

\[ \frac{\partial F_{\varphi}}{\partial M}(M_\varphi, m_\varphi) = 0. \quad (3.35) \]

On the other hand, we have

\[ \frac{\partial^2 F_u}{\partial M^2} = \frac{\partial}{\partial M} \left[ 2 \int_\xi^u u_x \sqrt{\frac{1}{2} u \left( u + \frac{1 + \sqrt{5}}{2} \right) (u - m)} \, dx \right] \]

\[ = -2 \sqrt{\frac{1}{2}} M \left( M + \frac{1 + \sqrt{5}}{2} \right) (M - m). \quad (3.36) \]

So

\[ \frac{\partial^2 F_u}{\partial M^2}(M_\varphi, m_\varphi) = -\sqrt{2}. \quad (3.37) \]

Similarly, we have

\[ \frac{\partial^2 F_u}{\partial M \partial m} = \frac{\partial}{\partial m} \left[ 2 \int_\xi^u u_x \sqrt{\frac{1}{2} u \left( u + \frac{1 + \sqrt{5}}{2} \right) (u - m)} \, dx \right] \]

\[ - \frac{1}{4} \int_\xi^u u_x^2 \, dx - \frac{1 + \sqrt{5}}{8} H_0[u]. \quad (3.38) \]
According to the derivation integral with parameters and referring to appendix B, we get

\[
\frac{\partial}{\partial m} \left[ 2 \int_{\xi}^{\eta} u_{\xi} \sqrt{\frac{1}{2} u \left( u + \frac{1 + \sqrt{5}}{2} \right) (u - m)} \, dx \right] = \frac{\sqrt{2}}{2} \int_{m}^{M} \sqrt{\frac{u \left( u + \frac{1 + \sqrt{5}}{2} \right)}{u - m}} \, du
\]

\[
= \frac{\sqrt{2} m \sqrt{m + \frac{1 + \sqrt{5}}{2}}}{3(1 - k^2)^2} \left[ 2(1 - k^2)u_1 + (k^2 - 2)E(u_1) + (2 - k^2)\text{tn}u_1 \text{dn}u_1 \right] (3.39)
\]

\[
+ (1 - k^2)\text{tn}u_1 \text{dn}u_1 \text{nc}^2 u_1 \right],
\]

where \( u_1 = \text{sn}^{-1}(\sqrt{\frac{m - m_1}{M}}) \), from (3.4), (3.7) and (3.39), we get

\[
\frac{\partial^2 F_\phi}{\partial M \partial m} (M_\phi, m_\phi) = \frac{\sqrt{2}}{2} \int_{M_\phi}^{M} \frac{\varphi \left( \frac{\varphi + 1 + \sqrt{5}}{2} \right)}{\varphi - m_\phi} \, d\varphi - \frac{1}{4} \int_{S} \varphi^2 \, dx - \frac{1 + \sqrt{5}}{8} H_0[\varphi]
\]

\[
= J_{30} \varphi_1 + J_{31} E(\varphi_1) + J_{32} \text{tn}(\varphi_1) \text{dn}(\varphi_1) + J_{33} \text{tn}(\varphi_1) \text{dn}(\varphi_1) \text{nc}^2(\varphi_1), \quad (3.40)
\]

where

\[
J_{30} = \frac{2\sqrt{2} m_\varphi \sqrt{m_\varphi + \frac{1 + \sqrt{5}}{2}}}{3} - \sqrt{2} 5^{-1} 4^{-1} - \frac{\sqrt{2} (\sqrt{5} - 1)^2 (2 - 3 k^2)}{12(1 - k^2)} 5^{-1} 4^{-1},
\]

\[
J_{31} = \frac{\sqrt{2}}{k^2 - 1} 5^{-1} 4^{-1} - \frac{\sqrt{2} (\sqrt{5} - 1)^2 (2 - 3 k^2)}{12(1 - k^2)^2} 5^{-1} 4^{-1} + \frac{\sqrt{2} m_\varphi (2 - k^2) \sqrt{m_\varphi + \frac{1 + \sqrt{5}}{2}}}{3(1 - k^2)},
\]

\[
J_{32} = \frac{\sqrt{2}}{k^2 - 1} 5^{-1} 4^{-1} - \frac{\sqrt{2} (\sqrt{5} - 1)^2 (2 - 4 k^2)}{12(1 - k^2)^2} 5^{-1} 4^{-1} + \frac{\sqrt{2} m_\varphi (2 - k^2) \sqrt{m_\varphi + \frac{1 + \sqrt{5}}{2}}}{3(1 - k^2)},
\]

\[
J_{33} = \frac{-\sqrt{2} (\sqrt{5} - 1)^2}{12(1 - k^2)} 5^{-1} 4^{-1} + \frac{\sqrt{2} m_\varphi \sqrt{m_\varphi + \frac{1 + \sqrt{5}}{2}}}{3}.
\]

We known \( J_{30}, J_{31}, J_{32} \) and \( J_{33} \) are 0. Thence

\[
\frac{\partial^2 F_\phi}{\partial M \partial m} (M_\phi, m_\phi) = 0. \quad (3.41)
\]

Further, we have

\[
\frac{\partial F_u}{\partial m} = M \left[ \frac{\partial}{\partial m} \left[ 2 \int_{\xi}^{\eta} u_{\xi} \sqrt{\frac{1}{2} u \left( u + \frac{1 + \sqrt{5}}{2} \right) (u - m)} \, dx \right] - \frac{1}{4} \int_{S} u^2 \, dx \right]
\]

\[
- \frac{1 + \sqrt{5}}{8} H_0[u] \right] - \frac{\partial}{\partial m} \left[ 2 \int_{\xi}^{\eta} u_{\xi} \sqrt{\frac{1}{2} u \left( u + \frac{1 + \sqrt{5}}{2} \right) (u - m)} \, dx \right] (3.42)
\]

\[
+ \frac{1}{4} \int_{S} u^3 \, dx + \frac{1 + \sqrt{5}}{8} \int_{S} u^2 \, dx,
\]
where
\[
\frac{\partial}{\partial m} \left[ 2 \int_0^m u_x \sqrt{\frac{1}{2} u \left( u + \frac{1 + \sqrt{5}}{4} \right)} (u - m) \, dx \right]
= \frac{\sqrt{2} m^2}{2} \int_m^M \frac{u^2 \left( u + \frac{1 + \sqrt{5}}{4} \right)}{\sqrt{\frac{1}{2} u \left( u + \frac{1 + \sqrt{5}}{4} \right)} (u - m)} \, du
\]
\[
= \frac{\sqrt{2} m^2}{(1 - k^2) \sqrt{m + \frac{1 + \sqrt{5}}{4}}} \left[ \left( \frac{4m(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)} \right) + \frac{3mk^2}{5} + \frac{(1 + \sqrt{5})(2 - 3k^2)}{6} \right] u_1 - \left( \frac{3mk^2}{5(1 - k^2)} + \frac{8m(1 - 2k^2)^2}{15(1 - k^2)} \right)
- \frac{2(1 + \sqrt{5})(2k^2 - 1)}{6(1 - k^2)} E(u_1) + \left( \frac{4m(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)} \right)
+ \frac{3mk^2}{5(1 - k^2)} + \frac{(1 + \sqrt{5})(2 - 4k^2)}{6(1 - k^2)} \right] \text{tnu}_1 \text{dn} u_1 + \left( \frac{1 + \sqrt{5}}{6} \right)
+ \frac{4m(1 - 2k^2)}{15(1 - k^2)} \right] \text{tnu}_1 \text{dn} u_1 \text{nc}^2 u_1 + \frac{m}{5} \text{tnu}_1 \text{dn} u_1 \text{nc}^4 u_1 \right].
\]

Using the formula (3.7), (3.9), (3.40) and (3.43), we obtain
\[
\frac{\partial F_\varphi}{\partial m} (M_\varphi, m_\varphi) = \frac{1 + \sqrt{5}}{8} \int_S \varphi^2 \, dx + \frac{1}{4} \int_S \varphi^3 \, dx
- \frac{\sqrt{2} m^2}{2} \int_{m_\varphi}^{M_\varphi} \varphi^2 \left( \varphi + \frac{1 + \sqrt{5}}{2} \right) \, d\varphi
= J_{40} \varphi_1 + J_{41} E(\varphi_1) + J_{42} \text{tn}(\varphi_1) \text{dn}(\varphi_1)
+ J_{43} \text{tn}(\varphi_1) \text{dn}(\varphi_1) \text{nc}^2(\varphi_1) + J_{44} \text{tn}(\varphi_1) \text{dn}(\varphi_1) \text{nc}^4(\varphi_1),
\]
where
\[
J_{40} = \frac{\sqrt{2} (\sqrt{5} - 1)(2 - 3k^2)}{6(1 - k^2)} \frac{5 - \frac{1}{4}}{4} + \frac{\sqrt{2} (\sqrt{5} - 1)^3}{8(1 - k^2)} \frac{5 - \frac{5}{4}}{3(1 - k^2)} \left( \frac{4(1 - 2k^2)(2 - 3k^2)}{3(1 - k^2)} \right)
+ \frac{3k^2}{6}
+ \frac{1 + \sqrt{5}}{6} \left( \frac{4m_\varphi(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)} \right) + \frac{3m_\varphi k^2}{5}
\]
\[
J_{41} = \frac{\sqrt{2} (\sqrt{5} - 1)(2k^2 - 1)}{3(1 - k^2)} \frac{5 - \frac{1}{4}}{4} - \frac{\sqrt{2} (\sqrt{5} - 1)^3}{8(1 - k^2)} \frac{5 - \frac{5}{4}}{3(1 - k^2)} \left( \frac{8(1 - 2k^2)^2}{3(1 - k^2)} \right)
+ \frac{3k^2}{1 - k^2} + \frac{\sqrt{2} m^2}{(1 - k^2) \sqrt{m_\varphi + \frac{1 + \sqrt{5}}{2}}} \left( \frac{3m_\varphi k^2}{5(1 - k^2)} + \frac{8m_\varphi(1 - 2k^2)^2}{15(1 - k^2)} \right)
- \frac{2(1 + \sqrt{5})(2k^2 - 1)}{6(1 - k^2)},
\]
\[ J_{42} = \frac{\sqrt{3}(\sqrt{3} - 1)(2 - 4k^2)}{6(1 - k^2)^2} 5 - \frac{1}{4} + \frac{\sqrt{3}(\sqrt{3} - 1)^3}{8(1 - k^2)} 5 - \frac{5}{4} \left( \frac{3k^2}{1 - k^2} \right) \\
+ \frac{4(1 - 2k^2)(2 - 4k^2)}{3(1 - k^2)^2} - \frac{\sqrt{2}m^2_\varphi}{(1 - k^2)\sqrt{m_\varphi + \frac{1 + \sqrt{3}}{2}}} \left( 3m_\varphi k^2 \right) \\
+ \frac{4m_\varphi(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)^2} + \frac{(1 + \sqrt{5})(2 - 4k^2)}{6(1 - k^2)} , \]

\[ J_{43} = \frac{\sqrt{3}(\sqrt{3} - 1)}{6(1 - k^2)^2} 5 - \frac{1}{4} + \frac{\sqrt{3}(\sqrt{3} - 1)^3(1 - 2k^2)}{6(1 - k^2)^2} 5 - \frac{5}{4} \\
- \frac{\sqrt{2}m^2_\varphi}{(1 - k^2)\sqrt{m_\varphi + \frac{1 + \sqrt{5}}{2}}} \left( 4m_\varphi(1 - 2k^2) + \frac{1 + \sqrt{5}}{6} \right), \]

\[ J_{44} = \frac{\sqrt{3}(\sqrt{3} - 1)^3}{8(1 - k^2)^2} 5 - \frac{5}{4} \frac{\sqrt{2}m^3_\varphi}{5(1 - k^2)\sqrt{m_\varphi + \frac{1 + \sqrt{5}}{2}}}. \]

Entering the value, we know \( J_{40}, J_{41}, J_{42}, J_{43} \) and \( J_{44} \) are 0. Then

\[ \frac{\partial F_\varphi}{\partial m}(M_\varphi, m_\varphi) = 0. \] (3.45)

Further differentiation yields

\[ \frac{\partial^2 F_u}{\partial m^2} = \frac{\sqrt{2}}{2} \frac{\partial}{\partial m} \left[ \int_m^M \frac{u(u + \frac{1 + \sqrt{3}}{2})}{\sqrt{u(u + \frac{1 + \sqrt{3}}{2})(u - m)}} \, du \right] \\
- \frac{\sqrt{2}}{2} \frac{\partial}{\partial m} \left[ \int_m^M \frac{u^2(u + \frac{1 + \sqrt{3}}{2})}{\sqrt{u(u + \frac{1 + \sqrt{3}}{2})(u - m)}} \, du \right] \\
= \frac{\sqrt{2}}{2} \frac{\partial}{\partial m} \left[ \int_m^M \frac{u(u + \frac{1 + \sqrt{3}}{2})(1 - u)}{\sqrt{u(u + \frac{1 + \sqrt{3}}{2})(u - m)}} \, du \right], \] (3.46)

where

\[ \int_m^M \frac{u(u + \frac{1 + \sqrt{3}}{2})(1 - u)}{\sqrt{u(u + \frac{1 + \sqrt{3}}{2})(u - m)}} \, du = \frac{2}{\sqrt{m + \frac{1 + \sqrt{3}}{2}}} \int_0^{u_1} \left[ \frac{m}{cn^2 u} \left( \frac{m}{cn^2 u} + \frac{1 + \sqrt{3}}{2} \right) \left( 1 - \frac{m}{cn^2 u} \right) \right] \, du \]

\[ = \frac{2}{(1 - k^2)\sqrt{m + \frac{1 + \sqrt{3}}{2}}} \left[ \left( 1 + \sqrt{3} \right)(1 - k^2) - \frac{4m^3(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)} \right. \\
+ \left. \frac{(1 - \sqrt{5})m^2(2 - 3k^2)}{6} - \frac{3m^3k^2}{5} \right) u_1 + \frac{(3m^3k^2)^2}{5(1 - k^2)} + \frac{(1 - \sqrt{3})m^3k^2}{15(1 - k^2)} \]
\[
\frac{\partial^2 F_u}{\partial m^2} = \frac{\partial}{\partial m} \left[ \frac{\sqrt{2}}{(1-k^2)\sqrt{m + \frac{1+\sqrt{5}}{2}}} \left( \frac{4m^3 \sqrt{m}(1-k^2)}{2} + \frac{(1+\sqrt{5})m^2(2 - 3k^2)}{6} \right) \right] \\
+ \frac{2m^2(1 - \sqrt{5})(2k^2 - 1)}{6(1-k^2)} - \frac{(1+\sqrt{5})m}{2} \right) E(u_1) + \frac{(1-\sqrt{5})m^2(2 - 4k^2)}{6(1-k^2)} \\
+ \frac{(1+\sqrt{5})m}{2} - \frac{3m^3k^2}{2(1-k^2)} - \frac{4m^3(1-2k^2)(2 - 4k^2)}{15(1-k^2)^2} \right) \right] tu_1 du_1 \\
+ \left[ \frac{(1-\sqrt{5})m^2}{6} - \frac{4m^3(1-2k^2)}{15(1-k^2)} \right] tu_1 du_1 nc^2u_1 - \frac{m^3}{5} tu_1 du_1 nc^4u_1 ] \right). \\
\right]
\]

We can obtain
\[
\frac{\partial^2 F_u}{\partial m^2} = \frac{\partial}{\partial m} \left[ \frac{\sqrt{2}}{(1-k^2)\sqrt{m + \frac{1+\sqrt{5}}{2}}} \left( \frac{4m^3 \sqrt{m}(1-k^2)}{2} + \frac{(1+\sqrt{5})m^2(2 - 3k^2)}{6} \right) \right] \\
+ \frac{2m^2(1 - \sqrt{5})(2k^2 - 1)}{6(1-k^2)} - \frac{(1+\sqrt{5})m}{2} \right) E(u_1) + \frac{(1-\sqrt{5})m^2(2 - 4k^2)}{6(1-k^2)} \\
+ \frac{(1+\sqrt{5})m}{2} - \frac{3m^3k^2}{2(1-k^2)} - \frac{4m^3(1-2k^2)(2 - 4k^2)}{15(1-k^2)^2} \right) \right] tu_1 du_1 \\
+ \left[ \frac{(1-\sqrt{5})m^2}{6} - \frac{4m^3(1-2k^2)}{15(1-k^2)} \right] tu_1 du_1 nc^2u_1 - \frac{m^3}{5} tu_1 du_1 nc^4u_1 ] \right). \\
\right]
\]

where \( u'_1 \) is the derivative of \( u \) and

\[
J_{50} = \frac{4\sqrt{2}}{(4m + 2\sqrt{5})\frac{3}{2}(1-k^2)} \left( \frac{(1+\sqrt{5})m(1-k^2)}{2} + \frac{(1-\sqrt{5})m^2(2-3k^2)}{6} \right) \\
+ \frac{3m^3k^2}{5} - \frac{4m^3(1-2k^2)(2 - 4k^2)}{15(1-k^2)} \right) \right] tu_1 du_1 nc^2u_1 - \frac{m^3}{5} tu_1 du_1 nc^4u_1 ] \right). \\
\right]
\]

\[
J_{51} = \frac{4\sqrt{2}}{(4m + 2\sqrt{5})\frac{3}{2}(1-k^2)} \left( \frac{3m^3k^2}{5(1-k^2)} - \frac{(1+\sqrt{5})m}{2} + \frac{8m^3(1-2k^2)^2}{15(1-k^2)^2} \right) \\
+ \frac{2m^2(1 - \sqrt{5})(2k^2 - 1)}{6(1-k^2)} \right) \right] tu_1 du_1 nc^2u_1 - \frac{m^3}{5} tu_1 du_1 nc^4u_1 ] \right). \\
\right]
\]
By calculations, we have

\[
J_{52} = -\frac{4\sqrt{2}}{(4m + 2 + 2\sqrt{5})^2 (1 - k^2)} \left( \frac{(1 + \sqrt{5})m}{2} - \frac{4m^3(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)^2} \right) + \frac{(1 - \sqrt{5})m^2(2 - 4k^2)}{6(1 - k^2)} - \frac{3m^3k^2}{5(1 - k^2)} + \frac{\sqrt{2}}{(1 - k^2)\sqrt{m + \frac{1 + \sqrt{5}}{2}}} \left( \frac{(1 + \sqrt{5})m}{2} - \frac{4m^3(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)} + \frac{(1 - \sqrt{5})m^2(2 - 3k^2)}{6} - \frac{3m^3k^2}{5} \right),
\]

\[
J_{53} = -\frac{4\sqrt{2}}{(4m + 2 + 2\sqrt{5})^2 (1 - k^2)} \left( \frac{(1 + \sqrt{5})m}{2} - \frac{4m^3(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)} + \frac{(1 - \sqrt{5})m^2(2 - 3k^2)}{6} - \frac{3m^3k^2}{5} \right),
\]

\[
J_{54} = \frac{4\sqrt{2}m^3}{5(4m + 2 + 2\sqrt{5})^2 (1 - k^2)} + \frac{3\sqrt{2}m^2}{5(1 - k^2)\sqrt{m + \frac{1 + \sqrt{5}}{2}}},
\]

\[
J_{55} = \frac{\sqrt{2}}{(1 - k^2)\sqrt{m + \frac{1 + \sqrt{5}}{2}}} \left( \frac{(1 + \sqrt{5})m}{2} - \frac{4m^3(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)} + \frac{(1 - \sqrt{5})m^2(2 - 3k^2)}{6} - \frac{3m^3k^2}{5} \right),
\]

\[
J_{56} = \frac{\sqrt{2}}{(1 - k^2)\sqrt{m + \frac{1 + \sqrt{5}}{2}}} \left( \frac{3m^3k^2}{5(1 - k^2)} - \frac{(1 + \sqrt{5})m}{2} + \frac{2m^2(1 - \sqrt{5})(2k^2 - 1)}{6(1 - k^2)} + \frac{8m^3(1 - 2k^2)}{15(1 - k^2)^2} \right),
\]

\[
J_{57} = \frac{\sqrt{2}}{(1 - k^2)\sqrt{m + \frac{1 + \sqrt{5}}{2}}} \left( \frac{(1 + \sqrt{5})m}{2} - \frac{3m^3k^2}{5(1 - k^2)} - \frac{4m^3(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)^2} + \frac{(1 - \sqrt{5})m^2(2 - 4k^2)}{6(1 - k^2)} \right),
\]

\[
J_{58} = \frac{\sqrt{2}}{(1 - k^2)\sqrt{m + \frac{1 + \sqrt{5}}{2}}} \left( -\frac{4m^3(1 - 2k^2)}{15(1 - k^2)} + \frac{(1 - \sqrt{5})m^2}{6} \right),
\]

\[
J_{59} = -\frac{\sqrt{2}m^3}{5(1 - k^2)\sqrt{m + \frac{1 + \sqrt{5}}{2}}},
\]

By calculations, we have

\[
\frac{\partial^2 F_\varphi}{\partial m^2}(M_\varphi, m_\varphi) \approx -0.6936. \quad (3.49)
\]

This proves the lemma. \qed
Lemma 3.4. It holds that
\[ \max_{x \in \mathbb{S}} |f(x)| \leq \frac{\sqrt{2}H_1|\varphi|}{\sqrt{2} - A} \|f\|_{H^1(\mathbb{S})}, \quad f \in H^1(\mathbb{S}), \]  
(3.50)

where \( A \approx 0.5349 \), equality holds in (3.50) if and only if \( f = \varphi(\cdot - \xi) \) for some \( \xi \in \mathbb{R} \) that is, if and only if has the shape of a peakon.

Proof. For \( x \in \mathbb{S} \), we have
\[
\langle \varphi(\cdot - x + T), f \rangle_{H^1(\mathbb{S})} = \int_{\mathbb{S}} (\varphi(y - x + T)f(y) + \varphi'(y - x + T)f'(y)) \, dy 
\]
\[
= \int_{\mathbb{S}} (\varphi - \varphi'')(y - x + T)f(y) \, dy 
\]
\[
= \int_{\mathbb{S}} \left( -\frac{3}{4}\varphi^2(y - x + T) + \frac{1}{2}\varphi(y - x + T) + \frac{1}{4} \right) f(y) \, dy 
\]
\[
+ \sqrt{2} \int_{\mathbb{S}} \delta(y - x)f(y) \, dy 
\]
\[
= \int_{\mathbb{S}} \left( -\frac{3}{4}\varphi^2(y - x + T) + \frac{1}{2}\varphi(y - x + T) + \frac{1}{4} \right) f(y) \, dy + \sqrt{2} f(x). 
\]

Then, we get
\[
\sqrt{2} f(x) = \langle \varphi(\cdot - x + T), f \rangle_{H^1(\mathbb{S})} + \int_{\mathbb{S}} \left( \frac{3}{4}\varphi^2(y-x+T) - \frac{1}{2}\varphi(y-x+T) - \frac{1}{4} \right) f(y) \, dy. 
\]

Whereupon, we have following inequality
\[
\sqrt{2} |f(x)| \leq |\langle \varphi(\cdot - x + T), f \rangle_{H^1(\mathbb{S})}| + \int_{\mathbb{S}} \left| \frac{3}{4}\varphi^2(y-x+T) - \frac{1}{2}\varphi(y-x+T) - \frac{1}{4} \right| |f(y)| \, dy. 
\]

Denoting
\[
A = 2 \int_{0}^{T} \left| \frac{3}{4}\varphi^2(y-x+T) - \frac{1}{2}\varphi(y-x+T) - \frac{1}{4} \right| \, dy 
\]
\[
= \left( \frac{\sqrt{2}(\sqrt{5} - 1)^2(2 - 3k^2)}{4(1 - k^2)} 5^{-\frac{1}{4}} - \sqrt{2}(\sqrt{5} - 1) 5^{-\frac{1}{4}} \right) \sqrt{\frac{\sqrt{5}}{8}} T 
\]
\[
+ \left( \frac{\sqrt{2}(\sqrt{5} - 1)^2(2k^2 - 1)}{2(1 - k^2)^2} 5^{-\frac{1}{4}} + \sqrt{2}(\sqrt{5} - 1) 5^{-\frac{1}{4}} \right) E \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) 
\]
\[
+ \frac{\sqrt{2}(\sqrt{5} - 1)^2}{4(1 - k^2)} 5^{-\frac{1}{4}} \tan \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \tan \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) 
\]
\[
+ \left( \frac{\sqrt{2}(\sqrt{5} - 1)^2(2 - 4k^2)}{4(1 - k^2)^2} 5^{-\frac{1}{4}} - \sqrt{2}(\sqrt{5} - 1) 5^{-\frac{1}{4}} \right) \tan \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) \tan \left( \sqrt{\frac{\sqrt{5}}{8}} T \right) 
\]
\[
- \left( \frac{\sqrt{2}(\sqrt{5} - 1)^2(2 - 3k^2)}{2(1 - k^2)} 5^{-\frac{1}{4}} - 2\sqrt{2}(\sqrt{5} - 1) 5^{-\frac{1}{4}} \right) \sqrt{\frac{\sqrt{5}}{8}} T. 
\]
we obtain
\[
- \left( \frac{\sqrt{5} - 1}{1 - k^2} \right) ^{2} \left( \frac{1}{5} \right) ^{2} + 2 \left( \frac{\sqrt{5} - 1}{1 - k^2} \right) \left( \frac{1}{5} \right) ^{2} \left( \frac{\sqrt{5} - 1}{1 - k^2} \right) + \left( \frac{\sqrt{5} - 1}{1 - k^2} \right) \left( \frac{1}{5} \right) ^{2} - \frac{1}{2} T + 1
\]
\[
= 4 \sqrt{2} \left( \frac{1}{5} \right) ^{2} \left( \frac{\sqrt{5} - 1}{1 - k^2} \right) \left( \frac{1}{5} \right) ^{2} - 2 \left( \frac{\sqrt{5} - 1}{1 - k^2} \right) \left( \frac{1}{5} \right) ^{2} \left( \frac{\sqrt{5} - 1}{1 - k^2} \right) + \left( \frac{\sqrt{5} - 1}{1 - k^2} \right) \left( \frac{1}{5} \right) ^{2} - \frac{1}{2} T + 1 \approx 0.5349,
\]
we have
\[
\sqrt{2} \max_{x \in S} |f(x)| \leq |\langle \varphi \cdot - x + T, f \rangle_{H^1(S)}| + A \max_{x \in S} |f(x)|,
\]
since,
\[
H_{1}[\varphi] = \frac{1}{2} \| \varphi \|_{H^1(S)}^2,
\]
we obtain
\[
\max_{x \in S} |f(x)| \leq \frac{|\langle \varphi \cdot - x + T, f \rangle_{H^1(S)}|}{\sqrt{2} - A} \leq \frac{\| \varphi \|_{H^1(S)} \| f \|_{H^1(S)}}{\sqrt{2} - A} = \frac{2 H_{1}[\varphi]}{\sqrt{2} - A} \| f \|_{H^1(S)}.
\]
The Lemma 3.4 is proved.

**Lemma 3.5.** If \( u \in C([0, T); H^1(S)) \), then \( M_{u(t)} = \max_{x \in S} u(x, t) \) and \( m_{u(t)} = \min_{x \in S} u(x, t) \) are continuous functions of \( t \in [0, T) \).

**Proof.** By Lemma 3.4, for \( t, s \in [0, T) \),
\[
| M_{u(t)} - M_{u(s)} | = | \max_{x \in S} u(x, t) - \max_{x \in S} u(x, s) | \leq \max_{x \in S} | u(x, t) - u(x, s) | \leq \frac{\sqrt{2} H_{1}[\varphi]}{\sqrt{2} - A} \| u(\cdot, t) - u(\cdot, s) \|_{H^1(S)}.
\]
This proves that \( M_{u(t)} \) is continuous. The continuity of \( m_{u(t)} \) follows at once since \( m_{u(t)} = - M_{-u(t)} \).
Lemma 3.6. Let $u \in C([0,T); H^1(\mathbb{S}))$ be a solution of (1.4). Given a small neighborhood $\mathcal{U}$ of $(M_\varphi, m_\varphi)$ in $\mathbb{R}^2$, there is a $\delta > 0$ such that

$$
(M_u(t), m_u(t)) \in \mathcal{U} \text{ for } t \in [0,T) \text{ if } \|u(\cdot, 0) - \varphi\|_{H^1(\mathbb{S})} < \delta.
$$

(3.53)

Proof. Suppose $H_i[u] = H_i[\varphi] + \varepsilon_i$, $i = 0, 1, 2$. Then

$$
F_u(M, m) = F_\varphi(M, m) + \frac{1}{4} \left( M - \frac{1 + \sqrt{5}}{2} + m \right) \int_{\mathbb{S}} (u^3 - \varphi^3) \, dx - \frac{1 + \sqrt{5}}{8} M m \varepsilon_0
$$

$$
+ \frac{1}{4} \left( \sqrt{5} - 3 \right) M - M m + \frac{1 + \sqrt{5}}{2} m \right) \int_{\mathbb{S}} (u^2 - \varphi^2) \, dx + M \varepsilon_1 - \varepsilon_2
$$

$$
\leq F_\varphi(M, m) + \frac{1}{4} \left( M^3 - M^2 - M - \frac{1 + \sqrt{5}}{2} + \frac{3 + \sqrt{5}}{2} m \right) \varepsilon_0
$$

$$
+ M \varepsilon_1 - \varepsilon_2,
$$

so $F_u$ is a small perturbation of $F_\varphi$. The effect of the perturbation near the point $(M_\varphi, m_\varphi)$ can be made arbitrarily small by choosing the $\varepsilon_i$'s small. Lemma 3.3 says that $F_\varphi(M_\varphi, m_\varphi) = 0$ and that $F_\varphi$ has a critical point with negative definite second derivative at $(M_\varphi, m_\varphi)$. By continuity of the second derivative, there is a neighborhood around $(M_\varphi, m_\varphi)$, where $F_\varphi$ is concave with curvature bounded away from zero. Therefore, after a small perturbation, the set where $F_u \geq 0$ near $(M_\varphi, m_\varphi)$ will be contained in a neighborhood of $(M_\varphi, m_\varphi)$. Let $\mathcal{U}$ be given as in the statement. Shrinking $\mathcal{U}$ if necessary, we infer the existence of a $\delta' > 0$ such that for $u$ with

$$
|H_i[u] - H_i[\varphi]| < \delta', \quad i = 0, 1, 2,
$$

(3.54)

it holds that the set where $F_u \geq 0$ near $(M_\varphi, m_\varphi)$ is contained in $\mathcal{U}$, and $\mathcal{U}$ is surrounded by a set where $F_u < 0$. Lemma 3.2 and Lemma 3.5 say that $M_u(t)$ and $m_u(t)$ are continuous functions of $t \in [0,T)$, and $F_u(M_u(t), m_u(t)) \geq 0$ for $t \in [0,T)$. We conclude that for $u$ satisfying (3.54), we have

$$
(M_u(t), m_u(t)) \in \mathcal{U} \text{ for } t \in [0,T) \text{ if } (M_u(0), m_u(0)) \in \mathcal{U}.
$$

(3.55)

However, the continuity of the conserved functionals $H_i : H^1(\mathbb{S}) \to \mathbb{R}$, $i = 0, 1, 2$, shows that there is a $\delta > 0$ such that (3.54) holds for all $u$ with

$$
\|u(\cdot, 0) - \varphi\|_{H^1(\mathbb{S})} < \delta.
$$

(3.56)

Moreover, by Lemma 3.4, taking a smaller $\delta$ if necessary, we may also assume that $(M_u(0), m_u(0)) \in \mathcal{U}$ if $\|u(\cdot, 0) - \varphi\|_{H^1(\mathbb{S})} < \delta$. This proves the lemma. \hfill \square

Proof of Theorem 1.1. Let $u \in C([0,T); H^1(\mathbb{S}))$ be a solution of (1.4) and suppose we are given an $\varepsilon > 0$. Pick a neighborhood $\mathcal{U}$ of $(M_\varphi, m_\varphi)$ small enough that $|M - M_\varphi| < \frac{\sqrt{5}}{10} \varepsilon$ if $(M, m) \in \mathcal{U}$. Choose a $\delta > 0$ as in Lemma 3.6 so that (3.53) holds. Taking a smaller $\delta$ if necessary, we may also assume that

$$
|H_0[u] - H_0[\varphi]| < \frac{4}{3(5\sqrt{5} - 9)} \varepsilon, \quad |H_1[u] - H_1[\varphi]| < \frac{\varepsilon}{6} \text{ if } \|u(\cdot, 0) - \varphi\|_{H^1(\mathbb{S})} < \delta.
$$
Using the hypothesis that \( u(x) - \varphi(x - \xi) \) preserves sign, from Lemma 3.1, we conclude that

\[
\frac{1}{2} \|u - \varphi(\cdot - \xi)\|_{H^1(S)}^2 = H_1[u] - H_1[\varphi] + \sqrt{2} (M_\varphi - u(\xi + T)) \\
+ \int_S \left( \frac{3}{4} \varphi^2(x) - \frac{1}{2} \varphi(x) - \frac{1}{4} \right) (u(x) - \varphi(x - \xi)) \, dx \\
\leq |H_1[u] - H_1[\varphi]| + \sqrt{2} (M_\varphi - u(\xi + T)) \\
+ \max_{x \in S} \frac{3}{4} \varphi^2(x) - \frac{1}{2} \varphi(x) - \frac{1}{4} \left| \int_S (u(x) - \varphi(x - \xi)) \, dx \right|
\]

therefore,

\[
\|u - \varphi(\cdot - \xi(t))\|_{H^1(S)}^2 \leq \frac{5\sqrt{5} - 9}{4} |H_0[u] - H_0[\varphi]| + 2 |H_1[u] - H_1[\varphi]| \\
+ 2\sqrt{2} \left( M_\varphi - M(u(t)) \right) \leq \varepsilon,
\]

where \( \max_{x \in S} \frac{3}{4} \varphi^2(x) - \frac{1}{2} \varphi(x) - \frac{1}{4} = \frac{5\sqrt{5} - 9}{4} \) and \( \xi(t) \in \mathbb{R} \) is any point where \( u(\xi(t) + T, t) = M(u(t)) \). This completes the proof of the theorem.

**Appendix A.** In this Appendix, some basic properties of Jacobian elliptic integrals (see [1]) is collected for the convenience of the reader. We started setting the \textit{normal elliptic integral of the first kind}

\[
\int_0^y \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^{u_1} du = \text{sn}^{-1}(y, k) = F(\varphi, k),
\]

where \( y = \sin \varphi \) and \( \varphi = am(u_1) \), however, the \textit{normal elliptic integral of the second kind} is

\[
\int_0^y \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} = \int_0^{u_1} \text{dn}^2 u \, du \\
= E(u_1) \equiv E(\text{am}(u_1, k) \equiv E(\varphi, k).
\]

The parameter \( k \) is called the \textit{modulus} and belongs to the interval \((0, 1)\). The number \( k' \) is referred to as the \textit{complementary modulus} and is related to \( k \) by \( k' = \sqrt{1 - k^2} \). The variable \( \varphi \) is called the \textit{argument} of the normal elliptic integrals.

It is usually understood that \( 0 \leq y \leq 1 \) or \( 0 \leq \varphi \leq \frac{\pi}{2} \).

For \( y = 1 \), It is said that the integrals above are \textit{complete}. In this case, one writes

\[
\int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = F(\pi/2, k) \equiv K(k),
\]

\[
\int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} = E(\pi/2, k) \equiv E(k).
\]

Clearly, we have \( K(0) = E(0) = \pi/2 \) and \( E(0, k) = 0 \).

We using the inverse function of the elliptic integral of the first kind define the \textit{Jacobian Elliptic Functions}. This inverse function exists because that

\[
u(y_1, k) \equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} = F(\varphi, k),
\]

which is a strictly increasing function of the variable \( y_1 \), in its algebraic form, this integral has the property of being finite for all values of \( y_1 \). Its inverse function may thus be defined by \( y_1 = \sin \varphi = \text{sn}(u, k) \) and \( \varphi = \text{am}(u, k) \), or simply \( y_1 = \text{sn}(u) \).
\(\varphi = \text{am}u\) when not necessary emphasize the modulus \(k\); these may be read \emph{sine amplitude} \(u\) and \emph{amplitude} \(u\). So, \(sn\) is an odd function. The other two basic elliptic functions, the \textbf{cnoidal} and \textbf{dnoidal} functions, are defined in terms of \(sn\) by

\[
\begin{align*}
\text{cn}(u, k) &= \sqrt{1 - y^2} = \sqrt{1 - sn^2(u, k)}, \\
\text{dn}(u, k) &= \sqrt{1 - k^2y^2} = \sqrt{1 - k^2sn^2(u, k)}.
\end{align*}
\]

These functions have a real period, namely 4\(K\) and 2\(K\), respectively. The most important properties of the Jacobian elliptic functions used in this work are summarized by the formulas given below.

1. Fundamental relations:

\[
\begin{align*}
\text{sn}^2u + \text{cn}^2u &= 1, \quad k^2\text{sn}^2u + \text{dn}^2u = 1, \\
\text{dn}^2u - k^2\text{cn}^2u &= k^2, \quad k^2\text{sn}^2u + \text{cn}^2u = \text{dn}^2u, \\
-1 &\leq \text{sn}u \leq 1, -1 &\leq \text{cn}u \leq 1, k^2 &\leq \text{dn}u \leq 1.
\end{align*}
\]

2. Special values:

\[
\begin{align*}
\text{sn}(-u) &= -\text{sn}u, \quad \text{cn}(-u) = \text{cn}u, \quad \text{dn}(-u) = \text{dn}u, \\
\text{sn}0 &= 0, \quad \text{cn}0 = 1, \quad \text{dn}0 = 1, \quad \text{am}0 = 0.
\end{align*}
\]

3. Differentiation of the Jacobian elliptic functions:

\[
\begin{align*}
\frac{\partial}{\partial u} \text{sn}u &= \text{cn}u \text{dn}u, \quad \frac{\partial}{\partial u} \text{cn}u = -\text{sn}u \text{dn}u, \\
\frac{\partial}{\partial u} \text{dn}u &= -k^2\text{sn}u \text{cn}u, \quad \frac{\partial}{\partial u} \text{tn}u = \text{nc}u \text{dc}u, \quad \frac{\partial}{\partial u} \text{nc}u = -k^2\text{tn}u \text{dc}u.
\end{align*}
\]

4. Glaisher’s notation:

\[
\begin{align*}
\text{tn}u &= \frac{\text{sn}u}{\text{cn}u}, \quad \text{dc}u = \frac{\text{dn}u}{\text{cn}u}, \quad \text{nc}u &= \frac{1}{\text{cn}u}.
\end{align*}
\]

5. Differentiation of elliptic integrals:

\[
\frac{\partial}{\partial \varphi} E(\varphi, k) = \sqrt{1 - k^2 \sin^2 \varphi}.
\]

6. Differentiation of the Jacobian inverse functions:

\[
\frac{d}{dy} \text{sn}^{-1}(y, k) = \frac{1}{\sqrt{(1 - y^2)(1 - k^2y^2)}}.
\]

7. Identities:

\[
\begin{align*}
\text{sn}^{-1}(y, k) &= \text{cn}^{-1} \left(\sqrt{1 - y^2}, k\right) = \text{dn}^{-1} \left(\sqrt{1 - k^2y^2}, k\right) = \text{tn}^{-1} \left(y/\sqrt{1 - y^2}, k\right), \\
\text{cn}^{-1}(y, k) &= \text{sn}^{-1} \left(\sqrt{1 - y^2}, k\right) = \text{dn}^{-1} \left(\sqrt{k^2 + k^2y^2}, k\right) = \text{tn}^{-1} \left(\sqrt{(1 - y^2)/y^2}, k\right).
\end{align*}
\]
Appendix B. In this Appendix, we mainly present some of the formulas used in this article to calculate the elliptic function, which are from the references [1].

Formula 1:
\[ D_2 = \int \frac{du}{cn^2 u} = \frac{1}{k'^2} \left[ k'^2 u - E(u) + t\nu du \right]. \]

Formula 2:
\[ D_4 = \int \frac{du}{cn^4 u} = \frac{1}{3k'^4} \left[ k'^2 (2k'^2 - k^2)u + 2(2k^2 - 1)E(u) + (2 - 4k^2 + k'^2 nc^2 u)t\nu du \right]. \]

Formula 3:
\[ D_{2m+2} = \frac{(2m - 1)k^2 D_{2m-2} + 2m(1 - 2k^2)D_{2m} + t\nu d\nu c^2 m u}{(2m + 1)k'^2}. \]

Formula 4:
\[ \int_y^a \sqrt{(t-a)(t-b)(t-c)} \, dt = (a-b)^2(a-c)g \int_0^{u_1} \frac{tn^2 unc^2 u dc^2 u du}{}, \]

Formula 5:
\[ \int_y^a \frac{R(t) \, dt}{\sqrt{(t-a)(t-b)(t-c)}} = g \int_0^{u_1} \frac{a - bsn^2 u}{cn^2 u} \, du, \]

where \( R(t) \) is any rational function of \( t \).

Formula 6:
\[ \int_y^a \frac{(t-b)(t-c)}{t-a} \, dt = (a-b)(a-c)g \int_0^{u_1} \frac{dc^2 unc^2 u du}{}, \]

for formula 4, 5 and 6, \( y > a > b > c \), \( g = \frac{2}{\sqrt{a-c}}, \varphi = amu_1 = \sin^{-1} \sqrt{\frac{y-a}{y-b}}, \)
\( snu_1 = \sin \varphi \) and \( k^2 = \frac{b-c}{a-c} \).

Formula 7:
\[ \int \frac{tn^2 unc^2 u dc^2 u du}{}, \]

Formula 8:
\[ \int_0^{u_1} \frac{dc^2 unc^2 u du}{2} = \frac{1}{3k'^2} \left[ 2k'^2 + (k'^2 - 2)E(u) + t\nu d\nu (2 - k^2 + k'^2 nc^2 u) \right] \]

Appendix C. In this appendix, we mainly give the specific calculation process of two integrals. Integral 1:
\[ \int_\xi^\eta \sqrt{\frac{u}{2}} \left( u + \frac{1 + \sqrt{5}}{2} \right) (u-m) \, dx \]
\[ = -\frac{\sqrt{2}}{2} \int_m^M \frac{u^2 \left( u + \frac{1 + \sqrt{5}}{2} \right) (u-m) \, du}{\sqrt{u \left( u + \frac{1 + \sqrt{5}}{2} \right) (u-m)}} \]
\[
\begin{align*}
&= -\sqrt{\frac{2}{m + \frac{1+\sqrt{5}}{2}}} \int_0^{u_1} \left[ \frac{m^2}{cn^2 u} \left( \frac{m}{cn^2 u} + \frac{1 + \sqrt{5}}{2} \right) \left( \frac{m}{cn^2 u} - m \right) \right] du \\
&= -\sqrt{\frac{2}{m + \frac{1+\sqrt{5}}{2}}} \left[ \left( \frac{24m^4(1 - 2k^2)^2(2 - 3k^2)}{105(1 - k^2)^3} + \frac{3m^3k^2(1+\sqrt{5}/2 - m)}{5(1 - k^2)^2} \right) + \frac{4m^3(1+\sqrt{5}/2 - m)(1 - 2k^2)(2 - 3k^2)}{15(1 - k^2)^2} + \frac{18m^4k^2(1 - 2k^2)}{35(1 - k^2)^2} \right. \\
&\quad \left. - \frac{1+\sqrt{5}/2 - m}{3(1 - k^2)} + \frac{5m^4k^2(2 - 3k^2)}{21(1 - k^2)^2} \right) u_1 + \left( \frac{10m^4k^2(2k^2 - 1)}{21(1 - k^2)^3} \right) \left( 1 + \sqrt{5}/2 - m \right) + \left( \frac{1+\sqrt{5}/2 - m}{3(1 - k^2)^2} \right) E(u_1) \\
&\quad \left. + \left( \frac{18m^4k^2(1 - 2k^2)}{35(1 - k^2)^3} + \frac{3m^3k^2(1+\sqrt{5}/2 - m)}{5(1 - k^2)^2} + \frac{5m^4k^2(2 - 4k^2)}{21(1 - k^2)^3} \right) + \frac{24m^4(1 - 2k^2)^2(2 - 4k^2)}{105(1 - k^2)^4} + \frac{4m^3(1+\sqrt{5}/2 - m)(1 - 2k^2)(2 - 4k^2)}{15(1 - k^2)^3} \right. \\
&\quad \left. - \frac{1+\sqrt{5}/2 - m}{3(1 - k^2)} + \frac{24m^4(1 - 2k^2)^2}{105(1 - k^2)^3} + \frac{5m^4k^2}{21(1 - k^2)^2} \right) \left( 1 + \sqrt{5}/2 - m \right) - \frac{3m^3k^2}{5(1 - k^2)^2} u_1 \\
&\quad \left. + \left( \frac{6m^4(1 - 2k^2)}{35(1 - k^2)^3} + \frac{m^3(1+\sqrt{5}/2 - m)}{5(1 - k^2)^2} \right) \left( 1 + \sqrt{5}/2 - m \right) \right) \left( 1 + \sqrt{5}/2 - m \right) + \left( \frac{1+\sqrt{5}/2 - m}{7(1 - k^2)^3} \right) \left( 1 + \sqrt{5}/2 - m \right) \left( 1 + \sqrt{5}/2 - m \right) \\
&\quad + \frac{m^4}{7(1 - k^2)^3} \left( 1 + \sqrt{5}/2 - m \right) \left( 1 + \sqrt{5}/2 - m \right) \left( 1 + \sqrt{5}/2 - m \right) \\
&\quad + \frac{m^4}{7(1 - k^2)^3} \left( 1 + \sqrt{5}/2 - m \right) \left( 1 + \sqrt{5}/2 - m \right) \left( 1 + \sqrt{5}/2 - m \right) \\
\end{align*}
\]
\[
\begin{align*}
&\frac{3k^2}{5(1-k^2)} + \frac{4(1-2k^2)(2-4k^2)}{15(1-k^2)^2} \tanh u_1 \cosh u_1 + \frac{4(1-2k^2)}{15(1-k^2)} \\
&\frac{2k^2-1}{3(1-k^2)} \tanh u_1 \cosh^2 u_1 + \frac{1}{5} \tanh u_1 \cosh^4 u_1 \Bigg].
\end{align*}
\]

REFERENCES

[1] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Second edition, Die Grundlehren der mathematischen Wissenschaften, Band 67 Springer-Verlag, New York-Heidelberg, 1971.

[2] R. Camassa and D. D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett., 71 (1993), 1661–1664.

[3] R. Camassa, D. D. Holm and J. Hyman, *A new integrable shallow water equation*, Adv. Appl. Mech., 31 (1994), 1–33.

[4] R. M. Chen, X. C. Liu, Y. Liu and C. Z. Qu, *Stability of the Camassa-Holm peakons in the dynamics of a shallow-water-type system*, Calc. Var. Partial Differential Equations, 55 (2016), Art. 34, 22 pp.

[5] R. M. Chen, J. Lenells and Y. Liu, *Stability of the \( \mu \)-Camassa-Holm Peakons*, J. Nonlinear Sci., 23 (2013), 97–112.

[6] A. Chen, T. Deng and W. Huang, *Orbital stability of trigonometric periodic peakons for the modified Camassa-Holm equation*, Preprint, 2019.

[7] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math., 181 (1998), 229–243.

[8] A. Constantin, Global existence of solutions and wave breaking waves for a shallow water equation: A geometric approach, Ann. Inst. Fourier (Grenoble), 50 (2000), 321–362.

[9] A. Constantin and J. Escher, *Global existence and blow-up for a shallow water equation*, Ann. Sc. Norm. Super. Pisa, 26 (1998), 303–328.

[10] A. Constantin and W. Strauss, *Stability of peakons*, Comm. Pure Appl. Math., 53 (2000), 603–610.

[11] A. Constantin and L. Molinet, *Orbital stability of solitary waves for a shallow water equation*, Phys. D, 157 (2001), 75–89.

[12] A. Daróo and L. K. Arruda, *On the instability of elliptic traveling wave solutions of the modified Camassa-Holm equation*, J. Differential Equations, 266 (2019), 1946–1968.

[13] K. El. Dika and L. Molinet, *Stability of multipeakons*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), 1517–1532.

[14] B. Fuchssteiner and A. S. Fokas, *Symplectic structures, their Backlund transformations and hereditary symmetries*, Phys. D, 4 (1981/82), 47–66.

[15] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal., 74 (1987), 160–197.

[16] Z. H. Guo, X. C. Liu, X. X. Liu and C. Z. Qu, *Stability of peakons for the generalized modified Camassa-Holm equation*, J. Differential Equations, 266 (2019), 7749–7779.

[17] S. Hakkaev, I. D. Iliev and K. Kirchev, *Stability of periodic travelling shallow-water waves determined by Newton's equation*, J. Phys. A: Math. Theor., 41 (2008), 085203, 31 pp.

[18] D. D. Holm and M. F. Staley, *Wave structure and nonlinear balance in a family of evolutionary PDEs*, SIAM J. Appl. Dyn. Syst., 2 (2003), 323–380.

[19] B. Khesin, J. Lenells and G. Misiolek, *Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms*, Math. Ann., 342 (2008), 617–656.

[20] J. Lenells, *A variational approach to the stability of periodic peakons*, J. Nonlinear Math. Phys., 11 (2004), 151–163.

[21] J. Lenells, *Stability of periodic peakons*, Int. Math. Res. Not., (2004), 485–499.

[22] J. Lenells, G. Misiolek and F. Tiğlay, *Integrable evolution equations on spaces of tensor densities and their peakon solutions*, Comm. Math. Phys., 299 (2010), 129–161.

[23] Z. W. Lin and Y. Liu, *Stability of peakons for the Degasperis-Procesi equation*, Comm. Pure Appl. Math., 62 (2009), 125–146.

[24] X. C. Liu, Y. Liu and C. Z. Qu, *Orbital stability of the train of peakons for an integrable modified Camassa-Holm equation*, Adv. Math., 255 (2014), 1–37.

[25] X. C. Liu, L. Yue and C. Z. Qu, *Stability of peakons for the Novikov equation*, J. Math. Pure Appl., 101 (2014), 17–187.
[26] Y. Liu, C. Z. Qu and Y. Zhang, Stability of periodic peakons for the modified $\mu$-Camassa-Holm equation, *Phy. D.*, **250** (2013), 66–74.

[27] C. Z. Qu, X. C. Liu and Y. Liu, Stability of peakons for an integrable modified Camassa-Holm equation with cubic nonlinearity, *Comm. Math. Phys.*, **322** (2013), 967–997.

[28] C. Z. Qu, Y. Zhang, X. C. Liu and Y. Liu, Orbital stability of periodic peakons to a generalized $\mu$-Camassa-Holm equation, *Arch. Rational Mech. Anal.*, **211** (2014), 593–617.

[29] Y. Wang and L. X. Tian, Stability of periodic peakons for the Novikov equation, (2018), arXiv:1811.05835.

[30] J. L. Yin, L. X. Tian and X. H. Fan, Stability of negative solitary waves for an integrable modified Camassa-Holm equation, *J. Math. Phys.*, **51** (2010), 053515, 6 pp.

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