Computing Teichmüller Maps between Polygons

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Abstract

By the Riemann-mapping theorem, one can bijectively map the interior of an $n$-gon $P$ to that of another $n$-gon $Q$ conformally. However, (the boundary extension of) this mapping need not necessarily map the vertices of $P$ to those $Q$. In this case, one wants to find the “best” mapping between these polygons, i.e., one that minimizes the maximum angle distortion (the dilatation) over all points in $P$. From complex analysis such maps are known to exist and are unique. They are called extremal quasiconformal maps, or Teichmüller maps.

Although there are many efficient ways to compute or approximate conformal maps, there is currently no such algorithm for extremal quasiconformal maps. This paper studies the problem of computing extremal quasiconformal maps both in the continuous and discrete settings.

We provide the first constructive method to obtain the extremal quasiconformal map in the continuous setting. Our construction is via an iterative procedure that is proven to converge quickly to the unique extremal map. To get to within $\varepsilon$ of the dilatation of the extremal map, our method uses $O(1/\varepsilon^4)$ iterations. Every step of the iteration involves convex optimization and solving differential equations, and guarantees a decrease in the dilatation. Our method uses a reduction of the polygon mapping problem to that of the punctured sphere problem, thus solving a more general problem.

We also discretize our procedure. We provide evidence for the fact that the discrete procedure closely follows the continuous construction and is therefore expected to converge quickly to a good approximation of the extremal quasiconformal map.

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1 Introduction

One of the foundational results in complex analysis, the Riemann mapping theorem, states that any non-empty simply connected domain $U \subset \mathbb{C}$ can be mapped bijectively and conformally to the unit disk $\mathbb{D}$. This implies that the interiors of two simple planar $n$-gons $P$ and $Q$ can be mapped bijectively and conformally to each other. By another result \[3\], such a map $f : P \to Q$ extends continuously to the boundary $\partial P$ of $P$ (the edges). Generally, the vertices of $Q$ do not map to the vertices of $Q$ under this extended mapping.

Assume we are given an ordering $\{v_i\}_{i=1}^n$ and $\{v'_i\}_{i=1}^n$ of the vertices of $P$ and $Q$, respectively. Consider the space of homeomorphisms $f$ that map $P$ to $Q$, such that $f(v_i) = v'_i$. Such an $f$ is bound to stretch angles (unless the polygons are linear images of each other), and a classical way to measure this angle stretch by $f$ at a point $p \in P$ is by $\mu_f(p) = f_z(p)/f_{\bar{z}}(p)$. This complex-valued function $\mu_f$ is called the Beltrami coefficient of $f$, and it satisfies $||\mu_f||_{\infty} < 1$. The problem we consider is computing the "best" homeomorphism $f_*$ in the above class, i.e., an $f_*$ such that the norm of its Beltrami differential $||\mu_*||_{\infty}$ is the smallest amongst all homeomorphisms satisfying the above conditions. These homeomorphisms that stretch angles but by a bounded amount are called quasiconformal homeomorphisms (q.c.h.), and the "best" q.c.h. $f_*$ is called the extremal quasiconformal map, or the Teichmüller map.

As an example, consider two rectangles $R_i = [a_i, b_i] \times [0, 1]$ in the plane. Starting from the origin, label the vertices of $R_1$ and $R_2$ counter-clockwise as $\{v_j\}_{j=1}^4$ and $\{v'_j\}_{j=1}^4$, respectively ($v_1 = v'_1 = (0, 0)$). Consider the space of all q.c.h. $f : R_1 \to R_2$ such that $f(v_i) = v'_i$. It was shown by Grötzsch \[14\] that the affine map $f_*(x, y) = (ax/a_1, by/b_1)$ is the unique extremal quasiconformal map; any other map $f$ would stretch angles at some point $p \in R_1$ more than $g$ (i.e., $\exists p \in R_1 : |\mu_f(p)| > |\mu_*(p)|$). For the general $n$-gon case mentioned above, such an analytic solution does not exist. However, the extremal map exists and is unique (these are the famous theorems by Teichmüller \[22\] and \[23\], proven rigorously later by Ahlfors \[2\]), and is of the form conformal $\circ$ affine $\circ$ conformal.

Computing a Riemann mapping from a given polygon to the disk has gathered a lot of attention in the past. Algorithms (e.g., CRDT \[8\]) based on finding the unknown parameters in the Schwarz-Christoffel mapping formula \[7\] for a Riemann map were proposed, and the latest result by Bishop \[3\] computes a $(1 + \varepsilon)$ quasiconformal map in $O(n \log(1/\varepsilon) \log \log(1/\varepsilon))$.

No such algorithm that computes (or approximates) the extremal quasiconformal map is known. In contrast to the Riemann mapping problem, where Riemann gave a constructive proof, the proof by Teichmüller/Ahlfors is an existence result, and no constructive proofs are available. Furthermore, the "formula" for Teichmüller maps analogous to the Schwarz-Christoffel mapping for Riemann mapping states that $\mu_* = k\phi/|\phi|$, for some integrable holomorphic function $\phi$, with at most simple poles at the vertices of $P$. Thus, given $P$, we know all the extremal maps with domain $P$; our problem is figuring out which one takes us to our target $Q$. Even though $\phi$ comes from a finite dimensional family, there is no direct search criterion. This should be contrasted with the known relation between the images of the vertices of the polygon in the Schwarz-Christoffel formula and the concept of harmonic measure \[13\]. In fact, to the authors’ knowledge, there does not exist a method that, given a starting $f$ between $P$ and $Q$, computes a $g$ with $||\mu_g||_{\infty} < ||\mu_f||_{\infty}$ if one exists.

This paper gives the first results for theoretically constructing and algorithmically computing Teichmüller maps for the polygon case stated above. Our procedure is iterative; we 1) start with a q.c.h. that sends the vertices of $P$ to the vertices of $Q$ in the prescribed order, 2) improve on it, and then 3) recurse on the improved map.

The problem of computing a Teichmüller map is synonymous with computing geodesics in the Teich-
m"uller space endowed with the Teichm"uller metric, which is the universal cover of the moduli space of Riemann surfaces (in which all (mutually homeomorphic) one-dimensional complex manifolds are quotiented under the equivalence relation of biholomorphism). Teichm"uller theory is an active area of research in mathematics, and it has connections to topology, dynamics, algebraic geometry, and number theory. Being able to compute the distance between two given points in a Teichm"uller space (two equivalence classes of marked Riemann surfaces) would help us learn more about the geometry of this interesting space. This work is therefore intended to be an introduction to this rich subject from a computational perspective, and we certainly feel that many computationally challenging open problems lie hidden.

Computing Teichm"uller maps is also an important problem in the fields of medical imaging, computer graphics and vision. In medical imaging, conformal and quasiconformal mapping has been applied for brain cortical surface registration (\cite{24}, \cite{16}). In computer vision, conformal geometry has been applied for shape analysis and dynamic surface registration and tracking (\cite{25}, \cite{27}), and in computer graphics, conformal geometry has been applied for surface parameterization (\cite{15}).

Surface registration refers to the process of finding an optimal one-to-one correspondence between surfaces that preserves the surface geometric structures and reduces the distortions as much as possible. Teichm"uller maps satisfy all these requirements. Thus being able to compute them would help one get a novel algorithm for surface registration. In \cite{26} various advantages of extremal quasiconformal maps over many existing methods were discussed in detail, and we refer the reader to it for an overview of how extremal quasiconformal maps are important in geometry processing.

Related work The only previous work to have considered the problem of computing extremal quasiconformal maps is \cite{26}. The authors consider a very similar version where a Dirichlet boundary condition is given on the disk, and one is required to compute the extremal map whose boundary values satisfy the given condition. The authors propose a heuristic; they obtain a "highly nonlinear" energy and minimize it using an alternate-descent method. There is no guarantee on how far the solution is from the true extremal map, as the solution obtained could be a local minima of the energy. Another possibly related work is \cite{20}, where the authors use the concept of conformal welding to get fingerprints for a simple closed curve.

Various eminent mathematicians (Teichm"uller, Ahlfors, Bers, Reich, Strebel, Krushkal, Hamilton, etc.) have contributed to Teichm"uller theory. We refer the reader to \cite{12} and \cite{17} for some excellent introductions to Teichm"uller theory. Most of the classical results we use can either be found in these books, or references contained therein.

2 Problem statements and results

In this section we first state rigorously what extremal quasiconformal map we want to compute, and what we mean by computing such a map. We will then state our main results.

2.1 Problem statements

The amount of angle stretch induced by a quasiconformal homeomorphism (abbreviated henceforth as q.c.h.) $f$ can be quantified using the Beltrami coefficient $\mu_f$ of $f$. Defining $f_z = f_x + i f_y$ and $f_{\bar{z}} = f_x - i f_y$, where $f_x$ and $f_y$ denote the partials of $f$ w.r.t. $x$ and $y$, the Beltrami coefficient $\mu_f$ is defined as $\mu_f = f_{\bar{z}} / f_z$. Intuitively, a q.c.h. maps the unit circle in the tangent space at a point $p$ in the domain to an ellipse in the

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\footnote{It has been used by Lipman Bers to give a simpler proof of Thurston’s classification theorem for surface homeomorphisms.}
Furthermore, why the punctured sphere problem requires a starting map is indeed general.

Theorem 3. [Punctured sphere problem] Given \( \{w_i\}_{i=1}^n \) in the plane. Let \( \{v_i\}_{i=1}^n \) be an ordering of the vertices of \( P \) and \( Q \), respectively. Observe that:

1. The polygons, or for that matter any simply connected domain (with boundary as a Jordan curve) is conformally equivalent to the upper half plane \( \mathbb{H} \), and

2. Composition by conformal maps does not change the dilatation (maximal angle stretch).

Therefore, an \( n \)-gon is the same as \( \mathbb{H} \) with \( n \) marked points on the boundary \( \partial \mathbb{H} = \mathbb{R} \).

Problem 1. [Polygon mapping problem] Given \( \{z_1, ..., z_n, w_1, ..., w_n\} \in \partial \mathbb{H} \), find \( f_* : \mathbb{H} \to \mathbb{H} \) (with Beltrami coefficient \( \mu_* \)) satisfying:

1. \( f_* \) is a quasiconformal homeomorphism of \( \mathbb{H} \) to itself.
2. \( f_*(z_i) = w_i, \ i \in \{1, ..., n\} \)
3. \( ||\mu_*||_\infty \leq ||\mu_f||_\infty \) for all \( f \) satisfying (1) and (2) above.

Note that by Teichmüller’s theorem, the above \( f_* \) exists and is unique. We state the punctured sphere problem next, and show that it is in fact a generalization of the polygon mapping problem.

Problem 2. [Punctured sphere problem] Given \( \{z_1, ..., z_n, w_1, ..., w_n\} \in \partial \mathbb{H} \), find \( f_* : \mathbb{H} \to \mathbb{H} \) (with Beltrami coefficient \( \mu_* \)) satisfying:

1. \( f_* \) is a quasiconformal homeomorphism of \( \mathbb{H} \) to itself.
2. \( f_* \) is isotopic to \( h \) relative to the points \( \{0, 1, \infty, z_1, ..., z_n\} \), i.e. \( f_*(z_i) = w_i \).
3. \( ||\mu_*||_\infty \leq ||\mu_f||_\infty \) for all \( f \) satisfying (1) and (2) above.

We call the base \( z_i \)-punctured sphere \( R \) and the target \( w_i \)-punctured sphere \( S \) from now on. The reason why the punctured sphere problem requires a starting map \( h \) as input is that by Teichmüller’s theorem, the extremal map exists and is unique within each homotopy class. The following theorem shows that Problem 2 is indeed general.

Theorem 3. An algorithm for Problem 2 can be used to give a solution to Problem 1.

Proof of Theorem 3. We take an instance of the polygon mapping problem and convert it to an instance of the punctured sphere problem first.

Let \( h_0 \) be any quasiconformal homeomorphism mapping \( P \) to \( Q \), such that \( h_0(v_i) = v'_i \). By conformally mapping \( P \) and \( Q \) to \( \mathbb{H} \) (denote the maps by \( \pi_P \) and \( \pi_Q \)), we get a quasiconformal self-homeomorphism \( h_\mathbb{H} \) of \( \mathbb{H} \), satisfying \( h_\mathbb{H}(z_i) = w_i \), where \( z_i \) and \( w_i \) are images (under \( \pi_P \) and \( \pi_Q \)) of \( v_i \) and \( v'_i \), respectively. Furthermore, \( h_\mathbb{H} \) can be normalized to fix 0, 1 and \( \infty \). Let \( \mathbb{H}^- \) denote the lower half plane, and define a quasiconformal self-homeomorphism \( h_\ell \) of \( \mathbb{H}^- \) by \( h_\ell(z_i) = h_\mathbb{H}(z_i) \). Now \( h_\ell \) and \( h_\mathbb{H} \) agree on \( \mathbb{R} \), and can be pieced together to get a quasiconformal self-homeomorphism \( h \) of \( \hat{\mathbb{C}} \) satisfying \( h(z_i) = w_i \). Note that \( h \) fixes 0, 1 and \( \infty \).

The next theorem shows how one can get back the answer to the polygon mapping problem from the answer to the punctured sphere problem.

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3We allow for \( \infty \) to be a vertex of the polygon.
Lemma 4. Let $f$ be the solution to Problem 2 when it is fed the input data $\{ z_i, w_i, h \}$ as above. Then:

1. $\mu_f(z) = \mu_f(\bar{z})$.

2. Let $f_u$ denote the restriction of $f$ to $\mathbb{H}$. Then $(\pi_Q)^{-1} \circ f_u \circ \pi_P$ is the solution to Problem 7 with data $P$ and $Q$.

Proof: We first prove that for all $z \in \mathbb{C}$, $f(z) = \overline{f(\bar{z})}$. Define another homeomorphism $g$ as $g(z) = \overline{f(\bar{z})}$. It is straightforward to check that $g$ is a self homeomorphism of $\mathbb{C}$ and satisfies $g(z_i) = w_i$.

Now $||\mu_f||_{\infty} = ||\mu_g||_{\infty}$. By uniqueness of the extremal quasiconformal mapping, $f$ is unique, and so must satisfy $f = g$ everywhere. Thus $f(z) = \overline{f(\bar{z})}$, which implies $\mu_f(z) = \mu_f(\bar{z})$.

To prove the second assertion, let $f^*$ denote the solution to Problem 1 with data $P$ and $Q$. Using the above construction, we get a self-homeomorphism $h^*$ of $\mathbb{C}$ which satisfies the same properties as $h$ and $f$. Uniqueness of $f$ now implies that $f = h^*$.

Ways to represent the Teichmüller map In theory, a normalized q.c.h. $f$ can be specified by specifying $\mu_f$. For computational purposes, unless a closed form expression for $f_*$ or $\mu_*$ is available, the best one can do is to evaluate $f_*$ or $\mu_*$ at a dense set of points inside the domain. Teichmüller’s characterization states that $\mu_*(\text{the Beltrami coefficient of the solution to either Problem 1 or Problem 2})$, equals $k|\phi|/\phi$, for some $0 \leq k < 1$, and some integrable holomorphic quadratic differential $\phi$ (Definition 11). $\phi$ comes from an $n - 3$ dimensional family, and a closed form expression for a basis $\{ \phi_1, \cdots, \phi_{n-3} \}$ is available. Therefore, by representing the coefficients $c_i$ in $\phi = \sum_{i=1}^{n-3} c_i \phi_i$ and $k$, one can represent $\mu_*$. The input and output complexity of both problems would be $O(n)$ in this case. The q.c.h. $f_*$ is the solution of the Beltrami equation for $\mu_*$, and can be represented as a series of singular operators applied to $\mu_*$ ([6], [11], [2]).

Our representation: We do not perform a search on the coefficients $c_i$ and adopt the first approach instead. If $k_* = ||\mu_*||_{\infty}$ is the maximal dilatation of the extremal map $f_*$, then our goal can be stated as follows.

Goal: Given an $\varepsilon > 0$, compute the values of $f$ on a given set of points inside the base polygon $P$, where the Beltrami differential $\mu_f$ of $f$ satisfies $||\mu_f||_{\infty} < k_* + \varepsilon$.

Complexity: To the best of the authors’ knowledge, even if the polygons $P$ and $Q$ have rational coordinates, there is no known way to represent the extremal map with finite precision (all representations may consist of transcendental numbers). Thus, it is not known whether the problem is in NP or not. We therefore do not address the actual complexity, and straightaway aim towards an approximation algorithm.

2.2 Our results

Continuous construction: Problem 2 asks for the extremal Beltrami differential on $R$ (the $z_i$ punctured sphere) that is isotopic to the starting map $h$. All Beltrami differentials of q.c.h. that are isotopic to $h$ (relative to the punctures) constitute what is called the global equivalence class (Definition 12) of $\mu_h$, and our task is to compute the Beltrami differential in this class with the least $L_\infty$ norm. Denote the vector space of all Beltrami differentials on the base $z_i$-punctured sphere $R$ by $B(R)$, and the unit ball (in the $L_\infty$ norm) of this vector space as $B_1(R)$.

The global class of $\mu_h$ cannot be described in a closed form (the only way to know if two differentials are globally equivalent is to solve their Beltrami equation). It lies inside $B_1(R)$, and except in trivial cases, is not convex.
Our main result is that we solve the problem by breaking the $L_\infty$ minimization over the global class into a sequence of $L_\infty$ minimizations over a convex domain $\mathcal{D}(\mu_h)$, described explicitly (in terms of $O(n)$ equalities) in terms of $\mu_h$. This convex domain will be the class of Beltrami differentials that are *infinitesimally equivalent* (Definition 14) to $\mu_h$.

Let $\mathcal{D}(\mu)$ denote the infinitesimal equivalence class of $\mu$, and $P(\mu)$ the Beltrami differential $\nu_0 \in \mathcal{D}(\mu)$ such that $\|\nu_0\|_\infty \leq \|\nu\|_\infty$ for all $\nu \in \mathcal{D}(\mu)$. $P(\mu)$ is called infinitesimally extremal (Definition 15).

**Theorem 5.** [Limiting procedure for Punctured Sphere Problem] There exists a sequence of q.c.h. $f_i$ s.t.

1. $f_1 = h$, the starting map in Problem 2
2. *Isotopic*: $f_i$ is homotopic to $h$, and $f_i(z_j) = w_j$, for all $i$ and $j$.
3. *Explicit* construction: Denote by $\mu_i$ the Beltrami coefficient of $f_i$. Then $\mu_{i+1}$ is an “explicit function” of $\mu_i$ and $P(\mu_i)$ in that it can be obtained by solving two differential equations depending only on $\mu_i$ and $P(\mu_i)$.
4. Uniform Convergence: $f_i \to f_*$ uniformly and $\|\mu_i\|_{L_\infty} \to \|\mu_*\|_{L_\infty}$ as $i \to \infty$.

**Theorem 6.** [Fast approximation] There exist constants $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and

$$\forall i \geq \frac{C}{\varepsilon^4(1 - \|\mu_1\|_\infty)^2},$$

$$\|\mu_i\|_\infty - k_* < \varepsilon,$$ where $\mu_i$ is the Beltrami differential of $f_i$ in Theorem 5 above.

**Discretization:** We represent all Beltrami differentials as piecewise constant differentials on a fine mesh. Every step of the continuous procedure mentioned above is shown to have a discrete analog.

The mesh we will be working on depends on the error tolerance $\delta$ required; near the punctures the mesh is made up of (triangulated) regular polygons, whose number of vertices and radii depend on $\delta$. The mesh is a triangulation with edge lengths bounded above by $\varepsilon$ (which is a function of $\delta$). We call this triangulation a canonical triangulation of size $\varepsilon$ (see Definition 22) and denote it by $\Delta_\varepsilon$.

The first theorem states that our discretization for the operator $P$ that returns the infinitesimally extremal Beltrami coefficient is in fact an approximation.

**Theorem 7.** [Discrete infinitesimally extremal] Given an error tolerance $0 < \delta < 1$, a collection of $n$ punctures $z_1, z_2, \ldots, z_n$, a triangulation $\Delta_\varepsilon$ and a piecewise constant Beltrami coefficient $\mu$ (where $\|\mu\| < 1$), there exists an algorithm $\text{INEXT}$ that computes a piecewise constant Beltrami coefficient $\hat{\nu}$ such that $\|\hat{\nu}\| = \|\nu\| < \|\nu\|_\delta$, where $\nu = P(\mu)$.

**Discrete algorithm:** Having discretized the main component of our procedure, all the other steps in our procedure can be easily implemented in practice. Computational quasiconformal theory is a field still in its infancy, and very few error estimates on these widely-used discretizations are known. We define two subroutines next that concern the discretization of compositions and inverses of quasiconformal maps.

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4In fact, the existence of the solution to the Beltrami equation of an arbitrary $\mu \in L^\infty$ with $\|\mu\|_\infty < 1$ was shown by 1) first showing the existence of the solution to a piecewise constant $\mu$, 2) sewing the individual piecewise q.c. maps along the boundary, and 3) taking a limit of such piecewise constant differential $\mu_n \to \mu$ and showing that the maps converge.
Definition 8. [Subroutine: PIECEWISE-COMP ]

Input: A triangulation $\Delta,\epsilon$, two piece-wise constant Beltrami coefficients $\mu_1$ and $\mu_2$ (corresponding to q.c.h $f_1$ and $f_2$ respectively), and error tolerances $\delta_1$ and $\delta_2$.

Output: A triangulation $\Delta_{\epsilon}'$ that is a refinement of $\Delta,\epsilon$, a piecewise constant Beltrami coefficient $\mu_{\text{comp}}$ that approximates the Beltrami coefficient of the composition $f_3 = f_1 \circ f_2$ within error $\delta_1$ in the $L^\infty$ topology, and the images $f_3(v_a)$ of the vertices $v_a$ of $\Delta_{\epsilon}'$ up to an error of $\delta_2$.

Assuming the existence of the subroutines PIECEWISE-COMP and PIECEWISE-INV, we construct an approximation algorithm for the Teichmüller map.

Theorem 10. [Teichmüller Map Algorithm] Given 1) a triangulation $T_0$ that includes punctures $z_1,\ldots,z_n$, 2) a mesh of sample points $S$, 3) an error tolerance $\delta$, and 4) a piece-wise constant Beltrami coefficient $\mu_0$, whose corresponding q.c.h. $f_0$ satisfies $f_0(z_j) = w_j$, there exists an algorithm EXTREMAL that computes $\Delta_{\epsilon}$ and the images of $S$ up to an error of $\delta$ under a q.c.h. $F$ having a piece-wise constant (in the computed triangulation) Beltrami coefficient $\mu_F$ such that

1. $\|\mu_F\|_{L^\infty} - \|\mu_*\|_{L^\infty} < \delta$ where $\mu_*$ is the Beltrami coefficient of the extremal quasiconformal map on the punctured sphere in the homotopy class of $f_0$.

2. $|F(z_i) - \hat{w}_i| = O(\delta)$.

An implementation of our algorithm will be presented in a forthcoming paper.

Structure of the paper: In Section 3 we define some terms that we use in our construction. In Section 4 we delve into the proofs of Theorems 7 and 10. Section 5 describes our discretized procedure and proves Theorems 7 and 11. We conclude in Section 6 with discussions on complexity and generalizations to arbitrary Riemann surfaces.

3 Preliminaries

Section 7.1 and Section 7.2 (in Appendix Section 12) provide the basic definitions of Riemann surfaces and quasiconformal maps, respectively. For the sake of completeness of the main body, in this section we define some of the concepts we will require for our continuous construction.

Definition 11. [Holomorphic quadratic differential] A holomorphic quadratic differential on a Riemann surface $R$ is an assignment of a function $\phi_i(z_i)$ on each chart $z_i$ such that if $z_j$ is another local coordinate, then $\phi_i(z_i) = \phi_j(z_j)(\frac{dz_i}{dz_j})^2$.

We will denote the space of such differentials on $R$ as $A(R)$. By the Riemann-Roch theorem, the complex dimension of this vector space for a genus $g$ closed compact surface with $n$ punctures is $3g - 3 + n$.

Fact: For $R = \mathbb{C} \setminus \{0,1,\ldots,z_{n-3}\}$ (the Riemann sphere with $n$ punctures),

$$\phi_k(z) = \frac{\eta_k}{z(z-1)(z-z_k)}, \quad 1 \leq k \leq n-3,$$  \hspace{1cm} (1)
form a basis of \((n - 3)\) dimensional complex vector space \(A(R)\). Here \(\eta_i\) is a constant, chosen such that the norm of \(\phi\) is 1, i.e., \(||\phi|| = \int_R |\phi| = 1\).

**Equivalence relations on Beltrami coefficients**

**Global equivalence:** This relation is defined only on Beltrami differentials of norm less than 1, i.e. those that belong to the unit ball \(B_1(R) = \{\mu \in B(R) : ||\mu||_\infty < 1\}\). Given two such differentials \(\mu\) and \(\nu\), denote the solution to their respective normalized Beltrami equations as \(f^\mu : R \rightarrow R_0\) and \(f^\nu : R \rightarrow R_1\). Both \(R_0\) and \(R_1\) are punctured spheres.

**Definition 12.** [Global equivalence] \(\mu\) and \(\nu\) are called globally equivalent (written \(\mu \sim_g \nu\)) if:

1. \(f^\mu(z_i) = f^\nu(z_i) \ \forall i\).

2. The identity map from \(R_0\) to \(R_1\) is homotopic to \(f^\nu \circ (f^\mu)^{-1}\) via a homotopy consisting of quasiconformal homeomorphisms.

**Definition 13.** [Trivial Beltrami differential] A Beltrami differential \(\nu\) is called trivial if it is globally equivalent to 0.

**Infinitesimal Equivalence:** This relation is defined on all of \(B(R)\).

**Definition 14.** [Infinitesimal equivalence] \(\mu\) and \(\nu\) are infinitesimally equivalent (written \(\mu \sim_i \nu\)) if \(\int_R \mu \phi = \int_R \nu \phi\) for all \(\phi \in A(R)\), with \(||\phi|| = 1\).

**Definition 15.** [Infinitesimally extremal] A Beltrami differential \(\nu\) is called infinitesimally extremal if \(||\nu||_\infty \leq ||\mu||_\infty\) for all \(\mu \sim_i \nu\).

**Definition 16.** [Infinitesimally trivial Beltrami differential] A Beltrami differential \(\nu\) is called infinitesimally trivial if it is infinitesimally equivalent to 0.

Section 7.4 lists all the theorems from Teichmüller theory that we will require, namely the Mapping theorem (Theorem 31), the composition formula (Equation (21)), the variational lemma (Lemma 32), the principle of Teichmüller contraction (Equation (22)) and most importantly, the Hamilton-Krushkal, Reich-Strebel, necessary and sufficient condition for optimality (Theorem 33).

### 4 Proofs of Theorems 5 and 6

At the heart of our construction of the sequence \(f_i\) in Theorems 5 and 6 lies the following lemma. Let \(h\) be any quasiconformal homeomorphism between \(R\) (the \(z_i\) punctured sphere) and \(S\) (the \(w_i\) punctured sphere) which is a valid input to Problem 2 and \(\mu_h\) denote its Beltrami differential.

**Lemma 17.** Let \(\nu_h\) be the infinitesimally extremal Beltrami differential in the infinitesimal class of \(\mu_h\). Let \(\mu_g(t)\) be a curve of Beltrami differentials with the following properties:

1. \(\mu_g(t)\) is globally trivial.
2. \(\mu_g(t) = t(\mu_h - \nu_h) + O(t^2)\).

Denote the solution to the Beltrami equation of \(\mu_g(t)\) by \(g_t\). Then \(\exists \delta > 0\) such that \(\forall t < \delta\), the map \(h_t = h \circ (g_t)^{-1}\) has smaller dilatation than \(h\).

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3 Fixing the points 0, 1 and \(\infty\). Hence the freedom of Möbius tranformation is accounted for.
Proof of Lemma 17. By the formula for composition of quasiconformal maps (21) in Section 7.4,

$$\mu_{ht}(gt(z)) = \frac{\mu_h - \mu_g(t)}{1 - \mu_g(t)\theta_t} \cdot \frac{1}{\theta_t},$$

where \(\theta_t = \frac{p_t}{p_{\mu}}\) and \(p_t = \frac{\partial g_t}{\partial z}\). (2) implies

$$|\mu_{ht} \circ g_t|^2 = \frac{|\mu_h|^2 - 2\text{Re}(\mu_h\mu_g(t)) + |\mu_g(t)|^2}{1 - 2\text{Re}(\mu_h\mu_g(t)) + |\mu_g(t)||\mu_h|^2}$$

Using the fact that

$$||\mu_g(t) - t(\mu_h - \nu_h)||_{\infty} = O(t^2)$$

and differentiating (3) with respect to \(t\) once and putting \(t = 0\), we get that

$$|\mu_{ht} \circ g_t| = |\mu_h| - t \frac{1}{|\mu_h|} \text{Re}(\mu_h^2 - \mu_h \mu_g) + O(t^2)$$

Let \(k_0 = ||\nu_h||_{\infty} < k = ||\mu_h||_{\infty}\). Define

\[S_1 = \{z \in \mathbb{R} : |\mu_h(z)| \leq (k + k_0)/2\}\]

and

\[S_2 = \{z \in \mathbb{R} : (k + k_0)/2 < |\mu_h(z)| \leq k\}\]

Clearly, \(S_1 \cup S_2 = \mathbb{R}\). Since in \(S_1\) the starting value of this curve at \(t = 0\) is \(|\mu|\), which is certainly less than \(k\), (2) implies there exists \(\delta_1 > 0\) and \(c_1 > 0\) such that for \(0 < t < \delta_1\),

$$|\mu_{ht} \circ g_t(z)| \leq k - c_1 t \quad \text{for } z \in S_1$$

For \(z\) in \(S_2\) the coefficient of \(t\) in (4) is bounded below by

$$\frac{1 - k^2}{k} \left[\left(\frac{k + k_0}{2}\right)^2 - k_0 k\right] = \frac{1 - k^2}{k} \left(\frac{k - k_0}{2}\right)^2 > 0$$

Therefore, (4) implies there exists \(\delta_2 > 0\) and \(c_2 > 0\) such that for \(0 < t < \delta_2\),

$$|\mu_{ht} \circ g_t(z)| \leq k - c_2 t \quad \text{for } z \in S_2$$

Putting together (5) and (6), we find that \(||\mu_{ht}||_{\infty} < k\) for sufficiently small \(t > 0\), proving the lemma.

The proof is similar to that of the Hamilton-Krushkal, Reich-Strebel necessary-and-sufficient condition for extremality (see Theorem 33), published in a sequence of papers. We refer the reader to [12] for a combined proof of this celebrated result, which is the one we adapt. To the best of the authors’ knowledge, the above lemma is the first result that describes, given a starting map, how to get a map with a smaller dilatation.

The proof of Theorem 5 is constructive. We summarize the construction first:
4.1 Summary of the construction

At step $i$, given a starting map $f_i : R \to S$ with Beltrami coefficient $\mu_i$, let $v_i$ denote the infinitesimally extremal Beltrami coefficient in the infinitesimal class of $\mu_i$. Let $k_i = ||\mu_i||_{\infty}$ and $k_i^0 = ||v_i||_{\infty}$. Observe that $\mu_i - v_i$ is infinitesimally trivial (Definition 16).

1. Choose $t$ such that

$$t = \min \left( \frac{3}{4}, C_1, \frac{\varepsilon}{4}, \sqrt{\frac{\varepsilon}{2C_2}}, \frac{(k_i - k_i^0)^2(1 - k_i^2)}{1 - k_i^2 + C_2} \right),$$

where $\varepsilon \leq \min(1/2, (k_i - k_i^0)/8)$, and $C_1$ and $C_2$ are two explicit constants to be derived later.

2. Use Section 4.3 to construct a quasiconformal self-homeomorphism $g_i$ of $R$ such that

- $\mu_g$ is globally trivial.
- $||\mu_g - t(\mu_i - v_i)||_{\infty} < C_2t^2$, where $C_2$ is the same constant as in (7).

3. Form $f_{i+1} = f_i \circ (g_i)^{-1}$ such that $f_{i+1}$ has smaller dilatation than $f_i$ (by Lemma 17).

4. Reiterate with $f_{i+1}$ as the starting map.

4.2 How the construction implies Theorems 5 and 6

The $\mu_g$ in step (2) above can be constructed by solving two differential equations involving $t$, $\mu_i$ and $v_i$ (Lemma 21 in Section 4.3). Assuming that, we have the following lemma that quantifies the progress made in step $i$. Recall that $k_i = ||\mu_i||_{\infty}$ and $k_i^0 = ||v_i||_{\infty}$. 

Lemma 18. [Decrease in one step] If $t$ is chosen as in (7), then $k_i - k_{i+1} > d$, where

$$d = \min \left( \frac{k_i - k_i^0}{4}, \frac{(k_i - k_i^0)^2 t(1 - k_i^2)}{8} \right).$$

Proof of Lemma 18. To simplify notation, we let $k = k_i$ and $k_0 = k_i^0$, since we will be assuming that we are in step $i$ of the iteration. As in the proof of Lemma 17 let $S_1$ be the region where $|\mu_i| \leq \frac{k + k_0}{2}$ and $S_2$ be such that $\frac{k + k_0}{2} \leq |\mu_i| < k$. Assuming that $t < \text{min}(3/4, C_1)$ implies Lemma 21, so we assume this condition on $t$.

Furthermore, on $S_1$, if $t < \min \left( \frac{3}{4}, C_1, \frac{\varepsilon}{4}, \sqrt{\frac{\varepsilon}{2C_2}} \right)$, by the composition formula for q.c. maps we get,

$$|\mu_{i+1} \circ g_i(z)| \leq \left| \frac{\mu_i - \mu_g}{1 - \mu_i \mu_g} \right|$$

$$\leq \left| \frac{\mu_i - t(\mu_i - v_i)}{1 - \mu_i \mu_g} \right| + \frac{C_2t^2}{1 - \mu_i \mu_g}$$

$$\leq \frac{1}{1 - \varepsilon} \left( \frac{k + k_0}{2} + 2t + C_2t^2 \right)$$

$$\leq \frac{1}{1 - \varepsilon} \left( \frac{k + k_0}{2} + \varepsilon \right)$$

where the last inequality follows by requiring $|\mu_i \mu_g| < 2t + C_2t^2 < \varepsilon$, which is true for the assumed value of $t$. Notice that $|\mu_i|, |v_i|$ are less than 1.
Therefore, on $S_1$,

$$k - |\mu_{i+1}| \geq \frac{k - k_0 - 2\varepsilon(1 + k)}{2(1 - \varepsilon)} > \frac{k - k_0}{4}$$

if $\varepsilon \leq \frac{k-k_0}{8}$. On $S_2$,

$$\frac{|\mu_i - \mu_g|}{|1 - \mu_i\mu_g|} \leq \frac{|\mu_i - t(\mu_i - \nu_i)|}{|1 - \mu_i\mu_g|} + \frac{C_2t^2}{|1 - \mu_i\mu_g|} \leq \frac{|\mu_i - t(\mu_i - \nu_i)|}{|1 - \mu_i\mu_g|} + \frac{C_2t^2}{|1 - \mu_i\mu_g|}$$

Now,

$$k - |\mu_{i+1}| \geq \frac{|\mu_i| - |\mu_{i+1}|}{|1 - \mu_i\mu_g|}$$

$$\geq \frac{|\mu_i|(|1 - \mu_i t(\mu_i - \nu_i)| - |\mu_i|C_2t^2) - |\mu_i - t(\mu_i - \nu_i)|}{|1 - \mu_i\mu_g|} - \frac{C_2t^2}{1 - \varepsilon}$$

$$\geq \frac{|1 - \mu_i t(\mu_i - \nu_i)| - |\mu_i|C_2t^2}{|1 - \mu_i\mu_g|} - \frac{C_2t^2}{1 - \varepsilon}$$

(8)

where $A = |\mu_i|(|1 - \mu_i t(\mu_i - \nu_i)|)$ and $B = |\mu_i|C_2t^2 - |\mu_i - t(\mu_i - \nu_i)|$. Using

$$A - B = \frac{A^2 - B^2}{A + B}$$

$$A + B < 4$$

$$A^2 - B^2 = (1 - |\mu_i|^2)(2\Re(t\mu_i(\mu_i - \nu_i)) - t^2|\mu_i|^2|\mu_i - \nu_i|^2(1 + |\mu_i|^2)) \geq (1 - |\mu_i|^2)(t(k - k_0)^2 - t^2|\mu_i|^2|\mu_i - \nu_i|^2(1 + |\mu_i|^2))$$

(9)

Using Equation 9 in Equation 8 and the fact that $\varepsilon < 1/2$ we see that

$$k - |\mu_{i+1}| \geq \frac{1 - k^2}{4} (t(k - k_0)^2 - 8t^2) - 2C_2t^2 \geq \frac{(k - k_0)^2 t(1 - k^2)}{8}$$

(10)

with the last equation holding if \( t < \frac{(k-k_0)^2(1-k^2)}{1-k^2+C_2} \), concluding the proof.

Now we apply the principle of Teichmüller contraction, which essentially bounds $k_i - k_i^0$ from below by a function of $k_i - k_s$ (how far we are from the infinitesimally extremal coefficient tells us how far we are from the extremal). Using Lemma 18 Equation (7) and the principle of Teichmüller contraction (Equation (22)), we get

**Lemma 19.** There exists a constant $C_3 > 0$, such that if $t$ is chosen as in (7), then

$$k_i - k_{i+1} > C_3(1 - ||\mu_1||_\infty)^2(k_i - k_s)^4$$
Proof of Lemma 19. The first three terms in the min expression in Equation (7) are independent of the iteration step. If the value of $t$ is one of these, then the lemma is evident. Similarly, if the value of $d$ is the first of the two terms in the min in Lemma 18, then the lemma is clear too.

Assume now that $t = \frac{(k_i - k_0)^2(1-k_i^2)}{1-k_i^2+C_2}$. Noting that $1 - k_i + C_2 < C_2 + 1$, and that $1 - k_0^2 > 1 - k_i$, we get $t > \frac{(k_i - k_0)^2(1-k_i)}{1+C_2}$. Using this value in Lemma 18 gives

$$d > \frac{(k_i - k_0)^4(1-k_i)^2}{8(1+C_2)},$$

and by Teichmüller contraction, $k_i - k_0 \geq (k_i - k_*)/10$, implying

$$d > \frac{(k_i - k_*)^4(1-k_i)^2}{80(1+C_2)},$$

Putting $C_3 = \frac{1}{80(1+C_2)}$, and noting that $1 - k_i > 1 - k_1$ completes the proof of the lemma.

□

Using Lemma 19, the proofs of Theorems 5 and 6 can now be completed.

Proof of Theorem 5: The first three assertions follow easily from our construction (notice that $\mu_\vartheta$ being trivial implies that we stay within the initial homotopy class). We now prove assertion (4) in the statement of the theorem.

The $k_i$ form a decreasing sequence and are bounded from above by $k_1$ and below by $k_*$. Hence the $k_i$ converge to some $k \geq 0$. But then $k_i - k_{i+1} \to 0$ as $i \to \infty$, and Lemma 19 now implies that $k_i - k_* \to 0$, and so $k = k_*$. Thus $k_i \to k_*$. Now the fact that $f_i$ converge uniformly to $f_*$ follows because of the convergence property of q.c.h.

The space of all q.c.h. $f$ with $\|\mu_f\|_\infty < k_1$ forms a compact space, and so there exists a subsequence that converges. By the arguments above and by uniqueness of $f_*$, this limit must be $f_*$. Furthermore, this is true of any convergent subsequence of the $f_i$. Thus we get that the entire sequence $f_i$ converges uniformly to $f_*$.

Proof of Theorem 6: Let $A = C_2(1-\|\mu_1\|_\infty^2)$ and define $y_n = \|\mu_n\|_\infty - k_*$. By Lemma 19 and Teichmüller contraction (Equation 22), $y_{n+1} \leq y_n - Ay_n^4$. If $y_1 < \varepsilon$, we are done. If not, then $y_1 - y_2 \geq Ay_1^4 \geq A\varepsilon^4$ and thus $y_2 \leq y_1 - A\varepsilon^4$. If $y_2 < \varepsilon$ we are done. If not, $y_3 \leq y_2 - A\varepsilon^4 \leq y_1 - 2A\varepsilon^4$. Continuing inductively we see that $y_n \leq y_1 - (n-1)A\varepsilon^4$ if $y_{n-1} > \varepsilon$. The right hand side is less than $\varepsilon$ if $n > \frac{1}{A\varepsilon^4}(\|\mu_1\|_L^\infty - k_* - \varepsilon)$. Since $(\|\mu_1\|_L^\infty - k_* - \varepsilon) < 2$, putting $C = 2/C_3$ proves the theorem.

4.3 Constructing self homeomorphisms $g_i$

Given a starting map $f_i$, we show how to construct the self homeomorphism $g_i$ of $R$ used in our construction. We simplify notation by suppressing the $i$, keeping in mind that this is the $i$th step of the procedure. Thus $\mu$ and $\mu_\vartheta$ will denote the Beltrami differentials of $f_i$ and $g_i$, respectively. Also, $v$ is the infinitesimally extremal Beltrami differential in the infinitesimal class of $\mu$.

Let $\alpha = \mu - v$, $t$ be as in Equation (7), and let $f^\alpha$ be the normalized solution to the Beltrami equation for $t\alpha$. We observe next that $f^\alpha$ moves the points $z_i$ only by a distance $O(t^2)$.

Lemma 20. Let $r = \max_{1 \leq i \leq n-3} |z_i|$, and let $f^\alpha$ be as above. Then there exists a constant $C_r$ depending only on $r$, and a constant $\delta > 0$ such that for all $i$, $|f^\alpha(z_i) - z_i| \leq C_r t^2$, $\forall t < \delta$. 

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Proof of Lemma 20. By (1),
\[
\phi_i(\zeta) = \frac{z_k(z_k - 1)}{\zeta(z - 1)(\zeta - z_k)}
\]
for \(1 \leq i \leq n - 3\) is a basis for the space of quadratic differentials on \(R\). Let \(\zeta = \xi + i\eta\). Infinitesimal equivalence of \(\tau\alpha\) now implies that
\[
\int \int_C \frac{\tau\alpha d\xi d\eta}{\zeta(z - 1)(\zeta - z_k)} = 0
\]
(11)

Now we use the mapping theorem (Theorem 51 in Section 7.4). In the notation of the theorem, \(V(z_i) = 0\) by (11). Existence of \(\delta, C\) and the statement now follows from the statement of the mapping theorem.

Denote \(f^{\tau\alpha}(z_i)\) by \(z_i'\). We will first construct another homeomorphism \(f\nu\) from \(\hat{C}\) to itself which satisfies \(f\nu(z_i') = z_i\). We then define the required self homeomorphism \(g_i = f\nu \circ f^{\tau\alpha}\). The construction of \(f\nu\) will be via a vector field method.

**Construction of \(f\nu\) by a vector field method:** Let \(\{D_1, \ldots, D_{n-3}\}\) denote disjoint open disks centered at \(z_i\). Choosing the radius of each disk to be \(r = d/4\), where \(d = \max_{1 \leq i, j \leq n-3} |z_i - z_j|\) ensures disjointness.

We will fix these disks once and for all.

**A single disk:** We first construct a self homeomorphism \(f\nu_i\) of \(\hat{C}\) which is the identity outside \(D_i\), and maps \(z_i'\) to \(z_i\). Now \(z_i \in \mathbb{R}\), and by a rotation we can assume that \(z_i'\) is real and greater than \(z_i\). Consider the vector field
\[
X(z) = p(z)(z_i - z_i)\frac{\partial}{\partial x},
\]
where \(p(z)\) is a \(C^\infty\) function identically zero outside \(D_i\), and identically 1 inside the disk of radius \(r/2\) around \(z_i\), denoted as \(D_i'\). Let \(\gamma\) be the one parameter family of diffeomorphisms associated with this vector field. We denote the time parameter by \(s\) and note that the diffeomorphism \(\gamma_1\) sends \(z_i'\) to \(z_i\). We denote this diffeomorphism \(\gamma\) at \(s = 1\) by \(f\nu_i\). Now define \(f\nu = f^{n-3}_\nu \circ f^{n-2}_\nu \cdots f^1\nu\), and \(g_i = f\nu \circ f^{\tau\alpha}\). We have

**Lemma 21.** There exist constants \(C_1 > 0\) and \(C_2 > 0\) such that \(\mu_g\) above is globally trivial, and for all \(t < \min(3/4, C_1)\), \(||\mu_g - t(\mu - v)||_\infty < C_2 t^2\).

The exact values of \(C_1\) and \(C_2\) can be inferred from the proof. They equal the values of \(\delta\) and \(C_r\), respectively, in the mapping theorem, when \(r = 1\).

**Proof of Lemma 21**

In what follows, we denote \(f_i\) (the quasiconformal map after \(i\) iterations of the algorithm) by \(f\) for convenience. In fact, we drop the subscript \(i\) altogether. Recall that from \(\mu_i\), we construct a new quadratic differential \(\nu_i\) that is infinitesimally extremal. Let \(\alpha_i = \mu_i - \nu_i\).

We then construct \(g_i\) that is a self homeomorphism of the base punctured sphere \(R\) using the vector field method. If we manage to prove that the dilatation of the map obtained by the vector field method \(|v|_{L^\infty} \leq G t^2\), then for the composition (here \(|A| = 1\)) \(g_i = f\nu \circ f^{\tau\alpha}\), if \(i\) is sufficiently large then \(|1 + t\alpha v A| > \frac{1}{2}
\[
|\frac{t\alpha + Av \circ f^{\tau\alpha}}{1 + t\alpha Av} - t\alpha| = \left|\frac{Av \circ f^{\tau\alpha} + t^2\alpha^2 Av \circ f^{\tau\alpha}}{1 + t\alpha v A}\right| \leq (2G + 1)t^2
\]
(12)
Therefore the Beltrami of the composition is $t \alpha + O(t^2)$ where the $O(t^2)$ term is bounded above by $(2G + 1)t^2 = Ct^2$. Our aim is to prove that $G$ exists and is bounded independent of $i$. In whatever follows we denote $f_\nu$ simply by $f$.

Recall that the vector field is $X = \sum_j \rho_j (z_j(t) - z_j)$. Let $\gamma(t, s, y, \bar{y})$ be the flow where $s$ is the “time parameter” for the flow and $y$ is the initial position. Notice that the vector field diffeo $\gamma(t, 1, y, \bar{y}) = f$ and the Beltrami $v$. Notice that

\[
v = \frac{f_y}{f_y} = \frac{f_y - f_y f_t y}{f_y - f_y f_t y} \quad v_t = \frac{(f_{yt} f_y - f_y f_f y) f_y^2 - 2 f_y f_y (f_y f_y - f_y f_t y)}{f_y^4}
\]

If $|f_y| > m$ and $f_t, |f_y|, |f_y t|, |f_y t_t|, |f_y t|, |f_y t_o|$ and $|f_y t|$ are bounded above by $M$, then $|v_t| \leq \frac{6M^4}{m^4} = G$. First we have to prove that indeed $|v| < 1$ so that $f$ is q.c. For all of these things we consider

\[
\frac{d\gamma}{ds} = \sum \rho(\gamma)(z_i(t) - z_i)
\]
\[
\frac{d\gamma_t}{ds} = \sum (\rho z \gamma_t + \rho z \bar{\gamma}_t)(z_i(t) - z_i) + \sum \rho z_i
\]
\[
\frac{d\gamma_{tt}}{ds} = \sum (\rho z \gamma_{tt} + \rho z \bar{\gamma}_t)(z_i(t) - z_i) + \sum \rho z_i + \sum (\rho z \gamma_t + \rho z \bar{\gamma}_t)z_i
\]
\[
\frac{d\gamma_y}{ds} = \sum (\rho z \gamma_y + \rho z \bar{\gamma}_y)(z_i(t) - z_i)
\]
\[
\frac{d\gamma_{yt}}{ds} = \sum (\rho z \gamma_{yt} + \rho z \bar{\gamma}_y + \rho z \gamma_{tt} \bar{\gamma}_y + \rho z \gamma_t \bar{\gamma}_y + \rho z \gamma_{yt} \bar{\gamma}_y + \rho z \gamma_{tt} \bar{\gamma}_y)(z_i(t) - z_i) + \sum (\rho z \gamma_y + \rho z \bar{\gamma}_y)z_i(t)
\]

and similarly for the other quantities. Notice that $|z_i(t) - z_i| \leq Et^2$. For future reference let $\|\rho\|_{C^2} \leq \frac{1}{10000}$ and $\max_i |z_i| = a$. Notice that by Cauchy’s estimates $|z_i'| \leq 4(a + E)$ and $|z_i''| \leq 16(a + E)$ for $t < \frac{3}{4}$. For a system of IDE of the type

\[
w(s) - w = \int_0^s A(t)w(t)dt
\]

by the Gronwall inequality, $|w(s)| \leq |w| \exp(s \max_{[0, s]} \sqrt{\mu}||A||)$. Without loss of generality, let $t <$
Recall that $a = \max_i |z_i|$ (not $z_i(t)$ but $z_i(0)$), $E$ is that constant such that $|z_i(t) - z_i| \leq El^2$, and $\mathcal{P} = 1000\|\rho\|_{C^2}$. Note that $t$ has an additional condition in that $t < \min\left(\frac{3}{4}, \exp(-EP/2)\frac{1}{\sqrt{100EP}}\right)$ so as to ensure that indeed $|\gamma_y| \geq \frac{1}{2}$ and that $|v| < 1$.

Until now the constants depended on $\max_i |z_i|$. However, note that the extremal map problem is invariant under Möbius transformations fixing the upper half plane. The constants $\delta$ and $C_r$ in the mapping theorem depend only on $r$ (where $r$ is the disk inside which these estimates are valid). For the polygon mapping problem, a-priori all the $z_i$s are on the real line, and three of them are 0, 1 and $\infty$. Assume that $\infty$ is the $n$th puncture (so $z_n = \infty$), and choose $z_{\min} = \min_{i\neq n} z_i$ and $z_{\max} = \max_{i\neq n} z_i$. We then find a Möbius transformation that maps $z_{\min}$, $z_{\max}$ and $z_n$ to $0, 1, \infty$, respectively. Now all the new punctures are in the interval $[0, 1]$, and the mapping theorem provides absolute constants that do not depend on the punctures anymore.

We showed in Theorem 8 how to reduce the polygon mapping problem to the punctured sphere problem. However, the above procedure can also be directly implemented on polygons, once we have the appropriate basis for the space of quadratic differentials. We give a simple well-known description of this basis in Section 8.3 in the appendix.

5 Discretization of the procedure

Before we discretize the procedure, we give the properties of the mesh we work on as promised in Section 2.2. Given an error tolerance $\delta$, let $\varepsilon = O(\delta^{6n-2})$.

Definition 22. [Canonical triangulation of size $\varepsilon$] A canonical triangulation of size $\varepsilon$, denoted as $\Delta_\varepsilon$ is a set of vertices and edges $(V_\varepsilon, E_\varepsilon)$, with $z_i \in V_\varepsilon$, satisfying the following.

1. It contains the edges and vertices of a regular polygon centered at 0 and of diameter $O(\delta^{-1})$, and line segments joining the vertices of this polygon to $\infty$. 

\[ \min \left( \frac{3}{4}, \exp \left( -\frac{E\rho^2}{2} \frac{1}{\sqrt{100E\rho}} \right) \right) \] and $\mathcal{P}, E > 1$. Therefore

\[
\|\gamma_y(t, 1, y, \bar{y})\|_{C^0} \leq \exp(\mathcal{P}t^2) \leq \exp(\mathcal{P}) \\
\|\gamma_y(t, 1, y, \bar{y})\|_{C^0} \leq \mathcal{P}Et^2 \exp(\mathcal{P}t^2) \leq \exp(\mathcal{P}) \\
|\gamma_y(s) - 1| \leq 2\mathcal{P}t^2 \exp(\mathcal{P}) \leq \frac{1}{2} \\
m = \frac{1}{2} \\
\|\gamma_y(t, 1, y, \bar{y})\|_{C^0} < \mathcal{P}(a + E) \exp(\mathcal{P}) \\
\|\gamma_y(t, 1, y, \bar{y})\|_{C^0} < (\mathcal{P}(a + E))^3 \exp(\mathcal{P}) \\
\|\gamma_y(t, 1, y, \bar{y})\|_{C^0} < (\mathcal{P}(a + E))^2 \exp(\mathcal{P}) \\
\|\gamma_y(t, 1, y, \bar{y})\|_{C^0} < (\mathcal{P}(a + E))^2 \exp(\mathcal{P}) \\
\|\gamma_y(t, 1, y, \bar{y})\|_{C^0} < (\mathcal{P}(a + E))^3 \exp(\mathcal{P}) \\
\|\gamma_y(t, 1, y, \bar{y})\|_{C^0} < (\mathcal{P}(a + E))^3 \exp(\mathcal{P}) \\
C = \frac{6M^4}{m^2} + 1 < 200(\mathcal{P}(a + E))^4 \exp(4\mathcal{P})
\]
2. Except for the line segments to \( \infty \), all the other sides of the triangulation have Euclidean length at most \( \varepsilon \).

3. It contains the edges and vertices of regular polygons of \( N = O(\delta^{1-2n}) \) sides centered at the punctures of diameter \( O(\delta) \), and lines joining the the vertices of these polygons to their centers, i.e. to the punctures.

4. \( \Delta_\varepsilon \) is a refinement of \( \Delta_{1/2} \).

We now describe what the algorithm \textsc{INEXT} does, after which we prove Theorem 7.

5.1 \textsc{INEXT} and the proof of Theorem 7

We want to discretize the operator \( \mathcal{P}(\mu) \) which returns \( \nu \) with the least \( L^\infty \) norm satisfying \( \int_R \nu \phi_i = \int_R \mu \phi_i \) for all \( \phi_i \) in Equation (1). Note that the starting \( \mu \) is piecewise constant at the start of every iteration.

\textbf{Observation 23.} The integral of \( \phi_i \) over any triangle \( t_j \) can be computed analytically.

We provide this formula (that involves taking the logarithm of a complex number) in Section 8.1.

We approximate \( \nu \) by piecewise constant Beltrami differentials. One can easily see then that the constraints for infinitesimal equivalence become linear constraints of the form \( Ax = b \), where \( A \) is the matrix whose \((i,j)\)th entry equals \( \int_{t_j} \phi_i \), \( x \) is the vector of unknown values of the piecewise constant \( \nu \) on a triangle, and \( b \) is the vector of \( \int_{t_j} \mu_j \phi_i \), where \( \mu_j \) is the value of \( \mu \) on triangle \( t_j \).

If \( A, \ x \) and \( b \) are real, an \( L^\infty \) minimization can be formulated as a linear program. In our case, we break the vectors and matrices into their real and complex parts, and then we can formulate the program as a quadratically constrained quadratic program. Although in general they are NP-hard to solve, we show that our program involves positive semi-definite matrices; and it is known that such instances can be solved in polynomial time using interior-point methods [18]. Details are in Section 8.2.

\textbf{Lemma 24.} \textit{[INEXT]} There exists an algorithm \textsc{INEXT} that, given a piecewise constant \( \mu \) on \( \Delta_\varepsilon \) returns a piecewise constant \( \hat{\nu} \) such that \( \max_{t_j} \hat{\nu}(t_j) \leq \max_{t_j} \beta(t_j) \), where \( \beta \) is any piecewise constant (on \( \Delta_\varepsilon \)) Beltrami differential that is infinitesimally equivalent to \( \mu \).

With this, we are now in a position to prove Theorem 7.

\textbf{Proof sketch for Theorem 7.} The main idea is to use \( \nu = \mathcal{P}(\mu) \) and produce a piecewise constant \( \nu_p \), which does not satisfy the integral constraints, but the error involved can be estimated. We then add a piecewise constant differential to \( \nu_p \) to produce \( \tilde{\nu}_p \), that is in the same infinitesimal class as \( \mu \) and whose dilatation is close to that of \( \nu \). This proves that \( \hat{\nu} \) (whose dilatation is smaller than that of \( \tilde{\nu}_p \)) satisfies the desired requirement.

\textbf{Proof of Theorem 7.} In whatever follows we assume that \( |z_i| \leq A \). Let \( \tilde{k} = \min(|z_i - z_j|/200, |z_i - 1|/200, |z_i|/200) \). Here we prove that \( \|\nu\| - \|\nu_k\| < \delta \). The strategy is to produce a piece-wise constant Beltrami coefficient \( \tilde{\nu}_p \) whose norm is \( \delta \) close to \( \nu_k \) and satisfies the integral constraint exactly. To do this we first produce a piece-wise constant Beltrami \( \nu_p \) whose norm is equal to that of the infinitesimally extremal one \( \|\nu_k\| \) and \( |\int (\nu_p - \nu_k)\phi_k| < \delta \forall k \). We claim that we may produce \( \tilde{\nu}_p \) by adding terms of magnitude at most \( \delta \) to \( \nu_p \) so as to make sure that the integral constraint is obeyed exactly. Indeed, by the hypothesis on the canonical triangulation we see that one may choose sufficiently many triangles of size \( \frac{1}{2} \) and solve linear equations that determine the constants to be added on these triangles so as to satisfy the integral constraint. Thus the problem is reduced to finding \( \nu_p \) and proving the error estimate.
The infinitesimally extremal \( \nu_* = k_* \frac{\phi}{\phi} = k_* \frac{\sum c_k \phi_k}{\sum c_k \phi_k} = k_* \frac{\sum \tilde{c}_k \phi_k}{\sum c_k \phi_k} \) where \(|\tilde{c}_k| \leq 1\) with \(|c_1| = 1\). Then

\[
\nu_* = k_* \frac{\sum \tilde{c}_k (z - z_1) \ldots (z - z_{k-1})(z - z_{k+1}) \ldots |z(z - 1)(z - z_k)\ldots|}{\sum \tilde{c}_k (z - z_1) \ldots (z - z_{k-1})(z - z_{k+1}) \ldots |z(z - 1)(z - z_k)\ldots|} 
= k_* \frac{\sum \tilde{c}_k h_k(z)}{\sum \tilde{c}_k h_k(z)} g(z)
\]

We first approximate \( h_k \) and \( g \) by piece-wise constant functions with an error of at most \( \beta \) within the large polygon. Replace the values of \( h_k, g \) by their values at the centers of the triangles in the triangulation. This gives us \( \nu_\beta \) in the large polygon. Outside it we set \( \nu_\beta \) to 0.

The error thus caused for \( h_k \) is less than \( \varepsilon 2^n R^n \). For \( g \) the error is more subtle. In the small polygons around the punctures, if \( N > \frac{12\pi}{\beta} \) then the error caused in \( g(z) \) is less than \( \frac{\beta}{3} \) in those regions. For future use it is useful to note that

\[
\left| \frac{\eta_2}{y_2} - \frac{\eta_1}{y_1} \right| \leq \left| y_2 - y_1 \right| \int_0^1 \left| \frac{1}{y_{2+(1-t)y_1}} \right| dt \text{ and the same inequality holds for } \frac{|y_1|}{y}.
\]

Outside the polygons the error in \( g \) is less than \( \frac{2^n \varepsilon 2^n R^n}{(k)^{n-1} - 2^n R^n} \). Since the radius of the polygons is \( \tilde{k} \), the error in \( \sum \tilde{c}_k h_k \) outside the polygons is less than \( \frac{\beta^2}{2\pi R^n} \). Notice that \( \varepsilon < \min(\tilde{k}^2, \frac{\beta}{2\pi R^n}) \). Next we want to estimate

\[
I_k = \int \frac{|\eta_2(\nu_* - \nu_\beta)|}{|z(z - 1)(z - z_k)|} = \int U_k.
\]

Indeed,

\[
I_k \leq 4k_\varepsilon \int_{|z| \geq 2R} \frac{1}{r^2} dr d\theta + \int_{|z| \leq 2R} U_k 
\leq k_\varepsilon \frac{\delta}{2} + J
\]

where \( 2|z|^{3} \geq |z(z - 1)(z - z_k)| \geq \frac{|z|^3}{2} \) if \( R > \max \frac{4}{\delta}, 1, 2^n (1 + A)^n \). Now we estimate \( J_k \).

\[
J_k = \int_{|z| \leq 2R^n} \frac{U_k}{\sum \tilde{c}_k h_k > \kappa} + \int_{|z| \leq 2R^n} \frac{U_k}{\sum \tilde{c}_k h_k > \kappa} 
< k_\kappa \frac{\beta}{\kappa - \beta} + k_\kappa \beta + \sum_{\xi} \int_{|z| \leq 2R^n} \frac{U_k}{\sum \tilde{c}_k h_k > \kappa} - \kappa |z - z\xi| > \kappa \forall \xi
\]

\[
< k_\kappa \frac{\beta}{\kappa - \beta} + k_\kappa \beta + k_\kappa (1 + A)^2 \frac{12\pi k}{(\tilde{k})^2} + k_\kappa (A + 1)^2 \frac{(\pi \kappa^2/(n-1))^n}{\tilde{k}^{-2}}
\]

where the last estimate is obtained by remembering that \( \sum \tilde{c}_k h_k = (z - \lambda_1) \ldots \) for some \( \lambda_i \) within the big polygon. One may choose \( \tilde{k}, \kappa \) and \( \beta \) so as to estimate \( I_k \) by \( \delta \). Indeed,

\[
\tilde{k} < \frac{\delta (\tilde{k})^2}{1000 (1 + A)^2}
\]

\[
\kappa < \left( \frac{\delta^4 (\tilde{k})^6}{1000 (1 + A)^8} \right)^{(n-1)/2}
\]

\[
\beta < \frac{\delta}{6 + \delta}
\]

\[
< O(\delta^{2n-1})
\]
5.2 \textbf{EXTREMAL} and the proof of Theorem\textsuperscript{10}

Apart from the subroutines \textsc{INEXT}, \textsc{PIECEWISE-COMP} and \textsc{PIECEWISE-INV}, we will require three more subroutines to discretize our procedure.

\textbf{Definition 25.} [Subroutine: \textsc{TRIANG}]

\textbf{Input:} a set of points $S$, a size $M$, and a triangulation $\Delta_{\varepsilon}$.

\textbf{Output:} A triangulation $\Delta'_{\varepsilon}$ of the given size $M$ containing $S$ such that $\Delta'_{\varepsilon}$ is a refinement of $\Delta_{\varepsilon}$.

\textbf{Definition 26.} [Subroutine: \textsc{BELTRAMI}]

\textbf{Input:} A triangulation $\Delta_{\varepsilon}$ of the plane, a piecewise constant Beltrami coefficient $\mu$, and error tolerance $\delta$.

\textbf{Output:} A triangulation $\Delta'_{\varepsilon}$ that is a refinement of $\Delta_{\varepsilon}$, and the images $\hat{f}(v_i)$ of the vertices $v_i \in \Delta'_{\varepsilon}$ such that $|f''(v_i) - \hat{f}(v_i)| < \delta$.

\textbf{Definition 27.} [Subroutine: \textsc{VECT-FIELD}]

\textbf{Input:} A $C^k$ ($k$ sufficiently large, e.g. $k > 10$) vector field $X$ (written as a formula in terms of elementary functions), a triangulation $\Delta_{\varepsilon}$, and an error tolerance $\delta$.

\textbf{Output:} A triangulation $\Delta'_{\varepsilon}$ that is a refinement of $\Delta_{\varepsilon}$, the images of $v_i \in \Delta_{\varepsilon}$ up to error $\delta$ under a $C^k$ diffeomorphism $\gamma_x$ corresponding to the flow along $X$, and a piecewise smooth Beltrami coefficient that approximates the one up to error $\delta$.

\textbf{Implementing TRIANG, BELTRAMI and VECT-FIELD:} We outline ways to implement the above three subroutines:

1. Given a set of $n$ points, we can obtain the Delaunay triangulation in $O(n \log n)$ time. While implementing \textsc{TRIANG}, we first compute the Delaunay triangulation of all the points falling inside a triangle of the given triangulation. The we connect the vertices on the convex hull of such a set of points to one of the three vertices of the triangle they lie in. If this complete triangulation is not yet size $M$, we make the mesh denser by adding points as in \cite{19} (points are added to either the circumcenters of triangles or mid-points of edges), until we reach the desired size.

2. The solution to the Beltrami equation for $\mu$ can be expressed as a series of singular operators applied to $\mu$. There are many efficient algorithms and implementations (\cite{6}, \cite{11}) existing for \textsc{BELTRAMI}. Most of them can bound the $\ell^p$ norm of the error, but the methods in \cite{6} can be used to bound the $L^\infty$ error too \cite{5}.

3. The idea of deforming a surface by a vector field has been applied extensively in computer graphics. We refer the reader to \cite{1} for an implementation.

\textbf{Description of EXTREMAL:} The algorithm summarized below is based on Section\textsuperscript{4.1}.

- Use \textsc{TRIANG} to produce a triangulation of size required by $\textsc{INEXT}$ to run within an error of $\delta^{10}$.

- Loop $i = 1$ to $N$ where $N$ is the number of iterations in Theorem\textsuperscript{6} to produce the result within an error of $\delta/2$.
  1. Use $\textsc{INEXT}$ to produce $\nu_i$ from $\mu_i$ within an error of $\delta^{10}$. If $\nu_i = \mu_i$ then stop.
  2. Find $t_i$ by Equation\textsuperscript{7}, using $k_0$ as $\|\nu_i\|_{\infty}$.  

3. Invoke BELTRAMI for the coefficient \( t_i(\mu_i - \nu_i) \) to find the images of the punctures within an accuracy of \( t_i^3 \).

4. Define the vector field \( X \) as in the continuous construction using a piecewise polynomial version of the bump function (that is \( C^{10} \) for instance). Then call VECT\-FIELD to find a piecewise constant Beltrami coefficient up to an error of \( t_i^3 \).

5. Use PIECEWISE\-COMP to compose the Beltrami coefficients of step 3 and step 4 within an error \( (\|\mu_i\| - \|\nu_i\|)^5 \) for the Beltrami coefficient and \( \delta/i^2 \) for the q.c.h.

6. Use PIECEWISE\-INV to find the Beltrami coefficient of the inverse of the q.c.h of step 5, up to the same error as that in step 5.

7. Call PIECEWISE\-COMP to compose \( \mu_i \) and the Beltrami coefficient of step 6 to form \( \mu_{i+1} \) (up to the same error as that in step 5).

The algorithm terminates by producing \( \mu_N \). The proof of Theorem 10 is similar to that of Theorem 6.

### 6 Discussions

We conjecture our algorithm to run in polynomial time. This is evidenced by the fact that 1) the number of iterations is a polynomial in \( 1/\varepsilon \), 2) INEXT (quadratic program), TRIANG, BELTRAMI and VECT\-FIELD run in polynomial time, and 3) we expect the existence of polynomial time subroutines PIECEWISE\-COMP and PIECEWISE\-INV.

Open problems abound. Apart from improving (complexity and approximation) the algorithm we propose, the extremal map problem can be further explored in many directions.

1. Most of the ideas presented here (notably Lemma 17) can be used to envision an algorithm for computing Teichmüller maps between arbitrary (finite analytic type) Riemann surfaces. The problem is challenging for multiple reasons—for instance, an explicit basis of holomorphic quadratic differentials may not be available.

2. The authors feel that building a discrete version of Teichmüller theory would be an important achievement. Given a triangulated Riemann surface, defining a discrete analog of dilatation that gives nice results (e.g. existence and uniqueness) about the extremal map would be the next step in this direction.

3. Most of the surfaces we see in everyday life can be regarded as Riemann surfaces. Being able to compute the "best" angle-preserving map between them is certainly of theoretical and practical interest. Our current efforts are aimed at being able to visualize geodesics in Teichmüller space. Seeing the base polygon (or Riemann surface) morphing (similar to what was accomplished in [20]) into the target polygon (surface) under the solution to \( t\mu_s \), would give us an idea of how shapes actually change while following a geodesic in this moduli space.

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Appendices

7 Appendix 1: Quasiconformal maps and essential theorems from Teichmüller Theory

All of the material in this section is classical and can be found in books on complex analysis and Riemann surfaces, such as [2][9][10][12].

7.1 Riemann mapping theorem and Riemann surfaces

Theorem 28 (Riemann Mapping). Let $\Omega$ be a simply connected domain in the complex plane $\mathbb{C}$, not equal to the entire complex plane. Then there exists a biholomorphic map $f : \mathbb{D} \rightarrow \Omega$. Further, $f$ is unique up to composition by a Möbius transformation.

That $f$ is biholomorphic implies it is conformal. One can therefore state as a corollary that any two simply connected domains in $\mathbb{C}$ (not equal to $\mathbb{C}$) can be mapped conformally and bijectively to each other.

Riemann surface  Let $M$ be a two dimensional real manifold. A complex chart on $M$ is a homeomorphism $\phi$ from an open subset $A \subset R$ to an open subset $B \subset \mathbb{C}$. Let $\phi_1 : A_1 \rightarrow B_1$ and $\phi_2 : A_2 \rightarrow B_2$ be two complex charts. $\phi_1$ and $\phi_2$ are said to be compatible if the map

$$
\phi_2 \circ \phi_1^{-1} : \phi_1(A_1 \cap A_2) \rightarrow \phi_2(A_1 \cap A_2)
$$

is biholomorphic.

A complex atlas on $M$ is a system of charts which cover $M$, and in which any two charts are compatible. Two complex atlases are regarded equivalent if all charts in the union of the atlases are pairwise compatible.

Definition 29 (Riemann surface). A Riemann surface $R$ is a pair $(M, \sigma)$, where $M$ is a connected two-manifold and $\sigma$ is an equivalence class of complex atlases on $M$.

Examples of Riemann surface include the complex plane, domains in the complex plane, the Riemann sphere $\hat{\mathbb{C}}$ and all Riemannian manifolds (oriented two-manifolds with a Riemannian metric).

Given two Riemann surfaces $M$ and $N$, a map $f : M \rightarrow N$ is conformal if its restriction on any local conformal parameters is holomorphic. Geometrically, a conformal map preserves angles, and transforms infinitesimal circles to infinitesimal circles, as shown in Figure 1 frame (a), (b) and (c).

7.2 Quasiconformal maps

A generalization of conformal maps are quasiconformal maps, which are orientation preserving homeomorphisms between Riemann surfaces with bounded conformality distortion, in the sense that their first order approximations takes small circles to small ellipses of bounded eccentricity, as shown in Figure 1 frame (d) and (e). Mathematically, $f : \mathbb{C} \rightarrow \mathbb{C}$ is quasiconformal provided that it satisfies the Beltrami equation:

$$
f_z = \mu(z)f_z.
$$

(14)

for some complex-valued function $\mu$ satisfying $||\mu||_\infty < 1$. $\mu$ is called the Beltrami coefficient, and is a measure of the non-conformality of $f$. In particular, the map $f$ is conformal around a small neighborhood
of $p$ when $\mu(p) = 0$. As shown in Figure 2, the orientation of the ellipse is double the argument of $\mu$. The dilatation of $f$ is defined as the ratio between the major axis and the minor axis of the infinitesimal ellipse. The maximal dilatation of $f$ is given either by:

$$k_f = ||\mu_f||_\infty,$$

or by

$$K(f) = \frac{1 + ||\mu||_\infty}{1 - ||\mu||_\infty}. (16)$$

A homeomorphism with dilatation less than or equal to $K$ is called a $K$-quasiconformal mapping.

A homeomorphism with dilatation less than or equal to $K$ is called a $K$-quasiconformal mapping.

![Figure 1: Conformal and quasiconformal mappings from a human face surface to the planar disk.](image1)

![Figure 2: Beltrami coefficient.](image2)

### 7.3 Quadratic differentials

**Definition 30** (Holomorphic quadratic differential). A holomorphic quadratic differential on a Riemann surface $R$ is an assignment of a function $\phi_i(z_i)$ on each chart $z_i$ such that if $z_j$ is another local coordinate, then $\phi_i(z_i) = \phi_j(z_j) \left( \frac{dz_j}{dz_i} \right)^2$.

We will denote the space of such differentials on $R$ as $A(R)$. By the Riemann-Roch theorem, the complex dimension of this vector space for a genus $g$ closed compact surface with $n$ punctures is $3g - 3 + n$. 

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Figure 3: Holomorphic quadratic differential bases on a pentagon.

Figure 4: Holomorphic quadratic differentials on a pentagon. (a) and (b) show \[0.2(\phi'_1)^2 + 0.8(\phi'_2)^2]dz^2\]. (c) and (d) show \[-0.2(\phi'_1)^2 + 1.2(\phi'_2)^2]dz^2\].

The Riemann surfaces of primary importance to us are the punctured Riemann sphere and the unit disk. For the unit disk, there is only one chart \(z\), and therefore any function holomorphic in the interior of the disk can be viewed as a quadratic differential (the transition condition is vacuous). For \(R = \hat{\mathbb{C}} \setminus \{0, 1, \infty, z_1, \ldots, z_{n-3}\}\) (the Riemann sphere with \(n\) punctures),

\[
\phi_i(z) = \frac{z_k(z_k - 1)}{z(z - 1)(z - z_k)}, \quad 1 \leq k \leq n - 3,
\]

form a basis of \((n - 3)\) dimensional complex vector space \(A(R)\).

Another vector space of importance to us is the space of polygon differentials. Let \(P\) be a polygon in the plane, normalized so that \(0, 1\) and \(\infty\) are three vertices of \(P\). Suppose \(\phi_k : P \rightarrow D_k\) is the conformal mapping, where \(D_k\) is a planar rectangle, such that \(\phi_k\) maps \(\{0, 1, \infty, z_k\}\) to the four corners of the rectangle \(D_k\). Then

\[
\{(\phi'_1)^2dz^2, (\phi'_2)^2dz^2, \ldots, (\phi'_{n-3})^2dz^2\}
\]

form the bases of \(A(R)\). As shown in Figure 4, the Riemann surface \(R\) is a pentagon with vertices \(\{z_1, z_2, z_3, z_4, z_5\}\), \(\phi_1\) maps \(R\) to planar rectangles \(R_1\), such that \(\{z_1, z_2, z_3, z_4\}\) are mapped to four corners. The checkerboard texture on \(R_1\) is pulled back to \(R\) and shown in (a). Similarly, \(\phi_2\) maps \(\{z_1, z_2, z_3, z_5\}\) to a rectangle \(R_2\). Then \(\{(\phi'_1)^2dz^2, (\phi'_2)^2dz^2\}\) form the bases of all holomorphic quadratic differentials on the pentagon. Figure 4 shows the linear combinations of these bases. Figures 5 and 6 show the bases and certain linear combinations of the bases on a hexagon, respectively.
Figure 5: Holomorphic quadratic differential bases on a hexagon.

Figure 6: Holomorphic quadratic differentials on a hexagon.

An excellent book for studying Quadratic differentials in further detail is [21].
7.4 Classical theorems used in our construction

We start with a theorem which explains the dependence of a Beltrami coefficient to the solution of its Beltrami equation.

**Mapping Theorem** [12], Theorem 1, Page 10] Let \( \mu(z) \) be a measurable complex-valued function defined on a domain \( \Omega \) for which \( ||\mu||_\infty = k < 1 \). Consider the Beltrami equation,

\[
f_{\bar{z}}(z) = \mu(z)f_z(z).
\]

Equation \(18\) gives a one to one correspondence between the set of quasiconformal homeomorphisms of \( \hat{\mathbb{C}} \) that fix the points 0, 1 and \( \infty \) and the set of measurable complex-valued functions \( \mu \) on \( \hat{\mathbb{C}} \) for which \( ||\mu||_\infty < 1 \). Furthermore, the normalized solution \( f^\mu \) of Equation \(18\) depends holomorphically on \( \mu \) and for any \( r > 0 \) there exists \( \delta > 0 \) and \( C(r) > 0 \) such that

\[
|f^{t\mu}(z) - z - tV(z)| \leq C(r)t^2 \text{ for } |z| < r \text{ and } |t| < \delta,
\]

where \( V(z) = \frac{z(z-1)}{\pi} \int \int_{\mathbb{C}} \frac{\mu(\zeta)d\zeta d\eta}{\zeta(\zeta-1)(\zeta - z)} \).

Composition of Quasiconformal Maps

Let \( \mu, \sigma \) and \( \tau \) be the Beltrami coefficients of quasiconformal maps \( f^\mu, f^\sigma \) and \( f^\tau \) with \( f^\tau = f^\sigma \circ (f^\mu)^{-1} \). Then

\[
\tau = \left( \frac{\sigma - \mu}{1 - \bar{\mu}\sigma} \right) \circ (f^\mu)^{-1},
\]

where \( p = \frac{\partial}{\partial z} f^\mu(z) \) and \( \theta = \frac{\zeta}{p} \). In particular, if \( f^\sigma \) is the identity, that is, if \( \sigma = 0 \), then

\[
\tau = -\left( \frac{\mu}{p} \right) \circ (f^\mu)^{-1}.
\]

The following lemma relates infinitesimally trivial Beltrami coefficients to globally trivial ones.

**Lemma 32.** [Variational lemma] [12], Theorem 6, Page 140] \( \mu \) is an infinitesimally trivial Beltrami differential if, and only if, there exists a curve \( \sigma_t \) of trivial Beltrami differentials for which \( \sigma_t(z) = t\mu_z + O(t^2) \) uniformly in \( z \).

Teichmüller contraction

The principle of Teichmüller contraction states that given a Beltrami coefficient \( \mu \), its distance to the globally extremal \( \mu^* \) is of the same order as its distance to the infinitesimally extremal \( \nu \). For a full statement and proof of the principle, see [12], Theorem 10, page 103.

We will restate the part of the principle relevant to us. Let \( k_0 = ||\mu^*||_\infty \) be the dilatation of the extremal Beltrami coefficient in the same global class as \( \mu \), and let \( \nu \) be the infinitesimally extremal Beltrami coefficient in the infinitesimal class of \( \mu \). Fix \( 0 < k_1 < 1 \). Then

\[
\frac{||\mu||_\infty - k_0}{4} \leq \frac{2}{(1-k_1)^2} (||\mu||_\infty - ||\nu||_\infty) \leq \frac{2}{(1-k_1)^4} (||\mu||_\infty - k_0).
\]

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Hamilton-Krushkal, Reich-Strebel condition

Theorem 33. [Hamilton-Krushkal, Reich-Strebel necessary-and-sufficient condition for extremality] A quasiconformal map $f$ has minimal dilatation in its Teichmüller class if and only if its Beltrami coefficient $\mu$ is extremal in its infinitesimal class.

8 Appendix for the discretization of the procedure

8.1 Formula for the integral of $\phi_i$ on a triangle $t_j$

Let $D$ be the triangle whose vertices are $\alpha, \beta, \gamma$ (in that order). Then

$$
\int_D \frac{1}{z(z-1)(z-a)} = \int_D \left( \frac{1}{az} + \frac{1}{(1-a)(z-1)} + \frac{1}{a(a-1)(z-a)} \right)
$$

$$
= \int_0^1 \int_0^u \left( \frac{1}{a(\alpha + u(\beta - \alpha) + v(\gamma - \beta))} + \frac{1}{(1-a)(\alpha - 1 + u(\beta - \alpha) + v(\gamma - \beta))} + \frac{1}{a(a-1)(\alpha - a + u(\beta - \alpha) + v(\gamma - \beta))} \right)dvdu
$$

$$
= \frac{1}{\gamma - \beta} \left[ I_1 + \frac{1}{1-a} I_2 + \frac{1}{a(a-1)} I_3 \right]
$$

$$
I_1 = \frac{1}{\gamma - \alpha} \left( \gamma \ln \gamma - \gamma - \alpha \ln \alpha + \alpha \right) - \frac{1}{\beta - \alpha} \left( \beta \ln \beta - \beta - \alpha \ln \alpha + \alpha \right)
$$

$$
I_2 = \frac{1}{\gamma - \alpha} \left( (\gamma - 1) \ln(\gamma - 1) - \gamma - (\alpha - 1) \ln(\alpha - 1) + \alpha \right)
$$

$$
- \frac{1}{\beta - \alpha} \left( (\beta - 1) \ln(\beta - 1) - \beta - (\alpha - 1) \ln(\alpha - 1) + \alpha \right)
$$

$$
I_3 = \frac{1}{\gamma - \alpha} \left( (\gamma - a) \ln(\gamma - a) - \gamma - (\alpha - a) \ln(\alpha - a) + \alpha \right)
$$

$$
- \frac{1}{\beta - \alpha} \left( (\beta - a) \ln(\beta - a) - \beta - (\alpha - a) \ln(\alpha - a) + \alpha \right)
$$

8.2 Details of the algorithm INEXT

$P(\mu)$ is the solution to the following program.

Program 34.

$$
\min ||\nu||_{\infty}
$$

subject to : $\int_R \nu \phi_i = \int_R \mu \phi_i \forall i \in \{1, 2...n-3\}$

Program[34] is an $L_\infty$ norm minimization subject to certain constraints. We will solve the above program when $\nu$ ranges over all piecewise constant Beltrami differentials. Let $\{\nu_i\}_{i=1}^T$, where $T$ is the number of triangles in the triangulation, be a basis of piecewise constant Beltrami differentials.

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Consider the $j$th triangle $t_j$; the integral of any basis element $\phi$ in Equation (1) can be computed analytically, and in a preprocessing step, we compute the matrix $A$ where $\int_{t_j} \phi_i = (a_{ij})$.

We write $\nu = \sum \lambda_i \nu_i$, and represent $\nu$ as a vector $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_T)$. Each $\lambda_k$ is a complex number; separating into real and imaginary parts we get $\lambda_k = \lambda_{rk} + i\lambda_{ik}$. Let the analogous vector for $\mu$ be $\lambda'$.

Then the above constraint can be written as $A\lambda = b$, where $b = A\lambda'$. Using the above separation into real and imaginary parts for the matrix $A = A_r + iA_i$ and $b = b_r + ib_i$, this is equivalent to

\textbf{Definition 35 (Constraints).}

\begin{align*}
A_r \lambda_r - A_i \lambda_i &= b_r \tag{24} \\
A_i \lambda_r + A_r \lambda_i &= b_i \tag{25}
\end{align*}

We introduce another variable $z \in \mathbb{R}^+$, making the number of variables $(2T + 1)$. Let the vector of unknowns be $\beta = (\lambda_{r1}, \cdots, \lambda_{rT}, \lambda_{i1}, \cdots, \lambda_{iT}, z)$.

\textbf{Program 36.}

\begin{equation*}
\min \quad z
\end{equation*}

subject to : Constraints [24] and [25] and \begin{align*}
\lambda^2_{rj} + \lambda^2_{ij} - z &\leq 0 \quad \forall 1 \leq j \leq T
\end{align*}

The last constraint uses the fact that the solution $\nu^*$ to Program 34 is of Teichmüller form. The objective function of Program 36 is linear in the unknown variables. Constraints [24] and [25] are also linear. The last set of constraints can be written as $\beta^t P_j \beta - z \leq 0$, where $P_j$ is a $(2T + 1)$ matrix of zeroes with its $(j, j)$th and $(T + j, T + j)$th entry being 1. $P_j$ has all but two eigenvalues as 0, and two eigenvalues are 1, implying that it is positive semi-definite.

Although solving a quadratically constrained quadratic program in general is NP-Hard, positive semi-definite instances of it are polynomial time solvable. Numerical solvers for these programs have been vastly studied, and efficient implementations exist. We refer the reader to Page 42 of [18] for a complete reference.

This completes the proof for Lemma 24.

\section{Holomorphic quadratic differentials on polygons}

Suppose $P$ is a polygon with vertices $\{z_0, z_1, \cdots, z_{n-1}\}$. For each $2 < k < n$, there exists a unique conformal map $\phi_k$, which maps $P$ to a rectangle $R = [0, 1] \times [0, h]$, and maps $z_0, z_1, z_2, z_k$ to the four vertices of $R$. Then $\{(\phi'_k)^2 dz^2\}$ form the basis of holomorphic quadratic differentials on $P$.

All the above proofs can be modified analogously for the polygon mapping problem with minimal effort.