Vanishing of the Bare Coupling in Four Dimensions

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\textbf{Abstract}

We examine two restructurings of the series relationship between the bare and renormalized coupling constant in dimensional regularization. In one of these restructurings, we are able to demonstrate via all-orders summation of leading and successive $\epsilon = 0$ (dimensionality = 4) poles that the bare coupling vanishes in the dimension-4 limit.

\footnotesize
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In the context of dimensional regularization [1] with minimal subtraction, the bare \((g_B)\) and renormalized \((g)\) coupling constants are related by a series of the form [2, 3]

\[
g_B = \mu^\epsilon g \left[ 1 + a_{1,1}g^2/\epsilon + a_{2,1}g^4/\epsilon^2 + a_{3,1}g^6/\epsilon + a_{3,2}g^6/\epsilon^2 + a_{3,3}g^6/\epsilon^3 + \ldots \right]
\]

\[
= \mu^\epsilon \sum_{\ell=0}^{\infty} \sum_{k=\ell}^{\infty} a_{k,\ell} g^{2k+1} \epsilon^{-\ell},
\]

where \(a_{k,0} \equiv \delta_{k,0}\) and \(\epsilon \equiv 2 - n/2\) in \(n\) dimensions. This double summation is meaningful provided \(g\) is sufficiently small and provided \(\epsilon^{-1}\) is finite. On the basis of eq. (1), however it is generally held that the bare coupling becomes infinite in the 4-dimensional limit (i.e. \(g_B \to \infty\) as \(\epsilon \to 0\)). We argue in this note that renormalization group (RG) methods may be used to show \(\lim_{n \to 4} g_B = 0\), a result consistent with asymptotic-freedom expectations in which the bare coupling is the renormalized coupling in the infinite cut-off limit.

We begin first by noting that the series (1) may be reorganized as follows:

\[
g_B = \mu^\epsilon g \sum_{n=0}^{\infty} g^{2n} S_n \left( g^2/\epsilon \right)
\]

where the functions \(S_n(u)\) have power series expansions

\[
S_n(u) \equiv \sum_{m=n}^{\infty} a_{m,m-n} u^{m-n}.
\]

In particular, the summation over leading order poles is just

\[
S_0 \left( g^2/\epsilon \right) = a_{0,0} + a_{1,1} g^2/\epsilon + a_{2,2} \left( g^2/\epsilon \right)^2 + \ldots,
\]

and that over next-to-leading poles is

\[
gS_1 \left( g^2/\epsilon \right) = a_{1,0} g + a_{2,1} g^3/\epsilon + a_{3,2} g^5/\epsilon^2 + \ldots.
\]

Explicit summation of the series \(S_n(u)\) is possible given full knowledge of the \(\beta\)-function characterising the scale dependence of the coupling constant \(g\). Since \(g_B\) is independent of
the mass-scale $\mu$ introduced for dimensional consistency [i.e. the action $\int d^nx \times \mathcal{L}$ must be dimensionless], we find that in $n$-dimensions [3]

$$
\mu \frac{dg_B}{d\mu} = 0 = \left( \mu \frac{\partial}{\partial \mu} + \tilde{\beta}(g) \frac{\partial}{\partial g} \right) g_B \\
= \epsilon g_B + \left( -\epsilon + \sum_{n=1}^{\infty} b_{2n+1} g^{2n} \right) g \frac{dg_B}{dg} 
$$

(6)

where $\tilde{\beta}(g)(= -\epsilon g + \beta(g))$ turns into the usual $\beta$ function series in the limit $\epsilon \to 0$:

$$
\mu \frac{dg}{d\mu} \equiv \tilde{\beta}(g) \longrightarrow \sum_{n=1}^{\infty} b_{2n+1} g^{2n+1}. 
$$

(7)

Upon substituting eq. (1) into eq. (6), we find that

$$
0 = \sum_{n=1}^{\infty} \left( \frac{1}{\epsilon} \right)^{n-1} \sum_{m=n}^{\infty} (-2m)a_{m,n} g^{2m+1} \\
+ \sum_{\ell=0}^{\infty} \left( \frac{1}{\epsilon} \right)^{\ell} \sum_{p=\ell}^{\infty} (2p+1)a_{p,\ell} \sum_{q=1}^{\infty} b_{2q+1} g^{2(p+q)+1}. 
$$

(8)

Such a relationship between $\beta$-function coefficients and coefficients of poles in eq. (1) has been noted by 't Hooft [2] and by Collins and Macfarlane [3]. In eq. (8), the aggregate coefficient of $g^{2\ell+1}(1/\epsilon)^{\ell-1}$ is just

$$
-2\ell a_{\ell,\ell} + b_3(2\ell - 1)a_{\ell-1,\ell-1} = 0, \ \ell \geq 1. 
$$

(9)

Ordinarily, one would use this equation to obtain $b_3$ from the calculated value of $a_{1,1}$ ($a_{0,0} = 1$, and $b_3 = 2a_{1,1}$). However, we see that this recursion relation also determines all coefficients $a_{m,m}$ within the summation (4) over leading-order poles. Similarly, the aggregate coefficient of $g^{2\ell+1}(1/\epsilon)^{\ell-2}$ within eq. (8),

$$
-2\ell a_{\ell,\ell-1} + b_3(2\ell - 1)a_{\ell-1,\ell-2} + b_5(2\ell - 3)a_{\ell-2,\ell-2} = 0, \ \ell \geq 2, 
$$

(10)

not only implies $b_5 = +4a_{2,1}$, ($a_{1,0} = 0$), but also determines all coefficients $a_{m,m-1}$ within the summation (5) of next-to-leading-order poles. Indeed, the aggregate coefficient of $g^{2\ell+1}(1/\epsilon)^{\ell-k}$
within eq. (8),

\[
-2\ell a_{\ell,\ell-k+1} + \sum_{q=1}^{k} b_{2q+1}(2\ell - 2q + 1)a_{\ell-q,\ell-k} = 0, \quad \ell \geq k \geq 1
\]  

(11)
serves as a recursion relation for the evaluation of \(S_{k-1}(u)\), as defined by the summation (3).

To evaluate explicitly the summations \(S_n\) within eq. (2), we begin by multiplying the recursion relation (9) by \(u^{\ell-1}\) and summing from \(\ell = 1\) to infinity:

\[
0 = -2 \sum_{\ell=1}^{\infty} \ell a_{\ell,\ell} u^{\ell-1} + b_3 \sum_{\ell=1}^{\infty} (2\ell - 1)a_{\ell-1,\ell-1} u^{\ell-1} \\
= -2 \frac{dS_0(u)}{du} + 2b_3 u \frac{dS_0(u)}{du} + b_3 S_0(u).
\]  

(12)
The final line of eq. (12) is obtained using the definition (4) for \(S_0(u)\). Moreover, we see from eq. (4) that \(S_0(0) = a_{0,0} = 1\), in which case the solution to the separable first-order differential equation (12) is just

\[
S_0(u) = (1 - b_3 u)^{-1/2}.
\]  

(13)
Similarly, we can obtain a differential equation for \(S_1(u)\) by multiplying the recursion relation (10) by \(u^{\ell-2}\) and then summing from \(\ell = 2\) to \(\infty\). One easily finds from the definitions (3) of \(S_0\) and \(S_1\) that

\[
2 (1 - b_3 u) \frac{dS_1}{du} + \left( \frac{2}{u} - 3b_3 \right) S_1 \\
= b_5 \left[ 2u \frac{dS_0}{du} + S_0 \right]
\]  

(14)
where \(S_0(u)\) is given by (13), and where \(S_1(0) = a_{1,0} = 0\). Upon making a change in variable to

\[
w = 1 - b_3 u,
\]  

(15)
one finds after a little algebra that

\[
\frac{dS_1}{dw} + \frac{(1 - 3w)}{2w(1 - w)} S_1 = -\frac{b_5}{2b_3} w^{-5/2},
\]  

(16)
with initial condition \( S_1|_{w=1} = S_1|_{u=0} = 0 \). The solution of this differential equation is

\[
S_1[w(u)] = -\frac{b_5}{2b_3w^{1/2}(w-1)} \left( \log w + \frac{1}{w} - 1 \right).
\]  

(17)

It is worthwhile to note that

\[
S_0 = w^{-1/2}
\]

(18)

and that as \( w \to \infty \),

\[
S_1 \to -\frac{b_5}{2b_3}w^{-3/2} \log(w).
\]

(19)

For an asymptotically-free theory \((b_3 < 0)\), the \( w \to \infty \) limit corresponds to the limit \( \epsilon \to 0^+ \) when \( u = g^2/\epsilon \). Thus the restriction \( w = 1 - b_3g^2/\epsilon > 0 \) on the domain of eq. (17) as a real function necessarily implies for asymptotically free theories that the dimensionality \( n = 4 - 2\epsilon \) approaches four from below.

Consider now the general relation (11), where the index \( k \) is taken to be greater than or equal to 2. If we multiply Eq. (11) by \( u^{\ell-k} \) and then sum from \( \ell = k \) to \( \infty \), we obtain the following differential equation via the definition (3):

\[
0 = -2 \sum_{\ell=k}^{\infty} \left[ (\ell - k + 1) + (k - 1) \right] a_{\ell,\ell-k+1} u^{\ell-k} + \sum_{q=1}^{k} b_{2q+1} \sum_{\ell=k}^{\infty} \left[ 2(\ell - k) + 2(k - q) + 1 \right] a_{\ell-q,\ell-k} u^{\ell-k}
\]

\[
= -2 \frac{dS_{k-1}}{du} - \frac{2(k-1)}{u} S_{k-1} + \sum_{q=1}^{k} b_{2q+1} \left[ 2u \frac{dS_{k-q}}{du} + (2(k-q) + 1)S_{k-q} \right].
\]

(20)

By letting \( k - 1 \to k \) and then making use of the change-of-variable (15), we obtain the following differential equation for \( S_k[w] \) with \( k \geq 2 \):

\[
\frac{dS_k}{dw} + \left[ \frac{(2k+1)w - 1}{2w(w-1)} \right] S_k
\]

\[
= -\frac{b_{2k+3}}{2b_3} w^{-5/2} - \frac{1}{2b_3} \sum_{n=1}^{k-1} \frac{b_{2n+3}}{w} \left[ 2(w - 1) \frac{dS_{k-n}}{dw} + (2k - 2n + 1)S_{k-n} \right].
\]

(21)

The first term on the right hand side of eq. (21) is obtained making explicit use of the expression (18) for \( S_0 \). Note that if \( k \geq 1 \), \( S_k(u = 0) = S_k[w = 1] = 0 \). Since \( S_0 \) and \( S_1 \)
are already known, one may solve the \( k = 2 \) case of eq. (21) for \( S_2 \), then use this solution within the \( k = 3 \) version of eq. (21) to solve for \( S_3 \), etc., so as to obtain all \( S_k \) explicitly.\(^1\) Given knowledge of \( \{S_{k-1}, S_{k-2}, \ldots, S_0\} \), one finds the solution to the differential equation (21) for \( S_k \) to be

\[
S_k[w] = -\frac{1}{2b_3} \int_1^w dr r^{1/2} (r - 1)^k \left[ \sum_{n=1}^{k} \frac{b_{2n+3}}{r} \left[ 2(r - 1) \frac{d}{dr} + 2(k - n) + 1 \right] S_{k-n}[r] \right] \frac{w^{1/2}}{w^{1/2}(w - 1)^k}.
\]

(22)

We note from eqs. (17) and (18) that

\[
\frac{1}{r} \left[ 2(r - 1) \frac{d}{dr} + 1 \right] S_0[r] = r^{-5/2}
\]

(23)

\[
\frac{1}{r} \left[ 2(r - 1) \frac{d}{dr} + 3 \right] S_1[r] = -\frac{b_5}{b_3} \left( r^{-5/2} + \mathcal{O}(r^{-7/2}) \right).
\]

(24)

Consequently, one easily finds from eq. (22) that in the large \( w \) limit

\[
S_2 \sim w^{-3/2}.
\]

(25)

Upon substituting this behaviour into Eq. (22) for the \( k = 3 \) case, we then find that as \( w \to \infty \), \( S_3 \sim w^{-3/2} \), in which case successive iterations of Eq. (22) in the large \( w \) limit necessarily reproduce this asymptotic behaviour, regardless of the index \( k \):

\[
S_k[w] \sim w^{-3/2}, \quad k \geq 2.
\]

(26)

We then see from eqs. (18), (19) and (26) that for all \( k \),

\[
\lim_{\epsilon \to 0} S_k \left( \frac{g^2}{\epsilon} \right) = \lim_{w \to \infty} S_k[w] = 0,
\]

(27)

which implies via eq. (2) that

\[
\lim_{\epsilon \to 0} g_B = 0.
\]

(28)

\(^1\) Such an iteration of solutions is possible only if all coefficients \( b_{2n+3} = 2(n+1)a_{n+1,1} \) are already known.
Recall in a cut-off regularization scheme that one defines the bare coupling \( g_B \) to be the infinite cut-off limit of the renormalized coupling \( g_R \):

\[
g_B = \lim_{\Lambda \to \infty} g_R(\Lambda) = 0. \tag{29}
\]

The result (28) confirms for asymptotically free theories that the infinite cut-off limit in four dimensions and the \( n = 4 \) limit within dimensional regularization are consistent.\(^2\) Moreover, we note that the results (18), (19) and (26), which lead to Eq. (28), are not contingent upon the sign of \( b_3 \); the bare coupling constant is seen to vanish in the four-dimensional limit even if the theory is not asymptotically free (\( \epsilon \to 0^- \)). Thus, the infinities which occur order-by-order in an expansion such as (1) disappear upon all-orders summation of the \( S_n(g^2/\epsilon) \) sub-series (3) within the reorganized expansion (2).\(^3\)

Finally, we note that the series (1) may be reorganized into a power series in \( \epsilon^{-1} \),

\[
g_B = \mu^\epsilon \sum_{k=0}^{\infty} B_k(g) \epsilon^{-k}, \tag{30}
\]

where

\[
B_0(g) = g, \tag{31}
\]

\[
B_k(g) = \sum_{\ell=k}^{\infty} a_{\ell,k} g^{2\ell+1}, \quad k \geq 1. \tag{32}
\]

\(^2\)We are grateful to V. A. Miransky for pointing out that the result (29) is formally correct for non-asymptotically-free theories (\( b_3 > 0 \)) as well. To one-loop order \( g^2_R(\Lambda) = g^2_R(\mu_0) / \left[ 1 - 2b_3 g^2_R(\mu_0) \log(\Lambda/\mu_0) \right] \). If \( b_3 > 0 \), \( g^2_R(\Lambda) \) still goes to zero as \( \Lambda \to \infty \). However, such large-\( \Lambda \) behaviour is on the unphysical \( (g^2_R(\Lambda) < 0) \) side of this expression’s Landau pole. Consequently, the infinite cut off limit is inappropriate for non-asymptotically free theories, as discussed in ref. [4]. An alternate discussion of how the bare coupling behaves in \( \phi^4_4 \) theory appears in ref. [9].

\(^3\)The use of RG-invariance to obtain such all-orders summations has been applied to other perturbative processes [5].
The RG-invariance of $g_B$ to changes in $\mu$ may be utilised to determine the coefficients $B_k(g)$ explicitly. The requirement that

$$0 = \mu \frac{dg_B}{d\mu} = \mu^\epsilon \left[ \epsilon B_0 + \sum_{k=1}^{\infty} B_k(g) \epsilon^{1-k} \right. + \left. \bar{\beta}(g) \sum_{k=0}^{\infty} \frac{dB_k}{dg} \epsilon^{-k} \right], \quad (33)$$

with $B_0 = g$ and $\bar{\beta}(g) = -\epsilon g + \beta(g)$, as before, leads to a differential recursion relation [6]

$$\left( \frac{d}{dg} - \frac{1}{g} \right) B_{k+1}(g) = \frac{1}{g} \beta(g) \frac{dB_k}{dg}. \quad (34)$$

Noting from eqs. (31) and (32) that $B_k(0) = 0$, we solve eq. (34) to obtain

$$B_{k+1}(g) = g \int_0^g \frac{\beta(s) B_k'(s)}{s^2} ds, \quad (35)$$

where $\beta(s) = \sum_{n=1}^{\infty} b_{2n+1} s^{2n+1}$ as before. We then find from eqs. (31) and (35) that

$$B_1(g) = \sum_{n=1}^{\infty} b_{2n+1} g^{2n+1}/2n, \quad (36)$$

$$B_2(g) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{b_{2n+1} b_{2k+1}(2k+1)}{4k(n+k)} g^{2(n+k+1)} = \frac{3b_5^2 g^5}{8} + \frac{11b_3 b_5 g^7}{24} + \left( \frac{8b_3 b_7}{3} + \frac{5b_5^2}{4} \right) \frac{g^9}{8} + \ldots, \quad (37)$$

$$B_3(g) = \frac{5b_5^2 g^7}{16} + \frac{7b_3 b_5 g^9}{12} + \ldots. \quad (38)$$

Of course, the process of iterating eq. (35) can be repeated to obtain $B_k(g)$ for arbitrarily large $k$, assuming as before that all $\beta$-function coefficients $b_{2n+1}$ are known. For example, the $\beta$-function for $N = 1$ supersymmetric Yang-Mills theory can be extracted to all orders via imposition of the Adler-Bardeen theorem on the anomaly supermultiplet [7], or via instanton calculus methods [8]; however, eq. (31) may no longer be valid since minimal subtraction is not explicit in either of these approaches. Alternatively, one can show via eq. (36) that terms of order $\epsilon^{-1}$ in eq. (1) are sufficient in themselves to determine $\beta(g)$ [see footnote 1], or even that $\beta$-function coefficients $b_{2\ell+1}$ can be extracted via Eq. (37) provided $g_B$ is known to order $g^{2\ell+3}/\epsilon^2$. 

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