Anti-regular graphs with loops and their spectrum

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February 21, 2022

Abstract

An anti-regular graph is a graph whose degree sequence has only two repeated entries. It is known that for each integer $n \geq 1$ there are two non-isomorphic anti-regular graphs on $n$ vertices and such graphs are examples of threshold graphs. We show that if one allows loops in a graph, then there are two non-isomorphic graphs whose degree sequence contains all distinct entries and we compute closed-form expressions for the adjacency eigenvalues of such graphs.

1 Introduction

Let $G = (V, E)$ be an $n$-vertex simple graph, that is, a graph without loops or multiple edges. If $n \geq 2$ then it is well-known that $G$ contains at least two vertices of equal degree. If all vertices of $G$ have equal degree then $G$ is called a regular graph. It is then natural to say that $G$ is anti-regular if $G$ has only two vertices of equal degree. If $G$ is anti-regular then the complement graph $\overline{G}$ is easily seen to be anti-regular, and moreover, if $G$ is connected then $\overline{G}$ is disconnected. It was shown in [4] that up to isomorphism, there is only one connected anti-regular graph on $n$ vertices and that its complement is the unique disconnected $n$-vertex anti-regular graph. We denote by $A_n$ the unique connected anti-regular graph on $n \geq 2$ vertices.

An anti-regular graph is an example of a threshold graph [12]. Threshold graphs were first studied by Chvátal and Hammer [6] (see also [8]) in the problem of aggregating a set of linear inequalities in integer programming problems. Threshold graphs have been studied extensively and have applications in parallel computing, linear programming, resource allocation problems, and psychology [12]. Threshold graphs are also employed as reasonable models of real-world social networks, friendship networks, peer-to-peer networks, and networks of computer programs [7, 9].

Within the family of threshold graphs, the anti-regular graph $A_n$ plays a prominent role. For instance, $A_n$ is uniquely determined by its independence polynomial [11]. It was shown in [14] that $A_n$ is universal for trees, that is, every tree graph on $n$ vertices is isomorphic to a subgraph of $A_n$. Also, the eigenvalues of the Laplacian matrix of $A_n$ are all distinct integers and the missing eigenvalue from $\{0,1,\ldots,n\}$ is $\lfloor (n+1)/2 \rfloor$. In [15], the characteristic and

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matching polynomial of $A_n$ are studied and several recurrence relations are obtained for these polynomials, along with some spectral properties of the adjacency matrix of $A_n$. Recently in [2], a fairly complete characterization of the eigenvalues of the adjacency matrix of $A_n$ was obtained. In particular, it was shown that the interval $\Omega = [-1, \sqrt{2}, -1+i\sqrt{2}]$ contains only the eigenvalues $-1$ or $0$ (depending on whether $n$ is even or odd) and that as $n \to \infty$, the eigenvalues are symmetric about $-\frac{1}{2}$ and are dense in $(-\infty, -1+i\sqrt{2}] \cup \{0, -1\} \cup [-1, \sqrt{2})$. It is conjectured in [2] that in fact the only eigenvalues of any threshold graph contained in $\Omega$ are $-1$ and/or $0$, which if true would improve the forbidden interval bound obtained in [10], and furthermore would show that the eigenvalues of $A_n$ are optimal when viewed within the family of threshold graphs.

In this note, we consider the anti-regular graph with loops. Random threshold graphs with loops have been considered in [11, 7, 9] where asymptotic properties such as the spectral distribution and the rank of the adjacency matrix were studied. In this paper, we show that if one allows loops in a graph then there are two non-isomorphic graphs whose vertex degrees are all distinct. Not surprisingly, these anti-regular graphs with loops have as their underlying simple graph the anti-regular simple graphs. Exploiting the structure of the adjacency matrix of an anti-regular graph, we perform similarity transformations that lead to closed-form expressions for the eigenvalues of the adjacency matrix of the anti-regular graphs with loops. As in the case of simple threshold graphs (i.e., with no loops), we conjecture that the spectral properties of anti-regular graph with loops will play an important role within the class of threshold graphs with loops. This will be considered in a future investigation.

2 The anti-regular graph with loops

By a graph with loops we mean a pair $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ is a finite vertex set and $E$ is a set containing 2-element multi-sets of $V$, that is, $E \subset \{(u, v) \mid u, v \in V\}$. Hence, the only extra freedom we allow from simple graphs is that some vertices may contain at most one self-loop. The degree of a vertex in a graph with loops is the number of edges incident to the vertex with loops counted once. As usual, if $G$ has no loops then we will call $G$ a simple graph.

A graph with loops $G = (V, E)$ is a threshold graph if there exists a vertex weight function $w : V \to [0, \infty)$ and $t \geq 0$, called the threshold, such that for every $X \subset V$, $X$ is an independent set if and only if $\sum_{v \in X} w(v) \leq t$. In resource allocation problems, the weight $w(v_i)$ is interpreted as the amount of resources used by vertex $v_i$ and thus a subset of the vertices $X$ is an admissible set of vertices if the total amount of resources needed by $X$ is no more than the allowable threshold $t$. Of the several characterizations of threshold graphs [12], the most convenient is a one-to-one correspondence between threshold graphs and binary sequences. Given a binary sequence $b = (b_1, b_2, \ldots, b_n) \in \{0, 1\}^n$, we use a recursive process using the join and union graph operations to construct a $n$-vertex graph $G = G(b)$ as follows. Start with a single vertex $v_1$ and add the loop $\{v_1, v_1\}$ if and only if $b_1 = 1$. Then add a new vertex $v_2$ and connect $v_2$ with all existing vertices $\{v_1, v_2\}$ if and only if $b_2 = 1$. Iteratively, if the vertices $\{v_1, v_2, \ldots, v_k\}$ have been added, add vertex $v_{k+1}$ and connect it to all existing vertices $\{v_1, v_2, \ldots, v_{k+1}\}$ if and only if $b_{k+1} = 1$. If $b_j = 1$ we call $v_j$ a dominating vertex; in this case there is a loop at $v_j$ and $v_j$ is adjacent to all vertices
Theorem 2.1. Theorem 2 in [4] where simple graphs are considered.

Lemma 2.1. \(d\) are two anti-regular graphs with loops on \(n\) vertices; one is the null graph with one vertex which we denote by \(H_1\) and the second is the graph with one vertex and one loop which we denote by \(G_1\). Thus, \(d(H_1) = (0)\) and \(H_1\) is disconnected, and \(d(G_1) = (1)\) and \(G_1\) is connected. If \(G\) is an anti-regular graph with loops on \(n \geq 2\) vertices with degree sequence \(d(G) = (d_1, d_2, \ldots, d_n)\) then if \(d_n = n\) then necessarily \(d_1 = 1\) and if \(d_1 = 0\) then necessarily \(d_n = n - 1\). We have therefore proved the following.

Lemma 2.1. If \(G\) is an anti-regular graph with loops on \(n \geq 1\) vertices then either \(d(G) = (1, 2, \ldots, n)\) or \(d(G) = (0, 1, \ldots, n - 1)\).

We now characterize the anti-regular graphs with loops. The proof is similar to that of Theorem 2 in [4] where simple graphs are considered.

Theorem 2.1. For each \(n \geq 1\) the following hold:

(i) There are two non-isomorphic anti-regular graphs with loops on \(n\) vertices, which we denote by \(G_n\) and \(H_n\).

(ii) The graphs \(G_n\) and \(H_n\) are complementary, and one of them is connected and the other is disconnected.

(iii) If \(G_n\) is the connected one, then \(d(G_n) = (1, 2, \ldots, n)\) and \(d(H_n) = (0, 1, \ldots, n - 1)\).

(iv) Both \(G_n\) and \(H_n\) are threshold graphs with loops whose binary sequences are alternating.

(v) The binary sequence of \(G_n\) begins with \(n \mod 2\).
(vi) The binary sequence of \( H_n \) begins with \((n + 1) \mod 2\) and is obtained by flipping the bits in the binary sequence of \( G_n \).

**Proof.** The proof is by induction. The base case \( n = 1 \) has been proved above. By induction, assume that we have proved the claim for some \( n \geq 1 \). Hence, \( G_n \) and \( H_n \) denote the anti-regular graphs with loops on \( n \) vertices, where \( G_n \) is connected, \( H_n \) is the complement of \( G_n \) and is disconnected, \( d(G_n) = (1, 2, \ldots, n) \), and \( d(H_n) = (0, 1, \ldots, n-1) \). The binary sequence of \( G_n \) begins with \( n \mod 2 \) and \( H_n \) has binary sequence beginning with \((n + 1) \mod 2\). Let \( G_{n+1} \) be the threshold graph obtained by adding the dominating vertex \( v_{n+1} \) to \( H_n \). Then clearly \( d(G_{n+1}) = (1, 2, \ldots, n, n + 1) \), \( G_{n+1} \) is connected, and \( G_{n+1} \) has binary sequence beginning with \((n + 1) \mod 2\). On the other hand, let \( H_{n+1} \) be the threshold graph obtained by adding the isolated vertex \( v_{n+1} \) to \( G_n \). Then clearly \( H_{n+1} \) is disconnected and after a relabelling of the vertices we have \( d(H_{n+1}) = (0, 1, 2, \ldots, n) \). The binary sequence of \( H_{n+1} \) begins with \( n \mod 2 \equiv (n + 1) + 1 \mod 2 \). By construction and the induction hypothesis, \( H_{n+1} \) is the complement of \( G_{n+1} \). If now \( G \) is an anti-regular graph with loops on \( n+1 \) vertices with \( d(G) = (1, 2, \ldots, n + 1) \) then the graph \( G - v_{n+1} \) has degree sequence \((0, 1, \ldots, n - 1)\) and therefore \( G \) is \( G_{n+1} \) by the induction hypothesis. On the other hand, if \( G \) has degree sequence \( d(G) = (0, 1, \ldots, n) \) then \( G - v_1 \) has degree sequence \((1, 2, \ldots, n)\) and thus \( G \) is \( H_{n+1} \) by the induction hypothesis. \( \square \)

### 3 The eigenvalues of anti-regular graphs with loops

In this section we show that, contrary to the case of the simple anti-regular graphs [2], the eigenvalues of the adjacency matrix of the anti-regular graph with loops have closed-form formulas. To that end we recall that \( G_n \) denotes the connected and \( H_n \) denotes the disconnected anti-regular graph with loops on \( n \geq 1 \) vertices. We begin with the following.

**Lemma 3.1.** Let \( G_n \) be the connected anti-regular graph with loops on \( n \geq 1 \) vertices. Then the adjacency matrix \( A(G_n) \) is permutation similar to the Hankel matrix

\[
M_n = \begin{bmatrix}
1 & & & & \\
& \ddots & & & \\
& & \ddots & & \\
& & & \ddots & \\
1 & 1 & \cdots & \cdots & 1
\end{bmatrix}
\]

**Proof.** We first note that for \( n \geq 3 \)

\[
M_n = \begin{bmatrix}
0 & 0^T & 1 \\
0 & M_{n-2} & 1 \\
1 & 1 & 1^T & 1
\end{bmatrix}
\]

where \( M_1 = 1 \). The proof of the lemma is by strong induction. For \( n = 1 \), we have \( A(G_1) = M_1 \) and for \( n = 2 \) we have \( A(G_2) = M_2 \), and so the claim is trivial for these cases.
Assume by strong induction that the claim is true for all $k \in \{1, \ldots, n\}$ where $n \geq 2$. Using the labelling of the vertices corresponding to the binary sequence of $G_{n+1}$ we have

$$A(G_{n+1}) = \begin{bmatrix} A(H_n) & 1 \\ 1^T & 1 \end{bmatrix}.$$  \hfill (1)

Similarly, using the labelling of the vertices corresponding to the binary sequence of $H_n$ we have

$$A(H_n) = \begin{bmatrix} A(G_{n-1}) & 0 \\ 0^T & 0 \end{bmatrix}.$$

It is not hard to see that $A(H_n)$ is permutation similar to

$$\begin{bmatrix} 0 & 0 \\ 0 & A(G_{n-1}) \end{bmatrix}.$$

By the induction hypothesis, $A(G_{n-1})$ is permutation similar to $M_{n-1}$ and therefore $A(H_n)$ is permutation similar to

$$\begin{bmatrix} 0 & 0 \\ 0 & M_{n-1} \end{bmatrix}.$$

Therefore, from (1) it follows that $A(G_{n+1})$ is permutation similar to

$$\begin{bmatrix} 0 & 0 \\ 0 & M_{n-1} \\ 1 & 1^T \end{bmatrix} = M_{n+1}.$$

\[\Box\]

We now determine the eigenvalues of $M_n$, and thus also of $A(G_n)$. First of all, it is straightforward to show by induction that

$$\det(M_n) = -\det(M_{n-2}) = (-1)^{(n-1)/2}.$$  

More importantly, by inspection we have

$$M_n^{-1} = \begin{bmatrix} -1 & 1 \\ \vdots & \vdots \\ \vdots & \vdots \\ -1 & \vdots \\ 1 & \vdots \end{bmatrix}.$$
If \( D = \text{diag}(1, -1, 1, \ldots, (-1)^{n+1}) \) then \( D^{-1} = D \) and it is straightforward to verify that

\[
Y_n = DM_n^{-1}D = \begin{bmatrix}
1 & 1 \\
& & \ddots & \\
& & & 1 & \\
1 & 1 & & & \\
& & & & & \end{bmatrix}.
\]

If \( n \) is even let \( \sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) be the permutation

\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & \cdots & \frac{n}{2} & \frac{n}{2} + 1 & \frac{n}{2} + 2 & \cdots & n - 1 & n \\
2 & 4 & 6 & \cdots & n & n - 1 & n - 3 & \cdots & 3 & 1
\end{pmatrix},
\]

that is, \( \sigma(k) = 2k \) for \( k = 1, 2, \ldots, \frac{n}{2} \) and \( \sigma(\frac{n}{2} + j) = n - (2j - 1) \) for \( j = 1, 2, \ldots, \frac{n}{2} \). If \( n \) is odd then let \( \sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) be the permutation

\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & \cdots & \frac{n-1}{2} & \frac{n-1}{2} + 1 & \cdots & n - 1 & n \\
2 & 4 & 6 & \cdots & n - 1 & n & n - 2 & \cdots & 3 & 1
\end{pmatrix}.
\]

Let \( P \) be the corresponding permutation matrix for \( \sigma \) such that \( Px = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \) for \( x \in \mathbb{R}^n \). A straightforward proof by induction shows that

\[
X_n = P^T Y_n P = (-1)^{n+1} \begin{pmatrix}
0 & 1 \\
1 & \ddots & & \\
& \ddots & \ddots & \ddots \\
& & \ddots & 0 & 1 \\
& & & 1 & 1
\end{pmatrix}.
\]

The eigenvalues of \( X_n \) are known explicitly [13, Theorem 2] and are given by

\[
\lambda_j = -2(-1)^n \cos \left( \frac{2j - 1}{2n + 1} \pi \right), \quad j = 1, 2, \ldots, n.
\]

As a consequence, we obtain the following.

**Theorem 3.1.** Let \( G_n \) be the connected anti-regular graph with loops on \( n \geq 1 \) vertices. The eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of the adjacency matrix of \( G_n \) are

\[
\lambda_j = \frac{(-1)^{n+1}}{2} \cos \left( \frac{2j - 1}{2n + 1} \pi \right), \quad j = 1, 2, \ldots, n.
\]

Consequently, the eigenvalues of \( G_n \) are simple.

Let \( \sigma(n) \) denote the set of the eigenvalues of \( G_n \). Then the eigenvalues of \( H_n \) are therefore \( \{0\} \cup \sigma(n-1) \), where \( \sigma(0) = \emptyset \). Finally, using the closed-form expressions for the eigenvalues of \( G_n \), the following is immediate.

**Corollary 3.1.** Let \( \sigma(n) \) denote the eigenvalues of the connected anti-regular graph with loops on \( n \geq 1 \) vertices. Then the closure of \( \bigcup_{n \geq 1} \sigma(n) \) is \( (-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty) \).
4 Acknowledgements

The authors acknowledge the support of the National Science Foundation under Grant No. ECCS-1700578.

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