Research Article

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On an equivalence between regular ordered \( \Gamma \)-semigroups and regular ordered semigroups

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Abstract: In this paper, we develop a technique which enables us to obtain several results from the theory of \( \Gamma \)-semigroups as logical implications of their semigroup theoretical analogues.

Keywords: ordered \( \Gamma \)-semigroup, regular ordered \( \Gamma \)-semigroup, ordered semigroup, regular ordered semigroup, quasi-ideal

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1 Introduction and preliminaries

The theory of \( \Gamma \)-semigroups has been around for more than three decades and counts hundreds of research papers and many PhD theses. Along with \( \Gamma \)-semigroups, other structures such as ordered \( \Gamma \)-semigroups and fuzzy \( \Gamma \)-semigroups have been studied in recent years. The majority of the results proved so far are \( \Gamma \)-analogues of the well-known results of ordinary semigroups which their authors pretend to be genuine generalizations of their semigroup counterparts. It should be noted that there is a striking similarity between the proofs of the original semigroup theorems and their \( \Gamma \)-semigroup analogues. It is this similarity that is causing a growing concern among \( \Gamma \)-skeptics that many of the results in \( \Gamma \)-semigroup theory are logically equivalent with their counterparts in ordinary semigroups. But so far there has been no evidence that this concern is mathematically based. The aim of this paper is to develop a technique whose purpose is to demonstrate the equivalence for a pair of analogue results from the two theories. This technique is a refinement of that developed in [1] and has the advantage that it works for regular \( \Gamma \)-semigroups endowed with a partial order. More specifically, given an ordered \( \Gamma \)-semigroup \((S, \Gamma, \leq_S)\), we construct an ordered semigroup \((\Omega, \cdot, \leq_\Omega)\) and prove that \( S \) is regular if and only if \( \Omega \) is regular. This shows that regularity in the theory of \( \Gamma \)-semigroups can be interpreted as the usual regularity of semigroups. We go on further to prove that two characterizations of regularity, one for ordered \( \Gamma \)-semigroups and the other for ordered semigroups are logically equivalent. The characterization of the regularity of ordered \( \Gamma \)-semigroups is Theorem 8(iii) of [2] and also Theorem 3 of [3], which states that an ordered \( \Gamma \)-semigroup \((S, \Gamma, \leq_S)\) is regular if and only if one-sided ideals of \((S, \Gamma, \leq_S)\) are idempotent, and for every right ideal \( R \) and every left ideal \( L \) of \((S, \Gamma, \leq_S)\), \((RL) \) is a quasi ideal of \((S, \Gamma, \leq_S)\). On the other hand, the characterization of the regularity of ordered semigroups is Theorem 3.1(iii) of [4], which states that an ordered semigroup \((S, \cdot, \leq_S)\) is regular if and only if, one-sided ideals of \((S, \cdot, \leq_S)\) are idempotent, and for every right ideal \( R \) and every left ideal \( L \) of...
(S, Γ, ≤), (RTL) is a quasi ideal of (S, ·, ≤). Proving that the above analogue theorems are equivalent gives points to the idea that producing Γ-analogues of known results from the semigroup theory brings nothing new to the theory as pretended, but simply replicates those results in a new setting.

In what follows, we give a few basic notions that will be used throughout the paper. Let S and Γ be two nonempty sets. Any map from \( S \times Γ \times S \) to S will be called a Γ-multiplication in S and is denoted by \((·)_Γ\). The result of this multiplication for \( a, b \in S \) and \( y \in Γ \) is denoted by \( ayb \). In 1986, Sen and Saha [5,6] introduced the concept of a Γ-semigroup \( S \) as an ordered pair \((S, (·)_Γ)\), where S and Γ are nonempty sets and \((·)_Γ\) is a Γ-multiplication on S, which satisfies the following property:

\[
\forall (a, b, c, a, β) \in S^3 \times Γ^3, (aab)βc = a(abβc).
\]

Here we give some necessary definitions from ordered semigroup and ordered Γ-semigroup theory. An ordered semigroup S is a semigroup \((S, ·)\) together with an order relation \( ≤ \) such that \( a ≤ b \) implies \( ac ≤ bc \) and \( ca ≤ cb \) for all \( c \in S \). An ordered semigroup S is called regular if for every \( a \in S \) there exists \( x \in S \) such that \( a ≤ axa \). Equivalently, if \( a \in (aSa) \) for every \( a \in S \) or if \( A ⊆ (ASA) \) for any subset \( A \) of S. A nonempty subset \( Q \) of an ordered semigroup S is called a quasi-ideal of S if (1) \( Q = Q \) and (2) \( (QS) \cap (SQ) \subseteq Q \).

An ordered Γ-semigroup (shortly po-Γ-semigroup) defined by Sen and Seth in [7] is a Γ-semigroup together with an order relation \( ≤ \) such that \( a ≤ b \) implies \( ayb ≤ byc \) and \( cya ≤ cyb \) for all \( c \in S \) and all \( y \in Γ \). A nonempty subset \( A \) of a po-Γ-semigroup S is called a right (resp. left) ideal of S if (1) \( ATS ⊆ A \) resp. \( (STA) ⊆ A \) and (2) if \( a \in A \) and \( S ⊇ b ≤ a \), then \( b \in A \). A right (left) ideal \( A \) can be obviously written as \( A = A \). An ordered Γ-semigroup S is called regular if for every \( a \in S \) there exist \( x, y \in S \) and \( y_1, y_2 \in Γ \) such that \( a ≤ ay_1yx_2a \). A po-Γ-semigroup S is regular if and only if \( a \in (aSTa) \) for all \( a \in S \), equivalently, if \( A ⊆ (AΓSTA) \) for all \( A ⊆ S \). A nonempty subset \( Q \) of an ordered Γ-semigroup S is called a quasi-ideal of S if (1) \( Q = Q \) and (2) \( (QS) \cap (SQ) \subseteq Q \).

2 Construction of \( Ω_{γ_0} \)

Given an ordered Γ-semigroup \((S, Γ, ≤S)\), we define an ordered semigroup \((Ω_{γ_0}, ·, ≤_{Ω_{γ_0}})\). To define \( Ω_{γ_0} \) we use the fact that we can always define a multiplication \( · \) on any nonempty set \( Γ \) in such a way that \((Γ, ·)\) becomes a group. This in fact is equivalent to the axiom of choice. Also, we use the concept of the free product of two semigroups. Material related to this concept can be found in [8, pp. 258–261]. Furthermore, let \((F; ·)\) be the free semigroup on S. Its elements are finite strings \((x_1,...,x_n)\), where each \( x_j \in S \) and the product \( · \) is the concatenation of words. Now we define \( Ω_{γ_0} \) as the quotient semigroup of the free product \( F∗Γ \) of \((F; ·)\) with \((Γ, ·)\) by the congruence generated from the set of relations

\[
((x, y), (x_0y), ((x, y, y), (x, y, y))
\]

for all \( x, y ∈ S, y ∈ Γ \) and with \( γ_0 ∈ Γ \) a fixed element. The following is Lemma 2.1 of [9]. We have included here for convenience. Readers unfamiliar with rewriting systems can find anything necessary to understand the proof in the monograph [10].

**Lemma 2.1.** Every element of \( Ω_{γ_0} \) can be represented by an irreducible word which has the form \((y, x, y')\), \((y, x)\), \((x, y)\), \((y, x)\), \((x, y), y \) or \( x \), where \( x ∈ S \) and \( y, y' ∈ Γ \).

**Proof.** First, we have to prove that the reduction system arising from the given presentation is Noetherian and confluent, and therefore any element of \( Ω_{γ_0} \) is given by a unique irreducible word from \( S ∪ Γ \). Second, we have to prove that the irreducible words have one of these five forms. So if \( ω \) is a word of the form \( ω = (u, x, y, v) \) for \( y ∈ Γ \), \( x, y ∈ S \) and \( u, v \) possibly empty words, then \( ω \) reduces to \( ω' = (u, x, y, v) \). And if \( ω = (u, x, y, v) \), then it reduces to \( ω' = (u, x, y, v) \). In this way, we obtain a reduction system which is length reducing and therefore it is Noetherian. To prove that this system is confluent, from Newman’s lemma, it is sufficient to prove that it is locally confluent. For this, we need to see only the overlapping pairs.
1. \((x, y, z) \to (x y_0 y, z)\) and \((x, y, z) \to (x, y y_0 z)\) which both reduce to \((x y_0 y_0 z)\);
2. \((x, y, y, z) \to (x y y, z)\) and \((x, y, y, z) \to (x, y y y_0 z)\) which both reduce to \((x y y y_0 z)\);
3. \((x, y, y, z) \to (x y y, y, z)\) and \((x, y, y, z) \to (x, y y z)\) which both reduce to \((x y y y z)\);
4. \((y, y, y, z) \to (x y y, y', z)\) and \((x, y, y', y, z) \to (x, y, y y' z)\) which both reduce to \((x y y y' z)\);
5. \((y, y, y, y) \to (y, y, y', y)\) and \((y, y, y, y) \to (y, y y y' y)\) which both reduce to \((y y y' y)\).

To complete the proof, we need to show that the irreducible word representing the element of \(\Omega_{0, y}\) has one of the five forms stated. If the word which has neither a prefix nor a suffix made entirely of letters from \(\Gamma\), then it reduces to an element of \(S\) by performing the appropriate reductions. If the word has the form \((\alpha, \omega, \alpha'),\)(\(\alpha, \omega\)) or \((\alpha, \alpha')\), where \(\omega\) is a word which has neither a prefix nor a suffix made entirely of letters from \(\Gamma\), and \(\alpha, \alpha'\) have only letters from \(\Gamma\), then it reduces to an element of one of the first three forms. \(\square\)

**Definition 2.1.** We define an order relation \(\leq_{\Omega_{0, y}}\) in terms of \(\leq_S\) as follows:

1. For every \(x, y \in S\), we let \(x \leq_{\Omega_{0, y}} y \iff x \leq_S y\);
2. For every \(x, y \in S\), and \(y \in \Gamma\) we let \(x y \leq_{\Omega_{0, y}} y y \iff x \leq_S y\);
3. For every \(x, y \in S\), and \(y \in \Gamma\) we let \(x y \leq_{\Omega_{0, y}} y z \iff x \leq_S y \leq_S z\);
4. For every \(x, y \in S\), and \(y \in \Gamma\) we let, \(x y y' \leq_{\Omega_{0, y}} y y y' \iff x \leq_S y\);
5. The restriction of the relation in \(\Gamma\) is taken to be the equality.

Using the fact that \(\leq_S\) is an order relation in the \(\Gamma\)-semigroup \(S\), we can prove that \(\leq_{\Omega_{0, y}}\) is an order relation in the semigroup \(\Omega_{0, y}\). It is obvious that \(\leq_{\Omega_{0, y}}\) is reflexive, and very easy to see that it is antisymmetric. We check for convenience the transitivity.

1. If \(x, y, z \in S\) are such that \(x \leq_{\Omega_{0, y}} y\) and \(y \leq_{\Omega_{0, y}} z\), then by the definition, \(x \leq_S y\) and \(y \leq_S z\), hence \(x \leq_S z\) because \(\leq_S\) is transitive, and then \(x \leq_{\Omega_{0, y}} z\).
2. If \(x, y, z \in S\) and \(y \in \Gamma\) are such that \(x y \leq_{\Omega_{0, y}} y y\) and \(y y \leq_{\Omega_{0, y}} y z\), then \(x \leq_S y\) and \(y \leq_S z\), hence \(x \leq_S z\) and as a result \(x y \leq_{\Omega_{0, y}} y z\).
3. If \(x, y, z \in S\) and \(y \in \Gamma\) are such that \(x y \leq_{\Omega_{0, y}} y y\) and \(y y \leq_{\Omega_{0, y}} y z\), then similarly with above \(x \leq_S y\) and \(y \leq_S z\), hence \(x \leq_S z\), consequently \(x y \leq_{\Omega_{0, y}} y z\).
4. If \(x, y, z \in S\) and \(y \in \Gamma\) are such that \(x y y' \leq_{\Omega_{0, y}} y y y'\) and \(y y y' \leq_{\Omega_{0, y}} y z y'\), then \(x \leq_S y\) and \(y \leq_S z\), hence \(x \leq_S z\), consequently \(x y y' \leq_{\Omega_{0, y}} y z y'\).
5. If \(x, y \in \Gamma\) are such that \(x \leq_{\Gamma} y\), then \(x = y\).

Next we prove that the compatibility of \(\leq_S\) in \(S\) implies that of \(\leq_{\Omega_{0, y}}\) in \(\Omega_{0, y}\). We obtain the proof only for relations of type (4) of Definition 2.1 since the proofs for the other types are analogous. So let \(x y y' \leq_{\Omega_{0, y}} y y y'\), and want to prove that the inequality is preserved after multiplying both sides of the above on the left (resp. on the right) by one of the following elements: \(x \in \Gamma\), \(z \in S\), \(a z \in \Gamma S\), \(z b \in S \Gamma\), \(a z b \in \Gamma S\). Since the proofs for the compatibility on the right are symmetric to those on the left, we obtain them only for the left multiplication.

1. \((a)(x(y y')) \leq_{\Omega_{0, y}} a(y y') \iff (a \cdot y) x y' \leq_{\Omega_{0, y}} (a \cdot y) x y \iff x \leq_S y \iff y x y' \leq_{\Omega_{0, y}} x y y'.
2. \((z)(x(y y')) \leq_{\Omega_{0, y}} z(y y') \iff (z)(x)(y y') \leq_{\Omega_{0, y}} (z)(x)(y y') \iff z y x \leq_S z y y,\) where the latter is true since \(x \leq_S y\).
3. \((a)(x)(y y') \leq_{\Omega_{0, y}} (a)(x)(y y') \iff (a)(z)(y y') \leq_{\Omega_{0, y}} (a)(z)(y y') \iff z y x \leq_{\Omega_{0, y}} z y y,\) where the latter is true since \(x \leq_S y\).
4. \((z)(b)(y x y') \leq_{\Omega_{0, y}} (z)(b)(y x y') \iff (z)(b)(y x y') \leq_{\Omega_{0, y}} (z)(b)(y x y) \iff z y x \leq_{\Omega_{0, y}} z(b \cdot y) y,\) where the last inequality is true since \(x \leq_S y\).
5. \((a)(b)(y y') \leq_{\Omega_{0, y}} (a)(b)(y y') \iff (a)(z)(b \cdot y) x \leq_{\Omega_{0, y}} (a)(z)(b \cdot y) x \leq_{\Omega_{0, y}} z(b \cdot y) x \leq_{\Omega_{0, y}} z(b \cdot y) y,\) where the last inequality is true since \(x \leq_S y\).

Therefore, \(\leq_{\Omega_{0, y}}\) is compatible with the multiplication of \(\Omega_{0, y}\). Summarizing, we have the following.

**Proposition 2.1.** The triple \((\Omega_{0, y}, \cdot, \leq_{\Omega_{0, y}})\) is an ordered semigroup.
Since in Section 3 we deal with ordered ideals in both structures, \((S, \Gamma, \leq_S)\) and \((\Omega_I, \mathcal{R}, \leq_{\Omega_I})\), we will not use the standard notation \(X\) to indicate the ordered ideal, but we introduce a new one as in the following definition.

**Definition 2.2.** For \(C \subseteq S\), we define \(L_C(D) = \{x \in S : \exists c \in C \text{ such that } x \leq_S c\}\), and for every \(D \subseteq \Omega_I\), \(L_{\Omega_I}(D) = \{w \in \Omega_I : \exists d \in D \text{ such that } w \leq_{\Omega_I} d\}\).

We remark by passing the following.

**Lemma 2.2.** \(L_{\Omega_I}(C) = L_\Gamma(C)\) for \(C \subseteq S\).

**Proof.** First, we prove that \(L_\Gamma(C) \subseteq L_{\Omega_I}(C)\). If \(y \in L_\Gamma(C)\), then \(y \leq_S c\) for \(c \in C\). Since \(y, c \in S\), then \(y \leq_S c\) by Definition 2.1 is equivalent to \(y \leq_{\Omega_I} c\), hence \(y \in L_{\Omega_I}(C)\).

Conversely, if \(\omega \in L_{\Omega_I}(C)\), then \(\omega \leq_{\Omega_I} c\) for \(c \in C\) and by Definition 2.1 we must have that \(\omega \in S\), and that \(\omega \leq_S c\), proving that \(\omega \in L_\Gamma\).

The following lemma gives a relationship between the principal ordered ideal in \(S\) generated by some \(x \in S\) and the principal ordered ideal in \(\Omega_I\) generated by the same element \(x\).

**Lemma 2.3.** Let \(x \in S\) be an arbitrary element. The following hold true.

(i) The principal left ordered ideal of \(\Omega_I\) generated by \(x\) is the set \((x)_l^{\Omega_I} = (x)_l^S \cup \Gamma(x)_l^S\), where \((x)_l^S = L_\Gamma(x \cup STx)\) is the left ordered ideal of \(S\) generated by \(x\) and \(\Gamma(x)_l^S = \{yy' | y \in \Gamma, y \in L_\Gamma(STx \cup x)\}\).

(ii) The principal right ordered ideal of \(\Omega_I\) generated by \(x\) is the set \((x)_r^{\Omega_I} = (x)_r^S \cup (x)_r^\Gamma\), where \((x)_r^S = L_\Gamma(xTS \cup x)\) is the right ordered ideal of \(S\) generated by \(x\) and \((x)_r^\Gamma = \{yy' | y \in \Gamma, y \in L_\Gamma(xTS \cup x)\}\).

**Proof.** We obtain the proof for (i) since the proof for (ii) is dual to that of (i). So we have to prove that \(A \in (x)_l^{\Omega_I}\) if and only if \(A \in (x)_l^S \cup \Gamma(x)_l^S\). Indeed, if \(A \in (x)_l^{\Omega_I}\), then \(A \subseteq_{\Omega_I} B\), where \(B \in L_{\Omega_I}(\Omega_I x \cup x)\). If \(B = x\), we have \(A \subseteq_{\Omega_I} x\) and by Definition 2.1 \(A \leq_S x\), so \(A \in (x)_l^S\). If \(B \in \Omega_I x\), then \(B\) may have these forms:

1. \(B = (ayb) x = a(ybx)\). In this case, \(A \subseteq_{\Omega_I} a(ybx)\) which forces \(A = axz\), where \(z \leq_S ybx\), hence \(z \in (x)_l^S\) and then \(A \in (x)_l^S\).
2. \(B = (ayb) x = a(yb') x\). In this case, \(A \subseteq_{\Omega_I} a(yb') x\). One can see that in the same way as above, \(A \in \Gamma(x)_l^S\).
3. \(B = (yax) x = yax\). In this case, \(A \subseteq_{\Omega_I} yax\) and by definition 2.1 we have that \(A \subseteq_S yax\), therefore, \(A \in (x)_l^S\).
4. \(B = (ax) x = ax\). In this case, \(A \subseteq_{\Omega_I} ax\), where \(z \leq_S x\), hence \(z \in (x)_l^S\) and then \(A \in \Gamma(x)_l^S\).

Conversely, if \(A \in (x)_l^S \cup \Gamma(x)_l^S\), then either \(A \leq_S B\) where \(B \in (x)_l^S\) or \(A \leq_{\Omega_I} B\), where \(B = aC\) with \(a \in \Gamma\) and \(C \subseteq (x)_l^\Gamma\). In the first case, it follows at once that \(A \in (x)_l^{\Omega_I}\). In the second case, when \(B = aC\) and \(C \subseteq (x)_l^S\), the inequality \(A \leq_{\Omega_I} aC\) implies that \(A = aA'\) with \(A' \subseteq_S C\), and as a consequence \(A' \subseteq (x)_l^S\). Thus, \(A \in \Gamma(x)_l^S\).

\[\square\]

### 3 Regularity in ordered \(\Gamma\)-semigroups as a consequence of regularity in ordered semigroups

The following proposition shows that the regularity of an ordered \(\Gamma\)-semigroup can be completely characterized as the regularity of an ordered semigroup.
Proposition 3.1. S is a regular ordered \( \Gamma \)-semigroup if and only if \( \Omega_{\kappa} \) is a regular ordered semigroup.

Proof. If \( S \) is a regular ordered \( \Gamma \)-semigroup, then for all \( a \in S \), \( \exists x \in S \) and \( y_1, y_2 \in \Gamma \), such that \( a \leq_S ay_1xy_2a \). To prove \( \Omega_{\kappa} \) is a regular ordered semigroup, we have to prove that every element of \( \Omega_{\kappa} \) have an ordered inverse in \( \Omega_{\kappa} \). By Lemma 2.1, we have that the elements of \( \Omega_{\kappa} \) can be represented by an irreducible word which has only five forms. We prove regularity for elements of each of these five forms. So let first \( a_\alpha a_\alpha \in \Omega_{\kappa} \). To find its ordered inverse we take \( a \leq_S ay_1xy_2a \) and then by Definition 2.1 we have \( a_\alpha a_\alpha \leq_{\Omega_{\kappa}} a_\alpha ay_1xy_2a \) which tells us that \( a_\alpha a_\alpha \) is regular in \( \Omega_{\kappa} \), and \( a_\alpha^{-1}y_1xy_2a_\alpha^{-1} \in \Omega_{\kappa} \) is its ordered inverse. Second, for showing that \( aa \in \Omega_{\kappa} \) is regular, we take \( a \leq_S ay_1xy_2a \) and then by Definition 2.1 we can write \( aa \leq_{\Omega_{\kappa}} a(ay_1xy_2a) = (aa)(ay_1xy_2a) \). This tells us that \( aa \) is regular and as its inverse we can take \( y_1xy_2a^{-1} \in \Omega_{\kappa} \). Hence, we showed that \( \Omega_{\kappa} \) is regular ordered semigroup. Conversely, if \( \Omega_{\kappa} \) is regular ordered semigroup, then every \( a \in S \) has an inverse in \( \Omega_{\kappa} \). To show that \( S \) is a regular ordered \( \Gamma \)-semigroup, we show that every \( a \in S \) has an inverse in \( S \). For this, we distinguish between the five following forms. First, if the inverse of \( a \) in \( \Omega_{\kappa} \) has the form \( ax^\beta \in \Omega_{\kappa} \), for \( x \in S \), then \( a \leq_{\Omega_{\kappa}} ax^\beta \) by Definition 2.1 implies that \( a \leq_S ax^\beta a \), showing that \( a \) is regular in \( S \). Second, if the inverse of \( a \) in \( \Omega_{\kappa} \) has the form \( ax \), then \( a \leq_{\Omega_{\kappa}} a(ax)a = aax^0a \), and then Definition 2.1 implies that \( a \leq_S aax^0a \), which means that \( a \) is regular in \( S \). Third, the inverse of \( a \) in \( \Omega_{\kappa} \) has the form \( ax \), this case is similar to the second case. Fourth, the inverse of \( a \) in \( S \) is \( x \), then \( a \leq_{\Omega_{\kappa}} axa = a_\alpha x_\alpha a_\alpha \) and by Definition 2.1 \( a \leq_{\Omega_{\kappa}} a_\alpha x_\alpha a_\alpha \), which means that \( a \) is regular in \( S \). Finally, if the inverse have the form \( a \in \Gamma \), then \( a \leq_{\Omega_{\kappa}} aya \) and by Definition 2.1 we have that \( a \leq_S ayaya \) and \( a \) is regular in the ordered \( \Gamma \)-semigroup \( S \).

Before we prove our main theorem we need this technical lemma which is the analogue of implication (iii) \( \Rightarrow \) (ii) of Theorem 9.4 of [11].

Lemma 3.1. If \( (S, \Gamma, \leq_S) \) is an ordered \( \Gamma \)-semigroup such that one-sided ideals of \( (S, \Gamma, \leq_S) \) are idempotent, and for every right ideal \( R \) and every left ideal \( L \) of \( S \), \( L \leq_S \Gamma \Gamma \) is a quasi ideal of \( (S, \Gamma, \leq_S) \), then for every \( a \in S \), \( (a)^{\leq_S} \cap (a)^{\leq_S} = L\delta((a)^{\leq_S}) \Gamma(a)^{\leq_S}) \).

Proof. Observe first that
\[
(a)^{\leq_S} = L\delta(a \cup a \Gamma S) = L\delta(a \cup a \Gamma S) \Gamma L\delta(a \cup a \Gamma S) \subseteq L\delta(a \Gamma S) \subseteq (a)^{\leq_S},
\]
from which we derive that \( (a)^{\leq_S} \subseteq L\delta(a \Gamma S) \). In a similar fashion, one can get that \( (a)^{\leq_S} = L\delta(S \Gamma a) \). Furthermore, since
\[
L\delta((a)^{\leq_S}) \Gamma(a)^{\leq_S}) = L\delta((a)^{\leq_S}) \Gamma L\delta(S \Gamma a)
\]
is a quasi ideal, we have
\[
L\delta(L\delta(a \Gamma S) \Gamma L\delta(S \Gamma a) \Gamma S) \cap L\delta(S \Gamma L\delta(L\delta(a \Gamma S) \Gamma L\delta(S \Gamma a) \Gamma S)) \subseteq L\delta(L\delta(a \Gamma S) \Gamma L\delta(S \Gamma a))
\]
From this and the previous assumptions, we obtain
\[
(a)^{\leq_S} \cap (a)^{\leq_S} = L\delta(a \Gamma S) \cap L\delta(S \Gamma a)
= (L\delta(a \Gamma S) \Gamma L\delta(a \Gamma S) \Gamma L\delta(a \Gamma S)) \cap (L\delta(S \Gamma a) \Gamma L\delta(S \Gamma a) \Gamma L\delta(S \Gamma a))
\subseteq L\delta(L\delta(a \Gamma S) \Gamma L\delta(S \Gamma a) \Gamma S) \cap L\delta(S \Gamma L\delta(L\delta(a \Gamma S) \Gamma L\delta(S \Gamma a) \Gamma S))
\subseteq L\delta(L\delta(a \Gamma S) \Gamma L\delta(S \Gamma a))
= L\delta((a)^{\leq_S} \Gamma(a)^{\leq_S}),
\]
proving thus the nonobvious inclusion \( (a)^{\leq_S} \cap (a)^{\leq_S} \subseteq L\delta((a)^{\leq_S} \Gamma(a)^{\leq_S}) \).
Theorem 3.1. The following are logically equivalent.
(i) An ordered $Γ$-semigroup $(S, Γ, ≤_S)$ is regular if and only if, one-sided ideals of $(S, Γ, ≤_S)$ are idempotent, and for every right ideal $R$ and every left ideal $L$ of $(S, Γ, ≤_S)$, $L_S(RL)$ is a quasi ideal of $(S, Γ, ≤_S)$.
(ii) An ordered semigroup $(S, ·, ≤_S)$ is regular if and only if, one-sided ideals of $(S, ·, ≤_S)$ are idempotent, and for every right ideal $R$ and every left ideal $L$ of $(S, Γ, ≤_S)$, $L_S(RL)$ is a quasi ideal of $(S, ·, ≤_S)$.

Proof. (i) $⇒ (ii)$ is trivial since any regular ordered semigroup can be regarded as an regular ordered $Γ$-semigroup, where $Γ$ is a singleton. Also, one-sided ideals and quasi ideals in ordered semigroups are the same as those in ordered $Γ$-semigroups when $Γ$ is a singleton.

(ii) $⇒ (i)$. Assume first that the ordered $Γ$-semigroup $S$ is regular. Then by Proposition 3.1, $Ω_{g_0}$ is a regular ordered semigroup. Theorem 3.2 of [4] implies that for every right ideal $R$ and every left ideal $L$ of $Ω_{g_0}$, $R \cap L = L_{Ω_{g_0}}(RL)$, $L_{Ω_{g_0}}(RL)$ is a quasi-ideal of $Ω_{g_0}$, and also every right and left ideal of the semigroup $Ω_{g_0}$ is idempotent. Let $A$ now be an ordered right ideal of $S$ and consider the subset $R = L_{Ω_{g_0}}(A \cup AΓ)$ of $Ω_{g_0}$. We show that $R = L_{Ω_{g_0}}(A \cup AΓ)$ is a right ideal of the ordered semigroup $Ω_{g_0}$. To this end, we have to prove that it satisfies the two conditions of right ideals: (1) $L_{Ω_{g_0}}(A \cup AΓ)Ω_{g_0} \subseteq L_{Ω_{g_0}}(A \cup AΓ)$, (2) if $a \in L_{Ω_{g_0}}(A \cup AΓ)$ and $b \leq_{Ω_{g_0}} a$, then $b \in L_{Ω_{g_0}}(A \cup AΓ)$. To show the first we have to show that for every $b \in L_{Ω_{g_0}}(A \cup AΓ)$ and every $C \in Ω_{g_0}$, $bC \in L_{Ω_{g_0}}(A \cup AΓ)$. Since $b \in L_{Ω_{g_0}}(A \cup AΓ)$, either $b \leq_{Ω_{g_0}} a$ where $a \in A$, or $b \leq_{Ω_{g_0}} ay$, in which case $b = a'y$ and $a' \leq_S a$. In the first case, when $a, b \in S$ and $b \leq_S a$, it follows immediately that $bC \leq_{Ω_{g_0}} aC$ whatever the value of $C$ is, since $≤_{Ω_{g_0}}$ is a compatible relation. But still we have to prove that $aC \in L_{Ω_{g_0}}(A \cup AΓ)$ and this depends on the value of $C$. Since $C \in S \cup ΓS \cup ST \cup ΓS \cup Γ$, then

$$aC \in aS \cup aΓS \cup aST \cup aΓS \cup aΓ \subseteq aΓS \cup aΓS \cup aΓ \subseteq L_{Ω_{g_0}}(A \cup AΓ),$$

where the last inclusion comes from the fact that $A$ is an ordered right ideal of $(S, Γ, ≤_S)$. It remains to prove that the same holds true in the second case when $b = a'y$ with $a' \in S$ and $a' \leq_S a'$, and $y \in Γ$. Depending on the value of $C$ we have to prove that $ayC \in L_{Ω_{g_0}}(A \cup AΓ)$. Indeed,

$$ayC \in (ay)S \cup (ay)ΓS \cup (ay)ST \cup (ay)ΓS \cup (ay)Γ \subseteq aΓS \cup aΓS \cup aΓ \subseteq L_{Ω_{g_0}}(A \cup AΓ).$$

All the above verifications prove the first condition, while the second condition is obvious. So $R = L_{Ω_{g_0}}(A \cup AΓ)$ is a right ideal of the ordered semigroup $Ω_{g_0}$, and from [4] it follows that $R = L_{Ω_{g_0}}(A \cup AΓ)$ is an idempotent. Passing now from the ordered semigroup $Ω_{g_0}$ to the ordered $Γ$-semigroup $S$, we show that if every right ideal of the ordered semigroup $Ω_{g_0}$ is idempotent, then so is every right ideal of the ordered $Γ$-semigroup $S$. Let $A$ be a right ideal of the ordered $Γ$-semigroup $S$, we have to prove $A$ is an idempotent in $S$, that is, $L_S(ΑΓA) = A$. Since $A$ is a right ideal, $L_S(ΑΓA) \subseteq L_S(ΑΓS) \subseteq A$. To prove the converse, we utilize the fact that $L_{Ω_{g_0}}(A \cup AΓ)$ is an idempotent in $Ω_{g_0}$, thus

$$A \subseteq L_{Ω_{g_0}}(A \cup AΓ) \subseteq L_{Ω_{g_0}}(A \cup AΓ) \subseteq L_{Ω_{g_0}}(A \cup AΓ).$$

This implies that every $a \in A$ is lower with respect to $≤_{Ω_{g_0}}$ than some element of $A$ or some element of $AΓ$. The second case is impossible from the way we have defined $≤_{Ω_{g_0}}$, so it remains that there is some $y \in Γ$, and $a', a'' \in A$ such that $a \leq_{Ω_{g_0}} a'ya''$. But this is the same as to say that $a \leq_S a'ya''$, so $a \in L_S(ΓAΓ)$, and as a result $A \subseteq L_S(ΓAΓ)$. One can show that left ideals of $S$ too are idempotent by first proving in a similar fashion to above that for every left ideal $B$ of $S$, the set $L_{Ω_{g_0}}(B \cup ΓB)$ is a left ideal of $Ω_{g_0}$. Finally, if $A$ is a right ideal and $B$ a left ideal of the ordered $Γ$-semigroup $S$, we have to prove that $L_S(ΑΒΓ)$ is a quasi ideal of $(S, Γ, ≤_S)$, which means that:

$$L_S(L_S(ΑΒΓ)ΓS \cap STL_S(ΑΓB)) \subseteq L_S(ΑΒΓ) \quad (1)$$

and

$$L_S(L_S(ΑΓB)) = L_S(ΑΓB). \quad (2)$$
From [4] we have that for the right ideal $L_{Ω}(A \cup \Gamma)$ and for the left ideal $L_{Ω}(B \cup \Gamma)$ of $Ω$, the set $L_{Ω}(L_{Ω}(A \cup \Gamma) \cdot L_{Ω}(B \cup \Gamma)) = L_{Ω}(A \cup \Gamma \cdot (B \cup \Gamma))$ is a quasi ideal of $Ω$. To prove the first condition (1), we see that

$$L_{Ω}(L_{Ω}(A \cup \Gamma) \cdot L_{Ω}(B \cup \Gamma)) \subseteq L_{Ω}(L_{Ω}(A \cup \Gamma) \cdot L_{Ω}(B \cup \Gamma)) \cdot L_{Ω}(A \cup \Gamma \cdot L_{Ω}(B \cup \Gamma))$$

$$= L_{Ω}(A \cup \Gamma \cdot (B \cup \Gamma)) = L_{Ω}(A \Gamma) = L_{Ω}(A \Gamma),$$

where the last equality follows from Lemma 2.2. The second condition (2) is obviously true since $L_{Ω}(A \Gamma)$ is an ordered ideal.

Conversely, we assume that every right and left ideal of $S$ is an idempotent, and for every right ideal of $S$, the set $L_{Ω}(A \Gamma)$ is a quasi ideal of $S$, and want to prove that $S$ is regular. The strategy is to prove that under the given conditions, $Ω$ is a regular ordered semigroup, and then from Proposition 3.1 we obtain straightaway that $S$ is a regular $\Gamma$-semigroup. To prove the regularity of $Ω$, it is enough to prove that all right ideals $R$ and all left ideals $L$ of $Ω$ are idempotent, and $L_{Ω}(R \cdot L)$ is a quasi ideal of $Ω$. Let $R$ be a right ideal of $Ω$, and want to prove that $RR = R$. The inclusion $RR \subseteq R$ is trivial. To prove the reverse inclusion $R \subseteq RR$, we need to prove that every $x \in R$ is of the form $x = x_1x_2$, where $x_1, x_2 \in R$. There are several possibilities for $x$.

(i) $x \in S$. Then, from Lemma 2.3 we have that $(x)^{S \gamma S} = (x)^{S \subseteq \Gamma}$. Recalling that $(x)^{S \subseteq \Gamma}$ is idempotent from the assumption, so $(x)^{S \subseteq \Gamma} = (x)^{S \Gamma(x)^{S \subseteq \Gamma}}$, hence $x = x_1y_0$, where $x_1, x_2 \in (x)^{S \subseteq \Gamma}$ and $y \in \Gamma$. Now $x_1y \in (x)^{S \subseteq \Gamma}$ and $x_2 \in (x)^{S \subseteq \Gamma}$, consequently

$$x = (x_1y) \cdot x_2 \in (x)^{S \gamma S} \subseteq R \cdot R.$$

(ii) Let the element of $R$ be of the form $ax$ with $x \in S$ and $a \in \Gamma$. Since $R$ is a right ideal of $Ω$, then

$$x\Gamma = x(a\bullet \Gamma) = (xa)\Gamma \subseteq (xa)\Omega_\gamma \subseteq R\Omega_\gamma \subseteq R,$$

and then we also obtain that

$$x\Gamma S = x\Gamma S \subseteq R \Omega_\gamma \subseteq R.$$

Now we prove that the element $x$ above is necessarily in $R$. For this, we use again the fact that $(x)^{S \subseteq \Gamma}$ is idempotent. It follows from this assumption that $x = x'y'x''$, where either $x' \subseteq S x$ or $x' \subseteq S x \beta S$ with $s \in S$. In the first case,

$$x = x'y'x'' \subseteq S x'y'x'' \subseteq x\Gamma S \subseteq R,$$

from which it follows that $x \in R$. Similarly, in the second case we see that

$$x = x'y'x'' \subseteq S x \beta S x'y'x'' \subseteq x\Gamma S \subseteq R,$$

and then $x \in R$. From the proof of (i) above we have that $x = (x_1y) \cdot x_2$, where $x_1y \in R$ and $x_2 \in R$. Then $xa = (x_1y) \cdot x_2a$, where again $x_1y \in R$ and $x_2a \in R$. Thus, we proved that $xa$ is expressed as a product of two elements from $R$ as desired.

(iii) The element of $R$ is of the form $ax$ with $a \in \Gamma$ and $x \in S$. In this case, we have to show first the equality $(ax)^{S \gamma S} = a(x)^{S \gamma S}$. Indeed,

$$\xi \in (ax)^{S \gamma S} \Leftrightarrow \xi \in L_{Ω}(ax \cup \alpha \Omega_\gamma) \Leftrightarrow \xi = ax', \text{ where } x' \in L_{Ω}(x \cup x\Omega_\gamma) \Leftrightarrow \xi \in a(x)^{S \gamma S},$$

which proves that $(ax)^{S \gamma S} = a(x)^{S \gamma S}$. From Lemma 2.3, we derive that

$$(ax)^{S \gamma S} = a((x)^{S \subseteq \Gamma} \cup (x)^{S \subseteq \Gamma}).$$
Now the right ideal \((x)_{r}^{S}\) is idempotent, which means that \((x)_{r}^{S} = (x)_{r}^{S} \Gamma (x)_{r}^{S}\), hence
\[
(ax)_{r}^{S \emptyset_0} = a((x)_{r}^{S} \Gamma (x)_{r}^{S} \cup (x)_{r}^{S} \Gamma).
\]
It follows from (3) that \(ax = ax_{1}y_{2}\), where \(x_{1}, x_{2} \in (x)_{r}^{S}\). Now if we rewrite this term as \(ax = (ax_{1}) (ya^{-1})(ax_{2})\), we see that
\[
ax = (ax_{1}) (ya^{-1})(ax_{2}) \\
\in a((x)_{r}^{S} \cup (x)_{r}^{S} \Gamma) a((x)_{r}^{S} \cup (x)_{r}^{S} \Gamma) \\
= (ax)_{r}^{S \emptyset_0} \cdot (ax)_{r}^{S \emptyset_0},
\]
which proves the claim.

(iv) The element of \(R\) is of the form \(ax\beta\). We first note that
\[
(ax\beta)_{r}^{S \emptyset_0} = L_{\Omega_{r}}(ax\beta \cup ax\beta(\Gamma \cup S \cup GS \cup ST \cup GST)) \\
= L_{\Omega_{r}}(ax \Gamma \cup ax GS \cup ax GST).
\]
Since \((x)_{r}^{S}\) is idempotent, \(x = x_{1}y_{2}\) where \(x_{1}, x_{2} \in (x)_{r}^{S}\). It follows that
\[
ax_{1}y_{2}a^{-1} \in L_{\Omega_{r}}(ax \Gamma \cup ax GS \cup ax GST)
\]
and similarly,
\[
ax_{2} \beta \in L_{\Omega_{r}}(ax \Gamma \cup ax GS \cup ax GST),
\]
hence
\[
ax\beta = (ax_{1}y_{2}a^{-1})(ax_{2}\beta) \in \left(L_{\Omega_{r}}(ax \Gamma \cup ax GS \cup ax GST)\right)^{2} = ((ax\beta)_{r}^{S \emptyset_0})^{2} \subseteq R^{2}.
\]
This shows that \(ax\beta\) is expressed as a product of two elements of \(R\).

(v) The final case is when the element of \(R\) is some \(y \in \Gamma\). Observe that
\[
\Gamma = y \Gamma \subseteq R \Omega_{r} \subseteq R.
\]
Now letting 1 be the unit element of \((\Gamma, \cdot)\) we have that
\[
y = y \cdot 1 \in RR,
\]
and once again, \(y \in R\) is expressed as a product of two elements in \(R\), namely, \(y\) and 1.

Recollecting, we have proved that any right ideal \(R\) of \(\Omega_{r}\) is idempotent. Similarly, we can show that any left ideal \(L\) is idempotent in \(\Omega_{r}\). Now if \(R\) is a right ideal and \(L\) a left ideal of the ordered semigroup \(\Omega_{r}\), we have to prove that \(L_{\Omega_{r}}(RL)\) is a quasi ideal of \(\Omega_{r}\). This would follow immediately if we prove that \(R \cap L \subseteq L_{\Omega_{r}}(RL)\), since on one hand, \(R \cap L\) is a quasi-ideal, and on the other hand, \(L_{\Omega_{r}}(RL) \subseteq R \cap L\). Let \(ax\beta \in R \cap L\) where \(a, \beta \in \Gamma\) are operators from \(\Gamma\), and \(x \in S\) such that \(ax\beta \leq \Omega_{r} aab\) where \(aab \in R \cap L\) and \(a \in S\). From Definition 2.1, we have that \(x \leq a\). Consequently,
\[
x \in (a)_{r}^{S} \cap (a)_{r}^{S} = L_{\Omega_{r}}((a)_{r}^{S} \Gamma (a)_{r}^{S}),
\]
where the equality follows from Lemma 3.1. Then there are \(\xi_{r}, \zeta_{r}, \xi_{r} \in \Gamma\) and \(s, t \in S\) such that \(x \leq a\xi_{r}s \xi_{r} t \zeta_{r} a\). Furthermore, we have that
\[
ax\beta \leq L_{\Omega_{r}}((aab)(b^{-1}\xi_{r})((s \xi_{r} t)(\xi_{r}^{-1}a)(aab)) \in L_{\Omega_{r}}(RL)
\]
since \((b^{-1}\xi_{r})((s \xi_{r} t)(\xi_{r}^{-1}a)(aab)) \in L\). The remaining cases for an element from \(R \cap L\) include elements of the form \(ax, x\beta\) or simply \(x\), where \(a, \beta \in \Gamma\) and \(x \in S\). These cases are dealt similarly as above. \(\square\)
The following is straightforward.

**Corollary 3.1.** Any of the characterizations of the regularity of an ordered $\Gamma$-semigroup given in Theorem 8 of [2] is logically equivalent to its corresponding characterization of the regularity of an ordered semigroup given in Theorem 3.1 of [4].

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