Polynomial chaos expansion for uncertainty analysis and global sensitivity analysis

Ming Chen¹, Xinhu Zhang*, Kechun Shen¹, Guang Pan¹

¹ School of Marine Science and Technology, Northwestern Polytechnical University, Xi’an, Shaanxi Province, 710072, China

*Corresponding author’s e-mail: xinhu_zhang@126.com

Abstract. Uncertainty analysis has received increasing attention across all kinds of scientific and engineering fields recently. Uncertainty analysis is often conducted by Monte Carlo simulation (MCS), while with low convergence rate. In this paper, numerical test examples as benchmarks and engineering problems in practice are studied by polynomial chaos expansion (PCE) and compared with the solutions got by MCS. Results show that PCE approach establishes accurate surrogate model for complicated original model with efficiency to conduct uncertainty analysis and global sensitivity analysis. What’s more, sparse PCE is able to tackle problem of high dimension with efficiency. Hence PCE approach can be applied in uncertainty analysis and global sensitivity analysis of engineering problems with efficiency and effectiveness.

1. Introduction

The past decades have witnessed an increasing number of complicated computational models being established for the simulation and prediction of the response of systems in a variety of fields of science and engineering, as a consequence of the development of computer science and technology. Amongst the wide range of complex systems, the uncertainty of model structures and inputs have received much attention since they reflect the properties of model structures under realistic conditions[1-4].

Uncertainty analysis of such complicated models may become time consuming when confronted with models on which the simulations are computationally expensive[5-6]. An approach to bypass the problem is to substitute complicated models with precise and efficient surrogate models, which is cheap for evaluation[7]. Several techniques have been proven effective by performing limited model evaluations for instance: Kriging meta-model[8-10], response surface method[11-12] and machine learning approaches[13-16].

PCE is a rising approach utilized in assessment of random input variates by specific polynomial[17-18]. By establishing a surrogate model through this approach, the mapping relationship between output and the stochastic input variates can be captured. With simple form and fast convergence, PCE is adopted in the uncertainty analysis and global sensitivity analysis as compared to MCS based on complex original model[19-21].There mainly exists two different approaches in PCE, namely regression and pseudo-spectral projection method. As for the case with multivariate stochastic model, sparse polynomial chaos expansion is adopted as it is efficient taking into account computation time and the rate of convergence[22].

This paper is structured as following: PCE theory and global sensitivity analysis are briefly introduced in Section 2. After which, numerical test examples as benchmarks and engineering problems in practice are studied for validation purpose in Section 3 respectively, and in the end comes the conclusion.
2. Methodology

2.1. PCE theory

First introduced by Wiener[23], polynomial chaos was utilized to tackle stochastic processes by using Hermite polynomials and random variates which obey normal distribution. The approach by employing Hermite polynomials serving as an orthogonal polynomials is validated later[24]. Inspired by Wiener’s approach of Hermite polynomial chaos, Hermite polynomials is adopted as an orthogonal polynomials basis to deal with stochastic processes, and the approach is successfully applied to deal with a lot of engineering problems[25-28]. Although mathematically sound, there presents difficulties when performing Hermite polynomials considering convergence and non-Gaussian variates[29-30]. Consequently, the Wiener-Askey based polynomial chaos was introduced. And the difficulties are solved by employing the appropriate orthogonal polynomials basis, as compared to the original Wiener Hermite polynomials[31-32].

Consider a stochastic system with input variate x and system output y. As is presented in table 1, PCE possesses an orthogonal polynomial family which is associated with specified input distribution according to Askey scheme. Linkage between Askey scheme concerning the orthogonal polynomials and stochastic variates with a certain probability distribution function (PDF) is referred to table 1.

The orthogonal polynomials are employed to establish a surrogate model of the output[33], which is referred to equation (1), where p is the truncation order, $c_i$ is the coefficient, and $\psi_i(x)$ is the i order orthogonal polynomial.

$$y(x) \approx \hat{y}(x) = \sum_{i=0}^{p} c_i \psi_i(x)$$ (1)

There exists two different approaches to establish a PCE model, namely regression and pseudospectral projection method. Regression method is implemented by fitting a PCE to a set of generated samples and evaluations[34]. While the latter consists in applying quadrature rules to approximate the correspondent coefficient.

| Distribution | Density function | Orthogonal polynomials basis | Weight function | Domain |
|--------------|------------------|-----------------------------|----------------|--------|
| Normal       | $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ | Hermite $He_n(x)$ | $e^{-\frac{x^2}{2}}$ | $[-\infty, +\infty]$ |
| Uniform      | $\frac{1}{\sqrt{2}}$ | Legendre $P_n(x)$ | 1 | [-1,1] |
| Beta         | $\frac{(1+x)^\alpha (1-x)^\beta}{2^{\alpha+\beta+1} \Gamma(\alpha+1, \beta+1)}$ | Jacobi $P_n^{(\alpha,\beta)}(x)$ | $(1+x)^\beta (1-x)^\alpha$ | $[-1,1]$ |
| Gamma        | $\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}$ | Generalized Laguerre | $x^\alpha e^{-x}$ | $[0, +\infty]$ |
| Exponential  | $\frac{e^{-x}}{\Gamma(\alpha+1)}$ | Laguerre $L_n(x)$ | $e^{-x}$ | $[0, +\infty]$ |

2.2. Global sensitivity analysis

Take into consideration a square-integrable function $y = f(x)$ with n input parameters, the domain is $[0,1]^n$.

When it comes to the Sobol decomposition for $f(x)$, it is given as below[35-36]:

$$f(x_1, ..., x_n) = f_0 + \sum_{i=1}^{n} f_i(x_i) + \sum_{1\leq i < j \leq n} f_{ij} (x_i, x_j) + ... + f_{1,2,...,n}(x_1, ..., x_n)$$ (2)

where $f_0$ is a constant. Integrating $f_{i_1,...,i_k}(x_{i_1}, ..., x_{i_k})$ of equation (2) over single independent variate is zero. According to equation (2), $f(x)$ is orthogonally decomposed as below:

$$E \left[ f_{i_1,...,i_k}(x_{i_1}, ..., x_{i_k}) \times f_{i_q,...,i_t}(x_{i_q}, ..., x_{i_t}) \right] = 0 \forall (i_1, ..., i_k) \neq (i_q, ..., i_t)$$ (3)
When integrating the square of equation (2), equation (4) is got.

\[
\int f^2(x)dx = f_0^2 + \sum_{i=1}^{n} \int f_i^2(x_i)dx_i + \sum_{i \neq j} \int f_{ij}^2(x_i,x_j)dx_i dx_j + \cdots + \int f_{1,2,\ldots,n}^2(x_1,\ldots,x_n)dx_1 \ldots dx_n
\]

(4)

There comes the deduced expression below:

\[
V = \sum_{i=1}^{n} V_i + \sum_{i \neq j} V_{ij} + \cdots + V_{1,2,\ldots,n}
\]

(5)

And \( V \) stands for variance of \( f(x) \). \( V_i \) stands for variance of a certain individual \( x_i \). \( V_{x_{i_1} \ldots x_{i_p}} \) stands for the interactions among \( \{x_{i_1}, \ldots, x_{i_p}\} \).

The Sobol sensitivity indices are as below:

\[
S_i = \frac{V_i}{V} = \frac{\mathbb{E}(y|x_i) - \mathbb{E}(y)}{\mathbb{E}(y)}
\]

(6)

where \( x_i \) is denoted as a set of stochastic input variates \( \{x_{i_1}, \ldots, x_{i_p}\} \). \( S_i \) in equation (6) represents the part of \( V(y) \) affected by \( x_i \). There are two Sobol sensitivity indices which are mostly used,

\[
S_i^T = \frac{V_i}{V} = \frac{\mathbb{E}(y|x_i) - \mathbb{E}(y)}{\mathbb{E}(y)}
\]

(7)

\( S_i^T \) stands for the total influence by \( x_i \), and \( x_{-i} \) represents other random variates excluding \( x_i \).

\[
S_i = \frac{V_i}{V} = \frac{\mathbb{E}(y|x_i)}{\mathbb{E}(y)}
\]

(8)

\( S_i \) stands for the main influence of \( x_i \), namely the effect of single \( x_i \) on the output.

Sudret first discovered the relationship between the Sobol sensitivity indices and the PCE coefficients[37]. It is found analytical solution of Sobol sensitivity indices can be got through PCE coefficients. Once the PCE model is established and the correspondent coefficients obtained, the statistical metrics and Sobol sensitivity indices of system output responses can be obtained. Therefore PCE based global sensitivity analysis is wildly used in a variety of engineering applications[38-40].

3. Numerical examples and practical engineering problems

PCE is employed for the uncertainty analysis and global sensitivity analysis of two numerical test examples and two practical engineering problems in this section. The solutions got by the PCE method are compared with Monte Carlo simulation.

3.1. Fortini’s clutch

Consider Fortini’s clutch as shown in figure 1. Fortini’s clutch test case is widely applied in uncertainty analysis literature[41].

The contact angle \( y \) contains four independent stochastic input variates \( x_1, x_2, x_3, x_4 \):

\[
y = \arccos \left( \frac{0.5(x_2+x_3)+x_1}{0.5(x_2+x_3)+x_4} \right)
\]

(9)

Distribution parameters of all stochastic input variates are provided in table 2, and results got by PCE and MCS for Fortini’s clutch are given in table 3.

Regression and pseudo-spectral projection method are utilized to construct polynomial chaos expansion model of response function \( y \) respectively. All the input variates obey normal distribution, therefore Hermite orthogonal polynomial is employed to build PCE model. The order of PCE model with regression method is 3, whereas pseudo-spectral projection method with order 2. The statistical metrics (expectation, standard deviation) of \( y \) are shown in table 3 with the application of regression, pseudo-spectral projection method and Monte Carlo simulation respectively. As is presented in table 3, the cost of computation is represented by number of times of model evaluation. The cost of computation is a million times by implementing Monte Carlo simulation which is often utilized as a benchmark for comparison. As for the method of regression and pseudo-spectral projection, the cost of computation is
70 and 81 respectively, and obtaining the desired accuracy. It is shown that the method of regression and pseudo-spectral projection is efficient dealing with uncertainty analysis problems.

### Table 2. Distribution parameters of Fortini’s clutch example.

| Random input variates | Expectation (mm) | Coefficient of variation | Standard deviation (mm) | Distribution |
|-----------------------|------------------|--------------------------|-------------------------|--------------|
| $x_1$                 | 55.29            | 0.143%                   | 0.0793                  | Normal       |
| $x_2$                 | 22.86            | 0.019%                   | 0.0043                  | Normal       |
| $x_3$                 | 22.86            | 0.019%                   | 0.0043                  | Normal       |
| $x_4$                 | 101.60           | 0.078%                   | 0.0793                  | Normal       |

### Table 3. Statistical metrics for Fortini’s clutch example.

| Method                      | MCS                  | Pseudo-spectral projection | Error (%) | Regression | Error (%) |
|-----------------------------|----------------------|----------------------------|-----------|------------|-----------|
| Order of PCE                | -                    | 2                          | -         | 3          | -         |
| Expectation (rad)           | 0.121915181          | 0.121915679                | 4.0774e-6 | 0.121916101| 7.50744e-6|
| Standard deviation (rad)    | 0.011835801          | 0.011831558                | 0.0358%   | 0.011862172| 0.2228%   |
| Model evaluations           | 1e6                  | 81                         | -         | 70         | -         |

![Figure 1. Fortini’s clutch.](image)

3.2. Ishigami function

Take the Ishigami function\[42\] into consideration:

$$ y = \sin x_1 + asin^2 x_2 + b x_3^4 \sin x_1 $$

where the stochastic input variates are independent identically distributed, and are subject to uniform distribution, namely $x_i \sim \text{Uniform}[-\pi, \pi]$, for $i = 1, 2, 3$.

Thanks to strong non-linearity and nonmonotonicity, the Ishigami function by Ishigami and Homma\[43\] is used as an benchmark test example performing global sensitivity analysis. Marrel\[44\] proposes the values of $a$ and $b$ as follows: $a = 7$ and $b = 0.1$. The theoretical solutions of variance are as below:

$$ V = \frac{a^2}{8} + \frac{b \pi^2}{5} + \frac{b^2 \pi^8}{18} + \frac{1}{2}, V_1 = \frac{b \pi^4}{5} + \frac{b^2 \pi^8}{50} + \frac{1}{2}, V_2 = \frac{a^2}{8}, V_3 = 0, V_{12} = V_{23} = 0, V_{13} = \frac{8 b^2 \pi^8}{225} $$

In this example, the Legendre polynomial $P_n(x)$ is adopted to construct the PCE surrogate model by regression method according to table 1. When PCE is constructed, one can get Sobol sensitivity indices through the established PCE. Sobol sensitivity indices got by PCE and given by Reference\[45\] are shown in table 4 for the propose of validation.

According to table 4, the utilized PCE serves as an more accurate surrogate model when compared with the results in Reference\[45\]. As for the utilized PCE model, biggest relative error provided by PCE
is 1.6%. While the result provided in Reference [45] is 11.8%. It is shown the utilized PCE model is accurate and effective dealing with nonlinear stochastic system.

Table 4. Global sensitivity analysis of Ishigami function.

| Sobol sensitivity indices | Theoretical solutions | PCE | Relative error (%) | [45] Relative error (%) |
|--------------------------|-----------------------|-----|-------------------|------------------------|
| $S_1$                    | 0.31                  | 0.314010834 | 1.3 | 0.29 | 8.6 |
| $S_2$                    | 0.44                  | 0.442068395 | 0.5 | 0.43 | 2.7 |
| $S_3$                    | 0                     | 2.06516480e-7 | -  | 0   | -   |
| $S_{13}$                 | 0.24                  | 0.243902999 | 1.6 | 0.25 | 4.3 |
| $S_{12}$                 | 0                     | 1.41383703e-6 | -  | 0.01 | -   |
| $S_{23}$                 | 0                     | 3.45311016e-6 | -  | 0   | -   |
| $S^T_1$                  | 0.56                  | 0.55792795 | 0.4 | 0.57 | 1.5 |
| $S^T_2$                  | 0.44                  | 0.44208596 | 0.5 | 0.46 | 3.8 |
| $S^T_3$                  | 0.24                  | 0.24391936 | 1.6 | 0.27 | 11.8 |
| Polynomial degree        | 10                    | 7               |

3.3. Ten-bar truss

The wildly used ten-bar truss [46-47] is selected to conduct global sensitivity analysis in this section as depicted in figure 2. Both each of the horizontal and vertical bar share a length L and the elastic modulus is denoted as E respectively. There are three concentrated loads $P_1$, $P_2$ and $P_3$ applied on the node 2 and 3 respectively. Each of the ten bars share a section denoted by $A_i (i = 1, \ldots, 10)$ respectively. Statistical information of random variates is summarized according to table 5.

Table 5. Statistical information of ten-bar truss structure.

| Random input variates | Expectation | Coefficient of variation | Standard deviation | Distribution |
|-----------------------|-------------|--------------------------|--------------------|--------------|
| $L/m$                 | 1           | 5\%                       | 0.05               | Normal       |
| $A_i (i = 1, \ldots, 10)/m^2$ | 0.001       | 5\%                       | 5e-5               | Normal       |
| $E/GPa$               | 100         | 5\%                       | 5                  | Normal       |
| $P_1/KN$              | 80          | 5\%                       | 4                  | Normal       |
| $P_2/KN$              | 10          | 5\%                       | 0.5                | Normal       |
| $P_3/KN$              | 10          | 5\%                       | 0.5                | Normal       |
### Table 6. Sobol sensitivity indices for ten-bar truss structure.

| Main Sobol sensitivity indices | MCS      | Sparse PCE | Total Sobol sensitivity indices | MCS      | Sparse PCE |
|--------------------------------|----------|------------|---------------------------------|----------|------------|
| $S_{A_1}$                      | 0.023    | 0.023      | $S_{A_1}^T$                     | 0.023    | 0.023      |
| $S_{A_2}$                      | 0.001    | 0.001      | $S_{A_2}^T$                     | 0.001    | 0.001      |
| $S_{A_3}$                      | 0.023    | 0.023      | $S_{A_3}^T$                     | 0.023    | 0.023      |
| $S_{A_4}$                      | 0.001    | 0.001      | $S_{A_4}^T$                     | 0.001    | 0.001      |
| $S_{A_5}$                      | 0.001    | 0.001      | $S_{A_5}^T$                     | 0.001    | 0.001      |
| $S_{A_6}$                      | 0.001    | 0.001      | $S_{A_6}^T$                     | 0.001    | 0.001      |
| $S_{A_7}$                      | 0.020    | 0.020      | $S_{A_7}^T$                     | 0.020    | 0.020      |
| $S_{A_8}$                      | 0.014    | 0.013      | $S_{A_8}^T$                     | 0.014    | 0.013      |
| $S_{A_9}$                      | 0.002    | 0.002      | $S_{A_9}^T$                     | 0.001    | 0.002      |
| $S_{A_{10}}$                   | 0.001    | 0.001      | $S_{A_{10}}^T$                  | 0.001    | 0.001      |
| $S_{P_1}$                      | 0.213    | 0.212      | $S_{P_1}^T$                     | 0.213    | 0.212      |
| $S_{P_2}$                      | 0.026    | 0.025      | $S_{P_2}^T$                     | 0.026    | 0.025      |
| $S_{P_3}$                      | 0.002    | 0.004      | $S_{P_3}^T$                     | 0.002    | 0.004      |
| $S_{L}$                        | 0.337    | 0.338      | $S_{L}^T$                       | 0.338    | 0.338      |
| $S_{E}$                        | 0.342    | 0.336      | $S_{E}^T$                       | 0.343    | 0.336      |

Model evaluations 1.7e6  136  

As is depicted in table 6 above, global sensitivity analysis is conducted through sparse PCE with 136 model evaluations. By conducting global sensitivity analysis, it is revealed sparse PCE approach is efficient for 15 dimensional practical engineering application with a desired accuracy.

### 3.4. Roof truss

Consider a roof truss as presented according to figure 3. Strengthened with concrete, compression bars and top beam share sectional area $A_c$ and elastic modulus $E_c$. Consisting of steel, Tension bars and bottom beam share sectional area $A_s$ and elastic modulus $E_s$. The length is denoted as $l$. Uniformly distributed load $q$ is applied on roof truss. When equivalently treated, uniformly distributed load is equivalent to concentrated force, refer to figure 3(b). The value of concentrated force is $P = q l / 4$ applied on the node C, D and F. Distribution parameters for stochastic variates are shown according to table 7. The vertical deflection $Y_c$ of node C is as follows:

$$Y_c = \frac{q l^2}{2} \left( \frac{3.81}{A_c E_c} + \frac{1.13}{A_s E_s} \right)$$  \hspace{1cm} (12)

Figure 3. Structure of the roof truss.

![Figure 3. Structure of the roof truss.](image)
Table 7. Statistical information of roof truss.

| Random input variates | Expectation | Coefficient of variation | Standard deviation | Distribution |
|------------------------|-------------|--------------------------|--------------------|--------------|
| Distributed load, \( q/Nm^{-1} \) | 20,000 | 7% | 1400 | Normal |
| Sectional area, \( A_s/m^2 \) | 9.82e-4 | 6% | 5.892e-5 | Normal |
| Sectional area, \( A_c/m^2 \) | 0.04 | 12% | 0.0048 | Normal |
| Elastic modulus, \( E_s/MPa \) | 2e11 | 6% | 1.2e10 | Normal |
| Elastic modulus, \( E_c/MPa \) | 3e10 | 6% | 1.8e9 | Normal |
| Length, \( l/m \) | 12 | 1% | 0.12 | Normal |

Table 8. Sobol sensitivity indices of roof truss.

| Sobol sensitivity indices | Sparse PCE | Full PCE | MCS |
|--------------------------|------------|----------|-----|
| \( S_q \) | 0.463 | 0.451 | 0.448 |
| \( S_l \) | 0.038 | 0.037 | 0.035 |
| \( S_{As} \) | 0.140 | 0.140 | 0.143 |
| \( S_{Ac} \) | 0.175 | 0.186 | 0.185 |
| \( S_{Es} \) | 0.140 | 0.140 | 0.138 |
| \( S_{Ec} \) | 0.044 | 0.043 | 0.042 |
| \( S_{qT} \) | 0.463 | 0.454 | 0.451 |
| \( S_{lT} \) | 0.038 | 0.037 | 0.037 |
| \( S_{AT} \) | 0.140 | 0.140 | 0.144 |
| \( S_{AT} \) | 0.175 | 0.187 | 0.188 |
| \( S_{ET} \) | 0.140 | 0.141 | 0.140 |
| \( S_{ET} \) | 0.044 | 0.044 | 0.044 |

Model evaluations

| 64 | 729 | 1e6 |

Main and total Sobol sensitivity indices obtained by full PCE, sparse PCE and MCS (1e6 samples as a reference) are presented in table 8. As is presented in tables 8, the full PCE needs 729 model evaluations, while sparse PCE method needs 64 model evaluations to yield accurate results for Sobol sensitivity indices. It is shown sparse PCE is capable of dealing with engineering application with efficiency and accuracy.

4. Conclusion
Through numerical test examples as benchmarks and engineering problems in practice, PCE method is validated by comparing with the results obtained by MCS. Results show that PCE can act as a surrogate model to substitute complicated original model with accuracy and efficiency. What’s more, PCE is applicable for complex non-linear problems with high dimension.

Acknowledgments
This work was supported by National Natural Science Foundation of China (51909219, 61803306), the National Key Research and Development Program of China (Grant No.2016YFC0301300), Fundamental Research Funds for the Central Universities (Grant No.3102019JC006), and China Postdoctoral Science Foundation (Grand No.2020M673492).

References
[1] Kumar, D., Raisee, M., Lacor, C. (2016) An efficient non-intrusive reduced basis model for high dimensional stochastic problems in CFD. Computers & Fluids, 138: 67–82.
[2] Wei, H., Matteo, D. I. E. Z., Campana, E. F., et al. (2013) A one-dimensional polynomial chaos method in CFD–Based uncertainty quantification for ship hydrodynamic performance. Journal
of Hydrodynamics, Ser. B, 25(5): 655-662.

[3] Wu, X., Zhang, W., Song, S. (2017) Uncertainty quantification and sensitivity analysis of transonic aerodynamics with geometric uncertainty. International Journal of Aerospace Engineering, 2017: 1-16.

[4] Hosder, S., Walters, R., Balch, M. (2007) Efficient sampling for non-intrusive polynomial chaos applications with multiple uncertain input variables. In: 48th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference. Hawaii. pp. 1-16.

[5] Liu, J., Song, W. P., Han, Z. H., et al. (2017) Efficient aerodynamic shape optimization of transonic wings using a parallel infilling strategy and surrogate models. Structural and Multidisciplinary Optimization, 55(3): 925-943.

[6] Bartoli, N., Lefebvre, T., Dubreuil, S., et al. (2019) Adaptive modeling strategy for constrained global optimization with application to aerodynamic wing design. Aerospace Science and technology, 90: 85-102.

[7] Konakli, K., Sudret, B. (2016) Polynomial meta-models with canonical low-rank approximations: Numerical insights and comparison to sparse polynomial chaos expansions. Journal of Computational Physics, 321: 1144-1169.

[8] Han, Z. H., Zhang, Y., Song, C. X., et al. (2017) Weighted gradient-enhanced kriging for high-dimensional surrogate modeling and design optimization. Aiaa Journal, 55(12): 4330-4346.

[9] Han, Z. H., Zimmermann, Götz, S. (2012) Alternative cokriging method for variable-fidelity surrogate modeling. AIAA journal, 50(5): 1205-1210.

[10] Han, Z. H., Görtz, S. (2012) Hierarchical kriging model for variable-fidelity surrogate modeling. AIAA journal, 50(9): 1885-1896.

[11] Öktem, H., Erzurumlu, T. U. N. C. A. Y., Kurtaran, H. A. S. A. N. (2005) Application of response surface methodology in the optimization of cutting conditions for surface roughness. Journal of materials processing technology, 170(1-2): 11-16.

[12] Youn, B. D., Choi, K. K. (2004) A new response surface methodology for reliability-based design optimization. Computers & structures, 82(2-3): 241-256.

[13] Park, J., Sandberg, I. W. (1991) Universal approximation using radial-basis-function networks. Neural computation, 3(2): 246-257.

[14] Vt S E, Shin Y C. (1994) Radial basis function neural network for approximation and estimation of nonlinear stochastic dynamic systems. IEEE transactions on neural networks, 5(4): 594-603.

[15] Schölkopf, B., Platt, J. C., Shawe-Taylor, J., et al. (2001) Estimating the support of a high-dimensional distribution. Neural computation, 13(7): 1443-1471.

[16] Forrester, A. I., Keane, A. J. (2009) Recent advances in surrogate-based optimization. Progress in aerospace sciences, 45(1-3): 50-79.

[17] Soize, C., Ghanem, R. (2004). Physical systems with random uncertainties: chaos representations with arbitrary probability measure. SIAM Journal on Scientific Computing, 26(2): 395-410.

[18] Le Maitre, O. P., Knio, O. M., Najm, H. N., et al. (2004) Uncertainty propagation using Wiener–Haar expansions. Journal of computational Physics, 197(1): 28-57.

[19] Choi, S. K., Grandhi, R. V., Canfield, R. A., et al. (2004) Polynomial chaos expansion with latin hypercube sampling for estimating response variability. AIAA Journal, 42(6): 1191-1198.

[20] Berveiller, M., Sudret, B., Lemaire, M. (2006) Stochastic finite element: a non intrusive approach by regression. European Journal of Computational Mechanics, 15(1-3): 81-92.

[21] Xiu, D., Hesthaven, J. S. (2005) High-order collocation methods for differential equations with random inputs. SIAM Journal on Scientific Computing, 27(3): 1118-1139.

[22] Blatman, G., Sudret, B. (2011) Adaptive sparse polynomial chaos expansion based on least angle regression. Journal of computational Physics, 230(6): 2345-2367.

[23] Wiener, N. (1938) The homogeneous chaos. American Journal of Mathematics, 60(4): 897-936.

[24] Cameron, R. H., Martin, W. T. (1947) The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals. Annals of Mathematics, 48: 385-392.

[25] Ghanem, R. (1999) Ingredients for a general purpose stochastic finite elements implementation.
Computer methods in applied mechanics and engineering, 168(1-4): 19-34.
[26] Spanos, P. D., Ghanem, R. (1989) Stochastic finite element expansion for random media. Journal of engineering mechanics, 115(5): 1035-1053.
[27] Ghanem, R. (1998) Scales of fluctuation and the propagation of uncertainty in random porous media. Water Resources Research, 34(9): 2123-2136.
[28] Ghanem, R. (1999) Stochastic finite elements with multiple random non-Gaussian properties. Journal of Engineering Mechanics, 125(1): 26-40.
[29] Chorin, A. J. (1974) Gaussian fields and random flow. Journal of Fluid Mechanics, 63(1): 21-32.
[30] Orszag, S. A., Bissonnette, L. R. (1967) Dynamical Properties of Truncated Wiener-Hermite Expansions. The Physics of Fluids, 10(12): 2603-2613.
[31] Xiu, D., Karniadakis, G. E. (2002) The Wiener–Askey polynomial chaos for stochastic differential equations. SIAM journal on scientific computing, 24(2): 619-644.
[32] Xiu, D., Karniadakis, G. E. (2003) Modeling uncertainty in flow simulations via generalized polynomial chaos. Journal of computational physics, 187(1): 137-167.
[33] Xiu, D. (2010) Numerical methods for stochastic computations. Princeton university press, Princeton.
[34] Cheng, K., Lu, Z. (2018) Adaptive sparse polynomial chaos expansions for global sensitivity analysis based on support vector regression. Computers & Structures, 194: 86-96.
[35] IM, S. (1993) Sensitivity estimates for nonlinear mathematical models. Math. Model. Comput. Exp, 1(4): 407-414.
[36] Sobol, I. M. (2001) Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates. Mathematics and computers in simulation, 55(1-3): 271-280.
[37] Sudret, B. (2008) Global sensitivity analysis using polynomial chaos expansions. Reliability engineering & system safety, 93(7): 964-979.
[38] Buzzard, G. T. (2012) Global sensitivity analysis using sparse grid interpolation and polynomial chaos. Reliability Engineering & System Safety, 107: 82-89.
[39] Garcia-Cabrero, O., Valocchi, A. (2014) Global sensitivity analysis for multivariate output using polynomial chaos expansion. Reliability Engineering & System Safety, 126: 25-36.
[40] Sandoval, E. H., Anstett-Collin, F., Basset, M. (2012) Sensitivity study of dynamic systems using polynomial chaos. Reliability Engineering & System Safety, 104: 15-26.
[41] Creveling, C. M. (1997) Tolerance design: a handbook for developing optimal specifications. Prentice Hall, Englewood Cliffs.
[42] Torre, E., Marelli, S., Embrechts, P., et al. (2019) Data-driven polynomial chaos expansion for machine learning regression. Journal of Computational Physics, 388: 601-623.
[43] Marrel, A., Iooss, B., Laurent, B., et al. (2009) Calculations of sobol indices for the gaussian process metamodel. Reliability Engineering & System Safety, 94(3): 742-751.
[44] Błatman, G., Sudret, B. (2010) Efficient computation of global sensitivity indices using sparse polynomial chaos expansions. Reliability Engineering & System Safety, 95(11): 1216-1229.
[45] Sonmez, M. (2011) Artificial Bee Colony algorithm for optimization of truss structures. Applied Soft Computing, 11(2): 2406-2418.
[46] Wei, P., Lu, Z., Song, J. (2014) Extended Monte Carlo simulation for parametric global sensitivity analysis and optimization. AIAA journal, 52(4): 867-878.