Perturbations of the local gravity field due to mass distribution on precise measuring instruments: a numerical method applied to a cold atom gravimeter

G D’Agostino¹, S Merlet², A Landragin² and F Pereira Dos Santos²

¹ INRIM, Istituto Nazionale di Ricerca Metrologica, 73 Strada delle Cacce, 10135 Turin, Italy
² LNE-SYRTE, Observatoire de Paris, CNRS et UPMC, 61 avenue de l’Observatoire, 75014 Paris, France

E-mail: sebastien.merlet@obspm.fr

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Abstract

We present a numerical method, based on a FEM simulation, for the determination of the gravitational field generated by massive objects, whatever geometry and space mass density they have. The method was applied for the determination of the self-gravity effect of an absolute cold atom gravimeter which aims at a relative uncertainty of \(10^{-9}\). The deduced bias, calculated with a perturbative treatment, is finally presented. The perturbation reaches \((1.3 \pm 0.1) \times 10^{-9}\) of the Earth’s gravitational field.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Gravitational forces from surrounding masses are usually negligible compared with electromagnetic forces and thus not considered as relevant in laboratory based experiments. Nevertheless, they can significantly impact on accurate force measurement [1] or even constitute the dominant effect [2, 3]. As an example, an aluminium disc with 50 cm diameter and 4 cm thickness generates a gravitational field ranging from 4 to 1 parts in \(10^9\) of the Earth’s gravity field \(g\), from its surface to 30 cm along its axis. Components with similar mass density and dimensions are commonly used in precise instruments developed for measuring the gravity field with uncertainties of a few parts in \(10^9\). The measurement principle of these instruments is based on the determination of the ballistic trajectory followed by a mass, i.e. corner cube retroreflectors and recently atoms, during their vertical free fall under the influence of gravity [4–6]. Due to constraints in their design, their mass distribution is in general not symmetric with respect to the trajectory of the test mass. The effect of parasitic attractions, which is called the self-gravity effect when restricted to the influence of the device itself, has to be carefully estimated and if necessary corrected for in order to guarantee the accuracy of the measurement.

In the case of the absolute corner cube gravimeter FG5, the authors in [4] indicate that the \(g\) measurement is corrected for the attraction of the apparatus, with an uncertainty of \(10^{-9}\) m s\(^{-2}\) without giving details on its calculation. An absolute bias of \((1.4 \pm 0.1) \times 10^{-8}\) m s\(^{-2}\) has been calculated for the FG5 self-gravity effect using algorithms developed for processing perturbed spacecraft orbits [7]. A similar level of uncertainty is also necessary in the case of the cold atom gravimeter (CAG) described in [8], currently operating and improved within the framework of the LNE watt balance experiment [9, 10].

In this paper we describe a user-friendly numerical method for computing the gravitational field generated by extended and continuous massive objects, whatever geometry and space mass density distribution they have. Application of the method for the quantitative estimate of the self-gravity error on the CAG is also given.

2. Analogy between gravitational and electrical interactions

Mass and electric charge are, respectively, the sources of the gravitation and electromagnetic interaction between bodies [11]. In particular, Newton’s and Coulomb’s laws
which describe, respectively, the gravitational and electric forces occurring between two masses and two charges have the same behaviour. This analogy can be exploited to compute gravitational forces using methods originally developed for electromagnetic forces, by replacing individual charges by individual masses, space charge density by mass space density and \(1/4\pi \epsilon_0 \) by \(G \) where \(\epsilon_0\) is the vacuum permittivity and \(G\) is the gravitational constant.

In general, the gravitational field can be computed by solving cumbersome integrals of three-dimensional vector functions. Use of multipole moments makes it possible, for certain geometries, to calculate the gravitational interaction between extended bodies without the need for numerical integrations [12]. The accuracy of the result depends on the convergence characteristics of the series expressing the multipole moments. Unbiased analytical solutions exist only for axially symmetric and homogeneous bodies, e.g. a spherical shell, a solid sphere, a right rectangular prism, a right polygonal prism and a polyhedron [13]. For more complex systems, which cannot be easily broken down into simplified parts, the finite element method (FEM) helps in finding an approximate numerical solution of the integrals. The essential characteristic of this method is the mesh discretization of a continuous domain into a set of discrete sub-domains. For this application we used the FEM software packages developed by COMSOL Multiphysics [14], in particular the electrostatic module.

3. FEM and expected accuracy

In our application, the three-dimensional model of the object in a frame of reference \(Oxyz\) defines a body domain \(BD\) filled with a given mass density, whereas the surrounding domain \(SD\) shapes a vacuum space where we are interested in calculating with a given mass density, whereas the surrounding domain \(SD\) in a frame of reference

\[
G \rho_m \sum_i \sum_j \sum_k \mu_{ijk} \left( x_i \ln(y_i + r_{ijk}) + y_j \ln(x_i + r_{ijk}) - z_k \arctan \frac{x_i y_j}{z_k r_{ijk}} \right),
\]

(1)

where \(x_i = x - L, y_j = y - L, z_k = z - L/2, r_{ijk} = \sqrt{x_i^2 + y_j^2 + z_k^2}\) and \(\mu_{ijk} = (-1)^i (-1)^j (-1)^k\). Figure 2 represents the expected gravitational field along two lines.

Figure 1. Equipotential surfaces generated by an ellipsoidal body in its surrounding. At sufficiently large distances from the object, these equipotentials are spherical.
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Figure 2. Gravitational field of a prism of dimensions $2L, 2L, L$, calculated with the FEM simulation, along two lines parallel to the $z$-axis passing through the centre of the prism (black line) and at the distance $L$ from the prism (grey line).

parallel to the $z$-axis, within a range equal to $\pm 20L$. The black and grey curves correspond to two vertical lines passing through points $(0, 0, 0)$ and $(2L, 0, 0)$, respectively. Data are normalized with the maximum vertical field $\Gamma_{\text{max}}$ occurring at point $(0, 0, -L/2)$. The FEM simulation was performed by limiting SD to a sphere having a radius equal to twenty times $L$. Both the domains were meshed with an adaptive mesh generation after a uniform constraint of 15 mesh elements every $L$ on the two lines where the solution is needed. The relative difference between the simulation results and the analytical solution $\Delta \Gamma / \Gamma$ is shown in figure 3. As in the previous figure, the black curve concerns the line passing through $(0, 0, 0)$ and the grey curve concerns the line passing through $(2L, 0, 0)$. The systematic error occurring near $S_E$ and the scattering near the body geometry are kept below 0.1% of the generated field. As expected, more constrained mesh elements along the vertical lines correspond to a better resolution. Figure 4 shows a plot of the maximum relative difference $\Delta \Gamma / \Gamma$ near the internal surface $S_I$ versus the number of mesh elements every $L$. It is worth noting that 0.1% relative accuracy is reached after about 160 s of calculation time (CPU 2.26 GHz, 2.86 GB RAM). Significantly better results can be achieved by running the FEM algorithm on a platform with better CPU characteristics, which allows us to increase the radius of $S_E$ and the number of mesh elements constrained along the lines.

4. The cold atomic gravimeter

A scheme of the CAG setup is presented in [6, 8]. It performs a cyclic measurement of the gravitational acceleration $g$ with a cloud of $^{87}\text{Rb}$ cold atoms used as a test mass [16]. The gravimeter is shown in figure 5 during ICAG’09 [17] at BIPM and its core is shown in figure 6.

A detailed description of the principle of the gravimeter can be found in [16]. Briefly, an atomic cloud is loaded in a 3D-MOT (magneto-optical trap) and is further cooled down to $2\mu$K before being released. While the atoms fall down, three Raman pulses separated by $T (\pi/2 - \pi - \pi/2)$ split, redirect and recombine the atomic wave packets. They are induced by two vertical counter-propagating laser beams of wave-vectors $\vec{k}_1$ and $\vec{k}_2$ which couple the hyperfine levels $|F=1\rangle$ and $|F=2\rangle$ of the $^5S_{1/2}$ ground state via a two-photon transition [18]. They are delivered to the atoms through a single collimator [16] and retroreflected by a mirror placed inside the vacuum chamber. Due to conservation of angular momentum and to the Doppler shift induced by the free fall of the atoms, only two counter-propagating of the four beams will drive the Raman transitions. This feature allows us to create interferometers using effective wave-vector $\vec{k}_{\text{eff}} = \vec{k}_1 - \vec{k}_2$ pointing upwards or downwards. Finally, thanks to the state labelling method [19], the interferometer phase shift $\Delta \Phi$, which is the difference of the atomic phases accumulated along the two paths I and II (figure 6), is deduced from a fluorescence measurement.
measurement of the populations of each of the two states. It is given by \[ \Delta \Phi = \pm |\vec{k}_{\text{eff}}| |g| T^2, \] (2)
where $|\vec{k}_{\text{eff}}| = |\vec{k}_1| + |\vec{k}_2|$ for counter-propagating beams. Performing the interferometer with initial atoms in state $| F = 1 \rangle$ leads to different paths I and II using $\vec{k}_{\text{eff},1}$ or $\vec{k}_{\text{eff},2}$ (in black and grey in figure 6). The interferometer takes place in between the centre of the MOT chamber ($z = 0$ m) and the detection setup area ($z = -0.16$ m), through the free-fall vacuum chamber (FF). The atoms travel in an empty vertical cylinder of 40 mm diameter. The actual trajectory depends on the delay $t_1$ of the first pulse and on the time $T$ separating the Raman pulses. In our case, $t_1 = 16$ ms and $T = 70$ ms, so that the interferometer occurs in the region ($-1.3$ mm; $-120.2$ mm).

5. Mass attraction along the trajectory of the gravimeter

The CAG apparatus consists of four main parts, (i) the gravimeter core presented in figure 6, (ii) an isolation platform used to filter the vibrations due to background noise [21], (iii) a thermal and acoustic insulating box [22] and (iv) an optical bench [23].

The level of detail required in the modelling of each of the four parts depends on their mass and location with respect to the atomic trajectories. Most of the attention was focused on the gravimeter core, made of several sub-components located close to the free-falling atoms. The isolation platform was modelled with a homogenous box approximating the inner mass distribution. Due to the distance to the atoms, details of the elements inside the platform would not change the results. The insulating box was easily modelled with four lateral walls and a roof. Although the mass of the optical bench is significant, it is far away and the direction of its gravitational attraction is nearly perpendicular to the free-falling motion. The numerical simulation of its effect was not necessary (maximum effect < $10^{-10}$ m s$^{-2}$).

Taking advantage of the linear additivity of the gravitational field, the perturbation was evaluated by splitting the whole setup into its sub-components, computing the vertical components of the field intensities along the atoms’ trajectory and adding the contributions.

All the objects and the results were modelled in a reference frame $Oxyz$, having the origin located at the start of the atom’s drop, with the $z$-axis vertical and oriented upwards.

We choose to use a $S_E$ radius 100 times larger than the maximum distance between the object surface and its centre of mass, significantly larger than the factor of 20 used in section 3 for which the maximum error was as low as 0.1%. On the other hand, the adaptive mesh and the elements constrained along the trajectory were used depending on the magnitude of the effect. In most cases, user selection of the mesh size without element constraint was enough. Figure 7 shows the mesh pattern used to compute the effect of the magneto-optical trap. From numerous tests carried out for the evaluation of the modelling uncertainty we estimate the net error well below 1% of the generated gravity field.

Figure 8 shows the average effect for each component, from the biggest to the lowest one. The larger attraction is due to the free-fall chamber which increases the $g$ value by more than $3 \times 10^{-8}$ m s$^{-2}$. The magneto-optical trap (MOT) vacuum chamber compensates for most of this effect as anticipated during the design. The third component modifying $g$ with an impact larger than $10^{-6}$ m s$^{-2}$ is the detection setup, other elements having a smaller impact. As an example, the insulating box acts upwards by about $0.3 \times 10^{-8}$ m s$^{-2}$. Figure 9 displays the perturbation due to the mass attraction along the trajectory of the atoms. The parasitic acceleration is oriented downwards and varies from $1.3 \times 10^{-8}$ m s$^{-2}$ to $2.9 \times 10^{-8}$ m s$^{-2}$ along the atoms’ trajectory with a minimum of $0.5 \times 10^{-8}$ m s$^{-2}$ between the second and the third Raman pulses. The effect is stronger after the last pulse, in the detection area ($<-12$ mm), and close to $3 \times 10^{-8}$ m s$^{-2}$.

6. Effect on the gravity measurement

When small enough, parasitic phase shifts of the interferometer can be accurately calculated using a perturbative path integral treatment [24]. This approach, already used in [25, 26] to treat the effect of a linear vertical gravity gradient, is used here to calculate the perturbation of the interferometer phase due to the mass attraction effect of the device itself ($\Delta \Phi_{\text{f}}$). It consists in integrating the perturbed Lagrangian $\mathcal{L}_{\text{pert}}$ along the unperturbed paths I and II:

\[
\Delta \Phi_{\text{f}} = \frac{\hbar}{\bar{h}} \left( \int_I \mathcal{L}_{\text{pert}}[z(t), \dot{z}(t)] \, dt - \int_{\text{II}} \mathcal{L}_{\text{pert}}[z(t), \dot{z}(t)] \, dt \right),
\] (3)

where $\bar{h}$ is the reduced Planck constant. In our case, the perturbed Lagrangian is linked to the perturbed potential energy which is obtained by integrating the attraction force:

\[
\mathcal{L}_{\text{pert}} = -\delta E_{P} = m \int \Gamma(z) \, dz = mf(z),
\] (4)
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Figure 6. Scheme of the interferometer. Left, vacuum chamber partly cut together with some parts of the apparatus, with the magnetic shields partially removed. Centre, enlarged view of the free-fall section. Right, atomic trajectories with Raman wave-vector pointing either downwards (black line) or upwards (grey lines). The Raman pulses are depicted with grey filled areas. For the sake of clarity the value of the wave-vector $k_{\text{eff}}$ has been multiplied by 15 to distinguish the separation between path I (OABD) (dash lines) and path II (OACD) (continuous lines).

Figure 7. Two views of the MOT chamber meshed.

where $m$ is the mass of the falling atoms. Figure 10 shows the function $f(z)$ obtained by integrating the mass attraction $\Gamma(z)$ plotted in figure 9.

The position of the falling atoms $z(t)$ is calculated for each path I and II taking into account the Raman pulses changing the velocity in B for path I and in A and C for path II, leading to the functions represented in figure 6:

$$z_{\text{AB}}(t), \; z_{\text{BD}}(t), \; z_{\text{AC}}(t) \; \text{and} \; z_{\text{CD}}(t).$$

Combining equations (4) and (5) in equation (3), $\Delta \Phi_T$ can be expressed as

$$\Delta \Phi_T = \frac{m}{\hbar} \left( \int_{t_1}^{t_1+T} f(z_{\text{AB}}(t)) \, dt - \int_{t_1}^{t_1+2T} f(z_{\text{BD}}(t)) \, dt - \int_{t_1}^{t_1+T} f(z_{\text{AC}}(t)) \, dt + \int_{t_1}^{t_1+2T} f(z_{\text{CD}}(t)) \, dt \right),$$

with $t_1$ the time of the first Raman pulse. With our experimental parameters, the bias obtained with equation (2) is $1.27 \times 10^{-8} \text{ m s}^{-2}$ whatever the direction of $k_{\text{eff}}$. The difference between the two directions is negligible ($10^{-11} \text{ m s}^{-2}$). Moreover, due to the finite temperature, the atoms also move radially while falling. At $2 \mu\text{K}$, a point-like atomic cloud reaches a $1/e^2$ diameter of $4 \text{ mm}$ in the detection zone. We thus calculate the mass attraction $\Gamma$ for trajectories off-centred by $5 \text{ mm}$ in the two horizontal directions $x$ and $y$, to calculate the corresponding shifts on the $g$ measurement. We found differences with respect to the centred trajectory as small as $4 \times 10^{-11} \text{ m s}^{-2}$.

The global uncertainty in the calculation can be estimated by summing quadratically the uncertainties corresponding of the individual pieces. We find $5 \times 10^{-10} \text{ m s}^{-2}$. In addition to the uncertainty in the calculation, which can be further reduced by improving the mesh, we have also to account for additional uncertainties due to modelling in the mass distribution. They arise from simplifications in the geometrical description of the considered sub-components, from the influence of neglected sub-components and from imperfect knowledge of the densities. We assign a relative uncertainty of 1% as a conservative estimate for these contributions, which is
still negligible. The total uncertainty for the self-gravity error is therefore increased to $10^{-9} \text{ms}^{-2}$ which is, up to now, much lower than the targeted accuracy of the CAG.

Finally we consider the $g$ data collected with the CAG to be corrected by $(1.3 \pm 0.1) \times 10^{-8} \text{ms}^{-2}$ due to the self-gravity effect. This result agrees with the preliminary rough estimation of the effect of $(0 \pm 2) \times 10^{-8} \text{ms}^{-2}$ performed for last ICAG’09 [17].

7. Conclusion

We have described a numerical method based on a FEM simulation for accurate determination of the gravitational attraction generated by any distribution of massive objects, applied to the self-gravity effect on an atomic gravimeter. Using a perturbative treatment we find a relatively small systematic effect of $(1.3 \pm 0.1) \times 10^{-8} \text{ms}^{-2}$ thanks to a symmetric design of the vacuum chamber. It shows this study was required in order to achieve the targeted accuracy.

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