INFINITELY MANY NON-RADIAL SOLUTIONS FOR THE PRESCRIBED CURVATURE PROBLEM OF FRACTIONAL OPERATOR

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ABSTRACT. We consider the following problem:

\[
\begin{aligned}
(-\Delta)^s u &= K(y) u^{p-1} \quad \text{in } \mathbb{R}^N, \\
u > 0, \quad y \in \mathbb{R}^N,
\end{aligned}
\]

(P)

where \( s \in (0, \frac{1}{2}) \) for \( N = 3 \), \( s \in (0, 1) \) for \( N \geq 4 \) and \( p = \frac{2N}{N-2s} \) is the fractional critical Sobolev exponent. Under the condition that the function \( K(y) \) has a local maximum point, we prove the existence of infinitely many non-radial solutions for the problem \( (P) \).

1. Introduction. In this paper, we are concerned with the following problem:

\[
\begin{aligned}
(-\Delta)^s u &= K(y) u^{p-1} \quad \text{in } \mathbb{R}^N, \\
u > 0, \quad y \in \mathbb{R}^N,
\end{aligned}
\]

(1)

where \( s \in (0, \frac{1}{2}) \) for \( N = 3 \), \( s \in (0, 1) \) for \( N \geq 4 \) and \( p = \frac{2N}{N-2s} \) is the fractional critical Sobolev exponent. For any \( s \in (0, 1) \), \( (-\Delta)^s \) is the nonlocal operator defined as

\[
(-\Delta)^s u = c(N, s) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy = c(N, s) \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(x, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,
\]

where \( P.V. \) is the principal value and \( c(N, s) = \pi^{-(2s+\frac{N}{2})} \frac{\Gamma(\frac{N}{2} + s)}{\Gamma(-s)} \) (for more details on the fractional Laplacian, we refer to [11] and the references therein).

In recent years, problems with the fractional Laplacian have been extensively studied, see for example, [2]-[5], [13], [19], [20], [23] and the references therein. In particular, in [2], [4], [13], [20] and [23], some critical problems for the fractional Laplacian are considered.

It is well known that when \( K(y) = 1 \), the following functions

\[
U_{x, \Lambda}(y) = (4^s \gamma)^{\frac{N-2s}{2}} \left( \frac{\Lambda}{1 + \Lambda^2 |y - x|^2} \right)^{\frac{N-2s}{2}}, \quad \Lambda > 0, \quad x \in \mathbb{R}^N,
\]

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where \(\gamma = \frac{\Gamma(N/2 + s)}{\Gamma(N/2 - s)}\), are the unique (up to the translation and scaling) solutions for the problem (see [16]):

\[
(-\Delta)^s u = u^\frac{N+2}{N-2}, \quad u > 0 \text{ in } \mathbb{R}^N. \tag{2}
\]

We are interested in the existence of non-radial positive solutions. Note that, in general, even for \(s = 1\), the problem (1) does not always admit a solution. The sufficient condition on \(K\), under which the following problem:

\[
\begin{cases}
-\Delta u = K(y)u^\frac{N+2}{2}, & \text{in } \mathbb{R}^N, \\
u > 0, \; y \in \mathbb{R}^N,
\end{cases} \tag{3}
\]

admits a solution, has been extensively studied, see [1], [6]-[9], [14], [15], [17], [18], [22] and references therein.

In this paper, following the idea from Wei and Yan [21] dealing with the problem (3), we use the techniques in dealing with the singularly perturbed elliptic problems and use the number of the bubbles solutions to the equation (2) as the parameter in the construction of non-radially symmetric solutions for problem (1).

We consider the Schwartz space \(\mathcal{S}\) of rapidly decaying \(C^\infty\) functions on \(\mathbb{R}^N\). For any \(\varphi \in \mathcal{S}\), we define the Fourier transformation of \(\varphi\) by:

\[
\mathcal{F}\varphi(\xi) = \frac{1}{(2\pi)^\frac{N}{2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} \varphi(x) dx.
\]

We will look for solutions of the problem (1) in the energy space:

\[
D^s_s(\mathbb{R}^N) = \{ u \in L^\frac{2N}{N-2s}(\mathbb{R}^N) : \|(-\Delta)^s u\|_{L^2(\mathbb{R}^N)} < +\infty \}
\]

with the norm:

\[
\|u\|_{D^s_s(\mathbb{R}^N)} = \|(-\Delta)^s u\|_{L^2(\mathbb{R}^N)},
\]

where \(\|(-\Delta)^s u\|_{L^2(\mathbb{R}^N)}\) is defined by \(\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi\).

We define the functional \(I\) on \(D^s_s(\mathbb{R}^N)\) by:

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dy - \frac{1}{p} \int_{\mathbb{R}^N} K(y)|u|^p dy. \tag{4}
\]

Then the solutions of problem (1) correspond to the critical points of the functional \(I\).

Set

\[
\mu = k^\frac{N-2s}{2s-m}
\]

to be the scaling parameter. Using the transformation \(u(y) \mapsto \mu^{-\frac{N-2s}{2s-m}} u(\frac{y}{\mu})\), problem (1) becomes:

\[
\begin{cases}
(-\Delta)^s u = K(\frac{|y|}{\mu})u^{p-1}, & u > 0, \; y \in \mathbb{R}^N, \\
u \in D^s_s(\mathbb{R}^N).
\end{cases} \tag{5}
\]

Let \(y = (y', y'')\), \(y' \in \mathbb{R}^2, y'' \in \mathbb{R}^{N-2}\). Define

\[
H_s = \{ u : u \in D^s_s(\mathbb{R}^N), u \text{ is even in } y_i, i = 3, \ldots, N, \}
\]

and

\[
u(r \cos(\theta + \frac{2\pi j}{k}), r \sin(\theta + \frac{2\pi j}{k}), y'') = u(r \cos \theta, r \sin \theta, y'').
\]

Let

\[
x_j = (r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0), \; j = 1, \ldots, k,
\]

where \(k\) is any positive integer.
Lemma 2.1. For any constants $0 < \sigma < N - 2s$, there exists a constant $C > 0$, such that

$$W_{r, \Lambda}(y) = \sum_{j=1}^{k} U_{x_j, \Lambda}(y).$$

We assume that:

(K) $K(y)$ is rotationally symmetric and bounded. There is a constant $r_0 > 0$, such that $K(r)$ has a local maximum at $r_0$ and

$$K(r) = K(r_0) - c_0 |r - r_0|^m + O(|r - r_0|^{m+\theta}), \ r \in (r_0 - \delta, r_0 + \delta),$$

where $c_0, \theta, \delta > 0$ are some constants, and the constant $m \geq 2$ satisfies $N - 2s - \frac{(N-2s)^2 - 2s}{N+2s} < m < N - 2s$.

Without loss of generality, we assume that $K(r_0) = 1$.

Our main result is:

**Theorem 1.1.** Suppose that $s \in (0, \frac{1}{2})$ for $N = 3$ and $s \in (0, 1)$ for $N \geq 4$. If $K(y)$ satisfies the condition (K), then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, problem (1) has a solution $u_k$ of the form

$$u_k = W_{r_k, \Lambda_k} + \phi_k,$$

where $\phi_k \in H_s, \|\phi_k\|_{L^\infty} \to 0 \ (k \to \infty)$, $r_k \in [r_0 \mu - \frac{1}{\mu^s}, r_0 \mu + \frac{1}{\mu^s}]$, and $L_0 \leq \Lambda_k \leq L_1$ for some constants $L_1 > L_0 > 0$.

Throughout the paper, $C$ denote some positive constants.

2. Some basic estimates. In this section, we give some basic estimates.

**Lemma 2.1.** For any constants $0 < \sigma < N - 2s$, there exists a constant $C > 0$, such that

$$\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2s}(1 + |z|)^{2s+\sigma}} \, dz \leq \frac{C}{(1 + |y|)^\sigma}.$$

**Proof.** The proof of this lemma is similar to that of Lemma B.2 in [21] and Lemma 2.2 in [12]. For the sake of completeness, we give a sketch of the proof. Without loss of generality, we set $|y| \geq 2$, and let $d = \frac{|y|}{2}$, then

$$\int_{B_d(0)} \frac{dz}{|y - z|^{N-2s}(1 + |z|)^{2s+\sigma}} \leq \frac{C}{d^{N-2s}} \int_0^d r^{N-1} \, dr \leq C,$$

and

$$\int_{B_d(y)} \frac{dz}{|y - z|^{N-2s}(1 + |z|)^{2s+\sigma}} \leq \frac{C}{d^{2s+\sigma}} \int_{B_d(y)} \frac{dz}{|y - z|^{N-2s}} \leq C.$$

For $z \in \mathbb{R}^N \setminus \{B_d(0) \cup B_d(y)\}$, we have $|y - z| \geq \frac{|y|}{2}$ and $|z| \geq \frac{|y|}{2}$. If $|z| \geq 2|y|$, then $|z - y| \geq |z| - |y| \geq \frac{|y|}{2}$, and if $|z| < 2|y|$, then $|y - z| \geq \frac{|y|}{2} > \frac{|z|}{4}$. Thus, we have

$$\int_{\mathbb{R}^N \setminus \{B_d(0) \cup B_d(y)\}} \frac{dz}{|y - z|^{N-2s}(1 + |z|)^{2s+\sigma}} \leq C \int_{\mathbb{R}^N \setminus \{B_d(0) \cup B_d(y)\}} \frac{dz}{(1 + |z|)^{2s+\sigma}|z|^{N-2s}} \leq C \frac{d^\sigma}{d^\sigma} = C.$$
Lemma 2.2. Suppose that $\tau \in (0, \frac{N-2s}{2})$. Then there is a small $\theta > 0$, such that

$$
\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2s}} W_{r,\Lambda}^{\frac{4s}{N-2s}}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z-x_j|)^{\frac{N-2s-\tau}{2}} + \tau + \theta} dz
\leq C \sum_{j=1}^{k} \frac{1}{(1 + |y-x_j|)^{\frac{N-2s-\tau}{2}} + \tau + \theta}.
$$

Proof. Since

$$
|x_j - x_1| = 2|x_1| \sin \left(\frac{(j-1)\pi}{k}\right), \quad j = 2, \ldots, k,
$$

for any $\lambda > 0$, we have

$$
\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^\lambda} = \frac{1}{(2|x_1|)^\lambda} \sum_{j=2}^{k} \frac{1}{(\sin \left(\frac{(j-1)\pi}{k}\right))^\lambda} = \begin{cases}
\frac{2}{(2|x_1|)^\lambda} \sum_{j=2}^{\frac{k}{2}} \frac{1}{(\sin \left(\frac{(j-1)\pi}{k}\right))^\lambda} + \frac{1}{(2|x_1|)^\lambda}, & \text{if } k \text{ is even;} \\
\frac{2}{(2|x_1|)^\lambda} \sum_{j=2}^{\frac{k}{2}+1} \frac{1}{(\sin \left(\frac{(j-1)\pi}{k}\right))^\lambda}, & \text{if } k \text{ is odd.}
\end{cases}
$$

Note that there exist two constants $C_1, C_2 > 0$ such that

$$
0 < C_1 \leq \frac{\sin \left(\frac{(j-1)\pi}{k}\right)}{k} \leq C_2, \quad j = 2, \ldots, \left\lfloor \frac{k}{2} \right\rfloor + 1. \tag{7}
$$

For any $\lambda = \tau_1 > \frac{N-2s-m}{N-2s}$, we have

$$
\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\tau_1}} \leq C \frac{k}{\lambda} \sum_{j=1}^{k} \frac{1}{|x_j - x_1|^{\tau_1}} \leq \begin{cases}
C \frac{(k)^{\tau_1}}{\mu^{\tau_1}} \leq C, & \tau_1 > 1, \\
C \frac{(k)^{\tau_1}}{\mu^{\tau_1}} \leq C, & \tau_1 \leq 1.
\end{cases} \tag{8}
$$

Define

$$
\Omega_j = \{y : y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}, \langle y', y'' \rangle \geq \cos \frac{\pi}{k}\}, \quad j = 1, \ldots, k. \tag{9}
$$

Since, for any $z \in \Omega_1$ and $j > 1$, $|z - x_j| \geq |z - x_1|$ and $|z - x_j| \geq \frac{1}{2}|x_j - x_1|$, we have

$$
\sum_{j=2}^{k} \frac{1}{(1 + |z-x_j|)^{N-2s}} \leq \frac{1}{(1 + |z-x_1|)^{N-2s-\tau_1}} \sum_{j=2}^{k} \frac{1}{(1 + |z-x_j|)^{\tau_1}} \leq \frac{C}{(1 + |z-x_1|)^{N-2s-\tau_1}} \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\tau_1}} \leq \frac{C}{(1 + |z-x_1|)^{N-2s-\tau_1}};
$$

and

$$
W_{r,\Lambda}^{\frac{4s}{N-2s}}(z) \leq C \sum_{j=1}^{k} \frac{1}{(1 + |z-x_j|)^{N-2s}} \leq \frac{C}{(1 + |z-x_1|)^{4s-\frac{\lambda}{N-2s}}}. \tag{7}
$$
Thus, for \( z \in \Omega_1 \), we have
\[
W_{r,A}^{\frac{4s}{N}}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{N-2s}{2}+\tau}} \leq \frac{C}{(1 + |z - x_1|)^{\frac{N-2s}{2}s}} + \frac{C}{(1 + |z - x_1|)^{\frac{N-2s}{2}s}} \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^r},
\]
Similar to the proof of Lemma 2.1, we have
\[
\int_{\Omega_1} \frac{1}{|y - z|^{N-2s}} W_{r,A}^{\frac{4s}{N}}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\frac{N-2s}{2}+\tau}} dz \leq \int_{\Omega_1} \frac{1}{|y - z|^{N-2s}} \frac{C}{(1 + |z - x_1|)^{\frac{N-2s}{2}s} - \frac{C}{(1 + |y - x_1|)^{\frac{N-2s}{2}s} + \tau}} dz \leq \frac{C}{(1 + |y - x_1|)^{\frac{N-2s}{2}s} - \frac{C}{(1 + |y - x_1|)^{\frac{N-2s}{2}s} + \tau}}.
\]
By the symmetry, we have
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2s}} W_{r,A}^{\frac{4s}{N}}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_i|)^{\frac{N-2s}{2}+\tau}} dz = \sum_{j=1}^{k} \int_{\Omega_j} \frac{1}{|y - z|^{N-2s}} W_{r,A}^{\frac{4s}{N}}(z) \sum_{i=1}^{k} \frac{1}{(1 + |z - x_i|)^{\frac{N-2s}{2}+\tau}} dz \leq \sum_{j=1}^{k} \frac{C}{(1 + |y - x_j|)^{\frac{N-2s}{2}s} - \frac{C}{(1 + |y - x_j|)^{\frac{N-2s}{2}s} + \tau}},
\]
where we choose \( \tau_1 < \frac{(N-2s)2s}{N+2s} \).

3. **Finite dimensional reduction.** In this section, we perform a finite dimensional reduction by using \( W_{r,A}(y) \) as the approximation solution and considering the linearization of the problem (5) around the approximation solution \( W_{r,A}(y) \). We first introduce the following norms:
\[
||u||_* = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2}+\tau}} \right)^{-1} |u(y)|
\]
and
\[
||f||_{**} = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2}+\tau}} \right)^{-1} |f(y)|,
\]
where \( \max\{\frac{6s-N}{2}, \frac{N-2s-m}{N-2s}\} < \tau < \min\{1 + \eta, 2s, \frac{N-2s}{2}\} \) and \( \eta > 0 \) is small.
Let
\[
Z_{1,1} = \frac{\partial U_{r,A}}{\partial r}, \quad Z_{1,2} = \frac{\partial U_{r,A}}{\partial \Lambda}.
\]
We consider the following problem:

\[
\begin{align*}
(-\Delta)^s \phi_k - (p-1)K(\frac{|x|}{r})W_{r,\Lambda}^{p-2}\phi_k &= h_k + \sum_{l=1}^{2} c_l \sum_{i=1}^{k} U_{x_i,\Lambda}^{p-2}Z_{i,l}, \quad \text{in } \mathbb{R}^N, \\
\phi_k &\in H_s, \\
\langle U_{x_i,\Lambda}^{p-2}Z_{i,l}, \phi_k \rangle &= 0, \; i = 1, \ldots, k, \; l = 1, 2,
\end{align*}
\]

for some numbers \(c_l\), where \(\langle u, v \rangle = \int_{\mathbb{R}^N} uv\).

**Lemma 3.1.** Assume that \(\phi_k\) solves problem (10). If \(\|h_k\|_\ast \to 0\) as \(k \to \infty\), then \(\|\phi_k\|_\ast \to 0\) as \(k \to \infty\).

**Proof.** We follow the idea in [21] and proceed the proof by contradiction arguments. Assume that there exist \(h_k\) with \(\|h_k\|_\ast \to 0\) as \(k \to \infty\), \(\|\phi_k\|_\ast \geq c > 0\) with \(\Lambda = \Lambda_k\), \(\Lambda_k \in [L_1, L_2]\) and \(r = r_k \in [r_0 \mu - \frac{1}{\mu^r}, r_0 \mu + \frac{1}{\mu^r}]\). Without loss of generality, we can assume that \(\|\phi_k\|_\ast \equiv 1\). We have

\[
\phi_k(y) = (p-1)\sigma_{N,s} \int_{\mathbb{R}^N} \frac{K(\frac{|z|}{r})}{|y-z|^{N-2s}} W_{r,\Lambda}^{p-2}(z)\phi_k(z)dz \\
+ \sigma_{N,s} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2s}} \left[h(z) + \sum_{l=1}^{2} c_l \sum_{i=1}^{k} U_{x_i,\Lambda}^{p-2}(z) \cdot Z_{i,l}(z)\right]dz \\
=: J_1 + J_2,
\]

for some explicit positive constant \(\sigma_{N,s}\).

For the first term \(J_1\), by Lemma 2.2, we have

\[
|J_1| \leq C\|\phi_k\|_\ast \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2s}(1 + |z-x_i|)^{\frac{N-2s}{2} + \tau}} dz \\
\leq C\|h_k\|_\ast \sum_{i=1}^{k} \frac{1}{(1 + |y-x_i|)^{\frac{N-2s}{2} + \tau + \theta}}.
\]

For the second term \(J_2\), we make use of Lemma 2.1, so that

\[
|J_2| \\
\leq C\|h_k\|_\ast \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{1}{|y-z|^{N-2s}(1 + |z-x_i|)^{\frac{N-2s}{2} + \tau}} dz \\
+ C \sum_{l=1}^{2} |c_l| \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{1}{|y-z|^{N-2s}(1 + |z-x_i|)^{N+2s}} dz \\
\leq C\|h_k\|_\ast \sum_{i=1}^{k} \frac{1}{(1 + |y-x_i|)^{\frac{N-2s}{2} + \tau}} \\
+ C \sum_{l=1}^{2} |c_l| \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{1}{|y-z|^{N-2s}(1 + |z-x_i|)^{\frac{N-2s}{2} + \tau}} dz \\
\leq C\|h_k\|_\ast \sum_{i=1}^{k} \frac{1}{(1 + |y-x_i|)^{\frac{N-2s}{2} + \tau}} + C \sum_{l=1}^{2} |c_l| \sum_{i=1}^{k} \frac{1}{(1 + |y-x_i|)^{\frac{N-2s}{2} + \tau}}.
\]
Then, we have
\[
\left(\sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N-2s}{2} + \tau}}\right)^{-1} |\phi_k| \leq C \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N-2s}{2} + \tau}} + C\|h_k\|_{**} + C \sum_{i=1}^{2} |c_i|.
\]

Multiply both sides of (10) by $Z_{1,t}$, we have
\[
\sum_{l=1}^{2} c_l \sum_{i=1}^{k} \langle U^{p-2}_{x_i, \Lambda} Z_{i,t}, Z_{1,t} \rangle = \langle (-\Delta)^s \phi_k - (p-1)K\left(\frac{|y|}{\mu}\right)W^{p-2}_{r, \Lambda} \phi_k, Z_{1,t} \rangle - \langle h_k, Z_{1,t} \rangle.
\]

First of all, there exists a constant $\overline{c} > 0$ such that
\[
\sum_{i=1}^{k} \langle U^{p-2}_{x_i, \Lambda} Z_{i,t}, Z_{1,t} \rangle = (\overline{c} + o(1))\delta_{1,t}.
\]

On the other hand, we have
\[
\|h_k, Z_{1,t}\| \leq C\|h_k\|_{**} \int_{\mathbb{R}^N} \frac{1}{(1 + |y - x_1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N-2s}{2} + \tau}}
\]
\[
\leq C\|h_k\|_{**} \left[ \int_{\mathbb{R}^N} \frac{1}{(1 + |y - x_1|)^{\frac{N}{2} - s + \tau}} + \sum_{j=1}^{k} \int_{\Omega_j} \frac{1}{(1 + |y - x_1|)^{N-2s}} \sum_{i=2}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N-2s}{2} + \tau}} \right].
\]

Note that $\frac{N-2s}{2} > 1$. For $j = 1$, by (8), we have
\[
\int_{\Omega_1} \frac{1}{(1 + |y - x_1|)^{N-2s}} \sum_{i=2}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N-2s}{2} + \tau}} \leq C \int_{\Omega_1} \frac{1}{(1 + |y - x_1|)^{N-2s}} \sum_{i=2}^{k} \frac{1}{|x_i - x_1|^{\frac{N-2s}{2}}}
\]
\[
\leq C\left(\frac{k}{\mu}\right)^{\frac{N-2s}{2}} \int_{\Omega_1} \frac{1}{(1 + |y - x_1|)^{N-2s}}.
\]

For $j > 1$, by the symmetry, $\sum_{i=1}^{k} \frac{1}{|x_i - x_j|^s} = \sum_{i=2}^{k} \frac{1}{|x_i - x_j|^s}$ for $\lambda > 0$. Since, for any $y \in \Omega_j$ and $i \neq j$, $|y - x_i| \geq |y - x_j|$ and $|y - x_i| \geq \frac{1}{2}|x_i - x_j|$, we have
\[
\int_{\Omega_j} \frac{1}{(1 + |y - x_1|)^{N-2s}} \sum_{i=2}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N-2s}{2} + \tau}} \leq \int_{\Omega_j} \frac{1}{(1 + |y - x_1|)^{N-2s}} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}}
\]
Moreover, one has

\[
\begin{align*}
&\frac{1}{(1 + |y - x_1|)^{N-2s}} \sum_{i=2}^{k} \frac{1}{(1 + |y - x_i|)^{N-2s} + \tau} \\
&\leq \int_{\Omega_j} \left[ \frac{1}{(1 + |y - x_1|)^{N-2s}} \frac{1}{(1 + |y - x_j|)^{N-2s}} \right. \\
&\quad \quad \quad \quad \quad \left. + \frac{1}{(1 + |y - x_j|)^{N-2s}} \sum_{i=2}^{k} \frac{1}{(1 + |y - x_i|)^{N-2s}} \right] \\
&\leq C \int_{\Omega_j} \left[ \frac{1}{(1 + |y - x_1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{|x_i - x_j|^{N-2s}} \right] \\
&\leq C \frac{N-2s}{\mu} \int_{\Omega_j} \frac{1}{(1 + |y - x_j|)^{N-2s}}.
\end{align*}
\]

Since \( \frac{N-2s-m}{N-2s} < 1 \leq \frac{m}{2} \), we have

\[
|h_k, Z_{1,t}| \leq C(1 + k \left( \frac{N-2s}{\mu} \right) \|h_k\|_{**} \leq C\|h_k\|_{**}.
\]

Moreover, one has

\[
\langle (-\Delta)^s \phi_k - (p-1)K\left( \frac{|y|}{\mu} \right) W_{r, \Lambda}^{p-2} \phi_k, Z_{1,t} \rangle
\]

\[
= \langle (-\Delta)^s Z_{1,t}, \phi_k \rangle - (p-1) \langle K\left( \frac{|y|}{\mu} \right) W_{r, \Lambda}^{p-2} Z_{1,t}, \phi_k \rangle
\]

\[
= (p-1) \langle (U_{x_1, \Lambda}^{p-2} - W_{r, \Lambda}^{p-2}) Z_{1,t}, \phi_k \rangle - (p-1) \langle (K\left( \frac{|y|}{\mu} \right) - 1) W_{r, \Lambda}^{p-2} Z_{1,t}, \phi_k \rangle
\]

\[
=: I_1 - I_2.
\]

If \( p > 3 \), then \( 2s > 1 \) and

\[
|I_1|
\]

\[
\leq C\|\phi_k\| \int_{\mathbb{R}^N} (W_{x_1, \Lambda}^{p-2} - U_{x_1, \Lambda}^{p-2}) |Z_{1,t}| \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s + \tau}}
\]

\[
\leq C\|\phi_k\| \int_{\mathbb{R}^N} \left( \sum_{i=2}^{k} U_{x_i, \Lambda}^{p-2} + U_{x_1, \Lambda}^{p-3} \sum_{i=2}^{k} U_{x_i, \Lambda} \right) |Z_{1,t}| \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s + \tau}}
\]

\[
\leq C\|\phi_k\| \sum_{i=1}^{k} \int_{\Omega_t} \left( \sum_{i=2}^{k} U_{x_i, \Lambda}^{p-2} + U_{x_1, \Lambda}^{p-3} \sum_{i=2}^{k} U_{x_i, \Lambda} \right) |Z_{1,t}| \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s + \tau}}.
\]

For \( l = 1 \), by (8), we have

\[
\int_{\Omega_1} \left( \sum_{i=2}^{k} U_{x_i, \Lambda}^{p-2} |Z_{1,t}| \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s}} \right)
\]

\[
\leq C \int_{\Omega_1} \left( \sum_{i=2}^{k} \frac{1}{(1 + |y - x_i|)^{N-2s}} \right)^{p-2} \frac{1}{(1 + |y - x_1|)^{\frac{N}{2} (N-2s) + \tau}}
\]
where \( m \leq C \). 

\[
\sum_{i=2}^{k} \frac{1}{(1 + |y - x_i|)^{N-2s}} \leq C \int_{\Omega_1} \frac{1}{(1 + |y - x_1|)^{\frac{N-2s}{2} + \tau}}
\]

\[
\sum_{i=2}^{k} \frac{1}{(1 + |y - x_i|)^{N-2s}} \leq C \int_{\Omega_1} \frac{1}{(1 + |y - x_1|)^{\frac{N-2s}{2} + \tau}}
\]

\[
\frac{1}{(1 + |y - x_1|)^{\frac{N-2s}{2} + \tau}} \leq C \left( \frac{k}{\mu} \right)^{2s} \int_{\Omega_1} \frac{1}{(1 + |y - x_1|)^{\frac{N-2s}{2}}}
\]

Note that \( N > 4s, 2s > 1 \) and \( \tau > \frac{6s-N}{2} \). For \( \ell > 1 \), we have

\[
\left( \sum_{i=2}^{k} U_{x_i, A} \right)^{p-2} \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N-2s}{2} + \tau}}
\]

\[
\int_{\Omega_1} \frac{1}{(1 + |y - x_1|)^{\frac{N-2s}{2} + \tau}}
\]

\[
\left( \sum_{i=2}^{k} U_{x_i, A} \right)^{p-2} \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N-2s}{2} + \tau}}
\]

\[
\left( \sum_{i=2}^{k} U_{x_i, A} \right)^{p-2} \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{\frac{N-2s}{2} + \tau}}
\]

\[
\leq C \left( \frac{k}{\mu} \right)^{2s} \int_{\Omega_1} \frac{1}{(1 + |y - x_1|)^{\frac{N-2s}{2}}}
\]

Since \( m \geq 2 \) and \( N < 6s, \frac{N-2s-m}{N-2s} < 1 < \frac{2sm}{N-2s} \). Thus, we have

\[
\sum_{l=1}^{k} \int_{\Omega_1} \frac{1}{U_{x_1, A} \sum_{i=2}^{k} U_{x_i, A}} \leq C \left( \frac{k}{\mu} \right)^{2s} \leq C(\frac{1}{\mu})^{\sigma},
\]

where \( \sigma > 0 \) is a small constant. Similarly, we have

\[
\sum_{l=1}^{k} \int_{\Omega_1} \frac{1}{U_{x_1, A} \sum_{i=2}^{k} U_{x_i, A} |Z_{1,l}|} \leq C \left( \frac{k}{\mu} \right)^{2s} \leq C(\frac{1}{\mu})^{\sigma},
\]
By the condition (K) and Lemma 2.2, we have

\[ |I_2| \leq C \|\phi_k\| \sup_{|y| - \mu r_0 \leq \sqrt{r}} \left| K\left(\frac{|y|}{\mu}\right) - 1 \right| \frac{1}{(1 + |y - x_1|)^{N-2s}} W_{r,\Lambda}^{p-2} \]

\[ + \frac{C \|\phi_k\| \sup_{|y| - \mu r_0 > \sqrt{r}} \left| K\left(\frac{|y|}{\mu}\right) - 1 \right|}{(1 + |y - x_1|)^{N-2s}} W_{r,\Lambda}^{p-2} \]

\[ \leq \frac{C \|\phi_k\|}{\mu^2} \sup_{|y| - \mu r_0 \leq \sqrt{r}} \left| K\left(\frac{|y|}{\mu}\right) - 1 \right| \frac{1}{(1 + |y - x_1|)^{N-2s}} W_{r,\Lambda}^{p-2} \]

\[ + \frac{C \|\phi_k\|}{\mu^2} \sup_{|y| - \mu r_0 > \sqrt{r}} \left| K\left(\frac{|y|}{\mu}\right) - 1 \right| \frac{1}{(1 + |y - x_1|)^{N-2s}} W_{r,\Lambda}^{p-2} \]

\[ \leq \frac{C \|\phi_k\|}{\mu^2} \sup_{|y| - \mu r_0 \leq \sqrt{r}} \left| K\left(\frac{|y|}{\mu}\right) - 1 \right| \frac{1}{(1 + |y - x_1|)^{N-2s}} W_{r,\Lambda}^{p-2} \]

At last, we get

\[ |(-\Delta)^s \phi_k - (p-1)K\left(\frac{|y|}{\mu}\right)W_{r,\Lambda}^{p-2} \phi_k, Z_{1,i} | \leq \frac{C \|\phi_k\|}{\mu^\sigma} \]

The case \( p \leq 3 \) can be discussed in a similar way. By (14), (15), (16) and (17), we have

\[ |c_i| \leq C \left( \frac{\|\phi_k\|}{\mu^\sigma} + \|h_k\|_{**} \right). \]

Then by (13) and \( \|\phi_k\|_{**} = 1 \), there is \( R > 0 \) such that

\[ \|\phi_k(y)\|_{L^\infty(B_R(x_i))} \geq \alpha > 0, \]

for some \( i \). The translated version \( \tilde{\phi}_k = \phi_k(y - x_i) \) converges uniformly in any compact set to a solution \( u \) of the following equation:

\[ (-\Delta)^s u - (p-1)U_{0,\Lambda}^{p-2} u = 0, \quad \text{in} \ \mathbb{R}^N. \]

Since \( u \) is perpendicular to the kernel of this equation, \( u = 0 \). This is a contradiction to (18). \( \square \)
Using the same argument as in the proof of Proposition 4.1 in [10], we have the following proposition.

**Proposition 1.** There exist $k_0 > 0$ and a constant $C > 0$, independent of $k$, such that for all $k \geq k_0$ and all $h \in L^\infty(\mathbb{R}^N)$, problem (10) has a unique solution $\phi = L_k(h)$. Besides,

$$\|L_k(h)\|_* \leq C\|h\|_{{**}}, \quad |c_i| \leq C\|h\|_{{**}}.$$  

(19)

Now we consider the following problem:

$$\begin{cases}
(-\Delta)^s(W_{r,\Lambda} + \phi) = K\left(\frac{|y|}{r}\right)(W_{r,\Lambda} + \phi)^{p-1} + \sum_{i=1}^{k} c_i \sum_{l=1}^{k} U_{x_i,\Lambda}^{p-2} Z_{i,l}, & \text{in } \mathbb{R}^N, \\
\phi \in H_s, \\
\langle U_{x_i,\Lambda}^{p-2} Z_{i,l}, \phi \rangle = 0, & i = 1, \ldots, k, \ l = 1, 2.
\end{cases}$$

(20)

In the rest of this section, we devote ourselves to prove the following proposition by using the contraction mapping theorem.

**Proposition 2.** There exist $k_0 > 0$ and a constant $C > 0$, independent of $k$, such that for all $k \geq k_0$, $L_0 \leq \Lambda \leq L_1$, $|r - \mu r_0| \leq \frac{1}{p^r}$, problem (20) has a unique solution $\phi = \phi(r, \Lambda)$ satisfying,

$$\|\phi\|_* \leq C\left(\frac{1}{\mu}\right)^{\frac{2}{p} + \sigma}, \quad |c_i| \leq C\left(\frac{1}{\mu}\right)^{\frac{2}{p} + \sigma},$$

(21)

where $\sigma > 0$ is a small constant.

We first rewrite (20) as

$$\begin{cases}
(-\Delta)^s \phi - (p - 1)K\left(\frac{|y|}{r}\right)W_{r,\Lambda}^{p-2} \phi = N(\phi) + l_k + \sum_{i=1}^{k} c_i \sum_{l=1}^{k} U_{x_i,\Lambda}^{p-2} Z_{i,l} \text{ in } \mathbb{R}^N, \\
\phi \in H_s, \\
\langle U_{x_i,\Lambda}^{p-2} Z_{i,l}, \phi \rangle = 0, & i = 1, \ldots, k, \ l = 1, 2,
\end{cases}$$

(22)

where

$$N(\phi) = K\left(\frac{|y|}{\mu}\right)[(W_{r,\Lambda} + \phi)^{p-1} - (p - 1)W_{r,\Lambda}^{p-2} \phi - W_{r,\Lambda}^{p-1}],$$

$$l_k = K\left(\frac{|y|}{\mu}\right)W_{r,\Lambda}^{p-1} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{p-1}.$$  

In order to use the contraction mapping theorem to prove Proposition 2, we need to estimate $N(\phi)$ and $l_k$. In the following, we assume that $\|\phi\|_*$ is small.

**Lemma 3.2.** If $s \in (0, \frac{1}{2})$ for $N = 3$ and $s \in (0, 1)$ for $N \geq 4$, then $\|N(\phi)\|_{{**}} \leq C\|\phi\|_{\min(2,p-1)}^0$.

**Proof.** We have

(i) $0 < s \leq \frac{1}{2}$, $|N(\phi)| \leq C|\phi|^{p-1}$, $N = 3$;

(ii) $0 < s \leq \frac{2}{3}$, $|N(\phi)| \leq C|\phi|^{p-1}$, $N \geq 4$;

(iii) $\frac{2}{3} < s \leq \frac{5}{6}$, $|N(\phi)| \leq \left\{ \begin{array}{ll}
C|\phi|^{p-1}, & N \geq 5, \\
 CW_{r,\Lambda}^{p-3} \phi^2 + C|\phi|^{p-1}, & N = 4;
\end{array} \right.$

(iv) $\frac{5}{6} < s < 1$, $|N(\phi)| \leq \left\{ \begin{array}{ll}
C|\phi|^{p-1}, & N \geq 6, \\
 CW_{r,\Lambda}^{p-3} \phi^2 + C|\phi|^{p-1}, & N = 4, 5.$
We only give the proof for \( s \in (\frac{5}{6}, 1) \). Firstly, we consider \( N \geq 6 \). From the symmetry, we can assume that \( y \in \Omega_1 \). One has, \(|y - x_j| \geq \frac{1}{2}|x_1 - x_j|\) for \( j > 1 \). Thus, by (8),
\[
\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^\tau} \leq 1 + C \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^\tau} \leq C.
\]

Using the Hölder inequality, we obtain:
\[
|N(\phi)| \leq C\|\phi\|^{p-1}(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}})^{p-1}
\]
\[
\leq C\|\phi\|^{p-1}\sum_{j=1}^{k} (1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^\tau})^{p-2}
\]
\[
\leq C\|\phi\|^{p-1}\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}}.
\]

When \( N = 4, 5 \), we have
\[
|N(\phi)| \leq C\|\phi\|^2(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}})^2(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s}})^{p-3}
\]
\[
+ C\|\phi\|^{p-1}(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}})^{p-1}
\]
\[
\leq C(\|\phi\|^2 + \|\phi\|^{p-1})(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}})^{p-1}
\]
\[
\leq C\|\phi\|^2\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2s}{2} + \tau}}.
\]

Hence, we obtain \( \|N(\phi)\|_{**} \leq C\|\phi\|_{\min \{2, p-1\}}^{\min \{2, p-1\}} \).

Next, we estimate \( l_k \).

**Lemma 3.3.** If \( s \in (0, \frac{1}{2}) \) for \( N = 3 \) and \( s \in (0, 1) \) for \( N \geq 4 \), then there exists a small \( \sigma > 0 \) such that \( \|l_k\|_{**} \leq C(\frac{1}{\mu})^{\frac{\sigma}{2}} \).

**Proof.** We have
\[
l_k = K(\frac{|y|}{\mu})(W_{r, \Lambda}^{p-1} - \sum_{j=1}^{k} U_{x_j, \Lambda}^{p-1}) + (K(\frac{|y|}{\mu}) - 1) \sum_{j=1}^{k} U_{x_j, \Lambda}^{p-1}
\]
\[
=: J_1 + J_2.
\]

By the symmetry, we can assume that \( y \in \Omega_1 \). Then \(|y - x_j| \geq |y - x_1|\), and
\[
|J_1| \leq C[(\sum_{j=2}^{k} U_{x_j, \Lambda}^{p-1} + U_{x_1, \Lambda}^{p-2}) \sum_{j=2}^{k} U_{x_j, \Lambda} + \sum_{j=2}^{k} U_{x_1, \Lambda}^{p-1}]
\]
\[
\leq C[(\sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s}})^{p-1} + \sum_{j=2}^{k} \frac{1}{(1 + |y - x_1|)^{4s}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s}}]
\]
Thus, we have

$$\sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N+2s}} \leq C \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s}} \right)^{p-1} + C \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{4s}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s}}.$$ 

Since $\frac{6s-N}{2} < \tau < 2s$, $N - 2s > \frac{N+2s}{2} - \tau > \frac{N-2s}{2} > 1$, we have

$$\frac{C}{(1 + |y - x_1|)^{N+2s}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s}} \leq C \frac{1}{(1 + |y - x_1|)^{N+2s - \tau}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2s - \tau}} \leq C \left( \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N+2s - \tau}} \right) \left( \sum_{j=2}^{k} \frac{1}{\kappa^2 (N+2s - \tau)} \right)^{\frac{N-2s}{2}} \leq C \left( \sum_{j=2}^{k} \frac{1}{\kappa^2 (N+2s - \tau)} \right) \left( \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N+2s - \tau}} \right)^{\frac{N-2s}{2}} \leq C \left( \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N+2s - \tau}} \right)^{\frac{N-2s}{2}} \leq C \left( \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N+2s - \tau}} \right)^{\frac{N-2s}{2}}.$$ 

Since $\tau < 1 + \eta$ for small $\eta > 0$, $\frac{N+2s}{4s} \left( \frac{N-2s}{2} - \frac{N-2s}{N+2s} \right) > 1$, using the Hölder inequality, we have

$$\left( \sum_{j=2}^{k} \frac{1}{|y - x_j|^N} \right)^{p-1} \leq \sum_{j=2}^{k} \frac{1}{|y - x_j|^N} \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N+2s}} \right)^{\frac{N-2s}{2}} \leq C \sum_{j=2}^{k} \frac{1}{|y - x_j|^N} \left( \frac{1}{\kappa^2 (N+2s - \tau)} \right)^{\frac{N-2s}{2}} \leq C \left( \frac{1}{\kappa^2 (N+2s - \tau)} \right)^{\frac{N-2s}{2}}.$$ 

Thus

$$\| J_1 \|_{**} \leq C \left( \frac{1}{\kappa^2 (N+2s - \tau)} \right)^{\frac{N-2s}{2}}.$$ 

Now, we estimate $J_2$. For $y \in \Omega_1$ and $j > 1$, we have

$$U_{x_j, A}^{p-1} \leq C \frac{1}{(1 + |y - x_1|)^{\frac{N+2s}{2} + \tau}} \frac{1}{(1 + |y - x_j|)^{\frac{N+2s}{2} - \tau}} \leq C \frac{1}{(1 + |y - x_1|)^{\frac{N+2s}{2} + \tau}} \frac{1}{|x_1 - x_j|^{\frac{N+2s}{2} - \tau}}.$$
which implies
\[
|K(\frac{|y|}{\mu}) - 1| \sum_{j=2}^{k} U_{x_j,\lambda}^{p-1} | \leq \frac{C}{(1 + |y - x_1|)^{N + 2\sigma}} \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N + 2\tau - r}} \leq \frac{C}{(1 + |y - x_1|)^{N + 2\sigma}} \left( \frac{1}{\mu} \right)^{m + \sigma}.
\]

We split the slice \( \Omega_1 \) into two parts, namely:

\( I := \{ y \in \Omega_1 : |y| - \mu r_0 \geq \delta \mu \} \) and \( II := \{ y \in \Omega_1 : |y| - \mu r_0 < \delta \mu \} \), where \( \delta > 0 \) is the constant in \( (K) \).

When \( y \in I \), we have
\[
|y - x_1| \geq |y| - |x_1| \geq |y| - \mu r_0 - |x_1| - \mu r_0 \geq \frac{1}{2} \delta \mu.
\]

Hence
\[
|K(\frac{|y|}{\mu}) - 1| U_{x_1,\lambda}^{p-1} \leq C \frac{1}{|y - x_1|^{N + 2\sigma + r}} \frac{1}{\mu^{N + 2\tau - r}} \leq C \frac{1}{(1 + |y - x_1|)^{N + 2\sigma}} \left( \frac{1}{\mu} \right)^{m + \sigma}.
\]

When \( y \in II \), by the condition \( (K) \), we have
\[
|K(\frac{|y|}{\mu}) - 1| \leq C |\frac{|y|}{\mu} - \mu r_0|^m \leq C \frac{|y| - \mu r_0|^m}{\mu^m} \leq C \frac{|y| - |x_1|^m + |x_1| - \mu r_0|^m}{\mu^m} \leq C \frac{|y| - |x_1|^m}{\mu^m} + \frac{C}{\mu^{m + m \sigma}}
\]
and \( |y| - |x_1| \leq |y| - \mu r_0 + |\mu r_0 - x_1| \leq 2\delta \mu \). As a result,
\[
\frac{|y| - |x_1|^m}{\mu^m} \leq C \frac{1}{(1 + |y - x_1|)^{N + 2\sigma}} \leq C \frac{1}{\mu^{2\sigma + \sigma}} \frac{|y - x_1|^\sigma}{(1 + |y - x_1|)^{N + 2\sigma}} \leq C \frac{1}{\mu^{2\sigma + \sigma}} \frac{1}{(1 + |y - x_1|)^{N + 2\tau + r}} \frac{1}{(1 + |y - x_1|)^{N + 2\tau - r - \frac{m}{2} - \sigma}} \leq C \left( \frac{1}{\mu} \right)^{\frac{m}{2} + \sigma},
\]
since \( \frac{N + 2s}{2} - \tau - \frac{m}{2} - \sigma > 2s - \tau - \sigma > 0 \) for small \( \sigma > 0 \). Thus, we get
\[
\|J_2\|_* \leq C \left( \frac{1}{\mu} \right)^{\frac{m}{2} + \sigma}.
\]
Combining with the obtained results, we obtain
\[
\|J_k\|_* \leq \|J_1\|_* + \|J_2\|_* \leq C \left( \frac{1}{\mu} \right)^{\frac{m}{2} + \sigma}.
\]
\( \square \)
Proof of Proposition 2. Let \( y = (y', y'') \), \( y' \in \mathbb{R}^2 \), \( y'' \in \mathbb{R}^{N-2} \), and let \( \hat{E} \) be the completion of \( C_0^\infty (\mathbb{R}^N) \) under the norm of \( \| \cdot \|_s \). Set

\[
E = \{ u | u \in \hat{E}, u \text{ is even in } y_i, i = 3, \ldots, N, \ u(r \cos(\theta + \frac{2\pi j}{k}), r \sin(\theta + \frac{2\pi j}{k}), y'') = u(r \cos(\theta), r \sin(\theta), y''), \ \| u \|_s \leq (\frac{1}{\mu})^{\frac{p}{2}}, \int_{\mathbb{R}^n} U^{p - \frac{n}{2}} u(x) \, dx \, 0, \ i = 1, 2, \ldots, k, \ l = 1, 2 \}.
\]

By Proposition 1, the solution \( \phi \) of (20) is equivalent to the following fixed point problem:

\[
\phi = A(\phi) = : L_k(N(\phi)) + L_k(\hat{l}_k).
\]

Hence, it is sufficient to prove that the operator \( A \) is a contraction map from the set \( E \) to itself. In fact, for any \( \phi \in E \), by Proposition 1, Lemma 3.2 and Lemma 3.3, we have

\[
\| A(\phi) \|_s \leq C \| L_k(N(\phi)) \|_s + C \| L_k(\hat{l}_k) \|_s,
\]

\[
\leq C \left[ \| N(\phi) \|_{s*} + \| \hat{l}_k \|_{s*} \right],
\]

\[
\leq C \left[ \| \phi \|_{s*}^{\min(p - 1, 2)} + \left(\frac{1}{\mu}\right)^{\frac{p-\sigma}{p}} \right]
\]

\[
\leq \left(\frac{1}{\mu}\right)^{\frac{p}{2}},
\]

which shows that \( A \) maps \( E \) to \( E \) itself and \( E \) is invariant under \( A \) operator.

On the other hand, we have \( |N'(t)| \leq C |t|^{p-2} \) for \( p \leq 3 \) and \( |N'(t)| \leq C(W_{p,A}^{p-3}) |t| + |t|^{p-2} \) for \( p > 3 \). If \( p \leq 3 \), \( \forall \phi_1, \phi_2 \in E \), by the Hölder inequality, we have

\[
|N(\phi_1) - N(\phi_2)| \leq C(|\phi_1| + |\phi_2|)^{p-2} |\phi_1 - \phi_2|
\]

\[
\leq C(\| \phi_1 \|_{s*}^{p-2} + \| \phi_2 \|_{s*}^{p-2}) \| \phi_1 - \phi_2 \|_s \left( \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{\frac{n+p-2}{2}}} \right)^{p-1}
\]

\[
\leq C(\| \phi_1 \|_{s*}^{p-2} + \| \phi_2 \|_{s*}^{p-2}) \| \phi_1 - \phi_2 \|_s \left( \sum_{j=1}^k \frac{1}{(1 + |y - x_j|)^{\frac{n+p-2}{2}}} \right).
\]

Hence

\[
\| A(\phi_1) - A(\phi_2) \|_s = \| L_k(N(\phi_1) - N(\phi_2)) \|_s,
\]

\[
\leq C \| N(\phi_1) - N(\phi_2) \|_{s*},
\]

\[
\leq C \left(\frac{1}{\mu}\right)^{\frac{p-\sigma}{p}} \| \phi_1 - \phi_2 \|_s
\]

\[
\leq \left(\frac{1}{\mu}\right)^{\frac{p}{2}} \| \phi_1 - \phi_2 \|_s.
\]

The case \( p > 3 \) can be discussed in a similar way. Hence \( A \) is a contraction map. The Banach fixed point theorem tells us that there exists a unique solution \( \phi \in E \) for the problem (20).

Finally, by Proposition 1, we have

\[
\| \phi \|_s \leq C \left(\frac{1}{\mu}\right)^{\frac{p}{2}} + \sigma \text{ and } |c_l| \leq C \| N(\phi) + l_k \|_{s*} \leq C \left(\frac{1}{\mu}\right)^{\frac{p}{2}} + \sigma.
\]

\( \Box \)
4. Energy expansion. Define the functional $F(r, \Lambda)$ by

$$F(r, \Lambda) := I(W_{r,\Lambda} + \phi),$$

where $\phi$ is the function obtained in Proposition 2 and $I$ is the functional of problem (5), that is:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{|(-\Delta)^\frac{s}{2} u|^2}{\mu} dy - \frac{1}{p} \int_{\mathbb{R}^N} K\left(\frac{|y|}{\mu}\right) |u|^p dy.$$ 

In this section, we give the energy expansion for $F(r, \Lambda)$ and $\frac{\partial F(r, \Lambda)}{\partial \Lambda}$.

We begin with

**Lemma 4.1.** For $k$ large enough, we have

$$I(W_{r,\Lambda}) = k\left[A + \frac{B_1}{\Lambda^m \mu^m} + \frac{B_2}{\Lambda^{m-2} \mu^m} (\mu r_0 - r)^2 - \sum_{j=2}^{k} \frac{B_3}{\Lambda^{N-2s} |x_1 - x_j|^{N-2s}} + O\left(\frac{1}{\mu^{m+\sigma}} + \frac{|\mu r_0 - r|^3}{\mu^m}\right)\right],$$

where $A$ and $B_i, i = 1, 2, 3$ are positive constants, and $r = |x_1|$.

**Proof.** By using the symmetry of slices $\Omega_j$ (defined in (9)), we have

$$\int_{\mathbb{R}^N} \frac{|(-\Delta)^\frac{s}{2} W_{r,\Lambda}|^2}{\mu} dy$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{k} \int_{\mathbb{R}^N} U_{r_{j,\Lambda}}^{p-1} U_{r_{i,\Lambda}} dy$$

$$= k \left( \int_{\mathbb{R}^N} U_{r_{1,\Lambda}}^{p} + \sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{r_{1,\Lambda}}^{p-1} U_{r_{i,\Lambda}} \right)$$

$$= k \left[ \int_{\mathbb{R}^N} U_{r_{0,1}}^{p} + \sum_{i=2}^{k} \left( (4^\gamma)^{\frac{N}{2}} \int_{\mathbb{R}^N} \Lambda^N \left( 1 + \Lambda^2 |y - x_1|^2 \right)^{\frac{N-2s}{2}} \left( 1 + \Lambda^2 |y - x_i|^2 \right)^{\frac{-N-2s}{2}} \right) \right]$$

$$= k \left[ \int_{\mathbb{R}^N} U_{r_{0,1}}^{p} + \sum_{i=2}^{k} \Lambda^{N-2s} |x_1 - x_i|^{N-2s} + O\left( k \sum_{i=2}^{k} \frac{1}{|x_1 - x_i|^{N-2s+\sigma}} \right) \right]$$

$$= k \left[ \int_{\mathbb{R}^N} U_{r_{0,1}}^{p} + \sum_{i=2}^{k} \Lambda^{N-2s} |x_1 - x_i|^{N-2s} + O\left( \frac{1}{\mu} \right)^{N-2s+\sigma} \right].$$

The curvature term can be rewritten into three terms as the following:

$$\int_{\mathbb{R}^N} K\left(\frac{|y|}{\mu}\right) |W_{r,\Lambda}(y)|^p$$

$$= k \left[ \int_{\Omega_1} K\left(\frac{|y|}{\mu}\right) U_{r_{1,\Lambda}}^{p} + \int_{\Omega_1} K\left(\frac{|y|}{\mu}\right) U_{r_{1,\Lambda}}^{p-1} \left( \sum_{i=2}^{k} U_{r_{i,\Lambda}} \right) \right]$$

$$+ O\left( \int_{\Omega_1} U_{r_{1,\Lambda}}^{p/2} \left( \sum_{i=2}^{k} U_{r_{i,\Lambda}} \right)^{p/2} \right).$$
For the first term, we have
\[
\int_{\Omega_1} K\left(\frac{|y|}{\mu}\right)U_{x_1,\Lambda}^p = \int_{\Omega_1} U_{x_1,\Lambda}^p + \int_{\Omega_1} \left(K\left(\frac{|y|}{\mu}\right) - 1\right)U_{x_1,\Lambda}^p
\]
\[
= \int_{\mathbb{R}^N} U_{0,1}^p + \int_{\Omega_1} \left(K\left(\frac{|y|}{\mu}\right) - 1\right)U_{x_1,\Lambda}^p + O\left(\frac{1}{\mu^{m+\sigma}}\right).
\]

We split the slice \(\Omega_1\) into two parts, namely,
\[
I := \{y \in \Omega_1 : \|y - \mu r_0\| \geq \delta \mu\}
\]
and
\[
II := \{y \in \Omega_1 : \|y - \mu r_0\| < \delta \mu\},
\]
where \(\delta > 0\) is the constant in \((K)\).

In the region \(I\), we have
\[
\|y - x_1\| \geq \|y\| - |x_1| \geq \|y - \mu r_0\| - |x_1| - \mu r_0 \geq \frac{\delta}{2}\mu.
\]
Thus
\[
\left| \int_I K\left(\frac{|y|}{\mu}\right) - 1\right)U_{x_1,\Lambda}^p \right| \leq C \int_I \left(\frac{1}{1 + |y - x_1|}\right)^{2N}
\]
\[
\leq \frac{C}{\mu^{N-\tau}} \int_I \left(\frac{1}{1 + |y - x_1|}\right)^{N+\tau}
\]
\[
= O\left(\frac{1}{\mu^{N-\tau}}\right).
\]

In the region \(II\), by the condition \((K)\), we have
\[
\int_{\Omega_1} K\left(\frac{|y|}{\mu}\right) - 1\right)U_{x_1,\Lambda}^p
\]
\[
= -\frac{c_0}{\mu^m} \int_{\Omega_1} \|y - \mu r_0\|^m U_{x_1,\Lambda}^p + O\left(\frac{1}{\mu^{m+\sigma}} \int_{\Omega_1} \|y - \mu r_0\|^{m+\theta} U_{x_1,\Lambda}^p\right)
\]
\[
= -\frac{c_0}{\mu^m} \int_{\mathbb{R}^N} \|y - x_1\| - \mu r_0\|^m U_{0,\Lambda}^p + O\left(\frac{1}{\mu^{m+\sigma}}\right).
\]

But
\[
\frac{1}{\mu^m} \int_{\mathbb{R}^N \setminus B_{|x_1/2}(0)} \|y - x_1\| - \mu r_0\|^m U_{0,\Lambda}^p
\]
\[
\leq C \int_{\mathbb{R}^N \setminus B_{|x_1/2}(0)} \left(\frac{|y - x_1|}{\mu^m} + 1\right) U_{0,\Lambda}^p
\]
\[
\leq C \frac{\mu^{N-\tau}}{\mu^{N-\tau}}.
\]

Set \(y = (y_1, y^*)\), \(y^* = (y_2, \ldots, y_N)\). If \(y \in B_{|x_1/2}(0)\), then \(|x_1 - y_1| \geq \frac{|x_1|}{2} > 0\) and
\[
|y - x_1| = |x_1| - y_1 + O\left(\frac{|y|^2}{|x_1| - y_1}\right) = |x_1| - y_1 + O\left(\frac{|y|^2}{|x_1|}\right).
\]

Then
\[
\|y - x_1\| - \mu r_0\|^m
\]
\[
= |x_1| - y_1 + O\left(\frac{|y|^2}{|x_1|}\right) - \mu r_0\|^m
\]
\[
\begin{align*}
=|y_1|^m + m|y_1|^{m-2}y_1(\mu r_0 - |x_1| + O\left(\frac{|y|^2}{|x_1|}\right)) \\
+ \frac{1}{2}m(m-1)|y_1|^{m-1}(\mu r_0 - |x_1| + O\left(\frac{|y|^2}{|x_1|}\right))^2 + O((\mu r_0 - |x_1| + O\left(\frac{|y|^2}{|x_1|}\right))^3).
\end{align*}
\]

Note that
\[
\int_{B|x_1/2(0)} |y_1|^{m-2}y_1 U_{0,\Lambda}^p = 0.
\]

We have
\[
\int_{B|x_1/2(0)} |y - x_1| - \mu r_0|^m U_{0,\Lambda}^p
\]
\[
= \int_{\mathbb{R}^N} |y_1|^m U_{0,\Lambda}^p + \frac{1}{2}m(m-1) \int_{\mathbb{R}^N} |y_1|^{m-2}U_{0,\Lambda}(\mu r_0 - |x_1|)^2 \\
+ O(|\mu r_0 - |x_1||^3 + \frac{1}{\mu}).
\]

Hence,
\[
\int_{\Omega_1} K\left(\frac{|y|}{\mu}\right) U_{p,1}^p = \int_{\mathbb{R}^N} U_{p,1}^p - \frac{c_0}{\Lambda m^m} \int_{\mathbb{R}^N} |y_1|^m U_{p,1}^p \\
- \frac{c_0}{\Lambda^{m-2}\mu^m} \frac{1}{2}m(m-1) \int_{\mathbb{R}^N} |y_1|^{m-2}U_{0,1}(\mu r_0 - |x_1|)^2 \\
+ O\left(\frac{1}{\mu^{m+\sigma}} + \frac{|\mu r_0 - |x_1||^3}{\mu^m}\right).
\]

For the second term, we have
\[
\int_{\Omega_1} K\left(\frac{|y|}{\mu}\right) \left(\sum_{i=2}^k U_{x_i,\Lambda}\right) U_{p,1}^{p-1}
\]
\[
= \int_{\Omega_1} \left(\sum_{i=2}^k U_{x_i,\Lambda}(y)\right) U_{p,1}^{p-1}(y) + \int_{\Omega_1} \left(K\left(\frac{|y|}{\mu}\right) - 1\right) \left(\sum_{j=2}^k U_{x_j,\Lambda}(y)\right) U_{p,1}^{p-1}(y)
\]
\[
= (4^\gamma \gamma)^{\frac{N}{2}} \left[ \int_{D_1} \sum_{i=2}^k \frac{\Lambda^N}{(1 + \Lambda^2 |y - x_i|^2)^{N-2s}} \right. \\
+ \left. \int_{D_2} \sum_{i=2}^k \frac{\Lambda^N}{(1 + \Lambda^2 |y - x_i|^2)^{N-2s}} \right]
\]
\[
\times \left[ \int_{D_3} \left(K\left(\frac{|y|}{\mu}\right) - 1\right) \sum_{i=2}^k \frac{\Lambda^N}{(1 + \Lambda^2 |y - x_i|^2)^{N-2s}} \right. \\
+ \left. \int_{D_4} \left(K\left(\frac{|y|}{\mu}\right) - 1\right) \sum_{i=2}^k \frac{\Lambda^N}{(1 + \Lambda^2 |y - x_i|^2)^{N-2s}} \right]
\]
\[
= \sum_{i=2}^k \frac{B_0}{\Lambda^{N-2s}|x_1 - x_i|^{N-2s}} + O\left(\frac{1}{\mu^{N-2s+\sigma}}\right),
\]
where
\[
D_1 := \{y \in \Omega_1 : \frac{1}{2}|x_i - x_i| \leq |y - x_i| \leq 2|x_i - x_i|\}, \quad D_2 := \{y \in \Omega_1 : |y - x_i| > 2|x_i - x_i|\}, \quad D_3 := \{y \in \Omega_1 : ||y| - \mu r_0| \leq \sqrt{\mu}\} \quad \text{and} \quad D_4 := \{y \in \Omega_1 : ||y| - \mu r_0| > \sqrt{\mu}\}.
\]
For the last term, for any \( \max\{1, \frac{(N-2s)^2}{N}\} < \alpha < N - 2s \), we have

\[
(U_{x_i,\Lambda} \sum_{i=2}^{k} U_{x_i,\Lambda})^{p/2} \leq C \left( \frac{1}{(1 + |y - x_1|)^{N-2s}} \sum_{i=2}^{k} \frac{1}{(1 + |y - x_i|)^{N-2s}} \right)^{p/2}
\]

\[
\leq C \left( \frac{1}{(1 + |y - x_1|)^{2(N-2s)-\alpha}} \sum_{i=2}^{k} \frac{1}{|x_1 - x_i|^\alpha} \right)^{p/2}
\]

\[
\leq C \left( \frac{k}{\mu} \right)^{\frac{N-2s}{\alpha}} \frac{1}{(1 + |y - x_1|)^{2N-\frac{2s}{\alpha}}}.
\]

Thus,

\[
\int_{\Omega^1} U_{x_i,\Lambda}^{p/2} \sum_{i=2}^{k} U_{x_i,\Lambda}^{p/2} \leq C \left( \frac{k}{\mu} \right)^{\frac{N-2s}{\alpha}} \int_{\Omega^1} \left( \frac{1}{(1 + |y - x_1|)^{2N-\frac{2s}{\alpha}}} \right)^{p/2}
\]

\[
= O \left( \left( \frac{k}{\mu} \right)^{N-2s+\sigma} \right). \tag{26}
\]

By summing the obtained results in (23), (24), (25) and (26), we have

\[
I(W_{r,\Lambda}) = k \left[ \left( 1 - \frac{1}{p} \right) \int_{\mathbb{R}^N} U_{0,1}^p + \frac{1}{\Lambda^m \mu^m} \int_{\mathbb{R}^N} |y_1|^m U_{0,1}^p \right]
\]

\[
+ \frac{1}{\Lambda^{m-2} \mu^m} \int_{\mathbb{R}^N} |y_1|^{m-2} U_{0,1}^p |\mu r_0 - r|^2 - \frac{1}{2} \frac{B_0}{\Lambda^{N-2s}} |x_1 - x_i|^{N-2s}
\]

\[
+ O \left( \frac{1}{\mu^{m+\sigma}} + \frac{|\mu_0 - r|^3}{\mu^m} \right)
\]

\[\square\]

Using the similar arguments, we can get the energy expansion of \( \frac{\partial I(W_{r,\Lambda})}{\partial \Lambda} \) as follows:

**Lemma 4.2.** For \( k \) large enough, we have

\[
\frac{\partial I(W_{r,\Lambda})}{\partial \Lambda} = k \left[ - \frac{m B_1}{\Lambda^{m+1} \mu^m} + \sum_{i=2}^{k} \frac{B_3(N-2s)}{\Lambda^{N-2s+1} |x_1 - x_i|^{N-2s}} + O \left( \frac{1}{\mu^{m+\sigma}} + \frac{1}{\mu^m} |\mu r_0 - |x_1||^2 \right) \right],
\]

where \( B_1 \) and \( B_2 \) are same positive constants in Lemma 4.1.

Now, we are ready to get the expansion of \( F(r,\Lambda) \) and \( \frac{\partial F(r,\Lambda)}{\partial \Lambda} \).

**Proposition 3.** We have

\[
F(r,\Lambda) = k \left[ A + \frac{B_1}{\mu^m \Lambda^m} + \frac{B_2}{\Lambda^{m-2} \mu^m} |\mu r_0 - r|^2 - \sum_{i=2}^{k} \frac{B_3}{\Lambda^{N-2s} |x_1 - x_i|^{N-2s}} \right]
\]

\[
+ O \left( \frac{1}{\mu^{m+\sigma}} + \frac{|\mu_0 - r|^3}{\mu^m} \right),
\]

where \( \sigma > 0 \) is a small constant, \( A \) and \( B_i > 0, i = 1, 2, 3 \) are same constants in Lemma 4.1.
Proof. By (20), we have $\langle I'(W_r, \Lambda + \phi), \phi \rangle = 0$ and the functional $F(r, \Lambda)$ can be expanded as the following:

$$F(r, \Lambda) = I(W_r, \Lambda) + \frac{1}{2} D^2 I(W_r, \Lambda + t\phi)(\phi, \phi)$$

$$= I(W_r, \Lambda) + \frac{1}{2} \int_{\mathbb{R}^N} (-\Delta)^s \phi \cdot \phi - \frac{p-1}{2} \int_{\mathbb{R}^N} \frac{|y|^s}{\mu} |(W_r, \Lambda + t\phi)^{p-2} \phi|^2$$

$$= I(W_r, \Lambda) - \frac{p-1}{2} \int_{\mathbb{R}^N} K(\frac{|y|}{\mu}) [(W_r, \Lambda + t\phi)^{p-2} - W_r^{p-2}] \phi^2(y) + \frac{1}{2} \int_{\mathbb{R}^N} (N(\phi) + t_k \phi)$$

$$= I(W_r, \Lambda) + O(\int_{\mathbb{R}^N} |\phi|^p + |N(\phi)||\phi| + |l_k||\phi|).$$

But

$$\int_{\mathbb{R}^N} (|N(\phi)||\phi| + |l_k||\phi|) \leq C(||N(\phi)||_{s^*} + ||l_k||_{s^*})||\phi|| \int_{\mathbb{R}^N} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N + \frac{2s}{2} + \tau}} \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{N - \frac{2s}{2} + \tau}}.$$

If $y \in \Omega_1$, for $\tau - \beta > \frac{N - 2s - m}{N - 2s}$, we have

$$\sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N + \frac{2s}{2} + \tau}} \leq \frac{C}{(1 + |y - x_1|)^{N + \frac{2s}{2} + \beta}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{\tau - \beta}}$$

$$\leq \frac{C}{(1 + |y - x_1|)^{N + \frac{2s}{2} + \beta}} \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^\tau - \beta}$$

$$\leq \frac{C}{(1 + |y - x_1|)^{N + \frac{2s}{2} + \beta}}$$

and

$$\sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N - \frac{2s}{2} + \tau}} \leq \frac{C}{(1 + |y - x_1|)^{N - \frac{2s}{2} + \beta}}.$$

Thus,

$$\int_{\mathbb{R}^N} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N + \frac{2s}{2} + \tau}} \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{N - \frac{2s}{2} + \tau}}$$

$$\leq C \int_{\Omega_1} \left( \frac{1}{(1 + |y - x_1|)^{N + \frac{2s}{2} + \tau}} + \frac{1}{(1 + |y - x_1|)^{N - \frac{2s}{2} + \beta}} \right)$$

$$\times \left( \frac{1}{(1 + |y - x_1|)^{N - \frac{2s}{2} + \tau}} + \frac{1}{(1 + |y - x_1|)^{N - \frac{2s}{2} + \beta}} \right)$$

$$\leq C k \int_{\Omega_1} \frac{1}{(1 + |y - x_1|)^{N + 2\beta}} \leq C k.$$

Therefore

$$\int_{\mathbb{R}^N} (|N(\phi)||\phi| + |l_k||\phi|) \leq C k(||N(\phi)||_{s^*} + ||l_k||_{s^*}||\phi||_s)$$

$$\leq C k \left( \frac{1}{\mu} \right)^{m + \sigma}.$$
By using the similar arguments, we have

\[
\int_{\mathbb{R}^N} |\phi|^p \leq C\|\phi\|^p \int_{\mathbb{R}^N} \left( \sum_{i=1}^{k} \frac{1}{(1 + |y - x_i|)^{N-2s+\gamma}} \right)^p
\]

\[
\leq Ck\|\phi\|^p \int_{\Omega} \left( \frac{1}{(1 + |y - x_1|)^{N-2s+\beta}} \right)^p
\]

\[
\leq Ck(\frac{1}{\mu})^{m+\sigma}.
\]

By Lemma 4.1, we get the desired result. \(\square\)

**Proposition 4.** We have

\[
\frac{\partial F(r, \Lambda)}{\partial \Lambda} = k \left[ - \frac{B_1 m}{\Lambda^{m+1} \mu^m} + \sum_{i=2}^{k} \frac{B_3 (N - 2s)}{\Lambda^{N-2s+1}|x_1 - x_i|^N} + O\left( \frac{1}{\mu^{m+\sigma}} + \frac{(\mu r_0 - |x_1|)^2}{\mu^m} \right) \right].
\]

**Proof.** A direct calculus shows that

\[
\frac{\partial F(r, \Lambda)}{\partial \Lambda} = \langle I'(W_{r, \Lambda} + \phi), \frac{\partial W_{r, \Lambda}}{\partial \Lambda} + \frac{\partial \phi}{\partial \Lambda} \rangle
\]

\[
= \frac{\partial I(W_{r, \Lambda})}{\partial \Lambda} + \langle I'(W_{r, \Lambda} + \phi) - I'(W_{r, \Lambda}), \frac{\partial W_{r, \Lambda}}{\partial \Lambda} \rangle + \sum_{i=1}^{k} \left( \sum_{l=1}^{2} \frac{c_l}{\mu} \langle U_{x_i, \Lambda}^{p-2} Z_{i,l}, \frac{\partial \phi}{\partial \Lambda} \rangle \right)
\]

\[
= \frac{\partial I(W_{r, \Lambda})}{\partial \Lambda} + \langle \phi, (-\Delta)^s U_{x, \Lambda} \rangle + \langle K\frac{|\phi|}{\mu} (W_{r, \Lambda} + \phi) - W_{r, \Lambda}^{-1}, \frac{\partial W_{r, \Lambda}}{\partial \Lambda} \rangle
\]

\[
- \sum_{l=1}^{2} \sum_{i=1}^{k} \left( \frac{\partial (U_{x_i, \Lambda}^{p-2} Z_{i,l})}{\partial \Lambda}, \phi \right)
\]

\[
= \frac{\partial I(W_{r, \Lambda})}{\partial \Lambda} + I_1 + I_2 + I_3.
\]

Since \((-\Delta)^s U_{x, \Lambda} = U_{x, \Lambda}^{p-1},\)

\[
I_1 = \langle \phi, \frac{\partial}{\partial \Lambda} \left( \sum_{j=1}^{k} U_{x, \Lambda}^{p-1} \right) \rangle = (p-1) \sum_{j=1}^{k} \langle \phi, U_{x, \Lambda}^{p-2} Z_{i,2} \rangle = 0.
\]

Note that \(\int_{\mathbb{R}^N} U_{x, \Lambda}^{p-2} \frac{\partial U_{x, \Lambda}}{\partial \Lambda} \phi = 0\) for \(j = 1, \ldots, k.\) Then we have

\[
|I_2| \leq (p-1) \left| \int_{\mathbb{R}^N} K\frac{|\phi|}{\mu} W_{r, \Lambda}^{p-2} \frac{\partial W_{r, \Lambda}}{\partial \Lambda} \phi \right| + O(\|\phi\|^2)
\]

\[
\leq (p-1) \left| \int_{\mathbb{R}^N} K\frac{|\phi|}{\mu} (W_{r, \Lambda}^{p-2} \frac{\partial W_{r, \Lambda}}{\partial \Lambda} - \sum_{j=1}^{k} U_{x, \Lambda}^{p-2} \frac{\partial U_{x, \Lambda}}{\partial \Lambda} \phi) \right|
\]

\[
+ (p-1) \left| \sum_{j=1}^{k} \int_{\mathbb{R}^N} (K\frac{|\phi|}{\mu} - 1) U_{x, \Lambda}^{p-2} \frac{\partial U_{x, \Lambda}}{\partial \Lambda} \phi \right| + O(\|\phi\|^2)
\]
Choose \( \Lambda \)

We rewrite the expansion of \( F \) such that

By Proposition 2, we have

By (6) and (7), there exists a constant \( B_4 > 0 \) such that

We rewrite the expansion of \( F_{r, \Lambda} \) and \( \frac{\partial F_{r, \Lambda}}{\partial \Lambda} \) as the following:

and

Choose \( \Lambda_0 = \left( \frac{B_4(N - 2s)}{B_1 \mu r_0 - \sigma} \right)^{\frac{1}{N - 2s}} \) such that

\[-\frac{B_4}{\Lambda_{m+1}} + \frac{B_4(N - 2s)}{\Lambda_{N + 1 - 2s} r_0 - N - 2s} = 0.\]
For \( \tilde{\theta} > 0 \) small, we define
\[
D = \{(r, \Lambda) : r \in [\mu r_0 - \frac{1}{\mu^\tilde{\theta}}, \mu r_0 + \frac{1}{\mu^\tilde{\theta}}], \Lambda \in \left[\Lambda_0 - \frac{1}{\mu^2\tilde{\theta}}, \Lambda_0 + \frac{1}{\mu^2\tilde{\theta}}\right]\}.
\]
Then for \((r, \Lambda) \in D\), we have \(r^{N-2s} = \mu^{N-2s}(r_0^{N-2s} + O(\mu^{1+\tilde{\theta}}))\). Hence, for \((r, \Lambda) \in D\), we have
\[
F(r, \Lambda) = k\left[A + \left(\frac{B_1}{\Lambda^m} - \frac{B_4}{\Lambda^{N-2s}r_0^{N-2s}}\right) \frac{1}{\mu^m} + \frac{B_2}{\Lambda^{m-2}\mu^m} |\mu r_0 - r|^2 + O\left(\frac{1}{\mu^{m+\tilde{\theta}}} + \frac{|\mu r_0 - r|^3}{\mu^m} + \frac{k}{\mu^{N-2s}}\right)\right] \quad (27)
\]
and
\[
\frac{\partial F(r, \Lambda)}{\partial \Lambda} = k\left[ -\frac{B_1 m}{\Lambda^{m+1}} + \frac{B_4 (N-2s)}{\Lambda^{N-2s+1}r_0^{N-2s}}\right] \frac{1}{\mu^m} + O\left(\frac{1}{\mu^{m+\tilde{\theta}}} + \frac{|\mu r_0 - r|^2}{\mu^m} + \frac{k}{\mu^{N-2s}}\right)\] \quad (28)
Let
\[
\bar{F}(r, \Lambda) = -F(r, \Lambda), \quad (r, \Lambda) \in D.
\]
Set
\[
\alpha_1 = k\left[-A - \left(\frac{B_1}{\Lambda_0^m} - \frac{B_4}{\Lambda_0^{N-2s}r_0^{N-2s}}\right) \frac{1}{\mu^m} - \frac{1}{\mu^{m+2\tilde{\theta}}}\right], \quad \alpha_2 = k(-A + \eta),
\]
where \(\eta > 0\) is a small constant.
Let
\[
F^\alpha := \{(r, \Lambda) \in D, \bar{F}(r, \Lambda) \leq \alpha\}.
\]
We consider the flow \((r(t), \Lambda(t))\) generated by:
\[
\begin{align*}
\frac{dr}{dt} &= -D_r \bar{F}, \quad t > 0 \\
\frac{d\Lambda}{dt} &= -D_\Lambda \bar{F}, \quad t > 0 \\
(r, \Lambda) &\in F^\alpha_2.
\end{align*}
\]
\textbf{Proposition 5.} The flow \((r(t), \Lambda(t))\) does not leave \(D\) before it reaches \(F^\alpha_1\).
\textbf{Proof.} Since \(|r - \mu r_0| \leq \frac{1}{\mu^{\tilde{\theta}}}, \) if \(\Lambda = \Lambda_0 + \frac{1}{\mu^2\tilde{\theta}}\), then there exists a constant \(C_1 > 0\) such that
\[
\frac{\partial \bar{F}(r, \Lambda)}{\partial \Lambda} = k\left[\frac{C_1}{\mu^{m+2\tilde{\theta}}} + O\left(\frac{1}{\mu^{m+2\tilde{\theta}}}\right)\right] > 0, \text{ for } |r - \mu r_0| \leq \frac{1}{\mu^{\tilde{\theta}}},
\]
On the other hand, if \(\Lambda = \Lambda_0 - \frac{1}{\mu^2\tilde{\theta}}\), there exists a constant \(C_2 > 0\) such that
\[
\frac{\partial F(r, \Lambda)}{\partial \Lambda} = k\left[-\frac{C_2}{\mu^{m+2\tilde{\theta}}} + O\left(\frac{1}{\mu^{m+2\tilde{\theta}}}\right)\right] < 0, \text{ for } |r - \mu r_0| \leq \frac{1}{\mu^{\tilde{\theta}}},
\]
Hence, the flow \((r(t), \Lambda(t))\) does not leave \(D\) if \(|r - \mu r_0| \leq \frac{1}{\mu^{\tilde{\theta}}}\).
The only possibility that the flow tends to leave $D$ lies in the position where $|r(t) - \mu r_0| = \frac{1}{\mu^g}$ and $|\Lambda - \Lambda_0| \leq \frac{1}{\mu^{\frac{3}{2}g}}$. Note that

$$
\frac{B_1}{\Lambda^m} - \frac{B_4}{\Lambda N - 2s \frac{N-2s}{r_0}} = \left[ \frac{B_1}{\Lambda_0^m} - \frac{B_4}{\Lambda_0^N - 2s \frac{N-2s}{r_0}} \right] + \left[ - \frac{B_1 m}{\Lambda_0^m + 1} + \frac{(N - 2s) B_4}{\Lambda_0^N + 1 - 2s \frac{N-2s}{r_0}} \right] (\Lambda - \Lambda_0) + O(|\Lambda - \Lambda_0|^2)
$$

Thus for $k$ large enough, we have

$$
F(r, \Lambda) = k \left[ - A - \frac{B_1}{\Lambda^m} - \frac{B_4}{\Lambda N - 2s \frac{N-2s}{r_0}} \right] \frac{1}{\mu^m} - \frac{B_2 |\mu r_0 - r|^2}{\Lambda^m - 2 \mu^m} + O\left( \frac{1}{\mu^{m+3g}} \right)
$$

$$
= k \left[ - A - \frac{B_1}{\Lambda_0^m} - \frac{B_4}{\Lambda_0^N - 2s \frac{N-2s}{r_0}} \right] \frac{1}{\mu^m} - \frac{B_2}{\Lambda_0^m - 2 \mu^m + 2^g} + O\left( \frac{1}{\mu^{m+3g}} \right)
$$

$$
< \alpha_1.
$$

As a result, the flow $(r(t), \Lambda(t))$ does not leave $D$ before it reaches $\bar{F}^{\alpha_1}$. \qed

Now we are ready to prove the main theorem.

**Proof of Theorem 1.1.** It is sufficient to prove that $\bar{F}$ has a critical point in $D$. Let

$$
\Gamma := \{ \gamma \in C(D, D) : \gamma(r, \Lambda) = (\gamma_1(r, \Lambda), \gamma_2(r, \Lambda)) \in D, (r, \Lambda) \in D, \gamma(r, \Lambda) = (r, \Lambda) \text{ if } |r - \mu r_0| = \frac{1}{\mu^g} \}
$$

and

$$
c := \inf_{\gamma \in \Gamma} \max_{(r, \Lambda) \in D} \bar{F}(\gamma(r, \Lambda)).
$$

We claim that $c$ is actually the critical value of $\bar{F}$. In fact, by the critical point theory, it is sufficient to prove:

(i) $\alpha_1 < c < \alpha_2$,

(ii) $\sup_{|r - \mu r_0| = \frac{1}{\mu^g}} \bar{F}(r, \Lambda) < \alpha_1, \forall \gamma \in \Gamma$.

We first prove (ii). If $|r - \mu r_0| = \frac{1}{\mu^g}$, then for any $\gamma \in \Gamma$, $\gamma(r, \Lambda) = (r, \Lambda)$. Thus, by (29), we have,

$$
\bar{F}(\gamma(r, \Lambda)) = \bar{F}(r, \Lambda) < \alpha_1.
$$

Now, we prove (i). It is easy to prove $\bar{F}(r, \Lambda) < \alpha_2$. For the lower bound, we fix $\Lambda_0$. For any $\gamma \in \Gamma$, we have $\gamma_1(r, \Lambda_0) = r$ for $|r - \mu r_0| = \frac{1}{\mu^g}$. Then by the continuity of $\gamma_1(r, \Lambda_0)$, there is $\bar{r} \in (\mu r_0 - \frac{1}{\mu^g}, \mu r_0 + \frac{1}{\mu^g})$ such that $\gamma_1(\bar{r}, \Lambda_0) = \mu r_0$. Let $\lambda = \gamma_2(\bar{r}, \Lambda_0)$. Then by (27)

$$
\max_{(r, \Lambda) \in D} \bar{F}(\gamma(r, \Lambda)) \geq \bar{F}(\gamma(\bar{r}, \Lambda_0)) \geq \bar{F}(\mu r_0, \lambda)
$$

$$
= \bar{F}(\mu r_0, \lambda).
$$
\[ \begin{align*}
&= k \left[ -A - \left( \frac{B_1}{A^m} - \frac{B_4}{A^{N-2s} r_0^{N-2s}} \right) \frac{1}{\mu^m} + O\left( \frac{1}{\mu^{m+\sigma}} + \frac{k}{\mu^{N-2s}} \right) \right] \\
&= k \left[ -A - \left( \frac{B_1}{\Lambda_0^m} - \frac{B_4}{\Lambda_0^{N-2s} r_0^{N-2s}} \right) \frac{1}{\mu^m} + O\left( \frac{1}{\mu^{m+3\sigma}} \right) \right]
\end{align*} \]

Finally, for each \( k \) large enough, we get the critical point \((r_k, \Lambda_k)\) for \( \overline{F}(r, \Lambda) \), hence the critical point \( \phi_k + W_{r_k, \Lambda_k} \) of the functional \( I \).

\[ \Box \]

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