A Small Model for the Cohomology of Some Principal Bundles

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Abstract

Let $G$ be a compact, connected and simply connected Lie group, and $\Omega G$ the space of the loops in $G$ based at the identity. This note shows a way to compute the cohomology of the total space of a principal $\Omega G$-bundle over a manifold $M$, from the cohomology of $G$, the differential forms on $M$ and the characteristic classes of the bundle. The equivariant situation is also treated.

1 Introduction

Let $P \to M$ be a principal $G$ bundle, where $G$ is a reductive Lie group and $M$ a differential manifold. Let us recall that the real cohomology ring of $G$ is an exterior algebra over a graded vector space $V_G$, with basis $\{x_i\}$: $H^*(G) \cong \Lambda V_G$ (in this paper, $\Lambda V$ denotes the free graded-commutative algebra over the graded vector space $V$). Now if $BG$ is a classifying space for $G$, then $H^*(BG) \cong \Lambda V_B$, where $V_B = V_G[-1]$ denotes the vector space isomorphic to $V_G$, but with grading shifted by one, so that there is a base $\{y_i\}$ of $V$ such that $\deg y_i = \deg x_i + 1$ and a map $\tau: V_G \to V_B; x_i \mapsto y_i$ called the distinguished transgression. Now for any given principal $G$-bundle $P \to M$, any choice of a connection provides a way to pull the generators of $H^*(BG)$ (that is, the elements of $V_B$) back to $\Omega^*(M)$ (that is, the differential forms on $M$). Such a pullback of the image of an element $x_i$ of $V_G$ via the distinguished transgression is a representative of the characteristic class corresponding to $x_i$, and denoted by $c_i$ (the class and the representative are denoted the same way).

Greub, Halperin and Vanstone prove the following

Theorem 1. Let $P \to M$ be a principal $G$-bundle over a smooth manifold $M$. The real cohomology of $P$ is isomorphic to the homology of the complex $\Omega^*(M) \otimes \Lambda V_B$ with the differential defined as follows:

$$d(\omega \otimes 1) = d_{dR} \omega \otimes 1$$
and
\[
d(\omega \otimes x_1 \wedge \ldots \wedge x_r) = d_{dR} \omega \otimes x_1 \wedge \ldots \wedge x_r + (-1)^{[\omega]} \sum_{j=1}^{r} (-1)^{j} c_j \cdot \omega \otimes x_1 \wedge \ldots \hat{x}_j \ldots \wedge x_r
\]

Using tools from rational homotopy theory, Félix, Halperin and Thomas proved a very similar result but for a much wider class of groups and base spaces: they only require $G$ to be a path connected topological group with finite dimensional rational homology, and $M$ to be a simply connected CW-complex (see chapter 15, § (f) in [FHT]).

These tools allow us to turn our interest towards another class of principal bundles. Now $G$ is a compact, connected and simply connected Lie group. If we denote by $\Omega eG$ the space of loops in $G$, based at the identity, and the space of paths in $G$ starting at the identity by $P eG$, every $\Omega eG$-bundle can be pulled back from the path-loop fibration $\Omega eG \to P eG \to G$. If $M$ is a connected, simply connected manifold, let $f : M \to G$ be the classifying map for the principal $\Omega eG$-bundle $\Omega eG \to P \to M$ and $c_i = f^* \hat{x}_i$, where $\hat{x}_i$ denotes the invariant representative of the cohomology class $x_i$. Then the following holds true.

**Theorem 2.** Let $P \to M$ be a principal $\Omega eG$-bundle over a smooth compact, connected and simply connected manifold $M$. The cohomology of $P$ is given by the homology of the complex $\Omega^\ast(M)[y_1, \ldots, y_r]$, where the $y_i$'s are generators corresponding to those of $H^\ast(G)$, but of degree one less, endowed with the differential

\[
d(\omega p(y)) = d(\omega) p(y) + (-1)^{[\omega]} \sum_{i=1}^{r} \omega \wedge c_i \frac{\partial}{\partial y_i} p(y)
\]

Note that the choice of $c_i$'s is not unique, as will be made precise in Section 3.

With not so much more effort, one can prove an equivariant version of this. First recall that the equivariant cohomology algebra of $G$ acting on itself by conjugation is exactly $\Lambda(V_G \oplus V_B)$ (see Lemma 3). If $p : P \to M$ is the pull-back of the universal principal $\Omega eG$-bundle by an equivariant map $f : M \to G$ with respect to the conjugation action, and if we choose closed elements $c_i(\xi)$ of the Cartan model $C_G(M)$ for the equivariant cohomology of $M$ such that $c_i(\xi)$ represents $f^*_G(x_i)$ (the pullback along $f$ in equivariant cohomology), then we have the following.
Theorem 3. The $G$-equivariant cohomology of $P$ is isomorphic, as an algebra, to the cohomology of the complex $C_G(M)[y_1,\ldots,y_r]$ with differential

$$d : w(\xi)p(y) \mapsto d_{DR}(w(\xi))p(y) - \iota_{\xi}w(\xi)p(y) + (w \wedge c_i)(\xi) \sum_{i=1}^{r} \frac{\partial}{\partial y_i}p(y)$$

This result is rather interesting since the equivariant cohomology of such a space offers a control of the cohomology of the symplectic quotient $P//LG$ (see [BTW] and [AMM]). E. Meinrenken mentioned this model in his lecture notes on the subject [Meinrenken].

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2 The tools

2.1 Polynomial differential forms

In this section, we recall the construction of polynomial differential forms, and one of the main results of interest for us.

We call cochain algebra a differential graded algebra concentrated in non-negative degrees, with a differential of degree $+1$.

For a topological space $X$, let $S_*(X)$ be the set of singular simplices in $X$, seen as a simplicial set (see e.g. [May], example 1.5).

Let us consider the free graded commutative algebra $\Lambda(t_0,\ldots,t_n,y_0,\ldots,y_n)$ (with real coefficients) where the basis elements $t_i$ have degree 0 and $y_j$ have degree 1. There is a unique derivation in this algebra specified by $t_i \mapsto y_i$ and $y_i \mapsto 0$. This derivation preserves the ideal $I_n$ generated by the two elements $\sum_{0}^{n} t_i - 1$ and $\sum_{0}^{n} y_j$, so we can define the quotient differential algebra

$$(APL)_n = \frac{\Lambda(t_0,\ldots,t_n,y_0,\ldots,y_n)}{(\sum_{0}^{n} t_i - 1, \sum_{0}^{n} y_j)}$$

$$dt_i = y_i \text{ and } dy_i = 0$$

The cochain algebra morphisms $\partial_i : (APL)_{n+1} \to (APL)_n$ and $s_j : (APL)_n \to (APL)_{n+1}$ uniquely specified by

$$\partial_i : t_k \mapsto \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases} \text{ and } s_j : t_k \mapsto \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j \end{cases}$$
give $A_{PL} = \bigoplus_{n \geq 0}(A_{PL})_n$ a structure of simplicial cochain algebra.

We can now define the functor $A_{PL}$ from the category of topological spaces to the category of cochain algebras: $A_{PL}(X)$ is the set of simplicial set morphism from $S_\ast(X)$ to $A_{PL}$ (the set of elements of degree $p$ in $A_{PL}$), with point-wise addition, scalar and internal multiplication, as well as differentiation. If $f : X \to Y$ is a continuous map, then $A_{PL}(f) : A_{PL}(Y) \to A_{PL}(X)$ is the morphism of cochain algebra defined by precomposition by $S_\ast(f)$.

For any topological space $X$, $A_{PL}(X)$ is weakly equivalent to $C^\ast(X)$, that is there are commutative cochain algebras $(C(0), d), \ldots, (C(k), d)$ such that

$$A_{PL}(X) \xrightarrow{\sim} (C(0), d) \xrightarrow{\subseteq} \cdots \xrightarrow{\subseteq} (C(k), d) \xrightarrow{\subseteq} C^\ast(X)$$

If $M$ is a smooth manifold, then $A_{PL}(M)$ is also weakly equivalent to $\Omega^\ast(M)$.

The elements of the cochain algebra $A_{PL}(X)$ are called polynomial differential forms.

### 2.2 Sullivan algebras and Sullivan models

The main tools of this paper are Sullivan algebras, Sullivan models, and their relative counterpart. In this section we expose briefly their definitions and some of their properties, following [FHT]. Unless stated otherwise, all results come from this book, and all algebras have a unit.

**Definition:** A relative Sullivan algebra is a commutative cochain algebra of the form $(B \otimes \Lambda V, d)$, where

- $(B, d) = (B \otimes 1, d)$ is a sub cochain algebra, and $H^0(B) = \mathbb{R}$
- $1 \otimes V = V = \bigoplus_{p \geq 1} V^p$ and $\Lambda V$ is the free commutative algebra on $V$
- $V = \bigcup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \cdots$ is an increasing sequence of graded subspaces such that $d : V(0) \to B$ and $d : V(k) \to B \otimes \Lambda(V(k-1)), k \geq 1$

We identify $B = B \otimes 1$ and $\Lambda V = 1 \otimes \Lambda V$. The sub-cochain algebra $(B, d)$ is called the base algebra of $(B \otimes \Lambda V, d)$.

Now let $\varphi : (B, d) \to (C, d)$ be a morphism of commutative cochain algebras with $H^0(B) = \mathbb{R}$.

**Definition:** A Sullivan model for $\varphi$ is a quasi-isomorphism of cochain algebras

$$m : (B \otimes \Lambda V, d) \xrightarrow{\sim} (C, d)$$
such that \((B \otimes \Lambda V, d)\) is a relative Sullivan algebra with base \((B, d)\) and \(m|_B = \varphi\).

If \(f : X \to Y\) is a continuous map then a Sullivan model for \(A_{PL}(f)\) is called a Sullivan model for \(f\).

**Definition:** A Sullivan algebra or a Sullivan model is called *minimal* if

\[ \text{Im } d \subset B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2}V \]

The special case where \((B, d) = \mathbb{R}\) and \(\varphi : (B, d) \to (C, d)\) is the canonical morphism \(\mathbb{R} \to (A, d); 1 \mapsto 1\) gets particular attention:

**Definitions:**

1. A relative Sullivan algebra with base \(\mathbb{R}\) is simply called a *Sullivan algebra*.
2. A Sullivan model for \(\mathbb{R} \to (C, d); 1 \mapsto 1\) is called a *Sullivan model for* \((C, d)\).
3. If \(X\) is a path-connected topological space, a Sullivan model for \(A_{PL}(X)\) is called a *Sullivan model for* \(X\).
4. A Sullivan algebra \((\Lambda V, d)\) is called *minimal* if

\[ \text{Im } d \subset \Lambda^{\geq 2}V. \]

We can assure the existence of a Sullivan model under some hypotheses:

**Proposition 1.** A morphism \(\varphi : (B, d) \to (C, d)\) of commutative cochain algebras admits a Sullivan model if \(H^0(B) = \mathbb{R} = H^0(C)\) and \(H^1(\varphi)\) is injective.

Moreover, any commutative cochain algebra \((A, d)\) with \(H^0(A) = \mathbb{R}\) (and any path-connected topological space) admits a unique minimal Sullivan model.

We can combine Propositions 12.8 and 12.9 in [FHT] to get the following

**Proposition 2.** Let \(\eta : (A, d) \stackrel{\sim}{\longrightarrow} (C, d)\) be a quasi-isomorphism of commutative cochain algebras. Let \((\Lambda V, d)\) be a Sullivan algebra, and \(\psi : (\Lambda V, d) \to (C, d)\) a morphism of cochain algebras. Then there is a morphism of commutative cochain algebras \(\varphi : (\Lambda V, d) \to (A, d)\) such that \(H(\eta \circ \varphi) = H(\psi)\). Any two such morphisms \(\varphi_1, \varphi_2 : (\Lambda V, d) \to (A, d)\) satisfy \(H(\varphi_1) = H(\varphi_2)\)
It is called the lifting lemma because it says that for any given \( \psi \) and \( \eta \) as in Diagram (1), there exists a \( \phi \) making the diagram commute.

As it stands, this proposition isn’t so useful to us, but it has two crucial corollaries:

**Corollary 1.** Let \((A, d)\) and \((B, d)\) be two commutative cochain algebras, and \(\psi : (\Lambda V, d) \to (A, d)\) a Sullivan model for \((A, d)\). If \((A, d)\) and \((B, d)\) are weakly equivalent, then there exists a Sullivan model \(\phi : (\Lambda V, d) \to (B, d)\) for \((B, d)\).

**Proof:** Recall that for \((A, d)\) and \((B, d)\) to be weakly equivalent means that there are cochain algebras \((C(0), d), \ldots, (C(k), d)\) so that
\[
(A, d) \cong (C(0), d) \to \cdots \to (C(k), d) \cong (B, d).
\]
Now use Diagram (1) to conclude.

**Corollary 2.** Let \((A, d)\) and \((B, d)\) be two commutative cochain algebras, and let \(m_A : (\Lambda V_A, d) \to (A, d)\) and \(m_B : (\Lambda V_B, d) \to (B, d)\) be Sullivan models for \(A\) and \(B\) respectively. Then for any morphism \(f : (B, d) \to (A, d)\) there is a Sullivan representative for \(f\), that is a morphism \(\phi : \Lambda V_B \to \Lambda V_A\) such that \(H(f \circ m_B) = H(m_A \circ \phi)\).

Now suppose \((B \otimes \Lambda V, d)\) is a relative Sullivan Algebra and \(\psi : (B, d) \to (B', d)\) is a morphism of commutative cochain algebras with \(H^0(B') = \mathbb{R}\). Then \(\psi\) gives \((B', d)\) a structure of \((B, d)\) module, and the cochain algebra
\[
(B', d) \otimes_{(B, d)} (B \otimes \Lambda V, d) = (B' \otimes \Lambda V, d)
\]
is a relative Sullivan algebra with base \((B', d)\). It is called the pushout of \((B \otimes \Lambda V, d)\) along \(\psi\).

This pushout construction has the really useful property that it preserves quasi-isomorphisms:

**Lemma 1.** If \(\psi\) is a quasi-isomorphism, then so is \(\psi \otimes id : (B \otimes \Lambda V, d) \to (B' \otimes \Lambda V, d)\).

If \(\varepsilon : B \to \mathbb{R}\) is any augmentation, pushing \((B \otimes \Lambda V, d)\) out along \(\varepsilon\) yields a Sullivan algebra \((\Lambda V, d)\), which is called the Sullivan fibre at \(\varepsilon\).
2.3 Application to fibrations

Let \( p : X \to Y \) be a fibration with fibre \( F \), and \( q : Z \to A \) be the pullback of \( p \) along a continuous mapping \( f : A \to Y \), where both \( A \) and \( Y \) are simply connected.

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
\downarrow{q} & & \downarrow{p} \\
A & \xrightarrow{f} & Y
\end{array}
\]  

Assume in addition that one of \( H_*(X) \), \( H_*(Y) \), \( H_*(A) \) or \( H_*(F) \) is of finite type. Let us choose Sullivan models

\[
m_Y : (\Lambda V_Y, d) \to A_{PL}(Y),
\]

\[
n_A : (\Lambda W_A, d) \to A_{PL}(A)
\]

and

\[
m : (\Lambda V_Y \otimes \Lambda V, d) \to A_{PL}(X) \text{ (modelling } p \circ m_Y),
\]

as well as a Sullivan representative for \( f \)

\[
\psi : (\Lambda V_Y, d) \to (\Lambda W_A, d)
\]

Then one can define the morphism

\[
\xi : (\Lambda W_A, d) \otimes (\Lambda V_Y, d) (\Lambda V_Y \otimes \Lambda V, d) \to A_{PL}(Z)
\]

to be the composition

\[
(\Lambda W_A, d) \otimes (\Lambda V_Y, d) (\Lambda V_Y \otimes \Lambda V, d) \xrightarrow{n_A \otimes m} A_{PL}(A) \otimes A_{PL}(X) \xrightarrow{A_{PL}(q) \otimes A_{PL}(f)} A_{PL}(Z) \otimes A_{PL}(Z) \to A_{PL}(Z)
\]

where \( \cdot \) is simply the multiplication in \( A_{PL}(Z) \). We rewrite \( (\Lambda W_A, d) \otimes (\Lambda V_Y, d) (\Lambda V_Y \otimes \Lambda V, d) \) as \( (\Lambda W_A \otimes \Lambda V, d) \). One can show the following

**Proposition 3.** \( \xi : (\Lambda W_A \otimes \Lambda V, d) \to A_{PL}(Z) \) is a Sullivan model for \( Z \).  

Morally, this means that the pushout of the models is a model for the pullback.

Note that if \( A \) is a point \( y_0 \) of \( Y \) and \( f \) is the inclusion (so that \( Z = F \)), then the minimal Sullivan model for \( A = y_0 \) is \( (\mathbb{R}, 0) \xrightarrow{\sim} A_{PL}(A) \), so that \( (\Lambda W_A \otimes \Lambda V, d) = (\Lambda V, d) \), where \( \bar{d} \) comes from the pushout along the augmentation \( \psi : (\Lambda V_Y, d) \to \mathbb{R} \) induced by \( f \). We can now formulate the following

**Corollary 3.** If \( Y \) is simply connected and \( X \to Y \) is a fibration with fibre \( F \), then

\[
\xi : (\Lambda V, \bar{d}) \xrightarrow{\sim} A_{PL}(F)
\]

is a Sullivan model for \( F \).  

which can be rephrased as ‘the fibre of the model is a model for the fibre’.

7
3 The result

Let $G$ be a simply connected Lie group, and $P_eG \xrightarrow{\pi} G$ the path fibration over $G$. In this situation, we have a map of commutative cochain algebras $A_{PL}(\pi) : A_{PL}(G) \to A_{PL}(P_eG)$.

The cohomology ring of $G$ is the free graded commutative algebra $\Lambda V_G$, where $V_G$ is a finite dimensional graded vector space concentrated in odd degrees, with $V_G^1 = 0$. Moreover, $H^*(G) = (\Lambda V_G, 0)$ is a Sullivan algebra, so $H^*(G) \xrightarrow{\simeq} \Omega^*(G)^G \xrightarrow{\simeq} \Omega^*(G)$ is a Sullivan model, and since $A_{PL}(G)$ and $\Omega^*(G)$ are weakly equivalent, their Sullivan models are identified, so there is a Sullivan model $m_G : (\Lambda V_G, 0) \xrightarrow{\simeq} A_{PL}(G)$. Now let $\{x_i\}_{i=1}^r$ be a basis for $V_G$, and $V$ be a vector space isomorphic to $V_G$ with grading shifted by one, and basis $\{y_i\}_{i=1}^r$ such that $\deg y_i = \deg x_i - 1$. Let us call this correspondence $\delta : V \to V_G$ and extend it to a derivation $d$ on $\Lambda(V_G \oplus V)$. Since $\delta$ is an isomorphism of vector spaces and $V^0 = 0$, $(\Lambda(V_G \oplus V), d)$ is acyclic. Now let us push this out along $m_G$ to get $(A_{PL}(G) \otimes \Lambda V, d)$. Since $m_G$ is a quasi-isomorphism, $m_G \otimes id : (\Lambda(V_G \oplus V), d) \to (A_{PL}(G) \otimes \Lambda V, d)$ is also a quasi-isomorphism, by Lemma 1.

Let us now define $m : (A_{PL}(G) \otimes \Lambda V, d) \to (A_{PL}(P_eG), d)$ such that $m$ coincides with $A_{PL}(\pi)$ on $A_{PL}(G)$. Since $P_eG$ is contractible, every cocycle of degree $\geq 1$ is a coboundary. In particular, $m \circ m_G(x_i) = d a_i$ for some $a_i \in A_{PL}(P_eG)$. We now define $m(y_i)$ to be $a_i$. These assignments extend to a morphism of cochain algebra $m : (A_{PL}(G) \otimes \Lambda V, d) \to (A_{PL}(P_eG), d)$ that restricts to $A_{PL}(\pi)$ on $A_{PL}(G)$.

**Lemma 2.** $m : (A_{PL}(G) \otimes \Lambda V, d) \to (A_{PL}(P_eG), d)$ is a Sullivan model for $A_{PL}(\pi) : A_{PL}(G) \to A_{PL}(P_eG)$.

**Proof:** By construction, $m$ is a morphism of differential graded commutative algebras extending $A_{PL}(\pi)$. Since both $A_{PL}(P_eG)$ and $(A_{PL}(G) \otimes \Lambda V, d)$ are acyclic, $m$ is a quasi-isomorphism. Now we only need to show that $(A_{PL}(G) \otimes \Lambda V, d)$ is a Sullivan algebra.

By construction, $(A_{PL}(G), d)$ is a sub cochain algebra and $V = \oplus_{p \geq 1} V^p$, and by connectedness of $G$, $H^0(A_{PL}(G)) = \mathbb{R}$. We only have to check the nilpotence condition, which is trivially satisfied by setting $V(0) = V$, since $d$ maps $V$ to $A_{PL}(G)$.

An immediate consequence of this and Corollary 2 is that there is a quasi-isomorphism $(\Lambda V, 0) \to A_{PL}(\Omega_e G)$, which means that we can formulate the following

**Corollary 4.** If $G$ is a connected, simply-connected Lie group, and if $\{x_i\}$ are the generators of its cohomology algebra, then the cohomology algebra of $\Omega_e G$ is the polynomial ring $\mathbb{R}[\{y_i\}]$, where $\deg(y_i) = \deg(x_i) - 1$.\hfill \blacksquare
Let us now examine our situation:

\[
\begin{array}{c}
P \xrightarrow{f} P_G \\
\pi_P \downarrow \quad \pi \downarrow \\
M \xrightarrow{f} G
\end{array}
\]

(4)

The quasi-isomorphism \(m_G : (\Lambda V_G,0) \to A_{PL}(G)\) is the minimal model for \(G\). Let us choose a minimal Sullivan model \(n_M : (\Lambda W_M,d) \to A_{PL}(M)\). Then by Proposition 3, if \(H^\pi(P)\) is injective (e.g. if \(M\) is simply connected), we have the following

**Proposition 4.** The morphism

\[
\xi = A_{PL}(\pi_P)n_M \cdot A_{PL}(\hat{f})m : (\Lambda W_M,d) \otimes_{AV_G} (\Lambda V_G \otimes \Lambda V,d) \to A_{PL}(P)
\]

is a Sullivan model for \(P\)

We will most often write \((\Lambda W_M \otimes \Lambda V,d)\) for \((\Lambda W_M,d) \otimes_{AV_G} (\Lambda V_G \otimes \Lambda V,d)\). Now since \(A_{PL}(M)\) and \(\Omega^\ast(M)\) are weakly equivalent, their Sullivan models are the same, hence there is a quasi-isomorphism \((\Lambda W_M,d) \xrightarrow{\sim} \Omega^\ast(M)\). Using Lemma 1 one can replace \((\Lambda W_M,d)\) by \(\Omega^\ast(M)\) in \(\xi\) to get a weak equivalence between \(A_{PL}(P)\) and \((\Omega^\ast(M) \otimes \Lambda V,d)\). The differential on this complex is the following: if \(n_{DR} : (\Lambda W_M,d) \xrightarrow{\sim} \Omega^\ast(M)\) is the Sullivan model for \(\Omega^\ast(M)\) corresponding to \(n_M : (\Lambda W_M,d) \to A_{PL}(M)\), \(\psi : (\Lambda W_M,d) \to (\Lambda W_M,d)\) is a Sullivan representative for \(f\), and we write \(c_i = n_{DR}\psi(x_i)\), we have that:

\[
d : y_i = 1 \otimes y_i \mapsto c_i \\
d : w = w \otimes 1 \mapsto d w \otimes 1
\]

for any element \(w\) of \(\Omega^\ast(M)\). We thus have proved the following

**Theorem 4.** In the situation of the diagram 4, the cohomology of \(P\) is isomorphic, as an algebra, to the cohomology of the complex

\[
\Omega^\ast(M)[y_1,\ldots,y_r]
\]

with differential

\[
d : wP(y) \mapsto d wP(y) + \sum_{i=1}^r w \wedge c_i \frac{\partial}{\partial y_i} P(y)
\]

where the \(c_i\)'s are defined as above, and \(\deg y_i = \deg x_i - 1\). \(\square\)
Note that the resulting graded algebra depends neither on the choice of the Sullivan representative for $f$, nor of the choice of the model $n_{DR} : (\Lambda W_M, d) \to \Omega^*(M)$; the only important thing is that $n_{DR} \circ \psi(x_i)$ represents the same cohomology class as $A_{PL}(f) \circ m_G(x_i)$. Therefore, one can choose for instance to map $x_i$ to the pullback by $f$ of the bi-invariant representatives of $x_i$.

Of course, when $M$ is formal, we can go one step further: since $H^*(M)$ is weakly equivalent to $A_{PL}(M)$, a Sullivan model for $A_{PL}(M)$ is also one for $H^*(M)$, so we can pushout $(\Lambda W_M \otimes \Lambda V, d)$ to $(H^*(M) \otimes \Lambda V, d)$, which is still a model for $A_{PL}(P)$. In other words:

**Theorem 5.** In the situation of the diagram[4] and if $M$ is formal, the cohomology of $P$ is isomorphic, as an algebra, to the cohomology of the complex

$$H^*(M)[y_1, \ldots, y_r]$$

with differential

$$d : wP(y) \mapsto \sum_{i=1}^r w \wedge c_i \frac{\partial}{\partial y_i} P(y)$$

where the $c_i$'s are the characteristic classes of $P$, and $\deg y_i = \deg x_i - 1$. □

### 4 An example

These results allow for computations of some not entirely trivial bundles. For instance, let $G$ be $SU(3)$. In this case, $H^*(SU(3)) = \bigwedge [\eta_3, \eta_5]$, with $\deg \eta_3 = 3$ and $\deg \eta_5 = 5$. Let us then define

$$M = \{ g \tilde{g}^{-1} | g \in SU(3) \} = \{ gg^\top | g \in SU(3) \}.$$ 

Let us note that $M$ is the orbit of the identity by the ‘twisted’ action $h \mapsto g h \tilde{g}^{-1}$, and a quick calculation shows that the stabilizer of the identity for this action is $SO(3)$, so that $M = SU(3)/SO(3)$. Of course, the restriction of the biinvariant representative of $\eta_3$ in $\Omega^*(SU(3))$ to $SO(3)$ is still biinvariant. To see this, write $\eta_3$ in matrix form $\text{tr}(g^{-1} d g)^3$ and see $SO(3)$ as a subspace of $SU(3)$ (see e.g. [GHV], chap. 6, §7. for a justification). This implies that $i^* : H^*(SU(3)) \to H^*(SO(3))$ is surjective, so one can deduce (see e.g. [GHV] section 11.6) that $H^*(M) = \mathbb{R}[x_5]/(x_5^2)$.

In particular, $M$ is formal: any choice of a representative of the class $x_5$ yields a quasi-isomorphism $H^*(M) \sim \Omega^*(M)$.

Now in order to be able to apply our result, we have to know $f^*$. Trivially, $f^* \eta_3 = 0$, so we only have to examine $f^* \eta_5$. Now the map $SU(3) \to SU(3)$,
$g \mapsto g\bar{g}^{-1}$ is non-zero in degree 5 cohomology, so $f^*\eta_5$ cannot be zero, which means that $f^*\eta_5 = x_5$.

Let now $P$ be the pull back of the path fibration of $SU(3)$ by the inclusion $f : M \to SU(3)$. Since $M$ is formal, we can use Theorem 5:

$$H^*(P) = H^*(H^*(M)[y_2, y_4], d = f^*x_3\frac{\partial}{\partial y_2} + f^*x_5\frac{\partial}{\partial y_4})$$

Where $\deg y_k = k$.

Let us now do concrete computations. The cocycles are linear combinations of expressions of the form $y_2^p$ or $\eta_5 \otimes y_2^p y_4^q$. The only coboundaries are the $d(y_2^p y_4^q) = \eta_5 \otimes y_2^{p-1} y_4^q$, which includes all $\eta_5 \otimes y_2^p y_4^q$. The cohomology of $P$ is therefore simply $\mathbb{R}[y_2]$.

5 The equivariant case

Of particular interest are the cases where $G$ acts on $M$, and the map $p : M \to G$ is equivariant with respect to the adjoint action of $G$ on itself. The action of $G$ by conjugation on $P_\mathbb{C}G$ descend to conjugation on $G$ via the projection to the endpoint $\pi : P_\mathbb{C}G \to G$, so that $\pi$ is a universal $G$-equivariant $\Omega_\mathbb{C}G$-bundle.

This implies that an equivariant map $M \to G$ induces a pullback diagram, that is entirely composed of $G$-equivariant mappings:

$$\begin{array}{ccc}
P & \longrightarrow & P_\mathbb{C}G \\
\downarrow & & \downarrow \\
M & \longrightarrow & G
\end{array}$$

(5)

Since the Borel construction is functorial, we can compute the $G$-equivariant cohomology from topological spaces as in this diagram:

$$\begin{array}{ccc}
P_G & \longrightarrow & P_\mathbb{C}G_G \\
\downarrow & & \downarrow \\
M_G & \longrightarrow & G_G
\end{array}$$

(6)

One checks easily that this is still a pullback diagram, so that we can use the results of Section 2.3.

Since we still have actual topological spaces at hand, we still have that the pushout of the models are a model for the pullback. So first, we have to establish a model for $A_{PL}(G_G)$:
Lemma 3. \( m_{G_G} : (\Lambda(V_G \oplus V), 0) \xrightarrow{\sim} A_{PL}(G_G) \) is a Sullivan model for \( G_G \).

Proof: First remark that the bundle \( G \to G_G \to BG \) has a section (because the identity of \( G \) is a fixed point for the conjugation action), so that \( H^*(BG) = H_G^*(\text{pt}) \to H_G^*(G) = H^*(G_G) \) is injective. On the other hand, by Proposition 3 and the structure of \( H^*(G_G) \), there is a Sullivan model of the form \( \Lambda V \otimes \Lambda V, d \xrightarrow{\sim} A_{PL}(G_G) \), but since the differentials inside \( V \) and \( V \) are zero, and \( H^*(BG) \to H^*(G_G) \) is injective, the global differential in \( \Lambda V \otimes \Lambda V \) must be zero: if not, some elements in \( V = H^*(BG) \) would not be cocycles, so \( H^*(BG) \to H^*(G_G) \) could not be injective.

Then a construction similar to the one above yields that \( \Lambda(V_G \oplus V) \otimes \Lambda V, d \) with essentially the same differential as in the non-equivariant case is a Sullivan model for \( A_{PL}(G_G) \to A_{PL}(P_G) \). Now the pushout

\[
(\Lambda W_M \otimes (\Lambda(V_G \oplus V), 0) \otimes \Lambda V, d)
\]

is a model for \( H^*(P_G) = H_G^*(P) \), which can be written more concisely as

\[
H_G^*(P) = H^*(\Lambda W_M \otimes \Lambda V, d)
\]

Since \( M \) is a manifold, we can compute its equivariant cohomology via the Cartan model (see [GS], [Cartan 1] and [Cartan 2]). So ideally, we would like to include this model in our calculations. Fortunately, this is possible, because \( C^*(M_G) \) is weakly equivalent to \( A_{PL}(M_G) \), and the following

Lemma 4. The Cartan model is weakly equivalent to \( C^*(M_G) \).

Proof:
This is an easy corollary of a number of results in [GS]. I will simply present the chain of quasi-isomorphism, and indicate the result in this book saying that it is a quasi-isomorphism.

The first few are present in their proof of the equivariant De Rham Theorem, section 2.5. \( \mathcal{E} \) denotes the set of orthonormal \( n \)-tuples in \( \mathbb{C}^\infty \) (for all values of \( n \)), which is the inductive limit of \( \mathcal{E}_k \), the set of orthonormal \( n \)-tuples in \( \mathbb{C}^\infty \) (for all \( n < k \)), and \( \Omega^*(\mathcal{E}) \) denotes the projective limit of the \( \Omega^*(\mathcal{E})_k \).

\[
C^*((M \times \mathcal{E})/G) \xrightarrow{\sim} \Omega^*((M \times \mathcal{E})/G)
\]
\[
\Omega^*((M \times \mathcal{E})/G) \xleftarrow{\sim} \Omega^*(M \times \mathcal{E})_{bas}
\]
\[
\Omega^*(M \times \mathcal{E})_{bas} \xrightarrow{\sim} (\Omega^*(M) \otimes \Omega^*(\mathcal{E}))_{bas}
\]

All of these results use the compatibility of cohomology with colimits. The fist line comes from the usual De Rham theorem, the second is well known,
and the third is a bit more involved, making use of a spectral sequence argument.

Then we apply Theorem 4.3.1 and its proof, as suggested in section 4.4 (here $\text{W}$ denotes the Weyl algebra of $G$) to get:

\[
(\Omega^\ast(M) \otimes \Omega^\ast(\mathcal{E}))_{\text{bas}} \cong (\Omega^\ast(M) \otimes \Omega^\ast(\mathcal{E})_{\text{hor}}) \otimes W)^G
\]

\[
((\Omega^\ast(M) \otimes \Omega^\ast(\mathcal{E}))_{\text{hor}}) \otimes \Omega^\ast(\mathcal{E}) \cong (\Omega^\ast(M) \otimes \Omega^\ast(\mathcal{E})_{\text{hor}} \otimes \Omega^\ast(\mathcal{E}))^G
\]

The main ingredients here are the Mathai-Quillen isomorphism, and a filtration argument on the acyclic component. The two central lines are just interchanging the roles of $W$ and $\Omega^\ast(\mathcal{E})$.

and finally, in section 4.2 we find the last quasi-isomorphism we need:

\[
(\Omega^\ast(M) \otimes W)_{\text{bas}} \cong (\Omega^\ast(M) \otimes S(g^*))^G
\]

Which again comes from the Mathai-Quillen isomorphism, and the fact that $W_{\text{hor}} \cong S(g^*)$.

So the Sullivan models for $A_{PL}(M_G)$ and $(\Omega^\ast(M) \otimes S(g^*))^G$ are identified, which implies that there is a quasi-isomorphism $(\Lambda W_{M_G}, d) \cong (\Omega^\ast(M) \otimes S(g^*))^G$ along which we can pushout our model to give

\[
H^G_{\ast}(P) = ((\Omega^\ast(M) \otimes S(g^*))^G \otimes \Lambda V, d)
\]

and there remains to identify the differential.

Let us write $c_i$ for the image of $x_i$ under the Sullivan representative for $f_G$, followed by the model $(\Lambda W_{M_G}, d) \cong (\Omega^\ast(M) \otimes S(g^*))^G$, and let $y_i$ denote the basis elements of $V$. We can then write:

**Theorem 6.** In the situation of the diagram, the $G$-equivariant cohomology of $P$ is isomorphic, as an algebra, to the cohomology of the complex

\[
(\Omega^\ast(M) \otimes S(g^*))^G[y_1, ... y_r]
\]

with differential

\[
d : w(\xi)P(y) \mapsto d_{DR}(w(\xi)) - \iota_{\xi} w(\xi) + \sum_{i=1}^{r} (w \wedge c_i)(\xi) \frac{\partial}{\partial y_i} P(y)
\]
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