Universal birational invariants and $\mathbb{A}^1$-homology

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Abstract

Let $k$ be a field admitting a resolution of singularities. In this paper, we prove that the functor of zeroth $\mathbb{A}^1$-homology $H^{0}_{\mathbb{A}^1}$ is universal as a functorial birational invariant of smooth proper $k$-varieties taking values in a category enriched by abelian groups. For a smooth proper $k$-variety $X$, we also prove that the dimension of $H^{0}_{\mathbb{A}^1}(X;\mathbb{Q})(\text{Spec } k)$ coincides with the number of $R$-equivalence classes of $X(k)$. We deduce these results as consequences of the structure theorem that for a smooth proper $k$-variety $X$, the sheaf $H^{0}_{\mathbb{A}^1}(X)$ is the free abelian presheaf generated by the birational $\mathbb{A}^1$-connected components $\pi^{b\mathbb{A}^1}_0(X)$ of Asok-Morel.

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Introduction

Let $k$ be a field. In [Mor05], Morel introduced the $\mathbb{A}^1$-homology theory of smooth (separated and of finite type) $k$-schemes. In the unstable $\mathbb{A}^1$-homotopy theory introduced by Morel-Voevodsky [MV99], the $\mathbb{A}^1$-homology plays a role of the ordinary homology of topological spaces. For each $n \geq 0$, the $n$-th $\mathbb{A}^1$-homology

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of a smooth $k$-scheme $X$ is denoted by $H^A_0(X)$. This is a Nisnevich sheaf of abelian groups on the category of smooth $k$-schemes $Sm_k$. As an application of $A^1$-homology, Asok [Asok12] proved that the zeroth $A^1$-homology is a birational invariant of smooth proper $k$-varieties if $k$ is infinite. In this paper, we study the structure of the zeroth $A^1$-homology of smooth proper $k$-varieties and give some applications. In particular, we prove that the zeroth $A^1$-homology functor $H^A_0$ is universal as a functorial birational invariant of smooth proper $k$-varieties.

First, we state the universal birational invariance of the zeroth $A^1$-homology. Let $S^p_k$ be the category of smooth proper $k$-schemes and $Im H^A_0$ be the full subcategory of the category of abelian presheaves on $Sm_k$ spanned by objects isomorphic to $H^A_0(X)$ for some $X \in S^p_k$. Our first theorem is stated as follows.

**Theorem 1** (see Theorem 5.5). Assume $k$ admits a resolution of singularities. Let $A$ be an arbitrary category enriched by abelian groups (e.g. an abelian category).

1. Let $F: S^p_k \to A$ be an arbitrary functor which sends each birational morphism to an isomorphism. Then there exists one and only one (up to a natural equivalence) additive functor $F_{Sb}: Im H^A_0 \to A$ such that the diagram

$$
\begin{array}{ccc}
S^p_k & \xrightarrow{F} & A \\
H^A_0 & \searrow^{F_{Sb}} & \\
Im H^A_0 & \nearrow & \\
\end{array}
$$

is 2-commutative, i.e., we have a natural equivalence $F \cong F_{Sb} \circ H^A_0$.

2. Let $F': (S^p_k)^{op} \to A$ be an arbitrary functor which sends each birational morphism to an isomorphism. Then there exists one and only one (up to a natural equivalence) additive functor $F'_{Sb}: (Im H^A_0)^{op} \to A$ such that the diagram

$$
\begin{array}{ccc}
(S^p_k)^{op} & \xrightarrow{F'} & A \\
(H^A_0)^{op} & \searrow^{F'_{Sb}} & \\
(Im H^A_0)^{op} & \nearrow & \\
\end{array}
$$

is 2-commutative.

In this theorem, we may replace $S^p_k$ with other full subcategory of $Sm^p_k$ like the category of projective varieties (see Definition 5.3 and Example 5.4). A similar result also holds in a motivic situation (see Theorem 5.10). In this case, $Sm^p_k$ is replaced with the full subcategory of the category of finite correspondences consisting of proper schemes and the $A^1$-homology with the zeroth homology of Voevodsky’s motives, called Suslin homology sheaves (cf. [Voe00]). The birational invariance theorem of Asok [Asok12] is assumed.
Infinite, but this assumption is not necessary for our proof. In general, we prove that a proper birational morphism of (not necessarily proper) varieties over an arbitrary field induces an isomorphism of zeroth $A^1$-homology sheaves. (see Proposition 5.12).

Second, we relate the zeroth $A^1$-homology to rational points. We consider the zeroth $A^1$-homology with $\mathbb{Q}$-coefficients $H^0_{A^1}(X; \mathbb{Q})$ for a smooth proper $k$-variety $X$.

**Theorem 2** (see Theorem 4.7). Assume $k$ admits a resolution of singularities. For a smooth proper $k$-variety $X$, we have

$$\dim_{\mathbb{Q}} H^0_{A^1}(X; \mathbb{Q})(\text{Spec } k) = \#(X(k)/R).$$

Moreover, $H^0_{A^1}(X)(\text{Spec } k) = 0$ if and only if $X(k) = \emptyset$.

Here $X(k)/R$ is the quotient set of $X(k)$ by the $R$-equivalence introduced by Manin [Man86]. Note that the abelian group $H^0_{A^1}(X)(\text{Spec } k)$ can also be expressed in terms of triangulated categories (see Remark 4.10).

Third, we state a structure theorem of zeroth $A^1$-homology sheaves. In [AM11], Asok-Morel constructed a Nisnevich sheaf of sets $\pi^b_{A^1}(X)$ on $Sm_k$, called the birational $A^1$-connected components, for each $X \in Sm_k^{prop}$. We generalize this construction for all smooth $k$-schemes. Our structure theorem is stated as follows.

**Theorem 3** (see Theorem 4.1). Assume $k$ admits a resolution of singularities. For every $X \in Sm_k$, there exists a natural epimorphism of presheaves

$$Z_{pre}(\pi^b_{A^1}(X)) \twoheadrightarrow H^1_{A^1}(X).$$

Moreover, this is an isomorphism if $X$ is proper.

Here $Z_{pre}(\pi^b_{A^1}(X))$ is the free abelian presheaf generated by $\pi^b_{A^1}(X)$. A similar result also holds for Suslin homology sheaves (see Theorem 4.11). Theorem 3 has some applications to $A^1$-homotopy theory (see [4]). Moreover, the Suslin homology version of Theorem 3 also has an application to zero cycles (see Corollary 4.12). Theorems 1 and 2 are consequences of Theorem 3.

This paper is organized as follows. In §1, we recall and prepare basic results on birational sheaves and the localization of some categories of smooth $k$-schemes by birational morphisms. In §2 we generalize birational $A^1$-connected components of Asok-Morel for all Nisnevich sheaves on $Sm_k$. In §3 we give a relationship between birational sheaves and strictly $A^1$-invariant sheaves. In §6 we prove Theorems 2 and 3. In §6 we prove Theorem 1. In §6 we give applications to $A^1$-homotopy theory.

**Conventions.** Throughout this paper, we fix a field $k$ and a commutative unital ring $\Lambda$. All $k$-varieties are assumed irreducible but not assumed geometrically irreducible. The field $k$ is said to admit a resolution of singularities, if $k$-varieties always have a resolution of singularities. Let $Sm_k$ be the category of separated
and smooth $k$-schemes of finite type. Objects of $Sm_k$ are simply called smooth $k$-schemes. We regard every smooth $k$-scheme $X$ as the presheaf of sets on $Sm_k$ represented by $X$. For a presheaf $\mathcal{F}$ on $Sm_k$ and an affine scheme $\text{Spec } A$ which is the limit of an inversed system $\{U_\lambda\}_\lambda$ in $Sm_k$, we write

$$\mathcal{F}(A) = \operatorname{colim}_\lambda \mathcal{F}(U_\lambda).$$

For a category $\mathcal{C}$, we denote $\mathcal{Presh}(\mathcal{C})$ (resp. $\mathcal{Presh}(\mathcal{C}, \Lambda)$) for the category of presheaves of sets (resp. $\Lambda$-modules) on $\mathcal{C}$. A diagram of categories

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
G' \downarrow & & \downarrow G \\
\mathcal{C}' & \xrightarrow{F'} & \mathcal{D}'
\end{array}$$

is called 2-commutative, if there exists a natural equivalence $G \circ F \cong F' \circ G'$. For a local ring $A$, we denote $\kappa_A$ for its residue field.

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1 Birational sheaves and localizations of categories

In this section, we consider localizations of categories in the sense of Gabriel-Zisman [GZ67]. Let $\mathcal{C}$ be a category and $S$ be a family of morphisms in $\mathcal{C}$. Recall that the localization of $\mathcal{C}$ by $S$ is a category $S^{-1}\mathcal{C}$ with a morphism $\mathcal{L} : \mathcal{C} \to S^{-1}\mathcal{C}$ such that

- for every $s \in S$, the image $\mathcal{L}(s)$ is an isomorphism in $S^{-1}\mathcal{C}$, and

- for every category $\mathcal{D}$ and every functor $\mathcal{C} \to \mathcal{D}$ which sends each $s \in S$ to an isomorphism in $\mathcal{D}$, there exists one and only one (up to a natural equivalence) functor $S^{-1}\mathcal{C} \to \mathcal{D}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
S^{-1}\mathcal{C} \downarrow & & \downarrow G
\end{array}$$

is 2-commutative.

Note that the induced functor $\mathcal{Presh}(S^{-1}\mathcal{C}) \to \mathcal{Presh}(\mathcal{C})$ is fully faithful and its essential image is spanned by presheaves which send each $s \in S$ for a bijection. We especially treat some cases where $\mathcal{C}$ is a category of smooth $k$-schemes and $S$ consists of all birational morphisms in $\mathcal{C}$. 
1.1 Localizations by birational morphisms

Let $\mathcal{S}_m^{\text{var}}$ (resp. $\mathcal{S}_m^{\text{prop}}$, $\mathcal{S}_m^{\text{pv}}$) be the full subcategory of $\mathcal{S}_m$ consisting of irreducible (resp. proper, proper and irreducible) $k$-schemes. For a subcategory $\mathcal{C} \subseteq \mathcal{S}_m$, we write $\mathcal{S}_m^{-1} \mathcal{C}$ for the localization of $\mathcal{C}$ by birational morphisms within $\mathcal{C}$. In [KS07], Kahn-Sujatha proved that the inclusion $\mathcal{S}_m^{\text{pv}} \hookrightarrow \mathcal{S}_m^{\text{var}}$ induces an equivalence of categories $\mathcal{S}_m^{-1} \mathcal{S}_m^{\text{pv}} \cong \mathcal{S}_m^{-1} \mathcal{S}_m^{\text{var}}$ (1.1) if $k$ admits a resolution of singularities (see [KS07, Prop. 8.5]).

Let $\text{Cor}_{k,\Lambda}$ be the category of finite correspondences of Voevodsky with coefficients in a commutative unital ring $\Lambda$ (see definition [MVW, Def. 1.5]). We denote $\Gamma$ for the canonical functor $\mathcal{S}_m \to \text{Cor}_{k,\Lambda}$ and write $\text{Cor}_{k,\Lambda}(X, Y)$ for the set of finite correspondences $X \to Y$ in $\text{Cor}_{k,\Lambda}$.

For a subcategory $\mathcal{C} \subseteq \text{Cor}_{k,\Lambda}$, we denote $\mathcal{S}_m^{-1} \mathcal{C}$ for the localization of $\mathcal{C}$ by finite correspondences associated with a birational morphism. Let $\text{Cor}_{k,\Lambda}^{\text{var}}$ (resp. $\text{Cor}_{k,\Lambda}^{\text{prop}}$, $\text{Cor}_{k,\Lambda}^{\text{pv}}$) be the full subcategory of $\text{Cor}_{k,\Lambda}$ consisting of irreducible (resp. proper, proper and irreducible) $k$-schemes. Next, we give an analogue of the equivalence (1.1) for categories of finite correspondences.

For this, we prove the following lemma.

**Lemma 1.1.** For all $X_0, X, Y \in \text{Cor}_{k,\Lambda}$ and every birational morphism $f : X_0 \to X$ in $\mathcal{S}_m$, the induced map

$$f^* : \text{Cor}_{k,\Lambda}(X, Y) \to \text{Cor}_{k,\Lambda}(X_0, Y); c \mapsto c \circ \Gamma(f)$$

is injective.

**Proof.** Let $i : U \hookrightarrow X$ be a dense open embedding that $f^{-1}(i(U)) \to i(U)$ is an isomorphism. By the commutativity of the diagram

$$\begin{array}{ccc}
\text{Cor}_{k,\Lambda}(X, Y) & \xrightarrow{f^*} & \text{Cor}_{k,\Lambda}(X_0, Y) \\
\downarrow i^* & & \downarrow \\
\text{Cor}_{k,\Lambda}(U, Y) & \cong & \text{Cor}_{k,\Lambda}(f^{-1}(U), Y),
\end{array}$$

we only need to show that $i^*$ is injective. Note that for every $c \in \text{Cor}_{k,\Lambda}(X, Y)$ the finite correspondence $i^* c \in \text{Cor}_{k,\Lambda}(U, Y)$ coincides with the pullback of $c$ by the open embedding

$$j : U \times Y \hookrightarrow X \times Y$$

as an algebraic cycle of $U \times Y$. We write $W = (X \times Y) - j(U \times Y)$. Now we obtain an equality

$$\ker(Z(X \times Y) \xrightarrow{j^*} Z(U \times Y)) = Z(W),$$

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5
where \( Z(\cdot) \) means the set of algebraic cycles with \( \Lambda \)-coefficients. Thus we have

\[
\text{Ker}(\text{Cor}_{k,\Lambda}(X, Y) \xrightarrow{i^*} \text{Cor}_{k,\Lambda}(U, Y)) = \text{Cor}_{k,\Lambda}(X, Y) \cap \text{Ker}(Z(X \times Y) \xrightarrow{j^*} Z(U \times Y)) = \text{Cor}_{k,\Lambda}(X, Y) \cap Z(W).
\]

On the other hand, since \( U \times Y \) is non-empty, the map \( W \to X \) is not surjective. Therefore, all non-zero elements of \( Z(W) \) are not finite correspondences \( X \to Y \). Thus we have

\[
\text{Ker} i^* = \text{Cor}_{k,\Lambda}(X, Y) \cap Z(W) = 0.
\]

We prove that \( S_b^{-1}\text{Cor}^pv_{k,\Lambda} \cong S_b^{-1}\text{Cor}^{\text{var}}_{k,\Lambda} \).

**Proposition 1.2.** Assume \( k \) admits a resolution of singularities. Then the inclusion \( \text{Cor}^pv_{k,\Lambda} \hookrightarrow \text{Cor}^{\text{var}}_{k,\Lambda} \) induces an equivalence of categories

\[
S_b^{-1}\text{Cor}^pv_{k,\Lambda} \cong S_b^{-1}\text{Cor}^{\text{var}}_{k,\Lambda}.
\]

**Proof.** By [KS07, Thm. 2.1 and 4.3], we only need to show that the functors \( \text{Cor}^pv_{k,\Lambda} \to \text{Cor}^{\text{var}}_{k,\Lambda} \) and \( \text{Cor}^{\text{prop}}_{k,\Lambda} \to \text{Cor}_{k,\Lambda} \) satisfy the conditions (b1)-(b3) in [KS07, Prop. 5.10]. Then (b1) follows from Lemma [11] and (b2) and (b3) follow from a similar argument as [KS07, proof of Prop. 8.5] (see also [KS07, proof of Prop. 8.4]).

**Remark 1.3.** By a similar proof, we also obtain an equivalence of categories

\[
S_b^{-1}\text{Cor}_{k,\Lambda} \cong S_b^{-1}\text{Cor}^{\text{prop}}_{k,\Lambda}.
\]

### 1.2 Birational sheaves

Following Asok-Morel [AMI], we call a presheaf \( \mathcal{F} \) on \( Sm_k \) a *birational sheaf* if

- **B1** the canonical map \( \mathcal{F}(U \sqcup V) \to \mathcal{F}(U) \times \mathcal{F}(V) \) is bijective for all \( U, V \in Sm_k \), and

- **B2** every open embedding \( U \hookrightarrow X \) in \( Sm_k \) induces a bijection \( \mathcal{F}(X) \cong \mathcal{F}(U) \).

Note that birational sheaves are always Nisnevich (see [AMI, Lem. 6.1.2]). Let \( Shv_k \) be the category of Nisnevich sheaves of sets on \( Sm_k \). We denote \( Shv^br_k \) for the full subcategory of \( Shv_k \) consisting of birational sheaves. We give a canonical equivalence \( Shv^br_k \cong \mathcal{Presh}(S_b^{-1}Sm^{\text{var}}_k) \).

**Lemma 1.4.** There exists an equivalence of categories

\[
Shv^br_k \cong \mathcal{Presh}(S_b^{-1}Sm^{\text{var}}_k).
\]
such that the diagram

\[
\begin{array}{ccc}
\text{Shv}_k^{br} & \xrightarrow{\cong} & \text{Presh}(S_b^{-1} Sm_{k}^{\text{var}}) \\
\downarrow & & \downarrow \\
\text{Shv}_k & \longrightarrow & \text{Presh}(Sm_{k}^{\text{var}})
\end{array}
\]

is 2-commutative.

**Proof.** Let \(\text{Presh}_{B2}(Sm_{k}^{\text{var}})\) be the full subcategory of \(\text{Presh}(Sm_{k}^{\text{var}})\) consisting of presheaves which satisfy \(B2\). Then \(\text{Presh}_{B2}(Sm_{k}^{\text{var}})\) is equivalent to \(\text{Presh}(S_b^{-1} Sm_{k}^{\text{var}})\) by the universality of localizations. On the other hand, since every object in \(Sm_{k}\) is a finite coproduct of objects in \(Sm_{k}^{\text{var}}\), the restriction functor \(\text{Shv}_k^{br} \to \text{Presh}_{B2}(Sm_{k}^{\text{var}})\) is an equivalence by \(B1\). Thus we have a 2-commutative diagram

\[
\begin{array}{ccc}
\text{Shv}_k^{br} & \cong & \text{Presh}_{B2}(Sm_{k}^{\text{var}}) \cong \text{Presh}(S_b^{-1} Sm_{k}^{\text{var}}) \\
\downarrow & & \downarrow \\
\text{Shv}_k & \longrightarrow & \text{Presh}(Sm_{k}^{\text{var}})
\end{array}
\]

\(\Box\)

For a category \(C\) enriched by \(\Lambda\)-modules, we denote \(\text{Lin}(C, \Lambda)\) for the category of \(\Lambda\)-linear presheaves of \(\Lambda\)-modules on \(C\). We write

\[
PST_k(\Lambda) = \text{Lin}(\text{Cor}_{k,\Lambda}, \Lambda).
\]

Objects in \(\text{PST}_k(\Lambda)\) are called a presheaves with transfers. A presheaf with transfers \(M\) is called Nisnevich (resp. birational), if so is the direct image \(\Gamma_* M\) by the canonical functor \(\Gamma : Sm_{k} \to \text{Cor}_{k,\Lambda}\). We denote \(\text{NST}_k(\Lambda)\) (resp. \(\text{NST}_k^{br}(\Lambda)\)) for the category of Nisnevich (resp. birational) sheaves with transfers. We prove an analogue of Lemma 1.4 for sheaves with transfers.

**Lemma 1.5.** There exists an equivalence of categories

\[
\text{NST}_k^{br}(\Lambda) \cong \text{Lin}(S_b^{-1}\text{Cor}_{k,\Lambda}^{\text{var}}, \Lambda)
\]

such that the diagram

\[
\begin{array}{ccc}
\text{NST}_k^{br}(\Lambda) & \xrightarrow{\cong} & \text{Lin}(S_b^{-1}\text{Cor}_{k,\Lambda}^{\text{var}}, \Lambda) \\
\downarrow & & \downarrow \\
\text{NST}_k(\Lambda) & \longrightarrow & \text{Lin}(\text{Cor}_{k,\Lambda}^{\text{var}}\Lambda)
\end{array}
\]

is 2-commutative.
Proof. Let $\text{Lin}_{\mathcal{B}_2}(\text{Cor}_{k,A}^{\text{var}}, \Lambda)$ be the full subcategory of $\text{Lin}(\text{Cor}_{k,A}^{\text{var}}, \Lambda)$ consisting of presheaves which satisfy $\mathcal{B}_2$. By a similar argument as the proof of Lemma 1.4, we also have a 2-commutative diagram

\[
\begin{array}{ccc}
\ NST_k^\text{br} (\Lambda) & \overset{\cong}{\longrightarrow} & \text{Lin}_{\mathcal{B}_2}(\text{Cor}_{k,A}^{\text{var}}, \Lambda) \\
\downarrow & & \downarrow \\
\ NST_k (\Lambda) & \longrightarrow & \text{Lin}(\text{Cor}_{k,A}^{\text{var}}, \Lambda) \end{array}
\]

The following lemma says that birational sheaves are $A^1$-invariant.

**Lemma 1.6.** For every $\mathcal{F} \in \text{Shv}_k^{\text{br}}$ and every $U \in \text{Sm}_k^{\text{var}}$, the canonical map

\[
\mathcal{F}(U) \to \mathcal{F}(U \times A^1)
\]

is an isomorphism.

**Proof.** By [KS15, Thm. 1.7.9], stable birational morphisms are isomorphisms in $S_b^{-1}\text{Sm}_k^{\text{var}}$. Since the projection $U \times A^1 \to U$ is stable birational, we have $\mathcal{F}(U) \cong \mathcal{F}(U \times A^1)$ by Lemma 1.4. \qed

## 2 Birational $A^1$-connected components

In [AM11], Asok-Morel introduced a birational sheaf $\pi_0^{\text{br}}(X)$, called the birational $A^1$-connected components, for each $X \in \text{Sm}_k^{\text{prop}}$. In this section, we generalize this construction for all Nisnevich sheaves on $\text{Sm}_k$. We first construct a functor $\pi_0^{\text{br}} : \text{Shv}_k \to \text{Shv}_k^{\text{br}}$ and secondly prove that $\pi_0^{\text{br}}(X) \cong \pi_0^{b_{A^1}}(X)$ for every $X \in \text{Sm}_k^{\text{prop}}$. Moreover, we also prove that the functor $\pi_0^{\text{br}}$ is left adjoint to the inclusion $\text{Shv}_k^{\text{br}} \hookrightarrow \text{Shv}_k$. Similarly, we construct a left adjoint functor of the inclusion $\text{NST}_k^{\text{br}}(\Lambda) \hookrightarrow \text{NST}_k(\Lambda)$.

### 2.1 Birational $A^1$-connected components of sheaves

First, we give a functor $\pi_0^{\text{br}} : \text{Shv}_k \to \text{Shv}_k^{\text{br}}$.

**Definition 2.1.** We define a functor $\pi_0^{\text{br}} : \text{Shv}_k \to \text{Shv}_k^{\text{br}}$ as the composition

\[
\text{Shv}_k \hookrightarrow \text{Presh}(\text{Sm}_k) \to \text{Presh}(\text{Sm}_k^{\text{var}}) \to \text{Shv}_k^{\text{br}},
\]

where the third functor is a left Kan extension of

\[
\text{Sm}_k^{\text{var}} \to S_b^{-1}\text{Sm}_k^{\text{var}} \hookrightarrow \text{Presh}(S_b^{-1}\text{Sm}_k^{\text{var}}) \cong \text{Shv}_k^{\text{br}}.
\]

Explicitly, every sheaf $\mathcal{F}$ on $\text{Sm}_k$ is canonically isomorphic to the colimit of a direct system of smooth $k$-varieties $\{X_\lambda\}_\Lambda$ (note that every object in $\text{Sm}_k$ is
a coproduct of smooth $k$-varieties). Then by the definition, we have a natural bijection
\[ \pi_0^{br}(\mathcal{F})(U) \cong \colim_{\lambda} \Hom_{S_k^{\text{sm}_{\text{var}}}}(U, X_{\lambda}) \]
for every $U \in S^m_k$. Thus the map
\[ \Hom_{S_k}(U, X_{\lambda}) \to \Hom_{S_k^{\text{sm}_{\text{var}}}}(U, X_{\lambda}) \]
induces a natural morphism
\[ \mathcal{F} \to \pi_0^{br}(\mathcal{F}) \]
in $\text{Sh}_{nk}$ which is functorial for $\mathcal{F}$.

Let $X$ be a smooth proper $k$-variety. Recall that two points in $X(K)$ for a field extension $K/k$ are called $R$-equivalence (in the sense of Manin [Man86]), if these points are connected by the image of a chain of $k$-morphisms $\mathbb{P}^1_K \to X$. The quotient set of $X(K)$ by the $R$-equivalence is denoted by $X(K)/R$. The birational $\mathbb{A}^1$-connected components $\pi_0^{br}(X)$ has the property that there exists a canonical bijection
\[ X(K)/R \cong \pi_0^{br}(X)(K) \]
for every finitely generated separable extension $K/k$ (see [AM11, Thm. 6.2.1]). Our sheaf $\pi_0^{br}(X)$ also has the same property.

**Lemma 2.2.** Let $X$ be a smooth proper $k$-scheme and $K/k$ be a finitely generated separable extension. Then the map $X(K) \to \pi_0^{br}(X)(K)$ induced by the canonical morphism $X \to \pi_0^{br}(X)$ factors through a bijection
\[ X(K)/R \cong \pi_0^{br}(X)(K). \]

**Proof.** This follows from the bijection
\[ X(K)/R \cong \Hom_{S_k^{\text{sm}_{\text{var}}}}(U, X) \]
in [KS15, Thm. 6.6.3], where $U$ is a smooth model of $K$. \hfill \Box

Next, we give an isomorphism $\pi_0^{br}(X) \cong \pi_0^{\text{rk}1}(X)$ for each $X \in S_k^{\text{prop}}$.

**Proposition 2.3.** For every $X \in S_k^{\text{prop}}$, there exists an isomorphism of birational sheaves
\[ \pi_0^{br}(X) \cong \pi_0^{\text{rk}1}(X). \]

Before proving this proposition, we recall the category $\mathcal{F}_k^{\text{fin}} - \text{Set}$ introduced by Asok-Morel [AM11]. Let $\mathcal{F}_k$ be the category of finitely generated separable extension fields over $k$. Each object $S \in \mathcal{F}_k - \text{Set}$ is a covariant functor
\[ \mathcal{F}_k \to \text{Set} \]
together with a map \( S(K) \to S(\kappa_A) \) for each \( K \in \mathcal{F}_k \) and its discrete valuation ring \( A \) with \( \kappa_A \in \mathcal{F}_k \). Moreover, a morphism \( S \to S' \) in \( \mathcal{F}_k \) is a natural transformation \( S \to S' \) such that the diagram

\[
\begin{array}{ccc}
S(K) & \longrightarrow & S(\kappa_A) \\
\downarrow & & \downarrow \\
S'(K) & \longrightarrow & S'(\kappa_A)
\end{array}
\]

commutes. The restriction of each \( \mathcal{F} \in \text{Shv}_k^{br} \) on \( \mathcal{F}_k \) together with the map

\[
\mathcal{F}(K) \cong \mathcal{F}(A) \to \mathcal{F}(\kappa_A)
\]

is an object of \( \mathcal{F}_k \) - Set. Thus we have the restriction functor

\[
\text{Res} : \text{Shv}_k^{br} \to \mathcal{F}_k \text{-Set}. \tag{2.1}
\]

**Proof of Proposition 2.3.** By Lemma 1.6 and [AM11, Thm. 6.1.7], the restriction functor (2.1) is fully faithful. Thus we only need to construct an isomorphism

\[
\text{Res}(\pi_{br}^0(X)) \cong \text{Res}(\pi_{bh}^1(X)) \tag{2.2}
\]

in \( \mathcal{F}_k \) - Set. Lemma [2.2] and [AM11] Thm. 6.2.1 give a commutative diagram

\[
\begin{array}{ccc}
X(K) & \longrightarrow & X(K) \\
\downarrow & & \downarrow \\
\pi_{br}^0(X)(K) & \cong & X(K)/R \cong \pi_{bh}^1(X)(K)
\end{array}
\]

for all \( K \in \mathcal{F}_k \). Then the vertical maps are surjective. Thus for every extension field \( L/K \) which is finitely generated and separable over \( k \), the obviously commutative diagram

\[
\begin{array}{ccc}
X(K) & \longrightarrow & X(K) \\
\downarrow & & \downarrow \\
X(L) & \longrightarrow & X(L)
\end{array}
\]

shows that

\[
\begin{array}{ccc}
\pi_{br}^0(X)(K) & \cong & X(K)/R \cong \pi_{bh}^1(X)(K) \\
\downarrow & & \downarrow \\
\pi_{br}^0(X)(L) & \cong & X(L)/R \cong \pi_{bh}^1(X)(L)
\end{array}
\]

commutes. Similarly, for every discrete valuation ring \( A \) of \( K \) with \( \kappa_A \in \mathcal{F}_k \), the diagram

\[
\begin{array}{ccc}
X(K) & \longrightarrow & X(K) \\
\downarrow & & \downarrow \\
X(\kappa_A) & \longrightarrow & X(\kappa_A)
\end{array}
\]
shows that
\[\pi_0^{br}(X)(K) \xrightarrow{\cong} X(K)/R \xleftarrow{\cong} \pi_0^{hk}(X)(K)\]
also commutes. Here the map \(X(K) \to X(\kappa_A)\) is the composition of the inverse of the bijection \(X(K) \cong X(A)\) (the bijectivity follows from the valuative criterion of properness) and the canonical map \(X(A) \to X(\kappa_A)\). Therefore, the bijection
\[\pi_0^{br}(X)(K) \cong \pi_0^{hk}(X)(K)\]
gives an isomorphism \([2.2]\).

From now on, we write \(\pi_0^{hk} = \pi_0^{br}\). We have an adjunction \(Shv_k \rightleftarrows Shv^{br}_k\).

**Lemma 2.4.** The functor \(\pi_0^{hk}\) is left adjoint to the inclusion \(Shv^{br}_k \hookrightarrow Shv_k\).

**Proof.** Let \(\mathcal{G}\) be an arbitrary birational sheaf. For every \(X \in Sm^\text{var}_k\), Yoneda’s lemma in \(Sm^\text{var}_k\) gives a natural isomorphism
\[\text{Hom}_{Shv_k}(X, \mathcal{G}) \cong \mathcal{G}(X)\]
Under the identification by the equivalence in Lemma 1.4 the presheaf \(\pi_0^{hk}(X)\) is represented by \(X \in S_b^{-1}Sm^\text{var}_k\). Thus Yoneda’s lemma in \(S_b^{-1}Sm^\text{var}_k\) also gives an isomorphism
\[\text{Hom}_{Shv^{br}_k}(\pi_0^{hk}(X), \mathcal{G}) \cong \mathcal{G}(X)\]
and we have
\[\text{Hom}_{Shv^{br}_k}(\pi_0^{hk}(X), \mathcal{G}) \cong \text{Hom}_{Shv_k}(X, \mathcal{G}).\] \(\square\)

Let \(\mathcal{F}\) be a Nisnevich sheaf on \(Sm_k\) which is the colimit of a direct system \(\{X_\lambda\}_\lambda\) in \(Sm^\text{var}_k\). Then \(\text{(2.3)}\) gives isomorphisms
\[\text{Hom}_{Shv^{br}_k}(\pi_0^{hk}(\mathcal{F}), \mathcal{G}) \cong \lim_{\lambda} \text{Hom}_{Shv^{br}_k}(\pi_0^{hk}(X_\lambda), \mathcal{G})\]
\[\cong \lim_{\lambda} \text{Hom}_{Shv_k}(X_\lambda, \mathcal{G})\]
\[\cong \text{Hom}_{Shv_k}(\text{colim} X_\lambda, \mathcal{G})\]
\[\cong \text{Hom}_{Shv_k}(\mathcal{F}, \mathcal{G}).\] \(\square\)
2.2 Birational $\mathbb{A}^1$-connected components with transfers

Next, we define a functor $\Lambda_{\pi^{b,1}_{0,\text{tr}}}: \text{NST}_k(\Lambda) \rightarrow \text{NST}^{br}_k(\Lambda)$. We denote $\Lambda_{tr}$ for the Yoneda embedding

$$\text{Cor}_{k,\Lambda} \rightarrow \text{NST}_k(\Lambda); X \mapsto \text{Cor}_{k,\Lambda}(-, X).$$

**Definition 2.5.** We define a functor $\Lambda_{\pi^{b,1}_{0,\text{tr}}}: \text{NST}_k(\Lambda) \rightarrow \text{NST}^{br}_k(\Lambda)$ as the composition

$$\text{NST}_k(\Lambda) \hookrightarrow \text{PST}_k(\Lambda) \rightarrow \text{Lin}(\text{Cor}_{k,\Lambda}^{\var}, \Lambda) \rightarrow \text{NST}^{br}_k(\Lambda),$$

where the third functor is a left Kan extension of

$$\text{Cor}_{k,\Lambda}^{\var} \rightarrow S^{-1}_b\text{Cor}_{k,\Lambda}^{\var} \rightarrow \text{Lin}(S^{-1}_b\text{Cor}_{k,\Lambda}^{\var}, \Lambda) \cong \text{NST}^{br}_k(\Lambda).$$

We simply write $\Lambda_{\pi^{b,1}_{0,\text{tr}}}(X) = \Lambda_{\pi^{b,1}_{0,\text{tr}}}(\Lambda_{tr}(X))$.

We have an adjunction $\text{NST}_k(\Lambda) \dashv \text{NST}^{br}_k(\Lambda)$.

**Lemma 2.6.** The functor $\Lambda_{\pi^{b,1}_{0,\text{tr}}}$ is left adjoint to $\text{NST}^{br}_k(\Lambda) \rightarrow \text{NST}_k(\Lambda)$.

**Proof.** Let $N$ be an arbitrary birational sheaf with transfers of $\Lambda$-modules. For every $X \in \text{Cor}_{k,\Lambda}^{\var}$, Yoneda's lemma in $\text{Cor}_{k,\Lambda}^{\var}$ gives an isomorphism

$$\text{Hom}_{\text{NST}^{br}_k(\Lambda)}(\Lambda_{\pi^{b,1}_{0,\text{tr}}}(X), N) \cong N(X).$$

Under the identification by the equivalence in Lemma 1.5, the presheaf $\Lambda_{\pi^{b,1}_{0,\text{tr}}}(X)$ is represented by $X$ in $S^{-1}_b\text{Cor}_{k,\Lambda}^{\var}$. Thus Yoneda's lemma in $S^{-1}_b\text{Cor}_{k,\Lambda}^{\var}$ shows that

$$\text{Hom}_{\text{NST}_k(\Lambda)}(\Lambda_{tr}(X), N) \cong N(X) \cong \text{Hom}_{\text{NST}^{br}_k(\Lambda)}(\Lambda_{\pi^{b,1}_{0,\text{tr}}}(X), N). \quad (2.4)$$

Let $M$ be a Nisnevich sheaf with transfers such that

$$M \cong \text{colim}_\Lambda \Lambda_{tr}(X_\lambda)$$

for a direct system $\{X_\lambda\}_\Lambda$ in $\text{Cor}_{k,\Lambda}^{\var}$. Then there exists a canonical isomorphism

$$\Lambda_{\pi^{b,1}_{0,\text{tr}}}(M) \cong \text{colim}_\Lambda \Lambda_{\pi^{b,1}_{0,\text{tr}}}(X_\lambda)$$

by the definition of the functor $\Lambda_{\pi^{b,1}_{0,\text{tr}}}$. Thus (2.4) gives isomorphisms

$$\text{Hom}_{\text{NST}^{br}_k(\Lambda)}(M, N) \cong \text{Hom}_{\text{NST}^{br}_k(\Lambda)}(\text{colim}_\Lambda \Lambda_{tr}(X_\lambda), N)$$

$$\cong \text{lim}_\Lambda \text{Hom}_{\text{NST}^{br}_k(\Lambda)}(\Lambda_{tr}(X_\lambda), N)$$

$$\cong \text{lim}_\Lambda \text{Hom}_{\text{NST}^{br}_k(\Lambda)}(\Lambda_{\pi^{b,1}_{0,\text{tr}}}(X_\lambda), N)$$

$$\cong \text{Hom}_{\text{NST}_k(\Lambda)}(\Lambda_{\pi^{b,1}_{0,\text{tr}}}(M), N).$$

$\square$
Remark 2.7. Lemma 2.6 also gives a left adjoint of the forgetful functor \( \text{NST}_k(\Lambda) \to \text{Shv}_k \). Indeed, a left Kan extension \( \text{Shv}_k \to \text{NST}_k(\Lambda) \) of the composition \( \text{Sm}_k \xrightarrow{\Gamma} \text{Cor}_{k,\Lambda} \xrightarrow{\Lambda_{tr}} \text{NST}_k(\Lambda) \) is left adjoint to the forgetful functor \( \text{NST}_k(\Lambda) \to \text{Shv}_k \).

3 Birational sheaves of modules

Recall that a Nisnevich sheaf of \( \Lambda \)-modules \( M \) on \( \text{Sm}_k \) is called \emph{strictly} \( \mathbb{A}^1 \)-invariant, if the map

\[
H^i_{\text{Nis}}(U, M) \to H^i_{\text{Nis}}(U \times \mathbb{A}^1, M)
\]

induced by the projection \( U \times \mathbb{A}^1 \to U \) is an isomorphism for all \( i \geq 0 \) and all \( U \in \text{Sm}_k \). Our aim of this section is to construct a birational sheaf of \( \Lambda \)-modules \( M_{br} \) with a monomorphism \( \mu_M : M_{br} \to M \) which induces an isomorphism \( M_{br}(X) \cong M(X) \) for all \( X \in \text{Sm}^{\text{prop}}_k \). This morphism \( \mu_M \) plays a key role in the proof of Theorem 3.

3.1 Strictly \( \mathbb{A}^1 \)-invariant sheaves and birational invariance

We denote \( \text{Mod}^k_{\mathbb{A}^1}(\Lambda) \) (resp. \( \text{Mod}^k_{\text{br}}(\Lambda) \)) for the category of strictly \( \mathbb{A}^1 \)-invariant (resp. birational) sheaves of \( \Lambda \)-modules. Then we have \( \text{Mod}^k_{\text{br}}(\Lambda) \subseteq \text{Mod}^k_{\mathbb{A}^1}(\Lambda) \). Indeed, all birational sheaves are \( \mathbb{A}^1 \)-invariant by Lemma 1.6. On the other hand, \( \mathbb{A}^1 \)-invariant birational sheaves of abelian groups are strictly \( \mathbb{A}^1 \)-invariant by \[AH11\] proof of Lem. 2.4]. The following lemma says that every proper birational morphism induces an isomorphism for all strictly \( \mathbb{A}^1 \)-invariant sheaves.

**Lemma 3.1.** Let \( M \) be a strictly \( \mathbb{A}^1 \)-invariant sheaf of \( \Lambda \)-modules and \( f : X \to Y \) be a proper birational morphism of smooth \( k \)-varieties. Then the induced map \( M(Y) \to M(X) \) is an isomorphism.

**Proof.** We write \( K = k(X) \) and identify \( K \) with \( k(Y) \) under the isomorphism induced by \( f \). For each \( U \in \text{Sm}_k \), the map \( M(U) \to M(k(U)) \) gives an isomorphism

\[
M(U) \cong \bigcap_{x \in U^{(1)}} \text{Im}(M(O_{U,x}) \to M(k(U)))
\]

by \[Aso12\] Lem. 4.2]. Thus we only need to show that

\[
\bigcap_{x \in X^{(1)}} \text{Im}(M(O_{X,x}) \to M(K)) = \bigcap_{y \in Y^{(1)}} \text{Im}(M(O_{Y,y}) \to M(K)).
\]

The inclusion \( \supseteq \) follows from the commutative diagram

\[
\begin{array}{ccc}
M(Y) & \xrightarrow{f^*} & M(k(Y)) \\
\downarrow & & \downarrow \cong \\
M(X) & \xrightarrow{} & M(k(X)).
\end{array}
\]
We prove $\subseteq$. By the valuative criterion of properness, for every $y \in Y^{(1)}$ the commutative diagram

\[
\begin{array}{ccc}
\text{Spec } K & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\text{Spec } \mathcal{O}_{Y,y} & \longrightarrow & Y
\end{array}
\]

has a lift $\phi_y : \text{Spec } \mathcal{O}_{Y,y} \rightarrow X$. Then the local ring $\mathcal{O}_{X,\phi_y(y')}$ for the closed point $y' \in \text{Spec } \mathcal{O}_{Y,y}$ coincides with the discrete valuation ring $\mathcal{O}_{Y,y}$ of $K$. In particular, $\phi_y(y')$ has codimension 1 in $X$. Therefore, we have

\[
\bigcap_{x \in X^{(1)}} \text{Im}(M(\mathcal{O}_{X,x}) \rightarrow M(K)) \subseteq \bigcap_{y \in Y^{(1)}} \text{Im}(M(\mathcal{O}_{X,\phi_y(y')}) \rightarrow M(K)) = \bigcap_{y \in Y^{(1)}} \text{Im}(M(\mathcal{O}_{Y,y}) \rightarrow M(K)).
\]

Remark 3.2. More generally, Lemma 3.1 holds for unramified sheaves of Morel [Mor12] by the same proof.

3.2 Birational construction for strictly $\mathbb{A}^1$-invariant sheaves

Assume $k$ admits a resolution of singularities. We define a canonical functor $\mathcal{M}od_k^k(\Lambda) \rightarrow \mathcal{M}od_k^b(\Lambda)$. Lemma 3.1 says that the restriction to $S_b^{-1}\text{Sm}_k^{pv}$ of a strictly $\mathbb{A}^1$-invariant sheaf induces a presheaf on $S_b^{-1}\text{Sm}_k^{pv}$. Thus we have a functor

\[
\mathcal{M}od_k^k(\Lambda) \rightarrow \mathcal{P}resh(S_b^{-1}\text{Sm}_k^{pv}, \Lambda).
\]

Moreover, Lemma 1.4 and (1.1) give equivalences of categories

\[
\mathcal{M}od_k^b(\Lambda) \xrightarrow{\cong} \mathcal{P}resh(S_b^{-1}\text{Sm}_k^{ar}, \Lambda) \xrightarrow{\cong} \mathcal{P}resh(S_b^{-1}\text{Sm}_k^{pv}, \Lambda). \quad (3.1)
\]

Definition 3.3. When $k$ admits a resolution of singularities, we define a functor

\[
\mathcal{M}od_k^k(\Lambda) \rightarrow \mathcal{M}od_k^b(\Lambda); M \mapsto M^b
\]

as the composition

\[
\mathcal{M}od_k^k(\Lambda) \rightarrow \mathcal{P}resh(S_b^{-1}\text{Sm}_k^{pv}, \Lambda) \xrightarrow{\cong} \mathcal{M}od_k^b(\Lambda).
\]

We construct a natural morphism of sheaves $\mu^M : M^b \rightarrow M$ for each $M \in \mathcal{M}od_k^k(\Lambda)$. By the construction, we have $M^b(X) = M(X)$ for all $X \in \text{Sm}_k^{pv}$. For every $U \in \text{Sm}_k^{ar}$, the map $M(U) \rightarrow M(k(U))$ induces a natural isomorphism

\[
M(U) \cong \bigcap_{x \in U^{(1)}} \text{Im}(M(\mathcal{O}_{U,x}) \rightarrow M(k(U))) \quad (3.2)
\]

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by [Aso12, Lem. 4.2]. Thus [CT95, Prop. 2.1.8] shows that
\[ M^{br}(X) \cong \bigcap_{x \in X^{(1)}} \text{Im}(M(O_{X,x}) \hookrightarrow M(k(X))) \]
\[ = \bigcap_{A \in \mathcal{P}(k(X)/k)} \text{Im}(M(A) \hookrightarrow M(k(X))). \]
For a smooth proper compactification \( \overline{U} \) of \( U \in S_{k}^{\text{var}} \), we have
\[ M^{br}(U) \cong M^{br}(\overline{U}) \cong \bigcap_{A \in \mathcal{P}(k(U)/k)} \text{Im}(M(A) \hookrightarrow M(k(U))). \] \hspace{1cm} (3.3)
Thus the sheaf \( M^{br} \) can be regarded as the subsheaf \( U \mapsto \bigcap_{A \in \mathcal{P}(k(U)/k)} \text{Im}(M(A) \hookrightarrow M(k(U))) \) of \( M \) under the isomorphism (3.2). Then \( \mu^{M} \) is defined as the embedding \( M^{br} \hookrightarrow M \). We prove some basic properties of the functor \((-)^{br}\) and the morphism \( \mu^{M} \).

**Proposition 3.4.** Assume \( k \) admits a resolution of singularities. Let \( M \) be a strictly \( \mathbb{A}^{1} \)-invariant sheaf.

1. The morphism \( \mu^{M} : M^{br} \rightarrow M \) is injective.
2. The induced map \( \mu_{X}^{M} : M^{br}(X) \rightarrow M(X) \) is an isomorphism for every \( X \in S_{k}^{\text{prop}} \).
3. The induced map
\[ \text{Hom}_{\text{Mod}_{k}^{\mathbb{A}^{1}}(\Lambda)}(N, M^{br}) \xrightarrow{(\mu^{M})^{*}} \text{Hom}_{\text{Mod}_{k}^{\mathbb{A}^{1}}}(N, M) \]
is an isomorphism for every \( N \in \text{Mod}_{k}^{br}(\Lambda) \). In particular, the functor \((-)^{br}\) is right adjoint to the inclusion \( \text{Mod}_{k}^{br}(\Lambda) \hookrightarrow \text{Mod}_{k}^{\mathbb{A}^{1}}(\Lambda) \).

**Proof.** The assertions 1 and 2 follow from the construction of \( M^{br} \) and \( \mu^{M} \). We prove 3. Since \( \mu^{M} \) is injective by 1, so is \((\mu^{M})^{*}\). On the other hand, for a morphism \( f : N \rightarrow M \) in \( \text{Mod}_{k}^{br}(\Lambda) \), we have a morphism \( f' : N \rightarrow M^{br} \) defined by
\[ N(U) \cong \bigcap_{A \in \mathcal{P}(k(U)/k)} \text{Im}(N(A) \hookrightarrow N(k(U))) \]
\[ \rightarrow \bigcap_{A \in \mathcal{P}(k(U)/k)} \text{Im}(M(A) \hookrightarrow M(k(U))) \cong M^{br}(U) \]
for each \( U \in S_{k}^{\text{var}} \) by using the isomorphism in 3.3. Then \( \mu_{k}^{M} \circ f' = f \) and thus \((\mu^{M})^{*}\) is surjective. \( \square \)
3.3 Birational construction for $\mathbb{A}^1$-invariant sheaves with transfers

Assume $k$ perfect. Let $\mathbf{HI}_k(\Lambda)$ be the category of $\mathbb{A}^1$-invariant sheaves with transfers of $\Lambda$-modules. Note that every object in $\mathbf{HI}_k(\Lambda)$ is strictly $\mathbb{A}^1$-invariant as a sheaf on $Sm_k$ (see [MVW] Thm. 13.8). Lemma 1.6 shows that $\mathbf{NST}^br_k(\Lambda)$ is a full subcategory of $\mathbf{HI}_k(\Lambda)$. In [KS17, §6], Kahn-Sujatha construct a right adjoint functor

$$\mathbf{NST}^br_k(\Lambda) \rightarrow \mathbf{HI}_k(\Lambda); M \mapsto M_{nr}$$

of the inclusion $\mathbf{NST}^br_k(\Lambda) \rightarrow \mathbf{HI}_k(\Lambda)$ which sends $M \in \mathbf{HI}_k(\Lambda)$ to the presheaf with transfers defined by

$$U \mapsto \bigcap_{A \in \mathcal{P}(k(U)/k)} \ker \left( M(k(U)) \xrightarrow{\partial_A} M_{-1}(\kappa_A) \right).$$

Here $M_{-1}$ is the contraction of $M$ and $\partial_A$ is the residue map associated with $A$ (see definitions [MVW] Lec. 23 and Ex. 24.6). On the other hand, the sequence

$$0 \rightarrow M(A) \rightarrow M(k(U)) \xrightarrow{\partial_A} M_{-1}(\kappa_A) \rightarrow 0$$

is exact by [MVW] Ex. 24.6]. Thus we have

$$M_{nr}(U) = \bigcap_{A \in \mathcal{P}(k(U)/k)} \text{Im}(M(A) \rightarrow M(k(U))). \quad (3.4)$$

Then [KS17] Lem. 6.2.3, 6.2.4 and 6.2.6] show that $M_{nr}$ can be regarded as the subsheaf

$$U \mapsto \bigcap_{A \in \mathcal{P}(k(U)/k)} \text{Im}(M(A) \rightarrow M(k(U)))$$

of $M$ under the isomorphism (3.2). Thus when $k$ admits a resolution of singularities, there exists a canonical isomorphism $\Gamma_*(M_{nr}) \cong (\Gamma_* M)^{br}$ in $\mathcal{M}od_k(\Lambda)$, i.e., the diagram

$$
\begin{array}{ccc}
\mathbf{HI}_k(\Lambda) & \xrightarrow{(-)_{nr}} & \mathbf{NST}^br_k(\Lambda) \\
\psi_* \downarrow & & \downarrow \psi_* \\
\mathcal{M}od^z_k(\Lambda) & \xrightarrow{(-)^{br}} & \mathcal{M}od^br_k(\Lambda)
\end{array}
$$

is 2-commutative, by the construction of $M^{br}$. Moreover, if we write $\nu^M$ for the embedding $M_{nr} \hookrightarrow M$, the diagram

$$
\begin{array}{ccc}
\Gamma_*(M_{nr}) & \xrightarrow{\Gamma_*(\nu^M)} & \Gamma_* M \\
\cong \downarrow & & \downarrow \\
(\Gamma_* M)_{nr} & \xrightarrow{\mu^M} & \Gamma_* M
\end{array}
$$

commutes. We obtain an analogue of Proposition 3.4 for the functor $(-)_{nr}$ without assuming a resolution of singularities.
Proposition 3.5. Assume $k$ perfect. Let $M$ be an $\mathbb{A}^1$-invariant sheaf with transfers.

(1) The map $\nu^M : M_{nr} \to M$ is injective.

(2) The induced map $\nu^M_X : M_{nr}(X) \to M(X)$ is an isomorphism.

Proof. The assertion (1) follows from the construction of $\nu^M$. By (3.4) and [CT95, Prop. 2.1.8], we have

$$M_{nr}(X) \cong \bigcap_{A \in \mathcal{P}(k(X)/k)} \text{Im}(M(A) \to M(k(X))) = \bigcap_{x \in X^{(1)}} \text{Im}(M(O_{X,x}) \hookrightarrow M(k(X))).$$

Thus (2) follows from (3.2). $\square$

4 Structure of zeroth $\mathbb{A}^1$- and Suslin homology

4.1 Structure theorem of zeroth $\mathbb{A}^1$-homology

Our aim of this subsection is to prove a structure theorem of the $\mathbb{A}^1$-homology of smooth proper varieties. In [Mor05], Morel defined the $\mathbb{A}^1$-homology sheaf $H^{A^1}_i(F)$ for $F \in S_{hv}^k$ as a strictly $\mathbb{A}^1$-invariant sheaf. Especially, the zeroth $\mathbb{A}^1$-homology functor $H^{A^1}_0(-; \Lambda) : S_{hv}^k \to \text{Mod}_{A^1}^k(\Lambda)$ can be characterized as a left adjoint of the forgetful functor $\text{Mod}_{A^1}^k(\Lambda) \to S_{hv}^k$ (see [Aso12, Lem. 3.3]). For $\mathcal{F} \in S_{hv}^k$, we denote $\Lambda_{\text{pre}}(\mathcal{F})$ for the presheaf of free $\Lambda$-modules generated by $\mathcal{F}$. Note that if $\mathcal{F}$ is birational, then so is $\Lambda_{\text{pre}}(\mathcal{F})$. Thus we have the functor

$$\Lambda_{\text{pre}} : S_{hv}^{br} \to \text{Mod}_{br}^k(\Lambda).$$

This is left adjoint to the forgetful functor $\text{Mod}_{br}^k(\Lambda) \to S_{hv}^{br}$ by the definition. Our structure theorem of zeroth $\mathbb{A}^1$-homology is stated as follows.

Theorem 4.1. Assume $k$ admits a resolution of singularities. For every $X \in S_{mk}$, there exists a natural epimorphism of sheaves

$$\Lambda_{\text{pre}}(\pi_{0A^1}^k(X)) \to H^{A^1}_0(X; \Lambda).$$

Moreover, this is an isomorphism if $X$ is proper.

Proof. By Lemmas 2.3 and 3.4 we obtain three adjunctions

$$S_{hv}^k \rightleftarrows S_{hv}^{br} \rightleftarrows \text{Mod}_{br}^k(\Lambda) \rightleftarrows \text{Mod}_{A^1}^k(\Lambda).$$

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Hence the composite functor
\[
\Shv_k \xrightarrow{\pi_0^{hk}} \Shv_k^{br} \xrightarrow{\Lambda_{pre}} \Mod_k^{br}(\Lambda) \hookrightarrow \Mod_k^{hk}(\Lambda)
\]
is left adjoint to the composition
\[
\Shv_k \hookrightarrow \Shv_k^{br} \hookrightarrow \Mod_k^{br}(\Lambda) \leftarrow \Mod_k^{hk}(\Lambda).
\]
By using this adjunction and Yoneda’s lemma in \( \Sm_k \), we have a natural isomorphism
\[
\Hom_{\Mod_k^{hk}(\Lambda)}(\Lambda_{pre}(\pi_0^{hk}(X)), M) \cong \Hom_{\Shv_k}(X, M^{br}) \cong M^{br}(X) \quad (4.1)
\]
for all \( X \in \Sm_k \) and all \( M \in \Mod_k^{hk}(\Lambda) \). On the other hand, the zeroth \( \aleph^1 \)-homology functor
\[
H_0^{\aleph^1}(-; \Lambda) : \Shv_k \to \Mod_k^{\aleph^1}(\Lambda)
\]
is left adjoint to the forgetful functor \( \Mod_k^{\aleph^1}(\Lambda) \to \Shv_k \). Therefore, we also obtain a natural isomorphism
\[
\Hom_{\Mod_k^{\aleph^1}(\Lambda)}(H_0^{\aleph^1}(X; \Lambda), M) \cong \Hom_{\Shv_k}(X, M) \cong M(X). \quad (4.2)
\]
By (4.1) and (4.2), the natural monomorphism \( \mu^M : M^{br} \to M \) induces a injection
\[
\Hom_{\Mod_k^{\aleph^1}(\Lambda)}(\Lambda_{pre}(\pi_0^{hk}(X)), M) \to \Hom_{\Mod_k^{\aleph^1}(\Lambda)}(H_0^{\aleph^1}(X; \Lambda), M).
\]
This map is an isomorphism when \( X \) is proper by Proposition 3.4. Thus Yoneda’s lemma in \( \Mod_k^{\aleph^1}(\Lambda) \) leads an epimorphism of sheaves
\[
\Lambda_{pre}(\pi_0^{hk}(X)) \to H_0^{\aleph^1}(X; \Lambda)
\]
which is an isomorphism for proper \( X \).

\[\square\]

**Corollary 4.2.** Assume \( k \) admits a resolution of singularities. For \( X \in \Sm_k^{prop} \) and \( U \in \Sm_k^{var} \), the section \( H_0^{\aleph^1}(X; \Lambda)(U) \) is the free \( \Lambda \)-module generated by \( X(k(U))/R \).

**Proof.** This follows from Theorem 4.1 and Lemma 2.2. \[\square\]

We obtain the Künneth formula and the universal coefficient theorem of \( \aleph^1 \)-homology in degree zero.

**Corollary 4.3.** Assume \( k \) admits a resolution of singularities. For \( X, Y \in \Sm_k^{prop} \), there exists a natural isomorphism
\[
H_0^{\aleph^1}(X \times Y; \Lambda) \cong H_0^{\aleph^1}(X; \Lambda) \otimes_{\Lambda} H_0^{\aleph^1}(Y; \Lambda),
\]
where \( \otimes_{\Lambda} \) means the tensor product of presheaves over \( \Lambda \).
Proof. This follows from Theorem 4.4 and the natural bijection

\[(X \times Y)(K)/R \cong (X(K)/R) \times (Y(K)/R)\]

for each \(K \in \mathcal{F}_k\).

\[\square\]

Corollary 4.4. Assume \(k\) admits a resolution of singularities. For \(X \in \text{Sm}_k^{\text{prop}}\), there exists a natural isomorphism

\[H^A_0(X; \Lambda) = H^A_0(X; \mathbb{Z}) \otimes \Lambda,\]

where \(\otimes\) means the tensor product of presheaves over \(\mathbb{Z}\).

Proof. This follows from \(\Lambda_{\text{pre}}(-) \cong \mathbb{Z}_{\text{pre}}(-) \otimes \Lambda\) and Theorem 4.4 \(\square\)

Remark 4.5. Assume \(k\) admits a resolution of singularities. By Theorem 4.4, there exists a pairing

\[H^A_0(X; \Lambda)(Y) \times H^A_0(Y; \Lambda)(Z) \to H^A_0(X; \Lambda)(Z)\]

defined by the map

\[\text{Hom}_{\text{Sm}_k^{-1}}(Y, X) \times \text{Hom}_{\text{Sm}_k^{-1}}(Z, Y) \to \text{Hom}_{\text{Sm}_k^{-1}}(Z, X); (f, g) \mapsto f \circ g\]

for all \(X, Y, Z \in \text{Sm}_k^{\text{prop}}\). In particular, this pairing gives a ring structure for \(H^A_0(X; \Lambda)(X)\).

4.2 Rational points and finiteness

In this subsection, we give some examples where \(H^A_0(X; \Lambda)(k)\) is finitely generated. We first introduce the following notation.

Definition 4.6. We write \(b^A_0(X) = \dim_\mathbb{Q} H^A_0(X; \mathbb{Q})(k)\) for \(X \in \text{Sm}_k\). This is called the zeroth \(A^1\)-Betti number of \(X\).

By Corollary 4.2, \(b^A_0(X)\) coincides with the rank of the free \(\Lambda\)-module \(H^A_0(X; \Lambda)(k)\) if \(X\) is proper. Our structure theorem of zeroth \(A^1\)-homology enables to relate the \(A^1\)-Betti number to \(k\)-rational points modulo \(R\)-equivalence.

Theorem 4.7. Assume \(k\) admits a resolution of singularities. For a smooth proper \(k\)-variety \(X\), we have

\[b^A_0(X) = \#(X(k)/R)\]

In particular, \(b^A_0(X) = 0\) if and only if \(X(k) = \emptyset\).

Proof. This follows from Corollary 4.2

The following proposition implies the finiteness of \(b^A_0(X)\) for a rationally connected smooth proper real variety \(X\).
Proposition 4.8. Let $X$ be a rationally connected smooth proper variety over $\mathbb{R}$. Then we have
\[ H_{A^1}^0(X; \Lambda)(\mathbb{R}) \cong H_0(X(\mathbb{R}); \Lambda). \]
In particular, $b_{A^1}^0(X)$ coincides with the ordinary zeroth Betti number of the real manifold $X(\mathbb{R})$.

Proof. By [Kol99, Cor. 1.7], the set of $\mathbb{R}$-equivalence classes $X(\mathbb{R})/\mathbb{R}$ coincides with the connected components of $X(\mathbb{R})$. Thus the assertion follows from Corollary 4.2. By using Theorem 4.7, we also obtain some examples where $b_{A^1}^0(X)$ is finite.

Example 4.9. (1) Let $X$ be a smooth proper variety with finitely many $k$-rational points. Then $b_{A^1}^0(X)$ is finite. Thus for example, $b_{A^1}^0(C)$ is finite for a smooth projective curve $C$ of genus $\geq 2$ over a number field by Mordell conjecture proved by Faltings [Fal83].

(2) Let $X$ be a rationally connected smooth proper variety over either of $\mathbb{C}$ or a $p$-adic field. Then $b_{A^1}^0(X)$ is also finite by [Kol99, Cor. 1.5].

Remark 4.10. For $X$ a smooth $k$-variety, $H_{A^1}^0(X; \mathbb{Z})(k)$ can be expressed in terms of triangulated categories as follows.

(1) There exist natural isomorphisms
\[ H_{A^1}^0(X; \mathbb{Z})(k) \cong \text{Hom}_{\text{Mod}_k}(\mathbb{Z}, H_{A^1}^0(X; \mathbb{Z})) \cong \text{Hom}_{D(A^1)}(\mathbb{Z}, \mathbb{Z}(X)). \]
Here $D(A^1)$ is the $A^1$-derived category defined by Morel [Mor05] and $\mathbb{Z}(X)$ is the Nisnevich sheafification of $\mathbb{Z}_{pre}(X)$. This follows from [Aso12, Lem. 3.3] and its proof.

(2) There also exist natural isomorphisms
\[ H_{A^1}^0(X; \mathbb{Z})(k) \cong \pi_{S^0}(\Sigma_{S^1}(X_+))(k) \cong \text{Hom}_{SH_{S^1}}(S^0, \Sigma_{S^1}(X_+)). \]
Here $\Sigma_{S^1}(X_+)$ is the infinity $S^1$-suspension of $X_+ = X \sqcup \text{Spec } k$, $\pi_{S^0}(-)$ is the $S^1$-stable zeroth $A^1$-homotopy sheaf, $SH_{S^1}(k)$ is the $A^1$-homotopy category of $S^1$-spectra, and $S^0$ is the sphere spectrum (see definitions [Mor05]). The first isomorphism is given by [AH11, Prop. 2.1] and the second isomorphism is given by the definition of $\pi_{S^0}(-)$.

4.3 Zeroth Suslin homology and zero cycles

In this subsection, we prove the Suslin homology version of Theorem 4.1. For $X \in Sm_k$, the Suslin homology sheaf $H_{A^1}^0(X; \Lambda)$ is defined as the $i$-th homology sheaf with transfers of the motive of $X$ with $\Lambda$-coefficients in the sense of Voevodsky [Voe00]. Then the section $H_{A^1}^0(X; \Lambda)(k)$ coincides with the singular homology of Suslin-Voevodsky [SV96], called Suslin homology group. Note that
$H^S_0(X; \Lambda)$ is $\mathbb{A}^1$-invariant by the $\mathbb{A}^1$-invariance of the motive of $X$. Thus we have a functor

$$H^S_0(-; \Lambda) : Cor_{k, \Lambda} \to HI_k(\Lambda).$$

As a counterpart of Theorem 4.1, we construct an isomorphism of sheaves with transfers $\Lambda\pi^{tr \mathbb{A}^1}(X) \cong H^S_0(X; \Lambda)$ for $X \in Sm^k_{\text{prop}}$.

**Theorem 4.11.** Assume $k$ perfect. For every $X \in Sm_k$, there exists a natural epimorphism of sheaves with transfers

$$\Lambda\pi^{tr \mathbb{A}^1}(X) \to H^S_0(X; \Lambda).$$

Moreover, this is an isomorphism if $X$ is proper.

**Proof.** By Lemma 2.6, the composite functor

$$NST_k(\Lambda) \xrightarrow{\Lambda\pi^{tr \mathbb{A}^1}} NST^b_k(\Lambda) \hookrightarrow HI_k(\Lambda)$$

is left adjoint to the functor $(-)_{nr} : HI_k(\Lambda) \to NST_k(\Lambda)$. For all $X \in Sm_k$ and all $M \in HI_k(\Lambda)$, thus we obtain natural isomorphisms

$$\text{Hom}_{HI_k(\Lambda)}(\Lambda\pi^{tr \mathbb{A}^1}(X), M) \cong \text{Hom}_{NST_k(\Lambda)}(\Lambda_{tr}(X), M_{nr}) \cong M_{nr}(X)$$

by Yoneda’s lemma in $Cor_{k, \Lambda}$. On the other hand, [Aso12, Lem. 3.3] gives an isomorphism

$$\text{Hom}_{HI_k(\Lambda)}(H^S_0(X; \Lambda), M) \cong M(X).$$

Thus the map

$$\text{Hom}_{HI_k(\Lambda)}(\Lambda\pi^{tr \mathbb{A}^1}(X), M_{nr}) \to \text{Hom}_{HI_k(\Lambda)}(H^S_0(X; \Lambda), M)$$

obtained by $\nu^{M} : M_{nr} \to M$ introduces an epimorphism of sheaves with transfers

$$\Lambda\pi^{tr \mathbb{A}^1}(X) \to H^S_0(X; \Lambda)$$

by Yoneda’s lemma in $HI_k(\Lambda)$. By Proposition 3.5, this morphism is an isomorphism if $X$ is proper. $\square$

Assume $k$ perfect. For every $X \in Sm^k_{\text{prop}}$, there exists a canonical isomorphism

$$H^S_0(X; Z)(K) \cong CH_0(X_K)$$

for all $K \in \mathcal{F}_k$ (see [Aso12, Ex. 4.9]). Thus we obtain an application to Chow groups of zero cycles. We simply write

$$S^{-1}_b Cor_k(X, Y) = \text{Hom}_S(-1 Cor_{k, Z})(X, Y)$$

for $X, Y \in Sm_k$.  

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Corollary 4.12. Assume $k$ perfect. Let $X$ be a smooth proper $k$-variety and $U$ be a smooth $k$-variety. Then we have

$$S_{b}^{-1}\text{Cor}_{k}(U, X) \cong \text{CH}_{0}(X_{k(U)}).$$

Proof. By Theorem 4.11 and the construction of $\Lambda\pi_{0,fr}^{b}(X)$, we have

$$S_{b}^{-1}\text{Cor}_{k}(U, X) \cong \mathbb{Z}_{A}^{b}\pi_{0,fr}^{b}(X)(U) \cong H_{0}^{S}(X; \mathbb{Z})(U) \cong H_{0}^{S}(X; \mathbb{Z}(k(U))).$$

On the other hand, we also have $H_{0}^{S}(X; \mathbb{Z})(k(U)) \cong \text{CH}_{0}(X_{k(U)})$ by [Aso12, Ex. 4.9].

Remark 4.13. Assume $k$ perfect. Like Remark 4.5, the composition of morphisms in $S_{b}^{-1}\text{Cor}_{k,\Lambda}$ also induces a pairing

$$H_{0}^{S}(X; \Lambda)(Y) \times H_{0}^{S}(Y; \Lambda)(Z) \to H_{0}^{S}(X; \Lambda)(Z)$$

and a ring structure of $H_{0}^{S}(X; \Lambda)(X)$ for all $X, Y, Z \in \text{Sm}_{k}^{\text{prop}}$. Thus we obtain a pairing

$$\text{CH}_{0}(X_{k(Y)}) \times \text{CH}_{0}(Y_{k(Z)}) \to \text{CH}_{0}(X_{k(Z)})$$

and a ring structure of $\text{CH}_{0}(X_{k(X)})$.

5 Universal birational invariance

5.1 Universal birational invariance of $\mathbb{A}_{1}$-homology

Our aim of this subsection is to prove the main theorem of this paper, i.e., the universal birational invariance of zeroth $\mathbb{A}_{1}$-homology. For a category $C$, we denote $\Lambda(C)$ for the category whose objects are same with $C$ and that $\text{Hom}_{\Lambda(C)}(A, B)$ for each $A, B \in \Lambda(C)$ is the free $\Lambda$-module generated by $\text{Hom}_{C}(A, B)$. Note that every functor from $C$ to an $\Lambda$-enriched category uniquely factors through the canonical $\Lambda$-linear functor $C \to \Lambda(C)$. Thus we have a functor

$$H_{0}^{a}(-; \Lambda) : \Lambda(S_{b}^{-1}\text{Sm}_{k}^{\text{prop}}) \to \text{Mod}_{k}(\Lambda).$$

Proposition 5.1. Assume $k$ admits a resolution of singularities. Then the functor 5.1 is fully faithful.

Proof. By [KS07] Thm. 6.4, we only need to show that the functor

$$H_{0}^{a}(-; \Lambda) : \Lambda(S_{b}^{-1}\text{Sm}_{k}^{\text{prop}}) \to \text{Mod}_{k}(\Lambda)$$

is fully faithful. The canonical functor $S_{b}^{-1}\text{Sm}_{k}^{\text{prop}} \to \Lambda(S_{b}^{-1}\text{Sm}_{k}^{\text{prop}})$ induces an equivalence of categories

$$\text{Pres}\text{h}(S_{b}^{-1}\text{Sm}_{k}^{\text{prop}}, \Lambda) \cong \text{Lin}(\Lambda(S_{b}^{-1}\text{Sm}_{k}^{\text{prop}}), \Lambda).$$
On the other hand, we obtain equivalences

\[ \mathcal{M}od_k^p(\Lambda) \cong \mathcal{P}resh(S^{-1}_b S^{\text{prop}}_k, \Lambda) \cong \mathcal{P}resh(S^{-1}_b S^{\text{prop}}_k, \Lambda) \]

by (3.1). Thus we have a functor

\[ H : \Lambda(S^{-1}_b S^{\text{prop}}_k) \cong \mathcal{L}in(\Lambda(S^{-1}_b S^{\text{prop}}_k), \Lambda) \cong \mathcal{M}od_k^p(\Lambda) \cong \mathcal{M}od_k(\Lambda), \]

where the first functor is the Yoneda embedding. Therefore, \( H \) is fully faithful by Yoneda’s lemma in \( \Lambda(S^{-1}_b S^{\text{prop}}_k) \). By Theorem 4.1, there exists a natural isomorphism

\[ H(X) \cong \text{Hom}_{\Lambda(S^{-1}_b S^{\text{prop}}_k)}(\cdot, X) \cong \Lambda_{\text{pre}}(\pi_0^k X) \cong \text{H}^{A1}_0(X; \Lambda) \]

for all \( X \in S^{\text{prop}}_k \). Thus the diagram

\[
\begin{array}{ccc}
S^{\text{prop}}_k & \xrightarrow{H^{A1}_0(\cdot; \Lambda)} & \mathcal{M}od_k(\Lambda) \\
\downarrow & & \downarrow \\
\Lambda(S^{-1}_b S^{\text{prop}}_k) & \xrightarrow{H} & \mathcal{M}od_k(\Lambda)
\end{array}
\]

is 2-commutative. This diagram shows that the functor (5.2) is naturally equivalent to \( H \). Since \( H \) is fully faithful, so is (5.2).

Proposition 5.1 shows that the zeroth \( A^1 \)-homology functor on \( S^{-1}_b S^{\text{prop}}_k \) is conservative and faithful.

**Corollary 5.2.** Assume \( k \) admits a resolution of singularities. Then the functor

\[ \text{H}^{A1}_0(\cdot; \Lambda) : S^{-1}_b S^{\text{prop}}_k \to \mathcal{M}od_k(\Lambda). \]  

(5.3)

is conservative and faithful.

**Proof.** Since the functors \( S^{-1}_b S^{\text{prop}}_k \to \Lambda(S^{-1}_b S^{\text{prop}}_k) \) and (5.1) are conservative and faithful by Proposition 5.1, so is (5.3).

For describing universal birational invariance, we introduce the following term.

**Definition 5.3.** Let \( \mathcal{C} \) be a full subcategory of \( S^{\text{prop}}_k \). Then \( \mathcal{C} \) is called a birationally fully faithful subcategory of \( S^{\text{prop}}_k \), if the functor \( S^{-1}_b \mathcal{C} \to S^{-1}_b S^{\text{prop}}_k \) is fully faithful.

We see some basic examples of birationally fully faithful subcategories of \( S^{\text{prop}}_k \).
Example 5.4. Clearly, $Sm_k^{\text{prop}}$ is a birationally fully faithful subcategory of itself. Let $Sm_k^{\text{proj}}$ (resp. $Sm_k^{\text{proj}}$) be the full subcategory of $Sm_k^{\text{prop}}$ spanned by smooth projective $k$-varieties (resp. $k$-schemes). When $k$ admits a resolution of singularities, $Sm_k^{\text{proj}}$ is a birationally fully faithful subcategory of $Sm_k^{\text{prop}}$ by the equivalence $Sm_k^{\text{proj}} \cong Sm_k^{\text{prop}}$ in [KS07, Thm. 8.8]. Moreover, $Sm_k^{\text{proj}}$ and $Sm_k^{\text{proj}}$ are also birationally fully faithful subcategories of $Sm_k^{\text{prop}}$ by [KS07, Thm. 6.4].

We prove the universal birational invariance of the $\mathbb{A}^1$-homology functor on birationally fully faithful subcategories of $Sm_k^{\text{prop}}$. For a subcategory $C \subseteq Sm_k^{\text{prop}}$, we denote $\text{Im}_C^\Lambda H_0^{\mathbb{A}^1}$ for the full subcategory of $\text{Mod}_k(\Lambda)$ spanned by sheaves isomorphic to $H_0^{\mathbb{A}^1}(X; \Lambda)$ for some $X \in C$.

Theorem 5.5. Assume $k$ admits a resolution of singularities. Let $C$ be a birationally full subcategory of $Sm_k^{\text{prop}}$ and $A$ be an arbitrary category enriched by $\Lambda$-modules.

1. Let $F : C \to A$ be an arbitrary functor which sends each birational morphism to an isomorphism. Then there exists one and only one (up to a natural equivalence) $\Lambda$-linear functor $F_{S_b} : \text{Im}_C^\Lambda H_0^{\mathbb{A}^1} \to A$ such that the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{F} & A \\
\downarrow & & \downarrow \\
\text{Im}_C^\Lambda H_0^{\mathbb{A}^1} & \xrightarrow{F_{S_b}} & A
\end{array}
$$

is 2-commutative.

2. Let $F' : C^{\text{op}} \to A$ be an arbitrary functor which sends each birational morphism to an isomorphism. Then there exists one and only one (up to a natural equivalence) $\Lambda$-linear functor $F'_{S_b} : (\text{Im}_C^\Lambda H_0^{\mathbb{A}^1})^{\text{op}} \to A$ such that the diagram

$$
\begin{array}{ccc}
C^{\text{op}} & \xrightarrow{F'} & A \\
\downarrow & & \downarrow \\
(\text{Im}_C^\Lambda H_0^{\mathbb{A}^1})^{\text{op}} & \xrightarrow{F'_{S_b}} & A
\end{array}
$$

is 2-commutative.

Proof. Since $S_b^{-1}C \to S_b^{-1}Sm_k^{\text{prop}}$ is fully faithful, so is the composition

$$
\Lambda(S_b^{-1}C) \to \Lambda(S_b^{-1}Sm_k^{\text{prop}}) \to \text{Mod}_k(\Lambda)
$$

by Proposition 5.1. On the other hand, the essential image of the functor (5.4) coincides with $\text{Im}_C^\Lambda H_0^{\mathbb{A}^1}$. Thus we have an equivalence $\Lambda(S_b^{-1}C) \cong \text{Im}_C^\Lambda H_0^{\mathbb{A}^1}$.
such that the diagram

\[
\begin{array}{ccc}
\mathcal{C} & \overset{\cong}{\longrightarrow} & \mathcal{C} \\
\downarrow & & \downarrow \mathbf{H}_0^i (-; \Lambda) \\
\Lambda(S_b^{-1}\mathcal{C}) & \overset{\cong}{\longrightarrow} & \text{Im}^C_\Lambda \mathbf{H}_0^i
\end{array}
\] (5.5)

is 2-commutative. Thus (1) follows from the universality on \(\mathcal{C} \to \Lambda(S_b^{-1}\mathcal{C})\). Similarly, we obtain (2) by taking the opposite categories in (5.5).

**Remark 5.6.** The functor \(F_{S_k}\) (resp. \(F'_{S_k}\)) in Theorem 5.5 is canonically extended to \(\text{Mod}^\text{br}_{S_k}(\Lambda) \to \mathcal{A}\) (resp. \((\text{Mod}^\text{br}_{S_k}(\Lambda))^{op} \to \mathcal{A}\)), if \(\text{Sm}^\text{pjv}_k \subseteq \mathcal{C}\) and \(\mathcal{A}\) is cocomplete. Indeed, the equivalences (3.1) and \(\text{Sm}^\text{pjv}_k \cong \text{Sm}^\text{var}_k\) in [KS07, Prop. 8.5] induce

\[
\text{Mod}^\text{br}_{S_k}(\Lambda) \cong \text{Lin}(\Lambda(S_b^{-1}\text{Sm}^\text{var}_k), \Lambda) \cong \text{Lin}(\Lambda(S_b^{-1}\text{Sm}^\text{pjv}_k), \Lambda).
\]

Under the identification by this equivalence, the category \(\text{Im}^C_\Lambda \mathbf{H}_0^i\) contains the full subcategory of \(\text{Mod}^\text{br}_{S_k}(\Lambda)\) consisting of representable presheaves on \(\Lambda(S_b^{-1}\text{Sm}^\text{pjv}_k)\) by Theorem 4.1. Thus the purpose extension is obtained as a left Kan extension of \(\Lambda(S_b^{-1}\text{Sm}^\text{pjv}_k) \to \mathcal{A}\) (resp. \((\Lambda(S_b^{-1}\text{Sm}^\text{pjv}_k))^{op} \to \mathcal{A}\)).

### 5.2 Universal birational invariance of Suslin homology

In this subsection we prove the Suslin homology version of the universal birational invariance property. We have a functor

\[
\mathbf{H}_0^i (-; \Lambda) : S_b^{-1}\text{Cor}^\text{prop}_{k, \Lambda} \to \text{NST}_{k}(\Lambda).
\]

**Proposition 5.7.** Assume \(k\) perfect and admits a resolution of singularities. Then the functor (5.6) is fully faithful.

**Proof.** By [KS07] Thm. 6.4, we only need to show that the functor

\[
\mathbf{H}_0^i (-; \Lambda) : S_b^{-1}\text{Cor}^\text{prop}_{k, \Lambda} \to \text{NST}_{k}(\Lambda)
\]

is fully faithful. By Lemma 1.5 and Proposition 1.2 we obtain equivalences of categories

\[
\text{NST}^\text{br}_{k}(\Lambda) \cong \text{Lin}(\Lambda(S_b^{-1}\text{Cor}^\text{var}_{k, \Lambda}), \Lambda) \cong \text{Lin}(\Lambda(S_b^{-1}\text{Cor}^\text{prop}_{k, \Lambda}), \Lambda).
\]

Thus we have a functor

\[
\mathcal{H}_{tr} : S_b^{-1}\text{Cor}^\text{prop}_{k, \Lambda} \to \text{Lin}(\Lambda(S_b^{-1}\text{Cor}^\text{prop}_{k, \Lambda}), \Lambda) \cong \text{NST}^\text{br}_{k}(\Lambda) \to \text{NST}_{k}(\Lambda)
\]

where the first functor is the Yoneda embedding. Hence \(\mathcal{H}_{tr}\) is fully faithful by Yoneda’s lemma in \(S_b^{-1}\text{Cor}^\text{prop}_{k, \Lambda}\). By Theorem 4.11 there exists a natural isomorphism

\[
\mathcal{H}_{tr}(X) \cong S_b^{-1}\text{Cor}_{k, \Lambda}(-, X) \cong \Lambda\pi^\text{tor}_{0, \text{tr}}(X) \cong \mathbf{H}_0^i(X; \Lambda)
\]
for all $X \in \text{Cor}^{\text{pv}}_{k,\Lambda}$. Thus the diagram

$$
\begin{array}{ccc}
\text{Cor}^{\text{pv}}_{k,\Lambda} & \xrightarrow{H_0^S(-;\Lambda)} & \text{NST}_k(\Lambda) \\
\downarrow & & \downarrow \\
S_b^{-1}\text{Cor}^{\text{pv}}_{k,\Lambda} & \xrightarrow{\mathcal{H}_\text{tr}} & \text{NST}_k(\Lambda)
\end{array}
$$

is 2-commutative. Therefore, since $\mathcal{H}_\text{tr}$ is fully faithful, so is the functor (5.7).

We also introduce birationally fully faithful subcategories of $\text{Cor}^{\text{prop}}_{k,\Lambda}$.

**Definition 5.8.** Let $\mathcal{C}$ be a full subcategory of $\text{Cor}^{\text{prop}}_{k,\Lambda}$. Then $\mathcal{C}$ is called a birationally fully faithful subcategory of $\text{Cor}^{\text{prop}}_{k,\Lambda}$, if the functor $S_b^{-1}\mathcal{C} \to S_b^{-1}\text{Cor}^{\text{prop}}_{k,\Lambda}$ is fully faithful.

**Example 5.9.** Clearly, $\text{Cor}^{\text{prop}}_{k,\Lambda}$ is a birationally fully faithful subcategory of itself. By [KS07, Thm. 6.4], the subcategory $\text{Cor}^{\text{pv}}_{k,\Lambda} \subseteq \text{Cor}^{\text{prop}}_{k,\Lambda}$ is also birationally fully faithful.

For a subcategory $\mathcal{C} \subseteq \text{Cor}^{\text{prop}}_{k,\Lambda}$, we denote $\text{Im}^c_{\Lambda}H_0^S$ for the full subcategory of $\text{NST}_k(\Lambda)$ spanned by sheaves with transfers isomorphic to $H_0^S(X; \Lambda)$ for some $X \in \mathcal{C}$. We prove universal birational invariance of the zeroth Suslin homology.

**Theorem 5.10.** Assume $k$ perfect and admits a resolution of singularities. Let $\mathcal{C}$ be a birationally fully faithful subcategory of $\text{Cor}^{\text{prop}}_{k,\Lambda}$ and $\mathcal{A}$ be an arbitrary category enriched by $\Lambda$-modules.

1. Let $F : \mathcal{C} \to \mathcal{A}$ be an arbitrary $\Lambda$-linear functor which sends each birational morphism to an isomorphism. Then there exists one and only one (up to a natural equivalence) $\Lambda$-linear functor $F_{S_b} : \text{Im}^c_{\Lambda}H_0^S \to \mathcal{A}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{A} \\
\downarrow & & \downarrow \\
\text{Im}^c_{\Lambda}H_0^S & \xrightarrow{F_{S_b}} & \mathcal{A}
\end{array}
$$

is 2-commutative.

2. Let $F' : \mathcal{C}^\text{op} \to \mathcal{A}$ be an arbitrary $\Lambda$-linear functor which sends each birational morphism to an isomorphism. Then there exists one and only one (up to a natural equivalence) $\Lambda$-linear functor $F'_{S_b} : \text{Im}^c_{\Lambda}H_0^S \to \mathcal{A}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{C}^\text{op} & \xrightarrow{F'} & \mathcal{A} \\
\downarrow & & \downarrow \\
\text{Im}^c_{\Lambda}H_0^S & \xrightarrow{F'_{S_b}} & \mathcal{A}
\end{array}
$$

is 2-commutative.
Proof. By Proposition 5.7, the functors $\mathcal{C} \to \text{Im}^C_{\Lambda} \mathcal{H}^S_0$ and $\mathcal{C}^{op} \to (\text{Im}^C_{\Lambda} \mathcal{H}^S_0)^{op}$ are equivalences of categories. Thus this theorem follows from the universality of localizations of categories.

Remark 5.11. Like Remark 5.6, the functor $F_S$ (resp. $F'_S$) in Theorem 5.10 is canonically extended to $\text{NST}^{br}_k (\Lambda) \to A$ (resp. $(\text{NST}^{br}_k (\Lambda))^{op} \to A$) if $\text{Cor}^{brk}_k \subseteq \mathcal{C}$ and $A$ is cocomplete.

5.3 Proper birational invariance

In this subsection, we prove that a proper birational morphism of smooth (not necessary proper) $k$-varieties induces an isomorphism of the zeroth $A^1$- and Suslin homology. This is a refinement of Asok’s result [Aso12 Thm. 3.9].

Proposition 5.12. Let $f : X \to Y$ be a proper birational morphism of smooth $k$-varieties.

(1) The induced morphisms

$$H_{11}^A (X; \Lambda) \to H_{11}^A (Y; \Lambda)$$

is an isomorphisms of sheaves.

(2) The induced morphisms

$$H^S_0 (X; \Lambda) \to H^S_0 (Y; \Lambda)$$

is an isomorphism of sheaves with transfers.

Proof. By Lemma 3.1 for every $M \in \text{Mod}^A_k (\Lambda)$ the induced map $M(Y) \to M(X)$ is an isomorphism. Thus [Aso12 Lem. 3.3] gives isomorphisms

$$\text{Hom}_{\text{Mod}^A_k (\Lambda)} (H^A_{11} (Y; \Lambda), M) \cong M(Y) \xrightarrow{\cong} M(X) \cong \text{Hom}_{\text{Mod}^A_k (\Lambda)} (H^A_{11} (X; \Lambda), M)$$

for all $M \in \text{Mod}^A_k (\Lambda)$. Thus Yoneda’s lemma in $\text{Mod}^A_k (\Lambda)$ induces an isomorphism

$$H^A_{11} (X; \Lambda) \xrightarrow{\cong} H^A_{11} (Y; \Lambda).$$

Similarly, [Aso12 Lem. 3.3] also gives an isomorphism

$$\text{Hom}_{\text{HI}_k (\Lambda)} (H^S_0 (Y; \Lambda), M') \xrightarrow{\cong} \text{Hom}_{\text{HI}_k (\Lambda)} (H^S_0 (X; \Lambda), M')$$

for all $M' \in \text{HI}_k (\Lambda)$. Thus Yoneda’s lemma in $\text{HI}_k (\Lambda)$ induces an isomorphism

$$H^S_0 (X; \Lambda) \xrightarrow{\cong} H^S_0 (Y; \Lambda)$$

in $\text{HI}_k (\Lambda)$. □
6 Applications to $\mathbb{A}^1$-homotopy theory

In this section, we give some applications to $\mathbb{A}^1$-homotopy theory. A morphism $X \to Y$ in $\text{Sm}_k$ is called a $S^1$-stable $\mathbb{A}^1$-0-equivalence, if the induced morphism

$$\pi_0^S(\Sigma_\infty S^1(X_+)) \to \pi_0^S(\Sigma_\infty S^1(Y_+))$$

is an isomorphism. Note that $X \to Y$ is a $S^1$-stable $\mathbb{A}^1$-0-equivalence if and only if the induced morphism $\mathbb{H}_{0}^{\mathbb{A}^1}(X; \mathbb{Z}) \to \mathbb{H}_{0}^{\mathbb{A}^1}(Y; \mathbb{Z})$ is an isomorphism by [AH11, Prop. 2.1]. When $X$ and $Y$ are proper, we also obtain other equivalent conditions.

**Proposition 6.1.** Assume $k$ admits a resolution of singularities. Let $f : X \to Y$ be a morphism of smooth proper $k$-schemes. Then the following conditions are equivalent.

1. The morphism $f$ is a $S^1$-stable $\mathbb{A}^1$-0-equivalence.
2. The morphism $f$ is an isomorphism in $S_{-1}^b\text{Sm}_k$.
3. The induced map $X(K)/R \to Y(K)/R$ is bijective for all $K \in \mathcal{F}_k$.

**Proof.** Corollary 5.2 shows that (1) $\Rightarrow$ (2). Moreover, (2) $\Rightarrow$ (3) follows from [KS15, Thm. 6.6.3]. On the other hand, (3) is equivalent to the condition that the induced morphism $\pi_0^{\mathbb{A}^1}(X) \to \pi_0^{\mathbb{A}^1}(Y)$ is an isomorphism by [AM11, Thm. 6.2.1]. Thus Theorem 4.1 shows that the induced map $\mathbb{H}_{0}^{\mathbb{A}^1}(X; \mathbb{Z}) \to \mathbb{H}_{0}^{\mathbb{A}^1}(Y; \mathbb{Z})$ is an isomorphism. Hence, we have (3) $\Rightarrow$ (1).

Next, we consider the $\mathbb{A}^1$-connectedness of smooth proper varieties (see definition [MV99, p.110]). In [Aso12], Asok proves that a smooth proper $k$-variety $X$ is $\mathbb{A}^1$-connected if and only if the structure morphism $X \to \text{Spec } k$ induces an isomorphism of zeroth $\mathbb{A}^1$-homology sheaves $\mathbb{H}_{0}^{\mathbb{A}^1}(X; \Lambda) \cong \Lambda$.

**Remark 6.2.** Proposition 6.1 implies that a smooth proper $k$-variety $X$ is $\mathbb{A}^1$-connected if and only if the structure morphism $X \to \text{Spec } k$ is an isomorphism in $S_{-1}^b\text{Sm}_k$, when $k$ admits a resolution of singularities. However, this assertion holds without the assumption on resolution of singularities. Indeed, $X$ is $\mathbb{A}^1$-connected if and only if $\#(X(K)/R) = 1$ for all $K \in \mathcal{F}_k$ by [AM11, Cor. 2.4.4]. On the other hand, by [KS15, Thm. 8.5.1] this is equivalent to the condition that the morphism $X \to \text{Spec } k$ is an isomorphism in $S_{-1}^b\text{Sm}_k$.

The following gives another equivalent condition of the $\mathbb{A}^1$-connectedness of $X$.

**Proposition 6.3.** Assume $k$ admits a resolution of singularities. A smooth proper $k$-variety $X$ is $\mathbb{A}^1$-connected if and only if

$$b_{0}^{\mathbb{A}^1}(X_{k(X)}) \leq 1 \leq b_{0}^{\mathbb{A}^1}(X).$$
Proof. “Only if” follows from [Aso12, Thm. 4.14]. We prove “if”. Since $b^1_0(X) \neq 0$, the variety $X$ has a $k$-rational point by Theorem 4.7. Moreover, since $X_{k(X)}$ also has a $k(X)$-rational point, we have $b^1_0(X_{k(X)}) = 1$. This implies $\#(X(k(X))/R) = \#(X_{k(X)}(k(X))/R) = 1$ by Corollary 4.2. Thus [KS15, Thm. 8.5.1.] shows that the structure morphism $X \to \text{Spec } k$ is an isomorphism in $S_{-1} S_{\text{sm}}$. Therefore, $X$ is $\mathbb{A}^1$-connected (see Remark 6.2).

Next, we prove a comparison result between $\mathbb{A}^1$-homotopy and ordinary or étale homotopy. For a complex variety $X$, we denote $X^{\text{an}}$ for the associated analytic space.

**Proposition 6.4.** Assume $k$ admits a resolution of singularities. Let $X \to Y$ be a $S^1$-stable $\mathbb{A}^1$-0-equivalence of smooth proper $k$-varieties.

1. The induced map $\pi^\text{\acute{e}t}_1(X) \to \pi^\text{\acute{e}t}_1(Y)$ is an isomorphism.
2. If $k = \mathbb{C}$, the continuous map $X^{\text{an}} \to Y^{\text{an}}$ is a 1-equivalence of topological spaces.

**Proof.** Since the functor of (étale) fundamental groups is birational invariant of smooth proper varieties, this is regarded as a functor on $S_{-1} S^{\text{prop}}_\text{sm}$. On the other hand, $X \to Y$ is an isomorphism in $S_{-1} S^{\text{prop}}_\text{sm}$ by Proposition 6.1. Thus the map of (étale) fundamental groups induced by $X \to Y$ is an isomorphism.

**Definition 6.5.** A smooth $k$-scheme $X$ is called $R$-rigid, if for all $K \in F_k$ the scalar extension $X_K$ has no rational curves over $K$.

By [AM11] Thm. 6.2.1], a smooth proper variety $X$ is $R$-rigid if and only if the canonical morphism of sheaves $X \to \pi^\text{\acute{e}t}_0(X)$ is an isomorphism. We see examples of $R$-rigid varieties.

**Example 6.6.** (1) Let $C$ be a geometrically irreducible smooth projective curve of genus $\geq 1$. Then $C$ is $R$-rigid. Indeed, for every $K \in F_k$ the scalar extension $C_K$ has genus $\geq 1$ and thus $C_K$ has no rational curves over $K$.

(2) An abelian variety $X$ is also $R$-rigid. Indeed, for every $K \in F_k$ the scalar extension $X_K$ is a coproduct of abelian varieties over $K$ and thus $X_K$ has no rational curves over $K$.

We give a classification of $R$-rigid varieties up to a $S^1$-stable $\mathbb{A}^1$-0-equivalences.

**Proposition 6.7.** Assume $k$ admits a resolution of singularities. Let $f : X \to Y$ be a morphism of $R$-rigid smooth proper $k$-varieties. Then the following conditions are equivalent.

1. $f$ is an isomorphism in $S^{\text{sm}}_k$.
(2) \( f \) is birational.

(3) \( f \) is stable birational.

(4) \( f \) is an \( \mathbb{A}^1 \)-weak equivalence.

(5) \( f \) is a \( S^1 \)-stable \( \mathbb{A}^1 \)-0-equivalence.

Proof. \( (1) \Rightarrow (2) \Rightarrow (3) \) and \( (1) \Rightarrow (4) \Rightarrow (5) \) are obvious. We first prove that \( (3) \Rightarrow (5) \). By [KS15 Thm. 1.7.2], \( X \to Y \) is an isomorphism in \( S_{S_{\mathbb{A}^1}} \). Thus by Proposition 6.1, \( X \to Y \) is a \( S^1 \)-stable \( \mathbb{A}^1 \)-0-equivalence. Next, we prove that \( (5) \Rightarrow (1) \). Then the induced map \( X(k(U))/R \to Y(k(U))/R \) is bijective for all \( U \in Sm_k^{\text{var}} \) by Proposition 6.1. Hence, Lemma 2.2 shows that the morphism of birational sheaves

\[
\pi^{\mathbb{A}^1}_0(X) \to \pi^{\mathbb{A}^1}_0(Y) \tag{6.1}
\]

is an isomorphism. On the other hand, since \( X \) and \( Y \) are \( R \)-rigid, there exist canonical isomorphisms of sheaves \( X \cong \pi^{\mathbb{A}^1}_0(X) \) and \( Y \cong \pi^{\mathbb{A}^1}_0(Y) \). Thus by Yoneda’s lemma in \( Sm_k \), (6.1) induces an isomorphism \( X \cong Y \). \( \square \)

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