On density of the zeros of Dedekind zeta-functions

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Abstract For any $\sigma$ with $0 \leq \sigma \leq 1$ and any $T > 10$ sufficiently large, let $N_\zeta(\sigma, K, T)$ be the number of zeros $\rho = \beta + i\gamma$ of $\zeta_K(s)$ with $|\gamma| \leq T$ and $\beta \geq \sigma$ and the zero being counted according to multiplicity. For $k \geq 3$, we have

$$N_\zeta(\sigma, K, T) \ll T^{\frac{2k}{6k-3}(1-\sigma)+\varepsilon},$$

where

$$\frac{2k + 3}{2k + 6} \leq \sigma < 1$$

and the implied constant may depend on the number field $K$ and $\varepsilon$. This improves previous results for $k \geq 3$ of certain range of $\sigma$. Some other related improved results are also given.

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1. Introduction

Let $K$ be a number field of degree $k = [K : Q]$, $k \geq 2$ and $k = r_1 + 2r_2$, where $r_1$ is the number of real conjugate fields and $2r_2$ is the number of complex conjugate fields. Also let $O_K$ be the ring of integers in $K$. For $\Re(s) > 1$, one defines the Dedekind zeta function associated to the number field $K$ as follows:

$$\zeta_K(s) = \sum_{a} \frac{1}{(Na)^s} = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}, \quad \Re(s) > 1,$$

where $a$ sums over the non-zero integral ideals $O_K$ of $K$, $Na$ is the norm of $a$, and $a_K(n) = \#\{a : Na = n\}$ is the ideal counting function of $K$. Next, we investigate estimate for the number of zeros of Dedekind zeta function $\zeta_K(s)$ in the strip $1/2 \leq \sigma \leq 1$. For any $\sigma$ with $0 \leq \sigma \leq 1$ and any $T > 10$ sufficiently large, let $N_\zeta(\sigma, K, T)$ be the number of zeros $\rho = \beta + i\gamma$ of $\zeta_K(s)$ with $|\gamma| \leq T$ and $\beta \geq \sigma$ and the zero being counted according to multiplicity. It is well known that

$$N_\zeta(\sigma, K, T) \sim \frac{k}{\pi} T \log T$$

as $T \to \infty$. For $k = 2$, and an absolute positive constant $C = C(\varepsilon, K) > 0$, Heath-Brown [5] showed that

$$N_\zeta(\sigma, K, T) \ll T^{A(\sigma)(1-\sigma)}(\log T)^C$$

with

$$A(\sigma) = \begin{cases} \frac{2}{\sigma} & \text{for } \frac{3}{4} \leq \sigma \leq 1 - \varepsilon, \\ \frac{2 - \varepsilon}{2 + \varepsilon} & \text{for } \frac{3}{4} \leq \sigma \leq \frac{3}{4}, \\ \frac{111}{124} & \text{for } \sigma \geq \frac{111}{124}, \end{cases}$$

where the implied constant depends only on $K$ and $\varepsilon$. In fact, for $k \geq 2$, Heath-Brown [5] showed that $A(\sigma) = 2 + \varepsilon$ for $\sigma \geq 3/4(1 - \mu)$ provided that

$$\zeta(1/2 + it) \ll t^{\mu}(\log t)^v,$$

where $\zeta(s)$ is the Riemann zeta function. Then the Bourgain’s recent result implies that for $\sigma > 63/71 \approx 0.88733$, one has

$$N_\zeta(\sigma, K, T) \ll T^{2(1-\sigma)+\varepsilon}.$$
If one adapts Ivić’s [10] idea for this problem, one can obtain that for \( \sigma > 53/60 \approx 0.88334 \), one has
\[
N_\zeta(\sigma, K, T) \ll T^{2(1-\sigma)+\varepsilon}.
\]
Recently, by adapting the ideas of Bourgain [1] and Heath-Brown [7], Chen-Debruyne-Vindas [3] improves the result of Ivić [10]. If one adapts the idea of [3] for this problem, one can obtain that for \( \sigma > 1407/1601 \approx 0.87883 \), one has
\[
N_\zeta(\sigma, K, T) \ll T^{2(1-\sigma)+\varepsilon}.
\]
The other results in Ivić’s [10] article cannot be translated to the case of \( k = 2 \) because there is no corresponding 6-th mean value estimate for Dedekind zeta functions of \( k = 2 \). In fact, only by the 2-th mean value estimate for Dedekind zeta functions of \( k = 2 \), the sharpest \( A(\sigma) \) only be \( 2 + \varepsilon \). In this paper, by a different of view, by the result of [13, 6], we can obtain a sharper \( A(\sigma) \) when \( \sigma \) close to 1.

**Theorem 1.1.** For \( k = 2 \), we have
\[
N_\zeta(\sigma, K, T) \ll T^{4(1-\sigma)+\varepsilon},
\]
where \((\kappa, \lambda)\) is any exponent pair with
\[
\frac{1 + \lambda - 6\kappa}{2 - 8\kappa} \leq \sigma < 1,
\]
\( \kappa < 1/4, \kappa + 1 \leq 5\lambda/2, \) and the implied constant may depend on the number field \( K \) and \( \varepsilon \).

**Remark 1.** For \( k = 2 \), if we choose \((\kappa, \lambda) = (1/14, 11/14)\), we can obtain that for \( 19/20 < \sigma \leq 1 \), we have
\[
N_\zeta(\sigma, K, T) \ll T^{4(1-\sigma)+\varepsilon},
\]
where the implied constant may depend on the number field \( K \) and \( \varepsilon \).

For \( k \geq 3 \) and a positive absolute constant \( C \), in 1968, Sokolovsky [18] showed that
\[
N_\zeta(\sigma, K, T) \ll (k+2-C/(k^2 \log(k+2)))(1-\sigma)+\varepsilon.
\]
Later, in 1977, Heath-Brown [5] improved the above result and showed that for any \( \varepsilon > 0 \), there exists a constant \( c = c(K, \varepsilon) \) such that
\[
N_\zeta(\sigma, K, T) \ll T^{k(1-\sigma)+\varepsilon}
\]
holds uniformly for \( 1/2 \leq \sigma \leq 1 \) when \( k \geq 3 \). Further assume that \( \zeta_K(1/2 + it) \ll t^\mu \) for some real number \( \mu > 0 \) and for all \( |t| \geq 10 \). Then, for any \( 1/2 \leq \sigma \leq 1 \), in [15], it is proved that
\[
N_\zeta(\sigma, K, T) \ll T^{2(1+2\mu)(1-\sigma)(\log T)^c(k)},
\]
where \( c(k) \) is a positive constant depending on \( k \). As an immediate corollary, one can get
\[
N_\zeta(\sigma, K, T) \ll T^{6+2k(1-\sigma)+\varepsilon},
\]
Since \((6 + 2k)/3 < k \) for any \( k > 6 \), hence for any number field \( K \) of degree \([K : Q] = k \geq 7\), the zero-density estimate strengthens a general result of Heath-Brown [5].

In this paper, we will show the following results. Our results improve the result of Heath-Brown [5] of \( 5/6 \leq \sigma \leq 1 \) for \( 3 \leq k \leq 6 \) and improve the result of [15] for \( k \geq 7 \).

**Theorem 1.2.** For \( k \geq 3 \), we have
\[
N_\zeta(\sigma, K, T) \ll T^{\frac{2k}{6+2k}(1-\sigma)+\varepsilon},
\]
where
\[
\frac{2k + 3}{2k + 6} \leq \sigma < 1
\]
and the implied constant may depend on the number field \( K \) and \( \varepsilon \).
The ideas of dealing with the above problems can also be used to deal with other problems. For example, by involving the ideas above, we are also interested in the zero-density problem for $L$-functions associated to a holomorphic cusp form $f$ with respect to the full modular group $SL_2(\mathbb{Z})$. $L(s, f)$ can be defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \lambda_f(p)p^{-s} + p^{-2s}\right)^{-1}, \quad \Re(s) > 1,$$

where $\lambda_f(n) \in \mathbb{R}$ ($n = 1, 2, \cdots$) are eigenvalues of the Hecke operators $T(n)$ and is normalized so that $\lambda_f(1) = 1$. It is well known that the above series converges absolutely for $\Re(s) > 1$, and $L(s, f)$ can be continued analytically to the whole complex plane. The Generalized Riemann Hypothesis predicts that the non-trivial zeros of $L(s, f)$ all lie on the critical line $\Re(s) = 1/2$. We can refer to [11] for the above preparatory knowledge. Ivić considered such type problems and showed in [10] and showed that $N_f(\sigma, T) \ll T^{8/3(1-\sigma)+\varepsilon}$, where

$$N_f(\sigma, T) := \sum_{\sigma \leq \beta \leq \frac{1}{2}-1, \; 10 \leq |\gamma| \leq T} 1.$$

And the zero density hypothesis implies that $A(\sigma) = 2$. Moreover, when $\sigma$ is close to 1, much shaper bounds were given. For example, in [10], Ivić proved that

$$N_f(\sigma, T) \ll T^{A(\sigma)(1-\sigma)+\varepsilon},$$

where

$$A(\sigma) = \frac{4}{8\sigma - 5} \quad \text{for} \quad \frac{11}{12} \leq \sigma \leq 1.$$

When $\sigma$ is near to 1, this upper bound is the best possible estimate so far in the existing literature. However, the so-called Weyl bound (especially the $L^6$-norm of Jutila [12]) was not exhausted. In this paper, we give the following estimate.

**Theorem 1.3.** Let $f$ be a holomorphic cusp form for $SL_2(\mathbb{Z})$ and let $L(s, f)$ be the corresponding $L$-function. Then we have

$$N_f(\sigma, T) \ll T^{A(\sigma)(1-\sigma)+\varepsilon}$$

with

$$A(\sigma) = \begin{cases} \frac{4}{3\sigma - 1} & \text{for} \quad \frac{1+\lambda-4\kappa}{2-6\kappa} \leq \sigma \leq 1, \\ \frac{4\kappa(1-\sigma)}{(1-2\kappa)\sigma+(3\kappa-\lambda-1)} & \text{for} \quad \frac{1+\lambda+\kappa}{2(1+\kappa)} \leq \sigma \leq \frac{1+\lambda-4\kappa}{2-6\kappa}, \end{cases}$$

where $(\kappa, \lambda)$ is any exponent pair with $\kappa < 1/3$, $\kappa + 1 \leq 4\lambda$ and the implied constant depends only on $f$ and $\varepsilon$.

If we choose $(\kappa, \lambda) = (1/14, 11/14)$, we will obtain the follows. When $\sigma$ is near to 1 (for $21/22 \leq \sigma \leq 1$), we obtain a best possible estimate with exhausted Weyl bounds.

**Corollary 1.4.** Let $f$ be a holomorphic cusp form for $SL_2(\mathbb{Z})$. Then we have

$$N_f(\sigma, T) \ll T^{A(\sigma)(1-\sigma)+\varepsilon} \quad (1.1)$$

with

$$A(\sigma) = \begin{cases} \frac{4}{3\sigma - 1} & \text{for} \quad \frac{21}{22} \leq \sigma \leq 1, \\ \frac{13\sigma-11}{15\sigma-11} & \text{for} \quad \frac{17}{18} \leq \sigma \leq \frac{21}{22}, \end{cases}$$

where the implied constant depends only on $f$ and $\varepsilon$. 
Remark 2. We remark that for the Wely’s bound, we obtain a best possible upper bound \( A(\sigma) = 4/(4\sigma - 1) \). And we give nontrivial improvement for the previous results in \( 17/18 \leq \sigma < 1 \).

Remark 3. One can also obtain a little better result by using the estimate
\[
\int_0^T |\zeta(5/7 + it)|^{196} dt \ll T^{14+\varepsilon}
\]
in [19]. By using such an idea, one may also improve some related results slightly.

2. Proof of Theorem 1.1 and 1.2

In order to estimate the number of zeros of
\[
\zeta_K(s)X = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}, \quad \Re s > 1,
\]
the usual procedure of zero-detecting technique will be used. For \( \zeta_K(s) \), we have
\[
\frac{1}{\zeta_K(s)} = \sum_{n=1}^{\infty} \frac{\mu_K(n)}{n^s}, \quad \Re s > 1.
\]
Write
\[
M_X(s, K) = \sum_{n \leq X} \frac{\mu_K(n)}{n^s},
\]
where \( X, Y > 2 \) are some powers of \( T \) which will be chosen later. Then we can obtain the following
\[
\zeta_K(s)M_X(s, K) = \sum_{n=1}^{\infty} \frac{C_X(n, K)}{n^s}, \quad \Re s > 1,
\]
where
\[
C_X(n, K) = \begin{cases} 
1 & \text{if } n = 1, \\
0 & \text{if } 2 \leq n \leq X, \\
D_X(n, K) & \text{if } n > X.
\end{cases} \quad (2.2)
\]
with
\[
D_X(n, K) = \sum_{d|n, \ d \leq X} \mu_K(d)a_K \left( \frac{n}{d} \right).
\]
By similar arguments as previous section, we find that a non-trivial zero counted of \( \zeta_K(s) \) denoted by \( N_K(\sigma, T) \) satisfies either
\[
\left| \sum_{X < n < Y \log^2 Y} \frac{D_X(n, K)}{n^\rho} e^{-\frac{1}{4} + \frac{i}{2} n^\rho} \right| \gg 1, \quad (2.3)
\]
or
\[
Y^{1/2-\beta} \int_{-\log^2 T}^{\log^2 T} \left| L\left( \frac{1}{2} + i\gamma + i\omega, K \right) \right| d\omega \gg 1. \quad (2.4)
\]
Denote by \( \mathcal{R}_1 \) the number of the class-1 zeros \( \rho = \beta + i\gamma \in A \) for which (2.3) is satisfied, and \( \mathcal{R}_2 \) for the class-2 zeros which satisfy (2.4). Moreover, we assume that any two zeros \( \rho = \beta + i\gamma, \rho' = \beta' + i\gamma' \) counted in \( \mathcal{R}_1 \) satisfy \( |\gamma - \gamma'| \geq 2 \log^4 T \). Then we have
\[
N_{\zeta}(\sigma, K, T) \ll (1 + \mathcal{R}_1 + \mathcal{R}_2) \log^5 T. \quad (2.5)
\]

Lemma 2.1. Let \( t > 1 \). Then we have
\[
\zeta_K(1/2 + it) \ll t^{k/6+\varepsilon}.
\]
Proof. See [13, 6].

**Lemma 2.2.** Taking $Y = T^{k/(6\sigma - 3) + \varepsilon}$, then we can rule out the situation of (2.4).

Proof. For $Y = T^{k/(6\sigma - 3) + \varepsilon}$, by Lemma 2.1 we have

$$ Y^{\frac{1}{2} - \beta} \int_{-\log^2 T}^{\log^2 T} |\zeta K(\frac{1}{2} + i\gamma + i\omega)| \, d\omega \ll T^\varepsilon. $$

This completes the proof. □

The following lemma can be proven by next section.

**Lemma 2.3.** For $\frac{1}{2} < \sigma < 1$, we have

$$ R_1 \ll T^\varepsilon \left( N^{2-2\sigma} + R_1 N^{1+\lambda-\kappa-2\sigma} T^{\kappa+\varepsilon} \right), $$

(2.6)

for some $N$ satisfying

$$ Y^{\frac{1}{2}} \log Y < N \leq Y \log^2 Y, $$

(2.7)

where $(\kappa, \lambda)$ is any exponent pair.

For $k = 2$, to bound $R_1$, the interval $[-T, T]$ can be divided into consecutive intervals of length $T_0$. Denote by $R_0$ the number of class-1 zero in $R_1$ with $|t| \leq T_0$. Then Huxley’s subdivision [8], we have

$$ R_1 \ll R_0 \left( 1 + \frac{T}{T_0} \right). $$

Set

$$ T_0 = N^{\frac{2\sigma - 1 - (\lambda - \kappa)}{\kappa} + \varepsilon}. $$

By Lemma 3.3, one can obtain

$$ R_0 \ll T^{\varepsilon} N^{2-2\sigma}. $$

Therefore, for

$$ \frac{1 + \lambda + \kappa}{2(1 + \kappa)} \leq \sigma \leq 1, $$

we have

$$ R_1 \ll T^{\varepsilon} N^{2-2\sigma} \left( 1 + \frac{T}{T_0} \right) \ll T^{\varepsilon} \left( N^{2-2\sigma} + T N^{2-2\sigma - \frac{2\sigma - 1 - (\lambda - \kappa)}{\kappa}} \right). $$

Setting

$$ Y = T^{2/(6\sigma - 3)}, $$

then for $k = 2$, we have $R_2 \ll T^{\varepsilon}$. For

$$ \frac{1 + \lambda - 6\kappa}{2 - 8\kappa} \leq \sigma \leq 1, $$

we get

$$ R_1 + R_2 \ll T^{\varepsilon} \left( N^{2-2\sigma} + T N^{2-2\sigma - \frac{2\sigma - 1 - (\lambda - \kappa)}{\kappa}} + 1 \right) \ll T^{\varepsilon} \left( Y^{2-2\sigma} + T Y^{1-\sigma - \frac{2\sigma - 1 - (\lambda - \kappa)}{2\kappa}} + 1 \right) \ll T^{\varepsilon} \left( T^{\frac{4(1-\sigma)}{6\sigma - 3}} + T^{1 + \frac{2(1-\sigma - \frac{2\sigma - 1 - (\lambda - \kappa)}{6\sigma - 3})}{6\sigma - 3}} \right) \ll T^{\varepsilon} T^{\frac{4(1-\sigma)}{6\sigma - 3} + \varepsilon}, $$

for
where $\kappa < 1/4$. Note that when $\kappa + 1 \leq 5\lambda/2$, we have

$$\frac{1 + \lambda - 6\kappa}{2 - 8\kappa} \geq \frac{1 + \lambda + \kappa}{2(1 + \kappa)}.$$ 

This completes the proof of Theorem 1.1.

For $k \geq 3$, to bound $\mathcal{R}_1$, the interval $[-T, T]$ can be divided into consecutive intervals of length $T_0$. Denote by $\mathcal{R}_0$ the number of class-1 zero in $\mathcal{R}_1$ with $|t| \leq T_0$. Then Huxley’s subdivision [8], we have

$$\mathcal{R}_1 \ll \mathcal{R}_0 \left(1 + \frac{T}{T_0}\right).$$

Set

$$T_0 = N^{4\sigma-2+\varepsilon}.$$ 

By Lemma 2.3, choosing $(\kappa, \lambda) = (1/2, 1/2)$, one can obtain

$$\mathcal{R}_0 \ll T^\varepsilon N^{2-2\sigma}.$$ 

Therefore, for

$$\frac{2}{3} \leq \sigma \leq 1,$$

we have

$$\mathcal{R}_1 \ll T^\varepsilon N^{2-2\sigma} \left(1 + \frac{T}{T_0}\right) \ll T^\varepsilon (N^{2-2\sigma} + TN^{4-6\sigma}).$$

Recall that

$$Y = T^{k/(6\sigma-3)},$$

by Lemma 2.2, we can obtain

$$\mathcal{R}_1 + \mathcal{R}_2 \ll T^{\frac{2k(1-\sigma)}{6\sigma-3} + \varepsilon},$$

provided that $k \geq 3$ for

$$\sigma \leq 1.$$ 

As

$$\frac{2k(1-\sigma)}{6\sigma-3} \leq 1$$

for $\frac{2k+3}{2k+6} \leq \sigma \leq 1$, we have

$$\mathcal{R}_1 + \mathcal{R}_2 \ll T^{\frac{2k(1-\sigma)}{6\sigma-3} + \varepsilon},$$

provided that

$$\max \left\{ \frac{2}{3}, \frac{2k+3}{2k+6} \right\} \leq \sigma \leq 1.$$ 

This completes the proof of Theorem 1.2. □

3. Proof of Theorem 1.3

In order to estimate the number of zeros of $L(s, f)$, the usual procedure of zero-detecting technique will be used. For $L(s, f)$, we have

$$\frac{1}{L(s, f)} = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s}, \quad \Re s > 1,$$

where the multiplicative function $\mu_f(n)$ is

$$\mu_f(p^a) = \begin{cases} 1 & \text{if } a = 0, 2, \\ -\lambda_f(p) & \text{if } a = 1, \\ 0 & \text{if } a \geq 3. \end{cases}$$
Write

\[ M_X(s, f) = \sum_{n \leq X} \frac{\mu_f(n)}{n^s}, \]

where \( X, Y > 2 \) are some powers of \( T \) which will be chosen later. Then we can obtain the following

\[ L(s, f)M_X(s, f) = \sum_{n=1}^{\infty} \frac{C_X(n)}{n^s}, \Re s > 1, \]

where

\[ C_X(n) = \begin{cases} 
1 & \text{if } n = 1, \\
0 & \text{if } 2 \leq n \leq X, \\
D_X(n) & \text{if } n > X
\end{cases} \tag{3.8} \]

with

\[ D_X(n) = \sum_{d|n, d \leq X} \mu_f(d)\lambda_f\left(\frac{n}{d}\right). \]

By the well known Mellin’s transform

\[ e^{-x} = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} \Gamma(w)x^{-w}dw \tag{3.9} \]

we have

\[ \sum_{n=1}^{\infty} \frac{C_X(n)}{n^s} e^{-\frac{n}{s}} = \frac{1}{2\pi i} \int_{(2)} \sum_{n=1}^{\infty} \frac{C_X(n)}{n^{s+w}} Y^w \Gamma(w)dw \]

\[ = \frac{1}{2\pi i} \int_{(2)} L(s+w, f)M_X(s+w, f)Y^w \Gamma(w)dw. \]

Shifting the integral to \( \Re w = 1/2 - \sigma \) where \( \sigma = \Re s > 1/2 \), and leaving the residue at \( w = 0 \), the above expression becomes

\[ L(s, f)M_X(s, f) + \frac{1}{2\pi i} \int_{(\frac{1}{2}-\sigma)} L(s+w, f)M_X(s+w, f)Y^w \Gamma(w)dw. \]

Let \( s = \rho = \beta + i\gamma \) be a zero of \( L(s, f) \) with \( \sigma \leq \beta \leq 1 \). Then by (3.8) we get

\[ e^{-\frac{\rho}{s}} + \sum_{n>X} \frac{D_X(n)}{n^\rho} e^{-\frac{n}{s}} \]

\[ = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} L\left(\frac{1}{2} + i\gamma + i\omega, f\right)M_X\left(\frac{1}{2} + i\gamma + i\omega, f\right) \Gamma\left(\frac{1}{2} - \beta + i\omega\right)Y^{\frac{1}{2}-\beta+i\omega}d\omega. \]

By Stirling’s formula, one finds that a non-trivial zero counted in \( N_f(\sigma, T) \) satisfies either

\[ \left| \sum_{X<n<Y \log^2 Y} \frac{D_X(n)}{n^\rho} e^{-\frac{n}{s}} \right| \gg 1, \tag{3.10} \]

or

\[ Y^{\frac{1}{2}-\beta} \left| \int_{-\log^2 T}^{\log^2 T} \left| L\left(\frac{1}{2} + i\gamma + i\omega, f\right)M_X\left(\frac{1}{2} + i\gamma + i\omega, f\right) \right| d\omega \gg 1. \tag{3.11} \]

Now we divide rectangle \( A = [\sigma, 1] \times [-T, T] \) into a series of consecutive rectangles \( A_i \) of height \( 3 \log^4 T \), starting from \( A_1 = [\sigma, 1] \times [0, 3 \log^4 T], A_2 = [\sigma, 1] \times [-3 \log^4 T, 0] \). By Theorem 1 in [16], each rectangle has at most \( O(\log^2 T) \) zeros of \( L(s, f) \). Denote by \( R_1 \) the number of the class-1 zeros \( \rho = \beta + i\gamma \in A \) for which (3.10) is satisfied, and \( R_2 \) for the class-2 zeros which
satisfy (3.11). Moreover, we assume that any two zeros \( \rho = \beta + i\gamma, \rho' = \beta' + i\gamma' \) counted in \( R_i \) satisfy \( |\gamma - \gamma'| \geq 2\log^4 T \). Then we have
\[
N_f(\sigma, T) \ll (1 + R_1 + R_2) \log^5 T. \tag{3.12}
\]
To bound \( N_f(\sigma, T) \), one need to bound \( R_1 \) and \( R_2 \), respectively. In order to bound \( R_2 \), we also need a result related to Jutila [12].

**Lemma 3.1.** Let \( T > 1 \) and \( |t_r| \leq T \) \((1 \leq r \leq R)\). Suppose \( |t_r - t_s| \geq (\log T)^2 \) for \( r \neq s \leq R \). Then we have
\[
\sum_{r=1}^{R} \left| L\left(\frac{1}{2} + it_r, f\right) \right|^6 \ll T^{2+\eps}. \tag{3.13}
\]

**Lemma 3.2.** We have
\[
R_2 \ll T^{2+\eps} Y^{3-6\sigma}.
\]

**Proof.** By Deligne’s bound \( \lambda_f(n) \ll n^\eps \), for \( X = T^\eps \) and \( |\omega| \leq \log^2 T \), we have
\[
M_X \left(\frac{1}{2} + i\gamma + i\omega, f\right) \ll T^\eps.
\]
By (3.11), there exists \( \gamma' \in (\gamma - \log^2 T, \gamma + \log^2 T) \) such that
\[
T^\eps (\log^2 T) Y^{\frac{1}{2} - \sigma} \left| L\left(\frac{1}{2} + i\gamma', f\right) \right| \gg 1.
\]
Taking the sixth power on both sides and summing over all the class-2 zeros in \( R_2 \) we get, by Lemma 3.1,
\[
R_2 \ll T^{2\eps} (\log^4 T) Y^{3-6\sigma} \sum_{r=1}^{R_2} \left| L\left(\frac{1}{2} + i\gamma_r, f\right) \right|^6 \ll T^{2+3\eps} Y^{3-6\sigma}.
\]
This finishes the estimate of \( R_2 \). \( \square \)

The following lemma can be obtained by similar argument of the proof of Lemma 2.3.

**Lemma 3.3.** For \( 1/2 < \sigma < 1 \), we have
\[
R_1 \ll T^\eps \left( N^{2-2\sigma} + R_1 N^{1+\lambda - \kappa - 2\sigma} T^{\kappa + \eps} \right), \tag{3.14}
\]
for some \( N \) satisfying
\[
Y^{\frac{1}{2} + \lambda} \log Y < N \leq Y \log^2 Y, \tag{3.15}
\]
where \((\kappa, \lambda)\) is any exponent pair.

To bound \( R_1 \), the interval \([-T, T]\) can be divided into consecutive intervals of length \( T_0 \). Denote by \( R_0 \) the number of class-1 zero in \( R_1 \) with \( |t| \leq T_0 \). Then Huxley’s subdivision [8], we have
\[
R_1 \ll R_0 \left(1 + \frac{T}{T_0}\right).
\]
Set
\[
T_0 = N^{\frac{2\sigma - 1 - (\lambda - \kappa)}{\kappa + \eps}}.
\]
By Lemma 3.3, one can obtain
\[
R_0 \ll T^\eps N^{2-2\sigma}.
\]
Therefore, for
\[
\frac{1 + \lambda + \kappa}{2(1 + \kappa)} \leq \sigma \leq 1,
\]
we have
\[ R_1 \ll T^c N^{2-2\sigma} \left( 1 + \frac{T}{T_0} \right) \ll T^c \left( N^{2-2\sigma} + TN^{2-2\sigma-\frac{2\sigma-1-(\lambda-\kappa)}{\kappa}} \right). \]

By Lemma 3.2,
\[ R_2 \ll T^{2+\varepsilon} Y^{3-6\sigma}. \tag{3.16} \]

Setting
\[ Y = T^{2/(4\sigma-1)}, \]
for
\[ \frac{1 + \lambda - 4\kappa}{2 - 6\kappa} \leq \sigma \leq 1, \]
we get
\[ R_1 + R_2 \ll T^c \left( N^{2-2\sigma} + TN^{2-2\sigma-\frac{2\sigma-1-(\lambda-\kappa)}{\kappa}} + T^{2+\varepsilon} Y^{3-6\sigma} \right) \]
\[ \ll T^c \left( Y^{2-2\sigma} + TY^{1-\sigma-\frac{2\sigma-1-(\lambda-\kappa)}{2\kappa}} + T^{2+\varepsilon} Y^{3-6\sigma} \right) \]
\[ \ll T^c \left( T^{\frac{4\kappa(1-\sigma)}{4\sigma-1}} + T^{1+\frac{2(1-\sigma-\frac{2\sigma-1-(\lambda-\kappa)}{2\kappa})}{4\sigma-1}} \right) \]
\[ \ll T^{\frac{4\kappa(1-\sigma)}{4\sigma-1} + \varepsilon}, \]
where we assume that \( \kappa < 1/3 \). Note that when \( \kappa + 1 \leq 4\lambda \), we have
\[ \frac{1 + \lambda - 4\kappa}{2 - 6\kappa} \geq \frac{1 + \lambda + \kappa}{2(1 + \kappa)}. \]

Setting
\[ Y = T^{\frac{12-2\kappa}{2\kappa + (3\kappa - \lambda - 1)}} \]
and apply (3.16), for
\[ \frac{1 + \lambda + \kappa}{2(1 + \kappa)} \leq \sigma \leq \frac{1 + \lambda - 4\kappa}{2 - 6\kappa}, \]
we get
\[ R_1 + R_2 \ll T^c \left( N^{2-2\sigma} + TN^{2-2\sigma-\frac{2\sigma-1-(\lambda-\kappa)}{\kappa}} + T^{2+\varepsilon} Y^{3-6\sigma} \right) \]
\[ \ll T^c \left( Y^{2-2\sigma} + TY^{1-\sigma-\frac{2\sigma-1-(\lambda-\kappa)}{2\kappa}} + T^{2+\varepsilon} Y^{3-6\sigma} \right) \]
\[ \ll T^c \left( T^{\frac{4\kappa(1-\sigma)}{2\kappa (3\kappa - \lambda - 1)}} + T^{2+\frac{6\kappa(1-2\sigma)}{2\kappa (3\kappa - \lambda - 1)}} \right) \]
\[ \ll T^{\frac{4\kappa(1-\sigma)}{2\kappa (3\kappa - \lambda - 1)} + \varepsilon}. \]

This finishes the proof of Theorem 1.3. □

4. Proof of Lemma 2.3

To prove Lemma 2.3, we will follow the argument in Ivić [9]. By dyadic interval we split the sum in into \( O(\log Y) \) sums of the form \( \sum_{M < n \leq 2M} \frac{D_X(n)}{n^\rho} e^{-\frac{n}{Y}} \). Hence each \( \rho = \beta + i\gamma \) counted in \( R_1 \) satisfies
\[ \sum_{M_j < n \leq 2M_j} \frac{D_X(n)}{n^\rho} e^{-\frac{n}{Y}} \gg \frac{1}{\log Y} \tag{4.17} \]
for some $M_j$ satisfying
\[ X \leq M_j = 2^{-j} Y \log^2 Y \leq Y \log^2 Y, \quad j = 1, 2, \ldots, J, \]
where $J = \lfloor \log(X^{-1} Y \log^2 Y) / \log(2) \rfloor + 1$. Denote by $R_j$ the number of zeros satisfying (4.17) in $R_1$, and let $R = R_{j_0}$ be the largest among $R_j$ for $1 \leq j \leq J$. Then
\[ R_1 \ll R \log Y. \quad (4.18) \]
Write $M = M_{j_0}$ and choose the large enough positive integer $l \geq r \geq 1$ such that ($l$ is a natural number depended on $M$ and $r$ is a fixed integer)
\[ M^l \ll Y^r \log^2 Y \ll M^{l+1}, \]
and then raise both sides of (4.17) to the power $l$, we get
\[ \left| \sum_{M^l < n \leq (2M)^l} \frac{E_X(n)}{n^\rho} \right| \gg \frac{1}{\log^l Y}, \]
where
\[ E_X(n) = \sum_{\substack{n_1 \cdots n_l = n \leq 2M}} D_X(n_1)D_X(n_2) \cdots D_X(n_l)e^{-n^{-\beta_r+\gamma_r}}. \]
Write $N = M^l, N_1 = (2M)^l$. Then
\[ Y^{\frac{r^2}{l+1}} \log^2 Y \ll N \ll Y^r \log^2 Y. \]
As the upper bound of 2-th power moment for the cusp forms, we know that choosing $r = 1$ is suitable. Then $R$ counts the number of $\rho$ which satisfies
\[ \left| \sum_{N < n \leq N_1} \frac{E_X(n)}{n^\rho} \right| \gg \frac{1}{\log^k Y}. \quad (4.19) \]
Now we show that
\[ \sum_{N < n \leq N_1} \frac{|E_X(n)|^2}{n^{2\sigma}} \ll N^{1-2\sigma+\epsilon}. \quad (4.20) \]
By Deligne’s [4] bound, we have
\[ \sum_{N < n \leq N_1} \frac{|E_X(n)|^2}{n^{2\sigma}} \ll N^{1-2\sigma+\epsilon}. \]
To bound $R$, we let $\rho = \beta_r + i\gamma_r$ in (4.19), where $\sigma \leq \beta_r \leq 1, |\gamma_r| \leq T$. Then
\[ R \ll \log^l Y \sum_{r \leq R} \left| \sum_{N < n \leq N_1} \frac{E_X(n)}{n^{\beta_r+\gamma_r}} \right| = \log^l Y \sum_{r \leq R} \left| \sum_{N < n \leq N_1} \frac{E_X(n)}{n^{\sigma+\gamma_r}} \right|. \]
By partial summation we get
\[ R \ll \log^l Y \max_{N \leq u \leq N_1} \sum_{r \leq R} \left| \sum_{N < n \leq u} \frac{E_X(n)}{n^{\sigma+\gamma_r}} \right|. \quad (4.21) \]
To bound the above double sum, we will use the following well known lemma, which can also be seen in [9, 19, 14].
Lemma 4.1. (Halász-Montgomery inequality) Let \( a_n, b_n \) \((n = 1, 2, \ldots)\) be complex numbers. For \( a = \{ a_n \}_{n=1}^\infty \in \mathbb{C}^\infty, \ b = \{ b_n \}_{n=1}^\infty \in \mathbb{C}^\infty, \) define the inner product

\[
(a, b) = \sum_{n=1}^\infty a_n \overline{b_n}, \quad \|a\|^2 = (a, a).
\]

Let \( \xi, \varphi_1, \varphi_2, \ldots, \varphi_R \) be arbitrary vectors in \( \mathbb{C}^\infty. \) Then we have

\[
\sum_{r \leq R} |(\xi, \varphi_r)| \leq \|\xi\| \left( \sum_{r,s \leq R} (\varphi_r, \varphi_s) \right)^{\frac{1}{2}}.
\]

For \( \xi = \{\xi_n\}_{n=1}^\infty, \) by Lemma 4.1, we have

\[
\xi_n = \begin{cases} 
E_X(n) e^{2\pi i n \sigma} & \text{if } N < n \leq u, \\
0 & \text{otherwise},
\end{cases}
\]

and \( \varphi_r = \{\varphi_{r,n}\}_{n=1}^\infty \) with \( \varphi_{r,n} = e^{-2\pi i n \gamma_r}. \) By (4.20) one has \( \|\xi\|^2 \ll N^{1-2\sigma+\varepsilon}. \) Thus by (4.21) and Lemma 4.1 we get

\[
R^2 \ll Y^{\varepsilon} \left( RN^{2-2\sigma} + N^{1-2\sigma} \sum_{r \neq s \leq R} |G(i\gamma_r - i\gamma_s)| \right).
\]

where \( G(it) = \sum_{n=1}^\infty e^{-\frac{2\pi i}{N} n it}. \) Here we have used the fact that \( G(0) \ll N \) for bounding the diagonal terms. By Mellin’s transform, we have

\[
\sum_{n=1}^\infty e^{-\frac{2\pi i}{N} n^{-s}} = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} \zeta(s+\omega) \Gamma(\omega) Y^{\omega} d\omega.
\]

Thus

\[
G(it) = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} \zeta(\omega+it) \Gamma(\omega) N^{\omega} d\omega.
\]

Let \( (\kappa, \lambda) \) be any exponent pair. Then move the integration line to \( \Re(\omega) = \lambda - \kappa, \) and note that the residue at the simple pole \( \omega = 1 - it \) is \( O(Ne^{-|t|}) \), we get

\[
G(it) = (2\pi i)^{-1} \int_{\lambda-\kappa-i\infty}^{\lambda-\kappa+i\infty} \zeta(\omega+it) \Gamma(\omega) N^{\omega} d\omega + O\left(Ne^{-|t|}\right).
\]

By Stirling’s formula, it is easy to check that the contribution from \( |v| \geq \log^2 T \) to the above integral is \( O(1), \) where \( v = \Im \omega. \) Moreover for \( |v| \leq \log^2 T, \) one has \( \Gamma(\lambda - \kappa + iv) \ll 1. \) Therefore, we get

\[
\sum_{r \neq s \leq R} |G(i\gamma_r - i\gamma_s)| 
\ll N \sum_{r \neq s \leq R} e^{-|\gamma_r - \gamma_s|} + R^2 + N^{\lambda-\kappa} \int_{-\log^2 T}^{\log^2 T} \sum_{r \neq s \leq R} |\zeta(\lambda - \kappa + it \gamma_r - i\gamma_s + iv)| dv
\ll RN + R^2 + N^{\lambda-\kappa} R^2 T^{\kappa+\varepsilon},
\]

where we have used

\[
\zeta(\lambda - \kappa + it) \ll (|t| + 1)^{\kappa+\varepsilon},
\]

which can be seen in [9, 19].
Back to (4.23), we get

\[ R \ll T^\varepsilon \left( N^2 - 2\sigma + N^{1-2\sigma} N^{\lambda - \kappa} T^{\kappa + \varepsilon} \right) \]

\[ \ll T^\varepsilon \left( N^2 - 2\sigma + N^{1+\lambda - \kappa - 2\sigma} T^{\kappa + \varepsilon} \right). \]

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