GLOBAL SMALL SOLUTIONS TO THE 3D COMPRESSIBLE VISCOUS NON-RESISTIVE MHD SYSTEM

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ABSTRACT. Whether or not smooth solutions to the 3D compressible magnetohydrodynamic (MHD) equations without magnetic diffusion are always global in time remains an extremely challenging open problem. No global well-posedness or stability result is currently available for this 3D MHD system in the whole space $\mathbb{R}^3$ or the periodic box $\mathbb{T}^3$ even when the initial data is small or near a steady-state solution. This paper presents a global existence and stability result for smooth solutions to this 3D MHD system near any background magnetic field satisfying a Diophantine condition.

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I. INTRODUCTION AND MAIN RESULT

1.1. Model and synopsis of related studies. We consider the 3D viscous compressible magnetohydrodynamic (MHD) system without magnetic diffusion,

$$\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \quad t > 0, \ x \in \mathbb{T}^3, \\
\rho \partial_t u + \rho u \cdot \nabla u - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla P &= (\nabla \times B) \times B, \\
\partial_t B + u \cdot \nabla B - B \cdot \nabla u &= -B \text{div} u, \\
\text{div} B &= 0, 
\end{align*}$$

(1.1)
where \( \mathbb{T}^3 \) is the 3D periodic box, and \( \rho = \rho(t,x), u = u(t,x) \) and \( B = B(t,x) \) denote the fluid density, velocity field and the magnetic field, respectively. The parameters \( \mu \) and \( \lambda \) are shear viscosity and volume viscosity coefficients, respectively, which satisfy the standard strong parabolicity assumption,

\[
\mu > 0 \quad \text{and} \quad \nu \overset{\text{def}}{=} \lambda + 2\mu > 0.
\]

The pressure \( P = P(\rho) \) is a smooth function of the density satisfying \( P' > 0 \) and \( P'(\rho) = 1 \) for some constant reference density \( \rho > 0 \). The MHD system concerned here arises in modeling fluids that can be treated as perfect conductors such as strongly collisional plasmas. In addition, the breakdown of ideal MHD is known to be the cause of solar flares, the largest explosions in the solar system [26].

This paper focuses on the small data global well-posedness and stability problem. This problem is extremely challenging due to the lack of magnetic diffusion. In fact, it remains an open problem to construct global solutions of the compressible non-resistive MHD system even with small initial data near background magnetic fields satisfying a Diophantine condition. Our research was inspired by a recent work of W. Chen, Z. Zhang and J. Zhou [4], which solved a stability problem on the 3D incompressible MHD equations.

A vector \( \mathbf{n} \in \mathbb{R}^3 \) is said to satisfy the Diophantine condition if, for any \( \mathbf{k} \in \mathbb{Z}^3 \setminus \{0\} \),

\[
|\mathbf{n} \cdot \mathbf{k}| \geq \frac{c}{|\mathbf{k}|r} \quad \text{for some } c > 0 \text{ and } r > 2. \tag{1.2}
\]

There are vectors that do not satisfy the Diophantine condition such as those with all three components being rational. But almost all vectors in \( \mathbb{R}^3 \) do satisfy (1.2), as demonstrated in [4].

We consider the evolution of perturbations near a background magnetic field \( \mathbf{n} \) satisfying the Diophantine condition defined above. For the sake of simplicity, we still use \( B \) to denote the perturbation \( \mathbf{B} - \mathbf{n} \). Then the perturbed equations can be rewritten as

\[
\begin{aligned}
&\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0, \\
&\rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \text{div} \mathbf{u} + \nabla P = \mathbf{n} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{B} - \mathbf{n} \nabla \mathbf{B} - \mathbf{B} \nabla \mathbf{B}, \\
&\partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{n} \cdot \nabla \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{n} \text{div} \mathbf{u} - \mathbf{B} \text{div} \mathbf{u}, \\
&\text{div} \mathbf{B} = 0, \\
&(\rho,\mathbf{u},\mathbf{B})|_{t=0} = (\rho_0,\mathbf{u}_0,\mathbf{B}_0),
\end{aligned} \tag{1.3}
\]

where \( \mathbf{n} \nabla \mathbf{B} = \sum_{i=1}^3 \mathbf{n}_i \nabla B_i \) and \( \mathbf{B} \nabla \mathbf{B} = \sum_{i=1}^3 B_i \nabla B_i \).

Before stating our main result, we briefly summarize some related work. When \( \mathbf{B} = 0 \), (1.1) reduces to the isentropic compressible Naiver-Stokes equations. Fundamental issues on the compressible Naiver-Stokes equations such as well-posedness and blowup problems have been extensively investigated (see, e.g., [9], [10], [17], [34], [35]). When the density \( \rho \) is a constant, (1.1) becomes the viscous non-resistive incompressible MHD system, which is the subject of numerous investigations (see, e.g., [1], [2], [7], [8], [19], [22], [23], [24], [25], [28], [30], [36], [37]). The compressible MHD equations with both dissipation and magnetic diffusion are also the focus of many studies (see, e.g., [3], [11], [12], [18], [29], [33]). Needless to say, the references listed above is just a very small portion of the vast literature on this subject. In contrast, not that many results
are currently available for the compressible viscous MHD equations without magnetic diffusion. The small data global existence and stability for the 2D compressible MHD equation without magnetic diffusion was studied by Wu and Wu [31] and Wu and Zhu [32]. Wu and Wu [31] presented a new and systematic approach on the well-posedness and stability problem on partially dissipated systems. Wu and Zhu [32] introduced a diagonalization process to understand the spectrum structure and large-time behavior. The work of Wu and Zhu [32] is the first to investigate such system on bounded domains and the first to solve this problem by pure energy estimates, which help reduce the complexity in other approaches. Very recently Dong, Wu and Zhai [6] proved the global existence of strong solutions to the 2-D compressible non-resistive MHD equations with small initial data. Jiang and Jiang [13] showed the stability/instability criteria for the stratified compressible magnetic Rayleigh-Taylor problem in Lagrangian coordinates in the three-dimensional case. Besides these stability results, several other existence and uniqueness results are also available. [14] studied the non-resistive limit and the magnetic boundary layer of the 1D compressible MHD equation. [20] obtained the global existence of weak solutions and their large-time behavior for this 1D MHD system. Zhong [38] obtained the local strong solutions without any Cho-Choe-Kim type compatibility conditions in \( \mathbb{R}^2 \). Li and Sun [21] obtained the existence of global weak solutions for the 2D non-resistive compressible MHD equations. [23] extended this global result to include density-dependent viscosity coefficient and non-monotone pressure law. We emphasize that no global well-posedness or stability result is currently available for this 3D MHD system in the whole space \( \mathbb{R}^3 \) or the periodic box \( T^3 \) even when the initial data is small or near a steady-state solution.

1.2. Main result. Attention is focused on the system (1.3) for \((t,x) \in [0,\infty) \times T^3\) with the volume of \( T^3 \) normalized to unity,
\[
|T^3| = 1.
\]
For notational convenience, we write
\[
\bar{\rho} = \int_{T^3} \rho_0 \, dx.
\]
Besides the regularity assumption on the initial data \((\rho_0, u_0, B_0)\), we also assume that
\[
\int_{T^3} \rho_0 u_0 \, dx = 0 \quad \text{and} \quad \int_{T^3} B_0 \, dx = 0.
\]
It is easy to check that, for sufficiently regular solutions, these averages are conserved in time,
\[
\int_{T^3} \rho \, dx = \bar{\rho}, \quad \int_{T^3} \rho u \, dx = 0, \quad \text{and} \quad \int_{T^3} B \, dx = 0.
\]

(1.4)

The main result of the paper is stated as follows.

**Theorem 1.1.** Assume \( n \) satisfies the Diophantine condition (1.2). Let \( N \geq 4r + 7 \) with \( r > 2 \). Consider the system (1.3) with the initial data \((\rho_0, u_0, B_0)\) satisfying
\[
\rho_0 - \bar{\rho} \in H^N(T^3), \quad c_0 \leq \rho_0 \leq c_0^{-1}, \quad u_0 \in H^N(T^3), \quad B_0 \in H^N(T^3)
\]
for some constant \( c_0 > 0 \). We further assume
\[
\int_{T^3} \rho_0 u_0 \, dx = \int_{T^3} B_0 \, dx = 0.
\]
Then there exists a small constant \( \varepsilon \) such that, if
\[
\|\rho_0 - \bar{\rho}\|_{H^N} + \|u_0\|_{H^N} + \|B_0\|_{H^N} \leq \varepsilon,
\]

then the system (1.3) admits a unique global solution \((\rho - \bar{\rho}, \mathbf{u}, \mathbf{B}) \in C([0, \infty); H^N)\). Moreover, for any \(t \geq 0\) and \(r + 4 \leq \beta \leq N\), there holds

\[
\| (\rho - \bar{\rho}) (t) \|_{H^\beta} + \| \mathbf{u} (t) \|_{H^\beta} + \| \mathbf{B} (t) \|_{H^\beta} \leq C \epsilon (1 + t)^{-\frac{3(N-\beta)}{2(N-r-4)}}.
\] (1.5)

As established in [4, Section 2], almost all vector fields \(\mathbf{n}\) in \(\mathbb{R}^3\) actually satisfy the Diophantine condition (1.2). Of course, there are vectors that do not satisfy this condition such as those with all components being rational numbers.

1.3. **Difficulties and scheme of the proof.** The local well-posedness of (1.3) can be shown via a procedure that is now standard for compressible equations (see, e.g., [16]). The focus of the proof is on the global bound of \((\rho - \bar{\rho}, \mathbf{u}, \mathbf{B})\) in \(H^N (\mathbb{T}^3)\). We use the bootstrapping argument and start by making the ansatz that

\[
\sup_{t \in [0,T]} \left( \| \rho - \bar{\rho} \|_{H^N} + \| \mathbf{u} \|_{H^N} + \| \mathbf{B} \|_{H^N} \right) \leq \delta,
\]

for suitably chosen \(0 < \delta < 1\). The main efforts are devoted to proving that, if the initial norm is taken to be sufficiently small, namely

\[
\| \rho_0 - \bar{\rho} \|_{H^N} + \| \mathbf{u}_0 \|_{H^N} + \| \mathbf{B}_0 \|_{H^N} \leq \varepsilon
\]

with sufficiently small \(\epsilon > 0\), then

\[
\sup_{t \in [0,T]} \left( \| \rho - \bar{\rho} \|_{H^N} + \| \mathbf{u} \|_{H^N} + \| \mathbf{B} \|_{H^N} \right) \leq \frac{\delta}{2}. \tag{1.6}
\]

It is not trivial to prove (1.6). The difficulty is due to the lack of dissipation or damping in the equations of \(\rho\) and \(\mathbf{B}\). Without any stabilizing mechanism, the norms of \(\rho\) and \(\mathbf{B}\) would grow in time and it would be impossible to establish (1.6). Therefore, it is crucial to exploit any potential smoothing and stabilizing effects due to the coupling and interaction of \(\rho, \mathbf{u}\) and \(\mathbf{B}\). This paper is able to discover the hidden dissipation in two quantities. The first one is the directional derivative of \(\mathbf{B}\) along \(\mathbf{n}\), namely \(\mathbf{n} \cdot \nabla \mathbf{B}\) while the second one is the combined quantity \(\rho - \bar{\rho} + \mathbf{n} \cdot \mathbf{B}\). To be more precise, we assume \(\bar{\rho} = 1\) and write \(a = \rho - 1\) and rewrite (1.3) as

\[
\begin{cases}
\partial_t a + \text{div} \mathbf{u} = f_1, \\
\partial_t \mathbf{u} - \text{div} (\bar{\mu} (\rho) \mathbf{v}) - \nabla (\bar{\lambda} (\rho) \text{div} \mathbf{u}) + \nabla a + \nabla (\mathbf{n} \cdot \mathbf{B}) - \mathbf{n} \cdot \nabla \mathbf{B} = f_2, \\
\partial_t \mathbf{B} - \mathbf{n} \cdot \nabla \mathbf{u} + \mathbf{n} \text{div} \mathbf{u} = f_3, \\
\text{div} \mathbf{B} = 0,
\end{cases}
\]

where \(\bar{\mu}, \bar{\lambda}, f_1, f_2\) and \(f_3\) are explicitly given in Subsection 3.3. \(f_1, f_2\) and \(f_3\) are essentially nonlinear terms. We outline the main steps in the proof of (1.6) and explain how the coupling and interaction leads to the dissipative and stabilizing effects in \(\mathbf{n} \cdot \nabla \mathbf{B}\) and \(a + \mathbf{n} \cdot \mathbf{B}\).

The first step estimates the norm \(\|(a, \mathbf{u}, \mathbf{B})\|_{H^\ell}\) with \(0 \leq \ell \leq N\) in terms of the \(L^\infty\)-norms of \(a, \mathbf{u}, \mathbf{B}\) and their gradients,

\[
\frac{1}{2} \frac{d}{dt} \| (a, \mathbf{u}, \mathbf{B}) \|_{H^\ell}^2 + \mu \| \nabla \mathbf{u} \|_{H^\ell}^2 + (\bar{\lambda} + \mu) \| \text{div} \mathbf{u} \|_{H^\ell}^2 \leq CY_0 (t) \|(a, \mathbf{u}, \mathbf{B})\|_{H^\ell}^2.
\]
where \( Y_\infty(t) \), as given in Lemma 3.3, defined by

\[
Y_\infty(t) \overset{\text{def}}{=} \| (\nabla a, \nabla u, \nabla B) \|_{L^\infty} + \| (\nabla a, \nabla u, \nabla B) \|_{L^2}^2 \\
+ \| (a, B) \|_{L^2}^2 + \| a \|_{L^2}^2 \| B \|_{L^2}^2 + \| B \|_{L^2}^2 \| \nabla B \|_{L^2}^2.
\]

(1.7)

In order to obtain a global bound for \( \| (a, u, B) \|_{H^3} \), we need to control the time integral of \( Y_\infty(t) \). Due to the lack of dissipation or damping terms in the equations of \( a \) and \( B \), the time integrals of \( \| a \|_{L^\infty}, \| B \|_{L^\infty}, \| \nabla a \|_{L^\infty} \) and \( \| \nabla B \|_{L^\infty} \) can not be bounded directly. Our idea is to exploit the hidden dissipation in the quantities \( n \cdot \nabla B \) and \( a + n \cdot B \).

The second main step is to combine the equation of \( \mathbb{P} u \) (the divergence part of \( u \)) and the equation of \( B \) to establish the stabilizing effect in \( n \cdot \nabla B \). Here \( \mathbb{P} = I - \nabla \Delta^{-1} \text{div} \) is the projection onto divergence-free vector fields. We take advantage of the special structure in the system

\[
\begin{align*}
\partial_t \mathbb{P} u - \mu \Delta \mathbb{P} u &= n \cdot \nabla B + \mathbb{P} f_4, \\
\partial_t B - n \cdot \nabla u + n \text{div} u &= f_3,
\end{align*}
\]

and consider the time evolution of the inner product

\[
\frac{d}{dt} \sum_{0 \leq s \leq r+3} \int_{\mathbb{T}^3} \Lambda^s \mathbb{P} u \cdot \Lambda^s (n \cdot \nabla B) \, dx.
\]

We obtain, as stated in Lemma 3.4, that

\[
\| n \cdot \nabla B(t) \|_{H^{r+3}}^2 - \frac{d}{dt} \sum_{0 \leq s \leq r+3} \int_{\mathbb{T}^3} \Lambda^s \mathbb{P} u \cdot \Lambda^s (n \cdot \nabla B) \, dx \leq C \| u(t) \|_{H^{r+5}}^2,
\]

(1.8)

which yields the time integrability of \( \| n \cdot \nabla B(t) \|_{H^{r+3}}^2 \). Combining with the Poincaré type inequality induced by the Diophantine condition on \( n \),

\[
\| B \|_{H^3} \leq C \| n \cdot \nabla B \|_{H^{r+3}},
\]

we are then able to control the time integrability of \( \| B \|_{L^\infty} \) and \( \| \nabla B \|_{L^\infty} \).

The third main step is to establish the stabilizing property of the combined quantity \( a + n \cdot B \). The idea is to make use of the interaction between the equation of \( a + n \cdot B \) and that of \( \mathbb{Q} u \),

\[
\begin{align*}
\partial_t (a + n \cdot B) &= n \cdot \nabla u - (|n|^2 + 1) \text{div} u + f_1 + f_3 \cdot n, \\
\partial_t \mathbb{Q} u - \nu \Delta \mathbb{Q} u + \nabla (a + n \cdot B) &= \mathbb{Q} f_4
\end{align*}
\]

(1.9)

where \( \mathbb{Q} = \nabla \Delta^{-1} \text{div} \) projects vectors onto their gradient parts. Noting that \( \text{div} \mathbb{Q} u = \text{div} u \), we observe that (1.9) involves a wave structure for \( a + n \cdot B \) and \( \mathbb{Q} u \). To make the wave structure more explicit, we can rewrite (1.9) as, after ignoring some non-essential terms,

\[
\begin{align*}
\partial_t (a + n \cdot B) &= -(|n|^2 + 1) \text{div} u, \\
\partial_t \mathbb{Q} u - \nu \Delta \mathbb{Q} u + \nabla (a + n \cdot B) &= 0,
\end{align*}
\]

which can be converted into

\[
\begin{align*}
\partial_t (a + n \cdot B) - \nu \Delta \partial_t (a + n \cdot B) - (|n|^2 + 1) \Delta (a + n \cdot B) &= 0, \\
\partial_t \text{div} \mathbb{Q} u - \nu \Delta \partial_t \text{div} \mathbb{Q} u - (|n|^2 + 1) \Delta \text{div} \mathbb{Q} u &= 0.
\end{align*}
\]
Alternatively we can make the stabilizing effect more explicit by considering the combined quantities

\[ d \overset{\text{def}}{=} a + n \cdot B \quad \text{and} \quad G \overset{\text{def}}{=} \mathbb{Q}u - \frac{1}{\nu} \Delta^{-1} \nabla d, \]

which satisfy

\[
\begin{aligned}
\partial_t d + \frac{1}{\nu}(|n|^2 + 1)d + (|n|^2 + 1) \text{div} G &= n \cdot \nabla u \cdot n + f_1 + f_3 \cdot n, \\
\partial_t G - \nu \Delta G &= \frac{1}{\nu}(|n|^2 + 1) \mathbb{Q}u - \frac{1}{\nu} \Delta^{-1} \nabla (n \cdot \nabla u \cdot n) + \mathbb{Q}f_4 - \frac{1}{\nu} \Delta^{-1} \nabla (f_1 + f_3 \cdot n).
\end{aligned}
\]

These equations clearly reveal the damping in \( d = a + n \cdot B \) and the dissipation in \( G \). We then estimate the Sobolev norms \( \|d\|_{H^{r+4}} \) and \( \|G\|_{H^{r+4}} \). \( a \) in some of the nonlinear terms in \( f_1 \) and \( f_4 \) is replaced by \( d - n \cdot B \). Combining with (1.7) with \( \ell = r + 4 \) and (1.8) in the previous steps, we then obtain a self-contained inequality.

The last step is to establish (1.6) and close the bootstrapping argument. The energy inequality obtained in the previous step, together with interpolation inequalities, allow us to show that

\[ \|(a, u, B)(t)\|_{H^3} \leq C(1 + t)^{-\frac{3}{2}} \]  

(1.10)

when \( \delta > 0 \) is taken to be sufficiently small. In particular, the time integral of \( Y_{\infty}(t) \) in (1.7) is time integrable,

\[ \int_0^\infty Y_{\infty}(t) \, dt \leq C < \infty. \]

Grönwall’s inequality then yields

\[ \|(a, u, B)(t)\|_{H^N} \leq C \|(a_0, u_0, B_0)\|_{H^N}. \]

Taking the initial norm to be sufficiently small, we achieve (1.6). The decay rate in (1.5) is a consequence of (1.10) and an interpolation inequality.

We point out some differences between the handling of this compressible MHD system and that of the corresponding incompressible counterpart. The compressible MHD system contains the density, which involves no damping or dissipation. A crucial component in the proof of Theorem 1.1 is how to obtain the decay estimate for \( \|a(t)\|_{H^3} \). It does not appear to be possible to work directly with the equation of \( a \) to obtain such decay estimate. As described above, our idea is to exploit the wave structure in the evolution of \( a + n \cdot B \) and \( \text{div} \mathbb{Q}u \). In contrast, the incompressible MHD equations do not contain the density equation. The second main difference is that the velocity \( u \) of the compressible MHD system contains the divergence-free part and the gradient part,

\[ u = P u + \mathbb{Q}u. \]

It is necessary to consider the equations of \( Pu \) and \( \mathbb{Q}u \) separately when we seek the hidden stabilizing effect in the quantities \( n \cdot \nabla B \) and \( a + n \cdot B \). The velocity \( u \) in the incompressible MHD system is already divergence-free and does not have the gradient part. Thus the compressible MHD system is much more challenging to deal with.
1.4. **Organization of the paper.** The rest of this paper is structured as follows. Section 2 recalls several functional inequalities to be used in the proof of Theorem 1.1. Section 3 proves Theorem 1.1. The long proof is accomplished in six subsections. Subsection 3.1 explains how to prove the local well-posedness and initiates the bootstrapping argument. Subsection 3.2 provides the basic $L^2$-energy estimate. Subsection 3.3 establishes the higher-order energy estimate (1.7). Subsection 3.4 explores the dissipation in $\mathbf{B}$ and proves (1.8). Subsection 3.5 discovers the dissipation of the quantity $\mathbf{a} \cdot \mathbf{n} \cdot \mathbf{B}$ and obtains a self-contained inequality for $(\mathbf{a}, \mathbf{u}, \mathbf{B})$ in $H^{r+4}$. Subsection 3.6 proves the decay estimate (1.10) and then (1.6) to close the bootstrapping argument. The decay rate in (1.5) is shown via (1.10) and an interpolation inequality.

2. **Preliminaries**

This section provides several functional inequalities to be used in the proof of our main result. We first recall a weighted Poincaré inequality first established by Desvillettes and Villani in [5].

**Lemma 2.1.** Let $\Omega$ be a bounded connected Lipschitz domain and $\bar{\rho}$ be a positive constant. There exists a positive constant $C$, depending on $\Omega$ and $\bar{\rho}$, such that for any nonnegative function $\rho$ satisfying

$$\int_{\Omega} \rho \, dx = 1, \quad \rho \leq \bar{\rho},$$

and any $\mathbf{u} \in H^1(\Omega)$, there holds

$$\int_{\Omega} \rho \left( \mathbf{u} - \int_{\Omega} \rho \, dx \right)^2 \, dx \leq C \|\nabla \mathbf{u}\|_{L^2}^2. \quad (2.1)$$

In order to remove the weight function $\rho$ in (2.1) without resorting to the lower bound of $\rho$, we need another variant of Poincaré inequality (see Lemma 3.2 in [9]).

**Lemma 2.2.** Let $\Omega$ be a bounded connected Lipschitz domain in $\mathbb{R}^3$ and $p > 1$ be a constant. Given positive constants $M_0$ and $E_0$, there is a constant $C = C(E_0, M_0)$ such that for any non-negative function $\rho$ satisfying

$$M_0 \leq \int_{\Omega} \rho \, dx \quad \text{and} \quad \int_{\Omega} \rho^p \, dx \leq E_0,$$

and for any $\mathbf{u} \in H^1(\Omega)$, there holds

$$\|\mathbf{u}\|_{L^2}^2 \leq C \left[ \|\nabla \mathbf{u}\|_{L^2}^2 + \left( \int_{\Omega} \rho \|\mathbf{u}\| \, dx \right)^2 \right].$$

The next lemma states a special Poincaré inequality involving a vector satisfying the Diophantine condition.

**Lemma 2.3.** Assume $\mathbf{n} \in \mathbb{R}^3$ satisfies the Diophantine condition (1.2). Let $s \in \mathbb{R}$. Then, for any function $f$ satisfies $\nabla f \in H^{r+\frac{1}{2}}(\mathbb{T}^3)$ and $\int_{\mathbb{T}^3} f \, dx = 0$,

$$\|f\|_{H^r(\mathbb{T}^3)} \leq C \|\mathbf{n} \cdot \nabla f\|_{H^{r+\frac{1}{2}}(\mathbb{T}^3)}.$$  

The proof of this lemma follows from the standard Poincaré inequality and the Diophantine condition. Finally we recall several calculus inequalities.
Lemma 2.4. ([15]) Let $s \geq 0$. Then there exists a constant $C$ such that, for any $f, g \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$, we have
\[
\|fg\|_{H^s} \leq C(\|f\|_{L^\infty}\|g\|_{H^s} + \|g\|_{L^\infty}\|f\|_{H^s}).
\]

Lemma 2.5. ([15]) Let $s > 0$. Then there exists a constant $C$ such that, for any $f \in H^s(\mathbb{T}^3) \cap W^{1,\infty}(\mathbb{T}^3), g \in H^{s-1}(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$, there holds
\[
\|\Lambda^s f \cdot \nabla g\|_{L^2} \leq C(\|\nabla f\|_{L^\infty}\|\Lambda^s g\|_{L^2} + \|\Lambda^s f\|_{L^2}\|\nabla g\|_{L^\infty}).
\]

Lemma 2.6. ([27]) Let $s > 0$ and $f \in H^s(\mathbb{T}^3) \cap L^\infty(\mathbb{T}^3)$. Assume that $F$ is a smooth function on $\mathbb{R}$ with $F(0) = 0$. Then we have
\[
\|F(f)\|_{H^s} \leq C(1 + \|f\|_{L^\infty}^{[s]+1}\|f\|_{H^s},
\]
where the constant $C$ depends on $\sup_{k \leq [s]+2, j \leq \|f\|_{L^\infty}} \|F^{(k)}(t)\|_{L^\infty}$.

3. PROOF OF THE MAIN THEOREM

This section is devoted to proving Theorem 1.1. The proof is long and is thus divided into several subsections for the sake of clarity.

3.1. Local well-posedness. Given the initial data $(\rho_0 - \bar{\rho}, u_0, B_0) \in H^N(\mathbb{T}^3)$, the local well-posedness of (1.3) could be proven by using the standard energy method (see, e.g., [16]). Thus, we may assume that there exists $T > 0$ such that the system (1.3) has a unique solution $(\rho - \bar{\rho}, u, B) \in C([0, T]; H^N)$. Moreover,
\[
\frac{1}{2}c_0 \leq \rho(t, x) \leq 2c_0^{-1}, \quad \text{for any } t \in [0, T]. \tag{3.1}
\]

We use the bootstrapping argument to show that this local solution can be extended into a global one. The goal is to derive a global a priori upper bound. To initiate the bootstrapping argument, we make the ansatz that
\[
\sup_{t \in [0, T]} (\|\rho - \bar{\rho}\|_{H^N} + \|u\|_{H^N} + \|B\|_{H^N}) \leq \delta, \tag{3.2}
\]
where $0 < \delta < 1$ obeys requirements to be specified later. In the following subsections we prove that, if the initial norm is taken to be sufficiently small, namely
\[
\|a_0\|_{H^N} + \|u_0\|_{H^N} + \|B_0\|_{H^N} \leq \epsilon, \quad a_0 = \rho_0 - \bar{\rho}
\]
with sufficiently small $\epsilon > 0$, then
\[
\sup_{t \in [0, T]} (\|\rho - \bar{\rho}\|_{H^N} + \|u\|_{H^N} + \|B\|_{H^N}) \leq \frac{\delta}{2}.
\]

The bootstrapping argument then leads to the desired global bound.
3.2. **Basic energy estimates.** Denote by \( g(\rho) \) the potential energy density, namely

\[
g(\rho) = \rho \int_0^\rho \frac{P(\tau) - P(\bar{\rho})}{\tau^2} \, d\tau.
\]

For any fixed positive constant \( c_0 \), if \( c_0 \leq \rho \leq c_0^{-1} \), then

\[
g(\rho) \sim (\rho - \bar{\rho})^2.
\]

The standard basic energy estimate gives

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} \left( 2g(\rho) + \rho |\nabla u|^2 + |\mathbf{B}|^2 \right) \, dx + \mu \|\nabla u\|^2_{L^2} + (\lambda + \mu) \|\text{div} u\|^2_{L^2} = 0, \tag{3.3}
\]

where we have used the following cancellations

\[
\int_{\mathbb{T}^3} (\mathbf{n} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{n} \cdot \nabla \mathbf{u} + \nabla \mathbf{u} \cdot \nabla \mathbf{B}) \cdot \mathbf{B} \, dx = 0,
\]

\[
\int_{\mathbb{T}^3} (\nabla \mathbf{B} + \mathbf{n} \nabla \mathbf{B} \cdot \mathbf{u} dx + \int_{\mathbb{T}^3} (\mathbf{n} \cdot \nabla \mathbf{u} + \nabla \mathbf{u} \cdot \mathbf{B}) \cdot \mathbf{B} dx = 0.
\]

Without loss of generality, we assume from now on that \( \bar{\rho} = 1 \). If we set

\[
a \overset{\text{def}}{=} \rho - 1,
\]

then (3.3) implies the result in the following lemma.

**Lemma 3.1.** Let \( \rho, u \) and \( \mathbf{B} \) be smooth functions to (1.3), then there holds

\[
\frac{1}{2} \frac{d}{dt} \| (a, u, B) \|^2_{L^2} + \mu \|\nabla u\|^2_{L^2} + (\lambda + \mu) \|\text{div} u\|^2_{L^2} = 0.
\]

In view of Lemmas 2.1, 2.2 and (1.4), we have the following Poincaré type inequalities, which will be frequently used in this paper.

**Lemma 3.2** (Poincaré type inequalities). Let \( \rho, u \) and \( \mathbf{B} \) be smooth functions to (1.3) on \([0, \infty) \times \mathbb{T}^3\) satisfying (1.4), then for any \( t \geq 0 \), there hold

\[
\| \mathbf{B}(t) \|^2_{L^2} \leq C \|\nabla \mathbf{B}(t)\|^2_{L^2}, \tag{3.4}
\]

\[
\| (\sqrt{\rho}u)(t) \|^2_{L^2} \leq C \|\nabla u(t)\|^2_{L^2}, \tag{3.5}
\]

and

\[
\| u(t) \|^2_{L^2} \leq C \|\nabla u(t)\|^2_{L^2}. \tag{3.6}
\]

**Proof.** Clearly, Due to (1.4), (3.4) is a standard Poincaré inequality. (3.5) follows from Lemma 2.1 and \( \int_{\mathbb{T}^3} \rho u \, dx = 0 \) in (1.4). (3.6) is a consequence of (3.5) and Lemma 2.2. \( \square \)

Throughout we make the assumption that

\[
\sup_{t \in \mathbb{R}_+, x \in \mathbb{T}^3} |a(t, x)| \leq \frac{1}{2}. \tag{3.7}
\]

Because of \( H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3) \), (3.7) is ensured by the fact that the solution constructed here has small norm in \( H^2(\mathbb{T}^3) \). It then follows from Lemma 2.6 that the following composition estimate holds,

\[
\| F(a) \|_{H^s} \leq C \| a \|_{H^s}, \quad \text{for} \ F(0) = 0 \text{ and any } s > 0. \tag{3.8}
\]
3.3. Higher order energy estimates. To obtain higher order energy estimates, we set
\[
\hat{\mu}(\rho) \overset{\text{def}}{=} \frac{\mu}{\rho}, \quad \hat{\lambda}(\rho) \overset{\text{def}}{=} \frac{\lambda + \mu}{\rho}, \quad I(a) \overset{\text{def}}{=} \frac{a}{1+a}, \quad \text{and} \quad k(a) \overset{\text{def}}{=} \frac{P'(1+a)}{1+a} - 1.
\]
Then (1.3) can be reformulated as
\[
\begin{aligned}
&\partial_t a + \text{div} u = f_1, \\
&\partial_t u - \text{div} (\hat{\mu}(\rho) \nabla u) - \nabla (\hat{\lambda}(\rho) \text{div} u) + \nabla a = n \cdot \nabla B - \nabla (n \cdot B) + f_2, \\
&\partial_t B = n \cdot \nabla u - n \text{div} u + f_3, \\
&\text{div} B = 0,
\end{aligned}
\]
\[
(a, u, B)|_{t=0} = (a_0, u_0, B_0),
\]
where
\[
\begin{aligned}
f_1 &= -u \cdot \nabla a - a \text{div} u, \\
f_2 &= -u \cdot \nabla u + B \cdot \nabla B + B \nabla B + k(a) \nabla a + \mu(\nabla I(a)) \nabla u \\
&\quad + (\lambda + \mu)(\nabla I(a)) \text{div} u - I(a)(n \cdot \nabla B + B \cdot \nabla B - n \nabla B - B \nabla B), \\
f_3 &= -u \cdot \nabla B + B \cdot \nabla u - B \text{div} u.
\end{aligned}
\]

The aim of this subsection is to prove the following key lemma.

**Lemma 3.3.** Let \((a, u, B) \in C([0, T]; H^N)\) be a solution to the system (3.9). For any \(0 \leq \ell \leq N\), there holds
\[
\frac{1}{2} \frac{d}{dt} \|(a, u, B)\|_{H^\ell}^2 + \mu \|\nabla u\|_{H^\ell}^2 + (\lambda + \mu) \|\text{div} u\|_{H^\ell}^2 \leq CY_\infty(t) \|(a, u, B)\|_{H^\ell}^2
\]
with
\[
Y_\infty(t) \overset{\text{def}}{=} \| (\nabla a, \nabla u, \nabla B) \|_{L^\infty} + \| (\nabla a, \nabla u, \nabla B) \|_{L^2} + \| (a, B) \|_{L^2} + \| a \|_{L^2}^2 + \| B \|_{L^2}^2 + \| \nabla B \|_{L^2}^2.
\]

**Proof.** (3.10) with \(\ell = 0\) is the basic energy inequality in Lemma 3.1. We consider the case when \(\ell \geq 1\). Writing \(\Lambda = \sqrt{-\Delta}\) and applying \(\Lambda^s\) with \(1 \leq s \leq \ell\) to (3.9) and then taking \(L^2\) inner product with \((\Lambda^s a, \Lambda^s u, \Lambda^s B)\) yields
\[
\frac{1}{2} \frac{d}{dt} \|(\Lambda^s a, \Lambda^s u, \Lambda^s B)\|_{L^2}^2 - \int_{T^3} \Lambda^s \text{div} (\mu(\rho) \nabla u) \cdot \Lambda^s u \, dx = \int_{T^3} \Lambda^s \nabla (\hat{\lambda}(\rho) \text{div} u) \cdot \Lambda^s u \, dx
\]
\[
= \int_{T^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx + \int_{T^3} \Lambda^s f_2 \cdot \Lambda^s u \, dx + \int_{T^3} \Lambda^s f_3 \cdot \Lambda^s B \, dx
\]
where we have used the following cancellations
\[
\int_{T^3} \Lambda^s \text{div} u \cdot \Lambda^s a \, dx + \int_{T^3} \Lambda^s \nabla a \cdot \Lambda^s u \, dx = 0;
\]
\[
\int_{T^3} \Lambda^s (n \cdot \nabla B) \cdot \Lambda^s u \, dx + \int_{T^3} \Lambda^s (n \cdot \nabla u) \cdot \Lambda^s B \, dx = 0;
\]
\[
\int_{T^3} \Lambda^s \nabla (n \cdot B) \cdot \Lambda^s u \, dx + \int_{T^3} \Lambda^s (\text{div} u) \cdot \Lambda^s B \, dx = 0.
\]
The second term on the left-hand side of (3.11) can be written as
\begin{align*}
- \int_{T^3} \Lambda^s \text{div} (\mu(\rho) \nabla u) \cdot \Lambda^s u &\, dx \\
= \int_{T^3} \Lambda^s (\mu(\rho) \nabla u) \cdot \nabla \Lambda^s u &\, dx \\
= \int_{T^3} \mu(\rho) \nabla \Lambda^s u \cdot \nabla \Lambda^s u &\, dx + \int_{T^3} [\Lambda^s, \mu(\rho)] \nabla u \cdot \nabla \Lambda^s u. \tag{3.12}
\end{align*}
Due to (3.1), we have for any \( t \in [0, T] \) that
\begin{equation}
\int_{T^3} \mu(\rho) \nabla \Lambda^s u \cdot \nabla \Lambda^s u \, dx \geq c_0^{-1} \mu \| \Lambda^{s+1} u \|^2_{L^2}. \tag{3.13}
\end{equation}

For the last term in (3.12), we first rewrite this term into
\begin{align*}
\int_{T^3} [\Lambda^s, \mu(a)] \nabla u \cdot \nabla \Lambda^s u &\, dx = \int_{T^3} [\Lambda^s, \mu(a) - \mu + \mu] \nabla u \cdot \nabla \Lambda^s u \\
&= - \int_{T^3} [\Lambda^s, \mu I(a)] \nabla u \cdot \nabla \Lambda^s u.
\end{align*}
Then, with the aid of Lemmas 2.5, 2.6 and (3.8), we have
\begin{align*}
\left| \int_{T^3} [\Lambda^s, \mu I(a)] \nabla u \cdot \nabla \Lambda^s u \, dx \right| \\
&\leq C \| \nabla \Lambda^s u \|_{L^2} \left( \| \nabla I(a) \|_{L^2} \| \Lambda^s u \|^2_{L^2} + \| \nabla u \|_{L^\infty} \| \Lambda^s I(a) \|_{L^2} \right) \\
&\leq \frac{c_0^{-1}}{2} \mu \| \Lambda^{s+1} u \|^2_{L^2} + C \left( \| \nabla a \|^2_{L^\infty} \| \Lambda^s u \|^2_{L^2} + \| \nabla u \|^2_{L^\infty} \| \Lambda^s a \|^2_{L^2} \right). \tag{3.14}
\end{align*}
Inserting (3.13) and (3.14) into (3.12) leads to
\begin{align*}
- \int_{T^3} \Lambda^s \text{div} (\mu(\rho) \nabla u) \cdot \Lambda^s u &\, dx \geq \frac{c_0^{-1}}{2} \mu \| \Lambda^{s+1} u \|^2_{L^2} \\
&- C \left( \| \nabla a \|^2_{L^\infty} \| \Lambda^s u \|^2_{L^2} + \| \nabla u \|^2_{L^\infty} \| \Lambda^s a \|^2_{L^2} \right).
\end{align*}
The third term on the left-hand side of (3.11) can be dealt with similarly. Hence,
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| (\Lambda^s a, \Lambda^s u, \Lambda^s B) \|^2_{L^2} &+ c_0^{-1} \mu \| \Lambda^{s+1} u \|^2_{L^2} + c_0^{-1} (\lambda + \mu) \| \Lambda^s \text{div} u \|^2_{L^2} \\
&\leq C \left( \| \nabla a \|^2_{L^\infty} \| \Lambda^s u \|^2_{L^2} + \| \nabla u \|^2_{L^\infty} \| \Lambda^s a \|^2_{L^2} \right) \\
&+ \int_{T^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx + \int_{T^3} \Lambda^s f_2 \cdot \Lambda^s u \, dx + \int_{T^3} \Lambda^s f_3 \cdot \Lambda^s B \, dx. \tag{3.15}
\end{align*}
We now estimate successively terms on the right hand side of (3.15). To bound the first term in \( f_1 \),
we rewrite it into
\begin{align*}
\int_{T^3} \Lambda^s (u \cdot \nabla a) \cdot \Lambda^s a &\, dx = \int_{T^3} (\Lambda^s (u \cdot \nabla a) - u \cdot \nabla \Lambda^s a) \cdot \Lambda^s a \, dx + \int_{T^3} u \cdot \nabla \Lambda^s a \cdot \Lambda^s a \, dx \\
&\overset{\text{def}}{=} A_1 + A_2. \tag{3.16}
\end{align*}
By Lemma 2.5,
\[
A_1 \leq C \left( \| [\Lambda^s, \mathbf{u} \cdot \nabla] a \|_{L^2} \right) \| \Lambda^s a \|_{L^2} \\
\leq C \left( \| \nabla \mathbf{u} \|_{L^\infty} \| \Lambda^s a \|_{L^2} + \| \Lambda^s \mathbf{u} \|_{L^2} \| \nabla a \|_{L^\infty} \right) \| \Lambda^s a \|_{L^2} \\
\leq C \left( \| \nabla \mathbf{u} \|_{L^\infty} + \| \nabla a \|_{L^\infty} \right) \left( \| \Lambda^s a \|_{L^2}^2 + \| \Lambda^s \mathbf{u} \|_{L^2}^2 \right). \quad (3.17)
\]
By integration by parts,
\[
A_2 \leq C \| \nabla \mathbf{u} \|_{L^\infty} \| \Lambda^s a \|_{L^2}^2. \quad (3.18)
\]
For the second term in \( f_1 \), it follows from Lemma 2.4 that
\[
\int_{T^3} \Lambda^s (\text{div} \mathbf{u}) \cdot \Lambda^s a \, dx \leq C \left( \| \text{div} \mathbf{u} \|_{L^\infty} \| a \|_{H^s} + \| \text{div} \mathbf{u} \|_{H^1} \| a \|_{L^\infty} \right) \| \Lambda^s a \|_{L^2} \\
\leq \frac{c_0^{-1} \mu}{16} \| \Lambda^{s+1} \mathbf{u} \|_{L^2}^2 + C \left( \| \nabla \mathbf{u} \|_{L^\infty} + \| a \|_{L^\infty} \right) \| \Lambda^s a \|_{L^2}^2, \quad (3.19)
\]
where we have used the inequalities
\[
\| a \|_{H^s} \leq C \| \Lambda^s a \|_{L^2} \quad \text{and} \quad \| \text{div} \mathbf{u} \|_{H^1} \leq C \| \Lambda^{s+1} \mathbf{u} \|_{L^2}.
\]
Collecting (3.17), (3.18) and (3.19), we can get
\[
\int_{T^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx \leq \frac{c_0^{-1} \mu}{16} \| \Lambda^{s+1} \mathbf{u} \|_{L^2}^2 \\
+ C \left( \| \nabla \mathbf{u} \|_{L^\infty} + \| a \|_{L^\infty} \right) \left( \| \nabla \mathbf{u} \|_{L^\infty} + \| a \|_{L^\infty} \right) \left( \| \Lambda^s a \|_{L^2}^2 + \| \Lambda^s \mathbf{u} \|_{L^2}^2 \right). \quad (3.20)
\]
For the first term in \( f_3 \), a similar process as in (3.17) and (3.18) yields
\[
\int_{T^3} \Lambda^s (\mathbf{u} \cdot \nabla \mathbf{B}) \cdot \Lambda^s \mathbf{B} \, dx \leq C \left( \| \nabla \mathbf{u} \|_{L^\infty} + \| \nabla \mathbf{B} \|_{L^\infty} \right) \left( \| \Lambda^s \mathbf{u} \|_{L^2}^2 + \| \Lambda^s \mathbf{B} \|_{L^2}^2 \right). \quad (3.21)
\]
For the last two terms in \( f_3 \), a derivation similar to (3.19) gives
\[
\int_{T^3} \Lambda^s (\mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \text{div} \mathbf{u}) \cdot \Lambda^s \mathbf{B} \, dx \leq \frac{c_0^{-1} \mu}{16} \| \Lambda^{s+1} \mathbf{u} \|_{L^2}^2 + C \left( \| \nabla \mathbf{u} \|_{L^\infty} + \| \mathbf{B} \|_{L^\infty} \right) \| \Lambda^s \mathbf{B} \|_{L^2}^2.
\]
Therefore,
\[
\int_{T^3} \Lambda^s f_3 \cdot \Lambda^s \mathbf{B} \, dx \leq \frac{c_0^{-1} \mu}{16} \| \Lambda^{s+1} \mathbf{u} \|_{L^2}^2 \\
+ C \left( \| \nabla \mathbf{u} \|_{L^\infty} + \| \nabla \mathbf{B} \|_{L^\infty} + \| \mathbf{B} \|_{L^\infty} \right) \left( \| \Lambda^s \mathbf{u} \|_{L^2}^2 + \| \Lambda^s \mathbf{B} \|_{L^2}^2 \right). \quad (3.21)
\]
In the following, we bound the terms in \( f_2 \). To do so, we write
\[
\int_{T^3} \Lambda^s f_2 \cdot \Lambda^s \mathbf{u} \, dx = \sum_{i=3}^{9} A_i \quad (3.22)
\]
with

\[ A_3 \overset{\text{def}}{=} \int_{\Omega^3} \Lambda^s (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \Lambda^s \mathbf{u} \, dx, \quad A_4 \overset{\text{def}}{=} \int_{\Omega^3} \Lambda^s (\mathbf{B} \cdot \nabla \mathbf{B}) \cdot \Lambda^s \mathbf{u} \, dx, \]

\[ A_5 \overset{\text{def}}{=} \int_{\Omega^3} \Lambda^s (k(a) \nabla a) \cdot \Lambda^s \mathbf{u} \, dx, \quad A_6 \overset{\text{def}}{=} \int_{\Omega^3} \Lambda^s (\mu (\nabla I(a)) \nabla \mathbf{u}) \cdot \Lambda^s \mathbf{u} \, dx, \]

\[ A_7 \overset{\text{def}}{=} \int_{\Omega^3} \Lambda^s (\lambda + \mu) (\nabla I(a)) \nabla \mathbf{u}) \cdot \Lambda^s \mathbf{u} \, dx, \]

\[ A_8 \overset{\text{def}}{=} \int_{\Omega^3} \Lambda^s (I(a) (\mathbf{n} \cdot \nabla \mathbf{B} - \mathbf{n} \cdot \nabla \mathbf{u})) \cdot \Lambda^s \mathbf{u} \, dx, \]

\[ A_9 \overset{\text{def}}{=} \int_{\Omega^3} \Lambda^s (I(a) (\mathbf{B} \cdot \nabla \mathbf{B} - \nabla \mathbf{B})) \cdot \Lambda^s \mathbf{u} \, dx. \]

The term \( A_3 \) can be bounded as in (3.16) to get

\[ A_3 \leq C \| \nabla \mathbf{u} \|_{L^2} \| \Lambda^s \mathbf{u} \|_{L^2}^2. \]

We next deal with the term \( A_4 \). In view of \( \text{div} \mathbf{B} = 0 \), one can write

\[ A_4 = \int_{\Omega^3} \Lambda^s \text{div} (\mathbf{B} \otimes \mathbf{B}) \cdot \Lambda^s \mathbf{u} \, dx \]

\[ \leq C \| \mathbf{B} \|_{L^\infty} \| \mathbf{B} \|_{H^s} \| \Lambda^s \mathbf{u} \|_{L^2} \]

\[ \leq \frac{c_s^{-1}}{16} \| \Lambda^s \mathbf{u} \|_{L^2}^2 + C \| \mathbf{B} \|_{L^\infty} \| \mathbf{B} \|_{H^s}^2. \]

By Lemmas 2.4 and 2.8, we have

\[ A_5 \leq C (\| \nabla a \|_{L^\infty} \| k(a) \|_{H^{s-1}} + \| \nabla a \|_{H^{s-1}} \| k(a) \|_{L^\infty} ) \| \Lambda^s \mathbf{u} \|_{L^2} \]

\[ \leq \frac{c_s^{-1}}{16} \| \Lambda^s \mathbf{u} \|_{L^2}^2 + C(\| \nabla a \|_{L^\infty} \| \Lambda^s \mathbf{u} \|_{L^2}^2 + \| \nabla \mathbf{u} \|_{L^\infty} \| a \|_{H^s}^2), \]

where we have used the fact that \( \| k(a) \|_{L^\infty} \leq \| a \|_{L^\infty} \). Similarly,

\[ A_6 + A_7 \leq C (\| \nabla I(a) \|_{L^\infty} \| \Lambda^s \mathbf{u} \|_{L^2} + \| \nabla I(a) \|_{H^{s-1}} \| \nabla \mathbf{u} \|_{L^\infty} ) \| \Lambda^s \mathbf{u} \|_{L^2} \]

\[ \leq \frac{c_s^{-1}}{16} \| \Lambda^s \mathbf{u} \|_{L^2}^2 + C(\| \nabla a \|_{L^\infty} \| \Lambda^s \mathbf{u} \|_{L^2}^2 + \| \nabla \mathbf{u} \|_{L^\infty} \| a \|_{H^s}^2 \]

and

\[ A_8 \leq C (\| I(a) \|_{L^\infty} \| \nabla \mathbf{B} \|_{H^{s-1}} + \| I(a) \|_{H^{s-1}} \| \nabla \mathbf{B} \|_{L^\infty} ) \| \Lambda^s \mathbf{u} \|_{L^2} \]

\[ \leq \frac{c_s^{-1}}{16} \| \Lambda^s \mathbf{u} \|_{L^2}^2 + C(\| a \|_{L^\infty} \| \mathbf{B} \|_{H^s}^2 + \| \nabla \mathbf{B} \|_{L^\infty} \| a \|_{H^s}^2). \]  \hspace{1cm} (3.23)

For the last term \( A_9 \), we use Lemmas 2.4 and 2.6 again to get

\[ A_9 \leq C (\| I(a) \|_{L^\infty} \| \mathbf{B} \|_{H^{s-1}} + \| I(a) \|_{H^{s-1}} \| \mathbf{B} \|_{L^\infty} ) \| \Lambda^s \mathbf{u} \|_{L^2} \]

\[ \leq \frac{c_s^{-1}}{16} \| \Lambda^s \mathbf{u} \|_{L^2}^2 + C(\| a \|_{L^\infty} \| \mathbf{B} \|_{H^s}^2 + \| \nabla \mathbf{B} \|_{L^\infty} \| a \|_{H^s}^2). \]  \hspace{1cm} (3.24)

Due to

\[ \| \mathbf{B} \|_{H^{s-1}} \leq C \| \mathbf{B} \|_{L^\infty} \| \mathbf{B} \|_{H^s}, \]

which, together with (3.24), leads to

\[ A_9 \leq \frac{c_s^{-1}}{16} \| \Lambda^s \mathbf{u} \|_{L^2}^2 + C(\| a \|_{L^\infty} \| \mathbf{B} \|_{H^s}^2 + \| \mathbf{B} \|_{L^\infty} \| \nabla \mathbf{B} \|_{L^\infty} \| a \|_{H^s}^2). \]
Applying where
Proof. Define
\( P \)
the desired estimate (3.10) by summing up for any \( 1 \leq s \leq \ell \). This completes the proof of Lemma 3.3.

3.4. The dissipation of the magnetic field. The MHD system concerned in this paper involves no magnetic diffusion. We need to exploit the hidden dissipation due to the background magnetic field. The goal of this subsection is to establish the upper bound stated in the following lemma.

**Lemma 3.4.** Assume \( \delta > 0 \) in (3.2) is taken to be sufficiently small, then, for any \( t \in [0, T] \),

\[
\| n \cdot \nabla B(t) \|_{H^{r+3}}^2 - \frac{d}{dt} \sum_{0 \leq s \leq r+3} \int_{T^3} \Lambda^s P u \cdot \Lambda^s (n \cdot \nabla B) \, dx \\
\leq C \| u(t) \|_{H^{r+5}}^2 + C \delta^2 \| a + n \cdot B \|_{H^{r+4}}^2 .
\]  

**Proof.** Define

\[ Q = \nabla \Delta^{-1} \text{div} \quad \text{and} \quad P = I - Q, \]

where \( P \) is the standard Leray projector operator. It is easy to check that

\[ P(n \cdot \nabla B) = n \cdot \nabla B, \quad P(\nabla(n \cdot B)) = 0, \]

\[ Q(n \cdot \nabla B) = 0, \quad Q(\nabla(n \cdot B)) = \nabla(n \cdot B). \]

Applying operator \( P \) to the second equation of (3.9) gives

\[
\partial_t P u - \mu \Delta P u = n \cdot \nabla B + P f_4,
\]  

(3.27)

where \( f_4 \) is slightly different from \( f_2 \), namely

\[
f_4 \overset{\text{def}}{=} -u \cdot \nabla u + B \cdot \nabla B + B \nabla B + k(a) \nabla a \\
- I(a)(\mu \Delta u + (\lambda + \mu) \nabla \text{div} u) - I(a)(n \cdot \nabla B + B \cdot \nabla - n \nabla B + B \nabla B). \]

Applying \( \Lambda^s \) (\( 0 \leq s \leq r+3 \)) to (3.27), and taking the \( L^2 \)-inner product with \( \Lambda^s (n \cdot \nabla B) \), we obtain

\[
\| \Lambda^s (n \cdot \nabla B) \|_{L^2}^2 = \int_{T^3} \Lambda^s \partial_t P u \cdot \Lambda^s (n \cdot \nabla B) \, dx \\
- \mu \int_{T^3} \Lambda^s \Delta P u \cdot \Lambda^s (n \cdot \nabla B) \, dx - \int_{T^3} \Lambda^s (P f_4) \cdot \Lambda^s (n \cdot \nabla B) \, dx.
\]  

(3.28)

By Hölder’s inequality,

\[
\int_{T^3} \Lambda^s \Delta P u \cdot \Lambda^s (n \cdot \nabla B) \, dx \leq \frac{1}{8} \| \Lambda^s (n \cdot \nabla B) \|_{L^2}^2 + C \| \Lambda^{s+2} u \|_{L^2}^2 ,
\]

\[
\int_{T^3} \Lambda^s (P f_4) \cdot \Lambda^s (n \cdot \nabla B) \, dx \leq \frac{1}{8} \| \Lambda^s (n \cdot \nabla B) \|_{L^2}^2 + C \| \Lambda^s f_4 \|_{L^2}^2 .
\]
Next we shift the time derivative in the first term on the right hand side of (3.28) and use the third equation in (3.9) to get

\[ \int_{T^3} \Lambda^s \partial_t \Pi u \cdot \Lambda^s (n \cdot \nabla B) \, dx \]

\[ = \frac{d}{dt} \int_{T^3} \Lambda^s \Pi u \cdot \Lambda^s (n \cdot \nabla B) \, dx - \int_{T^3} \Lambda^s \Pi u \cdot \Lambda^s (n \cdot \nabla \partial_t B) \, dx \]

\[ = \frac{d}{dt} \int_{T^3} \Lambda^s \Pi u \cdot \Lambda^s (n \cdot \nabla B) \, dx + \int_{T^3} \Lambda^s (n \cdot \nabla \Pi u) \cdot \Lambda^s \partial_t B \, dx \]

\[ = \frac{d}{dt} \int_{T^3} \Lambda^s \Pi u \cdot \Lambda^s (n \cdot \nabla B) \, dx + \int_{T^3} \Lambda^s (n \cdot \nabla \Pi u) \cdot \Lambda^s (n \cdot \nabla u) \, dx \]

\[ - \int_{T^3} \Lambda^s (n \cdot \nabla \Pi u) \cdot \Lambda^s (n \cdot \nabla u) \, dx + \int_{T^3} \Lambda^s (n \cdot \nabla \Pi u) \cdot \Lambda^s f_3 \, dx. \quad (3.29) \]

The last three terms in (3.29) can be bounded by

\[ \int_{T^3} \Lambda^s (n \cdot \nabla \Pi u) \cdot \Lambda^s f_3 \, dx \leq C \left( \| \Lambda^s f_3 \|^2_{L^2} + \| \Lambda^{s+1} u \|^2_{L^2} \right). \]

Collecting the estimates above, we can infer from (3.28) that

\[ \| \Lambda^s (n \cdot \nabla B) \|^2_{L^2} - \frac{d}{dt} \sum_{0 \leq s \leq r+3} \int_{T^3} \Lambda^s \Pi u \cdot \Lambda^s (n \cdot \nabla B) \, dx \]

\[ \leq C \left( \| \Lambda^{s+2} u \|^2_{L^2} + \| \Lambda^s f_3 \|^2_{L^2} + \| \Lambda^{s+4} f_4 \|^2_{L^2} \right). \quad (3.30) \]

In the following, we deal with the terms in \( f_3, f_4 \), respectively. At first, by Lemma 2.4,

\[ \| \Lambda^s (u \cdot \nabla B) \|^2_{L^2} \leq C \left( \| u \|_{H^s}^2 \| \nabla B \|_{H^r}^2 + \| u \|_{H^s}^2 \| \nabla B \|_{H^r}^2 \right) \]

\[ \leq C \left( \| u \|_{H^s}^2 \| B \|_{H^r}^2 + \| u \|_{H^s}^2 \| B \|_{H^r}^2 \right) \]

\[ \leq C \delta^2 \left( \| u \|_{H^s}^2 + \| B \|_{H^r}^2 \right). \quad (3.31) \]

Similarly,

\[ \| \Lambda^s (B \cdot \nabla u - B \text{div} u) \|^2_{L^2} \leq C \delta^2 \left( \| u \|_{H^3}^2 + \| B \|_{H^3}^2 \right). \]

Moreover, by Lemma 2.3,

\[ \| B \|_{H^3}^2 \leq C \| n \cdot \nabla B \|_{H^{r+3}}^2, \]

from which we get

\[ \| \Lambda^s f_3 \|^2_{L^2} \leq C \delta^2 \| u \|_{H^3}^2 + C \delta^2 \| n \cdot \nabla B \|_{H^{r+3}}^2. \quad (3.32) \]

We now deal with the terms in \( f_4 \). The term \( \| \Lambda^s (u \cdot \nabla u) \|^2_{L^2} \) can be bounded as in (3.31),

\[ \| \Lambda^s (u \cdot \nabla u) \|^2_{L^2} \leq C \delta^2 \| \Lambda^{s+1} u \|_{L^2}^2. \quad (3.33) \]
Thanks to Lemma 2.4 again,
\[
\|
\Lambda^s(B \cdot \nabla B + B \nabla B)\|_{L^2}^2 \leq C(\|B\|_{L^\infty}^2 \|\nabla B\|_{H^r}^2 + \|\nabla B\|_{L^\infty}^2 \|B\|_{H^r}^2)
\leq C(\|B\|_{H^2}^2 \|B\|_{H^{r+1}}^2 + \|\nabla B\|_{L^2}^2 \|B\|_{H^r}^2)
\leq C \|B\|_{H^{r+1}}^2 \|B\|_{H^3}^2
\leq C \delta^2 \|n \cdot \nabla B\|_{H^{r+3}}^2. \tag{3.34}
\]

The term \(\mathbb{P}(k(a) \nabla a) = 0\) since \(k(a) \nabla a\) can be written as a gradient. By Lemma 2.4,
\[
\|
\Lambda^s(I(a) \Delta u)\|_{L^2}^2 \leq C(\|I(a)\|_{H^r}^2 \|\Delta u\|_{H^r}^2 + \|\Delta u\|_{L^\infty}^2 \|I(a)\|_{H^r}^2)
\leq C(\|a\|_{H^3}^2 \|u\|_{H^{r+2}}^2 + \|u\|_{H^4}^2 \|a\|_{H^r}^2)
\leq C \delta^2 \|u\|_{H^{r+2}}^2 + C \delta^2 \|u\|_{H^4}^2. \tag{3.35}
\]

The term \(I(a) \nabla \text{div} u\) can be dealt with similarly. The last term in \(f_4\) can be bounded as in (3.23), (3.24) and (3.34) to get
\[
\|
\Lambda^s(I(a)n \nabla B)\|_{L^2}^2 \leq C(\|I(a)\|_{L^\infty}^2 \|n \nabla B\|_{H^r}^2 + \|n \nabla B\|_{L^\infty}^2 \|I(a)\|_{H^r}^2)
\leq C(\|a\|_{H^3}^2 \|n \nabla B\|_{H^r}^2 + \|B\|_{H^3}^2 \|a\|_{H^r}^2)
\leq C(\|a + n \cdot B - n \cdot B\|_{H^3}^2 \|B\|_{H^r}^2 \|\nabla B\|_{H^{r+3}}^2 \|a\|_{H^r}^2)
\leq C(\|a + n \cdot B\|_{H^3}^2 + \|B\|_{H^3}^2 \|B\|_{H^4}^2 \|n \cdot \nabla B\|_{H^{r+3}}^2 \|a\|_{H^4}^2)
\leq C \delta^2 \|a + n \cdot B\|_{H^{r+4}}^2 + C \delta^2 \|n \cdot \nabla B\|_{H^{r+3}}^2, \tag{3.36}
\]

and
\[
\|
\Lambda^s(I(a)(n \cdot \nabla B))\|_{L^2}^2 + \|\Lambda^s(I(a)(B \cdot \nabla B - B \nabla B))\|_{L^2}^2 \leq C \delta^2 \|n \cdot \nabla B\|_{H^{r+3}}^2. \tag{3.37}
\]

Combining (3.33), (3.34), (3.35) and (3.37) gives
\[
\|
\Lambda^s f_4\|_{L^2}^2 \leq C \delta^2 \|u\|_{H^{r+5}}^2 + C \delta^2 \|n \cdot \nabla B\|_{H^{r+3}}^2 + C \delta^2 \|a + n \cdot B\|_{H^{r+4}}^2. \tag{3.38}
\]

Inserting (3.32) and (3.38) in (3.30) and taking \(\delta\) small enough, we obtain (3.26). This proves Lemma 3.4.

3.5. The dissipation of the combined quantity \(a + n \cdot B\). The equations of \(a\) and \(B\) in (3.9) do not contain any dissipative or damping terms. But we do need these stabilizing effects in order to prove the desired stability results. This subsection explores the structure of (3.9) and discovers that the equation of the combined quantity
\[
a + n \cdot B
\]

and the equation of the gradient part \(\mathcal{Q} u\) of \(u\) form a system with smoothing and stabilizing effects. Combining the bound for \(a + n \cdot B\) and \(n \cdot \nabla B\) allows us to control \(a\).

We first derive the equation of
\[
d \overset{\text{def}}{=} a + n \cdot B \quad \text{and} \quad G \overset{\text{def}}{=} \mathcal{Q} u - \frac{1}{\nu} \Delta^{-1} \nabla d.
\]
It follows from the third equation in (3.9) that
\[
\partial_t (\mathbf{n} \cdot \mathbf{B}) = \mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n} - \mathbf{n} \cdot \mathbf{n} \text{div} \mathbf{u} + f_3 \cdot \mathbf{n}
\]
which, together with the equation of \(a\) in (3.9), gives
\[
\partial_t (a + \mathbf{n} \cdot \mathbf{B}) = \mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n} - (|\mathbf{n}|^2 + 1) \text{div} \mathbf{u} + f_1 + f_3 \cdot \mathbf{n}. \tag{3.39}
\]
Applying the operator \(\mathcal{Q}\) to the velocity equation in (3.9) yields
\[
\partial_t \mathcal{Q} \mathbf{u} - \nu \Delta \mathcal{Q} \mathbf{u} + \nabla a + \nabla (\mathbf{n} \cdot \mathbf{B}) = \mathcal{Q} f_4, \tag{3.40}
\]
with \(\nu \equiv \lambda + 2\mu\). By the definition of \(\mathcal{Q} = \nabla \Delta^{-1} \text{div}\), we note that
\[
\text{div} \mathbf{u} = \text{div} \mathcal{Q} \mathbf{u} = \text{div} \mathbf{G} + \frac{1}{\nu} d.
\]
(3.39) and (3.40) then yield
\[
\begin{cases}
\partial_t d + \frac{1}{\nu}(|\mathbf{n}|^2 + 1)d + (|\mathbf{n}|^2 + 1)\text{div} \mathbf{G} = \mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n} + f_1 + f_3 \cdot \mathbf{n}, \\
\partial_t \mathbf{G} - \nu \Delta \mathbf{G} = \frac{1}{\nu}(|\mathbf{n}|^2 + 1)\mathcal{Q} \mathbf{u} - \frac{1}{\nu} \Delta^{-1} \nabla (\mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n}) + \mathcal{Q} f_4 - \frac{1}{\nu} \Delta^{-1} \nabla (f_1 + f_3 \cdot \mathbf{n}).
\end{cases} \tag{3.41}
\]
On the one hand, for any \(m \geq 0\), applying \(\Lambda^m\) to the first equation in (3.41), and multiplying it by \(\Lambda^m d\) lead to
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^m d\|_{L^2}^2 + \frac{1}{\nu} (|\mathbf{n}|^2 + 1) \|\Lambda^m d\|_{L^2}^2 = \int_{T^3} \Lambda^m (\mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n}) \cdot \Lambda^m d \, dx \\
- (|\mathbf{n}|^2 + 1) \int_{T^3} \Lambda^m \text{div} \mathbf{G} \cdot \Lambda^m d \, dx + \int_{T^3} \Lambda^m (f_1 + f_3 \cdot \mathbf{n}) \cdot \Lambda^m d \, dx
\]
\[
\leq C (\|\Lambda^m \nabla \mathbf{u}\|_{L^2} \|\Lambda^m d\|_{L^2} + \|\Lambda^m \text{div} \mathbf{G}\|_{L^2} \|\Lambda^m d\|_{L^2} + \int_{T^3} \Lambda^m (f_1 + f_3 \cdot \mathbf{n}) \cdot \Lambda^m d \, dx)
\]
\[
\leq \frac{1}{8\nu} \|\Lambda^m d\|_{L^2}^2 + C (\|\Lambda^{m+1} \mathbf{u}\|_{L^2}^2 + \|\Lambda^{m+1} \mathbf{G}\|_{L^2}^2 + \|\Lambda^m f_1\|_{L^2}^2 + \|\Lambda^m f_3\|_{L^2}^2)
\]
from which we have
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^m d\|_{L^2}^2 + \frac{1}{\nu} (|\mathbf{n}|^2 + 1) \|\Lambda^m d\|_{L^2}^2
\]
\[
\leq C (\|\Lambda^{m+1} \mathbf{u}\|_{L^2}^2 + \|\Lambda^{m+1} \mathbf{G}\|_{L^2}^2 + \|\Lambda^m f_1\|_{L^2}^2 + \|\Lambda^m f_3\|_{L^2}^2). \tag{3.42}
\]
On the other hand, for the second equation in (3.41) and for any \(m \geq 0\), there holds similarly that
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^m \mathbf{G}\|_{L^2}^2 + \nu \|\Lambda^{m+1} \mathbf{G}\|_{L^2}^2
\]
\[
= \frac{1}{\nu} (|\mathbf{n}|^2 + 1) \int_{T^3} \Lambda^m \mathcal{Q} \mathbf{u} \cdot \Lambda^m \mathbf{G} \, dx - \frac{1}{\nu} \int_{T^3} \Lambda^m (\Delta^{-1} \nabla (\mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n})) \cdot \Lambda^m \mathbf{G} \, dx
\]
\[
+ \int_{T^3} \Lambda^m \mathcal{Q} f_4 \cdot \Lambda^m \mathbf{G} \, dx + \frac{1}{\nu} \int_{T^3} \Lambda^m \Delta^{-1} \nabla (f_1 + f_3 \cdot \mathbf{n}) \cdot \Lambda^m \mathbf{G} \, dx. \tag{3.43}
\]
For $m = 0$, we get by the Young inequality and the Poincaré inequality that

$$\frac{1}{2} \frac{d}{dt} \|G\|_{L^2}^2 + \nu \|
abla G\|_{L^2}^2 \leq \frac{1}{\nu} \left( |n|^2 + 1 \right) \int_{T^3} Q u \cdot G \, dx - \frac{1}{\nu} \int_{T^3} (\Delta^{-1} \nabla (n \cdot \nabla u \cdot n)) \cdot G \, dx + \int_{T^3} Q f_4 \cdot G \, dx + \frac{1}{\nu} \int_{T^3} \Delta^{-1} \nabla (f_1 + f_3 \cdot n) \cdot G \, dx \leq C \left( \|u\|_{L^2} + \|f_4\|_{L^2} + \|\Delta^{-1} \nabla (f_1 + f_3 \cdot n)\|_{L^2} \right) \|G\|_{L^2} \leq \frac{\nu}{2} \|
abla G\|_{L^2}^2 + C \left( \|u\|_{H^1}^2 + \|(f_1, f_3)\|_{H^{m-1}}^2 + \|f_4\|_{L^2}^2 \right). \tag{3.44}$$

For $1 \leq m \leq N$, we get by the integration by parts and the Young inequality that

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^m G\|_{L^2}^2 + \nu \|\Lambda^{m+1} G\|_{L^2}^2 \leq C \|\Lambda^{m-1} u\|_{L^2} \|\Lambda^{m+1} G\|_{L^2} + C \|\Lambda^{m-1} f_4\|_{L^2} \|\Lambda^{m+1} G\|_{L^2} + C \|\Lambda^{m-2} (f_1 + f_3 \cdot n)\|_{L^2} \|\Lambda^{m+1} G\|_{L^2} \leq \frac{\nu}{4} \|\Lambda^{m+1} G\|_{L^2}^2 + C \|\Lambda^{m-1} u\|_{L^2}^2 + C \|\Lambda^{m-2} f_1\|_{L^2}^2 + C \|\Lambda^{m-2} f_3\|_{L^2}^2 + C \|\Lambda^{m-1} f_4\|_{L^2}^2 \leq \frac{\nu}{4} \|\Lambda^{m+1} G\|_{L^2}^2 + C \|\Lambda^{m+1} u\|_{L^2}^2 + C \|\Lambda^{m-2} f_1\|_{L^2}^2 + C \|\Lambda^{m-2} f_3\|_{L^2}^2 + C \|\Lambda^{m-1} f_4\|_{L^2}^2$$

from which and (3.44), we have for any $0 \leq m \leq N$ that

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^m G\|_{L^2}^2 + \frac{\nu}{2} \|\Lambda^{m+1} G\|_{L^2}^2 \leq C \left( \|
abla u\|_{H^m}^2 + \|f_1\|_{H^m}^2 + \|f_3\|_{H^m}^2 + \|f_4\|_{H^{m-1}}^2 \right). \tag{3.45}$$

Multiplying (3.45) by a suitable large constant and then adding to (3.42), we get

$$\frac{d}{dt} \left( \|\Lambda^m d\|_{L^2}^2 + \|\Lambda^m G\|_{L^2}^2 \right) + \frac{1}{\nu} \|\Lambda^m d\|_{L^2}^2 + \nu \|\Lambda^{m+1} G\|_{L^2}^2 \leq C \left( \|
abla u\|_{H^m}^2 + \|f_1\|_{H^m}^2 + \|f_3\|_{H^m}^2 + \|f_4\|_{H^{m-1}}^2 \right). \tag{3.46}$$

Summing up $s$ from 0 to $m$ in (3.15) gives

$$\frac{d}{dt} \left( \|(a, u, B)\|_{H^m}^2 + \mu \|
abla u\|_{H^m}^2 + (\lambda + \mu) \|\text{div} u\|_{H^m}^2 \right) \leq C \left| \sum_{s=0}^{m} \int_{T^3} \Lambda^s f_1 \cdot A^s \nabla u \, dx \right| + C \left| \sum_{s=0}^{m} \int_{T^3} \Lambda^s f_4 \cdot A^s u \, dx \right| + C \left| \sum_{s=0}^{m} \int_{T^3} \Lambda^s f_3 \cdot A^s B \, dx \right|. \tag{3.47}$$
Multiplying (3.47) by a suitable large constant and then adding to (3.46) lead to
\[
\frac{d}{dt} \| (a, u, B, d, G) \|^2_{H^m} + \frac{1}{\nu} \| d \|^2_{H^m} + \mu \| \nabla u \|^2_{H^m} + (\lambda + \mu) \| \text{div} u \|^2_{H^m} + \nu \| \nabla G \|^2_{H^m} \\
\leq C \left( \| f_1 \|^2_{H^m} + \| f_3 \|^2_{H^m} + \| f_4 \|^2_{H^{m-1}} \right) + C \left| \sum_{s=0}^{n} \int_{T^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx \right| \\
+ C \left| \sum_{s=0}^{n} \int_{T^3} \Lambda^s f_3 \cdot \Lambda^s B \, dx \right| + C \left| \sum_{s=0}^{n} \int_{T^3} \Lambda^s f_4 \cdot \Lambda^s u \, dx \right| .
\] (3.48)

Thanks to the Young inequality and the Poincaré inequality, for \( s = 0 \), the last term in (3.48) can be bounded as
\[
\left| \int_{T^3} f_4 \cdot u \, dx \right| \leq \frac{\mu}{8} \| u \|^2_{L^2} + C \| f_4 \|^2_{L^2} \leq \frac{\mu}{8} \| \nabla u \|^2_{L^2} + C \| f_4 \|^2_{L^2} .
\]

Similarly, for \( 1 \leq s \leq m \), we have
\[
\left| \sum_{s=1}^{m} \int_{T^3} \Lambda^s f_4 \cdot \Lambda^s u \, dx \right| \leq \frac{\mu}{8} \| \nabla u \|^2_{H^m} + C \| f_2 \|^2_{H^{m-1}} .
\]

Inserting the above two inequalities into (3.48) gives
\[
\frac{d}{dt} \| (a, u, B, d, G) \|^2_{H^m} + \frac{1}{\nu} \| d \|^2_{H^m} + \mu \| \nabla u \|^2_{H^m} \\
+ (\lambda + \mu) \| \text{div} u \|^2_{H^m} + \nu \| \nabla G \|^2_{H^m} \\
\leq C \left( \| f_1 \|^2_{H^m} + \| f_3 \|^2_{H^m} + \| f_4 \|^2_{H^{m-1}} \right) \\
+ C \left| \sum_{s=0}^{n} \int_{T^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx \right| + C \left| \sum_{s=0}^{n} \int_{T^3} \Lambda^s f_3 \cdot \Lambda^s B \, dx \right| .
\] (3.49)

Taking \( m = r + 4 \) in (3.49) implies that
\[
\frac{1}{2} \frac{d}{dt} \| (a, u, B, d, G) \|^2_{H^{r+4}} + \frac{1}{\nu} \| d \|^2_{H^{r+4}} + \mu \| \nabla u \|^2_{H^{r+4}} \\
+ (\lambda + \mu) \| \text{div} u \|^2_{H^{r+4}} + \nu \| \nabla G \|^2_{H^{r+4}} \\
\leq C \left( \| f_1 \|^2_{H^{r+4}} + \| f_3 \|^2_{H^{r+4}} + \| f_4 \|^2_{H^{r+3}} \right) \\
+ C \left| \sum_{s=0}^{n} \int_{T^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx \right| + C \left| \sum_{s=0}^{n} \int_{T^3} \Lambda^s f_3 \cdot \Lambda^s B \, dx \right| .
\] (3.50)

Multiplying (3.50) by a suitable large constant \( \gamma \) and adding to (3.26) give rise to
\[
\frac{d}{dt} \left\{ \gamma \left( \| a \|^2_{H^{r+4}} + \| (d, u, B, G) \|^2_{H^{r+4}} \right) - \sum_{0 \leq s \leq r+3} \int_{T^3} \Lambda^s \text{P} u \cdot \Lambda^s (n \cdot \nabla B) \, dx \right\} \\
+ \| n \cdot \nabla B \|^2_{H^{r+3}} + \gamma \left( \frac{1}{\nu} \| d \|^2_{H^{r+4}} + \mu \| \nabla u \|^2_{H^{r+4}} + (\lambda + \mu) \| \text{div} u \|^2_{H^{r+4}} + \nu \| \nabla G \|^2_{H^{r+4}} \right) \\
\leq C \left( \| f_1 \|^2_{H^{r+4}} + \| f_3 \|^2_{H^{r+4}} + \| f_4 \|^2_{H^{r+3}} \right) \\
+ C \left| \sum_{s=0}^{n} \int_{T^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx \right| + C \left| \sum_{s=0}^{n} \int_{T^3} \Lambda^s f_3 \cdot \Lambda^s B \, dx \right| .
\] (3.51)
We now bound the terms on the right hand side of (3.51). By Lemma 2.4,
\[
\|f_1\|_{H^{r+4}}^2 \leq C(\|u\|_{H^{r+4}}^2 \|\nabla a\|_{H^{r+4}}^2 + \|a\|_{H^{r+4}}^2 \|\nabla u\|_{H^{r+4}}^2) \\
\leq C \|\nabla u\|_{H^{r+4}}^2 \|a\|_{H^N}^2 \\
\leq C \delta^2 \|\nabla u\|_{H^{r+4}}^2.
\]
Similarly,
\[
\|f_3\|_{H^{r+4}}^2 \leq C(\|u\|_{H^{r+4}}^2 \|\nabla B\|_{H^{r+4}}^2 + \|B\|_{H^{r+4}}^2 \|\nabla u\|_{H^{r+4}}^2) \\
\leq C(\|u\|_{H^{r+4}}^2 \|B\|_{H^N}^2 + \|B\|_{H^N}^2 \|\nabla u\|_{H^{r+4}}^2) \\
\leq C \|\nabla u\|_{H^{r+4}}^2 \|B\|_{H^N}^2 \\
\leq C \delta^2 \|\nabla u\|_{H^{r+4}}^2.\]

We turn to the term $\sum_{s=0}^{r+4} \int_{\mathbb{T}^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx$. By Lemma 2.5,
\[
\sum_{s=0}^{r+4} \int_{\mathbb{T}^3} (\Lambda^s (u \cdot \nabla a) - u \cdot \nabla \Lambda^s a) \cdot \Lambda^s a \, dx + \sum_{s=0}^{r+4} \int_{\mathbb{T}^3} u \cdot \nabla \Lambda^s a \cdot \Lambda^s a \, dx \\
\leq C \sum_{s=0}^{r+4} (\|\nabla u\|_{L^\infty} \|\Lambda^s a\|_{L^2} + \|\Lambda^s u\|_{L^2} \|\nabla a\|_{L^\infty}) \|a\|_{H^{r+4}} + C \|\nabla u\|_{L^\infty} \|a\|_{H^{r+4}}^2 \\
\leq C \|\nabla u\|_{H^{r+4}} \|a\|_{H^{r+4}}^2.
\]
By Lemma 2.4, there holds
\[
\sum_{s=0}^{r+4} \int_{\mathbb{T}^3} \Lambda^s (\text{div} u) \cdot \Lambda^s a \, dx \leq C \|\nabla u\|_{H^{r+4}} \|a\|_{H^{r+4}}^2.
\]
As a result, we have
\[
\left| \sum_{s=0}^{r+4} \int_{\mathbb{T}^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx \right| \leq C \|\nabla u\|_{H^{r+4}} \|a\|_{H^{r+4}}^2 \\
\leq \frac{\mu}{8} \|\nabla u\|_{H^{r+4}}^2 + C \|d\|_{H^{r+4}}^4 + C \|B\|_{H^{r+4}}^4 \\
\leq \frac{\mu}{8} \|\nabla u\|_{H^{r+4}}^2 + C \delta^2 \|d\|_{H^{r+4}}^2 + C \|B\|_{H^{r+4}}^4.
\]
Similarly, the last term in (3.51) can be bounded as
\[
\left| \sum_{s=0}^{r+4} \int_{\mathbb{T}^3} \Lambda^s f_3 \cdot \Lambda^s B \, dx \right| \leq \frac{\mu}{8} \|\nabla u\|_{H^{r+4}}^2 + C \|B\|_{H^{r+4}}^4.
\]
For any $N \geq 2r + 5$, from Lemma 2.3, we have
\[
\|B\|_{H^3} \leq C \|\n \cdot \nabla B\|_{H^{r+3}}, \quad \text{and} \quad \|B\|_{H^{r+4}}^2 \leq C \|B\|_{H^3} \|B\|_{H^N} \leq C \delta \|\n \cdot \nabla B\|_{H^{r+3}},
\]
which gives
\[
\|B\|_{H^{r+4}}^4 \leq C \delta^2 \|\n \cdot \nabla B\|_{H^{r+3}}^2.
\]

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As a result, we get
\[
\left| \sum_{s=0}^{r+4} \int_{T^3} \Lambda^s f_1 \cdot \Lambda^s a \, dx \right| + C \left| \sum_{s=0}^{r+4} \int_{T^3} \Lambda^s f_3 \cdot \Lambda^s B \, dx \right|
\leq \frac{\mu}{8} \| \nabla u \|_{H^{r+4}}^2 + C \delta^2 (\| n \cdot \nabla B \|_{H^{r+3}}^2 + \| d \|_{H^{r+4}}^2).
\] (3.58)

Finally we estimate \( \| f_4 \|_{H^{r+3}}^2 \) and start with the first term \( \| u \cdot \nabla u \|_{H^{r+3}}^2 \). By Lemma 2.4,
\[
\| u \cdot \nabla u \|_{H^{r+3}}^2 \leq C \| u \|_{H^{r+4}}^2 \| \nabla u \|_{H^{r+4}}^2
\leq C \| u \|_{H^N}^2 \| \nabla u \|_{H^{r+4}}^2
\leq C \delta^2 \| \nabla u \|_{H^{r+4}}^2.
\]

Similarly, by Lemma 2.3,
\[
\| B \cdot \nabla B + B \nabla B \|_{H^{r+3}}^2 \leq C \| B \|_{H^2}^2 \| \nabla B \|_{H^{r+3}}^2
\leq C \| n \cdot \nabla B \|_{H^{r+3}}^2 \| B \|_{H^N}^2
\leq C \delta^2 \| n \cdot \nabla B \|_{H^{r+3}}^2.
\]

With the aid of Lemma 2.4 again, we can deduce that
\[
\| k(a) \nabla a \|_{H^{r+3}}^2 \leq C \left( \| \nabla a \|_{H^{r+3}}^2 \| k(a) \|_{L^\infty}^2 + \| k(a) \|_{H^{r+3}}^2 \| \nabla a \|_{L^\infty}^2 \right)
\leq C \| a \|_{H^3}^2 \| d \|_{H^N}^2
\leq C \| d - n \cdot B \|_{H^3}^2 \| a \|_{H^N}^2
\leq C (\| d \|_{H^3}^2 + \| B \|_{H^3}^2) \| a \|_{H^N}^2
\leq C \delta^2 \| d \|_{H^{r+4}}^2 + C \delta^2 \| n \cdot \nabla B \|_{H^{r+3}}^2
\]

and
\[
\| I(a)(\mu \Delta u + (\lambda + \mu) \nabla \text{div} u) \|_{H^{r+3}}^2 \leq C \| a \|_{H^N}^2 \| \Delta u \|_{H^{r+3}}^2
\leq C \delta^2 \| \nabla u \|_{H^{r+4}}^2.
\]

By Lemmas 2.4 and 2.6, and (3.8),
\[
\| I(a)(n \cdot \nabla B - n \nabla B) \|_{H^{r+3}}^2 \leq C(\| I(a) \|_{L^\infty} \| \nabla B \|_{H^{r+3}}^2 + \| \nabla B \|_{L^\infty} \| I(a) \|_{H^{r+3}}^2)
\leq C (\| B \|_{H^N}^2 \| a \|_{H^3}^2 + \| B \|_{H^3}^2 \| a \|_{H^{r+4}}^2)
\leq C \delta^2 \| d - n \cdot B \|_{H^3}^2 + C \| n \cdot \nabla B \|_{H^{r+3}}^2 \| a \|_{H^N}^2
\leq C \delta^2 (\| d \|_{H^3}^2 + \| B \|_{H^3}^2) + C \| n \cdot \nabla B \|_{H^{r+3}}^2 \| a \|_{H^N}^2
\leq C \delta^2 \| d \|_{H^{r+4}}^2 + C \delta^2 \| n \cdot \nabla B \|_{H^{r+3}}^2. \] (3.59)

The last term in \( \| f_4 \|_{H^{r+3}}^2 \) can be dealt with similarly as (3.59). Collecting the estimates above yields
\[
\| f_4 \|_{H^{r+3}}^2 \leq C \delta^2 \| \nabla u \|_{H^{r+4}}^2 + C \delta^2 \| n \cdot \nabla B \|_{H^{r+3}}^2 + C \delta^2 \| d \|_{H^{r+4}}^2. \] (3.60)
Inserting (3.52), (3.53), (3.58) and (3.60) in (3.51) and taking $\delta > 0$ to be sufficiently small, we obtain

$$
\frac{d}{dt} \left\{ \gamma \left( \| (a, u, B, d, G) \|^2_{H^{r+4}} \right) - \sum_{0 \leq s \leq r+3} \int_{T^3} \Lambda^s \mathbb{P} u \cdot \Lambda^s (n \cdot \nabla B) \, dx \right\} + \| n \cdot \nabla B \|^2_{H^{r+3}} \\
+ \gamma \left( \frac{1}{V} \| d \|^2_{H^{r+4}} + \mu \| \nabla u \|^2_{H^{r+4}} + (\lambda + \mu) \| \text{div} u \|^2_{H^{r+4}} + v \| \nabla G \|^2_{H^{r+4}} \right) \leq 0.
$$

(3.61)

3.6. Completing the proof of Theorem 1.1. This subsection finishes the bootstrapping argument and thus completes the proof of Theorem 1.1. Let $\gamma > 1$. We set

$$
\mathcal{E}(t) = \gamma \left( \| a \|^2_{H^{r+4}} + \| (d, u, B, G) \|^2_{H^{r+4}} \right) - \sum_{0 \leq s \leq r+3} \int_{T^3} \Lambda^s \mathbb{P} u \cdot \Lambda^s (n \cdot \nabla B) \, dx,
$$

and

$$
\mathcal{D}(t) = \gamma \left( \frac{1}{V} \| d \|^2_{H^{r+4}} + \mu \| \nabla u \|^2_{H^{r+4}} + (\lambda + \mu) \| \text{div} u \|^2_{H^{r+4}} + v \| \nabla G \|^2_{H^{r+4}} \right) \\
+ \| n \cdot \nabla B \|^2_{H^{r+3}}.
$$

(3.61) then becomes

$$
\frac{d}{dt} \mathcal{E}(t) + \frac{1}{2} \mathcal{D}(t) \leq 0.
$$

(3.62)

Clearly, for $\gamma > 1$,

$$
\mathcal{E}(t) \geq \| (d, u, B, G) \|^2_{H^{r+4}}.
$$

For any $N \geq 4r + 7$, we invoke the interpolation inequality

$$
\| B \|^2_{H^{r+4}} \leq \| B \|^3_{H^{r+3}} \| B \|^2_{H^N} \leq C \delta^\frac{1}{2} \| n \cdot \nabla B \|^3_{H^{r+3}}
$$

to obtain

$$
\mathcal{E}(t) \leq C \left( \| d \|^2_{H^{r+4}} + \| (u, G) \|^2_{H^{r+4}} + \| B \|^2_{H^{r+4}} \right) \\
\leq C \| d \|^3_{H^{r+4}} \| d \|_{H^{r+4}} + C \| (u, G) \|^3_{H^{r+3}} \| (u, G) \|_{H^N} + C \| B \|^3_{H^{r+3}} \| B \|_{H^N} \\
\leq C \delta^\frac{1}{2} \| d \|^2_{H^{r+4}} + C \delta^\frac{1}{2} \| (u, G) \|^2_{H^{r+3}} + C \| (u, G) \|_{H^N} + C \| B \|^2_{H^{r+3}} \| B \|_{H^N} \\
\leq C \delta^\frac{1}{2} \| d \|^2_{H^{r+4}} + C \delta^\frac{1}{2} \| (u, G) \|^2_{H^{r+3}} + C \| (u, G) \|_{H^N} + C \delta^\frac{1}{2} \| n \cdot \nabla B \|^2_{H^{r+3}}.
$$

Inserting this inequality in (3.62) gives

$$
\frac{d}{dt} \mathcal{E}(t) + c(\mathcal{E}(t))^\frac{1}{2} \leq 0.
$$

It then follows easily that

$$
\mathcal{E}(t) \leq C(1+t)^{-3}.
$$

(3.63)

Taking $\ell = N$ in (3.10) and using Sobolev’s inequalities, we have

$$
\frac{d}{dt} \| (a, u, B) \|^2_{H^N} + \mu \| \nabla u \|^2_{H^N} + (\lambda + \mu) \| \text{div} u \|^2_{H^N} \leq CZ(t) \| (a, u, B) \|^2_{H^N}
$$

(3.64)
with
\[
Z(t) \overset{\text{def}}{=} \| (d, u, B) \|_{H^3} + \| (d, u, B) \|_{H^3}^2 + \| d \|_{H^3}^2 \| B \|_{H^3}^2 + \| B \|_{H^3}^4 .
\]
Clearly, the decay upper bound in (3.63) implies
\[
\int_0^t Z(\tau) d\tau \leq C.
\]
Grönwall’s inequality applied to (3.64) implies
\[
\| (a, u, B) \|_{H^N}^2 \leq C \| (a_0, u_0, B_0) \|_{H^N}^2 \leq Ce^2.
\]
By taking \( \varepsilon \) to be sufficiently small, say \( \sqrt{C} \varepsilon \leq \delta/2 \), we obtain
\[
\| (a, u, B) \|_{H^N} \leq \frac{\delta}{2}.
\]
The bootstrapping argument then implies that the local solution can be extended as a global one in time. Finally we prove the decay rate in (1.5). By (3.63),
\[
\| a(t) \|_{H^{r+4}} + \| u(t) \|_{H^{r+4}} + \| B(t) \|_{H^{r+4}} \leq C \left( 1 + t \right)^{-\frac{3}{2}}.
\] (3.65)
(1.5) is a consequence of (3.65) and the interpolation inequality, for any \( r + 4 \leq \beta < N \),
\[
\| f(t) \|_{H^\beta} \leq \| f(t) \|_{H^{r+4}} \| f(t) \|_{H^{r-4}}^{\frac{N-\beta}{r+4}} \| f(t) \|_{H^N}^{\frac{\beta-r-4}{r+4}} .
\]
This completes the proof of Theorem 1.1. □

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