CHANGE OF SCALE FORMULAS FOR A GENERALIZED CONDITIONAL WIENER INTEGRAL

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1. Introduction

Let $C[0,t]$ denote the space of real-valued continuous functions on $[0,t]$ and define a random vector $Z_n : C[0,t] \to \mathbb{R}^n$ by $Z_n(x) = (\int_0^{t_1} h(s)ds, \ldots, \int_0^{t_n} h(s)ds)$, where $0 < t_1 < \cdots < t_n = t$ is a partition of $[0,t]$ and $h \in L_2[0,t]$ with $h \neq 0$ a.e. Using a simple formula for a conditional expectation on $C[0,t]$ with $Z_n$, we evaluate a generalized analytic conditional Wiener integral of the function $G_r(x) = F(x)\Psi(\int_0^{t_1} v_1(s)ds, \ldots, \int_0^{t_r} v_r(s)ds)$ for $F$ in a Banach algebra and for $\Psi = f + \phi$ which need not be bounded or continuous, where $f \in L_p(\mathbb{R}^r)$ ($1 \leq p \leq \infty$), $\{v_1, \ldots, v_r\}$ is an orthonormal subset of $L_2[0,t]$ and $\phi$ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^r$. Finally we establish various change of scale transformations for the generalized analytic conditional Wiener integrals of $G_r$ with the conditioning function $Z_n$.
transform of a measure of bounded variation over $\mathbb{R}^n$. Furthermore the author and his coauthors [6, 8, 11] introduced various kinds of the change of scale formulas for the conditional Wiener integrals of the function of the form $F_1$ defined on $C_0[0, t]$, $C_0(\mathbb{R})$, the infinite dimensional Wiener space and $C[0, t]$, an analogue of Wiener space [9] which is the space of real-valued continuous paths on $[0, t]$.

Let $h \in L_2[0, t]$ with $h \neq 0$ a.e. on $[0, t]$. Define a stochastic process $Z : C[0, t] \times [0, t] \rightarrow \mathbb{R}$ by $Z(x, s) = \int_0^t h(u)dx(u)$ for $x \in C[0, t]$ and $s \in [0, t]$, where the integral denotes the Paley-Wiener-Zygmund integral, and let

$$Z_n(x) = (Z(x, t_1), \ldots, Z(x, t_n)).$$

On the space $C[0, t]$ the author [7] derived a simple formula for a generalized conditional Wiener integral given the vector-valued conditioning function $Z_n$.

Using the simple formula on $C[0, t]$ with the conditioning function $Z_n$, we evaluate a generalized analytic conditional Wiener integral of the function $G_r$ having the form

$$G_r(x) = F(x)\Psi\left(\int_0^t v_1(s)dx(s), \ldots, \int_0^t v_r(s)dx(s)\right)$$

for $F$ in a Banach algebra which corresponds to the Cameron-Storvick’s Banach algebra $S$ [4] and for $\Psi = f + \phi$ which need not be bounded or continuous, where $f \in L_p(\mathbb{R}^r)(1 \leq p \leq \infty)$, $\{v_1, \ldots, v_r\}$ is an orthonormal subset of $L_2[0, t]$ and $\phi$ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^r$. Finally we establish various kinds of new change of scale transformations for the generalized analytic conditional Wiener integral of $G_r$ with the conditioning function $Z_n$. We note that the results of this paper are different from those in [6, 8, 11].

2. A generalized conditional Wiener integral

Let $C$, $C_+$ and $C_-$ denote the sets of complex numbers, complex numbers with positive real parts and nonzero complex numbers with nonnegative real parts, respectively.

Let $(C[0, t], B(C[0, t]), w_\varphi)$ be the analogue of Wiener space associated with a probability measure $\varphi$ on the Borel class of $\mathbb{R}$, where $B(C[0, t])$ denotes the Borel class of $C[0, t]$ [9]. For $v \in L_2[0, t]$ and $x \in C[0, t]$ let $(v, x) = \int_0^t v(s)dx(s)$ denote the Paley-Wiener-Zygmund integral of $v$ according to $x$. The inner product on the real Hilbert space $L_2[0, t]$ is denoted by $\langle \cdot, \cdot \rangle$. Furthermore the dot product on the $r$-dimensional Euclidean space $\mathbb{R}^r$ is also denoted by $\langle \cdot, \cdot \rangle_{\mathbb{R}^r}$.

Let $F : C[0, t] \rightarrow \mathbb{C}$ be integrable and let $X$ be a random vector on $C[0, t]$. Then we have the conditional expectation $E[F|X]$ given $X$ from a well-known probability theory. Furthermore there exists a $P_X$-integrable function $\psi$ on the value space of $X$ such that $E[F|X](x) = (\psi \circ X)(x)$ for $w_\varphi$-a.e. $x \in C[0, t]$, where $P_X$ is the probability distribution of $X$. The function $\psi$ is called the conditional Wiener $w_\varphi$-integral of $F$ given $X$ and it is also denoted by $E[F|X]$. 
Let 0 = t_0 < t_1 < \cdots < t_n = t be a partition of [0,t], where n is a positive integer. Let h ∈ L_2[0,t] be of bounded variation with h \neq 0 a.e. For j = 1, \ldots, n let α_j = \|x(t_j-1,t_j)\|^{\frac{1}{2}}(t_j-1,t_j)h and let V be the subspace of L_2[0,t] generated by \{α_1, \ldots, α_n\}. Let V^⊥ be the orthogonal complement of V. Let \mathcal{P} : L_2[0,t] → V be the orthogonal projection given by

\[ \mathcal{P}v = \sum_{j=1}^{n} \langle v, α_j \rangle α_j \]

and \mathcal{P}^⊥ : L_2[0,t] → V^⊥ be the orthogonal projection. For x ∈ C[0,t] define the stochastic integral by

\[ Z(x,s) = \int_0^s h(u)dx(u), \quad 0 \leq s \leq t \]

and let Z_n : C[0,t] → \mathbb{R}^n be given by

(1) \[ Z_n(x) = (Z(x,t_1), \ldots, Z(x,t_n)). \]

Let b(s) = \int_0^s (h(u))^2du and for x ∈ C[0,t] define the polygonal function \([Z(x,·)]_b\) of \(Z(x,·)\) by

(2) \[ [Z(x,·)]_b(s) = \sum_{j=1}^{n} \chi_{(t_{j-1},t_j)}(s) \left( Z(x,t_{j-1}) + \frac{b(s) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} \right) \]

for \(s \in [0,t]\), where \(\chi_{(t_{j-1},t_j)}\) denotes the indicator function on the interval \((t_{j-1},t_j)\). Similarly for \(ξ = (ξ_1, \ldots, ξ_n) ∈ \mathbb{R}^n\) the polygonal function \([ξ]_b\) of \(ξ\) is given by (2) replacing \(Z(x,t_j)\) by \(ξ_j\) \((j = 1, \ldots, n)\) with \(ξ_0 = 0\). For a function \(F : C[0,t] → \mathbb{C}\) such that \(F(Z(x,·))\) is integrable over \(x\), we have by Theorem 2.12 in [7]

(3) \[ E[F(Z(x,·))|Z_n]\langle ξ\rangle = E[F(Z(x,·) - [Z(x,·)]_b + [ξ]_b)] \]

for \(P_{Z_n}\)-a.e. \(ξ ∈ \mathbb{R}^n\) (for a.e. \(\xi ∈ \mathbb{R}^n\)), where \(P_{Z_n}\) is the probability distribution of \(Z_n\) on the Borel class of \(\mathbb{R}^n\). For \(λ > 0\) let \(F^λ(x) = F(λ^{-\frac{1}{2}}Z(x,·))\) and \(Z^λ_n(x) = Z_n(λ^{-\frac{1}{2}}x)\) for \(x ∈ C[0,t]\), where \(Z_n\) is given by (1). Suppose that \(E[F^λ_Z]\) exists. By the definition of the conditional Wiener \(w_{\lambda}-\text{integral}\) and (3)

(4) \[ E[F^λ_Z|Z_n]\langle ξ\rangle = E[F(λ^{-\frac{1}{2}}(Z(x,·) - [Z(x,·)]_b + [ξ]_b))] \]

for \(P_{Z_n}\)-a.e. \(ξ ∈ \mathbb{R}^n\), where \(P_{Z^λ_n}\) is the probability distribution of \(Z^λ_n\) on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\). Let \(F^λ_Z(ξ)\) be the right-hand side of (4). If \(F^λ_Z(ξ)\) has the analytic extension \(J^λ_Z(ξ)\) on \(\mathbb{C}_+\), then it is called the conditional analytic Wiener \(w_{\lambda}-\text{integral}\) of \(F_Z\) given \(Z_n\) with the parameter \(λ\) and denoted by

\[ E_{\text{analyt}}[F_Z|Z_n]\langle ξ\rangle = J^λ_Z(ξ) \]
for $\bar{\xi} \in \mathbb{R}^n$. Moreover if for nonzero real $q$, $E^{anwx}[F_Z|Z_n](\bar{\xi})$ has the limit as $\lambda$ approaches $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman $w_\omega$-integral of $F_Z$ given $Z_n$ with the parameter $q$ and denoted by

$$E^{anw}[F_Z|Z_n](\bar{\xi}) = \lim_{\lambda \to -iq} E^{anwx}[F_Z|Z_n](\bar{\xi}).$$

**Lemma 2.1.** Let $v \in L_2[0,t]$. Then for $w_\omega$-a.e. $x \in C[0,t]$

$$(v,[Z(x,\cdot)]_h) = (P(vh),x).$$

**Proof.** By the definition of the Paley-Wiener-Zygmund integral

$$(v,[Z(x,\cdot)]_h)$$

$$= \sum_{j=1}^{n} Z(x,t_j) - Z(x,t_{j-1}) \int_{t_{j-1}}^{t_j} v(s)db(s)$$

$$= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} v(s)(h(s))^2ds$$

$$= \sum_{j=1}^{n} \chi_{[t_{j-1},t_j]}(h)^2 \left( \int_{0}^{t_j} h(s)dx(s) - \int_{0}^{t_{j-1}} h(s)dx(s) \right)$$

$$= \sum_{j=1}^{n} \langle vh, \alpha_j \rangle \langle \alpha_j, x \rangle = (P(vh),x)$$

which completes the proof. \qed

### 3. Generalized analytic conditional Feynman integrals

Throughout this paper let $h \in L_2[0,t]$ be of bounded variation with $h \neq 0$ a.e. and $\{v_1,v_2,\ldots,v_r\}$ be an orthonormal subset of $L_2[0,t]$ such that $\{P^\perp(hv_1),\ldots,P^\perp(hv_r)\}$ is an independent set. Let

$$\{e_1,\ldots,e_r\}$$

be the orthonormal set obtained from $\{P^\perp(hv_1),\ldots,P^\perp(hv_r)\}$ by the Gram-Schmidt orthonormalization process. Now for $l = 1,\ldots,r$ let $P^\perp(hv_l) = \sum_{j=1}^{r} \alpha_{lj} e_j$ be the linear combinations of the $e_j$s and let

$$A = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr}
\end{bmatrix}$$

be the coefficient matrix of the combinations. We can also regard $A$ as the linear transformation $T_A : \mathbb{R}^r \to \mathbb{R}^r$ given by $T_A(z) = \bar{z}A$, where $\bar{z}$ is an arbitrary row-vector in $\mathbb{R}^r$. We note that $A$ is invertible so that $T_A$ is an isomorphism.

**Remark 3.1.** An example of $h$ and $\{v_1,\ldots,v_r\}$ satisfying the above conditions can be obtained by the following process. Let

$$h(s) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j]}(s) \frac{2(-1)^j}{t_j-t_{j-1}} \left( s - \frac{t_{j-1} + t_j}{2} \right) + \chi_{[0]}(s)$$
and for \( l = 1, \ldots, r \) let
\[
h_l(s) = \sum_{j=1}^{n} \chi(t_{j-1}, t_j)(s) \left( \frac{(-1)^j 2^{2l-1}}{\left( t_j - t_{j-1} \right)^{2l-1}} \left( s - \frac{t_{j-1} + t_j}{2} \right)^{2l-1} \right) + \chi(0)(s)
\]
for \( s \in [0, t] \). For a.e. \( s \in [0, t] \) let \( \sum_{j=1}^{r} c_j h_l(s) = 0 \). Fix \( k \in \{1, \ldots, n\} \) and take distinct points \( a_1, \ldots, a_r \) in \( \left( \frac{t_{k-1} + t_k}{2}, t_k \right) \) satisfying the above equality.

Let \( b_m = \frac{2}{t_k - t_{k-1}} a_m - \frac{t_k + t_{k-1}}{t_k - t_{k-1}} \) for \( m = 1, \ldots, r \). Replacing \( s \) by \( a_m \) we have the linear equation system with unknowns \( c_1, \ldots, c_r \); \( \sum_{j=1}^{r} b_m^{2l-1} c_l = 0 \) for \( m = 1, \ldots, r \).

The determinant of the coefficient matrix is given by
\[
\begin{vmatrix}
    b_1 & b_1^2 & \cdots & b_1^{2r-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_r & b_r^2 & \cdots & b_r^{2r-1}
\end{vmatrix}
= \left( \prod_{m=1}^{r} b_m \right) \prod_{1 \leq j < k \leq r} (b_j - b_k^2) \neq 0
\]
so that \( c_1 = \cdots = c_r = 0 \), which shows that \( \{h_1, \ldots, h_r\} \) is an independent set. Let \( \{v_1, \ldots, v_r\} \) be the orthonormal set obtained from \( \{h_1, \ldots, h_r\} \) by the Gram-Schmidt orthonormalization process. Now let \( v_l = \sum_{j=1}^{r} \beta_{lj} h_j \) for \( l = 1, \ldots, r \). Then we have
\[
\mathcal{P}^\perp(hv_l) = \sum_{j=1}^{r} \beta_{lj} hh_j - \sum_{j=1}^{r} \sum_{k=1}^{n} \beta_{lj} (hh_j, \alpha_k) \alpha_k.
\]

We note that
\[
\langle hh_j, \alpha_k \rangle = \frac{1}{\|\chi(t_{k-1}, t_k)\|} \int_{t_{k-1}}^{t_k} (h(s))^2 h_j(s) ds = 0
\]
so that for a.e. \( s \in [0, t] \)
\[
\mathcal{P}^\perp(hv_l)(s) = \sum_{j=1}^{r} \sum_{p=1}^{n} \beta_{lj} \chi(t_{p-1}, t_p)(s) \left( \frac{2^{2j}}{(t_p - t_{p-1})^{2j}} \left( s - \frac{t_{p-1} + t_p}{2} \right)^{2j} \right) + \sum_{j=1}^{r} \beta_{lj} \chi(0)(s).
\]
To prove the independence of \( \{\mathcal{P}^\perp(hv_l) : l = 1, \ldots, r\} \) let
\[
\sum_{l=1}^{r} c_l^2(\mathcal{P}^\perp(hv_l))(s) = 0 \quad \text{for a.e.} \quad s \in [0, t].
\]
Fix \( \rho \in \{1, \ldots, n\} \) and take distinct points \( a'_1, \ldots, a'_r \) in \( \left( \frac{t_{\rho-1} + t_\rho}{2}, t_\rho \right) \) satisfying the above two equalities. Let \( b'_m = \frac{2}{t_\rho - t_{\rho-1}} a'_m - \frac{t_\rho + t_{\rho-1}}{t_\rho - t_{\rho-1}} \) for \( m = 1, \ldots, r \). Replacing \( s \) by \( a'_m \) we have \( \sum_{j=1}^{r} (\sum_{j=1}^{r} \beta_{lj}(b'_m)^{2j}) c'_l = 0 \) for \( m = 1, \ldots, r \). The determinant of the coefficient matrix is given by
\[
\begin{vmatrix}
    \sum_{j=1}^{r} (b'_1)^{2j} \beta_{1j} & \cdots & \sum_{j=1}^{r} (b'_1)^{2j} \beta_{rj} \\
    \vdots & \ddots & \vdots \\
    \sum_{j=1}^{r} (b'_r)^{2j} \beta_{1j} & \cdots & \sum_{j=1}^{r} (b'_r)^{2j} \beta_{rj}
\end{vmatrix}
\]
Let $\tilde{M}(\mathbb{R}^r)$ be the space of all functions $\phi$ on $\mathbb{R}^r$ defined by

\[(7) \quad \phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i(\vec{u}, \vec{z})\} d\rho(\vec{z}),\]

where $\rho$ is a complex Borel measure of bounded variation over $\mathbb{R}^r$. Let

\[\mathcal{M}(L_2[0,t])\]

be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L_2[0,t]$ and let $\mathcal{S}_{w_\nu}$ be the space of all functions $F$ which for $\sigma \in \mathcal{M}(L_2[0,t])$ have the form

\[(8) \quad F(x) = \int_{L_2[0,t]} \exp\{i(v, x)\} d\sigma(v)\]

for $w_\nu$-a.e. $x \in C[0,t]$. We note that $\mathcal{S}_{w_\nu}$ is a Banach algebra [4, 9].

Let $(\vec{v}, x) = ((v_1, x), \ldots, (v_r, x))$ and $(hv, x) = ((hv_1, x), \ldots, (hv_r, x))$ for $x \in C[0,t]$. For a complete orthonormal basis $\{e_1, \ldots, e_r, e_{r+1}, \ldots\}$ containing (5) and $v \in L_2[0,t]$ let

\[(9) \quad c_j(v) = \langle v, e_j \rangle \quad \text{for } j = 1, \ldots, r, r + 1, \ldots\]

**Theorem 3.2.** Let $\Psi(x) = \phi(\vec{v}, x) F(x)$, where $\phi$ and $F$ are given by (7) and (8), respectively. For $\lambda \in \mathbb{C}_+$, $v \in L_2[0,t]$, $\vec{\xi} \in \mathbb{R}^n$ and $\vec{\zeta} \in \mathbb{R}^r$ let

\[(10) \quad A_1(\vec{\xi}, v, \vec{\zeta}) = \exp\{i\langle v, [\vec{\xi}]_h \rangle + i\langle \vec{v}, [\vec{\xi}]_b \rangle + \langle \vec{\zeta}, \vec{\zeta} \rangle \}

and

\[(11) \quad A_2(\lambda, v, \vec{\zeta}) = \exp\left\{-\frac{1}{2\lambda} \left[\|\mathcal{P}^+(hv)\|^2 - \|\vec{c}(\mathcal{P}^+(hv))\|_{\mathbb{R}^r}^2 + \|\vec{e}(\mathcal{P}^+(hv))\|_{\mathbb{R}^r} + T_\lambda \vec{e}\right]\}

where $\vec{c} = (c_1, \ldots, c_r)$ and the $c_j$s are given by (9). Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$

\[E^{\text{ann}}[\Psi Z_n](\vec{\zeta}) = \int_{L_2(0,t)} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{\zeta}) A_2(\lambda, v, \vec{\zeta}) d\rho(\vec{\zeta}) d\sigma(v).\]

Moreover for a nonzero real $q$, $E^{\text{ann}}[\Psi Z_n](\vec{\zeta})$ is given by the right hand side of the above equality replacing $\lambda$ by $-iq$. 
By the Morera’s theorem and the dominated convergence theorem we have the following form

By the same process as used in the proof of Theorem 2.6 in [8] we can obtain

where for a functional \( g : L_2[0,t] \rightarrow L_2[0,t] \)

\( (g(hv), x) = ((g(hv_1), x), \ldots, (g(hv_r), x)). \)

By the same process as used in the proof of Theorem 2.6 in [8] we can obtain

By the Morera’s theorem and the dominated convergence theorem we have the theorem. \( \square \)

For \( 1 \leq p \leq \infty \) let \( A^{(p)} \) be the space of the cylinder functions having the following form

(12) \( F_r(x) = f(\bar{v}, x) \)

for \( w_\varphi \)-a.e. \( x \in C[0,t] \), where \( f \in L_p(\mathbb{R}^r) \). Without loss of generality we can take \( f \) to be Borel measurable.

**Theorem 3.3.** Let \( 1 \leq p \leq \infty \) and \( F_r \in A^{(p)} \) be given by (12). Then for \( \lambda \in \mathbb{C}_+ \) we have

\[
E^{aw\lambda}[(F_r)z|Z_n](\xi) = \left( \frac{1}{2\pi} \right)^\frac{p}{2} \int_{\mathbb{R}^r} f(uA^T + (\bar{v}, [\xi]_b)) \exp \left\{ -\frac{\lambda}{2} ||u||_{L^2}^2 \right\} du
\]

for a.e. \( \xi \in \mathbb{R}^n \), where \( A^T \) is the transpose of \( A \) given by (6). Furthermore if \( p = 1 \), then for a non-zero real \( q \) \( E^{aw\lambda}[(F_r)z|Z_n](\xi) \) is given by the right hand side of the above equality replacing \( \lambda \) by \(-iq\).

**Proof.** By the same process as used in the proof of Theorem 3.1 in [8]

\[
I^{(\lambda)}_{(F_r),x}(\xi) = E[F_r(\lambda^{-\frac{1}{2}}(Z(x, \cdot) - [Z(x, \cdot)]_b) + [\xi]_b)]
\]

\[
= \int_{C[0,t]} f(\lambda^{-\frac{1}{2}}(P^+(hv), x) + (\bar{v}, [\xi]_b))dw_\varphi(x)
\]

\[
= \left( \frac{1}{2\pi} \right)^\frac{p}{2} \int_{\mathbb{R}^r} f(uA^T + (\bar{v}, [\xi]_b)) \exp \left\{ -\frac{\lambda}{2} ||u||_{L^2}^2 \right\} du
\]
for $\lambda > 0$ and a.e. $\xi \in \mathbb{R}^n$. By the Morera’s theorem we have the first part of the theorem. If $p = 1$, then the final result follows from the dominated convergence theorem. □

**Theorem 3.4.** Let $G_r = F F_r$, where $F \in S_{w,\phi}$ and $F_r \in A_r(1 \leq p \leq \infty)$ are given by (8) and (12), respectively. For $\lambda \in \mathbb{C}^n$, $v \in L^2[0,t]$ and $\bar{u} \in \mathbb{R}^r$ let

$$
E_{\text{anw}}\lambda [G_r] Z_n(\xi) = E_{\text{anw}}\lambda [G_r] Z_n(\xi) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \int_{L^2[0,t]} \exp\{i(v, [\xi_x])\} \int_{\mathbb{R}^r} f(\bar{u} A^T + (v, [\xi_x])) A_3(\lambda, v, \bar{u}) d\bar{u} d\sigma(v),
$$

where $A_3(\lambda, v, \bar{u})$ and $A_3(\xi_x) b$ are given by (9). Then we have for $\lambda \in \mathbb{C}^n$ and a.e. $\xi \in \mathbb{R}^n$

$$
E_{\text{anw}}\lambda [G_r] Z_n(\xi) = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \int_{L^2[0,t]} \exp\{i(v, [\xi_x])\} \int_{\mathbb{R}^r} f(\bar{u} A^T + (v, [\xi_x])) A_3(\lambda, v, \bar{u}) d\bar{u} d\sigma(v),
$$

where $A^T$ is the transpose of $A$ given by (6). Furthermore if $p = 1$, then for a real $q E_{\text{anw}}\lambda [G_r] Z_n(\xi)$ is given by the right hand side of the above equality replacing $\lambda$ by $-iq$.

**Proof.** By the same process as used in the proof of Theorem 3.3 in [8]

$$
I_{(G_r)\xi}(\xi) = E[G_r, \lambda^{-\frac{1}{2}}(Z(x, \cdot) - [Z(x, \cdot)]) b + [\xi_x])]
$$

$$
= \int_{L^2[0,t]} \exp\{i(v, [\xi_x])\} \int_{\mathbb{C}[0,t]} \exp\{i\lambda^{\frac{1}{2}}(P^+(vh), x)\} f(\lambda^{-\frac{1}{2}}(v, [\xi_x])) A_3(\lambda, v, \bar{u}) d\bar{u} d\sigma(v)
$$

for $\lambda > 0$ and a.e. $\xi \in \mathbb{R}^n$. By the Morera’s theorem we have the first part of the theorem. If $p = 1$, then the final result follows from the dominated convergence theorem. □

From Theorems 3.2 and 3.4 we have the following corollary by the linealities of the generalized conditional Wiener and Feynman integrals on the analogue of Wiener space.

**Corollary 3.5.** Let $\phi$, $F$ and $F_r \in A_r(1 \leq p \leq \infty)$ be given by (7), (8) and (12), respectively. Furthermore let $q$ be a nonzero real number. Then
For nonzero real number, then $K$ where $r$ is given by 

\[ E^{\text{anw}}[(\phi(\tilde{v}, \cdot) + F_r)F_z | Z_n](\xi) \] 

exists for $\lambda \in \mathbb{C}_+$ and a.e. $\xi \in \mathbb{R}^n$, and it is given by 

\[
E^{\text{anw}}[(\phi(\tilde{v}, \cdot) + F_r)F_z | Z_n](\xi) = \int_{L^2[0,t]} \int_{\mathbb{R}^r} A_1(\tilde{\xi}, v, \tilde{z}) A_2(\lambda, v, \tilde{z}) d\rho(\tilde{z}) + \exp\{i(v, [\xi]_b)\} 
\times \left(\frac{\lambda}{2\pi}\right)^{\frac{\tilde{v}}{2}} \int_{\mathbb{R}^r} f(\tilde{v}, [\xi]_b) A_3(\lambda, v, \tilde{u}) d\tilde{u} d\sigma(v),
\]

where $A_1$, $A_2$ and $A_3$ are given by (10), (11) and (13), respectively. In particular if $F_r \in A^{(1)}$, then $E^{\text{anw}}[(\phi(\tilde{v}, \cdot) + F_r)F_z | Z_n](\xi)$ exists for a.e. $\xi \in \mathbb{R}^n$ and it is obtained with replacing $\lambda$ by $-iq$ in the right-hand side of the above equality.

4. A change of scale formula using the polygonal function

In this section we derive change of scale formulas for the generalized conditional Wiener integrals of unbounded functions on the analogue of Wiener space using the polygonal function.

Let $\{e_j : j = 1, 2, \ldots\}$ be a complete orthonormal basis for $L^2[0, t]$ containing $\{e_1, \ldots, e_r\}$ which is given by (5). For $m \in \mathbb{N}$, $\lambda \in \mathbb{C}_+$ and $x \in C[0, t]$ let

\[ K_m(\lambda, x) = \exp\left\{ \frac{1 - \lambda}{2} \sum_{j=1}^{m} (e_j, x)^2 \right\}. \] (14)

**Theorem 4.1.** Let $1 \leq p \leq \infty$ and $F_r$ be given by (12). Then for $\lambda \in \mathbb{C}_+$ and a.e. $\tilde{\xi} \in \mathbb{R}^n$ we have

\[ E^{\text{anw}}[(F_r)F_z | Z_n](\xi) = \lambda^\frac{2}{m} \int_{C[0, t]} K_r(\lambda, x) F_r(Z(x, \cdot) - [Z(x, \cdot)]_b + [\tilde{\xi}]_b) dw_{\tilde{\xi}}(x), \]

where $K_r$ is given by (14) replacing $m$ by $r$. Moreover if $p = 1$ and $q$ is a nonzero real number, then

\[ E^{\text{anw}}_{F_r}[(F_r)F_z | Z_n](\xi) = \lim_{m \to \infty} \lambda^\frac{2}{m} \int_{C[0, t]} K_r(\lambda_m, x) F_r(Z(x, \cdot) - [Z(x, \cdot)]_b + [\tilde{\xi}]_b) dw_{\tilde{\xi}}(x), \]

for any sequence $\{\lambda_m\}_{m=1}^{\infty}$ in $\mathbb{C}_+$ converging to $-iq$ as $m$ approaches $\infty$.

**Proof.** For $\lambda \in \mathbb{C}_+$ and a.e. $\tilde{\xi} \in \mathbb{R}^n$ we have by Lemma 2.1

\[ \Gamma(\lambda, r, \tilde{\xi}) \equiv \lambda^\frac{2}{m} \int_{C[0, t]} K_r(\lambda, x) F_r(Z(x, \cdot) - [Z(x, \cdot)]_b + [\tilde{\xi}]_b) dw_{\tilde{\xi}}(x) \]

\[ = \lambda^\frac{2}{m} \int_{C[0, t]} K_r(\lambda, x) f((\mathcal{P}^\perp(h\tilde{v}), x) + (\tilde{v}, [\tilde{\xi}]_b)) dw_{\tilde{\xi}}(x) \]
which completes the proof of the first part of the theorem. If $p = 1$, then the final result follows from the dominated convergence theorem.

By the generalized Wiener integration theorem [9, Theorem 3.5] and Theorem 3.3

\[ \Gamma(\lambda, r, \xi) \]

\[ = \left( \frac{\lambda}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(u) \exp\left\{ -\frac{1}{2} \|u\|^2 \right\} \exp\left\{ -\frac{1}{2} \|v\|^2 \right\} \, du \]

\[ = E^{\text{new}}[\langle F_r \rangle | Z_n](\xi), \]

which completes the proof of the first part of the theorem. If $p = 1$, then the final result follows from the dominated convergence theorem. \( \square \)

**Theorem 4.2.** Let $\Psi$ be as given in Theorem 3.2. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\xi \in \mathbb{R}^n$ we have

\[ E^{\text{new}}[\Psi_{\mathcal{L}} | Z_n](\xi) \]

\[ = \lim_{m \to \infty} \lambda^\frac{n}{2} \int_{C[0,t]} K_m(\lambda, x) \Psi(Z(x, \cdot) - [Z(x, \cdot)]_b + [\xi]_b) \, dw_{\varphi}(x), \]

where $K_m$ is given by (14). Moreover if $q$ is a nonzero real number and $\{\lambda_m\}_{m=1}^{\infty}$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as $m$ approaches $\infty$, then $E^{\text{new}}[\Psi_{\mathcal{L}} | Z_n](\xi)$ is given by the right hand side of (15) replacing $\lambda$ by $\lambda_m$.

**Proof.** For $m > r$, $\lambda \in \mathbb{C}_+$ and a.e. $\xi \in \mathbb{R}^n$ we have by Lemma 2.1

\[ \Gamma(\lambda, m, \xi) \equiv \int_{C[0,t]} K_m(\lambda, x) \Psi(Z(x, \cdot) - [Z(x, \cdot)]_b + [\xi]_b) \, dw_{\varphi}(x) \]

\[ = \int_{L_2[0,t]} \int_{\mathbb{R}^n} A_1(\xi, v, z) \int_{C[0,t]} K_m(\lambda, x) \exp\{i[(v, Z(x, \cdot) - [Z(x, \cdot)]_b + [\xi]_b)]_b \} \, dw_{\varphi}(x) \, d\sigma(v) \]

\[ = \int_{L_2[0,t]} \int_{\mathbb{R}^n} A_1(\xi, v, z) \int_{C[0,t]} K_m(\lambda, x) \exp\{i[(v, Z(x, \cdot) - [Z(x, \cdot)]_b + [\xi]_b)]_b \} \, dw_{\varphi}(x) \, d\sigma(v), \]

where $A_1$ and $K_m$ are given by (10) and (14), respectively. By the similar method as used in the proof of Lemma 8 in [11]

\[ \Gamma(\lambda, m, \xi) \]

\[ = \lambda^{-\frac{n}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^n} A_1(\xi, v, z) \exp\left\{ \frac{\lambda - 1}{2\lambda} \sum_{j=1}^{m} (c_j(P^\perp(\mathcal{L}))^2 - \frac{1}{\lambda}) \right\} \, d\sigma(v), \]

\[ \times \langle e(P^\perp(\mathcal{L})), T_{\mathcal{A}}z \rangle_{\mathbb{R}^n} - \frac{1}{2\lambda} \|T_{\mathcal{A}}z\|^2_{\mathbb{R}^n} - \frac{1}{2} \|P^\perp(\mathcal{L})\|^2 \right\} \, d\sigma(v), \]
where \( \vec{c} = (c_1, \ldots, c_r) \) and the \( c_j \)'s are given by (9). Now we have by the dominated convergence theorem and the Parseval's identity

\[
\lim_{m \to \infty} \lambda^\frac{m}{2} \Gamma(\lambda, m, \vec{\xi}) = \int_{L^2[0, t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \exp\left\{ -\frac{1}{2\lambda} \|P^+(hv)\|^2 - \frac{1}{\lambda} (\vec{c}(P^+(hv)), T_A \vec{z})_{\mathbb{R}^r} \right\}d\rho(\vec{z})d\sigma(v)
\]

where \( A_2 \) is given by (11). Now the proof of the first part of the theorem is completed by Theorem 3.2. The remainder part of the theorem immediately follows from the dominated convergence theorem.

**Theorem 4.3.** Let \( G_r \) be as given in Theorem 3.4. Then for \( \lambda \in \mathbb{C}_+ \) and a.e. \( \vec{z} \in \mathbb{R}^n \), \( E_n^{\nu,r}[|\langle G_r \rangle Z_n]|(\vec{\xi}) \) is given by the right hand side of (15) replacing \( \Psi \) by \( G_r \). Moreover if \( p = 1, q \) is a nonzero real number and \( \{\lambda_m\}_{m=1}^{\infty} \) is a sequence in \( \mathbb{C}_+ \) converging to \( -i\eta \) as \( m \) approaches \( \infty \), then \( E_n^{\nu,r}[|\langle G_r \rangle Z_n]|(\vec{\xi}) \) is given by the right hand side of (15), where \( \lambda \) and \( \Psi \) are replaced by \( \lambda_m \) and \( G_r \), respectively.

**Proof.** For \( m > r, \lambda \in \mathbb{C}_+ \) and a.e. \( \vec{c} \in \mathbb{R}^n \) we have by Lemma 2.1

\[
\Gamma(\lambda, m, \vec{\xi}) = \int_{C[0, t]} K_m(\lambda, x)G_r(Z(x, \cdot) - [Z(x, \cdot)]_b + [\vec{\xi}]_b)dw(\vec{\xi})
\]

\[
= \int_{L^2[0, t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{C[0, t]} K_m(\lambda, x) \exp\{i(P^+(vh), x)\}
\times f\big((P^+(hv), x) + (\vec{v}, [\vec{\xi}]_b)\big)dw(\vec{\xi})d\sigma(v).
\]

By the similar method as used in the proof of Lemma 7 in [11]

\[
\Gamma(\lambda, m, \vec{\xi}) = \lambda^\frac{m}{2} \left( \frac{\lambda}{2\pi} \right)^\frac{m}{2} \int_{L^2[0, t]} \exp\left\{ i(v, [\vec{\xi}]_b) + \frac{\lambda-1}{2\lambda} \sum_{j=1}^{m} (c_j(P^+(hv)))^2 \right\}
\times \exp\left\{ -\frac{1}{2} \|P^+(hv)\|^2 + \frac{1}{\lambda} \|\vec{c}(P^+(hv))\|^2_{\mathbb{R}^r} \right\}
\int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b))
\times \exp\left\{ -\frac{\lambda}{2} \|\vec{u}\|_{\mathbb{R}^r}^2 + i(\vec{c}(P^+(hv)), \vec{u})_{\mathbb{R}^r} \right\}d\sigma(\vec{v}),
\]

where \( \vec{c} = (c_1, \ldots, c_r) \), the \( c_j \)'s are given by (9) and \( A^T \) is the transpose of \( A \) given by (6). Now we have by the dominated convergence theorem and the Parseval's identity

\[
\lim_{m \to \infty} \lambda^\frac{m}{2} \Gamma(\lambda, m, \vec{\xi})
\]
\[ E_\mu \left[ \left( \frac{\lambda}{2\pi} \right)^\frac{T}{2} \int_{L_2(0, t)} \exp \{ i\{v, [\xi]\}_b \} \int_{\mathbb{R}^\prime} f(\bar{v}A^T + (\bar{v}, [\xi]_b)) \exp \left\{ -\frac{1}{2\lambda} \right\} \times \left[ \|P\|_{\mathbb{H}}^2 - \bar{v}[P\|_{\mathbb{H}}^2 \right] - \frac{\lambda}{2} \|v\|_{\mathbb{R}^\prime}^2 + i\langle\bar{v}[P\|_{\mathbb{H}}^2, \bar{u}\rangle_{\mathbb{R}^\prime} \right] \] \\\neq \left( \frac{\lambda}{2\pi} \right)^\frac{T}{2} \int_{L_2(0, t)} \exp \{ i\{v, [\xi]\}_b \} \int_{\mathbb{R}^\prime} f(\bar{v}A^T + (\bar{v}, [\xi]_b))A_3(\lambda, v, u)d\bar{u}d\sigma(v), \]

where \( A_3 \) is given by (13). Now the proof of the first part of the theorem is completed by Theorem 3.4. If \( p = 1 \), then the final result immediately follows from the dominated convergence theorem.

Combining Theorems 4.2 and 4.3 we have the following corollary by the linealities of the generalized conditional Wiener and Feynman integrals on the analogue of Wiener space.

**Corollary 4.4.** Let \((\phi(\bar{v}, \cdot) + F_r)F\) be as given in Corollary 3.5. Then for \( \lambda \in C_+ \) and a.e. \( \xi \in \mathbb{R}^n \) \( E^{\mu,\nu}_{\mathbb{R}^n} \left[ ((\phi(\bar{v}, \cdot) + F_r)F)_{\mu,\nu}[Z_n](\xi) \right] \) is given by the right hand side of (15) replacing \( \Psi \) by \((\phi(\bar{v}, \cdot) + F_r)F\). Moreover if \( p = 1 \), \( q \) is a nonzero real number and \( \{\lambda_m\}_{m=1}^{\infty} \) is a sequence in \( C_+ \) converging to \(-iq\) as \( m \) approaches \( \infty \), then \( E^{\mu,\nu}_{\mathbb{R}^n} \left[ ((\phi(\bar{v}, \cdot) + F_r)F)_{\mu,\nu}[Z_n](\xi) \right] \) is given by the right hand side of (15), where \( \lambda \) and \( \Psi \) are replaced by \( \lambda_m \) and \((\phi(\bar{v}, \cdot) + F_r)F\), respectively.

Letting \( \lambda = \gamma^{-2} \) in Corollary 4.4 we have the following change of scale formula for the generalized conditional Wiener integrals on the analogue of Wiener space using the polygonal function.

**Corollary 4.5.** Let \( F, F_r \) and \( \phi \) be as given in Corollary 4.4. Then for \( \gamma > 0 \) and a.e. \( \xi \in \mathbb{R}^n \)
\[ E[F(\gamma Z(x, \cdot))((\phi(\bar{v}, \gamma Z(x, \cdot)) + F_r(\gamma Z(x, \cdot)))|\gamma Z_n(x)](\xi) \]
\[ = \lim_{m \to \infty} \gamma^{-m} \int_{C[0, t]} \exp \left\{ \frac{2\gamma^2 - 1}{2\gamma^2} \sum_{j=1}^{m} (e_j, x)^2 \right\} F(Z(x, \cdot) - [Z(x, \cdot)]_b \]
\[ + [\xi]_b)(\phi(\bar{v}, Z(x, \cdot) - [Z(x, \cdot)]_b)_b + F_r(Z(x, \cdot) - [Z(x, \cdot)]_b)_b + [\xi]_b)dw_\gamma(x). \]

**5. A change of scale formula using the cylinder functions**

In this section we derive a change of scale formula for the generalized conditional Wiener integrals of unbounded functions on the analogue of Wiener space using the cylinder functions.

**Theorem 5.1.** Let \( 1 \leq p \leq \infty \) and \( A^T \) be the transpose of \( A \) given by (6). For an orthonormal set \( \{h_1, \ldots, h_r\} \) in \( L_2(0, t) \) let \( H_r(\lambda, x) = \exp \{ \frac{1}{2\lambda} \sum_{j=1}^{r} (h_j, x)^2 \} \). Let \( F_r \) and \( f \) be related by (12). Then for \( \lambda \in C_+ \) and a.e. \( \xi \in \mathbb{R}^n \) we have
\[ E^{\mu,\nu}_{\mathbb{R}^n} \left[ (F_r)_{\mu,\nu}[Z_n](\xi) \right] = \lambda^{\frac{T}{2}} \int_{C[0, t]} H_r(\lambda, x)f(\bar{h}, x)A^T + (\bar{v}, [\xi]_b))dw_\gamma(x), \]
where \((\vec{h}, x) = ((h_1, x), \ldots, (h_r, x))\). Moreover if \(p = 1\) and \(q\) is a nonzero real number, then
\[
E^{a_n}E[(F_r)Z|Z_n](\vec{\xi}) = \lim_{m \to \infty} \lambda^\frac{r}{2} \int_{C[0,t]} H_r(\lambda, x)f((\vec{h}, x)A^T + (\vec{v}, [\vec{\xi}])d\nu(x)
\]
for any sequence \(\{\lambda_m\}_{m=1}^{\infty}\) in \(\mathbb{C}_+\) converging to \(-iq\) as \(m\) approaches \(\infty\).

**Proof.** For \(\lambda \in \mathbb{C}_+\) and a.e. \(\vec{\xi} \in \mathbb{R}^n\) we have by Theorem 3.3
\[
\lambda^\frac{r}{2} \int_{C[0,t]} H_r(\lambda, x)f((\vec{h}, x)A^T + (\vec{v}, [\vec{\xi}])d\nu(x)
\]
\[
= \left(\frac{\lambda}{2\pi}\right)^\frac{r}{2} \int_{\mathbb{R}^r} \exp\left\{\frac{1}{2\lambda} \|\vec{u}\|^2_{2} - \lambda \|\vec{h}\|^2_{2}\right\} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}])d\nu(x)
\]
\[
= E^{a_n}E[(F_r)Z|Z_n](\vec{\xi}),
\]
which completes the proof of the first part of the theorem. If \(p = 1\), then the final result follows from the dominated convergence theorem. \(\square\)

**Theorem 5.2.** Let \(A\) be given by (6) and \(\Psi\) be as given in Theorem 3.2. Then for \(\lambda \in \mathbb{C}_+\) and a.e. \(\vec{\xi} \in \mathbb{R}^n\) we have
\[
E^{a_n}E[\Psi Z|Z_n](\vec{\xi}) = \lim_{m \to \infty} \lambda^\frac{r}{2} \int_{C[0,t]} K_m(\lambda, x) \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \exp
\]
\[
\{i(\|\mathcal{P}^+(v)\|, x) + (\langle \vec{e}, x \rangle, \vec{z}A)_{\mathbb{R}^r}) d\lambda \nu(v)dx\}
\]
\[
\text{where } \langle \vec{e}, x \rangle = ((e_1, x), \ldots, (e_r, x)), A_1 \text{ and } K_m \text{ are given by (10) and (14), respectively. Moreover if } q \text{ is a nonzero real number and } \{\lambda_m\}_{m=1}^{\infty} \text{ is a sequence in } \mathbb{C}_+ \text{ converging to } -iq \text{ as } m \text{ approaches } \infty, \text{ then } E^{a_n}E[\Psi Z|Z_n](\vec{\xi}) \text{ is given by the right hand side of the above equality, where } \lambda \text{ is replaced by } \lambda_m.\]

**Proof.** Let \(m > r\). For \(v \in L_2[0,t]\) let \(f_{m+1} = \mathcal{P}^+(v) - \sum_{j=1}^{m} c_j \mathcal{P}^+(v) e_j\) and let \(g_{m+1} = \int f_{m+1} \) if \(f_{m+1} \neq 0\), where \(c_j\) is given by (9). Let \(g_{m+1} = 0\) if \(f_{m+1} = 0\). For \(\lambda \in \mathbb{C}_+\) and a.e. \(\vec{\xi} \in \mathbb{R}^n\) we have by the generalized Wiener integration theorem [9, Theorem 3.5]
\[
\Gamma(\lambda, m, \vec{\xi})
\]
\[
= \int_{C[0,t]} K_m(\lambda, x) \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \exp\{i(\mathcal{P}^+(v) , x) + (\langle \vec{e}, x \rangle, \vec{z}A)_{\mathbb{R}^r}) d\lambda \nu(v)dx\}
\]
\[
= \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \int_{C[0,t]} K_m(\lambda, x) \exp\{i(\sum_{j=1}^{m} c_j \mathcal{P}^+(v) e_j , x) + \|f_{m+1}\|((g_{m+1}, x) + (\langle \vec{e}, x \rangle, \vec{z}A)_{\mathbb{R}^r}) dx\} d\nu(v)dx\}
\]
where $\vec{a} = (a_1, \ldots, a_r)$. Using the following well-known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + bu\} du = \left(\frac{\pi}{a}\right)^{\frac{r}{2}} \exp\left\{-\frac{b^2}{4a}\right\}$$

for $a \in \mathbb{C}^+$ and any real $b$.

$$\Gamma(\lambda, m, \tilde{\xi}) = \left(\frac{1}{2\pi}\right)^{\frac{r}{2}} \int_{L^2[0,t]} \int_{\mathbb{R}^r} A_1(\tilde{\xi}, v, z) \int_{\mathbb{R}^m} \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^{m} u_j^2 + i \left[\sum_{j=1}^{m} c_j(\mathcal{P}^+ (vh)) u_j \right] + \frac{1}{2} \|f_{m+1}\|^2 \right\} d(u_1, \ldots, u_m) d\rho(\tilde{z}) d\sigma(v)$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{r}{2}} \int_{L^2[0,t]} \int_{\mathbb{R}^r} A_1(\tilde{\xi}, v, z) \int_{\mathbb{R}^m} \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^{m} u_j^2 + i \left[\sum_{j=1}^{m} c_j(\mathcal{P}^+ (vh)) u_j \right] - \frac{1}{2} \|f_{m+1}\|^2 \right\} d(u_1, \ldots, u_m) d\rho(\tilde{z}) d\sigma(v),$$

where $\tilde{c}(\mathcal{P}^+ (vh)) = (c_1(\mathcal{P}^+ (vh)), \ldots, c_r(\mathcal{P}^+ (vh)))$. By (16)

$$\Gamma(\lambda, m, \tilde{\xi})$$

$$= \lambda^{-\frac{m}{2}} \int_{L^2[0,t]} \int_{\mathbb{R}^r} A_1(\tilde{\xi}, v, z) \exp\left\{-\frac{1}{2\lambda} \left[ \|\tilde{c}(\mathcal{P}^+ (vh)) + z \tilde{\xi}\|_{\mathbb{R}^r}^2 + \sum_{j=r+1}^{m} c_j(\mathcal{P}^+ (vh)) \|\mathcal{P}^+ (vh)\|^2 \right] - \frac{1}{2} \|\mathcal{P}^+ (vh)\|^2 - \sum_{j=1}^{m} c_j(\mathcal{P}^+ (vh))^2 \right\} d\rho(\tilde{z}) d\sigma(v).$$

By the dominated convergence theorem and the Parseval’s identity

$$\lim_{m \to \infty} \lambda^{\frac{m}{2}} \Gamma(\lambda, m, \tilde{\xi}) = \int_{L^2[0,t]} \int_{\mathbb{R}^r} A_1(\tilde{\xi}, v, z) A_2(\lambda, v, z) d\rho(\tilde{z}) d\sigma(v),$$

where $A_2$ is given by (11). Now the proof of the first part of the theorem is completed by Theorem 3.2. The second part of the theorem immediately follows from the dominated convergence theorem. \qed
Theorem 5.3. Let $A^T$ be the transpose of $A$ given by (6). Let $G_r$ be as given in Theorem 3.4. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\xi \in \mathbb{R}^n$ we have

$$E^{\text{new}}[[G_r]_Z | Z_n](\xi) = \lim_{m \to \infty} \int_{\mathbb{C}[0,t]} K_m(\lambda, x) \int_{L_2[0, t]} \exp\{i[(v, [\xi]_b)] + (\mathcal{P}^+(vh), x)]\} f((\bar{v}, x)A^T + (i, [\xi]_b))d\sigma(v)dw_r(x),$$

where $(\bar{v}, x) = ((e_1, x), \ldots, (e_r, x))$ and $K_m$ is given by (14). Moreover if $p = 1$, $q$ is a nonzero real number and $\{\lambda_m\}^{\infty}_{m=1}$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as $m$ approaches $\infty$, then $E^{\text{new}}[[G_r]_Z | Z_n](\xi)$ is given by the right hand side of the above equality, where $\lambda$ is replaced by $\lambda_m$.

Proof. For $m > r$, $\lambda \in \mathbb{C}_+$ and a.e. $\xi \in \mathbb{R}^n$

$$
\begin{align*}
\Gamma(\lambda, m, \xi) & = \int_{\mathbb{C}[0,t]} K_m(\lambda, x) \int_{L_2[0, t]} \exp\{i[(v, [\xi]_b)] + (\mathcal{P}^+(vh), x)]\} f((\bar{v}, x)A^T + (i, [\xi]_b))d\sigma(v)dw_r(x) \\
& = \left(\frac{1}{2\pi}\right) \int_{L_2[0, t]} \exp\{i[(v, [\xi]_b)]\} \int_{R^{m+1}} f(\bar{u}A^T + (i, [\xi]_b)) \exp\left\{\frac{1-\lambda}{2}\right\} \\
& \times \sum_{j=1}^{m} u_j^2 + i \sum_{j=1}^{m} c_j (\mathcal{P}^+(vh))u_j + \|f_{m+1}\|u_{m+1}\} - \frac{1}{2} \sum_{j=1}^{m} u_j^2 \right) d(u_1, \ldots, u_m, u_{m+1})d\sigma(v) \\
& = \left(\frac{1}{2\pi}\right) \int_{L_2[0, t]} \exp\{i[(v, [\xi]_b)]\} \int_{R^m} f(\bar{u}A^T + (i, [\xi]_b)) \exp\left\{\frac{\lambda}{2} \sum_{j=1}^{m} u_j^2 \right\} \\
& \times \sum_{j=1}^{m} c_j (\mathcal{P}^+(vh))u_j - \frac{1}{2}\|f_{m+1}\| \right) d(u_1, \ldots, u_m)d\sigma(v)
\end{align*}
$$

by the generalized Wiener integration theorem [9, Theorem 3.5] and (16), where $\bar{u} = (u_1, \ldots, u_r)$ and $f_{m+1}$ is as given in the proof of Theorem 5.2. By (16)

$$
\begin{align*}
\Gamma(\lambda, m, \xi) & = \lambda^{-\frac{1}{2\pi^2}} \int_{L_2[0, t]} \exp\{i[(v, [\xi]_b)]\} \int_{R^r} f(\bar{u}A^T + (i, [\xi]_b)) \left\{\frac{\lambda}{2\pi} \sum_{j=1}^{m} u_j^2 \right\} \\
& + i(\mathcal{P}^+(vh), \bar{u})_{R^r} - \frac{1}{2\lambda} \sum_{j=r+1}^{m} (c_j (\mathcal{P}^+(vh)))^2 - \frac{1}{2} \left\{\|\mathcal{P}^+(vh)\|^2 - \sum_{j=1}^{m} (c_j (\mathcal{P}^+(vh)))^2 \right\}
\end{align*}
$$

$d\sigma(v)$,
equality, where Corollary 5.5. Wiener space using the cylinder functions. 

\[ \lambda \frac{C_1}{2\pi} \int_{\mathbb{R}^n} \exp\{i\phi(v, [\xi]_b)\} f((\xi, [\xi]_b)) A_3(\lambda, v, \bar{u}) d\bar{u}dv, \]

where \( A_3 \) is given by (13). Now the proof of the first part of the theorem is completed by Theorem 3.4. The second part of the theorem immediately follows from the dominated convergence theorem. \( \square \)

Combining Theorems 5.2 and 5.3 we have the following corollary by the linearity of the generalized conditional Wiener and Feynman integrals on the analogue of Wiener space.

**Corollary 5.4.** Let \( \phi(\bar{v}, \cdot) + F_r \) be as given in Corollary 3.5. Then for \( \lambda \in \mathbb{C}_+ \) and a.e. \( \xi \in \mathbb{R}^n \)

\[ E^{w_n}|((\phi(\bar{v}, \cdot) + F_r)F)|Z_n|([\xi]) \]

\[ = \lim_{m \to \infty} \lambda^{\frac{2}{2}} \int_{C[0,t]} K_m(\lambda, x) \int_{\mathbb{R}^n} \exp\{i((\phi(\bar{v}, \cdot) + F_r)F)|Z_n|([\xi]) \]

\[ \times \exp\{i((\bar{e}, x), \bar{Z}A)|R|d\bar{Z} + \exp\{i(v, [\xi]_b)\} f((\bar{e}, x)AT + (\bar{v}, [\xi]_b)) \}
\]

\[ d\sigma(v)dw_{\rho}(x), \]

where \( (\bar{e}, x) = ((e_1, x), \ldots, (e_r, x)) \), \( A \), \( A_1 \) and \( K_m \) are given by (6), (10) and (14), respectively. Moreover if \( p = 1 \), \( q \) is a nonzero real number and \( \{\lambda_m\}_{m=1}^{\infty} \) is a sequence in \( \mathbb{C}_+ \) converging to \(-iq\) as \( m \) approaches \( \infty \), then \( E^{w_m}|((\phi(\bar{v}, \cdot) + F_r)F)|Z_n|([\xi]) \) is given by the right hand side of the above equality, where \( \lambda \) is replaced by \( \lambda_m \).

Letting \( \lambda = \gamma^{-2} \) in Corollary 5.4 we have the following change of scale formula for the generalized conditional Wiener integrals on the analogue of Wiener space using the cylinder functions.

**Corollary 5.5.** Let \( F \), \( F_r \) and \( \phi \) be as given in Corollary 4.4. Then for \( \rho > 0 \) and a.e. \( \xi \in \mathbb{R}^n \)

\[ E[F(\gamma Z(x, \cdot))(\phi(\bar{v}, \gamma Z(x, \cdot)) + F_r(\gamma Z(x, \cdot))]|\gamma Z_n(x)|([\xi]) \]

\[ = \lim_{m \to \infty} \gamma^{-m} \int_{C[0,t]} \exp\{2\gamma^2 - 1 \sum_{j=1}^{m} (e_j, x)^2\} \int_{\mathbb{R}^n} \exp\{i((\phi(\bar{v}, \cdot) + F_r)F)|Z_n|([\xi]) \]

\[ \times \left[ \int_{\mathbb{R}^n} A_1(\xi, v, \bar{Z}) \exp\{i((\bar{e}, x), \bar{Z}A)|R|d\bar{Z} + \exp\{i(v, [\xi]_b)\} f((\bar{e}, x)AT + (\bar{v}, [\xi]_b)) \}
\]

\[ + (\bar{v}, [\xi]_b) \}
\]

\[ d\sigma(v)dw_{\rho}(x). \]
Remark 5.6.  
(1) The choice of the orthonormal set \( \{ h_1, \ldots, h_r \} \) in Theorem 5.1 is independent of \( \{ e_1, \ldots, e_r \} \).

(2) The results of this paper are different from those in [6, 8, 11]. If \( h = 1 \) a.e. on \([0, t]\), then \( F(Z(x, \cdot)) = F(x - x(0)) \) and \( Z_n(x) = (x(t_1) - x(0), \ldots, x(t_n) - x(0)) \). In this case we can take an orthonormal subset \( \{ v_1, v_2, \ldots, v_r \} \) of \( L_2[0, t] \) such that \( P_{\perp v_1}, \ldots, P_{\perp v_r} \) are independent [11, Remark 1]. Furthermore if \( \phi = \delta_0 \), the Dirac measure concentrated at 0, then Theorems 4.2 and 4.3 generalize the equations (28) and (29) in [11].

(3) The results of this paper are independent of a particular choice of the probability measure \( \phi \).

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