PERPETUAL INTEGRALS FOR LÉVY PROCESSES

LEIF DÖRING AND ANDREAS E. KYPRIANOU

Abstract. Given a Lévy process $\xi$ we ask for necessary and sufficient conditions for almost sure finiteness of the perpetual integral $\int_0^\infty f(\xi_s) \, ds$, where $f$ is a positive locally integrable function. If $\mu = \mathbb{E}[\xi_1] \in (0, \infty)$ and $\xi$ has local times we prove the 0-1 law

$$
P\left( \int_0^\infty f(\xi_s) \, ds < \infty \right) \in \{0, 1\}
$$

with the exact characterization

$$
P\left( \int_0^\infty f(\xi_s) \, ds < \infty \right) = 0 \iff \int_0^\infty f(x) \, dx = \infty.
$$

The proof uses spatially stationary Lévy processes, local time calculations, Jeulin’s lemma and the Hewitt-Savage 0-1 law.

The study of perpetual integrals $\int f(X_s) \, ds$ with finite or infinite horizon for diffusion processes $X$ has a long history partially because of their use in the analysis of stochastic differential equations and insurance, financial mathematics as the present value of a continuous stream of perpetuities.

The main result of the present article is a characterization of finiteness for perpetual integrals of Lévy processes:

**Theorem 1.** Suppose that $\xi$ is a Lévy process that has strictly positive mean $\mu < \infty$, local times and is not a compound Poisson process. If $f$ is a measurable locally integrable positive function, then the following 0-1 law holds:

(T1) \(P\left( \int_0^\infty f(\xi_s) \, ds < \infty \right) = 1 \iff \int_0^\infty f(x) \, dx < \infty\)

and

(T2) \(P\left( \int_0^\infty f(\xi_s) \, ds < \infty \right) = 0 \iff \int_0^\infty f(x) \, dx = \infty\).

Let us briefly compare the theorem with the existing literature:

(i) If $\xi$ is a Brownian motion with positive drift, then results were obtained through the Ray-Knight theorem, Jeulin’s lemma and Khashminkii’s lemma by Salminen/Yor [11], [12].

(ii) For spectrally negative $\xi$, i.e. $\xi$ only jumps downwards, the equivalence was obtained in Khoshnevisan/Salminen/Yor [12], see also Example 3.9 of Schilling/Voncracek [9]. The spectrally negative case also turns out to be easier in our proof.

(iii) If $f$ is (ultimately) decreasing, results for general Lévy processes have been proved in Erickson/Maller [5]. In this case the result stated in Theorem 1 follows easily from the law of large numbers by estimating $2\mu t > \xi_t > \frac{1}{2}\mu t$ for $t$ big enough. The very same argument also shows that for $\mu = +\infty$ the integral test (T) fails in general. For a result in this case we again refer to [5].

**Remark 2.** It is not clear whether or not the assumption $\xi$ having local time plays a role. For (ultimately) decreasing $f$ the existence of local time is clearly not needed, whereas we have no conjecture for general $f$.

**Proof of Theorem 1**

Before going into the proof let us fix some notation and facts needed below. For more definitions and background we refer for instance to [1] or [8]. The law of $\xi$ issued from $x \in \mathbb{R}$ will be denoted by $\mathbb{P}^x$. 


abbreviating $\mathbb{P} = \mathbb{P}^{0}$, and the characteristic exponent is defined as
$$\Psi(\lambda) := -\log \mathbb{E} \left[ \exp(i\lambda X_1) \right], \quad \lambda \in \mathbb{R}.$$  

We recall from Theorem V.1 of [11] that $\xi$ has local times $(L_t(x))_{t \geq 0, x \in \mathbb{R}}$ if and only if
$$\int_{-\infty}^{\infty} \mathcal{R} \left( \frac{1}{1 + \Psi(r)} \right) dr < \infty. \quad (1)$$

This means that for any bounded measurable function $f : \mathbb{R} \to [0, \infty)$ the occupation time formula
$$\int_0^t f(\xi_s) ds = \int_{\mathbb{R}} f(x)L_t(x) dx, \quad t \geq 0,$$
holds almost surely.

A consequence of (1) is also that points are non-polar. More precisely, a Theorem of Kesten and Bretagnolle states that $\mathbb{P}(\tau_x < \infty) > 0$ for all $x > 0$ if $\tau_x = \inf\{ t : \xi_t = x \}$, see for instance Theorem 7.12 of [10].

Throughout we assume $\xi$ is transient so that $\int_{-\infty}^{\infty} \mathcal{R}(\frac{1}{\Psi(r)}) dr < \infty$ and, consequently, (1) implies $\int_{-\infty}^{\infty} \mathcal{R} \left( \frac{1}{\Psi(r)} \right) dr < \infty$. But then Theorem II.16 of [11] implies that the potential measure
$$U(dx) = \int_0^\infty \mathbb{P}(\xi_s \in dx) ds$$
has a bounded density $u(x)$ with respect to the Lebesgue measure.

We start with the easy direction of Theorem 1:

**Proof of Theorem 1. Sufficiency of Integral Test.** Suppose that $\int_{\mathbb{R}} f(x) dx < \infty$. Since we assume that $\xi$ is transient and has a local time we can use the existence and boundedness of the potential density to obtain
$$\mathbb{E} \left[ \int_0^\infty f(\xi_s) ds \right] = \int_{\mathbb{R}} f(x) \int_0^\infty \mathbb{P}(\xi_s \in dx) ds = \int_{\mathbb{R}} f(x)u(x) dx \leq \sup_{x \in \mathbb{R}} u(x) \int_{\mathbb{R}} f(x) dx < \infty.$$  

Since finiteness of the expectation implies almost sure finiteness the sufficiency of the integral test for almost sure finiteness of the perpetual integral is proved. \qed

For the reverse direction we use Jeulin’s lemma, here is a simple version:

**Lemma 3.** Suppose $(X_x)_{x \in \mathbb{R}}$ are non-negative, non-trivial and identically distributed random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then
$$\mathbb{P} \left( \int_{\mathbb{R}} f(x)X_x dx < \infty \right) = 1 \quad \implies \quad \int_{\mathbb{R}} f(x) dx < \infty.$$  

**Proof.** Since $X_x$ are identically distributed, we may choose $\varepsilon > 0$ so that $\mathbb{P}(X_x > \varepsilon) = \delta > 0$ for all $x \in \mathbb{R}$. Since $\int_{\mathbb{R}} f(x)X_x dx$ is almost surely finite, there is some $N \in \mathbb{N}$ so that $\mathbb{P}(A_N) > 1 - \delta/2$ with $A_N = \{ \int_{\mathbb{R}} f(x)X_x dx > N \}$. Hence, we have
$$\mathbb{E}[X_x1_{A_N}] \geq \varepsilon \mathbb{P}(\{X_x > \varepsilon\} \cap A_N) > \varepsilon \delta/2 > 0.$$  

But then
$$N \geq NP(A_N) \geq \mathbb{E} \left[ \int_{\mathbb{R}} f(x)X_x dx 1_{A_N} \right] = \int_{\mathbb{R}} f(x)\mathbb{E}[X_x1_{A_N}] dx \geq \varepsilon \delta/2 \int_{\mathbb{R}} f(x) dx.$$  

The proof is now complete. \qed

Note that there are different versions of Jeulin’s lemma (see for instance [10]). Most commonly, one refers to Jeulin’s lemma if $\mathbb{P}(\int_{\mathbb{R}} f(x)X_x dx < \infty) > 0$ but more assumptions on the $X_x$ are posed. Those extra assumptions on $X_x$ are not satisfied in our setting but we can employ a 0-1 law that allows us to work with $\mathbb{P}(\int_{\mathbb{R}} f(x)X_x dx < \infty) = 1.$
We would like to apply Jeulin’s lemma via the occupation time formula

\[
\int_0^\infty f(\xi_s) \, ds = \lim_{t \to \infty} \int_t^\infty f(\xi_s) \, ds = \lim_{t \to \infty} \int_0^\infty f(x) L_t(x) \, dx = \int_0^\infty f(x) L_\infty(x) \, dx
\]

with \( L_\infty(x) := \lim_{t \to \infty} L_t(x) \). The argument is too simplistic because the distribution of \( L_\infty(x) \) depends on \( x \). Only if \( \xi \) is spectrally negative the laws \( L_\infty(x) \) are independent of \( x \) by the strong Markov property. To make this idea work we work with randomized initial conditions instead. In what follows we chose a particularly convenient initial distribution motivated by a result from fluctuation theory (see Lemma 3 of [2]). Our assumption \( \mathbb{E}[\xi_1] < \infty \) implies that

\[
\mathbb{P}(\xi_{T_\infty} - x \in dy) \xrightarrow{\mathcal{L}} \rho(dy),
\]

where \( T_\infty = \inf\{t \geq 0 : \xi_t \geq z\} \) and \( \rho \) is a non-degenerate probability law, called the stationary overshoot distribution. The convergence in \( \mathcal{L} \) is a consequence of the quintuple law of [8] and the distribution \( \rho \) can be written down explicitly.

Since \( \rho \) is the stationary overshoot distribution we have

\[
\mathbb{P}^\rho(\xi_{T_\infty} - a \in dy) := \int \mathbb{P}^x(\xi_{T_\infty} - a \in dy) \rho(dx) = \rho(dy), \quad \forall a > 0,
\]

so that spatial stationarity holds due to the strong Markov property: under \( \mathbb{P}^\rho \)

\[
(\xi^{(a)}_t)_{t \geq 0} := (\xi_{T_\infty + t} - a)_{t \geq 0}
\]

has law \( \mathbb{P}^\rho \) for all \( a > 0 \). The stationarity property \( \mathcal{L} \) will be the key to apply Jeulin’s lemma.

**Lemma 4.** For any \( x > 0 \) we have

\[
\mathbb{P}^\rho(L_\infty(x) \in dy) = \mathbb{P}^\rho(L_\infty(1) \in dy).
\]

**Proof.** First note that \( \mathbb{P}^\rho(T_\infty < \infty) = 1 \) for all \( x > 0 \) and \( L(x) \) only starts to increase at some time at or after \( T_\infty \). Then

\[
\mathbb{P}^\rho(L_\infty(x) \in dy) = \int_0^\infty \mathbb{P}^x(L_\infty(x) \in dy) \mathbb{P}^\rho(\xi_{T_\infty} \in dz)
\]

\[
= \int_0^\infty \mathbb{P}^{x+z}(L_\infty(x) \in dy) \mathbb{P}^\rho(\xi_{T_\infty} - x \in dz)
\]

\[
= \int_0^\infty \mathbb{P}^{x+z}(L_\infty(x) \in dy) \rho(dz)
\]

\[
= \int_0^\infty \mathbb{P}^x(L_\infty(0) \in dy) \rho(dz),
\]

using the strong Markov property, the spatial stationarity of \( \mathbb{P}^\rho \) and spatial homogeneity of \( \xi \). Since the righthand side is independent of \( x \) the proof is complete. \( \square \)

Next, we use the Hewitt-Savage 0-1 law (see [4]) in order to get the weak version of Jeulin’s lemma going. If \( X^0, X^1, \ldots \) denotes a sequence of random variables taking values in some measurable space, then an event \( A \in \sigma(X^0, X^1, \ldots) \) is called exchangeable if it is invariant under finite permutations (i.e. only finitely many indices are changed) of the sequence \( X^0, X^1, \ldots \). The Hewitt-Savage 0-1 law states that any exchangeable event of an iid sequence has probability 0 or 1.

**Lemma 5.** \( \mathbb{P}\left( \int_0^\infty f(\xi_s) \, ds < \infty \right) \in \{0, 1\} \).

**Proof.** The idea of the proof is to write \( \Lambda := \{\int_0^\infty f(\xi_s) \, ds < \infty\} \) as an exchangeable event with respect to the iid increments of \( \xi \) on intervals \([n, n+1] \) so that \( \mathbb{P}(\Lambda) \in \{0, 1\} \). Let \( \mathcal{D} \) denote the RCLL functions \( w : [0, 1] \to \mathbb{R} \). If \( \xi \) is the given Lévy process, then define the increment processes as

\[
(\xi^n_t)_{t \in [0, 1]} = (\xi_{n+t} - \xi_n)_{t \in [0, 1]}.
\]
As a consequence we find that since clearly $\Lambda$ is exchangeable for $\xi$, we have

$$\xi_r = \xi_{r-n}^n + \sum_{i=0}^{n-1} \xi_1^i \quad \forall r \in [n, n+1).$$

Using that $g_1 : (w_t)_{t \in [0,1]} \mapsto (w_1)_{t \in [0,1]}$, $g_2 : (w, w')_{t \in [0,1]} \mapsto (w + w')_{t \in [0,1)}$ and $g_3 : (w_t)_{t \in [0,1]} \mapsto \int_0^1 f(w_s)ds$ are measurable mappings, there are measurable mappings $g^n : D^n \to \mathbb{R}$ such that

$$\int_0^1 f(\xi^r_n + \sum_{i=0}^{n-1} \xi^i_1) dr = g^n(\xi^0, ..., \xi^n).$$

As a consequence we find that

$$\left\{ \int_0^\infty f(\xi_s) ds < \infty \right\} = \left\{ \sum_{n=0}^{\infty} \int_0^{n+1} f(\xi_s) ds < \infty \right\}$$

$$= \left\{ \sum_{n=0}^{\infty} \int_0^1 f(\xi^r_n + \sum_{i=0}^{n-1} \xi^i_1) dr < \infty \right\}$$

$$= \left\{ \sum_{n=0}^{\infty} g^n(\xi^0, ..., \xi^n) < \infty \right\}$$

$$\in \sigma(\xi^0, \xi^1, ...).$$

Since clearly $\Lambda$ is exchangeable for $\xi^0, \xi^1, ...$ the Hewitt-Savage 0-1 law implies the claim. □

**Lemma 6.** Suppose $\mathbb{P}(\int_0^\infty f(\xi_s)ds < \infty) = 1$, then $\mathbb{P}^\rho(\int_0^\infty f(\xi_s)ds < \infty) = 1$.

**Proof.** The statement is obvious if $\xi$ is a subordinator, so we assume it is not.

Next we show that $\mathbb{P}^x(\int_0^\infty f(\xi_s)ds < \infty) = 1$ for any $x > 0$. To see this we use the strong Markov property at $\tau_0 = \inf\{t : \xi_t = 0\}$ which is finite with positive probability since points in $\mathbb{R}$ are non-polar:

$$\mathbb{P}^x\left( \int_0^\infty f(\xi_s) ds < \infty \right) \geq \mathbb{P}^x\left( \int_{\tau_0}^\infty f(\xi_s) ds < \infty, \tau_0 < \infty \right)$$

$$= \mathbb{P}^x\left( \int_0^\infty f(\xi_{s+\tau_0} - \xi_{\tau_0}) ds < \infty, \tau_0 < \infty \right)$$

$$= \mathbb{P}^0\left( \int_0^\infty f(\xi_s) ds < \infty \right) \mathbb{P}^x(\tau_0 < \infty)$$

$$> 0.$$  

But then the 0-1 law of Lemma 4 implies that $\mathbb{P}^x\left( \int_0^\infty f(\xi_s) ds < \infty \right) = 1$. Finally, we obtain

$$\mathbb{P}^\rho\left( \int_0^\infty f(\xi_s) ds < \infty \right) = \int_{\mathbb{R}} \mathbb{P}^x\left( \int_0^\infty f(\xi_s) ds < \infty \right) \rho(dx) = \int_{\mathbb{R}} \rho(dx) = 1$$

and the proof is complete. □

Now we are ready to prove the more delicate part of Theorem 1.

**Proof of Theorem 1.** Necessity of Integral Test. Suppose $\mathbb{P}(\int_0^\infty f(\xi_s)ds < \infty) > 0$ which then implies $\mathbb{P}(\int_0^\infty f(\xi_s)ds < \infty) = 1$ by Lemma 5. Hence, $\mathbb{P}^\rho(\int_0^\infty f(\xi_s)ds < \infty) = 1$ by Lemma 6. Using the occupation time formula we get

$$\int_0^\infty f(\xi_s) ds = \lim_{t \to \infty} \int_0^t f(\xi_s) ds = \lim_{t \to \infty} \int_{\mathbb{R}} f(x)L_t(x) dx = \int_{\mathbb{R}} f(x)L_\infty(x) dx \quad \mathbb{P}^\rho-a.s.$$  

In Lemma 4 we proved that $L_\infty(x)$ is independent of $x$ under $\mathbb{P}^\rho$ so that Jeulin’s Lemma implies $\int_{\mathbb{R}} f(x)dx < \infty$. □
Acknowledgement. The authors thank Jean Bertoin for several discussions on the topic.

References

[1] J. Bertoin: "Lévy processes." Cambridge Tracts in Mathematics 121, Cambridge University Press, 1996.
[2] J. Bertoin, M. Savov: "Some applications of duality for Lévy processes in a half-line." Bull. London Math. Soc. 43, pp. 97-110, 2011.
[3] D. Dufresne: "The distribution of a perpetuity, with applications to risk theory and pension funding." Scand. Actuarial J., pp. 39-79, 1990.
[4] E. Hewitt, L. J. Savage: "Symmetric measures on Cartesian products." Trans. Amer. Math. Soc., 80, pp. 470-501 (1955).
[5] K.B. Erickson, R.A. Maller: "Generalised Ornstein-Uhlenbeck processes and the convergence of Lévy integrals." Séminaire de Probabilités XXXVIII, 2005.
[6] D. Khoshnevisan, P. Salminen, M. Yor: "A note on a.s. finiteness of perpetual integral functionals of diffusions." Electr. Comm. Probab., Vol. 11, pp. 108-117.
[7] A. Kyprianou, J.C. Pardo, V. Rivero: "Exact and asymptotic n-tuple laws at first and last passage." Ann. Appl. Probab. Volume 20, pp. 522-564, 2010.
[8] A. Kyprianou: "Fluctuations of Lévy Processes with Applications." Springer (2013).
[9] R. Schilling, Z. Vondraček: "Absolute continuity and singularity of probability measures induced by a purely discontinuous Girsanov transform of a stable process." arXiv:1403.7364
[10] A. Matsumoto, K. Yano: "On a zero-one law for the norm process of transient random walk." Séminaire de Probabilités XLIII, pp. 105-126, 2011.
[11] P. Salminen, M. Yor: "Properties of perpetual integral functionals of Brownian motion with drift." Ann. I.H.P., 41, pp. 335-347, 2005.
[12] P. Salminen, M. Yor: "Perpetual integral functionals as hitting and occupation times." Electr. J. Probab., Vol. 10, p. 371-419, 2005.

Leif Döring, Departement Mathematik, ETH Zürich, Rämistrasse 10, 8092 Zürich, Switzerland

Andreas E. Kyprianou, Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, United Kingdom