DYSON-PAIRS AND ZERO-MASS BLACK HOLES

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Abstract

It has been argued by Dyson in the context of QED in flat spacetime that perturbative expansions in powers of the electric charge $e$ cannot be convergent because if $e$ is purely imaginary then the vacuum should be unstable to the production of charged pairs. We investigate the spontaneous production of such Dyson pairs in electrodynamics coupled to gravity. They are found to consist of pairs of zero-rest mass black holes with regular horizons. The properties of these zero rest mass black holes are discussed. We also consider ways in which a dilaton may be included and the relevance of this to recent ideas in string theory. We discuss accelerating solutions and find that, in certain circumstances, the ‘no strut’ condition may be satisfied giving a regular solution describing a pair of zero rest mass black holes accelerating away from one another. We also study wormhole and tachyonic solutions and how they affect the stability of the vacuum.

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1 Introduction

Recently there has been much interest in the role of Bogomol’nyi saturated states corresponding to extreme black holes in string theory. One particularly interesting suggestion is that at special points of the moduli space of string vacua these states may become massless and the theory might thereby exhibit enhanced symmetry [1]. Another related suggestion is that these states might appear at points in the space of Calabi-Yau vacua associated to singular geometries with conifold points [2]. Away from these special points there is a region of moduli space where these Bogomol’nyi black hole states are well described semi-classically by classical solutions of the Einstein equations coupled to an appropriate matter system, but presumably as one approaches a special point the semi-classical description breaks down.

Nevertheless attempts have been made to find classical solutions describing black holes with zero ADM mass [3]-[6]. These solutions are supersymmetric, i.e. they admit Killing spinors, but they are nakedly singular. By the Positive Energy Theorem for black holes these singularities are inevitable because the matter sources satisfy the Dominant Energy Condition. This is a pity because there are many questions one would like to ask about the properties of these putative zero-rest mass black holes. How do they move for instance? The solutions given in [3]-[6] are static and at rest. This is not possible for ordinary zero-rest mass particles. What happens if these zero rest mass black holes receive a small kick? Are they stable? What happens if one tries to accelerate them using an electric field? It is difficult to answer questions like this using singular solutions because the answers one obtains are dependent upon the boundary conditions one imposes at the singularities.

The aim of this paper is to study a related but hopefully simpler situation. Consider, to begin with, ordinary electrodynamics. Ignoring a possible theta term, the theory is specified by the Maxwell Lagrangian

$$-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$$

where the electric charge $e$ may be said to label the possible vacua. This is clear in string theory where it is an expectation value

$$e^2 = \langle \exp 2\phi \rangle$$

where $\phi$ is the dilaton. Studying the dependence of theory as a function of the coupling constant $e$ is therefore the same as studying the behaviour of
the theory as one moves around the moduli space of vacua. One immediate question is whether various properties of the theory depend analytically upon the coupling constant. This would be true in a neighbourhood of the origin if perturbation theory in powers of $e^2$ were uniformly convergent. It was argued long ago by Dyson [7] that this cannot be so. The argument is by contradiction. If the series converged in this way it would do so everywhere within a small enough disc about the origin. In particular perturbation theory would converge if $e^2$ were negative. But in such a world, like charges would attract and they would destabilize the vacuum.

In a gravity theory charged black holes, particularly the extreme black holes, behave very much like charged particles and so this beautiful argument of Dyson suggests looking at what happens to black holes in a world in which, instead of the standard Lagrangian

$$\frac{1}{16\pi} (R - F_{\mu\nu}F^{\mu\nu}), \quad (1.3)$$

we change the sign of the electromagnetic contribution and use

$$\frac{1}{16\pi} (R + F_{\mu\nu}F^{\mu\nu}) \quad (1.4)$$

instead. More generally we could consider a theory with a dynamical scalar $\sigma$. The action would be

$$\frac{1}{16\pi} \left( R - 2(\partial\sigma)^2 + e^{-2a\sigma} F_{\mu\nu}F^{\mu\nu} \right), \quad (1.5)$$

where $a$ is a dimensionless coupling constant. Note that the scalar $\sigma$ has positive kinetic energy. If $a = 0$ we may consistently set $\sigma = 0$ and we get back to the usual Einstein-anti-Maxwell case. If $a = \sqrt{3}$ we get anti-Kaluza-Klein theory in which the extra dimension is timelike rather than the usual spacelike case. Our results may therefore also be relevant to theories with more than one time direction [8]. The case $a = 1$ corresponds to string theory. Formally the action above is obtained by setting

$$\sigma = \phi + i \frac{\pi}{2a} \quad (1.6)$$

1In this paper we use units such that $G = c = 4\pi\varepsilon_0 = 1$
in the usual action. Because this is just a displacement of $\sigma$, it leaves the kinetic term unchanged. It may be of interest to note that Lindström and Rocek have drawn attention to vector fields with negative kinetic energies in Yang-Mills theories with zero mass monopoles \[9\].

Another application of these ideas is to the field outside a fundamental string in four spacetime dimensions \[10\] (or more generally an $(n - 3)$-brane in $n$ spacetime dimensions). The dilaton behaves as

$$e^{-2\phi} = 1 - 8\mu \ln r$$

where $r$ is the radial distance from the core of the string. Note that the $\phi$ used here is twice that used in \[10\]. At sufficiently large transverse distances $e^{2\phi}$ becomes negative which means that the effective string coupling constant becomes pure imaginary. The metric becomes singular at $r = e^{1/8\mu}$ and so it cannot be interpreted in a straightforward way as an ordinary horizon. In the Kaluza-Klein context a similar phenomenon is encountered and one may have an ordinary horizon in the higher dimensional space because the Killing field one is reducing on switches from being spacelike to being timelike.

Of course one may, if one wishes, consider a scalar field with negative kinetic energy. This also produces some exotic solutions even without an electromagnetic field, including an Einstein-Rosen bridge with $g_{00} = -1$ and thus no horizon separating the two sides. This is discussed briefly in section 2.2 and in more detail in section 5.

In what follows we shall sometimes use, as we did above, the prefix “anti” to describe quantities associated to an electromagnetic field of the opposite sign from usual. Thus, in the next section, we will consider black holes with “anti-charge”. Of course, like anti-charges attract rather than repel, as they do ordinarily. If we take the conventional sign for the scalar field kinetic term, then it also leads to an attractive force. As we shall see this means that extreme black holes are not possible in these theories. Moreover none of the solutions we shall discuss admit Killing spinors. Another difference is that the solutions are non-singular outside a regular event horizon. Thus the phenomena we are about to describe are not precisely the same as those considered in \[1]-\[6\]. However the solutions in this paper do share the common feature with those of \[3]-\[6\] that the total 4-momentum vanishes. It seems quite possible therefore that they may throw some light on the dynamical behaviour of such objects.
2 Static Zero Mass Black Holes

The relevant metrics and fields for electrically charged solutions in Einstein-anti-Maxwell-dilaton theory may be obtained from the usual case \[11, 14\] by setting the charge to be pure imaginary. The four-metric is

\[
ds^2 = - \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right) \frac{1 - a^2}{1 + a^2} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} \left(1 - \frac{r_-}{r}\right) \frac{a^2 - 1}{1 + a^2} dr^2
\]

\[
+ r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2a}{1 + a^2}} (d\theta^2 + \sin^2 \theta d\phi^2),
\]

(2.1)

with scalar field

\[
e^\sigma = \left(1 - \frac{r_-}{r}\right)^{\frac{a}{1 + a^2}}
\]

(2.2)

and Maxwell field

\[
F = \frac{Q}{r^2} dt \wedge dr.
\]

(2.3)

For the magnetic case the metric is the same but the sign of the scalar field \(\sigma\) must be reversed and the Maxwell field becomes \(F = P \sin \theta d\theta \wedge d\phi\). The ADM mass \(M\) is

\[
M = \frac{1}{2} \left( r_+ + \frac{1 - a^2}{1 + a^2} r_- \right)
\]

(2.4)

and the anticharge \(Q\) is given by

\[
|Q| = \sqrt{-r_+ r_-} \frac{1}{1 + a^2}.
\]

(2.5)

The scalar charge \(\Sigma\) is given by

\[
\Sigma = - \frac{ar_-}{1 + a^2},
\]

(2.6)

thus

\[
M^2 + \Sigma^2 + Q^2 = \frac{1}{4} (r_+ - r_-)^2.
\]

(2.7)

From the spacetime point of view these solutions will have a regular event horizon at \(r = r_+\) and behave like black holes provided that \(r_+\) is positive and \(r_-\) is negative (the usual case has both positive but with \(r_- \leq r_+\)).
the case \( a = 0 \), \( r_- \) would be a Cauchy horizon, or more generally it is a singularity, but because \( r_- \) is negative one passes through the horizon \( r_+ \) to reach the singularity at \( r = 0 \) without ever reaching \( r_- \). The causal structure is thus the same as the Schwarzschild solution. For the same reason there are no extreme holes. One may also check that one cannot analytically continue the usual multi-solutions to have imaginary values of the charge, and so anti-gravity is excluded.\[3]

The behaviour of the solutions depends in an essential way on whether \( a^2 \) is less or greater than unity. In the former case we may, by taking \( r_- \) sufficiently negative, violate the positive mass theorem. In fact, by choosing

\[
r_+ = \frac{a^2 - 1}{1 + a^2 r_-},
\]

we can obtain a solution with vanishing total ADM mass. One may ask what happens if \( r \) is allowed to be negative. One then has an asymptotically flat spacetime with negative ADM mass containing a naked singularity at \( r = r_- \).

By contrast if \( a^2 \geq 1 \) and \( r \) is positive then the ADM mass \( M \) is always positive. An explanation for this fact is presumably that the larger the coupling constant \( a \), the greater is the positive contribution of the scalar field to the total energy, and if \( a^2 \geq 1 \) this overwhelms the negative contribution of the negative energy vector field. In particular, in the case of anti-Kaluza-Klein theory (\( a^2 = 3 \)), the black holes always have positive mass.

The case \( a = -\sqrt{3} \) is associated with the five dimensional metric of signature \( ++--- \):

\[
ds^5 = -\left(1 - \frac{r_-}{r}\right)^{\frac{3}{2}} (d\tau + 2A_\mu dx^\mu)^2 + \left(1 - \frac{r_-}{r}\right)^{\frac{1}{2}} \left\{ -\left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{-\frac{1}{2}} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} \left(1 - \frac{r_-}{r}\right)^{\frac{1}{2}} dr^2 \right\}
\]

\[
+ r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{3}{2}} (d\theta^2 + \sin^2 \theta d\phi^2)
\]

\[2\text{In fact it may be possible to have anti-gravity in a limited sense: in principle, 2 such black holes with } Q = M \text{ and electric charges of opposite signs should be able to remain in a static equilibrium since unlike charges repel in this theory. It is obvious, however, that such a static equilibrium is not possible for more than 2 black holes, since they cannot all have charges of different signs from one another, and so this is not true anti-gravity.}\]
where the upper sign corresponds to the electric case and the lower to the magnetic case and the vector field $A_\mu$ must be chosen accordingly. In either case, since for $r > 0$ the metric component $g_{\tau\tau}$ never vanishes, the regularity properties are the same as for the four dimensional metric.

In fact, for $a^2 \geq 1$, one can show that $M$ is not just positive but it is bounded below by a positive quantity proportional to the charge:

$$M \geq |Q| \sqrt{a^2 - 1}.$$  \hspace{1cm} (2.10)

This is reminiscent of the Bogomol’nyi mass bound in ordinary EMD theory, however, the solutions saturating this bound are not extreme in the usual sense. They have 2 distinct horizons $r_\pm$ satisfying

$$r_+ = \frac{1 - a^2}{1 + a^2} r_-.$$  \hspace{1cm} (2.11)

The fact that these solutions are not extreme, i.e. $r_+ \neq r_-$, will have profound implications when we come to study their thermodynamic properties. It will be seen that they have non-zero Hawking temperature and the surface area of the event horizon is non-zero. Hence one might expect that they should lose mass by radiating neutral particles resulting in a decrease in the ratio $\frac{M}{|Q|}$. However, no static solutions exist with $\frac{M}{|Q|} < \sqrt{a^2 - 1}$! This will be discussed in more detail in Section 4.

The main conclusion is that if one reverses the sign of the coupling of the electromagnetic kinetic term in the action then static zero rest mass black hole solutions which are non-singular outside a regular event horizon are indeed possible for $a^2 < 1$. Since they are spherically symmetric the total spatial momentum vanishes as well as the mass and so they have vanishing total 4-momentum. In this respect they should be distinguished from conventional massless particles which have a non-zero but lightlike total 4-momentum. They should also not be confused with tachyonic excitations which are associated with a spacelike 4-momentum.

It is also interesting to consider what happens if the kinetic energy of the scalar field is taken to be negative. In this case, the scalar field produces a repulsive force and so extreme solutions may be possible and also multi-centre solutions satisfying a force balance, depending on the strength of the dilaton coupling $a$ and the sign of the Maxwell kinetic term. If the Maxwell kinetic term is given the opposite sign, then zero rest mass black holes again become a possibility.
It has become conventional to use the abbreviation EMD for the standard Einstein-Maxwell-dilaton theory in which both the Maxwell field and the scalar are given positive kinetic energies. For our purposes, it is therefore convenient to introduce the obvious abbreviations EMD, EMĐ and ĖMD for Einstein-anti-Maxwell-dilaton, Einstein-Maxwell-anti-dilaton and Einstein-anti-Maxwell-anti-dilaton theories respectively. We will now derive and investigate a class of static spherically symmetric black hole solutions with electric charge in each of these theories. These black holes will form a 2 parameter family of solutions labelled by their mass $M$ and electric charge $Q$, with the scalar dilaton charge $\Sigma$ being determined by $M$ and $Q$. We will also find a new wormhole solution which lies outside the class of black hole solutions.

2.1 EMĐ Theory

Here the Maxwell field is given the usual sign in the action but the scalar is given negative kinetic energy:

$$\frac{1}{16\pi} \left( R + 2(\partial \sigma)^2 - e^{-2a\sigma} F_{\mu\nu} F^{\mu\nu} \right).$$

(2.12)

Static spherically symmetric black holes solutions may be obtained by making use of internal symmetries of the dimensionally reduced 3-dimensional action. Writing the metric in the form

$$ds^2 = -e^{2u}dt^2 + e^{-2u}h_{ij}dx^idx^j,$$

(2.13)

where $i, j$ run from 1 to 3, and taking the Maxwell field strength 2-form to be the exterior derivative of the 1-form vector potential $A = \phi dt$ where $\phi(x^i)$ is the electric potential, the system may be described by the 3-dimensional non-linear $\sigma$-model action

$$\int d^3x \sqrt{h} \left\{ R(h) - 2(\partial u)^2 + 2(\partial \sigma)^2 + 2e^{-2u-2a\sigma}(\partial \phi)^2 \right\}.$$

(2.14)

Charged solutions may thus be generated from neutral ones using the symmetries of this action. The continuous symmetries of the action are given infinitesimally by the following 4 Killing vectors acting on the internal target space:

$$k^{(1)} = \frac{\partial}{\partial \phi}, \quad k^{(2)} = \phi \frac{\partial}{\partial \phi} + \frac{1}{a} \frac{\partial}{\partial \sigma}, \quad k^{(3)} = \frac{\partial}{\partial u} + \phi \frac{\partial}{\partial \phi},$$
\[ k^{(4)} = 2\phi \frac{\partial}{\partial u} + \left[ e^{2u+2a\sigma} + (1 - a^2)\phi^2 \right] \frac{\partial}{\partial \phi} - 2a\phi \frac{\partial}{\partial \sigma}. \] (2.15)

\( k^{(4)} \) is the generator of the required anti-dilaton-Harrison transformation which produces electrically charged solutions from neutral ones.

Alternatively, a class of static black holes may be obtained from the usual dilaton black holes by making the substitutions \( \sigma \rightarrow i\sigma \) and \( a \rightarrow -ia \). The new scalar field \( \sigma \) and \( a \) are required to be real and so in the original solution they must be made pure imaginary by analytic continuation. The new metric is thus obtained simply by the replacement \( a^2 \rightarrow -a^2 \):

\[
\begin{align*}
\ ds^2 &= - \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{r_-}{r} \right)^{\frac{1+ia^2}{1-a^2}} \ dt^2 + \left( 1 - \frac{r_+}{r} \right)^{-1} \left( 1 - \frac{r_-}{r} \right)^{\frac{1+ia^2}{a^2+1}} \ dr^2 \\
&\quad + \frac{a^2}{a^2+1} (d\theta^2 + \sin^2 \theta d\phi^2). \tag{2.16}
\end{align*}
\]

The scalar field becomes

\[ e^\sigma = \left( 1 - \frac{r_-}{r} \right)^{\frac{a^2}{a^2+1}} \] (2.17)

and the Maxwell field is

\[ F = \frac{Q}{r^2} dt \wedge dr, \] (2.18)

where

\[ Q = \sqrt{\frac{r_+r_-}{1-a^2}}. \] (2.19)

Thus for \( a^2 < 1 \) we must have \( 0 \leq r_- \leq r_+ \) and for \( a^2 > 1, r_- \leq 0 \leq r_+ \). Clearly then extreme solutions are only possible if \( a^2 < 1 \), and they satisfy

\[ M^2 = \frac{Q^2}{1-a^2}. \] (2.20)

The scalar charge is given by

\[ \Sigma = \frac{ar_-}{1-a^2} \] (2.21)

and the ADM mass is

\[ M = \frac{1}{2} \left( r_+ + \frac{1 + a^2}{1 - a^2} r_- \right). \] (2.22)
Thus we have
\[ M^2 - \Sigma^2 - Q^2 = \frac{1}{4}(r_+ - r_-)^2 \] (2.23)
and so these black holes cannot have zero mass.

The solutions for \( a^2 = 1 \) require more careful consideration. As we take the limit \( a^2 \to 1 \), it is clear that various metric components will blow up, unless \( r_- \to 0 \). However, we may obtain finite results by setting \( r_- = |\Sigma|(1 - a^2) \) and keeping \( \Sigma \) constant as we take the limit:
\[
\begin{align*}
    ds^2 &= -\left(1 - \frac{r_+}{r}\right)e^{-2|\Sigma|/r} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1}e^{2|\Sigma|/r} dr^2 \\
    &\quad + r^2 e^{2|\Sigma|/r} (d\theta^2 + \sin^2\theta d\phi^2). \text{ (2.24)}
\end{align*}
\]
The scalar field becomes
\[ \sigma = \frac{\Sigma}{r}, \text{ (2.25)} \]
where the sign of \( \Sigma \) is the same as the sign of \( a = \pm 1 \) and the Maxwell field is
\[
F = \frac{\sqrt{|\Sigma| r_+}}{r^2} dt \wedge dr. \text{ (2.26)}
\]
The ADM mass is now
\[ M = |\Sigma| + \frac{1}{2}r_+ \text{ (2.27)} \]
which is always positive.

For \( a^2 > 1 \) the causal structure is the same as for the Schwarzschild solution, with a spacelike singularity at \( r = 0 \) hidden behind a regular event horizon at \( r = r_+ \). The singularity at \( r = r_- \) is unreachable, since \( r_- \leq 0 \) in this case. For \( a^2 \leq 1 \), the singularity at \( r = r_- \geq 0 \) is null (except in the Einstein-Maxwell case, \( a = 0 \), when it becomes the inner Cauchy horizon).

### 2.2 EMD Theory

If we allow both the scalar and the Maxwell fields to have negative kinetic terms, then we expect once more to find regular zero rest mass black holes. The action is
\[
\frac{1}{16\pi} \left( R + 2(\partial\sigma)^2 + e^{-2\sigma} F_{\mu\nu} F^{\mu\nu} \right) \text{ (2.28)}
\]
and the solutions may be obtained from those above by taking $Q$ to be pure imaginary. The metric is the same but now

$$Q = \sqrt{\frac{r_+ r_-}{1 - a^2}}, \quad (2.29)$$

so if $a^2 < 1$, $r_- \leq 0$ and if $a^2 > 1$, $r_+ \geq 0$. Extreme solutions satisfy

$$M^2 = \frac{Q^2}{a^2 - 1} \quad (2.30)$$

and so are only possible for $a^2 > 1$.

Zero mass black holes are now possible once more, if we set

$$r_+ = \frac{1 + a^2}{a^2 - 1} r_- \quad (2.31)$$

for $a^2 \neq 1$. The case $a^2 = 1$ must be checked separately. Setting $r_- = |\Sigma|(a^2 - 1)$ and taking the limit $a^2 \to 1$ we obtain the metric

$$ds^2 = - \left(1 - \frac{r_+}{r}\right) e^{2|\Sigma|/r} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} e^{-2|\Sigma|/r} dr^2$$

$$+ r^2 e^{-2|\Sigma|/r} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.32)$$

with

$$F = \frac{\sqrt{|\Sigma|}}{r^{2}} dt \wedge dr \quad (2.33)$$

and

$$\sigma = \frac{\Sigma}{r}, \quad (2.34)$$

where the sign of $\Sigma$ is now opposite to that of $a$. The ADM mass is therefore

$$M = \frac{1}{2} r_+ - |\Sigma| \quad (2.35)$$

which may become zero or negative.

Thus we see that zero rest mass black holes, non-singular outside a regular event horizon, violating the Positive Energy Theorem, exist in $\text{EMD}$ theory for $a^2 < 1$ and in $\text{EMD}$ theory for all $a$, but not in $\text{EMD}$ theory. That is not to say that the Positive Energy Theorem holds in $\text{EMD}$ theory.
Since the scalar field has negative kinetic energy, it violates the Dominant Energy Condition (in fact it violates the Weak Energy Condition) and so we expect counter-examples to the Positive Energy Theorem. One such counter-example with $F_{\mu \nu} = 0$ may be obtained from the Schwarzschild solution by using the discrete duality between the metric function $u$ and the dilaton $\sigma$ in the action (2.14) with $\phi = 0$. The resulting solution describes a transparent massless wormhole:

$$ds^2 = -dt^2 + dr^2 + (r^2 + c^2)(d\theta^2 + \sin^2 \theta d\phi^2),$$

$$\sigma = \tan^{-1} \left( \frac{r}{c} \right). \quad (2.36)$$

Since $F_{\mu \nu} = 0$, this is a solution of both EMD and EMD theory, violating the Positive Energy Theorem. This is discussed in more detail in section 5.

It is interesting to ask whether such solutions are stable. Presumably they are not. In the case of zero mass black holes, some insight into this question may be gained by studying their thermodynamic properties and also by considering accelerating solutions.

## 3 Accelerating Solutions

In order to see how these black holes move we turn to the dilaton-C-metrics [12, 13]. In EMD theory these have the form

$$ds^2 = \frac{1}{A^2(x - y)^2} \left\{ F(x) \left[ G(y) dt^2 - \frac{dy^2}{G(y)} \right] + F(y) \left[ \frac{dx^2}{G(x)} + G(x) d\phi^2 \right] \right\}. \quad (3.1)$$

In the magnetic case the scalar field is

$$e^{2\alpha} = \frac{F(x)}{F(y)} \quad (3.2)$$

where

$$F(u) \equiv (1 + r_-Au)^{\frac{2}{1 + \alpha^2}} \quad (3.3)$$

and

$$G(u) \equiv \left[ 1 - u^2(1 + r_+Au) \right] (1 + r_-Au)^{\frac{1}{1 + \alpha^2}}. \quad (3.4)$$
The vector field is given by

\[ A = \sqrt{\frac{-r_+ r_-}{1 + a^2}} x \text{d}\phi. \]  

(3.5)

The electrically charged solutions may be obtained by a generalized duality transformation. This transformation leaves the metric unchanged. As before we are interested in the case when \( r_+ \) is positive but \( r_- \) negative. We take the acceleration parameter \( A \) to be positive. The behaviour of the solutions depends upon the cubic

\[ \overline{G}(u) \equiv 1 - u^2(1 + r_+ Au). \]  

(3.6)

Because \( r_- \) is negative the ordering of the roots is different from the standard one and so our labelling is different. For \( 0 < r_+ A < \frac{2}{\sqrt{27}} \) the cubic \( \overline{G}(u) \) has three real roots, which we label in increasing order:

\[-\frac{1}{Ar_+} < u_1 < -\sqrt{3} < u_2 < -1 \quad \text{and} \quad \frac{\sqrt{3}}{2} < u_3 < 1.\]  

(3.7)

Thus roots \( u_1 \) and \( u_2 \) are negative and \( u_3 \) is positive. In addition, if \( a^2 < 1 \) then \( G(u) \) has another zero corresponding to the root of the function

\[ \overline{F}(u) = 1 + r_- Au \]  

(3.8)

i.e. \( u = u_4 = -\frac{1}{Ar_-} \).

If we interpret \( x \) as an angular coordinate then the requirement that \( g_{\phi\phi} > 0 \) and that it have finite range dictates it must vary between the least negative and the least positive root of \( G(u) \), i.e. \( x \in [u_2, u_3] \). The end points of this interval will be regular axes of symmetry free of conical singularities if and only if the ‘no strut’ condition

\[ G'(u_2) + G'(u_3) = 0 \]  

(3.9)

holds. We are assuming of course that \((r_+, r_-, A)\) are chosen so that

\[ u_3 < u_4 = -\frac{1}{r_- A}. \]  

(3.10)

Because \( u_3 < 1 \) it suffices that \( |r_-| < \frac{1}{A} \) for this to be true.
If the coordinate $t$ is to be timelike then the radial variable $y$ must range between the two negative or the two positive roots of $G(u)$. In the limit $A \to 0$, the coordinate transformation $y \to -\frac{1}{\lambda r}$, $x \to \cos \theta$, $t \to At$, and $\phi \to \phi$ gives the EMD black hole solutions in the usual coordinates. Thus $y = u_1$ corresponds to the black hole event horizon and, for $A \neq 0$, $y = u_2$ is the acceleration horizon. Therefore we choose $y \in [u_1, u_2]$. The resulting solution will be asymptotically flat. Because it is boost-invariant the total ADM 4-momentum vanishes. The solution may be thought of as a pair of uniformly accelerating black holes with ‘mass parameter’ $M$ and anti-charge $Q$. However it is better to think of $M$ as a measure of the size of the horizon. As in the case of the static black holes considered in the last section, the Cauchy horizon/singularity at $r_-$ cannot be reached because $r_- < 0$ and there is a singularity at $r = 0$. The causal structure is thus equivalent to that of the usual vacuum C-metric [12].

It is well known that if the vector field has the usual sign in the Lagrangian then it is impossible to satisfy the regularity condition (3.9) for any value of the electric charge, including zero, and thereby eliminate the conical singularities. Indeed if this were possible then the solutions would violate the Positive Mass Theorem for black holes. Moreover one might then expect to be able to pick the acceleration parameter of the solution so as to render the surface gravities of the acceleration horizon and the event horizon equal. If this were possible it would be possible to analytically continue the time parameter to give a regular Euclidean instanton solution (periodic in imaginary time) which would mediate the instability of Minkowski spacetime. However, as we shall see, the situation is rather different if the vector field enters the Lagrangian with the opposite sign. This is not surprising because we have seen in the previous section that static non-singular configurations with vanishing total 4-momentum exist in such theories.

The simplest case to consider is the Einstein-anti-Maxwell one, $a = 0$. Then the function $G(u)$ becomes (after a suitable linear coordinate transformation of $x$ and $y$, see [12]) the quartic:

$$G(u) = 1 - u^2 - 2MAu^3 + Q^2A^2u^4. \quad (3.11)$$

Clearly if we set $M = 0$ this quartic polynomial will be symmetrical about the origin and the no strut condition is automatically satisfied by symmetry. Thus we have a regular Lorentzian solution representing two uniformly accelerating black holes with equal and opposite charges in an asymptotically flat
spacetime with vanishing total ADM mass. Note that ultimately the black holes will approach the velocity of light.

Because the solution without an applied electric field has no conical singularity, adding an electric field should produce a solution with a conical singularity. If this is correct it indicates that trying to accelerate a zero rest mass black hole with an electric field cannot give rise to a constant acceleration.

What about the Instanton? We need to check to see whether we can choose $QA$ so that the surface gravities of the event horizon and the acceleration horizon can be equal. This requires that the slopes at the two roots $u = u_1$ and $u = u_2$ be equal in magnitude and opposite in sign. This can only happen in the limiting case that they coincide $u_1 = u_2$. This requires that the charge and acceleration be related by

$$|Q| = \frac{1}{2} A. \quad (3.12)$$

Thus we see that although accelerating solutions without conical singularities exist in this case, there are no instanton solutions which would mediate the decay of the vacuum. We will discuss vacuum decay in more detail in section 7.

The situation with a scalar field present is only slightly more complicated. Remaining with EMD theory, so the scalar has positive kinetic energy, we recall that zero rest mass black holes were possible only if $a^2 < 1$. Thus we expect to be able to satisfy the no strut condition only if $a^2 < 1$ and obtain a regular solution representing a pair of zero rest mass EMD black holes accelerating away from one another.

Writing the function $G(u)$ as a product of its factors, we have

$$G(u) = C(u - u_1)(u - u_2)(u - u_3)(u - u_4)^p \quad (3.13)$$

where

$$p = \frac{1 - a^2}{1 + a^2} \quad (3.14)$$

and $u_1, u_2, u_3, u_4$ are ordered:

$$u_1 < u_2 < u_3 < u_4. \quad (3.15)$$

The no strut condition then becomes

$$(u_2 - u_1)(u_4 - u_2)^p = (u_3 - u_1)(u_4 - u_3)^p. \quad (3.16)$$
We may solve this to express \( u_4 \), say, in terms of the other roots:

\[
  u_4 = \frac{u_3(u_3 - u_1)^\frac{1}{p} - u_2(u_2 - u_1)^\frac{1}{p}}{(u_3 - u_1)^\frac{1}{p} - (u_2 - u_1)^\frac{1}{p}}. \tag{3.17}
\]

Finally, for consistency, we need to check that \( u_4 > u_3 \). If \( p > 0 \) (i.e. \( a^2 < 1 \)), then the denominator of the expression for \( u_4 \) is positive and we can multiply up. The inequality \( u_4 > u_3 \) then simplifies to

\[
  (u_3 - u_2)(u_2 - u_1)^\frac{1}{p} > 0, \tag{3.18}
\]

which holds automatically. Thus the no strut condition can be made to hold if \( a^2 < 1 \). If \( a^2 > 1 \) (i.e. \( p < 0 \)), the denominator in the expression for \( u_4 \) is negative, all inequalities must be reversed on multiplying up and the condition \( u_4 > u_3 \) cannot be satisfied. In the special case \( a^2 = 1, p = 0 \) and the no strut condition reduces to \( u_2 = u_3 \).

Thus, as predicted, the conical singularity in the EMD C-metric may only be removed if \( a^2 < 1 \), giving a regular solution representing a pair of zero rest mass black holes with equal and opposite charges accelerating away from one another.

### 3.1 EMD Accelerating Solutions

The EMD C-metric may be obtained from the usual dilaton C-metric by the replacement \( \sigma \to i\sigma, a \to -ia \). The form of the metric remains unchanged but the functions \( F \) and \( G \) become

\[
  F(u) = (1 + r_Au)^{\frac{2a^2}{a^2-1}},
  G(u) = \left[1 - u^2(1 + r_Au)\right](1 + r_Au)^{\frac{1+a^2}{1-a^2}}. \tag{3.19}
\]

The scalar field is again given by

\[
  e^{2\sigma} = \frac{F(x)}{F(y)}, \tag{3.20}
\]

and the vector potential 1-form is

\[
  A = \sqrt{\frac{r_+r_-}{1 - a^2}}xd\phi. \tag{3.21}
\]
For \( a^2 < 1 \), \( 0 \leq r_- < r_+ \) and so the new ordering of the roots of \( G(u) \) becomes
\[
    u_4 = -\frac{1}{Ar_-} < u_1 < u_2 < u_3. \tag{3.22}
\]
As before we restrict \( x \) to lie between \( u_2 \) and \( u_3 \) so that the metric component \( g_{\phi\phi} \) is positive and \( y \in [u_1, u_2] \). The no strut condition \( G'(u_2) + G'(u_3) = 0 \) is equivalent to
\[
    (u_2 - u_1)(u_2 - u_4)^p = (u_3 - u_1)(u_3 - u_4)^p \tag{3.23}
\]
where
\[
    p = \frac{1 + a^2}{1 - a^2} > 0. \tag{3.24}
\]
Thus solving for \( u_4 \), we obtain
\[
    u_4 = \frac{u_3(u_3 - u_1)^{\frac{1}{p}} - u_2(u_2 - u_1)^{\frac{1}{p}}}{(u_3 - u_1)^{\frac{1}{p}} - (u_2 - u_1)^{\frac{1}{p}}}. \tag{3.25}
\]
For consistency, we require that \( u_4 < u_1 \) which, after multiplying up, is equivalent to
\[
    (u_3 - u_1)^{1 + \frac{1}{p}} < (u_2 - u_1)^{1 + \frac{1}{p}}. \tag{3.26}
\]
But, since \( 1 + \frac{1}{p} = \frac{2}{1 + a^2} > 0 \), this inequality does not hold. Therefore, with the required ordering of the roots, the no strut condition cannot hold. A special case of this result is \( a = 0 \), ordinary Einstein-Maxwell theory, for which it is well known that one cannot remove the conical singularities of the C-metric.

For \( a^2 > 1 \), \( r_- \leq 0 < r_+ \), the ordering of the roots reverts to
\[
    u_1 < u_2 < u_3 < u_4 = -\frac{1}{Ar_-} \tag{3.27}
\]
and a similar argument can be used to show that the no strut condition cannot hold. In fact the argument is identical to the argument used in EM\( \bar{D} \) theory for \( a^2 > 1 \) above.

The special case \( a^2 = 1 \) requires more careful consideration. As in the last section, when we were considering EM\( \bar{D} \) black holes, we set \( r_- = |\Sigma|(1 - a^2) \) and take the limit \( a^2 \to 1 \). This gives
\[
    F(u) = e^{-2|\Sigma|Au},
    G(u) = \left[1 - u^2(1 + r_+Au)\right]e^{2|\Sigma|Au}. \tag{3.28}
\]
Thus the fourth root $u_4$ of $G(u)$ disappears and $G(u)$ just has the 3 roots $u_1 < u_2 < u_3$. The no strut condition $G'(u_2) + G'(u_3) = 0$ becomes
\[(u_2 - u_1)e^{2|\Sigma|Au_2} = (u_3 - u_1)e^{2|\Sigma|Au_3}.\] (3.29)

Since $u_2$ and $u_3$ are larger than $u_1$ and we are assuming here that $A > 0$, this condition may only be satisfied in the degenerate case $u_2 = u_3$.

Thus we see that, as in ordinary EMD theory, in EMD theory one can never remove the conical singularities in the C-metric to obtain a regular solution describing a pair of zero rest mass black holes accelerating away from one another. This is as expected, since the theory did not allow static zero rest mass black holes of this type and so, although we expect the vacuum to be unstable due to the presence of a scalar with negative kinetic energy, the instability must manifest itself in a different way. One possibility is that there may be a semi-classical instability leading to the production of transparent massless wormholes of the type discussed in section 2.2. This will be discussed in more detail in section 5.

### 3.2 EMD Accelerating Solutions

These solutions may be obtained from the previous solutions by analytic continuation, setting the charge to be pure imaginary. The metric is unchanged but the Maxwell potential becomes
\[A = \sqrt{\frac{r_+r_-}{a^2 - 1}}xd\phi.\] (3.30)

Therefore, for $a^2 < 1$, $r_- \leq 0$ and the proof that the no strut condition can be satisfied is identical to the EMD case. For $a^2 > 1$, $0 \leq r_- < r_+$ and it is easy to show that the conical singularities can still be removed by an appropriate choice of $r_+$ and $r_-$. In the special case $a^2 = 1$, we set $r_- = |\Sigma|(a^2 - 1)$ and take the limit $a^2 \to 1$ to obtain
\[F(u) = e^{2|\Sigma|Au},\]
\[G(u) = \left[1 - u^2(1 + r_+Au)\right]e^{-2|\Sigma|Au}.\] (3.31)

The no strut condition then becomes
\[(u_2 - u_1)e^{-2|\Sigma|Au_2} = (u_3 - u_1)e^{-2|\Sigma|Au_3}.\] (3.32)
It is easy to see that this does now have solutions satisfying \( u_1 < u_2 < u_3 \).

Thus, for all values of the dilaton coupling \( a \) in EMD theory, it is possible to remove the conical singularities from the C-metric to obtain a regular solution describing 2 zero rest mass black holes with equal and opposite charges accelerating away from one another. This is consistent with the result of the last section that, for all \( a \), EMD theory allows static zero rest mass black holes which are non-singular outside a regular event horizon.

\section{Actions and Thermodynamics}

In this section we study the thermodynamic properties of the black holes discussed in the last section. We do this first by calculating their classical Euclidean actions since this will also be useful in the later section on vacuum stability. Throughout this section, we will assume that the scalar field \( \sigma \) has positive kinetic energy. We will work with the EMD action, since the EMD solutions may be obtained simply by changing the sign of \( r_- \). The Euclidean action is given by

\begin{equation}
I_E = -\frac{1}{16\pi} \int_M d^4x \sqrt{g} \left\{ R - 2(\partial\sigma)^2 - e^{-2\sigma} F^2 \right\} - \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{h} [K] \quad (4.1)
\end{equation}

where the boundary term is added to cancel second derivatives of the metric appearing in the action and it ensures that \( I_E \) is additive over adjacent space-time regions. The Euclidean dilaton black hole solutions may be obtained from their Lorentzian counterparts by setting \( t = i\tau \):

\begin{equation}
ds^2 = \left( 1 - \frac{r_-}{r} \right) \left( 1 - \frac{r_+}{r} \right)^{1+a^2}{1+a^2} \ d\tau^2 + \left( 1 - \frac{r_-}{r} \right)^{-1} \left( 1 - \frac{r_+}{r} \right)^{-\frac{2a^2}{1+a^2}} \ dr^2 \\
+ r^2 \left( 1 - \frac{r_-}{r} \right)^{\frac{2a^2}{1+a^2}} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.2)
\end{equation}

Restricting ourselves to electrically charged solutions the electromagnetic field strength 2-form becomes

\begin{equation}
F = \frac{i}{r^2} \sqrt{\frac{r_+ r_-}{1+a^2}} d\tau \wedge dr. \quad (4.3)
\end{equation}
In the EMD case we have $0 \leq r_- < r_+$ so that $F$ is pure imaginary for the Euclidean solutions. In EMD theory we must replace $r_-$ by $-r_-$ in $F$ but now $r_- \leq 0 < r_+$ so that $F$ is still pure imaginary.

Note that the metric is apparently singular at $r = r_+$. To examine the metric just outside $r = r_+$ more closely, in the non-extreme case, set $r = r_+ + \varepsilon^2$ and look at the line element for small $\varepsilon$:

$$ds^2 \approx \frac{\varepsilon^2}{r_+} \left( 1 - \frac{r_-}{r_+} \right) \frac{1}{1+a^2} d\tau^2 + 4r_+ \left( 1 - \frac{r_-}{r_+} \right) \frac{a^2 - 1}{1+a^2} d\varepsilon^2,$$

(4.4)

neglecting the $\theta, \phi$ dependance. Clearly $r = r_+ (\varepsilon = 0)$ is a conical singularity which may be removed by identifying $\tau$ with period $\beta$ where

$$\beta = 4\pi r_+ \left( 1 - \frac{r_-}{r_+} \right) \frac{a^2 - 1}{1+a^2}.$$  

(4.5)

### 4.1 Actions for Non-Extreme Black Holes

The calculation of the action is simplified by making use of the Einstein equations which imply that for on-shell solutions $R = 2(\partial \sigma)^2$ and so the action reduces to

$$I_E = \frac{1}{16\pi} \int_M d^4 x \sqrt{g} e^{-2a\sigma} F^2 - \frac{1}{8\pi} \int_{\partial M} d^3 x \sqrt{h} [K].$$

(4.6)

The volume term may be evaluated very simply. We have spherical symmetry and $\tau \in [0, \beta], r \in [r_+, \infty)$ so, absorbing the dilaton factor into the integration measure, the integral reduces to

$$\int_M d^4 x \sqrt{g} e^{-2a\sigma} \cdots = 4\pi \beta \int_{r_+}^{\infty} dr \ r^2 \cdots.$$  

(4.7)

The scalar invariant $F^2$ is given by

$$F^2 = -\frac{2r_+ r_-}{(1+a^2) r^4}$$

(4.8)

and so the volume integral contribution to the action is

$$\frac{1}{16\pi} \int_M d^4 x \sqrt{g} e^{-2a\sigma} F^2 = -\frac{\beta r_-}{2(1+a^2)}.$$  

(4.9)
Horizon \( r = r_- \)

Boundary at Infinity

Figure 1: Geometry of non-extreme Euclidean dilaton black holes

In the EMD case, \( F^2 \) becomes \( \frac{2r+r_+}{r(1+a^2)r^4} \), with \( r_- \) negative, but the sign of the \( F^2 \) term in the action is changed and so the contribution to the action is given by the same formula (4.9).

To evaluate the boundary term, it is necessary to identify precisely where the boundaries are. It is here that the extreme solutions differ from the non-extreme solutions. We will deal with the non-extreme case first. In this case, the topology of the Euclidean solution is the well-known cigar shape, see Fig. 1 (\( \theta, \phi \) coordinates suppressed). Thus the only boundary is at \( r = \infty \). The contribution to the action is the integral over this boundary of \([K]\), the trace of the extrinsic curvature of the boundary minus a term to ensure that the action of flat space is zero. Shifting the radial coordinate \( r \rightarrow r - \frac{a^2r_+}{1+a^2} \) so that, for large \( r \), a sphere of radius \( r \) has surface area \( 4\pi r^2 + \mathcal{O}(1) \), the metric for large \( r \) becomes

\[
ds^2 \sim \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\quad (4.10)
\]

where

\[
M = \frac{1}{2} \left( r_+ + \frac{1-a^2}{1+a^2}r_- \right). \quad (4.11)
\]
The normal to the surface \( r = R \) is \( n_\mu = (0, N, 0, 0) \) where \( N = \sqrt{g_{rr}} = 1 + \frac{M}{R} + O(R^{-2}) \) is the lapse function. The trace of the extrinsic curvature is

\[
K = \frac{1}{2N} \left[ g^{rr} g_{rr,r} + g^{\theta\theta} g_{\theta\theta,r} + g^{\phi\phi} g_{\phi\phi,r} \right]
\]

\[
= \frac{2}{R} - \frac{M}{R^2} + O(R^{-3}). \tag{4.12}
\]

Thus we may identify the \( \frac{2}{R} \) term as the contribution from flat \( \mathbb{R}^4 \) which must be subtracted to ensure that the action of flat space is zero. This gives \( [K] = -\frac{M}{R^2} + O(R^{-3}) \). The surface integral over the surface \( r = R \) is

\[
\int_{r=R} d^3x \sqrt{h} = 4\pi\beta R^2 + O(R). \tag{4.13}
\]

Therefore the boundary integral contribution to the action is

\[
-\frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{h} [K] = \frac{1}{2} \beta M. \tag{4.14}
\]

Thus the total Euclidean action may be written as

\[
I_E = \frac{1}{4} \beta (r_+ - r_-) = \frac{1}{2} \beta (M - Q\Phi) \tag{4.15}
\]

where \( \Phi = \frac{Q}{r_+} \) is the electric potential on the horizon, in a gauge in which \( A_\tau \) vanishes at infinity. Using the expression for \( \beta \) (4.5) we may write \( I_E \) in terms of \( r_\pm \):

\[
I_E = \pi r_+^2 \left( 1 - \frac{r_-}{r_+} \right)^{\frac{2a^2}{1+a^2}}. \tag{4.16}
\]

This is precisely \( \frac{1}{4} A \) where \( A \) is the area of the black hole event horizon. This formula, in terms of \( r_\pm \), continues to be true for EMD black holes.

### 4.2 Thermodynamics

In ordinary Einstein-Maxwell theory the Reissner-Nordström black hole is expected to undergo Hawking radiation and thereby lose mass. Assuming that the black hole emits neutral particles, the black hole charge will remain constant and the mass will decrease. Eventually the black hole will approach extremality and the Hawking radiation will turn off. It is well known that this
process takes an infinitely long time and so an extreme Reissner-Nordström solution is never actually produced. In the EMD case, the situation is rather different [14]. In particular, for $a^2 > 1$ the Hawking temperature blows up as one approaches the extreme solution. Also, however, the area of the event horizon decreases to zero. It is thus interesting to ask which effect dominates and how long it takes to produce an extreme solution.

In the EMD case there are no extreme solutions and the Hawking radiation is never turned off. One is then lead naturally to ask what happens as such black holes evaporate. In the case $a^2 < 1$ when regular black holes with $M \leq 0$ are allowed, there seems to be no reason why the black holes cannot evaporate away all of their mass. But what happens next? Is there some lower bound on the mass or will it continue to decrease to $-\infty$? How long will this process take? In the case $a^2 \geq 1$ there is a lower bound on the mass, $M \geq |Q|\sqrt{a^2 - 1}$, and again there are no extreme solutions. So what is the endpoint of the evaporation process in this case?

In the study of thermodynamics it is useful to define thermodynamic potentials. Let us define the Gibbs Free Energy:

$$W = M - TS - Q\Phi = -T\log Z.$$  \hspace{1cm} (4.17)

In the semi-classical approximation we approximate the partition function $Z$ by $e^{-IE}$. Thus we have

$$W = TI_E = \frac{1}{2}(M - Q\Phi).$$  \hspace{1cm} (4.18)

This then gives the well known result

$$S = I_E = \frac{1}{4}A$$  \hspace{1cm} (4.19)

and also the Smarr formula for the mass:

$$M = 2TS + Q\Phi.$$  \hspace{1cm} (4.20)

Regarding $M$ as a homogeneous function of $S$ and $Q$ we obtain directly the First Law of black hole thermodynamics:

$$dM = TdS + \Phi dQ.$$  \hspace{1cm} (4.21)
Figure 2: Specific heats of EMD black holes for various values of $a$

Since the evaporation is via neutral particles, we must consider processes taking place at constant $Q$. For example, consider the specific heat at constant $Q$:

$$C_Q = \frac{\partial M}{\partial T} \bigg|_Q = T \frac{\partial S}{\partial T} \bigg|_Q .$$

In terms of $r_\pm$ this is

$$C_Q = -2\pi r_+^2 \left( 1 - \frac{1 - a^2 r_-}{1 + a^2 r_+} \right) \left( 1 - \frac{3 - a^2 r_-}{1 + a^2 r_+} \right)^{-1} \left( 1 - \frac{r_-}{r_+} \right)^{\frac{2a^2}{1 + a^2}}. \quad (4.23)$$

In ordinary EMD theory we have $0 \leq r_- < r_+$. For $a^2 \neq 1$ one may express $\frac{C_Q}{Q^2}$ as a function of $\frac{M}{Q}$, see Fig. 2. For $a^2 < 1$ the specific heat diverges when
\[ r_+ = \frac{3 - a^2}{1 + a^2} r_- \quad \text{i.e.} \quad \left| \frac{M}{Q} \right| = \frac{2 - a^2}{\sqrt{3 - a^2}}. \quad (4.24) \]

For \( a^2 > 1 \) this divergence is excluded by the requirement that \( r_+ > r_- \).

The origin of this divergence can be seen by considering behaviour of the temperature as \( M \) varies. Initially, for large \( M \), the temperature increases as \( M \) decreases. However, for \( a^2 < 1 \), the temperature ultimately approaches zero as one approaches extremality and so there must be a turning point in the temperature as a function of \( M \) at constant \( Q \). This then is the divergence of the specific heat \( C_Q \). In the case \( a^2 > 1 \), however, the temperature diverges as one approaches extremality and so there is no such turning point. The case \( a^2 = 1 \) is somewhat special. In this case the specific heat reduces to

\[ C_Q = -2\pi r_+^2 = -8\pi M^2 \quad (4.25) \]

independent of \( Q \). The temperature is \( \frac{1}{8\pi M} \), also independent of \( Q \). This is the case which arises in string theory and we have the interesting result that the thermodynamic properties at constant charge are identical to those of the the Schwarzschild solution in General Relativity. This may be seen as a result of the fact that for \( a^2 = 1 \) the metric is the same as the Schwarzschild solution except for the angular components of the metric which do not affect its thermodynamic properties. We may thus deduce immediately that the black hole will radiate away its mass faster and faster, reaching extremality in a finite time.

To estimate the time taken for the more general EMD black holes to radiate sufficient mass to become extreme, we use Stefan’s law to approximate the rate of loss of mass of the black hole:

\[ \frac{dM}{dt} \approx -\sigma A T^4 \quad (4.26) \]

which gives

\[ \frac{dM}{dt} \propto \frac{1}{r_+^2} \left( 1 - \frac{r_-}{r_+} \right)^{\frac{4-2a^2}{1+a^2}} \quad (4.27) \]

with \( Q \) held constant. For nearly extreme solutions, we may set \( M = \frac{Q}{\sqrt{1+a^2}} + \epsilon \) giving

\[ \frac{d\epsilon}{dt} \propto \epsilon^{\frac{4-2a^2}{1+a^2}}. \quad (4.28) \]
Thus the time taken to reach extremality will be infinite for $a^2 < 1$ and finite for $a^2 \geq 1$. Since the area of the event horizon vanishes for extreme EMD black holes, this semi-classical description of the black hole thermodynamics is expected to break down near extremality and a more complete quantum theory is needed to make further predictions about what happens near the end of the evaporation.

In EMD theory $r_- \leq 0 < r_+$ and there are no extreme solutions. So what is the endpoint of the evaporation? The specific heat at constant charge is again given by (4.23) but now $r_- \leq 0$ and the electric charge is given by $Q = \sqrt{-r_+ r_-}$. For $a^2 \neq 1$ the behaviour of $\frac{C_Q}{Q^2}$ as a function of $M/Q$ is shown in Fig. 3. Again the behaviour is qualitatively different depending on whether $a^2$ is greater than or less than 1. For $a^2 < 1$ there is no lower bound on the mass and the specific heat approaches zero as $M \to -\infty$. This can be seen as a result of the fact that the temperature $T$ increases without bound as $M \to -\infty$ with $Q$ held constant (see Fig. 4). A natural question to ask is then how quickly will the black hole radiate away mass and how long will it take for $M$ to reach $-\infty$. As $\frac{M}{|Q|} \to -\infty$, the outer horizon
Figure 4: Temperatures of EMD black holes for various values of $a$
shrinks as \( r_+ \sim \frac{1}{M} \) and the inner horizon approaches \(-\infty\) like \( M \). Thus the temperature \( T \) diverges like

\[
T \sim |M|^{\frac{3-a^2}{1+a^2}}
\]

and the area of the event horizon decreases to zero like

\[
\mathcal{A} \sim |M|^{-\frac{1}{1+a^2}}.
\]

Thus, if we approximate the rate of loss of mass by Stefan’s Law, we have

\[
\frac{dM}{dt} \sim -|M|^{\frac{3-a^2}{1+a^2}}.
\]

Therefore, since \( a^2 < 1 \), this rather crude approximation for the evaporation rate tells us that the black hole’s mass will reach \(-\infty\) within a finite time. Of course, as \( M \to -\infty \), the event horizon shrinks to zero size and so this semi-classical description will break down. Nonetheless this simple calculation is yet another indicator of the instability of these black holes.

For \( a^2 > 1 \) the situation is rather different. In this case the mass is bounded below: \( M \geq |Q|\sqrt{a^2 - 1} \) and no static spherically symmetric solutions exist with masses smaller than this. Also, as was pointed out in section 2, solutions saturating this bound are not extreme. They have \( r_+ \neq r_- \) and \( T \neq 0 \). Fig. 4 shows that as \( M \to |Q|\sqrt{a^2 - 1} \) the temperature \( T \) becomes finite and non-zero. Also the specific heat \( C_Q \) vanishes in this limit. How long does a black hole take to radiate away sufficient mass to saturate this bound? What happens to black holes which do saturate this mass bound? Since the temperature \( T \) and event horizon area \( \mathcal{A} \) both remain finite and non-zero in this limit, Stefan’s law predicts that EMD black holes with \( a^2 > 1 \) should evaporate and reach \( M = |Q|\sqrt{a^2 - 1} \) within a finite time. Once the black hole reaches this limit it can no longer lose mass at constant charge and remain static and spherically symmetric since no such solutions exist with \( M < |Q|\sqrt{a^2 - 1} \). Of course, when any black hole loses mass by the emission of a particle, the spacetime during the process is neither static nor spherically symmetric. However, in the case of a stable black hole such as the Schwarzschild solution, the spacetime may settle down to a new Schwarzschild solution with reduced mass after the emission of the particle. In the present case, however, there is no such static spherically symmetric solution for the spacetime to settle down to. We may thus expect some sort of
catastrophic quantum mechanical instability to be exhibited by these black holes.

The case $a^2 = 1$ is again somewhat special. The temperature and specific heat are given by
\begin{equation}
T = \frac{1}{8\pi M} \quad \text{and} \quad C_Q = -8\pi M^2,
\end{equation}

independent of $Q$. Note that once again these thermodynamic quantities are identical to those of the Schwarzschild solution. Also, as for the Schwarzschild solution, the mass is positive, $M > 0$. However, unlike the Schwarzschild solution, these black holes have electric charge and so we cannot set $M = 0$ unless $Q = 0$. If it were the case that these black holes with $a^2 = 1$ only emitted neutral particles then, as for the Schwarzschild black hole in the semi-classical approximation, they would radiate away all of their mass within a finite time. This is not possible, however, since no solution exists with $M = 0$ and $Q \neq 0$. This problem is easily resolved because as $M$ decreases with $Q$ held fixed, the ratio $Q / M$ increases without bound, as does the electric potential $\Phi$ on the horizon. Eventually this will become large enough to give a significant probability for the emission of charged particles by the black hole. Thus the black hole may radiate away all of its charge as well as all of its mass.

### 4.3 Actions for Extreme Black Holes

To calculate the Euclidean action for extreme dilaton black holes ($r_+ = r_- = r_H$) we need to take into account the fact that the topology of these solutions is different from that of non-extreme solutions and so there is potentially an extra contribution to the action from the inner boundary. The Euclidean metric is
\begin{equation}
\begin{aligned}
\text{ds}^2 &= \left(1 - \frac{r_H}{r}\right) \frac{2a^2}{1+a^2} d\tau^2 + \left(1 - \frac{r_H}{r}\right)^{-\frac{2}{1+a^2}} dr^2 \\
&\quad + r^2 \left(1 - \frac{r_H}{r}\right) \frac{2a^2}{1+a^2} (d\theta^2 + \sin^2 \theta d\phi^2).
\end{aligned}
\end{equation}

There are 2 distinct cases to be considered: $a = 0$ and $a \neq 0$. If $a = 0$ we have the extreme electric Euclidean Reissner-Nordström solution. The horizon $r = r_H$ is infinitely far away along any curve in this space and so
the topology is the the well-known pipette shape, see Fig. 5. Since the point $r = r_H$ is no longer a point in the space, we must add to the action a contribution from the extrinsic curvature integrated over an inner boundary just outside $r = r_H$. If $a \neq 0$ the point $r = r_H$ is a curvature singularity. We may deal with this by cutting it out of the space by introducing a boundary just outside $r = r_H$. In either case there is potentially a contribution to the Euclidean action due to the integral of the extrinsic curvature over this boundary. However, it is easy to check that this boundary term vanishes as the inner boundary is taken to $r_H$.

In the non-extreme case we were forced to identify $\tau$ with a particular period $\beta$ in order to remove a conical singularity on the horizon. In the extreme case, the horizon is no longer a part of the space and so we are free to identify $\tau$ with any period $\beta$ we choose, or not at all. Clearly then the Euclidean action vanishes in the extreme limit $r_+ \to r_-$ since the contributions from the boundary term at infinity $\frac{1}{2} \beta M$ and from the volume integral of $F^2$ cancel. Note that this would not be the case if we considered magnetically charged black holes since in that case $F$ would be real and the volume integral would give a positive contribution of $\frac{1}{2} \beta M$ to the action. The Euclidean action would then be $I_E = \beta M$. 

Figure 5: Geometry of extreme Euclidean Reissner-Nordström black hole
5 Massless Wormholes

If one considers a scalar field $\sigma$ with negative kinetic energy rather than a vector field then regular massless ultra-static wormhole solutions are possible. In four spacetime dimensions the transparent massless wormhole metric (2.36), in isotropic coordinates, is given by

$$ds^2 = -dt^2 + \frac{c^2}{4} \left(1 + \frac{1}{r^2}\right)^2 (dx^2 + dy^2 + dz^2),$$  \hspace{1cm} (5.1)

with $r^2 = x^2 + y^2 + z^2$ and the scalar field is

$$\sigma = \tan^{-1}\left(\frac{r^2 - 1}{2r}\right),$$  \hspace{1cm} (5.2)

where $c$ is a constant.

The spatial sections have the form of an Einstein-Rosen bridge joining two isometric regions each with vanishing ADM mass. The isometry interchanging the regions is the inversion:

$$r \rightarrow \frac{1}{r}. \hspace{1cm} (5.3)$$

Because the metric is ultra-static (i.e. $g_{00} = -1$) there is no horizon hiding the two sides of the bridge from one another as there is for black holes. Moreover the Newtonian gravitational potential $U = -\frac{1}{2} \ln(-g_{00})$ is constant and therefore the wormhole exerts no gravitational attraction.

Because the $g_{00} = -1$ we may regard the solution is an instanton of a three-dimensional theory. The four-dimensional Einstein equations are

$$R_{\mu\nu} = -2(\partial_\mu \sigma)(\partial_\nu \sigma)$$  \hspace{1cm} (5.4)

where $\mu, \nu = 0, 1, 2, 3$. The equations are trivially satisfied if either $\mu$ or $\nu = 0$. For $\mu, \nu = 1, 2, 3$, they give the Euler-Lagrange equations of a Euclidean three-dimensional theory with action (modulo boundary terms)

$$\frac{1}{16\pi} \int d^3x \sqrt{g} \left\{ R + 2(\partial \sigma)^2 \right\}$$  \hspace{1cm} (5.5)

in which the field $\sigma$ contributes negatively to the action. This unusual sign typically arises in wormhole theories when one analytically continues...
a Lorentzian theory (in the present case a (2+1)-dimensional theory) containing a pseudo-scalar field to a Riemannian one. An alternative viewpoint is to use Hodge duality to replace $d\sigma$ by a two-form

$$d\sigma = \star F.$$  \hspace{1cm} (5.6)

One then gets a theory of gravity coupled to abelian electrodynamics. In the absence of gravity it has been pointed out by Polyakov [13] that this theory admits (singular) instantons which correspond to Dirac monopoles. It is interesting that if one couples this system to gravity (which has no dynamical degrees of freedom in three dimensions) these Polyakov instantons become non-singular.

In theories without gravity one often says that non-singular instantons in $d$ spatial dimensions may be interpreted as non-singular solitons in $d+1$ spacetime dimensions. The present example with $d=3$, and its obvious generalization to higher $d$ show clearly that if gravity is involved then the soliton may be rather exotic. In particular it may be massless. Note, however, that if one really were considering gravity coupled to abelian electrodynamics, i.e. Einstein-Maxwell theory rather than gravity coupled to a scalar field, then the three-dimensional solutions cannot be trivially promoted to four-dimensional solutions. The field equations in four dimensions are

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 2F_{\alpha\mu}F_{\beta\nu}g^{\mu\nu} - \frac{1}{2}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}.$$  \hspace{1cm} (5.7)

The spatial equations agree with what one would get in three dimensions but the $0-0$ equation imposes an extra constraint which the Einstein-Polyakov instantons do not satisfy. This must be so because we know that Einstein-Maxwell theory satisfies the usual positive and dominant energy conditions and does not admit non-singular solutions with zero ADM mass.

It seems reasonable to conjecture that if the zero mass wormhole solutions were perturbed they would become singular and/or accelerate off to infinity. It would be interesting to know therefore whether there exist non-singular accelerating solutions. The obvious procedures seem only to lead accelerating solutions with singularities. It would also be interesting to know whether multi-solutions exist representing more than one wormhole at rest.

Another question relates to “tachyonic” solutions. Tachyons should be thought of as representing what happens to an unstable background after the instability has set in rather than indicating any violation of causality. It
is clear that the characteristics of a scalar field are still given by the metric regardless of whether it contributes negatively to the energy of the system. Tachyonic solutions will be discussed in greater detail later. For the time being we note that they should be invariant under $\mathbb{R} \times SO(2, 1)$ or $SO(2) \times SO(2, 1)$ rather than $\mathbb{R} \times SO(3)$. To obtain them in the present case one makes the replacement $t \to iz$ and $z \to it$ with $z$ and $t$ real and thus $r^2 \to x^2 + y^2 - t^2$. The resulting solutions

$$ds^2 = dz^2 + \frac{c^2}{4} \left( 1 + \frac{1}{r^2} \right)^2 (dx^2 + dy^2 - dt^2) \quad (5.8)$$

are invariant under translations in the new $z$ coordinate and are nakedly singular on the Cerenkov cone given by

$$x^2 + y^2 = t^2. \quad (5.9)$$

Note that if we set $t = 0$ we obtain time-symmetric initial data on $\mathbb{R} \times (\mathbb{R}^2 - \{0\})$, i.e. the product of a two-dimensional wormhole with the real line. The three-metric

$$ds^2 = dz^2 + \frac{c^2}{4} \left( 1 + \frac{1}{x^2 + y^2} \right)^2 (dx^2 + dy^2) \quad (5.10)$$

is non-singular but has zero ADM mass. The resulting spacetime subsequently becomes nakedly singular on the Cerenkov cone.

This behaviour is similar to that of $\lambda \phi^{\frac{n-2}{2}}$ scalar field theory in $n$ flat spacetime dimensions where the sign of $\lambda$ is such as to give a negative potential energy. There are solutions of the form

$$\frac{1}{(a^2 + x^2 - t^2)^{\frac{n-2}{2}}} \quad (5.11)$$

The solution has non singular time-symmetric initial data of finite energy at $t = 0$ but blows up on the spacelike hyperboloid

$$t = \sqrt{a^2 + x^2}. \quad (5.12)$$
We saw in the previous section the relevance of $SO(2,1)$ invariant solutions for the instability process. We shall refer to these as “tachyonic solutions” with the same understanding as before – they do not signal acausality but rather instability.

In vacuum or Einstein-Maxwell theory assuming that $SO(2,1)$ acts on two-dimensional orbits implies, by a simple generalization of Birkhoff’s theorem, the existence of an additional spacelike Killing vector. If this had non-compact orbits the spacetime would then have the same symmetry group as that of a spacelike world line in flat Minkowski spacetime. However there is a surprise. The replacement

\[ \theta \rightarrow \frac{\pi}{2} + it \]
\[ t \rightarrow iZ. \] (6.1)

in the standard Schwarzschild metric leads to

\[ ds^2 = \left(1 - \frac{r_+}{r}\right) dZ^2 + \left(1 - \frac{r_+}{r}\right)^{-1} dr^2 + r^2(-dt^2 + \cosh^2 t d\phi^2). \] (6.2)

This metric is indeed invariant under translations along the direction of motion of the “tachyon” (i.e. in the $Z$ direction) and reversal of the $Z$ coordinate. In fact the symmetries of the metric become more transparent if one introduces pseudo isotropic coordinates $T, X, Y$ by

\[ T = s \sinh t \]
\[ X = s \cosh t \cos \phi \] (6.3)
\[ Y = s \cosh t \sin \phi \]

with \( r = s(1 + \frac{r_+}{4s})^2 \). The metric then becomes

\[ ds^2 = \left(\frac{1 - \frac{r_+}{4s}}{1 + \frac{r_+}{4s}}\right)^2 dZ^2 + \left(1 + \frac{r_+}{4s}\right)^4 (-dT^2 + dX^2 + dY^2). \] (6.4)

This metric has been interpreted as that of a tachyon, the surface \( r = 2M \) being thought of as the analogue of a Cerenkov cone [16]. However this interpretation is not really tenable because if \( r_+ \) is taken to be positive then
the metric is complete and everywhere non-singular only as long as the spatial coordinate $Z$ is identified modulo $4\pi r_+:

$$0 \leq Z \leq 4\pi r_+. \quad (6.5)$$

The variable $s$ then runs from $s = \frac{1}{4}r_+$ to infinity and the pseudo isotropic coordinates $(T, Y, Y)$ are constrained to lie outside the hyperboloid

$$X^2 + Y^2 - T^2 \geq \frac{r_+^2}{4}. \quad (6.6)$$

The Killing horizon at $r = r_+$ (i.e. $s = \frac{1}{4}r_+$) is a null surface. One might be tempted to say that in some sense the presence of the tachyon has brought about the ‘compactification’ of space and the restriction of spacetime to the exterior of its Cerenkov cone.

In the light of our previous discussion in section 4 and the closely related case of the instability of the Kaluza-Klein vacuum [17] a more satisfactory interpretation is to regard this Lorentzian metric as the result of a tunnelling instability of the flat spacetime on $\mathbb{R}^3 \times S^1$ where the $S^1$ factor refers to the periodic spacelike coordinate $Z$. Alternatively one may think of it as providing the Cauchy development of the non-singular time symmetric initial data set on $S^1 \times \mathbb{R}^2$ obtained by putting $t = 0$ in (6.2).

7 Semi-Classical Vacuum Decay

The existence of negative energy, asymptotically Minkowskian black hole solutions in Einstein-anti-Maxwell theory is a strong indication that the vacuum of the theory is unstable. Further evidence for this instability was provided in the sections describing accelerating black holes and their thermodynamics. A more convincing demonstration of the instability of the vacuum would be provided by an instanton solution describing its decay. We therefore look for a Euclidean solution which approaches the vacuum at infinity, i.e. it must be asymptotically $\mathbb{R}^4$.

As we saw in the last section, the Euclidean Schwarzschild solution will not do because, to avoid a singularity at the horizon $r = r_+$, it was necessary to periodically identify the former time coordinate $Z$. The solution is thus asymptotically $\mathbb{R}^3 \times S^1$. Note that the period of this identification
of the imaginary time coordinate, $\beta = 4\pi r_+$, is precisely $\frac{1}{T}$ where $T$ is the Hawking temperature of the Lorentzian black hole solution. This suggests that we should consider the extreme Reissner-Nordström solution since this has $T = 0$ and so we would be free to identify $Z$ with whatever period we liked (or not at all). There is a slight complication in that, in order to obtain the Euclidean Reissner-Nordström solution, we need to take $Q$ to be pure imaginary. However, since this is precisely what we had to do to obtain the anti-Reissner-Nordström solution, we will obtain the desired solution of the Euclidean Einstein-anti-Maxwell equations with $Q$ real. To see this in more detail, consider the $Q = M$ Lorentzian anti-Reissner-Nordström solution:

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} - \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

$$A = \frac{Q}{r}dt.$$  \hspace{1cm} (7.1)

Here $Q = M$ is real and the horizons satisfy $r_- < 0 < r_+$, so this is not an extreme solution (in fact as we saw in section 2, there are no extreme solutions in this theory). Now we obtain the Euclidean solution by setting $t = iz$ and, in order to keep the vector potential 1-form $A$ real, we must also make the replacement $Q \rightarrow iQ$. We thus obtain the following Euclidean solution of the Einstein-anti-Maxwell equations

$$ds^2 = \left(1 - \frac{M}{r}\right)^2dz^2 + \left(1 - \frac{M}{r}\right)^{-2}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

$$A = \frac{M}{r}dz.$$  \hspace{1cm} (7.2)

which is our desired instanton. In isotropic coordinates, $r = \rho + M$, it may be written as

$$ds^2 = \left(1 + \frac{M}{\rho}\right)^{-2}dz^2 + \left(1 + \frac{M}{\rho}\right)^2[dx^2 + dy^2 + d\tau^2]$$

$$\hspace{1cm} (7.3)$$

where $\rho^2 = x^2 + y^2 + \tau^2$. Note that this space, restricted to $\rho > 0$, is everywhere non-singular and geodesically complete. The singularity at $\rho = 0$ is at an infinite proper distance along any geodesic. There is no need to periodically identify $z$, or alternatively, we are free to identify $z$ with any period we choose.
To see how the above instanton leads to the instability of the vacuum, we analytically continue to the Lorentzian solution by setting $\tau = it$:

$$ds^2 = \left(1 + \frac{M}{\rho}\right)^{-2} dz^2 + \left(1 + \frac{M}{\rho}\right)^2 \left[dx^2 + dy^2 - dt^2\right],$$

$$A = \frac{M}{\rho + M}dz$$

where $\rho^2 = x^2 + y^2 - t^2$. This then is the solution into which the vacuum decays. It is a tachyon solution, as in the last section, describing a charged particle moving with infinite speed along the $z$-axis. However, here we wish to interpret it differently as representing the instability of the vacuum in Einstein-anti-Maxwell theory. It is still everywhere non-singular and geodesically complete since the proper distance down the “infinite throat” at $\rho = 0$ is infinite along all geodesics. However, it is clear that this throat region expands outwards radially in all directions perpendicular to the $z$-axis at the speed of light.

Due to the cylindrical symmetry of the solution, it is best described in cylindrical polar coordinates, $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1} \frac{x}{y}$:

$$ds^2 = \left(1 + \frac{M}{\sqrt{r^2 - t^2}}\right)^{-2} dz^2 + \left(1 + \frac{M}{\sqrt{r^2 - t^2}}\right)^2 \left[dr^2 + r^2 d\theta^2 - dt^2\right].$$

Thus, at $t=0$, an infinite throat forms along the $z$ axis. The $z$-axis becomes infinitely far away along geodesics in the new spacetime and this throat region expands away from the $z$-axis at the speed of light. Actually it is this rapid expansion which is responsible for the $z$-axis being infinitely far away. If one considers a constant time slice, then it is easy to see that $r = 0$ is not at an infinite proper distance from points with $r > 0$. However, such curves in the constant time slice are not geodesics. After the formation of the throat, the topology of the spacetime changes to $\mathbb{R}^{1,1} \times (\mathbb{R}^2 - \{0\})$. The above solution describes one such throat region forming and expanding to infinity at the speed of light. It is more realistic, physically, to suppose that actually such throats will form throughout the vacuum at a given rate per unit volume and then expand outwards until they collide with one another. Of course, an observer in the spacetime will never see a throat form because, since it travels outwards at the speed of light, by the time he sees it, he will have
already fallen down it! To investigate the properties of the spacetime “down the throat”, i.e. near $r = t$, define new coordinates $u, T$ by

\begin{align*}
    r &= e^u \cosh T \\
    t &= e^u \sinh T
\end{align*}

(7.6)

then the metric becomes

\begin{equation}
    ds^2 \approx \frac{e^{2u}}{M^2} dz^2 + M^2 du^2 - M^2 dT^2 + M^2 \cosh^2 T d\theta^2.
\end{equation}

(7.7)

This is a completely non-singular, cylindrically symmetric spacetime ($\mathbb{R}^{1,2} \times S^1$) such that the radius of the $S^1$ increases rapidly as the time coordinate $T$ increases.

Finally, in order to show that the above mechanism for the decay of the Einstein-anti-Maxwell vacuum is important, we need to calculate the probability of the production of such infinite throats. According to the semi-classical approximation, this probability is $P = e^{-I_E}$ where $I_E$ is the action of the Euclidean instanton (7.2). But (7.2) is equivalent to the extreme Euclidean Reissner-Nordström solution of the Euclidean Einstein-Maxwell equations and so, as we saw in section 4.3, its action is zero. Therefore one expects these infinite throats to be copiously produced and hence, as expected, the Einstein-anti-Maxwell vacuum will be genuinely unstable.

A similar construction gives the decay process for the EMD vacuum. We start from the EMD black hole solutions of section 2:

\begin{equation}
    ds^2 = -\left(1 - \frac{r^+}{r}\right)\left(1 - \frac{r^-}{r}\right)^{\frac{a^2}{1 + a^2}} dt^2 + \left(1 - \frac{r^+}{r}\right)^{-1} \left(1 - \frac{r^-}{r}\right)^{\frac{2a^2}{1 + a^2}} dr^2 \\
    + r^2 \left(1 - \frac{r^-}{r}\right)^{\frac{a^2}{1 + a^2}} (d\theta^2 + \sin^2 \theta d\phi^2).
\end{equation}

(7.8)

The electromagnetic vector potential 1-form is

\begin{equation}
    A = \sqrt{-\frac{r^+ r^-}{1 + a^2}} \frac{dt}{r}.
\end{equation}

(7.9)

Note that $r^- \leq 0 < r^+$ and so $A$ is real. We euclideanize by setting $t = iz$ and change the sign of $r^-$ to keep the 1-form $A$ real. Since $r^-$ is now positive, we may obtain the extreme solution by setting $r^- = r^+$ giving

\begin{equation}
    ds^2 = \left(1 - \frac{r^+}{r}\right)^{\frac{2}{1 + a^2}} dz^2 + \left(1 - \frac{r^+}{r}\right)^{-\frac{2}{1 + a^2}} dr^2
\end{equation}

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\[ + r^2 \left( 1 - \frac{r_+}{r} \right)^{\frac{2a^2}{1 + a^2}} (d\theta^2 + \sin^2 \theta d\phi^2), \]  
\[ A = \frac{r_+}{\sqrt{1 + a^2}} \frac{dz}{r}. \]

Note that this is a Euclidean solution of the EMD equations. It has a curvature singularity at \( r = r_+ \) and so this point must be excluded from the space. This means that there is no need to compactify any of the coordinates and so the topology may be taken as \( \mathbb{R}^4 - \{0\} \). It is again convenient to introduce isotropic coordinates, \( r = \rho + r_+ \) giving

\[ ds^2 = \left( 1 + \frac{r_+}{\rho} \right)^{-\frac{2}{1 + a^2}} dz^2 + \left( 1 + \frac{r_+}{\rho} \right)^{\frac{2}{1 + a^2}} (dx^2 + dy^2 + dr^2), \]  
\[ (7.11) \]

where \( \rho^2 = x^2 + y^2 + r^2 \). This then is the instanton describing the decay process of the EMD vacuum. The Lorentzian solution into which the vacuum decays is obtained by setting \( \tau = it \). The curvature singularity at \( \rho = 0 \) now gives a singular lightcone \( x^2 + y^2 = t^2 \) expanding radially outwards at the speed of light from the \( z \)-axis. This may be viewed as the time evolution of the initial data given by the 3-metric (setting \( \tau = 0 \) in (7.11))

\[ ds^2 = \left( 1 + \frac{r_+}{\sqrt{x^2 + y^2}} \right)^{-\frac{2}{1 + a^2}} dz^2 + \left( 1 + \frac{r_+}{\sqrt{x^2 + y^2}} \right)^{\frac{2}{1 + a^2}} (dx^2 + dy^2) \]  
\[ (7.12) \]

which is completely non-singular. The resulting spacetime, however, is singular on the lightcone \( x^2 + y^2 = t^2 \):

\[ ds^2 = \left( 1 + \frac{r_+}{\sqrt{x^2 + y^2 - t^2}} \right)^{-\frac{2}{1 + a^2}} dz^2 \]
\[ + \left( 1 + \frac{r_+}{\sqrt{x^2 + y^2 - t^2}} \right)^{\frac{2}{1 + a^2}} (dx^2 + dy^2 - dt^2). \]  
\[ (7.13) \]

Thus a naked singularity spontaneously forms along the \( z \)-axis at \( t = 0 \) which then expands radially outwards at the speed of light. We saw in section 4.3 that the action for the Euclidean instanton (7.11) is zero and so one expects copious production of such line singularities and thus that the EMD vacuum will be extremely unstable. Note that this argument remains true equally well for all values of the dilaton coupling \( a \).
8 Conclusions

We have derived families of black holes in theories of gravity coupled to a Maxwell field and a scalar dilaton, in which the kinetic energies for either or both of the fields are allowed to be negative. In the case where just the Maxwell field is given negative kinetic energy, we found that regular black hole solutions with zero or negative mass were possible provided that the dilaton coupling $a$ was less than 1. If the dilaton is also given negative kinetic energy, then such black holes are possible for all values of $a$. In this case we also found a neutral massless wormhole solution characterized by a non-zero scalar charge. This wormhole was described as ‘transparent’ because it has no horizons and timelike or null geodesics can pass through it freely (it was shown not to exert any gravitational attraction) and so it would be possible to see through it from one universe into another isometric universe.

The analogue of the C-metric in these theories was derived, describing a pair black holes accelerating away from one another. An investigation into whether or not it is possible to satisfy the ‘no strut’ condition and thus remove the conical singularities of the C-metric showed that it was possible to do so in exactly those cases where the theory admitted zero rest mass black holes. We thus found regular solutions describing pairs of zero rest mass black holes accelerating away from one another. We failed, however, to find, from these solutions, an instanton which would describe the decay of the vacuum. However, by looking at tachyonic solutions, we were able to give a alternative way in which the vacuum would decay. The action for the instanton describing this instability was found to be zero so that this decay mode is not suppressed at all.

Dyson's original argument against the convergence of the perturbation series in QED relied on the production of pairs of oppositely charged particles which (if the electromagnetic coupling constant were taken to be pure imaginary) would repel one another thus destabilizing the vacuum. In this paper, we tried to repeat the argument with pairs of charged black holes in a theory of gravity coupled to a Maxwell field with negative kinetic energy. We were, however, unable to find an instanton describing the production of such pairs of black holes. However, by using other semi-classical arguments, we were able to show that in such a theory the vacuum is still unstable. Thus we may still argue that a perturbative theory describing gravity and electromagnetism cannot be uniformly convergent as a perturbative expansion in
the electromagnetic coupling constant $e$.

The results were also extended to include a scalar dilaton field and the vacuum in this case was also found to be unstable. This result holds for all values of the dilaton coupling $a$, in particular for $a = -\sqrt{3}$ which is the case which arises from Kaluza-Klein theory in which the extra dimension is taken to be timelike rather than spacelike. This may have some relevance to recent ideas in string theory, F-theory and other theories which consider the possibility of extra timelike dimensions [8]. The results imply that such theories appear to be very unstable and, as we have seen, their solutions may be rather pathological, as we might have expected.

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