Virtual light-by-light scattering and the $g$ factor of a bound electron

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The contribution of the light-by-light diagram to the $g$ factor of electron and muon bound in Coulomb field is obtained. For electron in a ground state, our results are in good agreement with the results of other authors obtained numerically for large $Z$. For relatively small $Z$ our results have essentially higher accuracy as compared to the previous ones. For muonic atoms, the contribution is obtained for the first time with the high accuracy in whole region of $Z$.

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I. INTRODUCTION

The progress in experimental investigations of the $g$ factor of a bound electron and muon in ions stimulated intensive theoretical investigation of various contributions to this quantity. The contributions of self-energy, vacuum polarization, and nuclear effects have been considered. An essential part of the theoretical uncertainty has been related to the contribution of the vacuum polarization of an external homogeneous magnetic field in the electric field of atom (so-called the “magnetic-loop” contribution). The corresponding diagram is shown in Fig. 1. In this diagram, double line in the fermion loop corresponds to the electron propagator in the Coulomb field. Note that the contribution of the free electron loop to the vacuum polarization of a homogeneous magnetic field vanishes due to the gauge invariance. The first non-vanishing terms of expansion with respect to the Coulomb field shown in Fig. 1 is the contribution of virtual light-by-light scattering with one of the quanta corresponding to the external magnetic field. The results of numerical calculations of the magnetic-loop contribution, which take into account all orders of the parameter $Z\alpha$ ($Z$ is the nuclear charge number, $\alpha = e^2$ is the fine-structure constant, $\hbar = c = 1$), are presented in Ref. 6. At present, the most accurate experimental data are obtained in the region of medium $Z$. Unfortunately, in this region the uncertainty of the results of Ref. 6 is very big, being, e.g., 100% for $Z = 12$. In Ref. 9, the leading in $Z\alpha$ magnetic-loop contribution to the $g$ factor of an electron in $S$ state of a hydrogen-like ion has been derived. It reads

$$\frac{\Delta g_0}{g_0} = \frac{\Delta g_0}{2} = \frac{7\alpha(Z\alpha)^5}{432n^4},$$

where $g_0$ is the Landé factor equal to two for $S$ state. One can compare this correction with the result of Ref. 6 for rather large $Z$ where the accuracy of the numerical calculation is reasonable. This comparison shows the noticeable difference which can be attributed to the contribution of the next-to-leading terms in $Z\alpha$-expansion, starting from $\alpha(Z\alpha)^6$. Since the numerical factor in Eq. 1 is very small ($\sim 1/30$), the next-to-leading contribution of magnetic loop to the $g$ factor even at small $Z$, if the corresponding numerical factor is of order of unity.

In the present paper, we generalize Eq. 1 to the case of arbitrary bound electron state. We also calculate the next-to-leading contribution of magnetic loop to the $g$ factor of the electron in arbitrary state (or the magnetic moment of the electron in this state). It has the form $\Delta g_1 = \alpha(Z\alpha)^6(a_1 \ln(1/Z\alpha) + a_2)$, where $a_{1,2}$ are some constants and $a_1$ is not zero only for $S$ states. In order to calculate this contribution, it is sufficient to take into account the diagrams of magnetic light-by-light scattering and use the nonrelativistic wave functions of the bound electron. Comparison of the correction $\Delta g_0 + \Delta g_1$ for $1S_{1/2}$ state with the results of Ref. 6 shows that the account of $\Delta g_1$ does not provide good agreement for relatively small $Z \sim 30$, where the numerical calculations were performed with sufficient accuracy.
Thus, for such $Z$ it is necessary to take into account next terms in $Z\alpha$. These terms have two different origins. First, they come from the relativistic corrections to the wave function of a bound electron. Next, they come from the higher-order contributions to the electron loop. Note that the diagram in Fig. 1 can be interpreted as the contribution of the scattering of the magnetic quantum in a Coulomb field (virtual Delbrück scattering) to the $g$ factor. It is known that the Coulomb corrections to the Delbrück amplitude for momentum of quantum $q \lesssim m$ ($m$ is the electron mass) are numerically small even for large $Z$ \cite{11, 12}. In contrast, the account of the corrections to the wave function is very important, starting from relatively small $Z$. We calculate the correction $\Delta g$ using the relativistic wave function and the leading approximation for the electron loop. As a result we have obtained good agreement with the numerical data of \cite{6} even for very large $Z$ (difference is 4\% for $Z = 92$). Using such approach, we have calculated the corresponding correction of the electron loop to the $g$ factor of a bound muon.

\[ \Delta g = \frac{e B \cdot \langle J \rangle}{2m} \left( 1 - \frac{2\kappa}{Z\alpha^2} \right), \]

where $\kappa = (J + 1/2)\text{sign}(L - J)$. In Eq. (2) we have used the relation

\[ i \kappa \times A_k = (2\pi)^3 \delta(k) B. \]

Note that a sign of $T^{(0)}$ is opposite to that of Hamiltonian. Substituting the radial wave functions $f_1(r)$ and $f_2(r)$ for the Coulomb field (see, e.g., \cite{14}), we obtain for the arbitrary bound state

\[ T^{(0)} = \frac{e B \cdot \langle J \rangle}{2m} g, \]

where $g = \frac{2\kappa}{1 - 4\kappa^2} \left( 1 - \frac{2\kappa \varepsilon}{m} \right) = \frac{2\kappa}{1 - 4\kappa^2} \left( 1 - \frac{2\kappa}{\sqrt{1 + (Z\alpha)^2/(\gamma + n_r)^2}} \right), \]

where $n_r$ is the radial quantum number, $\varepsilon$ is the binding energy, and $\gamma = \sqrt{\kappa^2 - (Z\alpha)^2}$. The particular cases of this formula obtained earlier are presented in \cite{13}. In the non-relativistic approximation ($Z\alpha \ll 1$), Eq. (2) turns to

\[ T_0^{(0)} = \frac{e B \cdot \langle J \rangle}{2m} g_0, \quad g_0 = \frac{2\kappa}{2\kappa + 1}. \]

We now pass to the calculation of the amplitude $T^{(1)}$ corresponding to the diagram shown in Fig. 1. It has the form

\[ T^{(1)} = e \int \frac{dk}{(2\pi)^3} \int \frac{dq}{(2\pi)^3} \frac{4\pi}{q^2} A_k^i M_l^i j_q^l, \]
where the amplitude $M_{li}$ of the virtual Delbrück scattering in the case $k \ll m$ has the form following from the gauge invariance

$$M_{li} = \frac{\alpha}{m^3} \delta^{ij}(k \cdot q) - q^i k^j \mathcal{F}(q/m, Z\alpha).$$

(8)

Note that $F$ is even function of $Z\alpha$. In the leading in $Z\alpha$ approximation (contribution of light-by-light scattering),

$$\mathcal{F}(q/m, Z\alpha) = (Z\alpha)^2 F(q/m),$$

(9)

with $F(0) = 7/1152$, see Ref. [9]. From Eqs. (4), (7), (8), and (9) we obtain

$$T^{(1)} = e^{4k\alpha(Z\alpha)^2 B \cdot (J)} \int_0^\infty dq F(q/m) \int_0^\infty drr_1 f_1(r) f_2(r) \left( \frac{\sin qr}{qr} - \cos qr \right).$$

(10)

Using the relation $M_{li} = [2\alpha(Z\alpha)^2(k \cdot q)/m^3] F(q/m)$, following from Eq. (8), and the gauge invariance of the light-by-light scattering amplitude, we can represent $F(q/m)$ in the following form

$$F(q/m) = \frac{m^3}{2\pi} \int \frac{dQ}{Q^2(q - Q)^2} \frac{q \cdot (\nabla_k M)_{k=0}}{Q^2},$$

$$M = 2i \int \frac{d^4p}{(2\pi)^4} \text{Sp} \left\{ G(p) \gamma^i G(p - k) \gamma^0 \left[ G(p + Q - q) \gamma^i G(p + Q) \gamma^0 + G(p + Q - q) \gamma^0 G(p - q) \gamma^i + G(p - Q - k) \gamma^0 G(p - q) \gamma^i \right] \right\},$$

(11)

where $G(p) = [\hat{p} - m]^{-1}$ is a free electron propagator. Straightforward calculation leads to the representation of the function $F$ in the form of two-fold integral with respect to the Feynman parameters. Resulting formulas being rather cumbersome are not presented here explicitly. For $x = q/m \ll 1$, the first two terms of expansion of the function $F(x)$ have the form

$$F(x) = \frac{7}{1152} (1 + \frac{8}{35} x).$$

(12)

The first term in this formula agrees with the result of Ref. [9]. For $x \gg 1$, the asymptotics of the function $F(x)$ reads

$$F(x) = \frac{1}{2x^3}.$$  

(13)

For arbitrary $x$, we performed the numerical tabulation of the function $F(x)$. The result is shown in Fig. 2 and in Table I.

III. CORRECTION TO $g$ FACTOR AT SMALL $Z\alpha$

In order to obtain the leading term of expansion in $Z\alpha$ of the amplitude $T^{(1)}$, it is sufficient to use Eq. (10) with the substitution $F(q/m) \to F(0)$ and the wave functions taken in the nonrelativistic approximation. In this approximation $f_1(r)$ coincides with $R(r)$, the radial part of the nonrelativistic wave function, and

$$f_2(r) = \frac{1}{2m} \left( R'(r) + \frac{1}{r} + \frac{\kappa}{r} R(r) \right).$$

(14)

The correction $\Delta g$ to the Landé factor is determined by the relation

$$\frac{\Delta g}{g_0} = \frac{T^{(1)}}{T_0^{(0)}}.$$

(15)

Taking in Eq. (10) the integral over $q$, and then over $r$, we obtain the leading contribution $\Delta g_0$ for the arbitrary state

$$\Delta g_0 = \frac{7\alpha(Z\alpha)^5}{144\kappa^3(2L + 1)(2\kappa - 1)} = \frac{7\alpha(Z\alpha)^5}{288\kappa^3J(J + 1)(2J + 1)},$$

(16)
where $n = n_r + |\kappa|$ is a principal quantum number. For $S$ states ($L = 0$, $\kappa = -1$), this result is in agreement with Eq. (1) obtained in Ref. [1].

The relativistic corrections to the wave function as well as the corrections to the magnetic loop have the relative magnitude $(Z\alpha)^2$. Therefore, the term $\Delta g_1$ of the order $\alpha(Z\alpha)^6$ can also be obtained with the use of the nonrelativistic wave functions and magnetic loop in the leading approximation (light-by-light scattering diagrams). For $L \neq 0$, it is sufficient to substitute the second term of expansion of $F(x)$, see Eq. (12), in Eq. (10). Then we obtain

$$\frac{\Delta g_1}{g_0} = \frac{2\alpha(Z\alpha)^6}{45\pi n^3(2L+1)(2\kappa-1)^2} \left( \frac{3}{L(L+1)} - \frac{1}{n^2} \right).$$

(17)

For $S$ states, calculation of $\Delta g_1$ is more complicated. For $nS$ state, $f_1(r)f_2(r) = (\pi/m)\rho_n'(r)$, where $\rho_n(r)$ is the electron density in the nonrelativistic approximation. Substitution of Eq. (12) in Eq. (10) leads to logarithmic divergence. Therefore, it is convenient to split the region of integration over $r$ in Eq. (10) into two: $[0,r_0]$ and $[r_0,\infty)$ with $1/m \ll r_0 \ll 1/(mZ\alpha)$. In the first region, we can replace $\rho'(r)$ by $\rho'(0)$ and take the integral over $r$. In the second region, we can use the expansion (12) and take the integral over $q$. Sum of these two contributions, as it should be, is independent of $r_0$. The final result reads

$$\frac{\Delta g_1}{g_0} = \frac{4\alpha(Z\alpha)^6}{135\pi n^3} \left( \ln \frac{1}{Z\alpha} - a - b_n \right),$$

$$a = -\frac{1}{2} + \frac{35}{8} \int_0^\infty dx \ln x F''(x) \approx 2.6,$$

$$b_n = -C + \frac{1}{\rho_n'(0)} \int_0^\infty dr \ln(mZ\alpha r)\rho_n'(r),$$

(18)

where $C = 0.577...$ is the Euler constant. For each $n$, the coefficient $b_n$ can be easily calculated so that $b_1 = \ln 2 \approx 0.693$, $b_2 = 5/8 = 0.625$, $b_3 = 55/54 + \ln 2/3 \approx 0.613$, $b_\infty = C + \ln 2 - 2/3 \approx 0.604$.

**IV. CORRECTION TO g FACTOR AT Z\alpha \sim 1**

As it was pointed out in the Introduction, the sum $\Delta g_0 + \Delta g_1$ gives a good approximation to $\Delta g$ only for small $Z$. For intermediate $Z$, it is necessary to account for the next terms in $Z\alpha$. The largest corrections are due to the significant difference between the relativistic wave function and the nonrelativistic one already at intermediate $Z$. At the same time, the difference between the function $F$ and its leading approximation $(Z\alpha)^2 F$ results in the corrections which are numerically small even for large $Z$. Using the numerical results for $F(x)$ and the relativistic wave functions, we have performed the tabulation of $\Delta g$ for various $Z$, using $T^{(1)}$ from Eq. (10) as an approximation to $T^{(1)}$. The
results of this tabulation for 1S₁/₂, 2S₁/₂, and 2P₁/₂ states are presented in Table III. For 1S₁/₂ state, we also present the contribution of the first two terms of expansion in Zα, Eqs. (18), (19), and the correction ∆gₙᵣ obtained with the use of nonrelativistic wave functions. The results for 1S₁/₂ are also shown in Fig. 3.

For Z < 10, both ∆g₀ + ∆g₁ and ∆gₙᵣ coincide with ∆g with accuracy better than one percent. The difference grows with Z, reaching 10% at Z ~ 30 for ∆g₀ + ∆g₁ and at Z ~ 50 for ∆gₙᵣ.

In Table III we also show the results of numerical tabulation from Ref. [6] for 1S₁/₂ state. For 30 < Z < 70, our result for ∆g agrees with that obtained in Ref. [6] within 1% percent. The difference between these two results for Z < 30 is due to poor accuracy of the numerical results of Ref. [6]. For Z > 70 the difference increases and becomes 8% for Z = 92. This difference corresponds to the contribution of next-to-leading terms in magnetic loop, that was taken into account in Ref. [6] and omitted in our paper. Thus, the effect of these terms is small in a wide region of Z, while the relativistic effects in the wave function become important already at relatively small Z.

V. THE CORRECTION ∆g FOR MUONIC ATOMS

The correction ∆g to the g factor of a bound muon due to the electron magnetic loop can be obtained from Eq. (10) with f₁(r) and f₂(r) being the wave functions of the muon. The asymptotics of ∆g for μZα/(mn²) ≈ 1.5Z/n² ≫ 1 (μ is the muon mass) can be calculated as follows. We split the region of integration over q in Eq. (10) into two: [0, q₀] and [q₀, ∞) with m ≪ q₀ ≪ μZα/n². In the first region we can replace [(qr)⁻¹ sin qr − cos qr] by (qr)²/3 and take the integral over r. In the second region we can use the asymptotics Eq. (13) and take the integral over q. Summing these two contributions, we obtain

\[ \Delta g_{as} = g \frac{2\alpha(Z\alpha)^2}{3\pi}[\ln(\mu Z\alpha/m) - A - B], \]

\[ A = 2 \int_0^\infty dy \ln y \theta_y (y^3 F(y)) \approx 2.24, \]

\[ B = C - \frac{4}{3} - \frac{Z\alpha(1 - 2\kappa^2)}{\mu} \int_0^\infty dx x^3 \hat{f}_1(x) \hat{f}_2(x) \ln x, \]

\[ \hat{f}_1(x) = (\mu Z\alpha)^{-3/2} f_1(x/\mu Z\alpha), \quad \hat{f}_2(x) = (\mu Z\alpha)^{-3/2} f_2(x/\mu Z\alpha), \quad \]  

(19)

\( g \) is defined in Eq. (6). For 1S₁/₂ state we obtain

\[ g = \frac{2}{3}(1 + 2\gamma), \quad B = C - \frac{4}{3} + \psi(2\gamma + 2) - \ln 2, \quad \gamma = \sqrt{1 - (Z\alpha)^2}. \]

(20)

For \( n = n_r + |\kappa| \gg 1 \), we have

\[ g = g_0 = \frac{2\kappa}{2\kappa + 1}, \quad B = C + \ln(n_r^2/2). \]

(21)

The formula (19) can be interpreted as follows. In Ref. [16] the logarithmic contribution of the electron vacuum polarization to the magnetic moment of a heavy nucleus was calculated. The result obtained has the form

\[ \frac{\Delta g}{g} = \frac{2\alpha(Z\alpha)^2 H(Z\alpha)}{3\pi} \ln(1/mR_{nucl}), \]

(22)
where $R_{\text{nucl}}$ is the nuclear radius, $R_{\text{nucl}} \ll 1/m$. The coefficient $(Z\alpha)^2 H(Z\alpha)$ was calculated exactly in $Z\alpha$, i.e., with the account of all Coulomb corrections to the electron loop. The function $H(Z\alpha)$ tends to unity when $Z\alpha \to 0$, and significantly differs from unity only for very large $Z$. Large logarithm $\ln(1/mR_{\text{nucl}})$ in Eq. (22) appears as a result of integration over distance $r$ in the region $R_{\text{nucl}} \ll r \ll 1/m$. We can consider the muonic atom as some nucleus with the effective radius $R_{\text{nucl}} \sim n^2/\mu Z\alpha$. In the case $\mu Z\alpha/(mn^2) \gg 1$, we have $R_{\text{nucl}} \ll 1/m$. Substituting this radius into Eq. (22) and replacing $H(Z\alpha) \to 1$ (that corresponds to the contribution of light-by-light scattering), we obtain the logarithmically amplified term in Eq. (19). Note that the coefficient $n^2$ in $R_{\text{nucl}}$ corresponds to the asymptotics of $B$ in Eq. (19) at $n \gg 1$. Strictly speaking, the charge of such effective nucleus is $Z \to 1$, but not $Z$. However, under the condition $\mu Z\alpha/(mn^2) \approx 1.5Z/n^2 \gg 1$, this difference is not important.

In Table I we present $\Delta g$ for $1S_{1/2}$ state of muonic atom calculated for arbitrary $Z$. For comparison, we present also asymptotics Eq. (19). As it should be, the accuracy of asymptotics (19) increases with $Z$ being $4\%$ for $Z = 40$ and $1\%$ for $Z = 92$.

In summary, we found higher-order magnetic loop corrections to the bound $g$ factor in order $\alpha(Z\alpha)^6$ for arbitrary state. Despite a small coefficient in the leading term of order $\alpha(Z\alpha)^3$, and the logarithmic enhancement of the higher-order contribution, the leading term still dominate for $Z = 6$ and $Z = 8$, important for experiment. Previously used numerical results show a certain underestimation of the magnetic-loop contribution for $Z < 20$. Theoretical description of this contribution presented in this paper is more reliable. The difference of less than a few percent between our analytic results and the numerical calculations of Ref. [8] at high $Z (80 \div 90)$ shows that the contribution of the higher-order terms in magnetic loop may be safely neglected for $Z \lesssim 50$. We also calculated the correction $\Delta g$ for the bound muon, and its behavior is very peculiar. All known contributions to the bound $g$ factor scale as $n^{-2}$ or $n^{-3}$. The correction found in this paper does not contain such a strong suppression factor. This correction is a dominant bound state QED correction for a bound muon, which even for $1S_{1/2}$ state supersedes the free vacuum polarization term [8]. The results obtained significantly diminish the uncertainty of the theoretical predictions for the $g$ factor of a bound particle.

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| $x$ | $F(x)/F(0)$ | $x$ | $F(x)/F(0)$ | $x$ | $F(x)/F(0)$ | $x$ | $F(x)/F(0)$ | $x$ | $F(x)/F(0)$ |
|-----|-------------|-----|-------------|-----|-------------|-----|-------------|-----|-------------|
| 0   | 1           | 0.42| 1.07       | 0    | 0.927       | 5   | 0.222       | 26  | $3.81 \times 10^{-3}$ |
| 0.05| 1.01        | 0.44| 1.07       | 1.5  | 0.897       | 5.5 | 0.184       | 28  | $3.1 \times 10^{-3}$  |
| 0.1 | 1.02        | 0.46| 1.07       | 1.7  | 0.867       | 6.  | 0.154       | 30  | $2.55 \times 10^{-3}$ |
| 0.15| 1.03        | 0.48| 1.08       | 1.8  | 0.837       | 6.5 | 0.13        | 32  | $2.12 \times 10^{-3}$ |
| 0.16| 1.03        | 0.5 | 1.08       | 1.9  | 0.806       | 7   | 0.11        | 36  | $1.52 \times 10^{-3}$ |
| 0.17| 1.04        | 0.55| 1.08       | 2.   | 0.776       | 7.5 | $9.45 \times 10^{-2}$| 40  | $1.12 \times 10^{-3}$ |
| 0.18| 1.04        | 0.6 | 1.08       | 2.1  | 0.746       | 8   | $8.15 \times 10^{-2}$| 44  | $8.55 \times 10^{-4}$ |
| 0.19| 1.04        | 0.65| 1.08       | 2.2  | 0.716       | 9   | $6.18 \times 10^{-2}$| 48  | $6.66 \times 10^{-4}$ |
| 0.2 | 1.04        | 0.7 | 1.08       | 2.3  | 0.687       | 10  | $4.79 \times 10^{-2}$| 52  | $5.28 \times 10^{-4}$ |
| 0.22| 1.04        | 0.75| 1.08       | 2.4  | 0.659       | 11  | $3.78 \times 10^{-2}$| 56  | $4.26 \times 10^{-4}$ |
| 0.24| 1.05        | 0.8 | 1.07       | 2.5  | 0.631       | 12  | $3.04 \times 10^{-2}$| 60  | $3.49 \times 10^{-4}$ |
| 0.26| 1.05        | 0.85| 1.07       | 2.6  | 0.605       | 13  | $2.47 \times 10^{-2}$| 64  | $2.89 \times 10^{-4}$ |
| 0.28| 1.05        | 0.9 | 1.06       | 2.7  | 0.579       | 14  | $2.04 \times 10^{-2}$| 72  | $2.05 \times 10^{-4}$ |
| 0.3 | 1.06        | 0.95| 1.05       | 2.8  | 0.554       | 15  | $1.7 \times 10^{-2}$ | 80  | $1.5 \times 10^{-4}$ |
| 0.32| 1.06        | 1   | 1.05       | 2.9  | 0.531       | 16  | $1.43 \times 10^{-2}$| 88  | $1.14 \times 10^{-4}$ |
| 0.34| 1.06        | 1.1 | 1.03       | 3   | 0.508       | 18  | $1.05 \times 10^{-2}$| 96  | $8.8 \times 10^{-5}$  |
| 0.36| 1.06        | 1.2 | 1.01       | 3.5  | 0.409       | 20  | $7.86 \times 10^{-3}$| 104 | $6.95 \times 10^{-5}$ |
| 0.38| 1.07        | 1.3 | 0.981      | 4.   | 0.331       | 22  | $6.05 \times 10^{-3}$| 112 | $5.58 \times 10^{-5}$ |
| 0.4 | 1.07        | 1.4 | 0.955      | 4.5  | 0.27        | 24  | $4.76 \times 10^{-3}$| 120 | $4.55 \times 10^{-5}$ |

TABLE I: Function $F(x)/F(0)$ versus $x = q/m$; $F(0) = 7/1152$. 


| $Z$ | $\Delta g_0 + \Delta g_1$ | $\Delta g_{nr}$ | $\Delta g$ | $\Delta g$(Ref. [6]) | $8\Delta g$ | $8\Delta g$ |
|-----|------------------|-----------------|-----------|-----------------|-------------|-------------|
| 1   | $4.935 \times 10^{-9}$ | $4.934 \times 10^{-9}$ | $4.934 \times 10^{-9}$ | $4.936 \times 10^{-9}$ | $1.638 \times 10^{-9}$ |           |
| 2   | $1.58 \times 10^{-7}$  | $1.58 \times 10^{-7}$  | $1.58 \times 10^{-7}$  | $1.58 \times 10^{-7}$  | $5.26 \times 10^{-8}$  |           |
| 3   | $1.2 \times 10^{-6}$   | $1.2 \times 10^{-6}$   | $1.2 \times 10^{-6}$   | $1.2 \times 10^{-6}$   | $4.01 \times 10^{-7}$   |           |
| 4   | $5.04 \times 10^{-6}$  | $5.04 \times 10^{-6}$  | $5.04 \times 10^{-6}$  | $5.05 \times 10^{-6}$  | $1.69 \times 10^{-6}$  |           |
| 5   | $1.53 \times 10^{-5}$  | $1.53 \times 10^{-5}$  | $1.54 \times 10^{-5}$  | $1.54 \times 10^{-5}$  | $5.18 \times 10^{-6}$  |           |
| 6   | $3.79 \times 10^{-5}$  | $3.8 \times 10^{-5}$   | $3.81 \times 10^{-5}$  | $3.82 \times 10^{-5}$  | $1.29 \times 10^{-5}$  |           |
| 7   | $8.16 \times 10^{-5}$  | $8.17 \times 10^{-5}$  | $8.2 \times 10^{-5}$   | $8.23 \times 10^{-5}$  | $2.81 \times 10^{-5}$  |           |
| 8   | $1.58 \times 10^{-4}$  | $1.58 \times 10^{-4}$  | $1.59 \times 10^{-4}$  | $1.6 \times 10^{-4}$   | $5.49 \times 10^{-5}$  |           |
| 9   | $2.83 \times 10^{-4}$  | $2.84 \times 10^{-4}$  | $2.86 \times 10^{-4}$  | $2.87 \times 10^{-4}$  | $9.94 \times 10^{-5}$  |           |
| 10  | $4.76 \times 10^{-4}$  | $4.78 \times 10^{-4}$  | $4.82 \times 10^{-4}$  | $4.84 \times 10^{-4}$  | $1.69 \times 10^{-4}$  |           |
| 11  | $7.61 \times 10^{-4}$  | $7.66 \times 10^{-4}$  | $7.72 \times 10^{-4}$  | $7.76 \times 10^{-4}$  | $2.73 \times 10^{-4}$  |           |
| 12  | $1.17 \times 10^{-3}$  | $1.18 \times 10^{-3}$  | $1.19 \times 10^{-3}$  | $1.19 \times 10^{-3}$  | $4.24 \times 10^{-4}$  |           |
| 13  | $1.72 \times 10^{-3}$  | $1.74 \times 10^{-3}$  | $1.76 \times 10^{-3}$  | $1.77 \times 10^{-3}$  | $6.35 \times 10^{-4}$  |           |
| 14  | $2.48 \times 10^{-3}$  | $2.51 \times 10^{-3}$  | $2.54 \times 10^{-3}$  | $1.4 \times 10^{-3}$   | $2.95 \times 10^{-4}$  |           |
| 15  | $3.46 \times 10^{-3}$  | $3.52 \times 10^{-3}$  | $3.57 \times 10^{-3}$  | $2(1) \times 10^{-3}$  | $3.6 \times 10^{-3}$   | $1.31 \times 10^{-3}$ |
| 16  | $4.74 \times 10^{-3}$  | $4.82 \times 10^{-3}$  | $4.9 \times 10^{-3}$   | $3(1) \times 10^{-3}$  | $4.95 \times 10^{-3}$  | $1.82 \times 10^{-3}$ |
| 17  | $6.35 \times 10^{-3}$  | $6.48 \times 10^{-3}$  | $6.6 \times 10^{-3}$   | $5(2) \times 10^{-3}$  | $6.67 \times 10^{-3}$  | $2.48 \times 10^{-3}$ |
| 18  | $8.36 \times 10^{-3}$  | $8.56 \times 10^{-3}$  | $8.73 \times 10^{-3}$  | $6(2) \times 10^{-3}$  | $8.83 \times 10^{-3}$  | $3.31 \times 10^{-3}$ |
| 19  | $1.39 \times 10^{-2}$  | $1.43 \times 10^{-2}$  | $1.46 \times 10^{-2}$  | $1.0 \times 10^{-2}$   | $1.48 \times 10^{-2}$  | $5.66 \times 10^{-3}$ |
| 20  | $3.28 \times 10^{-2}$  | $3.45 \times 10^{-2}$  | $3.56 \times 10^{-2}$  | $3 \times 10^{-2}$    | $3.63 \times 10^{-2}$  | $1.44 \times 10^{-2}$ |
| 21  | $6.72 \times 10^{-2}$  | $7.22 \times 10^{-2}$  | $7.53 \times 10^{-2}$  | $6 \times 10^{-2}$    | $7.7 \times 10^{-2}$   | $3.19 \times 10^{-2}$ |
| 22  | $0.123$           | $0.136$           | $0.144$          | $0.138$          | $0.148$          | $6.37 \times 10^{-2}$ |
| 23  | $0.207$           | $0.238$           | $0.254$          | $0.249$          | $0.262$          | $0.118$          |
| 24  | $0.325$           | $0.389$           | $0.421$          | $0.410$          | $0.437$          | $0.206$          |
| 25  | $0.481$           | $0.607$           | $0.665$          | $0.658$          | $0.695$          | $0.341$          |
| 26  | $0.676$           | $0.907$           | $1.01$           | $1.01$           | $1.06$           | $0.545$          |
| 27  | $0.904$           | $1.31$            | $1.48$           | $1.48$           | $1.56$           | $0.841$          |
| 28  | $1.15$            | $1.84$            | $2.1$            | $2.12$           | $2.24$           | $1.26$           |
| 29  | $1.41$            | $2.51$            | $2.92$           | $2.95$           | $3.13$           | $1.85$           |
| 30  | $1.63$            | $3.35$            | $3.97$           | $4.03$           | $4.29$           | $2.66$           |
| 31  | $1.77$            | $4.4$             | $5.3$            | $5.39$           | $5.77$           | $3.76$           |
| 32  | $1.78$            | $5.67$            | $6.96$           | $7.11$           | $7.62$           | $5.23$           |
| 33  | $1.55$            | $7.2$             | $9$              | $9.24$           | $9.93$           | $7.18$           |
| 34  | $0.983$           | $9.02$            | $11.5$           | $11.9$           | $12.8$           | $9.75$           |
| 35  | $0.252$           | $10.6$            | $13.7$           | $14.2$           | $15.3$           | $12.2$           |
| 36  | $-1.77$           | $13.7$            | $18.1$           | $18.9$           | $20.5$           | $17.5$           |
| 37  | $-4.34$           | $16.5$            | $22.5$           | $23.5$           | $25.5$           | $23.1$           |

**TABLE II:** The quantity $\gamma^3 \Delta g$ in units $10^{-6}$, calculated in various approximations for $1S_{1/2}$, $2S_{1/2}$, and $2P_{1/2}$ states. Our results are obtained with the account for the magnetic loop in the leading approximation (contribution of light-by-light scattering). The quantity $\Delta g_{nr}$ denotes the correction obtained with the use of Eq. (10) with the functions $f_1(r)$ and $f_2(r)$ taken in the nonrelativistic approximation, see Eq. (14).
| $Z$ | $\Delta g$ | $\Delta g_{as}$ | $Z$ | $\Delta g$ | $\Delta g_{as}$ |
|-----|----------|---------------|-----|----------|---------------|
| 1   | $1.043 \times 10^{-2}$ | $-0.2701$ | 24  | 159.5    | 145.9         |
| 2   | 0.1274   | $-0.6226$    | 28  | 233.7    | 217.9         |
| 3   | 0.484    | $-0.7983$    | 32  | 324.1    | 306.1         |
| 4   | 1.186    | $-0.6592$    | 36  | 430.8    | 411.7         |
| 5   | 2.317    | $-0.109$     | 40  | 554.4    | 532.8         |
| 6   | 3.944    | 0.9268       | 44  | 694.9    | 671.5         |
| 7   | 6.124    | 2.508        | 48  | 852.3    | 827.4         |
| 8   | 8.904    | 4.687        | 52  | 1026.0   | 1000.0        |
| 9   | 12.33    | 7.506        | 56  | 1217.0   | 1189.0        |
| 10  | 16.43    | 11.09        | 60  | 1424.0   | 1395.0        |
| 11  | 21.24    | 15.22        | 64  | 1646.0   | 1616.0        |
| 12  | 26.80    | 20.17        | 68  | 1883.0   | 1852.0        |
| 13  | 33.13    | 25.90        | 72  | 2134.0   | 2103.0        |
| 14  | 40.26    | 32.42        | 76  | 2398.0   | 2366.0        |
| 15  | 48.20    | 39.77        | 80  | 2673.0   | 2641.0        |
| 16  | 56.98    | 47.96        | 83  | 2886.0   | 2854.0        |
| 17  | 66.62    | 57.01        | 88  | 3251.0   | 3219.0        |
| 18  | 77.14    | 66.94        | 90  | 3400.0   | 3368.0        |
| 20  | 100.9    | 89.52        | 92  | 3550.0   | 3519.0        |

TABLE III: $\Delta g$ in units $10^{-6}$ for $1S_{1/2}$ state of muonic atom. $\Delta g_{as}$ is the asymptotics (19).