SPT indices emerging from translation invariance in two dimensional quantum spin systems

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Abstract

We consider SPT-phases with on-site $G$ (where $G$ is any discrete group) symmetry for two-dimensional quantum spin systems. We then impose translation invariance in one direction and observe that on top of the $H^3(G, \mathbb{T})$-valued index constructed in [10], an additional $H^2(G, \mathbb{T})$-valued index emerges. We also show that if we impose translation invariance in two directions, on top of the expected $H^3(G, \mathbb{T}) \oplus H^2(G, \mathbb{T})$ valued index, an additional $H^1(G, \mathbb{T})$-valued index emerges.

1 Introduction

The notion of symmetry protected topological phases of matter (SPT) was introduced by Chen, Liu and Wen [5] (in 1d), [4] (in 2d) and [3] (in any dimension). This last reference used a setup involving so-called nonlinear sigma models and provided an $H^{d+1}(G, U(1))$ valued index for any $d$-dimensional nonlinear sigma model. One other possible setup in which one can define SPT phases is in quantum spin systems (defined in section 2.1). To define SPT phases in this setting one needs to fix a discrete group $G$ with an on site group action $g \mapsto U_g$ (with $g \in G$), defining a global group action $\beta_g$ (also defined in section 2.1). One then calls a state $\omega$, $G$-invariant, if $\omega \circ \beta_g = \omega$. Then, one needs to impose a restriction on the space of states. We ask that our states are short range entangled (SRE), see definition 3.1. A state $\omega$ is short range entangled if there exists a disentangler $\gamma$. This is a locally generated automorphism generated by a one parameter family of interactions $\Phi \in B_{F_\rho}([0,1])$ ($\gamma = \gamma^\Phi_{0,1}$ as defined in section 2.3) satisfying that $\omega \circ \gamma$ is a product state. Combining these two definitions, we say that $\omega$ is an SPT state if it is short range entangled and $G$-invariant.

One then defines an equivalence relation on these SPT states by saying that $\omega_1$ is equivalent to $\omega_2$ with respect to the on site group action $U$ if there exists a $G$-invariant one parameter family of interactions $\Phi \in B_{F_\rho}([0,1])$ such that $\omega_1 \circ \gamma^\Phi_{0,1} = \omega_2$ (again, see section 2.3 for the definitions). One can then extend this equivalence to the stronger notion of being stably equivalent. This goes as follows: one defines an operation called stacking that takes two SPT states $\omega_1$ and $\omega_2$ and outputs a third one $\omega_1 \otimes_{\text{stack}} \omega_2$ (and similarly for the group action). This $\otimes_{\text{stack}}$ is defined in section 2.1. In short, it is the tensor product at the level of the on site Hilbert space. One then defines another equivalence relation and says that two SPT states with on site group actions $(\omega_1, U_1)$ and $(\omega_2, U_2)$ are stably equivalent if and only if there exist trivial SPTs with their group.

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1There are other definitions possible like for instance the concept of invertible $G$-invariant states used in [6] or the unique gapped ground state of a $G$ invariant interaction presented in the introduction of [10].

2A trivial state is a $G$-invariant product state that transforms trivially under the on site group action (see remark 5.11).
actions, \((\phi_1, \tilde{U}_1)\) and \((\phi_2, \tilde{U}_2)\) and a \((G\text{ independent})\)unitary \(V\) mapping between the respective on site Hilbert spaces\(^3\) such that

1. \(V^\dagger U_{1,g} \otimes \tilde{U}_{1,g} V = U_{2,g} \otimes \tilde{U}_{2,g}\) for all \(g \in G\).

2. \(\omega_1 \otimes_{\text{stack}} \phi_1 \circ i_V\) (where \(i_V\) is again defined in section \(\text{section 2.1}\)) is equivalent to \(\omega_2 \otimes_{\text{stack}} \phi_2\) with respect to the above on site group action.

In one spatial dimension so called matrix product states\(^5\) can be used to construct non-trivial SPT states. Later on and more generally it was shown in \(\text{section 9}\) that any \(G\) in one spatial dimension so called matrix product states\(^5\) can be used to construct non-trivial SPT states. Later on and more generally it was shown in \(\text{section 9}\) that any \(G\)-invariant state satisfying the split property (this includes any SRE state) carries an \(H^2(G, \mathbb{T})\)-valued\(^6\) index and that this index is constant on the (stable) equivalence classes. Later on in \(\text{section 6}\) it was then shown that this classification problem is complete (the index, seen as a map from the set of stable equivalence classes to \(H^2(G, \mathbb{T})\) is injective.). In two spatial dimensions, more recently in \(\text{section 11}\), it was proven there is an \(H^3(G, \mathbb{T})\)-valued index that is constant on the (stable) equivalence classes\(^7\).

We’ve used stacking to define the stable equivalence relation. Stacking however plays a more general role. It widely believed that it allows us to define an (abelian) group structure on the set of stable equivalence classes\(^8\). Although it is nowhere stated explicitly, the author thinks that the literature (like for example \(\text{section 6}\)) suggests the following conjecture:

**Conjecture 1.1.** Let \(S\) be the set of stable SPT classes for some fixed group and a fixed lattice. Let \(*\) be the map

\[
* : S^2 \to S : (\langle \omega_1, U_1 \rangle, \langle \omega_2, U_2 \rangle) \mapsto \langle \omega_1 \otimes_{\text{stack}} \omega_2, U_1 \otimes U_2 \rangle.
\]

Then \(*\) is well defined (independent of the choice of representative) and \(S\) together with the \(*\) operation forms an abelian group.

Before proceeding, we make three remarks on this conjecture:

1. The claim that \(*\) is well defined and abelian is trivial. Namely, by letting \(i_V\) be the automorphism that exchanges the two algebras that are being stacked, one can exchange the order of the stacking freely. By using

\[
\omega_1 \circ \gamma_{0;1}^{\Phi_1} \otimes_{\text{stack}} \omega_2 \circ \gamma_{0;1}^{\Phi_2} = (\omega_1 \otimes_{\text{stack}} \omega_2) \circ \gamma_{0;1}^{\Phi_1 \otimes 1 + \Phi_2} \sim \omega_1 \otimes_{\text{stack}} \omega_2,
\]

one can show that this is a well-defined map.

2. A representative of the inverse is sometimes called a \(G\)-inverse.

3. From a category point of view, if one assumes that this conjecture is true (for some subclass of groups that is a category), it is not hard to see that it gives rise to a contravariant functor from this category of groups to the category of abelian groups (say \(S[G]\)). More specifically, to each group morphism \(f : G_1 \to G_2\), one can find a group morphism \(\tilde{f} : S[G_2] \to S[G_1] : \langle \omega, U \rangle \mapsto \langle \omega, U \circ f \rangle\) and this \(\tilde{f}\) is indeed invariant on the choice of representative (if two states can be connected while preserving the symmetry action \(\beta\), they can certainly be connected while preserving the symmetry \(\beta \circ f\)).

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\(^3\)The reader can think of this \(V\) as being a unitary transformation of the on site group action.

\(^4\)By \(H^n(G, \mathbb{T})\) we mean the \(n\)-th (Borel) group cohomology of \(G\) with coefficients in the torus \(\mathbb{T}\) (seen as a (group) module under addition with trivial group action). Group cohomology is defined in section \(\text{section 2.1}\).

\(^5\)In \(\text{section 11}\), there is no mention of the stacking operation but using remark \(\text{4.29}\) and remark \(\text{4.31}\) one can clearly extend it to the stable equivalence class.

\(^6\)The stacking with a trivial state will then be related to the identity of this group.
In what follows, we will call $S$ with the above group structure the stable SPT classification. In this paper we consider the 2D case with the (stable) equivalence relation as presented here but with one notable difference. We will only consider states, interactions and on site symmetries that have a translation symmetry in one direction. There is a conjecture about the SPT classification of such states, see section 3.1.5.1 of [13] for a more general and detailed exposition (which was partially based on [3]).

**Conjecture 1.2.** The set of translation invariant SPTs under stable equivalence\(^7\) also satisfies conjecture [13]. Let $g$ be the inclusion map from the space of 2d translation invariant SPT states to the more general set of 2d SPT states. Let $f$ be the map that takes a 1d SPT state and outputs a translation invariant 2d SPT state by taking the tensor product in the direction of the translation symmetry. The sequence

$$0 \to \{\text{1d SPT states}\} \xrightarrow{f} \{\text{2d translation invariant SPT states}\} \xrightarrow{g} \{\text{2d SPT states}\} \to 0 \quad (1.3)$$

induces a sequence (of group morphisms) on the stable equivalence classes. By this we mean that the class of $f(\phi)$ only depends on the class of $\phi$ and similarly for $g$. Moreover this induced sequence is exact and split.

Clearly this implies that

$$\{\text{2d translation invariant SPT classification}\} \cong \{\text{1d SPT classification}\} \oplus \{\text{2d SPT classification}\}. \quad (1.4)$$

Since a 1d SPT state carries an $H^2(G, \mathbb{T})$ valued index and a 2d SPT state carries an $H^3(G, \mathbb{T})$ valued index, this means that 2d SPT states with a translation symmetry should carry an $H^2(G, \mathbb{T}) \oplus H^3(G, \mathbb{T})$ valued index. This is consistent with the result of section XII of [3].

The first result of this paper is the construction of an $H^2(G, \mathbb{T})$ valued index (which we will denote by Index) on the space of translation invariant SPT states that is consistent with conjectures 1.1 and 1.2. By being consistent with the conjecture it is meant (among other things) that, if $\text{Index}_1$ is the one dimensional SPT index from [3], then $\text{Index}(f(\omega)) = \text{Index}_1(\omega)$ for any 1d SPT $\omega$. This construction highly relies on the objects that are present in the construction of the 2d SPT index in [10].

We also look at the case where there is a second translation symmetry (so the system is translation invariant in both $x$ and $y$ directions). For such systems there is the following conjecture (see again [13] and [3]):

**Conjecture 1.3.** The space of SPTs, translation invariant in two directions under stable equivalence also satisfies conjecture [13]. Let $g$ be the inclusion map from the space of 2d SPT states that are translation invariant in both $x$ and $y$ directions to the more general set of 2d SPT states with translation invariance in the $x$ direction only. Let $f$ be the map that takes a 1d translation invariant SPT state and outputs a 2d SPT state that is translation invariant in both $x$ and $y$ directions by taking the tensor product in the $y$ direction. The sequence

$$0 \to \begin{cases} 1\text{d translation invariant} \\ \text{SPT states} \end{cases} \xrightarrow{f} \begin{cases} 2\text{d SPT states with two} \\ \text{translation symmetries} \end{cases} \xrightarrow{g} \begin{cases} 2\text{d SPT states translation} \\ \text{invariant in } x \text{ direction} \end{cases} \to 0 \quad (1.5)$$

induces a sequence (of group morphisms) on the stable equivalence classes. By this we mean that the class of $f(\phi)$ only depends on the class of $\phi$ and similarly for $g$. Moreover, this induced sequence is exact and split.

\(^7\)In the stable equivalence relation for translation invariant states we ask that the on site symmetry is translation invariant and that any path connecting two states and each trivial state is not only $G$-invariant but also translation invariant.
Translation invariant SPT states in one spatial dimension are known to carry an \( H^2(G, \mathbb{T}) \oplus H^1(G, \mathbb{T}) \) valued index (see appendix B). From this and the last two conjectures, we expect there is an
\[
H^3(G, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \oplus H^1(G, \mathbb{T})
\]
valued index. The first part of the index is just \( \text{Index}_{2d} \). The following two parts can be related to the case of a single translation symmetry as follows. Let \( \mu \) be the automorphism that rotates the lattice by \( 90^\circ \) and let \( \text{Index}(\omega) \) be the index as constructed before (this is the second part of 1.6). The third part of the index is now given by \( \text{Index}(\omega \circ \mu) \) (the state rotated by \( 90^\circ \)). The last part of the index requires a different construction altogether. It can be thought of as being the charge in the Brillouin zone.

A final remark:

**Remark 1.4.** As noticed in [3]. The groups in which these indices take values can be concisely written out. Using that
\[
H^0(\mathbb{Z}, \mathbb{T}) \cong \mathbb{T} \quad H^1(\mathbb{Z}, \mathbb{T}) \cong \mathbb{T} \quad H^2(\mathbb{Z}, \mathbb{T}) \cong 0 \quad H^3(\mathbb{Z}, \mathbb{T}) \cong 0
\]
and inserting this into the K"unneth formula for \( \mathbb{Z} \times G \) gives
\[
H^2(\mathbb{Z} \times G, \mathbb{T}) \cong \bigoplus_{i=0}^{2} H^i(\mathbb{Z}, \mathbb{T}) \otimes H^{2-i}(G, \mathbb{T}) \cong H^2(G, \mathbb{T}) \oplus H^1(G, \mathbb{T})
\]
(1.8)
\[
H^3(\mathbb{Z} \times G, \mathbb{T}) \cong \bigoplus_{i=0}^{3} H^i(\mathbb{Z}, \mathbb{T}) \otimes H^{3-i}(G, \mathbb{T}) \cong H^3(G, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \oplus H^1(G, \mathbb{T}).
\]
(1.9)

This result can in turn be used to work out the K"unneth formula for \( \mathbb{Z}^2 \times G \)
\[
H^3(\mathbb{Z}^2 \times G, \mathbb{T}) \cong \bigoplus_{i=0}^{3} H^i(\mathbb{Z}, \mathbb{T}) \otimes H^{3-i}(\mathbb{Z} \times G, \mathbb{T}) \cong H^3(G, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \oplus H^1(G, \mathbb{T}).
\]
(1.10)

The layout of this paper is as follows: First in section 2 we explain the setup, define the concept of locally generated automorphisms and give the algebraic definition of group cohomology. In section 3.1 we state the two results for SRE states. In section 3.2 we state the first result using the weaker assumption 3.6 and the second result using the weaker assumption 3.9. It is using these weaker assumptions that we then prove the statements. Section 4 provides a proof of the first statement whereas section 5 provides a proof of the second statement. The results in this paper rely heavily on the methods that where developed in [10].

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**Data availability statement**

No new data were created or analyzed in this study. Data sharing is not applicable to this article.
2 Setup and definitions

In this paper we work in the two dimensional lattice $\mathbb{Z}^2$. We will first need some specific subsets of $\mathbb{Z}^2$, so let

\begin{align*}
L &:= \{(x, y) \in \mathbb{Z}^2 | x < 0\}, \\
R &:= \{(x, y) \in \mathbb{Z}^2 | x \geq 0\} \\
U &:= \{(x, y) \in \mathbb{Z}^2 | y \geq 0\}, \\
D &:= \{(x, y) \in \mathbb{Z}^2 | y < 0\}
\end{align*}

(2.1) be the left, right, upper and lower half planes respectively and let

\begin{equation}
C_\theta := \{(x, y) \in \mathbb{Z}^2 | \tan(\theta) \leq \frac{|y|}{|x|}\}
\end{equation}

(2.3) be the horizontal cone (the green area in figure 1). We will use $\tau$ to denote the bijection that moves every element of $\mathbb{Z}^2$ one site upward. Similarly we let $\nu$ denote the bijection that translates every element of $\mathbb{Z}^2$ by one site to the right. In what follows we will sometimes need to widen our cone or other subsets of $\mathbb{Z}^2$ vertically by one site. For this purpose we define $W$ such that for any $\Gamma \subset \mathbb{Z}^2$,

\begin{equation}
W(\Gamma) := \Gamma \cup \tau(\Gamma) \cup \tau^{-1}(\Gamma).
\end{equation}

(2.4) For example, the red area in figure 1 is $W(C_\theta)^c$. In the later part we will also need to rotate our lattice by $90^\circ$ (clockwise) so call $\mu$ the bijection on $\mathbb{Z}^2$ that does precisely this.

2.1 Quasi local $C^*$ algebra

The setup in this paper will be very similar to the setup in [10] and is just the standard setup for defining quantum spin systems. For the rest of this paper, take $d \in \mathbb{N}_+$ arbitrary. This number will be called the on site dimension.\footnote{We will not discuss rotation invariant states in this paper. The $\mu$ is only defined to make certain constructions work.} For each $z \in \mathbb{Z}^2$, let $A_{\{z\}}$ be an isomorphic copy of $B(\mathbb{C}^d)$ (the bounded operators on $\mathbb{C}^d$). In what follows, let $\mathfrak{S}_{\mathbb{Z}^2}$ be the set of finite subsets of $\mathbb{Z}^2$. For any $\Lambda \in \mathfrak{S}_{\mathbb{Z}^2}$, we set $A_\Lambda = \bigotimes_{z \in \Lambda} A_{\{z\}}$. We define the local algebra as $A_{\text{loc}} = \bigcup_{\Lambda \in \mathfrak{S}_{\mathbb{Z}^2}} A_\Lambda$ and define the quasi local $C^*$ algebra as the norm closure of the local algebra ($A = \overline{A_{\text{loc}}}$). Similarly, for any (possibly infinite) subset $\Gamma \subset \mathbb{Z}^2$ we set $A_{\text{loc}, \Gamma} = \bigcup_{\Lambda \in \mathcal{G}_\Gamma} A_\Lambda$ and $A_\Gamma = \overline{A_{\text{loc}, \Gamma}}$.

We will need to define some automorphisms on this algebra using the bijections defined previously. To this end, define the isomorphisms

\begin{equation}
\tau : A_\Gamma \to A_{\tau(\Gamma)} \quad \nu : A_\Gamma \to A_{\nu(\Gamma)} \quad \mu : A_\Gamma \to A_{\mu(\Gamma)}
\end{equation}

(2.5) as the translation upwards, the translation to the right and the right-handed rotation respectively. By construction, these isomorphisms are automorphisms if $\Gamma$ is taken to be $\mathbb{Z}^2$ (because $d$ is a constant throughout the lattice).

We will need to define one additional set of automorphisms. For any set of unitaries $V_i \in U(\mathbb{C}^d)$ (labelled by $i \in \mathbb{Z}^2$), we define the (unique) automorphism $i^\Gamma_{V_i} \in \text{Aut}(A_\Gamma)$ (for all $\Gamma \subset \mathbb{Z}^2$) such that

\begin{equation}
i^\Gamma_{V_i}(A) = \text{Ad}(\otimes_{i \in I} V_i)(A)
\end{equation}

(2.6) for all $I \subset \Gamma$ finite and $A \in A_I$. More specifically, let $G$ be a discrete group and let $U_i \in \text{hom}(G, U(A_i))$ (for $i \in \mathbb{Z}^2$) be the on site group action. Let $\beta^\Gamma \in \text{hom}(G, \text{Aut}(A_\Gamma))$ (for any $\Gamma \subset \mathbb{Z}^2$) be such that

\begin{equation}
\beta^\Gamma_{U_i}(A) = i^\Gamma_{U_i(g)}(A) = \text{Ad}(\otimes_{i \in I} U_i(g))(A)
\end{equation}

(2.7)
for any \( g \in G, I \subset \Gamma \) finite and \( A \in \mathcal{A}_I \). Clearly, any unitary transformation of our representation \( U(\cdot) \to V^\dagger U(\cdot)V \) induces a transformation of the group action \( \beta_g \to i_V^{-1} \circ \beta_g \circ i_V \).

Throughout our paper when we are discussing translation invariance in the vertical direction we will ask that the group action is translation invariant in the vertical direction (\( \tau \circ \beta_{g}^{\Gamma} = \beta_{g}^{\Gamma} \circ \tau \)). If we also have translation invariance in the horizontal direction, we will also ask that the group action be translation invariant in that direction as well (\( \nu \circ \beta_{g}^{\Gamma} = \beta_{g}^{\nu(\Gamma)} \circ \nu \)). As it turns out, in showing that the \( H^1 \)-valued index is consistent with all the conjectures of the introduction, we will even require that the on site group action \( U_i(g) \) is the same at each site. In what follows we will always assume this.

Clearly by construction there is a tensor product operation on this \( \mathbb{C}^* \) algebra in the sense that for any \( \Gamma_1, \Gamma_2 \subset \mathbb{Z}^2 \) satisfying that \( \Gamma_1 \cap \Gamma_2 = \emptyset \) we can define a bilinear, surjective map

\[
\otimes: \mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2} \to \mathcal{A}_{\Gamma_1 \cup \Gamma_2}.
\]  

There is however a second tensor product operation on this \( \mathbb{C}^* \) algebra that we will use. We will call it the stacking operation. It is such that for any \( \Gamma \subset \mathbb{Z}^2 \) we define the bilinear, surjective map

\[
\otimes_{\text{stack}}: \mathcal{A}_{\Gamma} \times \mathcal{A}_{\Gamma} \to \mathcal{A}_{\Gamma}^2
\]  

where \( \mathcal{A}_{\Gamma}^2 \) is the quasi local \( \mathbb{C}^* \) algebra on \( \mathbb{Z}^2 \) constructed from an on site algebra \( \mathcal{B}(\mathbb{C}^d) \otimes \mathcal{B}(\mathbb{C}^d) \simeq \mathcal{B}(\mathbb{C}^{d \times d}) \) \(^{\text{10}}\). We will also need to define the stacking of the group action by which we mean

\[
(\beta_g \otimes_{\text{stack}} \beta_g)^{\Lambda} = \bigotimes_{z \in \Lambda} \text{Ad}(U(g) \otimes U(g)_z) \quad (2.10)
\]

for any \( \Lambda \in \mathfrak{S}_{\mathbb{Z}^2} \) (in fact stacking can be defined for any automorphism, not just the group action).

### 2.2 States on \( \mathcal{A} \)

We say that a linear functional \( \omega: \mathcal{A} \to \mathbb{C} \) is a state if it positive and normalized. This means that for any \( A \in \mathcal{A}, \omega(A^\dagger A) \geq 0 \) and that \( \omega(1) = 1 \). We denote the set of states on \( \mathcal{A} \) by \( \mathcal{S}(\mathcal{A}) \). In this paper we will work exclusively with states that are pure, in this setting this means that it is extremal in the sense that for any \( p \in ]0,1[ \), the only two other states \( \omega_1, \omega_2 \in \mathcal{S}(\mathcal{A}) \) that satisfy

\[
\omega = p\omega_1 + (1-p)\omega_2 \quad (2.11)
\]

must satisfy \( \omega_1 = \omega_2 = \omega \). The set of pure states on \( \mathcal{A} \) will be called \( \mathcal{P}(\mathcal{A}) \). In this paper we will often refer to the GNS triple. For its definition, construction and properties we refer to \([2]\). We will sometimes write the tensor product of states \( \omega \otimes \phi \) to be such that \( (\omega \otimes \phi)(a \otimes b) = \omega(a)\phi(b) \) and similarly for the stacking tensor product. We will call a pure state \( \phi \) on \( \mathcal{A}_\Gamma \) a product state if for any \( \Lambda \subset \Gamma \), the restriction \( \phi|_{\mathcal{A}_\Lambda} \) is still pure.

### 2.3 Interactions and locally generated automorphisms

An interaction \( \Phi \) is a map

\[
\Phi: \mathfrak{S}_{\mathbb{Z}^2} \to \mathcal{A}_{\text{loc}}: I \mapsto \Phi(I) \quad (2.12)
\]  

\(^{\text{10}}\)Clearly this can also be used to stack two \( \mathbb{C}^* \)-algebras with different on site Hilbert spaces, this was used in the introduction. In the rest of the paper we will for simplicity of notation (but without loss of generality) only stack two identical \( \mathbb{C}^* \)-algebras on top of each other.
where \( \Phi(I) \in \mathcal{A}_I \) is hermitian (\( \Phi(I) = \Phi(I)^* \)). We will sometimes require a norm on the space of interactions that indicates how local an interaction acts. Take \( F : \mathbb{N} \rightarrow \mathbb{R}^+ \) any monotonically decreasing positive function then we define an \( F \)-norm on the space of interactions by:

\[
\| \Phi \|_F := \sup_{x,y \in \mathbb{Z}^2} \frac{1}{F(|x-y|)} \sum_{Z \in \mathcal{A}_{x,y}, Z \ni x,y} \| \Phi(Z) \|.
\] (2.13)

Following \[10\] we will fix a specific family of monotonically decreasing positive functions by saying that for any \( 0 < \phi < 1 \) we define

\[
F_\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ : r \mapsto \exp\left(-r^\phi\right) \frac{1}{(1 + r)^4}.
\] (2.14)

We will sometimes use the restriction of an interaction. For some \( \Gamma \subset \mathbb{Z}^2 \) we define \( \Phi \) by

\[
\Phi : \mathcal{A}_{\mathbb{Z}^2} \rightarrow \mathcal{A}_{\text{loc}} : I \mapsto \begin{cases} \Phi(I) & \text{if } I \subset \Gamma, \\ 0 & \text{otherwise.} \end{cases}
\] (2.15)

The set of interactions with this norm (for a fixed \( F \)) is a Banach space. One of the properties of \( F \)-local interactions is that they can be used to generate automorphisms. Let

\[
\Phi : \mathcal{A}_{\mathbb{Z}^2} \times [0, 1] \rightarrow \mathcal{A}_{\text{loc}} : (I, t) \mapsto \Phi(I, t)
\] (2.16)

be a norm continuous\[11\] one parameter family of interactions such that the \( F \)-norm is uniformly bounded \((\sup_{t \in [0,1]} \| \Phi(., t) \|_F < \infty)\). We will denote the set of one parameter families of interactions that satisfy this property by \( \mathcal{B}_F([0,1]) \). For any \( \Phi \in \mathcal{B}_F([0,1]) \), we define the locally generated automorphism (LGA) \( \gamma_{\Phi}^{I,t} \) such that for any \( A \in \mathcal{A} \), \( \gamma_{\Phi}^{I,t}(A) \) is given as the solution to the differential equation

\[
\frac{d}{dt} \gamma_{\Phi}^{I,t}(A) = -i \sum_{I \in \mathcal{A}_{\mathbb{Z}^2}} \gamma_{\Phi}^{I,t}([\Phi(I, t), A])
\] (2.17)

with initial condition \( \gamma_{\Phi}^{I,s}(A) = A \) (the existence of this automorphism is proven in \[8\]). This satisfies the condition that if \( \sup_t \left\| \sum_{I \in \mathcal{A}_{\mathbb{Z}^2}} \Phi(I, t) \right\| < \infty \) then

\[
\gamma_{s,t}^{\Phi} = \text{Ad}(u_{s,t}^{\Phi}), \quad u_{s,t}^{\Phi} := T \exp \left(-i \int_t^s ds' \sum_{I \in \mathcal{A}_{\mathbb{Z}^2}} \Phi(s', t) \right).
\] (2.18)

Sometimes we will say that an interaction is \( G \)-invariant. By this we simply mean that \( \beta_g(\Phi(I)) = \Phi(I) \) (for all \( I \in \mathcal{A}_{\mathbb{Z}^2} \)). Similarly if we say an interaction is translation invariant we mean that \( \tau(\Phi(I)) = \Phi(\tau(I)) \) (for all \( I \in \mathcal{A}_{\mathbb{Z}^2} \)). It should be clear that if an interaction is \( G \)-invariant (translation invariant) that then the LGA it generates commutes with the group action (the translation automorphism) as well.

### 2.4 (Borel) Group cohomology groups

In this paper we define the Group cohomology groups using the algebraic approach. A standard reference for Group cohomology is \[1\]. Let \( G \) be an arbitrary discrete group. For any \( n \in \mathbb{N} \), let \( C^n(G, M) \) (in principle this can be for any (left) \( G \)-module \( M \) but in this paper we will always take \( M = \mathbb{T} \) with the addition and the trivial group action)\[12\] be the group of all functions from

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\[11\]By which we mean that \( \Phi(I, \cdot) \) is norm continuous for each \( I \in \mathcal{A}_{\mathbb{Z}^2} \).

\[12\]So in particular when we write \( g_1 \phi(g_2, \cdots, g_n) \) for any \( \phi \in C^{n-1}(G, M) \) we will sometimes identify it simply with \( \phi(g_2, \cdots, g_n) \).
\( g^n \) to \( M \). We will use additive notation on this space and denote the group operator by + and the inverse of an element of the group \( \phi \) by \(-\phi\). Now define the coboundary homomorphisms

\[
d^{n+1} : C^n(G, M) \mapsto C^{n+1}(G, M)
\]

such that

\[
(d^{n+1}\phi)(g_1, \cdots, g_{n+1}) = g_1\phi(g_2, \cdots, g_{n+1}) + \sum_{i=1}^n (-1)^i\phi(g_1, \cdots, g_{i-1}, g_i g_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1}\phi(g_1, \cdots, g_n).
\]

Since \( d^{n+1} \circ d^n = 0 \) this defines a cochain complex and we can define its cohomology as

\[
H^n(G, M) = Z^n(G, M)/B^n(G, M)
\]

where

\[
Z^n(G, M) := \ker(d^{n+1}) \quad B^n(G, M) := \begin{cases} 0 & \text{if } n = 0 \\ \operatorname{im}(d^n) & \text{if } n \geq 1. \end{cases}
\]

In what follows we will denote the group action on \( H^n(G, M) \) also in the same additive notation that we used for \( C^n(G, M) \). This notation makes sense because of the property that for any \( \phi_1, \phi_2 \in Z^n(G, M) \) we have that \( \langle \phi_1 + \phi_2 \rangle_{H^n(G, M)} = \langle \phi_1 \rangle_{H^n(G, M)} + \langle \phi_2 \rangle_{H^n(G, M)} \).

**Remark 2.1.** In [3], they argue that this definition of group cohomology is only the right approach for discrete groups. When we want to generalize this to any continuous group that has a well defined De Haar measure we additionally have to impose that the cochains are measurable. Group cohomology with measurable cochains is sometimes called Borel Group cohomology. In a future paper we will show how one can construct a Borel Group cohomology class from SPT states protected by any locally compact Lie group.

### 3 Results

We will present our claim starting from two different assumptions. The first assumption is related to short range entanglement whereas the second assumption is more technical but also more general (the first assumption will imply the second assumption).

#### 3.1 Statement of the results in terms of short range entanglement

We can use these locally generated automorphisms to define the concept of short range entanglement.

**Definition 3.1.** We say that a pure state \( \omega \in \mathcal{P}(A) \) is short range entangled (in short an SRE state) if and only if there exists a one parameter family of interactions \( \Phi \in B_{F_d}([0, 1]) \) (for some \( 0 < \phi < 1 \)) such that \( \omega \circ \gamma^\phi_{0,1} \) is a product state. We then call \( \gamma^\phi_{0,1} \) a disentangler for \( \omega \).

**Definition 3.2.** Let \( L_j = \{(i,j) | i \in \mathbb{Z}\} \subset \mathbb{Z}^2 \) be a horizontal line. Let \( \phi \in \mathcal{P}(A_{L_0}) \) be an SRE state (over \( \mathbb{Z} \)) that is \( G \)-invariant under a group action \( \beta^\phi_{L_0} \). Define \( \phi_i \in \mathcal{P}(A_{L_i}) \) as \( \phi_i := \phi_0 \circ \tau^{-i} \). We define the infinite tensor product state \( \omega_{\phi} \) as

\[
\omega_{\phi} := \bigotimes_{j \in \mathbb{Z}} \phi_j.
\]

This is a \( G \)-invariant SRE state over \( \mathbb{Z}^2 \) that is invariant under the automorphism \( \tau \).
We can now formulate our first result for the case where there is one translation symmetry.

**Theorem 3.3.** For any $A$ satisfying the construction from section [2.1] and any choice of on site group action $U \in \text{hom}(G, U(\mathbb{C}^d))$, there exists a map

$$\text{Index}^{A,U} : \{ \omega \in \mathcal{P}(A) | \omega \text{ is an SRE state, } \omega \circ \beta_g = \omega \text{ and } \omega \circ \tau = \omega \} \rightarrow H^2(G, \mathbb{T}) \quad (3.2)$$

that is well defined (doesn’t depend on the product state or the choice of disentangler). This map will be consistent with conjecture 1.2. By this we mean that:

1. For any $G$-invariant, translation invariant family of interactions $\Phi \in \mathcal{B}_{F_\phi}([0,1])$ (for some $0 < \phi < 1$) we have that

$$\text{Index}^{A,U}(\omega) = \text{Index}^{A,U}(\omega \circ \gamma_{0,1}). \quad (3.3)$$

2. For $\phi$, any one dimensional short ranged entangled $G$-invariant state, $\omega_\phi$ as defined in 3.2 satisfies

$$\text{Index}^{A_{id},U}(\phi) = \text{Index}^{A,U}(\omega_\phi) \quad (3.4)$$

where Index$_{id}$ is defined in appendix [B].

3. This index is a group homomorphism under the stacking operator. By this we mean that

$$\text{Index}^{A_1,U \otimes U_2}(\omega_1 \otimes \text{stack} \omega_2) = \text{Index}^{A_1,U}(\omega_1) + \text{Index}^{A_2,U}(\omega_2). \quad (3.5)$$

4. The index is independent on the choice of basis for the on site Hilbert space. By this we mean that for any on site unitary $V$ we get that

$$\text{Index}^{A,\text{Ad}(V^*) \circ U}(\omega \circ i_V) = \text{Index}^{A,U}(\omega). \quad (3.6)$$

**Proof.** See subsection [4.8]

Now for the second result I would like to define an index on the space of $G$-invariant SRE states that is translation invariant in both the horizontal and the vertical directions. In the introduction we already stated that the conjecture indicates that this is classified by $H^3(G, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \oplus H^1(G, \mathbb{T})$. The first index is just the index when there are no translations. Clearly there are two other indices one can define already. Namely, take

$$\omega \mapsto \text{Index}(\omega) \quad \text{and} \quad \omega \mapsto \text{Index}(\omega \circ \mu) \quad (3.7)$$

where $\mu$ was the rotation automorphism. This will give the $H^2(G, \mathbb{T}) \oplus H^2(G, \mathbb{T})$ part. The second result presented in this paper is that we can also define this $H^1(G, \mathbb{T})$ index.

**Theorem 3.4.** For any $A$ satisfying the construction from section [2.1] and any choice of on site group action $U \in \text{hom}(G, U(\mathbb{C}^d))$, there exists a map

$$\text{Index}^{A,U}_{\text{trans}} : \{ \omega \in \mathcal{P}(A) | \omega \text{ is an SRE state, } \omega \circ \beta_g = \omega, \omega \circ \tau = \omega \text{ and } \omega \circ \nu = \omega \} \rightarrow H^1(G, \mathbb{T}) \quad (3.8)$$

that is well defined (doesn’t depend on the product state or the choice of disentangler). This map will be consistent with conjecture 1.3. By this we mean that:

1. For any $G$ invariant family of interactions that is translation invariant in both directions $\Phi \in \mathcal{B}_{F_\phi}([0,1])$ we have that

$$\text{Index}^{A,U}_{\text{trans}}(\omega) = \text{Index}^{A,U}_{\text{trans}}(\omega \circ \gamma_{0,1}). \quad (3.9)$$

13There is a result from group cohomology that $H^1(G, \mathbb{T}) \cong \text{hom}(G, \mathbb{T})$. We will use this identification freely.
2. Let $\phi$ be a one dimensional short ranged entangled $G$-invariant, translation invariant state then $\omega_\phi$ defined in 3.2 satisfies

$$\text{Index}_{1\text{d\ trans}}^{A_{L_0\ U}}(\phi) = \text{Index}_{2\text{trans}}^{A\ U}(\omega_\phi)$$

(3.10)

where $\text{Index}_{1\text{d\ trans}}^{A\ U}$ is defined in appendix B.

3. Let $\mu$ be the automorphism that rotates every element of the $C^*$ algebra by 90$^\circ$ degrees then

$$\text{Index}_{2\text{trans}}^{A\ U}(\omega) = \text{Index}_{2\text{trans}}^{A\ U}(\omega \circ \mu).$$

(3.11)

4. This index is a group homomorphism under the stacking operator by which we mean that

$$\text{Index}_{2\text{trans}}^{A\ U_1 \otimes U_2}(\omega_1 \otimes \text{stack} \ \omega_2) = \text{Index}_{2\text{trans}}^{A\ U_1}(\omega_1) + \text{Index}_{2\text{trans}}^{A\ U_2}(\omega_2).$$

(3.12)

5. The index is independent on the choice of basis for the on site Hilbert space by which we mean that for any on site unitary $V$ we get that

$$\text{Index}_{2\text{trans}}^{A\ Ad(V^\dagger)\circ U}(\omega \circ i_V) = \text{Index}_{2\text{trans}}^{A\ U}(\omega).$$

(3.13)

Proof. See subsection 5.8.

Except when we need it explicitly we will not write the $A, U(g)$ superscript in the indices. This need will only arise when we deal with the proof of the stacking properties.

### 3.2 Statement of the result in terms of Q-automorphisms

Before we can start to prove the results we will first formulate the result starting from a weaker assumption. Similarly what was done in [10], we will define an index on the space of states that can be disentangled by a Q-automorphism. This class of automorphisms is defined as:
Definition 3.5. Take $\alpha \in \text{Aut}(\mathcal{A})$. We say that $\alpha \in \text{QAut}_1(\mathcal{A})$ if and only if $\forall \theta \in ]0, \pi/2[$ there exists an $\alpha_L \in \text{Aut}(\mathcal{A}_L), \alpha_R \in \text{Aut}(\mathcal{A}_R), V_1 \in \mathcal{U}(\mathcal{A})$ and a $\Theta \in \text{Aut}(\mathcal{A}_{W\theta})^{\mathbb{R}}$ such that

$$\alpha = \text{Ad}(V_1) \circ \alpha_L \otimes \alpha_R \circ \Theta.$$  

(3.14)

We will often write $\alpha_0$ to mean $\alpha_L \otimes \alpha_R$. In this paper we will consider states that satisfy the following property (see figure 1 for the support of the automorphisms):

Assumption 3.6. Let $\omega \in \mathcal{P}(\mathcal{A})$ be

1. such that there exists an automorphism $\alpha \in \text{QAut}_1(\mathcal{A})$ and a product state $\omega_0 \in \mathcal{P}(\mathcal{A})$ satisfying

$$\omega = \omega_0 \circ \alpha.$$  

(3.15)

2. such that there exists a $\theta \in ]0, \pi/2[$ for which there exists a map

$$\tilde{\beta} : G \to \text{Aut}(\mathcal{A}) : g \mapsto \tilde{\beta}_g$$

satisfying

$$\omega \circ \tilde{\beta}_g = \omega$$

$$\tilde{\beta}_g = \text{Ad}(V_{g,2}) \circ \eta_L^g \otimes \eta_R^g \circ \beta^U_g$$

(3.17)

for some $V_{g,2} \in \mathcal{U}(\mathcal{A}), \eta_L^g \in \text{Aut}(\mathcal{A}_{C\theta \cap L})$ and $\eta_R^g \in \text{Aut}(\mathcal{A}_{C\theta \cap R})$.

3. translation invariant ($\omega \circ \tau = \omega$).

Lemma 3.7. Take $\omega \in \mathcal{P}(\mathcal{A})$ a short range entangled state that is $G$-invariant and translation invariant then it satisfies assumption 3.6.

Proof. See subsection 4.8.

We now state the result that using this assumption we can define an $H^2(G, \mathbb{T})$-valued index.

Theorem 3.8. For any $\mathcal{A}$ satisfying the construction from section 2.1 and any choice of on site group action $U \in \text{hom}(G, U(\mathbb{C}^d))$, there exists a map

$$\text{Index}^{A,U} : \{ \omega \in \mathcal{P}(\mathcal{A}) | \omega \text{ satisfies assumption 3.6} \} \to H^2(G, \mathbb{T})$$

(3.18)

that is well defined (doesn’t depend on any of the choices in assumption 3.6). This map will be consistent with conjecture 1.2 by which we mean that:

1. For any $G$ invariant and translation invariant family of interactions $\Phi \in \mathcal{B}_{F_{\phi}}([0, 1])$ (for some $0 < \phi < 1$) we have that

$$\text{Index}^{A,U}(\omega) = \text{Index}^{A,U}(\omega \circ \gamma_{0,1}^\Phi).$$

(3.19)

2. For $\phi \in \mathcal{P}(\mathcal{A}_{L_0})$, any a one dimensional $G$-invariant state satisfying assumption 4.16, the state $\omega_\phi$ as defined in 3.2 satisfies

$$\text{Index}^{A_{id,U}}(\phi) = \text{Index}^{A,U}(\omega).$$

(3.20)

3. This index is a group homomorphism under the stacking operator. By this we mean that

$$\text{Index}^{A_{id,U} \otimes U}(\omega_1 \otimes_{\text{stack}} \omega_2) = \text{Index}^{A,U}(\omega_1) + \text{Index}^{A,U}(\omega_2).$$

(3.21)
4. The index is independent on the choice of basis for the on site Hilbert space by which we mean that for any on site unitary $V$ we get that

$$\text{Index}^{A, \text{Ad}(V^1)}_{\omega \circ i_V}(\omega) = \text{Index}^{A, U}_{\omega}(\omega).$$

(3.22)

**Proof.** This statement is proven in section 4.

Now we will present the assumption from which we can define the $H^1(G, \mathbb{T})$-valued index (see figure 2 for the support of the automorphisms).

**Assumption 3.9.** Let $\omega \in \mathcal{P}(A)$ be

1. such that there exists an automorphism $\alpha \in \text{QAut}_1(A)$ and a product state $\omega_0 \in \mathcal{P}(A)$ satisfying

$$\omega = \omega_0 \circ \alpha.$$  

(3.23)

2. such that there exists a $\theta \in [0, \pi/2]$ for which there exists a map

$$\tilde{\beta} : G \to \text{Aut}(A) : g \mapsto \tilde{\beta}_g$$

(3.24)

satisfying

$$\omega \circ \tilde{\beta}_g = \omega \quad \tilde{\beta}_g = \text{Ad}(V_{g,2}) \circ \eta_{L}^g \otimes \eta_{R}^g \circ \beta_{g}^U$$

(3.25)

for some $V_{g,2} \in \mathcal{U}(A)$, $\eta_{L}^g \in \text{Aut}(A_{\nu^{-1}(\mathbb{C}_g \cap L)})$ and $\eta_{R}^g \in \text{Aut}(A_{\nu(C_g \cap R)})$.

3. translation invariant in both directions

$$\omega \circ \tau = \omega \quad \omega \circ \nu = \omega.$$  

(3.26)

\[14\] Note that our definition of $\text{QAut}_1(A)$ is slightly different from the definition of $\text{QAut}(A)$ from [10] because of this widening $W$. 

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Figure 2: This figure indicates the support area of the different automorphisms when there are two translation directions. The angle $\theta$ still has to be smaller then or equal to what was indicated here so that the $\Theta$ and the $\eta_g$ commute but now we must be able to do both a horizontal and a vertical widening of the support of $\eta_g$. 

Now we will present the assumption from which we can define the $H^1(G, \mathbb{T})$-valued index (see figure 2 for the support of the automorphisms).
Lemma 3.10. Let \( \omega \) be a short range entangled state that is \( G \)-invariant and translation invariant in two directions then it satisfies assumption 3.9.

Proof. See subsection 5.8. \( \square \)

Theorem 3.11. For any \( A \) satisfying the construction from section 2.1 and any choice of on site group action \( U \in \text{hom}(G,U(\mathbb{C}^d)) \), there exists a map

\[
\text{Index}_{A_U}^{2, \text{trans}} : \{ \omega \in \mathcal{P}(A) | \omega \text{ satisfies assumption 3.9} \} \to H^1(G,T) \tag{3.27}
\]

that is well defined (doesn't depend on any of the choices made in assumption 3.9). This map will be consistent with conjecture 1.3 by which we mean the following:

1. For any \( G \) invariant family of interactions that is translation invariant in both directions \( \Phi \in \mathcal{B}_{P_\phi}([0,1]) \) we have that

\[
\text{Index}_{A_U}^{2, \text{trans}}(\omega) = \text{Index}_{A_U}^{2, \text{trans}}(\omega \circ \gamma_0^\Phi). \tag{3.28}
\]

2. For \( \phi \in \mathcal{P}(A_{L_0}) \), any a one dimensional \( G \)-invariant and translation invariant state satisfying assumption 4.16 the state \( \omega_\phi \) as defined in 3.2 satisfies

\[
\text{Index}_{A_{L_0}^{1, \text{trans}}}^{A_U}(\phi) = \text{Index}_{A_U}^{2, \text{trans}}(\omega_\phi). \tag{3.29}
\]

3. This index is a group homomorphism under the stacking operator by which we mean that

\[
\text{Index}_{A_U}^{2, \text{trans}}(\omega_1 \otimes \text{stack} \omega_2) = \text{Index}_{A_U}^{2, \text{trans}}(\omega_1) + \text{Index}_{A_U}^{2, \text{trans}}(\omega_2). \tag{3.30}
\]

4. The index is independent on the choice of basis for the on site Hilbert space by which we mean that for any on site unitary \( V \) we get that

\[
\text{Index}_{A_{U}^{\text{Ad}(V^\dagger)}}^{2, \text{trans}}(\omega \circ i_V) = \text{Index}_{A_U}^{2, \text{trans}}(\omega). \tag{3.31}
\]

Proof. This statement is proven in section 5. \( \square \)

3.3 Some remarks

The first thing we would like to remark is that in [10], a different equivalence class on states is used. First of all the set of states considered is a priori different. Namely, for any state \( \psi \in \mathcal{P}(A) \), we define two different conditions

1. There exists a \( \Phi \in \mathcal{B}_{P_\phi}([0,1]) \) such that \( \psi \) is the unique gapped groundstate of \( \Phi \).

2. \( \psi \) is an SRE state.

However in theorem 5.1 of [10] it is proven that the first item implies the second item. If one then looks at the definition of the index there, one could as well have started from the second condition. This is precisely what we did in this paper. As a side note, one should also be able to prove that the second item implies the first item by using theorem D.5 of [10] with \( K_t \) the disentangler and \( \Phi \) a gapped interaction that has the product state as its groundstate. However, we won’t work this out explicitly in this paper.

When we add the group, the story changes considerably. Let us define the two different equivalence classes. To this end, let \( \psi_1 \) and \( \psi_2 \) be two (pure) \( G \)-invariant states that are unique gapped groundstates of \( \Phi_0 \) and \( \Phi_1 \in \mathcal{P}_{SL,\beta} \) respectively. The following two conditions define different equivalence classes:

\[\text{This is defined in [10] but it essentially means bounded interaction that is connected to a trivial (without overlapping therms) interaction.}\]
1. There exists a bounded path of gapped interactions $\lambda \mapsto \tilde{\Phi}(\lambda)$ (see [10] for precise definitions) such that $\psi_1$ and $\psi_2$ are unique gapped groundstates of $\tilde{\Phi}(0)$ and $\tilde{\Phi}(1)$ respectively.

2. There exists a $G$-invariant LGA, $\gamma_{0,1}$ such that $\psi_1 \circ \gamma_{0,1} = \psi_2$.

A priori those two conditions are not equivalent. However in [10] it is proven that, the first equivalence implies the second one (I don’t think this implication requires a finite group). The proof that equivalent states have the same index only uses the second condition. This means that one could also have taken the second equivalence condition as the starting point. This is exactly what we do in this paper.

The second thing we would like to address is the fact that in our assumption the group is any discrete group whereas for instance in [10] the group is asked to be finite. We have two remarks to make on this subject:

1. There are certainly some additional results that one can prove if the group is finite. For instance proving that any $G$-invariant path of states (generated by any LGA) can be generated by a $G$-invariant LGA is something that can only be done (as far as I know) using a normalised De Haar measure on the group. In this setting, this means a finite group.

2. Notice that the construction of the index for non-compact groups remains consistent when the group action is in reality the group action of a finite group after a projection to this finite group. Take for example the $H^1$ index for the (non-compact) group $\mathbb{Z}$. Since our on site Hilbert space is finite and our group action is translation invariant in both directions there is just one representation on a finite Hilbert space defining the index. Suppose that that finite representation of $\mathbb{Z}$ is of the form that one just projects $\mathbb{Z}$ to $\mathbb{Z}_n$ and then defines a representation of $\mathbb{Z}_n$. One can then see the index simply as an index in $H^1(\mathbb{Z}_n, U(1)) \cong \mathbb{Z}_n$ that is then embedded into $H^1(\mathbb{Z}, U(1)) \cong U(1)$. It turns out that this construction is consistent with calculating the $H^1(\mathbb{Z}, U(1))$ directly. One can make similar remarks on the constructions for the $H^2$ index of the group $\mathbb{Z} \times \mathbb{Z}$. Namely, let us assume that the group action for column $i$ is formed out of the projection to a $\mathbb{Z}_{n_i} \times \mathbb{Z}_{m_i}$ action. Let $n$ be such that all $n_i$ are divisors of $n$ and similarly for $m$. Now we can see our index as taking values in $H^2(\mathbb{Z}_n \times \mathbb{Z}_m, U(1)) \cong \mathbb{Z}_{\gcd(n,m)}$ which can also be naturally embedded in $H^2(\mathbb{Z} \times \mathbb{Z}, U(1)) \cong U(1)$. It turns out again that this index too is consistent with calculating the $H^2(\mathbb{Z} \times \mathbb{Z}, U(1))$-valued index directly.$^{16}$

### 4 The $H^2(G, \mathbb{T})$-valued index

In this section we will define the $H^2(G, \mathbb{T})$-valued index starting from assumption 3.6. We will restate this assumption here for convenience:

**Assumption 3.6.** Let $\omega \in \mathcal{P}(\mathcal{A})$ be

1. such that there exists an automorphism $\alpha \in \text{QAut}_1(\mathcal{A})$ and a product state $\omega_0 \in \mathcal{P}(\mathcal{A})$ satisfying

$$\omega = \omega_0 \circ \alpha. \quad (3.15)$$

2. such that there exists a $\theta \in ]0, \pi/2[$ for which there exists a map

$$\tilde{\beta} : G \to \text{Aut}(\mathcal{A}) : g \mapsto \tilde{\beta}_g \quad (3.16)$$

$^{16}$In category theory, this consistency condition here is no more then the fact that the group cohomology group is a contravariant functor (see remark 2 of conjecture 1.1 in the introduction).
satisfying
\[ \omega \circ \tilde{\beta}_g = \omega \]
\[ \tilde{\beta}_g = \text{Ad}(V_{g,2}) \circ \eta^L_g \otimes \eta^R_g \circ \beta^U_g \] (3.17)

for some \( V_{g,2} \in \mathcal{U}(\mathcal{A}) \), \( \eta^L_g \in \text{Aut}(\mathcal{A}_{C \cap L}) \) and \( \eta^R_g \in \text{Aut}(\mathcal{A}_{C \cap R}) \).

3. translation invariant \((\omega \circ \tau = \omega)\).

In what follows, we will first fix certain objects to define the index and then in section 4.3 we will show that it is independent of all these choices. Fix an \( \omega_0 \) and an \( \alpha \). Fix \((\mathcal{H}_0 = \mathcal{H}_L \otimes \mathcal{H}_R, \pi_0 = \pi_L \otimes \pi_R, \Omega_0)\) a GNS triple (see theorem [A.1]) of \( \omega_0 \) where \((\forall \sigma \in \{ L, R \})\) \( \pi_\sigma : \mathcal{A}_\sigma \to \mathcal{B}(\mathcal{H}_\sigma) \) is a GNS representation of the restriction of \( \omega_0 \) to \( \sigma \). Fix as well \( V_1, \alpha_0 = \alpha_L \otimes \alpha_R \) and \( \Theta \) a choice of decomposition for \( \alpha \) such that \( \Theta \) and \( \eta_\theta \) commute (see figure [I]). Because we will be working with states that are translation invariant it make sense to define the translation action on the GNS space of \( \omega_0 \):

**Lemma 4.1.** There exists a unique \( v \in \mathcal{U}(\mathcal{H}_0) \) such that
\[ \text{Ad}(v) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \Theta \circ \tau \circ \Theta^{-1} \circ \alpha_0^{-1} \text{ and } \pi_0(V_1)v\pi_0(V_1^\dagger)\Omega_0 = \Omega_0. \] (4.1)

**Proof.** Since \( \omega \circ \tau = \omega \) we get by the uniqueness of the GNS triple (see corollary [A.2]) that there exists a unique \( \tilde{v} \in \mathcal{U}(\mathcal{H}_0) \) satisfying
\[ \text{Ad}(\tilde{v}) \circ \pi_0 \circ \alpha = \pi_0 \circ \alpha \circ \tau \text{ and } \tilde{v}\Omega_0 = \Omega_0. \] (4.2)
Since \( \alpha = \text{Ad}(V_1) \circ \alpha_0 \circ \Theta \) we get that
\[ \text{Ad} \left( \pi_0(V_1^\dagger)\tilde{v}\pi_0(V_1) \right) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \tau \text{ and } \tilde{v}\Omega_0 = \Omega_0. \] (4.3)
Choosing \( v = \pi_0(V_1^\dagger)\tilde{v}\pi_0(V_1) \) concludes the proof. \( \square \)

### 4.1 Cone operators

We now define a subgroup of \( \mathcal{U}(\mathcal{H}_0) \) that includes the representations of cone automorphisms:

**Definition 4.2.** Take \( 0 < \theta < \pi/2 \), \( \alpha_L \in \text{Aut}(\mathcal{A}_L) \), \( \alpha_R \in \text{Aut}(\mathcal{A}_R) \) and \( \alpha_0 := \alpha_L \otimes \alpha_R \). Take \( \sigma \in \{ L, R \} \) and \( x \in \mathcal{U}(\mathcal{H}_\sigma) \otimes 1_{\mathcal{H}_{\sigma^2}} \) then we say that \( x \) is a cone operator on \( \sigma \) (or in short \( x \in \text{Cone}_\sigma(\alpha_0, \theta) \)) if and only if there exists a \( \xi \in \text{Aut}(\mathcal{A}_{W(\mathcal{H})}) \) such that
\[ \text{Ad}(x) \circ \pi_0 \circ \alpha_0 = \pi_0 \circ \alpha_0 \circ \xi. \] (4.4)

\( \text{Cone}_\sigma(\alpha_0, \theta) \) is a subgroup of \( \mathcal{U}(\mathcal{H}_\sigma) \otimes 1_{\mathcal{H}_{\sigma^2}} \). We have that
\[ \text{Cone}_R(\alpha_0, \theta) \subseteq \text{Cone}_L(\alpha_0, \theta)' \quad \text{Cone}_L(\alpha_0, \theta) \subseteq \text{Cone}_R(\alpha_0, \theta)'. \] (4.5)

If additionally \( \xi \in \text{Aut}(\mathcal{A}_{C \cap \sigma}) \) we will say that \( x \in \text{Cone}_L^W(\alpha_0, \theta) \).

**Proof.** The proof of equation [4.5] just follows from the fact that we defined \( \text{Cone}_\sigma(\alpha_0, \theta) \) such that
\[ \text{Cone}_L(\alpha_0, \theta) \subseteq \mathcal{U}(\mathcal{H}_L) \otimes 1_{\mathcal{H}_R} \quad \text{Cone}_R(\alpha_0, \theta) \subseteq 1_{\mathcal{H}_L} \otimes \mathcal{U}(\mathcal{H}_R). \] (4.6)

We will also define a generalisation of this. We will define a subgroup of \( \mathcal{U}(\mathcal{H}_0) \) that includes both the group \( \pi_0 \circ \alpha_0 \circ \Theta(\mathcal{U}(\mathcal{A}_R)) \) and the representations of cone automorphisms:
**Definition 4.3.** Take $0 < \theta < \pi/2$, $\alpha_L \in \text{Aut}(A_L)$, $\alpha_R \in \text{Aut}(A_R)$ and $\Theta \in \text{Aut}(A_{W(C_0)})$. Take $\sigma \in \{L, R\}$ and take $x \in \mathcal{U}(\mathcal{H}_0)$ then we say that $x$ is an inner after cone operator on $\sigma$ (or in short $x \in \text{IAC}_\sigma(\alpha_0, \theta, \Theta)$) if there exists an $A \in \mathcal{U}(A_\sigma)$ and a $y \in \text{Cone}_\sigma(\alpha_0, \theta)$ such that

$$x = \pi_0 \circ \alpha_0 \circ \Theta(A) y.$$  \hfill (4.7)

$IAC_\sigma$ is a subgroup of $\mathcal{U}(\mathcal{H}_0)$. We get that

$$\text{IAC}_R(\alpha_0, \theta, \Theta) \subseteq \text{IAC}_L(\alpha_0, \theta, \Theta)' \quad \text{IAC}_L(\alpha_0, \theta, \Theta) \subseteq \text{IAC}_R(\alpha_0, \theta, \Theta)'.$$  \hfill (4.8)

If additionally $y \in \text{Cone}_L^W(\alpha_0, \theta)$ we will say that $x \in \text{IAC}_L^W(\alpha_0, \theta, \Theta)$.

**Proof.** Take $u_L \in \text{IAC}_L(\alpha_0, \theta, \Theta)$ and $u_R \in \text{IAC}_R(\alpha_0, \theta, \Theta)$ arbitrary. Take $A_\sigma \in \mathcal{U}(A_\sigma)$ and $v_\sigma \in \text{Cone}_\sigma(\alpha_0, \theta)$ (for $\sigma \in \{L, R\}$) such that $u_\sigma = \pi_0 \circ \alpha_0 \circ \Theta(A_\sigma)v_\sigma$. We then get

$$[u_L, u_R] = u_Lu_R - u_Ru_L \hfill (4.9)$$

$$= \pi_0 \circ \alpha_0 \circ \Theta(A_L)v_L\pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R - \pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_L \hfill (4.10)$$

$$= \pi_0 \circ \alpha_0 \circ \Theta(A_L)\text{Ad}(v_L)\pi_0 \circ \alpha_0 \circ \Theta(A_R)v_Lv_R - \pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_L. \hfill (4.11)$$

If we now take $\xi_\sigma \in \text{Aut}(A_{W(C_0)} \cap A_\sigma)$ such that $\text{Ad}(v_\sigma) \circ \pi_0 \circ \alpha_0 = \pi_0 \circ \alpha_0 \circ \xi_\sigma$ we get that

$$= \pi_0 \circ \alpha_0 \circ \Theta(A_L)\pi_0 \circ \alpha_0 \circ \Theta(A_R)v_Lv_R - \pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R\pi_0 \circ \alpha_0 \circ \ Theta(A_L)v_L \hfill (4.12)$$

$$= \pi_0 \circ \alpha_0 \circ \Theta(A_R)\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_Lv_R - \pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_L \hfill (4.13)$$

$$= \pi_0 \circ \alpha_0 \circ \Theta(A_L)v_Lv_R - \pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_L. \hfill (4.14)$$

Using property [4.5] we now get that

$$= \pi_0 \circ \alpha_0 \circ \Theta(A_R)\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_Lv_R - \pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_L \hfill (4.15)$$

$$= \pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_Lv_R - \pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_R - \pi_0 \circ \alpha_0 \circ \Theta(A_R)v_R\pi_0 \circ \alpha_0 \circ \Theta(A_L)v_L \hfill (4.16)$$

$$= 0 \hfill (4.17)$$

concluding the proof. \qed

The following lemma shows that if a unitary on the GNS space of $\omega_0$ has an adjoint action that is the product of cones then it is a product of cone operators:

**Lemma 4.4.** Take $x \in \mathcal{U}(\mathcal{H}_0)$ such that there exist $\xi_\sigma \in \text{Aut}(A_{C_0} \cap A_\sigma)$ (for $\sigma \in \{L, R\}$) satisfying

$$\text{Ad}(x) \circ \pi_0 \circ \alpha_0 = \pi_0 \circ \alpha_0 \circ \xi_\sigma \otimes \xi_R \hfill (4.18)$$

then there exist $x_\sigma \in \text{Cone}_\sigma(\alpha_0, \theta)$ such that

$$x = x_L \otimes x_R. \hfill (4.19)$$

These are unique up to a phase.

**Proof.** Using lemma [A.3] the result follows. \qed

Following Yoshiko Ogata [10] we now define objects using the following lemma:

**Lemma 4.5.** There unitaries $W_g \in \mathcal{U}(\mathcal{H}_0)$ and $u_\sigma(g, h) \in \mathcal{U}(\mathcal{H}_\sigma)$ for all $g, h \in G$ and for all $\sigma \in \{L, R\}$ satisfying

$$\text{Ad}(W_g) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \circ \beta^U_g \circ \Theta^{-1} \circ \alpha_0^{-1} \hfill (4.20)$$

$$\text{Ad}(u_\sigma(g, h)) \circ \pi_\sigma = \pi_\sigma \circ \alpha_\sigma \circ \eta^\sigma_g \circ \beta^U_g \circ \eta^\sigma_h \circ (\beta^U_g)^{-1} \circ (\eta^\sigma_h)^{-1} \circ \alpha_\sigma^{-1} \hfill (4.21)$$

$$u_L(g, h) \otimes u_R(g, h) = W_gW_hW_{gh}^{-1}. \hfill (4.22)$$

16.
Proof. The result follows from the fact that assumption 3.6 is sufficient to define the objects used in lemma 2.1 and definition 2.2 of [10].

These objects will be the starting point for our definition. Clearly the \( u_{\sigma}(g,h) \) are elements of \( \text{Cone}_r^W(\alpha_0, \theta) \). The \( W_g \) have the following property:

**Lemma 4.6.** Take \( \Xi^\sigma \in \text{Cone}_r(\alpha_0, \theta) \) (for all \( \sigma \in \{L, R\} \)) then \( \text{Ad}(W_g)(\Xi^\sigma) \in \text{Cone}_r(\alpha_0, \theta) \) (for all \( g \in G \)). If more generally we take \( \Xi^\sigma \in \text{IAC}_r(\alpha_0, \Theta) \) then \( \text{Ad}(W_g)(\Xi^\sigma) \in \text{IAC}_r(\alpha_0, \Theta) \). Similarly if \( \Xi^\sigma \in \text{Cone}_r^W(\alpha_0, \theta) \) then \( \text{Ad}(W_g)(\Xi^\sigma) \in \text{Cone}_r^W(\alpha_0, \theta) \).

Proof. Take \( \tilde{\Xi}^\sigma \) such that \( \Xi^\sigma = \tilde{\Xi}^\sigma \otimes 1_{H_{\alpha}} \). Take \( \xi_{\sigma} \in \text{Aut}(\mathcal{A}_{\text{C}_{\alpha}} \cap \mathcal{A}_{\sigma}) \) such that

\[
\text{Ad}(\tilde{\Xi}^\sigma) \circ \pi_\sigma \circ \alpha_\sigma = \pi_\sigma \circ \alpha_\sigma \circ \xi_{\sigma}.
\]
(4.23)

We have that

\[
\text{Ad}(\text{Ad}(W_g)(\Xi^R)) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \zeta_{g,R}.
\]
(4.24)

where

\[
\zeta_{g,\sigma} := \eta_g \circ \beta^U_\sigma \circ \xi_{\sigma} \circ (\beta^U_\sigma)^{-1} \circ \eta_g^{-1}.
\]
(4.25)

Since \( \zeta_{g,\sigma} \in \text{Aut}(\mathcal{A}_{\text{C}_{\alpha}} \cap \mathcal{A}_{\sigma}) \) we get that

\[
\text{Ad}(\text{Ad}(W_g)(\Xi^R)) \circ \pi_0 \circ \alpha_0 = \pi_L \circ \alpha_L \otimes \pi_R \circ \alpha_R \circ \zeta_{g,R}.
\]
(4.26)

Using lemma A.3 there exists a \( Z_{g,R} \in \mathcal{U}(H_R) \) satisfying that \( \text{Ad}(W_g)(\Xi^R) = 1_{H_L} \otimes Z_{g,R} \). To show the second result we only need to observe that for any \( A \in \mathcal{U}(A_R) \) we get that

\[
\text{Ad}(W_g)(\pi_0 \circ \alpha_0 \circ \Theta(A)\Xi^\sigma) = \text{Ad}(W_g)(\pi_0 \circ \alpha_0 \circ \Theta(A)\text{Ad}(W_g)(\Xi^R)) = \pi_0 \circ \alpha_0 \circ \Theta \circ \zeta_{R}(A)\text{Ad}(W_g)(\Xi^R).
\]
(4.27)
(4.28)

By the previous result, this is clearly in \( \text{IAC}_R(\alpha_0, \Theta, \Theta) \) concluding the proof.

We now have to define one more object (remember that \( v \) was defined in lemma 4.1):

**Lemma 4.7.** There exist \( K_g^L \in \text{Cone}_L(\alpha_0, \theta) \) and \( K_g^R \in \text{Cone}_R(\alpha_0, \theta) \) such that

\[
v^\dagger W_g vW_g^\dagger = K_g^L \otimes K_g^R = K_g
\]
(4.29)

and they are unique (up to a phase). They satisfy the identity

\[
\text{Ad}(K_g^R) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \tau^{-1} \circ \eta_g^R \circ \beta_g^{RU} \circ \tau \circ (\beta_g^{RU})^{-1} \circ (\eta_g^R)^{-1} \circ \alpha_0^{-1}.
\]
(4.30)

Proof. By some straightforward calculation we have that

\[
\text{Ad}(K_g) \circ \pi_0 = \text{Ad}(v^\dagger W_g vW_g^\dagger) \circ \pi_0
\]
(4.31)

\[
= \bigotimes_{\sigma=L,R} \pi_\sigma \circ \alpha_\sigma \circ \tau^{-1} \circ \eta_\sigma \circ \beta_\sigma^{SU} \circ \tau \circ (\beta_\sigma^{SU})^{-1} \circ (\eta_\sigma^S)^{-1} \circ \alpha_\sigma^{-1}.
\]
(4.32)

Using lemma 4.4 concludes the proof that these unitaries exist and are unique up to a \( G \)-dependent phase.

\[\text{Observe in particular that } \Theta \text{ has cancelled out}\]
4.2 Definition of the index

**Lemma 4.8.** There exists a $C : G^2 \to U(1)$ such that

$$K^R_g \text{Ad}(W_g) \left( K^R_h \text{Ad} \left( W_h W_{gh}^\dagger \right) (K^R_{gh})^\dagger \right) u_R(g, h) = C(g, h) v^\dagger u_R(g, h) v$$  \hspace{1cm} (4.33)

for all $g, h \in G$.

**Proof.** Since the GNS representation is irreducible this is equivalent to showing that the left and righthandside of equation (4.33) have the same adjoint action on the GNS representation. We first prove the result for the full tensor product. By using the definition of $u(g, h)$ we get

$$v^\dagger u_L(g, h) v v^\dagger u_R(g, h) v = v^\dagger (u_L(g, h) \otimes u_R(g, h)) v$$ \hspace{1cm} (4.34)

$$= v^\dagger W_g W_h W_{gh}^{-1} v.$$ \hspace{1cm} (4.35)

Now using the definition of $K_g$ gives:

$$\text{(4.34)} = K_g W_g v^\dagger W_h W_{gh}^{-1} v$$ \hspace{1cm} (4.36)

$$= K_g W_g K_h W_h v^\dagger W_{gh}^{-1} v$$ \hspace{1cm} (4.37)

$$= K_g W_g K_h W_h W_{gh}^\dagger K_{gh}^{-1}$$ \hspace{1cm} (4.38)

$$= K_g \text{Ad}(W_g) \left( K_h \text{Ad} \left( W_h W_{gh}^\dagger \right) \left( K_{gh}^\dagger \right) \right) u_L(g, h) \otimes u_R(g, h).$$ \hspace{1cm} (4.40)

Using lemma 4.6 we get that

$$(K^L_g \otimes K^R_g) \text{Ad}(W_g) \left( (K^L_h \otimes K^R_h) \text{Ad} \left( W_h W_{gh}^\dagger \right) \left( (K^L_{gh} \otimes K^R_{gh})^{-1} \right) \right)$$ \hspace{1cm} (4.41)

$$= K^L_g \text{Ad}(W_g) \left( K^L_h \text{Ad} \left( W_h W_{gh}^\dagger \right) \left( (K^L_{gh})^\dagger \right) \right) \otimes K^R_g \text{Ad}(W_g) \left( K^R_h \text{Ad} \left( W_h W_{gh}^\dagger \right) \left( (K^R_{gh})^\dagger \right) \right)$$ \hspace{1cm} (4.42)

concluding the proof. \hfill \Box

**Lemma 4.9.** The function $C$ as defined in lemma 4.8 is a 2-cochain.

**Proof.** Take lemma 2.4 from [10]. There it is stated that there exists a 3-cochain $C'$ such that

$$u_R(g, h) u_R(g, h) u_R(g, h) u_R(g, h) u_R(g, h) u_R(g, h) = C'(g, h, k) 1.$$ \hspace{1cm} (4.43)

It is clear that this 3-cochain is invariant under the substitution

$$W_g \to v^\dagger W_g v \quad \quad u_R(g, h) \to v^\dagger u_R(g, h) v.$$ \hspace{1cm} (4.44)

If we now prove that it is also invariant under the substitution

$$W_g \to K_g W_g \quad \quad u_R(g, h) \to K^R_g \text{Ad}(W_g) \left( K^R_h \text{Ad} \left( W_h W_{gh}^\dagger \right) \left( K^R_{gh})^\dagger \right) \right) u_R(g, h)$$ \hspace{1cm} (4.45)

we have proved our result because using equation (4.33) we then get that

$$C'(g, h, k) = C(h, k) C(g, h, k)^{-1} C(g, h, k) C(g, h, k)^{-1} C'(g, h, k)$$ \hspace{1cm} (4.46)

proving the result. Inserting substitution (4.45) into (4.43) gives

$$u_R(g, h) u_R(g, h) u_R(g, h) u_R(g, h) u_R(g, h) u_R(g, h) = C'(g, h, k) 1.$$ \hspace{1cm} (4.47)

$$\to K^R_g \text{Ad}(W_g) \left( K^R_h \text{Ad} \left( W_h W_{gh}^\dagger \right) \left( K^R_{gh})^\dagger \right) \right) u_R(g, h)$$ \hspace{1cm} (4.48)

$$K^R_{gh} \text{Ad}(W_{gh}) \left( K^R_k \text{Ad} \left( W_k W_{ghk}^\dagger \right) \left( K^R_{ghk})^\dagger \right) \right) u_R(g, h, k)$$ \hspace{1cm} (4.49)

$$u_R(g, h, k) \text{Ad}(W_g) \left( \text{Ad} \left( W_h W_{ghk}^\dagger \right) \left( K^R_{ghk})^\dagger \right) \right) \left( K^R_{gh})^\dagger \right) \left( K^R_{gh})^\dagger \right) \left( K^R_{gh})^\dagger \right) \left( K^R_{gh})^\dagger \right) \left( K^R_{gh})^\dagger \right) \left( K^R_{gh})^\dagger \right) \left( K^R_{gh})^\dagger \right).$$
Using the fact that $W_g W_h W_{gh}^\dagger = u_L \otimes u_R(g, h)$ and lemma 4.6 (for the substitution $K_g \rightarrow K_g^R$ in the last line) one now gets

$$= K_g^R \text{Ad}(W_g) \left( K_{g}^R \text{Ad} \left( W_h W_{gh}^\dagger \right) \left((K_{gh}^R)\dagger \right) \right) W_g W_h W_{gh}^\dagger u_L(g, h)$$

$$K_{gh}^R \text{Ad}(W_{gh}) \left( K_{h}^R \text{Ad} \left( W_k W_{gkh}^\dagger \right) \left((K_{gkh})\dagger \right) \right) W_{gh} W_k W_{gkh}^\dagger u_L(g, h, k)$$

$$W_{gkh} W_{hk}^\dagger W_g^\dagger u_L(g, h, k) \text{Ad}(W_g) \left( \text{Ad} \left( W_{hk} W_{gkh}^\dagger \right) \left((K_{gkh})\dagger \right) \left(K_{ghk}^R\dagger \right) \left(K_h^R\dagger \right) \right) (K_g^R)$$

$$= K_g^R \text{Ad}(W_g) \left( K_{h}^R \text{Ad} \left( W_h W_{gh}^\dagger \right) \left((K_{gh})\dagger \right) \right) W_g W_h W_{gh}^\dagger u_L(g, h)$$

$$K_{gh}^R \text{Ad}(W_{gh}) \left( K_{k}^R \text{Ad} \left( W_k W_{gkh}^\dagger \right) \left((K_{gkh})\dagger \right) \right) W_{gh} W_k W_{gkh}^\dagger u_L(g, h, k)$$

$$W_{gkh} W_{hk}^\dagger W_g^\dagger \text{Ad}(W_g) \left( \text{Ad} \left( W_{hk} W_{gkh}^\dagger \right) \left(K_{gkh}^R\dagger \right) \left(K_{ghk}^R\dagger \right) \right) (K_g^R)$$

We will now use the fact that the $u_L(g, h, k)$ commutes with everything that has support only on the right. Combining this with lemma 4.6 gives

$$= K_g^R \text{Ad}(W_g) \left( K_{h}^R \text{Ad} \left( W_h W_{gh}^\dagger \right) \left((K_{gh})\dagger \right) \right) W_g W_h W_{gh}^\dagger u_L(g, h)$$

$$K_{gh}^R \text{Ad}(W_{gh}) \left( K_{k}^R \text{Ad} \left( W_k W_{gkh}^\dagger \right) \left((K_{gkh})\dagger \right) \right) W_{gh} W_k W_{gkh}^\dagger u_L(g, h, k)$$

$$W_{gkh} W_{hk}^\dagger W_g^\dagger \text{Ad}(W_g) \left( \text{Ad} \left( W_{hk} W_{gkh}^\dagger \right) \left(K_{gkh}^R\dagger \right) \left(K_{ghk}^R\dagger \right) \right) (K_g^R)$$

Doing the same with $u_L(g, h, k)$ now gives

$$= K_g^R \text{Ad}(W_g) \left( K_{h}^R \text{Ad} \left( W_h W_{gh}^\dagger \right) \left((K_{gh})\dagger \right) \right) W_g W_h W_{gh}^\dagger u_L(g, h)$$

$$K_{gh}^R \text{Ad}(W_{gh}) \left( K_{k}^R \text{Ad} \left( W_k W_{gkh}^\dagger \right) \left((K_{gkh})\dagger \right) \right) \text{Ad} \left( W_{gkh} W_{hk}^\dagger W_{gh}^\dagger \right) \text{Ad}(W_g) \left( \text{Ad} \left( W_{hk} W_{gkh}^\dagger \right) \left(K_{gkh}^R\dagger \right) \left(K_{ghk}^R\dagger \right) \right) (K_g^R)$$

When also applying this to $u_L(g, h)$ one gets

$$= K_g^R \text{Ad}(W_g) \left( K_{h}^R \text{Ad} \left( W_h W_{gh}^\dagger \right) \left((K_{gh})\dagger \right) \right) W_g W_h W_{gh}^\dagger (K_g^R)$$

$$K_{gh}^R \text{Ad}(W_{gh}) \left( K_{k}^R \text{Ad} \left( W_k W_{gkh}^\dagger \right) \left((K_{gkh})\dagger \right) \right) \text{Ad} \left( W_{gkh} W_{hk}^\dagger W_{gh}^\dagger \right) \text{Ad}(W_g) \left( \text{Ad} \left( W_{hk} W_{gkh}^\dagger \right) \left(K_{gkh}^R\dagger \right) \left(K_{ghk}^R\dagger \right) \right) (K_g^R)$$

$$W_{gkh} W_{hk}^\dagger W_g^\dagger u_L(g, h) u_R(g, h, k) u_R(g, h, k) \text{Ad}(W_g) \left( u_L(g, h, k) \right)$$

$$\text{Ad}(W_g) \left( \text{Ad} \left( W_k W_{gh}^\dagger \right) \left(K_{kh}^R\dagger \right) \left(K_{h}^R\dagger \right) \right) (K_g^R)$$
Filling in the equation for the 3-cochain (equation 4.43) now gives

\[ C'(g, h, k)K_g^R \text{Ad}(W_g)(K_h^R \text{Ad}(W_h^\dagger)(K_{gh}^R)^\dagger)W_g W_h W_{gh}^\dagger \]  
\[ K_{gh}^R \text{Ad}(W_{gh})(K_h^R \text{Ad}(W_h^\dagger)(K_{gh}^R)^\dagger) \]
\[ \text{Ad}(W_{gh}W_{hk} W_{ghk}^\dagger)\left(\frac{1}{2}\text{Ad}(W_g)(\text{Ad}(W_h)(\text{Ad}(W_k W_{hk}^\dagger)(K_h^R)(K_k^R)^\dagger)(K_h^R)^\dagger)(K_g^R)^\dagger) \right). \]

Fully writing out the adjoints now gives:

\[ C'(g, h, k)K_g^R W_g K_h^R W_h W_{gh}^\dagger (K_{gh}^R)^\dagger W_{gh} W_h^\dagger W_g W_h W_{gh}^\dagger \]
\[ K_{gh}^R W_{gh} K_k^R W_{hk} W_{ghk}^\dagger (K_{ghk}^R)^\dagger W_{ghk} W_{hk}^\dagger W_{gh} W_{ghk}^\dagger W_{gh} \]
\[ W_{gh} W_{hk}^\dagger W_g W_k W_{hk}^\dagger K_{hk}^R W_{hk} W_{hk}^\dagger (K_h^R)^\dagger W_h^\dagger W_k^\dagger W_{hk}^\dagger (K_h^R)^\dagger W_g (K_g^R)^\dagger \]
\[ = 1C'(g, h, k) \]  
concluding the proof.

Our translation index is now defined as

**Definition 4.10.** Let \( \phi : G^2 \to U(1) \) be the 2-cochain defined in 4.8. Take \( \phi : G^2 \to \mathbb{T} \) such that \( C(g, h) = \exp(i\phi(g, h)) \) then we define the index as

\[ \text{Index}^\phi_{\mathbb{T}}(\theta, \tilde{\beta}_g, \eta_g, \alpha_0, \Theta, \omega, \omega_0) := \langle \phi \rangle \in H^2(G, \mathbb{T}) \]

and (as advertised) it is only a function of the automorphisms (and the product state) not on the choice of the GNS triple of \( \omega_0 \) or on the choice of phases in \( W_g, u_L(g, h), u_R(g, h), v, K_g^L \) and \( K_g^R \).

**Proof.** Clearly the construction is invariant under the choice of GNS triple since this simply amounts to an adjoint action by some unitary on every operator. Now we will show that it is invariant under the choice of phases of our operators. Clearly the 2-cochain \( C \) is invariant under

\[ u_L(g, h) \to \alpha(g)\alpha(g)\alpha(gh)^{-1}\beta(g, h)^{-1}u_L(g, h) \]
\[ u_R(g, h) \to \beta(g, h)u_R(g, h) \]
\[ v \to \gamma v. \]

Under the transformation

\[ K_g^L \to \delta(g)^{-1}K_g^L \]
\[ K_g^R \to \delta(g)K_g^R \]
we get \( C(g, h) \to \delta(g)\delta(h)\delta(gh)^{-1}C(g, h) \) which is clearly still in the same equivalence class concluding the proof.

### 4.3 The index is independent of the choices we made

In this section we will show that the index is only dependent on \( \omega \) and not on the choices of our automorphisms nor on \( \omega_0 \). First we show independence of \( \alpha \) and its decomposition.

\[ \text{Even though we explicitly wrote down the on site group action } U, \text{ this index is only dependent on } \text{Ad}(U). \]
Lemma 4.11. Take \( \omega_{01}, \omega_{02} \in \mathcal{P}(\mathcal{A}) \) product states and let \( \alpha_1, \alpha_2 \in \text{Aut}(\mathcal{A}) \) be such that
\[
\omega_{01} \circ \alpha_1 = \omega_{02} \circ \alpha_2 = \omega. \tag{4.61}
\]
Let \( V_{11}, V_{12} \in \mathcal{U}(\mathcal{A}), \alpha_{L/R,1}, \alpha_{L/R,2} \in \text{Aut}(\mathcal{A}_{L/R}) \) and \( \Theta_1, \Theta_2 \in \text{Aut}(\mathcal{A}_{W(C_0)}) \) be such that
\[
\alpha_1 = \text{Ad}(V_{11}) \circ \alpha_{01} \circ \Theta_1 \quad \quad \alpha_2 = \text{Ad}(V_{12}) \circ \alpha_{02} \circ \Theta_2 \tag{4.62}
\]
with \( \alpha_{0,i} = \alpha_{L,i} \otimes \alpha_{R,i} \), then
\[
\text{Index}^{\mathcal{A},\mathcal{U}}(\theta, \tilde{\beta}_g, \eta_g, \alpha_{0,1}, \Theta_1, \omega, \omega_{01}) = \text{Index}^{\mathcal{A},\mathcal{U}}(\theta, \tilde{\beta}_g, \eta_g, \alpha_{0,2}, \Theta_2, \omega, \omega_{02}). \tag{4.63}
\]

Proof. We will first prove the result in the case that \( \omega_0 = \omega_{01} = \omega_{02} \) and then generalise this result. Since \( \omega_0 \circ \alpha_2 \circ \alpha_1^{-1} = \omega_0 \) there exists a \( \tilde{x} \in \mathcal{U}(\mathcal{H}_0) \) such that
\[
\pi_0 \circ \alpha_2 \circ \alpha_1^{-1} = \text{Ad}(\tilde{x}) \circ \pi_0. \tag{4.64}
\]
Now define \( x \in \mathcal{U}(\mathcal{H}_0) \) to be
\[
x := \pi_0(V_{11}^\dagger \tilde{x} \pi_0(V_{11}) \tag{4.65}
\]
then
\[
\pi_0 \circ \alpha_{02} \circ \Theta_2 = \text{Ad}(x) \circ \pi_0 \circ \alpha_{01} \circ \Theta_1. \tag{4.66}
\]
Now take \( W_{g,1}, u_{R,1}(g, h) \) and \( K_{g,1}^R \) to be the operators belonging to the first choice (with arbitrary phases). We have (see [10] lemma 2.11)
\[
\text{Ad}(xW_{g,1} x^\dagger) \circ \pi_0 = \pi_0 \circ \alpha_{02} \circ \Theta_2 \circ \eta_g \circ \beta_g^U \circ \Theta_2^{-1} \circ \alpha_{02}^{-1}, \tag{4.67}
\]
\[
\text{Ad}(xu_{R,1}(g, h)x^\dagger) \circ \pi_0 = \pi_0 \circ \alpha_{02} \circ \eta_g^R \circ \beta_g^R \circ \Theta_2 \circ \eta_g^{-1} \circ \Theta_2^{-1} \circ \alpha_{02}^{-1}. \tag{4.68}
\]
and through similar arguments we get
\[
\text{Ad}(xv x^\dagger) \circ \pi_0 = \pi_0 \circ \alpha_{02} \circ \Theta_2 \circ \theta \circ \Theta_2^{-1} \circ \alpha_{02}^{-1} \tag{4.69}
\]
\[
\text{Ad}(xK_{g,1}^R x^\dagger) \circ \pi_0 = \pi_0 \circ \alpha_{02} \circ \theta \circ \Theta_2 \circ \eta_g^{-1} \circ \Theta_2^{-1} \circ \alpha_{02}^{-1}. \tag{4.70}
\]
This shows that \( xW_{g,1} x^\dagger, xu_{R,1}(g, h)x^\dagger \) and \( xK_{g,1}^R x^\dagger \) are operators belonging to the second choice (with our new translation operator). Since our index is invariant under this substitution this concludes the proof when \( \omega_0 = \omega_{01} = \omega_{02} \). Now suppose that \( \omega_{01} \neq \omega_{02} \). Since they are both product states there exists a \( \gamma \in \text{Aut}(\mathcal{A}) \) satisfying \( \omega_{02} = \omega_{01} \circ \gamma \) that is of the form \( \gamma = \gamma^L \otimes \gamma^R \). We now have
\[
\text{Index}(\theta, \tilde{\beta}_g, \eta_g, \alpha_{0,2}, \Theta_2, \omega, \omega_{02}) = \text{Index}(\theta, \tilde{\beta}_g, \eta_g, \alpha_{0,2}, \Theta_2, \omega, \omega_{01} \circ \gamma) = \text{Index}(\theta, \tilde{\beta}_g, \eta_g, \gamma \circ \alpha_{0,2}, \Theta_2, \omega, \omega_{01}) \tag{4.71}
\]
concluding the proof. \( \square \)

We now show that the index is invariant under some transformation parametrised by the group \( \text{IAC}_R^L \):

Lemma 4.12. Take \( \omega_0, \theta, \alpha_0, \Theta \) and \( \eta_g \) as usual, take \( v, W_{g,1}, u_{\sigma,1}(g, h) \) and \( K_{g,1}^\sigma \) the operators corresponding to these automorphisms and take \( \delta_g^\sigma \in \text{IAC}_\sigma^L(\alpha_0, \theta, \Theta) \) (for all \( \sigma \in \{L, R\} \) and all \( g \in G \)). Define
\[
W_{g,2} := \delta_g W_{g,1} \tag{4.72}
\]
\[
u_{\sigma,2}(g, h) := \delta_g^\sigma W_{g,1} \delta_g^{-\dagger} W_{g,1}^\dagger u_{\sigma,1}(g, h) (\delta_{gh})^\dagger \tag{4.73}
\]
\[
K_{g,2}^\sigma := v^\dagger \delta_g^\sigma v K_{g,1}^\sigma (\delta_g^\dagger)^\dagger \tag{4.74}
\]
then equation (4.33), defining the index of \( v, W_{g,1}, u_{\sigma,1}(g, h) \) and \( K_{g,1}^\sigma \), is still valid when we replace these operators by \( v, W_{g,2}, u_{\sigma,2}(g, h) \) and \( K_{g,2}^\sigma \) respectively. The two cochain also remains unchanged.
Proof. First we will prove that we still have that
\[ u_{L,2}(g, h) u_{R,2}(g, h) = W_{g,h,2} W_{h,g,2}^\dagger. \] (4.75)

To do this first insert the definition to obtain
\[ u_{L,2}(g, h) u_{R,2}(g, h) = \delta_g W_{g,2} \delta_h W_{h,2}^\dagger u_{L,1}(g, h)(\delta_{gh})^1 \delta_g W_{g,1} \delta_h W_{h,1}^\dagger u_{R,1}(g, h)(\delta_{gh})^1. \] (4.76)

Since by lemma 4.6 \( W_{g,1} \delta_g W_{g,1}^\dagger \in \text{IAC}_\sigma(\alpha, \theta, \Theta) \) (for \( \sigma \in \{L, R\} \) and \( g \in G \)) we get by repetitive use of equation (4.8) that
\[ = \delta_g W_{g,1} \delta_h W_{h,1}^\dagger u_{L,1}(g, h)(\delta_{gh})^1 u_{R,1}(g, h)(\delta_{gh})^1 \] (4.77)
\[ = \delta_g W_{g,1} \delta_h W_{h,1}^\dagger u_{L,1}(g, h) u_{R,1}(g, h)(\delta_{gh})^1 \delta_{gh} \] (4.78)
\[ = \delta_g W_{g,1} \delta_h W_{h,1}^\dagger W_{h,1} W_{g,1}(\delta_{gh})^1 \] (4.79)
\[ = W_{g,2} W_{h,2}^\dagger \] (4.80)

concluding the proof of equation (4.75). Using this we obtain:
\[ K_{g,2} W_{g,2} K_{h,2} W_{h,2}^\dagger (K_{g,2}^R)^1 W_{g,2} W_{h,2}^\dagger u_{R,2}(g, h) \] (4.81)
\[ = K_{g,2} W_{g,2} K_{h,2} W_{h,2}^\dagger u_{L,2}(g, h)^\dagger \] (4.82)
\[ = K_{g,2} W_{g,2} K_{h,2} W_{h,2}^\dagger u_{L,2}(g, h)^\dagger (K_{h,2}^R)^1 \] (4.83)
\[ = K_{g,2} W_{g,2} K_{h,2} W_{h,2}^\dagger W_{g,2} W_{h,2}^\dagger u_{R,2}(g, h)(K_{g,2}^R)^1. \] (4.84)

Filling this in the definition of the 2-cochain gives
\[ C_2(g, h) = K_{g,2} W_{g,2} K_{h,2} W_{h,2}^\dagger u_{R,2}(g, h)(K_{g,2}^R)^1 u_{L,2}(g, h)^\dagger v \] (4.85)
\[ = v^\dagger \delta_v K_{g,2} W_{g,1} (\delta_{gh})^1 \delta_h W_{g,1}^\dagger v K_{h,2} (\delta_{gh})^1 W_{h,1}^\dagger \] (4.86)
\[ = v^\dagger \delta_v K_{g,2} (\delta_{gh})^1 \delta_h W_{g,1}^\dagger K_{h,2} (\delta_{gh})^1 W_{h,1}^\dagger \] (4.87)
\[ = v^\dagger W_{g,1} (\delta_{gh})^1 W_{g,1}^\dagger \] (4.88)

We now insert \((K_{h,1}^R)^1 W_{g,1}(K_{g,1}^R)^1 K_{g,1} W_{g,1} K_{h,1} = 1:\)
\[ C_2(g, h) = v^\dagger \delta_v K_{g,1} (\delta_{gh})^1 \delta_h W_{g,1}^\dagger K_{h,1} (\delta_{gh})^1 W_{h,1}^\dagger \] (4.89)
\[ = v^\dagger W_{g,1} (\delta_{gh})^1 W_{g,1}^\dagger \] (4.90)

Using equation (4.8) we now get:
\[ C_2(g, h) = C_1(g, h) v^\dagger \delta_v K_{g,1} (\delta_{gh})^1 \delta_h W_{g,1}^\dagger (K_{g,1}^R)^1 \delta_h W_{g,1}^\dagger (\delta_{gh})^1 W_{g,1}^\dagger \] (4.91)
\[ = C_1(g, h) v^\dagger \delta_v K_{g,1} W_{g,1}^\dagger (\delta_{gh})^1 W_{g,1}^\dagger (\delta_{gh})^1 W_{g,1}^\dagger \] (4.92)
\[ = C_1(g, h) v^\dagger \delta_v W_{g,1} (\delta_{gh})^1 W_{g,1}^\dagger (\delta_{gh})^1 W_{g,1}^\dagger \] (4.93)

concluding the proof.
We will now show that the index is independent on the choice of \( \tilde{\beta}_g \) and its decomposition.

**Lemma 4.13.** Take \( \tilde{\beta}_{g,1}, \tilde{\beta}_{g,2} \in \text{Aut}(A), V_{g,21}, V_{g,22} \in \mathcal{U}(A), \eta_{g,1}^L, \eta_{g,2}^L \in \text{Aut}(A_L \cap A_{C_0}) \) and \( \eta_{g,1}^R, \eta_{g,2}^R \in \text{Aut}(A_R \cap A_{C_0}) \) such that there exist \( V_{g,21}, V_{g,22} \in \mathcal{U}(A) \) satisfying

\[
\tilde{\beta}_{g,1} = \text{Ad}(V_{g,21}) \circ \eta_{g,1} \circ \beta_g^U \quad \tilde{\beta}_{g,2} = \text{Ad}(V_{g,22}) \circ \eta_{g,2} \circ \beta_g^U
\]

and

\[
\omega \circ \tilde{\beta}_{g,1} = \omega \circ \tilde{\beta}_{g,2} = \omega
\]

then

\[
\text{Index}(\theta, \tilde{\beta}_{g,1}, \eta_{g,1}, \alpha_0, \Theta, \omega, \omega_0) = \text{Index}(\theta, \tilde{\beta}_{g,2}, \eta_{g,2}, \alpha_0, \Theta, \omega, \omega_0).
\]

**Proof.** Take

\[
\alpha = \text{Ad}(V_1) \circ \alpha_0 \circ \Theta
\]

the usual decomposition. Since

\[
\omega_0 \circ \alpha \circ (\tilde{\beta}_{g,2})^{-1} = \omega_0 \circ \alpha
\]

there exist \( \tilde{\delta}_g \in \mathcal{U}(H_0) \) such that

\[
\text{Ad}(\tilde{\delta}_g) \circ \pi_0 \circ \alpha = \pi_0 \circ \alpha \circ (\tilde{\beta}_{g,2})^{-1}.
\]

Inserting equation (4.94) and (4.97) in this gives

\[
\text{Ad}(\tilde{\delta}_g) \circ \pi_0 \circ \text{Ad}(V_1) \circ \alpha_0 \circ \Theta = \pi_0 \circ \text{Ad}(V_1) \circ \alpha_0 \circ \Theta \circ \text{Ad}(V_{g,21}) \circ \eta_{g,1} \circ \eta_{g,2} \circ \text{Ad}(V_{g,22}).
\]

Putting the \( V_1 \)'s to the other side gives:

\[
\text{Ad}(\pi_0(V_1)^\dagger \tilde{\delta}_g \pi_0(V_1)) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \text{Ad}(V_{g,21}) \circ \eta_{g,1} \circ \eta_{g,2} \circ \text{Ad}(V_{g,22}).
\]

Doing the same for the \( V_{g,21} \) and \( V_{g,22} \) now gives

\[
\text{Ad}(\pi_0 \circ \alpha_0 \circ \Theta(V_{g,21}) \pi_0(V_1)^\dagger \tilde{\delta}_g \pi_0(V_1)) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \text{Ad}(V_{g,22}) = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_{g,2} \circ (\eta_{g,1})^{-1}
\]

\[
\text{Ad}(\pi_0 \circ \alpha_0 \circ \Theta(V_{g,21}) \pi_0(V_1)^\dagger \tilde{\delta}_g \pi_0(V_1) \circ \alpha_0 \circ \Theta(V_{g,22})) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \alpha_0 \circ (\eta_{g,1})^{-1} \circ \alpha_0^{-1}.
\]

Since the last equation is split we have by lemma 4.4 that we can take \( \delta_g^L \in \text{Cone}_L^W(\alpha_0, \theta) \) and \( \delta_g^R \in \text{Cone}_R^W(\alpha_0, \theta) \) such that

\[
\delta_g^L \otimes \delta_g^R = \pi_0 \circ \alpha_0 \circ \Theta(V_{g,21}) \pi_0(V_1)^\dagger \tilde{\delta}_g \pi_0(V_1) \circ \alpha_0 \circ \Theta(V_{g,22}).
\]

Take \( W_{g,1}, u_{R,1}(g, h) \) and \( K_{g,1}^R \) to be the operators belonging to the first choice (with arbitrary phases). Define

\[
W_{g,2} := \delta_g W_{g,1}
\]

\[
u_{R,2}(g, h) := \delta_g^R W_{g,1} \delta_h W_{g,1} u_{R,1}(g, h)(\delta_{gh})^\dagger
\]

\[
K_{g,2}^R := v^\dagger \delta_g^R v K_{g,1}^R(\delta_{g})^\dagger
\]

then by construction \( W_{g,2}, u_{R,2}(g, h) \) and \( K_{g,2}^R \) are operators belonging to the second choice. Now using lemma 4.12 concludes the proof. \( \square \)

**Lemma 4.14.** The index is independent of the choice of angle \( \theta \).

**Proof.** Take \( 0 < \theta_1 < \theta_2 < \pi/2 \). Take \( \alpha \in \text{QAut}_1(A) \) and (for all \( \sigma \in \{L, R\} \)) take \( \eta_{g,1}^\sigma \in \text{Aut}(A_L \cap A_{C_{0}}) \) and \( \eta_{g,2}^\sigma \in \text{Aut}(A_L \cap A_{C_{02}}) \) to be operators belonging to \( \tilde{\beta}_{g,1} \) and \( \tilde{\beta}_{g,2} \) leaving \( \omega \) invariant. Since \( \text{Aut}(A_L \cap A_{C_{01}}) \subset \text{Aut}(A_L \cap A_{C_{02}}) \) the result now follows from lemma 4.13. \( \square \)

Due to all these considerations we will write the index as \( \text{Index}^A(\omega) \) from here on onward.
4.4 Example (consistency with conjecture 1.2)

Take $B = \overline{B_{\text{loc}}}$ the quasi local $C^*$ algebra in one spatial dimension such that

$$\mathcal{A} = \text{span}\left\{ \bigotimes_{j \in J} (B_{\text{loc}})_j \mid J \subset \mathbb{Z} \text{ finite} \right\}. \quad (4.108)$$

In words this is expressed as the 2d quasi local algebra is the norm closure of the product of the 1d local algebra. We will need one additional restriction on these operators

**Definition 4.15.** We say that $b \in \mathcal{U}(B)$ is summable if there exists a sequence $b_i \in \mathcal{U}(B_{[-i,i]}^\text{loc})$ such that

$$\sum_{i=n_0}^{\infty} \|b - b_i\| < \infty \quad (4.109)$$

for all $n_0 \in \mathbb{N}$.

**Assumption 4.16.** Take $\phi \in \mathcal{P}(B)$ such that

1. $\phi \circ \beta_g = \phi$.
2. there exist automorphisms $\tilde{\alpha}_L \in \text{Aut}(B_L), \tilde{\alpha}_R \in \text{Aut}(B_R)$ and a summable operator $b \in \mathcal{U}(B)$ such that

$$\phi_0 = \phi \circ \text{Ad}(b) \circ \tilde{\alpha}_L \otimes \tilde{\alpha}_R \quad (4.110)$$

is a product state.

All $G$–invariant SRE states satisfy this assumption:

**Lemma 4.17.** Suppose $\phi \in \mathcal{P}(A)$ is such that there exists a $\Phi \in \mathcal{B}_{F_{\varphi}}([0,1])$ (a one parameter family of interactions) for some $0 < \varphi < 1$, such that $\phi_0 = \phi \circ \eta_{0,\varphi}$ for some product state $\phi_0$ then if $\phi \circ \beta_g = \phi$, $\phi$ satisfies assumption 4.16

**Proof.** The proof simply uses lemma C.1.

**Lemma 4.18.** This $\phi$ satisfies the split property.

**Proof.** Take $(\mathcal{H}_L \otimes \mathcal{H}_R, \pi_L \otimes \pi_R, \Omega_L \otimes \Omega_R)$ a GNS triple of $\phi_0$. By construction $(\mathcal{H}_L \otimes \mathcal{H}_R, \pi_L \circ \tilde{\alpha}_L \otimes \pi_R \circ \tilde{\alpha}_R, \pi_L \otimes \pi_R(b^L)\Omega_L \otimes \Omega_R)$ is a GNS triple of $\phi$. This concludes the proof that $\phi$ satisfies the split property.

By the result of [9] this implies that $\phi$ has a well defined 1d SPT index (see appendix B). For any $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ take

$$L_j := \{(x,y) \in \mathbb{Z}^2 | y = j \} \quad  L_{j,n} := \{(x,y) \in \mathbb{Z}^2 | y = j, x \in [-n,n]\} \quad (4.111)$$

$$L_{<j} := \{(x,y) \in \mathbb{Z}^2 | y < j \} \quad L_{>j} := \{(x,y) \in \mathbb{Z}^2 | y > j \}. \quad (4.112)$$

We will now define the infinite tensor product state.

**Definition 4.19.** Take $\phi \in \mathcal{P}(B)$ arbitrary. There exists a unique state $\omega$ satisfying that for any $i \in \mathbb{Z}$ we have that for any $A \in \mathcal{A}_{L_i}$ and $B \in \mathcal{A}_{L_{-i}}$

$$\omega(A \otimes B) = \phi(A)\omega(B). \quad (4.113)$$

**Proof.** Clearly this condition gives a construction by induction for $\omega(A)$ for each $A \in \mathcal{A}_{\text{loc}}$. For general elements we define

$$\omega(A) := \lim_{n \to \infty} \omega(A_n) \quad (4.114)$$

where $A_n$ is a sequence in $\mathcal{A}_{\text{loc}}$ converging in $\mathcal{A}$. 

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We have that:

**Lemma 4.20.** Let $\omega$ be the infinite tensor product of a state $\phi$ that satisfies assumption [4.16] then $\omega$ satisfies assumption [3.6].

**Proof.** To show that $\tilde{\beta}_g$ exists note that we can take $\tilde{\beta}_g$ to be simply $\beta^U_g$. The translation invariance was true by construction. We now only have to find an $\alpha \in Q\text{Aut}_1(\mathcal{A})$ that disentangles $\omega$. We define $\alpha$ recursively such that for any $A_1 \in \mathcal{A}_{L_j}$ and $A_2 \in \mathcal{A}_{2^j/L_j}$ (for some $j$) we have that

$$
\alpha(A_1 \otimes A_2) = \text{Ad}(b) \circ \tilde{\alpha}_L \otimes \tilde{\alpha}_R(A_1)\alpha(A_2).
$$

(4.115)

This Automorphism indeed disentangles since

$$
\omega_0 := \omega \circ \alpha
$$

(4.116)

satisfies that for any $A \in \mathcal{A}_{L_i}$, $B \in \mathcal{A}_{L_j}$ with $i \neq j$ we have that

$$
\omega_0(A \otimes B) = \phi_0(A)\phi_0(B)
$$

(4.117)

so that it is clearly a product state. We will now show that $\alpha \in Q\text{Aut}_1(\mathcal{A})$. To do this, let $b_j$ be a summable sequence converging to $b$ such that $b_j \in L(\mathcal{B})_{[-n(j),n(j)]}$ where $n(j) \in \mathbb{N}$ is the largest number such that $L_{n(j)} \subset \mathcal{W}(\mathcal{C}_b)^e$. Take $0 < \theta < \pi/2$ and define $\Theta \in \text{Aut}(\mathcal{A}_{\mathcal{W}(\mathcal{C}_b)^e})$ through

$$
\Theta = \bigotimes_{j \in \mathbb{Z}} \text{Ad}(b_j)_{(-j)}.
$$

(4.118)

Let $\alpha_\sigma \in \text{Aut}(\mathcal{A}_\sigma)$ be

$$
\alpha_\sigma := \bigotimes_{j \in \mathbb{Z}} (\tilde{\alpha}_\sigma)_{(-j)}.
$$

(4.119)

Now define

$$
V_{1,m} := \bigotimes_{j \in \mathbb{Z}\setminus[-m,m]} (bb^j)_{(-j)}.
$$

(4.120)

We will now show that the limit

$$
V_1 := \lim_{m \to \infty} V_{1,m}
$$

(4.121)

exists (is an element of $\mathcal{A}$). By construction we have that for any $\epsilon$ there exists an $n_0 > 0$ such that

$$
\epsilon_n := \sum_{i=n}^{\infty} \|b - b_i\| < \epsilon
$$

(4.122)

for all $n > n_0$. We will now show that our sequence is a Cauchy sequence. Let $\epsilon > 0$ and take $n_0$ accordingly. For any $n, m \geq n_0$ we have that

$$
\|V_{1,n} - V_{1,m}\| \leq \|V_{1,n} - V_{1,n_0}\| + \|V_{1,m} - V_{1,n_0}\|
$$

(4.123)

$$
= \left\| V_{1,n}V_{1,n_0}^\dagger - 1 \right\| + \left\| V_{1,m}V_{1,n_0}^\dagger - 1 \right\|.
$$

(4.124)

We will find a bound on the first term as the bound of the second term is analogous. First notice that we have for any tensor product that

$$
A \otimes B - C \otimes D = (A - C + C) \otimes B - C \otimes D = (A - C) \otimes B + C \otimes (B - D).
$$

(4.125)

By using this property recursively we get

$$
\left\| \bigotimes_{i=-n}^{-n_0} (bb^i)_i \bigotimes_{i=n_0}^{n} (bb^i)_i - 1_{\mathcal{H}_{L_{-n,\ldots,-n}} \otimes 1_{\mathcal{H}_{L_{0,\ldots,n}}} \right\|
$$

(4.126)

$$
\leq 2 \sum_{i=n_0}^{n} \left\| bb^i - 1_{\mathcal{H}_{L_{i}}} \right\| < 2\epsilon.
$$

(4.127)

This proves that the sequence is a Cauchy sequence and hence the convergence follows. □
This shows that $\omega$ has a well defined 2d translation index. We will now show that $\text{Index}(\omega) = \text{Index}_{d}(\phi)$. We now fix a GNS triple of $\omega$ that is of the form

$$
(\mathcal{H}_0 = \bigotimes_{\rho \in \{L,R\}} \mathcal{H}_{\rho,\sigma}, \pi_0 = \bigotimes_{\rho \in \{L,R\}} \pi_{\rho,\sigma}, \Omega_0 = \bigotimes_{\rho \in \{L,R\}} \Omega_{\rho,\sigma})
$$

(4.128)

where the $\pi_{\rho,\sigma} : A_{\rho \cap \sigma} \to \mathcal{B}(\mathcal{H}_{\rho,\sigma})$ are all irreducible representations.

**Remark 4.21.** We clearly have (by construction) that

$$
\pi_{L_{-1},L} \circ \alpha_L|_{L_{-1}} \otimes \pi_{L_{-1},R} \circ \alpha_R|_{L_{-1}}
$$

is a GNS representation of $\phi$. This implies that if we take any $k^R : G \to \mathcal{U}(\mathcal{H}_{L_{-1},R})$ such that

$$
\text{Ad}(k^R(g)) \circ \pi_{L_{-1},R} \circ \alpha_R|_{L_{-1}} = \pi_{L_{-1},R} \circ \alpha_R|_{L_{-1}} \circ \beta_{g}^{L_{-1}}|R
$$

(4.130)

then there exists a 2 cochain $C$ satisfying that

$$
k^R(g)k^R(h)k^R(gh)^{-1} = C(g, h)1_{\mathcal{H}_{L_{-1},R}}
$$

(4.131)

and that the equivalence class this cochain is in, is given by $\text{Index}_{d}(\phi)$.

Let $\hat{k} : G \to \mathcal{U}(\mathcal{H}_{L_{-1}})$ be the unique operator that satisfies

$$
\text{Ad}(\hat{k}_g) \circ \pi_{L_{-1}} = \pi_{L_{-1}} \circ \alpha_{L_{-1}} \circ \beta_{g}^{L_{-1}} \circ \alpha_{-1}|_{L_{-1}}
$$

(4.132)

$$\hat{k}_g \Omega_{L_{-1}} = \Omega_{L_{-1}}.
$$

Similarly, let $\hat{w} : G \to \mathcal{U}(\mathcal{H}_{L_{-1},L} \otimes \mathcal{H}_{L_{-1},R})$ be the unique operator such that

$$
\text{Ad}(\hat{w}_g) \circ \pi_{L_{-1}} = \pi_{L_{-1}} \circ \alpha_{L_{-1}} \circ \beta_{g}^{L_{-1}}(U) \circ \alpha_{-1}|_{L_{-1}}
$$

(4.133)

$$\hat{w}_g \Omega_{L_{-1}} = \Omega_{L_{-1}}.
$$

Clearly both functions are (linear) representation of the group. Now define

$$
W_g = \pi_0(V_1)^{(\hat{w}_g \otimes 1_{\mathcal{H}_{L_{-1}}}) \otimes 1_{\mathcal{H}_{L_{-1}}}) \pi_0(V_1).
$$

(4.134)

This operator valued function is also a linear representation and satisfies the conditions indicated in lemma 4.5. Since $W_g$ is a linear representation we get that $W_g W_h W_h^{-1} = 1_{\mathcal{H}_0}$. This implies that we can take $u_{\sigma}(g, h) = 1_{\otimes_{\rho \in \{L_{-1},L_{-1},L_{-1}\}} \mathcal{H}_{\rho,\sigma}}$ (for any $\sigma \in \{L, R\}$). All that is now left is to calculate $K^\sigma_g \in \mathcal{U}(\mathcal{H}_\sigma)$. We get that

$$
K_g = v^\dagger W_g v W_g^\dagger
$$

(4.135)

$$
v^\dagger \pi_0(V_1) \hat{w}_g \otimes 1_{\mathcal{H}_{L_{-1}}} \otimes 1_{\mathcal{H}_{L_{-1}}} \pi_0(V_1) v W_g^\dagger
$$

(4.136)

$$
= \pi_0(V_1) \hat{v}^\dagger \hat{w}_g \otimes 1_{\mathcal{H}_{L_{-1}}} \otimes 1_{\mathcal{H}_{L_{-1}}} \pi_0(V_1) W_g^\dagger.
$$

(4.137)

Notice however that $\hat{v}^\dagger \hat{w}_g \otimes 1_{\mathcal{H}_{L_{-1}}} \otimes 1_{\mathcal{H}_{L_{-1}}} \hat{v}$ and $\hat{w}_g \otimes \hat{k}_g \otimes 1_{\mathcal{H}_{L_{-1}}}$ both leave the cyclic vector invariant and have the same adjoint action on the GNS representation. By the uniqueness of such operators we get that

$$
K_g = \pi_0(V_1)^{(\hat{w}_g \otimes \hat{k}_g \otimes 1_{\mathcal{H}_{L_{-1}}})} \pi_0(V_1) W_g^\dagger.
$$

(4.138)

Using the adjoint of equation (4.134) this gives

$$
K_g = \pi_0(V_1)^{(1_{\mathcal{H}_{L_{-1}}} \otimes \hat{k}_g \otimes 1_{\mathcal{H}_{L_{-1}}})} \pi_0(V_1)
$$

(4.139)

$$
= 1_{\mathcal{H}_{L_{-1}}} \otimes \pi_{L_{-1}}(b)^\dagger \hat{k}_g \pi_{L_{-1}}(b) \otimes 1_{\mathcal{H}_{L_{-1}}}
$$

(4.140)
where $\tilde{v} = \pi_0(V_1)^tv\pi_0(V_1)$. Now there exists a $k^L : G \to \mathcal{U}(\mathcal{H}_{L-1,L})$ and a $k^R : G \to \mathcal{U}(\mathcal{H}_{L-1,R})$ such that

$$k^L_g \otimes k^R_g = \pi_{L-1}(b)^t\tilde{k}_g\pi_{L-1}(b) \quad (4.141)$$

and these $k^g$ will satisfy what was written in remark 4.21. Define $K^g = k^g \otimes 1_{\mathcal{H}_{L-1,\sigma}} \otimes 1_{\mathcal{H}_{L-1,\sigma}}$. Notice that we have (by construction) that $\forall g, h \in G$

$$[K^g \otimes 1_{\mathcal{H}_{L-1,\sigma}}, W_h] = 0. \quad (4.142)$$

We now get (using this equation and the fact that $W_g$ is a representation)

$$K^g R^g W_g K^h R^h W_{gh} (K^R_{gh})^\dagger W_{gh} W^\dagger_{gh} W^\dagger_R (g, h) = \quad (4.143)$$

$$= W_g W^\dagger_{gh} K^g R^g (K^R_{gh})^\dagger \quad (4.144)$$

$$= k^g k^h (k^R_{gh})^\dagger \otimes 1_{\mathcal{H}_{L-1,\sigma}} \otimes 1_{\mathcal{H}_{L-1,\sigma}} \quad (4.145)$$

and by remark 4.21 this gives that $C(g, h) = \tilde{C}(g, h)$.

### 4.5 Index is invariant under locally generated automorphisms

The goal of this section is now to show that the index we constructed is invariant under locally generated automorphisms (the final proof of this statement is under theorem 4.27). To this end let $H \in \mathcal{B}_C([0, 1])$ be $G$-invariant, translation invariant (in the vertical direction) one parameter family of interactions. Define furthermore for notational simplicity in this section $H_{\text{split}} = H_L + H_R$ (with $H_L$ and $H_R$ the restrictions of $H$ as defined in 2.3). Clearly we have that $\gamma^{H_{\text{split}}}_{\mathcal{L}:s} = \gamma_{\mathcal{L}:s} H_L \otimes \gamma_{\mathcal{L}:s} H_R$ and we still have that $\gamma^{H_{\text{split}}}_{\mathcal{L}:s} \circ \tau = \tau \circ \gamma^{H_{\text{split}}}_{\mathcal{L}:s}$ (just like we have for $H$).

For the remainder of the section we will use a decomposition using the following lemma:

**Lemma 4.22.** Take $0 < \theta < \pi/2$ and $\Theta \in \text{Aut}(\mathcal{A}_{W(C_R)})$. There exist $\Phi^U_{0:1} \in \text{Aut}(\mathcal{A}_{W(C_R)\cap U})$ and $\Phi^D_{0:1} \in \text{Aut}(\mathcal{A}_{W(C_R)\cap D})$, both commuting with $\beta_g$, such that there exists some $a \in \mathcal{U}(\mathcal{A})$ satisfying that

$$\gamma^{H}_{0:1} \circ \gamma^{H_{\text{split}}}_{1:0} = \text{Ad}(a) \otimes \Phi^U_{0:1} \otimes \Phi^D_{0:1}. \quad (4.146)$$

**Proof.** This follows from the fact that $\gamma^{H}_{0:1} \circ \gamma^{H_{\text{split}}}_{1:0} \in \text{GVAut}_1(\mathcal{A})$ (see lemma D.4 part 2). \qed

We will then use this decomposition to define a morphism:

**Lemma 4.23.** Let $\tilde{a} = \pi_0 \circ \alpha_0 \circ \Theta(a)$ and define the map

$$\phi_{1,\sigma} : \text{IAA}_\sigma(\alpha_0, \theta, \Theta) \rightarrow \mathcal{U}(\mathcal{H}) \quad (4.147)$$

$$: x = \pi_0 \circ \alpha_0 \circ \Theta(A)y \mapsto \phi_1(x) = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^{H}_{0:1} \circ \gamma^{H_{\text{split}}}_{1:0}(A)\tilde{a}y\tilde{a}^\dagger$$

where $A \in \mathcal{U}(\mathcal{A}_\sigma)$ and $y \in \text{Cone}_{\sigma}(\alpha_0, \theta)$. This map satisfies that

1. for (the unique) $\xi \in \text{Aut}(\mathcal{A}_{W(C_R)\cap \sigma})$ such that

$$\text{Ad}(y) \circ \pi_0 \circ \alpha_0 = \pi_0 \circ \alpha_0 \circ \xi \quad (4.148)$$

we get that

$$\text{Ad}(\phi_{1,\sigma}(x)) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^{H}_{0:1} \circ \gamma^{H_{\text{split}}}_{1:0} = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^{H}_{0:1} \circ \gamma^{H_{\text{split}}}_{1:0} \circ \text{Ad}(A) \circ \xi. \quad (4.149)$$

2. it is well defined (independent of the choice of $A$ and $y$).
3. it is a group homomorphism.

4. \( \hat{a} \hat{\phi}_1(x) \hat{a} \in \text{IAC}_\sigma(\alpha_0, \theta, \Theta \circ \Phi^U \oplus \Phi^D). \)

5. if \( x \in \text{IAC}^W(\alpha_0, \theta, \Theta) \) then \( \hat{a} \hat{\phi}_1(x) \hat{a} \in \text{IAC}^W(\alpha_0, \theta, \Theta \circ \Phi^U \oplus \Phi^D). \)

**Proof.** In this proof we take \( \sigma = R \), the proof when \( \sigma = L \) is equivalent. Take \( x \in \text{IAC}_R(\alpha_0, \theta, \Theta) \) arbitrary. Take \( A \in U(\mathcal{A}_R) \) and \( y \in \text{Cone}_R(\alpha_0, \theta, \Theta) \) such that

\[
x = \pi_0 \circ \alpha_0 \circ \Theta(A)y
\]

(4.150)

and take \( \xi \in \text{Aut}(\mathcal{A}_C \cap \mathcal{A}_R) \) the automorphism such that

\[
\text{Ad}(y) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \xi.
\]

(4.151)

Take \( a \) and \( \Phi = \Phi^U \otimes \Phi^D \) as given in equation (4.146). We now define

\[
\phi_{1,R}(x) := \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} \circ \theta^\dagger (A) \hat{a}y\hat{a}^\dagger.
\]

(4.152)

We now have to prove three things:

1. It satisfies equation (4.149).
2. This map is well defined (independent of our choices).
3. This is a group homomorphism.

To show the first item just observe that

\[
\text{Ad}(\phi_{1,R}(x)) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} = \text{Id}
\]

(4.153)

\[
= \text{Ad}\left( \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} (A) \pi_0 \circ \alpha_0 \circ \Theta(a) y \pi_0 \circ \alpha_0 \circ \Theta(a) \right) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0}
\]

(4.154)

\[
= \text{Ad}\left( \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} (A) \right) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \text{Ad}(a) \circ \xi \circ \text{Ad}(a^\dagger) \circ \gamma^H_{0,1} \circ \gamma^H_{1,0}.
\]

(4.155)

Now we insert \( \gamma^H_{0,1} \circ \gamma^H_{1,0} \circ \gamma^H_{0,1} = \text{Id} \) and obtain:

\[
= \text{Ad}\left( \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} (A) \right) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} \circ \gamma^H_{0,1} \circ \gamma^H_{1,0}
\]

(4.156)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} \circ \text{Ad}(A) \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} \circ \text{Ad}(a) \circ \xi \circ \text{Ad}(a^\dagger) \circ \gamma^H_{0,1} \circ \gamma^H_{1,0}.
\]

(4.157)

Using equation (4.146) this gives

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} \circ \text{Ad}(A) \circ \Phi^{-1} \circ \xi \circ \Phi
\]

(4.158)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_{0,1} \circ \gamma^H_{1,0} \circ \text{Ad}(A) \circ \xi
\]

(4.159)

concluding the proof of the first item. To show the second item notice that since \( \pi_0 \circ \alpha_0 \circ \Theta \) is irreducible the solution (if it exists which we just showed) of equation (4.149) is unique up to a phase. This means that to show that our map is independent of the choices we only need to show that we can’t obtain another phase by picking a different representative. To show this, suppose we have two different representatives of \( x \), say

\[
x = \pi_0 \circ \alpha_0 \circ \Theta(A_1)y_1 = \pi_0 \circ \alpha_0 \circ \Theta(A_2)y_2.
\]

(4.160)
By construction, this implies that
\[ \pi_0 \circ \alpha_0 \circ \Theta(A_2^\dagger A_1) = y_2 y_1^\dagger \]  
(4.161)
and hence we require that \( A_2^\dagger A_1 \in \mathcal{U}(\mathcal{A}_W^{(C_\Theta)}) = \mathcal{U}(\mathcal{A}_{W(C_\Theta)}) \) (this last equality follows from section 3 of [2]). We know that there must exist some \( \alpha \in U(1) \) such that
\[
\phi_1(\pi_0 \circ \alpha_0 \circ \Theta(A_1)y_1)\phi_1(\pi_0 \circ \alpha_0 \circ \Theta(A_2)y_2)^\dagger = \alpha. 
\]
(4.162)
We will show that the expression on the lefthandside is self adjoint. Clearly, this implies that \( \alpha = 1 \). First notice that
\[
\phi_1(\pi_0 \circ \alpha_0 \circ \Theta(A_1)y_1)^\dagger \phi_1(\pi_0 \circ \alpha_0 \circ \Theta(A_2)y_2) = (\phi_1(\pi_0 \circ \alpha_0 \circ \Theta(A_1)y_1)^\dagger \phi_1(\pi_0 \circ \alpha_0 \circ \Theta(A_2)y_2))^\dagger 
\]
(4.163)
is true if and only if
\[
\tilde{a} y_2 y_1^\dagger \tilde{a} \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} (A_1^\dagger A_2) \tilde{a} y_2 y_1^\dagger \tilde{a}^\dagger = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} (A_2^\dagger A_1). 
\]
(4.164)
We can show that this last equation is true by using the fact that \( A_2^\dagger A_1 \in \mathcal{U}(\mathcal{A}_W^{(C_\Theta)}) \) combined with equation (4.161) gives us that
\[
\text{Ad}(\tilde{a})(y_2 y_1^\dagger) = \text{Ad}(\tilde{a}) \circ \pi_0 \circ \alpha_0 \circ \Theta(A_2^\dagger A_1) 
\]
(4.165)
\[= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} \circ \Phi^{-1}(A_2^\dagger A_1) \]
(4.166)
\[= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} (A_2^\dagger A_1). \]
(4.167)
Inserting this in equation (4.164) concludes the proof that our map is independent of the choice of representatives. To then show the last item take \( x_1, x_2 \in \text{IAC}_{R}(\alpha_0, \Theta) \) arbitrary. Take \( A_1, A_2 \in \mathcal{U}(\mathcal{A}_R), y_1, y_2 \in \text{Cone}_{R}(\alpha_0, \Theta) \) and \( \xi_1, \xi_2 \in \text{Aut}(\mathcal{A}_W^{(C_\Theta)} \cap \mathcal{A}_R) \) such that
\[
x_i = \pi_0 \circ \alpha_0 \circ \Theta(A_i)y_i \quad \text{Ad}(y_i) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \xi_i. 
\]
(4.168)
We have
\[
x_1 x_2 = \pi_0 \circ \alpha_0 \circ \Theta(A_1)y_1 \pi_0 \circ \alpha_0 \circ \Theta(A_2)y_2 
\]
(4.169)
\[= \pi_0 \circ \alpha_0 \circ \Theta(A_1)y_1 \pi_0 \circ \alpha_0 \circ \Theta(A_2)y_1^\dagger y_2 \]
(4.170)
\[= \pi_0 \circ \alpha_0 \circ \Theta(A_1 \xi_1(A_2))y_1 y_2 
\]
(4.171)
giving
\[
\phi_{1,R}(x_1 x_2) = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} (A_1 \xi_1(A_2)) \tilde{a} y_1 y_2 \tilde{a}^\dagger. 
\]
(4.172)
where \( \tilde{a} = \pi_0 \circ \alpha_0 \circ \Theta(a) \). Filling this in gives
\[
\phi_{1,R}(x_1 x_2)^\dagger \phi_{1,R}(x_1) = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} (A_1 \xi_1(A_2)) \tilde{a} y_1 y_2 \tilde{a}^\dagger \tilde{a} y_2 y_1^\dagger \tilde{a}^\dagger \]
(4.173)
\[
\pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} (A_2) \tilde{a} y_1 y_2 \tilde{a}^\dagger \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} (A_2^\dagger). 
\]
(4.174)
Writing out a part of this gives:
\[
\text{Ad}(\tilde{a} y_1 y_2 \tilde{a}^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} (A_2) 
\]
(4.175)
\[= \pi_0 \circ \alpha_0 \circ \Theta \circ \text{Ad}(a) \circ \xi_1 \circ \text{Ad}(a^\dagger) \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} (A_2). 
\]
(4.176)
Using equation (4.146) now yields:

\[ \begin{align*}
\pi_0 \circ \alpha_0 \circ \Theta \circ \text{Ad}(a) \circ \xi_1 \circ \Phi(A^1_2) \\
\pi_0 \circ \alpha_0 \circ \Theta \circ \text{Ad}(a) \circ \Phi \circ \xi_1(A^2_1) \\
\pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H \circ \gamma^H_{\text{split}} \circ \xi_1(A^2_2).
\end{align*} \] (4.177)

This shows that

\[ \text{Ad}(\tilde{a}y_i\tilde{a}^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_0 \circ \gamma^H_{\text{split}}(A^2_2) = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_0 \circ \gamma^H_{\text{split}} \circ \xi_1(A^2_2). \] (4.180)

Inserting this in equation (4.174) shows that

\[ \phi_{1,R}(x_1x_2)\phi_{1,R}(x_1)\phi_{1,R}(x_2) = 1 \] (4.181)

concluding the proof.

We now have to extend the above definition to the group

\[ \text{IAC}_{L \times R}(\alpha_0, \theta, \Theta) = \{ ab = ba | a \in \text{IAC}_L(\alpha_0, \theta, \Theta), b \in \text{IAC}_R(\alpha_0, \theta, \Theta) \}. \] (4.182)

**Lemma 4.24.** Define

\[ \phi_1 : \text{IAC}_{L \times R}(\alpha_0, \theta, \Theta) \to \mathcal{U}(\mathcal{H}) : ab \mapsto \phi_{1,L}(a)\phi_{1,R}(b) \] (4.183)

where \( a \in \text{IAC}_L(\alpha_0, \theta, \Theta) \) and \( b \in \text{IAC}_R(\alpha_0, \theta, \Theta) \). This is well defined, by which we mean that

\[ \phi_{1,L}(a)\phi_{1,R}(b) = \phi_{1,R}(b)\phi_{1,L}(a). \] (4.184)

This is still a morphism.

**Proof.** First notice that using the previous lemma (equation (4.149) in particular) one can show that indeed

\[ \text{Ad}(\phi_{1,L}(a)\phi_{1,R}(b)) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_0 \circ \gamma^H_{\text{split}} = \text{Ad}(\phi_{1,R}(b)\phi_{1,L}(a)) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_0 \circ \gamma^H_{\text{split}}. \] (4.185)

This already shows that they commute up to a phase. The proof that this phase is 1, is analogous to the morphism part of the proof of the above lemma. The last part, that this new map is a morphism as well now follows from the above lemma.

**Lemma 4.25.** Define again \( \tilde{a} \) through equation (4.146). Let

\[ \phi_2 : \{ \tau^{-p}W_gv^p | g \in G, p \in \{-1, 0, 1\} \} \to \mathcal{U}(\mathcal{H}) : \tau^{-p}W_gv^p \mapsto \tilde{a}\tau^{-p}W_gv^p\tilde{a}^\dagger. \] (4.186)

This satisfies

\[ \text{Ad}(\phi_2(\tau^{-p}W_gv^p)) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_0 \circ \gamma^H_{\text{split}} \circ \tau^{-p} \circ \eta_g \circ \beta_g^U \circ \Phi_{\text{T}}^{-1} \circ \eta_g \circ \beta_g^U \circ \Phi_{\text{T}}^{-1} \circ \gamma^H_0 \circ \Theta^{-1} \circ \alpha_0^{-1}. \] (4.187)

**Proof.** To show that this indeed satisfies equation (4.187) observe that using equation (4.146) we obtain:

\[ \text{Ad}(\phi_2(W_g)) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \Theta \circ \text{Ad}(a) \circ \eta_g \circ \beta_g^U \circ \text{Ad}(a^\dagger) \circ \Theta^{-1} \circ \alpha_0^{-1} \] (4.188)

\[ = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^H_0 \circ \gamma^H_{\text{T}} \circ \Phi_{\text{T}}^{-1} \circ \eta_g \circ \beta_g^U \circ \Phi_{\text{T}}^{-1} \circ \gamma^H_0 \circ \Theta^{-1} \circ \alpha_0^{-1}. \] (4.189)

Since \( \Phi \) commutes with both the \( \eta_g \) and with the \( \beta_g^U \) the result follows.

\[ \square \]
Lemma 4.26. The \( \phi_1 \) and \( \phi_2 \) defined previously satisfy that for any \( x \in \text{IAC}_R(\alpha_0, \theta, \Theta) \) we have that
\[
\phi_1(\text{Ad}(W_g)(x)) = \text{Ad}(\phi_2(W_g))(\phi_1(x)).
\] (4.190)
If additionally \( x \in \text{IAC}_R^W(\alpha_0, \theta, \Theta) \) we get that
\[
\phi_1(\text{Ad}(v)(x)) = \text{Ad}(v)(\phi_1(x)) \quad \quad \phi_1(\text{Ad}(v^\dagger)(x)) = \text{Ad}(v^\dagger)(\phi_1(x)).
\] (4.191)

Proof. Take \( x \in \text{IAC}_R(\alpha_0, \theta, \Theta) \) arbitrary. Take \( A \in \mathcal{U}(\mathcal{A}_R) \) and \( y \in \text{Cone}_R(\alpha_0, \theta) \) such that
\[
x = \pi_0 \circ \alpha_0 \circ \Theta(A)y
\] (4.192)
and \( \xi \in \text{Aut}(\mathcal{A}_{C_0} \cap \mathcal{A}_R) \) the automorphism such that
\[
\text{Ad}(y) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \xi.
\] (4.193)

Take \( a \) and \( \Phi \) as given in equation (4.146) and take \( \tilde{a} = \pi_0 \circ \alpha_0 \circ \Theta(a) \). We now get
\[
W_gxW_g^\dagger = \text{Ad}(W_g) \circ \pi_0 \circ \alpha_0 \circ \Theta(A)W_gyW_g^\dagger = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \circ \beta_g^U(A)W_gyW_g^\dagger.
\] (4.194)
(4.195)

Since \( W_gyW_g^\dagger \) satisfies that there exists an \( \tilde{\xi} \in \text{Aut}(\mathcal{A}_{C_0} \cap \mathcal{A}_R) \) such that
\[
\text{Ad}(W_gyW_g^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \tilde{\xi}
\] (4.196)
(one can just take \( \tilde{\xi} = \eta_g \circ \beta_g^U \circ \xi \circ \beta_g^U \circ \eta_g^{-1} \)) we get that
\[
\phi_1(W_gxW_g^\dagger) = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} \circ \eta_g \circ \beta_g^U(A)\tilde{a}W_gyW_g^\dagger\tilde{a}^\dagger.
\] (4.197)

Filling this in gives
\[
\phi_1(W_gxW_g^\dagger)\phi_2(W_g)\phi_1(x)\phi_2(W_g)^\dagger = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} \circ \eta_g \circ \beta_g^U(A)\tilde{a}W_gyW_g^\dagger\tilde{a}^\dagger.
\] (4.198)
(4.199)

To prove that this is the identity operator observe that
\[
\text{Ad}(\tilde{a}W_g\tilde{a}^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}}(A)
\] (4.200)
\[
= \text{Ad}(\tilde{a}W_g) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \text{Ad}(a^\dagger) \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}}(A)
\] (4.201)
\[
= \text{Ad}(\tilde{a}W_g) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \Phi(A)
\] (4.202)
\[
= \text{Ad}(\tilde{a}) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \circ \beta_g^U \circ \Phi(A)
\] (4.203)
\[
= \text{Ad}(\tilde{a}) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \Phi \circ \eta_g \circ \beta_g^U(A)
\] (4.204)
\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \text{Ad}(a) \circ \Phi \circ \eta_g \circ \beta_g^U(A)
\] (4.205)
\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} \circ \eta_g \circ \beta_g^U(A)
\] (4.206)

Filling this in equation (4.199) concludes the proof of the first item. The proof of the second item looks very similar. We now get that (the part with \( v \) replaced with \( v^\dagger \) and \( \tau \) by \( \tau^{-1} \) is equivalent)
\[
vxv^\dagger = \pi \circ \alpha_0 \circ \Theta \circ \tau(A)vv^\dagger.
\] (4.207)

When we now take \( \tilde{\xi} = \tau \circ \xi \circ \tau^{-1} \) we still obtain an equation of the form
\[
\text{Ad}(v^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \tilde{\xi}.
\] (4.208)
This again (in analogy to equation (4.197)) implies that we can take
\[ \phi_1(v x \tilde{a}^\dagger) = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} \circ \tau(A) \tilde{a} v y v^\dagger \tilde{a}^\dagger. \] (4.209)

Filling this in now gives
\[ \phi_1(v x v^\dagger) v \phi_1(x)^\dagger v^\dagger \]
\[ = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} \circ \tau(A) \tilde{a} v y v^\dagger \tilde{a}^\dagger \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} (A)^\dagger v^\dagger. \] (4.210)

Before we proceed we will first show the equation
\[ \text{Ad}(\tilde{a} v y v^\dagger \tilde{a}^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \tau. \] (4.212)

To prove this, observe that
\[ \text{Ad}(\tilde{a} v y v^\dagger \tilde{a}^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta = \text{Ad}(\tilde{a}^\dagger) \circ \xi \circ \text{Ad}(a). \] (4.213)

Now inserting equation (4.146) now yields:
\[ = \text{Ad}(\tilde{a} v y v^\dagger \tilde{a}^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} \circ \tau \circ \xi \circ \tau^{-1} \circ \tau \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} \circ \tau \circ \xi \circ \tau^{-1} \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} \circ \tau. \] (4.214)

Using again equation (4.146) now yields:
\[ = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} \circ \tau \circ \xi \circ \gamma_{0;1}^H \circ \tau \circ \xi \circ \tau^{-1} \circ \tau \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} \circ \tau. \] (4.215)

concluding the proof of equation (4.212). We will now show that the expression in equation (4.211) is the identity. First we will show that the A’s in equation (4.211) cancel. By equation (4.212) we get that
\[ \text{Ad}(\tilde{a} v y v^\dagger \tilde{a}^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} (A)^\dagger = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{split} \circ \tau(A)^\dagger. \] (4.220)

Filling this in shows that the A’s cancel. Now, what is left to show is that
\[ \tilde{a} v y v^\dagger \tilde{a}^\dagger v^\dagger = 1. \] (4.221)

We will show this in two steps, first notice that because of equation (4.212) we get that
\[ \text{Ad}(\tilde{a} v y v^\dagger \tilde{a}^\dagger v^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta. \] (4.222)

This already implies that there exists an \( \alpha \in U(1) \) such that
\[ \tilde{a} v y v^\dagger \tilde{a}^\dagger v^\dagger = \alpha. \] (4.223)

To show that \( \alpha = 1 \) we merely have to show that
\[ \tilde{a} v y v^\dagger \tilde{a}^\dagger v^\dagger = (\tilde{a} v y v^\dagger \tilde{a}^\dagger v^\dagger)^\dagger. \] (4.224)

This is equivalent to showing that
\[ \text{Ad}(v^\dagger \tilde{a}^\dagger v \tilde{a}) (\text{Ad}(y)(v^\dagger \tilde{a}^\dagger v \tilde{a})) = \text{Ad}(y)(v^\dagger \tilde{a}^\dagger v \tilde{a}). \] (4.225)
To show this we will fill in $v^\dagger a^\dagger v\hat{a} = \pi_0 \circ \alpha_0 \circ \Theta(\tau^{-1}(a^\dagger)a)$ in this which gives

$$\text{Ad}(v^\dagger a^\dagger v\hat{a}) \circ \pi_0 \circ \alpha_0 \circ \Theta(\tau^{-1}(a^\dagger)a)$$

(4.226)

$$= \pi_0 \circ \alpha_0 \circ \text{Ad}(\tau^{-1}(a^\dagger)a) \circ \xi(\tau^{-1}(a^\dagger)a).$$

(4.227)

However, since

$$\text{Ad}(\tau^{-1}(a^\dagger)a) = \tau^{-1} \circ \Phi \circ \gamma^H_{0;1} \circ \gamma^H_{1;0} \circ \tau \circ \gamma^H_{0;1} \circ \gamma^H_{1;0} \circ \Phi^{-1} = \tau^{-1} \circ \Phi \circ \tau \circ \Phi^{-1}$$

(4.228)

we get that $\text{Ad}(\tau^{-1}(a^\dagger)a)$ commutes with $\xi$. After filling this in equation (4.227) we get the desired result. This concludes the proof.

\[\square\]

**Theorem 4.27.** Index$(\omega) = $ Index$(\omega \circ \gamma^H_{0;1})$

**Proof.** Take $\alpha \in \text{QAut}_1(A)$ such that $\omega = \omega_0 \circ \alpha$ for some product state $\omega_0$. Take $0 < \theta_1 < \theta_2 < \theta_3 < \pi/2$ arbitrary. Take $\alpha_L \in \text{Aut}(A_L)$, $\alpha_R \in \text{Aut}(A_R)$, $V_1 \in \mathcal{U}(A)$ and $\Theta \in \text{Aut}(\mathcal{A}_{W(C_{\phi})^\tau})$ such that

$$\alpha = \text{Ad}(V_1) \circ \alpha_L \circ \alpha_R \circ \Theta.$$  

(4.229)

Take $\tilde{\beta}_{g,1}$ such that $\omega \circ \tilde{\beta}_{g,1} = \omega$ and such that there exists some $V_{2,g} \in \mathcal{U}(A)$ and some $\eta^L_R \in \text{Aut}(\mathcal{A}_{C_{\phi}} \cap \mathcal{A}_{U/R})$ satisfying

$$\tilde{\beta}_{g,1} = \text{Ad}(V_{2,g}) \circ \eta^L_R \circ \eta^R_R \circ \beta^U_g.$$  

(4.230)

Take $W_g, u_{L/R}(g, h), v$ and $K^L_R$ operators belonging to these automorphisms (see lemmas 4.5 and 4.7). Clearly $\alpha \circ \gamma^H_{0;1}$ satisfies $\omega \circ \gamma^H_{0;1} = \omega_0 \circ \alpha \circ \gamma^H_{0;1}$. We also have that $\alpha \circ \gamma^H_{0;1} \in \text{QAut}_1(A)$. To show this notice that

$$\omega \circ \gamma^H_{0;1} = \text{Ad}(V_1) \circ (\alpha_L \circ \gamma^H_{0;1} \circ \alpha_R) \circ \Theta$$

(4.231)

$$= \text{Ad}(V_1) \circ (\alpha_L \circ \gamma^H_{0;1} \circ \alpha_R \circ \Theta \circ \gamma^H_{0;1}).$$

(4.232)

Because of lemma D.5 and lemma D.4 part 2 there exists a $\tilde{\Theta} \in \text{Aut}(\mathcal{A}_{W(C_{\phi})^\tau})$ and some $\tilde{V}_1, A_1 \in \mathcal{U}(A)$ such that

$$\alpha \circ \gamma^H_{0;1} = \text{Ad}(\tilde{V}_1) \circ (\alpha_L \circ \gamma^H_{0;1} \circ \alpha_R) \circ \tilde{\Theta}$$

(4.233)

$$= \text{Ad}(\tilde{V}_1) \circ (\alpha_L \circ \gamma^H_{0;1} \circ \alpha_R \circ \Theta \circ \gamma^H_{0;1}).$$

(4.234)

If we now define the automorphism $\tilde{\beta}_{g,2} = \gamma^H_{1;0} \beta_{g,1} \gamma^H_{0;1}$ then this indeed satisfies that $\omega \circ \gamma^H_{0;1} \circ \tilde{\beta}_{g,2} = \omega \circ \gamma^H_{0;1}$. Define $\tilde{\eta}^\sigma \in \text{Aut}(\mathcal{A}_{C_{\phi} \cap \sigma})$ through lemma D.7. It satisfies that there exists a $\tilde{V}_{2,g} \in \mathcal{U}(A), A_{2,g} \in \mathcal{U}(\mathcal{A}_{\sigma})$ such that

$$\gamma^H_{1;0} \beta_{g,2} \gamma^H_{0;1} = \text{Ad}(\tilde{V}_{2,g}) \circ \tilde{\eta}^\sigma \circ \beta^U_g$$

(4.235)

$$\tilde{\eta}^\sigma = \text{Ad}(A_{2,g}) \circ \gamma^H_{1;0} \circ \gamma^H_{0;1} \circ \beta^U_g \circ (\beta^U_g)^{-1}.$$  

(4.236)

Take $\Phi$ like in equation (4.146) and define

$$\phi_1 : \text{IAC}_R(\alpha_0, \Theta) \to \mathcal{U}(\mathcal{H})$$

(4.237)

the group homomorphism defined from lemma 4.23. Take similarly

$$\phi_2 : \{W_g | g \in G\} \to \mathcal{U}(\mathcal{H})$$

(4.238)
the map defined in (4.25) with arbitrary phase. The following operators now belong to $\omega \circ \gamma^{H}_{0;1}$:

\[
\begin{align*}
\tilde{v} &= \pi_0(A_1)u_{\pi_0(A_1)}^\dagger \\
\tilde{W}_g &= \pi_0(A_1)\phi_1(\delta^L_g \otimes \delta^R_g)\phi_2(W_g)\pi_0(A_1^\dagger) \\
\tilde{u}_\sigma(g, h) &= \pi_0(A_1)\phi_1(\delta^L_g W_{\tilde{\omega}_h} W_{\tilde{g}}^\dagger u_{\sigma}(g, h)(\delta^L_{gh})^\dagger) \pi_0(A_1^\dagger) \\
\tilde{K}_g^\sigma &= \pi_0(A_1)\phi_1(v^\dagger \delta^L_g v K_g^\sigma(\delta^L_g)^\dagger) \pi_0(A_1^\dagger)
\end{align*}
\]

where $\delta^L_g = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^{H}_{0;1}(A^L_{2,g})$ and $A_1$ was defined in equation (4.234). The fact that the index is invariant under $\text{Ad}(\pi_0(A_1))$ and under $\phi(\cdot)$ is just due to the fact that both transformations are homomorphisms. To show that it is invariant under the transformation with de $\delta^L_g$ we invoke lemma 4.12. All that is left to show is that operators (4.239) through (4.242) are indeed operators belonging to $\omega \circ \gamma^{H}_{0;1}$. To do this we first state that a small calculation shows that

\[
\text{Ad}(x) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \xi
\]

we get that

\[
\text{Ad}(\pi_0(A_1)x\pi_0(A_1^\dagger)) \circ \pi_0 \circ \alpha_0 \circ \gamma^{H}_{0;1} \circ \Theta = \pi_0 \circ \alpha_0 \circ \gamma^{H}_{0;1} \circ \tilde{\Theta} \circ \gamma^{H}_{0;1} \circ \xi \circ \gamma^{H}_{0;1}
\]

This already proves that

\[
\text{Ad}(\pi_0(A_1)v\pi_0(A_1^\dagger)) \circ \pi_0 \circ \alpha_0 \circ \gamma^{H}_{0;1} \circ \tilde{\Theta} = \pi_0 \circ \alpha_0 \circ \gamma^{H}_{0;1} \circ \tilde{\Theta} \circ \gamma^{H}_{0;1} \circ \tau \circ \gamma^{H}_{0;1}
\]

To show the second one we get that

\[
\begin{align*}
\text{Ad}(\phi_1(\delta^L_g)\phi_2(W_g)) \circ \pi_0 \circ \alpha_0 \circ \Theta &= \text{Ad}(\phi_1(\delta^L_g)) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^{H}_{0;1} \circ \gamma^{H}_{1;0} \circ \eta_g \circ \beta_g \circ \gamma^{H}_{0;1} \circ \gamma^{H}_{1;0} \\
&= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^{H}_{0;1} \circ \gamma^{H}_{1;0} \circ \text{Ad}(\gamma^{H}_{0;1}(A^L_{2,g}) \otimes \gamma^{H}_{0;1}(A^R_{2,g})) \circ \eta_g \circ \beta_g \circ \gamma^{H}_{0;1} \circ \gamma^{H}_{1;0} \\
&= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^{H}_{0;1} \circ \text{Ad}(A^L_{2,g} \otimes A^R_{2,g}) \circ \gamma^{H}_{1;0} \circ \eta_g \circ \beta_g \circ \gamma^{H}_{0;1} \circ \gamma^{H}_{1;0}
\end{align*}
\]

Using equation (4.236) on this we obtain

\[
\begin{align*}
&= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma^{H}_{1;0} \circ \tilde{\eta}_g \circ \tilde{\beta}_g \circ \gamma^{H}_{1;0}
\end{align*}
\]

Inserting this in equation (4.244) with $x = \phi_1(\delta^L_g)\phi_2(W_g)$ and $\xi = \gamma^{H}_{0;1} \circ \tilde{\eta}_g \circ \tilde{\beta}_g \circ \gamma^{H}_{1;0}$ now proves the fact that $\tilde{W}_g$ is an operator belonging to $\omega \circ \gamma^{H}_{0;1}$. Proving that $\tilde{u}_\sigma(g, h)$ and $\tilde{K}_g^\sigma$ are operators belonging to $\omega \circ \gamma^{H}_{0;1}$ is completely analogous. This concludes the proof.

### 4.6 Stacking in the $H^2(G, \mathbb{T})$-valued index

Take $A$ a quasi local $C^*$ algebra on a two dimensional lattice and take $U \in \text{hom}(G, U(\mathbb{C}^d))$ an on site group action arbitrary. Let

\[
\omega_1, \omega_2 \in \{\omega \in \mathcal{P}(A) | \omega \text{ satisfies assumption } 3.6\}
\]

be two states. The goal of this section is to prove that

\[
\text{Index}^{A^2, U \otimes U}(\omega_1 \otimes \text{stack} \omega_2) = \text{Index}^{A^2, U}(\omega_1) + \text{Index}^{A^2, U}(\omega_2)
\]
Let $\omega_0$ and $\gamma_{0;1}$ (for all $i \in \{1, 2\}$) be product states and disentanglers satisfying that

$$\omega_i = \omega_0 \circ \gamma_{0;1}.$$  \hspace{1cm} (4.254)

Let

$$(\mathcal{H}_0, \pi_0, \Omega_0)$$ \hspace{1cm} (4.255)

be GNS triples for the $\omega_0$. Clearly

$$(\mathcal{H}_{01} \otimes \mathcal{H}_{02}, \pi_{01} \otimes \pi_{02}, \Omega_{01} \otimes \Omega_{02})$$ \hspace{1cm} (4.256)

is a GNS triple for $\omega_{01} \otimes_{\text{stack}} \omega_{02}$. The following lemma now holds:

**Lemma 4.28.** Let $W_{g,i}, u^R_1(g, h), K^R_{g,1}$ and $v_1$ be operators belonging to $\omega_i$ then $W_{g,1} \otimes W_{g,2}, u^R_1(g, h) \otimes u^R_2(g, h), K^R_{g,1} \otimes K^R_{g,2}$ and $v_1 \otimes v_2$ are operators belonging to $\omega_1 \otimes_{\text{stack}} \omega_2$.

**Proof.** To show this we simply have to apply the adjoint action of these operators on $\pi_{01} \otimes \pi_{02}$ and see that these are indeed implementations of the right automorphisms. This is so by construction. \qed

We are now ready to provide a proof of equation 4.253. We have

$$C(g, h) = (K^R_{g,1} \otimes K^R_{g,2}) \text{Ad}(W_{g,1} \otimes W_{g,2})((K^R_{g,1} \otimes K^R_{g,2}) \text{Ad}((W_{g,1} \otimes W_{g,2})(W_{gh,1} \otimes W_{gh,2}^\dagger))((K^R_{g,1} \otimes K^R_{g,2}^\dagger)$$ \hspace{1cm} (4.257)

$$(u^R_1(g, h) \otimes u^R_2(g, h))v_1^1(u^R_1(g, h) \otimes u^R_2(g, h))^\dagger v$$ \hspace{1cm} (4.258)

$$= C_1(g, h)C_2(g, h)$$ \hspace{1cm} (4.259)

concluding the proof.

**Remark 4.29.** Clearly the same argument can be used to argue that the $H^3$-valued index is invariant under stacking.

### 4.7 The $H^2$-valued index is invariant of choice of basis of on site Hilbert space

Let $V \in U(\mathbb{C}^d)$ be a unitary operator on the on site Hilbert space and take $i_V$ as defined in subsection 2.1. In this subsection we will prove that

\[
\text{Index}^A, \text{Ad}(V^\dagger)U(\theta, i_V^{-1} \circ \tilde{\beta}_g \circ i_V, i_V^{-1} \circ \eta_g \circ i_V, i_V^{-1} \circ \alpha_0 \circ i_V, i_V^{-1} \circ \Theta \circ i_V, \omega \circ i_V, \omega_0 \circ i_V) = \text{Index}^A, U(\theta, \tilde{\beta}_g, \eta_g, \alpha_0, \Theta, \omega, \omega_0).
\]  \hspace{1cm} (4.260)

We do this by proving the following lemma:

**Lemma 4.30.** Let $W_g, u_R(g, h), K^R_g$ be operators belonging to

$$\text{Index}^A, U(\theta, \tilde{\beta}_g, \eta_g, \alpha_0, \Theta, \omega, \omega_0)$$ \hspace{1cm} (4.262)

for a GNS triple for $\omega_0$, $(\mathcal{H}_0, \pi_0, \Omega_0)$ then $W_g, u_R(g, h), K^R_g$ are also operators belonging to

$$\text{Index}^A, \text{Ad}(V^\dagger)U(\theta, i_V^{-1} \circ \tilde{\beta}_g \circ i_V, i_V^{-1} \circ \eta_g \circ i_V, i_V^{-1} \circ \alpha_0 \circ i_V, i_V^{-1} \circ \Theta \circ i_V, \omega \circ i_V, \omega_0 \circ i_V)$$ \hspace{1cm} (4.263)

for the GNS triple for $\omega_0 \circ i_V$ given by $(\mathcal{H}_0, \pi_0 \circ i_V, \Omega_0)$.
Proof. To show this, we merely have to observe that for any \( x \in U(\mathcal{H}_0) \) satisfying that
\[
\text{Ad}(x) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \xi
\] (4.264)
for some \( \xi \in \text{Aut}(\mathcal{A}) \) we get that
\[
\text{Ad}(x) \circ \pi_0 \circ i_V \circ i_V^{-1} \circ \alpha_0 \circ i_V \circ i_V^{-1} \circ \Theta \circ i_V \circ i_V^{-1} \circ \xi \circ i_V
\] (4.265)
concluding the proof. \( \square \)

Clearly the above lemma implies that the \( H^2 \)-valued index is invariant under the desired transformation.

Remark 4.31. Clearly the same argument can be used to argue that the \( H^3 \)-valued index is invariant under the same transformation.

4.8 Proof of theorem 3.3
We will first prove lemma 3.7 which we state here for convenience:

Lemma 3.7. Take \( \omega \in \mathcal{P}(\mathcal{A}) \) a short range entangled state that is \( G \)-invariant and translation invariant then it satisfies assumption 3.6.

Proof. Let \( \gamma_{\Phi}^{\Phi'} \) be the disentangler with \( \Phi' \in B_{F_\phi}([0,1]) \) (for some \( 0 < \phi < 1 \)) a one parameter family of interactions. By lemma 3.4 part 5 (which is just a slight modification of section 5 from [10]) we have that \( \gamma_{\Phi}^{\Phi'} \in \text{SQAut}_1(\mathcal{A}) \) (by which we mean the definition of SQAut as presented in appendix D.1, not the one in [10]) since \( \text{SQAut}_1(\mathcal{A}) \subset \text{QAut}_1(\mathcal{A}) \) this implies that \( \gamma_{\Phi}^{\Phi'} \in \text{QAut}_1(\mathcal{A}) \). The existence of the \( \tilde{\beta}_g \) was done in [10] section 3, where one has to use the fact that \( \gamma_{0:1}^{\Phi'} \) is also an SQAut in the definition by [10]. This concludes the proof that assumption 3.6 is satisfied. \( \square \)

Now we are ready to provide the proof

Proof of theorem 3.3. Take
\[
\omega \in \{ \omega \in \mathcal{P}(\mathcal{A}) | \omega \text{ is an SRE state, } \omega \circ \beta_g = \omega \text{ and } \omega \circ \tau = \omega \} \quad (4.266)
\]
arbitrary. By lemma 3.7 \( \omega \) satisfies assumption 3.6. Using theorem 3.8 we get that there indeed exists a well defined index and that it is invariant under locally generated automorphisms and satisfies item 3. By lemma 4.17 we get that the \( \phi \) presented in part 2 of the theorem satisfies assumption 4.16. We can now use the result of section 4.4 to conclude the proof of item 2. This concludes the proof of the theorem. \( \square \)

5 The \( H^1(G, \mathbb{T}) \)-valued index

In this section we discuss translation invariance not just in the vertical direction but in two directions simultaneously. Let \( \nu \in \text{Aut}(\mathcal{A}) \) be the automorphism that translates the lattice to the right by one site. Let \( \mu \in \text{Aut}(\mathcal{A}) \) be the automorphism that rotates the lattice by 90° clockwise. There is a relation between these automorphisms. Specifically, the automorphism \( \tau \) that translated the lattice upwards by one site can now be written as \( \tau = \mu^{-1} \circ \nu \circ \mu \). We now let our state \( \omega \) satisfy assumption 3.9. We will restate this assumption here for convenience

Assumption 3.9. Let \( \omega \in \mathcal{P}(\mathcal{A}) \) be
Figure 3: This figure indicates the support area of the different automorphisms when there are two translation directions. The angle $\theta$ still has to be smaller then or equal to what was indicated here so that the $\Theta$ and the $\eta_g$ commute but now we must be able to do both a horizontal and a vertical widening of the support of $\eta_g$. Also the coordinate $(-1, -1)$ which will be of particular interest is indicated.

1. such that there exists an automorphism $\alpha \in \Aut_1(A)$ and a product state $\omega_0 \in \mathcal{P}(A)$ satisfying

$$\omega = \omega_0 \circ \alpha.$$

(3.23)

2. such that there exists a $\theta \in ]0, \pi/2[$ for which there exists a map

$$\tilde{\beta} : G \to \Aut(A) : g \mapsto \tilde{\beta}_g$$

(3.24)

satisfying

$$\omega \circ \tilde{\beta}_g = \omega \quad \tilde{\beta}_g = \Ad(V_{g,2}) \circ \eta^L_g \otimes \eta^R_g \circ \beta^U_g$$

(3.25)

for some $V_{g,2} \in \mathcal{U}(A)$, $\eta^L_g \in \Aut(A_{\nu^{-1}(C_\theta \cap L)})$ and $\eta^R_g \in \Aut(A_{\nu(C_\theta \cap R)})$.

3. translation invariant in both directions

$$\omega \circ \tau = \omega \quad \omega \circ \nu = \omega.$$  

(3.26)

For an overview of the supports see figure 3. In this section we will need a stronger version of the cone operators (from definition 1.2):

**Definition 5.1.** Take $0 < \theta < \pi/2$ and $\alpha_L \in \Aut(A_L)$ and $\alpha_R \in \Aut(A_R)$. Take $\sigma \in \{L, R\}$ and $x \in \mathcal{U}(\mathcal{H}_\sigma) \otimes I_{\mathbb{Z}^2/\sigma}$ then we say that $x$ is a cone operator on $\nu^\sigma(\sigma)^{19}$ (or in short $x \in \text{Cone}_{\nu^\sigma(\sigma)}(\alpha_0, \theta)$) if and only if there exists a $\xi \in \Aut(A_{W(\nu^\sigma(C_\theta \cap \sigma))})$ such that

$$\Ad(x) \circ \pi_0 \circ \alpha_0 = \pi_0 \circ \alpha_0 \circ \xi.$$  

(5.1)

$\text{Cone}_{\nu^\sigma(\sigma)}(\alpha_0, \theta)$ is a subgroup of $\mathcal{U}(\mathcal{H}_\sigma) \otimes I_{\mathbb{Z}^2/\sigma}$. If additionally $\xi \in \Aut(A_{\nu^\sigma(C_\theta \cap \sigma)})$ we will say that $x \in \text{Cone}_{\nu^\sigma(\sigma)}^W(\alpha_0, \theta)$.

$^{19}$Where we define $\nu^L := \nu^{-1}$ and $\nu^R := \nu$. 

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Similarly as before we will also need the generalisation of this definition:

**Definition 5.2.** Take $0 < \theta < \pi/2$, $\alpha_L \in \text{Aut}(A_L)$, $\alpha_R \in \text{Aut}(A_R)$ and $\Theta \in \text{Aut}(\mathcal{A}_{W(G)\gamma})$. Take $\sigma \in \{L, R\}$ and take $x \in \mathcal{U}(H_0)$ then we say that $x$ is an inner after cone operator on $\nu^*(\sigma)$ (or in short $x \in IAC_{\nu^*(\sigma)}(\alpha_0, \theta, \Theta)$) if there exists an $A \in \mathcal{U}(\mathcal{A}_{\nu^*(\sigma)})$ and a $y \in \text{Cone}_{\nu^*(\sigma)}(\alpha_0, \theta)$ such that

$$x = \pi_0 \circ \alpha_0 \circ \Theta(A)y. \quad (5.2)$$

### 5.1 Definition of the index

First notice that since this assumption is strictly stronger then the assumption in 3.6 we can still define all the objects we did before. So in what follows fix a $v \in \mathcal{U}(H_0)$, $W_g \in \mathcal{U}(H_0)$, $u_\sigma(g, h) \in \text{Cone}_{\nu^*(\sigma)}(\alpha_0, \theta)$ and $K^\sigma_g \in \text{Cone}_{\nu^*(\sigma)}(\alpha_0, \theta)$ (for every $g, h \in G$) and then later on we will show independence on the choice of these objects. We remind the reader of some equalities from section 4

$$\text{Ad}(v) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \tau \quad (5.3)$$

$$\text{Ad}(W_g) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g \circ \beta^U_g \quad (5.4)$$

$$\text{Ad}(u_\sigma(g, h)) \circ \pi_\sigma \circ \alpha_\sigma = \pi_\sigma \circ \alpha_\sigma \circ \eta^U_\sigma \circ \beta^U_g \circ \eta^R_h \circ (\beta^U_{g^{-1}}) \circ (\eta^U_{g^{-1}})^{-1} \quad (5.5)$$

$$u_L(g, h) \otimes u_R(g, h) = W_g W_h W_{gh}^{-1} \quad (5.6)$$

$$\text{Ad}(K^R_g) \circ \pi_0 \circ \alpha_0 = \pi_0 \circ \alpha_0 \circ \tau^{-1} \circ \eta^R \circ \beta^R_g \circ \tau \circ \eta^R_{g^{-1}} \circ (\beta^R_{g^{-1}})^{-1} \circ \tau \quad (5.7)$$

In what follows, we will need the vertical analogue of the $v$. We define the horizontal translation operator:

**Lemma 5.3.** There exists a unique $w \in \mathcal{U}(H_0)$ such that

$$\text{Ad}(w) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \Theta \circ \tau^{-1} \circ \alpha_0^{-1} \quad \text{and} \quad \pi_0(V_1)w\pi_0(V_1)^\dagger \Omega_0 = \Omega_0. \quad (5.8)$$

**Proof.** Analogous to the proof of 4.11. \qed

We can use these objects to define $b^\sigma_g$. 

**Lemma 5.4.** There exist a $b^L_g \in \text{Cone}_{\nu^*(\mathcal{L})}(\alpha_0, \theta)$ and a $b^R_g \in \text{Cone}_{\nu^*(\mathcal{R})}(\alpha_0, \theta)$ (both unique up to an exchange of a $G$-dependent phase) satisfying that

$$w^\dagger W_g w W_g^\dagger = b_g := b^L_g \otimes b^R_g. \quad (5.9)$$

It will satisfy

$$\text{Ad}(b^\sigma_g \otimes 1_{\mathcal{L}_z/\sigma}) \circ \pi_0 \circ \alpha_0 = \pi_0 \circ \alpha_0 \circ \nu \circ (\eta^\sigma_g)^{-1}. \quad (5.10)$$

**Proof.** It is easy enough to show that

$$\text{Ad}(w^\dagger W_g w W_g^\dagger) \circ \pi_0 = \pi_0 \circ (\alpha_L \circ \nu \circ (\eta^L_g)^{-1} \circ \alpha^{-1}_L) \otimes (\alpha_R \circ \nu \circ (\eta^R_g)^{-1} \circ \alpha^{-1}_R). \quad (5.11)$$

To do this one only has to use that $\beta^U_g$ commutes with $\nu$. By lemma A.3 the result now follows. \qed

**Lemma 5.5.** There exists an $\alpha : G \to U(1)$ satisfying

$$w^\dagger b^R_g v h_g K^R_g(b^R_g)^\dagger = \alpha(g)w^\dagger K^R_g w \quad (5.12)$$

where

$$h_g := \pi_0 \circ \alpha_0 \circ \Theta(U(-1, -1)(g)). \quad (5.13)$$
Proof. We have that

\[
\text{Ad}(v^\dagger K_g^R w) \circ \pi_0 \circ \alpha_0 \circ \Theta \quad (5.14)
\]

For all \( \beta_g^L \). We have that

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \tau^{-1} \circ \eta_g^R \circ \beta_g^R \circ \tau \circ (\beta_g^R)^{-1} \circ (\eta_g^R)^{-1} \circ \nu. \quad (5.15)
\]

Now we will use the fact that because of translation invariance of the group action, we have that \( \tau^{-1} \circ \beta_g^R \circ \tau \circ (\beta_g^R)^{-1} = \beta_g^R \). With \( L_j = \{(x, y) \in \mathbb{Z}^2 | y = j \} \). This leads to

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \tau^{-1} \circ \eta_g^R \circ \beta_g^R \circ \tau \circ (\beta_g^R)^{-1} \circ (\eta_g^R)^{-1} \circ \nu.
\]

Now we will use the fact that because of translation invariance of the group action, we have that \( \tau^{-1} \circ \beta_g^R \circ \tau \circ (\beta_g^R)^{-1} = \beta_g^R \). This leads to

\[
= \text{Ad}(v^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \eta_g^R \circ \tau \circ \beta_g^R \circ (\eta_g^R)^{-1} \circ \nu. \quad (5.16)
\]

Inserting \( b_g^R v v^\dagger(b_g^R)^\dagger \) now gives

\[
= \text{Ad}(v^\dagger b_g^R v v^\dagger(b_g^R)^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \eta_g^R \circ \tau \circ \beta_g^R \circ (\eta_g^R)^{-1} \circ \nu. \quad (5.17)
\]

Using the inverse of equation (5.10) we now get

\[
= \text{Ad}(v^\dagger b_g^R v) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_g^R \circ \tau \circ \eta_g^R \circ \tau \circ (\eta_g^R)^{-1} \circ \nu \quad (5.18)
\]

Now we will use the fact that \( \nu^{-1} \circ \beta_g^R \circ \nu = \text{Ad}(U_{(1,-1)}(g)) \circ \beta_g^R \circ \nu \) to obtain

\[
= \text{Ad}(v^\dagger b_g^R v) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \tau^{-1} \circ \eta_g^R \circ \tau \circ \text{Ad}(U_{(1,-1)}(g)) \circ \beta_g^R \circ \nu^{-1} \circ (\eta_g^R)^{-1} \circ \nu. \quad (5.19)
\]

Since (as you can check in figure 3) \( \tau^{-1}(\eta_g^R(\tau(U_{(1,-1)}(g)))) = U_{(1,-1)}(g) \) this gives

\[
= \text{Ad}(v^\dagger b_g^R v) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \tau^{-1} \circ \eta_g^R \circ \tau \circ \text{Ad}(U_{(1,-1)}(g)) \circ \beta_g^R \circ \nu^{-1} \circ (\eta_g^R)^{-1} \circ \nu \quad (5.20)
\]

Now using again that \( \tau \circ \beta_g^R \circ \nu = \beta_g^R \circ \tau \circ (\beta_g^R)^{-1} \) we get:

\[
= \text{Ad}(v^\dagger b_g^R v h_g) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \tau^{-1} \circ \eta_g^R \circ \tau \circ \beta_g^R \circ \nu^{-1} \circ (\eta_g^R)^{-1} \circ \nu \quad (5.21)
\]

After again using (5.10) we obtain:

\[
= \text{Ad}(v^\dagger b_g^R v h_g K_g^R(b_g^R)^\dagger) \circ \pi_0 \circ \alpha_0 \circ \Theta. \quad (5.22)
\]

By the irreducibility of \( \pi_0 \) this implies that there indeed exists such an \( \alpha \).

We have that \( h_g \in IAC_L(\theta, \alpha, \Theta) \). On top of that we also have that

**Lemma 5.6.** For all \( g, h \in G \) we have that

\[
[h_g, W_h] = 0 \quad (5.23)
\]

\[
[h_g, b_h^L] = 0. \quad (5.24)
\]

Proof. We have that (see figure 3)

\[
\text{Ad}(W_h)(h_g) = \pi_0 \circ \alpha_0 \circ \Theta \circ \eta_h \circ \beta_h^U(U_{(1,-1)}(g)) \quad (5.25)
\]

\[
= \pi_0 \circ \alpha_0 \circ \Theta(U_{(1,-1)}(g)) = h_g \quad (5.26)
\]

concluding the proof of the first item. The second proof is analogous. □
Finally before we can show that the $\alpha$ is a $U(1)$ representation we need to prove an analogue of the equation in the 2-cochain lemma 4.8.

**Lemma 5.7.** The equality

\[
\text{Ad}(W_g b_h W_h W_{gh} W_g^\dagger (b_{gh}^R)^\dagger W_g^\dagger W_{gh}^\dagger u_R(g, h)) \circ \pi_0 = \text{Ad}(w^\dagger u_R(g, h)w) \circ \pi_0
\]

(5.30)

holds.

**Proof.** The proof is almost completely analogous to the proof of 4.8. We first start by showing that

\[
\text{Ad}(b_g W_g b_h W_h W_{gh}^\dagger (b_{gh})^\dagger W_g^\dagger W_{gh}^\dagger u_L(g, h) \otimes u_R(g, h)) \circ \pi_0 = \text{Ad}(w^\dagger u_R(g, h) \otimes u_R(g, h) w) \circ \pi_0
\]

(5.31)

holds. This part is completely the same. The only difference is now that to prove that

\[
b_g W_g b_h W_h W_{gh}^\dagger (b_{gh})^\dagger W_g^\dagger W_{gh}^\dagger W_g^\dagger
\]

(5.32)

is split one needs to use the new (shifted) support of $\eta_0^g$.

Using this we can show that:

**Lemma 5.8.** The $\alpha$ defined previously is a $U(1)$-representation.

**Proof.** Take the definition of the 2-cochain (lemma 4.8)

\[
K^R_g W_g K^R_h W_h W_{gh}^\dagger (K^R_{gh})^\dagger W_g^\dagger W_{gh}^\dagger u_R(g, h) = C(g, h) v^\dagger u_R(g, h) v.
\]

(5.33)

Clearly this equation is invariant under the substitution

\[
K^R_g \mapsto w^\dagger K^R_g \quad W_g \mapsto w^\dagger W_g \quad u_R(g, h) \mapsto w^\dagger u_R(g, h) w.
\]

(5.34)

If we now also show that it is invariant under the substitution

\[
K^R_g \mapsto v^\dagger b_g^R v h_g^R \quad W_g \mapsto w^\dagger W_g \quad u_R(g, h) \mapsto w^\dagger u_R(g, h) w
\]

(5.35)

then we get using the definition of $\alpha$ that

\[
C(g, h) = C(g, h) \frac{\alpha(g) \alpha(h)}{\alpha(gh)}
\]

(5.36)

which would clearly conclude the proof. To show this, notice that we have by construction and by lemma 5.7 that

\[
\text{Ad}(b_g W_g) \circ \pi_0 = \text{Ad}(w^\dagger W_g w) \circ \pi_0
\]

(5.37)

\[
\text{Ad}(b_g W_g b_h^R W_h W_{gh}^\dagger (b_{gh}^R)^\dagger W_g^\dagger W_{gh}^\dagger u_R(g, h)) \circ \pi_0 = \text{Ad}(w^\dagger u_R(g, h) w) \circ \pi_0.
\]

(5.38)

This shows that there exists a $\beta : G \to U(1)$ and a $\gamma : G \times G \to U(1)$ such that

\[
\begin{align*}
 w^\dagger W_g w &= \beta(g) b_g^R b_g^R W_g \quad w^\dagger u_R(g, h) w = \gamma(g, h) b_g^R W_g b_h^R W_h W_{gh}^\dagger (b_{gh}^R)^\dagger W_g^\dagger W_{gh} W_g^\dagger u_R(g, h).
\end{align*}
\]

(5.39)

Since the equation is clearly invariant under the substitution with these two phases all that is left to do is to show that equation (5.33) is invariant under the substitution:

\[
K^R_g \mapsto v^\dagger b_g^R v h_g^R \quad W_g \mapsto b_g^R b_g^R W_g
\]

(5.40)
\[ u_R(g, h) \mapsto b_g^R W_g b_h^R W_h W_{gh}^\dagger (b_{gh}^R)^\dagger W_{gh} W_h^\dagger W_y^\dagger u_R(g, h). \] (5.41)

To show this first notice that by using the fact that \( h_g \in \text{IAC}_R(\theta, \alpha_0, \Theta) \)' and lemma 5.6 we get that

\[
K_g^R W_g K_h^R W_h W_{gh}^\dagger (K_{gh}^R)^\dagger
\]

\[ \mapsto v^\dagger b_g^R v h_g^\dagger K_g^R (b_g^R)^\dagger b_h^L W_g v^\dagger b_h^R v W_h W_{gh}^\dagger (b_{gh}^R)^\dagger (v^\dagger b_{gh}^R v h_{gh}^\dagger K_{gh}^R (b_{gh}^R)^\dagger)^\dagger 
\]

\[ = h_g h_h^\dagger v^\dagger b_g^R v K_g^R (b_g^R)^\dagger b_h^L b_h^R W_g v^\dagger b_h^R W_h W_{gh}^\dagger (b_{gh}^R)^\dagger (v^\dagger b_{gh}^R v K_{gh}^R (b_{gh}^R)^\dagger)^\dagger \] (5.43)

(all the \( h \)-operators can be put in front). Since \( h_g \) is an honest representation we get that the \( h \)-operators cancel out. This leaves us with checking that equation (5.33) is invariant under

\[
K_g^R \mapsto v^\dagger b_g^R v K_g^R (b_g^R)^\dagger 
\]

\[ W_g \mapsto b_g^L b_g^R W_g \] (5.45)

\[
u_R(g, h) \mapsto b_g^R W_g b_h^R W_h W_{gh}^\dagger (b_{gh}^R)^\dagger W_{gh} W_h^\dagger W_y^\dagger u_R(g, h). \] (5.46)

Using the fact that

\[
(b_{gh}^R)^\dagger W_{gh} W_h^\dagger W_y^\dagger u_R(g, h) = (b_{gh}^R)^\dagger u_L(g, h)^\dagger = u_L(g, h)^\dagger (b_{gh}^R)^\dagger 
\]

\[
W_{gh} W_h^\dagger W_y^\dagger u_R(g, h) (b_{gh}^R)^\dagger \] (5.47)

\[
\text{we see that we can prove that our equation is invariant under the substitution using lemma 4.12. This concludes the proof.} \]

**Definition 5.9.** Let \( \alpha \) be the \( U(1) \)-representation defined in lemma 5.5. Take \( \phi \in \text{hom}(G, \mathbb{T}) \) such that \( \alpha(g) = \exp(i\phi(g)) \). We define the 2 translation index as

\[
\text{Index}_{2\text{trans}}^{\mathcal{A}_U}(\theta, \tilde{\beta}_g, \eta_g, \alpha_0, \Theta, \omega, \omega_0) := \phi \in H^1(G, \mathbb{T}) \] (5.49)

and (as advertised) it is only a function of the automorphisms (and the product state) not on the choice of the GNS triple of \( \omega_0 \) or on the choice of phases in \( W_g, u_L(g, h), u_R(g, h), v, w, K_g^L, K_g^R, b_g^L \) and \( b_g^R \).

**Proof.** Clearly the construction is invariant under the choice of GNS triple since this simply amounts to an adjoint action by some unitary on every operator. The proof that it is independent of the choice of phases is just as trivial.

One can also define this index starting from operators acting on the left. As the following lemma shows this makes no difference (does not give us a new index):

**Lemma 5.10.** Define \( \tilde{\alpha}(g) \) such that

\[
(\alpha(g)w)^\dagger K_g^L w = \tilde{\alpha}(g)w^\dagger K_g^L w \] (5.50)

then \( \alpha(g)\tilde{\alpha}(g) = 1 \).

**Proof.** By multiplying the left hand side of equation (5.50) with the left hand side of equation (5.12) we obtain

\[
v^\dagger b_g^L v W_g b_h^R W_h W_{gh}^\dagger = v^\dagger b_g^L v W_g b_h^R W_h W_{gh}^\dagger = w^\dagger W_g v W_g^\dagger w \]

\[ = w^\dagger K_g w \] (5.53)

whereas by multiplying the right hand sides we obtain

\[
\tilde{\alpha}(g)w^\dagger K_g w = \alpha(g)\tilde{\alpha}(g)w^\dagger K_g w. \] (5.54)

These two results can only be consistent if the lemma holds.
5.2 The $H^1(G, \mathbb{T})$-valued index is invariant under choices

In this section we will show that the index is only dependent on $\omega$ and not on the choices of our automorphisms nor on $\omega_0$. On this last item we remark:

**Remark 5.11.** A product state can have a non-trivial $H^1(G, \mathbb{T})$ index and the index is not defined relative to $\omega_0$ so in particular

$$\text{Index}_{2 \text{trans}}^{\mathcal{A}U}(\theta, \beta_g^U, \text{Id}, \text{Id}, \omega_0, \omega_0)$$

(5.55)

can still be non-trivial. In fact in this case we get $\alpha(g) = \omega_0(U_{-1,-1}(g))$. To show this, notice that we can take $b_g = 1_H$ and we can chose $K_g^R$ so that it leaves the cyclic vector of $\omega_0, \Omega_0$ invariant. Applying the definition of the index on this cyclic vector then gives us $h_g\Omega_0 = \alpha(g)\Omega_0$. This obviously implies that $\langle \Omega_0| h_g\Omega_0 \rangle = \alpha(g)\langle \Omega_0| \Omega_0 \rangle$ giving us indeed the advertised equation.

We will now show independence of $\alpha$ and its decomposition.

**Lemma 5.12.** Take $\omega_{01}, \omega_{02} \in \mathcal{P}(\mathcal{A})$ product states and $\alpha_1, \alpha_2 \in \text{Aut}(\mathcal{A})$ such that

$$\omega_{01} \circ \alpha_1 = \omega_{02} \circ \alpha_2 = \omega.$$  

(5.56)

Let $V_{11}, V_{12} \in \mathcal{U}(\mathcal{A})$, $\alpha_{L,R,1}, \alpha_{L,R,2} \in \text{Aut}(\mathcal{A}_{L,R})$ and $\Theta_1, \Theta_2 \in \text{Aut}(\mathcal{A}_{W(g)})^c$ be such that

$$\alpha_1 = \text{Ad}(V_{11}) \circ \omega_{01} \circ \Theta_1 \quad \quad \quad \alpha_2 = \text{Ad}(V_{12}) \circ \omega_{02} \circ \Theta_2$$

(5.57)

with $\alpha_{0,i} = \alpha_{L,i} \otimes \alpha_{R,i}$, then

$$\text{Index}_{2 \text{trans}}^{\mathcal{A}U}(\theta, \beta_g, \eta_g, \alpha_{0,1}, \Theta_1, \omega, \omega_{01}) = \text{Index}_{2 \text{trans}}^{\mathcal{A}U}(\theta, \beta_g, \eta_g, \alpha_{0,2}, \Theta_2, \omega, \omega_{02}).$$

(5.58)

**Proof.** We will first prove the result in the case that $\omega_0 = \omega_{01} = \omega_{02}$ and then generalise this result. Since $\omega_0 \circ \alpha_2 \circ \alpha_1^{-1} = \omega_0$ there exists a $x \in \mathcal{U}(\mathcal{H}_0)$ such that

$$\pi_0 \circ \alpha_2 \circ \alpha_1^{-1} = \text{Ad}(x) \circ \pi_0.$$  

(5.59)

Now define $x \in \mathcal{U}(\mathcal{H}_0)$ to be

$$x := \pi_0(V_{11}^\dagger)x\pi_0(V_{11})$$

(5.60)

then

$$\pi_0 \circ \alpha_{02} \circ \Theta_2 = \text{Ad}(x) \circ \pi_0 \circ \alpha_{01} \circ \Theta_1.$$  

(5.61)

Let $v, w, K_g^\sigma$ and $b_g^\sigma$ be operators belonging to the first choice then by construction $xvx^\dagger, xwx^\dagger, xK_g^\sigma x^\dagger$ and $xb_g^\sigma x^\dagger$ are operators belonging to the second choice. Since the index is invariant under this substitution this concludes the proof in the case that $\omega_0 = \omega_{01} = \omega_{02}$. Now suppose that $\omega_{01} \neq \omega_{02}$. Since they are both product states there exists a $\gamma \in \text{Aut}(\mathcal{A})$ satisfying $\omega_{02} = \omega_{01} \circ \gamma$ that is of the form $\gamma = \gamma^L \otimes \gamma^R$. We now have

$$\text{Index}_{2 \text{trans}}^{\mathcal{A}U}(\theta, \beta_g, \eta_g, \alpha_{0,2}, \Theta_2, \omega, \omega_{02}) = \text{Index}_{2 \text{trans}}^{\mathcal{A}U}(\theta, \beta_g, \eta_g, \alpha_{0,2}, \Theta_2, \omega, \omega_{01} \circ \gamma)$$

(5.62)

$$= \text{Index}_{2 \text{trans}}^{\mathcal{A}U}(\theta, \beta_g, \eta_g, \gamma \circ \alpha_{0,2}, \Theta_2, \omega, \omega_{01})$$

(5.63)

concluding the proof.

In what follows we will need that the index is invariant under the following transformation:
**Lemma 5.13.** Take $\omega_0, \theta, \alpha_0, \Theta$ and $\eta_g$ as usual, take $v, w, W_{g,1}, u_{\sigma,1}(g,h), K_{g,1}$ and $b_{g,1}$ operators corresponding to these automorphisms and take $\delta_g^\sigma \in \text{IAC}_{v_\sigma(\sigma)}(\alpha_0, \Theta, \theta)$ (for all $\sigma \in \{L, R\}$ and all $g \in G$). Define

$$
W_{g,2} := \delta_g W_{g,1} \tag{5.64}
$$

$$
u_{\sigma,2}(g,h) := \delta_g^\sigma W_{g,1} \delta_h^\sigma W_{g,1}^\dagger u_{\sigma,1}(g,h)(\delta_{gh})^\dagger \quad \tag{5.65}
$$

$$
K_{g,2} := v^\dagger \delta_g^\sigma v K_{g,1}(\delta_g^\dagger) \quad \tag{5.66}
$$

$$
b_{g,2} := w^\dagger \delta_g^\sigma w b_{g,1}(\delta_g^\dagger) \quad \tag{5.67}
$$

then the index of $v, w, W_{g,1}, u_{\sigma,1}(g,h)$, $K_{g,1}$, $h_g$ and $b_{g,1}$ is equal to the index of $v, w, W_{g,2}, u_{\sigma,2}(g,h), h_g, K_{g,2}$ and $b_{g,2}$.

**Proof.** We define the index in function of the new variables:

$$
\alpha_2(g) := v^\dagger b_{g,2}^\dagger v h_g K_{g,2}^R(b_{g,2}^\dagger)^\dagger w^\dagger (K_{g,2}^R)^\dagger w \tag{5.68}
$$

$$
= v^\dagger (w^\dagger \delta_g^R w b_{g,1}(\delta_g^\dagger)^\dagger) v h_g (v^\dagger \delta_g^R v K_{g,1}(\delta_g^\dagger)^\dagger) (w^\dagger \delta_g^R w b_{g,1}(\delta_g^\dagger)^\dagger) w^\dagger (v^\dagger \delta_g^R v K_{g,1}(\delta_g^\dagger)^\dagger) w \tag{5.69}
$$

$$
= v^\dagger w^\dagger \delta_g^R w b_{g,1}(\delta_g^\dagger)^\dagger v h_g v^\dagger \delta_g^R v K_{g,1}(\delta_g^\dagger)^\dagger w^\dagger (\delta_g^\dagger)^\dagger w v K_{g,1} v^\dagger (\delta_g^\dagger)^\dagger v w. \tag{5.70}
$$

If we now insert $vv^\dagger$, we obtain

$$
\alpha_2(g) = v^\dagger w^\dagger \delta_g^R K_{g,1} b_{g,1}(\delta_g^\dagger)^\dagger w^\dagger (K_{g,1}^R)^\dagger v^\dagger (\delta_g^\dagger)^\dagger v w \tag{5.71}
$$

where the cancelation is due to the fact that $v^\dagger \delta_g^R v \in \text{IAC}(\alpha_0, \Theta, \Theta)$ whereas $h_g \in \text{IAC}(\alpha_0, \Theta, \Theta)$. Again after inserting $vv^\dagger$ and $wv^\dagger$ we now get

$$
\alpha_2(g) = v^\dagger w^\dagger \delta_g^R w v h_g K_{g,1} b_{g,1}(\delta_g^\dagger)^\dagger w^\dagger v^\dagger (\delta_g^\dagger)^\dagger v w \tag{5.72}
$$

$$
= v^\dagger w^\dagger \delta_g^R w \alpha_1(g) w^\dagger v^\dagger (\delta_g^\dagger)^\dagger v w \tag{5.73}
$$

$$
= \alpha_1(g) \tag{5.74}
$$

concluding the proof.

We will now show that the index is independent on the choice of $\tilde{\beta}_g$ and its decomposition.

**Lemma 5.14.** Take $\tilde{\beta}_{g,1}, \tilde{\beta}_{g,2} \in \text{Aut}(\mathcal{A}), V_{g,21}, V_{g,22} \in \mathcal{U}(\mathcal{A}), \eta_{g,1}, \eta_{g,2} \in \text{Aut}(\mathcal{A}_{(\mathcal{A} \cap L)})$ and $\eta_{g,1}^R, \eta_{g,2}^R \in \text{Aut}(\mathcal{A}_{\sigma(\mathcal{A} \cap R)})$ such that there exist $V_{g,21}, V_{g,22} \in \mathcal{U}(\mathcal{A})$ satisfying

$$
\tilde{\beta}_{g,1} = \text{Ad}(V_{g,21}) \circ \eta_{g,1} \circ \beta_g^U \quad \tilde{\beta}_{g,2} = \text{Ad}(V_{g,22}) \circ \eta_{g,2} \circ \beta_g^U \tag{5.75}
$$

and

$$
\omega \circ \tilde{\beta}_{g,1} = \omega \circ \tilde{\beta}_{g,2} = \omega \tag{5.76}
$$

then

$$
\text{Index}_2 \text{trans}(\theta, \tilde{\beta}_{g,1}, \eta_{g,1}, \alpha_0, \Theta, \omega, \omega_0) = \text{Index}_2 \text{trans}(\theta, \tilde{\beta}_{g,2}, \eta_{g,2}, \alpha_0, \Theta, \omega, \omega_0). \tag{5.77}
$$

**Proof.** Take

$$
\alpha = \text{Ad}(V_1) \circ \alpha_0 \circ \Theta \tag{5.78}
$$

the usual decomposition. Since

$$
\omega_0 \circ \alpha \circ \tilde{\beta}_{g,1} \circ (\tilde{\beta}_{g,2})^{-1} = \omega_0 \circ \alpha \tag{5.79}
$$

there exist $\tilde{\delta}_g \in \mathcal{U}(\mathcal{H}_0)$ such that

$$
\text{Ad}(\tilde{\delta}_g) \circ \pi_0 \circ \alpha = \pi_0 \circ \alpha \circ \tilde{\beta}_{g,2}(\tilde{\beta}_{g,1})^{-1}. \tag{5.80}
$$
Inserting equation (5.75) and (5.78) in this gives
\[
\text{Ad}\left(\tilde{\delta}_g\right) \circ \pi_0 \circ \text{Ad}(V_1) \circ \alpha_0 \circ \Theta = \pi_0 \circ \text{Ad}(V_1) \circ \alpha_0 \circ \Theta \circ \text{Ad}(V_{g,21}) \circ \eta_{g,1} \circ \eta_{g,2}^{-1} \circ \text{Ad}(V_{g,22}).
\]
(5.81)
Putting the \(V\)'s to the other side gives:
\[
\text{Ad}\left(\pi_0 \circ \alpha_0 \circ \Theta(V_{g,21})\right) \pi_0(V_1) \pi_0 \circ \alpha_0 \circ \Theta(V_{g,22}) \circ \pi_0 = \pi_0 \circ \alpha_0 \circ \eta_{g,2} \circ \eta_{g,1}^{-1} \circ \alpha_0^{-1}.
\]
(5.82)
Since the last equation is split we have again by the same argument used to prove lemma 4.4 that we can take \(\delta_g^L \in \text{Cone}_{\nu^{-1}(L)}(\alpha_0, \theta)\) and \(\delta_g^R \in \text{Cone}_{\nu(R)}(\alpha_0, \theta)\) such that
\[
\delta_g^L \otimes \delta_g^R = \alpha_0 \circ \Theta(V_{g,21}) \pi_0(V_1) \tilde{\delta}_g \pi_0(V_1) \pi_0 \circ \alpha_0 \circ \Theta(V_{g,22}).
\]
(5.83)
Take \(v, w, W_{g,1}, u_{\sigma,1}(g, h), K_{g,1}^\sigma\) and \(b_{g,1}^\sigma\) operators corresponding to the first choice (with arbitrary phases). Define
\[
W_{g,2} := \delta_g W_{g,1}
\]
(5.84)
\[
u_{\sigma,2}(g, h) := \delta_g^\sigma W_{g,1} \delta_h^\sigma W_{g,1}^{\dagger} u_{\sigma,1}(g, h) (\delta_{gh})^\dagger
\]
(5.85)
\[
K_{g,2}^\sigma := v^\dagger \delta_g^\sigma v K_{g,1}^\sigma (\delta_g^\sigma)^\dagger
\]
(5.86)
\[
b_{g,2}^\sigma := w^\dagger \delta_g^\sigma w b_{g,1}^\sigma (\delta_g^\sigma)^\dagger
\]
(5.87)
then \(W_{g,2}, u_{R,2}(g, h), K_{g,2}^R\) and \(b_{g,2}^R\) are operators belonging to the second choice. Now using lemma 5.13 concludes the proof. \(\square\)

**Lemma 5.15.** The index is independent of the choice of angle \(\theta\).

**Proof.** After what was done so far this result is trivial. \(\square\)

Due to all these considerations we will write this index as \(\text{Index}^{AU}_{2,\text{trans}}(\omega)\) from here on onward.

### 5.3 Example (consistency with conjecture 1.3)

This example will be very similar to what was done in section 4.4. We still define \(\mathcal{A}\) as the closure of the tensor product of the \(\mathcal{B}\). In this case however we will require that the translation morphism \(\nu\) acts as an automorphism on the \(C^*\) algebra \(\mathcal{B}\) and that our one dimensional state is invariant under this automorphism. We will need:

**Assumption 5.16.** Take \(\phi \in \mathcal{P}(\mathcal{B})\) such that

1. \(\phi \circ \beta_g = \phi.\)

2. there exist automorphisms \(\tilde{\alpha}_L \in \text{Aut}(\mathcal{B}_L), \tilde{\alpha}_R \in \text{Aut}(\mathcal{B}_R)\) and a summable (see definition 4.15) operator \(b \in \mathcal{U}(\mathcal{B})\) such that

\[
\phi_0 = \phi \circ \text{Ad}(b) \circ \tilde{\alpha}_L \otimes \tilde{\alpha}_R
\]

(5.88)

is a product state.

3. \(\phi \circ \nu = \phi.\)

Since this state satisfies the split property (see lemma 4.18) we can define an \(H^1(G, \mathbb{T})\) valued index for it (see appendix B). Now we will take again the infinite tensor product (see definition 4.19) of such states then we get:
Lemma 5.17. Let $\omega$ be the infinite tensor product of a state $\phi$ that satisfies assumption 5.16 then $\omega$ satisfies assumption 3.9.

Proof. The proof of this lemma is analogous to the proof of lemma 4.20.

This shows that $\omega$ indeed has a well defined $H^1(G, \mathbb{T})$ index. We will now show that $\text{Index}_{2\text{trans}}(\omega) = \text{Index}_{1\text{d trans}}(\phi)$. Just like in section 4.4 we fix a GNS triple of $\omega_0$ that is of the form

$$
\left( \mathcal{H}_0 = \bigotimes_{\rho \in \{L_{>1}, L_{-1}, L_{<1}\}} \mathcal{H}_\rho, \pi_0 = \bigotimes_{\rho \in \{L_{>1}, L_{-1}, L_{<1}\}} \pi_\rho, \Omega_0 = \bigotimes_{\rho \in \{L_{>1}, L_{-1}, L_{<1}\}} \Omega_\rho \right) (5.89)
$$

where the $\pi_\rho, \sigma : \mathcal{A}_\rho \rightarrow \mathcal{B}(\mathcal{H}_\rho)$ are all irreducible representations.

Remark 5.18. We clearly have (by construction) that

$$
\pi_{L_{-1}, L} \circ \alpha_{L}|_{L_{-1}} \otimes \pi_{L_{-1}, R} \circ \alpha_{R}|_{L_{-1}}
$$

is a GNS representation of $\phi$. This implies that if we take any $k^R : G \to \mathcal{H}_{L_{-1}, R}$ such that

$$
\text{Ad}(k^R(g)) \circ \pi_{L_{-1}, R} \circ \alpha_{R}|_{L_{-1}} = \pi_{L_{-1}, R} \circ \alpha_{R}|_{L_{-1}} \circ \beta_{g}^{L_{-1}\cap R}
$$

and $w_{L_{-1}}$ to be such that

$$
\text{Ad}(w_{L_{-1}}) \circ \pi_{L_{-1}, L} \otimes \pi_{L_{-1}, R} = \pi_{L_{-1}, L} \otimes \pi_{L_{-1}, R} \circ \nu
$$

then there exists a $U(1)$ representation $\alpha$ satisfying that

$$
(\pi_{L_{-1}, L}(U_{-1,1}(g)) \otimes 1_{\mathcal{H}_{L_{-1}, R}})(1_{\mathcal{H}_{L_{-1}, L}} \otimes k^R(g)) = \alpha(g) w^\dagger_{L_{-1}} (1_{\mathcal{H}_{L_{-1}, L}} \otimes k^R(g)) w_{L_{-1}}
$$

and that it is given by $\text{Index}_{1\text{d trans}}(\phi)$.

Just like in section 4.4 let $k : G \to \mathcal{U}(\mathcal{H}_{L_{-1}})$ be the unique operator that satisfies

$$
\text{Ad}(k^g) \circ \pi_{L_{-1}} = \pi_{L_{-1}} \circ \alpha|_{L_{-1}} \circ \beta_{g}^{L_{-1}} \circ \alpha^{-1}|_{L_{-1}} \quad k^g \Omega_{L_{-1}} = \Omega_{L_{-1}}.
$$

We can take

$$
K_g = 1_{\mathcal{H}_{L_{>1}}} \otimes \pi_{L_{-1}}(b)^\dagger \tilde{k}_g \pi_{L_{-1}}(b) \otimes 1_{\mathcal{H}_{L_{<1}}} = 1_{\mathcal{H}_{L_{>1}}} \otimes (b^g \otimes k^R_g) \otimes 1_{\mathcal{H}_{L_{<1}}}.
$$

Similarly we can take $w_\rho \in \mathcal{H}_\rho$ (where $\rho \in \{L_{>1}, L_{-1}, L_{<1}\}$) satisfying

$$
\text{Ad}(w_\rho) \circ \pi_\rho \circ \alpha|_\rho \circ \Theta|_\rho = \pi_\rho \circ \alpha|_\rho \circ \Theta|_\rho \circ \nu
$$

and then we get that $w = w_{L_{>1}} \otimes w_{L_{-1}} \otimes w_{L_{<1}}$. Since $\eta_g = \text{Id}_{\mathcal{A}}$ we can simply take $b^g = 1_{\mathcal{H}_\phi}$. We now have that

$$
\pi_0 \circ \alpha_0 \circ \Theta(U_{-1,1}(1_{\mathcal{H}_{L_{-1}}} \otimes K^R_g) = 1_{\mathcal{H}_{L_{>1}}} \otimes (\pi_{L_{-1}, L} \circ \tilde{\alpha}_L(U_{-1,1}) \otimes k^R(g)) \otimes 1_{\mathcal{H}_{L_{<1}}}.
$$

By what was written in remark 5.18 we get that

$$
= 1_{\mathcal{H}_{L_{>1}}} \otimes \alpha(g) w_{L_{-1}} (1_{\mathcal{H}_{L_{-1}, L}} \otimes k^R(g)) w^\dagger_{L_{-1}} \otimes 1_{\mathcal{H}_{L_{<1}}}
$$

$$
= \alpha(g) w^\dagger_{L_{-1}} (1_{\mathcal{H}_{L_{>1}}} \otimes 1_{\mathcal{H}_{L_{-1}, L}} \otimes k^R(g) \otimes 1_{\mathcal{H}_{L_{<1}}}) w
$$

$$
= \alpha(g) w^\dagger (1_{\mathcal{H}_{L}} \otimes K^R_g) w
$$

showing that indeed $\text{Index}_{2\text{trans}}(\omega) = \text{Index}_{1\text{d trans}}(\phi)$.
5.4 The $H^1(G, \mathbb{T})$-valued index is invariant under locally generated automorphisms

The goal of this section is to show that the index we constructed is invariant under locally generated automorphisms. To this end let $H \in \mathcal{B}_{F_0}(\{0,1\})$ be a $G$-invariant, translation invariant (in both directions) one parameter family of interactions. In the rest of this section we will define (for notational simplicity) $H^\text{split} := H_{r-1} + H_{r-1}(R)$. Clearly, this $H^\text{split}$ is translation invariant in the vertical direction but not in the horizontal direction. The proof is very similar to what was done in theorem 4.22 but with some notable differences. We first need a different version of lemma 4.23 (in fact it is almost completely identical only the definition of $H^\text{split}$ is now slightly different) and use this to define some objects used in the rest of the section:

**Lemma 5.19.** Take $0 < \theta < \pi/2$ and $\Theta \in \text{Aut}(\mathcal{A}_{W(C_a)^c})$. There exist $\Phi^U_{\alpha \beta} \in \text{Aut}(\mathcal{A}_{W(C_a)^c \cap U})$ and $\Phi^D_{\alpha \beta} \in \text{Aut}(\mathcal{A}_{W(C_a)^c \cap D})$, both commuting with $\beta_g$, such that there exists some $a \in \mathcal{U}(\mathcal{A})$ satisfying that

$$\gamma_{0;1}^H \circ \gamma_{1;0}^H \circ \gamma_{0;1}^H \circ \gamma_{1;0}^H = \text{Ad}(a) \circ \Phi^U_{0;1} \otimes \Phi^D_{0;1}. \quad (5.101)$$

**Proof.** This follows from the fact that because of $\mathbf{D.4}$ part 2 (or a slight modification thereof because of the new definition of $H^\text{split}$) $\gamma_{0;1}^H \circ \gamma_{1;0}^H \circ \gamma_{0;1}^H \circ \gamma_{1;0}^H \in \text{GVAut}_1(\mathcal{A})$.

We now need a slightly modified version of lemma 4.23

**Lemma 5.20.** Take $\tilde{a} := \pi_0 \circ \alpha_0 \circ \Theta(a)$ and define the map

$$\phi_{1,\sigma} : \text{IAC}_\sigma(\alpha_0, \theta, \Theta) \to \mathcal{U}(\mathcal{H})$$

$$: x = \pi_0 \circ \alpha_0 \circ \Theta(A)y \mapsto \phi_{1,\sigma}(x) = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^H \circ \gamma_{0;1}^H \circ \gamma_{1;0}^H \circ \text{Ad}(A) \circ \xi$$

where $A \in \mathcal{A}_\sigma$ and $y \in \text{Cone}_\sigma(\alpha_0, \theta)$. This map satisfies that

1. for (the unique) $\xi \in \text{Aut}(\mathcal{A}_{C_a \cap \sigma})$ such that

$$\text{Ad}(y) \circ \pi_0 \circ \alpha_0 = \pi_0 \circ \alpha_0 \circ \xi$$

we get that

$$\text{Ad}(\phi_1(x)) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^H \circ \gamma_{0;1}^H \circ \gamma_{1;0}^H \circ \text{Ad}(A) \circ \xi. \quad (5.104)$$

2. it is well defined (independent of the choice of $A$ and $y$).

3. it is a group homomorphism.

4. in general

$$\alpha^\dagger \phi_1(x) \tilde{a} \in \text{IAC}_\sigma(\alpha_0, \theta, \Theta \circ \Phi^U_{0;1} \otimes \Phi^D_{0;1}). \quad (5.105)$$

If on top of this, when $x \in \text{IAC}_{\sigma_\sigma(\sigma)}$ then

$$\tilde{a} \alpha^\dagger \phi_1(x) \tilde{a} \in \text{IAC}_{\sigma_\sigma(\sigma)}(\alpha_0, \theta, \Theta \circ \Phi^U_{0;1} \otimes \Phi^D_{0;1}). \quad (5.106)$$

A similar statement holds when $\text{IAC}_W(\alpha_0, \theta, \Theta \circ \Phi^U_{0;1} \otimes \Phi^D_{0;1})$ or when $\text{IAC}_W(\alpha_0, \theta, \Theta \circ \Phi^U_{0;1} \otimes \Phi^D_{0;1})$.

**Proof.** The proof of this lemma is analogous to the proof of lemma 4.23.

We now have to extend the above definition to the group

$$\text{IAC}_{L \times R}(\alpha_0, \theta, \Theta) = \{ab = ba|a \in \text{IAC}_L(\alpha_0, \theta, \Theta), b \in \text{IAC}_R(\alpha_0, \theta, \Theta)\}. \quad (5.107)$$
Lemma 5.21. Define

\[ \phi_1 : \mathrm{IAC}_{L \times R}(\alpha_0, \theta, \Theta) \to \mathcal{U}(\mathcal{H}) : ab \mapsto \phi_{1,L}(a)\phi_{1,R}(b) \]  

(5.108)

where \( a \in \mathrm{IAC}_L(\alpha_0, \theta, \Theta) \) and \( b \in \mathrm{IAC}_R(\alpha_0, \theta, \Theta) \). This is well defined, by which we mean that

\[ \phi_{1,L}(a)\phi_{1,R}(b) = \phi_{1,R}(b)\phi_{1,L}(a). \]  

(5.109)

This is still a morphism.

Proof. Identical to the proof of lemma 4.24. \( \Box \)

Lemma 5.22. Define again \( \tilde{a} := \pi_0 \circ \alpha_0 \circ \Theta(a) \), with \( a \) defined through equation (5.101). Let

\[ \phi_2 : \{ \text{Ad}(w^{-p}v^{-s})(W_g)|g \in G, p, s \in \{-1,0,1\} \} \to \mathcal{U}(\mathcal{H}) : \]  

(5.110)

\[ \text{Ad}(w^{-p}v^{-s})(W_g) \mapsto \text{Ad}(\tilde{a}w^{-p}v^{-s})(W_g). \]

This satisfies

\[ \text{Ad}(\phi_2(w^{-p}v^{-s}W_gv^pw^p)) \circ \pi_0 \circ \alpha_0 \circ \Theta \]  

(5.111)

\[ = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} \circ \nu^{-p} \circ \tau^{-s} \circ \eta_p \circ \beta_g \circ \tau^s \circ \nu^p \circ \gamma_{0;1}^{H_{\text{split}}} \circ \gamma_{1;0}^H. \]

Proof. The proof of this lemma is analogous to the proof of lemma 4.25. \( \Box \)

Lemma 5.23. The \( \phi_1 \) and \( \phi_2 \) defined previously satisfy that for any \( x \in \mathrm{IAC}_\sigma(\alpha_0, \theta, \Theta) \) we have that

\[ \phi_1(\text{Ad}(w^{-p}W_gw^p)(x)) = \text{Ad}(\phi_2(w^{-p}W_gw^p))(\phi_1(x)) \]  

(5.112)

\[ \phi_1(\text{Ad}(v^s)(x)) = \text{Ad}(v^s)(\phi_1(x)) \]  

(5.113)

\[ \phi_2(\text{Ad}(v^s)(w^{-p}W_gw^p)) = \text{Ad}(v^s)(\phi_2(w^{-p}W_gw^p)) \]  

(5.114)

for any \( p, s \in \{0,1\} \). For the second equation we need the stronger condition that \( x \in \mathrm{IAC}_\sigma^W(\alpha_0, \theta, \Theta) \).

Proof. The proof of this lemma is analogous to the proof of lemma 4.26. \( \Box \)

Lemma 5.24. There exist \( A_L \in \mathcal{U}(\mathcal{A}_L), A_R \in \mathcal{U}(\mathcal{A}_\nu(R)) \) and \( \Phi^{\mu\nu} \in \text{Aut}(\mathcal{A}_{W(G_{\mu})\cap \mu \nu}) \) where \( \mu \in \{U, D\} \) and \( \nu \in \{L, \nu(R)\} \) satisfying that

\[ \nu \circ \gamma_{0;1}^{H_{\text{split}}} \circ \nu^{-1} \circ \gamma_{1;0}^{H_{\text{split}}} = \gamma_{0;1}^{H} \otimes \gamma_{0;1}^{H_{\text{split}}} \circ \gamma_{1;0}^{H_{\text{split}}} \]  

(5.115)

\[ = \text{Ad}(A_L) \otimes \text{Ad}(A_R) \circ \bigotimes_{\mu \in \{U, D\}, \nu \in \{L, \nu(R)\}} \Phi^{\mu\nu} \]  

(5.116)

and that \( \Phi^{\mu\nu} \circ \beta_g = \beta_g \circ \Phi^{\mu\nu} \).

Proof. This follows from [D.3] part 2. \( \Box \)

Lemma 5.25. Define

\[ a_L := \pi_0 \circ \alpha_0 \circ \Theta(A_L) \]  

(5.117)

\[ a_R := \pi_0 \circ \alpha_0 \circ \Theta(A_R) \]

then for any \( K \in \text{Cone}_{L\times \nu(R)}(\alpha_0, \theta) \) we get that

\[ w^\dagger \phi_1(K)w = \phi_1(w^\dagger a_L \otimes a_R K a_L^\dagger \otimes a_R^\dagger w). \]  

(5.118)

Similarly we have that

\[ w^\dagger \phi_2(W_g)w = \phi_1(w^\dagger a_L \otimes a_R w)\phi_2(w^\dagger W_gw)\phi_1(w^\dagger a_L^\dagger \otimes a_R^\dagger w). \]  

(5.119)
Proof. We will only show the first equality as the second one is completely analogous. We need to show that
\[
\phi_1(K) = \Ad(w\phi_1(w^\dagger a_L \otimes a_R)w)(\phi_1(w^\dagger K w))
\]  
\[= \Ad(w\phi_1(w^\dagger a_L \otimes a_R)w)(\tilde{a}w^\dagger K w\tilde{a}^\dagger)
\]  
\[= \Ad(w\phi_1(w^\dagger a_L \otimes a_R)w)(\tilde{a}w^\dagger a_1^\dagger \phi_1(K)\tilde{a}^\dagger w^\dagger)
\]  
\[= \Ad(w\phi_1(w^\dagger a_L \otimes a_R)w)(\tilde{w}\tilde{a}w^\dagger a_1^\dagger)(\phi_1(K)).
\]  
(5.120)
(5.121)
(5.122)
(5.123)

To do this, we first find an \(A \in \mathcal{U}(A)\) such that
\[\pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{H}^{0;1} \circ \gamma_{1;0}^{H_{\text{split}}}(A) = w\phi_1(w^\dagger a_L \otimes a_R)w(\tilde{w}\tilde{a}w^\dagger a_1^\dagger)
\]  
(5.124)

and then we will prove that \(A \in \mathcal{U}(A_{W(C_g)})\). It should be clear that the result follows from this.

Finding this \(A\) can be done by working out
\[w\phi_1(w^\dagger a_L \otimes a_R)w(\tilde{w}\tilde{a}w^\dagger a_1^\dagger) = \pi_0 \circ \alpha_0 \circ \Theta \left( \nu \circ \gamma_{0;1}^H \circ \nu^{-1}(A_L \otimes A_R)\nu(a) a^\dagger \right).
\]  
(5.126)

First we insert \(\gamma_{H}^{0;1} \circ \gamma_{1;0}^{H_{\text{split}}} \circ \gamma_{0;1}^{H_{\text{split}}} \circ \gamma_{1;0}^{H} = \text{Id}\) to obtain
\[= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} \circ \gamma_{0;1}^{H_{\text{split}}} \circ \gamma_{1;0}^H \left( \nu \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} \circ \nu^{-1}(A_L \otimes A_R)\nu(a) a^\dagger \right).
\]  
(5.127)

Using the fact that \(H\) is translation invariant allows us to cancel it giving
\[= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} \circ \gamma_{0;1}^{H_{\text{split}}} \circ \gamma_{1;0}^H \left( \nu \circ \nu^{-1}(A_L \otimes A_R)\nu(a) a^\dagger \right)
\]  
(5.128)

\[= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}} \circ \nu \circ \nu^{-1}(A_L \otimes A_R)\nu(a) a^\dagger).
\]  
(5.129)

This shows that we can take
\[A := \gamma_{0;1}^{H_{\text{split}}} \circ \nu \circ \gamma_{1;0}^{H_{\text{split}}} \circ \nu^{-1}(A_L \otimes A_R)\gamma_{0;1}^{H_{\text{split}}} \circ \gamma_{1;0}^H(\nu(a) a^\dagger),
\]  
(5.130)

Now we only have to show that indeed \(A \in \mathcal{U}(A_{W(C_g)})\). To do this, take \(X \in A_{W(C_g)}\) arbitrary. If we can show that \(\Ad(A)(X) = X\) for any such \(X\) then we get that \(A \in \mathcal{U}(A_{W(C_g)})\). Since by corollary 3.1 of [24] we get that \(\mathcal{U}(A_{W(C_g)}) = \mathcal{U}(A_{W(C_g)})\) this is sufficient to conclude the proof. To show this, observe that
\[\Ad(A)(X) = \gamma_{0;1}^{H_{\text{split}}} \circ \nu \circ \gamma_{1;0}^{H_{\text{split}}} \circ \nu^{-1} \circ \Ad(A_L \otimes A_R) \circ \nu \circ \gamma_{0;1}^{H_{\text{split}}} \circ \nu^{-1} \circ \gamma_{1;0}^{H_{\text{split}}} \circ \gamma_{1;0}^H(\nu(a) a^\dagger)
\]  
(5.131)

where the cancellation is done using the fact that \(H\) is translation invariant. Using equation (5.116) we obtain
\[\Ad(A)(X) = (\bigotimes_{\mu} \Phi^{\mu})^{-1} \circ \nu \circ \gamma_{0;1}^{H_{\text{split}}} \circ \gamma_{1;0}^{H} \circ \Ad(a) \circ \nu^{-1} \circ \Ad(a^\dagger) \circ \gamma_{0;1}^{H} \circ \gamma_{1;0}^{H_{\text{split}}}(X).
\]  
(5.132)

Now using equation (5.101) we get
\[\Ad(A)(X) = (\bigotimes_{\mu} \Phi^{\mu})^{-1} \circ \nu \circ (\Phi^{U} \otimes \Phi^{D})^{-1} \circ \nu^{-1} \circ \Phi^{U} \otimes \Phi^{D}(X) = X.
\]  
(5.133)

concluding the proof.

---

\[\text{21This is most easily seen from}
\]
\[\Ad(\phi_1(K)) \circ \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}}(A) = \phi_1(\Ad(K) \circ \pi_0 \circ \alpha_0 \circ \Theta(A)) = \phi_1(\pi_0 \circ \alpha_0 \circ \Theta(A)) = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1}^H \circ \gamma_{1;0}^{H_{\text{split}}}(A).
\]  
(5.125)
Lemma 5.26. Define $e^L_g \in \text{IAC}_L(\alpha_0, \theta)$ and $e^R_g \in \text{IAC}_R(\alpha_0, \theta)$ through

$$e^\sigma_g := \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1}(A_\sigma \eta^\sigma_g \circ \beta^U_g(A^\sigma_g)) \quad (5.134)$$

(where $A_\sigma$ was defined in equation (5.116)) then

$$w^\dagger \phi_2(W_g)w = \phi_1(w^\dagger e^L_g \otimes e^R_g) \phi_2(w^\dagger W_g w) \quad (5.135)$$

$$w^\dagger \phi_1(K^\sigma_g)w = v^\dagger \phi_1(e^\sigma_g) v \phi_1(w^\dagger K^\sigma_g w) \phi_1(e^\sigma_g)^\dagger. \quad (5.136)$$

Proof. By equation (5.119) we get that

$$w^\dagger \phi_2(W_g)w = \phi_1(w^\dagger a_L \otimes a_R w) \phi_2(w^\dagger W_g w) \phi_1(w^\dagger a_L^\dagger \otimes a_R^\dagger w) \quad (5.137)$$

$$= \phi_1(w^\dagger a_L \otimes a_R w) \text{Ad}(\phi_2(w^\dagger W_g w)) \phi_1(w^\dagger a_L^\dagger \otimes a_R^\dagger w) \phi_2(w^\dagger W_g w). \quad (5.138)$$

Using result 2 and 3 of lemma 5.23 on this now gives

$$= \phi_1(w^\dagger a_L \otimes a_R \text{Ad}(W_g)(a^\dagger_L \otimes a^\dagger_R)w) \phi_2(w^\dagger W_g w) \quad (5.139)$$

$$= \phi_1(w^\dagger a_L \text{Ad}(W_g)(a^\dagger_L) \otimes a_R \text{Ad}(W_g)(a^\dagger_R)w) \phi_2(w^\dagger W_g w). \quad (5.140)$$

Since

$$a_\sigma \text{Ad}(W_g)(a_\sigma^\dagger) = \pi_0 \circ \alpha_0 \circ \Theta(A_\sigma \eta^\sigma_g \circ \beta^U_g(A^\sigma_g)) \quad (5.141)$$

$$w^\dagger a_\sigma \text{Ad}(W_g)(a_\sigma^\dagger)w = e^\sigma_g \quad (5.142)$$

the first result follows. For the second result we start by using equation (5.118) which gives us

$$w^\dagger \phi_1(K^\sigma_g)w = \phi_1(w^\dagger a_L \otimes a_R K^\sigma_g a^\dagger_L \otimes a^\dagger_R w) \quad (5.143)$$

$$= \phi_1(w^\dagger a_\sigma K^\sigma_g a_\sigma^\dagger w). \quad (5.144)$$

Inserting $v^\dagger W_g v a_\sigma^\dagger a_\sigma v^\dagger W_g^\dagger v = 1$ in this gives us

$$= \phi_1(w^\dagger a_\sigma \text{Ad}(v^\dagger W_g v)(a^\dagger_s) K^\sigma_g a^\dagger_s w) \quad (5.145)$$

$$= \phi_1(w^\dagger a_\sigma \text{Ad}(v^\dagger W_g v)(a^\dagger_s) K^\sigma_g \text{Ad}((K^\sigma_g)^\dagger v^\dagger W_g v)(a_\sigma)a^\dagger_s w). \quad (5.146)$$

We now use the fact that $K_g W_g = v^\dagger W_g v$ to get

$$= \phi_1(w^\dagger a_\sigma \text{Ad}(v^\dagger W_g v)(a^\dagger_s) K^\sigma_g \text{Ad}(W_g)(a_\sigma)a^\dagger_s w) \quad (5.147)$$

$$= \phi_1(w^\dagger a_\sigma \text{Ad}(v^\dagger W_g v)(a^\dagger_s) K^\sigma_g (e^\sigma_g)^\dagger w). \quad (5.148)$$

Inserting $ww^\dagger$ in this and using the second result of lemma 5.23 gives

$$= \phi_1(w^\dagger a_\sigma \text{Ad}(v^\dagger W_g v)(a^\dagger_s) w w^\dagger K^\sigma_g w (e^\sigma_g)^\dagger w) \quad (5.149)$$

$$= \phi_1(w^\dagger a_\sigma \text{Ad}(v^\dagger W_g v)(a^\dagger_s) w^\dagger K^\sigma_g w) \phi_1(w^\dagger (e^\sigma_g)^\dagger w) \quad (5.150)$$

$$= v^\dagger \phi_1(w^\dagger a_\sigma \text{Ad}(v^\dagger W_g v)(a^\dagger_s) w^\dagger) v \phi_1(w^\dagger K^\sigma_g w) \phi_1(w^\dagger (e^\sigma_g)^\dagger w). \quad (5.151)$$

The only thing we now have to prove is that

$$a_\sigma \text{Ad}(v^\dagger W_g v)(a^\dagger_s) = v^\dagger a_\sigma \text{Ad}(W_g)(a^\dagger_s) v = v^\dagger w e^\sigma_g w^\dagger v \quad (5.152)$$

since inserting this, concludes the proof. To do this notice that we can rewrite equation (5.152) as

$$\text{Ad}(v^\dagger W_g v)(v^\dagger a^\dagger_\sigma v a_\sigma) = v^\dagger a^\dagger_\sigma v a_\sigma. \quad (5.153)$$
We will prove this equality in two steps. First we will find some $B_σ \in \mathcal{U}(A)$ such that
\[ \pi_0 \circ \alpha_0 \circ \Theta(B_σ) = v^i a^i_{\sigma} v a_\sigma \quad (5.154) \]
then we will show that $B_σ \in \mathcal{A}(W(C_\sigma \cap \sigma))$ and that $\beta^U_g(B_σ) = B_σ$. To find $B_σ$, we write out
\[ v^i a^i_{\sigma} v a_\sigma = \pi_0 \circ \alpha_0 \circ \Theta(\nu^{-1}(A^\dagger_0) A_\sigma) \quad (5.155) \]
and observe that we can take $B_σ := \nu^{-1}(A^\dagger_0) A_\sigma$. Now we will prove that $B_σ \in \mathcal{U}(\mathcal{A}(W(C_\sigma \cap \sigma)^c)) = \mathcal{U}(\mathcal{A}(W(C_\sigma \cap \sigma)^c))$ arbitrary. Using equation \[\text{[5.116]}\] we have that
\[
\text{Ad}(B_σ)(X) = \tau^{-1} \circ \text{Ad}(A^\dagger_0) \circ \tau \circ \text{Ad}(A_σ)(X) = \tau^{-1} \circ \bigotimes_{\mu \nu} \Phi_{\mu \nu} \circ \alpha^{H_{\text{split}}}_{0;1} \circ \mu \circ \alpha^{H_{\text{split}}}_{0;1} \circ \alpha_{0;1} \circ \Theta^{-1} \circ \bigotimes_{\mu \nu} \Phi^{-1}_{\mu \nu}(X) = X
\]
concluding the proof. The fact that $\text{Ad}\left(\beta^U_g(B_σ)\right) = \text{Ad}(B_σ)$ follows from this same equation as well. This already shows that there exists a $G$-dependent phase $\alpha(g)$ such that $\beta^U_g(B_σ) B^\dagger_0 = \alpha(g) \mathbb{1}_A$. To show that $\alpha(g) = 1$ we again have to show that this expression is self adjoint which is a one line calculation. \[\square\]

**Theorem 5.27.** Index\text{trans}_2(\omega) = \text{Index}\text{trans}_2(\omega \circ \gamma^H_{0;1}).

**Proof.** The start of the proof is identical to the proof of theorem 4.27. Take $\alpha \in \text{QAut}_1(A)$ such that $\omega = \omega_0 \circ \alpha$ for some product state $\omega_0$. Take $0 < \theta_1 < \theta_2 < \theta_3 < \pi/2$ arbitrary. Take $\alpha_L \in \text{Aut}(A_L)$, $\alpha_R \in \text{Aut}(A_R)$, $V_1 \in \mathcal{U}(A)$ and $\Theta \in \text{Aut}(\mathcal{A}(W(C_\sigma \cap \sigma))$ such that
\[ \alpha = \text{Ad}(V_1) \circ \alpha_L \otimes \alpha_R \circ \Theta. \quad (5.159) \]
Take $\tilde{\beta}_{g,1}$ such that $\omega \circ \tilde{\beta}_{g,1} = \omega$ and such that there exists some $V_{2,g} \in \mathcal{U}(A)$ and some $\eta^g_\sigma \in \text{Aut}(\mathcal{A}_{\sigma}(W(C_\sigma \cap \sigma))$ (for $\sigma \in \{L, R\}$) satisfying
\[ \tilde{\beta}_{g,1} = \text{Ad}(V_{2,g}) \circ \eta^L_g \otimes \eta^R_g \circ \beta^U_g. \quad (5.160) \]
Take $W_0, v, w, K^{L/R}_g$ and $b^{L/R}_g$ operators belonging to these automorphisms. Clearly $\omega \circ \gamma^H_{0;1}$ satisfies $\omega \circ H_{0;1} = \omega_0 \circ \alpha \circ H_{0;1}$. We also have that $\alpha \circ \gamma^H_{0;1} \in \text{QAut}_1(A)$. To show this notice that
\[ \alpha \circ \gamma^H_{0;1} = \text{Ad}(V_1) \circ (\alpha_L \circ \gamma^L_{0;1} \otimes \alpha_R \circ \gamma^R_{0;1}) \circ \gamma^H_{0;1} \circ \Theta \circ \gamma^H_{0;1} = \text{Ad}(V_1) \circ (\alpha_L \circ \gamma^L_{0;1} \otimes \alpha_R \circ \gamma^R_{0;1}) \circ \gamma^H_{0;1} \circ \gamma^H_{1;0} \circ \Theta \circ \gamma^H_{0;1}. \quad (5.161) \]
Because of lemma \[\ref{D.5}\] and lemma \[\ref{D.4}\] part 2 there exists a $\tilde{\Theta} \in \text{Aut}(\mathcal{A}(W(C_\sigma \cap \sigma))$ and some $\tilde{V}_1, A_1 \in \mathcal{U}(A)$ such that
\[ \alpha \circ \gamma^H_{0;1} = \text{Ad}\left(\tilde{V}_1\right) \circ (\alpha_L \circ \gamma^H_{0;1} \otimes \alpha_R \circ \gamma^H_{0;1}) \circ \tilde{\Theta} = \text{Ad}(A_1) \circ \alpha_L \otimes \alpha_R \circ \Theta \circ \gamma^H_{0;1}. \quad (5.163) \]
If we now define the automorphism $\tilde{\beta}_{g,2} = \gamma^H_{1;0} \circ \tilde{\beta}_{g,1} \circ \gamma^H_{0;1}$ then this indeed satisfies that $\omega \circ \gamma^H_{0;1} \circ \tilde{\beta}_{g,2} = \omega \circ \gamma^H_{0;1}$. Define $\tilde{\eta}^g_\sigma$ through lemma \[\ref{D.12}\]. It satisfies that there exists a $\tilde{V}_{2,g} \in \mathcal{U}(A)$, $A_{2,g} \in \mathcal{U}(\mathcal{A}_{\sigma}(\sigma))$ such that
\[ \gamma^H_{1;0} \gamma^H_{0;1} \circ \tilde{\beta}_{g,2} = \text{Ad}\left(\tilde{V}_{2,g}\right) \circ \tilde{\eta}^L_g \otimes \tilde{\eta}^R_g \circ \beta^U_g \quad (5.165) \]
\[ \tilde{\eta}^g_\sigma = \text{Ad}(A_{2,g}) \circ \gamma^H_{1;0}(\sigma) \circ \tilde{\eta}^g_\sigma \circ \beta^U_g \circ \gamma^H_{0;1}(\sigma) \circ (\beta^U_g)^{-1}. \quad (5.166) \]
Let $\Phi$ satisfying equation (5.101) arbitrary and define

$$
\delta_g^\alpha := \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_0^H \text{split} (A_{2,g}^\alpha).
$$

Let

$$
\phi_1 : \text{IAC}_R(\alpha_0, \Theta) \to \mathcal{U}(\mathcal{H})
$$

be the homomorphism defined from lemma 5.20. Take similarly

$$
\phi_2 : \{\text{Ad}(\omega^{-p}v^{-s})|W_g|g \in G, p, s \in \{-1, 0, 1\}\} \to \mathcal{U}(\mathcal{H})
$$

the map defined in 5.22. The following operators now belong to $\omega \circ \gamma_0^H$:

$$
\begin{align*}
\dot{v} &= \pi_0(A_1)v\pi_0(A_1)^\dagger \\ \dot{w} &= \pi_0(A_1)w\pi_0(A_1)^\dagger \\ \dot{W}_g &= \pi_0(A_1)\phi_1(\delta_g^L \otimes \delta_g^R)\phi_2(W_g)\pi_0(A_1)^\dagger \\ \dot{K}_g^\alpha &= \pi_0(A_1)\phi_1(v^\dagger \delta_g^L vK_g^\alpha (\delta_g^R)^\dagger)\pi_0(A_1)^\dagger \\ \dot{b}_g^\alpha &= \pi_0(A_1)w^\dagger \phi_1(\delta_g^R)\phi_1(\epsilon_g^R)\phi_1(b_g^R)\phi_1(\delta_g^R)^\dagger \pi_0(A_1)^\dagger \\ \dot{h}_g &= \pi_0(A_1)\phi_1(h_g)\pi_0(A_1).
\end{align*}
$$

First we will show that this transformation indeed leaves the index invariant, then we will show that these operators indeed belong to $\omega \circ \gamma_0^H$. Let $\alpha \in \text{hom}(G, U(1))$ be the old index and $\tilde{\alpha} \in \text{hom}(G, U(1))$ the new one (of course I can only speak about the new index if these new operators are operators belonging to $\omega \circ \gamma_0^H$. Let us assume this for now and prove it later.). We have that

$$
\alpha(g) = v^\dagger b_g^R v h_g K_g^\alpha (b_g^R)^\dagger w^\dagger (K_g^R)^\dagger w \\
= \phi_1(v^\dagger b_g^R v h_g K_g^\alpha (b_g^R)^\dagger w^\dagger (K_g^R)^\dagger w) \\
= \phi_1(v^\dagger b_g^R v h_g K_g^\alpha (b_g^R)^\dagger) \phi_1(w^\dagger (K_g^R)^\dagger w).
$$

Using equation (5.136) on this gives us

$$
\begin{align*}
&= \phi_1(v^\dagger b_g^R v h_g K_g^\alpha (b_g^R)^\dagger) \phi_1(\epsilon_g^R) v w^\dagger \phi_1((K_g^R)^\dagger) w \\
&= \phi_1(v^\dagger \epsilon_g^R b_g^R v h_g K_g^\alpha (b_g^R)^\dagger) \epsilon_g^R v w^\dagger \phi_1((K_g^R)^\dagger) w.
\end{align*}
$$

This shows that the index is invariant under the $\phi_1$ and $\epsilon_g^R$ part (simultaneously). The fact that the index is invariant under the $\phi_1(\delta_g)$ part is now completely equivalent to the proof of lemma 5.13. The fact that the index remains unchanged by the $\text{Ad}(\pi_0(A_1))$ transformation is trivial. This concludes this part of the proof. Now we only need to show that these operators indeed belong to $\omega \circ \gamma_0^H$. The proof that the first four operators belong to $\omega \circ \gamma_0^H$ is completely analogous to what was done in the proof of 4.27 and we will therefore focus on the $\dot{b}_g^\alpha$ part. To show this, observe
that we have

\[
\text{Ad}(\phi_1(\epsilon g^R g)) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1} \circ H_{\text{split}} \circ \nu^{-1} \circ \text{Ad}(A_R \eta_g^R \circ \beta_g^R \nu \gamma_{1;0}) \circ \nu^{-1} \circ \eta_g^R \circ \nu
\]  

(5.181)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1} \circ H_{\text{split}} \circ \gamma_{1;0} \circ \nu^{-1} \circ \text{Ad}(A_R \eta_g^R \circ \beta_g^R \nu \gamma_{1;0}) \circ \nu_g^R \circ \beta_g^R \circ \nu
\]  

(5.182)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1} \circ H_{\text{split}} \circ \gamma_{1;0} \circ \nu^{-1} \circ \text{Ad}(A_R \eta_g^R \circ \beta_g^R \nu \gamma_{1;0}) \circ \nu_g^R \circ \beta_g^R \circ \nu
\]  

(5.183)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1} \circ H_{\text{split}} \circ \gamma_{1;0} \circ \nu^{-1} \circ \text{Ad}(A_R \eta_g^R \circ \beta_g^R \nu \gamma_{1;0}) \circ \nu_g^R \circ \beta_g^R \circ \nu
\]  

(5.184)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1} \circ H_{\text{split}} \circ \gamma_{1;0} \circ \nu^{-1} \circ \text{Ad}(A_R \eta_g^R \circ \beta_g^R \nu \gamma_{1;0}) \circ \nu_g^R \circ \beta_g^R \circ \nu
\]  

(5.185)

After using the restriction to \( R \) of equation (5.16), this leads to

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \gamma_{0;1} \circ H_{\text{split}} \circ \left( \text{Ad}(A_R \gamma_{1;0} \circ \nu_g^R \circ \beta_g^R \circ \nu \gamma_{1;0}) \circ \nu \right)
\]  

(5.186)

On the other hand we have that

\[
\text{Ad}(w^\dagger \phi_1(\delta_g)w) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \gamma_{0;1} \circ \text{Ad}(A_{2,g}^{\dagger}) \circ \gamma_{1;0} \circ \nu.
\]  

(5.187)

This leads us to

\[
\text{Ad}(w^\dagger \phi_1(\delta_g^R)w \phi_1(\epsilon g^R g)) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \gamma_{0;1} \circ H_{\text{split}} \circ \left( \text{Ad}(A_{2,g}^{\dagger}) \circ \gamma_{1;0} \circ \nu_g^R \circ \beta_g^R \circ \nu \gamma_{1;0} \circ \nu \right)
\]  

(5.188)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \gamma_{0;1} \circ H_{\text{split}} \circ \nu_g^R \circ \beta_g^R \circ \nu \gamma_{1;0} \circ \nu
\]  

(5.189)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \gamma_{0;1} \circ H_{\text{split}} \circ \nu_g^R \circ \beta_g^R \circ \nu \gamma_{1;0} \circ \nu
\]  

(5.190)

Now define \( S = \{(0, y) | y \in Z\} \) then

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1} \circ \nu^{-1} \circ \nu_g^R \circ \beta_g^R \circ \nu \gamma_{1;0} \circ \nu
\]  

(5.191)

Doing the same thing on the right will give us

\[
\text{Ad}(w^\dagger \phi_1(\delta_g^R)w \phi_1(\epsilon g^R g) \phi_1(\delta_g^R)^{\dagger}) \circ \pi_0 \circ \alpha_0 \circ \Theta = \pi_0 \circ \alpha_0 \circ \Theta \circ \nu^{-1} \circ \gamma_{0;1} \circ H_{\text{split}} \circ \nu_g^R \circ \beta_g^R \circ \nu \gamma_{1;0} \circ \nu
\]  

(5.192)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1} \circ \nu^{-1} \circ \nu_g^R \circ \beta_g^R \circ \nu \gamma_{1;0} \circ \nu \circ \gamma_{0;1} \circ H_{\text{split}} \circ \nu \circ \gamma_{1;0} \circ H_{\text{split}} \circ \nu \circ \gamma_{1;0} \circ H_{\text{split}} \circ \nu
\]  

(5.193)

\[
= \pi_0 \circ \alpha_0 \circ \Theta \circ \gamma_{0;1} \circ \nu^{-1} \circ \nu_g^R \circ \beta_g^R \circ \nu \gamma_{1;0} \circ \nu \circ \gamma_{0;1} \circ H_{\text{split}} \circ \nu \circ \gamma_{1;0} \circ H_{\text{split}} \circ \nu
\]  

(5.194)

The role that the \( A_1 \) will play is analogous to the proof of (5.27), concluding the proof. 

\[\square\]
5.5 The $H^1(G, \mathbb{T})$-valued index is invariant under rotations by $90^\circ$

Take

$$\omega \in \left\{ \omega \in \mathcal{P}(A) \middle| \omega, \omega \circ \tau = \omega \text{ and } \omega \circ \nu = \omega \right\}$$

(5.195)

arbitrary and let $\mu$ be the automorphism that rotates the lattice by $90^\circ$. This section will be dedicated to proving that

$$\text{Index}_{2 \text{ trans}}^{A,U(g)}(\omega) = \text{Index}_{2 \text{ trans}}^{A,U(g)}(\omega \circ \mu).$$

(5.196)

In what follows, let $\omega_0$ be the product state and let $\gamma^\Phi_{0,1}$ be the disentangler such that

$$\omega = \omega_0 \circ \gamma^\Phi_{0,1}.$$  (5.197)

Sometimes we will need a specific decomposition of this locally generated Automorphism:

**Lemma 5.28.** Let $\{1, 2, 3, 4\}$ be regions indicated in figure 4 (including the red areas) and let $A_{\text{red}}$ be the part of the $C^*$ algebra that has support in the red regions of figure 4, there exists a $B \in \mathcal{U}(A)$ and a $\Theta \in \text{Aut}(A_{\text{red}})$ such that $\beta^\mu_\nu \circ \Theta = \Theta \circ \beta^\mu_\nu$ (for all $\mu \in \{U, D\}$ and $\nu \in \{L, R\}$) satisfying

$$\gamma^\Phi_{0,1} = \text{Ad}(B) \circ \gamma^{\sum_{i=1}^4 \Phi_i}_{0,1} \circ \Theta$$

where $\Phi_i$ is the restriction of $\Phi$ to region $i \in \{1, 2, 3, 4\}$.

**Proof.** Completely analogous to the proof of (for example) 5.199

Let $A_{g,1}, A_{g,2} \in \mathcal{U}(A), \eta^g_\sigma \in \text{Aut}(A_{\sigma^{\mu}(C_\theta \cap \sigma)})$ and $\eta^g_\rho \in \text{Aut}(A_{\sigma^{\mu}(C_\theta \cap \rho)})$ be such that the automorphisms

$$\tilde{\beta}_{g,1} = \text{Ad}(A_{g,1}) \circ \eta^L_g \otimes \eta^R_g \circ \beta^U_g$$

and

$$\tilde{\beta}_{g,2} = \text{Ad}(A_{g,2}) \circ \eta^U_{2,g} \otimes \eta^D_{2,g} \circ \beta^L_g$$

(5.199)

satisfy $\omega \circ \tilde{\beta}_{g,1} = \omega \circ \tilde{\beta}_{g,2} = \omega$. Let $(\mathcal{H}_0 = \bigotimes_i \mathcal{H}_{0,i}, \pi_0 = \bigotimes_{i=1}^4 \pi_{0,i}, \bigotimes_i \Omega_{0,i})$ be a GNS triple for $\omega_0$ (see the caption of figure 4). The following now holds:

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**Lemma 5.29.** There exists an \( X_g \in \mathcal{U}(\mathcal{H}_0) \) satisfying that
\[
\text{Ad}(X_g) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \gamma^\Phi_{0;1} \circ \eta_\beta \circ \beta^U_g \circ \beta^R \circ \beta^L_g (5.200)
\]
where \( \eta_g = \beta^U_g \circ \eta_{2,g-1} \circ (\beta^U_g)^{-1} \).

**Proof.** Let \( W_g \) and \( \tilde{W}_g \) be such that
\[
\text{Ad}(W_g) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \gamma^\Phi_{0;1} \circ \eta_\beta \circ \beta^U_g \quad \text{Ad}(\tilde{W}_g) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \gamma^\Phi_{0;1} \circ \eta_{2,g} \circ \beta^L_g (5.201)
\]
then
\[
\text{Ad}(W_g \tilde{W}_g^{-1}) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \gamma^\Phi_{0;1} \circ \eta_\beta \circ \beta^U \circ \eta_{2,g} \circ (\beta^L_g)^{-1} (5.202)
\]
\[
= \pi_0 \circ \gamma^\Phi_{0;1} \circ \eta_\beta \circ \beta^RU \circ (\beta^LD_g)^{-1}. (5.203)
\]
Now we use lemma 5.28 and obtain
\[
\text{Ad}(W_g \tilde{W}_g^{-1}) \circ \tau \circ \text{Ad}(B) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \tau \circ \gamma^\Phi_{0;1} \circ \eta_\beta \circ \beta^R \circ (\beta^LD_g)^{-1} (5.204)
\]
\[
\text{Ad}(\tilde{W}_g) \circ \tau \circ \text{Ad}(B) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \tau \circ \gamma^\Phi_{0;1} \circ \eta_\beta \circ \beta^R \circ (\beta^LD_g)^{-1} (5.205)
\]
\[
\text{Ad}(\pi(0 \beta^1)W_g \tilde{W}_g^{-1} \pi_0(B)) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \tau \circ \gamma^\Phi_{0;1} \circ \eta_\beta \circ \beta^R \circ (\beta^LD_g)^{-1} (5.206)
\]
\[
\times (\pi_0 \circ \pi_0 \circ \gamma^\Phi_{0;1} \circ \eta_\beta \circ \beta^R \circ (\beta^LD_g)^{-1} \circ \gamma^\Phi_{0;1})
\]
By appendix \( \mathbf{A.3} \) there exist \( \tilde{X}_g \in \mathcal{U}(\mathcal{H}_{0,1} \otimes \mathcal{H}_{0,2}) \) and \( \tilde{Y}_g \in \mathcal{U}(\mathcal{H}_{0,3} \otimes \mathcal{H}_{0,4}) \) such that
\[
\tilde{X}_g \otimes \tilde{Y}_g = \pi_0(\beta^1)W_g \tilde{W}_g^{-1} \pi_0(B). (5.207)
\]
Define \( X_g = \pi_0(B)(\tilde{X}_g \otimes \mathbb{1}) \pi_0(\beta^1) \). This indeed satisfies equation (5.200) concluding the proof. \( \square \)

**Lemma 5.30.** Let \( v, w, K^R_g \) and \( b^R_g \) be such that
\[
\text{Ad}(v) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \gamma^\Phi_{0;1} \circ \tau (5.208)
\]
\[
\text{Ad}(w) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \gamma^\Phi_{0;1} \circ \nu (5.209)
\]
\[
\text{Ad}(K^R_g) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \tau \circ \gamma^\Phi_{0;1} \circ \tau^{-1} \circ \eta^R \circ \beta^R \circ (\beta^R)^{-1} \circ (\eta^R)^{-1} (5.210)
\]
\[
\text{Ad}(b^R_g) \circ \tau \circ \gamma^\Phi_{0;1} = \pi_0 \circ \gamma^\Phi_{0;1} \circ \nu^{-1} \circ \eta^R \circ \nu \circ (\eta^R)^{-1} (5.211)
\]
then
\[
v^t b^R_g v h_g K^R_g (b^R_g)^t = \alpha(g) w^t K^R_g w (5.212)
\]
holds. Moreover \( \alpha = \text{Index}_2 \text{trans}(\omega) \). If one now defines \( \tilde{b}^R_g \) and \( \tilde{K}^U_g \) such that
\[
\tilde{b}^U_g K^R_g X_g = v^t X_g v (5.213)
\]
\[
b^R_g \tilde{K}^U_g X_g = w^t X_g w (5.214)
\]
then
\[
w^t \tilde{b}^U_g w h_g \tilde{K}^U_g (\tilde{b}^U_g)^t = \tilde{\alpha}(g) v^t \tilde{K}^U_g v (5.215)
\]
holds. Moreover \( \tilde{\alpha} = \text{Index}_2 \text{trans}(\omega \circ \mu) \).
Proof. The existence of the \( v, w, K_g^R \) and \( b_g^R \) is due to the fact that \( \gamma_{0;1}^\Phi \in \text{QAut}_1(\mathcal{A}) \). This follows from lemma D.4 and from the fact that \( \text{SQAut}(\mathcal{A}) \subset \text{QAut}_1(\mathcal{A}) \). By construction this also implies that indeed \( \alpha = \text{Index}_{2 \text{ trans}}(\omega) \). We now only have to show the second part of the proof. We have that

\[
\text{Ad}\left(K_g^U \right) \circ \pi_0 \circ \gamma_{0;1}^\Phi = \text{Ad}\left((b_g^R)^{i}w^{i}X_{g}wX_{g}^{i} \right) \circ \pi_0 \circ \gamma_{0;1}^\Phi
\]

(5.216)

\[
= \text{Ad}\left((b_g^R)^{i}w^{i}X_{g}w \right) \circ \pi_0 \circ \gamma_{0;1}^\Phi \circ (\beta_g^{RU})^{-1} \circ (\eta_g^{U})^{-1}
\]

(5.217)

\[
= \text{Ad}\left((\hat{b}_g^U)^{i} \right) \circ \pi_0 \circ \gamma_{0;1}^\Phi \circ \nu^{-1} \circ \bar{\eta}_g^{U} \circ \eta_g^{R} \circ \nu \circ (\beta_g^{RU})^{-1} \circ (\eta_g^{U})^{-1}
\]

(5.218)

\[
= \pi_0 \circ \gamma_{0;1}^\Phi \circ \nu^{-1} \circ \bar{\eta}_g^{U} \circ \beta_g^{RU} \circ \nu \circ (\beta_g^{RU})^{-1} \circ (\eta_g^{U})^{-1}.
\]

(5.219)

By similar arguments we get that

\[
\text{Ad}\left(\hat{b}_g^U \right) \circ \pi_0 \circ \gamma_{0;1}^\Phi = \pi_0 \circ \gamma_{0;1}^\Phi \circ \tau^{-1} \circ \bar{\eta}_g^{U} \circ \tau \circ (\eta_g^{U})^{-1}.
\]

(5.220)

This also shows that

\[
\text{Ad}\left(K_g^U \right) \circ \pi_0 \circ \gamma_{0;1}^\Phi \circ \mu = \pi_0 \circ \gamma_{0;1}^\Phi \circ \mu \circ \mu^{-1} \circ \nu^{-1} \circ \bar{\eta}_g^{U} \circ \beta_g^{RU} \circ \nu \circ (\beta_g^{RU})^{-1} \circ (\eta_g^{U})^{-1} \circ \mu
\]

(5.221)

and similarly for \( \hat{b}_g^U \) and therefore these operators also belong to \( \omega \circ \mu \). This together with the independence of the choice of \( \bar{\eta}_g \) proves the second result.

Lemma 5.31. The following equalities hold

\[
[h_g, w^{i}\hat{b}_g^U w] = 0 \quad (5.222)
\]

\[
[w^{i}\hat{b}_g^U w, v^{i}\hat{b}_g^R v] = 0 \quad (5.223)
\]

\[
[h_g, v^{i}\hat{b}_g^R v] = 0 \quad (5.224)
\]

\[
[h_g, K_g^R] = 0 \quad (5.225)
\]

\[
[K_g^R, v^{i}\hat{K}_g^U] = 0 \quad (5.226)
\]

\[
[K_g^R, v^{i}\hat{K}_g^U v] = 0 \quad (5.227)
\]

Proof. All of the above commute as automorphisms on the \( C^* \) algebra, we now have to show that the operators in the GNS space commute as well. First we will start by proving equations (5.222), (5.224) and (5.226). For any operator \( x \in \mathcal{U}(\mathcal{H}_0) \) satisfying that there exists a \( \xi \in \text{Aut}(\mathcal{A}) \) such that

\[
\text{Ad}(x) \circ \pi_0 \circ \gamma_{0;1}^\Phi = \pi_0 \circ \gamma_{0;1}^\Phi \circ \xi
\]

(5.228)

we get that

\[
\text{Ad}(x)(h_g) = \pi_0 \circ \gamma_{0;1}^\Phi \circ \xi(U_{(-1,-1)}(g)).
\]

(5.229)

Applying this to these equations gives the desired result. To show the remaining equations decompose \( \pi_0 \) and \( \gamma_{0;1}^\Phi \) as

\[
\gamma_{0;1}^\Phi = \text{Ad}(B) \circ \bigotimes_{i=1}^{4} \gamma_{0;1}^\phi \circ \Theta
\]

(5.230)

\[
\pi_0 = \bigotimes_{i=1}^{4} \pi_{0;i}
\]

where the \( \pi_{0;i} \) are irreducible representations in \( \mathcal{A}_i \), the areas \( i \) are indicated in figure 4 and the \( \Theta \) have support in the areas shaded red. This shows that up to some inner automorphisms the operators with \( R \) on top act on different parts of the decomposition of \( \pi_0 \) then the operators with a \( U \) on top do. Combining this with lemma A.3 concludes the proof.

Lemma 5.32. \( \hat{\alpha}(g) = \alpha(g) \).
Proof. We will work out the expression

\[ h_g v^\dagger R v w^\dagger \tilde{b}_g^U w K_g^R \tilde{K}_g^U X_g \]  

in two different ways. On the one hand we will use (5.223) together with equation (5.222) to get

\[ \alpha(g) w^\dagger b_g^U w K_g^R \tilde{K}_g^U X_g \]  

Now using equation (5.212) we get

\[ \alpha(g) w^\dagger b_g^U w K_g^R \tilde{K}_g^U X_g \]

where we’ve used equation (5.214) in the first equality and (5.213) in the second equality. On the other hand, starting with equation (5.224), (5.225) and (5.226) gives

\[ \alpha(g) w^\dagger b_g^U w K_g^R \tilde{K}_g^U X_g \]

Now using equation (5.215) we get

\[ \alpha(g) w^\dagger b_g^U w K_g^R \tilde{K}_g^U X_g \]

where we’ve used equation (5.227) and (5.225) in the first equality, equation (5.213) in the second equality and equation (5.214) in the last equality. These two results can only be consistent if \( \alpha(g) = \tilde{\alpha}(g) \) for all \( g \in G \).

\[ \square \]

5.6 Stacking in the \( H^1(G, \mathbb{T}) \)-valued index

Take \( \mathcal{A} \) a quasi local \( C^* \) algebra on a two dimensional lattice and take \( U(g) \) an on site group action arbitrary. Let

\[ \omega_1, \omega_2 \in \{ \omega \in \mathcal{P}(\mathcal{A}) | \omega \text{ satisfies assumption } 3.9 \} \]

be two states. The goal of this section is to prove that

\[ \text{Index}_{2 \text{trans}}^{A, U(g) \otimes U(g)}(\omega_1 \otimes \text{stack } \omega_2) = \text{Index}_{2 \text{trans}}^{A, U(g)}(\omega_1) \text{Index}_{2 \text{trans}}^{A, U(g)}(\omega_2). \]

The proof of this is completely analogous to what was in section 4.6.

5.7 The \( H^1 \)-valued index is invariant of choice of basis of on site Hilbert space

Let \( V \in U(\mathbb{C}^d) \) be a unitary operator on the on site Hilbert space and take \( i_V \) as defined in subsection 2.1. In this subsection we will prove that

\[ \text{Index}_{2 \text{trans}}^{A, U(V)}(\theta, \tilde{\beta}_g, \eta_0, \alpha_0, \Theta, \omega, \omega_0) \]

The proof is completely analogous to the proof presented in subsection 4.7.
5.8 Proof of theorem 3.4

We will first prove lemma 3.10 which we restate here for convenience:

Lemma 3.10. Let \( \omega \) be a short range entangled state that is \( G \)-invariant and translation invariant in two directions then it satisfies assumption 3.9.

Proof. Completely analogous to the proof of lemma 3.7 except that the support area of the \( \eta_g \) automorphism has become smaller. To deal with this, replace the SQAut(\( \mathcal{A} \)) in theorem 3.1 of [10] by SQAut\(_g\)(\( \mathcal{A} \)) (see definition D.8) and observe that the resulting \( \eta_g \) has the right support. \( \square \)

Proof of theorem 3.4. Take \( \omega \in \{ \omega \in \mathcal{P}(\mathcal{A}) \mid \omega \text{ is an SRE state, } \omega \circ \beta_g = \omega, \omega \circ \tau = \omega \text{ and } \omega \circ \nu = \omega \} \) arbitrary. By lemma 3.10 it satisfies assumption 3.9. Using theorem 3.11 we get that there indeed exists a well defined index that is invariant under locally generated automorphisms and satisfies item 4. By lemma 4.17 we get that the \( \phi \) presented in part 2 of the theorem satisfies assumption 5.16. We can now use the result of section 5.3 to conclude the proof of item 2. Item 3 is proven in section 5.4 concluding the proof of the theorem. \( \square \)

A Representations and states

In this section we summarize all the results we need from [2].

Theorem A.1. Let \( \omega \) be a state on a \( C^* \)-algebra \( \mathcal{A} \). It follows that there exists a cyclic representation \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\) of \( \mathcal{A} \) such that

\[
\omega(A) = \langle \Omega_\omega | \pi_\omega(A) \Omega_\omega \rangle
\]

for all \( A \in \mathcal{A} \). Moreover, the representation is unique up to unitary equivalence.

Proof. Theorem 2.3.16 in [2]. \( \square \)

We will call this cyclic representation a GNS triple for \( \omega \).

Corollary A.2. Let \( \omega \) be a state over the \( C^* \)-algebra \( \mathcal{A} \) and \( \tau \) a \( * \)-automorphism of \( \mathcal{A} \) which leaves \( \omega \) invariant, i.e.,

\[
\omega \circ \tau = \omega.
\]

Let \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\) be a GNS triple for \( \omega \). It follows that there exists a uniquely determined unitary operator \( U_\omega \), such that

\[
U_\omega \pi_\omega(A) U_\omega^\dagger = \pi_\omega \circ \tau(A)
\]

for all \( A \in \mathcal{A} \), and

\[
U_\omega \Omega_\omega = \Omega_\omega.
\]

If furthermore, \( \omega \) is pure then (up to a phase) there even exists a unique \( U_\omega \) without requiring the second condition.

Proof. The proof of this corollary is a combination of Corollary 2.3.17 and Theorem 2.3.19 from [2]. \( \square \)

The following lemma does not come from Bratteli-Robinson but in stead follows from Wigner’s theorem:
Lemma A.3. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary unital $C^*$-algebras. Let $(\mathcal{H}_\mathcal{A}, \pi_\mathcal{A})$ and $(\mathcal{H}_\mathcal{B}, \pi_\mathcal{B})$ be arbitrary irreducible * representations on $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $U \in \mathcal{U}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})$ be such that there exists an $\alpha_\mathcal{A} \in \text{Aut}(\mathcal{A})$ and an $\alpha_\mathcal{B} \in \text{Aut}(\mathcal{B})$ satisfying

$$\text{Ad}(U) \circ (\pi_\mathcal{A} \otimes \pi_\mathcal{B}) = \pi_\mathcal{A} \circ \alpha_\mathcal{A} \otimes \pi_\mathcal{B} \circ \alpha_\mathcal{B}$$

then there exists a $U_\mathcal{A} \in \mathcal{U}(\mathcal{H}_\mathcal{A})$ and a $U_\mathcal{B} \in \mathcal{U}(\mathcal{H}_\mathcal{B})$ such that

$$U = U_\mathcal{A} \otimes U_\mathcal{B}.$$  \hfill (A.2)

Proof. From the assumptions we see that $\forall A \in \mathcal{A}$

$$\text{Ad}(U) \circ \pi_\mathcal{A} \otimes \pi_\mathcal{B}(\mathcal{A} \otimes \mathbb{1}) = \pi_\mathcal{A} \circ \alpha_\mathcal{A}(\mathcal{A}) \otimes \mathbb{1} \subset \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathbb{1}. \hfill (A.3)$$

Since $\text{Ad}(U)$ is continuous in the weak operator topology, it follows that we can extend the map

$$\text{Ad}(U)|_{\text{Im}(\pi_\mathcal{A})} : \text{Im}(\pi_\mathcal{A}) \otimes \mathbb{1} \to \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathbb{1} \hfill (A.4)$$

to the closure in weak operator topology. By irreducibility $\pi_\mathcal{A}(\mathcal{A})'' = \mathcal{B}(\mathcal{H}_\mathcal{A})$ and therefore we get a map

$$\text{Ad}(U)|_{\mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathbb{1}} : \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathbb{1} \to \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathbb{1}. \hfill (A.5)$$

By restriction this gives rise to an automorphism $\tau_\mathcal{A} : \mathcal{U}(\mathcal{H}_\mathcal{A}) \to \mathcal{U}(\mathcal{H}_\mathcal{A})$. By Wigners theorem any automorphism of $\mathcal{B}(\mathcal{H}_\mathcal{A})$ is inner and therefore there exists a $U_\mathcal{A}$ such that $\tau_\mathcal{A} = \text{Ad}(U_\mathcal{A})$. Doing the same on $\mathcal{B}$ gives us similarly a $\tau_\mathcal{B}$ and a $U_\mathcal{B}$. We now get

$$\text{Ad}(U) \circ (\pi_\mathcal{A} \otimes \pi_\mathcal{B}) = \tau_\mathcal{A} \circ \pi_\mathcal{A} \otimes \tau_\mathcal{B} \circ \pi_\mathcal{B} = \text{Ad}(U_\mathcal{A} \otimes U_\mathcal{B}) \circ (\pi_\mathcal{A} \otimes \pi_\mathcal{B}). \hfill (A.6)$$

By the irreducibility of $\pi_\mathcal{A} \otimes \pi_\mathcal{B}$ we get that (up to some irrelevant phase) we need that $U = U_\mathcal{A} \otimes U_\mathcal{B}$ concluding the proof. \hfill \qed

B  Translation invariance in one dimension

Let $\mathcal{B}$ be the quasi local $C^*$ algebra defined with on the one dimensional lattice. Let $U_i \in \text{hom}(G, \mathcal{U}(\mathcal{A}_i))$ (for $i \in \mathbb{Z}$) be the on site group action. Let $\beta_g$ be again so that for each finite $I \subset \mathbb{Z}$, $\beta_I^g = \text{Ad}(\bigotimes_{i \in I} U_i(g))$. In this section, we will look at states satisfying the split property. We say that $\omega \in \mathcal{P}(\mathcal{B})$ satisfies the split property if. it is quasi equivalent to $\omega|_L \otimes \omega|_R$. This implies that there exists a GNS triple of $\omega$ that is of the form

$$(\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R, \pi = \pi_L \otimes \pi_R; \Omega). \hfill (B.1)$$

We will denote the space of pure states satisfying the split property by $\mathcal{SP}(\mathcal{A})$. Now let us additionally ask that $\omega$ is invariant under the group action ($\omega \circ \beta_g = \omega$). The following now holds:

Lemma B.1. There exists maps $u_\sigma : G \to \mathcal{U}(\mathcal{H}_\sigma)$ (for all $\sigma \in \{ L, R \}$) satisfying

$$\text{Ad}(u_\sigma(g)) \circ \pi_\sigma = \pi_\sigma \circ \beta_g \quad u_L(g) \otimes u_R(g) \Omega = \Omega.$$ \hfill (B.2)

They are unique up to a $G$-dependent phase.

Proof. See [9]. \hfill \qed

This can now be used to define the one dimensional SPT index:
Lemma B.2. There exists a 2-cochain $C : G^2 \to U(1)$ that satisfies
\[ u_R(g)u_R(h)u_R(gh) = C(g, h)\mathbb{1}_{H_R} \quad u_L(g)u_L(h)u_L(gh) = C(g, h)^{-1}\mathbb{1}_{H_L}. \] (B.3)

Its second group cohomology class is independent of the choice of GNS triple or the choice of (a $G$-dependent) phase in $u_R(g)$. This index is invariant under locally generated automorphisms generated by $G$-invariant interactions.

Proof. Again see [9].

Now let us assume additionally (on top of the split property and the $G$-invariance) that $\omega$ be translation invariant. To this end, let $\nu$ be the automorphism that translates $B$ by one site to the right. We assume that $\omega \circ \nu = \omega$. The following now holds:

Lemma B.3. There exists a $w \in \mathcal{U}(\mathcal{H})$ such that
\[ \text{Ad}(w) \circ \pi = \pi \circ \nu \quad w\Omega = \Omega. \] (B.4)

This $w$ is unique up to a $G$-dependent phase.

Proof. Since $\omega \circ \nu = \omega$, we get that
\[ (\mathcal{H}, \pi \circ \nu, \Omega) \sim (\mathcal{H}, \pi, \Omega). \] (B.5)

The uniqueness of the GNS triple now implies the existence of this $w$. Its uniqueness follows from the irreducibility of $\pi$.

We can use this to define an additional 1d index for translation invariant states.

Lemma B.4. There exists a (unique) $\alpha \in \text{hom}(G, U(1))$ satisfying that
\[ \pi(U_{-1}(g))(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(g)) = \alpha(g)w^\dagger(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(g))w. \] (B.6)

This index is invariant under locally generated automorphisms generated by interactions that are both $G$ and translation invariant.

Proof. The fact that $\alpha$ exists follows from the fact that $\pi$ is irreducible and combined with the fact that
\[ \text{Ad}(\pi(U_{-1}(g))(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(g))) \circ \pi = \text{Ad}(w^\dagger(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(g))w) \circ \pi = \pi \circ \beta_{g}^{[-1,\infty[}. \] (B.7)

The fact that $\alpha$ is a $U(1)$—representation follows from the fact that $u_R$ is the lift of a projective representation. Suppose that $u_R(g)u_R(h) = C(g, h)u_R(gh)$ then
\[ \pi(U_{-1}(g))(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(g))\pi(U_{-1}(h))(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(h)) = C(g, h)\pi(U_{-1}(gh))(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(gh)) \] (B.8)
on the one hand whereas
\[ w^\dagger(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(g))ww^\dagger(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(h))w = C(g, h)w^\dagger(\mathbb{1}_{\mathcal{H}_L} \otimes u_R(gh))w. \] (B.9)

These two equations can only be consistent if $\alpha(g)\alpha(h) = \alpha(gh)$. The proof that this is independent of the choice of GNS triple is straightforward. Now we need to show that this index is invariant under LGA’s generated by $G$-invariant interactions. To this end let $F$ be a monotonically decreasing positive function decaying faster then any polynomial. Let $\Phi \in \mathcal{B}_F([0, 1])$ be a $G$-invariant, translation invariant interaction. For simplicity of notation let $\Phi_{\text{split}} = \Phi_{U^{-1}(L)} + \Phi_{U(R)}$. Let $A \in \mathcal{U}(\mathcal{A})$ be such that
\[ \gamma_{0, 1}^\Phi = \text{Ad}(A) \circ \gamma_{0, 1}^\Phi_{\text{split}} \quad \beta_g(A) = A. \] (B.10)
By construction \((\mathcal{H}, \pi \circ \gamma_{0:1}^{\Phi_{\text{split}}}, \pi(A^\dagger)\Omega)\) is a GNS triple of \(\omega \circ \gamma_{0:1}^{\Phi}\). Let \(\tilde{w} := \pi(A^\dagger)w\pi(A)\) then

\[
\tilde{\alpha}(g) := \pi(U_{-1}(g))u_R(g)\tilde{w}^1u_R(g)^\dagger \tilde{w} \tag{B.11}
\]

is the index of \(\omega \circ \gamma_{0:1}^{\Phi}\). Working this out gives

\[
\tilde{\alpha}(g) = \pi(U_{-1}(g))u_R(g)\pi(A^\dagger)w^1\pi(A)u_R(g)^\dagger \pi(A^\dagger)w\pi(A) \tag{B.12}
\]

\[
= \pi(U_{-1}(g))u_R(g)w^1\pi(\nu(A^\dagger)A)u_R(g)^\dagger \pi(A^\dagger)\nu(A)w \tag{B.13}
\]

\[
= \pi(U_{-1}(g))u_R(g)w^1u_R(g)^\dagger \pi(\nu(\beta_g^R(\nu(A^\dagger))A)\pi(A^\dagger)\nu(A))w \tag{B.14}
\]

\[
= \alpha(g)\pi \circ \beta_g^R(\nu(A^\dagger)A)\pi(A^\dagger)\nu(A). \tag{B.15}
\]

This shows that we conclude the proof if we can show that

\[
\beta_g^R(\nu(A^\dagger)A) = \nu(A^\dagger)A. \tag{B.16}
\]

We will do this proof in two steps. The first step is that we prove that

\[
\text{Ad}(\beta_g^R(\nu(A^\dagger)A)) = \text{Ad}(\nu(A^\dagger)A). \tag{B.17}
\]

We use the first equation of \((B.10)\) and obtain

\[
\text{Ad}(\beta_g^R(\nu(A^\dagger)A)) = \beta_g^R \circ \text{Ad}(\nu(A^\dagger)A) \circ (\beta_g^R)^{-1} \tag{B.18}
\]

\[
= \beta_g^R \circ \nu \circ \gamma_{0:1}^{\Phi_{\text{split}}} \circ \gamma_{1:0}^{\Phi} \circ \nu^{-1} \circ \gamma_{0:1}^{\Phi_{\text{split}}} \circ (\beta_g^R)^{-1} \tag{B.19}
\]

\[
= \beta_g^R \circ \gamma_{0:1}^{\Phi_{\text{split}}} \circ \gamma_{1:0}^{\Phi} \circ (\beta_g^R)^{-1} \tag{B.20}
\]

\[
= \text{Ad}(\nu(A^\dagger)A). \tag{B.21}
\]

This shows that there exists some \(\beta : G \to U(1)\) such that

\[
\beta(g) = \beta_g^R(\nu(A^\dagger)A)(A^\dagger\nu(A)). \tag{B.22}
\]

However because of equation \((B.17)\) we get that

\[
\beta(g) = \beta_g^R(\nu(A^\dagger)A)(A^\dagger\nu(A)) = (A^\dagger\nu(A))\text{Ad}(A^\dagger\nu(A)) (\beta_g^R(\nu(A^\dagger)A)) = (A^\dagger\nu(A))(\beta_g^R(\nu(A^\dagger)A)) = \beta(g) \tag{B.23}
\]

proving that \(\beta(g) = 1\) and concluding the proof.

All together we constructed a proof of the following theorem:

**Theorem B.5.** There exist maps

\[
\text{Index}_{1d} : \{\omega \in \mathcal{SP}(A) | \omega \circ \beta_g = \omega\} \to H^2(G, T) \tag{B.24}
\]

\[
\text{Index}_{1d \text{trans}} : \{\omega \in \mathcal{SP}(A) | \omega \circ \beta_g = \omega \text{ and } \omega \circ \nu = \omega\} \to H^1(G, T) \cong \text{hom}(G, T). \tag{B.25}
\]

For any \(G\)-invariant family of interactions \(\Phi \in \mathcal{B}_F([0, 1])\), we have that \(\text{Index}_{1d}(\omega) = \text{Index}_{1d}(\omega \circ \gamma_{0:1}^{\Phi})\). For any \(G\)-invariant family of interactions \(\Phi' \in \mathcal{B}_F([0, 1])\) that is also translation invariant, we have that \(\text{Index}_{1d \text{trans}}(\omega) = \text{Index}_{1d \text{trans}}(\omega \circ \gamma_{0:1}^{\Phi'})\).

**C  Properties of locally generated automorphisms: 1d**

In this section, let \(A\) be a quasi local \(C^*\) algebra over \(\mathbb{Z}\).
**Lemma C.1.** Take \( \Phi \in B_{F_{\phi}}([0, 1]) \) (with \( F_{\phi} \) as presented in \((2.14)\)) then there exists an \( A \in \mathcal{U}(\mathcal{A}) \) satisfying
\[
\text{Ad}(A) = \gamma_{0;1}^\Phi + \gamma_{1;0}^\Phi. \tag{C.1}
\]
Moreover, this \( A \) is summable (see definition 4.15).

**Proof.** We have for any \( B \in \mathcal{A} \) that
\[
\gamma_{0;t}^\Phi \circ \gamma_{t;0}^\Phi(B) = B + \int_0^t ds \frac{d}{ds} \left( \gamma_{0;s}^\Phi \circ \gamma_{s;0}^\Phi(B) \right) \tag{C.2}
\]
\[
= B - i \int_0^t ds \sum_{I \in \Phi} \gamma_{0;s}^\Phi \left( \left[ \sum_{I \in \Phi} \gamma_{s;0}^\Phi(\Phi_L(s, I) + \Phi_R(s, I) - \Phi(s, I)) \right], B \right). \tag{C.3}
\]
To simplify notation we will define an interaction
\[
\tilde{\Phi}(s, I) = \Phi_L(s, I) + \Phi_R(s, I) - \Phi(s, I). \tag{C.5}
\]
We will in essence have to find a 1D analogue of the proof of Theorem 5.2 of \([10]\) with a few exceptions. Most notably we need the summability condition. First let us still define the extensions
\[
X(m) := \{ x \in \mathbb{Z} | \exists y \in X : \text{dist}(x, y) \leq m \} \tag{C.6}
\]
and
\[
\Delta X(m) := \Pi X(m) - \Pi X(m-1) \tag{C.7}
\]
the differences between the conditional expectation values. Define an interaction
\[
\Xi(Z, t) := \sum_{m \geq 0} \sum_{X \in \mathbb{Z}} \Delta X(m) (\gamma_{0;1}^\tilde{\Phi}(X(t))) \tag{C.8}
\]
then we get by construction that
\[
\gamma_{0;1}^\Phi \circ \gamma_{t;0}^\Phi = \gamma_{0;t}^\Xi. \tag{C.9}
\]
Define some sets
\[
B_n := [-n, n] \cap \mathbb{Z} \quad B_{n,L} := ] -\infty, -n ] \cap \mathbb{Z} \quad B_{n,R} := ] n, \infty [ \cap \mathbb{Z}. \tag{C.10}
\]
In what follows we will find a bound on
\[
c_n := \sup_{t \in [0, 1]} \| \sum_{Z \in \mathcal{B}_Z} (\Xi(Z, t) - \Xi_{B_n}(Z, t)) \| = \sup_{t \in [0, 1]} \| \Xi_{B_n}(Z, t) \| \tag{C.11}
\]
that is summable. Finding this would conclude the proof. In analogy with equation (5.22) of \([10]\) we get
\[
c_n \leq \sum_{Z \in \mathcal{B}_Z} \sum_{m \geq 0} X_{\text{m} \cap B_{n,L} \neq \emptyset, Z \cap B_{n,R} \neq \emptyset} \| \tilde{\Phi}(X(t)) \| |X| G_F(m). \tag{C.12}
\]
Now using a trick analogous to what was done in equation (5.27) of \[10\] we get

\[
c_n \leq \|\Phi_1\|_F \left( \sum_{m \geq 0} G_F(m) \right) \sum_{x \in B_{n,L}} \sum_{y \in B_{n,R}} F(d(x, y)). \tag{C.14}
\]

To show that the last bound is summable we only have to observe that

\[
\sum_{n=0}^{\infty} \sum_{x \in B_{n,L}} \sum_{y \in B_{n,R}} F(d(x, y)) \leq \sum_{x \in C_1} \sum_{y \in C_2} F(d(x, y)). \tag{C.15}
\]

where \( C_1 \) is the cone one obtains by putting the \( B_{n,L} \) on top of each other whereas \( C_2 \) is the cone one gets by putting the \( B_{n,R} \) on top of each other. The proof that the sum over the cones is bounded is done explicitly in \[10\]. \( \square \)

### D Properties of locally generated automorphisms: 2d

In this section, let \( \mathcal{A} \) be a quasi local \( C^* \) algebra over \( \mathbb{Z}^2 \). Take \( H \in \mathcal{B}_{F_2}([0,1]) \) (for some \( 0 < \phi < 1 \)) and \( \gamma_{st}^H \) its locally generated automorphism (see section 2.3). We will now show some properties of this locally generated automorphism \( \gamma_{st}^H \). In this section we will heavily rely on the framework and theorems of Yosiko Ogata \[10\] who in her turn relies heavily on the framework developed in \[8\]. We will use the notation

\[
C_{[0,\theta]} := C_\theta \quad C_{[\theta_1,\theta_2]} := C_{\theta_2}/C_{\theta_1} \quad C_{[\theta,\pi/2]} := \mathbb{Z}^2/C_\theta
\]

where \( 0 < \theta < \pi/2 \) and \( 0 < \theta_1 < \theta_2 < \pi/2 \).

#### D.1 The one translation symmetry

This section will define the classes of automorphisms required in the assumptions and proofs about the \( H^2 \)-valued translation index. In analogy with section 2.1 from \[10\] we will define some classes of automorphisms (see figure 3):

**Definition D.1.** Take \( \alpha \in \text{Aut}(\mathcal{A}) \). We say that \( \alpha \in \text{SQAut}_1(\mathcal{A}) \) if and only if for any \( 0 < \theta_{0.8} < \theta_1 < \theta_{1.2} < \theta_{1.8} < \theta_3 < \theta_{2.8} < \theta_2 < \theta_{3.2} < \pi/2 \) there exist \( a \in \mathcal{A}, \alpha_{[0,\theta_1],r} \in \text{Aut}(\mathcal{A}_{[0,\theta_1]/C_\theta}), \alpha_{[\theta_1,\theta_2],r,\rho} \in \text{Aut}(\mathcal{A}_{[\theta_1,\theta_2]/C_\theta} \cap \mathcal{A}_{[\rho,\rho]}), \alpha_{[\theta_2,\theta_3],r} \in \text{Aut}(\mathcal{A}_{[\theta_2,\theta_3]/C_\theta} \cap \mathcal{A}_{[\rho,\rho]}), \alpha_{[\theta_3,\pi/2],r} \in \text{Aut}(\mathcal{A}_{[\theta_3,\pi/2]/C_\theta} \cap \mathcal{A}_{[\rho,\rho]}), \alpha_{[\theta_1,\theta_2,\theta_3],r,\rho,\sigma} \in \text{Aut}(\mathcal{A}_{[\theta_1,\theta_2,\theta_3]/C_\theta} \cap \mathcal{A}_{[\rho,\rho]}), \alpha_{[\theta_1,\theta_2,\theta_3,\pi/2],r,\rho,\sigma} \in \text{Aut}(\mathcal{A}_{[\theta_1,\theta_2,\theta_3,\pi/2]/C_\theta} \cap \mathcal{A}_{[\rho,\rho]}). \tag{D.1}
\]

\[
\alpha = \text{Ad}(a) \circ \bigotimes_{r} \alpha_{[0,\theta_1],r} \circ \bigotimes_{r,\rho} \alpha_{[\theta_1,\theta_2],r,\rho} \circ \bigotimes_{r,\rho,\sigma} \alpha_{[\theta_2,\theta_3],r,\rho,\sigma} \circ \bigotimes_{r,\rho,\sigma,\tau} \alpha_{[\theta_1,\theta_2,\theta_3],r,\rho,\sigma,\tau}. \tag{D.2}
\]

If additionally everything (except \( a \)) commutes with \( \beta^U \) we say that \( \alpha \in \text{GSAut}_1(\mathcal{A}) \).

As a remark, notice that this definition is not exactly identical to de definition given in \[10\] due to the translation operators.

**Definition D.2.** Take \( \alpha \in \text{Aut}(\mathcal{A}) \). We say that \( \alpha \in \text{HAut}_1(\mathcal{A}) \) if and only if for any \( 0 < \theta < \pi/2 \) there exists an \( a \in \mathcal{A} \) and some \( \alpha_{\sigma} \in \text{Aut}(\mathcal{A}_{[0,\theta]}/\mathcal{A}_{[\rho,\rho]}) \) for each \( \sigma \in \{L, R\} \) such that

\[
\alpha = \text{Ad}(a) \circ \alpha_L \otimes \alpha_R. \tag{D.3}
\]
We will need one additional class of automorphisms:

**Definition D.3.** Take $\alpha \in \text{Aut}(A)$. We say that $\alpha \in \text{VAut}_1(A)$ if and only if there exists some $a \in A$, $\alpha_U \in \text{Aut}(\tau(\mathcal{A}_C \cap \mathcal{A}_D))$ and an $\alpha_D \in \text{Aut}(\tau^{-1}(\mathcal{A}_C \cap \mathcal{A}_D))$ such that

$$\alpha = \text{Ad}(a) \circ \alpha_U \otimes \alpha_D.$$  

(D.4)

If furthermore $\alpha_U \circ \beta_g^U = \beta_g^U \circ \alpha_U$ we say that $\alpha \in \text{GVAut}_1(A)$.

Now we will state some properties of locally generated automorphisms:

**Lemma D.4.** Take $H$ an interaction such that there exists a $0 < \phi < 1$ satisfying that $\|H\|_{f_\phi} \leq 1$. The following statements now hold (for any $s, t \in \mathbb{R}$):

1. $\gamma_{s,t}^H \circ \gamma_{t,s}^D \otimes \gamma_{t,s}^D \in H\text{Aut}_1(A)$.
2. $\gamma_{s,t}^H \circ \gamma_{t,s}^D \otimes \gamma_{t,s}^D \in \text{VAut}_1(A)$. If additionally $H$ is a $G$-invariant interaction we even have $\gamma_{s,t}^H \circ \gamma_{t,s}^D \otimes \gamma_{t,s}^D \in \text{GVAut}_1(A)$.
3. $\gamma_{s,t}^H \otimes \gamma_{s,t}^D \in \text{SQAut}_1(A)$. If additionally $H$ is a $G$-invariant interaction we even have $\gamma_{s,t}^H \otimes \gamma_{s,t}^D \in \text{GSQAut}_1(A)$.
4. If $H$ is $G$-invariant then $\gamma_{s,t}^H \circ \beta_g^U \circ \gamma_{s,t}^D \circ (\beta_g^U)^{-1} \in H\text{Aut}_1(A)$.
5. $\gamma_{s,t}^H \in \text{SQAut}_1(A)$.

*Proof.* Part 1 is done in Proposition 5.5 in [10] and part 4 follows trivially from Part 1. Except for the translation operators in my definition of the $\text{SQAut}(A)$, part 3 follows from Theorem 5.2 in [10]. To show part 3 we therefore have to show that Theorem 5.2 in [10] still holds if we replace $\mathcal{C}_0$ and $\mathcal{C}_1$ in the proof by

$$\mathcal{C}_0 := \{ C_{[0,\theta_1],\sigma}, C_{[\theta_1,\theta_2],\sigma,\rho}, C_{[\theta_2,\theta_3],\sigma,\rho}, C_{[\theta_3,\pi/2],\sigma,\rho} \cup \tau^\rho(C_{[\theta_2,\theta_3],\sigma,\rho}), \tau^\rho(C_{[\theta_3,\pi/2],\sigma,\rho}) | \sigma = \{ L, R \}, \rho = \{ U, D \} \} \quad \text{(D.5)}$$

$$\mathcal{C}_1 := \{ C_{[0,\theta_1],\sigma} \cap \tau^\sigma \tau, C_{[\theta_1,\theta_2],\sigma} \cap \tau^\sigma \tau, \tau^\rho(C_{[\theta_2,\theta_3],\sigma} \cap \tau^\sigma \tau) \cap \tau^\rho \sigma \} = \{ L, R \}, \rho = \{ U, D \}. \quad \text{(D.6)}$$

Take the $\Psi$ from this proof to be our $H$, take $\Psi^{(0)}$ to be $\sum_{C \in \mathcal{C}_0} H_{C}$ (with our new definition of $\mathcal{C}_0$) and take $\Psi^{(1)} := \Psi - \Psi^{(0)}$. Now define $\Xi^{(s)}(Z,t)$ through

$$\Xi^{(s)}(Z,t) := \sum_{m=0}^{\infty} \sum_{X \in Z \cap \mathbb{N}} \sum_{X(m) = Z} \Delta_{X(m)}(\gamma_{s,t}^\Psi(\Psi^{(1)}(X,t))) \quad \text{(D.7)}$$

Figure 5: This figure shows the support of the different automorphisms present in the decomposition of an SQ$_1$-automorphism acting on the upper half plane.
with these new definitions of $\Psi(0)$ and $\Psi(1)$. We now want so show that for every $t$,

$$
\sum_{z \in \mathbb{Z}^2} \left( \Xi(z, t) - \sum_{C \in C_1} 1_{z \in C} \Xi(z, t) \right) \tag{D.8}
$$

is bounded. Following the arguments in equation (5.22) to (5.24) in [10] we still get that with these new definitions of $\Psi(0)$

is now defined using the new $\alpha$ item 3. The proof of item 5 just follows from the fact that for any $C$

Since the remainder of the proof can remain unchanged from [10], this concludes the proof of $\Psi(0)$ that

we begin with the latter. In this case we still have in complete analogy with the proof in [10] that

$$
M(C_1, C_2) := \sum_{m \geq 0} \sum_{X : X \cap C \neq \emptyset, X \cap C_2 \neq \emptyset} \left( \sup_{t \in [0, 1]} \|\Psi(1)(X; t)\| |X| G_F(m) \right) \tag{D.10}
$$

where

is now defined using the new $C_1$. To bound these $M(C_1, C_2)$ we will (just like in [10]) differentiate between two cases. That is, the case where $C_1$ and $C_2$ are adjacent and the case where they are not. We begin with the latter. In this case we still have in complete analogy with the proof in [10] that

$$
M(C_1, C_2) \leq b_0(C_1, C_2) := \sum_{m \geq 0} \sum_{X : X \cap C_1 \neq \emptyset} \left( \sup_{t \in [0, 1]} \|\Psi(1)(X; t)\| |X| G_F(m) \right) \tag{D.11}
$$

$$
\leq \|\Psi_1\|_F \sum_{x \in C_1} \sum_{y \in C_2} F(d(x, y)) \sum_{m = 0}^{\infty} G_F(m) < \infty. \tag{D.12}
$$

What is now left to show is the case where $C_1$ and $C_2$ are adjacent. Take $\tilde{C} \in C_1$ such that $C_1 \cap \tilde{C} \neq \emptyset$ and $C_2 \cap \tilde{C} \neq \emptyset$. Take $L_1 = \partial \tilde{C}/C_2$ and $L_2 = \partial \tilde{C}/C_1$. By following the same reasoning that led to equation (5.36) in [10] we get that

$$
M(C_1, C_2) \leq b_0(C_1, C_2/\tilde{C}) + b_0(C_1/\tilde{C}, C_2) + b_0(\tilde{C} \cap C_2, (C_1 \cup C_2)^c) + b_1(C_1 \cap \tilde{C}, C_2 \cap \tilde{C}, L_1, L_2) \tag{D.13}
$$

where

$$
b_1(C_1 \cap \tilde{C}, C_2 \cap \tilde{C}, L_1, L_2) := \sum_{m = 0}^{\infty} \sum_{X \in C_2 : X \cap C_1 \cap \tilde{C} \neq \emptyset} \sup_{t \in [0, 1]} \|\Psi(X; t)\| |X| G_F(m) \tag{D.14}
$$

$$
\leq \|\Psi_1\|_F \sum_{m = 0}^{\infty} G_F(m) \left( \sum_{x \in C_2 \cap \tilde{C}} + \sum_{x \in C_1 \cap \tilde{C}} \right) F(d(x, y)) < \infty. \tag{D.15}
$$

Since the remainder of the proof can remain unchanged from [10], this concludes the proof of item 3. The proof of item 5 just follows from the fact that for any $\alpha_1 \in \text{SQAut}(\mathcal{A})$ and for any

---

22 We say that $C_1$ and $C_2$ are adjacent if and only if $\#(C_1 \cap C_2) = \infty$.

23 Here $\Psi_1$ is defined through $\Psi_1(X) = |X|^1 \Psi(X)$.

24 Here $\partial \tilde{C}$ means the boundary of $\tilde{C}$. 

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\[ \alpha_2 \in H\text{Aut}(A) \] we have that \( \alpha_2 \circ \alpha_1 \in S\text{QAut}(A) \). We now only need to comment on item 2. We must show that
\[
(\gamma_{s,t}^H \circ \gamma_{t,s}^L \otimes \gamma_{t,s}^R)^{-1} = \gamma_{s,t}^H \otimes \gamma_{s,t}^H \circ \gamma_{t,s}^H \in \text{GVAut}(A).
\] (D.16)

The proof of this starts analogously to the proof of item 1 and 3. We take \( \Psi = H, \Psi^{(0)} = H_{r(C^\tau_0)} + H_{r^{-1}(C^\tau_0)} \) and \( \Psi^{(1)} = \Psi - \Psi^{(0)} \). Define \( \Xi^{(s)}(Z,t) \) again through equation (D.7). In analogy to what was done in equation (5.54) in \[10\] we obtain
\[
\sum_{Z \in \mathbb{Z} \cap (C^\tau_0)} \sup_{\tau \geq 0} \left\| \Xi^{(s)}(Z,t) \right\| \leq \frac{8}{C_F} (e^{2F(\Psi)} - 1) \sum_{m=0}^{\infty} \sum_{X : (m) \not\in \tau^{-1}(C^\tau_0)} \sup_{t \in [0,1]} \left\| \Psi(X;t) \right\| X|G_F(m).
\] (D.17)

If \( X \cap (W(C_0) \cap L) \neq \emptyset \) and \( X \cap R \neq \emptyset \).

1. \( X \cap (W(C_0) \cap R) \neq \emptyset \) and \( X \cap L \neq \emptyset \).

2. \( X \subseteq W(C_0)^c \) and
   (a) \( X \subseteq U, X \subseteq D \), or
   (b) \( X \subseteq U, X \subseteq L, X \subseteq R \) and \( X(m) \cap (\tau^{-1}(C^\tau_0)) \neq \emptyset \), or
   (c) \( X \subseteq D, X \subseteq L, X \subseteq R \) and \( X(m) \cap (\tau(C^\tau_0) \cap U) \neq \emptyset \).

This shows that we have a bound
\[
\sum_{Z \in \mathbb{Z} \cap (C^\tau_0)} \sup_{t \in [0,1]} \left\| \Xi^{(s)}(Z,t) \right\| \leq \frac{8}{C_F} (e^{2F(\Psi)} - 1) (\sum_{t \in [0,1]} \sum_{X : (m) \not\in \tau^{-1}(C^\tau_0)} \sup_{t \in [0,1]} \left\| \Xi^{(s)}(Z,t) \right\| X|G_C(m).\right.
\] (D.18)

This concludes the proof.

This implies certain things for our locally generated automorphisms. From these four statements we can prove the following results:

**Lemma D.5.** Take \( H \) an interaction such that there exists a \( 0 < \phi < 1 \) satisfying that \( \|H\|_{f_0} \leq 1 \). Take \( \theta_1 \) and \( \theta_2 \) such that \( 0 < \theta_1 < \theta_2 < \pi/2 \) then for all \( \Theta \in \text{Aut}(A_{W(C_{\theta_2})}) \) and \( s,t \in \mathbb{R} \) there exists an \( a_1 \in \mathcal{U}(A) \) and a \( \tilde{\Theta} \in \text{Aut}(A_{W(C_{\theta_1})}) \) such that
\[
\gamma_{t,s}^H \circ \Theta \circ \gamma_{s,t}^H = \text{Ad}(a_1) \circ \tilde{\Theta}.
\] (D.20)

**Proof.** We have that
\[
\gamma_{t,s}^H \circ \Theta \circ \gamma_{s,t}^H = \gamma_{t,s}^H \otimes \gamma_{t,s}^H \circ \gamma_{s,t}^H \otimes \gamma_{s,t}^H \circ \Theta \circ \gamma_{s,t}^H \circ \gamma_{t,s}^H \otimes \gamma_{t,s}^H \circ \gamma_{s,t}^H \otimes \gamma_{s,t}^H \otimes \gamma_{s,t}^H.
\] (D.21)

Using that \( \gamma_{s,t}^H \circ \gamma_{t,s}^H \otimes \gamma_{s,t}^H \in H\text{Aut}_1(A) \) we get that there exists some \( a \in A \) and \( \eta \in \text{Aut}(A_{C_0}) \) such that
\[
\gamma_{t,s}^H \circ \Theta \circ \gamma_{s,t}^H = \text{Ad}(a) \circ \gamma_{t,s}^H \otimes \gamma_{t,s}^H \circ \eta_{s,t}^{-1} \circ \Theta \circ \eta_{s,t} \circ \gamma_{s,t}^H \otimes \gamma_{s,t}^H
\] (D.22)
\[
= \text{Ad}(a) \circ \gamma_{t,s}^H \otimes \gamma_{t,s}^H \circ \Theta \circ \gamma_{s,t}^H \otimes \gamma_{s,t}^H.
\] (D.23)

Since by [D.4] part 3 \( \gamma_{s,t}^H \otimes \gamma_{s,t}^H \in \text{GSQAut}_1(A) \) the result follows.
Lemma D.6. Take $H$ an interaction such that there exists a $0 < \phi < 1$ satisfying that $\|H\|_{f_\phi} \leq 1$. Take $\theta_1$ and $\theta_2$ such that $0 < \theta_1 < \theta_2 < \pi/2$. Then for all $\eta_g^\sigma \in \text{Aut}\left(\mathcal{A}_{C_{\theta_1}} \cap \mathcal{A}_{\sigma}\right)$ (where $\sigma \in \{L, R\}$ and $g \in G$) and $s, t \in \mathbb{R}$ there exist $a_2, \in \mathcal{U}(\mathcal{A}), a_{3, \sigma} \in \mathcal{U}(\mathcal{A}_{\sigma})$ and some $\tilde{\eta}_g \in \text{Aut}\left(\mathcal{A}_{C_{\theta_1}} \cap \mathcal{A}_{\sigma}\right)$ such that

\[
\begin{align*}
\gamma_{t,s}^H \circ \eta_g^L \otimes \eta_g^R \circ \gamma_{s,t}^H &= \text{Ad}(a_2) \circ (\bar{\eta}_g^L \otimes \bar{\eta}_g^R) \quad \text{(D.24)} \\
\gamma_{t,s}^H \circ \eta_g^g \circ \gamma_{s,t}^H &= \text{Ad}(a_{3, \sigma}) \circ \tilde{\eta}_g \quad \text{(D.25)}
\end{align*}
\]

Proof. In this proof, take $H_{\text{split}} = H_L + H_R$. First we show that equation (D.25) implies equation (D.24). This is because using equation (D.24) we get that

\[
\begin{align*}
\gamma_{t,s}^H \circ \eta_g \circ \gamma_{s,t}^H &= \gamma_{t,s}^H \circ \gamma_{s,t}^{H_{\text{split}}} \circ \eta_g \circ \gamma_{s,t}^{H_{\text{split}}} \circ \eta_g \circ \gamma_{t,s}^{H_{\text{split}}} \circ \gamma_{s,t}^H \\
&= \gamma_{t,s}^H \circ \gamma_{s,t}^{H_{\text{split}}} \circ \text{Ad}(a_{3, L} \otimes a_{3, R}) \circ \tilde{\eta}_g \circ \gamma_{s,t}^{H_{\text{split}}} \circ \gamma_{t,s}^H.
\end{align*}
\]

If one now uses the fact that $\gamma_{t,s}^H \circ \gamma_{s,t}^{H_{\text{split}}} \in \text{GVAut}_1(\mathcal{A})$ (see lemma D.4 item 2) the implication follows. To finish the proof we now only have to use the fact that the $\gamma_{0; 1}^H$ are in SQAut$_1(\mathcal{A}_\sigma)$.

Additionally when we add the group action, we get:

Lemma D.7. Take $H$ a $G$-invariant interaction such that there exists a $0 < \phi < 1$ satisfying that $\|H\|_{f_\phi} \leq 1$. Take $\theta_1$ and $\theta_2$ such that $0 < \theta_1 < \theta_2 < \pi/2$. Then for all $\eta_g^\sigma \in \text{Aut}\left(\mathcal{A}_{C_{\theta_1}} \cap \mathcal{A}_{\sigma}\right)$ (where $\sigma \in \{L, R\}$ and $g \in G$) and $s, t \in \mathbb{R}$ there exist $a_2, \in \mathcal{U}(\mathcal{A}), a_{3, \sigma} \in \mathcal{U}(\mathcal{A}_{\sigma})$ and some $\tilde{\eta}_g \in \text{Aut}\left(\mathcal{A}_{C_{\theta_1}} \cap \mathcal{A}_{\sigma}\right)$ such that

\[
\begin{align*}
\gamma_{t,s}^H \circ \eta_g^L \otimes \eta_g^R \circ \beta_g^U \circ \gamma_{s,t}^H &= \text{Ad}(a_2) \circ (\bar{\eta}_g^L \otimes \bar{\eta}_g^R) \circ \beta_g^U \\
\gamma_{t,s}^H \circ \eta_g^g \circ \beta_g^U \circ \gamma_{s,t}^H &= \text{Ad}(a_{3, \sigma}) \circ \tilde{\eta}_g \circ \beta_g^U \circ \gamma_{s,t}^H.
\end{align*}
\]

Proof. In this proof, take again $H_{\text{split}} = H_L + H_R$. First we show that equation (D.29) implies equation (D.28). This is because using (the inverse of) equation (D.28) we get that

\[
\begin{align*}
\gamma_{t,s}^H \circ \eta_g \circ \beta_g^U \circ \gamma_{s,t}^H \circ \beta_g^{U^{-1}} \circ (\bar{\eta}_g)^{-1} \\
= (\text{Inner}) \circ \gamma_{t,s}^H \circ \eta_g \circ \beta_g^U \circ \gamma_{s,t}^H \circ \gamma_{t,s}^{H_{\text{split}}} \circ \beta_g^{U^{-1}} \circ (\bar{\eta}_g)^{-1} \circ \gamma_{s,t}^{H_{\text{split}}} \circ \gamma_{t,s}^H.
\end{align*}
\]

Now using the fact that by lemma D.4 item 2, $\gamma_{s,t}^H \circ \gamma_{t,s}^{H_{\text{split}}} \in \text{GVAut}_1(\mathcal{A})$ this gives us

\[
= (\text{Inner}) \circ \gamma_{t,s}^H \circ \eta_g \circ \beta_g^U \circ \beta_g^{U^{-1}} \circ (\bar{\eta}_g)^{-1} \circ \gamma_{s,t}^H.
\]

This concludes the proof that equation (D.29) implies equation (D.28). Now we only have to prove equation (D.29). To finish the proof we now only have to use the fact that the $\gamma_{0; 1}^H$ are in GSQAut$_1(\mathcal{A}_{\sigma})$.

D.2 The two translation symmetries

This section will define the classes of automorphisms required in the assumptions and proofs about the $H^1$-valued translation index. In analogy with section 2.1 from [10] we will define some classes of automorphisms (see figure [3]):

Definition D.8. Take $\alpha \in \text{Aut}(\mathcal{A})$. We say that $\alpha \in \text{SQAut}_2(\mathcal{A})$ if and only if for any $0 < \theta_{0, 8} < \theta_1 < \theta_{1, 2} < \theta_{1, 8} < \theta_2 < \theta_{2, 8} < \theta_3 < \theta_{3, 2} < \pi/2$ there exist $\alpha \in \mathcal{A}, \alpha_{[0, \theta_1]} \in \text{Aut}\left(\mathcal{A}_{\phi(C_{[0, \theta_1]} \cap \sigma)}\right), \alpha_{[\theta_1, \theta_2]} \in \text{Aut}\left(\mathcal{A}_{\phi(C_{[\theta_1, \theta_2]} \cap \sigma)}\right), \alpha_{[\theta_2, \theta_3]} \in \text{Aut}\left(\mathcal{A}_{\phi(C_{[\theta_2, \theta_3]} \cap \sigma)}\right), \alpha_{[\theta_3, \pi/2]} \in \text{Aut}\left(\mathcal{A}_{\phi(C_{[\theta_3, \pi/2]} \cap \sigma)}\right)$.
Figure 6: This figure shows the support of the different automorphisms present in the decomposition of an SQ₂-automorphism acting on the upper half plane.

\[ \alpha_{[\theta, \pi/2], \rho} \in \text{Aut} \left( \mathcal{A}_{\tau^\rho (C_{[\theta, \pi/2], \rho})} \right), \quad \alpha_{[\theta_0, \theta_1], \sigma, \rho} \in \text{Aut} \left( \mathcal{A}_{\tau^\rho \alpha \sigma (C_{[\theta_0, \theta_1], \sigma, \rho})} \right), \]

\[ \alpha_{[\theta, \theta_2], \sigma, \rho} \in \text{Aut} \left( \mathcal{A}_{\tau^\rho \alpha \sigma (C_{[\theta, \theta_2], \sigma, \rho})} \right) \quad \text{and} \quad \alpha_{[\theta_2, \theta_1], \sigma, \rho} \in \text{Aut} \left( \mathcal{A}_{\tau^\rho \alpha \sigma (C_{[\theta_2, \theta_1], \sigma, \rho})} \right) \quad \text{for any } \rho \in \{U, D\} \quad \text{and} \quad \sigma \in \{L, R\} \quad \text{here} \quad \tau^U = \tau, \quad \tau^D = \tau^{-1}, \quad \nu^R = \nu \quad \text{and} \quad \nu^L = \nu^{-1} \quad \text{such that} \]

\[ \alpha = \text{Ad}(a) \circ \bigotimes_{\sigma} \alpha_{[0, \theta], \sigma, \rho} \circ \bigotimes_{\rho, \sigma} \alpha_{[\theta_1, \theta_2], \sigma, \rho} \circ \bigotimes_{\rho, \sigma} \alpha_{[\theta_2, \theta_1], \sigma, \rho} \circ \bigotimes_{\rho, \sigma} \alpha_{[\theta_2, \theta_1], \sigma, \rho}. \]  

(D.33)

If additionally everything (except \(a\)) commutes with \(\beta^U_g\) we say that \(\alpha \in \text{GSQAut}_2(\mathcal{A})\).

We now have to define a different horizontal automorphism:

**Definition D.9.** Take \(\alpha \in \text{Aut}(\mathcal{A})\). We say that \(\alpha \in \text{HAut}_2(\mathcal{A})\) if and only if for any \(0 < \theta < \pi/2\) there exists an \(a \in \mathcal{A}\) and some \(\alpha_\sigma \in \text{Aut} \left( \mathcal{A}_{\tau^\rho \alpha \sigma} \right) \) for each \(\sigma \in \{L, R\}\) such that

\[ \alpha = \text{Ad}(a) \circ \alpha_L \otimes \alpha_R. \]  

(D.34)

We will let the definition of vertical automorphism stay the same, namely \(\text{VAut}_2(\mathcal{A}) = \text{VAut}_1(\mathcal{A})\) and \(\text{GVAut}_2(\mathcal{A}) = \text{GVAut}_1(\mathcal{A})\).

**Lemma D.10.** Lemma [D.4] will still be true when we exchange all the \(1\)'s for \(2\)'s.

**Proof.** Completely equivalent to the proof of [D.4].

**Lemma D.11.** Take \(\theta_1\) and \(\theta_2\) such that \(0 < \theta_1 < \theta_2 < \pi/2\). Then for all \(\eta_g^\sigma \in \text{Aut} \left( \mathcal{A}_{\tau^\rho \alpha \sigma} \right)\) (where \(\sigma \in \{L, R\}\) and \(g \in G\)) and \(s, t \in \mathbb{R}\) there exist \(a_2, s \in U(\mathcal{A}), a_\sigma \in U(\mathcal{A})\) and some \(\tilde{\eta}_g^\sigma \in \text{Aut} \left( \mathcal{A}_{\tau^\rho \alpha \sigma} \right)\) such that

\[ \gamma_{t, s} H^\rho \circ \eta_g^L \otimes \eta_g^R \circ \gamma_{s, t} = \text{Ad}(a_2) \circ (\tilde{\eta}_g^L \otimes \tilde{\eta}_g^R) \]  

(D.35)

\[ \gamma_{t, s} H^\rho \circ \eta_g^\sigma \gamma_{s, t} = \text{Ad}(a_\sigma) \circ \tilde{\eta}_g^\sigma. \]  

(D.36)

**Proof.** Analogous to the proof of [D.6].

Additionally when we add the group action, we get:

**Lemma D.12.** Take \(H\) a \(G\)-invariant interaction such that there exists a \(0 < \phi < 1\) satisfying that \(\|H\|_{\ell_\phi} \leq 1\). Take \(\theta_1\) and \(\theta_2\) such that \(0 < \theta_1 < \theta_2 < \pi/2\). Then for all \(\eta_g^\sigma \in \text{Aut} \left( \mathcal{A}_{C_{\theta_1} \cap \mathcal{A}_\sigma} \right)\) (where \(\sigma \in \{L, R\}\) and \(g \in G\)) and \(s, t \in \mathbb{R}\) there exist \(a_2, s \in U(\mathcal{A}), a_\sigma \in U(\mathcal{A}_\sigma)\) and some \(\tilde{\eta}_g^\sigma \in \text{Aut} \left( \mathcal{A}_{\tau^\rho \alpha \sigma} \right)\) such that

\[ \gamma_{t, s} H^\rho \circ \eta_g^L \otimes \eta_g^R \circ \beta_g^U \circ \gamma_{s, t} = \text{Ad}(a_2) \circ (\tilde{\eta}_g^L \otimes \tilde{\eta}_g^R) \circ \beta_g^U \]  

(D.37)

\[ \gamma_{t, s} \circ \eta_g^\sigma \circ \beta_g^U \circ \gamma_{s, t} = \text{Ad}(a_\sigma) \circ \tilde{\eta}_g^\sigma \circ \beta_g^U. \]  

(D.38)

**Proof.** Analogous to the proof of [D.7].

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