Realistic Results of Low-Lying Charmonium Using An Instanton Potential

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Abstract

We investigate the spectrum and decay rates of low lying charmonium states within the framework of the non relativistic quark model by employing a Coulomb like potential from the perturbative one gluon exchange and the linear confining potential along with the potential derived form instanton vacuum to account for the hyperfine mass splitting of charmonium states. We predict radiative E1, M1, two-photon, leptonic and two-gluon decay rates of low lying charmonium states. An overall agreement is obtained with the experimental masses and decay widths.

Keywords: Non-relativistic quark model; charmonium; radiative decay; two gluon decay; two photon decay; leptonic decay

1. Introduction

Charmonia are bound states of a charm and an anticharm quark (c\bar{c}), and represent an important testing ground for the properties of the strong interaction. There has been a great progress in the observation of the charmonium states from past few years. The discovery of the first charmonium state \(J/\psi\)\textsuperscript{[1, 2]} has revolutionized the field of hadron spectroscopy. This lead to a clear understanding of the prevailing theory of particle physics. Several quarkonium states have been observed after the discovery of the charmonium state \(J/\psi\) at BNL and SLAC\textsuperscript{3}. The first observation of singlet ground state of charmonium \(\eta_c\) was done by Mark II and crystal Ball experiments in the radiative decays of \(J/\psi\) and \(\psi'\textsuperscript{[3]}\). The discoveries of conventional states \(h_c(1P), h_c(2P), \chi_c(1P), \chi_c(2P), \eta_c(1S)\) and the observation of the exotic states like \(X(3872), X(3915), Y(4260), Z(3930)\) at Belle, BaBar, LHC, BESIII,CLEO, etc have lead to renewed interest in quarkonium physics\textsuperscript{3}. These new observations have given a deeper understanding of the charmonium physics and have unraveled many mysteries\textsuperscript{4}. Charmonium system is a powerful tool for the study of forces between quarks in QCD like non-perturbative interaction. Studies of charmonia production can improve our understanding of heavy quark production and the formation of bound states. The exploration and understanding of the substructure of hadrons, presented in terms of quarks and gluons by quantum chromodynamics(QCD), has lead to a considerable progress in the study of charmonium states.

The quark antiquark potential cannot be obtained from the first principles of QCD. Therefore, one has to use potential models to explain the observed hadronic properties. The QCD inspired potential models have been playing an important role in investigating heavy quarkonium, owing to the presence of large nonpertubative effects in the energy regions. Most of the quark potential models \[5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\] have common ingredients under the non relativistic limit. In the original Cornell model it was assumed as Lorentz scalar, which gives a vanishing long range magnetic contribution and agrees with the flux tube picture of quark confinement. Another possibility is that confinement may be a more complicated mixture of scalar and timelike vector, while the vector potential is anticonfining. In pure (c\bar{c}) models, the Lorentz nature of confinement is tested by multiplet splitting of orbitally excited charmonium states\textsuperscript{17}.

There have been numerous calculations of charmonium spectra such as the first principles calculations starting with the QCD Lagrangian such as Lattice QCD\textsuperscript{18, 19} and NRQCD which provide rigorous theoretical implications for the experimental observations on these states and quark model calculations provide more intuitive insights into these and supply important phenomenological guidance towards their study. The confinement problem makes it hard to calculate quantities for bound states within QCD as one cannot apply perturbative QCD. The NRQCD formalism is found to provide systematic treatment of the perturbative and non perturbative components of QCD at hadronic scale \[20, 21\]. These QCD inspired potential models use a short range part motivated by perturbative QCD (Coulomb like or One gluon exchange potential) \[20, 22, 23, 24, 25, 26, 27, 28, 29, 30\] and a phenomenological long range part accounting for confinement (i.e.

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linear, logarithmic or quadratic potential). The instanton potential model deduced by a non-relativistic reduction of the 't Hooft interaction has already been successfully applied in several studies of the hadron spectra. The main achievement of instantons in hadron spectroscopy is the resolution of the $U_A(1)$ problem, which leads to a good prediction of the masses of $\eta$ and $\eta'$ mesons.

It is observed in [38] that instanton effects are very much effective in reproducing the mass of quarkonia. The instanton effects come into play significantly in the heavy quark mass. They turn out to be rather small in describing the hyperfine mass splittings of the charmonia. This might be due to the fact that the spin-dependent part of the potential from the instanton vacuum is almost an order of magnitude smaller than the central part. The tensor interaction almost does not contribute to the results. In order to obtain the realistic results of the hyperfine mass splittings as well as of the charmonium masses, we need to include the Coulomb-Like Potential coming from the perturbative one gluon exchange and confining potential together with that from the instanton vacuum. Taking into consideration the above factors, we have developed a nonrelativistic potential model with instanton effects.

The paper is organized in 4 sections. In sec. 2 we briefly review the theoretical background for non relativistic model, the description of radiative and weak decay widths. In sec. 3 we present the results and discussions. Sec. 4 is devoted to conclusions and outlook.

2. Theoretical Background

2.1. The Model

In a potential approach the entire dynamics is governed by a Hamiltonian which is composed of a kinetic energy term $K$ and a potential energy term $V$ that takes into account the interaction between the quark and the antiquark,

$$H = K + V. \tag{1}$$

We calculate mass spectra of the charmonium states and their decays by solving explicitly the Schrödinger equation, using the heavy-quark potential derived from the instanton vacuum $V = V_{QQ}(\vec{r})$ with the confining $(V_{conf}(\vec{r}))$ and Coulomb potentials $(V_{coul}(\vec{r}))$.

The Kinetic energy is given by,

$$K = \sum_{i=1}^{2} \left( M_i + \frac{P_i^2}{2M_i} \right) \tag{2}$$

Where $M_i$ and $P_i$ are the mass and momenta of the $i^{th}$ quark. $K$ is the sum of kinetic energies including the rest mass minus the kinetic energy of the center of mass of the total system.[39]

The potential energy $V$ is given by,

$$V(\vec{r}) = V_{QQ}(\vec{r}) + V_{conf}(\vec{r}) + V_{coul}(\vec{r}) \tag{3}$$

where the heavy-quark potential $V_{QQ}(\vec{r})$ is,

$$V_{QQ}(\vec{r}) = V_C(\vec{r}) + V_{SD}(\vec{r}). \tag{4}$$

The explicit form of the central potential from the instanton vacuum is [38]

$$V_C(\vec{r}) \approx \frac{4\pi\rho^3}{R^4N_c} \left( 1.345 \frac{r^2}{\rho^2} - 0.501 \frac{r^4}{\rho^4} \right) \tag{5}$$

For the inter quark distance larger than the instanton size ($r >> \rho$) we get the appropriate central potential,

$$V_C(\vec{r}) \approx 2\Delta M_Q - \frac{g_{np}}{r} \tag{6}$$

where the second term can be understood as a non perturbative contribution to the perturbative one gluon exchange potential at large $r$. The non perturbative coupling constant $g_{np}$ could be regarded as non perturbative correction to the strong coupling constant $\alpha_s$. When $r$ tends to infinity the potential is saturated at the value $2\Delta M_Q$ which implies that the instanton vacuum cannot explain the quark confinement. $\Delta M_Q$ is the correction to the heavy-quark mass from the instanton vacuum [38].

The spin-dependent potentials $V_{SD}(\vec{r})$ can be the spin- spin interaction ($V_{SS}(\vec{r})$), the spin-orbit coupling term ($V_{LS}(\vec{r})$) and the tensor part ($V_T(\vec{r})$) and they are,

$$V_{SS}(\vec{r}) = \frac{1}{3m_Q^2} \nabla^2 V_C(\vec{r}); \quad V_{LS}(\vec{r}) = \frac{1}{2m_Q^2} \frac{1}{r} \frac{dV_C(\vec{r})}{dr}; \quad V_T(\vec{r}) = \frac{1}{3m_Q^2} \left( \frac{1}{r} \frac{dV_C(\vec{r})}{dr} - \frac{d^2V_C(\vec{r})}{dr^2} \right) \tag{7}$$
The expectation values of $\langle \vec{S}_Q \cdot \vec{S}_{\bar{Q}} \rangle$ depends on the total spin ($\vec{S}$) of the meson, and are given by,

$$\vec{S}_Q \cdot \vec{S}_{\bar{Q}} = \begin{cases} \frac{1}{4} & \text{for } \vec{S} = 1 \\ -\frac{3}{4} & \text{for } \vec{S} = 0 \end{cases}$$ (8)

The confinement term represents the non perturbative effect of QCD which includes the spin-independent linear confinement term $^{[39]}$

$$V_{\text{conf}}(\vec{r}) = -\left[ \frac{3}{4} V_0 + \frac{3}{4} c r \right] F_1 \cdot F_2$$ (9)

where $c$ and $V_0$ are constants. $F$ is related to the Gell-Mann matrix, $F_1 = \lambda^1_2$ and $F_2 = \lambda^2_2$. Especially, we have $F_1 \cdot F_2 = \frac{-3}{4}$ for the mesons.

The coulomb-like (perturbative) one gluon exchange part of the potential is given by

$$V_{\text{coul}}(\vec{r}) = -\frac{4\alpha_e}{3}$$ (10)

with the running strong coupling constant $\alpha_s$,

$$\alpha_s = \frac{4\pi}{\left(11 - \frac{2}{3} n_f\right) \left(\ln \frac{\mu^2}{\Lambda^2}\right)}$$ (11)

where $\Lambda$ is the QCD scale, which is taken as 120 MeV, $n_f$ is the number of flavors and $\mu$ is the renormalization scale related to the constituent quark mass.

2.2. Radiative Transitions

In calculating the radiative decay widths, we have assumed that in the non relativistic limit, the dipole radial matrix elements are independent of $J$, i.e all states within the same angular momentum multiplet have the same wave function $^{[40]}$. Radiative transitions could play an important role in the discovery and identification of charmonium states. They are sensitive to the internal structure of states, in particular to $^3L_L - ^1L_L$ mixing for states with $J = L$.

2.2.1. E1 Transitions

The partial width for an $E_1$ radiative transition between states in the non relativistic quark model is given by

$$\Gamma(i \rightarrow f + \gamma) = \frac{4\alpha e^2}{3} (2J' + 1) S_{ij} E_{ij} k_0^3 |\mathcal{E}_{ij}|^2$$ (12)

where $k_0$ is the energy of the emitted photon,

$$k_0 = m_i - m_f$$

$\alpha$ is the fine structure constant. $e_c = 2/3$ is the charge of the $c$ quark in units of $e$. $m_i$ and $m_f$ are the masses of initial and final mesons. The statistical factor $S_{ij} = \max(l, l') \left\{ \frac{J}{l'} \quad \frac{1}{s} \quad \frac{J'}{l} \right\}^2$, $J$, $J'$ are the total angular momentum of initial and final mesons, $l$, $l'$ are the orbital angular momentum of initial and final mesons and $s$ is the spin of initial meson.

$$\mathcal{E}_{ij} = \frac{3}{k_0} \int_0^\infty r^3 R_{nl}(r) R_{n'l'}^*(r) dr \left[ \frac{k_0 r}{2} j_0 \left( \frac{k_0 r}{2} \right) - j_1 \left( \frac{k_0 r}{2} \right) \right]$$ (13)

is the radial overlap integral which has the dimension of length, with $R_{nl}(r)$ and $R_{n'l'}^*(r)$ are the normalized radial wave functions for the corresponding states.
2.2.2. M1 Transitions

Radiative transitions which flip spin are described by magnetic dipole (M1) transitions. The rates for magnetic dipole transitions between S-wave $c\bar{c}$ states are given in the non relativistic approximation by [41, 42, 43, 44, 45, 46, 47].

\[
\Gamma(i \rightarrow f + \gamma) = \frac{4\alpha e c^2}{3m_c^2} \frac{2J_f + 1}{2L + 1} \delta_{LL'} \delta_{SS'} k_0^3 |M_{if}(r)|^2
\]

where $M_{if}$ is the radial overlap integral which has the dimension of length,

\[
M_{if} = \int_0^\infty 4\pi r^3 R_{nl}(r) j_0(kr/2) R'_{nl}(r) dr
\]

is the overlap integral for unit operator between the coordinate wave functions of the initial and the final meson states, $j_0(kr/2)$ is the spherical Bessel function, $m_c$ is the mass of charm quark. $J_f$ is the total angular momentum of final meson state, $L$ is the orbital angular momentum initial meson state.

2.3. Annihilation Decays

The annihilation decays of some charmonium states into gluons and light quarks make significant contributions to the total decay width of the states. The annihilation decays allow us to determine wave function at the origin. The annihilation decays of some $c\bar{c}$ states into photons can be used as a tool for the production and identification of resonances.

2.3.1. Two Gluon Decays

The charmonium states $1S_0$, $3P_0$, $3P_2$ and $1D_2$ can decay into two gluons, which account for a substantial portion of the hadronic decays for states below $c\bar{c}$ threshold. The two gluon decay widths are sensitive to the behaviour of the $q\bar{q}$ wave function and its derivatives near the origin. The two gluon decay widths are given by [48, 49]

\[
\Gamma(n^1S_0 \rightarrow 2g) = \frac{2\alpha s}{3m_c^2} |R_{nS}(0)|^2 \left( 1 + \frac{44\alpha s}{\pi} \right)
\]

\[
\Gamma(n^3P_0 \rightarrow 2g) = \frac{6\alpha s}{m_c^2} |R'_{nS}(0)|^2
\]

\[
\Gamma(n^3P_2 \rightarrow 2g) = \frac{8\alpha s}{5m_c^2} |R'_{nS}(0)|^2
\]

\[
\Gamma(n^1D_2 \rightarrow 2g) = \frac{2\alpha s}{3\pi m_c^2} |R''_{nD}(0)|^2
\]

2.3.2. Two Photon Decays

The $q\bar{q}$ quark pair in charge conjugation even states with $J \neq 1$ can annihilate into two photons. The expressions for the decay rates of $n^1S_0$, $n^3P_0$ and $n^3P_2$ states into two photons with the first order QCD radiative corrections are given by [49]

\[
\Gamma(n^1S_0 \rightarrow \gamma\gamma) = \frac{3e^4\alpha^2}{m_c^2} |R_{nS}(0)|^2 \left( 1 - \frac{3.4\alpha s}{\pi} \right)
\]

\[
\Gamma(n^3P_0 \rightarrow \gamma\gamma) = \frac{27e^4\alpha^2}{m_c^2} |R'_{nS}(0)|^2 \left( 1 + \frac{0.2\alpha s}{\pi} \right)
\]

\[
\Gamma(n^3P_2 \rightarrow \gamma\gamma) = \frac{36e^4\alpha^2}{5m_c^2} |R''_{nS}(0)|^2 \left( 1 - \frac{16\alpha s}{\pi} \right)
\]

The two photon decay widths of P wave charmonium states depend on the derivative of the radial wave function at the origin.

2.3.3. Leptonic Decays

The decay of vector meson into charged leptons proceeds through the virtual photon ($q\bar{q} \rightarrow l^+l^-$ where $l = e^-, \mu^-, \tau^-)$). The $3S_1$ and $3D_1$ states have quantum numbers of a virtual photon, $J^{PC} = 1^{--}$ and can annihilate into lepton pairs through one photon.

The leptonic decay width of the vector meson ($3S_1$ charmonium) including first order radiative QCD correction is given by [48, 49]

\[
\Gamma(n^3S_1 \rightarrow e^+e^-) = \frac{4\alpha^2 e^2 |R_{nS}(0)|^2}{M_{nS}^2} \left( 1 - \frac{16\alpha s}{3\pi} \right)
\]
where \( \alpha \approx \frac{1}{137} \) is the fine structure constant, \( M_{nS} \) is the mass of the decaying charmonium state and \( e_c = 2/3 \) is the charge of the charm quark in units of the electron charge. For D wave charmonium states the leptonic decay width with leading order QCD correction is given by

\[
\Gamma(n^3D_1 \to e^+ e^-) = \frac{25\alpha^2 e_c^2 |R''_{nD}(0)|^2}{2m^2 M_{nD}^2} \left( 1 - \frac{16\alpha_s}{3\pi} \right)
\]

where \( M_{nD} \) is the mass of the decaying charmonium state.

### 3. Results and Discussions

We have used harmonic oscillator wave function which has been extensively used in atomic and nuclear physics. This quark-antiquark wave function corresponding to the relative and center of mass coordinates have been expressed as

\[
\psi_{nlm}(r, \theta, \phi) = N \left( \frac{r}{b} \right)^l L_n^{l+1/2}(\frac{r}{b}) \exp \left(-\frac{r^2}{2b^2}\right) Y_{lm}(\theta, \phi)
\]

where \(|N|\) is the normalizing constant given by

\[
|N|^2 = \frac{2\alpha^3 n!}{\sqrt{\pi}} \frac{2^{2(n+l)+1}}{(2n+2l+1)!} (n+l)!
\]

and \( L_n^{l+1/2}(x) \) are the associated Laguerre polynomials given by

\[
L_n^{l+1/2}(\frac{r}{b}) = \frac{\exp(\frac{r}{b})}{n!} \frac{d^n}{dr^n} \left[ \exp \left(-\frac{r}{b}\right) \left(\frac{r}{b}\right)^{-l+1/2+n} \right]
\]

The main advantage of using the harmonic oscillator wave function is that it allows the separation of the motion of the center of mass and has been extensively used to classify the spectra of baryons and mesons and extending it to nucleon-nucleon interaction is straightforward. If the basic states are the harmonic oscillator wave functions it is straightforward to evaluate the matrix elements of few body systems such as mesons or baryons. Since the basic states are the products of the harmonic oscillator wave functions they can be chosen in a manner that allows the product wave functions to be expanded as a finite sum of the corresponding products for any other set of Jacobi coordinates.

We have two important parameters characterizing the dilute instanton liquid; the average size of the instanton \( \rho = \frac{1}{3} \) fm, the values of \( \rho \) is less effective in the spin-dependent parts of the potential. The average separation between instantons is \( \tilde{R} = (N/V)^{-1} \approx 1 \) fm. The strength of each part of the potential becomes stronger when smaller value of \( \tilde{R} \) is employed. The instanton density is given as \( N/V \approx (200 MeV)^4 \) and number of colors \( N_C \) is 3. This heavy quark potential from the instanton vacuum depends on \( r \). However the distance of the quark and the antiquark gets farther apart, certain non perturbative contributions should be taken into account in the potential. There is yet another non perturbative effect on the heavy quark potential from instantons, which is known to be one of the most important topological object in describing the QCD vacuum. The Wilson loop was averaged in the instanton ensemble to get the heavy-quark potential, which rises almost linearly as the relative distance between the quark and the antiquark increases, then it gets saturated. Though the instanton vacuum does not explain quark confinement, it will play a certain role in describing the characteristics of the quarkonia. One of the effects of instantons is to contribute a new term to the interaction energy of quarks. Instanton liquid model describes well the hadrons at low as well as at intermediate energies. As the separation of the quark increases, the potential increases rapidly, soon surpasses the ordinary coulomb energy describing quark interactions in the asymptotically free region. The spin-spin interaction from the instanton vacuum looks similar to a Gaussian type interaction. The spin-orbit potential behaves like the spin-spin potential. The tensor interaction however shows a different r dependence.

Other parameters in our potential model are, the running coupling constant \( \alpha_s \), harmonic oscillator size parameter \( b \), the charm quark mass \( m_c \), the confinement strength \( c \) and a constant \( V_0 \). We obtain the value ‘b’ by minimizing the expectation value of the Hamiltonian i.e, \( \frac{\delta \langle H \rangle}{\delta b} = 0 \). The confinement strength \( c \) is fixed by the stability condition for variation of mass of the vector meson against the size parameter \( b \). The running coupling constant \( \alpha_s \) is fixed using eqn. (11). To fix \( m_c \) and \( V_0 \), we start with a set of reasonable values of \( m_c \) and \( V_0 \) and diagonalize the matrix for \( c \bar{c} \) meson. Then we tune these parameters to obtain an agreement with the experimental value for the mass of \( c \bar{c} \) meson. We use the following set of parameter values.

\[
m_c = 1475 \text{ MeV}; \quad b = 0.29 \text{ fm}; \quad \alpha_s = 0.2 - 0.3; \quad c = 65 \text{ MeV fm}^{-1}; \quad V_0 = -125 \text{ MeV}; \quad (28)
\]
for QCD results ∆\( M_{QCD} \) predicts a 2S hyperfine splittings of 57 MeV. Other models. The lattice QCD calculations predict a mass of 4074 ± 24 MeV. We calculate the fine structure of P wave charmonium states. For 1\( ^1P_1 \) - 1\( ^3P_0 \) splitting we obtain

\[
\Delta M(1^3S_1 - 1^1S_0) = 111 \text{ MeV.}
\]

The spin-orbit and tensor potential in eqn(7) produce the fine structure of the charmonium levels. The hyperfine splitting of 1S state obtained in our model \( \Delta M(1^1S_1 - 1^1S_0) = 111 \text{ MeV.} \) The hyperfine mass splitting calculated in our model agrees with the experimental value \( \Delta M(1^1S_1 - 1^1S_0) = 113.2 \pm 0.7 \text{ MeV} \) and lattice QCD results \( \Delta M(1^3S_1 - 1^1S_0) = 114 \pm 1 \text{ MeV.} \) The hyperfine mass splitting of 2S states \( \Delta M(2^1S_1 - 2^1S_0) = 42 \text{ MeV} \) is in good agreement with the experimental one \( \Delta M(2^3S_1 - 2^1S_0) = 47 \pm 1 \text{ MeV} \). The lattice QCD predicts a 2S hyperfine splittings of 57.9 ± 2.0 MeV which is slightly higher than the experimental value. We calculate the fine structure of P wave charmonium states. For 1\( ^1P_1 \) - 1\( ^3P_0 \) splitting we obtain

In Table 1 we list the low lying charmonium masses and compare them with experimental data and other theoretical models. Our predictions for the masses agree with PDG data within a few MeV. Our model correctly reproduces the mass spectrum and fine and hyperfine splittings of charmonium levels. The mass of ground spin-singlet state \( \eta_c(1S) \) is found to be 2983 MeV which is in good agreement with the experimentally measured mass value 2983.6 ± 0.7 MeV. Lattice QCD calculations predict a mass of 2985(1) MeV for \( \eta_c(1S) \) state. The mass of the spin triplet state \( J/\psi(1S) \) calculated in our model is 3094 MeV which is in good agreement with experimental value 3096.96 ± 0.011 MeV and other theoretical models. The lattice QCD calculations predict a mass of 3099 ± 1 MeV for \( J/\psi(1S) \) state. The masses of radically excited Charmonium state \( \eta_c(2S) \) and its triplet partner \( \psi(2S) \) calculated in our model are in good agreement with both experimental value and with other theoretical models. The lattice QCD calculations predict a slightly less mass values for \( \eta_c(2S) \) \( (M(\eta_c(2S)) = 3162 ± 9 \text{ MeV and } \psi(2S)(M(\psi(2S)) = 3653 ± 12) \text{ MeV.} \) It is clear from the table that the mass of \( \eta_c(3S) \) and its triplet partner calculated in our model is reasonably agree with experimental values and with other models. The lattice QCD calculations predict a mass of 4074 ± 20 MeV for \( \eta_c(3S) \) and 4099 ± 24 MeV for \( \psi(3S) \).

The mass of orbitally excited spin-singlet charmonium \( h_c(1P) \) is in good agreement with experiment and other theoretical models. However the lattice QCD predicts a mass of 3506 ± 6 MeV for \( h_c(1P) \) state which is slightly lesser than the experimental value 3508.3 MeV. The spin triplet states \( \chi_{cJ} \) are in good agreement with both experiment and other phenomenological models. The lattice QCD obtains the masses 20 – 25 MeV lesser than the experimental values for these states. We have also predicted masses of low lying D wave states which reasonably agree with available experimental data and other theoretical models.

| \( n^2S + 1L_J \) | Name  | \( J^P \) | This Work | \( M_{exp} \) | \( \Delta M \) | \( S \) | \( 54 \) | \( 55 \) | \( 17 \) | \( 56 \) |
|-----------------|--------|-----------|-----------|-------------|-----------|-----|-----|-----|-----|-----|
| 1\( ^1S_0 \)    | \( \eta_c(1S) \) | 0⁻      | 2983      | 2983.6 ± 0.7 | 2970      | 2981 | 2981.7 | 2990.4 | 2990 |
| 2\( ^1S_0 \)    | \( \eta_c(2S) \) | 0⁻      | 3630      | 3639.2 ± 0.11 | 3620      | 3635 | 3619.2 | 3646.5 | 3643 |
| 3\( ^1S_0 \)    | \( \eta_c(3S) \) | 0⁻      | 4061      | ...         | 4060      | 3989 | 4052.5 | 4071.9 | 4054 |
| 1\( ^3P_1 \)    | \( J/\psi \) | 1⁻      | 3094      | 3096.96 ± 0.011 | 3100      | 3096 | 3096.92 | 3085.1 | 3096 |
| 2\( ^3P_1 \)    | \( \psi(2S) \) | 1⁻      | 3672      | 3686.108 ± 0.018 | 3680      | 3685 | 3686.1 | 3382.1 | 3703 |
| 3\( ^3P_1 \)    | \( \chi_{c3}(3S) \) | 1⁻      | 4052      | 4039 ± 1 | 4100 | 4039 | 4102 | 4100.2 | 4097 |
| 1\( ^1D_2 \)    | \( \eta_c(1D) \) | 2⁻      | 3811      | ...         | 3840      | 3807 | 3822.3 | 3807.3 | 3812 |
| 2\( ^1D_2 \)    | \( \eta_c(2D) \) | 2⁻      | 4170      | ...         | 4210      | 4196 | 4196.9 | 4173.7 | 4166 |
| 1\( ^3D_1 \)    | \( \psi(1D) \) | 1⁻      | 3788      | ...         | 3820      | 3783 | 3789.4 | 3785.3 | 3796 |
| 2\( ^3D_1 \)    | \( \psi(2D) \) | 1⁻      | 4182      | 4191±1.5 | 4190 | 4159 | 4159.2 | 4150.4 | 4153 |
| 1\( ^3D_2 \)    | \( \psi(1D) \) | 2⁻      | 3834      | 3823      | 3840      | 3795 | 3822.1 | 3807.7 | 3810 |
| 2\( ^3D_2 \)    | \( \psi(2D) \) | 2⁻      | 4276      | ...         | 4212      | 4190 | 4195.8 | 4173.7 | 4160 |
| 1\( ^3D_3 \)    | \( \psi(1D) \) | 3⁻      | 3849      | ...         | ...      | 3844.8 | 3814.6 | ... | ... |
| 2\( ^3D_3 \)    | \( \psi(2D) \) | 3⁻      | 4229      | ...         | ...      | 4218.9 | 4182.9 | ... | ... |

Table 1: Mass spectrum (in MeV).
\[ \Delta M(1^3P_1 - 1^3P_0) = 76 \text{ MeV} \] which is slightly lower than the experimental value \[ \Delta M(1^3P_1 - 1^3P_0) = 95.5 \pm 0.8 \text{ MeV} \]. The Lattice QCD calculations predict a 1P splitting of 68.4 \pm 5.0 + 11.8 - 3.0 MeV \cite{59} which reasonably agree with our calculations. For \( 1^3P_2 - 1^3P_1 \) splitting we obtain \( \Delta M(1^3P_2 - 1^3P_1) = 49 \text{ MeV} \) which is in good agreement with the experimental data \( \Delta M(1^3P_2 - 1^3P_1) = 45.7 \pm 0.2 \text{ MeV} \). The lattice QCD calculations predict \( \Delta M(1^3P_2 - 1^3P_1) = 31.4 \pm 8.4 \text{ MeV} \). Our calculations are in better agreement with the lattice QCD calculations.

We take the ratio of the two fine structures, \( \frac{\Delta M(1^3P_2 - 1^3P_1)}{\Delta M(1^3P_1 - 1^3P_0)} \) which will shed light on the nature of the confinement. Our estimate of the ratio is \( \frac{\Delta M(1^3P_2 - 1^3P_1)}{\Delta M(1^3P_1 - 1^3P_0)} = 0.64 \). The experimental value of the ratio is \( \frac{\Delta M(1^3P_2 - 1^3P_1)}{\Delta M(1^3P_1 - 1^3P_0)} = 0.48 \). This confirms that our potential has both Lorentz scalar and vector nature. This also implies that the Coulomb term in the potential dominates in our model. Another interesting quantity is the P-state hyperfine splitting \( \Delta M(1^1P_1 - 1^3P) \), where,

\[
M(1^3P) = \frac{5M(1^3P_2) + 3M(1^3P_1) + M(1^3P_0)}{9}
\]

is the center of gravity of the P-wave system. The P-state hyperfine splitting should be very much smaller than S-state hyperfine splitting since the P state wave function is zero at the origin. Our estimate of the P-state hyperfine splittings is \( \Delta M(1^1P_1 - 1^3P_0) = 0.23 \text{ MeV} \). The experimental value of P-state hyperfine splittings is \( \Delta M(1^1P_1 - 1^3P_0) = 0.9 \text{ MeV} \). The lattice QCD calculations obtain P-state hyperfine splitting of -1.4 MeV. The spin averaged masses are defined by

\[
M(nS) = \frac{3M(n^3S_1) + M(n^1S_0)}{4}
\]

\[
M(nP) = \frac{3M(n^1P_1) + 5M(n^3P_2) + 3M(n^3P_1) + M(n^3P_0)}{12}
\]

with \( n=1,2,3,... \) the radial quantum numbers. The spin averaged masses calculated in our model are \( M(1S) = 3666.25 \text{ MeV} \), \( M(2S) = 3661.5 \text{ MeV} \), \( M(1P) = 3521.83 \text{ MeV} \) and \( M(2P) = 3941.75 \text{ MeV} \). The experimental value of spin averaged masses \( M(1S) \), \( M(2S) \) and \( M(1P) \) are 3067.6 MeV, 3663 MeV and 3525.5 MeV respectively \cite{59}.

The calculated spin averaged mass splittings and mass splittings are listed in Table 2.

| Mass Splittings | This work | Exp | Lattice QCD | GI | 61 | 62 | 63 |
|-----------------|-----------|-----|-------------|----|----|----|----|
| \( M(1^3S_1 - 1^1S_0) \) | 111 | 117.1 | 59.2 \pm 18 | 113 | 114 | 118 |
| \( M(2^3S_1 - 2^1S_0) \) | 42 | 92 | 45 \pm 9 | 53 | 44 | 50 |
| \( M(1^3P_2 - 1^3P_1) \) | 49 | 45.7 | 17.4 \pm 41 | 40 | 36 | 44 |
| \( M(1^3P_1 - 1^3P_0) \) | 76 | 195.5 | 30.6 \pm 37 | 65 | 101 | 77 |
| \( M(1^1P_1 - 1^3P) \) | 23 | 0.9 | -1.4 | | | |
| \( M(1^1P_1 - 1^3S) \) | 455.75 | 458.5 | 448.8 \pm 29 | | | |
| \( M(1^3P_0 - 1^3S) \) | 360.75 | 347.1 | 419.8 \pm 47 | | | |
| \( M(1^3P_2 - 1^3S) \) | 436.75 | 442.9 | 448.8 \pm 34 | | | |
| \( M(1^3P_1 - 1^3S) \) | 485.75 | 488.6 | 448.8 \pm 29 | | | |
| \( M(1^1P - 1^3S) \) | 455.58 | 457.9 | 457.9 | | | |
| \( M(2^3P - 2^3S) \) | 280.25 | ... | 462 \pm 72 | | | |
| \( M(2^3S - 1^3S) \) | 595.25 | 595 | 671 \pm 21 | | | |
| \( M(2^3P - 1^3P) \) | 419.92 | ... | 675 \pm 76 | | | |

Radiative decays of excited charmonium states are the powerful tools which can be used to study the internal structure and they provide a good test of the predictions for the various models.

The possible E1 decay modes have been listed in Table 3 and the predictions for E1 decay widths are given. Also our predictions have been compared with other theoretical models. Most of the predictions for E1 transitions are in qualitative agreement. However, there are some differences in the predictions due to differences in phase space arising from different mass predictions and also from the wave function effects. We find our results are compatible with other theoretical model values for most of the channels.

The M1 transition rates of charmonium states have been calculated using eqn (13). Allowed M1 transitions correspond to triplet-singlet transition between S-wave states and between P-state of the same \( n \) quantum number, while hindered M1 transitions are either triplet-singlet or singlet-triplet transitions between S-wave states of different quantum numbers. In order to calculate decay rates of hindered transitions we need to include relativistic corrections. There are three main types of corrections: relativistic modification of the nonrelativistic
wave functions, relativistic modification of the electromagnetic transition operator, and finite-size corrections. In addition to these, there are additional corrections arising from the quark anomalous magnetic moment.

### Table 3: M1 Transition rates

| Transition | k (MeV) | $\Gamma_{\text{Exp}} (i \rightarrow f)$ (keV) | $61^{\text{I}}$ (keV) | $61^{\text{II}}$ (keV) | $17^{\text{I}}$ (keV) | $17^{\text{II}}$ (keV) |
|------------|---------|---------------------------------|-----------------|-----------------|-----------------|-----------------|
| $1^4P_0 \rightarrow 1^4S_1 + \gamma$ | 383 | 146.9 | 119.5±8 | 152 | 114 | 97 | 139.3 |
| $1^3P_1 \rightarrow 1^3S_1 + \gamma$ | 409 | 264.825 | 295.8±13 | 314 | 239 | 241 | 330 | 38.4 |
| $1^3P_2 \rightarrow 1^3S_1 + \gamma$ | 520 | 517.322 | 517.322 | 498 | 352 | 482 | 465 | 546.4 |
| $2^3S_1 \rightarrow 1^3P_0 + \gamma$ | 458 | 364.186 | 384.2±16 | 424 | 313 | 315 | 421 | 319.4 |
| $2^3S_0 \rightarrow 1^3P_1 + \gamma$ | 245 | 12.94 | 63 | 26 | 47 | 34 | 25.2 |
| $2^3S_1 \rightarrow 1^3P_1 + \gamma$ | 108 | 10.46 | 49 | 36.5 | 72 | 17.4 |
| $2^3S_1 \rightarrow 1^3P_2 + \gamma$ | 169 | 13.144 | 28.0±1.2 | 54 | 29 | 42.8 | 48 | 29.1 |
| $1^3D_1 \rightarrow 1^3P_0 + \gamma$ | 120 | 7.95 | 26.6±1.1 | 38 | 24 | 30.1 | 43 | 26.5 |
| $1^3D_2 \rightarrow 1^3P_1 + \gamma$ | 285 | 76.65 | 79.2±16 | 125 | 77 | 99 | 146 | 104.9 |
| $1^3D_2 \rightarrow 1^3P_2 + \gamma$ | 236 | 2.95 | 3.88 | <24.6 | 4.9 | 3.3 | 6.8 | 1.9 |
| $1^3D_2 \rightarrow 1^3P_1 + \gamma$ | 331 | 211.976 | 307 | 268 | 313 | 321 | 256.7 |
| $1^3D_2 \rightarrow 1^3P_1 + \gamma$ | 282 | 44.605 | 64 | 66 | 69.5 | 79 | 61.8 |
| $1^3D_2 \rightarrow 1^3P_1 + \gamma$ | 289 | 191.51 | 339 | 344 | 389 | 398 |
| $1^3D_3 \rightarrow 1^3P_2 + \gamma$ | 297 | 207.195 | 272 | 296 | 402 | 340 |

Corrections to the wave function that give contribution to the transition amplitude are of two categories: (1) higher order potential corrections, which are distinguished as (a) the zero recoil effect and (b) recoil effects of the final state meson, and (2) color octet effects. The color octet effects are not included in potential model formulation and are not considered so far in radiative transitions. The spherical Bessel function $j_0(k_0r/2)$, introduced in eqn(14), takes into account the so called finite size effect (equivalently, re-summing the multipole-expanded magnetic amplitude to all orders). For small $k_0$, $j_0(k_0r/2)$ goes to 1, so that transitions with $n' = n$ have favored matrix elements, and the corresponding partial decay widths are suppressed by smaller $k_0^2$ factors. For a large value of photon energy (k), transitions with $n \neq n'$ have favored the matrix element, since $j_0(k_0r/2)$ becomes very small. M1 transition rates are very sensitive to hyperfine splittings of the levels due to the $k_0^2$ factor in eqn(14). The resulting M1 radiative transition rates of these states are presented in Table 4. In this table we give calculated values for decay rates of M1 radiative transitions in comparison with the other theoretical quark models. The M1 transition rates calculated in our model agree very well with the values predicted by other theoretical models.

### Table 4: M1 Transition rates

| Transition | k (MeV) | $\Gamma_{\text{Exp}} (i \rightarrow f)$ (keV) | $61^{\text{I}}$ (keV) | $61^{\text{II}}$ (keV) | $17^{\text{I}}$ (keV) | $17^{\text{II}}$ (keV) |
|------------|---------|---------------------------------|-----------------|-----------------|-----------------|-----------------|
| $1^3S_1 \rightarrow 1^3S_0 + \gamma$ | 113 | 3.17 | 1.58±0.37 | 2.9 | 2.4 | 1.5 | 2.2 |
| $2^3S_2 \rightarrow 2^3S_0 + \gamma$ | 43 | 0.175 | 0.143±0.027 | 0.21 | 0.17 | 3.1 | 3.8 |
| $2^3S_2 \rightarrow 2^3S_0 + \gamma$ | 705 | 2.02 | 0.97±0.027 | 4.6 | 9.6 | 6.1 | 6.9 |
| $2^3S_2 \rightarrow 2^3S_1 + \gamma$ | 589 | 0.600 | 7.9 | 5.6 | 0.70 | 0.71 |
| $3^3S_1 \rightarrow 2^3S_2 + \gamma$ | 546 | 1.117 | 0.61 | 2.6 | 3.2 | 3.7 |
| $3^3S_2 \rightarrow 2^3S_1 + \gamma$ | 514 | 0.744 | 1.3 | 0.84 | 1.7 | 1.6 |
| $3^3S_2 \rightarrow 3^3S_0 + \gamma$ | 1103 | 0.620 | 6.3 | 6.9 | 5.9 | 6.5 |
| $2^3P_1 \rightarrow 1^3P_0 + \gamma$ | 624 | 29.73 | 0.071 | 0.11 | 1.3 | 1.2 |
| $2^3P_1 \rightarrow 1^3P_1 + \gamma$ | 574 | 10.172 | 0.058 | 0.36 | 0.16 | 0.13 |
| $2^3P_1 \rightarrow 1^3P_1 + \gamma$ | 653 | 8.05 | 0.033 | 1.5 | 5.6 | 5.3 |
| $2^3P_2 \rightarrow 1^3P_0 + \gamma$ | 162 | 0.0016 | 0.67 | 1.3 | 1.0 | 0.89 |
| $2^3P_1 \rightarrow 1^3P_1 + \gamma$ | 475 | 2.811 | 0.050 | 0.045 | 0.15 | 0.13 |
| $2^3P_0 \rightarrow 1^3P_1 + \gamma$ | 186 | 0.0042 | 0.029 | 0.50 | 4.8 | 4.8 |

1 Non relativistic quark model
2 Relativized quark model
3 Non relativistic quark model
4 Relativized quark model
5 Non relativistic quark model
6 Relativistic quark model
Using the Van-Royen-Weisskopf relation [48], we have calculated annihilation decay widths like leptonic decay width, two-photon and two-gluon decay widths with the inclusion of radiative corrections. The resulting leptonic decay widths are listed in Table 5. Our predictions for leptonic decay widths have been compared with experiment and other theoretical models and are found to be in good agreement with them. We give predictions for two photon and two-gluon decay widths of charmonium states in Table 6 and Table 7 respectively. The two photon and two-gluon decay widths for charmonium states reasonably agree with the available experimental data and other theoretical models.

### Table 5: Leptonic Decay widths (in keV)

| State | This Work $\Gamma_{l^+l^-}$ | Exp $\Gamma_{l^+l^-}$ | [39] | [55] | [64] |
|-------|-----------------------------|------------------------|-------|-------|-------|
| $J/\psi$ | 3.112 | 5.59±0.14 | 3.589 | 4.28 | 12.13 |
| $\psi(2S)$ | 2.197 | 2.33±0.07 | 1.440 | 2.25 | 5.03 |
| $\psi(3S)$ | 1.701 | 0.86±0.07 | 0.975 | 1.66 | 3.48 |
| $1^3D_1$ | 0.275 | 0.242±0.030 | 0.096 | 0.09 | 0.056 |
| $2^3D_1$ | 0.223 | 0.83±0.07 | 0.112 | 0.16 | 0.096 |

Using the Van-Royen-Weisskopf relation [48], we have calculated annihilation decay widths like leptonic decay width, two-photon and two-gluon decay widths with the inclusion of radiative corrections. The resulting leptonic decay widths are listed in Table 5. Our predictions for leptonic decay widths have been compared with experiment and other theoretical models and are found to be in good agreement with them. We give predictions for two photon and two-gluon decay widths of charmonium states in Table 6 and Table 7 respectively. The two photon and two-gluon decay widths for charmonium states reasonably agree with the available experimental data and other theoretical models.

### Table 6: Two-Photon Decay widths (in keV)

| State | This work $\Gamma$ | Exp | [39] | [65] | [66] |
|-------|-------------------|-----|-------|-------|-------|
| $\eta_c$(1S) | 6.96 | 7.2±0.7 | 6.812 | 3.50 | 7.18 |
| $\eta_c$(2S) | 10.45 | 7.0±3.5 | 2.625 | 1.38 | 1.71 |
| $\eta_c$(3S) | 1.03 | 1.760 | 0.94 | 1.21 |
| $\chi_c0$(1P) | 13.43 | 2.36±0.35 | 2.119 | 1.39 | 3.28 |
| $\chi_c0$(2P) | 2.67 | 1.308 | 1.11 |
| $\chi_c2$(1P) | 1.72 | 0.66±0.07 | 0.261 | 0.44 |
| $\chi_c2$(2P) | 0.343 | 0.168 | 0.48 |

### Table 7: Two-Gluon Decay widths (in keV)

| State | This work $\Gamma$ | Exp | [39] | [67] | [68] |
|-------|-------------------|-----|-------|-------|-------|
| $\eta_c$(1S) | 28.60 | 28.6±2.2 | 22.048 | 15.70 | 32.209 |
| $\eta_c$(2S) | 42.90 | 14±7 | 8.496 | 8.10 |
| $\eta_c$(3S) | 4.26 | 5.696 |
| $\chi_c0$(1P) | 47.76 | 10.3±0.6 | 6.114 | 4.68 | 10.467 |
| $\chi_c0$(2P) | 9.50 | 3.775 |
| $\chi_c2$(1P) | 5.27 | 1.97±0.11 | 0.633 | 1.72 | 1.169 |
| $\chi_c2$(2P) | 1.04 | 0.401 |

4. Conclusions and Outlook

In this work, we use a heavy quark potential derived from the instanton vacuum along with Coulomb and linear confinement potentials for low lying charmonium $c\bar{c}$ states. The higher lying charmonium states are significantly influenced from systematic uncertainties, mainly due to the long-range part of the spin-independent potential which is less known. Our calculations for the low lying charmonia are basically in good agreement with experimental measurements.

Once all spin-dependent terms of the inter quark potential are determined we will gain new and valuable insights on the charmonium states. The spin singlet states of quarkonia are of particular importance since it gives the direct measurement of the hyperfine splitting between the energy levels. The hyperfine separation is directly related to the spin-spin interaction. The hyperfine mass splitting also provides a test of the Lorentz nature of quarkonium spectra. We found that the theoretical predictions of our model calculations are remarkably consistent with well established experimental data for the conventional charmonium states. We can find that below the open charm threshold, our theoretical calculations from the NRQM model excellently agree with the lattice spectroscopy and experiments [57, 58]. The mass splittings between the radial excitations and the ground state also provides an important check on the validity of our potential model. The spin-spin potential receives large uncertainties due to the discretization artifacts more than the spin-independent central potential.

The parameters in our work, are fixed from the mass spectroscopy and the same set of a parameters have been used to obtain the decay widths. Different potentials can reproduce the same spectra. Hence, in a given
model, one must be able to calculate other observables like annihilation decay widths, the radiative decay widths, etc. Heavy quarkonium decays provide a deeper insight on the exact nature of the inter-quark forces and decay mechanisms. For example, the leptonic decay widths are a probe of the quarkonium system that provide important information complementary to level spacings. As a consequence of the higher excited states of charmonium systems being known, it is natural to study their E1 and M1 transitions rates. As there are very few of the pseudoscalar excited states ($n^1S_1$) known, the experimental data for M1 transitions are scarce compared to E1 transitions.

The model wave function, model parameters and the masses of the low lying charmonia states obtained have been used to study some of the decay properties of low lying charmonia. Using the predicted masses and the radial wave function at the origin leptonic decays, two photon and two gluon decays are computed using the Van Royen-Weisskopf relation. The calculated values including the correction factor agree with the experimental values within a few MeV. These results also demonstrate the importance of the QCD correction factor including the decay constants and other short range phenomenon using potential model. The predicted decay widths are comparable with the experimental values for the low lying charmonia states. From our calculations we conclude that the inclusion of QCD correction factors are of importance for obtaining accurate results. The recent results from B factories and new programs at BES, CLEO, and GSI have led to a resurgence of interest in the physics of charmonia. We argue that a detailed experimental investigation of the spectrum of excited charmonia states and their decay properties will considerably improve our understanding of the nonperturbative aspects of QCD.

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