Characterization of balls by generalized Riesz energy

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Abstract
We show that balls, circles and 2-spheres can be identified by generalized Riesz energy among compact submanifolds of the Euclidean space that are either closed or with codimension 0, where the Riesz energy is defined as the double integral of some power of the distance between pairs of points. As a consequence, we obtain the identification by the interpoint distance distribution.

KEYWORDS
convex geometry, integral geometry, Riesz energy

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1 | INTRODUCTION

Suppose $X$ is a compact submanifold of $\mathbb{R}^d$ which is either a compact body $\Omega$, i.e. the closure of a bounded open set of $\mathbb{R}^d$, or a closed submanifold $M$. Let us consider the integral

$$I_q(X) = \int_{X \times X} |x - y|^q \, dx \, dy,$$

where $dx$ and $dy$ are the Lebesgue measures of $X$. It is well-defined if $q > -\dim X$. It is called the Riesz $q$-energy of $X$ when $X$ is a compact body and $-d < q < 0$.

Fix a submanifold $X$ and consider the power $q$ in the integral as a complex number, denoted by $z$ in what follows. Then (1.1) is well-defined on a domain $\{z \in \mathbb{C} : \Re z > -\dim X\}$, where the map $z \mapsto I_z(X)$ is holomorphic. Extend the domain of (1.1) by analytic continuation to a region of $\mathbb{C}$, which depends on the regularity of $X$ (it is the whole complex plane $\mathbb{C}$ if $X$ is smooth). Then we obtain a meromorphic function with only simple poles at some negative integers. We denote it by $B_X(z)$ and call it Brylinski’s beta function of $X$, as it can be expressed by the beta function when $X$ is a circle, sphere or a ball. It was introduced by Brylinski [2] for knots, studied by Fuller and Vemuri [9] for closed (hyper-)surfaces, and by Solanes and the author [16] for compact bodies.

The beta function provides geometric quantities of $X$. For example, the volumes of $X$ and of the boundary $\partial X$ if exists, the total squared curvature of closed curves or the Willmore functional of closed surfaces as residues, and some kind energies as values at special $z$’s. With these quantities, we are inclined to ask a question to what extent a space $X$ can be identified by the beta function $B_X(z)$. We begin with introducing some preceding results on the identification by closely related geometric quantities.

Let $f_X(r)$ be the interpoint distance distribution of $X$, which is given by

$$f_X(r) = \text{Vol}((x, y) \in X \times X : |x - y| \leq r).$$

It is equivalent to the Riesz energy (1.1) through the (inverse) Mellin transform as we will see later in Proposition 2.1.


Figure 1 Mallows and Clark's counter-example

Figure 2 Let $I_1$ and $I_2$ be reflections in lines $L_1$ and $L_2$ respectively, which form the angle $2q\pi$ ($q \in \mathbb{Q}$). Then $R = I_1I_2$ is the rotation by angle $2q\pi$. Let $\Omega_1, \Omega_2$ and $\Omega_3$ be mutually disjoint regions satisfying $I_1\Omega_1 = \Omega_1$, $I_2\Omega_2 = \Omega_2$, $R\Omega_3 = \Omega_3$ and $I_1\Omega_1 \neq \Omega_1$. Then $X = \Omega_1 \cup \Omega_3 \cup \Omega_2$ and $X' = \Omega_1 \cup R(\Omega_3 \cup \Omega_2) = \Omega_3 \cup I_1(\Omega_1 \cup \Omega_2)$. This is a picture after Caelli's paper

The chord length distribution of a convex body $K$ is given by

$$g_K(r) = \mu\{\ell' \in \mathcal{E}_1 : L(K \cap \ell') \leq r\},$$

where $\mathcal{E}_1$ is the set of lines in $\mathbb{R}^d$, $\mu$ is a measure on $\mathcal{E}_1$ that is invariant under motions of $\mathbb{R}^d$, and $L$ means the length. It is equivalent to the interpoint distance distribution for convex bodies in the sense that $g_K$ uniquely determines and is uniquely determined by $f'_K$ (for example, [13, p. 25]), which is a consequence of the Blaschke–Petkantschin formula (for example, (4.2), [18, p. 46]).

Let us first consider the identification problem of $X$ by the interpoint distance distribution; whether $f_X(r) = f_{X'}(r)$ for any $r$ implies $X = X'$ up to motions of $\mathbb{R}^d$. The picture is quite different according to whether we assume the convexity of $X$ or not, although the answer is negative in both cases.

In fact, for convex bodies, Mallows and Clark [12] gave a pair of non-congruent convex planar polygons with the same chord length distribution as illustrated in Figure 1, whereas Waksman [20] pointed out that it is exceptional by showing that a "generic" planar convex polygon can be identified by the chord length distribution.

On the other hand, for general case, Caelli [3] gave a method to produce pairs of non-congruent subsets of $\mathbb{R}^2$, which are not convex in general, with the same interpoint distance distribution by using two axes of symmetry, as is illustrated in Figure 2.

Let us next consider a weaker problem, whether balls and spheres can be identified by the interpoint distance distribution. Again, the picture is different according to whether we assume convexity or not.

Among convex bodies $K$, balls can be identified by the interpoint distance distribution. It follows directly from the fact that only balls give the maximum of the Riesz energy $I_q(K)$ for $-d < q < 0$ among all convex bodies $K$ with a given volume.
2 | PRELIMINARIES

Let \( \phi(t) \) be a function on the positive real axis which decays rapidly at both 0 and \( \infty \). The Mellin transform of \( h \), denoted by \( \tilde{\phi}(s) = \mathcal{M}[\phi; s] \), is given by

\[
\tilde{\phi}(s) = \int_0^\infty \phi(t) t^{s-1} dt \quad (s \in \mathbb{C}).
\]

To be precise, the domain where the integral converges depends on the function \( \phi \). The function \( \phi(t) \) can be recovered from \( \tilde{\phi}(s) \) by the inverse Mellin transform as

\[
\phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{\phi}(s) t^s \, ds \quad (c \in \mathbb{R}).
\]

**Proposition 2.1.** The Riesz energy (1.1) can be obtained from the derivative of the interpoint distance distribution via the Mellin transform as

\[
I_q(X) = \mathcal{M}[f'_X; q + 1] = \int_0^\infty r^q f'_X(r) \, dr \quad (q > -\dim X),
\]

and vice versa via the inverse Mellin transform as

\[
f'_X(r) = \mathcal{M}^{-1}[I_{q-1}(X); r] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-z} I_{q-1}(X) \, dz \quad (c > 1 - \dim X).
\]

**Proof.** The equality (2.1) is a consequence of the coarea formula ([5, 3.4.3], [6, 3.2.12]).

The convergence of the integrals can be deduced as follows. Since \( f_X(r) \sim v_n r^m \text{Vol}(X) \) for \( r \to 0^+ \), where \( n = \dim X \) and \( v_n \) is the volume of the unit \( n \)-ball, and \( f'_X(r) = 0 \) if \( r \) is bigger than the diameter of \( X \), which is denoted by \( \delta \), we have \( f'_X(r) = O(r^{m-1}) \) as \( r \to 0^+ \) and \( f'_X(r) = 0 \) if \( r > \delta \). It implies the existence of the inverse Mellin transform (2.2) for \( c > 1 - n \).

Next we show that the argument in [16] goes almost parallel even if we weaken the assumption of regularity of \( X \), and introduce some preceding results on the residues of the beta function from [2, 9, 16]. We start with a technical lemma.

**Lemma 2.2.** Suppose \( g \) is a function of class \( C^{k+1} \) \((k \geq 1)\) from a neighbourhood of 0 to \( \mathbb{R} \). If \( g \) satisfies \( g(0) = g'(0) = 0 \) then \( \tilde{g} \) defined by

\[
\tilde{g}(t) = \begin{cases} 
\frac{g(t)}{t}, & t \neq 0, \\
0, & t = 0,
\end{cases}
\]

is of class \( C^k \).
Proof. Remark that \( \bar{g}(t) \) is of class \( C^{k+1} \) at \( t \neq 0 \). We first show

\[
\bar{g}(j)(0) = \frac{\bar{g}^{(j+1)}(0)}{j+1} \quad (0 \leq j \leq k)
\]

(2.3)

inductively. It is satisfied when \( j = 0 \) by definition of \( \bar{g} \). Assume (2.3) is satisfied up to \( j \) \((0 \leq j \leq k-1)\). Using l'Hôpital's rule at the equality * indicated below, we obtain

\[
\bar{g}^{(j+1)}(0) = \lim_{h \to 0} \frac{\bar{g}^{(j)}(h) - \bar{g}^{(j)}(0)}{h}
\]

(2.4)

\[
= \lim_{h \to 0} \frac{\left( (g(t) \cdot t^{-1})^{(j)}(h) - \frac{\bar{g}^{(j+1)}(0)}{j+1} \right)}{h}
\]

\[
= \lim_{h \to 0} \frac{\sum_{i=0}^{j} \frac{j}{i} g^{(j-i)}(h) \cdot (-1)^i i! h^{-(i+1)} - \frac{\bar{g}^{(j+1)}(0)}{j+1} h^{j+1}}{h}
\]

\[
= \lim_{h \to 0} \frac{\sum_{i=0}^{j} (-1)^i \frac{j}{(j-i)!} g^{(j-i)}(h) h^{j-i} - \frac{\bar{g}^{(j+1)}(0)}{j+1} h^{j+1}}{h^{j+2}}
\]

\[
= \lim_{h \to 0} \frac{\frac{j}{j+2} \bar{g}^{(j+2)}(0)}{(j+2) h^{j+1}} \quad (0 \leq j \leq k-1),
\]

(2.5)

which shows that (2.3) is satisfied at \( j + 1 \).

The continuity of \( \bar{g}^{(j)}(t) \) at \( t = 0 \) for \( j \leq k-1 \) also follows from the above, to be precise, from the equality of (2.4) and ((2.5)). The continuity when \( j = k \) can be proved similarly;

\[
\lim_{h \to 0} \left( \bar{g}^{(k)}(h) - \bar{g}^{(k)}(0) \right) = \lim_{h \to 0} \frac{\bar{g}^{(k+1)}(h) h^k - \bar{g}^{(k+1)}(0) h^k}{(j+1) h^k} = 0,
\]

which completes the proof.

Let \( M \) be an \( m \) dimensional closed submanifold of \( \mathbb{R}^d \) \((d > m)\) and \( x \in M \). Put

\[
\psi_{M,x}(t) = \text{Vol}(M \cap B^d_x(t)) \quad \text{and} \quad \varphi_{M,x}(t) = \psi_{M,x}'(t),
\]

where \( B^d_x(t) \) is a \( d \)-ball with center \( x \) and radius \( t \). Then,

\[
I_z(M) = \int_{M \times M} |x - y|^z \, dx \, dy = \int_0^{\infty} t^z \left( \int_M \varphi_{M,x}(t) \, dx \right) dt
\]

(2.6)

for \( \Re z > -m \) ([16, Proposition 3.3]).

Let \( \Omega \) be a compact body in \( \mathbb{R}^d \) and let \( x \in M \). Put

\[
\psi_{v,x}(t) = \int_{\partial \Omega \cap B^d_x(t)} \langle n_x, n_y \rangle \, dy \quad \text{and} \quad \varphi_{v,x}(t) = \psi_{v,x}'(t),
\]
where \(n_x\) and \(n_y\) are outer unit normal vectors to \(\Omega\) at \(x\) and \(y\). Then,

\[
I_z(\Omega) = \int_{\Omega \times \Omega} |x - y|^z \, dxdy
\]

\[
= \frac{-1}{(z + 2)(z + d)} \int_{\partial \Omega \times \partial \Omega} |x - y|^{z+2} \langle n_x, n_y \rangle \, dxdy
\]

\[
= \frac{-1}{(z + 2)(z + d)} \int_0^\infty t^{z+2} \left( \int_{\partial \Omega} \varphi_{x,t}(t) \, dx \right) dt
\]

(2.7)

for \(\mathfrak{Re} z > -d\) and \(z \neq -2\) ([16, Lemma 4.1]).

**Proposition 2.3.**

(1) If \(M\) is an \(m\) dimensional closed submanifold of class \(C^{k+2}\) \((k \geq 1)\), then

\[
\varphi_{M,x}(t) = t^{m-1} \overline{\varphi}_{M,x}(t)
\]

for some \(\overline{\varphi}_{M,x}\) of class \(C^k\). Moreover, \(\overline{\varphi}_{M,x}(t)\) satisfies

\[
\overline{\varphi}_{M,x}(0) = \sigma_{m-1}, \quad \overline{\varphi}_{M,x}^{(2i-1)}(0) = 0 \quad (1 \leq 2i - 1 \leq k).
\]

(2.9)

(2) If \(\Omega\) is a compact body of class \(C^{k+2}\) \((k \geq 1)\), then \(\varphi_{\nu,x}(t) = t^{d-2} \overline{\varphi}_{\nu,x}(t)\) for some \(\overline{\varphi}_{\nu,x}\) of class \(C^k\), which satisfies \(\overline{\varphi}_{\nu,x}(0) = \sigma_{d-2}\) and \(\overline{\varphi}_{\nu,x}^{(2i-1)}(0) = 0\) \((1 \leq 2i - 1 \leq k)\).

The statements (1) and (2) above are \(C^k\) analogues of Theorem 3.3 of [9], and Proposition 3.1 and Corollary 3.2 of [16] respectively. Apart from the regularity, essentially same arguments also work. Here we give explicit description.

**Proof.** (1) We may assume, without loss of generality, that the point \(x\) is the origin, \(M\) is tangent to \(\mathbb{R}^m\) at \(x = 0\) and that a neighbourhood of \(x = 0\) of \(M\) can be expressed as a graph of a function \(f\) of class \(C^{k+2}\) from \(\mathbb{R}^m\) to \(\mathbb{R}^{d-m}\). Put

\[
U_t = \{ x \in \mathbb{R}^m : |x|^2 + |f(x)|^2 \leq t \} \quad (t \geq 0),
\]

take a sufficiently small positive number \(t_0\), define \(F : U_{t_0} \rightarrow \mathbb{R}^d\) by \(F(x) = (x, f(x))\), and let \(J : U_{t_0} \rightarrow \mathbb{R}_{>0}\) be the Jacobian given by

\[
J(x) = \det \left( \frac{\partial F}{\partial x_i}(x), \frac{\partial F}{\partial x_j}(x) \right)_{i,j=1,\ldots,m}.
\]

For a unit vector \(v \in S^{m-1}\), define \(t(v, \cdot)\) on a neighbourhood of 0 by

\[
t(v, \xi) = \text{sgn}(\xi) |F(\xi v)| = \xi \sqrt{1 + \left| \frac{f(\xi v)}{\xi} \right|^2} \quad (\xi \neq 0),
\]

\[
t(v, 0) = 0,
\]

where \(\text{sgn}(\xi)\) is the signature of \(\xi\). Since \(f\) is of class \(C^{k+2}\), Lemma 2.2 implies that \(t(v, \cdot)\) is of class \(C^{k+1}\).

Let \(\xi(v, \cdot)\) be the inverse function of \(t(v, \cdot)\);

\[
\xi(v, t) = \begin{cases} 
\xi \geq 0 & \text{such that } \xi^2 + |f(\xi v)|^2 = t^2 \quad (t \geq 0), \\
\xi < 0 & \text{such that } \xi^2 + |f(\xi v)|^2 = t^2 \quad (t < 0),
\end{cases}
\]

then \(\xi(v, \cdot)\) is of class \(C^{k+1}\). We have

\[
\xi'(v, 0) = 1,
\]

\[
\xi(-v, t) = -\xi(v, -t).
\]

(2.10)
Since
\[ \psi_{M,x}(t) = \int_{U_t} J(x) \, dx \]
\[ = \int_{S^{m-1}} \int_0^{\xi(v,t)} J(\xi) \xi^{m-1} \, d\xi \, dv \]
\[ = \int_{S^{m-1}} \int_0^t J(\xi(v,\tau)v) \xi(v,\tau)^{m-1} \frac{\partial \xi}{\partial t}(v,\tau) \, d\tau \, dv \]
\[ = \int_0^t \tau^{m-1} \left[ \int_{S^{m-1}} J(\xi(v,\tau)v) \left( \frac{\xi(v,\tau)}{\tau} \right)^{m-1} \frac{\partial \xi}{\partial t}(v,\tau) \, dv \right] \, d\tau, \]
we have
\[ \varphi_{M,x}(t) = t^{m-1} \int_{S^{m-1}} J(\xi(v,t)v) \left( \frac{\xi(v,t)}{t} \right)^{m-1} \frac{\partial \xi}{\partial t}(v,t) \, dv, \]
where we remark that by substituting \( g(t) = \xi(v,t) - t \) in Lemma 2.2 we obtain
\[ \left. \frac{\xi(v,t)}{t} \right|_{t=0} = 1. \]
Therefore we can put
\[ \overline{\varphi}_{M,x}(t) = \int_{S^{m-1}} J(\xi(v,t)v) \left( \frac{\xi(v,t)}{t} \right)^{m-1} \frac{\partial \xi}{\partial t}(v,t) \, dv. \]
Since \( \xi(v,\cdot) \) is of class \( C^{k+1} \), both \( \frac{\xi(v,t)}{t} \) and \( \frac{\partial \xi}{\partial t}(v,t) \) are of class \( C^k \) as functions of \( t \) by Lemma 2.2, which implies that \( \overline{\varphi}_{M,x}(t) \) is of class \( C^k \).

Put
\[ \eta(v,t) = J(\xi(v,t)v) \left( \frac{\xi(v,t)}{t} \right)^{m-1} \frac{\partial \xi}{\partial t}(v,t). \]
Since \( \overline{\varphi}_{M,x}(t) \) can be expressed by
\[ \overline{\varphi}_{M,x}(t) = \frac{1}{2} \int_{S^{m-1}} \left[ J(\xi(v,t)v) \left( \frac{\xi(v,t)}{t} \right)^{m-1} \frac{\partial \xi}{\partial t}(v,t) + J(\xi(-v,t)v) \left( \frac{\xi(-v,t)}{t} \right)^{m-1} \frac{\partial \xi}{\partial t}(-v,t) \right] \, dv, \tag{2.11} \]
the relation (2.10) implies that the integrand of (2.11), which we denote by \( \Phi_{M,x}(t) \), is
\[ \Phi_{M,x}(t) = \frac{\eta(v,t) + \eta(v,-t)}{2}. \]
Since
\[ J(0) = 1, \left. \frac{\xi(v,t)}{t} \right|_{t=0} = 1 \text{ and } \frac{\partial \xi}{\partial t}(v,0) = 1, \]
\( \Phi_{M,x}(0) = 1 \) for any \( v \in S^{m-1} \), which implies \( \overline{\varphi}_{M,x}(0) = \sigma_{m-1} \), and since
\[ \left. \left( \frac{\partial}{\partial t} \right)^{2i-1} \Phi_{M,x}(t) \right|_{t=0} = \frac{1}{2} \left. \left( \frac{\partial}{\partial t} \right)^{2i-1} (\eta(v,t) + \eta(v,-t)) \right|_{t=0} = 0, \]
we have \( \overline{\varphi}_{M,x}^{(2i-1)}(0) = 0 \).

(2) The same statements for \( \overline{\varphi}_{v,x}(t) \) can be proved in the same way. \( \square \)
Since the formulae (2.6) and (2.8) imply
\[
I_\omega(M) = \int_0^\infty t^{z+m-1} \left( \int_M \psi_{M,\omega}(x) \, dx \right) \, dt,
\]
\[
I_\omega(\Omega) = \frac{-1}{(z+2)(z+d)} \int_0^\infty t^{z+d} \left( \int_M \psi_{\Omega,\omega}(x) \, dx \right) \, dt,
\]
the regularization of \( I_\omega(M) \) and \( I_\omega(\Omega) \) can be reduced to that of an integral of the form \( I_{w,\phi} = \int_0^\infty t^{w} \phi(t) \, dt \). If \( \phi(t) \) is of class \( C^k \) then the integrand of the first term of the right hand side of
\[
I_{w,\phi} = \int_0^\infty t^{w} \phi(t) \, dt
\]

\[
= \int_0^1 t^{w} \left[ \phi(t) - \phi(0) - \phi'(0)t - \cdots - \frac{\phi^{(k-1)}(0)}{(k-1)!} t^{k-1} \right] \, dt + \int_1^\infty t^{w} \phi(t) \, dt + \sum_{l \leq j \leq k} \frac{\phi^{(j)}(0)}{(j-1)!} \left( \frac{1}{(w+j)!} \int_0^\infty (t^{w+j}) \, dt \right)
\]

([11, Ch.1, 3.2]) can be estimated by \( t^{w+k} \), and hence the integral converges for \( \Re w > -k - 1 \). Therefore \( I_{w,\phi} = \int_0^\infty t^{w} \phi(t) \, dt \) is meromorphic on \( \Re w > -k - 1 \) having possible simple poles at \( w = -1, \ldots, -k \) with the residue at \( w = -j \) given by \( \phi^{(j)}(0)/(j-1)! \) for \( j = 1, \ldots, k \). Since \( \psi_{M,\omega} \) and \( \psi_{\Omega,\omega} \) are of class \( C^k \) and \( \psi_{M,\omega}^{(2)}(0) = \psi_{\Omega,\omega}^{(2)}(0) = 0 \), by putting \( w = z + m - 1 \) for \( M \) or \( w = z + d \) for \( \Omega \), we obtain the following.

**Corollary 2.4.**

1. Suppose \( M \) is an \( m \) dimensional closed submanifold of \( \mathbb{R}^d \) of class \( C^{k+2} \) \((k \geq 1)\). The beta function \( B_M(z) \) is meromorphic on \( \Re z > -m - k \) which has possible simple poles at \( z = -m - 2i \), where \( 0 \leq 2i \leq k - 1 \), with

\[
\text{Res}(B_M, -m - 2i) = \frac{1}{(2i)!} \int_M \psi_{M,\omega}^{(2i)}(0) \, dx \quad (0 \leq 2i \leq k - 1).
\]

In particular,

\[
\text{Res}(B_M, -m) = \sigma_{m-1} \text{Vol}(M),
\]

where \( \sigma_j \) is the volume of the unit \( j \)-sphere.

2. Suppose \( \Omega \) is a compact body in \( \mathbb{R}^d \) of class \( C^{k+2} \) \((k \geq 1)\). The beta function \( B_\Omega(z) \) is meromorphic on \( \Re z > -d - k - 1 \) which has possible simple poles at \( z = -d - (2i + 1) \), where \( 1 \leq 2i + 1 \leq k \), with

\[
\text{Res}(B_\Omega, -d - (2i + 1)) = \frac{-1}{(d+2i-1)(2i+1)!} \int_{\partial \Omega} \psi_{\Omega,\omega}^{(2i)}(0) \, dx \quad (1 \leq 2i + 1 \leq k).
\]

In particular,

\[
\text{Res}(B_\Omega, -d - 1) = -\frac{\sigma_{d-2}}{d-1} \text{Vol}(\partial \Omega).
\]

**Remark 2.5.**

- Equation (2.13) for smooth closed curves was given in [2]. The formulae (2.12), (2.14) and (2.15) for smooth case were given in [16].
- When \( M \) is a closed surface in \( \mathbb{R}^3 \), the second residue which appears at \( z = -4 \) is given by

\[
\text{Res}(B_M, -4) = \frac{8}{3} \int_M (\kappa_1 - \kappa_2)^2 \, dx,
\]

where \( \kappa_1 \) and \( \kappa_2 \) are principal curvatures of \( M \) (Theorem 4.1 of [9]; see also Proposition 3.8 of [16] for the correction of the coefficient).
• The first residue of \( B_{Ω}(z) \) which appears at \( z = -d \) is given by

\[
\text{Res}(B_{Ω}, -d) = \sigma_{d-1} \text{Vol}(Ω),
\]

which can be computed using (2.7) without using differentiability of \( \bar{q}_{σ, x}(r) \) ([16, Lemma 4.5]).

• The residues of the beta function do not indicate the number of the connected components of \( X \) immediately.

## 3 Identification of Balls and Spheres

Let \( B^0(r), S^1(r), \) and \( S^2(r) \) be an \( n \)-ball, circle, and a 2-sphere of radius \( r \) respectively.

**Lemma 3.1.** If \( X \) is a disjoint union of closed curves in \( \mathbb{R}^d, B_X(-2) \) ≥ 0 with equality if and only if \( X \) is a single circle.

*Proof.* Brylinski showed that \( B_C(-2) = E(C) - 4 \) for a single curve \( C \), where

\[
E(C) = \int_{C \times C} (|x - y|^2 - d_k(x, y)^2) \ dx \ dy
\]

\[
= 4 + \int_C \left[ \lim_{\varepsilon \to 0^+} \int_{d_k(x, y) \geq \varepsilon} |x - y|^2 \ dy - \frac{2}{\varepsilon} \right] dx,
\]

where \( d_k(x, y) \) is the (shortest) arc-length between \( x \) and \( y \) along \( C \).\(^1\) Freedman, He and Wang showed that \( E(C) \geq 4 \) for any single closed curve \( C \) in \( \mathbb{R}^3 \) with equality if and only if \( C \) is a circle ([8, Corollary 2.2]).

Since the definition of the energy and the proofs of the above statements do not use the condition that the dimension of the ambient space is equal to 3, the above argument holds regardless of the codimension.

Suppose \( X \) is a disjoint union of \( n \) closed curves; \( X = C_1 \cup \cdots \cup C_n \). We have

\[
B_X(-2) = \sum_{i=1}^n B_{C_i}(-2) + 2 \sum_{i<j} \int_{C_i} \int_{C_j} |x - y|^2 \ dx \ dy \geq \sum_{i=1}^n B_{C_i}(-2) \geq 0,
\]

where the second equality holds if and only if \( C_i \) is a circle for any \( i \) and the first equality holds if and only if \( n = 1 \). \( \square \)

**Lemma 3.2.** Let \( X = S^2_1(r_1) \cup S^2_2(r_2) \) be a disjoint union of two 2-spheres in \( \mathbb{R}^3 \) with radii \( r_1 \) and \( r_2 \) such that the diameter of \( X \) is not greater than 2. Put

\[
\Delta_{2-\varepsilon}^C = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x - y| > 2 - \varepsilon\}.
\]

Then there are positive constants \( \varepsilon_1 \) and \( C \) depending on \( r_1 \) and \( r_2 \) such that if \( 0 < \varepsilon < \varepsilon_1 \) then

\[
\frac{\text{Vol}(\{S^2_1(r_1) \times S^2_2(r_2) \cap \Delta_{2-\varepsilon}^C\})}{\text{Vol}(S^2_1(r_1) \times S^2_2(r_2))} < C\varepsilon^2.
\]

*Proof.* Put \( \varepsilon_1 = \min\{1, r_1, r_2\} \). Since the numerator of the left hand side of (3.3) is an increasing function of the distance between two spheres, we have only to show the inequality when the distance is equal to \( 1 - 2r_1 - 2r_2 \). Therefore we may assume both \( S^2_1(r_1) \) and \( S^2_2(r_2) \) are contained in the unit ball with center the origin.

If \( (x, y) \in \{S^2_1(r_1) \times S^2_2(r_2) \cap \Delta_{2-\varepsilon}^C\} \) then

\[
2 - \varepsilon < |x - y| \leq |x| + |y| \leq |x| + 1,
\]

which means that \( x \) is in the complement of the ball with center the origin and radius \( 1 - \varepsilon \), which we denote by \( (B^3(1 - \varepsilon))^C \).

Let us show

\[
A\left(S^2_1(r_1) \cap (B^3(1 - \varepsilon))^C\right) = 2\pi r_1 \frac{\varepsilon \left(1 - \frac{\varepsilon}{2}\right)}{1 - r_1}, \tag{3.4}
\]

where \( A \) means the area. Assume \( \varepsilon < 2r_1 \) so that \( S^2_1(r_1) \) intersects \( B^3(1 - \varepsilon) \).
FIGURE 3 The unit sphere, $B^1(1-\varepsilon)$ and $S^2_1(r_1)$

FIGURE 4 A spherical cap

Let $O$ be the origin and $C$ be the center of $S^2_1(r_1)$ (Figure 3). Let $A$ be the tangent point of $S^2_1(r_1)$ and the unit sphere with center $O$, and $B$ be a point in $S^2_1(r_1) \cap \partial B^1(1-\varepsilon)$. Then $S^2_1(r_1) \cap (B^1(1-\varepsilon))^c$ is a spherical cap (Figure 4) of hight $AH$, where $H$ is the foot of the perpendicular from $B$ to the line $AO$. Let $\theta$ be the angle $\angle BOC$. Since

$$\cos \theta = \frac{(1-r_1)^2 + (1-\varepsilon)^2 - r_1^2}{2(1-r_1)(1-\varepsilon)},$$

we have

$$AH = 1 - (1-\varepsilon) \cos \theta = \frac{\varepsilon(1-\frac{\varepsilon}{2})}{1-r_1}.$$ 

The area of the spherical cap is given by $2\pi r_1 \cdot AH$, which implies (3.4).
By (3.4) we have
\[
\frac{\text{Vol}(\{S_1^2(r_1) \times S_2^2(r_2)\} \cap \Delta_{2-\varepsilon})}{\text{Vol}(S_1^2(r_1) \times S_2^2(r_2))} \leq \frac{A(S_1^2(r_1) \cap (B^3(1-\varepsilon))^c \cdot A(S_2^2(r_2) \cap (B^3(1-\varepsilon))^c)}{A(S_1^2(r_1)) \cdot A(S_2^2(r_2))}
\]
\[
= \frac{\varepsilon^2 \left(1 - \frac{\varepsilon}{2}\right)^2}{4r_1r_2(1-r_1)(1-r_2)},
\]
which implies that if we put
\[
C = \frac{1}{4r_1r_2(1-r_1)(1-r_2)}
\]
then the inequality (3.3) is satisfied. □

**Lemma 3.3.** Suppose X is a disjoint union of n 2-spheres in \(\mathbb{R}^3\) that has the same area and diameter as \(S^2(r)\). If \(n > 1\) then X has a different interpoint distance distribution, and hence a different beta function, from \(S_1^2\).

**Proof.** We may assume without loss of generality that \(r = 1\). Assume that \(X = S_1^2(r_1) \cup \cdots \cup S_n^2(r_n)\) with \(n > 1\), \(r_1 \geq \cdots \geq r_n\), \(r_1^2 + \cdots + r_n^2 = 1\) and that the diameter of \(X\) is equal to 2. Put \(\varepsilon_0 = \min\{2 - 2r_1, r_n, 1\}\). Then if \(0 < \varepsilon < \varepsilon_0\) then \((S_1^2(r_1) \times S_2^2(r_2)) \cap \Delta_{2-\varepsilon} = \emptyset\) for any \(i\). Therefore, Lemma 3.2 implies
\[
\text{Vol}(X \times X) \cap \Delta^c_{2-\varepsilon} = \sum_{i \neq j} \text{Vol}\left((S_i^2(r_i) \times S_j^2(r_j)) \cap \Delta^c_{2-\varepsilon}\right) \leq C_0 \varepsilon^2 \sum_{i \neq j} \text{Vol}(S_i^2(r_i) \times S_j^2(r_j)),
\]
where
\[
C_0 = \max_{i \neq j} \frac{1}{4r_ir_j(1-r_i)(1-r_j)}.
\]
If we take \(\varepsilon > 0\) so that \((C_0 + \frac{1}{2})\varepsilon < 1\) then
\[
\text{Vol}(X \times X) \cap \Delta^c_{2-\varepsilon} < C_0 \varepsilon^2 \text{Vol}(X \times X) < \left(\varepsilon - \frac{\varepsilon^2}{4}\right) \text{Vol}(S^2 \times S^2) = \text{Vol}\left((S^2 \times S^2) \cap \Delta^c_{2-\varepsilon}\right),
\]
which completes the proof. □

Let \(A \cong B\) denote that \(A\) is congruent to \(B\).

**Theorem 3.4.** Let \(X\) be either a compact body or a closed submanifold of a Euclidean space.

1. If \(X\) is a compact body of class \(C^3\) then \(B_X(z) = B_{B^n(r)}(z) (\text{Re } z > -n - 2)\) for some natural number \(n\) implies \(X \cong B^n(r)\).
2. If \(X\) is of class \(C^4\) then \(B_X(z) = B_{B^n(r)}(z) (\text{Re } z > -n - 2)\) for some natural number \(n\) implies \(X \cong B^n(r)\).
3. If \(X\) is of class \(C^4\) then \(B_X(z) = B_{S^1(r)}(z) (\text{Re } z > -3)\) implies \(X \cong S^1(r)\).
4. If \(X\) is of class \(C^5\) and the codimension is either 0 or 1 then \(B_X(z) = B_{S^1(r)}(z) (\text{Re } z > -5)\) implies \(X \cong S^2(r)\).

**Proof.** (0) First note that Corollary 2.4 implies that the regularity of \(X\) in the assumption of each statement of the theorem guarantees that the beta function is well-defined on each indicated region.

Next, note that if \(X\) is of class \(C^4\) then one can judge from \(B_X(z)\) whether \(X\) is a compact body or a closed submanifold. To be precise, when the first pole occurs at \(z = z_0\) (which indicates that \(\dim X = -z_0\)), \(X\) is a compact body if and only if \(B_X(z)\) has a pole at \(z = z_0 - 1\). Remark that Corollary 2.4 implies that the residue at \(z = z_0 - 1\) is defined when \(X\) is of class \(C^4\).

(1) By Equations (2.17) and (2.15), the residues at \(z = -n\) and \(-n - 1\) imply that \(\dim X = n\), \(\text{Vol}(X) = \text{Vol}(B^n(r))\), and \(\text{Vol}(\partial X) = \text{Vol}(S^{n-1})\). Then the isoperimetric inequality in general dimension ([6, 3.2.43], [5, 5.6.2]) implies that \(X\) is an \(n\)-ball with radius \(r\).

(2) The above explanation (0) indicates that (2) can be reduced to (1).

(3) Suppose \(B_X(z) = B_{S^1}(z)\). Then \(X\) is a union of closed curves by (0) above. By Lemma 3.1, \(X\) is a single circle. By Equation (2.13), the residue at \(z = -1\) implies that \(L(X) = 2\pi r\), and hence \(X = S^1(r)\).
Suppose \( B_X(z) = B_{S^2(r)}(z) \). Then \( X \) is a union of 2-dimensional closed surfaces by (0) above, and hence, by the assumption on the codimension, \( X \) is a surface in \( \mathbb{R}^3 \). Since \( \text{Res}(B_X, -4) = 0 \), Equation (2.16) shows that \( X \) is totally umbilic, which implies that each connected component of \( X \) is part of either a sphere or a plane (Meusnier 1785). Since \( X \) is a closed surface, \( X \) is a union of 2-dimensional closed surfaces by (0) above, and hence, by the assumption on the codimension, \( X \) is a surface in \( \mathbb{R}^3 \). Since \( \text{Res}(B_X, -4) = 0 \), Equation (2.16) shows that \( X \) is totally umbilic, which implies that each connected component of \( X \) is part of either a sphere or a plane (Meusnier 1785). Since \( X \) is a closed surface, \( X \) is a union of spheres. By (2.13), \( \text{Res}(B_X, -2) = \text{Res}(B_{S^2(r)}, -2) \) implies that the area of \( X \) is the same as that of \( S^2(r) \). Since the diameter of \( X \) is given by \( \lim_{n \to \infty} (B_X(n))^{1/n} \), \( X \) has the same diameter as \( S^2(r) \). Now the conclusion follows from Lemma 3.3.

**Corollary 3.5.** Under the same assumptions on the regularity and the codimension as in Theorem 3.4, balls, circles, and 2-spheres can be identified by the interpoint distance distribution.

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## ENDNOTE
1. The formula (3.2) minus 4 was given in [15], and the formula (3.1) was first given in [14] and studied in [8]. The functional \( E(C) \) is called Möbius energy in [8] since it is invariant under Möbius transformation of \( \mathbb{R}^3 \cup \{ \infty \} \) ([8, Theorem 2.1]).

## REFERENCES
[1] D. Auckly and L. Sadun, *A family of Möbius invariant 2-knot energies*, Geometric Topology, Proceedings of the 1993 Georgia International Topology Conference (W. H. Kazez, ed.), AMS/IP Stud. Adv. Math., Amer. Math. Soc., Providence, RI, 1997, pp. 235–258.
[2] J.-L. Brylinski, *The beta function of a knot*, Internat. J. Math. 10 (1999), 415–423.
[3] T. Caelli, *On generating spatial configurations with identical interpoint distance distributions*, Combinatorial Mathematics, VII, Proceedings of the Seventh Australian Conference (Univ. Newcastle, Newcastle, 1979), Lecture Notes in Math., vol. 829, Springer, Berlin, 1980, pp. 69–75.
[4] P. Davy, *Inequalities for moments of secant length*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 68 (1984), no. 2, 243–246.
[5] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992.
[6] H. Federer, *Geometric measure theory*, Springer, 1969.
[7] W. Fenchel, *Über Krümmung und Windung geschlossener Raumkurven*, Math. Ann. 101 (1929), 238–252.
[8] M. H. Freedman, Z.-X. He, and Z. Wang, *Möbius energy of knots and unknots*, Ann. of Math. (2) 139 (1994), 1–50.
[9] E. J. Fuller and M. K. Vemuri, *The Brylinski beta function of a surface*, Geom. Dedicata 179 (2015), 153–160.
[10] R. J. Gardner, *Geometric tomography*, second edition, Cambridge University Press, New York, 2006.
[11] I. M. Gelfand and G. E. Shilov, *Generalized functions. Volume I: Properties and operations*, Academic Press, New York and London, 1967.
[12] C. L. Mallows and J. M. C. Clark, *Linear-intercept distributions do not characterize plane sets*, J. Appl. Prob. 7 (1970), 240–244.
[13] B. Matérn, *Spatial variation*, Springer-Verlag, Berlin, 1985.
[14] N. Nakauchi, *A remark on O’Hara’s energy of knots*, Proc. Amer. Math. Soc. 118 (1993), 293–296.
[15] J. O’Hara, *Energy of a knot*, Topology 30 (1991), 241–247.
[16] J. O’Hara and G. Solanes, *Regularized Riesz energies of submanifolds*, Math. Nachr. 291 (2018), no. 8–9, 1356–1373.
[17] L. A. Santaló, *On the measure of line segments entirely contained in a convex body*, Aspects of Mathematics and Its Applications, North-Holland Math. Library, vol. 34, North-Holland, Amsterdam, 1986, pp. 677–687.
[18] L. A. Santaló, *Integral geometry and geometric probability*, Addison-Wesley Publishing Company, 1976.
[19] R. Schneider, *Inequalities for random flats meeting a convex body*, J. Appl. Probab. 22 (1985), 710–716.
[20] P. Waksman, *Plane polygons and a conjecture of Blaschke’s*, Adv. Appl. Prob. 17 (1985), no. 4, 774–793.

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