ON THE HIT PROBLEM FOR THE STEENROD ALGEBRA
IN SOME GENERIC DEGREES AND APPLICATIONS

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ABSTRACT. Let $\mathcal{P}_n := H^*\left(\mathbb{RP}^\infty\right)^n \cong \mathbb{F}_2[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over the prime field of two elements, $\mathbb{F}_2$. We investigate the Peterson hit problem for the polynomial algebra $\mathcal{P}_n$, viewed as a graded left module over the mod-2 Steenrod algebra, $\mathcal{A}$. For $n > 4$, this problem is still unsolved, even in the case of $n = 5$ with the help of computers.

The purpose of this paper is to continue our study of the hit problem by developing a result in [17] for $\mathcal{P}_n$ in the generic degree

$$r(2^s - 1) + m,$$

where $r = n = 5$, $m = 13$, and $s$ is an arbitrary non-negative integer. Note that for $s = 0$, and $s = 1$, this problem has been studied by Phuc [16], and [17], respectively.

As an application of these results, we get the dimension result for the polynomial algebra in the generic degree

$$d = (n - 1)(2^n + u - 1) - 1,$$

where $u$ is an arbitrary non-negative integer, and $n = 6$.

One of the major applications of hit problem is in surveying a homomorphism introduced by Singer, which is a homomorphism

$$Tr_n : \text{Tor}_{n,n+d}^A(\mathbb{F}_2, \mathbb{F}_2) \rightarrow (\mathbb{F}_2 \otimes_\mathcal{A} \mathcal{P}_n)^{GL(n; \mathbb{F}_2)}$$

from the homology of the Steenrod algebra to the subspace of $(\mathbb{F}_2 \otimes_\mathcal{A} \mathcal{P}_n)^d$ consisting of all the $GL(n; \mathbb{F}_2)$-invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $\text{Tor}_{n,n+d}^A(\mathbb{F}_2, \mathbb{F}_2)$. The behavior of the fifth Singer algebraic transfer in degree $5(2^s - 1) + 13.2^s$ was also discussed at the end of this paper.

1. Introduction

Throughout the paper, we denote a prime field with two elements by $\mathbb{F}_2$. Let $\mathbb{RP}^\infty$ be the infinite dimensional real projective spaces. Then, $H^*\left(\mathbb{RP}^\infty\right)^n \cong \mathbb{F}_2[x_1]$, and therefore, the mod-2 cohomology algebra of the direct product of $n$ copies of $\mathbb{RP}^\infty$ is isomorphic to the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_n]$, reviewed as an unstable $\mathcal{A}$-module on $n$ generators $x_1, x_2, \ldots, x_n$, each of degree one.

The action of $\mathcal{A}$ on $\mathcal{P}_n$ is determined by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & i > 1. \end{cases}$$

and the Cartan formula $Sq^k(uv) = \sum_{i=0}^k Sq^i(u)Sq^{k-i}(v)$, where $u, v \in \mathcal{P}_n$ (see Steenrod and Epstein [23]).

One of central problems of Algebraic Topology is to determine a minimal set of generators for the ring of invariants $(\mathcal{P}_n)^{G_n}$, where $G_n \subset GL(n; \mathbb{F}_2)$, the general
linear group of invertible matrices. This ring is stable under the action of $\mathcal{A}$. The problem is so-called hit problem for the polynomial algebra. If we consider $\mathbb{F}_2$ as a trivial $\mathcal{A}$-module, then the hit problem is equivalent to the problem of finding a basis for the $\mathbb{F}_2$-graded vector space 

$$\{ (\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)^{G_n} \}_{d \geq 0}. \quad (1.1)$$

The structure of this space has first been studied by Peterson \[15\], Wood \[33\], Singer \[20\], Priddy \[18\], Janfada-Wood \[5, 6\] with Sum-Tin \[27, 28\], the present writer \[29\], \[30\] and others. Janfada \[7, 8\], Nam \[14\], Repka-Selick \[19\], Silverman \[22\], Wood \[33\], Sum \[24, 25\], Sum-Tin \[27, 28\] the problem is studied by Singer \[20\] for $G_2$, Kameko \[10\], who show its relationship to several classical problems in cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, stable homotopy type of the classifying space of finite groups. Then, this problem was investigated by Hung-Peters on \[4\], Kameko \[10\], for $G_2 \times G_2$, cobordism theory, modular representation theory, Adams spectral sequence for the Singer \[20\], Priddy \[18\], who show its relationship to several classical problems in $G_2$. In general, it is proved by Wood \[33\]. This is an extremely useful tool for determining $\mathcal{A}$-generators for $\mathcal{P}_n$.

For a natural number $d$, let $\alpha(d)$ be the number of digits 1 in the binary expansion of $d$. We define a function $\mu : \mathbb{N} \to \mathbb{N}$ is given by

$$\mu(0) = 0, \text{ and } \mu(d) = \min \{ m \in \mathbb{N} : d = \sum_{i=1}^{m} (2^{d_i} - 1), d_i > 0 \} = \min \{ m \in \mathbb{N} : \alpha(d + m) \leq m \}. $$

In \[15\], Peterson conjectured that as a module over the Steenrod algebra $\mathcal{A}$, $\mathcal{P}_n$ is generated by monomials in degree $d$ and satisfying the inequality $\alpha(d + n) \leq n$, where $\alpha(d)$ is the number of digits one in the binary expansion of $d$, and proved it for $n \leq 2$, in general, it is proved by Wood \[33\]. This is an extremely useful tool for determining $\mathcal{A}$-generators for $\mathcal{P}_n$.

One of the main tools in the study of the hit problem is Kameko’s squaring operation

$$\widetilde{S}_{q^0} := (\widetilde{S}_{q^0})_{(n; 2n + d)} : (\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_{2n + d} \to (\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d,$$

which is induced by an $\mathbb{F}_2$-linear map $S_n : \mathcal{P}_n \to \mathcal{P}_n$, given by

$$S_n(x) = \begin{cases} y, & \text{if } x = x_1 x_2 \ldots x_k y^2 \\ 0, & \text{otherwise} \end{cases}$$

for any monomial $x \in \mathcal{P}_n$. Clearly, $(\widetilde{S}_{q^0})_{(n; 2n + d)}$ is an $\mathbb{F}_2$-epimorphism.
From the results of Wood \[33\], Kameko \[9\], and Sum \[25\], the hit problem is reduced to the case of degree \(d\) of the form \(d = r(2^s - 1) + 2^t m\), where \(r, m, t\) are non-negative integers such that \(0 \leq \mu(m) < r \leq n\).

Now, the structure of the space \((1.2)\) was completely calculated for \(n \leq 4\), (see Peterson \[15\] for \(n = 1\), and \(n = 2\), see Kameko for \(n = 3\) in his thesis \[9\], see Sum \[25\] for \(n = 4\)). For \(n \geq 5\), it is still unsolved, even in the case of \(n = 5\) with the help of computers.

In the present paper, we develop a result in \[17\] on the hit problem for \(P_n\) in the generic degree \(r(2^s - 1) + m.2^s\), where \(r = n = 5\), \(m = 13\), and \(s\) is an arbitrary non-negative integer. Remarkably, for \(s = 0\), and \(s = 1\), this problem has been studied by Phuc \[16\], and \[17\], respectively. Moreover, as an application of the above results, we get the dimension results for the graded polynomial algebra in the generic degree \(d = (n - 1).(2^{n+u-1} - 1) + \ell.2^{n+u-1}\) where \(u\) is an arbitrary non-negative integer, \(\ell \in \{23, 67\}\), and \(n = 6\).

Note that the general linear group \(GL(n; \mathbb{F}_2)\) acts naturally on \(P_n\) by matrix substitution. Since the two actions of \(A\) and \(GL(n; \mathbb{F}_2)\) upon \(P_n\) commute with each other, there is an action of \(GL(n; \mathbb{F}_2)\) on \(\mathbb{F}_2 \otimes_A P_n\). From this event, the hit problem becomes one of the main tools for the studying of the general linear groups over \(\mathbb{F}_2\). Recently, the hit problem and its applications to representations of general linear groups have been presented in the books of Walker and Wood \[31, 32\].

On the other hand, one of the major applications of hit problem is in surveying a homomorphism introduced by W. M. Singer. It is a useful tool in describing the general linear groups have been presented in the books of Walker and Wood \[31, 32\].

Singer \[20\] defined the algebraic transfer, which is a homomorphism

\[
\psi_n : \mathbb{F}_2 \otimes_{GL(n; \mathbb{F}_2)} PH^*((\mathbb{R} \mathcal{P}^\infty)^n) \rightarrow Ext_A^{n,n+*}(\mathbb{F}_2, \mathbb{F}_2).
\]

Singer has indicated the importance of the algebraic transfer by showing that \(\psi_n\) is a isomorphism with \(n = 1, 2\) and at some other degrees with \(n = 3, 4\), but he also disproved this for \(\psi_5\) at degree 9, and then gave the following conjecture.

**Conjecture 1.1.** For any \(n \geq 0\), the algebraic transfer \(\psi_n\) is a monomorphism.

It could be seen from the work of Singer the meaning and necessity of the hit problem. In \[1\], Boardman confirmed this again by using the modular representation theory of linear groups to show that \(\psi_3\) is also an isomorphism.

For \(n \geq 4\), the Singer algebraic transfer was studies by many authors (See Boardman \[1\], Bruner-Ha-Hung \[2\], Minami \[13\], Sum-Tin \[26\] and others). However, Singer’s conjecture is still open for \(n \geq 4\).

The results of the hit problem are used to study and verify the Singer conjecture for the algebraic transfer. The behavior of the fifth Singer algebraic transfer in the degree \(d_5 = 5(2^s - 1) + 13.2^t\) was also discussed at the end of this paper.

Next, in Section \[2\] we recall some needed information on admissible monomials in \(P_n\). The main results are presented in Section \[3\]. The proofs of the main results will be presented in Section \[4\].
2. Preliminaries

In this section, we recall a few useful preliminaries on the admissible monomials, spike monomials, and the hit monomials from Kameko [9], and Sun [25], which will be used in the next section.

We will denote by $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and

$$X_\mathcal{J} = X_{\{j_1, j_2, \ldots, j_s\}} = \prod_{j \in \mathcal{J}} x_j, \quad \mathcal{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_n,$$

In particular, $X_{\mathbb{N}_n} = 1$, $X_{\emptyset} = x_1 x_2 \cdots x_n$, $X_j = x_1 \cdots \hat{x}_j \cdots x_n$, $1 \leq j \leq n$, and $X := X_n \in \mathcal{P}_{n-1}$.

Let $\alpha_t(d)$ be the $t$-th coefficient in dyadic expansion of $d$. Then, $d = \sum_{t \geq 0} \alpha_t(d) 2^t$ where $\alpha_t(d) \in \{0, 1\}$. Let $x = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \in \mathcal{P}_n$. Denote $\nu_j(x) = a_j, 1 \leq j \leq n$.

Let $\mathcal{J}_t(x) = \{j \in \mathbb{N}_n : \alpha_t(\nu_j(x)) = 0\}$, for $t \geq 0$. Then, we have $x = \prod_{t \geq 0} X_{\mathcal{J}_t(x)}^{2^t}$.

**Definition 2.1.** For a monomial $x$ in $\mathcal{P}_n$, define two sequences associated with $x$ by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_t(x), \ldots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_t(x)),$$

where $\omega_i(x) = \sum_{1 \leq j \leq n} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathcal{J}_{i-1}(x)}, \; i \geq 1$. The sequence $\omega(x)$ is called the weight vector of $x$.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_t, \ldots)$ be a sequence of non-negative integers. The sequence $\omega$ is called the weight vector if $\omega_i = 0$ for $i \gg 0$. For a weight vector $\omega$, we define $\deg \omega = \sum_{t \geq 0} 2^{t-1} \omega_t$.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

Denote by $\mathcal{P}_n(\omega)$ the subspace of $\mathcal{P}_n$ spanned by all monomials $y$ such that $\deg y = \deg \omega$, $\omega(y) \leq \omega$, and by $\mathcal{P}_n^-(\omega)$ the subspace of $\mathcal{P}_n$ spanned by all monomials $y \in \mathcal{P}_n(\omega)$ such that $\omega(y) < \omega$.

**Definition 2.2.** Let $\omega$ be a weight vector and $u, v$ two polynomials of the same degree in $\mathcal{P}_n$. We define the equivalence relations $\equiv$ and $\equiv_{\omega}$ on $\mathcal{P}_n$ by stating that

(i) $u \equiv v$ if and only if $u - v \in A^+ \mathcal{P}_n$.

(ii) $u \equiv_{\omega} v$ if and only if $u, v \in \mathcal{P}_n(\omega)$ and $u - v \in (A^+ \mathcal{P}_n \cap \mathcal{P}_n(\omega) + \mathcal{P}_n^-(\omega))$ denoting the set of all the $A$-decomposable elements, in which $Sq^2 : \mathcal{P}_{n-2r} \to \mathcal{P}_n$, and $A^+$ is the kernel of an epimorphism $A \to \mathbb{F}_2$ of graded $\mathbb{F}_2$-algebras. If $u$ is a linear combination of elements in the images of the Steenrod squaring operations $Sq^r$ for $r > 0$, then $u$ is called decomposable. Also, $u$ is said to be $\omega$-decomposable if $u \equiv_{\omega} 0$.

Then, we have an $\mathbb{F}_2$-quotient space of $\mathcal{P}_n$ by the equivalence relation $\equiv_{\omega}$ as follows

$$Q \mathcal{P}_n(\omega) = \mathcal{P}_n(\omega)/((A^+ \mathcal{P}_n \cap \mathcal{P}_n(\omega) + \mathcal{P}_n^-(\omega)).$$
Definition 2.3. Let $u$, $v$ be monomials of the same degree in $\mathcal{P}_n$. We say that $u < v$ if and only if one of the following holds:

(i) $\omega(u) < \omega(v)$;

(ii) $\omega(u) = \omega(v)$, and and $\sigma(u) < \sigma(v)$.

Definition 2.4. A monomial $u$ is said to be inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_i < u$ for $i = 1, 2, \ldots, m$ and $u - \sum_{i=1}^m v_i \in \mathcal{A}^+ \mathcal{P}_n$.

We say $u$ is admissible if it is not inadmissible. From these events, it is important to remark that the set of all the admissible monomials of degree $d$ in $\mathcal{P}_n$ is a minimal set of $\mathcal{A}$-generators for $\mathcal{P}_n$ in degree $d$. And therefore, $(\mathbb{F}_2 \otimes _\mathcal{A} \mathcal{P}_n)_d$ is an $\mathbb{F}_2$-vector space with a basis consisting of all the classes represent by the elements in $(\mathcal{P}_n)_d$.

Definition 2.5. Let $u \in \mathcal{P}_n$. We say $u$ is strictly inadmissible if and only if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_j < u$, for $j = 1, 2, \ldots, m$ and $u = \sum_{j=1}^m v_j + \sum_{i=1}^{2^r-1} Sq^i(f_i)$ with $s = \max\{k : \omega_k(u) > 0\}$, $f_i \in \mathcal{P}_n$.

It is easy to check that if $u$ is strictly inadmissible monomial, then it is inadmissible monomial.

Theorem 2.6 (Kameko [9], Sum [25]). Suppose that $u, v, w \in \mathcal{P}_n$ satisfying the conditions $\omega_t(u) = 0$ if $t > k > 0$, $\omega_t(w) \neq 0$ and $\omega_t(w) = 0$ if $t > r > 0$. Then,

(i) $uw^{a^k}$ is inadmissible if $w$ is inadmissible.

(ii) $wv^{a^r}$ is strictly inadmissible if $w$ is strictly inadmissible.

Definition 2.7. Let $z = x_1^{d_1}x_2^{d_2} \ldots x_n^{d_n} \in \mathcal{P}_n$. The monomial $z$ is called a spike if $d_j = 2t_j - 1$ for $t_j$ a non-negative integer and $j = 1, 2, \ldots, n$. Moreover, $z$ is called the minimal spike, if it is a spike such that $t_1 > t_2 > \ldots > t_r-1 > t_r > 0$ and $t_j = 0$ for $j > r$.

The following is a Singer’s criterion on the hit monomials in $\mathcal{P}_n$.

Theorem 2.8 (Singer [20]). Assume that $z$ is the minimal spike of degree $d$ in $\mathcal{P}_n$, and $u \in (\mathcal{P}_n)_d$ satisfying the condition $\mu(d) \leq n$. If $\omega(u) < \omega(z)$, then $u$ is hit.
3. Statement of the main results

In this section, we list the main results of this paper, proofs of the main results will be presented in the next section. First, we study the hit problem for \( K \). For \( k \) an arbitrary non-negative integer.

For \( s = 0 \), we have \( k_0 = 5(2^0 - 1) + 13 \cdot 2^0 = 13 \). Then, \( \dim(\mathbb{F}_2 \otimes \mathcal{P}_5)_{13} = 250 \). This result has been computed in [16] by explicitly determining all admissible monomials of \( \mathcal{P}_5 \) in degree 13 (see Phuc [16]).

Moreover, for \( s = 1 \), \( k_1 = 5(2^1 - 1) + 13 \cdot 2^1 = 31 \), the following result has been shown in [17] by Phuc.

**Theorem 3.1** (Phuc [17]). There exist exactly 866 admissible monomials of degree 31 in \( \mathcal{P}_5 \). Consequently, \( \dim(\mathbb{F}_2 \otimes \mathcal{P}_5)_{31} = 866 \).

One of our main contributions in this article is to study the \( \mathbb{F}_2 \)-graded vector space \( (\mathbb{F}_2 \otimes \mathcal{P}_5)_{5(2^s - 1) + 13 \cdot 2^s} \), where \( s \) is any positive integer greater than one.

For \( s = 2 \), we have \( k_2 = 5(2^2 - 1) + 13 \cdot 2^2 = 67 \). Since \( \mathcal{P}_n = \oplus_{d \geq 0}(\mathcal{P}_n)_d \) is the graded polynomial algebra, and Kameko’s homomorphism

\[
(\tilde{S}_{i_1^0})_{(5;67)} : (\mathbb{F}_2 \otimes \mathcal{P}_5)_{67} \rightarrow (\mathbb{F}_2 \otimes \mathcal{P}_5)_{31}
\]

is an \( \mathbb{F}_2 \)-epimorphism, it follows that

\[
(\mathbb{F}_2 \otimes \mathcal{P}_5)_{67} \cong (\mathbb{F}_2 \otimes \mathcal{P}_5^0)_{67} \bigoplus (\ker(\tilde{S}_{i_1^0})_{(5;67)} \cap (\mathbb{F}_2 \otimes \mathcal{P}_5^+)_{67}) \bigoplus \text{Im}(\tilde{S}_{i_1^0})_{(5;67)}
\]

Consider the homomorphism \( \Phi : \mathcal{P}_5 \rightarrow \mathcal{P}_5 \) is an \( \mathbb{F}_2 \)-homomorphism determined by \( \Phi(x) = \prod_{i=1}^{s} x_i x^2 \), for \( x \in \mathcal{P}_5 \). Suppose that \( \mathcal{C}^5_{31} = \{ a_i \in \mathcal{P}_5 : 1 \leq i \leq 866 \} \). Then, we set

\[
\mathcal{D}_{\text{Im}}^{5} = \{ [b_i] : b_i = \Phi(x), \text{ for all } x \in \mathcal{C}_{31}^5 \}.
\]

From this, we easily conclude the following theorem.

**Theorem 3.2.** \( \text{Im}(\tilde{S}_{i_1^0})_{(5;67)} \) is isomorphic to a subspace of \( (\mathbb{F}_2 \otimes \mathcal{P}_5)_{67} \) generated by all the classes of the form \( [b_i] \), with \( 1 \leq i \leq 866 \). That means, the set \( \mathcal{D}_{\text{Im}}^{5} \) is a basis of the \( \mathbb{F}_2 \)-vector space \( \text{Im}(\tilde{S}_{i_1^0})_{(5;67)} \).

For any \( 1 \leq t \leq 5 \), consider the homomorphism \( \mathcal{T}_t : \mathcal{P}_4 \rightarrow \mathcal{P}_5 \) of algebras by substituting

\[
\mathcal{T}_t(x_i) = \begin{cases} x_i, & \text{if } 1 \leq i < t, \\ x_{i+1}, & \text{if } t \leq i < 5. \end{cases}
\]

Then, \( \mathcal{T}_t \) is a homomorphism of \( \mathcal{A} \)-modules. Put \( \mathcal{D}_0^{\otimes 5} := \{ c_i : c_i \in \bigcup_{t=1}^{5} \mathcal{T}_t(C^5_0) \} \).

Using the result in Sum [25] (see Theorem 1.4), an easy computation shows that \( |\mathcal{D}_0^{\otimes 5}| = 460 \). More specifically, we have the following.

**Theorem 3.3.** The set \( \{ c_i : 1 \leq i \leq 460 \} \) is a basis of the \( \mathbb{F}_2 \)-vector space \( (\mathbb{F}_2 \otimes \mathcal{P}_5^0)_{5(2^2 - 1) + 13 \cdot 2^2} \). Consequently, \( \dim(\mathbb{F}_2 \otimes \mathcal{P}_5^0)_{5(2^2 - 1) + 13 \cdot 2^2} = 460 \).
Remark 3.4. We recall a result in Mothebe-Kaelo-Ramatebele [12] as follows:

Set $\mathcal{N}_{(n,t)} = \{I = (i_1, i_2, \ldots, i_t) : 1 \leq i_1 < \ldots < i_t \leq n\}$, $1 \leq t < n$. For $I \in \mathcal{N}_{(n,t)}$, define the homomorphism $f_I : P_t \to P_n$ of algebras by substituting $f_I(x_\ell) = x_{i_\ell}$ with $1 \leq \ell \leq t$. Then, $f_I$ is a monomorphism of $A$-modules. We have a direct summand decomposition of the $\mathbb{F}_2$-vector subspaces:

$$
\mathbb{F}_2 \otimes_A P_n^0 = \bigoplus_{1 \leq t \leq n-1} \bigoplus_{I \in \mathcal{N}_{(n,t)}} (Q f_I(P_t^+)),
$$

where $Q f_I(P_t^+) = \mathbb{F}_2 \otimes_A f_I(P_t^+)$.

Hence, $\dim(Q f_I(P_t^+)) = \dim(\mathbb{F}_2 \otimes_A P_t^+)d$ and $|\mathcal{N}_{(n,t)}| = \binom{n}{t}$. Combining with the results in Kameko [9] and Sum [25], we have

$$
\dim(\mathbb{F}_2 \otimes_A P_n^0)_d = \sum_{\mu(d) \leq t \leq n-1} \binom{n}{t} \dim(\mathbb{F}_2 \otimes_A P_t^+)_d.
$$

Since $\mu(67) = 3$, it follows that if $t < 3$ then the spaces $(\mathbb{F}_2 \otimes_A P_t^+)_{67}$ are trivial. Moreover, by using the results in Kameko [9] and Sum [25], we have

$$
\dim(\mathbb{F}_2 \otimes_A P_3^+)_{67} = \begin{cases} 14, & \text{if } t = 3, \\ 64, & \text{if } t = 4. \end{cases}
$$

From this, we get

$$
\dim(\mathbb{F}_2 \otimes_A P_5^0)_{67} = \binom{5}{3} \dim(\mathbb{F}_2 \otimes_A P_3^+)_{67} + \binom{5}{4} \dim(\mathbb{F}_2 \otimes_A P_4^+)_{67} = 460.
$$

Next, we explicitly determine the vector space $\operatorname{Ker}(\tilde{S}q_n^0)_{(5,67)} \cap (\mathbb{F}_2 \otimes_A P_5^+)^{67}$. We will denote by $Q P_n^+(\omega) = Q P_n(\omega) \cap (\mathbb{F}_2 \otimes_A P_n^+)$. Then, we have the following theorem.

Theorem 3.5. Let $\tilde{\omega}_1 := (3, 4, 2, 2, 2)$, $\tilde{\omega}_2 := (3, 4, 4, 3, 1)$, and $\tilde{\omega}_3 := (3, 2, 1, 1, 1)$.

(i) Suppose $u$ is an element of $(C_{67}^0 \cap P_5^+)$ such that $|u|$ does not belong to $\operatorname{Im}(\tilde{S}q_n^0)_{(5,67)}$, then $\omega(u)$ is one of the following sequences: $\tilde{\omega}_m$, with $m = 1, 2, 3$. Moreover, we have an isomorphism of the $\mathbb{F}_2$-vector spaces:

$$
\operatorname{Ker}(\tilde{S}q_n^0)_{(5,67)} \cap (\mathbb{F}_2 \otimes_A P_5^+)_{67} \cong 3 \bigoplus_{m=1} \bigoplus_{Q P_n^+(\tilde{\omega}_m)}.
$$

(ii) We have $\dim(\operatorname{Ker}(\tilde{S}q_n^0)_{(5,67)} \cap (\mathbb{F}_2 \otimes_A P_5^+)_{67}) = 161$.

From the results of Theorems 3.2, 3.3, and 3.5, we obtain the following corollary.

Corollary 3.6. There exist exactly 1487 admissible monomials of degree 67 in $P_5$. Consequently, $\dim(\mathbb{F}_2 \otimes_A P_5)_{5(2^2-1)+13, 2^2} = 1487$. 

Consider the degree \( \kappa_s = 5(2^s - 1) + 13.2^s \), for any \( s \geq 3 \). We set

\[
x - \max \{0, n - \alpha(d + n) - 
\zeta(d + n)\},
\]

where \( \zeta(u) \) the greatest integer \( v \) such that \( u \) is divisible by \( 2^v \). We recall a result in [28] the following.

**Theorem 3.7** (Tin-Sum [28]). Let \( d \) be an arbitrary non-negative integer. Then

\[
(S_q)^{r-t} : (F_2 \otimes A_5)_{n(2^r-1)+2^r d} \longrightarrow (F_2 \otimes A_5)_{n(2^r-1)+2^r d}
\]

is an isomorphism of \( GL(n; F_2) \)-modules for every \( r \geq t \) if and only if \( t \geq \xi(n; d) \).

As an application of Theorem 3.7, we get the following theorem.

**Theorem 3.8.** The set \( \{ [x] : x \in \Phi^{s-2} (\cdots) \} \) is a basis of the \( F_2 \)-vector space \( (F_2 \otimes A_5)_{5(2^s-1)+13.2^s} \), for any \( s > 2 \). This implies that the \( F_2 \)-vector space \( (F_2 \otimes A_5)_{5(2^s-1)+13.2^s} \) has dimension 1487, for any \( s > 2 \).

Now, we describe the next main results by studying the \( F_2 \)-graded vector space \( F_2 \otimes A_5 \) in some generic degrees of the form \( (n-1)(2^r-1) + m.2^r + \), where \( r \) is an arbitrary positive integer, \( m \in \{23, 67\} \), and in the case \( n = 6 \).

As is well known, after explicitly determining \( F_2 \otimes A_4 \), Sum [29] has established an inductive formula by \( n \) for the dimension of the vector space \( (F_2 \otimes A_5)_d \), where \( d \) is of general degree (see Theorem 1.3).

From the results in Sum [25], combined with the above results we get the following theorem.

**Theorem 3.9.** The vector space \( (F_2 \otimes A_5)_{5(2^r-1)+13.2^r} \) is \( 93681 \)-dimensional, for all positive integers \( s \).

On the other hand, based on the results in [31] for the hit problem of five variables, we also obtain the following.

**Theorem 3.10.** For any integer \( r > 0 \), there exist exactly 78435 admissible monomials of degree \( 5.2^r + 23.2^r \) in \( P_5 \). Consequently, \( \dim(F_2 \otimes A_5)_{5.2^r + 23.2^r} = 78435 \), for all \( r > 0 \).

### 4. Proofs of main results

#### 4.1. Proof of Theorem 3.5

We first prove Part (i) of the theorem. Let us denote by

\[
Q_{P_5}^+ := \text{Span} \{ [x] \in F_2 \otimes A_5 : x \text{ is admissible and } \omega(x) = \omega \}.
\]

Using the results in Walker-Wood [31], we see that

\[
(F_2 \otimes A_5)_d = \bigoplus_{\deg \omega = d} Q_{P_5}^+ \cong \bigoplus_{\deg \omega = d} Q_{P_5}(\omega),
\]

Since \( P_n = \bigoplus_{\deg \omega = d} (P_n)_d \) is the graded polynomial algebra, combined with the above results, it follows that \( (F_2 \otimes A_5)_67 = \bigoplus_{\deg \omega = 67} Q_{P_5}^+ \).
Suppose that \( u \) is an admissible monomial of degree 67 in \( P_5^+ \) such that \([u] \) belongs to \( \text{Ker}(\tilde{S}q_0^a)_{(5,67)} \). It is easy to check that \( z = x_1^4x_2^3x_3 \) is the minimal spike of degree 67 in \( P_5 \) and \( \omega(z) = \omega \). Using Theorem 2.3, we obtain \( \omega_1(u) \geq \omega_1(z) = 3 \). Since the degree of \( \omega \) is odd, it follows that either \( \omega_1(u) = 3 \) or \( \omega_1(u) = 5 \).

If \( \omega_1(u) = 5 \), then \( u = \prod_{i=1}^{5} x_i v_i^2 \) with \( v \) a monomial in \((P_5)_{31}\). By Theorem 2.6, \( v \) is an admissible monomial and \([v] \neq 0 \). Thus, \([v] = \text{Ker}(\tilde{S}q_0^a)_{(5,67)}([u]) \neq 0 \).

This contradicts the fact that \([u] \) belongs to \( \text{Ker}(\tilde{S}q_0^a)_{(5,67)} \). From this, one gets \( \omega_1(u) = 3 \). Then, we have \( u = x_i x_j x_k y^2 \) with \( 1 \leq i < j < k \leq 5 \), where \( y \in (P_5)_{32} \).

Using Theorem 2.6, \( y \) is also admissible. By a simple computation shows that \( \omega(y) = (4, 4, 3, 1) \) or \( \omega(y) = (4, 2, 2, 2) \) or \( \omega(y) = (2, 1, 1, 1, 1) \). And therefore, we get \( \omega(u) = \omega_m \), with \( m = 1, 2, 3 \).

From these above, combined with the aid of 4.2, we have a direct summand decomposition of the \( \mathbb{F}_2 \)-vector spaces:

\[
\text{Ker}(\tilde{S}q_0^a)_{(5,67)} \cap (\mathbb{F}_2 \otimes_A P_5^+)_{67} = \bigoplus_{m=1}^{3} QP_5^+(\tilde{\omega}_m).
\]

Part (i) is proved.

The proof of Part (ii) of the above theorem is too long and computationally very technical. We sketch its proof as follows:

Consider the set \( B_{67}^{\otimes 5} := \{X_{i,j} : F^2 : 1 \leq i < j \leq 5, F \in C_{\mathbb{F}_2}^{5}\} \). By using Theorem 2.6, we see that if \( X \) is an admissible monomial of degree 67 in \( P_5 \), then \( X \in B_{67}^{\otimes 5} \). Putting

\[
C_{67}^{\otimes 5} := \{X \in B_{67}^{\otimes 5} : X \text{ admissible monomial, and } \omega(X) = \omega_m, \text{ for } m = 1, 2, 3\}.
\]

By direct calculations, we filter out and eliminate the inadmissible monomials in \( B_{67}^{\otimes 5} \), one gets \( |C_{67}^{\otimes 5}(\omega)| = 161 \).

Note that the result dimension of \( \text{Ker}(\tilde{S}q_0^a)_{(5,67)} \cap (\mathbb{F}_2 \otimes_A P_5^+)_{67} \) has been verified by using a computer calculation program in SAGE by H-V Vu. We would like to say thank you for his support.

4.2. Proof of Theorem 3.8

It is easy to check that for \( n = 5 \) and \( d = 67 \) then \( \alpha(d+n) = \alpha(72) = 3 \), and \( \zeta(d+n) = \zeta(72) = 3 \), and therefore \( \xi(n;d) = 0 \). By Theorem 3.7, we get an isomorphism of \( \mathbb{F}_2 \)-graded vector spaces:

\[
(\mathbb{F}_2 \otimes_A P_5)_{5(2^2-1)+2^r.67} \cong (\mathbb{F}_2 \otimes_A P_5)_{5(2^0-1)+2^r.67}
\]

for all \( r \geq 0 \).

And therefore, we obtain

\[
(\mathbb{F}_2 \otimes_A P_5)_{5(2^2-1)+13.2^r} \cong (\mathbb{F}_2 \otimes_A P_5)_{5(2^2-1)+13.2^r} \text{ for all } s > 2.
\]

That means, \( \text{dim}(\mathbb{F}_2 \otimes_A P_5)_{5(2^2-1)+13.2^r} = \text{dim}(\mathbb{F}_2 \otimes_A P_5)_{5(2^2-1)+13.2^r} = 1487 \), for all \( s > 2 \).

Moreover, for every \( s > 2 \), the set \( \{[x] : x \in \Phi^s \} \) is a basis of the \( \mathbb{F}_2 \)-vector space \((\mathbb{F}_2 \otimes_A P_5)_{5(2^2-1)+13.2^r}\).
4.3. Proof of Theorems 3.3 and 3.10

Consider the degree \( d = (n - 1)(2^r - 1) + 2^r m \), with \( r, m \) positive integers such that \( 1 \leq n - 3 \leq \mu(m) \leq n - 2 \). If \( r \geq n - 1 \), then we have the following result, which was shown in Sum \([25]\).

\[
\dim(\mathbb{F}_2 \otimes_A \mathcal{P}_n)_d = (2^n - 1) \dim(\mathbb{F}_2 \otimes_A \mathcal{P}_{n-1})_m.
\]

For \( n = 6 \), and \( m = 67 \), we can easily see that \( \mu(67) = 3 = \alpha(67 + \mu(67)) = \alpha(70) \).

Hence, using the above result, one gets

\[
\left| C^{\otimes 6}_{5(2^r-1)+67.2^r} \right| = 63.\left| C^{\otimes 5}_{67} \right| = 93681, \text{ for any integer } r \geq n - 1 = 5.
\]

So, there exist exactly 93681 admissible monomials of degree \( 5(2^r - 1) + 67.2^r \) in \( \mathcal{P}_n \), for any integer \( r > 4 \). Consequently, \( \dim(\mathbb{F}_2 \otimes_A \mathcal{P}_6)_{5(2^{r+1}-1)+67.2^{r+1}} = 93681 \), for all positive integers \( s \). Theorem 3.3 is proved.

Moreover, Tin \([30]\) showed that \( (\mathbb{F}_2 \otimes_A \mathcal{P}_5)_{5(2^r-1)+9.2^r} \) is an \( \mathbb{F}_2 \)-vector space of dimension 1245, for any integer \( r > 0 \). Consequently, \( \left| C^{\otimes 5}_{5(2^r-1)+9.2^r} \right| = 1245 \), for any integer \( r > 0 \).

Consider the degree \( 5(2^u - 1) + 23.2^u \). It is easy to check that

\[
\mu(23) = \alpha(23 + \mu(23)) = \alpha(26) = n - 3 = 3.
\]

And therefore, using the result in Sum \([25]\) (see Theorem 1.3), we get

\[
\left| C^{\otimes 6}_{5(2^u-1)+23.2^u} \right| = (2^6 - 1)|C^{\otimes 5}_{23}|, \text{ for any integer } u \geq 5.
\]

This implies that the \( \mathbb{F}_2 \)-graded vector space \( (\mathbb{F}_2 \otimes_A \mathcal{P}_6)_{5(2^{r+1}-1)+23.2^{r+1}} \) has dimension 78435, for any integer \( v > 0 \). Theorem 3.10 is proved.

5. On Behavior of the Singer Algebraic Transfer

We first recall the definition of the Singer algebraic transfer, which is a homomorphism

\[
\psi_n : \mathbb{F}_2 \otimes_{GL(n;\mathbb{F}_2)} PH_*(\mathbb{R}\mathcal{P}^\infty)^n \rightarrow Ext_A^{n,n+s}(\mathbb{F}_2, \mathbb{F}_2).
\]

Here, \( (\mathbb{F}_2 \otimes_A \mathcal{P}_n)_{d}^{GL(n;\mathbb{F}_2)} \) is the subspace of \( (\mathbb{F}_2 \otimes_A \mathcal{P}_n)_d \) consisting of all the \( GL(n;\mathbb{F}_2) \)-invariant classes of degree \( d \), and \( \mathbb{F}_2 \otimes_{GL(n;\mathbb{F}_2)} PH_d((\mathbb{R}\mathcal{P}^\infty)^n) \) be dual to \( (\mathbb{F}_2 \otimes_A \mathcal{P}_n)_{d}^{GL(n;\mathbb{F}_2)} \).

Noting that in the dual case, we also have an algebraic homomorphism called Singer’s algebraic transfer, \( Tr_n := (\psi_n)^* \), which is a homomorphism from the homology of the Steenrod algebra, \( Tor_A^{n,n+s}(\mathbb{F}_2, \mathbb{F}_2) \), to the subspace of \( (\mathbb{F}_2 \otimes_A \mathcal{P}_n)_d \) consisting of all the \( GL(n;\mathbb{F}_2) \)-invariant classes (see Singer \([20]\)).

In \([20]\), and \([30]\), we based on the the results for the hit problem to study and verify the Singer conjecture for the algebraic transfer in degrees \( 5(2^s - 1) + 2^s m \), for \( s \) an arbitrary positive integer, and \( m \in \{1, 2, 3\} \), we obtain the following theorem.

**Theorem 5.1.** Let \( s \) be an arbitrary positive integer. Singer’s conjecture is true for \( n = 5 \) and the generic degrees \( d_s = 5(2^s - 1) + 2^s m \), where \( m \in \{1, 2, 3\} \).
In the current article, by using Theorem [6,7], we also see that
\[(F_2 \otimes_A \mathcal{P}_5)^{GL(5;F_2)}_{5(2^s-1)+13.2^2} \cong (F_2 \otimes_A \mathcal{P}_5)^{GL(5;F_2)}_{5(2^s-1)+13.2^2}, \text{ for all } s > 1.
\]

And therefore, we get
\[F_2 \otimes GL(5;F_2) PH_{5(2^s-1)+13.2^2}((\mathbb{R}P^\infty)^5) \cong F_2 \otimes GL(5;F_2) PH_{5(2^s-1)+13.2^2}((\mathbb{R}P^\infty)^5),\]
for all \(s > 1\). Hence, we need only to compute the dimension of vector space
\[F_2 \otimes GL(5;F_2) PH_{5(2^s-1)+13.2^2}((\mathbb{R}P^\infty)^5) \text{ for } s = 2.
\]

Noting that for \(s = 1\), then dim \(((F_2 \otimes_A \mathcal{P}_5)^{GL(5;F_2)}_{5(2^s-1)+13.2^2}) = 2\) (see Phuc [17]).
Moreover, since Kameko’s homomorphism
\[(\tilde{S}_q^0)_{(5,67)} : (F_2 \otimes_A \mathcal{P}_5)_{5(2^s-1)+13.2^2} \rightarrow (F_2 \otimes_A \mathcal{P}_5)_{5(2^s-1)+13.2^2}\]
is also an \(GL(5;F_2)\)-epimorphism, it follows that
\[\dim (F_2 \otimes GL(5;F_2) PH_{5(2^s-1)+13.2^2}((\mathbb{R}P^\infty)^5) \leq \dim(\tilde{S}_q^0)_{(5,67)}, \text{ for all } s > 1.\]

Remarkably, for any \(s > 1\), all elements of \(F_2 \otimes GL(5;F_2) PH_{5(2^s-1)+13.2^2}((\mathbb{R}P^\infty)^5)\) are of the form \(\langle \xi(\Phi(\nu)) + \gamma(\Phi(\beta)) + [f] \rangle^t\), where \(\xi, \gamma \in F_2\), \(\nu\) and \(\beta\) are two generators of \(F_2 \otimes A \mathcal{P}_5^{GL(5;F_2)}\), and \(f \in (\mathcal{P}_5)_{5(2^s-1)+13.2^2}\) such that \([f]\) belongs to \(\ker(\tilde{S}_q^0)_{(5,67)}\).
By using techniques of the hit problem, combining the computation of \(\text{Ext}_{A}^{n,n}(Z_2, Z_2)\) by Lin [11] and Chen [3], we will describe explicitly all these elements, and verify the Singer’s conjecture for the fifth algebraic transfer in the near future.

Acknowledgment. I would like to express my warmest thanks to my adviser, Prof. Nguyen Sum (Saigon University) for helpful conversations. I also thank Dr. Dang Yo Phuc (Khanh Hoa University) for a helpful discussion. This research is supported by Ho Chi Minh City University of Technology and Education (HCMUTE), Vietnam.

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