Remarks on Multiplicative Metric Spaces and Related Fixed Points

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Abstract

In this article we studied the relationship between metric spaces and multiplicative metric spaces. Also, we pointed out some fixed and common fixed point results under some contractive conditions in multiplicative metric spaces can be obtained from the corresponding results in standard metric spaces.

1 Introduction

The notion of multiplicative metric space was introduced by Bashirov et al. \cite{4}. In 2012, Ozavsar and Cervikel \cite{6} defined the notion of convergence in multiplicative metric spaces and studied some fixed point results in such space. After that, many researchers consider this space and many results on fixed point theory were considered.

We begin by introducing the definition of multiplicative metric space.

\textbf{Definition 1.} \cite{6} Let \(X\) be a nonempty set and \(p : X \times X \to [1, +\infty)\). We say that \((X, p)\) is a multiplicative metric space if for all \(x, y, z \in X\) we have:

1. \(p(x, y) \geq 1\) and \(x = y\) if and only if \(p(x, y) = 1\);
2. \( p(x, y) = p(y, x) \);
3. \( p(x, z) \leq p(x, y)p(y, z) \).

The definition of a multiplicative Cauchy sequence in a multiplicative metric space is given as follows:

**Definition 2.** \[6\] A sequence \( \{x_n\} \) in a multiplicative metric space \((X, p)\) is said to be multiplicative Cauchy sequence if for all \( \epsilon > 1 \), there exists \( N \in \mathbb{N} \) such that \( p(x_n, x_m) < \epsilon \) for all \( m, n \geq N \). Also, if every multiplicative Cauchy sequence is convergent, then \((X, p)\) is called a complete multiplicative metric space.

For the definitions of open balls and convergence in multiplicative metric spaces, we refer the reader to \[6\].

Ozavsar and Cervikel \[6\] introduced the concept of multiplicative contraction and proved that every multiplicative contraction in a complete multiplicative metric space has a unique fixed point.

**Definition 3.** \[6\] Let \((X, p)\) be a multiplicative metric space. A mapping \( f : X \to X \) is called multiplicative contraction if there exists a real number \( \lambda \in [0, 1) \) such that \( p(f(x_1), f(x_2)) \leq p(x_1, x_2)^\lambda \) for all \( x_1, x_2 \in X \).

**Theorem 1.** \[6\] Let \((X, p)\) be a complete multiplicative metric space and let \( f : X \to X \) be a multiplicative contraction. Then \( f \) has a unique fixed point.

The notion of weakly commuting mappings was introduced by Sessa \[3\] in 1982. While, Jungck \[2\] initiated the concept of weakly compatible mappings in 1996 as a generalization of the notion of weakly commuting mappings. Moreover, many authors studied many fixed point theorems for weakly commuting mappings in metric spaces. See \[2\], \[10\], \[11\], \[12\], \[13\], and \[14\].

**Definition 4.** \[2\] Let \( A \) and \( S \) be self-mappings on a metric space \((X, d)\). Then, \( A \) and \( S \) are said to be weakly compatible if they commute at their coincident point; that is, \( Ax = Sx \) for some \( x \in X \) implies \( ASx = SAx \).

**Definition 5.** \[3\] Let \( S \) and \( T \) be two self-mappings of a metric space \((X, d)\). Then \( S \) and \( T \) are said to be weak commutative mappings if

\[
d(STx, TSx) \leq d(Sx, Tx),
\]

for all \( x \in X \).

It is clear that if \( S \) and \( T \) are weak commutative mappings, then \( S \) and \( T \) are weakly compatible.

He et al. \[5\] employed the concept of weakly commutative mappings to introduce and prove the following common fixed point theorem in multiplicative metric spaces.

**Theorem 2.** \[5\] Let \((X, p)\) be a complete multiplicative metric space. Suppose that \( A, B, S \) and \( T \) are four self-mappings of \( X \) satisfying the following conditions:
1. $T(X) \subseteq A(X)$ and $S(X) \subseteq B(X)$;
2. The pairs $(S, A)$ and $(T, B)$ are weakly commutative;
3. One of $A, B, S$ and $T$ is continuous;
4. 
   
   $$p(Sx, Ty) \leq \{\max\{p(Ax, By), p(Ax, Sx), p(By, Ty), p(Ax, Ty), p(By, Sx)\}\}^\lambda, \lambda \in (0, \frac{1}{2}). \quad (1)$$

Then $A, B, S$ and $T$ have a unique common fixed point.

In this paper, we study the relationship between the multiplicative metric space and the standard metric space. Also, we show that the proof of Theorem 1 and Theorem 2 are obtained from the corresponding results in standard metric spaces.

## 2 Main Results

We start by giving the relationship between the multiplicative metric space and the standard metric space.

If we have a multiplicative metric space $(X, p)$, then the corresponding metric space $(X, d_p)$ is given by the following theorem.

**Theorem 3.** Let $(X, p)$ be a multiplicative metric space. Define $d_p : X \times X \to [0, +\infty)$ by

$$d_p(x, y) = \ln(p(x, y)).$$

Then $(X, d_p)$ is a metric space.

**Proof.** Follows from the properties of logarithms. □

Moreover, if we have a metric space $(X, d)$, then the corresponding multiplicative metric space $(X, p_d)$ is given by the following theorem.

**Theorem 4.** Let $(X, d)$ be a metric space. Define $p_d : X \times X \to [0, +\infty)$ by

$$p_d(x, y) = e^{d(x, y)}.$$

Then $(X, p_d)$ is a multiplicative metric space.

**Proof.** The proof follows from the properties of exponential functions. □

Now, we can transfer and prove many applications considered on multiplicative metric space to a standard metric space and use their proof in the metric space case. For instance, considering the definition of contraction on multiplicative metric space, the proof the following result is a straightforward.
Theorem 5. A sequence \( \{x_n\} \) is a multiplicative Cauchy sequence in a multiplicative metric space \((X, p)\) if and only if \( \{x_n\} \) is a Cauchy sequence in the corresponding metric space \((X, d_p)\).

Applying the logarithmic function to the multiplicative contraction inequality that have been defined in Definition 3 will give us the inequality

\[
d_p(f(x_1), f(x_2)) = \ln p(f(x_1), f(x_2)) \leq \lambda d_p(x_1, x_2).
\]

Note that if \((X, p)\) is a complete multiplicative metric space, then the corresponding metric space \((X, d_p)\) is also a complete metric space.

It is obvious we can get the regular contraction inequality which was introduce by Banach. Therefore, one can prove the result in Theorem \(1\) using the new metric space \((X, d_p)\) and the Banach contraction theorem.

We have furnished all the necessary backgrounds to present the proof of Theorem \(1\) from the standard metric space.

Proof of Theorem \(1\):

Since \((X, p)\) is a complete multiplicative metric space, the corresponding metric space \((X, d_p)\) is a complete metric space. Also, since \(f\) is a multiplicative contraction, it is a contraction in \((X, d_p)\). Therefore it satisfies the Banach contraction conditions and thus it has a unique fixed point. \(\square\)

Now, we furnish all the necessary backgrounds to prove Theorem \(2\) from the corresponding standard metric space.

Recall the following definition.

Definition 6. \(\square\) Let \(X\) be a nonempty set and let \(d : X \times X \rightarrow [0, +\infty)\) be a function satisfying the following conditions:

1. \(d(x, y) = d(y, x)\).
2. If \(d(x, y) = 0\) then we have \(x = y\).
3. \(d(x, y) \leq d(x, z) + d(z, y)\) for all \(x, y, z \in X\).

Then the pair \((X, d)\) is called the d-metric space. It also appeared under the name of metric-like space \(\square\).

It is clear that every metric space is a d-metric space.
Definition 7. \[ A\] A sequence \( \{x_n\} \) in a \( d \)-metric space \((X, d)\) is called a Cauchy sequence if for given \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( d(x_n, x_m) < \epsilon \) for all \( m, n \geq N \). Also, \((X, d)\) is called complete if every Cauchy sequence in it is convergent.

Definition 8. A function \( \phi : [0, \infty) \rightarrow [0, \infty) \) is said to be contractive modulus if \( \phi(t) < t \) for \( t > 0 \).

Definition 9. A real valued function \( \phi \) defined on \( X \subseteq \mathbb{R} \) is said to be upper semicontinuous if
\[
\lim_{n \to \infty} \phi(t_n) \leq \phi(t),
\]
for every sequence \( \{t_n\} \) with \( \lim_{n \to \infty} t_n = t \).

Panthi et al.\[1\] introduced and proved the following result.

Theorem 6. \[1\] Let \((X, d)\) be a complete \( d \)-metric space. Suppose that \( A, B, S \) and \( T \) are four self mappings of \( X \) satisfying the following conditions:

1. \( T(X) \subseteq A(X) \) and \( S(X) \subseteq B(X) \),
2. \( d(Sx, Ty) \leq \phi(m(x, y)) \) where \( \phi \) is an upper semicontinuous contractive modulus and
\[
m(x, y) \leq \max\{d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{1}{2}d(Ax, Ty), \frac{1}{2}d(By, Sx)\}
\]
3. The pairs \((S, A)\) and \((T, B)\) are weakly compatible.

Then \( A, B, S \) and \( T \) have a unique common fixed point.

Now, we utilize Theorem 6 to introduce and prove the following result.

Corollary 2.1. Let \((X, d)\) be a complete \( d \)-metric space. Suppose that \( A, B, S \) and \( T \) are four self mappings of \( X \) satisfying the following conditions:

1. \( T(X) \subseteq A(X) \) and \( S(X) \subseteq B(X) \),
2. suppose there exists \( \lambda \in [0, 1) \) such that
\[
d(Sx, Ty) \leq \lambda \max\{d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{1}{2}d(Ax, Ty), \frac{1}{2}d(By, Sx)\}
\]
3. The pairs \((S, A)\) and \((T, B)\) are weakly compatible.

Then \( A, B, S \) and \( T \) have a unique common fixed point.

Proof. Define \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
\phi(t) = \lambda t,
\]
where \( 0 < \lambda < 1 \). Then

1. \( \phi \) is a contractive modulus, and
2. $\phi$ is an upper semicontinuous on $\mathcal{R}$.

From Theorem 6 we get the result. \hfill\Box

Now, we are ready to present the proof of Theorem 2 from the corresponding standard metric space.

**Proof of Theorem 2:**

Take $\ln$ to both side in Inequality 1 in Theorem 2, we get

$$d_p(Sx, Ty) \leq \lambda \max\{d_p(Ax, By), d_p(Ax, Sx), d_p(By, Ty), d_p(Ax, Ty), d_p(By, Sx)\},$$

where $\lambda \in (0, \frac{1}{2})$ which is a special case of the contractive condition in Corollary 2.1. Since every pair of weakly commutative mappings is weakly compatible. All the hypotheses of Corollary 2.1 hold. Thus the four mappings $A, B, S$ and $T$ have a unique common fixed point. \hfill\Box

**Conclusion:**

From our discussions, we note that some fixed and common fixed point theorems in multiplicative metric spaces can be deduced from the corresponding standard metric spaces. Moreover, we can formulate many fixed and common fixed point theorems in multiplicative metric spaces from the corresponding results in standard metric spaces. So the researchers must be careful in working in multiplicative metric spaces.

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