CONFIGURATIONS OF POINTS WITH SUM 0

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ABSTRACT. We compute the virtual Poincaré polynomials of the configuration space of $n$ ordered points on an elliptic curve with sum 0.

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For any complex quasi-projective algebraic variety $X$, the virtual Poincaré polynomial $S(X) \in \mathbb{Z}[x]$ is defined [DK87], [Tot02] by the properties

- $S(X) = \sum \text{rk} H^i(X) x^i$ for smooth, projective $X$,
- $S(X) = S(X \setminus C) + S(C)$ for a closed subvariety $C \subset X$,
- $S(X \times Y) = S(X)S(Y)$.

For any space $X$, the ordered configuration space

$$F_n(X) = \{x_1, \ldots, x_n \in X^n | x_i \neq x_j \}$$

is the space of $n$ distinct points in $X$. Computing its cohomology is a classical, hard problem. As the configuration space is the complement of diagonals, determining the virtual Poincaré polynomials is simpler and was done for example by Getzler [Get95][Get99].

Let $E$ be an elliptic curve with neutral element 0. We will compute the virtual Poincaré polynomial of the space

$$F^n_0(E) = \{x_1, \ldots, x_n \in E^n | x_i \neq x_j \text{ and } \sum x_i = 0 \}.$$

The author was supported by the grant ERC-2012-AdG-320368-MCSK. I want to thank Karim Adiprasato, Emanuele Delucchi, Emmanuel Kowalski, Johannes Schmitt, Junliang Shen for very helpful discussions and especially Rahul Pandharipande for his invaluable support. This is part of the author’s PhD thesis.
Our approach is to decompose $F_n(X)$ in the Grothendieck ring of varieties. We use an elementary version of methods of Getzler that immediately generalizes to $F_n^0(E)$. The answer seems to be new.

The combinatorial tools are Stirling numbers and Möbius functions and we will review them first.

1. Stirling Numbers of First Kind

The Stirling number of first kind $s(n, k)$ counts the numbers of permutations in $S_n$ with exactly $k$ cycles (compare [Sta11, Chap. 1.3]). Write $\text{Part}(n, k)$ for all the partitions $\sigma$ of the set $\{1, \ldots, n\}$ into $k$ disjoint, non-empty subsets $\sigma_i$. We call the $k$ subsets $\sigma_1, \ldots, \sigma_k$ in no particular order. Then

$$s(n, k) = \sum_{\sigma \in \text{Part}(n, k)} \prod(|\sigma_i| - 1)!.$$ 

Let $x$ be a positive integer. In order to determine a generating series for $s(n, k)$, we look at the action of $S_n$ on sets of functions $\{1, \ldots, n\} \to \{1, \ldots, x\}$. The quotient consists of the multisets of size $n$ on $\{1, \ldots, x\}$ and has cardinality

$$\binom{n+x-1}{n} = \frac{x(x+1) \cdots (x+n-1)}{n!}.$$ 

On the other hand, any $\tau \in S_n$ with $k$ cycles has $x^k$ fixed points. By Burnside’s lemma

$$\frac{x(x+1) \cdots (x+n-1)}{n!} = \frac{1}{n!} \sum_{\tau \in S_n} |\text{Fix } \tau|$$

and we get

$$x(x+1) \cdots (x+n-1) = \sum s(n, k) x^k.$$ 

As it is true for all integers $x$, we have found a formal generating series.

2. The Möbius Function of the Partition Poset

We write $\text{Part}(n)$ for the partitions of the set $\{1, \ldots, n\}$. The number of parts of $\sigma \in \text{Part}(n)$ is called $l(\sigma)$. The set $\text{Part}(n)$ is partially ordered by setting $\sigma \leq \pi$ if $\sigma$ is finer than $\pi$. Write $\emptyset = \{\{1\}, \ldots, \{n\}\}$ for the minimal partition.

**Theorem 2.1 (Möbius Inversion).** For any finite poset $(M, \leq)$, the Möbius function $\mu: M \times M \to \mathbb{Z}$ on $M$ is defined by the relations

$$\mu(x, z) = 0 \text{ when } x \not\leq z \quad \sum_{x \leq y \leq z} \mu(x, y) = \delta(x, z) \text{ when } x \leq z.$$ 

Here $\delta$ is the Kronecker delta

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$
Let $f : M \to \mathbb{Z}$ a function on $M$ and

$$g(x) = \sum_{x \leq y} f(y).$$

Then we can reconstruct $f$ from $g$:

$$f(x) = \sum_{x \leq y} \mu(x, y)g(y).$$

Following [BG75], we will use Möbius inversion to compute the Möbius function for the poset of partitions. Let $x$ be a positive integer and $p : \{1, \ldots, n\} \to \{1, \ldots, x\}$ a function. The preimages of the elements of $\{1, \ldots, x\}$ induce a partition of $\{1, \ldots, n\}$, that we call the kernel of $p$. Let $f(\sigma)$ be the number of functions $\{1, \ldots, n\} \to \{1, \ldots, x\}$ with kernel $\sigma$. Then $f(0)$ counts all injective functions $\{1, \ldots, n\} \to \{1, \ldots, x\}$, hence it is

$$f(0) = x(x - 1) \cdots (x - n + 1).$$

On the other hand $g(\sigma) = \sum_{\sigma \leq \pi} f(\pi)$ allows the same values on different parts on $\sigma$. Hence we have

$$g(\sigma) = x^{l(\sigma)}.$$

By Möbius inversion

$$f(0) = \sum_{\sigma} \mu(0, \sigma)g(\sigma)$$

or

$$x(x - 1) \cdots (x - n + 1) = \sum_{\sigma \in \text{Part}(n)} \mu(0, \sigma)x^{l(\sigma)}.$$

As this holds for all values of $x$, it is valid as an identity for formal polynomials. So for the maximal partition $1 = \{1, \ldots, n\}$, we can immediately read off the constant term and get

$$\mu(0, 1) = (-1)^{n-1}(n - 1)!.$$

For general $\sigma$, the poset $\{ \pi \in \text{Part}(n) | \pi \leq \sigma \}$ is a product of posets

$$\{ \pi \in \text{Part}(n) | \pi \leq \sigma \} \simeq \{ \pi \in \text{Part}(|\sigma_1|) | \pi \leq \sigma_1 \} \times \cdots \times \{ \pi \in \text{Part}(|\sigma_{l(\sigma)}|) | \pi \leq \sigma_{l(\sigma)} \}$$

and hence

$$\mu(0, \sigma) = \mu(0, \sigma_1) \cdots \mu(0, \sigma_{l(\sigma)}) = (-1)^{n-l(\sigma)} \prod_{i}(|\sigma_i| - 1)!.$$

### 3. Virtual Poincaré Polynomials of Configuration Spaces

For any $X$, we write $[X]$ for the class of $X$ in the Grothendieck ring of varieties. We have maps

$$F_n(X) \to F_{n-1}(X)$$

with fiber $X \setminus (n - 1)$ [FH01]. This suggests – ignoring possible topological problems –

$$[F_n(X)] = [F_{n-1}(X)] \times [X - (n - 1)]$$
and hence
\[
[F_n(X) = [X]([X] - 1) \cdots ([X] - n + 1) = \sum_{k \geq 0} [X]^k (-1)^{n-k} s(n, k).
\]

We will prove this formula be a different approach using the Möbius function of the partition poset. It is inspired by Getzler [Get95, Get99], who even gave a description for the $S_n$ action on $S(F_n(X))$.

We look at the higher diagonals
\[
\Delta_\sigma = \{ x_1, \ldots, x_n \in X^n | x_i = x_j \text{ if } i \text{ and } j \text{ are in the same part of } \sigma \}
\]
for any partition $\sigma$ of $\{1, \ldots, n\}$. By the inclusion-exclusion principle we have a decomposition
\[
[F_n(X)] = [X^n] - \sum_{i \neq j} \{ x_i = x_j \} + \cdots = \sum_{\sigma \in \text{Part}(n)} m_\sigma [\Delta_\sigma].
\]
for some coefficients $m_\sigma \in \mathbb{Z}$. In order to be a valid decomposition of $F_n(x)$, the coefficients $m_\sigma$ have to satisfy the condition
\[
\sum_{\Delta_{\pi} \subseteq \Delta_\sigma} m_\sigma = \begin{cases} 1 & \text{if } \pi = 0 \\ 0 & \text{otherwise} \end{cases}
\]
for any partition $\pi \in \text{Part}(n)$. As $\Delta_{\pi} \subseteq \Delta_\sigma$ if and only if $\sigma \leq \pi$, these equations are exactly the definition of the Möbius function for the poset $\text{Part}(n)$:
\[
\sum_{\sigma \leq \pi} \mu(0, \sigma) = \begin{cases} 1 & \pi = 0, \\ 0 & \text{otherwise}. \end{cases}
\]
So we get
\[
m_\sigma = \mu(0, \sigma) = (-1)^{n-l(\sigma)} \prod_i (|\sigma_i| - 1)!
\]
and with $[\Delta_\sigma] = [X]^{l(\sigma)}$ we can compute:
\[
[F_n(X)] = \sum_{\sigma \in \text{Part}(n)} [X]^{l(\sigma)} (-1)^{n-l(\sigma)} \prod_i (|\sigma_i| - 1)! = \sum_{k \geq 1} [X]^k (-1)^{n-k} s(n, k).
\]

Now applying $S$ immediately proves:
\[
S(F_n(X)) = \sum_{k \geq 1} S(X)^k (-1)^{n-k} s(n, k)
\]

4. Configurations of Points with Sum 0

Let $E$ be an elliptic curve with neutral element 0. There is a map
\[
\Sigma : F_n(X) \to E, (x_1, \ldots, x_n) \mapsto \sum x_i.
\]
We look at the fiber $F^n_0(E) = \Sigma^{-1}(0) = \{ x_1, \ldots, x_n \in E^n | x_i \neq x_j, \sum x_i = 0. \}$

By intersecting the decomposition $[F_n(E)] = \sum_{\sigma} m_\sigma [\Delta_\sigma]$ with $\Sigma^{-1}(0)$ we get
\[
[F^n_0(E)] = \sum_{\sigma \in \text{Part}(n)} m_\sigma [\Delta_\sigma \cap \Sigma^{-1}(0)].
\]
These loci have a simpler description. Take a partition \( \sigma \) with \( l \) parts. We see:

\[
\Delta_\sigma \cap \Sigma^{-1}(0) = \{ y_1, \ldots, y_l \in E^l \mid \sum |\sigma_i| y_i = 0 \}
\]

By a coordinate change, we can compute the following solutions of this linear equation:

\[
\{ y_1, \ldots, y_l \in E^l \mid \sum |\sigma_i| y_i = 0 \} \simeq \{ z_1, \ldots, z_l \in E^l \mid \gcd(|\sigma_1|, \ldots, |\sigma_l|) z_i = 0 \}
\]

\[
\simeq E^{l-1} \times (\mathbb{Z}/\gcd(|\sigma_1|, \ldots, |\sigma_l|)\mathbb{Z})^2
\]

With the notation

\[
\gcd(\sigma) = \gcd(|\sigma_1|, \ldots, |\sigma_r|)
\]

we get

\[
[F_0^n(E)] = \sum_{\sigma \in \text{Part}(n)} (-1)^{n-l(\sigma)} [E]^{l(\sigma)-1} \gcd^2(\sigma) \prod_i (|\sigma_i| - 1)!
\]

Hence the following theorem is proven.

**Theorem 4.1.** Define

\[
s_m(n, k) = \sum_{\sigma \in \text{Part}(n,k)} \gcd^2(\sigma) \prod_i (|\sigma_i| - 1)!
\]

Then we have

\[
[F_0^n(E)] = \sum_{k \geq 1} [E]^{k-1} (-1)^{n-k} s_m(n, k)
\]

and

\[
S(F_0^n(E)) = \sum_{k \geq 1} S(E)^{k-1} (-1)^{n-k} s_m(n, k).
\]

The numbers \( s_m(n, k) \) are a form of modified Stirling numbers. Any \( \sigma \in \text{Part}(n) \) with \( l(\sigma) > \frac{n}{2} \) contains a part of length 1. So \( \gcd(\sigma) = 1 \) and

\[
s(n, k) = s_m(n, k) \text{ if } k > \frac{n}{2}.
\]

For a prime \( p \), the only partition \( \sigma \in \text{Part}(p) \) with \( \gcd(p) \neq 1 \) is \( \sigma = \{1, \ldots, p\} \). Hence

\[
s(p, k) = s_m(p, k) \text{ for } k > 1.
\]

In general,

\[
s(n, 1) = (n-1)! \quad s_m(n, 1) = n^2(n-1)!,
\]

as \( \{1, \ldots, n\} \) is the only partition of length 1.

Unfortunately, it is not straightforward to extend the methods of [Get95], [Get99] to describe the \( S_n \)-action on \( S(F_0^n(E)) \), because the identification

\[
\{ y_1, \ldots, y_l \in E^l \mid \sum |\sigma_i| y_i = 0 \} \simeq E^{l-1} \times (\mathbb{Z}/\gcd(\sigma)\mathbb{Z})^2
\]

is not compatible with the \( S_n \) and \( S_l \) actions.
Here we give the full formulas for $[F_n(E)]$ and $[F_0^0(E)]$ for all $n \leq 8$.

| $n$ | $[F_n(E)]$ |
|-----|------------|
| 2   | $E^2 - E$  |
| 3   | $E^3 - 3E^2 + 2E$ |
| 4   | $E^4 - 6E^3 + 11E^2 - 6E$ |
| 5   | $E^5 - 10E^4 + 35E^3 - 50E^2 + 24E$ |
| 6   | $E^6 - 15E^5 + 85E^4 - 225E^3 + 274E^2 - 120E$ |
| 7   | $E^7 - 21E^6 + 175E^5 - 735E^4 + 1624E^3 - 1764E^2 + 720E$ |
| 8   | $E^8 - 28E^7 + 322E^6 - 1960E^5 + 6769E^4 - 13132E^3 + 13068E^2 - 5040E$ |

| $n$ | $[F_0^0(E)]$ |
|-----|------------|
| 2   | $E - 4$   |
| 3   | $E^2 - 3E + 18$ |
| 4   | $E^3 - 6E^2 + 20E - 96$ |
| 5   | $E^4 - 10E^3 + 35E^2 - 50E + 600$ |
| 6   | $E^5 - 15E^4 + 85E^3 - 270E^2 + 864E - 4320$ |
| 7   | $E^6 - 21E^5 + 175E^4 - 735E^3 + 1624E^2 - 1764E + 35280$ |
| 8   | $E^7 - 28E^6 + 322E^5 - 1960E^4 + 7084E^3 - 16912E^2 + 42048E - 322560$ |

**References**

[BG75] E. A. Bender and J. R. Goldman. On the applications of Moebius inversion in combinatorial analysis. *Amer. Math. Monthly*, 82(8):789–803, 1975.

[DK87] V. Danilov and A. Khovanski. Newton polyhedra and an algorithm for computing hodge-deligne numbers. *Math. USSR Izv.*, 29, 1987.

[FH01] E. R. Fadell and S. Y. Husseini. *Geometry and topology of configuration spaces*. Springer, 2001.

[Get95] E. Getzler. Mixed Hodge structures of configuration spaces. In *arXiv:alg-geom/9510018*, October 1995.

[Get99] E. Getzler. Resolving mixed Hodge modules on configuration spaces. *Duke Math. J.*, 96(1):175–203, 1999.

[Sta11] R. P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, New York, NY, USA, 2nd edition, 2011.

[Tot02] B. Totaro. Topology of singular algebraic varieties. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 533–541. Higher Ed. Press, Beijing, 2002.