A NUMBER THEORETIC PROBLEM ON THE DISTRIBUTION OF POLYNOMIALS WITH BOUNDED ROOTS

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Abstract. Let $E_d^{(s)}$ denote the set of coefficient vectors $(a_1, \ldots, a_d) \in \mathbb{R}^d$ of contractive polynomials $x^d + a_1 x^{d-1} + \cdots + a_d \in \mathbb{R}[x]$ that have exactly $s$ pairs of complex conjugate roots and let $v_d^{(s)} = \lambda_d(E_d^{(s)})$ be its $(d$-dimensional) Lebesgue measure. We settle the instance $s = 1$ of a conjecture by Akiyama and Pethő, stating that the ratio $v_d^{(s)}/v_d^{(0)}$ is an integer for all $d \geq 2s$. Moreover we establish the surprisingly simple formula $v_d^{(1)}/v_d^{(0)} = (P_d(3) - 2d - 1)/4$, where $P_d(x)$ are the Legendre polynomials.

1. Introduction

Let $E_d$ denote the set of all coefficient vectors $(a_1, \ldots, a_d) \in \mathbb{R}^d$ of polynomials $x^d + a_1 x^{d-1} + \cdots + a_d$ with coefficients in $\mathbb{R}$ and all roots having absolute value less than 1, and let $E_d^{(s)}$ denote the subset of the coefficient vectors of those polynomials in $E_d$ that have exactly $s$ pairs of complex conjugate roots. Let furthermore $v_d = \lambda_d(E_d)$ and $v_d^{(s)} = \lambda_d(E_d^{(s)})$ denote the $d$-dimensional Lebesgue measures of the referring sets.

The sets $E_d$ have been studied by several authors in different context, compare e.g. Schur [11], Fam and Meditch [4] or Fam [3]. More recently, the regions $E_d$ have become of interest in the study of “shift radix systems”, since the regions where those systems have a certain periodicity property are in close connection with the regions $E_d$ (compare e.g. Kirschenhofer et al. [7]). Fam [3] established the formula

\begin{equation}
(1.1) \quad v_d = \begin{cases} 2^{2m^2} \prod_{j=1}^{m} \frac{(j-1)!^4}{(2j-1)!^2} & \text{if } d = 2m, \\ 2^{2m^2+2m+1} \prod_{j=1}^{m} \frac{j!(j-1)!^2}{(2j-1)!(2j+1)!} & \text{if } d = 2m + 1. \end{cases}
\end{equation}

In [1] Akiyama and Pethő gave a number of results on the quantities $v_d^{(s)}$, including an integral representation for general $s$ from which they derived an explicit formula in the...
instance $s = 0$ as well as a somewhat involved expression for $s = 1$ reading

$$v_d^{(0)} = \frac{2^{d+1/2}}{d!} S_d(1, 1, 1/2),$$

(1.2) $$v_d^{(1)} = 2^{d-1/2} \sum_{j=0}^{d-2} \sum_{k=0}^{d-2-j} \frac{(-1)^{d-k} 2^{d-2-2k-j}}{j!k!(d-2-j-k)!} B_{d-2}(d-2-k, d-2-k-j)$$

\[ \int_{z=0}^{1} \int_{y=-2\sqrt{z}}^{2\sqrt{z}} y^j(y+z+1)^k dy dz \]

for $d \geq 2$ and $0 \leq k \leq j \leq d$ where

(1.3) $$S_d(1, 1, 1/2) := \frac{1}{\prod_{i=0}^{d-1} \left( \frac{2i+1}{i} \right)}$$

is a special instance of the Selberg integral $S_n(\alpha, \beta, \gamma)$ and where

(1.4) $$B_d(j, k) := \prod_{i=1}^{k} \frac{2 + (d - i - 1)/2}{3 + (2d - i - 1)/2} \prod_{i=1}^{j} (1 + (d - i)/2) \prod_{i=1}^{k} (1 + (d - i)/2)$$

\[ \prod_{i=1}^{j+k} (2 + (2d - i - 1)/2) \]

is a special instance of Aomoto’s generalization of the Selberg integral (compare Andrews et al. [2, Section 8] for Selberg’s and Aomoto’s integrals).

Furthermore, Akiyama and Pethő in [1] proved that the ratios $v_d^{(s)}/v_d^{(0)}$ are rational, and, motivated by extensive numerical evidence, stated the following

**Conjecture 1.1.** [1, Conjecture 5.1] The quotient

$$v_d^{(s)}/v_d^{(0)}$$

is an integer for all non-negative integers $d, s$ with $d \geq 2s$.

In Section 2 of this paper we will prove this conjecture for the instance $s = 1$ and in addition give a surprisingly simple explicit formula for the quotient in this case involving the Legendre polynomials evaluated at $x = 3$. In the proof we will combine several transformations of binomial sums, one of them corresponding to a special instance of Pfaff’s reflection law for hypergeometric functions. We refer the reader in particular to the standard reference [6, Section 5] for the techniques that we will apply.

In Section 3 we will use our main theorem to establish a linear recurrence for the sequence $(v_d^{(1)}/v_d^{(0)})_{d \geq 0}$, and from its generating function will derive its asymptotic behaviour for $d \to \infty$. Combined with a result from [1], this also gives information on the asymptotic behaviour of the probability $p_d^{(1)} = v_d^{(1)}/v_d$ of a contractive polynomial of degree $d$ to have exactly one pair of complex conjugate roots.

In the final section we discuss possible generalizations of our results.
2. Main result

**Theorem 2.1.** The quotient $v_d^{(1)}/v_d^{(0)}$ is an integer for each $d \geq 2$. Furthermore we have

$$\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{P_d(3) - 2d - 1}{4},$$

where

$$P_d(x) := 2^{-d} \sum_{k=0}^{\lfloor d/2 \rfloor} (-1)^k \binom{d-k}{k} \binom{2d-2k}{d-k} x^{d-2k}$$

$$= \sum_{k=0}^{d} \binom{d+k}{2k} \binom{2k}{k} \left( \frac{x-1}{2} \right)^k$$

are the Legendre polynomials (cf. [10, p. 66]).

**Proof.** In a first step we solve the double integral in identity (1.2) for $v_d^{(1)}$. Let $j \geq 0$, $k \geq 0$. Then

$$\int_{y=-2}^{y=2} \int_{z=0}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz = \int_{y=0}^{y=2} \int_{z=y^2/4}^{z=2} y^j (y+z+1)^k \, dz \, dy$$

$$= \frac{1}{k+1} \left( \int_{y=2}^{y=0} y^j (y+2)^{k+1} \, dy - \int_{y=0}^{y=2} y^j (y/2 + 1)^{2k+2} \, dy \right)$$

$$= \frac{1}{k+1} \left( 2^{j+k+2} \int_{y=-1}^{y=1} y^j (y+1)^{k+1} \, dy - 2^{j+1} \int_{y=-1}^{y=1} y^j (y+1)^{2k+2} \, dy \right)$$

where we performed the substitution $y/2 \to y$ in the last step. By iterated partial integration we gain now from the last expression that

$$(2.1) \quad \int_{z=0}^{1} \int_{y=-2\sqrt{z}}^{2\sqrt{z}} y^j (y+z+1)^k \, dy \, dz = \frac{2^{j+2k+4}}{k+1} \left( \sum_{r=1}^{j+1} (-2)^{r-1}(j)_{r-1} \frac{1}{(k+r+1)_r} - \sum_{r=1}^{j+1} (-2)^{r-1}(j)_{r-1} \frac{1}{(2k+r+2)_r} \right)$$

with $(x)_j := \prod_{i=0}^{j-1}(x-i)$. 

In the following we insert (2.1) in formula (1.2) and perform stepwise a first evaluation of \( v^{(1)}_d / v^{(0)}_d \) mainly as a sum of products of factorials.

\[
\frac{v^{(1)}_d}{v^{(0)}_d} = \left( \frac{2^{d-1}(d-2)/2-2}{d-2} \sum_{j=0}^{d-2} \frac{d-2-j}{j!}!((d-2-j-k)! \prod_{i=1}^{d-2-k-j} (1 + (d-2-i)/2)^{-1} + (2(d-2) - i - 1)/2 \prod_{i=0}^{d-1} (2i+1)}{2^{d-2-k}k^{d-2-k-j}(2 + (2(d-2) - i - 1)/2) \prod_{i=0}^{d-2-k} (2i+1)} \right) \\
\sum_{j=0}^{d-1} \frac{(-1)^{d+k+2}d-j-2}{j!(k+1)!(d-j-k-2)!} \prod_{i=1}^{d-j-k-2} \frac{1}{(d-i)^{1/2}} \prod_{i=1}^{d-1} \frac{2^{d+i+1}}{d-i+1} \\
= \sum_{j=0}^{d-1} \frac{(-1)^{d+k+1}}{j!(k+1)!(d-j-k-2)!} \prod_{i=1}^{d-j-k-2} \frac{1}{(d-i)^{1/2}} \prod_{i=1}^{d-1} \frac{2^{d+i+1}}{d-i+1} \\
= \sum_{j=0}^{d-1} \frac{(-2)^r(-1)^{r-1}(j)_r}{k(r-1)!} \prod_{i=1}^{d-r-1} \frac{1}{(2k+r+2)!} \\
= \sum_{j=0}^{d-1} \frac{(-2)^r(-1)^{r-1}(j)_r}{k(r-1)!} \prod_{i=1}^{d-r-1} \frac{1}{(2k+r+2)!} \\
= \sum_{j=0}^{d-1} \frac{(-2)^r(-1)^{r-1}(j)_r}{k(r-1)!} \prod_{i=1}^{d-r-1} \frac{1}{(2k+r+2)!} \\
= \sum_{j=0}^{d-1} \frac{(-2)^r(-1)^{r-1}(j)_r}{k(r-1)!} \prod_{i=1}^{d-r-1} \frac{1}{(2k+r+2)!}
\]

In the next step we rewrite the last expression as a sum over products of binomial coefficients.
Using the substitution \( j + k + 2 \to a, k + 1 \to b \) the latter expression reads

\[
\sum_{a=2}^{d} \sum_{b=0}^{a-1} \sum_{r=1}^{a-b} (-1)^{d+b}(-2)^{r-2} \frac{a}{a+b} \binom{d}{a} \binom{d+a}{b} \binom{a+b}{2b+r} \binom{2b+r}{b} - \sum_{a=2}^{d} \sum_{b=0}^{a-1} \sum_{r=1}^{a-b} (-1)^{d+b}(-2)^{r-2} \frac{a}{a+b} \binom{d}{a} \binom{d+a}{b} \binom{a+b}{2b} \binom{2b}{b}
\]

so that

\[
\left( \begin{array}{c}
\sum_{a=2}^{d} (-1)^{d}a \binom{d}{a} \left( \binom{d+a}{b} \binom{a+b}{2b+r} \binom{2b+r}{b} \right) - \sum_{r=1}^{a} \left( -1 \right)^{r} \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b}{b} \right) = 0.
\]

(2.2)

In the following we will simplify the two innermost sums.

We start with the first sum. If \( r = a \) the sum trivially equals \( \frac{1}{a} \). Let us assume \( 1 \leq r < a \) now. Then we have

\[
\sum_{b=0}^{a-r} (-1)^{b} \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} = \frac{1}{a-r} \sum_{b=0}^{a-r} (-1)^{b} \binom{a-r}{b} \binom{a+b-1}{b+r} = \frac{(-1)^{r}}{a-r} \sum_{b=0}^{a-r} \binom{a-r}{b} \binom{r-a}{b+r} = \frac{(-1)^{r}}{a-r} \binom{0}{a} = 0,
\]

where we used

\[
\left( -1 \right)^{k} \binom{k-n-1}{k} = \binom{n}{k} \quad (n \in \mathbb{Z}, k \geq 0)
\]

for the second identity, and Vandermonde’s identity

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{s}{k+t} = \sum_{k=0}^{n} \binom{n}{k} \binom{s}{n+t-k} = \binom{n+s}{n+t} \quad (s \in \mathbb{Z}, n, t \geq 0)
\]

for the third one.

Altogether we have established

\[
\sum_{b=0}^{a-r} (-1)^{b} \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b+r}{b} = \frac{1}{a} \delta_{r,a} \quad (1 \leq r \leq a),
\]

(2.3)

where \( \delta_{r,a} \) denotes the Kronecker symbol.

Now we turn to the second sum in question. Since this is a sum reminiscent of a sum treated in [6, Section 5.2, Problem 7] we first try to adopt the strategy followed there and
use [6, Section 5.1, identity 5.26]

\[(2.4) \quad \binom{l+q+1}{m+n+1} = \sum_{0 \leq k \leq l} \binom{l-k}{m} \binom{q+k}{n} \quad (l, m \geq 0, n \geq q \geq 0).\]

With \(l = a + b - 1, q = 0, m = 2b, n = r - 1\) and \(k = s\) we get

\[
\sum_{b=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b}{b} = \sum_{b=0}^{a-r} \sum_{s=0}^{a+b-1} (-1)^b \frac{1}{a+b} \binom{a+b-s-1}{2b} \binom{s}{r-1} \binom{2b}{b}
\]

which by a change of summations yields

\[
\sum_{s=r-1}^{2a-r-1} \binom{s}{r-1} \sum_{b=0}^{a-s-1} (-1)^b \frac{1}{a+b} \binom{a+b-s-1}{2b} \binom{2b}{b}
\]

\[(2.5) \quad \sum_{s=r-1}^{a-1} \binom{s}{r-1} \sum_{b=0}^{a-s-1} (-1)^b \frac{1}{a+b} \binom{a+b-s-1}{2b} \binom{2b}{b}.
\]

Now we are ready to apply sum \(S_m\) from [6, Section 5.2, Problem 8]

\[(2.6) \quad S_m = \sum_{k=0}^{n} (-1)^k \frac{1}{k+m+1} \binom{n+k}{2k} \binom{2k}{k} = (-1)^n \frac{m!n!}{(m+n+1)!} \binom{m}{n} \quad (m, n \geq 0).
\]

With \(m = a - 1, n = a - s - 1\) and \(k = b\) we find that \(2.5\) from above equals

\[
\sum_{s=r-1}^{a-1} \binom{s}{r-1} (-1)^{a+s+1} (a-1)! (a-s-1)! \binom{a-1}{a-s-1}
\]

\[(2.7) \quad \frac{(-1)^{a+1} (a-1)! (a-s-1)!}{(2a-1)!} \frac{2a-1}{r-1} \binom{2a-1}{r-1} \sum_{s=r-1}^{a-1} (-1)^s \binom{2a-r}{s-r+1}
\]

\[
\frac{(-1)^{a+r} (a-1)! (a-1)!}{(2a-1)!} \frac{2a-1}{r-1} \sum_{s=0}^{a-r} (-1)^s \binom{2a-r}{s}
\]

(\text{where we applied the substitution } s-r+1 \to s \text{ in the last step}). Using the basic identity

\[
\sum_{j=0}^{k} (-1)^j \binom{n}{j} = (-1)^k \binom{n-1}{k} \quad (n, k \geq 0)
\]

to evaluate the last sum in \(2.7\) we finally get

\[
\sum_{b=0}^{a-r} (-1)^b \frac{1}{a+b} \binom{a+b}{2b+r} \binom{2b}{b} = \frac{(a-1)! (a-1)!}{(2a-1)!} \frac{2a-1}{r-1} \binom{2a-r-1}{a-r}
\]

\[(2.8) \quad \frac{1}{2a-r} \binom{a-1}{a-r}.
\]
Now we go on plugging the results (2.3) and (2.8) from above in (2.2) and find

\[
\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d} (-1)^d a \binom{d}{a} \binom{d+a}{d} \left( \frac{(-2)^{d-2}}{a} - \sum_{r=1}^{a} \frac{(-2)^{d-2}}{2a-r} \frac{a-1}{a-r} \right)
\]

(2.9)

\[
= \sum_{a=2}^{d} (-1)^{d+1} a \binom{d}{a} \binom{d+a}{d} \left( \sum_{r=0}^{a-1} \frac{(-2)^{d-2}}{2a-r-1} \frac{a-1}{r} - \frac{(-2)^{d-2}}{a} \right).
\]

In order to get rid of the inner sum we use an identity that may be proved as an application of the classical reflection law

(2.10)

\[
\frac{1}{(1-z)^a} F \left( \begin{array}{c} a, b \\ c \end{array} \bigg| \frac{-z}{1-z} \right) = F \left( \begin{array}{c} a, c-b \\ c \end{array} \bigg| \frac{z}{c} \right)
\]

for hypergeometric functions by J.F. Pfaff [8], namely

(2.11)

\[
\sum_{k=0}^{m} (-2)^{k} \frac{2^{m}+1}{2^{m}-k+1} \binom{m}{k} = \left( -1 \right)^{m} 2^{2m} \binom{2m}{m}
\]

compare [6, identity (5.104)]. In this way we find

\[
\frac{v_d^{(1)}}{v_d^{(0)}} = \sum_{a=2}^{d} (-1)^{d+a+1} a \binom{d}{a} \binom{d+a}{d} \left( 2^{2a-3} \frac{1}{2a-1} - \frac{1}{(2a-2)} - \frac{1}{a} \right)
\]

(2.12)

\[
= \sum_{a=2}^{d} (-1)^{d+a} \binom{d}{a} \binom{d+a}{d} \left( 2^{2a-2} - 2^{2a-2} \frac{1}{(2a)} \right)
\]

so that we have proved that the ratio \( v_d^{(1)}/v_d^{(0)} \) is an integer.

In the last step of the proof we establish the explicit formula for the ratios. Recall the Legendre polynomials \( P_d(x) \), as defined in the theorem, and let

(2.13)

\[
\rho_d(x) := \sum_{k=0}^{d} \binom{d+k}{d-k} x^k
\]

denote the associated Legendre polynomials (cf. [10] p. 66]). Then (2.12) yields

(2.14)

\[
\frac{v_d^{(1)}}{v_d^{(0)}} = (-1)^d \frac{P_d(-3) - \rho_d(-4)}{4}.
\]

Now (cf. [3] p. 158)]

(2.15)

\[
P_d(-x) = (-1)^d P_d(x).
\]
Furthermore \( \rho_d \) satisfies the recursive formula
\[
\rho_d(x) = (x + 2)\rho_{d-1}(x) - \rho_{d-2}(x)
\]
(2.16)
\[
\rho_0(x) = 0, \rho_1(x) = x + 1
\]
(compare [10, p. 66]) so that we have
\[
(-1)^d \rho_d(-4) = 2d + 1
\]
which completes the proof of (2.1).

3. Recurrence, asymptotic behaviour and probabilities

In this section we apply Theorem 2.1 in order to establish a recurrence for the quotients \( v_d^{(1)}/v_d^{(0)} \) as well as to establish the asymptotic behaviour of this sequence for \( d \to \infty \) and its consequence on the probabilities \( v_d^{(1)}/v_d^{(0)} \).

Since the Legendre polynomials satisfy the recursive formula
\[
dP_d(x) - (2d - 1)xP_{d-1}(x) + (d - 1)P_{d-2}(x) = 0
\]
(3.1)
\[
P_0(x) = 1, P_1(x) = x
\]
(cf. [9, p. 160]) we get the following second order linear recurrence for our ratios \( v_d^{(1)}/v_d^{(0)} \).

**Corollary 3.1.**
\[
d \frac{v_d^{(1)}}{v_d^{(0)}} - 3(2d - 1) \frac{v_{d-1}^{(1)}}{v_{d-1}^{(0)}} + (d - 1) \frac{v_{d-2}^{(1)}}{v_{d-2}^{(0)}} = 2d(d - 1), \quad (d \geq 2)
\]
(3.2)
\[
\frac{v_0^{(1)}}{v_0^{(0)}} = \frac{v_1^{(1)}}{v_1^{(0)}} = 0.
\]

We turn our attention now to the asymptotic behaviour of the ratios for \( d \to \infty \) and start by their generating function. The generating function of the Legendre polynomials is given by ([10, p. 78])
\[
\sum_{d \geq 0} P_d(x)z^d = \frac{1}{\sqrt{1 - 2xz + z^2}},
\]
(3.3)
so that the generating function of our ratios reads

**Corollary 3.2.**
\[
V_1(z) := \sum_{d \geq 0} \frac{v_d^{(1)}}{v_d^{(0)}} z^d = \frac{1}{4} \left( \frac{1}{\sqrt{1 - 6z + z^2}} - \frac{1 + z}{(1 - z)^2} \right).
\]
(3.4)

Performing singularity analysis the latter result allows to establish the asymptotic behaviour of the ratios for \( d \to \infty \) as follows.
Proposition 3.3. For \( d \to \infty \)

\[
\frac{v_d^{(1)}}{v_d^{(0)}} = \frac{1}{8\sqrt{2}\sqrt{\pi d}} (3 + 2\sqrt{2})^{d+\frac{1}{2}} \left( 1 + O\left(\frac{1}{d}\right) \right).
\]

Proof. We adopt the usual technique of singularity analysis of generating functions, compare e.g. \([3, \text{Chapter IV}]\) or \([12, \text{Chapter 8}]\). The dominating singularity of the generating function \( V_1(z) \) is given by the zero \( 3 - 2\sqrt{2} \) of \( 1 - 6z + z^2 \) closest to the origin, whereas the other zero of \( 1 - 6z + z^2 \) as well as the term \( \frac{1+z}{(1-z)^2} \) will give a contribution that is exponentially smaller than the contribution of the main term. The local expansion of \( V_1(z) \) about the dominating singularity reads

\[
V_1(z) = \frac{1}{8\sqrt{2}\sqrt{3 - 2\sqrt{2}}} \left( 1 - \frac{z}{3 - 2\sqrt{2}} \right)^{-1/2} \left( 1 + O\left(\frac{z}{3 - 2\sqrt{2}}\right) \right)
\]

for \( z \to 3 - 2\sqrt{2} \), from which the asymptotics is immediate. \( \square \)

In \([1]\) Akiyama and Pethó also discussed the probabilities

\[
p_d^{(s)} := \frac{v_d^{(s)}}{v_d}
\]

for a contractive polynomial of degree \( d \) in \( \mathbb{R}[x] \) to have \( s \) pairs of complex conjugate roots. In particular they derived (cf. \([1, \text{Theorem 6.1}]\))

\[
\log p_d^{(0)} = -\frac{\log 2}{2} d^2 + \frac{1}{8} \log d + \mathcal{O}(1), \quad \text{for} \quad d \to \infty,
\]

for the probability of totally real polynomials and, by numerical evidence for \( d \leq 100 \), conjectured that

\[
\log p_d^{(1)} \leq -\frac{\log 2}{2} d^2 + d \log q
\]

for some constant \( q \). Now, obviously,

\[
p_d^{(1)} = \frac{v_d^{(1)}}{v_d^{(0)}} p_d^{(0)},
\]

so that from \((3.7)\) and our Proposition 3.3 we gain

Corollary 3.4. The probability \( p_d^{(1)} \) for a contractive polynomial of degree \( d \) in \( \mathbb{R}[x] \) to have exactly one pair of complex conjugate roots fulfills

\[
\log p_d^{(1)} = -\frac{\log 2}{2} d^2 + d \log(3 + 2\sqrt{2}) + \mathcal{O}(\log d) \quad \text{for} \quad d \to \infty.
\]
4. Concluding remarks

In this paper we were able to settle the instance $s = 1$ of Conjecture 1.1. The question arises, whether our methods could be used to prove the conjecture for additional instances of $s \geq 2$ or even for general $s \geq 1$. A crucial point for a possible application of our method would be to establish a generalization of the Selberg-Aomoto integral for integrands that will occur with the evaluation of $v_d^{(s)}$ for $s \geq 2$ similar to formula (1.2) in the instance $s = 1$. Work is in progress on this question, but even the explicit evaluation of the integrals that appear in instance $s = 2$ seems to be very hard.

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