Heavy-quark contribution to the proton’s magnetic moment

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We study the contribution to the proton’s magnetic moment from a heavy quark sea in quantum chromodynamics. The heavy quark is integrated out perturbatively to obtain an effective dimension-6 magnetic moment operator composed of three gluon fields. The leading contribution to the matrix element in the proton comes from a quadratically divergent term associated with a light-quark tensor operator. With an approximate knowledge of the proton’s tensor charge, we conclude that a heavy sea-quark contribution to the proton’s magnetic moment is positive in the asymptotic limit. We comment on the implication of this result for the physical strange quark.

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1. Introduction. Naively, the proton is composed of three valence quarks bound by the strong interaction. However, quantum chromodynamics (QCD), the theory of strong interactions, implies a much more complex structure: The proton also contains a sea of virtual gluons and quark-antiquark pairs. These virtual components of the proton have measurable consequences on the proton’s macroscopic properties, such as its mass and magnetic moment. In this article, we study the contribution to the proton’s magnetic moment from a heavy quark sea in quantum chromodynamics. The heavy quark is integrated out perturbatively to obtain an effective magnetic moment operator made of pure gluon fields. We find that the leading contribution of the matrix element is related to the nucleon’s tensor charge and is positive. We thus conclude that a sea-quark contribution to the proton’s magnetic moment depends non-trivially on the sea-quark mass: It must change sign at some critical value of the sea-quark mass. We argue that this critical value could be smaller than the physical mass of the strange quark.

2. Effective gluonic operator from heavy quark current. Consider a proton of momentum $p$ and polarized in the $z$-direction, $|p \uparrow\rangle$. The $z$-component of its magnetic moment is defined as

$$
\mu_p = \frac{\langle p \uparrow| \frac{1}{2} \int d^4 \vec r (\vec r \times \vec j_{em})_z |p \uparrow\rangle}{\langle p \uparrow|p \uparrow\rangle},
$$

(1)

where $\vec j_{em}$ is the electromagnetic current. In order to carry out the spatial integration, it is useful to consider the off-diagonal matrix element and to take the forward limit after integrating over $\vec r$

$$
\mu_p = -i \frac{1}{2} \left( \vec \nabla_q \times \langle p' \uparrow|j(0)|p \uparrow\rangle \right)_{z=q=0},
$$

(2)

where $q = p' - p$ is the momentum transfer.

Now consider the contribution to the magnetic moment from a heavy quark with electromagnetic current $\vec j_Q = \vec Q \gamma^\mu Q$. (The quark also contributes to the magnetic moment through wave functions. However, this is conventionally regarded as a part of the light-quark contribution.) To lowest order in $1/m_Q$, its contribution to the form factor is given by the diagram shown in Fig. 1. When $m_Q \gg \Lambda_{QCD}$, the quark loop can be integrated out perturbatively. The heavy-quark electromagnetic current is matched to a leading effective operator consisting of pure gluon fields

$$
\vec j_{\mu Q} = C(m_Q) \partial^\alpha T_{\mu \alpha}(m_Q) + \ldots
$$

(3)
where ellipses denote higher order contributions in $1/m_Q$, and $C(m_Q) = g^3(m_Q)/(4\pi) 2.45 m_Q^2$ to leading order in the strong coupling $g(m_Q)$, and

$$T_{\mu\alpha} = 14 \text{Tr} G_{\mu\sigma} \{ G^{\sigma\tau}, G_{\tau\alpha} \} - 5 \text{Tr} G_{\sigma\tau} \{ G^{\sigma\tau}, G_{\mu\rho} \} ,$$

where $G_{\mu\rho}$ is the strong field-strength tensor. Notice that the tensor $T_{\mu\alpha}$ is antisymmetric and that the effective gluon operator is renormalized at the scale $m_Q$.

This is a striking result: The leading power dependence on the muon mass is $1/m_Q^2$, rather than $1/m_Q$ as implied by a naive heavy-$m_\mu$ expansion similar to (4-8). The discrepancy is resolved if one realizes that the effective operator $T_{yx}$ (obtained by replacing the gluon fields by photon fields in Eq. (5)) is renormalized at scale $m_\mu$. Power counting indicates that the matrix element of $T_{yx}$ in the electron state is quadratically divergent, and the divergence is cut-off naturally by the scale $m_\mu$. Since the matrix element of the effective operator has dimension 3, it must behave as $(\langle e|T_{yx}|e\rangle) = (2\pi)^3 \delta^3(0)

where the $m_e$ dependence comes from the chirality flip of the electron. This demonstrates that the matrix element of the effective operator is dominated by quantum fluctuations at the scale of the heavy fermion mass.

The QED result can be directly applied to an interesting case in QCD. Consider a spin-1/2 $\Lambda_b$ baryon made of a heavy bottom quark and two light up and down quarks coupled to zero spin and isospin. To leading order in $1/m_b$, the spin of $\Lambda_b$ is carried by the heavy $b$-quark. The magnetic moment of $\Lambda_b$ from the top quark sea can be calculated just like in the QED case. Including a color factor of $C_d = \sum_{abc} d_{abc}^2/48 = 5/18$, the leading order contribution in $\Lambda_{QCD}$/m_b and $m_b/m_t$ is

$$\delta\mu_{\Lambda_b}^e = \mu_{\Lambda_b} \left( \frac{3}{2} \frac{\alpha_s}{\pi} \right) C_d \left( \frac{m_b}{m_t} \right)^2 \left( \frac{3}{2} \frac{\zeta(3) - 19}{16} \right) + ... ,$$

where $\mu_{\Lambda_b} = e \hbar/2m_{\Lambda_b} c$. We thus conclude that the top quark contributes positively to the $\Lambda_b$ magnetic moment.

4. Gluonic matrix element in the proton. The above example is very instructive for the behavior of the effective gluonic operator matrix element in the proton: Its dependence on the renormalization scale $m_Q$ and $\Lambda_{QCD}$ must exhibit the form

$$\langle p|T_{yx}|p\rangle = a m_Q^2 \Lambda_{QCD} + b \Lambda_{QCD}^3 ,$$

where the possible logarithmic dependence on $m_Q$ is neglected. The $a$-term depends quadratically on $m_Q$ and is therefore dominant. It receives contributions mainly from quantum fluctuations at scale $m_Q$. Power counting of the diagrams involved in $\Lambda_{QCD}$ shows that this term comes from single quark contributions. Diagrams involving two or three quarks from the proton cannot produce a term quadratic in $m_Q$. The $b$-term is entirely determined by physics at the non-perturbative scale $\Lambda_{QCD}$.

We first focus on the $a$-term since it dominates the matrix element. Because it involves contributions from both large momentum flow and scale $\Lambda_{QCD}$, we factorize them by matching the dimension-6 operator $T_{\mu\nu}^\rho$ to a set of dimension-4 ones $\theta_{i\rho}^\mu^\nu$.

$$T_{\mu\nu}^\rho = m_Q^2 \sum_i C_i \theta_{i\rho}^\mu^\nu + ... ,$$

where $\mu$ is a momentum flowing from the quark line and $\nu$ is a momentum flowing from the photon line.
where $\theta^{\mu\nu}$ must be gauge-invariant second-order antisymmetric tensors with odd charge parity, and ellipses denote higher dimensional operators. Since we are interested in the forward matrix element, total derivative operators are ignored. Such gauge invariant dimension-4 operator cannot be constructed with pure gluonic fields. There are, however, two such operators made of quark fields,

$$\epsilon^{\mu\nu\alpha\beta} \bar{\psi} (i D_\alpha \gamma_\beta - i D_\beta \gamma_\alpha) \psi, \quad m \bar{\psi} \sigma^{\mu\nu} \psi. \quad (13)$$

Using the QCD equation of motion ($i \not{D} - m) \psi = 0$, the first operator can be reduced to the second one. Therefore, we find that

$$T^{\mu\nu} = am_3^2 \sum_f m_f \bar{\psi} \gamma^\mu \gamma^\nu \gamma^5 \psi_f + \ldots, \quad (14)$$

where $a = \frac{1}{2}(\alpha_s/\pi)^3 C_d[3/2\zeta(3) - 19/16]$ comes from the matching calculation for a single quark state, and the sum runs over the light-quark flavors.

It is now straightforward to calculate the leading contribution to the magnetic moment,

$$\delta \mu_p^Q = C(m_Q) m_3^2 a (m_3 \delta u + m_d \delta d) + \ldots, \quad (15)$$

where $\delta u$ and $\delta d$ are the tensor charges of the proton defined as $\langle P S \bar{\psi} \sigma^{\mu\nu} \sigma^5 \psi | P S \rangle = 2\delta \psi (S^\mu P^\nu - S^\nu P^\mu)$, with $P$ and $S$ the proton’s four-momentum and spin, respectively. The above equation is one of the main results of this paper. Using the PDG values $m_u = 2.8 \pm 1.3$ MeV and $m_d = 6 \pm 2$ MeV and the tensor charge from the MIT bag model $\delta u = 1.2$ and $\delta d = -0.3$, we find that $\delta \mu_p^Q$ is positive. Lattice calculations at higher renormalization scales yield a similar ratio $\delta u/\delta d$ which does not change the conclusion.

The strange quark also contributes to Eq. (16) through the proton wave function, but its contribution is suppressed compared to that of the light up and down quarks. The scenario we study here is that the strange quark is the fictitious heavy quark. The corresponding $m_Q \delta Q$ term in Eq. (16) can be calculated by matching $\bar{Q} \sigma^{\mu\nu} Q$ to the gluon operator $T^{\mu\nu}$ and then evaluating the leading contribution from the matrix element $T^{\mu\nu}$ in the proton state. The result is that although $m_Q \delta Q$ is a leading power, it is suppressed by $\alpha_s^3(m_Q)$ relative to the up and down quark contributions and is therefore significantly smaller.

5. An example of non-perturbative contribution. The subleading non-perturbative contribution is hard to calculate without a specific model of the proton. Here we present quark model calculations where the excitations of the quarks and the non-linear gluonic interactions are neglected. In this Abelian approximation, the gluon fields consist of eight copies of the electromagnetic field obeying the linear Maxwell equations. The chromo-electric fields are given by $E^a(r) = \frac{1}{2} \lambda^a \vec{E}(r) = -\frac{1}{2} \vec{\lambda} \vec{A}(r)$, with

$$\Phi(r) = \frac{g}{4\pi} \int d^3 r_1 \frac{\rho(r_1)}{|r - r_1|}, \quad (16)$$

where the index $a$ refers to its color, $\rho(r) = \langle \alpha | \rho(r) | \beta \rangle$ denotes the charge density matrix element between two states, $| \alpha \rangle$ and $| \beta \rangle$, and $\lambda^a$ are the Gell-Mann matrices. Similarly, the chromo-magnetic fields are given by $B^a(r) = \frac{1}{2} \lambda^a \vec{B}(r) = -\frac{1}{2} \vec{\lambda} \vec{A}(r)$ with

$$\vec{A}(r) = \frac{g}{4\pi} \int d^3 r_1 \frac{\vec{J}(r_1)}{|r - r_1|}, \quad (17)$$

where $\vec{J}(r) = \langle \alpha | \vec{J}(r) | \beta \rangle$ is the quark current density matrix element.

We compute the contribution to the magnetic moment due to a single quark polarized in the $+z$-direction, and is in the ground state of a quark model potential. The electric and magnetic fields can be calculated using the wave functions that describe the quark fields in the ground state of the model. We use the non-relativistic (NR) quark model and the MIT bag model [9, 10]. The dominant contribution to the proton magnetic moment is

$$\delta \mu_p^Q = C(m_Q) \int dr r^2 m(r), \quad (18)$$

$$m(r) = 7 \vec{E} \cdot \vec{B}_f + \vec{B} \cdot \vec{E}_f - 2 \vec{E} \cdot \vec{B} = \sum_f \left( \frac{2}{3} \bar{B}_f^\mu \cdot \vec{B}_f^\mu + \frac{1}{3} \bar{B}_f^\mu \cdot \vec{B}_f^\mu \right) + \frac{2}{3} \text{Re} \vec{B}_f \cdot \vec{B}_f$$

The numerical evaluation of these integrals results in $\delta \mu_p^Q / \mu_N C(m_Q) \approx 2.2 \text{ fm}^{-4}$ and $1.3 \text{ fm}^{-4}$ in the NR quark model and in the MIT bag model, respectively. If naively applying this result for the physical strange quark, with $m_s = 110$ MeV and $\alpha_s(m_s) \approx 1$, we get $\delta \mu_p^Q \approx 0.1 \mu_N$, where $\mu_N = e\hbar/2Mc$ is the nuclear magneton.

In both models, the $\delta \mu_p^Q$ is therefore positive. However, this result is reached for different reasons. In the NR model, the $\vec{B}$ field is generated by the color dipole of the quark and hence is correlated with its spin. The triple-$\vec{B}$ term can be neglected because the quark is non-relativistic. When averaging over the direction of the $\vec{B}$ and $\vec{E}$ inside the proton, the term $7(\vec{E} \cdot \vec{B})\vec{E}$ dominates over $2\vec{E}^2 \vec{B}$. In the MIT bag, on the other hand, the color current is generated from the orbital motion of the quark. The direction of motion is correlated with the spin. Here the triple-$\vec{B}$ term is non-negligible. After canceling the negative $\vec{E}^2 \vec{B}$ term, the result is again positive.

It would be nice, of course, to also calculate contributions from intermediate excited states, particularly the contribution with excitation energy of order $m_Q$. Because of the intermediate state sum, the single-quark matrix element dominates the $\bar{b}$-term. Such a calculation is beyond the scope of this paper. In the NR quark model, it can be shown that all excited state contribution is positive. For the MIT bag, on the other hand, we have only verified this for the first few excited states.
6. Mass-dependence of the sea quark contribution to the proton’s magnetic moment. Thus it appears that a heavy sea quark contributes positively to the proton’s magnetic moment. On the other hand, there are indications, that a light sea quark contributes negatively.

One sensible picture for the light quark sea is the meson cloud model [11, 12]. If the strange quark is light, the kaon is then a Goldstone boson. The proton has a certain probability to dissociate into a kaon, which contains an anti-strange quark, plus a Λ, which contains a strange quark. The kaon carries a unit of orbital angular momentum and its contribution to the magnetic moment is negative. The Λ’s spin and magnetic moment are dominated by the strange quark and the latter is positive. Therefore the total contribution from light strange and anti-strange quarks is negative.

There is also an indirect lattice QCD calculation for the nucleon sea contribution to the magnetic moment of the proton [12],

$$\delta \mu_{p}^{u \text{-loop}} = \delta \mu_{p}^{s} - 0.3(1) ,$$  \hspace{1cm} (19)

where $\delta \mu_{p}^{u \text{-loop}}$ is the up (or down) quark “sea contribution” defined in the same sense as the strange quark sea. Thus unless the real strange quark contribution is larger than $0.3 \mu_{N}$ (the recent world data average gives $\delta \mu_{p}^{s}(Q^{2} \sim 0.1 \text{GeV}^{2}) = (0.28 \pm 0.20) \epsilon_{s} \mu_{N}$ [13]), the light-sea contribution shall be negative.

7. Conclusion. In this paper, we advocate the study of the contribution to the magnetic moment of the proton from a heavy-quark sea. We show that the leading contribution can be calculated in perturbative QCD combined with the quark tensor charge and that the contribution is positive. If one believes that the contribution is negative when the quark mass is small, the magnetic moment has a nontrivial dependence on the sea-quark mass. Therefore the sign of the strange quark contribution to the proton’s magnetic moment is very sensitive to the QCD dynamics.

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![FIG. 2: A possible picture for a sea-quark contribution to the proton’s magnetic moment as a function of the sea-quark mass.](image-url)

The ineluctable consequence of these results obtained in the light and heavy quark limits is that the sea-quark contribution to the proton’s magnetic moment vanishes at least once at some critical value of the sea-quark mass. How large is this critical mass? A plausible answer is that its value is slightly above the masses of the up and down quarks, i.e. around 10 to 20 MeV. Indeed, this is where the chiral behavior seems to rapidly set in, as can be seen from chiral extrapolations of lattice data [12]. If this is correct, the strange quark contribution to the magnetic moment of the proton is positive.

[1] D. S. Armstrong et al. [G0 Collaboration], Phys. Rev. Lett. 95, 092001 (2005); K. A. Aniol et al. [HAPPEX Collaboration], arXiv:nucl-ex/0506011; D. T. Spayde et al. [SAMPLE Collaboration], Phys. Lett. B 583, 79 (2004); F. E. Maas et al. [A4 Collaboration], Phys. Rev. Lett. 93, 022002 (2004).
[2] For references on various models, see reviews, E. J. Beise, M. L. Pitt and D. T. Spayde, Prog. Part. Nucl. Phys. 54, 289 (2005); D. H. Beck and B. R. Holstein, Int. J. Mod. Phys. E 10, 1 (2001); D. H. Beck and R. D. McKeown, Ann. Rev. Nucl. Part. Sci. 51, 189 (2001).
[3] D. B. Kaplan and A. Manohar, Nucl. Phys. B 310, 527 (1988).
[4] H. Euler and B. Kockel, Naturwissenschaften 23, 246 (1935); J. D. Jackson, Classical Electrodynamics, John Wiley and Sons, New York, 1975.
[5] S. Laporta and E. Remiddi, Phys. Lett. B 301, 440 (1993).
[6] R. L. Jaffe and X. Ji, Phys. Rev. Lett. 67, 552 (1991); H. X. He and X. Ji, Phys. Rev. D 54, 6897 (1996).
[7] S. Eidelman et al. [Particle Data Group], Phys. Lett. B 592, 1 (2004).
[8] See for example, M. Gockeler et al. [QCDSF Collaboration], Phys. Lett. B 627, 113 (2005).
[9] T. A. DeGrand, R. L. Jaffe, K. Johnson and J. E. Kiskis, Phys. Rev. D 12, 2060 (1975).
[10] See e.g., R. K. Bhaduri, Models of the Nucleon, Addison Wesley, Reading, 1988.
[11] M. J. Musolf and M. Burkard, Z. Phys. C 61, 433 (1994).
[12] D. B. Leinweber and A. W. Thomas, Phys. Rev. D 62, 074505 (2000); D. B. Leinweber et al., Phys. Rev. Lett. 94, 212001 (2005).
[13] K. Paschke, Jefferson Lab seminar, April, 2006.