Limited resources in multi-lane stochastic transport system

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Abstract

The present study proposes a generalized mean-field approach to examine the significant effect of the finite supply of particles on multi-lane coupled system with non-conserving dynamics. The steady-state behavior is analyzed by exploring vital characteristics such as phase diagrams, density profiles, residence time and power spectra. Despite the fully asymmetrical coupling environment, symmetrical phases are identified along with asymmetrical phases. The emergence of shock results in the breaking of symmetry prevailing among the lanes for a critical value of the total number of particles in the system. Additionally, bulk induced phase transition results in the shifting from low density to high density regime. As expected, jamming length increases with increase in the total number of particles in the system. Particles follow the pseudo-Gaussian distribution with decreasing variance exhibiting the significant effect of limited resources on the system properties. For the lower values of the total number of particles, the current initially increases and then saturate beyond its critical value. Through power spectra damped oscillations are observed in the particles occupancy in one of the lanes while other lane and reservoir show undamped profile with non-conserving dynamics in the bulk.

1. Introduction

Many real life physical and biological phenomena depend upon resources that compete in a pool of limited availability, for instance, ribosomes involved in translation during protein synthesis and finite medium engrossed in vehicular and pedestrian dynamics, etc [1–5]. In particular, intracellular transport involving driven biological machines, molecular motors, is also far from being unlimited. Motor proteins such as kinesin, dynein and myosin, perform directed motion along cytoskeleton protofilaments serving as macromolecular highways to and from within distant locations [6–8]. The improper functioning of motors can lead to fatal diseases like left-right body determination and tumor suppression, etc [9, 10]. The transit progression of systems relying on the availability of specific resources, demands the balance between available amount and requirement. This situation results in the competition for limited resources, thereby affecting the system dynamics significantly [11–14].

The totally asymmetric simple exclusion process (TASEP), a specific driven lattice gas model, is an exemplary system to analyze such transport phenomena over last few decades [15]. Despite its simplicity, it has acquired paradigm status for explaining various complex non-equilibrium phenomena like boundary induced phase transition, phase separation and shock formation [16–19]. The majority of these systems has been solved analytically with prime focus on the pool of infinite availability of particles (standard TASEP) [15, 20–26]. Incorporating the scenario originating due to real-time dynamics invoked by limited resources, TASEP studies have been recently extended by coupling single trail to a finite reservoir (constrained TASEP) [11, 12]. Motivated by the multi-lane transport systems, including molecular motors proceeding simultaneously along multiple microtubules and man-made transport systems, few studies have been conducted on two-lane TASEP in the presence of finite resources [13, 14]. These models developed a theoretical framework to tackle in vitro and in vivo experimental observations of biological phenomena involving finite motor protein concentration and protein synthesis.
Experimental studies on the transport of motor proteins across microtubules suggest that along with the translational motion, individual motors interact with the surrounding environment by permitting particle absorption (desorption) to (from) the filament [27, 28]. The competing dynamics of the non-conserved particles can also be significantly visualized in vehicular as well as pedestrian flow. These findings have been quantitatively characterized by integrating the constrained availability of resources with Langmuir kinetics (LK) resulting in more general single lane TASEP with LK [11].

Further, to understand physical and biological complex processes more realistically, some studies on two-lane TASEP with coupling between lanes have been conducted under infinite reservoir. In this direction, Pronina et al [20] studied a two-lane TASEP with fully asymmetrical coupling conditions and concluded that even one sided coupling affects the system dynamics significantly. Recently, few studies have been conducted on both fully and partially asymmetric coupling in a two-lane standard TASEP with [21, 22, 24, 29] and without LK [30]. Nevertheless, the existing works on coupling between lanes focus only on the two-lane TASEP with infinite particles.

Motivated by the crucial effect of the lane changing process along with absorption–desorption scenario on the systems with infinite resources, we wish to explore two-lane coupled non-conserved TASEP with a finite pool of reservoir. As a first step towards the understanding of the two-lane constrained TASEP, we consider fully asymmetrical coupling among the lanes in the proposed system. Since, the existing studies on constrained TASEP mainly focus on domain wall theory [12, 13] and modified mean field theory [14] for single as well as multi-lane uncoupled conserved system, these theories are not directly applicable to capture the non-trivial effects of the finite resources on the system dynamics. We propose a generalized mean-field framework to address the incorporated phenomena and to manifest novel behaviors, including density profiles, phase diagrams emerging in the system. Besides, we concentrate on the on-site residence time of particles interpreted as the average time a particle spends at a specific site. The dynamical properties of the proposed system are also explored using power spectra of total particles occupancy in the lanes. Our other aim here is to probe the effect of limited resources on the vital characteristics, particularly, residence time, power spectra and particle distribution.

The proposed model serves as a natural medium to study many multi-lane transport systems interacting with the surrounding environment. The main objective of the present study is to propose a general theory to capture system dynamics of multi-lane constrained coupled system with LK.

The paper is organized as follows. The model with all the relevant parameters is defined in section 2. Further, the mean-field framework is presented in section 3 followed by the proposed theory in section 4. Several results and discussions on significant parameters including phase diagrams, residence time and power spectra based on theoretical and simulation studies are provided in sections 5, 6 and 7, respectively. Finally, the results are summarized and future scopes of the presented model are concluded in section 8.

2. Model, processes and parameters

We mimic the system dynamics considering particles moving unidirectionally on two coupled one-dimensional lanes each with L sites, denoted by Lane-1 and Lane-2, respectively (figure 1). The state of a site is characterized by a discrete occupation number, \( \tau_j^i \) \((i = 1, 2, \ldots, L \text{ and } j = 1, 2)\), either one or zero for occupied and vacant site, respectively. The particles are distributed under hard-core exclusion principle guaranteeing that no site can be occupied by more than one individual. Additionally, with forward hopping in the bulk, we consider the attachment and detachment of the particles to and from the filament along with the lane changing process. Further, motivated by the finite pool of resources in many biological and physical phenomena, we assume that both the lanes share the finite and constant number of particles, denoted by \( N \). Both the ends of Lane-1 and Lane-2 are connected to a single reservoir with finite particles, \( N_r(i) \), at any time \( t \). It is significant to mention that here reservoir represents the surroundings of the trails considered in the system. For each time step, a lane site \((i, j)\) is randomly chosen and particle hopping takes place as per the random sequential update rules. The progressive system dynamics for Lane-1 based on the standard TASEP are governed by following sub-processes.

(i) At entrance site \((i = 1)\), a particle hops from a reservoir onto the lane with a rate \( \alpha \), provided the site is empty.

(ii) At exit site \((i = L)\), particle leaves the lane and jump into the reservoir with another rate \( \beta \).

(iii) In the bulk \((1 < i < L)\), If the \( i \)th site is empty, then a particle from the reservoir attaches to the site with a rate \( \omega_{ii} \). If \( \tau_1^i = 1 \), firstly a particle tries to detach from the lane to the reservoir with a rate \( \omega_{iL} \). If this attempt fails, the particle moves forward in its own lane with unit rate, provided the target site is empty. In case of
unsuccessful forward movement, shifting to Lane-2 is attempted with a rate $\omega$ only if the corresponding site is empty.

The dynamics in Lane-2 are similar, with the only exception that the particles of Lane-2 are prohibited to switch to Lane-1 on account of fully asymmetrical coupling condition. The above progression rules are identical to standard two-lane TASEP with or without LK ($\omega_a = \omega_d = 0$) under the limiting case of $N \to \infty$ [20, 24]. Further, for the special case $\omega_a = \omega_d = \omega = 0$, this model converges to recently studied systems on the finite reservoir [13, 14]. Since in most of the real world phenomena only finite number of particles are available in the surroundings, we assume that the trails are coupled with the finite pool of resources. Seeing that the number of particles in the reservoir is limited at any time $t$, the ingress and the attachment rates will not be constant and will depend upon its density while detachment and removal rate will remain constant. We define the modified effective entry rate $\alpha_{eff}$ and effective attachment rate $\omega_{aeff}$ to depend on the number of particles in the reservoir, $N_r(t)$, as follows.

$$\alpha_{eff} = \alpha \psi(N_r(t)),$$

$$\omega_{aeff} = \omega_a \psi(N_r(t)),$$

where $\psi(.)$ is a monotonically increasing function with the boundary conditions as $\psi(0) = 0$ and $\psi(N) = 1$. Physically, the impulse for defining effective rates and $\psi(.)$ can be interpreted as: if there is no particle in the reservoir then $\alpha_{eff}$ should be zero, signifying $\psi(0) = 0$. Additionally, for significantly large reservoir, $N_r(t) \to N$, $\alpha_{eff}$ should tend to $\alpha$ suggesting $\lim_{N_r(t) \to N} \psi(N_r(t)) = 1$. Under this limiting case, all the rates will become independent of $N_r(t)$ and the resulting system has been extensively studied in [24]. As suggested in studies based on single lane finite reservoir TASEP, the exquisite of functional form involving effective rates does not affect the system dynamics [12]. In this direction, we define $\psi(N_r(t))$ as

$$\psi(N_r(t)) = \frac{N_r(t)}{N}.$$

Further, we have

$$N = N_f(t) + N_i(t), \quad N_i(t) = \sum_{j=1}^{2} N_{j}(t), \quad 0 \leq N_{j}(t) \leq L,$$

where, $N_{j}(t)$ ($j = 1, 2$) denotes the total number of particles in $j$th lane at any time $t$. In the limit of a substantial total number of particles, i.e., $N \to \infty$, we retrieve $N_{j}(t) \approx N_i$, resulting in the standard TASEP with constant entry and attachment rates. Note that the proposed model is more suitable to study a number of natural and man-made transport systems.

3. Mean-field equations

The evolution of motor density in the bulk ($1 < i < L$) for both the lanes is governed by

$$\frac{d \langle \tau^i \rangle}{dt} = \omega_{aeff} (1 - \langle \tau^i \rangle) + \langle \tau^{i-1} \rangle (1 - \langle \tau^{i+1} \rangle) - \omega_d \langle \tau^i \rangle \langle \tau^{i-1} \rangle - \langle \tau^{i+1} \rangle (1 - \langle \tau^{i+1} \rangle) - \omega \langle \tau^{i-1} \rangle (1 - \langle \tau^{i-1} \rangle),$$
The positive and negative terms on the right hand side denote the gain and loss terms arising due to attachment, detachment and forward hopping succeeded by lane changing processes. At the boundaries, particle densities evolve as

$$\frac{d\langle \tau_i^j \rangle}{dt} = \omega_{\text{eff}}(1 - \tau_i^j) + \langle \tau_i^{j-1}(1 - \tau_i^j) \rangle - \omega_{d}(\tau_i^j) - \langle \tau_i^j(1 - \tau_i^{j+1}) \rangle + \omega_{\text{eff}}(1 - \tau_i^j),$$

where \(\langle \ldots \rangle\) characterizes statistical average. A coarse grained description of the system is obtained to get densities in both the lanes under mean-field limit by defining lattice constant \(\epsilon = 1/L\), continuous variable \(x = i/L \in [0, 1]\) and rescaled time \(t' = t/L\). In order to get insight into the competing interplay between boundary and bulk dynamics, kinetic rates should decrease simultaneously with increase in lane length. For LK and lane changing rates with order of magnitude larger than \(O(1/L)\), the steady-state of the system will be that of coupled LK dynamics. In case of order being smaller than \(O(1/L)\), the effect of LK and coupling is negligible and the system will behave as the uncoupled conserved TASEP. More significantly, to visualize the impact of LK and lane changing processes on system dynamics, we define the reduced attachment, detachment and lane changing rates inversely proportional to the system size as:

$$\omega_{\text{eff}}L = \Omega_{\text{eff}}, \quad \omega_dL = \Omega_d, \quad \omega_L = \Omega.$$  

Based on the spatial homogeneity in the continuum limit, we replace \(\langle \tau_i^j \rangle\) by a continuous variable \(\rho_j(x) \in [0, 1]\) and get

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_2}{\partial x} - \frac{\epsilon}{2} \left[ \frac{\partial \rho_1}{\partial x} + \rho_1(1 - \rho_1) \right] = \left[ \Omega_{\text{eff}}(1 - \rho_1) - \Omega_d\rho_1 - \Omega_d\rho_2(1 - \rho_1) \right]$$

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_2}{\partial x} - \frac{\epsilon}{2} \left[ \frac{\partial \rho_2}{\partial x} + \rho_2(1 - \rho_2) \right] = \left[ \Omega_{\text{eff}}(1 - \rho_2) - \Omega_d\rho_2 + \Omega_d\rho_1(1 - \rho_2) \right].$$

Here, \(\rho_1\) and \(\rho_2\) denotes the average density in Lane-1 and Lane-2, respectively and right hand side is the source term originating due to lane changing transitions and Langmuir kinetics. Without loss of generality, we have denoted \(t'\) by \(t\) in equation (10). Under the special choice \(\Omega = \Omega_d\) the steady-state equations reduce to

$$\frac{\epsilon}{2} \frac{d^2\rho_1}{dx^2} + (2\rho_1 - 1) \frac{d\rho_1}{dx} - \Omega_d[\rho_1(1 + \psi(N_1(t))) - \psi(N_1(t))] - \Omega_{\text{eff}}(1 - \rho_2) = 0,$$

$$\frac{\epsilon}{2} \frac{d^2\rho_2}{dx^2} + (2\rho_2 - 1) \frac{d\rho_2}{dx} - \Omega_d[\rho_2(1 + \psi(N_2(t))) - \psi(N_2(t))] + \Omega_{\text{eff}}(1 - \rho_2) = 0,$$

with boundary conditions mapping to \(\rho_1(0) = \rho_2(0) = \alpha_{\text{eff}}\) and \(\rho_1(1) = \rho_2(1) = 1 - \beta\). It is worthwhile to mention here that \(\alpha_{\text{eff}}\) and \(\Omega_{\text{eff}}\) are variables and depend upon the number of particles in the reservoir. Due to lack of counsel about, \(N_j(t)\), singular perturbation technique [24, 31] cannot be applied here, which in the past has been found suitable to solve unconserved coupled systems. In the succeeding section, we discuss certain shortcomings of existing theories followed by generalized mean-field theory to handle the proposed coupled system.

### 4. Generalized mean-field theory

A simplified case of the proposed finite reservoir model without coupling and LK has been studied using domain wall [13] and modified mean-field theory [14]. The basic idea of domain wall theory is to assume a sharp shock located at any site between a high and low density region. The density profile is estimated based on the probability of locating this shock at a particular site. The probability is computed utilizing the drifting rates of the shock which cannot be computed in the presence of lane switching process, thereby making the domain wall theory insufficient to handle the proposed model. Further, modified mean-field theory (MMFT) is a simple and efficient technique that can generate phase diagrams of both single as well as multi-lane uncoupled systems. This theory requires the explicit expressions of density profiles in limiting case of infinite reservoir, which are not available for standard two-lane coupled TASEP with LK. Additionally, its inability to generate density profiles makes this theory inadequate for our system.

Now, we propose generalized mean-field technique to study multi-lane coupled TASEP with the attachment-detachment process. This theory not only generates phase diagrams and density profiles, but also determines significant characteristics like residence time and power spectra presented in upcoming sections. The underline idea is to compute variable rates, \(\alpha_{\text{eff}}\) and \(\Omega_{\text{eff}}\) in terms of particle densities, \(\rho_1\) and \(\rho_2\), implicitly.
The theory is formulated within the framework of mean-field studies resulting in the implicit relationship between variable rates and densities.

In this direction, we determine the total number of particles on the lanes, \( N_i(t) \), at any time \( t \) using mean-field theory. Since, \( \rho^i_j \) is a continuous function denoting average particle occupancy of any site \( i \) in lane \( j \), using the continuous Riemann sum we have:

\[
N_i(t) = N_i(0) + \int_0^t \rho^i_j(x, t) \, dx.
\]

Since \( N \) is shared by both the lanes and reservoir, we obtain

\[
N_i(t) = N - L \sum_{j=1}^{2} \int_0^1 \rho^i_j(x, t) \, dx.
\]  

Further, \( \alpha_{\text{eff}} \) and \( \Omega_{\text{eff}} \) can be computed as

\[
\alpha_{\text{eff}} = \alpha \left[ 1 - \frac{L \sum_{j=1}^{2} \int_0^1 \rho^i_j(x, t) \, dx}{N} \right],
\]

\[
\Omega_{\text{eff}} = \Omega_a \left[ 1 - \frac{L \sum_{j=1}^{2} \int_0^1 \rho^i_j(x, t) \, dx}{N} \right].
\]  

Using equation (14)–(15), we obtain the equations defining steady-state average densities for both the lanes in bulk as:

\[
\frac{\epsilon}{2} \frac{d^2 \rho^1_1}{dx^2} + (2\rho^1_1 - 1) \frac{d \rho^1_1}{dx} - \Omega_d \left[ \rho^1_1 + (\rho^1_1 - 1) \left[ 1 - \frac{L}{N} \left[ \int_0^1 [\rho^i_1(x, t) + \rho^i_2(x, t)] \, dx \right] \right] \right] = 0,
\]

\[
- \frac{\epsilon}{2} \frac{d^2 \rho^1_2}{dx^2} + (2\rho^1_2 - 1) \frac{d \rho^1_2}{dx} - \Omega_d \left[ \rho^1_2 + (\rho^1_2 - 1) \left[ 1 - \frac{L}{N} \left[ \int_0^1 [\rho^i_1(x, t) + \rho^i_2(x, t)] \, dx \right] \right] \right] = 0,
\]  

\[
\frac{\epsilon}{2} \frac{d^2 \rho^2_1}{dx^2} + (2\rho^2_1 - 1) \frac{d \rho^2_1}{dx} - \Omega_d \left[ \rho^2_1 + (\rho^2_1 - 1) \left[ 1 - \frac{L}{N} \left[ \int_0^1 [\rho^i_1(x, t) + \rho^i_2(x, t)] \, dx \right] \right] \right] = 0,
\]

\[
+ \frac{\epsilon}{2} \frac{d^2 \rho^2_2}{dx^2} + (2\rho^2_2 - 1) \frac{d \rho^2_2}{dx} - \Omega_d \left[ \rho^2_2 + (\rho^2_2 - 1) \left[ 1 - \frac{L}{N} \left[ \int_0^1 [\rho^i_1(x, t) + \rho^i_2(x, t)] \, dx \right] \right] \right] = 0,
\]

with additional boundary conditions \( \rho^1_1(0) = \rho^2_1(0) = \alpha \left[ 1 - \frac{L}{N} \left[ \sum_{j=1}^{2} \int_0^1 \rho^i_j(x, t) \, dx \right] \right] \) and \( \rho^1_1(1) = \rho^2_1(1) = 1 - \beta = \gamma \). Since the obtained system is independent of \( N_i(t) \), we can now apply singular perturbation technique \([29, 31]\) in order to explore the outcomes of above coupled and nonlinear system. For more detailed description of above analysis, we refer the readers to \([29, 31]\).

5. Phase diagrams and density profiles

In this section, to investigate the effect of total number of particles on steady-state system properties, we derive phase diagrams for specific values of \( N/L \in (0, \infty) \) exhibiting significant topological changes in the controlling parameter space \( (\alpha, 1 - \beta) \). Theoretical findings determined by generalized mean-field approximation along with singular perturbation technique \([29, 31]\) are verified by performing continuous time Monte Carlo simulation for system size \( L = 1000 \). The simulations are accomplished for \( 10^{10} \) time steps and initial 5% steps are discarded to secure the manifestation of the steady-state. The average densities in both the lanes are calculated by taking time averages over an interval of \( 10L \).

For a special case of \( \Omega_{\alpha} = \Omega_{\beta} = \Omega = 0 \), both the lanes become uncoupled and the steady-state dynamics of reduced system has already been studied extensively in the literature \([13, 14]\). Motivated by the findings of standard TASEP where fully asymmetric coupling has produced non-trivial effects on system properties, we wish to study constrained system under fully asymmetric coupling conditions. In this direction, the phase diagrams for fixed \( \Omega_{\alpha} = \Omega_{\beta} = 0.2 \) and \( \Omega = 1 \) with respect to \( N/L \) are shown in figure 2. In the phase diagrams, \( \chi/\gamma \) stands for phase \( \chi \) and \( \gamma \) in Lane-1 and Lane-2, respectively. \( LD, HD \) and \( S \) indicate low density, high density and shock phase, respectively. For a very small value of \( N/L \), symmetric \( LD/LD \) phase is observed as displayed in figure 2(a). Physically, at this stage, scarcity of the particles in the system leads to reduced effective arrival and
Figure 2. Phase diagram for $L = 1000$ (a) $N/L = 0.03$ (b) $N/L = 0.09$ (c) $N/L = 0.5$ (d) $N/L = 1.2$ (e) $N/L = 2.2$ (f) $N/L \rightarrow \infty$. Here $\Omega = 1$ and $\Omega_a = \Omega_b = 0.2$. Solid lines and circles denote theoretical and Monte Carlo simulation phase boundaries, respectively. Note that phase boundaries are calculated within an estimated error of less than 1%. Color code shows the number of particles in the reservoir. Details about the notations $LD$, $HD$ and $S$ are in the text.
attachment rates, thereby, resulting in low density in both the lanes. The phase diagram exhibits only one phase till $N/L = 0.05$. As $N/L$ increases, the symmetric $S/S$ phase appears along with existing $LD/LD$ phase (figure 2(b)). The presence of exhaustive symmetric phases suggests the negligible effect of lane switching process on the system dynamics till this instant.

Further, disrupting the symmetry beyond the critical $N/L = 0.15$, an asymmetric $LD/S$ phase starts emerging within two symmetric $LD/LD$ and $S/S$ phases. No significant topological changes in the phase diagram have been observed till $N/L = 1.6$, except shifting of phase boundaries resulting in expansion of $S/S$ and $LD/S$, and shrinkage of $LD/LD$ domain (see figures 2(c)–(d)). Further, after the crucial $N/L = 1.6$, both qualitative and quantitative changes have been observed in the phase diagram. A bulk induced shock starts emerging in Lane-2 resulting in transition from $LD/LD$ to $LD/HD$ domain for larger values of $\alpha$ and $\beta$. The observed phase transition for $\alpha = 0.8$, $\gamma = 0.3$, $\Omega = 0.2$ and $\Omega = 1$ is exhibited in figure 3. It is significant to mention here that such transitions have not been encountered in earlier studies on both finite [12–14] and two-lane system with infinite reservoir [24], though, the same has been reported in the three-lane open standard TASEP system with or without LK [32, 33].

With further increase in $N/L$, at critical value of 1.9, the topology of the phase diagram alters due to the appearance of a new asymmetrical $S/HD$ phase, resulting in the expansion of $LD/S$ phase and shrinkage of $LD/LD$ region, as shown in figure 2(e). Beyond this stage, $LD/HD$ and $S/HD$ phases expand while $LD/LD$, $LD/S$ and $S/S$ shrink. Some typical density profiles for $N/L = 2.2$ are displayed in figure 4. No quantitative changes in the phase topology have been observed till $N/L = 2.68$. A far $N/L = 2.68$, new $HD/HD$ phase appears altering the phase boundaries due to expansion and shrinkage of the existing phases. The topology of the phase diagram remains preserved for further values of $N/L$ except transitions in domain boundaries. In the limit $N/L \to \infty$ (figure 2(f)), as expected, the phase diagram for two-lane coupled system with unlimited resources has been retrieved [24].

Another noteworthy observation in the proposed system is that no buffering regime with constant number of particles in the reservoir has been encountered for any $N/L$ which is in disparity to other constrained systems [14]. This can be explained as follows: the buffering effect has been reported in the literature [14] in the scenario where either of the lane is in shock phase [14]. This shock absorbs increased particles, keeping $N_t$ fixed. But due to the significant effect of LK and lane changing processes in the proposed system, shock phase is not able to exhaust completely all increased particles, resulting in absence of buffering regime.

Additionally, it is notable that in contrast to uncoupled multi-lane finite reservoir system, the proposed model has encountered $HD/HD$ phase for comparative higher values of $N/L$ [14]. This observation is mainly due to lane switching and LK processes, which diminish the crowding of particles moving along the trails for lower values of $N/L$. Moreover, due to the functional dependency of attachment rate on $N_t(t)$, detachment rate dominates attachment rate in this scenario resulting in the delayed appearance of $HD/HD$ phase. Besides, in disparity to monotonic shrinkage in shock phase in single and multi-lane uncoupled system with finite resources, here region of $S/S$ phase firstly expands and then shrinks with an increasing $N/L$ [14]. In comparison to the two-lane standard system without LK [20] and earlier studies on finite reservoir, the maximal current phase is not encountered in our system for any value of $N/L$. The observed difference is primarily due to the
Figure 4. Density profiles with $\Omega_a = \Omega_d = 0.2$ and $\Omega = 1$. (a) LD/LD (b) LD/S (c) LD/HD (d) S/HD (e) S/S. $(\alpha, \gamma) = (0.2, 0.3), (0.4, 0.7), (0.85, 0.65), (0.85, 0.90), (0.60, 0.85)$ in (a)–(e), respectively. The density profiles obtained from generalized mean-field approximation are shown by dotted lines in pink (green) color for Lane-1 (2). The curves marked with squares (solid circles) show Monte Carlo simulation results for Lane-1 (2).
presence of asymmetric coupling and LK processes in the proposed system. This result is in accordance with the two-lane system with infinite number of particles [24].

Further, we analyze the indistinguishable features of shock in the proposed two-lane system. The effect of lane size on the shock profile has also been inspected. Figure 5(a) displays shock in Lane-2 for various system sizes with fixed value of \( N/L = 2 \). As observed bulk solution obtained through Monte Carlo simulations is independent of system size for a fixed \( N/L \). However, sharpness in steep rise of the shock escalates with an increase in system size. Additionally, we figure out the shock position in lane-\( j \) (\( x_j, j = 1, 2 \)) using steadiness of current across the shock as:

\[
\rho_{j,+}^2 - \rho_{j,-}^2 = \rho_{j,-}^2 - \rho_{j,-1}^2, \quad j = 1, 2,
\]

where,

\[
\rho_{j,-} = \lim_{x \to x_j^-} \rho_j(x),
\]

\[
\rho_{j,+} = \lim_{x \to x_j^+} \rho_j(x).
\]

In S/S phase, both the shocks originating due to the deconfinement of the right boundary layers, move towards left with increase in \( N/L \) and for its sufficiently large value, stabilize as displayed in figure 5(b). Irrespective of phase, shock in either lane, exhibit the same behavior and experiences a continuous change in its position with an increment in \( N/L \). Further, due to fully asymmetric coupling environment, shock in Lane-2 remains to the left of the shock in Lane-1. Besides, increase in \( N/L \) shifts the shock location to left, resulting in escalation of particles jam length denoted by, \( 1-x_j \). Note that below a critical \( N/L \), jam length becomes zero, indicating the transitions from 3 to \( LD \) phase (see inset of figure 5(b)). In addition, the observed shock heights (\( \Delta_{1,2} \)) are exhibited in figure 5(c), characterizing its decreasing behavior for both the lanes with respect to \( N/L \). Both the shocks display equal height for the system with very small number of particles. With further increase in \( N/L \) to a critical value, the shock height in Lane-1 increases and afar this value it is dominated by shock height in Lane-2.

Additionally, to get insight about fluctuations in \( N_{j}(t) \), \( N_j(t) \) and \( N_\delta(t) \), we plot probability distribution of particles, \( P(N_j) \), utilizing Monte Carlo simulations as shown in figures 6(a)–(b). Here \( N_j \) denotes the particles chosen situationally among Lane-1, Lane-2 and reservoir, respectively. The particles follow the pseudo-Gaussian distribution with lower variance in comparison to standard TASEP. This is mainly due to the transient nature of injection and attachment rates. Further, from figure 6, it is clear that the distribution of the number of particles in lanes for both finite and infinite reservoir exhibits varied spread against \( N_j \). Since for both the lanes, the distribution for finite reservoir has smaller deviation from mean in comparison to standard TASEP, the former has lower variance. Due to the deviation of the particles in Lane-1, Lane-2 and reservoir from their respective mean values, both \( \alpha_{eff} \) and \( \omega_{alpha} \) work as restoring force to ease or burden some the entry of a new particle in either lane. In contrast, standard TASEP system does not have any control on the injection and attachment rates, leading to the dearth of restoring forces.
6. Current and residence time

Current is a natural and vital characteristic of the dynamics which is regulated by steady-state densities and determines the flow of particles in the system. Since in the proposed model, the total number of particles administers the various characteristics, including densities in both lanes, so indirectly it influences the current as well. Moreover, it has also been observed that current in single lane TASEP is significantly affected by imposing the constraint on the total number of particles in the system \[12\]. Hence, it becomes crucial to analyze steady-state average current from the aspect of \( N \) in the presented two-lane model. In this direction, we firstly discuss the effect of the total number of particles, \( N \), on the steady-state average current \( (J) \) displayed in figure 7(a). For the lower values of \( N/L \) irrespective of \( N \), the current firstly increases and then saturate beyond a critical \( N/L \). It is notable to mention that the current achieves its maximum saturated value when \( LD/LD \) phase transits to \( LD/HD \) as shown in figure 7(a).

Further, for deep insight into the system properties, we study residence time, the average time a particle spends at a particular site before proceeding to next one \[34\]. The analysis of the residence time is equivalently...
relevant in intracellular and vehicular transport as the duration that a motor or vehicle spends on the lane can be experimentally determined and can be utilized further to explore important physical quantities like transport coefficient and diffusion constant. Moreover, it can significantly extract information on the crowding and traffic jams of molecular motors moving along microtubules. It is worthwhile to mention here that the residence time in non-conserved multi-lane coupled system with or without finite reservoir is not yet studied. Inspired by the significance of residence time, we examine it for both the lanes of the proposed system. Note that due to fully asymmetric coupling environment, particles of each lane will have different residence time. To begin with, we first calculate the theoretical expressions of residence time of a particle on a specific site in jth lane. In this direction, the explicit expression for residence time of Lane-1 can be written as:

\[
\tau_i^1 = \left[ (1 - \tau_i^{1+1}(t)) + \omega_d \tau_i^{1+1}(t) + \tau_i^{1+1}(t)(1 - \omega_d) \omega [1 - \tau_i^1(t)] \right] \\
+ [2\tau_i^{1+1}(t)(1 - \omega_d)[(1 - \omega)(1 - \tau_i^2(t)) + \tau_i^2(t)] \times [1 - \tau_i^{1+1}(t+1)] \\
+ \omega_d \tau_i^{1+1}(t+1) + \tau_i^{1+2}(t+1)(1 - \omega_d) \omega (1 - \tau_i^2(t+1))] \\
+ 3[\tau_i^{1+1}(t)\tau_i^{1+1}(t+1)(1 - \omega_d)(1 - \tau_i^2(t)) + \tau_i^2(t)]\omega [1 - \tau_i^1(t+1)] \\
+ \tau_i^2(t+1)) \times [(1 - \tau_i^{1+1}(t+2)) + \omega_d \tau_i^{1+1}(t+2) + \tau_i^{1+1}(t+2)(1 - \omega_d) \omega (1 - \tau_i^2(t+2))] + ... .
\] (21)

The above equation calculates the expected time a particle spends at an ith site by computing its probability distribution function. Here in the infinite series on right hand side, the first term includes the probability of all possible transitions for a particle to leave the ith site at time t. The second term in the series incorporates the probability of a particle to stay at the ith site at time t along with its chances to leave the same site at time t + 1. In general, the nth term involves the probability that a particle stays at ith site till time n - 1 and leaves the site at time n.

Simplifying equation (21), we have

\[
\tau_i^1 = (1 - \tau_i^{1+1}(t)(1 - \omega_d - \omega) - \omega \tau_i^{1+1}(t)\omega_d + \tau_i^2(t)(1 - \omega_d)] \\
+ \sum_{j=1}^{\infty} [1 + (1 + y)(1 - \tau_i^{1+1}(t) + y)(1 - \omega - \omega_d) - \omega \tau_i^{1+1}(t + y)] \\
\times \omega_d + \tau_i^2(t + y)(1 - \omega_d)]\omega (1 - \omega_d)^y \prod_{k=0}^{n-1} \tau_i^{1+1}(t + k)[1 - \omega(1 - \tau_i^1(t + k))]
\] .

(22)

Similarly, we compute expected residence time for Lane-2:

\[
\tau_i^2 = [(1 - \omega_d)(1 - \tau_i^{2+1}(t)) + \omega_d] + 2[\tau_i^{2+1}(t)(1 - \omega_d)\omega_d(1 - \tau_i^2(t) + 1)(1 - \omega_d + \omega_d)] \\
+ \omega_d \tau_i^2(t + 1) + \tau_i^{2+1}(t + 1)(1 - \omega_d)\omega_d(1 - \tau_i^2(t + 1)) + ... ,
\]

\[
\tau_i^2 = \left[(1 - \tau_i^{2+1}(t)(1 - \omega_d) + \sum_{j=1}^{\infty} [1 + (1 + y)(1 - \tau_i^{2+1}(t) + y)(1 - \omega - \omega_d)](1 - \omega_d)^y \right] \\
\times \prod_{k=0}^{n-1} \tau_i^{2+1}(t + k)
\] .

(23)

Further ignoring the time related correlation, we get

\[
\prod_{k=0}^{n} \tau_i^{1+1}(t + k) = \prod_{k=0}^{n} \tau_i^2(t + k) = \langle \tau_i^1(t + k) \rangle = \langle \tau_i^2(t + k) \rangle = \langle \tau_i^{n+1} \rangle .
\] (24)

Using above expression along with mean-field approximation, the residence time for both the lanes in terms of steady-state densities can be computed as:

\[
\tau_i^1 = (1 - \rho_i^{1+1}(1 - \omega_d - \omega) - \omega \rho_i^{1+1}[\omega_d + \rho_i^2(1 - \omega_d)] \\
+ 2[(\rho_i^{1+1}(1 - \omega_d)(1 - \omega)(1 - \rho_i^2)\omega)(1 - \rho_i^{1+1}(1 - \omega_d - \omega) - \omega \rho_i^{1+1}[\omega_d + \rho_i^2(1 - \omega_d)]) \\
+ 3[(\rho_i^{1+1}(1 - \omega_d)(1 - \omega)(1 - \rho_i^2)\omega)(1 - \rho_i^{1+1}(1 - \omega_d - \omega) - \omega \rho_i^{1+1}[\omega_d + \rho_i^2(1 - \omega_d)]) \\
= \sum_{n=1}^{\infty} n(\rho_i^{1+1}(1 - \omega_d)(1 - \omega)(1 - \rho_i^2)\omega)^{-1} \prod_{k=0}^{n-1} \rho_i^{1+1}(1 - \omega_d - \omega) - \omega \rho_i^{1+1}[\omega_d + \rho_i^2(1 - \omega_d)]) \\
= 1 - \rho_i^{1+1}(1 - \omega_d)(1 - \omega - \omega \rho_i^2(1 - \omega_d)).
\] (25)
\[ r_i^j = 1 - \rho_i^{j+1}(1 - \omega_d) + 2\rho_i^{j+1}(1 - \omega_d)(1 - \rho_i^{j+1}(1 - \omega_d)) + 3\rho_i^{j+1}(1 - \omega_d)^2[1 - \rho_i^{j+1} \times (1 - \omega_d)] + \ldots, \]
\[ = \sum_{n=1}^{\infty} n[\rho_i^{j+1}(1 - \omega_d)]^{n-1}[1 - \rho_i^{j+1}(1 - \omega_d)], \]
\[ = \frac{1}{1 - \rho_i^{j+1}(1 - \omega_d)}. \]

Note that under the special case of \( \omega = 0 \), both the lanes have the same residence time \( 1/1 - \rho_i^{j+1}(1 - \omega_d) \) which is in agreement with single lane TASEP including LK [34]. Further, the boundary conditions on the exit site of the both the lanes are given by
\[ r_i^L = \frac{1}{1 - (1 - \beta)(1 - \omega_d)(1 - \omega) - \omega p_i^{j-1}(1 - \omega_d)}, \]
\[ r_i^R = \frac{1}{\omega_d + \beta(1 - \omega_d)}. \]

Under the influence of LK, considering the possibility of a particle to detach from the lane before arriving at the exit site, the total residence time of a particle in jth lane, \( R_j \) is given by (for more details, see [34])
\[ R_j = r_j^L + \sum_{i=1}^{L} r_j^i \prod_{k=2}^{j-1} p_i^{k-1,j}, \]
where, \( P_i^{j-1,i} \) and \( P_i^{j-1,i} \) represent the probability that a particle hops from site \( i-1 \) to site \( i \) without undergoing detachment and lane switching for Lane-1 and Lane-2, respectively and can be calculated as follows:
\[ P_i^{j-1,i} = \langle (1 - \omega_d - \omega)(1 - \tau_i^j(t)) + (1 - \omega_d)\tau_i^j(t)\rangle \langle 1 - \omega(1 - \tau_i^{j-1}(t)) + \tau_i^{j-1}(t) \rangle \times \langle 1 - \omega - \omega_d[1 - \tau_i^{j+1}(t)] + (1 - \omega_d)\tau_i^{j+1}(t)\rangle \langle \tau_i^{j+1}(t) \rangle \times \langle 1 - \omega(1 - \tau_i^{j-2}(t + 1)) + \tau_i^{j-2}(t + 1) \rangle \langle 1 - \omega_d - \omega\rangle \langle 1 - \tau_i^{j+1}(t + 2) + \ldots \rangle, \]
\[ = (1 - \omega_d - \omega)(1 - \tau_i^j(t)) \times \sum_{y=1}^{\infty} \langle \tau_i^{j-1}(t + y) \rangle \times \sum_{k=0}^{y-1} \langle \tau_i^{j-1}(t + k) \rangle \times \sum_{n=1}^{\infty} n[\rho_i^{j-1}(1 - \omega_d)]^{n-1}[1 - \omega(1 - \rho_i^{j-2}(1 - \omega_d))]^{n-1}[1 - \rho_i^j], \]
\[ = \frac{(1 - \rho_i^j)(1 - \omega_d - \omega)}{1 - \rho_i^j(1 - \omega_d)(1 - \rho_i^{j-1})}. \]

Note that, to calculate \( P_i^{j-1,i} \), we have used the same approximation which was used to derive equation (25).

Similarly, \( P_i^{j-1,i} \) is calculated and given by:
\[ P_i^{j-1,i} = \sum_{n=1}^{\infty} n[\rho_i^{j-1}(1 - \omega_d)]^{n-1}[1 - \rho_i^{j-2}(1 - \omega_d) - \omega]\langle 1 - \omega(1 - \rho_i^{j-1}(t)) \rangle^{n-1}, \]
\[ = 1 - \frac{\omega_d}{1 - \rho_i^j(1 - \omega_d)}. \]

The expression of the total residence time (equation (29)) computed by multiplication of residence time of each site and the probability that a particle entering from leftmost site reaches this site without detaching and lane switching can be simplified as:
\[ R_1 = r_1^L + \sum_{i=1}^{L} r_1^i \prod_{k=2}^{i-1} \langle \frac{(1 - \rho_i^L)(1 - \omega_d - \omega)}{1 - \rho_i^L(1 - \omega_d)(1 - \rho_i^{L-1})} \rangle, \]
\[ R_2 = r_2^L + \sum_{i=1}^{L} r_2^i \prod_{k=2}^{i-1} \langle (1 - \omega_d) \rangle. \]

\( R_1 \) and \( R_2 \) give the total residence time in Lane-1 and Lane-2, respectively. For the simplified case of \( \omega = 0 \) and \( \omega_d = \omega_d = 0 \), \( R_1 \) agrees with \( R_2 \), which in turn matches with the total residence time for single lane system with and without LK, respectively [34].

Moreover, the effect of \( N/L \) on the total residence time, \( R_j (j = 1, 2) \), in both the lanes is displayed in figure 7(b). As expected, due to fully asymmetric coupling, particles of Lane-2 stay longer in comparison to those on Lane-1. For smaller values of \( N/L \), firstly both \( R_1 \) and \( R_2 \) increase and after a distinct crucial value of \( N/L \), they both saturate to different constants. As expected, it follows from the equation (32)–(33), the gap between \( R_1 \) and
R₂ decreases for ω → 0, which validates the findings of the [34]. Further, inset of figure 7(b) exhibits steady-state currents in both the lanes against N/L. Since with increase in N/L, system transits from LD/LD to LD/HD phase as shown in figure 2, currents in both the lanes initially increase and finally saturate for larger values of N/L. Both current and the total residence time displays similar behavior which can be understood as follows. Initially, for lower values of N/L, both the lanes are in LD phase. With the additional increase in number of particles in the system, the flow enhances, leading to increase in current. Moreover, the available empty space allows the particles to stay for longer time resulting in increase in the total residence time. On the other side, for higher values of N/L, both flow as well as the total residence time, saturates.

7. Power spectra

As the system evolves, number of particles in Lane-1, Lane-2, and reservoir fluctuates with time t. However, their time averages \( \langle N_l(t) \rangle, \langle N_l(t) \rangle \) and \( \langle N_l(t) \rangle \) remains constant in the steady-state raising curiosities about their properties. Progressing in this direction, we converted time domain to frequency domain and calculated the power spectra of number of particles in Lane-1, Lane-2 and reservoir with respect to frequency f. The power of these quantities namely, power spectra, can be computed based on Fourier transform of autocorrelations \( \langle N_l(t)N_l(t') \rangle, \langle N_l(t)N_l(t') \rangle \) and \( \langle N_l(t)N_l(t') \rangle \) for distant time steps t and t'.

Recently, a study highlighting the significant effect of finite reservoir on power spectra of total particle occupancy has been conducted on a single lane TASEP [35, 36]. However, no such study has been proposed on a coupled multi-lane system with or without LK. Motivated by the non-trivial outcomes of power spectra in single lane, we now introduce the average power spectra of a coupled multi-lane system with non-conserving dynamics. Langvein equation based technique has been utilized to understand features of power spectra and has been validated through Monte Carlo simulation. Through simulations we calculated the power spectra, \( I_j(f) \), for jth lane by determining Fourier transform of total number of particles in steady-state. We have

\[
\hat{N}_j(f) = \sum_{t=0}^{T} e^{it}N_j(t),
\]

where \( T \) is data size with

\[
f = \frac{2\pi m}{T}; \quad m = 0, 1, 2, \ldots
\]

Many such independent iterations are constructed to arrive at average power spectra, we get

\[
I_j(f) = \langle |\hat{N}_j(f)|^2 \rangle.
\]

Theoretically, the power spectra can be computed using Langvein equations for both the lanes as:

\[
\frac{\rho_1(1, t)}{dt} = \alpha_{\text{eff}}(1 - \rho_1(1, t)) - \rho_1(1, t)(1 - \rho_1(2, t)) + \xi_1(0, t) - \xi_1(1, t),
\]

\[
\frac{\rho_1(L, t)}{dt} = \rho_1(L - 1, t)(1 - \rho_1(L, t)) - \beta \rho_1(L, t) + \xi_1(L - 1, t) - \xi_1(L, t),
\]

\[
\frac{\rho_1(x, t)}{dt} = \rho_1(x - 1, t)(1 - \rho_1(x, t)) - \rho_1(x, t)(1 - \rho_1(x + 1, t)) + \Omega_{\text{eff}}(1 - \rho_1(x, t)) - \Omega_{\text{eff}} \sum_{y=2}^{L-1} \xi_1(y - 1, t) + \xi_1(x - 1, t)
\]

\[
- \xi_1(x, t), \quad x \in [2, L - 1],
\]

\[
\frac{\rho_1(0, t)}{dt} = \beta \rho_1(L, t) - \alpha_{\text{eff}}(1 - \rho_1(1, t)) + \Omega_{\text{eff}} \sum_{y=2}^{L-1} \rho_1(y, t) - \Omega_{\text{eff}} \sum_{y=2}^{L-1} (1 - \rho_1(y, t))
\]

\[
+ \sum_{y=2}^{L-1} \xi_1(y, t) + \xi_1(L, t) - \xi_1(0, t),
\]

\[
\frac{\rho_2(1, t)}{dt} = \alpha_{\text{eff}}(1 - \rho_2(1, t)) - \rho_2(1, t)(1 - \rho_2(2, t)) + \xi_2(0, t) - \xi_2(1, t),
\]

\[
\frac{\rho_2(L, t)}{dt} = \rho_2(L - 1, t)(1 - \rho_2(L, t)) - \beta \rho_2(L, t) + \xi_2(L - 1, t) - \xi_2(L, t),
\]

\[
\frac{\rho_2(x, t)}{dt} = \rho_2(x - 1, t)(1 - \rho_2(x, t)) - \rho_2(x, t)(1 - \rho_2(x + 1, t)) + \Omega_{\text{eff}}(1 - \rho_2(x, t)) - \Omega_{\text{eff}} \sum_{y=2}^{L-1} \xi_2(y - 1, t) + \xi_2(x - 1, t)
\]

\[
- \xi_2(x, t), \quad x \in [2, L - 1],
\]

\[
\frac{\rho_2(0, t)}{dt} = \beta \rho_2(L, t) - \alpha_{\text{eff}}(1 - \rho_2(1, t)) + \Omega_{\text{eff}} \sum_{y=2}^{L-1} \rho_2(y, t) - \Omega_{\text{eff}} \sum_{y=2}^{L-1} (1 - \rho_2(y, t))
\]

\[
+ \sum_{y=2}^{L-1} \xi_2(y, t) + \xi_2(L, t) - \xi_2(0, t),
\]
\[
\frac{\rho_2(x,t)}{dt} = \rho_2(x-1,t)(1-\rho_2(x,t)) - \rho_2(x,t)(1-\rho_2(x+1,t)) + \Omega_{a,\ell}(1-\rho_2(x,t)) \\
- \Omega_{d}\rho_2(x,t) + \xi_2(0,t) - \sum_{y=2}^{L-1} \xi_2(y,t) + \xi_2(x-1,t) \\
- \xi_2(x,t), x \in [2, L-1],
\]
(43)

\[
\frac{\rho_2(0,t)}{dt} = \beta\rho_2(L,t) - \alpha_{\text{eff}}(1-\rho_2(1,t)) + \Omega_{a,\ell}\sum_{y=2}^{L-1} \rho_2(y,t) - \Omega_{d}\sum_{y=2}^{L-1} (1-\rho_2(y,t)) \\
+ \sum_{y=2}^{L-1} \xi_2(y,t) + \xi_2(L,t) - \xi_2(0,t),
\]
(44)

where \(\xi\) is a Gaussian noise having following properties:

\[
\langle \xi_2(x,t)\xi_4(x,t') \rangle = 0, \\
\langle \xi_2(x,t)\xi_1(x,t') \rangle = A_i \delta_{xx'} \delta_{tt'}, \\
\langle \xi_2(x,t)\xi_4(x,t') \rangle = A_2 \delta_{xx'} \delta_{tt'}.
\]
(45)

Here, \(A_1 = \overline{\rho}_j(1 - \overline{\rho}_j)\) is variance of the noise in \(j\)th lane for \(j = 1, 2\) with \(\overline{\rho}_j\) as the average density in lane \(j\).

Now, defining the fluctuation \(\phi_j(x,t)\) in the density \(\rho_j\) for the \(j\)th lane by the difference of corresponding density and its average, we have

\[
\phi_1(x,t) = \rho_1(x,t) - \overline{\rho}_1, \\
\phi_2(x,t) = \rho_2(x,t) - \overline{\rho}_2.
\]
(46)

Now, we characterize Fourier transform of fluctuations as given below:

\[
\phi_1(x,t) = \sum_{k,f} e^{ikx+ft} \tilde{\phi}_1(k,f), \\
\phi_2(x,t) = \sum_{k,f} e^{ikx+ft} \tilde{\phi}_2(k,f),
\]
(47)

with

\[
k = \frac{2\pi u}{L+1}, \quad u \in [0, L], \\
f = \frac{2\pi m}{T}, \quad m \in [0, 5000].
\]
(48)

For the reservoir, fluctuation about its average value is given by

\[
\phi(0,t) = N_r(t) - \overline{N}_r, \\
= -N_r(t) + \overline{N}_r - N_2(t) + \overline{N}_2, \\
= \phi_1(0,t) + \phi_2(0,t).
\]
(49)

Here, \(\phi_1(0,t)\) and \(\phi_2(0,t)\) are fluctuations in the reservoir due to change in the corresponding lanes occupation. Expanding \(\alpha_{\text{eff}}\) and \(\Omega_{a,\ell}\) using Taylor series expansion about the average number of particles in reservoir, \(\overline{N}_r(t)\), and retaining terms upto first order, we obtain

\[
\alpha_{\text{eff}} = \alpha_0 \psi(\overline{N}_r) + [N_r(t) - \overline{N}_r] \frac{\partial \alpha_0 \psi(\overline{N}_r)}{\partial N_r} |_{N_r(\overline{N}_r)} + \ldots, \\
\Omega_{a,\ell} = \alpha_{\text{eff}} \psi(\overline{N}_r) + [N_r(t) - \overline{N}_r] \frac{\partial \alpha_{\text{eff}} \psi(\overline{N}_r)}{\partial N_r} |_{N_r(\overline{N}_r)} + \ldots
\]
(50)

Using equations (37)–(44) and (46), we obtain following equations in terms of fluctuations in lanes and reservoir:

\[
\frac{d\phi_{1,2}(x,t)}{dt} = (1 - \overline{\rho}_{1,2})[\phi_1(0,t) + \phi_2(0,t)] \frac{\partial \psi(\overline{N}_r(t))}{\partial N_r} |_{N_r(\overline{N}_r)} + (1 - \overline{\rho}_{1,2})\phi_{1,2}(x-1,t) \\
+ [-1 \pm \overline{\rho}_1 \Omega(1 - \overline{\rho}_2) - \Omega_d \psi(\overline{N}_r)]\phi_{1,2}(x,t) \pm \Omega_{a,\ell} \phi_{1,2}^2(x,t) \\
+ \overline{\rho}_{1,2} \phi_{1,2}(x+1,t) + (1 - \overline{\rho}_{1,2}) \Omega_d \psi(\overline{N}_r) \pm \overline{\rho}_1^2 \Omega(1 - \overline{\rho}_2) - \Omega_d \overline{\rho}_{1,2} \\
+ \xi_{1,2}(0,t) - \sum_{y=2}^{L-1} \xi_{1,2}(y,t) + \xi_{1,2}(x-1,t) - \xi_{1,2}(x,t).
\]
(51)
Here, \( \phi_1(x, t) \) and \( \phi_2(x, t) \) correspond to fluctuations in Lane-1 and Lane-2, respectively. Similar equations for reservoir and boundaries can be given as:

\[
\frac{d\phi_{1,2}(x, t)}{dt} = \left[ -(1 - \rho_{1,2})\alpha \frac{\partial\psi(N(t))}{\partial N_1(t)}|_{N_1(t)=0} + \rho_{1,2} \Omega_2 \frac{\partial\psi(N(t))}{\partial N_1(t)}|_{N_1(t)=\bar{N}} \right] \phi_1(0, t) + \phi_2(0, t) + \alpha\psi(N) \phi_{1,2}(1, t) + \beta \phi_{1,2}(L, t) + \rho_{1,2}[\beta + \Omega_d + \Omega_2 \psi(N)] - (1 - \rho_{1,2}) \alpha \psi(N) + \sum_{y=2}^{L-1} \xi_{1,2}(y, t) + \xi_{1,2}(L, t) - \xi_{1,2}(0, t),
\]

\[52\]

\[
\frac{d\phi_{1,2}(1, t)}{dt} = \alpha(1 - \rho_{1,2})[\phi_1(0, t) + \phi_2(0, t)] \frac{\partial\psi(N(t))}{\partial N_1(t)}|_{N_1(t)=\bar{N}} + [\rho_{1,2} - \alpha\psi(N) - 1] \phi_{1,2}(1, t) + \rho_{1,2}[\beta + \alpha\psi(N)] - \xi_{1,2}(1, t),
\]

\[53\]

\[
\frac{d\phi_{1,2}(L, t)}{dt} = (1 - \rho_{1,2})[\phi_{1,2}(L - 1, t) - (\beta + \rho_{1,2}) \phi_{1,2}(L, t)] + \rho_{1,2}[1 - \rho_{1,2} - \beta] + \xi_{1,2}(L, t) - \xi_{1,2}(L - 1, t).
\]

\[54\]

Considering the complexity involved in above coupled Langevin equations, we obtain the solution of above equations for special case \( \Omega_2 = \Omega_d = 0 \). For the reduced system, we use Fourier transform method to calculate power spectra of Lane-1, Lane-2 and reservoir. Now, by combining equations (51)–(54) for Lane-1 using kronecker delta function, we have

\[
\frac{d\phi_{1}(x, t)}{dt} = (1 - \rho_{1}) \phi_1(x - 1, t) - [1 + 2\rho_{1} \Omega(1 - \rho_{2})] \phi_1(x, t) + \rho_{1}^2 \Omega \phi_1(x, t)
\]

[\[55\]]

Writing \( \phi_1 \) in terms of \( \tilde{\phi}_1 \) and taking inverse Fourier transform of the above equation, we obtain:

\[
\tilde{\phi}_1(k, f) = \frac{1}{\lambda_f(k,f)} \left[ \rho_{1}^2 \Omega \tilde{\phi}_1(k, f) + (e^{-ik} - 1) \tilde{\xi}_1(k, f) + \frac{1}{(L + 1)T} \sum_{x,t} e^{-ik(x+\beta)t} \right],
\]

\[56\]

where \( \lambda_f(k,f) \) is explicitly given in the appendix and \( v \) is inhomogeneous part given by

\[
v = \delta_{x,0} \left[ -(1 - \rho_{1}) \alpha \frac{\partial\psi(N(t))}{\partial N_1(t)}|_{N_1(t)=\bar{N}} + [\alpha\psi(N) - \rho_{1}] \phi_1(1, t) - [-1 - 2\rho_{1} \Omega(1 - \rho_{2})]
\times \phi_1(0, t) - \rho_{1}^2 \Omega \phi_2(0, t) + (\beta - 1 + \rho_{1}) \phi_1(L, t) + \rho_{1} \beta - (1 - \rho_{1}) \alpha \psi(N) + \rho_{1}^2 \Omega(1 - \rho_{2}) \phi_1(1, t)
\]

\[57\]
Taking inverse Fourier transform of \( \nu \), we get

\[
\tilde{\phi}_f(k, f) = \frac{1}{(L + 1)\lambda_f(k, f)} \left[ e^{-ik} - 1 \right] \tilde{\xi}_f(k, f)(L + 1) + S_f(k) \sum_k \tilde{\phi}_f(k, f) + U_f(k) \sum_k \tilde{\phi}_2(k, f) \\
+ G_f(k) \sum_k e^{ik} \tilde{\phi}_f(k, f) + H_f(k) \sum_k e^{ik} \tilde{\phi}_2(k, f) + \frac{Q_f}{T} (k) \sum_t e^{-i\phi_t},
\]

(58)

where \( \lambda_f(k, f), S_f(k), U_f(k), G_f(k), H_f(k) \) and \( Q_f(k) \) are given in appendix.

Now, using \( \phi_f(0, t) \) from equation (47), we have

\[
\tilde{N}_f(f) = -\frac{1}{T} \sum_t e^{-i\phi_f(0, t)} = -\sum_k \tilde{\phi}_f(k, f).
\]

(59)

Similarly, we can obtain

\[
\tilde{N}_2(f) = -\frac{1}{T} \sum_t e^{-i\phi_2(0, t)} = -\sum_k \tilde{\phi}_2(k, f).
\]

(60)

Now we define,

\[
D_1\tilde{N}_f(f) = -\sum_k e^{ik} \tilde{\phi}_f(k, f), \quad D_2\tilde{N}_2(f) = -\sum_k e^{ik} \tilde{\phi}_2(k, f).
\]

(61)

Here, \( D_1 \) and \( D_2 \) are fitting parameters. Applying summation in equation (58) on both sides over \( k \), we obtain

\[
\tilde{N}_f(f) = \sum_k \left[ \frac{(e^{-ik} - 1) \tilde{\xi}_f(k, f)}{\lambda_f(k, f) B_f(k)} \right] - \frac{C_f(f)}{B_f(f)} \tilde{N}_1(f) + \frac{E_f(f) E_f(f)}{B_f(f)} + \frac{G_f(f)}{B_f(f)}
\]

(62)

where \( B_f, C_f, E_f \) and \( E_f \) are functions of frequency \( f \) and are explicitly given in the appendix. Further, for Lane-2, \( \tilde{N}_2(f) \) is given as:

\[
\tilde{N}_2(f) = \sum_k \left[ \frac{(e^{-ik} - 1) \tilde{\xi}_2(k, f)}{\lambda_2(k, f) B_2(f)} \right] - \frac{C_2(f)}{B_2(f)} \tilde{N}_2(f) + \frac{E_2(f) E_2(f)}{B_2(f)}.
\]

(63)

Solving equation (62) and equation (63), we obtain

\[
\tilde{N}_f(f) = \frac{B_1(f) B_2(f)}{C_1(f) C_2(f) - B_1(f) B_2(f)} \left[ \sum_k \left[ \frac{(e^{-ik} - 1) \tilde{\xi}_f(k, f)}{\lambda_f(k, f) B_f(k)} \right] + \frac{E_f(f) E_f(f)}{B_f(f)} + \frac{G_f(f)}{B_f(f)} \right] \times \left[ \sum_k \left[ \frac{(e^{-ik} - 1) \tilde{\xi}_2(k, f)}{\lambda_2(k, f) B_2(f)} \right] + \frac{E_2(f) E_2(f)}{B_2(f)} \right],
\]

(64)

and

\[
\tilde{N}_2(f) = \frac{B_1(f) B_2(f)}{C_1(f) C_2(f) - B_1(f) B_2(f)} \left[ \sum_k \left[ \frac{(e^{-ik} - 1) \tilde{\xi}_2(k, f)}{\lambda_2(k, f) B_2(f)} \right] + \frac{E_f(f) E_f(f)}{B_f(f)} + \frac{G_f(f)}{B_f(f)} \right] \times \left[ \sum_k \left[ \frac{(e^{-ik} - 1) \tilde{\xi}_2(k, f)}{\lambda_2(k, f) B_2(f)} \right] + \frac{E_2(f) E_2(f)}{B_2(f)} \right].
\]

(65)

Here, \( B_2, C_2, E_2 \) and \( \lambda_2 \) are expressions of frequency and can be corresponded as stated above. Further, power spectra of Lane-1 and Lane-2 can be computed using equation (36) as:

\[
I_f(f) = \langle |\tilde{N}_f(f)|^2 \rangle,
\]

(66)

\[
I_2(f) = \langle |\tilde{N}_2(f)|^2 \rangle.
\]

(67)

Simplifying above equations, we have

\[
I_f(f) = \frac{\left| B_1(f) B_2(f) \right|^2}{\left| C_1(f) C_2(f) - B_1(f) B_2(f) \right|^2} \left[ \sum_k \left[ \frac{A_1(2 - 2 \cos k)}{\left| \lambda_f(k, f) B_f(f) \right|^2} \right] + \left| E_f(f) \right|^2 \left| \frac{E_f(f)}{B_f(f)} \right|^2 \right] + \frac{\left| G_1(f) \right|^2}{\left| B_1(f) \right|^2} \times \left[ \sum_k \left[ \frac{A_1(2 - 2 \cos k)}{\left| \lambda_f(k, f) B_f(f) \right|^2} \right] + \left| E_f(f) \right|^2 \left| \frac{E_f(f)}{B_f(f)} \right|^2 \right],
\]

(68)
The power spectra of reservoir can be computed as
\[ I_\text{r}(f) = \frac{|B_1(f)B_2(f)|^2}{|C_1(f)C_2(f) - B_1(f)B_2(f)|^2} \sum_k \left| \frac{A_\ell(2 - 2 \cos k)}{|\lambda_\ell(k, f)B_1(f)|^2} \right|^2 + |E(f)|^2 \frac{|E_2(f)|^2}{|B_2(f)|^2} \]
\[ + \frac{|C_2(f)|^2}{|B_2(f)|^2} \times \left[ \sum_k \left| \frac{A_\ell(2 - 2 \cos k)}{|\lambda_\ell(k, f)B_1(f)|^2} \right|^2 + |E(f)|^2 \frac{|E_1(f)|^2}{|B_1(f)|^2} \right]. \]  

(69)

The power spectra of reservoir can be computed as
\[ I_\text{r}(f) = \langle |\hat{N}_\ell(f) + \hat{N}_{\ell^\prime}(f)|^2 \rangle. \]  

(70)

Since we have recorded data after every $\ell$ interval in simulation, we obtain $f$ as
\[ f_{m,\sigma} = \frac{2\pi m}{T}(1 + \frac{\sigma^T}{\ell}), \quad \ell = 100(L + 1), \quad M = \ell^\prime T, \quad \sigma \in [0, \ell^\prime - 1]. \]  

(71)

For the simplified case of $\Omega = 0$, with same functional form, the expressions of power spectra of both lanes reduce to the power spectra of single lane TASEP [35]. The future work aims at exploring the general solution along with detailed analysis of the non-conserved system.

The power spectra of particles occupancy without LK is displayed in figure 8(a). The result exhibits small damped oscillations even at the higher frequency for $I_1$ while $I_2$ and $I_r$ have negligible suppressions. This is in contrast to single lane constrained TASEP where significantly large suppressions have been observed in the power spectra of reservoir [35]. This variation originates due to lack of coupling between lanes and effect of $\alpha_{\text{eff}}$ on the power spectra of the system. The magnitude of suppressions depends upon the variability of injection rate which further varies with the number of particles in the reservoir. Figure 8(b) displays the power spectra of particles occupancy incorporating non-conserving dynamics. Since the detachment and removal rates are independent of $N_\ell(t)$ and the system is under the influence of coupling, damped oscillations for lower frequencies are observed in Lane-1, in comparison to very small oscillations for reservoir and Lane-2. These oscillations in Lane-1 dies out for the higher run of frequencies.

It is clear that the proposed approach not only provides density profiles and phase diagrams but also calculates other important characteristics such as residence time and power spectra which contribute in getting a deeper insight into system dynamics. Apart from this, the technique utilized in the paper can also be easily extended to analyze the system with more number of lanes with or without non-conserving dynamics under the constraint on total number of particles.

8. Conclusion

In this work, we have studied a two-lane totally asymmetric simple exclusion process with attachment-detachment and fully asymmetric coupling in both the lanes under the influence of finite pool of particles. As a result of the finite reservoir, the total number of particles, $N_\ell$, in the system remains constant at any time $t$. A generalized mean-field theory is proposed to analyze the outcomes of system dynamics on the steady-state
properties. The second order steady-state system of differential equations derived by employing continuum mean-field approximation is solved using singular perturbation technique. Theoretical findings are found to be in good agreement with Monte Carlo simulation results.

The non-trivial effect of limited resources on dynamics of transport systems, in particular, phase diagrams, density profiles, residence time and power spectra are examined. We have obtained the steady-state phase diagrams for varied values of \(N/L\) at which significant topological changes are observed. As anticipated the phase diagram depends crucially on the total number of particles in the system. Beyond a critical value of \(N/L\), the symmetry of phase diagram breaks down due to the appearance of shock phase. Interestingly, \(L/S\) and \(S/S\) phases manifest non-monotonic behavior with respect to the total number of particles in the system. No transition in the shock phase to the high density phase has been observed until \(N/L = 1.9\) due to the influence of \(N_r(t)\) on injection and attachment rates. There are no significant topological changes in the phase diagram for higher values of \(N/L\). The transition in the phase boundaries has only been observed due to expansion/ shrinkage of existing phases. Despite fully asymmetrical coupling environment, symmetrical phases are identified along with asymmetrical phases. Further, it has been observed that the length of traffic jams increases with the increment in the total number of particles in the system while the shock height decreases in contrast.

Additionally, it has been noticed that steady-state average current in system saturates beyond a critical value of \(N/L\). In order to get more insight into system dynamics, the analytical expressions of residence time are derived and it has been concluded that due to the asymmetrical coupling, the particles in Lane-2 stay longer in comparison to those on Lane-1. The theoretical expressions of power spectra for particles occupancy in both the lanes and reservoir are obtained using a Langevin equation based technique. As an effect of coupling and LK, it has been perceived that reservoir and Lane-2 experience an undamped profile while Lane-1 displays damped oscillations.

The present study not only explains the intracellular transport carried out by motor proteins in rate limiting motor factors, but also enlarge one’s insight into other constrained transport systems observed in nature. The important noteworthy observations arising due to the biased lane-changing rule can be generalized to more extensive case of asymmetric coupling environment. Future work aims at including more natural situations such as bi-directional lane switching, inhomogeneous hopping rate, bi-directional transport and varying system size revealing additional fascinating information about the system.

Appendix

Here, we present the expressions of \(B_1, C_1, E_1, \lambda_1\) and \(E\) obtained in equation (62).

\[
\lambda_1(k, f) = e^{if} - \cos k + i(1 - 2\beta_1)\sin k,
\]

\[
B_1(f) = 1 - \sum_k \frac{S_1(k) + G_1(k)D_1 + H_1(k)D_2}{(L + 1)P_1(k, f)},
\]

where \(S_1, G_1\) and \(H_1\) are given by:

\[
S_1(k) = -(1 - \beta_1)\alpha \frac{\partial \psi(N_1(t))}{\partial N_1} |_{N_1(t)=N_{01}} + 1 + 2\beta_2 \Omega(1 - \beta_1),
\]

\[
+ \left[ \alpha(1 - \beta_1) \frac{\partial \psi(N_1(t))}{\partial N_1} |_{N_1(t)=N_{01}} - \Omega\beta_1^2 \right] e^{-ik} - \beta_1 e^{-ik},
\]

\[
G_1(k) = \alpha \psi(N) - \beta_1 + [\beta_2 - \alpha \psi(N_01) + 2\beta_2 \Omega(1 - \beta_1)] e^{-ik},
\]

\[
H_1(k) = \beta - 1 + \beta_1 + [1 - \beta_1 - \beta + 2\beta_2 \Omega(1 - \beta_2) + \Omega\beta_1^2] e^{-ik},
\]

\[
G_2(f) = \frac{U_1(k)}{(L + 1)P_1(k, f)},
\]

with

\[
U_1(k) = -(1 - \beta_1)\alpha \frac{\partial \psi(N_1(t))}{\partial N_1} |_{N_1(t)=N_{01}} - \Omega\beta_1^2 + \alpha(1 - \beta_1) \frac{\partial \psi(N_1(t))}{\partial N_1} |_{N_1(t)=N_{01}} e^{-ik},
\]

\[
E_1(f) = \frac{Q_1(k)}{T(L + 1)P_1(k, f)}, E(f) = \frac{e^{-\Omega t} - 1}{e^{\Omega t} - 1},
\]

where

\[
Q_1(k) = \frac{1}{T} [\beta_1(1 - \beta_1) + \Omega\beta_1^2(1 - \beta_2)](1 + e^{-ik}) + (1 - \beta_1)[\alpha \psi(N_01) - \beta_2] e^{-ik}.
\]
All the above quantities with subscript 1 are for Lane-1 while without subscript is common for both lanes. Similarly, we can derive the required expressions for Lane-2.

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