Variants of SGD for Lipschitz Continuous Loss Functions in Low-Precision Environments

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Abstract

Motivated by neural network training in low-precision arithmetic environments, this work studies the convergence of variants of SGD using adaptive step sizes with computational error. Considering a general stochastic Lipschitz continuous loss function, an asymptotic convergence result to a Clarke stationary point is proven as well as the non-asymptotic convergence to an approximate stationary point. It is assumed that only an approximation of the loss function’s stochastic gradient can be computed in addition to error in computing the SGD step itself. Different variants of SGD are tested empirically, where improved test set accuracy is observed compared to SGD for two image recognition tasks.

1 Introduction

This paper studies the convergence of variants of stochastic gradient descent (SGD) using adaptive steps sizes in an environment with non-negligible computational error. The assumptions are given in a general form but are motivated by the error from using low-bit finite precision arithmetic for neural network training. Given the continuously increasing size of deep learning models, there is a strong motivation to do training in lower-bit formats. The majority of research in this area is focused on hardware design using number formats of different precision for different types of data (gradients, weights, etc.) to accelerate training and reduce memory requirements, while aiming to incur minimal accuracy degradation, see (Wang et al., 2022, Table 1). Our work is complementary to this line of research, with a focus on trying to gain a better understanding of low-bit training from the perspective of the training algorithm, namely, we seek to understand what level of error can be imposed while still maintaining a theoretical convergence guarantee, and to observe empirically to what extent simple variants of SGD can improve a model’s accuracy in low-bit environments.

The theoretical convergence analysis in Section 5 focuses on using variants of perturbed SGD

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(PSGD) with adaptive step sizes to find an (approximate) stationary point of the function
\[ f(w) := \mathbb{E}[F(w, \xi)], \]
where \( w \in \mathbb{R}^d, \xi \in \mathbb{R}^n \) is a random vector from a probability space \((\Omega, \mathcal{F}, P)\), and \( F(w, \xi) \) is assumed to be a stochastic Lipschitz continuous function in \( w \in \mathbb{R}^d \), with the precise details given in Section 2. Unlike assuming that \( F(w, \xi) \) is convex or that it has a Lipschitz continuous gradient, this assumption is much closer to reality as a wide range of neural network architectures are known to be at least locally Lipschitz continuous (Davis et al., 2020). Section 6 examines the empirical performance of variants of SGD, focusing on image recognition tasks in fixed-point environments, and presents a numerical substantiation of our main theoretical convergence result.

Before these results, an overview of fixed-point arithmetic using stochastic rounding is given in Section 3. In Section 4, relevant past work studying optimization with computational error is discussed, with the paper concluding in Section 7. The appendices contain a table of notation in Appendix A, all of the proofs of the theoretical results, as well as results and plots from the numerical experiments.

2 Lipschitz Continuous Loss Functions

This section contains the required assumptions and resulting properties for functions \( f(w) \) of form (1). It is assumed that \( F(w, \xi) \) is continuous in \( w \) for each \( \xi \in \mathbb{R}^n \), and Borel measurable in \( \xi \) for each \( w \in \mathbb{R}^d \). For almost all \( \xi \in \mathbb{R}^n \),
\[ |F(w, \xi) - F(w', \xi)| \leq L_0(\xi)\|w - w'\|_2 \]
for all \( w, w' \in \mathbb{R}^d \), where \( L_0(\xi) \) is a real-valued measurable function which is square integrable, \( Q := \mathbb{E}[L_0(\xi)^2] < \infty \). It follows that \( f(w) \) is \( L_0 := \mathbb{E}[L_0(\xi)] \)-Lipschitz continuous (Metel, 2023, Proposition 2). As is common for loss functions used in machine learning, we make the following assumption.

**Assumption 1.** The loss function \( f(w) \geq 0 \) for all \( w \in \mathbb{R}^d \).

Assumption 1 is without serious loss of generality: If \( \inf_{w \in \mathbb{R}^d} f(w) \geq -z > -\infty \) for some \( z > 0 \), \( f(w) \) can be redefined as \( f(w) := \mathbb{E}[F(w, \xi)] + z \). Let \( m(S) \) denote the Lebesgue measure of any measurable set \( S \). Let \( B_\epsilon^p(w) := \{x \in \mathbb{R}^d : \|x - w\|_p \leq \epsilon\} \) and \( \overline{B}_\epsilon^p(w) := \{x \in \mathbb{R}^d : \|x - w\|_p \leq \epsilon\}, \) for \( \epsilon \geq 0 \) and \( p \geq 1 \), and for simplicity let \( B_\epsilon^p := B_\epsilon^p(0) \) and \( \overline{B}_\epsilon^p := \overline{B}_\epsilon^p(0) \).

The Clarke \( \epsilon \)-subdifferential (Goldstein, 1977) has been defined using the Euclidean norm for a function \( h(w) \) as
\[ \partial h(w) := \text{co}\{\partial h(x) : x \in \overline{B}_\epsilon^p(w)\}, \]
where co denotes the convex hull, and \( \partial h(x) \) denotes the Clarke subdifferential, which for a locally Lipschitz continuous function \( h(w) \) is equal to
\[ \partial h(w) = \text{co}\{v : \exists x^k \rightarrow x, x^k \in D, \nabla h(x^k) \rightarrow v\}, \]
where $D$ is the domain of $\nabla h(w)$. The Clarke $\epsilon$-subdifferential is a commonly used relaxation of the Clarke subdifferential for the development and analysis of algorithms for the minimization of non-smooth non-convex Lipschitz continuous loss functions. In particular, for any $\epsilon_1, \epsilon_2 > 0$, algorithms have been developed with non-asymptotic convergence guarantees in expectation and with high probability for the approximate stationary point $\text{dist}(0, \partial_1 f(w)) \leq \epsilon_2$, see for example (Davis et al., 2022; Metel, 2023; Tian et al., 2022; Zhang et al., 2020d). At times it is more convenient to consider different norms, so we define a generalization of the Clarke $\epsilon$-subdifferential for $p \geq 1$ as

$$\partial^p h(w) := \text{co}\{\partial h(x) : x \in B^p_\epsilon(w)\}.$$ 

**Proposition 1.** Let $h(w)$ be a locally Lipschitz continuous function for $w \in \mathbb{R}^d$. For $\epsilon \geq 0$ and $p \geq 1$, $\partial^p h(w)$ is compact and outer semicontinuous for all $w \in \mathbb{R}^d$.

Let $\{\alpha_k\}$ be a sequence such that $\alpha_k > 0$ for all $k \in \mathbb{N}$ with $\lim_{k \to \infty} \alpha_k = 0$. The next proposition proves the continuous convergence (Rockafellar and Wets, 2009, Definition 5.41) of the sequence of set-valued mappings $\{\partial^p_{\alpha_k} h\}$ to the Clarke subdifferential of $h(w)$.

**Proposition 2.** Let $h(w)$ be a locally Lipschitz continuous function for $w \in \mathbb{R}^d$. The sequence of mappings $\{\partial^p_{\alpha_k} h\}$ converges continuously to $\partial h(w)$ for all $w \in \mathbb{R}^d$.

It is not assumed that $f(w)$ nor $F(w, \xi)$ are differentiable. We instead define $\tilde{\nabla} F(w, \xi)$ to be a Borel measurable function which equals the gradient of $F(w, \xi)$ almost everywhere it exists. This can be computed using back propagation for a wide range of neural network architectures made up of elementary functions, see (Bolte and Pauwels, 2020, Proposition 3 & Theorem 2) for more details.

In the convergence analysis in Section 5, iterate perturbation is used with samples of a random variable $u : \Omega \to \mathbb{R}^d$ which is uniformly distributed over $B^\infty_\alpha(w)$ for an $\alpha > 0$, denoted as $u \sim U(B^\infty_\alpha)$. Let $f_\alpha(w) := \mathbb{E}[f(w + u)]$ for $u \sim U(B^\infty_\alpha)$ be the expected value of the perturbed loss function (1). We will now list some useful properties.

**Proposition 3.** (Metel, 2023, Propositions 3 & 6) & (Metel and Takeda, 2022, Lemma 4.2)

1. For any $w \in \mathbb{R}^d$ and $\alpha > 0$, with $u \sim U(B^\infty_\alpha)$, $\mathbb{E}[\tilde{\nabla} F(w + u, \xi)] = \nabla f_\alpha(w)$ and

2. the gradient of $f_\alpha(w)$ is $L^\alpha_1 := 2\alpha^{-1} \sqrt{L_0}$-Lipschitz continuous.

3. For almost all $(w, \xi) \in \mathbb{R}^{d+n}$, $\|\tilde{\nabla} F(w, \xi)\|_2 \leq L_0(\xi)$.

We will also need the following proposition, connecting the gradient of $f_\alpha(w)$ with the $L_\infty$-norm Clarke $\frac{\alpha}{2}$-subdifferential of $f(w)$ for our convergence analysis.

**Proposition 4.** For all $w \in \mathbb{R}^d$ and $\alpha > 0$, it holds that $\nabla f_\alpha(w) \in \partial^\infty_{\frac{\alpha}{2}} f(w)$.

### 3 Fixed-point Arithmetic Environments

In this work we focus on fixed-point number formats, which are of greater practical interest given their simpler arithmetic resulting in reduced hardware complexity and energy con-
consumption, see (Wang et al., 2022, Table 2). We denote a general fixed-point arithmetic environment as \( \mathbb{F} \subset \mathbb{R} \) when further specification is not required. For \( m, n \in \mathbb{Z}_{\geq 0} \), with \( m \leq n \), let \([n]_m := [m, \ldots, n]\), and in particular \([n] := [n]_1\).

Following (Gupta et al., 2015), all \( y \in \mathbb{F} \) are represented in the form of
\[
[e_r e_{r-1}(\ldots) e_1 d_1 d_2(\ldots)d_t],
\]
written in radix complement (Weik, 2001, Page 1408), using \( r \in \mathbb{Z}_{\geq 0} \) digits to represent the integer part and \( t \in \mathbb{Z}_{\geq 0} \) digits to represent the fractional part of \( y \), with \( r + t > 0 \). Using a base \( \beta \in \mathbb{Z}_{> 1} \), \( \mathbb{Z}_{\geq 0} \ni e_i < \beta \) for all \( i \in [r] \) and \( \mathbb{Z}_{\geq 0} \ni d_i < \beta \) for all \( i \in [t] \).

For any \( \mathbb{F} \), let \( \Lambda^-, \lambda, \) and \( \Lambda^+ \) denote the smallest, the smallest positive, and the largest representable numbers, respectively, with its range defined as \( \mathcal{R}_\mathbb{F} := \{x \in \mathbb{R} : \Lambda^- \leq x \leq \Lambda^+\} \).

We will consider two forms of rounding: rounding to nearest and stochastic rounding. Given an \( x \in \mathbb{R}_\mathbb{F} \), let \( \lfloor x \rfloor_\mathbb{F} := \max\{y \in \mathbb{F} : y \leq x\} \) and \( \lceil x \rceil_\mathbb{F} := \min\{y \in \mathbb{F} : y \geq x\} \), and let \( R(x) \in \mathbb{F} \) denote a function which performs one of the two rounding methods. For rounding to nearest an \( x \in \mathbb{R}_\mathbb{F} \),
\[
R(x) \in \arg\min_{y \in \{\lfloor x \rfloor_\mathbb{F}, \lceil x \rceil_\mathbb{F}\}} |y - x|.
\]

When \( \lceil x \rceil_\mathbb{F} - x = x - \lfloor x \rfloor_\mathbb{F} \), this work does not depend on the use of a specific tie-breaking rule, but for simplicity we assume that it is deterministic, such as rounding to even or away (IEEE Computer Society, 2019, Section 4.3.1). For stochastic rounding,
\[
R(x) := \begin{cases} 
\lfloor x \rfloor_\mathbb{F} & \text{with probability } p = \frac{x - \lfloor x \rfloor_\mathbb{F}}{|\lfloor x \rfloor_\mathbb{F} - \lfloor x \rfloor_\mathbb{F}|} \\
\lceil x \rceil_\mathbb{F} & \text{with probability } 1 - p.
\end{cases}
\]

Considering the error \( \delta := R(x) - x \), it is well known that \( \mathbb{E}[\delta] = 0 \), e.g. (Connolly et al., 2021, Lemma 5.1). We will also need a bound on its variance.

**Proposition 5.** For an \( x \in \mathcal{R}_\mathbb{F} \), it holds that
\[
\mathbb{E}[\delta] = 0 \quad \text{and} \quad \text{Var}(\delta) = \mathbb{E}[\delta^2] \leq \frac{\beta^{-2t}}{4}.
\]

When \( x \notin \mathcal{R}_\mathbb{F} \), we assume that \( R(x) = \arg\min_{y \in \{\Lambda^-, \Lambda^+\}} |y - x| \) for both rounding methods, which is similar to how overflows are handled when using rounding towards zero (IEEE Computer Society, 2019, Section 7.4).

### 4 Past Work on Optimization with Computational Error

Research on optimization in environments with error is vast when considering stochastic optimization as a subset. The minimization of a stochastic function with further computational
error seems to be a topic much less explored. We highlight a few papers which were found to be most relevant to the current research.

An influential paper for this work was (Bertsekas and Tsitsiklis, 2000), where the convergence of a gradient method of the form \( w^{k+1} = w^k + \eta^k (s^k + \hat{e}^k) \) is studied, where \( \eta^k \) is a step size, \( s^k \) is a direction of descent, \( \hat{e}^k \) is a deterministic or stochastic error, and it is assumed that \( f(w) \) has a Lipschitz continuous gradient. It was proven that \( f(w^k) \) converges, and if the limit is finite, then \( \nabla f(w^k) \to 0 \), without any type of boundedness assumptions.

In (Solodov and Zavriev, 1998), a parallel projected incremental algorithm onto a convex compact set is proposed for solving finite-sum problems. It is assumed that there is non-vanishing bounded error when computing subgradients \( g_i(w) \in \partial f_i(w) \) of each subfunction, with a convergence result to an approximate stationary point with an error level relative to the error in computing the subgradients. Each subfunction \( f_i(w) \) is assumed to be Lipschitz continuous but regular, i.e. its one-sided directional derivative exists and for all \( v \in \mathbb{R}^d \)
\[ f'_i(w; v) = \max_{g \in \partial f_i(w)} \langle g, v \rangle \] (Clarke, 1990, Section 2.3), which precludes functions with downward cusps such as \( (1 - \text{ReLU}(x))^2 \) (Davis et al., 2020, Section 1).

Recent work studying the convergence of gradient descent for convex loss functions with a Lipschitz continuous gradient in a low-precision floating-point environment is presented in (Xia et al., 2024). Besides proposing biased stochastic rounding schemes which prevent small gradients from being rounded to zero, inequalities are provided involving the step size, the unit roundoff, and the norms of the gradient and iterates which guarantee either a convergence rate to the optimal solution, or at least the (expected) monotonicity of the loss function values.

We also mention the paper (Yang et al., 2019), which studies the algorithm \( w^{k+1} = R(w^k - \eta^k \nabla \tilde{f}(w^k)) \), where \( \nabla \tilde{f}(w) \) is a stochastic gradient and \( R(\cdot) \) performs stochastic rounding into an \( \mathbb{F} \). It is assumed that the loss function \( f(w) \) is strongly convex, with Lipschitz continuous gradient and Hessian, with \( \nabla \tilde{f}(w^k) \) being uniformly bounded from \( \nabla f(w^k) \) for all \( k \geq 1 \). Convergence to a neighbourhood of the optimal solution is proven which depends on the precision of \( \mathbb{F} \), with an improved dependence proven when considering an exponential moving average of iterates computed in full-precision.

## 5 Perturbed SGD with Computational Error

The PSGD algorithm is first described with infinite precision in order to more easily describe the modelling of PSGD with computational error. Given an initial iterate \( w^1 \in \mathbb{R}^d \), we consider a perturbed mini-batch SGD algorithm of the form

\[
    w^{k+1} = w^k - \frac{\eta_k}{M} \sum_{i=1}^{M} \tilde{\nabla} F(w^k + u^k, \xi^{k,i}),
\]

where \( \eta_k \geq 0 \) is the step size, \( M \in \mathbb{Z}_{>0} \) is the mini-batch size, \( u^k \sim U(B_{\alpha_k}^\infty) \) is a sample from a uniform distribution with parameter \( \alpha_k > 0 \), and \( \{\xi^{k,i}\} \) are \( M \) samples of \( \xi \).
In order to model PSGD with computational error we introduce the following notation:

1. $\hat{\nabla} F(w, \xi, b)$ is a Borel measurable function in $(w, \xi, b) \in \mathbb{R}^{d+n+s}$ which approximates the stochastic gradient $\nabla F(w, \xi)$, where $b \in \mathbb{R}^s$ is a discrete random vector which models the use of stochastic rounding,

2. $\hat{u}^k \in \mathbb{R}^d$ is an approximation of a sample from the continuous distribution $U(\mathbb{B}^\infty_{\frac{s}{2}})$, and

3. $\hat{e}^k \in \mathbb{R}^d$ is a random vector which models the error from computing the basic arithmetic operations in (3).

The proposed model of PSGD with computational error takes the form

$$w^{k+1} = w^k - \frac{\eta_k}{M} \sum_{i=1}^M \hat{\nabla} F(w^k + \hat{u}^k, \xi^{k,i}, b^{k,i}) + \hat{e}^k.$$  (4)

The sampling of $\hat{u}^k$, $\{\xi^{k,i}\}$, and $\{b^{k,i}\}$ is assumed to be done independently. We consider a filtration $\{F_k\}$ on the probability space $(\Omega, \mathcal{F}, P)$, where $F_k := \sigma(\hat{u}^j, \{\xi^{k,i}\}, \{b^{k,i}\}, \eta_j, \hat{e}^j : j \in [k])$ and a sequence of $\sigma$-algebras $\{G_k\}$, where $G_k := \sigma(\hat{u}^k, \{\xi^{k,i}\}, \{b^{k,i}\}, \eta_k)$. We note that $w^k$ is $F_{k-1}$-measurable, setting $F_0 := \{\emptyset, \Omega\}$ and assuming that $w^1 \in \mathbb{R}^d$ is given.

5.1 Assumptions concerning PSGD with Computational Error

The following material has been separated into subsections for easy reference.

5.1.1 Description of $\hat{u}^k$

The original $u^k \sim U(\mathbb{B}^\infty_{\frac{s}{2}})$ is replaced by a sample $\hat{u}^k \in \mathbb{R}^d$ from a probability distribution $\hat{P}^k$, where the sequence of probability distributions $\{\hat{P}^k\}$ and parameters $\{\alpha^k\}$ are assumed to be deterministic. In a finite precision environment, a natural choice for each $\hat{P}^k$ would be a discrete approximation of $U(\mathbb{B}^\infty_{\frac{s}{2}})$.

5.1.2 Description of $b^{k,i}$

The inclusion of the random vector $b \in \mathbb{R}^s$ in $\hat{\nabla} F(w, \xi, b)$ is to model the use of stochastic rounding. The size $s \in \mathbb{N}$ of $b$ is equal to the number of basic arithmetic operations required to approximately compute $\nabla F(w, \xi)$, see (Croci et al., 2022, Section 7) for an overview of the implementation of stochastic rounding in practice. It is assumed that for all $k \in \mathbb{N}$ and $j \in [s]$, $b^k_j \in \mathbb{R}$ is a discrete uniformly distributed random variable over a finite set $V^k_j \subset \mathbb{R}$. We will denote the distribution of $b^k_j$ as $U(V^k_j)$, where $V^k := \{\hat{b} : P(b^k = \hat{b}) > 0\}$ is the support of $b^k$. In (4), the set $\{b^{k,i}\} \subset \mathbb{R}^s$ contains $M$ samples of $b^k \sim U(V^k)$.

5.1.3 Assumptions on the error of $\hat{\nabla} F(w^k + \hat{u}^k, \xi, b^k)$

The required accuracy of the perturbed approximate stochastic gradient $\hat{\nabla} F(w^k + \hat{u}^k, \xi, b^k)$ is contained in the following assumption.
Assumption 2. There exists constants $c_1 > 0, c_2 > 0$, and $K \in \mathbb{N}$ such that for all $k \geq K$,
\begin{align}
\langle \mathbb{E}[\nabla F(w^k + \hat{u}^k, \xi, b^k)|\mathcal{F}_{k-1}], \nabla f_{\alpha_k}(w^k) \rangle &\geq c_1 \|\nabla f_{\alpha_k}(w^k)\|^2_2 \quad \text{and} \\
\mathbb{E}[\|\nabla F(w^k + \hat{u}^k, \xi, b^k)\|^2_2|\mathcal{F}_{k-1}] &\leq c_2 Q \tag{5}
\end{align}
almost surely, where $\hat{u}^k \sim \hat{P}^k$, $b^k \sim U(V^k)$, $f_{\alpha_k}(w) := \mathbb{E}[f(w + u^k)]$ for $u^k \sim U(B_{\infty}^\alpha)$, and recalling that $Q := \mathbb{E}[L_0(\xi)^2]$.

In our asymptotic convergence analysis, we will assume that $\lim_{k \to \infty} \alpha_k = 0$, so for any $\alpha > 0$, $K \in \mathbb{N}$ in Assumption 2 can be chosen such that (5) and (6) are only required to hold almost surely for $\alpha_k \leq \alpha$. Inequalities (5) and (6) are variants of classic error assumptions, see (Levitin and Polyak, 1966, Equations (4.3) & (4.4)) and (Bertsekas and Tsitsiklis, 2000, Equation (1.5)), tailored to our problem setting. Inequality (5) states that the conditional expectation of $-\nabla F(w^k + \hat{u}^k, \xi, b^k)$ must be a direction of descent for $f_{\alpha_k}$ at $w^k$ almost surely when $k \in \mathbb{N}$ is sufficiently large. When $\hat{\nabla} F(w, \xi, b) = \nabla F(w, \xi)$ for almost all $(w, \xi) \in B_{\infty}^\alpha(w^k) \times \mathbb{R}^n$ and all $b^k \in V^k$, and $\hat{u}^k \sim U(B_{\infty}^\alpha)$, inequalities (5) and (6) are satisfied with $c_1 = c_2 = 1$ from Propositions 3.1 and 3.3. Given that any $0 < c_1 < 1$ and $1 < c_2 < \infty$ are valid, Assumption 2 allows $\hat{\nabla} F(w^k + \hat{u}^k, \xi, b^k)$ to be an approximation of $\nabla F(w^k + u^k, \xi)$ with nontrivial error.

For simplicity let $\hat{\nabla} F^{k,i}(w^k) := \hat{\nabla} F(w^k + \hat{u}^k, \xi^{k,i}, b^{k,i})$ for $i \in [M]$, and $\hat{\nabla} F^k(w^k) := \frac{1}{M} \sum_{i=1}^M \hat{\nabla} F^{k,i}(w^k)$. We will require the following bound.

Proposition 6. For all $k \geq K$ from Assumption 2, $\mathbb{E}[\|\hat{\nabla} F^k(w^k)\|^2_2|\mathcal{F}_{k-1}] \leq c_2 Q$ almost surely.

5.1.4 Assumptions concerning $\eta_k$

We consider adaptive step sizes, motivated by methods such as gradient normalization and clipping, to stabilize the algorithm steps (4). We assume that for all $k \in \mathbb{N}$, $\eta_k = \hat{\eta}_k \psi_k$, where $\hat{\eta}_k > 0$ is deterministic and $\psi_k \geq 0$ is a random variable, with more details about $\{\psi_k\}_{k=1}^\infty$ given in the following assumption.

Assumption 3. We assume that

1. $\psi_k$ is essentially bounded by $\mathcal{F}_{k-1}$-measurable random variables $0 \leq \Psi_k^L \leq \Psi_k^U < \infty$ when conditioned on $\mathcal{F}_{k-1}$: $\mathbb{P}(\Psi_k^L \leq \psi_k \leq \Psi_k^U|\mathcal{F}_{k-1}) = 1$ almost surely for all $k \in \mathbb{N}$,

2. $\Psi_k^U$ is essentially uniformly bounded by constants $0 < \underline{\Psi}^U \leq \overline{\Psi}^U < \infty$: $\mathbb{P}(\underline{\Psi}^U \leq \Psi_k^U \leq \overline{\Psi}^U) = 1$ for all $k \in \mathbb{N}$, and

3. $\{\Delta_k\}_{k=1}^\infty$, where $\Delta_k := \Psi_k^U - \Psi_k^L$, almost surely uniformly converges (Rambaud, 2011, Proposition 1) to 0.

Assumption 3.3 is restrictive, requiring the length of the conditional essential range of $\psi_k, \Delta_k$, to decrease with $\lim_{k \to \infty} \psi_k = \lim_{k \to \infty} \Psi_k^U = \lim_{k \to \infty} \Psi_k^L$ almost surely. Our analysis allows adaptive step sizes, but in the limit the adaptiveness can only be with respect to, in essence, $\mathcal{F}_{k-1}$-measurable quantities. Assumption 3.3 stems from the difficulty in analyzing $\mathbb{E}[\psi_k \hat{\nabla} F(w^k + \hat{u}^k, \xi, b^k)]$.
\( \hat{u}^k, \xi^{k,i}, b^{k,i} | \mathcal{F}_{k-1} \) given that \( \psi_k \) can change the expected step direction. Our asymptotic convergence result requires the following bound on \( \Delta_k \).

**Assumption 4.** There exists a constant \( c_3 > 0 \) and a \( K \in \mathbb{N} \) such that for all \( k \geq K \),
\[
\Delta_k \leq c_3 \hat{\eta}_k^\alpha \text{ almost surely.}
\]

### 5.1.5 Assumptions concerning \( \hat{e}_k \)

The random vector \( \hat{e}_k \in \mathbb{R}^d \) in (4) models the error from computing the addition, subtraction, multiplication, and division with finite precision in (4) given \( w^k, \eta_k, M \), and \{\( \hat{\nabla}F(w^k + \hat{u}^k, \xi^{k,i}, b^{k,i}) \}\).

**Assumption 5.** There exists a constant \( c_4 > 0 \) and a \( K \in \mathbb{N} \) such that for all \( k \geq K \),
\[
\mathbb{E}[\hat{e}_k | \mathcal{F}_{k-1}, \mathcal{G}_k] = 0 \quad \text{and} \quad \mathbb{E}[\|\hat{e}_k\|^2 | \mathcal{F}_{k-1}] \leq c_4 \hat{\eta}_k^2.
\]

We believe that the assumptions placed \( \eta_k \) and \( \hat{e}_k \) require some justification, which has been deferred to Subsection 5.3 in order to first present the convergence analysis of PSGD.

### 5.2 Convergence analysis of PSGD with Computational Error

We now present our asymptotic convergence result to a Clarke stationary point.

**Theorem 1.** Assume that perturbed SGD (4) is run such that Assumption 2 holds for a non-increasing sequence \( \{\alpha_k\} \) using step sizes \( \eta_k = \hat{\eta}_k \psi_k \geq 0 \) such that Assumption 3 holds, with \( \{\alpha_k\} \) and \( \{\hat{\eta}_k\} \) chosen such that
\[
\sum_{k=1}^{\infty} \alpha_k \hat{\eta}_k = \infty, \quad \sum_{k=1}^{\infty} \alpha_k^{d-1} \hat{\eta}_k^2 < \infty,
\]
and \( \lim_{k \to \infty} \alpha_k = 0 \). Assuming in addition that Assumptions 4 and 5 hold, almost surely, there exists a subsequence of indices \( \{k_i\} \) such that
\[
\lim_{i \to \infty} \|\nabla f_{\alpha_{k_i}}(w^{k_i})\|_2 = 0
\]
and for every accumulation point \( w^* \) of \( \{w^{k_i}\} \),
\[
\text{dist}(0, \partial f(w^*)) = 0.
\]

The next proposition gives a family of sequences \( \{\alpha_k\} \) and \( \{\hat{\eta}_k\} \) which satisfy the conditions of Theorem 1 and Assumption 3.3 using Assumption 4, where the step sizes \( \hat{\eta}_k \) are assumed to lie in an interval to account for rounding error.

**Proposition 7.** For \( 0.5 < q < 1 \), \( p = \frac{1-q}{d} \), and constants \( 0 < c_5 \), and \( 0 < c_6 \leq c_7 < \infty \), setting \( \alpha_k = \frac{c_5}{k^p} \) and \( \hat{\eta}_k \in [\frac{c_6}{k^p}, \frac{c_7}{k^p}] \) for \( k \in \mathbb{N} \) satisfies the conditions of Theorem 1, as well as Assumption 3.3 given the bound on \( \Delta_k \) from Assumption 4.
Giving an asymptotic convergence result in Theorem 1 in a setting motivated by finite precision arithmetic may seem contradictory, in particular, how \( \lim_{k \to \infty} \hat{\eta}_k = 0 \) in Proposition 7. If we consider a sequence of fixed-point environments \( \{F_{t_j}\}_{j=1}^{\infty} \) with increasing fractional digits \( t_{j+1} > t_j \) for all \( j \in \mathbb{N} \), a schedule could be followed where we use \( F_{t_1} \) for iterations \([1, \hat{K}_1]\), \( F_{t_2} \) for iterations \([\hat{K}_1 + 1, \hat{K}_2]\), and so on for a predetermined sequence \( \{\hat{K}_j\}_{j=1}^{\infty} \subset \mathbb{N} \), which could accommodate decreasing step sizes. This idea of increasing the number of fractional digits through time was successfully used in (Gupta et al., 2015, Figure 3), where neural network training in an \( F_{12} \) was performed until stagnation occurred, after which the fractional digits were increased to \( t = 16 \), resulting in a rapid accuracy improvement. Another, perhaps more practical approach is to simply consider a fixed \( k \). Assume further that Assumptions 4 and 5 hold for all \( k \in \mathbb{N} \), and that for a constant \( c_9 \), where \( 0 < c_9 < c_1 \Psi_U \), it holds that \( K \geq \left( \frac{c_8 c_9}{2 \alpha (c_1 \Psi_U - c_9)} \right)^2 \). For \( \hat{w} := w^{k+1} \),

\[
\mathbb{E}[\text{dist}(0, \partial_{\frac{c_1}{2}} f(\hat{w}))^2] \leq \frac{f_\alpha(w)}{c_8 c_9 \sqrt{K}} + \frac{c_3 c_8 c_2 Q}{2 \alpha c_9 \sqrt{K}} + \frac{\sqrt{d} L_0 c_8}{\alpha c_9 \sqrt{K}} (\Psi_U'^2 c_2 Q + c_4),
\]

and to guarantee that

\[
\mathbb{E}[\text{dist}(0, \partial_{\frac{c_1}{2}} f(\hat{w}))] \leq \nu
\]

for any \( \nu > 0 \) requires \( K = O(\alpha^{-2} \nu^{-4}) \).

### 5.3 Results supporting the assumptions on \( \eta_k \) and \( \hat{\psi}_k \)

The following proposition considers the rounding error when computing \( \eta_k = \hat{\eta}_k \psi_k \) and shows how Assumption 3 can be satisfied in a fixed-point environment \( F \).

**Proposition 8.** Assume that rounding to nearest is used in an \( F \) with \( \Lambda^+ \geq 1 \), \( \hat{\eta}_k = R(\eta'_k) \) for \( \eta'_k \in \mathbb{R}_{\geq \lambda} \), \( \psi'_k \in \mathbb{R}_{\geq 0} \) is a random variable, and \( F_{\geq 0} \ni \hat{\Psi}_k^L \leq \hat{\Psi}_k^U \in F_{\geq 1} \) are \( F_{k-1} \)-measurable random variables for all \( k \in \mathbb{N} \). Considering \( \hat{\psi}_k := \max(\hat{\Psi}_k^L, \min(R(\psi'_k)), \hat{\Psi}_k^U) \) \( \in F_{\geq 0} \), there exists a random variable \( \psi_k \in \mathbb{R}_{\geq 0} \) such that \( \eta_k = \hat{\eta}_k \psi_k = R(\hat{\eta}_k \psi_k) \), and \( F_{k-1} \)-measurable random variables \( 0 \leq \Psi_k^L \leq \Psi_k^U < \infty \) such that \( \Psi_k^L \leq \psi_k \leq \Psi_k^U \) for all \( k \in \mathbb{N} \) satisfying Assumption 4.1. There exists constants \( 0 < \Psi_k^U \leq \Upsilon_k^U < \infty \) such that \( \Psi_k^L \leq \hat{\psi}_k \leq \Psi_k^U \) for all \( k \in \mathbb{N} \) satisfying Assumption 4.2, and setting \( \hat{\Psi}_k^L = \hat{\Psi}_k^U = \Psi_k^U \) for all \( k \geq K \) for a \( K \in \mathbb{N} \) satisfies Assumption 3.3.

We now compare Assumption 3 to the assumptions used in four papers which studied gradient clipping. All of the papers proved a non-asymptotic convergence result for non-convex stochastic optimization problems after running for \( K \in \mathbb{N} \) iterations. Different variants of
gradient clipping were studied, so we will first consider a general gradient clipping algorithm,

\[ w^{k+1} = w^k - \hat{\eta} \min \left( h \left( \frac{1}{\|g^k\|_2} \right), 1 \right) g^k, \quad (7) \]

where \( g^k \) is a stochastic gradient of a loss function \( f \) sampled at \( w^k \), and \( h(\cdot) \) is a non-negative function of potentially other omitted parameters. Taking \( \psi_k = \min \left( h \left( \frac{1}{\|g^k\|_2} \right), 1 \right) \), \( \Psi^L_k = 0 \) and \( \Psi^U_k = \Psi^U = 1 \) for \( k \in \mathbb{N} \) are valid bounds for Assumptions 3.1 and 3.2. When the gradient is not clipped, i.e. \( h \left( \frac{1}{\|g^k\|_2} \right) \geq 1 \), (7) takes the form of SGD with a fixed step size, \( w^{k+1} = w^k - \hat{\eta} g^k \). If there exists a \( K' \in \mathbb{N} \leq K \) such that gradient clipping does not occur almost surely for all \( k \geq K' \), then for all \( k \geq K' \), \( \Psi^L_k = 1 \) is valid, \( \Delta_k = 0 \), and Assumption 3.3 is satisfied.

For the papers (Zhang et al. (2020c), Zhang et al. (2020b), & Koloskova et al. (2023)), Assumption 3 holds under the following bounded stochastic gradient assumption.

**Assumption 6.** There exists a constant \( G > 0 \) such that \( \|g^k\|_2 \leq G \) almost surely for all \( k \in \mathbb{N} \).

In (Zhang et al. (2020b) & Zhang et al. (2020a)), the following is assumed by the authors.

**Assumption 7.** There exists a constant \( \sigma > 0 \) such that \( \|g - \nabla f(w)\|_2 \leq \sigma \) almost surely for all \( w \in \mathbb{R}^d \).

Assumption 3 then holds assuming that their algorithms converge to within a neighbourhood of a stationary point based on the following assumption.

**Assumption 8.** There exists a \( G > 0 \) and a \( K' \in \mathbb{N} \leq K \) such that for all \( k \geq K' \), \( \|\nabla f(w^k)\|_2 \leq G \) almost surely.

For the sake of brevity, we will only present the detailed analysis for (Zhang et al., 2020b).

**Proposition 9.** For the gradient clipping algorithm studied in (Zhang et al., 2020b, Theorem 7), if Assumption 6 or 8 holds, taking \( K \) sufficiently large no gradient clipping will be performed almost surely for all \( k \in \mathbb{N} \) or for all \( k \geq K' \), respectively.

For the algorithm studied in (Zhang et al., 2020c, Theorem 2) using the parameter values in the proof on page 14 in the appendix of their work, a similar analysis as was shown for (Zhang et al., 2020b, Theorem 7) using Assumption 6 can be performed. For the convergence bound proven in (Koloskova et al., 2023, Theorem 3.3), after choosing parameter values which ensure it converges to 0, e.g. \( \hat{\eta} = K^{-\frac{1}{4}} \) and \( c = K^{\frac{1}{4}} \) (their clipping radius parameter), Assumption 6 can also be used. The algorithm studied in (Zhang et al., 2020a) allows for mixing gradient and momentum clipping. We considered their algorithm with the momentum parameter \( \beta = 0 \), resulting in a gradient clipping algorithm. The analysis in (Zhang et al., 2020a, Theorem 3.2) uses Assumption 7, and if Assumption 8 holds for \( G \leq 4\sigma \), then for all \( k \geq K' \) no gradient clipping will be performed almost surely.

We now show how Assumption 5 can hold in a fixed-point environment \( \mathbb{F} \).
Proposition 10. Assume that

\[ w^k ⊙ (η_k ⊙ M) ⊙ (\hat{∇} F^{k,1}(w^k) ⊕ ... ⊕ \hat{∇} F^{k,M}(w^k)) = w^k - \frac{η_k}{M} \sum_{i=1}^{M} \hat{∇} F^{k,i}(w^k) + e^k, \]

(8)

where the “o” symbols represent the corresponding operation in a fixed-point environment \( \mathbb{F}_r^\beta \subset \mathbb{R} \) in base \( \beta \in \mathbb{Z}_{>1} \), with \( t \in \mathbb{Z}_{\geq 0} \) fractional and \( r \in \mathbb{Z}_{\geq 0} \) integer digits such that \( r + t > 0 \), using stochastic rounding. Assume that \( w^k, \hat{∇} F^{k,i}(w^k) \in (\mathbb{F}_r^\beta)^d \) for \( i \in [M] \), and that \( η_k, M \in \mathbb{F}_r^\beta \). Assume that \( r ≥ 0 \) is chosen sufficiently large such that no overflow will occur in the computation of the left-hand side of (8). Assumption 5 holds with \( c_4 = \frac{M}{4}(M c_2 Q + 1)(\Ψ^U)^2 \).

6 Empirical Analysis of SGD Variants

This section compares different variants of SGD to improve the test set accuracy for image recognition tasks, and presents a numerical substantiation of Theorem 1. The focus is on two Resnet models: Training Resnet 20 on CIFAR-10 (R20C10) and Resnet 32 on CIFAR-100 (R32C100). The experiments were conducted using QPyTorch (Zhang et al., 2019b), which enabled the simulation of training using fixed-point arithmetic with stochastic rounding. We focus our experiments on using stochastic rounding given that it is the rounding method of choice for low-precision deep learning (Gupta et al., 2015; Wang et al., 2022; Yang et al., 2019).

For all experiments we train for 200 epochs, with an initial SGD step size of \( \hat{η}_k = 0.1 \), which is divided by 10 after 100 epochs, with a mini-batch size of \( M = 128 \). We evaluated algorithms based on their mean and minimum accuracy over 10 runs, placing value on algorithms’ robustness to quantization error. As the choice of 200 epochs is arbitrary, the average of the mean and minimum accuracies over the last 30 epochs were used for evaluation.

Let \( \mathbb{F}_{X/Y} \) denote an \( \mathbb{F} \) with \( \beta = 2 \), using \( X \) fractional bits, \( Y \) bits in total, and stochastic rounding. Our use of QPyTorch followed closely the CIFAR10 Low Precision Training Example found at (Zhang et al., 2019a). Assuming that the training is done in \( \mathbb{F}_{X/Y} \), in our implementation all weight, gradient, and momentum quantization is done using \( \mathbb{F}_{X/Y} \), stochastic rounding is used throughout, no gradient accumulator is used, no gradient scaling is performed, batch statistics are used to calculate the mean and variance for batch normalization, and input parameters \( \hat{η}_k \) and \( \alpha_k \) are quantized.

6.1 Gradient Normalization

A simple variant that was found to improve test set accuracy was gradient normalization (GN) as presented in Algorithm 1, where \( μ > 0 \) is a small positive constant to avoid division by 0, and \( \{\hat{η}_k\}_{k=1}^\infty \) is a baseline deterministic step size sequence.\(^1\) Initial experiments found that using the L1-norm and a simple moving average with \( c = 10 \) outperformed using an

\(^1\)When \( k = 1 \), \( \eta_1 \) is set to \( \hat{η}_1 \).
L2-norm or an exponential moving average with a weight parameter equal to \(R(0.1)\). We will refer to SGD using GN as NSGD.

**Algorithm 1** GN: Gradient Normalization (by step size update for iteration \(k > 1\))

**Input:** \(\nabla F(w^k) \in \mathbb{R}^d; \{g_{nrm}^i\}_{i=\max(1,k-c)}^{k-1} \subset \mathbb{R}_{>0}; \hat{\eta}_k, \mu \in \mathbb{R}_{>0}\)

\[
\begin{align*}
g_{nrm}^k &= \max(R(\|\nabla F(w^k)\|_1), \mu) \\
m_{ave}^{k-1} &= R(\frac{1}{\min(k-1,c)} \sum_{i=\max(1,k-c)}^{k-1} g_i^{nrm}) \\
\eta_k &= R(\hat{\eta}_k * m_{ave}^{k-1}/g_{nrm}^k)
\end{align*}
\]

If the norm of the gradient is larger (smaller) than average, the baseline step size \(\hat{\eta}_k\) is decreased (increased), which is intended to stabilize the norm of the algorithm’s updates \(\|w^{k+1} - w^k\|_2\) through time. For the more common form of normalized SGD, \(\eta_k = \hat{\eta}_k/g_{nrm}^k\) (Shor, 1998, Equation (2.7), Nesterov, 2004, Section 3.2.3), it seems unclear how to choose \(\hat{\eta}_k\) a priori, whereas with NSGD, assuming that \(\mathbb{E}[\eta_k - \hat{\eta}_k] \approx 0\), the need to tune \(\{\hat{\eta}_k\}\) can be avoided given that for any training task \(\{\hat{\eta}_k\}\) can be set to what has previously been used for SGD, which also allows for a clearer comparison between SGD with and without GN.

In our implementation of GN, only one rounding operation is performed in each line of Algorithm 1. This implicitly assumes that intermediate steps are stored in sufficiently high precision such that no additional rounding errors are observable in the final output. This choice is consistent with the implementation of rounding using QPyTorch, where a quantization layer is added after each layer which induces rounding errors.

We also consider a variant of Algorithm 1 named Delayed GN (DGN), which replaces \(g_{nrm}^k\) with \(g_{nrm}^{k-1}\) in the output of Algorithm 1, i.e. \(\eta_k = R(\hat{\eta}_k * m_{ave}^{k-1}/g_{nrm}^{k-1})\). GN and DGN can be seen as two extremes of Restricted Gradient Normalization (RGN), presented in Algorithm 2. The quantity \(m_{ave}^{k-1}/g_{nrm}^{k-1}\) is \(\mathcal{F}_{k-1}\)-measurable and is used to construct \(\mathcal{F}_{k-1}\)-measurable bounds \(\hat{\Psi}^L_k \geq 0\) and \(\hat{\Psi}^U_k > 0\) to clip \(m_{ave}^{k-1}/g_{nrm}^{k-1}\). GN is the step size which occurs when \(\Delta_k \geq 2\Lambda^*/\lambda\), with no clipping occurring when computing \(\hat{\psi}_k^2\) and DGN is the step size when \(\Delta_k = 0\). The values of \(\hat{\Psi}^L_k\) and \(\hat{\Psi}^U_k\) are chosen as evenly and as far apart as possible from \(m_{ave}^{k-1}/g_{nrm}^{k-1}\). Assuming that \(\mathbb{E}[m_{ave}^{k-1}/g_{nrm}^{k-1}|\mathcal{F}_{k-1}] \approx m_{ave}^{k-1}/g_{nrm}^{k-1}\), Algorithm 2 approximately maximizes \(\min(\mathbb{E}[m_{ave}^{k-1}/g_{nrm}^{k-1}|\mathcal{F}_{k-1}] - \hat{\Psi}^L_k, \hat{\Psi}^U_k - \mathbb{E}[m_{ave}^{k-1}/g_{nrm}^{k-1}|\mathcal{F}_{k-1}])\), minimizing the probability of clipping assuming that the distribution of \(m_{ave}^{k-1}/g_{nrm}^{k-1}\) is symmetric and unimodal.

\(^2\)Given that \(m_{ave}^{k-1}, g_{nrm}^{k-1} \in \mathbb{R}_{>0}\), \(m_{ave}^{k-1}/g_{nrm}^{k-1} \leq \Lambda^*/\lambda\) since \(m_{ave}^{k-1} \leq \Lambda^*\) and \(g_{nrm}^{k-1} \geq \lambda\).
Algorithm 2 RGN: Restricted Gradient Normalization (for $k > 1$)

**Input:** $m_{\text{ave}}^{k-1}, g_{\text{nrm}}^{k-1}, g_{\text{nrm}}^k, \hat{\eta}_k \in \mathbb{F}_{>0}; \frac{\Delta_k}{2} \in \mathbb{R}_{\geq 0}$

$v_k = \min\left(\frac{\Delta_k}{2}, \frac{m_{\text{ave}}^{k-1}}{g_{\text{nrm}}^k}\right)$

$\hat{\Psi}_L^k = \frac{m_{\text{ave}}^{k-1}}{g_{\text{nrm}}^k} - v_k$

$\hat{\Psi}_U^k = \frac{m_{\text{ave}}^{k-1}}{g_{\text{nrm}}^k} + \Delta_k - v_k$

$\hat{\psi}_k = \min\left(\max(\hat{\Psi}_L^k, \frac{m_{\text{ave}}^{k-1}}{g_{\text{nrm}}^k}), \hat{\Psi}_U^k\right)$

**Output:** $\eta_k = R(\hat{\eta}_k \ast \hat{\psi}_k)$

6.2 Training in Fixed-Points Environments

We now compare the performance of SGD, NSGD, DNSGD, PSGD, PNSGD, and PDNSGD, where PSGD, PNSGD, and PDNSGD are SGD, NSGD, and DNSGD (SGD with DGN) with iterate perturbation. For the perturbed algorithms, the perturbation level was always kept at $\alpha_k = 0.1\hat{\eta}_k$, with the finite-precision forward and back propagation computed at $u^k + \mathbf{u}^k$ for $u^k = R(u)$, where $u \sim U(B_{\mathbb{F}}^\alpha)$. To determine the appropriate ratio of fractional bits, we were guided by the results of (Gupta et al., 2015), and experimented with a majority of bits being fractional, given that in their experiments with $F_{X/16}$, the best accuracy occurred with $X=14$, with further improvement using $F_{16/20}$ (Gupta et al., 2015, Figures 1, 2, & 3). The choice of each fixed-point environment $F_{X/Y}$ was determined by finding the smallest $Y$ which did not result in all algorithms collapsing to random guessing.

Algorithms were ranked based on their Sum of Accuracies (SoA), where their mean and minimum accuracies were summed over all experiments. A list of each experiment conducted and each algorithm’s accuracy is presented in Table 1. It was difficult to interpret a plot of all six algorithms, so for both experiments SGD and the algorithms with the highest and lowest accuracies were plotted, which are found in Figure 1 in Appendix D. According to SoA, PNSGD was the best performing algorithm overall, with SGD being the worst.

| Acc     | SGD | NSGD | DNSGD | PSGD | PNSGD | PDNSGD |
|---------|-----|------|-------|------|-------|--------|
| R20C10 $\mathbb{F}_{15/20}$ Mean | 0.8640 | 0.8601 | 0.8588 | 0.8608 | 0.8641 | 0.8564 |
| R20C10 $\mathbb{F}_{15/20}$ Min  | 0.8456 | 0.8507 | 0.8475 | 0.8484 | 0.8566 | 0.8326 |
| R32C100 $\mathbb{F}_{17/24}$ Mean | 0.5902 | 0.5982 | 0.5973 | 0.5953 | 0.6015 | 0.6013 |
| R32C100 $\mathbb{F}_{17/24}$ Min  | 0.5749 | 0.5754 | 0.5850 | 0.5794 | 0.5872 | 0.5873 |
| SoA     | 2.8745 | 2.8845 | 2.8886 | 2.8840 | 2.9094 | 2.8777 |

Table 1: Accuracies of all algorithms in fixed-point stochastic rounding environments, with their Sum of Accuracies (SoA). Acc indicates whether each row contains mean or minimum accuracies. Bold and underline indicate the highest and lowest value in each column.
6.3 Numerical substantiation of Theorem 1

The previous section evaluated the performance of variants of SGD based on test set accuracy using a step size schedule \( \{ \hat{\eta}_k \} \) similar to what is typically used to train Resnet models. We now attempt to substantiate the asymptotic convergence result of Theorem 1 using PSGD and PSGD with RGN (PRNSGD). In these experiments we record the loss and the gradient norm on the training set over 200 epochs. To make the experiments easier to interpret, the step sizes and iterate perturbation were computed in single precision floating-point. The sequences \( \{ \alpha_k \} \) and \( \{ \hat{\eta}_k \} \) were chosen according to Proposition 7 with \( q = 0.51, c_5 = 0.01 \) and \( c_6 = c_7 = 0.1 \), following the initial step size and perturbation level used in the experiments of Section 6.2. For PRNSGD, a monotonically decreasing \( \{ \Delta_k \} \) was chosen according to Assumption 4 with \( c_3 \) chosen such that in the last iteration \( \Delta_k = 0.1 \).

We trained R20C10 and R32C100 in the same fixed-point environments with stochastic rounding used in Section 6.2. The plots of the training set loss and gradient norm are plotted in Figure 2 in Appendix D. Given that the convergence result in Theorem 1 is with respect to the infinite precision function, these plots were computed in single precision floating-point. We observe that the training set losses are consistently decreasing. For the plots of the gradient norms, for the R20C10 experiments, the mean gradient norms are decreasing from approximately the 10\(^{th}\) epoch, and for the R32C100 experiments, the mean gradient norms are gradually decreasing from approximately the 60\(^{th}\) epoch. In order to make our observations more precise, we fit linear functions, \( y = ax + b \), to the mean gradient norms over the whole experiment (epochs 0-200), over epochs 101-200, and over epochs 151-200, presented in Table 3 in Appendix D, where we recorded the estimated values of \( a \) and \( b \), and the residual sum of squares (RSS). We observe that the mean gradient norms are decreasing over all tested subsets according to the fitted equations.

7 Conclusion

This paper studied the theoretical and empirical convergence of variants of SGD with computational error. A new asymptotic convergence result to a Clarke stationary point, as well as the non-asymptotic convergence to an approximate stationary point were presented for perturbed SGD with adaptive step sizes, applied to a stochastic Lipschitz continuous loss function with error in computing its stochastic gradient, as well as the SGD step itself. Variants of SGD using gradient normalization and iterate perturbation were compared empirically to SGD in fixed-point environments using stochastic rounding, where improved test set accuracy compared to SGD was observed, in particular using PNSGD.
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# Appendix A  Table of Notation

Table 2: Table of notation used in the body of the paper divided by section.

| Symbol | Description                                                                 | Page |
|--------|-----------------------------------------------------------------------------|------|
| Section 1 |                                                                                     |      |
| $f$   | Loss function                                                                | 2    |
| $w$   | Decision variables of $f$                                                      | 2    |
| $F$   | Stochastic loss function                                                      | 2    |
| $\xi$ | Random vector argument of $F$                                                 | 2    |
| Section 2 |                                                                                     |      |
| $L_0(\xi)$ | Lipschitz constant of $F(w, \xi)$ for almost all $\xi$             | 2    |
| $Q$   | $Q := E[L_0(\xi)^2]$                                                         | 2    |
| $L_0$ | $L_0 := E[L_0(\xi)]$                                                         | 2    |
| $m$   | Lebesgue measure                                                              | 2    |
| $B_p^\epsilon(w)$ | $p$-norm $\epsilon$-open ball centered at $w$ | 2    |
| $\overline{B}_p^\epsilon(w)$ | $p$-norm $\epsilon$-closed ball centered at $w$ | 2    |
| $\partial h$ | Clarke subdifferential of a function $h$                                    | 2    |
| $\partial_p^\epsilon h$ | $p$-norm Clarke $\epsilon$-subdifferential of a function $h$ | 3    |
| $\nabla F$ | Function equal to $\nabla F$ almost everywhere it exists                | 3    |
| $u$   | Random vector uniformly distributed over $\overline{B}_2^\epsilon(w)$       | 3    |
| $\alpha$ | Diameter of ball that $u$ is sampled from                                   | 3    |
| $f_\alpha$ | $f_\alpha(w) := E[f(w + u)]$                                                | 3    |
| $L_1^\alpha$ | Lipschitz constant of gradient of $f_\alpha$ | 3    |
| Section 3 |                                                                                     |      |
| $\mathbb{F}$ | A fixed-point arithmetic environment                                           | 3    |
| $[n]_m$ | $[n]_m := [m, ..., n]$                                                        | 4    |
| $[n]$  | $[n] := [n]_1$                                                                | 4    |
| $r$   | Number of integer digits of a fixed-point number                            | 4    |
| $t$   | Number of fractional digits of a fixed-point number                         | 4    |
| $\beta$ | Base of $\mathbb{F}$                                                         | 4    |
| $d_i$ | Value of the $i^{th}$ fractional digit of a fixed-point number              | 4    |
| $e_i$ | Value of the $i^{th}$ integer digit of a fixed-point number                 | 4    |
| $\Lambda^-$ | Smallest representable number in $\mathbb{F}$                             | 4    |
| $\lambda$ | Smallest positive representable number in $\mathbb{F}$                    | 4    |
| $\Lambda^+$ | Largest representable number in $\mathbb{F}$                            | 4    |
| $\mathcal{R}_\mathbb{F}$ | $\mathcal{R}_\mathbb{F} := \{x \in \mathbb{R} : \Lambda^- \leq x \leq \Lambda^+\}$ | 4    |
| $[x]_\mathbb{F}$ | $[x]_\mathbb{F} : = \max\{y \in \mathbb{F} : y \leq x\}$                | 4    |
| $[x]_\mathbb{F}$ | $[x]_\mathbb{F} : = \min\{y \in \mathbb{F} : y \geq x\}$                | 4    |
| $R$   | Rounding to nearest or stochastic rounding                                    | 4    |
| Section 5 |                                                                                     |      |
| $\eta_k$ | Step size of PSGD in the $k^{th}$ iteration                                 | 5    |
Appendix B  Proofs of Sections 2 & 3

Proof. (Proposition 1): Let \( \mathcal{N}_r \) be a neighbourhood of the set \( \overline{B}_\epsilon^p(\mathbf{w}) \), with \( L_\epsilon \) being a Lipschitz constant of \( h(\mathbf{x}) \) restricted to \( \mathcal{N}_r \). Given that \( \|\phi\|_2 \leq L_\epsilon \) for all \( \phi \in \partial h(\mathbf{x}) \), for all \( \mathbf{x} \in \overline{B}_\epsilon^p(\mathbf{w}) \) \cite{clarke1990}, the set \( \partial_p^\prime h(\mathbf{w}) := \{ \partial h(\mathbf{x}) : \mathbf{x} \in \overline{B}_\epsilon^p(\mathbf{w}) \} \) is bounded, which equals \( \partial_p^\prime h(\mathbf{w}) \) without taking the convex hull. Since \( \partial h(\mathbf{w}) \) is outer semicontinuous \cite{clarke1990}, \( \partial h(\mathbf{c}) \) is closed if \( \mathcal{C} \) is compact \cite{rockafellar2009}, hence \( \partial^p_\epsilon h(\mathbf{w}) = \partial h(\overline{B}_\epsilon^p(\mathbf{w})) \) is compact. Since the convex hull of a compact set is compact \cite{rockafellar2009}, it also holds that \( \partial^p_\epsilon h(\mathbf{w}) \) is compact.

We next prove that \( \hat{\partial}^p_\epsilon h(\mathbf{w}) \) is outer semicontinuous following \cite{rockafellar2009, definition}. Let \( \mathbf{w}^k \to \mathbf{w} \) and \( z^k \to z \), with \( z^k \in \hat{\partial}^p_\epsilon h(\mathbf{w}^k) \). There exists at least one
Proof. (Proposition 2): The proof uses (Rockafellar and Wets, 2009, Proposition 5.49) and (Rockafellar and Wets, 2009, Inequality 4(13)). Consider any \( \mathbf{w} \in \mathbb{R}^d \) and any \( \epsilon > 0 \). For any \( \alpha_k > 0 \), the Pompeiu-Hausdorff distance (Rockafellar and Wets, 2009, Example 4.13) between \( \hat{\partial}_{\alpha_k}^p h(\mathbf{w}) := \{ \partial h(x) : x \in \overline{B}_{\epsilon}^p(\mathbf{w}) \} \) and \( \partial h(\mathbf{w}) \) with respect to the chosen \( p \)-norm\(^3\) equals

\[
\begin{align*}
d_{p,\infty}(\hat{\partial}_{\alpha_k}^p h(\mathbf{w}), \partial h(\mathbf{w})) &= \inf \{ \gamma > 0 : \hat{\partial}_{\alpha_k}^p h(\mathbf{w}) \subseteq \partial h(\mathbf{w}) + \overline{B}^p_\gamma(\mathbf{w}), \partial h(\mathbf{w}) \subseteq \hat{\partial}_{\alpha_k}^p h(\mathbf{w}) + \overline{B}^p_\gamma(\mathbf{w}) \} \\
&= \inf \{ \gamma > 0 : \hat{\partial}_{\alpha_k}^p h(\mathbf{w}) \subseteq \partial h(\mathbf{w}) + \overline{B}^p_\gamma(\mathbf{w}) \}.
\end{align*}
\]

By the outer semicontinuity of \( \partial h(\mathbf{w}) \), there exists a \( \delta > 0 \), such that \( \partial h(\overline{B}^p_\delta(\mathbf{w})) \subseteq \partial h(\mathbf{w}) + \overline{B}^p_\epsilon(\mathbf{w}) \). There exists a \( K \in \mathbb{N} \) such that for all \( j \geq K, \alpha_j \leq \delta \), implying that \( \hat{\partial}_{\alpha_j}^p h(\mathbf{w}) = \partial h(\overline{B}^p_{\alpha_j}(\mathbf{w})) \subseteq \partial h(\mathbf{w}) + \overline{B}^p_\epsilon(\mathbf{w}) \).

\(^3\)This is done for simplicity, with the result holding for any norm on \( \mathbb{R}^d \) given their equivalence.
For any \( j \in \mathbb{N} \), there exists a \( K' \in \mathbb{N} \) such that for all \( i \geq K' \), \( \alpha_i \leq \frac{\alpha}{2} \), from which it holds that for all \( i \geq K' \) and for all \( x \in \overline{B}_{\alpha_{K'}}(w) \), \( \partial_{\alpha_i} h(x) \subseteq \partial_{\alpha_{K'}} h(w) \). In particular for \( K \), there exists a \( K' \geq K \) such that for all \( i \geq K' \) and all \( x \in \overline{B}_{\alpha_{K'}}(w) \), \( \partial_{\alpha_i} h(x) \subseteq \partial_{\alpha_{K'}} h(w) \subseteq \partial h(w) + \overline{B}_{\epsilon}(w) \), from which \( d_{\infty}(\partial_{\alpha_i} h(x), \partial h(w)) \leq \epsilon \). Given that \( \partial h(w) \) and \( \overline{B}_{\epsilon}(w) \) are convex sets, taking the convex hull of both sides, it also holds that for all \( i \geq K' \) and \( x \in \overline{B}_{\alpha_{K'}}(w) \), \( \partial_{\alpha_i} h(x) \subseteq \partial h(w) + \overline{B}_{\epsilon}(w) \), implying that \( d_{\infty}(\partial_{\alpha_i} h(x), \partial h(w)) \leq \epsilon \) as well, proving that \( \{ \partial_{\alpha_i} h \} \) converges continuously to \( \partial h(w) \) for all \( w \in \mathbb{R}^d \).

**Proof. (Proposition 4):** Let \( \tilde{\nabla} f(w) \) be a Borel measurable function equal to the gradient of \( f(w) \) almost everywhere it exists, see (Mettel and Takeda, 2022, Example A.1) for a method of its construction. It holds that \( \nabla f(w + u) \in \partial f(w + u) \) when \( f \) is differentiable at \( w + u \in \mathbb{R}^d \) (Clarke, 1990, Proposition 2.2.2), which is for almost all \( u \in \overline{B}_{\frac{\alpha}{2}} \) by Rademacher’s theorem. It follows that for almost all \( u \in \overline{B}_{\frac{\alpha}{2}} \), \( \tilde{\nabla} f(w + u) \in \partial_{\frac{\alpha}{2}} f(w) \), hence \( \mathbb{E}[\tilde{\nabla} f(w + u)] \in \partial_{\frac{\alpha}{2}} f(w) \). The result follows given that \( \nabla f_\alpha(w) = \mathbb{E}[\tilde{\nabla} f(w + u)] \) (Mettel, 2023, Proposition 3).

**Proof. (Proposition 5):** For completeness we will give a proof that \( \mathbb{E}[\delta] = 0 \). Let \( \omega := [x]_F - [x]_F \), noting that \([x]_F - x = \omega - (x - [x]_F)\).

\[
\mathbb{E}[\delta] = ([x]_F - x)\frac{[x]_F - x}{\omega} + ([x]_F - x)(1 - \frac{x - [x]_F}{\omega})
= ([x]_F - x)\frac{[x]_F - x}{\omega} + (\omega - (x - [x]_F))\frac{\omega}{\omega}
= ([x]_F - x)\frac{[x]_F - x}{\omega} + (\omega - (x - [x]_F))\frac{[x]_F - x}{\omega} = 0.
\]

Letting \( \kappa := x - [x]_F \),

\[
\operatorname{Var}[\delta] = \mathbb{E}[\delta^2] - \mathbb{E}[\delta]^2
= ([x]_F - x)^2\frac{[x]_F - x}{\omega} + ([x]_F - x)^2(1 - \frac{x - [x]_F}{\omega})
= (\omega - \kappa)^2\frac{\kappa}{\omega} + \kappa^2\frac{\omega - \kappa}{\omega}
= \frac{\kappa}{\omega}(\omega^2 - 2\omega\kappa + \kappa^2 + \kappa\omega - \kappa^2)
= \frac{\kappa}{\omega}(\omega^2 - \omega^2 - 2\omega\kappa + \kappa^2 + \kappa\omega - \kappa^2)
= \kappa\omega - \kappa^2
\leq \frac{\omega^2}{2} - \frac{\omega^2}{4} = \frac{([x]_F - [x]_F)^2}{4} = \frac{\beta^{-2t}}{4},
\]

where the inequality holds given that \( k = \frac{\omega}{2} \) maximizes the strongly concave function (9), and the final result holds given that from (2), \([x]_F - [x]_F = \beta^{-t}\). \(\Box\)
Appendix C  Proofs of Section 5

Proof. (Proposition 6):

\[ E[\|\nabla F(w^k)\|^2_2 | \mathcal{F}_{k-1}] = E[\|\frac{1}{M} \sum_{i=1}^{M} \nabla F(w^k + \hat{u}^k, \xi^{k,i}, b^{k,i})\|^2_2 | \mathcal{F}_{k-1}] \]

\[ = E[\sum_{j=1}^{d} \left( \frac{1}{M} \sum_{i=1}^{M} \nabla F_j(w^k + \hat{u}^k, \xi^{k,i}, b^{k,i}) \right)^2 | \mathcal{F}_{k-1}] \]

\[ \leq E[\sum_{j=1}^{d} \frac{1}{M} \sum_{i=1}^{M} \nabla F_j(w^k + \hat{u}^k, \xi^{k,i}, b^{k,i})^2 | \mathcal{F}_{k-1}] \]

\[ = \frac{1}{M} \sum_{i=1}^{M} E[\|\nabla F(w^k + \hat{u}^k, \xi^{k,i}, b^{k,i})\|^2_2 | \mathcal{F}_{k-1}] \stackrel{a.s.}{=} c_2 Q, \]

where the first inequality uses Jensen’s inequality and the second uses (6).

Lemma 1. (Robbins and Siegmund, 1971, Theorem 1) For all \( k \geq 1 \), let \( z_k, \theta_k \), and \( \zeta_k \) be non-negative \( \mathcal{F}_{k-1} \)-measurable random variables such that

\[ E[z_{k+1} | \mathcal{F}_{k-1}] \leq z_k + \theta_k - \zeta_k \]

almost surely, and assume that \( \sum_{k=1}^{\infty} \theta_k < \infty \) almost surely. It holds almost surely that \( z_k \) converges to a random variable, \( \lim_{k \to \infty} z_k = z_\infty < \infty \), and \( \sum_{k=1}^{\infty} \zeta_k < \infty \).

Proof. (Theorem 1): This proof requires a Robbins-Siegmund inequality which is given directly above as Lemma 1. We begin the analysis by assuming that the algorithm has been run sufficiently long such that \( k \geq K \in \mathbb{N} \) and that for all \( k \geq K \) the (in)equalities in Assumptions 2, 4, and 5 hold, and \( \Delta_k \leq c_1 \Psi_k \) almost surely using Assumption 3. By the \( L_1^0 \)-smoothness of \( f_\alpha(w) \) (Proposition 3.2 & Nesterov (2004, Lemma 1.2.3)),

\[ f_{\alpha_k}(w^{k+1}) \leq f_{\alpha_k}(w^k) + \langle \nabla f_{\alpha_k}(w^k), w^{k+1} - w^k \rangle + \frac{L_{\alpha_k}^0}{2} \|w^{k+1} - w^k\|^2 \]

\[ = f_{\alpha_k}(w^k) + \langle \nabla f_{\alpha_k}(w^k), -\eta_k \nabla \mathcal{F}^k(w^k) + \hat{e}^k \rangle + \frac{L_{\alpha_k}^0}{2} \|w^{k+1} - w^k\|^2 \]  \( \Rightarrow \) \( f_{\alpha_k+1}(w^{k+1}) \leq f_{\alpha_k}(w^k) + f_{\alpha_k+1}(w^{k+1}) - f_{\alpha_k}(w^{k+1}) - \eta_k \langle \nabla f_{\alpha_k}(w^k), \nabla \mathcal{F}^k(w^k) \rangle \]

\[ + \langle \nabla f_{\alpha_k}(w^k), \hat{e}^k \rangle + \frac{L_{\alpha_k}^0}{2} \|w^{k+1} - w^k\|^2. \]
Focusing on \( f_{\alpha_{k+1}}(w^{k+1}) - f_{\alpha_k}(w^{k+1}) \),

\[
 f_{\alpha_{k+1}}(w^{k+1}) - f_{\alpha_k}(w^{k+1}) \\
= f_{\alpha_{k+1}}(w^{k+1}) - \int_{-\alpha/2}^{\alpha/2} \frac{f(w + u)}{\alpha^d_k} du + \int_{-\alpha/2}^{\alpha/2} \frac{f(w + u)}{\alpha^d_k} du \\
= f_{\alpha_{k+1}}(w^{k+1}) - \int_{-\alpha/2}^{\alpha/2} \frac{f(w + u)}{\alpha^d_k} du + \int_{-\alpha/2}^{\alpha/2} \frac{f(w + u)}{\alpha^d_k} du \\
\leq f_{\alpha_{k+1}}(w^{k+1}) \left( 1 - \frac{\alpha^d_{k+1}}{\alpha^d_k} \right),
\]

where the assumption that \( \alpha_{k+1} \leq \alpha_k \) was used for the third equality, and Assumption 1 was used for the inequality at the end. Plugging into (11),

\[
 f_{\alpha_{k+1}}(w^{k+1}) = f_{\alpha_k}(w^k) + f_{\alpha_{k+1}}(w^{k+1}) \left( 1 - \frac{\alpha^d_{k+1}}{\alpha^d_k} \right) - \eta_k \langle \nabla f_{\alpha_k}(w^k), \nabla F^k(w^k) \rangle \\
+ \langle \nabla f_{\alpha_k}(w^k), e^k \rangle + \frac{L^0}{2} \| w^{k+1} - w^k \|^2
\]

\[
= f_{\alpha_k}(w^k) + f_{\alpha_{k+1}}(w^{k+1}) \left( 1 - \frac{\alpha^d_{k+1}}{\alpha^d_k} \right) - \eta_k \langle \nabla f_{\alpha_k}(w^k), \nabla F^k(w^k) \rangle \\
+ \langle \nabla f_{\alpha_k}(w^k), e^k \rangle + \frac{L^0}{2} \| w^{k+1} - w^k \|^2
\]

\[
\Rightarrow \frac{\alpha^d_{k+1}}{\alpha^d_k} f_{\alpha_{k+1}}(w^{k+1}) \leq f_{\alpha_k}(w^k) - \eta_k \langle \nabla f_{\alpha_k}(w^k), \nabla F^k(w^k) \rangle \\
+ \langle \nabla f_{\alpha_k}(w^k), e^k \rangle + \frac{L^0}{2} \| w^{k+1} - w^k \|^2
\]

where the value of \( L^0 \) from Proposition 3 was used in the second inequality. Taking the conditional expectation of (12) with respect to \( \mathcal{F}_{k-1} \),

\[
\mathbb{E}[\alpha^d_{k+1} f_{\alpha_{k+1}}(w^{k+1}) | \mathcal{F}_{k-1}] \\
\leq \alpha^d_k f_{\alpha_k}(w^k) - \alpha^d_k \eta_k \mathbb{E}[\psi_k \langle \nabla f_{\alpha_k}(w^k), \nabla F^k(w^k) \rangle | \mathcal{F}_{k-1}] + \alpha^d_k \langle \nabla f_{\alpha_k}(w^k), \mathbb{E}[e^k | \mathcal{F}_{k-1}] \rangle \\
+ \alpha^d_k \mathbb{E}[e^k | \mathcal{F}_{k-1}] | \mathcal{F}_{k-1}] - 2 \mathbb{E}[[\eta_k \langle \nabla F^k(w^k), e^k \rangle | \mathcal{F}_{k-1}] + \mathbb{E}[\| e^k \|^2 | \mathcal{F}_{k-1}])
\]

It holds that \( \mathbb{E}[[e^k | \mathcal{F}_{k-1}] = \mathbb{E}[\mathbb{E}[e^k | \mathcal{F}_{k-1}, \mathcal{G}_k] | \mathcal{F}_{k-1}] = 0 \) almost surely by Assumption 5,

\[
\mathbb{E}[\psi_k^2 \| \nabla F^k(w^k) \|^2 | \mathcal{F}_{k-1}] \leq (\Psi_k^2)^2 \mathbb{E}[\| \nabla F^k(w^k) \|^2 | \mathcal{F}_{k-1}] \leq (\Psi^2)^2 c_2 Q
\]

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using Assumption 3 and Proposition 6,
\[
\mathbb{E}[\eta_k \langle \nabla F_k^k(w^k), e^k \rangle | F_{k-1}] \\
= \mathbb{E}[\mathbb{E}[\eta_k \langle \nabla F_k^k(w^k), e^k \rangle | G_k, F_{k-1}] | F_{k-1}] \\
= \mathbb{E}[\eta_k \langle \nabla F_k^k(w^k), e^k | G_k, F_{k-1} \rangle | F_{k-1}] \overset{a.s.}{=} 0,
\]
and \(\mathbb{E}[\| e^k \|^2_2 | F_{k-1}] \leq c_4 \hat{\eta}_k^2\) almost surely by Assumption 5. Applying these results in (13),
\[
\mathbb{E}[\alpha_{k+1}^d f_{\alpha_{k+1}}(w^{k+1}) | F_{k-1}] \overset{a.s.}{\leq} \alpha_{k}^d f_{\alpha_k}(w^k) - \alpha_{k}^d \hat{\eta}_k \mathbb{E}[\psi_k \langle \nabla f_{\alpha_k}(w^k), \nabla F_k^k(w^k) \rangle | F_{k-1}] \\
+ \alpha_{k}^{d-1} \hat{\eta}_k^2 \Delta L_0((\nabla U)^2) c_2 Q + c_4). (14)
\]
Focusing now on the conditional expectation \(\mathbb{E}[\psi_k \langle \nabla f_{\alpha_k}(w^k), \nabla F_k^k(w^k) \rangle | F_{k-1}]\):
\[
\mathbb{E}[\psi_k \langle \nabla f_{\alpha_k}(w^k), \nabla F_k^k(w^k) \rangle | F_{k-1}] \\
= \mathbb{E}[\psi_k^U (\| \nabla f_{\alpha_k}(w^k) - \nabla F_k^k(w^k) \|_2^2 - \| \nabla f_{\alpha_k}(w^k) \|_2^2 - \| \nabla F_k^k(w^k) \|_2^2) | F_{k-1}] \\
\overset{a.s.}{\leq} \frac{\Psi_k^U}{2} \mathbb{E}[\| \nabla f_{\alpha_k}(w^k) - \nabla F_k^k(w^k) \|_2^2 | F_{k-1}] - \frac{\Psi_k^L}{2} \mathbb{E}[\| \nabla f_{\alpha_k}(w^k) \|_2^2 | F_{k-1}] \\
= \frac{\Psi_k^U}{2} (\| \nabla f_{\alpha_k}(w^k) \|_2^2 - 2 \| \nabla f_{\alpha_k}(w^k), \mathbb{E}[\| \nabla F_k^k(w^k) \|_2^2 | F_{k-1}) + \mathbb{E}[\| \nabla F_k^k(w^k) \|_2^2 | F_{k-1})] \\
- \frac{\Psi_k^L}{2} \mathbb{E}[\| \nabla f_{\alpha_k}(w^k) \|_2^2 | F_{k-1}] \\
\overset{a.s.}{\leq} \frac{\Psi_k^U}{2} (\| \nabla f_{\alpha_k}(w^k) \|_2^2 - 2 \| \nabla f_{\alpha_k}(w^k) \|_2^2 + \mathbb{E}[\| \nabla F_k^k(w^k) \|_2^2 | F_{k-1})] \\
- \frac{\Psi_k^L}{2} \mathbb{E}[\| \nabla f_{\alpha_k}(w^k) \|_2^2 | F_{k-1}] \\
= \frac{1}{2} \Psi_k^U (c_1 \Psi_k^U + \frac{\Psi_k^L}{2}) \| \nabla f_{\alpha_k}(w^k) \|_2^2 + \left( \frac{\Psi_k^U}{2} - \frac{\Psi_k^L}{2} \right) \mathbb{E}[\| \nabla F_k^k(w^k) \|_2^2 | F_{k-1}] \\
\overset{a.s.}{\leq} \frac{1}{2} \Psi_k^U \| \nabla f_{\alpha_k}(w^k) \|_2^2 + \left( \frac{\Psi_k^U}{2} - \frac{\Psi_k^L}{2} \right) c_2 Q \overset{a.s.}{\leq} \frac{c_1}{2} \Psi_k^U \| \nabla f_{\alpha_k}(w^k) \|_2^2 + \frac{c_2}{c_2} Q, (15)
\]
where the first inequality uses Assumption 3.1, the second inequality uses inequality (5) of Assumption 2, and the last inequality uses the assumption that \(\Delta_k \leq c_5 \Psi_k^U\) almost surely for \(k \geq K\) and Assumption 4. Plugging (16) into (14),
\[
\mathbb{E}[\alpha_{k+1}^d f_{\alpha_{k+1}}(w^{k+1}) | F_{k-1}] \overset{a.s.}{\leq} \alpha_{k}^d f_{\alpha_k}(w^k) - \frac{\alpha_k^d \hat{\eta}_k}{2} (c_1 \Psi_k^U \| \nabla f_{\alpha_k}(w^k) \|_2^2 - c_3 \hat{\eta}_k \| \nabla f_{\alpha_k}(w^k) \|_2^2) \\
+ \alpha_{k}^{d-1} \hat{\eta}_k^2 \Delta L_0((\nabla U)^2) c_2 Q + \frac{c_4}{c_4}
\]
\[
= \alpha_{k}^d f_{\alpha_k}(w^k) - \frac{\alpha_k^d \hat{\eta}_k}{2} c_1 \Psi_k^U \| \nabla f_{\alpha_k}(w^k) \|_2^2 \\
+ \alpha_{k}^{d-1} \hat{\eta}_k^2 \Delta L_0((\nabla U)^2) c_2 Q + c_4 + \frac{c_3}{c_3} \hat{\eta}_k \| \nabla f_{\alpha_k}(w^k) \|_2^2.
\]
Lemma 1 can now be applied (redefining the index from $k = \overline{K}, \overline{K} + 1, \ldots$ to $k = 1, 2, \ldots$) with 
$z_k = \alpha_k^d f_{\alpha_k}(w^k)$, $\theta_k = \alpha_k^d \hat{\eta}^2_k (\sqrt{d}L_0((\Psi^U)^2c_2Q + c_4) + \frac{c_3}{2}c_2Q)$, and $\zeta_k = \alpha_k^d \hat{\eta}^2_k \frac{c_4}{2} \| \nabla f_{\alpha_k}(w^k) \|^2_2$, given that 
$$
\sum_{k=K}^{\infty} \alpha_k^d \hat{\eta}^2_k (\sqrt{d}L_0((\Psi^U)^2c_2Q + c_4) + \frac{c_3}{2}c_2Q)
\leq (\sqrt{d}L_0((\Psi^U)^2c_2Q + c_4) + \frac{c_3}{2}c_2Q) \sum_{k=1}^{\infty} \alpha_k^d \hat{\eta}^2_k < \infty
$$
by assumption, proving that almost surely 
$$
\sum_{k=K}^{\infty} \alpha_k^d \hat{\eta}^2_k \frac{c_4}{2} \| \nabla f_{\alpha_k}(w^k) \|^2_2 < \infty. \quad (17)
$$

It follows that $\liminf_{k \to \infty} \| \nabla f_{\alpha_k}(w^k) \|^2_2 = 0$ almost surely, as for any $\epsilon > 0$ if there exists a $\overline{K}_2 \geq \overline{K}$ such that $\| \nabla f_{\alpha_k}(w^k) \|^2_2 \geq \epsilon$ almost surely for all $k \geq \overline{K}_2$, 
$$
\sum_{k=\overline{K}_2}^{\infty} \alpha_k^d \hat{\eta}^2_k \frac{c_4}{2} \| \nabla f_{\alpha_k}(w^k) \|^2_2 \overset{a.s.}{\geq} \frac{c_4}{2} \| \nabla f_{\alpha_k}(w^k) \|^2_2 \geq k_{\overline{K}_2} \alpha_k^d \hat{\eta} = \infty,
$$
given that $\sum_{k=1}^{\infty} \alpha_k^d \hat{\eta}_k = \infty$ by assumption and $\sum_{k=1}^{\overline{K}_2-1} \alpha_k^d \hat{\eta}_k$ is finite, contradicting (17). There exists almost surely a subsequence of indices $\{k_i\}$ for which $\lim_{i \to \infty} \| \nabla f_{\alpha_{k_i}}(w^{k_i}) \|^2_2 = \liminf_{k \to \infty} \| \nabla f_{\alpha_k}(w^k) \|^2_2 = 0$. If $w^*$ is an accumulation point of $\{w^{k_i}\}$, let $\{k_{ij}\}$ be a subsequence of $\{k_i\}$ such that $\lim_{j \to \infty} w^{k_{ij}} = w^*$. Given that $\partial_{0,5\alpha_{k_{ij}}} f(w^{k_{ij}})$ converges continuously to $\partial f(w^*)$ by Proposition 2, it holds that 
$$
\lim_{j \to \infty} \text{dist}(0, \partial_{0,5\alpha_{k_{ij}}}^\infty f(w^{k_{ij}})) = \text{dist}(0, \partial f(w^*))
$$
(Rockafellar and Wets, 2009, Exercise 5.42 (b)). Since $\nabla f_{\alpha_{k_{ij}}}(w^{k_{ij}}) \in \partial_{0,5\alpha_{k_{ij}}} \infty f(w^{k_{ij}})$ from Proposition 4, 
$$
\text{dist}(0, \partial f(w^*)) = \lim_{j \to \infty} \text{dist}(0, \partial_{0,5\alpha_{k_{ij}}}^\infty f(w^{k_{ij}})) \leq \lim_{j \to \infty} \| \nabla f_{\alpha_{k_{ij}}}(w^{k_{ij}}) \|^2_2 = 0.
$$
\qed

Proof. (Proposition 7): Setting $\alpha_k = \frac{c_5}{k^p}$ and $\hat{\eta}_k \in \left[\frac{c_6}{k^q}, \frac{c_7}{k^q}\right]$, $\{\alpha_k\}$ is non-increasing with $\lim_{k \to \infty} \alpha_k = 0$ for $p > 0$. The summation conditions hold when 
$$
\sum_{k=1}^{\infty} \alpha_k^d \hat{\eta}_k \geq c_5c_6 \sum_{k=1}^{\infty} k^{-dp}k^{-q} = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \alpha_k^d \hat{\eta}_k \leq c_5c_7 \sum_{k=1}^{\infty} k^{-(d-1)p}k^{-2q} < \infty,
$$

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which is true when $0 < dp + q \leq 1$ and $(d - 1)p + 2q > 1$, which holds when $0.5 < q < 1$ and $p = \frac{(1 - q)}{d}$. Defining $\hat{q} := 2q - 1 > 0$ and using Assumption 4 for $k$ sufficiently large,

$$\Delta_k \leq c_3 \frac{\hat{q} k}{\alpha_k} \leq c_3 c_5 c_7 k^{p - q} \leq c_3 c_5 c_7 k^{1 - 2q} = c_3 c_5 c_7 \hat{q},$$

considering $d = 1$ for the third inequality, showing that Assumption 3.3 holds.

**Proof. (Theorem 2):** Taking the conditional expectation with respect to $F_{k-1}$ of inequality (10) in the proof of Theorem 1, and simplifying the notation, letting $\alpha_k = \alpha$ and $\eta_k = \hat{\eta} \psi_k$,

$$\mathbb{E}[f_{a}(w^{k+1})|F_{k-1}]$$

$$\leq f_a(w^{k}) - \hat{\eta} \mathbb{E}[\psi_k(\nabla f_a(w^{k}), \hat{\nabla} F^k(w^{k}))|F_{k-1}] + \langle \nabla f_a(w^{k}), \mathbb{E}[\hat{e}^k|F_{k-1}] \rangle + \frac{\sqrt{dL_0}}{\alpha} (\hat{\eta}^2 \mathbb{E}[\psi_k^2(\hat{\nabla} F^k(w^{k}))^2]_2 F_{k-1} - 2 \mathbb{E}[\eta_k(\nabla F^k(w^{k}), \hat{e}^k)|F_{k-1}] - \mathbb{E}[\|\hat{e}^k\|_2^2|F_{k-1}]])$$

$$\leq f_a(w^{k}) - \hat{\eta} \mathbb{E}[\psi_k(\nabla f_a(w^{k}), \hat{\nabla} F^k(w^{k}))|F_{k-1}] + \frac{\sqrt{dL_0}}{\alpha} \hat{\eta}^2 (\mathbb{E}[\psi_k^2]|F_{k-1}) + \frac{dL_0}{\alpha} \hat{\eta}^2 (\mathbb{E}[\psi_k^2]|F_{k-1}) + c_4$$

$$\leq f_a(w^{k}) - \hat{\eta} (c_1 \hat{\Psi}^U - \frac{c_3 \hat{\eta} k}{2c_2}) \|\nabla f_a(w^{k})\|_2^2 + \hat{\eta} (\frac{c_3 \hat{\eta} k}{2c_2}) c_2 Q + \frac{\sqrt{dL_0}}{\alpha} \hat{\eta}^2 (\mathbb{E}[\psi_k^2]|F_{k-1}) + c_4$$

$$= f_a(w^{k}) - \hat{\eta} (c_1 \hat{\Psi}^U - \frac{c_3 \hat{\eta} k}{2c_2}) \|\nabla f_a(w^{k})\|_2^2 + \hat{\eta} (\frac{c_3 \hat{\eta} k}{2c_2}) c_2 Q + \frac{\sqrt{dL_0}}{\alpha} \hat{\eta}^2 (\mathbb{E}[\psi_k^2]|F_{k-1}) + c_4$$

$$\leq f_a(w^{k}) - \frac{c_8}{\sqrt{K}} (c_1 \hat{\Psi}^U - \frac{c_3 \hat{\eta} k}{2c_2}) \|\nabla f_a(w^{k})\|_2^2 + \frac{c_3 \hat{\eta} k}{2c_2} c_2 Q + \frac{\sqrt{dL_0}}{\alpha} \hat{\eta}^2 (\mathbb{E}[\psi_k^2]|F_{k-1}) + c_4$$

where the second inequality holds using the same simplifications used to get inequality (14), and the third inequality was shown as inequality (15), both in the proof of Theorem 1. The fourth inequality uses Assumption 4, and the last inequality holds using the assumption that

$$K = \left(\frac{c_8}{2c_2(c_1 \Psi^U - c_9)}\right)^2.$$  Dividing by $\frac{c_8}{\sqrt{K}}$ and rearranging,

$$\|\nabla f_a(w^{k})\|_2^2 \leq \frac{c_8}{c_9} (f_a(w^{k}) - \mathbb{E}[f_a(w^{k+1})|F_{k-1}]) + \frac{c_3 \hat{\eta} k}{2c_2} c_2 Q + \frac{\sqrt{dL_0}}{\alpha} \hat{\eta}^2 (\mathbb{E}[\psi_k^2]|F_{k-1}) + c_4.$$  Taking the expectation, summing the inequalities over $k \in [K]$, and dividing by $K$,

$$\frac{1}{K} \sum_{k=1}^{K} \mathbb{E}[\|\nabla f_a(w^{k})\|_2^2] \leq \frac{f_a(w^{1}) - \mathbb{E}[f_a(w^{K+1})]}{c_8 c_9 \sqrt{K}} + \frac{c_3 \hat{\eta} k}{2c_2} c_2 Q + \frac{\sqrt{dL_0}}{\alpha} \hat{\eta}^2 (\mathbb{E}[\psi_k^2]|F_{k-1}) + c_4.$$  Noting that $\mathbb{E}[\|\nabla f_a(w^{k})\|_2^2] = \frac{1}{K} \sum_{k=1}^{K} \mathbb{E}[\|\nabla f_a(w^{k})\|_2^2]$, $\mathbb{E}[\text{dist}(0, \partial_{\nabla} f(\hat{w}))^2] \leq \mathbb{E}[\|\nabla f_a(w^{k})\|_2^2]$ from Proposition 4, and that $\mathbb{E}[f_a(w^{K+1})] \geq 0$ by Assumption 1,
Given that $\mathbb{E}[\text{dist}(0, \partial_2^\infty f(\tilde{w}))] \leq \mathbb{E}[\text{dist}(0, \partial_2^\infty f(\tilde{w}))^2]$ by Jensen’s inequality, the requirement that $\mathbb{E}[\text{dist}(0, \partial_2^\infty f(\tilde{w}))] \leq \nu$ is satisfied when

$$\sqrt{dL_0c_8}((\hat{\Psi}^U)^2c_2Q + c_4) \leq \nu^2. \tag{1}$$

After rearranging, this requires that

$$\frac{1}{\alpha^2\nu^4} \left( \frac{\alpha f_a(w^1)}{c_8c_9} + \frac{c_3c_8c_2Q}{2c_9} + \frac{\sqrt{dL_0c_8}(\hat{\Psi}^U)^2c_2Q + c_4}{\alpha c_9} \right)^2 \leq K. \tag{2}$$

**Proof. (Proposition 8):** The sequence $\{\hat{\eta}_k\}_{k=1}^\infty$ remains a deterministic positive sequence by the monotonicity of rounding to nearest (Higham, 2002, Page 38) and the assumption that $\hat{\eta}_k \in \mathbb{R}_{\geq \lambda}$. Let $\eta_k = R(\hat{\eta}_k \psi_k) = \hat{\eta}_k \psi_k(1 + \delta_{\hat{\psi}_k})$ with $\psi_k = \psi_k(1 + \delta_{\hat{\psi}_k})$, where $\delta_x$ is the relative rounding error from multiplying $\hat{\eta}_k$ with $x \in \mathbb{R}$. It holds that $\psi_k \in [\Psi^L_k, \Psi^U_k]$, where $\Psi^L_k := \hat{\Psi}^L_k(1 + \delta_{\hat{\psi}_k})$ and $\Psi^U_k := \hat{\Psi}^U_k(1 + \delta_{\hat{\psi}_k})$ are $\mathcal{F}_{k-1}$-measurable, satisfying Assumption 3.1. Setting $\overline{\Psi}^U = \frac{\lambda}{\lambda}$ is valid for Assumption 3.2 given that

$$\hat{\eta}_k \Psi^U_k = R(\hat{\eta}_k \hat{\Psi}^U_k) \geq R(\hat{\eta}_k) \geq \lambda \quad \Rightarrow \quad \Psi^U_k \geq \frac{\lambda}{\hat{\eta}_k} \geq \frac{\lambda}{\hat{\lambda}},$$

where the first inequality holds since $\hat{\Psi}^U_k \in \mathbb{F}_{\geq 1}$. Setting $\overline{\Psi}^U = \frac{\Lambda^+}{\lambda}$ is valid given that

$$\hat{\eta}_k \Psi^U_k = R(\hat{\eta}_k \hat{\Psi}^U_k) \leq \Lambda^+ \quad \Rightarrow \quad \Psi^U_k \leq \frac{\Lambda^+}{\hat{\eta}_k} \leq \frac{\Lambda^+}{\lambda},$$

where the first inequality holds since that for all $y \in \mathbb{R}$, $R(y) \leq \Lambda^+$ by assumption in Section 3. Setting $\Psi^L_k = \hat{\Psi}^L_k$ for all $k \geq K$ sets $\Psi^L_k = \hat{\Psi}^L_k(1 + \delta_{\hat{\psi}_k}) = \hat{\Psi}^U_k(1 + \delta_{\hat{\psi}_k}) = \Psi^U_k$ with $\Delta_k = 0$, satisfying Assumption 3.3. \qed

**Proof. (Proposition 9):** In (Zhang et al., 2020b, Theorem 7), gradient clipping of the form

$$w^{k+1} = w^k - \hat{\eta} \min \left( \frac{1}{16\hat{\eta}^2L_1(\|g^k\|_2 + \sigma)}, 1 \right) g^k$$

is considered, where Assumption 7 is assumed. The step size $\hat{\eta}$ is chosen as

$$\hat{\eta} = \min \left( \frac{1}{20L_0}, \frac{1}{128L_1\sigma}, \frac{1}{\sqrt{K}} \right).$$

Taking $\psi_k = \min \left( \frac{1}{16\hat{\eta}^2L_1(\|g^k\|_2 + \sigma)}, 1 \right)$, for $K$ sufficiently large, $\hat{\eta} = \frac{1}{\sqrt{K}}$ and

$$\psi_k = \min \left( \frac{K}{16L_1(\|g^k\|_2 + \sigma)}, 1 \right).$$
If Assumption 6 holds, taking $K$ sufficiently large such that $\frac{K}{16L_1(G + \sigma)} \geq 1$, no gradient clipping will be performed almost surely for all $k \in \mathbb{N}$. If Assumption 6 does not hold, for all $w \in \mathbb{R}^d$, almost surely,

$$\sigma \geq \|g - \nabla f(w)\|_2 \geq \|g\|_2 - \|
abla f(w)\|_2 \Rightarrow \|g\|_2 \leq \sigma + \|
abla f(w)\|_2,$$

using Assumption 7 and the reverse triangle inequality, hence

$$\frac{K}{16L_1(\|g^k\|_2 + \sigma)} \geq \frac{K}{16L_1(\|
abla f(w^k)\|_2 + 2\sigma)}$$

almost surely. Given any $G > 0$ for which Assumption 8 could hold, assume that $K \geq 16L_1(G + 2\sigma)$ is sufficiently large such that $K' \leq K$. It holds almost surely for $k \geq K'$ that

$$\psi_k \geq \min \left( \frac{K}{16L_1(\|
abla f(w^k)\|_2 + 2\sigma)}, 1 \right) \geq \min \left( \frac{16L_1(G + 2\sigma)}{16L_1(G + 2\sigma)}, 1 \right) \geq 1,$$

and no gradient clipping will be performed.

Proof. (Proposition 10): We adapt the results of (Wilkinson, 1965, Page 4 & 5), which for the basic operations $\{+,-,\times,\div\}$ with rounding to nearest applied to $x, y \in \mathbb{R}_t^\beta$ gives absolute errors bounded by $\{0, 0, 0.5\beta^{-t}, 0.5\beta^{-t}\}$, respectively, assuming no overflow. With the adoption of stochastic rounding these errors are increased to $\{0, 0, \beta^{-t}, \beta^{-t}\}$, but with the benefit of the errors from the operations $\{\times, \div\}$ being unbiased.

Evaluating the left-hand side of (8), following the order of operations,

$$w^k \ominus (\eta_k \odot M) \ominus (\hat{\nabla} F^{k,1}(w^k) \oplus \ldots \oplus \hat{\nabla} F^{k,M}(w^k)) = w^k \ominus (\frac{\eta_k}{M} + \delta_0) \ominus (\hat{\nabla} F^{k,1}(w^k) \oplus \ldots \oplus \hat{\nabla} F^{k,M}(w^k)) = w^k \ominus (\frac{\eta_k}{M} + \delta_0) \ominus \sum_{i=1}^M \hat{\nabla} F^{k,i}(w^k) = w^k - (\frac{\eta_k}{M} + \delta_0) \sum_{i=1}^M \hat{\nabla} F^{k,i}(w^k) + \delta^1 \geq w^k - \frac{\eta_k}{M} \sum_{i=1}^M \hat{\nabla} F^{k,i}(w^k) - \delta_0 \sum_{i=1}^M \hat{\nabla} F^{k,i}(w^k) - \delta^1,$$
where $\delta_0 \in \mathbb{R}$ is the error from the division, and $\delta^1 \in \mathbb{R}^d$ is the vector of errors from the multiplication. Setting $\hat{\epsilon}^k = -\delta_0 \sum_{i=1}^M \hat{\nabla}F^{k,i}(w^k) - \delta^1$,

$$
\mathbb{E}[\hat{\epsilon}^k | \mathcal{F}_{k-1}, G_k] = -M \mathbb{E}[\delta_0 \hat{\nabla}F^{k}(w^k) | \mathcal{F}_{k-1}, G_k] = -M \mathbb{E}[\delta_0 | \mathcal{F}_{k-1}, G_k, \delta_0] | \mathcal{F}_{k-1}, G_k]
= -M \mathbb{E}[\delta_0 | \mathcal{F}_{k-1}, G_k] \hat{\nabla}F^{k}(w^k) = 0.
$$

Considering now $\mathbb{E}[\|\hat{\epsilon}^k\|^2_2 | \mathcal{F}_{k-1}]$,

$$
\mathbb{E}[\|\hat{\epsilon}^k\|^2_2 | \mathcal{F}_{k-1}]
= \mathbb{E}[\|\hat{\epsilon}^k\|^2_2 | \mathcal{F}_{k-1}]
= \mathbb{E}[\|\delta_0 M \hat{\nabla}F^{k}(w^k) + \delta^1\|^2_2 | \mathcal{F}_{k-1}]
= \mathbb{E}[\|\delta_0 M \hat{\nabla}F^{k}(w^k)\|^2_2 | \mathcal{F}_{k-1}] + 2\mathbb{E}[\langle \delta_0 M \hat{\nabla}F^{k}(w^k), \delta^1 \rangle | \mathcal{F}_{k-1}] + \mathbb{E}[\|\delta^1\|^2_2 | \mathcal{F}_{k-1}]
= M^2 \mathbb{E}[\|\delta_0 \hat{\nabla}F^{k}(w^k)\|^2_2 | \mathcal{F}_{k-1}] + 2M \mathbb{E}[\langle \delta_0 \hat{\nabla}F^{k}(w^k), \delta^1 \rangle | \mathcal{F}_{k-1}] + \mathbb{E}[\|\delta^1\|^2_2 | \mathcal{F}_{k-1}].
$$

(18)

Focusing on $\mathbb{E}[\|\hat{\nabla}F^{k}(w^k)\|^2_2 | \mathcal{F}_{k-1}]$,

$$
\mathbb{E}[\|\hat{\nabla}F^{k}(w^k)\|^2_2 | \mathcal{F}_{k-1}]
= \mathbb{E}[\|\hat{\nabla}F^{k}(w^k)\|^2_2 | \mathcal{F}_{k-1}]
\leq \mathbb{E}[\|\delta_0 \hat{\nabla}F^{k}(w^k)\|^2_2 | \mathcal{F}_{k-1}] + \frac{\beta^{-2t}}{4} \mathbb{E}[\|\hat{\nabla}F^{k}(w^k)\|^2_2 | \mathcal{F}_{k-1}] \leq \frac{\beta^{-2t}}{4} c_2 Q,
$$

using Propositions 5 and Proposition 6. The inequality in Proposition 6 holds surely given that $\mathbb{E}[\|\hat{\nabla}F^{k}(w^k)\|^2_2 | \mathcal{F}_{k-1}]$ can only take on a finite number of values. Considering now $\mathbb{E}[\langle \delta_0 \hat{\nabla}F^{k}(w^k), \delta^1 \rangle | \mathcal{F}_{k-1}]$,

$$
\mathbb{E}[\langle \delta_0 \hat{\nabla}F^{k}(w^k), \delta^1 \rangle | \mathcal{F}_{k-1}]
= \mathbb{E}[\langle \delta_0 \hat{\nabla}F^{k}(w^k), \delta^1 \rangle | \mathcal{F}_{k-1}, G_k, \delta_0] | \mathcal{F}_{k-1}]
= \mathbb{E}[\langle \delta_0 \hat{\nabla}F^{k}(w^k), \mathbb{E}[\delta^1 | \mathcal{F}_{k-1}, G_k, \delta_0] | \mathcal{F}_{k-1} \rangle | \mathcal{F}_{k-1} \rangle = 0.
$$

Focusing on the final term of (18),

$$
\mathbb{E}[\|\delta^1\|^2_2 | \mathcal{F}_{k-1}]
= \mathbb{E}\left[\sum_{i=1}^M (\delta^1_i)^2 | \mathcal{F}_{k-1}\right]
= \sum_{i=1}^M \mathbb{E}[\|\delta^1_i\|^2_2 | \mathcal{F}_{k-1}, G_k, \delta_0] | \mathcal{F}_{k-1}]
\leq \sum_{i=1}^M \frac{\beta^{-2t}}{4} = \frac{M}{4} \beta^{-2t}.
$$

Continuing from (18),

$$
\mathbb{E}[\|\hat{\epsilon}^k\|^2_2 | \mathcal{F}_{k-1}]
\leq \frac{M}{4} (M c_2 Q + 1) \beta^{-2t} \leq \frac{M}{4} (M c_2 Q + 1) \eta_k^2
= \frac{M}{4} (M c_2 Q + 1) (\eta_k \psi_k)^2 \leq \frac{M}{4} (M c_2 Q + 1) M \eta_k^2 (\Psi^T)^2,
$$

where the first inequality holds since $\beta^{-t}$ is the smallest positive element of $\mathbb{F}_{r,t}$. The final inequality holds since $\psi_k \leq \Psi^T$ surely using Assumption 3, the fact that $\eta_k \in \mathbb{F}_{r,t}$ can only take on a finite number of values, and that $\eta_k$ is deterministic. 

\qed
Figure 1: (Section 6.2) Plots of SGD, PNSGD, and PDNSGD. The mean (thick solid), minimum (thin solid), and maximum (dotted) test set accuracy for R20C10 in $\mathbb{F}_{15/20}$ (top), and R32C100 in $\mathbb{F}_{17/24}$ (bottom) over 10 runs.

Table 3: (Section 6.3) Fitted linear functions $y = ax + b$ to the mean gradient norms.

| epochs (x) | a     | b     | RSS   |
|------------|-------|-------|-------|
| R20C10 experiments |       |       |       |
| PSGD        | [0, 200] | -0.0039 | 1.2850 | 4.5268 |
| PRNSGD      | [0, 200] | -0.0037 | 1.2669 | 4.3350 |
| PSGD        | [101, 200] | -0.0015 | 0.9381 | 0.1339 |
| PRNSGD      | [101, 200] | -0.0014 | 0.9325 | 0.1550 |
| PSGD        | [151, 200] | -0.0011 | 0.8691 | 0.0454 |
| PRNSGD      | [151, 200] | -0.0016 | 0.9745 | 0.0768 |
| R32C100 experiments |       |       |       |
| PSGD        | [0, 200] | -0.0010 | 1.4232 | 4.2259 |
| PRNSGD      | [0, 200] | -0.0008 | 1.3970 | 4.3180 |
| PSGD        | [101, 200] | -0.0019 | 1.5401 | 0.5717 |
| PRNSGD      | [101, 200] | -0.0019 | 1.5559 | 0.4987 |
| PSGD        | [151, 200] | -0.0015 | 1.4625 | 0.2789 |
| PRNSGD      | [151, 200] | -0.0013 | 1.4314 | 0.1833 |
Figure 2: (Section 6.3) Plots of PSGD and PRNSGD. The mean (thick solid), minimum (thin solid), and maximum (dotted) training set loss and gradient norm for R20C10 in $\mathbb{F}_{15/20}$ (top) and R32C100 in $\mathbb{F}_{17/24}$ (bottom) over 10 runs.