Particle acceleration in Kerr–(anti-)de Sitter black hole backgrounds

Yang Li, Jie Yang, Yun-Liang Li, Shao-Wen Wei and Yu-Xiao Liu

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, People’s Republic of China

E-mail: liyang09@lzu.cn, yangjiev@lzu.edu.cn, liyunl09@lzu.cn, weishaow06@lzu.cn and liuyx@lzu.edu.cn

Received 20 April 2011, in final form 9 September 2011
Published 14 October 2011
Online at stacks.iop.org/CQG/28/225006

Abstract

Bañados, Silk and West (BSW) proved that Kerr black holes could act as particle accelerators with arbitrarily high center-of-mass energy, if two conditions are satisfied: (1) these black holes are extremal, (2) one of the colliding particles has critical angular momentum. In this paper, we extend the research to the cases of Kerr–(anti-)de Sitter black holes and find that the cosmological constant has an important effect on the result. In order for the case of Kerr–anti-de Sitter black holes (with negative cosmological constant) to get arbitrary high center-of-mass energy, we need an additional condition besides the above two for Kerr ones. While, for the case of general Kerr–de Sitter black holes (with positive cosmological constant), the collision of two particles can take place on the outer horizon of the black holes and the center-of-mass energy of collision can blow up arbitrarily if the above second condition is satisfied. Hence, non-extremal Kerr-de Sitter black holes could also act as particle accelerators with arbitrarily high center-of-mass energy.

PACS numbers: 97.60.Lf, 04.70.-s

(Some figures may appear in colour only in the online journal)

1. Introduction

Two years ago, Banados, Silk and West (BSW) reported a process (BSW process) that two particles may collide on the horizon of the extremal Kerr black hole with the arbitrarily high center-of-mass (CM) energy [1]. Although it has been pointed out in [2, 3] that there are astrophysical limitations preventing a Kerr black hole to be extremal, and the gravitational radiation and backreaction effects should be counted in this process, similar processes have been found in other kinds of black holes or naked singularities and the BSW process of the Kerr black hole has been studied more deeply [3–15]. On the other hand, a general analysis
of this BSW process has been done for rotating black holes [16] and for most general black holes [17, 18]. Some efforts have also been made to draw some implications concerning the effects of gravity generated by colliding particles in [19].

In this paper, we investigate the BSW process of the Kerr–(anti-)de Sitter black hole, and our goal is to see the effect of the cosmological constant on the BSW process. There are good reasons to believe that our results can be reduced to the ones of BSW given in [1] as the cosmological constant turns to zero. Besides, because the Kerr–(anti-)de Sitter black hole does not have a simple horizon structure as the previous studied black holes, we have to use a different method to study the BSW process.

This paper is organized as follows. In section 2, we study the horizon structure of Kerr–(anti-)de Sitter black holes. In section 3, we calculate the CM energy of the particle collision on the horizon of the black holes, and derive the critical angular momentum to make the CM energy to blow up. In section 4, we find the BSW process requirements for the black hole and the colliding particle from the geodesic motion of the colliding particle. The conclusion is given in the last section.

2. Extremal Kerr–(anti-)de Sitter black holes

In this section, we would like to study the extremal the Kerr–(anti-)de Sitter black holes. First, the vacuum metric of the Kerr–anti-de Sitter (Kerr–AdS) black holes in the Boyer–Lindquist coordinate system with units $c = G = 1$ is given by

$$\mathrm{d}s^2 = -\frac{\Delta_r}{\rho^2} \left( \frac{\mathrm{d}t}{\Sigma} - \frac{a}{\Sigma} \sin^2 \theta \, \mathrm{d}\phi \right)^2 + \frac{\rho^2 \mathrm{d}r^2}{\Delta_r} + \frac{\rho^2 \mathrm{d}\theta^2}{\Delta_\theta} + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left( a \, \mathrm{d}r - \frac{r^2 + a^2}{\Sigma} \, \mathrm{d}\phi \right)^2,$$

where

$$\Delta_r = (r^2 + a^2) \left( 1 + \frac{r^2}{l^2} \right) - 2Mr,$$

$$\Delta_\theta = 1 - \frac{a^2 \cos^2 \theta}{l^2},$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$

$$\Sigma = 1 - \frac{a^2}{l^2}.$$

$M$ is related to the mass of the black hole, $a$ is related to the black hole’s spin angular momentum per mass by $a = J/M$ and $l^2$ is related to the cosmological constant $\Lambda$ by $l^2 = -\Lambda/3$. And for the Kerr–de Sitter (Kerr–dS) black hole, the form of the vacuum metric will remain the same, but $l^2$ in $\Delta_\Sigma$ and $\Sigma$ should be replaced by $-l^2$.

The horizon $r_h$ is given by $\Delta_r|_{r=r_h} = 0$. We can make a comparison of coefficients in $\Delta_r = l^{-2}r^4 + (1 + a^2 l^{-2})r^2 - 2Mr + a^2 = l^{-2} \prod_{i=1}^{4} (r - r_{hi})$ for the Kerr–AdS case, where $r_{hi}$ ($i = 1, 2, 3, 4$) denotes the zeros of $\Delta_r$ [20]. From this comparison we can conclude that for the Kerr–AdS black hole, there are two separated positive horizons at most, and $\Delta_r$ is positive outside the outer horizon of the Kerr–AdS black hole. In the same way, we can conclude for the Kerr–dS black hole that there are three separated positive horizons at most, and $\Delta_r$ is negative outside the outer horizon of the black hole. For the Kerr–AdS (Kerr–dS) black hole, when two horizons of the black hole coincide, the black hole is extremal.

2
If we consider the extremal Kerr–AdS black hole, we have to make a comparison of coefficients in
\[\Delta_1 = l^2 + a^2 I^2 - 2M r^2 + a^2 = (r - x)^2 (a^2 r^2 + a_1 r + a_2) \] with
\[a_0, a_1, a_2\) being real \[20\]. From this comparison we can obtain
\[\frac{a_2^2 l^2}{x^2} - 3x^2 = a^2 + I^2, \quad (6)\]
\[2x^3 - 2 \frac{a_2^2 l^2}{x} = -2MI^2. \quad (7)\]
In these equations, \(x\) is positive and related to the coincided horizon of the extremal Kerr–AdS black hole. Then, \(x\) can be solved as
\[x = \sqrt{-\frac{a^2}{6} - \frac{l^2}{6} + \frac{1}{6} \sqrt{a^4 + 14a_2^2 l^2 + I^4}}. \quad (8)\]
Analogously we can obtain these equations for the Kerr–dS case:
\[\frac{a_2^2 l^2}{x^2} + 3x^2 = -a^2 + I^2, \quad (9)\]
\[2x^3 + 2 \frac{a_2^2 l^2}{x} = 2MI^2, \quad (10)\]
where \(x\) is positive and related to the coincided horizon of the extremal Kerr–dS black hole. Also \(x\) can be solved as
\[x_1 = \sqrt{-\frac{a^2}{6} + \frac{l^2}{6} - \frac{1}{6} \sqrt{a^4 - 14a_2^2 l^2 + I^4}}. \quad (11)\]
\[x_2 = \sqrt{-\frac{a^2}{6} + \frac{l^2}{6} + \frac{1}{6} \sqrt{a^4 - 14a_2^2 l^2 + I^4}}. \quad (12)\]
These two solutions are both the coincided horizons of the extremal Kerr–dS black hole. By computing \(d^2 \Delta_1/dr^2\) at \(r = x_1\) and \(r = x_2\), we can see that \(x_1\) and \(x_2\) are related to the inner and outer coincided horizons, respectively.

We can also solve \(M\) and \(a\) from equations (9) and (10):
\[M = \frac{x (l^2 - x^2)}{l^2 (l^2 + x^2)}, \quad (13)\]
\[a^2 = \frac{x^2 (l^2 - 3x^2)}{l^2 + x^2}, \quad (14)\]
from which we can see that \(x^2 < l^2/3\) and \(a^2\) has a range of \(0 < a^2 < (7 - 4\sqrt{3})l^2\). So for an extremal Kerr–dS black hole, there are upper limits for the extremal horizon and the angular momentum of the black hole, while for a Kerr–AdS black hole, \(a\) also has an upper limit \(l\). When \(a\) reaches \(l\), the metric will be singular, and when \(a\) exceeds \(l\), the Kerr–AdS black hole will be unstable due to the superradiance \[21\].

3. The CM energy of the collision on the horizon of the Kerr–(anti-)de Sitter black hole

To investigate the CM energy of the collision on the horizon of the Kerr–(anti-)de Sitter black hole, we have to derive the 4-velocity of the colliding particle. And we only study the particle motion on the equatorial plane \((\theta = \frac{\pi}{2}, \rho^2 = r^2\).

The generalized momenta \(P_\mu\) can be given as
\[P_\mu = g_{\mu \nu} \dot{\gamma}^\nu. \quad (15)\]
where the dot denotes the derivative with respect to the affine parameter $\lambda$ and $\mu, \nu = t, r, \phi, \theta$. Thus, in equatorial motion, generalized momenta $P_t$ and $P_\phi$ are turned out to be
\begin{equation}
P_t = g_{tt} \dot{t} + g_{t\phi} \dot{\phi},
\end{equation}
\begin{equation}
P_\phi = g_{\phi\phi} \dot{\phi} + g_{\phi t} \dot{t}.
\end{equation}
$P_t$ and $P_\phi$ are constants of motion. In fact, $P_t$ and $P_\phi$ correspond to the test particle’s energy per unit mass $E$ and the angular momentum parallel to the symmetry axis per unit mass $L$, respectively. And in the following discussion we will just regard these two constants of motion as $E \equiv P_t$ and $L \equiv P_\phi$ [20].

The affine parameter $\lambda$ can be related to the proper time by
\begin{equation}
\mu = \lambda \tau,
\end{equation}
where $\tau$ is given by the normalization condition $-\mu^2 = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ with $\mu = 1$ for time-like geodesics and $\mu = 0$ for null geodesics. For a time-like geodesic, the affine parameter can be identified with the proper time, and thus from equations (16) and (17), we can solve 4-velocity components $\dot{t}$ and $\dot{\phi}$:
\begin{equation}
\frac{d}{d\tau} \frac{d}{d\tau} = \frac{E((a^2 + r^2)^2 - \Delta_a a^2) + L \Sigma a(\Delta_r - a^2 - r^2)}{r^2 \Delta_r},
\end{equation}
\begin{equation}
\frac{d}{d\tau} \dot{\phi} = \frac{E \Sigma a(r^2 + a^2 - \Delta_r) + L \Sigma^2 (\Delta_r - a^2)}{r^2 \Delta_r}.
\end{equation}
For the remained component $\dot{r} = \frac{dr}{d\tau}$, of the equatorial motion, we can obtain it from the Hamilton–Jacobi equation of the time-like geodesic
\begin{equation}
\frac{\partial S}{\partial \tau} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}
\end{equation}
with the ansatz
\begin{equation}
S = \frac{1}{2} \tau - E t + L \phi + S_r(r),
\end{equation}
where $S_r(r)$ is the function of $r$. Inserting the ansatz into (20), and with the help of the metric (1), we obtain
\begin{equation}
\left( \frac{dS_r(r)}{dr} \right)^2 = \frac{K^2 - \Delta_r(r^2 + (L \Sigma - aE)^2)}{\Delta_r^2},
\end{equation}
\begin{equation}
K = a^2 E + E r^2 - a L \Sigma.
\end{equation}
On the other hand, we have
\begin{equation}
\frac{dS_r(r)}{dr} = P_r = g_{rr} \dot{r} = \frac{\mu^2}{\lambda} \dot{r}.
\end{equation}
Thus, we obtain the square of the radial 4-velocity component
\begin{equation}
\left( \frac{dr}{d\tau} \right)^2 = \frac{K^2 - \Delta_r(r^2 + (L \Sigma - aE)^2)}{r^4}.
\end{equation}
Here, we have obtained all nonzero 4-velocity components for the equatorial motion geodesic. Next we would like to study the CM energy of the two-particle collision in the backgrounds of Kerr–(anti-)de Sitter black holes. We assume that the two particles have the angular momentum per unit mass $L_1, L_2$ and energy per unit mass $E_1, E_2$, respectively. For simplicity, the particles in consideration have the same rest mass $m_0$. The expression of the CM energy $E_{CM}$ of this two-particle collision is given by [1]
\begin{equation}
E_{CM} = \sqrt{2} m_0 \sqrt{1 - g_{\mu\nu} u^\mu_1 u^\nu_2},
\end{equation}
where \( u^\mu_1, u^\mu_2 \) are the 4-velocity vectors of the two particles \((u = (\dot{r}, \ddot{r}, 0, \dot{\phi}))\). With the help of equations (18), (19), and (25), we obtain the CM energy

\[
\frac{1}{2m_0^{CM}} \mathcal{E}^2_{CM} = \frac{1}{r^2 \Delta r} \left( (E_2 L_2 + E_1 L_1) \Sigma a (\Delta_r - a^2 - r^2) + L_1 L_2 \Sigma^2 (a^2 - \Delta_r) \right) + \frac{1}{r^2 \Delta r} (E_1 E_2 ((a^2 + r^2)^2 - a^2 \Delta_r) + r^2 \Delta r - H_1 H_2),
\]

(27)

where

\[
H_i = \sqrt{E_i^2 (a^2 + r^2)^2 - a^2 E_i^2 \Delta_r - r^2 \Delta_r - 2 a E_i L_i (a^2 + r^2 - \Delta_r) \Sigma + L_i^2 (a^2 - \Delta_r) \Sigma^2}
\]

\times (i = 1, 2).

(28)

For simplicity, we can rescale the CM energy as \( \bar{E}_{CM}^2 \equiv \frac{1}{\Sigma} \mathcal{E}^2_{CM} \). We would like to see \( \bar{E}_{CM}^2 \) when the particles collide on the horizon. So we have to make \( \Delta_r = 0 \) at equation (27). The denominator of \( \bar{E}_{CM}^2 \) is zero, and the numerator of it is

\[
K_i K_2 = \sqrt{K_i^2 K_2^2},
\]

(29)

\[
K_i = K |E = E_i, L = L_i|, \quad i = 1, 2.
\]

(30)

When \( K_i K_2 \geq 0 \), the numerator will be zero and the value of \( \bar{E}_{CM}^2 \) on the horizon will be undetermined; but when \( K_i K_2 < 0 \), the numerator will be a negative finite value and \( \bar{E}_{CM}^2 \) on the horizon will be negative infinity. So it should have \( K_i K_2 \geq 0 \), and for the CM energy on the horizon, we have to compute the limiting value of equation (27) as \( r \to r_h \), where \( r_h \) is related to the horizon of the black hole.

We can make \( \Delta_r = (r - r_h) (a_1 r^3 + a_2 r^2 + a_1 r + a_0) \) in equation (27). Then, we expand equation (27) at \( r_h \), which is the horizon under the consideration. When \( r = r_h \), the remaining term in the expansion of (27) is the zero-order term. In fact, the zero-order term is of the lowest order in the expansion of (27). So the limiting value of \( \bar{E}_{CM}^2 \) as \( r \to r_h \) is given by the zero-order term of (27) as

\[
\bar{E}_{CM}^2 (r \to r_h) = \frac{A}{2K_1 K_2},
\]

(31)

where

\[
A = \left[ (a^2 + r_h^2) (E_1 + E_2) - a \Sigma (L_1 + L_2) \right]^2 + r_h^2 \Sigma^2 (E_2 L_1 - E_1 L_2)^2.
\]

(32)

When \( K_1 = a^2 E_1 + E_1 r_h^2 - a L_1 \Sigma = 0 \), \( A \) will be

\[
A = \frac{(a^2 + r_h^2) K_2^2}{a^2}.
\]

(33)

So we can see that when \( K_1 = 0 \) and \( K_2 \neq 0 \), the CM energy on the horizon will blow up. We will call the angular momentum per unit mass that makes \( K_i = 0 \) the critical angular momentum \( L_{Ci} \), and \( L_{Ci} \) is given as

\[
L_{Ci} = \frac{(a^2 + r_h^2) E_i}{a \Sigma}, \quad i = 1, 2.
\]

(34)

We can also prove that when \( K_1 = 0 \) and \( K_2 = 0 \), the CM energy will not blow up. So if we need the CM energy to be arbitrarily high, one of the colliding particles must have the critical angular momentum and the other particle must not have the critical angular momentum.

We can see that the critical angular momentum depends on the horizon \( r_h \), and when we consider different horizons of the black hole, the critical angular momenta corresponding to
the horizons will be different. This result can reduce to the critical angular momentum for the case of the Kerr black hole when the cosmological constant becomes zero.

In order to get arbitrarily high CM energy on the horizon of the Kerr–AdS(dS) black hole, the colliding particle with the critical angular momentum must be able to reach the outer horizon of the black hole. We will study this part in next section.

4. The radial motion of the particle with the critical angular momentum near the outer horizon of the black hole.

In this section, we will study the conditions under which the particle with the critical angular momentum can reach the outer horizon of the black hole. In order for a particle to reach the horizon of the black hole, the square of the radial component of the 4-velocity \( \left( \frac{dr}{d\tau} \right)^2 \) in equation (25) has to be positive in the neighborhood outside of the black hole’s horizon.

We denote \( \left( \frac{dr}{d\tau} \right)^2 \) as \( R(r) \). Obviously, when the particle has the critical angular momentum, \( R(\text{r}_{\text{h}}) = 0 \) on the horizon of the Kerr–AdS or Kerr–dS black hole. So if the particle with the critical angular momentum can reach the horizon of the black hole, the derivative of \( R(r) \) with respect to \( r \) must be positive at the horizon \( r_{\text{h}} \), i.e.

\[
\left. \frac{dR(r)}{dr} \right|_{r=\text{r}_{\text{h}}} > 0. \tag{35}
\]

Before doing the computation, we would like to make a parameter replacement

\[
M = \frac{\left( r'_{\text{h}} \right)^2 + a^2 \left( 1 + \frac{l^2 r'_{\text{h}}^2}{l^2} \right)}{2(r'_{\text{h}})} \tag{36}
\]

for the Kerr–AdS black hole, and

\[
M = \frac{\left( r'_{\text{h}} \right)^2 + a^2 \left( 1 - \frac{l^2 r'_{\text{h}}^2}{l^2} \right)}{2(r'_{\text{h}})} \tag{37}
\]

for the Kerr–dS black hole. After this parameter replacement, \( r'_{\text{h}} \) will be the horizon of the black hole \( (\Delta r\big|_{r=\text{r}'_{\text{h}}} = 0) \). So we can start to discuss the black hole’s horizon \( r'_{\text{h}} \) and let \( r'_{\text{h}} \) be identified with \( r_{\text{h}} \). Thus, we will only write \( r_{\text{h}} \) in the following discussion. For the Kerr–dS black hole, \( r_{\text{h}} \) must not exceed \( l \) to avoid a negative \( M \). After this parameter replacement, computing \( \frac{dR(r)}{dr} \) at \( r = r_{\text{h}} \) for the particle with the critical angular momentum will give

\[
\left. \frac{dR(r)}{dr} \right|_{r=r_{\text{h}}} = W_{\text{AdS}} \cdot B \tag{38}
\]

for the Kerr–AdS black hole, and

\[
\left. \frac{dR(r)}{dr} \right|_{r=r_{\text{h}}} = W_{\text{dS}} \cdot B \tag{39}
\]

for the Kerr–dS black hole, where

\[
W_{\text{AdS}} = a^2 \left( l^2 - r_{\text{h}}^2 \right) - r_{\text{h}}^2 \left( l^2 + 3r_{\text{h}}^2 \right), \tag{40}
\]

\[
W_{\text{dS}} = a^2 \left( l^2 + r_{\text{h}}^2 \right) - r_{\text{h}}^2 \left( l^2 - 3r_{\text{h}}^2 \right), \tag{41}
\]

\[
B = \frac{a^2 + E^2}{a^2 r_{\text{h}}^2 l^2}. \tag{42}
\]
rh

region 1: W_{\text{AdS}} < 0

region 2: W_{\text{AdS}} > 0

boundary case: W_{\text{AdS}} = 0

Figure 1. The shape of rh(a^2, l^2) in (43) with l = 10^4. The point (a^2, rh) in region 2 will make \( \frac{dR(r)}{dr} \bigg|_{r=r_h} \) positive, which means the particle with critical angular momentum can reach the horizon rh.

Note that because B > 0, whether \( \frac{dR(r)}{dr} \bigg|_{r=r_h} \) is positive only depends on the sign of W_{\text{AdS}} or W_{\text{dS}}. Both W_{\text{dS}} and W_{\text{AdS}} only depend on the parameters of the black hole. Next, we will discuss the cases of the Kerr–AdS and Kerr–dS black holes, respectively.

4.1. The Kerr–AdS case

In the Kerr–AdS black hole case, by solving W_{\text{AdS}} = 0, we obtain

\[
r_h = \sqrt{-\frac{a^2}{6} - \frac{l^2}{6} + \frac{1}{6} \sqrt{a^4 + 14a^2l^2 + l^4}}.
\]  (43)

We draw the shape of rh(a^2, l^2) in (43) with l = 10^4 in figure 1, in which every point is related to a combination of the black hole horizon rh and the black hole spin a, and the point (a^2, rh) on the line means that the corresponding W_{\text{AdS}} is zero. Because W_{\text{AdS}} is a continuous function of rh and a^2, the different regions in figure 1 separated by the line (W_{\text{AdS}} = 0) relate to different signs of W_{\text{AdS}}. So we call that W_{\text{AdS}} = 0 is the boundary case. We can verify that above the line, rh and a^2 make W_{\text{AdS}} < 0; and below the line, rh and a^2 make W_{\text{AdS}} > 0.

When W_{\text{AdS}} > 0, the particle with critical angular momentum can reach the horizon rh. But we have to make sure that rh is the outer horizon of the black hole. Note that equation (43) is just the same as equation (8), which means rh in the boundary case of \( \frac{dR(r)}{dr} \bigg|_{r=r_h} = 0 \) is the extremal horizon x of the Kerr–AdS black hole. This means when the point (a^2, rh) is on the line in figure 1, the black hole is extremal. But when (a^2, rh) is off the line, is the black hole extremal or not? To answer that, we must make the parameter replacement (36) in the extremal horizon equations (6) and (7), and solve them for rh. Equation (43) must be one of the solutions, and this solution is related to the extremal horizon. Recall that the Kerr–AdS black hole can have two positive horizons at most. So if the black hole is extremal, these two horizons must coincide and the black hole will only have one horizon, namely rh. Thus, equation (43) is the only solution. This means that only if (a^2, rh) is on the line in figure 1, the Kerr–AdS black hole can be extremal.

So the line in figure 1 can also be regarded as the boundary case of the horizon situation of the black hole. This is because when we pick a point (a^2, rh) off the line in figure 1, if we find that rh is the inside horizon of the black hole, rh cannot turn into the outer horizon by crossing the other horizon or the number of the black hole horizons cannot change unless the
point crosses the line. In this case, when the point \((a^2, r_h)\) is above the line, \(r_h\) is the outer horizon and the black hole has two horizons; when \((a^2, r_h)\) is on the line, \(r_h\) is the extremal horizon and the black hole has only one extremal horizon; when \((a^2, r_h)\) is below the line, \(r_h\) is the inner horizon and the black hole has two horizons. This means that if we want the particle with critical angular momentum to reach the outer horizon of the Kerr–AdS black hole, the only chance is that the black hole is extremal. Thus, we must choose these points \((a^2, r_h)\) on the line in figure 1. But in this boundary case, \(\frac{dR(r)}{dr}|_{r=r_h}=0\) and we must calculate \(\frac{dR(r)}{dr}|_{r=r_h}\):

\[
\frac{d^2R(r)}{dr^2}|_{r=r_h} = G \cdot E^2 + J, \tag{44}
\]

where

\[
G = \frac{a^8 + 3a^6l^2 - 81a^4l^4 - 65a^2l^6 - 5l^8}{54a^2} + \frac{\sqrt{a^2 + 14a^2l^2 + l^4}(-a^6 + 6a^4l^2 + 30a^2l^4 + 5l^6)}{54a^2}, \tag{45}
\]

and

\[
J = \frac{1}{18} \left(-a^6 - a^4l^2 + 13a^2l^4 + l^6\right) + \frac{1}{18} \left(a^4 - 6a^2l^2 - l^4\right)\sqrt{a^2 + 14a^2l^2 + l^4}. \tag{46}
\]

In the above calculation we have already used equation (43). If \(\frac{dR(r)}{dr}|_{r=r_h}>0\), the particle with the critical angular momentum can reach the only horizon of the extremal Kerr–AdS black hole. It can be proved that if \(\frac{dR(r)}{dr}|_{r=r_h}=0\),

\[
0 < a^2 < (5 - 2\sqrt{5})l^2, \tag{47}
\]

\[
E > \sqrt{-\frac{J}{G}}. \tag{48}
\]

And if \(\frac{dR(r)}{dr}|_{r=r_h}=0\), we can prove that \(\frac{dR(r)}{dr}|_{r=r_h}\) must be negative. Note that the upper limit of the black hole spin \(a\) in equation (47) is still below \(l\).

Now we summarize the result for the case of the Kerr–AdS black hole and give a comparison to the case of the Kerr black hole. We find that, for a non-extremal Kerr–AdS black hole, the particle with the critical angular momentum cannot reach the outer horizon of the black hole, which is the same as with the case of the Kerr black hole. However, for an extremal Kerr–AdS black hole, if the additional conditions (47) and (48) are satisfied, the particle with the critical angular momentum can reach the outer horizon of the black hole. While, for an extremal Kerr black hole, this process can always occur.

4.2. The Kerr–dS case

Analogously to the Kerr–AdS case, we solve the boundary case \(W_{dS}=0\) and obtain

\[
r_{h_1} = \sqrt{\frac{a^2}{6} + \frac{l^2}{6} - \frac{1}{6} \sqrt{a^2 - 14a^2l^2 + l^4}}. \tag{49}
\]

\[
r_{h_2} = \sqrt{\frac{a^2}{6} + \frac{l^2}{6} + \frac{1}{6} \sqrt{a^2 - 14a^2l^2 + l^4}}. \tag{50}
\]

We draw these two boundary lines in figure 2 with \(l = 10^4\). We can see that these two boundary lines join together at \(a^2 = (7 - 4\sqrt{5})l^2\). So actually there is only one boundary line in
Figure 2. The shapes of $r_{h1}(a^2, l^2)$ in (49) (the below line) and $r_{h2}(a^2, l^2)$ in (50) (the above line) with $l = 10^4$. In region 2, $(a^2, r_h)$ can make $\frac{\partial W_{dS}}{\partial r_h} \bigg|_{r_h} > 0$, which means that when $(a^2, r_h)$ is in region 2, the particle with the critical angular momentum can reach the black hole horizon $r_h$. In contrast, when $(a^2, r_h)$ is in region 1, the particle with the critical angular momentum cannot reach the black hole horizon $r_h$.

Figure 3. Four boundary lines and the two axes separate the plane into four different regions. The horizontal line at the top refers to the upper limit of $r_h$.

Like the Kerr–AdS case, we verify that inside the boundary line, $(a^2, r_h)$ makes $W_{dS} < 0$; and outside the boundary line, $(a^2, r_h)$ makes $W_{dS} > 0$.

When $W_{dS} > 0$, we still have to make sure that $r_h$ is the outer horizon of the black hole. Note that equations (49) and (50) are the same as equations (11) and (12). This means that the boundary line in figure 2 is also the boundary line of the horizon situation of the black hole. But unlike the Kerr–AdS case, the Kerr–dS black hole can have three positive horizons at most and this means there are other boundary lines of the horizon situation of the black hole. To find them, we make the parameter replacement (37) in equations (9) and (10), and solve it for $r_h$. Obviously, equations (49) and (50) are two solutions. And there are two other solutions which relate to the situations that the black hole is extremal but $r_h$ is not the extremal horizon. We draw all these boundary lines of the horizon situation of the black hole in figure 3 with $l = 10^4$. 


Figure 4. The different types of the horizon situation of the Kerr–dS black hole. The coordinates in this figure have no meaning. $A_1$–$A_9$ are related to the points $A_1$–$A_9$ in figure 3.

To see the effect of the boundary lines, we draw a vertical line crossing all the boundary lines in figure 3 and we let $(a^2, r_h)$ moving along this line to see the change of the horizon situation of the black hole. On this vertical line, we choose one point in each different region (denoted by $A_1$–$A_9$), and for each point, we draw the horizon situation of the black hole in figure 4.

Comparing figures 2, 3 and 4, we can find that when $(a^2, r_h)$ is in region 1, region 2 or on line 1 in figure 3, the particle with critical angular momentum can reach the outer horizon of the Kerr–dS black hole. When $(a^2, r_h)$ is on line 2, $r_h$ is the outer extremal horizon and \( \frac{d^2R}{dr^2} \bigg|_{r=r_h} = 0 \). By computing \( \frac{d^2R}{dr^2} \bigg|_{r=r_h} \), it can be proved that \( \frac{d^2R}{dr^2} \bigg|_{r=r_h} \) is positive. So when $(a^2, r_h)$ is on line 2, the particle with critical angular momentum can reach the outer horizon of the Kerr–dS black hole.

As a summary, we find that as long as $r_h$ is the outer horizon of the Kerr–dS black hole, the particle with critical angular momentum can always reach the horizon $r_h$. This means the particle with critical angular momentum can always reach the outer horizon of the Kerr–dS black hole without constraints coming from the geodesic motion of the particle. This is very different from the case of Kerr and Kerr–AdS black holes.

4.3. From the Kerr–AdS case to the Kerr–dS case

Here, we analyze in detail that how the Kerr–AdS (Kerr–dS) case changes into the Kerr case when the cosmological constant changes from negative to positive.

When the cosmological constant turns from negative to zero, equation (43) becomes

\[
r_h = a.
\]  

(51)
We denote $W_{\text{Kerr}} \equiv W_{\text{AdS}}|_{\Lambda \to 0}$. We draw $r_h = a$ in figure 5. So we can see that $r_h = a$ can serve as a boundary line. Thus, the particle with critical angular momentum can reach Kerr black hole’s outer horizon only if the Kerr black hole is extremal.

When the cosmological constant turns from positive to zero, the right part of equation (49) also changes into $a$ and the right part of equation (50) becomes positive infinite.

Recalling figures 1, 2 and 5, we find that the curved boundary line in figure 1 will become the straight boundary line in figure 5 when the cosmological constant turns from negative to zero. And the straight boundary line in figure 5 will bend back in figure 2. So it can surround region 1 in figure 2 completely when the cosmological constant turns from zero to positive.

In this way, region 1 in figure 2 where $W_{\text{dS}} < 0$ can be bounded and outside the boundary, $W_{\text{dS}} > 0$ and $r_h$ can still be the outer horizon. This is why particles with critical angular momentum can reach the outer horizon of the Kerr–dS black hole without requiring the black hole to be extremal.

In fact, $\frac{dR}{dr}|_{r=r_h}$ can be rewritten as

$$\frac{dR}{dr}|_{r=r_h} = \frac{(a^2 + E^2 r_h^2)}{a^2 r_h^2} \left( \frac{d\Delta_r}{dr} \right)_{r=r_h},$$

and when $\frac{dE}{dr}|_{r=r_h} = 0$,

$$\frac{d^2R}{dr^2}|_{r=r_h} = \left( \frac{8}{r_h^2} - \frac{1}{a^2} \frac{d^2\Delta_r}{dr^2} |_{r=r_h} \right) E^2 - \frac{1}{r_h^2} \frac{d^2\Delta_r}{dr^2} |_{r=r_h}.$$  \hspace{1cm} (53)

For the Kerr–AdS or the Kerr black hole, on the outer non-extremal horizon, $\frac{d\Delta_r}{dr} > 0$; and on the extremal horizon, $\frac{d\Delta_r}{dr} = 0$ and $\frac{d^2\Delta_r}{dr^2} > 0$. So as $- (a^2 + E^2 r_h^2) < 0$, $\frac{dE}{dr}$ cannot be positive on the outer non-extremal horizon. Thus, the outer horizon has to be extremal, and as $\frac{dE}{dr}$ must be positive on the extremal horizon, from equation (53), we can see that the parameters of the black hole and the particle must be confined. In fact, we can still obtain (47) and (48) in this way.
For the Kerr–dS black hole, on the outer non-extremal horizon, $\frac{dA}{dr} < 0$; and on the outer extremal horizon, $\frac{dA}{dr} = 0$ and $\frac{d^2A}{dr^2} < 0$. So from equations (52) and (53), we can see that on the outer non-extremal horizon, $\frac{dR}{dr}$ is positive; and on the outer extremal horizon, $\frac{dR}{dr} = 0$ and $\frac{d^2R}{dr^2} > 0$. Thus, the Kerr–dS black hole need not be extremal and there is no additional condition needed.

From the above analysis, we know why the Kerr–AdS and Kerr cases are similar and why the Kerr–dS case is so different.

5. Conclusion

In this work, we have analyzed the possibility that Kerr–(anti-)de Sitter black holes could act as particle accelerators. We find that the result is different from the case of Kerr black holes because of the non-vanishing cosmological constant in the background spacetime. In order for two particles to collide in the outer horizon of the Kerr, Kerr–AdS, or Kerr–dS black holes and to reach arbitrary high CM energy, one and only one of the colliding particles should have a critical angular momentum. Besides, for the case of the Kerr black hole, it has to be extremal. For the Kerr–AdS one, it has to be extremal, and an additional condition should be satisfied. However, for the case of the Kerr–dS black hole, it does not need to be extremal and no additional condition need to be satisfied. Hence, non-extremal Kerr-de Sitter black holes could also act as particle accelerators with arbitrarily high CM energy, which is very different from the cases of the Kerr and Kerr–AdS black holes. By analyzing how the Kerr–AdS (Kerr–dS) case changes into the Kerr case when the cosmological constant vanishes, we have seen exactly why the Kerr–dS case is so different.

Acknowledgments

YL is grateful to Dr Pujian Mao for the valuable discussion. This work was supported by the National Natural Science Foundation of China (no 11075065), the Doctoral Program Foundation of Institutions of Higher Education of China (no 20070730055), the Fundamental Research Funds for the Central Universities (lzujbky-2010-171) and the Fundamental Research Fund for Physics and Mathematic of Lanzhou University (LZULL200907).

References

[1] Banados M, Silk J and West S M 2009 Phys. Rev. Lett. 103 111102
[2] Berti E, Cardoso V, Gualtieri L, Pretorius F and Sperhake U 2009 Phys. Rev. Lett. 103 239001
[3] Jacobson T and Sotiriou T P 2010 Phys. Rev. Lett. 104 021101
[4] Grib A A and Pavlov Y V 2011 Astropart. Phys. 34 581
[5] Lake K 2010 Phys. Rev. Lett. 104 211102
[6] Lake K 2010 Phys. Rev. Lett. 104 259903 (erratum)
[7] Grib A A and Pavlov Y V 2010 JETP Lett. 92 125
[8] Wei S-W, Liu Y-X, Guo H and Fu C-E 2010 Phys. Rev. D 82 103005
[9] Grib A A and Pavlov Y V 2010 On particle collisions near Kerr’s black holes arXiv:1007.3222 [gr-qc]
[10] Zaslavskii O B 2010 JETP Lett. D 92 571
[11] Wei S-W, Liu Y-X, Li H-T and Chen F-W 2010 J. High Energy Phys. JHEP12(2010)066
[12] Mao P-J, Li R, Jia L-Y and Ren J-R 2010 Acceleration of particles in Einstein–Maxwell–Dilaton black hole arXiv:1008.2660 [hep-th]
[13] Patil M and Joshi P S 2010 Phys. Rev. D 82 104049
[16] Zaslavskii O B 2010 Phys. Rev. D 82 083004
[17] Zaslavskii O B 2011 Class. Quantum Grav. 28 105010
[18] Zhu Y, Wu S-F, Liu Y-X and Jiang Y 2011 Phys. Rev. D 84 043006
[19] Kimura M, Nakao K and Tagoshi H 2011 Phys. Rev. D 83 044013
[20] Hackmann E and Lämmerzahl C 2010 Phys. Rev. D 81 044020
[21] Hawking S W and Reall H S 1999 Phys. Rev. D 61 024014