IRREDUCIBILITY OF ENUMERABLE BETTING-STRATEGIES

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Abstract. We study the problem of whether a betting strategy can be reduced to a set of restricted betting strategies, such as betting on a restricted set of stages or bet on a restricted of favorable outcomes. We show that the class of effectively enumerable betting strategies has irreducible members (which cannot be reduced to a set of restricted betting strategies). We answer questions of Kastermans and Hitchcock by constructing a real on which no kastergale (left-c.e. supermartingale with effectively determined favorable outcomes) succeeds, but some general left-c.e. supermartingale succeeds on it. We also generalize Muchnik’s paradox by showing that there is a non-Martin-Löf -random real such that no muchgale (left-c.e. supermartingale betting on restricted stages) succeeds on it. Our methodology is general enough to strongly support the metaconjecture that if a natural class of left-c.e. supermartingales defines Martin-Löf randomness, then a single member of that class can do so; thus, the class of left-c.e. supermartingales cannot be reduced to a simpler, natural subclass of it. For example, we show that there is a non-Martin-Löf -random real such that no kastergale or muchgale succeeds on it.

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1. Introduction

In betting games, a player is repeatedly allocates capital on an array of options, at the risk of loss or the possibility of gain, depending on the outcomes of an unpredictable process. Such games have proved to be very useful models of a variety of problems, and an indispensable tool for understanding the nature of information that has its roots in frequentist foundation for probability by von Mises [1919, 1957]. A betting strategy for the player has two components: (a) the favored outcome at each bet; and (b) the amount of capital, the wager that is placed on the preferred outcome, which reflects the odds of the bet with respect to the information that is available to the player. For simplicity we restrict our attention to the generic case of binary outcomes, where the wager is doubled or lost depending whether the actual outcome was favored or not. Typical restricted betting strategies include the self-explanatory fixed-wager, the martingale where each wager is twice the previous one, and its variations, and proportional betting in which wagers are set as a fixed proportion of the existing capital. A prominent example of the latter is the ubiquitous investment strategy known as the Kelly criterion, which maximizes the expectation of the logarithm of the capital, and was discovered by Kelly [1956] as an interpretation of Shannon’s concept of information rate.

The success of a betting strategy may lie on its capital allocation, its dynamic choice of favorable outcomes, a strategic choice of the stages where a bet is placed, or a combination of the above. In order to examine the dependence of a betting strategy on the above components, we can ask if it can be reduced \footnote{Here “reduce” means that it succeeds on a real } to a set of restricted betting strategies, which do not alternate favorable outcomes, or restrict their wagers on a specific subset of the stages:
(a) can it be reduced to fixed-outcome betting strategies?
(b) can it be reduced to betting strategies that bet on alternate stages?

Property (a) indicates that the success is not based on strategic alternations of favorable outcomes, but on its dynamic capital allocation. The possibility of such betting strategy reduction is the topic of the present work; the answer depends on the information that is available in the game. We show that there are enumerable betting strategies that are not reducible to restricted enumerable betting strategies

where restricted will be made precise and includes the above examples of fixed-outcome and stage-restricted strategies. In this way, we answer a question of Kastermans and a question of Hitchcock, as reported by Downey and Hirschfeldt [2010, §7.9] and Downey [2012], as well as extending work of Muchnik [2009].

**Kasterman’s question.** Following Ville [1939] we formalize betting strategies in terms of *martingales*: non-negative functions \( M \) on \( 2^{<\omega} \) such that

\[
2 \cdot M(\sigma) = M(\sigma \ast 0) + M(\sigma \ast 1)
\]

for all \( \sigma \in 2^{<\omega} \)

where \( \omega \) is the set of natural numbers, \( 2^{<\omega} \) is the set of binary strings and \( \ast \) denotes concatenation. Martingales model the capital of a player who plays against a casino, by placing wagers on binary outcomes. If the equality in (1) is replaced with \( \geq \) we get the notion of *supermartingale*, which often model inflation environments as well as savings. In terms of feasibility, we also need to qualify the betting strategies from a constructive point of view: a standard choice advocated by Schnorr [1971] is to consider computable betting strategies, namely martingales that can be fully defined by a Turing machine, and their effective countable mixtures. The latter can be equivalently defined as the *left-c.e. (left-computably-enumerable) martingales, supermartingales*, which means that there exist computable array \( (M[t] : t \in \omega) \) of (super)martingales such that \( M[t] \) dominates \( M[t - 1] \) and their point-wise limit is \( M \). Left-c.e. supermartingales represent games where betting decisions are not fully transparent to a constructive observer, being infinitary and revealed only gradually, through monotone approximations that converge to the limit wagers. We say that a (super)martingale \( M \) succeeds on an infinite binary sequence \( x \) (also called a *real*) if it is unbounded on the initial segments of \( x \). In this case we may say that \( x \) is predictable with respect to \( M \). Left-c.e. martingales and supermartingales are indispensable tools in the study of algorithmic information: \( x \) is algorithmically random in the sense of Martin-Löf [1966], also known as *Martin-Löf-random* or *1-random*, iff no left-c.e. (super)martingale succeeds on it.

We formally define fixed-outcome betting strategies and their generalizations.

**Definition 1.1** (Sided betting strategies). Let \( i < 2 \), \( M : 2^{<\omega} \to \mathbb{R} \) and let \( p \) be a partial boolean function on binary strings. We say that \( M \) is *i-sided at \( \sigma \)* if \( M(\sigma \ast i) \geq M(\sigma \ast (1 - i)) \). We say that \( M \) is *p-sided* if it is \( p(\sigma) \)-sided on each \( \sigma \) in the domain of \( p \), and \( M(\sigma \ast i) = M(\sigma \ast (1 - i)) \) otherwise. For a p-sided function \( M \), we say \( M \) is

- *single-sided* if \( p \) is a boolean constant;
• partially-computably-sided if $p$ is partial computable.

The interplay between the flow of information in the game and the strength of a betting strategy has been the core of several problems in algorithmic information, most notably the question of whether computable non-monotonic betting can simulate left-c.e. supermartingales, which has its roots in the study of non-monotonic selection rules by Kolmogorov [1998] and Loveland [1966] and remains open to this day despite numerous attempts by Merkle et al. [2006], Bienvenu et al. [2009] among others. In this spirit, Kastermans asked the following generalized version of (a) for left-c.e. supermartingales:

\[ Kastermans’ question: \text{given a left-c.e. supermartingale } M \text{ which} \]
\[ \text{succeeds on } x, \text{does there exist a partially-computably-sided } M’ \]
\[ \text{which succeeds on } x? \]

Hitchcock asked if the above reduction holds in the special case where the wagers the ratios $M(\sigma * i)/M(\sigma)$ are uniformly left-c.e., which means that a commitment of a betting strategy to bet a certain proportion of the capital on some outcome at some stage of the approximation is irreversible, and unaffected by any potential future increase of the existing capital. This subclass of the left-c.e. supermartingales is known as hitchgales, while the partially-computably-sided left-c.e. supermartingales are known as kastergales. These problems were reported by Downey and Hirschfeldt [2010, §7.9] and Downey [2012].

1.1. Our contribution. Our main technical result is a negative answer to the questions of Kastermans and Hitchcock, along with several extensions which give negative answers to questions like (a), (b) about reducibility into restricted betting strategies, for the case of left-c.e. supermartingales. We reach the solution to (2) gradually through a systematic analysis of the expectation and variance of supermartingales in a game-theoretic framework, which differentiates our approach to previous attempts that are detailed below. Our main innovation is developed in §2 where we prove the irreducibility to the fixed-outcome betting strategies:

**Theorem 1.2** (Fixed-outcome). There exists a real $x$ such that no single-sided left-c.e. supermartingale succeeds on $x$, but $x$ is not 1-random, so some left-c.e. supermartingale succeeds on $x$.

In §3 we obtain a negative answer to the questions of Kastermans and Hitchcock.

**Theorem 1.3.** There exists a real $x$ such that no kastergale succeeds on $x$, but $x$ is not 1-random, so some left-c.e. supermartingale succeeds on it.

We demonstrate the generality of our methodology in §4, by generalizing a result of Muchnik [2009] that is known as Muchnik’s paradox and has been discussed by Chernov et al. [2008] and Bauwens [2014] in the context of online complexity and prediction. Consider the following restriction of left-c.e. supermartingales, where they are required to restrain their betting to certain partitions of the stages.

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2 Clearly, for every left-c.e. partially-computably-sided supermartingale $M$, there is a computable array of supermartingales $(M[t] : t \in \omega)$ and a partial computable function $p : \subseteq 2^{\omega} \rightarrow 2$ such that $\lim_{t \rightarrow \omega} M[t](\sigma) = M(\sigma)$ for all $\sigma \in 2^{\omega}$ and $M[t]$ is $p[t]$-sided where $p[t]$ is the approximation of $p$ at time $t$, i.e., for each $\sigma \in 2^{\omega}$, $M$ has only one chance to decide its sidedness at $\sigma$ and before it makes that decision, it has to be both 0, 1-sided at $\sigma$. 

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**Definition 1.4 (muchgale).** Given \( i < l \), we say that \( M \) is \((l, i)\)-betting if
\[
|\sigma| \equiv i \mod l \quad \Rightarrow \quad M(\sigma) \geq \max_{j < 2} M(\sigma \ast j).
\]
We refer to \((l, i)\)-betting left-c.e. supermartingales as muchgales.

Muchnik [2009] considered the \((2, i)\)-betting left-c.e. supermartingales and showed that there exists a real \( x \) such that no \((2, i)\)-betting left-c.e. supermartingale succeeds on it, but \( x \) is not 1-random. In §4 we generalize this result and show that some reals are unpredictable with respect to both all muchgales and all kastergales, yet they are not 1-random:

**Theorem 1.5.** There exists a real \( x \) such that no kastergale or muchgale succeeds on \( x \), but \( x \) is not 1-random, so some left-c.e. supermartingale succeeds on it.

Finally we put forth and give strong evidence toward the following:

**Conjecture:** if a natural subclass of left-c.e. supermartingales defines 1-randomness, then a single member of that class can do so.

The definition of “natural” will be given in section 4. The kastergale, hitchgales as well as muchgales are examples of natural subclasses of left-c.e. supermartingales. A consequence of (3) is that left-c.e. supermartingales cannot be reduced to restricted supermartingales; that is: no simpler natural subclass of left-c.e. supermartingales defines 1-randomness.

### 1.2. Previous work.
Our irreducibility results are interesting because they only occur in the context of enumerable strategies. Indeed, every computable martingale is the product of:

- two computable fixed-outcome martingales [Barmpalias et al., 2020, §3]
- two computable martingales that bet on alternate stages [Muchnik, 2009, §2]

so in a fully constructive environment, our initial questions (a), (b) have positive answers. Muchnik [2009] was the first to demonstrate a negative answer, by showing that there exists a real \( x \) which is unpredictable with respect to left-c.e. supermartingales that only bet on even stages and those which only bet on odd stages, but is not 1-random, so some left-c.e. supermartingale succeeds on it. Hence such restricted betting strategies do not characterize 1-randomness. Bauwens [2014] provides expressions of Muchnik's paradox in terms of online complexity.

Barmpalias et al. [2020] examined the case of single-sided strategies, which turned out to be more powerful than muchgales. Using different methods, they obtained a weak analogue of Theorem 1.2, which is restricted to effective countable mixtures of computable martingales; this is a subclass of left-c.e. martingales that corresponds to the condition that the wagers are left-c.e. in a uniform way. The dependence of their method to the martingale condition and the enumerability of the wagers is discussed by Barmpalias et al. [2020, §5] where the need for a new method toward an answer to Kasterman’s question is anticipated. Barmpalias and Liu [2022] use different methods to deal with similar questions in terms of fractal dimension.

The problem of restricted betting has also been studied by Schweinsberg [2005] in a purely probabilistic fashion, in the context of red and black games of Dubins and Savage.
A distinguished characteristic of our line of work with the probabilistic tradition is the enumerability of the strategies, which corresponds to imperfect information during the game. While variance analysis is a standard tool in probability, our methodology is novel in the study of effective strategies as all of the previous works relied on recursion-theoretic methods.

1.3. Organization. The rest of this paper is devoted to proving Theorems 1.2 - 1.5. Theorems 1.2, 1.3, 1.5 are proved in §2, §3, §4 respectively. In §4, we demonstrate strong evidence toward (3). The proof of Theorem 1.2 in §2 already includes the bulk of our method, which is then used for Theorems 1.2 - 1.5. Many lemmas, claims and definitions in §3, are analogues of those in §2. We repeat them in order to make it convenient for readers to check the proof.

1.4. Notation. We maintain some consistency in our notation, reserving variables $i, j, k, n, m, t$, for non-negative integers, $\sigma, \tau, \rho$ for finite strings, $x, y, z$ for reals (infinitely long binary sequence), and $A, B$ for sets of reals or strings.

For $\sigma, \rho \in 2^{<\omega}$:
- $\rho \preceq \sigma, \sigma \succeq \rho$ denote that $\rho$ is a prefix of $\sigma$;
- let $[\sigma] \equiv \{x \in 2^{\omega} : x \preceq \sigma\}$ be the set of binary strings extending $\sigma$; for a set $A \subseteq 2^{\omega}$, let $[A] \equiv \cup_{\sigma \in A} [\sigma]$;
- let $[\sigma] := \{x \in 2^{\omega} : x \preceq \sigma\}$ and $[A] = \bigcup_{\sigma \in A} [\sigma]$;
- let $\sigma \upharpoonright n$ denote the string $\sigma(0) \cdots \sigma(n-1)$; let $\sigma^n$ denote the string $\sigma^n(0) \cdots \sigma^n(n)$;
- let $\emptyset$ also denote the empty string.

For a set $A \subseteq 2^{<\omega}$, the measure of $A$, denoted as $\mu(A)$, refers to the Lebesgue measure of $[A]$; for $\sigma \in 2^{<\omega}$, we write $\mu(\sigma)$ for $\mu(\{\sigma\})$; we write $\mu(A|B)$ for $\mu(A \cap B)/\mu(B)$. For a vector $v \in \mathbb{R}^k$, we use $\|v\|_p$ to denote the $L^p$-norm of $v$.

For functions $M, \tilde{M}$,
- we say $\tilde{M}$ dominates $M$ on a set $A$ if $\tilde{M}(\sigma) \geq M(\sigma)$ for all $\sigma \in A$;
- we denote the average of $M$ on a set $A$ of strings by:

$$\int_{\sigma \in A} M(\sigma) := \int_A M := \sum_{\sigma \in A} M(\sigma) \cdot \mu(\sigma).$$

2. DEFEAT SINGLE-SIDED SUPERMARTINGALES

As many assertions in computability theory, Theorem 1.2 can be seen as a game between two players (hence called Alice and Baby) where Alice controls a Martin-Löf test and Baby controls infinitely many i-sided left-c.e. supermartingales. It is usually very helpful to study the finite version of such games. In the game, Alice tries to provide (enumerate) a $\sigma^*$ so that $\sum_{i < \omega} M_i(\tilde{\sigma})$ does not reach certain threshold for all $\tilde{\sigma} \preceq \sigma^*$ (where $M_i$ is the left-c.e. i-sided supermartingale controlled by Baby). This gives rise to the following finite game. Let $0 \leq c \leq d, n \in \omega$.

**Definition 2.1** $(c, d, n)$-sided-game. At each round $t \in \omega$, Alice firstly enumerates a $\sigma^* \in 2^n$ (that has not been enumerated before); then Baby presents $i$-sided supermartingale $M_i[r]$ (for $i < 2$) such that the following hold: let $M[r] := (M_0[r], M_1[r])$,

- $\|M[r](\tilde{\sigma})\|_1 \geq d$ for some $\tilde{\sigma} \preceq \sigma$;
- $M_i[r]$ dominates $M_i[r - 1]$ on $2^{<n}$ (we set $M_i[-1] \equiv 0$).
Alice wins the game if for some $t$, $||M[t](\emptyset)||_1 \geq c$.

We call $\hat{\sigma}$ a catching-point of $\sigma$. Let $A$ be the set of $\sigma$ that Alice enumerated during the game 2.1. We show that given $c < d$, $\varepsilon > 0$, Alice can win the $(c,d,n)$-sided-game 2.1 with a cost $\mu(A) \leq \varepsilon$ (for some $n$). By scaling,

- $(c,d,n)$-sided-game is equivalent to $(c\hat{c},d\hat{c},n)$-sided-game (for all $\hat{c} > 0$).

When $d = 1$ the $(c,d,n)$-sided-game is called $(c,n)$-sided-game. We prove

**Lemma 2.2.** For every $c < 1, \varepsilon > 0$, there is an $n \in \omega$ such that Alice has a winning strategy for the $(c,n)$-sided-game 2.1 such that $\mu(A) \leq \varepsilon$.

We emphasize that there is no complexity notion involved in game 2.1 or Lemma 2.2. The proof of Lemma 2.2 consists of several separate ideas listed below (which will consist most of the ingredients needed for Theorem 1.3). The first three ingredients reduce Lemma 2.2 to a weaker assertion (these three ingredients reduce the winning strategy of game 2.1 to a series of other games with each being easier for Alice to win than the previous one).

1. **Nesting argument.** This is similar as how we apply Lemma 2.2 iteratively to prove Theorem 1.2. This, roughly speaking, allows Alice to enumerate much more $\sigma$ in trade of a reasonably stronger goal (i.e., $c$ is increased reasonably while $\mu(A)$ is allowed to be close to 1). The nesting argument reduce Lemma 2.2 to Lemma 2.5.

2. **Dynamic goal argument.** A dynamic goal version of game 2.1 has the same rule as game 2.1 but a different winning criterion (of Alice). The dynamic winning criterion allows Alice to achieve a dynamic goal depending on how much resource she cost, namely $\mu(A)$ (the more she cost the better goal she is supposed to achieve). The argument shows that a winning strategy for the dynamic goal version of game 2.1 (namely game 2.6) gives rise to a winning strategy of game 2.1. Thus Lemma 2.5 is subsequently reduced to Lemma 2.9.

3. **Imposing restriction on the sided player.** The dynamic goal game (game 2.6) will allow us to impose further restrictions on Baby’s action: $||M[t](\sigma)||_1 \geq 1$ (instead of $||M[t](\hat{\sigma})||_1 \geq 1$ for some $\hat{\sigma} \leq \sigma$) and $||M[t](\rho)||_1 \leq 1 + \delta$ for all $\rho \in 2^{\Sigma n}$ (where $\delta$ is a pre-chosen small parameter). These restrictions force that the more capital Baby allocates, the more $M_1[t]$ is alike a martingale. Thus we reduce game 2.6 to its restricted version (game 2.12) and the winning strategy of game 2.6, namely Lemma 2.9, is reduced to Lemma 2.15.

4. A winning strategy for the restricted dynamic game 2.12 (proof of Lemma 2.15). The following diagram illustrates the framework of the proof $^5$.\[\begin{array}{ll}
\text{Lemma 2.2} & \iff_{2.4} \text{Lemma 2.5 (more resource allowed, §2.1)}; \\
\text{Lemma 2.5} & \iff_{2.7} \text{Lemma 2.9 (dynamic goal, §2.2)}; \\
\text{Lemma 2.9} & \iff_{2.13} \text{Lemma 2.15 (imposing restrictions on Baby, §2.3)}; \\
\text{Proof of Lemma 2.15} & \text{(section 2.4)}. \\
\end{array}\]

$^5$Where $\phi \iff \psi$ means assertion $\phi$ reduce assertion $\psi$ to $\phi$, i.e., $\phi \land \psi$ implies $\psi$. 

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Notably, other than the proof of Lemma 2.15, none of these arguments take advantage of \(i\)-sidedness but the mere fact that \(M_t[t]\) is supermartingale and non-decreasing with respect to \(t\). The rest of §2 is devoted to prove Lemma 2.2. Before that, we prove Theorem 1.2.

Proof of Theorem 1.2 using Lemma 2.2. Note that given \(i\)-sided left-c.e. supermartingale \(M_t\), let \(M = (M_0, M_1)\), if \(\|M(\emptyset)\|_1 < c\) and \(c < d\), then for some \(n \in \omega\), we can (effectively) play the winning strategy for the \((c, d, n)\)-sided-game 2.1 on \(2^n\) (against \(M_0, M_1\)) to computably enumerate a sequence \(V = (\sigma[0], \ldots, \sigma[s-1])\) of strings (with \(\mu(V) \leq 1/2\)) such that let \(\sigma^* = \sigma[s-1]\), we have \(\|M(\hat{\sigma})\|_1 \leq d\) for all \(\hat{\sigma} \leq \sigma^*\). This is clearly done by taking \(M_t[t]\) as the \(i\)-sided supermartingale presented by Baby. We refer to this process as controlling \(M_0, M_1\) on \(2^{[0,n]}\) and \(\sigma^*\) as a non-catchable-string. Roughly speaking, the non-1-random real \(x\) is produced as follows.

1. On \(2^{n_0}\), play the winning strategy for sided-game 2.1 to control \(M_0, M_1\) on \(2^{[0,n_0]}\).
2. Upon obtaining \(\sigma^*_0\), play the winning strategy for the sided-game 2.1 (on \([\sigma^*_0]^{c, d}\) \(2^{n_0+n_1}\) to control \(M_0 + M_1, M_1 + M_1\) on \(2^{[0,n_1]}\) and so forth. Where \(M_t\) is (some scaling) of another \(i\)-sided left-c.e. supermartingale.

In the end, the non-1-random \(x\) will be \(\cup_k \sigma^*_k\). Of course, when the winning strategy tells us to enumerate \(\sigma \in 2^n\), we have no idea whether \(\sigma\) is a non-catchable-string. We will simply take \(\sigma\) as if it is (unless found otherwise later) and keep doing the above process (item (2)) over \(\sigma\). If at some point it is found that \(\sigma\) is not a non-catchable-string, then go back to the sided-game 2.1 on \(2^{n_0}\) (give up whatever is done over \(\sigma\)), obtain (invoking the winning strategy) the next \(\hat{\sigma} \in 2^n\) and proceed with \(\sigma\). The more concrete argument goes as follows.

Let \(n_0, n_1, \ldots, 0 < c_0 < d_0 < c_1 < \cdots < 2\) be computable sequences such that:

- there is a winning strategy of Alice for the \((c_k, d_k, n_k)\)-sided-game
  such that \(\mu(A) \leq 1/2\) for all \(k\).

Let \(M_{i,0}, M_{i,1}, \cdots\) be an effective list of all \(i\)-sided left-c.e. supermartingale; let \(M_k = (M_{0,k}, M_{1,k})\). For convenience, suppose

\[\|M_k(\emptyset)\|_1 \leq \delta_k\]

for all \(k \in \omega, i \in 2\),

where \(\delta_k \in Q : k \in \omega\) is a computable sequence so that each is sufficiently small (to be specified).

We will define a sequence \((V_k \subseteq 2^{n_0+\cdots+n_k} : k \in \omega)\) of uniformly c.e. sets with \(\mu(V_k) \leq 2^{-k}\); and the non-1-random real \(x\) will be (the unique) element of \(\cap_k V_k\).

To make sure that no \(i\)-sided supermartingale succeeds on \(x\), we satisfy the requirement:

\[\sum_{k \in \omega} \|M_k(x \upharpoonright m)\|_1 < 2\]

for all \(m \in \omega\).

Construction.

1. On \(2^{n_0}\), play the winning strategy for the \((c_0, d_0, n_0)\)-sided-game 2.1 to control \(M_0\) (namely \(\|M_0(\hat{\sigma})\|_1 \leq d_0\) for all \(\hat{\sigma} \leq \sigma^*\)). The set \(V_0\) consists of the strings enumerated during this game.
2. When the strategy tells us to enumerate \(\sigma\), take \(\sigma\) as if it is a non-catchable-string. Then, on \([\sigma]^{c, d, n_1}\), play the winning strategy for the \((c_1, d_1, n_1)\)-sided-game 2.1 to control \(\sum_{k < 2} M_k\) (namely \(\sum_{k < 2} \|M_k(\hat{\sigma})\|_1 \leq d_1\) for all \(\hat{\sigma} \leq \sigma^*\)).
\[ \sigma \preceq \sigma^* \). The set \( V_1 \) consists of strings in \( 2^{n_0+n_1} \) enumerated during the above game.

3. If at some point it is found that \( \sigma \in 2^{n_0} \) is not a non-catchable-string, then go back to the game on \( 2^{n_0} \), obtain the next string \( \tilde{\sigma} \) by the winning strategy and proceed with \( \tilde{\sigma} \) (instead of \( \sigma \)) as in step (2).

4. Similarly for \( V_2, V_3, \ldots \).

Note that if \( \sigma_0^* \) is a non-catchable-string on \( 2^{n_0} \), then it follows that \( |M_k(\sigma_0^*)|_1 \leq d_0 \). Since \( \delta_1 \) is sufficiently small, we have

\[ \sum_{k<2} |M_k(\sigma_0^*)|_1 < c_1. \]

Thus the game on \( [\sigma_0^*] \preceq 2^{n_0+n_1} \) will produce a \( \sigma_1^* \in [\sigma_0^*] \preceq 2^{n_0+n_1} \) such that

\[ \sum_{k<2} |M_k(\sigma)|_1 \leq d_1 \]

for all \( \sigma_0^* \preceq \sigma \preceq \sigma_1^* \). In other words, each \( \sigma_k^* \) exist, verifying (4). On the other hand, it’s clear that \( \mu(V_k) \leq 2^{-k} \). \( \square \)

2.1. Nesting. First we notice a nesting property of the \((c, n)\)-sided-game 2.1.

Claim 2.3. Suppose Alice has a winning strategy for \((c_j, n_j)\)-sided-game 2.1 such that \( \mu(A) \leq \epsilon_j \) for each \( j < 2 \). Then Alice has a winning strategy for \((c_0c_1, n_0+n_1)\)-sided-game 2.1 such that \( \mu(A) \leq \epsilon_0\epsilon_1 \).

Proof. In order to win the \((c_0c_1, n_0+n_1)\)-sided-game we nest \((c_1, n_1)\)-strategy into the \((c_0c_1, c_1, n_0)\)-strategy.\(^6\) For each \( \rho \in 2^n \) let \((c_1, n_1)\)-sided-game denote the shift of \((c_1, n_1)\)-sided-game by \( \rho \), in the sense that each string \( \sigma \) in the \((c_1, n_1)\)-sided-game is replaced by \( \rho \cdot \sigma \). Let’s see what happens if Alice plays the \((c_1, n_1)\)-strategy (during a \((c_0c_1, n_0+n_1)\)-sided-game).

- If for each \( \sigma \in [\rho] \preceq 2^{n_0+n_1} \) Alice enumerates, \( \sigma \) admits a catching-point \( \hat{\sigma} \) (by the round \( \sigma \) is enumerated) such that \( \hat{\sigma} \geq \rho \), then the \((c_1, n_1)\)-strategy will forces \(^7\) that at some round \( t \), \( |M[t](\rho)|_1 \geq c_1 \).

- Otherwise, for some \( \sigma \) Alice enumerates, the catching-point satisfies \( \hat{\sigma} \preceq \rho \).

Which ever is the case, we have

\[ |M[t](\hat{\rho})|_1 \geq c_1 \text{ for some } \hat{\rho} \preceq \rho \]

while the cost satisfies \( \mu(A_\rho) \leq \epsilon_1 \mu(\rho) \).

\[ |M[t](\hat{\rho})|_1 \geq c_1 \text{ for some } \hat{\rho} \preceq \rho \]

The strategy for \((c_0c_1, n_0+n_1)\)-sided-game is to play \((c_0c_1, c_1, n_0)\)-sided-game (at level \( n_0 \)), but each time an enumeration \( \sigma \in 2^{n_0} \) is instructed by the strategy, Alice instead play and win the shifted \((c_1, n_1)\)-sided-game, which we may view as a sub-game. Note that the outcome \( |M(\hat{\rho})|_1 \geq c_1 \) for some \( \hat{\rho} \preceq \rho \) of the sub-game \((c_1, n_1)\rho \) that is won (see (5)) corresponds to a valid response by Baby on the fictitious move \( \rho \) in \((c_0c_1, c_1, n_0)\)-sided-game by Alice. Since the strategy for \((c_0c_1, c_1, n_0)\)-sided-game that we simulate is winning, it follows that the nested

\(^6\)We call the winning strategy for \((c_j, n_j)\)-sided-game with \( \mu(A) \leq \epsilon_j \) as \((c_j, n_j)\)-strategy.

\(^7\)In case of any doubt, here “a strategy forces \( \varphi^* \) does not have anything to do with the forcing notion in logic. It simply means the strategy will ensure that at some round \( t \), the state of the game will satisfy \( \varphi \).
strategy is a winning strategy for \((c_0c_1, c_1, n_0)\)-sided-game. If \(A_0 \subseteq 2^{m_0}\) is the set of strings corresponding to the fictitious moves by Alice in \((c_0c_1, c_1, n_0)\)-sided-game, then by (5), the cost of the nested strategy is at most \(\varepsilon_1 \cdot \mu(A_0) \leq \varepsilon_0 \varepsilon_1\).

\[\Box\]

Claim 2.4. Suppose for every \(a > 0\), there is a \(\delta(a) > 0\) (depending on \(a\)) such that for every \(0 < \delta < \delta(a)\), there is an \(n \in \omega\) such that Alice has a winning strategy for \((1 - \delta, n)\)-sided-game 2.1 such that \(\mu(A) \leq 1 - a\delta\). Then for every \(c < 1, \varepsilon > 0\), there is an \(n \in \omega\) such that Alice has a winning strategy for \((c, n)\)-sided-game 2.1 such that \(\mu(A) \leq \varepsilon\).

**Proof.** Fix \(c < 1\). Let \(a, \delta > 0\) be such that for some \(k \in \omega\)
\[(1 - a\delta)^k \leq \varepsilon, (1 - \delta)^k \geq c\] and let \(n \in \omega\) so that
Alice has a winning strategy for \((1 - \delta, n)\)-sided-game 2.1
such that \(\mu(A) \leq 1 - a\delta\).

Now the conclusion follows by applying Claim 2.3 \(k\) times.

By Claim 2.4, to prove Lemma 2.2, it suffices to prove the following.

Lemma 2.5. For every \(a > 0\), there is a \(\delta(a) > 0\) (depending on \(a\)) such that for every \(0 < \delta < \delta(a)\), there is an \(n \in \omega\) such that Alice has a winning strategy for \((1 - \delta, n)\)-sided-game 2.1 such that \(\mu(A) \leq 1 - a\delta\).

2.2. Dynamic goal. For technical reason, we consider the following game which allows Alice to achieve a dynamic goal depending on the resource she cost (the more she cost the better goal she is supposed to achieve). Let \(a > 0, n \in \omega\).

Definition 2.6 (Dynamic \((a, n)\)-sided-game). At each round \(t \in \omega\), Alice firstly enumerates a \(\sigma \in 2^n\) (that has not been enumerated before); then Baby presents \(i\)-sided supermartingale \(M_i[t]\) such that the following hold:

- \(|\|M[t](\hat{\sigma})\|_1| \geq 1\) for some \(\hat{\sigma} \leq \sigma\);
- \(M_i[t]\) dominates \(M_i[t - 1]\) on \(2^{n_i}\).

Let \(A[t]\) denote the set of \(\sigma\) Alice enumerated by round \(t\). Alice wins the game if for some \(t, 1 - ||M[t](\theta)||_1| \leq \frac{1}{a}(1 - \mu(A[t]))\).

In case of any confusion, we emphasize that \(1 - ||M[t](\theta)||_1| \leq 0, \mu(A[t]) < 1\) satisfies the winning criterion of game 2.6. Clearly, dynamic game 2.6 has the same rule as game 2.1 but a different winning criterion.

Claim 2.7. Let \(a, \varepsilon > 0, n \in \omega\). Suppose Alice has a winning strategy for the dynamic \((a, n)\)-sided-game 2.6 such that \(\mu(A) \leq 1 - \varepsilon\). Then for every \(0 < \delta \leq \varepsilon/2a\), there is an \(n \in \omega\) such that Alice has a winning strategy for the \((1 - 2\delta, n)\)-sided-game 2.1 such that \(\mu(A) \leq 1 - a\delta\).

**Proof.** Let \(\hat{n}\) be sufficiently large, say \(2^{-\hat{n}} < a\delta\). Roughly speaking, the winning strategy goes like this. Alice play the winning strategy of the dynamic game 2.6 over each \(\rho \in 2^n\) (while monitoring the cost \(\mu(\hat{A}_\rho)\) over each \(\rho\)) until \(\int_{\rho \in A[t]} (1 - \mu(A_\rho|\rho)) \approx a\delta\) (where \(A[t]\) is the set of \(\rho \in 2^n\) over which the strategy has been played by round \(t\)). Then Alice simply enumerates \(2^n \setminus A[t]\). If for each \(\rho \in A[t]\), the dynamic winning criterion is reached, then

\[1 - ||M[t](\theta)||_1| \leq \int_{\rho \in \hat{A}[t]} (1 - ||M[t](\rho)||_1) \leq \int_{\rho \in \hat{A}[t]} \frac{1}{\alpha}(1 - \mu(\hat{A}_\rho|\rho)) \lesssim \delta\].
A small difficulty is that it’s not necessary that for each \( \rho \in \tilde{A}[t] \), the dynamic winning criterion is reached since some string over \( \rho \) Alice enumerates may admit catching-point that is a prefix of \( \rho \) (i.e., the action of Baby may not be a valid action with respect to the sub-game over \( \rho \)). To overcome this, note that (like we argued in the proof of Claim 2.3), for each \( \rho \in 2^{\tilde{n}} \), Alice could play the winning strategy for the dynamic \((a, \tilde{n})\)-sided-game 2.6 (on \( [\rho]^{\tilde{n}} \cap 2^{\tilde{n}+\tilde{n}} \)) so as to force

\[(6) \text{ either } \rho \text{ admits a catching-point, namely } \|M[t](\hat{\rho})\|_1 \geq 1 \text{ for some } \hat{\rho} \leq \rho;\]

\[(7) \text{ or } 1 - \|M[t](\rho)\|_1 \leq \frac{1}{a}(1 - \mu(\hat{A}_\rho | \rho)),\]

where \( \hat{A}_\rho \) is the set of strings Alice enumerated on the game over \( \rho \).

We design the following strategy of Alice for the \((1 - 2\delta, \tilde{n} + \tilde{n})\)-sided-game 2.1.

1. Until certain criterion (namely (8)) is reached, for each \( \rho \in 2^{\tilde{n}} \) (in whatever order): play the winning strategy for the dynamic \((a, \tilde{n})\)-sided-game 2.6 (on \( [\rho]^{\tilde{n}} \cap 2^{\tilde{n}+\tilde{n}} \)) so as to force either (6) or (7).  

2. Let \( \tilde{A}[t] \) denote the set of \( \rho \in 2^{\tilde{n}} \) that the above action has been done with respect to \( \rho \) (by round \( t \)). Alice do the above action to each \( \rho \in 2^{\tilde{n}} \) until at some round \( \tilde{t} \),

\[(8) \int_{\rho \in \tilde{A}[\tilde{t}]} (1 - \mu(\hat{A}_\rho | \rho)) \geq a\delta \]

3. Now (right after the round \( \tilde{t} \)) Alice simply enumerates \( 2^{\tilde{n}} \setminus \tilde{A}[\tilde{t}] \) (enumerate \( B \) means enumerate every elements in \( [B]^{\tilde{n}} \cap 2^{\tilde{n}+\tilde{n}} \)).

Clearly, the round \( \tilde{t} \) must exist since, by the hypothesis of this claim, \( 1 - \mu(\hat{A}_\rho | \rho) \geq \varepsilon \geq a\delta \). Since \( \tilde{n} \) is sufficiently large and during each sub-game, \( \int_{\rho \in \tilde{A}[\tilde{t}]} (1 - \mu(\hat{A}_\rho | \rho)) \) increases at most \( 2^{-\tilde{n}} \), we may assume

\[(9) \int_{\rho \in \tilde{A}[\tilde{t}]} (1 - \mu(\hat{A}_\rho | \rho)) \leq 2a\delta.\]

By observation (6), (7) (and the nondecreasing, supermartingale of \( M[t] \), let \( t \) be the round everything is done), actions (1), (3) force that

- for every \( \rho \in 2^{\tilde{n}} \setminus \tilde{A}[\tilde{t}] \), (6) follows;
- for every \( \rho \in \tilde{A}[\tilde{t}] \), either (6) or (7) follows.

Thus, there must be a prefix-free set \( B \subseteq 2^{\tilde{n}} \) with \( [B]^{\tilde{n}} \supseteq 2^{\tilde{n}} \setminus \tilde{A}[\tilde{t}] \) such that

\[(10) 1 - \|M[t](\rho)\|_1 \leq \frac{1}{a}(1 - \mu(\hat{A}_\rho | \rho)) \quad \text{for all } \rho \in \tilde{A}[\tilde{t}] \setminus [B]^{\tilde{n}};\]

\[(11) \|M[t](\hat{\rho})\|_1 \geq 1 \quad \text{for all } \hat{\rho} \in B.\]

\(^8\)For example, if a catching-point of \( \rho \) already exist (i.e., (6) is already true), then Alice does nothing over \( \rho \) and \( \hat{A}_\rho = \emptyset \).
Thus, by supermartingale of $M_t[r]$, 

$$1 - ||M_t[\emptyset]||_1 \leq \int_{\rho \in \tilde{A}[r]\setminus \{\emptyset\}^+} \left(1 - ||M_t[\rho]||_1\right) (11) \leq \int_{\rho \in \tilde{A}[r]\setminus \{\emptyset\}^+} \frac{1}{a}(1 - \mu(\tilde{A}_\rho|\rho)) (10) \leq \frac{1}{a}(1 - \mu(\tilde{A}_\rho|\rho)) \leq 2\delta. (9)$$

On the other hand, the cost of Alice, namely $\mu(A)$, satisfies:

$$1 - \mu(A) = \int_{\rho \in \tilde{A}[r]} (1 - \mu(\tilde{A}_\rho|\rho)) \geq a\delta. (8)$$

Thus we are done. $\square$

**Remark 2.8.** The proof of Claim 2.7 does not take advantage of $i$-sidedness but the mere fact that $M_t[r]$ is a supermartingale nondecreasing with respect to $t$.

By Claim 2.7, Lemma 2.5 is reduced to the following assertion:

- For every $a > 0$, there exists a $\varepsilon > 0$ and $n \in \omega$ such that Alice has a winning strategy for the dynamic $(a,n)$-sided-game 2.6 such that $\mu(A) \leq 1 - \varepsilon$.

Combine with the upward closure of game 2.6 (i.e., if Alice has a winning strategy for the dynamic $(a,n)$-sided-game 2.6 such that $\mu(A) \leq 1 - \varepsilon$, then she also has a winning strategy for the dynamic $(a,\hat{n})$-sided-game 2.6 such that $\mu(A) \leq 1 - \varepsilon$ provided $\hat{n} \geq n$), Lemma 2.5 is reduced to the following:

**Lemma 2.9.** For every $a > 0$, there is an $n \in \omega$ such that Alice has a winning strategy for the dynamic $(a,n)$-sided-game 2.6 such that $\mu(A) < 1$.

## 2.2.1. The power of dynamic goal —— winning attention.

The dynamic goal argument is arguably the most important ingredients of the proof of Lemma 2.2. Let’s briefly explain the power of dynamic goal (i.e., why it is considerably easier to win the dynamic game 2.6). During the dynamic $(a,n)$-sided-game 2.6 (or some similar game),

**Definition 2.10** (winning attention). We say $\rho \in 2^{\leq n}$ receives winning attention at round $t$ iff:

$$1 - ||M_t[\rho]||_1 \leq \frac{1}{a}(1 - \mu(A[t]|\rho)) \quad \text{while} \quad \mu(A[t]|\rho) < 1. (12)$$

The point is,

**Fact 2.11.** If some $\rho$ receives winning attention, then Alice wins the dynamic $(a,n)$-sided-game 2.6 (with $\mu(A) < 1$) by enumerating (whatever is left in) $2^n \setminus \{\rho\}^\perp$.

**Proof.** To see this, upon doing so, it forces every $\tilde{\rho} \in 2^{\leq n} \setminus \{\rho\}$ to admit a catching-point. Therefore, we have, by supermartingale of $M_t[r]$, (12) and $A[t+1] = A[t] \cup \tilde{A}[r] \setminus \{\rho\}$.
\[2^n \setminus [\rho]^\omega,\]

\[
1 - ||M[t + 1](\emptyset)||_1 \le (1 - ||M[t + 1](\rho)||_1)\mu(\rho)
\le \frac{1 - \mu(A[t]|\rho)}{a}\mu(\rho)
= \frac{1}{a}(1 - \mu(A[t + 1])).
\]

On the other hand, \(A[t + 1] \cap [\rho]^\omega = A[t] \cap [\rho]^\omega \neq \emptyset\). Thus \(\mu(A[t + 1]) < 1\). \(\square\)

Note: whether \(\rho\) receives winning attention only relies on what happens in \([\rho]^\omega\).

2.3. **Imposing restrictions on the sided player.** The dynamic winning criterion allows us to impose very strong restriction on the sided player’s action.

**Definition 2.12** (Restricted dynamic \((a, \delta, n)\)-sided-game). At each round \(t \in \omega\), Alice firstly enumerates a \(\sigma \in 2^n\) (that has not been enumerated before); then Baby presents \(i\)-sided supermartingale \(M_i[t]\) such that the following hold:

- \(||M[t](\sigma)||_1 \ge 1\);
- \(||M[t](\rho)||_1 \le 1 + \delta\) for all \(\rho \in 2^{\le n}\);
- \(M_i[t]\) dominates \(M_i[t-1]\) on \(2^{\le n}\) (we set \(M_i[-1] \equiv 0\)).

Let \(A[t]\) denote the set of \(\sigma\) Alice enumerated by round \(t\). Alice wins the game if for some \(t\), \(1 - ||M[t](\emptyset)||_1 \le \frac{1}{a}(1 - \mu(A[t]))\).

Clearly the restricted dynamic game 2.12 has the same winning criterion as dynamic game 2.6 but different rules. We refer to the additional requirement \(||M[t](\sigma)||_1 \ge 1\) (instead of \(||M[t](\hat{\sigma})||_1 \ge 1\) for some \(\hat{\sigma} \le \sigma\)) and \(||M[t](\rho)||_1 \le 1+\delta\) as restriction rule I, restriction rule II respectively. Let’s firstly focus on the restriction rule II. The intuition (how a winning strategy for the restricted game helps her win the non restricted game) is the following. If, in the non restricted dynamic game, at some round \(t\), Baby is forced to set \(||M[t](\rho)||_1 > 1\), then it is likely that \(\rho\) receives winning attention (at round \(t\)) since we now have \(1 - ||M[t](\rho)||_1 < 0\). So upon this (namely \(||M[t](\rho)||_1 > 1\)) happening,

\[
(13) \quad \text{as long as some string in } [\rho]^\omega \cap 2^n \text{ is not enumerated by round } t,
\]

\(\rho\) receives winning attention.

How do Alice ensure (13). That is simple, whenever she wants to enumerate \(\sigma\), instead of enumerating \(\sigma\), she enumerates \((|\sigma|^\omega \cap 2^{\ge n}) \setminus \{\sigma 0^n\}\). i.e., she reserves the string \(\sigma 0^n\) to prepare for the winning attention: \(1 - ||M[t](\sigma)||_1 < 0\). By doing so, she could no longer force \(||M[t](\hat{\sigma})||_1 \ge 1\) (for some \(\hat{\sigma} \le \sigma\)), but \(||M[t](\hat{\sigma})||_1 \ge 1 - 2^{-n}\) (for some \(\hat{\sigma} \le \sigma\)). Clearly, letting \(\hat{n}\) to be sufficiently large, this small difference can be ignored. The intuition of restriction rule I is similar. The concrete proof goes as follows.

**Claim 2.13.** Let \(a, \delta > 0, \hat{n} \in \omega\). Suppose Alice has a winning strategy for the restricted dynamic \((a, \delta, \hat{n})\)-sided-game 2.12 such that \(\mu(A) < 1\). Then there is an \(n \in \omega\) such that Alice has a winning strategy for the dynamic \((a/2, n)\)-sided-game 2.6 such that \(\mu(A) < 1\).

**Proof.** Let \(\hat{n}\) be sufficiently large so that \(\Delta = 2^{-\hat{n}}\) is sufficiently small (depending on \(a, \delta, \hat{n}\) and to be specified). Firstly, we note that for \(\rho \in 2^{\hat{n}}\), upon enumerating
either $\hat{\rho}$ admits a catching-point. i.e.,
\[
||M[t](\hat{\rho})||_1 \geq 1 \text{ for some } \hat{\rho} \leq \rho;
\]
(15) or 
\[
1 - ||M[t](\rho)||_1 \geq 1 - \Delta.
\]
A winning strategy for dynamic $(a/2, \hat{n} + \hat{n})$-sided-game \ref{2.6} proceeds as follows:

1. On $2^\hat{n}$, Alice plays the winning strategy for the $((1 - \Delta)$-scaled $9^\text{th})$ restricted dynamic $(a, \delta, \hat{n})$-sided-game \ref{2.12} until some string $\rho \in 2^{\leq \Delta + \hat{n}}$ receives winning attention (with respect to the dynamic $(a/2, \hat{n} + \hat{n})$-sided-game \ref{2.6});
2. But when the strategy tells her to enumerate $\rho \in 2^\hat{n}$, instead of enumerating $\rho$, she enumerates $D_\rho$, which forces either (14) or (15).
3. When some $\rho \in 2^{\leq \Delta + \hat{n}}$ receives winning attention, Alice then enumerates $2^{\Delta + \hat{n}} \setminus \{\rho\}^\Delta$.

Clearly, there are three cases depending on whether Baby breaks the restriction rule I, II of the $((1 - \Delta)$-scaled) restricted dynamic game \ref{2.12} played on $2^\hat{n}$.

**Case 1:** Baby breaks the restriction rule I. i.e., for some $\rho \in 2^\hat{n}$, where $D_\rho$ is enumerated (at round $t$), (15) does not follow.

As we observed, this means (14) follows. But if catching-point $\hat{\rho}$ of $\rho \in 2^\hat{n}$ appears, then $\hat{\rho}$ clearly receives winning attention (with respect to the dynamic $(a/2, \hat{n} + \hat{n})$-sided-game \ref{2.6}), since $1 - ||M[t](\hat{\rho})||_1 \leq 0$ while $\mu(A[t][\hat{\rho}]) < 1$ (since $\rho^0\rho \notin A[t]$). As we argued in Fact \ref{2.11}, action (3) makes Alice win the dynamic $(a/2, \hat{n} + \hat{n})$-sided-game \ref{2.6} (with $\mu(A) < 1$).

**Case 2:** Baby breaks the restriction rule II. i.e., (since $\Delta$ is sufficiently small depending on $\delta$) for some $\rho \in 2^{\leq \Delta}$
\[
||M[t](\rho)||_1 > (1 - \Delta)(1 + \delta)
\]
\[
\geq 1 + \delta/2.
\]
However, that implies $\rho$ receives winning attention since $1 - ||M[t](\rho)||_1 < 0$ while $\mu(A[t][\rho]) < 1$. By Fact \ref{2.11}, action (3) makes Alice win the dynamic $(a/2, \hat{n} + \hat{n})$-sided-game \ref{2.6} (with $\mu(A) < 1$).

**Case 3:** Baby does not break the restriction rules.

Let $A[t]$ denote the set of $\rho \in 2^\hat{n}$ such that $D_\rho$ is enumerated by round $t$. The winning strategy (of the $(1 - \Delta)$-scaled restricted dynamic $(a, \hat{n})$-sided-game \ref{2.12} on $2^\hat{n}$) forces that at some round $t$,
\[
(1 - \Delta) - ||M[t](\emptyset)||_1 \leq (1 - \Delta) \cdot \frac{1}{\hat{\delta}}(1 - \mu(A[t])).
\]

---

9In a $(1 - \Delta)$-scaled restricted dynamic $(a, \delta, \hat{n})$-sided-game, the restriction rule I, II are
\[
||M[t](\sigma)||_1 \geq 1 - \Delta, ||M[t](\rho)||_1 \leq (1 - \Delta)(1 + \delta) \text{ respectively. The winning criterion is}
\]
\[
(1 - \Delta) - ||M[t](\emptyset)||_1 \leq (1 - \Delta) \cdot \frac{1}{\hat{\delta}}(1 - \mu(A[t])).
\]
Then we have (since $\Delta$ is sufficiently small depending on $a, \tilde{n}$, so $\frac{1}{a}(1 - \mu(A[r])) \geq 2^{-\tilde{n}/a} \geq \Delta$)

\[
1 - ||M[r](\emptyset)||_1 \leq (1 - \Delta) \cdot \frac{1}{a}(1 - \mu(A[r])) + \Delta \\
\leq \frac{2}{a}(1 - \mu(A[r])) \text{ while } \\
\mu(A[r]) < 1
\]

So Alice wins the dynamic $(a/2, \tilde{n} + \tilde{n})$-sided-game with $\mu(A) < 1$. Thus we are all done. \hfill \Box

Remark 2.14. The proof of Claim 2.13 does not take advantage of $i$-sidedness but the mere fact that $M_t[r]$ is a supermartingale nondecreasing with respect to $t$.

By Claim 2.13, to prove Lemma 2.9, it suffices to prove the following.

Lemma 2.15. For every $a > 0$, there are $n \in \omega, \delta > 0$ such that Alice has a winning strategy for restricted dynamic $(a, \delta, n)$-sided-game 2.12 such that $\mu(A) < 1$.

2.4. Proof of Lemma 2.15. Fix $a > 0$, a sufficiently small $\delta > 0$ (depending on $a$) and a sufficiently large $n$ (depending on $a$). We start with the following observation (16). For a nonempty prefix-free set $B \subseteq 2^{<\omega}$, a function $M : 2^{<\omega} \to \mathbb{R}^k$ (not necessarily a vector-martingale), we write

\[
\mathbb{E}(M|B) = \frac{1}{\mu(B)} \int_B M \text{ and } \\
\text{Var}(M|B) = \frac{1}{\mu(B)} \int_{\sigma \in B} ||M(\sigma) - \mathbb{E}(M|B)||_2^2.
\]

In §2.4, for a $\rho \in 2^{<\omega}$, let $B_\rho = \{\rho 11, \rho 0\}, \tau_\rho = \rho 10$. The point is, (16) if $B_\rho$ is enumerated while nothing in $[\tau_\rho]^2$ is enumerated, then either $\rho$ receives winning attention; or Baby is forced to allocate a constant large variance on $B_\rho$.

Claim 2.16. Let $M_t$ be $i$-sided supermartingale (not necessarily martingale). Suppose $||M(11)||_1, ||M(0)||_1 \geq 1$. Then, let $B = \{0, 11\}$, we have $1 - ||M(\emptyset)||_1 \leq C\sqrt{\text{Var}(M|B)}$ where $C$ is an absolute constant.

Proof. This is because if the variance is small, then $\min\{M_t(0), M_t(11)\}$ is close to $\mathbb{E}(M_t|B)$; and by $i$-sided of $M_t$, $M_t(\emptyset) \geq \min\{M_t(0), M_t(11)\}$. More specifically, $\text{Var}(M_t|B) \geq \frac{1}{4} \left(\min\{M_t(0), M_t(11)\} - \mathbb{E}(M_t|B)\right)^2$. So, suppose $\sqrt{\text{Var}(M|B)} = \Delta$. We have: for some absolute constant $C_0$ (say $C_0 = 2$),

\[
\mathbb{E}(M_t|B) - \min\{M_t(0), M_t(11)\} \leq C_0\Delta.
\]

By $i$-sided of $M_t$, $M_t(\emptyset) \geq \min\{M_t(0), M_t(11)\}$. Thus

\[
1 - ||M(\emptyset)||_1 \leq \mathbb{E}(||M||_1|B) - \sum_i \min\{M_t(0), M_t(11)\} \\
\leq 2C_0\Delta.
\]

\hfill \Box
The strategy of Alice is to enumerate $\sigma \in 2^n$ in some fixed order so that for many $\rho \in 2^{\leq n}$, there is a round $t$ such that $B_\rho$ has been enumerated (by round $t$) while nothing in $[\tau_r]^c$ is enumerated (by round $t$). This forces Baby to allocate constant large variance on $B_\rho$ for many $\rho$ (if no $\rho$ receives winning attention), which contradicts the variance analysis (Claim 2.17).

Let $\tilde{B}_0, \ldots, \tilde{B}_m \subseteq 2^{\leq n}$ be a sequence of prefix-free sets such that $[\tilde{B}_m] = 2^\omega$, $\tilde{B}_m \subseteq [\tilde{B}_{m-1}]^c$. For each $\rho \in \tilde{B}_{m-1}$, let $\tilde{B}_\rho = [\rho]^c \cap \tilde{B}_m$. Let $\epsilon > 0$.

**Claim 2.17.** Let $(M_j, j < k)$ be supermartingales (not necessarily single-sided) such that $1 - \epsilon \leq \int_{\sigma \in 2^n} ||M(\sigma)||_1$, $||M(\emptyset)||_1 \leq 1$ and $M_j(\rho) \leq 2$ for all $\rho \in 2^{\leq n}$. Then, for some constant $C(k)$ (depending on $k$),

$$\sum_{m<n} \int_{\rho \in \tilde{B}_m} \text{Var}(M|\tilde{B}_\rho) \leq C(k)(1 + m\epsilon).$$

**Proof.** Let $\tilde{M}_j$ be a martingale generated by $M_j \uparrow 2^n$ (i.e., $\tilde{M}_j$ agrees with $M_j$ on $2^n$). Note that $M_j$ dominates $\tilde{M}_j$ on $2^{\leq n}$. Let $\Delta M_j = M_j - \tilde{M}_j$ and $\Delta M = (\Delta M_0, \ldots, \Delta M_{k-1})$. Note that (by $(z + y)^2 \leq 2z^2 + 2y^2$ for any prefix-free set $B$),

$$\text{Var}(M|B) \leq 2\text{Var}(\tilde{M}|B) + 2\text{Var}(\Delta M|B).$$

We need to estimate each term of (17).

Concerning the first term, recall that the variance of a random variable equals to the expectation of its conditional variance plus the variance of its expectation. More specifically, let $z$ be a random variable, $F$ be a filter (say a finite collection of mutually disjoint events whose union is the whole probability space), let $y = \mathbb{E}(z|F)$; then $\mathbb{E}(\text{Var}(z|F)) = \text{Var}(z) - \text{Var}(y)$. Using this\(^{10}\) and by martingale of $M_j$, we have

$$\int_{\rho \in \tilde{B}_m} \text{Var}(\tilde{M}|\tilde{B}_\rho) = \text{Var}(\tilde{M}|\tilde{B}_{m+1}) - \text{Var}(\tilde{M}|\tilde{B}_m).$$

Therefore, since range($\tilde{M}_j$) $\subseteq [0, 2]$, for some constant $C_0(k)$ depending on $k$ (say $C_0(k) = 4k$),

$$\sum_{m<n} \int_{\rho \in \tilde{B}_m} \text{Var}(M|\tilde{B}_\rho) \leq C_0(k).$$

Concerning the second term, since range($\Delta M_j$) $\subseteq [0, 2]$, we have for every prefix-free set $B$:

$$\text{Var}(\Delta M|B) \leq \mathbb{E}(||\Delta M||^2_2|B) \leq 2\mathbb{E}(||\Delta M||_1|B).$$

By supermartingale of $\Delta M_j$,

$$\int_{\rho \in \tilde{B}_{m+1}} \mathbb{E}(||\Delta M||_1|\tilde{B}_\rho) = \mathbb{E}(||\Delta M||_1|\tilde{B}_m) \leq ||\Delta M(\emptyset)||_1 \leq \epsilon.$$

Combining (17), (18), (19), (20) concludes the proof. \(\square\)

\(^{10}\)Where we take $\tilde{B}_{m+1}$ as the probability space, $\tilde{M} \uparrow \tilde{B}_{m+1}$ as the random variable $z$, $\{\tilde{B}_\rho : \rho \in \tilde{B}_m\}$ as the filter.
Alice will enumerate strings in $2^n$ by a specific fixed order so that for many $\rho \in 2^{sn}$, there is a round $t$ such that $B_\rho$ has been enumerated (by round $t$) while nothing in $[\tau_\rho]^\leq$ is enumerated (by round $t$). To define this order, we fix an embedding $e : 3^{sn/2} \to 2^{sn}$ inductively defined as follows:

$$e(\emptyset) = \emptyset; \text{ and suppose } e(\alpha) = \rho, \text{ then }$$

$$e(\alpha0) = \rho0, e(\alpha1) = \rho11, e(\alpha2) = \rho10.$$  

Let $\leq_{\text{lex}}$ denote the lexicographical order on $3^{n/2}$ (for example $01 <_{\text{lex}} 02 <_{\text{lex}} 10$).

**Strategy.** Alice enumerates $(e(\alpha) : \alpha \in 3^{n/2})$ in the lexicographical order on $3^{n/2}$ until some $\rho \in 2^{sn}$ receives winning attention. When that happens, Alice then enumerates $2^n \setminus \rho^\leq$. (End of Strategy.)

Clearly there are two cases.

**Case (a):** Some $\rho$ receives winning attention during the game.

As we argued in Fact 2.11, this means Alice win the game with $\mu(A) < 1$.

**Case (b):** Otherwise.

We will derive a contradiction. The strings Alice expects to receive winning attention are the elements of $e(3^{sn/2})$. Now since they don’t, each of them will force Baby to allocate a constant large variance. To see this, for each $\rho \in e(3^{sn/2})$, by the lexicographical order, there is a round $t$ such that

$$(21) \quad B_\rho \text{ is enumerated while nothing in } [\tau_\rho]^\leq \text{ is enumerated.}$$

We refer to (21) as $\rho$ receives potentially-winning attention. Clearly

$$1 - \mu(A[t]|\rho) = \mu(\tau_\rho|\rho) = 2^{-2}.$$  

Now since $\rho$ does not receive winning attention\(^{11}\)

$$1 - ||M[t](\rho)||_1 \geq 2^{-2}/a.$$  

By Claim 2.16, for some constant $C(a)$ (say $C(a) = \frac{1}{2^{C�a^3}}$ where $C$ is the constant in Claim 2.16) depending on $a$,

$$\text{Var}(M[t]|B_\rho) \geq C(a).$$

But $||M[t](\tau)||_1 \geq 1$ for each $\tau \in B_\rho$, which means by the restriction rule II (game 2.12 second item), $\text{Var}(M[t]|B_\rho) \geq C(a) - 4\delta$ for all $t \geq t$. Let $\hat{B}_\rho = B_\rho \cup \{\tau_\rho\}$, since $\mu(B_\rho) \geq \mu(\hat{B}_\rho)/2$ and $\delta$ is sufficiently small (depending on $a$), we have

$$\text{Var}(M[t]|\hat{B}_\rho) \geq C(a)/4 \text{ for all } t \geq t.$$  

Let $\hat{B}_m = e(3^m)$ (which verifies the setting of $\hat{B}_m$ in Claim 2.17) we have: for some round $t$ by the near end of the game,

$$\text{sufficiently many } \rho \text{ in } e(3^{sn/2}), \text{ say every } \rho \in \cup_{m \leq n/4} e(3^m),$$

$$\text{have received potentially-winning attention (see (21)); and}$$

$$\text{Alice has enumerated sufficiently many strings.}$$

---

\(^{11}\)This, namely (22), (23), is why it is crucial that nothing in $[\tau_\rho]^\leq$ is enumerated when $B_\rho$ has been enumerated. And that’s why we need to enumerate in the lexicographical order.
such that

\[ \sum_{m < n/2 - 1} \int_{\rho \in B_m} \text{Var}(M[t]|\tilde{B}_\rho) \geq \frac{C(a)n}{16} \]  

while

\[ \int_{\sigma \in 2^n} ||M[t](\sigma)||_1 \geq 1 - \epsilon. \]

Where \( \epsilon \) is sufficiently small (say \( o(C(a)) \), since it can be chosen to be \( 2^{-n} \) and \( n \) is sufficiently large depending on \( a \)). Since \( \emptyset \) does not receive winning attention, so

\[ ||M[t](\emptyset)||_1 \leq 1. \]

But (28), (29) together with Claim 2.17 implies (for some absolute constant \( C \))

\[ \sum_{m < n/2 - 1} \int_{\rho \in B_m} \text{Var}(M[t]|\tilde{B}_\rho) \leq C(1 + ns). \]

Thus a contradiction to (27). By the choice of \( C(a) \), we can choose \( n = O(a^2) \).

3. Defeat Kastergales

Again, Theorem 1.3 is reduced to the winning strategy of the following game. Let \( 0 \leq c \leq 1, n \in \omega \). We will use \( p \) to denote partial function on \( 2^{<\omega} \) with \( \text{range}(p) \subseteq 2 \); we say function \( p_1 \) extends \( p_0 \) if \( \text{dom}(p_0) \subseteq \text{dom}(p_1) \) and \( p_1 \) agrees with \( p_0 \) on \( \text{dom}(p_0) \).

**Definition 3.1** ((\( c, n, k \))-partial-sided-game). At each round \( t \in \omega \), Alice firstly enumerates a \( \sigma \in 2^n \) (that has not been enumerated before); then Baby presents a partial function \( p_j[t] \) (for each \( j < k \)), \( p_j[t] \)-sided supermartingale \( M_j[t] \) (for each \( j < k \)) such that the following hold:

- \( p_j[t] \) extends \( p_j[t - 1] \) (we set \( \text{dom}(p_j[-1]) = \emptyset \));
- \( ||M[t](\tilde{\sigma})||_1 \geq 1 \) for some \( \tilde{\sigma} \leq \sigma \);
- \( M_j[t] \) dominates \( M_j[t - 1] \) on \( 2^{<n} \).

Alice wins the game if for some \( t \), \( ||M[t](\emptyset)||_1 \geq c \).

Until §3.3, fix a \( k \in \omega \) (unless claimed otherwise).

**Lemma 3.2.** For every \( c < 1, \epsilon > 0 \), there is an \( n \in \omega \) such that Alice has a winning strategy for \((c, n, k)\)-partial-sided-game 3.1 such that \( \mu(A) \leq \epsilon \).

Again, we reduce (by nesting, dynamic goal and restricting sided player's action) partial-sided-game 3.1 to the corresponding restricted dynamic partial-sided-game 3.9 (and Lemma 3.2 to Lemma 3.11). This part, §3.1, is the same as §2.1-§2.3. The new ingredients are the following.

1. The idea of Lemma 2.15 is to force Baby to allocate a constant large variance on \( 2^{\rho/t^2} \cap [\rho]^{<n} \) for many \( \rho \in 2^{<n} \). This is done by enumerating \( B_\rho \) while nothing in \( [\tau_\rho]^{<n} \) is enumerated. For kastergale, there isn’t such a simple strategy (to force Baby to allocate variance on \( 2^{\rho/t^2} \cap [\rho]^{<n} \)). However, assuming a strategy to force that, we show that Alice can win the restricted dynamic partial-sided-game 3.9 (so Lemma 3.11–a winning strategy for restricted dynamic game 3.1, is reduced to Lemma 3.15–a winning strategy for variance game 3.12).

2. An inductive (in \( k \)) proof that Alice has a winning strategy for variance game 3.12.
The following diagram illustrates the framework of the new ingredients.

Lemma 3.11 $\iff$ Lemma 3.15 (reduce to variance game 3.12, §3.2);
A winning strategy for the variance game 3.12 (proof of Lemma 3.15, §3.3).

Proof of Theorem 1.3 using Lemma 3.2. The same as how Theorem 1.2 is proved using Lemma 2.2.

Notably, other than a small portion of the proof of Lemma 3.15 (that Alice has a winning strategy for the variance game 3.12 when $k = 1$), none of the ingredients relies on partial-sidedness but the nondecreasing and supermartingale of $M_j[r]$.

3.1. Nesting, dynamic goal and restrictions. We first note a nesting property of the $(c,n,k)$-partial-sided-game 3.1. By the same argument we used for Claim 2.3 we have:

Claim 3.3. Suppose Alice has a winning strategy for the $(c_j,n_j,k)$-partial-sided-game 3.1 such that $\mu(A) \leq \varepsilon_j$ (for each $j < 2$). Then Alice has a winning strategy for $(c_0c_1,n_0+n_1,k)$-partial-sided-game 3.1 such that $\mu(A) \leq \varepsilon_0\varepsilon_1$.

By the argument we used for Claim 2.4 we have:

Claim 3.4. Suppose for every $a > 0$, there is a $\delta(a) > 0$ (depending on $a$) such that for every $0 < \delta < \delta(a)$, there is an $n \in \omega$ such that Alice has a winning strategy for the $(1-\delta,n,k)$-partial-sided-game 3.1 such that $\mu(A) \leq 1 - a\delta$. Then for every $c < 1, \varepsilon > 0$, there is an $n \in \omega$ such that Alice has a winning strategy for $(c,n,k)$-partial-sided-game 3.1 such that $\mu(A) \leq \varepsilon$.

By Claim 3.4, to prove Lemma 3.2, it suffices to prove the following.

Lemma 3.5. For every $a > 0$, there is a $\delta(a) > 0$ (depending on $a$) such that for every $0 < \delta < \delta(a)$, there is an $n \in \omega$ such that Alice has a winning strategy for $(1-\delta,n)$-partial-sided-game 3.1 such that $\mu(A) \leq 1 - a\delta$.

Again, we reduce the $(c,n)$-partial-sided-game 3.1 to a dynamic game.

Definition 3.6 (Dynamic $(a,n,k)$-partial-sided-game). Let $a > 0, n \in \omega$. At each round $t \in \omega$, Alice firstly enumerates a $\sigma \in 2^n$ (that has not been enumerated before); then Baby presents $p_j[t]$, $p_j[t]$-sided supermartingale $M_j[t]$ such that the following hold:

- $p_j[t]$ extends $p_j[t-1]$;
- $||M[t](\hat{\sigma})||_1 \geq 1$ for some $\hat{\sigma} \preceq \sigma$;
- $M_j[t]$ dominates $M_j[t-1]$ on $2^{\leq n}$.

Let $A[t]$ denote the set of $\sigma$ Alice enumerated by round $t$. Alice wins the game if for some $t, 1 - ||M[t](\emptyset)||_1 \leq \frac{1}{a}(1 - \mu(A[t]))$.

Claim 3.7. Let $a, \varepsilon > 0, \hat{n} \in \omega$. Suppose Alice has a winning strategy for the dynamic $(a,\hat{n},k)$-partial-sided-game 3.6 such that $\mu(A) \leq 1 - \varepsilon$. Then for every $0 < \delta \leq \varepsilon/2a$, there is an $n$ such that Alice has a winning strategy for $(1-2\delta,n,k)$-partial-sided-game 3.1 such that $\mu(A) \leq 1 - a\delta$.

Proof. Exactly as Claim 2.7 since in Claim 2.7 we merely use the fact that $M_j[t]$ is supermartingale and nondecreasing. □
Thus Lemma 3.5 is reduced to the following:

**Lemma 3.8.** For every \(a > 0\), there is an \(n \in \omega\) such that Alice has a winning strategy for dynamic \((a,n,k)\)-partial-sided-game 3.6 such that \(\mu(A) < 1\).

Again, we simplify Lemma 3.8 by imposing restrictions on Baby’s action.

**Definition 3.9** (Restricted dynamic \((a,\delta,n,k)\)-partial-sided-game). At each round \(t \in \omega\), Alice firstly enumerates a \(\sigma \in 2^n\) (that has not been enumerated before); then Baby present \(p_j[t]\), \(p_j[t]\)-sided supermartingale \(M_j[t]\) such that the following hold:

- \(p_j[t]\) extends \(p_j[t-1]\);
- ||\(M[r](\sigma)\)||_1 \geq 1;
- ||\(M[r](\rho)\)||_1 \leq 1 + \delta\) for all \(\rho \in 2^\delta_n\);
- \(M_j[t]\) dominates \(M_j[t-1]\) on \(2^\delta_n\).

Let \(A[r]\) denote the set of \(\sigma\) Alice enumerated by round \(t\). Alice wins the game if for some \(t\), \(1 - ||M[r](\emptyset)||_1 \leq \frac{1}{n}(1 - \mu(A[r]))\).

By the same argument we used for Claim 2.13 we have:

**Claim 3.10.** Let \(a,\delta > 0, n \in \omega\). Suppose Alice has a winning strategy for the restricted dynamic \((a,\delta,n,k)\)-partial-sided-game 3.9 such that \(\mu(A) < 1\). Then there is an \(n \in \omega\) such that Alice has a winning strategy for the dynamic \((a/2,n,k)\)-partial-sided-game 3.6 such that \(\mu(A) < 1\).

By Claim 3.10, to prove Lemma 3.8, it suffices to prove the following.

**Lemma 3.11.** For every \(a > 0\), there are \(n \in \omega, \delta > 0\) such that Alice has a winning strategy for the unrestricted dynamic \((a,\delta,n,k)\)-partial-sided-game 3.9 such that \(\mu(A) < 1\).

In §3.2, we reduce the restricted dynamic partial-sided-game to yet another game.

### 3.2. Reducing to variance game

As commented below Lemma 3.2, it suffices to show that Alice can force Baby to allocate large variance on the set of strings she enumerates. This gives rise to the following variance game.

**Definition 3.12** (Variance \((a,\Delta,m,k)\)-partial-sided-game). At each round \(t \in \omega\), Alice firstly enumerates a \(\sigma \in 2^n\) (that has not been enumerated before); then Baby presents \(p_j[t]\), \(p_j[t]\)-sided supermartingale \(M_j[t]\) such that the following hold:

- \(p_j[t]\) extends \(p_j[t-1]\);
- ||\(M[r](\sigma)\)||_1 \geq 1;
- \(M_j[t]\) dominates \(M_j[t-1]\) on \(2^\delta_n\).

Let \(A[r]\) be the set of \(\sigma\) Alice enumerated by round \(t\). Alice wins if for some \(t\),

- \((\text{type-(a)})\) either \(1 - ||M[r](\emptyset)||_1 \leq \frac{\Delta}{n}(1 - \mu(A[r]))\);
- \((\text{type-(b)})\) or \(\text{Var}(M[r]A[r]) \geq \Delta\).

Later, we will show that Alice does have a winning strategy for the variance game 3.12 such that \(\mu(A) < 1\) (Lemma 3.15). Now we reduces Lemma 3.10 to Lemma 3.15. Let \(a,\Delta > 0, m \in \omega\).

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12The variance game does not have restriction rule II for Baby. Indeed, Alice can still win it without that rule (see §3.3).

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Claim 3.13. Suppose Alice has a winning strategy for the variance \((a, \Delta, m, k)\)-partial-sided-game 3.12 such that \(\mu(A) < 1\). Then there are \(n \in \omega, \hat{\delta} > 0\) such that Alice has a winning strategy for restricted dynamic \((a, \hat{\delta}, n, k)\)-partial-sided-game 3.9 such that \(\mu(A) < 1\).

Proof. For each \(l \in \omega\), we inductively define a \(l\)-level-variance-strategy. For each \(l\), it is a strategy played on \(2^{ml}\). We show that for sufficiently large \(l\) (depending on \(a, \Delta, m, k\)), the strategy wins the restricted dynamic \((a, \hat{\delta}, ml, k)\)-partial-sided game 3.9 (where \(\hat{\delta}\) is sufficiently small depending on \(m, \Delta\) and to be specified).

**Strategy** (1-level-variance-strategy). This is simply the winning strategy for the variance \((a, \Delta, m, k)\)-partial-sided-game 3.12 except that when some \(\rho\) receives winning attention (recall from (12) the definition of winning attention), Alice then enumerates \(2^{m} \setminus [\rho]^{\hat{\delta}}\) and terminates. Moreover, after reaching the winning criterion of the variance \((a, \Delta, m, k)\)-partial-sided-game 3.12\(^{13}\), Alice keep enumerating whatever is left in \(2^{m}\). (End of Strategy.)

The round \(t\) Alice wins the variance \((a, \Delta, m, k)\)-partial-sided-game 3.12 is the round \(0\) receives potentially-winning attention (compare to (21)). If no \(\rho\) (in particular \(0\) does not) receives winning attention, it means Alice win the variance \((a, \Delta, m, k)\)-partial-sided-game 3.12 in type-(b) way, namely

\[ Var(M[t]|A[t]) \geq \Delta \]

(otherwise \(0\) receives winning attention since by the hypothesis of Claim 3.13, \(\mu(A[t]) < 1\)). Since \(\hat{\delta}\) is sufficiently small (depending on \(m, \Delta\)) and \(M_{j}[t]\) is nondecreasing, we have

\[ Var(M[t]|A[t]) \geq \Delta/2 \text{ for all } t \geq 1; \text{ thus} \\
Var(M[t]|2^{m}) \geq 2^{-2m} \Delta \text{ for all } t \geq 1. \]

We emphasize that these variance strategies are not about winning the variance game 3.12, but the restricted dynamic \((a, \hat{\delta}, m, k)\)-partial-sided-game 3.9. The variance strategy, if not interrupted by winning attention, will enumerate all strings in the end.

**Strategy** (\(l\)-level-variance-strategy). On \(2^{m}\), Alice plays the 1-level-variance-strategy. But when the strategy tells her to enumerate \(\rho\), instead of enumerating \(\rho\) directly, Alice plays the \((l-1)\)-level-variance-strategy on \([\rho]^{\hat{\delta}} \cap 2^{ml}\). Again, during this process (main game or sub-game), if some \(\rho \in 2^{\leq ml}\) receives winning attention, Alice then enumerates \(2^{ml} \setminus [\rho]^{\hat{\delta}}\) and terminates. (End of strategy.)

We show that when \(l\) is sufficiently large, the \(l\)-level-variance-strategy wins the restricted dynamic \((a, \hat{\delta}, ml, k)\)-partial-sided-game 3.9.

**Case (a):** Some \(\rho\) receives winning attention.

This is the same as case (a) of the proof of Lemma 2.15 (see Fact 2.11).

**Case (b):** Otherwise.

We derive a contradiction. Note that if a sub-strategy over \(\rho\) calls for winning attention, then so does the main game (since winning attention of \(\rho\) only depends

\(^{13}\)Since they are actually playing the restricted dynamic game 3.9, now Baby even has to conform to stronger restrictions (i.e., restriction rule II). So the variance game winning strategy still leads to a winning state (of the variance game).
on what happened in $[\rho]^{\leq}$. Thus if some sub-strategy calls for winning attention, the whole strategy terminates and Alice wins the restricted dynamic $(a, \Delta, m, k)$-partial-sided-game 3.9 (as we argued in Fact 2.11). In other words, in this case, for every $\hat{l} < l$, every $\rho \in 2^{ml}$,

$$\text{(32) the (sub) variance } (a, \Delta, m, k)\text{-partial-sided-game 3.12 on } [\rho]^{\leq} \cap 2^{[\rho]+m}\text{ is won in a type-(b) way (i.e., (30)).}$$

Also note that in a $\hat{l}$-level-variance-strategy, when Alice wins the main variance game (the one on $2^{\hat{l}}$), it means $[\hat{A}[r]]^{\leq} \cap 2^{ml}$ has been enumerated (where $\hat{A}[r] \subseteq 2^m$ is the set of strings Alice consumed to win the variance game on $2^m$). So (as (31)) by restriction rule

$$\text{(33) once } M[r] \text{ allocate a constant large variance on } \hat{A}[r],$$
the variance won’t change too much ever since (i.e., (31)).

By the near end of the game (when $\hat{M}(r)$ is close to 1),

$$\text{(34) for sufficiently many } \rho \in \cup_{l<l} 2^{ml} \text{ (say every } \rho \in \cup_{l<l} 2^{\rho+l}),$$
Alice has won the variance game 3.12 on $[\rho]^{\leq} \cap 2^{[\rho]+m}$.

Combining (31), (32), (33), (34) (see case (b) of Lemma 2.15) and letting $\hat{B}_i = 2^{ml}$, if for each $\rho \in \hat{B}_{i-1}$ we let $\hat{B}_i = [\rho]^{\leq} \cap \hat{B}_i$, there exists a round $t$ (near end of the game) such that:

$$\sum_{l<l} \int_{\rho \in \hat{B}_i} \text{Var}(M[r]|\hat{B}_i) \geq 2^{-3m} \Delta \cdot l \text{ while}$$

$$\int_{\sigma \in 2^{ml}} ||M[r](\sigma)||_1 \geq 1 - \varepsilon.$$

where $\varepsilon$ is sufficiently small (say $\varepsilon = 2^{-ml}$). Since $\emptyset$ does not receive winning attention,

$$||M[r](\emptyset)||_1 \leq 1.$$  

Therefore, by Claim 2.17,

$$\sum_{l<l} \int_{\rho \in \hat{B}_i} \text{Var}(M[r]|\hat{B}_i) \leq C(k)(1 + \varepsilon l)$$

Since we can choose $\varepsilon = 2^{-ml} = o(2^{-3m} \Delta/C(k))$ and $l$ is sufficiently large, this is a contradiction. \qed

**Remark 3.14.** Again, Claim 3.13 does not rely on sidedness but the supermartingale and nondecreasing of $M_f[r]$.

By Claim 3.13, it remains to prove

**Lemma 3.15.** For every $a > 0$, there are $m \in \omega$, $\Delta > 0$ such that Alice has a winning strategy for variance $(a, \Delta, m, k)$-partial-sided-game 3.12 such that $\mu(A) < 1$.

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3.3. Winning the variance game. The difficulty to win the partial-sided-game 3.1 is that $M_j$ cooperate with each other. But in the variance game 3.12 they have to behave like constant (on $A[t]$) functions (otherwise Alice wins in the type-(b) way), so there is no cooperation among $M_j$. Therefore we could simply defeat them one by one.

We prove Lemma 3.15 by induction on $k$. For $k = 1$. Let $a, \Delta > 0$; let $m \in \omega$ be sufficiently large (depending on $a$).

**Strategy 3.16 ($k = 1$).** Enumerate each $\sigma \in ([1]^2 \cap 2^m) \setminus \{1 \cdots 1\}$ until

(35) $p_0[t](\emptyset)$ is defined; or

(36) every $\sigma \in ([1]^2 \cap 2^m) \setminus \{1 \cdots 1\}$ is enumerated (while $p_0[t] \uparrow$).

If (35) happens, Alice then enumerates $i$ where $p_0[t](\emptyset) = 1 - i$.

(End of Strategy.)

If (35) occurs, clearly Alice wins the game since now $M_0[t](\emptyset) \geq 1$ while $\mu(A[t]) < 1$ (since either $1 \cdots 1 \not\in A[t]$ or $A[t] \cap [0]^2 = \emptyset$). In case (36), Alice wins since

$$1 - M_0[t](\emptyset) \uparrow \leq 1 - M_0[t](1) \leq 2^{-(m-1)} \text{ while } 1 - \mu(A[t]) \geq 1/2.$$

Note that now we do not only have that Lemma 3.15 is true when $k = 1$, but by what we have done in §3.1 - 3.2, we have

- Lemma 3.2 is true when $k = 1$.

In other words, by induction, it suffices to prove that

- Lemma 3.2 (for $k = k - 1$) $\Rightarrow$ Lemma 3.15 (for $k = k$).

For $k = k$: given $a > 0$. Let $\varepsilon > 0, c < 1$ and $m \in \omega$ be such that $\varepsilon \approx 0, c \approx 1$ and Alice has a winning strategy for the $(c, m/2, k - 1)$-partial-sided-game 3.1 such that $\mu(A) \leq \varepsilon$. Let $\Delta > 0$ be sufficiently small (depending on $a, k$) so that for some $\hat{\Delta} > 0$: $2^{-m}(\hat{\Delta}/2k)^2 \geq \Delta$ and $1 - c + 2\hat{\Delta} \leq (1 - 2\varepsilon)/a$. The following strategy wins the variance $(a, \Delta, m, k)$-partial-sided-game 3.12 (for $k = k$).

**Strategy ($k = k$).** Until reaching type-(b) winning criterion, Alice do the following.

1. In phase 1: Alice plays the winning strategy for the (scaled) $(c, m/2, k - 1)$-partial-sided-game 3.1 against $M_1, \ldots, M_{k-1}$ on $2^{m/2}$. Let $\hat{A}$ denote the set of strings in $2^{m/2}$ Alice has enumerated when she wins (the $(c, m/2, k - 1)$-partial-sided-game 3.1 against $M_1, \ldots, M_{k-1}$ on $2^{m/2}$).

2. In phase 2, for each $\rho \in 2^{m/2} \setminus \hat{A}$ (in whatever order), play the winning strategy for the (scaled) $(c, m/2, k - 1)$-partial-sided-game 3.1 against $M_0$ on $[\rho]^2 \cap 2^m$.

(End of Strategy.)

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14 We emphasize that $\Delta$ can be arbitrarily small depending on $a, m$.  
15 Alice will play the winning strategy assuming type-(b) winning criterion won’t happen. If that is the case, then let $c_0 = M_0[0](\sigma_0)$ where $\sigma_0 \in 2^m$ is the first string Alice enumerates, we have $\sum_{0 < j < k} M_j[t](\sigma) \geq 1 - c_0 - \Delta$ for each $\sigma \in 2^m$ Alice enumerates (since $2^{-m}(\hat{\Delta}/2k)^2 \geq \Delta$). So she will play a $(1 - c_0 - \Delta)$-scaled $(c, m/2, k - 1)$-partial-sided-game 3.1. It will be seen from the proof later that it doesn’t matter if $1 - c_0 - \Delta < 0$. 

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Assuming type-(b) winning criterion is not reached (otherwise we are done), we prove that type-(a) winning criterion will be reached. When Alice wins the \((c,m/2,k-1)\)-partial-sided-game \(3.1\) against \(M_1, \ldots, M_{k-1}\) on \(2^{m/2}\), by scaling,
\[
\sum_{0 < j < k} M_j[t](\emptyset) \geq c(1 - c_0 - \hat{\Delta}).
\]
Here we are clearly using the crucial fact that \(\sum_{0 < j < k} M_j[t](\rho) \geq 1 - c_0 - \hat{\Delta}\) for all \(\rho \in 2^{m/2}\) enumerated by Alice in phase 1 (since type-(b) winning criterion does not occur). This is the advantage (for Alice) in the variance partial-sided-game \(3.12\), which the original partial-sided-game \(3.1\) does not have (so this induction argument cannot prove Lemma \(3.2\)). Let \(t\) be the round everything is done. Obviously, for each \(\rho \in A\), \(M_0[t](\rho) \geq c_0 - \hat{\Delta}\) (since type-(b) winning criterion does not occur). Meanwhile, for each \(\rho \in 2^{m/2} \setminus \hat{A}\), by scaling, \(M_0[t](\rho) \geq c(c_0 - \hat{\Delta})\). Thus
\[
M_0[t](\emptyset) \geq c(c_0 - \hat{\Delta}).
\]
Therefore, \(\sum_j M_j[t](\emptyset) \geq c - 2\hat{\Delta}\) and the cost of Alice satisfies \(\mu(A) \leq 2\epsilon\). Thus we are done.

Remark 3.17. The induction step, does not rely on partial-sidedness but only the nondecreasing and supermartingale properties of \(M_j[t]\). In combination with remarks 2.8, 2.14, 3.14, strategy 3.16 is the only part of the proof of Lemma \(3.2\) concerning partial-sidedness.

4. Subclasses of left-c.e. supermartingale

We address the question: is there a proper subclass of left-c.e. supermartingale defining 1-randomness. First, we need to formalize the question. A supermartingale-approximation is a sequence \(M[\leq t] = (M[0], \ldots, M[t])\) of supermartingales. We use \(\mathcal{M}\) to denote a set of supermartingale-approximations. An \(\mathcal{M}\)-gale is an infinite sequence \((M[t] : t \in \omega)\) such that \(M[\leq t] \in \mathcal{M}\) for each \(t \in \omega\), and \(\lim_{t \to \omega} M[t](\sigma)\) exist for all \(\sigma \in 2^{<\omega}\); it is computable iff \((M[t] : t \in \omega)\) is computable.

Example 4.1. For muchgale, let \(\mathcal{M}_0\) be the class of nondecreasing supermartingale-approximations \(M[\leq t]\) such that for some \(i < l\), \(M[i]\) is \((l,i)\)-betting for all \(i \leq t\). Let \(\mathcal{M} = \bigcup \mathcal{M}_i\), then muchgale is computable \(\mathcal{M}\)-gale. For kastergale, let \(\mathcal{M}\) be the class of nondecreasing supermartingale approximations \(M[\leq t]\) such that for every \(\sigma \in 2^{<\omega}\), if there is a \(i \leq t\) such that \(M[i](\sigma 0) > M[i](\sigma 1)\), then for every \(i \leq t\), \(M[i](\sigma 0) \geq M[i](\sigma 1)\). Then kastergale is computable \(\mathcal{M}\)-gale.

Game 2.1 can be defined with respect to a class \(\mathcal{M}\) of supermartingale-approximations.

Definition 4.2 ((\(c,n,k\))-\(\mathcal{M}\)-game). At each round \(t \in \omega\), Alice firstly enumerates a \(\sigma \in 2^n\); then Baby presents \(M_j[t]\) with \(M_j[\leq t]\) for each \(j < k\) such that:
- for every \(\sigma \in A[t]\), \(||M[t](\sigma)\||_1 \geq 1\) for some \(\tilde{\sigma} \subseteq \sigma\);
Alice wins the game if for some \(t\), \(||M[t](\emptyset)\||_1 \geq c\).

We say \(\mathcal{M}\) is nondecreasing if for each \(M[\leq t] \in \mathcal{M}\), \(M[i]\) dominates \(M[\leq i]\) for all \(1 \leq i \leq t\). We say \(\mathcal{M}\) is homogeneous iff: for every \(\rho \in 2^{<\omega}\), let \(h_\rho\) denote the isomorphism \(\sigma \mapsto \rho \sigma\), we have \(M[\leq t] \in \mathcal{M}\) implies \(M[\leq t] \circ h_\rho \in \mathcal{M}\). We say \(\mathcal{M}\) is subsequence-closed iff for every \(M[\leq t] \in \mathcal{M}\), every \(t_0 < \cdots < t_{s-1} \leq t\),
We say $M$ is scale-closed iff for every $M[\leq t] \in M$, every $c > 0$, $cM[\leq t] \in M$. Abusing terminology, we say $M$ is $\Pi_1^0$-class iff for every $t \in \omega$, the set of $M[\leq t] \in M$ is a $\Pi_1^0$-class uniformly in $t$.

**Remark 4.3.** The homogeneity property is implicitly used in all ingredients (as all those finite games are defined on $[\emptyset]^{\leq \varepsilon} \cap 2^{\varepsilon[\rho]}$ instead of $[\rho]^{\leq \varepsilon} \cap 2^{\varepsilon[\rho]}$). However, this property is not essential, but for notation convenience since those finite games can be defined on $[\rho]^{\leq \varepsilon} \cap 2^{\varepsilon[\rho]}$ and all ingredients (nesting, dynamic goal, restriction and reduction to variance game) can be generalized accordingly.

Let classes of supermartingale-approximation ($M_l : l \in \omega$) be $\Pi_1^0$-class (uniformly in $l$), subsequence-closed, scale-closed, nondecreasing and homogeneous with $M_l \subseteq M_{l+1}$ for all $l \in \omega$; and $M = \cup_l M_l$. As the proof of Theorem 1.2 using Lemma 2.15,

**Claim 4.4.** Suppose for every $l \in \omega$, every $c < 1, \varepsilon > 0$ and $k \in \omega$, there is an $n \in \omega$, such that Alice has a winning strategy for $(c, n, k)$-$M_l$-game 4.2 such that $\mu(A) \leq \varepsilon$. Then there is a non-1-random real $x \in 2^{\omega}$ on which no computable $M$-game succeeds.

**Remark 4.5.** We note that subsequence-closed and $\Pi_1^0$-class of $M_l$ are essential for Claim 4.4 (see footnote 16). On the other hand, homogeneity is only for convenience (see remark 4.3). Actually, nondecreasing may not be essential for Claim 4.4. But discussing this issue does seem worth while.

As commented in remarks 2.8, 2.14, 3.14 and 4.3 (that nesting, dynamic goal, restriction and reducing to variance game only concern nondecreasing, supermartingale and homogeneity of $M$), the winning strategy of $M$-game 4.2 (with arbitrarily small cost) is reduced to the winning strategy of the corresponding variance game (with a cost smaller than 1).

**Definition 4.6** (variance $(a, \Delta, m, k)$-$M$-game). At each round $t \in \omega$, Alice firstly enumerates a $\sigma \in 2^{\omega}$; then Baby presents $M_j[t]$ with $M_j[\leq t] \in M$ for each $j < k$ such that

- for every $\sigma \in A[t]$, $||M[t](\sigma)||_1 \geq 1$.

Alice wins the game if for some $t$,

- (type-(a)) either $1 - ||M[t](\emptyset)||_1 \leq \frac{1}{n}(1 - \mu(A[t]))$;
- (type-(b)) or $\text{Var}(M[t]|A[t]) \geq \Delta$.

By remark 3.17, a winning strategy for variance game 4.6 is reduced to the special case $k = 1$ and that reduction only depends on nondecreasing, supermartingale and homogeneity so:

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16 The subsequence-closed together with $\Pi_1^0$-class of $M$ ensures that the winning strategy of Alice is computable, which is needed when applying the winning strategy in, say, the proof of Theorem 1.2. To this end, it suffices that Alice computably knows all the possible action of Baby at round $t$ given the game history. Note that $\Pi_1^0$-class of $M$ is not enough. Because when Alice plays against computable $M$-gales and enumerates a string (at time $t$), the valid action of Baby may come very late, say at time $\hat{t}$ where $\hat{t}$ is much larger than $t$. For example, consider the class $(M^*)$ where $M^*$ is the universal left-c.e. supermartingale. For this class, Alice has a winning strategy for the $(c, n, l)$-$M$-game played at $[\rho]^{\leq \varepsilon} \cap 2^{\varepsilon[\rho]}$ by enumerating one string in $[\rho]^{\leq \varepsilon} \cap 2^{\varepsilon[\rho]}$, provided $M^*(\rho) < c$. But the winning strategy is not computable (uniformly in $\rho$) as the class is not subsequence-closed.
Claim 4.7. Let $\mathcal{M}$ be nondecreasing and homogeneous and suppose for every $a > 0$, there are $m \in \omega$, $\Delta > 0$ such that Alice has a winning strategy for variance $(a, \Delta, m, 1)$-$\mathcal{M}$-game 4.6 such that $\mu(A) < 1$. Then for every $c < 1, \varepsilon > 0$ and $k \in \omega$, there is an $n \in \omega$ such that Alice has a winning strategy for $(c, n, k)$-$\mathcal{M}$-game 4.2 such that $\mu(A) \leq \varepsilon$.

We already demonstrated how to defeat a single kastergale (in the variance game) in §3.3. Thus, to prove Theorem 1.5, it suffices to show how to defeat a single muchgale. However, this is obvious: for the class of $(l, i)$-betting supermartingales, Alice could simply enumerate the set \{\sigma \in 2^\omega : \sigma(i) = 0\}, forcing $M[r](\emptyset) \geq 1$. By Claims 4.4, 4.7, we get:

**Theorem 1.5.** There exists a real $x$ such that no kastergale or muchgale succeeds on $x$, but $x$ is not 1-random, so some left-c.e. supermartingale succeeds on it.

Remark 4.8. On the other hand, if Alice could not even win Baby (in the variance game 4.6) when $k = 1$, by the definition of variance game 4.6, it is readily seen that some member (the one Baby used against Alice) of $\mathcal{M}$ is almost as flexible as a general left-c.e. supermartingale. This strongly supports conjecture (3): if a natural subclass of left-c.e. supermartingale defines 1-randomness, then a single member of that class can do so.

5. Conclusion

We have shown that a wide class of restricted enumerable strategies including kastergales and muchgales cannot define 1-randomness, answering questions by Kastermans and Hitchcock as well as generalizing Muchnik’s paradox. In this process we developed a general methodology, which strongly supports the thesis that the universal left-c.e. supermartingales is irreducible in the class of left-c.e. supermartingales, in the sense that no restricted class of left-c.e. supermartingales succeed on all reals on which the universal left-c.e. supermartingales succeed. We have noted that simple special cases of this result, such as the case of single-sided betting strategies, were immune to existing methods which were largely recursion-theoretic.

One drawback of our methodology is that it does not give any information about the effective Hausdorff dimension of the constructed non-1-random reals. In the same way that non-1-randomness is related to the success of left-c.e. supermartingales, effective Hausdorff dimension relates to the growth-rate of the capital in betting strategies: Lutz [2000, 2003] showed that the reals $x$ with effective Hausdorff dimension $< 1$ are exactly the reals on which a left-c.e. supermartingale succeeds with exponential growth of the capital. In this sense, effective Hausdorff dimension can gauge the power of a subclass of left-c.e. betting strategies in a more refined way, by examining the complexity of the reals $x$ where they fail. For example, in the case of the class $\mathcal{M}$ of effective mixtures of single-sided martingales, Barmpalias et al. [2020] showed that there exists a real $x$ of effective Hausdorff dimension $\dim_H(x) = 1/2$ such that no betting strategy in $\mathcal{M}$ succeeds on $x$; moreover, this is optimal in the sense that some betting strategy in $\mathcal{M}$ succeeds on all reals $y$ with $\dim_H(y) < 1/2$. For the case of muchgales, a similar result was obtained by Barmpalias and Liu [2022]. However, the question for kastergales or even
single-sided left-c.e. supermartingales is wide-open:

Is there a real with effective Hausdorff dimension < 1 such that no single-sided left-c.e. supermartingale (alt. no kastergale) succeeds on it?

The fact that this question remains open indicates the power of single-sided enumerable betting strategies is over other forms of restricted betting strategies such as muchgales.

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