AN IDENTIFICATION PROBLEM FOR A LINEAR EVOLUTION EQUATION IN A BANACH SPACE

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Abstract. We study a problem of a parameter identification related to a linear evolution equation in a Banach space, using an additional information about the solution. For sufficiently regular data we provide an exact solution given by a Volterra integral equation, while for less regular data we obtain an approximating solution by an optimal control approach. Under certain hypotheses, the characterization of the limit of the sequence of the approximating solutions reveals that it is a solution to the original identification problem. An application to an inverse problem arising in population dynamics is presented.

1. Problem statement. The problem of parameter identification in a dynamic mathematical model describing a real process is of particular interest in a large range of sciences, from engineering to economy, biology, medicine or environmental sciences. Its mathematical approach may be challenging and the problem is not one of a particular estimation technique. Of course, the ideal solution is to obtain a close identification formula for the parameter, but generally this can be done under strong hypotheses. Let us explain this by using an abstract formulation, considering the evolution equation

\[ y'(t) = Ay(t) + u(t)z(t), \quad t \in (0, T), \]

\[ y(0) = y_0, \]

where \( A \) is an infinitesimal operator of a \( C_0 \)-semigroup on a Banach space \( X \), and \( z : [0, T] \to X \). Given the demand (1)-(2), the requirement is to recover the unknown function \( u : [0, T] \to \mathbb{R} \), from the supply

\[ \phi(y(t)) = g(t), \quad t \in [0, T], \]

where \( g : [0, T] \to \mathbb{R} \) is known and \( \phi \in X^* \), the dual of \( X \).

A large class of identification problems for linear evolution equations of first-order in the hyperbolic and parabolic cases and of second-order has been studied in papers by Angelo Favini and co-authors. A particular attention has been given to strongly

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degenerate equations and recently to integro-differential equations. Sufficient conditions for the existence and uniqueness of the solution to the identification problem have been found. They include the condition $\phi(z(t)) \neq 0$. Without exhausting the list we can cite a few results: [1], [2], [3], [4], [5], [8], [9], [10], [11], [12].

In this paper, we shall discuss the possibility of solving such a problem in two situations. The general assumptions, besides the fact that $A$ is a generator of a $C_0$-semigroup on the Banach space $X$, are:

(i) $y_0 \in X$

(ii) $z \in L^2(0, T; X)$

(iii) $g \in L^2(0, T; \mathbb{R})$.

This paper addresses the reconstruction of $u$, and consequently of $y$, as a solution to (1)–(2), from the observation (3). We shall call it problem $(IP)$. A first result is to solve this problem in an exact form, under stronger conditions, in Section 2. For more relaxed hypotheses, including also $\phi(z(t)) = 0$ on some intervals, we shall introduce an optimal control approach providing an approximating solution to $(IP)$. The sequence of these solutions will tend to the solution to $(IP)$ if this exists in this case. These will be detailed in Section 3. Finally, in Section 4, we shall give an example of application to an inverse problem related to population dynamics.

2. A direct approach.

**Definition 2.1.** Let $u \in C[0, T]$ and $z \in C([0, T]; X)$. A strong solution to (1)–(2) is a function $y \in C^1([0, T]; X) \cap C([0, T]; D(A))$ which satisfies (1) for all $t \in [0, T]$.

**Theorem 2.2.** Let us assume that $y_0 \in D(A), z \in C([0, T]; D(A))$, $\phi(z(t)) \neq 0$, for all $t \in [0, T]$, $g \in C^1([0, T]; \mathbb{R})$.

Then, problem $(IP)$ has a unique solution

$$(u, y) \in C[0, T] \times C^1([0, T]; X) \cap C([0, T]; D(A)).$$

**Proof.** Since $A$ is a generator of a $C_0$-semigroup on $X$, the Cauchy problem (1)–(2) has a unique mild solution $y \in C([0, T]; X)$ given by

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}z(s)u(s)ds, \text{ for all } t \in [0, T].$$

By (4) this turns out to be actually a strong solution. Applying $\phi$ we get

$$\phi(y(t)) = \phi(e^{At}y_0) + \int_0^t \phi(e^{A(t-s)}z(s))u(s)ds.$$

Denoting

$$G(t, s) := \phi(e^{A(t-s)}z(s)), \ 0 \leq s \leq t \leq T,$$

$$\Gamma u(t) := \int_0^t G(t, s)u(s)ds$$

and recalling (3), we can write (8) as

$$\Gamma u(t) = g(t) - \phi(e^{At}y_0).$$

We note that $(t, s) \to G(t, s)$ is differentiable on $[0, T] \times [0, T]$ and the right-hand side of (9) is differentiable on $[0, T]$. This is an integral equation of the first kind.
which can be reduced to a linear Volterra equation of second kind by a standard way. Namely, by differentiating (9) with respect to \( t \) we obtain

\[
G(t, t)u(t) - G(t, 0)u(0) + \int_0^t G_t(t, s)u(s)ds + \phi(e^{At}Ay_0) = g'(t).
\]

Next, writing

\[
f(t) := (g'(t) - \phi(e^{At}Ay_0) + \phi(e^{At}z(0))u(0))(G(t, t))^{-1},
\]

and observing that \( G(t, t) = \phi(z(t)) \neq 0 \) for all \( t \in [0, T] \), we get

\[
u(t) = \int_0^t K(t, s)u(s)ds + f(t), \quad \text{for all } t \in [0, T],
\]

where

\[
K(t, s) = -\frac{G_t(t, s)}{G(t, t)}, \quad G_t(t, s) = \phi(e^{A(t-s)}Az(s)), \quad 0 \leq s \leq t \leq T.
\]

For the continuity of \( u \) we should choose \( u(0) = 0 \). Indeed, by (10) we have

\[
u(0) = f(0) = (g'(0) - \phi(Ay_0) + \phi(z(0))u(0))(\phi(z(0)))^{-1},
\]

which yields \( g'(0) = \phi(Ay_0) \), and differentiating (3) we get \( u(0) = 0 \), because

\[
g'(0) = \phi(Ay_0) + u(0)\phi(z(0)).
\]

Under our hypotheses \( f \in C[0, T], \mathbb{R} \) and so the Volterra integral equation of second kind (10) has a unique solution \( u \in C[0, T] \), given by

\[
u(t) = f(t) + \int_0^t K(t, s)f(s)ds, \quad \text{for all } t \in [0, T],
\]

with the resolvent kernel

\[
K(t, s) = \sum_{j=0}^{\infty} K_j(t, s), \quad \text{for all } 0 \leq s \leq T, \quad 0 \leq t \leq T,
\]

where

\[
K_n(t, s) = \int_s^t K(t, \tau)K_{n-1}(\tau, s)d\tau, \quad K_0(t, s) = K(t, s).
\]

As also known, \( u(t) \) can be iteratively obtained as

\[
u_{n+1}(t) = \int_0^t K(t, s)u_n(s)ds + f(t), \quad u_0(t) = u(0),
\]

and the sequence \( u_n(t)_{n \geq 1} \) converges strongly to \( u(t) \), the solution to (10), as \( n \to \infty \). By (7) it follows that the corresponding solution \( y(t) \) is unique, hence problem (IP) has a unique strong solution \( (u, y) \), as claimed.

The following results is easily observed by (12).

**Corollary 2.3.** Under the hypotheses of Theorem 2.2, if \( u(0) \geq 0, f(t) \geq 0 \) for all \( t \in [0, T] \) and \( K(t, s) \geq 0 \) for all \( t, s \in [0, T] \), it follows that \( u(t) \geq 0 \) for all \( t \in [0, T] \). Moreover, if \( u(0) \leq 0, f(t) \leq 0 \) for all \( t \in [0, T] \) and \( K(t, s) \leq 0 \) for all \( t, s \in [0, T] \), it follows that \( u(t) \leq 0 \) for all \( t \in [0, T] \).
3. An optimal control approach. An alternative for solving problem \((IP)\) is to use an optimal control approach by considering the minimization problem

\[
\min \left\{ J(u) = \frac{1}{2} \int_0^T (\phi(y(t)) - g(t))^2 \, dt; \; u \in C[0,T] \right\} \tag{P}
\]

for all \((u,y)\) satisfying \((1)–(2)\).

It is obvious that if \((1)–(2)\) has a unique solution, this turns out to be the unique solution to \((P)\), but the converse assertion is not generally true. It is not clear if \((P)\) may have a solution, especially if we do not assume that \(\phi(z(t)) \neq 0\). The way to get however a solution is to use an approximating control problem \((P_\lambda)\) which provides an approximating solution \((u^*_\lambda, y^*_\lambda)\) and to check if this could tend to the solution to \((P)\), if the latter has one. In this approximating problem we shall require less regularity for the data than in the first case of Section 2.

The hypotheses used in this part are (i)–(iii) and \(X\) is a reflexive Banach space. We stress that we do not require that \(\phi(z(t))\) is nonzero.

Let \(\lambda\) be positive and let us introduce the family of optimal control problems

\[
\min \left\{ J_\lambda(u) = \frac{1}{2} \int_0^T (\phi(y(t)) - g(t))^2 \, dt + \frac{\lambda}{2} \| u\|_{L^2(0,T)}^2 \right\} \tag{P_\lambda}
\]

subject to \((1)–(2)\), where

\[U = \{ u \in L^2(0,T); \; u(t) \in [a,b] \subset \mathbb{R}, \; a.e. \; t \in (0,T)\}.
\]

Here, \(u\) acts as a control.

**Proposition 3.1.** Problem \((P_\lambda)\) has a unique solution \((u^*_\lambda, y^*_\lambda)\), where \(y^*_\lambda\) is the mild solution to \((1)–(2)\).

**Proof.** Since \(J_\lambda(u) \geq 0\) it follows that its infimum \(d_\lambda\) exists and it is nonnegative. Let us consider a minimizing sequence \((u^n_\lambda, y^n_\lambda)\) with \(u^n_\lambda \in U\) and so

\[
d_\lambda \leq J(u^n_\lambda, y^n_\lambda) \leq d_\lambda + \frac{1}{n} \quad \text{for} \quad n \geq 1. \tag{13}
\]

It follows that we can select a subsequence, denoted still by \(n\) such that

\[
u^n_\lambda \to u^*_\lambda \text{ weak-* in } L^\infty(0,T),
\]

and \(u^*_\lambda\) is in \(U\). For \(u = u^n_\lambda\) problem \((1)–(2)\) has a unique mild solution (since \(u^n_\lambda z \in L^1(0,T;X)\) and \(y_0 \in X\), that is

\[
y^n_\lambda(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} z(s) u^n_\lambda(s) \, ds, \quad \text{for all} \quad t \in [0,T],
\]

whence we deduce that

\[
y^n_\lambda(t) \to y^*_\lambda(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} z(s) u^*_\lambda(s) \, ds
\]

weakly in \(X\) for all \(t \in [0,T]\). Thus, \(y^*_\lambda\) follows to be the solution to \((1)–(2)\). Next, since \(\phi \in X^*\) we have

\[
\phi(y^n_\lambda(t)) \to \phi(y^*_\lambda(t)), \quad \text{for all} \quad t \in [0,T]
\]

and passing to the limit in (13) we get \(J_\lambda(u^*_\lambda) = d_\lambda\). We have got that \((P_\lambda)\) has a solution. Finally, we note that the functional is strictly convex, so that the minimum \(u^*_\lambda\) is unique. \(\square\)
Proposition 3.2. Let \( u^*_\lambda, y^*_\lambda \) be the solution to \((P_\lambda)\). Then, the first order necessary conditions of optimality are

\[
u^*_\lambda(t) = P_{[a,b]} \left( \frac{1}{\lambda} \langle p_\lambda(t), z(t) \rangle_{X^*,X} \right), \quad \text{a.e. } t \in (0,T),
\]

where \( p_\lambda \) is the solution to the dual backward problem

\[
\begin{align*}
\frac{dp_\lambda}{dt} & = A^* p_\lambda(t) - (\phi(y^*_\lambda(t)) - g(t))\phi, \quad t \in (0,T), \\
p_\lambda(T) & = 0,
\end{align*}
\]

and \( A^* \) is the adjoint of \( A \).

Proof. Let \( u^*_\lambda, y^*_\lambda \) be the solution to \((P_\lambda)\) and for \( \sigma > 0 \) let us denote

\[
u^*_\lambda = u^*_\lambda + \sigma w, \quad \text{with } w = v - u^*_\lambda \text{ and } v \in U.
\]

Now, we consider the equation in variations

\[
\begin{align*}
\frac{dY}{dt}(t) & = AY(t) + w(t)z(t), \quad t \in (0,T), \\
Y(0) & = 0,
\end{align*}
\]

which has a unique mild solution \( Y \in C([0,T];X) \). It is easily seen by \((1)\)–\((2)\) that the function

\[
Y := \lim_{\sigma \to 0} \frac{y^{\sigma} - y^*}{\sigma}
\]

strongly in \( C([0,T];X) \)

is the unique mild solution to the Cauchy problem \((16)\). We specify that \( y^{\sigma} \) and \( y^\sigma \) are the solutions to \((1)\)–\((2)\) corresponding to \( u^\sigma \) and \( u^* \), respectively.

Now, since \( (u^*_\lambda, y^*_\lambda) \) is optimal in \((P_\lambda)\) it satisfies

\[
J_\lambda(u^*_\lambda) \leq J_\lambda(u^*_\lambda),
\]

whence by performing a short calculation, dividing by \( \lambda \) and passing to the limit as \( \lambda \to 0 \), we get the optimality relation

\[
\int_0^T (\phi(y^*_\lambda(t)) - g(t))\phi(Y(t))dt + \lambda \int_0^T u^*_\lambda(t)w(t)dt \geq 0.
\]

We recall that, since \( X^* \) is reflexive, \( A^* \) is generating a \( C_0 \)-semigroup on \( X^* \) (see [14], p. 41). Then, since \( \phi \in X^* \) equation \((15)\) has a unique mild solution \( p_\lambda \in C([0,T];X^*) \),

\[
p_\lambda(t) = \int_t^T e^{A^*(s-t)}(\phi(y^*_\lambda(s)) - g(s))\phi ds.
\]

Now, if \( p_\lambda \) and \( Y \) would be strong solutions (if e.g. \( \phi \in D(A^*) \) and \( wz \in D(A) \)), we could multiply \((16)\) by \( p_\lambda \) and to integrate by parts to get

\[
\begin{align*}
\langle p_\lambda(T), Y(T) \rangle_{X^*,X} - \langle p_\lambda(0), Y(0) \rangle_{X^*,X} - \int_0^T \langle (p_\lambda)_*(t), Y(t) \rangle_{X^*,X} dt \\
= \int_0^T \langle p_\lambda(t), AY(t) \rangle_{X^*,X} dt + \int_0^T \langle p_\lambda(t), w(t)z(t) \rangle_{X^*,X} dt.
\end{align*}
\]
Since in our case the data are not regular, the same result of integration by parts can actually follow by passing to the limit in regularizing problems for \( Y \) and \( p_\lambda \), corresponding to regular data.

Next, by some calculations taking into account the initial and final conditions in (16) and (15) we get

\[
\int_0^T (-(p_\lambda)_t(t) - A^*p_\lambda(t), Y(t))_{X^*,X} dt = \int_0^T (p_\lambda(t), z(t))_{X^*,X} w(t) dt,
\]

which implies by (15)

\[
- \int_0^T \phi(y_\lambda^*(t) - g(t))\phi(Y(t)) dt = \int_0^T (p_\lambda(t), z(t))_{X^*,X} w(t) dt.
\]

By comparison with (17) we find

\[
\int_0^T (-(p_\lambda(t), z(t))_{X^*,X} + \lambda u_\lambda^*(t)) w(t) dt \geq 0,
\]

whence taking into account that \( w = v - u_\lambda^* \), we get

\[
\int_0^T (p_\lambda(t), z(t))_{X^*,X} - \lambda u_\lambda^*(t)(u_\lambda^*(t) - v(t)) dt \geq 0, \quad \text{for any} \ v \in U.
\]

Thus, we deduce that

\[
\langle p_\lambda(t), z(t)\rangle_{X^*,X} - \lambda u_\lambda^*(t) \in N_{[a,b]}(u_\lambda^*(t)) = \partial I_{[a,b]}(u_\lambda^*(t)), \ \text{a.e.} \ t \in (0, T),
\]

where \( N_{[a,b]}(u_\lambda^*(t)) \) is the normal cone to \([a,b]\) at \( u_\lambda^*(t) \) and \( \partial I_{[a,b]} : \mathbb{R} \to 2^\mathbb{R} \) is the subdifferential of the indicator function of \([a,b]\). Then,

\[
u_\lambda^*(t) + \frac{1}{\lambda} \partial I_{[a,b]}(u_\lambda^*(t)) \ni \frac{1}{\lambda} \langle p_\lambda(t), z(t)\rangle_{X^*,X}
\]

which implies (14), as claimed. \( \square \)

Let

\[
\mathcal{M} := \{ u \in L^2(0,T); \ u(t) \in [a,b] \ \text{a.e.}, \ (u,y) \text{ solves (1)-(3)} \}.
\]

It is easily seen that \( \mathcal{M} \) a convex and close subset. Let us denote by \( \text{Pr}_{\mathcal{M}}(0) \) the projection of 0 on \( \mathcal{M} \).

**Theorem 3.3.** Let \((u_\lambda^*, y_\lambda^*)\) be a solution to \((P_\lambda)\). If \( \mathcal{M} \neq \emptyset \), then for \( \lambda \to 0 \) we have

\[
u_\lambda^* \to \bar{u} \ \text{strongly in} \ L^2(0,T), \tag{18}
\]

\[
y_\lambda^* \to \bar{y} \ \text{strongly in} \ C([0,T]; X), \tag{19}
\]

where \((\bar{u}, \bar{y})\) is a solution to the identification problem (1)-(3). Moreover,

\[
\bar{u} = \text{Pr}_{\mathcal{M}}(0).
\]

**Proof.** Let \((u_\lambda^*, y_\lambda^*)\) be a solution to \((P_\lambda)\). By the optimality condition we have

\[
J_\lambda(u_\lambda^*) = \frac{1}{2} \int_0^T (\phi(y_\lambda^*(t) - g(t))^2 dt + \frac{\lambda}{2} \| u_\lambda^* \|^2_{L^2(0,T)}
\]

\[
\leq J_\lambda(u) = \frac{1}{2} \int_0^T (\phi(y(t) - g(t))^2 dt + \frac{\lambda}{2} \| u \|^2_{L^2(0,T)}.
\]
for all \( u \in U \). In particular, let us \( u = u^* \) and \( y = y^* \) where \((u^*, y^*)\) is any solution to (1)-(3), if \( \mathcal{M} \neq \emptyset \). Hence \( \phi(y^*(t)) = g(t) \), a.e. \( t \in (0, T) \) and we have
\[
\|u^*_{\lambda_n}\|_{L^2(0, T)} \leq \|u^*\|_{L^2(0, T)}. \tag{20}
\]
Hence, on a subsequence \( \lambda_n \to 0 \), we get \( u^*_{\lambda_n} \to \tilde{u} \) weakly in \( L^2(0, T) \). Consequently, \( y^*_{\lambda_n}(t) \to \tilde{y}(t) \) weakly in \( X \), for all \( t \in [0, T] \), where
\[
\tilde{y}(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}z(s)\tilde{u}(s)ds, \ t \in [0, T]
\]
and so \( \tilde{y}(t) \) is a solution to (1)-(2). Moreover,
\[
\phi(y^*_{\lambda_n}) \to \phi(\tilde{y}) \text{ as } \lambda \to 0
\]
and so \((\tilde{u}, \tilde{y})\) is a mild solution to (1)-(3), that is \( \tilde{u} \in \mathcal{M} \).

On the one hand, by (20) we have
\[
\|\tilde{u}\|_{L^2(0, T)} \leq \liminf_{\lambda_n \to 0} \|u^*_{\lambda_n}\|_{L^2(0, T)} \leq \|u^*\|_{L^2(0, T)}
\]
for any \( u^* \in \mathcal{M} \) and so we deduce that the distance from \( \tilde{u} \) to 0 is the smallest, that is
\[
\tilde{u} = \text{Pr}_{\mathcal{M}}(0). \tag{21}
\]
Namely, \( \tilde{u} = 0 \) if \( 0 \notin \mathcal{M} \) and \( \tilde{u} \in \partial \mathcal{M} \), if \( 0 \notin \mathcal{M} \). On the other hand, it follows that
\[
u
u
\]
\[
\|\tilde{u}\|_{L^2(0, T)} \leq \limsup_{\lambda_n \to 0} \|u^*_{\lambda_n}\|_{L^2(0, T)} \leq \|u^*\|_{L^2(0, T)} \text{ and } u^*_{\lambda_n} \to \tilde{u}
\]
weakly in \( L^2(0, T) \) (which is strictly convex). Finally, by the uniqueness of \( \tilde{u} \) defined by (21) it follows that (22) holds for all \( \{\lambda_n\} \to 0 \) and so (19) follows, as claimed. Moreover, this implies (18), too. This ends the proof. \( \square \)

4. Application to population dynamics control. In this section, we present an application to a problem arising in the control of population dynamics. Let us consider a population structured with respect to the age \( a \), diffusing in a habitat \( \Omega \), whose dynamics is governed by a supplementary mortality rate \( \mu(a, x) \), a fertility rate \( \beta(a) \) and a free term \( u(t)z(a, x) \) showing a change of population, possibly due to demographic events. Both these rates are nonnegative and essentially bounded, \( 0 \leq \beta(a) \leq \beta_+ \), a.e. \( a \in (0, a^+) \), \( 0 \leq \mu(a, x) \leq \mu_+ \), a.e. \( (a, x) \in (0, a^+) \times \Omega \). The equations describing the dynamics of the population of density \( y \) are:
\[
y_t + y_a - \Delta y + \mu(a, x)y = u(t)z(a, x), \text{ in } (0, T) \times (0, a^+) \times \Omega, \tag{23}
\]
\[
y(t, 0, x) = \int_0^{a^+} \beta(a)y(t, a, x)da, \text{ in } (0, T) \times \Omega, \tag{24}
\]
\[
- \nabla y \cdot \nu = 0, \text{ on } (0, T) \times (0, a^+) \times \partial \Omega, \tag{25}
\]
\[
y(0, a, x) = y_0, \text{ in } (0, a^+) \times \Omega, \tag{26}
\]
where \( a^+ \) is the maximum life age, \( \nu \) is the outer unit normal to the boundary of \( \Omega \) and (24) is the renewal equation in population dynamics, providing the number of newborns. The problem is to find the rate \( u(t) \) such that to ensure the value \( g(t) \) of a prescribed total population with ages in a certain age range \([a_1, a_2] \subset [0, a^+]\),
\[
\phi(y(t)) = \int_{a_1}^{a_2} \int_{\Omega} y(t, a, x)dxda = g(t), \text{ a.e. } t \in (0, T). \tag{27}
\]
We shall study the problem in the following functional framework. Let us denote

\[ H := L^2(\Omega), \quad V = H^1(\Omega), \quad V' = (H^1(\Omega))' \]

\[ \mathcal{H} = L^2(0, a^+; H), \quad V = L^2(0, a^+; V), \quad V' = L^2(0, a^+; V'). \]

We introduce \( B_0 : V \to V' \) by

\[ \langle B_0 y, \psi \rangle_{V', V} = \int_0^{a^+} \int_\Omega (\nabla y \cdot \nabla \psi + \mu(a, x)y\psi)dxda, \quad \text{for all } \psi \in V, \]

and \( B : D(B) \subset \mathcal{H} \to \mathcal{H} \), by \( By = y_a + B_0 y \) for all \( y \in D(B) \), where

\[ D(B) = \left\{ y \in V; \ y_a \in V', \ By \in \mathcal{H}, \ y(0, x) = \int_0^{a^+} \beta(y)g(y, x)da \right\}. \]

Then, problem (23)–(26) can be written as

\[ \frac{dy}{dt}(t) = Ay(t) + u(t)z(a, x), \quad \text{a.e. } t \in (0, T), \]

\[ y(0) = y_0, \]

where \( A = -B \). We have

**Proposition 4.1.** Let

\[ y_0 \in D(B), \ \ z \in \mathcal{H}, \ g \in C^1([0, T]; \mathbb{R}), \]

\[ \int_{a_1}^{a_2} \int_\Omega z(a, x)dxda \neq 0. \]

Then, the identification problem (23)–(27) has a unique strong solution given by (11), where

\[ K(t, s) = -\frac{\int_{a_1}^{a_2} \int_\Omega e^{A(t-s)}Azdxda}{\int_{a_1}^{a_2} \int_\Omega z(a, x)dxda}, \]

\[ f(t) = \frac{g'(t) - \int_{a_1}^{a_2} \int_\Omega e^{At}Aydxda + \int_{a_1}^{a_2} \int_\Omega e^{At}zu(0)dxda}{\int_{a_1}^{a_2} \int_\Omega z(a, x)dxda}, \]

\[ u(0) = 0. \]

**Proof.** One applies the results in Section 2, for the operator \( A = -B \), after showing that \( A \) generates a \( C_0 \)-semigroup on the Hilbert space \( X = \mathcal{H} \). Equivalently, we show that \( B = -A \) is quasi \( m \)-accretive on \( \mathcal{H} \). Indeed, \( B \) is quasi monotone

\[ \langle \lambda' + B \rangle y, y \rangle_{V', V} = \lambda' \|y\|_\mathcal{H}^2 + \frac{1}{2} \int_\Omega y^2(a^+, x)dx \]

\[ -\frac{1}{2} \int_\Omega \left( \int_0^{a^+} \beta(a)y(a, x)da \right)^2 dx \]

\[ + \|\nabla y\|_\mathcal{H}^2 + \int_0^{a^+} \int_\Omega \mu(a, x)y^2da \]

\[ \geq \left( \lambda' - \frac{1}{2} \beta^2(a^+) \right) \|y\|_\mathcal{H}^2 \geq 0, \quad \text{for } \lambda' > \frac{1}{2} \beta^2(a^+). \]

Moreover, \( B \) is quasi \( m \)-accretive because \( \text{Range}(\lambda' + B) = \mathcal{H} \). To prove this we consider the equation

\[ \lambda' y + y_a - \Delta y + \mu(a, x)y = h, \quad \text{in } (0, a^+) \times \Omega, \]

(29)
with the boundary conditions

$$\nabla y \cdot v = 0 \text{ on } (0, a^+) \times \partial \Omega, \ y(0, x) = \int_0^{a^+} \beta(a)y(a, x)da$$  \hspace{1cm} (30)$$

and \( h \in \mathcal{H} \) and show that it has a solution \( y \in D(B) \), by the Banach fixed point theorem. Namely, let us fix \( \zeta \in \mathcal{H} \) and consider the equation (29) with the boundary conditions

$$\nabla y \cdot v = 0 \text{ on } (0, a^+) \times \partial \Omega, \ y(0, x) = \int_0^{a^+} \beta(a)\zeta(a, x)da.$$  \hspace{1cm} (31)$$

By the general results concerning parabolic boundary-value systems, e.g., by Lions’ theorem. Namely, let us fix \( A \in \mathcal{H} \) to (29), (31). By a direct computation, one can prove that \( \Phi \) is a contraction on \( H \) and show that it has a solution \( y \), which we denote by \( \tilde{y} \), for \( \lambda^* \) large enough, hence \( \Phi(\zeta) = y^* = \zeta \) and so we can replace \( \zeta \) in (31) by \( y^* \). Indeed, considering two solutions \( y_1 \) and \( y_2 \) corresponding to \( \zeta_1 \) and \( \zeta_2 \) we multiply the difference equation for \( y_1 - y_2 \) by \( y_1 - y_2 \) and integrate over \( (0, a^+) \times \Omega \). We get

$$\frac{1}{2} \| (y_1 - y_2)(a^+) \|^2_H + \lambda' \| y_1 - y_2 \|^2_H \leq \frac{1}{2} \int_0^{a^+} \beta(a)(\zeta_1 - \zeta_2)(a, x) da \left( \int_0^{a^+} \beta(a)(\zeta_1 - \zeta_2)(a, x) da \right) dx,$$

whence

$$\lambda' \| y_1 - y_2 \|^2_H \leq \frac{1}{2} \beta^2_{\zeta^*} a^+ \| \zeta_1 - \zeta_2 \|^2_H,$$

showing that for \( \lambda' \) large enough, \( \Phi \) is a contraction on \( \mathcal{H} \). It follows that system (29), (30) has a unique solution in the previous indicated spaces. Moreover, \( By = h - \lambda y \in \mathcal{H} \), hence \( y \in D(B) \) for each \( h \in \mathcal{H} \). Thus, \( A \) quasi \( m \)-accretive and so \( A = -B \) generates a \( C_0 \)-semigroup on \( \mathcal{H} \), so that Theorem 2.2 can be applied.

For weaker regularity of the problem data we resort to the optimal control approach by solving the minimization problem

$$\min \left\{ J_\lambda(u) = \frac{1}{2} \int_0^T \left( \int_0^{a^+} \int_\Omega y(t, a, x) dx da - g(t) \right)^2 dt + \frac{\lambda}{2} \int_0^T u^2(t) dt \right\}$$  \hspace{1cm} (\tilde{P}_\lambda)$$

for all \( u \in U = \{ v \in L^2(0, T); \ v(t) \in [a, b] \subset \mathbb{R} \text{ a.e. } t \in (0, T) \} \), and applying the results in Section 3.

**Proposition 4.2.** Let \( y_0 \in \mathcal{H}, \ z \in \mathcal{H}, \ g \in L^2(0, T; \mathbb{R}) \). Then, problem (\( \tilde{P}_\lambda \)) has a unique solution \( u^*_\lambda \) given by

$$u^*_\lambda(t) = P_{[a, b]} \left( \frac{1}{\lambda} (p(t), z(t))_{\nu', \nu} \right), \text{ a.e. } t \in (0, T),$$

where \( p \) is the solution to the dual backward problem

$$p_t + p_a + \Delta y - \mu(a, x)p + \beta(a)p(0, x) = F(t), \ \text{in } (0, T) \times (0, a^+) \times \Omega,$$

$$p(t, a^+, x) = 0, \ \text{in } (0, T) \times \Omega,$$

$$\nabla p \cdot \nu = 0, \ \text{on } (0, T) \times (0, a^+) \times \partial \Omega,$$

$$p(T, a, x) = 0, \ \text{in } (0, a^+) \times \Omega,$$

$$F(t) = \int_0^{a^+} \int_\Omega \tilde{y}^*(t, a, x) dx da - g(t).$$
and \( y^*_k \) is the solution to (28) corresponding to \( u^*_k \). If \( \mathcal{M} \neq \emptyset \), then \( u^*_k \rightarrow \bar{u} \) strongly in \( L^2(0,T) \) and \((\bar{u}, \bar{y})\) is a solution to (23)–(27). Moreover, \( \bar{u} = \Pr_{\mathcal{M}}(0) \).

**Proof.** The state system (28) has a unique solution \( y \in C([0,T]; \mathcal{H}) \cap L^2(0,T; \mathcal{V}) \cap C([0,a^+]; L^2(0,T; \mathcal{H})) \). The proof can be led as in [7], [13]. The existence of the control follows by Proposition 3.1. In Proposition 3.2, the optimality relation (17) reads

\[
\int_0^T \int_0^{a^+} \int_\Omega F(t)Y(t,a,x)dx_1 dt + \lambda \int_0^T u^*_k(t)w(t)dt \geq 0
\]

and the system in variations is

\[
\frac{dY}{dt}(t) = AY(t) + w(t)z(a,x), \quad t \in (0,T),
\]

\[
Y(0) = 0,
\]

with \( A = -B \) previously defined in this section. Like the state system, it has a unique solution

\[
Y \in C([0,T]; \mathcal{H}) \cap L^2(0,T; \mathcal{V}) \cap C([0,a^+]; L^2(0,T; \mathcal{H})).
\]

We set \( \bar{A}p = p_a + \Delta p - \mu(a,x)p + \beta(a)p(0,\cdot) \), \( \bar{A} : D(\bar{A}) \subset \mathcal{H} \rightarrow \mathcal{H} \) with

\[
D(\bar{A}) = \{ p \in \mathcal{V}; \ p_a \in \mathcal{V}, \ \bar{A}p \in \mathcal{H}, \ p(a^+,x) = 0, \ \nabla p \cdot \nu = 0 \text{ on } (0,a^+) \times \Omega \}.
\]

First, we prove that \( \bar{A} \) is actually \( A^* \), the dual of \( A \). Indeed, we easily see that

\[
\langle \bar{A}p, y \rangle_{\mathcal{V}', \mathcal{V}} = \langle p, Ay \rangle_{\mathcal{V}, \mathcal{V}'} , \quad \forall p \in D(\bar{A}), \ y \in D(A),
\]

but this is not sufficient to ensure that \( \bar{A} \) is the adjoint of \( A \), because \( A^* \) may be an extension of \( \bar{A} \). It suffices however to prove that \( \bar{B} = \bar{A} \) is quasi \( m \)-accretive on \( \mathcal{H} \) and so, \( \bar{B} \) being maximal it follows that \( \bar{B} = B^* \) which implies that \( \bar{A} = A^* \).

We have

\[
\langle \lambda' I + \bar{B}q, q \rangle_{\mathcal{V}', \mathcal{V}} = \lambda' \| p \|_{\mathcal{H}}^2 - \frac{1}{2} \int_\Omega p^2(a^+,x)dx + \frac{1}{2} \int_\Omega p^2(0,x)dx + \| \nabla p \|_{\mathcal{H}}^2 + \int_0^{a^+} \int_\Omega \mu p^2dxda - \int_0^{a^+} \int_\Omega \beta(a)p(0,x)pdxda \\
\geq (\lambda' - 4\beta^2a^+) \| p \|_{\mathcal{H}}^2 + \frac{1}{4} \int_\Omega p^2(0,x)dx > 0 \text{ for } \lambda' > 4\beta^2a^+.
\]

For the quasi \( m \)-accretiveness, we consider the system

\[
\lambda' p - p_a - \Delta y + \mu(a,x)p - \beta(a)p(0,\cdot) = h, \in \ (0,a^+) \times \Omega, \\
p(a^+,x) = 0, \in \Omega, \\
\nabla p \cdot \nu = 0, \ \text{on} \ (0,a^+) \times \partial \Omega,
\]

in which we make the transformation \( \Delta = a^+ - a, \ p(a^+ - a, x) = q(a^+,x) \), getting

\[
\lambda' q + q_a - \Delta q + \mu(a^+,x)q - \beta(a^+)q(a^+,x) = h, \in \ (0,a^+) \times \Omega, \\
q(0,x) = 0, \in \Omega, \\
\nabla q \cdot \nu = 0, \text{ on } (0,a^+) \times \partial \Omega.
\]
Here, $\tilde{\beta}(a') = \beta(a^+ - a')$, $\mu(a',x) = \mu(a^+ - a', x)$. For $h \in \mathcal{H}$ we should have a solution $q \in D(B)$. We proceed again by a fixed point theorem, by taking $v \in C([0,a^+];H)$, fixing $v(a^+,x)$ in the equation

$$X'q + q \omega - \Delta q + \mu(a',x)q = h + \tilde{\beta}(a')v(a^+,x) \in \mathcal{H}$$

and showing that the function $\Psi(v(a^+,\cdot)) = q(a^+\cdot)$ maps $H$ into $H$ and it is a contraction.

The equation (33) together with the conditions in (32) has a unique solution $q \in C([0,a^+];H) \cap L^2(0,a^+;H^2(\Omega)) \cap W^{1,2}(0,a^+;V')$. Now, writing the difference $\omega$ of two solutions $q$ and $\overline{q}$, corresponding to $v$ and $\overline{v}$ we have

$$\omega_{a'} - \Delta \omega + \mu \omega + X\omega = \tilde{\beta}(a')\overline{v}(a^+,x),$$

with $\nabla \omega \cdot \nu = 0$ and $\omega(0,x) = 0$ in $\Omega$, $\overline{v}(a^+,x) = v(a^+,x) - \mu(a^+,x)$. We multiply the equation by $\omega$ and integrate over $(0,a^+) \times \Omega$. We get

$$\frac{1}{2} \int_{\Omega} \omega^2(a^+,x) dx + \lambda \|\omega\|^2_{\mathcal{H}} \leq \frac{a^+}{\varepsilon} \int_0^{a^+} \int_{\Omega} \tilde{\beta}^2(a') \omega^2 dx da + \frac{\varepsilon}{a^+} \int_0^{a^+} \int_{\Omega} \overline{v}^2(a^+,x) dx da,$$

which yields

$$\frac{1}{2} \int_{\Omega} \omega^2(a^+,x) dx + \left(\lambda - \frac{a^+}{\varepsilon} \beta^2\right) \|\omega\|^2_{\mathcal{H}} \leq \varepsilon \int_{\Omega} \overline{v}^2(a^+,x) dx.$$

For $2\varepsilon < 1$ we get that $\Psi$ is a contraction on $H$. Moreover, it is obvious that the solution $q \in D(B)$ and the proof of the quasi $m$-accretiveness is ended. Thus, the adjoint system

$$-\frac{dp}{dt}(t) = \tilde{A}p(t) + F(t), \quad t \in (0,T),$$

$$p(T) = 0$$

has a unique solution $p \in C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \cap C([0,a^+];L^2(0,T;H))$.

Now, we can apply Proposition 3.2 and Theorem 3.3 to get the assertions in Proposition 4.2.

Finally, as a remark, since $y$ is a density it should be positive. Assuming that $y_0 \geq 0$ and that the free term in (23) is nonnegative, then by multiplying (23) by the negative part $(y)^-$, making some computations and applying the Stampacchia lemma, we get that the solution is positive. Taking into account the expression (14) for $u^*_0$, in order to have a nonnegative free term in (23) we must have $z(a,x) \langle p(t), z(t) \rangle_{\mathcal{V},\mathcal{V}} \geq 0$ for a.e. $t \in (0,T)$. $\square$

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