Moment bounds of a class of stochastic heat equations driven by space-time colored noise in bounded domains.

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Abstract
We consider the fractional stochastic heat type equation
\[ \partial_t u(t, x) = -(-\Delta)^{\alpha/2} u(t, x) + \xi \sigma(u(t, x)) F(t, x), \quad x \in D, \quad t > 0, \]
with nonnegative bounded initial condition, where \( \alpha \in (0, 2] \), \( \xi > 0 \) is the noise level, \( \sigma : \mathbb{R} \to \mathbb{R} \) is a globally Lipschitz function satisfying some growth conditions and the noise term behave in space like the Riesz kernel and is possibly correlated in time and \( D \) is the open ball of radius \( R > 0 \), centered at the origin. When the noise term is not correlated in time, we establish a change in the growth of the solution of these equations depending on the noise level \( \xi \). On the other hand when the noise term behaves in time like the fractional Brownian motion with index \( H \in (1/2, 1) \), We also derive explicit bounds leading to a well-known weakly \( \rho \)-intermittency property.

Keywords: Stochastic partial differential equations, space-time correlated Gaussian noise, weakly \( \rho \)-intermittent, phase transition.
1 Introduction

Stochastic Partial Differential Equations (SPDEs) have been studied a lot recently due to many challenging open problems in the area but also due to their deep applications in disciplines that range from applied mathematics, statistical mechanics, and theoretical physics, to theoretical neuroscience, theory of complex chemical reactions [including polymer science], fluid dynamics, and mathematical finance, see for example [15] for an extensive list of literature devoted to the subject. On the other hand SPDEs driven by a random noise which is white in time but colored in space have increasingly received a lot of attention recently, following the foundational work of [8]. One difference with SPDEs driven by space-time white noise is that they can be used to model more complex physical phenomena which are subject to random perturbations. Two phenomena of interest are usually observed when studying these SPDEs, "intermittency" and "phase transition". See for example [1], [2], [3], [4], [9] and [13] for the former and [10], [12], [14] and [21] for the latter.

In this article, we consider an SPDE driven by a space-time colored noise. This type of equation has received a lot of attention recently, see for example [1], [2], [4], [9] and the references therein. The novelty is that we assume the space to be a proper bounded open subset of \( \mathbb{R}^d \).

Consider the fractional stochastic heat equation on the open ball \( D \) subset of \( \mathbb{R}^d, d \geq 1 \) with zero exterior Dirichlet boundary conditions:

\[
\begin{cases}
\partial_t u_t(x) = -(-\Delta)^{\alpha/2} u_t(x) + \xi \sigma(u_t(x)) \hat{F}(t,x) & x \in D, \ t > 0, \\
u_t(x) = 0 & x \in D^c
\end{cases}
\]  

(1.1)

where \( \alpha \in (0, 2], -(-\Delta)^{\alpha/2} \) is the \( L^2 \)-generator of a symmetric \( \alpha \)-stable process killed upon exiting the domain \( D \). The coefficient \( \xi \) denotes the level of the noise, \( \sigma : \mathbb{R} \to \mathbb{R} \) is a globally Lipschitz function. We set \( D := B_R(0) \), the ball of radius \( R \), centered at the origin. The mean zero Gaussian process \( \hat{F} \) is a space-time colored noise, i.e.

\[
\mathbb{E} \left( \hat{F}(t,x) \hat{F}(s,y) \right) = \gamma(t-s) \Lambda(x-y),
\]

(1.2)

where \( \gamma : \mathbb{R} \to \mathbb{R}_+ \) and \( \Lambda : \mathbb{R}^d \to \mathbb{R}_+ \) are general nonnegative and nonnegative definite (generalized) functions satisfying some integrability conditions. The Fourier transform of the latter, \( \hat{\Lambda} = \mu \) is a tempered measure. We first focus our attention on the case where the noise term is uncorrelated in time.

The objective of this paper is to provide lower and upper bounds for the moments of the stochastic fractional heat equation (1.1). But first, let us define some terms and expressions we will use in this paper.

**Definition 1.1.** Assume \( \gamma = \delta_0 \). Following [20], a random field \( \{u_t(x)\}_{t \geq 0, x \in D} \) is called a mild solution of (1.1) in Walsh-Dalang sense if

1. \( u_t(x) \) is jointly measurable in \( t \geq 0 \) and \( x \in D \);

2. for all \( (t,x) \in \mathbb{R}_+ \times D \), the stochastic integral\( \int_0^t \int_D p_D(t-s,x,y) \sigma(u_s(y)) F(dy,ds) \) is well-defined in \( L^2(\Omega) \); by the Walsh-Dalang isometry, this is equivalent to

\[
\sup_{t > 0} \sup_{x \in D} \mathbb{E}[|u_t(x)|^2] < \infty.
\]
3. The following integral equation holds in $L^2(\Omega)$:

\[
(1.3) \quad u_t(x) = (\mathcal{G}u_0)_t(x) + \xi \int_0^t \int_D p_D(t-s,x,y)\sigma(s,y)F(dy,ds),
\]

where

\[
(\mathcal{G}u_0)_t(x) := \int_D p_D(t,x,y)u_0(y)dy
\]

and $p_D(t,x,y)$ denotes the Dirichlet heat kernel of the stable Lévy process. It is the transition density of the stable Lévy process killed in the exterior of $D$. Please refer to Section 2 for a short description of the latter.

When $\gamma = \delta_0$, following Dalang [8], it is well-known that if the spectral measure satisfies the following condition:

\[
(1.4) \quad \int_{\mathbb{R}^d} \frac{\mu(\zeta)}{1+|\zeta|^\alpha} < \infty,
\]

then there exists a unique random field solution of (1.1).

Some examples of space correlation functions satisfying condition (1.4) include:

- **Space-time white noise**: $\Lambda = \delta_0$ in which case $\mu(d\zeta) = d\zeta$ and (1.4) holds only when $\alpha > d$ which implies $d = 1$ and $1 < \alpha \leq 2$.

- **Riesz Kernel**: $\Lambda(x) = |x|^{-\beta}$, $0 < \beta < d$. Here $\mu(d\zeta) = c|\zeta|^{-(d-\beta)}d\zeta$ and (1.4) holds whenever $\beta < \alpha$.

- **Bessel kernel**: $\Lambda(x) = \int_0^\infty y^{\frac{d-d\beta}{2}}e^{-\frac{|x|^2}{4y}}dy$. $\mu(d\zeta) = c(1 + |\zeta|^2)^{-\frac{\eta}{2}}d\zeta$ and (1.4) implies $\eta > d - \alpha$.

- **Fractional Kernel**: $\Lambda(x) = \prod_{i=1}^d |x_i|^{2H_i-2}$. $\mu(\zeta) = c\prod_{i=1}^d |x_i|^{1-2H_i}d\zeta$ and (1.4) holds whenever $\sum_{i=1}^d H_i > d - \frac{\alpha}{2}$.

We now turn our attention on the case where the noise term is also correlated in time.

**Definition 1.2.** Assume $\sigma = Id$, the identity map. An adapted random field $\{u_t(x)\}_{t>0,x\in D}$ such that $\mathbb{E}[|u_t(x)|^2] < \infty$ for all $(t,x)$ is a mild solution to (1.1) in the Skorohod sense if for any $(t,x) \in \mathbb{R}_+ \times D$, the process $\{p_D(t-s,x,y)u_s(y)I_{[0,t]}(s) : s \geq 0, y \in D\}$ is Skorohod integrable and the following integral equation holds:

\[
(1.5) \quad u_t(x) = (\mathcal{G}u_0)_t(x) + \xi \int_0^t \int_D p_D(t-s,x,y)u_s(y)F(\delta s,\delta y).
\]

It is well-known that a unique mild solution (1.5) exists in the Skorohod sense provided that the time correlation $\gamma$ is locally integrable and the space correlation $\Lambda$ satisfies condition (1.4). When handling the mild solution in the Skorohod sense, we shall make use of the Wiener-chaos expansion.

Recall that the covariance given by (1.2) is a mere formal notation. Let $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ be the space of test functions on $\mathbb{R}_+ \times \mathbb{R}^d$. Then on a complete probability space $(\Omega, \mathcal{F}, P)$, we
consider a family of centered Gaussian random variables indexed by the test function \( \{ F(\varphi), \varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \} \) with covariance

\[
E[\dot{F}(\varphi)\dot{F}(\psi)] = \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(t,x)\psi(s,y)\gamma(t-s)f(x-y)dxdydt\,ds.
\]

We write equation (1.6) formally as (1.2). Let \( \mathcal{H} \) be the completion of \( C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \) with respect to the inner product

\[
\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \varphi(t,x)\psi(s,y)\gamma(t-s)f(x-y)dxdydt\,ds.
\]

The mapping \( \varphi \mapsto F(\varphi) \in L^2(\Omega) \) is an isometry which can be extended to \( \mathcal{H} \). We denote this map by

\[
F(\varphi) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi(t,x)F(dt,dx), \quad \varphi \in \mathcal{H}.
\]

Note that if \( \varphi, \psi \in \mathcal{H} \),

\[
E[\dot{F}(\varphi)\dot{F}(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}}.
\]

Furthermore, \( \mathcal{H} \) contains the space of measurable functions \( \varphi \) on \( \mathbb{R}_+ \times \mathbb{R}^d \) such that

\[
\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} |\varphi(t,x)|\varphi(s,y)|\gamma(t-s)f(x-y)dxdydt\,ds < \infty.
\]

For \( n \geq 0 \), denote by \( \mathcal{H}_n \) the \( n \)th Wiener-chaos of \( F \). Recall that \( \mathcal{H}_0 \) is just \( \mathbb{R} \) and for \( n \geq 1 \), \( \mathcal{H}_n \) is the closed linear subspace of \( L^2(\Omega) \) generated by the random variables \( \{ H_n(F(h)), \ h \in \mathcal{H}, \| h \|_{\mathcal{H}} = 1 \} \) where \( H_n \) is the \( n \)th Hermite polynomial. For \( n \geq 1 \), we denote by \( \mathcal{H}^{\otimes n} \) (resp. \( \mathcal{H}^n \)) the \( n \)th tensor product (resp. the \( n \)th symmetric tensor product) of \( \mathcal{H} \). Then, the mapping \( I_n(h^{\otimes n}) = H_n(F(h)) \) can be extended to a linear isometry between \( \mathcal{H}^n \) (equipped with the modified norm \( \sqrt{n!}\| \cdot \|_{\mathcal{H}^{\otimes n}} \)) and \( \mathcal{H}_n \), see for example [16] and [18] and the references therein.

Consider now a random variable \( X \in L^2(\Omega) \) measurable with respect to the \( \sigma \)-field \( \mathcal{F}^F \) generated by \( F \). This random variable can be expressed as

\[
X = E[X] + \sum_{n=1}^{\infty} I_n(f_n),
\]

where the series converges in \( L^2(\Omega) \) and the elements \( f_n \in \mathcal{H}_n, \ n \geq 1 \) are determined by \( X \). This identity is known as the Wiener-chaos expansion. Please refer to [16] and [18] for a complete description on the matter.
We will need the following assumptions:

**Assumption 1.3.** $\gamma : \mathbb{R} \to \mathbb{R}_+$ is locally integrable.

**Assumption 1.4.** There exist constants $C_1$ and $C_2$ and $0 < \beta < \alpha \wedge d$ such that for all $x \in \mathbb{R}^d$,
$$C_1|x|^{-\beta} \leq \Lambda(x) \leq C_2|x|^{-\beta}.$$  

**Assumption 1.5.** There exist positive constants $C_{1\sigma}$ and $C_{2\sigma}$ and $0 < \beta < \alpha \wedge d$ such that for all $x \in \mathbb{R}^d$,
$$C_{1\sigma}|x| \leq |\sigma(x)| \leq C_{2\sigma}|x|.$$  

**Assumption 1.6.** There exist constants $l_\sigma$ and $L_\sigma$ such that for all $x \in \mathbb{R}^d$,
$$l_\sigma|x| \leq |\sigma(x)| \leq L_\sigma|x|.$$  

Throughout the remainder of this paper, $\alpha \in (0, 2\wedge d)$, the letter $C$ or $c$ with or without subscript(s) denotes a constant with no major importance to our study. Assumption 1.4 and assumption 1.6 hold unless stated otherwise. For simplicity we also fix $R = 1$. We are now ready to state our main results.

**Theorem 1.7.** Assume $\gamma = \delta_0$ and $\sigma$ satisfies assumption 1.5. Then for all $t > 0$ and $p \geq 2$, there exist positive constants $c_1, c_2(\alpha, \beta, d, l_\sigma), C_1$ and $C_2(\alpha, \beta, d, L_\sigma)$ such that for all $x \in \mathbb{R}^d$,
$$c_1^p e^{pt \left( c_2 \xi^{\alpha \wedge d - \beta} - \mu_1 \right)} \leq \inf_{x \in D_\epsilon} \mathbb{E}|u_t(x)|^p \leq \sup_{x \in D_\epsilon} \mathbb{E}|u_t(x)|^p \leq C_1^p e^{pt \left( C_2 \xi^{\alpha \wedge d - \beta} z_p^{-\alpha} - (1-\delta)\mu_1 \right)},$$
where $z_p$ is the constant in the Burkholder-Davis-Gundy’s inequality.

This theorem shows that the rate at which the moments of the solution to equation (1.1) exponentially grow or decay depends explicitly on the non-local operator $-(-\Delta)^{\alpha/2}$, the noise level $\xi$ and the noise term via the quantity $\xi^{\alpha \wedge d - \beta}$. This result provides an extension to [17] where the author used equation (1.1) with $\sigma = \text{Id}$, an essential assumption when using the Wiener-Chaos expansion in the proofs. However, the proof we provide for this theorem uses a different argument. This theorem also provides an extension to [11] where similar bounds were obtained but only for the second moments of the solution to equation (1.1). While showing explicitly the dependence of moments of solution of equation (1.1) with the noise level $\xi$, it also implies that there exist $\xi_0(p) > 0$ such that for all $\xi < \xi_0$ and $x \in D$,
$$-\infty < \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}|u(t, x)|^p < 0,$$
and there exists $\xi_1(p)$ such that for all $\xi > \xi_1$ and for all $\epsilon > 0$, $x \in D_\epsilon$,
$$0 < \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}|u(t, x)|^p < \infty.$$  

These results were proved in [14, for the case $\alpha = 2$] and [10, for $0 < \alpha < 2$] but without showing the explicit dependence of the moments on $\xi$.

The next result is concerned with the space-time colored noise case.
Theorem 1.8. Assume \( \sigma(x) = x \) and \( \gamma \) satisfies assumption 1.3. Then for all \( t > 0 \) and \( p \geq 2 \), there exist constants \( \overline{c}_1, \overline{c}_2(\alpha, \beta), \overline{C}_1 \) and \( \overline{C}_2(\alpha, \beta) \) such that for all \( \xi > 0 \) and \( \delta > 0 \),

\[
\overline{c}_1 e^{2p(\frac{2}{\alpha-\beta} - \mu)} \leq \inf_{x \in \mathcal{D}_t} \mathbb{E}|u_t(x)|^p \leq \sup_{x \in \mathcal{D}_t} \mathbb{E}|u_t(x)|^p \leq \overline{C}_1 e^{p(\frac{3}{\alpha-\beta} - (1-\delta)\mu_1)}
\]

where \( \kappa(t) := 2 \int_0^t \gamma(r)dr \) and \( \eta(t) := \int_0^{t/3} \gamma(r)dr \).

Though these bounds might not be very sharp, to the best of our knowledge, this is the first paper ever to examine the moments of the solution of SPDEs driven by such type of noise in bounded domains. Notice again the dependence of moments with the noise level. Here however, little can be said about the behavior of the random field solution with time since the functions \( \eta \) and \( \kappa \) are also dependent of time.

One of the time correlation functions that has received a lot of attention lately is the correlation function of the so-called fractional Brownian motion (of index \( H \)) i.e

\[
(1.7) \quad \gamma(r) = C_H |r|^{2H-2}, \quad \text{for } H \in (1/2, 1) \text{ and } C_H = H(2H - 1).
\]

We refer the interested reader to [1] and the references therein for more information about this function. The following result is an easy consequence of Theorem 1.8.

Corollary 1.9. Suppose conditions of Theorem 1.8 hold and assume in addition that the time correlation is given by (1.7). Then for all \( p \geq 2 \), there exist constants \( \overline{c}_1, \overline{c}_2(\alpha, \beta), \overline{C}_1 \) and \( \overline{C}_2(\alpha, \beta) \) such that for all \( \xi > 0 \) and \( \delta > 0 \),

\[
\overline{c}_1 e^{2p(\frac{2}{\alpha-\beta} - \mu_1 t)} \leq \inf_{x \in \mathcal{D}_t} \mathbb{E}|u_t(x)|^p \leq \sup_{x \in \mathcal{D}_t} \mathbb{E}|u_t(x)|^p \leq \overline{C}_1 e^{p(\frac{3}{\alpha-\beta} - (\mu_1 t))}
\]

A similar result was obtained in [1] but the authors worked on the Euclidean space \( \mathbb{R}^d \). Furthermore, the bounds were not obtained for all values of \( t \). The results provided in this Corollary lead to yet another phenomenon known as intermittency.

Define the \( p^{th} \) upper Liapounov moment of the random field \( u := \{u_t(x)\}_{t>0,x \in D} \) at \( x_0 \in D \) as

\[
\overline{\gamma}(p) := \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}|u_t(x_0)|^p \text{ for all } p \in (0, \infty).
\]

Following [13], the random field \( u \) is said to be weakly intermittent if:

for all \( x \in D \), \( \overline{\gamma}(2) > 0 \) and \( \overline{\gamma}(p) < \infty \) for all \( p \in (2, \infty) \).

It is said to be fully intermittent if:

\[
p \mapsto \frac{\overline{\gamma}(p)}{p} \text{ is strictly increasing for all } p \geq 2 \text{ and } x \in D.
\]

It is also known that weak-intermittency can sometimes imply full intermittency, see for example [13] and the references therein. This corollary shows in fact that for all \( \xi > 0 \) and for all \( p \in (0, \infty) \), there exists finite constants \( C_1, C_2 > 0 \) such that

\[
C_1 < \gamma(p, \rho) := \liminf_{t \to \infty} \frac{1}{tp} \log \mathbb{E}|u_t(x_0)|^p \leq \overline{\gamma}(p, \rho) := \limsup_{t \to \infty} \frac{1}{tp} \log \mathbb{E}|u_t(x_0)|^p < C_2.
\]
Since $\rho = \frac{2H\alpha - \beta}{\alpha - \beta} > 1$. Hence the solution of equation (1.1) is weakly $\rho$-intermittent: see, for example, [1]. In terms of comparison with Theorem 1.7, when $\xi < \left(\frac{\mu_1}{C(p, \delta)}\right)^\frac{\alpha-\beta}{2\alpha}$, then the solution $u$ in Theorem 1.7 is not weakly-intermittent. However, quite the opposite situation occurs for the same random field $u$ when $\xi > \left(\frac{\mu_1}{C_1(p)}\right)^\frac{\alpha-\beta}{2\alpha}$.

The rest of the paper is organized as follows: in section 2, we provide several estimates needed for the proofs of our results; section 3 is devoted to the proofs of our main results and this paper ends with an Appendix where useful results from other authors are compiled.

2 Preliminaries

The Dirichlet heat kernel will play a major role in the proof of our results. Here we give a few details about it. We define the "killed process":

\[ X^D_t = \begin{cases} X_t & t < \tau_D \\ 0 & t \geq \tau_D, \end{cases} \]

where $\tau_D = \inf\{t > 0 : X_t \notin D\}$ is the first exiting time.

Define

\[ r^D(t, x, y) := \mathbb{E}^x[p(t - \tau_D, X_{\tau_D}, y) ; \tau_D < t], \]

then

\[ p^D(t, x, y) = p(t, x, y) - r^D(t, x, y), \]

where $p(t, \ldots)$ is the transition density of the "unkilled process" $X_t$. When $\alpha = 2$, $X_t$ corresponds to a Brownian motion (Wiener process) with variance $2t$ and in this case $p(t, \ldots)$ is explicitly given by

\[ p(t, x, y) = \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{d/2}} \] for all $x, y \in \mathbb{R}^d$.

When $\alpha \in (0, 2)$, no explicit expression is known for $p(t, \ldots)$. But the following approximation holds:

\[ C_1 \min\left(t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}}\right) \leq p(t, x, y) \leq C_2 \min\left(t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}}\right) \]

for some positive constants $C_1$ and $C_2$. See for example [7] and the references therein. One important property of the heat kernel $p(\cdot)$ is the Chapman-Kolmogorov identity (also known as the semigroup property), i.e.

\[ \int_{\mathbb{R}^d} p(t, x, z)p(s, y, z)dz = p(t + s, x, y) \] for all $x, y \in \mathbb{R}^d$ and $s, t > 0$. 

It is an easy fact that $p_D(.)$ also satisfies the Chapman-Kolmogorov identity. Recall that the Dirichlet heat kernel $p_D(t,x,y)$ has the spectral decomposition

$$p_D(t,x,y) = \sum_{n=1}^{\infty} e^{-\mu_n t} \phi_n(x) \phi_n(y), \quad \text{for all } x, y \in D, \ t > 0,$$

where $\{\phi_n\}_{n \geq 1}$ is an orthonormal basis of $L^2(D)$ and $0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \ldots$ is a sequence of positive numbers satisfying, for all $n \geq 1$:

$$-(\Delta)^{n/2} \phi_n(x) = -\mu_n \phi_n(x) \quad x \in D$$

$$\phi_n(x) = 0 \quad x \in D^c.$$

It is well-known that

$$c_1 n^{\alpha/d} \leq \mu_n \leq c_2 n^{\alpha/d}$$

for some constants $c_1, c_2 > 0$. See for example [5, Theorem 2.3], for more details. Moreover by [6, Theorem 4.2], for all $x \in D$,

$$c^{-1} (1 - |x|)^{\alpha/2} \leq \phi_1(x) \leq c(1 - |x|)^{\alpha/2}, \quad \text{for some } c > 1.$$

For example when $\alpha = 2$, and $d = 1$, i.e $D = (-1, 1)$, we get for $n = 1, 2, \ldots$

$$\phi_n(x) = \sin \left( \frac{n \pi x}{2} \right) \quad \text{and} \quad \mu_n = \left( \frac{n \pi}{2} \right)^2.$$

We shall need the following estimates to prove our main results. The first two follow from applications of Theorems 4.5 and 4.4.

**Proposition 2.1.** Fix $\epsilon \in (0, \frac{\pi}{2})$. Then for any $x, y \in D_\epsilon$ such that $|x - y| < t^{1/\alpha}$, we have

$$p_D(t,x,y) \geq c t^{-d/\alpha} e^{-\mu_1 t} \quad \text{for all } t > 0$$

and some positive constant $c$.

**Proof.** We first prove the Lemma for $\alpha = 2$. Assume $|x - y| < \sqrt{t}$. We apply Theorem 4.5 to get

$$p_D(t,x,y) \geq C_1 \min \left( 1, \frac{\phi_1(x) \phi_1(y)}{1 \wedge t} \right) e^{-\mu_1 t} \frac{e^{-c \frac{|x-y|^2}{t}}}{1 \wedge t^{d/2}}$$

$$\geq C_2 e^{-\mu_1 t} \left\{ \min \left( 1, \frac{\epsilon^2}{t} \right) e^{-c \frac{|x-y|^2}{t}} 1_{\{t < 1\}} + \min \left( 1, \epsilon^2 \right) e^{-c \frac{|x-y|^2}{t}} 1_{\{t \geq 1\}} \right\}$$

$$= C_2 e^{-\mu_1 t} \left\{ e^{-c \frac{|x-y|^2}{t}} 1_{\{t < 1\}} + \epsilon^2 e^{-c \frac{|x-y|^2}{t}} 1_{\{t < \epsilon^2 t\}} + e^{-c \frac{|x-y|^2}{t + \epsilon^2 t}} 1_{\{t \geq \epsilon^2 t < 1\}} + \min \left( 1, \epsilon^2 \right) e^{-c \frac{|x-y|^2}{t}} 1_{\{t \geq 1\}} \right\}$$

$$\geq C_3 e^{-\mu_1 t} t^{-d/2} \left\{ 1_{\{t < \epsilon^2 t\}} + \frac{\epsilon^2}{t} 1_{\{t \geq \epsilon^2 t < 1\}} + c(\epsilon) t^{d/2} 1_{\{t \geq 1\}} \right\}$$

$$\geq C_4 e^{-\mu_1 t} t^{-d/2}.$$

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Note the use of (2.5) in the second inequality above since \(x, y \in D\). This proves the inequality for \(\alpha = 2\).

Now suppose \(0 < \alpha < 2\). Assuming \(|x - y| < t^{1/\alpha}\), we apply Theorem 4.4 to get

\[
p_D(t, x, y) \geq C_1 e^{-\mu_1 t} \left[ \min \left( 1, \frac{\phi_1(x)}{\sqrt{t}} \right) \min \left( 1, \frac{\phi_1(y)}{\sqrt{t}} \right) \min \left( \frac{t^{-d/\alpha}}{|x - y|^{\alpha + d}} \right) 1_{\{t < 1\}} + \phi_1(x) \phi_1(y) 1_{\{t \geq 1\}} \right]
\]

\[
\geq C_2 e^{-\mu_1 t} \left\{ \min \left( 1, \frac{e^2}{t} \right) \min \left( \frac{t^{-d/\alpha}}{|x - y|^{\alpha + d}} \right) 1_{\{t < 1\}} + e^2 1_{\{t \geq 1\}} \right\}
\]

\[
\geq C_3 e^{-\mu_1 t} \left\{ \min \left( 1, \frac{t^{1/\alpha}}{|x - y|} \right) \frac{\alpha + d}{\alpha} 1_{\{t < 1\}} + e^2 1_{\{t \geq 1\}} \right\}
\]

\[
C_3 e^{-\mu_1 t} \left\{ \min \left( 1, \frac{t^{1/\alpha}}{|x - y|} \right) \frac{\alpha + d}{\alpha} 1_{\{t < 1\}} \right\} + e^2 \frac{\alpha + d}{\alpha} 1_{\{t \geq 1\}} \right\}
\]

\[
= C_4 e^{-\mu_1 t} \left\{ \min \left( 1, \frac{t^{1/\alpha}}{|x - y|} \right) \frac{\alpha + d}{\alpha} 1_{\{t < 1\}} \right\} + e^2 \frac{\alpha + d}{\alpha} 1_{\{t \geq 1\}} \right\}
\]

Again note the use of (2.5) in the second inequality above. This concludes the proof. \( \square \)

**Lemma 2.2.** For all \( \delta > 0 \), there exists \( c_2(\delta) > 0 \) such that for all \( x, w \in D \) and \( s, t > 0 \),

\[
\int_{D \times D} p_D(t, x, y) p_D(s, w, z) \Lambda(y - z) dydz \leq c_2 e^{-(1 - \delta) \mu_1 (t + s)} (s + t)^{-\beta/\alpha}
\]

**Proof.** As usual, we first prove the result for \( \alpha = 2 \). By Theorem 4.5, we have

\[
\int_{D \times D} p_D(t, x, y) p_D(s, w, z) \Lambda(y - z) dydz \leq C_1 e^{-\mu_1 (t+s)} \int_{D \times D} e^{-c_1 |x - y|^2} \frac{1}{1 \wedge t^{d/2}} e^{-c_2 |w - z|^2} \frac{1}{1 \wedge s^{d/2}} \Lambda(y - z) dydz
\]

\[
\leq C_2 e^{-\mu_1 (t+s)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t, x, y) p(s, w, z) \Lambda(y - z) dydz 1_{\{t < 1, s < 1\}} \right\}
\]

\[
+ \int_{\mathbb{R}^d} p(t, x, y) \Lambda(y - z) dy 1_{\{t < 1, s \geq 1\}} + \int_{\mathbb{R}^d} p(s, w, z) \Lambda(y - z) dz 1_{\{t \geq 1, s < 1\}} + c 1_{\{t \geq 1, s \geq 1\}} \right\}
\]

\[
\leq C_2 e^{-\mu_1 (t+s)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t, x, y) \Lambda(y) dy 1_{\{t < 1, s < 1\}} \right\}
\]

\[
+ \int_{\mathbb{R}^d} p(t, x, y) \Lambda(y - z) dy 1_{\{t < 1, s \geq 1\}} + \int_{\mathbb{R}^d} p(s, w, z) \Lambda(y - z) dz 1_{\{t \geq 1, s < 1\}} + c 1_{\{t \geq 1, s \geq 1\}} \right\}
\]

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\[
\begin{aligned}
&\leq C_3 e^{-\mu(t+s)} \left\{ c_1(t+s)^{-\beta/2} 1_{\{t<s<1\}} + c_2 t^{-\beta/2} 1_{\{t<s\geq 1\}} \right. \\
&\quad \left. + c_3 s^{-\beta/2} 1_{\{t\geq 1, s<1\}} + c_1 \{t \geq 1, s \geq 1\} \right\} \\
&= C_3 e^{-\mu(t+s)} (t+s)^{-\beta/2} \left\{ c_1 1_{\{t<s<1\}} + c_2 \left(1 + \frac{s}{t}\right)^{\beta/2} 1_{\{t<s\geq 1\}} \\
&\quad + c_3 \left(1 + \frac{t}{s}\right)^{\beta/2} 1_{\{t\geq 1, s<1\}} + (t+s)^{\beta/2} 1_{\{t\geq 1, s \geq 1\}} \right\} \\
&\leq C_5 e^{-(1-\delta)\mu(t+s)} (t+s)^{-\beta/2} \text{ for all } \delta > 0.
\end{aligned}
\]

Note the use of (2.1) in the second inequality, the Chapman-Kolmogorov identity (2.3) in the first integral in the third inequality and Proposition 4.8 in the fourth inequality.

The proof for the case $0 < \alpha < 2$ follows a very similar argument. By Theorem 4.4, we have

\[
\begin{aligned}
\int_{D \times D} p_D(t, x, y)p_D(s, w, z)\Lambda(y - z)dydz \\
&\leq C_1 e^{-\mu(t+s)} \left\{ \int_{D \times D} \min \left( t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}} \right) \min \left( s^{-d/\alpha}, \frac{s}{|w-z|^{\alpha+d}} \right) \Lambda(y - z)dydz 1_{\{t<s<1\}} \\
&\quad + \int_{D \times D} \min \left( t^{-d/\alpha}, \frac{t}{|x-y|^{\alpha+d}} \right) \Lambda(y - z)dydz 1_{\{t<s\geq 1\}} \\
&\quad + \int_{D \times D} \min \left( s^{-d/\alpha}, \frac{t}{|w-z|^{\alpha+d}} \right) \Lambda(y - z)dydz 1_{\{t\geq 1, s\geq 1\}} + 1_{\{t \geq 1, s \geq 1\}} \right\} \\
&\leq C_2 e^{-\mu(t+s)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t, x, y)p(s, w, z)\Lambda(y - z)dydz 1_{\{t<s<1\}} \\
&\quad + \int_{\mathbb{R}^d} p(t, x, y)\Lambda(y - z)dy 1_{\{t<s\geq 1\}} + \int_{\mathbb{R}^d} p(s, w, z)\Lambda(y - z)dz 1_{\{t\geq 1, s<1\}} + 1_{\{t \geq 1, s \geq 1\}} \right\} \\
&= C_2 e^{-\mu(t+s)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t+s, x-w, y)\Lambda(y)dy 1_{\{t<s<1\}} \\
&\quad + \int_{\mathbb{R}^d} p(t, x, y)\Lambda(y - z)dy 1_{\{t<s\geq 1\}} + \int_{\mathbb{R}^d} p(s, w, z)\Lambda(y - z)dz 1_{\{t\geq 1, s<1\}} + 1_{\{t \geq 1, s \geq 1\}} \right\} \\
&\leq C_4 e^{-\mu(t+s)} \left\{ c_1 (t+s)^{-\alpha/\beta} 1_{\{t<s<1\}} + c_2 t^{-\alpha/\beta} 1_{\{t<s\geq 1\}} \\
&\quad + c_3 s^{-\alpha/\beta} 1_{\{t\geq 1, s<1\}} + c_1 \{t \geq 1, s \geq 1\} \right\} \\
&\leq C_5 e^{-\mu(t+s)} (t+s)^{-\alpha/\beta} \left\{ c_1 1_{\{t<s<1\}} + c_2 \left(1 + \frac{s}{t}\right)^{\beta/\alpha} 1_{\{t\geq 1, s \geq 1\}} \right\}
\end{aligned}
\]
\[ + c_3 \left( \frac{1 + \frac{t}{s}}{s} \right)^{\beta/\alpha} 1_{\{t \geq 1, s < 1\}} + (t + s)^{\beta/\alpha} 1_{\{t \geq 1, s \geq 1\}} \right) \]

\[ \leq C_6 e^{-(1-\delta)\mu_1 (t+s)} (t+s)^{-\beta/\alpha} \] for all \( \delta > 0 \).

Again notice the use of (2.2) in the second inequality, the semigroup property (2.3) in the first integral in the third inequality and Proposition 4.8 in the fourth inequality. This concludes the proof.

\[ \square \]

**Lemma 2.3.** Suppose \( a \geq 0 \) and \( \zeta > -1 \). Then

\[ I_\zeta^n(a,b) := \int_{\{a<r_1<r_2<\cdots<r_n<b\}} [(r_2-r_1)(r_3-r_2)\cdots(b-r_n)]^\zeta dr_1dr_2\cdots dr_n = \frac{\Gamma(1+\zeta)^{n+1}(b-a)^{n(1+\zeta)}}{\Gamma(n(1+\zeta)+1)}, \]

where \( \Gamma(.) \) is the Euler’s gamma function.

**Proof.** We shall consider two cases here:

When \( a = 0 \), this is just [4, Lemma 3.5].

Assume now that \( a > 0 \), then integrating iteratively yields:

starting with

\[ \int_a^{r_2} (r_2 - r_1)^\zeta dr_1 = \frac{(r_2 - a)^{1+\zeta}}{1+\zeta}. \]

Next,

\[ \int_a^{r_3} (r_2 - a)^{1+\zeta}(r_3 - r_2)^\zeta dr_2 = \int_0^{r_3-a} r_2^{1+\zeta}(r_3 - a - r_2)^\zeta dr_2 = (r_3 - a)^{2(1+\zeta)} B\left( (1+\zeta) + 1, \zeta + 1 \right), \]

where we have used successively the change of variables \( r_2 \to r_2 - a \) and \( r_2 \to \frac{r_2}{r_3-a} \) and \( B(.,.) \) is the Euler’s Beta function, i.e

\[ B(c, d) = \int_0^1 u^{c-1}(1-u)^{d-1} du, \quad c > 0, d > 0. \]

Continuing this way, we end up with
\[ I_n(a, b) = \frac{1}{1 + \zeta} \left[ B\left((1 + \zeta + 1, \zeta + 1\right) B\left(2(1 + \zeta) + 1, \zeta + 1\right) \cdots B\left((n - 2)(1 + \zeta), \zeta + 1\right) \right] \]
\[ \times \int_a^b (r_n - a)^{(n-1)(1+\zeta)} (b - r_n)^{\zeta} dr_n \]
\[ = \frac{(b - a)^{n(1+\zeta)}}{1 + \zeta} \left[ B\left((1 + \zeta + 1, \zeta + 1\right) B\left(2(1 + \zeta) + 1, \zeta + 1\right) \cdots B\left((n - 1)(1 + \zeta), \zeta + 1\right) \right]. \]

The fact that \( \Gamma(z + 1) = z\Gamma(z) \) for all \( z > 0 \) together with \( B(c, d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)} \) concludes the proof.

The following result is essential for the proof of the lower bound in Theorem 1.7.

**Proposition 2.4.** Fix \( \epsilon > 0 \). Then for all \( x \in D_\epsilon \), we have
\[
\mathbb{E}|u_t(x)|^2 \geq ce^{-2\mu_1 \epsilon t} \sum_{n=1}^{\infty} \left( C\xi_{l, \sigma} \right)^{2n} \left( \frac{t^n}{n!} \right)^2 \left( \frac{\alpha-\beta}{\alpha} \right),
\]
for some positive constants \( c \) and \( C = C(\alpha, \beta, d) \).

**Proof.** By squaring the mild solution (1.3), we get
\[
\mathbb{E}|u_t(x)|^2 = (\mathcal{G}u_0)|^2_t(x) + \xi^2 \int_0^t \int_{D^2} p_D(t-s, x, y)p_D(t-s, x, z) \mathbb{E}|\sigma(u_s(y))\sigma(u_s(z))|\Lambda(y-z)dydzds.
\]

Now using Assumption 1.5, we get
\[
\mathbb{E}|u_t(x)|^2 \geq (\mathcal{G}u_0)|^2_t(x) + \xi^2 \int_0^t \int_{D^2} p_D(t-s, x, y)p_D(t-s, x, z) \mathbb{E}|u_s(y)u_s(z)|\Lambda(y-z)dydzds.
\]

But we also have from the mild solution and Assumption 1.5 that
\[
\mathbb{E}|u_s(y)u_s(z)| \geq \left| (\mathcal{G}u_0)_s(y)(\mathcal{G}u_0)_s(z) \right|
\]
\[ + \xi^2 \int_0^1 \int_{D^2} p_D(s-s_1, y_1, y_1)p_D(s-s_1, z_1, z_1) \mathbb{E}|u_{s_1}(y_1)u_{s_1}(z_1)|\Lambda(y_1-z_1)dy_1dz_1ds_1.
\]

Thus, combining this inequality with the previous one, we get
\[ E[u_t(x)]^2 \geq (G_{u_0})^2_t(x) \]

\[ \leq \int_0^t \int_{D_2} p_D(t-s,x,y)p_D(t-s,x,z)|G_{u_0}(y)(G_{u_0}(z)|\Lambda(y-z)dydzds \]

\[ \times \mathbb{E}[u_{s_1}(y_1)u_{s_1}(z_1)|\Lambda(y_1-z_1)dy_1dz_1ds_1dydzds]. \]

Continuing this iteration and possibly relabeling the variables, we end up with

\[ E[u_t(x)]^2 \geq (G_{u_0})^2_t(x) \]

\[ + \sum_{n=1}^{\infty} \left( \xi^2/\sigma^2 \right)^n \int_0^t \int_{D_2} \int_0^{s_1} \int_{D_2} \int_0^{s_2} \ldots \int_{D_2} \int_0^{s_{n-1}} \int_{D_2} |G_{u_0}(y_n)(G_{u_0}(z_n)| \Lambda(x_i-y_i)dy_idz_ids_i \]

where we have set \( y_0 := x =: z_0 \) and \( s_0 := t \). Now for \( x \in D_\varepsilon \), choose for \( i = 1, 2, \ldots, n \), \( x_i \) and \( y_i \) such that

\[ y_i \in B\left(x, \frac{(s_{i-1} - s_i)^{1/\alpha}}{3}\right) \cap B\left(y_{i-1}, \frac{(s_{i-1} - s_i)^{1/\alpha}}{3}\right) \]

and

\[ z_i \in B\left(x, \frac{(s_{i-1} - s_i)^{1/\alpha}}{3}\right) \cap B\left(z_{i-1}, \frac{(s_{i-1} - s_i)^{1/\alpha}}{3}\right) \]

so that

\[ |z_i - z_{i-1}| < (s_{i-1} - s_i)^{1/\alpha} \quad \text{and} \quad |y_i - y_{i-1}| < (s_{i-1} - s_i)^{1/\alpha}. \]

Furthermore,

\[ |z_i - y_i| < (s_{i-1} - s_i)^{1/\alpha}. \]

These estimates will ensure that, for all \( i = 1, 2, \ldots, n \),

\[ p_D(s_{i-1} - s_i, y_{i-1}) \geq C_1(s_{i-1} - s_i)^{-d/\alpha}e^{-\mu_1(s_{i-1} - s_i)}, \]

\[ p_D(s_{i-1} - s_i, z_{i-1}) \geq C_2(s_{i-1} - s_i)^{-d/\alpha}e^{-\mu_1(s_{i-1} - s_i)} \]

and

\[ \Lambda(y_i - z_i) \geq C_3(s_{i-1} - s_i)^{-\beta/\alpha}. \]
for some positive constants $C_1, C_2$ and $C_3$, thanks to Proposition 2.1 and Assumption 1.4. Moreover, since the initial solution $u_0$ is bounded, using Lemma 4.2, we get

$$\left| (Gu_0)_{s_n}(y_n)(Gu_0)_{s_n}(z_n) \right| \geq C_4 e^{-2\mu t_n}.$$

Combining these estimates yields

$$\mathbb{E}|u_t(x)|^2 \geq C_7 e^{-2\mu t} \sum_{n=1}^{\infty} \left( \frac{\xi^2}{\sigma^2} \right)^n \int_{\Theta_n(t)} \int_{A_1 \times B_1} \cdots \int_{A_n \times B_n} \prod_{i=1}^{n} (s_{i-1} - s_i)^{-\beta/\alpha} (s_{i-1} - s_i)^{-2d/\alpha} ds_i dy_i dz_i ds_i$$

Where $\Theta_n(t) := \{(s_0, s_1, \ldots, s_{n-1}) \in \mathbb{R}^n_+ : s_0 > s_1 > \ldots > s_{n-1}\}$,

$A_i := \left\{ y_i \in B \left( x, \frac{(s_{i-1} - s_i)^{1/\alpha}}{d} \right) \cap B \left( y_{i-1}, \frac{(s_{i-1} - s_i)^{1/\alpha}}{d} \right) \right\}$

and $B_i := \left\{ z_i \in B \left( x, \frac{(s_{i-1} - s_i)^{1/\alpha}}{d} \right) \cap B \left( z_{i-1}, \frac{(s_{i-1} - s_i)^{1/\alpha}}{d} \right) \right\}$.

It is not hard to see that $\text{Volume}(A_i) \cap \text{Volume}(B_i) \geq C_6 (s_{i-1} - s_i)^{d/\alpha}$ for all $i = 1, 2, \ldots, n$. Taking into account the latter gives

$$\mathbb{E}|u_t(x)|^2 \geq C_7 e^{-2\mu t} \sum_{n=1}^{\infty} \left( \frac{\xi^2}{\sigma^2} \right)^n \int_{\Theta_n(t)} \prod_{i=1}^{n} (s_{i-1} - s_i)^{-\beta/\alpha} ds_i$$

$$= C_8 e^{-2\mu t} \sum_{n=1}^{\infty} \left( \frac{C_9 \xi^2}{\sigma^2} \right)^n \frac{t^{n(1-\beta/\alpha)}}{\Gamma \left( n(1-\beta/\alpha) + 1 \right)}.$$ 

where we have used Lemma 2.3 with $a = 0$ and $b = t$. Finally applying the approximation 4.1 yields the desired result.

Armed with all the necessary tools, we can now prove our main results. 

\section{Proofs of the main results}

\textit{Proof of Theorem 1.7.} For the upper bound, we combine the Burkhölder-Davis-Gundy’s, Minkowski’s and Jensen’s inequalities after taking the $p^{th}$ power of the mild solution to get
\[ E|u_t(x)|^p \]
\[ \leq 2^{p-1} \left\{ \left( (G u_0)_t(x) \right)^p \right\} \]
\[ + \xi_{p}^{\nu} \left( \int_0^t \int_{\Omega} p_{D} (t-s,x,y) p_{D} (t-s,x,z) \Lambda(y-z) E|\sigma(u_s(y))\sigma(u_s(z))| dy dz ds \right)^{p/2} \]
\[ \leq 2^{p-1} \left\{ \left( (G u_0)_t(x) \right)^p \right\} \]
\[ + \xi_{p}^{\nu} \left( \int_0^t \left( \sup_{y \in \Omega} E|\sigma(u_s(y))|^p \right)^{2/p} \int_{\Omega \times \Omega} p_{D} (t-s,x,y) p_{D} (t-s,x,z) \Lambda(y-z) dy dz ds \right)^{p/2} \]

Where \( z_p \) is as in Theorem 1.7, See for example [13]. Note that we have also used the following fact straight from Hölder’s inequality:
\[ E|\sigma(u_s(y))\sigma(u_s(z))| \leq \left[ E|\sigma(u_s(y))|^2 \right]^{1/2} \left[ E|\sigma(u_s(z))|^2 \right]^{1/2} \]
\[ \leq \sup_{y \in \Omega} E|\sigma(u_s(y))|^2. \]

Because \( u_0 \) is bounded, using Assumption 1.5 and Lemma 4.3, we get
\[ \int_0^t \left( \sup_{y \in \Omega} E|u_s(y)|^p \right)^{2/p} \int_{\Omega \times \Omega} p_{D} (t-s,x,y) p_{D} (t-s,x,z) \Lambda(y-z) dy dz ds \]
\[ \leq \left. (f(t)) := \left( \sup_{x \in \Omega} E|u_t(x)|^p \right)^{2/p} \right. \]

Thus, defining a new function \( F(t) := e^{(2-\delta)\mu_1 t} f(t) \), we get for all \( t > 0 \),
\[ F(t) \leq c_1 + c_2 \xi_{p}^{\nu} \int_0^t F(s)(t-s)^{-\beta/\alpha} ds. \]

Finally applying Proposition 4.1 with \( \rho = 1 - \beta/\alpha \) yields the desired upper bound.

For the lower bound, we combine Proposition 2.4 and Proposition 4.7 with \( \nu = \frac{\alpha - \beta}{\alpha} > 0 \), together with Jensen’s inequality to get the expected bound. \[ \square \]

**Proof of Theorem 1.8.** The solution to (1.1) (when \( \sigma = Id \)) has the following Wiener-chaos expansion in \( L^2(\Omega) \):
\[ u_t(x) = (G u_0)_t(x) + \sum_{n=1}^{\infty} \xi^n I_n(h_n(.,t,x)) := \sum_{n=0}^{\infty} \xi^n I_n(h_n(.,t,x)), \]

(3.1)
where \( I_0 \) is the identity map on \( \mathbb{R} \), \( h_0(t, x) = (G u_0)_t(x) \) and \( I_n \) denotes the multiple Wiener integral with respect to \( F \) in \( \mathbb{R}^n \times D^n \) for any \( n \geq 1 \), and for any \( (t_1, ..., t_n) \in \mathbb{R}^n \), \( x_1, ..., x_n \in D \\

\[
\begin{align*}
&h_n(t_1, x_1, ..., t_n, x_n, t, x) = p_D(t - t_n, x_n) p_D(t_n - t_{n-1}, x_n, x_{n-1}) ... p_D(t_2 - t_1, x_2, x_1) (G u_0)_t(x_1) 1_{\{0 < t_1 < ... < t_n < t\}}.
\end{align*}
\]

See for example [1], [9] or [18] and references therein for more details. It follows that

\[
\mathbb{E}|u_t(x)|^2 = |(G u_0)_t(x)|^2 + \sum_{n=1}^\infty 2^n n! \|\tilde{h}_n(., t, x)\|^2_{H^\otimes 2},
\]

where \( \tilde{h}_n \) is the symmetrization of \( h_n \), i.e.

\[
\begin{align*}
n!\|\tilde{h}_n(., t, x)\|^2_{H^\otimes 2} &= \int_{T_n(t) \times T_n(s)} \int_{D^{2n}} p_D(t - t_n, x_n) p_D(t - s_n, x_n - y_n) \gamma(t_n - s_n) \Lambda(x_n - y_n) \\
&\times p_D(t_n - t_{n-1}, x_n, x_{n-1}) p_D(s_{n-1} - s_n, y_{n-1}, y_n-1) \gamma(t_n - s_n-1) \Lambda(x_n - y_{n-1}) ... \\
&\times p_D(t_2 - t_1, x_2, x_1) p_D(s_2 - s_1, y_2, y_1) \gamma(t_1 - s_1) \Lambda(x_1 - y_1) (G u_t(x_1)) \\
&\times (G u_0)_1(y_1) dx_1 dx_2 ... dx_n dy_1 ... dy_n dt_1 ... dt_n ds_1 ... ds_n,
\end{align*}
\]

where

\[
T_n(r) := \{(r_1, ..., r_n) \in \mathbb{R}^n : 0 < r_1 < r_2 < ... < r_n < r\}.
\]

For simplicity, we write \( dt = dt_1 ... dt_n, ds = ds_1 ... ds_n, dx = dx_1 ... dx_n \) and \( dy = dy_1 ... dy_n \). Let us first take care of the upper bound. To this end we apply Lemma 2.2 iteratively to get

\[
n!\|\tilde{h}_n(., t, x)\|^2_{H^\otimes 2} \leq C_1 e^{-(2-\delta)\mu_t} \int_{T_n(t) \times T_n(s)} \prod_{i=1}^n (t_i - s_i) \prod_{i=1}^n \gamma(t_i - s_i) \prod_{i=1}^n (t_{i+1} + s_{i+1} - (t_i + s_i))^{-\beta/\alpha} dt ds,
\]

where we have set \( s_{n+1} := t =: t_{n+1} \). It follows that

\[
n!\|\tilde{h}_n(., t, x)\|^2_{H^\otimes 2} \leq C_1 e^{-(2-\delta)\mu_t} \int_{T_n(t) \times T_n(s)} \prod_{i=1}^n \gamma(t_i - s_i) \prod_{i=1}^n (t_{i+1} - t_i)^{-\beta/\alpha} dt ds.
\]

We first take care of the integrals \( \int_{T_n(s)} \prod_{i=1}^n \gamma(t_i - s_i) ds \). We have

\[
\int_{T_n(s)} \prod_{i=1}^n \gamma(t_i - s_i) ds \leq \int_{[0,t]^n} \prod_{i=1}^n \gamma(t_i - s_i) ds.
\]

Now for \( i = 1, 2, ..., n \),

\[
\begin{align*}
\int_0^t \gamma(t_i - s_i) ds_i &= \int_{t_{i-t}}^t \gamma(r) dr \\
&\leq \int_0^t \gamma(r) dr + \int_0^t \gamma(r) dr, \\
&\leq 2 \int_0^t \gamma(r) dr,
\end{align*}
\]

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since $\gamma$ satisfies Assumption 1.3. Therefore, setting $\kappa(t) := 2 \int_0^t \gamma(r)dr$, we have

$$\int_{T_n(t) \times T_n(s)} \prod_{i=1}^n \gamma(t_i - s_i) \prod_{i=1}^n (t_{i+1} - t_i)^{-\beta/\alpha} dt ds \leq \kappa^n \int_{T_n(t)} \prod_{i=1}^n (t_{i+1} - t_i)^{-\beta/\alpha} dt \leq \frac{C_1^{n+1} \kappa n^{(1-\beta)/\alpha}}{\Gamma(n(1-\beta/\alpha) + 1)} \cdot C_1 = C_1(\alpha, \beta).$$

Note the use of Proposition 2.3 with $a = 0$ and $b = t$ in the second inequality. Now using Stirling’s approximation (4.1), we have for $n = 0, 1, 2, \ldots$

$$\mathbb{E}|u_t(x)|^2 \leq C_2 e^{(2-\delta)\mu t} \sum_{n \geq 0} \frac{\left(C_1 \xi^2 \kappa \right)^n}{(n!)^{1-\beta/\alpha}}.$$

Next, using Minkowski’s inequality and the equivalence of norms in a fixed Wiener chaos space, it follows that

$$\left( \mathbb{E}|u_t(x)|^p \right)^{1/p} \leq \sum_{n=0}^{\infty} (p-1)^{n/2} \xi^n \left( n! \|	ilde{h}_n(\cdot, t, x)\|_{\mathcal{H}^{p/2}} \right)^{1/2} \leq C_2 e^{(1-\delta)\mu t} \sum_{n=0}^{\infty} (p-1)^{n/2} \xi^n \frac{K^{n/2} p^{n(\alpha-\beta)/2\alpha}}{(n!)^{(\alpha-\beta)/2\alpha}}.$$

Finally, using Proposition 4.6 with $\nu = (\alpha - \beta)/2\alpha$ yields the desired bound.

We now turn our attention to the proof of the lower bound. From equation (3.2) we have,

$$n! \|\tilde{h}_n(\cdot, t, x)\|_{\mathcal{H}^{p/2}}^2 \geq \int_{T(n,s,t)} \int_{D^n} p_D(t - t_n, x, x_n) p_D(t - s_n, x, y_n) \gamma(t_n - s_n) \Lambda(x_n - y_n) \times p_D(s_n - s_{n-1}, y_n, y_{n-1}) \gamma(t_{n-1} - s_{n-1}) \Lambda(x_{n-1} - y_{n-1}) \ldots p_D(t_2 - t_1, x_2, x_1) \times p_D(s_2 - s_1, y_2, y_1) \gamma(t_1 - s_1) \Lambda(x_1 - y_1) \left( (\mathcal{G}u_0)_{t_1}(x_1) \right) \left( (\mathcal{G}u_0)_{s_1}(y_1) \right) dx dy dt ds.$$

where

$$T(n, s, t) := \{ T_n(t) \times T_n(s) \} \cap \left\{ \frac{t}{3} < t_1, \text{ for } i = 1, 2, \ldots, n, \frac{t}{3} \leq t_{i+1} - s_i, s_{i+1} - s_i \leq t_{i+1} - t_i \right\}$$

Since $x \in D_t$, choose, for $i = 1, \ldots, n$, $x_i, y_i \in D_t$ satisfying:

$$x_i \in B \left( x, \frac{1}{3} (t_{i+1} - t_i)^{1/\alpha} \right) \cap B \left( x_{i-1}, \frac{1}{3} (t_{i+1} - t_i)^{1/\alpha} \right)$$

and

$$y_i \in B \left( x, \frac{1}{3} (s_{i+1} - s_i)^{1/\alpha} \right) \cap B \left( y_{i-1}, \frac{1}{3} (s_{i+1} - s_i)^{1/\alpha} \right).$$

with $x_0 := x =: y_0$. Thus, for $i = 1, 2, \ldots, n$, it follows that

$$|x_i - x_{i-1}| < (t_{i+1} - t_i)^{1/\alpha} \quad \text{and} \quad |y_i - y_{i-1}| < (s_{i+1} - s_i)^{1/\alpha}.$$
Furthermore,

\[ |x_i - y_i| < (t_i - t_{i-1})^{1/\alpha} \]  

This gives us \( \Lambda(x_i - y_i) \geq c_1(t_{i+1} - t_i)^{-\beta/\alpha} \).

Now appealing to Proposition 2.1, we get

\[ p_D(t_{i+1} - t_i, x_i, x_{i-1}) \geq c_2(t_{i+1} - t_i)^{-d/\alpha}e^{-\mu_1(t_{i+1} - t_i)} \]

and

\[ p_D(s_{i+1} - s_i, y_i, y_{i-1}) \geq c_3(s_{i+1} - s_i)^{-d/\alpha}e^{-\mu_1(s_{i+1} - s_i)}. \]

Next, denoting \( A_i := \{ x_i \in B(x, \frac{1}{\sqrt{3}}(t_{i+1} - t_i)^{1/\alpha}) \cap B(x_{i-1}, \frac{1}{\sqrt{3}}(t_{i+1} - t_i)^{1/\alpha}) \} \) and

\( B_i := \{ y_i \in B(x, \frac{1}{\sqrt{3}}(s_{i+1} - s_i)^{1/\alpha}) \cap B(y_{i-1}, \frac{1}{\sqrt{3}}(s_{i+1} - s_i)^{1/\alpha}) \} \), it follows that

\[ n! \| \mathcal{H}_n(., t, x) \|_{\mathcal{H}^2}^2 \geq C_4 e^{-2\mu_1 t} \int_{T(n,s,t)} \int_{A_1 \times B_1} \cdots \int_{A_n \times B_n} \prod_{i=1}^n \gamma(t_i - s_i) \times \prod_{i=1}^n (t_{i+1} - t_i)^{-\beta/\alpha}(t_{i+1} - t_i)^{-d/\alpha}(s_{i+1} - s_i)^{-d/\alpha} \, dx \, dy \, dt \, ds. \]

Note the use of Lemma 4.2-a) and the fact that \( u_0 \) is bounded in the above display. It’s not hard to see that, for \( i = 1, 2, ..., n \), \( \text{Volume}(A_i) \geq C_i(t_{i+1} - t_i)^{d/\alpha} \) and \( \text{Volume}(B_i) \geq c_i(s_{i+1} - s_i)^{d/\alpha} \) for some positive constants \( c_i \) and \( C_i \) independent of \( t_i \) and \( s_i \) respectively. This will ensure that

\[ n! \| \mathcal{H}_n(., t, x) \|_{\mathcal{H}^2}^2 \geq C_5 e^{-2\mu_1 t} \int_{T(n,s,t)} \prod_{i=1}^n \gamma(t_i - s_i) \prod_{i=1}^n (t_{i+1} - t_i)^{-\beta/\alpha} \, dt \, ds. \]

The iterative integrals with time correlation integrands deserve a particular attention. We take care of them first. Assuming \( t/3 < t_1 \) and \( t_i \geq t_{i+1} - s_i \) for \( i = 1, 2, ..., n \), we start with

\[ \int_{s_{n-1}}^t \gamma(t_n - s_n) \, ds_n = \int_{t_{n-t}}^{t_n-s_{n-1}} \gamma(r) \, dr \]

\[ \geq \int_0^{t/3} \gamma(r) \, dr, \]

since \( t_n - t < 0 \) and \( t_n - s_{n-1} \geq t/3 \).

Next,

\[ \int_{s_{n-2}}^t \gamma(t_{n-1} - s_{n-1}) \, ds_{n-1} = \int_{t_{n-1}-t}^{t_{n-1-s_{n-2}} \gamma(r) \, dr \]

\[ \geq \int_0^{t/3} \gamma(r) \, dr, \]

where, again \( t_{n-1} - t < 0 \) and \( t_n - s_{n-2} \geq t/3 \).
Thus, setting
\[ \eta(t) := \int_0^{t/3} \gamma(r) dr \]
and continuing this way, we get
\[
\int_{T(n,s,t)} \prod_{i=1}^n \gamma(t_i - s_i) ds \geq C_6 \eta^{n-1} \int_0^t \gamma(t_1 - s_1) ds_1
\]
\[ \geq C_6 \eta^{n-1} \int_{t_1-t}^{t_1} \gamma(r) dr \]
\[ \geq c_7 \gamma \eta^n \]
since \( t_1 - t < 0 \) and \( t/3 < t_1 \). Setting \( T_n(t/3, t) := \{(t_1, \cdots, t_n) : t/3 < t_1 < t_2 < \cdots t_n < t\} \), it follows that
\[
n!\| \tilde{h}_n(. , t, x) \|_{H^2} \geq C_8 e^{-2\mu_1 t} \eta^n \int_{T_n(t/3, t)} \prod_{i=1}^n (t_{i+1} - t_i) - \beta/\alpha dt
\]
\[ \geq \frac{C_9^{n+1} e^{-2\mu_1 t} \eta^n t^{n(\frac{\alpha-\beta}{\alpha})}}{\Gamma\left(n(1 - \beta/\alpha) + 1\right)} \]
\[ \geq \frac{C_{10} e^{-2\mu_1 t} \lambda^n \eta^n t^{n(\frac{\alpha-\beta}{\alpha})}}{(n!)^{\frac{\alpha-\beta}{\alpha}}}, \]
where we have used Lemma 2.3 with \( a = t/3 \) and \( b = t \) and the approximation (4.1). Therefore, applying Proposition 4.7 with \( v = \frac{\alpha-\beta}{\alpha} \) yields,
\[
E|u_t(x)|^2 \geq C_9 e^{-2\mu_1 t} \sum_{n \geq 1} \left( \frac{\xi^2 \lambda \eta}{n!^{\frac{\alpha-\beta}{\alpha}}} t^{n(\frac{\alpha-\beta}{\alpha})} \right)
\]
\[ \geq C_{10} e^{t \left( c \xi^{\alpha-\beta} - 2\mu_1 \right)}. \]
Finally an application of Jensen’s inequality concludes the proof. \( \square \)

**Proof of Corollary 1.9.** This follows immediately from Theorem 1.8 with
\[ \kappa(t) = 2C_H \int_0^t r^{2H-2} = C_1 t^{2H-1} \quad \text{and} \quad \eta(t) = C_H \int_0^{t/3} r^{2H-2} = C_2 t^{2H-1}, \]
for positive constants \( C_1 = C_1(H) \) and \( C_2 = C_2(H) \). \( \square \)
4 Appendix

We compile in this section some results from other authors that we have used in our paper.

**Proposition 4.1.** [11, Proposition 2.5] Let $\rho > 0$ and suppose that $f(t)$ is a locally integrable function satisfying

$$f(t) \leq c_1 + \kappa \int_0^t (t-s)^{\rho-1} f(s) \, ds \quad \text{for all } t > 0,$$

where $c_1$ is some positive constant. Then, we have

$$f(t) \leq c_2 e^{c_3 (\Gamma(\rho) \kappa)^{1/\rho} t} \quad \text{for all } t > 0,$$

for some positive constants $c_2$ and $c_3$.

**Lemma 4.2.** [17, Proposition 3.1] For any $\epsilon \in (0, \frac{1}{2})$, there exist positive constants $c_1(\epsilon)$ such that for all $x, w \in D_{\epsilon}$ and $t > 0$,

$$a) \int_{D_{\epsilon}} p_D(t, x, y) dy \geq c_1 e^{-\mu_1 t}.$$

If we further impose $|x - w| \leq t^{1/\alpha}$, then there exists a positive constant $c_2(\epsilon)$ such that

$$b) \int_{D_{\epsilon} \times D_{\epsilon}} p_D(t, x, y) p_D(t, w, z) \Lambda(y - z) dydz \geq c_2 e^{-2\mu_1 t - \beta/\alpha}.$$

**Lemma 4.3.** [17, Proposition 3.2] For all $\delta > 0$, there exists $c_2(\delta) > 0$ such that for all $x, w \in D$ and $t > 0$

$$a) \int_D p_D(t, x, y) dy \leq ce^{-\mu_1 t}$$

$$b) \int_{D \times D} p_D(t, x, y) p_D(t, w, z) \Lambda(y - z) dydz \leq c_2 e^{-(2-\delta)\mu_1 t - \beta/\alpha}.$$

**Theorem 4.4.** [7, theorem 1.1] Assume $\alpha \in (0, 2)$. There exists a positive constant $C$ such that for all $x, y \in D$ and $t > 0$,

$$C^{-1} e^{-\mu_1 t} \left[ \min \left( 1, \frac{\phi_1(x)}{\sqrt{t}} \right) \min \left( 1, \frac{\phi_1(y)}{\sqrt{t}} \right) \min \left( t^{-d/\alpha}, \frac{t}{|x - y|^{\alpha + d}} \right) 1_{\{t < 1\}} + \phi_1(x) \phi_1(y) 1_{\{t \geq 1\}} \right] \leq p_D(t, x, y) \leq$$

$$Ce^{-\mu_1 t} \left[ \min \left( 1, \frac{\phi_1(x)}{\sqrt{t}} \right) \min \left( 1, \frac{\phi_1(y)}{\sqrt{t}} \right) \min \left( t^{-d/\alpha}, \frac{t}{|x - y|^{\alpha + d}} \right) 1_{\{t < 1\}} + \phi_1(x) \phi_1(y) 1_{\{t \geq 1\}} \right]$$

**Theorem 4.5.** [19, theorem 2.2] Assume $\alpha = 2$. Then there exist positive constants $c_1, C_1, c_2$ and $C_2$ such that for all $x, y \in D$ and $t > 0$,

$$C_1 \min \left( 1, \frac{\phi_1(x) \phi_1(y)}{1 \wedge t} \right) e^{-\mu_1 t} e^{-c_1 |x-y|^2 / t} \leq p_D(t, x, y) \leq C_2 \min \left( 1, \frac{\phi_1(x) \phi_1(y)}{1 \wedge t} \right) e^{-\mu_1 t} e^{-c_2 |x-y|^2 / t \wedge t^{d/2}}$$

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Proposition 4.6. [2, Lemma A.1] For any $a > 0$,
\[
\sum_{k=0}^{\infty} \frac{x^k}{(k!)^\nu} \leq C_1 e^{c_1 x^{1/\nu}}, \quad x > 0,
\]
for some constants $c_1(\nu)$ and $C_1(\nu) > 0$.

Moreover,
\[
\Gamma(n\tau + 1) \sim C_n (n!)^\tau, \quad \tau > 0,
\]
where $C_n$ is such that $\lambda^{-n} \leq C_n \leq \lambda^n$ for some $\lambda(\alpha, \beta) > 1$.

Proposition 4.7. [3, Lemma 5.2] For any $b > 0$,
\[
\sum_{k=1}^{\infty} \frac{x^k}{(k!)^\nu} \geq c_1 e^{c_2 x^{1/\nu}}, \quad x > 0,
\]
for some constants $c_1(\nu) > 0$ and $c_2(\nu) > 0$.

Proposition 4.8. [1, Lemma 2.2] For any $t > 0$ and $w \in \mathbb{R}^d$,
\[
\int_{\mathbb{R}^d} e^{-t|v|^\alpha} |\omega - v|^{-d+\beta} dv \leq K_{d,\alpha,\beta} t^{-\beta/\alpha},
\]
where
\[
K_{d,\alpha,\beta} := \sup_{w \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v|^{-d+\beta}}{1 + |\omega - v|^\alpha} dv.
\]
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