CONVEX BODIES ASSOCIATED TO ACTIONS OF REDUCTIVE GROUPS

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ABSTRACT. We associate convex bodies to a wide class of graded $G$-algebras where $G$ is a connected reductive group. These convex bodies give information about the Hilbert function as well as the multiplicities of irreducible representations appearing in the graded algebra. We extend the notion of Duistermaat-Heckman measure to graded $G$-algebras and prove a Fujita type approximation theorem as well as a Brunn-Minkowski inequality for this measure. This in particular applies to arbitrary $G$-line bundles giving an equivariant version of the theory of volumes of line bundles. We generalize the Brion-Kazarnovskii formula for the degree of a spherical variety to arbitrary $G$-varieties. And finally we generalize the Bernstein theorem regarding the intersection numbers of divisors in a toric variety to a larger class of $G$-varieties. Our approach follows some of the previous works of A. Okounkov. We use the asymptotic theory of semigroups of integral points and Newton-Okounkov bodies developed in [Kaveh-Khovanskii09].

Key words: Reductive group action, multiplicity of a representation, Duistermaat-Heckman measure, moment map, graded $G$-algebra, $G$-line bundle, volume of a line bundle, semigroup of integral points, convex body, mixed volume, Brunn-Minkowski inequality.

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Note: This is a preliminary version and may contain some typos.

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INTRODUCTION

The general idea in the theory of Newton polytopes is that one can read off many geometric/topological invariants of the hypersurfaces/complete intersections in $(\mathbb{C}^*)^n$ from the Newton polytopes of their defining Laurent polynomials (see [Khovanskii84] for a survey). This fits directly into the theory of toric varieties and torus actions. Motivated by this, in the 80’s the second author had posed the question of extending the notion of a Newton polytope to actions of other reductive groups. Since then much work has been done in this direction and several of the theorems about toric varieties have been generalized to other cases e.g. spherical varieties. See [Alexeev-Brion04], [Brion89], [Brion87], [Kaveh03], [Kaveh04], [Kazarnovskii87], [Kiritchenko06], [Kiritchenko07], [Okounkov97], [Timashev06] for some of these works.

The purpose of the present paper is to associate convex bodies to a large class of reductive group actions, far generalizing the Newton polytopes of a toric variety and its analogue for spherical varieties.

Motivated by the work of Okounkov, in the papers [Lazarsfeld-Mustata08] and [Kaveh-Khovanskii08-1, Kaveh-Khovanskii09], the authors associate convex bodies to line bundles (more generally to graded linear series) on varieties, without requiring presence of a group action. These convex bodies encode information about the asymptotic of the linear series. In particular, when the line bundle is ample their volumes give the self-intersection index of the divisor class of the line bundle.

In fact, in [Kaveh-Khovanskii09] a more general situation is addressed. The authors develop an asymptotic theory of semigroups of integral points. To a large class of semigroups of integral points (which are not necessarily

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1A convex body is a compact convex subset of the Euclidean space.
finitely generated) they associate a convex body and show that it is responsible for the asymptotic behaviour of the semigroup. They call it the **Newton-Okounkov body of the semigroup**.

This theory of semigroups is then used to prove results about the Hilbert functions of a large class of graded algebras of polynomials (in one variable and with coefficients in the field of rational functions on an \( n \)-dimensional variety). This is made possible via a choice of a \( \mathbb{Z}^n \)-valued valuation on the field of rational functions. Applied to algebras of sections of line bundles (and more generally to graded linear series), these results give new proofs of some well-known results as well as some new theorems in algebraic geometry and convex geometry. In particular, the authors prove a far reaching generalization of the Ku\'snirenko theorem ([Kushnirenko76]) in toric geometry for arbitrary varieties.

In the present paper we consider the above general theory for the case of varieties equipped with an action of a reductive group. We apply the asymptotic theory of semigroups of integral points and Newton-Okounkov bodies developed in [Kaveh-Khovanskii09] to the semigroups naturally associated to varieties with a group action.

Let us start with explaining the case of invariant subvarieties of projective space. Let a connected reductive algebraic group \( G \) act linearly on a vector space \( V \). This induces an action of \( G \) on the projective space \( \mathbb{P}(V) \). Let \( X \) be a \( G \)-invariant projective subvariety of \( \mathbb{P}(V) \). Consider the homogeneous coordinate ring \( R = R(X) = \bigoplus_{k \geq 0} R_k \). It is a graded \( G \)-algebra, i.e. \( G \) acts linearly on \( R \) respecting the multiplication and the grading. Let \( \Lambda \) and \( \Lambda^+ \) denote the lattice of weights and the semigroup of dominant weights for \( G \).

For a dominant weight \( \lambda \in \Lambda^+ \) let \( V_\lambda \) denote the irreducible representation with highest weight \( \lambda \), and let \( m_{k,\lambda} \) denote the multiplicity of \( V_\lambda \) in \( R_k \). It is well-known that the set

\[
\Delta(X) = \text{conv}\left( \bigcup_{k > 0} \{ \lambda/k \mid m_{k,\lambda} \neq 0 \} \right),
\]

is a convex polytope usually called the **moment polytope of \( X \)** (see [Brion87]). For each \( k > 0 \), define a finitely supported measure \( d\mu_k \) on \( \Lambda_\mathbb{R} \), the linear span of \( \Lambda \), by:

\[
d\mu_k = \sum_{\lambda \in \Lambda^+} m_{k,\lambda} \delta_{\lambda/k},
\]

where \( \delta_x \) denotes the Dirac measure centred at the point \( x \). One knows that, for large values of \( k \), the total mass of \( d\mu_k \) (i.e. \( \sum_{\lambda} m_{k,\lambda} \)) is a polynomial whose degree we denote by \( d \). Moreover, the measures \( d\mu_k/k^{d-\dim \Delta(X)} \) converge weakly to a measure \( d\mu \) supported on the convex polytope \( \Delta(X) \). The measure \( d\mu \) is called the **Duistermaat-Heckman measure of \( X \)**, as it coincides with the Duistermaat-Heckman measure of \( X \) regarded as a Hamiltonian space: let \( X \) be smooth. Take a maximal compact subgroup \( K \) of \( G \). For this one proves that the ring \( R^U \) of unipotent invariants is finitely generated.
Fix a $K$-invariant Hermitian inner product on $V$. This induces a symplectic structure on $\mathbb{P}(V)$ and hence on $X$ which makes $X$ into a $K$-Hamiltonian manifold. Let $\phi : X \to \text{Lie}(K)^*$ denote its moment map. Then one shows that the polytope $\Delta(X)$ coincides with the image of the moment map $\phi(X)$ intersected with the positive Weyl chamber $\Lambda^+$. Moreover, let $\lambda \in \Delta(X)$ be a regular value for $\phi$ and let $X_\lambda = \phi^{-1}(\lambda)/K_\lambda$ (where $K_\lambda$ is the $K$-stabilizer of $\lambda$ in $\text{Lie}(K)^*$) be the so-called symplectic reduction at $\lambda$. Then the measure $d\mu$ coincides with the Duistermaat-Heckman measure $\text{Vol}(X_\lambda)d\gamma$ where $d\gamma$ is the (properly normalized) Lebesgue measure on $\Delta(X)$ and $\text{Vol}$ is the symplectic volume (see [Guillemin-Sternberg84]).

In [Okounkov90], Okounkov associates a convex body to the homogeneous coordinate ring of $X$ such that it projects to the moment polytope of $X$ and the push-forward of the Lebesgue measure on this convex body gives the Duistermaat-Heckman measure on $\Delta(X)$.

In the present paper we generalize the above situation. Instead of a projective $G$-variety we take an arbitrary $G$-variety $X$, and instead of the homogeneous coordinate ring, we consider a graded $G$-subalgebra $A$ of the algebra of polynomials $\mathbb{C}(X)[t]$ (in one variable $t$ and with coefficients in the field of rational functions $\mathbb{C}(X)$).

To include the algebra of sections of $G$-line bundles in our discussion, we will consider a slightly more general type of actions of $G$ on $\mathbb{C}(X)$, namely the \textit{twisted actions of $G$} (see Section 2.2). Let $\varphi : G \to \mathbb{C}(X)^*$ be a map which satisfies the group cocycle condition. We define the \textit{action of $G$ on $\mathbb{C}(X)$ twisted by $\varphi$} by $g \ast \varphi f = (g \cdot f)\varphi(g)$, where $g \in G$, $f \in \mathbb{C}(X)$ and $g \cdot f$ is the usual action of $G$ on $\mathbb{C}(X)$. This is motivated by the following: let $\mathcal{L}$ be a $G$-line bundle on $X$ with a non-zero section $\tau \in H^0(X, \mathcal{L})$. Put $D = \text{Div}(\tau)$. The map $f \mapsto f\tau$ identifies the vector space $L(D) = \{f \in \mathbb{C}(X) \mid (f) + D \geq 0\}$ with $H^0(X, \mathcal{L})$. Under this identification the action of $G$ on $H^0(X, \mathcal{L})$ corresponds to the action of $G$ on $L(D)$ twisted by a cocycle $\varphi$ where $\varphi(g) = g \cdot \tau/\tau$.

Let $X$ be an irreducible $G$-variety of dimension $n$. Let $A$ be a graded $G$-invariant subalgebra of the polynomial algebra $\mathbb{C}(X)[t]$. We denote the $k$-th homogeneous component of $A$ by $A_k$. We have $A_k = L_k t^k$ for a subspace $L_k \subset \mathbb{C}(X)$. We call $L_k$ the $k$-th subspace of $A$. For simplicity we will assume that for any large $k > 0$, $L_k$ is non-zero. (This assumption is not necessary but makes the statements of the main results simpler.) We will consider three types of graded $G$-algebras:

- Algebra $A_L$ generated by constants and a $G$-invariant finite dimensional subspace $L \subset \mathbb{C}(X)$ in degree $1$. That is, $A_L = \bigoplus_{k \geq 0} L^k t^k$.
- An algebra of $G$-\textit{integral type}, i.e. a $G$-algebra $A$ which is a finite module over a $G$-algebra $A_L$ for some $G$-invariant subspace $L$.
- An algebra of $G$-\textit{almost integral type}, i.e. a $G$-algebra $A$ which is contained in an algebra of $G$-integral type.
Homogeneous coordinate rings of projective $G$-varieties are typical examples of algebras of $G$-integral type and rings of sections of $G$-line bundles are typical examples of algebras of almost $G$-integral type. Throughout the rest of introduction, unless otherwise stated, $A$ is an algebra of almost integral type.

To $A$ we associate three convex bodies:

- The moment body $\Delta(A) \subset \Lambda \mathbb{R}$, the linear span of $\Lambda$. The maximum dimension of $\Delta(A)$ is $r = \text{rank}(X)$.

- The multiplicity body $\hat{\Delta}(A) \subset \mathbb{R}^n$. The maximum dimension of $\hat{\Delta}(A)$ is $\hat{r}$, the transcendence degree of the field $\mathbb{C}(X)^U$ of unipotent invariants.

- The string body $\tilde{\Delta}(A) \subset \mathbb{R}^n + N$, where $N$ is the number of positive roots. The maximum dimension of $\tilde{\Delta}(A)$ is $n$.

Also we have natural linear projections $\hat{\pi} : \hat{\Delta}(A) \to \Delta(A)$ and $\tilde{\pi} : \tilde{\Delta}(A) \to \hat{\Delta}(A)$. These bodies encode information respectively about the irreducible representations appearing in the homogeneous components of $A$, their multiplicities, and the Hilbert function of $A$. Below we briefly explain their construction and properties.

As before, for $\lambda \in \Lambda^+$ let $m_{k,\lambda}$ denote the multiplicity of the irreducible representation $V_\lambda$ in the $k$-th homogeneous component $A_k$. We define $\Delta(A)$ by

$$\Delta(A) = \text{conv}(\bigcup_{k>0} \{\lambda/k \mid m_{k,\lambda} \neq 0\}).$$

It can be shown that $\Delta(A)$ is a convex body. We call it the moment body of the algebra $A$ (see Section 3.1). When $A$ is finitely generated it is a convex polytope.

Next, we define a convex body $\hat{\Delta}(A)$ and a linear projection $\hat{\pi} : \hat{\Delta}(A) \to \Delta(A)$. The construction of $\hat{\Delta}(A)$ depends on a choice of a Borel invariant $\mathbb{Z}^n$-valued valuation on $\mathbb{C}(X)$ (See Section 4.1). Our construction follows that of Okounkov which constructed this convex body for the homogeneous coordinate ring of a projective $G$-variety ([Okounkov96]). Let $p = \dim \Delta(A)$ and $\hat{p} = \dim \hat{\Delta}(A)$. Also for $k > 0$ let the finitely supported measure $d\mu_k$ be as in (II).

**Theorem 1.** 1) The measures $d\mu_k/k^{\hat{p}-p}$ converge weakly to a measure $d\mu_A$ which we call the Duistermaat-Heckman measure of the $G$-algebra $A$. 2) The measure $d\mu_A$ coincides with the push-forward (under the linear projection $\hat{\pi}$) of the Lebesgue measure on the multiplicity body $\hat{\Delta}(A)$ to the moment body $\Delta(A)$.

There is a natural product on the collection of subspaces of rational functions. For two subspaces $L', L'' \subset \mathbb{C}(X)$, the product $L'L''$ is the linear span of $fg$ where $f \in L'$, $g \in L''$. One also defines a componentwise product of graded algebras. Let $A', A'' \subset \mathbb{C}(X)[t]$ be graded algebras. For each $k \geq 0$ write $A'_k = L'_kt^k$, $A''_k = L''_kt^k$ for subspaces of rational functions $L'_k$, $L''_k$. 


For each dominant weight \( \lambda \in \Lambda^+ \) one constructs a so-called

\[ L''_k. \] Then the product \( A = A'A'' \) is the subalgebra whose \( k \)-th homogeneous component is \( L'_k L''_k \).

**Theorem 2** (Brunn-Minkowski type inequality for Duistermaat-Heckman measure). 1) Let \( \bar{f} \) be the transcendence degree (over \( \mathbb{C} \)) of the subfield \( \mathbb{C}(X)^U \) of unipotent invariants. Let \( A', A'' \) be two graded \( G \)-subalgebras in \( \mathbb{C}(X)[[t]] \) with \( A = A'A'' \) their componentwise product. Let \( d\mu_A = f_A d\gamma \), \( d\mu_{A'} = f_{A'} d\gamma \) and \( d\mu_{A''} = f_{A''} d\gamma \) denote the corresponding Duistermaat-Heckman measures with the density functions \( f_A, f_{A'} \) and \( f_{A''} \) respectively. (Here \( d\gamma \) is a Lebesgue measure.) Then for \( \lambda' \in \Delta(A'), \lambda'' \in \Delta(A'') \) we have:

\[ f_{A'}(\lambda')^{1/\bar{f}} + f_{A''}(\lambda'')^{1/\bar{f}} \leq f_A(\lambda' + \lambda'')^{1/\bar{f}}. \]

2) In particular, let \( L', L'' \) be \( G \)-line bundles on a projective \( G \)-variety \( X \). Also assume that for large \( k \), \( L' \otimes^k \) and \( L'' \otimes^k \) have non-zero sections. Then the same inequality as in 1) holds for the density functions of the Duistermaat-Heckman measures of \( L' \otimes L'', L' \) and \( L''. \)

This inequality generalizes the log-concavity result in [Okounkov96].

Let \( L \subset \mathbb{C}(X) \) be a finite dimensional subspace of rational functions. To \( L \) one naturally associates a rational map \( \Phi_L : X \to \mathbb{P}(L^*) \) called the Kodaira map of \( L \). Here \( \mathbb{P}(L^*) \) denotes the projectivization of the dual vector space of \( L \) (see Section 13). Let \( Y_L = \Phi_L(X) \) be the closure of the image of \( X \). Then the homogeneous coordinate ring of \( Y_L \) as a subvariety of \( \mathbb{P}(L^*) \) is the graded algebra \( A_L \) whose \( k \)-th homogeneous component is \( L^k \otimes^k \). When \( L \) is \( G \)-invariant, \( \Phi_L \) is a \( G \)-equivariant map and thus \( Y_L \) is a \( G \)-invariant subvariety and its homogeneous coordinate ring \( A_L \) is a \( G \)-algebra.

The Fujita approximation theorem in the theory of divisors states that the so-called volume of a divisor can be approximated arbitrarily closely by the self-intersection numbers of ample divisors ([Fujita94], [Lazarsfeld94] Section 11.4). We prove an analogue of the Fujita approximation theorem for the Duistermaat-Heckman measure defined above. For \( k > 0 \) let \( dp_k \) denote the Duistermaat-Heckman measure associated to the graded \( G \)-algebra \( A_{L_k} \) (equivalently to the projective \( G \)-variety \( Y_{L_k} \subset \mathbb{P}(L^*_k) \)). Also let \( O_{1/k} : \Lambda_R \to \Lambda_R \) denote the multiplication by the scalar \( 1/k \).

**Theorem 3** (Fujita approximation type theorem for Duistermaat-Heckman measure). The measures \( O_{1/k}^*(dp_k)/k^{dp-p} \) weakly converge to the measure \( d\mu_A \) on \( \Delta(A) \). Here \( O_{1/k}^* \) denotes the push-forward measure under the map \( O_{1/k} \).

When \( A \) is the algebra of sections of a \( G \)-line bundle \( L \), the above theorem asserts that the Duistermaat-Heckman measure of \( L \) can be approximated arbitrarily closely by the Duistermaat-Heckman measure of very ample line bundles.

The third and last convex body is the string body of the \( G \)-algebra \( A \) (see Section 5.2). For each dominant weight \( \lambda \in \Lambda^+ \) one constructs a so-called
string polytope $\Delta(\lambda)$ (see [Littelmann98]). A string polytope has the property that the set of integral points in it are in one-to-one correspondence with the elements of a so-called crystal basis for the irreducible representation $V_\lambda$. One easily extends the construction of $\Delta(\lambda)$ to any $\lambda \in \Lambda^+_W$, the positive Weyl chamber. The string polytopes are a generalization of the construction of more well-known Gelfand-Cetlin polytopes for GL$(n, \mathbb{C})$. The construction of string polytopes for a reductive group $G$ depends on the choice of a reduced decomposition of the longest element $w_0$ in the Weyl group.

The convex body $\tilde{\Delta}(A)$ is the convex body fibred over $\hat{\Delta}(A)$ with string polytopes as fibres. More precisely, we have a projection $\tilde{\pi}: \tilde{\Delta}(A) \to \hat{\Delta}(A)$ such that for each $a \in \hat{\Delta}(A)$, the fibre $\tilde{\pi}^{-1}(a)$ is the string polytope $\Delta(\lambda)$, where $\lambda = \hat{\pi}(a)$.

The idea of using Gelfand-Cetlin polytopes in this context goes back to Okounkov. Following a question of the second author to extend the notion of Newton polytope of a toric variety to other reductive group actions, Okounkov used Gelfand-Cetlin polytopes to construct such polytopes for spherical varieties for actions of classical groups ([Okounkov97]). Later Alexeev and Brion used string polytopes to generalize Okounkov’s construction to arbitrary reductive groups ([Alexeev-Brion04]).

One can show that similar to the multiplicity body $\hat{\Delta}(A)$, the moment body $\Delta(A)$ and the string body $\tilde{\Delta}(A)$ also fit into the general construction of a convex body associated to a graded algebra and a choice of a valuation on the field of rational functions $\mathbb{C}(X)$ (see Section 1.4). In the present paper we are not concerned with this point.

The string convex body is a special case of the convex bodies considered in [Lazarsfeld-Mustata08], [Kaveh-Khovanskii08-1] and [Kaveh-Khovanskii09]. When the $G$-variety under consideration is spherical (and hence any $G$-algebra $A \subset \mathbb{C}(X)[t]$ is multiplicity-free) the multiplicity body $\hat{\Delta}(A)$ coincides with the moment body $\Delta(A)$ (see Section 6.2). Moreover, if the algebra $A$ is finitely generated (e.g. the homogeneous coordinate ring of a spherical $G$-variety) we see that the string body of $A$ is a polytope. These construct a rich class of examples for which it is guaranteed that the convex bodies in [Lazarsfeld-Mustata08] and [Kaveh-Khovanskii08-1] are polytopes.

From the main theorems on the asymptotic of semigroups of integral points (Section 1.1) and the construction of the string body of $A$ it follows that the volume of the string body is responsible for the asymptotic of the Hilbert function of $A$:

**Theorem 4.** Let $H_A(k) = \dim A_k$ be the Hilbert function of $A$. Let $q = \dim \hat{\Delta}(A)$. Then: 1) $H_A$ has growth degree $q$, i.e. $\lim_{k \to \infty} H_A(k)/k^q$ exists and is a positive number $a_q$ (called the $q$-th growth coefficient of $H_A$). 2) We have the following formulae for $a_q$ in terms of the three convex bodies
associated to $A$:

\[
\begin{align*}
a_q &= \text{Vol}_q(\hat{\Delta}(A)), \\
&= \int_{\hat{\Delta}(A)} \hat{\pi}^* f(\lambda) d\hat{\gamma}, \\
&= \int_{\Delta(A)} f(\lambda) d\mu_A.
\end{align*}
\]

In the above, $\text{Vol}_q$ is the $q$-dimensional (normalized) volume of $\hat{\Delta}(A)$, $d\hat{\gamma}$ is a (normalized) Lebesgue measure on $\hat{\Delta}(A)$, $d\mu_A$ is the Duistermaat-Heckman measure associated to $A$, and the polynomial $f : \Delta(A) \to \mathbb{R}$ is given by:

\[f(\lambda) = \prod_{\alpha \in R^+ \setminus E} \frac{\langle \lambda, \alpha \rangle}{\langle \rho, \alpha \rangle}.
\]

Here $E$ is the set of positive roots which are orthogonal to the moment body $\Delta(A)$ and $\rho$ is half of the sum of positive roots.

The above in particular gives formulae for the so-called volume of a $G$-linearized line bundle.

Let $X$ be an $n$-dimensional irreducible variety. Let $L_1, \ldots, L_n$ be finite dimensional subspaces of $\mathbb{C}(X)$. In [Kaveh-Khovanskii08-2] the authors define an intersection index $[L_1, \ldots, L_n]$ as the number of solutions in $X$ of a system of equations $f_1 = \cdots = f_n = 0$ where each $f_i$ is a generic function from the space $L_i$. In counting the solutions, we neglect the solutions $x$ at which all the functions in some space $L_i$ vanish as well as the solutions at which at least one function from some subspace $L_i$ has a pole. The intersection index $[L_1, \ldots, L_n]$ has all the properties of the intersection index of divisors and can be considered as an extension of the intersection theory of divisors (see Section 1.3). If the Kodaira map $\Phi_L$ of a subspace $L$ gives a birational isomorphism between $X$ and its image, then the self-intersection index $[L, \ldots, L]$ is in fact the degree of $Y_L$ in the projective space $\mathbb{P}(L^*)$.

In Section 6 we consider finite dimensional subspaces of rational functions which are moreover invariant under the group action. We apply Theorem 3 to the $G$-algebra $A_L$ associated to $L$ to give formulae for the self-intersection index $[L, \ldots, L]$:

**Corollary 5.** 1) Let $(L, \varphi) \in K_G(X)$ be an invariant subspace of rational functions. Moreover, assume that $\Phi_L$ gives a birational isomorphism between $X$ and its image. Then:

\[
[L, \ldots, L] = n! \text{Vol}_n(\tilde{\Delta}(A_L)), \\
= n! \int_{\tilde{\Delta}(A_L)} \tilde{\pi}^* f(\lambda) d\tilde{\gamma}, \\
= n! \int_{\Delta(A_L)} f(\lambda) d\mu_{A_L}.
\]
In the above, \( \text{Vol}_n \) is (normalized) \( n \)-dimensional volume, \( d\hat{\gamma} \) is a (normalized) Lebesgue measure on \( \Delta(A_L) \) and \( d\mu_{A_L} \) is the Duistermaat-Heckman measure associated to \( A_L \). 2) Similar formulae hold for the self-intersection index of a divisor of a very ample \( G \)-line bundle. 3

Notice that since \( A_L \) is finitely generated then \( \Delta(A_L) \) is in fact a polytope (moment polytope). Moreover, if \( X \) is a spherical \( G \)-variety then \( \Delta(A_L) = \hat{\Delta}(A_L) \) and thus \( \tilde{\Delta}(A_L) \) is also a polytope (which in fact can be concretely described in many cases). This generalizes the Brion-Kazarnowskii formula for the degree of a spherical variety embedded in a projective space ([Brion89], [Kazarnovskii87]) to arbitrary \( G \)-varieties (see Section 6.2). The Brion-Kazarnowskii formula itself is a generalization of the Kusnirenko theorem from toric geometry.

Finally, in Section 7, for certain class of \( G \)-varieties (the so-called \( S \)-varieties) we observe that the map \( L \mapsto \Delta(A_L) \) is an additive map with respect to the product of subspaces. This then gives a generalization of the well-known Bernstein theorem in toric geometry ([Bernstein75]):

**Theorem 6.** Let \( X \) be an \( S \)-variety for the action of \( G \). 1) Let \( (L_i, \varphi_i) \in K_G(X), i = 1, \ldots, n \), be invariant subspaces of rational functions. Then the intersection index \( [L_1, \ldots, L_n] \) can be computed as a mixed integral of the convex bodies \( \Delta(A_{L_1}), \ldots, \Delta(A_{L_n}) \). 2) Similar formulae hold for the intersection index of divisors of very ample \( G \)-line bundles on \( X \).

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1. Preliminaries

1.1. Semigroups of integral points and Newton-Okounkov bodies.

Let \( S \subset \mathbb{Z}^n \) be a semigroup of integral points, i.e. \( 0 \in S \) and \( S \) is closed under addition.

Let \( C(S) \) be the closure of the convex hull of \( S \), that is, the smallest closed convex cone (with apex at the origin) containing \( S \). Also let \( G(S) \) be the subgroup of \( \mathbb{Z}^n \) generated by \( S \) and, \( L(S) \) the linear subspace of \( \mathbb{R}^n \) spanned by \( S \). The sets \( C(S) \) and \( G(S) \) lie in \( L(S) \). To \( S \) we associate its regularization which is the semigroup \( \text{Reg}(S) = C(S) \cap G(S) \). The regularization \( \text{Reg}(S) \) is a simpler semigroup with more points containing the semigroup \( S \). In [Kaveh-Khovanski09] Section 1.1] it is proved that the regularization \( \text{Reg}(S) \) asymptotically approximates the semigroup \( S \). We call this the approximation theorem. More precisely:

**Theorem 1.1** (Approximation theorem). Let \( C' \subset C(S) \) be a strongly convex cone which intersects the boundary (in the topology of the linear space \( L(S) \)) of the cone \( C(S) \) only at the origin. Then there exists a constant \( N > 0 \) (depending on \( C' \)) such that each point in the group \( G(S) \) which lies in \( C' \) and whose distance from the origin is bigger than \( N \) belongs to \( S \).

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3 A cone is strongly convex if it does not contain any line.
Consider the Euclidean space \( \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n \) and let \( \pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) denote the projection on the first factor. Let \( S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n \) be a semigroup and let \( S_k = S \cap \pi^{-1}(k) \) be the set of points in \( S \) at level \( k \). Then \( \pi(S) \) consists of \( k \) such that \( S_k \neq \{0\} \). It is a subsemigroup in \( \mathbb{Z}_{\geq 0} \). Let \( m(S) \) be the index of the subgroup generated by \( \pi(S) \) in \( \mathbb{Z} \). One shows that for sufficiently large \( k \), we have \( k \in \pi(S) \) if and only if \( k \) is divisible by \( m(S) \).

**Definition 1.2.** We call a semigroup \( S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n \) a **non-negative semigroup** if it is not contained in the hyperplane \( \pi^{-1}(0) \). Moreover we assume for simplicity that \( m(S) = 1 \).

**Remark 1.3.** The assumption \( m(S) = 1 \) is not crucial and one can slightly modify all the statements that follow so that they hold without this assumption.

As above let \( C(S) \) be the smallest closed convex cone containing \( S \), \( G(S) \) the subgroup of \( \mathbb{Z}^{n+1} = \mathbb{Z} \times \mathbb{Z}^n \) generated by \( S \), and \( L(S) \) the rational subspace in \( \mathbb{R}^{n+1} \) spanned by \( S \). If in addition the cone \( C(S) \) intersects the hyperplane \( \pi^{-1}(0) \) only at the origin, \( S \) is called a **strongly non-negative semigroup**. We denote the group \( G(S) \cap \pi^{-1}(0) \) by \( \Lambda(S) \) and call it the **lattice associated to the non-negative semigroup** \( S \). The index of this sublattice in \( \{0\} \times \mathbb{Z}^n \) will be denoted by \( \text{ind}(\Lambda(S)) \) or simply \( \text{ind}(S) \). Finally, the number of points in \( S_k \) is denoted by \( H_S(k) \). \( H_S \) is called the **Hilbert function of the semigroup** \( S \).

**Definition 1.4 (Newton-Okounkov convex set).** We call the projection of the convex set \( C(S) \cap \pi^{-1}(1) \) to \( \mathbb{R}^n \) (under the projection on the second factor \((1, x) \mapsto x\)), the **Newton-Okounkov convex set of the semigroup** \( S \) and denote it by \( \Delta(S) \). In other words,

\[
\Delta(S) = \text{conv}\left( \bigcup_{k>0} \{x/k \mid (k, x) \in S_k\} \right).
\]

If \( S \) is strongly non-negative then \( \Delta(S) \) is compact and hence a convex body.

**Remark 1.5.** In [Kaveh-Khovanskiii09] the Newton-Okounkov convex set is defined as \( C(S) \cap \pi^{-1}(1) \) (instead of its projection to \( \mathbb{R}^n \)). For the purposes of this paper we prefer to work with its projection.

Let us define the notion of volume normalized with respect to a lattice.

**Definition 1.6 (Normalized volume).** Let \( \Lambda \subset \mathbb{R}^n \) be a lattice of full rank \( n \). Let \( E \subset \mathbb{R}^n \) be a rational affine subspace of dimension \( q \). That is, \( E \) is parallel to a linear subspace of dimension \( q \) which is rational with respect to \( \Lambda \). The **Lebesgue measure normalized with respect to the lattice** \( \Lambda \) in \( E \) is the translation invariant Lebesgue measure \( d\gamma \) in \( E \) normalized such that the smallest measure of a \( q \)-dimensional parallelepiped with vertices in \( E \cap \Lambda \) is equal to 1. The measure of a subset \( A \subset E \) will be called its **normalized volume** and denoted by \( \text{Vol}_q(A) \) (whenever the lattice \( \Lambda \) is clear from the context).
Let \( \text{Reg}(S) \) be the regularization of \( S \) and let \( H_{\text{Reg}(S)}(k) \) be its Hilbert function. It follows from the approximation theorem (Theorem 1.1) that \( H_S(k) \) and \( H_{\text{Reg}(S)}(k) \) have the same asymptotic as \( k \) goes to infinity. Thus the Newton-Okounkov convex set \( \Delta(S) \) is responsible for the asymptotic of the Hilbert function of \( S \):

**Theorem 1.7.** The function \( H_S(k) \) grows like \( a_q k^q \) where \( q \) is the dimension of the convex body \( \Delta(S) \) and the \( q \)-th growth coefficient \( a_q = \lim_{k \to \infty} H_S(k)/k^q \) is equal to \( \text{Vol}_q(\Delta(S))/\text{ind}(S) \).

More generally, in [Kaveh-Khovanskii09, Theorem 1.13] the above theorem is extended to the sum of values of a function on the point \( s \) of the semigroup \( S \).

**Theorem 1.8.** Let \( S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n \) be a strongly non-negative semigroup with the Newton-Okounkov convex body \( \Delta(S) \). Put \( q = \dim \Delta(S) \). Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a polynomial of degree \( d \) and \( f^{(d)} \) the homogeneous component of \( f \) of degree \( d \). Then

\[
\lim_{k \to \infty} \sum_{(k,x) \in S_k} \frac{f(x)}{k^{q+d}} = \left(\frac{1}{\text{ind}(S)}\right) \int_{\Delta(S)} f^{(d)}(x) d\gamma,
\]

where \( d\gamma \) is the Lebesgue measure on \( \Delta(S) \) normalized with respect to the lattice \( \Lambda(S) \).

One defines a levelwise addition operation on the subsets of \( \mathbb{R} \times \mathbb{R}^n \): for each subset \( A \subset \mathbb{R} \times \mathbb{R}^n \) and \( k \in \mathbb{R} \), let \( A_k \) denote the set of points of \( A \) in level \( k \), i.e. \( A_k = A \cap \pi^{-1}(k) \).

**Definition 1.9 (Levelwise addition of subsets).** Let \( A, B \subset \mathbb{R} \times \mathbb{R}^n \). Define the set \( A \oplus_t B \subset \mathbb{R} \times \mathbb{R}^n \) by:

\[
A \oplus_t B = \{(x+y,k) \mid k \in \mathbb{R}, (x,k) \in A_k, (y,k) \in B_k\}.
\]

Let \( S', S'' \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n \) be two non-negative semigroups. One sees that \( S = S' \oplus_t S'' \) is a non-negative semigroup. Moreover,

\[
\Delta(S) = \Delta(S') + \Delta(S''),
\]

where the addition in the right-hand side is the Minkowski sum of convex sets defined by \( A + B = \{a+b \mid a \in A, b \in B\} \).

**Example 1.10.** 1) Let \( S \) be the non-negative semigroup consisting of all the integral points in \( \mathbb{Z}_{\geq 0} \times \mathbb{Z} \) lying to the right of the broken line \( |y| = x \) (where \( x \) and \( y \) are the first and second coordinates respectively). The subspace \( L \) is the whole \( \mathbb{R}^2 \). The cone \( C \) is the cone generated by the vectors \((1,1)\) and \((1,-1)\). The Newton-Okounkov convex body \( \Delta(S) \) is the line segment \([-1,1]\).

2) Let \( S \) be the non-negative semigroup consisting of all the integral points lying to the right of the curve \( \sqrt{|y|} = x \). Then the cone \( C \) of \( S \) is the
whole right half-plane \( \{ x \geq 0 \} \) and thus \( S \) is not strongly non-negative. The Newton-Okounkov convex set \( \Delta(S) \) is the whole line \( \mathbb{R} \) which is unbounded.

1.2. **Linear transformations between semigroups.** Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear map where \( n \geq m \). Let \( \tilde{T} = \text{Id} \oplus T : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R}^m \), that is, for \( x \in \mathbb{R}^n, x_1 \in \mathbb{R} \) we have \( \tilde{T}(x_1, x) = (x_1, T(x)) \).

Let \( S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^m \) and \( S' \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n \) be strongly non-negative semigroups such that \( \tilde{T}(S') = S \). As usual let

- \( \Delta(S), \Delta(S') \) denote the Newton-Okounkov bodies of \( S \) and \( S' \),
- \( L(S), L(S') \), the (real) subspaces spanned by \( S \) and \( S' \),
- \( C(S), C(S') \), the cones associated to \( S \) and \( S' \),
- \( G(S), G(S') \), the groups generated by \( S \) and \( S' \) respectively.

Also let \( q = \dim \Delta(S) \) and \( q' = \dim \Delta(S') \). Since \( \tilde{T}(S') = S \) we see that \( \tilde{T}(L(S')) = L(S), \tilde{T}(C(S')) = C(S), \tilde{T}(G(S')) = G(S) \), and \( T(\Delta(S')) = \Delta(S) \).

For a point \( p \in \mathbb{R}^n \), let \( \delta_p \) denote the Dirac measure supported at the single point \( p \). Given \( k > 0 \), define the \( k \)-th multiplicity measure \( d\mu_k \) on \( \Delta(S) \) by:

\[
d\mu_k = \sum_{(k, x) \in S_k} \#(\tilde{T}^{-1}(k, x)) \delta_{x/k}.
\]

It is a finitely supported measure where a point \( x \) has non-zero measure if \( (k, kx) \in S_k \), in which case the measure of \( x \) is equal to the number of points in the preimage \( \tilde{T}^{-1}(k, x) \). In other words, take a subset \( U_0 \subset \Delta(S) \) and let \( U = \{ 1 \} \times U_0 \) be its shift to level 1. We have:

\[
(2) \quad \int_{U_0} d\mu_k = \#(\tilde{T}^{-1}(kU) \cap S'_k).
\]

It is clear that the total mass of \( d\mu_k \) is \( \#S'_k \).

**Theorem 1.11.** 1) The measures \( d\mu_k / k^{q'-q} \) weakly converge to a measure \( d\mu \) supported on \( \Delta(S) \). Moreover, \( d\mu \) is the push-forward of the Lebesgue measure on \( \Delta(S') \) to \( \Delta(S) \) (normalized with respect to the lattice \( \Lambda(S) \)).

2) Let \( d\gamma = fd\gamma \) where \( d\gamma \) is the Lebesgue measure on \( \Delta(S) \) (normalized with respect to the lattice \( \Lambda(S) \)). Then:

\[
\text{Vol}_{q'}(\Delta(S')) = \int_{\Delta(S)} fd\gamma.
\]

**Proof.** 1) We show that the measures \( d\mu_k / k^{q'-q} \) converge to the push-forward of the normalized Lebesgue measure on \( \Delta(S') \). To this end, let \( U_0 \subset \Delta(S) \) be a convex open subset which does not intersect the boundary of \( \Delta(S) \). Let \( U = \{ 1 \} \times U_0 \) be the shift of \( U_0 \) to the level 1. Let \( U' = \tilde{T}^{-1}(U) \). It suffices to show that

\[
\lim_{k \to \infty} \left( 1 / k^{q'} \right) \int_{U_0} d\mu_k = \text{Vol}_{q'}(U').
\]
One knows that
\[
\text{Vol}_q(U') = \lim_{k \to \infty} \frac{\#(kU' \cap G(S'))}{k^d}.
\]
Let \(C' \subset \mathbb{R} \times \mathbb{R}^n\) be the cone over the set \(U'\). Applying the approximation theorem (Theorem \[1.11\]) to the semigroup \(C(S')\) and the cone \(C'\) we see that there is \(N > 0\) such that for \(k > N\), \(kU' \cap G(S') = kU' \cap S_k'\). Hence
\[
\text{Vol}_q(U') = \lim_{k \to \infty} \frac{\#(kU' \cap S_k')}{k^d} = \lim_{k \to \infty} \frac{\int_{U_k} d\mu_k}{k^d} \quad \text{(from (2))},
\]
which proves the claim. 2) follows immediately from 1). \(\square\)

Finally, we prove a theorem about the relation between the measures associated to \((S, S')\) and their subsemigroups. Take an integer \(k > 0\) such that \(S_k \neq \emptyset\). Let \(S_k\) (respectively \(S_k'\)) denote the subsemigroup of \(S\) (respectively \(S'\)) generated by the level \(S_k\) (respectively \(S_k'\)). Since \(T(S') = S\) we see that \(T(S_k') = S_k\). Let \(T_k : S_k' \to S_k\) denote the restriction of \(T\). Similar to above, let \(d\rho_k\) denote the measure associated to the pair \((S_k, S_k')\), that is:
\[
d\rho_k = \lim_{\ell \to \infty} \frac{1}{\ell^{d-q}} \sum_{(k\ell, x) \in S_k} \delta_{x/\ell}.
\]
(As in Theorem \[1.11\] one shows that the above limit of measures exists.) Note that for large \(k\), the subspace \(L(S_k')\) coincides with \(L(S')\) and the lattice \(G(S_k')\) coincides with \(G(S')\). Let \(O_{1/k} : \mathbb{R}^m \to \mathbb{R}^m\) denote the multiplication by the scalar \(1/k\).

**Theorem 1.12.** As \(k \to \infty\), the measures \(O_{1/k}^*(d\rho_k)/k^{d-q}\) converge weakly to the measure \(d\mu\) associated to the pair \((S', S)\). Here \(O_{1/k}^*\) denote the push-forward measure by the map \(O_{1/k}\).

**Proof.** By Theorem \[1.11\](1) we know that the measure \(d\mu\) is the push-forward (under the linear transformation \(T\)) of the normalized Lebesgue measure on the body \(\Delta(S')\) to \(\Delta(S)\). Let \(\Delta_k'\) (respectively \(\Delta_k\)) denote the convex hull of \(\{x' \mid (k, x') \in S_k'\}\) (respectively \(\{x \mid (k, x) \in S_k\}\)). Similar to the proof of Theorem \[1.11\] one sees that the measure \(O_{1/k}^*(d\rho_k)/k^{d-q}\) is the push-forward of the normalized Lebesgue measure on \((1/k)\Delta_k'\) to \((1/k)\Delta_k\). But as \(k \to \infty\), the polytopes \(\Delta_k'\) (respectively \(\Delta_k\)) converge to the body \(\Delta(S')\) (respectively \(\Delta(S)\)) with respect to the Hausdorff metric on subsets. The claim follows easily from this. \(\square\)

### 1.3. Intersection index of subspaces of rational functions.

The material in this section are a quick review of the results in [Kaveh-Khovanski08-2].

Let \(X\) be an irreducible variety over \(\mathbb{C}\). Consider the collection \(K_{rat}(X)\) of all the finite dimensional subspaces of rational functions on \(X\). The set \(K_{rat}(X)\) has a natural multiplication: the product \(L_1L_2\) of two subspaces
Theorem 1.15. The intersection index $U_Y$ is the subspace spanned by all the products $fg$ where $f \in L_1$, $g \in L_2$. With respect to this multiplication, $K_{rat}(X)$ is a commutative semi-group.

To each subspace $L \in K_{rat}(X)$ one associates a Kodaira rational map $\Phi_L : X \dashrightarrow \mathbb{P}(L^*)$, where $\mathbb{P}(L^*)$ is the projectivization of the dual space $L^*$. Let $x \in X$ and assume that $f(x)$ is defined for all $f \in L$. Consider the linear functional $\xi \in L^*$ given by $\xi(f) = f(x)$ for all $f \in L$. Then $\Phi_L(x)$ is the element of $\mathbb{P}(L^*)$ represented by $\xi \in L^*$. We will denote the closure of the image of $X$ in $\mathbb{P}(L^*)$, under the Kodaira map $\Phi_L$, by $Y_L$.

**Example 1.13** (Subspace associated to a Cartier divisor). Let us assume that $X$ is projective and let $D$ be a (Cartier) divisor on $X$. To $D$ one associates the subspace $L(D)$ defined by:

$$L(D) = \{ f \in \mathbb{C}(X) \mid (f) + D \geq 0 \}.$$ 

It is well-known that $L(D)$ is finite dimensional. Next suppose $\mathcal{L}$ is a globally generated line bundle on $X$ and $D$ is the divisor of a non-zero global section $\tau \in H^0(X, \mathcal{L})$. Since $\tau$ does not have poles, $D > 0$ and hence $L(D)$ contains the constants. The map $f \mapsto f\tau$ gives a vector space isomorphism between $L(D)$ and $H^0(X, \mathcal{L})$, in particular, they have the same dimension.

**Definition 1.14.** The intersection index $[L_1, \ldots, L_n]$ of $L_1, \ldots, L_n \in K_{rat}(X)$ is the number of solutions in $X$ of a generic system of equations $f_1 = \cdots = f_n = 0$, where $f_1 \in L_1, \ldots, f_n \in L_n$. In counting the solutions, we neglect the solutions $x$ at which all the functions in some space $L_i$ vanish as well as the solutions at which at least one function from some space $L_i$ has a pole.

More precisely, let $\Sigma \subset X$ be a hypersurface which contains: 1) all the singular points of $X$; 2) all the poles of the functions from any of the $L_i$; 3) for any $i$, the set of common zeros of all the $f \in L_i$. Then for a generic choice of $(f_1, \ldots, f_n) \in L_1 \times \cdots \times L_n$, the intersection index $[L_1, \ldots, L_n]$ is equal to the number of the solutions $\{x \in X \setminus \Sigma \mid f_1(x) = \cdots = f_n(x) = 0\}$.

From the definition one sees that if the Kodaira map $\Phi_L$ is a birational map onto its image $Y_L$ then the self-intersection index $[L, \ldots, L]$ is equal to the degree of the projective subvariety $Y_L \subset \mathbb{P}(L^*)$.

**Theorem 1.15.** The intersection index $[L_1, \ldots, L_n]$ is well-defined. That is, there is a Zariski open subset $U$ in the vector space $L_1 \times \cdots \times L_n$ such that for any $(f_1, \ldots, f_n) \in U$ the number of solutions $x \in X \setminus \Sigma$ of the system $f_1(x) = \cdots = f_n(x) = 0$ is the same (and hence equal to $[L_1, \ldots, L_n]$). Moreover, the above number of solutions is independent of the choice of $\Sigma$ containing 1)-3) above.

The following properties of the intersection index are trivial consequences of the definition: 1) $[L_1, \ldots, L_n]$ is a symmetric function of the n-tuples $L_1, \ldots, L_n \in \mathcal{K}_{rat}(X)$, (i.e. takes the same value under a permutation of $L_1, \ldots, L_n$); 2) The intersection index is monotone, (i.e. if $L'_1 \subseteq L_1, \ldots, L'_n \subseteq L_n$, then $[L'_1, \ldots, L'_n] \geq [L_1, \ldots, L_n]$).
$L_n$, then $[L_1, \ldots, L_n] \geq [L_1', \ldots, L_n']$; and 3) The intersection index is non-negative (i.e. $[L_1, \ldots, L_n] \geq 0$).

The next theorem contains one of the main properties of the intersection index.

**Theorem 1.16** (Multi-linearity). Let $L_1', L_1'' , L_2, \ldots, L_n \in K_{rat}(X)$ and put $L_1 = L_1' L_1''$. Then

$$[L_1, \ldots, L_n] = [L_1', \ldots, L_n] + [L_1'', \ldots, L_n].$$

**Example 1.17** (Intersection number of divisors). Let $X$ be an irreducible projective variety of dimension $n$. Let $D_1, \ldots, D_n$ be very ample divisors on $X$. As in Example 1.13 to each $D_i$ there corresponds a subspace $L(D_i) = \{ f \in \mathbb{C}(X) \mid (f) + D_i > 0 \}$. One verifies that the (classical) intersection number of the divisors $D_1, \ldots, D_n$ coincides with the intersection index $[L(D_1), \ldots, L(D_n)]$ of the subspaces $L(D_i)$.

As with any other commutative semi-group, there corresponds a Grothendieck group to the semi-group $K_{rat}(X)$. For a commutative semi-group $K$, the Grothendieck group $G(K)$ is the abelian group defined as follows: for $x, y \in K$ we say $x \sim y$ if there is $z \in K$ with $xz = yz$. Then $G(K)$ is the group of formal quotients of the equivalence classes of $\sim$. There is a natural homomorphism $\phi : K \to G(K)$. The Grothendieck group satisfies the following universal property: for any abelian group $G'$ and a homomorphism $\phi' : K \to G'$, there exist a unique homomorphism $\psi : G(K) \to G'$ such that $\phi' = \psi \circ \phi$. From the multi-linearity of the intersection index it follows that the intersection index extends to the Grothendieck group of $K_{rat}(X)$.

The Grothendieck group of $K_{rat}(X)$ and its intersection index can be considered as an extension of the group of Cartier divisors (on a projective variety) and their intersection index. As is discussed in [Kaveh-Khovanskii08,2], the intersection theory on the Grothendieck group of $K_{rat}(X)$ enjoys properties similar to that of the mixed volume of convex bodies in the Euclidean space $\mathbb{R}^n$.

### 1.4. Graded algebras and valuations

Let $X$ be an irreducible variety with the field of rational functions $F = \mathbb{C}(X)$. In this section we recall some basic notions regarding graded algebras and their Hilbert functions. In particular we introduce certain classes of subalgebras of the polynomial ring $F[t]$ whose Hilbert functions have bounded growth degree. An important example of a graded algebra in $F[t]$ is the homogeneous coordinate ring of $X$ embedded in a projective space or more generally the algebra of sections of a line bundle/divisor on $X$ (See Example 1.18).

Let $A = \bigoplus_{k \geq 0} A_k$ be a graded subalgebra of $F[t]$. Then each homogeneous component $A_k$ is of the form $L_k t^k$ for $L_k \subset F$. We call $L_k$ the $k$-th subspace of $A$. One defines the Hilbert function, $H_A$ by $H_A(k) = \dim L_k$.

We will be interested in the asymptotic behavior of the Hilbert functions of graded algebras.
We now define three classes of graded subalgebras which will play main roles later:

1. To each non-zero finite dimensional linear subspace \( L \subset F \) one associates the graded algebra \( A_L \) defined as follows: its zero-th homogeneous component is \( \mathbb{C} \) and for each \( k > 0 \) its \( k \)-th subspace is \( L^k \) (spanned by all the products \( f_1 \cdots f_k \) with \( f_i \in L \)). That is,
   \[
   A_L = \bigoplus_{k \geq 0} L^k t^k.
   \]

2. We call a graded subalgebra \( A \subset F[t] \), an algebra of integral type, if there is an algebra \( A_L \), for some non-zero finite dimensional subspace \( L \) over \( \mathbb{C} \), such that \( A \) is a finitely generated \( A_L \)-module.

3. We call a graded subalgebra \( A \subset F[t] \), an algebra of almost integral type, if it is contained in an algebra of integral type. Equivalently, \( A \) is of almost type if it is contained in an algebra \( A_L \) for some finite dimensional subspace \( L \).

For a graded algebra \( A \subset F[t] \), the set of \( k \) with \( L_k \neq \{0\} \) is a subsemigroup of \( \mathbb{Z}_{\geq 0} \). Let \( m(A) \) denote the index of the subgroup generated by this set in \( \mathbb{Z} \). One can show that for sufficiently large \( k \), we have \( L_k \neq \{0\} \) if and only if \( k \) is divisible by \( m(A) \).

For simplicity, for the rest of the paper, we will only deal graded algebras \( A \) with \( m(A) = 1 \).

If \( A \) is an algebra of integral type, its Hilbert function \( H_A(k) \) coincides with a polynomial for large values of \( k \). If \( A \) is not of integral type its Hilbert function is not necessarily a polynomial (for large values of \( k \)). Although when \( A \) is of almost integral type the Hilbert function \( H_A \) has polynomial growth, i.e. there is an integer \( m > 0 \) and a real number \( a_m > 0 \) such that the limit \( \lim_{k \to \infty} H_A(k)/k^m \) exists and is equal to \( a_m \) (see Theorem \([1, 21] \) and \([\text{Kaveh-Khovanskii09}] \) Theorem 2.32). We call \( m \) and \( a_m \), the growth degree of \( H_A \) and the \( m \)-th growth coefficient of \( H_A \) respectively.

Let \( A' = \bigoplus_{k \geq 0} L'_k t^k \) and \( A'' = \bigoplus_{k \geq 0} L''_k t^k \) be two graded subalgebras of \( F[t] \). One defines the algebra \( A = \bigoplus_{k \geq 0} L_k t^k \) where \( L_k = L'_k L''_k \). It is called the componentwise product of \( A' \) and \( A'' \) and denoted by \( A = A'A'' \).

It is easy to see that for two subspaces \( L', L'' \in \mathbb{K}_{rat}(X) \) we have \( A_{L'}A_{L''} = A_{L'/L''} \). Also if \( A' \) and \( A'' \) are of integral type (respectively almost integral type) then \( A = A'A'' \) is also of integral type (respectively almost integral type).

**Example 1.18** (Algebra of sections of a divisor). Let \( X \) be a projective variety and \( \mathcal{L} \) a globally generated line bundle on \( X \) (i.e. \( H^0(X, \mathcal{L}) \neq \{0\} \)). Take a non-zero section \( \tau \in H^0(X, \mathcal{L}) \) and let \( D \) be the divisor of zeros of \( \sigma \). As in Example \([1, 13] \) to each divisor \( kD \), where \( k > 0 \) is an integer, there corresponds a subspace \( L(kD) = \{ f \in \mathbb{C}(X) \mid (f) + kD > 0 \} \). The elements in \( L(kD) \) can naturally be identified with the section in \( H^0(X, \mathcal{L}^{\otimes k}) \). To the
divisor $D$ we associate its \textit{algebra of sections} $R(D) \subset \mathbb{C}(X)[t]$ defined by

$$R(D) = \bigoplus_{k \geq 0} L(kD),$$

where by convention $L(0) = \mathbb{C}$. One can show that $R(D)$ is always an algebra of almost integral type.\footnote{If $X$ is a so-called projectively normal subvariety of a projective space and $D$ is a hyperplane section divisor, then $R(D)$ can be identified with the homogeneous coordinate ring of $X$.}

We denote the collection of all subalgebras of almost integral type in $F = \mathbb{C}(X)$ by $A_{\text{rat}}(X)$. It is a semigroup with respect to componentwise product of graded subalgebras.

We know briefly introduce the notion of a valuation on a field of rational functions of an algebraic variety $X$. It enables us to convert problems about the asymptotic of Hilbert function of a graded algebra to problems about asymptotic of semigroups of integral points ([Kaveh-Khovanskii09 Part II]).

Let $X$ be a variety of dimension $n$ over $\mathbb{C}$. Let $\Gamma$ be an ordered abelian group, i.e. an abelian group equipped with a total order compatible with the group operation (which we write additively).

\begin{definition}
A function $v : \mathbb{C}(X)^* \to \Gamma$ is a \textit{valuation} with values in $\Gamma$ if:

1. $v(fg) = v(f) + v(g)$, for all non-zero $f, g \in \mathbb{C}(X)$.
2. $v(f + g) \geq \min(v(f), v(g))$, for all non-zero $f, g \in \mathbb{C}(X)$.
3. $v(\lambda f) = v(f)$, for all non-zero $f \in \mathbb{C}(X)$ and non-zero $\lambda \in \mathbb{C}$.

For any $a \in \Gamma$ consider the quotient vector space,

$$F_a = \{ f \mid v(f) \geq a \}/\{ f \mid v(f) > a \}.$$  

We call this the leaf at $a$. The valuation $v$ is called a valuation with one-dimensional leaves if for any $a \in \Gamma$, the leaf $F_a$ has dimension at most 1. Equivalently, $v$ has one-dimensional leaves, if whenever $v(f) = v(g)$, for some $f, g \in \mathbb{C}(X)$, then there is $\lambda \neq 0$ such that $v(g - \lambda f) > v(g)$.

For different examples of valuations on the field of rational functions see [Kaveh-Khovanskii09 Section 2.2].

Fix a total order on $\mathbb{Z}^n$ compatible with the addition and take a valuation $v : \mathbb{C}(X)^* \to \mathbb{Z}^n$. Let $A = \bigoplus_{k \geq 0} L_k t^k, L_k \subset \mathbb{C}(X)$, be a graded algebra. For any positive integer $k$, let $S_k = \{(k, v(f)) \mid f \in L_k \setminus \{0\}\}$ and put $S_0 = \{(0, 0)\} \subset \mathbb{Z} \times \mathbb{Z}^n$. Define

$$S = S(A) = \bigcup_{k \geq 0} S_k.$$  

It is a semigroup in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$. As in Section 1.1, we associate the following objects to the semigroup $S$:
- \( \Delta(S) \), the Newton-Okounkov convex set of the non-negative semigroup \( S \).
- \( \Lambda(S) \), the lattice associated to \( S \).
- \( \text{ind}(S) \), the index of the lattice \( \Lambda(S) \) in \( \mathbb{Z}^n \).

The following are the key properties of valuation for us. We will assume that the valuation \( v \) has one-dimensional leaves.

**Proposition 1.20.** Let \( A \subset F[t] \) be a graded algebra with \( S = S(A) \) the corresponding valuation semigroup. 1) For any integer \( k > 0 \) we have \( \dim L_k = \#S_k \). 2) If \( A \) is of almost integral type then \( S \) is a strongly non-negative semigroup and hence the Newton-Okounkov convex set \( \Delta(S) \) is a convex body.

Finally, using Proposition 1.20 it is shown in [Kaveh-Khovanskiii09, Section 1.3] that the Newton-Okounkov convex body \( \Delta(S) \) is responsible for the growth of the Hilbert function of the graded algebra \( A \). Let \( q \) be the dimension of the Newton-Okounkov body \( \Delta(S) \) (which is equal to the rank of the lattice \( \Lambda(S) \)), and let \( \text{Vol}_q \) denote the normalized Lebesgue measure.

**Theorem 1.21.** 1) The growth degree of the Hilbert function \( H_A \) of \( A \) is equal to \( q \). 2) \[
\lim_{k \to \infty} \frac{H_A(k)}{k^q} = n!\text{Vol}_q(\Delta(S))/\text{ind}(S).
\]
3) In particular, if \( A = A_L \) is the algebra associated to a subspace \( L \in K_{\text{rat}}(X) \) of rational functions then the self-intersection number \([L, \ldots, L]\) is given by

\[
\frac{n!d}{\text{ind}(S)}\text{Vol}_n(\Delta(S)),
\]

where \( \text{Vol}_n \) is the \( n \)-dimensional Euclidean measure in \( \mathbb{R}^n \) and \( d \) is the mapping degree of the Kodaira map \( \Phi_L \) (if finite). If \( \text{ind}(S) \) or the degree \( d \) are not finite then both \([L, \ldots, L]\) and \( \text{Vol}_n(\Delta(S)) \) are 0.

In the rest of the present paper we will consider varieties equipped with an action of a reductive group \( G \). We will be concerned with graded algebras in \( A_{\text{rat}}(X) \) which are invariant under the \( G \)-action. We will associate different semigroups and their Newton-Okounkov convex bodies to these algebras which give us information about the \( G \)-action as well as the Hilbert function of the algebra.

**2. Generalities on reductive group actions**

**Notation:** Throughout the rest of the paper we will use the following notation. \( G \) denotes a connected reductive algebraic group over \( \mathbb{C} \).

- \( B \) denotes a Borel subgroup of \( G \) and \( T \) and \( U \) the maximal torus and maximal unipotent subgroups contained in \( B \) respectively.
- \( R = R(G,T) \) is the root system with \( R^+ = R^+(G,T) \) the subset of positive roots for the choice of \( B \). We denote by \( \alpha_1, \ldots, \alpha_r \) the corresponding simple roots where \( r \) is the semi-simple rank of \( G \).
We refer to this as the natural action of $f \in G$ on $X$. That is, $X$ is equipped with a reductive algebraic group over $G$, the action of $g \in G$ of regular functions on $X$ of rational functions (Example 1.13): Let $X$ be an irreducible projective $G$-variety and let $L$ be a $G$-linearized line bundle on $X$. The (Cartier) divisor of $\tau$ identifies the space of sections of $L$ with a subspace of rational functions. For every section $\sigma \in H^0(X, L)$ we can write $\sigma = f_\sigma \tau$. The map $\sigma \mapsto f_\sigma$ identifies $H^0(X, L)$ with the subspace of rational functions $L(D) = \{ f \in \mathbb{C}(X) \mid (f) + D > 0 \}$. Since $D$ is not necessarily $G$-invariant the subspace $L(D)$ is not in general invariant under the natural action of $G$ on $\mathbb{C}(X)$. Although, as we see below, it is invariant under a twisted action of $G$ which has to do with the action of $G$ on $L$. For every $g \in G$ define $\varphi_g \in \mathbb{C}(X)^*$ by $\varphi_g = f_g \tau$, i.e. $g \cdot \tau = \varphi_g \tau$. Then for any section $\sigma \in H^0(X, L)$ we have

$$g \cdot \sigma = (g \cdot f_\sigma)(g \cdot \tau) = (g \cdot f_\sigma)\varphi_g \tau,$$

2.1. $G$-varieties and invariant subspaces of rational functions. Let $G$ be a reductive algebraic group over $\mathbb{C}$ and let $X$ be an irreducible $G$-variety, that is $X$ is equipped with an algebraic action of $G$. We will denote the action of $g \in G$ on $x \in X$ by $g \cdot x$. The action of $G$ on $X$ induces an action of $G$ on $\mathbb{C}(X)$, the field of rational functions on $X$: for $g \in G$ and $f \in \mathbb{C}(X)$, $g \cdot f$ is the rational function defined by:

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

We refer to this as the natural action of $G$ on $\mathbb{C}(X)$.

The above action of $G$ on $\mathbb{C}(X)$ restricts to an action of $G$ on the ring of regular functions $\mathcal{O}(X)$. It is well-known that, with this action, the ring $\mathcal{O}(X)$ is a rational $G$-module, that is every $f \in \mathcal{O}(X)$ lies in a finite dimensional $G$-submodule. In general the field $\mathbb{C}(X)$ is not a rational $G$-module.

We will be interested in more general actions of $G$ on the field $\mathbb{C}(X)$. We call them the twisted actions of $G$ on $\mathbb{C}(X)$. This allows us to include in our discussion, the actions of $G$ on the spaces of sections of $G$-linearized line bundles on $X$. Let us start by observing how the $G$-action is affected during the identification of the space of sections of a line bundle with a subspace of rational functions (Example 1.13): Let $X$ be an irreducible projective $G$-variety and let $L$ be a $G$-linearized line bundle on $X$. That is, $L$ has a $G$-action which lifts the action of $G$ on $X$ and is linear on each fibre. As in Example 1.13 fix a non-zero global section $\tau \in H^0(X, L)$ and let $D$ be the (Cartier) divisor of $\tau$. For any other section $\sigma \in H^0(X, L)$ we can write $\sigma = f_\sigma \tau$. The map $\sigma \mapsto f_\sigma$ identifies $H^0(X, L)$ with the subspace of rational functions $L(D) = \{ f \in \mathbb{C}(X) \mid (f) + D > 0 \}$. Since $D$ is not necessarily $G$-invariant the subspace $L(D)$ is not in general invariant under the natural action of $G$ on $\mathbb{C}(X)$. Although, as we see below, it is invariant under a twisted action of $G$ which has to do with the action of $G$ on $L$. For every $g \in G$ define $\varphi_g \in \mathbb{C}(X)^*$ by $\varphi_g = f_g \tau$, i.e. $g \cdot \tau = \varphi_g \tau$. Then for any section $\sigma \in H^0(X, L)$ we have

$$g \cdot \sigma = (g \cdot f_\sigma)(g \cdot \tau) = (g \cdot f_\sigma)\varphi_g \tau,$$
where $g \cdot f_x$ is the natural action of $G$ on $\mathbb{C}(X)$. Thus under the identification of $H^0(X,L)$ with $L(D)$, the action of $G$ on $H^0(X,L)$ corresponds to the action of $G$ on $L(D)$ defined by

$$g \ast f = (g \cdot f)\varphi_g.$$ 

One verifies the following: a) The map $\varphi : G \to \mathbb{C}(X)^*$, $g \mapsto \varphi_g$, satisfies the group cocycle condition:

$$\varphi_{g_1 g_2} = (g_1 \cdot \varphi_{g_2})\varphi_{g_1}, \quad \forall g_1, g_2 \in G.$$ 

b) If we replace $\tau$ with another section $h\tau$, with $h \in L(D)$, then $\varphi_g$ is replaced with $(g \cdot h/h)\varphi_g$.

Now let us generalize the above situation by replacing the sections of a line bundle with rational functions. Let $X$ be any irreducible $G$-variety (not necessarily projective) with the field of rational functions $\mathbb{C}(X)$. Let $\varphi : G \to \mathbb{C}(X)^* := \mathbb{C}(X) \setminus \{0\}$ be a map satisfying the condition (3), and such that for every $x \in X$, $g \mapsto \varphi_g(x)$ gives a rational function on $G$. Such a function is called a group cocycle.

Given a group cocycle as above one can define an action of $G$ on $\mathbb{C}(X)$ twisted by the cocycle $\varphi$: let $f \in \mathbb{C}(X)$ and $g \in G$. Define the $\varphi$-twisted action $g \ast f$ by:

$$g \ast f = (g \cdot f)\varphi_g,$$

where in the right-hand side we have the multiplication of rational functions $g \cdot f$ and $\varphi_g$. The following is straightforward to verify:

**Proposition 2.1.** For $g \in G$ and $f \in \mathbb{C}(X)$ we have: 1) $\ast_\varphi$ defines a linear action of $G$ on $\mathbb{C}(X)$, 2) The set of functions satisfying (3) is a group with respect to multiplication of functions (i.e. if $\varphi_1, \varphi_2$ satisfy (3), then the product $\varphi_1 \varphi_2$ also satisfies (3), and if $\varphi$ satisfies (3) then the reciprocal $1/\varphi$ also satisfies (3)).

**Remark 2.2.** Let $Z(G,X)$ denote the group of all cocycles $\varphi : G \to \mathbb{C}(X)^*$ and let $B(G,X)$ denote the subgroup of coboundaries, i.e. cocycles of the form $\varphi_g = g \cdot h/h$ for some function $h \in \mathbb{C}(X)^*$. The quotient $Z(X,G)/B(X,G)$ is usually called the first group cohomology of $G$ with coefficients in $\mathbb{C}(X)^*$ and denoted by $H^1(G,\mathbb{C}(X)^*)$.

**Definition 2.3.** Let $L \subset \mathbb{C}(X)$ be a finite dimensional subspace of rational functions. If $L$ is stable under the action of $G$ twisted by a cocycle $\varphi$, we call $L$ a $\varphi$-invariant subspace. We denote the collection of all the pairs $(L, \varphi)$, where $L$ is a finite dimensional $\varphi$-invariant subspace, by $\mathbf{K}_G(X)$. By abuse of terminology, we call a pair $(L, \varphi) \in \mathbf{K}_G(X)$ an invariant subspace.

It is easy to see that if $L$ is a $\varphi$-invariant subspace and $h \in \mathbb{C}(X)^*$ then the subspace $hL$ is $\varphi'$-invariant where $\varphi' = (g \cdot h/h)\varphi$.

The following is straightforward:
Proposition 2.4. Let \((L_1, \varphi_1), (L_2, \varphi_2) \in K_G(X)\), then \((L_1L_2, \varphi_1\varphi_2) \in K_G(X)\). That is, \(K_G(X)\) is a semigroup with respect to this multiplication of pairs.

Let \((L, \varphi) \in K_G(X)\) and let \(\Phi_L : X \rightarrow \mathbb{P}(L^*)\) be the Kodaira map. The \(\varphi\)-twisted action of \(G\) on \(L\) induces an action of \(G\) on \(L^*\) and hence on \(\mathbb{P}(L^*)\). We have the following:

**Proposition 2.5.** The Kodaira map \(\Phi_L\) is \(G\)-equivariant.

**Proof.** Let \(x \in X\) and \(g \in G\). We need to show that \(g \ast \varphi \Phi_L(x) = \Phi_L(g \cdot x)\) (whenever the Kodaira map is defined). Let us assume that \(\Phi_L(x)\) is defined i.e. \(f(x)\) is defined for all \(f \in L\). Let \(\xi \in L^*\) be the linear functional defined by \(\langle \xi, f \rangle = f(x)\). Then \(\Phi_L(x) = [\xi]\) is the element of the projective space \(\mathbb{P}(L^*)\) represented by \(\xi \in L^*\). Also let \(g \in G\) and let \(\eta \in L^*\) be the linear functional defined by \(\langle \eta, f \rangle = f(g \cdot x)\). Then \(\Phi_L(g \cdot x) = [\eta]\). Now by the definition of the \(G\)-actions \(\ast\) and \(\cdot\) and their dual actions on \(L^*\) we have:

\[
\langle g \ast \xi, f \rangle = \langle \xi, (g^{-1} \ast f) \rangle, \\
\quad = \langle (g^{-1} \ast f)(x), \\
\quad = \langle ((g^{-1} \cdot f)(\varphi_g))(x), \\
\quad = f(g \cdot x)\varphi_g(x), \\
\quad = \langle \eta, f \rangle \varphi_g(x).
\]

Thus \(g \ast \xi\) is a scalar multiple of \(\eta\) and hence they represent the same element in \(\mathbb{P}(L^*)\). That is, \(g \ast \Phi_L(x) = \Phi_L(g \cdot x)\) and the Kodaira rational map is \(G\)-equivariant. \(\square\)

Similar to the Grothendieck group of \(K(X)\) we can define the Grothendieck group of invariant subspaces \(K_G(X)\). The intersection index of subspaces then extend to this Grothendieck group.

2.2. **Graded \(G\)-algebras.** An algebra \(A\) is called a \(G\)-algebra if \(G\) acts on \(A\) respecting the algebra operations. If \(A\) is graded we require that the action of \(G\) respects the grading. We will mainly deal with graded subalgebras in the polynomial ring \(F[t]\) where \(F = \mathbb{C}(X)\) is the field of rational functions on a \(G\)-variety \(X\).

**Definition 2.6 (\(G\)-algebras of (almost) \(G\)-integral type).** Let \(A = \bigoplus_{k \geq 0} L_k t^k\), \(L_k \subset F\) be a graded subalgebra of \(F[t]\). We say \(A\) is a graded \(G\)-algebra with \(G\)-action twisted by \(\varphi\), if for each \(k > 0\), the subspace \(L_k \subset F\) is \(\varphi^k\)-invariant. For simplicity we will assume that \(m(A) = 1\), i.e. for any large integer \(k\), \(L_k\) is non-zero. We define three classes of graded \(G\)-algebras:

1. Algebra \((A_L, \varphi)\) where \((L, \varphi) \in K_G(X)\) is an invariant subspace.
2. \((A, \varphi)\) is an algebra of \(G\)-integral type if there is an invariant subspace \((L, \varphi) \in K_G(X)\) such that \(A\) is a finite module over \(A_L\).
3. \((A, \varphi)\) is an algebra of almost \(G\)-integral type if \(A\) is contained in an algebra of \(G\)-integral type \((B, \varphi)\).
We denote the collection of all pairs \((A, \varphi)\) of almost \(G\)-integral type by \(A_G(X)\). By abuse of terminology we may refer to \(A\) (instead of \((A, \varphi)\)) as an algebra of (almost) \(G\)-integral type.

**Remark 2.7.** The assumption that \(m(A) = 1\) is not crucial and is made to make the statements in the rest of paper simpler. One can slightly modify the statements so that they hold without this assumption.

**Example 2.8.**
1) (\(G\)-algebra associated to an invariant subspace) Let \((L, \varphi) \in K_G(X)\) be an invariant subspace of rational functions on \(X\). Then \((A_L, \varphi)\) is a graded algebra of almost \(G\)-integral type i.e. \((A_L, \varphi) \in A_G(X)\).

2) (Integral closure) Let \((A, \varphi) \in A_G(X)\) be a graded algebra of almost \(G\)-integral type. One verifies that the integral closure \((\overline{A}, \varphi)\) is also a graded algebra of almost \(G\)-integral type i.e. \((\overline{A}, \varphi) \in A_G(X)\).

**Example 2.9** (Algebra of sections of a \(G\)-line bundle). Let \(X\) be a projective irreducible \(G\)-variety and let \(L\) be a \(G\)-linearized line bundle on \(X\). As in Section 2.1 fix a non-zero section \(\tau \in H^0(X, L)\) with divisor \(D\) and let the cocycle \(\varphi : G \to \mathbb{C}(X)^*\) be defined by \(\varphi_g = (g \cdot \tau)/\tau\). One shows that \((R(D), \varphi)\) is a graded algebra of almost \(G\)-integral type.

It is easy to verify that if \((A', \varphi'), (A'', \varphi'') \in A_G(X)\) are two graded \(G\)-subalgebras, then \((A'A'', \varphi'\varphi'') \in A_G(X)\), i.e. the componentwise product \(A'A''\) is a \(G\)-algebra with the action twisted by \(\varphi'\varphi''\) and is of almost \(G\)-integral type.

### 3. Moment convex body of a \(G\)-algebra

3.1. **Semigroup of highest weights and moment convex body.** In this section we discuss the semigroup of highest weights and moment convex body for a graded \(G\)-algebra, \((A, \varphi) \in A_G(X)\), where as before \(X\) is an irreducible \(G\)-variety.

One associates a lattice and a semigroup of weights to \(X\) which measure the highest weights that can appear in the \(G\)-modules consisting of functions on \(X\).

**Definition 3.1.** For any variety \(X\) let \(\Lambda(X) \subset \Lambda\) denote the lattice of weights of \(B\)-eigenfunctions of \(\mathbb{C}(X)\) for the natural action of \(G\). It is called the \textit{weight lattice} of \(X\). Also if \(X\) is quasi-affine let \(\Lambda^+(X)\) denote the semigroup of weights of \(B\)-eigenfunctions in the algebra of regular functions \(O(X)\). It is called the \textit{weight semigroup} of \(X\). The rank of the lattice \(\Lambda(X)\) is denoted by \(r(X)\) and is called the \textit{rank of the \(G\)-variety} \(X\). The linear span of \(\Lambda(X)\) will be denoted by \(\Lambda_\mathbb{R}(X)\).
Now let \( A = \bigoplus_{k \geq 0} A_k \) be a graded \( G \)-algebra. Let us write \( A_k \) as the sum of its isotypic components
\[
A_k = \bigoplus_{\lambda \in \Lambda^+} A_{k, \lambda},
\]
where \( A_{k, \lambda} \) is the sum of all the copies of the irreducible representation \( V_{\lambda} \) in \( A_k \). The following statement about the tensor product of irreducible representations is well-known:

**Theorem 3.2.** Let \( \lambda, \mu \) be dominant weights. Then 1) \( V_{\lambda+\mu} \) appears in \( V_{\lambda} \otimes V_{\mu} \) with multiplicity 1. 2) If for some dominant \( \nu \), \( V_{\nu} \) appears in \( V_{\lambda} \otimes V_{\mu} \) with non-zero multiplicity then \( \nu \) is equal to \( \lambda + \mu - \sum_{\alpha \in R^+} c_\alpha \alpha \) with \( c_\alpha \geq 0 \) for every root \( \alpha \). 3) In particular, \( \nu \) lies in the convex hull of the \( W \)-orbit of \( \lambda + \mu \) intersected with the positive Weyl chamber \( \Lambda^+_R \).

**Corollary 3.3** (Multiplication of isotypic components). Let \( \lambda, \mu \) be dominant weights and \( k, \ell > 0 \). If \( A_{k, \lambda} \) and \( A_{\ell, \mu} \) are non-zero then \( V_{\lambda+\mu} \) appears in \( A_{k, \lambda} A_{\ell, \mu} \) with non-zero multiplicity. Moreover, if \( V_{\nu} \) appears in \( A_{k, \lambda} A_{\ell, \mu} \) with non-zero multiplicity then \( \nu \) lies in the convex hull of the \( W \)-orbit of \( \lambda + \mu \) intersected with the positive Weyl chamber \( \Lambda^+_R \).

**Proof.** Let \( f \in A_{k, \lambda} \) and \( g \in A_{\ell, \mu} \) be highest weight vectors. Then \( fg \in A_{k+\ell, \lambda+\mu} \) is clearly a highest weight vector of weight \( \lambda + \mu \) and hence \( A_{k+\ell, \lambda+\mu} \neq 0 \). Let \( V_{\lambda}, V_{\mu} \) be copies of the irreducible representations of highest weights \( \lambda, \mu \) in \( A_{k, \lambda}, A_{\ell, \mu} \) respectively. Then \( V_{\lambda} V_{\mu} \subset A_{k+\ell, \lambda+\mu} \) is a quotient of the tensor product \( V_{\lambda} \otimes V_{\mu} \). The claim now follows from Theorem 3.2. \( \square \)

**Definition 3.4** (Semigroup of highest weights and moment convex set). 1) Define the set \( S_G(A) \subset \mathbb{Z}_{\geq 0} \times \Lambda^+ \) by
\[
S_G(A) = \{(k, \lambda) \mid A_{k, \lambda} \neq \{0\}\}.
\]
Note that if \( f \) and \( g \) are \( B \)-eigenfunctions with weights \( \lambda \) and \( \mu \) respectively, then \( fg \) is a \( B \)-eigenfunction with weight \( \lambda + \mu \). It follows that \( S_G(A) \) is a semigroup (under addition). We call \( S_G(A) \) the semigroup of highest weights of \( A \) or simply weight semigroup of \( A \). When there is no chance of confusion we will drop the subscript \( G \) and write \( S(A) \) instead of \( S_G(A) \). 2) We call the Newton-Okounkov convex set of the semigroup \( S(A) \) the moment convex set or simply the moment set of \( A \) and denote it by \( \Delta_G(A) \) or simply \( \Delta(A) \). 3) We call the lattice \( \Lambda(S(A)) \) associated to the semigroup \( S(A) \), the weight lattice of \( A \) and denote it by \( \Lambda(A) \). Also the linear subspace spanned by \( \Lambda(A) \) will be denoted by \( \Lambda_R(A) \).

**Remark 3.5.** Consider the subalgebra \( A^U \) of \( U \)-invariants in \( A \). Since the action of \( U \) respects the grading we can write \( A^U = \bigoplus_{k \geq 0} A^U_k \). Because \( T \) normalizes \( U \), the algebra \( A^U \) is stable under the action of \( T \), i.e. is a \( T \)-algebra. One can alternatively define \( S(A) \) to be the semigroup of \( T \)-weights of the subalgebra \( A^U \), i.e. \( S_T(A^U) = S_G(A) \).
Remark 3.6 (Connection with moment polytope in symplectic geometry). Let $X$ be a Hamiltonian $K$-manifold with the moment map $\mu : X \to \text{Lie}(K)^*$. It is a well-known result due to F. Kirwan that the intersection of the image of the moment map with the positive Weyl chamber is a convex polytope usually called the moment polytope or Kirwan polytope of the Hamiltonian $K$-space $X$.

Let $(L, \varphi) \in K_G(X)$ be an invariant subspace of rational functions with the Kodaira map $\Phi_L : X \to \mathbb{P}(L^*)$. As usual let $Y_L \subset \mathbb{P}(L^*)$ denote the closure of the image of $\Phi_L$. It is a closed irreducible $G$-invariant subvariety of the projective space $\mathbb{P}(L^*)$ with homogeneous coordinate ring $A_L$. Let us assume that $Y_L$ is smooth. Fix a $K$-invariant inner product on $L^*$ where $K$ is a maximal compact subgroup of $K$. This induces a $K$-invariant symplectic structure on $\mathbb{P}(L^*)$ and hence on $Y_L$. One has:

1) With this symplectic structure, $Y_L$ is a Hamiltonian $K$-manifold.
2) From the principle of quantization commutes with reduction it follows that this convex polytope coincides with $\Delta(A_L)$ (see [Guillemin-Sternberg84] and [Brion87]).

The following proposition shows that the weight lattices of all the graded $G$-algebras are in fact contained in the weight lattice of $X$.

Proposition 3.7. 1) Let $(A, \varphi)$ be a graded $G$-algebra. Then the lattice of weights $\Lambda(A)$ associated to $A$ is contained in the lattice of weights $\Lambda(X)$. It follows that the moment body $\Delta(A)$ is parallel to the linear space $\Lambda_{\mathbb{R}}(X)$. 2) Suppose $L \in K_G(X)$ is such that the Kodaira map $\Phi_L$ gives a birational isomorphism between $X$ and its image. Then the weight lattice $\Lambda(A_L)$ coincides with $\Lambda(X)$.

Proof. 1) Let $\lambda - \mu$ be an element of $\Lambda(A)$ where $\lambda, \mu$ are weights for two $B$-eigenfunctions $f, g \in A_k$ respectively, for some $k$ and with respect to the $\varphi^k$-twisted action of $G$ on $A_k$. Then $f/g$ is a $B$-eigenfunction of weight $\lambda - \mu$ in $\mathbb{C}(X)$ with respect to the natural action of $G$. (This proves 1). 2) Let $\gamma$ be a weight in $\Lambda(X)$ and let $h \in \mathbb{C}(X)$ be a $B$-eigenfunction with weight $\gamma$ (for the natural action of $G$ on $\mathbb{C}(X)$). Since the Kodaira map $\Phi_L$ is a birational isomorphism there is $k > 0$ and $f, g \in L^k$ such that $h = f/g$. Using Lie-Kolchin theorem (as in [Timashev06, Proposition 5.5] or [Popov-Vinberg89, Theorem 3.3]) one can find $B$-eigenfunctions $f_0, g_0 \in L^k$ with $h = f_0/g_0$. Let $\lambda, \mu$ be the weights of $f_0, g_0$ respectively. Then $\gamma = \lambda - \mu$ and hence $h \in \Lambda(A)$. This finishes the proof.

One shows that if $A$ is finitely generated then the semigroup $S(A)$ is also finitely generated (see [Popov86]):

Theorem 3.8. 1) If $A$ is a finitely generated $G$-algebra then $A^U$ is also finitely generated. 2) If $A$ is a finitely generated graded $G$-algebra then the semigroup $S(A)$ is a finitely generated semigroup.
Corollary 3.9. 1) If $A$ is a finitely generated $G$-algebra then the moment convex body $\Delta(A)$ is in fact a polytope which we call the moment polytope of $A$. In particular for any invariant subspace $(L,\varphi) \in K_G(X)$, the moment convex body $\Delta(A_L)$ is a polytope. 2) Let $(A,\varphi) \in A_G(X)$, i.e. $A$ is an algebra of almost $G$-integral type. Then $S(A)$ is a strongly non-negative semigroup. It follows that the moment convex set $\Delta(A)$ is a convex body which we call the moment convex body or simply the moment body of $A$.

Proof. 1) Follows from Theorem 3.8(2). 2) Let $(A,\varphi) \in A_G(X)$ be an algebra of almost $G$-integral type. From definition there is an invariant subspace $(L,\varphi) \in K_G(X)$ such that $A \subset A_L$. By 1) we know that $S(A_L)$ is finitely generated. As none of the generators lie in $\Lambda \times \{0\}$, the semigroup $S(A_L)$ is a strongly non-negative semigroup. Now since $S(A) \subset S(A_L)$ we conclude that $S(A)$ is also a strongly non-negative semigroup.

The following example concerns the situation in the Bernstein-Kušnirenko theorem on the number of solutions of a system of Laurent polynomials in $(\mathbb{C}^*)^n$.

Example 3.10 ($T$-invariant subspaces of Laurent polynomials). Let $T = X = (\mathbb{C}^*)^n$ be the algebraic torus acting on itself by multiplication. In $T$, the weight lattice $\Lambda$ and the semigroup of dominant weights $\Lambda^+$ both coincide with the lattice $\chi(T)$ of characters of $T$. Fixing a set of coordinates $(x_1,\ldots,x_n)$ on $X$ we identify $\chi(T)$ with $\mathbb{Z}^n$. Let $I \subset \mathbb{Z}^n$ be a finite set of characters. To $I$ we can associate a subspace $L(I)$ consisting of all Laurent polynomials with exponents in $I$. The subspace $L(I)$ is a $T$-invariant finite dimensional subspace of rational functions on $X$ for the natural action of $T$. The dimension of $L(I)$ is equal to $\# I$. Let $A = A_{L(I)}$ be the corresponding graded algebra. One sees that the semigroup of weights $S(A)$ is the semigroup in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$ generated by $\{1\} \times I$ and the moment convex body $\Delta(A)$ is the convex hull of the points in $I$.

From another point of view, the above example describes the moment polytope of an equivariant very ample line bundle on a toric variety.

The next example concerns the generalized Plücker embedding of the flag variety into a projective space.

Example 3.11 (Line bundles on flag variety). 1) Let $X = G/B$ be the variety of complete flags for a reductive algebraic group $G$. Let $\lambda \in \Lambda^+$ be a dominant weight and let $L_\lambda$ denote the $G$-linearized line bundle on $X$ associated to $\lambda$. One knows that for any integer $k > 0$, $H^0(X,L_\lambda^{\otimes k}) \cong V_{k\lambda}^*$ as $G$-modules. Let $\tau_\lambda$ be a highest weight section in $H^0(X,L_\lambda)$ with divisor $D_\lambda$. As in Example 3.18 consider the algebra of sections

$$A = R(D_\lambda) = \bigoplus_{k \geq 0} L(kD_\lambda) \cong \bigoplus_{k \geq 0} V_{k\lambda}^*,$$

where multiplication between the $V_{k\lambda}^*$ is the Cartan multiplication. Let us regard $A$ as a graded $G$-algebra. For a dominant weight $\lambda$ let $\lambda^*$ be the
dominant weight $-w_0 \lambda$. Then one knows that $V_{k\lambda}^* \cong V_{k\lambda^-}$ (as $G$-modules) and hence the semigroup $S_G(A)$ is given by

$$S_G(A) = \{(k\lambda^*, k) \mid k \in \mathbb{N}\}.$$  

Thus the moment convex body coincides with the single point $\{\lambda^*\}$.

2) On the other hand let us regard the algebra of sections $A$ above as a $T$-algebra and describe the moment body $\Delta_T(A)$. It is well-known that the convex hull of $T$-weights in $V_\lambda$ coincides with the convex hull of Weyl group orbit $\{w\lambda \mid w \in W\}$. It thus follows that the moment convex body $\Delta_T(A)$ is the convex hull of the Weyl group orbit $\{w\lambda^* \mid w \in W\}$.

The following is related to the situation addressed in [Kazarnovskii87] generalizing the Bernstein-Kušnirenk theorem to representations of reductive groups. ([Kazarnovskii87] uses a symplectic geometric approach.)

**Example 3.12** (Moment polytope of a representation). Let $\pi : G \to \text{GL}(n, \mathbb{C}) \subset \text{Mat}(n, \mathbb{C})$ be a finite dimensional representation of $G$. Let $\pi_{ij} : G \to \mathbb{C}$, $i, j = 1, \ldots, n$ be the matrix elements, i.e. the entries of $\pi$. Let $L$ be the subspace of regular functions on $G$ spanned by the $\pi_{ij}$. Consider the action of $G \times G$ on $G$ by the multiplication from left and right. Then the subspace $L$ is a $(G \times G)$-invariant subspace. As in [Kazarnovskii87], one can show that the moment polytope of the $(G \times G)$-algebra $A_L$ (which lives in $\Lambda_\mathbb{R}^+ \times \Lambda_\mathbb{R}^+$) can be identified with the convex hull of the $W$-orbit of the highest weights of the representation $\pi$ intersected with the positive Weyl chamber $\Lambda_\mathbb{R}^+$.

We now give lower and upper estimates for the moment polytope of an algebra $A_L$ associated to an invariant subspace $(L, \varphi) \in K_G(X)$. Note that $A_L$ is finitely generated and hence its moment body is a polytope (Corollary 3.9). We define two polytopes which are lower and upper estimates for the moment polytope $\Delta(A_L)$. Let $I \subset \Lambda^+$ be the finite set of all dominant weights $\lambda$ where $V_\lambda$ appears in the $G$-module $L$ with non-zero multiplicity. Also let $Q(I)$ be the intersection of the convex polytope obtained by taking the convex hull of the union of the $W$-orbits of all the $\lambda \in I$ with the positive Weyl chamber $\Lambda_\mathbb{R}^+$.

**Proposition 3.13.** With notation as above, we have

$$\text{conv}(I) \subset \Delta(A) \subset Q(I).$$

**Proof.** Follows immediately from Theorem 3.3. \qed

Later in Section 7 we will consider a class of varieties where the lower estimate happens. Namely, for horospherical varieties the moment polytope is equal to $\text{conv}(I)$. The upper estimate $Q(I)$ happens for the so-called symmetric varieties. It turns out that in both cases the moment polytope is additive (see [Kaveh03] for the case of symmetric varieties).
3.2. **Superadditivity of the moment convex body.** Let \( X \) be an irreducible \( G \)-variety. In this section we will deal with the graded \( G \)-algebras \((A, \varphi) \in A_G(X)\) and will address the additivity of the mapping \( A \mapsto \Delta(A) \) with respect to the componentwise product of \( G \)-algebras in \( A_G(X) \). We will show that in general this map is superadditive but in the case of a torus action \( G = T \) is additive.

Let us start with the case of a torus action. Let \( T \) be a torus and \( X \) an irreducible \( T \)-variety. Proposition 3.14.

1) Let \((A', \varphi'), (A'', \varphi'') \in A_T(X)\) be two graded \( T \)-algebras. Then \( \Delta_T(A) = \Delta_T(A') + \Delta_T(A'') \), where \((A, \varphi) = (A'A'', \varphi'\varphi'')\) is the componentwise product of \( A', A'' \), and the addition in the right-hand side is the Minkowski sum of convex bodies.

2) Let \( L \) be a \( T \)-invariant subspace of rational functions and let \( I \) be the finite set of \( T \)-weights of \( L \). Then the moment convex body of the algebra \( A_L \) coincides with the convex hull of \( I \).

3) The map \( L \mapsto \Delta_T(A_L) \) is additive with respect to the multiplication of subspaces. That is, if \( L_1, L_2 \) are two \( T \)-invariant subspaces then

\[
\Delta_T(A_{L_1L_2}) = \Delta_T(A_{L_1}) + \Delta_T(A_{L_2}).
\]

Proof. 1) If \( f \) and \( g \) are \( T \)-eigenfunctions of weights \( \lambda \) and \( \gamma \) respectively then \( fg \) is a \( T \)-eigenfunction of weight \( \lambda + \gamma \). It implies that the semigroup \( S_T(A) \) is the levelwise addition of semigroups \( S_T(A') \) and \( S_T(A'') \). It then follows that \( \Delta_T(A) = \Delta_T(A') + \Delta_T(A'') \) (see Section 1.9). 2) The semigroup \( S(A_L) \) is generated by the finite set \( \{1\} \times I \subset \mathbb{Z}_{\geq 0} \times \chi(T) \) and hence its moment convex body coincides with the convex hull of \( I \). 3) Follows immediately from 1).

Now we address the case of a reductive group action. Let \((A', \varphi), (A'', \varphi'') \in A_G(X)\) be two graded \( G \)-algebras. We have the following inclusion of the algebras of \( U \)-invariants:

\[
(A'^U)(A''^U) \subset (A'A'')^U,
\]
where \( A'A'' \) denotes the componentwise product of algebras (with action twisted by \( \varphi'\varphi'' \)). In general \((A'^U)(A''^U)\) might be strictly smaller than \((A'A'')^U\), and hence the map \( A \mapsto \Delta(A) \) is only superadditive with respect to the componentwise product of algebras (see Remark 3.5):

Proposition 3.15 (Superadditivity). With notation as above, we have

\[
\Delta(A') + \Delta(A'') \subset \Delta(A).
\]

In particular, if \((L', \varphi'), (L'', \varphi'') \in K_G(X)\) are invariant subspaces then we have

\[
\Delta(A_{L'}) + \Delta(A_{L''}) \subset \Delta(A_{L'L''}).
\]
4. Multiplicity convex body and Duistermaat-Heckman measure

4.1. Multiplicity convex body of a $G$-algebra. As usual let $X$ be an irreducible $G$-variety with $\dim X = n$. Following Okounkov [Okounkov96], in this section we define a larger semigroup $\hat{S}(A)$ lying over the weight semigroup $S(A)$ such that it encodes information about the multiplicities of irreducible representations appearing in $A$.

In [Okounkov96] the author deals with the algebras of sections of $G$-linearized very ample line bundles. Here we deal with the larger class of graded algebras of almost $G$-integral type, which already includes the algebra of sections of arbitrary $G$-linearized line bundles.

Fix a $B$-invariant valuation $v : \mathbb{C}(X)^* \to \mathbb{Z}^n$ with one-dimensional leaves. It is shown in [Okounkov96] that such a valuation exists.

We associate a lattice to $X$ containing the values of $v$ at the $U$-invariant functions on $X$. More precisely,

**Definition 4.1.** Let $\hat{\Lambda}(X) = v(\mathbb{C}(X)^U \setminus \{0\})$ be the image of the subfield of rational $U$-invariants under the valuation $v$. It is a lattice in $\mathbb{Z}^n$. The rank of $\hat{\Lambda}(X)$ is equal to the transcendence degree of the field $\mathbb{C}(X)^U$. We will denote this rank by $\hat{r} = \hat{r}(X)$. We also denote the linear span of $\hat{\Lambda}(X)$ by $\hat{\Lambda}_R(X)$.

Let $(A, \varphi)$ be a graded $G$-algebra and write $A = \bigoplus_{k \geq 0} L_k t^k$. Define the semigroup $\hat{S}(A)$ to be the valuation semigroup of the algebra $A^U$ of $U$-invariants. More precisely,

**Definition 4.2.** 1) Define the set $\hat{S}_G(A) \subset \Lambda^+ \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_m$ by

$$\hat{S}_G(A) = \bigcup_{k \geq 0} \{ (k, v(f)) \mid f \in L_k^U \setminus \{0\} \}.$$ 

We call the semigroup $\hat{S}_G(A)$, the *multiplicity semigroup of the algebra* $A$. When there is no ambiguity we will simply write $\hat{S}(A)$ instead of $\hat{S}_G(A)$. 2) Denote the Newton-Okounkov convex set of the semigroup $\hat{S}_G(A)$ by $\hat{\Delta}_G(A)$ or simply $\hat{\Delta}(A)$ and call it the *multiplicity convex set* of $A$. 3) Denote the lattice $\Lambda(\hat{S}(A))$ associated to the semigroup $\hat{S}(A)$ by $\hat{\Lambda}(A)$.

The following are some basic properties of the semigroup $\hat{S}(A)$.

**Proposition 4.3.** 1) If $A$ is a graded algebra of almost $G$-integral type then $\hat{S}(A)$ is a strongly non-negative semigroup and hence the multiplicity convex set $\hat{\Delta}(A)$ is a convex body. 2) The lattice $\hat{\Lambda}(A)$ associated to the semigroup $\hat{S}(A)$ is contained in the lattice $\hat{\Lambda}(X)$. It follows that the multiplicity body $\hat{\Delta}(A)$ is parallel to the linear space $\hat{\Lambda}_R(X)$. 3) Suppose $L \in K_G(X)$ is such that the Kodaira map $\Phi_L$ gives a birational isomorphism between $X$ and its image. Then the lattice $\Lambda(A_L)$ coincides with $\hat{\Lambda}(X)$. 
Proof. 1) If \(A\) is an algebra of almost \(G\)-integral type then the algebra \(A^U\) of \(U\)-invariants is of almost integral type. Thus by Proposition 1.20(2), \(\hat{S}(A)\) is a non-negative semigroup. 2) Let \((\lambda - \mu, v(f) - v(g))\) be an element of the lattice \(\Lambda(A)\), where \(f, g\) are \(B\)-eigenfunctions in \(A^U_k\), for some \(k > 0\), of weights \(\lambda, \mu\) respectively (with respect to the twisted action of \(G\) on \(A\)). Then \(f/g\) is a \(B\)-eigenfunction with weight \(\lambda - \mu\) with respect to the natural action of \(G\) on \(\mathbb{C}(X)\). Since \(v(f/g) = v(f) - v(g)\), it follows that \((\lambda - \mu, v(f) - v(g))\) lies in \(\Lambda(X)\) which proves 2). 3) Let \(a \in \Lambda(X) = v(\mathbb{C}(X)^U \setminus \{0\})\) be a value of the valuation on \(U\)-invariants. Take \(h \in \mathbb{C}(X)^U\) with \(v(h) = a\). Since \(\Phi_L\) is a birational isomorphism, for some \(k > 0\) there are \(f, g \in L^k\) with \(h = f/g\). As in the proof of Proposition 3.7(2) we can find \(U\)-invariant functions \(f_0, g_0 \in L^k\) such that \(h = f_0/g_0\). But \(v(h) = v(f_0) - v(g_0)\) which shows that \(a \in \hat{\Lambda}(A)\) as required. \(\square\)

Next we will show that the semigroup \(\hat{S}(A)\) contains information about the multiplicities of the irreducible representations appearing in \(A\). Following Okounkov, we show that there is a natural projection from \(\hat{S}(A)\) onto the weight semigroup \(S(A)\). Write

\[
A = \bigoplus_{k \geq 0} \bigoplus_{\lambda \in \Lambda^+} L_{k,\lambda}^k t^k,
\]

where \(L_{k,\lambda}\) is the \(\lambda\)-isotypic component in \(L_k\), i.e. the sum of all copies of the irreducible representation \(V_\lambda\) in \(L_k\). Let us recall the following well-known facts: 1) The subspace of \(U\)-invariants \(V_\lambda^U\) in an irreducible representation \(V_\lambda\) is one-dimensional (which in fact consists of the highest weight vectors). 2) The dimension of the subspace \(L_{k,\lambda}^U\) is equal to the multiplicity \(m_{k,\lambda}\) of the highest weight \(\lambda\) in the \(G\)-module \(L_k\).

**Lemma 4.4.** Let \(k > 0\) be an integer. 1) Let \(f \in L_{k,\lambda}, g \in L_{k,\mu}\) be two \(B\)-eigenfunctions of weights \(\lambda, \mu\) respectively and assume that \(\lambda \neq \mu\). Then \(v(f) \neq v(g)\). 2) For any value \(a \in v(L_{k,\lambda}^U \setminus \{0\})\) there is a dominant weight \(\lambda \in \Lambda^+\) and a \(B\)-eigenfunction \(f \in L_{k,\lambda}^U\) such that \(v(f) = a\).

**Proof.** 1) By contradiction suppose that \(a = v(f) = v(g)\). Consider the leaf

\[
F_a = \{h \mid v(h) \geq a\} \setminus \{h \mid v(h) > a\}.
\]

By Property (4) in definition of a valuation (Definition 1.19) \(F_a\) has dimension 1. On the other hand, since the valuation \(v\) is \(B\)-invariant, \(F_a\) is a \(B\)-module. But then \(F_a\) can not have two distinct weights \(\lambda\) and \(\mu\). The contradiction proves the claim. 2) We know that \(\dim L_k^U = \sum_{\lambda \in \Lambda^+} \dim L_{k,\lambda}^U\), moreover (by Proposition 1.20(1)) \(\dim L_k^U = \#v(L_k^U \setminus \{0\})\) and \(\dim L_{k,\lambda}^U = \#v(L_{k,\lambda}^U \setminus \{0\})\). But by 1) if \(\lambda \neq \mu\) then \(v(L_{k,\lambda}^U \setminus \{0\}) \cap v(L_{k,\mu}^U \setminus \{0\}) = \emptyset\). Thus \(v(L_k^U \setminus \{0\})\) is equal to \(\bigcup_{\lambda \in \Lambda^+} v(L_{k,\lambda}^U \setminus \{0\})\). This proves 2). \(\square\)

Let \(a \in v(L_k^U \setminus \{0\})\) be a value of the valuation on the \(k\)-th subspace \(L_k\). By Lemma 4.4(2) there exists a unique weight \(\lambda(a)\) such that \(a = v(f)\)
for some $B$-eigenfunction $f$ of weight $\lambda(a)$. Consider the map $(k,a) \mapsto (k,\lambda(a))$. It is easy to see that this gives a surjective additive map $\hat{\pi}$ from the semigroup $\hat{S}(A)$ onto $S(A)$: let $f$, $g$ be two $B$-eigenfunctions of weights $\lambda$, $\mu$ respectively. Also let $v(f) = a$ and $v(g) = b$. Then clearly $fg$ is a $B$-eigenfunction of weight $\lambda + \mu$ and $v(fg) = a + b$. That is, $\lambda(a + b) = \lambda(a) + \lambda(b)$. The map $\hat{\pi}$ then extends to a linear map from $L(\hat{S}(A))$ to $L(S(A))$ which by abuse of notation we denote again by $\hat{\pi}$.

The number of points in the fibre of $\hat{\pi}$ over a point $(k,\lambda) \in S(A)$ gives the multiplicity of the corresponding highest weight in $L_k$:

**Proposition 4.5.** For each $(\lambda,k) \in SG(A)$, the number of points in the inverse image $\hat{\pi}^{-1}(\lambda,k)$ is equal to the multiplicity $m_{k,\lambda}$ of the irreducible representation $V_\lambda$ in $L_k$.

**Proof.** From definition the number of points in $\hat{\pi}^{-1}(\lambda,k)$ is equal to the number of points in the image of $L_{U,\lambda,k} \setminus \{0\}$ under the valuation $v$. But by Proposition 1.20(1), this is equal to the dimension of $L_{U,k,\lambda}$ which in turn is equal to the $\lambda$-multiplicity of $L_k$. $\square$

Let $M(k) = \sum_{\lambda \in \Lambda_+} m_{k,\lambda}$ be the sum of multiplicities in $A_k$ (i.e. $M(k)$ is the Hilbert function $H_{A^U}(k)$ of the graded algebra $A^U$). From Proposition 1.3(2), it follows that $M(k)$ is equal to the number of points in the semigroup $\hat{S}(A)$ at level $k$. Applying Theorem 1.7 to the semigroup $\hat{S}(A)$ we obtain:

**Corollary 4.6.** Let $A \in AG(X)$ be a graded algebra of almost $G$-integral type. Let $\hat{p} = \dim \hat{\Delta}(A)$ be the (real) dimension of the multiplicity body of $A$. Then the function $M(k)$ has growth degree $\hat{p}$ and its $\hat{p}$-th growth coefficient is equal to the $\hat{p}$-dimensional volume of $\hat{\Delta}(A)$ (where the volume is normalized with respect to the lattice $\hat{\Lambda}(A))$. That is,

$$\lim_{k \to \infty} \frac{M(k)}{k^{\hat{p}}} = \text{Vol}_{\hat{p}}(\hat{\Delta}(A)).$$

In Section 1.3 we will give a formula for the self-intersection index of an invariant subspace $L$ of rational functions in terms of the integral of an (explicitly defined) polynomial function over the multiplicity convex body of the algebra $A = A_L$.

**Remark 4.7.** In a similar fashion, we can describe the asymptotic of the multiplicities $m_{k,k,\lambda}$ as $k \to \infty$, in terms of the dimension and volume of the fibres $\hat{\pi}^{-1}(\lambda)$ of the projection $\hat{\pi} : \hat{\Delta}(A) \to \Delta(A)$.

### 4.2. Duistermaat-Heckman measure for graded $G$-algebras

Let $X$ be an irreducible $G$-variety. In this section we extend the definition of Duistermaat-Heckman measure (for projective $G$-varieties and $G$-linearized very ample line bundles) to graded $G$-algebras $A \in AG(X)$.

Let us recall the Duistermaat-Heckman measure for a Hamiltonian action from symplectic geometry. Let $K$ be a compact Lie group and $X$ a
Hamiltonian $K$-manifold with the moment map $\phi : X \to \text{Lie}(K)^*$. Let us denote the moment polytope which is the intersection of $\phi(X)$ with the positive Weyl chamber by $\Delta(X)$. Let $\lambda \in \Delta(X)$ be a regular value for the moment map and let $K_\lambda$ denote its $K$-stabilizer. The reduced space $X_\lambda = \phi^{-1}(\lambda)/K_\lambda$ is a symplectic manifold with respect to a natural symplectic form. The Duistermaat-Heckman measure on the polytope $\Delta(X)$ is the measure $\text{Vol}(X_\lambda)d\gamma$, where $\text{Vol}$ is the symplectic volume and $d\gamma$ is the Lebesgue measure on $\Delta(X)$ (normalized with respect to the lattice $\Lambda$).

Let $(A, \varphi) \in A_G(X)$ be a graded algebra of almost $G$-integral type. Let $p = \dim \Delta(A)$ and $\hat{p} = \dim \hat{\Delta}(A)$ be the (real) dimensions of the moment body $\Delta(A)$ and the multiplicity body $\hat{\Delta}(A)$ respectively. From Corollary 4.6 we know that the sum of multiplicities $M(k)$ (that is, the Hilbert function of the algebra $A^k_{\Lambda}$) has growth degree equal to $\hat{p}$.

For each integer $k > 0$ consider the measure $d\mu_k$ with finite support defined on the positive Weyl chamber $\Lambda^+_{\mathbb{R}}$ by

$$d\mu_k = \sum_{\lambda \in \Lambda^+} m_{k,\lambda} \delta_{\lambda/k},$$

where $\delta_x$ denotes the Dirac measure centered at the point $x$. Let $d\hat{\gamma}$ be the Lebesgue measure on the multiplicity body $\hat{\Delta}(A)$ normalized with respect to the lattice $\hat{\Lambda}(A)$ associated to the semigroup $\hat{S}(A)$.

Consider the linear map $\hat{\pi} : \hat{S}(A) \to S(A)$. With notations as in Section 1.2, the measure $d\mu_k$ is the measure associated to the pair $(S(A), \hat{S}(A))$ and the linear map $\hat{\pi}$. Applying Theorem 1.11 we obtain the following:

**Theorem 4.8.** 1) The sequence $d\mu_k/k^{\hat{p} - p}$ weakly converges (as $k \to \infty$) to a (finite) measure $d\mu_A$ supported on the moment convex body $\Delta(A)$. 2) The measure $d\mu_A$ is equal to the push-forward of the Lebesgue measure on $\hat{\Delta}(A)$ to the moment body $\Delta(A)$.

**Definition 4.9** (Duistermaat-Heckman measure for $G$-algebras). Let $(A, \varphi) \in A_G(X)$ be a graded algebra of almost $G$-integral type. In analogy with the case of Hamiltonian spaces, we call $d\mu_A$ the Duistermaat-Heckman measure associated to the $G$-algebra $A$.

In Section 6.1 we will give a formula for the self-intersection index of an invariant subspace $L$ of rational functions in terms of the integral of an (explicitly defined) polynomial over the moment body of $A = A_L$ with respect to the Duistermaat-Heckman measure of the algebra $A$.

**Remark 4.10** (Duistermaat-Heckman measure for projective $G$-subvarieties). Let $(L, \varphi) \in K_G(X)$ be an invariant subspace of rational functions with the Kodaira map $\Phi_L : X \to \mathbb{P}(L^*)$. As usual let $Y_L \subset \mathbb{P}(L^*)$ denote the closure of the image of $\Phi_L$. It is a closed irreducible $G$-invariant subvariety of the projective space $\mathbb{P}(L^*)$. Let us assume $Y_L$ is smooth. As in Remark 3.6, fix a $K$-invariant Hermitian inner product on $L^*$ where $K$ is a maximal compact subgroup of $G$. This induces a $K$-invariant symplectic
structure on $\mathbb{P}(L^*)$ and hence on $Y_L$. With this $Y_L$ becomes a Hamiltonian $K$-space. Using the principle of quantization commutes with reduction one proves that (up to multiplication by a constant) the Duistermaat-Heckman measure of the Hamiltonian space $Y_L$ coincides with the measure $d\mu_{A_L}$ (see [Guillemin-Sternberg84, Theorem 6.5]).

Similar to the moment body, the multiplicity body also enjoys a superadditivity property. Let $(A', \varphi), (A'', \varphi'') \in A_G(X)$ be two graded $G$-algebras. We have the following inclusion of the algebras of $U$-invariants:

$$(A'^U)(A''^U) \subset (A'A'')^U,$$

where $A'A''$ denotes the componentwise product of algebras (with action twisted by $\varphi\varphi''$). In general $(A'^U)(A''^U)$ might be strictly smaller than $(A'A'')^U$ and thus the map $A \mapsto \Delta(A)$ is in general only superadditive:

**Proposition 4.11** (Superadditivity for multiplicity body). With notation as above, we have

$$\hat{\Delta}(A') + \hat{\Delta}(A'') \subset \hat{\Delta}(A).$$

In particular, if $(L', \varphi'), (L'', \varphi'') \in K_G(X)$ are invariant subspaces then we have

$$\hat{\Delta}(A_{L'}) + \hat{\Delta}(A_{L''}) \subset \hat{\Delta}(A_{L'L''}).$$

**Proof.** Let $(k, a') \in \hat{S}(A')_k$ and $(k, a'') \in \hat{S}(A'')_k$ be two elements at level $k$ in the semigroups $\hat{S}(A')$ and $\hat{S}(A'')$ respectively. We can find $f' \in L'^U_k$ and $f'' \in L''^U_k$ with $v(f') = a'$ and $v(f'') = a''$. Then $f'f'' \in L^U_k$ has valuation $a' + a''$. Thus the point $(k, a' + a'')$ lies in $\hat{S}(A)_k$. This proves that the levelwise addition of $\hat{S}(A')$ and $\hat{S}(A'')$ is contained in $\hat{S}(A)$. The superadditivity immediately follows from this inclusion. \qed

Recall that by Proposition 4.3(1), the multiplicity body $\hat{\Delta}(A)$ is parallel to the linear space $\hat{A}_\mathbb{R}(X)$. We denote the dimension of $\hat{A}_\mathbb{R}(X)$ (equivalently the rank of $\hat{A}(X)$) by $\hat{r} = \hat{r}(X)$. Now applying superadditivity (Proposition 4.11) and the classical Brunn-Minkowski inequality we obtain the following:

**Corollary 4.12** (Brunn-Minkowski inequality for D-H measure of algebras). Let $(A', \varphi), (A'', \varphi'') \in A_G(X)$ be two graded $G$-algebras. Also assume that $m(A') = m(A'') = 1$. Let $d\mu_{A'} = f_{A'}d\gamma$, $d\mu_{A''} = f_{A''}d\gamma$ and $d\mu_A = f_Ad\gamma$ denote the Duistermaat-Heckman measures/functions for the algebras $A'$, $A''$ and $A = A'A''$ respectively. Then for $\lambda' \in \Delta(A')$, $\lambda'' \in \Delta(A'')$ we have

$$f_{A'}(\lambda')^{1/\hat{r}} + f_{A''}(\lambda'')^{1/\hat{r}} \leq f_A(\lambda' + \lambda'')^{1/\hat{r}}.$$

**Proof.** The corollary follows from Proposition 4.11 and the classical Brunn-Minkowski inequality applied to $\Delta(A')$ and $\Delta(A'')$. \qed

**Corollary 4.13** (Brunn-Minkowski inequality for D-H measure of line bundles). Let $L'$ and $L''$ be two $G$-line bundles on $X$ with divisors $D'$ and $D''$ respectively. Let $R(D')$, $R(D'')$ and $R(D' + D'')$ denote the corresponding algebra of sections (see Example 2.7). Also assume that $m(R(D')) =
\( m(R(D'')) = 1 \). Let \( d\mu_L = f_L \, d\gamma \), \( d\mu_{L''} = f_{L''} \, d\gamma \) and \( d\mu = f \, d\gamma \) denote the Duistermaat-Heckman measures/functions for the algebras \( R(D') \), \( R(D'') \) and \( R(D' + D'') \) respectively. Then for \( \lambda' \in \Delta(R(D')) \), \( \lambda'' \in \Delta(R(D'')) \) we have
\[
 f_{L'}(\lambda')^{1/\ell} + f_{L''}(\lambda'')^{1/\ell} \leq f(\lambda' + \lambda'')^{1/\ell}.
\]

**Proof.** Follows from Proposition 4.12 and the fact that \( R(D')R(D'') \subset R(D' + D'') \). \( \square \)

### 4.3. Fujita approximation for Duistermaat-Heckman measure

The Fujita approximation theorem in the theory of divisors states that the so-called volume of a big divisor can be approximated arbitrarily closely by the Duistermaat–Heckman measure of graded algebras. That is, we prove that the Duistermaat-Heckman measure of a graded \( G \)-algebra can be approximated arbitrarily closely by the Duistermaat-Heckman measures of \( G \)-algebras of type \( A_L \) for finite dimensional invariant subspaces \( L \). This will follow from the analogous statement (Theorem 1.12) for semigroups.

Now we apply Theorem 1.12 to prove a version of Fujita approximation theorem for the Duistermaat-Heckman measure of graded \( G \)-algebras. That is, we prove that the Duistermaat-Heckman measure of a graded \( G \)-algebra can be approximated arbitrarily closely by the Duistermaat-Heckman measures of \( G \)-algebras of type \( A_L \) for finite dimensional invariant subspaces \( L \). This will follow from the analogous statement (Theorem 1.12) for semigroups.

Take an integer \( k > 0 \) such that \( A_k \neq \{0\} \). Let \( L_k \subset \mathbb{C}(X) \) be the \( k \)-th subspace of \( A \) i.e. \( A_k = L_k^k \). Consider the graded \( G \)-algebra \( A_{L_k} \) associated to \( L_k \). Let \( d\rho_k \) be the Duistermaat-Heckman measure associated to \( A_{L_k} \). In other words, \( d\rho_k \) is the Duistermaat-Heckmann measure of the projective \( G \)-subvariety \( Y_{L_k} \subset \mathbb{P}(L_k^\ell) \). Let \( \hat{m}_{k,\ell,\lambda} \) be the multiplicity of the irreducible representation \( V_\lambda \) in \( (L_k^\ell)^\ell \), i.e. the \( \ell \)-th subspace of \( A_{L_k} \). Then:
\[
 d\rho_k = \lim_{\ell \to \infty} (1/\ell^p) \sum_{\lambda \in \Lambda^\ell} \hat{m}_{k,\ell,\lambda} \delta_{\lambda/\ell}.
\]

The Duistermaat-Heckman measure \( d\rho_k \) is supported on the convex polytope \( \Delta(A_{L_k}) \) (which is contained in the convex body \( k \Delta(A) \)).

Let \( O_{1/k} : \Lambda_R \to \Lambda_R \) denote the multiplication by the scalar \( 1/k \).

**Theorem 4.14** (Fujita approximation type theorem for Duistermaat-Heckman measure of algebras). Let \( (A, \varphi) \in A_G(X) \) be a graded algebra of almost \( G \)-integral type. Then, as \( k \to \infty \), the measures \( O_{1/k}^*(d\mu_k)/k^{\beta-p} \) converge weakly to the Duistermaat-Heckman measure \( d\mu_A \) associated to the \( G \)-algebra \( A \). Here \( O_{1/k}^* \) denotes the push-forward of the measure \( d\mu_k \) on \( \Delta(A_{L_k}) \) to the convex body \( \Delta(A) \).
Proof. The claim follows from Theorem 1.12 applied to $\tilde{\pi} : \hat{S}(A) \to S(A)$. \hfill $\square$

5. String convex body of a $G$-algebra

5.1. String polytopes for irreducible representations. In their now classical paper [Gelfand-Cetlin50], Gelfand and Cetlin constructed natural bases for irreducible representations of $GL(n, \mathbb{C})$ and showed how to parameterize the elements of these bases with integral points in certain convex polytopes. Let us explain their work in more details: a dominant weight $\lambda$ for $G = GL(n, \mathbb{C})$ can be represented as a decreasing sequence of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$.

Then the elements of the Gelfand-Cetlin basis for the irreducible representation $V_\lambda$ are parameterized by the integral solutions $x_{ij}$ of the following set of inequalities:

\[
\begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n-2} & \lambda_{n-1} & \lambda_n \\
x_{1,1} & x_{2,1} & \cdots & x_{n-2,1} & x_{n-1,1} \\
x_{1,2} & x_{2,2} & \cdots & x_{n-3,2} & x_{n-2,2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
x_{1,n-1} & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

where the notation $a \geq c \geq b$ means $a \geq c \geq b$. The set of real solutions for the $x_{ij}$ in the above inequalities is a convex polytope called the Gelfand-Cetlin polytope (or simply G-C polytope) $\Delta_{GC}(\lambda)$. From the construction of Gelfand and Cetlin, the dimension of the irreducible representation $V_\lambda$ is equal to the number of integral points in the convex polytope $\Delta_{GC}(\lambda)$.

The Gelfand-Cetlin approach has been generalized to all finite dimensional irreducible representations of any reductive group by the works of Littelmann ([Littelmann98]) and Bernstein-Zelevinsky ([Bernstein-Zelevinsky01]).

Firstly, similar constructions can be done for the other classical groups $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$ and $SP(2n, \mathbb{C})$ to obtain natural bases for the irreducible representations together with a parametrization by integral points satisfying similar (explicit) inequalities as in (4) (see [Berenstein-Zelevinsky88]). Also one defines G-C polytopes for a direct product of the above groups simply by defining it to be the Cartesian product of the corresponding G-C polytopes. We will denote all the above G-C polytopes for the classical groups or direct products of them by $\Delta_{GC}(\lambda)$.

Secondly and more generally one can construct natural bases for all finite dimensional irreducible representations of any connected reductive algebraic group $G$ (the so-called crystal bases) and give a parametrization (string
parametrization) of these bases (see [Littelmann98 and Berenstein-Zelevinsky01}). Unlike the classical cases of $\text{GL}(n, \mathbb{C})$, $\text{SO}(n, \mathbb{C})$ or $\text{SP}(2n, \mathbb{C})$, in general, the defining equations for the corresponding polytopes for a general reductive group are not as easy to write down. Below we give a brief account of the string polytopes. As we won’t deal with crystal bases later, we will skip their discussion and will only discuss their string parametrization.

The string parametrization depends on a choice of a reduced decomposition for $w_0$, the longest element in the Weyl group $W$. More precisely, a sequence of simple reflections $w_0 = (s_{i_1}, \ldots, s_{i_N})$ (of course with possible repetitions) is called a reduced decomposition for $w_0$ if

$$w_0 = s_{i_1} \cdots s_{i_N},$$

and $N = \ell(w_0)$ is the length of $w_0$, \(\text{dim} V\) (which is equal to the number of positive roots).

Fix a reduced decomposition $w_0$ for $w_0$. Let $\lambda \in \Lambda^+$ be a dominant weight. For $f \in V_\lambda$ put $\iota_{w_0}(f) = (\lambda, a_1, \ldots, a_N)$ where $a_1, \ldots, a_N$ are defined as follows: for each root $\alpha$ let $E_\alpha \in \text{Lie}(G)$ denote the Chevalley generator for the root subspace of $\alpha$. Then

$$a_1 = \max\{a | E_{\alpha_{i_1}}^a \cdot f \neq 0\},$$

$$a_2 = \max\{a | E_{\alpha_{i_2}}^a E_{\alpha_{i_1}}^a \cdot f \neq 0\},$$

$$a_3 = \max\{a | E_{\alpha_{i_3}}^a E_{\alpha_{i_2}}^a E_{\alpha_{i_1}}^a \cdot f \neq 0\},$$

and so on. The $n$-tuple of non-negative integers $(a_1, \ldots, a_N)$ is called the string parameters associated to $f$ (for the choice of the reduced decomposition $w_0$).

Let $\pi : \Lambda_\mathbb{R} \times \mathbb{R}^N \to \Lambda_\mathbb{R}$ be the projection on the first factor. The following remarkable result due to Littelmann (see [Littelmann98]) describes the image of the string parametrization:

**Theorem 5.1.** 1) For each dominant weight $\lambda$, the dimension of $V_\lambda$ coincides with the number of points in the image $\iota_{w_0}(V_\lambda \setminus \{0\})$. 2) There exists a convex rational polyhedral cone $C_{w_0} \subset \Lambda_\mathbb{R} \times \mathbb{R}^N$ such that for each dominant weight $\lambda \in \Lambda^+$ the image of $V_\lambda \setminus \{0\}$ under $\iota_{w_0}$ coincides with $\pi^{-1}(\lambda) \cap C_{w_0} \cap (\Lambda \times \mathbb{Z}^N)$. Thus $\text{dim} V_\lambda$ is equal to the number of integral points in the slice $\pi^{-1}(\lambda) \cap C_{w_0}$ of the cone $C_{w_0}$.

**Definition 5.2.** The intersection of the cone $C_{w_0}$ with the plane $\pi^{-1}(\lambda)$ is a convex polytope called the string polytope associated to $\lambda$ and denoted by $\Delta_{w_0}(\lambda)$. Theorem 5.1 states that the number of integral points in the string polytope $\Delta_{w_0}(\lambda)$ is equal to $\text{dim} V_\lambda$.

**Remark 5.3.** In [Littelmann98] it is shown that when $G = \text{GL}(n, \mathbb{C})$ and for a natural choice of a reduced decomposition $w_0$, after a fixed linear change

\(\text{i.e. the minimum number of simple reflections needed to write } w_0 \text{ as a product of simple reflections.}
of parameters independent of \( \lambda \), the string polytope \( \Delta_{w_0}(\lambda) \) coincides with the Gelfand-Cetlin polytope \( \Delta_{GC}(\lambda) \).

In general the string polytopes are only superadditive with respect to sum of weights. Although in the case of Gelfand-Cetlin polytopes they are actually additive:

**Proposition 5.4** (Superadditivity of string polytopes). *Fix a reduced decomposition \( \overrightarrow{w_0} \). Then for any two dominant weights \( \lambda, \gamma \in \Lambda^+ \) we have*

\[
\Delta_{w_0}(\lambda) + \Delta_{w_0}(\gamma) \subset \Delta_{w_0}(\lambda + \gamma).
\]

*Proof.* Follows from the fact that \( C_{w_0} \) is a convex cone. \( \square \)

The following is a corollary of the defining inequalities of the G-C polytopes for classical groups (e.g. the inequalities (1) for \( G = GL(n, \mathbb{C}) \)).

**Proposition 5.5** (Additivity of G-C polytopes). *Let \( G \) be one of the groups \( GL(n, \mathbb{C}) \), \( SL(n, \mathbb{C}) \), \( SO(n, \mathbb{C}) \), \( SP(2n, \mathbb{C}) \) or a direct product of them. Then for any two dominant weights \( \lambda, \gamma \in \Lambda^+ \) we have*

\[
\Delta_{GC}(\lambda) + \Delta_{GC}(\gamma) = \Delta_{GC}(\lambda + \gamma).
\]

5.2. **String convex body of a \( G \)-algebra.** Let \( (A, \varphi) \) be a graded \( G \)-algebra. We now define a semigroup lying over the multiplicity semigroup \( \hat{S}(A) \) and which encodes information both about multiplicities and the dimensions of the isotypic components.

Fix a reduced decomposition \( \overrightarrow{w_0} \) for the longest element \( w_0 \) in the Weyl group.

**Definition 5.6.** Define the set \( \tilde{S}_G(A) \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^{n+N} \) by

\[
\tilde{S}_G(A) = \{(k, a, b) \mid (k, a) \in \hat{S}(A), \ b \in \Delta_{w_0}(\lambda(a)) \cap \mathbb{Z}^N \},
\]

where \( \Delta_{w_0}(\lambda) \) is the string polytope associated to the weight \( \lambda \) and the reduced decomposition \( \overrightarrow{w_0} \). When there is no ambiguity we will write \( \tilde{S}(A) \) instead of \( \tilde{S}_G(A) \).

The map \( (k, a, b) \mapsto (k, a) \) induces a surjective map \( \tilde{\pi} : \tilde{S}(A) \to \hat{S}(A) \).

**Proposition 5.7.** 1) The set \( \tilde{S}_G(A) \) is a semigroup (under addition). 2) For every \( (k, a) \in \hat{S}(A) \), the number of points in \( \tilde{\pi}^{-1}(k, a) \) is equal to the dimension of irreducible representation \( V_\lambda \), where \( \lambda = \lambda(a) \). 3) Consider the linear map \( \tilde{\pi} \circ \hat{\pi} : \hat{S}(A) \to S(A) \). For every \( (k, \lambda) \in S(A) \), the number of points in \( (\tilde{\pi} \circ \hat{\pi})^{-1}(k, \lambda) \) is equal to the dimension of \( \lambda \)-isotopic component \( A_{\lambda, k} \) in \( A_k \). If \( A \) is an algebra of almost \( G \)-integral type then the semigroup \( \hat{S}(A) \) is a strongly non-negative semigroup and hence its Newton-Okounkov convex set is a convex body.

*Proof.* 1) From the fact that \( C_{w_0} \) is a convex cone it follows that if \( b_1, b_2 \) are integral points in the string polytopes \( \Delta_{w_0}(\lambda) \) and \( b_2 \in \Delta_{w_0}(\gamma) \) respectively, then \( b_1 + b_2 \) is an integral point in the string polytope \( \Delta_{w_0}(\lambda + \gamma) \). This
implies that $\tilde{S}(A)$ is a semigroup. 2) The number of points in $\pi^{-1}(k,a)$ is equal to the number of integral points in the string polytope $\Delta_{w_0}(\lambda)$ (where $\lambda = \lambda(a)$), and the latter is equal to $\dim V_\lambda$. 3) By Proposition 4.3(1) we know that the semigroup $\hat{S}(A)$ is a strongly non-negative semigroup. Also since the cone $C_{w_0}$ is strongly convex, the set of integral points in it is a strongly non-negative semigroup. These two facts imply the claim. □

Definition 5.8. We denote the Newton-Okounkov convex set of the semigroup $\tilde{S}(A)$ by $\tilde{\Delta}(A)$ and call it the string convex set of $(A, \varphi)$. By above, when $A$ is of almost $G$-integral type, $\tilde{\Delta}(A)$ is a convex body.

Proposition 5.9 (Superadditivity of the string body). Let $(A', \varphi'), (A'', \varphi'') \in A_G(X)$ be two graded $G$-algebras. Then we have

$$\tilde{\Delta}(A') + \tilde{\Delta}(A'') \subset \tilde{\Delta}(A).$$

In particular, if $(L', \varphi'), (L'', \varphi'') \in K_G(X)$ are invariant subspaces then we have

$$\tilde{\Delta}(A_{L'}) + \tilde{\Delta}(A_{L''}) \subset \tilde{\Delta}(A_{L'L''}).$$

Proof. It follows immediately from the superadditivity of the string polytope (Proposition 5.4) and the superadditivity of the multiplicity body (Proposition 4.11). □

Finally, analogous to the weight lattice $\Lambda(X)$ and the multiplicity lattice $\hat{\Lambda}(X)$, we can associate a sublattice $\hat{\Lambda}(X)$ of $\mathbb{Z}^{n+N}$ to the variety $X$ which is the largest possible lattice that can appear as $\hat{\Lambda}(A)$ for a $G$-algebra $A$. Define the lattice $\hat{\Lambda}(X)$ to be the lattice generated by all $(a,b)$ where $a \in \hat{\Lambda}(X)$ and $b \in \Delta_{w_0}(\lambda(a)) \cap \mathbb{Z}^N$. The next proposition follows from the definition and Proposition 4.3(2) and (3).

Proposition 5.10. 1) Let $(A, \varphi) \in A_G(X)$ be a $G$-algebra. Then the lattice $\hat{\Lambda}(A)$ is contained in $\hat{\Lambda}(X)$. 2) Suppose $(L, \varphi) \in K_G(X)$ is an invariant subspace such that the Kodaira map $\Phi_L$ is a birational isomorphism between $X$ and its image. Then $\hat{\Lambda}(A_L)$ coincides with $\hat{\Lambda}(X)$.

6. Self-intersection index of invariant subspaces

In this section we give formulae for the growth of the Hilbert function of a graded $G$-algebra which then implies formulae for the self-intersection index of an invariant subspace in terms of the convex bodies $\Delta(A), \hat{\Delta}(A)$ and $\tilde{\Delta}(A)$.

6.1. Growth of Hilbert function and self-intersection index. Let $X$ be an irreducible $G$-variety of dimension $n$ and let $(A, \varphi) \in A_G(X)$ be a graded subalgebra of almost $G$-integral type. In Section 2.2 we associated three semigroups to $(A, \varphi)$:
The semigroup of weights $S(A) \subset \mathbb{Z}_{\geq 0} \times \Lambda^+$, which encodes information about the irreducible representations appearing in the homogeneous components of $A$. We called the convex body of this semigroup, the moment body of $A$ and denoted by $\Delta(A)$.

The multiplicity semigroup $\hat{S}(A) \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$, which encodes information about multiplicity of the irreducible representations appearing in the homogeneous components of $A$. We called the convex body of this semigroup, the multiplicity body of $A$ and denoted by $\hat{\Delta}(A)$. The maximum dimension of $\hat{\Delta}(A)$ is the transcendence degree (over $\mathbb{C}$) of the field $\mathbb{C}(X)^U$ of $U$-invariants. There is a natural projection $\hat{\pi} : \hat{\Delta}(A) \to \Delta(A)$.

The string semigroup $\tilde{S}(A) \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n + \mathbb{N}$, which parameterizes a natural basis (with respect to the $G$-action) for the homogeneous components of $A$ (here $N$ is the dimension of $U$). We called the convex body of this semigroup, the string convex body of $A$ and denoted by $\tilde{\Delta}(A)$. There is a natural projection $\tilde{\pi} : \tilde{\Delta}(A) \to \Delta(A)$.

The maximum dimension of $\tilde{\Delta}$ is $n = \dim X$.

Suppose $A$ is a graded subalgebra of almost $G$-integral type. The Hilbert function $H_A$ of the graded algebra $A$ has polynomial growth, i.e. $H_A$ is $O(q)$ for some $q > 0$. Let $a_q$ denote the $q$-th growth coefficient of the Hilbert function $H_A$, i.e.

$$a_q = \lim_{k \to \infty} H_A(k)/k^q.$$ 

We give formulae for the growth coefficient $a_q$ of the Hilbert function of the graded algebra $A$ in terms of the convex bodies $\Delta, \hat{\Delta}$ and $\tilde{\Delta}$. From this one then obtains formulae for the self-intersection index of an invariant subspace (as well as self-intersection index of a $G$-linearized line bundle) in terms of these convex bodies.

We start with the largest semigroup namely the semigroup $\tilde{S}$. Let $\tilde{S}_k$ denote the points at level $k$ in the semigroup $\tilde{S}$. As in Section 5.2 we have $\dim A_k = \# \tilde{S}_k$. Applying Theorem 1.7 to the semigroup $\tilde{S}$ we obtain:

**Theorem 6.1.** 1) The growth degree $q$ is equal to the real dimension of the convex body $\tilde{\Delta}$. 2) The $q$-th growth coefficient $a_q$ is equal to $\Vol_q(\tilde{\Delta})$ where volume is normalized with respect to the lattice $\tilde{\Lambda}(A)$ associated to the semigroup $\tilde{S}$.

The above theorem is a particular case of Theorem 1.7 when $A$ is a $G$-algebra. In this case the semigroup associated to $A$ is more structured which enables us to give a more concrete description of its Newton-Okounkov body.

Let $\lambda \in \Lambda^+$ be a dominant weight. By Weyl dimension formula we have

$$F(\lambda) = \dim V_\lambda = \prod_{\alpha \in R^+} \langle \lambda + \rho, \alpha \rangle / \langle \rho, \alpha \rangle,$$
where \( \langle \cdot, \cdot \rangle \) is the Killing form, \( R^+ \) is the set of positive roots and \( \rho \) is half the sum of positive roots. Consider \( F \) as a function on the lattice of weights \( \Lambda(A) \) (generated by all the differences \( \lambda - \mu \) for all \( k > 0 \) and \( \lambda, \mu \in S_k \)). Then the homogeneous component of highest degree of \( F \), as a function on the subspace \( \Lambda_{\mathbb{R}}(A) \) spanned by the lattice \( \Lambda(A) \), is given by:

\[
(5)\quad f(\lambda) = \prod_{\alpha \in R^+ \setminus E} \frac{\langle \lambda, \alpha \rangle}{\langle \rho, \alpha \rangle}.
\]

Here \( E \) is the set of positive roots which are orthogonal to the moment polytope \( \Delta \).

Let \( \hat{\pi}^*F \) and \( \hat{\pi}^*f \) denote the pull-backs of \( F \) and \( f \) to the semigroup \( \hat{S} \) via the projection \( \hat{\pi} : \hat{S} \to S \) respectively. Applying Theorem \ref{thm:main} to the semigroup \( \hat{S} \) and the polynomial \( \hat{\pi}^*F \) we get:

**Corollary 6.2.** The \( q \)-th growth coefficient \( a_q \) of the Hilbert function \( H_A \) is equal to the integral

\[
\int_{\hat{\Delta}} \hat{\pi}^* f d\hat{\gamma},
\]

where \( d\hat{\gamma} \) is the Lebesgue measure on the real span of the convex body \( \hat{\Delta} \) normalized with respect to the lattice \( \hat{\Lambda}(A) \).

Finally we can give a formula for the growth coefficient \( a_q \) as an integral over the moment convex body \( \Delta \). Recall that \( d\mu \) denotes the Duistermaat-Heckman measure for the \( G \)-algebra \( A \).

**Corollary 6.3.** The \( q \)-th growth coefficient \( a_q \) of the Hilbert function \( H_A \) is equal to the integral

\[
\int_{\Delta} f d\mu,
\]

where \( d\mu \) is the Duistermaat-Heckman measure of \( A \).

Applying the above results we get formulae for the self-intersection number of an invariant subspace. Let \( (L, \varphi) \in K_G(X) \) be an invariant subspace of rational functions on \( X \). Let \( A = A_L \) be the algebra \( A_L \) associated to \( L \). Recall that since \( A_L \) is finitely generated, \( \Delta \) is a polytope (i.e. the moment polytope).

Let \( d(L) \) denote the mapping degree of the Kodaira map \( \Phi_L : X \dashrightarrow \mathbb{P}(L^*) \). Also let \( \bar{s}(L) \) be the index of the lattice \( \Lambda(A) \) (associated to the semigroup \( \hat{S}(A) \)) in \( \Lambda(X) \) (see Section \ref{sec:semigroup}).

**Theorem 6.4.** 1) We have the following formula for the self-intersection index of an invariant subspace \( (L, \varphi) \):

\[
[L, \ldots, L] = \frac{n!d(L)}{\bar{s}(L)} \text{Vol}_{\text{n}}(\hat{\Delta}),
\]
where \( \text{Vol}_n \) is the Euclidean measure in \( \hat{\Lambda}_\mathbb{R}(X) = \hat{\Lambda}(X) \otimes \mathbb{R} \) normalized with respect to the lattice \( \hat{\Lambda}(X) \). 2) If moreover we assume that the Kodaira map \( \Phi_L \) is a birational isomorphism between \( X \) and its image, then

\[
[L, \ldots, L] = n! \text{Vol}_n(\hat{\Delta}).
\]

**Proof.** 1) Let \( Y_L = \Phi_L(X) \). By Hilbert’s theorem, the Hilbert function \( H_{A_L} \) grows of degree \( \dim Y_L \) and its leading coefficient is \( \deg(Y_L)/(\dim Y_L)! \). On the other hand, by Theorem 6.1, the growth degree of \( H_{A_L} \) is equal to \( q = \dim \hat{\Delta} \) and its leading coefficient is equal to \( (1/\hat{s}(L))\text{Vol}_{\hat{\Delta}} \). Now, if \( \dim Y_L < \dim X \) then the self-intersection index \( [L, \ldots, L] = 0 \). Also the Hilbert function \( H_{A_L} \) has growth less than \( n \) and thus \( \text{Vol}_n(\hat{\Delta}) = 0 \). If \( \dim Y_L = n \), then the Kodaira map has finite mapping degree \( d(L) \) and we have \( [L, \ldots, L] = d(L)\deg(Y_L) \). Since \( \deg(Y_L) = \frac{n!}{\hat{s}(L)}\text{Vol}_n(\hat{\Delta}) \) the claim follows. 2) Follows from 1) and Proposition 5.10(2). \( \square \)

Similarly let \( \hat{s}(L) \) and \( s(L) \) denote the indices of the lattices \( \hat{\Lambda}(A) \) and \( \Lambda(A) \) in \( \hat{\Lambda}(X) \) and \( \Lambda(X) \) respectively (see Section 3.1 and Section 4.1).

**Corollary 6.5.**

1) \[
[L, \ldots, L] = \frac{n!d(L)}{\hat{s}(L)} \int_{\hat{\Delta}} \hat{\pi}^* f d\hat{\gamma},
\]

where the measure \( d\hat{\gamma} \) is the Euclidean measure in \( \hat{\Lambda}_\mathbb{R}(X) \) normalized with respect to the lattice \( \hat{\Lambda}(X) \).

2) \[
[L, \ldots, L] = \frac{n!d(L)}{\hat{s}(L)} \int_{\Delta} f d\mu,
\]

where the measure \( d\mu \) is the Duistermaat-Heckman measure of the algebra \( A_L \).

3) Moreover, if \( \Phi_L \) is a birational isomorphism between \( X \) and its image, then we have

\[
[L, \ldots, L] = n! \int_{\hat{\Delta}} \hat{\pi}^* f d\hat{\gamma},
\]

and

\[
[L, \ldots, L] = n! \int_{\Delta} f d\mu.
\]

**Proof.** Follows directly from Theorem 6.4. \( \square \)

We have analogous statements for the self-intersection index of divisors on a projective \( G \)-variety \( X \).

**Corollary 6.6.** Let \( X \) be a projective \( G \)-variety of dimension \( n \) and let \( L \) be a \( G \)-linearized very ample line bundle on \( X \). Let \( D \) be a divisor of \( L \) and let \( R(D) \) be the corresponding algebra of sections regarded as a \( G \)-algebra (as in
Example 6.9. Then the self-intersection index \( D^n \) of the divisor \( D \) is equal to
\[
n! \int_{\Delta} f(\lambda) d\mu.
\]

6.2. Case of a spherical variety. A \( G \)-variety \( X \) is called spherical if a Borel subgroup (and hence every Borel subgroup) has a dense orbit. Some authors require that a spherical variety be normal. Here we do not need the normality assumption.

Remark 6.7. 1) When \( G = T \) is a torus, spherical varieties are exactly toric varieties.

2) By Bruhat decomposition, partial flag varieties \( G/P \) are spherical.

3) Again by Bruhat decomposition, \( G \) is a spherical variety for the action of \( G \times G \) given by multiplication from left and right.

It is well-known that the spaces of sections of \( G \)-line bundles over spherical varieties are multiplicity-free \( G \)-modules. The following is the analogous statement for the invariant subspaces of rational functions.

**Proposition 6.8.** If \( X \) is spherical and \((L, \varphi) \in K_G(X)\) then \( L \) is a multiplicity-free \( G \)-module.

**Proof.** For a dominant weight \( \lambda \) let \( f, g \in L \) be two \( B \)-eigenvectors with weight \( \lambda \) in the \( G \)-module \( L \). Then \( f/g \in \mathbb{C}(X) \) is a \( B \)-invariant function (for the natural (non-twisted) action of \( G \) on \( \mathbb{C}(X) \)). But as \( X \) is spherical it has a dense \( B \)-orbit which then implies that \( f/g \) is a constant function. Thus \( f \) is a scalar multiple of \( g \) which shows that every highest weight representation \( V_{\lambda} \) appears in \( L \) with multiplicity at most 1.

Since invariant subspaces of functions over spherical varieties are always multiplicity-free, we observe that, when \( X \) is spherical, for any algebra \((A, \varphi) \in A_G(X)\) the multiplicity body \( \Delta(A) \) coincides with the moment body \( \Delta(X) \). In this case, Theorem 6.4 and Corollary 6.5 get nicer and more explicit forms.

**Corollary 6.9** (Self-intersection index of invariant subspaces for spherical varieties). Let \( X \) be a spherical variety of dimension \( n \) and let \((L, \varphi) \in K_G(X)\) be an invariant subspace of rational functions. We have
\[
[L, \ldots, L] = \frac{n!d(L)}{s(L)} \text{Vol}_n(\Delta(A_L)) = \frac{n!d(L)}{s(L)} \int_{\Delta(A_L)} f(\lambda) d\gamma,
\]
where \( d\gamma \) is the Lebesgue measure (normalized with respect to \( \Lambda(X) \)) and \( f(\lambda) \) is as in the paragraph preceding Corollary 6.2. Moreover, if \( \Phi_L \) is a birational isomorphism between \( X \) and its image then:
\[
[L, \ldots, L] = n!\text{Vol}_n(\Delta(A_L)) = n! \int_{\Delta(A_L)} f(\lambda) d\gamma.
\]
Corollary 6.10 (self-intersection index of divisors for spherical varieties). Let $X$ be a projective spherical variety of dimension $n$ and let $\mathcal{L}$ be a $G$-linearized very ample line bundle on $X$. Let $D$ be a divisor of $\mathcal{L}$ and let $R(D)$ be the corresponding algebra of sections regarded as a $G$-algebra (as in Example 2.9). Then the self-intersection index $D^n$ of the divisor $D$ is equal to:

$$n! \text{Vol}_n(\Delta(R(D))) = n! \int_{\Delta(R(D))} f(\lambda) d\gamma,$$

where $d\gamma$ and $f(\lambda)$ are as above.

Remark 6.11. The algebras $R(D)$ and $A_L$ are both finitely generated and hence both their moment bodies are convex polytopes. Also, since for spherical varieties, the multiplicity body coincides with the moment body, it follows that their string convex bodies are also convex polytopes. This makes the formulae for the growth of Hilbert functions and self-intersection indices much more concrete.

Corollaries 6.9 and 6.10 are slight generalizations of the Brion-Kazarnovskii formula for the degree of a $G$-linearized line bundle over a spherical variety (see [Alexeev-Brion04, Brion89, Kazarnovskii87]).

7. $S$-varieties

In this section we consider a class of $G$-varieties, namely $S$-varieties, which behave very similar to toric varieties among $T$-varieties. These varieties have the smallest moment polytopes, i.e. for them the lower bound for the moment polytope in Proposition 3.13 is attained. We will see that this implies additivity for the moment polytope.

The celebrated Bernstein theorem in toric geometry gives the number of solutions of a system of Laurent polynomials in terms of mixed volume of polytopes: let $\Delta_1, \ldots, \Delta_n$ be $n$ integral polytopes in $\mathbb{R}^n$. Let $f_1, \ldots, f_n$ be generic Laurent polynomials in $x = (x_1, \ldots, x_n)$ with Newton polytopes $\Delta_1, \ldots, \Delta_n$ respectively. Then the number of solutions of the system $f_1(x) = \cdots = f_n(x) = 0$ in $(\mathbb{C}^*)^n$ is equal to $n! V(\Delta_1, \ldots, \Delta_n)$ where $V$ denotes the mixed volume of convex bodies in $\mathbb{R}^n$.

From the additivity of the moment polytope we will prove an extension of the Berstein theorem for the $S$-varieties.

We start with the analogous (and slightly more general concept) for $G$-algebras. Let $A = \bigoplus_{k \geq 0} A_k$ be a graded $G$-algebra and write $A$ as the sum of its isotypic components

$$A = \bigoplus_{k \geq 0} \bigoplus_{\lambda \in \Lambda^+} A_{k,\lambda}.$$

Definition 7.1. We call a graded algebra $A$ a Popov $G$-algebra if for any $k, \ell > 0$ and $\lambda, \gamma \in \Lambda^+$ we have

$$A_{k,\lambda} A_{\ell,\gamma} \subset A_{k+\ell,\lambda+\gamma}.$$
Let $X$ be an irreducible $G$-variety. For Popov algebras in $A_G(X)$, the moment convex body $\Delta(A)$ is additive with respect to componentwise product of graded algebras:

**Proposition 7.2.** Let $X$ be an irreducible $G$-variety. Let $A', A'' \in A_G(X)$ be two Popov graded $G$-algebras and denote by $A = A'A''$ their componentwise product. Let $S' = S(A')$, $S'' = S(A'')$ and $S = S(A)$ be the semigroups of weights corresponding to the algebras $A'$, $A''$ and $A = A'A''$ respectively. Also let $S'_k, S''_k$ and $S_k$ denote the set of points at level $k$ in the corresponding semigroups. Then 1) $S_k = S'_k + S''_k$. 2) $\Delta(A) = \Delta(A') + \Delta(A'')$.

**Proof.** 1) Follows from condition (6). 2) is a direct consequence of 1). □

Now we consider the varieties for which the condition (6) holds for all the $G$-algebras in $A_G(X)$.

**Definition 7.3.** Let $X$ be an irreducible $G$-variety. If the stabilizer of every point in $X$ contains a maximal unipotent subgroup then $X$ is called a horospherical variety. In particular if $X = G/H$ is a homogeneous $G$-space then $X$ is horospherical if and only if $H$ contains a maximal unipotent subgroup.

**Remark 7.4.** For a $G$-variety to be horospherical it is enough that the stabilizer of every point in a non-empty Zariski open set contains a maximal unipotent subgroup.

The following result of Popov (see [Popov86, §4]) characterizes horospherical varieties in terms of the properties of multiplication in the ring of regular functions:

**Theorem 7.5.** Let $X$ be an affine $G$-variety with the ring of regular functions $O(X)$, and for any dominant weight $\lambda$ let $O(X)_\lambda$ denote the $\lambda$-isotypical component of $O(X)$. Then for every $\lambda, \gamma \in \Lambda^+$, we have $O(X)_\lambda O(X)_\gamma \subseteq O(X)_{\lambda + \gamma}$ if and only if $X$ is a horospherical $G$-variety.

From this we can conclude that the graded $G$-algebras associated to horospherical varieties are Popov algebras (in the sense of Definition 7.1).

**Corollary 7.6.** Let $X$ be a horospherical $G$-variety. Then 1) For any invariant subspace $(L, \varphi) \in K_G(X)$, the algebra $A_L$ is a Popov algebra. 2) More generally, any graded $G$-subalgebra $A \in A_G(X)$ is a Popov $G$-algebra. 3) Let $A', A'' \in A_G(X)$ be two graded $G$-algebras. Then $\Delta(A'A'') = \Delta(A') + \Delta(A'')$

**Proof.** 1) Recall that $\Phi_L : X \rightarrow \mathbb{P}(L^*)$ is $G$-equivariant and hence $Y_L$, the closure of the image of the Kodaira map, is a $G$-invariant projective subvariety of $\mathbb{P}(L^*)$. Moreover if $x \in X$ and $y = \Phi_L(x)$ then the $G$-stabilizer of $y$ contains the $G$-stabilizer of $x$ and hence by Remark 7.4 the variety

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The notions of spherical variety and horospherical variety are different and should not be confused with each other. In particular, a horospherical variety may not be spherical and vice versa.
$Y_L$ is also horospherical. Now consider the variety $\tilde{Y}_L$ which is the cone over the projective variety $Y_L$. The variety $\tilde{Y}_L$ is an affine variety with the action of $G \times \mathbb{C}^*$. The ring of regular functions on $\tilde{Y}_L$ can be identified with $A_L = \bigoplus_{k \geq 0} L^k$. Since $Y_L$ is horospherical for the action of $G$ it follows that $\tilde{Y}_L$ is horospherical for the action of $G \times \mathbb{C}^*$. Let $(\lambda, k)$ be a dominant weight for the group $G \times \mathbb{C}^*$, where $\lambda \in \Lambda^+$ and $k \geq 0$. The $(\lambda, k)$-th isotypic component of the ring $O(\tilde{Y}_L) = A_L$ is the $\lambda$-isotypic component $(L^k)_\lambda$ of the subspace $L^k$. The claim now follows from Theorem 7.5. 2) Follows from 1). 3) Follows from 2) and Proposition 7.2.

**Corollary 7.7.** Let $X$ be a horospherical $G$-variety and let $(L, \varphi) \in K_G(X)$ be a $G$-invariant subspace and let $A$ be the finite set of highest weights of the $G$-module $L$. Then: 1) The moment convex body of the $G$-algebra $A_L$ coincides with the convex hull of $A$. 2) The map $L \mapsto \Delta(A_L)$ is additive with respect to the multiplication of subspaces. That is, if $(L_1, \varphi_1), (L_2, \varphi_2)$ are two invariant subspaces then

$$\Delta(A_{L_1L_2}) = \Delta(A_{L_1}) + \Delta(A_{L_2})$$

where the addition on the right-hand side is the Minkowski sum of convex polytopes.

**Proof.** 1) By Corollary 7.6 the algebra $A_L$ satisfies the condition in [6]. Let $S = S(A_L)$ be the semigroup of weights associated to $A_L$. From this it follows that $S_k$ coincides with $k \ast A = \{a_1 + \cdots + a_k \mid a_i \in A\}$. It then implies that $\Delta(A_L)$ is the convex hull of $A$. 2) follows from Corollary 7.6(3) and the fact that $A_{L_1A_{L_2}} = A_{L_1L_2}$. □

Finally we consider the horospherical varieties which contain an open $G$-orbit. These are exactly the horospherical varieties which are also spherical.

**Definition 7.8.** A horospherical $G$-variety $X$ which contains an open $G$-orbit is called an $S$-variety.

**Proposition 7.9.** An $S$-variety is spherical.

**Proof.** Let $x \in X$ be a point in the open $G$-orbit and let the stabilizer $H$ of $x$ contain a maximal unipotent subgroup $U$. Let $B$ be the Borel subgroup containing $U$. By Bruhat decomposition $Bw_0B = Bw_0U$ is dense in $G$ and hence the $B$-orbit of $w_0x \in X$ is dense in $X$, i.e. $X$ is spherical. □

**Remark 7.10.** (1) If $G = T$ is a torus then $S$-varieties are exactly toric varieties (possibly non-normal).

(2) An important example of an $S$-variety is the homogeneous space $G/U$. It can be shown that $G/U$ is an affine variety (which amounts to the fact that the ring of $U$-invariants in $\mathbb{C}[G]$ is finitely generated).
The additivity of the moment polytope implies a Bernstein theorem for $S$-varieties, i.e. we can compute the intersection index of invariant subspaces of rational functions using mixed volumes/integrals.

Let $\mathcal{V}$ be the vector space of virtual convex bodies in $\Lambda_\R(X)$, i.e. the vector space spanned by the collection of convex bodies in $\Lambda_\R(X)$. We define a polynomial $\text{Int}$ on this vector space as follows: for a convex body $\Delta \subset \Lambda_\R(X)$ put

$$\text{Int}(\Delta) = \int_\Delta f d\gamma,$$

where $f$ is as in (5) and $d\gamma$ is the Lebesgue measure normalized with respect to $\Lambda(X)$. The function $\text{Int}$ is a homogeneous polynomial of degree $n$ on $\mathcal{V}$.

Let $I(\Delta_1, \ldots, \Delta_n)$ denote the polarization of this homogeneous polynomial, that is, an $n$-linear function on $\mathcal{V}$ such that $\text{Int}(\Delta) = I(\Delta_1, \ldots, \Delta_n)$.

The following is a generalization of Bernstein theorem ([Bernstein75]):

**Corollary 7.11.** Let $X$ be an $n$-dimensional $S$-variety. Let $(L_1, \varphi_1), \ldots, (L_n, \varphi_n) \in K_G(X)$ be invariant subspaces such that for each $i$ the Kodaira map $\Phi_{L_i}$ is a birational isomorphism between $X$ and its image. We have:

$$[L_1, \ldots, L_n] = n! I(\Delta_1, \ldots, \Delta_n),$$

**Proof.** Follows from Theorems 7.12 and Theorem 6.9.

Next we consider the case of classical groups and their Gelfand-Cetlin polytopes. Recall that a special feature in this case is that the Gelfand-Cetlin polytopes are also additive.

**Theorem 7.12.** Let $G$ be one of the groups: algebraic torus $T = (\C^*)^{n_1}$, $\text{SL}(n_2, \C)$, $\text{SP}(n_3, \C)$, $\text{SO}(n_4, \C)$ or a direct product of these groups. Let $X$ be an $n$-dimensional $S$-variety for the action of $G$ and let $(L_1, \varphi_1), (L_2, \varphi_2) \in K_G(X)$ be invariant subspaces of rational functions. Let $\tilde{\Delta}_1 = \tilde{\Delta}(A_{L_1})$, $i = 1, 2$ and $\tilde{\Delta} = \tilde{\Delta}(A_{L_1L_2})$ denote the Newton-Okounkov polytopes as constructed in Section 3 using G-C polytopes. Then:

$$\tilde{\Delta} = \tilde{\Delta}_1 + \tilde{\Delta}_2.$$

**Proof.** Follows from additivity of the moment polytope (Corollary 7.11) and additivity of G-C polytopes (Proposition 5.5).

**Corollary 7.13.** Let $G$ be as above and $X$ an $n$-dimensional $S$-variety for the action of $G$. Let $(L_1, \varphi_1), \ldots, (L_n, \varphi_n) \in K_G(X)$ be invariant subspaces such that for each $i$ the Kodaira map $\Phi_{L_i}$ is a birational isomorphism between $X$ and its image. Let $\tilde{\Delta}_i = \tilde{\Delta}(A_{L_1})$, $i = 1, \ldots, n$ denote the Newton-Okounkov polytopes as constructed in Section 3 using G-C polytopes. We then have

$$[L_1, \ldots, L_n] = n! V(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n),$$

where $V$ is the mixed volume of convex bodies in $\R^n$.

**Proof.** Follows from Theorems 7.12 and Theorem 6.9.
Remark 7.14. Let $X$ be an $S$-variety and $(L, \varphi) \in K_G(X)$ an invariant subspace. Assume that $\Phi_L$ has finite mapping degree. In this case one can show that the index $s(L)$ is equal to the mapping degree of $\Phi_L$. This implies that, in fact, the above corollary holds without the assumption that the $\Phi_L$ give birational isomorphisms.

Corollary 7.15. Let $G$ be as above and let $L_1, \ldots, L_n$ be very ample $G$-line bundles on the $S$-variety $X$. Let $D_1, \ldots, D_n$ be divisors for $L_1, \ldots, L_n$ respectively. Then the intersection index $D_1 \cdot \cdots \cdot D_n$ of these divisors is given by the mixed volume:

$$n! V(\tilde{\Delta}(R(D_1)), \ldots, \tilde{\Delta}(R(D_n))).$$

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