The spin correlation function in 2D statistical mechanics models with inhomogeneous line defects

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Abstract. We consider the critical spin–spin correlation function of the 2D Ising model with a line defect whose strength is an arbitrary function of the position. By using path integral techniques in the continuum description of this model in terms of fermion fields, we obtain an analytical expression for the correlator as a functional of the position-dependent coupling. Thus, our result provides one of the few analytical examples that allows one to illustrate the transit of a magnetic system from scaling to non-scaling behavior in a critical regime. We also show that the non-scaling behavior obtained for the spin correlator along a nonuniformly altered line of an Ising model remains unchanged in the Ashkin–Teller model.

Keywords: correlation functions, classical phase transitions (theory), correlation functions (theory), defects (theory)

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1. Introduction

Two-dimensional statistical mechanics systems play a central role in our present understanding of phase transitions and critical phenomena. Outstanding members of this family of theories are the Ising model, the Ashkin–Teller model [1] and the eight-vertex model [2]. These models are useful for shedding light on a variety of phenomena, in both classical and quantum physics, ranging from biological applications [3] to the theory of cuprate superconductors [4]. Moreover, important advances in material science, accomplished over recent decades, have developed the ability to grow and experimentally explore ultrathin ferromagnetic films [5], giving us the opportunity to test some of the theoretical predictions. One of the fundamental questions concerning these essentially 2D materials is that of the role of defects and impurities in the critical properties of magnetic systems. Apart from its intrinsic academic interest, a detailed knowledge on the influence of defects on physical properties is always useful on general grounds, since all real materials are, to some extent, defected. In some cases of applied interest, such as ultrahigh-density magnetic recording media, it has been shown that linear defects can be used to efficiently control domain wall pinning, thus stabilizing the large area domain structure of ultrathin films [6]. Linear charge defects may also appear in graphene grown by chemical vapor deposition on Ni surfaces [7].

On the theoretical side, very little is known exactly about the behavior of planar systems in the presence of line defects [8]. For the simple square Ising lattice with an altered row (Bariev’s model [9]) it has been shown that the scaling index of the magnetization varies continuously with the defect strength [9,10], whereas the critical exponent of the energy density at the defect line remains unchanged [11]–[13]. Taking this model as a workbench, much insight has been obtained about the origin of nonuniversal critical behavior [14]. More recently, by using path integrals within the continuous formulation of Ashkin–Teller and Baxter models, it was shown that the magnetic exponent depends on the strength of the defect in exactly the same way as in Bariev’s model [15].

From another perspective, due to the well-known connection between the classical 2D Ising model and the quantum field theory of Dirac fermions in 1+1 dimensions, the study of...
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defects as perturbations of conformal field theories has led to very important results in the
area of integrable quantum field theories [16]–[18]. This line of research was later focused
on the problem of conductance in quantum wires [19]. The analysis of more mathematical
aspects concerning the role of impurities and defects in the renormalization group flows
of conformal models is currently under intensive investigation [20,21]. Very recently, the
entanglement between two pieces of a quantum chain was analyzed by exploiting the
connection with an Ising model with a defect line [22].

All these advances were achieved for the case of homogeneous defects, i.e. when the
defect strength is constant along the altered line. The case of nonuniform couplings
has been analyzed in the context of extended defects at surfaces [23]–[25] and in the
bulk [26,27], displaying a rich variety of behaviors in the local critical properties.

In this work we consider a narrow inhomogeneous defect and study the spin–spin
correlator on the altered line. In other words, we analyze the extension of Bariev’s model
to the case in which the strength of the line defect is a function of the position on the
column with modified couplings. Then, our result for the critical spin–spin correlator is
a generalization of the result first obtained in [10] for a uniform line defect. By using
a path integral approach in the continuum limit, we have obtained a formula that gives
the spin–spin correlation function as a functional of an arbitrary defect distribution. This
allows us to explore the effect of different kinds of specific alterations in a straightforward
way. We have also shown that the results remain valid for the Ashkin–Teller model, i.e. we
found that in these altered systems the non-scaling behavior of magnetic correlations for
the inhomogeneous defect coincides with the one obtained in the Ising case.

The paper is organized as follows. In section 2 we explain our computational
procedure for the well-known defect-free Ising model. In section 3 we show how to extend
the method when a line of altered couplings is included in the system. We emphasize how
the case of inhomogeneous defect strength can be naturally considered with our technique.
In section 4 we illustrate the use of our result, showing the predictions for two specific
defect functions. In section 5 we extend the procedure to the more complex Ashkin–
Teller and Baxter models. Finally in section 6 we summarize our findings and present our
conclusions.

2. The method: the defect-free case

For completeness and illustrative purposes, we start by describing the computational
procedure for the homogeneous defect-free case. The Hamiltonian of the original lattice
model is given by

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_2 \sigma_i \sigma_j$$

(1)

where $\langle ij \rangle$ means that the sum runs over nearest neighbors of a square lattice ($\sigma = \pm 1$).

As shown in [28] the scaling regime of the 2D IM can be described in the continuum
limit in terms of a model of Majorana fermions with Lagrangian density:

$$\mathcal{L}[\alpha] = \bar{\alpha} i \gamma \alpha$$

(2)

where $\alpha$ represents a Majorana spinor with components $\alpha_{1,2}$. Let us recall that this
components are connected to fermion annihilation and creation operators $c_r$ and $c_r^\dagger$.
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attached to site $r$ ($c_r = (e^{-ir/4}/\sqrt{2})(\alpha_1(r) + i\alpha_2(r))$). It is also useful for later convenience
to define the energy density as $\epsilon_\alpha = \alpha_1\alpha_2$. The symbol $\phi$ stands for $\gamma_\nu \partial_\nu$, with $\gamma_\nu$ the
usual Euclidean Dirac matrices ($\nu = 0, 1$, associated with space directions).

Similar manipulations, based on the Jordan–Wigner transformation [29], allow us to
write the on-line spin–spin correlation function in the form [28]

$$\langle \sigma(0) \sigma(R) \rangle_{\text{Ising}} = \exp \left( \pi \int_0^R dx (\epsilon_\alpha(x) + \epsilon_{\alpha'}(x)) \right)$$

where the vacuum expectation value must now be computed with respect to an Euclidean
action with Lagrangian density $\tilde{L}[\alpha, \alpha'] = L[\alpha] + L[\alpha']$, the $\alpha'$ being the replicated fermion
fields. Following [30], we can build Dirac fermions $\Psi$ as

$$\Psi = \alpha + i\alpha'$$

In terms of these new fields we can write the Lagrangian density $\tilde{L}[\alpha, \alpha']$ in the form

$$\tilde{L}[\Psi] = \bar{\Psi}i/\partial \Psi,$$

where $\gamma_5 = i\gamma_0\gamma_1$. On the other hand equation (4) can be expressed as

$$\langle \sigma(0) \sigma(R) \rangle_{\text{Ising}} = \exp \left( \pi \int d^2x \bar{\Psi} A \Psi \right).$$

Gathering the above results we can write

$$\langle \sigma(0) \sigma(R) \rangle_{\text{Ising}}^2 = \frac{Z[g = \pi]}{Z[g = 0]},$$

where

$$Z[g] = \int D\bar{\Psi} D\Psi \exp \left( -\int d^2x \left( \tilde{L}[\Psi] + g\bar{\Psi} A \Psi \right) \right).$$

The continuum limit of the squared two-point spin correlation function is exactly
expressed in terms of the vacuum to vacuum functional of a quantum field theory
describing a Dirac fermion interacting with a classical background $A_\nu$. Now we make
the following change of path integral variables in the numerator of equation (9), with
chiral and gauge parameters $\Phi$ and $\eta$, respectively:

$$\Psi = e^{-\pi(\gamma_5\Phi - in)} \zeta, \quad \bar{\Psi} = \bar{\zeta}e^{-\pi(\gamma_5\Phi + in)}.$$
The integration measures $D\Psi$ and $D\zeta$ are related through the so called Fujikawa Jacobian $J$, $D\Psi D\Psi = J[\Phi, \eta] D\zeta D\zeta$. If the parameters of the transformation are related to the previously introduced vector field $A_{\nu}$ in the form

$$A_{\nu} = \epsilon_{\nu\rho} \partial_{\rho} \Phi + \partial_{\nu} \eta$$

one easily gets $Z[g = \pi] = J Z[g = 0]$, which leads to

$$\langle \sigma(0)\sigma(R) \rangle_{\text{Ising}}^2 = J(R). \quad (13)$$

As explained in [32], the Jacobian $J(R)$ must be computed with a gauge-invariant regularization prescription in order to avoid an unphysical linear divergence. Following this procedure one finds that $J$ depends on the $\Phi$-field only, as follows:

$$J(R) = \exp -\pi/2 \int d^2x \partial_{\nu} \Phi(x, R) \partial^{\nu} \Phi(x, R). \quad (14)$$

The explicit form of $\Phi(x, R)$ is determined by combining equations (8) and (12) which gives the following partial differential equation for $\Phi$:

$$\Box \Phi(x_0, x_1, R) = -\delta(x_0) \frac{d}{dx_1} [\theta(x_1) \theta(R-x_1)] \quad (15)$$

where $\Box = \partial^2_0 + \partial^2_1$. The solution of this equation is easily obtained by using the Green’s function of the D’Alembertian: $G_0(z_0, z_1) = (1/4\pi) \ln(z_0^2 + z_1^2 + a^2)$, with $a$ an ultraviolet cutoff related to the original lattice spacing. Replacing in (14) and considering the limit $R \gg a$, we find the well-known result $\langle \sigma(0)\sigma(R) \rangle_{\text{Ising}} \simeq (a/R)^{1/4}$.

3. The inhomogeneous line defect

Now we include a line defect in the original Ising lattice. To be specific we consider the so called chain defect (here we employ the terminology of [8], which corresponds to Bariev’s second type of defect, in which bonds along the same column are replaced: $J_2 \to J'_2$). In previous studies the altered coupling $J'_2$ was taken as a constant. From now on we allow $J'_2$ to vary from site to site, i.e. we make $J'_2 \to J'_2(x_1)$.

We will study the two-spin correlation function in the column of altered bonds $(x_0 = 0)$ [10]. It is known that the continuous version of the classical model is modified, due to the defect, by the addition in equation (2) of a term $2\pi \mu(x_1) \delta(x_0) \epsilon_{\alpha}(x)$, with $\mu = J'_2(x_1) - J_2$ (see for instance [13]). By carefully examining the fermionic representation of $\sigma$-spin operators on the lattice, following the lines of [30], one also finds that in the continuum limit each spin operator on the defect line picks up a similar $\mu$-dependent factor, in such a way that the correlator for the defective model is given by a simple modification of equation (3):

$$\langle \sigma(0)\sigma(R) \rangle_{\text{inhom}} = \left\langle \exp \left( \pi \int dx_1 (1 + 4\mu(x_1)) \epsilon_{\alpha}(x_1) \right) \right\rangle_\mu. \quad (16)$$

It is evident that the squared correlator can be written again as in equation (7). The presence of the inhomogeneous defect manifests itself in the form of the $A_{\nu}$-field which is

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The spin correlation function in 2D statistical mechanics models with inhomogeneous line defects now rescaled by a factor \((1 + 4 \mu(x_1))\). Thus equation (8) becomes

\[
A_0(x_0, x_1) = -(1 + 4 \mu(x_1)) \delta(x_0) \theta(x_1) \theta(R - x_1)
\]

(17)

\[
A_1(x_0, x_1) = 0.
\]

(18)

The implementation of the change of variables given by (11) and (12) leads to the generalization of (13):

\[
\langle \sigma(0) \sigma(R) \rangle_{\text{inhom}}^2 = J_{\text{inhom}}(R).
\]

(19)

Formally \(J_{\text{inhom}}(R)\) is still given by (14), but the effects coming from the nonuniformity of the defect strength enter the game through the function \(\Phi(x_0, x_1, R)\), which now obeys a non-trivial differential equation depending on \(\mu(x_1)\):

\[
\Box \Phi(x_0, x_1, R) = -\delta(x_0) \frac{d}{dx_1} \left[ (1 + 4 \mu(x_1)) \theta(x_1) \theta(R - x_1) \right].
\]

(20)

The formal solution of this equation is

\[
\Phi(x_0, x_1, R) = \frac{1}{4\pi} \ln \frac{x_0^2 + a^2 + (x_1 - R)^2}{x_0^2 + a^2 + x_1^2} + \frac{1}{\pi} \int_0^R dx_1' \mu(x_1') \frac{d}{dx_1} \ln \left[ x_0^2 + (x_1 - x_1')^2 + a^2 \right].
\]

(21)

Replacing in the corresponding expression for \(J_{\text{inhom}}(R)\) we obtain

\[
\langle \sigma(0) \sigma(R) \rangle_{\text{inhom}} = \left( \frac{a^2}{a^2 + R^2} \right)^{1/8 + (\mu(0) + \mu(R)/4)} e^{F(R)},
\]

(22)

where

\[
F(R) = \frac{1}{4} \int_0^R dx \mu(x) \frac{d}{dx} \left[ \ln \left( \frac{(a^2 + (x - R)^2)^{1 + 4\mu(R)}}{(a^2 + x^2)^{1 + 4\mu(0)}} \right) \right]
\]

\[
- \frac{1}{4} \int_0^R \int_0^R dx dy (1 + 4 \mu(x)) \frac{d}{dy} \mu(y) \frac{d}{dx} \left[ \ln \left( a^2 + (x - y)^2 \right) \right].
\]

(23)

In the above integrals we have dropped the subscript 1 in the integration variables, in order to simplify the notation \((x_1 \to x\) and \(y_1 \to y\). It is easy to check that in the special case \(\mu(x) \to \mu = \text{constant}\), one obtains

\[
\langle \sigma(0) \sigma(R) \rangle_{\mu} \simeq \left( \frac{a}{R} \right)^{2\Delta_{\sigma}},
\]

(24)

with \(\Delta_{\sigma} = 1/8(1 + 4\mu)^2\), which is the well-known result first obtained by McCoy and Perk [10].

Formulas (22) and (23) constitute the main formal result of this paper. They give the critical spin–spin correlation on the altered line of an Ising model, as a functional of an arbitrarily varying defect strength. In section 4 we will show some specific predictions for definite defect distributions.
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Figure 1. Spin–spin correlation as a function of \( r \) for constant (solid line) and variable (point line) defect strength. We set \( \beta = 10 \). The dashed line indicates the defect-free universal behavior \( r^{-1/4} \).

4. Application to some specific defects

Let us now consider some specific defect functions for which \( F(R) \) can be analytically evaluated. We start with the following defect distribution:

\[
\mu(x) = \mu_0 \frac{1}{1 + |x|/b},
\]

where \( b \) is a characteristic length scale. This function is similar to the one considered by Bariev in his study of horizontal large scale inhomogeneities [26]. Passing to dimensionless variables \( r = R/a \) and \( \beta = b/a \), and considering weak defect strengths \( (\mu_0 \ll 1) \), for \( R \gg a \) and \( b \gg a \) we obtain

\[
\langle \sigma(0)\sigma(r) \rangle_{\text{inhom}} = \left( \frac{1}{r} \right)^{1/4 + \mu_0(2\beta+r)/(\beta+r)} \left( \frac{\beta}{\beta + r} \right)^{-\mu_0 r/(\beta+r)}
\times \exp\left(\mu_0r(2\beta+r)/(\beta(\beta+r)^2)\right) \arctan(r).
\]

We see that the magnetic correlation exhibits non-scaling behavior, as expected for a local inhomogeneity. This result is in qualitative agreement with the analysis of [26]. However we should stress that we are considering a different situation here. Indeed, the present case corresponds to a standard 2D Ising model in which just one column \( (x_0 = 0) \) is altered in a nonuniform way, whereas in [26] the couplings along columns are kept constant, while the couplings along all rows are modified in a nonuniform fashion. In figure 1 we compare the decays of correlations for a constant defect (solid line), a non-constant defect with a decay law (25) (point line) and the universal defect-free behavior (dashed line). In agreement with physical intuition the correlation decays monotonically with distance, in an intermediate way, faster than in the defect-free case and slower than in the case in which the defect strength is constant. Another expected feature, well reproduced by our solution, concerns the behavior with \( \beta = b/a \): for increasing \( \beta \) the non-scaling decay becomes faster, being indistinguishable from that in the uniform case for large enough \( \beta \).
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\[ \langle \sigma(0) \sigma(r) \rangle \]

Figure 2. Spin–spin correlation as a function of \( r \) for a line defect given by a slab starting at \( x/a = 10 \) and ending at \( x/a = 50 \), for \( \mu_0 = 0.1 \) (point line). The dashed line indicates the defect-free universal behavior \( r^{-1/4} \). The solid line corresponds to a uniform defect with \( \mu_0 = 0.1 \).

Let us now study a different function \( \mu(x) \), which represents a non-monotonic alteration of the line \( x_0 = 0 \). For simplicity we consider a sequence of \( N \) slabs of heights \( \mu_0_i \) \((i = 1, \ldots, N)\). Each slab starts at \( x = ac_i \) and ends at \( x = ad_i \):

\[
\mu(x) = \sum_{i=1}^{N} \mu_0_i \theta(x - ac_i) \theta(ad_i - x),
\]

where \( \theta(x) \) is Heaviside’s function. Evaluating \( F(R) \) and replacing in (22), in the weak coupling regime \((\mu_0_i \ll 1)\) and for \( r, c_i, d_i \gg 1 \) we obtain

\[
\langle \sigma(0)\sigma(r) \rangle_{\text{inhom}} = \left(\frac{1}{r}\right)^{1/4} \prod_{i=1}^{N} \left[ \left(\frac{1}{r}\right)^2 \left(\frac{c_i^2}{(r^2 + 1)}\right)^{\mu_0_i (d_i - r)(r - c_i)/2} \right] \times \left[ \left(\frac{c_i}{d_i}\right)^2 \left(\frac{(d_i - r)^2 + 1}{(c_i - r)^2 + 1}\right)^{\mu_0_i (r - d_i)/2} \right].
\]

(28)

In figure 2 we display the result given by the above formula for the simplest case: one slab or ‘barrier’ starting at \( x/a = c = 10 \) and ending at \( x/a = d = 50 \). For \( r < c \) the critical two-spin correlation coincides with the standard, non-defected correlation. In the presence of the defect, for \( c < r < d \), it exhibits a faster decay. The correlation reaches a local minimum at \( r = d \), and then it starts growing, approaching again the universal behavior corresponding to the magnetic critical index \( 1/8 \), asymptotically. In figure 3, taking into account that (28) is valid for both positive and negative values of \( \mu_0 \), we show the critical correlation for a defect which is oscillatory along a certain portion of the line \( x_0 = 0 \), a sequence of five alternating slabs \((\mu_0 = 0.1)\) and wells \((\mu_0 = -0.1)\). As before, the spin–spin function coincides with the non-defected one, for small distances \((r < c_1)\). For \( c_1 < r < d_5 \) there is an oscillatory behavior around the universal curve \( r^{-1/4} \). For large distances the correlation tends to the universal decay.

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5. Extension to the Ashkin–Teller model

In this section we show how to extend the treatment of inhomogeneous linear impurities to the Ashkin–Teller system consisting of two Ising subsystems with spin variables $\sigma_i$ and $\tau_i$ coupled by a quartic interaction [1, 2]. The corresponding lattice Hamiltonian reads

$$H = - \sum_{\langle ij \rangle} [J_2(\sigma_i \sigma_j + \tau_i \tau_j) + J_4 \sigma_i \sigma_j \tau_i \tau_j] \quad (29)$$

where $\langle ij \rangle$ means that the sum runs over nearest neighbors of a square lattice ($\sigma, \tau, = \pm 1$). As is well known, in the vicinity of the critical point this model can be described in the continuum limit in terms of two Majorana fermions interacting via their energy densities:

$$L[\alpha, \beta] = \bar{\alpha} \gamma_\alpha + \bar{\beta} \gamma_\beta - \lambda \epsilon_\alpha \epsilon_\beta \quad (30)$$

where $\alpha$ and $\beta$ are the Majorana spinors with components $\alpha_{1,2}, \beta_{1,2}$ respectively. $\epsilon_\alpha = \alpha_1 \alpha_2$ and $\epsilon_\beta = \beta_1 \beta_2$ are the corresponding energy densities. The coupling constant $\lambda$ is proportional to $J_4/J_2$. Let us now include, as before, a linear defect affecting one of the original Ising lattices, say the one with spins $\sigma$. If this impurity is placed at column $x_0 = 0$, in the continuum limit we have to add to $L$ a term $2\pi \mu(x_1) \delta(x_0) \epsilon_\alpha(x)$, with $\mu = J_2 - J_2$. As shown in [15], in order to compute the spin–spin correlator on the altered line it is still possible to use the doubling technique depicted in section 2. However, in spite of the formal analogy, the situation is much more complex here. First of all, since we have two sets of spins, we have to introduce two Dirac fields: $\Psi = \alpha + i\alpha'$ and $\chi = \beta + i\beta'$. We then obtain

$$\langle \sigma(0) \sigma(R) \rangle^2_{AT} = \left\langle \exp \left( \pi \int d^2x \bar{\Psi} \mathcal{A} \Psi \right) \right\rangle_{\mu}. \quad (31)$$

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Here the background field \( A_\nu \) is given by (17) and the vacuum expectation value must be computed with respect to a Euclidean action with Lagrangian density \( \mathcal{L}[\Psi, \chi] \):

\[
\tilde{\mathcal{L}}[\Psi, \chi] = \bar{\Psi} i \nabla \Psi + \bar{\chi} i \nabla \chi - \frac{\lambda}{8} [\bar{\chi} \gamma_5 \chi \bar{\Psi} \gamma_5 \Psi + \text{Im}(\chi^T \gamma_1 \chi) \text{Im}(\bar{\Psi}^T \gamma_1 \Psi)],
\]

(32)

where \( \gamma_5 = i \gamma_0 \gamma_1 \) and \( \chi^T, \gamma^T \) are the transposed spinors.

The implementation of the change of variables given by (11) and (12) leads to

\[
\langle \sigma(0) \sigma(R) \rangle_{\text{AT}}^2 = \langle \sigma(0) \sigma(R) \rangle_{\text{inhom}}^2 F(\lambda, R, \mu)
\]

(33)

where \( \langle \sigma(0) \sigma(R) \rangle_{\text{inhom}} \) is the defected Ising correlator given in (22) and

\[
F(\lambda, R, \mu) = \mathcal{N}(\lambda) \langle \exp[S_\phi(\zeta, \chi) + S_\eta(\zeta, \chi)] \rangle_0
\]

(34)

where \( \langle \cdot \rangle_0 \) means the vacuum expectation value with respect to the model of free \( \chi \) and \( \zeta \) fermions. \( \mathcal{N}(\lambda) \) is a normalization constant independent of \( R \). Since the analysis of the dependence of \( F(\lambda, \mu, R) \) on \( R \) is more easily done in momentum space, we have Fourier transformed \( S_\phi(\zeta, \chi) \) and \( S_\eta(\zeta, \chi) \) in the above equation:

\[
S_\phi(\zeta, \chi, \mu) = \frac{\lambda}{8} \int \prod_{j=1}^4 \frac{d^2 p_j}{(2\pi)^2} [\chi(p_1) \gamma_5 \chi(p_2) \bar{\gamma}(p_3) \gamma_5 G(P, R, \mu) \zeta(p_4)],
\]

(35)

with \( G(P, R, \mu) \) being a diagonal 2 \times 2 matrix given by

\[
G(P, R, \mu) = \begin{pmatrix} g_+(P, R, \mu) & 0 \\ 0 & g_-(P, R, \mu) \end{pmatrix},
\]

(36)

where \( g_{\pm}(P, R, \mu) = \pm \int d^2 x e^{i P x} e^{\mp 2\pi \Phi(x, \mu, R)} \) and \( P = p_1 + p_2 + p_3 + p_4 \). A similar expression is obtained for \( S_\eta \) with \( G(P, R) \) replaced by

\[
H(P, R, \mu) = \begin{pmatrix} h(P, R, \mu) & 0 \\ 0 & h(P, R, \mu) \end{pmatrix},
\]

(37)

with \( h(P, R, \mu) = \int d^2 x e^{i P x} e^{2 \pi \eta(x, R, \mu)} \). The explicit functional forms of \( \Phi(x, R, \mu) \) and \( \eta(x, R, \mu) \) can be determined following the same steps as were described in previous sections, yielding

\[
\Phi(x_0, x_1, R, \mu) = -\frac{1}{4\pi} \ln \left( \frac{x_0^2 + a^2 + (x_1 - R)^2}{x_0^2 + a^2 + x_1^2} \right) + \frac{2}{\pi} \int_0^R dx'_1 \mu(x'_1) \frac{(x_1 - x'_1)}{(x_0^2 + (x_1 - x'_1)^2 + a^2)}
\]

(38)

and

\[
\eta(x_0, x_1, R, \mu) = \frac{x_0}{2\pi} \int_0^R dy \frac{(1 + 4\mu(y))}{(x_0^2 + a^2 + (y - x_1)^2)}.
\]

(39)

Then, \( g_{\pm}(P, R, \mu) \) becomes

\[
g_{\pm}(P, R, \mu) = \pm \int d^2 x e^{i P x} \left( \frac{x_0^2 + a^2 + (x_1 - R)^2}{x_0^2 + a^2 + x_1^2} \right)^{\pm 1/2} e^{\pm 4 \int_0^R dy \mu(y)((x_1 - y)/(x_0^2 + a^2 + (y - x_1)^2))}
\]

(40)

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and \( h(P, R) \) becomes

\[
h(P, R, \mu) = \int d^2 x e^{i P x e^{i x_0 R}}/\sqrt{x_0^2 + a^2} \arctan(R \sqrt{x_0^2 + a^2}/(x_0^2 + x_1^2 + a^2 - Rx_1)) \\
\quad \times e^{i 4 x_0} \int_0^R dy (\mu(y)/(x_0^2 + a^2 + (y-x_1)^2))
\]

(41)

Since any possible dependence on \( R \) of the function \( F(\lambda, R, \mu) \) comes from \( g_{\pm}(P, R, \mu) \) and \( h(P, R, \mu) \), our problem is reduced to the analysis of these integrals. Let us first introduce a cutoff \( L \), which can be interpreted as the size of the system, in order to avoid infrared divergences (the thermodynamic limit will be recovered at the end of the computation by setting \( L \to \infty \)). In terms of the dimensionless variable \( u_\rho = x_\rho/L \), (\( \rho = 0, 1 \)) we obtain

\[
g_{\pm}(P, R, \mu) = \lim_{L \to \infty} \pm L^2 \int_{|u|<1} d^2 u e^{i P L u} \left( \frac{u_0^2 + a^2/L^2 + (u_1 - (R/L))^2}{u_0^2 + a^2/L^2 + u_1^2} \right)^{+1/2} \\
\quad \times e^{i \pi(4/L)} \int_0^R dy \mu(y)/(u_0^2 + a^2/L^2 + (y/L-u_1)^2) = \pm (2\pi)^2 \delta^2(P)
\]

(42)

and a similar result for \( h(P, R) \). Then, in the thermodynamic limit (\( a \ll R \ll L \)) \( \lambda(P, R, \mu) \) becomes independent of \( R \) and the critical behavior coincides with that of the 2D Ising model in the presence of an arbitrary inhomogeneous defect.

6. Summary and conclusions

We have considered the critical behavior of the two-spin correlation function in the continuum, field theory version of the 2D Ising model with a line defect placed at the column \( x_0 = 0 \). In contrast to previous studies, ours has taken into account possible variations of the defect strength with the position on the line. Our main result (equations (22) and (23)) provides an analytical expression for the critical spin–spin correlation as a functional of an arbitrary defect distribution. From this one can explore the effect of different kinds of nonuniform impurity distributions on the magnetization. In particular, our finding can be used to analyze, within the critical regime, the transition from scaling to non-scaling behavior. We have discussed, as examples, in order to illustrate the approach and check its validity, two special cases: a defect strength decaying monotonically with distance from a given point, and a sequence of slabs. Finally, we extended the analysis to a nonhomogeneous line defect placed at one column of an Ashkin–Teller system, showing that the spin correlator on the altered line decays, in the thermodynamic limit, in the same way as in the Ising model.

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