Maximal green sequences for preprojective algebras

Magnus Engenhurst*
Mathematical Institute, University of Bonn
Endenicher Allee 60, 53115 Bonn, Germany

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Abstract

Maximal green sequences were introduced as combinatorical counterpart for Donaldson-Thomas invariants for 2-acyclic quivers with potential by B. Keller. We take the categorical notion and introduce maximal green sequences for preprojective algebras for quivers without loops. We show that a quiver has a maximal green sequence if and only if it is of Dynkin type.

1 Introduction

The motivation for this paper arose from the connection from maximal green sequences of a 2-acyclic quiver $Q$ to stable modules over the Jacobi algebra $J(Q,W)$ for some non-degenerate potential $W$ [3]. Maximal green sequences were introduced by B. Keller in [8] as sequences of mutations of 2-acyclic quivers $Q$ that correspond to sequences of simple tilts in the finite-dimensional derived category of the Ginzburg algebra of $(Q,W)$. We are interested in the categorical side of this correspondence and introduce maximal green sequences associated to Abelian subcategories of triangulated categories which are 'nice'. More precisely, a maximal green sequence is a sequence of simple tilts of a algebraic heart $\mathcal{A}$ of a bounded t-structure of a triangulated category that we can tilt indefinitely (cf. Definition 3.4). An example is the category of finite-dimensional nilpotent modules $\mathcal{A} = \mathcal{P}(Q) - \text{nil}$ over the preprojective algebra $\mathcal{P}(Q)$ of a non-Dynkin quiver

*engenhurst@math.uni-bonn.de
Q without loops inside the bounded derived category \( \mathcal{D}^b(A) \). In the case of a Dynkin quiver we replace the derived category by a 'better behaved' triangulated category. \[1\] Stable modules over \( \mathcal{P}(Q) \) will provide us with examples of maximal green sequences and they will play a key role in the proof of our main result (Propositions \[3.2\] and \[3.3\]).

**Theorem 1.1.** Let \( Q \) be a quiver without loops. Then there is a maximal green sequence of \( \mathcal{P}(Q) \) if and only if \( Q \) is of Dynkin type.

In fact, stable modules over preprojective algebras play a central role in the proof of the Kac conjecture for indivisible dimension vectors by W. Crawley-Boevey and M. van den Bergh in \[19\]. Their key step in the formulation of Proposition \[4.1\] is also crucial for this paper. Another motivation comes from work on spaces of stability conditions for preprojective algebras \[1\] \[20\]. Further, stable modules over preprojective algebras show up in the work of S. Cecotti on BPS states in \[2\]. The following result (remark \[4.1\]) could also be of interest in physics:

**Proposition 1.1.** Let \( Q \) be a quiver without loops that is not of Dynkin type. Then there is no discrete central charge on \( \mathcal{P}(Q) - \text{nil} \) with finitely many stables.

We fix the notation: Given a set of objects or full subcategories \( E_i \) for \( i \in I \) for some index set \( I \) \( \langle E_i : i \in I \rangle \) will denote the extension-closed full subcategory generated by \( E_i \) with \( i \in I \).

## 2 Preprojective algebras

In this section we review results on preprojective algebras that will be used in the next sections. For more details see e.g. \[15\] or \[17\].

Let \( Q = (Q_0, Q_1) \) be a quiver without loops \( \cup \) and with \( n \) vertices and let \( h, t : Q_1 \to Q_0 \) be the head and tail maps. We obtain the double quiver \( \overline{Q} \) from \( Q \) by adding for every arrow \( a : i \to j \) in \( Q_1 \) an arrow \( a^* : j \to i \) in the opposite direction. Let \( \mathbb{C} \overline{Q} \) be the path algebra of \( \overline{Q} \).

**Definition 2.1.** \[9\] The preprojective algebra \( \mathcal{P}(Q) \) of \( Q \) is defined by

\[
\mathcal{P}(Q) := \mathbb{C} \overline{Q} / (c)
\]

where \( c \) is the ideal generated by \( \sum_{a \in Q_1} (aa^* - a^*a) \).
Example 2.1. Let us consider the $A_2$-quiver $\begin{array}{c}1 \xrightarrow{a} 2 \xleftarrow{a^*} \end{array}$. The associated double quiver is the following

$$\begin{array}{c}1 \xrightarrow{a} 2 \xleftarrow{a^*} \end{array}$$

and the relations are $aa^* = a^*a = 0$.

Example 2.2. If we consider the $A_3$-quiver $\begin{array}{c}1 \xrightarrow{a} 2 \xrightarrow{b} 3 \end{array}$, then the associated double quiver is

$$\begin{array}{c}1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xleftarrow{b^*} \xleftarrow{a^*} \end{array}$$

and the relations are $a^*a = bb^* = 0$ and $aa^* - b^*b = 0$.

The preprojective algebra does not depend on the orientation of $Q$. Let $\mathcal{P}(Q) - \text{mod}$ be the category of finite-dimensional left $\mathcal{P}(Q)$-modules and $\mathcal{P}(Q) - \text{nil}$ the category of nilpotent finite-dimensional left $\mathcal{P}(Q)$-modules. A $\mathcal{P}(Q)$-module $M$ is nilpotent if a composition series of $M$ contains only the simple modules $S_1, \ldots, S_n$ associated to the $n$ vertices of $Q$. The category $\mathcal{P}(Q) - \text{nil}$ is of finite length with $n$ simple modules $S_1, \ldots, S_n$. If $Q$ is a Dynkin quiver the algebra $\mathcal{P}(Q)$ is finite-dimensional and all finite-dimensional $\mathcal{P}(Q)$-modules are nilpotent.

The modules in $\mathcal{P}(Q) - \text{mod}$ can be identified with the finite-dimensional representations $V = (V_i, \phi_a)$ of the quiver $\overline{Q} = (\overline{Q}_0; \overline{Q}_1)$ in which the linear maps $\phi_a, a \in \overline{Q}_1$ fulfill the relations

$$\sum_{a \in \overline{Q}_1; b(a) = i} \phi_a \phi_a^* - \sum_{a \in \overline{Q}_1; t(a) = i} \phi_a^* \phi_a = 0$$

for all $i \in \overline{Q}_0$.

Let $(\ , \ )$ be the symmetric bilinear form defined on the root lattice

$$\mathbb{Z}Q_0 = \mathbb{Z}[S_1] \oplus \cdots \oplus \mathbb{Z}[S_n]$$

by

$$(x, y) := 2 \sum_{i \in \overline{Q}_0} x_i y_i - \sum_{a \in \overline{Q}_1} x_{at} y_j.$$

We have the following useful result:
Proposition 2.1. Let $Q$ be a quiver without loops and let $M, N$ be two finite-dimensional $\mathcal{P}(Q)$-modules. Then we have

$$(\dim M, \dim N) = \Hom(M, N) + \Hom(N, M) - \dim \Ext^1(M, N).$$

In particular, $\Ext^1(M, N) = \Ext^1(N, M)$.

A brick is a module $M$ with $\Hom(M, M) = \mathbb{C}$. The following is well-known:

Lemma 2.1. Let $Q$ be a Dynkin quiver and $M$ be a brick in $\mathcal{P}(Q) \mod$. Then $\Ext^1(M, M) = 0$ and $\dim M$ is a root, i.e. $(\dim M, \dim M) = 2$.

Proof. Since $Q$ is Dynkin the symmetric bilinear form (2.1) is positive definite. $(M, M)$ is even and therefore $\Ext^1(M, M)$ vanishes by Proposition 2.1. \qed

Note that there are only finitely many bricks in $\mathcal{P}(Q) \mod$ for a Dynkin quiver $Q$ since all bricks $M$ are rigid, i.e. $\Ext^1(M, M) = 0$ (cf. [16]).

3 Maximal green sequences

We want to consider $\mathcal{P}(Q) - nil$ as an algebraic heart of a t-structure of a triangulated category $\mathcal{D}$ such that we can tilt indefinitely (cf. Definition 3.3). For $Q$ not of Dynkin type we take $\mathcal{D}^b(\mathcal{P}(Q) - nil)$. B. Keller proved that $\mathcal{D}^b(\mathcal{P}(Q) - nil)$ has a Serre functor $[2]$, i.e. is a 2-Calabi-Yau category in this case [12]. In the case of a Dynkin quiver $Q$ we replace the derived category by a better-behaved category $\hat{\mathcal{D}}$ described in [1]: Let $G \subset SL_2(\mathbb{C})$ be a finite subgroup and let $\text{Coh}_G(\mathbb{C}^2)$ denote the category of $G$-equivariant coherent sheaves on $\mathbb{C}^2$. Consider the full subcategory $\mathcal{A} \subset \text{Coh}_G(\mathbb{C}^2)$ consisting of equivariant sheaves with no non-trivial $G$-equivariant sections. Then $\hat{\mathcal{D}}$ is the full subcategory of $\mathcal{D}^b(\text{Coh}_G(\mathbb{C}^2))$ consisting of complexes whose cohomology sheaves lie in $\mathcal{A}$. The important fact for this paper is that $\hat{\mathcal{D}}$ is equivalent to $\mathcal{P}(Q) - mod$ where $Q$ is a Dynkin quiver and $\hat{\mathcal{D}}$ is 2-Calabi-Yau.

Let $\mathcal{D}$ be the triangulated category $\hat{\mathcal{D}}$ described above in the case of a Dynkin quiver $Q$ and the derived category $\mathcal{D}^b(\mathcal{P}(Q) - nil)$ else. Every simple module $S$ of $\mathcal{P}(Q) - nil$ is a 2-spherical object in $\mathcal{D}$, i.e.

$$\Hom_{\mathcal{D}}^i(S, S) = \begin{cases} \mathbb{C} & \text{if } i = 0, 2 \\ 0 & \text{else} \end{cases}.$$

By [13] every spherical object defines an auto-equivalence $\Phi_S$ of $\mathcal{D}$, the Seidel-Thomas twist, such that for every $E \in \mathcal{D}$ there is an exact triangle:

$$\Hom^\bullet_{\mathcal{D}}(S, E) \otimes S \to E \to \Phi_S(E) \to.$$

(3.1)
We identify throughout the Grothendieck groups $K(\mathcal{P}(Q) - nil)$ and $K(\mathcal{D})$ with the root lattice $\mathbb{Z}Q_0$. The induced linear map on the Grothendieck group $K(\mathcal{D})$ gives

$$[\Phi_S(E)] = [E] = \chi(S, E)[S]$$

where $\chi : K(\mathcal{D}) \times K(\mathcal{D}) \to \mathbb{Z}$ is the Euler form

$$\chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \text{Hom}^i_{\mathcal{D}}(E, F).$$

We have

$$\chi(S_i, S_j) = \begin{cases} 2 & \text{if } i = j \\ \#(\text{arrows } i \to j \text{ in } Q) - \#(\text{arrows } j \to i \text{ in } Q) & \text{if } i \neq j \end{cases}$$

and thus the lattice $(K(\mathcal{P}(Q) - nil), \chi(\cdot, \cdot))$ can be identified with the root lattice $(\mathbb{Z}Q_0, (\cdot, \cdot))$ associated to the quiver $Q$.

**Definition 3.1.** We call the heart $\mathcal{A}$ of a bounded t-structure of a triangulated category $\mathcal{D}$ algebraic if 1. it has finite length, i.e. there are no infinite chains of inclusions or quotients for all objects and 2. it has finitely many simple objects. We call a heart $\mathcal{A}$ rigid if all its simple objects $S$ are rigid, i.e. $\text{Ext}^1_{\mathcal{A}}(S, S) = 0$.

Given a simple object $S$ in an algebraic heart $\mathcal{A}$ there is a well-known construction to define a new heart $\mathcal{A}_S$ of a bounded t-structure of $\mathcal{D}$, see [5, 6]. We review it in the following:

**Definition 3.2.** A torsion pair in an Abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ satisfying

1. $\text{Hom}_{\mathcal{A}}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
2. every object $E \in \mathcal{A}$ fits into a short exact sequence

$$0 \to T \to E \to F \to 0$$

for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The objects of $\mathcal{T}$ are called torsion and the objects of $\mathcal{F}$ are called torsion-free, $\mathcal{T}$ is called torsion class and $\mathcal{F}$ torsion-free class.
Proposition 3.1. [5] Let $\mathcal{A}$ be the heart of a bounded t-structure on a triangulated category $\mathcal{D}$. Denote by $H^i(E) \in \mathcal{A}$ the $i$-th cohomology object of $E$ with respect to this t-structure. Let $(\mathcal{F}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Then the full subcategory

$$\mathcal{A}^* = \{ E \in \mathcal{D} | H^i(E) = 0 \text{ for } i \notin \{0, 1\}, H^0(E) \in \mathcal{F}, H^1(E) \in \mathcal{F} \}$$

is the heart of a bounded t-structure on $\mathcal{D}$.

We say $\mathcal{A}^*$ is obtained from $\mathcal{A}$ by (left) tilting with respect to the torsion pair $(\mathcal{F}, \mathcal{F})$. The pair $(\mathcal{F}, \mathcal{F}[−1])$ is a torsion pair in $\mathcal{A}^*$.

Suppose $\mathcal{A} \subset \mathcal{D}$ is an algebraic heart of a bounded t-structure on $\mathcal{D}$. Given a simple object $S \in \mathcal{A}$ we can view $\langle S \rangle$ as the torsion class of a torsion pair on $\mathcal{A}$ with torsion-free class

$$\mathcal{F} = \{ E \in \mathcal{A} | \text{Hom}_\mathcal{A}(S, E) = 0 \}. \quad (3.2)$$

This gives the simple (left) tilt of $\mathcal{A}$ at $S$. If the heart $\mathcal{A}_S$ is again algebraic we can repeat this construction. The composition of left tilts is described by

Lemma 3.1. [14] Let $\mathcal{A}$ be the heart of a bounded t-structure of a triangulated category. Let $(\mathcal{F}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$ and $(\mathcal{F}', \mathcal{F}')$ a torsion pair in $\mathcal{A}^* = \langle \mathcal{F}, \mathcal{F}[-1] \rangle$. If $\mathcal{F} \subset \mathcal{F}'$, then the left-tilt $\mathcal{A}^{**} = \langle \mathcal{F}', \mathcal{F}'[-1] \rangle$ of $\mathcal{A}^*$ equals a left-tilt of $\mathcal{A}$.

Definition 3.3. Let $\mathcal{A}$ be an algebraic heart of a bounded t-structure of a triangulated category $\mathcal{D}$ with $n$ simple objects. We say we can tilt $\mathcal{A}$ indefinitely if any heart obtained from $\mathcal{A}$ by a finite sequence of simple tilts is again algebraic with $n$ simple objects.

Let $\mathcal{D}$ be the triangulated category $\hat{\mathcal{D}}$ described above in the case of a Dynkin quiver $Q$ and the derived category $\mathcal{D}^b(\mathcal{P}(Q) − \text{nil})$ else. Then $\mathcal{A} = \mathcal{P}(Q) − \text{nil}$ is an algebraic heart of a t-structure in $\mathcal{D}$ and it is well-known that we can tilt $\mathcal{A}$ indefinitely, i.e. the hearts obtained by any finite sequence of simple tilts of $\mathcal{A}$ have finite length with $n$ simple objects. Further, these simple objects are again 2-spherical.

Definition 3.4. Let $\mathcal{A}$ be an algebraic heart of a bounded t-structure of a triangulated category $\mathcal{D}$ that we can tilt indefinitely. We call a finite sequence of simple tilts of the heart $\mathcal{A}$ such that we strictly tilt at objects in $\mathcal{A}$ a green sequence of $\mathcal{A}$. If the last heart in the sequence is the shifted heart $\mathcal{A}[-1]$ we call it a maximal green sequence. We call the number of simple tilts in a green sequence its length.
Note that by Lemma 3.1 all simple objects of a heart $\mathcal{A}'$ appearing in a green sequence lie in $\mathcal{A}$ or $\mathcal{A}[-1]$.

**Lemma 3.2.** Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ an algebraic heart of a bounded t-structure of $\mathcal{D}$ with simple objects $S_1, \ldots, S_n$ that we can tilt indefinitely. Then we have the following:

(i) We tilt in a green sequence of $\mathcal{A}$ at an indecomposable module of $\mathcal{A}$ at most once.

(ii) We tilt in a maximal green sequence of $\mathcal{A}$ at all $n$ simple objects $S_1, \ldots, S_n$ of $\mathcal{A}$.

**Proof.** Ad (i). Note that there are no non-zero morphisms from $\mathcal{A}[i]$ to $\mathcal{A}[j]$ for $i > j$. If we tilt a heart at an indecomposable $S$ then $S[-1]$ remains in all following hearts in the green sequence since $\text{Hom}(E, S[-1]) = 0$ for $E \in \mathcal{A}$ by (3.2). Then the claim follows from the fact that we can not have $S$ and $S[-1]$ in only one heart.

Ad (ii). By Lemma 3.1 any heart $\mathcal{A}'$ appearing in a green sequence is given by the tilt at some torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, i.e., $\mathcal{A}' = \langle \mathcal{F}, \mathcal{T}[-1] \rangle$. Thus we have for every object $C \in \mathcal{A}$ a short exact sequence

$$0 \rightarrow A \rightarrow C \rightarrow B \rightarrow 0$$

with $A \in \mathcal{T}$ and $B \in \mathcal{F}$. Thus the object $S_i$ or the object $S_i[-1]$ lie in $\mathcal{A}'$ for all $i$. There must be two hearts coming after each other in this sequence such that $\mathcal{A}'$ contains the simple $S_i$ for $i = 1, \ldots, n$ and the consecutive heart $\mathcal{A}'_S$ obtained from tilting $\mathcal{A}'$ at some simple object $S'$ of $\mathcal{A}'$ contains the object $S_i[-1]$. Thus in this case we have the short exact sequence

$$0 \rightarrow E \rightarrow S_i[-1] \rightarrow F \rightarrow 0$$

in $\mathcal{A}'_S$ with $E \in \langle S' \rangle^\perp \subset \mathcal{A}'$ and $F \in \langle S' \rangle [-1]$. The morphism $S_i[-1] \rightarrow F$ is non-zero, otherwise $S_i[-1]$ would be in $\langle S' \rangle^\perp \subset \mathcal{A}'$. But we can not have $S_i$ and $S_i[-1]$ in $\mathcal{A}'$. Thus there is a non-zero morphism $f : S_i \rightarrow F'$ in $\mathcal{A}$ with $F' = F[1] \in \langle S' \rangle \subset \mathcal{A}$. Since $S_i$ is a simple object in $\mathcal{A}$ $f$ is injective with cokernel $\text{coker } f$. From the exact triangle

$$S_i \rightarrow F' \rightarrow E[2] \rightarrow$$

follows that $\text{coker } f \cong E[2]$. Thus $E \cong 0$ since $E$ is generated by objects in $\mathcal{A}$ and $\mathcal{A}[-1]$ and we have $S_i \cong S'$. 

**Proposition 3.2.** Let $Q$ be a quiver of Dynkin type with $m$ positive roots. Then any green sequence can be completed to a maximal green sequence of $\mathcal{P}(Q)$. The length of any maximal green sequence is $m$ and we tilt at $m$ objects $E_1, \ldots, E_m$ whose classes are the $m$ positive roots.
Proof. Note that we have only finitely many bricks in $\mathcal{P}(Q) - \text{nil}$. By Lemma 3.1 all simple objects of hearts appearing in a green sequence are of the form $E$ or $E[-1]$ with $E$ a brick in $\mathcal{A}$. Now the first claim follows from Lemma 3.2(i).

It remains to show the second claim. For this we will anticipate notions from the next section. Let $\mathcal{D}$ be the triangulated category $\hat{\mathcal{D}}$ described above. In a maximal green sequence we tilt at a simple module say $S_1$ of $\mathcal{A} = \mathcal{P}(Q) - \text{mod}$ first. Given an algebraic heart $\mathcal{A}'$ in $\mathcal{D}$ with $n$ simple objects $S_1, \ldots, S_n$ we can define a Bridgeland stability condition $\sigma = (Z, \mathcal{P})$ by choosing a complex number in the upper half-plane $\mathbb{H}$ for any simple $S_1, \ldots, S_n$ by Lemma 5.2 in [6]. This defines a central charge $Z$ on the Grothendieck group $K(\mathcal{A}') = K(\mathcal{D})$ and the subcategories $\mathcal{P}(\phi)$ of $\sigma$-semistable objects of phase $\phi$ with $\phi \in (0, 1]$ are exactly the semistable objects of $\mathcal{A}'$ with respect to $Z$ together with the zero objects. Let us choose a central charge on $\mathcal{P}(Q) - \text{mod}$ such that the central charge $Z(S_1)$ is left to all central charges $Z(S_2), \ldots, Z(S_n)$ in the upper halfplane $\mathbb{H}$. Note that all roots are indivisible since there are no imaginary roots fora Dynkin quiver. By Proposition 4.2 for any positive root $\alpha$ there is a semistable module with class $\alpha$. With the chosen central charge there are two possibilities: 1. The central charge of a semistable module $E$ with class $[E] = \alpha$ lies right to $Z(S_1)$ in the upper halfplane and $E$ is contained in the tilted heart $\mathcal{A}_S$ by Lemma 4.1 and (3.2). 2. $Z(E)$ and $Z(S_1)$ have the same phase and thus $E = S_1$ since in this case $E \in \langle S_1 \rangle$ and the class $[E]$ is indivisible. Note that there can not be an object in a heart with class $\alpha$ if there is already an object with class $-\alpha$. We tilt next at a simple object $S'$ of $\mathcal{A}'_{S_1}$. We can again choose a central charge such that is $S'$ left-most. By Proposition 4.2 in the tilted heart $\mathcal{A}_S$ there is a semistable object $E'$ with class a positive root $\beta$. By the same arguments we have $S' = E'$ or $E'$ is contained in the next heart in the sequence. Going on in this way we see that we tilt at $m$ indecomposables with classes the positive roots. Since an object we tilt at in a maximal green sequence is a brick in $\mathcal{P}(Q) - \text{mod}$ and thus has class a positive root we tilt exactly at $m$ objects with classes the positive roots.

We give examples of maximal green sequences for quivers of Dynkin type in the next section. In particular, we will see that for two maximal green sequences it can happen that we tilt at exactly the same objects but in a different order. First we note the following remarkable result:

**Proposition 3.3.** Let $Q$ be a quiver without loops. If there exists a maximal green sequence of $\mathcal{P}(Q)$ then $Q$ is of Dynkin type.

*Proof.* This follows from the proof of Proposition 3.2 since we have infinitely many indecomposable roots in the non-Dynkin case and thus have in any heart $\mathcal{A}'$ appearing in a green sequence of $\mathcal{P}(Q)$ infinitely many objects with class a positive indecomposable root. \qed
4 Stable modules over preprojective algebras

In this section we recall the notion of stability for Abelian categories and use it to construct examples of maximal green sequences for preprojective algebras. Further, we review results that are used in section 3 for the proof of the main result.

**Definition 4.1.** A central charge (or stability function) on an Abelian category $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that for any nonzero $E \in \mathcal{A}$, $Z(E)$ lies in the upper halfplane

$$\mathbb{H} := \{r \cdot \exp(i\pi \phi)|0 < \phi \leq 1, r \in \mathbb{R}_{>0}\} \subset \mathbb{C}. \quad (4.1)$$

Every object $E \in \mathcal{A}$ has a phase $0 < \phi(E) \leq 1$ such that $Z(E) = r \cdot \exp(i\pi \phi(E))$ with $r \in \mathbb{R}_{>0}$. We say a nonzero object $E \in \mathcal{A}$ is (semi)stable with respect to the central charge $Z$ if every proper subobject $0 \neq A \subset E$ satisfies $\phi(A) < \phi(E)$ ($\phi(A) \leq \phi(E)$). A central charge is called discrete if different stable object have different phases.

The following useful Lemma is well-known:

**Lemma 4.1.** Let $E$ and $F$ be semistable objects with respect to a central charge on an Abelian category $\mathcal{A}$. If we have $\phi(E) > \phi(F)$, then $\text{Hom}_{\mathcal{A}}(E, F) = 0$.

We consider stable modules in $\mathcal{A} = \mathcal{P}(Q) - \text{nil}$. Let $\alpha$ be a class in $K(\mathcal{A})$. We call a central charge $Z : K(\mathcal{A}) \to \mathbb{C}$ generic with respect to $\alpha$ if $\Im(Z(\beta)/Z(\alpha)) \neq 0$ for all $0 < \beta < \alpha$.

A root $\alpha$ in the root lattice of $Q$ is called indivisible if there is no root $\beta$ with $\alpha = m\beta$ for an integer $m$ with $|m| > 1$. The real roots $\alpha$ are indivisible since we have $(\alpha, \alpha) = 2$ in this case. Further, every imaginary root is a multiple of an indivisible root and all non-zero multiples are roots (cf. [18]).

We have the following important result:

**Proposition 4.1.** [19] Let $\alpha$ be a positive indivisible root and let $Z : K(\mathcal{P}(Q) - \text{nil}) \to \mathbb{C}$ be a generic central charge with respect to $\alpha$, then there is a stable module in $\mathcal{P}(Q) - \text{nil}$ with respect to $Z$ with class $\alpha$. 

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For the definition of Bridgeland stability conditions on a triangulated category we refer to [6]. Let $\text{Stab}^\circ(D)$ be the connected component of the space of stability conditions of the category $D$ associated to a preprojective algebra containing the stability conditions with heart $A = P(Q) - \text{nil}$. Using the description of $\text{Stab}^\circ(D)$ given in [1] for the Dynkin and affine case and in [20] for the non-Dynkin case we can generalize this Proposition to any stability condition in $\text{Stab}^\circ(D)$:

**Proposition 4.2.** [20] Given a stability condition $\sigma = (Z, P)$ in $\text{Stab}^\circ(D)$ and a indivisible root $\alpha$. Then there is a $\sigma$-semistable object with class $\alpha$.

The connection of stable modules to maximal green sequences is given by the following

**Proposition 4.3.** Let $Q$ be a quiver without loops. Let $Z : K(P(Q) - \text{nil}) \to \mathbb{C}$ be a discrete central charge with finitely many stable modules. Then the stable objects of $P(Q) - \text{nil}$ in the order of decreasing phase define a maximal green sequence.

**Proof.** This follows immediately from the proof of Proposition 4.1 in [3].

**Remark 4.1.** It follows from Propositions 3.3 and 4.3 that there is no discrete central charge with finitely many stable objects for non-Dynkin quivers. Note that Dynkin quivers automatically have finitely many stable modules since all stables are bricks.

We could calculate maximal green sequences directly by using the Seidel-Thomas twists [3.1] Instead we use Proposition 4.3 to easily find discrete central charges that induce maximal green sequences.

**Example 4.1.** Let us consider example 2.1 again. In this case we have 4 indecomposable $P(Q)$-modules that are all bricks.[15] We define a central charge on $P(Q) - \text{mod}$ by choosing two complex numbers $Z(S_1), Z(S_2)$ in the upper half-plane $\mathbb{H}$ for the two simple modules $S_1$ and $S_2$ associated to the vertices 1 and 2. If $\phi(S_1) > \phi(S_2)$ then the stable modules are

\[
\begin{array}{c}
0 & \rightarrow & \mathbb{C} \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C} \\
\mathbb{C} & \rightarrow & 0
\end{array}
\]

and if $\phi(S_2) > \phi(S_1)$ the stable modules are

\[
\begin{array}{c}
0 & \rightarrow & \mathbb{C} \\
\mathbb{C} & \xrightarrow{0} & \mathbb{C} \\
\mathbb{C} & \rightarrow & 0
\end{array}
\]
These central charges are indeed discrete and thus define two maximal green sequences. In fact, these are all maximal green sequences in this case. Note that all maximal green sequences are of length 3.

**Example 4.2.** Let $Q$ be a quiver of type $A_3$ with orientation as in example 2.2. In this case we have 12 indecomposables, 11 out of these are bricks \([13]\). We choose a discrete central charge on $\mathcal{P}(Q) - \text{mod}$ with $\phi(S_1) > \phi(S_2) > \phi(S_3)$. The stable modules with respect to any such central charge are the three simples together with the modules:

$\begin{align*}
\begin{array}{c}
\text{C} \\
0 \\
\text{id} \\
\text{C} \\
\text{C} \\
\text{C} \end{array} \quad \begin{array}{c}
\text{0} \\
\text{id} \\
\text{C} \\
\text{C} \\
\text{id} \\
\text{id} \end{array} \quad \begin{array}{c}
\text{C} \\
0 \\
\text{id} \\
\text{C} \\
\text{C} \\
\text{id} \end{array}.
\end{align*}$

For the choice $\phi(S_3) > \phi(S_2) > \phi(S_1)$ the stable modules together with the three simple modules are

$\begin{align*}
\begin{array}{c}
\text{C} \\
0 \\
\text{id} \\
\text{C} \\
\text{C} \\
\text{C} \end{array} \quad \begin{array}{c}
\text{0} \\
\text{id} \\
\text{C} \\
\text{C} \\
\text{id} \\
\text{id} \end{array} \quad \begin{array}{c}
\text{C} \\
0 \\
\text{id} \\
\text{C} \\
\text{C} \\
\text{id} \end{array}.
\end{align*}$

Note that the induced maximal green sequences depend on whether the central charge of the stable object with class $(1,1,1)$ lies to the left or to the right of $Z(S_2)$ in the upper halfplane. Therefore we get two maximal green sequences at a time given by ordered tuples of indecomposables obtained from each other by commuting two consecutive objects.

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**References**

[1] T. Bridgeland: Stability conditions and Kleinian singularities, Int. Math. Res. Notices 21 (2009), 4142-4157.

[2] S. Cecotti: Categorical tinkertoys for N=2 gauge theories, Int. J. Mod. Phys. A28 (2013), no. 5-6, 1330006.

[3] M. Engenhorst: Tilting and refined Donaldson-Thomas invariants, J. Algebra 400 (2014), 299-314.
[4] A. Beilinson, J. Bernstein, P. Deligne: Faisceaux Pervers, Astérisque 100 (1983).

[5] D. Happel, I. Reiten, S.O. Smalø: Tilting in abelian categories and quasi-tilted algebras, Mem. Amer. Math. Soc. 120, no. 575 (1996).

[6] T. Bridgeland: Spaces of stability conditions, in: D. Abramovich et al. (eds.): Algebraic Geometry: Seattle 2005, Proc. of Symposia in Pure Mathematics, AMS, (2009), 1-22.

[7] A. King, Y. Qiu: Exchange Graphs of acyclic Calabi-Yau categories, arXiv preprint (arXiv:1109.2924), 2011.

[8] B. Keller: On cluster theory and quantum dilogarithm identities, Representation Theory of Algebras and Related Topics (A. Skowronski, K. Yamagata eds.), European Mathematical Society, 2011, 85-116.

[9] I. Gelfand, V. Ponomarev: Model algebras and representations of graphs, Functional Anal. Appl. 13 (1979), 157-165.

[10] V. Dlab, C.M. Ringel: The module theoretical approach to quasi-hereditary algebras, in: Representations of algebras and related topics (Kyoto, 1990), CUP, (1992), 200-224.

[11] W. Crawley-Boevey: On the exceptional fibres of Kleinian singularities, Amer. J. Math. 122 (2000), 1027-1037.

[12] B. Keller: Calabi-Yau triangulated categories, in: Trends in Representation Theory of Algebras and Related Topics (A. Skowronski ed.), European Mathematical Society, 2008, 467-489.

[13] P. Seidel, R. Thomas: Braid group actions on derived categories, Duke Math. J. 108 (2001), 37-108.

[14] K. Nagao: Donaldson-Thomas theory and cluster algebras, Duke Math. J. 162 (2013), 1313-1367.

[15] C. Geiss, B. Leclerc, J. Schröer: Semicanonical bases and preprojective algebras, Ann. Scient. Ec. Norm. Sup., 38 (2005), 193-253.

[16] C. Geiss, B. Leclerc, J. Schröer: Rigid modules over preprojective algebras, Invent. Math. 165 (2006), 589-632.

[17] W. Crawley-Boevey, M. Holland: Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (1998), 605-635.
[18] V. Kac: Infinite-dimensional Lie algebras, Cambridge University Press 1990.

[19] W. Crawley-Boevey, M. van den Bergh: Absolutely indecomposable representations and Kac-Moody Lie algebras, Invent. Math. 155 (2004), 537-559.

[20] A. Ikeda: Stability conditions for preprojective algebras and root systems of Kac-Moody Lie algebras, arXiv preprint (arXiv:1402.1392), 2014.