GENERALIZED CLOSE-TO-CONVEXITY RELATED WITH BOUNDED BOUNDARY ROTATION

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Abstract. The class \( P_{\alpha,m}[A,B] \) consists of functions \( p \), analytic in the open unit disc \( E \) with \( p(0) = 1 \) and satisfy
\[
p(z) = \left( \frac{m}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{m}{4} - \frac{1}{2} \right) p_2(z), \quad m \geq 2,
\]
and \( p_1, p_2 \) are subordinate to strongly Janowski function \( \left( \frac{1+Bz}{1+Bz} \right)^\alpha \), \( \alpha \in (0,1] \) and \( -1 \leq B < A \leq 1 \). The class \( P_{\alpha,m}[A,B] \) is used to define \( V_{\alpha,m}[A,B] \) and \( T_{\alpha,m}[A,B;0;B_1] \), \( B_1 \in [-1,0) \). These classes generalize the concept of bounded boundary rotation and strongly close-to-convexity, respectively. In this paper, we study coefficient bounds, radius problem and several other interesting properties of these functions. Special cases and consequences of main results are also deduced.

1. Introduction

Let \( A \) denote the class of analytic functions defined in the open unit disc \( E = \{ z : |z| < 1 \} \) and be given by
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E.
\]

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Let $S \subset A$ be the class of univalent functions in $E$ and let $C$, $S^*$ and $K$ be the subclasses of $S$ consisting of convex, starlike and close-to-convex functions, respectively. For details, see [3].

For $f, g \in A$, we say $f$ is subordinate to $g$ in $E$, written as $f(z) \prec g(z)$, if there exists a Schwartz function $w(z)$ such that

$$f(z) = g(w(z)), \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$ 

Furthermore, if the function $g$ is univalent in $E$, then we have the following equivalence

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(E) \subset g(E).$$

Convolution of $f$ and $g$ is defined as

$$(g * f)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$ 

The class $P_\alpha[A, B]$ of strongly Janowski functions is defined as follows.

**Definition 1.1.** Let $p$ be analytic in $E$ with $p(0) = 1$. Then $p \in P_\alpha[A, B]$, if

$$p(z) \prec (1 + Az)(1 + Bz)$$

and $\alpha \in (0, 1]$ and $-1 \leq B < A \leq 1$.

We denote $P_\alpha[0, B_1]$ as $P_\alpha[B_1]$, $-1 \leq B_1 < 0$.

The class $P_\alpha[A, B]$ is generalized as:

**Definition 1.2.** An analytic function $p : p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is in the class $P_{\alpha,m}[A, B]$, if and only if, there exist $p_1, p_2 \in P_\alpha[A, B]$ such that

$$(1.2) \quad p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z), \quad m \geq 2.$$ 

It is obvious $P_{\alpha,2}[A, B] = P_\alpha[A, B]$. For the class $P_1[A, B] = P[A, B]$, we refer to [6].

About the class $P_\alpha[A, B]$, we observe the following.

(i) $p(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^\alpha$ implies $p \in P_\alpha[A, B]$ and it can easily be shown that $\phi_\alpha(A, B; z) = \left(\frac{1 + Az}{1 + Bz}\right)^\alpha$ is convex univalent in $E$. In fact simple calculation yield that

$$Re\phi_\alpha'(A, B; z) \geq \alpha(A - B) |1 - A|^{-1} |1 - B|^{-1} > 0, \quad z \in E.$$ 

This shows $\phi_\alpha(A, B; z)$ is univalent in $E$.

Also

$$Re \left\{ \frac{(z\phi_\alpha'(A, B; z))'}{\phi_\alpha'(A, B; z)} \right\} \geq \frac{T(r)}{(1 + Ar)(1 + Br)},$$

where

$$T(r) = 1 - \alpha(A - B)r - ABr^2.$$
is decreasing on \((0, 1)\) and \(T(0) = 1\).

This implies \(\text{Re} \left[ \frac{(z \phi_{\alpha}(A; B; z))'}{(\phi_{\alpha}(A; B; z))'} \right] \geq 0\) in \(E\).

(ii) For \(A = 1, B = -1, p \in \phi_{\alpha}(1, -1; z)\) implies

\[ |\arg p(z)| \leq \frac{\alpha \pi}{2}, \quad z \in E. \]

**Definition 1.3.** Let \(f, g \in A, \frac{(g * f)'(z)}{z} \neq 0, z \in E\). Then \(f \in V_{\alpha, m}[A, B; g]\), if and only if,

\[ \frac{(z(g * f)')'}{(g * f)'} \in P_{\alpha, m}[A, B], \quad z \in E, \]

with \(F = zf', F \in R_{\alpha, m}[A, B; g]\), if and only if, \(f \in V_{\alpha, m}[A, B; g]\) in \(E\).

**Special Cases.**

(i) \(V_{1,m}[A, B; \frac{1}{1-z}] = V_m[A, B] \subset V_m[1, -1] = V_m\), where \(V_m\) is the well known class of functions of bounded boundary rotation. See, for example, \([2, 10, 12]\).

(ii) \(R_{1,m}[A, B; \frac{1}{1-z}] = R_m[A, B] \subset R_m\) and \(R_m\) is the class of functions with bounded radius rotation, see \([9]\).

(iii) \(V_{\alpha, m}[A, B; \frac{1}{1-z}^\alpha] = R_{\alpha, m}[A, B; \frac{1}{1-z}] = R_{\alpha, m}[A, B]\).

**Definition 1.4.** Let \(f, g \in A\) with \((g * f)(z) \neq 0\). Then \(f \in T_{\alpha, m}[A, B; 0; B_1; g]\), if there exists \(\psi \in V_{\alpha, m}[A, B; g]\) such that, for \(B_1 \in [-1, 0]\),

\[ \frac{(g * f)'}{(g * \psi)'} \in P_{\alpha}[B_1], \quad z \in E. \]

We note that \(T_{1,1}[A, B; 0; -1; \frac{1}{1-z}] = T_m[A, B]\). For certain special cases, see \([8, 11, 12]\).

2. **The class** \(V_{\alpha, m}[A, B; g]\)

**Theorem 2.1.** Let \(f \in V_{\alpha, m}[A, B; g]\) and let \(g(z) = z + \sum_{n=2}^{\infty} b_n z^n\). Then, with \(f\) given by (1.1), \(A_n = a_n b_n\),

\[ A_n = O(1)n^\sigma, \quad \sigma = \left\{ \left( \frac{m}{2} + 1 \right)(1 - \rho) - (\rho + 2) \right\}, \]

where \(\rho = \left( \frac{1-A}{1-B} \right)^\alpha\), \(m \geq 2(1+\rho)\frac{1}{1-\rho}\) and \(O(1)\) denotes a constant.

**Proof.** Let \(F = f * g\). Then \(F \in V_{\alpha, m}[A, B]\). Since \(p \in P_{\alpha}[A, B]\) implies \(\text{Re} \ p(z) > \rho, \rho = \left( \frac{1-A}{1-B} \right)^\alpha\), it follows that \(V_{\alpha, m}[A, B] \subset V_m(\rho)\).

Now, \(F \in V_m(\rho)\), we can write

\[ F_1'(z) = (F_1'(z))^{1-\rho}, \quad F_1 \in V_m, \]
Using a result due to Brannan [2], we can write

\[ zF'_1(z) = \frac{(s_1(z))(\frac{1}{2} + \frac{1}{2})(1 - \rho)}{(s_2(z))(\frac{1}{2} - \frac{1}{2})(1 - \rho)}, \quad s_1, s_2 \in S^*. \]

Therefore, from (2.1), (2.2) and Cauchy Theorem with \( z = re^{i\theta}, \) we have

\[ n^2|A_n| \leq \frac{1}{2\pi} \int_0^{2\pi} |F'_1(z)h(z)|^{1-\rho} d\theta, \quad h \in P_{\alpha,m}[A,B] \subset P_m(\rho) \]

\[ n^2|A_n| = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{s_1(z)}{s_2(z)} \right| \left( \frac{1}{2\pi} \int_0^{2\pi} |s_1(z)| \left( \frac{1}{2} + \frac{1}{2} \right)(1 - \rho) \frac{1}{2\pi} d\theta \right)^{\frac{1}{1-\rho}} \cdot \left( \frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \right)^{\frac{1}{1-\rho}}. \]

Applying distortion result for \( s_2 \in S^* \) and Holder’s inequality in (2.3), we get

\[ n^2|A_n| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| s_1(z) \right| \left( \frac{1}{2\pi} \int_0^{2\pi} \left| s_1(z) \right| \left( \frac{1}{2} + \frac{1}{2} \right)(1 - \rho) \frac{1}{2\pi} d\theta \right)^{\frac{1}{1-\rho}} \cdot \left( \frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \right)^{\frac{1}{1-\rho}}. \]

Now, for \( h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \), we use Parsval identity to have

\[ \frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \]

\[ \leq 1 + m^2(1 - \rho)^2 \sum_{n=1}^{\infty} r^{2n} \]

\[ = \frac{1 + [m^2(1 - \rho)^2 - 1]r^2}{1 - r^2}, \]

where we have used coefficient bounds \( |c_n| \leq m(1 - \rho), \) for \( h \in P_m(\rho). \)

From (2.5) together with subordination for starlike functions, and a result due to Hayman [5] for \( m \geq \frac{2(1+\rho)}{1-\rho}, \) we have

\[ n^2|A_n| \leq c_1(m, \rho) \left( \frac{1}{1-r} \right)^{\left( \frac{1}{2} + 1 \right)(1 - \rho) - \rho}, \]

where \( c_1(m, \rho) \) denotes a constant.

Taking \( r = 1 - \frac{1}{n} \) in (2.6), we obtain the required result.

**Special Cases.**

(i) Let \( g(z) = \frac{z}{1-z}, \) then \( A_n = a_n. \) Take \( A = 0, \) and in this case \( f \in V_m. \) This leads us to a known coefficient result that \( a_n = O(1)n^{(\frac{1}{2} - 1)}. \)

(ii) Let \( f \in V_{1,m} \left[ 0, -1, \frac{z}{(1-z)^2} \right] = R_m \left( \frac{1}{2} \right). \) Then \( a_n = O(1)n^{\frac{1}{2} - 2}, m \geq 6. \)
Theorem 2.2. Let $f \in V_{\alpha,m}[A, B; g]$. Then, for $F = f * g$, $z = re^{i\theta}$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$, we have

$$
\int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{(zf'(z))^\prime}{F'(z)} \right\} d\theta > -\left( \frac{m}{2} - 1 \right) (1 - \rho)\pi, \quad \rho = \left( \frac{1 - A}{1 - B} \right)^\alpha.
$$

Proof. Proof is straightforward, since for $\alpha, m \geq 1$ and $F \in V_m(\rho)$ implies there exist $F_1 \in V_m$ with $F'(z) = (F_1'(z))^{(1 - \rho)}$. Now, using essentially the same method given in [2], the required result follows. \qed

Remark 2.1. Let $\beta \left( \frac{m}{2} - 1 \right) (1 - \rho)$. Then, from a result of Goodman [4] and from (2.7), it follows that $F = f * g \in V_{\alpha,m}[A, B]$ is univalent for $\beta = \left( \frac{m}{2} - 1 \right) (1 - \rho) \leq 1$. That is $F \in S$ for $m \leq \frac{2(2 - \rho)}{1 - \rho}$. As a special case, with $g(z) = \frac{z}{1 - z}$, $A = 0$, $B = -1$ and $\alpha = 1$, we have $F = f$, $\rho = \frac{1}{2}$. Then $f \in V_{1,m}[0, -1]$ implies

$$
\int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{(zf'(z))^\prime}{F'(z)} \right\} d\theta > -\left( \frac{m}{4} - \frac{1}{2} \right)\pi
$$

For this, we can conclude that

$$V_{1,m}[0, -1] \subset S \quad \text{for} \quad 2 \leq m \leq 6.
$$

Also, with $g(z) = \frac{z}{1 - z}$, $A = 1$, $B = -1$, we have a well known result that $f \in V_m$ is univalent for $2 \leq m \leq 4$.

Theorem 2.3. Let $f \in V_{\alpha,m}[A, B; g]$, $m \leq \frac{2(2 - \rho)}{1 - \rho}$ and $\rho = \left( \frac{1 - A}{1 - B} \right)^\alpha$. Then $F(E)$ with $F = f * g$, contains the disc $d$:

$$d = \left\{ w : |w| < \frac{4}{8 + \alpha m |A - B|} \right\}
$$

Proof. From Theorem 2.2, $F$ is univalent in $E$. Let $w_0$ ($w_0 \neq 0$) be any complex number such that $F(z) \neq w_0$ for $z \in E$. Then the function

$$F_1(z) = \frac{w_0 F(z)}{w_0 - F(z)} = z + \left( A_2 + \frac{1}{w_0} \right) z^2 + \ldots
$$

is analytic and univalent in $E$. Using the well known Bieberbach Theorem for the best bound for second coefficient of univalent functions, see [3], we have

$$\frac{1}{|w_0|} - |A_2| \leq |A_2 + \frac{1}{w_0}| \leq 2.
$$

This gives us

$$\frac{1}{|w_0|} \leq 2 + |A_2| \leq 2 + \frac{\alpha m |A - B|}{4} = 8 + \alpha m |A - B|.
$$

This completes the proof. \qed
Special Cases.

(i) Let $A = 1$, $B = -1$, $\alpha = 1$; $(\rho = 0)$ and so $F(E)$ contains the disc $|w| < \frac{2}{1+m}$, $m \leq 4$.

(ii) With $A = 0$, $B = -1$, $\alpha = \frac{1}{2}$, we have $\rho = \frac{1}{4}$, and $F(E)$ contains the disc $|w| < \frac{8}{16+m}$, $m \leq \frac{14}{3}$.

The following properties of the class $V_{\alpha,m}[A,B;g]$ can easily be proved with simple computations and well known results and therefore we omit the proof.

**Theorem 2.4.**

(i) The class $V_{\alpha,m}[A,B;g]$ is preserved under the integral operator $L : A \rightarrow A$ defined as

$$L(z) = \int_0^z (L'_1(\xi))^\beta (L'_2(\xi))^\gamma d\xi,$$

where $L_i \in V_{\alpha,m}[A,B;g]$, $i = 1,2$ and $\beta, \gamma$ are positively real with $\beta + \gamma = 1$.

(ii) Let $f \in V_{\alpha,m}[A,B;\frac{z}{1-z}]$. Then, with $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$, $z \in E$ and $z = re^{\theta}$, we have

$$\frac{(1-B\rho)^{(1-\rho)(\frac{\theta}{2} + \frac{1}{2})}}{(1+B\rho)^{(1-\rho)(\frac{\theta}{2} - \frac{1}{2})}} \leq |f'(z)| \leq \frac{(1+B\rho)^{(1-\rho)(\frac{\theta}{2} + \frac{1}{2})}}{(1-B\rho)^{(1-\rho)(\frac{\theta}{2} - \frac{1}{2})}}.$$

For $\alpha = 1$, $f \in V_m[A,B]$ and $A = 1$, $B = -1$, the result reduces to $f \in V_m$ studied in [2].

(iii) Let $f \in V_{\alpha,2}[A,B;\frac{z}{1-z}]$ and define $F \in A$ as

$$F(z) = \frac{\beta + 1}{z^\beta} \int_0^z t^{\beta-1} f(t) dt, \beta > 0.$$

Then $F$ is convex of order $\gamma(\rho)$, $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$, where

$$\gamma = \gamma(\rho) = \left\{ \frac{(\beta + 1)}{\Gamma(2(1-\rho), 1; (\beta + 2); \frac{1}{2})} - \beta \right\},$$

the $2F_1$ represents Gauss hypergeometric function.

(iv) The set of all points $\log f'(z)$ for a fixed $z \in E$ and $f$ ranging over the class $V_{\alpha,m}[A,B;g]$ is convex.

(v) Let $f \in V_{\alpha,m}[A,B;\frac{z}{1-z}]$, $B \neq 0$. Then $f$ is close-to-convex for $|z| < r_1$, where

$$r_1 = \left\{ \sin \left( \frac{\pi}{B(\gamma - 2)} \right) \right\}, \quad B \neq 0, \quad m > \frac{2}{\gamma}, \quad \gamma = 1 - \left(\frac{1-A}{1-B}\right)^\alpha.$$

(vi) Let $f \in V_{\alpha,m}[A,B;g]$, and let $F = f \ast g$. Then $F$ is convex of order $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$ for $|z| < r_m$, where

$$r(m) = \frac{m - \sqrt{m^2 - 4}}{2}, \quad m \geq 2.$$

**Theorem 2.5.** Let $f_1, f_2 \in V_{\alpha,m}[A,B;g]$, $\beta, \delta, c$ and $\nu$ be positively real, $c \geq \beta \geq 1$, $(\nu + \delta) = \beta$. Let $F = F_1 \ast g$, $G_i = f_i \ast g$, $i = 1,2$ and define

$$[F(z)]^\beta = cz^{(\beta-c)} \int_0^z t^{\nu-1} \left( G_1^\delta(t) G_2^\nu(t) \right) dt.$$
Then, for \( z = re^{i\theta}, \ 0 \leq \theta_1 < \theta_2 \leq 2\pi, \ \frac{zF'}{F} = p, \) we have
\[
\int_{\theta_1}^{\theta_2} \text{Re} \left\{ p(z) + \frac{1}{p} \frac{i \beta z^p(z)}{p(z) + \frac{1}{p}(c - \beta)} \right\} d\theta > -(1 - \rho) \left( \frac{m}{2} - 1 \right) \pi, \quad \rho = \left( \frac{1 - A}{1 - B} \right)^{\alpha}. 
\]

Proof. First we show that there exists a function \( F \in A \) satisfying (2.8). We assume \( F_1 \ast g \neq 0, \ f_i \ast g \neq 0, \ z \in E. \) Let
\[
Q(z) = (G_1'(z))^\delta (G_2'(z))^\nu = 1 + d_1z + d_2z^2 + \ldots
\]
and choose the branches which equal 1, when \( z = 0. \)

For \( K(z) = z^{c^{-1}}(G_1'(z))^\delta (G_2'(z))^\nu = z^{c^{-1}}Q(z), \) we have
\[
N(z) = \frac{c}{z^c} \int_0^z K(t)dt = 1 + \frac{c}{c+1}d_1z + \ldots
\]
Hence \( N \) is well defined and analytic.

Now let
\[
F(z) = \left[ z^\delta N(z) \right]^\frac{1}{\beta} = z \left[ N(z) \right]^\frac{1}{\beta},
\]
where we choose the branch of \( [N(z)]^\frac{1}{\beta} \) which equal 1 when \( z = 0. \) Thus \( F \in A \) and satisfies (2.8). We write
\[
(2.9) \quad \frac{zF'(z)}{F(z)} = p(z), \quad F = F_1 \ast g.
\]

From (2.8) and (2.9) with some calculations
\[
\beta p(z) + \frac{\beta z^p(z)}{(c - \beta) + \beta p(z)} = \delta \left[ (zG_1'(z))^\nu \right] + \nu \left[ \frac{zG_1'(z)}{G_2'(z)} \right].
\]
That is
\[
p(z) + \frac{1}{p} \frac{i \beta z^p(z)}{p(z) + \frac{1}{p}(c - \beta)} = \frac{1}{\delta} \beta \left[ (zG_1'(z))^\nu \right] + \nu \left[ \frac{zG_1'(z)}{G_2'(z)} \right].
\]
We now apply Theorem 2.2 and obtain the required result.

For \( m \leq \frac{2(2 - \rho)}{1 - \rho} \) and applying a result proved in [14], it can easily be deduced that
\[
\int_{\theta_1}^{\theta_2} \text{Re} \left\{ p(z) \right\} d\theta > -\pi, \quad p(z) = \frac{z(F_1 \ast g)'}{F_1 \ast g}.
\]
Taking \( g(z) = \frac{z}{(1 - \beta)z}, \) it follows that \( F_1 \in S \) in \( E, \) see [4].

3. The Class \( T_{a,m}[A, B; 0; B_1; g] \)

Theorem 3.1. Let \( f \in T_{a,m} \left[ A, B; 0; B_1; \frac{1}{1 - z} \right] = T_{a,m}[A, B; 0; B_1]. \) Then, for \( z = re^{i\theta}, \ 0 \leq \theta_1 < \theta_2 \leq 2\pi, \ \rho_1 = \left( \frac{1}{2} \right)^{\alpha}, \ \rho = \left( \frac{1 - A}{1 - B} \right)^{\alpha}, \)
\[
\int_{\theta_1}^{\theta_2} \text{Re} \left\{ \frac{(z^p(z))'}{f(z)} \right\} d\theta > -\beta \pi, \quad \beta = \left[ (1 - \rho_1) + \left( \frac{m}{2} - 1 \right) (1 - \rho) \right].
\]
\textbf{Proof.} For } f \in T_{\alpha,m}[A,B;0;B_1], \text{ we can write}
\begin{equation}
\frac{f'(z)}{\psi'(z)} = h(z), \quad \psi \in V_{\alpha,m}[A,B], \quad h \in P_\alpha[0,B_1].
\end{equation}

To prove this result, we shall essentially follow the method due to Kaplan [4].

For } \psi \in V_{\alpha,m}[A,B], \text{ it implies that } \psi \in V_m(\rho), \text{ where } \rho = \left(\frac{1-A}{1-B}\right)^\alpha.

Also } h \in P_\alpha[0,B_1], B_1 \in [-1,0) \text{ is equivalent to } h \sim \left(\frac{1}{1+B_1^2}\right)^\alpha. \text{ That is, } h \in P(\alpha_1) \subset P, \quad \alpha_1 = \left(\frac{1}{2}\right)^\alpha.

Now, with } z = re^{i\theta}, \text{ write } p(z) = \arg f'(z) \text{ and } q(z) = \arg \psi'(z). \text{ Then}
\begin{equation}
|p(z) - q(z)| < \left(1 - \left(\frac{1}{2}\right)^\alpha\right) \frac{\pi}{2}
\end{equation}

Let } P(r,\theta) = p(re^{i\theta}) + \theta, \quad Q(r,\theta) = q(re^{i\theta}) + \theta \text{ be defined for } 0 \leq r < 1 \text{ and for all } \theta. \text{ This gives us}
\begin{equation}
|P(r,\theta) - Q(r,\theta)| < \left(1 - \left(\frac{1}{2}\right)^\alpha\right) \frac{\pi}{2}.
\end{equation}

From Theorem 2.2, for } \psi \in V_{\alpha,m}[A,B] \subset V_m(\rho), \text{ we have}
\int_{\theta_1}^{\theta_2} \text{Re} \left\{ \left(\frac{z\psi'(z)}{\psi'(z)}\right)' \right\} d\theta > -\left(m - 1\right) \left(1 - \left(\frac{A}{1-B}\right)^\alpha\right) \pi, \quad (z = re^{i\theta}).

Thus
\begin{equation}
|Q(r,\theta_1) - Q(r,\theta_2)| < \left(1 - \left(\frac{A}{1-B}\right)^\alpha\right) \left(\frac{m}{2} - 1\right) \pi.
\end{equation}

From (3.2) and (3.3), it follows that
\begin{align*}
|P(r,\theta_1) - P(r,\theta_2)| &= |\{P(r,\theta_1) - Q(r,\theta_1)\} - \{P(r,\theta_2) - Q(r,\theta_2)\} + \{Q(r,\theta_1) - Q(r,\theta_2)\}| \\
&< \left(1 - \left(\frac{1}{2}\right)^\alpha\right) \frac{\pi}{2} + \left(1 - \left(\frac{1}{2}\right)^\alpha\right) \frac{\pi}{2} + \left(1 - \left(\frac{A}{1-B}\right)^\alpha\right) \left(m - 1\right) \pi \\
&= \left[\left(1 - \left(\frac{1}{2}\right)^\alpha\right) + \left(1 - \left(\frac{A}{1-B}\right)^\alpha\right) \left(\frac{m}{2} - 1\right)\right] \pi = \beta \pi,
\end{align*}

and this proves our result. \hfill \Box

\textbf{Special Cases.}

(i) Let } \alpha = 1, A = 1 \text{ and } B = -1. \text{ Then } \beta = \frac{m-1}{2} = 1 \text{ for } m = 3. \text{ This implies}
\begin{equation}
f \in T_{1,m}[1,-1;0;-1] \text{ is univalent for } 2 \leq m \leq 3.
\end{equation}

(ii) For } A = 0, B = -1, \alpha = 1 \text{ we have } \beta = \frac{m}{4} \text{ and, in this case, } f \text{ is univalent for } 2 \leq m \leq 4.
Remark 3.1. For $F \in A$, Goodman [4] introduced a class $K(\beta)$ as
\[
\int_{\theta_1}^{\theta_2} \Re \left\{ \frac{(zF'(z))'}{F'(z)} \right\} d\theta > -\beta \pi, \quad z = re^{i\theta}, \quad 0 \leq \theta_1 < \theta_2 \leq 2\pi,
\]
and $\beta \geq 0$. When $0 \leq \beta \leq 1$, $K(\beta)$ consists of univalent functions (close-to-convex), while for $\beta > 1$, $F$ need not even be finitely-valent, see [4].

We note that, for $\rho_1 = \left(\frac{1}{2}\right)^{\alpha}$, $\rho = \left(\frac{1-\alpha}{1-\beta}\right)^{\alpha}$.

\[
T_{\alpha,m}[A,B;0;B_1] \subset K\left(\frac{m}{2}(1-\rho) + (\rho - \rho_1)\right).
\]

This implies $F \in T_{\alpha,m}[A,B;0;-1]$ is univalent for $m \leq 2\left[1 + \frac{\rho_1}{1-\rho}\right]$.

Theorem 3.2. For $g(z) = \frac{z}{1-z}$, let $f \in T_{\alpha,2}[A,B,0,B_1]$ and for $\gamma, \beta > 0$, let $F_1$ be defined by
\[
F_1(z) = \left[ (1 + \beta)z^{-\beta} \int_0^z t^{\beta-1}f(t)dt \right]^{\frac{1}{\gamma}}.
\]

Then $F_1 \in T_{1,2}[A,B;0;B_1]$ in $E$.

Proof. We can write (3.4) as
\[
F_1(z) = \left[ \left( \frac{f(z)}{z} \right)^\gamma \left( \frac{\phi_{\gamma,\beta}(z)}{z} \right) \right]^{\frac{1}{\gamma}},
\]
where
\[
\phi_{\gamma,\beta}(z) = \sum_{n=1}^{\infty} \left( \frac{z^n}{n + \gamma + \beta} \right)
\]
is convex in $E$.

Since $f \in T_{\alpha,2}[A,B;0;B_1]$, there exists $\psi_1 = z\psi' \in R_{\alpha,2}[A,B]$ such that $\frac{f'}{\psi'} \in P_{\alpha}[0,B_1]$, $\psi = V_{\alpha,2}[A,B]$ in $E$. Let
\[
G_1(z) = \left[ (\beta + 1)z^{-\beta} \int_0^z t^{\beta-1}\psi_1(t)dt \right]^{\frac{1}{\gamma}}, \quad G_1 = zG'.
\]

We first show that $G \in V_{\alpha,2}[A,B]$.

From (3.7), it follows that
\[
\{z^\beta G_1^\gamma(z)\}' = z^{\beta-1}(\psi_1(z))
\]
That is
\[
(G_1^\gamma(z)) [\beta + \gamma H_1(z)] = \psi_1(z), \quad H_1(z) = \frac{zG_1'(z)}{G_1(z)}
\]
Logarithmic differentiation of (3.9) and simple computations give us
\[
H_1(z) + \frac{zH_1'(z)}{\gamma H_1(z) + \beta} = \frac{z\psi_1'}{\psi_1(z)} < \left( \frac{1 + Az}{1 + Bz} \right)^\alpha < \left( \frac{1 + Az}{1 + Bz} \right).
\]
Now, using Theorem 3.3 of [7, p: 109], it follows from (3.10) that \( H_1 \in P[A, B] \) and \( G_1 = zG' \) belongs to \( R_{1,2}[A, B] = S^*[A, B] \). Therefore \( G \in V_{1,2}[A, B] = C[A, B] \).

From (3.4), we have

\[
\frac{zF_1'(z)}{G_1'(z)} = \frac{\phi_{\gamma, \beta}(z) * z \left( \frac{\psi_1(z)}{z} \right)^{\gamma} \left( z f'(z) - \int_{\psi_1(z)}^{z} \frac{\psi_1(z)}{z} \right)}{\phi_{\gamma, \beta}(z) * z \left( \frac{\psi_1(z)}{z} \right)^{\gamma}} = \frac{\phi_{\gamma, \beta}(z) * z \left( \frac{\psi_1(z)}{z} \right)^{\gamma} h(z)}{\phi_{\gamma, \beta}(z) * z \left( \frac{\psi_1(z)}{z} \right)^{\gamma}}, \quad h \in P_\alpha(B_1).
\]

Since \( h(z) \) is analytic in \( E \), \( h(0) = 1 \), and \( \phi_{\gamma, \beta}(z) \) is convex, \( \psi_1 \in \mathcal{S}^* \), we use a result due to Ruscheweyh and Sheil-Small [17] to conclude that \( \left( \frac{zF_1'(z)}{G_1'(z)} \right)(E) \subset \overline{\text{Coh}}(E) \), where \( \overline{\text{Coh}}(E) \) denotes convex hull of \( h(E) \). This implies \( F_1 \in T_{1,2}[A, B; 0; B_1] \) in \( E \) or \( \gamma = 1 \) in (3.4), we obtain the well known Bernardi integral operator, see [7].

**Theorem 3.3.** Let \( F = f * g, f \in T_{\alpha,m}[A, B; 0; B], B \neq 0 \). Then with \( \rho = \left( \frac{1-a}{1-B} \right)^{\alpha} \) and \( \gamma = \frac{A-B}{\lambda B} \),

\[
(i) \quad \left( \frac{1}{1+Br} \right)^{\alpha} (1-Br)^{(1-\rho)(\frac{m+2}{4})} \leq |F'(z)| \leq \left( \frac{1+Br}{1-Br} \right)^{(1-\rho)(\frac{m-2}{4})} \left( \frac{1}{1-Br} \right)^{\alpha}
\]

\[
(ii) \quad \frac{2^{\gamma(1-\rho)}}{1+B} \left[ G_{12}(a, b; c; -1) - r_1^{-a}G_{12}(a, b; c; -r_1) \right] \leq |F(z)| \leq \frac{2^{\gamma(1-\rho)}}{1+B} \left[ G_{12}(a, b; c; -1) - r_1^{-a}G_{12}(a, b; c; -r_2) \right],
\]

where \( r_1 = -r_2^{-1} = \frac{1+Br}{1-Br} \), \( m \leq \left[ \frac{4(1-\alpha)}{\gamma(1-\rho)} + 2 \right] \) and \( a \) is given in (3.16).

**Proof.** We can write for \( F \in T_{\alpha,m}[A, B; 0; B] \),

\[
F'(z) = G'(z)h(z), \quad h \in P_\alpha[B_1], \quad G = \psi * g \in V_{\alpha,m}[A, B].
\]

Since \( h \in P_\alpha[B] \), it easily follows that

\[
(iii) \quad \left( \frac{1}{1+Br} \right)^{\alpha} \leq |h(z)| \leq \left( \frac{1}{1-Br} \right)^{\alpha}
\]

From Theorem 2.4 (ii) and (3.12), the proof of (i) is established.

We now proceed to prove (ii).

Let \( d_r \) denote the radius of the largest schlicht disc centered at the origin contained in the image of \( |z| < r \) under \( F(z) \). Then there is a point \( z_0, |z_0| = r \), such that \( |F(z_0)| = d_r \). The ray from 0 to \( F(z_0) \) lies entirely
in the image and the inverse image of this ray is a curve in $|z| < r$.

Using (3.11), we have

$$d_r = |F(z_0)| = \int_{C} |F'(z)| dz, \quad r = \frac{A - B}{2B}$$

$$\geq \int_{0}^{1} |z| \left[ \frac{(1 - Bs)\gamma((1-\rho)(\frac{m}{4} + \frac{1}{2}))}{(1 + Bs)\gamma((1-\rho)(\frac{m}{4} - \frac{1}{2}) + \alpha)} \right] ds$$

$$= \int_{0}^{1} \left[ \frac{(1 - Bs)\gamma(\frac{1}{2} + \alpha)}{(1 + Bs)\gamma(1-\rho)(\frac{m}{4} - \frac{1}{2}) + \alpha} \right] \cdot (1 - Bs)\gamma(1-\rho - \alpha) ds,$$

(3.13)

Let $\frac{1 + Bs}{1 - Bs} = t$. Then $\frac{2B}{(1 - Bs)^2} = dt$, and $1 - Bs = \frac{2}{1 + t}$. This implies $ds = \frac{2B}{(1 + t)^2} dt$. Therefore, from (3.13), we have

$$|F(z_0)| \geq \int_{1}^{\frac{1 + Bs}{1 - Bs}} t^{-((1-\rho)(\frac{m}{4} - \frac{1}{2}) - \alpha)} \cdot \left( \frac{2}{1 + t} \right)^{1-\rho - \alpha} \cdot \frac{2B}{1 + t} \left( \frac{1}{1 + t} \right)^{2} dt$$

$$= -\frac{2}{|B|} \left[ \int_{0}^{\frac{1 + Bs}{1 - Bs}} t^{(1-\rho)(\frac{m}{4} - \frac{1}{2}) - \alpha} \cdot (1 + t)^{(1-\rho - \alpha)} dt \right]$$

$$- \int_{0}^{1} t^{(1-\rho)(\frac{m}{4} - \frac{1}{2}) + \alpha} \cdot (1 + t)^{(1-\rho - \alpha)} dt$$

$$= \frac{2\gamma(1-\rho)}{|B|} \left[ I_1 + I_2 \right].$$

(3.14)

Now put $t = r_1 u$ with $r_1 = \frac{1 + Bs}{1 - Bs}$. Then $dt = r_1 du$ and

$$I_1 = \int_{0}^{1} (r_1 u)^{-\gamma(1-\rho)(\frac{m}{4} - \frac{1}{2}) - \alpha} \cdot (1 + r_1 u)^{1-\rho - \alpha} du$$

$$= r_1^{-\gamma(1-\rho)(\frac{m}{4} - \frac{1}{2}) + \alpha - 1} \cdot \int_{0}^{1} u^{-\gamma(1-\rho)(\frac{m}{4} - \frac{1}{2}) - \alpha} \cdot (1 + r_1 u)^{-\gamma(1-\rho) + \alpha} du$$

$$= r_1^{-\gamma(1-\rho)(\frac{m}{4} - \frac{1}{2}) + \alpha - 1} \cdot \frac{\Gamma(a)\Gamma(c - a)}{\Gamma(c)} \cdot G_{12}(a, b; c; -r_1),$$

(3.15)

where $\Gamma$ and $G_{12}$, respectively denote gamma and Gauss hypergeometric functions. Also, here, $b, c$ are positively real for $m \leq 2 \left\{ 1 + \frac{2(1-\alpha)}{1-\rho} \right\}$ and are given as

$$a = -\gamma(1-\rho) \left( \frac{m}{4} - \frac{1}{2} \right) - \alpha + 1, \quad \gamma = \frac{A - B}{2B}, \quad B \neq 0$$

$$b = -\gamma(1-\rho) + \alpha,$$

$$c = -\gamma(1-\rho) \left( \frac{m}{4} - \frac{1}{2} \right) - \alpha + 2, \quad (c - a) > 0.$$

(3.16)

Similarly, we calculate $I_2$ and have

$$I_2 = \frac{\Gamma(a)\Gamma(c - a)}{\Gamma(c)} \cdot G_{12}(a, b; c; -1).$$

(3.17)
Using (3.15), (3.16) and (3.17) in (3.14), we obtain the lower bound of $|F(z)|$. For the upper bound, we proceed in similar way and have

$$|F(z)| \leq \int_0^{|z|} \frac{1 + Bs}{1 - Bs} \gamma (1 - \rho) (\frac{m}{2} + \frac{1}{2}) \cdot (1 + Bs)^{\alpha} ds$$

Now similar computations yield the required bound and the proof is complete. \[\square\]

By choosing suitable and permissible values of involved parameters, we obtain several new and also known results.

**Remark 3.2.**

(i) We use a result of Pommerenke [16] together with Theorem 3.1 and easily deduce that the class $T_{\alpha,m}[A,B;0;−1]$, $m \leq 2 \left\{ 1 + \frac{\rho}{1 - \rho} \right\}$, $\rho_1 = \left( \frac{1}{2} \right)^\alpha$, $\rho = \left( \frac{1 - A}{1 - B} \right)^\alpha$, is a linearly invariant family of order $B_2 = \left\{ \frac{m}{2} (1 - \rho) + (\rho - \rho_1) + 1 \right\}$. With similar argument given in [16], we have the covering result for $T_{\alpha,m}[A,B;0;−1]$ as:

The image of $E$ under $F = f \ast g \in T_{\alpha,m}[A,B;0;−1]$ contains the Schlicht disc $|z| = \frac{1}{2B_2}$, where $B_2 = \left\{ \frac{m}{2} (1 - \rho) + 1 + \rho - \rho_1 \right\}$, and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$.

(ii) Let $F_\ast$ be defined as

$$F_\ast(z) = \frac{1}{B_1} \left[ \left( \frac{1 + z}{1 - z} \right)^{B_2} - 1 \right] = z + \sum_{n=2}^{\infty} A_n^* z^n,$$

where

$$B_1 = \left\{ \frac{m}{2} (1 - \rho) + (\rho - \rho_1) + 2 \right\},$$

$$B_2 = \left\{ \frac{m}{2} (1 - \rho) + (\rho - \rho_1) + 1 \right\}.$$

It can be shown, with some computations, that $F_\ast$ belongs to the linearly invariant family of $T_{\alpha,m}[A,B;0;−1]$.

Using this concept, together with the same argument of Pommerenke [16], we have $|A_n| \leq |A_n^*|$, $n \geq 1$ and $L_r(F) \leq L_r(F_\ast)$, $F \in T_{\alpha,m}[A,B;0;−1]$ when $L_r(F)$ is the length of the image of the circle $|z| = r$ under $F$, $0 \leq r < 1$.

**Theorem 3.4.** Let $f \in T_{\alpha,m}[A,B;0;−1;g]$ and let $F = f \ast g \neq 0$ in $E$ with

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n.$$

Then, for $m > 2$,

$$A_n = O(1) \cdot n^{\gamma_1}, \quad \gamma_1 = \left\{ \frac{m}{2} (1 - \rho) + [\rho_1 - (1 + \rho)] \right\},$$

where $O(1)$ is a constant depending on $m$, $\alpha$, $A$ and $B$ only.
Proof. For $F \in T_{\alpha,m}[A, B; 0; -1]$, we can write

$$F' = G'h, \quad G \in V_{\alpha,m}[A, B], \quad G = \psi * g, \quad \psi \in V_{\alpha,m}[A, B; g].$$

Since $V_{\alpha,m}[A, B] \subset V_m(\rho)$, $\rho = \left(\frac{1-A}{1-B}\right)^\alpha$, and it is well known that there exists $G_i \in V_m$ such that $G'(z) = (G'(z))^{1-\rho}$ for $z \in E$.

Also $h \prec \left(\frac{1}{1+z}\right)^\alpha$, which implies $|\arg h(z)| < \frac{\rho_1 \pi}{2}$, $\rho_1 = \left(\frac{1}{2}\right)^\alpha$.

Therefore we have

$$F' = (G_1')^{1-\rho} (h_1)^{\rho_1}, \quad Reh_1 > 0$$

in $E$.

For $G_1 \in V_m$, there exists $s \in S^*$ such that $G'_1 = sh_2^{(\frac{2m}{2}-1)}$, $m > 2$ and $Reh_2 > 0$ in $E$, see [1].

Thus, for $F \in T_{\alpha,m}[A, B; 0; -1]$, it follows that

$$F' = (s)^{1-\rho} (h_2)^{(1-\rho)(\frac{m}{2}-1)} (h_1)^{\rho_1}, \quad h_i \in P, \quad i = 1, 2$$

(3.18)

So, by Cauchy Theorem and (3.18), we have for $z = re^{i\theta}$,

$$n|A_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} |s|^{1-\rho} |h_1|^{\rho_1} |h_2|^{(1-\rho)(\frac{m}{2}-1)} d\theta$$

$$\leq \frac{1}{r^n} \left(\frac{r}{(1-r)^2}\right)^{(1-\rho)} \left[\left(\frac{1}{2\pi} \int_0^{2\pi} |h_1|^2 d\theta\right)^{\frac{\rho_1}{2}} \cdot \left(\frac{1}{2\pi} \int_0^{2\pi} |h_2|^{\frac{2m}{2}-\rho_1} d\theta\right)^{\frac{2-\rho_1}{2}}\right],$$

where $\delta = (1-\rho)\left(m - \frac{1}{2}\right)$ and we have used distortion result for $s \in S^*$ and Holder inequality.

Now, for $m > \left\{2 + \frac{2-\rho_1}{1-\rho}\right\}$, we apply a result due to Hayman [5] for $h_i \in P$ and obtain

$$n|A_n| \leq c(\rho, \rho_1, m) \cdot \left(\frac{1}{1-r}\right)^{1+\delta + \rho_1 - 2\rho}$$

(3.19)

where $c(\rho, \rho_1, \delta)$ is a constant.

Setting $r = 1 - \frac{1}{n}$, $n \rightarrow \infty$ in (3.19), the required result follows as

$$A_n = O(1) \cdot n^{\left\{\frac{\alpha}{2} (1-\rho) + [\rho_1 - (\rho + 1)]\right\}}, \quad \rho_1 = \left(\frac{1}{2}\right)^\alpha, \quad \rho = \left(\frac{1-A}{1-B}\right)^\alpha,$$

and $m > \left\{2 + \frac{2-\rho_1}{1-\rho}\right\}$, $n \geq 2$.

Special Cases.

(i) $A = 1$ implies that $\rho = 0$ and for $\alpha = 1$, $\rho_1 = \frac{1}{2}$. Then

$$A_n = O(1) \cdot n^{\frac{\alpha}{2} - \frac{1}{2}}, \quad m > \frac{7}{2}$$

Taking $m = 4$, we have $A_n = O(1) \cdot n^2$. 
(ii) $A = \frac{1}{2}$, $B = -1$, $\alpha = 1 \Rightarrow \rho_1 = \frac{1}{4}$. Also $\rho_1 = \frac{1}{2}$. Then $m = 5 > 4$ implies $A_n = \mathcal{O}(1) \cdot n^{2\pi}$.

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