Macroscopic quantum tunneling and resonances in coupled Bose-Einstein condensates with oscillating atomic scattering length

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Abstract

We study the macroscopic quantum tunneling, self-trapping phenomena in two weakly coupled Bose-Einstein condensates with periodically time-varying atomic scattering length.

The resonances in the oscillations of the atomic populations are investigated. We consider oscillations in the cases of macroscopic quantum tunneling and the self-trapping regimes. The existence of chaotic oscillations in the relative atomic population due to overlaps between nonlinear resonances is showed. We derive the whisker-type map for the problem and obtain the estimate for the critical amplitude of modulations leading to chaos. The diffusion coefficient for motion in the stochastic layer near separatrix is calculated. The analysis of the oscillations in the rapidly varying case shows the possibility of stabilization of the unstable \( \pi \) - mode regime.

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I. INTRODUCTION

Recently much attention has been paid to the investigation of macroscopic tunneling phenomena between coupled Bose-Einstein condensates (BEC) [1,2]. The theory has been developed for weakly as well as for strongly overlapped condensates [3]- [8]. As a result, it was shown that, in oscillations of the relative atomic population, periodic processes of two types can exist. One which is characterized by a zero mean atomic imbalance function (<z(t)> = 0), corresponding to the periodic flux of atoms between two condensates due to the overlaps of the wavefunctions. The second type has a net atomic population balance <z(t)> ≠ 0, corresponding to the so-called macroscopic self-trapping, with localization of atoms in one of the condensates, and a periodically varying population around a constant value. The different regimes are connected with the value of the initial phase differences θ = φ1 − φ2 between the condensates and the effective nonlinearity Λ ∼ (α1 + α2)N, where αi are the nonlinearity parameters depending on the atomic scattering length (which in turn depends on the two-atoms interactions), and N is the total number of atoms.

If a periodic variation of the parameters of the trap and of the BEC is introduced, new phenomena show up. Examples of this are the resonant phenomena with a periodically varying trap potential considered recently for the case of weakly coupled BEC in [9] and for the strongly overlaped BEC in [10]. The optical analog of this process is the electromagnetic wave transmission in a nonlinear coupler with periodic variation of tunnel coupling [11,12]. Further interesting cases occur when a time-dependent atomic scattering length in BEC is considered. This is made possible due to the fact that as depends resonantly on the external magnetic field or on the laser field [13] - [18]. This means that the scattering length can be varied in time, in particular, periodically. The main interest now is to investigate the influence of this periodic variation of atomic scattering length on the macroscopic interference phenomena. The central point here consists in that, as the interference phenomena are time-periodic processes in a nonlinear system, we can expect a nonlinear resonant response of the system under time-periodic perturbations, leading to a variety of phenomena, such as
II. FORMULATION OF PROBLEM. SLOWLY TIME-VARYING ATOMIC SCATTERING LENGTH

The problem of weakly coupled BEC’s in a double-well trap potential with time-dependent atomic scattering length \(a_s\) can be described by the two-modes model

\[
\begin{align*}
\imath \hbar \frac{\partial \psi_1}{\partial t} &= (E_1 + \alpha_1(t)|\psi_1|^2)\psi_1 - K\psi_2, \\
\imath \hbar \frac{\partial \psi_2}{\partial t} &= (E_2 + \alpha_2(t)|\psi_2|^2)\psi_2 - K\psi_1,
\end{align*}
\]

(1)

where the parameters \(E_i, \alpha_i, K\) are defined by the overlaps integrals of the time dependent Gross-Pitaevsky eigenfunctions and derived in [4]. In particular, \(\alpha_i(t)\) are the parameters characterizing the nonlinear interactions between atoms and \(\alpha_i \sim g_0 = 4\pi\hbar^2a_s(t)/m\), where \(a_s(t)\) is the time dependent atomic scattering length. This system of equations is valid in the approximation of a weak link between condensates. As comparison with the numerical solution of GPE shows that it is a good approximation for \(z \leq 0.6\). (see the definition of \(z(t)\) below). Another limitation is connected with the time dependence of \(a_s\). For a harmonic modulation in time we should require that the perturbation does not introduce transitions between the ground state and excited states in the trap, whose energy gap is of the order of \(\hbar \omega_0\), where \(\omega_0\) is the harmonic frequency of trap. Then we have the restriction \(\Omega \ll \omega_0\). Another characteristic frequency is given by the coupling between condensates and is defined as \(\omega_L = 2K\). For the weak coupling case when \(\omega_L \ll \omega_0\) the regimes of resonant \(\Omega \sim \omega_L\) and rapid modulations of \(a_s\) can be realized. The periodic modulation of nonlinearity in the Gross-Pitaevsky equation can also lead to the parametric instability of the collective excitations of the condensate [19,20], that, in turn, leads to the break down of the two-modes approximation. Because these frequencies are far from the ones considered below we will use the model (1) in our analysis.

Introducing new variables \(\psi_i = \sqrt{N_i} \exp(i\theta_i), z = (N_1 - N_2)/N_T, N_T = N_1 + N_2, \Phi = \)
θ_1 - θ_2, where \( N_i, \theta_i \) are the number of atoms and phases in the i-th trap, we get the system

\[
\dot{z}_t = -\sqrt{1 - z^2} \sin \Phi, \\
\dot{\Phi}_t = \nu \Lambda(t) z + \frac{z}{\sqrt{1 - z^2}} \cos \Phi + \Delta E_0,
\]

(2) (3)

where \( \Lambda = (\alpha_1(t) + \alpha_2(t)) N_T / 2K, \) \( t = \omega_L t, \) \( \omega_L = 2K, \) \( \omega_L \) being the frequency of the linear Rabi oscillations, \( \Delta E_0 = \Delta E / 2K, \) and \( \nu = \pm 1 \) for the positive and negative atomic scattering length, respectively.

The Hamiltonian of the unperturbed system \((a_s(t) = \text{const}, \Delta E_0 = \text{const}, \eta = 0)\) is

\[
H = \frac{\nu \Lambda z^2}{2} - \sqrt{1 - z^2} \cos(\Phi) + \Delta E_0 z.
\]

Below, if not specified otherwise, we will consider the case of positive scattering length \( \nu = 1, \) and \( \Delta E_0 = 0. \) Consider the initial phase difference to be zero, \( \Phi(0) = 0 \) – the zero-phase mode. Depending on the number of atoms, \( N, \) and two-atoms interaction, \( \alpha, \) i.e. \( \Lambda, \) there may exist a macroscopic quantum tunneling regime with \( \langle z(t) \rangle = 0, \) corresponding to \( \Lambda < \Lambda_c \) and a self-trapped regime with \( \langle z(t) \rangle \neq 0, \) corresponding to \( \Lambda > \Lambda_c. \) The same is valid for the initial phase difference, \( \Phi(0) = \pi \) – the \( \pi \)-phase mode.

Let us consider now the case of periodic modulations of the atomic scattering length, when \( a_s(t) = a_0(1 + \epsilon \cos \Omega t), \) inducing a variation of \( \Lambda, \) i.e. \( \Lambda = \Lambda_0 + \Lambda_1 \cos(\Omega t). \) Such a variation can be achieved by the variation of the external magnetic field \[18\] - a magnetic field induced Feshbach resonances for example. Near the resonance the scattering length is varied dispersively and can have positive and negative values. We will consider situations when we are not close to resonance and variations of scattering length are small, that is, \( \Lambda_1 / \Lambda_0 = \epsilon \ll 1. \)

A. Resonances in zero phase modes

Let us first consider the case where \( \Phi(0) = 0. \) Take the case of \( z^2 \ll 1 \) and \( \Lambda_1 > z. \)

Taking into account that \( z_t \sim \Phi \sim z^2, \) assumed to be small, we obtain the equation for the relative atomic population
\[ z_{tt} + (\Lambda(t) + 1)z - \frac{\Lambda(t) + 1}{2}z^3 = 0. \]  \hspace{1cm} (5)

where, \( \Lambda(t) = \Lambda_0(1 + \epsilon \sin(\Omega t)) \). For \( \epsilon \ll 1 \) we can reduce the equation to

\[ z_{tt} + \omega_0^2 z - \beta z^3 + \epsilon \Lambda_0 \cos(\Omega t)z = 0, \]  \hspace{1cm} (6)

where \( \omega_0 = \sqrt{1 + \Lambda_0} \) is the frequency of linear oscillations, and \( \beta = (\Lambda_0 + 1)/2 \). It is useful to introduce a new variable \( y = \sqrt{(\Lambda_0 + 1)}z \leq 1 \) and a scaled time \( \tau = t\omega_0 \). Then we have the equation

\[ y_{\tau\tau} + y - \beta_1 y^3 + \varepsilon_1 \cos(\Omega_1 \tau)y = 0, \]  \hspace{1cm} (7)

where \( \beta_1 = 1/(2\sqrt{\Lambda_0 + 1}) \), \( \varepsilon_1 = \varepsilon \Lambda_0/(\Lambda_0 + 1) \), \( \Omega_1 = \Omega/\omega_0 \).

Let us consider the dynamics in the parametric resonance region when \( \Omega_1 = 2 + \Delta \). Using the results of perturbation theory [21], we find that the parametric resonance occurs when the condition for the square of increment \( s^2 > 0 \) is satisfied, where

\[ s^2 = \frac{1}{4}[(\varepsilon_1 - \frac{3}{8}\beta_1 a^2)^2 - \Delta^2]. \]  \hspace{1cm} (8)

We have included here a nonlinear correction to the frequency \( 3\beta_1 a^2/8 \). Then we obtain the limit for the amplitude for parametric resonance to occur,

\[ a_p^2 \leq \frac{8(\varepsilon_1 - 2\Delta)}{3\beta_1 \varepsilon_1}, \]  \hspace{1cm} (9)

and \( z_p = a_p/\sqrt{\Lambda_0 + 1} \).

In Fig.(1) we present the oscillations of \( z(t) \) when the parameters are in the region of the parametric resonance, by integrating numerically Eq.(2).

**B. Resonances in \( \pi \) - phase modes**

We now come to the case \( \phi(0) = \pi \), with for \( \Lambda < 1 \) and \( z^2 \ll 1 \). The system (2) is simplified and reduces to one equation for \( \Phi \), having the form of the equation of a double pendulum. Inspection of this equation shows that there exists a valley in the effective energy
$V(\Phi)$ around $\Phi = \pi$. The maximum of the valley depth is achieved when $\Lambda_0 \to 1$. Thus we can search the solution of the system in this region of parameters assuming $z \ll 1, \Phi = \pi + \delta(t)$. It results that we obtain the following equation for $z(t)$

$$z_{tt} + (1 - \Lambda_0)z - \Lambda_1 \cos(\Omega t)z = 0. \quad (10)$$

Here $\Lambda_0 < 1$. The parametric resonance in the oscillations of the atomic population appears when $\Omega = 2\sqrt{1 - \Lambda_0} + \Delta$. The parametric instability occurs when $\Lambda_1/(2\sqrt{1 - \Lambda_0}) \geq \Delta$. The increment $s$ is, in this case,

$$s = \frac{1}{2} \sqrt{\frac{\Lambda_1}{2\sqrt{1 - \Lambda_0}^2} - \Delta^2}. \quad (11)$$

For example, when $\Lambda_0 = 0.36, \Delta = 0.1$ we have $\Omega = 1.7$. In Fig.(2) we plot the oscillations of $z(t)$ for this choice of parameters, and with $\Lambda_1 = 0.162$, by a numerical integration of Eq.(2).

Note that the frequency of linear Rabi oscillations is $\omega_L = 2K$, or one in the units used. Thus the resonance in $\pi$-mode occurs at frequencies lower than the Rabi frequency. It is worth mentioning that the metastable $\pi$-phase mode has been recently observed in the weak link separating two reservoirs of superfluid He$_3$ [22].

### III. Chaotic Dynamics in Oscillations of the Relative Atomic Population

The oscillations induced by the periodic perturbations of the scattering length may be complex and become chaotic, for certain regions of the parameters. Note that the unperturbed system is equivalent to the Duffing oscillator. Thus, as is known, the resonance overlaps under periodic perturbations can lead to chaotic variations of $z(t)$ [23][25]. In such cases, to study the chaotic motion, it is useful to look for the motion near the separatrix of the unperturbed system. The total Hamiltonian is $H = H_0 + V$, where $H_0$ is unperturbed Hamiltonian (4) and the perturbation is $V = \Lambda_1 \cos(\Omega t)z^2/2$. The value of the Hamiltonian
on separatrix is $H_s = 1$. The separatrix divides the regions with macroscopic quantum tunnelling, with $<z> = 0$, from the self-trapped regions where $<z> \neq 0$. The expressions for the separatrix solution are:

$$z_s(t) = \sqrt{\frac{a}{b}} \text{sech}(\sqrt{2at}),$$
$$\sin^2(\Phi_s) = \frac{a^2 \text{sech}^2(\sqrt{2at}) \tanh^2(\sqrt{2at})}{2K_0^2(b - \text{sech}^2(\sqrt{2at}))},$$

where $a = |1 - \Lambda|/2$, $b = \Lambda^2/8$. For example, for the zero-phase mode the separatrix is given by the hyperbolic fixed points $\Phi = \pm \pi, z = 0$.

The existence of chaos can be proved by calculating the Melnikov function. The calculations are similar to the ones performed in [9] and shows that the Melnikov function has an infinite number of zeros, so the existence of chaotic regimes in the atomic population oscillations is proved. In Fig.(3) we plot the Melnikov distance versus the frequency $\Omega$ for $\Lambda = 10$. We see that the maximum value is achieved for $\Omega \approx 3.89$. In Fig.4 we plot the difference of atomic population $z(t)$ as function of time for $\Lambda_0 = 10, z(0) = 0.4, \Lambda_1 = 0.2, \Omega = 3.89$.

It is interesting to obtain an analytical estimate for the critical amplitude of modulation $\Lambda_1$ leading to the chaotic behavior in the relative atomic population. Using the expression for the separatrix (12) we can calculate the Melnikov-Arnold integral and find that the energy change for a perturbation $\Lambda_1 \cos(\Omega t + \psi)(\text{where } \psi \text{ is the phase}) \text{ is}$

$$\Delta H = \int_{-\infty}^{\infty} \left( \frac{\partial H_0}{\partial t} + [H_0, V] \right) dt = \alpha \sin(\psi), \quad \alpha = \frac{\pi \varepsilon |1 - \Lambda|^2 \Omega^2}{\Lambda \sinh(\pi \Omega/2\sqrt{\Lambda - 1})},$$

where $[...]$ denotes the Poisson bracket.

The period of the unperturbed motion is $T = 4\kappa K(\kappa)/(CA)$, where [13]:

$$\kappa^2 = \frac{1}{2} \left[ 1 + \frac{H_0 \Lambda - 1}{\sqrt{\Lambda^2 + 1 - 2H_0 \Lambda}} \right],$$
$$C^2 = \frac{2}{\Lambda^2} \left[ (H_0 \Lambda - 1) + \sqrt{\Lambda^2 + 1 - 2H_0 \Lambda} \right],$$

where $K(\kappa)$ is the complete elliptic integral of the first kind. Near the separatrix, when $H_0 \to 1$, we have $\kappa^2 \to 1$. Introducing the parameter $\delta = 1 - H_0$, $H_0 < 1, \delta << 1$, and
taking into account that $K(\kappa) \to \ln(4/\sqrt{1-\kappa^2})$, when $\kappa^2 \to 1$, we obtain the estimate for the period of oscillations near the separatrix

$$T \approx \frac{2}{\sqrt{\Lambda - 1}} \ln \left( \frac{4\sqrt{2(\Lambda - 1)}}{\sqrt{2(\Lambda - 1)}} \right). \quad (16)$$

Using the expressions (13) and (16) we find the whisker-type map for our problem

$$\delta_{n+1} = \delta_n + \alpha \sin(\psi_n),$$
$$\psi_{n+1} = \psi_n + \gamma \ln \left( \frac{\mu}{\sqrt{\delta_{n+1}}} \right). \quad (17)$$

where $\gamma = 2\Omega/\sqrt{\Lambda - 1}$ and $\mu = 4\sqrt{2(\Lambda - 1)}/\sqrt{\Lambda}$.

This map can be simplified, using the linearization around the fixed points [24]. The fixed points are defined by

$$2\pi l = \gamma \ln \left( \frac{\mu}{\sqrt{\delta^{(l)}_f}} \right), \quad l = 1, 2... \quad (18)$$

Let us introduce the dimensionless energy $I_n$, defined by

$$I_n = -\frac{\gamma}{\delta^{(l)}_f} (\delta_n - \delta^{(l)}_f). \quad (19)$$

Substituting this expression into Eq.(17), and redefining $I_n \to I_n/2$, we obtain the standard map

$$I_{n+1} = I_n - K \sin(\psi_n),$$
$$\psi_{n+1} = \psi_n + I_{n+1}, \quad (20)$$

where the parameter $K$ is

$$K = \frac{\alpha \gamma}{2\delta^{(l)}_f}. \quad (21)$$

As is well known, chaos appears when $K \geq 1$ (exact value is $K^* = 0.9716$). Then we have

the estimate for the critical amplitude of modulation

$$\Lambda_1 \geq \frac{\delta^{(l)}_f \Lambda^3 \sinh(\pi \Omega/2\sqrt{\Lambda - 1})}{\pi \sqrt{\Lambda - 1} \Omega^3}. \quad (22)$$
From the above, we may calculate the diffusion coefficient for the motion in the stochastic layer
\[ D = \frac{< (\Delta E)^2 >}{t}. \tag{23} \]
It results that we have \( D = K^2/2 \). For frequencies \( \Omega \leq 2\sqrt{\Lambda - 1}/\pi \) we have the estimate
\[ D \approx \frac{2^{10}\pi^2\varepsilon^2(\Lambda - 1)^3\Omega^4}{\Lambda^2} \exp\left(-\frac{4\pi l\sqrt{\Lambda - 1}}{\Omega}\right). \tag{24} \]
For \( \varepsilon = 0.1, \Omega = 2.5, \Lambda = 9, l = 1 \), the time of exit from the stochastic layer is 10 i.e. \( \sim 4 \) periods of the oscillations of the atomic scattering length.

IV. RAPIDLY VARYING SCATTERING LENGTH

In the case when the frequency of oscillations of the atomic scattering length is much larger than the tunneling frequency \( \omega_L = 2K \). i.e. \( \Lambda = \Lambda(t/\epsilon) \), where \( \epsilon = \omega_L/\Omega << 1 \), it is useful to derive an averaged set of equations. Using a multiscale approach \[26\] and expanding \( z = \bar{z} + \epsilon z_1 + ..., \Phi = \bar{\Phi} + \epsilon \Phi_1 .... \) we can derive the system of equations for the slowly varying functions \( \bar{z}, \bar{\Phi} \). The limits of validity is \( \bar{z}^2 << 1 \).

The averaged system turns out to be:
\[ \bar{z}_t = -\sqrt{1 - \bar{z}^2}\sin(\bar{\Phi})(1 - \delta \bar{z}^2), \tag{25} \]
\[ \bar{\Phi}_t = \nu < \Lambda > + \bar{z}\cos(\bar{\Phi})\sqrt{1 - \bar{z}^2}(1 + 2\delta - 3\delta \bar{z}^2). \tag{26} \]
where \( \delta \) is proprotional to \( \epsilon^2 \), and for the harmonic modulation is \( \Lambda_t^2/(4\Omega^2) \). The corresponding Hamiltonian is
\[ H = \frac{\nu < \Lambda >}{2} \bar{z}^2 - \sqrt{1 - \bar{z}^2}\cos(\bar{\Phi})(1 - \delta \bar{z}^2). \tag{27} \]

With respect to the unperturbed case, a new stable point can occur. A bifurcation is observed when \( \delta \) is increased. The unstable point (in the undisturbed case) at \( z = 0, \Phi = \pi \) may become stable under rapid perturbations. This phenomenon is the analog of the Kapitza
stabilization of the unstable fixed point of the pendulum by rapidly oscillating the suspension point. By passing through the bifurcation, a change in the topology of the phase portrait happens. The value of $\delta$ for this to occur depends on $<\Lambda>$. To have an estimate of the critical value of $\delta$, consider $z^2 << 1$ and $\Phi = \pi + \psi, \psi << \pi$. The equation for $z$ reads:

$$z_{tt} - (<\Lambda> - 1 - 2\delta)z = 0.$$ \hspace{1cm} (28)

Thus, when $\delta > \delta_c = (<\Lambda> - 1)/2$ we have oscillatory motion around $(0, \pi)$ in the phase plane. When $\delta < \delta_c$ we have an hyperbolic point, and $(0, \pi)$ is no longer stable. For example, with $\Lambda = 2$, the critical $\delta$ is 0.50. For $<\Lambda> = 1$, the point is always stable, as $\delta_c = 0$. To support these results, we have numerically integrated the original systems given by Eq. (2) with an explicit harmonic fast-time variation for $\Lambda(t)$. The above described features immediately show up.

Note also that we can cross the sign of $a_s$ from positive to negative. Experiments with very a large monotonic change of sign show that the BEC shrinks very fast. It would be interesting to check if the condensate with attractive interaction of atoms stable under the rapid variation of scattering length. This problem requires, however, separate investigations. In any case in quasi 1D geometries collapse is suppressed and this analysis is relevant.

In order to better understand the physical consequences of the averaging, we note that for small $z^2 << 1$ we can rewrite the Hamiltonian in the form

$$H = \frac{\nu \Lambda_r z_r^2}{2} - \sqrt{1 - z_r^2} \cos (\Phi),$$

where $z_r = (1 + \delta)\bar{z}$, $\Lambda_r = (1 - 2\delta) <\Lambda>$. Thus we can conclude that the result of the averaging consists, for a fixed initial phase difference between condensates, in the lowering of the critical relative population for self-trapping regime to occurred and in the increase of the critical value of the nonlinear parameter $\Lambda$. We come to the picture of a more rigid pendulum in comparison with the constant case. The threshold for the switching from the macroscopic quantum tunneling regime to the self-trapping regime is shifted to lower initial values of the atomic imbalance.
V. CONCLUSION

In this paper we have studied the new effects occurring due to time-periodic variations of the atomic scattering length. The resonances in oscillations of the atomic imbalance for the different regimes have been analyzed.

The interaction of resonances occurring under oscillating scattering length can lead to complicated behavior of the atomic oscillations and gives rise to the phenomenon of chaotic macroscopic quantum tunneling. In this work we prove the existence of such regime, using the Melnikov function approach and calculating the Melnikov distance, characterizing the width of the stochastic layer near the separatrix of the unperturbed system. We derive a whisker type map for the problem and obtain the estimate for the critical value of amplitude $\Lambda_1$ leading to the chaotic motion. The diffusion coefficient for the motion in the stochastic layer is calculated and shown that the time to cross the stochastic layer is the order of few periods of modulations. We also consider the evolution of the system under rapidly varying oscillations of the scattering length and derive the averaged system for the slow-time variations of the relative atomic population $\tilde{z}(t)$ and the phase $\tilde{\Phi}(t)$. The analysis of the fixed points shows that the stabilization of the system under rapid perturbation, when the unperturbed is unstable, is possible in the $\pi$ -phase mode.

VI. ACKNOWLEDGMENTS

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FIG. 1. The parametric resonance in zero-phase mode. The parameters are: $\Lambda_0 = 3$, $\Lambda_1 = 0.12$, $\Delta = 0.1$. 
FIG. 2. The parametric resonance for \( \pi \)-phase mode: The parameters are: \( \Lambda_0 = 0.36, \Lambda_1 = 0.155, \Delta = 0.1, \Omega = 1.7, \Phi(0) = \pi \).
FIG. 3. Melnikov $D(\Omega)$ distance versus frequency for $\epsilon = 0.1, \Lambda = 2, \phi(0) = 0$
FIG. 4. The chaotic oscillations of the relative atomic population $z(t)$ for

$\Omega = 3.89, \Lambda = 2, \Lambda_0 = 0.2, \Phi(0) = 0, z(0) = 0.6$