UNCONDITIONALLY SECURE QUANTUM BIT COMMITMENT IS POSSIBLE

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Abstract

Bit commitment involves the submission of evidence from one party to another so that the evidence can be used to confirm a later revealed bit value by the first party, while the second party cannot determine the bit value from the evidence alone. It is widely believed that unconditionally secure quantum bit commitment is impossible due to quantum entanglement cheating, which is codified in a general impossibility theorem. In this paper, the scope of this general impossibility proof is extended and analyzed, and gaps are found. Three specific protocols are described for which the entanglement cheating as given in the impossibility proof fails to work. One of these protocols, QBC2, is proved to be unconditionally secure.

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NOTE

(1) In this v7 of the paper, which is really version 3, the two previous versions are subsumed and complete proofs are given for all the claims. The “history” of some of the protocols discussed can be traced from the previous v1-v6 of this paper.

(2) Many of the points made in this version were mentioned in my Capri talk in July 2000. However, the paper prepared for that Proceedings volume, which is available at quant-ph/0009113, concentrates on anonymous-key cryptography with only passing remarks on bit commitment.

(3) The reader interested only in an unconditionally secure quantum bit commitment protocol can go directly form section II to section VI.
I Introduction

Quantum cryptography [1], the study of information security systems involving quantum effects, has recently been associated almost exclusively with the cryptographic objective of key distribution. This is due primarily to the nearly universal acceptance of the general impossibility of secure quantum bit commitment (QBC), taken to be a consequence of the Einstein-Podolsky-Rosen (EPR) type entanglement cheating which rules out QBC and other quantum protocols that have been proposed for various other cryptographic objectives [2].

In a bit commitment scheme, one party, Adam, provides another party, Babe, with a piece of evidence that he has chosen a bit b (0 or 1) which is committed to her. Later, Adam would “open” the commitment: revealing the bit b to Babe and convincing her that it is indeed the committed bit with the evidence in her possession. The usual concrete example is for Adam to write down the bit on a piece of paper which is then locked in a safe to be given to Babe, while keeping for himself the safe key that can be presented later to open the commitment. The evidence should be binding, i.e., Adam should not be able to change it, and hence the bit, after it is given to Babe. It should also be concealing, i.e., Babe should not be able to tell from it what the bit b is. Otherwise, either Adam or Babe would be able to cheat successfully.

In standard cryptography, secure bit commitment is to be achieved either through a trusted third party or by invoking an unproved assumption on the complexity of certain computational problem. By utilizing quantum effects, various QBC schemes not involving a third party have been proposed that were supposed to be unconditionally secure, in the sense that neither Adam nor Babe can cheat with any significant probability of success as a matter of physical laws. In 1995-1996, a general proof on the impossibility of unconditionally secure QBC and the insecurity of previously proposed protocols were described [3]-[6]. Henceforth, it has been accepted that secure QBC and related objectives are impossible as a matter of principle [7]-[8].

There is basically just one impossibility proof, which gives the EPR attacks for the cases of equal and unequal density operators that Babe has for the two different bit values. The proof shows that if Babe’s successful cheating probability $P_c^B$ is close to the value 1/2, which is
obtainable from pure guessing of the bit value, then Adam’s successful cheating probability $P^A_c$ is close to the perfect value 1. This result is stronger than the mere impossibility of unconditional security, namely that it is impossible to have both $P^B_c \sim 1/2$ and $P^A_c \sim 0$. Since there is no known characterization of all possible QBC protocols, logically there can really be no general impossibility proof, strong or not, even if it were indeed impossible to have an unconditionally secure QBC protocol. This problem of scope of the impossibility proof can be seen from the following simple example.

Suppose Adam commits a state $|\phi\rangle$ of a single qubit (two-dimensional quantum state space) for the bit value 0 and $|\phi'\rangle$ for 1. Adam opens by declaring the bit value, and Babe verifies by measuring the corresponding projection, $|\phi\rangle\langle\phi|$ or $|\phi'\rangle\langle\phi'|$. It is intuitively clear, but will be formalized as local state invariance in this paper, that Adam can launch no effective EPR cheat. Of course, it is true in this case that if $P^B_c \sim 1/2$ then $|\langle\phi|\phi'\rangle|^2 \sim 1$, so $P^A_c \sim 1$ simply by declaring the bit value 1 even when $|\phi\rangle$ is committed. However, it is a priori possible for a protocol to have the property that $P^B_c \sim 1/2$ while Adam cannot form any effective cheating entanglement as in this example but with $P^A_c \sim 0$. To have a general impossibility proof, one has to show that this property cannot be obtained in any QBC protocol or that any unconditionally secure QBC protocol would contradict some known principle. The mere absence of counterexamples does not constitute a proof.

The general questions of scope of the impossibility proof will be addressed specifically in Section IV. Three QBC schemes not covered by the impossibility proof will be described in Sections V-VII, although only one of them, QBC2 in Section VI, is proved to be unconditionally secure in this paper. The results are developed within nonrelativistic quantum mechanics, unrelated to relativistic protocols [9] or cheat-sensitive protocols [10]. The essential point is that the flow of classical information between Adam and Babe in the protocol is crucial to the possible operations they can carry out, hence fundamentally affecting the security level of the scheme. In the impossibility proof, it is basically assumed that both Adam and Babe possess full information at each stage of the protocol, an unwarranted assumption.

In Section II the impossibility proof will be reviewed. Since the issues involved in quantum cryptography, or classical cryptography for that matter, are often subtle, it is the policy of this paper to give complete proofs for its claims. Thus, the gap between the quantitative
impossibility claim and the result available in the literature will be filled. An in-principle insecure protocol QBC0 is also described that underlies QBC1 and QBC3. In Section III, the impossibility proof in the original formulation is extended to cover the situation in which Babe applies a superoperator transformation to Adam’s committed state before perfect verification. Another insecure protocol QBC01, related to QBC0, is described as an illustration. The reader who just wants to see an unconditionally secure QBC protocol can go directly from Section II to Section VI. Note that the results in this paper are valid in infinite-dimensional spaces. Also, the same index symbols $i, j$, etc., may denote different quantities in different sections.
II The Impossibility Proof

In this Section we review the standard formulation of the impossibility proof and then recast it in a form more suitable for quantitative analysis and extension, and describe a protocol QBC0. The development of this section will be used in the rest of the paper.

According to the impossibility proof, Adam would generate $|\Phi_0\rangle$ or $|\Phi_1\rangle$ depending on $b = 0$ or $1$,

$$|\Phi_0\rangle = \sum_i \sqrt{p_i}|e_i\rangle|\phi_i\rangle,$$  

$$|\Phi_1\rangle = \sum_i \sqrt{p'_i}|e'_i\rangle|\phi'_i\rangle$$

where the states $\{|\phi_i\rangle\}$ and $\{|\phi'_i\rangle\}$ in $H_B$ are openly known, $i \in \{1, \ldots, M\}$, $\{p_i\}$ and $\{p'_i\}$ are known probabilities, while $\{|e_i\rangle\}$ and $\{|e'_i\rangle\}$ are two complete orthonormal sets in $H_A$.

All Dirac kets are normalized in this paper. Adam sends Babe $H_B$ while keeping $H_A$ to himself. He opens by measuring the basis $\{|e_i\rangle\}$ or $\{|e'_i\rangle\}$ in $H_A$ according to his committed state $|\Phi_0\rangle$ or $|\Phi_1\rangle$, resulting in a specific $|\phi_i\rangle$ or $|\phi'_i\rangle$ on $H_B$, and telling Babe which $i$ he has obtained. Babe verifies by measuring the corresponding projector and will obtain the value 1 (yes) with probability 1. Adam can, as was argued, switch between $|\Phi_0\rangle$ and $|\Phi_1\rangle$ by operation on $H_A$ alone, and thus alter the evidence to suit his choice of $b$ before opening the commitment. In the case $\rho_0^B \equiv \text{tr}_A|\Phi_0\rangle\langle\Phi_0| = \rho_1^B \equiv \text{tr}_A|\Phi_1\rangle\langle\Phi_1|$, the switching operation is to be obtained by using the so-called “Schmidt decomposition [11],” the expansion of $|\Phi_0\rangle$ and $|\Phi_1\rangle$ in terms of the eigenstates $|\hat{\phi}_k\rangle$ of $\rho_0^B = \rho_1^B$ with eigenvalues $\lambda_k$ and the eigenstates $|\hat{e}_k\rangle$ and $|\hat{e}'_k\rangle$ of $\rho_0^A$ and $\rho_1^A$,

$$|\Phi_0\rangle = \sum_k \sqrt{\lambda_k}|\hat{e}_k\rangle|\hat{\phi}_k\rangle, \quad |\Phi_1\rangle = \sum_k \sqrt{\lambda_k}|\hat{e}'_k\rangle|\hat{\phi}_k\rangle$$

By applying a unitary $U_A$ that brings $\{|\hat{e}_k\rangle\}$ to $\{|\hat{e}'_k\rangle\}$, Adam can select between $|\Phi_0\rangle$ or $|\Phi_1\rangle$ any time before he opens the commitment but after he supposedly commits. When $\rho_0^B$ and $\rho_1^B$ are not equal but close, it was shown that one may transform $|\Phi_0\rangle$ by an $U_A$ to a $|\tilde{\Phi}_0\rangle$ with $|\langle\Phi_1|\tilde{\Phi}_0\rangle|$ as close to 1 as $\rho_0^B$ is close to $\rho_1^B$ according to the fidelity $F$ chosen, and thus the state $|\tilde{\Phi}_0\rangle$ would serve as the effective EPR cheat.

In addition to the above quantitative relations, the gist of the impossibility proof is supposed to lie in its generality – that any QBC protocol could be fitted into its formulation,
as a consequence of various arguments advanced in [3]-[8]. Among other reasons to be elaborated in Section IV, it appeared to the present author from his development of a new cryptographic tool, anonymous quantum key technique [12], that the impossibility proof is not sufficiently general. First of all, there is no need for Adam to entangle anything in an honest protocol. When Adam picks b=0, he can just send Babe a state $|\phi_i\rangle$ with probability $p_i$. When he picks b=1, he sends $|\phi'_i\rangle$ with probability $p'_i$. If the anonymous key technique is employed, $|\phi_i\rangle$ and $|\phi'_i\rangle$ are to be obtained from applying $U_i$ or $U'_i$ from some fixed openly known set of unitary operators $\{U_i\}$ and $\{U'_i\}$ on $H_B$ by Adam to the states $|\psi\rangle$ sent to him by Babe and known only to her. As a consequence, Adam would not be able to determine the cheating unitary transformation $U^A$ as in protocol QBC1, to be described in Section V after the impossibility proof is first analyzed generally.

In a QBC protocol, the $\{|\phi_i\rangle\}$ and $\{|\phi'_i\rangle\}$ are chosen so that they are concealing as evidence, i.e. Babe cannot reliably distinguish them in optimum binary hypothesis testing [13]. (The role of quantum detection theory in QBC together with some new results used in this paper are elaborated in Appendix A). They would also be binding if Adam is honest and sends them as they are above, which he could not change after Babe receives them. Babe can always guess the bit with a probability of success $P^B_c = 1/2$, while Adam should not be able to change a committed bit at all. However, it is meaningful and common to grant unconditional security when the best $P^B_c$ Babe can achieve is arbitrarily close to 1/2 and Adam’s best probability of successfully changing a committed bit $P^A_c$ is arbitrarily close to zero even when both parties have perfect technology and unlimited resources including unlimited computational power [4]. To facilitate the quantitative analysis of these performance measures, the impossibility proof would first be reformulated.

Before proceeding, note the following basic property of entanglement important in QBC. Theorem (Local State Invariance): Let $\rho_{AB}$ be a state on $H_A \otimes H_B$ with marginal states $\rho^A \equiv tr_B \rho_{AB}$, $\rho^B$. The individual or combined effects of any state transformation and quantum measurement (averaged over the measurement results) on $H^A$ alone leaves $\rho^B$ invariant.

See Appendix B for a proof and a discussion of its role in the impossibility of superluminal communication via quantum entanglement.

As a consequence of this theorem, Adam cannot cheat by changing the $\rho^B_0 \neq \rho^B_1$ case to
the $\rho_0^B = \rho_1^B$ case whatever the $\rho$’s are. In particular, a single pure state as in the example of Section I cannot be changed.

The operation of unitary transformation with subsequent measurement of an orthonormal basis is equivalent to the mere measurement of another orthonormal basis $\{|\tilde{e}_i\rangle\}$ on the system. Thus, the net cheating operation can be described by writing

$$|\Phi_0\rangle = \sum_i \sqrt{p_i}|\tilde{e}_i\rangle|\tilde{\phi}_i\rangle,$$

(4)

$$\sqrt{p_i}|\tilde{\phi}_i\rangle \equiv \sum_j \sqrt{p_j}V_{ji}|\phi_j\rangle$$

(5)

for a unitary matrix $V$ defined by $|e_i\rangle = \sum_j V_{ij}|\tilde{e}_j\rangle$, and then measuring $|\tilde{e}_i\rangle$. For convenience, we may still in the rest of the paper refer to the cheating operation as a $U^A$ transformation described at the beginning of this Section. Local state invariance is a property complementary to the fact that the $|\tilde{\phi}_i\rangle$ obtainable by operation on $H^A$ alone are some proper linear combinations of the $|\phi_i\rangle$ from (5). The quantitative expression for $P_e^A$ can now be given. If Babe verifies the individual $|\phi'_i\rangle$, the Adam’s successful cheating probability is

$$P_e^A = \sum_i \hat{p}_i |\langle \tilde{\phi}_i|\phi'_i\rangle|^2.$$  

(6)

In general, the optimal cheating probability $\bar{P}_c^B$ for Babe is given by the probability of correct decision for optimally discriminating between two density operators $\rho_0^B$ and $\rho_1^B$ by any quantum measurement. From (A4) with $p_0 = 1/2$,

$$\bar{P}_c^B = \frac{1}{4}(2 + \|\rho_0^B - \rho_1^B\|_1)$$

(7)

where $\| \cdot \|_1$ is the trace norm, $\| \tau \|_1 \equiv tr(\tau^\dagger \tau)^{1/2}$, for a trace-class operator $\tau$ [14]. In terms of a security parameter $n$ that can be made arbitrarily large, the statement of unconditional security (US) can be quantitatively expressed as

(US) \quad \lim_n \bar{P}_c^B = \frac{1}{2} \quad \text{and} \quad \lim_n \bar{P}_e^A = 0.$$

(8)

Condition (US) is equivalent to the statement that for any $\epsilon > 0$, there exists an $n_0$ such that for all $n > n_0$, $\bar{P}_c^B - \frac{1}{2} < \epsilon$ and $\bar{P}_e^A < \epsilon$, i.e. $\bar{P}_c^B - \frac{1}{2}$ and $\bar{P}_e^A$ can both be made
arbitrarily small for sufficiently large \( n \). The impossibility proof claims a lot more than the mere impossibility of (US), it asserts \(^4\) the following statement (IP):

\[
\text{(IP)} \quad \lim_{n} \overline{P}_B^c = \frac{1}{2} \Rightarrow \lim_{n} \overline{P}_A^c = 1.
\]

In the \( \rho_0^B = \rho_1^B \) case, the EPR cheat shows that \( \overline{P}_B^c = \frac{1}{2} \) implies \( \overline{P}_A^c = 1 \). Thus (IP) generalizes it to the assertion that the function \( P_c^A(\overline{P}_B^c) \), obtained by varying \( n \), is continuous from above at \( \overline{P}_B^c = \frac{1}{2} \). Note the considerable difference between the truth of (IP) and the much weaker statement that (US) is impossible. In the middle ground that \( \lim_{n} \overline{P}_B^c = \frac{1}{2} \) implies just \( 0 < \lim_{n} P_c^A < 1 \), the protocol would be concealing for Babe and cheat-sensitive for Adam.

The key observation \(^3\)-\(^4\) in the proof of (IP) is the use of Uhlmann’s theorem, that there exist purifications \( |\Phi_0\rangle \) and \( |\Phi_1\rangle \) of any given \( \rho_0 \) and \( \rho_1 \) such that \( |\langle \Phi_0|\Phi_1\rangle|^2 \) attains the maximum possible value given by \( F(\rho_0, \rho_1) \equiv |\text{tr}\sqrt{\sqrt{\rho_0, \rho_1} \sqrt{\rho_0}}|^2 \). The conclusion is drawn, without supporting details, that if \( F(\rho_0^B, \rho_1^B) \) is close to 1, then so is \( \overline{P}_c^A \). This conclusion can be related to (IP) via the bound \(^{15}\)

\[
2\left[1 - \sqrt{F(\rho_0, \rho_1)}\right] \leq \|\rho_0 - \rho_1\|_1.
\]

as follows. Let \( \|\rho_0^B - \rho_1^B\|_1 \leq \epsilon \), so that \( F(\rho_0^B, \rho_1^B) \geq (1 - \frac{\epsilon}{2})^2 \) from \(^{14}\). From Uhlmann’s theorem, choose \( |\Phi_0\rangle \) and \( |\Phi_1\rangle \) of (1)-(2) to be the purifications that achieve the maximum \( F(\rho_0^B, \rho_1^B) \) so that \( |\langle \Phi_0|\Phi_1\rangle| \geq 1 - \frac{\epsilon}{2} \). The cheating operation on \( |\Phi_0\rangle \) is given by (5), and Adam’s successful cheating probability is given by the following

\textbf{Lemma 1:} For probabilities \( \alpha_i \) and complex numbers \( \lambda_i \),

\[
\sum_i \alpha_i |\lambda_i|^2 \geq \left| \sum_i \alpha_i \lambda_i \right|^2
\]

(the sums can be over infinite sets).

\textit{Proof.} When \( \lambda_i \) are real, \(^{11}\) follows from Jensen’s inequality \(^{16}\) and the concavity of the function \( x \mapsto x^2 \). The complex case follows by expanding each \( \lambda_i \) into real and imaginary parts. \( \square \)

Since \( \langle \Phi_0|\Phi_1\rangle = \sum_i \sqrt{p_i p_i'} \langle \tilde{\phi}_i|\phi_i'\rangle \), it follows from \(^{11}\) with \( \lambda_i = \sqrt{p_i p_i'} \langle \tilde{\phi}_i|\phi_i'\rangle \) (no need to include the \( p_i' = 0 \) terms) and \( \alpha_i = p_i' \) that \( \overline{P}_c^A \geq 1 - \epsilon \) whenever \( \overline{P}_c^B \leq \frac{1}{2} + \frac{\epsilon}{4} \). Thus, the
statement (IP) is proved. In particular, one has the convergence rate

$$P^B_c = \frac{1}{2} + O\left(\frac{1}{n}\right) \Rightarrow P^A_c = 1 - O\left(\frac{1}{n}\right).$$

(12)

As an illustration, consider the following protocol, in which hashing via the parity function is used to make $\rho^B_0$ close to $\rho^B_1$ in a sequence generalization of the example in Section I. 

**PROTOCOL QBC0:**

(i) Adam sends Babe a sequence of $n$ qubits, each is either one of $\{|\phi\rangle, |\phi'\rangle\}$, such that an even number of $|\phi'\rangle$ corresponds to $b = 0$ and an odd number to $b = 1$, with probability $1/2^{n-1}$ for each sequence of either parity.

(ii) Adam opens the commitment by revealing the sequence of $n$ states. Babe verifies by measuring the corresponding projection on each qubit to see that the entire sequence is correct.

To show that this scheme can be made concealing, note that $\rho^B_0 - \rho^B_1$ factorizes into products of individual qubit parts as follows. Let $j = (j_1, \ldots, j_n) \in \{0,1\}^n$, $P_{l0} = |\phi\rangle\langle\phi|$, $P_{l1} = |\phi'\rangle\langle\phi'|$, $l \in \{1, \ldots, n\}$. Let $\Lambda_0 = \{j| \bigoplus_{l=1}^n j_l = 0\}$, $\Lambda_1 = \{j| \bigoplus_{l=1}^n j_l = 1\}$ be the even and odd parity $n$-bit sets. Then

$$\rho^B_i = \frac{1}{2^{n-1}} \sum_{j \in \Lambda_i} \bigotimes_{l=1}^n P_{lj}, \quad i \in \{0,1\}$$

(13)

and so

$$\rho^B_0 - \rho^B_1 = \frac{1}{2^{n-1}} \bigotimes_{l=1}^n (P_{l0} - P_{l1}).$$

(14)

Thus, Babe’s optimum quantum decision reduces to optimally deciding between $|\phi\rangle$ and $|\phi'\rangle$ for each qubit individually and then seeing whether there is an even or odd number of $|\phi'\rangle$’s. The optimum error probability $p_e$ for each qubit is given in (A5), and the optimum error probability $\bar{P}_c^B$ of correct bit decision on the sequence is, from the even and odd binomial sums (cf. Appendix C),

$$\bar{P}_c^B = \frac{1}{2} + \frac{1}{2}(1 - 2p_e)^n.$$  

(15)

Thus, $\bar{P}_c^B$ is close to $\frac{1}{2}$ exponentially in $n$ independently of $\frac{1}{2} \geq p_e > 0$. However, Adam can now cheat by forming entanglement as in (1)-(2), with $\bar{P}_c^A$ exponentially close to 1 in accordance with (IP).
An Extension of the Impossibility Proof

In this Section, a protocol QBC01 will be described in which Babe introduces a lossy transformation on Adam’s committed state while still being able to verify perfectly. While it may be argued that such transformation cannot succeed in obtaining a secure protocol on qualitative grounds, it may also be argued otherwise. Specifically, the coherence of the states (1)-(2) can be deliberately destroyed by Babe with such a CP map, reducing the entangled states to incoherent superpositions in her observation space. It turns out that if she does that, which she can emphatically do, the resulting condition on the number \( n \) of modes would not fit with the other requirements of the protocol. Indeed, the impossibility proof will be extended to cover all such possibilities of Babe introducing a CP-map transformation.

The following protocol is closely related to QBC0.

PROTOCOL QBC01.

(i) Adam sends Babe a sequence of \( n \) states \( |\alpha_i\rangle \in \mathcal{H}_B \), \( \mathcal{H}^B = \bigotimes_l \mathcal{H}_B^l \), each \( |\alpha_i\rangle \) being either one of two coherent states \( \{|\alpha\rangle, |\alpha'\rangle\} \), such that an even number of \( |\alpha'\rangle \) corresponds to \( b = 0 \) and an odd number to \( b = 1 \), with probability \( 1/2^{n-1} \) for each sequence of either parity.

(ii) Babe splits each state \( |\alpha_i\rangle \) to \( |\sqrt{\eta}\alpha_i\rangle \) on \( \mathcal{H}_B^l \), \( \eta < 1 \).

(iii) Adam opens the commitment by revealing the sequence of \( n \) states. Babe verifies by measuring the corresponding projection on each \( |\sqrt{\eta}\alpha_i\rangle \) to see that the entire sequence is correct.

The cheating transformation on this protocol would produce from (5) a superposition of coherent states with large energy difference when \( |\alpha - \alpha'| >> 1 \). As explained in version 2 of this paper (v4-v6), such superpositions are supersensitive to loss [17]-[19], thus offering the possibility that a lossy transformation, which would not affect perfect verification on coherent states, would destroy the necessary entanglement for Adam to cheat successfully. However, in this multimode situation, in order to destroy the coherence one needs to have a loss of one photon per mode, not just one photon, and the protocol cannot be made secure. Indeed, this condition on the destruction of coherence is what makes fault-tolerant quantum computing in the presence of loss possible.
We now give the impossibility proof that defeats such a maneuver by Babe. Let $\mathcal{J}_B$ be any completely positive (CP) map (superoperator) on density operators introduced by Babe. Let $X_{i1}^B$ be the measurement operator that perfectly verifies the $b = 1$ case given $i$, i.e. $X_{i1}^B$ is the $\Pi_1$ operator part of a POM for the “1” or “not 1” decision in quantum hypothesis testing as described in Appendix A, with perfect verification corresponding to the condition

$$\text{tr}X_{i1}^B \mathcal{J}_B |\phi_i'\rangle\langle\phi_i'| = 1. \quad (16)$$

The $P_A^c$ then becomes

$$P_A^c = \sum_i \tilde{p}_i \text{tr}X_{i1}^B \mathcal{J}_B |\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|. \quad (17)$$

The following lemma and all other results in this paper are valid in infinite-dimensional spaces.

**Lemma 2** [14]: For any bounded operator $X$ and any trace-class operator $\tau$,

$$|\text{tr}X \tau| \leq \|X\| \|\tau\|_1, \quad (18)$$

where $\|\cdot\|$ is the ordinary operator norm.

Since $\|X_{i1}^B\| \leq 1$, from (18) we get

$$P_A^c \geq 1 - \sum_i \tilde{p}_i \|\mathcal{J}_B (|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i| - |\phi_i'\rangle\langle\phi_i'|)\|_1. \quad (19)$$

From the original $P_A^c = \sum_i \tilde{p}_i |\langle\tilde{\phi}_i|\phi_i'\rangle|^2 \geq 1 - \epsilon$ for $\|\rho_0^B - \rho_1^B\| \leq \epsilon$ proved in Section II, one obtains, by relating inner product and trace norm for pure states as in (A4)-(A5),

$$\sum_i \tilde{p}_i \| |\tilde{\phi}_i\rangle\langle\tilde{\phi}_i| - |\phi_i'\rangle\langle\phi_i'|\|_1^2 \leq 4\epsilon. \quad (20)$$

The following theorem is actually valid for any positive trace-preserving map $\mathcal{J}$.

**Theorem 20**: 

$$\|\mathcal{J}(\rho_0 - \rho_1)\|_1 \leq \|\rho_0 - \rho_1\|_1. \quad (21)$$

From (20) and (21), $\sum_i \tilde{p}_i \|\mathcal{J}_B (|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i| - |\phi_i'\rangle\langle\phi_i'|)\|_1^2 \leq 4\epsilon$ and, using (11),

$$\sum_i \tilde{p}_i \|\mathcal{J}_B (|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i| - |\phi_i'\rangle\langle\phi_i'|)\|_1 \leq 2\sqrt{\epsilon}. \quad (22)$$
Putting (22) into (19) yields $P_c^A \geq 1 - 2\sqrt{\epsilon}$, completing the proof of (IP). It appears that the use of the trace norm cannot be avoided here, in contrast to the $J_B = I^B$ case, which is responsible for the weakening of the $\bar{P}_c^A$ convergence rate from $1 - O\left(\frac{1}{n}\right)$ to $1 - O\left(\frac{1}{\sqrt{n}}\right)$.

The perfect verification condition (16), preserved in protocol QBC01, is not necessary for a secure QBC protocol. This point and the entanglement destruction strategy of protocol QBC01 will be exploited in protocol QBC3 of Section VII. These possibilities also suggest that it is now appropriate to examine the assumptions underlying the impossibility proof.
IV The Limited Scope of the Impossibility Proof

The generality of the scope of the impossibility proof is analyzed in this section on general grounds. This is an important issue because unconditionally secure bit commitment is too useful to give up easily, and the available impossibility proof has many weaknesses that can be exploited for secure QBC protocols. At the very least, one may hope that hidden assumptions, perhaps practically valid, may be revealed. Indeed one such assumption is that the quantum and classical communications involved are over a perfect channel, which should be considered different from the assumption that the parties have perfect technology. This is a good assumption for some situations, but not others such as long-distance fiber-optic communications. Another example in which this assumption is not valid involves satellite-to-satellite optical communications where the receivers’ fields of view have to be opened up, perhaps because the signals are deliberately spread, so that the sun’s background contributes a significant amount of noise. In both of these cases, one can stretch the meaning of “perfect technology” to say that no unavoidable classical disturbance needs to be present – say, by throwing the sun to another galaxy. (And what about the cosmic background radiation?) But then the relevance of such results to reality is quite questionable. In this paper, a perfect channel is granted. Since it is widely believed that there is a complete impossibility proof in such a case, I would try to show otherwise independently of the protocols of the paper.

The major problem is, of course, to decide whether the formulation given in [3]-[8] is sufficiently broad to include all possible QBC protocols. Typically, one proves general impossibility by showing that any concretely suggested possibility would lead to a contradiction. The simplest example is that the possibility of superluminal communication via quantum entanglement would contradict local state invariance (cf. Appendix B). Another example would be the quantum no-clone theorem, where cloning contradicts unitarity on a sufficiently large Hilbert space [21] as well as quantum detection theory (cf. Appendix A). In von Neumann’s famous no-hidden-variable theorem [22], a contradiction is derived from what he considered to be the requirements for a hidden-variable theory. Perhaps more significant and illuminating is the impossibility proof of certain geometric constructions by straight-edge and compass developed in the first half of the nineteenth century, in which any such
construction is characterized by the membership of a certain number lying in a tower of quadratic extension fields \[23\]. This example is significant because it is nontrivial to capture enough of the essence of any straightedge-and-compass construction to be able to produce a mathematical contradiction when the construction is impossible. Thus, for a general impossibility proof of unconditionally secure QBC, one would expect that the general essence of any such protocol would be extracted to yield a contradiction. Clearly the impossibility proof does not do that, but rather relies on the claim that any possible QBC protocol can be reduced to its formulation. It is not a priori impossible to exhaustively describe and classify all operations of a certain kind, say, in quantum key distribution one typically characterizes all possible attacks Eve can launch. However, it is much more difficult to characterize all possible protocols than all possible attacks for any cryptographic objective because an arbitrary interactive flow of information between users is possible in a protocol. Indeed, no characterization of all protocols for a specific objective is known in standard (classical) cryptography. The scope problems of the impossibility proof are numbered as follows.

1. One justification for the all-encompassing nature of the formulation is that Adam is proceeding exactly as if he were honest, except right before opening, in carrying out his EPR cheat. This is not true because there is no need for him to entangle anything in an honest protocol. He can just pick a \( |\phi_i \rangle \) or \( |\phi_i' \rangle \) and send it.

2. Because of this, it is not clear why Adam must be able to form the entanglement he needs for any possible protocol.

3. Furthermore, it is not clear why Adam must be able to determine the cheating transformation, even apart from complexity questions, for any possible protocol. Protocol QBC1 of section V provides a direct challenge in this situation, while protocol QBC2 of Section VII can also be considered to pose this problem.

4. The formulation postpones any measurement to the end of the commitment phase and claims that it entails no loss of generality. But why wouldn’t it affect the quantitative cheating probabilities? Protocol QBC2 provides an example in which the timing of the measurement has substantial consequence.

5. The density operators \( \rho_0^B \) and \( \rho_1^B \) for Babe are not necessarily the marginal states obtained from the states generated by Adam because of Babe’s possible lack of information,
a situation that is built into the protocol. Thus, Adam’s EPR cheat may not correspond to the $\rho_0^B \sim \rho_1^B$ case. An example is provided by QBC2.

(6) It is clearly possible to avoid EPR cheats, as in the example described in Section I. While (IP) holds in this case, it holds not because of EPR cheats. The question is: why is it that an EPR-cheat-free protocol necessarily cannot satisfy (US)? Protocol QBC2 is an explicit example that (US) is possible in such a protocol.

(7) It is not clear why perfect verification is necessary, the only performance measures here being the cheating probabilities. This freedom in a QBC protocol is exploited in QBC3.

(8) It is not clear why Babe is necessarily unable to destroy Adam’s entanglement by her action alone. Despite the failure of QBC01 of Section III, this possibility is manifested in protocol QBC3.

The list could be continued. Note that the burden is on the impossibility proof to resolve these points in its favor with convincing arguments, which have not been provided. Indeed, all three protocols QBC1 to QBC3, and even protocol QBC01 to a lesser extent, lie outside the framework of the impossibility proof, and no impossibility argument has been given for this kind of protocols. While there are various underlying reasons on the limited scope of the impossibility proof formulation, a major one is that the interactive flow of information between Adam and Babe may prevent cheating because of each party’s lack of relevant information at any particular stage of the protocol. Such information flow is what makes the Yao model of two-party protocols [24] not sufficiently specific to characterize all QBC protocols, which he did not claim to have done. Furthermore, modification of the Yao model to have measurements at the end of the commitment phase, perhaps thought to be equivalent by the Lo-Popescu theorem [25], is not justified with the use of anonymous states because the state needs to be known to guarantee the validity of that theorem. The basic problem of a general impossibility proof lies in the characterization of the essence of any possible QBC protocol that makes it insecure. The information flow problem that makes it so difficult to characterize all classical protocols surely carries over to the quantum domain.

There are well-known and widely accepted claims in the literature [26]-[27] that classical noisy channels would make unconditionally secure bit commitment possible. While I believe the specific protocols described in [26]-[27] are not proved unconditionally secure, I also be-
lieve unconditionally secure ones can indeed be based on noisy channels, a subject to be discussed elsewhere. Such results are not considered to be contradictory to the QBC impossibility proof presumably for the following reasons. First, classical noise is often thought to be part of an imperfect channel, i.e. it does not have to be present in principle. Apart from the points made at the beginning of this section, such a viewpoint is not correct. The quantum noise in any given quantum signaling scheme for classical communication, the minimum amount of which is determined through the optimum quantum measurement via quantum detection theory, is in principle unavoidable and functions exactly like classical noise in the optimal quantum detector [28]-[29]. As will be shown elsewhere, this crucial point opens up the possibility of developing unconditionally secure, practical, and efficient optical-speed cryptographic systems for all the standard cryptographic objectives via quantum states that are not superpositions of one another. Secondly, a truly classical noise system would not entail the possibility of quantum entanglement and EPR cheating. However, there are many ways to suppress EPR cheats, such as the example in Section I and the QBC2 in Section VI. While it is not easy to restore unconditional security with such suppression in a perfect channel, a noisy channel, even one created with quantum noise, would provide a powerful way for such restoration. Indeed, the development of such protocols will be the subject of a future treatment.
V Protocol QBC1

In this Section we consider the use of anonymous states in a QBC protocol which is essentially the one in version 1 (v1-v3) of this paper. In this protocol QBC1, the bit value is encoded in the parity of a sequence as in QBC0 of Section II, except that each individual state is obtained with Adam applying the openly known $U_0$ or $U_1$ to the states $|\psi\rangle$ sent to him by Babe, corresponding to the 0 or 1 bit position in the sequence. For example, $|\psi\rangle$ could be any state on a fixed great circle of the Bloch-Poincaré sphere of a qubit, with $U_0 = I$ and $U_1$ being a rotation by a fixed angle on the great circle independently of the bit position, say with $\langle \psi | U_1^\dagger U_0 | \psi \rangle = \lambda > 0$. See Ref. [12] for further discussion of anonymous-key cryptography. Coherent-state implementation is also possible, as in QBC01.

**PROTOCOL QBC1:**

(i) Babe sends Adam a sequence of $n$ qubit states $|\psi_l\rangle \in \mathcal{H}_B^l$, $\mathcal{H}_B^l = \bigotimes_l \mathcal{H}_B^l$, $l \in \{1, \ldots, n\}$, unknown to Adam.

(ii) Adam commits via the parity of the sequence $j = (j_1, \ldots, j_n) \in \{0, 1\}^n$ by applying $U_{lj_l}$ to $|\psi_l\rangle$ for openly known $U_{l0}$ and $U_{l1}$, with $\langle \psi_l | U_{l1}^\dagger U_{l0} | \psi_l \rangle = \lambda > 0$ independently of $l$.

(iii) Adam opens by revealing his $j$ sequence. Babe checks every state $U_{lj_l}|\psi_l\rangle$.

This scheme can be made concealing exactly as in QBC0, (14)-(15). As for its binding behavior, consider first the situation in which Adam can only entangle each qubit individually. He cannot switch any committed $U_{l0}|\psi_l\rangle$ or $U_{l1}|\psi_l\rangle$ to any other state due to local state invariance which applies to each of the states he sends separately for that state, expressing the obvious fact that there is no entanglement to a single state. If he were to entangle $U_{l0}|\psi_l\rangle$ or $U_{l1}|\psi_l\rangle$ to another state anyway, he would just present a mixed state for that qubit to Babe for that $j_l$. In this case, a different criterion needs to be used as discussed below. If he sticks to committing first a correct state for the bit, the best cheating probability he can get it

$$P_c^A = \left| \langle \psi_l | U_{l1}^\dagger U_{l0} | \psi_l \rangle \right|^2 = \lambda^2$$

by generating any sequence of $n - 1$ states, picking the last one for the bit commitment, and declaring it to be otherwise when desired. From (15) and (23), one can make $P_c^A = O(m^{-1})$ and $\bar{P}_c^A - 1/2 = O(2^{-m})$ with $n = O(m^2)$. Hence unconditional security is obtained for
large $m$ if (23) is indeed the overall best Adam can do. In addition to $P^A_c$, one can use another criterion, $P^A_a$, the average probability that Adam’s committed evidence is accepted by Babe after he opens, which is always at least $1/2$ similar to $P^B_c$ with $P^A_a = (1 + P^A_c)/2$ when (1) is used as an initial state $|\Phi_0\rangle$ by Adam. For a general $|\Phi_0\rangle$, (6) can be simply generalized to give an expression for $P^A_a$ with optimization for $P^A_a$ to be performed also over initial $\{\phi^0_i\}, \{p^0_i\}$. In the present situation, since single-qubit entanglement by Adam would just lead to a mixed presented state from local state invariance, $P^A_a$ is obtained by a fixed $|\phi^0_i\rangle = |\phi^0\rangle$ with $P^A_a = (1 + \lambda)/2$. Thus, $P^B_c - \frac{1}{2} = O(2^{-m})$ and $P^A_a - \frac{1}{2} = O(m^{-1})$ are achieved for $n = O(m^3)$.

Adam can, however, form the entanglement without knowing the $|\psi_i\rangle$’s, by applying the unitary operator $U$ on $H^A \otimes H^B$,

$$U = \sum_i |e_i\rangle \langle e_i| \otimes U_i$$

with initial state $|A\rangle \in H^A$ satisfying $\sqrt{p_i} = \langle e_i|A\rangle$, as was indicated in version 1 of this paper. On the other hand, contrary to the claim in that version, Adam can also entangle qubit by qubit via, for each $i = (i_1, \ldots, i_n)$ in (25),

$$U_i = \bigotimes_l U_{l_{i_l}} = (I_1 \otimes \ldots \otimes U_{n_{i_{n_i}}} \ldots (U_{i_{i_1}} \otimes \ldots \otimes I_n).$$

By applying (24)-(25), Adam can form the proper entangled state (1) or (2) without knowing the $|\psi_i\rangle$’s. However, he cannot determine the cheating transformation $U^A$. In general such a cheating transformation for the $\rho^B_0 \neq \rho^B_1$ case is determined by Uhlmann’s theorem as follows [31].

Let $|\lambda_i\rangle$ and $|\mu_i\rangle$ be the eigenstates of $\rho^B_0$ and $\rho^B_1$ with eigenvalues $\lambda_i$ and $\mu_i$. The Schmidt normal forms of the purifications $|\Phi_0\rangle$ and $|\Phi_1\rangle$ of $\rho^B_0$ and $\rho^B_1$ are given by

$$|\Phi_0\rangle = \sum_i \sqrt{\lambda_i} |f_i\rangle |\lambda_i\rangle,$$

$$|\Phi_1\rangle = \sum_i \sqrt{\mu_i} |g_i\rangle |\mu_i\rangle$$

for complete orthonormal sets $\{|f_i\rangle\}$ and $\{|g_i\rangle\}$ on $H^A$. Define the unitary operators $U_0, U_1$ and $U_2$ by

$$U_0 |\lambda_i\rangle = |\mu_i\rangle.$$
\[ U_1 |\lambda_i \rangle = |f_i \rangle, \quad (29) \]
\[ U_2 |\mu_i \rangle = |g_i \rangle. \quad (30) \]

Since one can always pick \( \mathcal{H}^A \) to be isomorphic to \( \mathcal{H}^B \), one can identify them via the isomorphism. Let \( U \) be the unitary operator for the polar decomposition of \( \sqrt{\rho_0^B} \sqrt{\rho_1^B} \) \[ \sqrt{\rho_0^B} \sqrt{\rho_1^B} = \left| \sqrt{\rho_0^B} \sqrt{\rho_1^B} \right| U. \quad (31) \]

Then \( |\langle \Phi_0 | \Phi_1 \rangle|^2 \) assumes its maximum value \( F(\rho_0^B, \rho_1^B) \) when
\[ U U_2^T U_0^T U_1^T = I \quad (32) \]

where \( T \) denotes the transpose operation. Thus, when \( \rho_0^B, \rho_1^B \), and \( |e_i \rangle \) are given, \( |g_i \rangle = |e'_i \rangle \) of \( |\Phi_1 \rangle \) is determined from (30) via solving for \( U \) from (32), which required detailed explicit knowledge of \( \rho_0^B \) and \( \rho_1^B \). In terms of the notation for (13)-(14), the density operators are
\[ \rho_i^B = \frac{1}{2^{n-1}} \sum_{j \in \Lambda_i} \otimes_{l=1}^{n} U_{lj} |\psi_l \rangle \langle \psi_l | U_{lj}^* \quad i \in \{0, 1\}, \quad (33) \]

which is unknown to Adam through the \( |\psi_l \rangle \) uncertainty. If Adam picks a cheating transformation for a particular \( |\psi_l \rangle \) sequence, and then the \( |\psi_l \rangle \) sequence is randomly varied, it is easily seen that the resulting \( P^A_c \) can be very small, as e.g. when the corresponding odd-parity state is actually of even parity. However, it is not easy to develop an unconditional security proof because Adam has many other possible actions, including committing states which are not exactly correct for the bit value as mentioned above. Nevertheless, the protocol clearly shows in a simple way that the impossibility proof fails to work as intended.

Note that this anonymous-key strategy also works in the case \( \rho_0^B = \rho_1^B \) if \( \rho_0^B \) is not highly degenerate, e.g., not proportional to the identity \( I^B \), such that its eigenstates cannot be readily determined as in the case of (33). Indeed, for \( n = \infty \) the \( \rho_0^B \) and \( \rho_1^B \) from (33) are equal and not proportional to \( I^B \). Note that the strategy of this protocol, namely the use of anonymous states, is applicable to any QBC protocol, and will be employed next for protocol QBC2.
VI Protocol QBC2

In this section, protocol QBC2 is developed with a complete unconditional security proof by exploiting the following point: the states $\rho_0^B$ and $\rho_1^B$ that enter into (7) are not necessarily the marginal states obtained from (1)-(2) due to Babe’s lack of information built into a QBC protocol. This situation is actually easy to obtain, but then Adam can usually cheat successfully with this information. The anonymous-key technique can be utilized to prevent both Adam and Babe from cheating to yield an unconditionally secure protocol to be explained in successive steps as follows.

In anonymous-key encryption [12], Babe transmits to Adam a state $|\psi\rangle$ only known to herself. Adam sends a bit $b$ back to Babe via modulating $|\psi\rangle$ by openly known unitary operators $U_b$. For the present purpose, the following would suffice – $|\psi\rangle \in S_0$ is one of the four possible BB84 states of a qubit, $S_0 = \{|\uparrow\rangle, \mid \rightarrow\rangle, \mid \nearrow \rangle, \mid \swarrow \rangle\}$, (e.g. the vertically, horizontally, and diagonally polarized states). Adam sends back $U_b|\psi\rangle$ with $U_0 = I^B$ and $U_1$ being a clockwise rotation by $\pi/2$ on the polarization circle, so that Babe can always tell the bit from the state. Let her send Adam a set $S$ of the above four different states on four qubits in a random order known only to herself, with each state named by its order. Thus, $S = \{\mid \lambda_1\rangle_1, \mid \lambda_2\rangle_2, \mid \lambda_3\rangle_3, \mid \lambda_4\rangle_4\}$ where the subscript $j$ on $\mid \rangle_j$ denotes the name of the state and $\{\lambda_j\}$ is a random permutation of the set $S_0 = \{\uparrow, \rightarrow, \nearrow, \swarrow\}$. Adam picks randomly one of these four named states in $S$, keeping the name to himself, modulates it and sends it to Babe as the commitment. For example, he chooses $|\lambda_2\rangle_2$ with subscript 2 on $\mid \rangle_2$ known to him, rotates $\lambda_2$ unknown to him clockwise by $\pi/2$ for $b = 1$, and sends it back to Babe who does not know the state name ”2” yet. He opens by revealing the state name and the bit value. Without knowing the state name, it is easy to check that $\rho_0^B = \rho_1^B = I^B/2$ for Babe. When she learns the state name from Adam’s opening, she knows the corresponding state for each bit value and can verify by measuring the corresponding projection. The actual permutation of the $S_0$-states in $S$ has to be hidden from Adam because if he knows, he can cheat by committing any state in $S$ and announcing it to be another appropriate state from $S$.

Consider first Adam’s possibility of cheating. When he picks a specific named state $|\lambda_j\rangle_j$,
he cannot apply the EPR cheat as a consequence of local state invariance or the fact that there is no entanglement for a single state. He can announce a different name of the state from the one he actually sent, with a probability of successfully reversing $b$ (i.e. getting it accepted by Babe in her verification) given by $3/4$. He can use his own state instead of the one sent by Babe; the best way to do that is by trying to determine which name corresponds to which state in $S$ by optimally processing the set $S$ from $M$-ary quantum detection theory (cf. Appendix A). In each case he attains a probability of success bounded away from zero. Let $p_A$ be his maximum probability of success, which is determined by the optimal $M$-ary quantum detector because his openings amount to a decision making that consists in matching each $\lambda_j$ with an element of $S_0$. The exact value of $p_A$ is not relevant for the security proof of our final protocol. The only relevant point here is that $p_A$ is a fixed number less than one. Hence, in an independent $m$-sequence, his probability of successful cheating, $\bar{P}_c^A = p_A^m$, goes to zero exponentially in $m$.

To show that $p_A < 1$, assume that Adam can cheat perfectly with $p_A = 1$. This implies that he can determine $\lambda_j$ for each $|\lambda_j\rangle_j$ from the set $S$ with certainty without knowing the random permutation. However, the different possible permutations yield nonorthogonal (mixed) states on the different qubit sets. By Theorem A2 in Appendix A, $p_A = 1$ is impossible. Indeed, the optimum $p_A$ is a fixed number bounded away from zero, not arbitrarily small in a parameter $n$ that grows with the number of such randomly permuted four-state sets.

It is possible for Adam to consider EPR cheats by permuting the contents of the states to be used later with a single qubit while keeping track of the state name. In this way, he can form the entanglement (1)-(2), but he cannot transform one into the other without knowing the specific permutation of the $\lambda_j$ in the set $S$ presented to him. And, of course, if he knows the permutation, he can cheat by proper announcement without the need for entanglement. Note also that if he can entangle and transform without knowing the actual permutation, local state invariance would be violated by permuting the states back to the given order. Indeed, this and all other possible attacks by Adam are accounted for in the above argument that $p_A < 1$ holds in any of Adam’s possible cheating schemes as a consequence of optimum quantum detection theory.
The only way that Babe can cheat is to send Adam a different set $S'$ of states, e.g. the same polarization state on the polarization circle for all four qubits, which would yield $\bar{P}_c^B = 1$. This is to be prevented statistically via testing by having Babe send Adam a total number of $n$ sets of $S$-states, all named by their order. Consider first the case in which Adam only commits a single qubit, and Babe sends a total of $4n$ states $|\lambda_{jl}\rangle$, $j \in \{1, \ldots, n\}$, $l \in \{1, \ldots, 4\}$. If Babe is honest, then, for each $j$, $\{|\lambda_{jl}\rangle\}$ is a random permutation of $S_0$. To prevent Babe from cheating, Adam would randomly set aside one set $j_0$ and ask Babe for the state identities in the other $n-1$ sets. After Babe reveals the state identities from their names provided by Adam, he can verify that Babe indeed sent him sets of proper states and proceed to pick one from the $j_0$ set to commit his bit. If Babe sends a set $S' = \{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$, which is not a random permutation of $S_0$, then there is a probability $p_1$ that it will pass Adam’s testing verification,

$$p_1 = |\langle 1 | \uparrow\rangle|^2 \cdot |\langle 2 | \rightarrow\rangle|^2 \cdot |\langle 3 | \nearrow\rangle|^2 \cdot |\langle 4 | \swarrow\rangle|^2,$$  \hspace{1cm} (34)

and a corresponding optimum probability $p_2$ that Babe can determine the bit knowing the qubit is from $S'$. The value of $p_2$ is determined by the optimum binary quantum detector. For example, if Babe sends all states at the angle $\pi/8$ from $\uparrow$, $p_1 = \cos^2 \frac{\pi}{8} \cos^2 \frac{\pi}{8} \cos^2 \frac{3\pi}{8} \sin^2 \frac{\pi}{8}$ and $p_2 = 1$. As far as the existence of an unconditionally secure protocol is concerned, the only thing we need to know is that $p_1 = 1$ implies $p_2 = \frac{1}{2}$ from (34). It is clear that any entanglement used by Babe on the state she sent would not help her cheat, because Adam is doing everything on the individual qubit level determined by the individual marginal qubit states. Indeed, Babe’s entanglement would only make $p_2$ smaller. For Babe to get $\bar{P}_c^B$ away from $\frac{1}{2}$, she needs to send state sets with $p_2 = \frac{1}{2} + \epsilon$ where $\epsilon$ is bounded away from zero, i.e. not arbitrarily small as a function of $n$, and send enough of them so that the chance that one of them is picked as $j_0$ by Adam is also not arbitrarily small for large $n$. In such a situation where Adam retains one of Babe’s cheating state sets which constitute a nonzero fraction $\gamma$ of the total number $n$, the probability that Babe’s cheating would not be found out is $p_1^{n-1}$, assuming Adam indeed sets aside one of the cheating state sets, which goes to zero exponentially. This argument is essentially correct and will be presented rigorously in the more general situation of the protocol in the following. Here we tried to indicate the simple
intuitive picture of the situation, and the fact that our scheme so far already contradicts the (IP) statement, although it falls short of the (US) statement. It should be evident that regardless of whether (US) can be obtained in this kind of schemes, the are not covered by the formulation of the impossibility proof.

Were Babe found to be cheating, the protocol would of course abort, which is equivalent to one party aborting in the middle of any protocol, something each party can always choose to do. Thus, our scheme is no different in this respect from any other cryptographic protocol and is essentially different from the cheat-sensitive QBC protocols [10] in that it has nothing to do with detecting possible cheating by Adam and Babe after Adam commits as prescribed in the definition of cheat-sensitive protocols. Indeed, Adam can discover the cheating before he commits the bit. Even though he could postpone the cheating detection measurement in our protocol, such a move would have betrayed his bit to Babe, cf. point (4) in Section IV. More significantly, the cheating probabilities were not quantified precisely in Ref. [10] – presumably if the successful cheating probability is bounded away from zero, then so is the cheat-detection probability. In the present case, arbitrarily small successful cheating probabilities can be obtained in the next protocol, the parameters \( n, m \) of which are determined as shown in the following security proof.

**PROTOCOL QBC2:**

(i) Babe sends Adam \( n \) sets of qubit states, each set a random permutation of the four BB84 states on four different qubits, in a random order only known to herself. The states are named by their order in the sequence.

(ii) Adam randomly puts \( m \) sets of such states aside and asks Babe to identify the rest of the states from their names. After checking that the states are correct, he commits the bit by picking one state randomly out of each of the \( m \) sets, modulates them by the same \( U_b \), and sends them to Babe.

(iii) Adam opens by revealing the names of the states he sent and the bit value. Babe verifies by measuring the corresponding projections.

It should be clear that no entanglement cheating would be effective in this protocol: as discussed above, entanglement cheating by Adam or Babe serves no purpose as the qubits are processed individually. For each bit value Adam commits, there is only one product marginal
state for Babe and thus no cheating transformation for Adam. If Adam entangles anyway, he would merely send back mixed marginal states to Babe as she verifies on individual qubits. If he does not commit a correct state as discussed after (23) in Section V, it merely changes $p_A$, the optimum value of which is not 1 as shown above. If Babe entangles anyway, she would just get back mixed states for herself. Consequence of such a situation, however, is also covered in the following. Similarly, introducing any classical correlation would serve no purpose. The protocol is binding because Adam’s $\bar{P}_c^A = p_A^m \to 0$ for large $m$. It is concealing basically for the same reason as the single-qubit case, a systematic proof given as follows.

Let $N$ be the number of state sets Babe sends to Adam with probabilities $p_1$ of passing Adam’s detection, $p_1 < 1$ with corresponding $p_2 > \frac{1}{2}$. Consider first the case in which these probabilities are uniform among the $N$ sets so that Babe can have the best possible $p_2$ given $p_1$ among the $m$ different committed qubits. The other $n - N$ sets have $p_1 = 1$ and $p_2 = \frac{1}{2}$. The probability that $k$ of these $N$ sets fall into the $m$ choices by Adam is given by the hypergeometric distribution,

$$P_k(N, n, m) = \binom{N}{k} \binom{n-N}{m-k} \binom{n}{m}. \quad (35)$$

The probability that none of these $N$ sets fall into the chosen $m$ group is $P_0(N, n, m)$, a decreasing function of $N$ and an increasing function of $n$. Let $m$ be the smallest integer that yields $\bar{P}_c^A = p_A^m \leq \epsilon$ for given $\epsilon > 0$. The idea is that $N$ must be large enough that at least one of the $N$ sets needs to fall into the $m$ group to get $\bar{P}_c^B > \frac{1}{2}$, but then by making $n$ large, $N$ would have to be so large that the probability $P_u$ that Babe’s cheating sets are undetected becomes too small. Recall that $\bar{P}_c^B$ is the optimal probability Babe succeeds in identifying the bit from measurements on $m$ committed qubits. It will be shown that the condition

$$\bar{P}_c^B \geq \frac{1}{2} + \epsilon \quad (36)$$

would imply $P_u \leq \epsilon$ by proper choice of $n$, thus ensuring unconditional security. Since Babe must have at least one of the $N$ sets picked up by Adam among his $m$ sets in order to satisfy (36),

$$\bar{P}_c^B \leq \frac{1}{2} P_0 + (1 - P_0) P(p_2, m) \leq 1 - \frac{P_0}{2}. \quad (37)$$
By equating the upper and lower bounds (36) and (37) on \( \bar{P}_{c} \), \( N \) must satisfy
\[
N \geq f(\epsilon, n, m(\epsilon, p_{A}))
\]
(38)

where \( f \) is defined through \( P_{0}(N, n, m) \) and is an increasing function of \( n \). For any \( N, n, \)
\[
P_{0}(N, n, m) = \binom{n - m}{N} \binom{n}{N}
\]
can be made arbitrarily small with \( n \) large. Thus, \( N \) can be forced to be arbitrarily small from (38) with \( n \) sufficiently large. If there is an a priori maximum \( \bar{p}_{1} \) among the qubits in the \( N \) sets, which is proved in the following, one would have \( P_{u} \leq \bar{p}_{1}^{N-m} \). So \( n \) can be chosen to make \( N \) large enough from (38) to yield \( \bar{p}_{1}^{N-m} = \epsilon \). As a consequence, \( \bar{P}_{c}^{A} \leq \epsilon \) and \( \bar{P}_{c}^{B} \leq \epsilon \), proving (US).

To put an a priori limit on \( p_{1} \) independent of \( n \) and less than one, consider first the case where all qubits in the \( N \) sets have the same underlying \( S' \) so that Babe knows what measurement to make on each. Let \( P(S', m) \) be the optimum probability that Babe succeeds in identifying \( b \) from measurements on the \( m \) qubit sets. Thus, \( P(S', m) \) is a continuous function of the \( S' \) that gives rise to the \( p_{2} \) as it is a trace norm of the states from (A4). (All norm topologies are equivalent in finite-dimensional spaces). In order for (36) to be satisfied, one must have
\[
P(S', m) \geq \frac{1}{2} + \epsilon
\]
(39)
for some \( \epsilon > 0 \). The maximum \( p_{1} \) that Babe can have is determined among all the qubit sets \( S' \) that satisfy (39) and \( 1 \geq P(S', m) \). The maximum \( \bar{p}_{1} = \max_{S'} p_{1}(S') \) exists for the following reason. Thus the set of \( S' \) obeying (39) and \( P(S', m) \leq 1 \) is closed and thus compact. The function \( p_{1}(S') \) of (34) is continuous. The existence of \( \bar{p}_{1} \) thus follows from the Weierstrass theorem. That is, a maximum \( \bar{p}_{1} \) is achieved by some \( S'_0 \) in the constraint set and so \( \bar{p}_{1} < 1 \). Now suppose Babe has formed entanglements among the sets she sends to Adam. The \( N \) sets are defined according to whether each marginal state, as checked and modulated by Adam, would have \( p_{j1} = 1, j \in \{1, \ldots, n\} \). Thus, instead of \( P(S', m) \) one has \( P(S', m) \) that includes optimization over all possible entangled states \( S' \), which provides an
upper bound to $\bar{P}_c^B$ and is still given through the trace norm (A4). Let $\bar{p}_1 = \max_{S_j} p_1(S'_j)$ for all marginal $S'_j$ obtained from $S'$ that satisfy $1 \geq P(S', m) \geq \frac{1}{2} + \epsilon$. All $S'_j$ in the $N$-set lead to $p_{j2} < \frac{1}{2}$ by definition of the $N$-set. Thus, the existence of $\bar{p}_1 < 1$ follows as in the uniform $S'_j$ case. We have now exhausted all possible actions by Adam and Babe.

In order to execute this protocol in accordance with the above proof in choosing $m$ and $n$, one needs to know $p_A$ and $\bar{p}_1$. These appear to be difficult to obtain analytically, and numerical solutions would need to be used in an actual implementation. In such a situation, the above technicality on the existence of $\bar{p}_1 < 1$ would not occur. While it is easily shown that no four large-energy coherent states can approximate the behavior of the four BB84 states in $S_0$, it may still be possible to develop large-energy coherent-state implementation of this protocol because not all properties of the BB84 states are needed.
Protocol QBC3

The points (7)-(8) in Section IV are now exploited to create a protocol that defeats Adam’s EPR cheat. Consider the following addition to protocol QBC0 in Section II: after Adam commits, Babe picks randomly $N$ out of the $n$ qubits and measures randomly on each either $|\phi\rangle\langle\phi|$ or $|\phi'\rangle\langle\phi'|$, but does not tell Adam which qubits she picked and what measurement results she obtained. When Adam opens, she would verify among the $N$ qubits those that match Adam’s announcement and the rest $n-N$ qubits, and take those that don’t match Adam’s announcement as correct. Thus, she does not have a perfect verification, but Adam cannot cheat successfully by changing one bit position in his announcement when $N/n$ is small. On the other hand, this action by Babe effectively destroys the entanglement that Adam may have formed for the EPR cheat, as shown below. Babe needs to keep secret which $N$ qubits she made measurements upon, or else Adam can alter his basis $|e_i\rangle$ to entangle properly to the other $n-N$ qubits. Condition on the parameters will be given.

**PROTOCOL QBC3**

(i) Adam sends Babe a sequence of $n$ qubits, each in either one of $\{|\phi\rangle, |\phi'\rangle\}$, and commits $b$ via the parity of the sequence with uniform probability.

(ii) Babe randomly picks $N$ out of $n$ qubits, randomly measures either $|\phi\rangle\langle\phi|$ or $|\phi'\rangle\langle\phi'|$ on each, and keeps the results secret from Adam.

(iii) After Adam reveals the sequence commitment, Babe verifies those states that match among the $N$ measured qubits and the $n-N$ unmeasured ones.

The protocol can be made concealing as in QBC0 and QBC1, but Adam can now cheat in more ways. Similar to QBC1, he can pick one qubit and announce it otherwise, which now has a higher probability of success because of Babe’s measurements. From the union bound on the probability of two possible events,

\[ P_c^A \leq |\langle\phi|\phi'\rangle|^2 + \frac{N}{n}. \]  

(40)

Thus one may pick

\[ |\langle\phi|\phi'\rangle|^2 = O(m^{-1}) \]  

(41)
similar to (23) and also
\[ \frac{N}{n} = O(m^{-1}), \tag{42} \]
so that \( P^A_c = O(m^{-1}) \). Adam can, in view of Babe’s possible measurements, entangle as small a number of qubits as possible. If he wants an entanglement cheating probability of
\[ \hat{P}^A_c = 1 - O(m^{-1}), \tag{43} \]
he would need to entangle \( n' = O(m \log m) \), so that the resulting \( \hat{P}^B_c = \frac{1}{2} + O(m^{-1}) \) would guarantee (13) through (12). Thus, to maintain just this order of \( \hat{P}^B_c \), \( n \) should be reduced to \( n = O(m \log m) \) compared to QBC1, and so \( N = O(\log m) \) from (43).

If Adam just cheats as if Babe has made no measurement, a direct computation shows that, for \( p_i = p'_i = 1/M \),
\[ |\langle \Phi''_0 | \Phi_1 \rangle|^2 = 2^{-N} \tag{44} \]
where \( |\Phi''_0 \rangle \) is the cheating entangled state from \( |\Phi_0 \rangle \) after Babe made her \( N \) qubit measurements as follows. The state \( |\Phi''_0 \rangle \) can be written, from (5),
\[ |\Phi''_0 \rangle = \sqrt{\frac{N}{M}} \sum_j \sum_{i_{N+1}...i_{n-1}} V_{i_{N+1}...i_{n-1}} |e'_{j} \rangle |\phi_{i_{N+1}...i_{n-1}} \rangle \tag{45} \]
where \( i_N \) are the fixed indices corresponding to Babe’s measurements results and \( \mathcal{N} \) is a normalization constant determined to be \( \mathcal{N} = 2^N \). Then (44) follows from (45) and the unitarity of \( V_{ij} \). However, this does not yet constitute an unconditional security proof for the following reasons. Adam does not have to apply the cheating transformation as if Babe has made no measurements. It remains to be demonstrated that his optimal cheating transformation, particularly in the case he does not generate an exactly correct initial state for the bit value as discussed after Eq. (23), would lead to an arbitrarily small \( P^A_c \). Furthermore, Adam may aim lower than \( \hat{P}^A_c = 1 - O(m^{-1}) \) by optimizing differently, just to defeat (US).

I believe QBC3 is in fact unconditionally secure, as I believe QBC1 is, and a new formulation of the QBC problem is being developed to facilitate further analysis of the \( P^A_c \) behavior in QBC protocols with possible entanglement attacks. Such general treatment is important because the strategy of this protocol is applicable to all QBC protocols in which the bit value is obtained from a correlated function of the individual bit positions, and the strategy of QBC1, namely the use of anonymous states, is applicable to any QBC protocol.
VIII Conclusion

I hope this paper leaves no doubt that not only is there no general impossibility proof for unconditionally secure quantum bit commitment, but that, in fact, an unconditionally secure QBC protocol has actually been provided. The intuitive reasons and a complete proof that QBC2 satisfies (US) have been described in Section VI. The protocols QBC1 and QBC3, while not proved to be unconditionally secure in this paper, already demonstrate the failure of the impossibility proof given in the literature. Additional gaps of the impossibility proof are indicated in Section IV and can be exploited for further secure QBC schemes.

Some comments on the practicality of our protocols are in order. Protocols QBC0, QBC1, and QBC3 can be readily implemented with large-energy coherent states. However, there is a sensitivity problem that results from $|\langle \phi | \phi' \rangle| \sim 0$, which obscures the difference in practice between the two cases of detection for verification versus cheating corresponding to the cases when the state is known or unknown. An investigation into sensitive detection schemes would be timely. Also, it is expected that this and other practical difficulties can be alleviated by the use of error-correcting codes or hash functions more complicated than parity. Perhaps a large-energy coherent-state scheme similar to QBC2 can also be developed. Another promising avenue is the utilization of the irreducible quantum noise in quantum signal detection schemes to achieve unconditionally secure bit commitment. The loss in fiber-optic communications, especially for the established Internet backbone, can also be used to generate irreducible quantum noise. The resulting protocols, together with similarly possible quantum key distribution and encryption schemes, may open the exciting possibility of optical-speed unconditionally secure cryptography for widespread applications.
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Appendix A
Quantum Detection Theory

Quantum detection theory [13], [28] is concerned with the determination of the optimum quantum measurement and the resulting optimum performance for discriminating a finite number $M$ of alternative hypotheses according to a given performance criterion linear in the density operators $\rho_j$, $j \in \{1, \ldots, M\}$, describing the quantum states of the different alternatives. It has not been used in the previous quantum cryptography literature other than my papers [12], [19], [32], [33], although it actually has a crucial role, especially in QBC. Thus, Babe’s optimum probability of cheating is given by the optimum binary quantum detector for $\rho^B_0$ and $\rho^B_1$.

In binary quantum hypothesis testing with a priori probabilities $p_0$ and $p_1 = 1 - p_0$, the decision is made on the basis of measuring a POM (positive operator-valued measure) described by $\Pi_0$ and $\Pi_1 = I - \Pi_0$, $\Pi_0 \geq 0$ (an operator inequality $A \geq B$ means that $A - B$ is positive semidefinite). The hypothesis $i$ is chosen correctly from the measurement result with probability $\text{tr} \Pi_i \rho_i$, so that the total probability of correct decision is given by

$$P_{C_2} = p_0 \text{tr} \Pi_0 \rho_0 + p_1 \text{tr} \Pi_1 \rho_1.$$  \hfill (A1)

In $M$-ary hypothesis testing, (A1) generalizes to

$$P_{CM} = \sum_{i=1}^{M} p_i \text{tr} \Pi_i \rho_i$$  \hfill (A2)

where the $\{\Pi_i\}$ form the $M$-outcome POM

$$\sum_{i=1}^{M} \Pi_i = I, \quad \Pi_i \geq 0.$$  \hfill (A3)

An operator $\tau$ is called trace-class if its trace norm $\|\tau\|_1 \equiv \text{tr} \sqrt{\tau^\dagger \tau}$ is defined (finite); thus all operators on finite-dimensional spaces are trace-class. Density operators are trace-class. The optimum $\bar{P}_{C_2}$ among all POM’s can be written as follows.

Lemma A1:

$$\bar{P}_{C_2} = \frac{1}{2} + \frac{1}{2} \|p_0 \rho_0 - p_1 \rho_1\|_1.$$  \hfill (A4)
Proof: Write $p_0 \rho_0 - p_1 \rho_1 = \sigma_+ - \sigma_-$, the positive and negative eigenvalue parts, so that $|p_0 \rho_0 - p_1 \rho_1| \equiv \sqrt{(p_0 \rho_0 - p_1 \rho_1)^2} = \sigma_+ + \sigma_-$. From $\text{tr}(p_0 \rho_0 - p_1 \rho_1) = p_0 - p_1$, one has $\text{tr}\sigma_+ = \text{tr}\sigma_- + p_0 - p_1$. Now, from (A1),

$$\hat{P}_{C^2} = p_1 + \max_{0 \leq \Pi \leq I} \text{tr}\Pi (p_0 \rho_0 - p_1 \rho_1),$$

while

$$\max_{0 \leq \Pi \leq I} \text{tr}\Pi (p_0 \rho_0 - p_1 \rho_1) = \max_{0 \leq \Pi \leq I, \Pi\sigma_- = 0} \text{tr}\Pi\sigma_+ = \text{tr}\sigma_+ = \frac{1}{2} \|p_0 \rho_0 - p_1 \rho_1\|_1 + \frac{1}{2} (p_0 - p_1),$$

and (A4) follows. $\Box$

For two pure states, $|\psi_0\rangle$ and $|\psi_1\rangle$, (A4) reduces to

$$P_{C^2} = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4p_0 p_1 |\langle \psi_0 | \psi_1 \rangle|^2}. \quad (A5)$$

The use of “information” e.g. as in Ref. [34], is not sufficient in QBC because it is not the relevant performance measure, and the optimum detectors for $P_c^B$ and mutual information are usually not the same. Indeed, generally in cryptography, the use of mutual information is often not sufficiently precise because it has only asymptotic significance in a noisy system, and at least Eve has no possibility of coding. Thus, the performance resulting from attacks by Eve or by cheating among users in QBC should be measured by their respective probabilities of success. In some cases, including many quantum key-distribution situations, the mutual information could be used to bound the successful eavesdropping probability. But even in those situations the resulting system design may be overly pessimistic when the mutual information criterion is employed.

An important condition whose validity seems clear intuitively is that $P_{C^2} = 1$ in binary quantum detection if and only if the states satisfy $\rho_0 \rho_1 = 0$, i.e. the ranges of $\rho_0$ and $\rho_1$ are orthogonal subspaces of the state Hilbert space. The “if” part is immediate and the “only if” part, which follows from (A5) when $\rho_0$ and $\rho_1$ are pure states, seems to be a consequence of the general no-clone theorem. Specifically, one would be able to clone two nonorthogonal states if one could discriminate between them perfectly. However, the unitarity argument used for no-cloning is not sufficient to include measurement transformations – at least many physicists
believe that a quantum measurement transformation with a specific reading is not describable by a unitary transformation on any larger Hilbert space. Nor is linearity sufficient. Thus, the impossibility of perfectly discriminating nonorthogonal pure states, expressed as \( \rho_0 \rho_1 \neq 0 \) for general mixed states, is a separate property to be demonstrated, indeed even just for completing the no-clone argument. That such a property can be demonstrated from quantum detection theory, as done below, appears to me to be another manifestation of the “magical unity” or consistency of the quantum formalism.

The proof of the following theorem generalizes a finite-dimensional proof for the case \( \lambda_0 = \lambda_1 = 1 \) first communicated to the author by Masanao Ozawa.

**Theorem A1:** For positive constants \( \lambda_0, \lambda_1 \) and density operators \( \rho_0, \rho_1 \), the maximum value of \( \| \lambda_0 \rho_0 - \lambda_1 \rho_1 \|_1 \) among all possible \( \rho_0, \rho_1 \) occurs only when \( \rho_0 \rho_1 = 0 \), with

\[
\| \lambda_0 \rho_0 - \lambda_1 \rho_1 \|_1 = \lambda_0 + \lambda_1. \tag{A6}
\]

**Proof:** In the finite-dimensional case, the polar decomposition of

\[
\rho' \equiv \lambda_0 \rho_0 - \lambda_1 \rho_1 = U|\lambda_0 \rho_0 - \lambda_1 \rho_1| \tag{A7}
\]

always exists for a unitary \( U \). In the infinite-dimensional case, \( U \) is only a partial isometry in general \[^35\]. Since \( \rho' \) on \( \mathcal{H} \) has an eigenvector decomposition as it is trace-class, \( U \) becomes an isometry when restricted to the space \( \mathcal{H}_r \subset \mathcal{H} \), the range of \( \rho' \). Thus, \( U^\dagger U = I_{\mathcal{H}_r} \), and we can write

\[
|\lambda_0 \rho_0 - \lambda_1 \rho_1| = V(\lambda_0 \rho_0 - \lambda_1 \rho_1) \tag{A8}
\]

where \( V = U^\dagger, \|V\| = 1 \). From (A8),

\[
\|\rho'\|_1 = \text{tr} V(\lambda_0 \rho_0 - \lambda_1 \rho_1) = \text{tr} V\lambda_0 \rho_0 - \text{tr} V\lambda_1 \rho_1. \tag{A9}
\]

Now, from (A8), for any trace-class operator \( A \geq 0 \) and any \( V \) with \( \|V\| = 1 \), the real part

\[
\text{Re} \text{tr}(VA) \leq |\text{tr}(VA)| \leq \|VA\|_1 \leq \|V\|\|A\|_1 = \text{Re} \text{tr}A,
\]

leading to

\[
\text{Re} \text{tr}(VA) \leq \text{Re} \text{tr}A \tag{A10}
\]
from which it follows that
\[ -\lambda_0 \leq \text{Re} \text{tr}V \lambda_0 \rho_0 \leq \lambda_0, \quad -\lambda_1 \leq -\text{tr}U \lambda_1 \rho_1 \leq \lambda_1. \] (A11)

From (A9) and (A11), we have
\[ \max_{\rho_0, \rho_1} \|\rho\|_1 = \lambda_0 + \lambda_1 \] (A12)
which occurs when
\[ \text{Re} \text{tr}V \rho_0 = 1 \quad \text{and} \quad \text{Re} \text{tr}V \rho_1 = -1. \] (A13)

Let \( \rho_0 \) have the spectral decomposition \( \rho_0 = \sum_n \nu_n \langle \phi_n | \phi_n \rangle \). Then (A13) implies
\[ \sum_n \nu_n \langle \phi_n | V | \phi_n \rangle = 1. \]
Since \( 0 \leq \nu_n \leq 1 \) and \( \sum_n \nu_n = 1 \), if \( \nu_n \neq 0 \) we have \( \text{Re} \langle \phi_n | V | \phi_n \rangle = 1 \) and hence \( V | \phi_n \rangle = | \phi_n \rangle \).

Let \( \rho_1 \) have the spectral decomposition \( \rho_1 = \sum_m \mu_m \langle \psi_m | \psi_m \rangle \). Similarly, if \( \mu_m \neq 0 \), then \( V | \psi_m \rangle = -| \psi_m \rangle \). Since \( VV^\dagger = I_{\mathcal{H}_r} \), the eigenvectors of \( V \) with different eigenvalues are mutually orthogonal and hence
\[ \rho_0 \rho_1 = \sum_{m,n} \nu_n \mu_m \langle \phi_n | \psi_m \rangle \langle \phi_n | \psi_m \rangle = 0. \] (A14)

\[ \square \]

**Corollary A1:** \( P_{C2} = 1 \) if and only if \( \rho_0 \rho_1 = 0 \).

I would like to emphasize that by itself, without the need for unitarity, Corollary A1 already implies the no-clone theorem for arbitrary \( \rho_0 \rho_1 \neq 0 \). This is because if one can clone, one can obtain an indefinitely large number of copies of the state, which would make it possible to determine the state arbitrarily accurately and hence contradicting the corollary. On the other hand, an argument using the physical interpretation of density operator as an ensemble would show, in conjunction with the pure-state result from (A5), that the eigenstates of \( \rho_0 \) and \( \rho_1 \) must be mutually orthogonal to ensure \( \bar{P}_{C2} = 0 \), thus proving the corollary without Theorem A1. While this can be considered a new kind of mathematics, proving mathematical theorems from physical arguments, it is appropriate to separate physical interpretation from what the mathematical formalism says by itself, if only to check whether they are compatible.
The above theorem can be generalized to $M$-ary hypothesis testing.

**Theorem A2:** $P_{CM} = 1$ if and only if $\rho_i \rho_j = 0$ for all $i \neq j$.

**Proof:** If one pair is not orthogonal, say $\rho_1 \rho_2 \neq 0$, then

$$P_{CM} = p_1 \text{tr} \Pi_1 \rho_1 + p_2 \text{tr} \Pi_2 \rho_2 + \sum_{i=3}^{M} p_i \text{tr} \Pi_i \rho_i \leq p_1 \text{tr} \Pi_1 \rho_1 + p_2 \text{tr} \Pi_2 \rho_2 + 1 - p_1 - p_2$$

since (A3) implies $\|\Pi_i\| \leq 1$ so that $\text{tr} \Pi_i \rho_i \leq 1$ from (18) and (A3). Because the maximum over $\Pi_0$ of the expression (A1) is given by (A4) for any positive $p_0, p_1$, as can be seen from the proof of Lemma A1, it follows from Theorem A1 that $p_1 \text{tr} \Pi_1 \rho_1 + p_2 \text{tr} \Pi_2 \rho_2 = 1$ if and only if $\rho_1 \rho_2 = 0$. Thus $P_{CM} < 1$. The contraposition of this conclusion is the nontrivial part of the theorem. □
Appendix B
Local State Invariance

The local state invariance theorem is conceptually significant and has a simple proof.

Theorem (Local State Invariance): Let $\rho^{AB}$ be a state on $\mathcal{H}^{A} \otimes \mathcal{H}^{B}$ with marginal states $\rho^{A} \equiv \text{tr}_{B}\rho^{AB}, \rho^{B}$. The individual or combined effects of any state transformation and quantum measurement (averaged over the measurement results) on $\mathcal{H}^{A}$ alone leaves $\rho^{B}$ invariant.

Proof: It suffices to consider a pure state $|\Phi\rangle \in \mathcal{H}^{A} \otimes \mathcal{H}^{B}$ in Schmidt form $|\Phi\rangle = \sum_{k} \alpha_{k} |e_{k}\rangle |\phi_{k}\rangle$, $\langle e_{k}|e_{k'}\rangle = \langle \phi_{k}|\phi_{k'}\rangle = \delta_{kk'}$ so that $\rho^{B} = \sum_{k} |\alpha_{k}|^{2} |\phi_{k}\rangle \langle \phi_{k}|$. The most general operation on $\mathcal{H}^{A}$ can be represented by extending $\mathcal{H}^{A}$ to $\mathcal{H}^{A} \otimes \mathcal{H}^{A'}$ with initial state $|A'\rangle \in \mathcal{H}^{A'}$, and applying a unitary $U$ and measuring a complete orthonormal basis $\{|n\rangle \langle n|\}$ on $\mathcal{H}^{A} \otimes \mathcal{H}^{A'}$ \[36\]. This results in $\tilde{\rho}^{B} = \sum_{n} |\langle n|U|\Phi\rangle \langle A'|\langle \Phi|U^{|}|n\rangle so that $\langle \phi_{k}|\tilde{\rho}^{B}|\phi_{k'}\rangle = |\alpha_{k}|^{2} \delta_{kk'} = \langle \phi_{k}|\rho^{B}|\phi_{k'}\rangle$. The same result obtains when either $U$ or the measurement on $\{|n\rangle \langle n|\}$ is omitted. □

The Schmidt decomposition in the above proof only simplifies notation and is not essential. This theorem implies that superluminal communication via quantum entanglement is impossible, which would be obtained if and only if $\rho^{B}$ is changed so that a binary communication channel of classical information with nonzero channel capacity is created. Observe that the averaging over measurement results in the theorem is a crucial condition for application to superluminal communication in which the specific measurement result on $\mathcal{H}^{A}$ is unknown to the party with $\mathcal{H}^{B}$. While there are many proofs on the impossibility of entanglement induced superluminal communication in the literature, see, e.g., \[37\], none appears to be as complete and simple as the proof just given. In particular, the impossibility of cloning quantum states in some such proofs is not sufficient to establish the impossibility of superluminal communication.
Appendix C
Even and Odd Binomial Sums

The even and odd binomial sums used in obtaining (15) are derived as follows. Let $P_m$ be the odd sum

$$P_m = \sum_{r \text{ odd}}^m \binom{m}{r} p^r (1-p)^{m-r} \quad (C1)$$

where $p \leq \frac{1}{2}$, and let $Q_m$ be the even sum, $Q_m = 1 - P_m$. Using the identity

$$\binom{m+1}{r} = \binom{m}{r} + \binom{m}{r-1},$$

the following difference equation for $P_m$ can be derived from (C1):

$$P_{m+1} - P_m = p(Q_m - P_m) = p(1 - 2P_m). \quad (C2)$$

Eq. (C2) with the initial condition $P_1 = p$ is solved to yield

$$P_m = \frac{1}{2} - \frac{1}{2} (1 - 2p)^m. \quad (C3)$$
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Apart from EPR cheats, there are other serious problems with this BCJL protocol. A major one is that Babe’s optimum quantum measurement does not factorize into individual qubit ones when a code is employed because $\rho_B^0 - \rho_B^1$ does not then factorize. As discussed in Appendix A, mutual information is not the appropriate criterion, but its optimizing measurement also does not factorize in this case anyway. Another problem is that there is no known polynomial time algorithm for the NP-hard problem of determining the minimum distance of a randomly chosen binary linear code, even probabilistically or approximately, which would be needed to execute the protocol and to prevent cheating.

[35] P.R. Halmos, A Hilbert Space Problem Book, Springer-Verlag, 1982, ch. 15-16.

[36] This is clearly all one can do according to the standard formulation of quantum mechanics. When restricted to $\mathcal{H}^A$, this gives rise to CP map “superoperator” and POM “generalized” measurement, which indeed characterize all possible state transformations and measurements because they can be so represented in an extended space as shown by M. Ozawa, J. Math Phys. 25, 79 (1984).

[37] H. Scherer and P. Busch, Phys. Rev. A 47, 1647 (1993).