Radial Basis Function Approximation with Distributively Stored Data on Spheres

Han Feng · Shao-Bo Lin · Ding-Xuan Zhou

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Abstract This paper proposes a distributed weighted regularized least squares algorithm (DWRLS) with radial basis functions to tackle spherical data that are stored across numerous local servers and cannot be shared with each other. Via developing a novel integral operator approach based on spherical quadrature rules, we succeed in deriving optimal approximation rates for DWRLS and theoretically demonstrate that DWRLS performs similarly as running a weighted regularized least squares algorithm on the whole data stored on a large enough machine. This interesting finding implies that distributed learning is capable of sufficiently exploiting potential values of distributively stored spherical data, even though local servers cannot access the whole data.

Keywords Distributed learning · Scattered data approximation · Sphere · Integral operator

1 Introduction

In geophysics, solar system, climate prediction, environment governance and meteorology, and image rendering, samples formed as input-output pairs are collected over spheres [13][15][49], such as the surface of the earth and the direction of radiation. Due to the storage bottleneck and data privacy, these spherical data are often distributively stored across numerous computational servers. Typical examples include the CHAMP (Challenging Mini-satellite Payload) data [42] that involve billions of gravity and magnetic field measurements and cannot be stored on a single server, and the nuclear energy data [12] that record the nuclear energy distribution for some countries and cannot be shared with others. The classical fitting schemes such as spherical harmonics [38], spherical basis functions [40], spherical wavelets [15], spherical needlets [39], spherical kernel methods [32] and spherical filtered hyperinterpolation [45] are incapable of tackling these distributively stored data since they require to access the whole data on a single server.

Distributed learning [52], based on a divide-and-conquer approach, provides a promising way to tackle distributively stored spherical data. This strategy applies a specific learning algorithm to a data subset on each local server to produce a local estimator (function), and then synthesizes a global estimator by utilizing some weighted average of the obtained local estimators. Using some integral operator approaches, the feasibility...
of distributed learning has been verified in Euclidean spaces for distributed kernel ridge regression [30, 52], distributed kernel-based gradient descents [24, 31], distributed kernel-based spectral algorithms [29, 37] and distributed local average regression [6] in the sense that distributed learning can achieve the optimal approximation rates of its batch counterpart, i.e., running corresponding algorithms on the whole data, provided the number of local servers is not so large and the samples are collected via a random manner. However, these interesting results do not apply to spherical data, mainly due to the fact that spherical data such as CHAMP and nuclear energy data, gathered by satellites, are often sampled at fixed positions to save resources, making the existing analysis framework based on random sampling and concentration inequalities no more available. Furthermore, the spectrum of a kernel-based integral operator defined on the sphere, one of the simplest examples of homogeneous manifolds, is totally different from that defined on Euclidean spaces, implying that the integral operator approach developed in [24, 29, 31, 37] is infeasible for spherical data. In a word, there lacks a unified theoretical analysis framework to provide a springboard to understanding and designing distributed learning schemes for spherical data.

The purpose of this paper is to develop a distributed learning scheme to handle distributively stored spherical data and provide a theoretical analysis framework to verify its feasibility. Our study stems from three interesting observations. At first, though deterministic sampling on spheres excludes the usability of concentration inequalities [34] for random variables and makes the existing integral operator approach in [24, 29, 31, 37] infeasible, the well developed quadrature rules on spheres [3, 4, 35] that quantify the difference between integrals and their discretizations, provide an alternative way to develop an exclusive integral operator approach for spherical data. Then, if a spherical quadrature rule is adopted to ease the theoretical analysis, the classical discrete (regularized) least-squares approaches on spheres [21, 22] should be replaced by a weighted (regularized) least-squares scheme, just as [25] did for spherical harmonics approximation. Finally, once the integral operator theory is established, the standard error decomposition technique for distributed learning [7, 16] that connects the approximation errors of the global estimator and local estimators is sufficient to derive the approximation error estimate of the distributed learning scheme. Motivated by these observations, we develop a distributed weighted regularized least squares algorithm (DWRLS) associated with some spherical radial basis function to fit distributively stored spherical data.

Our main theoretical contributions are three folds. Firstly, we succeed in developing an exclusive integral operator approach for deterministic sampling on spheres, in which numerous bounds that describe differences between an integral operator and its empirical counterpart are derived. The derived bounds for deterministic sampling are similar to those for random sampling in Euclidean spaces, without using any concentration inequalities in statistics. Secondly, adopting the developed integral operator approach, we deduce optimal approximation rates of DWRLS, even when the data are heavily contaminated, that is, the noise of outputs is large. Finally, we rigorously prove that DWRLS performs similarly as its batch counterpart in the sense that they achieve the same optimal order of approximation error, provided the number of local servers is not so large, showing the feasibility of distributed learning to fit distributively stored spherical data.

The rest of the paper is organized as follows. In Section 2, we firstly introduce spherical radial basis functions and then introduce the distributed weighted regularized least squares algorithm (DWRLS) on the sphere $S^d$. In Section 3, we provide approximation error estimates for DWRLS. As a byproduct, we also derive optimal approximation error estimates for weighted regularized least squares algorithm (WRLS) with the whole data stored on a single server. In Section 4, we develop a novel integral operator approach for spherical data based on spectrum analysis on the sphere and a spherical quadrature formula. Section 5 gives proofs of the main results. In Section 6, we conduct several numerical simulations to show the power of DWRLS in practice.

## 2 Distributed Weighted Regularized Least Squares on the Sphere

In this section, we propose a distributed weighted regularized least squares (DWRLS) algorithm to tackle distributively stored scattered data on the unit sphere $S^d$ of the $d + 1$ Euclidean space $\mathbb{R}^{d+1}$. Assume that there are $m \in \mathbb{N}$ servers, each of which possesses a data set $D_j := \{x_{i,j}, y_{i,j} \}_{i=1}^{|D_j|}$ of cardinality $|D_j|$, where $A_j := \{x_{i,j}\}_{i=1}^{|D_j|} \subset S^d$, $y_{i,j} = f^*(x_{i,j}) + \epsilon_{i,j}, \quad \forall i = 1, \ldots, |D_j|, j = 1, \ldots, m,$ \quad (1)
A harmonic of degree $k_D$ in the premise that the data in $D_j \cap D_{j'} = \emptyset$ for $j \neq j'$, implying that different servers possess different data. Our aim is to design a fitting scheme based on the magnitude of noise, the fitting problem in (1) is different from that in [21] where the noise is assumed to be deterministic and extremely small. Without loss of generality, we assume further $D_j \cap D_{j'} = \emptyset$ for $j \neq j'$, in the premise that the data in $D_j$ cannot be shared with each other, to yield an estimator $\tilde{f}_D$ such that $\tilde{f}_D$ is near to $f^*$.

2.1 Spherical basis function and native space

For integer $k \geq 0$, the restriction to $S^d$ of a homogeneous harmonic polynomial of degree $k$ is called a spherical harmonic of degree $k$. Denote by $\mathbb{H}_k^d$ and $\Pi_k^d$ the classes of all spherical harmonics of degree $k$ and all spherical polynomials of degree $k \leq s$, respectively. It can be found in [38] that the dimensions of $\mathbb{H}_k^d$ and $\Pi_k^d$ are

$$d_k^d := \begin{cases} \frac{2k+d-1}{k+1} \binom{k+d-1}{k}, & \text{if } k \geq 1, \\ 1, & \text{if } k = 0 \end{cases}$$

with $d_k^d \sim k^{d-1}$ and $\sum_{s=0}^k d_k^d = d_s^d \sim s^d$, respectively. Throughout this paper, $a \sim b$ for $a, b \in \mathbb{R}$ means that there are absolute constants $c_1, c_2$ such that $c_1 b \leq a \leq c_2 b$.

Let $\{Y_{k,\ell}\}_{j=1}^d$ be an arbitrary orthonormal basis of $\mathbb{H}_k^d$ and $P_k^{d+1}$ be the normalized Legendre polynomial, i.e., $P_k^{d+1}(1) = 1$ and

$$\int_{-1}^1 P_k^{d+1}(t)P_j^{d+1}(t)(1-t^2)^{d/2}dt = \frac{\Omega_d}{\Omega_{d-1}} d_k^{d+1}\delta_{k,j},$$

where $\Omega_d = \frac{2^{2d+1}}{d!(2d+1)}$ denotes the volume of $S^d$ and $\delta_{k,j}$ is the usual Kronecker symbol. It can be found in [48] that

$$|P_k^{d+1}(t)| \leq 1, \quad \forall k \in \mathbb{N}, t \in [-1, 1].$$

The classical addition formula [38] establishes a relation between $Y_{k,\ell}$ and $P_k^{d+1}$ via

$$\sum_{\ell=1}^{d_k^d} Y_{k,\ell}(x)Y_{k,\ell}(x') = \frac{d_k^d}{\Omega_d} P_k^{d+1}(x \cdot x').$$

We say that a function $\phi \in L^2([-1, 1])$ is a spherical basis function (SBF) if its expansion $\phi(t) = \sum_{k=0}^{\infty} \hat{\phi}_k \frac{d_k^d}{\Omega_d} P_k^{d+1}(t)$ has all Fourier-Legendre coefficients

$$\hat{\phi}_k := \Omega_{d-1} \int_{-1}^1 P_k^{d+1}(t)\phi(t)(1-t^2)^{d/2}dt > 0.$$ 

It is well known that each SBF $\phi$ corresponds to a native space $\mathcal{N}_\phi$ that is defined by

$$\mathcal{N}_\phi := \left\{ f(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k^d} \hat{f}_{k,\ell} Y_{k,\ell}(x) : \sum_{k=0}^{\infty} \hat{\phi}_k^{-1} \sum_{\ell=1}^{d_k^d} |\hat{f}_{k,\ell}|^2 < \infty \right\}$$

with inner product $\langle f, g \rangle_\phi := \sum_{k=0}^{\infty} \hat{\phi}_k^{-1} \sum_{\ell=1}^{d_k^d} \hat{f}_{k,\ell} \overline{\hat{g}_{k,\ell}}$ and norm $\| f \|_\phi := \left( \sum_{k=0}^{\infty} \hat{\phi}_k^{-1} \sum_{\ell=1}^{d_k^d} |\hat{f}_{k,\ell}|^2 \right)^{1/2}$, where $\hat{f}_{k,\ell} := \int_{\partial S^d} f(x)Y_{k,\ell}(x)d\omega(x)$ is the Fourier coefficient of $f$ with respect to $Y_{k,\ell}$ and $d\omega$ denotes the Lebesgue measure of the sphere.
If an SBF $\phi$ satisfies further $\sum_{k=0}^{\infty} \hat{\phi}(x, \hat{x}) d^d < \infty$, then $\phi$ is said to be (semi-)positive definite. Under this circumstance, $\mathcal{K}_{\phi}$ is a reproducing kernel Hilbert space with reproducing kernel $(x, x') \mapsto \phi(x \cdot x')$. Typical positive definite functions used for spherical data are the Sobolev-type functions with smoothness index $\gamma > d/2$.

$$S_{\gamma}(t) = \sum_{k=0}^{\infty} \left(k(k + d - 1) + 1\right) f^{d+1}(t)$$

and the Gaussian function with width $\tau > 0$

$$G_{\tau}(t) = e^{-\frac{t^2}{\tau^2}} = \sum_{k=0}^{\infty} e^{-2/\tau^2} \frac{\tau^{d-1}}{\Gamma((d/2))} \frac{d^d}{\Omega_d} f^{d+1}(t),$$

where $\Gamma(\cdot)$ is the Gamma function and $I_{\nu}(\cdot)$ is modified Bessel function of the first kind, defined by

$$I_{\nu}(t) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{t^{\nu+j}}{(\nu+j+1)} \left(\frac{1}{2}\right)^{v+2j}.$$  

### 2.2 Distributed weighted regularized least squares

To introduce the distributed algorithm based on a positive definite kernels $\phi(x, x') = \phi(x \cdot x')$, we need spherical quadrature rules for scattered data. For an arbitrary $1 \leq j \leq m$, define the mesh norm, separation radius and mesh ratio of $A_j$ by $h_{A_j} := \max_{x \in \mathcal{S}^d} \min_{i, j \in A_j} \text{dist}(x, x_{i,j})$, $q_{A_j} := \frac{1}{2} \min_{i, j \in A_j} \text{dist}(x_{i,j}, x_{j})$ and $\rho_{A_j} := \frac{h_{A_j}}{q_{A_j}}$, respectively. The mesh ratio $\rho_{A_j} \geq 1$ measures how uniformly the points of $A_j$ are distributed on $\mathcal{S}^d$. We say that $A_j$ is $\tau_j$-quasi uniform, if there is a constant $\tau_j \geq 2$ such that $\rho_{A_j} \leq \tau_j$. The existence of $\tau_j$-quasi uniform set with $\tau_j \geq 2$ has been verified in [40].

For $s_j \in \mathbb{N}$, a set $\mathcal{D}_{A_j,s_j} := \{(w_{i,j,s_j}, x_{i,j}) : w_{i,j,s_j} \geq 0 \text{ and } x_{i,j} \in A_j\}$ is said to be a positive quadrature rule on $\mathcal{S}^d$ with degree $s_j \in \mathbb{N}$, if

$$\int_{\mathcal{S}^d} P(x) d\omega(x) = \sum_{i,j \in A_j} w_{i,j,s_j} P(x_{i,j}), \quad \forall P \in \mathcal{P}_{s_j}^d.$$  

The following positive quadrature rule can be found in [3, Theorem 3.1] or [35].

**Lemma 1** For every $1 \leq j \leq m$, if $A_j = \{x_{i,j}\}_{i,j}^{\left|D_j\right|}$ is $\tau_j$-quasi uniform and $s_j \leq c\left|D_j\right|^{1/d}$, then there exists a quadrature rule $\mathcal{D}_{A_j,s_j} = \{(w_{i,j,s_j}, x_{i,j}) : w_{i,j,s_j} \geq 0 \text{ and } x_{i,j} \in A_j\}$ satisfying $0 \leq w_{i,j,s_j} \leq c_1\left|D_j\right|^{-1}$, where $c, c_1$ are constants depending only on $\tau_j$ and $d$.

With the help of the spherical quadrature rule, we can proceed our description as follows. On the $j$-th server, we take an $s_j \in \mathbb{N}$ to admit a quadrature rule $\mathcal{D}_{A_j,s_j} = \{(w_{i,j,s_j}, x_{i,j}) : w_{i,j,s_j} \geq 0 \text{ and } x_{i,j} \in A_j\}$ with $0 \leq w_{i,j,s_j} \leq c_1\left|D_j\right|^{-1}$. Then, the weighted regularized least squares (WRLS) on the $j$-th server, with a regularization parameter $\lambda_j > 0$, is defined by

$$f_{D_j, W_{i,j,s_j}} \lambda_j = \arg\min_{f \in \mathcal{P}_{s_j}^d} \sum_{(i,j) \in D_j} w_{i,j,s_j} (f(x_{i,j}) - y_{i,j})^2 + \lambda_j \|f\|^2_\phi.$$  

Finally, all these local estimators are transmitted to the global serve to synthesize a global estimator, called as distributed weighted regularized least squares (DWRLS), as

$$f_{D_j, W_{i,j,s_j}} \lambda_j = \sum_{(i,j) \in D_j} \left|D_j\right|^{-1} f_{D_j, W_{i,j,s_j}} \lambda_j.$$  

It should be mentioned that all local estimators defined by (8) possess closed-form solutions. Denote by $W_{D_j, s_j}$ the $\left|D_j\right| \times \left|D_j\right|$ diagonal matrix with diagonal elements $w_{1,j,s_j}, \ldots, w_{\left|D_j\right|, s_j}$ and $\Phi_{A_j} = \{\phi(x_{i,j}, x_{j})\}_{i,j}^{\left|D_j\right|}$.
Lemma 2 Let \( f_{D_j,w_{i,j}} \) be defined by \( f_{D_j,w_{i,j}} = \sum_{(x_i,y_i) \in D_j} a_{i,j} \phi_{i,j} \). Then \( f_{D_j,w_{i,j}} = \sum_{i,j=1}^n a_{i,j} \phi_{i,j} \) with

\[
(a_{1,j}, \ldots, a_{D,j})^T = (W_{D,j} \Phi_{D_j} + \lambda_j I)^{-1} W_{D,j} y_{D_j},
\]

where \( \phi_{i,j} = \phi(x_{i,j}, \cdot) \) and \( y_{D_j} = (y_{1,j}, \ldots, y_{D,j})^T \).

3 Main Results

In this section, we derive approximation rates of DWRLS for noisy data \( (1) \).

3.1 Approximation capability of WRLS

Before presenting the approximation rates of DWRLS, we should provide a baseline for analysis, where the approximation error of WRLS is needed. Define

\[
f_{D,W,\lambda} = \arg \min_{f \in \Lambda} \sum_{(x_i,y_i) \in D} w_i \| f(x_i) - y_i \|^2 + \lambda \| f \|_{\phi}^2 \tag{10}
\]

as the estimator derived by WRLS, where \( D = \{ x_i \}_{i=1}^{|D|} \), \( \Lambda := \cup_{i=1}^{|D|} \{ w_i \} \) and \( \{ w_i \}_{i=1}^{|D|} \) for \( s \in \mathbb{N} \) is the quadrature weights of the quadrature rule \( Q_{D,s} := \{ (w_{i,s}, x_{i,s}) \} \) with \( 0 \leq w_{i,s} \leq c_1 |D|^{-1} \). It is easy to see that the WRLS estimator in \( (10) \) is a batch version of DWRLS, which assumes that all data are stored on a single large server and WRLS is capable of handling them. According to Lemma 2, since the matrix-inversion is involved in WRLS, it requires \( O(|D|^2) \) memory requirements and \( O(|D|^3) \) float-computations to solve the optimization problem in \( (10) \), which is infeasible when the data size is huge even if all the data could be collected without considering the data privacy issue. The study of approximation capability of WRLS \( (10) \) is necessary, since it enhances the understanding of WRLS by means of determining which conditions are sufficient to guarantee that distributed learning performs similarly to its batch counterpart.

Let \( \phi \) be a positive definite function with \( 0 < \phi_k < 1 \) for \( k = 1, 2, \ldots \) and \( \psi(t) = \sum_{k=0}^\infty \psi_k \frac{d^k}{dx_k} P_{d+1}(t) \) be another SBF satisfying

\[
\psi_k = \phi_k^r, \quad 0 \leq r \leq 1.
\]

Therefore, we have \( \phi_k \leq \psi_k \) and consequently \( N_{\psi} \leq N_{\phi} \). Our following theorem whose proof will be given in Section 5 presents an estimate for the approximation error of WRLS \( (10) \) under the metric of \( N_{\psi} \).

**Theorem 1** Let \( 0 < \delta < 1 \), \( \Lambda \) be \( \tau \)-quasi uniform and \( Q_{D,s} := \{ (w_{i,s}, x_{i,s}) \} \) be a quadrature rule satisfying \( 0 < w_{i,s} \leq c_1 |D|^{-1} \). Under \( (7) \) with \( m = 1 \) and \( f^* \in N_{\psi} \) and \( (11) \) with \( 0 \leq r \leq 1 \), if \( \phi_k \sim k^{-2\gamma} \), with \( \gamma > d/2 \), \( \lambda \sim |D|^{-\frac{3\gamma}{d}} \), and \( s \geq \lambda^{-1/\gamma} \), then with confidence \( 1 - \delta \), there holds

\[
\| f_{D,W,\lambda} - f^* \|_{N_{\psi}} \leq C |D|^{-\frac{(1+\gamma)r\sigma^2}{d}} \log \frac{3}{\delta},
\]

where \( C \) is a constant depending only on \( c_1, \tau, d, M \) and \( \| f^* \|_{\phi} \).

Setting \( r = 0 \) in Theorem 1 we have \( N_{\psi} = L^2(\mathbb{S}^d) \). Then it follows from \( (11) \) that

\[
\| f_{D,W,\lambda} - f^* \|_{L^2(\mathbb{S}^d)} \leq C |D|^{-\frac{7\gamma}{d}} \log \frac{3}{\delta},
\]

holds with confidence \( 1 - \delta \), which coincides with the approximation bounds derived in \( [32, 33] \) for random samples and cannot be improved further according to the theory in \( [5, \text{Theorem 2}] \). Theorem 1 requires \( \lambda \sim |D|^{-\frac{3\gamma}{d}} \) and \( s \geq \lambda^{-1/\gamma} \), implying \( s \geq C_1 |D|^{\frac{3\gamma}{d}} \) for some absolute constant \( C_1 \). Such a restriction is mild according to Lemma 1 from which we can deduce that any \( s \leq C_1 |D|^{\frac{3\gamma}{d}} \) admits a positive quadrature formula as required. In this way, we can choose any \( s \) satisfying \( C_1 |D|^{\frac{3\gamma}{d}} \leq s \leq C_1 |D|^{\frac{3\gamma}{d}} \) in WRLS \( (10) \) by noting \( 2\gamma > d \).
Scattered data fitting on the sphere is a hot topic in approximation theory and numerical analysis. Numerous fitting schemes [14] have been developed for this purpose. The radial basis function approach has triggered enormous research activities [18–20, 22, 26, 28, 36, 40, 41]. In particular, [22] studied the approximation capability of discrete least squares algorithm and the approximation error of order $O(|D|^{-\gamma/d})$ is derived for noiseless data; [40] considered the native space barrier problem of SBF approximation and established a Bernstein-type inequality for shifts of SBF; [19, 20] studied the Lebesgue constant for kernel approximation on the sphere. It should be mentioned that these interesting results focused on interpolation problems, in which the collected data are assumed to be clean, i.e., $\varepsilon_{i,j} = 0$ in (1). Differently, Theorem 1 studies a fitting problem with noisy data, which imposes strict requirements on the stability of the fitting scheme. The most related work in this direction is [21], where approximation error for kernel-based regularized least squares algorithm was derived for noisy spherical data. Our derived error in (12) is a little bit worse than that of [21] at the first glance. However, it should be highlighted that we do not impose the strict restriction that the magnitude of noise is small. In fact, the noise to guarantee an approximation rate of order $O(|D|^{-\gamma/d})$ should be extremely small in [21], while that to guarantee an approximation rate of order $O(|D|^{-\gamma/(2\gamma+d)})$ can be very large. Furthermore, our result focuses on the Sobolev-type error estimate while that of [21] is only carried out in $L^2(S^d)$. As mentioned above, the rate $O(|D|^{-\gamma/(2\gamma+d)})$ cannot be improved further under the same setting as Theorem 1. In short, different approximation rates between [21] and our work are due to the different types of data rather than the algorithm selection or proof skills.

Another line of related work is [5, 7, 23, 30], where a similar order of approximation error was derived for similar kernel-based algorithms for random samples. There are two important differences between our work and the results in [5, 7, 23, 30]. On one hand, we consider data deterministically sampled over quasi-uniform points while the existing work focus on random samples. The deterministic setting makes the widely used concentration inequalities [5] no more available for our analysis. Instead, we develop a novel integral operator approach based on spherical quadrature rules. On the other hand, since our analysis involves spherical quadrature rules, we are interested in the weighted regularized least squares algorithm with quadrature weights rather than the regularized least squares in [5, 7, 23, 30]. That is, our adopted algorithm is different from those in [5, 7, 23, 30].

It should be mentioned that in the existing literature we do not find any similar results as Theorem 1 under the same setting of this paper, though the statement is a little bit standard. The main reason is that we are concerned with scattered data fitting with large noise that is different from the classical scattered data fitting problem [21, 40, 41] and deterministic samples which is different from the kernel learning problem in statistical learning theory [5, 7, 30].

### 3.2 Approximation capability of DWRLS

In the previous subsection, we present an optimal approximation error estimate of WRLS when the data satisfy (1) with $m = 1$, showing that WRLS can provide a perfect estimator, assuming that the whole data can be gathered together on a single server and the size of data is not so large. Facing distributively stored data, WRLS fails to achieve the approximation error bounds as (12) since there are only $|D_j|$ data available to the $j$-th local server. In this part, we show that DRWLS can yield similar approximation error bounds, provided the number of local servers is not so large. The following theorem presents our main result of this paper.

**Theorem 2** Let $1 \leq j \leq m$, $s_j \in \mathbb{N}$, $A_j = \{x_{i,j}\}_{i=1}^{|D_j|}$ be $\tau_j$-quasi uniform and $\mathcal{P}_{A_j,s_j} := \{(w_i,s_j) : w_i,s_j \geq 0 \text{ and } x_{i,j} \in A_j\}$ be a quadrature rule satisfying $0 < w_{i,j} \leq c_{1,j}|D_j|^{-1}$ for some $c_{1,j}$ depending only on $d$ and $\tau_j$. Under (11) with $f^* \in \mathcal{N}_0$ and (11) with $0 \leq r \leq 1$, if $\hat{\phi}_k \sim k^{-2\gamma}$ with $\gamma > d/2$, $\lambda_j \sim |D|^{-\gamma/(2\gamma + d)}$ and $s_j \geq \lambda_j^{-1/\gamma}$, then

$$E[\|f_{D,W,\lambda} - f^*\|_\psi] \leq C|D|^{-\frac{1}{(1-r)\gamma}}\frac{\gamma}{2\gamma + d},$$

where $C$ is a constant depending only on $\gamma, r, \tau_j, d, M$ and $\|f^*\|_\psi$.

It seems that there is no restriction on the number of local servers in Theorem 2. However, Lemma 1 shows that to admit a positive quadrature rule in local servers, $s_j$ for every $1 \leq j \leq m$ should satisfy $s_j \leq |D_j|^{1/d}$. But
$s_j$ in Theorem 2 should satisfy
\[ s_j \geq \lambda_j^{-1/\gamma} \geq C_2 |D|^{2\gamma/d}, \]

implying
\[ C_2 |D|^{2\gamma/d} \leq |D_j|^{1/d}, \quad \forall j = 1, 2, \ldots, m, \]

where $C_2$ is an absolute constant. If $|D_1| \sim |D_2| \sim \cdots \sim |D_m|$, the above inequality yields
\[ m \leq C_3 |D|^{2\gamma/d} \tag{15} \]

for some absolute constant $C_3$. This illustrates that to guarantee the conditions of Theorem 2 the number of local servers should not be so large, which alternately implies that $|D_j|$ should not be small for every $j \in \{1, \ldots, m\}$.

Comparing Theorem 2 with Theorem 1 we find that under (15) and $|D_1| \sim |D_2| \sim \cdots \sim |D_m|$, DWRLS performs similarly to WRLS in the sense that their approximation rates are of the same order, showing the power of DWRLS to fit distributively stored spherical data. In Theorem 2 we also find that regularization parameters in all local servers are of the same order and are similar to that for WRLS in Theorem 1. It is practically difficult to determine such a perfect $\hat{\lambda}$ under the distributed learning framework, since $\gamma$ is difficult to quantify in practice. As shown in Theorem 1 the theoretically optimal regularization parameter satisfies $\lambda \sim |D|^{-1/(2\gamma+\delta)}$ and it can be realized by using the well known cross-validation approach \cite{17, Chap.7}. However, in the distributed learning setting, there are only $|D_j|$ data in the $j$-th local server and we are capable of getting a regularization $\lambda_j \sim |D_j|^{-2\gamma/(2\gamma+d)}$ via cross-validation in the $j$-th server. Practically, we can set $\hat{\lambda}_j = (\lambda_j)^{\log |D_j|/|D|}$. Then $\hat{\lambda}_j \sim |D_j|^{-2\gamma/(2\gamma+d)}$ implies $\hat{\lambda}_j \sim |D|^{-2\gamma/(2\gamma+d)}$, which is theoretically optimal as shown in Theorem 2.

A distributed filter hyperinterpolation scheme has already been developed for distributively stored spherical data in our recent work \cite{33}, where optimal approximation error estimates were derived under the random sampling setting. There are mainly four differences between Theorem 2 and results in \cite{33}. First, the learning schemes are different. In particular, we study radial basis function approximation based on weighted regularized least squares while \cite{33} considered spherical polynomial approximation that can be constructed directly. Second, the types of data are different. To be detailed, we study deterministic and quasi-uniform sampling while \cite{33} focused on random sampling. Third, the measurements of error are different. Indeed, we consider Sobolev-type error estimates but \cite{33} conducted the analysis only in $L^2(S^d)$. Finally, the analysis frameworks are different. In this paper, we develop a novel integral operator approach to analyze the feasibility of DWRLS while the analysis in \cite{33} follows from the standard concentration inequality approach in \cite{6, 30, 52}.

### 4 Integral Operator Approach Based on Spherical Positive Quadrature Rules

In this section, we propose an integral operator approach based on spherical positive quadrature rules to derive approximation error of WRLS and DWRLS.

#### 4.1 Spectrum of spherical basis function and spherical quadrature formulas

Let $\phi$ be an SBF. Define the integral operator $L_\phi : L^2(S^d) \rightarrow L^2(S^d)$ by
\[ L_\phi f(x) := \int_{S^d} \phi(x \cdot x') f(x') d\omega(x'), \quad f \in L^2(S^d). \]

The Funk-Hecke formula \cite{38}
\[ L_\phi Y_{k,j}(x) = \hat{\phi}_k Y_{k,j}(x), \quad \forall j = 1, \ldots, d_k^2, k = 0, 1, \ldots \tag{16} \]

shows that the eigen-pairs of $L_\phi$ are
\[ (\hat{\phi}_0, Y_{0,1}), (\hat{\phi}_1, Y_{1,1}), \ldots, (\hat{\phi}_1, Y_{1,d_1^2}), \ldots, (\hat{\phi}_k, Y_{k,1}), \ldots, (\hat{\phi}_k, Y_{k,d_k^2}), \ldots \tag{17} \]
This implies
\[
L\phi f(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d^d} \hat{f}_{k,\ell} \phi Y_{k,\ell}(x) = \sum_{k=0}^{\infty} \phi_k \sum_{\ell=1}^{d^d} \hat{f}_{k,\ell} Y_{k,\ell}(x), \quad f \in L^2(S^d)
\]
and
\[
\|f\|_{L^2(S^d)}^2 = \sum_{k=0}^{\infty} \phi_k^{-1} \sum_{\ell=1}^{d^d} |\hat{f}_{k,\ell}|^2 = \sum_{k=0}^{\infty} \phi_k \sum_{\ell=1}^{d^d} |\phi_k^{1/2} \hat{f}_{k,\ell}|^2 = \|L\phi f\|_{L^2(S^d)}^2,
\]
where \(\psi(\ell)\) is another SBF with Fourier-Legendre coefficients satisfying (11), and \(\eta(L\phi)\) is defined by spectrum calculus, i.e.,
\[
\eta(L\phi)f(x) = \sum_{k=0}^{\infty} \eta(\phi_k) \sum_{\ell=1}^{d^d} \hat{f}_{k,\ell} Y_{k,\ell}(x).
\]
If \(r = 0\) in (11), then (18) implies \(\|f\|_{L^2(S^d)} = \|L\phi f\|_{L^2(S^d)}\).

Let \(\mathcal{L}_{\phi}\) be the integral operator on \(\mathcal{N}_\phi\) defined (17) by
\[
\mathcal{L}_{\phi} f(x) := \int_{S^d} \phi(x \cdot x') f(x') d\omega(x'), \quad f \in \mathcal{N}_\phi.
\]
Then, it follows from (4) and (17) that the eigen-pairs of \(\mathcal{L}_{\phi}\) are
\[
(\hat{\phi}_0, \sqrt{\phi_0}Y_{0,1}), (\hat{\phi}_1, \sqrt{\phi_0}Y_{1,1}), \ldots, (\hat{\phi}_k, \sqrt{\phi_0}Y_{k,1}), \ldots, (\hat{\phi}_k, \sqrt{\phi_0}Y_{k,d}), \ldots
\]
(19)
Therefore, for any \(f \in \mathcal{N}_\phi\), it follows from (4) and (16) that
\[
\mathcal{L}_{\phi} f = \mathcal{L}_{\phi} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d^d} (f, \sqrt{\phi_0}Y_{k,\ell})_\phi \sqrt{\phi_0}Y_{k,\ell} = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d^d} (\hat{f}_{k,\ell})^{-1/2} \hat{f}_{k,\ell} \mathcal{L}_{\phi} \sqrt{\phi_0}Y_{k,\ell}
\]
\[
= \sum_{k=0}^{\infty} \sum_{\ell=1}^{d^d} \hat{f}_{k,\ell} \mathcal{L}_{\phi}Y_{k,\ell} = \sum_{k=0}^{\infty} \phi_k \sum_{\ell=1}^{d^d} \hat{f}_{k,\ell} Y_{k,\ell} = L\phi f.
\]
This implies
\[
\eta(\mathcal{L}_{\phi}) f = \eta(L\phi)f, \quad \forall f \in \mathcal{N}_\phi,
\]
and
\[
\eta(\mathcal{L}_{\phi})(f + g) = \sum_{k=0}^{\infty} \eta(\phi_k) \sum_{\ell=1}^{d^d} (f + g)_{k,\ell} Y_{k,\ell} = \sum_{k=0}^{\infty} \eta(\phi_k) \sum_{\ell=1}^{d^d} (f_{k,\ell} + g_{k,\ell}) Y_{k,\ell}
\]
\[
= \eta(\mathcal{L}_{\phi}) f + \eta(\mathcal{L}_{\phi}) g, \quad \forall f, g \in \mathcal{N}_\phi.
\]
(21)

We then deduce a quadrature rule which will play a crucial role in our integral operator approach in the following proposition.

**Proposition 1** Let \(\mathcal{S} = \{x_i\}_{i=1}^{\mathcal{S}} \subset S^d\) be a set of scattered data and \(\mathcal{Q}_\mathcal{S}, s := \{(w_{i,j}, x_i) : w_{i,j} > 0, x_i \in \mathcal{S}\}\) be a positive quadrature rule on \(S^d\) with degree \(s \in \mathbb{N}\). If \(\phi_k \sim k^{-2r}\) with \(r > d/2\) and \(\eta_{s,u}(t) = (t + \lambda)^{-u}\), then for any \(f, g \in \mathcal{N}_\phi\), any \(u, v \in [0, 1)\) satisfying \(u + v \leq 1\) and any \(\lambda \geq s^{-2r}\), there holds
\[
\left| \int_{S^d} (\eta_{s,u}(L\phi)f)(x)(\eta_{s,v}(L\phi)g)(x) d\omega(x) - \sum_{x_i \in \mathcal{S}} w_{i,j} \left[ (\eta_{s,u}(L\phi)f)(x_i)(\eta_{s,v}(L\phi)g)(x_i) \right] \right|
\]
\[
\leq \tilde{c} \|f\|_\phi \|g\|_\phi \left\{ \begin{array}{ll}
\lambda^{-u} s^{-Y} + \lambda^{-v} s^{-Y} + \lambda^{-u-v} s^{-2Y} + s^{-(1-u-v)Y}, & \text{if } u + v < 1, \\
\lambda^{-u} s^{-Y} + \lambda^{-v} s^{-Y} + \lambda^{-u-v} s^{-2Y} + \log s, & \text{if } u + v = 1,
\end{array} \right.
\]
where \(\tilde{c}\) is a constant depending only on \(\gamma, u, v\) and \(d\).

The proof of Proposition 1 is a little bit standard, requiring the usage of the aforementioned spectrum analysis and spherical polynomials approximation. We move it to Appendix for the sake of brevity. If we set \(u = v = 0\) and \(g(x) \equiv 1\), Proposition 1 is the classical spherical positive quadrature rule for Sobolev space established in [4].
4.2 Operator representation and operator differences

Let \( \phi \) be a positive definite function. Define \( S_D f = (f(x_1), \ldots, f(x_{|D|}))^T \) and

\[
S_{D,W_i}^T c = \sum_{i=1}^{|D|} w_{i,s} c_i \phi_i.
\]

Write

\[
L_{\phi,D,W_i} f := S_{D,W_i}^T S_D f = \sum_{i=1}^{|D|} w_{i,s} f(x_i) \phi_i.
\]

Then it is easy to check that \( L_{\phi,D,W_i} : \mathcal{N}_\phi \rightarrow \mathcal{N}_\phi \) is a positive operator of finite rank. The following proposition presents the operator representation of \( f_{D,\lambda,W_i} \).

**Proposition 2** Let \( f_{D,W_i,\lambda} \) be defined by (15) with \( D_j \) being replaced by \( D \). Then, \( L_{\phi,D,W_i} : \mathcal{N}_\phi \rightarrow \mathcal{N}_\phi \) is a positive operator and

\[
f_{D,W_i,\lambda} = (L_{\phi,D,W_i} + \lambda I)^{-1} S_{D,W_i}^T y_D.
\]

Proposition 2 is a standard result in kernel-based learning [46,47]. We provide its proof in Appendix for the sake of completeness. We then derive tight bounds for differences between the operator \( \mathcal{L}_\phi \) and its empirical counterpart \( L_{\phi,D,W_i} \), which is the core in our analysis. Our first result concerns an upper bound of \( \| \mathcal{L}_\phi - L_{\phi,D,W_i} \| \), where \( \|A\| \) denotes the spectral norm of the operator \( A \).

**Proposition 3** Let \( \mathcal{D}_{A,s} := \{(w_{i,s}, x_i) : w_{i,s} \geq 0 \text{ and } x_i \in \Lambda\} \) be a positive quadrature rule on \( \mathbb{S}^d \) with degree \( s \in \mathbb{N} \). If \( \hat{\phi}_k \sim k^{-2\gamma} \) with \( \gamma > d/2 \), then for any \( 0 \leq \nu < 1 \) and \( 0 < \lambda < 1 \), there holds

\[
\| (\mathcal{L}_\phi + \lambda I)^{-\nu} (L_{\phi,D,W_i} - \mathcal{L}_\phi) \| \leq C(\lambda^{-s-\gamma} + s^{-(1-v)\gamma}).
\]

**Proof** Due to the definition of operator norm and \( (\mathcal{L}_\phi + \lambda I)^{-\nu} g \in \mathcal{N}_\phi \), we have

\[
\| (\mathcal{L}_\phi + \lambda I)^{-\nu} (L_{\phi,D,W_i} - \mathcal{L}_\phi) \| = \sup_{\|f\|_{\phi} \leq 1} \| (\mathcal{L}_\phi + \lambda I)^{-\nu} (L_{\phi,D,W_i} - \mathcal{L}_\phi) f \|_{\phi}
\]

\[= \sup_{\|f\|_{\phi} \leq 1} \sup_{\|g\|_{\phi} \leq 1} (\mathcal{L}_\phi + \lambda I)^{-\nu} (L_{\phi,D,W_i} - \mathcal{L}_\phi) f, g \phi_{\phi}
\]

\[= \sup_{\|f\|_{\phi} \leq 1} \sup_{\|g\|_{\phi} \leq 1} (L_{\phi,D,W_i} - \mathcal{L}_\phi) f, (\mathcal{L}_\phi + \lambda I)^{-\nu} g \phi_{\phi}
\]

\[= \sup_{\|f\|_{\phi} \leq 1} \sup_{\|g\|_{\phi} \leq 1} \left| \int_{\mathbb{S}^d} f(x') \phi_{x'} d\omega(x') - \sum_{i=1}^{|D|} w_{i,s} f(x_i) \phi_{x_i}, (\mathcal{L}_\phi + \lambda I)^{-\nu} g \phi_{\phi} \right|
\]

\[= \sup_{\|f\|_{\phi} \leq 1} \sup_{\|g\|_{\phi} \leq 1} \left| \int_{\mathbb{S}^d} f(x') (\mathcal{L}_\phi + \lambda I)^{-\nu} g(x') d\omega(x') - \sum_{i=1}^{|D|} w_{i,s} f(x_i) (\mathcal{L}_\phi + \lambda I)^{-\nu} g(x_i) \right|
\]

Then, it follows from Proposition 1 with \( u = 0 \) that

\[
\| (\mathcal{L}_\phi + \lambda I)^{-\nu} (L_{\phi,D,W_i} - \mathcal{L}_\phi) \| \leq C(\lambda^{-s-\gamma} + s^{-(1-v)\gamma}).
\]

This completes the proof of Proposition 3.

Our next tool concerns bounds of operator products.
Proposition 4 Let \(0 \leq v \leq 1/2, \, 0 < \lambda < 1\) and \(\mathcal{L}_{\lambda,s} := \{(w_{i,s}, x_i) : w_{i,s} \geq 0 \text{ and } x_i \in A\}\) be a positive quadrature rule on \(\mathbb{S}^d\) with degree \(s \in \mathbb{N}\). If \(\hat{\phi}_k \sim k^{-2\gamma}\) with \(\gamma > d/2\), then
\[
\|(\mathcal{L}_{\phi} + \lambda I)^{-1}(\mathcal{L}_{\phi,D,w_i} + \lambda I)\| \leq \tilde{c}(\lambda^{-1} s^{-1} + \lambda^{-1 + v} s^{-(1-v)\gamma} + 1)
\] (25)
and
\[
\|(L_{\phi,D,w_i} + \lambda I)^{-1}(\mathcal{L}_{\phi} + \lambda I)\| \leq 2\tilde{c}^{2} s^{-2} + 2\tilde{c}^{2} \lambda^{-2} s^{-2} + 2\tilde{c}^{2} \lambda^{-2} s^{-2} + \tilde{c}\lambda^{-1 + v} s^{-(1-v)\gamma} + \tilde{c}\lambda^{-1} s^{-1} + 1.\] (26)

Proof For positive operators \(A, B\), since
\[A^{-1}B = (A^{-1} - B^{-1})B + I = A^{-1}(B - A) + I,
\]
we have
\[
(\mathcal{L}_{\phi} + \lambda I)^{-1}(L_{\phi,D,w_i} + \lambda I) = (\mathcal{L}_{\phi} + \lambda I)^{-1}(L_{\phi,D,w_i} - \mathcal{L}_{\phi}) + I.
\]
Then it follows Proposition 3 that
\[
\|(\mathcal{L}_{\phi} + \lambda I)^{-1}(L_{\phi,D,w_i} + \lambda I)\| \leq \|(\mathcal{L}_{\phi} + \lambda I)^{-1}(L_{\phi,D,w_i} - \mathcal{L}_{\phi})\| + 1
\]
\[
\leq \|(\mathcal{L}_{\phi} + \lambda I)^{-1+v}\|(\mathcal{L}_{\phi} + \lambda I)^{-1}(L_{\phi,D,w_i} - \mathcal{L}_{\phi})\| + 1 \leq \tilde{c}\lambda^{-1 + v} s^{-(1-v)\gamma} + \tilde{c}\lambda^{-1} s^{-1} + 1.
\]
This proves (25). Concerning (26), we use the second order decomposition developed in [30], i.e.,
\[A^{-1}B = A^{-1}(B - A)B^{-1}(B - A) + B^{-1}(B - A) + I\]
and obtain
\[
(L_{\phi,D,w_i} + \lambda I)^{-1}(\mathcal{L}_{\phi} + \lambda I) = (L_{\phi,D,w_i} - \mathcal{L}_{\phi})(\mathcal{L}_{\phi} + \lambda I)^{-1}(L_{\phi,D,w_i} - \mathcal{L}_{\phi}) + I.
\]
Therefore, if \(v \leq 1/2\), we have from (25) and Proposition 3 that
\[
\|(L_{\phi,D,w_i} + \lambda I)^{-1}(\mathcal{L}_{\phi} + \lambda I)\| \leq 2\tilde{c}^{2} \lambda^{-2} s^{-2} + 2\tilde{c}^{2} \lambda^{-2} s^{-2} + \tilde{c}\lambda^{-1 + v} s^{-(1-v)\gamma} + \tilde{c}\lambda^{-1} s^{-1} + 1.
\]
This completes the proof of Proposition 4.

Our next bound is on the difference between \(L_{\phi,D,w_i} f_\rho\) and \(S_{D,w_i,YD}^\rho\).

Proposition 5 Let \(0 < \delta < 1\) and \(\mathcal{L}_{\lambda,s} := \{(w_{i,s}, x_i) : w_{i,s} \geq 0 \text{ and } x_i \in A\}\) be a quadratic rule on the sphere with \(0 \leq w_{i,s} \leq c_1|D|\). If \(\hat{\phi}_k \sim k^{-2\gamma}\) with \(\gamma > d/2\) and \(y_i = f^*(x_i) + \varepsilon_i\) with \(\varepsilon_i\) i.i.d. random noise satisfying \(E[\varepsilon_i] = 0\) and \(|\varepsilon_i| \leq M\) for some \(M > 0\), then with confidence \(1 - \delta\), there holds
\[
\|(\mathcal{L}_{\phi} + \lambda I)^{-1/2}(L_{\phi,D,w_i} f^* - S_{D,w_i,YD}^\rho)\|_{\phi} \leq \tilde{c} M\lambda^{-\frac{d}{2\gamma}} |D|^{-1/2} \log \frac{3}{\delta},
\]
where \(\tilde{c}\) is a constant depending only on \(d\) and \(c_1\).

To prove Proposition 5, we need two tools. The first one is the well known Hoeffding lemma [34].

Lemma 3 Let \(X\) be a random variable with \(E[X] = 0, a \leq X \leq b\). Then for \(u > 0\), there holds
\[
E\left[e^{uX}\right] \leq e^{u^2(b-a)^2/8}.
\]

The other is a variant of Hoeffding’s tail inequality.

Lemma 4 Let \(0 \leq w_{i,s} \leq c_1|D|^{-1}\). Then for any \(t > 0\) and \(k = 0, 1, \ldots\), we have
\[
P\left[\sum_{i=1}^{[D]} \sum_{f \neq i} w_{i,f'} (f^*(x_i) - y_i) (f^*(x_{f'}) - y_{f'}) P_{kD}^{t+1}(x_i, x_{f'}) \geq t\right] \leq \exp\left(-2M^{-4} c_1^{-4} t^2 |D|^2\right).
\]
Proof At first, we recall the well known Chernoff’s inequality, showing that for any \( t, b > 0 \) and random variable \( \xi \), there holds

\[
P[\xi \geq t] \leq \frac{E[e^{b\xi}]}{e^{bt}}.
\]

Set \( \xi_k = \sum_{i=1}^{[D]} \sum_{j \neq i} w_{i,j} w_{j,k}(f^+(x_i) - y_i)(f^+(x_j) - y_j)P_{k}^{d+1}(x_i \cdot x_j) \). Since \( \{\xi_i\} \) are independent random variables and

\[
E[y_i] = E[f^+(x_i) + \xi_i] = f^+(x_i) + E[\xi_i] = f^+(x_i),
\]

we have \( E[\xi_k] = 0 \) for any \( k = 0, 1, \ldots \). Then Chernoff’s inequality implies

\[
P[\xi_k \geq t] \leq e^{-bt}E[\exp(b\xi_k)]
\]

But \( 0 \leq w_{i,j} w_{j,k} \leq c_1|D|^{-1} \) and (2) yield

\[
|w_{i,j} w_{j,k}(f^+(x_i) - y_i)(f^+(x_j) - y_j)P_{k}^{d+1}(x_i \cdot x_j)| \leq c_1^2 M^2 |D|^{-2}.
\]

Hence, Lemma 3 implies for any \( i = 1, \ldots, |D| \),

\[
E\left[\exp\left(\sum_{i=1}^{[D]} \sum_{j \neq i} w_{i,j} w_{j,k}(f^+(x_i) - y_i)(f^+(x_j) - y_j)P_{k}^{d+1}(x_i \cdot x_j)\right)\right] \leq \exp(M^4 c_1^4 b^2 |D|^{-4}/2).
\]

Therefore, we obtain

\[
P[\xi_k \geq t] \leq e^{-bt} \prod_{i=1}^{[D]} \prod_{j \neq i} \exp(M^4 c_1^4 b^2 |D|^{-4}/2) \leq e^{-bt} e^{M^4 c_1^4 |D|^{-2}/2}.
\]

Setting \( b = 2M^4 c_1^{-4}|D|^2 t \), we obtain

\[
P[\xi_k \geq t] \leq \exp\left(-\frac{2t^2 |D|^2}{M^4 c_1^4}\right), \quad \forall k = 0, 1, \ldots
\]

This completes the proof of Lemma 4.

Based on the above lemma, we can prove Proposition 5 as follows.

Proof (Proof of Proposition 5) Due to definitions of \( L_{\phi,D,W_i} \) and \( S_{D,W_i}^T \), we have

\[
L_{\phi,D,W_i} f^* - S_{D,W_i}^T y_D = \sum_{i=1}^{[D]} w_{i,s}(f^+(x_i) - y_i) \phi_{x_i}.
\]

According to (27), we then get

\[
\left\| (L_{\phi} + \lambda I)^{-1/2}(L_{\phi,D,W_i} f^* - S_{D,W_i}^T y_D) \right\|_\phi^2
\]

\[
= \left( \sum_{i=1}^{[D]} \sum_{j \neq i} w_{i,j} w_{j,k}(f^+(x_i) - y_i)(L_{\phi} + \lambda I)^{-1} \phi_{x_i} \cdot \sum_{i=1}^{[D]} \sum_{j \neq i} w_{i,j} w_{j,k}(f^+(x_j) - y_j) \phi_{x_j} \right)_\phi
\]

\[
= \sum_{i=1}^{[D]} \left( \sum_{j \neq i} w_{i,j} w_{j,k}(f^+(x_i) - y_i)(f^+(x_j) - y_j)(L_{\phi} + \lambda I)^{-1} \phi_{x_i} \phi_{x_j} \right)_\phi
\]

\[
= \sum_{i=1}^{[D]} \left( \sum_{j \neq i} w_{i,j} w_{j,k}(f^+(x_i) - y_i)(f^+(x_j) - y_j)(L_{\phi} + \lambda I)^{-1} \phi_{x_i} \phi_{x_j} \right)_\phi
\]

\[
+ \sum_{i=1}^{[D]} w_{i,s}^2 (f^+(x_i) - y_i)^2 ((L_{\phi} + \lambda I)^{-1} \phi_{x_i} \phi_{x_i})_\phi.
\]
But \( \phi(t) = \sum_{k=0}^{\infty} \hat{\phi}_k \frac{d^d}{dt^d} P_{k+1}^d(t) \) and (3) yield
\[
\phi(x \cdot x') = \sum_{k=0}^{\infty} \hat{\phi}_k \sum_{\ell=1}^{d^d} Y_{k,\ell}(x) Y_{k,\ell}(x'), \quad \forall x, x' \in S^d.
\]

Then,
\[
(\mathcal{L}_\phi + \lambda I)^{-1} \phi = \sum_{k=0}^{\infty} (\hat{\phi}_k + \lambda I)^{-1} \hat{\phi}_k \sum_{\ell=1}^{d^d} Y_{k,\ell}(x) Y_{k,\ell}(x).
\]

Therefore, it follows from the reproducing property of \( \phi \) and (3) that
\[
\left\langle (\mathcal{L}_\phi + \lambda I)^{-1} \phi, \phi \right\rangle = \left\langle \sum_{k=0}^{\infty} (\hat{\phi}_k + \lambda I)^{-1} \hat{\phi}_k \sum_{\ell=1}^{d^d} Y_{k,\ell}(x) Y_{k,\ell}(x), \phi \right\rangle
\]
\[
= \sum_{k=0}^{\infty} (\hat{\phi}_k + \lambda I)^{-1} \hat{\phi}_k \sum_{\ell=1}^{d^d} Y_{k,\ell}(x) Y_{k,\ell}(x) = \sum_{k=0}^{\infty} (\hat{\phi}_k + \lambda I)^{-1} \hat{\phi}_k \frac{d^d}{\Omega_d} P_{k+1}^d(x \cdot x').
\]

Plugging the above equations into (27) and noting \( P_{k+1}^d(1) = 1 \), we have
\[
\left\| (\mathcal{L}_\phi + \lambda I)^{-1/2} (L_{\phi,D,W} f^* - S_{D,W}^T) \right\|_\phi^2 \leq \sum_{k=0}^{\infty} (\hat{\phi}_k + \lambda I)^{-1} \hat{\phi}_k \frac{d^d}{\Omega_d} \sum_{i=1}^{D} w_{i,s}^2 (f^*(x_i) - y_i)^2.
\]

Hence, Lemma 4 together with \( w_{i,s} \leq c_1 |D|^{-1} \) implies that with confidence \( 1 - \exp \left( -\frac{2^{2^2} |D|^2}{c_1^2} \right) \), there holds
\[
\left\| (\mathcal{L}_\phi + \lambda I)^{-1/2} (L_{\phi,D,W} f^* - S_{D,W}^T) \right\|_\phi^2 \leq \left( c_1^2 M^2 |D|^{-1} + t \right) \sum_{k=0}^{\infty} (\hat{\phi}_k + \lambda I)^{-1} \hat{\phi}_k \frac{d^d}{\Omega_d}.
\]

Noting further \( \hat{\phi}_k \sim k^{-2\gamma}, d^d \sim k^{d-1} \) and \( \gamma > d/2 \), we obtain
\[
\sum_{k=0}^{\infty} (\hat{\phi}_k + \lambda I)^{-1} \hat{\phi}_k \frac{d^d}{\Omega_d} \leq c_4 \sum_{k=0}^{\infty} \frac{k^{-2\gamma + d-1}}{k^{-2\gamma + \lambda}} \leq c_4 \int_1^{\infty} \frac{t^{d-1}}{1 + \lambda t^{2\gamma}} dt = c_5 \lambda^{-\frac{d}{2\gamma}},
\]
where \( c_4, c_5 \) are constants depending only on \( d \). Thus, with confidence \( 1 - \exp \left( -\frac{2^{2^2} |D|^2}{c_1^2} \right) \), there holds
\[
\left\| (\mathcal{L}_\phi + \lambda I)^{-1/2} (L_{\phi,D,W} f^* - S_{D,W}^T) \right\|_\phi^2 \leq c_6 \lambda^{-\frac{d}{2\gamma}} (t + |D|^{-1}),
\]
for \( c_6 \) a constant depending only on \( d \). Setting \( \delta = \exp \left( -\frac{2^{2^2} |D|^2}{c_1^2} \right) \), we obtain \( t = \frac{M^2}{\sqrt{2} |D|} \). Therefore, with confidence \( 1 - \delta \), there holds
\[
\left\| (\mathcal{L}_\phi + \lambda I)^{-1/2} (L_{\phi,D,W} f^* - S_{D,W}^T) \right\|_\phi^2 \leq c_7 M \lambda^{-\frac{d}{2\gamma} |D|^{-1/2} \log \frac{3}{\delta}}.
\]
where \( c_7 \) is a constant depending only on \( d \) and \( c_1 \). This completes the proof of Proposition 5.
5 Proofs

In this section, we prove our main results by using the integral operator approach established in Section 4.

5.1 Proof of Theorem 1

For \( f^* \in \mathcal{N}_\phi \), define

\[
f^*_{D,W_i} = \arg \min_{f \in \mathcal{N}_\phi} \sum_{(x_i,y_i) \in D} w_i \cdot (f(x_i) - f^*(x_i))^2 + \lambda \| f \|_\phi^2
\]  

(28)

as the noise-free version of \( f_{D,W_i} \). Then, it follows from Proposition 2 that

\[
f^*_{D,W_i} = (L_{\phi,D,W_i} + \lambda I)^{-1}L_{\phi,D,W_i}f^*.
\]  

(29)

Therefore, we have

\[
\left\| f_{D,W_i} - f^* \right\|_\psi \leq \left\| f^*_{D,W_i} - f^* \right\|_\psi + \left\| f_{D,W_i} - f^*_{D,W_i} \right\|_\psi.
\]  

(30)

The following lemma provides an estimate for the approximation error.

Lemma 5 Let \( \mathcal{D}_{\lambda,s} := \{ (w_i,x_i) : w_{i} \geq 0 \text{ and } x_i \in \Lambda \} \) be a positive quadrature rule on \( \mathbb{S}^d \) with degree \( s \in \mathbb{N} \). If \( \hat{\phi}_k \sim k^{-2\gamma} \) with \( \gamma > d/2 \), \( f^* \in \mathcal{N}_\phi \) and (11) holds with \( 0 \leq r \leq 1 \), then

\[
\left\| f^*_{D,W_i} - f^* \right\|_\psi \leq (4\epsilon^2 \lambda^{-1}s^{-2\gamma} + 2\epsilon s^{-\gamma} + \lambda)^{(1-r)/2}\| f^* \|_\phi.
\]  

(31)

Proof Since \( f^* \in \mathcal{N}_\phi \) and (11) holds, we have from (18), (20) and (29) that

\[
\left\| f^* - f^*_{D,W_i} \right\|_\psi = \left\| (L_{\phi,D,W_i} + \lambda I)^{-1}(L_{\phi,D,W_i} + \lambda I)f^* \|_\phi
\]  

\[
= \lambda \left\| (L_{\phi,D,W_i} + \lambda I)^{-1}(L_{\phi,D,W_i} + \lambda I)f^* \right\|_\phi
\]  

\[
\leq \lambda \left\| (L_{\phi,D,W_i} + \lambda I)^{-(r+1)/2}f^* \right\|_\phi
\]  

\[
= (4\epsilon^2 \lambda^{-1}s^{-2\gamma} + 2\epsilon s^{-\gamma} + \lambda)^{(1-r)/2}\| f^* \|_\phi.
\]

This completes the proof of Lemma 5.

Next, we aim to bound the estimate error, as shown in the following lemma.

Lemma 6 Let \( 0 < \delta < 1 \) and \( \mathcal{D}_{\lambda,s} := \{ (w_i,x_i) : w_{i} \geq 0 \text{ and } x_i \in \Lambda \} \) be a positive quadrature rule on \( \mathbb{S}^d \) with degree \( s \in \mathbb{N} \) satisfying \( 0 < w_{i} \leq c_1 \| D \|^{-1} \). If \( \hat{\phi}_k \sim k^{-2\gamma} \) with \( \gamma > d/2 \), \( f^* \in \mathcal{N}_\phi \), (11) holds with \( 0 \leq r \leq 1 \) and \( y_i = f(x_i) + \epsilon_i \) with \( \epsilon_i \) i.i.d. random noise satisfying \( E[\epsilon_i] = 0 \) and \( |\epsilon_i| \leq M \) for some \( M > 0 \), then with confidence \( 1 - \delta \), there holds

\[
\left\| f_{D,W_i} - f^*_{D,W_i} \right\|_\psi = \epsilon^2 M \lambda^{-\frac{2\gamma+1}{2}}(4\epsilon^2 \lambda^{-1}s^{-2\gamma} + 2\epsilon s^{-\gamma} + 1)\| D \|^{-1/2}\log \frac{3}{\delta}.
\]  

(33)
Proof Since \( f^* \in \mathcal{N}_\phi \) and \((11)\) holds, we have from \((24)\) and \((29)\) that
\[
\|f_{D,W_i,\lambda} - f_{D,W_i,\lambda}^*\| = \|L_{\phi}^{(1-r)/2}(f_{D,W_i,\lambda} - f_{D,W_i,\lambda}^*)\| \phi
= \|L_{\phi}^{(1-r)/2}(L_{\phi,D,W_i,\lambda + \lambda I})^{-1}(S_{D,W_i,YD} - \mathcal{L}_{\phi} f^*)\| \phi
\leq \lambda^{-r/2}\|(L_{\phi} + \lambda I)^{1/2}(S_{D,W_i,YD} - \mathcal{L}_{\phi} f^*)\| \phi
\leq \lambda^{-r/2}\|(L_{\phi} + \lambda I)^{1/2}(L_{\phi,D,W_i,\lambda + \lambda I})^{-1/2}\|(S_{D,W_i,YD} - \mathcal{L}_{\phi} f^*)\| \phi.
\]
Therefore, Proposition \(4\) with \( v = 0 \), Proposition \(5\) and \((32)\) yield that with confidence \( 1 - \delta \), there holds
\[
\|f_{D,W_i,\lambda} - f_{D,W_i,\lambda}^*\| = \lambda^{-r/2}(4\epsilon^2\lambda^{-2s^{-2}\gamma} + 2\epsilon\lambda - 1s^{-\gamma} + 1)c'\lambda^{-\frac{\gamma}{2}}|D|^{-1/2}\log \frac{3}{\delta}.
\]
This completes the proof of Lemma \(6\).

Now we are in a position to prove Theorem \(1\).

Proof (Proof of Theorem \(7\)) Plugging \((33)\) and \((31)\) into \((30)\), we obtain that with confidence \( 1 - \delta \), there holds
\[
\|f_{D,W_i,\lambda} - f^*\| \leq \left(4\epsilon^2\lambda^{-2s^{-2}\gamma} + 2\epsilon\lambda - 1s^{-\gamma} + 1\right)c'\lambda^{-\frac{\gamma}{2}}|D|^{-1/2}\log \frac{3}{\delta}.
\]
Then for \( s \geq \lambda^{-1/2} \), the above estimate yields
\[
\|f_{D,W_i,\lambda} - f^*\| \leq c_8 \left(\lambda^{\frac{1}{2}} + \lambda^{-\frac{2\gamma d}{D}}|D|^{-1/2}\right)\log \frac{3}{\delta},
\]
where \( c_8 := (1 + 4\epsilon^2 + 2\epsilon)\max\{\|f^*\|, c'M\} \). Noting further \( \lambda \sim |D|^{-\frac{2\gamma d}{D}} \), we get that with confidence \( 1 - \delta \), there holds
\[
\|f_{D,W_i,\lambda} - f^*\| \leq C|D|^{\frac{1}{2}}\lambda^{-\frac{2\gamma d}{D}}\log \frac{3}{\delta},
\]
where \( C \) is a constant depending only on \( d, M \) and \( \|f^*\| \). The proof of Theorem \(1\) is completed.

5.2 Proof of Theorem \(2\)

To prove Theorem \(2\) we need the following error decomposition strategy.

Lemma 7 Let \( \tilde{f}_{D,W_i,\lambda} \) be defined by \((9)\). Then, we have
\[
E[\|\tilde{f}_{D,W_i,\lambda} - f^*\|^2] \leq \sum_{j=1}^{m} \frac{|D_j|^2}{|D|^2} E \left[\|f_{D_j,W_j,\lambda_j} - f^*\|^2\right] + \sum_{j=1}^{m} \frac{|D_j|^2}{|D|^2} \left[\|f_{D_j,W_j,\lambda_j} - f^*\|^2\right],
\]
where \( f_{D_j,W_j,\lambda_j}^* \) is the noise-free version of \( f_{D_j,W_j,\lambda_j} \) defined by
\[
f_{D_j,W_j,\lambda_j}^* = (L_{\phi,D_j,W_j,\lambda_j} + \lambda_j I)^{-1}L_{\phi,D_j,W_j,\lambda_j}f^*.
\]
Proof} Due to \( \sum_{j=1}^m \frac{|D_j|}{|D|} = 1 \), we have

\[
\| \hat{T}_{D,w_j, \lambda} - f^* \|^2 = \left\| \sum_{j=1}^m \frac{|D_j|}{|D|} (f_{D_j, w_{j,s}, \lambda_j} - f^*) \right\|^2
\]

\[
= \sum_{j=1}^m \frac{|D_j|^2}{|D|^2} \| f_{D_j, w_{j,s}, \lambda_j} - f^* \|^2 + \sum_{j=1}^m \frac{|D_j|}{|D|} \left( f_{D_j, w_{j,s}, \lambda_j} - f^* \sum_{k \neq j} \frac{|D_k|}{|D|} (f_{D_k, w_{k,s}, \lambda_k} - f^*) \right).
\]

Taking expectations, we have

\[
E \left[ \left\| \hat{T}_{D,w_j, \lambda} - f^* \right\|^2 \right] = \sum_{j=1}^m \frac{|D_j|^2}{|D|^2} E \left[ \left\| f_{D_j, w_{j,s}, \lambda_j} - f^* \right\|^2 \right]
\]

\[
+ \sum_{j=1}^m \frac{|D_j|}{|D|} \left( E_{D_j} [ f_{D_j, w_{j,s}, \lambda_j} - f^* ] , E [ \hat{T}_{D,w_j, \lambda} - f^* ] - f^* - \frac{|D_j|}{|D|} E_{D_j} [ f_{D_j, w_{j,s}, \lambda_j} - f^* ] \right).
\]

But

\[
\sum_{j=1}^m \frac{|D_j|}{|D|^2} \left( E_{D_j} [ f_{D_j, w_{j,s}, \lambda_j} - f^* ] , E [ \hat{T}_{D,w_j, \lambda} - f^* ] - f^* \right) = \| E [ \hat{T}_{D,w_j, \lambda} ] - f^* \|^2.
\]

Then,

\[
E \left[ \left\| \hat{T}_{D,w_j, \lambda} - f^* \right\|^2 \right] = \sum_{j=1}^m \frac{|D_j|^2}{|D|^2} E \left[ \left\| f_{D_j, w_{j,s}, \lambda_j} - f^* \right\|^2 \right]
\]

\[
- \sum_{j=1}^m \frac{|D_j|^2}{|D|^2} \left( E [ f_{D_j, w_{j,s}, \lambda_j} ] - f^* \right)^2 + \| E [ \hat{T}_{D,w_j, \lambda} ] - f^* \|^2.
\]

Noting further from \( \xi \) that

\[
E [ \hat{T}_{D,w_j, \lambda} ] = \sum_{j=1}^m \frac{|D_j|}{|D|} f_{D_j, w_{j,s}, \lambda_j},
\]

we obtain from \( \sum_{j=1}^m \frac{|D_j|}{|D|} = 1 \) that

\[
\| E [ \hat{T}_{D,w_j, \lambda} ] - f^* \|^2 = \left\| \sum_{j=1}^m \frac{|D_j|}{|D|} (f_{D_j, w_{j,s}, \lambda_j} - f^*) \right\|^2 \leq \sum_{j=1}^m \frac{|D_j|}{|D|} \| f_{D_j, w_{j,s}, \lambda_j} - f^* \|^2,
\]

which proves the bound in \( \xi \). The proof of Lemma 7 is completed.

For further analysis, the following lemma is needed.

**Lemma 8** Let \( \xi \) be a random variable with nonnegative values. If \( \xi \leq \mathcal{A} \log^b \frac{\mathcal{A}}{\delta} \) holds with confidence \( 1 - \delta \) for some \( \mathcal{A}, b, c > 0 \) and any \( 0 < \delta < 1 \), then

\[
E [ \xi ] \leq c \Gamma ( b + 1 ) \mathcal{A},
\]

where \( \Gamma ( \cdot ) \) is the Gamma function.

**Proof** Since \( \xi \leq \mathcal{A} \log^b \frac{\mathcal{A}}{\delta} \) holds with confidence \( 1 - \delta \), we have for any \( t > 0 \),

\[
P [ \xi > t ] \leq c \exp \{ - \mathcal{A} t^{-1/b} \}
\]

Using the probability to expectation formula

\[
E [ \xi ] = \int_0^\infty P [ \xi > t ] dt
\]
Furthermore, we obtain from (34) with \( D \) being replaced by \( D_j \) that

This completes the proof of Theorem 2.

Based on the above lemmas, we can prove Theorem 2 as follows.

**Proof (Proof of Theorem 2)** For any \( 1 \leq j \leq m \), since \( \hat{f}_k \sim k^{-2\gamma} \) with \( \gamma > d/2 \), \( f^* \in \mathcal{M}_0 \), (11) holds, \( \Lambda_j = \{ x_{i,j} \}_{i=1}^{|D_j|} \) be \( \tau_i \)-quasi uniform and \( \mathcal{L}_{A_j,s_j} := \{ (w_{i,j}, s_{i,j}) : w_{i,j,s} \geq 0 \text{ and } x_{i,j} \in A_j \} \) be a quadrature rule satisfying \( 0 < w_{i,j,s} \leq c_1 |D_j|^{-1} \), it follows from Lemma 5 with \( \Lambda \) being replaced by \( \Lambda_j \) that

\[
\| f^*_D, w_{i,j,s}, \lambda_j - f^* \|_w \leq (4\varepsilon^2 \lambda_j^{-2} s_j^{-2\gamma} + 2\varepsilon s_j^{-\gamma} + \lambda_j)^{(1-r)/2} \| f^* \|_\phi. \tag{37}
\]

Furthermore, we obtain from (34) with \( D \) being replaced by \( D_j \) that with confidence \( 1 - \delta \), there holds

\[
\| f^*_D, w_{i,j,s}, \lambda_j - f^* \|_w \leq (4\varepsilon^2 \lambda_j^{-2} s_j^{-2\gamma} + 2\varepsilon s_j^{-\gamma} + \lambda_j)^{(1-r)/2} \| f^* \|_\phi + c M \lambda_j^{-\frac{2\gamma+d}{2r}} (4\varepsilon^2 \lambda_j^{-2} s_j^{-2\gamma} + 2\varepsilon s_j^{-\gamma} + \lambda_j)^{(1-r)+1} |D_j|^{-1/2} \log^3 \frac{3}{\delta}. \tag{38}
\]

Applying Lemma 8 with \( \varepsilon^* = 2(\varepsilon^r)^2 M^2 \lambda_j^{-\frac{2\gamma+d}{2r}} (4\varepsilon^2 \lambda_j^{-2} s_j^{-2\gamma} + 2\varepsilon s_j^{-\gamma} + \lambda_j)^{(1-r)+1} |D_j|^{-1} \), \( b = 2 \) and \( c = 3 \), we get from (38) that

\[
E[\| f^*_D, w_{i,j,s}, \lambda_j - f^* \|_w^2] \leq 2(4\varepsilon^2 \lambda_j^{-2} s_j^{-2\gamma} + 2\varepsilon s_j^{-\gamma} + \lambda_j)^{(1-r)+1} \| f^* \|_\phi^2 + 12(\varepsilon^r)^2 M^2 \lambda_j^{-\frac{2\gamma+d}{2r}} (4\varepsilon^2 \lambda_j^{-2} s_j^{-2\gamma} + 2\varepsilon s_j^{-\gamma} + \lambda_j)^{(1-r)+1} |D_j|^{-1}. \tag{39}
\]

Plugging (39) and (37) into (35), we have

\[
E[\| \bar{f}_{D,W_j,\lambda} - f^* \|_w^2] \leq 2 \sum_{j=1}^m \left( \frac{|D_j|^2}{|D|^2} + \frac{|D_j|}{|D|} \right) (4\varepsilon^2 \lambda_j^{-2} s_j^{-2\gamma} + 2\varepsilon s_j^{-\gamma} + \lambda_j)^{(1-r)+1} \| f^* \|_\phi^2 + 12(\varepsilon^r)^2 M^2 \sum_{j=1}^m \frac{|D_j|^2}{|D|^2} \lambda_j^{-\frac{2\gamma+d}{2r}} (4\varepsilon^2 \lambda_j^{-2} s_j^{-2\gamma} + 2\varepsilon s_j^{-\gamma} + \lambda_j)^{(1-r)+1} |D_j|^{-1}.
\]

Since \( s_j \geq \lambda_j^{-1/\gamma} \), we have

\[
E[\| \bar{f}_{D,W_j,\lambda} - f^* \|_w^2] \leq c_9 \sum_{j=1}^m \frac{|D_j|}{|D|} \left( \lambda_j^{1-r} + \frac{1}{|D|} \lambda_j^{-\frac{2\gamma+d}{2r}} \right),
\]

where \( c_9 \) is a constant depending only on \( d, r, \gamma, M \) and \( \| f^* \|_\phi \). Noting further \( \lambda_j \sim |D|^{-\frac{2\gamma}{2\gamma+d}} \), the above estimate yields

\[
E[\| \bar{f}_{D,W_j,\lambda} - f^* \|_w^2] \leq 2 c_9 |D|^{-\frac{2(1-r)\gamma}{2\gamma+d}}.
\]

This completes the proof of Theorem 2.
6 Simulations

In this section, we conduct several numerical simulations to verify our theoretical statements and show the excellent performance of DWRLS. We compare the following three methods: distributed filtered hyperinterpolation (DFH) proposed in [53], WRLS, and DWRLS, where WRLS with training all samples in a batch mode is considered as a baseline. Two functions are used to generate samples for simulation. The first function is considered as a baseline. Two functions are used to generate samples for simulation. The first function is constructed via the Wendland function [8]

\[ \hat{\Psi}(u) = (1-u)^8 (32u^3 + 25u^2 + 8u + 1), \]

where \( u_+ = \max\{u, 0\} \), and it is defined by

\[ f_1(x) = \sum_{i=1}^{10} \hat{\Psi}(|x - z_i|_2), \]

where \( z_i \ (i = 1, \cdots, 10) \) are the center points of the regions of an equal area partitioned by Leopardi’s recursive zonal sphere partitioning procedure [27]. The second function is the Franke function, modified by Renka [43].

\[ f_2(x) = 0.75 \exp\left(-\frac{(9x^{(1)} - 2)^2}{4} - \frac{(9x^{(2)} - 2)^2}{4} - \frac{(9x^{(3)} - 2)^2}{4}\right) + 0.75 \exp\left(-\frac{(9x^{(1)} + 1)^2}{49} - \frac{(9x^{(2)} + 1)^2}{10} - \frac{(9x^{(3)} + 1)^2}{10}\right) + 0.5 \exp\left(-\frac{(9x^{(1)} - 7)^2}{4} - \frac{(9x^{(2)} - 3)^2}{4} - \frac{(9x^{(3)} - 5)^2}{4}\right) - 0.2 \exp\left(-\frac{(9x^{(1)} - 4)^2}{4} - \frac{(9x^{(2)} - 7)^2}{4} - \frac{(9x^{(3)} - 5)^2}{4}\right), \]

where \( x = (x^{(1)}, x^{(2)}, x^{(3)})^T \).

Before describing the simulations, we introduce the generation process of the simulation data as follows: First, Womersley’s symmetric spherical 45-designs [51] are used to generate 1038 points \( \{x_i\}_{i=1}^{1038} \) on the unit sphere. Second, the points \( \{x_i\}_{i=1}^{1038} \) are rotated by the rotation matrix

\[ A_j := \begin{pmatrix} \cos(j\pi/10) & -\sin(j\pi/10) & 0 \\ \sin(j\pi/10) & \cos(j\pi/10) & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

to obtain new points \( \{x_{ij}\}_{j=1}^{10} \) for \( j = 1, \cdots, 10 \), i.e., \( x_{ij} = A_j \hat{x}_i \), and the points \( \{x_{ij}\}_{i=1,j=1}^{1038,10} \) are used as the inputs of training samples; the corresponding outputs \( \{(y_{ij})_{i=1,j=1}^{1038,10}\} \) are generated by

\[ y_{ij} = f_j(x_{ij}) + \varepsilon_{ij}, \quad i = 1, \cdots, 1038, \quad j = 1, \cdots, 10, \]

where \( \varepsilon_{ij} \) is the independent Gaussian noise \( \mathcal{N}(0, 0.1^2) \). In this way, there are a total of \( N = 10380 \) training samples. Finally, the inputs \( \{x_i^r\}_{i=1}^{N_r} \) of testing samples are \( N_r = 10000 \) generalized spiral points on the unit sphere, and the corresponding outputs \( \{y_i^r\}_{i=1}^{N_r} \) are generated by \( y_i^r = f_j(x_i^r) \).

Some implementation details of simulations are described as follows. For DFH, the training samples are equally distributed to 10 local machines, i.e., the samples \( D_j := \{(x_{ij}, y_{ij})\}_{i=1}^{1038} \) obtained by the \( j \)-th rotation matrix are located on the \( j \)-th local machine, and the parameter \( L \) related to the polynomial degree is selected from the set \( \{2, 4, \cdots, 40\} \). For DWRLS, the training samples are distributed to \( m \) \((m \geq 10)\) local machines in the following way: Let \( \tau := \text{mod}(m, 10) \). If \( \tau = 0 \), i.e., \( m \) can be divided by 10, then the samples of each set \( D_j \) are randomly and equally distributed to \( m/10 \) local machines. If \( r > 0 \), we randomly choose \( r \) sets from \( \{D_j\}_{j=1}^{10} \): the samples of each chosen set are equally distributed to \( \lfloor m/10 \rfloor \) local machines; the samples of each set of the remaining \( 10 - \tau \) sets are equally distributed to \( \lfloor m/10 \rfloor \) local machines. In the execution of WRLS and DWRLS, we use the positive definite function \( \phi_2(x, x') = \Psi(||x - x'||_2) \) with the regularization parameter \( \lambda \) being chosen from the set \( \{\frac{\lambda}{2}\}_{\frac{\lambda}{2}} > 10^{-10}, q = 0, 1, 2, \cdots\} \) for the approximation of function \( f_1 \). For the approximation of function \( f_2 \), the positive definite function is defined as \( \phi_1(x, x') = \exp\left(-\frac{||x - x'||^2}{2\sigma^2}\right) \) with the regularization parameter \( \lambda \) being chosen from the set \( \{\frac{\lambda}{2}\}_{\frac{\lambda}{2}} > 10^{-10}, q = 0, 1, 2, \cdots\} \) and the width \( \sigma \) being
chosen from 10 values which are drawn in a logarithmic, equally spaced interval $[0.1, 1]$. All parameters in the simulations are selected by grid search.

Our first aim of simulation is to compare DWRLS with DFH. The result is shown in Figure 1. It can be found in Figure 1 that for the same partitions of data, DWRLS is at least as good as DFH. This is not a surprising phenomenon since our theoretical assertions are made under the Sobolev error estimate while the result of DFH \cite{33} is only derived in $L^2(S^d)$, showing that our analysis is available to numerous types of data.

Our second aim is to show the relation between the fitting performance of DWRLS and the number of local servers. The numerical results is shown in Figure 2. There are two interesting findings from Figure 2: 1) The RMSE of DWRLS is somewhat stable with the number of servers $m$. Taking the Franke function for example, the RMSE changes from 0.013 to 0.020 when the number of servers increases from 1 to 100; 2) When $m$ is not so large, DWRLS performs similarly as WRLS, which verifies our theoretical assertions. Both findings show the excellent performance of DWRLS in fitting noisy spherical data.

Our third purpose is to explicitly demonstrate the performance of DWRLS via plotting the ground truth functions, the fitted functions and the approximation error. The first rows of Figures 3 and 4 concern the ground truth functions and fitted functions, while the second rows exhibit the noise of ground truth and approximation errors. In this simulation, the number of local servers of DFH is always set to be 10 while that of DWRLS is set to be 10, 50, 100, respectively. From these two figures, it can be found that the fitted functions are very similar as the ground truth functions for DWRLS with not so large $m$, i.e., $m = 10, 50$. Furthermore, the fitted RMSE is near to zero for DWRLS with $m = 10, 50$, though it is different from the ground truth noise. The main reason is that the regularization term $\lambda \|f\|_0^3$ in the definition of DWRLS provides a trade-off between the approximation error and estimate error in (30). Under this circumstance, the fitted error can be much smaller than the ground truth error, provided the regularization parameter $\lambda$ is appropriately tuned. This also verifies the power of DWRLS in fitting noisy spherical data.
Appendix: Proofs of Proposition 1 and Proposition 2

To prove Proposition 1, we need the following spherical Marcinkiewicz-Zygmund inequalities for spherical polynomials [10, 11].

**Lemma 9** Let $\mathcal{Q}_{A,s} := \{ (w_{i,s}, x_i) : w_{i,s}, x_i \in \Lambda \}$ be a positive quadrature rule on $S^d$ with degree $s \in \mathbb{N}$. For any $P \in \Pi_{s'}^d$ with $s' \in \mathbb{N}$, there holds

$$\sum_{x_i \in A} w_{i,s} |P(x_i)|^2 \leq \tilde{c}_1 \left( \frac{s'}{s} \right)^d \|P\|_{L^2(S^d)}^2, \quad s' > s,$$

$$\tilde{c}_2 \|P\|_{L^2(S^d)}^2 \leq \sum_{x_i \in A} w_{i,s} |P(x_i)|^2 \leq \tilde{c}_3 \|P\|_{L^2(S^d)}^2, \quad s' \leq s,$$

where $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ are constants depending only on $d$.

**Proof (Proof of Proposition 1)** For any $h \in \mathcal{N}_\varphi$, we get $\eta_{\lambda,a}(L\varphi) h \in \mathcal{N}_\varphi$. Set $h_{-1,\lambda,a}(x) = 0$ and

$$h_{j,\lambda,a}(x) := \sum_{k=0}^{2^j} \eta_{\lambda,a}(\hat{\phi}_k) \sum_{\ell=1}^{Z(d,k)} \hat{h}_{k,\ell} Y_{k,\ell}(x).$$
We have from $\eta_{\lambda,u}(t) = (\lambda + t)^{-u}$ and $\hat{\phi}_k \sim k^{-2\gamma}$ with $\gamma > d/2$ that

$$\eta_{\lambda,u}(L_\phi)h = \sum_{j=0}^\infty h_{j,\lambda,u}(x) - h_{j-1,\lambda,u}(x)$$

and

$$\|h_{j,\lambda,u} - h_{j-1,\lambda,u}\|_{L^2(\mathbb{S}^d)} \leq \|\eta_{\lambda,u}(L_\phi)h\|_{L^2(\mathbb{S}^d)} + \|h_{j,\lambda,u} - \eta_{\lambda,u}(L_\phi)h\|_{L^2(\mathbb{S}^d)}$$

$$\leq 2 \left( \sum_{k=2^{j-1}+1}^{2^j} \phi_k \right) \left( \sum_{\ell=1}^{Z(d,k)} (h_{\ell,\ell})^2 \right)^{1/2}$$

$$\leq \tilde{c}_4 \left( \frac{2^{-\gamma}}{(2^{-2\gamma} + \lambda^{2\gamma})} \right) \|h\|_\phi,$$

where $\tilde{c}_4$ is a constant depending only on $\gamma$. For any $f, g \in \mathcal{A}_\phi$ and $u, v \in [0, 1]$, we get from (44) that

$$(\eta_{\lambda,u}(L_\phi)f)(x)(\eta_{\lambda,v}(L_\phi)g)(x) = \sum_{j=0}^L (f_{j,\lambda,u} - f_{j-1,\lambda,u})(g_{\ell,\lambda,v} - g_{\ell-1,\lambda,v})$$

$$+ \sum_{j=L+1}^{2^L} (f_{j,\lambda,u} - f_{j-1,\lambda,u})(g_{\ell,\lambda,v} - g_{\ell-1,\lambda,v}),$$

where $L$ is the unique integer satisfying

$$2^L \leq s < 2^{L+1}$$

and $f_{j,\lambda,u}, g_{\ell,\lambda,v}$ are similar as $h_{j,\lambda,u}$ with $h$ being replacing by $f, g$, respectively. Denote

$$\mathcal{A}_{j,\lambda,u,v} := \left| \int_{\mathbb{S}^d} \left( f_{j,\lambda,u}(x) - f_{j-1,\lambda,u}(x) \right) \left( g_{\ell,\lambda,v}(x) - g_{\ell-1,\lambda,v}(x) \right) d\omega(x) \right.$$

$$- \sum_{x_i \in \Xi} w_{i,s} \left( f_{j,\lambda,u}(x_i) - f_{j-1,\lambda,u}(x_i) \right) \left( g_{\ell,\lambda,v}(x_i) - g_{\ell-1,\lambda,v}(x_i) \right).$$

Since $\mathcal{P}_{\Xi,s} = \{(w_{i,s}, x_i) : w_{i,s} > 0 \text{ and } x_i \in \Xi\}$ is a positive quadrature rule on $\mathbb{S}^d$ with degree $s \in \mathbb{N}$, we have

$$\mathcal{A}_{j,\lambda,u,v} = 0, \quad \forall 2^{j+\ell} \leq 2L \leq s.$$

Hence,

$$\left| \int_{\mathbb{S}^d} (\eta_{\lambda,u}(L_\phi)f)(x)(\eta_{\lambda,v}(L_\phi)g)(x) d\omega(x) - \sum_{x_i \in \Xi} w_{i,s} (\eta_{\lambda,u}(L_\phi)f)(x_i)(\eta_{\lambda,v}(L_\phi)g)(x_i) \right|$$

$$\leq \sum_{j=L}^{2^L} \mathcal{A}_{j,\lambda,u,v} = \left( \sum_{j=0}^{L} \sum_{\ell=0}^{L} + \sum_{j=L}^{2^L} \sum_{\ell=0}^{L} + \sum_{j=L}^{2^L} \sum_{\ell=L+1}^{2^L} \right) \mathcal{A}_{j,\lambda,u,v}$$

$$=: I_1 + I_2 + I_3 + I_4.$$

But the Hölder inequality implies

$$\mathcal{A}_{j,\lambda,u,v} \leq \|f_{j,\lambda,u} - f_{j-1,\lambda,u}\|_{L^2(\mathbb{S}^d)} \|g_{\ell,\lambda,v} - g_{\ell-1,\lambda,v}\|_{L^2(\mathbb{S}^d)}$$

$$+ \left( \sum_{x_i \in A} w_{i,s} \left( f_{j,\lambda,u}(x_i) - f_{j-1,\lambda,u}(x_i) \right)^2 \right)^{1/2} \left( \sum_{x_i \in A} w_{i,s} \left( g_{\ell,\lambda,v}(x_i) - g_{\ell-1,\lambda,v}(x_i) \right)^2 \right)^{1/2}.$$
Then we get from Lemma 9 and (45) that

\[
\mathcal{A}_{f,t,\ell} \leq \tilde{c}_1 \left( \frac{2^{-y(j+\ell)}}{(2-2y+\lambda)^y}\|f\|_\phi \|g\|_\phi \right) + \tilde{c}_2 \begin{cases} \\
\end{cases}
\]

where \(\tilde{c}_1, \tilde{c}_2\) are constants depending only on \(d, \gamma\). Since \(u + v \leq 1\), we have for any \(\lambda > 0\) that

\[
I_1 = \sum_{j+\ell > L, j, \ell \leq L} \mathcal{A}_{f,t,\ell} \leq \tilde{c}_3 \sum_{j+\ell > L, j, \ell \leq L} 2^{-y(j+\ell)} \|f\|_\phi \|g\|_\phi \leq \tilde{c}_3 \left\{ \begin{array}{l} 2^{-y(j+\ell)} \|f\|_\phi \|g\|_\phi \quad \text{if } j + \ell > L, j, \ell \leq L, \\
2^{-y(j+\ell)} \|f\|_\phi \|g\|_\phi \quad \text{if } j > L, \ell \leq L, \\
2^{-y(j+\ell)} \|f\|_\phi \|g\|_\phi \quad \text{if } j \leq L, \ell > L, \\
2^{-y(j+\ell)} \|f\|_\phi \|g\|_\phi \quad \text{if } j > L, \ell > L, \\
\end{array} \right. \\
\]

where \(\tilde{c}_3\) are constants depending only on \(u, v, \gamma, d\). Furthermore, for any \(u, v \in [0,1]\) satisfying \(u + v \leq 1\), and any \(\lambda \geq 2^{-2yL}\), we have from \(\gamma > d/2\) that

\[
I_2 \leq \tilde{c}_{4,1} \sum_{k=L+1} 2^{-y(j+\ell)} \|f\|_\phi \|g\|_\phi \leq \tilde{c}_{4,2} \sum_{k=L+1} 2^{-y(j+\ell)} \|f\|_\phi \|g\|_\phi \\
\]

where \(\tilde{c}_{4,j}\) for \(j = 2, 3, 4\) are constants depending only on \(d, \gamma, u, v\). Combining all the above estimates, we obtain the desired estimate with \(\tilde{c} := \tilde{c}_{3,1} + \tilde{c}_{3,2} + \tilde{c}_{3,3} + \tilde{c}_{3,4}\). This completes the proof of Proposition 1.

**Proof (Proof of Proposition 2)** Since the functional derivative of (8) is

\[
2 \sum_{j=1}^{D} w_{i,j} (f(x_j) - y_j) \phi_{y_j} + 2\lambda f,
\]

we get

\[
\sum_{j=1}^{D} w_{i,j} (y_i - f_{D,W,\lambda}(x_i)) \phi_{y_i} = \lambda f_{D,W,\lambda}.
\]

This together with (23) and (24) implies

\[
(L_{D,W} + \lambda I) f_{D,W,\lambda} = S_{D,W} f_D,
\]

Noting further \(L_{D,W}\) is a positive operator, we then have

\[
f_{D,W,\lambda} = (L_{D,W} + \lambda I)^{-1} S_{D,W} f_D.
\]

This completes the proof of Proposition 2.
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References

1. R. Bathis. Matrix Analysis, Volume 169 of Graduate Texts in Mathematics. Springer, Berlin, 1997.
2. Å. Björck, Numerical Methods for Least Squares Problem. SIAM, Philadelphia, 1996.
3. G. Brown, F. Dai, Approximation of smooth functions on compact two-point homogeneous spaces, J. Funct. Anal., 220: 401-423, 2005.
4. J. S. Brauchart, K. Hesse, Numerical integration over spheres of arbitrary dimension, Constr. Approx., 25: 41-71, 2007.
5. A. Caponnetto, E. De Vito, Optimal rates for the regularized least squares algorithm, Found. Comput. Math., 7: 331-368, 2007.
6. X, Chang, S. B. Lin, Y. Wang, Divide and conquer local average regression, Electron. J. Statist., 11: 1326-1350, 2017.
7. X. Chang, S. B. Lin, D. X. Zhou, Distributed semi-supervised learning with kernel ridge regression, J. Mach. Learn. Res., 18 (46): 1-22, 2017.
8. A. Chernih, I. H. Sloan, R. S. Womersley, Wendland functions with increasing smoothness converge to a Gaussian, Adv. Comput. Math., 40: 185-200, 2014.
9. F. Cucker, D. X. Zhou, Learning Theory: An Approximation Theory Viewpoint, Cambridge University Press, Cambridge, 2007.
10. F. Dai, Multivariate polynomial inequalities with respect to doubling weights and $A^*$ weights, J. Funct. Anal., 235 (1): 137-170, 2006.
11. F. Dai, On generalized hyperinterpolation on the sphere, Proc. Amer. Math. Soc., 2931-2941, 2006.
12. M. Dittmar, Nuclear energy: Status and future limitations, Energy, 37 (1): 35-40, 2012.
13. S. Dodelson, Modern Cosmology, Academic Press, London, 2003.
14. G. E. Fasshauer, L. L. Schumaker, Scattered data fitting on the sphere, Math. Methods Curves & Surfaces II, 117-166, 1998.
15. W. Freeden, T. Gervens, M. Schreiner, Constructive Approximation on the Sphere, Oxford University Press Inc., New York, 1998.
16. Z. C. Guo, S. B. Lin, D. X. Zhou, Distributed learning with spectral algorithms, Inverse Probl., 33: 074009, 2017.
17. L. Györfi, M. Kohler, A. Krzyzak, H. Walk, A Distribution-Free Theory of Nonparametric Regression, Springer, Berlin, 2002.
18. T. Hangelbroek, F. J. Narcowich, J. D. Ward, Kernel approximation on manifolds I: bounding the Lebesgue constant, SIAM J. Math. Anal., 42 (4): 1760-1762, 2010.
19. T. Hangelbroek, F. J. Narcowich, X. Sun, J. D. Ward, Kernel approximation on manifolds II: the $L_p$ norm of the $L_2$ projector, SIAM J. Math. Anal., 43 (2): 662-684, 2011.
20. T. Hangelbroek, F. J. Narcowich, X. Sun, J. D. Ward, Polyharmonic and related kernels on manifolds: interpolation and approximation, Found. Comput. Math., 12 (5): 625-670, 2012.
21. K. Hesse, I. H. Sloan, R. S. Womersley, Radial basis function approximation of noisy scattered data on the sphere, Numer. Math., 137: 579-605, 2017.
22. Q. T. Le Gia, F. J. Narcowich, J. D. Ward, H. Wendland, Continuous and discrete least-squares approximation by radial basis functions on spheres, J. Approx. Theory, 143: 124-133, 2007.
23. X. Guo, L. Li, Q. Wu, Modeling interactive components by coordinate kernel polynomial models. Math. Found. Comput., 3(4): 263-277, 2020.
24. T. Hu, Q Wu, D. X. Zhou, Distributed kernel gradient descent algorithm for minimum error entropy principle, Appl. Comput. Harmonic Anal., 49(1): 229-256, 2020.
25. J. Keiner, S. Kunis, D. Potts, Efficient reconstruction of functions on the sphere from scattered data, J. Fourier Anal. Appl., 13: 435-458, 2007.
26. K. Jetter, J. Stöckler, J. D. Ward, Error estimates for scattered data interpolation on spheres, Math. Comput., 68: 743-747, 1999.
27. P. Leopardi, A partition of the unit sphere into regions of equal area and small diameter, Electronic Trans. Numer. Anal., 25: 309-327, 2006.
28. J. Levesley, X. Sun, Approximation in rough native spaces by shifts of smooth kernels on spheres, J. Approx. Theory, 133: 269-283, 2005.
29. J. Lin, A. Rudi, L. Rosasco, V. Cevher, Optimal rates for spectral algorithms with least-squares regression over Hilbert spaces, Appl. Comput. Harmon. Anal., 48: 868-890, 2020.
30. S. B. Lin, X. Guo, D. X. Zhou, Distributed learning with regularized least squares, J. Mach. Learn. Res., 18(92): 1-31, 2017.
31. S. B. Lin, D. X. Zhou, Distributed kernel-based gradient descent algorithms, Constr. Approx., 47: 249-276, 2018.
32. S. B. Lin, Nonparametric regression using needlet kernels for spherical data, J. Complex., 50: 60-83, 2019.
33. S. B. Lin, Y. G. Wang, D. X. Zhou, Distributed filtered hyperinterpolation for noisy data on the sphere, SIAM J. Numer. Anal., 59: 634-659, 2021.
34. P. Massart, Concentration inequalities and model selection, Ecole d’Et de Probabilits de Saint-Flour XXXIII-2003. Springer, 2007.
35. H. N. Mhaskar, F. J. Narcowich, J. D. Ward, Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature, Math. Comput., 70: 1113-1130, 2001.
36. H. N. Mhaskar, F. J. Narcowich, J. Prestin, J. D. Ward, $L^p$ Bernstein estimates and approximation by spherical basis functions, Math. Comput., 79 (2010), 1647-1679.
37. N. Mücke, G. Blanchard, Parallelizing spectrally regularized kernel algorithms. J. Mach. Learn. Res., 19: 1-29, 2018.
38. C. Müller, Spherical Harmonics, Lecture Notes in Mathematics, Vol. 17, Springer, Berlin, 1966.
39. F. J. Narcowich, P. Petrushev and J. D. Ward, Localized tight frames on spheres, SIAM J. Math. Anal., 38: 574-594, 2006.
40. F. J. Narcowich, X. P. Sun, J. D. Ward, H. Wendland, Direct and inverse sobolev error estimates for scattered data interpolation via spherical basis functions, Found. Comput. Math., 7: 369-370, 2007.
41. F. J. Narcowich, J. D. Ward, Scattered data interpolation on spheres: Error estimates and locally supported basis functions, SIAM J. Math. Anal., 33: 1393-1410, 2002.
42. C. H. Reigber, H. Luehr, P. Schwintzer, CHAMP mission status, Adv. Space Res., 30 (2): 129-134, 2002.
43. R. J. Renka, Multivariate interpolation of large sets of scattered data, ACM Trans. Math. Software, 14: 139-148, 1988.
44. I. J. Schoenberg, Positive definite functions on spheres, Duke Math. J., 9: 96-108, 1942.
45. I. H. Sloan, Polynomial interpolation and hyperinterpolation over general regions, J. Approx. Theory, 83: 238-254, 1995.
46. S. Smale, D. X. Zhou, Shannon sampling II: Connections to learning theory, Appl. Comput. Harmonic Anal., 19: 285-302, 2005.
47. S. Smale, D. X. Zhou, Learning theory estimates via integral operators and their approximations, Constr. Approx. 26: 153–172, 2007.
48. G. Szego, Orthogonal Polynomials, American Mathematical Society, New York, 1967.
49. Y. T. Tsai, Z. C. Shih, All-frequency precomputed radiance transfer using spherical radial basis functions and clustered tensor approximation, ACM Trans. Graph., 25: 967-976, 2006.
50. D. L. Wang, H. L. Xu, Q. Wu, Averaging versus voting: A comparative study of strategies for distributed classification, Math. Found. Comput., 3: 185–193, 2020.
51. R. S. Womersley, Efficient spherical designs with good geometric properties, In Contemporary computational mathematics-A celebration of the 80th birthday of Ian Sloan (pp. 1243-1285). Springer, Cham.
52. Y. Zhang, J. C. Duchi, M. J. Wainwright, Divide and conquer kernel ridge regression: A distributed algorithm with minimax optimal rates, J. Mach. Learn. Res., 16: 3299-3340, 2015.