Cubical rectangles are defined and explored formally here; even though, they have appeared implicitly in quite many distinct areas of discrete mathematics. They originally were applied in threshold logic as the fundamental elements to define completely monotonic functions [12,15,21]. Elsewhere, they appeared in study of a hierarchy of pseudo-Boolean functions [11] or in the introduction of c-complexes in [8], and finally they have been surfaced as very special sublattices in Boolean lattices [9].

We enumerate and characterize cubical rectangles in order to construct new posets, transforming into special lattices that will be called rectangular lattices here and are denoted by $\mathcal{R}_n$. They are closely related to the cubical lattices $\mathcal{C}_n$, that is the face lattice of the $n-$cube. The latter lattices have been highlighted in a landmark paper of Metropolis-Rota, where they presented a dimension-free characterization of $\mathcal{C}_n$ [13,14]. The rectangular lattices $\mathcal{R}_n$ are the main topics of Theorems 4, 5 and 6; where, the previous results on cubical rectangles are applied and the connection between $\mathcal{R}_n$ and $\mathcal{C}_n$ is explored.

There are indications that geometry of these basic combinatorial objects, that is the cubical rectangles, has not received that much of attention in threshold logic. For instance in Muroga [15], where the author refers to them as parallelogram on page 127. In fact, there is no parallelogram in the $n$-cube that is not a rectangle, but the figure on the same page of [15] gives the opposite impression.

The c-complexes and their relatives cut-complexes of [8] show a different path of adventures on cubical rectangles. They are closely related to the geometry of threshold logic and the cut-number problem over the $n$–cube that were the initial root and the main guideline for this line of research, see [5,6,17,22]. Also for different but related research in threshold logic see [1,4,16].

The hypercube or the unit $n$–cube embedded in the Euclidean space $R^n$ has been a major core of study in many different disciplines such as combinatorics, coding theory, Boolean optimization, computer architectures, and many more. In particular they manifest themselves in various formats; for instance, $C^n = \{(x_1, \ldots, x_n) \in R^n | 0 \leq x_i \leq 1 \}$ is a convex polytope whose vertex set $B^n = \{(x_1, \ldots, x_n) \in R^n | x_i = 0, 1 \}$ is the $n$-dimensional Boolean cube. The vertices and edges of the polytope $C^n$ form the well-known geometric graph $Q_n$. In the latter graph, two vertices are adjacent if their coordinates are the same except exactly one. Thus, $Q_n$ contains $2^n$ vertices and $n \cdot 2^{n-1}$ edges, moreover it is a compound of lower dimensional subcubes that are called $k$–faces for...
0 ≤ k ≤ n. Overall \( Q_n \) has \( 2^{n-k} \binom{n}{k} \) k-faces, where vertices and edges are 0, and 1-dimensional faces respectively. For more on the basics and terminology of the graphs, polytopes, lattices, and threshold logic, see [20,10,9,15].

Here in this note, by \( n \)-cube we mean \( C^n \) embedded in the Euclidean space \( R^n \) whose graph is \( Q_n \) and its center is \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) \in R^n \). Avoiding the trivial cases throughout the paper, we always assume \( n ≥ 3 \). Our main problem here is to enumerate and characterize cubical rectangles, that is those subsets of \( B^n \) consisting of four co-planar vertices that form a rectangle (counting squares just as special regular rectangles). A word of caution is that the classical squares or 2-faces of \( C^n \) or equivalently 4-cycles of \( Q_n \) are all cubical rectangles, but they are not the only squares in this new class. In fact, cubical rectangles form a larger class than the family of 4-cycles of \( Q_n \). For instance, they also include all central rectangles that pass through the center of \( k \)-dimensional faces for each \( k, 3 ≤ k ≤ n \).

Let \( (x, y, z, w) \) be an ordered 4-tuple of elements in \( B^n \). Then the convex hull \( \text{conv}(x, y, z, w) = \text{conv}\{x, y, z, w\} \) is called a cubical rectangle if the vector \( \overrightarrow{zw} \) is a translation of \( \overrightarrow{xy} \), or equivalently \( \overrightarrow{yw} \) is a translation of \( \overrightarrow{xz} \). In other words \( \text{conv}(x, y, z, w) \) is a cubical rectangle if and only if \( w - z = y - x \). The cubical rectangle \( \text{conv}(x, y, z, w) \) will be denoted by \( < x; y; z, w > \), where the four vertices form a plane rectangle whose vertex set is \( \{x, y, z, w\} \) and the segments \( xw, yz \) are its diagonals.

A central rectangle of the \( n \)-cube is a cubical rectangle passing through the center of the cube. A chord is a closed segment joining two vertices of the \( n \)-cube. In particular, a diagonal of the \( n \)-cube is a chord joining two opposite vertices and an edge is a chord joining two adjacent vertices. Here we enumerate the cubical rectangles by various methods and study their incidence relations with vertices and chords.

Let \( x, y \in B^n \) be two distinct vertices, then the Hamming distance between \( x \) and \( y \) is defined by \( d_H(x, y) = |\{i|1 ≤ i ≤ n, x_i ≠ y_i\}| \). A chord with the end vertices \( x, y \in B^n \) and \( d_H(x, y) = i \) is said to be of Hamming length \( i \), and is called an \( i \)-chord. Hence, a diagonal of \( Q_n \), that is a chord of Hamming length \( n \), is an \( n \)-chord and evidently there are \( 2^{n-1} \) diagonals in the \( n \)-cube. Every two distinct diagonals determine uniquely a central rectangle, and so there are exactly \( \binom{2^{n-1}}{2} \) central rectangles. A cubical rectangle of type \( i \) is one
with two parallel sides of Hamming length \( i \). So a cubical rectangle of type 1 contains at least one pair of parallel edges of the \( n \)-cube. Evidently, a cubical rectangle whose sides are of Hamming length \( i \), and \( j \) belongs to two cubical rectangles of types \( i \), and \( j \) simultaneously. The first enumeration result as follows:

**Theorem 1.**

(a) The total number of cubical rectangles \( r_n \) in the \( n \)-cube is

\[
r_n = 2^{n-3}(3^n - 2^{n+1} + 1),
\]

(b) The total number of cubical rectangles containing a given vertex \( x \) of the \( n \)-cube is

\[
r_n(x) = \frac{1}{2}(3^n - 2^{n+1} + 1),
\]

**Proof.**

(a): To count the total number of cubical rectangles of part (a), we begin with rectangles of type \( i \) and consider first the easy case of \( i = 1 \). Fixing an edge and its parallel class of \( 2^{n-1} \) elements, there are \( \binom{2^{n-1}}{2} \) rectangles formed by different pairs of edges in this parallel class. Overall there are \( n \) different parallel classes of edges, that is counting

\[
n \binom{2^{n-1}}{2}
\]

rectangles that includes repeated squares of type 1. In fact any 2-face of the \( n \)-cube has been counted twice that will be corrected in the rest of the proof.

For the general case that is counting the cubical rectangles of type \( i \), consider a fixed \( i \)-face and its parallel class that includes \( 2^{n-i} \) distinct \( i \)-faces. Each \( i \)-face contain \( 2^{i-1} \) distinct diagonals of Hamming length \( i \), so for each pair of parallel \( i \)-faces there are \( 2^{i-1} \) cubical rectangles of type \( i \). All together, we obtain

\[
2^{i-1} \binom{2^{n-i}}{2}
\]

rectangles of type \( i \), but there are \( \binom{n}{i} \) distinct parallel classes of \( i \)-faces. Consequently, the total number of rectangles of type \( i \) will be

\[
\binom{n}{i} 2^{i-1} \binom{2^{n-i}}{2}.
\]

Now, adding them up and considering that each rectangle is counted twice in two different directions, we obtain \( r_n \), the total number of cubical rectangles in the \( n \)-cube:
\[ r_n = \frac{1}{2} \sum_{i=1}^{n-1} \binom{n}{i} 2^{i-1} \binom{2n-i}{2} = \sum_{i=1}^{n-1} \binom{n}{i} 2^{i-3}(2^{n-i} - 1)2^{n-i} \]

\[ = 2^{n-3} \sum_{i=1}^{n-1} \binom{n}{i} (2^{n-i} - 1). \quad (*) \]

\[ = 2^{n-3} \left( \sum_{i=1}^{n-1} \binom{n}{i} 2^{n-i} - \sum_{i=1}^{n-1} \binom{n}{i} \right) = 2^{2n-3} \left( \sum_{i=1}^{n-1} \binom{n}{i} 2^{-i} \right) - 2^{n-3}(2^n - 2) \]

\[ = 2^{2n-3}((1 + \frac{1}{2})^n - 1 - 2^{-n}) - 2^{n-3}(2^n - 2) \]

\[ = 2^{n-3}(3^n - 2^{n+1} + 1) \]

(b) In this part we prove part (b) first and then apply the result to obtain an alternative proof of (a). We begin by counting the cubical rectangles containing a given vertex \(x\) and a fixed chord \(e\) containing \(x\). Considering first the case when the rectangles are of type 1, that is \(e\) is an edge of the \(n\)-cube, there are \(2^{n-1} - 1\) rectangles containing \(e\). Since there exists exactly \(n\) edges containing \(x\), there are overall \(n(2^{n-1} - 1)\) such rectangles with repeated squares that have been counting twice. However, the double counting also occurs for all rectangles of different types and will be corrected by a factor \(\frac{1}{2}\) before the total number in the following argument. For the general case of rectangles of type \(i\), there are \(\binom{n}{i} (2^{n-i} - 1)\) rectangles containing the vertex \(x\), including the double counting. Hence the total number of distinct rectangles of all types containing the vertex \(x\) will be:

\[ \frac{1}{2} \sum_{i=1}^{n-1} \binom{n}{i} (2^{n-i} - 1), \]

where, the factor \(\frac{1}{2}\) corrects the double counting error; that each rectangle has been counted twice. However, the latter sum is \(r_n(x) = \frac{1}{2}(3^n - 2^{n+1} + 1)\), applying the results of the first part, following the computations next to the identity \((*)\). This proves part (b).

(b) \(\rightarrow\) (a):

Now, to conclude (a) from (b), we count all the rectangles for all the \(2^n\) vertices,
considering that each rectangle has been counted 4 times, we have:

\[ r_n = \frac{1}{4}2^n r_n(x) = 2^{n-3}(3^n - 2^{n+1} + 1). \]

To grasp a view of the geometric nature of these rectangles, two basic and natural questions are raised. First, are the vertices of a cubical rectangle \( T \) the only vertices of the \( n \)-cube that belong to the 2-dimensional plane of the cubical rectangle \( T \)?, that is exactly the affine hull of \( T \) denoted by \( \text{aff}(T) \). Second, are the central rectangles the only one that intersect with the interior of the \( n \)-cube \( C^n \). In other words, Is there any cubical rectangle that meets the interior of the cube, but does not pass through the center. These questions will be answered in Theorem 2, but beforehand we should recall more on basics of the hypercube. The convex polytope \( C^n \) is homeomorphic to a closed \( n \)-dimensional ball in \( R^n \), and has \( 2n \) facets \(((n-1)\text{-dimensional faces)}\), where all are \((n-1)\)-dimensional cubes embedded in the relative boundary of \( C^n \). All the facets have unique centers that determine uniquely the facets. This is true for all the faces and their centers, that is two faces of the \( n \)-cube are identical if and only if they have the same center. Intersections of any two faces is also a face, and this also defines the meet operation in the cubical lattice \( R_n \). The cube hull of a vertex set \( S \), denoted by \( \text{cub} S \) is the smallest face containing \( S \), that is exactly the join of \( S \) in the cubical lattice. For an \( i \)-chord \( e \), \( \text{cub}(e) \) is the \( i \)-face whose center is the midpoint of \( e \). The diagonal dimension, or simply the dimension, of a cubical rectangle \( T \) is \( \text{dim}(\text{cub} T) \). A cubical rectangle of dimension \( i \) is said to be an \( i \)-rectangle, and evidently the \( n \)-rectangles are the same as central ones.

**Theorem 2.**

(a) The central rectangles of the \( n \)-cube are the only cubical rectangles that have nonempty intersection with the interior of the polytope \( C^n \).

(b) Let \( T \) be a cubical rectangle in the \( n \)-cube, then \( T = (\text{aff} T) \cap C^n \) in other words, the convex polygon \((\text{aff} T) \cap C^n \) is a rectangle and coincides with \( T \).

**Proof.**

(a) Let \( T = < x, y; z, w > \) be a cubical rectangle of the \( n \)-cube that is not central, that is the Hamming length of its diagonals \( xw \) and \( yz \) are at most \( n - 1 \). Hence, \( \text{cub}\{x, w\} \) and \( \text{cub}\{y, z\} \) are proper faces of the \( n \)-cube whose common centers is the center of \( T \), so they are identical faces. This implies that \( \text{cub}\{x, w\} = \text{cub}\{y, z\} = \text{cub}\{x, w, y, z\} = \text{cub} T \) is a proper face of dimension at most \( n - 1 \). Therefore, \( T \) is contained in a facet of the cube and has no interior points of the \( n \)-cube.

(b) A consequence of (a) is that any cubical rectangle \( T \) is central for the face \( \text{cub} T \), so we need to prove the assertion of the theorem only for central rectangles. Let \( T \) be a central rectangle such that \((\text{aff} T) \cap C^n \) is not a rect-
angle, that is a convex $k$-gon with $k \geq 5$. It is clear that all sides of $T$ have Hamming length of at most $n - 1$ and so lie in some facet of the $n$-cube. However, the polygon $(\text{aff } T) \cap C^n$ contains $T$ and its interior must contain at least one side of $T$, since $k \geq 5$. This contradicts the fact that all sides of $T$ lie in some facet of the $n$-cube.

We proceed to give another short proof of Theorem 1(a), applying the geometric result of Theorem 2. In any $i$-face of the $n$-cube there are exactly $inom{2i-1}{2}$ central rectangles of dimension $i$, therefore the total number of rectangles is obtained by

$$r_n = \sum_{i=2}^{n} \binom{n}{i} 2^{n-i} \binom{2i-1}{2} = 2^{n-2} \sum_{i=2}^{n} \binom{n}{i} 2^{i-1} 2^{n-2} \sum_{i=2}^{n} \binom{n}{i}, \text{ by applying } \binom{2i-1}{2} = \frac{1}{2} 2^{i-1}(2^{i-1}-1).$$

$$= 2^{n-3}[(\sum_{i=0}^{n} \binom{n}{i} 2^i) - 1 - 2n] - 2^{n-2}[(\sum_{i=0}^{n} \binom{n}{i}) - 1 - n]$$

$$= 2^{n-3}(3^n - 1 - 2n) - 2^{n-2}(2^n - 1 - n) = 2^{n-3}(3^n - 2^{n+1} + 1)$$

**Chords and their incidence relations.**

Chords form another class of combinatorial objects that include all the edges of the $n$-cube and are closely related to the cubical rectangles and vertices of the $n$-cube. Vertices, chords, and cubical rectangles are connected to each others via three incidence relations between them. The way they are related is the essential theme of Theorem 3. To explore these incidence relations, we first need to recall a few observations. Theorem 1(b) explains the incidence relation between vertices and the rectangles. It is also easy to see that each vertex is contained in $2^{n-1}$ chords, and of course each chord contains exactly 2 vertices. Consequently, what is only left to uncover is the relation between chords and rectangles that is the subject of the next Theorem 3. Evidently, each cubical rectangle includes 6 chords, but there are two different ways that this inclusion could occur. A chord in general can be included, as a side or a diagonal, in different rectangles, except when the chord is an edge or a diagonal of the $n$-cube, where only one of the two cases could occur. This leads us to two new types of inclusions. A chord, that is contained in in a given cubical
rectangle $T$, is said to be $d$-included (d for diagonally) in $T$ if it is a diagonal of $T$. And it is said to be $s$-included (s for sidewise) in $T$ if it is a side of the rectangle $T$.

**Theorem 3.**

(a) Each $i$–chord of the $n$–cube is included in $2^{i-1}+2^{n-i}-2$ cubical rectangles, from which $2^{i-1}-1$ are $d$-inclusions and the remaining $2^{n-i}-1$ are $s$-inclusions.

(b) Each of the three numbers of part (a) leads to a different enumeration of the total number of rectangles stated in Theorem 1 (a).

**Proof.**

(a) An $i$–chord $e$ is contained in the unique $i$–face $F = \text{cub}(e)$ that contains all the $i$-rectangles that include $e$, as a diagonal, by the argument given in part (a) of Theorem 2. There are $2^{i-1} - 1$ such central rectangles of $F$, since every pair of diagonals in $F$ uniquely determine one rectangle of this type and the number of $i$–chords in $F$ different from $e$, is exactly $2^{i-1} - 1$. Now consider again $F = \text{cub}(e)$. There are $2^{n-i}-1$ of $i$–faces parallel to $F$, and each contains a unique $i$–chord parallel to $e$. These are all the cubical rectangles that contain $e$ as a side and that proves part (a).

(b) Each $i$–chord is contained in a unique $i$–face and each $i$–face includes $2^{i-1}$ of $i$-chords as its diagonals. Then there are a total of $2^{i-1} \binom{n}{i} 2^{n-i} = \binom{n}{i} 2^{n-1}$ of $i$–chords, since $\binom{n}{i} 2^{n-i}$ is the total number of $i$–faces.

Combining the result of part (a) and the fact that each cubical rectangle contains 6 chords, we recount again the total of cubical rectangles:

$$r_n = \frac{1}{6} \sum_{i=1}^{n} \binom{n}{i} 2^{n-1}(2^{i-1} + 2^{n-i} - 2) =$$

$$= \frac{1}{6} 2^{n-1}\left(\sum_{i=1}^{n} \binom{n}{i} 2^{i-1} + \sum_{i=1}^{n} \binom{n}{i} 2^{n-i} - 2\sum_{i=1}^{n} \binom{n}{i}\right) =$$

$$= 2^{n-2}(3^n - 1) + 2^{2n-1}\left(\frac{3}{2}n - 1\right) + 2^n(2^n - 1) =$$

$$= \frac{1}{6}(\frac{3}{4})(6^n - 2(4^n) + 2^n) = 2^{n-3}(3^n - 2^{n+1} + 1).$$

The latter computations have been based on the usual inclusion of chords in
cubical rectangles. We may replace this inclusion by d-inclusion or s-inclusion. That means there are two slightly different proofs that are obtained from the one of part (b) by replacing \((2^{i-1} + 2^{n-i} - 2)\) with \(2^{i-1} - 1\) and \(2^{n-i} - 1\) respectively. Moreover, we also need to interchange the factor \(\frac{1}{6}\) with \(\frac{1}{2}\) and \(\frac{1}{4}\) respectively.

**Rectangular Lattices.**

Constructing a new and different application of cubical rectangles in this section, a notational convenience is in order. Here, we consider chords or cubical rectangles only as finite sets of vertices of the \(n\)-cube. In other words, cubical rectangles are 4 element subsets of \(B^n\) that form a plane rectangle and chords are 2 element subsets of the set of vertices. In the following theorem, \(V = B^n\) denotes the set of vertices, \(C\) stands for the set of all chords and \(R\) represents the set of all cubical rectangles of the \(n\)-cube. Let us define a new poset \(P\) over the \(n\)-cube, where \(P := V \cup C \cup R\) that is ordered by inclusion. The poset \(P\) will be called VCR poset over the \(n\)-cube, where all the sets \(V, C, R\) and \(P\), are actually dependent on \(n\). As long as there is no confusion and \(n\) is fixed, in order to simplify the notations, we replace \(V(n), C(n), R(n)\) and \(P(n)\), by \(V, C, R\) and \(P\), respectively. Evidently, the set \(V\), the set of chords \(C\), and \(R\) form 3 levels of maximal anti-chains in the poset. Let \(R_n\) be the poset that is obtained from \(P\) by joining top and bottom elements to it. The following theorem shows that \(R_n\) is a lattice and will be called the rectangular lattice of the \(n\)-cube.

**Theorem 4.**

(a) The VCR is a poset of size \(2^{n-3}(3^n + 2^{n+2} - 2^{n+1} + 5)\) that is transformed into the rectangular lattice \(R_n\), if it is joined with 0 and 1. Moreover, the resulting rectangular lattice \(R_n\) is graded, atomic and nondistributive.

(b) The rectangular lattice \(R_n\) is the smallest lattice, ordered by inclusion, that contains both the set of vertices and the cubical rectangles.

**Proof. (a)** The intersection of any two distinct but non-disjoint cubical rectangles either is a vertex or a chord. In general, the intersection of any two non-disjoint elements of \(P\) is in \(P\). Hence the poset will become a lattice if it is joined with zero and one elements. The resulting lattice \(R_n\) is obtained by attaching the empty set and the entire vertex set to the poset, serving as 0 and 1 respectively. The meet operation in the resulting lattice is the intersection and the join \(x \vee y\), of \(x, y\) will be the smallest element in \(P\) that contains both \(x\) and \(y\).

Any chord covers two vertices and is covered by at least two cubical rectangles. Consequently, the covering relation between chords and rectangles or between chords and vertices are the only coverings in the VCR poset, and then any maximal chain has the same length in the resulting rectangular lattice \(R_n\).
that is of rank 4 and so is graded. A chord is the join of its end vertices and any cubical rectangle is the join of a pair of parallel chords. Thus, any element of the lattice is the join of a set of vertices and so the lattices is atomic. On the other hand, the set of vertices $V$ is the set atoms of $P$, and it is also the set of join irreducible elements $\mathcal{J}(P)$ of $P$. Then $O(\mathcal{J}(P)) = O(V)$, where $O(V)$ denotes the poset of down-sets (or order ideals) of $V$. However, $O(V)$ is a Boolean lattice of $2^2n$ elements and is different from $P$, that concludes the $VCR$ lattice $P$ is nondistributive, applying Birkhoff’s representation theorem.

(b) We must show that each rectangular lattice $\mathcal{R}_n$, as a subposet of the Boolean lattice $\mathcal{P}(B^n)$ that is ordered by inclusion, is the smallest lattice that contains $V \cup R$. Since any finite lattice includes 0 and 1, then to complete the proof we need only to show the claim that $C \subset L$ for any lattice $L$ that is a subposet of $\mathcal{P}(B^n)$ with $V \cup R \subset L$. Recall that the intersection of any two distinct non-disjoint cubical rectangles either is a single vertex or a chord. Now, consider any given chord containing two end vertices $x$ and $y$, then there exist 2 distinct cubical rectangles $R_1$ and $R_2$ containing both $x$ and $y$. So $\{x, y\} \subset R_1 \cap R_2 \in L$, since the intersection is the meet operation in $L$. Therefore, $\{x, y\} = R_1 \cap R_2 \in L$ and our claim $C \subset L$ holds.

**Remark 1. Cubical lattices and the Metropolis-Rota’s conjecture.**
The rectangular lattices $\mathcal{R}_n$ of Theorem 4 are closely related to the cubical lattices $\mathcal{C}_n$; the connection that leads us to a fruitful discussion on cubical lattices. New interests in cubical lattices were raised by a well-known paper of Metropolis-Rota, where the authors have presented a dimension-free characterization of these non-distributive lattices [13,14]. The article includes an informal conjecture on a broad application of cubical lattices; the informality that overshadows the existence of the conjecture. In fact, what the authors have published contains more than a specific problem to solve, but they laid out a proposal for a new area of research similar to the Boolean algebra that is called cubical algebra in this article. Here is what they have stated in [13,14]:

"We are led to surmise that the second such face structure, or cubical algebra as we shall call it, ought to play a complementary role to the Boolean algebra."

The authors also explained how they expected their characterization result should play a complementary role:

"The resulting algebraic structure is suited for application to synthesis problems for Boolean functions."

Since then, many articles have been appeared on the cubical lattices citing this paper, for instance see [2,3], however in connection with the main conjecture much more research remains to be done. Back in Nineties, we applied cubical lattice techniques to establish a geometric connection to threshold logic via
geometric flavors from convex polytopes [6]. And then later on, again applying cubical lattice methods, we have presented a characterization result on cut-complexes in [7] that is exactly recognizing a class of Boolean functions and thus moving forward toward the conjecture of Metropolis-Rota on the cubical lattices. Thus, more research on the connection between rectangular lattices $\mathcal{R}_n$ above and the cubical lattices $\mathcal{C}_n$ is natural and will be interesting to explore. Next two theorems provide some clues in this direction.

The nondistributivity of Theorem 4 has been improved to nonmodularity in the following theorem.

**Theorem 5.** The two lattices $\mathcal{R}_n$ and $\mathcal{C}_n$, $n \geq 2$, are both nonmodular, and in fact there exists a common nonmodular sublattice $N_5$ of 5 elements contained in both of them.

**Proof.** There are two cases here, where the $M_3-N_5$-Theorem will be applied. First, we consider the case of $n = 2$. Suppose that the $n$-cube is square $abcd$, where $ab$ and $cd$ are two parallel edges. Here, $\mathcal{R}_2$ and $\mathcal{C}_2$ are vertex-edge lattice of the complete graph $K_4$ and a 4-cycle respectively. Then, a common sublattice $N_5$ is $\{0, 1, a, ab, cd\}$. For $n \geq 3$, suppose $ef$ is any 2-face of the $n$-cube, where $e$ and $f$ are parallel edges. Let $e'f'$ be the square opposite to $ef$. In other words $ee'$ and $ff'$ form two central cubical rectangles. Then in this case a common sublattice $N_5$ that works for both $\mathcal{R}_n$ and $\mathcal{C}_n$, will be $\{0, 1, e, ef, e'f'\}$.

**Remark 2.**
Defining a new relation $\sim$ over $\mathcal{R}_n$, we say $S \sim T$ if $\text{cub } S = \text{cub } T$ with $S, T \in \mathcal{R}_n$. It is easy to observe that $\sim$ is an equivalence relation over $\mathcal{R}_n$. Recall that by any chord or cubical rectangle $T$ of dimension $i$, we mean $\text{dim}(\text{cub } T) = i$. Thus, any $i$-rectangle is equivalent to an $i$-chord if they have a common center. In fact in any face $F$, all the central chords (diagonals) and all the central rectangles are all equivalent to the face $F$ itself. And conversely, any two equivalent elements $S$ and $T$ with $S, T \in \mathcal{C} \cup \mathcal{R}$ must share the same center, where $\mathcal{C}$ and $\mathcal{R}$ are the set of chords and rectangles respectively. In other words, the elements of an equivalence class $[T]$ of $T \in \mathcal{R}_n$ are $\text{cub } T$, all of the central rectangles of $\text{cub } T$ and all of the central chords in $\text{cub } T$. Considering the equivalence relation $\sim$ above, we define an order over the the set of all equivalence classes in $\mathcal{R}_n/\sim$ as follows: Let $S, T \in \mathcal{R}_n$, then we define $[S] \prec [T]$ if $\text{cub } S \subset \text{cub } T$. Finally, we can state the theorem that describes the main relation between $\mathcal{R}_n$ and $\mathcal{C}_n$. 

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Theorem 6.
The quotient set $\mathcal{R}_n / \sim$ equipped with the order $\prec$ above is a lattice that is isomorphic to $\mathcal{C}_n$.

Proof. The structure of the equivalence classes described in Remark 2 can be applied to show that the map $\varphi : \mathcal{R}_n / \sim \rightarrow \mathcal{C}_n$ defined by $\varphi([S]) = \text{cub } S$, is an order isomorphism and hence a lattice isomorphism between two lattices. The isomorphism is in fact induced from the order-preserving map $\varphi : \mathcal{R}_n \rightarrow \mathcal{C}_n$ defined by $\varphi(S) = \text{cub } S$.

The meet and join operations over $\mathcal{R}_n / \sim$ are defined naturally from the order $\prec$ in this lattice. Let $[S] \land [T] = [Y]$, where $Y \in \mathcal{R}_n$, then we define $\text{cub } Y = \text{cub } S \cap \text{cub } T$. And similarly Let $[S] \lor [T] = [Z]$, where $Z \in \mathcal{R}_n$, then we have here $\text{cub } Z = \text{cub } (\text{cub } S \cup \text{cub } T)$.

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