Polynomial of Best Uniform Approximation to $1/x$ and Smoothing in Two-level Methods

Johannes Kraus · Panayot Vassilevski · Ludmil Zikatanov

In memory of Sergei Nepomnyaschikh - a pioneer in domain decomposition methods

Abstract — We derive defect correction scheme for constructing the sequence of polynomials of best approximation in the uniform norm to $1/x$ on a finite interval with positive endpoints. As an application, we consider two-level methods for scalar elliptic partial differential equation (PDE), where the relaxation on the fine grid uses the aforementioned polynomial of best approximation. Based on a new smoothing property of this polynomial smoother that we prove, combined with a proper choice of the coarse space, we obtain as a corollary, that the convergence rate of the resulting two-level method is uniform with respect to the mesh parameters, coarsening ratio and PDE coefficient variation.

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1. Introduction

The polynomial of best approximation in uniform norm to $1/x$ on a finite interval can be found in different forms in many classical texts on approximation theory, for example, see [10, p. 33, Eq. (4.25)], [11, Exercise 1.20]. In fact, the approximating polynomial for $1/(t-a)$, $a > 1$, has already been discovered by Chebyshev in 1887, see [4]. Here, we derive a defect correction algorithm for constructing the sequence of polynomials of best approximation and show several results important for the applications, namely, sufficient conditions for positivity and the monotonicity of this sequence.

As an application we study two-level methods with smoothers based on the polynomial of best approximation to $1/x$ on a finite interval $[\lambda_{\min}, \lambda_{\max}]$, $0 < \lambda_{\min} < \lambda_{\max}$, in the $\| \cdot \|_\infty$ (uniform) norm. We base our iterations on a three term recurrence which is given

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For such smoothers we show a smoothing property of the polynomial and as a corollary, based on an abstract two-level estimate we derive two-level (TL) or two-grid (TG) convergence estimates in the case of discretized elliptic PDE with heterogeneous coefficients. The estimate explicitly depends on the degree of the polynomial (or on the range of the spectrum which needs to be resolved by the smoother). We prove that if coarse spaces with stability and approximation properties that are robust with respect coefficient variation are used, then the two-level methods with polynomial smoothers based on the polynomial of best approximation to \(1/x\) are robust with respect to the variation in the coefficients of the PDE. Several examples of coarse spaces that provide the required contrast independent approximation property are available in the literature, cf., e.g., [6], [12], and earlier [1] as modified recently in [3]).

The paper is organized as follows. In Section 2 we introduce the three-term recurrence relation for the polynomial of best approximation to \(1/x\) and give description of algorithms for computing such approximations with matrix arguments. We then discuss the properties of the sequence of polynomials of best approximation to \(1/x\) in Section 3. In Section 4 we discuss and prove the smoothing property of the polynomial smoother. This property explicitly involves the polynomial degree and we use it in an abstract two-level convergence result. As a corollary, we derive an estimate for the convergence rate in case of finite element discretization of scalar elliptic PDE with coarse spaces that provide contrast independent approximation resulting in a contrast independent two-grid convergence. This convergence behavior is illustrated also with numerical tests in Section 5.

2. Best polynomial approximation to \(1/x\) in uniform norm

We begin with notation and some simple and well known definitions related to Chebyshev polynomials. We consider a finite interval, \([\lambda_{\text{min}}, \lambda_{\text{max}}]\), with \(0 < \lambda_{\text{min}} < \lambda_{\text{max}} < \infty\). We denote

\[
\kappa = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}, \quad \sigma = \frac{1}{\lambda_{\text{max}} - \lambda_{\text{min}}}, \quad a = \frac{\lambda_{\text{max}} + \lambda_{\text{min}}}{\lambda_{\text{max}} - \lambda_{\text{min}}} = \frac{\kappa + 1}{\kappa - 1}.
\] (2.1)

Note that \(a > 1\) and \(\sigma > 0\). The change of variables

\[
t = \frac{2}{\lambda_{\text{max}} - \lambda_{\text{min}}} \left( x - \frac{\lambda_{\text{max}} + \lambda_{\text{min}}}{2} \right) = 2\sigma x - a,
\]

maps the interval \([\lambda_{\text{min}}, \lambda_{\text{max}}]\) to \([-1, 1]\). The inverse map is \(x = \frac{1}{2\sigma}(t + a)\), and hence, \(\frac{1}{x} = \frac{2\sigma}{t + a}\). We thus aim to find the polynomial of degree less than or equal to \(m\) of best approximation in the norm \(\| \cdot \|_{\infty, [-1, 1]}\) of \(f(t) = \frac{1}{t + a}, a > 1\). We note that if \(Q_m(t)\) is the polynomial of best approximation to \(1/(t + a)\) on \([-1, 1]\), and the error of approximation is

\[
E_{[-1,1]} = \min_{Q \in P_m} \left\| \frac{1}{t + a} - Q \right\|_{L_\infty[-1,1]},
\]

then

\[
q_m(x) := 2\sigma Q_m(2\sigma x - a), \quad \text{and} \quad E = \min_{q \in P_m} \left\| \frac{1}{x} - q \right\|_{L_\infty[\lambda_{\text{max}}, \lambda_{\text{min}}]} = 2\sigma E_{[-1,1]} \quad (2.2)
\]
are the polynomial of best approximation in $L^\infty$-norm on $[\lambda_{\min}, \lambda_{\max}]$ and the error of approximation, respectively.

We denote the (first kind) Chebyshev polynomial of degree $j$ by $T_j$. For $T_j(\xi) \in \mathcal{P}_j$ we have

$$T_j(\xi) = \frac{1}{2} [(\xi + \sqrt{\xi^2 - 1})^j + (\xi - \sqrt{\xi^2 - 1})^{-j}] = \frac{1}{2} [(\xi + \sqrt{\xi^2 - 1})^j + (\xi - \sqrt{\xi^2 - 1})^j].$$

We recall that

$$T_j(t) = \cos j \arccos(t), \quad t \in [-1,1]$$

and denote

$$\delta := a - \sqrt{a^2 - 1} = \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}, \quad \eta = -\delta. \quad (2.3)$$

Evidently, $0 \leq \delta < 1$, $\delta^{-1} = a + \sqrt{a^2 - 1}$, $\eta < 0$ and $\delta = |\eta|$.

With this notation in hand, we have the following identities,

$$a = -\frac{1}{2}(\eta + \eta^{-1}), \quad \frac{1}{t + a} = \frac{2}{2t - \eta - \eta^{-1}}, \quad (2.4)$$

and directly from the expression for $T_j(\xi)$ given above, we also have

$$T_j(a) = \frac{1}{2}(-1)^k(\eta^k + \eta^{-k}), \quad T_j(-a) = \frac{1}{2}(\eta^k + \eta^{-k}). \quad (2.5)$$

### 2.1. Approximation error and three-term recurrence

We now give a representation of the best polynomial approximation to $\frac{1}{t + a}$ in the $L^\infty$-norm on the interval $[-1,1]$.

**Theorem 2.1.** For the polynomials of best uniform approximation to $\frac{1}{t + a}$ on $[-1,1]$, the following three-term recurrence relation holds:

$$\eta^{-1}Q_{m+2}(t) - 2tQ_{m+1}(t) + \eta Q_m(t) = -2, \quad m = 0,1,\ldots \quad (2.6)$$

with

$$Q_0(t) = \frac{a}{a^2 - 1}, \quad Q_1(t) = \frac{1}{\sqrt{a^2 - 1}} - \frac{t}{a^2 - 1}.$$  

The error of approximation is:

$$E_{[-1,1]} = \min_{Q \in \mathcal{P}_m} \left\| \frac{1}{t + a} - Q \right\|_{L^\infty[-1,1]} = \frac{\delta^m}{a^2 - 1}. \quad$$

**Proof.** The proof of the three term recurrence relation follows from [9, Theorem 1], where such recurrence relation for the best approximation to $\frac{1}{t - a}$ is shown. How to derive the approximation for $\frac{1}{t + a}$ from the one for $\frac{1}{t - a}$ is straightforward by showing that $Q_m(t)$, (the polynomial of best uniform approximation to $\frac{1}{t - a}$ on $[-1,1]$) satisfies $Q_m(t) = -\tilde{Q}_m(-t)$, where $\tilde{Q}_m(t) \in \mathcal{P}_m$, is the polynomial of best uniform approximation to $\frac{1}{t - a}$, $a > 1$ on
[-1, 1]. The error of approximation is also straightforward to estimate by showing that $Q_m(t)$ also can be written as

$$Q_m(t) = \frac{1}{t+a} - \frac{2\eta^m}{(\eta-\eta^{-1})^2} \frac{R_{m+1}(t)}{t+a},$$

where

$$R_{m+1}(t) = \eta^{-1}T_{m+1}(t) - 2T_m(t) + \eta T_{m-1}(t).$$

This form of $Q_m(t)$ is derived by showing that the right hand side of (2.7) satisfies the three term recurrence given in (2.6).

Next lemma gives an estimate on $|R_{m+1}(t)|$ by a linear polynomial, which is used later to derive a sufficient condition for the positivity of $Q_m(t)$.

**Lemma 2.1.** The following estimate holds for the polynomial $R_{m+1}(t)$ defined in the proof of Theorem 2.1 (equation (2.8)):

$$-2(t+a) \leq R_{m+1}(t) \leq 2(t+a), \quad t \in [-1, 1].$$

**Proof.** Recall that by the definition of $\eta$ and $\delta$ (see (2.3)), we have that $\eta < 0$, and $|\eta| = \delta$. Let $t = \cos \alpha$, for $\alpha \in [0, \pi]$. Then we find that

$$R_{m+1}(t) + 2t - \eta - \eta^{-1} = \eta^{-1}(T_{m+1}(t) - 1) - 2(T_m(t) - t) + \eta(T_{m-1}(t) + 1)$$

$$= -2\eta^{-1} \sin^2 \frac{m+1}{2}\alpha + 4\sin \frac{m+1}{2}\alpha \sin \frac{m-1}{2}\alpha$$

$$- 2\eta \sin^2 \frac{m-1}{2}\alpha$$

$$= -2\eta^{-1} \left( \sin \frac{m+1}{2}\alpha - \eta \sin \frac{m-1}{2}\alpha \right)^2$$

$$= 2\delta^{-1} \left( \sin \frac{m+1}{2}\alpha + \delta \sin \frac{m-1}{2}\alpha \right)^2 \geq 0. \tag{2.10}$$

In an analogous fashion we obtain

$$R_{m+1}(t) - 2t + \eta + \eta^{-1} = \eta^{-1}(T_{m+1}(t) + 1) - 2(T_m(t) + t) + \eta(T_{m-1}(t) + 1)$$

$$= 2\eta^{-1} \cos^2 \frac{m+1}{2}\alpha - 4\cos \frac{m+1}{2}\alpha \cos \frac{m-1}{2}\alpha$$

$$+ 2\eta \cos^2 \frac{m-1}{2}\alpha$$

$$= 2\eta^{-1} \left( \cos \frac{m+1}{2}\alpha - \eta \cos \frac{m-1}{2}\alpha \right)^2$$

$$= -2\delta^{-1} \left( \cos \frac{m+1}{2}\alpha + \delta \cos \frac{m-1}{2}\alpha \right)^2 \leq 0. \tag{2.11}$$

Combining (2.10) and (2.11) and using $2t - \eta - \eta^{-1} = 2(t+a)$ yields the desired result. □
2.2. Stationary iteration with the polynomial of best approximation to $1/x$

After rescaling, the result in Theorem 2.1 gives us the polynomial approximation on the interval $[\lambda_{\max}, \lambda_{\min}]$. Indeed, the recurrence relation for $q_{m+1}(x) = 2\sigma Q_{m+1}(2\sigma x - a)$ is:

$$Q_{m+1}(2\sigma x - a) = \eta[-2 + 2(2\sigma x - a)Q_{m}(2\sigma x - a) - \eta Q_{m-1}(2\sigma x - a)].$$

Multiplying by $2\sigma$ then gives

$$q_{m+1}(x) = \eta[-4\sigma + 2(2\sigma x - a)Q_{m}(2\sigma x - a) - 2\sigma Q_{m-1}(2\sigma x - a)]. \quad (2.12)$$

This formula can be used to perform stationary iterations towards solving $Au = f$ for a given symmetric and positive definite matrix $A$ and a given symmetric positive definite preconditioner $D$ to $A$. We first write $\eta$, $\sigma$ and $a$ in terms of $\mu_0 = 1/\lambda_{\max}$ and $\mu_1 = 1/\lambda_{\min}$ and $\delta$ (defined in (2.3)). The reason for choosing these parameters is that the constants in the algorithm, with the exception of $\delta$, are symmetric with respect to $\mu_0$ and $\mu_1$. From the three term recurrence relation for $q_{m}(x)$, (2.12), it is straightforward to get the following identity

$$q_{m+1}(x) - q_{m}(x) = \delta^2(q_{m}(x) - q_{m-1}(x)) + \frac{4\mu_0\mu_1}{(\sqrt{\mu_0} + \sqrt{\mu_1})^2}[1 - xq_{m}(x)]. \quad (2.13)$$

Note that this identity allows us to write $q_{m+1}$ as a correction to $q_{m}$ and therefore computing $q_{m+1}$ looks like iteration in a defect-correction method: First computing the residual $[1 - q_{m}(x) x]$, and then trying to correct it by adding an additional term. One can also easily see that for any initial $q_0$ and $q_1$, if the sequence $q_{m}(x)$ converges, then it converges to $x^{-1}$. In other words, choosing $q_0$ and $q_1$ different from what they are above, will not generate the sequence of best approximations to $x^{-1}$, but still this sequence will converge to $x^{-1}$.

We now move on to consider a standard stationary iterative method of the form: Given an approximation $v$ to the solution $u$ of the linear system in hand, the next approximation $w$ is defined as

$$w = v + R(f - Av).$$

A sequence of such approximations, approaching $u$ (when the method is convergent) is obtained by applying this iteration with $w = u_{j+1}$, $v = u_j$, $j = 0, \ldots$, and, with $u_0$, a given initial guess.

In our focus are iterative methods with $R$ which is polynomial in $A$ and we define

$$R = q_m(D^{-1}A)D^{-1},$$

where $q_m$ is the polynomial of best approximation to $1/x$ on the interval $[1/\kappa, \lambda]$ with $\lambda$ an upper bound for the largest eigenvalue of $D^{-1}A$ and $\kappa > 1$, a parameter controlling the length of the interval. In general, iterations with polynomial $R$ are known as semi-iterative methods (see [7] or earlier, [14]).

**Algorithm 2.2** (Polynomial Preconditioning with $R = q_m(D^{-1}A)D^{-1}$).

Given $r$, the following steps provide $R r = q_m(D^{-1}A)D^{-1} r$:

0. Initially, compute $\overline{r} = D^{-1} r$ and set

$$v_0 = \frac{1}{2}(\mu_0 + \mu_1)\overline{r} \quad \text{and} \quad v_1 = \frac{1}{2}(\sqrt{\mu_0} + \sqrt{\mu_1})^2 \overline{r} - \mu_0\mu_1 D^{-1} A \overline{r}. $$
1. For $j = 1, 2, \ldots, m - 1$, compute the current and preconditioned residuals,
\[ r_j = r - Av_j, \quad \overline{r}_j = D^{-1}r_j. \]

Next, $v_{j+1}$ is computed based on the recurrence formula (2.13) with matrix argument
\[ v_{j+1} = v_j + \delta^2(v_j - v_{j-1}) + \frac{4\mu_0\mu_1}{(\sqrt{\mu_0} + \sqrt{\mu_1})^2}r_j. \]

2. Finally, set $Rr = v_m$.

### 3. Properties of the sequence of polynomials

To simplify the presentation, we now set $\lambda = \lambda_{\max}$ and in this notation we have $\lambda_{\min} = \frac{\lambda}{\kappa}$ (recall the definition of $\kappa$ given in §2). We thus consider the best approximation $q_m(x)$ to $\frac{1}{x}$ on the interval $\left[\frac{\lambda}{\kappa}, \lambda\right]$. We prove several results on the positivity of the polynomial $q_m(x)$, and the monotonicity of the sequence $\{q_m\}$ for sufficiently large $m$.

We first note the following identity
\[
 x \, q_m(x) = 2\sigma x Q_m(2\sigma x - a) = (t + a) \, Q_m(t) \\
 = 1 - \frac{2\eta^m}{(\eta - \eta^{-1})^2} R_{m+1}(t) = 1 - \frac{2(-1)^m \delta^m}{(\delta - \delta^{-1})^2} R_{m+1}(t), \quad t \in [-1, 1] \tag{3.1}
\]

This gives
\[
 1 - x q_m(x) = \frac{2(-1)^m \delta^m}{(\delta - \delta^{-1})^2} R_{m+1}(t). \tag{3.2}
\]

Next Lemma shows that $(1 - q_m(x)) > 0$ for all $x \in \left[0, \frac{\lambda}{\kappa}\right]$.

**Lemma 3.1.** Let $q_m(x)$ be the polynomial of degree less than or equal to $m$, which furnishes the best approximation to $\frac{1}{x}$ in the $L^\infty$-norm on the interval $\left[\frac{\lambda}{\kappa}, \lambda\right]$, $\kappa > 1$. Then the following inequality holds:
\[
 0 < 1 - x q_m(x), \quad \forall x \in \left(0, \frac{\lambda}{\kappa}\right] \tag{3.3}
\]

**Proof.** Consider the polynomial
\[
 p(x) = 1 - x q_m(x).
\]

Note that $p(x)$ is of degree at most $(m + 1)$. Since we have
\[
 p(x) = x \left(\frac{1}{x} - q_m(x)\right),
\]
and $x > 0$ in the intervals of interest, we may conclude that the sign changes in the function $\left(\frac{1}{x} - q_m(x)\right)$ are the same as the sign changes in $p(x)$ for any $x > 0$. However, $q_m(x)$ is the
polynomial of best uniform approximation to \( \frac{1}{x} \), and hence there are at least \((m+2)\) points of Chebyshev alternance in the interval \( \left[ \frac{\lambda}{\kappa}, \lambda \right] \). Thus, there exist points \( \{x_j\}_{j=1}^{m+2} \) such that

\[
\frac{\lambda}{\kappa} \leq x_1 < x_2 < \ldots < x_{m+1} < x_{m+2} \leq \lambda,
\]

and also such that

\[
\left( \frac{1}{x_j} - q_m(x_j) \right) = - \left( \frac{1}{x_{j+1}} - q_m(x_{j+1}) \right), \quad j = 1, \ldots, (m+1).
\]

We define now \( \varepsilon := \left( \frac{1}{x_1} - q_m(x_1) \right) \), and use the alternation property to get that

\[
p(x_j)p(x_{j+1}) = -x_j(x_{j+1}\varepsilon^2 < 0, \quad j = 1, \ldots, (m+1).
\]

Hence, we may conclude that all the roots of \( p(x) \) are disjoint, and that each of them lies in the open interval \((x_j, x_{j+1})\), \( j = 1, \ldots, (m+1) \). We may also conclude that there are no roots of \( p(x) \) outside of the open interval \( \left( \frac{\lambda}{\kappa}, \lambda \right) \) and there are no roots of its first derivative outside this interval. This is so by the Rolle’s theorem: the first derivative \( p'(x) \) clearly has \( m \) distinct roots, each lying between the roots of \( p(x) \). Hence, \( p(x) \) is either strictly increasing or strictly decreasing on the interval \([0, \frac{\lambda}{\kappa}]\) and also it cannot have a zero in this interval. Recall that \( 0 < \delta = -\eta < 1 \) and that \( T_j(-1) = (-1)^j \). Using the definition of \( R_{m+1}(t) \) from the proof of Theorem 2.1 (Equation (2.8)), and the relation (3.2) it follows that

\[
p \left( \frac{\lambda}{\kappa} \right) = \frac{2(-1)^m\delta^m}{(\delta - \delta^{-1})^2} R_{m+1}(-1)
\]

\[
= \frac{2(-1)^m\delta^m}{(\delta - \delta^{-1})^2} \left[ (-\delta^{-1})(-1)^{m+1} - 2(-1)^m + (-\delta)(-1)^{m-1} \right]
\]

\[
= \frac{2\delta^m}{(\delta - \delta^{-1})^2} (\delta^{-1} + \delta - 2) = \frac{2\delta^m}{(\delta + \delta^{-1} + 2)} < 1 = p(0).
\]

Here we have used that

\[
(\delta - \delta^{-1})^2 = \left[ (\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}})^2 (\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}}) \right]^2 = (\delta + \delta^{-1} - 2)(\delta + \delta^{-1} + 2).
\]

We thus conclude that \( p(0) > p \left( \frac{\lambda}{\kappa} \right) \) and therefore \( p(x) \) must be decreasing on \( \left( 0, \frac{\lambda}{\kappa} \right) \), and this leads to

\[
0 < \frac{2\delta^m}{(\delta + \delta^{-1} + 2)} = p \left( \frac{\lambda}{\kappa} \right) \leq p(x) \leq 1,
\]

which concludes the proof. \( \square \)

Next lemma shows that for \( x \in \left[ 0, \frac{\lambda}{\kappa} \right] \) the sequence of polynomials of best approximation of increasing degree is monotone.
Lemma 3.2 (Monotonicity). For all \( x \in \left[ 0, \frac{\lambda}{\kappa} \right] \) we have that \( q_m(x) < q_{m+1}(x) \), where 
\( q_j(x), \ j = m, (m+1) \) is the best polynomial approximation of degree at most \( j \) to \( \frac{1}{x} \) in the \( L^\infty \)-norm on the interval \( \left[ \frac{\lambda}{\kappa}, \lambda \right] \), \( \kappa > 1 \).

Proof. The proof amounts to showing that \( [q_{m+1}(x) - q_m(x)] > 0 \) for all \( x \in \left[ 0, \frac{\lambda}{\kappa} \right] \). For such values of \( x \) we have \( x \leq \frac{\lambda}{\kappa} = \mu_1^{-1} \), and, hence

\[
q_1(x) - q_0(x) = \frac{1}{2} (\mu_0 + \mu_1 + 2\sqrt{\mu_0\mu_1}) - \mu_0\mu_1x - \frac{1}{2} (\mu_0 + \mu_1) = \sqrt{\mu_0\mu_1}(1 - x\sqrt{\mu_0\mu_1}) \geq \sqrt{\mu_0\mu_1}(1 - \mu_1^{-1}\sqrt{\mu_0\mu_1}) = \frac{\sqrt{\kappa} - 1}{\lambda} > 0.
\]

Further, from (2.13) and Lemma 3.1 we have

\[
q_{m+1}(x) - q_m(x) = \frac{4\mu_0\mu_1}{(\sqrt{\mu_0} + \sqrt{\mu_1})^2}[1 - q_m(x)x] + \delta^2[q_m(x) - q_{m-1}(x)]
\]

\[
= \frac{4\kappa}{\lambda(1 + \sqrt{\kappa})^2}[1 - q_m(x)x] + \delta^2[q_m(x) - q_{m-1}(x)]
\]

\[
\geq \frac{8\kappa\delta^m}{\lambda(1 + \sqrt{\kappa})^2(\delta + \delta^{-1} + 2)} + \delta^2[q_m(x) - q_{m-1}(x)].
\]

Noticing that \( (\delta + \delta^{-1} + 2) = \frac{4\kappa}{\kappa - 1} \) then leads to:

\[
q_{m+1}(x) - q_m(x) \geq \frac{2}{\lambda}\delta^{m+1} + \delta^2[q_m(x) - q_{m-1}(x)]. \tag{3.6}
\]

Clearly,

\[
q_{m+1} - q_m(x) > 0, \quad \text{if} \quad q_m(x) - q_{m-1}(x) > 0,
\]

and a standard induction argument concludes the proof of the lemma. \( \square \)

Next lemma is a straightforward corollary of Lemma 3.1.

Lemma 3.3. Let \( q_m(x) \) be the best polynomial approximation of degree at most \( m \) to \( \frac{1}{x} \) in \( L^\infty \)-norm on the interval \( \left[ \frac{\lambda}{\kappa}, \lambda \right] \), \( \kappa > 1 \). Suppose that \( q_m(x) \) is positive on the interval \( \left[ \frac{\lambda}{\kappa}, \lambda \right] \). Then \( q_m(x) \) is positive on the whole interval \( x \in (0, \lambda] \).

Proof. We have already shown in the previous lemma that \( q_m(x) > q_0(x) > 0 \), for all \( m \geq 1 \) and \( x \in \left[ 0, \frac{\lambda}{\kappa} \right] \). Since, by assumption \( q_m(x) \) is positive on the interval \( \left[ \frac{\lambda}{\kappa}, \lambda \right] \) the proof is complete. \( \square \)

In the two-level method convergence estimates in the next section, we will use the following result (which also includes a sufficient condition for the positivity of \( q_m(x) \)).
Lemma 3.4. Assume that $\kappa$ and $m$ are such that
\[
\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^m \leq \frac{\omega}{\kappa - 1}, \quad \text{for some } \omega \in (0, 2).
\] (3.7)

Then the following inequality holds for all $x \in (0, \lambda]$:
\[
\frac{1}{2} \min \left\{ \frac{\kappa + 1}{2} - \frac{2 - \omega}{x} \right\} \leq q_m(x) \leq \frac{1}{x} \left(1 + \frac{\omega}{2}\right).
\] (3.8)

Proof. Lower bound: We prove first the lower bound when $x \in \left[\frac{\lambda}{\kappa}, \lambda\right]$. Let $R_{m+1}(t)$ be the polynomial that has been defined in Theorem 2.1. We use the relation (3.1) and Lemma 2.1. Note that $-1 \leq t \leq 1$ for $x \in \left[\frac{\lambda}{\kappa}, \lambda\right]$, and we estimate below $xq_m(x)$ as follows
\[
xq_m(x) = 1 - \frac{2(-1)^m \delta^m}{(\delta - \delta^{-1})^2} R_{m+1}(t) \geq 1 - \frac{2\delta^m}{(\delta - \delta^{-1})^2} |R_{m+1}(t)|
\geq 1 - \frac{2\delta^m}{(\delta - \delta^{-1})^2} (2t + \delta + \delta^{-1})
\geq 1 - \frac{2\delta^m}{(\delta - \delta^{-1})^2} (2 + \delta + \delta^{-1})
= 1 - \frac{2\delta^m}{\delta + \delta^{-1} - 2} = 1 - \delta^m \frac{\kappa - 1}{2} \geq \frac{2 - \omega}{2}.
\]

In the last two steps we have used the identity (3.4) and the definition of $\delta$, given in (2.3). We thus have shown that $q_m(x) \geq \frac{2 - \omega}{2x}$ for all $x \in \left[\frac{\lambda}{\kappa}, \lambda\right]$. Next, we apply Lemma 3.2 and we have that
\[
q_m(x) \geq q_0(x) = \frac{\kappa + 1}{2\lambda}, \quad \text{for } x \in \left[0, \frac{\lambda}{\kappa}\right],
\]
which concludes the proof of the lower bound.

Upper bound: To prove the upper bound, we need to consider only the case $x \in \left[\frac{\lambda}{\kappa}, \lambda\right]$, because from Lemma 3.1 we already know that $xq_m(x) < 1$ for $x \in \left[0, \frac{\lambda}{\kappa}\right]$. Now, for $x \in \left[\frac{\lambda}{\kappa}, \lambda\right]$, we apply an argument analogous to the one for the lower bound using the relation (3.1) and Lemma 2.1 (just changing "−" to "+"):
\[
xq_m(x) = 1 - \frac{2(-1)^m \delta^m}{(\delta - \delta^{-1})^2} R_{m+1}(t) \leq 1 + \delta^m \frac{\kappa - 1}{2} \leq 1 + \frac{\omega}{2}.
\]

Remark 3.1. Note that this lemma implies that the polynomial of best approximation is positive on $[0, \lambda]$ as long as (3.7) is satisfied with $\omega \in (0, 2)$. 
\[\square\]
To conclude this section, we discuss conditions relating $\kappa$ and the degree of the polynomial $m$ so that (3.7) holds. In what follows, without loss of generality we assume that $\ln((\kappa - 1)/\omega) > 1$. In applications (particularly for analysis of convergence of two-level methods) we are interested in large values of $\kappa$ (resp. $m$). Since $\omega \in (0, 2)$, such condition is clearly satisfied for $\kappa > 2e + 1$.

For fixed and sufficiently large $\kappa$, (as we assumed above), let $m$ satisfy

$$\frac{\sqrt{\kappa} + 1}{2} \ln[(\kappa - 1)/\omega] \leq m \leq 1 + \frac{\sqrt{\kappa} + 1}{2} \ln[(\kappa - 1)/\omega].$$

(3.9)

We will now show that the lower bound in (3.9) implies (3.7) (and therefore also the conclusion of Lemma 3.4). Since $0 < \delta < 1$ we have

$$\delta^m = \left(1 - \frac{2}{\sqrt{\kappa} + 1}\right)^m \leq \left[\left(1 - \frac{2}{\sqrt{\kappa} + 1}\right)^{\sqrt{\kappa} + 1}\right]^\frac{1}{2} \ln[(\kappa - 1)/\omega].$$

On the other hand, the function $(1 - 2/\xi)^\xi$ is increasing for all $\xi > 2$, and hence

$$\delta^m < \left[\lim_{\xi \to \infty} \left(1 - \frac{2}{\xi}\right)^\xi\right]^\frac{1}{2} \ln[(\kappa - 1)/\omega] = \exp\left(-\ln \frac{\kappa - 1}{\omega}\right) = \frac{\omega}{\kappa - 1}.$$

Thus, if $\kappa$ is given, the polynomial degree $m$ for which (3.7) holds is bounded below by the right hand side of (3.9).

In addition, it is easy to show that if (3.9) holds, then we also have

$$\frac{1}{\kappa + 1} \leq c_\omega \left(\frac{\ln m}{m}\right)^2,$$

with $c_\omega = \frac{1}{2} \sup_{\kappa > 1, \omega \in (0, 2)} \left(\frac{\ln(\kappa/2)}{1 + \ln(\kappa/\omega)}\right)^2$. (3.10)

Note that $c_\omega$ is finite. The inequality (3.10) is seen as follows. Since the logarithm is an increasing function on its domain we get

$$\ln m \geq \ln[\sqrt{\kappa} + 1/2] + \ln[(\kappa - 1)/\omega] \geq \frac{1}{2} \ln(\kappa/2).$$

Also, from (3.9), since $\sqrt{\kappa} + 1 \geq 2$ we have:

$$m^2 \leq \left(1 + \frac{\sqrt{\kappa} + 1}{2} \ln[(\kappa - 1)/\omega]\right)^2 \leq \left(\frac{\sqrt{\kappa} + 1}{2}(1 + \ln[(\kappa - 1)/\omega]\right)^2 \leq \frac{1}{2}(\kappa + 1)(1 + \ln \kappa/\omega)^2.$$

Combining the last two estimates then gives (3.10).

**4. An application to two-level methods**

We consider the linear system of equations

$$Au = f,$$

(4.1)
where \( A \in \mathbb{R}^{N \times N} \) is a symmetric and positive definite matrix, and \( f \in \mathbb{R}^N \) is a given right hand side vector. To describe a general two-level multiplicative method, we denote \( V = \mathbb{R}^N \), and also introduce a coarse space \( V_H, V_H \subset V, N_H = \dim V_H, N_H < N \). In the following we will always assume that \( V_H = \text{range}(P) \), where \( P : \mathbb{R}^{N_H} \rightarrow V \) and its matrix representation in the canonical basis of \( \mathbb{R}^{N_H} \) is given by the coefficients in the expansion of the basis in \( V_H \) via the basis in \( V \). Clearly, \( P \) is a full rank operator and its matrix representation is oftentimes called prolongation or interpolation matrix. The restriction of \( A \) on the coarse space is denoted by \( A_H = P^T A P \).

### 4.1. Convergence rate estimates

In this subsection we prove convergence estimates for the classical multiplicative two-level iteration, with polynomial smoother which is used to define a preconditioner \( B \approx A^{-1} \). In a recent work [2] the properties of special polynomial smoothers have been exploited in order to conduct an improved convergence analysis of smoothed aggregation algebraic multigrid methods. Here, only for completeness, we include a two-level convergence result presented in [3]. The only difference is that we use a polynomial smoother with polynomial defined via Algorithm 2.2. As in [3] we show explicit dependence of the estimates on the degree of the polynomial.

The results up to and including Theorem 4.3 hold for general SPD \( A, V \) and \( V_H \), provided that the smoother is constructed using the polynomials of best approximation to \( 1/x \) on a suitably chosen interval.

In this subsection, by \( \rho(X) \) we denote the spectral radius of a matrix \( X \). If, in addition, \( X \) is symmetric and positive definite matrix, we denote the \( X \)-norm by \( \|v\|_X^2 = v^T X v \).

We define the two-grid (or TG) preconditioner using a classical two-level algorithm which reads as follows.

**Algorithm 4.1.** Given \( w \in V \) which approximates the solution of (4.1) we define the next approximation \( v \in V \) to \( u \) via the following two steps:

1. Coarse grid correction: \( y := w + PA_H^{-1} P^T (f - Aw) \)
2. Smoothing: \( v := y + R(f - Ay) \).

We assume that \( R \) is symmetric and positive definite and \( A \)-norm convergent, namely

\[
\|I - RA\|_A^2 < 1. \tag{4.2}
\]

The error propagation operator for the two-level iteration above is

\[
E_{TL} = (I - RA)(I - \pi_A), \quad \pi_A = PA_H^{-1} P^T A.
\]

We then define the two-level preconditioner as:

\[
B = (I - E_{TL} E_{TL}^*) A^{-1}.
\]

Here \( E_{TL}^* \) denotes the adjoint with respect to the inner product defined by \( A \). Introducing \( \tilde{R} \) such that

\[
(I - \tilde{R} A) = (I - RA)^2 \quad \text{and hence} \quad \tilde{R} = 2R - RAR. \tag{4.3}
\]
it is straightforward then to compute that (see, e.g., [15]):

\[
B = \bar{R} + (I - RA) P A_H^{-1} P^T (I - AR). \tag{4.4}
\]

Recall a necessary and sufficient condition for \( R \) to be a convergent smoother in \( A \)-norm, i.e., (4.2) to hold is that \( \bar{R} \) is SPD.

Our goal will be to prove a convergence rate estimate for the two-level method with polynomial smoother. First, let us denote with \( D \) the diagonal of \( A \) and set

\[
R = q_m(D^{-1}A)D^{-1},
\]

where \( q_m(x) \) is the polynomial of best approximation to \( 1/x \) on a fixed interval \([\lambda/\kappa, \lambda] \). Both \( \lambda \) and \( \kappa \) are to be specified later.

One may also write \( R \) in the form

\[
R = D^{-1/2}q_m(\widehat{A})D^{-1/2}, \quad \widehat{A} = D^{-1/2}AD^{-1/2}. \tag{4.5}
\]

Using the notation from Section 3, we set \( \lambda = \|\widehat{A}\|_{\ell_\infty} \). In what follows, we hold \( \lambda \) fixed and we vary \( \kappa \) and the degree of the polynomial \( m \). However, \( \kappa \) and \( m \) do not vary independently and we assume that \( \kappa \) and \( m \) satisfy the condition (3.7). With such choice of \( \lambda \), \( \kappa \) and \( m \), one can easily show that \( \bar{R} \) is a contraction (a convergent smoother) in \( A \)-norm and we do so by showing that \( \bar{R} \) is SPD, which, as we mentioned earlier, is both necessary and sufficient condition for (4.2) to hold. Clearly, \( \bar{R} \) can be written (see (4.3)) as

\[
\bar{R} = D^{-1/2}[2q_m(\widehat{A}) - q_m^2(\widehat{A})]D^{-1/2}.
\]

From the upper bound in Lemma 3.4 we immediately get that for all \( x \in (0, \lambda] \) we have \( xq_m(x) \leq \frac{2 + \omega}{2} \). Therefore, for all \( w \in V \) we get

\[
w^T(2q_m(\widehat{A}) - [q_m(\widehat{A})]^{2\widehat{A}})w \geq (2 - \|xq(x)\|_{\infty,(0,\lambda)})w^Tq_m(\widehat{A})w \geq \frac{2 - \omega}{2}w^Tq_m(\widehat{A})w.
\]

Applying the inequality above with \( w = D^{-1/2}y \) then shows that for all \( y \in V \)

\[
y^T \bar{R}y \geq \frac{2 - \omega}{2}y^T Ry \geq \frac{2 - \omega}{2} \min_{x \in (0,\lambda]} q_m(x) y^T D^{-1} y. \tag{4.6}
\]

From the lower bound in Lemma 3.4, we conclude that \( \bar{R} \) is SPD.

We further note that each of the off-diagonal entries of \( (D^{-1/2}AD^{-1/2}) \) is less than 1 and the diagonal entry is equal to 1. Therefore, we have that

\[
1 \leq \|D^{-1/2}AD^{-1/2}\| = \rho(D^{-1/2}AD^{-1/2}) \leq \|D^{-1/2}AD^{-1/2}\|_{\ell_\infty} = \lambda \leq n_z, \tag{4.7}
\]

where \( n_z \) is the maximal number of non-zeros in a row of \( A \).

The convergence rate estimates are derived from the following theorem (two-level version of the XZ-identity, cf. [5, 15]).

**Theorem 4.2.** Assume that \( \bar{R} \) is SPD. Then the following identity holds:

\[
v^T B^{-1} v = \inf_{v_H \in V_H} [\|v_H\|_A^2 + \|v - v_H\|_{\bar{R}_H}^2]. \tag{4.8}
\]
Based on Theorem 4.2, we now state and prove a convergence result involving the polynomial smoother.

**Theorem 4.3.** Let $A$ be a symmetric positive definite matrix and $D$ be its diagonal. Let $\lambda = \|D^{-1/2}AD^{-1/2}\|_{\ell_\infty}$, and also $\kappa > 1$ and $m$ satisfy (3.7). If $R = q_m(D^{-1}A)D^{-1}$, with $q_m(x)$ the polynomial of best approximation to $1/x$ on the interval $[\lambda/\kappa, \lambda]$, then the following estimate holds for all $v \in V$:

$$v^T B^{-1} v \leq \frac{4}{(2 - \omega)} \inf_{v_H \in V_H} \left[ \|v_H\|_A^2 + \frac{\lambda}{(\kappa + 1)} \|v - v_H\|_D^2 + \frac{1}{2 - \omega} \|v - v_H\|_A^2 \right].$$  \hspace{1cm} (4.9)

**Proof.** First, we see that from (4.6) we have that

$$y^T \bar{R} y \geq \frac{2 - \omega}{2 - \omega} y^T R y \quad \text{and hence} \quad y^T \bar{R}^{-1} y \leq \frac{2}{2 - \omega} y^T R^{-1} y.$$  \hspace{1cm} (4.10)

Under the assumptions we made in the statement of the theorem we can apply Lemma 3.4, and get that for all $x \in (0, \lambda]$,

$$\frac{1}{q_m(x)} \leq 2 \max \left\{ \frac{\lambda}{\kappa + 1}, \frac{x}{2 - \omega} \right\} \leq \left( \frac{2\lambda}{\kappa + 1} + \frac{2x}{2 - \omega} \right).$$  \hspace{1cm} (4.11)

Since $\hat{A}$ and $q_m(\hat{A})$ commute, and have the same set of orthonormal eigenvectors, we have

$$w^T [q_m(\hat{A})]^{-1} w \leq \frac{2\lambda}{\kappa + 1} \|w\|_{\ell_2}^2 + \frac{2}{2 - \omega} \|w\|_A^2.$$

Taking $y = D^{1/2} w$ in the inequality above and using the estimate given in (4.10)

$$y^T \bar{R}^{-1} y \leq \frac{2}{2 - \omega} y^TR^{-1}y \leq \frac{4}{(2 - \omega)} \left[ \frac{\lambda}{(\kappa + 1)} \|y\|_D^2 + \frac{1}{2 - \omega} \|y\|_A^2 \right].$$  \hspace{1cm} (4.12)

The proof is concluded by taking $y = (v - v_H)$ and applying Theorem 4.2. \hfill $\Box$

Without loss of generality, we set now $\omega = 1$ and use that in equation (3.10) $c_\omega = c_1 \leq \frac{1}{2}$. The estimate in the Theorem 4.3 takes the form.

**Corollary 4.4.** Under the assumptions of Theorem 4.3, with $\omega = 1$ we have

$$v^T B^{-1} v \leq 4 \inf_{v_H \in V_H} \left[ \|v_H\|_A^2 + \frac{\lambda}{(\kappa + 1)} \|v - v_H\|_D^2 + \|v - v_H\|_A^2 \right].$$

In addition, if $\kappa$ and $m$ satisfy (3.9) we have

$$v^T B^{-1} v \leq 2 \inf_{v_H \in V_H} \left[ 2\|v_H\|_A^2 + \frac{\lambda \ln^2 m}{m^2} \|v - v_H\|_D^2 + 2\|v - v_H\|_A^2 \right].$$

To stress the fact that estimate (4.12) is purely algebraic, we formulate it separately, as this is our main new result.
Theorem 4.5. Let $A$ be an s.p.d. matrix and $D$ a given s.p.d. preconditioner for $A$ such that $\|D^{-\frac{1}{2}}AD^{-\frac{1}{2}}\| \leq \lambda$. Consider the polynomial preconditioner

$$R = q_m(D^{-1}A)D^{-1},$$

where $q_m$ is the polynomial of best approximation of $1/x$ over the interval $[\frac{\lambda}{\kappa}, \lambda]$. The parameter $\kappa$ is chosen depending on $m$ such that (3.7) holds for a given $\omega \in (0, 2)$. Then the following smoothing property holds for $R$ and its symmetrized version $\bar{R}$ (see (4.3)):

$$\frac{2 - \omega}{2} v^T R^{-1} v \leq v^T R^{-1} v \leq \frac{2\lambda}{\kappa + 1} v^T Dv + \frac{2}{2 - \omega} v^T Av.$$

In addition, if $\kappa$ and $m$ satisfy (3.9), we have

$$\frac{2 - \omega}{2} v^T \bar{R}^{-1} v \leq v^T \bar{R}^{-1} v \leq 2\lambda \frac{\ln^2 m}{m^2} v^T Dv + \frac{2}{2 - \omega} v^T Av.$$

Remark 4.1. Theorem 4.5 provides an estimate which is different from the classical definition of smoothing properties of semi-iteration methods found in [7, Section 6.2.5] and [8, Section 10.8.1], see also [13, Section 4.1, and Section 4.3.2]. The above estimate combined with a weak approximation property of the coarse space implies two-grid convergence, whereas the classical smoothing property of Hackbusch combined with a strong approximation property of the Galerkin coarse-grid projection implies two-grid convergence that improves with the number of smoothing steps. We note also that the parameter $\kappa$ can be used to control which part of the spectrum of $A$ is resolved by the smoothing iteration. This is so, because the properties of the sequence of polynomials given in Section 3 show that the smoothing iteration is convergent as long as the approximating polynomial is positive on the interval $[\frac{\lambda}{\kappa}, \lambda]$. In another words, we can “approximate” well on a relatively small interval and this correspond to quick damping of the eigenmodes of $A$ corresponding to large eigenvalues (the ones close to $\lambda$), that is we have a good smoothing property. Of course, other polynomial sequences can be used to provide such smoothing iterations. We refer to Hackbusch [7, Section 6.2.5] and also to a recent work by Brezina and Vassilevski [3] for further discussions. In the next section we show how $\kappa$, or, equivalently, the polynomial degree, can be chosen in conjunction with the approximation properties of coarse spaces to yield a uniformly convergent two-grid method for finite element discretization of a PDE.

4.2. Two-level method for discretized PDE

In this section we apply the abstract two–level result to the case of a two-level iterative method with large coarsening ratio for the solution of a system of linear algebraic equations arising from a discretization of scalar elliptic equation with heterogeneous coefficients similarly to the presentation in [3], now for the case of a different polynomial smoother from Theorem 4.5. We consider the following variational problem: Find $u \in H^1_D(\Omega)$, for a given polygonal (polyhedral) domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $3$) and a source term $f \in L^2(\Omega)$, such that

$$a(u, v) \equiv \int_{\Omega} \alpha(x) \nabla u \cdot \nabla v = \int_{\Omega} f(x)v(x) = (f, v), \quad \text{for all} \quad v \in H^1_D(\Omega). \quad (4.13)$$
Here, \( \Omega \subset \mathbb{R}^d \) \( d = 2, 3 \) is a given domain whose boundary \( \Gamma = \partial \Omega \) is partitioned as \( \Gamma = \Gamma_D \cup \Gamma_N \). We assume that \( \Gamma_D \neq \emptyset \) is closed as a subset of \( \Gamma \) and also has a nonzero \((d - 1)\) dimensional measure. We refer to \( \Gamma_D \) as the Dirichlet part of the boundary and \( \Gamma_N \) as the Neumann part of the boundary. In the variational problem (4.13), \( H^1_D(\Omega) \) denotes the space of functions in \( H^1(\Omega) \) whose traces vanish on \( \Gamma_D \).

We are interested in the case when the diffusion coefficient \( \alpha = \alpha(x) \) is a piecewise constant function, that may have large variations within \( \Omega \). We thus assume that \( \bar{\Omega} = \cup_{l=1}^{m_0} \bar{\Omega}_l \), with polygonal (polyhedral) subdomains \( \Omega_l \), and that \( \alpha(x) = \alpha_l \), for all \( x \in \Omega_l \) and \( l = 1, \ldots, m_0 \). We introduce the following energy norm:

\[
\|v\|_{a}^2 = \int_{\Omega} \alpha(x)|\nabla v|^2 = \sum_{l=1}^{m_0} \alpha_l \int_{\Omega_l} |\nabla v|^2. \tag{4.14}
\]

We also need the weighted \( L_2 \) norm:

\[
\|v\|_{0,a}^2 = \int_{\Omega} \alpha(x)v^2 = \sum_{l=1}^{m_0} \alpha_l \int_{\Omega_l} v^2. \tag{4.15}
\]

We consider a standard discretization of the variational problem (4.13) with piecewise linear continuous finite elements. To define the finite element spaces and the approximate solution, we assume that we have a locally quasi–uniform, simplicial triangulation \( T_h \) of \( \Omega \). We assume that this triangulation also resolves \( \Omega_l \), namely, for \( l = 1, \ldots, m_0 \) we have:

\[
\bar{\Omega} = \cup_{\tau \in T_h} \tau, \quad \bar{\Omega}_l = \cup_{\tau \in \mathcal{T}_h} \tau, \tag{4.16}
\]

where \( \mathcal{T}_h \subset T_h \), for \( l = 1, \ldots, m_0 \). The standard space of piecewise linear (w.r.t \( T_h \)) and continuous functions vanishing on the boundary of \( \Omega \) is denoted by \( V_h \).

The discrete problem then reads: Find \( u \in V_h \) such that

\[
a(u,v) = (f,v), \quad \text{for all} \quad v \in V_h. \tag{4.17}
\]

The notation and constructions in the previous section are suitable for the finite element setting as well. Indeed, a coarse space corresponding to \( V_H \) (denoted here with \( V_H \)) as \( V_H = \text{range}(P) \), with the same \( P \) as before, but this time representing the coefficients in the expansion of the basis in \( V_H \), \( \{\varphi_j^H\}_{j=1}^{N_H} \) via the canonical Lagrange basis \( \{\varphi_j\}_{j=1}^{N} \) in \( V_h \). Evaluating the bilinear form on the basis for \( V_h \) and the basis for \( V_H \) defines the stiffness matrix \( A \) and the matrix \( A_H \):

\[
A_{kj} = a(\varphi_j, \varphi_j), \quad (A_H)_{kj} = (AP^TA)_{jk} = a(\varphi_j^H, \varphi_j^H). \]

According to the considerations in the previous section, we use bold face to represent vectors of degrees of freedom and normal font for functions. Thus a function \( v \in V_h \) is represented by the vector \( \mathbf{v} \in \mathbf{V} \).

We make the following assumption for the stability and approximation properties of the coarse function space \( V_H \).

**Approximation and stability assumption:** For any \( v \in V_h \) there exists \( v_H \in V_H \) such that

\[
H^{-2}\|v - v_H\|_{0,a}^2 + \|v - v_H\|_{a}^2 \leq c_{\text{as}}\|v\|_{a}^2, \tag{4.18}
\]

where \( H \) is the diameter of the support of a typical basis function in \( V_H \), and the constant \( c_{\text{as}} \) is independent of the variations of the coefficient \( \alpha(x) \).
Construction of coarse spaces satisfying this assumption is possible as already mentioned, and we refer to [6], [12], and earlier [1] as modified recently in [3] for such constructions.

We next introduce a well-known inequality relating the weighted \( L^2 \) norm on the function space \( V_h \) and the norm provided by the diagonal of the stiffness matrix on the space of degrees of freedom (nodal values of the piece-wise linear functions). Let \( \{ \lambda_{j,T} \}_{j=1}^{d+1} \) be the barycentric coordinates in an element \( T \in \mathcal{T}_h \) and \( \alpha_T \) be the value of the coefficient on \( T \) (recall that \( \alpha(x) \) is piece-wise constant). Let \( v \in V_h \) with corresponding vector of degrees of freedom \( v \in V \). We have the following simple inequality

\[
\| v \|_D^2 = \sum_{T \in \mathcal{T}_h} \alpha_T \sum_{j=1}^{d+1} v_{j,T}^2 |\nabla \lambda_{j,T}|^2 \leq \sum_{T \in \mathcal{T}_h} c_T h_T^{-2} \alpha_T \sum_{j=1}^{d+1} v_{j,T}^2 |\lambda_{j,T}|^2
\]

In the inequalities above, we have used standard inverse inequality, and also that the local \( \| v \| \) bound follows directly from Corollary 4.4 together used in conjunction with the simple equivalence provided by Theorem 4.5.

\[\text{The lower bound is immediate, since } E_{TL} \text{ is a contraction in the } A\text{-norm. The upper bound follows directly from Corollary 4.4 together used in conjunction with the simple inequalities relating the function space } V_h \text{ and } V \text{ (see (4.19)). Given } v \in V_h, \text{ let } v \in V \text{ be the corresponding vector of degrees of freedom. We have}
\]

\[
\| v \|_T^2 \leq 4 \inf_{v_H \in V_h} \left[ \| v_H \|_A^2 + \frac{\lambda}{(\kappa + 1)} \| v - v_H \|_D^2 + \| v - v_H \|_A^2 \right] + 4 \inf_{v_H \in V_h} \left[ 2 \| v \|_a^2 + \frac{c_M n_z h^2}{(\kappa + 1)^2} \| v - v_H \|_0,\alpha^2 + 3 \| v - v_H \|_a^2 \right] + 4 \left[ 2 + \frac{c_6 c_M n_z}{(\kappa + 1)} \left( \frac{H}{h} \right)^2 + 3 c_6 \right] \| v \|_a^2 = K_{TG} \| v \|_T^2.
\]
Clearly, for \( (\sqrt{\kappa} + 1) \geq \frac{H}{h} \), and \( m \) satisfying (3.9), for example, \( m \geq \frac{H}{h} \ln(H/h) \), the spectral equivalence is uniform with respect to mesh size and coefficient variation.

5. Choice of coarse spaces and numerical tests

In this section, we present a number of tests that illustrate the robustness of the two–level methods with the polynomial smoother analyzed in the present paper all in accordance with Theorem 4.6. We consider the second order elliptic equation (4.13) with a mixture of Neumann and Dirichlet boundary conditions. The Dirichlet boundary conditions are imposed on the “east” and “west” vertical boundaries, i.e., \( \Gamma_D = \Gamma_E \cup \Gamma_W \) of \( \Omega \). As we pointed out, the coefficient \( \alpha(x) \) is piecewise constant and we assume that the fine triangulation of \( \Omega \) is aligned with (resolves) all the coefficient discontinuities. In Fig. 5.1 we show an example of a fine grid \( T_h \), aligned with discontinuities.

![Figure 5.1. Checkerboard coefficient distribution on a mesh with 25600 elements and 13041 vertices.](image)

5.1. Coarse spaces

We use element agglomeration to define “coarse elements” as illustrated in Fig. 5.2, and a variant of the spectral AMGe method (see, e.g., [15]) in the form presented in [3]. Briefly the main steps in such coarse space construction are:

- Partitioning of the degrees of freedom as a union of non-overlapping sets, \( \{A\} \) called aggregates. This is achieved by first partitioning the set of elements into agglomerated elements \( \{\tau\} \) (union of fine-grid elements). We use graph partitioner (metis) applied to the graph having vertices the fine-grid elements with edges between two elements if they share a common interface. Then, we form aggregates \( A \), where each aggregate (a set of fine degrees of freedom) corresponds a unique agglomerated element \( \tau = \tau_A \) by distributing the shared fine degrees of freedom (fine-grid element vertices belonging to two or more agglomerated elements) to a unique aggregate.
Constructing a tentative interpolation matrix $\mathbf{P}$, defined for an agglomerate $\mathbf{\tau}$. Consider the local generalized eigenproblem,

$$A_\mathbf{\tau} \varphi_j = \theta_j D_\mathbf{\tau} \varphi_j,$$

where $A_\mathbf{\tau}$ is the local stiffness matrix corresponding to the agglomerated element $\mathbf{\tau}$, $D_\mathbf{\tau}$ is its diagonal and $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_{n_\mathbf{A}}$ with $n_\mathbf{A} = |\mathbf{A}|$ (cardinality of $\mathbf{A}$). Given a spectral tolerance $\theta$, we select the eigenvectors $\{\varphi_j\}_{j=1}^{n_\theta}$, where $n_\theta$ is the largest integer for which the inequality $\theta_{n_\theta} < \theta$ holds. Extended by zero outside each $\mathbf{A}$, the vectors $\{\varphi_j\}_{j=1}^{n_\theta}$ form $n_\theta$ columns of the global tentative interpolation operator $\mathbf{P}$.

Constructing the coarse space as the range of the interpolation matrix $\mathbf{P}$, which is defined as

$$\mathbf{P} = s_m \left( \lambda^{-1} D^{-1} \mathbf{A} \right) \mathbf{P},$$

Here, as in the previous section, $D$ is the diagonal of $\mathbf{A}$, $\lambda \geq \|D^{-\frac{1}{2}} A D^{-\frac{1}{2}}\|$ (e.g., $\lambda = \|D^{-1/2} A D^{-1/2}\|_\infty$), and $s_m(t)$ is the smoothed aggregation (SA) polynomial (cf., e.g., [2])

$$s_m(t) = \frac{(-1)^m}{(2m + 1)} \frac{T_{2m+1}(\sqrt{t})}{\sqrt{t}}.$$

5.2. Numerical tests

We recall some of the notations and definitions which are used in the tables and figures in this section.

- $N$ is the number of fine grid degrees of freedom;
- $N_H$ is the number of coarse degrees of freedom;
- $\text{nnz}(X)$ is the number of the nonzero elements in a matrix $X$;
• $\tilde{\eta}_{TG}$ is the asymptotic convergence factor of the two grid method;

• $oc(B)$ is the operator complexity measure of the two-grid preconditioner $B$, defined as

$$oc(B) = \frac{nnz(A) + nnz(A_H)}{nnz(A)}.$$ 

The first set of experiments are on a mesh with 102,400 elements and $N = 51,681$ vertices using 300 agglomerated elements (AEs). We stop the iterations when the relative preconditioned residual norm is reduced by a factor of $\varepsilon = 10^{-8}$. The piecewise constant coefficient $\alpha(x)$ is distributed in a checkerboard fashion with the values 1 (blue) and $10^6$ (red) as illustrated in Fig. 5.1.

The experiments are performed for $m = 2, 4, 6, 8$, $a \equiv \frac{1}{\kappa} = 0.158, 0.085, 0.055, 0.04$, and $a = 0.2, 0.1, 0.08, 0.06$ respectively. They are chosen such that the inequality (3.7) (with $\omega = 1$) holds:

$$\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^m = \left(\frac{1 - \sqrt{a}}{1 + \sqrt{a}}\right)^m < \frac{a}{1 - a} = \frac{1}{\kappa - 1}, \quad a = \frac{1}{\kappa}.$$ 

The same degree $m$ is used for the polynomial smoother in the two-level algorithm and the smoother of the tentative interpolation matrix (it is smoothed out by $s_m(D^{-1}A)$). The number of non-zero entries of $A$, is $nnz(A) = 359,841$. We also show how the spectral tolerance $\theta$ and the polynomial degree $m$ influences the convergence versus operator complexity. The results are presented in Tables 5.1–5.4. It is evident from the results that the method can become fairly fast (in terms of convergence factors) at the expense of large operator complexity.

### Table 5.1. Two-grid convergence, $m = 2$. 

| $\theta$ | $N_H$ | $nnz(A_H)$ | $oc(B)$ | $a = 0.158$ | $a = 0.2$ |
|---|---|---|---|---|---|
| 0.010 | 774 | 15,930 | 1.04 | 0.995 | 0.995 |
| 0.077 | 3,629 | 342,515 | 1.95 | 0.879 | 0.889 |
| 0.149 | 6,557 | 1,115,207 | 4.10 | 0.393 | 0.492 |

### Table 5.2. Two-grid convergence when $m = 4$. 

| $\theta$ | $N_H$ | $nnz(A_H)$ | $oc(B)$ | $a = 0.085$ | $a = 0.1$ |
|---|---|---|---|---|---|
| 0.010 | 774 | 22,092 | 1.06 | 0.985 | 0.986 |
| 0.077 | 3,629 | 472,907 | 2.31 | 0.531 | 0.538 |
| 0.149 | 6,557 | 1,538,845 | 5.28 | 0.188 | 0.084 |
Polynomial of best uniform approximation to $1/x$ and two-level methods

| $\theta$ | $N_H$ | $\text{nnz}(A_H)$ | $\text{oc}(B)$ | $a = 0.055$ | $a = 0.08$ |
|----------|-------|-------------------|----------------|-------------|-------------|
| 0.010    | 774   | 29,448            | 1.08           | 0.965       | 0.969       |
| 0.077    | 3,629 | 636,671           | 2.78           | 0.205       | 0.179       |
| 0.149    | 6,557 | 2,074,291         | 6.76           | 0.202       | 0.026       |

Table 5.3. Two-grid convergence when $m = 6$.

| $\theta$ | $N_H$ | $\text{nnz}(A_H)$ | $\text{oc}(B)$ | $a = 0.04$ | $a = 0.06$ |
|----------|-------|-------------------|----------------|-------------|-------------|
| 0.010    | 774   | 37,618            | 1.10           | 0.926       | 0.933       |
| 0.077    | 3,629 | 808,357           | 3.25           | 0.197       | 0.111       |
| 0.149    | 6,557 | 2,632,755         | 8.32           | 0.193       | 0.028       |

Table 5.4. Two-grid convergence when $m = 8$.

In the last experiment shown in Table 5.5, we illustrate the behavior of the method with respect to varying the contrast $10^c$ again distributed in a checkerboard fashion. As it is clearly seen, the two-grid method exhibits very good uniform two-grid convergence with operator complexity less than two.

| $c$ | -12 | -9 | -6 | -3 | 0 | 3 | 6 | 9 | 12 |
|-----|-----|----|----|----|---|---|---|---|----|
| $N_H$ | 2336 | 2336 | 2336 | 2339 | 2322 | 2322 | 2322 | 2322 | 2322 |
| $\text{oc}(B)$ | 1.94 | 1.94 | 1.94 | 1.94 | 1.93 | 1.93 | 1.93 | 1.93 | 1.93 |
| $n_{it}$ | 17 | 17 | 17 | 17 | 17 | 16 | 16 | 16 | 16 |
| $\tilde{\rho}_{TG}$ | 0.219 | 0.219 | 0.219 | 0.219 | 0.219 | 0.200 | 0.198 | 0.197 | 0.198 |

Table 5.5. Contrast independent two-grid convergence; coefficient jumps are $10^c$. The method corresponds to spectral threshold $\theta = 0.045$, $m = 8$, and $a = 0.04$.

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