Combational Differential Algebra of $x^p$

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Abstract. We link $n$-jets of the affine monomial scheme defined by $x^p$ to the stable set polytope of some perfect graph. We prove that, as $p$ varies, the dimension of the coordinate ring of a certain subscheme of the scheme of $n$-jets as a $\mathbb{C}$-vector space is a polynomial of degree $n + 1$, namely the Ehrhart polynomial of the stable set polytope of that graph. One main ingredient for our proof is a result of Zobnin who determined a differential Gröbner basis of the differential ideal generated by $x^p$. We generalize Zobnin’s result to the bivariate case. We study $(m, n)$-jets, a higher-dimensional analog of jets, and relate them to regular unimodular triangulations.

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Introduction

Differential algebra—an infinite version of polynomial algebra in a sense—studies polynomial partial differential equations with tools from commutative algebra. Differential algebraic geometry studies varieties that are defined by a system of polynomial PDEs. An upper bound for the number of components of such a variety was recently constructed in [19]. Differential algebraic geometry comes with an own version of the Nullstellensatz, the differential Nullstellensatz, relating points of a differential variety with formal power series solutions of the defining system of equations. Lower and upper bounds for the effective differential Nullstellensatz are provided in [14]. In this article, we transfer the combinatorial flavor of commutative algebra [21] to differential algebra and undertake first steps in combinatorial differential algebra. We present a case study of the fat point $x^p$ on the affine line.

Denote by $C_{p,n}$ the ideal in $R_n = \mathbb{C}[x_0, \ldots, x_n]$ generated by the coefficients of $f_{p,n} = (x_0 + x_1 t + \cdots + x_n t^n)^p$, read as a polynomial in $t$ with coefficients in $R_n$. The affine scheme defined by $C_{p,n}$ is a subscheme of the scheme of $n$-jets of the fat point $x^p$ on the affine line. Bernd Sturmfels suggested investigating the following question on the dimension of the coordinate ring of that scheme.

Question 2.1. For fixed $n \in \mathbb{N}$, is the sequence $(\dim\mathbb{C}(R_n/C_{p,n}))_{p \in \mathbb{N}}$ a polynomial in $p$ of degree $n + 1$?

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The point of departure is experimental observations. A first main result of this article is Theorem 2.5, proving that the answer to this question is yes. The dimension of this quotient ring provides a dimensional bound in the study of the multiplicity structure of the arc space of the fat point which was undertaken by the first-named author and Pogudin in [2].

One main tool for the proof is a result from differential algebra. The object of study is the ring of differential polynomials in x over C. The latter is the differential ring \( \mathbb{C}[x^{(\infty)}] = (\mathbb{C}[x, x^{(1)}, x^{(2)}, \ldots], \partial) \), i.e., the polynomial ring in the countably infinitely many variables \( \{x^{(k)}\}_{k \in \mathbb{N}} \) with the derivation \( \partial \) acting as \( \partial(x^{(k)}) = x^{(k+1)} \), \( \partial|_C \equiv 0 \).

An ideal \( I \) in \( \mathbb{C}[x^{(\infty)}] \) is a differential ideal if \( \partial(I) \subseteq I \). Zobnin [31] proved that the singleton \( \{x^p\} \) is a differential Gröbner basis of the differential ideal generated by \( x^p \) with respect to certain monomial orderings, namely so called \( \beta \)-orderings.

Denote by \( I_{p,n} \) the differential ideal generated by \( x^p \) and \( x^{(n)} \). Then the map

\[
R_{p,n}/C_{p,n} \xrightarrow{\sim} \mathbb{C}[x^{(\infty)}]/I_{p,n+1}, \quad x_k \mapsto \frac{1}{k!} x^{(k)}
\]

is an isomorphism and Zobnin’s result can be used to investigate \( C_{p,n} \). An investigation of the initial ideal of \( C_{p,n} \) then reveals the following.

**Theorem 2.5.** As \( p \) varies, the number \( \dim_\mathbb{C}(R_{p,n}/C_{p,n}) \) is polynomial of degree \( n + 1 \). It is the Ehrhart polynomial of the convex polytope

\[
P_n := \{(u_0, \ldots, u_n) \in (\mathbb{R}_{\geq 0})^{n+1} | u_i + u_{i+1} \leq 1 \text{ for all } 0 \leq i \leq n - 1 \}
\]

evaluated at \( p - 1 \), i.e., it counts the lattice points of the polytope \( P_n \) dilated by \( p - 1 \).

In [5], the authors study the arc scheme of a variety, which is defined to be the projective limit of the jet schemes. Arc schemes are an important tool from algebraic geometry to study formal power series solutions to the defining equations of a variety. Via Hilbert–Poincaré series, they bring a combinatorial flavor in, which is different from the one we present here; among others, they relate the double point to partitions of integers. A study of jet schemes of monomial ideals was also undertaken in [12]. Therein, it is shown that jet schemes of monomial ideals are in general not monomial, but their reduced subschemes are. A study of the multiplicity of jet schemes of simple normal crossing divisors was undertaken by Yuen in [30]. In [29], she introduced truncated \( m \)-wedges, a two-dimensional analog of jets, studying differentials in two variables whose orders add up to \( m \) at most. In Definition 1.8, we introduce another generalization of jets to higher dimensions, namely \( (m, n) \)-jets, allowing for derivatives in the variables up to order \( m \) and \( n \), respectively.

We extend our studies of \( \dim_\mathbb{C}(R_{p,n}/C_{p,n}) \) to the case of two independent variables and give a link to regular unimodular triangulations. For the theory of triangulations, we refer our readers to [8, 28]. We study the ring of partial differential polynomials in \( x \) over \( \mathbb{C} \), i.e., \( \mathbb{C}[x^{(\infty, \infty)}] := (\mathbb{C}[x^{(k,\ell)}]_{k,\ell \in \mathbb{N}}, \partial_s, \partial_t) \) in two independent variables \( s,t \) and consider the differential ideal \( I_{p,(m,n)} \) generated by \( x^p, x^{(m,0)} \), and \( x^{(0,n)} \). Denote by \( C_{p,(m,n)} \) the ideal in \( \mathbb{C}[\{x_{k,\ell}\}_{0 \leq k \leq m, 0 \leq \ell \leq n}] \) generated by the coefficients of

\[
f_{p,(m,n)} := \left( \sum_{k=0}^m \sum_{\ell=0}^n x_{k,\ell} t^k s^\ell \right)^p,
\]

read as a bivariate polynomial in \( s \) and \( t \). In analogy to the one-dimensional case, we refer to points of the affine scheme associated to \( C_{p,(m,n)} \) as \( (m, n) \)-jets of \( x^p \). The ideals \( I_{p,(m,n)} \) and \( C_{p,(m,n)} \) then are related just as in the univariate case.
For a triangulation $T$ of the $m \times n$ rectangle and fixed $p$, we define $T$-orderings on the truncated partial differential ring $\mathbb{C}[x^{(\leq m, \leq n)}]$ as those monomial orderings for which the leading monomials of $\{(x^p)^{(k,l)}\}_{k=0,\ldots,m_p,\ell=0,\ldots,n_p}$ are supported on the triangles of $T$. Note that this is in contrast to the usual occurrence of regular triangulations in combinatorial commutative algebra, where the leading monomials are supported on non-faces (see Sturmfels’ correspondence [8, Theorem 9.4.5]). We consider the placing triangulation $T_{m,2}$ of the point configuration

$$[((0,0), (0,1), (0,2), (1,0), (1,1), (1,2), \ldots, (m,0), (m,1), (m,2))].$$

This is a regular unimodular triangulation of the $m \times 2$ rectangle induced by the vector $(1, 2, 2^2, \ldots, 2^{3m+2})$ in the lower hull convention.

We prove the following generalization of Zobnin’s result to the partial differential ring in two independent variables.

**Theorem 1.13.** For all $m, p \in \mathbb{N}$, $\{(x^p)^{(k,l)}\}_{0 \leq k \leq mp, 0 \leq \ell \leq 2p}$ is a Gröbner basis of the differential ideal generated by $x^p$ in the truncated partial differential ring $\mathbb{C}[x^{(\leq m, \leq 2)}]$ with respect to any $T_{m,2}$-ordering.

This theorem is the main ingredient for the following theorem.

**Theorem 2.6.** Let $m, p \in \mathbb{N}$. The number $\dim_{\mathbb{C}}(R_{m,2}/C_{p,(m,2)})$ is the Ehrhart polynomial of the $3(m + 1)$-dimensional lattice polytope

$$P_{(m,2)} := \{(u_{00}, u_{01}, u_{02}, \ldots, u_{m0}, u_{m1}, u_{m2}) \in (\mathbb{R}_{\geq 0})^{3(m+1)} | u_{k1,l1} + u_{k2,l2} + u_{k3,l3} \leq 1 \text{ for all indices } s.t. \{(k1,l1), (k2,l2), (k3,l3)\} \text{ is a triangle of } T_{m,2}\}$$

evaluated at $p - 1$.

In Section 3, we study regular unimodular triangulations of the $m \times n$ rectangle. We consider the weighted reverse lexicographical ordering on $\mathbb{C}[[x_k^l]_{0 \leq k \leq m, 0 \leq \ell \leq n}]$ for vectors inducing those triangulations in the upper hull convention. We show that for some of them, the coefficients of $f_{p,(m,n)}$ are a Gröbner basis of the ideal $C_{p,(m,n)}$.

We end our article with an outlook to future work. Our results suggest to further develop combinatorial differential algebra.

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1. Differential ideals and jets

1.1. One independent variable. In this section, we repeat basics from differential algebra and give a link to the theory of jet schemes. For further background on differential algebra, we refer the reader to the books [16, 27].

Consider the polynomial ring $\mathbb{C}[x, x^{(1)}, x^{(2)}, \ldots]$ in the countably infinitely many variables $\{x^{(k)}\}_{k \in \mathbb{N}}$. Denoting $x := x^{(0)}$, let $\mathbb{C}[x^{(\infty)}]$ be the differential ring

$$\mathbb{C}[x^{(\infty)}] := \left(\mathbb{C}[x, x^{(1)}, x^{(2)}, \ldots], \partial\right),$$

where the derivation acts as $\partial(x^{(k)}) = x^{(k+1)}$ and $\partial|_\mathbb{C} \equiv 0$.

**Definition 1.1.** An ideal $I \subset \mathbb{C}[x^{(\infty)}]$ is called a differential ideal if $\partial(I) \subseteq I$. For a subset $J$ of $\mathbb{C}[x^{(\infty)}]$, we denote by $\langle J \rangle^{(\infty)}$ the differential ideal generated by $J$. 

We denote by $I_{p,n} := \langle x^p, x^{(n)} \rangle^{(\infty)}$ the differential ideal in $\mathbb{C}[x^{(\infty)}]$ generated by $x^p$ and $x^{(n)}$ and by $\mathbb{C}[x^{[\leq n]}]$ the truncated differential polynomial ring $\mathbb{C}[x^{(\infty)}]/\langle x^{(n+1)} \rangle^{(\infty)}$.

For $n \in \mathbb{N}$, denote by 

$$
R_n := \mathbb{C}[x_0, \ldots, x_n]
$$

the polynomial ring in $n+1$ variables with coefficients in the complex numbers. Consider $f_{p,n} = (x_0 + x_1 t + \cdots + x_n t^n)^p \in R_n[t]$. By the multinomial theorem,

$$
f_{p,n} = \sum_{k_0 + \cdots + k_n = p} \binom{p}{k_0, k_1, \ldots, k_n} x_0^{k_0} \cdots x_n^{k_n} t^{k_1 + 2k_2 + \cdots + nk_n},
$$

where

$$
\binom{p}{k_0, k_1, \ldots, k_n} = \frac{p!}{k_0! \cdots k_n!}.
$$

Denote by $C_{p,n} \triangleleft R_n$ the ideal generated by the coefficients of $f_{p,n}$. The affine scheme $\text{Spec}(R_n/C_{p,n})$ is a subscheme of the scheme of $n$-jets of $\text{Spec}(\mathbb{C}[x]/\langle x^p \rangle)$.

Up to constants, the coefficient of $t^k$ in $f_{p,n}$ recovers the $k$-th derivative of the monomial $x^p$, giving rise to the following relation between the differential ideal $I_{p,n}$ and the ideal $C_{p,n}$ in the polynomial ring $R_n$.

**Proposition 1.2.** The following map is an isomorphism of $\mathbb{C}$-algebras:

$$
R_n/C_{p,n} \xrightarrow{\cong} \mathbb{C}[x^{(\infty)}]/I_{p,n+1}, \quad x_k \mapsto \frac{1}{k!} x^{(k)}.
$$

**Proof.** Notice that $(x^p)^{(k)}$ is given as follows:

$$
(x^p)^{(k)} = \sum_{j_0 + \cdots + j_{p-1} = k} \binom{k}{j_0, \ldots, j_{p-1}} x^{(j_0)} \cdots x^{(j_{p-1})}.
$$

Let us consider its image in the truncated differential ring $\mathbb{C}[x^{(\leq n)}]$. We denote by $i_\ell$ the multiplicity of $\ell$ in the multiset $\{j_0, \ldots, j_{p-1}\}$, so that $i_0 + \cdots + i_n = p$ and $i_1 + 2i_2 + \cdots + ni_n = k$. Let $y_i := x^{(i)}$ for all $0 \leq i \leq n$. Then

$$
(x^p)^{(k)} = \sum_{j_0 + \cdots + j_{p-1} = k} \binom{k}{j_0, \ldots, j_{p-1}} y_0^{i_0} \cdots y_n^{i_n}.
$$

In the previous sum, there are some repeated terms: for each $\{j_0, \ldots, j_{p-1}\}$ by exchanging the order of $j_i$ and respecting the numbers $i_0, \ldots, i_n$, we get the same term. We have $\binom{p}{i_0}$ possibilities to choose $i_0$ many places for $0$ in the multiset $\{j_0, \ldots, j_{p-1}\}$. We have $\binom{p-i_0}{i_1}$ possibilities to choose $i_1$ many places for $1$ from the remaining places in the set $\{j_0, \ldots, j_{p-1}\}$. We continue like this and obtain

$$
(x^p)^{(k)} = \sum_{(i_0, \ldots, i_n) \in I} \binom{p}{i_0, \ldots, i_n} \frac{k!}{(0!)^{i_0} \cdots (n!)^{i_n}} y_0^{i_0} \cdots y_n^{i_n},
$$

where $I = \{(i_0, i_1, \ldots, i_p) \mid i_0 + \cdots + i_n = p$ and $i_1 + \cdots + ni_n = k\}$. Denote by $\varphi$ the following homomorphism of rings:

$$
\varphi : \mathbb{C}[x^{(\leq n)}] \to R_n/C_{p,n}, \quad x^{(k)} \mapsto k! \cdot x_k.
$$

The kernel of $\varphi$ is the ideal generated by $\{(x^p)^{(k)}\}_{k \in \mathbb{N}}$. Thus,

$$
\mathbb{C}[x^{(\infty)}]/I_{p,n+1} \cong \mathbb{C}[x^{[\leq n]}]/\langle \{(x^p)^{(k)} \mid k \in \mathbb{N}\} \rangle \cong R_n/C_{p,n},
$$

concluding the proof. □
Remark 1.3. The statement of Proposition 1.2 is contained in the literature, such as [23, Proposition 2.3] or [22, Proposition 5.12]. To make this article self-contained, we decided to provide a proof.

Following [31], we now repeat the concept of differential Gröbner bases.

Definition 1.4. Fix a monomial ordering \( \prec \) on \( \mathbb{C}[x^{(\infty)}] \) and let \( I \triangleleft \mathbb{C}[x^{(\infty)}] \) be a differential ideal. A subset of polynomials \( G \subseteq I \) s.t. \( \langle G \rangle^{(\infty)} = I \) is a differential Gröbner basis of \( I \) if \( \{ \partial^k(g) \mid k \in \mathbb{N}, g \in G \} \) is an algebraic Gröbner basis of \( I \triangleleft \mathbb{C}[x, x^{(1)}, x^{(2)}, \ldots] \) with respect to \( \prec \).

Remark 1.5. We thank the anonymous referee for pointing us to the related articles [15, 24, 17] and Arthur Bik and Aida Maraj for a helpful discussion of those. The setup of \( \text{Inc}(\mathbb{N}) \)-stable ideals has parallels to the theory we study. Yet, the results cannot be immediately transferred from one to the other; in differential algebra, one has, for instance, fewer Noetherianity properties at one’s disposal and Leibniz’ rule instead of multiplicativity, which leads to a structurally different behavior. It would be intriguing to investigate in which extent results from one area transfer to the other.

Zobnin studied the differential ideal \( \langle x^p \rangle^{(\infty)} \) and proved the following.

Theorem 1.6 ([31]). The singleton \( \{ x^p \} \) is a differential Gröbner basis of \( \langle x^p \rangle^{(\infty)} \) with respect to the reverse lexicographical ordering.

Remark 1.7. Zobnin proved this result for so called \( \beta \)-orderings, i.e., monomial orderings on \( \mathbb{C}[x^{(\infty)}] \) for which the leading monomial of \( \langle x^p \rangle^{(k)} \) is of the form \( (x^{(i)})^a(x^{(i+1)})^{p-a} \) (see [18]). Since, in this article, we do not need the statement in its full generality, we just point out that the reverse lexicographical ordering is such a \( \beta \)-ordering. Note moreover that \( \langle x^p \rangle^{(k)} \) is bihomogeneous with respect to the vectors \( (1, 1, 1, \ldots) \) and \( (0, 1, 2, 3, \ldots) \), i.e., every monomial summand \( \prod_i (x^{(i)})^{a_i} \) in \( \langle x^p \rangle^{(k)} \) satisfies \( \sum_i a_i = p \) and \( \sum_i ia_i = k \).

1.2. Two independent variables. We now generalize Proposition 1.2 to two independent variables. We denote by \( \mathbb{C}[x^{(\infty,\infty)}] \) the ring of partial differential polynomials in \( x \) over \( \mathbb{C} \), i.e., the differential ring

\[
\mathbb{C}[x^{(\infty,\infty)}] := \left( \mathbb{C}[x^{(k,\ell)}]_{k,\ell \in \mathbb{N}}, \partial_s, \partial_t \right)
\]

in the two independent variables \( s, t \) and the commuting derivations \( \partial_s, \partial_t \) acting as

\[
\partial_s(x^{(k,\ell)}) = x^{(k+1,\ell)}, \quad \partial_t(x^{(k,\ell)}) = x^{(k,\ell+1)}, \quad \partial_s|_c \equiv \partial_t|_c \equiv 0.
\]

For \( m, n \in \mathbb{N} \), denote by \( I_{p,(m,n)} \) the differential ideal \( \langle x^p, x^{(m,0)}, x^{(0,n)} \rangle^{(\infty,\infty)} \) in \( \mathbb{C}[x^{(\infty,\infty)}] \) generated by \( x^p, x^{(m,0)}, \) and \( x^{(0,n)} \).

Denote by \( R_{m,n} \) the polynomial ring in the \( (m+1)(n+1) \) many variables \( \{ x^{(k,\ell)} \}_{0 \leq k \leq m, 0 \leq \ell \leq n} \) and let \( f_{p,(m,n)} \) be the bivariate polynomial

\[
f_{p,(m,n)} := \left( \sum_{k=0}^{m} \sum_{\ell=0}^{n} x^{(k,\ell)} s^k t^\ell \right)^p \in R_{m,n}[s,t].
\]

By the multinomial theorem,

\[
f_{p,(m,n)} = \sum_{\sum_{k,\ell} p} \left( \binom{p}{i_0,0,\ldots,i_{m,n}} \prod_{k,\ell} x^{i_k,\ell} s^{k-i_k,\ell} t^{\ell-i_k,\ell} \right),
\]

where \( (k,\ell) \in \{0,\ldots,m\} \times \{0,\ldots,n\} \) and \( i_{k,\ell} \in \mathbb{N} \) for all \( (k,\ell) \). Let \( C_{p,(m,n)} \triangleleft R_{m,n} \) denote the ideal generated by the the coefficients of \( f_{p,(m,n)} \).
Definition 1.8. We refer to points of \( \operatorname{Spec}(R_{m,n}/C_{p,(m,n)}) \) as \((m,n)\)-jets of the affine monomial scheme defined by \( x^p \).

Proposition 1.9. The following map is an isomorphism of \( \mathbb{C} \)-algebras:
\[
R_{m,n}/C_{p,(m,n)} \xrightarrow{\cong} \mathbb{C}[x^{(\infty,\infty)}]/I_{p,(m+1,n+1)}, \quad x_{k,\ell} \mapsto \frac{1}{k!\ell!} \cdot x^{(k,\ell)}.
\]

Proof. The coefficient \( f_{a,b} \) of \( s^a t^b \) in \( f_{p,(m,n)} \) is given as
\[
f_{a,b} = \sum_{(i_k,\ell_k)\in I} \left( \prod_{k,\ell} i_k \right) \left( p \sum_{i_0,\ldots,i_m} \cdot \prod_{k,\ell} i_k \right),
\]
where \( I = \{(i_k,\ell_k) \mid \sum_{k=0}^n (k \sum_{\ell=0}^\infty i_k,\ell_k) = a, \sum_{\ell=0}^\infty (\ell \sum_{k=0}^n i_k,\ell_k) = b, \sum i_k,\ell_k = p \} \).

By the symmetry of the second derivatives, we obtain
\[
(x^p)^{(a,b)} = \left( (x^p)^{(a,0)} \right)^{(0,b)}.
\]

For all \( 0 \leq i \leq m \) and \( 0 \leq s \leq n \), let \( j_i^s \) be the multiplicity of \( s \) in the multiset \( \{k_0+\ldots+k_{i-1}, \ldots, k_0+\ldots+k_{i-1}\} \). Thus \( k_i = \sum_{s=0}^n j_i^s \), \( \sum_{i,s} j_i^s = p \), \( \sum_{i,s} ij_i^s = a \), and \( \sum_{i,s} sj_i^s = b \). Let \( J \) denote the set of all those \((j_i^s)_{s,i}\). Then \( (x^p)^{(a,b)} \) equals
\[
\sum_{(j_i^s)\in J} \left( \prod_{(j_i^s)_{s,i}} i_s \right) \left( \sum_{j_0^0} b_0! \cdot \prod_{j_0^0} \left( x^{(i,s)} \right)^{j_i^0} \right).
\]

In order to generalize Theorem 1.6 to partial differential rings, we first generalize the concept of \( \beta \)-orderings. We denote by \( \mathbb{C}[x^{(\leq m,\leq n)}] \) the truncated differential ring \( \mathbb{C}[x^{(\infty,\infty)}]/(x^{(m,0)}, x^{(0,n)})^{(\infty,\infty)} \). A monomial is supported on a triangle of \( T \) if the indices of the variables of that monomial are among the vertices of a triangle of \( T \).

Definition 1.10. Fix \( m, n, p \in \mathbb{N} \) and a triangulation \( T \) of the \( m \times n \) rectangle. A monomial ordering \( \prec \) on \( \mathbb{C}[x^{(\leq m,\leq n)}] \) is a \( T \)-ordering if the leading monomial of each \( (x^p)^{(k,\ell)} \), \( 0 \leq k \leq mp, 0 \leq \ell \leq mp \), is supported on a triangle of \( T \).

Remark 1.11. By identifying \( x^{(k,\ell)} \) with \( k!\ell! x_{k,\ell} \), one equivalently defines a \( T \)-ordering as a monomial ordering on \( R_{m,n} = \mathbb{C}[\{x_{k,\ell}\}_{0\leq k\leq m, 0\leq \ell\leq n}] \) such that the leading monomial of each coefficient of \( f_{p,(m,n)} \in R_{m,n}[s, t] \) is supported on a triangle of \( T \).

Denote by \( T_{m,2} \) the unimodular triangulation of the \( m \times 2 \) rectangle depicted in Figure 1. This is the placing triangulation of the point configuration
\[
[(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), \ldots, (m,0), (m,1), (m,2)].
\]
Then from the first two lines in (1), we obtain

\[ M_{\text{monomial}} \]

tracted on a triangle of \( T \).

**Proof.** Consider \((x_1, x_2, \ldots)\). Moreover, the following identities hold:

\[ i \leq \sum_{j, n} x_{ij} + x_{in} \quad \text{for all} \quad i \leq \sum_{j, n} x_{ij} + x_{in} \]

Proposition 1.12. For all \( 0 \leq k \leq mp \) and \( 0 \leq \ell \leq np \), \((x^p)^{\langle k, \ell \rangle}\) has a unique monomial summand supported on a triangle of \( T_{m,n} \). Moreover, the reverse lexicographical ordering \( \prec \) on \( \mathbb{C}[x^{(0,0)}, x^{(0,1)}, \ldots, x^{(m,0)}, \ldots, x^{(m,n)}] \) is a \( T \)-ordering for \( T = T_{m,n} \) for all \( p \), where we order the variables as \( x^{(0,0)} \prec x^{(0,1)} \prec \cdots \prec x^{(0,n)} \prec \cdots \prec x^{(m,0)} \prec \cdots \prec x^{(m,n)} \).

**Proof.** Consider \((x^p)^{\langle k, \ell \rangle}\) and let us suppose that it has a monomial summand supported on a triangle of \( T \) of the form \( x_{h,n}^a x_{h+1,s}^b x_{h+1,s+1}^c \). Suppose that there exists a monomial \( M = \prod_j x_{ij}^{i_j} \) such that \( x_{h,n}^a x_{h+1,s}^b x_{h+1,s+1}^c \prec M \). Then, it follows that \( i_{h,n} \leq a, i_{h,0} = \cdots = i_{h,n-1} = 0, \) and \( i_{j,0} = \cdots = i_{j,n} = 0 \) for all \( j < h \). Moreover, the following identities hold:

\[ a + b + c = \sum_{j \geq h} i_{j,0} + \cdots + i_{j,n} = p, \]

\[ ha + (h + 1)b + (h + 1)c = \sum_{j \geq h} j(i_{j,0} + \cdots + i_{j,n}) = k, \]

\[ na + sb + (s + 1)c = \sum_{j \geq h} j_{j,1} + \cdots + nj_{j,n} = \ell. \]

Then from the first two lines in (1), we obtain

\[ (a - i_{h,n}) + (h + 1)(\sum_{j \geq h} (i_{j,0} + \cdots + i_{j,n}) - p) + \sum_{j \geq h+2} (j - h - 1)(i_{j,0} + \cdots + i_{j,n}) = 0. \]
Note that each of the three summands on the left hand side is nonnegative. Thus
\[ M = x_{h,n}^{i_{h,n}} x_{h+1,0}^{i_{h+1,0}} \cdots x_{h+1,n}^{i_{h+1,n}}, \quad i_{h,n} = a, \quad i_{h+1,0} + \ldots + i_{h+1,n} = b + c. \]
Since \( x_{h,n}^{a} x_{h+1,n}^{b} x_{h+1,s+1}^{c} < M \), we have \( i_{h+1,s} \leq b \), and for all \( r < s, i_{h+1,r} = 0 \). Then from the third line in (1) we have
\[ s(b + c) + c = s(i_{h+1,s} + \ldots + i_{h+1,n}) + i_{h+1,s+1} + \ldots + (n - s)i_{h+1,n}. \]
Thus \( b = i_{h+1,s} \) and \( c = i_{h+1,s+1} \). We conclude that \( M = x_{h,n}^{a} x_{h+1,s+1}^{b} x_{h+1,s+1}^{c} \).

Now suppose there exists a monomial summand of \((x^p)^{(k,\ell)}\) that is supported on a triangle of \( T_{m,n} \) of the form \( x_{h,s}^{a} x_{h+1,s+1}^{b} x_{h+1,0}^{c} \), and suppose that there exists a monomial \( M = x_{h,n}^{a} x_{h,s+1}^{b} x_{h+1,0}^{c} \) \( \neq \) \( M \). Then \( i_{h,s} \leq a, i_{h,r} = 0 \) for all \( r < s \), and
\[ a + b + c = \sum_{j \geq h} i_{j,0} + \ldots + i_{j,n} = p, \]
\[ ha + hb + (h + 1)c = \sum_{j \geq h} j(i_{j,0} + \ldots + i_{j,n}) = k, \]
\[ sa + (s + 1)b = \sum_{j \geq h} i_{j,1} + \ldots + ni_{j,n} = \ell. \]
Suppose \( a + b < i_{h,s} + \ldots + i_{h,n} \). Then \( b < i_{h,s+1} + \ldots + i_{h,n} \). By the third line in (2),
\[ (a + b) + b = s(i_{h,s} + \ldots + i_{h,n}) + (i_{h,s+1} + \ldots + (n - s)i_{h,n}) + \sum_{j \geq h} i_{j,1} + \ldots + ni_{j,n}, \]
which is a contradiction to our assumption. From the first two lines in (2) we obtain
\[ (a + b - (i_{h,s} + \ldots + i_{h,n})) + (h + 1) \left( \sum_{j \geq h} (i_{j,0} + \ldots + i_{j,n}) - p \right) + \sum_{j \geq h + 2} (j - h - 1)(i_{j,0} + \ldots + i_{j,n}) = 0. \]
Thus \( a + b = i_{h,s} + \ldots + i_{h,n} \), and \( c = i_{h+1,0} + \ldots + i_{h+1,n} \). Therefore, from (3) we conclude that \( a = i_{h,s}, b = i_{h,s+1} \), and \( c = i_{h+1,0} \) which means \( M = x_{h,s}^{a} x_{h,s+1}^{b} x_{h+1,0}^{c} \).

We have proved that if \((x^p)^{(k,\ell)}\) contains a monomial summand supported on a triangle of \( T_{m,n} \), then this monomial is its leading monomial. Hence for all \( 0 \leq k \leq mp, \) \( 0 \leq \ell \leq np, (x^p)^{(k,\ell)} \) has at most one monomial summand that is supported on a triangle of \( T_{m,n} \). The triangles of \( T_{m,n} \) are given by \{ \( (j,n), (j+1,s), (j+1,s+1) \) \} and \{ \( (j+1,0), (j,s), (j,s+1) \) \} for \( 0 \leq j \leq m - 1 \) and \( s = 0, \ldots, n - 1 \). The number of monomials of degree \( p \) that are supported on these triangles is \((mp+1)(np+1)\). Indeed, we have \( 2mn \) triangles, \((3n+1)m+n\) edges, and \((n+1)(m+1)\) vertices on \( T_{m,n} \). The number of monomials which can be formed by the \( 2mn \) triangles containing all three corresponding variables is \( 2mn \cdot \# \{ a+b+c = p \mid a, b, c > 0 \} = 2mn(\frac{p-1}{2})(p-2) \). The edges give rise to \((3n+1)m+n)(p-1)\) monomials of degree \( p \) in which both variables appear. The vertices give rise to \((n+1)(m+1)\) monomials of degree \( p \) containing only this variable. Thus, \( 2mn(\frac{p-1}{2})(p-2) + ((3n+1)m+n)(p-1) + (n+1)(m+1) = (mp+1)(np+1) \) monomials are supported on triangles of \( T_{m,n} \). Each of these monomials belongs to the set of monomials appearing in the expression of \((x^p)^{(k,\ell)}\) for some \( 0 \leq k \leq mp \) and \( 0 \leq \ell \leq np \). We conclude that every \((x^p)^{(k,\ell)}\) has exactly one monomial that is supported on a triangle of \( T_{m,n} \) and this monomial is its leading monomial.

We generalize Zobnin’s result to the case of two independent variables as follows.
Theorem 1.13. For all \( m, p \in \mathbb{N}, \{(x^p)^{(k,l)}\}_{0 \leq k \leq mp, 0 \leq l \leq 2p} \) is a Gröbner basis of the differential ideal generated by \( x^p \) in the truncated partial differential ring \( \mathbb{C}[x^{\leq m, \leq 2}] \) with respect to any \( T_{m,2} \)-ordering.

Proof. We pick up and generalize the construction from the proof of [25, Lemma 1]. Consider a \( \mathbb{C} \)-vector space \( V_p \) with \( p-1 \) pairs of countable series of basis vectors \( \xi_{i,s} \) and \( \eta_{i,s} ^r \), where \( r = 0, \ldots, p-2 \), and \( i, s \in \mathbb{N} \). Let \( \Lambda(V_p) \) denote the exterior algebra of \( V_p \). The two derivations increase the first (resp. second) index of the variables \( \xi_{i,s} \) and \( \eta_{i,s} ^r \) by one. Extending this rule by Leibniz’ rule endows \( \Lambda(V_p) \) with the structure of a differential algebra. Denote by \( V \) differential ideal generated by \( x \).

Claim. The variables \( x_{i_1 s_1}, x_{i_2 s_2}, \ldots, x_{i_n s_n} \) appearing in the monomial \( K \) can be distributed to sets \( S_0, \ldots, S_v \) with \( v \leq p-1 \) such that each \( S_v \) contains neither a repeated variable nor two variables that are supported on an edge of \( T_{m,2} \).

Proof. We prove the claim by induction over \( m \). For \( m = 0 \), suppose \( K = x_{0,0} ^{b} x_{0,1} ^{c} \). Since \( K \in M \), we have \( a + b \leq p-1 \) and \( b + c \leq p-1 \). In this case we can construct \( \text{max}(a,c) + b \leq p-1 \) sets \( S_r \) that satisfy the hypothesis of the claim. Suppose that the claim is true for some \( m \), and \( K = K' x_{m,0} ^{a} x_{m,1} ^{b} x_{m,2} ^{c} x_{m+1,0} ^{a'} x_{m+1,1} ^{b'} x_{m+1,2} ^{c'} \) where \( K' \in \mathbb{C}[x^{\leq m-1, \leq 2}] \). By the induction hypothesis, there exist \( v \leq p-1 \) sets \( S_0, \ldots, S_v \) that satisfy the hypothesis of the claim for the variables appearing in \( K' x_{m,0} ^{a} x_{m,1} ^{b} x_{m,2} ^{c} \) and distribute them over the sets that do not contain \( x_{m,0}, x_{m,1}, \) or \( x_{m,2} \). If the number of these sets is less than \( a' \), construct new sets containing only the single element \( x_{m+1,0} ^{a} \). Since \( a' + b + \text{max}(a,c) \leq p-1 \), the total number of the sets \( S_v \) is still \( p-1 \) at most. For the \( b' \) copies of \( x_{m+1,1} ^{a'} \), let us distribute them over the sets that do not contain \( x_{m,2}, x_{m+1,0} ^{a} \). If the total number of these latter sets is smaller than \( b' \), then construct new sets containing the single element \( x_{m+1,1} ^{a'} \). Since \( a' + b' + c \leq p-1 \), the total number of the sets \( S_v \) is still \( p-1 \) at most. We apply the same argumentation to \( c' \), concluding the proof.

Let us fix \( S_v = \{x_{i_1 s_1} \ldots, x_{i_n s_n}\} \) such that \( x_{i_1 s_1} < x_{i_2 s_2} < \cdots < x_{i_n s_n} \). Therefore, the elements of \( S_v \) satisfy the following:

\( (i_{j+1} - i_j > 1 \) or \( (i_{j+1} = i_j + 1) \) and \( s_j \neq 2 \) and \( s_j' \neq 0 \) \) or \( (i_{j+1} = i_j) \) and \( s_j = 0 \) and \( s_j' = 2 \).

We claim that \( \phi(x_{i_1 s_1} x_{i_2 s_2} \cdots x_{i_n s_n}) \) contains a monomial summand \( \alpha_r \) of the form

\[ \alpha_r = (\eta_{i_1 b_1} ^{r_1} \xi_{c_1 d_1} ^{r_1}) \wedge (\eta_{i_2 b_2} ^{r_2} \xi_{c_2 d_2} ^{r_2}) \wedge \cdots \wedge (\eta_{i_n b_n} ^{r_n} \xi_{c_n d_n} ^{r_n}), \]

where \( x_{b_j} < x_{a_j+1} + b_j \) and \( x_{c_j} < x_{c_{j+1} + d_j} \). Moreover, \( a_j + c_j = i_j \) and \( b_j + d_j = s_j' \). Let \( (a_1, b_1) = (i_1', s_1') \) and \( (c_1, d_1) = (0, 0) \). Suppose that \( \eta_{i_1 b_1} ^{r_1} \xi_{c_1 d_1} ^{r_1} \cdots \wedge (\eta_{i_n b_n} ^{r_n} \xi_{c_n d_n} ^{r_n}) \) is constructed and let us construct \( \eta_{a_{j+1}+1} b_{j+1} \wedge \xi_{c_{j+1}+1} d_{j+1} \).
Case 1. If \( i'_\ell + 1 - i'_{\ell} > 1 \), then set \((a_{\ell+1}, b_{\ell+1}) = (a_{\ell} + i'_{\ell+1} - i'_{\ell} - 1, s'_{\ell+1})\) and \((c_{\ell+1}, d_{\ell+1}) = (c_{\ell} + 1, 0)\).

Case 2. If \( i'_\ell = i'_{\ell} + 1 \), then:
- If \( b_{\ell} = 0 \) and \( d_{\ell} = 1 \), then set \((a_{\ell+1}, b_{\ell+1}) = (a_{\ell}, 1)\) and \((c_{\ell+1}, d_{\ell+1}) = (c_{\ell} + 1, s_{\ell+1} - 1)\).
- Otherwise set \((a_{\ell+1}, b_{\ell+1}) = (a_{\ell} + 1, s_{\ell+1} - 1)\) and \((c_{\ell+1}, d_{\ell+1}) = (c_{\ell}, 1)\).

Case 3. If \( i'_\ell = i'_{\ell} \), which implies \( s'_{\ell} = b_{\ell} = d_{\ell} = 0 \), then set \((a_{\ell+1}, b_{\ell+1}) = (a_{\ell}, 1)\) and \((c_{\ell+1}, d_{\ell+1}) = (c_{\ell}, 1)\).

We repeat the same construction for every \( r \). Then \( \alpha = \alpha_0 \land \alpha_1 \land \cdots \land \alpha_{\nu} \) is a monomial summand of \( \phi(K) \). Now suppose that there exists another monomial \( K^* \) of \( L \) such that \( \phi(K^*) \) contains \( \alpha \) as a monomial summand. Let \( x_{i_1^*, s_1^*} \) be the smallest variable appearing in \( K^* \). Then \( x_{i_1^*, s_1^*} \preceq x_{i_1, s_1} \). The expression of \( \phi(x_{i_1^*, s_1^*}) \) has a summand of the form \( \eta_{u,v}^r \land \xi_{u,v}^s \) for some \( r \) that appears in \( \alpha \) where \( u + \tilde{u} = i_1^* \) and \( v + \tilde{v} = s_1^* \).

This means that \( x_{i_1^*, s_1^*} \) appears in the monomial \( K \), and therefore \( x_{i_1^*, s_1^*} = x_{i_1, s_1} \). By going through all the variables of \( K^* \) and repeating the same argument, we conclude \( K = K^* \). Thus \( \phi(L) \neq 0 \) and \( L \notin I_{p,(m+1,3)} \).

\( \square \)

**Proposition 1.14.** If \( m \geq 1 \), \( n \geq 3 \), and \( p \geq 2 \), the family \( \{(x^p)_0^{(k,\ell)}\}_{0 \leq k \leq mp, 0 \leq \ell \leq np} \) is not a Gröbner basis of the differential ideal generated by \( x^p \) in the ring \( \mathbb{C}[x^{(\leq m, \leq n)}] \) with respect to any \( T_{m,n} \)-ordering.

**Proof.** If \( \{(x^p)_0^{(k,\ell)}\}_{0 \leq k \leq mp, 0 \leq \ell \leq np} \) is a Gröbner basis of the differential ideal generated by \( x^p \) w.r.t. the \( T_{m,n} \)-ordering \( < \), the same statement holds for the \( T_{m-1,n} \)-ordering \( < \).

Therefore, we restrict our proof to the case \( m = 1 \). Consider the differential polynomials \((x^p)^{(p-1,3)}\) and \((x^p)^{(p-1,0)}\). We will show that their \( S \)-polynomial does not have an LCM-representation. By [7, Theorem 2.9.6], the \( (x^p)^{(k,\ell)} \) then are not a Gröbner basis.

Note that \( \text{lm}((x^p)^{(p-1,3)}) = x_{0,3}x_{1,0}^p \) and \( \text{lm}((x^p)^{(p-1,0)}) = x_{0,0}x_{1,0}^p \). Their least common multiple is \( \text{LCM}(\text{lm}((x^p)^{(p-1,3)}), \text{lm}((x^p)^{(p-1,0)})) = x_{0,0}x_{0,3}x_{1,0}^p \). We proceed with the proof by contradiction. Suppose that

\[
S((x^p)^{(p-1,3)}), (x^p)^{(p-1,0)}) = \sum_{a,b} (x^p)^{(a,b)}g_{a,b},
\]

where \( \text{lm}((x^p)^{(a,b)}g_{a,b}) < x_{0,0}x_{0,3}x_{1,0}^p \). Since all monomials in \( S((x^p)^{(p-1,3)}), (x^p)^{(p-1,0)} \) are of degree \( p + 1 \) and homogeneous w.r.t. both derivatives \( \partial_x \) and \( \partial_t \), we can write

\[
S((x^p)^{(p-1,3)}), (x^p)^{(p-1,0)}) = \sum_{p-2 \leq a \leq p-1, 0 \leq b \leq 3} c_{a,b}(x^p)^{(a,b)}x_{p-1-a,3-b},
\]

where \( c_{a,b} \) are constants and \( (x^p)^{(a,b)}x_{p-1-a,3-b} < x_{0,1}x_{0,3}x_{1,0}^p \). We now list the polynomials that can show up in the previous equality with their leading monomials:

- \((x^p)^{(p-2,0)}x_{1,3}, \text{lm}((x^p)^{(p-2,0)}x_{1,3}) = x_{0,0}^2x_{1,0}x_{1,3}, \)
- \((x^p)^{(p-2,1)}x_{1,2}, \text{lm}((x^p)^{(p-2,1)}x_{1,2}) = x_{0,0}x_{0,1}x_{1,0}x_{1,2}, \)
- \((x^p)^{(p-2,2)}x_{1,1}, \text{lm}((x^p)^{(p-2,2)}x_{1,1}) = x_{0,1}^2x_{1,0}x_{1,1}, \)
- \((x^p)^{(p-2,3)}x_{1,0}, \text{lm}((x^p)^{(p-2,3)}x_{1,0}) = x_{0,1}x_{0,2}x_{1,0}x_{1,1}, \)
- \((x^p)^{(p-1,0)}x_{0,3}, \text{lm}((x^p)^{(p-1,0)}x_{0,3}) = x_{0,0}x_{1,0}^2x_{0,3}, \)
- \((x^p)^{(p-1,1)}x_{0,2}, \text{lm}((x^p)^{(p-1,1)}x_{0,2}) = x_{0,1}x_{1,0}x_{0,2}, \)
- \((x^p)^{(p-1,2)}x_{0,1}, \text{lm}((x^p)^{(p-1,2)}x_{0,1}) = x_{0,2}x_{1,0}^2x_{0,1}, \)
- \((x^p)^{(p-1,3)}x_{0,0}, \text{lm}((x^p)^{(p-1,3)}x_{0,0}) = x_{0,3}x_{1,0}^2x_{0,0}. \)
Among all these polynomials, only \((x^p)^{(p-2,0)}x_{1,3}\) and \((x^p)^{(p-2,1)}x_{1,2}\) have leading monomials smaller than \(x_{0,0,0,3}x_{p,-1}\) under the ordering \(<\). Thus,

\[(4) \quad S((x^p)^{(p-1,3)}, (x^p)^{(p-1,0)}) = c_{p-2,0}(x^p)^{(p-2,0)}x_{1,3} + c_{p-2,1}(x^p)^{(p-2,1)}x_{1,2}.\]

Note that \(x_{0,2}x_{1,0}^{p-2}x_{1,1}\) is a monomial summand of the polynomial \((x^p)^{(p-1,3)}\). Then \(x_{0,0,2}x_{1,0}^{p-2}x_{1,1}\) shows up in \(S((x^p)^{(p-1,3)}, (x^p)^{(p-1,0)})\) but not in \(c_{p-2,0}(x^p)^{(p-2,0)}x_{1,3} + c_{p-2,1}(x^p)^{(p-2,1)}x_{1,2}\), which is in contradiction to Equation (4). □

2. Linking \(\dim\mathbb{C}(R_n/C_{p,n})\) and \(\dim\mathbb{C}(R_{m,n}/C_{p,(m,n)})\) to lattice polytopes

We now link the sequences \(\dim\mathbb{C}(R_n/C_{p,n})\) and \(\dim\mathbb{C}(R_{m,n}/C_{p,(m,n)})\)—both considered as sequences in \(p\) to lattice polytopes.

2.1. Polynomiality of \(\dim\mathbb{C}(R_n/C_{p,n})\).

We investigate the following question.

**Question 2.1.** Fix \(n \in \mathbb{N}\). As \(p\) varies, is \((\dim\mathbb{C}(R_n/C_{p,n}))_{p \in \mathbb{N}}\) a polynomial in \(p\) of degree \(n + 1\)?

Before turning to the proof that the answer to this question is \(yes\), we present an explicit example.

**Example 2.2** \((\dim\mathbb{C}(R_N/C_{P,6}))_{p \in \mathbb{N}}\). Computations in \textsc{Singular} [9] reveal the first 13 entries of the sequence \(\dim\mathbb{C}(R_6/C_{P,6})_{p \in \mathbb{N}}\) to be

\[0, 1, 34, 353, 2037, 8272, 26585, 72302, 173502, 377739, 760804, 1437799, 2576795, \]

coinciding with the sequence \texttt{www.oeis.org/A244881}. With \textsc{Mathematica}, we compute the interpolating polynomial on the values for \(p = 1, \ldots, 20\) to be

\[
\frac{17}{315} p^7 + \frac{17}{90} p^6 + \frac{53}{180} p^5 + \frac{19}{72} p^4 + \frac{13}{90} p^3 + \frac{17}{360} p^2 + \frac{1}{140} p,
\]

which is indeed of degree \(7 = 6 + 1\). △

Let \(<\) denote the reverse lexicographical ordering on \(R_n = \mathbb{C}[x_0, \ldots, x_n]\). The following lemma, as also presented in [5], determines the initial ideal of \(C_{p,n}\) w.r.t. \(<\).

We present a proof building on Zobin’s result and Proposition 1.2.

**Lemma 2.3.** The initial ideal of \(C_{p,n}\) with respect to \(<\) is generated by the family

\[
\{x_i u_i x_{i+1}^{u_i+1} \mid u_i + u_{i+1} = p, \quad 0 \leq i \leq n - 1\}.
\]

**Proof.** Let us first prove that the leading monomials of our family of generators are \(x_i^n x_{i+1}^{u_i+1}\). Let \(0 \leq k < np\) be of the form \(k = mp + (p - a)\), where \(1 \leq a \leq p\) and \(0 \leq m \leq n - 1\). For \(k = np\), the leading term of \(f_k\) is \(x^n_p\), where \(f_k\) denotes the coefficient of \(t^k\) in the polynomial \(f_{p,n}\). We claim that the leading monomial of the polynomial \(f_{k}\) is \(x_{m}^{n}x_{m+1}^{p-a}\). Suppose that \(x_{i_0}^{u_0} \cdots x_{i_n}^{u_n} \succ x_{m}^{n}x_{m+1}^{p-a}\) for some monomial summand \(x_{i_0}^{u_0} \cdots x_{i_n}^{u_n}\) in \(f_{k}\). This implies that \(i_0 = \cdots = i_{m-1} = 0\). Then \(i_m + \cdots + i_n = p\) and \(m_i + \cdots + m_{n} = mp + p - a\). Since \(i_m \leq a\), from

\[(a - i_m) + (m + 1)(i_m + \cdots + i_n - p) + (i_{m+2} + \cdots + (n - m - 1)i_n) = 0\]

we conclude that \(i_m = a\), \(i_{m+1} = p - a\), and \(i_{m+2} = \cdots = i_n = 0\). Therefore, \(x_{m}^{n}x_{m+1}^{p-a}\) is indeed the leading monomial of \(f_k\). We now consider the truncated differential ring \(\mathbb{C}[x^{\leq n}]\). As rings, \(\mathbb{C}[x^{\leq n}] \cong \mathbb{C}[x_0, \ldots, x_n] = R_n\). Then the following holds:

\[
in_{\prec}\{(x^p)^{(k)} \mid 0 \leq k \leq np\} = \langle\{\lim_{k=0}^{np} r_k(x^p)^{(k)} \} \mid r_k \in \mathbb{C}[x^{(\infty)}]\},
\]
The fractional stable set polytope of graph, $Q_{\text{Stab}}(G)$, is defined as $\text{Stab}(G) := \text{conv} \{ \chi^S \in \mathbb{R}^{|V|} \mid S \subseteq V \text{ stable} \}$, where the incidence vectors $\chi^S = (\chi_v^S)_{v \in V} \in \mathbb{R}^{|V|}$ are defined as

\[
\chi_v^S := \begin{cases} 
1 & \text{if } v \in S, \\
0 & \text{else}.
\end{cases}
\]

The fractional stable set polytope of $G$ is defined as

\[
Q_{\text{Stab}}(G) := \left\{ x \in \mathbb{R}^{|V|} \mid 0 \leq x(v) \forall v \in V, \sum_{v \in Q} x(v) \leq 1 \text{ for all cliques } Q \text{ of } G \right\}.
\]

Hence $\text{Stab}(G) = \text{conv}\{(0,1)^{|V|} \cap Q_{\text{Stab}}(G)\}$. Chvátal [6, Theorem 3.1] proved that a graph $G$ is perfect if and only if $\text{Stab}(G) = Q_{\text{Stab}}(G)$. It follows from Fulkerson’s theory of anti-blocking polyhedra [10] that this result is equivalent to the perfect graph theorem. The latter was conjectured by Berge [3] and proven by Lovász [20].

**Proof of Lemma 2.4.** Consider the graph $G = (\{0,1,\ldots,n\},\{[i,i+1]\}_{i=0,\ldots,n-1})$. Observe that $P_n$ is precisely the fractional stable set polytope of $G$. Since $G$ is a perfect graph, $Q_{\text{Stab}}(G) = \text{Stab}(G)$ and $P_n$ has zero-one vertices as claimed. 

For a $d$-dimensional polytope $P \subseteq \mathbb{R}^n$ with integer vertices and $t \in \mathbb{N}$, denote by $L_P(t) := |tP \cap \mathbb{Z}^n|$ the number of lattice points of the dilated polytope $tP$. Ehrhart proved that this number is a rational polynomial in $t$ of degree $d$, i.e., there exist rational numbers $l_{P,0}, \ldots, l_{P,d}$, s.t.

\[
L_P(t) = l_{P,d}t^d + \cdots + l_{P,1}t + l_{P,0}.
\]

The polynomial $L_P \in \mathbb{Q}[t]$ is called the Ehrhart polynomial of $P$.

**Theorem 2.5.** The number $\dim_{\mathbb{C}}(R_n/C_{P,n})$ is the Ehrhart polynomial of the polytope $P_n$ defined in Lemma 2.4 evaluated at $p - 1$.

**Proof.** From Lemma 2.3 we read that $x_0^{u_0} \cdots x_n^{u_n}$ is a standard monomial if and only if $u_i + u_{i+1} < p$ for all $0 \leq i \leq n - 1$. The claim then follows from Lemma 2.4. $\square$
2.2. Investigation of $\dim_\mathbb{C}(R_{m,2}/C_{p,m})$. In this section, we generalize the results found for $R_n/C_{p,n}$ to two independent variables, i.e., to $R_{m,n}/C_{p,n}$.

**Theorem 2.6.** For all $m, p \in \mathbb{N}$, $\dim_\mathbb{C}(R_{m,2}/C_{p,m})$ is the Ehrhart polynomial of the $3(m + 1)$-dimensional lattice polytope

$$P_{m,2}(u_0, u_1, \ldots, u_m) \in (\mathbb{R}_{\geq 0})^{3(m+1)}|u_{k_1, l_1} + u_{k_2, l_2} + u_{k_3, l_3} \leq 1$$

for all indices s.t. \{(k_1, l_1), (k_2, l_2), (k_3, l_3)\} is a triangle of $T_{m,2}$

evaluated at $p - 1$.

**Proof.** Let $G$ be the edge graph of the regular triangulation from Figure 1 for $m = 2$ with $3(m + 1)$ vertices and $2 + 7m$ edges. Since this graph is perfect and the maximal cliques are precisely the triangles of $T_{m,2}$, $\text{Stab}(G) = Q\text{Stab}(G) = P_{m,2}$. By Theorem 1.13, $x_{m0}^u \cdots x_{m2}^u$ is a standard monomial if and only if for all triples of indices \{(i_1, j_1), (i_2, j_2), (i_3, j_3)\} that are a triangle of $T_{m,2}$, $u_{i_1, j_1} + u_{i_2, j_2} + u_{i_3, j_3} \leq p - 1$. \hfill $\square$

In terms of integer programming, Theorem 2.6 translates as follows.

**Corollary 2.7.** For all $m \in \mathbb{N}$, $\dim_\mathbb{C}(R_{m,2}/C_{p,m})$ is polynomial in $p$ of degree $3(m + 1)$. For fixed $p$, this is the number of non-negative integer solutions of the $2m^2$ linear inequalities $x_{i_1, j_1} + x_{i_2, j_2} + x_{i_3, j_3} \leq p - 1$, where \{(i_1, j_1), (i_2, j_2), (i_3, j_3)\} runs over the $2m^2$ many triangles of $T_{m,2}$.

3. Regular unimodular triangulations of the $m \times n$ rectangle

We now outline how regular unimodular triangulations of the $m \times n$ rectangle give rise to $T$-orderings on the truncated partial differential ring $C[x^{\leq m}, x^{\leq n}]$, or, equivalently, on the polynomial ring $\mathbb{C}[\{x_{k, \ell}\}_{0 \leq k \leq m, 0 \leq \ell \leq n}]$.

**Example 3.1** ($m = n = 2$). Again, denote by $C_{p,(2,2)}$ the ideal in

$$R_{2,2} = \mathbb{C}[x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}, x_{20}, x_{21}, x_{22}]$$

generated by the $(2p + 1)^2$ many coefficients $f_{k, \ell}$ of $s^k t^\ell$ in

$$f_{p,(2,2)} = (x_0 + x_0 t + x_2 t^2 + x_{10} s + x_{11} s t + x_{12} s t^2 + x_{20} s^2 + x_{21} s^2 t + x_{22} s^2 t^2)^p.$$ 

Let $< \text{ denote the weighted reverse lexicographical ordering on } R_{2,2} \text{ for the weight}$

$$w_{2,2} := (2^8 + 1, \ldots, 2^8 + 1) - (1, 2, 2^2, \ldots, 2^8) \in \mathbb{N}^9,$$

i.e., assigning weight 128 to $x_{00}$, weight 127 to $x_{01}$, and so on. Note that $w_{2,2}$ induces the triangulation $T_{2,2}$ in the upper hull convention. For $p = 3$, we find that within the monomials of the $f_{k, \ell}$, the following 8 triples of pairwise different variables show up:

$$\{x_{00}, x_{01}, x_{10}\}, \{x_{01}, x_{02}, x_{10}\}, \{x_{02}, x_{10}, x_{11}\}, \{x_{02}, x_{11}, x_{12}\}, \{x_{10}, x_{11}, x_{20}\}, \{x_{11}, x_{12}, x_{20}\}, \{x_{12}, x_{20}, x_{21}\}, \{x_{12}, x_{21}, x_{22}\},$$

the indices of each of which define a triangle of $T_{2,2}$. Computations in **Singular** prove that the coefficients of $f_{3,(2,2)}$ are a Gröbner basis of $C_{3,(2,2)} \triangleleft R_{2,2}$ with respect to the weighted reverse lexicographical ordering for $w_{2,2}$. Our computations for $p \leq 25$ approve the same statements, confirming Proposition 1.12 and Theorem 1.13. There are 64 regular unimodular triangulations of the $2 \times 2$ square in total, four of which give rise to a Gröbner basis in the sense above. Those are depicted in Figure 3. \hfill $\Delta$
Figure 3. The four regular unimodular triangulations of the $2 \times 2$ square giving rise to a Gröbner basis, the first of which is $T_{2,2}$. Note that they all arise from $T_{2,2}$ by rotating and flipping.

Example 3.2 ($m = 3, n = 2$). Again, denote by $C_{p,(3,2)}$ the ideal in

$$R_{3,2} = \mathbb{C}[x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}, x_{20}, x_{21}, x_{22}, x_{30}, x_{31}, x_{32}]$$

generated by the $(3p + 1)(2p + 1)$ coefficients of $f_{p,(3,2)}$. Let $\prec$ denote the weighted reverse lexicographical ordering on $R_{3,2}$ for the weight vector

$$w_{3,2} := (2^{11} + 1, \ldots, 2^{11} + 1) - (2^0, 2^1, \ldots, 2^{11}) \in \mathbb{N}^{12}.$$

For $p = 3$, the following 12 triples of pairwise different variables show up within the leading monomials of the 70 coefficients:

$$\{x_{00}, x_{01}, x_{10}\}, \{x_{01}, x_{02}, x_{10}\}, \{x_{02}, x_{10}, x_{11}\}, \{x_{02}, x_{11}, x_{12}\},$$
$$\{x_{10}, x_{11}, x_{20}\}, \{x_{11}, x_{12}, x_{20}\}, \{x_{12}, x_{20}, x_{21}\}, \{x_{12}, x_{21}, x_{22}\},$$
$$\{x_{20}, x_{21}, x_{30}\}, \{x_{21}, x_{22}, x_{30}\}, \{x_{22}, x_{30}, x_{31}\}, \{x_{22}, x_{31}, x_{32}\},$$

the indices of each of which define a triangle of $T_{3,2}$. Computations in Singular prove that the 70 coefficients $f_{k,\ell}$ of $s^k t^\ell$ in $f_{3,(3,2)}$ are indeed a Gröbner basis of $C_{3,(3,2)} \prec R_{3,2}$ w.r.t. $\prec$, turning $\prec$ into a $T_{3,2}$-ordering for $p = 3$. Note that there are 852 regular unimodular triangulations of the $3 \times 2$ rectangle in total, four of which give rise to a Gröbner basis in the sense above. Those four are depicted in Figure 4. △

Figure 4. The four regular unimodular triangular regulations of the $3 \times 2$ rectangle giving rise to a Gröbner basis, the first of which is $T_{3,2}$.

Remark 3.3. Computations finish only for small values of $m$ and $p$. For $m = 5$, we validated with Singular that the coefficients of $f_{p,(5,2)}$ are a Gröbner basis with respect to the weighted reverse lexicographical ordering $\prec$ for a vector inducing $T_{5,2}$ in the upper hull convention up to $p = 9$, approving that $\prec$ is a $T_{5,2}$-ordering for $p \leq 9$.

For $m = 8$, we validated this for $p \leq 6$. For greater values, even though computing over finite characteristic, the computations did not terminate within several days. △

Question 3.4. For which $m, n, p \in \mathbb{N}$ does there exist a regular unimodular triangulation $T$ of the $m \times n$ rectangle such that the coefficients of $f_{p,(m,n)}$ are a Gröbner basis of $C_{p,(m,n)}$ with respect to the weighted reverse lexicographical ordering for a vector inducing that triangulation in the upper hull convention?
As before, let $T_{m,n}$ be the placing triangulation of the point configuration $[(0,0),(0,1),\ldots,(0,n),(m,0),\ldots,(m,n)]$. We point out that this triangulation does not lead to a positive answer of Question 3.4 in general. For instance, $T_{1,3}$ does not give rise to a Gröbner basis; only the four triangulations depicted in Figure 5 do.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The four regular unimodular triangular regulations of the $1 \times 3$ rectangle that give rise to a Gröbner basis.}
\end{figure}

For $m = n = 3$, the question has a negative answer. There are in total 46 452 regular unimodular triangulations of the $3 \times 3$ square. The coefficients of $f_{3,(3,3)}$ are not a Gröbner basis of $C_{3,(3,3)}$ w.r.t. the weighted reverse lexicographical ordering for any of the vectors in the strict interior of the secondary cone of those triangulations.

**Remark 3.5.** As pointed out in [4], there are—up to symmetries—5 941 regular unimodular triangulations of the $3 \times 3$ square. It would actually be sufficient to check the Gröbner basis property for each of those.

It would be intriguing to find the reason for this failure and to determine all $m, n \in \mathbb{N}$ for which Question 3.4 has a positive answer. Let us point out that this problem gets computationally expensive quickly: for the $4 \times 2$ rectangle, there are 12 170 regular unimodular triangulations, whereas for the $4 \times 3$ rectangle, there are already 2 822 146.

Now let $T$ be a triangulation of the $m \times n$ rectangle as asked for in Question 3.4. We end this article with two questions.

**Question 3.6.** Which unimodular regular triangulations of the $m \times 2$ rectangle give rise to a Gröbner basis—can they be deduced from the triangulations found for $m = 2, 3$ (cf. Figures 3 and 4)?

**Question 3.7.** As $p$ varies, is $\dim_{\mathbb{C}}(R_{m,n}/C_p(m,n))$ the Ehrhart polynomial of the fractional stable set polytope of the edge graph of $T$ and is this graph perfect?

**References**

[1] P. Afsharjoo. Looking for a New Version of Gordon’s Identities. *Ann. Comb.*, March 2021.

[2] R. Ait El Manssour and G. Pogudin. Multiplicity structure of the arc space of a fat point. arXiv:2111.10446, 2021.

[3] C. Berge. Sur une conjecture relative au problème des codes optimaux. *Comm. 13e assemblée générale de l’URSI*, Tokyo, 1962.

[4] S. Brodsky, M. Joswig, R. Morrison, and B. Sturmfels. Moduli of tropical plane curves. *Res. Math. Sci.*, 2(4), 2015.

[5] C. Bruschek, H. Mourtada, and J. Schepers. Arc spaces and Rogers–Ramanujan identities. *Ramanujan J.*, 30(1):9–38, 2013.

[6] V. Chvátal. On certain polytopes associated with graphs. *J. Combin. Theory Ser. B*, 18:138–154, 1975.

[7] D. Cox, J. Little, and D. O’Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergrad. Texts Math. Springer New York, 2007.

[8] J. A. De Loera, J. Rambau, and F. Santos. *Triangulations*, volume 25 of *Algorithms Comput. Math*. Springer-Verlag Berlin Heidelberg, 2010.
W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 4-1-3—A computer algebra system for polynomial computations. http://www.singular.uni-kl.de, 2020.

D. R. Fulkerson. Anti-blocking polyhedra. J. Combin. Theory Ser. B, 12:50–71, 1972.

E. Gawrilow and M. Joswig. polymake: A framework for analyzing convex polytopes. In Polytopes—combinatorics and computation (Oberwolfach, 1997), volume 29 of DMV Sem., pages 43–73. Birkhäuser, Basel, 2000.

R. A. Gowdard Jr. and K. E. Smith. The jet scheme of a monomial scheme. Comm. Algebra, 34(5):1591–1598, 2006.

M. Grötschel. My favorite theorem. OPTIMA, 62:2–5, 1999.

R. Gustavsson, M. Kondratieva, and A. Ovchinnikov. New effective differential Nullstellensatz. Adv. Math., 290:1138–1158, 2016.

C. Hillar and S. Sullivant. Finite Gröbner bases in infinite dimensional polynomial rings and applications. Adv. Math., 229, 2009.

E. Kolchin. Differential Algebra and Algebraic Groups. Pure Appl. Math. Academic Press, 1973.

R. Krone, A. Leykin, and A. Snowden. Hilbert series of symmetric ideals in infinite polynomial rings via formal languages. J. Algebra, 485(1), 2016.

H. Levi. On the structure of differential polynomials and on their theory of ideals. Trans. Amer. Math. Soc., 51(3):532–568, 1942.

W. Li, A. Ovchinnikov, G. Pogudin, and T. Scanlon. Algorithms yield upper bounds in differential algebra. arXiv:2005.01608, 2020.

L. Lovász. Normal hypergraphs and the perfect graph conjecture. Discrete Math., 2:253–267, 1972.

E. Miller and B. Sturmfels. Combinatorial commutative algebra, volume 227 of Univ. Lecture Ser. Springer-Verlag, New York, 2005.

R. Moosa and T. Scanlon. Jet and prolongation spaces. J. Inst. Math. Jussieu, 9:391–430, 2010.

H. Mourtada. Jet schemes of complex plane branches and equisingularity. Ann. Inst. Fourier (Grenoble), 61(6):2313–2336, 2011.

U. Nagel and T. Römer. Equivariant Hilbert Series in non-Noetherian Polynomial Rings. J. Algebra, 486:204–245, 2017.

G. Pogudin. Primary differential nil-algebras do exist. Moscow Univ. Math. Bull., 69:33–36, 2014.

J. Rambau. TOPCOM: Triangulations of point configurations and oriented matroids. In A. M. Cohen, X.-S. Gao, and N. Takayama, editors, Mathematical Software—ICMS 2002, pages 330–340. World Scientific, 2002.

J. F. Ritt. Differential Algebra, volume 14 of Amer. Math. Soc. Colloq. Publ. American Mathematical Society, 1950.

B. Sturmfels. Gröbner bases and convex polytopes, volume 8 of Univ. Lecture Ser. American Mathematical Society, Providence, R.I., 1996.

C. Yuen. One higher dimensional analog of jet schemes. arXiv:math/0608633, 2006.

C. Yuen. The multiplicity of jet schemes of a simple normal crossing divisor. Comm. Algebra, 35(12):3909–3911, 2007.

A. I. Zobnin. One-element differential standard bases with respect to inverse lexicographical orderings. J. Math. Sci. (N. Y.), 163(5):523–533, 2009.

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