On the adjacency matrix of complex unit gain graph

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Abstract

A complex unit gain graph is a simple graph in which each orientation of an edge is given a complex number with modulus 1 and its inverse is assigned to the opposite orientation of the edge. In this article, first we establish bounds for the eigenvalues of the complex unit gain graphs. Then we study some of the properties of the adjacency matrix of complex unit gain graph in connection with the characteristic and the permanental polynomials. Then we establish spectral properties of the adjacency matrices of complex unit gain graphs. In particular, using Perron-Frobenius theory, we establish a characterization for bipartite graphs in terms of the set of eigenvalues of gain graph and the set of eigenvalues of the underlying graph. Also, we derive an equivalent condition on the gain so that the eigenvalues of the gain graph and the eigenvalues of the underlying graph are the same.

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1 Introduction

Let \( G = (V, E) \) be a simple, undirected, finite graph with the vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and the edge set \( E(G) \subseteq V \times V \). If two vertices \( v_i \) and \( v_j \) are adjacent, we write \( v_i \sim v_j \), and the edge between them is denoted by \( e_{ij} \). The degree of the vertex \( v_i \) is denoted by \( d_i \). The \((0,1)\)-adjacency matrix or simply the adjacency matrix of \( G \) is an \( n \times n \) matrix, denoted by \( A(G) = [a_{ij}] \), whose rows and columns are indexed by the vertex set of the graph and the entries are defined by

\[
a_{ij} = \begin{cases} 
1 & \text{if } v_i \sim v_j, \\
0 & \text{otherwise}.
\end{cases}
\]

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The adjacency matrix of a graph is one of the well studied matrix class in the field of spectral graph theory. For more details about the study of classes of matrices associated with graphs, we refer to [1, 3, 4, 5, 6, 12].

The notion of gain graph was introduced in [19]. For a given graph \( G \) and a group \( \mathcal{G} \), first orient the edges of the graph \( G \). For each oriented edge \( e_{ij} \) assign a value (the gain of the edge \( e_{ij} \)) \( g \) from \( \mathcal{G} \) and assign \( g^{-1} \) to the orientated edge \( e_{ji} \). If the group is taken to be the multiplicative group of unit complex numbers, the graph is called the complex unit gain graph.

Now let us recall the definition of complex unit gain graphs [14]. The set of all oriented edges of the graph \( G \) is denoted by \( \overrightarrow{E}(G) \).

**Definition 1.1.** A \( \mathbb{T} \)-gain graph (or complex unit gain graph) is a triple \( \Phi = (G, \mathbb{T}, \varphi) \), where

(i) \( G = (V, E) \) is a simple finite graph,

(ii) \( \mathbb{T} \) is the unit complex circle, i.e., \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \), and

(iii) the map \( \varphi: \overrightarrow{E}(G) \rightarrow \mathbb{T} \) is such that \( \varphi(e_{ij}) = \varphi(e_{ji})^{-1} \).

Since, we consider \( \mathbb{T} \)-gain graphs throughout this paper, we use \( \Phi = (G, \varphi) \) instead of \( \Phi = (G, \mathbb{T}, \varphi) \).

The study of the spectral properties of \( \mathbb{T} \)-gain graphs is interesting because this generalizes the theory of adjacency matrix for undirected graphs. In [14], the author introduced the adjacency matrix \( A(\Phi) = [a_{ij}]_{n \times n} \) for a \( \mathbb{T} \)-gain graph \( \Phi \) and provided some important spectral properties of \( A(\Phi) \). The entries of \( A(\Phi) \) are given by

\[
a_{ij} = \begin{cases} \varphi(e_{ij}) & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}
\]

If \( v_i \) is adjacent to \( v_j \), then \( a_{ij} = \varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})} = \overline{a_{ji}} \). Thus the matrix \( A(\Phi) \) is Hermitian, and its eigenvalues are real. Let \( \sigma(A(\Phi)) \) denote the set of eigenvalues of the matrix \( A(\Phi) \).

Particular cases of the notion of adjacency matrix of \( \mathbb{T} \)-gain graphs were considered with different weights in the literature [2, 10]. In [10], the authors considered complex weighted graphs with the weights taken from \( \{ \pm 1, \pm i \} \), and characterized unicyclic graph having strong reciprocal eigenvalue property. In [16], the authors studied some of the properties of the characteristics polynomial for gain graphs. For some interesting spectral properties of gain graphs, we refer to [14, 15, 16, 21]. When \( \varphi(e_{ij}) = 1 \) for all \( e_{ij} \), then \( A(\Phi) = A(G) \). Thus we can consider \( G \) as a \( \mathbb{T} \)-gain graph and we write this by \( (G, 1) \). Recently the notion of incident matrix and Laplacian matrix for \( \mathbb{T} \)-gain graphs have been studied [14, 17]. From the above discussion it is clear that, the spectral theory of \( \mathbb{T} \)-gain graphs generalizes the spectral theory of undirected graphs and some weighted graphs.

Next we recall some of the definitions and notation which we needed. For more details we refer to [14, 15, 18, 19, 20, 21].

**Definition 1.2.** The gain of a cycle (with some orientation) \( C = v_1v_2 \ldots v_lv_1 \), denoted by \( \varphi(C) \), is defined as the product of the gains of its edges, that is

\[
\varphi(C) = \varphi(e_{12})\varphi(e_{23})\cdots\varphi(e_{(l-1)l})\varphi(e_{1l}).
\]

A cycle \( C \) is said to be neutral if \( \varphi(C) = 1 \), and a gain graph is said to be balanced if all its cycles are neutral. For a cycle \( C \) of \( G \), we denote the real part of the gain of \( C \) by \( \Re(C) \), and it is independent of the orientation.
Definition 1.3. A function from the vertex set of $G$ to the complex unit circle $T$ is called a switching function. We say that, two gain graphs $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ are switching equivalent, written as $\Phi_1 \sim \Phi_2$, if there is a switching function $\zeta : V \to T$ such that

$$\varphi_2(e_{ij}) = \zeta(v_i)^{-1} \varphi_1(e_{ij}) \zeta(v_j).$$

The switching equivalence of two gain graphs can be defined in the following equivalent way: Two gain graphs $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ are switching equivalent, if there exists a diagonal matrix $D_\zeta$ with diagonal entries from $T$, such that

$$A(\Phi_2) = D_\zeta^{-1} A(\Phi_1) D_\zeta.$$

Definition 1.4. A potential function for $\varphi$ is a function $\psi : V \to T$, such that for each edge $e_{ij}$, $\varphi(e_{ij}) = \psi(v_i)^{-1} \psi(v_j)$.

Theorem 1.1. [19] Let $\Phi = (G, \varphi)$ be a $T$-gain graph. Then the following statements are equivalent:

(i) $\Phi$ is balanced,

(ii) $\Phi \sim (G, 1)$,

(iii) $\varphi$ has a potential function.

The following necessary condition for switching equivalence is known.

Theorem 1.2. [14] Let $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ be $T$-gain graphs. If $\Phi_1 \sim \Phi_2$, then $\sigma(A(\Phi_1)) = \sigma(A(\Phi_2))$.

In Section 4 we construct a counter example to show that the above necessary condition is not sufficient.

The next theorem gives a necessary condition for the for two $T$-gain graphs to be switching equivalent.

Theorem 1.3. [15] Let $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ be two $T$-gain graphs. If $\Phi_1 \sim \Phi_2$, then for any cycle $C$ in $G$, $\varphi_1(C) = \varphi_2(C)$ holds.

Definition 1.5. The characteristic polynomial of a gain graph, denoted by $P_{\Phi}(x)$, is defined as $P_{\Phi}(x) = \det(xI - A(\Phi)) = x^n + a_1x^{n-1} + \cdots + a_n$. The permanental polynomial of a gain graph, denoted by $Q_{\Phi}(x)$, is defined as $Q_{\Phi}(x) = \per(xI - A(\Phi)) = x^n + b_1x^{n-1} + \cdots + b_n$. The characteristic and the permanental polynomials of the underlying graph $G$ is denoted by $P_G(x)$ and $Q_G(x)$, respectively. Some important and interesting properties of $P_G(x)$ and $Q_G(x)$ can be found in [6, 13].

Definition 1.6. A matching in a graph $G$ is a set of edges such that no two of them have a vertex in common. The number of edges in a matching is called the size of that matching. The collection of all matchings of size $k$ in a graph $G$ is denoted by $M_k(G)$. We define $m_k(G) = |M_k(G)|$ with the convention that $m_0(G) = 1$. A matching is called maximal if it is not contained in any other matching. The largest possible cardinality among all matchings is called the matching number of $G$.

Let $\mathbb{C}^{n \times n}$ denote the set of all $n \times n$ matrices with complex entries. For a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, define $|A| = (|a_{ij}|)$. Let $\rho(A)$ denote the spectral radius of the matrix $A$. The following results about nonnegative matrices will be useful in Section 4.
Theorem 1.4. [9, Theorem 8.1.18] Let $A, B \in \mathbb{C}^{n \times n}$ and suppose that $B$ is nonnegative. If $|A| \preceq B$, then $\rho(A) \preceq \rho(|A|) \preceq \rho(A)$.

Theorem 1.5. [9, Theorem 8.4.5] Let $A, B \in \mathbb{C}^{n \times n}$. Suppose $A$ is nonnegative and irreducible, and $A \succeq |B|$. Let $\lambda = e^{i\theta} \rho(B)$ be a given maximum-modulus eigenvalues of $B$. If $\rho(A) = \rho(B)$, then there is a diagonal unitary matrix $D \in \mathbb{C}^{n \times n}$ such that $B = e^{i\theta} D A D^{-1}$.

This article is organized as follows: In Section 2, we provide some results on the eigenvalue bounds for the adjacency matrix of $T$-gain graphs. In Section 3, we study some of the properties of the coefficients of characteristic and permanental polynomials of gain adjacency matrices. In Section 4, we focus on the study of the spectral properties of $T$-gain graphs. We establish an equivalent condition for the equality of set of eigenvalues of a $T$-gain graph and that of the underlying graph. Finally, we give a characterization for the bipartite graphs in terms of the eigenvalues of the gains and the eigenvalues of the underlying graph.

2 Eigenvalue bounds for $T$-gain graphs

For any complex square matrix $B$, we use $\lambda(B)$ to denote its eigenvalues(or simply $\lambda$ when there is only one matrix under consideration). Let $B$ be a complex square matrix of order $n$ with real eigenvalues. We arrange the eigenvalues of $B$ as

$$
\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \lambda_1.
$$

We now recall some important results associated to complex square matrices having real eigenvalues.

Theorem 2.1. [8, Theorem 2.1] Let $B$ be an $n \times n$ complex matrix with real eigenvalues, and let $r = \frac{\text{trace } B}{n}$ and $s^2 = \frac{\text{trace } B^2}{n} - r^2$.

Then

$$
r - s(n-1)^{\frac{1}{2}} \leq \lambda_n \leq r - s/(n-1)^{\frac{1}{2}}, \text{ and }
$$

$$
r + s/(n-1)^{\frac{1}{2}} \leq \lambda_1 \leq r + s(n-1)^{\frac{1}{2}}.
$$

Theorem 2.2. [9] Let $B$ be a Hermitian matrix of order $n$ and let $B_r$ be any principal submatrix of $B$ of order $r$. Then for $1 \leq k \leq r$,

$$
\lambda_{n+k-r}(B) \leq \lambda_k(B_r) \leq \lambda_k(B).
$$

It is well known that([1, 6]), the $(i,j)$-th entry of the $k$-th power adjacency matrix of a simple graph provide the number of $k$-walks (walks of length $k$) from the vertex $i$ to $j$. The next lemma is the counter part of the above statement for $T$-gain graphs.

Lemma 2.1. Let $\Phi$ be a $T$-gain graph. Then the $(i,j)$-th entry $a_{ij}^{(k)}$ of $A(\Phi)^k$ is the sum of gains of all $k$-walks from the vertex $i$ to the vertex $j$.

Proof. Let $A(\Phi)^k = (a_{ij}^{(k)})$. Then

$$
a_{ij}^{(k)} = \sum a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-2} i_{k-1}} a_{i_{k-1} j},
$$

where $1 \leq i_1, i_2, \ldots, i_{k-1} \leq n$. Now, any term in the right side is nonzero if and only if $v_{i_1} \sim v_{i_2}$, $v_{i_1} \sim v_{i_2}$, ..., $v_{i_{k-1}} \sim v_j$. Hence the result follows. \qed
Next, we derive lower and upper bounds for the largest and smallest eigenvalues of $T$-gain graph in terms of the number of edges and vertices of the underlying graph.

**Theorem 2.3.** Let $\Phi$ be a $T$-gain graph with the underlying graph $G$. If $G$ has $n$ vertices and $m$ edges, then the smallest and largest eigenvalues of $A(\Phi)$ satisfy

$$-\sqrt{\frac{2m(n-1)}{n}} \leq \lambda_n \leq -\sqrt{\frac{2m}{n(n-1)}}.$$

and

$$\sqrt{\frac{2m}{n(n-1)}} \leq \lambda_1 \leq \sqrt{\frac{2m(n-1)}{n}}.$$

Both of the above bounds are tight.

**Proof.** Since the $i$-th diagonal entry of $A(\Phi)^2$ is $\sum_{v_i \sim v_j} \varphi(e_{ij})\varphi(e_{ji}) = d_i$, we have $r = \frac{\text{trace } A(\Phi)}{n} = 0$ and $s^2 = \frac{\text{trace } A(\Phi)^2}{n} - r^2 = \frac{2m}{n}$. Thus, by Theorem 2.1 we have

$$-\sqrt{\frac{2m(n-1)}{n}} \leq \lambda_n \leq -\sqrt{\frac{2m}{n(n-1)}}.$$

and

$$\sqrt{\frac{2m}{n(n-1)}} \leq \lambda_1 \leq \sqrt{\frac{2m(n-1)}{n}}.$$

Hence the proof is completed.

Let $\Phi = (K_n, \varphi)$ be a $T$-gain graph with $\varphi(e) = -1$ for all edges, then both of the left equality occurs and if $\varphi(e) = 1$ for all edges then both right equality attains.

In the next theorem, we derive a lower bound for the largest eigenvalue of the $T$-gain graph.

**Theorem 2.4.** Let $\Phi$ be a $T$-gain graph with the underlying graph $G$. Then

$$\lambda_1 \geq \sqrt[3]{\frac{6}{n} \sum_{C \in \mathcal{C}_3(G)} \Re(C)},$$

where $\mathcal{C}_3(G)$ denotes the collection of all cycles of length 3 in $G$.

**Proof.** If $\lambda_1$ is the largest eigenvalue of $A(\Phi)$, then $\lambda_1^3$ is the largest eigenvalue of $A(\Phi)^3$. So, we have $\lambda_1^3 \geq \frac{1}{n} \text{trace } A(\Phi)^3$. By Lemma 2.1, the $i$-th diagonal entry of $A(\Phi)^3$ is $a^{(3)}_{ii} = \sum_{v_i \sim v_j \sim v_k} \varphi(e_{ij})\varphi(e_{jk})\varphi(e_{ki}) = 2 \sum_{C \in \mathcal{C}_3(i)} \Re(C)$, where $\mathcal{C}_3(i)$ denotes collection of all triangles which contains the vertex $i$. Now, since each triangle contains 3 vertices, we have

$$\lambda_1^3 \geq \frac{1}{n} \text{trace } A(\Phi)^3 = \frac{1}{n} \sum_{i=1}^{n} a^{(3)}_{ii} = \frac{6}{n} \sum_{C \in \mathcal{C}_3(G)} \Re(C).$$

The result follows by taking cube root on both sides.

Next result gives a lower bound for the spectral radius of the $T$-gain graph in terms of the degrees of its vertices. This extends [11, Theorem 1] for the gain graphs.
Theorem 2.5. Let $\Phi$ be a $T$-gain graph with the underlying graph $G$. Then
\[
\sigma \geq \frac{1}{\sqrt{2}} \max_{i<j} \left( d_i + d_j + \sqrt{(d_i - d_j)^2 + 4|a_{ij}^{(2)}|} \right),
\]
where $\sigma = \max |\lambda_i|$ and $|a_{ij}^{(2)}|$ is defined as in Lemma 2.1.

Proof. We have, $\sigma^2 = \lambda_1(A(\Phi)^2)$. Let $A(\Phi)^2[i, j] = \begin{bmatrix} a_{ii}^{(2)} & a_{ij}^{(2)} \\ a_{ji}^{(2)} & a_{jj}^{(2)} \end{bmatrix}$ be a principal submatrix of $A(\Phi)^2$. Then, by Theorem 2.2 we have $\lambda_1(A(\Phi)^2) \geq \lambda_1(A(\Phi)^2[i, j])$. By Lemma 2.1, $a_{ii}^{(2)} = d_i$, $a_{jj}^{(2)} = d_j$ and $a_{ij}^{(2)} = a_{ji}^{(2)} = \sum_{v_i \sim v_k} \varphi(e_{ik}) \varphi(e_{kj})$. Thus,
\[
\lambda_1(A(\Phi)^2[i, j]) = \frac{1}{2} \left[ d_i + d_j + \sqrt{(d_i + d_j)^2 - 4(d_i d_j - |a_{ij}^{(2)}|^2)} \right],
\]
\[
= \frac{1}{2} \left[ d_i + d_j + \sqrt{(d_i - d_j)^2 + 4|a_{ij}^{(2)}|^2} \right].
\]
Since the above relation holds for all $i \neq j$, we have
\[
\sigma \geq \frac{1}{\sqrt{2}} \max_{i<j} \left( d_i + d_j + \sqrt{(d_i - d_j)^2 + 4|a_{ij}^{(2)}|^2} \right).
\]

\[\square\]

3 Characteristic and permanental polynomial of $T$-gain graphs

In this section first we recall the some known definitions. Then, we compute the coefficients of the characteristic and permanental polynomials in terms of the gains of the edges.

Definition 3.1. Let $K_n$ denote the complete graph on $n$ vertices, and $K_{p,q}$ denote the complete bipartite graph on $p + q$ vertices with the vertex partition $V = V_1 \cup V_2$, $|V_1| = p$ and $|V_2| = q$. A graph $G$ is called an elementary graph, if each of its component is either a $K_2$ or a cycle.

Let $H(G)$ denote the collection of all spanning elementary subgraphs of a graph $G$, and for any $H \in H(G)$, let $C(H)$ denote the collection of cycles in $H$. In [16], authors considered gain graphs with gains are taken from an arbitrary group. The following two results can be proved by taking the gains from the multiplicative group $T$ in Corollary 2.3 and Theorem 2.2 of [16], respectively.

Theorem 3.1. Let $\Phi$ be a $T$-gain graph with the underlying graph $G$. Then
\[
\det A(\Phi) = \sum_{H \in H(G)} (-1)^{n-p(H)} 2^{c(H)} \prod_{C \in C(H)} \mathbb{R}(C),
\]
where $p(H)$ is the number of components in $H$ and $c(H)$ is the number of cycles in $H$.

Corollary 3.1. Let $\Phi$ be any $T$-gain graph with the underlying graph $G$. Let $P_{\Phi}(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ be the characteristic polynomial of $\Phi$. Then
\[
a_i = \sum_{H \in H_i(G)} (-1)^{p(H)} 2^{c(H)} \prod_{C \in C(H)} \mathbb{R}(C),
\]
where $H_i(G)$ is the set of elementary subgraphs of $G$ with $i$ vertices.
Proof. We have
\[ a_i = (-1)^i \sum i \times i \text{ principal minors}. \]
Now, the result follows from Theorem 3.1. \qed

In the next theorem, we compute coefficients of the permanental polynomial of the gain graphs. The proof is similar to that of [6, Theorem 2.3.2] and [7]. For the sake of completeness we include a proof here.

**Theorem 3.2.** Let \( \Phi \) be a \( T \)-gain graph with the underlying graph \( G \). Then
\[
\text{per } A(\Phi) = \sum_{H \in \mathcal{H}(G)} 2^{c(H)} \prod_{C \in \mathcal{C}(H)} \Re(C),
\]
where \( c(H) \) is the number of cycles in \( H \).

**Proof.** We have
\[
\text{per } A(\Phi) = \sum_{\sigma \in S_n} b_\sigma,
\]
where \( b_\sigma = a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)} \), and \( S_n \) denotes the collection of all permutations on the set \( \{1, 2, \ldots, n\} \). Since \( a_{ii} = 0 \) for all \( i = 1, 2, \ldots, n \), we have \( b_\sigma \neq 0 \) only if \( v_i \sim v_{\sigma(i)} \) for all \( i = 1, 2, \ldots, n \). Let \( \gamma_1\gamma_2 \cdots \gamma_r \) be the cycle decomposition of the permutation \( \sigma \), where \( \gamma_i \)'s are disjoint cycles of length at least two. Thus, the decomposition of \( \sigma \) determines an elementary spanning subgraph \( H \) of \( G \), whenever \( b_\sigma \neq 0 \). Now, let us calculate the value of \( b_\sigma \). For this we consider each \( \gamma_i \)'s in the decomposition of \( \sigma \).

If \( \gamma_i = (jk) \) is a transposition, then \( a_{jk} \) and \( a_{kj} \) occurs in the expression of \( b_\sigma \). Also note that \( a_{jka_{kj}} = 1 \). If \( \gamma_i = (i_1i_2 \cdots i_k) \) is a \( k \)-cycle, then \( b_\sigma \) contains \( a_{i_1i_2}a_{i_2i_3} \cdots a_{i_ki_1} \).

Let \( C \) denote the cycle \( v_{i_1}v_{i_2} \cdots v_kv_{i_1} \), then \( a_{i_1i_2}a_{i_2i_3} \cdots a_{i_ki_1} = \varphi(C) \). By combining all these possibilities, we get \( b_\sigma = \prod_{C \in \mathcal{C}(H)} \varphi(C) \). Let \( \gamma_1, \gamma_2, \ldots, \gamma_s \) be the cycles of length at least 3 in the decomposition of \( \sigma \). Now if we replace any of \( \gamma_1, \gamma_2, \ldots, \gamma_s \) by \( \gamma_1^{-1}, \gamma_2^{-1}, \ldots, \gamma_s^{-1} \), then the sign of the obtained permutation is same that of \( \sigma \). Let \( \sigma' \) be the any of the \( 2^{c(H)} \) permutations, namely \( \gamma_1^{\pm 1}\gamma_2^{\pm 1} \cdots \gamma_s^{\pm 1} \gamma_{s+1} \cdots \gamma_r \), which have the same sign that of \( \sigma \). Then the contribution of \( b_{\sigma'} \) to the sum (3) is \( \prod_{C \in \mathcal{C}(H)} \varphi(C)^{\pm 1} \).

Therefore,
\[
\text{per } A(\Phi) = \sum_{H \in \mathcal{H}(G)} 2^{c(H)} \prod_{C \in \mathcal{C}(H)} [\varphi(C) + \varphi(C)^{-1}],
\]
\[
= \sum_{H \in \mathcal{H}(G)} 2^{c(H)} \prod_{C \in \mathcal{C}(H)} \Re(C).
\]
\[ \qed \]

**Corollary 3.2.** Let \( \Phi \) be any \( T \)-gain graph with the underlying graph \( G \). Let \( Q_\Phi(x) = x^n + b_1x^{n-1} + \cdots + b_n \) be the permanental polynomial of \( \Phi \). Then
\[
b_i = (-1)^i \sum_{H \in \mathcal{H}_i(G)} 2^{c(H)} \prod_{C \in \mathcal{C}(H)} \Re(C),
\]
where \( \mathcal{H}_i(G) \) is the set of elementary subgraphs of \( G \) with \( i \) vertices.

Now, let us calculate the characteristic polynomial of certain \( T \)-gain graph using the previous results.
Example 3.1 (Star of triangles). Let \( S_m^\Delta \) denote the star with \( m \) triangles, that is, the end vertices of \( m \) copies of \( K_2 \) are joined to single vertex (see Figure 1). We label the vertices of \( S_m^\Delta \) such that for each \( l \in \{1, \ldots, m\} \), \( v_1 v_2 v_{2l+1} v_1 \) denote a triangle in it. Let \( \Phi = (S_m^\Delta, \varphi) \) be a \( T \)-gain graph with \( \varphi(v_1 v_2 v_{2l+1} v_1) = e^{i\theta_l} \), for \( 1 \leq l \leq m \). Let \( \alpha = 2[\cos \theta_1 + \cos \theta_2 + \cdots + \cos \theta_l] \).

Let \( P_{\Phi}(x) = x^n + a_1 x^{n-1} + \cdots + a_n \) be the characteristic polynomial of \( \Phi \). We have \( a_1 = 0 \), and we calculate \( a_i \) for \( 1 \leq i \leq 2m+1 \). Since all the triangles in \( S_m^\Delta \) shares the vertex \( v_1 \). Thus any elementary subgraph with even number (say \( 2l \)) of vertices must be a matching of size \( l \). A \( l \)-matching in \( S_m^\Delta \) is either a set of \( l \) edges of the form \( v_2 v_{2l+1} \) or a set consisting of an edge of the form \( v_1 v_{2j} \) (or \( v_1 v_{2j+1} \)) together with \( l-1 \) edges of the form \( v_2 v_{2l+1}, j \neq i \).

Thus, by using Corollary 3.1, we have

\[
a_{2l} = (-1)^i m_l(S_m^\Delta) = (-1)^i \left[ \binom{m}{l} + 2m \binom{m-1}{l-1} \right],
\]

where \( m_l(S_m^\Delta) \) denote the number of \( l \) matchings of \( S_m^\Delta \).

On the other hand an elementary subgraph with odd number (say \( 2l+1 \)) of vertices must contain a triangle \( v_1 v_2 v_{2l+1} v_1 \) and \( l-1 \) edges of the form \( v_2 v_{2l+1}, j \neq i \). Therefore \( a_{2l+1} = (-1)^i \binom{m-1}{l-1} \alpha \). As a special case if \( \alpha = 0 \), then \( a_{2l+1} = 0 \) so in this case eigenvalues of \( \Phi \) are symmetric about 0.

In a similar way, we can show that \( b_2 = \binom{m}{l} + 2m \binom{m-1}{l-1} \) and \( b_{2l+1} = -\binom{m-1}{l-1} \alpha \).

**Figure 1**: The graph \( S_4^\Delta \).

Example 3.2. Let \( G = K_4 \) and \( \varphi \) is taken in such a way that any three vertices \( v_{i_1}, v_{i_2}, v_{i_3} \) with \( i_1 < i_2 < i_3 \) we have

\[
\varphi(v_{i_1} v_{i_2} v_{i_3} v_{i_1}) = e^{i\theta}.
\]

Then for any cycle \( C \) with three vertices we have \( \Re(C) = \cos \theta \) and the gain of any cycle which can be written as a product of gains of two cycles of order three. The gains of the cycles of order four are

\[
\begin{align*}
\varphi(v_{i_1} v_{i_2} v_{i_3} v_{i_4} v_{i_1}) &= \varphi(v_{i_1} v_{i_2} v_{i_3} v_{i_1}) \varphi(v_{i_1} v_{i_4} v_{i_1}) = e^{2i\theta}, \\
\varphi(v_{i_1} v_{i_2} v_{i_4} v_{i_3} v_{i_1}) &= \varphi(v_{i_1} v_{i_2} v_{i_4} v_{i_1}) \varphi(v_{i_1} v_{i_4} v_{i_1}) = 1, \quad \text{and} \\
\varphi(v_{i_1} v_{i_3} v_{i_2} v_{i_4} v_{i_1}) &= \varphi(v_{i_1} v_{i_3} v_{i_2} v_{i_1}) \varphi(v_{i_1} v_{i_2} v_{i_1}) = 1.
\end{align*}
\]

Therefore, the characteristic polynomial of \( \Phi \) is

\[
P_{\Phi}(x) = x^4 - 6x^2 - 8x \cos \theta + (1 - 4 \cos^2 \theta).
\]
Next, we study some of the relationships between the characteristic and permanental polynomials of the adjacency matrix of the T-gain graph and that of the underlying graph \( G \), respectively.

**Theorem 3.3.** Let \( \Phi = (G, \varphi) \) be a T-gain graph. If \( G \) is a tree, then

(i) \( P_\Phi(x) = P_G(x) \), and

(ii) \( Q_\Phi(x) = Q_G(x) \).

**Proof.** The result follows from Corollary 3.1 and Corollary 3.2.

**Theorem 3.4.** Let \( G \) be an unicyclic graph with the cycle \( C \) of length \( m \). If \( \Phi = (G, \varphi) \) be a T-gain graph such that \( \varphi(C) = e^{i\theta} \). If \( P_\Phi(x) \) and \( P_G(x) \) denote the characteristic polynomial of \( \Phi \) and \( G \), respectively, then

\[
P_\Phi(x) = P_G(x) + 2(1 - \cos \theta) \sum_{i=0}^{k} (-1)^i m_i(G - C)x^{n-m+2i},
\]

where \( k \) is the matching number of \( G - C \).

**Proof.** Let \( a_i(\Phi) \) and \( a_i(G) \) denote the coefficients of \( x^{n-i} \) in \( P_\Phi(x) \) and \( P_G(x) \), respectively. We have \( a_i(\Phi) = a_i(G) \), for all \( i < m \), and \( a_m(\Phi) = a_m(G) + 2(1 - \cos \theta) \).

Also note that, for \( 1 \leq i \leq 2k \),

\[
a_{m+i}(\Phi) = \begin{cases} 
a_{m+i}(G), & \text{if } i \text{ is odd}, 
a_{m+i}(G) + (-1)^{\frac{i}{2}} m_\frac{i}{2}(G - C)2(1 - \cos \theta), & \text{if } i \text{ is even}. \end{cases}
\]

Again \( G - C \) has matching number \( k \) implies \( G \) has no elementary subgraph of order greater than \( m + 2k \) which contains a cycle. Thus, for all \( i > m + 2k \), we have \( a_i(\Phi) = a_i(G) \). Therefore,

\[
P_\Phi(x) = P_G(x) + 2(1 - \cos \theta) \sum_{i=0}^{k} (-1)^i m_i(G - C)x^{n-m+2i},
\]

which completes the proof.

**Theorem 3.5.** Let \( G \) be an unicyclic graph with the cycle \( C \) of length \( m \). If \( \Phi = (G, \varphi) \) is a T-gain graph such that \( \varphi(C) = e^{i\theta} \). If \( Q_\Phi(x) \) and \( Q_G(x) \) denote the permanental polynomial of \( \Phi \) and \( G \), respectively, then

\[
Q_\Phi(x) = Q_G(x) + (-1)^{m+1}2(1 - \cos \theta) \sum_{i=0}^{k} m_i(G - C)x^{n-m+2i},
\]

where \( k \) is the matching number of \( G - C \).

**Proof.** Similar to the proof of Theorem 3.4.

**Corollary 3.3.** Let \( G \) be unicyclic and \( \Phi \) be any gain graph with the underlying graph \( G \). Then

\[
det A(\Phi) = \begin{cases} 
det A(G), & \text{if } 2k \neq n - m, 
det A(G) + (-1)^k 2m_k(G - C)(1 - \cos \theta), & \text{if } 2k = n - m, \end{cases}
\]

and, similarly,

\[
\operatorname{per} A(\Phi) = \begin{cases} 
\operatorname{per} A(G), & \text{if } 2k \neq n - m, 
\operatorname{per} A(G) + (-1)^{m+1} 2m_k(G - C)(1 - \cos \theta), & \text{if } 2k = n - m. \end{cases}
\]
Spectral properties of bipartite $\mathbb{T}$-gain graphs

In this section we study some of the spectral properties of the bipartite $\mathbb{T}$-gain graphs. First, let us establish that for a bipartite $\mathbb{T}$ gain graph $A(\Phi)$, the set of all eigenvalues $\sigma(A(\Phi))$ is symmetric about 0. Proof of the unweighted case can be found in [1].

Theorem 4.1. If $G$ is bipartite $\mathbb{T}$-gain graph, then the eigenvalues of $A(\Phi)$ are symmetric about 0.

Proof. Let $V = \{X, Y\}$ be the bipartition of the vertex set of $G$ such that $|X| = p$. Let $\lambda$ be an eigenvalue of $A(\Phi)$ and $x = [x_1 \cdots x_p \ x_{p+1} \cdots \ x_n]^T$ be a corresponding eigenvector. Then the vector $x' = [x_1 \cdots \ x_p - x_{p+1} \cdots - x_n]^T$ is non-zero and satisfies $A(\Phi)x' = -\lambda x'$. Therefore, $-\lambda$ is also an eigenvalue of $A(\Phi)$.

Remark 4.1. The converse of the above theorem need not be true for gain graphs. Consider the complete graph $K_3$ on three vertices with edge weights are equal to $i$. Then $A(\Phi) = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & i \\ -i & -i & 0 \end{bmatrix}$. The eigenvalues of $A(\Phi)$ are 0, $\pm \sqrt{3}$. But the underlying graph is not bipartite.

Remark 4.2. It is known that $r$-regular graph has the eigenvalue $r$. The example in Remark 4.1 also shows that, $r$ may not be an eigenvalue for a $r$-regular $\mathbb{T}$-gain graph.

It is known that the eigenvalues of $A(G)$ are symmetric about 0 if and only if $G$ is bipartite (see [2, 1]). But Remark 4.1 shows that this result need not true for $\mathbb{T}$-gain graphs. In the following theorem, we establish a sufficient condition for a gain graph with eigenvalues are symmetric with respect to origin to be bipartite.

Theorem 4.2. Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph such that the eigenvalues of $A(\Phi)$ are symmetric about 0. If

$$\sum_{C \in \mathcal{C}_i(G)} \Re(C) \neq 0,$$

where $\mathcal{C}_i(G)$ denotes the set of all cycles on $i$ vertices, then $G$ is bipartite.

Proof. It is sufficient to prove that $G$ does not have any odd cycles. Let $P_b(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ be the characteristics polynomial of $A(\Phi)$. Since, the eigenvalues of $A(\Phi)$ are symmetric about 0, we have $a_i = 0$ whenever $i$ is odd. Now, $a_3 = -2 \sum_{C \in \mathcal{C}_3(G)} \Re(C) = 0$, and by the assumption $\sum_{C \in \mathcal{C}_3(G)} \Re(C) \neq 0$. Thus $G$ does not have any triangle. Since $G$ does not contain $K_3$, the cycles with 5 vertices are the only elementary subgraphs on 5 vertices. A similar argument shows that $G$ does not have any cycles of length 5. Proceeding in this way, we can prove $G$ does not contain any odd cycles.

In the next theorem, we establish an upper bound for the largest eigenvalue of a bipartite graph.

Theorem 4.3. Let $\Phi$ be a $\mathbb{T}$-gain graph with the underlying graph $G$. If $G = K_{p,q}$, then $\lambda_1(A(\Phi)) \leq \sqrt{pq}$. Equality holds if and only if $\Phi$ is balanced.

Proof. We have, $\sum \lambda_i^2 = \left( \sum \lambda_i \right)^2 - 2 \sum_{i \neq j} \lambda_i \lambda_j = 2pq$. Since, the eigenvalues of $A(\Phi)$ are symmetric about 0, we get $2\lambda_1^2 \leq 2pq$, and hence $\lambda_1 \leq \sqrt{pq}$. Now, let us prove the necessary
and sufficient condition for the equality. If \( \Phi \) is balanced, then \( \sigma(A(\Phi)) = \sigma(K_{p,q}) \), and hence \( \lambda_1(A(\Phi)) = \sqrt{pq} \). Conversely, let \( \lambda_1(A(\Phi)) = \sqrt{pq} \). Suppose that \( \Phi \) is not balanced. Then there exists a smallest number \( l \geq 2 \) such that \( \Phi \) contains cycles \( C_1, C_2, \ldots, C_k \) of length \( 2l \) with \( \varphi(C_i) \neq 1 \). Now computing coefficient of \( x^{n-2l} \) in the characteristics polynomial of \( \Phi \), we get

\[
a_{2l} = 2k - 2 \sum_{i=1}^{k} \Re(C_i) \neq 0.
\]

Thus \( \lambda_2 > 0 \), and hence \( \lambda_1 < \sqrt{pq} \). This contradicts the fact that \( \lambda_1 = \sqrt{pq} \). Therefore \( \Phi \) must be balanced.

From Theorem 1.2, it is known that if two gain graphs are switching equivalent, then they have the same set of eigenvalues. The converse of this statement is not true in general, i.e., if the set of all eigenvalues of two \( T \)-gain graphs (with the same underlying graph \( G \)) are the same, then they need not be switching equivalent. Consider \( G \) as in the Figure 4. The graph

![Figure 2: A graph with 5 vertices.](image)

\( G \) contains two cycles, namely, \( C_1 = \{v_1, v_2, v_3, v_1\} \) and \( C_1 = \{v_1, v_4, v_5, v_1\} \). We construct \( \Phi_1 \) and \( \Phi_2 \) so that

\[
\varphi_1(C_1) = i, \quad \varphi_1(C_2) = 1 \quad \varphi_2(C_1) = \frac{1}{\sqrt{2}}(1 + i), \quad \varphi_2(C_2) = \frac{1}{\sqrt{2}}(1 + i).
\]

Let \( a_i(1) \) and \( a_i(2) \) denote the coefficient of \( x^{n-i} \) in the characteristics polynomial of \( \Phi_1 \) and \( \Phi_2 \), respectively. Then, by using Corollary 3.1, we have

\[
a_1(1) = a_1(2) = 0, \\
a_2(1) = a_2(2) = -6, \\
a_3(1) = a_3(2) = -2, \\
a_4(1) = a_4(2) = 1, \\
a_5(1) = a_5(2) = 2.
\]

Therefore \( \sigma(A(\Phi_1)) = \sigma(A(\Phi_2)) \). But, by Theorem 1.3, \( \Phi_1 \) and \( \Phi_2 \) are not switching equivalent.

In the remaining part of this section, we study when the set of all eigenvalues or the spectral radius of \( A(\Phi) \), for some \( \Phi \), equals to the the set of all eigenvalues or the spectral radius of the underlying graph, respectively.

**Lemma 4.1.** Let \( \Phi = (G, \varphi) \) be a \( T \)-gain(connected) graph, then \( \rho(A(\Phi)) \leq \rho(A(G)) \).

**Proof.** Since \( |A(\Phi)| \leq A(G) \), then, by Theorem 1.4, we have \( \rho(A(\Phi)) \leq \rho(A(G)) \). \( \square \)
In the next theorem, we establish an equivalent condition for $\rho(A(\Phi)) = \rho(A(G))$.

**Theorem 4.4.** Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain(connected) graph, then $\rho(A(\Phi)) = \rho(A(G))$ if and only if either $\Phi$ or $-\Phi$ is balanced.

**Proof.** If $\Phi$ or $-\Phi$ is balanced, then $\rho(A(\Phi)) = \rho(A(G))$. Conversely, suppose that $\rho(A(\Phi)) = \rho(A(G))$. Let $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ be the eigenvalues of $A(\Phi)$. Since $A(\Phi)$ is Hermitian, either $\rho(A(\Phi)) = 1$ or $\rho(A(\Phi)) = -\lambda_n$.

Now we have the following two cases:

**Case 1:** Suppose that $\rho(A(\Phi)) = \lambda_1$. Then, by Theorem 1.5, there is a diagonal unitary matrix $D \in \mathbb{C}^{n \times n}$ such that $A(\Phi) = DA(G)D^{-1}$. Hence $\Phi \sim (G, 1)$. Therefore, by Theorem 1.1, $\Phi$ is balanced.

**Case 2:** If $\rho(A(\Phi)) = -\lambda_n$, then $\lambda_n = e^{i\pi}\rho(A(\Phi))$. By Theorem 1.5, we have $A(\Phi) = e^{i\pi}DA(G)D^{-1}$, for some diagonal unitary matrix $D \in \mathbb{C}^{n \times n}$. Thus $A(-\Phi) = DA(G)D^{-1}$. Hence, $(-\Phi) \sim (G, 1)$. Thus, $-\Phi$ is balanced. □

**Theorem 4.5.** Let $G$ be a connected graph. Then

(i) If $G$ is bipartite, then whenever $\Phi$ is balanced implies $-\Phi$ is balanced.

(ii) If $\Phi$ is balanced implies $-\Phi$ is balanced for some gain $\Phi$, then the graph is bipartite.

**Proof.** (i) Suppose $G$ is bipartite and $\Phi$ is balanced. Then due to the absence of odd cycles, $-\Phi$ is balanced.

(ii) Let $\Phi$ be a balanced cycle such that $-\Phi$ is balanced. Suppose that $G$ is not bipartite. Then, any odd cycle in $G$ can not be balanced with respect to $-\Phi$, which contradicts the assumption. Thus $G$ must be bipartite. □

In the next theorem, we answer the following problem: which gains adjacency matrices are cospectral to the adjacency matrix of underlying graph.

**Theorem 4.6.** Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain(connected) graph. Then, $\sigma(A(\Phi)) = \sigma(A(G))$ if and only if $\Phi$ is balanced.

**Proof.** If $\sigma(A(\Phi)) = \sigma(A(G))$, then $\rho(A(\Phi)) = \rho(A(G))$. Now, by Theorem 4.4, we have either $\Phi$ or $-\Phi$ is balanced. If $\Phi$ is balanced, then we are done. Suppose that $-\Phi$ is balanced, then $-A(G)$ and $A(\Phi)$ have the same set of eigenvalues. Hence $\sigma(A(G)) = \sigma(-A(G))$. Thus, we have $G$ is bipartite. Therefore, by Theorem 1.5, $\Phi$ is balanced. □

In the next theorem, we derive a characterization for bipartite graphs in terms gains.

**Theorem 4.7.** Let $G$ be a connected graph. Then, $G$ is bipartite if and only if $\rho(A(\Phi)) = \rho(A(G))$ implies $\sigma(A(\Phi)) = \sigma(A(G))$ for every gain $\varphi$.

**Proof.** Suppose $\rho(A(\Phi)) = \rho(A(G))$ implies $\sigma(A(\Phi)) = \sigma(A(G))$ for any gain $\varphi$. Let $\Phi$ be balanced. We shall prove that $-\Phi$ is also balanced. By Theorem 4.6, we have $\sigma(A(\Phi)) = \sigma(A(G))$. Thus $\rho(A(\Phi)) = \rho(A(G))$. Also $\rho(A(\Phi)) = \rho(-A(\Phi))$ implies $\rho(A(\Phi)) = \rho(A(G))$. Thus $\sigma(A(-\Phi)) = \sigma(A(G))$, and hence, by Theorem 4.6, $-\Phi$ is balanced. Now, by Theorem 4.5, $G$ is bipartite.

Conversely, let $G$ be a bipartite graph, and $\Phi$ be such that $\rho(A(\Phi)) = \rho(A(G))$. Now, by Theorem 4.4 and Theorem 4.5, we have $\Phi$ to be balanced. Hence $\sigma(A(\Phi)) = \sigma(A(G))$. □
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