Activation of zero-error classical capacity in low-dimensional quantum systems

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Channel capacities of quantum channels can be nonadditive even if one of two quantum channels has no channel capacity. We call this phenomenon activation of the channel capacity. In this paper, we show that when we use a quantum channel on a qubit system, only a noiseless qubit channel can generate the activation of the zero-error classical capacity. In particular, we show that the zero-error classical capacity of two quantum channels on qubit systems cannot be activated. Furthermore, we present a class of examples showing the activation of the zero-error classical capacity in low-dimensional systems.

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I. INTRODUCTION

Zero-error channel capacity is quite different with the ordinary channel capacity, and it has unique properties in both classical and quantum systems [1, 2]. In particular, while the ordinary capacity of classical channels is additive, the zero-error capacity $C_0$ of classical channels is nonadditive [3]: there are two classical channels $E_1$ and $E_2$ such that

$$C_0(E_1 \times E_2) > C_0(E_1) + C_0(E_2) \text{ and } C_0(E_{1,2}) > 0.$$

In quantum systems, a stronger form of nonadditivity is possible [4]: there is a quantum channel $N$ with $C_0(N) = 0$ such that

$$C_0(I_2 \otimes N) > C_0(I_2) + C_0(N),$$

where $I_2$ is a noiseless qubit channel. We may think that a noiseless qubit channel $I_2$ activates the ability of the useless channel $N$ to transmit classical information, and so we call this nonadditivity activation. However, for classical channels $E_{1,2}$, the condition that $C_0(E_1) = 0$ implies $C_0(E_1 \times E_2) = C_0(E_2)$. Thus, the activation is a quantum phenomenon which never occur in classical channels.

Furthermore, nonadditivity can happen even when two channels have no capacities, and such a nonadditivity is called superactivation [4, 5]. However, the superactivation of the zero-error classical capacity can occur only in high-dimensional quantum systems; indeed, the input dimensions of quantum channels must be greater than or equal to 4 [6, 7]. It has been known that many extraordinary features of quantum channel capacities are revealed in high-dimensional or sufficiently large dimensional quantum systems [8–10], but it is not clear that those features can also happen in low-dimensional cases. Hence, it could be important to concern how quantum phenomena occur in low-dimensional quantum systems, especially qubit systems.

In this section, we investigate the activation of the zero-error classical capacity when we have a quantum channel on $\mathbb{C}^2$. We show that the activation cannot happen unless the quantum channel on $\mathbb{C}^2$ is noiseless.

For a quantum channel $N$ with Kraus operators $E_i$, the one-shot zero-error classical capacity $C_0^{(1)}(N)$ is defined as

$$C_0^{(1)}(N) \equiv \log \alpha(N),$$

where $\alpha(N)$ is the maximum number of (orthogonal) vectors $|\psi_1\rangle, \ldots, |\psi_m\rangle$ such that

$$|\psi_s\rangle \langle \psi_t| \perp S \equiv \text{span}\{E_i^\dagger E_j : i, j\}; \forall s \neq t.$$ (1)
The (asymptotic) zero-error classical capacity $C_0(N)$ is defined by

$$C_0(N) \equiv \lim_{n \to \infty} \frac{C_0^{(1)}(N \otimes n)}{n} = \lim_{n \to \infty} \frac{C_0^{(1)}(S \otimes n)}{n}. \quad (2)$$

We note that $C_0^{(1)}(N)$ depends only on its associated subspace $S$ called the noncommutative graph of $N$ [11].

**Remark 1.** The noncommutative graph $S(N)$ of a quantum channel $N$ has two properties: $S(N) = S(N)^\dagger$ and $I \in S(N)$. Conversely, any subspace $S \leq L(C^n)$ such that $S = S^\perp$ and $I \in S$ is indeed a noncommutative graph of some quantum channel [11]. Hence, we will use $C_0^{(1)}(S)$ and $C_0(S)$ without referring to any specific quantum channel.

**Remark 2.** In order to measure in the number of bits, we define $C_0^{(1)}(N)$ as $\log \alpha(N)$, where the base of the logarithm is 2.

We can easily obtain from Eq. (1) the following characterization [4].

**Proposition 3.** Let $S$ be a noncommutative graph. Then $C_0^{(1)}(S) = 0$ if and only if $S^\perp$ has no rank-one matrices.

Let $S$ and $T$ be noncommutative graphs, and $C_0^{(1)}(T) = 0$. The one-shot zero-error classical capacity of $S$ and $T$ can be activated if and only if

$$C_0^{(1)}(S \otimes T) > C_0^{(1)}(S) + C_0^{(1)}(T).$$

We can see the noncommutative graph $L(C^n)$ as an extremely noisy channel, and indeed $C_0^{(1)}(L(C^n)) = 0$ by Proposition 3. However, the following theorem says that $L(C^n)$ cannot cause the activation. Thus, we may think that channels should not be too noisy in order to be activated.

**Theorem 4.** For any noncommutative graph $S$, $C_0^{(1)}(S \otimes L(C^n)) = C_0^{(1)}(S) + C_0^{(1)}(L(C^n)).$

**Proof.** If $C_0^{(1)}(S \otimes L(C^n)) = 0$, clearly, $C_0^{(1)}(S) = 0$. We assume that $C_0^{(1)}(S \otimes L(C^n)) > 0$. For any $|\Phi\rangle$ and $|\Psi\rangle$ satisfying

$$|\Phi\rangle \langle \Psi| \perp S \otimes L(C^n),$$

let

$$|\Phi\rangle = \sum_i \sqrt{\lambda_i} |\phi_i\rangle, \quad |\Psi\rangle = \sum_j \sqrt{\mu_j} |\psi_j\rangle$$

in the Schmidt decompositions, where $\lambda_i$’s and $\mu_j$’s are positive. Then for any $A \in S$ and $t$, $s$,

$$0 = \text{Tr}[(|\Phi\rangle \langle \Psi|)^t (A \otimes |\phi_s\rangle \langle \psi_j|)]$$

$$= \sum_{i,j} \sqrt{\lambda_i \mu_j} \langle \lambda_i | A | \phi_i \rangle \langle \phi_i | \phi_s \rangle \langle \psi_i | \psi_j \rangle$$

$$= \sqrt{\lambda_s \mu_i} \langle \lambda_s | A | \mu_i \rangle.$$

So, $|\lambda_s\rangle \langle \mu_i| \perp S$ for any $s, t$. Thus, $C_0^{(1)}(S \otimes L(C^n)) \leq C_0^{(1)}(S)$, and hence $C_0^{(1)}(S \otimes L(C^n)) = C_0^{(1)}(S)$. □

Now, we consider noncommutative graphs $S$ and $T$ in $L(C^2)$ and $L(C^n)$, respectively. Using Proposition 3, we can see the following proposition which is used in the proof of Theorem 6.

**Proposition 5.** Let $S \leq L(C^2)$ and $T \leq L(C^n)$ be noncommutative graphs and $C_0^{(1)}(T) = 0$. Suppose that $|\psi_i\rangle = |0\rangle |v_i\rangle + |1\rangle |w_i\rangle$ are orthogonal states satisfying Eq. (7) with respect to $S \otimes T$, where $|v_i\rangle, |w_i\rangle \in C^n$ and $1 \leq i \leq 3$. Then for any $i \neq j$,

(i) $|v_i\rangle$’s and $|w_i\rangle$’s are nonzero.

(ii) $|v_i\rangle$ and $|w_i\rangle$ are linearly independent.

(iii) $|w_i\rangle$ and $|w_j\rangle$ are linearly independent.

**Proof.** Since $I_2 \in S$ and $|\psi_i\rangle = |0\rangle |v_i\rangle + |1\rangle |w_i\rangle$ satisfy Eq. (1) with respect to $S \otimes T$,

$$A_{ij} \equiv \langle v_j | v_i \rangle + \langle w_i | w_j \rangle \in T^\perp$$

for any $1 \leq i \neq j \leq 3$. We note that $T^\perp$ has no rank-one matrices from Proposition 3.

(i) Suppose that $|v_i\rangle = 0$. Then $A_{ij} = |w_i\rangle \langle w_j| \in T^\perp$ for $j = 2, 3$. Since $|\psi_i\rangle \neq 0$, $|v_i\rangle \neq 0$, and so $|w_2\rangle = |w_3\rangle$. Then $A_{23} = |v_2\rangle \langle v_3| \in T^\perp$, and hence $|v_2\rangle = 0$ or $|v_3\rangle = 0$. Thus, $|v_2\rangle = 0$ or $|v_3\rangle = 0$ which is a contradiction. Similarly, we can see that $|w_i\rangle$’s are nonzero.

(ii) Suppose that $|v_1\rangle = \alpha |w_1\rangle$ for some $\alpha$. Then $A_{13} = |w_1\rangle \langle w_3| + \langle w_1 | w_3 \rangle = \alpha |v_1\rangle \langle v_3| + \langle w_1 | w_3 \rangle = 0$ for $j = 2, 3$, and so $\langle v_1 | + \langle w_1 | = 0$ by (i). Then $A_{23} = (1 + |\alpha|^2) \langle v_2 | v_3 \rangle = \alpha |v_1\rangle \langle v_3| + \langle w_1 | w_3 \rangle = 0$, and hence $|v_2\rangle = 0$ or $|v_3\rangle = 0$ which is a contradiction to (i).

(iii) Suppose that $a |w_i\rangle + b |w_j\rangle = 0$. Then $aA_{ik} + bA_{jk} = (a |v_i\rangle + b |v_j\rangle) \langle v_i | v_j \rangle = 0$ for $j = 2, 3$. By (i), $a |v_i\rangle + b |v_j\rangle = 0$, and so $a |v_i\rangle + b |v_j\rangle = 0$. Since $|v_i\rangle$ and $|v_j\rangle$ are linearly independent, $a = 0 = b$, and hence $|w_i\rangle$ and $|w_j\rangle$ are linearly independent. □

We note that $CI_2$ is associated with a noiseless qubit channel (up to unitary equivalence). The following theorem says that the only quantum channel on $C^2$ causing the activation is a noiseless qubit channel.

**Theorem 6.** Let $S$ and $T$ be noncommutative graphs in $L(C^2)$ and $L(C^n)$, respectively. If the one-shot zero-error classical capacity of $S$ and $T$ can be activated, then $S = CI_2$.

**Proof.** Since any qubit channel cannot cause the superactivation of the zero-error classical capacity [4], $C_0^{(1)}(S) > 0$ or $C_0^{(1)}(T) > 0$. Moreover, we note that any noncommutative graph in $L(C^2)$ is $CI_2$, span $\{I_2, \sigma_3\}$, span $\{I_2, \sigma_1, \sigma_3\}$, or $L(C^2)$ (up to unitary equivalence), and its one-shot zero-error classical capacity is 1, 1, 0, 0, respectively [11].
We first assume that $C_0^{(1)}(S) = 0 < C_0^{(1)}(T)$. When $S = L(C^2)$, by Theorem 2, $C_0^{(1)}(S \otimes T) = C_0^{(1)}(S) + C_0^{(1)}(T)$. Suppose that $S = \text{span}\{I_2, \sigma_1, \sigma_3\}$. Let
\[ |\psi_i\rangle = |0\rangle |v_i⟩ + |1\rangle |w_i⟩ \in \mathbb{C}^2 \otimes \mathbb{C}^n \]
satisfy Eq. (1) with respect to $S \otimes T$. Then for any $i \neq j$,
\[ \langle \psi_i | (P \otimes Q) | \psi_j \rangle = 0, \forall P, Q \in S, Q \in T. \]
Choosing $P = I_2, \sigma_3$, and $\sigma_1$, we can see that
\[ |v_i⟩ \langle v_j| \perp T, \quad |v_i⟩ \langle v_j| \perp T. \]
Define
\[ |\phi_i⟩ = \begin{cases} |v_i⟩ & \text{if } |v_i⟩ \neq 0, \\ |w_i⟩ & \text{if } |v_i⟩ = 0. \end{cases} \]
Then from Eqs. (2) and (3), we can see that $|\phi_i⟩ ⟨\phi_j| \perp T$ for any $i \neq j$. Thus, $C_0^{(1)}(S \otimes T) = C_0^{(1)}(T)$, and hence $C_0^{(1)}(S \otimes T) = C_0^{(1)}(T)$.

We now assume that $C_0^{(1)}(S) > 0 = C_0^{(1)}(T)$. Suppose that $S = \text{span}\{I_2, \sigma_3\}$. Then there are
\[ |\psi_i⟩ = |0\rangle |v_i⟩ + |1\rangle |w_i⟩ \in \mathbb{C}^2 \otimes \mathbb{C}^n \]
satisfying Eq. (1) with respect to $S \otimes T$, where $1 \leq i \leq 3$. It is not hard to show that for any $i \neq j$,
\[ |v_i⟩ \langle v_j| \perp T^+, \quad |v_i⟩ \langle v_j| \in T^+. \]
and so $|v_i⟩ \langle v_j| \in T^+$. Since $C_0^{(1)}(T) = 0$, by Proposition 3, $T^+$ has no rank-one matrices. Thus, $|v_i⟩ = 0$ or $|v_i⟩ = 0$. This is a contradiction by Proposition 4 and we can conclude that $S = \mathbb{C}I_2$.

A necessary condition of activation in Theorem 6 can be extended to the asymptotic case.

**Theorem 7.** Let $S$ and $T$ be noncommutative graphs in $L(C^2)$ and $L(C^n)$, respectively. If the zero-error classical capacity of $S$ and $T$ can be activated, then $S = CI_2$.

**Proof.** We note that $C_0(S) = C_0^{(1)}(S)$ for any noncommutative graph $S \leq L(C^2)$ [11]. For the case of $C_0(S) = 0 < C_0(T)$, applying Theorem 6 recursively, we obtain
\[ C_0^{(1)}(S^\otimes k \otimes T^\otimes k) = C_0^{(1)}(S^\otimes (k-1) \otimes T^\otimes k) = \ldots = C_0^{(1)}(T^\otimes k). \]
Hence, $C_0(S \otimes T) = C_0(T)$.

We now consider the case of $C_0(S) > 0 = C_0(T)$, where $S = \text{span}\{I_2, \sigma_1, \sigma_3\}$. We will show that
\[ C_0^{(1)}(S^\otimes k \otimes T) = C_0^{(1)}(S^\otimes k) \]
which implies $C_0(S \otimes T) = C_0(S)$. Let
\[ |\psi_i⟩ = \sum_{t=0}^{2^k-1} |t⟩ |v_{i,t}⟩ \in \mathbb{C}^{2^k} \otimes \mathbb{C}^n \]
satisfy Eq. (1) with respect to $S^\otimes k \otimes T$. We note that $S^\otimes k = \text{span}\{I_2, \sigma_3^\otimes k\} = \text{span}\{|t⟩ ⟨t| : 0 \leq t \leq 2^k - 1\}$. Then for any $i \neq j$,
\[ 0 = \langle \psi_i | (t⟩ \langle t| \otimes M) | \psi_j⟩ = \langle v_{i,t} | M | v_{j,t}⟩ \]
for any $t$ and $M \in T$. By Proposition 5,
\[ |v_{i,t}⟩ ⟨v_{j,t}| = 0 \]
for any $t$ and $i \neq j$.

We now use the induction on $k$ to prove Eq. (6). For $k = 1$, it holds by Theorem 5. Assume that Eq. (5) holds for $k \geq 1$. Suppose that
\[ C_0^{(1)}(S^\otimes (k+1) \otimes T) > C_0^{(1)}(S^\otimes (k+1)) \]
Then there exist orthogonal vectors
\[ |\psi_i⟩ = \sum_{t=0}^{2^{k+1}-1} |t⟩ |v_{i,t}⟩ \in \mathbb{C}^{2^{k+1}} \otimes \mathbb{C}^n \]
satisfying Eq. (1) with respect to $S^\otimes (k+1) \otimes T$, where $i = 1, \ldots, 2^{k+1} + 1$. Let us consider
\[ |\psi_i⟩_u = \sum_{d=0}^{2^k-1} |d⟩ |v_{i,d}⟩ \in \mathbb{C}^{2^k} \otimes \mathbb{C}^n, \quad |\psi_i⟩_i = \sum_{d=2^k}^{2^{k+1}-1} |d⟩ |v_{i,d}⟩ \in \mathbb{C}^{2^k} \otimes \mathbb{C}^n. \]
Since $C_0^{(1)}(S^\otimes k \otimes T) = C_0^{(1)}(S^\otimes k) = k$, by Eq. (6), there exist at least $(2^k+1)$ zero $|\psi_i⟩_u$’s. However, $|\psi_i⟩_i$’s, for which $|\psi_i⟩_u$’s are zero, are nonzero and satisfy Eq. (1) with respect to $S^\otimes k \otimes T$. Therefore,
\[ C_0^{(1)}(S^\otimes k \otimes T) \geq \log(2^k + 1) > k = C_0^{(1)}(S^\otimes k). \]
This is a contradiction to the induction hypothesis, and hence Eq. (6) holds for all $k$.

**Remark 8.** Some noisy qubit channel can have a positive zero-error classical capacity; for example, the dephasing channel $N(p) = (1-p)ρ + pσ_2ρ\sigma_2$, where $0 < p < 1$. However, by Theorem 6, such a noisy qubit channel cannot generate the activation even with a small amount of noise.

**III. NONACTIVATION ON $C^2 \otimes C^2$**

We here show that the one-shot zero-error classical capacity of two quantum channels on $C^2$ cannot be activated.

**Corollary 9.** For any pair of quantum channels whose input systems are on $C^2$, the one-shot zero-error classical capacity cannot be activated.
proof. Since no qubit channel can cause the superactivation of the one-shot zero-error classical capacity \( C_0 \), let \( S \) and \( T \) be noncommutative graphs in \( \mathcal{L}(\mathbb{C}^2) \), and let \( C_0^{(1)}(S) = 0 \) \(<\) \( C_0^{(1)}(T) \). In the first part of the proof of Theorem \ref{thm:asymCapacity} we have shown that if a noncommutative graph \( S \leq \mathcal{L}(\mathbb{C}^2) \) has \( C_0^{(1)}(S) = 0 \), then
\[
C_0^{(1)}(S \otimes \tilde{T}) = C_0^{(1)}(\tilde{T})
\]
for any noncommutative graph \( \tilde{T} \in \mathcal{L}(\mathbb{C}^n) \) with \( C_0^{(1)}(\tilde{T}) > 0 \). Hence, \( C_0^{(1)}(S \otimes T) = C_0^{(1)}(T) \).

We can see that Corollary \ref{cor:asymCapacity} can be extended to the case of asymptotic capacity. Let \( S \) and \( T \) be noncommutative graphs in \( \mathcal{L}(\mathbb{C}^2) \) with \( C_0(S) = 0 \) \(<\) \( C_0(T) \). Then we can show that
\[
C_0^{(1)}((S \otimes T)^{\otimes k}) = C_0^{(1)}(S \otimes (S^{\otimes (k-1)} \otimes T^{\otimes k}))
\]
\[
= C_0^{(1)}(S^{\otimes (k-1)} \otimes T^{\otimes k})
\]
\[
\vdots
\]
\[
= C_0^{(1)}(T^{\otimes k})
\]
by applying Eq. \( \ref{eq:asymCapacity} \) recursively. Hence, we can see that \( C_0(S \otimes T) = C_0(T) \). Therefore, we obtain the following corollary.

**Corollary 10.** For any pair of quantum channels whose input systems are on \( \mathbb{C}^2 \), the zero-error classical capacity cannot be activated.

**Remark 11.** We can view that the results in corollaries \ref{cor:asymCapacity} and \ref{cor:asymCapacity} are an extension of the results in Ref. \[6\], in which it was shown that any qubit channel cannot cause the superactivation of \( C_0^{(1)} \) and \( C_0 \).

**IV. A CLASS OF EXAMPLES**

In this section, we construct noncommutative graphs which generate the activation of the zero-error classical capacity.

**Theorem 12.** For each \( m \geq 3 \), there is a noncommutative graph \( T \leq \mathcal{L}(\mathbb{C}^{m+1}) \) with \( C_0^{(1)}(T) = 0 \) such that \( C_0^{(1)}(CI_2 \otimes T) \geq \log m \). Let \( C_0^{(1)}(CI_2 \otimes T) = \log m \). \( C_0^{(1)}(CI_2 \otimes T) = \log m \). 

**Proof.** Define
\[
B_{ij} \equiv |i \rangle \langle j| + |i + 1 \rangle \langle j + 1| \in \mathbb{C}^{(m+1) \times (m+1)},
\]
where \( 0 \leq i \neq j \leq m - 1 \). Then we can easily see that
\[ T = \text{span}\{B_{ij} : 0 \leq i \neq j \leq m - 1\}^\perp \]
is a noncommutative graph.

We first show that
\[ T^\perp = \text{span}\{B_{ij} : 0 \leq i \neq j \leq m - 1\} \]
has no rank-one matrices. *i.e.*, \( C_0^{(1)}(T) = 0 \). Assume to the contrary that \( T^\perp \) has a rank-one matrix
\[
B \equiv \sum_{0 \leq i \neq j \leq m - 1} \alpha_{ij} B_{ij}
\]
for some \( \alpha_{ij} \). On the other hand, we can write
\[
B = |\psi\rangle \langle \phi|,
\]
where \( |\psi\rangle = \sum_{i=0}^{m} a_i |i\rangle \) and \( |\phi\rangle = \sum_{i=0}^{m} b_i^* |i\rangle \) are nonzero vectors. Let the \( q \)-th column of \( B \) be the right nonzero column and the \( p \)-th entry \( a_p b_q \) of the \( q \)-th column be the upper most nonzero entry; we use zero-based numbering. Then we can see
\[
a_0 b_q = a_1 b_q = \cdots = a_{p-1} b_q = 0.
\]
Since \( b_q \neq 0 \), \( a_0 = a_1 = \cdots = a_{p-1} = 0 \), and so first \( p \) rows of \( B \) are all zero. Consider the diagonal passing the \( (p, q) \) entry. Without loss of generality, let \( p < q \), then we obtain
\[
\alpha_{0, q-p} = \cdots = \alpha_{p, q} = 0.
\]
However, \( 0 \neq \alpha_{p, q} = \alpha_{p-1, q-1} + \alpha_{p, q} = 0 \), this is a contradiction. Thus, \( T^\perp \) has no rank-one matrices, and hence \( C_0^{(1)}(T) = 0 \).

We now show that \( C_0^{(1)}(CI_2 \otimes T) \geq \log m \). Let
\[
|\psi_i\rangle = |0\rangle |i\rangle + |1\rangle |i + 1\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^{m+1},
\]
where \( 0 \leq i \leq m - 1 \). Then for any \( R \in T \),
\[
\langle \psi_i | (I_2 \otimes R) |\psi_j\rangle = \langle i | R | j\rangle + \langle i + 1 | R | j + 1\rangle = 0
\]
Hence, \( C_0^{(1)}(CI_2 \otimes T) \geq \log m \).

**Remark 13.** When \( m = 3 \) in Theorem \ref{thm:asymCapacity}, we see that the one-shot zero-error classical capacity can be activated on \( \mathbb{C}^2 \otimes \mathbb{C}^4 \). This result shows a lower dimensional case than the example in Ref. \[4\] in which the input system is \( \mathbb{C}^2 \otimes \mathbb{C}^6 \). Moreover, this example has the smallest input dimensions to be activated so far.

Next, we show that the activation in theorem \ref{thm:asymCapacity} also holds in the asymptotic setting. To do this, we need the following lemma based on Ref. \[12\].

**Lemma 14.** Let \( S \leq \mathbb{C}^{m_1 \times n_1} \) and \( T \leq \mathbb{C}^{m_2 \times n_2} \) be subspaces. Then \( (S \otimes T)^\perp \) has a rank-one matrix if and only if there exist nonzero matrices \( A \) and \( B \) such that \( S \perp ATB \).

**Proof.** Suppose that \( (S \otimes T)^\perp \) has a rank-one matrix. Then there exist nonzero vectors \( |\psi\rangle \) and \( |\phi\rangle \)
\[
|\psi\rangle = \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} a_{ij} |i\rangle |j\rangle
\]
\[
|\phi\rangle = \sum_{k=0}^{n_1-1} \sum_{l=0}^{n_2-1} b_{kl} |k\rangle |l\rangle
\]
such that $|\psi\rangle\langle\phi| \in (S \otimes T)^\perp$. Then we can obtain for any $P \in S$ and $Q \in T$,

$$
\sum_{i,j,k,l} a_{ij} b_{kl}^* \langle k | P^i | i \rangle \langle j | Q^l | l \rangle = 0. \tag{8}
$$

Define two nonzero matrices

$$
A = \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} a_{ij} |i\rangle \langle j|
$$

$$
B = \sum_{k=0}^{n_1-1} \sum_{l=0}^{n_2-1} b^*_{kl} |k\rangle \langle l|.
$$

Then by Eq. (8), we obtain

$$
\text{Tr}[P^i A Q^l B] = \sum_{i,j,k,l} a_{ij} b^*_{kl} \langle k | P^i | i \rangle \langle j | Q^l | l \rangle = 0 \tag{9}
$$

for any $P \in S$ and $Q \in T$. Similarly, we can readily see the converse. \hfill \Box

**Theorem 15.** The noncommutative graphs $T$ in the proof of Theorem 12 cannot cause the superactivation. In particular, $C_0(T) = 0$ and $C_0(C_I \otimes T) \geq \log m > 1 = C_0(C_I) + C_0(T)$.

**Proof.** Let

$$
T = \text{span}\{B_{ij} : 0 \leq i \neq j \leq m - 1\}^\perp,
$$

where

$$
B_{ij} = |i\rangle \langle j| + |i+1\rangle \langle j+1| \in \mathbb{C}^{(m+1) \times (m+1)}.
$$

Suppose that $C^{(1)}_0(S \otimes T) > 0$ for some noncommutative graph $S$ with $C^{(1)}_0(S) = 0$. By Lemma 13 there exist nonzero matrices $A$ and $B$ such that $AB \subseteq S^\perp$. Since $C^{(1)}_0(S) = 0$,

$$
\text{rank}(AQB) \neq 1 \tag{10}
$$

for any $Q \in T$.

Let $Q = \sum_{u,v=0}^m a_{uv} |u\rangle \langle v|$ be any element in $T$. Then

$$
0 = \text{Tr} B_{ij}^\dagger Q = a_{i,j} + a_{i+1,j+1}
$$

for $0 \leq i \neq j \leq m-1$. From this, we can see that the followings belong to $T$:

$$
\sum_{k=0}^{m-j} (-1)^k |k\rangle \langle k|, \ j = 1, \ldots, m, \tag{11}
$$

$$
|\psi\rangle \langle \phi|, \ i = 0, \ldots, m, \tag{12}
$$

$$
\sum_{k=0}^{m-j} (-1)^k |j+k\rangle \langle k|, \ j = 1, \ldots, m. \tag{13}
$$

Putting matrices in Eqs. (11), (12), and (13) into Eq. (10), we can see that there is $0 \leq c \leq m$ such that $A |i\rangle = 0$ for $i = 0, \ldots, c$ and $|j\rangle B = 0$ for $j = c + 1, \ldots, m$. Similarly, there is $0 \leq d \leq m$ such that $A |i\rangle = 0$ for $i = d + 1, \ldots, m$ and $|j\rangle B = 0$ for $j = 0, \ldots, d$. Then $A = 0$ if $c \geq d$, and $B = 0$ if $c \leq d$. This is a contradiction since $A$ and $B$ are nonzero matrices. Therefore, $C^{(1)}_0(S \otimes T) = 0$ for any noncommutative graph $S$ with $C^{(1)}_0(S) = 0$. \hfill \Box

**Remark 16.** When the zero-error classical capacity can be activated, we can raise the following question: how much can it be activated? In other words, how large can $C^{(1)}_0(S \otimes T) - C^{(1)}_0(S)$ be for any noncommutative graph $T$ such that $C^{(1)}_0(T) = 0$? In Theorems 12 and 15 as well as examples in Ref. [4], the capacity of the combined channel can be unbounded above, and so it may need to be regularized by the dimensions of systems. Then the (regularized) largest value could measure the ultimate ability to activate another useless quantum channel. The above-mentioned question is related with the concept of potential capacity in Ref. [13].

**V. CONCLUSIONS**

We have considered when the activation of the zero-error classical capacity happens in low-dimensional input systems. First, we have shown that when one of two quantum channels is on a qubit system, the zero-error classical capacity of the combined channel can be activated only if the quantum channel on a qubit system is noiseless; that is, only a noiseless qubit channel can generate the activation. Moreover, we have shown that the zero-error classical capacity of two quantum channels on qubit systems cannot be activated. Finally, we have presented a class of examples showing the activation of the zero-error classical capacity in low-dimensional input systems. In particular, we have constructed an example having the smallest input dimensions so far.

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