Elaborating the word problem for free idempotent-generated semigroups over the full transformation monoid

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Abstract
With each semigroup one can associate a partial algebra, called the biordered set, which captures important algebraic and geometric features of the structure of idempotents of that semigroup. For a biordered set $\mathcal{E}$, one can construct the free idempotent-generated semigroup over $\mathcal{E}$, $IG(\mathcal{E})$, which is the free-est semigroup (in a definite categorical sense) whose biorder of idempotents is isomorphic to $\mathcal{E}$. Studies of these intriguing objects have been recently focusing on their particular aspects, such as maximal subgroups, the word problem, etc. In 2012, Gray and Ruškuc pointed out that a more detailed investigation into the structure of the free idempotent-generated semigroup over the biorder of $T_n$, the full transformation monoid over an $n$-element set, might be worth pursuing. In 2019, together with Gould and Yang, the present author showed that the word problem for $IG(\mathcal{E}, T_n)$ is algorithmically soluble. In a recent work by the author, it was showed that, for a wide class of biorders $\mathcal{E}$, the algorithmic solution of the word problem revolves around the so-called vertex groups, which arise as certain subgroups of direct products of pairs of maximal subgroups of $IG(\mathcal{E})$. In this paper we determine these vertex groups for the case when $\mathcal{E}$ is the biorder of idempotents of $T_n$.

Keywords Free idempotent-generated semigroup · Biordered set · Word problem · Full transformation monoid

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1 Introduction

For a semigroup $S$, let $E(S)$ denote the set of its idempotents. However, merely recording the collection of idempotent elements of a semigroup is frequently not enough, as a significant amount of information about the mutual relationships of these elements, as well as their impact to the general structure, is lost in this way. Therefore, as it turns out, it is useful to consider a partial algebra $E_S = (E, ·)$, where $E = E(S)$, obtained by retaining products of idempotents $ef$ such that $\{ ef, fe \} \cap \{ e, f \} \neq \emptyset$. (This amounts to saying that the product of $e$ and $f$, multiplied in some order, results in one of the factors. It is then easily verified that the product of $e$ and $f$ in the reverse order is an idempotent, too, although not necessarily equal to one of $e, f$.) Such a pair $\{ e, f \}$ is called a basic pair.

The partial algebra $E_S$ obtained in this way is called the biordered set of $S$. The name derives from the fact that it is convenient to define two quasi-orders on $E_S$: namely, let $e \leq_{\ell} f$ if and only if $ef = e$, and let $e \leq_{r} f$ if and only if $ef = f$. This effectively captures the basic pairs of $S$; in addition, the intersection $\leq_{\ell} \cap \leq_{r}$ is precisely the natural order of idempotents of $S$ (see e.g. [19]). It was shown by Nambooripad [21] and Easdown [12] that these partial structures can be finitely axiomatised: there is a finite set of formulæ such that for any abstract structure $E$ satisfying these axioms there is a semigroup $S$ such that $E \cong E_S$. Of course, the biordered set of any finite semigroup is finite, while the converse is not necessarily the case: there are finite biorders not stemming from any finite semigroup [11].

Crucial in the study of idempotent-generated semigroups (semigroups $S$ with the property that $S = \langle E(S) \rangle$), a very natural and omnipresent class of semigroups, is the notion of a free idempotent-generated semigroup on a biordered set $E$. It is defined by the presentation

$$\text{IG}(E) = \langle \bar{E} \mid \bar{e}f = e \cdot \bar{f} \text{ whenever } \{ e, f \} \text{ is a basic pair in } E \rangle,$$

where $\overline{E} = \{ \overline{e} : e \in E \}$ is an alphabet in a one-to-one correspondence with $E$, the set of elements of the biorder $E$. This is, in a quite definite sense, the “freest” idempotent-generated semigroup with biordered set isomorphic to $E$ (in the case of $\text{IG}(E)$, its biorder $\overline{E}$ is formed by elements of $\overline{E}$). More precisely, if $S$ is any semigroup such that $E_S \cong E$, with $\phi : E \rightarrow E_S$ being a biordered set isomorphism, then $\phi' = \iota^{-1}\phi : \overline{E} \rightarrow E_S$ (where $\iota : E \rightarrow \overline{E}$, defined by $e = \overline{e}$, $e \in E$, is also a biorder isomorphism) can be (uniquely) extended to a semigroup homomorphism $\Psi_{\phi} : \text{IG}(E) \rightarrow S$ (here the index $\phi$ intends to indicate that the homomorphism $\Psi$ depends on the choice of $\phi$; in other cases, when the initial isomorphism $\phi$ is irrelevant, we will just suppress this index). The image of this homomorphism is precisely the idempotent-generated part of $S$, namely its subsemigroup $S' = \langle E(S) \rangle$. 

—A. de Saint-Exupéry, Le petit prince
Free idempotent-generated semigroups were introduced by Nambooripad in [21] within a wider framework of a general study of regular semigroups (see also [22, 23]). Since then, they have been an object of fascination of an array of algebraists. The most recent resurgence of interest in this topic was initiated by papers [2] and [16]. Namely, for some time, a folklore conjecture (recorded officially only in [20]) was in circulation that the maximal subgroups of free idempotent-generated semigroups must necessarily be free groups. This conjecture proved to be wrong in a rather strong fashion: first, Brittenham, Margolis, and Meakin [2] constructed a 73-element semigroup $S$ (containing 37 idempotents) such that $\text{IG}(E_S)$ contains a maximal subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (so, not a free group), and then, in a ground-breaking paper [16], Gray and Ruškuc showed that for any group $G$ there is a suitable biorder $E$ such that $\text{IG}(E)$ contains a maximal subgroup isomorphic to $G$. This was followed by a series of papers studying these maximal subgroups, see e.g. [3, 6, 8, 10, 15, 17], after which the focus shifted to other structural features, and, primarily, to the question of the word problem. The pioneering paper in this sense were [5, 9], where the later exhibited the first example of a finite biorder $E$ (stemming, by the way, from a finite idempotent semigroup) such that all maximal subgroups of $\text{IG}(E)$ have decidable word problems (in fact, they were all free or trivial) while the word problem for $\text{IG}(E)$ is algorithmically unsolvable. The true nature of these problems was revealed in the papers [4] and [7], where it was shown that the word problem of $\text{IG}(E)$ is in fact equivalent to a specific type of a constraint satisfaction problem related to certain subgroups (called the vertex groups in [7]) of direct products of pairs of maximal subgroups of $\text{IG}(E)$.

The aim of the present paper is to determine these groups for the free idempotent-generated semigroup over $E_{T_n}$, the biorder of the full transformation monoid $T_n$ over an $n$-element set. We recall that the maximal subgroups of $\text{IG}(E_{T_n})$ were previously computed in [17]: with trivial exceptions, these are symmetric groups. With this knowledge at hand, it was then shown in [4] that the word problem of $\text{IG}(E_{T_n})$ is decidable for all finite $n$. Furthermore, it can be amply seen from [7] (see Theorem 3.9 and Theorem 2.4, supplemented by remarks preceding Theorem 3.6) that in the case when the word problem of $\text{IG}(E)$ is algorithmically soluble for a finite biorder $E$, the only real obstacle towards the goal of routinely writing, say, a GAP code [14] implementing this word problem is the knowledge of the corresponding vertex groups (or, to be more precise, their specific cosets). It was noted in [7, Remark 3.7] that given a finite biorder $E$ there exists an algorithm which outputs a finite generating set for any of the required vertex groups (within the direct product of corresponding maximal subgroups) as well as the necessary coset representatives. However (as we shall see below), the brute force methods for such computations can be very involved. It is thus the purpose of this paper to bypass such methods by providing combinatorial analysis and arguments sufficient to get hold of these vertex groups and their coset representatives directly. This reduces the word problem for $\text{IG}(E_{T_n})$ and the explicit specification of the corresponding algorithm to a sequence of standard computational tasks in finite group theory (see Subsect. 2.6 below).

The remainder of the paper has two parts. One aims at making this paper reasonably self-contained, and is devoted to the summary of all the main notions and results needed to explain the algorithmic problem where the mentioned vertex groups arise, turning out to be equivalent to the word problem of semigroups of the form $\text{IG}(E)$. In the other
half of the paper, in Proposition 3.6 and Theorem 3.13 we determine the vertex groups for $\text{IG}(E_T)$ (with few exceptions that are irrelevant to the word problem). Typically, these groups will be subgroups of direct products of finite symmetric groups; they will be closely related to a class of permutations preserving certain nice combinatorial configurations.

2 Preliminaries

2.1 General background

Throughout we assume familiarity with the basic notions and techniques of semigroup theory, and for these we refer to [19] as a standard textbook in the area. In particular, one of the most elementary tools are Green’s relations: $R$ relates elements that generate the same principal right ideal of a semigroup, and $L, J$ are respectively the left and the two-sided analogues; also, we have $H = R \cap L$ and $D = R \lor L = R \circ L$ (as $R \circ L = L \circ R$ holds). Furthermore, we have $H \subseteq R, L \subseteq D \subseteq J$, and, in general, all of these containments might be proper. On the other hand, it should be noted that $D = J$ holds in many natural examples of semigroups: for example, this is true for all finite (and more generally for all periodic) semigroups. Also, as proved in [7, Theorem 4.2(5)], $D = J$ also holds in $\text{IG}(E)$ whenever $E$ is a finite biorder.

These definitions naturally give rise to partial orders on the sets of $R$-/$L$-/$J$-classes of a semigroup (and thus to quasi-orders on the semigroup itself). Namely, for two $R$-classes we may write $Ra \leq_R Rb$ (or, alternatively, $a \leq_R b$) if and only if $aS^1 \subseteq bS^1$; in a similar fashion, one can order the $L$-classes and the $J$-classes, too.

Example 2.1 As our main concern in this paper is with the biorders of $T_n$, the finite full transformation monoids, here is the description of Green’s relations in $T_X$ (which are valid for a non-empty set $X$ of any cardinality), assuming that the functions in $T_X$ are acting on $X$ from the right and are thus composed left-to-right:

- $f \mathrel{R} g$ if and only if $\ker f = \ker g$;
- $f \mathrel{L} g$ if and only if $\im f = \im g = Xf = Xg$;
- $f \mathrel{J} g$ if and only if $\rank f = \rank g$.

In addition, we always have $D = J$ in $T_X$.

Example 2.2 In $T_n, n \geq 1$, the $J$-/$D$-classes form a chain of length $n$, as transformations are classified by their rank; so, it is convenient to denote these classes by $D_n, D_{n-1}, \ldots, D_1$. Here, $D_n \cong S_n$ is the group of units consisting of all permutations (= transformations of rank $n$), while at the other extreme, $D_1$ consists of all constant mappings (forming a semigroup of right zeros). The principal factor associated with $D_m, 2 \leq m \leq n$, is isomorphic to the Rees matrix semigroup $\mathcal{M}^0[S_m; I_m; \Lambda_m; P^{(m)}]$, where $I_m$ is the collection of all partitions of $\{1, n\}$ into $m$ classes, $\Lambda_m$ is the collection of all $m$-element subsets of $\{1, n\}$, while the entry $p_{PA}$ of the sandwich matrix $P^{(m)}$ is obtained in the following way. If $A \perp P$ (which means that $A \in \Lambda_m$ is a cross-section of $P \in I_m$) then this entry is set to be the label $\lambda(P, A) \in S_m$ of the pair $(P, A)$ [17], computed as described below; otherwise, it is 0. As for the permutation $\lambda(P, A)$,
assume that $P = \{P_1, \ldots, P_m\}$ and $A = \{a_1, \ldots, a_m\}$, with indexing done in such a way that $\min P_1 < \cdots < \min P_m$ and $a_1 < \cdots < a_m$. Now, the assumption $A \perp P$ ensures that each $P$-class contains a unique element of $A$: say, for each $1 \leq i \leq m$, we have that $a_{r_i} \in P_i$. Then the mapping

$$
\begin{pmatrix}
1 & 2 & \ldots & m \\
1 & 2 & \ldots & m
\end{pmatrix}
$$

is a permutation, and this is precisely $\lambda(P, A)$.

### 2.2 Basic structural facts about $\GammaG(\mathcal{E})$

Let $\mathcal{E}$ be a biordered set – arising from a semigroup $S$, so that $\mathcal{E} \cong \mathcal{E}_S$ – and let $\Psi : \GammaG(\mathcal{E}) \to S$ be the homomorphism, mentioned in the introduction, extending the map $\overline{e} \mapsto e, e \in E(S)$. There is a great degree of similarity between certain aspects of the structure of $\GammaG(\mathcal{E})$ and $S' = \langle E(S) \rangle$. Here we list some of them (see [16] for references corresponding to individual results):

- For any $e \in E$, $\Psi$ maps the $\mathcal{D}$-class of $\overline{e}$ in $\GammaG(\mathcal{E})$ precisely onto the $\mathcal{D}$-class of $e$ in $S'$; it is in this sense that we say that the regular $\mathcal{D}$-classes in $\GammaG(\mathcal{E})$ and $S'$ are in bijective correspondence and refer to corresponding regular $\mathcal{D}$-classes (in $\GammaG(\mathcal{E})$ and $S'$, respectively).
- $\Psi$ maps the $\mathcal{R}$-class of $\overline{e}$ onto the $\mathcal{R}$-class of $e$, the $\mathcal{L}$-class of $\overline{e}$ onto the $\mathcal{L}$-class of $e$.
- Consequently, the restriction of $\Psi$ to $H_{\mathcal{F}}$, the maximal subgroup of $\GammaG(\mathcal{E})$ containing the idempotent $\overline{e}$, is a surjective group homomorphism onto $He$, the maximal subgroup of $S'$ containing $e$. In other words, the maximal subgroup in a regular $\mathcal{D}$-class of $\GammaG(\mathcal{E})$ is a pre-image of the maximal subgroup in the corresponding regular $\mathcal{D}$-class of $S'$.

The most fundamental result of the seminal paper [16] provides a presentation for these maximal subgroups $H_{\mathcal{F}}$ (based on the structural data about $S'$ as input). This presentation if defined on the set of generators $\{f_{i\lambda} : (i, \lambda) \in \mathcal{H} \subseteq I \times \Lambda\}$, where $I, \Lambda$ are index sets for the collections of $\mathcal{R}$-/$\mathcal{L}$-classes within $D_\mathcal{F}$ (or within $D_e$ in $S'$, which is the same, as just explained), and $\mathcal{H}$ is the set of all pairs with the property that the $\mathcal{H}$-class $H_{i\lambda} = R_i \cap L_\lambda$ is a group, i.e. that it contains an idempotent (again, it is irrelevant whether we are looking at this within $\GammaG(\mathcal{E})$ or $S'$). As shown in [9, Theorem 3.10], there is an algorithm which, given a finite biordered set $\mathcal{E}$, computes a (finite) presentation for the maximal subgroup $H_{\mathcal{F}}$ of $\GammaG(\mathcal{E})$.

### 2.3 Regular elements in $\GammaG(\mathcal{E})$ and $\mathcal{D}$-fingerprints

Since $\GammaG(\mathcal{E})$ is defined in terms of a presentation over a generating set $\overline{E}$, every element of $\GammaG(\mathcal{E})$ can be represented by a word from $E^+$, in the sense of the natural (surjective) homomorphisms $E^+ \to \GammaG(\mathcal{E})$ extending the map $e \mapsto \overline{e}, e \in E$. So, for every element of $\GammaG(\mathcal{E})$, there is at least one word over the alphabet $E$ representing it. The problem
is – and this gives rise to the word problem – this representation is not necessarily unique: there might be multiple ways to represent an element of $\text{IG}(\mathcal{E})$. Thus the word problem (in this case for $\text{IG}(\mathcal{E})$) asks: is there an algorithm which, presented with two words from $E^+$, decides whether they represent the same element of $\text{IG}(\mathcal{E})$? Of course, all along the way we assume that $\mathcal{E}$ is a finite biorder.

Also, a relevant algorithmic question is the following one: given a word $w = e_1 \ldots e_m$, decide if it represents a regular element of $\text{IG}(\mathcal{E})$. A regularity criterion is found in [9], where in Theorem 3.6 it was proved that $\overline{w} \in \text{IG}(\mathcal{E})$ is regular if and only if $w$ contains a letter $e$ (called the seed) so that with the corresponding factorisation $w = uev$ we have $\overline{ue} \leq \mathcal{R} \overline{e} \mathcal{v}$, in which case $\mathcal{D} \overline{w}$. In a certain sense, a sort of a converse statement is true as well: whenever we have $w \equiv uev$ such that $\overline{e} \mathcal{D} \overline{w}$, then $e$ is necessarily a seed for $w$, with $\overline{w}$ being a regular element of $\text{IG}(\mathcal{E})$, so that $\overline{ue} \leq \mathcal{R} \overline{e} \mathcal{v}$. Furthermore, it was then argued in Theorem 3.7. of the same paper that this criterion can be effectively tested, so that there is an algorithm which establishes regularity of elements represented by given words.

For words representing regular elements of $\text{IG}(\mathcal{E})$, seeds are not necessarily unique. In fact, the main result of [13] shows not only that it might happen that every letter is a seed, but that in fact whenever $w = e_1 \ldots e_m$ represents a regular element, then there are $e'_1, \ldots, e'_m \in E$ such that we have $\overline{e'_i} \in D_w$ for all $1 \leq i \leq m$, and $\overline{w} = e'_1 \ldots e'_m$ holds in $\text{IG}(\mathcal{E})$. In other words, any word representing a regular element of $\text{IG}(\mathcal{E})$ can be rewritten in terms of idempotents all of which belong to the same $\mathcal{D}$-class as the regular element itself.

Now let $\overline{w}$ be a regular element. We have already mentioned that $\mathcal{D} = \mathcal{J}$ holds whenever $\mathcal{E}$ is finite; hence, we have $D_{\overline{w}} = J_{\overline{w}}$, and the principal factor arising from this $\mathcal{J}$-class must be a completely 0-simple semigroup, as only finitely many idempotents are involved. So, we can identify this principal factor with $\mathcal{M}^0[G; I, \Lambda; P]$, where $G$ is the maximal subgroup of $\text{IG}(\mathcal{E})$ in the regular $\mathcal{D}$-class $D_{\overline{w}}$ – a presentation of which is given e.g. in [9, Theorem 4.2], based on the data from an idempotent-generated semigroup $S'$ such that $\mathcal{E} \cong \mathcal{E}' - I$, $\Lambda$ is index sets for the corresponding principal factor ($\mathcal{D}$-class) of $S'$, and $P = [f_{i\lambda}^{-1}]_{I \times \Lambda}$ (see [16]). Consequently, it is possible to write

$$\overline{w} = (i, g, \lambda)$$

for some $g \in G$ (written as a word over $f_{i\lambda}$’s) and $i \in I$, $\lambda \in \Lambda$. Furthermore, as shown in [4, Theorem 4.3] (and noted in the subsequent Remark 4.4), there is an algorithm which, presented with a finite biorder $\mathcal{E}$ and a word $w \in E^+$, computes $i$, $\lambda$, and a word representing $g$.

However, in general, a word $u$ need not to represent a regular element of $\text{IG}(\mathcal{E})$. Yet, what we can do in this case is to consider the coarsest factorisation $u \equiv u_1 \ldots u_k$ into subwords such that each factor represents a regular element; that is to say that whenever $\overline{u_1} \ldots \overline{u_j}$ is regular for some $1 \leq i \leq j \leq k$ then necessarily $i = j$. Such a factorisation is in [4] called a minimal $r$-factorisation, and it was explained in [4, Section 3] that, given $u$, one can always effectively find one. Of course, minimal $r$-factorisations need not to be unique – there can be others, for the same word $u$. Furthermore, there might be another word $v$ such that $\overline{u} = \overline{v}$ holds in $\text{IG}(\mathcal{E})$, and this
word might have a host of its own minimal r-factorisations. Nevertheless, a striking result was proved in [4, Theorem 3.4]: if \( u, v \in E^+ \) are two words such that \( \overline{u} = \overline{v} \), with minimal r-factorisations \( u = u_1 \ldots u_k \) and \( v \equiv v_1 \ldots v_r \), then necessarily \( k = r \) and for all \( 1 \leq i \leq k \) we have \( u_i D v_i \) (in fact, we even have \( u_i R v_i \) and \( u_i L v_i \)).

So, in other words, there is a sequence \((D_1, \ldots, D_k)\) of regular \( D \)-classes of \( IG(E) \) which is an invariant of an element of \( IG(E) \): no matter what word we consider that represents the element in question, and no matter what minimal r-factorisation of that word we take, the (regular) elements represented by the factors will, in the given order, belong to these regular \( D \)-classes. Later on, in [7, Theorem 4.2(4)], it was proved that the assumption \( u D v \) already suffices to arrive at the same conclusion.

As already explained above, given a word representing a regular element, there is an algorithmic procedure of transforming it into a triple of the form \((i, g, \lambda)\). Thus if we have a general word \( w \in E^+ \) representing an element with \( D \)-fingerprint \((D_1, \ldots, D_m)\), there is a routine way to write up this element as a product

\[
\overline{w} = (i_1, g_1, \lambda_1) \ldots (i_m, g_m, \lambda_m),
\]

where \((i_s, g_s, \lambda_s) \in D_s\) for all \( 1 \leq s \leq m \). Hence, solving the word problem in \( IG(E) \) (and, more generally, sorting out its basic structure) essentially comes down to comparing products of the above form and, in particular, finding a way to establish whether they are equal in \( IG(E) \). In the following subsection we are going to introduce the main technical vehicle to express succinctly the gist of the word problem for \( IG(E) \). This vehicle is also useful in characterising the main structural properties of \( IG(E) \), such as its Green’s relations.

### 2.4 Contact graphs, vertex groups, the map \( \theta \)

First of all, let us note that in this subsection and in the remainder of this section the definition of the map \( \theta \), as well as the formulation of all the relevant results, are slightly modified with respect to the original ones (as they appeared in [4, 7]). This is done in order to avoid the notion of dual groups and thus to contribute slightly to the “aesthetic appeal” of the approach. However, it is but an easy exercise to see that the two approaches are completely equivalent.

Let us start with the following setup. Assume we have given a sequence of groups \( G_1, \ldots, G_m, m \geq 2 \). Furthermore, assume that for \( 1 \leq k < m \) we have given relations

\[
\rho_k \subseteq G_k \times G_{k+1},
\]

as well as two sequences of elements \( a_k, b_k \in G_k, 1 \leq k \leq m \). From these data, we define a new relation \( \rho \subseteq G_1 \times G_m \) by setting that \((g, h) \in \rho\) if and only if there exist \( x_r \in G_r, 2 \leq r \leq m, \) such that
\[(a_1^{-1}gb_1, x_2) \in \rho_1,\]
\[(a_2^{-1}x_2b_2, x_3) \in \rho_2,\]
\[\vdots\]
\[(a_{m-1}^{-1}x_{m-1}b_{m-1}, x_m) \in \rho_{m-1},\]
\[a_m^{-1}x_mb_m = h.\]

This is the general setting we are going to use to describe the map \(\theta\) associated with a relation obtained in this way from two elements of \(\text{IG}(\mathcal{E})\) of a given \(\mathcal{D}\)-fingerprint, maximal subgroups of \(\mathcal{D}\)-classes involved, and very specific relations obtained from group-labelled graphs we are about to describe. (By a map associated with a relation \(\rho \subseteq X \times Y\) we mean a function \(\varphi_\rho : X \to \mathcal{P}(Y)\) defined by \(y \in x\varphi_\rho\) if and only if \((x, y) \in \rho\). This is then easily extended to a function \(\mathcal{P}(X) \to \mathcal{P}(Y)\) by \(A\varphi_\rho = \bigcup_{x \in A} x\varphi_\rho\).)

Assume now that the (finite) biorder \(\mathcal{E}\) comes from an idempotent-generated semigroup \(S'\), so that (up to isomorphism) \(\mathcal{E} = \mathcal{E}'\). Let \(D_1, D_2\) be two (not necessarily distinct) regular \(\mathcal{D}\)-classes of \(S'\), whose \(\mathcal{R}\)-\(\mathcal{L}\)-classes are indexed by sets \(I_1, I_2\) and \(\Lambda_1, \Lambda_2\), respectively. We are going to define a graph \(H(D_1, D_2)\) on the vertex set \(\Lambda_1 \times I_2\) whose edges are labelled by elements of the group \(G_1 \times G_2\), where \(G_1, G_2\) are the maximal subgroups of \(\text{IG}(\mathcal{E})\) in its \(\mathcal{D}\)-classes corresponding to \(D_1\) and \(D_2\), respectively. This is going to be the contact graph of \(D_1\) and \(D_2\).

To define this graph, a crucial observation is that elements of \(\bar{E}\), the idempotents of \(\text{IG}(\mathcal{E})\), exercise left and right actions by partial transformations on index sets \(I\) and \(\Lambda\), respectively, of \(\mathcal{R}\)-\(\mathcal{L}\)-classes of a regular \(\mathcal{D}\)-class \(D\) of \(\text{IG}(\mathcal{E})\) (and thus of a corresponding \(\mathcal{D}\)-class of \(S'\)). Namely, for \(i, i' \in I\) we set \(\bar{e} \cdot i = i'\) if

\[\bar{e}(i, g, \lambda) = (i', g', \lambda')\]

holds in \(\text{IG}(\mathcal{E})\) for some \(g, g' \in G\), where \(G\) is a maximal subgroup contained in \(D\) (with generators \(f_{i\lambda}\), as described before), and some \(\lambda, \lambda' \in \Lambda\). As it transpires from [4, Proposition 4.1], we then necessarily have \(\lambda' = \lambda\) (and the above relation will hold whenever \(\lambda\) is replaced by any other index from \(\Lambda\)), and \(g' = cg\), where the coefficient \(c\) depends solely on \(\bar{e}\) and \(i\) (but not on \(g\) or \(\lambda\)). Similarly, we set \(\lambda \cdot \bar{e} = \lambda'\) if

\[(i, g, \lambda)\bar{e} = (i, gd', \lambda')\]

for some (or all) \(i \in I\), and some \(g, d \in G\) (where again \(d\) depends only on \(\bar{e}\) and \(\lambda\)). Clearly, these actions are vacuous (i.e. correspond to empty partial maps) unless \(\bar{e}\) comes from a \(\mathcal{D}\)-class that is \(\mathcal{J}\)-above \(D\). Also, [4, Proposition 4.1] shows that the two partial maps induced by a given idempotent \(\bar{e}\) are simultaneously empty or non-empty, and so the non-emptiness of one of them implies the existence of fixed points of the other one, and vice versa. The same result supplies the exact information about the coefficients \(c, d\); namely,

\[c = f_{i'\lambda_0}^{-1} f_{i\lambda_0}^{-1},\]
where \( \lambda_0 \) is any fixed point of the right action of \( \bar{e} \) upon \( \Lambda \), \( \lambda_0 \cdot \bar{e} = \lambda_0 \), such that both \( H_{\lambda_0} \) and \( H_{i'\lambda_0} \) are groups, i.e. contain idempotents \( e_{\lambda_0} \) and \( e_{i'\lambda_0} \), respectively. Two remarks are important here: the existence of such \( \lambda_0 \) is guaranteed, by [9, Proposition 2.2], by the mere existence of fixed points of the right action of \( e \) upon \( \Lambda \), such that both \( \text{Hi}_{\lambda_0} \) and \( \text{Hi'}_{\lambda_0} \) are groups, i.e. contain idempotents \( e_{\lambda_0} \) and \( e_{i'\lambda_0} \), respectively. Two remarks are important here: the existence of such \( \lambda_0 \) is guaranteed, by [9, Proposition 2.2], by the mere existence of fixed points of the right action of \( e \) upon \( \Lambda \) (and, in fact, by the non-emptiness of that action); and secondly, the choice of such \( \lambda_0 \) is irrelevant because if \( \mu_0 \) is another such fixed point then the presentation of \( G \) contains a relation of the form \( f_{i'\lambda_0} f_{i\mu_0}^{-1} = f_{i'\mu_0} f_{i\mu_0}^{-1} \) arising from an up-down singular square \( (i, i'; \lambda_0, \mu_0) \), see [16]. Analogously,

\[
d = f_{i_0\lambda}^{-1} f_{i_0\lambda'},
\]

where \( i_0 \) is any fixed point of the left action of \( e \) on \( I \) such that both \( H_{i_0\lambda} \) and \( H_{i_0\lambda'} \) contain idempotents.

Returning to the definition of the contact graph \( \mathcal{A}(D_1, D_2) \), for any idempotent \( \bar{e} \in \bar{E} \) such that \( \lambda = \mu \cdot \bar{e} \) and \( \bar{e} \cdot i = j \) we draw an edge

\[
(\lambda, i) \longrightarrow (\mu, j),
\]

and label it with \((a, b^{-1}) \in G_1 \times G_2\), where, assuming that the generators of \( G_1 \) are written as \( f_{i\lambda}^{(1)} \) and the generators of \( G_2 \) as \( f_{j\lambda}^{(2)} \),

\[
a = (f_{i_0\lambda}^{(1)})^{-1} f_{i_0\mu}^{(1)} \tag{2.1}
\]

for any fixed point \( i_0 \) of the left action of \( \bar{e} \) on \( I \) such that both \( H_{i_0\lambda} \) and \( H_{i_0\lambda'} \) contain idempotents, and

\[
b = f_{j_0\lambda}^{(2)} (f_{i_0\lambda}^{(2)})^{-1} \tag{2.2}
\]

for any fixed point \( \lambda_0 \) of the right action of \( \bar{e} \) on \( \Lambda \) such that both \( H_{i\lambda_0} \) and \( H_{i'\lambda_0} \) contain idempotents. In fact, this edge can be traversed in the opposite direction, too, with the amendment that its label is then considered to be \((a^{-1}, b) = (a, b^{-1})^{-1}\). As is usually the case, the label of a walk is the product of labels of edges along that walk. For \((\lambda, i) \in \Lambda_1 \times I_2\) we denote by \( W_{(\lambda, i)} \) the collection of labels of all closed walks based at the vertex \((\lambda, i)\). As noted in [7, Lemma 3.2], this collection is actually a subgroup of \( G_1 \times G_2 \), and we call it the \( \text{vertex group} \) at \((\lambda, i)\).

Now, let us fix a \( \mathcal{D} \)-fingerprint \((D_1, \ldots, D_m)\) in \( \text{IG}(\mathcal{E}) \), and let

\[
x = (i_1, a_1, \lambda_1) \ldots (i_m, a_m, \lambda_m)
\]

and

\[
y = (j_1, b_1, \mu_1) \ldots (j_m, b_m, \mu_m)
\]
be two elements of $\mathcal{LG}(\mathcal{E})$ of this $\mathcal{D}$-fingerprint. For $1 \leq k < m$, put
\[
\rho_k = \begin{cases} 
W(\lambda_k, i_{k+1})(g_k, h_k) & \text{if there exists a walk } (\lambda_k, i_{k+1}) \rightsquigarrow (\mu_k, j_{k+1}), \\
\emptyset & \text{otherwise.}
\end{cases}
\]

where $W(\lambda_k, i_{k+1})$ is the vertex group of $\mathcal{A}(D_k, D_{k+1})$ at $(\lambda_k, i_{k+1})$, and $(g_k, h_k)$ is the label of any walk $(\lambda_k, i_{k+1}) \rightsquigarrow (\mu_k, j_{k+1})$ (it is immaterial which walk we take, for if $(g'_k, h'_k)$ is the label of another such walk then $(g_k, h_k)(g'_k, h'_k)^{-1}$ is the label of a closed walk based at $(\lambda_k, i_{k+1})$ and so it belongs to $W(\lambda_k, i_{k+1})$, thus yielding the same right coset). As described previously, these parameters define a relation between $G_1$ and $G_m$, and the corresponding mapping $\mathcal{P}(G_1) \to \mathcal{P}(G_m)$ is denoted by $(\cdot, x, y)\theta$.

2.5 Putting it all together

In this (modified) setting just described, Theorem 3.9 of [7], characterising the word problem of $\mathcal{LG}(\mathcal{E})$, reads as follows.

**Theorem 2.3** $x = y$ holds in $\mathcal{LG}(\mathcal{E})$ if and only if $i_1 = j_1, \lambda_m = \mu_m$, and

$$1 \in ([1], x, y)\theta.$$  

Since by [7, Theorem 3.8(1)] if $A \subseteq G_1$ is a coset of a subgroup of $G_1$ (it doesn’t matter if it is left or right, because a left coset is of a subgroup is always a right coset of a conjugated subgroup), $(A, x, y)\theta$ is either empty or again a coset of a subgroup, it follows that the condition in the previous theorem is equivalent to saying that $([1], x, y)\theta$ is a subgroup of $G_m$.

Similarly, by adapting Corollary 4.3 of [7], Green’s relations in $\mathcal{LG}(\mathcal{E})$ can be expressed in terms of the map $\theta$ as follows.

**Theorem 2.4** Let $x, y \in \mathcal{LG}(\mathcal{E})$. If these elements are not of the same $\mathcal{D}$-fingerprint, they cannot be $\mathcal{J}$-related. Otherwise, if they are, we have:

(i) $x \mathcal{R} y$ if and only if $i_1 = j_1$ and $([1], x, y)\theta \neq \emptyset$;
(ii) $x \mathcal{L} y$ if and only if $\lambda_m = \mu_m$ and $1 \in (G_1, x, y)\theta$;
(iii) $x \mathcal{D} y$ if and only if $(G_1, x, y)\theta \neq \emptyset$.

As already mentioned, $\mathcal{D} = \mathcal{J}$ whenever $\mathcal{E}$ is finite.

The case when $x = y$ is particularly interesting. Here we have that $(H, x, x)\theta$ is a subgroup of $G_m$ whenever $H$ is a subgroup of $G_1$. In addition, $([1], x, x)\theta$ is a normal subgroup of $(G_1, x, x)\theta$, and the corresponding quotient is isomorphic precisely to the Schützenberger group of the $\mathcal{H}$-class of $x$ (see [19]). In fact, exactly along the lines of the proof of Proposition 4.6 of [7] it can be proved that whenever $H, K$ are two subgroups of $G_1$ such that $K$ is normal in $H$, then $(K, x, x)\theta$ is normal in $(H, x, x)\theta$.

2.6 The word problem of $\mathcal{LG}(\mathcal{E})$ reduced to group theory

Now, let us take a slightly more detailed look at the process of computing $(Ht, x, y)\theta$, where $H$ is a subgroup of $G_1$ and $t \in G_1$. We do this to make it abundantly clear that
this process entirely relies on the knowledge of vertex groups of the contact graphs, along with the information about their connected components and the choice of coset representatives arising from walks within these components. We remind, once again, that the result is either the empty set or a coset of a subgroup of $G_m$.

We define two sequences of subgroups $H_k, L_k$ of $G_k$, and two sequences of elements $t_k, z_k \in G_k$, $1 \leq k \leq m$, in a recursive fashion (more precisely, at some point, some of the sets $H_k$ may become empty, at which point all further $L_k$ are also empty, and the definitions of $t_k, z_k$ become irrelevant). First we set $H_1 = H$ and $t_1 = t$, and then, assuming $H_k$ and $t_k$ have already been defined, let

$$L_k = a_k^{-1} H_k a_k = H_k^a,$$

$$z_k = a_k^{-1} t_k b_k = t_k a_k^{-1} b_k,$$

$$H_{k+1} t_{k+1} = (L_k z_k) \varphi_{\rho_k},$$

where $\rho_k$ is either $W_{(\lambda_k, i_k+1)}(g_k, h_k)$, the coset of the vertex group, or the empty relation (depending whether a walk $(\lambda_k, i_k+1) \rightsquigarrow (\mu_k, j_k+1)$ exists or not). As already remarked (and shown in [7, Theorem 3.8]), the last of these recurrences makes sense because its right hand side is the second projection of the intersection of cosets

$$(L_k \times G_{k+1})(z_k, 1) \cap W_{(\lambda_k, i_k+1)}(g_k, h_k),$$

which itself is either empty, or a coset of $(L_k \times G_{k+1}) \cap W_{(\lambda_k, i_k+1)}$. Thus, $H_{k+1}$ is either empty, or the second projection of the subgroup

$$M_k = (L_k \times G_{k+1}) \cap W_{(\lambda_k, i_k+1)}$$

of $G_k \times G_{k+1}$, and in the latter case the second projection in question is indeed a coset of $H_{k+1}$ (with $t_{k+1}$ chosen arbitrarily such that $(\gamma, t_{k+1})$ belongs to the intersection (2.3) for some $\gamma \in G_k$). Finally, from the very definition of $\theta$ it follows that $(Ht, x, y) \theta = L_m z_m$.

3 Computing the vertex groups for $\text{IG}(\mathcal{E}_n)$

3.1 General observations

We start by recalling a very important remark from [16] (made at the beginning of Section 3 of that paper) that the action that elements of $\mathcal{E} \in \mathcal{E}$ exercise on $\mathcal{H}$-classes contained in an $\mathcal{R}$-class of an idempotent $\mathcal{T}$ in $\text{IG}(\mathcal{E})$ (so, an $\mathcal{R}$-class from a regular $\mathcal{D}$-class) is equivalent the the action that elements of $e \in \mathcal{E}$ exercise on $\mathcal{H}$-classes of the $\mathcal{R}$-class $R_f$ in an idempotent-generated semigroup $S$ such that $\mathcal{E} \cong \mathcal{E}_S$. (An analogous statement is true for $\mathcal{H}$-classes contained in fixed regular $\mathcal{L}$-class.) This follows from property (IG3) from that paper, which in turn is a consequence of [13]. This will substantially facilitate our considerations and computations, as it means that the (partial) action of $\mathcal{E}$ on the index sets $I, \Lambda$ associated with the principal factor.
corresponding to a \( \mathcal{R} \)-class \( D \) such that \( D \subseteq D_\varnothing \) can be “read off” already from the semigroup \( S \) itself. More precisely, we can formalise this via the following statement.

**Lemma 3.1** Let \( D \) be a regular \( \mathcal{R} \)-class of \( \text{IG}(\mathcal{E}) \), with \( I, \Lambda \) being the index sets of the collections of \( \mathcal{R} \)- and \( \mathcal{L} \)-classes, respectively, contained in \( D \). Let \( \varnothing \in \varnothing \). Then \( \varnothing \cdot i = i' \) (and so \( \varnothing(i, g, \lambda) = (i', h, \lambda) \) holds for some \( \lambda \in \Lambda \) and elements \( g, h \) of the maximal subgroup in \( D \)) if and only if \( eH_\lambda = H_{i'\lambda} \) holds in \( S \). Similarly, we have \( \lambda \cdot \varnothing = \lambda' \) if and only if \( H_{j\lambda}e = H_{j\lambda'} \) holds in \( S \) for some \( j \in I \).

We remind to the facts explained in Example 2.2 that in \( \mathcal{T}_n \), transformations on the \( n \)-element set \( [1, n] = \{1, \ldots, n\} \) are classified into (regular) \( \mathcal{R} \)-classes \( D_m \) according to their rank \( m \) (the size of their image), and the corresponding index sets are \( I_m \), consisting of all partitions of \( [1, n] \) into \( m \) classes (the kernels of transformations), and \( \Lambda_m \), consisting of all \( m \)-element subsets of \( [1, n] \) (the images of transformations). The role of \( S \) (with respect to \( \text{IG}(\mathcal{E}^n) \)) is taken by the idempotent-generated subsemigroup of \( \mathcal{T}_n \), which is, by the main result of [18], just \( \mathcal{T}_n \) stripped from all the non-trivial permutations, \( (\mathcal{T}_n \setminus \mathcal{S}_n) \cup \{e\} \). As for the maximal subgroups [17], they are the same (in regular \( \mathcal{R} \)-classes of \( \text{IG}(\mathcal{E}^n) \) and \( \mathcal{T}_n \)), the symmetric group \( \mathcal{S}_m \), whenever \( m \leq n - 2 \). The only differences arise when \( m = n \), when, of course, in \( \text{IG}(\mathcal{E}^n) \) the corresponding maximal subgroup is trivial, and when \( m = n - 1 \) when the maximal subgroup in \( \text{IG}(\mathcal{E}^n) \) is free of rank \( \binom{n}{2} - 1 \) (and not \( \mathcal{S}_{n-1} \)). Therefore, a typical regular element of \( \text{IG}(\mathcal{E}^n) \) from \( \overline{D}_m = D_m \Psi \) may be written as a triple \( (P, g, A) \), where \( P \) is a partition of \( [1, n] \) into \( m \) classes, \( A \) and \( m \)-element subset of \( [1, n] \) and \( g \) a member of the maximal subgroup contained in \( \overline{D}_m \).

We begin with what is essentially a restatement of the previous lemma in the context of \( \mathcal{T}_n \) and \( \text{IG}(\mathcal{E}^n) \), and its proof is an easy exercise for the reader. For a subset \( A \subseteq [1, n] \) and a partition \( P \) of \( [1, n] \) we say that \( A \) saturates \( P \) if every \( P \)-class contains at least one element of \( A \). Also, we say that \( P \) separates \( A \) if every \( P \)-class contains at most one element of \( A \). (Clearly, \( A \perp P \) if and only if both \( A \) saturates \( P \) and is separated by \( P \).)

**Lemma 3.2** Let \( A, B \) be \( m \)-element subsets of \( [1, n] \) and \( P, Q \) partitions of \( [1, n] \) into \( r \) classes. Let \( e \) be an idempotent transformation on \( [1, n] \).

1. \( A = B \cdot \varnothing \) exists if and only if \( \ker e \) separates \( B \) (in which case \( A = Be \)).
2. \( Q = \varnothing \cdot P \) exists if and only if \( \text{im } e \) saturates \( P \) (in which case the classes of \( Q \) are the inverse images of those of \( P \) under \( e \), each of which being a union of certain \( (\ker e) \)-classes).

Besides supplying essential information about the edges in the contact graph \( \mathcal{M}(\overline{D}_m, \overline{D}_r) \), this enables us to formulate a regularity criterion within \( \text{IG}(\mathcal{E}^n) \) that provides us with information beyond that following from [4, Lemma 6.2].

**Lemma 3.3** Let \( (P, g, A) \in \overline{D}_m \) and \( (P', g', A') \in \overline{D}_r \) be two regular elements of \( \text{IG}(\mathcal{E}^n) \). Then the product \( (P, g, A)(P', g', A') \) is regular if and only if

- either \( m \geq r \) and \( A \) saturates \( P' \), or
- \( m \leq r \) and \( P' \) separates \( A \).
Proof \((\Rightarrow)\) Assume that \((P, g, A)(P', g', A')\) is regular and that \(m \geq r\). As shown in the proof of [4, Proposition 4.1], and following from [13], each of the elements can be rewritten as a product of idempotents from their own \(\mathcal{D}\)-classes: so, there are \(\bar{e}_1, \ldots, \bar{e}_s \in \overline{D}_m\) and \(\bar{f}_1, \ldots, \bar{f}_t \in \overline{D}_r\) such that
\[
(P, g, A) = \overline{e}_1 \ldots \overline{e}_s,
(P', g', A') = \overline{f}_1 \ldots \overline{f}_t.
\]
Now, since the product \(\overline{e}_1 \ldots \overline{e}_s \overline{f}_1 \ldots \overline{f}_t\) is regular, it must contain a seed; since \(r \leq m\), this must be some (and thus any) of the \(f_1, \ldots, f_t\). In particular, \(f_1\) is a seed letter. But then, by [4, Remark 2.6], \(e_s f_1 \mathcal{L} \overline{f}_1\). On the other hand, \(\overline{f}_1 \mathcal{R} \overline{f}_1 \ldots \overline{f}_t = (P', g', A')\), which means that \(\ker f_1 = P'\) and so \(\overline{f}_1\) has a representation of the form \((P', h_1, B)\) (for some group element \(h_1\) and an \(r\)-element subset \(B\)). It follows that \(\overline{e}_s \cdot P'\) exists, which by the previous lemma means that \(\im e_s\) saturates \(P'\). However, \(\overline{e}_1 \ldots \overline{e}_s \mathcal{L} \overline{e}_s\), so \(\im e_s = A\), and the claim follows. We argue in a very similar fashion when \(m \leq r\).

\((\Leftarrow)\) Assume that \(m \geq r\) and that \(A\) saturates \(P'\); the other case is handled analogously. As in the previous part of the proof, each of \((P, g, A)\), \((P', g', A')\) can be written in \(\IG(\mathcal{S}_n)\) as a product of idempotents from their \(\mathcal{D}\)-classes, just as above. So, if we write \(\overline{e}_i = (P_i, g_i, A_i)\) for \(1 \leq i \leq s\) and \(\overline{f}_j = (Q_j, h_j, B_j)\) for \(1 \leq j \leq t\), then by [4, Lemma 6.2] we must have that \(A_i \perp P_{i+1}\) for all \(i < s\) and \(B_j \perp Q_{j+1}\) for all \(j < t\). Also, \(A_s = A\) and \(Q_1 = P'\). Since the former saturates the latter, by the previous lemma, \(\overline{e}_s (P', h_1, B_1) = (P'', h'_1, B_1)\), where the classes of \(P'' = \overline{e}_s \cdot P'\) arise as unions of \(P_s\)-classes. Hence, \(A_{s-1}\) saturates \(P''\). Proceeding in this fashion, we conclude that \((P, g, A)(P', g', A') \in \overline{D}_r\), a regular element of \(\IG(\mathcal{S}_n)\).

3.2 The connected components of contact graphs

As is well-known [18, 19], the idempotent-generated submonoid of \(\mathcal{I}_n\) (also called the singular part of \(\mathcal{I}_n\) and denoted by \(\Sing(\mathcal{I}_n)\)) is generated solely by the idempotent transformations of rank \(n - 1\). These are of the form \(\epsilon_{ij}\) for \(1 \leq i \neq j \leq n\), so that
\[
k \epsilon_{ij} = \begin{cases} k & k \neq j, \\ i & k = j. \end{cases}
\]

There are \(2(n-1)\) such idempotents, two per each \(\mathcal{R}\)-class, \(n - 1\) of them in each of the \(\mathcal{L}\)-classes. In particular, every non-identity idempotent transformation in \(\mathcal{I}_n\) can be expressed as a product of rank \(n - 1\) idempotents. This is reflected in contact graphs in the following way.

Lemma 3.4 Let \(m, r \leq n - 1\) and let \((A, P), (B, Q)\) be two vertices in the contact graph \(\mathcal{S}(\overline{D}_m, \overline{D}_r)\) of \(\IG(\mathcal{S}_n)\). Then there exists an edge in this graph \((A, P) \rightarrow (B, Q)\) labelled by \(e \in E(\mathcal{I}_n)\) if and only if there exist a sequence of vertices \((A_1, P_1), \ldots, (A_{k-1}, P_{k-1})\) and edges
\[
(A, P) \rightarrow (A_1, P_1) \rightarrow \ldots \rightarrow (A_{k-1}, P_{k-1}) \rightarrow (B, Q)
\]
labelled, respectively, by $e_{i_1 j_1}, \ldots, e_{i_k j_k}$ such that $e = e_{i_1 j_1} \ldots e_{i_k j_k}$ holds in $\mathcal{T}_n$.

**Proof** Assume first that $\mathcal{A}(\overline{D}_n, \overline{D}_r)$ contains an edge $(A, P) \rightarrow (B, Q)$ labelled by $e \in E(\mathcal{T}_n)$. As we may safely assume that $e \neq \text{id}_n$, by the main result of [18], $e$ can be written as a product of rank $n - 1$ idempotents, say $e = e_{i_1 j_1} \ldots e_{i_k j_k}$. Now define the following sequence of subsets of $[1, n]$:

\[
A_1 = B \cdot e_{i_1 j_1} \ldots e_{i_{k-1} j_{k-1}}, \\
A_2 = B \cdot e_{i_1 j_1} \ldots e_{i_{k-2} j_{k-2}}, \\
\vdots \\
A_{k-1} = B \cdot e_{i_1 j_1},
\]

and partitions of $[1, n]$:

\[
P_1 = e_{i_k j_k} \cdot P, \\
P_2 = e_{i_{k-1} j_{k-1}} \cdot e_{i_k j_k} \cdot P, \\
\vdots \\
P_{k-1} = e_{i_{k-2} j_{k-2}} \ldots e_{i_k j_k} \cdot P.
\]

It takes only a short reflection (upon multiple applications of Lemma 3.2) to see that all of these sets and partitions exist, in the sense that they are all $m$-element subsets and $r$-element partitions of $[1, n]$, respectively. Also, it is now straightforward to see that we have $A = A_1 \cdot e_{i_k j_k}$ and $e_{i_k j_k} \cdot P = P_1$, witnessing the existence of an edge $(A, P) \rightarrow (A_1, P_1)$ labelled by $e_{i_k j_k}$. Furthermore, for all $1 \leq s \leq k - 2$ we have $A_s = A_{s+1} \cdot e_{i_s j_s} \ldots e_{i_{k-s} j_{k-s}}$ and $e_{i_s j_s} \ldots e_{i_{k-s} j_{k-s}} \cdot P_s = P_{s+1}$, showing that there is an edge $(A_s, P_s) \rightarrow (A_{s+1}, P_{s+1})$ labelled by $e_{i_s j_s} \ldots e_{i_{k-s} j_{k-s}}$. Finally, the fact that $A_{k-1} = B \cdot e_{i_1 j_1}$ and $e_{i_1 j_1} \cdot P_{k-1} = Q$ verifies the existence of an edge $(A_{k-1}, P_{k-1}) \rightarrow (B, Q)$ labelled by $e_{i_1 j_1}$.

Conversely, assume that there is a walk

\[ (A, P) \rightarrow (A_1, P_1) \rightarrow \ldots \rightarrow (A_{k-1}, P_{k-1}) \rightarrow (B, Q) \]

where the edges are labelled, respectively, by $e_{i_k j_k}, \ldots, e_{i_1 j_1}$, such that the product $e = e_{i_1 j_1} \ldots e_{i_k j_k}$ is an idempotent in $\mathcal{T}_n$. Then we have $A = A_1 \cdot e_{i_k j_k}$ and $e_{i_k j_k} \cdot P = P_1$, as well as $A_s = A_{s+1} \cdot e_{i_s j_s} \ldots e_{i_{k-s} j_{k-s}}$ and $e_{i_s j_s} \ldots e_{i_{k-s} j_{k-s}} \cdot P_s = P_{s+1}$ for $1 \leq s \leq k - 2$, and $A_{k-1} = B \cdot e_{i_1 j_1}$ and $e_{i_1 j_1} \cdot P_{k-1} = Q$. The cumulative effect of these equalities is that we have $A = B \cdot e$ and $e \cdot P = Q$, thus proving the existence of an edge $(A, P) \rightarrow (B, Q)$ labelled by $e$.

A direct consequence of the previous lemma is that, if we wish to investigate the connected components of contact graphs (which is the purpose of this subsection) it completely suffices to focus solely on edges labelled by rank $n - 1$ idempotents. So, let us stop for a moment to take a closer look what does it means that we have an edge $(A, P) \rightarrow (B, Q)$ labelled, say, by $e_{i j}$.
Indeed, then we have \( A = B \varepsilon_{ij} \) and \( \overline{\varepsilon_{ij}} \cdot P = Q \). As for the first equality, we have two cases to discuss. First, if \( i \in B \), then necessarily \( j \notin B \) (for otherwise it would follow that \( |B| > |A| \)). However, in such a case it follows that \( A = B \). Otherwise, \( i \notin B \). Now, if \( j \notin B \) also, then again \( A = B \); however, if \( j \) does belong to \( B \), then \( A \) is obtained from \( B \) by removing \( j \) from it and replacing it with \( i \): \( A = (B \setminus \{j\}) \cup \{i\} \). In other words, \( B = (A \setminus \{i\}) \cup \{j\} \).

Now let us analyse the second equation. If \( i \) and \( j \) belong to the same \( P \)-class, then it is clear that \( P = Q \). Otherwise, \( i \in P_{k_i} \) and \( j \in P_{k_j} \) for some indices \( k_i \neq k_j \). Then the partition \( Q \) is obtained from \( P \) by adding \( j \) to the class \( P_{k_i} \) (as both \( i \), \( j \) belong to the inverse image of \( i \) under \( \varepsilon_{ij} \)), and consequently, by removing \( j \) from \( P_{k_j} \). However, for this latter to be possible (i.e. that the described operation does not change the rank of the partition), we must have \( |P_{k_j}| > 1 \), that is, \( P_{k_j} \) must contain at least one additional element except \( j \); in other words, \( P_{k_j} = \{j\} \) would exclude the possibility of existence of the edge we are considering.

Therefore, we can sum up that traversing an edge labelled by a rank \( n - 1 \) idempotent \( \varepsilon_{ij} \) originating from a vertex \((A, P)\) amounts to performing the following “elementary step”:

- Pick \( j \notin A \) not comprising a singleton class of \( P \), and move it from its \( P \)-class to another one (or possibly the same one), say \( P_s \).
- If you wish, remove an element \( i \in A \cap P_s \) from \( A \) and replace it by \( j \).

Of course, there is also the “reverse step”, against the arrow, which goes as follows:

- Remove an element \( j \in B \) from \( B \) and replace it by (possibly the same) element \( i \) from the same \( Q \)-class.
- If the previous step is performed with \( i \neq j \), then, if you wish, move \( j \) from its \( Q \)-class to another one.

These conditions can be made even more compact by saying that moving forth and back along the edges labelled by rank \( n - 1 \) idempotents allows us to do the following two types of moves with combinatorial “subset-partition structures” of the form \((A, P)\), where \( A \) is an \( m \)-element subset of \([1, n]\) and \( P \) is a partition of \([1, n]\) into \( r \) pieces:

1. Move around the subset elements within the given partition class.
2. Remove a point currently not belonging to the subset from a non-singleton partition class and add it to another one.

(Some rank \( n - 1 \) idempotent allow moves of type (1) and (2) simultaneously, but the same effect can be also achieved by traversing two consecutive edges as well.)

With this in mind, we can now proceed to formulate the connectedness criterion in \( \mathcal{A}(\overline{\mathcal{D}_m}, \overline{\mathcal{D}_r}) \). For a pair \((A, P)\), \(|A| = m\), \(|P| = r\), we call its type the sequence of numbers \( |A \cap P_1|, \ldots, |A \cap P_r| \) sorted in non-increasing order; for example, if \( n = 9 \), \( A = \{1, 3, 5, 7\} \) and \( P = \{\{1, 2, 6\}, \{3, 5, 7, 9\}, \{4, 8\}\} \), then the type of \((A, P)\) is \((3, 1, 0)\). When \((A, P)\) and \((B, Q)\) are of the same type, we say that they are homeomorphic and write \((A, P) \sim (B, Q)\). This is the same as saying that there is a pair of bijections \( \phi : A \to B \) and \( \psi : P \to Q \) such that for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq r \) we have

\[
a_i \in P_j \quad \text{if and only if} \quad a_i \phi \in P_j \psi.
\]
The pair \( (\phi, \psi) \) is then called a homeomorphism between \((A, P)\) and \((B, Q)\).

A pair \((A, P)\) is called stationary if all \(P\)-classes containing elements not in \(A\) are singletons.

**Proposition 3.5** Let \(m, r \leq n - 1\). Two different vertices \((A, P)\) and \((B, Q)\) of the graph \(\mathcal{A}(\overline{D}_m, \overline{D}_r)\) are connected if and only if they are homeomorphic and none of them is stationary.

**Proof** The direct implication of this proposition is immediately clear. Namely, if \((A, P)\) and \((B, Q)\) are indeed connected, then there is a sequence of edges connecting them, each of which is labelled by a rank \(n - 1\) idempotent. Hence, \((B, Q)\) can be obtained from \((A, P)\) by performing a sequence of steps of the type (1) and (2) above. However, note that none of these steps can change the type of a pair to which it is applied. Thus \((A, P)\) and \((B, Q)\) must be of the same type. Furthermore, none of them are stationary, for otherwise it is clear that it would not be possible to apply any of the steps (1) or (2) to a stationary pair in a nontrivial fashion.

Conversely, assume that \((A, P) \sim (B, Q)\) and that none of these two pairs is stationary. Upon fixing a homeomorphism \((\phi, \psi) : (A, P) \mapsto (B, Q)\), we are going to describe a sequence of steps (1),(2) that turns \((A, P)\) into \((B, Q)\). Our first aim is to describe a process that, whenever \(a \phi \neq a\), “moves” the point \(a \in A\) to \(a \phi\). We have three cases to consider; throughout, we use the assumption that these pairs are non-stationary. For a non-image point \(x \in [1, n] \setminus A\) we are going to use the term free if its current partition class is not a singleton (so both \((A, P)\) and \((B, Q)\) have at least one free point each).

- **\(a \phi\) is also in \(A\).** Let \(x\) be a free point in the current subset-partition structure. Remove it from its partition class and add it to the class containing \(a \phi\) (this is a move of type (2)). Then apply a step of type (1) to remove \(a \phi\) from the current subset and replace it by \(x\). Now, \(a \phi\) becomes a free point, so apply a step of type (2) to add it to the partition class containing \(a\). Finally, apply (1) to remove \(a\) from the subset and add \(a \phi\). (Note that this makes \(a\), at the moment, a free point.)
- **\(a \phi \notin A\) is a free point.** Remove \(a \phi\) from its partition class and add it to the class containing \(a\) (this is a step of type (2)). Then (by applying (1)) remove \(a\) from the subset and add \(a \phi\) to it. (Once again, this makes \(a\) a free point.)
- **\(a \phi \notin A\) currently comprises a singleton partition class.** Take a free point \(x\), remove it from its current partition class, and, by applying (2), add it to \(a \phi\), thus making it a class consisting of two free points. This creates a situation from the previous case, so proceed accordingly.

Note that one such step creates a new pair \((A', P')\) (where \(A' = (A \setminus \{a\}) \cup \{x\}\) in the first case and \(A' = (A \setminus \{a\}) \cup \{a \phi\}\) in the other two), which is of course still homeomorphic to \((B, Q)\), via \((\phi', \psi')\) such that

\[
y \phi' = \begin{cases} 
  a \phi & y = a \phi, \\
  (a \phi) \phi & y = x, \\
  y \phi & y \notin \{a \phi, x\}
\end{cases}
\]
in the first case, and

\[ y\phi' = \begin{cases} a\phi \ y = a\phi, \\ y\phi \ y \neq a\phi \end{cases} \]

in the other two. In any case, \( \phi' \) has more fixed points than \( \phi \). Therefore, either by employing an inductive argument, or by simply iterating the step described above (applying it now to \( (A', P') \) etc.), we arrive at the conclusion that \( (A, P) \) can be transformed, by a series of applications of (1) and (2), into a subset-partition pair of the form \( (B, Q') \) (where the partition \( Q' \) is possibly different from \( Q \), but induces the same partition on \( B \) as \( Q \) does).

So, now it remains to argue that we can use steps of type (2) (moving around points not belonging to \( B \)) in order to transform \( Q' \) into \( Q \). To be more precise, assume that \( B = B_1 \cup \cdots \cup B_t \) is the partition induced on \( B \) both by \( Q \) and \( Q' \) (so that both of the latter contain \( r - t \) classes not intersecting \( B \)). Furthermore, let

\[ B_1 \cup B'_1, \ldots, B_t \cup B'_t, C'_1, \ldots, C'_{r-t} \]

be the partition classes of \( Q' \), while the classes of \( Q \) are

\[ B_1 \cup B''_1, \ldots, B_t \cup B''_t, C_1, \ldots, C_{r-t}, \]

with \( B'_k, B''_k, C_l, C'_l \subseteq [1, n] \setminus B, 1 \leq k \leq t, 1 \leq l \leq r - t \). To prove the required assertion, since all steps involved are reversible, we are going to show that both \( (B, Q) \) and \( (B, Q') \) can be transformed into a fixed pair \( (B, Q'') \), where the classes of the partition \( Q'' \) are

\[ B_1 \cup X, B_2, \ldots, B_t, \{x_1\}, \ldots, \{x_{r-t}\}, \]

where \( x_1, \ldots, x_{r-t} \) are some arbitrary but fixed elements of \( [1, n] \setminus B \) and \( X = [1, n] \setminus (B \cup \{x_1, \ldots, x_{r-t}\}) \). We show this for \( (B, Q) \), the proof for \( (B, Q') \) being completely analogous. Now, some of the elements \( x_k, 1 \leq k \leq r - t \), may already form singleton classes among \( C_1, \ldots, C_{r-t} \); without loss of generality (and upon renumbering, if necessary) we may assume that \( C_k = \{x_k\} \) for all \( k < s \) for some \( s \). Other classes, not intersecting \( B \), namely \( C_s, \ldots, C_{r-t} \) are either not singletons, or are singletons but do not contain any of \( x_1, \ldots, x_{r-t} \). This, in particular, means that all the elements from the latter list are free in \( (B, Q) \). However, this very fact allows us to use (2) to remove them from their respective classes and put each \( x_l, s \leq l \leq r - t \), into \( C_l \) (transforming them into \( C_l \cup \{x_l\} \)). But then, at that moment, all members of \( C_s, \ldots, C_{r-t} \) become free, so all their elements can be sent to the class containing \( B_1 \) (leaving singleton classes \( \{x_s\}, \ldots, \{x_{r-t}\} \) behind). The same can be done with all elements of \( B'_2, \ldots, B'_t \) (as they are obviously free), so we are done. \( \Box \)

This result means that each stationary pair is an isolated vertex in \( \mathcal{A}(\overline{D}_m, \overline{D}_r) \), and so we can immediately conclude that its vertex group is trivial. Other, non-stationary pairs are classified into connected components according to their type.
3.3 The degenerate cases (i.e. involving rank $n - 1$)

It is at this point that we are going to explain in full detail the “with few exceptions” disclaimer made at the end of the introduction. These exceptions arise because we can discard some of the pairs/vertices $(A, P)$ of contact graphs, as their vertex groups never appear in the course of computing the map $\theta$ (and thus deciding the word problem and computing Green’s relations in $\text{IG}(\mathcal{F}_n)$); so, some of these “superfluous” vertex groups will not be computed here.

Namely, whenever working with elements of $\text{IG}(\mathcal{F}_n)$, we assume that they are given via some of their minimal $r$-factorisations (and, as discussed, we can always routinely extract at least one such factorisation from a given word over $E(\mathcal{F}_n)$). So, a typical such element will be of the form

$$x = (P_1, g_1, A_1) \ldots (P_s, g_s, A_s),$$

and it will be of the $\mathcal{D}$-fingerprint $(\overline{D}_m_1, \ldots, \overline{D}_m_s)$. Such a setting entails that the product of any two consecutive factors above is a non-regular element of $\text{IG}(\mathcal{F}_n)$.

Bearing in mind Lemma 3.3, this means that for all pairs $(A_t, P_{t+1}), 1 \leq t < s$, we have that $A_t$ neither saturates $P_{t+1}$, nor is separated by it. So, whenever we are asked to compute $(\cdot, x, y)\theta$, the process, as can be amply seen, will never involve a vertex group $W(A, P)$ such that $A$ saturates $P$ or is separated by $P$. This motivates a definition: pairs $(A, P)$ with either of these two properties will be called regular; otherwise, they are non-regular.

Therefore, the conclusion is that the vertex groups of regular pairs $(A, P)$ are completely irrelevant for the process of computing the map $\theta$; it is only the vertex groups and their cosets of non-regular pairs that can possibly be interesting for us from the practical point of view. However, then we have the following result.

**Proposition 3.6** If either $m = n - 1$ or $r = n - 1$ then a vertex $(A, P)$ in the graph $\mathcal{G}(\overline{D}_m, \overline{D}_r)$ is non-regular if only if it is stationary. Consequently, in such a case, its vertex group is trivial.

**Proof** First let $m = n - 1$. Then $A = [1, n] \setminus \{i\}$ for some $i$. So, the pair $(A, P)$ is regular unless $\{i\}$ is a singleton class in $P$. But this is precisely the case when $(A, P)$ is stationary under the assumption $m = n - 1$. Similarly, assume now that $r = n - 1$. Then all classes of $P$ are singletons except one, which contains two elements, say $\{i, j\}$. Now the only way $(A, P)$ can be non-regular is that both $i$ and $j$ belong to $A$. But this is also precisely the condition that makes all the $P$-classes not containing elements of $A$ singletons.

Hence, if either $m = n - 1$ or $r = n - 1$, the non-regular pairs are all isolated vertices of their corresponding contact graphs. On the other hand, when $m = r = n - 1$, all regular pairs form a single connected component. Computing $W_{(A, P)}$ for regular pairs $(A, P)$ when one of $m, r$ is equal to $n - 1$ would involve dealing with the $2^{(n)}$ generators $f_{Q, B}$ (with $|Q| = |B| = n - 1$ and $B \perp Q$) of the presentation from [17] for the maximal subgroup of $\overline{D}_{n-1}$, delving into the combinatorial conditions which of these generators are equal to 1 according to this presentation (these are the only...
defining relations appearing in that presentation), and then performing tedious yet unnecessary computations of subgroups within groups of one of the forms \( F_k \times F_k \), \( F_k \times S_r \), and \( S_m \times F_k \) for \( k = n^2 - 1 \), whereas, from the standpoint of (computational) applications in the required context, the previous proposition is all we need.

So, in the remainder or the paper we may safely assume that \( m, r \leq n - 2 \), which eliminates the appearance of free groups. In that case, we however will compute the vertex groups \( W_{(A, P)} \) even for the regular pairs \((A, P)\), for the simple reason that the regular case turns out to be not one bit different from the non-regular one; said otherwise, (non-)regularity of the pair has no impact whatsoever on proving the general result.

### 3.4 Vertex groups for \( \mathbb{S}(D_m, D_r) \) when \( m, r \leq n - 2 \)

We start by discussing the group labels associated with a general edge in \( \mathbb{S}(D_m, D_r) \).

**Proposition 3.7** Let \( A, B \) be \( m \)-element subsets of \([1, n]\), and let \( P, Q \) be partitions of \([1, n]\) into \( r \) pieces such that there is an edge in \( \mathbb{S}(D_m, D_r) \) directed from \((A, P)\) to \((B, Q)\) and labelled by \( e \in E \). Then this edge carries the group label \((\pi, \pi')\), where \( \pi \in S_m \) and \( \pi' \in S_r \) are permutations such that

\[
bi\pi e = ai
\]

holds for all \( 1 \leq i \leq m \) (assuming that \( a_1 < \cdots < a_m \) and \( b_1 < \cdots < b_m \)), and

\[
Pj e^{-1} = Qj\pi'
\]

holds for all \( 1 \leq j \leq r \) (assuming that \( \min P_1 < \cdots < \min P_r \) and \( \min Q_1 < \cdots < \min Q_r \)).

**Proof** Let us begin by recalling that, in the general case, the label of an edge within the contact graph of two regular \( D \)-classes corresponding to \( e \in E \) is \((a, b^{-1})\), where \( a, b \) are given by Eqs. (2.1) and (2.2). Here, \( f^{(k)}_{i, k}, k = 1, 2, \) are the generators of the maximal subgroups in the two \( D \)-classes involved, appearing in the presentation for these groups as described in [16, Theorem 5]. In the concrete case, for the biorder of \( T_n \), these generators will be of the form \( f^{(k)}_{P, A} \), where for \( k = 1 \) the partition \( P \) and the subset \( A \) of \([1, n]\) are of cardinality \( m \) for \( k = 1 \), and of cardinality \( r \) for \( k = 2 \) (in both cases we must have \( A \perp P \)). As it is explained at the beginning of [17, Section 3] (outlining the plan of the proof of the main result of that paper), in the end, when the presentation in question (applied to \( T_n \)) is sorted out – and identified to be a presentation for a symmetric group – the generator \( f^{(k)}_{P, A} \) will represent the label \( \lambda(P, A) \), see Example 2.2, a permutation in \( S_m \) or \( S_r \), respectively. This fact will be crucially taken into account in the reminder of the proof.

Concerning \( \pi \), the first component of the label of the considered edge, let us begin by noting that if \( 1 \leq i, i' \leq m \) are such that \( b_{i'} e = a_i \), then \( a_i e = a_i \). Therefore, \( a_i \) and \( b_{i'} \) belong to the same (ker \( e \))-class, and, furthermore, each \( a_i \) is a fixed point of \( e \), so \( A \) is separated by ker \( e \). These observations suffice to justify the existence of a
partition \( P_0 \) of rank \( m \), coarser than \( \ker e \), separating \( A \) (and thus \( B \)). Bearing in mind (2.1) and the previous remarks, we now have

\[
\pi = \left( f_{P_0 \cdot A}^{(1)} \right)^{-1} f_{P_0 \cdot B}^{(1)} = \lambda(P_0, A)^{-1} \lambda(P_0, B).
\]

Assume now that the classes of \( P_0 \) are \( P_0^{(s)} \), \( 1 \leq s \leq m \), so that \( \min P_0^{(1)} < \cdots < \min P_0^{(m)} \). If we write \( \sigma = \lambda(P_0, A) \) and \( \tau = \lambda(P_0, B) \), then \( a_{\sigma \tau} \), \( b_{\sigma \tau} \in P_0^{(s)} \) holds for all \( 1 \leq s \leq m \). We have already argued that the action of \( e \), restricted to \( B \), maps each element of \( B \) into the elements of \( A \) in the same (\( \ker e \))-class (and thus in the same \( P_0 \)-class). The conclusion is that we have \( b_{\sigma \tau} e = a_{\sigma \tau} \) for all \( 1 \leq s \leq m \). Upon re-indexing \( i = s \sigma \), we arrive at

\[
b_{i \pi} e = b_{i \sigma^{-1} \tau} e = a_i
\]

holding for all \( 1 \leq i \leq m \), which is precisely what we wanted to show.

We proceed by discussing the second label, \( \pi' \). Now, we have \( \overline{e} \cdot P = Q \), and so \( \im e \) saturates \( P \) by Lemma 3.2. If \( l \) is an image point of \( e \) such that \( l \in P_j \) (for some \( 1 \leq j \leq r \)), then \( le = l \) and so \( l \in P_j e^{-1} \) as well, the latter inverse image being equal to \( Q_{j'} \) for some \( 1 \leq j' \leq r \). It follows from these considerations that there is a subset \( A_0 \subseteq \im e \) which is a joint transversal for both \( P \) and \( Q \). By (2.2),

\[
\pi' = \left( f_{Q, A_0}^{(2)} \left( f_{P, A_0}^{(2)} \right)^{-1} \right)^{-1} = \lambda(P, A_0) \lambda(Q, A_0)^{-1}.
\]

If we write \( A_0 = \{ a_1 < \cdots < a_r \} \), \( \sigma = \lambda(P, A_0) \), and \( \tau = \lambda(Q, A_0) \), then we have \( a_t \in P_{t \sigma^{-1}} \cap Q_{t \tau^{-1}} \) for all \( 1 \leq t \leq r \). Similarly as in the previous paragraph, corresponding \( P \)-classes and \( Q \)-classes (with respect to the action of \( \overline{e} \)) are identified by containing the same element of their joint transversal \( A_0 \). Hence, we have \( \overline{e} \cdot P_{t \sigma^{-1}} = P_{t \sigma^{-1}} e^{-1} = Q_{t \tau^{-1}} \) for all \( 1 \leq t \leq r \). Again, by re-indexing \( j = t \sigma^{-1} \), we obtain

\[
P_j e^{-1} = Q_{j \sigma^{-1}} = Q_{j \pi'},
\]

just as required. \( \square \)

Clearly, whenever we are presented with a pair of permutations \((\pi, \pi') \in S_m \times S_r\) and two pairs \((A, P), (B, Q)\) such that \( |A| = |B| = m \) and \( |P| = |Q| = r \), we can construct two bijections \( \phi_{\pi} : A \rightarrow B \) and \( \psi_{\pi'} : B \rightarrow Q \) (by defining \( a_i \phi_{\pi} = b_{i \pi} \) and \( (B, \psi_{\pi'}) : (A, P) \sim (B, Q) \) (and it is straightforward to see that any homeomorphism arises in this way, from a pair of permutations). By far the most important such situation for us is described in the following lemma.
Lemma 3.8 If \((\pi, \pi') \in S_m \times S_r\) labels an edge \((A, P) \rightarrow (B, Q)\) in \(\mathcal{A}(D_m, D_r)\), then \((\phi\pi, \psi\pi')\) is a homeomorphism between the pairs involved. Consequently, the same conclusion holds for the group label of any walk in \(\mathcal{A}(D_m, D_r)\) and its endpoints.

**Proof** Proving this lemma amounts to showing that the following equivalence holds

\[ a_i \in P_j \text{ if and only if } b_{i\pi} \in Q_{j\pi'} \]

However, this is now routine, bearing in mind the previous proposition. Also, the second part of the lemma follows easily from the fact that the group label along any walk is the product of labels of its edges, as well as the fact that \(\phi\pi_1\pi_2 = \phi\pi_1\phi\pi_2\) and \(\psi\pi_1\pi_2' = \psi\pi_1\psi\pi_2'\) holds for any \(\pi_1, \pi_2 \in S_m, \pi_1', \pi_2' \in S_r\). \(\square\)

In particular, in the case when \((B, Q) = (A, P)\), a homeomorphism \((\phi, \psi)\) of \((A, P)\) to itself is called an auto-homeomorphism of \((A, P)\). A direct consequence of the previous lemma reads as follows.

**Corollary 3.9** The group label of any loop (closed walk) based at \((A, P)\) gives rise to an auto-homeomorphism of \((A, P)\).

Formulated in a descriptive way, it is pretty clear what an auto-homeomorphism of a pair \((A, P)\) does: it permutes the partition classes containing the same number of elements of \(A\), and then establishes bijections between elements of \(A\) belonging to the corresponding partition classes. If the auto-homeomorphism in question arose from the pair \((\pi, \pi') \in S_m \times S_r\), then this first permutation is completely determined by \(\pi'\): it is only subject to the restriction that we must have \(|P_j| = |P_{j\pi'}|\) for all \(1 \leq j \leq r\). Then, to choose \(\pi\) (which is in fact the totality of the bijections between the intersections of \(A\) with the classes of \(P\)), we need to comply with the condition that \(a_{i\pi} \in P_{j\pi'}\), whenever \(a_i \in P_j\), for all \(1 \leq i \leq m\) and \(1 \leq j \leq r\). So, basically, \(\pi\) is a permutation of \(A\) preserving a partition that is induced on it by \(P\). Now let \(\text{AHom}(A, P)\) denote the subgroup of \(S_m \times S_r\) consisting of all pairs \((\pi, \pi')\) inducing an auto-homeomorphism of \((A, P)\) in the described way (it is routine to show that such pairs indeed form a group). This leads us to the concrete description of this permutation group.

**Proposition 3.10** Let \(A\) be an \(m\)-element subset of \([1, n]\), and let \(P\) be a partition of \([1, n]\) into \(r\) classes, such that \(m_s, 1 \leq s \leq k\), denote the distinct sizes of non-empty intersections \(A \cap P_j, 1 \leq j \leq r\), \(\mu_s, 1 \leq s \leq k\), denotes the number of these intersections of size \(m_s\), and \(v\) denotes the number of empty intersections.

(i) The first projection \(\Gamma_1\) of \(\text{AHom}(A, P)\) (that is, the range of first components \(\pi\)) is isomorphic to the direct product of wreath products

\[ (S_{m_1} \wr S_{\mu_1}) \times \cdots \times (S_{m_k} \wr S_{\mu_k}). \]

(ii) The second projection \(\Gamma_2\) of \(\text{AHom}(A, P)\) (i.e. the range of second components \(\pi'\)) is isomorphic to the direct product

\[ S_{\mu_1} \times \cdots \times S_{\mu_k} \times S_v. \]
(iii) For \( \pi \in \Gamma_1 \), let \( \overline{\pi} \) be the permutation on the set

\[
J_{(A, P)} = \{ j \in [1, r] : A \cap P_j \neq \emptyset \}
\]

uniquely determined by \( \pi \) by \( j \overline{\pi} = j' \) if and only if \( a_{ij} \in P_j \) for some (and thus any) \( i \in [1, m] \) such that \( a_i \in P_j \). Then \( (\pi, \pi') \in \text{AHom}(A, P) \) if and only if \( \pi' = \pi \oplus \pi'' \), where \( \pi'' \) is any permutation of the set \([1, r] \setminus J_{(A, P)}\) (describing the part of \( \pi' \) corresponding to the permutation of \( P \)-classes not intersecting \( A \)).

**Proof** First of all, if \( (\pi, \pi') \in \text{AHom}(A, P) \) then \( \phi_\pi \) preserves the partition that \( P \) induces on \( A \): indeed, \( a_i, a_{i'} \in P_j \) for some \( 1 \leq i, i' \leq m, 1 \leq j \leq r \), if and only if \( a_i \pi, a_{i'} \pi' \in P_j \pi' \). Furthermore, by the compatibility condition just invoked, we must have \( \pi' = \pi \oplus \pi'' \) for some permutation \( \pi'' \) of the set \([1, r] \setminus J_{(A, P)}\). Conversely, if \( \pi \in \mathbb{S}_m \) is any permutation inducing a \( P \)-preserving permutation of \( A \), then it is straightforward to see that

\[
(\pi, \pi \oplus \pi'') \in \text{AHom}(A, P)
\]

for any permutation \( \pi'' \) of the set \([1, r] \setminus J_{(A, P)}\). Note that these remarks already show (iii). The statement (i) also follows immediately, as the structure of the group of partition-preserving permutations is well known, see [1, Lemma 2.1].

Also, we already know that any \( \pi' \in \Gamma_2 \) permutes the partition classes of \( P \) whose intersection with \( A \) have the same cardinality. Conversely, if \( \pi' \) is any such permutation, it is easy to construct a permutation \( \pi \in \Gamma_1 \) such that \( \pi' = \pi \oplus \pi'' \), where \( \pi'' \) is the restriction of \( \pi' \) to the classes not containing any element of \( A \) (for example, let \( \pi \) be the union of all monotone bijections \( A \cap P_j \mapsto A \cap P_{j' \pi'} \), \( 1 \leq j \leq r \)). Hence, (ii) follows. \( \Box \)

**Remark 3.11** In the notation introduced in Subsect. 2.4 (and then crucially used in Subsect. 2.6), the statement (iii) from the previous proposition can be expressed as follows. If \( \rho = \text{AHom}(A, P) \subseteq \mathbb{S}_m \times \mathbb{S}_r \) is considered as a relation, then

\[
\pi \varphi_\rho = \{ \pi \oplus \sigma : \sigma \in \mathbb{S}_{[1, r] \setminus J_{(A, P)}} \}.
\]

It is immediately seen that the latter set is just a coset of \( \text{Stab}(J_{(A, P)}) \), the pointwise stabiliser of \( J_{(A, P)} \) (which is isomorphic to \( \mathbb{S}_r \)), corresponding e.g. to \( \overline{\pi} \oplus \text{id}_{[1, r] \setminus J_{(A, P)}} \).

Notice that the previous Corollary 3.9 can be now reformulated in the following way.

**Lemma 3.12** For any vertex \( (A, P) \) of the graph \( \mathcal{A}(\overline{D}_m, \overline{D}_r) \) we have \( W_{(A, P)} \leq \text{AHom}(A, P) \).

However, our aim is to prove that, unless \( (A, P) \) is a stationary pair, equality holds in the previous lemma. This is actually the principal result of this paper.
Theorem 3.13 If the vertex \((A, P)\) is not stationary in \(\mathcal{A}(\overline{D}_m, \overline{D}_r)\), then its vertex group \(W_{(A, P)}\) coincides with \(\text{AHom}(A, P)\), the auto-homeomorphism group of \((A, P)\). Otherwise, \(W_{(A, P)}\) is trivial.

Bearing in mind the preceding lemma, the strategy for the proof of this theorem is first to identify the generators of \(\text{AHom}(A, P)\) (taking Proposition 3.10 into account), and then (in the non-stationary case), for each of these generators, constructing a loop in \(\mathcal{A}(\overline{D}_m, \overline{D}_r)\) based at \((A, P)\) whose group label is precisely the generator in question.

Lemma 3.14 Let \(A\) be an \(m\)-element subset of \([1, n]\) and let \(P\) be a partition of \([1, n]\) into \(r\) classes. Let \(\tau_{\alpha, \beta}\) denote the transposition of points \(\alpha, \beta\). Then \(\text{AHom}(A, P)\) is generated by the following elements:

(i) for any two \(1 \leq i \neq i' \leq m\) such that \(a_i, a_{i'}\) belong to the same \(P\)-class, the pairs

\[(\tau_{i, i'}, \text{id}_r);\]

(ii) for any two \(1 \leq j \neq j' \leq r\) such that \(|P_j| = |P_{j'}| = q \neq 0\), the pairs

\[(\tau_{i_1, i_1'}, \ldots, \tau_{i_q, i_q'}, \tau_{j, j'}),\]

where \(A \cap P_j = \{a_{i_1} < \cdots < a_{i_q}\}\) and \(A \cap P_{j'} = \{a_{i_1}' < \cdots < a_{i_q}'\}\);

(iii) for any two \(1 \leq j \neq j' \leq r\) such that \(A \cap P_j = A \cap P_{j'} = \emptyset\), the pairs

\[(\text{id}_m, \tau_{j, j'}).\]

Proof Firstly, it is clear that all the listed pairs indeed belong to \(\text{AHom}(A, P)\). Conversely, assume that \((\pi, \pi') \in \text{AHom}(A, P)\). Then, by Proposition 3.10(iii), for some permutation \(\pi''\) of \(N = [1, r] \setminus J_{(A, P)}\) we have

\[(\pi, \pi') = (\pi, \overline{\pi} \oplus \pi'') = (\pi, \overline{\pi} \oplus \text{id}_N)(\text{id}_m, \text{id}_{J_{(A, P)}} \oplus \pi'').\]

It is immediately clear that the second factor on the right-hand side is generated by pairs of type (iii). As for the first factor, it is generated by pairs of the form

\[(\gamma, \overline{\gamma} \oplus \text{id}_N),\]

where \(\gamma\) runs through the generating set of \(\Gamma_1\). This is so because the pairs of the form \((\pi, \overline{\pi} \oplus \text{id}_N)\) form a subgroup of \(\text{AHom}(A, P)\) isomorphic to \(\Gamma_1\), as the bar mapping is a group homomorphism of \(\Gamma_1\) into \(\Gamma_2: \overline{\pi_1} \overline{\pi_2} = \overline{\pi_1 \pi_2}\) holds for all \(\pi_1, \pi_2 \in \Gamma_1\). However, we already know the structure of \(\Gamma_1\), from Proposition 3.10(i): it is a direct product of wreath products of symmetric groups. The standard knowledge on generating sets of wreath products, considered as semidirect products, see [24], immediately implies the result of the lemma. \(\square\)
We may now proceed by proving our main theorem.

**Proof of Theorem 3.13** We begin with an observation that will greatly simplify the arguments in the remainder of the proof. Namely, the statement of Proposition 3.7 extends to the labels of arbitrary walks, in the following sense. The next result is easily verified by repeated applications of this proposition.

**Claim.** Let \((A, P)\) and \((B, Q)\) be two vertices in \(\mathcal{A}(D_m, D_r)\) connected by a walk

\[(A, P) \rightarrow (A_1, P^{(1)}) \rightarrow \ldots \rightarrow (A_{k-1}, P^{(k-1)}) \rightarrow (B, Q),\]

where the edges correspond to idempotent transformations \(e_1, \ldots, e_k \in E(\mathcal{T}_n)\), with labels \((\pi_s, \pi'_s), 1 \leq s \leq k\), respectively. Let \(f = e_k \ldots e_1 \in \mathcal{T}_n\). Then the label \((\pi, \pi') = (\pi_1, \pi'_1) \ldots (\pi_k, \pi'_k)\) of this walk is (uniquely) determined by the conditions

\[b_i \pi f = a_i \quad \text{and} \quad P_j f^{-1} = Q_j \pi',\]

for all \(1 \leq i \leq m\) and \(1 \leq j \leq r\).

When \((B, Q) = (A, P)\), this applies to closed walks, too. So, our aim is to exhibit, for each generator of \(\text{AHom}(A, P)\) listed in Lemma 3.14, a closed walk based at (a non-stationary pair) \((A, P)\) that corresponds to the considered generator in the sense of the previous claim. This will then complete the proof that \(W_{(A, P)} = \text{AHom}(A, P)\). We opt rather to present the moves corresponding to edges of these walks in a descriptive, combinatorial manner than to write out the sequences of steps (and calculate the labels) formally – as this would obscure a great deal the essentially simple ideas behind these constructions.

1. **Generators of the form** \((\tau_{i,i'}, \text{id}_r)\) where \(a_i, a_i' \in P_j\) for some \(i, i', j\). Here, we should present a sequence of steps (1),(2) – as specified in Subsect. 3.2 – starting and ending with the pair \((A, P)\), such that the resulting transformation \(f\) switches \(a_i\) and \(a_i'\) and leaves all the other elements of \(A\) intact. Along the way, we assume that \(p \in [1, n] \setminus A\) is a free point for \((A, P)\) (such point exists as \((A, P)\) is assumed to be not stationary). To allow easier tracking of the movement of points, let us call the initial point \(a_i\) red, and \(a_i'\) blue. Since \(p\) is free, we can remove it from its \(P\)-class, say \(P_{j'}\) and add it to \(P_j\) (unless it is already there, in case \(j' = j\)) – this is a move of type (2). Now we can use \(p\) to perform the switch within \(P_j\) by a sequence of moves of type (1): move the red point from \(a_i\) to \(p\), then the blue point from \(a_i'\) to \(a_i\), and finally the red point from \(p\) to \(a_i'\). At the end, if \(j' \neq j\), since \(p\) is free at this moment, we may apply (2) to move it back to \(P_{j'}\).

2. **Generators of the form** \((\tau_{i_1,i'_1} \ldots \tau_{i_q,i'_q}, \tau_{j,j'})\) where the classes \(P_{j_1}, P_{j'}\) are of the same cardinality, and the elements of \(A\) contained in them are switched in a monotone manner. In other words, we should find the way to switch the classes \(P_j\) and \(P_{j'}\) and their elements \(a_{i_t}, a_{i'_t}, 1 \leq t \leq q\), respectively. Again, as in the previous case, let \(p\) be a free point with respect to \((A, P)\), and, for tracking purposes, let us call the class \(P_j\), as well as its elements belonging to \(A\) red, and the class \(P_{j'}\) and its elements from \(A\) blue. We start by adding \(p\) to \(P_j\) (a move of type (2)). Then, we proceed by moving the first red point (currently at \(a_{i_1}\)) to \(p\). In this moment, \(a_{i_1}\) becomes a free point in the red class, so we may move it (by (2)) to the blue class. After this, we move the first
blue point from \( a_i' \) to \( a_i \). Now, \( a_i' \) becomes a free point in the blue class, so we move it to the red class, and, subsequently, move the red point currently at \( p \) to \( a_i' \). The cumulative effect of this part of the process is that \( a_i \) became a blue point belonging to the blue class, and \( a_i' \) a red point belonging to the red class (and \( p \) remained in the red class, being a free point again). But it is now clear that in the same fashion this process can be repeated for the (red-blue) pairs \((a_{i_2}, a_{i_2}')\), \( \ldots \), \((a_{i_q}, a_{i_q}')\). At the very end, as \( q > 0 \), all points of the involved classes \( P_j, P_j' \) not belonging to \( A \) are free, so they might be freely exchanged between the red and blue classes; also, \( p \) might be returned to the \( P \)-class it initially belonged to (as it is free at the end of the described process). Thus we again arrive at the pair \((A, P)\) with the only difference that now \( P_j \) is blue and \( P_j' \) red, and the pairs of their \( A \)-points \( a_{i_t}, a_{i_t}' \), \( 1 \leq t \leq q \), exchanged colours. This accounts for the required label.

Our theorem is now proved. \(\Box\)

It remains to comment on the choice of coset representatives \((g_k, h_k)\) in the context of the word problem for \( \text{IG}(ET_n) \) and, more generally, in the course of computing the map \( \theta \). As explained in Subsect. 2.4, for these it suffices to choose the label of any walk \((A_k, P_{k+1}) \sim (B_k, Q_{k+1})\) whenever such a walk exists. If it does, we have already argued in this paper that then we must have \((A_k, P_{k+1}) \sim (B_k, Q_{k+1})\) and so the label associated to any homeomorphism will do. Such a homeomorphism is very easy to compute: for the two considered pairs, one needs to match up partition classes of \( P_{k+1} \) and \( Q_{k+1} \) containing the same number of elements from \( A_k \) and \( B_k \), respectively, and then to (arbitrarily) choose bijections between the elements of these sets from matching classes.

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