A generalized Alon-Boppana bound and weak Ramanujan graphs

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Abstract

A basic eigenvalue bound due to Alon and Boppana holds only for regular graphs. In this paper we give a generalized Alon-Boppana bound for eigenvalues of graphs that are not required to be regular. We show that a graph \( G \) with diameter \( k \) and vertex set \( V \), the smallest nontrivial eigenvalue \( \lambda_1 \) of the normalized Laplacian \( \mathcal{L} \) satisfies

\[ \lambda_1 \leq 1 - \sigma \left( 1 - \frac{c}{k} \right) \]

for some constant \( c \) where \( \sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2 \) and \( d_v \) denotes the degree of the vertex \( v \).

We consider weak Ramanujan graphs defined as graphs satisfying \( \lambda_1 \geq 1 - \sigma \). We examine the vertex expansion and edge expansion of weak Ramanujan graphs and then use the expansion properties among other methods to derive the above Alon-Boppana bound.

1 Introduction

The well-known Alon-Boppana bound [8] states that for any \( d \)-regular graph with diameter \( k \), the second largest eigenvalue \( \rho \) of the adjacency matrix satisfies

\[ \rho \geq 2 \sqrt{d - 1} \left( 1 - \frac{2}{k} \right) - \frac{2}{k}, \tag{1} \]

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A natural question is to extend Alon-Boppana bounds for graphs that are irregular. Hoory [6] showed that for an irregular graph, the second largest eigenvalue $\rho$ of the adjacency matrix satisfies
\[
\rho \geq 2\sqrt{d-1} \left( 1 - \frac{c \log r}{r} \right)
\]
if the average degree of the graph after deleting a ball of radius $r$ is at least $d$ where $r, d > 2$.

For irregular graphs, it is often advantageous to consider eigenvalues of the normalized Laplacian for deriving various graph properties. For a graph $G$, the normalized Laplacian $\mathcal{L}$, defined by
\[
\mathcal{L} = I - D^{-1/2} A D^{-1/2}
\]
where $D$ is the diagonal degree matrix and $A$ denotes the adjacency matrix of $G$. One of the main tools for dealing with general graphs is the Cheeger inequality which relates the least nontrivial eigenvalue $\lambda_1$ to the Cheeger constant $h_G$:
\[
2h_G \geq \lambda_1 \geq h_G^2
\]
where $h_G = \min_S |\partial(S)|/\text{vol}(S)$ for $S$ ranging over all vertex subsets with volume $\text{vol}(S) = \sum_{u \in S} d_u$ no more than half of $\sum_{u \in V} d_u$ and $\partial(S)$ denotes the set of edges leaving $S$. For $k$-regular graphs, we have $\lambda_1 = 1 - \rho/k$ where $\rho$ denotes the second largest eigenvalue of the adjacency matrix. In general,
\[
\frac{\rho}{\max_v d_v} \leq 1 - \lambda_1 \leq \frac{\rho}{\min_v d_v}
\]
which can be used to derive a version of the Cheeger inequality involving $\rho$ which is less effective than (2) for irregular graphs.

In this paper, we will show that for a connected graph $G$ with diameter $k$, $\lambda_1$ is upper bounded by
\[
\lambda_1 \leq 1 - \sigma (1 - \frac{c}{k})
\]
for a constant $c$ where $\sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2$. The above inequality will be proved in Section 6.

The above bound of Alon-Boppana type improves a result of Young [10] who derived a similar eigenvalue bound using a different method. In [10] the notion of $(r, d, \delta)$-robust graphs was considered and it was shown that for a $(r, d, \delta)$-robust graph, the least nontrivial eigenvalue $\lambda_1$ satisfies
\[
\lambda_1 \leq 1 - \frac{2d \sqrt{d-1}}{\delta} \left( 1 - \frac{c}{r} \right).
\]
Here $(r, d, \delta)$-robustness means for every vertex $v$ and the ball $B_r(v)$ consisting of all vertices with distance at most $r$, the induced subgraph on the complement of $B_r(v)$ has
average degree at least \(d\) and \(\sum_{v \in B_r(v)} d_v^2/|V \setminus B_r(v)| \leq \delta\). We remark that our result in (3) does not require the condition of robustness.

We define weak Ramanujan graphs to be graphs with eigenvalue \(\lambda_1\) satisfying

\[
\lambda_1 \geq 1 - \sigma \geq \frac{1}{2}
\]

(5)

where \(\sigma = 2 \sum_v d_v \sqrt{d_v - 1}/\sum_v d_v^2\).

To prove the Alon-Boppana bound in (3), it suffices to consider only weak Ramanujan graphs. Weak Ramanujan graphs satisfy various expansion properties. We will describe several vertex-expansion and edge-expansion properties involving \(\lambda_1\) in Section 3, which will be needed later for proving a diameter bound for weak Ramanujan graphs in Section 4. The diameter bound and related properties of weak Ramanujan graphs are useful in the proof of the Alon-Boppana bound for general graphs.

We will also show that the largest eigenvalue \(\lambda_{n-1}\) of the normalized Laplacian satisfies

\[
\lambda_{n-1} \geq 1 + \sigma(1 - \frac{c}{k}).
\]

(6)

The proof will be given in Section 7.

2 Preliminaries

For a graph \(G = (V, E)\), we consider the normalized Laplacian

\[
\mathcal{L} = I - D^{-1/2}AD^{-1/2}
\]

where \(A\) denotes the adjacency matrix and \(D\) denotes the diagonal degree matrix with \(D(v, v) = d_v\), the degree of \(v\). We assume that there is no isolated vertex throughout this paper. For a vertex \(v\) and a positive integer \(l\), let \(B_l(v)\) denote the ball consisting of all vertices within distance \(l\) from \(v\). For an edge \(\{x, y\} \in E\) we say \(x\) is adjacent to \(y\) and write \(x \sim y\).

Let \(\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}\) denote eigenvalues of \(\mathcal{L}\), where \(n\) denotes the number of vertices in \(G\). It can be checked (see [2]) that \(\lambda_1 > 0\) if \(G\) is connected. The Alon-Boppana bound obviously holds if \(\lambda_1 = 0\). In the remainder of this paper, we assume \(G\) is connected.

Let \(\varphi_i\) denote the orthonormal eigenvector associated with eigenvalue \(\lambda_i\). In particular, \(\varphi_0 = D^{1/2}1/\sqrt{\text{vol}(G)}\) where \(1\) is the all 1’s vector and \(\text{vol}(G) = \sum_{v \in V} d_v\). We can then write

\[
\lambda_1 = \inf_{g \perp \varphi_0} \frac{\langle g, \mathcal{L}g \rangle}{\langle g, g \rangle} = \inf_{f \perp D^1} \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum z f^2(z) d_z} = \inf_{f \perp D^1} R(f)
\]

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where $f$ ranges over all functions satisfying $\sum_u f(u)d_u = 0$ and the sum $\sum_{x\sim y}$ ranges over all unordered pairs $\{x, y\}$ where $x$ is adjacent to $y$. Here $R(f)$ denote the Rayleigh quotient of $f$, which can be written as follows:

$$R(f) = \frac{\int |\nabla f|}{\int \|f\|^2}$$

where $\int \|f\|^2 = \sum_x f^2(x)dx$

and $\int |\nabla f| = \sum_{x\sim y} (f(x) - f(y))^2$.

For eigenfunction $\phi_i$, the function $f_i = D^{-1/2}\phi_i$, called the combinatorial eigenfunction associated with $\lambda_i$, satisfies

$$\lambda_i f(u)d_u = \sum_{v\sim u} (f(u) - f(v))$$

for each vertex $u$. In particular, for $f$ satisfying $\sum_u f(u)d_u = 0$, we have

$$\langle f, Af \rangle \leq (1 - \lambda_1)\langle f, Df \rangle$$

and

$$|\langle f, Af \rangle| \leq \max_{i \neq 0} (1 - \lambda_i)\langle f, Df \rangle.$$
Lemma 1. Let $S$ be a subset of vertices in $G$. Then

$$\frac{\lvert \partial(S) \rvert}{\text{vol}(S)} \geq \lambda_1 \left(1 - \frac{\text{vol}(S)}{\text{vol}(G)}\right).$$

Proof. Suppose $f$ is defined by

$$f = \frac{1_S}{\text{vol}(S)} - \frac{1_{\overline{S}}}{\text{vol}(\overline{S})}$$

where $1_S$ denotes the characteristic function defined by $1_S(v) = 1$ if $v \in S$ and 0 otherwise.

The Rayleigh quotient $R(f)$ satisfies

$$\lambda_1 \leq R(f) = \frac{\lvert \partial(S) \rvert}{\text{vol}(S)} \cdot \frac{\text{vol}(G)}{\text{vol}(\overline{S})}.$$

For the expansion of the vertex boundary, the Tanner bound [9] for regular graphs can be generalized as follows.

Lemma 2. Let $\lambda = \min_{\lambda_i \neq 0} |1 - \lambda_i|$. Then for any vertex subset $S$ in a graph,

$$\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{1 - \lambda^2}{\lambda^2 + \frac{\text{vol}(S)}{\text{vol}(G)}}. \tag{10}$$

The proof of the above inequality is by using the following discrepancy inequality (as seen in [2]).

Lemma 3. In a graph $G$, for two subset $X$ and $Y$ of vertices, the number $e(X, Y) = \lvert E(X, Y) \rvert$ of edges between $X$ and $Y$ satisfies

$$\left| e(X, Y) - \frac{\text{vol}(X) \text{vol}(Y)}{\text{vol}(G)} \right| \leq \bar{\lambda} \sqrt{\frac{\text{vol}(X) \text{vol}(Y) \text{vol}(\overline{X}) \text{vol}(\overline{Y})}{\text{vol}(G)}} \tag{11}$$

where $\bar{\lambda} = \min_{\lambda_i \neq 0} |1 - \lambda_i|$. The proof of Lemma 3 follows from (9) and can be found in [2]. The proof of (12) results from (11) by setting $X = S$ and $Y = S \cup \delta(S)$.

Here we will give a version of the vertex-expansion bounds for general graphs which only rely on $\lambda_1$ and are independent of other eigenvalues.

Lemma 4. In a graph $G$ with vertex set $V$ and the first nontrivial eigenvalue $\lambda_1$, for a subset $S$ of $V$ with $\text{vol}(S \cup \delta(S)) \leq \text{evol}(G) \leq \text{vol}(G)/2$, the vertex boundary of $S$ satisfies

(i) $$\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{2\lambda_1}{1 - \lambda_1 + 2\epsilon}. \tag{12}$$

(ii) If $1/2 \leq \lambda_1 \leq 1 - 2\epsilon$, then

$$\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{1}{(1 - \lambda_1 + 2\epsilon)^2}. \tag{13}$$
Proof. The proof of (i) follows from Lemma 1 since
\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{|\partial(S \cup \delta(S))| + |\partial(S)|}{\text{vol}(S)} \\
\geq \frac{\lambda_1(1-\epsilon)(\text{vol}(S) + \text{vol}(\delta(S))) + \lambda_1(1-\epsilon)\text{vol}(S)}{\text{vol}(S)}
\]
Therefore
\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{2\lambda_1(1-\epsilon)}{1 - \lambda_1(1-\epsilon)} \geq \frac{2\lambda_1}{1 - \lambda_1 + 2\epsilon}
\]
To prove (ii), we set \( f = 1_S + \gamma 1_{\delta(S)} \) where \( \gamma = 1 - \lambda_1 \). Consider \( g = f - c 1_V \) where \( c = \sum_u f(u) d_u / \text{vol}(G) \). By the Cauchy-Schwarz inequality, we have
\[
c^2 = \frac{1}{(\text{vol}(G))^2} \left( \sum_{u \in S \cup \delta(S)} f(u) d_u \right)^2 \leq \frac{\text{vol}(S \cup \delta(S))}{(\text{vol}(G))^2} \sum_u f^2(u) d_u \\
\leq \frac{\epsilon}{\text{vol}(G)} \sum_u f^2(u) d_u.
\]
Using the inequality in (8), we have
\[
\langle f, Af \rangle \leq \langle g, Ag \rangle + c^2 \text{vol}(G) \\
\leq \gamma \langle g, Dg \rangle + c^2 \text{vol}(G) \\
= \gamma \langle f, Df \rangle + (1 - \gamma) c^2 \text{vol}(G) \\
\leq (\gamma + \epsilon) \langle f, Df \rangle \\
= (\gamma + \epsilon) (\text{vol}(S) + \gamma^2 \text{vol}(\delta(S))).
\]
Let \( e(S, T) \) denote the number of ordered pairs \( (u, v) \) where \( u \in S, v \in T \) and \( \{u, v\} \in E \). Since \( \gamma = 1 - \lambda \leq 1/2 \), we have
\[
\langle f, Af \rangle \geq e(S, S) + 2\gamma e(S, \delta(S)) \\
\geq (1 - 2\gamma) e(S, S) + 2\gamma \text{vol}(S) \\
\geq 2\gamma \text{vol}(S)
\]
Together we have
\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{\gamma - \epsilon}{\sigma^2(\gamma + \epsilon)} \\
\geq \frac{1}{(\gamma + 2\epsilon)^2}
\]
since \( \gamma \geq 2\epsilon \).
\[\square\]
Recall that weak Ramanujan graphs have eigenvalue \( \lambda_1 \) satisfying
\[
\lambda_1 \geq 1 - \sigma
\]  
(14)
where \( \sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2 \). Lemma 1 implies that for \( S \) with \( \text{vol}(S \cup \delta(S)) \leq \epsilon \text{vol}(G) \),
\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} \geq \frac{1}{(\sigma + 2\epsilon)^2}.
\]

For \( k \)-regular Ramanujan graphs with eigenvalue \( \lambda_1 = 1 - 2\sqrt{k-1}/k \), the above inequality is consistent with the bound
\[
\frac{\text{vol}(\delta(S))}{\text{vol}(S)} = \frac{|\delta(S)|}{|S|} \geq \frac{1}{(\frac{2\sqrt{k-1}}{k} + 2\epsilon)^2}
\]
which is about \( k/4 \) when \( \text{vol}(S) \) is small. The factor \( k/4 \) in the above inequality was improved by Kahale [4] to \( k/2 \). There are many applications (see [1]) that require graphs having expansion factor to be \( (1 - \epsilon)k \). Such graphs are called lossless expanders. In [1], lossless graphs were constructed explicitly by using the zig-zag construction but the method for deriving the expansion bounds does not use eigenvalues. In this paper, the expansion factor as in Lemma 4 is enough for our proof later.

4 Weak Ramanujan graphs

We recall that a graph is said to be a weak Ramanujan graph as in (14) if
\[
\lambda_1 \geq 1 - \sigma \geq \frac{1}{2}
\]
where
\[
\sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2.
\]  
(15)
To prove the Alon-Boppana bound, it is enough to consider only weak Ramanujan graphs.

Lemma 5. As defined in (15), \( \sigma \) satisfies
\[
\frac{2\sqrt{\bar{d} - 1}}{\bar{d}} \leq \sigma \leq \frac{2\sqrt{\bar{d} - 1}}{\bar{d}}
\]
where \( \bar{d} \) denotes the average degree in \( G \) and \( \bar{d} \) denote the second order degree, i.e.,
\[
\bar{d} = \frac{\sum_v d_v}{n} \quad \text{and} \quad \bar{d} = \frac{\sum_v d_v^2}{\sum_v d_v}.
\]
Proof. The proof is mainly by using the Cauchy-Schwarz inequality. For the upper bound, we note that

$$\sigma = 2 \frac{\sum_v d_v \sqrt{d_v} - 1}{\sum_v d_v^2} \leq 2 \frac{\sqrt{\sum_v d_v^2 \sum_v (d_v - 1)}}{\sum_v d_v^2}$$

$$= 2 \frac{\sqrt{\sum_v (d_v - 1)}}{\sqrt{\sum_v d_v^2}}$$

$$\leq 2 \frac{\sqrt{\sum_v (d_v - 1)}}{\sum_v d_v/\sqrt{n}}$$

$$\leq 2 \frac{\sqrt{\sum_v (d_v - 1)}}{\frac{d}{\sqrt{n}}} \leq 2 \frac{\sqrt{d-1}}{d}.$$

For the upper bound, we will use the fact that for $a, b > 1$ and $a + b = c$,

$$a \sqrt{a - 1} + b \sqrt{b - 1} \geq c \sqrt{\frac{c}{2} - 1}$$

and therefore

$$\sum_v d_v \sqrt{d_v - 1} \geq \sum_v d_v \sqrt{\frac{\sum_v d_v}{n} - 1}.$$ 

Consequently, we have

$$\sigma = 2 \frac{\sum_v d_v \sqrt{d_v} - 1}{\sum_v d_v^2} \geq 2 \frac{\sum_v d_v \sqrt{\frac{\sum_v d_v}{n} - 1}}{\sum_v d_v^2 \sum_v d_v} \geq 2 \frac{\sqrt{d-1}}{d}$$

as desired. \qed

We remark that for graphs with average degree at least 20, we have $\sigma < 1/2 < \lambda_1$.

**Theorem 6.** Suppose a weak Ramanujan graph $G$ has diameter $k$. Then for any $\epsilon > 0$, we have

$$k \leq (1 + \epsilon) \frac{2 \log \text{vol}(G)}{\log \sigma^{-1}}$$

provided that the volume of $G$ is large, i.e., $\text{vol}(G) \geq c \sigma^\log(\sigma)/\epsilon$ for some small constant $c$.

**Proof.** We set

$$t = \lceil (1 + \epsilon) \frac{\log(\text{vol}(G))}{\log \sigma^{-1}} \rceil.$$ 

It suffices to show that for every vertex $v$, the ball $B_t(v)$ has volume more than $\text{vol}(G)/2$.

Suppose $\text{vol}(B_t(v)) \leq \text{vol}(G)/2$. Let

$$s_j = \frac{\text{vol}(B_j(u))}{\text{vol}(G)}.$$
By part (i) of Lemma 4, we have $\text{vol}(\delta(B_u(j))) \geq 0.5\text{vol}(B_u(j))$ for $j \leq t - 1$ and therefore $s_{j+1} \geq 1.5s_j$. Thus, if $j \leq t - c_1 \log(\sigma^{-1})$, then $s_j \leq \sigma^4$ where $c_1$ is some small constant satisfying $c_1 \leq 4(\log 1.5)^{-1}$.

Now we apply part (ii) of Lemma 4 and we have, for $j \leq t - c_1 \log(\sigma^{-1})$,

$$\frac{s_{j+1}}{s_j} = \frac{\text{vol}(B_{j+1}(u))}{\text{vol}(B_j(u))} \geq \frac{\text{vol}(\delta(B_j(u)))}{\text{vol}(B_j(u))} \geq \frac{1}{(\sigma + 2s_j)^2} \geq \frac{1}{(\sigma + 2\sigma^4)}.$$ 

This implies, for $l \leq t - c_1 \log(\sigma^{-1})$,

$$\frac{s_l}{s_0} \geq \prod_{0 < j < l} \frac{1}{(\sigma + 2s_j)^2} \geq \prod_{0 < j < l} \frac{1}{(\sigma + 2\sigma^4)^2} \geq \frac{1}{\sigma^{2l}(1 + 2\sigma^4)^{2l}}.$$ 

Since $s_0 \geq 1/\text{vol}(G)$ and $s_t \leq s_t \leq 1/2$, we have

$$\text{vol}(G) \geq \frac{1}{\sigma^{2l}(1 + 2\sigma^4)^{2l}}.$$ 

Hence

$$l \leq \frac{\log(\text{vol}(G))}{\log(\sigma^{-1}) + 2\sigma^4}.$$ 

However,

$$(1 + \epsilon)\frac{\log(\text{vol}(G))}{\log(\sigma^{-1})} \leq t \leq c_1 \log(\sigma^{-1}) + \frac{\log(\text{vol}(G))}{\log(\sigma^{-1}) + 2\sigma^4}$$

which is a contradiction for $G$ with $\text{vol}(G)$ large, say, $\text{vol}(G) \geq \sigma^{2c_1 \log \sigma} / \epsilon$. Thus we conclude that $s_t \geq 1/2$ and Theorem 6 is proved. 

**Theorem 7.** For a weak Ramanujan graph with diameter $k$, for any vertex $v$ and any $l \leq k/4$, the ball $B_v(l)$ has volume at most $\epsilon \text{vol}(G)$ if $k \geq c \log \epsilon^{-1}$, for some constants $c$.

**Proof.** We will prove by contradiction. Suppose that for $j_0 = \lceil k/4 \rceil$, there is a vertex $u$ with $\text{vol}(B_v(j_0)) > \epsilon \text{vol}(G)$. Let $r$ denote the largest integer such that

\[ s_r = \frac{\text{vol}(B_u(r))}{\text{vol}(G)} > \frac{1}{2}. \]

By the assumption, we have $r > k/4$ and $s_{j_0} > \epsilon$. There are two possibilities:

**Case 1:** $r \geq k/2$.

By part (i) of Lemma 4, we have $\text{vol}(\delta(B_u(j))) \geq 0.5\text{vol}(B_u(j))$ for $j \leq k/2$ and therefore $s_{j+1} \geq 1.5s_j$. Thus, for $j \leq k/2 - c_1 \log \epsilon^{-1}$, we have $s_j \leq \epsilon$ where $c_1 = 1/\log 1.5$. Since $k/4 \leq k/2 - c_1 \log \epsilon^{-1}$, we have a contradiction.
Case 2: $r < k/2$.

We define

$$s_j = \frac{\text{vol}(V \setminus B_u(j))}{\text{vol}(G)}.$$

Thus $s_j < 1/2$ for all $j \geq k/2$. We consider two subcases.

Subcase 2a: Suppose $\bar{s}_j \geq \epsilon$ for $j \geq k/2$.

Using Lemma 4, for $j$ where $r \leq j \leq k/2$, we have $\bar{s}_j \geq 1.5\bar{s}_{j+1}$. Thus, for some $j_1 \geq k/2 - c_1 \log \epsilon^{-1}$, we have $\bar{s}_j \geq 1/2$ or equivalently, $s_j \leq 1/2$. By using Lemma 4 again, for $j \leq j_1$, we have $s_{j+1} \geq 1.5s_j$ and therefore for any $j \leq j_1 - c_1 \log \epsilon^{-1}$ we have $s_j \leq \epsilon$. Since $j_1 - c_1 \log \epsilon^{-1} \geq k/2 - 2c_1 \log \epsilon^{-1} \geq k/4$, we again have a contradiction to the assumption $s_{j_0} \geq \epsilon$.

Subcase 2b: Suppose $\bar{s}_j < \epsilon$ for $j \geq k/2$

We apply part (ii) of Lemma 4 and we have, for $j \geq k/2$,

$$\frac{s_j}{s_{j+1}} \geq \frac{1}{(\sigma + 2\epsilon)^2}.$$

This implies, for $j_2 = \lceil k/2 \rceil$,

$$\frac{s_{j_2}}{s_k} \geq \prod_{k/2 < j \leq k} \frac{1}{(\sigma + 2s_j)^2} \geq \frac{1}{(\sigma + 2\epsilon)^k}.$$

Since $s_k \geq 1/\text{vol}(G)$, we have

$$\bar{s}_{j_1} \geq \frac{1}{\text{vol}(G)(\sigma + 2\epsilon)^k}.$$

Since the assumption of this subcase is $\bar{s}_{j_1} < \epsilon$, we have

$$k \geq \frac{\log n + \log \epsilon^{-1}}{\log \sigma^{-1}}.$$

We now use Lemma 4 and we have, for $j = k/2 - j' \geq r$,

$$\bar{s}_j \geq \frac{1}{\text{vol}(G)(\sigma + 2\epsilon)^{k+2j'}}.$$

Therefore, for some $j \leq k/2 - \log \epsilon^{-1}/\log \sigma^{-1}$, we have $s_j > 1/2$ which implies $r \geq k/2 - \log \epsilon^{-1}/\log \sigma^{-1}$.

Now we use the same argument as in Case 1 except shifting $r$ by $\log \epsilon^{-1}/\log \sigma^{-1}$. For some $j \leq r - c_1 \log \epsilon^{-1} \leq k/2 - \log \epsilon^{-1}/\log \sigma^{-1} - c_1 \log \epsilon^{-1}$, we have $s_j < \epsilon$. Since $\log \epsilon^{-1}/\log \sigma^{-1} + c_1 \log \epsilon^{-1} < k/4$, this leads to a contradiction and Theorem 7 is proved.
5 Non-backtracking random walks

Before we proceed to the proof of the Alon-Boppana bound, we will need some basic facts on non-backtracking random walks.

A non-backtracking walk is a sequence of vertices \( p = (v_0, v_1, \ldots, v_t) \) for some \( t \) such that \( v_{i-1} \sim v_i \) and \( v_{i+1} \neq v_{i-1} \) for \( i = 1, \ldots, t - 2 \). The non-backtracking random walk can be described as follows: For \( i \geq 1 \), at the \( i \)th step on \( v_i \), choose with equal probability a neighbor \( u \) of \( v_i \) where \( u \neq v_i - 1 \), move to \( u \) and set \( v_{i+1} = u \). To simplify notation, we call a non-backtracking walk an NB-walk. The modified transition probability matrix \( \tilde{P}_k \), for \( k = 0, 1, \ldots, t - 1 \), is defined by

\[
\tilde{P}_k(u, v) = \begin{cases} 
P^k(u, v) & \text{if } k = 0 \\
\sum_{p \in \mathcal{P}^{(k)}_{u,v}} w(p) & \text{if } k \geq 1
\end{cases}
\]

where the weight \( w(p) \) for an NB-walk \( p = (v_0, v_1, \ldots, v_t) \) with \( t \geq 1 \) is defined to be

\[
w(p) = \frac{1}{d_{v_0} \prod_{i=1}^{t-1} (d_{v_i} - 1)}
\]

and \( \mathcal{P}^{(k)}_{u,v} \) denotes the set of non-backtracking walks from \( u \) to \( v \). For a walk \( p = (v_0) \) of length 0, we define \( w(p) = 1 \).

Although a non-backtracking random walk is not a Markov chain, it is closely related to an associated Markov chain as we will describe below (also see [6]).

For each edge \( \{u, v\} \in E \), we consider two directed edges \((u, v)\) and \((v, u)\). Let \( \hat{E} \) denote the set consisting of all such directed edges, i.e. \( \hat{E} = \{(u, v) : \{u, v\} \in E\} \). We consider a random walk on \( \hat{E} \) with transition probability matrix \( P \) defined as follows:

\[
P((u, v), (u', v')) = \begin{cases} 
\frac{1}{d_{u} - 1} & \text{if } v = u'\text{and } u \neq v' \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( 1_E \) denote the all 1’s function defined on the edge set \( E \) as a row vector. From the above definition, we have

\[
1_E P = 1_E.
\]

In addition, we define the vertex-edge incidence matrix \( B \) and \( B^* \) for \( a \in V \) and \((b, c) \in \hat{E}\) by

\[
B(a, (b, c)) = \begin{cases} 
1 & \text{if } a = b, \\
0 & \text{otherwise}
\end{cases}
\]

\[
B^*((b, c), a) = \begin{cases} 
1 & \text{if } c = a, \\
0 & \text{otherwise.}
\end{cases}
\]
Let $1_V$ denote all 1’s vector defined on the vertex set $V$. Then
\[ 1_V B = 1_E. \] (19)

Although $\tilde{P}_k$ is not a Markov chain, it is related to the Markov chain determined by $P$ on $\tilde{E}$ as follows:

**Fact 1:** For $l \geq 1$,
\[ \tilde{P}_l = D^{-1} B P^l B^* \] (20)
and for the case of $l = 0$, we have $\tilde{P}_0 = I$.

By combining (19) and (20), we have

**Fact 2:**
\[ 1_V D \tilde{P}_l = 1_E B^* = 1_V D. \] (21)

Note that $1_V D$ is just the degree vector for the graph $G$. Therefore (21) states that the degree vector is an eigenvector of $\tilde{P}_l$. Using Fact 1 and 2, we have the following:

**Lemma 8.**

(i) For a fixed vertex $x$ and any integer $j \geq 0$, we have
\[ \sum_u d_u \sum_{p \in \mathcal{P}_{u,x}^{(j)}} w(p) = d_x \] (22)

(ii) For a fixed vertex $u$, we have
\[ \sum_x \sum_{p \in \mathcal{P}_{u,x}^{(j)}} w(p) = 1_u (I + \tilde{P}_1 + \ldots + \tilde{P}_l) 1^* = l + 1 \] (23)

where $1_u$ denotes the characteristic function which assumes value 1 at $u$ and 0 else where.

**Proof.** The proof of (22) and (23) follows from the fact that
\[ 1_V D \tilde{P}_j(x) = 1_V D (D^{-1} B P^j B^*) = 1_E P^j B^* = 1_V (I + \tilde{P}_1 + \ldots + \tilde{P}_l) 1^* = l + 1 \]
and $1_u \tilde{P}_j(x) = w(p)$ for $p \in \mathcal{P}_{u,x}^{(j)}$. \qed

6 An Alon-Boppana bound for $\lambda_1$

**Theorem 9.** In a graph $G = (V, E)$ with diameter $k$, the first nontrivial eigenvalue $\lambda_1$ satisfies
\[ \lambda_1 \leq 1 - \sigma \left( 1 - \frac{c}{k} \right) \]
where $\sigma$ is as defined in (15), provided $k \geq c' \log \sigma^{-1}$ and $\text{vol}(G) \geq c'' \sigma^{\log \sigma}$ for some absolute constants $c'$s.

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Proof. If $G$ is not a weak Ramanujan graph, we have $\lambda_1 \leq 1 - \sigma$ and we are done. We may assume that $G$ is weak Ramanujan.

From the definition of $\lambda_1$, we have

$$\lambda_1 \leq \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x f^2(x) dx} = R(f)$$

where $f$ satisfies $\sum_x f(x) dx = 0$.

We will construct an appropriate $f$ satisfying $R(f) \leq 1 - \sigma(1 - c/k)$ and therefore serve as an upper bound for $\lambda_1$. We set

$$t = \left\lfloor \frac{\log(\text{vol}(G))}{\log \sigma^{-1}} \right\rfloor$$

and choose $\epsilon$ satisfying

$$\epsilon \leq \frac{\sigma}{t} \leq \frac{c \sigma}{k}$$

by using Theorem 6 where $\sigma$ is as defined in (15).

We consider a family of functions defined as follows. For a specified vertex $u$ and an integer $l = \lceil k/4 \rceil$, we consider a function $g_u : V \to \mathbb{R}^+$, defined by

$$g_u(x) = \left(1_u (I + \tilde{P}_1 + \ldots + \tilde{P}_l)(x)\right)^{1/2}$$

$$= \left(\sum_{j=0}^{l} \sum_{p \in \mathcal{P}_u^{(j)}} w(p)\right)^{1/2}$$

where $\tilde{P}_j$ is as defined in (20) and $1_u$ is treated as a row vector. In other words, $g_u$ denotes the square root of the sum of non-backtracking random walks starting from $u$ taking $i$ steps for $i$ ranging from 0 to $l$.

Claim A:

$$\sum_u d_u \sum_x g_u^2(x) dx = \sum_{j=0}^{l} \sum_x d_u w(p) dx = (l + 1) \sum_x d_x^2$$

where the weight $w(p)$ of a walk $p$ is as defined in (17).

Proof of Claim A: From the definition of $g_u$ and (16), we have

$$\sum_u d_u \sum_x g_u^2(x) dx = \sum_{j=0}^{l} \sum_x d_u w(p)$$

$$= \sum_u d_u 1_u B(I + \tilde{P}_1 + \ldots + \tilde{P}_l)(x)$$
= \sum_u d_u \sum l_u \mathbf{1}_u D^{-1} \mathbf{BP}^i B^*(x) d_x + \sum_x d_x^2

= \sum l_u \sum u \mathbf{1}_u \mathbf{BP}^i B^*(x) d_x + \sum_x d_x^2

= \sum l \mathbf{1}_E \mathbf{BP}^i B^*(x) d_x + \sum_x d_x^2

= l \mathbf{1}_E B^*(x) d_x + \sum_x d_x^2

= (l + 1) \sum_x d_x^2.

Claim A is proved.

**Claim B:**

\[
\sum_u d_u \sum_{x \sim y} (g_u(x) - g_u(y))^2 \leq (l + 1 - l\sigma) \sum_x d_x^2,
\]

where \(\sum_{x \sim y}\) denotes the sum ranging over unordered pairs \(\{x, y\}\) where \(x\) is adjacent to \(y\).

**Proof of Claim B:**

We will use the following fact for \(a_i, b_i > 0\).

\[
\left(\sqrt{\sum_i a_i} - \sqrt{\sum_i b_i}\right)^2 \leq \sum_i \left(\sqrt{a_i} - \sqrt{b_i}\right)^2
\]

which can be easily checked.

For a fixed vertex \(u\), we apply Claim B:

\[
\sum_{x \sim y} (g_u(x) - g_u(y))^2
\]

\[
= \sum_{x \sim y} \left(\sqrt{\sum_{p \in \mathcal{P}^{(t)}_{u,x}} w(p)} - \sqrt{\sum_{p' \in \mathcal{P}^{(t)}_{u,y}} w(p')}\right)^2
\]

\[
\leq \sum_{t \leq l - 1} \sum_r \sum_{p \in \mathcal{P}^{(t)}_{u,r}} \left(\sqrt{w(p)} - \sqrt{w(p')}\right)^2 + \sum_{p \in \mathcal{P}^{(l)}_{u,r}} w(p)(d_x - 1)
\]

\[
\leq \sum_{t \leq l - 1} \sum_r \sum_{x} \sum_{p \in \mathcal{P}^{(t)}_{u,r}} \left(\sqrt{w(p)} - \sqrt{w(p)}\right)^2 (d_x - 1) + \sum_{p \in \mathcal{P}^{(l)}_{u,r}} \sqrt{w(p)}(d_x - 1)
\]
\[
\sum_{t \leq l - 1} \sum_{x} \sum_{p \in \mathcal{P}^{(t)}_{u,x}} w(p) \left( 1 + \frac{1}{d_x - 1} - \frac{2}{\sqrt{d_x - 1}} \right) (d_x - 1) + \sum_{p \in \mathcal{P}^{(l)}_{u,x}} w(p) (d_x - 1)
\]

\[
\sum_{t \leq l - 1} \sum_{x} \sum_{p \in \mathcal{P}^{(t)}_{u,x}} w(p) \left( d_x - 2 \sqrt{d_x - 1} \right) + \sum_{p \in \mathcal{P}^{(l)}_{u,x}} w(p) (d_x - 1).
\]

Using Fact 3, we have

\[
\sum_u d_u \sum_{x \sim y} (g_u(x) - g_u(y))^2 
\]

\[
\leq \sum_{t \leq l - 1} \sum_u d_u \sum_{p \in \mathcal{P}^{(t)}_{u,x}} w(p) \left( d_x - 2 \sqrt{d_x - 1} \right) + \sum_u d_u \sum_{p \in \mathcal{P}^{(l)}_{u,x}} w(p) (d_x - 1)
\]

\[
= l \sum_x d_x (d_x - 2 \sqrt{d_x - 1}) + \sum_x d_x^2
\]

\[
= l(1 - \sigma) \sum_x d_x^2 + \sum_x d_x^2
\]

\[
= (l + 1 - l\sigma) \sum_x d_x^2
\]

This proves Claim B.

**Claim C:** There is a vertex \( u \) satisfying

\[
R(g_u) \leq 1 - \sigma \left( 1 - \frac{1}{l + 1} \right)
\]

**Proof of Claim C:**
Combining Claim A and B, we have

\[
\sum_u d_u \sum_{x \sim y} (g_u(x) - g_u(y))^2 
\]

\[
\leq \left( l + 1 - l\sigma \right) \sum_x d_x^2
\]

\[
\leq \left( l + 1 - l\sigma \right) \left( \frac{1}{l + 1} \right) \sum_u d_u \sum_x g_u^2(x) d_x
\]

\[
= \left( 1 - \frac{l\sigma}{l + 1} \right) \sum_u d_u \sum_x g_u^2(x) d_x 
\]

(26)

Thus we deduce that there is a vertex \( u \) such that

\[
R(g_u) = \frac{\sum_{x \sim y} (g_u(x) - g_u(y))^2}{\sum_x g_u^2(x) d_x}
\]

\[
\leq 1 - \frac{l\sigma}{l + 1}.
\]

(27)
We define
\[ \alpha_v = \frac{\sum_x g_v(x)dx}{\sum_x dx} = \frac{\sum_x g_v(x)dx}{\text{vol}(G)} \]

We consider the function \( g'_u \) defined by
\[ g'_u(x) = g_u(x) - \alpha_u \]

Clearly, \( g'_u \) satisfies the condition that
\[ \sum_x g'_u(x)dx = 0 \]

Hence, we have
\[ \lambda_1 \leq R(g'_u) = \frac{\sum_{x \sim y} (g'_u(x) - g'_u(y))^2}{\sum_x g'^2_u(x)dx} = \frac{\sum_{x \sim y} (g_u(x) - g_u(y))^2}{\sum_x g^2_u(x)dx - \alpha^2 u \text{vol}(G)}. \quad (28) \]

Note that by the Cauchy-Schwarz inequality, we have
\[ \left( \sum_{x \in B_u(l)} g_u(x)dx \right)^2 \leq \text{vol}(B_u(l)) \sum_{x \in B_u(l)} g^2_u(x)dx. \]

and therefore
\[ \alpha^2_u \leq \frac{\text{vol}(B_u(l))}{\text{vol}(G)^2} \sum_x g'^2_u(x)dx. \]

By substitution into (28) and using (35), we have
\[ \lambda_1 \leq R(g'_u) \leq \frac{R(g)}{1 - \frac{\text{vol}(B_u(l))}{\text{vol}(G)}} \leq \frac{1 - \sigma(1 - \frac{1}{l+1})}{1 - \frac{\text{vol}(B_u(l))}{\text{vol}(G)}} \quad (29) \]
\[ \leq 1 - \sigma \left( 1 - \frac{1}{l+1} \right) + \frac{\text{vol}(B_u(l))}{\text{vol}(G)} \quad (30) \]
\[ \leq 1 - \sigma \left( 1 - \frac{c}{l+1} \right) \quad (31) \]

The last inequality follows from Theorem 7 and the choice of \( \epsilon = \sigma/k \). This completes the proof of Theorem 9. \( \square \)
7 A lower bound for $\lambda_{n-1}$

If a graph is bipartite, it is known (see [2]) that $\lambda_i = 2 - \lambda_{n-i-1}$ for all $0 \leq i \leq n - 1$ and, in particular, $\lambda_{n-1} = 2 - \lambda_0 = 2$. If $G$ is not bipartite, it is easy to derive the following lower bound:

$$\lambda_{n-1} \geq 1 + 1/(n-1)$$

by using the fact that the trace of $L$ is $n$. This lower bound is sharp for the complete graph. However if $G$ is not the complete graph, is it possible to derive a better lower bound? The answer is affirmative. Here we give an improved lower bound for $\lambda_{n-1}$.

**Theorem 10.** In a connected graph $G = (V, E)$ with diameter $k$, the largest eigenvalue $\lambda_{n-1}$ of the normalized Laplacian $L$ of $G$ satisfies

$$\lambda_{n-1} \geq 1 + \sigma \left(1 - \frac{c}{k}\right)$$

where $\sigma$ is as defined in (15), provided $k \geq c' \log \sigma^{-1}$ and $\text{vol}(G) \geq c'' \sigma \log \sigma$ for some absolute constants $c'$s.

**Proof.** By definition, $\lambda_{n-1}$ satisfies

$$\lambda_{n-1} \geq \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x f^2(x)dx} = R(f)$$

for any $f : V \rightarrow \mathbb{R}$.

We will construct an appropriate $f$ such that $R(f) \geq 1 + \sigma (1 - c/\gamma)$ by considering the following function $f_u : V \rightarrow \mathbb{R}^+$, for a fixed vertex $u$, defined by

$$\eta_u(x) = \begin{cases} 
(-1)^t \chi_u (\tilde{P}_t(x))^{-1/2} & \text{if dist}(u, x) = t \leq l \\
0 & \text{otherwise}
\end{cases}$$

where $l \leq \gamma/2$. Note that $|\eta_u(x)| = g_u(x)$ since we assume that $l \leq \gamma/2$. Using the same proof in Claim A, we have

**Claim A’:**

$$\sum_u d_u \sum_x \eta_u^2(x)dx = \sum_{j=0}^l \sum_x \sum_{p \in \mathcal{P}_{u,x}^{(j)}} d_u w(p)dx = (l + 1) \sum_x d_x^2.$$ 

**Claim B’:**

$$\sum_u d_u \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2 \geq (l + 1 + l\sigma) \sum_x d_x^2.$$
Proof of Claim B’: The proof is quite similar to that of Claim B. For a fixed vertex \( u \), the sum over unordered pair \( \{x, y\} \) where \( x \sim y \),

\[
\sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2
\]

\[
\leq \sum_{t \leq l-1} \sum_{r \in V} \sum_{p \in \mathcal{R}_{u,r}^{(t)}} \left( \sqrt{w(p)} + \sqrt{w(p')} \right)^2 - \sum_{p \in \mathcal{R}_{u,r}^{(t+1)}} w(p)(d_x - 1)
\]

\[
\leq \sum_{t \leq l-1} \sum_{x} \sum_{p \in \mathcal{R}_{u,x}^{(t)}} \left( \sqrt{w(p)} + \sqrt{w(p)} \right)^2 (d_x - 1) - \sum_{p \in \mathcal{R}_{u,x}^{(t)}} \sqrt{w(p)}(d_x - 1)
\]

\[
\leq \sum_{t \leq l-1} \sum_{x} \sum_{p \in \mathcal{R}_{u,x}^{(t)}} w(p)\left(1 + \frac{1}{d_x - 1} + \frac{2}{\sqrt{d_x - 1}}\right)(d_x - 1) - \sum_{p \in \mathcal{R}_{u,x}^{(t)}} w(p)(d_x - 1)
\]

\[
\leq \sum_{t \leq l-1} \sum_{x} \sum_{p \in \mathcal{R}_{u,x}^{(t)}} w(p)(d_x + 2\sqrt{d_x - 1}) - \sum_{p \in \mathcal{R}_{u,x}^{(t)}} w(p)(d_x - 1).
\]

Using Fact 3, we have

\[
\sum_{u} d_u \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2
\]

\[
\geq \sum_{t \leq l-1} \sum_{u} d_u \sum_{p \in \mathcal{R}_{u,x}^{(t)}} w(p)(d_x + 2\sqrt{d_x - 1}) - \sum_{u} d_u \sum_{p \in \mathcal{R}_{u,x}^{(t)}} w(p)(d_x - 1)
\]

\[
= l \sum_{x} d_x(d_x + 2\sqrt{d_x - 1}) - \sum_{x} d_x^2
\]

\[
= l(1 + \sigma) \sum_{x} d_x^2 - \sum_{x} d_x^2
\]

\[
= (l - 1 + l\sigma) \sum_{x} d_x^2
\]

This proves Claim B’.

Combining Claims A’ and B’, we have

\[
\sum_{u} d_u \sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2
\]

\[
\geq (l - 1 + l\sigma) \sum_{x} d_x^2
\]

\[
\geq (l - 1 + l\sigma) \left( \frac{1}{l+1} \right) \sum_{u} d_u \sum_{x} \eta_u^2(x)d_x
\]

\[
= \left( 1 + \frac{l\sigma}{l-1} \right) \sum_{u} d_u \sum_{x} \eta_u^2(x)d_x
\]

(34)
Thus we deduce that there is a vertex $u$ such that

$$R(\eta_u) = \frac{\sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2}{\sum_x \eta_u^2(x)d_x} \leq 1 + \frac{l\sigma}{l-1}. \quad (35)$$

We consider the function $\eta'_u$ defined by

$$\eta'_u(x) = \eta_u(x) - \alpha_u$$

where

$$\alpha_v = \frac{\sum_x \eta_v(x)d_x}{\sum_x d_x} = \frac{\sum_x \eta_v(x)d_x}{\text{vol}(G)}$$

so that $\eta'_u$ satisfies the condition that

$$\sum_x \eta'_u(x)d_x = 0$$

Hence, we have

$$\lambda_{n-1} \geq R(\eta'_u) = \frac{\sum_{x \sim y} (\eta'_u(x) - \eta'_u(y))^2}{\sum_x \eta'_u^2(x)d_x} \geq \frac{\sum_{x \sim y} (\eta_u(x) - \eta_u(y))^2}{\sum_x \eta_u^2(x)d_x - \alpha_u^2 \text{vol}(G)} \geq 1 + \sigma(1 + \frac{c}{l}) - \frac{\text{vol}(B_u(l))}{\text{vol}(G)}.$$ 

This completes the proof of Theorem 10. \qed

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Corrigendum – added 3th November 2017

1. In the abstract, line 6-8, the statement of the main result should be replaced by

\[ \lambda_1 \leq 1 - \sigma \left( 1 - \frac{5}{k} \right) \]

provided \( \sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2 \leq 1/2 \) and \( k(1.5)^k \geq \sigma^{-1} \) where \( d_v \) denotes the degree of the vertex \( v \) with minimum degree at least 2.

Also, page 12, line -3 to -1, the statement of Theorem 9 should be similarly replaced as above.

2. Page 3, line 13, the constant \( c \) should be replaced by 5.

3. Page 9, line -9. “... for some constant \( c \).” should be replaced by “... for \( c = 1/\log 1.5 \).”

4. Page 9, line -6. Replace “... largest ...” by ”... least ...”.

5. Page 10, line 3, \( \bar{s}_j \) should be replaced by \( \bar{s}_{j+1} \).

6. Page 3, line 7 to 11. Delete “We set ... as defined in (15).” Note that \( \epsilon \) was defined later near the end of the proof of Theorem 9.

7. Page 16, line -6, replace “... using (35), ...” by “... using (27), ...”.

8. Page 16, line -3. Replace “\( c/(l+1) \)” by “\( 5/k \)”.

9. Page 16, line -2. Replace “... the choice of \( \epsilon = \sigma/k \)” by “... the choice of \( \epsilon = \sigma/k \) which satisfies \( k \geq (\log \epsilon^{-1}) / \log 1.5 \)”

10. Page 17, line 11 to line 13, the statement of Theorem 10 should be replaced by

\[ \lambda_{n-1} \geq 1 + \sigma \left( 1 - \frac{5}{k} \right) \]

provided \( \sigma = 2 \sum_v d_v \sqrt{d_v - 1} / \sum_v d_v^2 \leq 1/2 \) and \( k(1.5)^k \geq \sigma^{-1} \) where \( d_v \) denotes the degree of the vertex \( v \) with minimum degree at least 2.