Continuous characterizations of Besov-Lizorkin-Triebel spaces 
and new interpretations as coorbits

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September 29, 2010

Abstract

We give characterizations for homogeneous and inhomogeneous Besov-Lizorkin-Triebel spaces \([28, 30, 31]\) in terms of continuous local means for the full range of parameters. In particular, we prove characterizations using tent spaces (Lusin functions) and spaces involving the Peetre maximal function in order to apply the classical coorbit space theory due to Feichtinger and Gröchenig \([7, 8, 9, 13, 14]\). This results in atomic decompositions and wavelet bases for homogeneous spaces. In particular we give sufficient conditions for suitable wavelets in terms of moment, decay and smoothness conditions.

Key Words: Besov-Lizorkin-Triebel type spaces, coorbit space theory, local means, Peetre maximal function, tent spaces, atomic decompositions, wavelet bases.

AMS Subject classification: 42B25, 42B35, 46E35, 46F05.

1 Introduction

This paper deals with Besov-Lizorkin-Triebel spaces \(\dot{B}^{s}_{p,q}(\mathbb{R}^d)\) and \(\dot{F}^{s}_{p,q}(\mathbb{R}^d)\) on the Euclidean space \(\mathbb{R}^d\) and their interpretation as coorbits. For this purpose we prove a number of characterizations for homogeneous and inhomogeneous spaces for the full range of parameters. Classically introduced in Triebel’s monograph \([28, 2.3.1]\) by means of a dyadic decomposition of unity, we use more general building blocks and provide in addition continuous characterizations in terms of Lusin and maximal functions. Equivalent (quasi-)normings of this kind were first given by Triebel in \([29]\). His proofs use in an essential way the fact that the function under consideration belongs to the respective space. Therefore, the obtained equivalent (quasi-)norms could not yet be considered as a definition or characterization of the space. Later on, Triebel was able to solve this problem partly in his monograph \([30, 2.4.2, 2.5.1]\) by restricting to the Banach space case. Afterwards, Rychkov \([23]\) completed the picture by simplifying a method due to Bui, Paluszyński, and Taibleson \([3, 4]\). However, \([24]\) contains some problematic arguments. One aim of the present paper is to provide a complete and self-contained reference for general characterizations of discrete and continuous type by avoiding these arguments. We use a variant of a method from Rychkov’s subsequent papers \([24, 25]\) which is originally due to Strömberg and Torchinsky developed in their monograph \([27, \text{Chapt. 5}]\).

In a different language the results can be interpreted in terms of the continuous wavelet transform (see Appendix A.1) belonging to a function space on the \(ax + b\)-group \(G\). Spaces on \(G\) considered here are mixed norm spaces like tent spaces \([5]\) as well as Peetre type spaces.
The latter are indeed new and received their name from the fact that quantities related to the classical Peetre maximal function are involved. This leads to the main intention of the paper. We use the established characterizations for the homogeneous spaces in order to embed them in the abstract framework of coorbit space theory originally due to Feichtinger and Grochenig [7, 8, 9, 13, 14] in the 80s. This connection was already observed by them in [7, 13, 14]. They worked with Triebel’s equivalent continuous normings from [29] and the results on tent spaces which were introduced more or less at the same time by Coifman, Meyer, Stein [5] to interpret Lizorkin-Triebel spaces as coorbits. On the one hand the present paper gives a late justification and on the other hand we observe that Peetre type spaces on $G$ are a much better choice for this issue. Their two-sided translation invariance is immediate and much more transparent as we will show in Section 4.1. Furthermore, generalizations in different directions are now possible. In a forthcoming paper we will show how to apply a generalized coorbit space theory due to Fornasier and Rauhut [11] in order to recover inhomogeneous spaces based on the characterizations given here. Moreover, the extension of the results to quasi-Banach spaces using a theory developed by Rauhut in [21, 22] is possible.

Once we have interpreted classical homogeneous Besov-Lizorkin-Triebel spaces as certain coorbits, we are able to benefit from the achievements of the abstract theory in [7, 8, 9, 13, 14]. The main feature is a powerful discretization machinery which leads in an abstract universal way to atomic decompositions. We are now able to apply this method which results in atomic decompositions and wavelet bases for homogeneous spaces. More precisely, sufficient conditions in terms of vanishing moments, decay, and smoothness properties of the respective wavelet function are given. Compact support of the used atoms does not play any role here. In particular, we specify the order of a suitable orthonormal spline wavelet system depending on the parameters of the respective space.

The paper is organized as follows. After giving some preliminaries we start in Section 2 with the definition of classical Besov-Lizorkin-Triebel spaces and their characterization via continuous local means. In Section 3 we give a brief introduction to abstract coorbit space theory which is applied in Section 4 on the $ax + b$-group $G$. We recover the homogeneous spaces from Section 2 as coorbits of certain spaces on $G$. Finally, several discretization results in terms of atomic decompositions and wavelet isomorphisms are established. The underlying decay result of the continuous wavelet transform and some basic facts about orthonormal wavelet bases are shifted to the appendix.

Acknowledgement: The author would like to thank Holger Rauhut, Martin Schäfer, Benjamin Scharf, and Hans Triebel for valuable discussions, a critical reading of preliminary versions of this manuscript and for several hints how to improve it.

1.1 Notation

Let us first introduce some basic notation. The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ denote the real numbers, complex numbers, natural numbers, natural numbers including 0 and the integers. The dimension of the underlying Euclidean space for function spaces is denoted by $d$, its elements will be denoted by $x, y, z, ...$ and $|x|$ is used for the Euclidean norm. We will use $|k|$ for the $\ell^1_d$-norm of a vector $k$. For a multi-index $\alpha$ and $x \in \mathbb{R}^d$ we write

$$x^{\alpha} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$$
and define the differential operators $D^\bar{\alpha}$ and $\Delta$ by

$$
D^\bar{\alpha} = \frac{\partial^{\bar{\alpha}}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \quad \text{and} \quad \Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}.
$$

If $X$ is a (quasi-)Banach space and $f \in X$ we use $\|f|X\|$ or simply $\|f\|$ for its (quasi-)norm. The space of linear continuous mappings from $X$ to $Y$ is denoted by $\mathcal{L}(X,Y)$ or simply $\mathcal{L}(X)$. Operator (quasi-)norms of $A \in \mathcal{L}(X,Y)$ are denoted by $\|A : X \to Y\|$, or simply $\|A\|$. As usual, the letter $c$ denotes a constant, which may vary from line to line but is always independent of $f$, unless the opposite is explicitly stated. We also use the notation $a \lesssim b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that $a \leq cb$. If $a \lesssim b$ and $b \lesssim a$ we will write $a \asymp b$.

## 2 Function spaces on $\mathbb{R}^d$

### 2.1 Vector valued Lebesgue spaces

The space $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, denotes the collection of complex-valued functions (equivalence classes) with finite (quasi-)norm

$$
\|f|L_p(\mathbb{R}^d)\| = \left( \int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p},
$$

with the usual modification if $p = \infty$. The Hilbert space $L_2(\mathbb{R}^d)$ plays a separate role for our purpose (Section 3). Having a sequence of complex-valued functions $\{f_k\}_{k \in I}$ on $\mathbb{R}^d$, where $I$ is a countable index set, we put

$$
\left\| \{f_k\}_{k \in I} \right\|_{L_p(\mathbb{R}^d)} = \left( \sum_{k \in I} \|f_k|L_p(\mathbb{R}^d)\|^q \right)^{1/q}
$$

and

$$
\left\| \{f_k\}_{k \in I} \right\|_{L_p(\ell_q,\mathbb{R}^d)} = \left( \left( \sum_{k \in I} |f_k(x)|^q \right)^{1/q} \right)^{1/p}.
$$

where we modify appropriately in the case $q = \infty$.

### 2.2 Maximal functions

For a locally integrable function $f$ we denote by $Mf(x)$ the Hardy-Littlewood maximal function defined by

$$
(Mf)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_{Q} |f(y)| \, dy, \quad x \in \mathbb{R}^d,
$$

where the supremum is taken over all cubes centered at $x$ with sides parallel to the coordinate axes. The following theorem is due to Fefferman and Stein [6].
Theorem 2.1. For $1 < p < \infty$ and $1 < q \leq \infty$ there exists a constant $c > 0$, such that
\[
\| \{ Mf_k \}_k \|_{L_p(\ell_q, \mathbb{R}^d)} \leq c \| \{ f_k \}_k \|_{L_p(\ell_q, \mathbb{R}^d)}
\]
holds for all sequences $\{ f_k \}_{k \in \mathbb{Z}}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

Let us recall the classical Peetre maximal function, introduced in [19]. Given a sequence of functions $\{ \Psi_k \}_{k \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ and a positive number $a > 0$ we define the system of maximal functions
\[
(\Psi^*_k f)_a(x) = \sup_{y \in \mathbb{R}^d} \frac{|(\Psi_k * f)(x + y)|}{(1 + 2^k|y|)^a}, \quad x \in \mathbb{R}^d, k \in \mathbb{Z}.
\]
Since $(\Psi_k * f)(y)$ makes sense pointwise (see the following paragraph) everything is well-defined. However, the value “$\infty$” is also possible for $(\Psi^*_k f)_a(x)$. This was the reason for the problematic arguments in [23] mentioned in the introduction. We will often use dilates $\Psi_k(x) = 2^{kd}(\Psi(2^k x))$ of a fixed function $\Psi \in \mathcal{S}(\mathbb{R}^d)$, where $\Psi_0(x)$ might be given by a separate function. Also continuous dilates are needed. Let the operator $\mathcal{D}_t^{L_p}, t > 0$, generate the $p$-normalized dilates of a function $\Psi$ given by $\mathcal{D}_t^{L_p} \Psi := t^{-d/p} \Psi(t^{-1} \cdot)$. If $p = 1$ we omit the super index and use additionally $\Psi_t := \mathcal{D}_t \Psi := \mathcal{D}_t^{L_1} \Psi$. We define $(\Psi^*_t f)_a(x)$ by
\[
(\Psi^*_t f)_a(x) = \sup_{y \in \mathbb{R}^d} \frac{|(\Psi_t * f)(x + y)|}{(1 + |y|/t)^a}, \quad x \in \mathbb{R}^d, t > 0. \tag{2.1}
\]
We will refer to this construction later on. It turned out that this maximal function construction can be used to interpret classical smoothness spaces as coorbits of certain Banach function spaces on the $ax + b$-group, see Section 4

2.3 Tempered distributions, Fourier transform

As usual $\mathcal{S}(\mathbb{R}^d)$ is used for the locally convex space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$ where its topology is generated by the family of semi-norms
\[
\| \varphi \|_{k, \ell} = \sup_{x \in \mathbb{R}^d, |\alpha|_1 \leq \ell} |D^{\alpha} \varphi(x)|(1 + |x|)^k, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \; k, \ell \in \mathbb{N}_0.
\]
The space $\mathcal{S}'(\mathbb{R}^d)$, the topological dual of $\mathcal{S}(\mathbb{R}^d)$, is also referred as the set of tempered distributions on $\mathbb{R}^d$. Indeed, a linear mapping $f : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ if and only if there exist numbers $k, \ell \in \mathbb{N}_0$ and a constant $c = c_f$ such that
\[
|f(\varphi)| \leq c_f \sup_{x \in \mathbb{R}^d, |\alpha|_1 \leq \ell} |D^{\alpha} \varphi(x)|(1 + |x|)^k \tag{2.2}
\]
for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The space $\mathcal{S}'(\mathbb{R}^d)$ is equipped with the weak$^*$-topology.

The convolution $\varphi \ast \psi$ of two integrable (square integrable) functions $\varphi, \psi$ is defined via the integral
\[
(\varphi \ast \psi)(x) = \int_{\mathbb{R}^d} \varphi(x - y)\psi(y) \, dy. \tag{2.3}
\]
If \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \) then (2.3) still belongs to \( \mathcal{S}(\mathbb{R}^d) \). The convolution can be extended to \( \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \) via \( (\varphi * f)(x) = f(\varphi(x - \cdot)) \). It makes sense pointwise and is a \( C^\infty \)-function in \( \mathbb{R}^d \) of at most polynomial growth.

As usual the Fourier transform defined on both \( \mathcal{S}(\mathbb{R}^d) \) and \( \mathcal{S}'(\mathbb{R}^d) \) is given by \( \mathcal{F}(f)(\varphi) := f(\mathcal{F}\varphi) \), where \( f \in \mathcal{S}'(\mathbb{R}^d) \), \( \varphi \in \mathcal{S}(\mathbb{R}^d) \), and

\[
\mathcal{F}\varphi(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \varphi(x) \, dx.
\]

The mapping \( \mathcal{F} \) is a bijection (in both cases) and its inverse is given by \( \mathcal{F}^{-1}\varphi = \mathcal{F}\varphi(-\cdot) \).

In order to deal with homogeneous spaces we need to define the subset \( \mathcal{S}_0(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \).

Following [28, Chapt. 5] we put

\[
\mathcal{S}_0(\mathbb{R}^d) = \{ \varphi \in \mathcal{S}(\mathbb{R}^d) : D^\alpha(\mathcal{F}\varphi)(0) = 0 \text{ for every multi-index } \alpha \in \mathbb{N}_0^d \}.
\]

The set \( \mathcal{S}'_0(\mathbb{R}^d) \) denotes the topological dual of \( \mathcal{S}_0(\mathbb{R}^d) \). If \( f \in \mathcal{S}'(\mathbb{R}^d) \), the restriction of \( f \) to \( \mathcal{S}_0(\mathbb{R}^d) \) clearly belongs to \( \mathcal{S}'_0(\mathbb{R}^d) \). Furthermore, if \( P(x) \) is an arbitrary polynomial in \( \mathbb{R}^d \), we have \( (f + P(\cdot))(\varphi) = f(\varphi) \) for every \( \varphi \in \mathcal{S}_0(\mathbb{R}^d) \). Conversely, if \( f \in \mathcal{S}'_0(\mathbb{R}^d) \), then \( f \) can be extended from \( \mathcal{S}_0(\mathbb{R}^d) \) to \( \mathcal{S}(\mathbb{R}^d) \), i.e., to an element of \( \mathcal{S}'(\mathbb{R}^d) \). However, this fact is not trivial and makes use of the Hahn-Banach theorem in locally convex topological vector spaces. We may identify \( \mathcal{S}'_0(\mathbb{R}^d) \) with the factor space \( \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d) \), since two different extensions differ by a polynomial.

### 2.4 Besov-Lizorkin-Triebel spaces

Let us first introduce the concept of a dyadic decomposition of unity, see also [28, 2.3.1].

**Definition 2.2.** (a) Let \( \Phi(\mathbb{R}^d) \) be the collection of all systems \( \{ \varphi_j(x) \}_{j \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^d) \) with the following properties

(i) \( \varphi_j(x) = \varphi(2^{-j}x) \) , \( j \in \mathbb{N} \) ,

(ii) \( \text{supp } \varphi_0 \subset \{ x \in \mathbb{R}^d : |x| \leq 2 \} \), \( \text{supp } \varphi \subset \{ x \in \mathbb{R}^d : 1/2 \leq |x| \leq 2 \} \), and

(iii) \( \sum_{j=0}^{\infty} \varphi_j(x) = 1 \) for every \( x \in \mathbb{R}^d \).

(b) Moreover, \( \check{\Phi}(\mathbb{R}^d) \) denotes the collection of all systems \( \{ \varphi_j(x) \}_{j \in \mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^d) \) with the following properties

(i) \( \varphi_j(x) = \varphi(2^{-j}x) \) , \( j \in \mathbb{Z} \) ,

(ii) \( \text{supp } \varphi = \{ x \in \mathbb{R}^d : 1/2 \leq |x| \leq 2 \} \), and

(iii) \( \sum_{j=-\infty}^{\infty} \varphi_j = 1 \) for every \( x \in \mathbb{R}^d \setminus \{0\} \).

**Remark 2.3.** If we take \( \varphi_0 \in \mathcal{S}(\mathbb{R}^d) \) satisfying

\[
\varphi_0(x) = \begin{cases} 
1 & : |x| \leq 1 \\
0 & : |x| > 2
\end{cases}
\]

and define \( \varphi(x) = \varphi_0(x) - \varphi_0(2x) \) then the system \( \{ \varphi_j(x) \}_{j \in \mathbb{N}_0} \) belongs to \( \check{\Phi}(\mathbb{R}^d) \) and the system \( \{ \varphi_j(x) \}_{j \in \mathbb{Z}} \) with \( \varphi_0 := \varphi \) belongs to \( \check{\Phi}(\mathbb{R}^d) \).
Now we are ready for the definition of the Besov and Lizorkin-Triebel spaces. See for instance [28, 2.3.1] for details and further properties.

**Definition 2.4.** Let \( \{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^d) \) and \( \Phi_j = F^{-1}\varphi_j, j \in \mathbb{N}_0 \). Let further \( -\infty < s < \infty \) and \( 0 < q \leq \infty \).

(i) If \( 0 < p \leq \infty \) then
\[
B^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f \ast B^s_{p,q}(\mathbb{R}^d)\| = \left( \sum_{j=0}^{\infty} 2^{jsq} \|\Phi_j \ast f\|_{L_p(\mathbb{R}^d)} \right)^{1/q} < \infty \right\}.
\]

(ii) If \( 0 < p < \infty \) then
\[
F^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f \ast F^s_{p,q}(\mathbb{R}^d)\| = \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |(\Phi_j \ast f)(x)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} < \infty \right\}.
\]

In case \( q = \infty \) we replace the sum by a supremum in both cases.

The homogeneous counterparts are defined as follows. For details, further properties and how to deal with occurring technicalities we refer to [28, Chapt. 5].

**Definition 2.5.** Let \( \{\varphi_j(x)\}_{j \in \mathbb{Z}} \in \hat{\Phi}(\mathbb{R}^d) \) and \( \Phi_j = F^{-1}\varphi_j \). Let further \( -\infty < s \leq \infty \) and \( 0 < q \leq \infty \).

(i) If \( 0 < p \leq \infty \) then
\[
\dot{B}^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f \ast \dot{B}^s_{p,q}(\mathbb{R}^d)\| = \left( \sum_{j=-\infty}^{\infty} 2^{jsq} \|\Phi_j \ast f\|_{L_p(\mathbb{R}^d)} \right)^{1/q} < \infty \right\}.
\]

(ii) If \( 0 < p < \infty \) then
\[
\dot{F}^s_{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f \ast \dot{F}^s_{p,q}(\mathbb{R}^d)\| = \left\| \left( \sum_{j=-\infty}^{\infty} 2^{jsq} |(\Phi_j \ast f)(x)|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)} < \infty \right\}.
\]

In case \( q = \infty \) we replace the sum by a supremum in both cases.

### 2.5 Inhomogeneous spaces

Essential for the sequel are functions \( \Phi_0, \Phi \in \mathcal{S}(\mathbb{R}^d) \) satisfying
\[
|\langle F\Phi_0 \rangle(x) \rangle | > 0 \quad \text{on} \quad \{ |x| < 2\varepsilon \}
\]
\[
|\langle F\Phi \rangle(x) \rangle | > 0 \quad \text{on} \quad \{ \varepsilon / 2 < |x| < 2\varepsilon \}, \tag{2.4}
\]
for some \( \varepsilon > 0 \), and
\[
D^{\bar{a}}(F\Phi)(0) = 0 \quad \text{for all} \quad |\bar{a}| \leq R. \tag{2.5}
\]

We will call the functions \( \Phi_0 \) and \( \Phi \) kernels for local means. Recall that \( \Phi_k = 2^{kd}\Phi(2^k \cdot) \), \( k \in \mathbb{N} \), and \( \Psi_t = D_t \Psi \). The upcoming four theorems represent the main results of the first part of the paper.
Theorem 2.6. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $a > d/p \min\{p,q\}$ and $R+1 > s$. Let further $\Phi_0, \Phi \in \mathcal{S}(\mathbb{R}^d)$ be given by \eqref{2.4} and \eqref{2.5}. Then the space $F_{p,q}^s(\mathbb{R}^d)$ can be characterized by
\[
F_{p,q}^s(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \| f | F_{p,q}^s(\mathbb{R}^d) \|_i < \infty \} , \quad i = 1, \ldots, 5,
\]
where
\[
\| f | F_{p,q}^s \|_1 = \| \Phi_0 * f | L_p(\mathbb{R}^d) \| + \left\| \left( \int_0^1 t^{-sq} |(\Phi_t * f)(x)|^{q dt/t} \right)^{1/q} | L_p(\mathbb{R}^d) \right\| ,
\]
(2.6)
\[
\| f | F_{p,q}^s \|_2 = \| (\Phi_0 f)_a | L_p(\mathbb{R}^d) \| + \left\| \left( \int_0^1 t^{-sq} \sup_{z \in \mathbb{R}^d} \left| (\Phi_t * f)(x+z) \right| \left| \frac{q dt}{1+|z|/t} \right|^{1/q} | L_p(\mathbb{R}^d) \right\| ,
\]
(2.7)
\[
\| f | F_{p,q}^s \|_3 = \| \Phi_0 * f | L_p(\mathbb{R}^d) \| + \left\| \left( \int_0^1 \int_{|z|<t} |(\Phi_t * f)(x+z)|^{q dt/td} \right)^{1/q} | L_p(\mathbb{R}^d) \right\| ,
\]
(2.8)
\[
\| f | F_{p,q}^s \|_4 = \| \left( \sum_{k=0}^\infty 2^{skq} \sup_{z \in \mathbb{R}^d} \left| (\Phi_k * f)(x+z) \right| \right)^{1/q} | L_p(\mathbb{R}^d) \right\| ,
\]
(2.9)
\[
\| f | F_{p,q}^s \|_5 = \| \left( \sum_{k=0}^\infty 2^{skq} \left| (\Phi_k * f)(x) \right| \right)^{1/q} | L_p(\mathbb{R}^d) \right\| ,
\]
(2.10)
with the usual modification in case $q = \infty$. Furthermore, all quantities $\| f | F_{p,q}^s(\mathbb{R}^d) \|_i$, $i = 1, \ldots, 5$, are equivalent (quasi-)norms in $F_{p,q}^s(\mathbb{R}^d)$.

For the inhomogeneous Besov spaces we obtain the following.

Theorem 2.7. Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$, $a > d/p$ and $R+1 > s$. Let further $\Phi_0, \Phi \in \mathcal{S}(\mathbb{R}^d)$ be given by \eqref{2.4} and \eqref{2.5}. Then the space $B_{p,q}^s(\mathbb{R}^d)$ can be characterized by
\[
B_{p,q}^s(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \| f | B_{p,q}^s(\mathbb{R}^d) \|_i < \infty \} , \quad i = 1, \ldots, 4,
\]
where
\[
\| f | B_{p,q}^s \|_1 = \| \Phi_0 * f | L_p(\mathbb{R}^d) \| + \left\| \left( \int_0^1 t^{-sq} |(\Phi_t * f)(x)|^{q dt/t} \right)^{1/q} ,
\]
\[
\| f | B_{p,q}^s \|_2 = \| (\Phi_0 f)_a | L_p(\mathbb{R}^d) \| + \left\| \left( \int_0^1 t^{-sq} \sup_{z \in \mathbb{R}^d} \left| (\Phi_t * f)(x+z) \right| \left| \frac{q dt}{1+|z|/t} \right|^{1/q} | L_p(\mathbb{R}^d) \right\| ,
\]
\[
\| f | B_{p,q}^s \|_3 = \| \Phi_0 * f | L_p(\mathbb{R}^d) \| + \left\| \left( \int_0^1 \int_{|z|<t} |(\Phi_t * f)(x+z)|^{q dt/td} \right)^{1/q} | L_p(\mathbb{R}^d) \right\| ,
\]
\[
\| f | B_{p,q}^s \|_4 = \| \left( \sum_{k=0}^\infty 2^{skq} \sup_{z \in \mathbb{R}^d} \left| (\Phi_k * f)(x+z) \right| \right)^{1/q} | L_p(\mathbb{R}^d) \right\| ,
\]
with the usual modification if $q = \infty$. Furthermore, all quantities $\| f | B_{p,q}^s(\mathbb{R}^d) \|_i$, $i = 1, \ldots, 4$, are equivalent quasi-norms in $B_{p,q}^s(\mathbb{R}^d)$.
2.6 Homogeneous spaces

The homogeneous spaces can be characterized similar. Here we do not have a separate function \( \Phi_0 \) anymore. We put \( \Phi_0 = \Phi \).

**Theorem 2.8.** Let \( s \in \mathbb{R}, 0 < p < \infty, 0 < q \leq \infty, a > d/\min\{p,q\} \) and \( R + 1 > s \). Let further \( \Phi \in \mathcal{S}(\mathbb{R}^d) \) be given by (2.4) and (2.5). Then the space \( \dot{F}_{p,q}^s(\mathbb{R}^d) \) can be characterized by

\[
\dot{F}_{p,q}^s(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \| f \|_{\dot{F}_{p,q}^s(\mathbb{R}^d)} < \infty \}, \quad i = 1, \ldots, 5,
\]

where

\[
\| f \|_{\dot{F}_{p,q}^s(\mathbb{R}^d)} = \left( \int_0^\infty t^{-sq} \| (\Phi_t \ast f)(x) \|_{L_p(\mathbb{R}^d)}^q \frac{dt}{t} \right)^{1/q},
\]

with the usual modification if \( q = \infty \). Furthermore, all quantities \( \| f \|_{\dot{F}_{p,q}^s(\mathbb{R}^d)} \), \( i = 1, \ldots, 5 \), are equivalent quasi-norms in \( \dot{F}_{p,q}^s(\mathbb{R}^d) \).

For the homogeneous Besov spaces we obtain the following.

**Theorem 2.9.** Let \( s \in \mathbb{R}, 0 < p, q \leq \infty, a > d/p \) and \( R + 1 > s \). Let further \( \Phi \in \mathcal{S}(\mathbb{R}^d) \) be given by (2.4) and (2.5). Then the space \( \dot{B}_{p,q}^s(\mathbb{R}^d) \) can be characterized by

\[
\dot{B}_{p,q}^s(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \| f \|_{\dot{B}_{p,q}^s(\mathbb{R}^d)} < \infty \}, \quad i = 1, \ldots, 4,
\]

where

\[
\| f \|_{\dot{B}_{p,q}^s(\mathbb{R}^d)} = \left( \int_0^\infty t^{-sq} \| (\Phi_t \ast f)(x) \|_{L_p(\mathbb{R}^d)}^q \frac{dt}{t} \right)^{1/q},
\]

with the usual modification if \( q = \infty \). Furthermore, all quantities \( \| f \|_{\dot{B}_{p,q}^s(\mathbb{R}^d)} \), \( i = 1, \ldots, 4 \), are equivalent quasi-norms in \( \dot{B}_{p,q}^s(\mathbb{R}^d) \).
with the usual modification if \( q = \infty \). Furthermore, all quantities \( \| f | \dot{B}^s_{p,q}(\mathbb{R}^d) \|_i, \ i = 1, \ldots, 4 \) are equivalent quasi-norms in \( \dot{B}^s_{p,q}(\mathbb{R}^d) \).

**Remark 2.10.** Observe, that the (quasi-)norms \( \| \cdot | \dot{F}^s_{p,q}(\mathbb{R}^d) \|_3 \) and \( \| \cdot | F^s_{p,q}(\mathbb{R}^d) \|_3 \) are characterizations via Lusin functions, see [30, 2.4.5] and [28, 2.12.1] and the references given there. We will return to it later when defining tent spaces, see Definition 4.1 and (4.1).

### 2.7 Particular kernels

For more details concerning particular choices for the kernels \( \Phi_0 \) and \( \Phi \) we refer mainly to Triebel [30, 3.3].

The most prominent nontrivial examples (besides the one given in Remark 2.3) of functions \( \Phi_0 \) and \( \Phi \) satisfying (2.4) and (2.5) are the classical local means. The name comes from the compact support of \( \Phi_0, \Phi \), which is admitted in the following statement.

**Corollary 2.11.** Let \( p,q,s \) as in Theorem 2.6. Let further \( k_0, k^0 \in S(\mathbb{R}^d) \) such that

\[
\mathcal{F}k_0(0), \mathcal{F}k^0(0) \neq 0
\]

and define

\[
\Phi_0 = k_0 \quad \text{and} \quad \Phi = \Delta^N k^0
\]

with \( N \in \mathbb{N} \) such that \( 2N > s \). Then (2.6), (2.7), (2.8), (2.9) and (2.10) characterize \( F^s_{p,q}(\mathbb{R}^d) \).

**Corollary 2.12.** Let \( p,q,s \) as in Theorem 2.6. Let further \( \varphi_0 \in S(\mathbb{R}^d) \) be a non-increasing radial function satisfying

\[
\varphi_0(0) \neq 0 \quad \text{and} \quad \mathcal{D}^\alpha \varphi_0(0) = 0
\]

for \( 1 \leq |\alpha|_1 \leq R \), where \( R + 1 > s \). Define \( \varphi := \varphi_0(\cdot) - \varphi_0(2\cdot) \) and put \( \Phi_0 := \mathcal{F}^{-1} \varphi_0 \) and \( \Phi := \mathcal{F}^{-1} \varphi \). Then (2.9) and (2.10) characterize \( F^s_{p,q}(\mathbb{R}^d) \).

### 2.8 Proofs

We give the proof for Theorem 2.6 in full detail. The proof of Theorem 2.8 is similar and even less technical. Let us refer to the respective paragraph for the necessary modifications. The proofs in the Besov scale are analogous, so we omit them completely. The proof technique is a modification of the one in Rychkov [23], where he proved the discrete case, i.e., that (2.9) and (2.10) characterize \( F^s_{p,q}(\mathbb{R}^d) \). However, Hansen [15, Rem. 3.2.4] recently observed that the arguments used for proving (34) in [23] are somehow problematic. The finiteness of the Peetre maximal function is assumed which is not true in general under the stated assumptions. Consider for instance in dimension \( d = 1 \) the functions

\[
\Psi_0(t) = \Psi_1(t) = e^{-t^2}
\]

and, if \( a > 0 \) is given, the tempered distribution \( f(t) = |t|^a \) with \( a < n \in \mathbb{N} \). Then \( (\Psi_1 f)_a(x) \) is infinite in every point \( x \in \mathbb{R} \). The mentioned incorrect argument was inherited to some subsequent papers dealing with similar topics, for instance [1], [17] and [33]. Anyhow, the stated results hold true. There is an alternative method to prove the crucial inequality (34) which avoids Lemma 3 in [23]. It is given in Rychkov [24] as well as [25]. A variant of this method,
which is originally due to Strömberg, Torchinsky [27, Chapt. V], is also used in our proof below.

We start with a convolution type inequality which will be often needed below. The following lemma is essentially Lemma 2 in [23].

**Lemma 2.13.** Let $0 < p, q \leq \infty$ and $\delta > 0$. Let $\{g_k\}_{k \in \mathbb{N}_0}$ be a sequence of non-negative measurable functions on $\mathbb{R}^d$ and put

$$G_\ell(x) = \sum_{k \in \mathbb{Z}} 2^{-|k-\ell|\delta} g_k(x), \quad x \in \mathbb{R}^d, \ell \in \mathbb{Z}.$$ 

Then there is some constant $C = C(p, q, \delta)$, such that

$$\|\{G_\ell\}|_{L_p(\mathbb{R}^d)}\| \leq C \|\{g_k\}|_{L_p(\mathbb{R}^d)}\|$$

and

$$\|\{G_\ell\}|_{L_p(\mathbb{R}^d)}\| \leq C \|\{g_k\}|_{L_p(\mathbb{R}^d)}\|$$

hold true.

**Proof of Theorem 2.6**

To begin with we prove the equivalence of the characterizations (2.6), (2.7), (2.9) and (2.10) for the same system $(\Phi_0, \Phi)$. The next step is to change from the system $(\Phi_0, \Phi)$ to a second one $(\Psi_0, \Psi)$ satisfying (2.4), (2.5) within the characterization (2.9). The equivalence of (2.9) and (2.10) was the original proof by Rychkov in [23]. Since Definition 2.4 can be seen as a special case of (2.10), we have that (2.6), (2.7), (2.9) and (2.10) generate the same space for all pairs $(\Phi_0, \Phi)$ satisfying (2.4) and (2.5), namely $F_{s, p, q}(\mathbb{R}^d)$. It remains to prove that (2.8) is equivalent to the rest.

**Step 1.**

We are going to prove the relations

$$\|f|_{F_{p, q}^s(\mathbb{R}^d)}\|_1 \asymp \|f|_{F_{p, q}^s(\mathbb{R}^d)}\|_2 \asymp \|f|_{F_{p, q}^s(\mathbb{R}^d)}\|_4 \asymp \|f|_{F_{p, q}^s(\mathbb{R}^d)}\|_5$$  \hspace{1cm} (2.12)

for every $f \in \mathcal{S}'(\mathbb{R}^d)$. We just give the proof of $\|f|_{F_{p, q}^s(\mathbb{R}^d)}\|_1 \asymp \|f|_{F_{p, q}^s(\mathbb{R}^d)}\|_2$ in detail since the remaining equivalences are analogous.

**Substep 1.1.**

Put $\varphi_0 = \mathcal{F}\Phi_0$ and $\varphi_\ell = (\mathcal{F}\Phi)(2^{-\ell} \cdot)$ if $\ell \geq 1$. Because of (2.4) it is possible to find functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^d)$ with $\text{supp} \psi_0 \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2 \varepsilon\}$, $\text{supp} \psi \subset \{\xi \in \mathbb{R}^d : \varepsilon/2 \leq |\xi| \leq 2 \varepsilon\}$ and $\psi_\ell(x) = \psi(2^{-\ell}x)$ such that

$$\sum_{\ell \in \mathbb{N}_0} \varphi_\ell(\xi) \cdot \psi_\ell(\xi) = 1.$$ 

We need a bit more. Fix a $1 \leq t \leq 2$. Clearly, we also have

$$\sum_{\ell \in \mathbb{N}_0} \varphi_\ell(t\xi) \cdot \psi_\ell(t\xi) = 1.$$
for all \( \xi \in \mathbb{R}^d \). With \( \Psi_0 = F^{-1} \psi_0 \) and \( \Psi = F^{-1} \psi \) we obtain then
\[
g = \sum_{m \in \mathbb{N}_0} (\Psi_m)_t \ast (\Phi_m)_t \ast g.
\]

We dilate this identity with \( 2^\ell \), i.e., \( g_\ell(\eta) = g(2^{-\ell d} \eta (2^{-\ell} \cdot)) \) for \( \eta \in \mathcal{S}(\mathbb{R}^d) \). An elementary calculation gives
\[
g_\ell = \sum_{m \in \mathbb{N}_0} (\Psi_m)_{t2^{-\ell}} \ast (\Phi_m)_{t2^{-\ell}} \ast g_\ell \tag{2.13}
\]
for every \( g \in \mathcal{S}'(\mathbb{R}^d) \). Obviously, we can rewrite (2.13) to obtain
\[
g = \sum_{m \in \mathbb{N}_0} (\Psi_m)_{t2^{-\ell}} \ast (\Phi_m)_{t2^{-\ell}} \ast g \tag{2.14}
\]
for all \( g \in \mathcal{S}'(\mathbb{R}^d) \). Let us now choose \( g = (\Phi_\ell)_t \ast f \) which gives for all \( f \in \mathcal{S}'(\mathbb{R}^d) \) the identity
\[
(\Phi_\ell)_t \ast f = \sum_{m \in \mathbb{N}_0} (\Phi_\ell)_t \ast (\Psi_m)_{t2^{-\ell}} \ast (\Phi_m)_{t2^{-\ell}} \ast f. \tag{2.15}
\]

For \( m, \ell \in \mathbb{N}_0 \) we define
\[
\Lambda_{m,\ell}(x) = \begin{cases} 2^{\ell d} \Phi_0(2^\ell x) & : m = 0 \\ \Phi_\ell(x) & : m > 0 \end{cases}, \quad x \in \mathbb{R}^d. \tag{2.16}
\]

Clearly, we have
\[
(\Phi_\ell)_t \ast (\Phi_m)_{t2^{-\ell}} = (\Lambda_{m,\ell})_t \ast (\Phi_{m+\ell})_t.
\]

Plugging this into (2.15) we end up with the pointwise representation \( (\ell \in \mathbb{N}) \)
\[
((\Phi_\ell)_t \ast f)(y) = \sum_{m \in \mathbb{N}_0} ((\Psi_m)_{2^{-\ell}} \ast (\Lambda_{m,\ell})_t \ast (\Phi_{m+\ell})_t \ast f)(y)
\]
\[
= \sum_{m \in \mathbb{N}_0} [(\Psi_m)_{2^{-\ell}} \ast (\Lambda_{m,\ell})_t] \ast ((\Phi_{m+\ell})_t \ast f)(y) \tag{2.17}
\]
\[
= \sum_{m \in \mathbb{N}_0} \int \left[ (\Psi_m)_{2^{-\ell}} \ast (\Lambda_{m,\ell})_t \right](y - z) \cdot ((\Phi_{m+\ell})_t \ast f)(z) \, dz
\]
for all \( y \in \mathbb{R}^d \). Let us mention that the case \( \ell = 0 \) plays a particular role. In this case we have to replace \( (\Phi_\ell)_t \) by \( \Phi_\ell \) in (2.15) and (2.17), (\( \Phi_{m+\ell})_t \) by \( \Phi_{m+\ell} \) in (2.17) if \( m = 0 \) and finally \( (\Lambda_{m,\ell})_t \) by \( \Lambda_{m,\ell} \) if \( m > 0 \).

**Substep 1.2.** Let us prove the following important inequality first. For every \( r > 0 \) and every \( N \in \mathbb{N}_0 \) we have
\[
|((\Phi_\ell)_t \ast f)(x)|^r \leq c \sum_{k \in \mathbb{N}_0} 2^{-kN} r 2^{k+\ell} \int_{\mathbb{R}^d} \left| \frac{((\Phi_{k+\ell})_t \ast f)(y))^r}{(1 + 2^{\ell} |x - y|)^{Nr}} \right| dy, \tag{2.18}
\]
where \(c\) is independent of \(f \in \mathcal{S}'(\mathbb{R}^d)\), \(x \in \mathbb{R}^d\) and \(\ell \in \mathbb{N}_0\). Again the case \(\ell = 0\) has to be treated separately according to the remark after (2.17). The representation (2.17) will be the starting point to prove (2.18). Namely, we have for \(y \in \mathbb{R}^d\)

\[
|((\Phi_\ell) * f)(y)| \leq \sum_{m \in \mathbb{N}_0} \int_{\mathbb{R}^d} \left|((\Psi_m)_{2^{-\ell}} * (\Lambda_{m,\ell})_t)(y - z) \cdot |((\Phi_{m+\ell})_t * f)(z)| dz
\]

\[
\leq \sum_{m \in \mathbb{N}_0} S_{m,\ell,t} \int_{\mathbb{R}^d} \frac{|((\Phi_{m+\ell})_t * f)(z)|}{(1 + 2^\ell |y - z|)^N} dz,
\]

(2.19)

where

\[
S_{m,\ell,t} = \sup_{x \in \mathbb{R}^d} \left|((\Psi_m)_{2^{-\ell}} * (\Lambda_{m,\ell})_t)(x) \cdot (1 + 2^{\ell}|x|)^N \right|.
\]

Elementary properties of the convolution yield (compare with (2.34))

\[
S_{m,\ell,t} = \frac{2^{\ell d}}{\ell^d} \sup_{x \in \mathbb{R}^d} |((\Psi_m * (\Lambda_{m,\ell})_t)(x)2^{\ell t}| \cdot (1 + 2^{\ell}|x|)^N
\]

\[
= \frac{2^{\ell d}}{\ell^d} \sup_{x \in \mathbb{R}^d} |((\Psi_m * \eta_{m,\ell})(x)| \cdot (1 + |t x|)^N,
\]

where

\[
\eta_{m,\ell}(x) = \begin{cases} 
\Phi(x) & \text{if } m > 0, \ell > 0, \\
\Phi_0(x) & \text{otherwise}.
\end{cases}
\]

With Lemma A.3 we see

\[
S_{m,\ell,t} \leq C_N 2^{\ell d} 2^{-mN}
\]

and put it into (2.19) to obtain

\[
|((\Phi_\ell) * f)(y)| \leq C_N \sum_{m \in \mathbb{N}_0} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}|((\Phi_{m+\ell})_t * f)(z)|}{(1 + 2^{\ell}|y - z|)^N} dz
\]

(2.20)

with the appropriate modification in case \(\ell = 0\). To continue we prefer the strategy used by Rychkov in [24 Thm. 3.2] and [25 Lem. 2.9]. Let us replace \(\ell\) by \(k + \ell\) in (2.20) and multiply on both sides with \(2^{-kN}\). Then we can estimate

\[
2^{-kN}|((\Phi_{k+\ell})_t * f)(y)| \leq C_N \sum_{m \in \mathbb{N}_0} 2^{-kN} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+k+\ell)d}|((\Phi_{m+k+\ell})_t * f)(z)|}{(1 + 2^{k+\ell}|y - z|)^N} dz
\]

(2.21)

\[
\leq C_N \sum_{m \in \mathbb{N}_0} 2^{-(m+k)N} \int_{\mathbb{R}^d} \frac{2^{(m+k+\ell)d}|((\Phi_{m+k+\ell})_t * f)(z)|}{(1 + 2^{k+\ell}|y - z|)^N} dz
\]

\[
= C_N \sum_{m \in \mathbb{N}_0} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}|((\Phi_{m+\ell})_t * f)(z)|}{(1 + 2^{\ell}|y - z|)^N} dz
\]

\[
\leq C_N \sum_{m \in \mathbb{N}_0} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}|((\Phi_{m+\ell})_t * f)(z)|}{(1 + 2^{\ell}|y - z|)^N} dz.
\]

(2.22)
Next, we apply the elementary inequalities
\[
(1 + 2^\ell |y - z|) \cdot (1 + 2^\ell |x - y|) \geq (1 + 2^\ell |x - z|),
\]
where \(0 < r \leq 1\). Let us define the maximal function
\[
M_{\ell,N}(x,t) = \sup_{k \in \mathbb{N}_0} \sup_{y \in \mathbb{R}^d} 2^{-kN} \frac{|((\Phi_{m+\ell})_t * f)(y)|}{(1 + 2^\ell |x - y|)^N}, \quad x \in \mathbb{R}^d, \tag{2.24}
\]
and estimate
\[
M_{\ell,N}(x,t) \leq C_N \sum_{m \in \mathbb{N}_0} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}|((\Phi_{m+\ell})_t * f)(z)|}{(1 + 2^\ell |x - z|)^N} \, dz \tag{2.25}
\]
\[
\leq C_N \sum_{m \in \mathbb{N}_0} 2^{-mNr} \left( \sup_{y \in \mathbb{R}^d} \frac{|((\Phi_{m+\ell})_t * f)(y)|}{(1 + 2^\ell |x - y|)^N} \right)^{1-r} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}|((\Phi_{m+\ell})_t * f)(z)|^r}{(1 + 2^\ell |x - z|)^{Nr}} \, dz. \tag{2.26}
\]
Observe that we can estimate the term \((...)^{1-r}\) in the right-hand side of (2.26) by \(M_{\ell,N}(x,t)^{1-r}\). Hence, if \(M_{\ell,N}(x,t) < \infty\) we obtain from (2.26)
\[
M_{\ell,N}(x,t)^r \leq C_N \sum_{m \in \mathbb{N}_0} 2^{-mNr} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}|((\Phi_{m+\ell})_t * f)(z)|^r}{(1 + 2^\ell |x - z|)^{Nr}} \, dz, \tag{2.27}
\]
where \(C_N\) is independent of \(x, f, \ell\) and \(t \in [1, 2]\). We claim that there exists \(N^f \in \mathbb{N}_0\) such that \(M_{\ell,N}(x,t) < \infty\) for all \(N \geq N^f\). Indeed, we use that \(f \in \mathcal{S}'(\mathbb{R}^d)\), i.e., there is an \(M \in \mathbb{N}_0\) and \(c_f > 0\) such that
\[
|((\Phi_{k+\ell})_t * f)(y)| \leq c_f \sup_{|\alpha|_1 \leq M} \sup_{z \in \mathbb{R}^d} |D^\alpha \Phi_{k+\ell}(z)| \cdot (1 + |y - z|)^M,
\]
see (2.2). Assuming \(N > M\) we estimate as follows
\[
|((\Phi_t)_t * f)(x)| \leq M_{\ell,N}(x,t) \tag{2.28}
\]
\[
\leq c \sup_{k \in \mathbb{N}_0} \sup_{y \in \mathbb{R}^d} 2^{-kN} \frac{|((\Phi_{k+\ell})_t * f)(y)|}{(1 + |x - y|^2)^{\frac{N}{2}}} \]
\[
\leq c \sup_{k \in \mathbb{N}_0} \sup_{y \in \mathbb{R}^d} 2^{(k+\ell)(M+d)} \sup_{z \in \mathbb{R}^d} |D^\alpha \gamma_{k+\ell}(z)| \cdot (1 + |y - z|)^M \sup_{|\alpha|_1 \leq M} \frac{1}{(1 + |x - y|)^N}
\]
\[
\leq c2^{(M+d)} \sup_{k \in \mathbb{N}_0} \sup_{z \in \mathbb{R}^d} |D^\alpha \gamma_{k+\ell}(z)| (1 + |x - z|)^N,
\]
where we again used the inequality (compare with (2.28))
\[
1 + |y - z| \leq (1 + |x - y|)(1 + |x - z|)
\]
and put
\[
\gamma_\ell(t) = \begin{cases} 
\Phi_0(t) & : \ell = 0 \\
\Phi(t) & : \ell > 0 
\end{cases}.
\]
Hence \(\gamma_{k+\ell}\) gives us only two different functions from \(S(\mathbb{R}^d)\). This implies the boundedness of \(M_{\ell,N}(x,t)\) for \(x \in \mathbb{R}^d\) if \(N > M = N_f\). Therefore, (2.27) together with (2.28) yield (2.18) with \(c = C_N\), independent of \(x, f\) and \(\ell\), for all \(N \geq N_f\). But this is not yet what we want. Observe that the right-hand side of (2.18) decreases as \(N\) increases. Therefore, we have (2.18) for all \(N \in \mathbb{N}_0\) but with \(c = c(f) = C_{N_f}\) depending on \(f\). This is still not yet what we want. Now we argue as follows: Starting with (2.18) where \(c = c(f)\) and \(N \in \mathbb{N}_0\) arbitrary, we apply the same arguments as used from (2.21) to (2.22), switch to the maximal function (2.24) with the help of (2.23) and finish with (2.27) instead of (2.25) but with a constant that depends on \(f\). But this does not matter now. Important is, that a finite right-hand side of (2.27) (which is the same as rhs(2.18)) implies \(M_{\ell,N}(x,t) < \infty\).

We assume rhs(2.18) < \(\infty\). Otherwise there is nothing to prove in (2.18). Returning to (2.26) and having in mind that now \(M_{\ell,N}(x,t) < \infty\), we end up with (2.27) for all \(N\) and \(C_N\) independent of \(f\). Finally, from (2.27) we obtain (2.18) and are done in case \(0 < r \leq 1\).

Of course, (2.18) also holds true for \(r > 1\) with a much simpler proof. In that case, we use (2.20) with \(N + d + \varepsilon\) instead of \(N\) and apply Hölder’s inequality with respect to \(1/r + 1/r' = 1\) first for integrals and then for sums.

**Substep 1.3.**

The inequality (2.18) implies immediately a stronger version of itself. Using (2.23) again we obtain for \(a \leq N\) and \(\ell \in \mathbb{N}\)
\[
(\Phi_{2^{-\ell}t}f)_a(x) \leq c \sum_{k \in \mathbb{N}_0} 2^{-kNr} 2^{(k+\ell)d} \int_{\mathbb{R}^d} \frac{|(\Phi_{k+\ell})_t * f)(y)|^r}{(1 + 2^{|x - y|})^a} dy.
\]
(2.29)

In case \(\ell = 0\) we have to replace \((\Phi_{2^{-\ell}t}f)_a(x)\) by \((\Phi_0 f)_a(x)\) on the left-hand side and \((\Phi_{k+\ell})_t\) by \(\Phi_{k+\ell} = \Phi_0\) for \(k = 0\) on the right-hand side. We proved, that the inequality (2.29) holds for all \(t \in [1, 2]\) where \(c > 0\) is independent of \(t\). If we choose \(r < \min\{p, q\}\), we can apply the norm
\[
\left( \int_{1}^{2} \frac{|\cdot|^{q/r} dt}{t} \right)^{r/q}.
\]
on both sides and use Minkowski’s inequality for integrals, which yields
\[
\left( \int_{1}^{2} |(\Phi_{2^{-\ell}t}f)_a(x)|^q dt \right)^{r/q} \leq c \sum_{k \in \mathbb{N}_0} 2^{-kNr} 2^{(k+\ell)d} \int_{\mathbb{R}^d} \frac{\left( \int_{1}^{2} |(\Phi_{k+\ell})_t * f)(y)|^q dt \right)^{r/q}}{(1 + 2^{|x - y|})^a} dy.
\]
(2.30)

If \(ar > d\) then we have
\[
g_\ell(y) = \frac{2^{dt}}{(1 + 2^{|y|a})} \in L_1(\mathbb{R}^d)
\]
and we observe 
\[
\left( \int_1^2 |\mathcal{L} (\Phi^*_{s-\ell t})_a(x)|^q \frac{dt}{t} \right)^{r/q} \leq c \sum_{k \in \mathbb{N}} 2^{-kN_r q_{\ell} d_{\ell s r}} \left[ g_{\ell} \left( \int_1^2 |\mathcal{L} (\Phi^*_{k+\ell t})_a(\cdot)|^q \frac{dt}{t} \right)^{r/q} \right](x) .
\]

Now we use a well-known majorant property in order to estimate the convolution on the right-hand side by the Hardy-Littlewood maximal function (see Paragraph 2.2 and [26, Chapt. 2]). This yields 
\[
\left( \int_1^2 |\mathcal{L} (\Phi^*_{s-\ell t})_a(x)|^q \frac{dt}{t} \right)^{r/q} \leq c \sum_{k \in \mathbb{N}} 2^{r_{\ell s} 2^{k(-N_r + d)}} M \left[ \left( \int_1^2 |((\Phi^*_{k+\ell t})_a(\cdot)|^q \frac{dt}{t} \right)^{r/q} \right](x) .
\]
An index shift on the right-hand side gives
\[
\left( \int_1^2 |\mathcal{L} (\Phi^*_{s-\ell t})_a(x)|^q \frac{dt}{t} \right)^{r/q} \leq c \sum_{k \in \ell + \mathbb{N}} 2^{\ell_{\ell s} 2^{k(-N_r + d)}} M \left[ \left( \int_1^2 |((\Phi^*_{k+\ell t})_a(\cdot)|^q \frac{dt}{t} \right)^{r/q} \right](x) .
\]
Choose now 
\[
d/a < r < \min\{p, q\}, \quad N > \max\{0, -s\} + a \quad \text{and put} \quad \delta = N + s - d/r > 0 .
\]
We obtain for \( \ell \in \mathbb{N} \)
\[
\left( \int_1^2 |\mathcal{L} (\Phi^*_{s-\ell t})_a(x)|^q \frac{dt}{t} \right)^{r/q} \leq c \sum_{k \in \mathbb{N}} 2^{-\delta r_{\ell k} 2^{kr_{\ell s} \ell}} M \left[ \left( \int_1^2 |((\Phi^*_{k+\ell t})_a(\cdot)|^q \frac{dt}{t} \right)^{r/q} \right](x) .
\]
Now we apply Lemma 2.13 in \( L_{p/r}(\ell_{q/r}, \mathbb{R}^d) \) which yields
\[
\left\| \left( \int_1^2 |\mathcal{L} (\Phi^*_{s-\ell t})_a(x)|^q \frac{dt}{t} \right)^{r/q} \right\|_{L_{p/r}(\ell_{q/r})} \leq c \left\| M \left[ \left( \int_1^2 |2^{k_{\ell s}} ((\Phi^*_{k+\ell t})_a(\cdot)|^q \frac{dt}{t} \right)^{r/q} \right] \right\|_{L_{p/r}(\ell_{q/r})} .
\]
The Fefferman-Stein inequality (see Paragraph 2.2/Theorem 2.1, having in mind that $p/r, q/r > 1$) gives

$$\left\| \left( \int_1^2 |2^{ks}(\Phi^*_2 - \ell f)_a(x)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\ell_q)}$$

$$\lesssim \left\| M \left[ \left( \int_1^2 |2^{ks}(\Phi^*_2 - \ell f)_a(x)|^q \frac{dt}{t} \right)^{r/q} \right] \right\|_{L_{p/r}(\ell_{q/r})}$$

$$\lesssim \left\| \left( \int_1^2 |2^{ks}(\Phi^*_2) \ast f(\cdot)|^q \frac{dt}{t} \right)^{r/q} \right\|_{L_{p/r}(\ell_{q/r})}$$

$$= \left\| \left( \int_1^2 |2^{ks}(\Phi^*_2) \ast f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\ell_q)}.$$

Hence, we obtain

$$\left\| \left( \int_0^1 |\lambda^{-\ell q}(\Phi^*_\lambda f)_a(x)|^q \frac{d\lambda}{\lambda} \right)^{1/q} \right\|_{L_p(\ell_q)} \lesssim \left\| \left( \sum_{\ell=1}^{\infty} \int_1^2 |2^{ks}(\Phi^*_2 - \ell f)_a(x)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\ell_q)}$$

$$= \left\| \left( \int_1^2 |2^{ks}(\Phi^*_2 - \ell f)_a(x)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\ell_q)}$$

$$\leq \left\| \left( \int_0^1 \lambda^{-\ell q}(\Phi^*_\lambda f(x)) \frac{d\lambda}{\lambda} \right)^{1/q} \right\|_{L_p(\ell_q)}.$$

The summand $\|(\Phi^*_\lambda f)_a|_{L_p(\ell_q)}\|$ can be estimated similar using (2.29) in case $\ell = 0$. This proves $\|f|_{F_{p,q}(\ell_q)}\|_2 \lesssim \|f|_{F_{p,q}(\ell_q)}\|_1$. With slight modifications of the argument we prove as well $\|f|_{F_{p,q}(\ell_q)}\|_2 \lesssim \|f|_{F_{p,q}(\ell_q)}\|_5$, $\|f|_{F_{p,q}(\ell_q)}\|_4 \lesssim \|f|_{F_{p,q}(\ell_q)}\|_1$, and $\|f|_{F_{p,q}(\ell_q)}\|_4 \lesssim \|f|_{F_{p,q}(\ell_q)}\|_5$. The inequalities $\|f|_{F_{p,q}(\ell_q)}\|_5 \lesssim \|f|_{F_{p,q}(\ell_q)}\|_4$ and $\|f|_{F_{p,q}(\ell_q)}\|_4 \lesssim \|f|_{F_{p,q}(\ell_q)}\|_2$ are immediate. This finishes the proof of (2.12).

**Step 2.** Let $\Psi_0, \Psi \in S(\ell_q)$ be functions satisfying (2.5). Indeed, we do not need (2.4) for the following inequality

$$\|f|_{F_{p,q}(\ell_q)}\|_4^\Psi \lesssim \|f|_{F_{p,q}(\ell_q)}\|_4^\Psi \quad (2.31)$$

which holds true for all $f \in S(\ell_q)$. We decompose $f$ similar as in Step 1. Exploiting the property (2.4) for the system $(\Phi_0, \Phi)$ we find $S(\ell_q)$-functions $\lambda_0, \lambda \in S(\ell_q)$ such that supp $\lambda_0 \subset \{\xi \in \ell_q^d : |\xi| \leq 2\varepsilon\}$ and supp $\lambda \subset \{\xi \in \ell_q^d : \varepsilon/2 \leq |\xi| \leq 2\varepsilon\}$ and

$$\sum_{k \in \mathbb{N}_0} \lambda_k(\xi) \cdot \varphi_k(\xi) = 1$$

for $\xi \in \ell_q^d$. Putting $\Lambda_0 = \mathcal{F}^{-1} \lambda_0$ and $\Lambda = \mathcal{F}^{-1} \lambda$ we obtain the decomposition

$$g = \sum_{k \in \mathbb{N}_0} \Lambda_k \ast \Phi_k \ast g \quad (2.32)$$
for every \( g \in S'(\mathbb{R}^d) \). We put \( g = \Psi_\ell * f \) for \( \ell \in \mathbb{N}_0 \) and see

\[
\Psi_\ell * f = \sum_{k \in \mathbb{N}_0} \Psi_\ell * \Lambda_k * \Phi_k * f.
\]  

(2.33)

Now we estimate as follows

\[
\left| (\Psi_\ell * \Lambda_k) * (\Phi_k * f) \right|(y) \leq \int_{\mathbb{R}^d} \left| (\Psi_\ell * \Lambda_k)(z) \cdot (\Phi_k * f)(y - z) \right| dz
\]

\[
\leq (\Phi^*_k f)_a(y) \int_{\mathbb{R}^d} \left| (\Psi_\ell * \Lambda_k)(z) \right| \cdot (1 + 2^k |z|)^a dz
\]

\[
\leq (\Phi^*_k f)_a(y) J_{\ell,k},
\]

where

\[
J_{\ell,k} = \int_{\mathbb{R}^d} \left| (\Psi_\ell * \Lambda_k)(z) \right| (1 + 2^k |z|)^a dz.
\]

We first observe that for \( x \in \mathbb{R}^d \) and functions \( \mu, \eta \in S(\mathbb{R}^d) \) the following identity holds true for \( u, v > 0 \)

\[
(\mu * \eta_v)(z) = \frac{1}{u^d} \tilde{[\mu * \eta_v/u]}(z/u) = \frac{1}{v^d} \tilde{[\mu_{u/v} * \eta]}(z/v).
\]  

(2.34)

This yields in case \( \ell \geq k \) (with a minor change if \( k = 0 \))

\[
J_{\ell,k} \leq \sup_{z \in \mathbb{R}^d} \left| (\Psi_\ell * \Lambda_k)(z) \right| (1 + |z|)^a + d + 1
\]

\[
\lesssim 2^{(k-\ell)(L+1-a)},
\]

where we used Lemma A.3 for the last estimate.

If \( k > \ell \) we change the roles of \( \Psi \) and \( \Lambda \) to obtain again with Lemma A.3 (minor change if \( \ell = 0 \))

\[
J_{\ell,k} \leq \int_{\mathbb{R}^d} \left| (\Psi * \Lambda_{k-\ell})(z) \right| (1 + |2^{k-\ell} z|)^a dx
\]

\[
\lesssim 2^{(k-\ell)a} \sup_{z \in \mathbb{R}^d} \left| (\Psi * \Lambda_{k-\ell})(z) \right| (1 + |z|)^a + d + 1
\]

\[
\lesssim 2^{(\ell-k)(L+1-a)},
\]

where \( L \) can be chosen arbitrary large since \( \Lambda \) satisfies \((M_L)\) for every \( L \in \mathbb{N} \) according to its construction. Let us further use the estimate

\[
(\Phi^*_k f)_a(y) \leq (\Phi^*_k f)_a(x)(1 + 2^k |x - y|)^a
\]

\[
\lesssim (\Phi^*_k f)_a(x)(1 + 2^d |x - y|)^a \max\{1, 2^{(k-\ell)a}\}.
\]
Consequently,
\[
\sup_{y \in \mathbb{R}^d} \frac{2^{k_s}|(\Psi_f \ast A_k \ast (\Phi_k \ast f))(y)|}{(1 + 2^{k_s}|x - y|)^a} \lesssim 2^{k_s}(\Phi_k^\ast f)_a(x)2^{(\ell-k)s} \max\{1, 2^{(k-\ell)a}\} J_{k,k}
\]

\[
\leq 2^{k_s}(\Phi_k^\ast f)_a(x) \left\{ \begin{array}{ll}
2^{(\ell-k)(L+1-a+s)} & : k > \ell \\
2^{(\ell-k)(R+1-s)} & : \ell \geq k
\end{array} \right.
\]

Plugging this into (2.33), choosing \( L \geq a + |s| \) and \( \delta = \min\{1, R + 1 - s\} \) we obtain the inequality
\[
2^{\ell s}(\Psi^\ast f)_a(x) \lesssim \sum_{k=0}^{\infty} 2^{-|k-\ell|\delta} 2^{k_s}(\Phi_k^\ast f)_a(x). \quad (2.35)
\]

for all \( x \in \mathbb{R}^d \). Applying Lemma 2.13 gives (2.31).

**Step 3.** What remains is to show that (2.8) is equivalent to the rest.

**Substep 3.1.** Let us prove
\[
\|f\|_{F^s_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{F^s_{p,q}(\mathbb{R}^d)}. \quad (2.36)
\]

We return to (2.29) in Substep 1.3. If \(|z| < 2^{-(\ell+k)t}\) formula (2.29) implies by shift in the integral the following
\[
(\Phi_{2-\ell t}^\ast f)_a(x)^r \leq C_N \sum_{k \in \mathbb{N}_0} 2^{-k(N-a)r} 2^{(k+\ell)d} \int_{\mathbb{R}^d} \left| (\Phi_{k+\ell} \ast f)(y + z) \right|^r \frac{d}{(1 + 2^r|x - y|)} dy. \quad (2.37)
\]

Indeed, we have
\[
1 + 2^r|x - y| \leq 1 + 2^r(|x - (y + z)| + |z|) \lesssim 1 + 2^r(|x - (y + z)|) + 2^{-k} \lesssim 1 + 2^r(|x - (y + z)|).
\]

Where the last estimate follows from the fact that \( k \in \mathbb{N}_0 \) in the sum. Instead of the integral \((\int_1^2 | \cdot |^{q/r} dt/t)^{r/q}\) we now take on both sides of (2.37) the norm
\[
\left( \int_1^2 \int_{|z| < t} | \cdot |^{q/r} \frac{dz}{t^{d+1}} \right)^{r/q}.
\]

The integration over \( z \) does not influence the left-hand side. Instead of (2.30) we obtain
\[
\left( \int_1^2 |(\Phi_{2-\ell t}^\ast f)_a(x)|^r \frac{dt}{t} \right)^{r/q} \leq c \sum_{k \in \mathbb{N}_0} 2^{-kNs} 2^{(k+\ell)d} \int_{\mathbb{R}^d} \left( \int_1^2 \int_{|z| < t} |(\Phi_{k+\ell}^\ast f)(y)|^q \frac{d}{(1 + 2^r|x - y|)} \right)^{r/q} dy.
\]

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We continue with analogous arguments as after (2.30) and end up with (2.36).

**Substep 3.2.** We prove \( \|f|_{F_{p,q}^s(\mathbb{R}^d)} \|_2 \lesssim \|f|_{F_{p,q}^s(\mathbb{R}^d)} \|_2 \). Indeed, it is easy to see, that we have for all \( t > 0 \)
\[
\frac{1}{t^d} \int_{|z| < t} |(\Phi_t * f)(x + z)| \, dz \lesssim \sup_{|z| < t} \frac{|(\Phi_t * f)(x + z)|}{(1 + |z|/t)^a} \lesssim (\Phi_t f)_a(x),
\]
and we are done. The proof is complete.

**Proof of Theorem 2.8**

The proof of Theorem 2.8 is almost the same as the previous one. It is less technical since we do not have to deal with a separate function \( \Phi_0 \) which causes several difficulties. However, there are still some technical obstacles which have to be discussed.

1. Although we are in the homogeneous world, we use the same decomposition as used in (2.14), even with the inhomogeneity \( \Phi_0 \). In the definition of \( \Lambda_{m,\ell}(x) \) in (2.16) we have to put in addition \( \Phi(x) \), if \( \ell = 0 \) and \( m > 0 \). The consequence is equation (2.17) for every \( \ell \in \mathbb{Z} \). Hence, the inhomogeneity is shifted to \( \Lambda_{m,\ell} \). This yields (2.29) for all \( \ell \in \mathbb{Z} \), where \( k \) still runs through \( \mathbb{N}_0 \). We need this for the argument in Substep 3.1.

2. In contrast to the previous decomposition, we use (2.32), (2.33) now for \( k,\ell \in \mathbb{Z} \), where \( \Phi_0 = \Phi \) and \( \Lambda_0 = \Lambda \). This works since we assume \( g \in S'_0(\mathbb{R}^d) \). Now we can even prove \( \|f|_{F_{p,q}^s(\mathbb{R}^d)} \|_2 \lesssim \|f|_{F_{p,q}^s(\mathbb{R}^d)} \|_2 \) and vice versa.

**Proof of Corollary 2.11 and 2.12**

1. The proof of Corollary 2.11 is immediate. We know that \( \Delta^N \) gives \( (\sum_{k=1}^{d} |\xi_k|^2)^N \) as factor on the Fourier side. This gives (2.5) immediately and together with (2.11) we have (2.4) for \( \varepsilon > 0 \) small enough.

2. In the case of Corollary 2.12 the situation is a bit more involved. Clearly, Condition (2.5) holds true. But the problem here is, that (2.4) may be violated for all \( \varepsilon > 0 \). However, we argue as follows. In Step 2 in the proof above we have seen, that we do not need (2.4) for the system \( (\Psi_0, \Psi) \). Hence, we can estimate (2.9) and (2.10) from above by a further characterization of \( F_{p,q}^s(\mathbb{R}^d) \). For the remaining estimates we apply Theorem 2.6 with the system \( (\Phi_0, \tilde{\Phi}) \) where
\[
\tilde{\Phi} = \Phi_0(x) - \frac{1}{2kd} \Phi_0(x/2^k),
\]
and \( k \in \mathbb{N} \) is chosen in such a way that (2.4) is satisfied. What remains is a consequence of the fact that
\[
\tilde{\Phi} = \Phi + \Phi_{-1} + \ldots + \Phi_{-(k-1)}.
\]
This type of argument is due to Triebel [30, 3.3.3].
3 Classical coorbit space theory

In [7, 8, 9, 14] a general theory of Banach spaces related to integrable group representations has been developed. The ingredients are a locally compact group \( G \) with identity \( e \), a Hilbert space \( H \) and an irreducible, unitary and continuous representation \( \pi : G \to L(H) \), which is at least integrable. One can associate a Banach space \( \text{Co}Y \) to any solid, translation-invariant Banach space \( Y \) of functions on the group \( G \). The main achievement of this abstract theory is a powerful discretization machinery for \( \text{Co}Y \), i.e., a universal approach to atomic decompositions and Banach frames. It allows to transfer certain questions concerning Banach space or interpolation theory from the function space to the associated sequence space level, see [8, 9, 18]. In connection with smoothness spaces of Besov-Lizorkin-Triebel type the philosophy of this approach is to measure smoothness of a function in decay properties of the continuous wavelet transform \( W_gf \) which is studied in detail in the appendix. Indeed, homogeneous Besov and Lizorkin-Triebel type spaces turn out to be coorbits of properly chosen spaces \( Y \) on the \( ax + b \)-group \( G \).

There are some more examples according to this abstract theory. One main class of examples refers to the Heisenberg group \( H \), the short-time Fourier transform and leads to the well-known modulation spaces as coorbits of weighted \( L^p(H) \) spaces, see [7, 7.1] and also [10].

3.1 Function spaces on \( G \)

Integration on \( G \) will always be with respect to the left Haar measure \( d\mu(x) \). The Haar module on \( G \) is denoted by \( \Delta \). We define further \( L_xF(y) = F(x^{-1}y) \) and \( R_xF(y) = F(yx) \), \( x, y \in G \), the left and right translation operators. A Banach function space \( Y \) on the group \( G \) is supposed to have the following properties

(i) \( Y \) is continuously embedded in \( L^1_{loc}(G) \),

(ii) \( Y \) is invariant under left and right translation \( L_x \) and \( R_x \), which represent in addition continuous operators on \( Y \),

(iii) \( Y \) is solid, i.e., \( H \in Y \) and \( |F(x)| \leq |H(x)| \) a.e. imply \( F \in Y \) and \( \|F|Y\| \leq \|H|Y\| \).

The continuous weight \( w \) is called sub-multiplicative if \( w(xy) \leq w(x)w(y) \) for all \( x, y \in G \). The space \( L^w_p(G) \), \( 1 \leq p \leq \infty \), of functions \( F \) on the group \( G \) is defined via the norm

\[
\|F|L^w_p(G)\| = \left( \int_G |F(x)w(x)|^p \, d\mu(x) \right)^{1/p},
\]

where we use the essential supremum in case \( p = \infty \). If \( w \equiv 1 \) then we simply write \( L_p(G) \).

It is easy to show that these spaces provide left and right translation invariance if \( w \) is sub-multiplicative. Later, in Paragraph 4.1 we are going to introduce certain mixed norm spaces where the translation invariance is not longer automatic.

3.2 Sequence spaces

**Definition 3.1.** Let \( X = \{x_i\}_{i \in I} \) be some discrete set of points in \( G \) and \( V \) be a relatively compact neighborhood of \( e \in G \).
(i) $X$ is called $V$-dense if $G = \bigcup_{i \in I} x_i V$.

(ii) $X$ is called relatively separated if for all compact sets $K \subset G$ there exists a constant $C_K$ such that
\[
\sup_{j \in I} \{ i \in I : x_i K \cap x_j K \neq \emptyset \} \leq C_K.
\]

(iii) $X$ is called $V$-well-spread (or simply well-spread) if it is both relatively separated and $V$-dense for some $V$.

**Definition 3.2.** For a family $X = \{ x_i \}_{i \in I}$ which is $V$-well-spread with respect to a relatively compact neighborhood $V$ of $e \in G$ we define the sequence space $Y^b$ and $Y^s$ associated to $Y$ as
\[
Y^b = \left\{ \{ \lambda_i \}_{i \in I} : \| \{ \lambda_i \}_{i \in I} \| Y^b = \left\| \sum_{i \in I} |\lambda_i| \mu(x_i V)^{-1} \chi_{x_i V} |Y \right\| \leq \infty \right\},
\]
\[
Y^s = \left\{ \{ \lambda_i \}_{i \in I} : \| \{ \lambda_i \}_{i \in I} \| Y^s = \left\| \sum_{i \in I} |\lambda_i| \chi_{x_i V} |Y \right\| \leq \infty \right\}.
\]

**Remark 3.3.** For a well-spread family $X$ the spaces $Y^b$ and $Y^s$ do not depend on the choice of $V$, i.e. different sets $V$ define equivalent norms on $Y^b$ and $Y^s$, respectively. For more details on these spaces we refer to [8].

### 3.3 Coorbit spaces

Having a Hilbert space $H$ and an integrable, irreducible, unitary and continuous representation $\pi : G \to L(H)$ then the general voice transform of $f \in H$ with respect to a fixed atom $g$ is defined as the function $V_g f$ on the group $G$ given by
\[
V_g f(x) = \langle \pi(x)g, f \rangle,
\]
where the brackets denote the inner product in $H$.

**Definition 3.4.** For a sub-multiplicative weight $w(\cdot) \geq 1$ on $G$ we define the space $A_w \subset H$ of admissible vectors by
\[
A_w = \{ g \in H : V_g g \in L^w_1(G) \}.
\]

If $A_w \neq \{0\}$ and $g \in A_w$ we define further
\[
H^1_w(\mathbb{R}^d) = \{ f \in H : \| f \| H^1_w = \| V_g f \| L^w_1(G) \}.
\]

Finally, we denote with $(H^1_w)^\sim$ the canonical anti-dual of $H^1_w$, i.e., the space of conjugate linear functionals on $H^1_w$.

We see immediately that $A_w \subset H^1_w \subset H$. The voice transform (3.1) can now be extended to $H^\wedge \subset (H^1_w)^\sim$ by the usual dual pairing. The space $H^\wedge$ can be considered as the space of test functions and the reservoir $(H^1_w)^\sim$ as distributions.

Let now $Y$ be a space on $G$ such that (i) - (iii) in Paragraph 3.1 hold true. We define further
\[
w_Y(x) = \max\{ \| L_x \|, \| L_{x^{-1}} \|, \| R_x \|, \| R_{x^{-1}} \|, \Delta(x^{-1}) \| R_x \| \}, \quad x \in G,
\]
where the operator norms are considered from $Y$ to $Y$. 

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Definition 3.5. Let $Y$ be a space on $G$ satisfying (i)-(iii) in Paragraph 3.1 and let the weight $w(x)$ be given by (3.2). Let further $g \in A_w$. We define the space $CoY$, which we call coorbit space of $Y$, through

$$CoY = \{ f \in (H^1_w)\sim : V_gf \in Y \} \quad \text{with} \quad \|f|CoY\| = \|V_gf|Y\|. \quad (3.3)$$

The following basic properties are proved for instance in [20, Thm. 4.5.13].

Theorem 3.6. (i) The space $CoY$ is a Banach space independent of the analyzing vector $g \in A_w$.

(ii) The definition of the space $CoY$ is independent of the reservoir in the following sense: Assume that $S \subset H^1_w$ is a non-trivial locally convex vector space which is invariant under $\pi$. Assume further that there exists a non-zero vector $g \in S \cap A_w$ for which the reproducing formula

$$V_gf = V_gg * V_gf$$

holds true for all $f \in S\sim$. Then we have

$$CoY = \{ f \in (H^1_w)\sim : V_gf \in Y \} = \{ f \in S\sim : V_gf \in Y \}.$$ 

3.4 Discretizations

This section collects briefly the basic facts concerning atomic (frame) decompositions in coorbit spaces. We are interested in atoms of type $\{\pi(x_i)g\}_{i \in I}$, where $\{x_i\}_{i \in I} \subset G$ represents a discrete subset, whereas $g$ denotes a fixed admissible analyzing vector.

Definition 3.7. A family $\{g_i\}_{i \in I}$ in a Banach space $B$ is called an atomic decomposition for $B$ if there exists a family of bounded linear functionals $\{\lambda_i\}_{i \in I} \subset B'$ (not necessary unique) and a Banach sequence space $B^\sharp = B^\sharp(I)$ such that:

(a) We have $\{\lambda_i(f)\}_{i \in I} \in B^\sharp$ for all $f \in B$ and there exists a constant $C_1 > 0$ with

$$\|\{\lambda_i(f)\}_{i \in I}\|_{B^\sharp} \leq C_1\|f\|_B.$$

(b) For all $f \in B$ we have

$$f = \sum_{i \in I} \lambda_i(f)g_i$$

in some suitable topology.

(c) If $\{\lambda_i\}_{i \in I} \in B^\sharp$ then $\sum_{i \in I} \lambda_i g_i \in B$ and there exists a constant $C_2 > 0$ such that

$$\left\| \sum_{i \in I} \lambda_i g_i \right\|_B \leq C_2\|\{\lambda_i\}_{i \in I}\|_{B^\sharp}.$$

Definition 3.8. A family $\{h_i\}_{i \in I} \subset B'$ is called a Banach frame for $B$ if there exists a Banach sequence space $B^b = B^b(I)$ and a linear bounded reconstruction operator $\Theta : B^b \to B$ such that:

(a) We have $\{h_i(f)\}_{i \in I} \in B^b$ for all $f \in B$ and there exist constants $C_1, C_2$ such that

$$C_1\|f\|_B \leq \|\{h_i(f)\}_{i \in I}\|_{B^b} \leq C_2\|f\|_B,$$
Remark 3.9. This setting differs slightly from the understanding of Triebel in [30, 31].

The following abstract result for the atomic decomposition in $\text{CoY}$ is due to Feichtinger and Gröchenig (see [8, Thm. 6.1]).

**Theorem 3.10.** Let $Y$ be a function space on the group $G$ satisfying the hypotheses (i)-(iii) from Paragraph 3.1 and let $w(x)$ be given by (3.2). Furthermore, the element $g \in A_w$ is supposed to satisfy

$$
\int_G \left( \sup_{y \in xV} |\langle \pi(y)g, g \rangle| \right) w(x, t) \, d\mu(x) < \infty.
$$

Then there exists a neighborhood $U$ of $e \in G$ and constants $C_0, C_1 > 1$ such that for every $U$-well-spread discrete set $X = \{x_i\}_{i \in I} \subset G$ the following is true.

(i) (Analysis) Every $f \in \text{CoY}$ has a representation

$$
f = \sum_{i \in I} \lambda_i \pi(x_i)g
$$

with coefficients $\{\lambda_i\}_{i \in I}$ depending linearly on $f$ and satisfying the estimate

$$
\|\{\lambda_i\}_{i \in I}|Y^2| \leq C_0 \|f\|_Y.
$$

(ii) (Synthesis) Conversely, for any sequence $\{\lambda_i\}_{i \in I} \in Y^2$ the element $f = \sum_{i \in I} \lambda_i \pi(x_i)g$ is in $\text{CoY}$ and one has

$$
\|f\|_\text{CoY} \leq C_1 \|\{\lambda_i\}_{i \in I}|Y^2|.
$$

In both cases, convergence takes place in the norm of $\text{CoY}$ if the finite sequences are norm dense in $Y^2$, and in the weak$^*$-sense of $(H^1_{w^*})^\sim$ otherwise.

**Remark 3.11.** According to Definition 3.7 the family $\{\pi(x_i)g\}_{i \in I}$ represents an atomic decomposition for $\text{CoY}$.

**Theorem 3.12.** Under the same assumptions as in Theorem 3.10 the system $\{\pi(x_i)g\}_{i \in I}$ represents a Banach frame for $\text{CoY}$, i.e.,

$$
\|f\|_{\text{CoY}} \simeq \|\{\langle \pi(x_i)g, f \rangle\}|Y^b\|, \quad f \in \text{CoY}.
$$

The following powerful result goes back to Gröchenig [13] and was generalized by Rauhut [21].

**Theorem 3.13.** Suppose that the functions $g_r, \gamma_r, r = 1, \ldots, n$, satisfy (3.4). Let $X = \{x_i\}_{i \in I}$ be a well-spread set such that

$$
f = \sum_{r=1}^n \sum_{i \in I} \langle \pi(x_i)\gamma_r, f \rangle \pi(x_i)g_r
$$

for all $f \in \mathcal{H}$. Then expansion (3.5) extends to all $f \in \text{CoY}$. Moreover, $f \in (\mathcal{H}^1_w)^\sim$ belongs to $\text{CoY}$ if and only if $\{\langle \pi(x_i)\gamma_r, f \rangle\}_{i \in I}$ belongs to $Y^b$ for each $r = 1, \ldots, n$. The convergence is considered in $\text{CoY}$ if the finite sequences are dense in $Y^b$. In general we have weak$^*$-convergence.

**Proof.** The proof of this result relies on the fact that there exists an atomic decomposition $\{\pi(y_i)g\}_{i \in I}$ by Theorem 3.10 with a certain $g$ satisfying (3.4) and a corresponding sequence of points $Z = \{y_i\}_{i \in I}$. This has to be combined with Theorem 3.12 and Theorem 3.10/(ii) and we are done. See [13] for the details.
4 Coorbit spaces on the $ax+b$-group

Let $\mathcal{G} = \mathbb{R}^d \times \mathbb{R}_+^*$ the $d$-dimensional $ax+b$-group. Its multiplication is given by

$$(x,t)(y,s) = (x + ty, st).$$

The left Haar measure $\mu$ on $\mathcal{G}$ is given by $d\mu(x,t) = dx\ dt/|t|^{d+1}$, the Haar module is $\Delta(x,t) = t^{-d}$. Giving a function $F$ on $\mathcal{G}$ the left and right translation $L_y = L_{(y,r)}$ and $R_y = R_{(y,r)}$ are given by

$$L_{(y,r)}F(x,t) = F((y,r)^{-1}(x,t)) = F\left(\frac{x - y}{r}, \frac{t}{r}\right)$$

and

$$R_{(y,r)}F(x,t) = F((x,t)(y,r)) = F(x + ty, rt).$$

### 4.1 Peetre type spaces on $\mathcal{G}$

The present paragraph is devoted to the definition of certain mixed norm spaces on the group. Such spaces have been considered in various papers, see [5, 7, 13, 14]. In particular, so-called tent spaces have some important applications in harmonic analysis. Indeed, it is possible to recover Lizorkin-Triebel spaces as coorbits of tent spaces.

Here we use a different approach and define a new scale of function spaces on the group. We call them Peetre type spaces since a quantity related to the Peetre maximal function (4.1) is involved in its definition. It turned out that they are straight forward to handle in connection with translation invariance. In contrast to the tent space approach they represent the more natural choice for considering Lizorkin-Triebel spaces as coorbits. Additionally, they seem to be suitable for inhomogeneous spaces and more general situations like weighted spaces and general 2-microlocal spaces, which will be studied in a further contribution to the subject.

**Definition 4.1.** Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$, and $a > 0$. We define the spaces $L_{p,q}^s(\mathcal{G})$, $T_{p,q}^s(\mathcal{G})$, and $\dot{P}_{p,q}^{s,a}(\mathcal{G})$ on the group $\mathcal{G}$ via the finiteness of the following (quasi-)norms

$${\| F \|}_{\dot{L}_{p,q}^s(\mathcal{G})} = \left( \int_0^\infty t^{-sq} \| F(\cdot,t) \|_{L_p(\mathbb{R}^d)}^q \frac{dt}{t^{d+1}} \right)^{1/q},$$

$${\| F \|}_{\dot{T}_{p,q}^s(\mathcal{G})} = \left( \int_0^\infty t^{-sq} \int_{B(0,t)} |F(x + z,t)|^q \frac{dt}{t^{d+1}} \right)^{1/q} \| L_p(\mathbb{R}^d) \|,$$

$${\| F \|}_{\dot{P}_{p,q}^{s,a}(\mathcal{G})} = \left( \int_0^\infty t^{-sq} \sup_{y \in \mathbb{R}^d} \left[ \frac{|F(x + y,t)|}{(1 + |y/t|^a)} \right]^q \frac{dt}{t^{d+1}} \right)^{1/q} \| L_p(\mathbb{R}^d) \|,$$

using the usual modification in case $q = \infty$.

**Proposition 4.2.** The spaces $\dot{L}_{p,q}^s(\mathcal{G})$, $\dot{T}_{p,q}^s(\mathcal{G})$ and $\dot{P}_{p,q}^{s,a}(\mathcal{G})$ are left and right translation invariant. Precisely, we have

$$\| L_{(z,r)} : \dot{L}_{p,q}^s(\mathcal{G}) \to \dot{L}_{p,q}^s(\mathcal{G}) \| = r^{d(1/p - 1/q) - s},$$

$$\| R_{(z,r)} : \dot{L}_{p,q}^s(\mathcal{G}) \to \dot{L}_{p,q}^s(\mathcal{G}) \| = r^{s+d/q},$$

$$\| L_{(z,r)} : \dot{T}_{p,q}^s(\mathcal{G}) \to \dot{T}_{p,q}^s(\mathcal{G}) \| = r^{d/p - s},$$

$$\| R_{(z,r)} : \dot{T}_{p,q}^s(\mathcal{G}) \to \dot{T}_{p,q}^s(\mathcal{G}) \| \leq C r^{d/q + s} \max\{1, r^{-b} (1 + |z|)^b\},$$

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where \( b > 0 \) is a constant depending on \( d, p \) and \( q \). Furthermore, we have

\[
\|L(z,r) : \hat{P}_{p,q}^{s,a}(G) \to \hat{P}_{p,q}^{s,a}(G)\| = r^{d(1/p-1/q) - s},
\]

\[
\|R(z,r) : \hat{P}_{p,q}^{s,a}(G) \to \hat{P}_{p,q}^{s,a}(G)\| \leq r^{s+d/q} \max\{1, r^{-a}\}(1 + |z|)^a.
\]

**Proof.** Step 1. The left and right translation invariance of \( \hat{L}_{p,q}^s(G) \) and \( \hat{T}_{p,q}^s(G) \) was shown in \[20\] Lem. 4.7.10.

Step 2. Let us consider \( \hat{P}_{p,q}^{s,a}(G) \). Clearly, we have for \( F \in \hat{P}_{p,q}^{s,a}(G) \)

\[
\|L(z,r)F : \hat{P}_{p,q}^{s,a}(G) \to \hat{P}_{p,q}^{s,a}(G)\| = \left\| \left( \int_0^\infty t^{-sq} \left[ \sup_{y \in \mathbb{R}^d} \left| F((x + y - z)/r, t/r) \right| q \frac{dt}{t^{d+1}} \right] \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}
\]

\[
= r^{d/p} \left\| \left( \int_0^\infty t^{-sq} \left[ \sup_{y \in \mathbb{R}^d} \left| F((x + y - r/t, t/r) \right| q \frac{dt}{t^{d+1}} \right] \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}
\]

\[
= r^{d(1/p-1/q) - s} \left\| \left( \int_0^\infty t^{-sq} \left[ \sup_{y \in \mathbb{R}^d} \left| F((x + y, t) \right| q \frac{dt}{t^{d+1}} \right] \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}.
\]

Hence, we obtain

\[
\|L(z,r) : \hat{P}_{p,q}^{s,a}(G) \to \hat{P}_{p,q}^{s,a}(G)\| = r^{d(1/p-1/q) - s}.
\]

The right translation invariance is obtained by

\[
\|R(z,r)F : \hat{P}_{p,q}^{s,a}(G) \to \hat{P}_{p,q}^{s,a}(G)\| = \left\| \left( \int_0^\infty t^{-sq} \left[ \sup_{y \in \mathbb{R}^d} \left| F((x + y/t, t/r) \right| q \frac{dt}{t^{d+1}} \right] \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}
\]

\[
= \left\| \left( \int_0^\infty t^{-sq} \left[ \sup_{y \in \mathbb{R}^d} \left| F((x + y, t) \right| q \frac{dt}{t^{d+1}} \right] \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}
\]

\[
= r^{s+d/q} \left\| \left( \int_0^\infty t^{-sq} \left[ \sup_{y \in \mathbb{R}^d} \left| F((x + y, t) \right| q \frac{dt}{t^{d+1}} \right] \right)^{1/q} \right\|_{L_p(\mathbb{R}^d)}.
\]

Observe that

\[
\sup_{y \in \mathbb{R}^d} \left| F(x + y, t) \right| q \frac{dt}{t^{d+1}} = \sup_{y \in \mathbb{R}^d} \left| F(x + y, t) \right| q \frac{dt}{t^{d+1}} \left( 1 + \frac{|y|}{t} \right)^a \left( 1 + \frac{y - tz}{r/t} \right)^a
\]

and

\[
\frac{(1 + |y|/t)^a}{(1 + |y - tz|/r/t)^a} \leq \frac{(1 + |y - tz|/t + |z|)^a}{(1 + |y - tz|/r/t)^a} = \frac{(1 + |y - tz|/t)^a}{(1 + |y - tz|/r/t)^a} \leq \frac{(1 + |y - tz|/t)^a}{(1 + |y - tz|/r/t)^a}.
\]

This yields

\[
\sup_{y \in \mathbb{R}^d} \left| F(x + y, t) \right| q \frac{dt}{t^{d+1}} \leq \max\{1, r^{-a}\}(1 + |z|)^a \sup_{y \in \mathbb{R}^d} \left| F(x + y, t) \right| q \frac{dt}{t^{d+1}}
\]

and consequently

\[
\|R(z,r) : \hat{P}_{p,q}^{s,a}(G) \to \hat{P}_{p,q}^{s,a}(G)\| \leq r^{s+d/q} \max\{1, r^{-a}\}(1 + |z|)^a.
\]
Remark 4.3. Note, that we did neither use the translation invariance of the Lebesgue measure nor any change of variable in order to prove the right translation invariance of $\dot{B}^{s,a}_{p,q}(G)$. This gives room for further generalizations, i.e., replacing the space $L_p(\mathbb{R}^d)$ by some weighted Lebesgue space $L_p(\mathbb{R}^d, \omega)$ for instance.

4.2 New old coorbit spaces

We start with $\mathcal{H} = L_2(\mathbb{R}^d)$ and the representation

$$\pi(x,t) = T_x \mathcal{D}_{L_2}^t,$$

where $T_x f = f(\cdot - x)$ and $\mathcal{D}_{L_2}^t f = t^{-d/2} f(\cdot/t)$ has been already defined in Paragraph 2.2. This representation is unitary, continuous and square integrable on $\mathcal{H}$ but not irreducible. However, if we restrict to radial functions $g \in L_2(\mathbb{R}^d)$ then $\mathbb{C}\{\pi(x,t)g : (x,t) \in G\}$ is dense in $L_2(\mathbb{R}^d)$.

Another possibility to overcome this obstacle is to extend the group by $SO(d)$, which is more or less equivalent, see [7, 8] for details. The voice transform in this special situation is represented by the so-called continuous wavelet transform $W_g f$ which we study in detail in Paragraph A.1 in the appendix.

Recall the abstract definition of the space $H^1_w$ and $A_w$ from Definition 3.4. The following result implied by our Lemma A.3 on the decay of the continuous wavelet transform. It states under which conditions on the weight $w$ the space $H^1_w$ is nontrivial.

Lemma 4.4. If the weight function $w(x,t) \geq 1$ satisfies the condition

$$w(x,t) \leq (1 + |x|)^r (t^s + t^{-s'})$$

for some $r, s, s' \geq 0$ then

$$S_0(\mathbb{R}^d) \hookrightarrow H^1_w.$$

This is a kind of minimal condition which is needed in order to define coorbit spaces in a reasonable way. Instead of $(H^1_w)^\sim$ one may use $S'_0(\mathbb{R}^d)$ as reservoir and a radial $g \in S_0(\mathbb{R}^d)$ as analyzing vector. Considering [3,2] we have to restrict to such function spaces $Y$ on $G$ satisfying (i),(ii),(iii) in Paragraph 3.1 where additionally

(iv) $$w(x,t) = w_Y(x,t) \lesssim (1 + |x|)^r (t^s + t^{-s'})$$

holds true for some $r, s, s' \geq 0$. The following theorem shows, how the spaces of Besov-Lizorkin-Triebel type from Section 2 can be recovered as coorbit spaces with respect to $G$.

Theorem 4.5. (i) For $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ we have

$$\dot{B}^s_{p,q}(\mathbb{R}^d) = Co\dot{B}^{s+d/2-d/q}(G).$$

(ii) for $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$ we have

$$\dot{B}^{s}_{p,q}(\mathbb{R}^d) = Co\dot{B}^{s+d/2}(G).$$

(iii) and if additionally $a > \frac{d}{\min(p,q)}$ we obtain

$$\dot{B}^{s}_{p,q}(\mathbb{R}^d) = Co\dot{B}^{s+d/2-d/q,a}(G).$$
Proof. Theorem 4.5 is a direct consequence of Definition 3.5, formula (A.1), Proposition 4.2, Theorems 2.8, 2.9 and the abstract result in Theorem 3.6.

Remark 4.6. (a) The assertions (i) and (ii) are not new. They appear for instance in [4, 13, 14] and rely on the characterizations given by Triebel in [29] and [30, 2.4, 2.5], see in particular [30, 2.4.5] for the variant in terms of tent spaces which were invented in [5]. From the deep result in [5, Prop. 4] it follows that \( \dot{T}^{s}_{p,q}(G) \) are translation invariant Banach function spaces on \( G \), which makes them feasible for coorbit space theory.

(b) Assertion (iii) is indeed new and makes the rather complicated tent spaces \( \dot{T}^{s}_{p,q}(G) \) obsolete for this issue. We showed that \( Y = P^{s,a}_{p,q}(G) \) is a much better choice since the right translation invariance is immediate and gives more transparent estimates for its norm. Once we are interested in reasonable conditions for atomic decompositions this is getting important, see Section 4.5.

4.3 Sequence spaces

In the sequel we consider a compact neighborhood of the identity element in \( G \) given by \( U = [0,a]^{d} \times [\beta^{-1},1] \), where \( a > 0 \) and \( 1 < \beta \). Furthermore, we consider the discrete set of points

\[
\{x_{j,k} = (ak\beta^{-j},\beta^{-j}) : j \in \mathbb{Z}, k \in \mathbb{Z}^{d}\}.
\]

This family is \( U \)-well-spread. Indeed,

\[
x_{j,k}U = Q_{j,k} \times [\beta^{-(j+1)},\beta^{-j}],
\]

where

\[
Q_{j,k} = [ak_{1}\beta^{-j},\alpha(k_{1} + 1)\beta^{-j}] \times \cdots \times [ak_{d}\beta^{-j},\alpha(k_{d} + 1)\beta^{-j}].
\]

Note that in this case the spaces \( Y^{b} \) and \( Y^{b} \) coincide. We will further use the notation

\[
\chi_{j,k}(x) = \begin{cases} 1 & : x \in Q_{j,k} \\ 0 & : \text{otherwise} \end{cases}.
\]

Definition 4.7. Let \( Y \) be a function space on \( G \) as above. We put

\[
Y^{\sharp}(\alpha,\beta) = \{\{\lambda_{j,k}\}_{j,k} : \|\{\lambda_{j,k}\}_{j,k}\|_{Y^{\sharp}(\alpha,\beta)} < \infty\},
\]

where

\[
\|\{\lambda_{j,k}\}_{j,k}\|_{Y^{\sharp}(\alpha,\beta)} = \left\| \sum_{j,k} |\lambda_{j,k}| \chi_{j,k}(x) \chi_{[\beta^{j},\beta^{j+1}]}(t) \right\|_{Y}.
\]

Theorem 4.8. Let \( 1 \leq p,q \leq \infty \), \( s \in \mathbb{R} \) and \( a > d/\min\{p,q\} \). Then

\[
\|\{\lambda_{j,k}\}_{j,k}\|_{(\dot{P}^{s,a}_{p,q})^{\sharp}(\alpha,\beta)} \asymp \left( \sum_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^{d}} \beta^{(s+d/q)q} |\lambda_{\ell,k}|^{q} \chi_{\ell,k}(x) \right)^{1/q} |L_{p}(\mathbb{R}^{d})|
\]

and

\[
\|\{\lambda_{j,k}\}_{j,k}\|_{(\dot{L}^{s}_{p,q})^{\sharp}(\alpha,\beta)} \asymp \left( \sum_{\ell \in \mathbb{Z}} \beta^{(s+d/q-d/p)q} \left( \sum_{k \in \mathbb{Z}^{d}} |\lambda_{\ell,k}|^{p} \right)^{q/p} \right)^{1/q}.
\]
**Proof.** We prove the first statement. The proof for the second one is even simpler. Let

$$F(x, t) = \sum_{j, k} |\lambda_{j,k}| \chi_{j,k}(x) \cdot \chi_{[\beta^{-j+1}, \beta^{-j}]}(t).$$

Discretizing the integral over \( t \) by \( t \propto \beta^{-\ell} \) we obtain

$$
\|F\| \leq \left\| \left( \int_0^{\infty} t^{-sq} \sup_{w} \left[ \frac{|F(x + w, t)|^{q/2}}{(1 + |w|/t)^{a}} \right] \right)^{1/q} \left\| L_p(\mathbb{R}^d) \right\| \times \left\| \left( \sum_{\ell \in \mathbb{Z}} \beta^{(s+d)/q} \int_{\beta^{-\ell}}^{\beta^{-\ell+1}} \left[ \sup_{w} \left[ \frac{|F(x + w, t)|^{q/2}}{(1 + \beta^{t}|w|)^{a}} \right] \right] \right)^{1/q} \left\| L_p(\mathbb{R}^d) \right\|. \tag{4.2}\n$$

With \( t \in [\beta^{-\ell+1}, \beta^{-\ell}] \) we observe

$$F(x, t) = \sum_{k} |\lambda_{\ell,k}| \chi_{\ell,k}(x) \tag{4.3}$$

and estimate

$$
\|F\| \leq \left\| \left( \sum_{\ell \in \mathbb{Z}} \beta^{(s+d)/q} \sup_{w \in \mathbb{R}^d} \frac{1}{(1 + \beta^{t}|w|)^{a}} \left[ \sum_{k} \chi_{\ell,k}(x + w) \right] \right)^{1/q} \left\| L_p(\mathbb{R}^d) \right\|. \tag{4.4}\n$$

In order to include also the situation \( \min\{p, q\} \leq 1 \) we use the following trick. Obviously, we can rewrite and estimate (4.4) with \( 0 < r < 1 \) in the following way

$$
\|F\| \leq \left\| \left( \sum_{\ell \in \mathbb{Z}} \left[ \sum_{k} \beta^{(s+d)/q} |\lambda_{\ell,k}|^{r} \sup_{w \in \mathbb{R}^d} \frac{\chi_{\ell,k}(x + w)}{(1 + \beta^{t}|w|)^{ar}} \right] \right)^{1/q} \left\| L_p(\mathbb{R}^d) \right\|. \tag{4.5}\n$$

We continue with the useful estimate

$$
\sup_{w} \frac{|\chi_{\ell,k}(x + w)|}{(1 + \beta^{t}|w|)^{ar}} \lesssim \frac{1}{(1 + \beta^{t}|x - k\beta^{-\ell}|)^{ar}} \lesssim \left( \chi_{\ell,k}(\cdot) * \frac{\beta^{ed}}{(1 + \beta^{t}|\cdot|)^{ar}} \right)(x). \tag{4.6}\n$$

Indeed, the first estimate is obvious. Let us establish the second one

$$
\left( \chi_{\ell,k}(\cdot) * \frac{1}{(1 + \beta^{t}|\cdot|)^{ar}} \right)(x) = \int (1 + \beta^{t}|x - y|)^{ar} d\nu
\begin{align*}
&\gtrsim \int_{|y| \leq c\beta^{-\ell}} \frac{1}{(1 + \beta^{t}|x - k\beta^{-\ell} - y|)^{ar}} d\nu \\
&\gtrsim \int_{|y| \leq c\beta^{-\ell}} \frac{1}{(1 + \beta^{t}|x - k\beta^{-\ell}| + \beta^{t}|y|)^{ar}} d\nu \\
&\gtrsim \beta^{-ed} \int_{0}^{\frac{u^{d-1}}{(1 + \beta^{t}|x - k\beta^{-\ell}| + u)^{ar}}} \beta^{-\ell d} d\nu \lesssim \beta^{-ed} \tag{4.7}\n\end{align*}
$$

\( \frac{1}{(1 + \beta^{t}|x - k\beta^{-\ell}| + u)^{ar}}. \)
Note, that the functions
\[ g_\ell(x) = \frac{\beta^{d\ell}}{(1 + \beta^{\ell})^a} \]
belong to \( L_1(\mathbb{R}^d) \) with uniformly bounded norm, where we need that \( ar > d \). Putting (4.7) and (4.6) into (4.5) we obtain
\[
\| F \|_{Y}^r \leq \left\| \left( \sum_{\ell \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}^d} \beta^{(s+d/q)\ell} |\lambda_{\ell,k}|^r \chi_{\ell,k}(x) \right]^{q/r} \right)^{r/q} |L_p/\tau(\mathbb{R}^d)|^r \right\|.
\]

Now we are in a position to use the majorant property of the Hardy-Littlewood maximal operator (see Paragraph 2.2 and [26, Chapt. 2]), which states that a convolution of a function \( f \) with a \( L_1(\mathbb{R}^d) \)-function (having norm one) can be estimated from above by the Hardy-Littlewood maximal function of \( f \). We choose \( r < \min\{p, q\} \) and apply Theorem 2.1 for the \( L_p/\tau(\ell_{q/r}) \) situation. This gives
\[
\| F \|_{Y}^r \lesssim \left\| \left( \sum_{\ell \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}^d} \beta^{(s+d/q)\ell} |\lambda_{\ell,k}|^q \chi_{\ell,k}(x) \right]^{q/r} \right)^{r/q} |L_p(\mathbb{R}^d)|^r \right\|^r.
\]
and finishes the upper estimate. Both conditions, \( ar > d \) and \( r < \min\{p, q\} \), are compatible if \( a > d/\min\{p, q\} \) is assumed at the beginning.

For the estimate from below we go back to (4.2) and observe
\[ \sup_w \frac{|F(x + w, t)|}{(1 + \beta^{\ell}|w|)^a} \geq |F(x, t)|, \]
which results in
\[
\| F \|_{Y} \gtrsim \left\| \left( \sum_{\ell \in \mathbb{Z}} \beta^{s+d/q} \int_{\beta^{-\ell+1}}^{\beta^{-\ell}} |F(x, t)|^q \frac{dt}{t} \right)^{1/q} |L_p(\mathbb{R}^d)| \right\|.
\]
A further use of (4.3) gives finally
\[
\| F \|_{Y} \gtrsim \left\| \left( \sum_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \beta^{(s+d/q)\ell} |\lambda_{\ell,k}|^q \chi_{\ell,k}(x) \right)^{1/q} |L_p(\mathbb{R}^d)| \right\|.
\]
The proof is complete.

**4.4 Atomic decompositions**

The following theorem is a direct consequence of the abstract results in Theorems 3.10 3.12.
Theorem 4.9. Let $1 \leq p, q \leq \infty$, $a > d / \min \{p, q\}$ and $s \in \mathbb{R}$. Let further $g \in S_0(\mathbb{R}^d)$ be a radial function. Then there exist numbers $\alpha_0 > 0$ and $\beta_0 > 1$ such that for all $0 < \alpha \leq \alpha_0$ and $1 < \beta \leq \beta_0$ the family

$$\{g_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} = \{T_{\alpha \beta} \mathcal{D}_\beta g\}$$

has the following properties:

(i) $\{g_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^d}$ forms a Banach frame for $\text{Co} \hat{L}_{p,q}(\mathcal{G})$ and $\text{Co} \hat{P}_{p,q}^{s,a}(\mathcal{G})$, i.e., we have a dual frame $\{e_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} \subset S_0(\mathbb{R}^d)$ with $f = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} (g_{j,k}, f) e_{j,k}$ and the norm equivalences

$$\|f|\text{Co} \hat{L}_{p,q}(\mathcal{G})\| \asymp \|\sum_{j,k \in \mathbb{Z}^d} \langle x, g_{j,k} \rangle \hat{L}_{p,q}^s(\mathcal{G})\|^{\alpha, \beta} , \quad f \in \text{Co} \hat{L}_{p,q}(\mathcal{G})$$

as well as

$$\|f|\text{Co} \hat{P}_{p,q}^{s,a}(\mathcal{G})\| \asymp \|\sum_{j,k \in \mathbb{Z}^d} \langle x, g_{j,k} \rangle \hat{P}_{p,q}^{s,a}(\mathcal{G})\|^{\alpha, \beta} , \quad f \in \text{Co} \hat{P}_{p,q}^{s,a}(\mathcal{G}).$$

(ii) $\{g_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^d}$ is an atomic decomposition, i.e., for $f \in \text{Co} \hat{P}_{p,q}^{s,a}(\mathcal{G})$ we have a (not necessarily unique) decomposition $\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} \lambda_{j,k}(f) g_{j,k}$ such that

$$\|\{\lambda_{j,k}(f)\}_{j,k} \hat{P}_{p,q}^{s,a}(\mathcal{G})\|^{\alpha, \beta} \lesssim \|f|\text{Co} \hat{P}_{p,q}^{s,a}(\mathcal{G})\|.$$

Conversely, if $\{\lambda_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} \in \hat{P}_{p,q}^{s,a}(\mathcal{G})$ then $f = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} \lambda_{j,k} g_{j,k}$ converges and belongs to $\text{Co} \hat{P}_{p,q}^{s,a}(\mathcal{G})$ and moreover,

$$\|f|\text{Co} \hat{P}_{p,q}^{s,a}(\mathcal{G})\| \lesssim \|\{\lambda_{j,k}\}_{j,k} \hat{P}_{p,q}^{s,a}(\mathcal{G})\|^{\alpha, \beta}$$

(analogously for $\text{Co} \hat{L}_{p,q}^s(\mathcal{G})$). Convergence is considered in the strong topology if the finite sequences are dense in $\hat{P}_{p,q}^{s,a}(\mathcal{G})$ and in the weak* topology otherwise.

Remark 4.10. (i) Since the analyzing function or atom $g$ can be chosen arbitrarily we allow more flexibility here than in the results given in Frazier/Jawerth [12] and Triebel [30, 31].

(ii) Instead of regular families of sampling points $(\alpha \beta^{-j} k, \beta^{-i})$ rather irregular families of points in $\mathcal{G}$ are allowed as long as they are distributed sufficiently dense, see Theorem 3.1.

4.5 Wavelet frames

In the sequel we consider wavelet bases on $\mathbb{R}^d$ in the sense of Lemma A.5 in the appendix. We have given an orthonormal scaling function $\Psi^0$ and the associated wavelet $\Psi^1$ on $\mathbb{R}$ and consider the tensor products $\Psi^{e,c}$, $e \in E$. Our aim is to specify, i.e., give sufficient conditions to $\Psi^0$, $\Psi^1$, such that $\{\Psi^1\}$ represents an unconditional basis in $B_{p,q}^s(\mathbb{R}^d)$ and $F_{p,q}^s(\mathbb{R}^d)$, respectively. We intend to apply our abstract Theorem 3.13 and need therefore to have (3.4) for all functions $\Psi^{e,c}$. To ensure this we impose certain smoothness $(S_K)$, decay $(D)$, and moment conditions $(M_{L})$ to $\Psi^1$ and $\Psi^0$, which are specified in Definition A.1.

Proposition 4.11. Let $L \in \mathbb{N}$, $K > 0$, and $\Psi^0$ be an orthogonal scaling function with associated wavelet $\Psi^1$ on $\mathbb{R}$. The function $\Psi^0$ is supposed to satisfy $(D)$ and $(S_K)$ and $\Psi^1$ is supposed to satisfy $(D)$, $(S_K)$ and $(M_{L-1})$. Let $V = [-1, 1]^d \times (1/2, 1] \subset \mathcal{G}$ a neighborhood of the identity $e \in \mathcal{G}$. Suppose further that for $r_1, r_2 \in \mathbb{R}$ the weight $w(x,t)$ is given by

$$w(x,t) = (1 + |x|)^r(t^{r_2} + t^{-r_1}) , \quad (x,t) \in \mathcal{G}.$$
If now
\[ r_1 < \min\{L, K\} - d/2 \ , \ r_2 < \min\{L, K\} + d/2 - v \] (4.8)
then we have
\[ \int_\mathbb{R}^d \int_0^\infty \sup_{(y, s) \in (x, t) V} \left| \langle \pi(y, s) \Psi^c, \Psi^c \rangle \right| w(x, t) \frac{dt}{t^{d+1}} \, dx < \infty . \]

**Proof.** With Lemma A.3 we obtain for \( W_{\Psi^1} \Psi^1 \) the following estimates
\[ |(W_{\Psi^1} \Psi^1)(s, t)| \lesssim \frac{t^{\min\{L, K\}+1/2}}{(1+t)^{2\min\{L, K\}+1}} \cdot \frac{1}{(1+|s|/(1+t))^N} . \]

And in addition
\[ |(W_{\Psi^1} \Psi^i)(s, t)| \lesssim \frac{t^{1/2}}{(t+1) (1+|s|/(1+t))^N} , \quad i = 1, 2. \]

Hence, for any \( c \in E \) the tensor product structure gives (assume without restriction that \( c_d = 1 \))
\[ |W_{\Psi^c} \Psi^c(x, t)| \lesssim \frac{t^{\min\{L, K\}}}{(1+t)^{2\min\{L, K\}}+1} \cdot \frac{t^{d/2}}{(1+t)^d} \prod_{i=1}^d \frac{1}{(1+|x_i|/(1+t))^N} . \]

The expression \( \sup_{(y, s) \in (x, t) V} |W_{\Psi^c} \Psi^c(y, s)| \) can be estimated similarly
\[ \sup_{(y, s) \in (x, t) V} |W_{\Psi^c} \Psi^c(y, s)| = \sup_{|y_s - x_s| \leq t} \sup_{t/2 \leq s \leq t} |W_{\Psi^c} \Psi^c(y, s)| \]
\[ \lesssim \frac{t^{\min\{L, K\}}}{(1+t)^{2\min\{L, K\}}+1} \cdot \frac{t^{d/2}}{(1+t)^d} \prod_{i=1}^d \frac{1}{(1+|x_i|/(1+t))^N} \]
\[ \lesssim \frac{t^{\min\{L, K\}}}{(1+t)^{2\min\{L, K\}}+1} \cdot \frac{t^{d/2}}{(1+t)^d} \prod_{i=1}^d \frac{1}{(1+|x_i|/(1+t))^N} \]

Fubini’s theorem and a change of variable yields
\[ \int_\mathbb{R}^d \int_0^\infty \sup_{(y, s) \in (x, t) V} \left| \langle \pi(y, s) \Psi^c, \Psi^c \rangle \right| w(x, t) \frac{dt}{t^{d+1}} \, dx \lesssim \int_0^\infty \frac{t^{\min\{L, K\}}}{(1+t)^{2\min\{L, K\}}+1} \cdot \frac{t^{d/2}}{(1+t)^d} \prod_{i=1}^d \frac{1}{(1+|x_i|/(1+t))^N} \frac{dt}{t^{d+1}} . \]

Finally it is easy to see that the latter is finite if the conditions in (4.8) are valid. This proves Proposition 4.11. \( \square \)

**Theorem 4.12.** Let \( L \in \mathbb{N} \), \( K > 0 \), and \( \Psi^0 \) be an orthogonal scaling function with associated wavelet \( \Psi^1 \) on \( \mathbb{R} \). The function \( \Psi^0 \) is supposed to satisfy (D) and \( (S_K) \) and \( \Psi^1 \) is supposed to satisfy (D), \( (S_K) \) and \( (M_{L-1}) \).

(a) If \( 1 \leq p, q \leq \infty \) and
\[ -\min\{L, K\} + \frac{d}{p} < s < \min\{L, K\} - d \left( 1 - \frac{1}{p} \right) \]
then \( (A.6) \) is a Banach frame for \( \hat{B}_{p,q}^s(\mathbb{R}^d) \) in the sense of (3.5).
(b) If \(1 \leq p < \infty, 1 \leq q \leq \infty\), and
\[- \min\{L, K\} + 2d \max\left\{\frac{1}{p}, \frac{1}{q}\right\} < s < \min\{L, K\} - d \max\left\{\frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{p}\right\}\]
then \((A.6)\) is a Banach frame for \(\hat{F}_{p,q}^{s}(\mathbb{R}^d)\) in the sense of \((3.5)\).

**Proof.** Let us prove (a). First of all, we apply Theorem 4.5/(i). Afterwards, we use Proposition 4.2 in order to estimate the weight \(w_Y(x, t)\) for \(Y = \hat{F}_{p,q}^{s+d/2-d/q} (G)\). We obtain
\[w_Y(x, t) = \max\{t^{d/(1/p - 1/2)} - s, ts^{d/(1/p - 1/2)}, ts^{d + 2}, t^{-s + 2/d}\}\]
\[\leq \left\{ \begin{array}{ll}
t^{-r_1} & : 0 < t < 1 \\
t^r_2 & : t \geq 1.
\end{array} \right.\]

Let us distinguish the cases \(s \geq 0\) and \(s < 0\). In the first case we can put \(r_1 = \max\{s - d(1/p - 1/2), -s + d(1/p - 1/2), s - d/2\}, r_2 = \max\{s + d/2, -s + d(1/p - 1/2)\}\) and \(v = 0\). Now we apply first Proposition 4.11. This gives the condition
\[0 \leq s < \min\{L, K\} - d(1 - 1/p).\]  
(4.9)

In the second case we put \(r_1 = \max\{s - d(1/p - 1/2), -s + d(1/p - 1/2), -s - d/2\}, r_2 = \max\{-s + d/2, -s + d(1/p - 1/2)\}\) and \(v = 0\). With Proposition 4.11 we obtain the condition
\[- \min\{L, K\} + d/p < s < 0.\]  
(4.10)

Finally \((4.9), (4.10)\) and Theorem 3.13 yield (a).

**Step 2.** We prove (b). We apply Theorem 4.5/(iii) and afterwards Proposition 4.2 and obtain for \(Y = \hat{F}_{p,q}^{s+d/2-d/q} (G)\)
\[w_Y(x, t) = \max\{t^{d/(1/p - 1/2)} - s, ts^{d/(1/p - 1/2)},
\]
\[ts^{d/2} \max\{1, t^{-a}\}(1 + |x|^a), ts^{d/2} \max\{t^{-a}, t^a\}(1 + |x|^a)\}\]
\[\leq (1 + |x|^a) \left\{ \begin{array}{ll}
t^{-r_1} & : 0 < t < 1 \\
t^r_2 & : t \geq 1.
\end{array} \right.\]

First, we consider the case \(s \geq 0\). We can put \(r_1 = \max\{s + a - d/2, s + d/2 - d/p\}, r_2 = \max\{s + d/2, -s + d/2 + a\}\) and \(v = a\). Proposition 4.11 gives the condition
\[0 \leq s < \min\{L, K\} - \max\{a, d(1 - 1/p)\}\]
which can be rewritten to
\[0 \leq s < \min\{L, K\} - d \max\{1/p, 1/q, 1 - 1/p\}\]

since \(a\) can be chosen arbitrarily greater than \(d \max\{1/p, 1/q\}\). This gives the upper bound in (b). Now we consider \(s < 0\). We put \(r_1 = \max\{-s + d/p - d/2, s + d/2 - d/p\}, r_2 = -s + d/2 + a\). This yields
\[- \min\{L, K\} + 2a < s < 0\]
and can be rewritten to
\[- \min\{L, K\} + 2d \max\{1/p, 1/q\} < s < 0.\]

This yields the lower bound in (b) and we are done. 

The following corollary is a consequence of Theorem 4.12 and the facts in Section A.2.
Corollary 4.13. Let $m > 0$ and $(\Psi^0, \Psi^1) = (\varphi_m, \psi_m)$ the spline wavelet system of order $m$. Then

(a) If $1 \leq p, q \leq \infty$ and

$$-m + 1 + \frac{d}{p} < s < m - 1 - d \left(1 - \frac{1}{p}\right)$$

then (A.6) is a Banach frame for $\dot{B}_{p,q}^s(\mathbb{R}^d)$ in the sense of (3.5).

(b) If $1 \leq p < \infty$, $1 \leq q \leq \infty$, and

$$-m + 1 + 2d \max \left\{\frac{1}{p}, \frac{1}{q}\right\} < s < m - 1 - d \max \left\{\frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{p}\right\}$$

then (A.6) is a Banach frame for $\dot{F}_{p,q}^s(\mathbb{R}^d)$ in the sense of (3.5).

Remark 4.14. The (optimal) smoothness conditions in [2] are slightly weaker than (a) in case $d = 1$. However, compared to the approach of Triebel [31, 32], we admit some more degree of freedom. The wavelet or atom does not have to be compactly supported. Additionally, in case $d = 1$ we do not need that $\psi \in C^u(\mathbb{R})$ where $u > s$. Indeed, the conditions in (a) and (b) are slightly weaker.

Remark 4.15. More examples can be obtained by using compactly supported Daubechies wavelets of a certain order or Meyer wavelets. Based on the underlying abstract result in Theorem 3.13 even biorthogonal wavelet systems providing sufficiently high smoothness and vanishing moments are suitable for this issue.

A Appendix: Wavelets

A.1 The continuous wavelet transform

The vector $g$ is said to be the analyzing vector for a function $f \in L_2(\mathbb{R}^d)$. The continuous wavelet transform $W_g f$ is then defined through

$$W_g f(x,t) = \langle T_x \mathcal{D}_t^{L^2} g, f \rangle, \quad x \in \mathbb{R}^d, t > 0,$$

where the bracket $\langle \cdot, \cdot \rangle$ denotes the inner product in $L_2(\mathbb{R}^d)$. We can write it in terms of the convolution (2.3) via

$$W_g f(x,t) = \left[ (\mathcal{D}_t^{L^2} g)(\cdot) * \tilde{f} \right](x) = t^{d/2} [\mathcal{D}_t g(\cdot) * \tilde{f}](x). \quad \text{(A.1)}$$

We call $g$ an admissible wavelet if

$$c_g := \int_{\mathbb{R}^d} \frac{|\mathcal{F}g(\xi)|^2}{|\xi|^d} \, d\xi < \infty.$$

If this is the case, then the family $\{T_x \mathcal{D}_t^{L^2} g\}_{t > 0, x \in \mathbb{R}^d}$ represents a tight continuous frame in $L_2(\mathbb{R}^d)$. For a proof we refer to Theorem 1.5.1 in [20].

Let us now specify the conditions $(M_L)$, $(D)$, and $(S_K)$ which we intend to impose on functions $\Phi, \Psi \in L_2(\mathbb{R}^d)$ in order to obtain a proper decay of the continuous wavelet transform $|W_{\Psi} \Phi(x,t)|$. 

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Definition A.1. Let $L + 1 \in \mathbb{N}_0$, $K > 0$ and fix the conditions $(D)$, $(M_L)$ and $(S_K)$ for a function $\Psi \in L_2(\mathbb{R}^d)$.

$(D)$ For every $N \in \mathbb{N}$ there exists a constant $c_N$ such that
\[ |\Psi(x)| \leq \frac{c_N}{(1 + |x|)^N}. \]

$(M_L)$ We have vanishing moments
\[ D^\alpha \mathcal{F}\Psi(0) = 0 \]
for all $|\alpha|_1 \leq L$.

$(S_K)$ The function
\[ (1 + |\xi|)^K |D^\alpha \mathcal{F}\Psi(\xi)| \]
belongs to $L_1(\mathbb{R}^d)$ for every multi-index $\bar{\alpha} \in \mathbb{N}_0^d$.

Remark A.2. If a function $g \in L_2(\mathbb{R}^d)$ satisfies $(S_K)$ for some $K > 0$ then by well-known properties of the Fourier transform we have $g \in C^{\lfloor K \rfloor}(\mathbb{R}^d)$.

The following lemma provides a useful decay result for the continuous wavelet transform under certain smoothness, decay and moment conditions, see also [12, 23, 16] for similar results in a different language. It represents a continuation of [23, Lem. 1] where one deals with $S(\mathbb{R}^d)$-functions.

Lemma A.3. Let $L \in \mathbb{N}_0$, $K > 0$ and $\Phi, \Psi, \Phi_0 \in L_2(\mathbb{R}^d)$.

(i) Let $\Phi$ satisfy $(D)$, $(M_{L-1})$ and let $\Phi_0$ satisfy $(D)$, $(S_K)$. Then for every $N \in \mathbb{N}$ there exists a constant $C_N$ such that the estimate
\[ |(W_\Phi \Phi_0)(x,t)\rangle \leq C_N \frac{t^{\min\{L,K\}+d/2}}{(1 + |x|)^N} \] holds true for $x \in \mathbb{R}^d$ and $0 < t < 1$.

(ii) Let $\Phi, \Psi$ satisfy $(D)$, $(M_{L-1})$ and $(S_K)$. For every $N \in \mathbb{N}$ there exists a constant $C_N$ such that the estimate
\[ |(W_\Phi \Psi)(x,t)\rangle \leq C_N \frac{t^{\min\{L,K\}+d/2}}{(1 + |x|)^N} \left(1 + \frac{|x|}{1 + t}\right)^{-N} \]
holds true for $x \in \mathbb{R}^d$ and $0 < t < \infty$.

Proof. Step 1. Let us prove (i). We follow the proof of Lemma 1 in [23]. This reference deals with $S(\mathbb{R}^d)$-functions, which makes the situation much more easy. Assume without loss of generality $\Phi = \Phi(-\cdot)$ and $\Phi_0$ to be real-valued. Formula (A.1) gives
\[ |(W_\Phi \Phi_0)(x,t)\rangle = t^{d/2}|(D_t \Phi) \ast \Phi_0|(x)\rangle. \]
Fix $0 < t < 1$. Obviously, the convolution $(D_t \Phi) * \Phi_0$ satisfies $(D)$. By well-known properties of the Fourier transform the derivative $D^\alpha F((D_t \Phi) * \Phi_0)(\xi)$ exists for every multi-index $\alpha \in \mathbb{N}_0^d$. For fixed $\alpha$ we estimate by using Leibniz’ formula

$$|D^\alpha F((D_t \Phi) * \Phi_0)(\xi)| = |D^\alpha (F\Phi(t \xi) \cdot F\Phi_0(\xi))| \leq c_\alpha \sum_{\beta \leq \alpha} |D^\beta (F\Phi(t \xi) \cdot D^{\alpha-\beta} F\Phi_0(\xi))| \leq c_\alpha t^L (1 + |\xi|)^L \sum_{\beta \leq \alpha} |D^{\alpha-\beta} F\Phi_0(\xi)|.$$  \hfill (A.4)

In the last step we used property $(M_{L-1})$. Assuming $K \geq L$ and exploiting $(S_K)$ we obtain that the left-hand side of (A.4) belongs to $L_1(\mathbb{R}^d)$ and

$$\|D^\alpha F((D_t \Phi) * \Phi_0)(\xi)|_{L_1(\mathbb{R}^d)}\| \leq c_\alpha t^L.$$  \hfill (A.5)

We proceed as follows

$$\max_{|\alpha| \leq N+1} \|D^\alpha F((D_t \Phi) * \Phi_0)(\xi)|_{L_1(\mathbb{R}^d)}\| \geq \max_{|\alpha| \leq N+1} \|F^{-1}[D^\alpha F((D_t \Phi) * \Phi_0)]|_{L_\infty(\mathbb{R}^d)}\| \geq c_N (1 + |x|)^N (|D_t \Phi| * \Phi_0)(x)|_{L_\infty(\mathbb{R}^d)}.$$  \hfill (A.6)

This estimate together with (A.3) and (A.5) yields (A.2).

Let us finally assume $K < L$ and return to (A.4). Clearly, the resulting inequality remains valid if we replace the exponent $L$ by $L' \in \mathbb{N}_0$ with $L' \leq L$. It is even possible to extend (A.4) to every $0 \leq L'' < L$ by the following argument. Let $L'' \notin \mathbb{N}$. We have on the one hand

$$\text{LHS}(A.4) \leq c_\alpha t^{|L''| + 1} (1 + |\xi|)^{|L''| + 1} G(\xi)$$

and on the other hand

$$\text{LHS}(A.4) \leq c_\alpha t^{|L''| + 1} (1 + |\xi|)^{|L''| + 1} G(\xi),$$

where $G(\xi) = \sum_{\beta \leq \alpha} |D^{\alpha-\beta} F\Phi_0(\xi)|$. Choosing $0 < \theta < 1$ such that $L'' = (1 - \theta)|L''| + \theta(|L'') + 1$ we obtain by a kind of interpolation argument

$$\text{LHS}(A.4) = \text{LHS}(A.3)^{1-\theta} \text{LHS}(A.4)^{\theta} \leq c_\alpha t^{L''} (1 + |\xi|)^{L''} G(\xi).$$

In particular, we obtain instead of (A.4)

$$|D^\alpha F((D_t \Phi) * \Phi_0)(\xi)| \leq c_\alpha t^K (1 + |\xi|)^K \sum_{\beta \leq \alpha} |D^{\alpha-\beta} F\Phi_0(\xi)|, \quad \xi \in \mathbb{R}^d.$$  \hfill (A.7)

We exploit property $(S_K)$ for $\Phi_0$ and proceed analogously as above. This proves (A.2).

**Step 2.** The estimate in (ii) is an immediate consequence of (A.2) and the fact

$$(W_\Phi \Psi)(x,t) = (W_\Psi \Phi)(-x/t, 1/t).$$

This completes the proof. ■
**Corollary A.4.** Let $\Phi, \Psi$ belong to the Schwartz space $S_0(\mathbb{R}^d)$. By Lemma A.3/(ii) for every $L, N \in \mathbb{N}$ there is a constant $C_{L,N} > 0$ such that

$$|(W_\Phi \Psi)(x,t)| \leq C_{L,N} \frac{t^{L+d/2}}{(1+t)^{2L+d}} \left(1 + \frac{|x|}{1+t}\right)^{-N}, \quad x \in \mathbb{R}^d, t > 0.$$ 

Additionally, we obtain for $\Phi \in S_0(\mathbb{R}^d)$ and $\Phi_0 \in S(\mathbb{R}^d)$ that

$$|(W_\Phi \Phi_0)(x,t)| \leq C_{L,N} t^{L+d/2} (1 + |x|)^{-N}, \quad x \in \mathbb{R}^d, 0 < t < 1.$$ 

**A.2 Orthonormal wavelet bases**

The following Lemma is proved in Wojtaszczyk [34, 5.1].

**Lemma A.5.** Suppose we have a multiresolution analysis in $L^2(\mathbb{R})$ with scaling functions $\Psi^0(t)$ and associated wavelets $\Psi^1(t)$. Let $E = \{0,1\}^d \setminus \{(0,\ldots,0)\}$. For $c = (c_1, \ldots, c_d) \in E$ let $\Psi^c = \bigotimes_{j=1}^d \Psi^{c_j}$. Then the system

$$\left\{2^d \frac{d}{2} \Psi^c(2^j x - k)\right\}_{c \in E, j \in \mathbb{Z}, k \in \mathbb{Z}^d}$$

is an orthonormal basis in $L^2(\mathbb{R}^d)$.

**Spline wavelets**

As a main example we will consider the spline wavelet system. The normalized cardinal B-spline of order $m+1$ is given by

$$N_{m+1}(x) := N_m \ast \chi(x), \quad x \in \mathbb{R}, \quad m \in \mathbb{N},$$

beginning with $N_1 = \chi$, the characteristic function of the interval $(0,1)$. By

$$\varphi_m(x) := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left[\frac{\mathcal{F}N_m(\xi)}{\left(\sum_{k=-\infty}^{\infty} |\mathcal{F}N_m(\xi + 2\pi k)|^2\right)^{1/2}}\right](x), \quad x \in \mathbb{R},$$

we obtain an orthonormal scaling function which is again a spline of order $m$. Finally, by

$$\psi_m(x) := \sum_{k=-\infty}^{\infty} \langle \varphi_m(t/2), \varphi_m(t-k) \rangle (-1)^k \varphi_m(2x + k + 1)$$

the generator of an orthonormal wavelet system is defined. For $m = 1$ it is easily checked that $-\psi_1(x-1)$ is the Haar wavelet. In general these functions $\psi_m$ have the following properties:

- $\psi_m$ restricted to intervals $[k/2, k+1/2]$, $k \in \mathbb{Z}$, is a polynomial of degree at most $m - 1$.
- $\psi_m \in C^{m-2}(\mathbb{R})$ if $m \geq 2$.
- $\psi_m^{(m-2)}$ is uniformly Lipschitz continuous on $\mathbb{R}$ if $m \geq 2$.
- The function $\psi_m$ satisfies a moment condition of order $m - 1$, i.e.

$$\int_{-\infty}^{\infty} x^\ell \psi_m(x) \, dx = 0, \quad \ell = 0, 1, \ldots, m - 1.$$ 

In particular, $\psi_m$ satisfies $(M_L)$ for $0 < L \leq m$ and $\psi_m, \varphi_m$ satisfy $(D)$ and $(S_K)$ for $K < m - 1$. 

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