HECKE OPERATORS AND THE COHERENT COHOMOLOGY OF
SHIMURA VARIETIES

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(Received 8 October 2019; revised 1 February 2021; accepted 2 February 2021; first published online 8 April 2021)

Abstract We consider the problem of defining an action of Hecke operators on the coherent cohomology of certain integral models of Shimura varieties. We formulate a general conjecture describing which Hecke operators should act integrally and solve the conjecture in certain cases. As a consequence, we obtain \(p\)-adic estimates of Satake parameters of certain nonregular self-dual automorphic representations of \(GL_n\).

Keywords and phrases: automorphic forms, Shimura varieties, Hecke operators

2020 Mathematics Subject Classification: Primary 11F46
Secondary 11G18

Contents

1 Introduction 2

2 Correspondences and coherent cohomology 6

3 The Satake isomorphism for unramified reductive groups 11

4 Shimura varieties 16

5 Shimura varieties of symplectic type 24

6 Local model in the symplectic case 33

7 Unitary Shimura varieties 45

8 Local models in the linear case 51
9 Weakly regular automorphic representations and Galois representations

1. Introduction

This article studies the problem of defining an action of Hecke operators on certain natural integral coherent cohomology of Shimura varieties.

Let us start by describing the situation for modular curves. Let $N \geq 4$ be an integer, and let $X \to \text{Spec } \mathbb{Z}[1/N]$ be the compactified modular curve of level $\Gamma_1(N)$. Let $k \in \mathbb{Z}$ and let $\omega^k$ be the sheaf of weight $k$ modular forms. Let $p$ be a prime number, $(p,N) = 1$. When $k \geq 1$, there is a familiar Hecke operator $T_p$ acting on the $\mathbb{C}$-vector space of weight $k$ modular forms. On $q$-expansions, the operator $T_p$ is given (on forms of nebentypus a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Z}^\times$) by the formula $T_p(\sum_{n \geq 0} a_n q^n) = \sum_{n \geq 0} a_n p q^n + p^{k-1} \chi(p) a_n q^{np}$. This formula is integral and the $q$-expansion principle implies that the action of $T_p$ actually arises from an action on $H^0(X, \omega^k)$. We now give a more geometric construction of $T_p$.

Assume first that we work over $\mathbb{Z}[1/Np]$. Then, viewing a modular form $f$ as a rule on triples $(E, \alpha, \omega)$ where $E$ is an elliptic curve, $\alpha$ is a point of order $N$ and $\omega$ is a nowhere vanishing differential form (and satisfying a growth condition near the cusps), there is a geometric formula defining $T_p$ [38, Formula 11.1.0.2]:

$$T_p(f(E, \alpha, \omega)) = \frac{1}{p} \sum_{H \in E[p]} f(E/H, \alpha', \omega'),$$

where $H \subset E[p]$ runs over all subgroups of $E[p]$ of order $p$, and if we let $\pi_H : E \to E/H$ be the isogeny then $\pi_H(\alpha) = \alpha'$ and $\pi_H^* \omega' = \omega$.

The above formula does not really make sense over $\mathbb{Z}[1/N]$, but a slight modification of it will be meaningful. We explain this modification.

There is a correspondence $X_0(p)$ over $X$ corresponding to the level $\Gamma_1(N) \cap \Gamma_0(p)$ (so $X_0(p)$ is a compactification of the moduli of triples $(E, \alpha, H)$ where $H \subset E[p]$ is a subgroup of order $p$), and there are two projections $p_1, p_2 : X_0(p) \to X$ (induced by $(E,H) \mapsto E$ and $(E,H) \mapsto E/H$). Then $T_p$ originates from a cohomological correspondence $T_p : p_2^* \omega^k \to p_1^* \omega^k$, where $p_1^!$ is the right adjoint functor to $(p_1)_!$.

The cohomological correspondence $T_p$ is obtained using the differential of the universal isogeny $\pi_H^* : p_2^* \omega \to p_1^* \omega$, the trace of $p_1$ and a suitable normalisation by a power of $p$ that makes everything integral (this corresponds to the coefficient $\frac{1}{p}$ in the displayed formula). More precisely, we first define a (rational if $k \leq 0$) map over $X_0(p)$,

$$T_p^{\text{naive}} : p_2^* \omega^k \xrightarrow{(\pi_H^*)^k} p_1^* \omega^k \xrightarrow{\text{tr}} p_1^! \omega^k$$

and then set $T_p = p^{-1} T_p^{\text{naive}}$ when $k \geq 1$ and $T_p = p^{-k} T_p^{\text{naive}}$ if $k \leq 1$.

The first basic result for elliptic modular forms is the following (see also [15, §4.5] and [19, Proposition 3.11]):

**Proposition 1.1.** For any $k \in \mathbb{Z}$, we have a cohomological correspondence $T_p : p_2^* \omega^k \to p_1^* \omega^k$ over $X_0(p)$, inducing a Hecke operator $T_p \in \text{End}(\mathbb{R}(X,\omega^k))$. 
Given the cohomological correspondence, the operator on the cohomology is simply obtained using pullback and pushforward as follows:

\[ T_p : H^*(X, \omega^k) \to H^*(X_0(p), p_1^* \omega^k) \xrightarrow{T_p} H^*(X_0(p), p_1^* \omega^k) \to H^*(X, \omega^k). \]

The action of \( T_p \) on formal \( q \)-expansions (of level \( \Gamma_1(N) \) and nebentypus \( \chi \)) is given by

- \( T_p(\sum_{n \geq 0} a_n q^n) = \sum_{n \geq 0} a_n p^{k-1} \chi(p) a_n p^k q^n \) if \( k \geq 1 \),
- \( T_p(\sum_{n \geq 0} a_n q^n) = \sum_{n \geq 0} p^{-1-k} a_n q^n + \chi(p) a_n q^n \) if \( k \leq 1 \);

therefore, \( T_p \) appears to be optimally integral.

One would like to generalise this to other Shimura varieties. To do this, we introduce some notations and recall the fundamental results concerning the cohomology of automorphic vector bundles on Shimura varieties. A standard reference for this material is [29].

Let \( (G, X) \) be a Shimura datum, let \( K \subset G(\mathbb{A}_f) \) be a neat compact open subgroup, and let \( Sh_K \) be the associated Shimura variety. This is a smooth scheme defined over the reflex field \( E \). Over \( Sh_K \) there is a large supply of automorphic vector bundles \( \mathcal{V}_{\kappa,K} \), naturally parametrised by weights \( \kappa \) of a maximal torus in \( G \), dominant for the compact roots. For a choice of polyhedral cone decomposition \( \Sigma \), one can construct a toroidal compactification \( Sh_{K,\Sigma}^{tor} \) of \( Sh_K \), such that \( D_{K,\Sigma} = Sh_{K,\Sigma}^{tor} \setminus Sh_{K,\Sigma} \) is a Cartier divisor. Moreover, there is a canonical extension \( \mathcal{V}_{\kappa,K,\Sigma} \) of the vector bundle \( \mathcal{V}_{\kappa,K} \), as well as a subcanonical extension \( \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}) \).

The coherent cohomology complexes we are interested in (which generalise the classical notion of modular forms) are

\[ \text{RG}(Sh_{K,\Sigma}^{tor}, \mathcal{V}_{\kappa,K,\Sigma}) \text{ and } \text{RG}(Sh_{K,\Sigma}^{tor}, \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma})). \]

These cohomology complexes are independent of \( \Sigma \), they have good functorial properties in the level \( K \) and they carry an action of the Hecke algebra \( \mathcal{H}_K = C_c^\infty(K \setminus G(\mathbb{A}_f)/K, \mathbb{Z}) \).

Over \( \mathbb{C} \) this coherent cohomology can be computed via the \((\mathfrak{p}, K)\)-cohomology of the space of automorphic forms on the group \( G \) [37]. Automorphic representations contributing to these cohomology groups therefore possess a rational structure. We now fix a prime \( p \). In order to study \( p \)-adic properties of automorphic forms contributing to the coherent cohomology, we introduce integral structures on \( \text{RG}(Sh_{K,\Sigma}^{tor}, \mathcal{V}_{\kappa,K,\Sigma}) \) and \( \text{RG}(Sh_{K,\Sigma}^{tor}, \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma})) \).

Let \( \lambda \) be a place of \( E \) above \( p \). If the Shimura datum \( (G, X) \) is of abelian type, \( G \) is unramified at \( p \), and \( K = K^p K_p \), where \( K^p \subset G(\mathbb{A}_f^p) \) and \( K_p \subset G(\mathbb{Q}_p) \) is hyperspecial, we have natural integral models \( \mathcal{S}h_K \) for \( Sh_K \), as well as an integral model of \( \mathcal{V}_{\kappa,K} \), over Spec \( \mathcal{O}_{E,\lambda} \) [52], [42], [40], [41]. Moreover, there is a theory of toroidal compactifications (at least in the Hodge case) \( \mathcal{S}h_{K,\Sigma}^{tor} \) of \( \mathcal{S}h_K \), with integral canonical extension \( \mathcal{V}_{\kappa,K,\Sigma} \) and integral subcanonical extension \( \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma}) \) [20], [46], [45], [55].

We are therefore led to consider the cohomology complexes \( \text{RG}(\mathcal{S}h_{K,\Sigma}^{tor}, \mathcal{V}_{\kappa,K,\Sigma}) \) as well as \( \text{RG}(\mathcal{S}h_{K,\Sigma}^{tor}, \mathcal{V}_{\kappa,K,\Sigma}(-D_{K,\Sigma})) \). Our work investigates the question of extending the action of the Hecke algebra \( \mathcal{H}_K \) to these cohomology groups. The action of the prime to \( p \) Hecke algebra extends without much difficulty, so our main task is to investigate
the action of (a suitable subalgebra of) the local Hecke algebra $\mathcal{C}_c^\infty(K_p \backslash G(\mathbb{Q}_p)/K_p,\mathbb{Q})$ on $\text{R}\Gamma(\mathcal{S}\text{h}^\text{tor}_{K,\Sigma},\mathcal{V}_{\kappa,K,\Sigma})$.

We formulate a conjecture (see Conjecture 4.16, as well as Conjectures 4.7 and 4.15) precisely describing a sub-$\mathbb{Z}$-algebra $\mathcal{H}^{\text{int}}_{p,\kappa,i}$ (where $i$ is an isomorphism of $\mathbb{C}$ and $\mathbb{Q}_p$) of $\mathcal{C}_c^\infty(K_p \backslash G(\mathbb{Q}_p)/K_p,\mathbb{Q})$ that should act on the cohomology complexes $\text{R}\Gamma(\mathcal{S}\text{h}^\text{tor}_{K,\Sigma},\mathcal{V}_{K,\Sigma})$ and $\text{R}\Gamma(\mathcal{S}\text{h}^\text{tor}_{K,\Sigma},\mathcal{V}_{K,\Sigma}(-D_{K,\Sigma}))$.

The definition of $\mathcal{H}^{\text{int}}_{p,\kappa,i}$ is given in Definition 4.8, with the help of the Satake basis $[V_\lambda]$ of $\mathcal{H}_p$ (where $\lambda$ runs through the set of dominant and Galois invariant characters of the dual group $\hat{G}$ of $G$) normalised by a certain power of $p$ determined by $\lambda$ and the weight $\kappa$.

In the modular curve case, the description of this subalgebra precisely reflects the normalising factors $p^{-\inf\{1,k\}}$ in the definition of the $T_p$-operator on the cohomology in weight $k$.

This conjecture is a translation for the coherent cohomology of Shimura varieties of results obtained by V. Lafforgue on the Betti cohomology of locally symmetric spaces in [44]. It is inspired by Katz-Mazur inequality: For $X$ a proper smooth scheme defined over $\mathbb{Z}_p$, Katz and Mazur gave $p$-adic estimates for the eigenvalues of the geometric Frobenius acting on the cohomology $H^i(X_{\overline{\mathbb{Q}}_p},\mathbb{Q}_\ell)$ ($\ell \neq p$). The estimates say that $p$-adic Newton polygon of the characteristic polynomial of Frobenius lies above the Hodge polygon (determined by the filtration on the de Rham cohomology of $X_{\overline{\mathbb{Q}}_p}$).

One believes that similar estimates hold for algebraic automorphic representations because of the conjectural correspondence between motives and automorphic representations [13], [8]. Roughly speaking, the weight $\kappa$ determines the Hodge polygon, and the characters of $\mathcal{H}_p$ determine the characteristic polynomials of Frobenii. We postulate the Hodge-Newton inequality to determine the precise shape of the maximal subalgebra $\mathcal{H}^{\text{int}}_{p,\kappa,i} \subset \mathcal{H}_p$ that should act integrally on the cohomology.

We then try to prove our conjecture in certain special cases. To the characteristic functions of all double cosets $K_pgK_p$ in the local Hecke algebra, one can associate Hecke correspondences $p_1,p_2 : Sh_{K \cap gK_p} \to Sh_K$ over $Sh_K$. These correspondences rarely admit integral models whose geometry is understood, except when $g$ is associated with a minuscule coweight. In this case, $K_p \cap gK_p g^{-1}$ is a parahoric subgroup and there is a good theory of integral models whose local geometry is described by the local model theory. We are thus led to work with Hecke operators associated to minuscule coweights (this is a serious restriction).

At this point a second obstacle arises: The projections $p_i$ almost never extend integrally to finite flat morphisms. This means that defining the necessary trace maps in cohomology is in principle complicated. We develop, using Grothendieck-Serre duality [32], a formalism of cohomological correspondences in coherent cohomology that solves the problem under some assumptions on the correspondences. In particular, we assume that our correspondences are given by Cohen-Macaulay schemes. This suffices for our purposes because using the theory of local models (see, e.g., [56]), one can often prove that integral models of Shimura varieties with parahoric level structure are Cohen-Macaulay.

In order to prove Conjecture 4.16, our strategy is to switch to the local model, where we can make all of the computations and then transfer back the information to the Shimura variety.
We completely solve our conjecture for Hilbert modular varieties ($G = \text{Res}_{F/Q} \text{GL}_2$), when $p$ is unramified in the totally real field $F$. To formulate the result precisely in this case, let $E'$ be the finite extension of $E = Q$ equal to the Galois closure of $F$. Weights for Hilbert modular forms are tuples of integers $\kappa = ((k_\sigma)_{\sigma \in \text{Hom}(F,E')}; k)$ where $k_\sigma$ and $k$ all have the same parity. We let $\iota : E' \to \overline{Q}_p$ be an embedding. Let $p = \prod_i p_i$ be the decomposition of $p$ in $\mathcal{O}_F$ as a product of prime ideals.

**Theorem 1.2** (Theorem 5.9 and Subsection 5.6.4). The Hecke algebra $\mathcal{H}_{p,\kappa,\iota}^{\text{int}} = \bigotimes_i \mathbb{Z}[T_{p_i}, S_{p_i}, S_{p_i}^{-1}]$ acts on $\text{RG}(\mathfrak{H}_{K,\Sigma}^{\text{tor}}, V_{\kappa, K, \Sigma})$ and $\text{RG}(\mathfrak{H}_{K,\Sigma}^{\text{tor}}, V_{\kappa, K, \Sigma}(-D_{K, \Sigma}))$.

Under the assumption that the weight $\kappa$ belongs to a certain cone (conjecturally the ample cone), the theorem was first proved in [19, Proposition 3.11] by different methods.

We can be explicit about normalisation factors. Let $I_i = \{ \sigma \in \text{Hom}(F,E') \mid \iota \circ \sigma(p_i) \subset m_{\overline{\mathbb{Z}}_{p_i}} \}$ and let

$$T_{p_i}^{\text{naive}} = \text{GL}_2(\mathcal{O}_{F_{p_i}})\text{diag}(p_i^{-1}, 1)\text{GL}_2(\mathcal{O}_{F_{p_i}})$$
and

$$S_{p_i}^{\text{naive}} = \text{GL}_2(\mathcal{O}_{F_{p_i}})\text{diag}(p_i^{-1}, p_i^{-1})\text{GL}_2(\mathcal{O}_{F_{p_i}})$$
be the naive, ‘unnormalised’ version of the Hecke operators attached to the familiar double classes (the reason we have to use these double classes and not their inverse is that we set up the theory in such a way that the Hecke algebra acts naturally on the left and not on the right on the cohomology). Our normalised operators are

$$T_{p_i} = p^{\sum_{\sigma \in I_i} \sup \{ \frac{e_{\sigma} + k}{2}, \frac{k - e_{\sigma}}{2} \}} T_{p_i}^{\text{naive}}$$
and

$$S_{p_i} = p^{\sum_{\sigma \in I_i} k} S_{p_i}^{\text{naive}}.$$  

For more general Shimura varieties of symplectic type we only have partial results because there is essentially only one minuscule coweight (see Theorem 5.9). However, the situation is better for unitary Shimura varieties. Let $F$ be a totally real field, and let $L$ be a totally imaginary quadratic extension of $F$. We let $G \subset \text{Res}_{L/Q} \text{GL}_n$ be a unitary group of signature $(p_r, q_f) \tau, F \to \mathbb{C}$. We assume that $p$ is unramified in $L$ and we let $\mathfrak{H}_{K,\Sigma}^{\text{tor}}$ be a toroidal compactification of the (smooth) integral model of the unitary Shimura variety attached to $G$.

**Theorem 1.3** (Theorem 7.5). Assume that all finite places $v \mid p$ in $F$ split in $L$. Let $\mathcal{H}_{p,\kappa,\iota}^{\text{int}} = \bigotimes_{0 \leq j \leq m} \mathbb{Z}[T_{p,j}, 0 \leq j \leq n, T_{p,0}^{-1}, T_{p,n}^{-1}]$ be the normalised integral Hecke algebra in weight $\kappa$ (where the $T_{p,j}$ are standard generators of this Hecke algebra). All of the operators $T_{p,j}$ act on $\text{RG}(\mathfrak{H}_{K,\Sigma}^{\text{tor}}, V_{\kappa, K, \Sigma})$ and $\text{RG}(\mathfrak{H}_{K,\Sigma}^{\text{tor}}, V_{\kappa, K, \Sigma}(-D_{K, \Sigma}))$. In particular, $\text{Im}(H^i(\mathfrak{H}_{K,\Sigma}^{\text{tor}}, V_{\kappa, K, \Sigma}) \to H^i(\mathfrak{H}_{K,\Sigma}^{\text{tor}}, V_{\kappa, K, \Sigma}) \otimes \mathbb{Z}_p \otimes \mathbb{Q}_p)$ is a lattice that is stable under the action of $\mathcal{H}_{p,\kappa,\iota}^{\text{int}}$.

**Remark 1.4.** The limitation of this last result is that we have not been able to prove that the operators $T_{p,j}$ commute with each other in general.
Let $L$ be a CM or a totally real number field. One can realise regular algebraic essentially (conjugate) self-dual cuspidal automorphic representations of $\text{GL}_n/L$ in the Betti cohomology of Shimura varieties. The interest of coherent cohomology is that it captures more automorphic representations. Namely, one can weaken the regularity condition to a condition that we call weakly regular odd (the oddness property is automatically satisfied in the regular case). Weakly regular, algebraic, odd, essentially (conjugate) self-dual, cuspidal automorphic representations on $\text{GL}_n/L$ admit a compatible system of Galois representations, but at the moment these compatible systems are not known to be de Rham in general (and local–global compatibility is not known). We can nevertheless prove the following result, which is to be viewed as the Katz-Mazur inequality.

**Theorem 1.5** (Theorem 9.11). Let $\pi$ be a weakly regular, algebraic, odd, essentially (conjugate) self-dual, cuspidal automorphic representation of $\text{GL}_n/L$ with infinitesimal character $\lambda = (\lambda_{i,\tau}, 1 \leq i \leq n, \tau \in \text{Hom}(L, \overline{\mathbb{Q}}))$ and $\lambda_1, \tau \geq \cdots \geq \lambda_n, \tau$. Let $p$ be a prime unramified in $L$ and $w$ be a finite place of $L$ dividing $p$. Assume also that $\pi_w$ is spherical and corresponds to a semi-simple conjugacy class $\text{diag}(a_1, \cdots, a_n) \in \text{GL}_n(\overline{\mathbb{Q}})$ by the Satake isomorphism. We let $\iota : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ be an embedding and $v$ the associated $p$-adic valuation normalised by $v(p) = 1$. After permuting we assume that $v(a_1) \leq \cdots \leq v(a_n)$. Let $I_w \subset \text{Hom}(L, \overline{\mathbb{Q}})$ be the set of embeddings $\tau$ such that $\iota \circ \tau$ induces the $w$-adic valuation on $L$. Then we have

$$\sum_{i=1}^{k} v(a_i) \geq \sum_{\tau \in I_w} \sum_{\ell=1}^{k} -\lambda_{\ell, \tau},$$

for $1 \leq k \leq n$, with equality if $k = n$.

### 1.1. Organisation of the article

In Section 2 we develop a formalism of cohomological correspondences in coherent cohomology. In Section 3 we give a number of classical results concerning the structure of the local spherical Hecke algebra of an unramified group. In Section 4 we introduce Shimura varieties and their coherent cohomology and formulate Conjecture 4.16 on the action of the integral Hecke algebra on the integral coherent cohomology. In Sections 5 and 6 we consider the case of Shimura varieties of symplectic type and their local models. In Sections 7 and 8 we consider the case of Shimura varieties of unitary type and their local models. Finally, the last section deals with applications to automorphic representations and Galois representations.

### 2. Correspondences and coherent cohomology

#### 2.1. Preliminaries on residues and duality

We start by recalling some results of Grothendieck duality theory for coherent cohomology. The original reference for this, which we use below, is [32]; although this and [14], which is based on it, suffice for our purposes, the more abstract approaches of [61], [53] and [48] give more general results with arguably more efficient proofs. In particular, the latter two references show that the noetherian and finite Krull dimension hypotheses of [32] can be eliminated and that most results extend to the unbounded derived category.
For a scheme $X$ we let $D_{qcoh}(\mathcal{O}_X)$ be the subcategory of the derived category $D(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules whose objects have quasi-coherent cohomology sheaves. We let $D^+_{qcoh}(\mathcal{O}_X)$ (respectively $D^-_{qcoh}(\mathcal{O}_X)$) be the full subcategory of $D_{qcoh}(\mathcal{O}_X)$ whose objects have 0 cohomology sheaves in sufficiently negative (respectively positive) degree. We let $D^b_{qcoh}(\mathcal{O}_X)$ be the full subcategory of $D_{qcoh}(\mathcal{O}_X)$ whose objects have 0 cohomology sheaves for all but finitely many degrees. We remark that if $X$ is locally noetherian $D_{qcoh}(\mathcal{O}_X)$ is also the derived category of the category of bounded below complexes of quasi-coherent sheaves on $X$ [32, I, Corollary 7.19]. We let $D^b_{qcoh}(\mathcal{O}_X)_{ftd}$ be the full subcategory of $D^b_{qcoh}(\mathcal{O}_X)$ whose objects are quasi-isomorphic to bounded complexes of flat sheaves of $\mathcal{O}_X$-modules [32, II, Definition 4.13]. We fix for the rest of this section a noetherian affine scheme $S$.

2.1.1. Embeddable morphisms. Let $X$, $Y$ be two $S$-schemes and $f : X \to Y$ be a morphism of $S$-schemes. The morphism $f$ is embeddable if there exist a smooth $S$-scheme $P$ and a finite map $i : X \to P \times_S Y$ such that $f$ is the composition of $i$ and the second projection [32, p. 189]. The morphism $f$ is projectively embeddable if it is embeddable and $P$ can be taken to be a projective space over $S$ [32, p. 206].

2.1.2. The functor $f^!$. For $f : X \to Y$ a morphism of $S$-schemes, there is a functor $Rf_* : D_{qcoh}(\mathcal{O}_X) \to D_{qcoh}(\mathcal{O}_Y)$. By [32, III, Theorem 8.7] if $f$ is embeddable, we can define a functor $f^! : D^+_{qcoh}(\mathcal{O}_Y) \to D^+_{qcoh}(\mathcal{O}_X)$. If $f$ is projectively embeddable, by [32, III, Theorem 10.5] there is a natural transformation (trace map) $Rf_* f^! \Rightarrow Id$ of endofunctors of $D^+_{qcoh}(\mathcal{O}_Y)$. Moreover, by [32, III, Theorem 11.1] this natural transformation induces a duality isomorphism:

$$\text{Hom}_{D_{qcoh}(\mathcal{O}_X)}(\mathcal{F}, f^!\mathcal{G}) \simeq \text{Hom}_{D_{qcoh}(\mathcal{O}_Y)}(Rf_* \mathcal{F}, \mathcal{G})$$ (2.1A)

for $\mathcal{F} \in D^-_{qcoh}(\mathcal{O}_X)$ and $\mathcal{G} \in D^+_{qcoh}(\mathcal{O}_Y)$.

The functor $f^!$ for embeddable morphism enjoys many good properties. Let us record one that will be crucially used.

Proposition 2.1. [[32, III, Proposition 8.8]] If $\mathcal{F} \in D^+_{qcoh}(\mathcal{O}_Y)$ and $\mathcal{G} \in D^b_{qcoh}(\mathcal{O}_Y)_{ftd}$, there is a functorial isomorphism $f^! \mathcal{F} \otimes^L Lf^* \mathcal{G} = f^!(\mathcal{F} \otimes^L \mathcal{G})$.

2.2. Cohen-Macaulay, Gorenstein and local complete intersection morphisms

2.2.1. Cohen-Macaulay morphisms. Recall that a morphism $h : X \to S$ is called Cohen-Macaulay (CM) if $h$ is flat, locally of finite type and has CM fibres.

Lemma 2.2. For a CM morphism of pure relative dimension $n$, $h^! \mathcal{O}_S = \omega_{X/S}[n]$, where $\omega_{X/S}$ is a coherent sheaf that is flat over $S$. Assume moreover that $S$ is CM. Then $\omega_{X/S}$ is a CM sheaf.

Proof. For the first point, see [14, Theorem 3.5.1]. For the second point, first observe that if $S$ is Gorenstein (e.g., $S$ is the spectrum of a field), so $\mathcal{O}_S[0]$ is a dualising complex for $S$,
then \( \omega_{X/S}[n] = h^! \mathcal{O}_S \) is a dualising complex for \( X \) [32, V, §§2,10]. Using local duality [7, Theorem 1.2.8], it follows from [7, Corollary 3.5.11] that \( \omega_{X/S} \) is a CM sheaf. In general, for a CM map \( h \) as above, the formation of \( \omega_{X/S} \) is compatible with base change [14, Theorem 3.6.1], so it follows from [26, Proposition 6.3.1] that \( \omega_{X/S} \) is a CM sheaf.

2.2.2. Gorenstein morphisms. A local ring \( A \) is Gorenstein if it satisfies the equivalent properties of [32, V, Theorem 9.1] (for instance, if \( A \) is a dualising complex for itself). Note that local complete intersection rings, in particular regular rings, are Gorenstein. Furthermore, Gorenstein rings are CM. A locally noetherian scheme is Gorenstein if all of its local rings are Gorenstein. Finally, a morphism \( h : X \to S \) is Gorenstein if it is flat and all of its fibres are Gorenstein.

Lemma 2.3. For a proper Gorenstein morphism \( h : X \to S \) of pure relative dimension \( n \), \( h^! \mathcal{O}_S = \omega_{X/S}[n] \), where \( \omega_{X/S} \) is an invertible sheaf on \( X \).

Proof. A Gorenstein morphism is CM, so by Lemma 2.2, \( h^! \mathcal{O}_S = \omega_{X/S}[n] \), where \( \omega_{X/S} \) is a coherent sheaf on \( X \). Because for a CM map \( h \) the formation of \( \omega_{X/S} \) is compatible with base change [14, Theorem 3.6.1], for any point \( s \in S \) \( \omega_{X/S}|_{X_s} = \omega_{X_s/s} \). By [32, V, Proposition 2.4, Theorem 8.3] (and the remark after Theorem 8.3), \( \omega_{X_s/s} \) is a dualising sheaf for \( X_s \). By [32, V, Theorem 3.1], the Gorenstein hypothesis on \( X_s \) implies that \( \omega_{X_s/s} \) is an invertible sheaf. Because \( \omega_{X/s} \) is flat over \( S \) by Lemma 2.2, it is flat over \( X \) by the fibrewise flatness criterion [27, Théorème 11.3.10], so it is an invertible sheaf over \( X \).

2.2.3. Local complete intersection morphisms. A morphism \( h : X \to S \) is called a local complete intersection (lci) morphism if locally on \( X \) we have a factorisation \( h : X \xrightarrow{i} Z \to S \) where \( i \) is a regular immersion [27, Définition 16.9.2] and \( Z \) is a smooth \( S \)-scheme. If \( h \) is lci, then the cotangent complex of \( h \), denoted by \( \mathbb{L}_{X/S} \), is a perfect complex concentrated in degree \(-1\) and \( 0 \) [36, Proposition 3.2.9]. Its determinant in the sense of [41] is denoted by \( \omega_{X/S} \).

Lemma 2.4. If \( h : X \to S \) is an embeddable morphism and an lci of pure relative dimension \( n \), then \( h^! \mathcal{O}_S = \omega_{X/S}[n] \) where \( \omega_{X/S} \) is the determinant of the cotangent complex \( \mathbb{L}_{X/S} \).

Proof. This follows from the very definition of \( h^! \) given in [32, III, Theorem 8.7].

2.3. An application: construction of a trace map

Proposition 2.5. Let \( X,Y \) be two embeddable \( S \)-schemes and let \( f : X \to Y \) be an embeddable morphism. Assume that \( X \to S \) is CM and that \( Y \to S \) is Gorenstein. Assume that \( X \) and \( Y \) have the same pure relative dimension over \( S \). Then \( f^! \mathcal{O}_Y := \omega_{X/Y} \) is a coherent sheaf. If, moreover, \( S \) is assumed to be CM, then \( \omega_{X/S} \) is a CM sheaf. If \( X \) and \( Y \) are smooth over \( S \), \( \omega_{X/S} = \det \Omega^1_{X/S} \otimes f^{-1} \mathcal{O}_Y (f^{-1} \det \Omega^1_{Y/S})^{-1} \).
Proof. We have $h^!\mathcal{O}_S = \omega_{X/S}[n]$. On the other hand,

\[
\begin{align*}
 h^!\mathcal{O}_S &= f^!(g^!\mathcal{O}_S) \\
 &= f^!(\omega_{Y/S}[n]) \\
 &= f^!(\mathcal{O}_Y \otimes \omega_{Y/S}[n]) \\
 &= f^!(\mathcal{O}_Y) \otimes f^*\omega_{Y/S}[n].
\end{align*}
\]

We observe that by adjunction we have a universal trace map $Rf_*\omega_{X/Y} \to \mathcal{O}_Y$. In particular, for any section $s \in H^0(X,\omega_{X/Y})$ (or, equivalently, morphism $\mathcal{O}_X \to \omega_{X/Y}$), we get a corresponding trace $\text{Tr}_s$:

\[
Rf_*\mathcal{O}_X \to \mathcal{O}_Y.
\]

Proposition 2.6. Let $X,Y$ be two embeddable $S$-schemes and let $f : X \to Y$ be an embeddable morphism. Assume that

1. $X \to S$ is CM,
2. $Y \to S$ is Gorenstein,
3. $S$ is CM,
4. $X$ and $Y$ have the same pure relative dimension over $S$ and
5. there are open sets $V \subset X, U \subset Y$ such that $f(V) \subset U$, $U$ and $V$ smooth over $S$ and $X \setminus V$ is of codimension 2 in $X$.

Then there is a canonical morphism $\Theta : \mathcal{O}_X \to \omega_{X/Y}$ called the fundamental class.

Moreover, if $W \subset Y$ is an open subscheme and $X \times_Y W \to W$ is finite flat, then the trace $\text{Tr}_\Theta$ restricted to $W$ is the usual trace map for the finite flat morphism $X \times_Y W \to W$.

Proof. It is enough to specify the fundamental class over $V$ because it will extend to all of $X$ by Lemma 2.2. Then over $V$, we have a map $df : f^*\Omega^1_{U/S} \to \Omega^1_{V/S}$ and we define the fundamental class as the determinant $\det(df) \in H^0(V, \det \Omega^1_{V/S} \otimes (f^* \det \Omega^1_{U/S})^{-1})$.

To prove the second claim, we can assume that $X,Y$ are smooth over $S$ and the map $X \to Y$ is finite flat. In this situation, $X \to Y$ is lci. We claim that the cotangent complex $\mathcal{L}_{X/Y}$ is represented by the complex in degree $-1$ and $0$:

\[
[f^*\Omega^1_{Y/S} \xrightarrow{df} \Omega^1_{X/S}]
\]

and that the determinant $\det(df) \in H^0(X,\omega_{X/Y}) = \text{Hom}(\mathcal{O}_X, f^!\mathcal{O}_Y)$ is the trace map $\text{tr}_f$. We have a closed embedding $i : X \hookrightarrow X \times_S Y$ of $X$ into the smooth $Y$-scheme $X \times_S Y$. We have an exact sequence

\[
0 \to \mathcal{I}_Y \to \mathcal{O}_{Y \times_S Y} \to \mathcal{O}_Y \to 0
\]

that gives after tensoring with $\mathcal{O}_X$ above $\mathcal{O}_Y$

\[
0 \to \mathcal{I}_X \to \mathcal{O}_{X \times_S Y} \to \mathcal{O}_X \to 0,
\]

where $\mathcal{I}_X$ is the ideal sheaf of the immersion $i$. It follows that $\mathcal{I}_X/\mathcal{I}_X^2 = \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes_{\mathcal{O}_Y} \mathcal{O}_X = \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_X$.

On the other hand, $i^*\Omega^1_{X \times_S Y} = \Omega^1_{X/S}$. The cotangent complex is represented by $[\mathcal{I}_X/\mathcal{I}_X^2 \to i^*\Omega^1_{X \times_S Y/Y}]$, which is the same as $[f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}]$. 

\[
\square
\]

\[
\]
The morphism \( f_* \det L_{X/Y} = \text{Hom}(f_* \mathcal{O}_X, \mathcal{O}_Y) \rightarrow \mathcal{O}_Y \) is the residue map that associates to \( \omega \in f_* \Omega^1_{X/S} \) and to \((t_1, \ldots, t_n)\) local generators of the ideal \( I_X \) over \( Y \) the function \( \text{Res}[\omega, t_1, \ldots, t_n] \). It follows from [32, Property (R6), p. 198] that the determinant of \([f^* \Omega^1_{Y/S} \rightarrow \Omega^1_{X/S}]\) maps to the usual trace map.

2.3.1. **Fundamental class and divisors.** We need a slight variant of the last proposition, where we also have some divisors.

**Proposition 2.7.** Let \( X, Y \) be two embeddable \( S \)-schemes and let \( f : X \rightarrow Y \) be an embeddable morphism. Let \( D_X \hookrightarrow X \) and \( D_Y \hookrightarrow Y \) be relative (with respect to \( S \)) effective Cartier divisors.

Assume that

1. \( X \rightarrow S \) is CM,
2. \( Y \rightarrow S \) is Gorenstein,
3. \( S \) is CM,
4. \( X \) and \( Y \) have the same pure relative dimension over \( S \),
5. there are open sets \( V \subset X, U \subset Y \) such that \( f(V) \subset U \), \( U \) and \( V \) are smooth over \( S \) and \( X \backslash V \) is of codimension 2 in \( X \),
6. the morphism \( f \) restricts to a surjective map from \( D_X \) to \( D_Y \) and
7. \( D_X \cap V \) and \( D_Y \cap U \) are relative normal crossings divisors.

The fundamental class \( \Theta : \mathcal{O}_X \rightarrow f^! \mathcal{O}_Y \) constructed in Proposition 2.6 induces a map

\[
\mathcal{O}_X(-D_X) \rightarrow f^! \mathcal{O}_Y(-D_Y).
\]

**Proof.** We may assume that \( X \) and \( Y \) are smooth, and \( D_X \) and \( D_Y \) are relative normal crossing divisors. In that case, we have a well-defined differential map \( df : f^* \Omega^1_{Y/S}(\log D_Y) \rightarrow \Omega^1_{X/S}(\log D_X) \). Taking the determinant yields \( \det df : f^* \det \Omega^1_{Y/S}(D_Y) \rightarrow \det \Omega^1_{X/S}(D_X) \) or, equivalently, \( \det df : \mathcal{O}_X(-D_X) \rightarrow f^! \mathcal{O}_Y(-D_Y) \).

2.4. **Cohomological correspondences**

Let \( X, Y \) be two \( S \)-schemes.

**Definition 2.8.** A correspondence \( C \) over \( X \) and \( Y \) is a diagram of \( S \)-morphisms:

\[
\begin{array}{ccc}
X & \xleftarrow{p_2} & C \\
& p_1 & \rightarrow \\
& & Y
\end{array}
\]

where \( X, Y, C \) have the same pure relative dimension over \( S \) and the morphisms \( p_1 \) and \( p_2 \) are projectively embeddable.

**Remark 2.9.** In practice, the maps \( p_1, p_2 \) will often be surjective and generically finite.
Let \( F \in D_{\text{qcoh}}(\mathcal{O}_X) \) and \( G \in D_{\text{qcoh}}^+(\mathcal{O}_Y) \).

**Definition 2.10.** A cohomological correspondence from \( F \) to \( G \) is the data of a correspondence \( C \) over \( X \) and \( Y \) and a map \( T : R(p_1)_!Lp_2^*F \to G \).

The map \( T \) can be seen, by (2.1.A), as a map \( Lp_2^*F \to p_1^!G \). Note that if \( F \) and \( G \) are coherent sheaves (in degree 0), then \( Lp_2^*F \) is concentrated in degrees \( \leq 0 \) and \( p_1^!G \) is concentrated in degrees \( \geq 0 \) (as follows from the construction in [32, III, Theorem 8.7]), so any map \( Lp_2^*F \to p_1^!G \) factors uniquely through the natural map \( Lp_2^*F \to p_2^*F \). It gives rise to a map, still denoted by \( T \), on cohomology

\[
\Gamma(X,F) \xrightarrow{Lp_2^*} \Gamma(C,p_2^*F) = \Gamma(Y,R(p_1)_!p_2^*F) \xrightarrow{T} \Gamma(Y,G).
\]

**Example 2.11.** Let \( C \) be a correspondence over \( X \) and \( Y \). We assume that the map \( p_1 : C \to Y \) satisfies the assumptions of Proposition 2.6, so there is a fundamental class \( \mathcal{O}_C \to p_1^!\mathcal{O}_Y \). Let \( F \) and \( G \) be locally free sheaves of finite rank over \( X \) and \( Y \). We assume that there is a map \( p_2^*F \to p_1^!G \). Twisting the fundamental class by \( p_1^!G \), we get a morphism \( p_1^!G \to p_1^!\mathcal{O}_Y \otimes p_1^!G \), and using the isomorphism of Proposition 2.1, we get a map \( p_1^!G \to p_1^!G \). Composing everything, we obtain \( T : p_2^*F \to p_1^!G \).

3. The Satake isomorphism for unramified reductive groups

3.1. The dual group

Let \( K \) be a non-Archimedean local field, \( \mathcal{O}_K \) its ring of integers and \( \pi \in \mathcal{O}_K \) a uniformising element. Let \( q \) be the cardinality of the residue field \( \mathcal{O}_K/\langle \pi \rangle \). Let \( G \) be a reductive group over \( \text{Spec} \mathcal{O}_K \). We assume that \( G \) is quasi-split and splits over an unramified extension of \( K \) and fix a Borel subgroup \( B \) and a maximal torus \( T \subset B \). We denote by \( X_*(T) \) and \( X^*(T) \) the groups of cocharacters and characters of \( T \). These groups carry an action of \( \Gamma = \hat{\mathbb{Z}} \), the unramified Galois group. We denote by \( T_d \subset T \) the maximal split torus inside \( T \). Its cocharacter group is \( X_*(T)^1 \) and we have \( X^*(T_d) = (X^*(T))_\Gamma \).

We now introduce the Langlands dual of \( G \), following [7, Chapter I]. Let \( \hat{G} \) be the dual group of \( G \), defined over \( \overline{Q} \), and let \( \hat{T} \) be a maximal torus in \( \hat{G} \) (so that \( X_*(\hat{T}) = X^*(\hat{T}) \)). We denote by \( \Phi^+ \) the set of positive roots for \( B \) in \( X^*(T) \) and by \( \hat{\Phi}^+ \subset X_*(\hat{T}) = X^*(\hat{T}) \) the corresponding set of positive coroots. This is the set of positive roots for a Borel subgroup \( \hat{B} \) of \( \hat{G} \) that contains \( \hat{T} \).

We let \( W \) be the Weyl group of \( G \) and \( \hat{G} \) and we denote by \( w_0 \) the longest element in \( W \) (which maps the Borel \( B \) to the opposite Borel). We let \( W_d \) be the subgroup of \( W \) that stabilises \( T_d \); this is also the fixed point set of the action of \( \Gamma \) on \( W \) [7, §6.1].

The set of dominant weights for \( G \) is denoted by \( P^+ \). This is the cone in \( X^*(T) \) of characters \( \lambda \) such that \( \langle \lambda, \alpha \rangle \geq 0 \) for all \( \alpha \in \Phi^+ \). We let \( P^+_\mathbb{R} \subset X^*(T)_\mathbb{R} \) be the positive Weyl chamber, which is the \( \mathbb{R}_{\geq 0} \)-linear span of \( P^+ \). The cone \( P^+_\mathbb{R} \) is a fundamental domain for the action of \( W \) on \( X^*(T) \otimes \mathbb{R} \). We have similar notations for \( \hat{G} \). We denote by \( \rho \) the half sum of elements of \( \Phi^+ \).

We choose a pinning to define an action of \( \Gamma \) on \( \hat{G} \), preserving \( \hat{B} \) and \( \hat{T} \), and we then let \( L^G = \hat{G} \rtimes \Gamma \).
3.2. Hecke algebras

Whenever we have a unimodular locally compact group $G$ and a compact open subgroup $K$, we denote by $\mathcal{H}(G, K)$ the algebra of compactly supported left and right $K$-invariant functions from $G$ to $\mathbb{Z}$. The product in $\mathcal{H}(G, K)$ is the convolution product (the Haar measure on $G$ is normalised by $\text{vol}(K) = 1$, so $1_K$ is the unit in $\mathcal{H}(G, K)$).

We have an isomorphism of algebras $\mathcal{H}(T_d(K), T_d(O_K)) = \mathbb{Z}[X_\lambda(T_d)]$. Moreover, the restriction map $\mathcal{H}(T(K), T(O_K)) \to \mathcal{H}(T_d(K), T_d(O_K))$ is an isomorphism of algebras because $T(K)/T(O_K) = T_d(K)/T_d(O_K)$ by [7, §9.5].

Because $G$ is quasi-split, the centraliser of $T_d$ in $G$ is $T$, so $W_d$ is equal to $N(T_d)/T$. We have a Levi decomposition $B = TN$. Let $\delta$ be the modulus character for $B$.

We now study $\mathcal{H} := \mathcal{H}(G(K), G(O_K))$, the spherical Hecke algebra for $G$. The characteristic functions $T_\lambda = 1_{G(O_K)}(\lambda(\pi) G(O_K))$ for $\lambda \in (\hat{P}^+)^G$ form a $\mathbb{Z}$-basis of $\mathcal{H}$ by the Cartan decomposition.

We define the Satake transform:

\[
S : \mathcal{H} \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \to \mathcal{H}(T(K), T(O_K)) \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]
\]

\[
f \mapsto \delta(t)^{\frac{1}{2}} \int_{N(K)} f(tn) dn,
\]

where the measure on $N(K)$ is normalised so that $N(K) \cap G(O_K)$ has measure 1. It induces an isomorphism [9, Theorem 4.1]

\[
S : \mathcal{H} \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \simeq \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}][X_\lambda(T_d)]^{W_d}.
\]

For any $\lambda \in (\hat{P}^+)^G$ let $V_\lambda$ be the irreducible representation of $\hat{G}$ with highest weight $\lambda$. Because $\Gamma$ preserves a pinning, the action of $\hat{G}$ on $V_\lambda$ extends to an action of $L^G$. This extension is determined uniquely by the condition that $\Gamma$ preserves a highest weight vector and any other extension differs from it by tensoring with a character of $\Gamma$. We continue to denote this extended representation by $V_\lambda$. If we consider (as we may) $\hat{G}$ and $L^G$ as being defined over $\mathbb{Q}$, then this extension of $V_\lambda$ is also defined over $\mathbb{Q}$.

Let $[V_\lambda]$ be the the trace of the representation $V_\lambda$ on $\hat{T} \rtimes \sigma$, where $\sigma$ is the generator of $\Gamma$ given by Frobenius. For $\lambda \in (\hat{P}^+)^G$, let $[\lambda]$ denote the formal sum of the elements in the $W_d$-orbit of $\lambda$, viewed as an element of $\mathbb{Z}[X_\lambda(T_d)]^{W_d}$.

**Lemma 3.1.** The trace $[V_\lambda]$ belongs to $\mathbb{Z}[X_\lambda(T_d)]^{W_d}$. Moreover,

$[V_\lambda] = [\lambda] + \sum_{\mu \subset \lambda} a_\mu [\mu]$ with $\mu \in (\hat{P}^+)^G$ and $a_\mu \in \mathbb{Z}$.\[1\]

**Proof.** We have a decomposition $V_\lambda = \bigoplus_{\mu \subset \lambda} X_\lambda(T) V_\lambda^\mu$, where $V_\lambda^\mu$ is the weight $\mu$ eigenspace of $V_\lambda$ for the action of $\hat{T}$. The action of $\sigma$ permutes the spaces $V_\lambda^\mu$, so if $\mu$ is not fixed by $\sigma$ then the trace of any element $t \rtimes \sigma$ restricted to $\sum_{\mu' \in O_T(\mu)} V_\lambda^{\mu'}$ is zero. Here $O_T(\mu)$ denotes the $\Gamma$-orbit of $\mu$. It follows that $[V_\lambda] \in \mathbb{Q}[X_\lambda(T_d)]$. Because the weight spaces are permuted by $W$ and $\sigma$ acts trivially on $W_d$, $[V_\lambda] \in \mathbb{Q}[X_\lambda(T_d)]^{W_d}$.

\[1\] It can be shown that $a_\mu \geq 0$, but we do not need this.
As noted earlier, we may consider $\hat{G}$, $LG$ and $V_\lambda$ as being defined over $\mathbb{Q}$. For each $\mu \in X_*(T_d)$, the eigenspace $V_\lambda^\mu$ is defined over $\mathbb{Q}$ and $\sigma$ acts on this space as a finite-order operator, so its trace must be in $\mathbb{Z}$.

Finally, because the highest weight space is one-dimensional, we have $[V_\lambda] = [\lambda] + \sum_{\mu < \lambda} a_\mu[\mu] \mu \in (\hat{\mathbb{P}}^+)^\Gamma$ with $a_\mu \in \mathbb{Z}$.

**Corollary 3.2.** The elements $[V_\lambda]_{\lambda \in (\hat{\mathbb{P}}^+)^\Gamma}$ provide a $\mathbb{Z}$-basis of $\mathbb{Z}[X_*(T_d)]^{W_d}$.

**Proof.** This follows immediately from Lemma 3.1 and the fact that the set $\{[\lambda]\}_{\lambda \in (\hat{\mathbb{P}}^+)^\Gamma}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[X_*(T_d)]^{W_d}$.

We may now relate the two bases $\{[V_\lambda]\}_{\lambda \in (\hat{\mathbb{P}}^+)^\Gamma}$ and $\{T_\lambda\}_{\lambda \in (\hat{\mathbb{P}}^+)^\Gamma}$ of the unramified Hecke algebra.

**Proposition 3.3.** We have the following properties:

1. $S(T_\lambda) = q^{(\lambda,\rho)}[V_\lambda] + \sum_{\mu < \lambda} b_\lambda(\mu)(q^{(\mu,\rho)}[V_\mu])$ for integers $b_\lambda(\mu)$,
2. $q^{(\lambda,\rho)}[V_\lambda] = S(T_\lambda) + \sum_{\mu < \lambda} d_\lambda(\mu)S(T_\mu)$ for integers $d_\lambda(\mu)$.

**Proof.** The formulae (1) and (2) are clearly equivalent. If $G$ is split, these formulae are (3.9) and (3.12) of [25], though a complete proof is not given there, so we provide the missing references.

By (6.8) of [58], for $\lambda$, $\mu$ dominant coweights in $X_*(T_d)$, $S(T_\lambda)(\mu(\pi)) = \delta(\mu(\pi))a_\lambda(\mu)$, where $a_\lambda(\mu)$ is an integer. By [58, Remark 2, p. 30], $a_\lambda(\lambda) = 1$ iff

$$\left( G(\mathcal{O}_K)\lambda(\pi)^N G(\mathcal{O}_K) \right) \cap G(\mathcal{O}_K)\lambda(\pi) G(\mathcal{O}_K) = G(\mathcal{O}_K)\lambda(\pi).$$

(3.2.A)

This holds by part (ii) of the proposition in (4.4.4) of [6]. Part (i) of the same proposition implies that $a_\lambda(\mu) = 0$ unless $\mu \leq \lambda$. (We note that these results in the unramified quasi-split case follow from the case of split groups by base changing to an unramified extension over which the group splits, applying the results there and then taking Galois invariants. Also, all cases that we actually use later are already proved in full in [58].)

Because $\delta(\mu(\pi)) = q^{(\mu,\rho)}$ (as $B$ is a Borel subgroup), (1) now follows from Lemma 3.1.

**3.3. Conjugacy classes**

We denote by $\hat{T}_d$ the torus with cocharacter group $X^*(T_d)$. Note that there is a map $\hat{T} \to \hat{T}_d$. It follows from the Satake isomorphism that there is a bijection

$$\text{Hom}(\mathcal{H}, \mathcal{Q}) = \hat{T}_d(\mathcal{Q})/W_d.$$

We have a surjective map $N_{G}(\hat{T}) \to W$. Let us denote by $\widetilde{W}_d$ the inverse image of $W_d$. By Lemmas 6.4 and 6.5 of [4], we have natural bijections

$$\hat{T}(\mathcal{Q}) \times \sigma/\text{Int}(\widetilde{W}_d) \to G(\mathcal{Q})^{ss}/\sigma - \text{conj}$$

(3.3.A)

and

$$\hat{T}(\mathcal{Q}) \times \sigma/\text{Int}(\widetilde{W}_d) \to \hat{T}_d(\mathcal{Q})/W_d,$$

(3.3.B)
so we deduce that there is a bijection

$$\text{Hom}(\mathcal{H}, \overline{\mathbb{Q}}) \sim \hat{G}(\overline{\mathbb{Q}})^{ss}/\sigma - \text{conj}$$

that associates to any $\chi : \mathcal{H} \to \overline{\mathbb{Q}}$ a semi-simple $\sigma$-conjugacy class $c \in \hat{G}(\overline{\mathbb{Q}})^{ss}$ characterised by the property that

$$\chi([V_\lambda]) = \text{Tr}(c \rtimes \sigma|V_\lambda)$$

for all $\lambda \in (\hat{P}^+)^\Gamma$.

### 3.4. The Newton map

Let us now fix a valuation $v$ on $\overline{\mathbb{Q}}$ extending the $p$-adic valuation of $\mathbb{Q}$ for some prime number $p$ and normalised by $v(p) = 1$. The valuation $v$ defines a homomorphism $\mathbb{G}_m(\overline{\mathbb{Q}}) \to \mathbb{R}$ and induces the Newton polygon map

$$\text{Newt}_v : \hat{G}(\overline{\mathbb{Q}})^{ss}/\sigma - \text{conj} = \hat{T}_d(\overline{\mathbb{Q}})/W_d \overset{v}{\to} X^*(T_d)_R/W_d \overset{\phi}{\to} (P^+_R)^\Gamma,$$

where the last map $\phi$ is defined as follows: We have a canonical identification of $(X^*(T)_R)^\Gamma$ with $(X^*(T_d)_R)^\Gamma = X^*(T_d)_R$ induced by the inclusion of $(X^*(T)_R)^\Gamma$ in $X^*(T)_R$. Given $y \in X^*(T_d)_R/W_d$, let $x \in (X^*(T)_R)^\Gamma$ be a lift of $y$, let $w \in W$ be such that $w(x) \in P^+_R$ and then define $\phi(y)$ to be the $\Gamma$-average of $w(x)$, which is an element of $(P^+_R)^\Gamma$. Namely,

$$\phi(y) = \frac{1}{|\Gamma/\text{Stab}_\Gamma w(x)|} \sum_{\gamma \in \Gamma/\text{Stab}_\Gamma w(x)} \gamma \cdot w(x).$$

This is independent of the choice of the lift $x$ of $y$, because any other lift is of the form $w'(x)$ with $w' \in W_d$.

**Lemma 3.4.** Let $c \in \hat{G}(\overline{\mathbb{Q}})^{ss}/\text{conj}$ and $\lambda \in (\hat{P}^+)^\Gamma$. Then the minimal valuation of an eigenvalue of $c \rtimes \sigma$ acting on $V_\lambda$ is equal to $\langle w_0(\lambda), \text{Newt}_v(c) \rangle$.

**Proof.** Because the set of eigenvalues is invariant under conjugation, we may assume by (3.3.A) that $c \in \hat{T}(\overline{\mathbb{Q}})$. Let $\hat{T}^\Gamma$ be the largest subtorus of $\hat{T}$ on which $\Gamma$ acts trivially. The map $\hat{T}^\Gamma \to \hat{T}_d$ induced by the inclusion of $\hat{T}^\Gamma$ in $\hat{T}$ is an isogeny, so the map $\hat{T}^\Gamma(\overline{\mathbb{Q}}) \to \hat{T}_d(\overline{\mathbb{Q}})$ is surjective. Thus, by (3.3.B) we may assume that $c \in \hat{T}^\Gamma(\overline{\mathbb{Q}})$.

Let $\mu$ be a weight of $\hat{T}$ occurring in $V_\lambda$. Then $\sigma$, and hence also $c \rtimes \sigma$, preserves the subspace $\sum_{\mu' \in O_T(\mu)} V_\lambda^{\mu'}$ of $V_\lambda$. Because $c$ is invariant under $\Gamma$, $c$ acts on this space by the scalar $\mu(c)$. On the other hand, $\sigma$ acts on this space by a finite-order automorphism. It follows that all of the eigenvalues of $c \rtimes \sigma$ on this space have valuation equal to $v(\mu(c))$.

The choice of the element $c$ gives an obvious lift $x \in (X^*(T)_R)^\Gamma$ of the image of $c$ in $X^*(T_d)_R/W_d$. Let $w \in W$ be such that $w(x) \in P^+_R$. Then $v(\nu(w(c))) \geq 0$ for all positive coroots $\nu \in \hat{\Phi}^+$. The eigenvalue of smallest valuation of $w(c)$ on $V_\lambda$ is then clearly the one corresponding to the lowest weight $w_0(\lambda)$, so it has valuation $\langle w_0(\lambda), w(x) \rangle$. Because $\lambda$, hence $w_0(\lambda)$, and also the pairing $\langle \cdot, \cdot \rangle$ are invariant under $\Gamma$, we have

$$\langle w_0(\lambda), w(x) \rangle = \langle w_0(\lambda), \text{Newt}_v(c) \rangle.$$
On the other hand, the set of eigenvalues of \( w(c) \) and \( c \) on \( V_\lambda \) is the same. The lemma follows because we have seen that the set of valuations of the eigenvalues of \( c \) and \( c \times \sigma \) on \( V_\lambda \) is the same.

Recall that for two elements \( \nu_1, \nu_2 \in P^+_\mathbb{R} \) we write \( \nu_1 \leq \nu_2 \) if \( \nu_2 - \nu_1 \) is a linear combination with coefficients in \( \mathbb{R}_{\geq 0} \) of elements of \( \Phi^+ \). We use the same notation for the restriction of this ordering to \((P^+_\mathbb{R})^\Gamma\).

**Remark 3.5.** Assume that \( G \) is \( \text{GL}_n \), \( T \) is the diagonal torus and \( B \) is the upper triangular subgroup. We identify \( X^*(T) \) with \( \mathbb{Z}^n \) and \( P^+_\mathbb{R} = \{ (\lambda_n, \cdots, \lambda_1) \in \mathbb{R}^n, \lambda_n \geq \cdots \geq \lambda_1 \} \). To any \( \lambda \in P^+_\mathbb{R} \) we can associate a convex polygon in \( \mathbb{R}^2 \) whose vertices are \( (i, \lambda_1 + \cdots + \lambda_i) \) for \( 0 \leq i \leq n \). If \( \mu, \nu \in P^+_\mathbb{R} \), then \( \mu \leq \nu \) means that the polygon of \( \mu \) is above the polygon of \( \nu \) with the same ending point.

**Lemma 3.6.** Let \( \nu \in (P^+_\mathbb{R})^\Gamma \) and \( c \in \hat{G}(\mathbb{Q})^{ss}/\text{conj} \). Then

\[
\text{Newt}_\nu(c) \leq \nu
\]

if and only if for all \( \lambda \in (\hat{P}^+)\Gamma \),

\[
v(\text{Tr}(c \times \sigma|V_\lambda)) \geq \langle w_0(\lambda), \nu \rangle.
\]

Note that the second inequality in the lemma does not depend on the choice of extension of \( V_\lambda \) that we have made above, because any other choice differs from it by tensoring with a character of \( \Gamma \).

**Proof.** When \( G \) is split, this is Lemme 1.3 of [44]; we show that Lafforgue’s proof extends to our setting without much difficulty. By (3.3.A) we may and do assume that \( c \in \hat{T}(\mathbb{Q}) \).

We first assume that for all \( \lambda \in (\hat{P}^+)\Gamma \), \( v(\text{Tr}(c \times \sigma|V_\lambda)) \geq \langle w_0(\lambda), \nu \rangle \). For \( \lambda \in (\hat{P}^+)\Gamma \), let \( r_\lambda = \dim(V_\lambda) \) and consider the representations \( \wedge^i V_\lambda \) for \( 1 \leq i \leq r_\lambda \). The action of \( \sigma \) on \( V_\lambda \) maps \( V_\lambda^\mu \) to \( V_\lambda^{\sigma(\mu)} \) and so the analogous statement holds for the weight spaces of \( \wedge^i V_\lambda \). As a representation of \( \hat{G} \), this breaks up as a sum of representations \( V_\mu \) with \( \mu \in P^+ \) and \( \mu \leq \lambda \). If the line generated by the highest weight vector of such a representation is not preserved by \( \sigma \), which always holds if \( \mu \notin (\hat{P}^+)\Gamma \), then this summand of \( \wedge^i V_\lambda \) is not preserved by \( \sigma \) and the trace of \( c \) on the sum of all such summands (which is preserved by \( c \)) is 0. If \( \mu \in (\hat{P}^+)\Gamma \) and the line spanned by the highest weight vector is preserved by \( \sigma \), then this \( \hat{G} \)-irreducible summand isomorphic to \( V_\mu \) is preserved by \( L^G \). It is not clear whether this representation is always isomorphic to our chosen extension but, as noted above, the condition in the lemma is independent of the choice.

Because the lowest weight occurring in all of the representations above is \( \geq w_0(i\lambda) \) and only the invariant \( \mu \) contribute to the trace of \( c \), we deduce from the assumption at the beginning that

\[
v(\text{Tr}(c \times \sigma|\wedge^i V_\lambda)) \geq \langle w_0(\lambda), \nu \rangle
\]

for all \( i \geq 0 \).

Let \( \alpha_1, \alpha_2, \ldots, \alpha_{r_\lambda} \) be the eigenvalues of \( c \) acting on \( V_\lambda \) ordered so that \( v(\alpha_1) \leq \cdots \leq v(\alpha_{r_\lambda}) \). Let \( i \in \{1, \ldots, r_\lambda\} \) be such that \( v(\alpha_1) = \cdots = v(\alpha_i) < v(\alpha_{i+1}) \). Then \( v(\text{Tr}(c \times \sigma|\wedge^i V_\lambda)) \geq \langle w_0(\lambda), \nu \rangle \)
$V_\lambda) = iv(\alpha_1)$ so we deduce from (3.4.A) that $v(\alpha_1) \geq \langle w_0(\lambda),\nu \rangle$. On the other hand, by Lemma 3.4, $v(\alpha_1) = \langle w_0(\lambda),\text{Newt}_v(c) \rangle$ because $w_0(\lambda)$ is the lowest weight of $V_\lambda$. We thus have

$$\langle w_0(\lambda),\text{Newt}_v(c) \rangle \geq \langle w_0(\lambda),\nu \rangle$$ (3.4.B)

for all $\lambda \in (P^+)^\Gamma$. Because $\lambda \mapsto -w_0(\lambda)$ is a bijection of $(\hat{P}^+)^\Gamma$ into itself and the cone generated by the elements of $(\hat{P}^+)^\Gamma$ is dual to the $\Gamma$-invariants of the cone generated by the positive coroots of $G$, we deduce that $\text{Newt}_v(c) \leq \nu$.

The converse, not explicitly stated in [44], is simpler: If $\text{Newt}_v(c) \leq \nu$, then $\nu - \text{Newt}_v(c)$ is a $\Gamma$-invariant element of the cone generated by the positive coroots of $\hat{G}$. The result then follows from the formula $v(\text{Tr}(c \times \sigma|V_\lambda)) = v(\alpha_1) = \langle w_0(\lambda),\text{Newt}_v(c) \rangle$ used above and the fact that $\langle w_0(\lambda),\alpha \rangle \leq 0$ for any $\alpha \in \Phi^+$.

4. Shimura varieties

4.1. Shimura varieties in characteristic zero

Let $(G,X)$ be a Shimura datum [17, §1]. This means that $G$ is a reductive group over $\mathbb{Q}$, and $X$ is a $G(\mathbb{R})$-conjugacy class of morphisms

$$h : \text{Res}_{\mathbb{C}/\mathbb{R}}G_m \rightarrow G_\mathbb{R}$$

satisfying the following conditions:

(1) Let $g$ be the Lie algebra of $G$. Then for any $h \in X$, the adjoint action determines on $g$ a Hodge structure of type $(-1,1)$, $(0,0)$, $(1,-1)$.

(2) For all $h \in X$, $Adh(i)$ is a Cartan involution on $G^{ad}(\mathbb{R})$.

(3) $G^{ad}$ has no factor $H$ defined over $\mathbb{Q}$ such that $H(\mathbb{R})$ is compact.

Example 4.1. The most fundamental example is the Siegel Shimura datum $(\text{GSp}_{2g}, \mathcal{H}_g^\pm)$. Let $g \in \mathbb{Z}_{\geq 1}$, let $V = \mathbb{Q}^{2g}$ and let $\psi$ be the symplectic form with $\psi(e_i,e_{2g+1-i}) = 1$ if $1 \leq i \leq g$ and $\psi(e_i,e_j) = 0$ if $i + j \neq 2g + 1$. Let $\text{GSp}_{2g}$ be the group of symplectic similitudes of $(V,\psi)$. We take $\mathcal{H}_g^\pm = \{ M \in M_{g \times g}(\mathbb{C}), M = ^t\bar{M}, \text{Im}(M) \text{ is definite} \}$ to be the Siegel space. Observe that $\mathcal{H}_g^\pm = \mathcal{H}_g^+ \cup \mathcal{H}_g^-$, where $\mathcal{H}_g^+$ (respectively $\mathcal{H}_g^-$) is defined by the condition that $\text{Im}(M)$ is positive (respectively negative) definite. Then $\mathcal{H}_g^\pm$ is the $\text{GSp}_{2g}(\mathbb{R})$-orbit of the morphism

$$h_0 : \text{Res}_{\mathbb{C}/\mathbb{R}}G_m \rightarrow G_\mathbb{R}, \quad a + ib \in \mathbb{C}^\times \mapsto \begin{pmatrix} a1_g & -bS_g \\ bS_g & a1_g \end{pmatrix},$$

where $1_g$ is the identity $g \times g$ matrix and $S_g$ is the $g \times g$ matrix with 1’s on the anti-diagonal and 0’s otherwise.

Because $\text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$, the choice of $h \in X$ determines, by projection to the first factor, a cocharacter $\mu : \mathbb{C}^\times \rightarrow G_\mathbb{C}$. We denote by $P_\mu \subset G_\mathbb{C}$ the parabolic defined by $P_\mu = \{ g \in G_\mathbb{C}, \lim_{t \rightarrow +\infty} \text{Ad}(\mu(t))g \text{ exists} \}$ and by $M_\mu = \text{Cent}_{G_\mathbb{C}}(\mu)$ its Levi factor.

Let $\text{FL}_{G,X}$ be the flag variety parametrising parabolic subgroups in $G$ of type $P_\mu$. The Borel embedding is the map $X \hookrightarrow \text{FL}_{G,X}$ given by $h \mapsto P_\mu$. Using this map, we endow $X$ with a complex structure [16, Proposition 1.1.14].
Let \( K \subset G(\mathbb{A}_f) \) be a compact open subgroup. We let \( Sh_K(\mathbb{C}) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K) \) be the associated Shimura variety. This is a complex manifold as soon as \( K \) is neat and in fact it has a structure of algebraic variety over \( \mathbb{C} \).

The conjugacy classes of \( \mu, P_\mu \) and \( M_\mu \) are defined over a number field \( E = E(G,X) \) called the reflex field. We deduce that \( FL_{G,X} \) is defined over \( E \). Moreover, it has been proven by Shimura, Deligne, Borovoi and Milne (see, e.g., [49]) that \( Sh_K(\mathbb{C}) \) has a canonical model \( Sh_K \to \text{Spec } E \).

**Example 4.2.** In the Siegel case \((GSp_{2g}, \mathcal{H}_g^+)\), \( Sh_K \) has a moduli interpretation: It is the moduli space of abelian varieties of dimension \( g \), with a polarisation and a level structure (prescribed by \( K \)).

### 4.2. Compactifications

For any choice \( \Sigma \) of rational polyhedral cone decomposition, one can construct a toroidal compactification \( Sh_K^{tor},\Sigma \) of \( Sh_K \). In general, this is a proper algebraic space over \( E \) and we have the property that \( D_{K,\Sigma} = Sh_K^{tor,\Sigma} \backslash Sh_K \) is a Cartier divisor (see [2] and [57]). Furthermore, for a suitable choice of \( \Sigma \), \( Sh_K^{tor,\Sigma} \) is smooth and projective.

### 4.3. Automorphic vector bundles

Let \( Z_s(G) \) be the greatest subtorus of the center of \( G \) that has no split subtorus over \( \mathbb{Q} \) but that splits over \( \mathbb{R} \). Let \( G^c = G/Z_s(G) \).

**Remark 4.3.** If \( F \) is a totally real field and \( G = \text{Res}_{F/\mathbb{Q}} \text{GL}_n \), with center \( Z = \text{Res}_{F/\mathbb{Q}} \text{GL}_1 \), then \( Z_s(G) \) is the kernel of the norm map \( Z \to \text{GL}_1 \).

One can define ([50, III, § 3]) an analytic space \( P_K(\mathbb{C}) \to S_K(\mathbb{C}) \), called the principal \( G^c \)-bundle, by

\[
P_K(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G^c(\mathbb{C}) \times G(\mathbb{A}_f)/K.
\]

There is a natural, \( G(\mathbb{C}) \)-equivariant map \( P_K(\mathbb{C}) \to FL_{G,X} \) defined by sending \((x,g,g') \in G(\mathbb{Q}) \backslash X \times G^c(\mathbb{C}) \times G(\mathbb{A}_f)/K \) to \( g^{-1}x \). We therefore have a diagram of analytic spaces:

\[
\begin{array}{ccc}
P_K(\mathbb{C}) & \xrightarrow{\beta} & FL_{G,X} \\
\downarrow & & \downarrow \\
Sh_K(\mathbb{C}) & \xrightarrow{\alpha} & FL_{G,X}
\end{array}
\]

By [51, III, Theorem 4.3 a)], \( P_K(\mathbb{C}) \) is the analytification of an algebraic variety \( P_K \) defined over \( E \), and there is a diagram of schemes

\[
\begin{array}{ccc}
P_K & \xrightarrow{\beta} & FL_{G,X} \\
\downarrow & & \downarrow \\
Sh_K & \xrightarrow{\alpha} & FL_{G,X}
\end{array}
\]

where \( \alpha \) is \( G \)-equivariant and \( \beta \) is a \( G^c \)-torsor.
Example 4.4. In the Siegel case, let $A \to Sh_K$ be the universal abelian scheme (only well defined up to quasi-isogenies, a representative of the isogeny class can be fixed by the choice of an integral PEL-datum). Then $P_K$ is the $\text{GSp}_{2g}$-torsor of trivialisations of $\mathcal{H}_{1,dR}(A/\text{Sh}_K)$ and the map $\alpha$ is given by the Hodge filtration on $\mathcal{H}_{1,dR}(A/\text{Sh}_K)$.

Let $E'$ be a finite extension of $E$ such that the conjugacy class of $\mu$, $M_\mu$ and $P_\mu$ has representatives defined over $E'$ and all finite dimensional algebraic representations of $M_\mu$ are defined over $E'$. We now consider all of the schemes $P_K$, $\text{Sh}_K$, $\text{FL}_{G,X} = G/P_\mu$ over $E'$. Denote by $\mathcal{V}B_G(\text{FL}_{G,X})$ the category of $G$-equivariant vector bundles on $\text{FL}_{G,X}$ and by $\text{Rep}_{E'}(M_\mu)$ the category of finite-dimensional algebraic representations of $M_\mu$ on $E'$-vector spaces. There is a functor

$$\text{Rep}_{E'}(M_\mu) \to \mathcal{V}B_G(\text{FL}_{G,X})$$

$$V \mapsto V$$

that is defined by $V = G \times V/\sim$ where $\sim$ is the equivalence relation $(gx,v) \sim (g,xv)$ for all $(g,v,x) \in G \times V \times P_\mu$, and we let $P_\mu$ act on $V$ through its projection $P_\mu \to M_\mu$.

Let $\mathcal{V}B(\text{Sh}_K)$ be the category of vector bundles over $\text{Sh}_K$ and $\text{Rep}_{E'}(M_\mu/Z_s(G))$ the category of finite-dimensional algebraic representations of $M_\mu/Z_s(G)$ on $E'$-vector spaces. We deduce that there is a functor

$$\text{Rep}_{E'}(M_\mu/Z_s(G)) \to \mathcal{V}B(\text{Sh}_K)$$

$$V \mapsto V_K$$

that is defined as follows: A representation $V$ of $M_\mu/Z_s(G)$ defines a representation of $P_\mu$ (by letting the unipotent radical act trivially) and therefore a $G$-equivariant vector bundle $V$ over $\text{FL}_{G,X}$. We can pull back this vector bundle by the map $\alpha$ to $P_K$ and descend it to $\text{Sh}_K$ using $\beta$.

4.4. Automorphic vector bundles and compactifications

By [28, Theorem 4.2], for a choice $\Sigma$ of rational polyhedral cone decomposition, there is a $G^c$-torsor $P_{K,\Sigma} \to Sh_{K,\Sigma}^{\text{tor}}$ extending $P_K$ and a diagram

$$\begin{array}{ccc}
P_{K,\Sigma} & \xrightarrow{\alpha} & \text{FL}_{G,X} \\
\downarrow{\beta} & & \\
Sh_{K,\Sigma}^{\text{tor}} & & \\
\end{array}$$

where $\alpha$ is $G$-equivariant and $\beta$ is a $G^c$-torsor. Therefore, the functor $V \mapsto V_K$ extends to a functor

$$\text{Rep}_{E'}(M_\mu/Z_s(G)) \to \mathcal{V}B(Sh_{K,\Sigma}^{\text{tor}})$$

$$V \mapsto V_{K,\Sigma}$$

and $V_{K,\Sigma}$ is called the canonical extension of $V_K$. Moreover, $V_{K,\Sigma}(-D_{K,\Sigma})$ is called the subcanonical extension of $V_K$. 
4.5. Cohomology of vector bundles and Hecke operators

For any $\Sigma$, $K$ and $V \in \text{Rep}(M_\mu/Z_s(G))$, we identify the vector bundle $\mathcal{V}_{K,\Sigma}$ with its associated locally free sheaf of sections and we consider the cohomology groups

$$H^*(\mathcal{V}_{K,\Sigma}) \quad \text{and} \quad H^*(\mathcal{V}_{K,\Sigma}(-D_{K,\Sigma})).$$

By \cite[Proposition 2.4]{[31]} these groups are independent of the choice of $\Sigma$, and we therefore simplify notations and let $H^i(K, V) = H^i(\mathcal{V}_{K,\Sigma})$ and $H_{\text{cusp}}^i(K, V) = H^i(\mathcal{V}_{K,\Sigma}(-D_{K,\Sigma}))$.

Let $\mathcal{H}_K$ be the Hecke algebra of functions $G(\mathbb{A}_f) \to \mathbb{Z}$, which are compactly supported and bi-$K$-invariant. By \cite[Proposition 2.6]{[31]}, the Hecke algebra $\mathcal{H}_K$ acts on the cohomology groups $H^i(K, V)$ and $H_{\text{cusp}}^i(K, V)$.

Observe that the Hecke algebra is generated by the characteristic functions $g \in G(\mathbb{A}_f)$. We spell out the action of $T_g$ by writing the corresponding cohomological correspondence. For any $g \in G(\mathbb{A}_f)$, we have a correspondence (for suitable choices of polyhedral cone decomposition $\Sigma$, $\Sigma'$ and $\Sigma''$)

$$\xymatrix{ \text{Sh}_{gKg^{-1}}^\text{tor} \ar[r]^{p_2} & \text{Sh}_{K,\Sigma'}^\text{tor} \ar[r]^{p_1} & \text{Sh}_{K,\Sigma}^\text{tor} }$$

where $p_1$ is simply the forgetful map (induced by the inclusion $gKg^{-1} \subset K$), and $p_2$ is the composite of the action map $g : \text{Sh}_{gKg^{-1}}^\text{tor} \to \text{Sh}^\text{tor}_{Kg^{-1}Kg,\Sigma''}$ and the forgetful map $\text{Sh}^\text{tor}_{Kg^{-1}Kg,\Sigma''} \to \text{Sh}_{K,\Sigma'}^\text{tor}$ (induced by the inclusion $g^{-1}Kg \cap K \subset K$). There is a corresponding cohomological correspondence $T_g : p_2^*\mathcal{V}_{K,\Sigma'} \to p_1^*\mathcal{V}_{K,\Sigma}$ that is simply obtained by composing the natural isomorphism $p_2^*\mathcal{V}_{K,\Sigma'} \to p_1^*\mathcal{V}_{K,\Sigma}$ (see \cite[\S2.5]{[31]}) and the map $p_2^*\mathcal{V}_{K,\Sigma} \to p_1^*\mathcal{V}_{K,\Sigma}$ that is deduced from the fundamental class $p_2^*\mathcal{O}_{\text{Sh}_{gKg^{-1}}^\text{tor}} \to p_1^*\mathcal{O}_{\text{Sh}_{K,\Sigma}^\text{tor}}$ (see Proposition 2.6).

4.6. The infinitesimal character

Let $(G,X)$ again be a Shimura datum. The reductive group $G$ is defined over $\mathbb{Q}$. We have chosen an extension $E'$ of the reflex field $E$ over which all representations of $M_\mu$ are defined and we now assume that $E'$ is large enough so that $G_{E'}$ is split. Let $S$ be a (split) maximal torus in $G_{E'}$ and let $X^*(S)$ be its character group. We assume that $S \subset (M_\mu)_{E'}$. The roots for $G$ that lie in the Lie algebra of $M_\mu$ are by definition the compact roots. The other roots are called noncompact. We make a choice of positive roots for $M_\mu$. We also make a choice of positive roots for $G$ by declaring that the noncompact positive roots are those corresponding to $g/\mathfrak{p}_\mu$. We denote by $\rho$ the half sum of all of the positive roots. Let $\kappa$ be a highest weight for $M_\mu$.

**Definition 4.5.** We define the infinitesimal character of $\kappa$, denoted $\infty(\kappa)$, by the formula

$$\infty(\kappa) = -\kappa - \rho \in X^*(S)_{\mathbb{Q}}.$$
Remark 4.6. This is a representative of the infinitesimal character of automorphic representations contributing to the cohomology $H^i(K, V_\kappa)$ or $H^i_{\text{cusp}}(K, V_\kappa)$ [31, Proposition 4.3.2].

We now choose a prime $p$ such that the group $G_{\mathbb{Q}_p}$ is unramified at $p$. As in Subsection 3.1, we denote by $T \subset G_{\mathbb{Q}_p}$ a maximal torus, split over an unramified extension of $\mathbb{Q}_p$ and contained in a Borel subgroup $B \subset G_{\mathbb{Q}_p}$. We let $X^*(T)$ be the character group of $T$ and let $P^+ \subset X^*(T)$ be the cone of dominant weights. We fix an embedding $\iota : E' \hookrightarrow \mathbb{Q}_p$.

The tori $S \times_{\text{Spec} E'} \text{Spec} \mathbb{Q}_p$ and $T \times_{\text{Spec} \mathbb{Q}_p} \text{Spec} \mathbb{Q}_p$ are conjugated by some element $g$ of $G(\mathbb{Q}_p)$. Conjugation by $g$ defines an isomorphism $X^*(S) \rightarrow X^*(T)$; this isomorphism depends on $g$, but the composite map $X^*(S) \rightarrow X^*(T) \rightarrow X^*(T)/W \simeq P^+$ is independent of the choice of $g$.

We therefore get a canonical element $\infty(\kappa, \iota) \in P^+_\mathbb{R}$ (which only depends on $\iota$), which is by definition the image of $\infty(\kappa)$ via the map $X^*(S)_{\mathbb{R}} \rightarrow P^+_\mathbb{R}$.

4.7. Newton and Hodge polygons

We assume that $K = K_p K^p$ is neat and $K_p$ is hyperspecial. Let $V_\kappa$ be the irreducible representation of $M_\mu$ defined over $E'$ with highest weight $\kappa$.

Let $\mathcal{H}_p = \mathcal{H}(G(\mathbb{Q}_p), K_p)$ be the Hecke algebra at $p$. It acts on the groups $H^i(K, V_\kappa)$ and $H^i_{\text{cusp}}(K, V_\kappa)$. We put $H^i(K, V_\kappa)_{\mathbb{Q}_p} = H^i(K, V_\kappa) \otimes_{E', \iota} \mathbb{Q}_p$ and $H^i_{\text{cusp}}(K, V_\kappa)_{\mathbb{Q}_p} = H^i_{\text{cusp}}(K, V_\kappa) \otimes_{E', \iota} \mathbb{Q}_p$.

Using the results of Subsections 3.3 and 3.4, we have a Newton map:

$$\text{Newt}_i : \text{Hom}(\mathcal{H}_p, \mathbb{Q}_p) \rightarrow \mathcal{G}(\mathbb{Q}_p)/\sigma - \text{conj} \rightarrow (P^+_\mathbb{R})^\Gamma.$$ 

Conjecture 4.7. Let $\chi : \mathcal{H}_p \rightarrow \mathbb{Q}_p$ be a character occurring in $H^i(K, V_\kappa)_{\mathbb{Q}_p}$ or $H^i_{\text{cusp}}(K, V_\kappa)_{\mathbb{Q}_p}$. Then we have the inequality in $(P^+_\mathbb{R})^\Gamma$:

$$\text{Newt}_i(\chi) \leq \frac{1}{|\Gamma/\text{Stab}_\Gamma(\infty(\kappa, \iota))|} \sum_{\gamma \in \Gamma/\text{Stab}_\Gamma(\infty(\kappa, \iota))} -w_0(\gamma \cdot \infty(\kappa, \iota)),$$

where $w_0$ is the longest element of the Weyl group.

We now explain that this conjecture is compatible with existing conjectures on the existence and properties of Galois representations attached to automorphic representations. Our convention is that the Artin reciprocity law is normalised by sending uniformising element to geometric Frobenius element and that the Hodge-Tate weight of the cyclotomic character is $-1$.

This inequality is to be viewed as an inequality between a Newton and a Hodge polygon. According to the work of [37] (see also [30]), all cohomology $H^i(K, V_\kappa)$ and $H^i_{\text{cusp}}(K, V_\kappa)$ can be represented by automorphic forms. Let $\pi$ be an automorphic representation contributing to these cohomology groups. We fix an embedding $E \rightarrow \mathbb{C}$ and an isomorphism $\iota : \mathbb{C} \rightarrow \mathbb{Q}_p$ extending our embedding $\iota : E \hookrightarrow \mathbb{Q}_p$. The automorphic representation $\pi$ is $\mathbb{C}$-algebraic with infinitesimal character $\infty(\kappa) = -\kappa - \rho$.

Let us assume that $\pi$ is also $L$-algebraic (for simplicity). Then, according to [8, Conjecture 3.2.2], there should be a geometric Galois representation $\rho_{\pi, \iota} : G_{\mathbb{Q}} \rightarrow L G(\mathbb{Q}_p)$.
satisfying a list of conditions. In particular, it should be crystalline at \( p \) (because \( \pi \) has spherical vectors at \( p \)) with Hodge-Tate weights \(-\infty(\kappa, \iota)\). (Note that our sign convention concerning the Hodge-Tate weight of the cyclotomic character is opposite to [8].) Let \( \chi_{\pi_p} : \mathcal{H}_p \to \mathbb{C} \) be the character describing the action of \( \mathcal{H}_p \) on \( \pi_p \). The conjugacy class of the crystalline Frobenius should be given by \( \iota \circ \chi_{\pi_p} \) via the Satake isomorphism. Then the inequality in \( (P^+)_\Gamma \)

\[
\text{Newt}_\iota(\iota \circ \chi_{\pi_p}) \leq \frac{1}{|\Gamma / \text{Stab}_\Gamma(\infty(\kappa, \iota))|} \sum_{\gamma \in \Gamma / \text{Stab}_\Gamma(\infty(\kappa, \iota))} -w_0(\gamma \cdot \infty(\kappa, \iota))
\]

is the Katz-Mazur inequality (see also [44, Lemma 4.2] and Theorem 9.11).

Motivated by this conjecture, we can introduce a modified 'integral structure' on the Hecke algebra \( \mathcal{H}_p \).

**Definition 4.8.** Let \( \mathcal{H}_{p, \kappa, \iota}^{\text{int}} \) be the \( \mathbb{Z} \)-subalgebra of \( \mathcal{H}_p \otimes \mathbb{Z}[\frac{1}{2}, \frac{-1}{2}] \) generated by the elements

\[
[V_\lambda]p^{(\lambda, \infty(\kappa, \iota))}
\]

**Lemma 4.9.** Conjecture 4.7 holds if and only if, for any character \( \chi : \mathcal{H}_p \to \overline{\mathbb{Q}}_p \) occurring in \( H^i(K, V_\kappa)_{\overline{\mathbb{Q}}_p} \) or \( H^i_{\text{cusp}}(K, V_\kappa)_{\overline{\mathbb{Q}}_p} \), we have \( \chi(\mathcal{H}_{p, \kappa, \iota}^{\text{int}}) \subset \mathbb{Z}_p \).

**Proof.** This follows from Lemma 3.6. \( \square \)

**Lemma 4.10.** The algebra \( \mathcal{H}_{p, \kappa, \iota}^{\text{int}} \) is a \( \mathbb{Z} \)-algebra of finite type.

**Proof.** Let \( \{\lambda_1, \ldots, \lambda_n\} \) be dominant weights of \( \hat{G} \) that generate the cone \( (\hat{P}^+)_{\Gamma} \) as a monoid and is closed under \( \leq \), in the sense that if \( \lambda \in (\hat{P}^+)_{\Gamma} \) satisfies \( \lambda \leq \lambda_i \) for some \( 1 \leq i \leq n \), then \( \lambda \in \{\lambda_1, \ldots, \lambda_n\} \). We claim that the map \( \mathbb{Z}[T_1, \ldots, T_n] \to \mathcal{H}_{p, \kappa, \iota}^{\text{int}} \) where \( T_i \) goes to \( [V_{\lambda_i}]q^{(\lambda_i, \infty(\kappa, \iota))} \) is surjective. Let \( \lambda \in (\hat{P}^+)_{\Gamma} \), and let us prove that \( [V_{\lambda}]q^{(\lambda, \infty(\kappa, \iota))} \) is in the image. We argue by induction and assume that this holds for all \( \lambda' \in (\hat{P}^+)_{\Gamma} \) with \( \lambda' < \lambda \). We can write \( \lambda = \sum_{i=1}^n k_i \lambda_i \) with \( k_i \in \mathbb{Z}_{\geq 0} \) and we have that

\[
\prod_{i=1}^n T_i^{k_i} = [V_{\lambda}]q^{(\lambda, \infty(\kappa, \iota))} + \sum_{\mu \in \hat{P}^+, \mu < \lambda} c_\mu [V_\mu]q^{(\lambda, \infty(\kappa, \iota))} \quad \text{with} \quad c_\mu \in \mathbb{Z}_{\geq 0}
\]

\[
= [V_{\lambda}]q^{(\lambda, \infty(\kappa, \iota))} + \sum_{\mu \in \hat{P}^+, \mu < \lambda} c_\mu [V_\mu]q^{(\mu, \infty(\kappa, \iota))}q^{(\lambda - \mu, \infty(\kappa, \iota))},
\]

and we can conclude because \( (\lambda - \mu, \infty(\kappa, \iota)) \in \mathbb{Z}_{\geq 0} \) because \( \lambda - \mu \) is a finite sum of positive roots with nonzero integral coefficients for \( \hat{G} \). \( \square \)

**Proposition 4.11.** Conjecture 4.7 holds if and only if both \( H^i(K, V_\kappa)_{\overline{\mathbb{Q}}_p} \) and \( H^i_{\text{cusp}}(K, V_\kappa)_{\overline{\mathbb{Q}}_p} \) contain \( \mathbb{Z}_p \)-lattices that are stable under \( \mathcal{H}_{p, \kappa, \iota}^{\text{int}} \).

**Proof.** We deduce from Lemma 4.9 that we can find a basis for \( H^i(K, V_\kappa)_{\overline{\mathbb{Q}}_p} \), such that the elements \( \mathcal{H}_{p, \kappa}^{\text{int}} \) act via upper triangular matrices with integral diagonal coefficients. After conjugating this basis by a diagonal matrix \( \text{diag}(p^{k_1}, \ldots, p^{k_n}) \) with \( k_1 \gg k_2 \cdots \gg k_n \) and using that \( \mathcal{H}_{p, \kappa, \iota}^{\text{int}} \) is a finite-type algebra, we can suppose that it acts via integral matrices. \( \square \)
4.8. Shimura varieties of abelian type

The theory of integral models of Shimura varieties produces (in many cases) integral structures on the coherent cohomology of automorphic vector bundles. In view of Proposition 4.11, it is natural to ask (see Conjectures 4.15 and 4.16) whether these integral structures are stable under the action of our integral Hecke algebras.

A Shimura datum \((G,X)\) is of Hodge type if there is an embedding \((G,X) \hookrightarrow (\text{GSp}_{2g},\mathcal{H}^+_g)\). A Shimura datum \((G,X)\) is of abelian type if there is a Shimura datum of Hodge type \((G_1,X_1)\) and a central isogeny \(G^{\text{der}}_1 \rightarrow G^{\text{der}}\) that induces an isomorphism \((G_1^{\text{ad}},X_1^{\text{ad}}) \simeq (G^{\text{ad}},X^{\text{ad}})\), where \(X^{\text{ad}}\) is the \(G^{\text{ad}}(\mathbb{R})\)-conjugacy class that contains \(X_1\) (and similarly for \(X^{\text{ad}}\)).

Example 4.12. Here is an important example of abelian Shimura datum, which we call a Shimura datum of symplectic type. Let \(F\) be a totally real field. We let \(G = \text{Res}_{F/\mathbb{Q}}\text{GSp}_{2g}\) and \(X = (\mathcal{H}^+_g)_{[F: \mathbb{Q}]}\). This datum is related to the following Hodge-type datum (which is actually of PEL type): Take \(G_1 \subset G\) to be the subgroup of elements whose similitude factor is in \(\text{GL}_1\) and \(X_1 = (\mathcal{H}^+_g)_{[F: \mathbb{Q}]} \cup (\mathcal{H}^-_g)_{[F: \mathbb{Q}]}\).

We fix \((G,X)\) a Shimura datum of abelian type and \(K \subset G(\mathbb{A}_f)\) a neat, compact, open subgroup. Let \(p\) be a prime such that \(G\) is unramified at \(p\), \(K = K^pK_p\) and \(K_p\) is hyperspecial. The group \(G\) has a reductive model over \(\mathbb{Z}(p)\). Let \(E'\) be a finite extension of \(\mathbb{Q}\) unramified at \(p\), which splits \(G\) (necessarily \(E \hookrightarrow E'\)). Let \(\lambda\) be a prime of \(\mathcal{O}_E\) dividing \(p\) and \(\lambda'\) a prime of \(\mathcal{O}_{E'}\) dividing \(\lambda\). We denote by \(\mathcal{O}_{E,\lambda}\) and \(\mathcal{O}_{E',\lambda}\) the localisations of \(\mathcal{O}_E\) and \(\mathcal{O}_{E'}\) at \(\lambda\) and \(\lambda'\), respectively. The cocharacter \(\mu\) is defined over \(E'\), and the parabolic \(P_\mu\) as well as its Levi subgroup \(M_\mu\) have a model over \(\text{Spec} \mathcal{O}_{E',\lambda'}\). Moreover, the conjugacy class of \(P_\mu\) is defined over \(\text{Spec} \mathcal{O}_{E,\lambda}\). The flag variety \(\text{FL}_{G,X}\) therefore has a proper smooth canonical integral model over \(\text{Spec} \mathcal{O}_{E,\lambda}\). To ease notations, we keep denoting by \(G\) the reductive model of \(G\) over \(\mathbb{Z}(p)\), by \(\text{FL}_{G,X} \rightarrow \text{Spec} \mathcal{O}_{E,\lambda}\) the canonical integral model of \(\text{FL}_{G,X} \rightarrow \text{Spec} E\) and by \(P_\mu\) and \(M_\mu\) the models over \(\text{Spec} \mathcal{O}_{E',\lambda'}\) of \(P_\mu\) and \(M_\mu\).

Theorem 4.13 ([40], [39]). There is a canonical model \(\mathbb{G}_K \rightarrow \text{Spec} \mathcal{O}_{E,\lambda} \text{ of Sh}_K\).

Theorem 4.14. Assume that \((G,X)\) is a Shimura datum of Hodge type or that \((G,X)\) is a Shimura datum of symplectic type. Let \(\Sigma\) be a polyhedral cone decomposition.

1. There is a canonical integral model \(\mathbb{G}^{\text{tor}}_{K,\sqrt{\Sigma}} \rightarrow \text{Spec} \mathcal{O}_{E,\lambda}\) for \(\text{Sh}^{\text{tor}}_{K,\Sigma}\) that is smooth for suitable choices of \(\Sigma\).

2. There is a canonical integral model \(\mathcal{P}_{K,\Sigma}\) for the principal \(G^c\)-torsor \(P_{K,\Sigma}\) and there is a diagram

\[
\begin{array}{ccc}
\mathbb{G}^{\text{tor}}_{K,\Sigma} & \xrightarrow{\alpha} & \text{FL}_{G,X} \\
\beta & & \\
\text{Sh}^{\text{tor}}_{K,\Sigma} & \xrightarrow{\gamma} & \\
\end{array}
\]

where \(\alpha\) is \(G\)-equivariant and \(\beta\) is a \(G^c\)-torsor.
**Proof.** The Siegel case is [20], the PEL case is [46] and [45] and the Hodge case is [55]. To our knowledge, there is no reference for the abelian case in general. We will explain in Section 5 the proof for the case of a symplectic type Shimura datum by a simple reduction to our knowledge, there is no reference for the abelian case in general. We will explain in [5, §3] in the case of the groups Res\(_{E/\mathbb{Q}}\)GSp\(_4\).

Let Rep\(_{\mathcal{O}_{E',\lambda'}}(M_\mu/Z_s(G))\) be the category of algebraic representations of \(M_\mu/Z_s(G)\) (\(M_\mu/Z_s(G)\) is viewed as a reductive group over Spec \(\mathcal{O}_{E',\lambda'}\)) over finite free \(\mathcal{O}_{E',\lambda'}\)-modules. Using Theorem 4.14 we get a functor

\[
\text{Rep}_{\mathcal{O}_{E',\lambda'}}(M_\mu/Z_s(G)) \rightarrow VB(\mathfrak{sh}^{\text{tor}}_{K,\Sigma})
\]

\[
V \mapsto \mathcal{V}_{K,\Sigma},
\]

which is an integral version of the functor of Subsection 4.4. Depending on the context, \(\mathcal{V}_{K,\Sigma}\) will mean the locally free sheaf over \(\mathfrak{sh}^{\text{tor}}_{K,\Sigma}\) attached to an object \(V \in \text{Rep}_{\mathcal{O}_{E',\lambda'}}(M_\mu/Z_s(G))\) or the locally free sheaf on \(Sh^{\text{tor}}_{K,\Sigma}\) attached to \(V_{E'} := V \otimes_{\mathcal{O}_{E',\lambda'}} E'.\)

The cohomology complexes \(R^i(\mathfrak{sh}^{\text{tor}}_{K,\Sigma}, \mathcal{V}_{K,\Sigma})\) and \(R^i(\mathfrak{sh}^{\text{tor}}_{K,\Sigma}, V_{\text{int}}(−D_{K,\Sigma}))\) are independent of \(\Sigma\) (this is a standard computation using the structure of the boundary; see [47, Theorem 8.6] in the PEL case), and these are perfect complexes of \(\mathcal{O}_{E',\lambda'}\)-modules. We observe that

\[
\text{Im} \left( H^i(\mathfrak{sh}^{\text{tor}}_{K,\Sigma}, \mathcal{V}_{K,\Sigma}) \otimes_{\mathcal{O}_{E',\lambda',\ell}} \mathbb{Z}_p \rightarrow H^i(K, V_{E'})_{\mathbb{Z}_p} \right)
\]

and

\[
\text{Im} \left( H^i(\mathfrak{sh}^{\text{tor}}_{K,\Sigma}, V_{\text{int}}(−D_{K,\Sigma})) \otimes_{\mathcal{O}_{E',\lambda',\ell}} \mathbb{Z}_p \rightarrow H^i_{\text{cusp}}(K, V_{E'})_{\mathbb{Z}_p} \right)
\]

are lattices that we denote respectively by \(H^i(K, V)_{\mathbb{Z}_p}\) and \(H^i_{\text{cusp}}(K, V)_{\mathbb{Z}_p}\).

**4.9. Conjectures on the action of the integral Hecke algebra**

Let \(\kappa\) be a dominant weight for \(M_\mu/Z_s(G)\) and let \(V_\kappa\) be the corresponding Weyl representation, defined over \(\mathcal{O}_{E',\lambda'}\). Namely, let \(\kappa^\vee\) be the dominant weight \(-w_{M_\mu}\kappa\) for \(w_{M_\mu}\) the longest element of the Weyl group of \(M_\mu\). Let \(B_{M_\mu}\) be the Borel of \(M_\mu\) (corresponding to our choice of positive roots). We let \(V_{\text{int}}\) be the space of functions \(f : M_\mu \rightarrow \mathbb{A}^1\) with the transformation property \(f(mb) = \kappa^\vee(b)f(m)\) for all \(b \in B_{M_\mu}\). The right action of \(G\) on itself given by \(g \mapsto L(g^{-1})\) (where \(L\) is the left translation) induces a left action on the space \(V_\kappa\), and \((V_\kappa)_{E'}\) is an irreducible highest weight representation of weight \(\kappa\).

**Conjecture 4.15.** The lattices \(H^i(K, V_\kappa)_{\mathbb{Z}_p}\) and \(H^i_{\text{cusp}}(K, V_\kappa)_{\mathbb{Z}_p}\) are stable under \(\mathcal{H}^{\text{int}}_{p,\kappa,\ell}\).

**Conjecture 4.16.** The algebra \(\mathcal{H}^{\text{int}}_{p,\kappa,\ell}\) acts on \(R^i(\mathfrak{sh}^{\text{tor}}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma})\) and \(R^i(\mathfrak{sh}^{\text{tor}}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma}(−D_{K,\Sigma}))\).

Observe that Conjecture 4.15 implies Conjecture 4.7 by Proposition 4.11, and Conjecture 4.16 obviously implies Conjecture 4.15.
5. Shimura varieties of symplectic type

This section is dedicated to Shimura varieties of symplectic type. We prove the missing part of Theorem 4.14, and we state in Theorem 5.9 some partial results towards Conjecture 4.16 that are proved in Section 6.

5.1. Group-theoretic data

Let \( \nu \) be a totally real field with \( [F:\mathbb{Q}] = d \) and let \((V, \Psi)\) be a symplectic \( F \)-vector space of dimension \( 2g \), with basis \( e_1, \ldots, e_{2g} \) and \( \Psi(e_i, e_j) = 0 \) if \( j \neq 2g+1-i \) and \( \Psi(e_i, e_{2g-i+1}) = 1 \) if \( 1 \leq i \leq g \). Let \( V_0 = \langle e_1, \ldots, e_g \rangle \) and \( V_1 = \langle e_{g+1}, \ldots, e_{2g} \rangle \) be sub-\( F \)-vector spaces of \( V \). The pairing \( \Psi \) on \( V \) restricts to a perfect pairing between \( V_0 \) and \( V_1 \). Let \( G \) be the algebraic group over \( \mathbb{Q} \) defined by

\[
G(R) = \{(g, \nu) \in \text{GL}_F(V \otimes \mathbb{Q} R) \times (F \otimes \mathbb{Q} R)^\times | \forall v, w \in V \otimes R, \, \Psi(gv, gw) = \nu \Psi(v, w)\}
\]

for any \( \mathbb{Q} \)-algebra \( R \). Let \( G_1 \) be the subgroup of \( G \) whose elements have their similitude factor \( \nu \) in \( \mathbb{G}_m \) embedded diagonally in \( \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \). Let also \( G^\text{der} \) be the derived subgroup of \( G \) defined by the condition \( \nu = 1 \).

Let \( T^\text{der} \) be the diagonal maximal torus of \( G^\text{der} \); \( T^\text{der} = \{ \text{diag}(t_1, \ldots, t_g, t_g^{-1}, \ldots, t_1^{-1}) \} \) with \( t_i \in \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \). The center \( Z \) of \( G \) is the group \( \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \) embedded diagonally in \( G \). Let \( T \) be the maximal torus of \( G \), which is generated by \( T^\text{der} \) and \( Z \). Let \( Z_1 \) be the center of \( G_1 \), the image of \( \mathbb{G}_m \) embedded diagonally in \( G_1 \). The maximal torus of \( G_1 \) is generated by \( Z_1 \) and \( T^\text{der} \).

Let \( X^*(T) \) be the group of characters of \( T \) defined over \( \mathbb{Q} \), identified with tuples \( \kappa = (k_{1, \sigma}, \ldots, k_{g, \sigma}, k_{\sigma}) \in \text{Hom}(F, \mathbb{Q}) \in \mathbb{Z}^{(g+1)d} \) satisfying the condition \( k_{\sigma} = \sum_i k_{i, \sigma} \mod 2 \), via the pairing

\[
\langle \kappa, (zt_1, \ldots, zt_g, zt_g^{-1}, \ldots, zt_1^{-1}) \rangle = \prod_{\sigma} \left( \sigma(z)^{k_{\sigma}} \prod_i \sigma(t_i)^{k_{i, \sigma}} \right).
\]

We denote by \( P \subset G \) the Siegel parabolic; that is, the stabiliser of the Lagrangian subspace \( V_0 \). We denote by \( M \) its Levi quotient (which we identify with the standard Levi subgroup of \( P \)). Note that \( M \simeq \text{Res}_{F/\mathbb{Q}} \text{GL}_g \times \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \).

The roots of \( G \) corresponding to the Lie algebra of \( M \) are by definition the compact roots; the other roots are called noncompact. We declare that a character of \( T \) is dominant for \( M \) if for all \( \sigma \in \text{Hom}(F, \mathbb{Q}) \), \( k_{1, \sigma} \geq \cdots \geq k_{g, \sigma} \). So our choice of Borel subgroup in \( M \) is the usual upper triangular Borel.

A character of \( T \) is dominant for \( G \) if for all \( \sigma \in \text{Hom}(F, \mathbb{Q}) \), \( 0 \geq k_{1, \sigma} \geq \cdots \geq k_{g, \sigma} \) (so our choice of noncompact positive roots are those corresponding to \( \mathfrak{g}/\mathfrak{p} \)).

We have similar definitions for \( G_1 \) and we denote by \( T_1, M_1, \ldots \), the intersections of \( T, M, \ldots \), with \( G_1 \). Weights for \( T_1 \) are labelled by

\[
((k_{1, \sigma}, \cdots, k_{g, \sigma})_{\sigma \in \text{Hom}(F, \mathbb{Q})}; k) \in \mathbb{Z}^{gd} \times \mathbb{Z}
\]
with the condition that \( k = \sum_{i} k_{i, \sigma} \mod 2 \) and a weight \( \kappa \) as above pairs with an element \( t \in T_1 \) via the formula
\[
\langle \kappa, (zt_1, \cdots, zt_g, zt_g^{-1}, \cdots, zt_1^{-1}) \rangle = z^k \prod_{i=1}^{g} (\prod_{\sigma} \sigma(t_i)^{k_{i, \sigma}}).
\]

### 5.2. Shimura varieties of symplectic type in characteristic 0

Let \( \mathcal{H}_g^\pm \) be the Siegel space of symmetric matrices \( M = A + iB \in M_{g \times g}(\mathbb{C}) \) with \( B \) definite (positive or negative). Let \( X = (\mathcal{H}_g^+ \cup \mathcal{H}_g^-)_{\text{Hom}(F, \overline{\mathbb{Q}})} \) and \( X_1 = (\mathcal{H}_g^+ \cup \mathcal{H}_g^-)_{\text{Hom}(F, \overline{\mathbb{Q}})} \subset X \). The group \( G(\mathbb{R}) \) acts on \( X \) and its subgroup \( G_1(\mathbb{R}) \) stabilizes \( X_1 \). The pairs \((G, X)\) and \((G_1, X_1)\) are Shimura data. The Siegel parabolic \( P \) is a representative of the conjugacy class of \( P_\mu \).

Let \( K \subset G(\mathbb{A}_f) \) be a compact open subgroup. We assume that \( K = \prod \ell \ell K_\ell \) and that \( K \) is neat. We also assume that \( p \) is unramified in \( F \) so \( G(\mathbb{Q}_p) = \prod \ell \ell GSp_{2g}(F_v) \) for unramified extensions \( F_v \) of \( \mathbb{Q}_p \). We further assume that \( K_p = \prod K_v \subset G(\mathbb{Z}_p) \) where \( K_v \) is either \( GSp_{2g}(O_{F_v}) \) or \( \text{Si}(\nu) \), the Siegel parahoric subgroup of elements with reduction mod \( p \) in \( P(O_{F_v}/p) \).

We let \( Sh_K(\mathbb{C}) = G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f)/K) \) be the quotient. Let \( G(\mathbb{Q})^+ \subset G(\mathbb{Q}) \) be the subgroup of elements whose similitude factor is totally positive.

By strong approximation, we can write
\[
G(\mathbb{A}_f) = \prod c G(\mathbb{Q})^+ cK,
\]
where the elements \( c \in G(\mathbb{A}_f) \) are such that the elements \( \nu(c) \) range through a set of representatives of \( F^\times \backslash (\mathbb{A}_f \otimes F)^\times /\nu(K) \) and we find that \( Sh_K(\mathbb{C}) = \prod c \Gamma(c, K) \backslash H_g^{\text{Hom}(F, \overline{\mathbb{Q}})} \), where \( \Gamma(c, K) = G(\mathbb{Q})^+ c cK^{-1} \).

This is an algebraic variety over \( \mathbb{C} \), and it has a canonical model \( Sh_K \) over \( \mathbb{Q} \). The Shimura variety is not of PEL type, and it is therefore useful to introduce another Shimura variety that is of PEL type. We begin by rewriting \( Sh_K(\mathbb{C}) = (G(\mathbb{Q}) \cap K_p) \backslash (X \times G(\mathbb{A}_f^p)/K^p) \) and then consider the Shimura variety of PEL type (in fact an infinite union of such) \( \widetilde{Sh}_K(\mathbb{C}) = (G(\mathbb{Q}) \cap K_p) \backslash (X_1 \times G(\mathbb{A}_f^p)/K^p) \).

By strong approximation, we find that
\[
G(\mathbb{A}_f^p) = \prod c (G(\mathbb{Q})^+ \cap K_p) cK^p,
\]
where \( c \in G(\mathbb{A}_f^p) \) ranges through a set of representatives of \( \mathbb{Z}_{(p)}^\times \backslash (\mathbb{A}_f^p \otimes F)^\times /\nu(K^p) \) and we find that \( \widetilde{Sh}_K(\mathbb{C}) = \prod c \Gamma_1(c, K) \backslash H_g^{\text{Hom}(F, \overline{\mathbb{Q}})} \), where \( \Gamma_1(c, K) = G(\mathbb{Q})^+ c K^{-1} \). It has a canonical model \( \widetilde{Sh}_K \rightarrow \text{Spec } \mathbb{Q} \) and we have a natural morphism
\[
\widetilde{Sh}_K \rightarrow Sh_K.
\]

---

\[2\] We could actually allow any parahoric level structure as in [5, Section 3.3.1], but in the sequel we shall only need to work at hyperspecial level or Siegel parahoric level.
On the set of geometric connected components, this map is given by: $\mathbb{Z}^{\times,+}_{(p)} \setminus (\mathbb{A}_f^P \otimes F)_{x} / \nu(K^P) \to \mathcal{O}_{F,p}^{\times,+(\mathbb{A}_f^P \otimes F)_{x}} / \nu(K^P)$.

If we let $c \in G(\mathbb{A}_f^P)$ so that $\nu(c)$ defines geometrically connected components $(\widetilde{Sh}_K)_c$ of $\widetilde{Sh}_K$ and $(Sh_K)_c$ of $Sh_K$, then the corresponding map $(\widetilde{Sh}_K)_c \to (Sh_K)_c$ is a finite Galois cover with group

$$\Delta(K) = (\mathcal{O}_F^{\times,+} \cap \nu(K^P)) / \nu(\mathcal{O}_F^\times \cap K^P),$$

where the first intersection is taken in $(\mathbb{A}_f^P \otimes F)^\times$ and the second in $G(\mathbb{A}_f^P)$, with $\mathcal{O}_F^\times$ embedded diagonally in the center of $G$.

5.3. Integral models

We observe that $\widetilde{Sh}_K$ has a canonical model $\widetilde{Gh}_K$ over $\mathbb{Z}(p)$ that is a moduli space of abelian varieties of dimension $gd$ with $K$-level structure, prime to $p$ polarisation, and an action of $\mathcal{O}_F$. More precisely, $\widetilde{Gh}_K$ represents the functor over $\mathbb{Z}(p)$ that parametrises equivalence classes of $(A, \iota, \lambda, \eta, \eta_p)$, where

(1) $A \to \text{Spec } R$ is an abelian scheme,
(2) $\iota: \mathcal{O}_F \to \text{End}(A) \otimes \mathbb{Z}(p)$ is an action,
(3) $\lambda: A \to A^d$ is a prime to $p$, $\mathcal{O}_F$-linear quasi-polarisation,
(4) $\eta$ is a $K^p$-level structure and
(5) $\eta_p$ is a $K_p$-level structure.

Let us spell out the definition of $K^p$-level structure. We may assume without loss of generality that $S = \text{Spec } R$ is connected, and we fix $\overline{s}$ a geometric point of $S$. The adelic Tate module $H_1(A|_{\overline{s}}, A^{\infty,p})$ carries a symplectic Weil pairing

$$\langle , \rangle: H_1(A|_{\overline{s}}, A^{\infty,p}) \times H_1(A|_{\overline{s}}, A^{\infty,p}) \to H_1(\mathbb{G}_m|_{\overline{s}}, A^{\infty,p})$$

or, equivalently, an $F$-linear symplectic pairing

$$\langle , \rangle_{1,\lambda}: H_1(A|_{\overline{s}}, A^{\infty,p}) \times H_1(A|_{\overline{s}}, A^{\infty,p}) \to H_1(\mathbb{G}_m|_{\overline{s}}, A^{\infty,p}) \otimes F.$$

The level structure $\eta$ is a $K^p$-orbit of pairs of isomorphisms $(\eta_1, \eta_2)$, where (with $V$ the standard symplectic space defined above)

(1) An $\mathcal{O}_F$-linear isomorphism of $\Pi_1(S, \overline{s})$-modules $\eta_1: V \otimes_{\mathbb{Z}} A^{\infty,p} \simeq H_1(A|_{\overline{s}}, A^{\infty,p})$.
(2) An $\mathcal{O}_F$-linear isomorphism of $\Pi_1(S, \overline{s})$-modules $\eta_2: F \otimes_{\mathbb{Z}} A^{\infty,p} \simeq F \otimes_{\mathbb{Z}} H_1(\mathbb{G}_m|_{\overline{s}}, A^{\infty,p})$. 
We moreover impose that the following diagram is commutative:

\[
\begin{array}{ccc}
V \otimes_{\mathbb{Z}} \mathbb{A}^\infty, p \times V \otimes_{\mathbb{Z}} \mathbb{A}^\infty, p & \xrightarrow{\eta_1 \times \eta_1} & H_1(A|_\mathbb{A}, \mathbb{A}^\infty, p) \times H_1(A|_\mathbb{A}, \mathbb{A}^\infty, p) \\
\downarrow{\langle \cdot, \cdot \rangle_1} & & \downarrow{\langle \cdot, \cdot \rangle_1} \\
F \otimes_{\mathbb{Z}} \mathbb{A}^\infty, p & \xrightarrow{\eta_2} & F \otimes_{\mathbb{Z}} H_1(\mathbb{G}_m|_\mathbb{A}, \mathbb{A}^\infty, p)
\end{array}
\]

The \( K_p \) level structure \( \eta_p \) is the data, for each \( v | p \) such that \( K_v = \text{Si}(v) \), of a maximal totally isotropic subgroup \( H_v \subset A[v] \).

A map between quintuples \((A, t, \lambda, \eta, \eta_p)\) and \((A', t', \lambda', \eta', \eta'_p)\) is an \( \mathcal{O}_F \)-linear prime to \( p \) quasi-isogeny (in the sense of [46, Definition 1.3.1.17]) \( f : A \rightarrow A' \) such that

- \( f^* \lambda = r \lambda' \) for a locally constant function \( r : S \rightarrow \mathbb{Z}_{(p)}^{x, +} \),
- \( f(\eta_p) = \eta'_p \) and
- \( H_1(f) \circ \eta = \eta' \).

This last condition means that \( \eta' \) is defined by \( H_1(f) \circ \eta_1 = \eta'_1 \) and \( \eta'_2 = r^{-1} \eta_2 \). Also, we have denoted \( \mathbb{Z}^{x, +}_{(p)} = \mathbb{Q}_{>0}^{x, +} \cap \mathbb{Z}_{(p)}^{x, +} \).

**Remark 5.1.** Note that we allow the similitude factor in the level structure to be in \( \mathbb{A}^\infty, p \otimes_{\mathbb{Q}} F(1) \), but we only allow quasi-isogenies with similitude factor in \( \mathbb{A}^\infty, p(1) \).

We now define an action of \((\mathcal{O}_F)^{x, +}_{(p)}\) on \( \widetilde{Sh}_K \) by scaling the polarisation. Namely, \( x \in (\mathcal{O}_F)^{x, +}_{(p)} \) sends \((A, t, \lambda, \eta, \eta_p)\) to \((A, t, x \lambda, x \eta, \eta_p)\) where \( x \eta = (\eta_1, x \eta_2) \).

This action restricts to a trivial action on the subgroup \( \nu(K^p \cap \mathcal{O}_F^{x, +}_{(p)}) \) (where \( \mathcal{O}_F^{x, +}_{(p)} \) is embedded diagonally in \( G(\mathcal{O}_F) \)), because for any \( x \in K^p \cap \mathcal{O}_F^{x, +}_{(p)} \) the multiplication by \( x : A \rightarrow A \) identifies the points \((A, t, \lambda, \eta, \eta_p)\) and \((A, t, x^{-2} \lambda, x^{-2} \eta, \eta_p)\).

We therefore get an action of \( \Delta = (\mathcal{O}_F^{x, +}_{(p)})^\times / \nu(K^p \cap \mathcal{O}_F^{x, +}_{(p)}) \). One can show that this action is free [5, Lemma 3.3.13]. The group \( \Delta \) is infinite, but the stabiliser of any connected component of \( \widetilde{Sh}_K \) is finite.

The étale surjective map \( \widetilde{Sh}_K \rightarrow Sh_K \) identifies \( Sh_K \) as the quotient of \( \widetilde{Sh}_K \) by the group \( \Delta \). The action of the group \( \Delta \) extends to a free action on \( \tilde{\mathcal{S}}h_K \). We can form the quotient of \( \tilde{\mathcal{S}}h_K \) by \( \Delta \), and this defines an integral model \( \mathcal{S}h_K \) for \( Sh_K \) over \( \mathbb{Z}_{(p)} \) (see [5, Section 3.3]).

### 5.4. Compactifications

We have smooth toroidal compactifications \( \tilde{\mathcal{S}}h_{K, \Sigma}^{tor} \) for suitable choices of polyhedral cone decompositions. The action of \( \Delta \) extends to an action on \( \tilde{\mathcal{S}}h_{K, \Sigma}^{tor} \), and this action is free so we get a smooth integral toroidal compactification \( \mathcal{S}h_{K, \Sigma}^{tor} \) of \( \mathcal{S}h_K \). See Subsection 3.5 of [5].
5.5. Integral automorphic sheaves

Let $E'$ be a Galois closure of $F$, $\lambda'$ be a place of $E'$ above $p$ and $\mathcal{O}_{E',\lambda'}$ the localisation of $\mathcal{O}_{E'}$ at $\lambda$. Let $\text{Rep}_{\mathcal{O}_{E',\lambda'}}(M_1)$ be the category of representations of $M_1$ over finite free $\mathcal{O}_{E',\lambda'}$-modules. Let $\text{FL}_{G,X} = G/Q = G_1/Q_1$ be the flag variety.

By [45, Proposition 6.9], over $\tilde{\text{Sh}}_{\text{tor}} K$, $\Sigma$, the first relative de Rham homology group has a canonical extension $H^1_{dR}(A/\tilde{\text{Sh}}_{\text{tor}} K, \Sigma)$ and it carries the Hodge filtration

$$0 \to \omega_A \to H^1_{dR}(A/\tilde{\text{Sh}}_{\text{tor}} K, \Sigma) \to \text{Lie}(A) \to 0$$

and a pairing $\langle , \rangle_\lambda$ induced by the polarisation. We can consider the principal $G_1$-torsor $\tilde{P}_{K, \Sigma} \to \tilde{\text{Sh}}_{\text{tor}} K, \Sigma$ of isomorphisms between $H^1_{dR}(A/\tilde{\text{Sh}}_{\text{tor}} K, \Sigma)$ and $(V, \Psi)$.

We therefore obtain the following diagram:

\[ \begin{array}{ccc}
\tilde{\text{Sh}}_{\text{tor}} K, \Sigma & \xrightarrow{\beta} & \text{FL}_{G,X} \\
\downarrow{\alpha} & & \downarrow{\beta'} \\
\tilde{P}_{K, \Sigma} & & \\
\end{array} \]

where $\alpha$ is $G_1$-equivariant and $\beta$ is a $G_1$-torsor. Using this $G_1$-torsor we can define a functor $\text{Rep}_{\mathcal{O}_{E',\lambda'}}(M_1) \to \text{VB}(\tilde{\text{Sh}}_{\text{tor}} K, \Sigma)$.

Let $\kappa = ((k_1, \sigma, \ldots, k_g, \sigma); k)$ be a dominant weight for $M_1$. There is an associated Weyl representation $V_\kappa$ and we denote by $V_{\kappa, K, \Sigma}$ the locally free sheaf corresponding to it via the above functor.

Example 5.2. Over $\mathcal{O}_{E',\lambda}$ we have $\omega_A = \oplus_\sigma(\omega_A)_\sigma$. Let $\sigma_0 \in \text{Hom}(F, E')$. Let $\kappa = ((k_1, \sigma, \ldots, k_g, \sigma); k)$ with $k_i, \sigma = 0$ if $\sigma \neq \sigma_0$ and $(k_1, \sigma_0, \ldots, k_g, \sigma_0) = (0, \ldots, 0, -1)$ and $k = 1$, then $V_{\kappa, K, \Sigma} = \text{Lie}(A)_{\sigma_0}$. Therefore, if $\kappa = ((k_1, \sigma, \ldots, k_g, \sigma); k)$ with $k_i, \sigma = 0$ if $\sigma \neq \sigma_0$ and $(k_1, \sigma_0, \ldots, k_g, \sigma_0) = (1, 0, \ldots, 0)$ and $k = -1$, then $V_{\kappa, K, \Sigma} = (\omega_A)_{\sigma_0}$.

Let $Z_s(G)$ be the subgroup of the centre of $G$ equal to the kernel of the norm map and we let $G^c = G/Z_s(G)$. We now restate and prove the missing part of Theorem 4.14.

Proposition 5.3. There is a natural commutative diagram

\[ \begin{array}{ccc}
\tilde{P}_{K, \Sigma} & \xrightarrow{\beta'} & \text{FL}_{G,X} \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
\tilde{\text{Sh}}_{\text{tor}} K, \Sigma & & \\
\end{array} \]

where $\alpha'$ is $G$-equivariant and $\beta'$ is a $G^c$-torsor.
Proof. We have a commutative diagram

\[
\begin{array}{c}
\widetilde{P}_{K,\Sigma} \\
\downarrow_{\beta} \\
\widetilde{\mathfrak{h}}_{K,\Sigma} \\
\downarrow_{\alpha} \\
\mathbb{F}L_{G,X}
\end{array}
\]

We consider the $G^c$-torsor $\widetilde{P}_{K,\Sigma} \times_{G^c} G^c$. We claim that this torsor descends to $\widetilde{\mathfrak{h}}_{K,\Sigma}^{tor}$. In order to prove this, we must exhibit a descent datum for the action of $\Delta$.

Let $(A,\iota,\lambda,\eta,\eta_p,\Psi)$ be an $R$-point of $\widetilde{P}_{K,\Sigma} \times_{G^c} G^c$, where $\Psi: V \otimes R \to \mathcal{H}_{1,dR}(A/R)^{can}$ is a symplectic isomorphism up to a similitude factor in $(F \otimes R)^\times$. For any $x \in K^p \cap \mathcal{O}_{F,(p)}$, the multiplication by $x^{-1}: A \to A$ induces a natural map

$$\mathcal{H}_{1,dR}(A/R)^{can} \to \mathcal{H}_{1,dR}(A/R)^{can}$$

that is scalar multiplication by $x^{-1}$ in the trivialisation $\Psi$. We observe that $x \in Z_s(G)(\mathcal{O}_{F})$, and therefore in $\widetilde{P}_{K,\Sigma} \times_{G^c} G^c$ the multiplication by $x^{-1}$ induces a canonical isomorphism between $(A,\iota,\lambda,\eta,\eta_p,\Psi)$ and $(A,\iota,x\lambda,\eta,\eta_p,\Psi)$. Therefore, we can simply define an action of $(\mathcal{O}_{F})^\times_{(p)}$ on $\widetilde{P}_{K,\Sigma} \times_{G^c} G^c$ by sending $(A,\iota,\lambda,\eta,\eta_p,\Psi)$ to $(A,\iota,\lambda,x\eta,\eta_p,\Psi)$, and this action passes to the quotient to an action of $\Delta$.

We can therefore descend the torsor $\widetilde{P}_{K,\Sigma} \times_{G^c} G^c$ to a $G^c$-torsor $P_{K,\Sigma} \to \mathfrak{h}_{K,\Sigma}^{tor}$. Moreover, one descends similarly the $P^c$-reduction of $\widetilde{P}_{K,\Sigma} \times_{G^c} G^c$, and therefore get a map $\beta': P_{K,\Sigma} \to \mathbb{F}L_{G,X}$. \hfill $\square$

Corollary 5.4. There is a functor $\text{Rep}^{\nu}_{O_{E'},\lambda'}(M/Z_s(G)) \to V B(\mathfrak{h}_{K,\Sigma}^{tor})$ which makes the following diagram commute:

\[
\begin{array}{ccc}
\text{Rep}^{\nu}_{O_{E'},\lambda'}(M_1) & \longrightarrow & V B(\widetilde{\mathfrak{h}}_{K,\Sigma}^{tor}) \\
\uparrow & & \uparrow \\
\text{Rep}^{\nu}_{O_{E'},\lambda'}(M/Z_s(G)) & \longrightarrow & V B(\mathfrak{h}_{K,\Sigma}^{tor})
\end{array}
\]

Remark 5.5. Let $\kappa = (k_1,\sigma,\cdots,k_g,\sigma;k_\sigma)_{\text{Hom}(F,\overline{Q})}$ be a dominant weight for $M$, with associated Weyl representation $V_\kappa$. The representation $V_\kappa$ belongs to $\text{Rep}^{\nu}_{O_{E'},\lambda'}(M/Z_s(G))$ if and only if $k_\sigma = k$ is independent of $\sigma$.

5.6. Minuscule coweights and the main theorem in the symplectic case

5.6.1. The general formula for minuscule coweights. We take $E'$ to be the Galois closure of $F$. We let $\iota: E' \to \overline{Q}_p$. Let $p$ be a prime unramified in $F$ and denote by $p_1,\cdots,p_m$ the prime ideals above $p$ in $F$. We let $I = \text{Hom}(F,E')$ and for each $1 \leq i \leq n$, let

$$I_i = \{ \sigma \in \text{Hom}(F,E'), \iota \circ \sigma \text{ induces the } p_i\text{-adic valuation on } F \}.$$
We consider a weight $\kappa = ((k_1, \sigma, \ldots, k_g, \sigma)_{\sigma \in I}; k)$ for $G$, where we assume that the parity of $\sum_i k_i, \sigma$ is independent of $\sigma$, and we choose $k \in \mathbb{Z}$ such that $\sum_i k_i, \sigma = k \mod 2$. We have, by Corollary 5.4, a sheaf $\mathcal{V}_{K, \Sigma}$ on $\mathcal{G}_K^{\text{tor}}$.

The spherical Hecke algebra at $p$, denoted $\mathcal{H}_p$, is a tensor product of the spherical Hecke algebras $\mathcal{H}_{p_i}$ at each prime $p_i$ dividing $p$. Each of the algebras $\mathcal{H}_{p_i}$ contains the following familiar characteristic functions of double cosets

$$T_{p_i} = \text{GSp}_{2g}(\mathcal{O}_{F_{p_i}})\text{diag}(p_i^{-1}g, 1_g)\text{GSp}_{2g}(\mathcal{O}_{F_{p_i}})$$

and $S_{p_i} = \text{GSp}_{2g}(\mathcal{O}_{F_{p_i}})\text{diag}(p_i^{-1}, p_i^{-1})\text{GSp}_{2g}(\mathcal{O}_{F_{p_i}})$, and to each of these double cosets we can associate a cohomological correspondence over $\mathbb{Q}$ on the sheaf $\mathcal{V}_{K, \Sigma}$ (see Subsection 4.5). We have added the superscript ‘naive’ because the cohomological correspondence is not suitably normalised in general.

By definition $T_{p_i}^{\text{naive}} = T_{\lambda}$ (see Subsection 3.2) for the cocharacter $\lambda : t \mapsto (\prod_{\sigma \in I_1} \text{diag}(t^{-1}g, 1_g)) \times (\prod_{\sigma \notin I_1} \text{diag}(1_g, 1_g))$, which is given in coordinates by

$$\left(\prod_{\sigma \notin I_1} (0, \cdots, 0; 0)_{\sigma}\right) \times \left(\prod_{\sigma \in I_2} (-\frac{1}{2}, \cdots, -\frac{1}{2}; \frac{1}{2})\right),$$

and $S_{p_i}^{\text{naive}} = T_\mu$ for the cocharacter $\mu : t \mapsto (\prod_{\sigma \in I_1} \text{diag}(t^{-1}g, 1_g, 1_g)) \times (\prod_{\sigma \notin I_1} \text{diag}(1_g, 1_g))$, which is given in coordinates by

$$\left(\prod_{\sigma \notin I_1} (0, \cdots, 0; 0)_{\sigma}\right) \times \left(\prod_{\sigma \in I_1} (0, \cdots, 0; -1)\right).$$

**Remark 5.6.** The goal of this remark is to justify the use of the double class

$$\text{GSp}_{2g}(\mathcal{O}_{F_{p_i}})\text{diag}(p_i^{-1}g, 1_g)\text{GSp}_{2g}(\mathcal{O}_{F_{p_i}})$$

rather than the double class

$$\text{GSp}_{2g}(\mathcal{O}_{F_{p_i}})\text{diag}(p_i^1g, 1_g)\text{GSp}_{2g}(\mathcal{O}_{F_{p_i}}),$$

which appears in some classical references. The difference can be explained as follows: The adelic points of $G$ act on the right on the tower of Shimura varieties and therefore on the left on the cohomology; in the classical theory of modular forms, one usually defines a right action of the Hecke algebra on the space of modular forms. Let us give some more details. To avoid complications with non-PEL Shimura varieties, we will assume that our totally real field is $\mathbb{Q}$. For simplicity, we also assume that $K \subset \text{GSp}_4(\hat{\mathbb{Z}})$. Associated to the element $g = \text{diag}(p^{-1}g, 1_g)$ we have a correspondence over $\mathbb{Q}$:

![Diagram](https://via.placeholder.com/150)

and the relation between $p_1^*A$ and $p_2^*A$ is given as follows (at least away from the boundary). There are symplectic isomorphisms $\psi_1 : \hat{\mathbb{Z}}^{2g} \to H_1(p_1^*A, \hat{\mathbb{Z}}) \mod gKg^{-1} \cap K$
and \( \psi_2 : \hat{\mathbb{Z}}^{2g} \to H_1(p_2^* A, \hat{\mathbb{Z}}) \) mod \( g^{-1} K g \cap K \) and a commutative diagram

\[
\begin{array}{c}
\hat{\mathbb{Z}}^{2g} \otimes \mathbb{Q} \xrightarrow{g} \hat{\mathbb{Z}}^{2g} \otimes \mathbb{Q} \\
\downarrow \psi_2 \downarrow \quad \downarrow \psi_1 \\
H_1(p_2^* A, \hat{\mathbb{Z}}) \otimes \mathbb{Q} \xrightarrow{\pi} H_1(p_1^* A, \hat{\mathbb{Z}}) \otimes \mathbb{Q}
\end{array}
\]

Therefore, the lattice \( H_1(p_2^* A, \hat{\mathbb{Z}}) \) contains the lattice \( H_1(p_1^* A, \hat{\mathbb{Z}}) \). This means that \( p_2^* A \) appears to be the quotient of \( p_1^* A \) by a Lagrangian subgroup of \( p_1^* A[p] \). The Shimura variety \( Sh_{q K g^{-1} \cap K} \) is therefore parametrising all Lagrangian subgroups of \( p_1^* A \) and the projection \( p_2 \) is given by ‘taking the quotient by the Lagrangian subgroup’. If we had made the choice of \( g = \text{diag}(1_g, p_{1g}) \), then we would have obtained the transposed correspondence.

Motivated by Definition 4.8, we now let

\[
T_{p_i} = [V_{\lambda}] p^{\langle \lambda, \infty(\kappa, \iota) \rangle} = p^{\langle \lambda, \infty(\kappa, \iota) \rangle - \langle \lambda, \rho \rangle} T_{p_i}^{\text{naive}},
\]

where the last equality follows from the fact that \( \lambda \) is minuscule. We now determine the value of the coefficient \( \langle \lambda, \infty(\kappa, \iota) \rangle - \langle \lambda, \rho \rangle \).

**Lemma 5.7.** We have \( \langle \lambda, \infty(\kappa, \iota) \rangle - \langle \lambda, \rho \rangle = \sum_{\sigma \in I_i} \sup_{1 \leq j \leq g} \left\{ \frac{\sum_{\ell=1}^{j} k_{\ell, \sigma} - \left( \sum_{\ell=j+1}^{g} k_{\ell, \sigma} \right)}{2} + k \left( j(j+1) \right) \right\} \).

**Proof.** We have \( \rho = (-1, -2, \cdots, -g; 0)_{\sigma \in I_i} \), so \( (\kappa + \rho)_\sigma = (-1, -2, \cdots, -g; k) \) if \( \sigma \notin I_i \) and \( (\kappa + \rho)_\sigma = (k_1, \sigma - 1, k_2, \sigma - 2, \cdots, k_g, \sigma - g; k) \) if \( \sigma \in I_i \). By definition, \( \infty(\kappa, \iota)_\sigma \) is the Weyl translate of \( -(\kappa + \rho)_\sigma \) in the dominant cone. We have \( \lambda_\sigma = (0, \cdots, 0; 0) \) if \( \sigma \notin I_i \), and \( \lambda_\sigma = (-\frac{1}{2}, \cdots, -\frac{1}{2}; -\frac{1}{2}) \). We clearly have

\[
\langle \lambda, \infty(\kappa, \iota) \rangle - \langle \lambda, \rho \rangle = \sum_{\sigma \in I} \langle \lambda_\sigma, \infty(\kappa, \iota)_\sigma \rangle - \langle \lambda_\sigma, \rho_\sigma \rangle
\]

and the contribution to the sum of any \( \sigma \notin I_i \) is 0.

Let us fix \( \sigma \in I_i \) and compute the corresponding pairing at \( \sigma \). We need to put each \( -(\kappa + \rho)_\sigma \) in the dominant cone (i.e., the coordinates on the left of the semicolon need to be nonpositive and in decreasing order) and the pairing with \( \lambda_\sigma \) will amount to taking the sum of the coordinates and multiplying it by \( -\frac{1}{2} \).

Now there will be some integer \( 0 \leq j \leq g \) such that the first \( j \) coordinates of \( -(\kappa + \rho)_\sigma \) are nonpositive and the next \( g - j \) coordinates are nonnegative. In that case, the weight will be put in the dominant form by first changing it to

\[
(-k_1, \sigma + 1, \cdots, -k_j, \sigma + j, k_{j+1}, \sigma - j - 1, \cdots, k_g, \sigma - g; -k)
\]

(so that all entries on the left of the semicolon are nonpositive) and then applying an element of the Weyl group of the Levi to put it in the dominant form. Of course, this
last operation is irrelevant for computing the pairing because we will take the sum of all coordinates. We therefore find that in this case
\[
\langle \lambda_\sigma, \infty(\kappa, \iota)_\sigma \rangle = \sum_{j=1}^{g} k_{\ell, \sigma} - \left( \sum_{g_{\ell+1}}^{g} k_{\ell, \sigma} \right) + \frac{k}{2} \sum_{\ell=1}^{j} \ell + \frac{g}{2} \sum_{\ell=j+1}^{g} \ell.
\]

Now we note that for any \( j' \neq j \), the corresponding sum on the right-hand side above, with \( j \) replaced by \( j' \), is less than or equal to the sum for \( j \). The formula then follows from the observation that \( -\langle \lambda_\sigma, \rho_\sigma \rangle = -\frac{k}{2} \sum_{g_{\ell+1}}^{g} \ell \). \( \square \)

Similarly, motivated by Definition 4.8, we also set
\[
S_{p_i} = [V_{p_i}]p^{\mu, \infty(\kappa, \iota)} = p^{\langle \lambda, \infty(\kappa, \iota) \rangle} - \langle \lambda, \rho \rangle S_{p_i}^{naive}
\]
and it is elementary to check that \( S_{p_i} = p\sum_{\sigma \in I_i} k_{p_i} S_{p_i}^{naive} \).

**Remark 5.8.** The coweights \( \lambda \) and \( \mu \) (for varying primes \( p_i \)) are the only minuscule coweights (up to twist by a central coweight). Unfortunately, our techniques do not allow us to deal with nonminuscule coweights well, and this is why we do not consider the entire Hecke algebra.

5.6.2. The main result in the symplectic case. We have the following partial result towards Conjecture 4.16:

**Theorem 5.9.** There are algebra morphisms
\[
\otimes_{0 \leq i \leq m} \mathbb{Z}[T_{p_i}, S_{p_i}, S_{p_i}^{-1}] \rightarrow \text{End}(\Gamma(\mathfrak{H}^{tor}_{K, \Sigma}, \mathcal{V}_{\kappa_0, K, \Sigma}))
\]
and
\[
\otimes_{0 \leq i \leq m} \mathbb{Z}[T_{p_i}, S_{p_i}, S_{p_i}^{-1}] \rightarrow \text{End}(\Gamma(\mathfrak{H}^{tor}_{K, \Sigma}, \mathcal{V}_{\kappa_0, K, \Sigma}(-D_{K, \Sigma})))
\]
extending the action over \( \mathbb{Q} \) (see Subsection 4.5).

We now discuss some special cases of the theorem.

5.6.3. \( G = \text{GL}_2/\mathbb{Q} \). The sheaf of weight \( k \) modular forms corresponds to the weight \( (k; -k) := \kappa \). From the formula of Lemma 5.7, we deduce that \( T_{p_i} = p^{-\inf\{1, k\}}T_{p_i}^{naive} \) and \( S_{p_i} = p^{-k}S_{p_i}^{naive} \) and that
\[
\mathcal{H}^{int}_{p, \kappa} = \mathbb{Z}[T_{p_i}, S_{p_i}, S_{p_i}^{-1}].
\]
It follows from Theorem 5.9 that \( \mathcal{H}^{int}_{p, \kappa} \) acts on the cohomology complex of weight \( k \) modular forms. Conjecture 4.16 is thus proven in this case.

5.6.4. \( G = \text{GL}_2/F \). We let \( \kappa = ((k_\sigma)_\sigma; k) \) with the property that \( k \) and all of the \( k_\sigma \) have the same parity. Then we find that
\[
T_{p_i} = p^{\sum_{\sigma \in I_i} \sup \{\frac{k_{\sigma} + k}{2}, \frac{k - k_\sigma}{2}\}}T_{p_i}^{naive}.
\]
and

\[ S_p = p^{\sum_{\ell \in \mathfrak{I}_i} k} S^{\text{naive}}_p. \]

We deduce that \( \mathcal{H}^{\text{int}}_{p, \kappa, \iota} = \bigotimes_i \mathcal{H}^{\text{int}}_{p, \kappa} \) where

\[ \mathcal{H}^{\text{int}}_{p, \kappa} = \mathbb{Z}[T_p, S_p, S_p^{-1}]. \]

It follows from Theorem 5.9 that \( \mathcal{H}^{\text{int}}_{p, \kappa, \iota} \) acts on the cohomology complex of weight \( \kappa \) modular forms. Conjecture 4.16 is thus also proven in this case.

5.6.5. \( G = \text{GSp}_{2g}/\mathbb{Q} \). We take \( \kappa = (k_1, \cdots, k_g; -\sum k_i) \). Our choice of the central character is the standard choice in the theory of Siegel modular forms. Indeed, the sheaf \( \mathcal{V}_{\kappa, K, \Sigma} \) has the following elementary description. First, on \( \mathcal{S}_{\text{tor}}^{K, \Sigma} \), we have a semi-abelian scheme \( A \) of dimension \( g \), and we denote by \( \omega_A \) the conormal sheaf. We denote by \( T \) the torsor of trivialisations of \( \omega_A \). This is a \( \text{GL}_g \)-torsor and we let \( \pi : T \to \mathcal{S}_{\text{tor}}^{K, \Sigma} \) be the projection. Then, unravelling the definitions, we find that \( \mathcal{V}_{\kappa, K, \Sigma} = \pi_* \mathcal{O}_T[\kappa^\vee] \) where \( \kappa^\vee = (-k_g, \cdots, -k_1) \) (and \( \pi_* \mathcal{O}_T[\kappa^\vee] \) is the subsheaf of \( \pi_* \mathcal{O}_T \) of sections that transform by the character \( \kappa^\vee \) under the action of the Borel). We finally obtain that

\[ T_p = p^{\sup_{1 \leq j \leq g} \left\{ -\sum_{\ell = j+1}^g k_i - \frac{j(j+1)}{2} \right\}} T_{p, \text{naive}} \]

and \( S_p = p^{-\sum k_i} S^{\text{naive}}_p \).

6. Local model in the symplectic case

In this section we will prove Theorem 5.9. The actions of the normalised Hecke operators will be defined using the results of Section 2, in particular the construction in Example 2.11. In order to do this, we need to understand the integrality properties of the Hecke correspondences with respect to automorphic vector bundles and also the pullback maps on differentials. This will be done by using the theory of local models of Shimura varieties. Variants of the local models we consider in this section were introduced in \([10, 18, 16]\) and studied further by Görtz \([23]\); for a general introduction to the theory of local models the reader may consult \([54]\).

After recalling the basic facts about the local models for symplectic groups, we prove two results, Proposition 6.5 (on the integrality properties of differentials) and Lemma 6.7 (on the integrality properties on automorphic bundles), which are used to construct normalised Hecke operators on the local model in Proposition 6.8. We then transport these computations to the Shimura variety side using Theorem 6.9 and thereby prove Theorem 5.9.

6.1. Definition

Let \( g \in \mathbb{Z}_{\geq 1} \) be an integer. For any positive integer \( t \), we let \( \text{Id}_t \) be the identity \( t \times t \) matrix and we let \( K_t \) be the anti-diagonal matrix of size \( t \times t \), with coefficients 1 on the anti-diagonal. When the context is clear, we sometimes write \( K \) instead of \( K_t \) and \( \text{Id} \) instead of \( \text{Id}_t \).
Let $V_0 = \mathbb{Z}^{2g}$. We equip $V_0$ with the symplectic pairing $\psi$ given by the matrix

$$J = \begin{pmatrix} 0 & K_g \\ -K_g & 0 \end{pmatrix}.$$ 

We now consider modules $V_1, \ldots, V_{2g-1} = \mathbb{Z}^{2g}$ and the following chain:

$$V_\bullet : V_0 \to V_1 \to \cdots \to V_{2g-1} \to V_0,$$

where the map from $V_i$ to $V_{i+1}$ is given in the canonical basis $(e_1, \ldots, e_{2g})$ of $\mathbb{Z}^{2g}$ by the map $e_j \mapsto e_j$ if $j \neq i+1$ and $e_{i+1} \mapsto pe_{i+1}$. Whenever necessary, indices are taken modulo $2g$ so that $V_{2g} := V_0$.

This chain is self-dual. The pairing $\psi$ and the maps in the chain induce pairings $\psi : V_r \times V_{2g-r} \to \mathbb{Z}$, which can be written as $p^2\psi'$, for a perfect pairing $\psi'$.

Let us fix a set $\emptyset \neq I \subset \{0, 1, \ldots, g\}$. We define the local model functor $M_I : \mathbb{Z} - \text{ALG} \to \text{SETS}$ that associates to an object $R$ of $\mathbb{Z} - \text{ALG}$ the set of isomorphism classes of commutative diagrams

$$V_{i_0} \otimes_\mathbb{Z} R \to V_{i_1} \otimes_\mathbb{Z} R \to \cdots \to V_{i_m} \otimes_\mathbb{Z} R$$

$$F_{i_0} \to F_{i_1} \to \cdots \to F_{i_m}$$

where $i_0 < i_1 \cdots < i_m$ are such that $\{i_0, \ldots, i_m\} = I \cup \{2g - i | i \in I\}$, the modules $F_{i_j}$, for $0 \leq j \leq m$, are rank $g$ locally direct factors of $V_{i_j} \otimes_\mathbb{Z} R$ and they are self-dual in the sense that $F_{2g-j} = F_i$ for all $i \in I$ (with respect to the pairing $\psi'$).

The functor $M_I$ is represented by a projective scheme that we denote by $M_I$. It is a closed subscheme of a product of Grassmannians, the embedding being given by the vertical maps of diagrams as above.

When $\emptyset \neq J \subset I$, there is an obvious map $M_I \to M_J$ given by forgetting the modules $F_j$ for $j \in I \setminus J$. There is a canonical isomorphism $M_{\{0\}} \simeq M_{\{g\}}$ given by taking $F_g \subset V_g \otimes_\mathbb{Z} R$ to be $F_0$ via the tautological identification of $V_0$ and $V_g$.

### 6.2. The affine Grassmannian

Let $\mathcal{V}_0 = \mathbb{F}_p[[t]]^{2g}$. We equip $\mathcal{V}_0$ with the symplectic pairing $\psi$ given by the matrix

$$J = \begin{pmatrix} 0 & K_g \\ -K_g & 0 \end{pmatrix}.$$ 

We now consider modules $\mathcal{V}_1, \ldots, \mathcal{V}_{2g-1} = \mathbb{F}_p[[t]]^{2g}$ and the chain

$$\mathcal{V}_\bullet = \mathcal{V}_0 \to \mathcal{V}_1 \to \cdots \to \mathcal{V}_{2g-1} \to \mathcal{V}_0,$$

where the map from $\mathcal{V}_i$ to $\mathcal{V}_{i+1}$ is given in the canonical basis $(e_1, \ldots, e_{2g})$ of $\mathbb{F}_p[[t]]^{2g}$ by the map $e_j \mapsto e_j$ if $j \neq i+1$ and $e_{i+1} \mapsto te_{i+1}$. Whenever necessary, indices are taken modulo $2g$ so that $\mathcal{V}_{2g} := \mathcal{V}_0$. Observe that $\mathcal{V}_k \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p = \mathcal{V}_k \otimes_\mathbb{Z} \mathbb{F}_p$.

Let $LG$ denote the loop group of $\text{GSp}_{2g}$ over $\mathbb{F}_p$. The group $LG$ acts naturally on $\mathcal{V}_0 \otimes_{\mathbb{F}_p[[T]]} \mathbb{F}_p((T))$ and therefore it acts on the chain $\mathcal{V}_\bullet \otimes_{\mathbb{F}_p[[T]]} \mathbb{F}_p((T))$. 


Let \( \emptyset \neq I \subset \{0,1,\ldots,g\} \). Let \( \mathcal{V}_I^* \) be the subchain of \( \mathcal{V}_* \) where we keep only the modules indexed by elements \( i \in I \) and \( i' = 2g - i \) for \( i \in I \). We denote by \( \mathcal{P}_I \) the parahoric subgroup of \( LG \) of automorphisms of the chain \( \mathcal{V}_I^* \). For any \( I \) as above, we define the affine flag variety as the ind-scheme \( \mathcal{F}_I := LG/\mathcal{P}_I \).

### 6.3. Stratification of the local model

It was observed by Goertz [23, §5] that the fibre \( \overline{M}_I \) of \( M_I \) over \( \text{Spec} \mathbb{F}_p \) embeds as a finite union of \( \mathcal{P}_I \) orbits in \( \mathcal{F}_I \). We recall the description of the map \( \overline{M}_I \to \mathcal{F}_I \). Given a diagram

\[
V_{i_0} \otimes \mathbb{Z} R \longrightarrow V_{i_1} \otimes \mathbb{Z} R \longrightarrow \cdots \longrightarrow V_{i_m} \otimes \mathbb{Z} R
\]

\[
F_{i_0} \longrightarrow F_{i_1} \longrightarrow \cdots \longrightarrow F_{i_m}
\]

where all of the vertical maps are inclusions and each \( \mathcal{F}_{i_j} \) is determined by the property that \( \mathcal{F}_{i_j}/tV_{i_j} \otimes \mathbb{Z} R = F_{i_j} \leftarrow (V_{i_j}/tV_{i_j}) \otimes \mathbb{Z} R = V_{i_j} \otimes R \). The chain \( \mathcal{F}_* \) determines an \( R \)-point of \( \mathcal{F}_I \).

We now recall the combinatorial description of the image of \( \overline{M}_I \) in \( \mathcal{F}_I \): Fix a Borel subgroup \( B \) of \( GSp_{2g} \) and a maximal torus \( T \subset B \). This gives a base for the root datum of \( GSp_{2g} \) with a corresponding Dynkin diagram with \( g \) vertices and Weyl group \( W \) generated by reflections \( s_1, s_2, \ldots, s_g \). Let \( \tilde{W} \) be the extended affine Weyl group of \( GSp_{2g} \). This is the semi-direct product of the Weyl group \( W \) and the cocharacter group \( X_*(T) \). It contains as a subgroup \( W_{af} \), the affine Weyl group of \( GSp_{2g} \), which is a Coxeter group with simple reflections \( s_1, s_2, \ldots, s_g \) and one affine reflection \( s_0 \).

We give a concrete realisation of \( W \) and \( \tilde{W} \) following [43]. The group \( W \) can be realised as the subgroup of \( S_{2g} \), the group of permutations of \( \{1,2,\ldots,2g\} \), which satisfy the condition \( w(i) + w(2g + i - 1) = 2g + 1 \) for all \( i = 1, \ldots, 2g \). The extended affine Weyl group of \( GL_{2g} \) is \( \mathbb{Z}^{2g} \rtimes S_{2g} \). It can be realised as a subgroup of the group of affine transformations of \( \mathbb{Z}^{2g} \) with \( \mathbb{Z}^{2g} \) acting by translation and \( S_{2g} \) by permutation of the coordinates. For an element \( v \in \mathbb{Z}^{2g} \) we will denote the corresponding translation by \( t_v \). The group \( \tilde{W} \) is realised as the centraliser in \( \mathbb{Z}^{2g} \rtimes S_{2g} \) of the element \((1,2g)(2,2g-1)\cdots(g,g+1) \in S_{2g} \). We have \( s_i = (i,i+1)(2g+1-i,2g-i) \) for \( 1 \leq i \leq g-1 \), \( s_g = (g,g+1) \) and \( s_0 = t_{(-1,0,\ldots,0,1)} \rtimes (1,2g) \). We let \( \mu = (1,\ldots,1,0,\cdots,0) \in \mathbb{Z}^{2g} \) (with both 1 and 0 repeated \( g \) times) be the minuscule coweight corresponding to our situation.
For each \( I \) as above, we let \( W_I \) be the subgroup of \( \widetilde{W} \) generated by the simple reflexions \( s_i, i \notin I \). This is a finite group because \( I \neq \emptyset \). The \( \mathcal{P}_I \) orbits in \( \mathcal{F}_I \) are parametrised by the double cosets \( W_I \backslash \widetilde{W} / W_I \). For any \( w \in \widetilde{W} \) we denote the orbit corresponding to the double coset \( W_I w W_I \) by \( U_{I,w} \) and the orbit closure by \( X_{I,w} \). The orbits included in \( \overline{M}_I \) are parametrised by the finite subset \( \text{Adm}_I(\mu) \) of \( W_I \backslash \widetilde{W} / W_I \) of \( \mu \)-admissible elements as defined in [43, Introduction]. The open orbits – these are the only ones we will need to know explicitly – are parametrised by the double cosets corresponding to elements in the \( W \)-orbit of \( \mu \) (viewed as translations in \( \widetilde{W} \)).

As noted earlier, whenever \( J \subset I \) we have a map \( M_I \to M_J \). This map is surjective and \( \text{Adm}_J(\mu) \) is the image of \( \text{Adm}_I(\mu) \) in \( W_J \backslash \widetilde{W} / W_J \).

When \( I = \{0, \cdots , g\} \), Kottwitz and Rapoport give a description of the set \( \text{Adm}_I(\mu) \). This is a subset of \( W_{af} c \subset \widetilde{W} \) where \( c = t_{(0, \ldots , 0,1,\ldots,1)} \cdot (1,g+1)(2,g+2)\cdots (g,2g) \). We can transport the Bruhat order and the length function from \( W_{af} \) to \( W_{af} c \) via the bijection \( W_{af} \cong W_{af} c \) of multiplication by \( c \) on the right. We observe that \( \mu \in W_{af} c \); indeed, \( \mu = w_{\mu} c \), where \( w_{\mu} = (s_0 s_{g-1} \cdots s_1)(s_g \cdots s_2)\cdots (s_0 s_{g-1}) s_g \in W_{af} \). Observe also that the length of \( \mu \) is \( \frac{g(g+1)}{2} \). The set \( \text{Adm}_I(\mu) \) is precisely the subset of \( W_{af} c \) of elements that are \( \leq \) a translation \( t_{w_{\mu}} \) for some \( w \in W \) [43, Theorem 4.5]. Furthermore, for each \( w \in \text{Adm}_I(\mu) \), the stratum \( U_{I,w} \) has dimension \( \ell(w) \).

### 6.4. Irreducible components

#### 6.4.1. The case that \( I = \{0, \cdots , g\} \).

There are \( 2^g \) translations in the \( W \)-orbit of \( \mu \) and they correspond to the open stratum in each of the \( 2^g \) irreducible components of \( \overline{M}_I \) when \( I = \{0, \cdots , g\} \). These \( 2^g \) translations are parametrised by \( W/W_c \) where \( W_c \) is the subgroup of \( W \) of elements that stabilise \( \mu \). It identifies with the elements in \( W \subset S_{2g} \) that preserve the sets \( \{1, \cdots , g\} \) and \( \{g+1, \cdots ,2g\} \), so this group is isomorphic to \( S_g \). The quotient \( W/W_c \) has a set of representatives in \( W \), denoted \( W_c \), and called Kostant representatives. An element \( w \in W_c \) is characterised by the property that it is the element of minimal length in the coset \( w W_c \). The elements in \( W_c \) are exactly the permutations \( w \in W \) for which \( w^{-1}(g) \geq \cdots \geq w^{-1}(1) \). Such representatives are in bijection with functions \( s : \{1, \cdots , g\} \to \{1, \cdots , 2g\} \), which are increasing and take exactly once one of the values \( i,2g+1-i \) for all \( 1 \leq i \leq g \). (Just set \( s_w = w^{-1}(i) \).) Moreover, the length of an element \( w \) with the corresponding function \( s \) is \( \frac{g(g+1)}{2} - \sum_{i \in \text{Im}(s)} (g-i) + \sum_{i \in \text{Im}(s)} i \).

We can concretely determine an element of each of the orbits in \( \overline{M}_I \) corresponding to these \( t_{w_{\mu}} \) for \( I = \{0, \cdots , g\} \) as follows. The group \( \widetilde{W} \) can be viewed as a subgroup of \( \text{GSp}_{2g}(\mathbb{F}_p((t))) \), with \( W \) being represented by permutation matrices and the elements of \( X_*(\widetilde{T}) \) as diagonal matrices by \( \chi \in X_*(\widetilde{T}) \mapsto \chi(t) \). The element \( t_{\mu} \) is thereby identified with \( \text{diag}(t, \cdots , t,1, \cdots , 1) \).

We now consider the inclusion of chains

\[ t\mathcal{V}_* \subset t_{w_{\mu}}(\mathcal{V}_*) \subset \mathcal{V}_*. \]
By reduction modulo $t$ and using the identification $V_\bullet \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p = V_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p$, we deduce that $t_{w\mu}(V_\bullet)/tV_\bullet \hookrightarrow V_\bullet$ defines an $\mathbb{F}_p$ point of $M_I$, which represents the $w\mu$-orbit.

**Remark 6.1.** Let $F_\bullet \subset V_\bullet \otimes k$ be a $k$-point in the $w\mu$ orbit. Let $s : \{1, \cdots, g\} \to \{1, \ldots, 2g\}$ be the associated function. We actually get that the map

$$F_{i-1} \to F_i$$

is an isomorphism if and only if $V_{i-1}/F_{i-1} \to V_i/F_i$ has kernel and cokernel of dimension 1 and if and only if $i \in \text{Im}(s)$.

**6.4.2. The case that $I = \{0\}$.** The special fibre $\overline{M}_{\{0\}}$ of $M_{\{0\}}$ is smooth and irreducible. Moreover, there is a single orbit.

**6.4.3. The case that $I = \{0, g\}$.

**Lemma 6.2.** The special fibre $\overline{M}_{\{0, g\}}$ of $M_{\{0, g\}}$ has $g + 1$ irreducible components. There are $g + 1$ open strata for the Kottwitz-Rapoport stratification, indexed by the integers $0 \leq s \leq g$. For each $0 \leq s \leq g$, a representative of the $s$-stratum is given by taking

$$F_0(s) = F_g(s) = \langle e_{s+1}, \cdots, e_g, e_{2g-s+1}, \cdots, e_{2g} \rangle.$$

This corresponds to the element $\text{diag}(t\text{Id}_s, \text{Id}_{g-s}, t\text{Id}_{g-s}, \text{Id}_s) \in LG$.

**Proof.** It is easy to see that for all $w \in W$, there exists a $w' \in W_I$ and $0 \leq s \leq g$ such that

$$w't_{w\mu} = \text{diag}(t\text{Id}_s, \text{Id}_{g-s}, t\text{Id}_{g-s}, \text{Id}_s).$$

This simply follows from the fact that $S_g = W_I$. It is easy to see that all of the orbits corresponding to these elements are disjoint and of dimension $\frac{g(g+1)}{2}$.

**Remark 6.3.** We have the equality $\text{diag}(t\text{Id}_s, \text{Id}_{g-s}, t\text{Id}_{g-s}, \text{Id}_s) = w(s)(t\text{Id}_g, \text{Id}_g)$ where $w(s)(i) = i$ if $i \leq s$, $w(s)(i) = 2g + 1 - i$ if $s+1 \leq i \leq g$.

**6.5. Local geometry of the local model**

The local model $M_{\{0\}}$ is smooth of relative dimension $\frac{g(g+1)}{2}$. The other local models $M_I$ for $I \neq \{0\}$ and $I \neq \{g\}$ are isomorphic to $M_{\{0\}}$ over $\text{Spec } \mathbb{Z}[1/p]$ but they have singular special fibre at $p$. Nevertheless, we have the following important result, the first part due to Görtz [23, Theorem 2.1] and the second to He [33, Theorem 1.2].

**Theorem 6.4.** The local models $M_I$ are flat over $\mathbb{Z}$ and $\overline{M}_I$ is reduced. Furthermore, $M_I$ is Cohen-Macaulay.
6.6. Hecke correspondence on the affine Grassmannians

We consider the correspondence

\[ \mathcal{F}_{\{0,g\}} \xrightarrow{p_2} \mathcal{F}_{\{g\}} \xrightarrow{p_1} \mathcal{F}_{\{0\}} \]

We now pick the element \( w(s)\mu \in \tilde{W} \) and restrict this map to a map

\[ \mathcal{U}_{\{0,g\},w(s)\mu} \xrightarrow{p_2} \mathcal{U}_{\{g\},w(s)\mu} \xrightarrow{p_1} \mathcal{U}_{\{0\},w(s)\mu} \]

**Proposition 6.5.** The map on differentials \( dp_1 : p_1^* \Omega^1_{U_{\{0,g\},w(s)\mu}} \to \Omega^1_{U_{\{0\},w(s)\mu}} \) has kernel and cokernel a locally free sheaf of rank \( \frac{(g-s)(g-s+1)}{2} \).

**Proof.** We have

\[ \mathcal{U}_{\{0,g\},w(s)\mu} \simeq \mathcal{P}_{\{0,g\}} / (\mathcal{P}_{\{0,g\}} \cap t_{w(s)\mu} \mathcal{P}_{\{0,g\}} t_{w(s)\mu}^{-1}) \]

\[ \mathcal{U}_{\{0\},w(s)\mu} \simeq \mathcal{P}_{\{0\}} / (\mathcal{P}_{\{0\}} \cap t_{w(s)\mu} \mathcal{P}_{\{0\}} t_{w(s)\mu}^{-1}) \]

and \( p_1 \) is the obvious \( \mathcal{P}_{\{0,g\}} \)-equivariant projection

\[ \mathcal{P}_{\{0,g\}} / (\mathcal{P}_{\{0,g\}} \cap t_{w(s)\mu} \mathcal{P}_{\{0,g\}} t_{w(s)\mu}^{-1}) \to \mathcal{P}_{\{0\}} / (\mathcal{P}_{\{0\}} \cap t_{w(s)\mu} \mathcal{P}_{\{0\}} t_{w(s)\mu}^{-1}) \].

Because the map is \( \mathcal{P}_{\{0,g\}} \)-equivariant, it suffices to prove the claim in the tangent space at the identity.

We first determine the shape of \( \mathcal{P}_{\{0\}} \) and \( \mathcal{P}_{\{0,g\}} \). The group \( \mathcal{P}_{\{0\}} \) is the hyperspecial subgroup of \( LG \), whose \( R \)-points are \( G(R[[t]]) \). The group \( \mathcal{P}_{\{0,g\}} \) is the Siegel parabolic group

\[ \mathcal{P}_{\{0,g\}}(R) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in LG(R) \right\}, \]

where \( a,b,c,d \in M_{g \times g}(R[[t]]) \).

We now determine that \( \mathcal{P}_{\{0\}} \cap t_{w(s)\mu} \mathcal{P}_{\{0\}} t_{w(s)\mu}^{-1} \) consists of matrices with the following shape (the \( * \) have integral values, the rows and columns are of size \( s, g-s, g-s, \) and \( s \)):

\[ \begin{pmatrix} * & t* & * & t* \\ \ast & \ast & \ast & \ast \\ \ast & t* & \ast & t* \\ \ast & \ast & \ast & \ast \end{pmatrix} \]
We also determine that $\mathcal{P}_{\{0,g\}} \cap t_{w(s)\mu} \mathcal{P}_{\{0,g\}} t_{w(s)\mu}^{-1}$ consists of matrices with the following shape:

$$
\begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}.
$$

Passing to the Lie algebras, we easily see that the kernel of the map

$$
\mathcal{P}_{\{0,g\}} / (\mathcal{P}_{\{0,g\}} \cap t_{w(s)\mu} \mathcal{P}_{\{0,g\}} t_{w(s)\mu}^{-1}) \to \mathcal{P}_{\{0\}} / (\mathcal{P}_{\{0\}} \cap t_{w(s)\mu} \mathcal{P}_{\{0\}} t_{w(s)\mu}^{-1})
$$

is the set of matrices of the form

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & tA & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

with $A \in M_{g-s \times g-s}(\mathbb{F}_p)$ satisfies $K_{g-s}^t A K_{g-s} = A$ (the symplectic condition).

6.7. The Hecke correspondence on the local model

We consider the correspondence

$$
\begin{array}{ccc}
M_{\{0,g\}} & \xrightarrow{p_2} & M_{\{g\}} \\
\downarrow{p_1} & & \downarrow{p_2} \\
M_{\{0\}} & & M_{\{g\}}
\end{array}
$$

6.7.1. Sheaves on the local model. Let $X$ be a scheme and $L$ be a locally free sheaf of rank $g$ over $X$. We let $T_L = \text{Isom}_X(\mathcal{O}_X^g, L)$ be the associated torsor. We let $\omega$ be the universal trivialisation. The group $\text{GL}_g$ acts on the right by $\omega_\gamma = \omega \circ \gamma$. Let $T$ be the standard diagonal torus in $\text{GL}_g$ and let $B$ be the upper triangular Borel.

Let $X^*(T)$ be the character group of $T$. We have $X^*(T) \simeq \mathbb{Z}^g$ via $(k_1, \ldots, k_g) \mapsto [\text{diag}(t_1, \ldots, t_g) \mapsto \prod t_1^{k_1}]$ and $P^+$, the cone of dominant weights, is given by $k_1 \geq k_2 \geq \cdots \geq k_g$.

For all $\kappa \in X^+(T)$ we denote by $L_\kappa = \pi_* \mathcal{O}_{T_L}[\kappa^\vee]$ where $\pi_* \mathcal{O}_{T_L}[\kappa^\vee]$ is the subsheaf of $\pi_* \mathcal{O}_{T_L}$ of sections $f(\omega)$ (omega a trivialisation of $L$) that satisfy $f(\omega b) = \kappa^\vee(b)f(\omega)$ and $\kappa^\vee = (-k_g, \ldots, -k_1)$ and $b \in B$. This is a locally free sheaf over $X$.

We can in particular apply this construction to the sheaves $L_0 = (V_0/F_0)^\vee$ on $M_{\{0\}}$ and $L_g = (V_g/F_g)^\vee$ on $M_{\{g\}}$ to obtain sheaves $L_{0,\kappa}$ on $M_{\{0\}}$ and $L_{g,\kappa}$ on $M_{\{g\}}$.

Remark 6.6. We have isomorphisms $L_g \simeq F_g$ and $L_0 \simeq F_0$ using the pairing, but these isomorphisms are not equivalent for the action of the center of the group $G$ (there is a ‘Tate’ twist).

6.7.2. The map $L_{g,\kappa} \to L_{0,\kappa}$. The natural map $V_0 \to V_g$ induces a map $p_1^* V_0/F_0 \to p_2^* V_g/F_g$ over $M_{\{0,g\}}$ and by duality a map $p_2^* L_g \to p_1^* L_0$ that we denote by $\alpha$. The
map $\alpha$ is an isomorphism on the generic fibre of $M_{\{0,g\}}$, so it induces an isomorphism $\alpha^*: p_2^*L_{g,\kappa} \to p_1^*L_{0,\kappa}$ on the generic fibre for each $\kappa$. We now investigate the integral properties of this map.

**Lemma 6.7.** Let $\kappa = (k_1, \cdots, k_g)$. Let $0 \leq s \leq g$. Let $\xi$ be the generic point of $U_{\{0,g\}}(s)$. The map $\alpha^*$ induces a map $\alpha^*: (p_2^*L_{g,\kappa})_\xi \to p^{k_g+\cdots+k_{g-s+1}}(p_1^*L_{0,\kappa})_\xi$ over the local ring $O_{M_{0,g},\xi}$.

**Proof.** We first check that over $U_{\{0,g\}}(s)$, the map $\alpha$ has kernel and cokernel a locally free sheaf of rank $s$. Indeed, it is enough to check this at the point corresponding to $t_{w(s)\mu}$, in which case the corresponding diagram is

$$
\begin{array}{c}
\mathcal{V}_0 \\
t_{w(s)\mu}
\end{array}
\begin{array}{c}
\mathcal{V}_0 \\
\text{diag}(t_{1g},1_g)
\end{array}
\begin{array}{c}
\mathcal{V}_g \\
t_{w(s)\mu}
\end{array}
\begin{array}{c}
\mathcal{V}_g \\
\text{diag}(t_{1g},1_g)
\end{array}
$$

and our claim is simply that the map $\mathcal{V}_0/t_{w(s)\mu}\mathcal{V}_0 \to \mathcal{V}_g/t_{w(s)\mu}\mathcal{V}_g$ has kernel of rank $s$. This is obvious.

We can work over the completion $R$ of $O_{M_{0,g},\xi}$, which has uniformising element $p$. We also denote by $v$ the $p$-adic valuation on $R$ normalised by $v(p) = 1$. We fix isomorphisms $L_0 \simeq R^g$ and $L_g \simeq R^g$ such that $\alpha = \text{diag}(p_{1s,1_g-s})$ in these bases. We have

$$GL_g(R) = \coprod_{w \in S_g} Iw U(R)$$

the Iwahori decomposition with $Iw$ the matrices that are upper triangular mod $p$ and $U$ the unipotent radical of $B$. Let $f \in L_{r,\kappa}$. Then for $i \in Iw$ and $w \in S_g$,

$$\alpha^* f(iw) = f(\alpha^{-1}iw) = f(\alpha^{-1}i\alpha w w^{-1}\alpha^{-1}w) = f(\alpha^{-1}i\alpha w)\kappa^\vee(w^{-1}\alpha^{-1}w).$$

Because $\alpha^{-1}i\alpha w \in GL_g(R)$, we deduce that $v(f(\alpha^{-1}i\alpha w)\kappa^\vee(w^{-1}\alpha^{-1}w)) \geq v(\kappa^\vee(w^{-1}\alpha^{-1}w)) \geq k_g + k_{g-1} + \cdots + k_{(g-s+1)}$ for all $w$ because $k_1 \geq k_2 \geq \cdots \geq k_g$.

**6.7.3. The cohomological correspondence.** We may now construct a cohomological correspondence. By Proposition 2.6 and Theorem 6.4, we have a fundamental class $p_1^*\mathcal{O}_{M(0)} \to p_1^*\mathcal{O}_{M(0)}$. Moreover, the sheaf $p_1^*\mathcal{O}_{M(0)}$ is a CM sheaf.

There is also a map $\alpha: p_2^*L_{g,\kappa} \to p_1^*L_{0,\kappa}$ defined on the generic fibre, so that putting everything together, we have a generically defined map (the naive cohomological correspondence)

$$T^\text{naive}: p_2^*L_{g,\kappa} \to p_1^*L_{0,\kappa}.$$
Proposition 6.8. Let \( T = p^{-\inf_j \left\{ \sum_{\ell=j+1}^{g} k_{\ell} + \frac{j(j+1)}{2} \right\}} T_{\text{naive}} \). Then \( T \) is a true cohomological correspondence:

\[
T : p^g L_{g, \kappa} \rightarrow p^L L_{0, \kappa}.
\]

Proof. Because \( p^g L_{g, \kappa} \) is a CM sheaf, any generically defined map from a locally free sheaf into \( p^L L_{0, \kappa} \) is defined globally if it is defined in codimension 1. So it is enough to check that \( T \) is defined at the generic points of all components of \( M^{0, \kappa} \).

Let \( 0 \leq s \leq g \) and let \( \xi \) be the generic point of the stratum \( U^{0, \kappa} (s) \). At this point, we see that

\[
T_{\text{naive}} : (p^g L_{g, \kappa})_\xi \rightarrow p^g \sum_{\ell=g-s+1}^{g} k_{\ell} \frac{(g-s)(g-s+1)}{2} (p^L L_{0, \kappa})_\xi
\]

by combining Lemma 6.7 and Proposition 6.5.

6.7.4. Proof of Theorem 5.9. The main point of the theorem is to construct the action of \( T_{p_i} \). The action of \( S_{p_i} \) is by automorphisms (they are some generalised diamond operators), and the commutativity of the various operators is rather formal.

We fix some prime \( p_i \). We let \( K(p_i) = K \cap t_i K t_i^{-1} \) for \( t_i = \text{diag}(\varpi_i^{-1} 1_g, 1_g) \subset G(A_f) \) where \( \varpi_i \) is the finite ad`ele, which is 1 at all places different from \( p_i \) and \( p \) at \( p_i \).

We claim that there is a Hecke correspondence

\[
\tilde{\mathcal{H}}^\text{tor}_{K(p_i), \Sigma} \rightarrow \tilde{\mathcal{H}}^\text{tor}_{K, \Sigma}
\]

which extends the usual Hecke correspondence on the generic fibre. In order to construct this correspondence we use the PEL Shimura variety. We claim that there is a diagram

\[
(6.7.A)
\]

where the lower hat is simply the quotient of the top hat by the action of \( \Delta \). The right square is Cartesian by definition. We explain how to define \( \tilde{p}_2 \) in order to make the left square Cartesian. We take a totally positive element \( x_i \in F^{x,+} \) that has the property that its \( p_i \)-adic valuation is exactly 1 but that its \( p_j \)-adic valuation is 0 for all \( j \neq i \). Given a point \( (A, t, \lambda, \eta, \eta_p) \) in \( \tilde{\mathcal{H}}^\text{tor}_{K(p_i)} \) where \( \eta_p \) corresponds to a maximal totally isotropic subgroup \( H \) of \( A[p_i] \) we define \( \tilde{p}_2 ((A, t, \lambda, \eta, \eta_p)) = (A', t', \lambda', \eta') \) where \( A' = A/H, \lambda' \) and \( \eta' \) have the obvious definitions and \( \lambda' \) is defined by descending the polarisation \( x_i \lambda \) to \( A/H \).

This indeed defines a prime to \( p \) polarisation.
The vertical maps in the above diagram are étale and surjective. We note that $\tilde{p}_2$ is not canonical (because of the ambiguity in the choice of $x_i$) but that $p_2$ is canonical.

Let $\kappa = ((k_1, \sigma), \cdots, (k_g, \sigma))$ be a dominant weight for $M_\mu / \mathbb{Z}_c(G)$: $k_1, \sigma \geq \cdots \geq k_g, \sigma$, and both $k$ and $\sum_i k_i, \sigma$ have the same parity. After inverting $p_i$ there is a map (denoted $T_i$ in Subsection 4.5) $T_{p_i}^{naive}: p_i^* \nabla_{\kappa, \Sigma} \rightarrow p_i^* \nabla_{\kappa, \Sigma}$ respecting the cuspidal subsheaves and therefore inducing $p_2^* \nabla_{\kappa, \Sigma}(-D_{K, \Sigma}) \rightarrow p_1^* \nabla_{\kappa, \Sigma}(-D_{K, \Sigma})$.

We will prove that $p_1^* \nabla_{\kappa, \Sigma}$ and $p_1^* \nabla_{\kappa, \Sigma}(-D_{K, \Sigma})$ are CM sheaves over $\mathcal{O}_{E', \lambda'}$ and that

$$\sum_{\sigma \in I} \sup_{t \leq j} \left\{ \sum_{\ell=1}^{t-1} k_{\ell, \sigma} - \sum_{\ell=1}^{t+1} k_{\ell, \sigma} + k \right\} T_{p_i}^{naive}$$

is a well-defined map integrally.

Actually, once we prove that $p_1^* \nabla_{\kappa, \Sigma}$ is a CM sheaf, it will be enough to check that the map $T_{p_i}$ is defined in codimension 1. Because it is well defined in characteristic 0 and the boundary is flat over $\mathbb{Z}_p$, it will be enough to check that the map $T_{p_i}$ is defined on the interior $\mathfrak{Sh}_{K(p_i)}$ of $\mathfrak{Sh}_{K(p_i), \Sigma'}$.

The main idea is to reduce everything to local model computations. This is slightly delicate because our Shimura datum is only of abelian type, but we can reduce to working with a PEL Shimura datum.

We can pull back the cohomological correspondence $T_{p_i}^{naive}$ over $\mathbb{Q}$ to a cohomological correspondence $\tilde{T}_{p_i}^{naive}: \tilde{p}_2^* \nabla_{\kappa} \rightarrow \tilde{p}_1^* \nabla_{\kappa}$. It is enough to prove everything for the latter correspondence.

Now we have a local model diagram of correspondences

$$\begin{array}{ccc}
\mathfrak{Sh}_{K(p_i)} & \xrightarrow{\tilde{p}_1} & \tilde{\mathfrak{H}}_{K(p_i)} \\
\downarrow q_1 & & \downarrow q_2 \\
\tilde{\mathfrak{H}}_K & \xrightarrow{h} & \mathfrak{H}_K \\
\downarrow t_1 & & \downarrow t_2 \\
M_{K}^{loc} & \xrightarrow{e} & M_{K(p_i)}^{loc}
\end{array}$$

By definition, $M_{K}^{loc} = \prod_{\sigma \in I} M_{\{0\}}$ and $M_{K(p_i)}^{loc} = \prod_{\sigma \in I_i} M_{\{0, g\}} \prod_{\sigma \notin I_i} M_{\{0\}}$. The projection $t_1$ is the product of the projections $p_1: M_{\{0, g\}} \rightarrow M_{\{0\}}$ at places $\sigma \in I_i$ and the identity otherwise. The projection $t_2$ is the product of the projections $p_2: M_{\{0, g\}} \rightarrow M_{\{g\}} \simeq M_{\{0\}}$ at $\sigma \in I_i$ and the identity if $\sigma \notin I_i$. The map $h$ is the torsor of symplectic trivialisations of $\mathcal{H}_{1, dR}(A / \mathfrak{Sh}_K)$ for $A$ the universal abelian scheme. The map $f$ is the torsor of symplectic trivialisations of the chain $\mathcal{H}_{1, dR}(A / \mathfrak{Sh}_K) \rightarrow \mathcal{H}_{1, dR}(A / H / \mathfrak{Sh}_K) \rightarrow \mathcal{H}_{1, dR}(A / \mathfrak{Sh}_K)$ (i.e., isomorphisms with the chain $\prod_{\sigma \in I_i} (V_0 \rightarrow V_g \rightarrow V_0) \times \prod_{\sigma \notin I_i} (V_0 \rightarrow V_0 \rightarrow V_0)$). The maps $g$ and $e$ are given by the Hodge filtration.

This diagram is commutative, the diagonal maps are smooth and the diagonal maps going to the left are surjective, but the squares are not Cartesian! We have the following theorem.
Theorem 6.9. Let \( \bar{x} : \text{Spec}(k) \to \mathfrak{H}_{K(p)} \). Let \( \bar{y} = f(\bar{x}), \bar{z} = \bar{p}_1(\bar{y}), \bar{y}' = g(\bar{x}), \bar{z}' = t_1(\bar{y}') \). Then there are isomorphisms between the strict henselisations

\[
\mathcal{O}_{\mathfrak{H}_{K}, \bar{z}} \simeq \mathcal{O}_{M_{K}^{loc}, \bar{z}'}
\]

and

\[
\mathcal{O}_{\mathfrak{H}_{K(p), \bar{y}}} \simeq \mathcal{O}_{M_{K(p)}^{loc}, \bar{y}'}.
\]

Moreover, there is a commutative diagram between the maps on Zariski cotangent spaces at \( \bar{y}, \bar{y}' \) and \( \bar{z}, \bar{z}' \):

\[
\begin{array}{ccc}
\mathfrak{m}_{\mathcal{O}_{\mathfrak{H}_{K}, \bar{z}}} / \mathfrak{m}_{\mathcal{O}_{\mathfrak{H}_{K}, \bar{z}}}^2 & \sim & \mathfrak{m}_{\mathcal{O}_{M_{K}^{loc}, \bar{z}'}} / \mathfrak{m}_{\mathcal{O}_{M_{K}^{loc}, \bar{z}'}}^2 \\
\downarrow \text{d}\bar{p}_1 & & \downarrow \text{d}t_1 \\
\mathfrak{m}_{\mathcal{O}_{\mathfrak{H}_{K(p), \bar{y}}} / \mathfrak{m}_{\mathcal{O}_{\mathfrak{H}_{K(p), \bar{y}}}}^2} & \sim & \mathfrak{m}_{\mathcal{O}_{M_{K(p)}^{loc}, \bar{y}'} / \mathfrak{m}_{\mathcal{O}_{M_{K(p)}^{loc}, \bar{y}'}}^2}
\end{array}
\]

Proof. The first point is the main result of local model theory. The second point is an immediate consequence of Grothendieck-Messing deformation theory (see [16, Thm. 2.1] for a precise statement of this theorem).

Corollary 6.10. The map \( \mathfrak{H}_{K, \Sigma}^{tor} \to \text{Spec } \mathcal{O}_{E', \chi'} \) is smooth and the map \( \mathfrak{H}_{K(p), \Sigma}^{tor} \to \text{Spec } \mathcal{O}_{E', \chi'} \) is a CM map.

Proof. Over the interior of the moduli space, this follows from the previous theorem. The description of the integral toroidal compactification in [55] shows that the property holds everywhere.

It follows that the cohomological correspondence can be extended to a rational map from a locally free sheaf to a CM sheaf \( T^{\text{naive}}_{p_i} : p_2^* \mathcal{V}_{\kappa, \Sigma} \to p_1^! \mathcal{V}_{\kappa, \Sigma} \). In particular, this corollary implies that it is enough to work with the interior of the Shimura variety.

For any \( \mu = (\mu_1, \cdots, \mu_g) \), with \( \mu_1 \geq \cdots \geq \mu_g \), we have already defined two sheaves \( p_1^* L_{0, \mu} \) and \( p_2^* L_{g, \mu} \) over \( M_{\{0, g\}} \) and a rational map \( \alpha^* : p_2^* L_{g, \mu} \to p_1^* L_{g, \mu} \).

Let us write \( \kappa_{\sigma} = (k_1, \sigma, \cdots, k_{g, \sigma}) \) for all \( \sigma \). We define sheaves \( L_{0, \kappa_{\sigma}}, L_{g, \kappa_{\sigma}} \) and \( L_{g, \kappa} = \bigotimes_{\sigma \in I_{\ell}} L_{g, \kappa_{\sigma}} \bigotimes_{\ell = 1} \bigotimes_{\kappa_{\ell}} L_{0, \kappa_{\ell}} \) on \( M_{K}^{loc} \) and a map

\[
\beta^* = \bigotimes_{\sigma \in I_{\ell}} \alpha^* \bigotimes_{\ell} \text{Id} : t_2^* L_{g, \kappa} \to t_1^* L_{0, \kappa}.
\]

Lemma 6.11. Over \( \mathfrak{H}_{K(p_i)} \), we have a commutative diagram

\[
\begin{array}{ccc}
g^* t_2^* L_{g, \kappa} & \sim & g^* t_1^* L_{0, \kappa} \\
\downarrow & & \downarrow \\
f^* p_2^* \mathcal{V}_{\kappa, K} & \sim & f^* p_1^* \mathcal{V}_{\kappa, K}
\end{array}
\]
\textbf{Proof.} The sheaf \((V_0/L_0)^\vee\) on the \(\sigma\)-component of the local model \(M^\text{loc}_K\) corresponds to the sheaf \(\omega_{A,\sigma}\) by definition. Therefore, the sheaves \(L_{0,\kappa}\) and \(L_{g,\kappa}\) correspond to the representations of \(M_\mu\) of highest weight \((k_{1,\sigma}, \ldots, k_{g,\sigma}; -\sum_i k_{i,\sigma})_{\sigma \in I}\). There are isomorphisms of sheaves over \(\tilde{\mathfrak{h}}_K(p_i)\): \(g^*t_2^*L_{g,\kappa} \simeq f^*p_2^*V_{\kappa,K}\) and \(g^*t_1^*L_{0,\kappa} \simeq f^*p_1^*V_{\kappa,K}\) but these are not \(G\)-equivariant isomorphisms. We can make them \(G\)-equivariant as follows. Over \(M_{\{0\}} = M_{\{g\}} = G/P\) we have a \(G\)-equivariant sheaf \(\mathcal{L}\) corresponding to the similitude character of \(G\) (viewed as a \(P\)-representation). This sheaf has a trivialisation (given by the similitude character of \(G\)), but its \(G\)-equivariant structure is not trivial. There is a canonical map \(p_2^*\mathcal{L} \to p_1^*\mathcal{L}\) over \(M_{\{0\},\{g\}}\) that is multiplication by \(p^{-1}\) in the trivialisations.

We can twist \(L_{0,\kappa} = \bigotimes_{\sigma} L_{0,\kappa,\sigma}\) to \(L'_{0,\kappa} = \bigotimes_{\sigma} L_{0,\kappa,\sigma} \otimes \mathcal{L}^{k_{i,\sigma}}\) and \(L_{g,\kappa}\) to

\[
L'_{g,\kappa} = \bigotimes_{\sigma \in I, \sigma \not\in I, \kappa} L_{0,\kappa,\sigma} \otimes \mathcal{L}^{k_{i,\sigma}} \bigotimes_{\sigma \not\in I, \kappa} L_{0,\kappa,\sigma} \otimes \mathcal{L}^{k_{i,\sigma}}.
\]

Therefore, we have a commutative diagram over \(M^\text{loc}_{K(p_i)}\):

\[
t_2^*L_{g,\kappa} \xrightarrow{p^-_{\sigma \in I, \sigma \not\in I, \kappa} k_{i,\sigma}} t_1^*L_{0,\kappa}
\]

for \((\beta')^*\) the natural map coming from the \(G\)-equivariant structure. After twisting, we have a commutative diagram

\[
g^*t_2^*L_{g,\kappa} \xrightarrow{g^*\beta'} g^*t_1^*L_{0,\kappa}
\]

We can now conclude the proof of Theorem 5.9. Let \(\xi\) be a generic point of the special fibre of \(\tilde{\mathfrak{h}}_K(p_i)\). It corresponds on the local model \(M^\text{loc}_{K(p_i)}\) to a point in the stratum \(\prod_{\sigma \in I, \sigma \not\in I, \kappa} U_{\{0\}, \{g\}, \mu} \times \prod_{\sigma \not\in I, \kappa} U_{\{g\}}\).

Using the definition of \(\beta\) in terms of \(\alpha\), Lemma 6.7, Theorem 6.9 and Lemma 6.11, we deduce that on the local ring at \(\xi\), we have a map

\[
\tilde{T}_p^\text{naive} : (p_2^*V_{\kappa,\xi}) \to p^* \left( \sum_{\sigma \in I, \kappa} \left( g_{-\alpha,-\beta} g_{-\alpha,-\beta+1} + \sum_{\ell=1}^{g} k_{\ell,\sigma} + \sum_{\ell=g+2}^{g} k_{\ell,\sigma+k} \right) (p_1^*V_{\kappa}) \right)_{\xi}.
\]

We conclude using the CM property in Corollary 6.10 that we have a well-defined cohomological correspondence

\[
\tilde{T}_p := p^* \left( \sum_{\sigma \in I, \kappa} \left( \sum_{i=1}^{\kappa} k_{i,\sigma} + \sum_{\ell=g+2}^{g} k_{\ell,\sigma+k} - \frac{j(j+1)}{2} \right) \right) \tilde{T}_p^\text{naive} : p_2^*V_{\kappa} \to p_1^*V_{\kappa}.
\]
which, using the diagram (6.7.A), shows that we have a cohomological correspondence

\[ T_p, := p \sum_{\sigma \in \tau, 1 \leq j \leq g} \left\{ \frac{\sum_{j}^{2} k_{\ell, \sigma} - \sum_{j+1}^{g} k_{\ell, \sigma + k} + j(j+1)}{2} \right\} T_{p_i}^{naive} : p_2^1 V_{\kappa} \to p_1^1 V_{\kappa}. \]

7. Unitary Shimura varieties

This section is dedicated to unitary Shimura varieties.

7.1. The Shimura datum

Let \( F \) be a totally real field, and let \( L \) be a totally imaginary quadratic extension of \( F \). We denote by \( c \in \text{Gal}(L/F) \) the complex conjugation. We let \( I = \text{Hom}(F, \overline{Q}) \) and for all \( \sigma \in I \) we choose an extension \( \tau : L \to \overline{Q} \) of \( \sigma \) to \( L \). Therefore, \( \text{Hom}(L, \overline{Q}) = I \prod I \circ c \).

Let \( V \) be a \( K \) vector space of dimension \( n \), together with a hermitian form \( \langle \cdot, \cdot \rangle \). We assume that this form is not definite at at least one real place of \( F \). Let \( G \) be the reductive group over \( Q \) of similitudes of \( (V, \langle \cdot, \cdot \rangle) \). Namely:

\[ G = \{(g,c) \in \text{Res}_{L/Q} GL(V) \times \mathbb{G}_m, \langle gv, gw \rangle = c(v,w) \ \forall v,w \in V \}. \]

We have natural isomorphisms \( F \otimes_Q R = R^I \) and \( L \otimes_Q R = C^I \) given by \( x \otimes y \mapsto (\tau(x)y)_{\tau \in I} \). We let \( V_R = V \otimes Q R \oplus V_{R,\tau} \) where \( V_{R,\tau} = V \otimes F,\tau R \). We choose an isomorphism \( V_{R,\tau} \simeq C^n \) where \( L \) acts on \( C^n \) via \( \tau \) and the hermitian pairing induced on \( C^n \), denoted by \( \langle \cdot, \cdot \rangle_{\tau} \), is of signature \( (p_\tau, q_\tau) \) and is in the standard form \( \sum_{i=1}^{p_\tau} z_i \bar{z}_i - \sum_{i=p_\tau+1}^{n} z_i \bar{z}_i \).

We deduce that \( G_R = G(\prod_{\tau \in I} U(p_\tau, q_\tau)) \). The natural action of \( G_R \) on \( V_R \simeq \oplus_{\tau \in I} C^n \) gives an embedding \( G_R \subset \prod_{\tau \in I} \text{Res}_{C/R} GL_n \).

Our Shimura datum is \( (G, X) \) where \( X \) is the \( G(R) \)-orbit of the homomorphism \( h_0 : \text{Res}_{C/R} \mathbb{G}_m \to G_R \) given by \( h_0(z) = \prod_{\tau} z_{p_\tau, q_\tau} \) where \( z_{p_\tau, q_\tau} \in GL_n(C) \) is the diagonal matrix \( \text{diag}(z_1_{p_\tau, 1_{q_\tau}}) \).

The centraliser of \( h_0 \) is \( K_\infty \times R^{X, +} \) where \( K_\infty = \prod_{\tau} U(p_\tau)(R) \times U(q_\tau)(R) \) is a maximal compact subgroup and \( R^{X, +} \) is the connected component of the identity in the center of \( G(R) \).

7.2. The flag variety

The embedding \( G_R \to \prod_{\tau \in I} \text{Res}_{C/R} GL_n \) induces, after extending scalars to \( C \) and projecting \( \text{Res}_{C/R} GL_n \times \text{Spec } R \text{ Spec } C = GL_n \times GL_n \) onto the first factor, a morphism \( G_C \to (\prod_{\tau \in I} GL_n) \) of algebraic groups over \( \text{Spec } C \). We thus get an isomorphism \( G_C \to (\prod_{\tau \in I} GL_n) \times \mathbb{G}_m \) whose second component is the similitude factor. The cocharacter \( \mu_0 \) attached to \( h_0 \) is given by \( \mu_0(z) = \prod_{\tau} \text{diag}(z_1_{p_\tau}, 1_{q_\tau}) \times z \).

We deduce that a representative of \( P_\mu \) is given by the group \( (\prod_{\tau} P_{p_\tau, q_\tau}) \times \mathbb{G}_m \subset G_C \) with \( P_{p_\tau, q_\tau} \) the standard parabolic subgroup of \( GL_n \) of lower triangular matrices with blocks of size \( p_\tau \) and \( q_\tau \), with Levi \( GL_{p_\tau} \times GL_{q_\tau} \).

The Borel embedding is the map

\[ X \to FL_G, X \]
sending \( h \) to the Hodge filtration \( \text{Fil}_h = \oplus \text{Fil}_{h,\tau} \) (i.e., the subspace stabilised by \( P_\mu \)) on

\[
V_R \otimes_R \mathbb{C} = \oplus_{\tau} V_{\tau} \otimes R \mathbb{C}.
\]

We have \( V_{\tau} \otimes_R \mathbb{C} \cong \mathbb{C}^n \otimes_R \mathbb{C} \cong \oplus_{\tau} (\mathbb{C}^n \oplus \mathbb{C}^n) \) where the last map is given for each \( \tau \) by

\[
\mathbb{C}^n \otimes_R \mathbb{C} \rightarrow \mathbb{C}^n \oplus \mathbb{C}^n \\
v \otimes x \mapsto (vx, \bar{vx}).
\]

We denote by \( V_{\tau,+,} \) and \( V_{\tau,-} \) the two factors in this isomorphism. The pairing \( \langle \cdot, \cdot \rangle_\tau \) induces a perfect pairing between \( V_{\tau,+,} \) and \( V_{\tau,-} \). We can therefore think of \( \text{FL}_{G,X} \) as a product of Grassmannians parametrising for each \( \tau \) a direct summand \( \text{Fil}_\tau = \text{Fil}_{\tau,+,} \oplus \text{Fil}_{\tau,-} \subset V_{\tau,+,} \oplus V_{\tau,-} \), where \( \text{Fil}_{\tau,+,} \) has rank \( q_\tau \), \( \text{Fil}_{\tau,-} \) has rank \( p_\tau \), and they are orthogonal with each other for the pairing \( \langle \cdot, \cdot \rangle_\tau \) (therefore, \( \text{Fil}_\tau \) is determined by \( \text{Fil}_{\tau,+,} \)).

The filtration at the point \( h_0 \), is given by \( \langle e_{p_\tau}, \ldots, e_n \rangle > \langle e_{p_\tau}, \ldots, e_n \rangle \subset \mathbb{C}^n \oplus \mathbb{C}^n \) for each \( \tau \) (in the canonical basis \( e_1, \ldots, e_n \) of \( \mathbb{C}^n \)).

We now choose the diagonal maximal torus \( S \subset G_R \). Then \( S = \left( \prod_{\tau} (U(1))^n \right) \times \mathbb{G}_m / \mu_2 \) and its character group is the subgroup of \((\mathbb{Z}^n)^I \times \mathbb{Z}\) of elements \((a_1, \tau, \ldots, a_{n,\tau}; k)\) with the condition that \( \sum_{\tau} a_{i,\tau} = k \mod 2 \). We have \( U(1) \times \text{Spec} \mathbb{R} \text{Spec} \mathbb{C} = \{(z_1, z_2) \in \mathbb{G}_m \times \mathbb{G}_m, z_1 z_2 = 1\} \), and the projection on the first coordinate induces an isomorphism \( U(1) \times_R \mathbb{C} \cong \mathbb{G}_m \). We have\n
\[
S_{\mathbb{C}} \cong \left( \prod_{\tau} \mathbb{G}_m^R \right) \times \mathbb{G}_m / \mu_2 \hookrightarrow G_{\mathbb{C}} \cong \left( \prod_{\tau} \text{GL}_n \right) \times \mathbb{G}_m,
\]

and this map is given explicitly by \((\prod x_{1,\tau}, \ldots, x_{n,\tau}; t) \mapsto \prod_{\tau} \text{diag}(tx_{1,\tau}, \ldots, tx_{n,\tau}) \times t^2 \).

We choose the upper triangular Borel in \( G_\mathbb{C} \) (this choice is compatible with our conventions in Subsection 4.6) and the corresponding dominant cone in \( X^*(S_{\mathbb{C}}) \) is given by the condition \( a_1,\tau \geq \cdots \geq a_{n,\tau} \) for all \( \tau \). There is also an associated dominant cone for the Levi \( \prod_{\tau} (\text{GL}_{p_\tau} \times \text{GL}_{q_\tau}) \times \mathbb{G}_m \), which is given by the conditions \( a_1,\tau \geq \cdots \geq a_{p_\tau,\tau} \) and \( a_{p_\tau+1,\tau} \geq \cdots \geq a_{n,\tau} \) for all \( \tau \).

The (trivial) vector bundle \( V_R \otimes_R \mathbb{C} \) over \( \text{FL}_{G,X} \) is associated with a representation of \( G_\mathbb{C} \). This representation is the direct sum of the standard \( n \)-dimensional representation and its complex conjugate. For any \( \tau_0 \in I \), the direct factors \( V_{\tau,0,+,} \) and \( V_{\tau,0,-} \) correspond respectively to the representations of \( G_\mathbb{C} \) with highest weight \(((1, \cdots, 0)_{\tau_0}, (0, \cdots, 0)_{\tau \neq \tau_0}; 1)\) and \(((0, \cdots, 1)_{\tau_0}, (0, \cdots, 0)_{\tau \neq \tau_0}; 1)\).

The vector bundles \( V_{\tau,0,+,}/\text{Fil}_{\tau,0,+,} \) and \( V_{\tau,0,-}/\text{Fil}_{\tau,0,-} \) correspond respectively to the two irreducible representations of the Levi \( \prod_{\tau} (\text{GL}_{p_\tau} \times \text{GL}_{q_\tau}) \times \mathbb{G}_m \) with highest weights: \(((1, \cdots, 0)_{\tau_0}, (0, \cdots, 0)_{\tau \neq \tau_0}; 1)\) and \(((0, \cdots, 1)_{\tau_0}, (0, \cdots, 0)_{\tau \neq \tau_0}; 1)\).

### 7.3. Integral model

The reflex field \( E \) is the fixed field in \( \overline{\mathbb{Q}} \) of the subgroup of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) consisting of all elements acting trivially on the set \( \{ (p_\tau, q_\tau) \}_\tau \).

We now let \( p \) be a prime unramified in \( L \). We choose an \( \mathcal{O}_K \)-lattice \( V_Z \subset V \) and we assume that \( V_Z \otimes \mathbb{Z}(p) \) is self-dual for the pairing \( \langle , \rangle \). The choice of this lattice gives an
integral model for $G$, and this model is reductive over $\mathbb{Z}_{(p)}$. Let $K = K^pK_p \subset G(\mathbb{A}_f)$ be a compact open subgroup with $K_p$ hyperspecial. Let $\lambda$ be a finite place of $E$ over $p$.

The Shimura variety $\mathcal{SH}_K$ represents the functor on the category of noetherian $\mathcal{O}_{E,\lambda}$-algebras that sends a noetherian $\mathcal{O}_{E,\lambda}$-algebra to the set of equivalence classes of $(A,\iota,\lambda,\eta)$, where:

1. $A \to \text{Spec } R$ is an abelian scheme.
2. $\lambda : A \to A^\iota$ is a prime to $p$ quasi-polarisation.
3. $\iota : \mathcal{O}_L \to \text{End}(A) \otimes \mathbb{Z}_{(p)}$ is a homomorphism of algebras with involution.
4. $\text{Lie}(A)$ satisfies the determinant condition (see Remark 7.1).
5. $\eta$ is a $K^p$-level structure.

**Remark 7.1.** The determinant condition is the following. We let $E'$ be an extension of $E$ that contains the Galois closure of $L$. We let $\lambda'$ be a place of $E'$ above $\lambda$. It is enough to spell out the condition when $R$ is an $\mathcal{O}_{E',\lambda}$-algebra. In that situation, we have

$$\text{Lie}(A) = \bigoplus_{\tau} \text{Lie}(A)_{\tau,+} \oplus \text{Lie}(A)_{\tau,-},$$

where $\text{Lie}(A)_{\tau,+} = \text{Lie}(A) \otimes_{\mathcal{O}_L} \mathbb{R}_{\tau} R$ and $\text{Lie}(A)_{\tau,-} = \text{Lie}(A) \otimes_{\mathcal{O}_L} \mathbb{R}_{\tau,\text{oc}} R$. The condition is that $\text{Lie}(A)_{\tau,+}$ is a locally free $R$-module of rank $p_\tau$ and $\text{Lie}(A)_{\tau,-}$ is locally free of rank $q_\tau$.

### 7.4. The vector bundle dictionary

This section follows [21, Section 5.5]. We now work over $\mathcal{O}_{E',\lambda}$ as in the preceding remark. We have $\omega_A = \bigoplus_{\tau} \omega_{A,\tau,+} \oplus \omega_{A,\tau,-}$. It follows from the discussion towards the end of Subsection 7.2 that $\omega_{A,\tau_0,+}$ corresponds to the irreducible representation of $(\prod_\tau P_{\tau,q_\tau}) \times \mathbb{G}_m$ of highest weight $\kappa = ((a_1, \tau, \cdots, a_n, \tau); k)$ with $(a_1, \tau_0, \cdots, a_n, \tau_0) = (0, \cdots, 0, -1, 0, \cdots, 0)$ (with $-1$ in position $p_\tau$), $k = -1$, $(a_1, \tau, \cdots, a_n, \tau) = (0, \cdots, 0)$ if $\tau \neq \tau_0$. We deduce similarly that $\omega_{A,\tau_0,-}$ corresponds to the weight $\kappa = ((a_1, \tau, \cdots, a_n, \tau); k)$ with $(a_1, \tau_0, \cdots, a_n, \tau_0) = (0, \cdots, 0, 1, 0, \cdots, 0)$ (with 1 in position $p_\tau + 1$), $k = -1$, $(a_1, \tau, \cdots, a_n, \tau) = (0, \cdots, 0)$ if $\tau \neq \tau_0$.

We now choose integers $a_{1,\tau} \geq \cdots \geq a_{p_\tau,\tau}$ and $b_{1,\tau} \geq \cdots \geq b_{q_\tau,\tau}$ for all $\tau \in I$ and consider the representation of the Levi $M_\mu$ of the parabolic $P_\mu$ with highest weight $\kappa = ((-a_{p_\tau,\tau}, \cdots, -a_{1,\tau}, b_{1,\tau}, \cdots, b_{q_\tau,\tau}); -\sum_{\ell,\tau} a_{\ell,\tau} - \sum_{\ell,\tau} b_{\ell,\tau})$. The sheaf $\mathcal{V}_{\kappa,K}$ associated with this weight has the following description. We denote by $\pi_{\tau,+} : \mathcal{T}_{\tau,+} \to \mathcal{SH}_K$ and $\pi_{\tau,-} : \mathcal{T}_{\tau,-} \to \mathcal{SH}_K$ the GL$_{p_\tau}$ and GL$_{q_\tau}$ torsors of trivialisation of $\omega_{A,\tau,+}$ and $\omega_{A,\tau,-}$. We let $\omega_{A,\tau,+} = (\pi_{\tau,+})_* (\mathcal{O}_{\mathcal{T}_{\tau,+}})[-a_{p_\tau,\tau}, \cdots, -a_{1,\tau}]$ and $\omega_{A,\tau,-} = (\pi_{\tau,+})_* (\mathcal{O}_{\mathcal{T}_{\tau,+}})[-b_{q_\tau,\tau}, \cdots, -b_{1,\tau}]$ (where the bracket means the isotropic part for the representation of the upper triangular Borel given by the characters $(-a_{p_\tau,\tau}, \cdots, -a_{1,\tau})$ and $(-b_{q_\tau,\tau}, \cdots, -b_{1,\tau})$ respectively).
Then we find that \( \mathcal{V}_{a, K} \) is simply \( \otimes_{\tau} \omega_{A, \tau, +} \otimes \omega_{A, \tau, -} \) that is usually considered in the theory of modular forms on unitary groups (we mean that our normalisation of the central character is the usual one).

### 7.5. Dual group

Recall that we have \( G_{\mathbb{C}} \cong (\prod_{\tau} \text{GL}_n) \times \mathbb{G}_m \) with diagonal torus \( S_{\mathbb{C}} = (\prod_{\tau} \mathbb{G}_m) \times \mathbb{G}_m/\mu_2 \) embedded via \( ((x_1, \tau, \cdots, x_n, \tau); t) \mapsto \prod_{\tau} \text{diag}(tx_1, \tau, \cdots, tx_n, \tau) \times t^2 \). Moreover, our choice of positive roots is given by the upper triangular Borel. The character group \( X^*(S_{\mathbb{C}}) \) is the subgroup of \((\mathbb{Z}^n)^{t} \times \mathbb{Z}\) of elements \( (a_1, \tau, \cdots, a_n, \tau) \) with the condition that \( \sum a_i, \tau = k \) mod 2. We have \( (a_1, \tau, \cdots, a_n, \tau) \) with \( (x_1, \tau, \cdots, x_n, \tau); t) = \prod_{\tau} x_i, \tau, t^k \).

The cocharacter group \( X_*(S_{\mathbb{C}}) \) identifies with the subgroup of \((\mathbb{Q}_2^+)^{t} \times \mathbb{Q}_2^+\) of elements \((r_1, \tau, \cdots, r_n, \tau); r)\) with \( r + r_i, \tau \in \mathbb{Z}\) for all \( (i, \tau) \). To an element \((r_1, \tau, \cdots, r_n, \tau); r)\) we associate the cocharacter \( t \mapsto (t^r_1, \tau, \cdots, t^n-r, \tau); t^r) = \prod_{\tau} \text{diag}(t^{r_1}, \tau, \cdots, t^{r_n}-r, \tau) \times t^{2r} \).

The pairing between characters and cocharacters is given by

\[
\langle (a_1, \tau, \cdots, a_n, \tau); (r_1, \tau, \cdots, r_n, \tau); r) \rangle = \sum_{i, \tau} a_i, \tau r_i, \tau + r k.
\]

We have the usual identification \( \hat{G} \cong G_{\mathbb{C}} = \prod_{\tau} (\text{GL}_n) \times \mathbb{G}_m \) with standard torus \( \hat{S} = S_{\mathbb{C}} \). We use the standard pinning. The complex conjugation acts on \( \hat{S} \) by

\[
((x_1, \tau, \cdots, x_n, \tau); t) \mapsto ((x_1, \tau, \cdots, x_n, \tau); t^{-1})
\]

and on the full \( \hat{G} \) by \( (g, c) \mapsto (\Phi_N, t g^{-1} \Phi_N^{-1} c, c) \) where \( \Phi_N \) is the anti-diagonal matrix with alternating 1 and \(-1\)'s on the anti-diagonal.

### 7.6. Formulas for the minuscule coweights

We let \( \iota : E' \hookrightarrow \overline{\mathbb{Q}}_p \) be an embedding continuous for the \( \lambda \)-adic topology on the source and \( p \)-adic topology on the target. Let \( p_1, \cdots, p_m \) be the primes above \( p \) in \( F \). We assume that they all split in \( L \). We have a partition \( I = \bigsqcup I_i \) where \( I_i \) is the set of embeddings \( \tau : F \rightarrow E' \) for which \( \iota \circ \tau \) induces the \( p_i \)-adic topology. The local \( L \)-group at \( p \) is simply a product \( \prod_{i}(\text{GL}_n)^{I_i} \times \mathbb{G}_m \times \Gamma \), where \( \Gamma = \hat{\mathbb{Z}} \) is the absolute Galois group of \( \mathbb{F}_p \). It acts by permutation on each of the sets \( I_i \).

We fix \( 0 \leq i \leq m \) and \( 1 \leq j \leq n \). We denote by \( V_{j, i} \) the representation of \( \hat{G} \) whose underlying vector space is \( \otimes_{\tau \in I_i} (\Lambda^j \mathbb{C}^n)^{\vee} \), with action of \( \hat{G} = (\prod_{\tau} \text{GL}_n) \times \mathbb{G}_m \) given by the dual of the \( j \)th exterior standard action of the \( \tau \)-factor for all \( \tau \in I_i \) and the inverse scalar multiplication by \( \mathbb{G}_m \). The corresponding highest weight \( \lambda_{i,j} \) is \((a_1, \tau, \cdots, a_n, \tau); -2 - 2 I_i \) with \((a_1, \tau, \cdots, a_n, \tau) = (0, \cdots, 0, -1, \cdots, -1) \) (with \( j \) many \(-1\)'s) if \( \tau \in I_i \), \((a_1, \tau, \cdots, a_n, \tau) = (0, \cdots, 0) \) if \( \tau \in I_i \). This representation extends to a representation of the local \( L \)-group.

The corresponding cocharacter of \( G_{\mathbb{C}} \) is

\[
t \mapsto \left( \prod_{\tau \in I_i} 1_n \right) \times \left( \prod_{\tau \in I_i} \text{diag}(1_{-j}, t^{-1}1_j) \right) \times t^{-1},
\]
which is given in coordinates by \((r_1, \cdots, r_n, -\frac{1}{2})\) with \(r_1, \cdots, r_n, \tau) = (\frac{1}{2}, \cdots, \frac{1}{2} - \frac{1}{2}, \cdots, - \frac{1}{2})\) (with \(n - j\) many \(\frac{1}{2}\)s) if \(\tau \in I_i\), \((a_1, \cdots, a_n, -\frac{1}{2})\) if \(\tau \notin I_i\).

We let \(T^{\text{naive}}_{p_i, j}\) be the Hecke operator associated to \(T_{\lambda, i, j}\). We have an embedding \(G(Q_p) \hookrightarrow \prod_i GL_n(F_{p_i}) \times GL_n(F_{p_i})\), and the operator \(T^{\text{naive}}_{p_i, j}\) is associated to the double coset

\[
G(Z_p) \left( \prod_{l \neq i} 1_n \times p_l^{-1} 1_n \right) \times (\text{diag}(1_{n-j}, p_i^{-1} 1_j) \times \text{diag}(p_i^{-1} 1_{n-j}, 1_j)) G(Z_p).
\]

**Remark 7.2.** Our definition of the Hecke operators \(T^{\text{naive}}_{p_i, j}\) depends on the choice of a prime above each \(p_k\) in \(L\). This means that some symmetry has been broken. For an explanation of the use of the double class

\[
G(Z_p) \left( \prod_{l \neq i} 1_n \times p_l^{-1} 1_n \right) \times (\text{diag}(1_{n-j}, p_i^{-1} 1_j) \times \text{diag}(p_i^{-1} 1_{n-j}, 1_j)) G(Z_p)
\]

rather than its inverse, we refer to Remark 5.6.

As in Subsection 7.4, we choose integers \(a_{1, \tau} \geq \cdots \geq a_{p_r, \tau}\) and \(b_{1, \tau} \geq \cdots \geq b_{q_r, \tau}\) for all \(\tau \in I_i\) and we consider the embedding of the Levi \(M_\mu\) of the parabolic \(P_\mu\) with highest weight \(\kappa = ((-a_{p_r, \tau}, \cdots, -a_{1, \tau}, b_{1, \tau}, \cdots, b_{q_r}); -\sum_{\ell, \tau} a_{\ell, \tau} - \sum_{\ell, \tau} b_{\ell, \tau})\).

**Remark 7.3.** In the rest of this section, we work with the weight \(\kappa\), and therefore we fix a particular normalisation of the central character, as justified in Subsection 7.4. We leave it to the reader to formulate Lemma 7.4 for another choice of central character. Theorem 7.5 holds for any normalisation of the central character.

Motivated by Definition 4.8, we let \(T^{\text{naive}}_{p_i, j} = p^{(\lambda_{i, j}, \infty(\kappa, \ell)) - (\lambda_{i, j}, \rho)} T^{\text{naive}}_{p_i, j}\). We now find a formula for the coefficient \((\lambda_{i, j}, \infty(\kappa, \ell)) - (\lambda_{i, j}, \rho)\).

**Lemma 7.4.** We have \((\lambda_{i, j}, \infty(\kappa, \ell)) - (\lambda_{i, j}, \rho) =

\[
\sum_{\tau \notin I_i} \left( -\sum_{\ell = 1}^{q_r} b_{\ell, \tau} \right)
\]

\[
+ \sum_{\tau \in I_i} \sup_{0 \leq r_s \leq p_r, 0 \leq s_r \leq q_r, s_r + r_s = n - j} \left\{ -\sum_{\ell = r_s + 1}^{p_r} a_{\ell, \tau} - \sum_{\ell = q_r - s_r + 1}^{q_r} b_{\ell, \tau} - r_\tau (q_r - s_\tau) \right\}. \]

**Proof.** We observe that \(\rho = ((\frac{n-1}{2}, \cdots, \frac{1-n}{2}); 0)\), so \(\kappa + \rho\) is equal to

\[
(-a_{p_r, \tau} + \frac{n-1}{2}, \cdots, -a_{1, \tau} + \frac{n-1}{2}, p_r + 1, b_1, \tau - \frac{n-1}{2} + q_r - 1, \cdots, b_{q_r} - \frac{n-1}{2});
\]

\[-\sum_{\ell, \tau} a_{\ell, \tau} - \sum_{\ell, \tau} b_{\ell, \tau}).
\]

Now we need to compute the pairing \((\lambda_{i, j}, \infty(\kappa, \ell)) - (\lambda_{i, j}, \rho)\). The product \((\lambda_{i, j}, \infty(\kappa, \ell))\) decomposes as

\[
\sum_{\tau} \langle (\lambda_{i, j})_\tau, \infty(\kappa, \ell)(\tau) \rangle - \left( \frac{1}{2} \sum_{\ell, \tau} a_{\ell, \tau} + \sum_{\ell, \tau} b_{\ell, \tau} \right).
\]
For all $\tau \notin I_i$, we have

$$\langle (\lambda_{i,j})_\tau, \infty(\kappa,\ell)_\tau \rangle_\tau = \frac{1}{2} \left( \sum_{\ell} a_{\ell,\tau} - \sum_{\ell} b_{\ell,\tau} \right).$$

For all $\tau \in I_i$, there will be integers $0 \leq r_{\tau} \leq p_{\tau}$ and $0 \leq s_{\tau} \leq q_{\tau}$ such that $s_{\tau} + r_{\tau} = n - j$ and such that the first $n - j$ coordinates of $\infty(\kappa,\ell)_\tau$ put in the dominant form will be (possibly not in this order)

$$\{-a_{1,\tau} + \frac{n-1}{2} - p_{\tau} + 1, \ldots, -a_{r_{\tau},\tau} + \frac{n-1}{2} - p_{\tau} + r_{\tau}, b_{q_{\tau}} - \frac{n-1}{2}, b_{q_{\tau} - s_{\tau} + 1} - \frac{n-1}{2} + s_{\tau} - 1\}.$$

In this case, an easy computation shows that

$$\langle (\lambda_{i,j})_\tau, \infty(\kappa,\ell)_\tau \rangle_\tau = \frac{1}{2} \left( \sum_{\ell=1}^{r_{\tau}} a_{\ell,\tau} - \sum_{\ell=r_{\tau}+1}^{p_{\tau}} a_{\ell,\tau} + \sum_{\ell=1}^{q_{\tau}-s_{\tau}} b_{\ell,\tau} - \sum_{\ell=q_{\tau} - s_{\tau} + 1}^{q_{\tau}} b_{\ell,\tau} \right) +$$

$$- \frac{n-1}{2} r_{\tau} + \frac{r_{\tau}(r_{\tau} - 1)}{2} + (p_{\tau} - r_{\tau}) r_{\tau} + \frac{n-1}{2} s_{\tau} - \frac{s_{\tau}(s_{\tau} - 1)}{2}.$$  

Moreover, we have the pairing $\langle \lambda_{i,j}, \rho \rangle_\tau = \frac{i(n-j)}{2}$ if $\tau \in I_i$ and $\langle \lambda_{i,j}, \rho \rangle_\tau = 0$ otherwise. So we finally get that for all $\tau \in I_i$,

$$\langle (\lambda_{i,j})_\tau, \infty(\kappa,\ell)_\tau \rangle_\tau - \langle \lambda_{i,j}, \rho \rangle_\tau = \frac{1}{2} \left( \sum_{\ell=1}^{r_{\tau}} a_{\ell,\tau} - \sum_{\ell=r_{\tau}+1}^{p_{\tau}} a_{\ell,\tau} + \sum_{\ell=1}^{q_{\tau}-s_{\tau}} b_{\ell,\tau} - \sum_{\ell=q_{\tau} - s_{\tau} + 1}^{q_{\tau}} b_{\ell,\tau} \right) - r_{\tau}(q_{\tau} - s_{\tau}),$$

and for all $\tau \notin I_i$, we have

$$\langle (\lambda_{i,j})_\tau, \infty(\kappa,\ell)_\tau \rangle_\tau - \langle \lambda_{i,j}, \rho \rangle_\tau = \frac{1}{2} \left( \sum_{\ell} a_{\ell,\tau} - \sum_{\ell} b_{\ell,\tau} \right).$$

It remains to add the factor $\sum_{\tau} - \frac{1}{2} (\sum_{\ell} a_{\ell,\tau} + \sum_{\ell} b_{\ell,\tau})$ to conclude.

\hfill \square

\subsection{The main theorem for unitary groups.}

We have that $H_{p,\kappa,\ell}^{int} = \otimes_{0 \leq i \leq m} \mathbb{Z}[T_{p_{i,j}}, 0 \leq j \leq n, T_{p_{i,0}}^{-1}, T_{p_{i,n}}^{-1}]$ and the following result gives some evidence towards Conjecture 4.16.

\textbf{Theorem 7.5.} The operators $T_{p_{i,j}}, 0 \leq j \leq n, 1 \leq i \leq m$, act on $\text{R} \Gamma(\mathcal{H}_{\kappa,\Sigma}^{\text{tor}}, \mathcal{V}_{\kappa,\Sigma})$ and $\text{R} \Gamma(\mathcal{H}_{\kappa,\Sigma}^{\text{tor}}, \mathcal{V}_{\kappa,\Sigma}(-D_{\kappa,\Sigma}))$.

\textbf{Remark 7.6.} We do not know how to prove the commutativity of the operators $T_{p_{i,j}}$ in general. The main point is that although the operators $T_{p_{i,j}}$ are associated to minuscule coweights, the composition of two such operators will not in general be associated to a minuscule coweight.

\textbf{Corollary 7.7.} Conjecture 4.15 holds in this case.
Hecke operators and the coherent cohomology of Shimura varieties

Proof. Conjecture 4.15 asserts that

\[ \text{Im}(H^i(\mathcal{H}^{\text{tor}}_{K,\Sigma}, \mathcal{V}_{K,\Sigma}) \to H^i(\mathcal{H}^{\text{tor}}_{K,\Sigma}, \mathcal{V}_{K,\Sigma}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \]

is a lattice in \( H^i(\mathcal{H}^{\text{tor}}_{K,\Sigma}, \mathcal{V}_{K,\Sigma}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) stable under the action of \( \mathcal{H}^{\text{int}}_{p,\kappa,t} \). This follows from Theorem 7.5. \( \square \)

7.6.2. Example: the group \( \text{GU}(2,1) \). Let us assume that \( F = \mathbb{Q} \) and that the signature \( (p_r,q_r) = (2,1) \) for a chosen embedding \( \tau : L \to \mathbb{Q} \). Let \( p \) be a prime that splits in \( L \). We choose an embedding \( \iota : L \to \mathbb{Q}_p \) that corresponds to a prime \( p \) above \( p \). We have \( pp^* = p \). We have an isomorphism \( G_{\mathbb{Q}_p} = \text{GL}_3 \times \mathbb{G}_m \).

The spherical Hecke algebra at \( p \) is the polynomial algebra generated by the characteristic functions (identified with double cosets)

1. \( T_0^{\text{naive}} = K_p(\text{diag}(1,1,1) \times p^{-1})K_p, (T_0^{\text{naive}})^{-1} \),
2. \( T_1^{\text{naive}} = K_p(\text{diag}(1,1,1) \times p^{-1})K_p, \)
3. \( T_2^{\text{naive}} = K_p(\text{diag}(1,1,1) \times p^{-1})K_p \) and
4. \( T_3^{\text{naive}} = K_p(\text{diag}(1,1,1) \times p^{-1})K_p, (T_3^{\text{naive}})^{-1} \).

There are associated representations of the dual group \( V_j = (\Lambda^j \mathbb{C}^3)^\vee \) (and action of \( \mathbb{G}_m \) by the inverse character) for \( 0 \leq j \leq 3 \). The relation given by the Satake transform is (observe that all of these representations are minuscule)

1. \( T_0^{\text{naive}} = [V_0,\tau] \),
2. \( T_1^{\text{naive}} = [V_1,\tau] \),
3. \( T_2^{\text{naive}} = [V_2,\tau] \) and
4. \( T_3^{\text{naive}} = [V_3,\tau] \).

We now pick integers \( (k_1,k_2,k_3) \) and consider the automorphic vector bundle

\[ \text{Sym}^{k_1-k_2} \omega_{A,\tau,+} \otimes \text{det} \omega_{A,\tau,+} \otimes \text{det} \omega_{A,\tau,-} \]

It is the sheaf \( \mathcal{V}_{K,\Sigma} \) for \( \kappa = (-k_2, -k_1,k_3; -k_2-k_1-k_3) \). It follows from the construction that the sheaf \( \text{det} \mathcal{H}_{1,\text{dR}}(A)_{\tau,+} \) is trivial (viewed as a sheaf, the equivariant action is not trivial). It is therefore harmless to assume that \( k_3 \) is constant and we assume that \( k_3 = 1 \).

We conclude that

\[ \mathcal{H}^{\text{int}}_{p,\kappa,t} = \mathbb{Z}[(p^{-1}T_0^{\text{naive}})^{\pm 1}, p^{-1-\inf\{1,k_2\}}T_1^{\text{naive}}, p^{-1-\inf\{k_2,k_1+k_2\}}T_2^{\text{naive}}, (p^{-k_1-k_2}T_3^{\text{naive}})^{\pm 1}] \].

8. Local models in the linear case

In this section we will prove Theorem 7.5. As in the symplectic case, the actions of the normalised Hecke operators will be defined using the results of Section 2, in particular the construction in Example 2.11. In order to do this, we need to understand the integrality properties of the Hecke correspondences with respect to automorphic vector bundles and also the pullback maps on differentials. This will be done by using the theory of local models of Shimura varieties.
After recalling the basic facts about the local models for general linear groups, we prove two results, Proposition 8.3 (on the integrality properties of differentials) and Lemma 8.4 (on the integrality properties on automorphic bundles), which are used to construct normalised Hecke operators on the local model in Proposition 8.5. These computations imply Theorem 7.5 in the same way as in the proof of Theorem 5.9.

8.1. Definition
Let \( n,p,q \in \mathbb{Z}_{\geq 1} \) be integers such that \( n = p + q \). We now consider modules \( V_0, \ldots, V_{n-1} = \mathbb{Z}^n \) and the following chain:

\[
V_0 : V_0 \to V_1 \to \cdots \to V_{n-1} \to V_0,
\]

where the map from \( V_i \) to \( V_{i+1} \) is given in the canonical basis \( (e_1, \ldots, e_n) \) of \( \mathbb{Z}^n \) by the map \( e_j \mapsto e_j \) if \( j \neq i+1 \) and \( e_{i+1} \mapsto pe_{i+1} \). Whenever necessary, indices are taken modulo \( n \) so that \( V_n := V_0 \) and the chain \( V_0 \) can be seen as an infinite chain.

Fix a set \( \emptyset \neq I \subset \{0,1,\ldots,n-1\} \). We define the local model functor \( \mathbf{M}_I : \mathbb{Z} - \mathbf{ALG} \to \mathbf{SETS} \) that associates to an object \( R \) of \( \mathbb{Z} - \mathbf{ALG} \) the set of isomorphism classes of commutative diagrams

\[
\xymatrix{ V_{i_0} \otimes \mathbb{Z} R \ar[r] & V_{i_1} \otimes \mathbb{Z} R \ar[r] \ar[u] & \cdots \ar[r] \ar[u] & V_{i_m} \otimes \mathbb{Z} R \ar[r] & \cdots \ar[u] & F_{i_0} \ar[u] & \cdots \ar[u] & F_{i_m} \ar[u] }
\]

where \( i_0 < i_1 \cdots < i_m \) are such that \( \{i_0, \ldots, i_m\} = I \) and the modules \( F_{i_j} \) for \( 0 \leq j \leq m \) are rank \( q \) locally direct factors.

The functor \( \mathbf{M}_I \) is represented by a projective scheme \( M_I \), which is a closed subscheme of a product of Grassmannians.

When \( \emptyset \neq J \subset I \), there is an obvious map \( M_I \to M_J \) given by forgetting the modules \( F_j \) for \( j \in I \setminus J \).

8.2. The affine Grassmannian
We now consider modules \( \mathcal{V}_0, \ldots, \mathcal{V}_{n-1} = \mathbb{F}_p[[t]]^n \) and the chain

\[
\mathcal{V}_0 : \mathcal{V}_0 \to \mathcal{V}_1 \to \cdots \to \mathcal{V}_{n-1} \to \mathcal{V}_0,
\]

where the map from \( \mathcal{V}_i \) to \( \mathcal{V}_{i+1} \) is given in the canonical basis \( (e_1, \cdots, e_n) \) of \( \mathbb{F}_p[[t]]^n \) by the map \( e_j \mapsto e_j \) if \( j \neq i+1 \) and \( e_{i+1} \mapsto te_{i+1} \). Whenever necessary, indices are taken modulo \( n \) so that \( \mathcal{V}_n := \mathcal{V}_0 \) and the chain \( \mathcal{V}_0 \) is made infinite. Observe that \( \mathcal{V}_0 \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p = \mathcal{V}_0 \otimes_{\mathbb{Z}} \mathbb{F}_p \).

Let \( LG \) denote the loop group of \( \text{GL}_n \) over \( \mathbb{F}_p \). The group \( LG \) acts naturally on \( \mathcal{V}_0 \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p((T)) \) and therefore it acts on the chain \( \mathcal{V}_0 \otimes_{\mathbb{F}_p[[t]]} \mathbb{F}_p((T)) \).

Let \( \emptyset \neq I \subset \{0,1,\ldots,n-1\} \) and let \( \mathcal{V}_I \) be the subchain of \( \mathcal{V}_0 \) where we keep only the modules indexed by elements \( i \in I \) (modulo \( n \)).

We denote by \( \mathcal{P}_I \) the parahoric subgroup of \( LG \) of automorphisms of the chain \( \mathcal{V}_I \). For any \( I \), we define the flag variety as the ind-scheme \( \mathcal{F}_I := LG/\mathcal{P}_I \).
8.3. Stratification of the local model

As in the case of symplectic groups, the special fibre $\overline{M}_I$ of $M_I$ at $p$ embeds as a finite union of $P_I$ orbits in $F_I$.

We recall the description of the map $\overline{M}_I \to F_I$. Given a diagram

$$
\begin{array}{ccccccc}
V_{i_0} \otimes_{\mathbb{Z}} R & \longrightarrow & V_{i_1} \otimes_{\mathbb{Z}} R & \longrightarrow & \cdots & \longrightarrow & V_{i_m} \otimes_{\mathbb{Z}} R \\
& & & & & & \\
F_{i_0} & \longrightarrow & F_{i_1} & \longrightarrow & \cdots & \longrightarrow & F_{i_m}
\end{array}
$$

corresponding to an $R$-point of $\overline{M}_I$, we can construct a new diagram

$$
\begin{array}{ccccccc}
\mathcal{V}_{i_0} \otimes_{\mathbb{Z}} R & \longrightarrow & \mathcal{V}_{i_1} \otimes_{\mathbb{Z}} R & \longrightarrow & \cdots & \longrightarrow & \mathcal{V}_{i_m} \otimes_{\mathbb{Z}} R \\
& & & & & & \\
\mathcal{F}_{i_0} & \longrightarrow & \mathcal{F}_{i_1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_{i_m} \\
& & & & & & \\
t\mathcal{V}_{i_0} \otimes_{\mathbb{Z}} R & \longrightarrow & t\mathcal{V}_{i_1} \otimes_{\mathbb{Z}} R & \longrightarrow & \cdots & \longrightarrow & t\mathcal{V}_{i_m} \otimes_{\mathbb{Z}} R
\end{array}
$$

where all the vertical maps are inclusions and each $\mathcal{F}_{i_j}$ is determined by the property that $\mathcal{F}_{i_j} / t\mathcal{V}_{i_j} \otimes_{\mathbb{Z}} R = F_{i_j} \leftrightarrow (\mathcal{V}_{i_j} / t\mathcal{V}_{i_j}) \otimes_{\mathbb{Z}} R = V_{i_j} \otimes_{\mathbb{Z}} R$. The chain $\mathcal{F}_\bullet$ determines an $R$-point of $F_I$.

We now recall the combinatorial description of the image of $\overline{M}_I$ in $F_I$: Fix a Borel subgroup $B$ of $\text{GL}_n$ and a maximal torus $T \subset B$. This gives a base for the root datum of $\text{GL}_n$. Let $\tilde{W}$ be the extended affine Weyl group of $\text{GL}_n$. This is the semi-direct product of the Weyl group $W = S_n$ and the cocharacter group $X_*(T) = \mathbb{Z}^n$. It can be realised as a subgroup of the group of affine transformations of $\mathbb{Z}^n$ with $\mathbb{Z}^n$ acting by translation and $S_n$ by permutation of the coordinates. Let $s_i = (i,i+1)$ for $1 \leq i \leq g - 1$ and $s_0 = t_{(-1,0,\ldots,0,1)} \times (1,n)$ be the usual choice of simple reflections. We let $\mu = (1,\ldots,1,0,\ldots,0) \in \mathbb{Z}^n$ (where 1 occurs $p$ times) be the minuscule coweight corresponding to our situation.

For each $\emptyset \neq I$ as above, we let $W_I$ be the subgroup of $\tilde{W}$ generated by the simple reflexions $s_{i_1}, i \notin I$. This is a finite group. The $P_I$ orbits in $F_I$ are parametrised by the double cosets $W_I \backslash \tilde{W} / W_I$. For any $w \in \tilde{W}$ we denote the orbit corresponding to the double coset $W_I w W_I$ by $U_I w$ and the orbit closure by $X_I w$. The orbits included in $\overline{M}_I$ are parametrised by the finite subset $\text{Adm}_I(\mu)$ of $W_I \backslash \tilde{W} / W_I$ of $\mu$-admissible elements.

Whenever $J \subset I$ we have a map $M_I \to M_J$. This map is surjective and $\text{Adm}_J(\mu)$ is the image of $\text{Adm}_I(\mu)$ in $W_J \backslash \tilde{W} / W_J$.

When $I = \{0,\ldots,n-1\}$, Kottwitz and Rapoport give a description of the set $\text{Adm}_I(\mu)$. The set $\text{Adm}_I(\mu)$ is precisely the subset of $\tilde{W}$ of elements that are $\leq$ (in the Bruhat order) a translation $t_{w\mu}$ for an element $w \in W$. In particular, the orbits that are open in $\overline{M}_I$ correspond to the translations $t_{w\mu}$. 
8.4. Irreducible components

8.4.1. The case that $I = \{0, \cdots, n-1\}$. There are actually $\frac{n!}{p!q!}$ translations $w \cdot \mu$ and they correspond to the open stratum in each of the $\frac{n!}{p!q!}$ irreducible components of $M_I$ when $I = \{0, \cdots, n-1\}$.

These $\frac{n!}{p!q!}$ translations are parametrised by $W/W_c$ where $W_c$ is the subgroup of $W$ of elements that stabilise $\mu$. It identifies with the elements in $W \subset S_n$ that preserve the sets $\{1, \cdots, p\}$ and $\{p+1, \cdots, n\}$. This group is isomorphic to $S_p \times S_q$.

We can concretely determine an element of each of the orbits corresponding to these $t \cdot w \cdot \mu$ in $M_I$ for $I = \{0, \cdots, n-1\}$ as follows. The group $\tilde{W}$ can be viewed as a subgroup of $GL_n(F_t)$. The element $t \cdot w \cdot \mu$ is identified with $\text{diag}(t, \cdots, t, 1, \cdots, 1)$.

We now consider the inclusion of chains:

$$tV_\bullet \subset t_w \cdot V_\bullet \subset V_\bullet.$$  

By reduction modulo $t$ and using the identification $V_\bullet \otimes_{F_t[[t]]} F_p = V_\bullet \otimes_{\mathbb{Z}} F_p$, we deduce that $t_w \cdot V_\bullet / t \cdot V_\bullet \hookrightarrow V_\bullet$ defines an $F_p$-point of $M_I$, which represents the $w \cdot \mu$-orbit.

8.4.2. The case that $\# I = 1$. For any $j, j' \in I$, the spaces $M_{\{j\}}$ and $M_{\{j'\}}$ are canonically isomorphic. The special fibre $\overline{M}_{\{0\}}$ of $M_{\{0\}}$ is smooth and irreducible of dimension $pq$. Moreover, there is a single orbit.

8.4.3. The case that $I = \{0, j\}$.

Lemma 8.1. The special fibre $\overline{M}_{\{0, j\}}$ of $M_{\{0, j\}}$ has $\inf\{p, j\} - \sup\{0, j - q\} + 1$ irreducible components, indexed by integers $\sup\{0, j - q\} \leq r \leq \inf\{p, j\}$. For each $\sup\{0, j - q\} \leq r \leq \inf\{p, j\}$, a representative of the $r$-stratum is given by taking

$$F_0(r) = F_j(r) = \langle e_{r+1}, \cdots, e_p, e_{n-j+r+1}, \cdots, e_n \rangle.$$  

This corresponds to the element $t_w \cdot \mu = \text{diag}(t \cdot \text{Id}_r, \text{Id}_{j-r}, t \cdot \text{Id}_{p-r}, \text{Id}_{q-j+r}) \in LG$.

8.5. Local geometry of the local model

The local model $M_{\{0\}}$ is smooth of relative dimension $pq$. The other local models $M_I$ are isomorphic to $M_{\{0\}}$ over Spec $\mathbb{Z}[1/p]$ but they have singular special fibre at $p$. Nevertheless, we have the following analogue of Theorem 6.4, the first part due to Görtz [23] and the second to He [33, Theorem 1.2]

Theorem 8.2. The local models $M_I$ are flat over $\mathbb{Z}$ and $\overline{M}_I$ is reduced. Furthermore, $M_I$ is Cohen-Macaulay.
8.6. Hecke correspondence of the affine Grassmannians

We consider the correspondence

\[ \begin{array}{ccc}
\mathcal{F}_{\{0,j\}} & \xrightarrow{p_2} & \mathcal{F}_{\{j\}} \\
& p_1 & \downarrow \\
\mathcal{F}_{\{0\}} & \xrightarrow{p_1} & \mathcal{F}_{\{0\}}
\end{array} \]

We now pick the element \( w(r)\mu \in \tilde{W} \) and restrict this diagram to get maps

\[ \begin{array}{ccc}
\mathcal{U}_{\{0,j\},w(r)\mu} & \xrightarrow{p_2} & \mathcal{U}_{\{j\},w(r)\mu} \\
& p_1 & \downarrow \\
\mathcal{U}_{\{0\},w(r)\mu} & \xrightarrow{p_1} & \mathcal{U}_{\{0\},w(r)\mu}
\end{array} \]

**Proposition 8.3.** The map on differentials \( dp_1 : p_1^*\Omega^1_{\mathcal{U}_{\{0\},w(r)\mu}/\mathbb{F}_p} \rightarrow \Omega^1_{\mathcal{U}_{\{0,j\},w(r)\mu}/\mathbb{F}_p} \) has kernel and cokernel a locally free sheaf of rank \((j-r)(p-r)\).

**Proof.** We have

\[ \mathcal{U}_{\{0,j\},w(r)\mu} \simeq \mathcal{P}_{\{0,j\}}/(\mathcal{P}_{\{0,j\}} \cap t_{w(r)\mu}\mathcal{P}_{\{0,j\}} t_{w(r)\mu}^{-1}) \]

\[ \mathcal{U}_{\{0\},w(r)\mu} \simeq \mathcal{P}_{\{0\}}/(\mathcal{P}_{\{0\}} \cap t_{w(r)\mu}\mathcal{P}_{\{0\}} t_{w(r)\mu}^{-1}), \]

and \( p_1 \) is the obvious \( \mathcal{P}_{\{0,j\}} \)-equivariant projection:

\[ \mathcal{P}_{\{0,j\}}/(\mathcal{P}_{\{0,j\}} \cap t_{w(r)\mu}\mathcal{P}_{\{0,j\}} t_{w(r)\mu}^{-1}) \rightarrow \mathcal{P}_{\{0\}}/(\mathcal{P}_{\{0\}} \cap t_{w(r)\mu}\mathcal{P}_{\{0\}} t_{w(r)\mu}^{-1}). \]

Because the map is \( \mathcal{P}_{\{0,j\}} \)-equivariant, it suffices to prove the claim in the tangent space at the identity.

We first determine the shape of \( \mathcal{P}_{\{0\}} \) and \( \mathcal{P}_{\{0,j\}} \). The group \( \mathcal{P}_{\{0\}} \) is the hyperspecial subgroup of \( LG \), whose \( R \)-points are \( G(R[[t]]) \). The group \( \mathcal{P}_{\{0,j\}} \) is the parahoric group with shape

\[ \mathcal{P}_{\{0,j\}}(R) = \{ M = \begin{pmatrix} a & b \\ tc & d \end{pmatrix} \in LG(R) \}, \]

where \( a \in M_{j \times j}(R[[t]]), d \in M_{n-j \times n-j}(R[[t]]), c \in M_{n-j \times j}(R[[t]]) \) and \( d \in M_{j \times n-j}(R[[t]]) \).

One computes that \( \mathcal{P}_{\{0\}} \cap t_{w(r)\mu}\mathcal{P}_{\{0\}} t_{w(r)\mu}^{-1} \) consists of matrices of the following shape (the \( \ast \) have integral values, the rows and columns are of size \( r, j-r, p-r, \) and \( q-j+r \)):

\[ \begin{pmatrix}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast 
\end{pmatrix}. \]
One also computes that \( P_{\{0,j\} \cap t_w(r) \mu} P_{\{0,j\} t_w^{-1}(r) \mu} \) consists of matrices with the following shape:

\[
\begin{pmatrix}
* & t* & * & t*\\
* & * & * & *\\
t* & t^2* & * & t* \\
t* & t* & * & *
\end{pmatrix}.
\]

Passing to the Lie algebras, we easily see that the kernel of the map

\[
P_{\{0,j\} \cap t_w(r) \mu} P_{\{0,j\} t_w^{-1}(r) \mu} \rightarrow P_{\{0\} \cap t_w(r) \mu} P_{\{0\} t_w^{-1}(r) \mu}
\]

is the set of matrices of the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & tA & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

with \( A \in M_{p-r \times j-r}(\mathbb{F}_p) \).

\[
\square
\]

8.7. The Hecke correspondence on the local model

We consider the correspondence

\[
\begin{array}{ccc}
M_{\{0,j\}} & \xrightarrow{p_2} & M_{\{0\}} \\
M_{\{j\}} & \xleftarrow{p_1} & M_{\{0\}}
\end{array}
\]

8.7.1. Sheaves on the local model. Let \( \kappa = (a_1, \ldots, a_p, b_1, \ldots, b_q) \), with \( a_1 \geq \cdots \geq a_p \) and \( b_1 \geq \cdots \geq b_q \), be a weight. We have a locally free rank \( p \) sheaf \( (V_0/F_0)^\vee \) and a locally free rank \( q \) sheaf \( F_0 \) over \( M_0 \) and similarly over \( M_j \). We let \( L_0 = (V_0/F_0)^\vee \oplus F_0 \) and \( L_j = (V_j/F_j)^\vee \oplus F_j \). To the weight \( \kappa \) we can naturally attach locally free sheaves \( L_{0, \kappa} \) and \( L_{j, \kappa} \) using the procedure described in Subsection 6.7.1.

8.7.2. The map \( L_{j, \kappa} \rightarrow L_{0, \kappa} \). The natural map \( V_0 \rightarrow V_j \) induces a map \( V_0/F_0 \rightarrow V_j/F_j \) and by duality a map \( V_j/F_j \rightarrow V_0/F_0 \) that we denote by \( \alpha_1 \). The map \( V_j \rightarrow V_0 \) induces a map \( F_j \rightarrow F_0 \) denoted by \( \alpha_2 \). We let \( \alpha = (\alpha_1, \alpha_2) : L_j \rightarrow L_0 \). The map \( \alpha \) is an isomorphism over \( \mathbb{Z}[1/p] \) and it induces an isomorphism \( \alpha^* : p_2^* L_{j, \kappa} \rightarrow p_1^* L_{0, \kappa} \). We now investigate the integral properties of this map.

Lemma 8.4. Let \( \kappa = (k_1, \ldots, k_n) \). Let \( \sup \{0,j-q\} \leq r \leq \inf \{p,j\} \). Let \( \xi \) be the generic point of \( U_{\{0,j\}, w(r) \mu} \). The map \( \alpha^* \) induces a map

\[
\alpha^* : (p_2^* L_{j, \kappa})_\xi \rightarrow p^{a_{p-r} + \cdots + a_{p-r}} p^{b_j + \cdots + b_{j-r}} (p_1^* L_{0, \kappa})_\xi
\]

over the local ring \( \mathcal{O}_{M_{0,j}, \xi} \).
Hecke operators and the coherent cohomology of Shimura varieties

Proof. We first check that over $U_{\{0,j\},w(r)\mu}$ the map $\alpha_1$ has kernel and cokernel a locally free sheaf of rank $r$ and the map $\alpha_2$ has kernel and cokernel a locally free sheaf of rank $q - j + r$. Indeed, it is enough to check this at the point corresponding to $t_{w(r)\mu}$, in which case the corresponding diagram is

$$
\begin{array}{c}
V_0 \xrightarrow{\text{diag}(t_{1j},1_{n-j})} V_j \\
V_0 \xleftarrow{\text{diag}(t_{1j},1_{n-j})} V_j \\
V_0 \xrightarrow{\text{diag}(1_j,1_{n-j})} V_j \\
V_0 \xleftarrow{\text{diag}(1_j,1_{n-j})} V_j \\
\end{array}
$$

and our claim is simply that the map $V_0/t_{w(r)\mu} V_0 \to V_j/t_{w(r)\mu} V_j$ has kernel of rank $r$ and $t_{w(r)\mu} V_j/tV_j \to t_{w(r)\mu} V_0/tV_0$ has kernel of rank $q - j + r$. This is obvious. We now conclude the proof as in Lemma 6.7.

8.7.3. The cohomological correspondence. We may now construct a cohomological correspondence. By Proposition 2.6 and Theorem 8.2, we have a fundamental class $p^! \mathcal{O}_{M\{0\}} \to p^* \mathcal{O}_{M\{0\}}$. Moreover, the sheaf $p^! \mathcal{O}_{M\{0\}}$ is a CM sheaf.

There is also a rational map $\alpha : p^* L_{j,\kappa} \to p^! L_{0,\kappa}$, so that putting everything together, we have a rational map (the naive cohomological correspondence)

$$
T^{\text{naive}} : p^* L_{j,\kappa} \to p^! L_{0,\kappa}
$$

We may now normalise this correspondence.

Proposition 8.5. Let $T = p^! \inf \left\{ \sum_{l=p-r+1}^p a_l + \sum_{l=r-j}^q b_l + (j-r)(p-r) \right\} T^{\text{naive}}$. Then $T$ is a true cohomological correspondence:

$$
T : p^* L_{j,\kappa} \to p^! L_{0,\kappa}.
$$

Proof. Because $p^! L_{0,\kappa}$ is a CM sheaf, any rational map from a locally free sheaf into $p^! L_{0,\kappa}$ is well defined if it is well defined in codimension 1. So it is enough to check that it is well defined on the generic points of all the components in the special fibre. Let $\sup \{0,j-q\} \leq r \leq \inf \{j,p\}$. Let $\xi$ be the generic point of the stratum $U_{\{0,j\}}(r)$. We then see that $T^{\text{naive}} : (p^* L_{j,\kappa})_\xi \to p^! \sum_{l=p-r+1}^p a_l + \sum_{l=r-j}^q b_l + (j-r)(p-r) (p^! L_{0,\kappa})_\xi$ by combining Lemma 8.4 and Proposition 8.3.

8.7.4. Proof of Theorem 7.5. Given Proposition 8.5, this is very similar to the proof of the main theorem in the symplectic case (see Subsection 6.7.4) and is left to the reader.

9. Weakly regular automorphic representations and Galois representations

In this section we study a class of automorphic forms that realise in the coherent cohomology of Shimura varieties and have associated Galois representations.
9.1. Weakly regular, odd, essentially (conjugate) self-dual algebraic cuspidal automorphic representations

Let $L$ be a $CM$ or totally real field, $F$ the maximal totally real subfield of $L$ and $c$ the complex conjugation (trivial if $L = F$ is a totally real field). We let $I = \text{Hom}(F, \mathbb{C})$ and $J = \text{Hom}(L, \mathbb{C})$.

Let $\pi$ be a cuspidal automorphic representation of $GL_n/L$ (we do not assume that the central character of $\pi$ is unitary). We define below certain properties (1), (2), (3) and (4) of such representations. We will say that a $\pi$ satisfying these properties is a weakly regular, algebraic, odd, essentially (conjugate) self-dual, cuspidal automorphic representation.

(1) **Essentially (conjugate) self-dual.** We say that $\pi$ is essentially self-dual when $L$ is totally real and essentially conjugate self-dual when $L$ is $CM$ if

$$\pi^c = \pi^\vee \otimes \chi,$$

where $\chi_0: \mathbb{A}^\times_F / F^\times \rightarrow \mathbb{C}^\times$ is a character such that $\chi_0(-1_v)$ is independent of $v$ for all $v | \infty$, and $\chi = \chi_0 \circ N_{L/F} \circ \det$. We note that in the $CM$ case, the character $\chi_0$ is not unique because we can multiply $\chi_0$ by the character associated to the quadratic extension $L/F$ without changing $\chi$. This will, however, change the sign of $\chi_0(-1_v)$.

(2) **$C$-algebraic.** We say that $\pi$ is $C$-algebraic if the infinitesimal character $\lambda = (\lambda_{1,\tau}, \cdots, \lambda_{n,\tau})_{\tau \in J}$ of $\pi_\infty$ lies in $((\mathbb{Z}^n + \frac{2n-1}{2} \mathbb{Z}^n) / S_n)^J$.

(3) **Weakly regular.** We say that a $C$-algebraic $\pi$ is weakly regular if for each $\tau \in J$ there are no distinct indices $1 \leq i < j < k \leq n$ such that $\lambda_{i,\tau} = \lambda_{j,\tau} = \lambda_{k,\tau}$.\footnote{Equally, after applying a permutation in $S_n$ to the indices, we have $\lambda_{1,\tau} > \cdots > \lambda_{[n/2],\tau}$ and $\lambda_{[n/2]+1,\tau} > \cdots > \lambda_{n,\tau}$.} We say that a $C$-algebraic $\pi$ is regular if for each $\tau \in J$, the elements $\{\lambda_{i,\tau}\}_{1 \leq i \leq n}$ are all distinct.\footnote{Equally, after applying a permutation in $S_n$ we have $\lambda_{1,\tau} > \cdots > \lambda_{n,\tau}$.}

(4) **Odd.** We consider an oddness condition on a $C$-algebraic, essentially (conjugate) self dual $(\pi, \chi)$ which is given by the existence of a pole at $s = 1$ for a certain $L$-function.

If $L$ is $CM$, the oddness condition is that $L(s,\text{Asai}(-1)^{n-1} \epsilon(\chi_0)(\pi) \otimes \chi_0^{-1})$ has a pole at $s = 1$.

If $L = F$ is totally real, the oddness condition is that $L(s,\Lambda^2 \pi \otimes \chi^{-1})$ has a pole at $s = 1$ if $\epsilon(\chi_0) = 1$ and $n$ is even, and that $L(s,\text{Sym}^2 \pi \otimes \chi^{-1})$ has a pole at $s = 1$ is $\epsilon(\chi_0) = -1$ and $n$ is even or $\epsilon(\chi_0) = 1$ and $n$ is odd. We will see below that $\epsilon(\chi_0) = 1$ when $n$ is odd and $L$ is totally real.

Let us explain the definition of the above $L$-functions and of the sign $\epsilon(\chi_0)$. We first recall the definition of the Asai representation. Assume that $L$ is $CM$. Consider the group $(GL_n(\mathbb{C}) \times GL_n(\mathbb{C})) \rtimes \text{Gal}(L/F)$ with $\text{Gal}(L/F)$ acting by permutation of the factors (in other words, this is (a finite form of) the $L$-group over $F$ of $\text{Res}_{L/F} GL_n$).
The tensor product of the standard representations gives a representation $\mathbb{C}^n \otimes \mathbb{C}^n$ of $\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$. For $\epsilon \in \{-1, 1\}$, it extends to representations $\text{Asai}^\epsilon : (\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})) \rtimes \text{Gal}(L/F) \to \text{GL}(\mathbb{C}^n \otimes \mathbb{C}^n)$ by putting $\text{Asai}^\epsilon(c)(v \otimes w) = \epsilon w \otimes v$.

We now define the sign of $\chi_0$: It follows from Theorem 9.7.

**Remark 9.2.** Let $\pi$ be a cuspidal automorphic representation of $\text{GL}_n/\mathbb{Q}$, both $L$-algebraic and with trivial infinitesimal character. Assume that $(\pi_1)_\infty$ is a limit of discrete series (and therefore corresponds to a nontrivial parameter $W_R \to \text{GL}_2(\mathbb{C})$ via the local Langlands correspondence), and $(\pi_2)_\infty$ corresponds to the trivial parameter $W_\mathbb{R} \to \text{GL}_2(\mathbb{C})$. Then $\pi_1 \otimes |\det|^\frac{1}{2}$ is $C$-algebraic, arises from a weight one modular form and is odd, and $\pi_2 \otimes |\det|^\frac{1}{2}$ arises from a Maass form of Galois type and is not odd. Thus, $\pi_1$ is an automorphic form that has a realisation in the coherent cohomology of modular curves, whereas $\pi_2$ has no such realisation. We observe that we cannot distinguish $(\pi_1)_\infty$ from $(\pi_2)_\infty$ by looking at the infinitesimal character; it is the oddness property of $\pi_1$ or $\pi_2$ that distinguishes them.

**Theorem 9.1.** Let $\pi$ be a weakly regular, algebraic, essentially (conjugate) self-dual, cuspidal automorphic representation of $\text{GL}_n/L$. Then $\pi$ is odd unless possibly when $n$ is even and for all $\tau \in J$, there is an ordering of the infinitesimal character $\lambda = (\lambda_1, \tau, \ldots, \lambda_n, \tau)_{\tau \in J}$ of $\pi_\infty$ such that $\lambda_i, \tau = \lambda_i + n/2, \tau$ for all $1 \leq i \leq n/2$ and $\tau \in J$.

**Proof.** Lemma 9.5, reduces the proof of the theorem to the $CM$ case. The $CM$ case follows from Theorem 9.7.

**Remark 9.2.** Let $\pi_1$ and $\pi_2$ be two cuspidal automorphic representations of $\text{GL}_2/\mathbb{Q}$, both $L$-algebraic and with trivial infinitesimal character. Assume that $(\pi_1)_\infty$ is a limit of discrete series (and therefore corresponds to a nontrivial parameter $W_R \to \text{GL}_2(\mathbb{C})$ via the local Langlands correspondence), and $(\pi_2)_\infty$ corresponds to the trivial parameter $W_\mathbb{R} \to \text{GL}_2(\mathbb{C})$. Then $\pi_1 \otimes |\det|^\frac{1}{2}$ is $C$-algebraic, arises from a weight one modular form and is odd, and $\pi_2 \otimes |\det|^\frac{1}{2}$ arises from a Maass form of Galois type and is not odd. Thus, $\pi_1$ is an automorphic form that has a realisation in the coherent cohomology of modular curves, whereas $\pi_2$ has no such realisation. We observe that we cannot distinguish $(\pi_1)_\infty$ from $(\pi_2)_\infty$ by looking at the infinitesimal character; it is the oddness property of $\pi_1$ or $\pi_2$ that distinguishes them.

### 9.1.1. Oddness and base change

In this section we let $L$ be a $CM$ field and $F$ its maximal totally real subfield. If $\pi$ is a cuspidal automorphic representation of $\text{GL}_n/F$, we let $\text{Res}_L(\pi)$ be the base change lift of $\pi$ to an automorphic representation of $\text{GL}_n/L$ (see [1]).

**Proposition 9.3.** Let $L$ be a $CM$ field and $F$ its maximal totally real subfield. Let $\chi_{L/F}$ be the associated quadratic character. Let $\pi$ be a weakly regular, algebraic, odd, essentially self-dual cuspidal automorphic representation of $\text{GL}_n/F$. Assume that $\pi \neq \pi \otimes \chi_{L/F}$. Then $\text{Res}_L(\pi)$ is a weakly regular, algebraic, odd, essentially conjugate self-dual, cuspidal automorphic representation of $\text{GL}_n/L$.

**Proof.** This follows from Lemma 9.4 and Lemma 9.5.
Lemma 9.4. Let $L$ be a CM field and $F$ be its maximal totally real subfield. Let $\chi_{L/F}$ be the associated quadratic character. Let $\pi$ be a weakly regular, algebraic, essentially self-dual, cuspidal automorphic representation of $GL_n/F$. Assume that $\pi \neq \pi \otimes \chi_{L/F}$. Then $Res_L(\pi)$ is a weakly regular, algebraic, essentially conjugate self-dual, cuspidal automorphic representation of $GL_n/L$.

Proof. Because $\pi \neq \pi \otimes \chi_{L/F}$, we deduce from [1, Theorem 4.2] that $Res_L(\pi)$ is cuspidal. Because $Res_L(\pi) = Res_L(\pi)^e$, we deduce that $Res_L(\pi)$ is essentially conjugate self-dual. Let $v | \infty$ be a place of $F$. Let $w$ be a place of $L$ above $v$. Then $Res_L(\pi)_w$ and $\pi_v$ have the same infinitesimal character that is $C$-algebraic and weakly regular.

Lemma 9.5. Let $\pi$ be a cuspidal automorphic representation of $GL_n/F$ satisfying the assumptions of Lemma 9.4, with base change $Res_L(\pi)$. Then $\pi$ is odd if and only if $Res_L(\pi)$ is odd.

Proof. We have an $L$-group ‘diagonal’ embedding $GL_n(\mathbb{C}) \times \text{Gal}(L/F) \hookrightarrow (GL_n(\mathbb{C}) \times GL_n(\mathbb{C})) \rtimes \text{Gal}(L/F)$, where $GL_n(\mathbb{C}) \times \text{Gal}(L/F)$ is (a finite form of) the $L$-group of $GL_n/F$. We have the decomposition

$$\text{Asai}^+|_{GL_n(\mathbb{C}) \times \text{Gal}(L/F)} = \Lambda^2 \mathbb{C}^n \otimes \chi_{L/F} \bigoplus \text{Sym}^2 \mathbb{C}^n$$

and

$$\text{Asai}^-|_{GL_n(\mathbb{C}) \times \text{Gal}(L/F)} = \Lambda^2 \mathbb{C}^n \bigoplus \text{Sym}^2 \mathbb{C}^n \otimes \chi_{L/F}.$$ 

It follows that

$$L(s, \text{Asai}^{(-1)^{n-1}}(\text{Res}_L(\pi)) \otimes \chi_0^{-1}) = L(s, \Lambda^2 \pi \otimes \chi_{L/F})^{e(\chi_0)(-1)^n \otimes \chi_0^{-1}} L(s, \text{Sym}^2 \pi \otimes \chi_{L/F}^{e(\chi_0)(-1)^n+1 \otimes \chi_0^{-1}}).$$

By [59, Theorem 4.1], neither of the $L$-functions $L(s, \Lambda^2 \pi \otimes \chi_{L/K})^{e(\chi_0)(-1)^n \otimes \chi_0^{-1}}$ and $L(s, \text{Sym}^2 \pi \otimes \chi_{L/K}^{e(\chi_0)(-1)^n+1 \otimes \chi_0^{-1}})$ vanishes at $s = 1$. We therefore deduce that $Res_L(\pi)$ is odd if $\pi$ is odd. Conversely, if we assume that $Res_L(\pi)$ is odd, then at least one of $L(s, \Lambda^2 \pi \otimes \chi_{L/K}^{e(\chi_0)(-1)^n \otimes \chi_0^{-1}})$ or $L(s, \text{Sym}^2 \pi \otimes \chi_{L/K}^{e(\chi_0)(-1)^n+1 \otimes \chi_0^{-1}})$ has a pole at $s = 1$. But notice that $L(s, \pi \otimes \chi_{L/K} \otimes \chi_0^{-1})$ is holomorphic at $s = 1$ because $\pi \neq \pi \otimes \chi_{L/K}$. It is now easy to deduce that $\pi$ is odd (by examining the parity of $n$ and the sign $e(\chi_0)$).

9.1.2. Descent to a unitary group. In this section we consider a CM field $L$. We say that $\pi$ on $GL_n/L$ is conjugate self-dual (as opposed to essentially conjugate self-dual) if $\pi^c = \pi^\vee$. It is not hard to prove that if $\pi$ is essentially conjugate self-dual, there exists an algebraic Hecke character $\psi: A_L^X/L^X \rightarrow \mathbb{C}^\times$ such that $\pi \otimes \psi$ is conjugate self-dual.

Theorem 9.6. Let $L$ be a CM-field and $\pi$ a weakly regular, algebraic, odd, conjugate self-dual, cuspidal automorphic representation of $GL_n/L$. Then there exists a cuspidal automorphic representation $\tilde{\pi}$ of the quasi-split unitary group $U(n)/F$ such that $\pi$ is the transfer of $\tilde{\pi}$ for the standard base change embedding and $\tilde{\pi}_\infty$ is a nondegenerate limit of discrete series.
Proof. This follows from the main results of [51]. We give some explanations.

Let $U(n)/F$ be the quasi-split unitary group. Its $L$-group $L^1U(n)/F$ (over $F$) is isomorphic to $GL_n(\mathbb{C}) \times Gal(L/F)$ with the complex conjugation $c$ acting by $g \mapsto \Phi_n^c t g^{-1} \Phi_n^{-1}$ where $\Phi_n$ is the anti-diagonal matrix with alternating 1 and $-1$ on the anti-diagonal.

Let $v$ be a place of $F$ and let $W_{F_v}$ be the Weil group of $F_v$. First assume that $v$ does not split in $L$. The local $L$-group $L^1U(n)/F_v$ is isomorphic to $GL_n(\mathbb{C}) \times W_{F_v}$ where an element of $W_{F_v} \setminus W_{L_v}$ acts via $g \mapsto \Phi_n^c t g^{-1} \Phi_n^{-1}$. If $v$ splits, the local $L$-group $L^1U(n)/F_v$ is isomorphic to $GL_n(\mathbb{C}) \times W_{F_v}^\sigma$.

Let us denote by $G = Res_{L/F}GL_n$. The $L$-group of $G$ is isomorphic to $(GL_n(\mathbb{C}) \times GL_n(\mathbb{C})) \times Gal(L/F)$ with the complex conjugation acting by permuting the factors. We similarly have local $L$-groups $L^1G/F_v = (GL_n(\mathbb{C}) \times GL_n(\mathbb{C})) \times W_{F_v}$ if $v$ does not split and $L^1G/F_v = (GL_n(\mathbb{C}) \times GL_n(\mathbb{C})) \times W_{F_v}^\sigma$ if $v$ splits.

We have the standard base change embedding of $L$-groups $\xi : L^1U(n)/F \rightarrow L^1G/F$ that sends $g \mapsto (g, t g^{-1} \Phi_n^c, \Phi_n)$ and $1 \times c \mapsto (\Phi_n, \Phi_n^{-1}) \times c$. We have similar local versions $\xi_v : L^1U(n)/F_v \rightarrow L^1G/F_v$ for any place $v$ of $F$.

Let $v$ be a place of $F$. Denote by $L_{F_v}$ the local Langlands group of $F_v$. We let $\Phi(n)_v$ be the set of isomorphism classes of parameters $L_{F_v} \rightarrow L^1G/F_v(\mathbb{C})$ and we let $\Phi(U(n))_v$ be the set of isomorphism classes of parameters $L_{F_v} \rightarrow L^1U(n)/F_v(\mathbb{C})$.

In the case that $v$ does not split in $L$, we recall how one can identify parameters $L_{L_v} \rightarrow GL_n(\mathbb{C})$ with parameters $L_{F_v} \rightarrow L^1G/F_v(\mathbb{C})$. We choose an element $w_c \in L_{F_v} \setminus L_{L_v}$. To $\rho : L_{L_v} \rightarrow GL_n(\mathbb{C})$ we associate the parameter $\rho' : L_{F_v} \rightarrow L^1G/F_v(\mathbb{C})$ defined by $\rho'(\sigma) = (\rho(\sigma), \rho(w_c^{-1} \sigma w_c)) \times 1$ if $\sigma \in L_{L_v}$ and $\rho'(w_c) = (\rho(w_c^2), 1) \times c$.

By [51, Lemma 2.2.1], if $v$ does not split, the natural map $\xi_v^* : \Phi(U(n))_v \rightarrow \Phi(n)_v$, given by $\phi \mapsto \xi_v \circ \phi$, induces a bijection between parameters $\phi : L_{F_v} \rightarrow L^1U(n)/F_v(\mathbb{C})$ and parameters $\rho : L_{L_v} \rightarrow GL_n(\mathbb{C})$ for which there exists an invertible matrix $A$ with

1. $t \rho \circ A \rho = A$, where $\rho(\sigma) = \rho(w_c^{-1} \sigma w_c)$.
2. $t A = (-1)^{n-1} A \rho(w_c^2)$.

Indeed, to such a parameter $\rho$ we associate the parameter $\phi$ defined by $\phi(\sigma) = \rho(\sigma) \times 1$ if $\sigma \in L_{L_v}$ and $\phi(w_c) = C \times c$ where $C = A^{-1} \Phi_n$.

Now let us consider a place $v$ of $F$ that splits in $L$. Let $w$ and $\tilde{w}$ be the two places in $L$ above $v$. We have canonical isomorphisms $L_{L_w} = L_{F_v}$ and $L_{L_{\tilde{w}}} = L_{F_v}$. A parameter $\rho : L_{F_v} \rightarrow L^1G/F_v(\mathbb{C})$ can be written as $\rho(\sigma) = (\rho_w(\sigma), \rho_{\tilde{w}}(\sigma)) \times \sigma$ for all $\sigma \in L_{F_v}$, where $\rho_w : L_{L_w} \rightarrow GL_n(\mathbb{C})$ and $\rho_{\tilde{w}} : L_{L_{\tilde{w}}} \rightarrow GL_n(\mathbb{C})$.

In this case, the natural map $\xi_v^* : \Phi(U(n))_v \rightarrow \Phi(n)_v$, given by $\phi \mapsto \xi_v \circ \phi$, induces a bijection between parameters $\phi : L_{F_v} \rightarrow L^1U(n)/F_v(\mathbb{C})$ and parameters $\rho : L_{L_v} \rightarrow GL_n(\mathbb{C})$ for which there exists an invertible matrix $A$ with $t \rho \circ A \rho = A$.

For any place $v$ of $F$ there is associated to $\pi_v$ a local Langlands parameter $\phi_{\pi_v} : L_{F_v} \rightarrow L^1G/F_v(\mathbb{C})$ ([31], [34]). Because $\pi$ is conjugate self-dual and odd, it follows from [51, Theorems 2.4.10, 2.5.4] that this parameter arises from a unique parameter $\tilde{\phi}_{\pi_v} : L_{F_v} \rightarrow L^1U(n)/F_v(\mathbb{C})$. By [51, Theorem 2.5.1], there is a local packet $\Pi_v$ associated to $\tilde{\phi}_{\pi_v}$. 
By [51, Theorem 2.5.1], because $\pi$ is cuspidal automorphic, any representation $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ with $\tilde{\pi}_v \in \Pi_v$ is cuspidal automorphic on $U(n)/F$.

Let us describe in more details the parameter $\tilde{\phi}_{\pi_v}$ at a place $v | \infty$. Associated to $\pi_v$, there is a parameter

$$\rho_v : L_{L_v} = \mathbb{C}^X \to \text{GL}_n(\mathbb{C})$$

that is conjugated to

$$z \mapsto \text{diag}((z/\bar{z})^{\lambda_1, v}, \ldots, (z/\bar{z})^{\lambda_n, v}).$$

This parameter is conjugate self-dual with respect to the standard orthogonal form given by $A = \text{Id}$.

Using the recipe of [51, Lemma 2.2.1] (observe that $(-1)^{n-1} \rho_v(w_2^2) = 1$), we deduce that the parameter $\tilde{\phi}_{\pi_v}$ corresponding to $\Pi_v$ is the nondegenerate limit of discrete series parameter given by

$$L_{F_v} = W_R \to L^1 U(n)/\mathbb{R}(\mathbb{C})$$

with $\phi(z) = \text{diag}((z/\bar{z})^{\lambda_1, v}, \ldots, (z/\bar{z})^{\lambda_n, v}) \times 1$ and $\phi(j) = \Phi_n \times c$ where $j \in W_R \setminus \mathbb{C}^X$ satisfies $j^2 = -1$.

**Theorem 9.7.** Let $\pi$ be a weakly regular, algebraic, conjugate self-dual, cuspidal automorphic representation of $\text{GL}_n/L$. Let $\lambda = (\lambda_i, v)$ be its infinitesimal character. Then $\pi$ is automatically odd unless possibly when $n$ is even and for all $\tau$, $\lambda_i, \tau = \lambda_{i+n/2}, \tau$ for some ordering of the infinitesimal character.

**Proof.** This again follows from the results of [51]. If $\pi$ is not odd, we deduce that $L(s, \text{Asai}(-1)^n(\pi))$ has a pole at $s = 1$. Let us fix a character $\chi_- : \Lambda_L^X/L^X \to \mathbb{C}^X$ verifying that $\chi_-^T = \chi_+^{-1}$ and $\chi_-|A_F^X$ corresponds to the quadratic character of $L/F$. We can define a twisted $L$-group embedding $\xi_- : L^1 U(n) \to L^1 G$ (for the Weil group form of the $L$-groups) that sends $g \times 1$ to $(g, t g^{-1}) \times 1$, $1 \times \sigma$ to $(\chi_- (\sigma), \chi_-^{-1}(\sigma)) \times \sigma$ if $\sigma \in W_F$ and $1 \times c$ to $(-\Phi_n, \Phi_n^{-1}) \times w_c$ for $w_c \in W_L \setminus W_F$. A similar argument as in the proof of Theorem 9.6 shows that $\pi$ descends via the twisted $L$-group embedding to an automorphic representation of $U(n)$. For each place $v | \infty$ the parameter of $\pi_v$, $\rho_v : L_{L_v} = \mathbb{C}^X \to \text{GL}_n(\mathbb{C})$ is conjugated to $z \mapsto \text{diag}((z/\bar{z})^{\lambda_1, v}, \ldots, (z/\bar{z})^{\lambda_n, v})$. For $\pi_v$ to descend via the twisted $L$-group embedding, there should exist a nondegenerate symplectic form $A$ on $\mathbb{C}^n$ such that $t \rho_v(\bar{z}) A \rho_v(z) = A$ for all $z \in \mathbb{C}^X$. It is easy to see that there is no such symplectic form, unless when $n$ is even and $\lambda_i, v = \lambda_{i+n/2}, v$ for some ordering of the $(\lambda_i, v)$. \(\square\)

### 9.2. Automorphic Galois representations

In this section we again let $L$ be a $CM$ or totally real field. We first recall the following result concerning Galois representations attached to regular, essentially (conjugate) self-dual, cuspidal automorphic representations of $\text{GL}_n/L$ (see, e.g., [12], [3]).
Theorem 9.8 (Bellaiche, Caraiani, Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin, Taylor, ...). Let \( \pi \) be a regular, algebraic, (essentially) conjugate self-dual cuspidal automorphic representation of \( \text{GL}_n / L \). In particular, \( \pi^c = \pi^\vee \otimes \chi \) and the infinitesimal character of \( \pi \) is \( \lambda = (\lambda_{1,\tau}, \ldots, \lambda_{n,\tau})_{\tau \in J} \) with \( \lambda_{1,\tau} > \cdots > \lambda_{n,\tau} \). Let \( \iota : \mathbb{C} \cong \overline{\mathbb{Q}}_p \). There is a continuous Galois representation \( \rho_{\pi,\iota} : G_L \to \text{GL}_n(\overline{\mathbb{Q}}_p) \) such that

1. \( \rho_{\pi,\iota}^c \simeq \rho_{\pi,\iota}^\vee \otimes \epsilon_p^{1-n} \otimes \chi_\iota \) where \( \chi_\iota \) is the \( p \)-adic realisation of \( \chi \) and \( \epsilon_p \) is the cyclotomic character,

2. \( \rho_{\pi,\iota} \) is pure,

3. \( \rho_{\pi,\iota} \) is de Rham at all places dividing \( p \), with \( \iota^{-1} \circ \tau \)-Hodge-Tate weights: \( -\lambda_{n,\tau} + \frac{n-1}{2}, \ldots, -\lambda_{1,\tau} + \frac{n-1}{2} \) and

4. for all finite place \( v \) one has

\[
WD(\rho_{\pi,\iota} |_{G_{F_v}})^{F-ss} = \text{rec}(\pi_v \otimes |_{v^{1/2}}) \text{det}(1 - |_v^{1/2})].
\]

Remark 9.9. Our convention is that the reciprocity law is normalised by sending geometric Frobenius to a uniformising element. Moreover, the cyclotomic character has Hodge-Tate weight \(-1\).

In the situation that regular is replaced by the weaker assumption of being weakly regular and odd, we have the following weaker theorem.

Theorem 9.10. Let \( \pi \) be a weakly regular, algebraic, odd, (essentially) conjugate self-dual, cuspidal automorphic representation of \( \text{GL}_n / L \). In particular, \( \pi^c = \pi^\vee \otimes \chi \). There is a continuous Galois representation \( \rho_{\pi,\iota} : G_L \to \text{GL}_n(\overline{\mathbb{Q}}_p) \) such that

1. \( \rho_{\pi,\iota}^c \simeq \rho_{\pi,\iota}^\vee \otimes \epsilon_p^{1-n} \otimes \chi_\iota \) where \( \chi_\iota \) is the \( p \)-adic realisation of \( \chi \) and

2. \( \rho_{\pi,\iota} \) is unramified at all finite places \( v \) for which \( \pi_v \) is unramified and one has

\[
WD(\rho_{\pi,\iota} |_{G_{F_v}})^{F-ss} = \text{rec}(\pi_v \otimes |_{v^{1/2}}) \text{det}(1 - |_v^{1/2})].
\]

Proof. By the patching technique of [60], we may reduce to the case of a CM field. Using Theorem 9.6, there is a \( \tilde{\pi} \) on the quasi-split unitary group \( U(n) / F \), which transfers to \( \pi \). Moreover, \( \tilde{\pi}_\infty \) is a nondegenerate limit of discrete series and therefore realises in the coherent cohomology of a unitary Shimura variety. We can then apply the main result of [56] or [22] to conclude. The point is that the Hecke eigensystem of \( \tilde{\pi} \) is a \( p \)-adic limit of Hecke eigensystems of regular, essentially conjugate self-dual, automorphic representations to which Theorem 9.8 applies.

One conjectures that \( \rho_{\pi,\iota} \) in Theorem 9.10 satisfies the stronger properties of Theorem 9.8. In particular, it should be de Rham with Hodge-Tate weights \( -\lambda_{n,\tau} + \frac{n-1}{2}, \ldots, -\lambda_{1,\tau} + \frac{n-1}{2} \), for \( \lambda = ((\lambda_{1,\tau}, \ldots, \lambda_{n,\tau})_{\tau \in J} \) the infinitesimal character of \( \pi_\infty \), and \( WD(\rho_{\pi,\iota} |_{G_{F_v}})^{F-ss} = \text{rec}(\pi_v \otimes |_{v^{1/2}}) \text{det}(1 - |_v^{1/2}) \) for all finite place. This has been verified in some special cases (e.g., for weight 1 modular forms).

We can prove the following very weak instance of local–global compatibility at places dividing \( p \).
Theorem 9.11. Let $\pi$ be a weakly regular, algebraic, odd, essentially (conjugate) self-dual, cuspidal automorphic representation of $GL_n/L$ with infinitesimal character $\lambda = (\lambda_i, \tau, 1 \leq i \leq n, \tau \in \text{Hom}(L, \mathbb{Q}))$ and $\lambda_1, \tau \geq \cdots \geq \lambda_n, \tau$. Let $p$ be a prime unramified in $L$ and let $w \mid p$ be a finite place in $L$. Assume also that $\pi_w$ is spherical and corresponds to a semi-simple conjugacy class $\text{diag}(a_1, \cdots, a_n) \in GL_n(\mathbb{Q})$ by the Satake isomorphism. We let $\iota : \mathbb{Q} \to \overline{\mathbb{Q}}_p$ be an embedding and $v$ the associated $p$-adic valuation normalised by $v(p) = 1$. After permuting we assume that $v(a_1) \leq \cdots \leq v(a_n)$. Let $I_w \subset \text{Hom}(L, \mathbb{Q})$ be the set of embeddings $\tau$ such that $\iota \circ \tau$ induces the $w$-adic valuation on $L$. Then we have

$$\sum_{i=1}^k v(a_i) \geq \sum_{\tau \in I_w} \sum_{\ell=1}^k -\lambda_{\ell, \tau}$$

for $1 \leq k \leq n$, with equality if $k = n$.

Proof. We first reduce to the $CM$ case because if $L$ is totally real, we can consider a $CM$ quadratic extension $L'$ of $L$ such that $w$ splits in $L'$ and $Res_{L'}(\pi)$ is cuspidal (for example, it suffices to take $L'$ to be ramified over $L$ at some finite place $v_0$ of $L$ where $\pi_{v_0}$ is unramified). Now assume $L$ is $CM$ and let $F$ be its maximal totally real subfield. We next explain how one can reduce to the case where all primes $v \mid p$ in $F$ split in $L$. We can construct a totally real quadratic extension $F'$ of $F$ with the property that $p$ is unramified in $F'$ and for all places $w \mid p$ of $F$, $w$ splits in $F'$ if and only if $w$ splits in $L$. We may also impose the additional condition that $F'$ ramifies at some place $v_0$ of $F$ that is inert in $L$ and such that $\pi_{v_0}$ is unramified. We now set $L' = F'L$. It is clearly sufficient to prove the result for the base change of $\pi$ to $L'$. Note that this base change is still weakly regular, algebraic, odd, essentially conjugate self-dual, cuspidal. The cuspidality follows from our choice of $L'$, it being ramified at $v_0$. The oddness is another application of Shahidi’s theorem that $L(s,\text{Asai}^{-1}(\lambda_0)\chi_0^{-1} \otimes \chi_0^{-1} \otimes \chi_{L'/L})$ does not vanish at $s = 1$.

We have thus reduced to the situation that $L$ is a $CM$ field and all primes $v \mid p$ in $F$ split in $L$. It is useful to make the further restriction that there exists a quadratic imaginary extension $L_0$ of $\mathbb{Q}$ such that $L = L_0 F$. We can achieve this by considering a quadratic imaginary extension $L_0$ of $\mathbb{Q}$ such that $p$ splits in $L_0$ and ramifies at a prime $q$ such that there is a place $v_0$ of $L$ above $q$ that is unramified over $\mathbb{Q}$ and such that $\pi_{v_0}$ is spherical. We may now replace $L$ by $L_0$, which is a $CM$ field containing a quadratic imaginary field and $\pi$ by $Res_{L_0}(\pi)$.

By Theorem 9.6 there is a cuspidal automorphic representation $\tilde{\pi}$ of $U(n)/F$ that transfers to $\pi$ for the standard base change embedding and is a limit of discrete series at infinity (with infinitesimal character given by $\lambda$; see the end of the proof of Theorem 9.6). The result now essentially follows from Corollary 7.7, but there is a subtlety in that Corollary 7.7 applies to the group $GU(n)$ and not $U(n)$. We will reduce to this case.

Let $\tilde{Z}$ be the center of $GU(n)$. We observe that $\tilde{Z} \times U(n) \to GU(n)$ is a surjective map of algebraic groups ($U(n)$ is viewed as an algebraic group over $\mathbb{Q}$ by Weil restriction). Let $c$ be the central character of $\tilde{\pi}$. Let $c'$ be an extension of $c$ to an algebraic automorphic character of $Z$ that is unramified at all places dividing $p$. We claim that the cuspidal automorphic representation $\tilde{\pi}$ admits an ‘extension’ to a cuspidal automorphic representation $\tilde{\pi}'$ of $GU(n)$ with central character $c'$. We explain the meaning
of this ‘extension’. By definition, \( \tilde{\pi}' \) will realise in the space of cusp forms with central character \( c' \) on \( \mathrm{GU}(n) \), say \( \mathfrak{A}_{cusp}(\mathrm{GU}(n))_{c'} \). There is a well-defined restriction map \( \text{res} : \mathfrak{A}_{cusp}(\mathrm{GU}(n))_{c} \to \mathfrak{A}_{cusp}(U(n))_{c} \) and we say that \( \tilde{\pi}' \) extends \( \tilde{\pi} \) if \( \tilde{\pi} \) is a constituent of \( \text{res}(\tilde{\pi}') \). Observe that the very general result \cite[Theorem 4.14]{35} implies that \( \pi \) admits an extension for some choice of \( c' \). We will prove the slightly stronger result that in our case, for any choice of \( c' \) lifting \( c \), the map \( \text{res} \) is surjective. Our argument follows \cite{11}, which proves a similar result under slightly different hypotheses. We take \( \phi \in \mathfrak{A}_{cusp}(U(n))_{c} \). One first extends \( \phi \) to a function \( \phi_1 \) on \( \bar{Z}(\mathbb{A}_{Q})U(n)(\mathbb{A}_{Q}) \subset \mathrm{GU}(n)(\mathbb{A}_{Q}) \) satisfying \( \phi_1(zg) = c'(z)\phi(g) \) for all \((z,g) \in \bar{Z}(\mathbb{A}_{Q}) \times U(n)(\mathbb{A}_{Q}) \). We then extend \( \phi_1 \) to a function \( \phi_2 \) on \( \mathrm{GU}(n)(\mathbb{Q})\bar{Z}(\mathbb{A}_{Q})U(n)(\mathbb{A}_{Q}) \subset \mathrm{GU}(n)(\mathbb{A}_{Q}) \) by letting \( \phi_2(\gamma g) = \phi_1(g) \) for all \((\gamma,g) \in \mathrm{GU}(n)(\mathbb{Q}) \times \bar{Z}(\mathbb{A}_{Q})U(n)(\mathbb{A}_{Q}) \). To check that \( \phi_2 \) is well defined, it suffices to prove that \( \mathrm{GU}(n)(\mathbb{Q})\cap \bar{Z}(\mathbb{A}_{Q})U(n)(\mathbb{A}_{Q}) \subset \bar{Z}(\mathbb{Q})U(n)(\mathbb{Q}) \). This actually amounts to proving that \( \mathbb{Q}^{x} \cap \nu(\bar{Z}(\mathbb{A}_{Q})) \subseteq \nu(\bar{Z}(\mathbb{Q})) \), which follows from Hasse’s theorem that in a quadratic extension any local norm is a global norm. Finally, we claim that \( \mathrm{GU}(n)(\mathbb{Q})\bar{Z}(\mathbb{A}_{Q})U(n)(\mathbb{A}_{Q}) \) is of finite index in \( \mathrm{GU}(n)(\mathbb{A}_{Q}) \) (the quotient is dominated by \( (\mathbb{Q}^{x}N_{L_{0}/Q}\mathbb{A}_{L_{0}}^{\times}) \mathbb{A}_{Q}^{\times}) \)). We can therefore extend \( \phi_2 \) by zero to a function \( \phi_3 \) defined on \( \mathrm{GU}(n)(\mathbb{A}_{Q}) \). The verification that \( \phi_3 \in \mathfrak{A}_{cusp}(\mathrm{GU}(n))_{c'} \) is made in \cite{11}. By construction, \( \text{res}(\phi_3) = \phi \).

Because all places \( v \mid p \) in \( F \) split in \( L \), we have that \( \mathrm{GU}(n)(\mathbb{Q}_{p}) = U(n)(\mathbb{Q}_{p})Z(\mathbb{Q}_{p}) \), so we deduce that \( \tilde{\pi}'_{v} \) is spherical at all places \( v \mid p \). Moreover, the Satake parameters of \( \tilde{\pi}'_{v} \) and \( \tilde{\pi}_{v} \) are related via the map on dual groups \( \mathrm{GU}(n) = \mathbb{G}_{m} \times \mathrm{GL}_{n}^{[F:\mathbb{Q}]} \to \bar{U}(n) = \mathrm{GL}_{n}^{[F:\mathbb{Q}]} \) (which forgets the \( \mathbb{G}_{m} \)). There is a similar story at Archimedean places and we deduce that \( \tilde{\pi}'_{v} \) is a limit of discrete series. We may apply Corollary 7.7 to \( \tilde{\pi}' \), and we deduce the validity of Conjecture 4.7 in this case, which is the inequality

\[
\text{Newt}_{c}(\chi) \leq \frac{1}{[\Gamma/\text{Stab}_{\Gamma}(\infty(\kappa,\iota))]} \sum_{\gamma \in \Gamma/\text{Stab}_{\Gamma}(\infty(\kappa,\iota))} -w_{0}(\gamma \cdot \infty(\kappa,\iota)).
\]

We project this identity under the map \( \mathrm{GU}(n) \to \bar{U}(n) \) and then further via the map \( \mathrm{GL}_{n}^{[F:\mathbb{Q}]} \to \mathrm{GL}_{n}^{[F:\mathbb{Q}]} \) for a chosen prime \( v \in F \) below the prime \( w \in L \). We can now unravel the meaning of this identity.

The local \( L \)-group of \( \text{Res}_{F_{c}/Q_{p}} \mathrm{GL}_{n} \) is \( \mathrm{GL}_{n}^{[F_{c}:Q_{p}]} \times \Gamma \) where \( \Gamma = \hat{\mathbb{Z}} \) acts by permutation of the factors.

The Satake parameter of \( \pi_{w} = \tilde{\pi}_{v} \) is given by the conjugacy class of \( \text{diag}(a_{1}, \cdots, a_{n}) \), and via the Newton map it goes to

\[
\frac{1}{[F_{v} : \mathbb{Q}_{p}]} (v(a_{n}), \cdots, v(a_{1}))^{[F_{v} : \mathbb{Q}_{p}]}
\]

in \( (P_{R}^{+})^{\Gamma} \) where \( (P_{R}^{+})^{\Gamma} \) is the subset of \( \Gamma \)-invariants in the cone of dominant weights \( P_{R}^{+} \subset (\mathbb{R}^{n})^{[F_{v} : \mathbb{Q}_{p}]} \). The projection of the infinitesimal character is \( (\lambda_{i,\tau})_{\tau \in I_{w}} \) and the average \( \sum_{\gamma \in \Gamma/\text{Stab}_{\Gamma}(\infty(\kappa,\iota))} -w_{0}(\gamma \cdot \infty(\kappa,\iota)) \) gives

\[
\frac{1}{[F_{v} : \mathbb{Q}_{p}]} \left( \sum_{\tau \in I_{w}} -\lambda_{n,\tau}, \cdots, \sum_{\tau \in I_{w}} -\lambda_{1,\tau} \right)^{[F_{v} : \mathbb{Q}_{p}]}.
\]

The theorem is proven.
Acknowledgements  We thank Arvind Nair, George Boxer and Sandeep Varma for useful conversations. We also thank Olivier Taibi for directing us to the paper [44] and for many useful exchanges and suggestions. We thank the TIFR and the ENS of Lyon for hosting us during this project. The first author was supported by DAE, Government of India, UID No. RTI2005, and also thanks Laurent Fargues for supporting his visit to Lyon in 2018 during the GeoLang programme via his ERC grant. The second author was supported by the ANR Percolator and the ERC-2018-COG-818856-HiCoShiVa.

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Hecke operators and the coherent cohomology of Shimura varieties

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