BIFURCATION SETS AND GLOBAL MONODROMIES OF NEWTON NON-DEGENERATE POLYNOMIALS ON ALGEBRAIC SETS

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Abstract. Let $S \subset \mathbb{C}^n$ be a non-singular algebraic set and $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. It is well-known that the restriction $f|_S : S \to \mathbb{C}$ of $f$ on $S$ is a locally trivial fibration outside a finite set $B(f|_S) \subset \mathbb{C}$. In this paper, we give an explicit description of a finite set $T_\infty(f|_S) \subset \mathbb{C}$ such that $B(f|_S) \subset K_0(f|_S) \cup T_\infty(f|_S)$, where $K_0(f|_S)$ denotes the set of critical values of the $f|_S$. Furthermore, $T_\infty(f|_S)$ is contained in the set of critical values of certain polynomial functions provided that the $f|_S$ is Newton non-degenerate at infinity. Using these facts, we show that if $\{f_t\}_{t \in [0,1]}$ is a family of polynomials such that the Newton polyhedron at infinity of $f_t$ is independent of $t$ and the $f_t|_S$ is Newton non-degenerate at infinity, then the global monodromies of the $f_t|_S$ are all isomorphic.

1. Introduction

Let $S \subset \mathbb{C}^n$ be a non-singular algebraic set and let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. In the seventies Thom [36], Varchenko [38], Verdier [39] and Wallace [40] proved that there exists a finite set $B \subset \mathbb{C}$ such that the restriction map

$$f : S \setminus f^{-1}(B) \to \mathbb{C} \setminus B$$

is a locally trivial $C^\infty$-fibration. We call the smallest such $B$ the bifurcation set of the restriction of $f$ on $S$ and we denote it by $B(f|_S)$. This fibration permits us to introduce the global monodromy of $f|_S$. Namely, for $r > \max\{|c| : c \in B(f|_S)\}$ and $S_r^1 := \{c \in \mathbb{C} : |c| = r\}$, this is the restriction map

$$f : S \cap f^{-1}(S_r^1) \to S_r^1.$$
The problem of studying the bifurcation set and global monodromy of polynomial functions has been extensively studied in several papers: for the case \( S = \mathbb{C}^n \) we refer the reader to \([14, 17, 13, 15, 18, 23, 26, 28, 31, 35, 37] \), etc., and for the general case to \([14, 20, 21] \).

Since the restriction \( f|_S : S \to \mathbb{C} \) is not proper, the bifurcation set \( B(f|_S) \) of \( f|_S \) contains not only the set \( K_0(f|_S) \) of critical values of \( f|_S \), but also other values due to the asymptotical “bad” behaviour at infinity. To control the set \( B(f|_S) \), we use the set \( T_\infty(f|_S) \) of \textit{tangency values at infinity} of the \( f|_S \) (see the definition in Section 3). It will be shown in Section 3 that \( T_\infty(f|_S) \) is a finite set and that \( B(f|_S) \subset K_0(f|_S) \cup T_\infty(f|_S) \), which means that \( f|_S \) is a locally trivial fibration over the complement of \( K_0(f|_S) \cup T_\infty(f|_S) \). Furthermore, the set \( T_\infty(f|_S) \) is contained in the set of critical values of certain polynomial functions provided that the restriction \( f|_S \) is Newton non-degenerate at infinity. These results generalize those given in \([21]\); for related results we refer the reader to \([3, 5, 19–21, 23, 27, 41]\).

In Section 4, using the results mentioned above, we will prove a stability theorem, which states that if \( \{f_t\}_{t \in [0,1]} \) is a family of polynomial functions on \( \mathbb{C}^n \) such that the Newton polyhedron at infinity of \( f_t \) is independent of \( t \) and the restriction \( f_t|_S \) is Newton non-degenerate at infinity, then the global monodromies of the \( f_t|_S \) are all isomorphic. This generalizes \([25, \text{Theorem 17}] \) and \([32, \text{Theorem 1.1}] \), where the case \( S = \mathbb{C}^n \) was studied.

2. Notations and Definitions

In this section we present some notations and definitions, which are used throughout this paper.

2.1. Notations. We suppose \( 1 \leq n \in \mathbb{N} \) and abbreviate \((x_1, \ldots, x_n)\) by \( x \). Let \( \mathbb{K} := \mathbb{R} \) or \( \mathbb{C} \). The inner product (resp., norm) on \( \mathbb{K}^n \) is denoted by \( \langle x, y \rangle \) for any \( x, y \in \mathbb{K}^n \) (resp., \( \|x\| := \sqrt{\langle x, x \rangle} \) for any \( x \in \mathbb{K}^n \)). The real part and complex conjugate of a complex number \( c \in \mathbb{C} \) are denoted by \( \Re c \) and \( \overline{c} \), respectively.

For each \( r > 0 \), we will write \( D_r := \{ c \in \mathbb{C} : |c| < r \} \) for the open disc and write \( S_r^{2n-1} := \{ x \in \mathbb{C}^n : \|x\| = r \} \) for the sphere.

Given nonempty sets \( I \subset \{1, \ldots, n\} \) and \( A \subset \mathbb{K}^n \), we define \( A^I := \{ x \in A : x_i = 0 \text{ for all } i \notin I \} \).

Let \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) and we denote by \( \mathbb{Z}_+ \) the set of non-negative integer numbers. If \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n \), we denote by \( x^\alpha \) the monomial \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

The gradient of a polynomial function \( f : \mathbb{C}^n \to \mathbb{C} \) is denoted by \( \nabla f \) as usual, i.e.,

\[
\nabla f(x) := \left( \frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right),
\]

so the chain rule may expressed by the inner product \( \partial f/\partial \mathbf{v} = \langle \mathbf{v}, \nabla f \rangle \).
2.2. Newton polyhedra and non-degeneracy conditions. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. Suppose that $f$ is written as $f = \sum_{\alpha} a_{\alpha} x^\alpha$. Then the support of $f$, denoted by $\text{supp}(f)$, is defined as the set of those $\alpha \in \mathbb{Z}^n_+$ such that $a_{\alpha} \neq 0$. The Newton polyhedron (at infinity) of $f$, denoted by $\Gamma(f)$, is defined as the convex hull in $\mathbb{R}^n$ of the set $\text{supp}(f)$. The polynomial $f$ is said to be convenient if $\Gamma(f)$ intersects each coordinate axis in a point different from the origin $0$ in $\mathbb{R}^n$. For each (closed) face $\Delta$ of $\Gamma(f)$, we will denote by $f_\Delta$ the polynomial $\sum_{\alpha \in \Delta} a_{\alpha} x^\alpha$; if $\Delta \cap \text{supp}(f) = \emptyset$ we let $f_\Delta := 0$.

Given a nonzero vector $q \in \mathbb{R}^n$, we define

$$d(q, \Gamma(f)) := \min\{\langle q, \alpha \rangle : \alpha \in \Gamma(f)\},$$
$$\Delta(q, \Gamma(f)) := \{\alpha \in \Gamma(f) : \langle q, \alpha \rangle = d(q, \Gamma(f))\}.$$  

By definition, for each nonzero vector $q \in \mathbb{R}^n$, $\Delta(q, \Gamma(f))$ is a closed face of $\Gamma(f)$. Conversely, if $\Delta$ is a closed face of $\Gamma(f)$ then there exists a nonzero vector $q \in \mathbb{R}^n$ such that $\Delta = \Delta(q, \Gamma(f))$.

**Remark 2.1.** The following statements follow immediately from definitions:

(i) For each nonempty subset $I$ of $\{1, \ldots, n\}$, if the restriction of $f$ on $\mathbb{C}^I$ is not identically zero, then $\Gamma(f) \cap \mathbb{R}^I = \Gamma(f|_{\mathbb{C}^I})$.

(ii) Let $\Delta := \Delta(q, \Gamma(f))$ for some nonzero vector $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$. By definition, $f_\Delta = \sum_{\alpha \in \Delta} a_{\alpha} x^\alpha$ is a weighted homogeneous polynomial of type $(q, d := d(q, \Gamma(f)))$, i.e., we have for all $t > 0$ and all $x \in \mathbb{C}^n$,

$$f_\Delta(t^{q_1} x_1, \ldots, t^{q_n} x_n) = t^d f_\Delta(x_1, \ldots, x_n).$$

This implies the Euler relation

$$\sum_{i=1}^n q_i x_i \frac{\partial f_\Delta}{\partial x_i}(x) = df_\Delta(x).$$

In particular, if $d \neq 0$ and $\nabla f_\Delta(x) = 0$, then $f_\Delta(x) = 0$.

For the rest of this section, let $g_1, \ldots, g_p : \mathbb{C}^n \to \mathbb{C}$ be polynomial functions and set

$$S := \{x \in \mathbb{C}^n : g_1(x) = 0, \ldots, g_p(x) = 0\}.$$  

The following definition of non-degeneracy is inspired from the work of Kouchnirenko [22], where the case $S = \mathbb{C}^n$ was considered.

**Definition 2.1.** We say that the restriction of $f$ on $S$ is *Newton non-degenerate at infinity* if, and only if, for every nonempty set $I \subset \{1, \ldots, n\}$ with $f|_{\mathbb{C}^I} \neq 0$, for every (possibly

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1Note that we do not include the origin in the definition of the Newton polyhedron $\Gamma(f)$.

2Since $\Gamma(f)$ is an integer polyhedron, we can assume that all the coordinates of $q$ are rational numbers.
empty) set \( J \subset \{ j \in \{1, \ldots, p \} : g_j|_{C^I} \neq 0 \} \), and for every vector \( q \in \mathbb{R}^n \) with \( \min_{i \in I} q_i < 0 \), the following conditions hold:

(i) the set

\[ \{ x \in \mathbb{C}^{*n} : g_{j,\Delta_j}(x) = 0 \text{ for } j \in J \} \]

is a reduced smooth complete intersection variety in the torus \( \mathbb{C}^{*n} \), i.e., the system of gradient vectors \( \nabla g_j, \Delta_j(x) \) for \( j \in J \) is \( \mathbb{C} \)-linearly independent on this variety;

(ii) if \( d(q, \Gamma(f|_{C^I})) < 0 \), then the set

\[ \{ x \in \mathbb{C}^{*n} : f_{\Delta_0}(x) = 0 \text{ and } g_{j,\Delta_j}(x) = 0 \text{ for } j \in J \} \]

is a reduced smooth complete intersection variety in the torus \( \mathbb{C}^{*n} \);

where \( \Delta_0 := \Delta(q, \Gamma(f|_{C^I})) \) and \( \Delta_j := \Delta(q, \Gamma(g_j|_{C^I})) \) for \( j \in J \).

Finally, following [24], we introduce a set, which plays an important role in the sequel. Namely, let \( \Sigma_\infty(f|_S) \) denote the set of all values \( c \in \mathbb{C} \) for which there exist a nonempty set \( I \subset \{1, \ldots, n\} \) with \( f|_{C^I} \neq 0 \), a (possibly empty) set \( J \subset \{ j \in \{1, \ldots, p\} : g_j|_{C^I} \neq 0 \} \), a vector \( q \in \mathbb{R}^n \) with \( \min_{i \in I} q_i < 0 \) and \( d(q, \Gamma(f|_{C^I})) = 0 \), a point \( x \in \mathbb{C}^{*I} \), and scalars \( \lambda_j \in \mathbb{C} \) for \( j \in J \), such that the following conditions hold:

\[
\begin{align*}
c &= f_{\Delta_0}(x), \\
g_{j,\Delta_j}(x) &= 0 \text{ for } j \in J, \\
\nabla f_{\Delta_0}(x) + \sum_{j \in J} \lambda_j \nabla g_{j,\Delta_j}(x) &= 0,
\end{align*}
\]

where \( \Delta_0 := \Delta(q, \Gamma(f|_{C^I})) \) and \( \Delta_j := \Delta(q, \Gamma(g_j|_{C^I})) \) for \( j \in J \).

We observe that the above value \( c \in \Sigma_\infty(f|_S) \) is indeed a critical value of the restriction of the polynomial \( f_{\Delta_0} \) on the variety

\[ \{ x \in \mathbb{C}^{*I} : g_{j,\Delta_j}(x) = 0 \text{ for } j \in J \} \].

Hence, by the Bertini–Sard theorem, \( \Sigma_\infty(f|_S) \) is a finite set provided that the restriction \( f|_S \) is Newton non-degenerate at infinity.

3. The bifurcation set of a polynomial function

From now on, let \( g_1, \ldots, g_p : \mathbb{C}^n \to \mathbb{C} \) be polynomial functions such that the algebraic set

\[ S := \{ x \in \mathbb{C}^n : g_1(x) = 0, \ldots, g_p(x) = 0 \} \]

is a reduced smooth complete intersection variety, i.e., the system of gradient vectors

\[ \nabla g_1(x), \ldots, \nabla g_p(x) \]

is \( \mathbb{C} \)-linearly independent for all \( x \in S \).
Lemma 3.1. There exists a real number $R_0 > 0$ such that for all $R \geq R_0$, the set $S$ intersects transversally with the sphere $S^2_{R}$.

Proof. We argue by contradiction. Suppose that there exist sequences $\{x^k\}_{k \in \mathbb{N}} \subset \mathbb{C}^n$ and $\{\lambda_j^k\}_{k \in \mathbb{N}} \subset \mathbb{C}, j = 1, \ldots, p + 1$, such that

(a1) $\|x^k\| \to \infty$ as $k \to \infty$;
(a2) $g_j(x^k) = 0$ for all $j = 1, \ldots, p$, and all $k \in \mathbb{N}$;
(a3) $\sum_{j=1}^{p} \lambda_j^k \nabla g_j(x^k) = \lambda_{p+1}^k x^k$;
(a4) The numbers $\lambda_j^k, j = 1, \ldots, p + 1$, are not all zero for all $k \in \mathbb{N}$.

By the Curve Selection Lemma at infinity (see [25] or [17]), there exist analytic curves $\phi: (0, \epsilon) \to \mathbb{C}^n$ and $\lambda_j: (0, \epsilon) \to \mathbb{C}, j = 1, \ldots, p + 1$, such that

(a5) $\|\phi(s)\| \to \infty$ as $s \to 0$;
(a6) $g_j(\phi(s)) = 0$ for all $j = 1, \ldots, p$, and all $s \in (0, \epsilon)$;
(a7) $\sum_{j=1}^{p} \lambda_j(s) \nabla g_j(\phi(s)) = \lambda_{p+1}(s) \phi(s)$ for $s \in (0, \epsilon)$;
(a8) $\lambda_j(s), j = 1, \ldots, p + 1$, are not all zero for $s \in (0, \epsilon)$.

We have

$$\left\langle \frac{d\phi(s)}{ds}, \sum_{j=1}^{p} \lambda_j(s) \nabla g_j(\phi(s)) \right\rangle = \sum_{j=1}^{p} \lambda_j(s) \left\langle \frac{d\phi(s)}{ds}, \nabla g_j(\phi(s)) \right\rangle = \sum_{j=1}^{p} \lambda_j(s) \frac{d}{ds} (g_j \circ \phi)(s) = 0.$$

Combined with the condition (a7), this implies that

$$0 = \lambda_{p+1}(s) \Re \left\langle \frac{d\phi(s)}{ds}, \phi(s) \right\rangle = \lambda_{p+1}(s) \frac{d\|\phi(s)\|^2}{2ds}.$$

But $\lambda_{p+1} \not\equiv 0$, which follows from the non-singularity of $S$ and the condition (a7). Hence,

$$\frac{d\|\phi(s)\|^2}{ds} = 0$$

for all $s > 0$ small enough, which contradicts the condition (a5). \qed

For the rest of this section, let $f: \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. It is well known that the bifurcation set $B(f|_S)$ of the restriction $f|_S: S \to \mathbb{C}$ contains the set $K_0(f|_S)$. Recall
that we write \( K_0(f|_S) \) for the set of critical values of the restriction of \( f \) on \( S \), i.e.,

\[
K_0(f|_S) := \{ c \in \mathbb{C} : \exists x \in S, \exists \lambda_j \in \mathbb{C}, j = 1, \ldots, p, \text{ such that } f(x) = c \text{ and } \nabla f(x) + \sum_{j=1}^{p} \lambda_j \nabla g_j(x) = 0 \}.
\]

By the Bertini–Sard theorem, \( K_0(f|_S) \) is a finite set.

Before formulating our first theorem, we also need the following concept (see also [17, Chapter 2]).

**Definition 3.1.** By the set of tangency values at infinity of the \( f|_S \) we mean the set

\[
T_\infty(f|_S) := \{ c \in \mathbb{C} : \exists \{x^k\} \subset S, \exists \{\lambda^k_j\} \subset \mathbb{C}, j = 1, \ldots, p + 1, \|x^k\| \to \infty, f(x^k) \to c, \nabla f(x^k) + \sum_{j=1}^{p} \lambda^k_j \nabla g_j(x^k) = \lambda^k_{p+1} x^k \text{ for all } k \in \mathbb{N} \}.
\]

Notice that for \( S = \mathbb{C}^n \) the set \( T_\infty(f|_S) \) coincides to the set \( S_f \) defined by Nemethi and Zaharia [24].

**Theorem 3.1.** \( T_\infty(f|_S) \) is a finite set and the following inclusion holds

\[
B(f|_S) \subset K_\infty(f|_S) \cup T_\infty(f|_S).
\] (1)

**Proof.** In order to prove the set \( T_\infty(f|_S) \) is finite, we use the set of asymptotic critical values at infinity of \( f|_S \) (see [20,21,23,33]):

\[
K_\infty(f, S) := \{ c \in \mathbb{C} : \exists \{x^k\} \subset S, \|x^k\| \to \infty, f(x^k) \to c, \|x^k\| \nu(x^k) \to 0 \text{ as } k \to \infty \},
\]

where \( \nu : \mathbb{C}^n \to \mathbb{R} \) is the Rabier function defined by

\[
\nu(x) := \inf \left\{ \left\| \nabla f(x) + \sum_{j=1}^{p} \lambda_j \nabla g_j(x) \right\| : \lambda_j \in \mathbb{C}, j = 1, \ldots, p \right\}.
\]

We will show that

\[
T_\infty(f|_S) \subset K_\infty(f, S).
\] (2)

This, of course, implies immediately that \( T_\infty(f|_S) \) is a finite set because we know from [21, Theorem 3.3] that \( K_\infty(f|_S) \) is a finite set.

In order to prove the inclusion (2), take any \( c \in T_\infty(f|_S) \). By definition, there exist sequences \( \{x^k\}_{k \in \mathbb{N}} \subset \mathbb{C}^n \) and \( \{\lambda^k_j\}_{k \in \mathbb{N}} \subset \mathbb{C}, j = 1, \ldots, p + 1, \) such that

(a1) \( \|x^k\| \to \infty \) as \( k \to \infty \);
(a2) \( f(x^k) \to c \) as \( k \to \infty \);
(a3) \( g_j(x^k) = 0 \) for all \( j = 1, \ldots, p, \) and all \( k \in \mathbb{N} \);
(a4) $\nabla f(x^k) + \sum_{j=1}^p \lambda_j^k \nabla g_j(x^k) = \lambda_{p+1}^k x^k$ for all $k \in \mathbb{N}$.

By the Curve Selection Lemma at infinity (see [25] or [17]), there exist analytic curves
\[
\phi: (0, \epsilon) \to \mathbb{C}^n \quad \text{and} \quad \lambda_j: (0, \epsilon) \to \mathbb{C}, \; j = 1, \ldots, p + 1,
\]
such that
\[
(a5) \|\phi(s)\| \to \infty \quad \text{as} \; s \to 0;
\]
\[
(a6) f(\phi(s)) \to c \quad \text{as} \; s \to 0;
\]
\[
(a7) g_j(\phi(s)) = 0 \quad \text{for all} \; j = 1, \ldots, p, \; \text{and all} \; s \in (0, \epsilon);
\]
\[
(a8) \nabla f(\phi(s)) + \sum_{j=1}^p \lambda_j(s) \nabla g_j(\phi(s)) = \lambda_{p+1}(s) \phi(s) \quad \text{for} \; s \in (0, \epsilon).
\]

If $\lambda_{p+1} \equiv 0$, then it is clear that $c \in K_\infty(f, S)$ and there is nothing to prove. So we may assume that $\lambda_{p+1}$ is not identically zero. It follows from (a7) and (a8) that
\[
0 \neq \frac{d \|\phi(s)\|^2}{2ds} = \Re \left( \frac{d\phi(s)}{ds}, \phi(s) \right) = \Re \left( \frac{d\phi(s)}{ds}, \frac{1}{\lambda_{p+1}(s)} \left[ \nabla f(\phi(s)) + \sum_{j=1}^p \lambda_j(s) \nabla g_j(\phi(s)) \right] \right)
= \Re \left( \frac{d\phi(s)}{ds}, \frac{1}{\lambda_{p+1}(s)} \left[ \frac{d}{ds} (f \circ \phi)(s) + \sum_{j=1}^p \lambda_j(s) \frac{d}{ds} (g_j \circ \phi)(s) \right] \right)
= \Re \left( \frac{1}{\lambda_{p+1}(s)} \left[ \frac{d}{ds} (f \circ \phi)(s) \right] \right).
\]

In particular, $f \circ \phi \neq c$.

On the other hand, we may write
\[
\|\phi(s)\| = as^\alpha + \text{higher-order terms in} \; s,
\]
\[
f(\phi(s)) = c + bs^\beta + \text{higher-order terms in} \; s,
\]
where $a \neq 0, b \neq 0$ and $\alpha, \beta \in \mathbb{Q}$. By the conditions (a5) and (a6) respectively, then $\alpha < 0$ and $\beta > 0$. Therefore, we have asymptotically as $s \to 0^+$,
\[
|\lambda_{p+1}(s)| \approx s^{\beta - 2\alpha}.
\]

It turns out from (a8) that
\[
\|\phi(s)\| \|\nabla f(\phi(s)) + \sum_{j=1}^p \lambda_j(s) \nabla g_j(\phi(s))\| \approx s^\beta \quad \text{as} \; s \to 0^+,
\]
which yields $c \in K_\infty(f, S)$. Hence the inclusion (2) holds.

For the proof of the inclusion (1) we fix $c^* \in \mathbb{C} \setminus (K_0(f|S) \cup T_\infty(f|S))$ and $D$ a small open disc centered at $c^*$, with the closure $\overline{D} \subset \mathbb{C} \setminus (K_0(f|S) \cup T_\infty(f|S))$. Then it is not hard to see that there exists a real number $R_0 > 0$ such that for all $c \in D$ and all $R \geq R_0$, 

the fiber \((f|_S)^{-1}(c)\) is non-singular and intersects transversally with the sphere \(S^{2n-1}_R\) (this is possible if \(D\) is small enough). By continuity, there exists an open neighbourhood \(U\) of \((f|_S)^{-1}(D)\cap\{x\in\mathbb{C}^n: \|x\|\geq R_0\}\) in \(\mathbb{C}^n\) such that the vectors \(\nabla f(x), \nabla g_1(x), \ldots, \nabla g_p(x)\), and \(x\) are \(\mathbb{C}\)-linearly independent for all \(x\in U\). Therefore, we can find a smooth vector field \(v_1\) on \(U\) satisfying the following conditions

\[
\begin{align*}
\text{(a1)} \quad & \langle v_1(x), \nabla f(x) \rangle = 1; \\
\text{(a2)} \quad & \langle v_1(x), \nabla g_j(x) \rangle = 0 \text{ for } j = 1, \ldots, p; \\
\text{(a3)} \quad & \langle v_1(x), x \rangle = 0.
\end{align*}
\]

(We can construct such a vector field locally, then extend it over \(U\) by a smooth partition of unity.)

We now fix \(\epsilon > 0\). Since \(D \cap K_0(f, S) = \emptyset\), the vectors \(\nabla f(x), \nabla g_1(x), \ldots, \nabla g_p(x)\) are \(\mathbb{C}\)-linearly independent for all \(x\) belonging to some open neighbourhood \(V\) of \((f|_S)^{-1}(D)\cap\{x\in\mathbb{C}^n: \|x\|\leq R_0 + \epsilon\}\) in \(\mathbb{C}^n\). Consequently, there exists a smooth vector field \(v_2\) on \(V\) such that the following conditions hold

\[
\begin{align*}
\text{(a4)} \quad & \langle v_2(x), \nabla f(x) \rangle = 1; \\
\text{(a5)} \quad & \langle v_2(x), \nabla g_j(x) \rangle = 0 \text{ for } j = 1, \ldots, p.
\end{align*}
\]

(We can construct such a vector field locally, then extend it over \(V\) by a smooth partition of unity.)

Next, we fix a partition of unity \(\theta_1\) and \(\theta_2\) subordinated to the covering

\[
\left\{ x \in U : \|x\| > R_0 + \frac{\epsilon}{3} \right\} \quad \text{and} \quad \left\{ x \in V : \|x\| < R_0 + \frac{2\epsilon}{3} \right\}
\]

of \((f|S)^{-1}(D)\), and define the smooth vector field \(v\) on \((f|S)^{-1}(D)\) by

\[
v := \theta_1 v_1 + \theta_2 v_2.
\]

Then we can see that the following conditions hold:

\[
\begin{align*}
\text{(a6)} \quad & \langle v(x), \nabla f(x) \rangle = 1; \\
\text{(a7)} \quad & \langle v(x), \nabla g_j(x) \rangle = 0 \text{ for } j = 1, \ldots, p; \\
\text{(a8)} \quad & \langle v(x), x \rangle = 0 \text{ provided that } \|x\| \geq R_0 + \epsilon.
\end{align*}
\]

Finally, integrating the vector field \(v\) we have that the restriction \(f: (f|S)^{-1}(D) \to D\) is a trivial \(C^\infty\)-fibration, which means that \(c^* \notin B(f|_S)\).

\[\square\]

Remark 3.1. (i) The inclusion (1) provides an extension to algebraic sets of Theorem 1 in [24], where the case \(S = \mathbb{C}^n\) was studied.

(ii) The inclusions (1) and (2) may be strict in general, see [29, 30] and [16].
The proof of Theorem 3.1 also implies the following inclusion, which was proved in [20, 21, 33],

\[ B(f|S) \subset K_0(f|S) \cup K_\infty(f|S). \]

(iv) A straightforward modification shows that Lemma 3.1 and Theorem 3.1 still hold in the case where \( S \) does not have the explicit form as it was assumed; in fact, it suffices to suppose that \( S \) is a non-singular constructive subset of \( \mathbb{C}^n \). As we shall not use this “improve” statement, we leave the proof as an exercise.

Under the non-degeneracy condition of Definition 2.1, we obtain the following bound of tangency values at infinity of \( f|S \) in terms of critical values of certain polynomial functions.

**Theorem 3.2.** Assume that the restriction \( f|S \) of \( f \) on \( S \) is Newton non-degenerate at infinity. Then

\[ T_\infty(f|S) \subset \Sigma_\infty(f|S) \cup K_0(f|S) \cup \{0\}. \]

Moreover, if the polynomial \( f: \mathbb{C}^n \to \mathbb{C} \) is convenient, then \( T_\infty(f|S) = \emptyset \).

**Proof.** For convenience we will write \( g_0 \) instead of \( f \).

Take arbitrary \( c \in T_\infty(g_0|S) \setminus (K_0(g_0|S) \cup \{0\}) \). We will show that \( c \in \Sigma_\infty(f|S) \). Indeed, by definition, there exist sequences \( \{x^k\}_{k \in \mathbb{N}} \subset \mathbb{C}^n \) and \( \{\lambda_j^k\}_{k \in \mathbb{N}} \subset \mathbb{C}, j = 1, \ldots, p + 1 \), such that

(a1) \( \|x^k\| \to \infty \) as \( k \to \infty \);
(a2) \( g_0(x^k) \to c \) as \( k \to \infty \);
(a3) \( g_j(x^k) = 0 \) for all \( j = 1, \ldots, p \), and all \( k \in \mathbb{N} \);
(a4) \( \nabla g_0(x^k) + \sum_{j=1}^{p} \lambda_j^k \nabla g_j(x^k) = \lambda_{p+1}^k x^k \) for all \( k \in \mathbb{N} \).

By the Curve Selection Lemma at infinity (see [25] or [17]), there exist analytic curves

\[ \phi: (0, \epsilon) \to \mathbb{C}^n \quad \text{and} \quad \lambda_j: (0, \epsilon) \to \mathbb{C}, \ j = 1, \ldots, p + 1, \]

such that

(a5) \( \|\phi(s)\| \to \infty \) as \( s \to 0 \);
(a6) \( g_0(\phi(s)) \to c \) as \( s \to 0 \);
(a7) \( g_j(\phi(s)) = 0 \) for all \( j = 1, \ldots, p \), and all \( s \in (0, \epsilon) \);
(a8) \( \nabla g_0(\phi(s)) + \sum_{j=1}^{p} \lambda_j(s) \nabla g_j(\phi(s)) = \lambda_{p+1}(s)\phi(s) \) for \( s \in (0, \epsilon) \).

Put \( I := \{i : \phi_i \neq 0\} \). By the condition (a5), \( I \neq \emptyset \). For \( i \in I \), we can write the curve \( \phi_i \) in terms of parameter, say

\[ \phi_i(s) = x_i^0 s^{q_i} + \text{higher-order terms in } s, \]

where \( x_i^0 \neq 0 \) and \( q_i \in \mathbb{Q} \). We have \( \min_{i \in I} q_i < 0 \), because of the condition (a5).
If $\lambda_{p+1} \equiv 0$, then it follows from the conditions (a7) and (a8) that
\[
\frac{d}{ds}(g_0 \circ \phi)(s) = \left\langle \frac{d\phi(s)}{ds}, \nabla g_0(\phi(s)) \right\rangle = -\sum_{j=1}^{p} \lambda_j(s) \left\langle \frac{d\phi(s)}{ds}, \nabla g_j(\phi(s)) \right\rangle
= -\sum_{j=1}^{p} \lambda_j(s) \frac{d}{ds}(g_j \circ \phi)(s) = 0.
\]

Consequently, $g_0(\phi(s)) = c$ for $s \in (0, \epsilon)$, and so $c \in K_0(g_0|_s)$, which is a contradiction. Therefore, $\lambda_{p+1} \neq 0$. Put $J := \{j \in \{1, \ldots, p\} : \lambda_j \neq 0\}$. For $j \in J \cup \{p + 1\}$, we can write
\[
\lambda_j(s) = c_j s^{m_j} + \text{higher-order terms in } s,
\]
where $c_j \neq 0$ and $m_j \in \mathbb{Q}$.

Put $J_1 := \{j \in \{0\} \cup J : g_j|_{cI} \neq 0\}$. The condition (a6) and the assumption that $c \neq 0$ together imply that $0 \in J_1$, and so $J_1 \neq \emptyset$. For each $j \in J_1$, let $d_j$ be the minimal value of the linear function $\sum_{i \in I} \alpha_i q_i$ on $\mathbb{R}^I \cap \Gamma(g_j)$ and $\Delta_j$ be the maximal face of $\mathbb{R}^I \cap \Gamma(g_j)$, where this linear function takes its minimum value, respectively. A simple calculation shows that
\[
g_j(\phi(s)) = g_j(\Delta_j(x^0)) s^d_j + \text{higher-order terms in } s,
\]
where $x^0 := (x_1^0, \ldots, x_n^0)$ with $x_i^0 = 1$ for $i \notin I$ and $g_j(\Delta_j)$ is the face function associated with $g_j$ and $\Delta_j$. The condition (a6) and the assumption that $c \neq 0$ together imply that
\[
d_0 \leq 0 \quad \text{and} \quad d_0 g_{0, \Delta_0}(x^0) = 0. \quad (3)
\]
Furthermore, it follows from the condition (a7) that
\[
g_{j, \Delta_j}(x^0) = 0 \quad \text{for all } j \in J_1 \setminus \{0\}. \quad (4)
\]

On the other hand, we have for all $i \in I$ and all $j \in J_1$,
\[
\frac{\partial g_j(\phi(s))}{\partial x_i} = \frac{\partial g_{j, \Delta_j}(x^0)}{\partial x_i} s^{d_j - q_i} + \text{higher-order terms in } s.
\]

Combined with the condition (a8), this equation implies that for all $i \in I$,
\[
\left(\sum_{j \in J_2} c_j \frac{\partial g_{j, \Delta_j}(x^0)}{\partial x_i}\right) s^{\ell - q_i} + \cdots = c_{p+1} x_i^0 s^m + \cdots,
\]
where $c_0 := 1, m_0 := 0, \ell := \min\{m_j + d_j : j \in J_1\}, J_2 := \{j \in J_1 : \ell = m_j + d_j\}$, and the dots stand for the higher-order terms in $s$. Clearly, $\ell - q_i \leq m_{p+1} + q_i$ for all $i \in I$. Therefore,
\[
\ell - m_{p+1} \leq 2 \min_{i \in I} q_i < 0. \quad (5)
\]
We next show that the set $I_1 := \{ i \in I : \ell - q_i = m_{p+1} + q_i \}$ is empty. To see this, we observe that

$$\sum_{j \in J_2} c_j \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0) x^0_i q_i = \begin{cases} c_{p+1} x^0_i & \text{if } i \in I_1, \\ 0 & \text{if } i \in I \setminus I_1, \\ 0 & \text{if } i \not\in I, \end{cases}$$

where the last equation holds because for all $i \not\in I$ and all $j \in J_2$, the polynomial $g_j, \Delta_j$ does not depend on the variable $x_i$. Consequently,

$$\sum_{i=1}^n \left( \sum_{j \in J_2} c_j \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0) x^0_i q_i \right) x^0_i q_i = \sum_{i \in I_1} \left( \sum_{j \in J_2} c_j \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0) x^0_i q_i \right) x^0_i q_i = \sum_{i \in I_1} c_{p+1} x^0_i \ell - m_{p+1}. \frac{\ell}{2}.$$

On the other hand, by the Euler relation, we have for all $j \in J_2$,

$$\sum_{i=1}^n \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0) x^0_i q_i = d_j g_j, \Delta_j(x^0).$$

It follows that

$$\sum_{i=1}^n \left( \sum_{j \in J_2} c_j \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0) x^0_i q_i \right) x^0_i q_i = \sum_{j \in J_2} c_j \left( \sum_{i=1}^n \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0) x^0_i q_i \right) = \sum_{j \in J_2} c_j d_j g_j, \Delta_j(x^0).$$

Therefore,

$$\sum_{i \in I_1} c_{p+1} x^0_i \ell - m_{p+1}. \frac{\ell}{2} = \sum_{j \in J_2} c_j d_j g_j, \Delta_j(x^0).$$

This, together with (3), (4), and (5), gives $I_1 = \emptyset$. Thus,

$$\sum_{j \in J_2} c_j \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0) = 0 \text{ for all } i = 1, \ldots, n.$$

Since the restriction of $g_0$ on $S$ is Newton non-degenerate at infinity, we deduce easily from (3) and (4) that $d_0 = 0$ and $0 \in J_2$, hence that $c = g_0, \Delta_0(x^0) \in \Sigma_\infty(g_0|_S)$.

Finally, assume that the polynomial $f: \mathbb{C}^n \to \mathbb{C}$ is convenient. Then $d_0 < 0$, which is a contradiction. Hence $T_\infty(f|_S) = \emptyset$. \hfill \Box

For $S = \mathbb{C}^n$, the next statement was shown in [24 Theorem 2].

**Corollary 3.1.** Under the assumption of Theorem 3.2, we have

$$B(f|_S) \subset \Sigma_\infty(f|_S) \cup K_0(f|_S) \cup \{0\}.$$
Moreover, if the polynomial \( f : \mathbb{C}^n \to \mathbb{C} \) is convenient, then \( B(f|_S) = K_0(f|_S) \).

Proof. This is an immediate consequence of Theorems 3.1 and 3.2 and the fact that the bifurcation set \( B(f|_S) \) contains the set \( K_0(f|_S) \) of critical values of the restriction \( f|_S \). \( \square \)

4. THE STABILITY OF GLOBAL MONODROMIES

Recall that the (non-singular) algebraic set \( S \) is given by

\[
S := \{ x \in \mathbb{C}^n : g_1(x) = 0, \ldots, g_p(x) = 0 \}.
\]

In what follows, let \( f(t, x) \) be a polynomial in \( x \in \mathbb{C}^n \) with coefficients which are smooth (i.e., \( C^\infty \)) complex valued functions of \( t \in [0, 1] \). We will write \( f_t(x) := f(t, x) \) and assume that for each \( t \in [0, 1] \), the restriction \( f_t|_S : S \to \mathbb{C} \) is dominant (i.e., the image set \( f_t(S) \) is dense in \( \mathbb{C} \)). With these preparations, we have the following stability result, which generalizes [25, Theorem 17] and [32, Theorem 1.1].

**Theorem 4.1.** Let the following conditions are satisfied:

(i) The Newton polyhedron of \( f_t \) is independent of \( t \);

(ii) For each \( t \in [0, 1] \), the restriction \( f_t|_S \) is Newton non-degenerate at infinity.

Then the global monodromies of the \( f_t|_S \) are all isomorphic.

The proof of Theorem 4.1 will be divided into several steps, which, for convenience, will be called lemmas.

**Lemma 4.1** (Boundedness of affine singularities). There exists a real number \( r > 0 \) such that

\[
K_0(f_t|_S) \subset D_r \quad \text{for all} \quad t \in [0, 1].
\]

Proof. Suppose the lemma were false. Then by the Curve Selection Lemma at infinity (see [25] or [17]), there exist analytic curves

\[
\phi : (0, \epsilon) \to \mathbb{C}^n, \quad t : (0, \epsilon) \to [0, 1], \quad \text{and} \quad \lambda_j : (0, \epsilon) \to \mathbb{C}, j = 1, \ldots, p,
\]

such that

(a1) \( \|\phi(s)\| \to \infty \) as \( s \to 0 \);

(a2) \( t(s) \to t_0 \in [0, 1] \) as \( s \to 0 \);

(a3) \( f_{t(s)}(\phi(s)) \to \infty \) as \( s \to 0 \);

(a4) \( g_j(\phi(s)) = 0 \) for all \( j = 1, \ldots, p \), and all \( s \in (0, \epsilon) \);

(a5) \( \nabla f_{t(s)}(\phi(s)) + \sum_{j=1}^p \lambda_j(s) \nabla g_j(\phi(s)) = 0 \) for \( s \in (0, \epsilon) \).
Put $I := \{ i : \phi_i \neq 0 \}$. By the condition (a1), $I \neq \emptyset$. For $i \in I$, we can write the curve $\phi_i$ in terms of parameter, say
\[ \phi_i(s) = x_i^0 s^q_i + \text{higher-order terms in } s, \]
where $x_i^0 \neq 0$ and $q_i \in \mathbb{Q}$. Observe that $\min_{i \in I} q_i < 0$ because of the condition (a1).

Recall from our assumptions that the Newton polyhedron $\Gamma(f_t)$ of $f_t$ does not depend on $t$. By the condition (a3), $\mathbb{R}^I \cap \Gamma(f_t) \neq \emptyset$. Let $d_0$ be the minimal value of the linear function $\sum_{i \in I} \alpha_i q_i$ on $\mathbb{R}^I \cap \Gamma(f_t)$ and $\Delta_0$ be the maximal face of $\mathbb{R}^I \cap \Gamma(f_t)$ where this linear function takes its minimum value. We can write
\[ f_t(s)(\phi(s)) = f_{t_0, \Delta_0}(x^0) s^{d_0} + \text{higher-order terms in } s, \]
\[ \frac{\partial f_t(s)(\phi(s))}{\partial x_i} = \frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} s^{d_0-q_i} + \text{higher-order terms in } s \quad \text{for all } i \in I, \]
where $x^0 := (x_i^0, \ldots, x_n^0)$ with $x_i^0 = 1$ for $i \notin I$ and $f_{t_0, \Delta_0}$ denotes the face function corresponding to $f_{t_0}$ and $\Delta_0$. By the condition (a3), $d_0 < 0$. Furthermore, for $i \notin I$, the function $f_{t_0, \Delta_0}$ does not depend on the variable $x_i$, and so
\[ \frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} = 0 \quad \text{for all } i \notin I. \quad \text{(6)} \]

Put $J := \{ j \in \{1, \ldots, p\} : \lambda_j \neq 0 \}$. If $J = \emptyset$, then from the condition (a5) we deduce for all $i \in I$ that $\frac{\partial f_t(s)}{\partial x_i}(\phi(s)) = 0$, and hence that $\frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} = 0$. It turns out from (6), the Euler relation, and the inequality $d_0 < 0$ that $f_{t_0, \Delta_0}(x^0) = 0$, which contradicts the non-degeneracy condition. Therefore, $J \neq \emptyset$. For $j \in J$, we can write
\[ \lambda_j(s) = c_j s^{m_j} + \text{higher-order terms in } s, \]
where $c_j \neq 0$ and $m_j \in \mathbb{Q}$.

Put $J_1 := \{ j \in J : g_j|c^j \neq 0 \}$. If $J_1 = \emptyset$, then
\[ \frac{\partial g_j}{\partial x_i}(\phi(s)) = 0 \quad \text{for all } i \in I \text{ and all } j \in J. \]

We deduce from the condition (a5) that
\[ \frac{\partial f_t(s)}{\partial x_i}(\phi(s)) = 0 \quad \text{for all } i \in I. \]
Consequently,
\[ \frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} = 0 \quad \text{for all } i \in I. \]
It follows from (6), the Euler relation, and the inequality $d_0 < 0$ that $f_{t_0, \Delta_0}(x^0) = 0$, which contradicts the non-degeneracy condition. Hence $J_1 \neq \emptyset$. For each $j \in J_1$, let $d_j$ be the
minimal value of the linear function $\sum_{i \in I} \alpha_i q_i$ on $\mathbb{R}^I \cap \Gamma(g_j)$ and $\Delta_j$ be the maximal face of $\mathbb{R}^I \cap \Gamma(g_j)$ where this linear function takes its minimum value. We can write

$$g_j(\phi(s)) = g_{j,\Delta_j}(x^0)s^{d_j} + \text{higher-order terms in } s,$$

where $g_{j,\Delta_j}$ is the face function associated with $g_j$ and $\Delta_j$. By the condition (a4), then

$$g_{j,\Delta_j}(x^0) = 0 \quad \text{for all } j \in J_1.$$  \hspace{1cm} (7)

On the other hand, for $i \in I$ and $j \in J_1$,

$$\frac{\partial g_j}{\partial x_i}(\phi(s)) = \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0)s^{d_j-q_i} + \text{higher-order terms in } s.$$

For $i \notin I$ and $j \in J_1$, the function $g_{j,\Delta_j}$ does not depend on the variable $x_i$, and hence,

$$\frac{\partial g_{j,\Delta_j}}{\partial x_i}(x_0) = 0.$$  \hspace{1cm} (8)

The condition (a5) implies that for all $i \in I$,

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0)s^{d_0-q_i} + \sum_{j \in J_2} c_j \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0)s^{\ell-q_i} + \cdots = 0,$$

where $\ell := \min_{j \in J_1}(m_j + d_j)$, $J_2 := \{ j \in J_1 : \ell = m_j + d_j \}$ and the dots stand for the higher-order terms in $s$. There are three cases to be considered.

Case 1: $\ell > d_0$. By (6) and (9), we have

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) = 0 \quad \text{for} \quad i = 1, \ldots, n.$$

This, together with the Euler relation, implies that

$$d_0 f_{t_0,\Delta_0} = 0.$$

Hence, $f_{t_0,\Delta_0}(x^0) = 0$ because of $d_0 < 0$. This contradicts the non-degeneracy condition.

Case 2: $\ell = d_0$. We deduce from (6), (8) and (9) that

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} c_j \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) = 0 \quad \text{for} \quad i = 1, \ldots, n.$$
Consequently,

\[
0 = \sum_{i=1}^{n} q_i x_i^0 \frac{\partial f_{t_0, \Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J} \sum_{i=1}^{n} c_j q_i x_i^0 \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0)
\]

\[
= \sum_{i=1}^{n} q_i x_i^0 \frac{\partial f_{t_0, \Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J} c_j \sum_{i=1}^{n} q_i x_i^0 \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0)
\]

\[
= d_0 f_{t_0, \Delta_0}(x^0) + \sum_{j \in J} c_j d_j g_j, \Delta_j(x^0)
\]

\[
= d_0 f_{t_0, \Delta_0}(x^0),
\]

where the last equation follows from (7). Since \(d_0 < 0\), we get \(f_{t_0, \Delta_0}(x^0) = 0\), which contradicts the non-degeneracy condition.

**Case 3:** \(\ell < d_0\). By (8) and (9), we obtain

\[
\sum_{j \in J} c_j \frac{\partial g_j, \Delta_j}{\partial x_i}(x^0) = 0 \quad \text{for} \quad i = 1, \ldots, n.
\]

This fact and (7) combined give a contradiction with the non-degeneracy condition. \(\square\)

**Lemma 4.2** (Boundedness of singularities at infinity). There exists a real number \(r > 0\) such that

\[
\Sigma_\infty(f_{t}|s) \subset D_r \quad \text{for all} \quad t \in [0, 1].
\]

**Proof.** Suppose the assertion of the lemma is false. By the Curve Selection Lemma at infinity (see [25] or [17]), we can find a nonempty set \(I \subset \{1, \ldots, n\}\) with \(f_{t|C^I} \neq 0\), a (possibly empty) set \(J \subset \{j \in \{1, \ldots, p\} : g_j|C^I \neq 0\}\), a vector \(q \in \mathbb{R}^n\) with \(\min_{i \in I} q_i < 0\) and \(0 = d(q, \Gamma(f_{t|C^I}))\), and analytic curves

\[
\phi: (0, \epsilon) \to (\mathbb{C}^*)^I, \quad t: (0, \epsilon) \to [0, 1], \quad \text{and} \quad \lambda_j: (0, \epsilon) \to \mathbb{C}, j \in J,
\]

such that the following conditions hold

(a1) \(|\phi(s)|| \to \infty\) as \(s \to 0\);

(a2) \(t(s) \to t_0 \in [0, 1]\) as \(s \to 0\);

(a3) \(f_{t(s), \Delta_0}(\phi(s)) \to \infty\) as \(s \to 0\);

(a4) \(g_j, \Delta_j(\phi(s)) = 0\) for all \(j \in J\) and all \(s \in (0, \epsilon)\);

(a5) \(\nabla f_{t(s), \Delta_0}(\phi(s)) + \sum_{j \in J} \lambda_j(s) \nabla g_j, \Delta_j(\phi(s)) = 0\) for all \(s \in (0, \epsilon)\),

where \(\Delta_0 := \Delta(q, \Gamma(f_{t|C^I}))\) and \(\Delta_j := \Delta(q, \Gamma(g_j|C^I))\) for \(j \in J\).

For \(i \in I\), we can write the curve \(\phi_i\) in terms of parameter, say

\[
\phi_i(s) = x_i^0 s^q_i + \text{higher-order terms in} \ s,
\]

where \(x_i^0 \neq 0\) and \(q_i^0 \in \mathbb{Q}\). Observe that \(\min_{i \in I} q_i^0 < 0\) because of the condition (a1).
Let $d_0$ be the minimal value of the linear function $\sum_{i \in I} \alpha_i q_i'$ on $\mathbb{R}^I \cap \Delta_0 (= \Delta_0)$ and $\Delta'_0$ be the maximal face of $\Delta_0$ where this linear function takes its minimum value. As the Newton polyhedron $\Gamma(f_t)$ of $f_t$ does not depend on $t$, we can write

$$f_{t(s), \Delta_0}(\phi(s)) = f_{t_0, \Delta'_0}(x^0)s^{d_0} + \text{higher-order terms in } s,$$

$$\frac{\partial f_{t(s), \Delta_0}(\phi(s))}{\partial x_i} = \frac{\partial f_{t_0, \Delta'_0}(x^0)}{\partial x_i} s^{d_0 - q'_i} + \text{higher-order terms in } s \quad \text{for } i \in I,$$

where $x^0 := (x^0_1, \ldots, x^0_n)$ with $x^0_i = 1$ for $i \notin I$. By the condition (a3), $d_0 < 0$. Furthermore, for $i \notin I$, the function $f_{t_0, \Delta'_0}$ does not depend on the variable $x_i$, and so

$$\frac{\partial f_{t_0, \Delta'_0}(x^0)}{\partial x_i} = 0 \quad \text{for all } i \notin I. \quad \text{(10)}$$

Put $J_1 := \{ j \in J : \lambda_j \neq 0 \}$. If $J_1 = \emptyset$, then the condition (a5) implies that

$$\frac{\partial f_{t(s), \Delta_0}(\phi(s))}{\partial x_i}(x^0) \equiv 0 \quad \text{for all } i \in I.$$

Consequently,

$$\frac{\partial f_{t_0, \Delta'_0}(x^0)}{\partial x_i} = 0 \quad \text{for all } i \in I.$$

Since $d_0 < 0$, it follows from (10) and the Euler relation that

$$f_{t_0, \Delta'_0}(x^0) = 0,$$

which contradicts the non-degeneracy condition. Thus, $J_1 \neq \emptyset$. For $j \in J_1$, we can write

$$\lambda_j(s) = c_j s^{m_j} + \text{higher-order terms in } s,$$

where $c_j \neq 0$ and $m_j \in \mathbb{Q}$.

For each $j \in J_1$, let $d_j$ be the minimal value of the linear function $\sum_{i \in I} \alpha_i q_i'$ on $\mathbb{R}^I \cap \Delta_j (= \Delta_j)$ and $\Delta'_j$ be the maximal face of $\Delta_j$ where this linear function takes its minimum value. We can write

$$g_{j, \Delta_j}(\phi(s)) = g_{j, \Delta'_j}(x^0)s^{d_j} + \text{higher-order terms in } s.$$

By the condition (a4), we have

$$g_{j, \Delta'_j}(x^0) = 0 \quad \text{for all } j \in J_1. \quad \text{(11)}$$

On the other hand, a direct calculation shows that for $i \in I$ and $j \in J_1$,

$$\frac{\partial g_j}{\partial x_i}(\phi(s)) = \frac{\partial g_{j, \Delta'_j}}{\partial x_i}(x^0)s^{d_j - q'_i} + \text{higher-order terms in } s.$$

For $i \notin I$ and $j \in J_1$, the function $g_{j, \Delta'_j}$ does not depend on the variable $x_i$, and so

$$\frac{\partial g_{j, \Delta'_j}}{\partial x_i}(x^0) = 0 \quad \text{for all } i \notin I \quad \text{and } j \in J_1. \quad \text{(12)}$$
The condition (a5) implies that for all \( i \in I \),
\[
\frac{\partial f_{t_0, \Delta'_0}(x^0)}{\partial x_i} s^{l-\ell'} + \cdots + \sum_{j \in J_2} c_j \frac{\partial g_j, \Delta'_0}{\partial x_i}(x^0) s^{\ell'-\ell'} + \cdots = 0,
\]
where \( \ell := \min_{j \in J_1} (m_j + d_j) \), \( J_2 := \{ j \in J_1 : \ell = m_j + d_j \} \) and the dots stand for the higher-order terms in \( s \). There are three cases to be considered.

**Case 1:** \( \ell > d_0 \). By (10) and (13), we have
\[
\frac{\partial f_{t_0, \Delta'_0}(x^0)}{\partial x_i} = 0 \quad \text{for} \quad i = 1, \ldots, n.
\]
This, together with the Euler relation, implies that
\[
d_0 f_{t_0, \Delta'_0}(x^0) = 0.
\]
Hence, \( f_{t_0, \Delta'_0}(x^0) = 0 \) because of \( d_0 < 0 \). This contradicts the non-degeneracy condition.

**Case 2:** \( \ell = d_0 \). We deduce from (10), (12) and (13) that
\[
\frac{\partial f_{t_0, \Delta'_0}(x^0)}{\partial x_i} + \sum_{j \in J_2} c_j \frac{\partial g_j, \Delta'_0}{\partial x_i}(x^0) = 0 \quad \text{for} \quad i = 1, \ldots, n.
\]
Consequently,
\[
0 = \sum_{i=1}^n q'_i x_i \frac{\partial f_{t_0, \Delta'_0}}{\partial x_i}(x^0) + \sum_{i=1}^n \sum_{j \in J_2} c_j q'_i x_i \frac{\partial g_j, \Delta'_0}{\partial x_i}(x^0)
= \sum_{i=1}^n q'_i x_i \frac{\partial f_{t_0, \Delta'_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} c_j \sum_{i=1}^n q'_i x_i \frac{\partial g_j, \Delta'_0}{\partial x_i}(x^0)
= d_0 f_{t_0, \Delta'_0}(x^0) + \sum_{j \in J_2} c_j d_j g_j, \Delta'_0(x^0)
= d_0 f_{t_0, \Delta'_0}(x^0),
\]
where the last equation follows from (11). Since \( d_0 < 0 \), we get \( f_{t_0, \Delta'_0}(x^0) = 0 \), which contradicts the non-degeneracy condition.

**Case 3:** \( \ell < d_0 \). By (12) and (13), we obtain
\[
\sum_{j \in J_2} c_j \frac{\partial g_j, \Delta'_0}{\partial x_i}(x^0) = 0 \quad \text{for} \quad i = 1, \ldots, n.
\]
This fact, together with (11), gives a contradiction with the non-degeneracy condition. □

**Lemma 4.3** (Transversality in the neighbourhood of infinity). Let \( r \) be a positive real number such that the conclusions of Lemmas 4.1 and 4.2 are fulfilled. Then there exists a real number \( R_0 > 0 \) such that for all \( t \in [0, 1] \), all \( R \geq R_0 \) and all \( c \in S^1_r \), we have the fiber \( (f_{t, S^1_r})^{-1}(c) \) intersects transversally with the sphere \( S^{2n-1}_R \).
Proof. If the assertion is not true, then by the Curve Selection Lemma at infinity (see \cite{25} or \cite{17}), there exist \( t_0 \in [0, 1], c \in \mathbb{S}_r \) and analytic curves

\[
\phi: (0, \epsilon) \rightarrow \mathbb{C}^n, \quad t: (0, \epsilon) \rightarrow [0, 1], \quad \text{and} \quad \lambda_j: (0, \epsilon) \rightarrow \mathbb{C}, j = 0, 1, \ldots, p + 1,
\]
satisfying the following conditions

(a1) \( \|\phi(s)\| \rightarrow \infty \) as \( s \rightarrow 0 \);

(a2) \( t(s) \rightarrow t_0 \) as \( s \rightarrow 0 \);

(a3) \( f_{t(s)}(\phi(s)) \rightarrow c \) as \( s \rightarrow 0 \);

(a4) \( g_j(\phi(s)) = 0 \) for all \( j = 1, \ldots, p \), and all \( s \in (0, \epsilon) \);

(a5) \( \lambda_0(s)\nabla f_{t(s)}(\phi(s)) + \sum_{j=1}^{p} \lambda_j(s)\nabla g_j(\phi(s)) = \lambda_{p+1}(s)\phi(s) \) for all \( s \in (0, \epsilon) \).

Put \( I := \{ i : \phi_i \neq 0 \} \). By the condition (a1), \( I \neq \emptyset \). For \( i \in I \), we can write the curve \( \phi_i \) in terms of parameter, say

\[
\phi_i(s) = x_i^0s^n + \text{higher-order terms in } s,
\]

where \( x_i^0 \neq 0 \) and \( q_i \in \mathbb{Q} \). Observe that \( \min_{i \in I} q_i < 0 \), because of the condition (a1).

By the condition (a3) and the fact that \( |c| = r > 0 \), we have \( \mathbb{R}^I \cap \Gamma(f_t) \neq \emptyset \). Let \( d_0 \) be the minimal value of the linear function \( \sum_{i \in I} \alpha_i q_i \) on \( \mathbb{R}^I \cap \Gamma(f_t) \) and \( \Delta_0 \) be the maximal face of \( \mathbb{R}^I \cap \Gamma(f_t) \) where this linear function takes its minimum value. As the Newton polyhedron \( \Gamma(f_t) \) of \( f_t \) does not depend on \( t \), we can write

\[
\frac{\partial f_{t(s)}(\phi(s))}{\partial x_i} = \frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} + \text{higher-order terms in } s,
\]

where \( x^0 := (x_1^0, \ldots, x_n^0) \) with \( x_i^0 = 1 \) for \( i \not\in I \). The condition (a3) and the fact that \( |c| = r > 0 \) together imply that

\[
d_0 \leq 0 \quad \text{and} \quad d_0 f_{t_0, \Delta_0}(x^0) = 0.
\]  

Furthermore, for \( i \not\in I \), the function \( f_{t_0, \Delta_0} \) does not depend on the variable \( x_i \), and so

\[
\frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} = 0 \quad \text{for all } i \not\in I.
\]

On the other hand, we deduce from the condition (a5), Lemmas \( \ref{lem:5.1} \) and \( \ref{lem:6.1} \) that \( \lambda_0 \neq 0 \) and \( \lambda_{p+1} \neq 0 \) (perhaps reducing \( \epsilon \)). Replacing \( \lambda_j \) by \( \frac{\lambda_j}{\lambda_0} \) if necessary, we may assume that \( \lambda_0 \equiv 1 \). Put \( J := \{ j \in \{1, \ldots, p\} : \lambda_j \neq 0 \} \). For \( j \in J \cup \{ p + 1 \} \), we can write

\[
\lambda_j(s) = c_j s^{m_j} + \text{higher-order terms in } s,
\]

where \( c_j \neq 0 \) and \( m_j \in \mathbb{Q} \).

Put \( J_1 := \{ j \in J : g_j|_{c^i} \neq 0 \} \). There are two cases to be considered.
Case 1. \( J_1 = \emptyset \). The condition (a5) implies that for \( i \in I \),
\[
\frac{\partial f_{t_0, \Delta_0}(x^0)_{s^{d_0 - q_i}}}{\partial x_i} = \cdots = \frac{1}{p+1} x_i^{m_{p+1} + q_i} + \cdots ,
\]
where the dots stand for higher-order terms in \( s \). Clearly, \( d_0 - q_i \leq m_{p+1} + q_i \) for all \( i \in I \), and so
\[
d_0 - m_{p+1} \leq 2 \min_{i \in I} q_i < 0.
\] (16)

Put \( I_1 := \{ i \in I : d_0 - q_i = m_{p+1} + q_i \} \). Observe that \( i \in I \setminus I_1 \) if, and only if,
\[
\frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} = 0,
\] (17)
and in this case \( d_0 - q_i < m_{p+1} + q_i \).

If \( I_1 = \emptyset \), then we get from (15) and (17)
\[
\frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} = 0 \quad \text{for all} \quad i = 1, \ldots, n.
\]
Hence, (14) and the non-degeneracy condition together imply that \( d_0 = 0 \). By the condition (a3), \( c = f_{t_0, \Delta_0}(x^0) \in \Sigma_{\infty}(f_{t_0}|s) \), which contradicts our assumption.

If \( I_1 \neq \emptyset \), then for all \( i \in I_1 \),
\[
\frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} = \frac{1}{c_{p+1} x_i^0} \quad \text{and} \quad d_0 - m_{p+1} = 2q_i.
\]
Hence, the Euler relation, (15) and (17) together imply that
\[
d_0 f_{t_0, \Delta_0} = \sum_{i=1}^{n} \frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} x_i^0 q_i = \sum_{i \in I_1} \frac{\partial f_{t_0, \Delta_0}(x^0)}{\partial x_i} x_i^0 q_i = \sum_{i \in I_1} |x_i^0|^2 \frac{d_0 - m_{p+1}}{2 c_{p+1}} \neq 0,
\]
where the last inequality follows from (16). This, combined with (14), implies a contradiction.

Case 2. \( J_1 \neq \emptyset \). For each \( j \in J_1 \), let \( d_j \) be the minimal value of the linear function \( \sum_{i \in I} \alpha_i q_i \) on \( \mathbb{R}^I \cap \Gamma(g_j) \) and \( \Delta_j \) be the maximal face of \( \mathbb{R}^I \cap \Gamma(g_j) \) where this linear function takes its minimum value. We can write
\[
g_j(\phi(s)) = g_j,\Delta_j(x^0) s^{d_j} + \text{higher-order terms in } s.
\]
By the condition (a4), then
\[
g_j,\Delta_j(x^0) = 0 \quad \text{for all} \quad j \in J_1,
\] (18)
On the other hand, for \( i \in I \) and \( j \in J_1 \),
\[
\frac{\partial g_j}{\partial x_i}(\phi(s)) = \frac{\partial g_j,\Delta_j}{\partial x_i}(x^0) s^{d_j - q_i} + \text{higher-order terms in } s.
\]
For \( i \notin I \) and \( j \in J_1 \), the function \( g_j,\Delta_j \) does not depend on the variable \( x_i \), and hence,
\[
\frac{\partial g_j,\Delta_j}{\partial x_i}(x^0) = 0.
\] (19)
Form the condition (a5), for \(i \in I\) we have
\[
\frac{\partial f_{t_0, \Delta_0}}{\partial x_i}(x^0) s_{d_0 - q_i} + \ldots + \sum_{j \in J_2} c_j \frac{\partial g_{j, \Delta_j}}{\partial x_i}(x^0) s_{\ell - q_i} + \ldots = c_{p+1} x_i^0 s_{m_{p+1} + q_i} + \ldots \tag{20}
\]
where \(\ell := \min_{j \in J_1}(m_j + d_j)\) and \(J_2 := \{ j \in J_1 : m_j + d_j = \ell \}\) and the dots stand for higher-order terms in \(s\). There are three cases to be considered.

Case 2.1. \(\ell > d_0\). The same argument as in Case 1 yields a contradiction.

Case 2.2. \(\ell = d_0\). From (20) we have \(d_0 - q_i \leq m_{p+1} + q_i\) for all \(i \in I\). Therefore
\[
d_0 - m_{p+1} \leq 2 \min_{i \in I} q_i < 0.
\]
Put \(I_2 := \{ i \in I : d_0 - q_i = m_{p+1} + q_i \}\). Hence, \(i \in I \setminus I_2\) if, and only if,
\[
\frac{\partial f_{t_0, \Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} c_j \frac{\partial g_{j, \Delta_j}}{\partial x_i}(x^0) = 0,
\]
and in this case \(d_0 - q_i < m_{p+1} + q_i\).

If \(I_2 = \emptyset\), then
\[
\frac{\partial f_{t_0, \Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} c_j \frac{\partial g_{j, \Delta_j}}{\partial x_i}(x^0) = 0 \quad \text{for all} \quad i = 1, \ldots, n.
\]
Hence, the non-degeneracy condition, (14) and (18) together imply that \(d_0 = 0\). Consequently, by the condition (a2), \(c = f_{t_0, \Delta_0}(x^0) \in \Sigma_\infty(f_{t_0}|s)\), which contradicts our assumption.

If \(I_2 \neq \emptyset\), then from (20) we have for all \(i \in I_2\),
\[
\frac{\partial f_{t_0, \Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} c_j \frac{\partial g_{j, \Delta_j}}{\partial x_i}(x^0) = \frac{c_{p+1} x_i^0}{2}
\]
\[
d_0 - m_{p+1} = 2q_i.
\]
This, together with the Euler relation, (14), (15), (18) and (19), yields
\[
0 = d_0 f_{t_0, \Delta_0}(x^0) + \sum_{j \in J_2} c_j d_j g_{j, \Delta_j}(x^0)
\]
\[
= \sum_{i=1}^n q_i x_i^0 \frac{\partial f_{t_0, \Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} \sum_{i=1}^n c_j q_i x_i^0 \frac{\partial g_{j, \Delta_j}}{\partial x_i}(x^0)
\]
\[
= \sum_{i=1}^n q_i x_i^0 \left( \frac{\partial f_{t_0, \Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} c_j \frac{\partial g_{j, \Delta_j}}{\partial x_i}(x^0) \right)
\]
\[
= \sum_{i \in I_2} q_i x_i^0 \left( \frac{\partial f_{t_0, \Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} c_j \frac{\partial g_{j, \Delta_j}}{\partial x_i}(x^0) \right)
\]
\[
= \sum_{i \in I_2} |x_i^0|^2 \frac{d_0 - m_{p+1}}{2} c_{p+1} \neq 0,
\]
which is impossible.

Case 2.3. $\ell < d_0$. From (20) we have $\ell - q_i \leq m_{p+1} + q_i$ for all $i \in I$. Therefore

$$\ell - m_{p+1} \leq 2 \min_{i \in I} q_i < 0.$$ 

Put $I_3 := \{i \in I : \ell - q_i = m_{p+1} + q_i\}$. Hence, $i \in I \setminus I_3$ if, and only if,

$$\sum_{j \in J_2} \overline{c_j} \frac{\partial g_j \Delta_j}{\partial x_i}(x^0) = 0,$$

and in this case $\ell - q_i < m_{p+1} + q_i$.

If $I_3 = \emptyset$, then

$$\sum_{j \in J_2} \overline{c_j} \frac{\partial g_j \Delta_j}{\partial x_i}(x^0) = 0 \quad \text{for all} \quad i = 1, \ldots, n,$$

which, together with (18), leads to a contradiction with the non-degeneracy condition.

If $I_3 \neq \emptyset$, then from (20) we have for all $i \in I_3$,

$$\sum_{j \in J_2} \overline{c_j} \frac{\partial g_j \Delta_j}{\partial x_i}(x^0) = \overline{c_{p+1}} x_i^0,$$

$$\ell - m_{p+1} = 2q_i.$$

This, together with the Euler relation and (18), yields

$$0 = \sum_{j \in J_2} \overline{c_j} d_j g_j \Delta_j(x^0)$$

$$= \sum_{j \in J_2} \sum_{i=1}^n \overline{c_j} q_i x_i^0 \frac{\partial g_j \Delta_j}{\partial x_i}(x^0)$$

$$= \sum_{i=1}^n \sum_{j \in J_2} \overline{c_j} q_i x_i^0 \frac{\partial g_j \Delta_j}{\partial x_i}(x^0)$$

$$= \sum_{i \in I_3} q_i x_i^0 \left( \sum_{j \in J_2} \overline{c_j} \frac{\partial g_j \Delta_j}{\partial x_i}(x^0) \right)$$

$$= \sum_{i \in I_3} \|x^0\|^2 \frac{\ell - m_{p+1}}{2} \overline{c_{p+1}} \neq 0,$$

which is impossible. \hfill \Box

We now can complete the proof of the Theorem 4.1.

**Proof of Theorem 4.1.** Let $r$ and $R_0$ be the positive real numbers such that the conclusions of Lemmas 3.1, 4.1, 4.2 and 4.3 are fulfilled. By Corollary 3.1, then $B(f_{t|S}) \subset D_r$ for all $t \in [0, 1]$. Furthermore, for all $(t, x) \in X := \{(t, x) \in [0, 1] \times S : f(t, x) \in S^1, \|x\| \geq R_0\}$,
the vectors $\nabla f_t(x), \nabla g_1(x), \ldots, \nabla g_p(x)$, and $\overline{X}$ are $C$-linearly independent. Therefore, we can find a smooth map $v_1: X \rightarrow \mathbb{C}^n, (t, x) \mapsto v_1(t, x)$, satisfying the following conditions

(a1) $(v_1(t, x), \nabla f_t(x)) = -\frac{\partial f_t}{\partial t}(x);$
(a2) $(v_1(t, x), \nabla g_j(x)) = 0$ for $j = 1, \ldots, p;$
(a3) $(v_1(t, x), x) = 0.$

We take arbitrary (but fixed) $\epsilon > 0$. Since $S^1_t \cap K_0(f_t|S) = \emptyset$ for all $t \in [0, 1]$, the vectors $\nabla f_t(x), \nabla g_1(x), \ldots, \nabla g_p(x)$ are $C$-linearly independent for all $(t, x) \in Y := \{(t, x) \in [0, 1] \times S : f(t, x) \in S^1_t, ||x|| \leq R_0 + \epsilon\}$. Consequently, there exists a smooth map $v_2: Y \rightarrow \mathbb{C}^n, (t, x) \mapsto v_2(t, x)$, such that the following conditions hold

(a4) $(v_2(t, x), \nabla f_t(x)) = -\frac{\partial f_t}{\partial t}(x);$
(a5) $(v_2(t, x), \nabla g_j(x)) = 0$ for $j = 1, \ldots, p.$

Next, by patching the maps $v_1$ and $v_2$ together using a smooth partition of unity, we get a smooth map

$v: \{(t, x) \in [0, 1] \times S : f(t, x) \in S^1_t\} \rightarrow \mathbb{C}^n, (t, x) \mapsto v(t, x),$

such that the following conditions hold:

(a6) $(v(t, x), \nabla f_t(x)) = -\frac{\partial f_t}{\partial t}(x);$
(a7) $(v(t, x), \nabla g_j(x)) = 0$ for $j = 1, \ldots, p;$
(a8) $(v(t, x), x) = 0$ provided that $||x|| \geq R_0 + \epsilon.$

Finally we can check that for each $x \in f_0^{-1}(S^1_r) \cap S$, there exists a unique $C^\infty$-map $\varphi: [0, 1] \rightarrow \mathbb{C}^n$ such that

$\varphi'(t) = v(t, \varphi(t)), \quad \varphi(0) = x.$

Moreover, for each $t \in [0, 1]$, the map

$\Phi_t: f_0^{-1}(S^1_t) \cap S \rightarrow f_t^{-1}(S^1_t) \cap S, \quad x \mapsto \varphi(t),$

is well-defined and is a $C^\infty$-diffeomorphism, which makes the following diagram commutes

\[
\begin{array}{ccc}
 f_0^{-1}(S^1_t) \cap S & \xrightarrow{\Phi_t} & f_t^{-1}(S^1_t) \cap S \\
 f_0 \downarrow & & \downarrow f_t \\
 S^1_r & \xrightarrow{id} & S^1_r
\end{array}
\]

where id denotes the identity map.

References

[1] E. Artal-Bartolo, I. Luengo, and A. Melle-Hernández. Milnor number at infinity, topology and Newton boundary of a polynomial function. Math. Z., 233(4):679–696, 2000.
[2] A. Bodin. Invariance of Milnor numbers and topology of complex polynomials. *Comment. Math. Helv.*, 78(1):134–152, 2003.
[3] A. Bodin. Newton polygons and families of polynomials. *manuscripta math.*, 113(3):371–382, 2004.
[4] Broughton. Milnor numbers and the topology of polynomial hypersurfaces. *Invent. Math.*, 92:217–242, 1988.
[5] Y. Chen, L. Dias, K. Takeuchi, and M. Tibăr. Invertible polynomial mappings via Newton non-degeneracy. *Ann. Inst. Fourier*, 64(5):1807–1822, 2014.
[6] Y. Chen and M. Tibăr. Bifurcation values of mixed polynomials. *Math. Res. Lett.*, 19(1):59–79, 2012.
[7] A. Dimca and A. Némethi. On the monodromy of complex polynomials. *Duke Math. Journal*, 108(2):199–209, 2001.
[8] A. Durfee. Five definitions of critical points at infinity. In *Singularities*, volume 162 of *The Brieskorn anniversary volume*, pages 345–360. Birkhäuser, Basel, 1998.
[9] H. V. Hà. Sur la fibration globale des polynômes de deux variables complexes. *C. R. Acad. Sci., Série I. Math.*, 309:231–234, 1989.
[10] H. V. Hà. Nombres de Lojasiewicz et singularités à l’infini des polynômes de deux variables complexes. *C. R. Acad. Sci., Paris, Série I*, 311:429–432, 1990.
[11] H. V. Hà. Sur l’irrégularité du diagramme splice pour l’entrelacement à l’infini des courbes planes. *C. R. Acad. Sci., Paris, Série I*, 313(5):277–280, 1991.
[12] H. V. Hà and D. T. Lê. Sur la topologie des polynômes complexes. *Acta Math. Vietnam.*, 9:21–32, 1984.
[13] H. V. Hà and L. A. Nguyễn. Le comportement géométrique à l’infini des polynômes de deux variables complexes. *C. R. Acad. Sci., Paris, Série I*, 309(3):183–186, 1989.
[14] H. V. Hà and T. T. Nguyễn. On the topology of polynomial functions on algebraic surfaces in $\mathbb{C}^n$. In J. P. Brasselet, J. L. Cisneros-Molina, D. Massey, J. Seade, and B. Teissier, editors, *Singularities II: Geometric and Topological Aspects*, volume 475 of *Contemp. Math.*, pages 61–67. Amer. Math. Soc., Providence, RI, 2008.
[15] H. V. Hà and T. S. Pham. Invariance of the global monodromies in families of polynomials of two complex variables. *Acta Math. Vietnam.*, 22(2):515–526, 1997.
[16] H. V. Hà and T. S. Pham. Critical values of singularities at infinity of complex polynomials. *Vietnam J. Math.*, 36(1):1–38, 2008.
[17] H. V. Hà and T. S. Pham. *Genericity in polynomial optimization*, volume 3 of *Series on Optimization and Its Applications*. World Scientific Publishing, Singapore, 2017.
[18] H. V. Hà and A. Zaharia. Families of polynomials with total Milnor number constant. *Math. Ann.*, 313:481–488, 1996.
[19] M. Ishikawa. The bifurcation set of a complex polynomial function of two variables and the Newton polygons of singularities at infinity. *J. Math. Soc. Japan*, 54(1):161–196, 2002.
[20] Z. Jelonek. On asymptotic critical values and the Rabier theorem. *Banach Center Publ.*, 65:125–133, 2004.
[21] Z. Jelonek and K. Kurdyka. Quantitative generalized Bertini–Sard theorem for smooth affine varieties. *Discrete Comput. Geom.*, 34(4):659–678, 2005.
[22] A. G. Kouchnirenko. Polyhedres de Newton et nombre de Milnor. *Invent. Math.*, 32:1–31, 1976.
[23] K. Kurdyka, P. Orro, and S. Simon. Semialgebraic Sard theorem for generalized critical values. *J. Differential Geom.*, 56:62–92, 2000.
[24] A. Némethi and A. Zaharia. On the bifurcation set of a polynomial function and Newton boundary. *Publ. Res. Inst. Math. Sci.*, 26(4):681–689, 1990.

[25] A. Némethi and A. Zaharia. Milnor fibration at infinity. *Indag. Math.*, 3:323–335, 1992.

[26] W. D. Neumann and P. Norbury. Monodromy and vanishing cycles of complex polynomials. *Duke Math. Journal*, 101(4):487–497, 2000.

[27] T. T. Nguyen. Bifurcation set, M-tameness, asymptotic critical values and Newton polyhedrons. *Kodai Math. J.*, 36(1):77–90, 2013.

[28] A. Parusiński. On the bifurcation set of a complex polynomial with isolated singularities at infinity. *Compos. Math.*, 97:369–384, 1995.

[29] L. Păunescu and A. Zaharia. On the Lojasiewicz exponent at infinity for polynomial functions. *Kodai Math. J.*, 20:269–274, 1997.

[30] L. Păunescu and A. Zaharia. Remarks on the Milnor fibration at infinity. *manuscripta math.*, 103:351–361, 2000.

[31] T. S. Pham. On the topology of the Newton boundary at infinity. *J. Math. Soc. Japan*, 60(4):1065–1081, 2008.

[32] T. S. Pham. Invariance of the global monodromies in families of nondegenerate polynomials in two variables. *Kodai Math. J.*, 33(2):294–309, 2010.

[33] P. J. Rabier. Ehresmann fibrations and Palais–Smale conditions for morphisms of Finsler manifolds. *Ann. of Math.*, 146:647–691, 1997.

[34] D. Siersma and M. Tibăr. Singularities at infinity and their vanishing cycles. *Duke Math. J.*, 80(3):771–783, 1995.

[35] D. Siersma and M. Tibăr. Topology of polynomial functions and monodromy dynamics. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(9):655–660, 1998.

[36] R. Thom. Ensembles et morphismes stratifiés. *Bull. Amer. Math. Soc.*, 75:240–284, 1969.

[37] M. Tibăr. On the monodromy fibration of polynomial functions with singularities at infinity. *C. R. Acad. Sci. Paris Sér. I Math.*, 9(1):1031–1035, 1997.

[38] A. N. Varchenko. Theorems on the topological equisingularity of families of algebraic varieties and families of polynomial mappings. *Math. USSR Izv.*, 6:949–1008, 1972.

[39] J. L. Verdier. Stratifications de Whitney et théorème de Bertini–Sard. *Invent. Math.*, 36:295–312, 1996.

[40] A. H. Wallace. Linear sections of algebraic varieties. *Indiana Univ. Math. J.*, 20:1153–1162, 1971.

[41] A. Zaharia. On the bifurcation set of a polynomial function and Newton boundary II. *Kodai Math. J.*, 19:218–233, 1996.

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