On a Vizing-type integer domination conjecture

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Abstract

Given a simple graph \(G\), a dominating set in \(G\) is a set of vertices \(S\) such that every vertex not in \(S\) has a neighbor in \(S\). Denote the domination number, which is the size of any minimum dominating set of \(G\), by \(\gamma(G)\). For any integer \(k \geq 1\), a function \(f : V(G) \to \{0, 1, \ldots, k\}\) is called a \(\{k\}\)-dominating function if the sum of its function values over any closed neighborhood is at least \(k\). The weight of a \(\{k\}\)-dominating function is the sum of its values over all the vertices. The \(\{k\}\)-domination number of \(G\), \(\gamma_{\{k\}}(G)\), is defined to be the minimum weight taken over all \(\{k\}\)-domination functions.

Brešar, Henning, and Klavžar (On integer domination in graphs and Vizing-like problems. Taiwanese J. Math. 10(5) (2006) pp. 1317–1328) asked whether there exists an integer \(k \geq 2\) so that \(\gamma_{\{k\}}(G \Box H) \geq \gamma(G)\gamma(H)\).

In this note we prove that if \(G\) is a claw-free graph and \(H\) is an arbitrary graph, then \(\gamma_{\{2\}}(G \Box H) \geq \gamma(G)\gamma(H)\). We also show \(\gamma_2(G \Box H) \geq \gamma(G)\gamma(H)\), where \(\gamma_2(G)\) is the 2-domination number of \(G\).

Keywords: dominating set, domination number, \(\{k\}\)-domination number, integer domination, weak \(\{k\}\)-domination number, 2-dominating set, 2-domination number, Cartesian product, Vizing’s conjecture

AMS subject classification: 05C69
1 Introduction

Given a graph $G$, a dominating set is a set of vertices $S$ in $G$ with the property that every vertex not in $S$ has a neighbor in $S$. The domination number of $G$, written $\gamma(G)$, is the cardinality of a minimum dominating set in $G$. One of the most influential and widely studied conjectures of domination in graphs is Vizing’s conjecture, originally posed by V.G. Vizing in 1963 [7]. This conjecture states that the domination number of the Cartesian product of graphs $G$ and $H$ is bounded below by the product of the domination numbers of $G$ and $H$.

Conjecture 1 (Vizing’s conjecture [7]) For every pair of finite graphs $G$ and $H$,

$$\gamma(G \square H) \geq \gamma(G)\gamma(H).$$

Attempts to prove Vizing’s conjecture are numerous. Indeed, there is a myriad of partial results and weaker formulations. For more on the history of Vizing’s conjecture, we refer the reader to the excellent survey [2].

In this paper we focus on two domiantion functions which produce values at least as large as the classical domination function: The 2-domination function and the integer domination function, first defined in [4], and applied to Cartesian products in [3]. In that paper, the authors asked two related Vizing-type questions:

Question 1 (Brešar, Henning, Klavžar [3]) For any graphs $G$ and $H$, is it true that

$$\gamma_2(G \square H) \geq \gamma(G)\gamma(H)?$$

and the weaker question

Question 2 (Brešar, Henning, Klavžar [3]) Is there a natural number $k$ such that for any pair of graphs $G, H$,

$$\gamma_k(G \square H) \geq \gamma(G)\gamma(H)?$$

These questions also appear in [2].

In this note we answer these questions in the affirmative for claw-free $G$ and arbitrary $H$ by a vertex labeling method first introduced in [5]. We also show that for claw-free $G$ and any $H$, $\gamma_2(G \square H) \geq \gamma(G)\gamma(H)$, where $\gamma_2(G)$ is the 2-domination number of $G$. In doing so, we provide more evidence for the validity of Vizing’s conjecture.

Definitions and Notation. In this note, all graphs will be considered finite and simple. Specifically, let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of $G$ will be denoted by $n(G) = |V(G)|$ and $m(G) = |E(G)|$, respectively.
Two vertices $v$ and $w$ in $G$ are adjacent, or neighbors, if $vw \in E(G)$. A set of pairwise non-adjacent vertices in $G$ is an independent set, or stable set. The open neighborhood of a vertex $v \in V(G)$, written $N_G(v)$, is the set of all neighbors of $v$, whereas the closed neighborhood of $v$ is $N_G[v] = N_G(v) \cup \{v\}$. Let $S \subseteq V(G)$ and $v \in S$. The open $S$-private neighborhood of $v$ is defined as $\text{pn}(v, S) = \{w \in V(G) : N_G(w) \cap S = \{v\}\}$. A graph $G$ is called claw-free if $G$ contains no $K_{1,3}$ as an induced subgraph.

As mentioned previously, a set of vertices $S \subseteq V(G)$ is dominating if every vertex not in $S$ has a neighbor in $S$. If $S$ is a dominating set with the additional property that $S$ is also an independent set, then $S$ is a independent dominating set. The cardinality of a minimum independent dominating set in $G$ is the independent domination number of $G$, denoted $i(G)$.

A set of vertices $S$ in a graph $G$ is 2-dominating if every vertex not in $S$ has at least 2 neighbors in $S$. The 2-domination number of $G$, written $\gamma_2(G)$, is the cardinality of a minimum 2-dominating set in $G$. With this definition, it is clear that every 2-dominating set is also a dominating set, and so, $\gamma_2(G) \geq \gamma(G)$.

For any integer $k \geq 1$, a function $f : V(G) \rightarrow \{0, 1, \ldots, k\}$ is called a $\{k\}$-dominating function if the sum of its function values over any closed neighborhood is at least $k$. The weight of a $\{k\}$-dominating function is the sum of its values over all the vertices. The $\{k\}$-domination number of $G$, $\gamma_{\{k\}}(G)$, is defined to be the minimum weight taken over all $\{k\}$-domination functions.

For other graph theoretic terminology and definitions, we will typically follow [6].

For a given positive integer $k$, we will also make use of the standard notation $[k] = \{1, \ldots, k\}$.

## 2 Main Result

In this section we will prove our main result, but before doing so we will need a useful theorem of Allan and Laskar [1] that states equivalence between the domination number and independent domination number of claw-free graphs. We state this theorem formally as follows.

**Theorem 1** (Allan and Laskar [1]) If $G$ is a claw-free graph, then $\gamma(G) = i(G)$.

We will also make use of the following observation.

**Observation 2** If $G$ is a claw-free graph and $S$ is a minimum independent dominating set in $G$, then any vertex not in $S$ is adjacent to either one or two vertices in $S$.

**Definition 1** For any integer $k \geq 1$, a function $f : V(G) \rightarrow \{0, 1, \ldots, k\}$ is called a weak $\{k\}$-dominating function if for $v \in V(G)$ such that $f(v) = 0$, the sum of the...
function values over the closed neighborhood of \(v\) is at least \(k\). The weight of a weak \(\{k\}\)-dominating function is the sum of its values over all the vertices. The weak \(\{k\}\)-domination number of \(G\), \(\gamma_{w\{k\}}(G)\), is defined to be the minimum weight taken over all weak \(\{k\}\)-dominating functions.

We note that we can make an equivalent formulation of a weak \(\{k\}\)-dominating function on \(G\) by giving multiplicity \(\ell\) to a dominating vertex \(v\) when \(f(v) = \ell\), for \(1 \leq \ell \leq k\). We can think of this as having \(G\) copies of \(v\). Call such a distribution of dominating vertices for a weak \(\{k\}\)-dominating function, a weak \(\{k\}\)-dominating set of \(G\). If \(f\) is a weak \(\{k\}\)-domination function of minimum weight, call the distribution of dominating vertices a minimum weak \(\{k\}\)-dominating set.

We are now ready to present our main result.

**Theorem 3** If \(G\) is a claw-free graph and \(H\) is a graph, then

\[
\gamma_{w\{2\}}(G \Box H) \geq \gamma(G)\gamma(H).
\]

**Proof.** Let \(G\) be a claw-free graph and \(H\) be a graph. By Theorem 1 we may choose an independent dominating set \(S \subseteq V(G)\) with \(|S| = \gamma(G)\). For notational simplicity, suppose \(|S| = k\), and label the vertices of \(S\) by \(v_1, \ldots, v_k\). Let \(D \subseteq V(G \Box H)\) be a minimum weak \(\{2\}\)-dominating set of \(G \Box H\), and so, \(|D| = \gamma_{w\{2\}}(G \Box H)\).

We next devise a labeling scheme for the vertices in \(D\) which is split into two separate parts; an initial labeling and a finishing labeling. The following labeling is the initial labeling, and we note that by Observation 2, the following labeling will assign at most 2 entries to each label.

1. For \(i \in [k]\), if \((v, h) \in D\) with \(v \in \{v_i\} \cup \text{pn}(v_i, S)\), then label \((v, h)\) by \(\{i\}\).

2. For distinct \(i, j \in [k]\), if \((v, h) \in D\) with \(v\) adjacent to both \(v_i\) and \(v_j\) in \(G\), where \((v_i, h') \cap D = \emptyset\) with \(h' \in N_H[h]\), and \((v_j, h'') \cap D = \emptyset\) with \(h'' \in N_H[h]\), then label \((v, h)\) by \(\{i, j\}\). If \((v, h)\) contains a vertex of \(D\) with multiplicity 2, label one copy by \(\{i\}\) and the other by \(\{j\}\).

3. For distinct \(i, j \in [k]\), if \((v, h) \in D\) with \(v\) adjacent to both \(v_i\) and \(v_j\) in \(G\), where \((v_i, h') \cap D = \emptyset\) with \(h' \in N_H[h]\), and \((v_j, h'') \cap D \neq \emptyset\) with \(h'' \in N_H[h]\), then label \((v, h)\) by \(\{i\}\).

4. For distinct \(i, j \in [k]\), if \((v, h) \in D\) with \(v\) adjacent to both \(v_i\) and \(v_j\) in \(G\), where \((v_i, h') \cap D \neq \emptyset\) with \(h' \in N_H[h]\), and \((v_j, h'') \cap D \neq \emptyset\) with \(h'' \in N_H[h]\), then label \((v, h)\) by \(\{i\}\) or \(\{j\}\) arbitrarily.

If \(S\) is a perfect independent dominating set of \(G\), then all vertices in \(G \Box H\) are assigned labels according to (1) in the above process. In this case the final labeling scheme is not necessary and we may bypass it. The following is the finishing labeling.
(5) If \((u, h)\) and \((v, h')\) are vertices in \(D\) that are both labeled \(\{i, j\}\) for distinct \(i, j \in [k]\), and \(hh' \in E(H)\), then relabel \((u, h)\) by \(\{i\}\), and \((v, h')\) by \(\{j\}\).

(6) If \((u, h)\) and \((v, h')\) are vertices in \(D\) that are labeled \(\{i\}\) and \(\{i, j\}\), respectively, for distinct \(i, j \in [k]\), and \(hh' \in E(H)\), then relabel \((v, h')\) by \(\{j\}\).

(7) If \((u, h)\) and \((v, h)\) are vertices in \(D\) that are both labeled \(\{i, j\}\) for distinct \(i, j \in [k]\), then relabel \((u, h)\) by \(\{i\}\), and \((v, h)\) by \(\{j\}\).

(8) If \((u, h)\) and \((v, h)\) are vertices in \(D\) with labels \(\{i\}\) and \(\{i, j\}\), respectively, then relabel \((v, h)\) by \(\{j\}\).

(9) If \((u, h)\) and \((v, h)\) are vertices in \(D\) with labels \(\{i, j\}\) and \(\{j, \ell\}\), respectively, then relabel \((v, h)\) by \(\{\ell\}\).

(10) If \((u, h)\) and \((v, h)\) are vertices of \(D\) both labeled \(\{i\}\), then we may relabel one of \((u, h)\) or \((v, h)\) by any other label.

**Claim 1** We may apply labelings (1) - (10) to \(D\) and produce a labeling such that each vertex has a label with exactly one entry.

**Proof.** Suppose there exists a vertex of \(D\), say \((x_{i_1}, h)\), which has been assigned the labeling \(\{i_1, i_2\}\). Then, according to labeling (2), \((v_{i_2}, h) \cap D = \emptyset\) and \((v_{i_1}, h') \cap D = \emptyset\) for all vertices \(h\) and \(h'\) for which \(hh' \in E(H)\). Since \(D\) is a 2-dominating set of \(G \square H\), and so, all vertices not in \(D\) have at least two neighbors in \(D\), it follows that both \((v_{i_2}, h)\) and \((v_{i_1}, h')\) have at least one other neighbor distinct from \((x_{i_1}, h)\) in \(D\). Without loss of generality, we consider the case when the neighbor of \((v_{i_2}, h)\) in \(D\) which is different from \((x_{i_1}, h)\) is the vertex \((v_{i_2}, h')\). Since \((x_{i_1}, h)\) is adjacent to \((v_{i_2}, h')\) in \(G \square H\), it is clear that \(hh' \in E(H)\), and so, by labeling (1), \((v_{i_2}, h')\) will have the label \(\{i_2\}\). This yields a contradiction since according to labels (3) and (4), \((x_{i_1}, h)\) could not have been assigned the labeling \(\{i_1, i_2\}\).

We next consider when a vertex \((x_{i_2}, h)\) \(\in D\) for some \(x_{i_2} \in V(G)\), different from \((x_{i_1}, h)\), is the neighbor of \((v_{i_2}, h)\). First suppose that \((x_{i_2}, h)\) is assigned a label that contains \(\{i_2\}\) (or \(\{i_1\}\)). By labeling (7) and (8), two vertices such as \((x_{i_2}, h)\) and \((x_{i_1}, h)\) would receive labels with one entry, which contradicts the possibility of the label \(\{i_1, i_2\}\) on \((x_{i_1}, h)\).

Finally, we suppose that \((x_{i_2}, h)\) has been assigned the label \(i_3\), which is distinct from either \(i_1\) or \(i_2\), for some \(i_3 \in [k]\). Let \(n\) be the minimal index so that for \(2 \leq \ell \leq n\), \((x_{i_{\ell-1}}, h)\) and \((x_{i_\ell}, h)\) are adjacent to \((v_{i_{\ell-1}}, h)\) and \((x_{i_m}, h)\) is adjacent to some vertex \((x_{i_m}, h)\) for some \(m \in [n-2]\), where \((x_{i_\ell}, h)\) is labeled \(i_\ell + 1\) for \(2 \leq \ell \leq n - 1\).

We consider the cycle \((v_{i_m}, h), (x_{i_m}, h), (v_{i_{m+1}}, h), \ldots, (x_{i_n}, h)\). Vertex \((x_{i_n}, h)\) may be labeled by \(\{i_{n-1}\}, \{i_m\}\), or \(\{i_m, i_{n-1}\}\). If the label on \((x_{i_n}, h)\) contains \(\{i_{n-1}\}\), then by labeling (10), we may relabel \((x_{i_{n-1}}, h)\) by \(\{i_{n-1}\}\), and continue relabeling vertex...
$(x_{i_\ell}, h)$ by $\{i_\ell\}$ for $2 \leq \ell \leq n - 1$. However, by labeling (8), this means $(x_{i_1}, h)$ could be labeled by $\{i_1\}$. If the label on $(x_{i_m}, h)$ contains $\{i_m\}$, then by labeling (10), we may relabel $(x_{i_\ell}, h)$ by $\{i_\ell\}$ for $2 \leq \ell \leq m$. Again, by labeling (8), this means $(x_{i_1}, h)$ could be labeled by $\{i_1\}$.

Thus, $(x_{i_1}, h)$ could be relabeled by a label with one entry. (c)

According to Claim 1, each vertex of $D$ has been assigned a label with a single entry. Choose $i \in [k]$, project all vertices of $D$ labeled $i$ onto $H$, and call the projected vertices $U = \{u_1, \ldots, u_\ell\}$.

**Claim 2** The set $U$ is a dominating set of $H$.

**Proof.** By way of contradiction, suppose $U$ is not a dominating set of $H$; that is, there exists $h \in V(H)$ such that $h \notin N[U]$. This means that in $G \Box H$, $(v_i, h)$ is dominated by some vertex $(v, h)$ of $D$ labeled by $\{j\}$ for some $j \in [k]$ with $j \neq i$. Labelings (3) and (4) could not have been applied in this case, since $h \notin N[U]$. If labeling (2) had been applied to $(v, h)$, and then any or none of the labelings (5), (6), (7), (8), (9), or (10) had been applied, then $(v, h)$ would have been adjacent to some vertex $(u, h)$ labeled $\{i\}$. Since $(v, h)$ is adjacent to $(v_i, h)$ and $(v_j, h)$, labeling (1) does not apply. This produces a contradiction since no labeling could have been applied to $(v, h)$ but all vertices of $D$ are labeled by Claim 1. Thus, $U$ is a dominating set of $H$, and the proof of the claim is finished. (c)

By Claim 2, for each $i \in [k]$, we may project $D$ onto $H$ and obtain a dominating set of $H$. Recalling $\gamma_{w(2)}(G \Box H) = |D|$, $\gamma(G) = k$, and $|U| \geq \gamma(H)$, we observe the following,

$$\gamma_{w(2)}(G \Box H) = |D| \geq \sum_{i=1}^{k} \gamma(H) = \gamma(G) \gamma(H).$$

Thus, the proof of the theorem is complete. □

Since a 2-dominating set produces a weak $\{2\}$-dominating function, for any graph $G$, $\gamma_{w(2)}(G) \leq \gamma_2(G)$. Furthermore, $\gamma_{w(2)}(G) \leq \gamma_{(2)}(G)$ since the $\{2\}$-dominating function is a minimum over a set with more restrictions than the weak $\{2\}$-dominating function. These two observations lead to the following.

**Corollary 4** For any claw-free graph $G$ and any graph $H$,

$$\gamma_{(2)}(G \Box H) \geq \gamma(G) \gamma(H)$$

$$\gamma_2(G \Box H) \geq \gamma(G) \gamma(H).$$
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