Low Complexity Algorithms for Transmission of Short Blocks over the BSC with Full Feedback

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Abstract—Building on the work of Horstein, Shaeveitz and Feder, and Naghshvar et al., this paper presents algorithms for low-complexity sequential transmission of a $k$-bit message over the binary symmetric channel (BSC) with full, noiseless feedback. To lower complexity, this paper shows that the initial $k$ binary transmissions can be sent before any feedback is required and groups the messages with equal posteriors to reduce the number of posterior updates from exponential in $k$ to linear in $k$. Simulations results demonstrate the achievable rates for this full, noiseless feedback system approach capacity rapidly as a function of $k$, significantly faster than the achievable rate curve of Polyanskiy et al. for a stop feedback system.

I. INTRODUCTION

Shannon [1] showed that feedback cannot improve the capacity of a discrete memoryless channel (DMC). However, Burnashev [2] showed that feedback combined with variable length coding can significantly increase the exponent with which the frame error rate (FER) decreases with blocklength. Polyanskiy et al. [3], [4] derived lower bounds on finite blocklength achievable rates with and without feedback that demonstrate the benefit to achievable rate of “stop feedback,” which is feedback that can only inform the transmitter when transmission should be terminated.

Even better performance should be attainable when stop feedback is replaced by feedback of all received symbols. For the binary symmetric channel (BSC) with noiseless feedback, Horstein [5] presented a simple and elegant one-phase transmission scheme that uses full feedback to achieve the capacity of the BSC [6]. Since Horstein’s work, several authors proposed various transmission schemes for BSC with full noiseless feedback under variant settings, in order to achieve the capacity or Burnashev’s optimal error exponent, e.g., [7]–[10]. Naghshvar et al. [11], [12] presented a finite-blocklength version of Horstein’s scheme, which they show attains the capacity and Burnashev’s optimal error exponent.

This paper focuses on Horstein’s scheme and the finite blocklength version pioneered in [11], [12]. Horstein’s scheme works as follows: For a set of $M$ messages and a given target error probability $\epsilon$, consider the unit interval initially partitioned into $M$ equal sub-intervals. Each sub-interval represents a message, and the length of each sub-interval denotes the posterior of the message. After each transmission, the receiver and transmitter (utilizing the full feedback) both use the channel output to compute new posteriors for each messages and update the sub-interval lengths accordingly.

The transmitter sends bit 0 if the sub-interval corresponding to the true message lies entirely above the midpoint, and sends bit 1 if it lies entirely below the midpoint. However, if the midpoint lies within the sub-interval of the true message, the transmitter sends 0 or 1 randomly according to the fraction of the portion of sub-interval that is above or below the midpoint. The transmission terminates when the length of the sub-interval of any message exceeds $1 - \epsilon$. Although the encoder behavior is essentially the same, we consider the communication phase to be when no message has a posterior greater than 0.5 and the confirmation phase to be when any message has a posterior greater than 0.5.

Unlike the Horstein scheme that sets the midpoint as a hard decision threshold for the transmitter, Naghshvar et al. [11], [12] assigns each message to one of two sets $S_0$, $S_1$. The two sets must satisfy the requirement that that the difference of the sums of the posteriors in each set is less than any individual posterior in $S_0$, which is required to have the larger sum. We refer to this transmission scheme as the small-enough-difference (SED) encoder because at each transmission, the algorithm seeks a two-way partitioning with a bounded small difference.

The transmitter sends a 0 if the true message is in $S_0$, and a 1 otherwise. Actual implementation of the SED encoder requires significant complexity. Perhaps for this reason, Naghshvar et al. did not present any simulation results but rather provide bounds on how a theoretical implementation would perform.

As its primary contribution, this paper provides algorithms for transmission of short blocks (on the order of 50 bits) on the BSC that can be implemented and presents simulation results. These algorithms are made possible by three primary insights: 1) The first $k$ transmissions can be sent with transmissions that achieve $P(S_0) = P(S_1)$ exactly without requiring any feedback. 2) After the initial $k$ transmissions, even though there are $2^k$ different messages, there are only $k+1$ different posterior probabilities. Grouping messages according to their posterior probabilities significantly reduces complexity since one computation computes the posterior for all messages in the group. 3) While an SED encoder can be implemented with relatively low complexity using this grouping as a starting point, an even simpler algorithm that relaxes the requirements on the maximum difference in the probabilities of $S_0$ and $S_1$ achieves essentially the same performance. These new implementations allow simulations that demonstrate how, for
a fixed target FER, a quantization effect leads to a non-monotonic rate increase as \( k \) grows. This non-monotonic behavior can be avoided by using randomization to overcome the quantization effect. As a final contribution this paper shows how achievable rate changes as a function of target FER.

The rest of this paper is organized as follows: Sec. II presents the system model and two tools used in the initial operation of the new algorithms. Sec. III-B shows that the first \( k \) transmissions can be sent before any feedback is required. Sec. III-C presents a technique of ordering and labeling possible messages according to their Hamming distance from the initial \( k \) received bits. Sec. III describes how the messages that are ordered and labeled as in Sec. II can be sorted and partitioned into two sets that either meet the SED criterion of [11], [12] or a more relaxed criterion that requires only one split of a labeled group of equal-posterior messages per transmission. Simulation results show that the relaxed criterion has a negligible effect on the rate as compared to the SED criterion. Sec. IV uses the threshold randomization to mitigate the rate penalty incurred for some small values of \( k \) when integer thresholds significantly exceed the required FER performance. Sec. V explores the tradeoff between FER and rate, and Sec. VI concludes the paper.

II. Initial Transmission and Labeling

A. System Model and the SED Encoder

The basic system model is depicted in Fig. 1, in which the forward channel is a DMC described by an ordered triple \( (\mathcal{X}, \mathcal{Y}, P(Y|X)) \) and the feedback channel noiselessly provides the received channel outputs to the receiver. Let \( \theta \) be the true message uniformly drawn from a message set \( \Omega = \{1, 2, \ldots, M\} \). At each time instant \( t, \ t = 1, 2, \ldots, \tau \), the transmitter is aware of both the true message \( \theta \) and the received symbols \( Y^{t-1} = (Y_0, Y_1, \ldots, Y_{t-1}) \), thanks to the noiseless feedback. The total transmission time (or the number of channel uses, or blocklength) \( \tau \) is a random variable that is governed by a stopping rule that is a function of the observed channel outputs.

In order to communicate \( \theta \) from the transmitter to the receiver, the transmitter produces channel inputs \( X_t, \ t = 1, 2, \ldots, \tau \), as a function of \( \theta \) and \( Y^{t-1} \), i.e.,

\[
X_t = e_t(\theta, Y^{t-1}), \quad t = 1, 2, \ldots, \tau, \tag{1}
\]

for some encoding function \( e_t : \Omega \times Y^{t-1} \rightarrow \mathcal{X} \). After observing \( \tau \) channel outputs \( Y^\tau \), the receiver makes a final estimate \( \hat{\theta} \) of the true message \( \theta \), which is a function of \( Y^\tau \), i.e.,

\[
\hat{\theta} = d(Y^\tau), \tag{2}
\]

for some decoding function \( d : \mathcal{Y}^\tau \rightarrow \Omega \). An error occurs if \( \hat{\theta} \neq \theta \) and the probability of error is given by \( P_e = \Pr(\hat{\theta} \neq \theta) \).

For a given target error probability \( \epsilon, \epsilon > 0 \), the fundamental problem of variable-length coding is to design the encoding function \( e_t(\cdot) \), decoding function \( d(\cdot) \), a stopping rule that defines the stopping time \( \tau \), such that \( P_e \leq \epsilon \) and the average blocklength \( E[\tau] \) is minimized.

In [11] and [12], Naghshvar et al. considered the following encoding rule (called the SED encoder), the decoding rule, and the stopping rule for the BSC(\( p \)), \( 0 < p < 1/2 \).

The SED encoding rule: at each time \( t, t = 1, 2, \ldots, \tau \), with the full, noiseless feedback \( Y^{t-1} \), the transmitter considers the belief state \( \rho(t) \) at time \( t - 1 \)

\[
\rho(t - 1) = [\rho_1(t - 1), \rho_2(t - 1), \ldots, \rho_M(t - 1)], \tag{3}
\]

where

\[
\rho_i(t) \triangleq \Pr(\theta = i|Y^t), \tag{4}
\]

with the convention that \( \rho_i(0) = 1/M \). Using Bayes rule, \( \rho(t) \) can be updated recursively from \( \rho(t - 1) \) upon receiving \( y_t \), i.e.,

\[
\rho_i(t) = \frac{\rho_i(t - 1)P(Y = y_t|X = e_t(i, Y^{t-1}))}{\sum_{j=1}^{M} \rho_j(t - 1)P(Y = y_t|X = e_t(j, Y^{t-1}))}. \tag{5}
\]

Next, the transmitter partitions \( \Omega \) into two subsets \( S_0(t-1) \) and \( S_1(t-1) \) such that

\[
0 \leq \sum_{i \in S_0(t-1)} \rho_i(t - 1) - \sum_{i \in S_1(t-1)} \rho_i(t - 1) \leq \min_{i \in S_0(t-1)} \rho(t - 1). \tag{6}
\]

Then, \( X_t = 0 \) if \( \theta \in S_0(t-1) \) and \( X_t = 1 \) otherwise.

The stopping rule and decoding rule: the stopping time \( \tau \) and the estimate \( \hat{\theta} \) are given by

\[
\tau = \min \{ t : \max_{i \in \Omega} \rho_i(t) \geq 1 - \epsilon \} \tag{7}
\]

\[
\hat{\theta} = \arg \max_{i \in \Omega} \rho_i(\tau). \tag{8}
\]

Clearly, the probability of error under stopping rule [7] meets the desired constraints,

\[
P_e = E[1 - \max_{i \in \Omega} \rho_i(\tau)] \leq \epsilon. \tag{9}
\]

We remark that if \( M = 2^k \), \( k = 1, 2, \ldots \), the partitioning algorithm for the SED encoder described in Naghshvar et al. [11], [12] requires exponential complexity in \( k \), making it difficult to implement in practice. Thus, a low complexity partitioning algorithm that can still guarantee a similar or equal performance as the SED encoder is desired.

B. Sending the First \( k \) Transmissions Without Feedback

Consider the BSC(\( p \)), \( 0 < p < 1/2 \) and define \( q = 1 - p \) henceforth. For brevity, let us define the probabilities of \( S_0(t) \) and \( S_1(t) \) after partitioning \( \Omega \) at time \( t \) by

\[
\pi_x(t) = \sum_{i \in S_x(t)} \rho_i(t), \quad x \in \{0, 1\}. \tag{10}
\]
We first demonstrate that, for $M = 2^k$, regardless of the feedback, a systematic transmission of the true message that also meets the SED condition of (6) over the BSC(p) is possible in the first $k$ transmissions.

**Theorem 1.** Let $\theta$ be a $k$-bit message uniformly drawn from $\Omega = \{1, 2, \ldots, M\}$, $M = 2^k$. Then, for $t \leq k$, there always exists an equal partitioning of $\Omega$ into $S_0(t)$ and $S_1(t)$ such that $\pi_0(t) = \pi_1(t)$.

**Proof:** First, let us consider the following decomposition of 1. For $t \leq k$, we have

$$1 = 2p\pi_x(t - 1) + 2q\pi_{1-x}(t - 1),$$

(11)

$$= \sum_{i=0}^{t-1} (t-1+i)2^{k-i}(2p)^i(2q)^{t-1-i} \cdot (2p + 2q)$$

(12)

$$= 2\sum_{i=0}^{t-1} (t+i)2^{k-(t+1)}(2p)^i(2q)^{t-1-i},$$

(13)

$$= \pi_x(t) + \pi_{1-x}(t) \text{ (valid iff } t \leq k - 1).$$

(14)

where $\pi_x(t) = \pi_x(t) = 1/2$ and $x, x' \in \{0, 1\}$ can be determined arbitrarily at time $t$. The above decomposition suggests that an equal partitioning of $\Omega$ is always possible as long as $t \leq k$, which is demonstrated in what follows.

First, (11) indicates that if $\pi_0(t - 1) = \pi_1(t - 1) = 1/2$, then after transmission, one subset is boosted by $2q$ and the other subset is attenuated by $2p$, according to Bayes’ rule.

Next, (12) demonstrates that if the structure of $S_0(t - 1)$ and $S_1(t - 1)$ are identical before the $t$-th transmission. Then it is possible to equally partition the $2^k$ posteriors into two identical subsets after the $t$-th transmission. One can see that $S_0(t - 1)$ and $S_1(t - 1)$ both share the same structure given by $t$ types indexed by $0 \leq i \leq t - 1$ each element in the type having probability $\frac{(2p)^i(2q)^{t-1-i}}{2^k}$ and each type containing $\binom{t-1}{i}2^{k-t}$ elements.

Finally, (13) indicates that after $t$ transmissions, if $t \leq k - 1$, we can again partition $2^t$ posteriors into two identical subsets with the same structure given by $t + 1$ types of posteriors $\frac{(2p)^i(2q)^{t+1-i}}{2^k}$, each type containing $\binom{t}{i}2^{k-(t+1)}$ elements. The equal partition may not proceed anymore once $t = k$.

An immediate consequence of Theorem 1 is that we can transmit the first $k$ bits systematically while maintaining SED condition of (6) in the first $k$ transmissions even without feedback. That is, if the binary representation of $\theta$ is $b(\theta) = (b_0^{(i)}, b_1^{(i)}, \ldots, b_{k-1}^{(i)})$, $b_i^{(i)} \in \{0, 1\}$, $i = 0, 1, \ldots, k - 1$, then $X_t = b_{t-1}$ is always possible as long as $t \leq k$, since we can always label the subset including $\theta$ by $b_{t-1}$.

**C. Ordering and Labeling Possible Messages**

After transmitting the first $k$ bits systematically, the receiver possesses a noisy version $y^k = (y_1, y_2, \ldots, y_k)$ of the $k$-bit true message $\theta$ over the BSC(p) and the transmitter is aware of the received bits thanks to the noiseless, full feedback.

First, we note that, after the $k$-th transmission, the posterior of each message $i \in \Omega$ can be explicitly computed according to the Hamming distance to the received sequence $y^k$. Formally speaking, define

$$b^{(i)} = (b_0^{(i)}, b_1^{(i)}, \ldots, b_{k-1}^{(i)})_2$$

(15)

as the binary representation of message $i, i \in \Omega$. Thus, if the Hamming distance between $b^{(i)}$ and $y^k$ is $d_H(b^{(i)}, y^k) = d_{i,y^k}$, the posterior of message $j, j \in \Omega$, after the $k$-th transmission is given by

$$p_j(k) = p^{d_{i,j}}q^{k-d_{i,j}}.$$  

(16)

Thus, each message with distance $d_{j,y^k}$ can be categorized into one of $(k + 1)$ groups $G_d(k), d = 0, 1, \ldots, k$, with group $G_d(k)$ having the same posterior $p^{d}q^{k-d}$. The cardinality of group $G_d(k)$ after the $k$-th transmission is given by $\binom{k}{d}$. If we introduce the lexicographical ordering for each group, then there is a one-to-one correspondence between message $b$ in $G_d(k)$ to an index $n_d(b)$, which later greatly simplifies the group split and list merge operations.

Next, we show that the index $n_d(b)$ can be calculated efficiently, which has been proposed and studied in the context of enumerative source coding [13]. For completeness of this paper, we introduce it in what follows. In general, consider the function

$$n_d(b) : \mathcal{U}_d \to \{0, 1, \ldots, \binom{k}{d} - 1\},$$

(17)

where $\mathcal{U}_d = \{b \in \{0, 1\}^k, w_H(b, y^k) = d\}$ consists of all messages whose binary representation is of distance $d$. Let $0 \leq i_1 < i_2 < \ldots < i_d \leq k - 1$ denote the position of 1’s for message $b$. Thus, $n_d(b)$ is given by

$$n_d(b) = \binom{i_1}{1} + \binom{i_2}{2} + \ldots + \binom{i_d}{d}.$$  

(18)

Conversely, given $n_d(b)$, we can easily recover message $j$ by sequentially determining $i_d, i_{d-1}, \ldots, i_1$. Namely, $i_d$ is determined by the largest integer such that $\binom{i_d}{d} \leq n_d(j)$; next, $i_{d-1}$ is determined by the largest integer such that $\binom{i_{d-1}}{d-1} \leq n_d(j) - \binom{i_d}{d}$; so on and so forth.

Hence, each group $G_j(k)$ can be compactly described by an ordered triple

$$G_j(k) = \{d, n_{\text{start}}, N, \delta\}$$

(19)

where $d$ is the Hamming distance from $y^k$, $n_{\text{start}}$ is the starting index, $N$ is the total elements in $G_j(t)$ and $\delta$ is the posterior associated with $G_j(t)$. For example, after the $k$-th transmission, $j = 0, 1, \ldots, k$,

$$G_j(k) = \{d, n_{\text{start}}, N, \delta\} = \{j, 0, \binom{k}{d}, p^{d}q^{k-d}\}.$$  

(20)

The number of groups at time $t, t \geq k$, depends on the partitioning algorithm and $Y^t$; if no group is split, then the number of groups remains $k + 1$ over time.
III. SORTING, GROUPING, AND SPLITTING POSTERIORS

We propose a system that transmits $b^{(g)}$ in the first $k$ transmissions. After the $k$-th transmission, the transmitter first generates a list of $(k + 1)$ groups $G_j(k) = \{d, n_{\text{start}}, N, \delta\}$, $d = 0, 1, \ldots, k$ in the order of decreasing posteriors $\delta$.

At the $t$-th transmission, $t > k$, the transmitter aims at partitioning $\Omega$ into two subsets $S_0(t-1)$, $S_1(t-1)$, by only using group movement and group split operations. Assume that the group $G = \{d, n_{\text{start}}, N, \delta\}$ is to be split at $(n_{\text{start}} + N_1)$-th position, $N_1 \in \{1, 2, \ldots, N - 1\}$. The resultant two subgroups are readily given by

$$G^{(1)} = \{d, n_{\text{start}}, N_1, \delta\}, \quad G^{(2)} = \{d, n_{\text{start}} + N_1, N - N_1, \delta\}.$$ (21)

After the $t$-th transmission, we update the posteriors by updating the associated posterior in each group. For example, if $S_0$ is boosted by $w_0$ and $S_1$ is attenuated by $w_1$, then the groups in $S_0, S_1$ are updated to

$$G = \{d, n_{\text{start}}, N, w_0\delta\}, \quad \text{if} \ G \in S_0$$ (23)
$$G = \{d, n_{\text{start}}, N, w_1\delta\}, \quad \text{if} \ G \in S_1.$$ (24)

A. Achieving the Small-Enough-Difference Criterion

In order to achieve optimal partitioning of the list into $S_0$ and $S_1$, the two new lists need to meet the small-enough-difference (SED) criterion of $[11], [12]$ given by (6). We implement Algorithm II $[11], [12]$ in an equivalent way, with one modification. The equivalent method is to assign the whole list to $S_1$ first instead of $S_0$, and move the message with largest probability to $S_0$ instead of the message with smallest probability to $S_1$. When we first have that next message assigned to $S_0$ will cause $\pi_0 \geq 0.5$, which might require splitting one group, if we cannot meet SED criterion, instead of swapping the list, we test if the whole or a splitting of the next group would be enough to meet SED criterion, and use it if it does, else proceed as we would have otherwise.

B. Reconstructing the Decoded Message

Once the confirmation phase is finished, a unique group $\hat{G} = \{d, n_{\text{start}}, N, \delta\}$ contains a single error case for which we need to determine the decoded message $\hat{b}$, i.e., $N = 1$, $\delta \geq 1 - \epsilon$. This is accomplished by the inverse of $n_d(b)$ as discussed in Sec. IIIC.

C. A Relaxed Criterion that Minimizes Splits

We next evaluate the performance of a system that limits the number of group splits to a maximum of one per transmission, which might prevent the system from meeting the SED criterion. But it guarantees a relaxed version of SED criterion given by:

$$0 \leq \sum_{i \in S_0(t-1)} \rho_i(t-1) - \sum_{i \in S_1(t-1)} \rho_i(t-1) \leq 2 \min_{i \in S_0(t-1)} \rho_i(t-1).$$ (25)

To implement this relaxed condition, we start the procedure as before, moving messages into $S_0$ in descending order. We continue until one message with posterior $\rho$ is moved into $S_0$ such that $\pi_0 \geq 0.5$. Since the last movement yields $\pi_0 \leq 0.5 + \rho$, we conclude $\pi_1 > 0.5 - \rho$ and thus, $\pi_0 - \pi_1 \leq 2\rho$. Note that $\rho$ is the smallest posterior in $S_0$, hence, the relaxed criterion is met.

IV. RANDOMIZED THRESHOLDS FOR PRECISE FERS

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This section explores the “notch” in rate visible in Fig. 2 at $k = 11$. The confirmation phase concludes when the posterior for the message being confirmed exceeds $1 - \epsilon$. However, the threshold is effectively an integer describing a target number of received bits consistent with the message being confirmed. The notch occurs because the integer thresholds quantize the posteriors that can be achieved so that for certain small values of $k$ the first time the the posterior exceeds $1 - \epsilon$ it achieves an FER well below $\epsilon$. This extra reliability incurs a rate penalty that induces the notch in Fig. 2.

Randomization of the threshold number of bits consistent with the message being confirmed can ameliorate this rate loss. To ease the simulation burden required to estimate FERs, we explore this issue for a target FER of $10^{-2}$ in this paper. Fig. 3 shows the rate achieved by the algorithm that achieves the SED criterion and also the rate achieved when the integer threshold is set to one less than the threshold that ensures a target FER of $10^{-2}$. Fig. 4 shows the corresponding FERs.

Compare the two figures in terms of the standard SED performance when the integer threshold is set to ensure a target FER of $10^{-2}$. The peak in rate at $k = 3$ (where $E[\tau]$ is approximately 4) corresponds to a peak in FER. The deep notch in rate that occurs at $k = 4$ (where $E[\tau]$ is approximately $10^{-3}$)
Fig. 3. Rate as a function of average blocklength over the BSC(0.05) for the SED criterion implemented with the threshold that guarantees FER $10^{-2}$ (but for certain small values of $k$ achieves a much lower FER), the threshold that is one less than that, which does not achieve FER $10^{-2}$, and a randomized choice between these two thresholds that closely approximates FER $10^{-2}$.

Fig. 4. Frame error rate as a function of average blocklength over the BSC(0.05) for the SED criterion implemented with the threshold that guarantees FER $10^{-2}$ (but for certain small values of $k$ achieves a much lower FER), the threshold that is one less than that, which does not achieve FER $10^{-2}$, and a randomized choice between these two thresholds that closely approximates FER $10^{-2}$.

7) corresponds to a notch in FER. To recover the rate and avoid over-achieving the FER target, consider an algorithm that randomizes the choice of threshold between the integer threshold that ensures a target FER of $10^{-2}$ and a threshold one less, so that the expected value of FER is precisely $10^{-2}$. Fig. 5 shows how a more strict requirement on the FER target affects the rate performance as a function of FER for the algorithm that achieves the SED criterion using a non-randomized threshold. Also shown for comparison are the SED lower bounds on achievable rate from [14]. Fig. 5 uses only 1000 trials to produce the rate curves even for FERs as low as $10^{-12}$. However, note that the FER target is necessarily achieved by the threshold which requires the posterior to be sufficiently large to ensure the FER before terminating the algorithm. Thus the number of trials does not need to be large enough to estimate the FER, only large enough to estimate the rate. As described in Appendix E of [15], 1000 trials is sufficient for a reasonable estimate of rate.

V. THE TRADEOFF BETWEEN FER AND RATE

Fig. 5 shows how a more strict requirement on the FER target affects the rate performance as a function of FER for the algorithm that achieves the SED criterion using a non-randomized threshold. Also shown for comparison are the SED lower bounds on achievable rate from [14]. Fig. 5 uses only 1000 trials to produce the rate curves even for FERs as low as $10^{-12}$. However, note that the FER target is necessarily achieved by the threshold which requires the posterior to be sufficiently large to ensure the FER before terminating the algorithm. Thus the number of trials does not need to be large enough to estimate the FER, only large enough to estimate the rate. As described in Appendix E of [15], 1000 trials is sufficient for a reasonable estimate of rate.

VI. CONCLUSIONS

This paper introduces algorithms for low-complexity sequential transmission of a $k$-bit message over the binary symmetric channel (BSC) with full, noiseless feedback. The initial $k$ binary transmissions can be sent before any feedback is required. A technique for managing posterior updates by grouping messages with equal-value posteriors lowers complexity. Relaxing the SED criterion further lowers complexity without sacrificing performance. Threshold randomization avoids the rate penalty incurred by integer thresholds that force an FER well below the target. Simulations results agree with the SED lower bound of [14] and show the trade-off of rate vs. target FER.

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