NEAR RESONANT APPROXIMATION OF THE BOUSSINESQ SYSTEM
ON A THREE-DIMENSIONAL TORUS

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Abstract. We introduce a new approximation for the rotating stratified Boussinesq system on a three-dimensional torus with anisotropic horizontal periods, based on the study of near resonant interactions. We obtain global existence for the proposed approximate system for arbitrary initial data in the Sobolev space $H^\ell$ with $\ell \geq 1$. This approximation presents novel regularity properties and is sufficiently accurate, with an important feature of coupling effects between slow and fast modes. The restricted convolution estimates, which are key to global existence, essentially hinge on a sharp counting of the relevant number of interactions. An additional regularity advantage arises from a careful examination of some mixed type interaction coefficients. In a wider context, the near resonant approach allows finding a balance between the inclusion of more interacting modes and the improvement of regularity properties, compared to the well-studied singular limit approach based on exact resonance.

1. Introduction

Let $T^3 := [0, 2\pi L_1] \times [0, 2\pi L_2] \times [0, 2\pi]$ be the three-dimensional flat torus with anisotropic horizontal periods $2\pi L_1, 2\pi L_2 > 0$. We are interested in approximating the unforced, rotating stratified Boussinesq system

$$\begin{cases}
\partial_t U^\epsilon + U^\epsilon \cdot \nabla U^\epsilon - \nu_1 \Delta U^\epsilon + \Omega e_3 \times U^\epsilon - N \rho e_3 = -\nabla p \\
\partial_t \rho + U^\epsilon \cdot \nabla \rho - \nu_2 \Delta \rho + N(U^\epsilon \cdot e_3) = 0 \\
\nabla \cdot U^\epsilon = 0,
\end{cases}$$

for unknown $(U^\epsilon, \rho) : T^3 \times \mathbb{R}^+ \to \mathbb{R}^4$, with zero-mean initial data $(U^\epsilon_0, \rho_0) \in H^\ell(T^3; \mathbb{R}^4)$, $\ell \geq 1$ and $\nabla \cdot U^\epsilon_0 = 0$. As an important model in the study of geophysical fluid dynamics, system (1) describes the dynamics of an incompressible fluid with velocity $U^\epsilon$ and density deviation $\rho$ from a linear background state, with $p$ standing for the pressure. In this context, the influence of rotation is measured through $\Omega > 0$ and that of stratification via the Brunt-Väisälä frequency $N > 0$. The positive parameters $\nu_1, \nu_2$ represent the viscosity and heat conductivity, respectively. The relative strength of the effects of rotation and stratification is measured via $\eta = \frac{\Omega}{N}$, which we assume to be fixed and satisfy $\eta \neq 1$. In the case $N = \rho = 0$, the Boussinesq system reduces to the incompressible rotating Navier-Stokes equations. More details on the mathematical background of (1) can be found in e.g. [14].

The pressure term can be removed from (1) using the projection $P = \begin{pmatrix} \mathcal{P}^\epsilon & 0 \\ 0 & 1 \end{pmatrix}$, where $\mathcal{P}^\epsilon$ is the ordinary Leray projection to divergence-free fields in three dimensions. Then, using a

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four-component field \( \mathbf{U} := (U^i, \rho) \), we obtain the system

\[
\partial_t \mathbf{U} + \mathbf{P}(\mathbf{U}^\top \cdot \nabla \mathbf{U}) - \nu \Delta \mathbf{P} \mathbf{U} = N \mathbf{L} \mathbf{U} \quad \text{with} \quad \nabla \cdot \mathbf{U} = 0 \quad \text{and} \quad \mathbf{U}_0 = (U^i_0, \rho_0),
\]

where

\[
\mathcal{L} = \mathbf{P} \begin{pmatrix} \eta J & 0 \\ 0 & J \end{pmatrix} \mathbf{P}, \quad \nu = \begin{pmatrix} \nu_1 1_{3\times 3} & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

and \( \mathbf{U}_0 \in H^\ell(\mathbb{T}^3; \mathbb{R}^4) \), \( \ell \geq 1 \), is divergence-free. Here and below, a 4-vector field is said to be divergence-free if its first 3 components, i.e., the velocity field, are divergence-free. Since the linear operator on the right-hand side of (2) is wave generating, the evolution operator \( e^{Nt \mathcal{L}} \) for the corresponding inviscid, linear problem can be used to transform our system into a more convenient form. In particular, setting \( \mathbf{u} = e^{-Nt \mathcal{L}} \mathbf{U} \) and \( B(\mathbf{U}, \mathbf{U}) = \mathbf{P}(\mathbf{U}^\top \cdot \nabla \mathbf{U}) \), an equivalent system to (2) can be obtained:

\[
\partial_t \mathbf{u} + B(Nt, \mathbf{u}, \mathbf{u}) + A \mathbf{u} = 0 \quad \text{with} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \mathbf{u}_0 = \mathbf{U}_0,
\]

where

\[
B(Nt, \mathbf{u}, \mathbf{u}) = e^{-Nt \mathcal{L}} B(e^{Nt \mathcal{L}} \mathbf{u}, e^{Nt \mathcal{L}} \mathbf{u}) \quad \text{and} \quad A = -e^{-Nt \mathcal{L}} \mathbf{P} \nu \Delta e^{Nt \mathcal{L}} \mathbf{u}.
\]

In this so-called modulated form, nonlinear effects are expressed through the transformed bilinearity \( B(Nt, \cdot, \cdot) \). This filtering technique has been widely used as a method to tackle problems for oscillatory perturbations to evolution equations, see e.g. [19].

The effects of nonlinear interactions in systems (2) and (3) can be analysed in terms of Fourier series. For a zero-mean field \( \mathbf{U} \in L^2(\mathbb{T}^3; \mathbb{C}^4) \), we have \( \mathbf{U}(x) = \sum_{k \in (\mathbb{Z}^3 \setminus \{0\})} e^{ik \cdot x} U_k \), with

\[
\hat{k} = (k_1/L_1, k_2/L_2, k_3)^T
\]

and \( U_k \) for the Fourier coefficients of \( \mathbf{U} \). As illustrated in Section 3, the symbol of \( \mathcal{L} \) is given by a \( 4 \times 4 \) anti-Hermitian matrix. Its spectrum consists of 0 with multiplicity two, and

\[
\pm i\omega_k = \pm i\frac{\sqrt{k_1^2 + k_2^2 + \eta^2 k_3^2}}{|\hat{k}|} \quad \text{for} \quad k = (k_1, k_2, k_3)^T \in \mathbb{Z}^3 \setminus \{0\},
\]

see e.g. [8]. Moreover, there exist corresponding eigenvectors \( \{ r_k^0, r_k^+, r_k^-, r_k^r \} \) that form an orthonormal basis of \( \mathbb{C}^4 \). Note that all our study does not involve the subspace spanned by \( e^{ik \cdot x} r_k^0 \), which on the physical side consists of any 4-vector field of which the velocity component is a potential flow and the density component vanishes. Then define projections

\[
\mathcal{P}_k^\sigma \mathbf{U}(x) = e^{ik \cdot x} \left( U_k \cdot r_k^r \right) r_k^\sigma \quad \text{for} \ \sigma \in \{0, \pm\},
\]

that mutually cancel each other, and express the bilinearity in (2) as a convolution sum

\[
B(\mathbf{U}, \mathbf{V}) = \sum_{k, m, n \in \mathbb{Z}^3 \setminus \{0\}} \sum_{\sigma_1, \sigma_2, \sigma_3 \in \{0, \pm\}} \mathcal{P}^-_{-n} B(\mathcal{P}_k^\sigma \mathbf{U}, \mathcal{P}_m^{\sigma_2} \mathbf{V}),
\]

where the arguments \( \mathbf{U}, \mathbf{V} \) are divergence-free with zero mean.

Quantities related to \( r_k^0 \) are customarily labelled slow, whereas those related to \( r_k^\pm \) fast. Then, for any weakly divergence-free vector field \( \mathbf{U} \in L^2(\mathbb{T}^3; \mathbb{C}^4) \), we can define slow and fast projections as follows:

\[
\mathbf{U}_s = \sum_{k \in \mathbb{Z}^3} \mathcal{P}_k^0 \mathbf{U} \quad \text{and} \quad \mathbf{U}_f = \mathbf{U} - \mathbf{U}_s.
\]
The slow part $U_s$ is in the kernel of $L$, whereas the fast part $U_f$ is in the orthogonal complement of the kernel of $L$ with the orthogonality in terms of the $L^2(\mathbb{T}^3; \mathbb{C}^4)$ inner product.

The eigenexpansion formalism can also be used for the modulated bilinearity of (3). In fact, we have the following expansion for divergence-free and zero-mean vector fields $U, \mathbf{V}$

$$B(\tau, U, \mathbf{V}) = \sum_{k,m,n \in \mathbb{Z}^3 \setminus \{0\}} \sum_{\sigma_1, \sigma_2, \sigma_3 \in \{0, \pm\}} e^{i\omega_{k,m,n} \tau} P_{-\sigma} B(\mathcal{P}_{k}^\sigma U, \mathcal{P}_{m}^{\sigma_2} \mathbf{V}),$$

where $\tau = N\tau$ and

$$\omega_{k,m,n}^\sigma := \sigma_1 \omega_k + \sigma_2 \omega_m + \sigma_3 \omega_n, \quad \text{for} \quad \sigma = (\sigma_1, \sigma_2, \sigma_3) \in \{0, \pm\}^3.$$

Under our notational convention, the input to the bilinearity is represented via the modes with wavevectors $k, m,$ and the output wavevector $-n$.

Exact resonances correspond to triplets $(k, m, n) \in (\mathbb{Z}^3 \setminus \{0\})^3$ satisfying $\omega_{k,m,n}^\sigma = 0$ and $k + m + n = 0$. Restrictions of nonlinear interactions to resonant modes play a decisive role in systems that arise from (2), in the limits $N, \Omega \to \infty$ with fixed $\eta$. [3] Lemma 4.1. The slow-fast dichotomy, reflected in the sign of each of the three interacting modes, results in a decoupling in the aforementioned resonant limits. In the case of purely slow interactions, the 3D quasigeostrophic (3D-QG) system is obtained. Its nonlinear part consists of a restricted decoupling in the aforementioned resonant limits. In particular, 3D-QG has been shown to approximate the exact resonance setting [8]. In particular, 3D-QG has been shown to approximate the dynamics of (2) in various limiting settings [5], [3], [2], [4], [8], [9], [10], [11]. Nevertheless, the fast ageostrophic (AG) part of the solution, containing fast-fast-fast (FFF) wave interactions, plays a significant role as well in the exact resonance setting [3].

Our main goal is to accurately approximate (2), under no limiting considerations, with a well-posed system. The latter will be based on a near resonant bilinearity $\tilde{B}(N\tau, \cdot, \cdot)$, which takes into account fundamentally more input and output interactions between the $(k, m, n)$ triplet modes in comparison to that of an exact resonance system.

1.1. Near resonant approximation based on restricted interactions. The eigenmode expansion of $B(N\tau, \cdot, \cdot)$ in (5) allows us to rigorously define a restricted bilinearity $\tilde{B}(N\tau, \cdot, \cdot)$, based on a relaxation of the exact resonance notion. In more detail, the proposed restriction on the level of interaction sets is quantified by two so-called bandwidths that depend on $\max\{|\tilde{k}|, |\tilde{m}|, |\tilde{n}|\}$, and satisfy

$$\delta(k, m, n), \delta^*(k, m, n) \in \left[0, \min\left\{\frac{\eta}{2}, \frac{1}{2}\right\}\right].$$

The purpose of $\delta$ and $\delta^*$ is to control the magnitude $|\omega_{k,m,n}^\sigma|$ in two different settings. On one hand, we define the restricted fast-fast-fast (FFF) interactions set

$$\mathcal{N}^{FFF} := \{(k, m, n) \in (\mathbb{Z}^3 \setminus \{0\})^3 : k + m + n = 0, \min_{\sigma \in \{\pm\}^3} |\omega_{k,m,n}^\sigma| \leq \delta(k, m, n)\},$$

with

$$\delta(k, m, n) \lesssim \min\{|\tilde{k}|^{-1}, |\tilde{m}|^{-1}, |\tilde{n}|^{-1}\}.$$

On the other hand, we define mixed near resonant subsets, for fast-fast-slow (FFS) and fast-slow-fast (FSF) interactions, as follows

$$\mathcal{N}^{FFS} := \{(k, m, n) \in (\mathbb{Z}^3 \setminus \{0\})^3 : k + m + n = 0, |\omega_k - \omega_m| \leq \delta^*(k, m, n)\},$$

$$\mathcal{N}^{FSF} := \{(k, m, n) \in (\mathbb{Z}^3 \setminus \{0\})^3 : k + m + n = 0, |\omega_k - \omega_n| \leq \delta^*(k, m, n)\},$$
under the assumption that
\[ \delta^*(k, m, n) \lesssim \min\{|\bar{k}|^{-\zeta}, |\bar{m}|^{-\zeta}, |\bar{n}|^{-\zeta}\}, \text{ for a parameter } \zeta > 0. \]

Under our notational convention, the first two superscripts of the mixed sets stand for the signs of the input modes, with the last one reserved for that of the output. By (7) and \( \min\{\eta, 1\} \leq \omega_k \) for all \( k \in \mathbb{Z}^3 \), there are no near resonant interactions between one fast and two slow modes, namely there are no FSS, SFS and SSF interactions. Also, in view of (6), all slow-slow-slow (SSS) interactions are trivially exact resonances. Finally, we will make no restrictions on slow-fast-fast (SFF) interactions since this is permitted as far as global existence is concerned.

In order to use our newly introduced sets to restrict the bilinearity of (3), it is necessary to keep track of the interaction signs involved. Hence, we introduce the following:

\[
B^\sigma_{k,m,n}(\tau, U, V) := e^{i\omega_{k,m,n}\tau} \mathcal{P}_{n}^{-\sigma_2} B(\mathcal{P}_{k}^{\sigma_1} U, \mathcal{P}_{m}^{\sigma_2} V),
\]

with the convention that \( B^\sigma_{k,m,n}(U, V) = B^\sigma_{k,m,n}(0, U, V) \). Here and in what follows, for the sake of brevity, we write \( k, m, n; \text{conv} \) in order to designate the summing index, \( k, m, n \in \mathbb{Z}^3 \setminus \{0\} \), \( k + m + n = 0 \), of any convolution sum that occurs throughout the text. In addition, the default range for any sign occurring in the following sums is \( \{0, \pm\} \), up to the specified constraints. Then, the slow output part of the original bilinearity can be expanded as follows

\[
B_s(\tau, U, V) = \sum_{k,m,n;\text{conv}} B_{k,m,n}^{000}(U, V) + \sum_{k,m,n;\text{conv}} \sum_{(\sigma_1, \sigma_2) \neq (0,0)} B_{k,m,n}^{\sigma_1 \sigma_2 0}(\tau, U, V).
\]

In an analogous manner, the fast output terms can be expanded into terms with purely fast and mixed input as follows

\[
B_f(\tau, U, V) = \sum_{k,m,n;\text{conv}} \sum_{\sigma_1 \sigma_2 \sigma_3 \neq 0} B_{k,m,n}^{\sigma_1 \sigma_2 \sigma_3}(\tau, U, V) + \sum_{k,m,n;\text{conv}} \sum_{\sigma_1 \sigma_2 = 0} \sum_{\sigma_3 \neq 0} B_{k,m,n}^{\sigma_1 \sigma_2 \sigma_3}(\tau, U, V).
\]

Equipped with this notation, we define the approximate terms with slow output as follows:

\[
\tilde{B}_s(\tau, U, V) = \sum_{k,m,n;\text{conv}} B_{k,m,n}^{000}(U, V) + \sum_{k,m,n;\text{conv}} \sum_{\sigma_1 \sigma_2 < 0} B_{k,m,n}^{\sigma_1 \sigma_2 0}(\tau, U, V) 1_{N_{FFS}}(k, m, n),
\]

where \( 1_{N_{FFS}}(\cdot, \cdot, \cdot) \) is the characteristic of the set FFS. In contrast to the original slow bilinearity, the input fields are now either both fast or both slow since, again, the restriction on \( \delta^* \) in (7) trivially excludes SFS and FSS terms in our approximation. We emphasize that the \( \tau \)-independent SSS part of the bilinearity is unrestricted in (12) and must all be considered in any notion of resonances. Finally, we note that FFS interactions with the same fast sign have been excluded from our considerations, as the corresponding values of \( |\omega_{k,m,n}^\sigma| \) are bounded from below by \( 2\min\{\eta, 1\} \) which is greater than the upper bound in (7).

In a similar manner, we define an approximate fast bilinearity as follows

\[
\tilde{B}_f(\tau, U, V) := \sum_{k,m,n;\text{conv}} \sum_{\sigma_1 \sigma_2 \sigma_3 \neq 0} B_{k,m,n}^{\sigma_1 \sigma_2 \sigma_3}(\tau, U, V) 1_{N_{FFS}}(k, m, n)
+ \sum_{k,m,n;\text{conv}} \sum_{\sigma_1 \sigma_2 < 0} B_{k,m,n}^{\sigma_1 \sigma_2 0}(\tau, U, V) 1_{N_{FSS}}(k, m, n) + \sum_{k,m,n;\text{conv}} \sum_{\sigma_2 \sigma_3 \neq 0} B_{k,m,n}^{0 \sigma_2 \sigma_3}(\tau, U, V).
\]
Our fast approximation only excludes SSF interactions, with restrictions posed on the FFF and FSF subsets. The latter is also subjected to a sign restriction, similarly to the slow mixed term. Nevertheless, the SFF part of the bilinearity is identical to the original.

Since $\nu_1 \neq \nu_2$ in general, a further approximation is needed for the Laplacian terms. In particular, the non-commutativity of the matrix $\nu$ with fast and slow projections introduces unwanted mixing effects. Thus, we consider the modified diagonal Laplacian operator

\[(14) \quad \tilde{A} \tilde{U} := -\tilde{\nu}_{11} \Delta \tilde{U}_s - \tilde{\nu}_{22} \Delta \tilde{U}_f.\]

Here, $\tilde{\nu}_{11}$ and $\tilde{\nu}_{12}$ denote scalar pseudodifferential operators of degree 0, depending linearly on $\nu_1$ and $\nu_2$, which result from eliminating the fast scale in $A$. In other words, $\tilde{A}$ is independent of $\tau = Nt$. Finally, we note that $\tilde{A}$ is elliptic with ellipticity constant $\nu_{\min} := \min\{\nu_1, \nu_2\}$.

We set

\[(15) \quad \tilde{B}(\tilde{U}, \tilde{V}) := \tilde{B}(0, \tilde{U}, \tilde{V}), \quad \tilde{B}_f(\tilde{U}, \tilde{V}) := \tilde{B}_f(0, \tilde{U}, \tilde{V}), \quad \tilde{B}_s(\tilde{U}, \tilde{V}) := \tilde{B}_s(0, \tilde{U}, \tilde{V})\]

for the restricted bilinearity of the original system [2]. In particular,

\[\tilde{B}(\tilde{U}, \tilde{U}) = \tilde{B}_s(\tilde{U}, \tilde{U}) + \tilde{B}_f(\tilde{U}, \tilde{U}),\]

via (12), (13), and (15) so that we arrive at the near resonant approximation of the Boussinesq system

\[(16) \quad \partial_t \tilde{U} + \tilde{B}(\tilde{U}, \tilde{U}) + \tilde{A} \tilde{U} = NL\tilde{U}.\]

In the absence of a dissipative term, system (16) would satisfy the fundamental property of $L^2$ energy conservation, since the FFS and FSF sets crucially share a common bandwidth $\delta^*$.  

1.2. Main results. The existence part for our proposed system is covered by the following a-priori estimate. Here and below, the dependence of a constant on other quantities is always in the form of a positive-valued, smooth function. Dependence on $T^3$ is just a shorthand notation for dependence on $L_1, L_2$.

**Theorem 1.** Let $\nu_1, \nu_2, T > 0$ and $\tilde{U}_0 \in H^1(T^3; \mathbb{R}^4)$ be a divergence-free and zero-mean vector field. Consider a solution $\tilde{U}$ of the near resonant approximation to the Boussinesq system (16) for $t \in [0, T]$, with initial data $\tilde{U}_0$. Moreover, let $\delta, \delta^* \in [0, \min\{\frac{7}{8}, \frac{3}{4}\})$ be the bandwidths for the sets $\mathcal{N}^{FFF}$ and $\mathcal{N}^{FFS}$, respectively. If

\[\delta \lesssim \min\{\frac{\bar{k}}{-1}, |\bar{n}|^{-1}, |\bar{n}|^{-1}\} \quad \text{and} \quad \delta^* \lesssim \min\{\frac{\bar{k}}{-\frac{6}{8}}, |\bar{n}|^{-\frac{6}{8}}, |\bar{n}|^{-\frac{6}{8}}\},\]

then there exist constants $C_1 = C_1(\eta, T^3)$ and $C_2 = C_2(\eta, T^3, \nu_{\min}^{-1}, \|\tilde{U}_0\|_{L^2}, \|\tilde{U}_s(0)\|_{H^1})$ both of which also depend on the implied constants in the above conditions, so that

\[(17) \quad \|\tilde{U}_s(T)\|_{H^1}^2 + 2\nu_{\min} \int_0^T \|\tilde{U}_s\|_{H^2}^2 dt \leq C_1 \left(\nu_{\min}^{-1}\|\tilde{U}_0\|_{L^2}^2 + \|\tilde{U}_s(0)\|_{H^1}\right)^2\]

and

\[(18) \quad \|\tilde{U}_f(T)\|_{H^1}^2 + \nu_{\min} \int_0^T \|\tilde{U}_f\|_{H^2}^2 dt \leq C_2 \|\tilde{U}_f(0)\|_{H^1}^2.\]
Standard existence results on local in time strong solutions

\[ \tilde{U} \in C \left( \left[ 0, T \right); H^{\ell}(\mathbb{T}^3; \mathbb{R}^4) \right) \cap L^2 \left( \left[ 0, T \right); H^{\ell+1}(\mathbb{T}^3; \mathbb{R}^4) \right) \quad \ell \geq \frac{1}{2}, \]

for systems with Navier-Stokes type bilinearity \[6, \text{Theorem 3.5}\] are applicable to our system. In particular, the modified viscosity matrix \( \tilde{\nu} \) is appropriately controlled due to Lemma 5, so that all the corresponding energy estimates go through. Crucially, our modified bilinearity obeys better product rules than an ordinary Navier-Stokes one in three dimensions, as we will prove in the sequel. Thus, well-posedness of the approximate system follows from Theorem 1, by arguments similar to \[6, \text{Theorem 3.5 and its remark, Corollary 3.1}\] and \[7, \text{Lemma 6.5}\].

**Remark 1.** An interesting partial decoupling property can be observed in (17): the \( L_1^\infty H_2^1 \) and \( L_1^2 H_2^2 \) norms of the slow part are bounded independently of \( \| U_f(0) \|_{H^1} \). This contrasts with the coupling effect due to near resonance approximation of the bilinearities.

The proof of Theorem 1 is based on a separate treatment for the slow and fast parts of the solution, taking into account some good properties of the slow equation. In more detail, we equivalently write the slow output part of (16) as a transport equation for the linear potential vorticity \( Q = \partial_1 U_2 - \partial_2 U_1 - \eta \partial_3 \rho \) which corresponds to the linear part of Ertel’s potential vorticity. On the Fourier side, the linear potential vorticity is expressed as

\[
Q = i \sum_{k \in \mathbb{Z}^3} e^{i k \cdot x} \sqrt{k_1^2 + k_2^2 + \eta^2 k_3^2} \left( U_k \cdot r_0^0 \right).
\]

By introducing operator \( L_{pv} : H^1(\mathbb{T}^3; \mathbb{R}^4) \to L^2(\mathbb{T}^3; \mathbb{R}) \) such that \( L_{pv} U = L_{pv} U_s = Q \), we derive the potential vorticity formulation of the slow equation as

\[
\partial_t Q + L_{pv} \tilde{B}_s(U, U) - \tilde{\nu}_{11} \Delta Q = 0.
\]

The slow bilinearity \( L_{pv} \tilde{B}_s(U, U) \) splits into two parts: \( L_{pv} B_s(U_s, U_s) \) and \( L_{pv} \tilde{B}_s(U_f, U_f) \), while the slow-fast terms vanish by definition of \( \tilde{B}_s \). The purely slow part of the bilinearity has a favourable transport structure. On the other hand, the slow mixed part, which is absent in the exact resonance setting, benefits from interaction coefficients that are proportional to \( \omega_{k_{mn}}^{\delta^*} \) and counteract coupling, see Section 6.3. In fact, smaller values of \( \delta^* \) have a stronger effect on the slow part.

Further regularity gains, for both the slow and fast output parts of (16), are established via lattice point counting methods. In particular, by a standard harmonic analysis argument, the number of interactions in a near resonance set determines the regularity properties in the estimates of the corresponding restricted convolution. This approach is standard by now in the exact resonance literature, see \[11, \text{Lemma 3.1}\] and \[6, \text{Lemma 6.2}\], but presents new challenges in the near resonance case.

The number of interactions in the sets \( \mathcal{N}^{FFF}, \mathcal{N}^{FFS} \) and \( \mathcal{N}^{FSF} \) is estimated via two different approaches. For purely fast interactions included in \( \mathcal{N}^{FFF} \), it suffices to investigate the cardinality of the localised set

\[
\left\{ k \in \mathbb{Z}^3 \setminus \{ \vec{0} \} : \min_{\sigma \in \{\pm\}^3} |\omega_{k_{mn}}^\delta| \leq \delta, \quad \frac{1}{2} \mid \vec{n} \mid \leq |\vec{k}| \leq \mid \vec{n} \mid \right\} \quad \text{for fixed} \quad n \in \mathbb{Z}^3,
\]
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which turns out to be $O(\delta |\tilde{n}|^3)$. In detail, the lattice counting method in this case utilises tools developed in [7], in the context of a near resonant approximation to the rotating Navier-Stokes equations, which reduce the counting problem to volume estimates for sublevel sets. In order to obtain further control on the volume in question, we rely on tools from the theory of elliptic integrals. Finally, we remark that our result differs from the corresponding result in [7], for the rotating Navier-Stokes system, by a logarithmic factor in $\delta$

On the other hand, for mixed interactions included in $\mathcal{N}^{FFS}$ and $\mathcal{N}^{FSF}$, we essentially need to investigate the cardinality of the following set:

$$\left\{ k \in \mathbb{Z}^3 \setminus \{0\} : \left| f_\eta \left( \frac{k_3}{|k|} \right) - f_\eta \left( \frac{n_3}{|n|} \right) \right| \leq \delta^*, M \leq |\tilde{k}| < 2M, |\tilde{k}| \leq |\tilde{n}| \right\}$$

for fixed $M > 0, n \in \mathbb{Z}^3 \setminus \{0\}$, where

$$f_\eta(x) := \sqrt{1 + (\eta^2 - 1)x^2}.$$ 

The cardinality turns out to be $O(\sqrt{\delta^* M^3} + M^2)$, which is sharp as shown in Appendix [3].

Theorem [1] is the analog of the recently established result [7, Theorem 1.1] on the rotating Navier-Stokes system, with some crucial differences. In that work, well-posedness results were obtained for a near resonant restricted Navier-Stokes type system, with bandwidth bounded as $\delta_{NS} \log(\delta_{NS}) \leq \min\{c, |\tilde{k}|^{-1}, |\tilde{n}|^{-1}, |\tilde{n}|^{-1}\}$, for an absolute constant $c$. Several subtleties arise in comparison. First, for the rotating Navier-Stokes system, slow modes are characterized by a null vertical component, i.e. $k_3 = 0$, whereas in [2] slow modes occur for every wavevector, thus SSS resonance occurs for every triplet $k, m, m$ in the convolution sum. Second, whereas the corresponding result on the rotating Navier-Stokes near resonance condition is:

$$\left| \frac{k_3}{|k|} \pm \frac{m_3}{|m|} \pm \frac{n_3}{|n|} \right| \leq \delta_{NS},$$

the non-linearity of $f_\eta$ further complicates things. In fact, non-SSS near resonances are characterized by:

$$\left| \sigma_1 f_\eta \left( \frac{k_3}{|k|} \right) + \sigma_2 f_\eta \left( \frac{m_3}{|m|} \right) + \sigma_3 f_\eta \left( \frac{n_3}{|n|} \right) \right| \leq \delta_\sigma,$$

where

$$\delta_\sigma = \begin{cases} 
\delta, & \text{when } \sigma_1 \sigma_2 \sigma_3 \neq 0 \\
\delta^*, & \text{when } (\sigma_1, \sigma_2, \sigma_3) = (\pm, \mp, 0), (\pm, 0, \mp) \\
2\max\{\eta, 1\}, & \text{when } \sigma_1 = 0, \sigma_2 \sigma_3 \neq 0 \\
\min\{\eta, 1\} - \epsilon, & \text{for any remaining cases with } \bar{\sigma} \neq \bar{0} 
\end{cases}$$

with arbitrarily small $\epsilon > 0$. Note that, a strictly positive gap between $\delta_\sigma$ and 0 substitutes small-divisor estimates, in terms of control of $|\omega_{kmn}^\sigma|^{-1}$. Lastly, a-priori estimates for [16] have to take into account the slow-fast splitting of the system that will be exploited in the proof of Theorem [1] in contrast to the unified approach of the rotating Navier-Stokes result.

We now proceed to the statement of our improved convolution sum estimates. First, we present a result on the restricted FFF terms, in the spirit of [3, Theorem 4.1] for the exact resonance case. We also mention the corresponding results on the rotating Navier-Stokes system, [1, Theorem 3.1] and [7, Theorem 1.3], in the exact and near resonance settings, respectively. The regularity for the restricted convolutions is measured in Sobolev spaces, with the notation $D^\ell U = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} e^{ik \cdot x} |\tilde{k}|^\ell U_k$. 

Theorem 2. Let $\ell \in (0, 1]$ and $u, v \in H^{\ell+1}(\mathbb{T}^3; \mathbb{R}^4)$ be divergence-free and zero-mean vector fields. Moreover, in the FFF near resonance set, let $\delta \in \{0, \min\{\frac{\eta}{2}, \frac{1}{2}\}\}$ satisfy
\[
\delta(k, m, n) \leq C_\delta \min\{|k|^{-1}, |m|^{-1}, |n|^{-1}\}.
\]
Then, the following estimate holds true
\[
|\left\langle D^\ell \widetilde{B}_s(u_f, v_f), D^\ell v_f \right\rangle| \lesssim \|u_f\|_{H^\ell} \|v_f\|_{H^\ell} \|v_f\|_{H^{\ell+1}},
\]
with the implied constant depending on $\eta, \ell, L_1, L_2, C_\delta$.

Theorem 2 allows us to gain $\frac{1}{2}$ derivatives when performing bilinear estimates in three dimensions, thus making the interactions 2D-like. In more detail, given two scalar functions $u, v$ defined on $\mathbb{R}^d$ or $\mathbb{T}^d$, and $a, b \in \{0, \frac{d}{2}\}$, with $0 < a + b$, we have
\[
\|uv\|_{H^{a+b-\frac{d}{2}}} \lesssim \|u\|_{H^a} \|v\|_{H^b},
\]
see Appendix A and [6] Lemma 6.2. Thus, the derivative cost of estimating a product in homogeneous Sobolev spaces with positive index, which are not algebras, is given by $\frac{d}{2}$.

Restrictions to the set $\mathcal{N}^{FFS}$ present more subtleties, due to the corresponding interaction coefficients being connected to the mixed interaction bandwidth, $\delta^*$. This phenomenon is not observed in the context of exact resonance, as the interaction coefficient is identically 0 when restricted to the exact resonance set – see e.g. [4], [8]. Furthermore, the importance of the interaction coefficients for the mixed terms in near resonance has not been extensively noted, to the best of our knowledge. An exception is [23], which focuses on long-time asymptotics for the Oceanic Primitive Equations and identifies the importance of interaction coefficients, as far as near resonant interactions are concerned. Moreover, in [18], among various results on numerics and multiscale asymptotic analysis for three fluid models, including (1), the smallness of interaction coefficients is highlighted, in a different near resonant setting. Finally we mention [17], concerning the connection of interaction coefficients with higher order wave interactions.

The following Theorem provides convolution sum estimates for FFS terms.

Theorem 3. Let $\ell \geq 0$ and $u, w \in H^{\ell+1}(\mathbb{T}^3; \mathbb{R})$ be divergence-free and zero-mean vector fields. Consider any $\zeta \in \{0, \frac{3}{2}\}$ and $\delta^* \in \{0, \min\{\frac{\eta}{2}, \frac{1}{2}\}\}$ that satisfy
\[
\delta^*(k, m, n) \leq C_{\delta^*} \min\{|k|^{-\zeta}, |m|^{-\zeta}, |n|^{-\zeta}\}.
\]
Then the FFS mixed terms satisfy the following estimate
\[
\left|\left\langle D^\ell \widetilde{B}_s(u_f, u_f), D^\ell u_f \right\rangle\right| \lesssim \|D^{\frac{3}{2}-\frac{\zeta}{2}} u_f\|_{L^2} \|D^{1+\frac{\zeta}{2}} u_f\|_{L^2} \|D^{\ell} w\|_{L^2},
\]
for any $a \in (0, \frac{3}{2} - \frac{\zeta}{4})$ and $\ell_1 \geq 0, \ell_2 \in \mathbb{R}$ with $\ell_1 + \ell_2 = 2\ell - 1$, where the implied constant depends on $\eta, L_1, L_2, \ell_1, \ell_2, a, \zeta, C_{\delta^*}$.

A straightforward modification of the proof of the previous theorem yields the following result on the level of linear potential vorticity.

Lemma 1. Under the assumptions of Theorem 3, the following estimate holds true
\[
\left|\left\langle L_{p_0} \widetilde{B}_s(u_f, u_f), L_{p_0} w \right\rangle\right| \lesssim \|D^{\frac{3}{2}-\frac{\zeta}{2}} u_f\|_{L^2} \|D^{1+\frac{\zeta}{2}} u_f\|_{L^2} \|L_{p_0} w\|_{L^2},
\]
for all $a \in (0, \frac{3}{2} - \frac{\zeta}{4})$, where the implied constant depends on $\eta, L_1, L_2, \zeta, a, C_{\delta^*}$. 
Theorem 3 highlights the importance of the interaction coefficients, which will act as a smoothing factor in the convolution sums. In fact, the parameter \(3 - \frac{\zeta}{2}\) provides an upper bound for the exponent in the power law for the cardinality of \(\mathcal{N}^{FFS}\). Whereas we can already save some derivatives via Lemma 8, further calculations on the interaction coefficients, as in Lemma 11, allow us to lower the derivative cost even more. In particular, the FFS estimate contains \(2\ell - \frac{5}{4} - \frac{5\zeta}{4}\) derivatives. As a consequence, our growth bound for \(\delta^*\) is justified as follows. After the standard \(L^2\) energy equality for (16) is obtained, an \(L^\infty L^2\) estimate for \(Q = L_{p\epsilon} u\) can only require an estimate for \(u_f\) in \(L^\infty H^4\), but not in any higher regularity space. Therefore, in view of Lemma 11 which is essentially the case \(\ell = 1\), we can only afford \(\frac{3}{2} - \frac{5\zeta}{4}\leq 0\).

The remaining mixed interactions are covered in the following.

**Theorem 4.** Let \(\ell \geq 0\) and \(u, v \in H^{\ell+1}(\mathbb{T}^3; \mathbb{R}^4)\) be divergence and zero-mean vector fields. Consider any \(\zeta \in \left[\frac{6}{5}, 2\right]\) and \(\delta^* \in \left[0, \min\left\{\frac{\ell}{2}, \frac{\zeta}{2}\right\}\right]\) that satisfy (22). Then the FFS mixed terms satisfy the following estimates

\[
\left| \left\langle D^\ell \widetilde{B}_f(u_f, v_s), D^\ell u_f \right\rangle \right| \leq \|D v_s\|_{L^2} \|D^{3 + \ell_1 - \frac{\xi}{2} - a} u_f\|_{L^2} \|D^{2\ell - \ell_1 + a} u_f\|_{L^2} + \|D^{\ell_1 + 1} v_s\|_{L^2} \|D^{\frac{3}{2} - \frac{\xi}{2} - a'} u_f\|_{L^2} \|D^{2\ell - \ell_1 + a'} u_f\|_{L^2},
\]

for any \(a, a' \in (0, \frac{3}{2} - \frac{\xi}{2})\) and \(\ell, \ell_1 > 0\) with \(\ell_1 < 2\ell\), where the implied constant depends on \(\eta, L_1, L_2, \ell, \ell_1, a, a', \zeta, C_{\delta^*}\).

The difference of solutions between the original and the restricted system can be estimated under appropriate lower bounds for the slow and fast bandwidths. The result depends on the ratio \(\nu_R := \frac{\nu_{\min}}{\nu_{\max}}\) with \(\nu_R \in [1, \infty)\).

**Theorem 5.** Let \(\zeta \in \left[\frac{6}{5}, 2\right]\), \(\ell > \max\{3 + \zeta, \frac{9}{2}\}\) and \(\ell' \in [1, \ell - 2 - \zeta]\). Moreover, consider two divergence free and zero-mean vector fields \(U_0, \tilde{U}_0 \in H^{\ell}(\mathbb{T}^3; \mathbb{R}^4)\). Let \(U, \tilde{U}\) be solutions of the Boussinesq system (2) and the approximate system (16), with initial data \(U_0, \tilde{U}_0\), respectively. Suppose, for positive constants \(E_0, c_s, c_f\), that

\[
\|U_0\|_{H^\ell} \leq E_0, \quad \|\tilde{U}_0\|_{H^\ell} \leq E_0,
\]

and suppose

\[
\delta^*(k, m, n) \geq c_s \min\{|k|^{-\zeta}, |m|^{-\zeta}, |n|^{-\zeta}\}, \quad \delta(k, m, n) \geq c_f \min\{|k|^{-1}, |m|^{-1}, |n|^{-1}\}.
\]

Then for all \(\nu_1, \nu_2 > 0\) there exists a constant \(C = C(c_s, c_f, \ell, \ell', \zeta, \eta, \mathbb{T}^3, \nu_{\max}, \nu_R)\), which remains bounded as \(\nu_{\max}\) vanishes, and a time \(T = T(E_0, \ell, \eta, \mathbb{T}^3)\) such that

\[
\|U - \tilde{U}\|_{H^\ell}^2 \leq \|U_0 - \tilde{U}_0\|_{H^\ell}^2 + CN^{-2} \quad \text{for all} \quad t \in [0, T].
\]

Theorem 5 is in line with the corresponding result from [7]. An extra subtlety comes from the different lower bounds for the bandwidths. This is nevertheless expected, due to the presence of more mixed interactions, as is shown by our arguments in Appendix B.

**Remark 2.** A significant phenomenon in the exact resonance case is the dependence of error estimates on \(L_1, L_2\). [3, Theorem 5.3], [10, Theorem 2]. In contrast to that, all our results, including the error estimates of Theorem 5, depend smoothly on the domain parameters.
Remark 3. We remark that Theorems 1, 5 continue to hold true, up to certain modifications, in the presence of a forcing term $F$. In more detail, local in time regularity assumptions, like $F \in L^2 ([T_0, T_0 + 1]; H^\ell (\mathbb{T}^3))$, for all $T_0 > 0$ and suitable $\ell > 0$, would yield similar results, after taking into consideration bounds for the relevant norms of $F$. In the exact resonance literature, results in that direction have been obtained in [2], [3], with [10] covering some different regularity assumptions on the forcing.

1.3. Literature review and outline. A non-exhaustive list of some previous works, concerning mainly the periodic case, follows. In the inviscid case with well-prepared data and a specific domain geometry, a rigorous study of convergence results for the asymptotic limit of (1) appears in [10]. On the other hand, a detailed study of the limit equations in the resonant setting has been carried in [4], in the absence of external forces, and [2], [3] in the forced case. In this series of works, rigorous existence results were proved for the limit quasigeostrophic and ageostrophic systems, together with estimates of error in Sobolev spaces. An asymptotic description for the weak limit of solutions to the Boussinesq equations was given in [8], [9], including the case of an axis of rotation different from $e_3$ and that of arbitrary initial data. In this series of works, the limiting arguments were carried in the cases of high stratification and finite rotation, and that of high stratification and rotation. Moreover, the fast-slow splitting and interaction coefficients for the Boussinesq system, in the context of exact resonance, are extensively used.

In [10], global well-posedness of (1) was shown for initial data in $H^\ell (\mathbb{T}^3; \mathbb{R}^4)$, $\ell > 1$, in the presence of a forcing term. Those results hold true for almost all domain ratios and arbitrary initial data. The almost-periodic case, under stratification effects only, has been examined in [12]. As far as works related to multiscale asymptotic analysis are concerned, we mention [24], [25] and [18]. Finally, in [15], singular three-scale limit methods have been applied to (1) in the periodic case, with well-prepared initial data.

We remark that, within [2], Section 5], a different concept of near resonance is presented. In more detail, in the so-called quasiresonance framework of that work, the near resonance condition is expressed via the distance of a ratio $\eta$ to a distinguished value $\eta^*$, which corresponds to exact resonance. Then, the obtained results are used in some small-divisor estimates. Nevertheless, the aforementioned analysis leads to results which feature a discontinuous dependence on the domain parameters, due to the methods used therein.

On the physical front, the important effect of near resonance is analysed for the forced Boussinesq system in the context of rotating stratified turbulence in, amongst other works, [20], [21], [16].

The remaining of this work is organized as follows. We first review our notational conventions in Section 2 with Section 3 containing some necessary preliminary results. The restricted convolution results that will be used in the proofs of Theorems 2, 3 and 4 are proved in Section 4. Then, Section 5 contains estimates on the number of FFF, FFS and FSF near resonant interactions, via the reduction of lattice point counting to volume estimates. Finally, Section 6 contains the proofs for the main results stated in the introduction.

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2. Notation

We briefly summarize some of the notation and conventions used throughout the text. First, a sum \( \sum_{(k,m,n) \in \mathbb{Z}^3} \) will be simply denoted by \( \sum_{k,m,n;\text{conv}} \). With the domain-adjusted wavevector \( \tilde{k} \) already defined in the Introduction, we further define \( \tilde{k}_H := (\tilde{k}_1, \tilde{k}_2)^\top \), corresponding to the horizontal part of \( \tilde{k} \), with \( k_H \) reserved for the case \( L_1 = L_2 = 1 \). The same conventions carry over to any \( m, n \in \mathbb{Z}^3 \) occurring throughout the text. We will write \( \tilde{k}_\eta := (\tilde{k}_1, \tilde{k}_2, \eta k_3)^\top \), for the modification of a given vector \( k \in \mathbb{Z}^3 \) in the vertical direction, so that the dispersion relation for (1) is given by \( k \). Moreover, we will write \( \Delta_\eta \) for the modified Laplacian

\[
\Delta_\eta = \partial^2_{x_1} + \partial^2_{x_2} + \eta^2 \partial^2_{x_3}.
\]

As far as the viscosity and heat conductivity constants in the dissipative terms are concerned, we define \( \nu_{\text{min}} := \min\{\nu_1, \nu_2\} \) and \( \nu_{\text{max}} := \max\{\nu_1, \nu_2\} \). Finally, if \( \tilde{k} \in \mathbb{Z}^3 \) is a 3D domain-adjusted wavevector, then we define \( \tilde{k}' := (\tilde{k}_1, \tilde{k}_2, k_3, 0)^\top \).

If the variables \( A_1, \ldots, A_k \) are known, we will write \( A \lesssim A_1, \ldots, A_k \) \( B \) in order to denote an inequality of the form

\[
A \leq cB, \quad \text{with} \quad c := c(A_1, \ldots, A_k).
\]

Unless otherwise stated, the implied constant in estimates of the aforementioned form will depend on \( \eta \). The letter \( C \) is reserved for constants, whose possible dependencies will be made explicit in each case.

The standard inner product in \( L^2(\mathbb{T}^3; \mathbb{C}^4) \) will be denoted by

\[
\langle f, g \rangle = \int_{\mathbb{T}^3} (f(x) \cdot \overline{g(x)}) \, dx,
\]

with \( (f \cdot g) \) standing for the ordinary dot product for vectors. The version of the Fourier transform that we use throughout this work is

\[
g_n := \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} g(x) e^{-in \cdot x} \, dx,
\]

for \( n \in \mathbb{Z}^3 \), so that \( g(x) = \sum_n g_n e^{in \cdot x} \). The corresponding Parseval’s identity is given by:

\[
\langle f(x), g(x) \rangle = |\mathbb{T}^3| \sum_{n \in \mathbb{Z}^3} f_n \cdot \overline{g_n}.
\]

Given a vector field \( u \in L^2(\mathbb{T}^3; \mathbb{R}^4) \), with Fourier coefficients \( u_n \), we write \( u_n^\sigma := (u_n \cdot \overline{r_n}) \), for \( n \in \mathbb{Z}^3 \) and a choice of sign \( \sigma \in \{0, +, -\} \). In a similar manner, the Fourier coefficients of the potential vorticity \( Q \) will be denoted by \( Q_n \). Finally, we recall that in the class of fields \( U : \mathbb{T}^3 \rightarrow \mathbb{R}^4 \) we have \( \overline{U_n} = U_{-n} \).

Since the spherical coordinate system will be widely used in what follows, we reserve the notation \((\theta_k, \phi_k)\) for the polar and azimuthal angles of the wavevector \( k \in \mathbb{Z}^3 \), respectively. In addition, we set \( c_k := \cos \theta_k \), with a similar convention for \( m, n \). Then the dispersion relation can be expressed in a simpler manner as \( \omega_k = \sqrt{1 + (\eta^2 - 1)c_k^2} \).

The dependence of the mixed and fast bandwidths \( \delta^\ast, \delta_\ast \), on the wavevectors will be explicitly given, unless otherwise stated.
Finally, we assume, without loss of generality, that the velocity field has a zero mean. This holds true up to a translation of the velocity by the mean drift, see e.g. [7] for more details. In the class of zero-mean fields $U$ defined on $\mathbb{T}^3$, we have $\|U\|_{H^L} = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |\hat{k}|^2 |U_k|^2$.

3. Preliminaries

3.1. The linear problem and the eigenvalues of $L$. We recall some elementary facts on the spectrum of the linear operator $L$. These will be used to further examine the fast and slow parts of a solution of (12). We begin with considering the linear initial value problem corresponding to (2), ignoring the viscosity and heat effects, namely:

\begin{equation}
\partial_\tau V = LV, \quad \text{with divergence free} \quad V(0) = V_0 \in H^L(\mathbb{T}^3; \mathbb{C}^4),
\end{equation}

for some $\ell \geq 0$. In particular, we have the following.

**Proposition 1.** Let $V_0 \in H^L(\mathbb{T}^3; \mathbb{C}^4)$ be divergence free with zero-mean. Then, a unique solution $V \in C^\infty(\mathbb{R}; H^L(\mathbb{T}^3; \mathbb{C}^4))$ for the system (27) is given by $V(\tau) = e^{\tau L}V_0$.

The claim can be shown by arguing on the level of the Fourier transform, since $e^{\tau L}$ is a unitary operator. Also, one can easily deduce that $e^{\tau L}$ commutes with $(-\Delta)^{\ell}$, for $\ell \in \mathbb{R}^+ \cup \{0\}$, via the use of Fourier series.

On the Fourier side, we have $P_k = 1 - \frac{k\omega_k}{|k|^2}$, for $k \in \mathbb{Z}^3 \setminus \{0\}$. Thus,

$$P_k = \begin{pmatrix} P_k^T & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L_k = P_k \begin{pmatrix} \eta J & 0 \\ 0 & J \end{pmatrix} P_k.$$

We recall from [9] that the eigenvalues of $L_k$ are given by 0 with multiplicity 2 and

$$\pm \omega_k = \pm i \sqrt{|\hat{k}_H|^2 + \eta^2 k_3^2 |\hat{k}|^2}, \quad \text{for} \quad k \in \mathbb{Z}^3 \setminus \{0\}.$$

For eigenvectors, we introduce the following vectors depending on wavevector $k \in \mathbb{Z}^3$:

\begin{equation}
\alpha_k := \begin{cases} 
(-\eta \tilde{k}_2 k_3, \eta \tilde{k}_1 k_3, 0, |\hat{k}_H|^2)^T + i \omega_k \left(\tilde{k}_1 k_3, \tilde{k}_2 k_3, -|\hat{k}_H|^2, 0\right)^T, & \text{when } |\hat{k}_H| \neq 0, \\
(i, 1, 0, 0)^T, & \text{else,}
\end{cases}
\end{equation}

Then, the normalized eigenvectors for $L$, when $|\hat{k}_H| \neq 0$, are given by

\begin{equation}
\begin{cases} 
\begin{aligned}
 r_{k_0}^0 &= |\hat{k}|^{-1} \tilde{k}_1 k_3^T, \\
 r_0^0 &= |\hat{k}_H|^{-1} \tilde{k}_0^0,
\end{aligned} & \text{for the zero eigenvalue,} \\
\begin{aligned}
 r_k^+ &= (\sqrt{2} |\hat{k}_H| |\hat{k}_\eta|)^{-1} \alpha_k, \\
 r_k^- &= (\sqrt{2} |\hat{k}_H| |\hat{k}_\eta|)^{-1} \overline{\alpha_k},
\end{aligned} & \text{for the } \omega_k \text{ eigenvalue,}
\end{cases}
\end{equation}

using the usual notation for the complex conjugate. On the other hand, for a purely vertical wavevector with $k_H = 0$, we have the following normalized eigenvectors

\begin{equation}
r_k^0 = (0, 0, 1, 0), \quad r_k^0 = \tilde{e}_k^0, \quad r_k^+ = \frac{1}{\sqrt{2}} \alpha_k, \quad r_k^- = \overline{r_k^+}.
\end{equation}
In both cases, the set of vectors \( \{ r^0_k, r^+_k, r^-_k \} \) forms an orthonormal basis of the subspace of \( \mathbb{C}^4 \) that corresponds to incompressible vector fields with wavevector \( k \), namely any \( (v_1, v_2, v_3, v_4)^\top \) with \( (v_1, v_2, v_3) \cdot \hat{k} = 0 \).

We briefly state some simple properties of it, taking into account the convolution condition as well.

**Lemma 2.** Let \( k, m, n \in \mathbb{Z}^3 \), then the following properties hold true:

1. \( r^\pm_k = r^\pm_k \)
2. \( e^0_n + e^0_k + e^0_m = 0 \), when \( k + m + n = 0 \)

**Proof.** First, the presence of an even number of components of \( k \) in \( \alpha \) immediately implies the first statement. The last statement follows from the linearity of the eigenvector, combined with the convolution condition. \( \square \)

We remark that
\[
\sum_{k \in \mathbb{Z}^3} \sum_{\sigma \in \{0, \pm\}} \mathcal{P}_k^\sigma \mathcal{P} u = \sum_{k \in \mathbb{Z}^3} \sum_{\sigma \in \{0, \pm\}} \mathcal{P} \mathcal{P}_k^\sigma u = \mathcal{P} u,
\]
where \( \mathcal{P} \) stands for the trivially extended Leray projection, directly following from the incompressibility of the basis. Using the eigendecomposition formalism, the action of the evolution operator on a divergence-free vector field \( U \in L^2(\mathbb{T}^3, \mathbb{R}^4) \) can then be expressed as follows
\[
e^{\tau L} U = \sum_{k \in \mathbb{Z}^3} \sum_{\sigma \in \{0, \pm\}} e^{i \omega^\sigma_k \tau} \mathcal{P}_k^\sigma U.
\]

**Remark 4.** In the class of real-valued vector fields \( u \) in \( \mathbb{T}^3 \), we have:
\[
(u_k \cdot \hat{r}^\sigma_k) r^\sigma_k = (u_{-k} \cdot \hat{r}^{-\sigma}_{-k}) r^{-\sigma}_{-k},
\]
for all \( \sigma \in \{\pm, 0\} \), as an immediate consequence of Lemma 2.

The dependence of the linear potential vorticity \( Q \) on the slow part of \( U \in H^1(\mathbb{T}^3, \mathbb{R}^4) \) can be made explicit, thanks to the fact that the Fourier symbol of operator \( L_{pv} \) is simply \( e^0_k \) given in (28). Therefore,
\[
L_{pv} U = i \sum_{k \in \mathbb{Z}^3} |\hat{k}| (U_k \cdot r^0_k) e^{i k \cdot x} = L_{pv} U_s.
\]

The conjugation on \( r^0_k \in \mathbb{R}^4 \) is omitted for brevity. Then orthogonality of the basis vectors \( r^0_k \) and \( r^\pm_k \) readily implies the following.

**Lemma 3.** Let \( \ell \in \mathbb{R} \) and \( U, V \in H^4(\mathbb{T}^3; \mathbb{R}^4) \) be divergence-free and zero-mean vector fields. Then the following statements hold true

i) \( \langle L_{pv} U, L_{pv} V \rangle = \langle U_s, (-\Delta^\eta) V_s \rangle \),

ii) \( \langle U_s, (-\Delta^\ell) V_f \rangle = 0 \).

**Proof.** In order to prove the first statement, we expand using (26) and (33):
\[
\langle L_{pv} U, L_{pv} V \rangle = |\mathbb{T}^3| \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |\hat{k}|^2 (U_k \cdot r^0_k) (V_k \cdot r^0_k).
\]
We use (26) once more for the second statement
\[
\left\langle U_s, (-\Delta)^t U_f \right\rangle = |T^3| \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{2t} \left( U_k \cdot r_k^0 \right) \left[ \left( r_k^0 \cdot r_k^+ \right) \left( V_k \cdot r_k^+ \right) + \left( r_k^0 \cdot r_k^- \right) \left( V_k \cdot r_k^- \right) \right] = 0.
\]
\[\square\]

3.2. The restricted bilinear and elliptic operators. The following result will prove crucial in obtaining the standard $L^2$ identity for (12).

**Lemma 4.** Let $U, V \in H^1(T^3; \mathbb{R}^3)$ be divergence-free and zero-mean vector fields. Then, for any $\tau \in \mathbb{R}$, the following properties hold true

i) $\bar{B}(\tau, U, V) \in \mathbb{R}^3$

ii) $\left\langle \bar{B}_s(\tau, U_s, V), V \right\rangle = 0$

iii) $\left\langle \bar{B}_f(\tau, U_s, V_f), V \right\rangle = \left\langle \bar{B}(\tau, U_f, V_f), V \right\rangle = 0$

iv) $\left\langle \bar{B}_f(\tau, U_f, V_s), V \right\rangle = -\left\langle \bar{B}_s(\tau, U_f, V_f), V \right\rangle$.

**Proof.** It suffices to prove the Lemma in the case $\tau = 0$. We prove (i) for the slow part $B_s$ only. In particular, taking the complex conjugate yields
\[
\bar{B}_s(U, V) = \sum_{k,m,n;\text{conv}} B^{000}_{k,m,n}(U, V) + \sum_{k,m,n;\text{conv}} \sum_{\sigma_1, \sigma_2 < 0} B^{\sigma_1, \sigma_2 0}_{k,m,n}(U, V) 1_{N_{NFS}}.
\]

For the SSS term, we change $(k, m) \rightarrow -(k, m)$ in the sum, so that we get
\[
\sum_{k,m,n;\text{conv}} B^{000}_{k,m,n}(U, V) = - \sum_{k,m,n;\text{conv}} (U_{-k} \cdot r_k^0) (r_k^0 \cdot i m') (V_{-m} \cdot r_m^0) (r_m^0 \cdot r_{-n}) r_{-n}^0
\]
\[
= \sum_{k,m,n;\text{conv}} B^{000}_{k,m,n}(U, V).
\]

For the mixed FFS term, we take into account that $r_k^{\sigma_1} = \bar{r}_k^{\sigma_2}$ and $r_k^{-\sigma_1} = r_k^{\sigma_2}$, for nonzero $\sigma_1 \neq \sigma_2$, and change $(k, m) \rightarrow -(k, m)$ in the sum, so that
\[
\sum_{k,m,n;\text{conv}} B^{\sigma_1, \sigma_2 0}_{k,m,n}(U, V) = - \sum_{k,m,n;\text{conv}} (U_{-k} \cdot r_k^{\sigma_1}) (r_k^{\sigma_1} \cdot i m') (V_{-m} \cdot r_m^{\sigma_2}) (r_m^{\sigma_2} \cdot r_{-n}) r_{-n}^0
\]
\[
= \sum_{k,m,n;\text{conv}} \left( U_k \cdot \bar{r}_k^{\sigma_2} \right) (r_k^{\sigma_2} \cdot i m') \left( V_m \cdot \bar{r}_m^{\sigma_1} \right) (r_m^{\sigma_1} \cdot r_{-n}) r_{-n}^0
\]
\[
= \sum_{k,m,n;\text{conv}} B^{\sigma_2, \sigma_1 0}_{k,m,n}(U, V).
\]

Since $1_{N_{NFS}}$ is an even function, the first claim of the Lemma follows.

As (ii), (iii) and (iv) all follow the same reasoning, we only prove the last one. In that direction, we recall that
\[
1_{N_{NFS}}(k, m, n) = 1_{N_{FSS}}(k, n, m),
\]
directly from the definition of the mixed sets. Then, we expand using the incompressibility of $r_k^\pm$:
\[
\left\langle \bar{B}_s(U_f, V_f), V \right\rangle = - \sum_{k,m,n;\text{conv}} (U_k \cdot r_k^-) (r_k^+ \cdot i m') (V_m \cdot r_m^+)(r_m^- \cdot r_n^0) (V_n \cdot r_n^0) 1_{N_{NFS}}(k, m, n)
\]
Let

\[ \sum_{k,m,n,\text{conv}} (U_k \cdot r_k^+ \cdot i \bar{\nu}^r) (V_m \cdot r_m^- \cdot i \bar{\nu}^r) (r_n^+ \cdot r_n^0 \cdot V_n \cdot r_n^0) \mathbf{1}_{\text{NFFS}}(k, m, n) \]

so that changing between \( m, n \) in the sum yields (iv).

The fact that

\[ \tilde{B}(U, V) = \tilde{B}_s(U, V) + \tilde{B}_f(U, V) \]

\[ = \tilde{B}_s(U, V_s) + \tilde{B}_s(U_f, V_f) + \tilde{B}_f(U_s, V_f) + \tilde{B}_f(U_f, V_s) \]

immediately implies the following.

**Corollary 1.** Under the assumptions of Lemma \([4]\) we have

\[ \langle \tilde{B}(U, V), V \rangle = 0. \]

We now present the expanded form of the Laplacian terms so that the motivation for introducing \( \tilde{\nu}_1, \tilde{\nu}_2 \) is highlighted. In particular, the slow and fast projections of the original Laplacian terms \( \tilde{A}_u \) in \((3)\), respectively, are (recall again \( r_k^0 \in \mathbb{R}^4 \))

\[ \bullet \quad - (\nu \Delta u)_s = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{\sigma \in \{\pm, 0\}} e^{i(k-x+\sigma \omega_k \tau)} |\tilde{k}|^2 (\nu r_k^0 \cdot r_k^0) (u_k \cdot r_k^0) r_k^0 \]

\[ \bullet \quad - (\nu \Delta u)_f = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{\sigma \in \{\pm, 0\}} e^{i(k-x-\sigma \omega_k \tau+\omega_k \tau)} |\tilde{k}|^2 (\nu r_k^0 \cdot \overline{r_k^0}) (u_k \cdot \overline{r_k^0}) r_k^{\sigma_1}, \]

whereas their respective approximations are given by:

\[ \bullet \quad - \tilde{\nu}_1 \Delta u_s = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} e^{i k \cdot x} |\tilde{k}|^2 (\nu r_k^0 \cdot r_k^0) (u_k \cdot r_k^0) r_k^0 = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |\tilde{k}|^2 (\nu r_k^0 \cdot \overline{r_k^0}) P_k^0 u_k \]

\[ \bullet \quad - \tilde{\nu}_2 \Delta u_f = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{\sigma_1 \in \{\pm\}} e^{i k \cdot x} |\tilde{k}|^2 (\nu r_k^{\sigma_1} \cdot \overline{r_k^{\sigma_1}}) (u_k \cdot \overline{r_k^{\sigma_1}}) r_k^{\sigma_1} \]

\[ = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{\sigma_1 \in \{\pm\}} |\tilde{k}|^2 (\nu r_k^{\sigma_1} \cdot \overline{r_k^{\sigma_1}}) P_k^{\sigma_1} u_k. \]

Thus, the effect of the fast time \( \tau \) is not present in this part of our approximation. Observe that \( \tilde{\nu}_1 \) and \( \tilde{\nu}_2 \) are scalar pseudodifferential operators of degree 0, hence commute with Laplacian, slow and fast projections, the evolution operator \( e^{\tau \tilde{\mathcal{L}}} \) and the linear PV operator \( L_{\text{pu}} \). In particular, we have

\[ (34) \quad - L_{\text{pu}} (\tilde{\nu}_1 \Delta u_s) = - \tilde{\nu}_1 \Delta Q = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} e^{i k \cdot x} |\tilde{k}|^2 (\nu r_k^0 \cdot r_k^0) Q_k \]

Finally, we record the following ellipticity property of the modified Laplacian operators.

**Lemma 5.** Let \( \ell \in \mathbb{R} \), \( \widetilde{U} \in H^{2\ell+2}(\mathbb{T}^3; \mathbb{R}^4) \), and \( \nu_1, \nu_2 \geq 0 \), then the following estimate holds true

\[ \nu_{\text{min}} \| \tilde{U}_s \|^2_{H^{\ell+1}} \leq - \langle \tilde{\nu}_1 \Delta \tilde{U}_s, (-\Delta)^\ell \tilde{U}_s \rangle, \]

\[ \nu_{\text{min}} \| \tilde{U}_f \|^2_{H^{\ell+1}} \leq - \langle \tilde{\nu}_2 \Delta \tilde{U}_f, (-\Delta)^\ell \tilde{U}_f \rangle, \]

\[ \nu_{\text{min}} \| \tilde{Q} \|^2_{H^{\ell+1}} \leq - \langle \tilde{\nu}_1 \Delta \tilde{Q}, (-\Delta)^\ell \tilde{Q} \rangle, \]

where \( \tilde{Q} = L_{\text{pu}} \tilde{U} \).
Proof. The first and third statement follow from (26),(34) and the fact that the Fourier symbol of $\tilde{\nu}_{11}$ is

\[
\left(\nu r_k^0 \cdot r_k^0\right) = \frac{\nu_1 |\hat{k}_H|^2 + \nu_2 \eta^2 |\hat{k}_3|^2}{|\hat{k}_\eta|^2}.
\]

The second statement follows similarly, after taking into account the orthogonality $r_k^+ \cdot \overline{r_k^-} = 0$ and the form of the Fourier symbol of $\tilde{\nu}_{22}$

\[
\left(\nu r_k^+ \cdot \overline{r_k^-}\right) = \left(\nu r_k^- \cdot \overline{r_k^+}\right) = \frac{\nu_1 \eta^2 |\hat{k}_3|^2 + 2(\nu_2 + \nu_1) |\hat{k}_H|^2}{2|\hat{k}_\eta|^2}.
\]

□

4. Restricted Convolution

In this section we present two restricted convolution lemmas, one to be applied to FFF interactions and the other to be applied to FFS and FSF interactions, respectively.

4.1. Restricted convolution under full symmetry. We recall the following version of [7, Lemma 3.1], which we will use in the proof of the FFF estimate Theorem 2.

Lemma 6. Let $\mathcal{N} \subset (\mathbb{Z}^3 \setminus \{\tilde{0}\})^3$, with $1_{\mathcal{N}}(\cdot, \cdot, \cdot)$ symmetric with respect to any permutation of its arguments, and

\[
\mathcal{N}_0 := \{(k, m, n) \in \mathcal{N} : |\hat{m}| \leq |\hat{k}| \leq |\hat{n}|\}.
\]

If there exist a constant $\beta \in [0, 3]$ and a constant $C_\mathcal{N}$ so that the counting condition

\[
(35) \quad \sum_{k \in \mathbb{Z}^3} 1_{\mathcal{N}_0}(k, -n - k, n) \leq C_\mathcal{N} |\hat{n}|^\beta, \quad \forall n \in \mathbb{Z}^3 \setminus \{\tilde{0}\}
\]

holds, then for zero-mean $u, v \in H^{3/2}(\mathbb{T}^3)$ and $w \in L^2(\mathbb{T}^3)$,

\[
(36) \quad \sum_{k,m,n; \text{conv}} |u_n| |v_k| |w_m| 1_{\mathcal{N}}(n, k, m) \lesssim \left(\|u\|_{H^{3/2}} \|v\|_{H_{\frac{3}{2}}} + \|u\|_{H_{\frac{3}{2}}} \|v\|_{L^2}\right) \|w\|_{L^2},
\]

with the implied constant depending on $C_\mathcal{N}, \mathbb{T}^3$.

4.2. Restricted convolution under reduced symmetry. We now move on to the following crucial result for controlling the mixed FFS and FSF energy estimates. The main difference to the related rotating Navier-Stokes results, [11, Lemma 3.1] and [7, Lemma 3.1] for FFF interactions, is a reduced symmetry. The proof is still based on a Paley-Littlewood decomposition, under our restricted interaction setting.

In more detail, in both cases of FSF and FFS interactions, there is a permutation symmetry only between two fast modes, e.g. $k$ and $n$. Thus, without loss of generality, we assume that $|\hat{k}| \leq |\hat{n}|$, and proceed to a Paley-Littlewood decomposition for $k$ in annuli of the form

\[
\mathcal{A}_i := \{k \in \mathbb{Z} : 2^{i-1} \leq |\hat{k}| < 2^i\}.
\]

If we further restrict $|\hat{m}| \leq |\hat{k}|$, then the proof of the trilinear estimate in [7, Lemma 3.1] goes through unchanged, with $k \in \mathcal{A}_i$ implying that $n \in \mathcal{A}_i \cup \mathcal{A}_{i+1}$. However, we must also consider the case $|\hat{m}| > |\hat{k}|$, a condition which does not provide us with an upper bound for $|\hat{n}|$, hence explaining the difference between (35) and (37) – also see the corresponding condition in [11, Lemma 3.1].
In what follows, we will use the following lemma concerning the sum of a sequence of weighted sums of Fourier coefficients that are restricted outside a geometric sequence of balls.

**Lemma 7.** Let $u$ be a sufficiently regular function on $\mathbb{T}^3$, with Fourier coefficients $u_n$, for $n \in \mathbb{Z}^3$. Moreover, define set $B_i = \{n \in \mathbb{Z}^3 : |\tilde{n}| < 2^i\}$. Then, the following inequality holds true for all $\ell > 0$:

$$\sum_{i=0}^{\infty} \sum_{n \in B_i} 2^{2i\ell} |u_n|^2 \lesssim \ell \|u\|_{H^\ell}^2.$$ 

In fact, a more delicate relation of the form

$$\left\| \left( \sum_{i=0}^{\infty} \sum_{n \in B_i} 2^{2i\ell} |u_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \sim \|(-\Delta)^{\frac{\ell}{2}} u\|_{L^p}$$

holds true for $p \in (1, \infty)$ and $\ell > 0$, see e.g. [13]. Nevertheless, we present a proof of the weaker statement that we later need, due to its simplicity.

**Proof.** First, we note that $i \leq \log_2 |\tilde{n}|$, whereas the geometric sum satisfies

$$\sum_{i=0}^{\log_2 |\tilde{n}|} 2^{i\ell} \leq \frac{1 - 2^{\ell(\log_2 |\tilde{n}| + 1)}}{1 - 2^\ell} \approx \ell |\tilde{n}|^\ell.$$

Thus,

$$\sum_{i=0}^{\infty} \sum_{n \in B_i} 2^{2i\ell} |u_n|^2 = \sum_{n \in B_i} \sum_{i=0}^{\log_2 |\tilde{n}|} 2^{i\ell} \lesssim \ell \sum_{n \in B_i} |\tilde{n}|^{2\ell} \lesssim \|u\|_{H^\ell}^2.$$

□

Then, the following lemma will allow us to control the crucial mixed trilinear term occurring in the energy estimates.

**Lemma 8.** Let $1_{X^c}(\cdot, \cdot, \cdot)$ be the characteristic of a subset of $(\mathbb{Z}^3 \setminus \{\tilde{0}\})^3$, which is symmetric with respect to permutations in its first two arguments. Moreover, let $A_i$ be the annuli

$$X_i = \{k \in \mathbb{Z}^3 : 2^{i-1} \leq |\tilde{k}| < 2^i\}, \quad \forall i \in \mathbb{Z}^+,$$

and suppose

$$\sum_{k \in A_i} 1_{X^c}(k, -n - k, n) \leq c_0 2^{\mu i}, \quad \forall i \in \mathbb{Z}^+, \quad \forall n \in \mathbb{Z}^3,$$

for some $c_0, \mu \geq 0$, not depending on $i$ and $n$. Then, for every

$$a \in (0, \frac{\mu}{2})$$

and for every zero-mean

$$u \in H^{\frac{\mu}{2} - a}(\mathbb{T}^3), \quad v \in H^a(\mathbb{T}^3), \quad w \in L^2(\mathbb{T}^3),$$

we have

$$\sum_{k, m, n: \text{conv}} 1_{X^c}(k, -n - k, n) |u_k v_n w_m| \lesssim \|u\|_{H^{\frac{\mu}{2} - a}} \|v\|_{H^a} \|w\|_{L^2}. $$
Remark 5. Lemma 8 can be generalized to sets $\mathcal{N}' \subset (\mathbb{Z}^3 \setminus \{ \vec{0} \})^3$, whose characteristic function possesses a permutation symmetry between any two arguments.

Proof. First, we set $\widetilde{\mathcal{N}} := \mathcal{N}' \cap \{(k, -n - k, n) \in (\mathbb{Z}^3 \setminus \{ \vec{0} \})^3 : |k| \leq |n|\}$. Then, due to the symmetry of $1_{\mathcal{N}'}$ in the variables $k$ and $n$, we have

$$\sum_{k,m,n;\text{conv}} |v_n u_k w_m| 1_{\mathcal{N}'}(k, -n - k, n) \leq \sum_{k,m,n;\text{conv}} (|v_n u_k| + |v_k u_n|)|w_m| 1_{\widetilde{\mathcal{N}}}(k, -n - k, n).$$

We will use a Paley-Littlewood type dyadic decomposition in order to estimate the first term on the right, expanding the sum with respect to $k$, and the full result will follow via reversing the roles of $k$ and $n$. In that direction, we have

$$\sum_{k,m,n;\text{conv}} |u_k w_m v_n| 1_{\widetilde{\mathcal{N}}}(k, -n - k, n) = \sum_{i \in \mathbb{Z}^+} \sum_{k \in A_i} \sum_{n \in B_{i-1}} |u_k w_m v_n| 1_{\widetilde{\mathcal{N}}}(k, -n - k, n).$$

For the sake of brevity, we suppress the input arguments to the characteristic function in the rest of this proof. Switching the order of summation between $k$ and $n$ and using the Cauchy-Schwarz inequality in the $n$-variable we have

$$\sum_{i \in \mathbb{Z}^+} \sum_{k \in A_i} \sum_{n \in B_{i-1}} |u_k w_{-n-k} v_n| 1_{\widetilde{\mathcal{N}}} \lesssim \sum_{i \in \mathbb{Z}^+} \left( \sum_{n \in B_{i-1}} \sum_{k \in A_i} |u_k w_{-n-k}| 1_{\widetilde{\mathcal{N}}} \right)^2 \left( \sum_{n \in B_{i-1}} \sum_{k \in A_i} |v_n| \right)^2 \lesssim \sum_{i \in \mathbb{Z}^+} \left( \sum_{n \in B_{i-1}} \sum_{k \in A_i} |u_k w_{-n-k}| 1_{\widetilde{\mathcal{N}}} \right)^2 \left( \sum_{n \in B_{i-1}} \sum_{k \in A_i} |v_n| \right)^2.$$

Then, using (37) and the Cauchy-Schwarz inequality in the $k$-variable, we obtain

$$\sum_{i \in \mathbb{Z}^+} \left( \sum_{n \in B_{i-1}} \sum_{k \in A_i} |u_k w_{-n-k}| 1_{\widetilde{\mathcal{N}}} \right)^2 \lesssim \sum_{i \in \mathbb{Z}^+} \sum_{n \in B_{i-1}} \left( \sum_{k \in A_i} |u_k w_{-n-k}| 1_{\widetilde{\mathcal{N}}} \right)^2 \lesssim \sum_{i \in \mathbb{Z}^+} \left( \sum_{n \in B_{i-1}} \sum_{k \in A_i} |v_n| \right)^2 \sum_{i \in \mathbb{Z}^+} \left( \sum_{n \in B_{i-1}} \sum_{k \in A_i} |u_k w_{-n-k}| 1_{\widetilde{\mathcal{N}}} \right)^2.$$

Finally, we insert the last estimate into (39) and combine it with the Cauchy-Schwarz inequality, in order to derive that

$$\sum_{k,m,n;\text{conv}} |u_k w_{-n-k} v_n| 1_{\widetilde{\mathcal{N}}}(k, -n - k, n) \lesssim \sum_{i \in \mathbb{Z}^+} \sum_{n \in B_{i-1}} \left( \sum_{k \in A_i} |u_k w_{-n-k}| 1_{\widetilde{\mathcal{N}}} \right)^2 \lesssim \sum_{i \in \mathbb{Z}^+} \left( \sum_{n \in B_{i-1}} \sum_{k \in A_i} |u_k w_{-n-k}| 1_{\widetilde{\mathcal{N}}} \right)^2 \lesssim \sum_{i \in \mathbb{Z}^+} \left( \sum_{n \in B_{i-1}} \sum_{k \in A_i} |u_k w_{-n-k}| 1_{\widetilde{\mathcal{N}}} \right)^2.$$

after switching the sums containing $k$ and $n$. By Parseval’s identity, we have uniform bound $\sum_{n \in B_{i-1}} |w_{-n-k}|^2 \lesssim \|w\|_{L^2}^2$. Thus, (38) follows from Lemma 7.
5. Counting near resonant interactions

We will address the counting of mixed FSF and FFS interactions in a short section before the much more challenging case of FFF interactions.

5.1. Counting mixed interactions. By fixing one of the fast modes in the mixed interaction set and limiting the length of the other fast mode, the counting problem is then reduced to a corresponding volume integral estimate, up to a lower order error term. The volume integral that comes up is expressed in rescaled spherical coordinates, with a modified radial component. Crucially, the range of the polar angle variable is restricted due to the near resonance condition. We first prove the volume estimates for the mixed sets, with the reduction of lattice point counting to the volume following in Corollary \[2\,\] Finally, we note that the next Lemma is posed in terms of the FSF set, but still applies to the FFS one.

**Lemma 9.** Let \( n \in \mathbb{Z}^3 \setminus \{0\}, \delta^* \in [0, \min\{\frac{\eta}{2}, \frac{1}{2}\}] \) and \( M > 0 \), be fixed and consider the set

\[
V_1 = \left\{ k \in \mathbb{R}^3 \setminus \{0\} : \left| \frac{\tilde{k}}{|k|} - \frac{\tilde{n}}{|n|} \right| \leq \delta^* \quad \text{and} \quad |\tilde{k}| \leq M \right\}.
\]

Then, the following estimate holds true:

\[
\text{vol}(V_1) \leq C(\eta)L_1L_2\sqrt{\delta^*}M^3,
\]

with \( C \) independent of \( L_1, L_2, M, n \).

**Proof.** First, note the volume element scaling property \( L_1L_2d\tilde{k} = dk \). Thus, it suffices to prove the estimate for \( L_1 = L_2 = 1 \). We introduce modified spherical coordinates for the wavevector \( k \); \((\lambda'_k, \theta_k, \phi_k) \in [0, 1] \times [0, \pi] \times [0, 2\pi] \), with a modified radial component \( \lambda'_k : = \frac{|k|}{M} \). In particular, we have

\[
k = M\lambda'_k(\sin \theta_k \cos \phi_k, \sin \theta_k \sin \phi_k, \cos \theta_k), \quad \text{for} \quad k \in \mathbb{Z}^3.
\]

The near resonance condition in the mixed context implies the following range for \( c_k := \cos \theta_k \)

\[
(\eta^2 - 1)c_k^2 \in \left[ (\omega_n - \delta^*)^2 - 1, (\omega_n + \delta^*)^2 - 1 \right],
\]

after using the definition of the dispersion relation directly. It immediately follows that \( |c_k| \) is constrained to an interval \( I \), with length \( |I| = \frac{4\omega_n\delta^*}{|\eta^2 - 1|} \lesssim \eta \sqrt{\delta^*} \), after using the trivial upper bound on \( \omega_n \). Then, using the change of variable \( |k| = \lambda'_k \) and also substituting \( \cos \theta_k \) instead of \( \theta_k \), we have

\[
\text{vol}(V_1) = M^3 \int_0^1 \int_0^{2\pi} \int_0^\pi 1_{V_1} \sin \theta_k d\theta_k d\phi_k d\lambda'_k \leq M^3 \int_0^1 \int_0^{2\pi} I_{I} dc_k d\phi_k d\lambda'_k
\]

\[
\leq C(\eta)\sqrt{\delta^*}M^3,
\]

via Fubini’s Theorem. \( \square \)

The following Corollary provides an estimate on the number of mixed interactions, if one of the fast wavenumbers is fixed.

**Corollary 2.** The number of integers in the set \( V_1 \) from Lemma \[2\] satisfies

\[
\sum_{k \in \mathbb{Z}^3} 1_{V_1}(k) \leq C \left( L_1L_2\sqrt{\delta^*}M^3 + L_1L_2M^2 + (L_1 + L_2)M \right),
\]

for a constant \( C \) that is independent of \( L_1, L_2, M, n \).
Proof. We first fix \( k_H \in \mathbb{Z}^2 \setminus \{ \vec{0} \} \). Then, defining \( \tilde{F} : \mathbb{R}^3 \setminus \{ \vec{0} \} \to \mathbb{R} \) with \( \tilde{F}(k_3) := \frac{|\hat{k}_\eta|}{|k|} - \frac{|\hat{n}_\eta|}{|\hat{n}|} \), we have
\[
\frac{\partial \tilde{F}}{\partial k_3} = (\eta^2 - 1) \frac{k_3|\hat{k}_H|^2}{|k_\eta||k|^3}.
\]
Since \( k_H \neq \vec{0}, \) there are at most two intervals of monotonicity for \( \tilde{F} \). On each such interval, the number of \( k_3 \in \mathbb{Z} \) can be estimated by \( \int_{\mathbb{R}} 1_{V_1}(k_H, k_3)dk_3 + 1, \) thus:
\[
\sum_{k_3 \in \mathbb{Z}} 1_{V_1}(k_H, k_3) \leq 2 + \int_{\mathbb{R}} 1_{V_1}(k_H, k_3)dk_3.
\]
The limits of the last integral depend linearly on \( |\hat{k}_H| \), as (40) holds true. Hence, we sum over \( \hat{k}_H \), with \( |\hat{k}_H| \leq M^2 \), so that estimating the left Riemann sums with the corresponding integrals, via [7, Lemma A.2], yields
\[
\sum_{k \in \mathbb{Z}^3} 1_{V_1}(k) \leq C \left( L_1L_2M^2 + (L_1 + L_2)M + \text{vol}(V_1) \right).
\]

5.2. Initial reduction to two-dimensional counting for FFF interactions. In order to estimate the number of FFF modes, under the symmetry assumptions of Lemma 6, we follow the strategy introduced in [7, Section 4.1] for the respective approximation of the rotating Navier-Stokes system. First, we define the function \( F_{n,\sigma_1,\sigma_2} : \mathbb{R}^3 \setminus \{ \vec{0}, -n \} \to \mathbb{R} \) with:
\[
F_{n,\sigma_1,\sigma_2}(k) = \frac{|\hat{m}_\eta|}{|\hat{m}|} + \sigma_1 \frac{|\hat{n}_\eta|}{|\hat{n}|} + \sigma_2 \frac{|\hat{k}_\eta|}{|\hat{k}|},
\]
for fixed \( n \in \mathbb{Z}^3 \setminus \{ \vec{0} \} \) and \( \sigma_1, \sigma_2 \in \{ \pm \} \). In what follows, we simply write \( F \) for the sake of brevity. In addition, since any wavevector \( k \in \mathcal{N}_0 \) satisfies \( \frac{|\hat{n}|}{2} \leq |\hat{k}| \leq |\hat{n}| \), we restrict our attention to the set
\[
V_{n,\sigma_1,\sigma_2} = \left\{ k \in \mathbb{R}^3 \setminus \{ \vec{0}, -n \} : |F| \leq \delta \quad \text{and} \quad \frac{|\hat{n}|}{2} \leq |\hat{k}| \leq |\hat{n}| \right\},
\]
with \( \delta \) depending on \( \max\{||\hat{k}|, |\hat{m}|, |\hat{n}|\} = |\hat{n}| \), hence fixed. Note that \( V_{n,\sigma_1,\sigma_2} \subset \mathbb{R}^3 \setminus \{ \vec{0}, -n \} \), so that we are not a-priori restricted to integer points. Similarly to our convention for \( F \), we will simply write \( V \) for the remainder of this subsection.

We now aim to reduce the lattice counting problem for \( V \) into a two-dimensional geometric problem. This requires some more information on the stationary points of \( F \) in each direction. In more detail, for fixed \( k_H \in \mathbb{Z}^2 \), we have:
\[
\frac{\partial F}{\partial k_3} = (\eta^2 - 1) \left[ \frac{\sigma_3 k_3|\hat{k}_H|^2}{|k_\eta||k|^3} - \frac{m_3|\hat{m}_H|^2}{|\hat{m}_\eta||\hat{m}|^3} \right].
\]
Then, the critical points of \( F \) for fixed \( n \) and \( \hat{k}_H \) are included in the set of solutions of the following equation
\[
k_3^2|\hat{k}_H|^4 \left[ |\hat{m}_H|^2 + \eta^2(k_3+n_3)^2 \right] \left[ |\hat{m}_H|^2 + (k_3+n_3)^2 \right]^3
\]
\[- (k_3+n_3)^2|\hat{m}_H|^4|\hat{k}_\eta|^2 |\hat{k}_H|^2 + k_3^2 \right]^3 = 0.
\]
The expression on the left-handside is a polynomial of degree 10 in $k_3$, with leading order coefficient $\eta^2(|\hat{k}_H|^4 - |\hat{m}_H|^4)$. If the latter is 0, then the critical points of $F$ are included in the roots of a polynomial of degree 9 in $k_3$, with leading order coefficient $6\eta^2 n_3 \neq 0$. In any case, for fixed $k_H \in \mathbb{Z}^2$, it follows that $\mathbb{R}$ can be split in at most 11 intervals where $F$ is strictly monotonic in $k_3$. Thus, we have

$$\sum_{k_3 \in \mathbb{Z}} 1_V(k_H, k_3) \leq 11 + \int_{\mathbb{R}} 1_V(k_H, k_3) \, dk_3.$$  

Then, using [7, Lemma A.2] we sum over $(k_1, k_2) \in \mathbb{Z}^2$ with $|\hat{k}_H| \leq |\hat{n}|$ to deduce that:

$$\sum_{k \in \mathbb{Z}^3} 1_V(k) \leq C \left[ L_1 L_2 |\hat{n}|^2 + (L_1 + L_2) |\hat{n}| \right] + \sum_{k_H \in \mathbb{Z}^2 \atop |k_H| \leq |\hat{n}|} \int_{\mathbb{R}} 1_V(k_H, k_3) \, dk_3. \quad (41)$$

In order to estimate the last sum, we define the set $S(k_3) := \{ k_H \in \mathbb{R}^2 : (k_H, k_3) \in V \}$, so that

$$\sum_{k_H \in \mathbb{Z}^2 \atop |k_H| \leq |\hat{n}|} \int_{\mathbb{R}} 1_V(k_H, k_3) \, dk_3 \leq \int_{\mathbb{R}} \# (\mathbb{Z}^2 \cap S(k_3)) \, dk_3.$$  

Thus we need to estimate the lattice points in $S(k_3)$, in order to deduce upper bounds for our counting problem.

5.3. **Lattice points in $S(k_3)$**. We now proceed by deriving an upper bound on the number of integer points in the two-dimensional set $S(k_3)$, via the following.

**Lemma 10.** Let $\delta \geq 0$, $n \in \mathbb{R}^3 \setminus \{0\}$ and $k_3 \in (-|\hat{n}|, |\hat{n}|)$ be fixed and satisfy

$$n_3(k_3 + n_3)(2k_3 + n_3) \neq 0.$$  

Then there exist constants $C_1, C_2$ so that

$$\# (\mathbb{Z}^2 \cap S(k_3)) \leq Area(S(k_3)) + C_1(L_1 + L_2) \sqrt{|\hat{n}|^2 - k_3^2} + C_2,$$  

with $Area(S(k_3)) = 0$, when $\delta = 0$.

The proof of Lemma 10 relies on the strategy of the proof of Lemma 4.1 of [7]. In particular, the two dimensional lattice counting problem for $S(k_3)$ is posed in a cylindrical coordinate setting. Then, the Implicit Function Theorem is used in order to express our set as a union of disjoint Jordan curves. It then remains to apply [7, Lemma A.2] in order to conclude. Since the dispersion relation for the Boussinesq system is different to that of the rotating Navier-Stokes one, we mainly sketch the technical differences in the proofs.

**Proof.** First, we remark that under our assumptions, and for a fixed choice of $k_3$, the sublevel set $\{(\hat{k}_1, \hat{k}_2) \in \mathbb{R}^2 : 0 < |F(\hat{k}_1, \hat{k}_2, k_3)| < \delta \}$ is nonempty and open. We will work with cylindrical coordinates $(r_k, \phi_k, k_3) \in [0, \infty) \times [0, 2\pi] \times \mathbb{R}$, with a modified radial variable. In particular, the horizontal component of $k$ can be expressed as $k_H = (L_1 r_k \cos \phi_k, L_2 r_k \sin \phi_k)$. Then, for fixed $n \in \mathbb{R}^3 \setminus \{0\}$ and $k_3 \in \mathbb{R} \setminus \{0\}$, we express the function $F$ in cylindrical coordinates through $G : [0, \infty) \times [0, 2\pi] \to \mathbb{R}$ with

$$G(r_k, \theta_k) = \sqrt{1 + \left(\frac{\eta^2 - 1}{|\hat{m}|^2} \right) m_3^2} + \sigma_1 \sqrt{1 + \left(\frac{\eta^2 - 1}{|\hat{n}|^2} \right) n_3^2} + \sigma_2 \sqrt{1 + \left(\frac{\eta^2 - 1}{|k|^2} \right) k_3^2},$$  

where $\sigma_1$ and $\sigma_2$ are constants.
where
\[ |\hat{m}|^2 = r_k^2 + r_n^2 + 2r_k r_n \cos(\phi_k - \phi_n) + (k_3 + n_3)^2. \]

The annular region on which \( k \) lies, restricts the range of \( r_k \) in a set of the form \([r_{in}, r_{out}]\), where
\[ r_{in} := \sqrt{\max\{0, \frac{1}{4}|\hat{m}|^2 - k_3^2\}} \quad \text{and} \quad r_{out} := \sqrt{|\hat{m}|^2 - k_3^2}. \]

Thus, we can express the set \( S(k_3) \) as \( S(k_3) = S^+(k_3) \cap S^-(k_3) \), where
\[ S^+ := \{ (L_1 r_k \cos \phi_k, L_2 r_k \sin \phi_k) \in \mathbb{R}^2 : (r_k, \phi_k) \in [r_{in}, r_{out}] \times [0, 2\pi], G(r_k, \phi_k) \leq \delta \} \]
and \( S^- \) is the set corresponding to \(-G\).

We first examine the case \( r_n \equiv 0 \). Then, we have \( \partial S(k_3) \) included in the union of the sets:
\[ \{ k_H \in \mathbb{Z}^2 : |\hat{k}_H| = r_{in} \} \cup \{ k_H \in \mathbb{Z}^2 \setminus \{0\} : |\hat{k}_H| = r_{out} \} \cup \{ k_H \in \mathbb{Z}^2 : |G| = \delta \}. \]

We claim that each of the aforementioned sets is a set of the form \( \{ k_H \in \mathbb{Z}^2 \setminus \{0\} : |\hat{k}_H| = c \} \), for \( c \geq 0 \), i.e. an ellipse. Indeed, when \( c = r_{in}, r_{out} \) the claim is clearly true, even when \( r_{in} = 0 \). On the other hand, when \( |G| = \delta \), we have
\[ \sigma_2 \sqrt{1 + \frac{(\eta^2 - 1)k_3^2}{|\hat{m}|^2}} + \sqrt{1 + \frac{(\eta^2 - 1)m_3^2}{|\hat{m}|^2}} = \pm \delta - \sigma_1 \eta, \]
with the result following after elimination of radicals. Thus, in the case of a fully vertical vector \( n \in \mathbb{R}^3 \setminus \{0\} \), the fact that \( \partial S(k_3) \) is contained in a union of ellipses suffices in order to use [7, Lemma A.2] and derive (43).

For \( r_n \neq 0 \), the azimuthal derivative of \( G \) is given by
\[ \frac{\partial G}{\partial \phi_k} = (\eta^2 - 1) \frac{|m_3|^2}{|\hat{m}_n||\hat{m}|^3} r_k r_n \sin(\phi_k - \phi_n). \]

Apparently, it changes sign at points of the form \( \phi_k = \phi_n + 4\pi \), for \( n \in \mathbb{Z} \). Also, if we fix \( r_k > 0 \), then there exist at most two solutions of the equation \( G(\phi_k) = 0 \), i.e. \( \text{Area}(S(k_3)) = 0 \), when \( \delta = 0 \).

As far as the radial derivative is concerned, we have
\[ \frac{\partial G}{\partial r_k} = (1 - \eta^2) \left[ \sigma_2 \frac{r_k k_3^2}{|\hat{m}_n||\hat{m}|^3} + \frac{(k_3 + n_3)^2 (r_k + r_n \cos(\phi_k - \phi_n))}{|\hat{m}_n||\hat{m}|^3} \right], \]
where \( r_m^2 = r_k^2 + r_n^2 + 2r_k r_n \cos(\phi_k - \phi_n) \) and \( |\hat{m}_n|^2 = r_k^2 + r_n^2 + 2r_k r_n \cos(\phi_k - \phi_n) + \eta^2 (k_3 + n_3)^2 \).

Thus, we have \( \frac{\partial G}{\partial r_k} = 0 \) implying that a polynomial equation in \( r_k \) is satisfied, with leading order term \( r_k^{10}(k_3^4 - m_3^4) \). Under our assumptions, the leading order coefficient is non-vanishing and there are at most 10 critical points in \( r_k \).

We define
\[ T := \bigcup_{\iota = 0,1} \{ G(r_k, \phi_n + \iota \pi) : r_k \text{ is a stationary point of } G(r_k, \phi_n + \iota \pi), \text{ or } r_k \in \{0, r_{in}, r_{out}\} \}, \]
and we assume that \( \delta \notin T \). Then, we will use [7, Lemma A.2], in order to prove the existence of constants \( C, C' \) independent of \( n, k_3, \delta, L_1, L_2 \) with the property that
\[ (6) \quad \text{if } \delta \notin T \cup \{ \pm \frac{|\hat{m}_n|}{|\hat{m}|} : 0 \}, \text{ then } \#(S^2 \cap S_+(k_3)) \leq \text{Area}(S_+(k_3)) + C(L_1 + L_2)r_{out} + C'. \]
We assume that \((G(r_k, \phi_n) - \delta)(G(r_k, \phi_n + \pi) - \delta) \leq 0\), for at least one \(r_k \in [r_{in}, r_{out}]\). If that is not the case, then \(S_k^+\) is either empty or the whole annulus \(r_{in} \leq r_k \leq r_{out}\) and follows from \([7, \text{Lemma A.2}]\). For a fixed azimuthal angle \(\phi\), there are at most 11 roots of the equation \(G(r_k, \phi_k) = \delta\), in view of \([14]\), with none of them equal to an element of the set \(\{0, r_{in}, r_{out}\}\). Thus, we also take into account the endpoints \(r_{in}\) and \(r_{out}\), so that all the zeroes of \(H\) in the interval \([r_{in}, r_{out}]\) are at most 24, namely:

\[
r_{in} \leq r_k^{(1)} < r_k^{(2)} < \ldots < r_k^{(2J)} \leq r_{out} \quad \text{for} \quad J \leq 12.
\]

In particular, \(G(r_k^{(i)}, \phi_n) = 0\) or \(G(r_k^{(i)}, \phi_n + \pi) = 0\), for an integer \(i \in [1, 24]\). Moreover, recalling \([14]\), we can apply the Implicit Function Theorem for \(G\) that is not the case, then \(\phi\) is close to \(\phi_n, \phi_n + \pi\) one can always apply the Implicit Function Theorem in order to express \(r_k\) as a \(C^1\) function of \(\phi_k\).

We define the family of curves

\[\Gamma_\delta := \{(L_1r_k \cos \Phi(r_k), L_2r_k \sin \Phi(r_k) : r_k \in \bigcup_{j=1}^J (r_k^{(2j)}, r_k^{(2j+1)})\},\]

which are differentiable in view of the Implicit Function Theorem and simple due to the restriction of the domain in \(r_k\). For \(j_0 \in [1, 12] \cap \mathbb{N}\), we will show that a restricted curve \(\gamma_\delta : (r_k^{(2j_0)}, r_k^{(2j_0+1)}) \to [0, \pi]\) belonging to \(\Gamma_\delta\) is rectifiable. Indeed,

\[
|\gamma_\delta| = \int_{(r_k^{(2j_0)}, r_k^{(2j_0+1)})} \sqrt{[(L_1r_k \cos \Phi(r_k))]^2 + [(L_2r_k \sin \Phi(r_k))]^2} \, dr_k
\]

\[
\leq C(L_1 + L_2) \left(1 + \int_{(r_k^{(2j_0)}, r_k^{(2j_0+1)})} |\Phi'| \, dr_k\right) r_{out}.
\]

In order to estimate the last improper integral, we proceed to an examination of the stationary points of \(\Phi\). First, we rewrite the relation \(G = \delta\) as follows

\[
\frac{|\tilde{m}|}{|m_3|} = \frac{|\tilde{k}|}{(C|k| - \sigma_2|k_\eta|)},
\]

where \(C = \delta - \sigma_2|\tilde{n}|\). An equivalent expression for \((47)\) is

\[
\frac{|\tilde{m}|}{|m_3|} = \sqrt{\frac{|\eta^2 - 1|}{(C|k| - \sigma_2|k_\eta|)^2 - |k|^2}}|\tilde{k}| := A_{kn}.
\]

We also recall that

\[
|\tilde{k}|^2 = |\tilde{k}_\eta|^2 + (1 - \eta^2)k_3^2.
\]

The expression of \(|\tilde{m}|\) in cylindrical coordinates with the azimuthal angle written in terms of \(\Phi\) is:

\[
|\tilde{m}|^2 = r_k^2 + r_n^2 + 2r_kr_n \cos (\Phi(r_k) - \phi) + m_3^2.
\]

Then, using \((48), (49), (50)\), and restricting our considerations to \(\Phi'(r_k) = 0\), we deduce that

\[
\frac{d}{dr_k} \frac{|\tilde{m}|^2 - r_n^2 - m_3^2}{r_k} - 1 = r_k^{-1} m_3^2 \frac{d}{dr_k} A_{kn}^2 - \frac{A_{kn}^2 m_3^2 - r_n^2 - m_3^2}{r_k^2} - 1 = 0.
\]
We focus on the derivative term involving $A_{kn}$:

$$\frac{1}{r_k} \frac{d}{dr_k} A_{kn}^2 = - \frac{2\sigma_2(\eta^2 - 1)^2(C|\bar{k}| - \sigma_2|\bar{k}_n|)k_3^2}{(|C|\bar{k} - \sigma_2|\bar{k}_n|)^2 - |k|^2 |\bar{k}_n|^2}.$$  

We multiply (51) by $[(C|\bar{k}| - \sigma_2|\bar{k}_n|)^2 - |k|^2 |\bar{k}_n|^2]$ and eliminate any remaining odd powers of $|\bar{k}_n|$, also using (19). It follows that $|\bar{k}|$ satisfies a polynomial equation of degree 14, with nonvanishing leading order coefficient since $\eta \neq 1$. Thus, the stationary points of $\Phi(r_k)$ are contained in the roots of a polynomial with degree 14. As a result, the curves contained in $\Gamma_\delta$ are rectifiable.

Then, the arguments in the proof of [7, Lemma 4.1] apply to our case as well, allowing us to extend $\Gamma_\delta$ to $\partial S^+(k_3) \cap ([r_{in}, r_{out}] \times [\phi_n, \phi_n + \pi])$. Moreover, we note the symmetry $G(r_k, \phi_n) = G(r_k, 2\phi_n - \phi_k)$ which allows us to transfer our results to the remaining part of the annulus, namely to $[r_{in}, r_{out}] \times [\phi_n - \pi, \phi_n]$. Thus, we can apply [7, Lemma A.2] to complete the proof of (46), up to the excluded values of $\delta$. The latter can be addressed by a simple density argument, which then completes the counting estimates for $\#(\mathbb{Z}^2 \cap S^+(k_3))$.

An application of the exclusion-inclusion principle to $\#(\mathbb{Z}^2 \cap S(k_3))$, where $S(k_3) = S^+(k_3) \cap S^-(k_3)$, identical to the one in the proof of [7, Lemma 4.1], completes the proof.

A combination of (41) and Lemma 10 yields the following Theorem, which reduces the lattice counting problem for the set $N_0$ to volume estimates for $V$. The result holds under a fixed choice of $n$ and is valid up to a remainder term of order $|\hat{n}|^2$, which takes into account the excluded values of $k$ from Lemma 10.

**Theorem 6.** Let $n \neq 0$ and $\delta \in \left[0, \min\{\frac{\eta}{\tau}, \frac{1}{\tau}\}\right]$ be fixed. Then there exists a constant $C > 0$ independent of $n, \delta, \sigma_1, \sigma_2, L_1, L_2$ such that the cardinality of the set

$$N_0 := \{(n, k, m) \in \mathcal{N} : |\hat{n}| \geq |\bar{k}| \geq |\hat{m}|\}$$

satisfies the following estimate

$$\sum_{k \in \mathbb{Z}^3} 1_{N_0}(n, k, -n - k) \leq C|\hat{n}|^2(L_1L_2 + L_1 + L_2) + \sum_{(\sigma_1, \sigma_2) \in \{\pm\}^2} vol(V_{n, \sigma_1, \sigma_2}).$$

### 5.4. Some properties of FFF interactions under near resonance condition

We begin this subsection with a lemma on the possible choices of signs for a near resonant FFF triplet. As it turns out, the simple bound $\omega_k > \delta$, in the Boussinesq approximation context, leads to a simplification in comparison to the rotating Navier-Stokes case in [7].

**Lemma 11.** Consider $\sigma_1, \sigma_2, \sigma_3 \in \{\pm\}$ and the ordered dispersion relation values $\omega_1 \geq \omega_2 \geq \omega_3 > \delta \geq 0$. If the near resonance condition

$$|\sigma_1\omega_1 + \sigma_2\omega_2 + \sigma_3\omega_3| \leq \delta$$

holds true, then we necessarily have

$$\sigma_1 \neq \sigma_2 \text{ and } \sigma_2 = \sigma_3,$$

namely, $(\sigma_1, \sigma_2, \sigma_3) = (+, -, -)$ or $(-, +, +)$.

**Proof.** It suffices to prove with the additional assumption $\sigma_1 = +$. 

We will argue by contradiction. First, we suppose instead that $\sigma_1 = \sigma_2$. Then, the near resonance condition implies that $\omega_1 + \omega_2 + \sigma_2\omega_3 \leq \delta$. Since $\omega_2 + \sigma_3\omega_3 \geq 0$ by the ordering assumption, we infer that $\omega_1 \leq \delta$, leading to a contradiction.

Similarly, we suppose instead $\sigma_2 \neq \sigma_3$. Then the first part of the conclusion guarantees that $\sigma_2 = -$, hence $\sigma_3 = +$. Then $\omega_1 - \omega_2 + \omega_3 \leq \delta$, by the near resonance condition. Since $\omega_1 - \omega_2 \geq 0$ by the ordering assumption, we infer that $\omega_3 \leq \delta$, leading to a contradiction. □

**Corollary 3.** Let $\delta \in [0, \min\{\frac{\eta}{2}, \frac{1}{2}\})$ be fixed. Then a necessary condition for the set of FFF near resonances to be non-empty is

$$\begin{cases} \eta \geq 2 - \delta, & \text{if } \eta > 1, \\ \eta \leq \frac{\delta + 1}{2}, & \text{if } \eta < 1. \end{cases}$$

**Proof.** Let $\omega_1 \geq \omega_2 \geq \omega_3$ be the ordered eigenvalues, in decreasing order, from Lemma [11]. Let $\delta > 0$ be fixed. Then a necessary condition for the set of FFF near resonances to be non-empty is

$$\omega_3 \leq \frac{\max\{\eta, 1\} + \delta}{2}$$

**Proof.** The near resonance condition and Lemma [11] directly imply that $-\delta \leq \omega_1 - 2\omega_3$, with the claim following from the trivial upper bound on $\omega_1$. □

5.5. **Elliptic integrals.** In order to calculate the volume of the set $V_{n,\sigma_1,\sigma_2}$, we follow the strategy of [7, Section 5] which leads to the study of certain elliptic integrals. First, we recall that $n \in \mathbb{Z}^3 \setminus \{0\}$ and $\delta > 0$ are fixed. Since the volume element of the integral under consideration scales like $L_1L_2d\hat{k} = dk$, it suffices to compute the volume for the case $L_1 = L_2 = 1$.

The volume integral

$$\int_{\mathbb{R}^3} 1_{V_{n,\sigma_1,\sigma_2}}(k)dk$$

is expressed in spherical coordinates $(\lambda_k, \theta_k, \phi_k) \in [\frac{\pi}{2}, 1] \times [0, \pi] \times [0, 2\pi]$, with a rescaled radial component $\lambda_k$, similarly to the proof of Lemma [11] but with $M = |n|$. In particular, $\lambda_k := \frac{|k|}{|n|}$, thus

$$k = |n|\lambda_k(\sin \theta_k \cos \phi_k, \sin \theta_k \sin \phi_k, c_k), \quad \text{for } k \in \mathbb{Z}^3.$$ 

First, a change of variable from the azimuthal angle to $c_m$ is performed. We define the set

$$A^F := \{(c_k, c_m) \in [-1, 1]^2 : \min_{(\sigma_1, \sigma_2) \in \{\pm\}^2} |\sigma_1\omega_k + \sigma_2\omega_n + \omega_m| \leq \delta\}.$$ 

An initial estimate

$$\text{vol}(V_{n,\sigma_1,\sigma_2}) \leq 16|n|^3 \int_0^\pi \int_{-1}^1 \int_{-1}^1 1_{A^F} q^*(\lambda_k, \phi_k, \theta_k)|c_m| \sin \theta_k dc_m d\lambda_k d\theta_k$$

can be then obtained, after examining the constraints on the magnitude of the wavevectors. In that context, the expression $q^*|c_m|$, which depends on all three variables $(\lambda_k, \phi_k, \theta_k)$, is
a result of the non-zero Jacobian after our change of variable, due to the behaviour of the azimuthal angle already observed in the proof of Lemma 10. Then, we define

\[ Q_{EL} := |c_m| \int_{\frac{1}{2}}^1 q^* d\lambda_k, \]

so that it suffices to estimate an expression of the form

\[ 16|n|^3 \int_{-1}^1 \int_{0}^{\pi} 1_{AF} Q_{EL}(\theta_k, c_m) \sin \theta_k d\theta_k dc_m, \]

due to the Fubini-Tonelli theorem. The behaviour of \( Q_{EL} \) can be further quantified using the theory of elliptic integrals. In particular, we recall [7, Lemma 5.4].

**Lemma 12.** Let \( c_m, c_k, c_n \in (-1, 1) \setminus \{0\} \) be pairwise distinct. Moreover, let \( \varsigma_i, i = 1, \ldots, 4, \) denote the elements of the set \( \{1, |c_k|, |c_m|, |c_n|\} \) arranged in decreasing order. Then the following statement holds true:

\[ (52) \quad Q_{EL} \leq \frac{C}{\sqrt{(1 - \varsigma_3)(\varsigma_2 - \varsigma_4)}} \left[ 1 + \log \frac{(1 - \varsigma_3)(\varsigma_2 - \varsigma_4)}{(1 - \varsigma_1)(\varsigma_2 - \varsigma_3)} \right], \]

for a constant \( C > 0 \). In addition, we have the following estimate:

\[ (53) \quad Q_{EL} \lesssim \frac{1}{\sqrt{(1 - \varsigma_3)(\varsigma_2 - \varsigma_4)}}, \quad \text{when} \quad |c_n| \leq \min\{|c_m|, |c_n|\}. \]

**Remark 6.** Combining (52) with the inequality \( \log x \leq x - 1 \), we obtain the following estimate

\[ (54) \quad Q_{EL} \lesssim \frac{1}{\sqrt{1 - \varsigma_i \sqrt{\varsigma_2 - \varsigma_3}}} \lesssim \eta \, \frac{1}{\sqrt{\varsigma_2 - \varsigma_3}}, \]

with the second inequality holding true in view of Lemma 4.

A technical necessity that stems from the previous Lemma is the consideration of the possible orderings of \( |c_k|, |c_m|, |c_n| \). However, the number of cases we need to take into account is limited, as the sign of the term corresponding to the wavevector labelled with \( m \) is always positive. In particular, Lemma 11 yields 6 possible cases.

5.6. **The volume estimate for FFF interactions.** In view of Lemmas 11 and 12 we can carry out a more refined estimate compared to that of [7]. In more detail, \( \omega_k, \omega_m, \omega_n \) are strictly bounded away from zero, a property which was not available in [7], while their ordering with respect to \( c_k, c_m, c_n \) is either fully preserved or reversed.

**Theorem 7.** Let \( \delta \in (0, \min\{\frac{3}{2}, \frac{1}{2}\}) \), \( \eta \neq 1 \), \( \sigma_1, \sigma_2 \in \{\pm\} \) and \( n \in \mathbb{Z}^3 \setminus \{\vec{0}\} \) be fixed. Moreover, consider the set

\[ V_{n,\sigma_1,\sigma_2} := \left\{ k \in \mathbb{R}^3 \setminus \{\vec{0}, n\} : |F_{n,\sigma_1,\sigma_2}(k)| \leq \delta \quad \text{and} \quad \frac{1}{2} |\vec{n}| \leq |\vec{k}| \leq |\vec{n}| \right\}. \]

Then, the following estimate holds true

\[ (55) \quad \text{vol}(V_{n,\sigma_1,\sigma_2}) \leq C(\eta)\delta L_1 L_2 |\vec{n}|^3, \]

for a positive constant \( C \) independent of \( \delta, L_1, L_2, n, \sigma_1, \sigma_2 \).
Proof. First, we recall the fact that $\omega = f(|c_k|)$ for an either strictly increasing function or strictly decreasing function $f_\eta$, namely $f_\eta(x) := \sqrt{1 + (\eta^2 - 1)x^2}$. The monotonicity of $f_\eta$ depends on the region of $\eta$ under consideration, when $x > 0$. We initially focus on the case $1 < \eta$, where $f_\eta$ is strictly increasing. Then, a reversal in monotonicity for the remaining $\eta$’s will still yield the result, up to some necessary permutations of the wavenumbers involved.

We introduce the following sets:

$A_0 := \{(c_k, c_m) \in [-1, 1]^2 : c_k c_m (1 - c_k^2) (1 - c_m^2) (c_k^2 - c_m^2) (c_k^2 - c_m^2) = 0\},$

$A_1 := \{(c_k, c_m) \in [-1, 1]^2 : |c_k| < \min\{|c_n|, |c_m|\} \setminus A_0,$

$A_2 := \{(c_k, c_m) \in [-1, 1]^2 : |c_m| < \min\{|c_n|, |c_k|\} \setminus A_0,$

$A_3 := \{(c_k, c_m) \in [-1, 1]^2 : |c_n| < \min\{|c_k|, |c_m|\} \setminus A_0,$

$B_1 := \{(c_k, c_m) \in [-1, 1]^2 : |c_n| < |c_m|\}

$B_2 := \{(c_k, c_m) \in [-1, 1]^2 : |c_k| < |c_m|\}.$

It suffices to prove (55) for $n$ in a dense subset of $\mathbb{R}^3$, so that we can exclude some problematic values. In particular, we fix $n \in \mathbb{Z}^3 \setminus \{0\}$, with $n_1 n_2 n_3 \neq 0$. Then, for $i = 1, 2, 3$, we define $A_i^F := A_i \cap A^F$.

We will extensively use the following relations in what follows:

$$dc_k = \pm \frac{1}{\sqrt{|\eta^2 - 1| \sqrt{|\omega^2 - 1|}}} \omega_k d\omega_k$$

and

$$c_k^2 - c_m^2 = \frac{\omega_k^2 - \omega_m^2}{\eta^2 - 1},$$

which hold true for any combination of $c_k, c_n, c_m$. Finally, we recall that due to Corollary 3 we need to examine the cases rotation dominated regime, $\eta \geq 2 - \delta$, and the stratification dominated regime, $\eta \leq \frac{2 + 1}{2}$.

The rotation dominated regime

When $I_{A_1} = 1$, the estimate (54) is applicable. Nevertheless, we need to take into account whether $|c_n| < |c_m|$ or not. In the former case, the only possible choice of signs for $(\omega_n, \omega_k)$ is $(-, -)$, due to Lemma 11. Then, the following relations hold true

$$\omega_m - \omega_n \in (\omega_k - \delta, \omega_k + \delta) \quad \text{and} \quad \sqrt{2 \sqrt{(\omega_k - \delta)} < \sqrt{\omega_m^2 - \omega_n^2},}$$

as a consequence of the near resonance condition and our ordering considerations. As all the estimates depend on the absolute value of the quantities under consideration, we restrict our attention to the case $c_k, c_m > 0$. Then we use (54), substituting $c_m$ with $\omega_m$ in the volume integral, in conjunction with (56) and (57), so that

$$\int_0^1 \int_{|c_n|}^1 \frac{1}{\sqrt{\omega_m^2 - c_n^2}} dc_m d\omega_k = \int_{|c_n|} |c_n| \int_{\omega_k + \omega_n + \delta}^{\omega_k + \omega_n - \delta} \frac{1}{\sqrt{\omega_m^2 - \omega_n^2} \sqrt{\omega_m^2 - 1}} d\omega_m d\omega_k \frac{\omega_m}{\omega_n - \omega_k - \delta} dc_k.$$ 

Since $\delta \leq \frac{1}{2}$ and $1 \leq \omega_k, \omega_n$, the last integrand is uniformly bounded, thus proving our claim in that context.

The case where $(\omega_m, \omega_n, \omega_k)$ have sign $(+, - +)$ requires a more delicate treatment, under the restriction that $\omega_m < \omega_n$. First, we have the analog of (57)

$$\omega_n - \omega_m \in (\omega_k - \delta, \omega_k + \delta) \quad \text{and} \quad \sqrt{2 \sqrt{(\omega_k - \delta)} < \sqrt{\omega_n^2 - \omega_m^2},}$$
Moreover, the near resonance condition together with (53), the ordering imposed on the magnitude of the wavevectors, and the fact that $\delta < \frac{1}{2}$ imply that:

$$\frac{\sqrt{2}}{2\sqrt{\eta^2 - 1}} < \frac{\sqrt{(2 - \delta)^2 - 1}}{\sqrt{\eta^2 - 1}} \leq |c_n| \leq \frac{|k_3|}{|\hat{n}|} + \frac{|m_3|}{|\hat{n}|} \leq |c_k| + 2|c_m| \leq 3|c_m|,$$

i.e. $\omega_m > \frac{\sqrt{2}}{\sqrt{6}}$. We recall that (54) is applicable. Then, our last estimate on $\omega_m$, combined with the fact that $\omega_k - \delta > \frac{1}{2}$, a change of variable from $c_m$ to $\omega_m$, (56) and (58), allow us to estimate:

$$\int_0^1 \int_{-1}^1 \frac{1_{A_k} \cap B_1}{c_m^2 - c_n^2} \frac{1}{\sqrt{1 - c_k^2}} d\omega_m d\omega_k < \int_0^1 \int_{\omega_k + \omega_n + \delta}^{\omega_k + \omega_n + \delta} \frac{1}{\sqrt{\omega_m^2 - \omega_n^2}} \frac{\omega_m}{\sqrt{\omega^2 - \omega_m^2 - 1}} d\omega_m d\omega_k$$

$$< C\delta \int_{|c_n|} \frac{1}{\sqrt{(\omega_k - \delta)}} d\omega_k < C\delta,$$

for an absolute constant $C > 0$.

The case $(c_k, c_m) \in A_k^c$ can be handled analogously, by permuting the roles of the three wavevectors where necessary. As far as the case $1_{A_1} = 1$ is concerned, we further distinguish cases. If $|c_k| < |c_m|$, then the signs for $(\omega_m, \omega_n, \omega_k)$ are $(+, +, -)$. Thus, we have

$$\omega_m - \omega_n \in (\omega_k - \delta, \omega_k + \delta).$$

We use (53), change variables from $c_m$ to $\omega_m$ and recall (56), (59), in order to deduce that

$$\int_0^1 \int_{-1}^1 \frac{1_{A_k} \cap B_2}{c_m^2 - c_n^2} \frac{1}{\sqrt{1 - c_k^2}} d\omega_m d\omega_k = \int_0^1 \int_{\omega_k + \omega_n - \delta}^{\omega_k + \omega_n + \delta} \frac{1}{\sqrt{\omega_m^2 - \omega_n^2}} \frac{\omega_m}{\sqrt{\omega^2 - \omega_m^2 - 1}} d\omega_m d\omega_k$$

$$< \sqrt{2}\delta \int_{|c_n|} \frac{1}{\sqrt{(\omega_k - \delta)}} \frac{\omega_k + \omega_n - \delta}{\sqrt{(\omega_k + \omega_n - \delta)^2 - 1}} \frac{1}{1 - c_k^2} d\omega_k$$

$$\leq \sqrt{2}C\delta,$$

for a constant $C > 0$ independent of $\delta, n, \eta$. Finally, when $|c_m| < |c_k|$ our last estimate still goes through, up to reversing the roles of $k$ and $m$.

**The stratification dominated regime**

The case $2\delta < \eta < 1$ presents a reversal in monotonicity.

In more detail, when $1_{A_1} = 1$ and $|c_n| < |c_m|$, then Lemma 11 implies that $(\omega_n, \omega_k)$ necessarily have opposite signs and

$$\omega_k - \omega_n \in (\omega_m - \delta, \omega_m + \delta).$$

We will use (52) and change coordinates first from $(c_k, c_m)$ to $(\omega_k, \omega_m)$ and then from $(\omega_k, \omega_m)$ to $(\omega_k, -\omega_n + \omega_k + \delta')$, so that $\delta' \in (-\delta, \delta)$ by (60). We also have

- $\omega_k^2 - \omega_m^2 < \omega_k^2 - \eta^2$, using the trivial lower bound for $\omega_m$,
- $\omega_k^2 - \omega_m^2 > (\omega_k^2 - \eta^2) > 0$, by (60) and the fact that $\delta < \frac{\eta}{2}$,
- $\frac{\omega_m}{\sqrt{1 - \omega_m^2}} \leq C$ by Corollary 4,
- $\omega_k \in (\eta + \omega_n + \delta', \min\{1, 2\omega_n + \delta'\}) := (b, a)$, due to the fact that $\omega_m < \omega_n$.
Thus, via (52), (56) and our change of variables

\[ \int_0^1 \int_0^1 1_{A^c_k \cap B_1} \left[ 1 + \log \frac{(1 - c_n^2)(c_m^2 - c_k^2)}{(1 - c_k^2)(c_m^2 - c_n^2)} \right] \frac{dc_k dc_m}{\sqrt{(1 - c_k^2)(c_m^2 - c_k^2)}} \]

\[ = \int_0^1 \int_0^1 1_{A^c_k \cap B_1} \left[ 1 + \log \frac{(\omega_n^2 - \eta^2)(\omega_m^2 - \omega_k^2)}{(\omega_m^2 - \eta^2)(\omega_k^2 - \omega_m^2)} \right] \frac{\omega_k \omega_m}{\sqrt{1 - \omega_k^2}} \frac{dc_k dc_m}{\sqrt{1 - \omega_m^2}} \]

\[ = \frac{1}{1 - \eta^2} \int_\eta^{\omega_n} \int_\eta^{\omega_m} 1_{A^c_k} \left[ 1 + \log \frac{(\omega_n - \eta)}{\omega_m - \omega_k} \right] \frac{1}{\sqrt{1 - \omega_k}} \frac{dc_k dc_m}{\sqrt{1 - \omega_m}} \]

\[ \leq C \int_{-\delta}^\delta \int_{\omega_n - \eta}^{\omega_n} 1_{A^c_k} \left[ 1 + \log \frac{(\omega_n - \eta)}{\omega_m - \omega_k + \delta} \right] \frac{1}{\sqrt{1 - \omega_k}} \frac{dc_k d\delta}{\omega_n - \eta} \]

\[ \leq C \int_{-\delta}^\delta \int_{0}^{\frac{1 - \omega_k^2}{\omega_n - \eta}} [C_1 - \log y] dy d\delta' \leq C \delta, \]

where we further changed variables from \( \sqrt{\frac{a - \omega_k}{\omega_n - \eta}} \) to \( y \). Note that the inner integral has upper limit not exceeding 1.

When \( 1_{A^c_k} = 1 \) and \( |c_m| < |c_n| \), we follow a similar procedure to the above. In particular, we take into account (60) and change coordinates from \((\omega_k, \omega_m)\) to \((\omega_k, -\omega_n + \omega_k + \delta')\). Then, (60) implies that

- \( \omega_m < 1 - \frac{\eta}{2} \),
- \( \omega_m - \omega_n \in (\max\{0, \omega_k - 2\omega_n - \delta\}, \omega_k - 2\omega_n + \delta) \),

so that we can estimate

\[ \int_0^1 \int_0^1 1_{A^c_k \cap B_1} \frac{dc_k dc_m}{c_n^2 - c_m^2} = \int_0^1 \frac{1}{\sqrt{1 - \eta^2}} \int_{\omega_n}^{\omega_k} \frac{1}{\sqrt{\omega_m^2 - \omega_n^2 \sqrt{1 - \omega_k^2}}} \right] \frac{dc_k dc_m}{\sqrt{1 - \omega_m^2}} \]

\[ < \frac{C}{\sqrt{1 - \eta^2}} \int_{\omega_n}^{\omega_k} \int_{\omega_m}^{\omega_k} \frac{1}{\sqrt{\omega_m - \omega_n \sqrt{1 - \omega_k}}} \frac{dc_k dc_m}{\sqrt{1 - \omega_m^2}} \]

\[ = \frac{C}{\sqrt{1 - \eta^2}} \int_{-\delta}^\delta \int_{\omega_n - \delta'}^{\omega_k} \frac{1}{\sqrt{2\omega_n - \delta' \sqrt{1 - \omega_k}}} \frac{1}{\sqrt{1 - \omega_k}} d\omega_k d\delta' \]

\[ = \frac{C}{\sqrt{1 - \eta^2}} \int_{-\delta}^\delta \int_{0}^{\frac{1 - \omega_k^2}{\sqrt{1 - y^2}}} \frac{1}{\sqrt{1 - y}} \frac{d\delta'}{\sqrt{y}} \leq C \delta, \]

where we further changed variables as \( y = \frac{1 - \omega_k}{\sqrt{1 - y^2}} \).

Finally, when \( 1_{A^c_k} = 1 \), we give the details on the estimate for the case \( |c_k| < |c_m| \), i.e. when \( \omega_m < \omega_k < \omega_n \). Hence, Corollary 4 implies that we have \( \omega_m \leq \frac{3}{4} \). In addition, it holds
that
\begin{equation}
\omega_m \in (\omega_n - \omega_k - \delta, \omega_n - \omega_k + \delta),
\end{equation}
due to the near resonance condition. We change variables from $c_m$ to $\omega_m$, using \((63), (61)\), the aforementioned boundedness of $\omega_m$ away from 1 and the fact that $2\eta < \omega_m + \omega_n$, in order to deduce:

\[
\int_{-1}^{1} \int_{-1}^{1} \frac{1}{A_k \cap B_2} \frac{1}{(c_m - c_n)^2} \sqrt{1 - c_k^2} \sqrt{1 - \omega_m^2} \sqrt{1 - \omega_n^2} \omega_m \omega_n \, d\omega_m \, d\omega_n < C \int_{|c_m|}^{1} \frac{1}{\sqrt{\omega_n - \omega_k}} \sqrt{1 - \omega_n^2} \sqrt{1 - \omega_k^2} \, dc_n < C \delta.
\]

\section{Proof of the main results}

\subsection{Proof of Theorem 2 (FFF estimates)}

We prove Theorem 2 using a similar strategy to \[7, \text{Theorem 1.3}\] in the rotating Navier-Stokes context.

\textit{Proof of Theorem 2.} First, we recall the sign convention for the bilinearity from \((11)\) and the conjugation property of Remark 4. Then, we use Parseval’s identity and the symmetry between $m,n$ in the resulting sum, due to incompressibility, in order to get:

\[
2 \left\langle \mathcal{D}' \tilde{B}_f(u_f, v_f), \mathcal{D}' v_f \right\rangle = i |T|^3 \sum_{k,m,n; \text{conv } \sigma_1 \sigma_2 \sigma_3 \neq 0} \left( |\tilde{n}|^2 - |\tilde{m}|^2 \right) \left( r_k^{\sigma_1} \cdot \tilde{m} \right) \left( r_m^{\sigma_2} \cdot r_n^{\sigma_3} \right) u_k^{\sigma_1} v_m^{\sigma_2} v_n^{\sigma_3} 1_{XFF}\(k,m,n\).
\]

In the range of $\ell$ under consideration, we use the triangle inequality and the mean value theorem to obtain $|\tilde{n}|^l - |\tilde{m}|^l \leq \ell |\tilde{k}| \max\{|\tilde{n}|^{\ell - 1}, |\tilde{m}|^{\ell - 1}\}$. In addition, $|r_k^{\sigma_1} \cdot \tilde{m}| \leq \min\{|\tilde{n}|, |\tilde{m}|\}$ holds true, due to incompressibility. Then, we complete the proof by combining Lemma 6 for $\beta = 2$ and our counting results from Theorems 6, 7 together with the choice of $\delta$ specified in Theorem 2.

\subsection{Energy estimates in $L^2$.}

As a starting step towards the proof of Theorem 1 we show that the standard $L^2$ energy inequality holds true for the restricted system \((16)\).

\textbf{Lemma 13.} Let $\tilde{U}_0 \in H^1(T^3; \mathbb{R}^3)$ with $\ell \geq 1$ be a divergence free and zero-mean vector field. If $\tilde{U}$ is a solution of \((16)\) with initial data $\tilde{U}_0$ for $t \in [0,T)$, then:

\begin{equation}
\|\tilde{U}(T)\|_{L^2}^2 + 2 \nu_{\min} \int_0^T \|\tilde{U}\|_{H^1}^2 \, dt \leq \|\tilde{U}_0\|_{L^2}^2.
\end{equation}

\textit{Proof.} We test \((16)\) with $\tilde{U}$, so that

\[
\partial_t \|\tilde{U}\|_{L^2}^2 + 2 \left\langle \tilde{A} \tilde{U}, \tilde{U} \right\rangle + 2 \left\langle \tilde{B}(\tilde{U}, \tilde{U}), \tilde{U} \right\rangle = 0
\]
follows from the skew-symmetry of $L$. Moreover, Lemma 3 and Lemma 5 imply that $\nu_{\min} \Vert \tilde{U} \Vert^{2}_{H^1} \leq \left \langle A\tilde{U}, \tilde{U} \right \rangle$. We also have $\left \langle \tilde{B}(\tilde{U}, \tilde{U}), \tilde{U} \right \rangle = 0$, due to Corollary 1 with (62) following after an integration in time.

### 6.3. Proof of Theorem 3 and slow output estimates.

We now present the proof of the FFS estimates of Theorem 3. Its main ingredients are Lemma 8, which is applicable due to Corollary 2 and a growth bound on the interaction coefficients for the mixed terms, which is based on the analysis of Appendix C.

**Proof of Theorem 3.** We recall (11) and Remark 4. Then, due to the $k, m$ symmetry of $1_{\mathcal{N}_{FFS}}(k, m, n)$, the mixed part of our slow approximation is given by

$$\tilde{B}_{s}(u_f, u_f) = \sum_{k, n, m; \text{conv}} (B^+_k u_k u_m^0 + B^+_m u_k^0 u_m) 1_{\mathcal{N}_{FFS}}(k, m, n)$$

$$= i|\mathbb{T}^3| \sum_{k, n, m; \text{conv}} S^{+0}_{kmn} u_k^+ u_m^- r^0_n 1_{\mathcal{N}_{FFS}}(k, m, n),$$

with the interaction coefficients defined via

$$S^{+0}_{kmn} = (r^+_k \cdot \tilde{m}^0)(r^-_m \cdot r^0_n) + (r^-_m \cdot \tilde{k}^0)(r^+_k \cdot r^0_n).$$

Using Parseval’s identity, we have

$$\left \langle \mathcal{D}' \tilde{B}_{s}(u_f, u_f), \mathcal{D}' w \right \rangle = i|\mathbb{T}^3| \sum_{k, n, m; \text{conv}} |\tilde{n}|^{2\ell} S^{+0}_{kmn} u_k^+ u_m^- r^0_n 1_{\mathcal{N}_{FFS}}(k, m, n).$$

Moreover, we have

$$|S^{+0}_{kmn}| \lesssim |(\omega_k - \omega_m)| |\tilde{k}||\tilde{m}||\tilde{n}|^{-1}$$

via Lemma 10 and $|\tilde{n}|^\ell \lesssim |\tilde{k}||\tilde{m}||\tilde{n}|^\ell$ via the triangle inequality, when $\ell > 0$. Then we combine the preceding estimates, (22), and Lemma 8 for the resulting convolution sum, with $\mu = 3 - \frac{\ell}{2}$, in order to obtain

$$\left \langle \mathcal{D}' \tilde{B}_{s}(u_f, u_f), \mathcal{D}' w \right \rangle \lesssim_{\eta, \mathbb{T}^3} \sum_{k, n, m; \text{conv}} |\omega_k - \omega_m| |\tilde{k}||\tilde{m}||\tilde{n}|^{2\ell-1} |u_k^+||u_m^-||w^0_n 1_{\mathcal{N}_{FFS}}(k, m, n)$$

$$\lesssim_{\eta, \ell} \sum_{k, m, n; \text{conv}} \delta^*(k, m, n) \left ( |\tilde{k}|^{\ell_1} + |\tilde{m}|^{\ell_1} \right ) |\tilde{k}||\tilde{m}||\tilde{n}|^{\ell_2} |u_k^+||u_m^-||w^0_n 1_{\mathcal{N}_{FFS}}(k, m, n)$$

$$\lesssim_{\eta, \ell} \sum_{k, m, n; \text{conv}} \left ( |\tilde{k}|^{\ell_1} + |\tilde{m}|^{\ell_1} \right ) |\tilde{k}|^{1-\frac{\ell_1}{2}} |\tilde{m}|^{1-\frac{\ell_1}{2}} |\tilde{n}|^{\ell_2} |u_k^+||u_m^-||w^0_n 1_{\mathcal{N}_{FFS}}(k, m, n)$$

$$\lesssim \|D^{\frac{\ell}{2} - \frac{\lambda}{4} - a + \ell_1} u_f\|_{L^2} \|D^{1+a-\frac{\ell}{2}} u_f\|_{L^2} \|\mathcal{D}' w\|_{L^2}.$$  \[ \square \]

**Proof of Lemma 7.** Using Parseval’s identity, and Lemma 3 we have

$$\left \langle L_{p_u} \tilde{B}_{s}(u_f, u_f), L_{p_u} w \right \rangle = i|\mathbb{T}^3| \sum_{k, n, m; \text{conv}} |\tilde{n}|^{2\ell} S^{+0}_{kmn} u_k^+ u_m^- r^0_n 1_{\mathcal{N}_{FFS}}(k, m, n).$$

The result then follows from adapting the proof of Theorem 3 by using a Fourier multiplier with symbol $|\tilde{n}|$ instead of $D$ and setting $\ell_1 = 0$ and $\ell_2 = 1$. \[ \square \]
We proceed by examining the SSS term \( \tilde{B}_s(\tilde{U}_s, \tilde{U}_s) \). In that direction, we remark that the following norm equivalence
\[
\min\{\eta^{-1}, 1\} \|\tilde{Q}\|_{H^{\ell-1}} \leq \|\tilde{U}_s\|_{H^{\ell}} \leq \max\{\eta^{-1}, 1\} \|\tilde{Q}\|_{H^{\ell-1}}
\]
holds true due to (33), where \( \tilde{Q} = L_{ps} \tilde{U} \). Thus, in view of Lemma 8 and the commutability properties of \( \tilde{\nu}_{11}, \tilde{\nu}_{22} \) in Section 3.2, it suffices to derive estimates for the following system
\[
\partial \tilde{Q} + L_{ps} \tilde{B}_s(\tilde{U}, \tilde{U}) - \tilde{\nu}_{11} \Delta \tilde{Q} = 0
\]
\[
\partial \tilde{U}_f + \tilde{B}_f(\tilde{U}, \tilde{U}) - \tilde{\nu}_{22} \Delta \tilde{U}_f = N \mathcal{L} \tilde{U}_f
\]
instead of (16). An advantage of working with the linear potential vorticity equation is that the SSS term, even though present in the PDE (65), plays no part on the \( L^2 \) energy level. In more detail, we have the following result from [2].

Lemma 14. The SSS transport term \( B_s(\tilde{U}_s, \tilde{U}_s) \) satisfies the following identities
\[
L_{ps} B_s(\tilde{U}_s, \tilde{U}_s) = - \left( (\Delta_\eta)^{-1} \nabla_H \tilde{Q} \right) \cdot \nabla_H \tilde{Q} = - \nabla_H \left( \tilde{Q} (\Delta_\eta)^{-1} \nabla_H \tilde{Q} \right).
\]
In particular,
\[
\langle L_{ps} B_s(\tilde{U}_s, \tilde{U}_s), \tilde{Q} \rangle = 0.
\]
In addition, the following estimate holds true
\[
\left| \left\langle L_{ps} B_s(\tilde{U}_s, \tilde{U}_s), (\Delta)^{\ell} \tilde{Q} \right\rangle \right| \lesssim \|\tilde{Q}\|_{H^{\ell}} \|\tilde{Q}\|_{H^{\ell+1}},
\]
for all \( \ell \in \mathbb{R}^+ \), with the implied constant independent of \( \tilde{U} \).

6.4. Proof of Theorem 4 and fast output estimates. In this section we prove the convolution sum estimate of Theorem 4 which follows a similar procedure to the proof of Theorem 3. In particular, we use Lemma 8 once more, since Corollary 2 holds true for the FSF interactions, but without any extra help from the interaction coefficients involved.

Proof of Theorem 4. Using Remark 4 and Parseval’s identity, we derive
\[
\left| \left\langle D^\ell \tilde{B}_f(u_f, v_s), D^\ell u \right\rangle \right| = i\left| T^3 \right| \sum \sum |\tilde{m}|^{2\ell} S_{k,m,n; conv \sigma_1 \sigma_3 < 0}^\sigma_1 \sigma_3 u_k^{\sigma_1} v_m^{\sigma_3} u_n^{\sigma_3} 1_{N_{FSF}}(k, m, n),
\]
with \( S_{k,m,n}^{\sigma_1 \sigma_3} = (r_{k}^\sigma \cdot \tilde{m}) (r_{m}^{\sigma_3} \cdot r_{n}^{\sigma_3}) \). We note that \( |S_{k,m,n}^{\sigma_1 \sigma_3}| \leq |\tilde{m}| \), for all possible choices of sign. Then, using the fact that \( |\tilde{n}|^{\ell} \lesssim \ell \), \( |\tilde{k}|^{\ell} \), \( |\tilde{m}|^{\ell} \) due to the convolution condition, we have
\[
\left| \left\langle D^\ell \tilde{B}_f(u_f, v_s), D^\ell u \right\rangle \right| \lesssim \left| T^3 \right| \sum \sum (|\tilde{k}|^{\ell} + |\tilde{m}|^{\ell}) |\tilde{m}| |\tilde{n}|^{2\ell-\ell} |v_m^{\sigma_3}| |u_k^{\sigma_3}| |u_n^{\sigma_3}| 1_{N_{FSF}}(k, m, n).
\]
Finally, we apply Lemma 8.2 twice, with \( \mu = 3 - \frac{\ell}{2} \), in order to conclude, also recalling Remark 5. In particular, the indicator function \( 1_{N_{FSF}}(\cdot, \cdot, \cdot) \) is symmetric with respect to permutations in its first and third arguments.

We now give a simple estimate for the SFF terms that we will encounter in the proof of Theorem 4. The result is posed in terms of an unrestricted bilinearity, as the operator \( \tilde{B}(\cdot, \cdot) \) includes all SFF interactions.
Lemma 15. Let \( \beta_1 \in (0, \frac{3}{2}) \). Moreover, let \( u \in H^{\frac{3}{2} - \beta_1}(\mathbb{T}^3; \mathbb{R}^4) \) and \( v \in H^{\beta_1 + 1}(\mathbb{T}^3; \mathbb{R}^4) \) be divergence-free and zero-mean fields. Then the following estimate holds true:

\[
|\langle B_f(u, v) , \Delta v \rangle| \lesssim_{\mathbb{T}^3, \beta_1} \|u\|_{H^{\frac{3}{2} - \beta_1}} \|v\|_{L^2} \|v\|_{H^{\beta_1 + 1}}.
\]

Proof. We use Parseval's identity, Remark 4 and the incompressibility of \( u \) to get:

\[
-2 \langle B_f(u, v) , \Delta v \rangle = i \int_{\mathbb{T}^3} \sum_{k,m,n; \text{conv} \sigma_2 \sigma_3 \neq 0} u_k^0 \nu_m^2 v_n^3 (r_k^0 \cdot \tilde{m}) (r_m^2 \cdot r_n^3) (\|\tilde{n}\|^2 - \|\tilde{m}\|^2).
\]

We have \( \|\tilde{n}\| - \|\tilde{m}\| \leq |\tilde{k}| \) via the triangle inequality and \( |(r_k^0 \cdot \tilde{m})| \leq \min\{\|\tilde{m}\|, \|\tilde{n}\|\} \), due to incompressibility. Also, we clearly have \( \|\tilde{n}\| + \|\tilde{m}\| < \max\{\|\tilde{n}\|, \|\tilde{m}\|\} \). Thus:

\[
|\langle B_f(u, v) , \Delta v \rangle| \leq \|\tilde{n}\| \|\tilde{m}\| \sum_{k,m,n; \text{conv} \sigma_2 \sigma_3 \neq 0} u_k^0 \|v_m^2\| \|v_n^3\| |\tilde{k}| \|\tilde{m}\| \|\tilde{n}\|.
\]

The result follows from (92).

6.5. Proof of Theorem 1 (Global energy bounds). Equipped with Theorems 2, 3 and Lemma 15 and the \( L^2 \) identity (62), we are now in a position to prove Theorem 1. We will extensively use the standard interpolation inequality

\[
\|u\|_{H^\ell} \lesssim \|u\|_{H^1} \|u\|_{L^2}^{1-\ell}, \quad \text{with} \quad \ell = \theta_1 \ell_1 + (1-\theta)\ell_2
\]

throughout the following proof.

Proof of Theorem 1. Estimates on the slow part We first test (65) with \( \tilde{Q} \) and estimate the slow component in \( L^2 \), recalling (33) and Lemma 3. In particular, the mixed slow interactions only consist of FFS triplets due to our choice of \( \delta^* \). Then, we use (68), (70), Lemma 1 with \( \zeta = \frac{\beta}{2} \), \( a = \frac{3}{2} \), and Lemma 5 to obtain:

\[
\partial_t \|\tilde{Q}\|^2_{L^2} + 2\nu_{\text{min}} \|\tilde{Q}\|^2_{H^1} \leq \partial_t \|\tilde{Q}\|^2_{L^2} - 2 \left\langle \tilde{\nu}_{11} \Delta \tilde{Q}, \tilde{Q} \right\rangle \leq 2 \left\langle \tilde{L}_{\tilde{\nu}} \tilde{B}_s(\tilde{U}_f, \tilde{U}_f), \tilde{Q} \right\rangle \leq 2 C \|\tilde{U}_f\|^2_{H^1} \|\tilde{Q}\|_{L^2},
\]

for a constant \( C \) that only depends on \( C_{\delta^*}, \eta, \mathbb{T}^3 \). We immediately infer that

\[
\partial_t \|\tilde{Q}\|_{L^2} \leq C \|\tilde{U}_f\|^2_{H^1}.
\]

Integrating the latter over \([0, T]\) and using (62), we obtain:

\[
\|\tilde{Q}(T)\|_{L^2} \leq \frac{C \nu_{\text{min}}^{-1}}{2} \|\tilde{U}_0\|^2_{L^2} + \|\tilde{Q}_0\|_{L^2},
\]

so that the \( L^\infty_t H^1_x \) estimate in (17) follows from (64). We insert the last estimate in (71) and use (62) in order to deduce

\[
2 \nu_{\text{min}} \int_0^T \|\tilde{Q}\|_{H^1} dt \leq C \nu_{\text{min}}^{-1} \|\tilde{U}_0\|^2_{L^2} (C \nu_{\text{min}}^{-1} \|\tilde{U}_0\|^2_{L^2} + \|\tilde{Q}_0\|_{L^2}) + \|\tilde{Q}_0\|^2_{L^2}
\]

and the \( L^2_t H^2_x \) estimate in (17) follows from (64).

Estimates on the fast part As far as the \( H^1 \) fast estimate is concerned, we test (73) with \( -\Delta \tilde{U}_f \) and use Lemma 3 and Lemma 5 to obtain:

\[
\partial_t \|\tilde{U}_f\|_{H^1} + 2 \nu_{\text{min}} \|\tilde{U}_f\|^2_{H^2} \leq \partial_t \|\tilde{U}_f\|^2_{H^1} + 2 \left\langle \tilde{\nu}_{22} \Delta \tilde{U}_f, \Delta \tilde{U}_f \right\rangle.
\]
for an absolute constant $C$. We follow a similar procedure for the FSF terms, with the help of Theorem 4 instead. In particular, (75) with $\zeta = \frac{6}{3}, \ell = 1, \ell_1 = \frac{3}{5}, a = a' = \frac{3}{5}, \ (64), \ (70), \ (72)$ and Young’s inequality yield
\begin{align}
(76) \quad \left| \left\langle \mathcal{B}_f(\tilde{U}_f, \tilde{U}_s), \Delta \tilde{U}_f \right\rangle \right| & \lesssim \eta, T > C \left[ \left\| \tilde{U}_s \right\|_{\dot{H}^1} \left\| \tilde{U}_f \right\|_{\dot{H}^1} + \left\| \tilde{U}_f \right\|_{\dot{H}^1} \right] \left\| \tilde{U}_f \right\|_{H^2} \\
& \lesssim \left( \left\| \tilde{Q} \right\|_{H^1} \left\| \tilde{U}_f \right\|_{H^1} + \left\| \tilde{Q} \right\|_{L^2} \left\| \tilde{U}_f \right\|_{\dot{H}^1} \right) \left\| \tilde{U}_f \right\|_{H^2} \\
& \leq \frac{C}{\nu_{\min}} \left( \left\| \tilde{Q} \right\|_{H^1}^2 + \left\| \tilde{U}_f \right\|_{H^1}^2 \right) \left\| \tilde{U}_f \right\|_{H^1}^2 + \frac{\nu_{\min}}{6} \left\| \tilde{U}_f \right\|_{H^2}^2,
\end{align}
for an absolute constant $C$. Finally, for the SFF terms we use Lemma 15 with $\beta_1 = 1,$ (61), (70) and Young’s inequality:
\begin{align}
(77) \quad \left| \left\langle B_f(\tilde{U}_s, \tilde{U}_f), \Delta \tilde{U}_f \right\rangle \right| & \lesssim \eta, T > 2 \left\| \tilde{U}_s \right\|_{H^2} \left\| \tilde{U}_f \right\|_{H^1} \left\| \tilde{U}_f \right\|_{H^2} \\
& \lesssim \left\| \tilde{U}_s \right\|_{H^2} \left\| \tilde{U}_f \right\|_{H^1} \left\| \tilde{U}_f \right\|_{H^2} \\
& \leq \frac{C}{\nu_{\min}} \left( \left\| \tilde{Q} \right\|_{H^1}^2 + \left\| \tilde{U}_f \right\|_{H^1}^2 \right) \left\| \tilde{U}_f \right\|_{H^2}^2,
\end{align}
for an absolute constant $C.$ Combining (74) with (75), (76) and (77) yields
\begin{align}
(78) \quad \partial_t \left\| \tilde{U}_f \right\|_{H^1}^2 + \nu_{\min} \left\| \tilde{U}_f \right\|_{H^2}^2 \leq C(\eta, T^3, C_\delta, C_\delta^2) \nu_{\min}^{-1} \left( \left\| \tilde{Q} \right\|_{H^1}^2 + \left\| \tilde{U}_f \right\|_{H^1}^2 \right) \left\| \tilde{U}_f \right\|_{H^1}^2.
\end{align}

Then, the $L^\infty_t H^1_x$ estimate in (18) follows by integrating the last inequality in time, (62), (64), (72) and Grönwall’s inequality. Finally, the $L^2_t H^2_x$ estimate in (18) follows by combining this estimate, (62), (64), (72) and (78).

6.6. Proof of Theorem 5 (Error estimates). In this section, we prove Theorem 5 on the difference of our approximation and the modulated system in an initial time interval. The estimate that we obtain depends on $\nu_1, \nu_2$ only via upper bounds for the ratio $\nu_R = \nu_{\max}/\nu_{\min}$ and $\nu_{\max}.$ Similar results have appeared concerning the proximity of the exact resonant dynamics to that of the full Boussinesq approximation in [2], [3], [4] and [10]. An interesting phenomenon is the higher regularity loss occurring in the difference equation, due to the presence of mixed interactions.

The strategy of the proof consists of conveniently expanding the difference of the approximate modulated and modulated systems, along with standard tools, like Grönwall’s inequality. In order to apply the latter, our control on $\omega_{knn}^2$ outside the mixed near resonant set proves crucial. Then, the estimation concerning the fast terms proceeds in a similar manner to [7]. On the other hand, a different lower bound for the bandwidth is implemented for the mixed terms. This is reflected on the larger derivative gap, $\ell - \ell'$, compared to the one of [7, Theorem 1.4]. Nevertheless, we only need standard bilinear estimates throughout the proof.
Proof of Theorem 5. Under the notation of Theorem 5, let \( u = e^{-\tau L} U \) be a solution to (3), and let \( \tilde{u} := e^{-\tau L} \tilde{U} \) be a solution to the corresponding approximate system

\[
\partial_t \tilde{u} + \tilde{B}(\tau, \tilde{u}, \tilde{u}) + \tilde{A}\tilde{u} = 0
\]

which is the modulated version of our proposed approximate system (16). Note that \( \tilde{A} \) is acting on \( \tilde{u} \) in the same way as in (14) due to the commutability properties of the restricted elliptic operators in Section 3.2. We set \( w = u - \tilde{u} \) and \( W = U - \tilde{U} \).

The equation for the difference

First, we derive the equation for \( w \):

\[
\partial_t w + B(\tau, u, w) + B(\tau, \tilde{u}, \tilde{w}) + B(\tau, \tilde{u}, \tilde{u}) - \tilde{B}(\tau, \tilde{u}, \tilde{u}) + \tilde{A}w + (A - \tilde{A})u = 0.
\]

The difference between the original and modified bilinearities, occurring in the difference equation, can be expressed as follows

\[
B(\tau, \tilde{u}, \tilde{u}) - \tilde{B}(\tau, \tilde{u}, \tilde{u}) = (B_s - \tilde{B}_s)(\tau, \tilde{u}_f, \tilde{u}_f) + (B_f - \tilde{B}_f)(\tau, \tilde{u}_f, \tilde{u}_f) + (B_f - \tilde{B}_f)(\tau, \tilde{u}_s, \tilde{u}_s) + B_s(\tau, \tilde{u}_f, \tilde{u}_s) + B_s(\tau, \tilde{u}_s, \tilde{u}_f) + \tilde{B}_s(\tau, \tilde{u}_f, \tilde{u}_s) + \tilde{B}_s(\tau, \tilde{u}_s, \tilde{u}_f).
\]

Elimination of oscillatory factors

We now focus on terms in the difference equation containing time oscillations. For the sake of brevity, we write \( \mathbf{1}_{\sigma_1 \sigma_2 < 0}, \mathbf{1}_{\sigma_1 \sigma_2 > 0} \) instead of introducing separate summation signs under these restrictions. In particular, we use the product rule in order to derive

\[
N(B_s - \tilde{B}_s)(\tau, \tilde{u}_f, \tilde{u}_f) = \partial_t r_1 + r_1'
\]

\[
:= \partial_t \sum_{k,m,n; \text{conv} \sigma_1, \sigma_2} \sum_{\sigma_1 \sigma_2} (\mathbf{1}_{\sigma_1 \sigma_2 < 0} \mathbf{1}_{N^{FFS}} \mathbf{1}_{\sigma_1 \sigma_2 > 0}) (i \omega_{k,m,n}^\sigma)^{-1} e^{i \omega_{k,m,n}^\sigma \tau} B_{k,m,n}^{\sigma_1 \sigma_2 0}(\tilde{u}_f, \tilde{u}_f)
\]

\[
- \sum_{k,m,n; \text{conv} \sigma_1, \sigma_2} \sum_{\sigma_1 \sigma_2} (\mathbf{1}_{\sigma_1 \sigma_2 < 0} \mathbf{1}_{N^{FFS}} \mathbf{1}_{\sigma_1 \sigma_2 > 0}) (i \omega_{k,m,n}^\sigma)^{-1} e^{i \omega_{k,m,n}^\sigma \tau} \partial_t B_{k,m,n}^{\sigma_1 \sigma_2 0}(\tilde{u}_f, \tilde{u}_f).
\]

A similar calculation for the FSF terms yields

\[
N(B_f - \tilde{B}_f)(\tau, \tilde{u}_f, \tilde{u}_s) = \partial_t r_2 + r_2'
\]

\[
:= \partial_t \sum_{k,m,n; \text{conv} \sigma_1, \sigma_3} \sum_{\sigma_1 \sigma_3} (\mathbf{1}_{\sigma_1 \sigma_3 < 0} \mathbf{1}_{N^{FFS}} \mathbf{1}_{\sigma_1 \sigma_3 > 0}) (i \omega_{k,m,n}^\sigma)^{-1} e^{i \omega_{k,m,n}^\sigma \tau} B_{k,m,n}^{\sigma_1 \sigma_2 0}(\tilde{u}_f, \tilde{u}_s)
\]

\[
- \sum_{k,m,n; \text{conv} \sigma_1, \sigma_3} \sum_{\sigma_1 \sigma_3} (\mathbf{1}_{\sigma_1 \sigma_3 < 0} \mathbf{1}_{N^{FFS}} \mathbf{1}_{\sigma_1 \sigma_3 > 0}) (i \omega_{k,m,n}^\sigma)^{-1} e^{i \omega_{k,m,n}^\sigma \tau} \partial_t B_{k,m,n}^{\sigma_1 \sigma_2 0}(\tilde{u}_f, \tilde{u}_s).
\]

Finally, the FFF terms can be expressed as follows

\[
N(B_f - \tilde{B}_f)(\tau, \tilde{u}_f, \tilde{u}_f) = \partial_t r_3 + r_3'
\]

\[
:= \partial_t \sum_{k,m,n; \text{conv} \sigma_1 \sigma_2 \sigma_3 \neq 0} 1_{N^{FFS}} (i \omega_{k,m,n}^\sigma)^{-1} e^{i \omega_{k,m,n}^\sigma \tau} B_{k,m,n}^{\sigma_1 \sigma_2 \sigma_3}(\tilde{u}_f, \tilde{u}_f).
\]
for the bilinear term, so that

\[ N B_f(r, u_s, u_s) = \partial_t r_4 + r_4' := \partial_t \sum_{k, m, n, \text{conv} \sigma_3 \neq 0} (i \omega_{kmn}^\sigma)^{-1} e^{-i \omega_{kmn}^\sigma \tau} B_{kmn}^{\sigma_1 \sigma_2 \sigma_3}(\tilde{u}_f, \tilde{u}_f) \]

As far as the SSF are concerned, we have

\[ NB_f(r, u_s, u_s) = \partial_t r_4 + r_4' := \partial_t \sum_{k, m, n, \text{conv} \sigma_3 \neq 0} (i \omega_{kmn}^\sigma)^{-1} e^{-i \omega_{kmn}^\sigma \tau} B_{kmn}^{000\sigma_3}(\tilde{u}_s, \tilde{u}_s) \]

We also define in a similar manner

\[ B_s(\tilde{u}_f, \tilde{u}_s) := \partial_t r_5 + r_5' \quad \text{and} \quad B_s(\tilde{u}_s, \tilde{u}_f) := \partial_t r_6 + r_6'. \]

Finally, we turn our attention to the oscillating viscosity and heat conductivity terms which appear in the difference equation, via setting

\[ N (A - \tilde{A}) u = \partial_t r_7 + r_7' \]

\[ \begin{align*}
&:= -\partial_t \Delta \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{\sigma \neq \sigma_1} (i \sigma_1 \omega_k - i \sigma \omega_k)^{-1} e^{i (k \cdot x + \sigma_1 \omega_k \tau - \sigma \omega_k \tau)} (\nu_k^{\sigma_1} \cdot \bar{r}_k) u_k^{\sigma_1} u_k^{\sigma} \\
&+ \Delta \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{\sigma \neq \sigma_1} (i \sigma_1 \omega_k - i \sigma \omega_k)^{-1} e^{i (k \cdot x + \sigma_1 \omega_k \tau - \sigma \omega_k \tau)} (\nu_k^{\sigma_1} \cdot \bar{r}_k) \partial_t u_k^{\sigma_1} u_k^{\sigma}.
\end{align*} \]

In order to proceed, we set

\[ w_r := w - N^{-1} \sum_{i=1}^7 r_i. \]

Then, \( w_r \) satisfies the following equation

\[ \begin{align*}
\partial_t w_r + A w_r + B(\tau, u, w_r) + B(\tau, w_r, \tilde{u}) + N^{-1} \sum_{i=1}^7 \{ A r_i + B(\tau, u, r_i) + B(\tau, r_i, \tilde{u}) + r_i' \} &= 0.
\end{align*} \]

We remark that the presence of a negative power of the dispersion relation \( \omega_{kmn}^\sigma \) in the denominators of some of the previous expressions is not problematic, as the lower bound on the bandwidth provides us with sufficient control. In more detail, we have an estimate of the form

\[ |\omega_{kmn}^\sigma|^{-1} < c_f^{-1}(|\tilde{k}| + |\tilde{m}|), \quad \text{when} \quad (k, m, n) \in (N^{EFF})^c. \]

On the other hand, we have a corresponding estimate

\[ |\omega_{kmn}^\sigma|^{-1} < c_s^{-1} C(\zeta)(|\tilde{k}| + |\tilde{m}|)^{\zeta}, \quad \text{when} \quad (k, m, n) \in (N^{FFS})^c, \]

with \( \zeta \in [\frac{g}{15}, 2] \) and \( C(\zeta) > 0 \). In addition, an identical estimate holds true in \( (N^{FES})^c \).

**Estimates on the time derivative**

Next, we examine the regularity cost of estimating the time derivative of the solution to either the modulated or the approximate modulated systems. In that direction, we use \( (90) \) for the bilinear term, so that

\[ \begin{align*}
\| \partial_t \tilde{u} \|_{H^t} &\leq \nu_{\text{max}} \| \tilde{u} \|_{H^{t+2}} + \| \tilde{B}(\tau, \tilde{u}, \tilde{u}) \|_{H^t} \\
&\leq \nu_{\text{max}} \| \tilde{u} \|_{H^{t+2}} + C(\ell') \| \tilde{u} \|_{H^{t+2}} + \| \tilde{u} \|_{H^{t+1}},
\end{align*} \]
for all $\gamma, \ell > 0$, with an identical estimate holding true for $u$ and $B(\tau, \cdot, \cdot)$.

**Estimates on the remainder terms**

We now examine the effect of (82) and (83) on the terms $r_i$, $r'_i$, for $i = 1 \ldots 6$. Since $\zeta > 1$, we estimate $r_1$ and $r'_1$ only, with $r_2, r'_2$ obeying similar estimates. The terms $r_3$ and $r'_3$ can be treated analogously, up to a substitution of $\zeta$ with 1.

First, we claim that

$$\|r_1\|_{H^\ell} \lesssim_{\ell, c_0, c_{-1}} \|\tilde{u}\|_{\zeta + \ell + 1} \|\tilde{u}\|_{\ell + \gamma}, \text{ for all } \gamma > 0.$$  

Indeed, we use (83) and the convolution condition, in order to deduce that:

$$\|r_1\|_{H^\ell} \lesssim_{\ell, c_0, c_{-1}} \sum_{k, m, n; \text{conv}} |\tilde{u}_k|^2 |\tilde{u}_m|^2 |\tilde{m}|^2 (|k| + |\tilde{m}|)^2 |\tilde{n}|^{2\ell}.$$

with the claim following via (90).

Similarly, using (84) and (90) once more, together with (70), we deduce

$$\|r'_1\|_{H^\ell} \lesssim_{\ell, c_0, c_{-1}} \left|\partial_t \tilde{u}\right|_{H^\ell + \zeta + 1} \|\tilde{u}\|_{\ell + \gamma} + \|\partial_t \tilde{u}\|_{H^\ell + \gamma} \|\tilde{u}\|_{H^\ell + \zeta + 1} \|\tilde{u}\|_{H^\ell + \zeta + 1} \|\tilde{u}\|_{H^\ell + \gamma},$$

for all $\gamma > 0$.

We now turn our attention to the remainder terms that do not come from a restriction on the level of interactions. Since $|\omega_k|^{-1} \leq \max\{1, \eta^{-1}\}$, for $\sigma \in \{\pm\}$, we have bounds

$$\|r_3\|_{H^\ell} \leq C(\eta, \ell, c_{-1}) \|\tilde{u}\|_{H^{\ell + 1} + \gamma} \|\tilde{u}\|_{H^{\ell + 1} + \gamma}, \text{ for all } \gamma > 0,$$

using (90) and the definition of $r_3$ directly. Moreover, $r_5, r_6$ can be bounded in a similar manner. The remaining terms, $r'_4, r'_5, r'_6$ can be estimated without the bandwidth cost of order $\zeta$. In particular, we have:

$$\|r'_4\|_{H^{\ell + 1}} \lesssim_{\ell, c_0, c_{-1}} \left|\partial_t \tilde{u}\right|_{H^{\ell + 1} + \gamma} \|\tilde{u}\|_{\ell + \gamma} + \|\partial_t \tilde{u}\|_{H^\ell + \gamma} \|\tilde{u}\|_{H^{\ell + 1} + \gamma} \|\tilde{u}\|_{H^{\ell + 1} + \gamma},$$

via (70), (84), (90) and the definition of $r'_4$ directly, with $r'_5$ and $r'_6$ satisfying similar estimates.

**Estimates on the dissipative remainder terms**

As far as $r_7$ and $r'_7$ are concerned, we have no derivative losses, since $|\sigma_1 \omega_k - \sigma_\omega_k|^{-1} \leq \frac{1}{2} \max\{1, \eta^{-1}\}$ when $\sigma \neq \sigma_1$. Thus, the following estimates hold true

$$\|r_7\|_{H^\ell} \leq C_{\nu_{max}} \|u\|_{\ell + 2} \text{ and } \|r'_7\|_{H^\ell} \leq C_{\nu_{max}} \|\partial_t u\|_{\ell + 2},$$

for all $\ell \in \mathbb{R}$ and a constant $C > 0$ depending on $\eta, \ell$.

**Final arguments**

We claim that the fields $u$ and $\tilde{u}$ satisfy local in time estimates in $L^\infty([0, T_0); H^\ell(\mathbb{T}^3; \mathbb{R}^4)) \cap L^2([0, T_0); H^{\ell + 1}(\mathbb{T}^3; \mathbb{R}^4))$, for a $T_0 = T_0(E_0, \ell, \eta, T^3)$, as solutions to (83) and (79), respectively. In particular, there exists a constant $C > 0$ and a time $T_0 > 0$, depending on $E_0$, so that

$$\sup_{t \in [0, T_0]} (\|u\|_{H^\ell} + \|\tilde{u}\|_{H^\ell}^2) + 2 \nu_{min} \int_0^{T_0} \left(\|u\|_{H^{\ell + 1}}^2 + \|\tilde{u}\|_{H^{\ell + 1}}^2\right) dt \lesssim_\ell E_0.$$
Indeed, the original Boussinesq system has a Navier-Stokes type bilinearity and for the range of \( \ell \) under consideration standard results apply, see e.g. \cite{22}. On the other hand, the approximation \((79)\) can be handled similarly, due to the \( L^2 \) cancellation property of Corollary \cite{11}. In more detail, a \((\eta, \nu)\)-independent local \( H^\ell \) estimate for the approximate system requires the cancellation property \( \langle \tilde{B}(U, D^\ell V), D^\ell V \rangle \) \( = 0 \) to handle the \((\ell + 1)^{th} \) order derivatives, which is guaranteed by Corollary \cite{11}. Next, we set

\[
\begin{align*}
\ell_\gamma & := \max \{ \ell', \frac{5}{2} + \gamma \} \\
\bar{\ell}_\gamma & := \max \{ \ell' + 1, \frac{5}{2} + \gamma \},
\end{align*}
\]

for \( \gamma > 0 \). The bilinear interactions of \((81)\) can be estimated similarly to the one in \((84)\). Testing the difference equation \((81)\) with \( (-\Delta)^{\ell'} \bar{w} \), we obtain

\[
\partial_t \| w_r \|_{H^{\ell'}}^2 + 2\nu_{\min} \| w_r \|_{H^{\ell'+1}}^2 \leq C \left( \| \bar{u} \|_{H^{\ell'}} + \| u \|_{H^{\ell'}} \right) \| w_r \|_{H^{\ell'}}^2 \tag{87}
\]

\[
+ CN^{-1} \sum_{i=1 \ldots 6} \left( \| u \|_{H^{\ell'-1}} \| r_i \|_{H^{\ell'+1}} + \| \bar{u} \|_{H^{\ell'}} \| r_i \|_{H^{\ell'}} \right) \| w_r \|_{H^{\ell'}}^2
\]

\[
+ CN^{-1} \left( \sum_{i=1 \ldots 7} \| r_i \|_{H^{\ell'-1}} + \| Ar_i \|_{H^{\ell'+1}} \right) \| w_r \|_{H^{\ell'+1}}
\]

\[
+ CN^{-1} \left( \| u \|_{H^{\ell'-1}} \| r_7 \|_{H^{\ell'}} + \| \bar{u} \|_{H^{\ell'}} \| r_7 \|_{H^{\ell'+1}} \right) \| w_r \|_{H^{\ell'+1}}
\]

for some \( C = C(\eta, \ell', \zeta, c_7^{-1}, c_8^{-1}) \), via \((90)\) and Hölder’s inequality. We examine the terms on the right separately. As far as the last term on the right is concerned, we use Young’s inequality and \((85)\) to get:

\[
\tag{88} N^{-1} (\| r_7 \|_{H^{\ell'-1}} + \| Ar_7 \|_{H^{\ell'+1}}) \| w_r \|_{H^{\ell'+1}} \leq \nu_{\max} (\| \partial_t u \|_{H^{\ell'+1}} + \| u \|_{H^{\ell'+1}}) \| w_r \|_{H^{\ell'+1}}
\]

\[
\leq \nu_{\max} \nu_R \left( \| u \|_{H^{\ell+3}}^2 + \| u \|_{H^{\ell+2}}^2 \right) \| w_r \|_{H^{\ell'+1}} + \frac{\nu_{\min}}{3} \| w_r \|_{H^{\ell'+1}}^2.
\]

The remaining terms containing factors that depend on \( r_7 \) and \( r_i \) can be estimated using \((84)\), \((85)\) and Young’s inequality:

\[
\tag{89} N^{-1} (\| r_i \|_{H^{\ell'-1}} + \| Ar_i \|_{H^{\ell'+1}}) \| w_r \|_{H^{\ell'+1}} \leq N^{-1} (\| r_i \|_{H^{\ell'-1}} + \nu_{\max} \| r_i \|_{H^{\ell'+1}}) \| w_r \|_{H^{\ell'+1}}
\]

\[
\leq N^{-1} \left( \nu_{\max} \| u \|_{H^{\ell'}} + \| u \|_{H^{\ell'+1}} \right) \| u \|_{H^{\ell'+1}} \| w_r \|_{H^{\ell'+1}}
\]

\[
\leq \nu_{\max} \nu_R \left( \| u \|_{H^{\ell+3}}^2 + \| u \|_{H^{\ell+2}}^2 \right) \| w_r \|_{H^{\ell'+1}} + \frac{\nu_{\min}}{20} \| w_r \|_{H^{\ell'+1}}^2.
\]

Utilizing \((81)\), \((85)\) and Young’s inequality once more, we obtain:

\[
\tag{90} N^{-1} \left( \sum_{i=1 \ldots 6} (\| r_i \|_{H^{\ell'-1}} + \| Ar_i \|_{H^{\ell'+1}}) \right) \| w_r \|_{H^{\ell'+1}}
\]

\[
\leq \nu_{\max} \nu_R \left( \| u \|_{H^{\ell+3}}^2 + \| u \|_{H^{\ell+2}}^2 \right) \| w_r \|_{H^{\ell'+1}} + \frac{\nu_{\min}}{3} \| w_r \|_{H^{\ell'+1}}^2.
\]
Inserting (87), (88), (89) back into our energy estimates, estimating any remaining terms that contain \( r_i \) and using (86) yields
\[
\partial_t \| u_r \|_{H^{\nu}}^2 + \nu_{\min} \| u_r \|_{H^{\nu+1}}^2 \lesssim E_0 \| u_r \|_{H^{\nu}}^2 + N^{-2}.
\]
Then, in view of (80) and our bounds on the remainder terms \( r_i \), an integration in time and an appeal to Grönwall’s inequality complete the proof.

**Appendix A. Some Product Estimates**

We prove a weaker instance of the classical homogeneous Kato-Ponce fractional Leibniz rule, which is still sufficient for our purpose. In order to proceed, and for a given \( f : \mathbb{T}^d \to \mathbb{R} \), we define:
\[
f_{ab} := \sum_{n \in \mathbb{Z}^n} e^{i\tilde{n} \cdot x} |f_n|.
\]

**Lemma 16.** Let \( \ell \geq 0 \) and \( f, g : \mathbb{T}^d \to \mathbb{R} \) be sufficiently smooth functions with zero-mean. Then for all \( 2 \leq q, r, q', r' \leq \infty \) with \( \frac{1}{q} = \frac{1}{q'} + \frac{1}{r} = \frac{1}{r'} + \frac{1}{T} \) and all \( a, b \geq 0 \), the following estimate holds true
\[
\| fg \|_{H^\ell} \lesssim \| D^{\ell+a} f_{ab} \|_{L^q} \| D^{-a} g_{ab} \|_{L^{q'}} + \| D^{-b} f_{ab} \|_{L^{r'}} \| D^{\ell+b} g_{ab} \|_{L^{r'}},
\]
with the implied constant only depending on \( \ell \).

**Proof.** We use Plancherel’s Theorem and a decomposition into high-low modes, in order to deduce that
\[
\begin{align*}
\| fg \|_{H^\ell}^2 &= \sum_{n \in \mathbb{Z}^n} \left( \sum_{k \in \mathbb{Z}^n} |\tilde{n} |^\ell f_k g_m \right)^2 \\
&\lesssim \sum_{n \in \mathbb{Z}^n} \left( \sum_{|\tilde{n}| \leq |\tilde{k}|} |\tilde{k} |^{\ell+a} |\tilde{n} |^{-a} |f_k| |g_m| \right)^2 + \sum_{n \in \mathbb{Z}^n} \left( \sum_{|\tilde{n}| \leq |\tilde{k}|} |\tilde{n} |^{\ell+b} |\tilde{k} |^{-b} |f_k| |g_m| \right)^2 \\
&\leq C(\ell) \left[ \| D^{\ell+a} f_{ab} \|_{L^2}^2 + \| D^{-a} g_{ab} \|_{L^{2}}^2 \right] + C(\ell) \left[ \| (D^{\ell+a} f_{ab}) (D^{-a} g_{ab}) \|_{L^2}^2 + \| (D^{-b} f_{ab}) (D^{\ell+b} g_{ab}) \|_{L^2}^2 \right].
\end{align*}
\]
Then the result follows by Hölder’s inequality.

**Corollary 5.** Let \( f, g : \mathbb{T}^d \to \mathbb{R} \) with zero-mean, and \( \beta_1, \beta_2 \in [0, \frac{d}{2}] \) with \( 0 < \beta_1 + \beta_2 \). Then the following estimate holds true:
\[
\| fg \|_{H^{\beta_1+\beta_2-\frac{d}{2}}} \lesssim \| f \|_{H^{\beta_1}} \| g \|_{H^{\beta_2}},
\]
with the implied constant depending on \( b_1, b_2, d \).

**Proof.** We use (90) with \( q = \frac{d}{\beta_2}, r = \frac{2d}{d-2\beta_2}, q' = \frac{d}{\beta_1}, r' = \frac{2d}{d-2\beta_1}, \) and \( a = b = 0 \). Then, we have
\[
\| fg \|_{H^{\beta_1+\beta_2-\frac{d}{2}}} \lesssim \| D^{\beta_1+\beta_2-\frac{d}{2}} f \|_{L^{\frac{2d}{d-2\beta_2}}} \| g \|_{L^{\frac{2d}{d-2\beta_2}}} + \| f \|_{L^{\frac{2d}{d-2\beta_1}}} \| D^{\beta_1+\beta_2-\frac{d}{2}} g \|_{L^{\frac{2d}{d-2\beta_1}}}.
\]
The result follows from Sobolev embedding, since the Sobolev norms of \( f, g \) are equal to those of \( f_{ab}, g_{ab} \) respectively.

Finally, we recall the following trilinear estimate.
Lemma 17. Let $f, g, h : \mathbb{T}^3 \to \mathbb{R}$ be zero-mean functions of sufficient regularity and $\beta \in (0, \frac{3}{2})$. Then the following estimate holds true

$$\sum_{k,m,n,\text{conv}} |f_k| |g_m| |h_n| \lesssim \|f\|_{H^{3/2-\beta}} \|g\|_{H^\beta} \|h\|_{L^2}. \quad (92)$$

Proof. The Cauchy-Schwarz inequality implies

$$\left( \sum_{k,m,n,\text{conv}} |f_k| |g_m| |h_n| \right)^2 \leq \sum_{k,m,n,\text{conv}} (|f_k| |g_m|)^2 \|h\|^2_{L^2}$$

and the result follows from Corollary \textbullet.

\[ \square \]

Appendix B. Lower Bounds

We assume $L_1 = L_2 = 1$, as the general case is done similarly.

Lemma 18. Consider $\eta \neq 1$ and $\delta^* \in (0, \min\{\frac{3}{2}, \frac{1}{2}\})$. Then, for any sufficiently large number $M$, the cardinality of the set of wavevectors

$$\{ k \in \mathbb{Z}^3 : M \leq |k| < 2M, |\omega_k - 1| \leq \delta^* \}$$

is bounded from below by $\frac{C\sqrt{\delta^*}}{|\eta^2 - 1|} M^3 + CM^2$ for an absolute constant $C > 0$.

Apparently, for any nonzero wavevector $n$ with $n_3 = 0$, any member of the above set satisfies $|\omega_k - \omega_n| \leq \delta^*$.

Proof. Let

$$c_{\delta^*} = \begin{cases} \sqrt{-\delta^*} \sqrt{1-\eta^2}, & \text{if } \eta < 1 \\ \sqrt{(\delta^*)^2 + 2\delta^*}, & \text{if } \eta > 1. \end{cases}$$

Consider the family of $k$ defined via

$$\left\{ k = (k_1, k_2, k_3) \in \mathbb{Z}^3 : \frac{3}{5}M \leq k_1 \leq M, \frac{4}{5}M \leq k_2 \leq M, 0 \leq k_3 \leq c_{\delta^*} |k_H| \right\}.$$

Clearly, $M \leq |k| < 2M$ and $\frac{|k_3|}{|k|} < c_{\delta^*}$, the latter of which is equivalent to $|\omega_k - 1| \leq \delta^*$.

For given $k_H \in \mathbb{Z}^2$, the possible integer choices for $k_3$ are $1 + \lfloor c_{\delta^*} |k_H| \rfloor > \frac{1}{2} \left(1 + c_{\delta^*} |k_H| \right)$. We sum over $k_1, k_2 \in \left[\frac{3}{5}M, M\right] \times \left[\frac{4}{5}M, M\right]$ in order to obtain a lower bound for the required cardinality

$$\sum_{k_1, k_2} (1 + c_{\delta^*}) |k_H| \gtrsim \int_{\frac{3}{5}M}^{M} \int_{\frac{4}{5}M}^{M} (1 + c_{\delta^*}) |k_H|dk_1dk_2 > \frac{C\sqrt{\delta^*}}{|\eta^2 - 1|} M^3 + CM^2.$$

\[ \square \]
Appendix C. Interaction coefficients

Lemma 19. Let \((k, m, n) \in \mathbb{Z}^3\). The interaction coefficients \(S_{kmn}^{+0}\) defined in (63) for FFS interactions are given as follows.

(i) If \(\vec{k}_H \neq \vec{0}\) and \(\vec{m}_H \neq \vec{0}\) then
\[
S_{kmn}^{+0} = i \left[\omega_m^2 - \omega_k^2\right] \left\{ \frac{\left|k\right|^2 \left|m\right|^2 (1 - \eta^2)^{-1} (\vec{k}_H \times \vec{m}_H)(\eta^2 - \omega_k \omega_m)}{2|\vec{k}_H||\vec{m}_H||\vec{n}_H||\vec{m}_H|} \right. \\
+ \left. \left[\omega_m - \omega_k\right] \eta \left(\frac{k_m^2 \left|m\right|^2 + m_m^2 \left|k\right|^2 (\vec{m}_H \cdot \vec{k}_H) - 2k_m m_m \left|\vec{k}_H\right|^2 \left|\vec{m}_H\right|^2}{2|\vec{k}_H||\vec{m}_H||\vec{n}_H||\vec{m}_H|} \right) \right\}.
\]

(ii) If \(\vec{k}_H = \vec{0}\) and \(\vec{m}_H \neq \vec{0}\) then
\[
S_{kmn}^{+0} = i \left[\omega_m^2 - \omega_k^2\right] \frac{k_m^3 \left|m\right|^2 (1 - \eta^2)^{-1} (1 - \omega_m)(im_1 + \tilde{m}_2)}{2|\vec{m}_H||\vec{n}_H||\vec{m}_H|}.
\]
The case for \(\vec{k}_H \neq \vec{0}\) and \(\vec{m}_H = \vec{0}\) is similar due to the \(k, m\) symmetry of \(1_{N_{FFS}}(k, m, n)\).

(iii) If \(\vec{k}_H = \vec{m}_H = \vec{0}\) then \(S_{kmn}^{+0} = 0\).

Proof. We will use the eigenvectors given in (30) and (31).

Case (i).

We proceed in estimating the two summands in the interaction coefficients separately.

First, using Lemma 2 and the incompressibility of \(r_m^0\), we have
\[
2|\vec{k}_h||\vec{m}_h||\vec{n}_h||\vec{m}_h| (r_m^+ \cdot \vec{m}_h) (r_m^- \cdot r_n) = - (\alpha_k \cdot \vec{m}_h) (\vec{m}_h \cdot r_n^0)
\]
\[
= \left[ k_3 \left( \eta \vec{m}_h \times \vec{k}_h - \omega_k + (\vec{k}_h \cdot \vec{m}_h) \right) + \omega_k |\vec{k}_h|^2 m_3 \left[ m_3 \left( \omega_m (\vec{m}_h \times \vec{k}_h) + \eta (\vec{m}_h \cdot \vec{k}_h) \right) - \eta k_3 |\vec{m}_h|^2 \right] \right] = A_1 + A_2 + A_3 + A_4,
\]
where

- \(A_1 = k_3 m_3 \left( \vec{k}_h \cdot \vec{m}_h \right) \left[ \eta^2 \vec{m}_h \times \vec{k}_h + \omega_k \omega_m \right]\)
- \(A_2 = i \eta \omega_k \left( \vec{k}_h \cdot \vec{m}_h \right) \left[ k_3^2 |\vec{m}_h|^2 + \eta m_3^2 |\vec{k}_h|^2 \right]\)
- \(A_3 = - (\vec{m}_h \times \vec{k}_h) \left[ \eta^2 k_3^2 |\vec{m}_h|^2 + \omega_k \omega_m m_3^2 |\vec{k}_h|^2 \right]\)
- \(A_4 = i \eta k_3 m_3 \left[ \omega_m |\vec{k}_h \times \vec{m}_h|^2 - \omega_k (|\vec{k}_h \cdot \vec{m}_h|^2 + |\vec{k}_h|^2 |\vec{m}_h|^2) \right]\).

As far as the second part of the sum is concerned, we have:
\[
2|\vec{k}_h||\vec{m}_h||\vec{n}_h||\vec{m}_h| (r_m^- \cdot \vec{k}_h^*) (r_m^+ \cdot r_n) = - (\vec{m}_h \cdot \vec{k}_h) (\omega_m \cdot r_m^0)
\]
\[
= \left[ m_3 \left( \eta \vec{k}_h \times \vec{m}_h + \omega_m (\vec{k}_h \cdot \vec{m}_h) \right) - i \omega_m |\vec{m}_h|^2 k_3 \left[ k_3 \left( i \omega_k (\vec{m}_h \times \vec{k}_h) + \eta (\vec{m}_h \cdot \vec{k}_h) \right) - \eta m_3 |\vec{k}_h|^2 \right] \right] = B_1 + B_2 + B_3 + B_4,
\]
with

- \(B_1 = k_3 m_3 \left( \vec{k}_h \cdot \vec{m}_h \right) \left[ \eta^2 \vec{k}_h \times \vec{m}_h - \omega_k \omega_m \right]\)
- \(B_2 = - i \omega_m \left( \vec{k}_h \cdot \vec{m}_h \right) \left[ \eta k_3^2 |\vec{m}_h|^2 + m_3^2 |\vec{k}_h|^2 \right]\)
- \(B_3 = (\vec{m}_h \times \vec{k}_h) \left[ \eta^2 m_3^2 |\vec{k}_h|^2 + \omega_k \omega_m k_3^2 |\vec{m}_h|^2 \right]\)
- \(B_4 = - i \eta k_3 m_3 \left[ \omega_k |\vec{k}_h \times \vec{m}_h|^2 + \omega_m (|\vec{k}_h \cdot \vec{m}_h|^2 + |\vec{k}_h|^2 |\vec{m}_h|^2) \right]\).

The symmetry between \(k, m\) then implies that:
\[
A_3 + B_3 = (\eta^2 - \omega_k \omega_m) (\vec{k}_h \times \vec{m}_h) \left[ k_3^2 |\vec{m}_h|^2 - m_3^2 |\vec{k}_h|^2 \right].
\]
Following [18], we observe that the definition of $\omega$ allows us to rewrite
\[
\omega^2_k - \omega^2_m = (1 - \eta^2) \frac{|\tilde{k}_H|^2 |\tilde{m}_H|^2 - |\tilde{m}_H| |\tilde{k}_H|^2}{|\tilde{k}_H|^2 |\tilde{m}_H|^2} = (1 - \eta^2) \frac{|\tilde{k}_H|^2 m^2_3 - |\tilde{m}_H| |\tilde{k}_H|^2}{|\tilde{k}_H|^2 |\tilde{m}_H|^2}.
\]
Thus, we obtain:
\[
A_3 + B_3 = \frac{|\tilde{k}_H|^2 |\tilde{m}_H|^2}{(1 - \eta^2)} (\tilde{k}_H \times \tilde{m}_H)(\eta^2 - \omega_k \omega_m) [\omega^2_m - \omega^2_k].
\]
Finally, the cancellation between $A_1, B_1$, together with the sums of the remaining terms
\[
A_2 + B_2 = -i \eta (\tilde{k}_H \cdot \tilde{m}_H)(m^3_3 |\tilde{k}_H|^2 + k^3_3 |\tilde{m}_H|^2)(\omega_m - \omega_k)
\]
and
\[
A_4 + B_4 = 2i \eta k m_3 |\tilde{k}_H|^2 |\tilde{m}_H|^2(\omega_m - \omega_k),
\]
yield the result.

Case (ii).
We have
\[
2 |\tilde{m}_n||\tilde{n}_n||\tilde{m}_H| \left( r^+ r^0 \right) = - (\alpha_k \cdot m') (\tilde{m}_1 + \tilde{m}_2) k_3 |\tilde{m}_H|^2
\]
and
\[
2 |\tilde{m}_n||\tilde{n}_n||\tilde{m}_H| \left( r^- r^0 \right) = - (\tilde{m}_1 + \tilde{m}_2) k_3 |\tilde{m}_H|^2 (\omega_m).
\]
The result then follows, since the wavevectors under consideration satisfy
\[
\omega^2_m - \omega^2_k = \frac{(1 - \eta^2) |\tilde{m}_H|^2}{|\tilde{m}_H|^2}.
\]

Case (iii). Trivial. 

\[\square\]

References

[1] Anatoli Babin, Alex Mahalov, and Basil Nicolaenko. Global regularity of 3D rotating Navier-Stokes equations for resonant domains. Indiana Univ. Math. J., 48(3):1133–1176, 1999.

[2] Anatoli Babin, Alex Mahalov, and Basil Nicolaenko. On the regularity of three-dimensional rotating Euler-Boussinesq equations. Math. Models Methods Appl. Sci., 9(7):1089–1121, 1999.

[3] Anatoli Babin, Alex Mahalov, and Basil Nicolaenko. Fast singular oscillating limits and global regularity for the 3D primitive equations of geophysics. ESAIM: Mathematical Modelling and Numerical Analysis, 34(2):201–222, 2000.

[4] Anatoli Babin, Alex Mahalov, Basil Nicolaenko, and Ye Zhou. On the asymptotic regimes and the strongly stratified limit of rotating Boussinesq equations. Theoretical and Computational Fluid Dynamics, 9(3-4):223–251, December 1997.

[5] Alfred J. Bourgeois and J. Thomas Beale. Validity of the quasigeostrophic model for large-scale flow in the atmosphere and ocean. SIAM J. Math. Anal., 25(4):1023–1068, 1994.

[6] Jean-Yves Chemin, Benoît Desjardins, Isabelle Gallagher, and Emmanuel Grenier. Mathematical Geophysics: An introduction to rotating fluids and the Navier-Stokes equations. Oxford University Press, 2006.

[7] Bin Cheng and Zisis N. Sakellaris. Near resonance approximation of rotating Navier-Stokes equations. https://arxiv.org/abs/2110.04927 2021.
[8] Pedro F. Embid and Andrew J. Majda. Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity. *Comm. Partial Differential Equations*, 21(3-4):619–658, 1996.

[9] Pedro F. Embid and Andrew J. Majda. Low Froude number limiting dynamics for stably stratified flow with small or finite Rossby numbers. *Geophys. Astrophys. Fluid Dynam.*, 87(1-2):1–50, 1998.

[10] Isabelle Gallagher. Applications of Schochet’s methods to parabolic equations. *J. Math. Pures Appl. (9)*, 77(10):989–1054, 1998.

[11] Isabelle Gallagher. Asymptotic of the solutions of hyperbolic equations with a skew-symmetric perturbation. *J. Differential Equations*, 150(2):363–384, 1998.

[12] Slim Ibrahim and Tsuyoshi Yoneda. Long-time solvability of the Navier-Stokes-Boussinesq equations with almost periodic initial large data. *J. Math. Sci. Univ. Tokyo*, 20(1):1–25, 2013.

[13] Herbert Koch, Daniel Tataru, and Monica Vișan. *Dispersion equations and nonlinear waves*, volume 45 of *Oberwolfach Seminars*. Birkhäuser/Springer, Basel, 2014. Generalized Korteweg-de Vries, nonlinear Schrödinger, wave and Schrödinger maps.

[14] Andrew Majda. *Introduction to PDEs and waves for the atmosphere and ocean*. Courant Institute of Mathematical Sciences American Mathematical Society, New York Providence, R.I, 2003.

[15] Pengcheng Mu and Qiangchang Ju. Three-scale singular limits of the rotating stratified Boussinesq equations. *Applicable Analysis*, 100(11):2405–2417, November 2019.

[16] D. Oks, Pablo Daniel Mininni, Raffaele Marino, and Annick Pouquet. Inverse cascades and resonant triads in rotating and stratified turbulence. *Physics of Fluids*, 29(11):111109, November 2017.

[17] Alex Owen, Roger Grimshaw, and Beth Wingate. Quartet, higher order and near resonant interactions in nonlinear wave equations. *arXiv preprint arXiv:1903.06265*, 2019.

[18] Alexander Owen. *Resonant effects in weakly nonlinear geophysical fluid dynamics*. PhD thesis, University of Exeter, 2019.

[19] Steven Schochet. Fast singular limits of hyperbolic PDEs. *J. Differential Equations*, 114(2):476–512, 1994.

[20] Leslie M. Smith and Fabian Waleffe. Generation of slow large scales in forced rotating stratified turbulence. *Journal of Fluid Mechanics*, 451:145–168, January 2002.

[21] Jai Sukhatme and Leslie M. Smith. Vortical and wave modes in 3D rotating stratified flows: random large-scale forcing. *Geophysical and Astrophysical Fluid Dynamics*, 102(5):437–455, September 2008.

[22] Michael E. Taylor. *Partial differential equations III, Nonlinear equations*. Springer-Verlag, 1996.

[23] Roger Temam and Djoko Wirosoetisno. Slow manifolds and invariant sets of the primitive equations. *Journal of the Atmospheric Sciences*, 68(3):675–682, March 2011.

[24] Jared P. Whitehead, Terry Haut, and Beth A. Wingate. The effect of two distinct fast time scales in the rotating, stratified Boussinesq equations: variations from quasi-geostrophy. *Theor. Comput. Fluid Dyn.*, 32(6):713–732, 2018.

[25] Jared P. Whitehead and Beth A. Wingate. The influence of fast waves and fluctuations on the evolution of the dynamics on the slow manifold. *J. Fluid Mech.*, 757:155–178, 2014.

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