Nontrivial Solutions of Dirac-Laplace Equation on Compact Spin Manifolds

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Abstract We apply the Fountain theorem to a class of nonlinear Dirac-Laplace equation on compact spin manifold. We show the standard Ambrosetti-Rabinowitz condition can be replaced by a more natural super-quadratic condition that is enough to obtain the Cerami condition under certain conditions. Multiple solutions of nonlinear Dirac-Laplace equation are obtained in this note.

Keywords: Dirac operator, Cerami condition, Fountain theorem

1 Introduction and main results

Motivated by quantum physics, Esteban-Séré [7,8] studied existence and multiplicity of solutions of nonlinear Dirac equations on $\mathbb{R}^3$. In this case, a large number of existence of solution has been obtained. Dirac operators on compact spin manifolds play a prominent role in the geometry and mathematical physics, such as the generalized Weierstrass representation of the surface in three manifolds [10] and the supersymmetric nonlinear sigma model in quantum field theory [11]. For the nonlinear Dirac equations on a general compact spin manifold, some results were recently obtained, see Amann [2], Isobe [13,14], Maalaoui and V. Martino [19,20], Gong and Lu [11,12] and [24,25] by authors. In addition, be different with these existing works, Ding and Li [6] studied a class of boundary value problem on a compact spin manifold $M$ with smooth boundary. The problem is a general relativistic model of confined particles by means of nonlinear Dirac fields on $M$. In this paper, we are concerned with a nonlinear square Dirac equation on compact spin manifolds, and deal with some existence results.

Assume $m \geq 2$ is an integer. Given an $m$-dimensional compact oriented Riemannian manifold $(M,g)$ equipped with a spin structure $\rho : P_{Spin(M)} \to P_{SO(M)}$, let $\Sigma M = \Sigma(M,g) = P_{Spin(M)} \times_{\sigma} \Sigma_m$ denote the complex spinor bundle on $M$, which is a complex vector bundle of rank $2^{[m/2]}$ endowed with the spinorial Levi-Civita connection $\nabla$ and a pointwise Hermitian scalar product $\langle \cdot , \cdot \rangle$. Write the point of $\Sigma M$ as $(x,\psi)$, where $x \in M$ and $\psi \in \Sigma_x M$. Laplace operator on spinors $\Delta : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M)$ is defined by

$$\Delta \psi = - \sum_{i=1}^n (\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i}) \psi,$$
it is independent of the choice of local orthonormal frame and is a second order differential elliptic operator.

The Dirac operator $D$ is an elliptic differential operator of order one,

\[ D = D_g : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M), \]

locally given by $D\psi = \sum_{i=1}^m \epsilon_i \cdot \nabla_{e_i} \psi$ for $\psi \in C^\infty(M, \Sigma M)$ and a local $g$-orthonormal frame $\{e_i\}^m_{i=1}$ of the tangent bundle $TM$. It is an unbounded essential self-adjoint operator in $L^2(M, \Sigma M)$ with the domain $C^\infty(M, \Sigma M)$, and its spectrum consists of an unbounded sequence of real numbers. These mean that the closure of $D$ is a self-adjoint operator in $L^2(M, \Sigma M)$ with the domain $H^1(M, \Sigma M)$.

The square $D^2$ of the Dirac operator as well as that of the Laplace operator $\Delta$ is second order differential operator, which is called Dirac-Laplace operator. On the Dirac operator and square Dirac operator under boundary condition has been discussed a lot in the literature, c.f. [6, 9, 10, 21, 23]. The connection between $D^2$ and $\Delta$ is given by Schrödinger-Lichnerowicz formula

\[ D^2 = \Delta + \frac{R}{4} \text{Id}, \]

where $R$ is the scalar curvature of $(M, g)$.

For a fiber preserving nonlinear map $h : \Sigma M \to \Sigma M$, we consider the following nonlinear Dirac-Laplace equations

\[ D^2 \psi(x) = h(x, \psi(x)) \quad \text{on} \ M, \quad (1) \]

where $\psi \in C^\infty(M, \Sigma M)$ is a spinor. We assume $h$ has a $\psi$-potential, namely there exist real valued function $H : \Sigma M \to \mathbb{R}$ such that $H_\psi = h$, then equation (1) has a variational structure. It is the Euler-Lagrange equation of the functional

\[ \Sigma(\psi) = \frac{1}{2} \int_M \langle D\psi, D\psi \rangle dx - \int_M H(x, \psi(x)) dx \quad (2) \]

on the Sobolev space $H^1(M, \Sigma M)$, see [11] and [13]. Here $dx$ is the Riemannian volume measure on $M$ with respect to the metric $g$, and $\langle \cdot, \cdot \rangle$ is the compatible metric on $\Sigma M$.

The functional (2) is strongly indefinite in the sense that the Dirac-Laplace operator $D^2$ takes infinitely many positive eigenvalues. This kind of problem similar to elliptic equation $\Delta \psi = h(x, \psi)$ on $\mathbb{R}^n$. In [3], Ambrosetti and Rabinowitz established the Mountain Pass Theorem which successfully applied to the elliptic equation on $\mathbb{R}^n$. It plays a key role not only in establishing the mountain pass geometry of the functional but also in obtaining Palais-Smale condition. It has been an open question that whether the more natural superlinear condition is enough to obtain the A-R condition for variational problem. In [17], the existence of solutions is given on $\mathbb{R}^n$ without Ambrosetti-Rabinowitz condition. Recently, Isobe [13] consider Dirac equations on compact spin manifold via Galerkin type approximations and linking arguments, the existence of infinitely many solutions is obtained under Ambrosetti-Rabinowitz condition. As showed by Liu and
Li[18], the Fountain theorem under Cerami condition also provided a powerful method to deal with strongly indefinite variational problems. In this note our aim is to study (1) with this method. Our techniques work for the existence of solutions to Dirac-Laplace equation under Cerami condition replace Palais-Smale condition.

For the nonlinearity $H$, we make the following hypotheses:

1. $H$ is $\alpha$-Hölder continuous in the direction of the base, where $0 < \alpha < 1$.

2. There exist a constant $C > 0$ such that
   \[ |H_\psi(x,\psi)| \leq C (1 + |\psi|^{q-1}) \]
   for any $(x,\psi) \in \Sigma M$, where $2 < q < 2^*$, $2^* = \frac{2m}{m-2}$ if $m \geq 3$ and $2^* = \infty$ if $m = 2$.

3. \[ \langle H_\psi(x,\psi),\psi \rangle \rightarrow +\infty \text{ as } |\psi| \rightarrow +\infty \text{ uniformly for } x \in M. \]

4. There exist $\theta \geq 1$ such that
   \[ \theta h(x,\psi) \geq h(x,\mu \psi), \]
   where $(x,\psi) \in \Sigma M$, $\mu \in [0,1]$ and $h(x,\psi) = \langle H(x,\psi),\psi \rangle - 2H(x,\psi)$.

5. \[ H(x,-\psi) = H(x,\psi) \]
   for any $(x,\psi) \in \Sigma M$.

Our main result is as follow.

**Theorem 1.** If the above $H$ satisfies $(H_1) - (H_4)$, then there exist a sequence of solutions $\{\psi_k\}_{k=1}^\infty \subset C^2(M,\Sigma M)$ to (1) with $L(\psi_k) \rightarrow \infty$ as $k \rightarrow \infty$.

**Notice:** The standard Ambrosetti-Rabinowitz superlinear condition has appeared in Isobe[13] as following:

1. There exist $\mu > 2$ and $R_1 > 0$ such that
   \[ 0 < \mu H(x,\psi) \leq \langle H_\psi(x,\psi),\psi \rangle \]
   for any $(x,\psi) \in \Sigma M$ with $|\psi| \geq R_1$. It implies there is $C > 0$ such that $F(x,\psi) \geq C |\psi|^p$ for $|\psi|$ large. Isobe[13] obtained existence and multiplicity of solutions to nonlinear Dirac equations on compact spin manifolds. However, in this paper, we use a more natural and weaker superlinear condition is that $(H_2)$:

   \[ \frac{\langle H_\psi(x,\psi),\psi \rangle}{|\psi|^2} \rightarrow +\infty \]

   as $|\psi| \rightarrow +\infty$ uniformly for $x \in M$.

In this paper, condition $(H_3)$ refers to the condition of Liu and Li[18]. Notice that $H(x,\psi) = H(x)|\psi|^p$ satisfies these conditions $(H_1) - (H_4)$. We also remark that the Hölder condition of $H$ is only used to prove the $C^2$-regularity of $\psi$.

Finally, as a concluding remark we point out that in $\mathbb{R}^n$, the scalar curvature vanished, i.e. $R = 0$. Then $D^2 = \Delta$ with $\Delta = -\sum \frac{\partial^2}{\partial x^2}$. The Dirac-Laplace equations (1) reduces to the following the elliptic equation

\[ \Delta \psi = h(x,\psi). \]

For this elliptic equation, the existence of solutions has been obtained under assumption Ambrosetti-Rabinowitz condition and different additional conditions.
2 Preliminaries

According to [10,16], we next give the definition of Laplace operator on spinors:

**Definition 1.** If \( \psi \in C^\infty(M, \Sigma M) \), then \( \Delta : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M) \) is defined by

\[
\Delta \psi = - \sum_{i=1}^{n} (\nabla e_i, \nabla e_i - \nabla \nabla e_i) \psi.
\]

Using Stokes’ theorem we have

\[
\int_M \langle \Delta \psi, \psi \rangle = \int_M \langle \nabla \psi, \nabla \psi \rangle = \int_M \langle \psi, \Delta \psi \rangle
\]

for any \( \psi \in C^\infty(M, \Sigma M) \). Here, \( \langle \nabla \psi, \nabla \psi \rangle \) is the scalar product on \( 1 \)-forms, i.e.

\[
\langle \nabla \psi, \nabla \psi \rangle = \sum_{i=1}^{n} \langle \nabla e_i \psi, \nabla e_i \psi \rangle,
\]

where \( \{e_i\}_{i=1}^{\infty} \) is a local orthonormal frame on the spin manifold.

The first Sobolev norm [10] of a smooth spinor field \( \psi \in C^\infty(M, \Sigma M) \) is given by

\[
\| \psi \|_{H^1}^2 = \| \psi \|^2 + \| \nabla \psi \|^2
\]

and the corresponding Sobolev space \( H^1(M, \Sigma M) \) is the completion of \( \psi \in C^\infty(M, \Sigma M) \) with respect to this norm, where \( \| \cdot \|^2 := \int_M | \psi |^2 \, dx \).

The Dirac operator \( D \) acts on spinors on \( M, D : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M) \) in the following definition:

**Definition 2.** The composition

\[
D = c \circ \nabla : C^\infty(M, \Sigma M) \to C^\infty(M, T^*M \oplus \Sigma M)
\]

\[
= C^\infty(M, TM \oplus \Sigma M) \to C^\infty(M, \Sigma M)
\]

is called the Dirac operator.

It is locally given by \( D\psi = \sum_{i=1}^{m} e_i \cdot \nabla e_i \psi \) for \( \psi \in C^\infty(M, \Sigma M) \). It is a first order elliptic operator in \( L^2(M, \Sigma M) \) with the domain \( C^\infty(M, \Sigma M) \). For a vector \( X \in TM \), its symbol \( \sigma(D)(X) : \Sigma M \to \Sigma M \) is given by Clifford multiplication: \( \sigma(D)(X) = X \cdot \psi \). By the Dirac operator definition, we know the Dirac operator is a symmetric operator on compact spin manifold, i.e.

\[
\int_M \langle D\psi, \phi \rangle dx = \int_M \langle \psi, D\phi \rangle dx,
\]

where the smooth spinor field \( \psi, \phi \in C^\infty(M, \Sigma M) \).

Another Sobolev norm is given by

\[
\| \psi \|^{2,2} = \| \psi \|^2 + \| D\psi \|^2.
\]

Thus we will obtain the next Lemma:
Lemma 1. The norm $\| \psi \|_{H^1}$ and the norm $\| \psi \|_{1,2}$ are equivalent norms.

Proof: Firstly, from the estimate below for the norm of $D\psi$:

$$
\| D\psi \|_2^2 = \sum_{i,j=1}^{n} \int_M \langle e_i \cdot \nabla e_i \psi, e_j \cdot \nabla e_j \psi \rangle \, dx \leq \sum_{i,j=1}^{n} \int_M \| \nabla e_i \psi \| \| \nabla e_j \psi \| \, dx \\
\leq \frac{1}{2} \sum_{i,j=1}^{n} \int_M (| \nabla e_i \psi |^2 + | \nabla e_j \psi |^2) \, dx = n \| \nabla \psi \|_2^2 \leq n \| \psi \|_{H^1}^2.
$$

This shows the Dirac operator $D : H^1(M, \Sigma M) \to L^2(M, \Sigma M)$ is a continuous operator and

$$
\| \nabla \psi \|_2^2 \geq \frac{1}{n} \| D\psi \|_2^2.
$$

(4)

On the other hand, by Schrödinger-Lichnerowicz formula, rewrite $\| D\psi \|_2^2$, we have as follows:

$$
\| D\psi \|_2^2 = \int_M \langle D^2 \psi, \psi \rangle \, dx = \| \nabla \psi \|_2^2 + \frac{R}{4} \| \psi \|_2^2.
$$

(5)

Set $R_{\min} = \min \{ R(x) : x \in M \}$ is the minimum of the scalar curvature and $R_{\max}$ its maximum. By the (5), we have

$$
\| \psi \|_{H^1}^2 + (\frac{R_{\min}}{4} - 1) \| \psi \|_2^2 \leq \| D\psi \|_2^2 \leq \| \nabla \psi \|_2^2 + (\frac{R_{\max}}{4} - 1) \| \psi \|_2^2.
$$

(6)

From (4) and (6), we obtain

$$
\frac{1}{n} (\| \psi \|_{H^1}^2 + \| D\psi \|_2^2) \leq (1 - \frac{R_{\min}}{4}) \| \psi \|_2^2 + (\frac{R_{\max}}{4} - 1) \| \psi \|_2^2.
$$

Thus $\| \psi \|_{H^1}$ and $\| \psi \|_{1,2}$ are equivalent norms. In other words, the Sobolev space $H^1(M, \Sigma M)$ can be defined as the completion of the $C^\infty(M, \Sigma M)$ with respect to the norm $\| \psi \|_{1,2}$. In the following, we will use the norm $\| \psi \|_{1,2}$ respect to the $H^1(M, \Sigma M)$.

The operator $D$ and $D^2$ are essentially self-adjoint in $L^2(M, \Sigma M)$. The eigenspinor $\psi_k$ of $D$ (with eigenvalue $\lambda_k$) are also eigenspinor of $D^2$ (with eigenvalue $\lambda_k^2$), i.e.

$$
\text{spec}(D^2) = \{ \lambda_k^2 | \lambda_k \in \text{spec}(D) \},
$$

where $\text{spec}(D)$ is expressed as a set of all eigenvalues of $D$ on the $H^1(M, \Sigma M)$.

In addition, the following Lemma c.f.[10] is showed that the kernels of $D$ and $D^2$ in $L^2(M, \Sigma M)$ is coincide, i.e. $\ker(D) = \ker(D^2)$.

Lemma 2. Let $M$ be a compact spin manifold and $\psi \in C^\infty(M, \Sigma M)$. Then for any number $t > 0$,

$$
\| D\psi \|_2^2 \leq t \| D^2 \psi \|_2^2 + \frac{1}{t} \| \psi \|_2^2.
$$
In the next, we will prove the conclusion $\ker(D) = \ker(D^2)$.

Let $\psi \in L^2(M, \Sigma M)$ satisfy $D^2\psi = 0$. By the regularity theorem for solution of elliptic differential equations we first conclude that $\psi$ is smooth. By the Lemma[2], we obtain

$$\|D\psi\|^2 \leq t \|D^2\psi\|^2 + \frac{1}{t} \|\psi\|^2 = \frac{1}{t} \|\psi\|^2 .$$

Since $\|\psi\|^2 < \infty$ imply for $t \to \infty$ that $D\psi \equiv 0$. Hence, $\ker(D) = \ker(D^2)$.

We introduce notation which will be frequently used throughout this paper. The complete orthonormal basis $\{\psi_k\}$ of $L^2(M, \Sigma M)$ consisting of the eigenspinors of $D^2$ is decomposed into two parts: $\{\psi_k\}_k \cup \{\psi^+_k\}_k \cup \{\psi^0_k\}_k$, where where $D^2\psi^+_k = \lambda^2_k \psi^+_k$ with $\lambda^2_k > 0$; $D^2\psi^0_k = 0$ and $\kappa = \dim(\ker D) = \dim(\ker D^2) < \infty$. By the elliptic regularity, we have any eigenspinor $\psi_k \in C^\infty(M, \Sigma M)$. Hence, we also set $H^+$ be the subspace spanned by eigenspinors $\{\psi^+_k\}_k$ with positive eigenvalues $\{\lambda^2_k\}_k$, and $H^0$ the nullspace of $D^2$. Then we have the decomposition of the Hilbert space $H^1(M, \Sigma M)$:

$$H^1(M, \Sigma M) = H^0 \oplus H^+ .$$

Under the condition $(H_1)$, by Sobolev embedding theorem, it is easily checked that $L$ is a $C^1$ functional on $H^1(M, \Sigma M)$ and we obtain the following proposition:

**Proposition 1.** Under the condition $(H_1)$ the functional $\mathcal{H} : H^1(M, \Sigma M) \to \mathbb{R}$ defined by

$$\mathcal{H}(\psi) = \int_M H(x, \psi(x)) dx ,$$

is of class $C^1$, and at each $\psi \in H^1(M, \Sigma M)$ derivations $\mathcal{H}'(\psi)$ was given by

$$\mathcal{H}'(\psi) \xi = \int_M \langle H\psi(x, \psi(x)), \xi(x) \rangle dx \ \forall \xi \in H^1(M, \Sigma M) .$$

In the view of the calculus of variations, the weak solutions to the problem (1) are obtained as critical points of the following Euler-Lagrange functional

$$\mathcal{L}(\psi) = \frac{1}{2} \int_M \langle D\psi, D\psi \rangle dx - \int_M H(x, \psi(x)) dx .$$

By the standard elliptic regularity theory, such a weak solution is in fact $C^2$ and a classical solution to problem (1).

### 3 The Cerami condition for $\mathcal{L}$

Let $F$ be a $C^1$ functional on a Banach space $E$, $c \in \mathbb{R}$. Recall that a sequence $\{u_n\} \subset E$ is called a Cerami condition if
(i) There exists a convergent subsequence for any boundedness sequence \( \{u_n\} \) with \( F(u_n) \to c \) and \( \| dF(u_n) \|_{E^*} \to 0 \) as \( n \to \infty \).

(ii) There exists \( \delta, r, \beta > 0 \) such that

\[
\| dF(u_n) \|_{E^*} \| u_n \| \geq \beta
\]

for any \( u_n \in F^{-1}[c - \delta, c + \delta] \) with \( \| u_n \| \geq r \).

In this section we prove the Cerami condition for \( \mathcal{L} \).

**Lemma 3.** Suppose \( H \) satisfies \((H_1),(H_2)\) and \((H_3)\). Then for any \( c \in \mathbb{R}, \mathcal{L} \) satisfies the Cerami condition with respect to \( H^1(M, \Sigma M) \).

**Proof.** We prove first Cerami condition (i) is satisfied. Let \( \psi_n \) be bounded in \( H^1(M, \Sigma M) \). By the Sobolev embedding theorem, we have the compact embedding

\[
H^1(M, \Sigma M) \hookrightarrow L^p(M, \Sigma M)
\]

for \( 1 \leq p < 2^* \). Passing to a subsequence, we may assume that for some \( \psi \in H^1(M, \Sigma M), \psi_n \rightharpoonup \psi \) weakly in \( H^1(M, \Sigma M) \) and \( \psi_n \rightharpoonup \psi \) strongly in \( L^p(M, \Sigma M) \) for any \( 1 \leq p < 2^* \). Setting \( \psi_n = \psi_n^0 + \psi_n^+ \) and \( \psi = \psi^0 + \psi^+ \) according to the \( H^1(M, \Sigma M) = H^0 \oplus H^+ \).

since \( \dim(H^0) < \infty, \|\cdot\|_{1,2} \) and \( \|\cdot\|_2 \) on \( H^0 \) are equivalent. Hence, we have

\[
\| \psi_n^0 - \psi^0 \|_{1,2} \leq C \| \psi_n^0 - \psi^0 \|_2 \to 0
\]

as \( n \to \infty \).

From the condition \((H_1)\), we get

\[
\begin{align*}
\| \psi_n^+ - \psi^+ \|_{1,2}^2 &= \| D\psi_n^+ - D\psi^+ \|_2^2 + \| \psi_n^+ - \psi^+ \|_2^2 \\
&\leq \| D\psi_n^+ - D\psi^+ \|_2^2 + (\lambda_1^+)^{-2} \| D\psi_n^+ - D\psi^+ \|_2^2 \\
&\leq C(d\mathcal{L}(\psi_n^+ - \psi^+), \psi_n^+ - \psi^+) + C \int_M (H_\psi(x, \psi_n^+ - \psi^+), \psi_n^+ - \psi^+) dx \\
&\leq C \| d\mathcal{L}(\psi_n^+ - \psi^+) \|_{H^1(M, \Sigma M)} \| \psi_n^+ - \psi^+ \|_{1,2} + C \int_M (1 + | \psi_n^+ - \psi^+ |^{p-1}) | \psi_n^+ - \psi^+ | dx \\
&\leq o(1) + C(\| \psi_n^+ - \psi^+ \|_1 + \| \psi_n^+ - \psi^+ \|_p) \to 0,
\end{align*}
\]

as \( n \to \infty \).

\((9)\) and \((10)\) imply the subsequence of \( \psi_n \) that converges \( \psi \) in \( H^1(M, \Sigma M) \).

So the Cerami condition (i) is verified.

To prove the Cerami condition (ii), we assume that

\[
\mathcal{L}(\psi_n) \to c, \quad \| \psi_n \|_{1,2} \to \infty, \quad \| d\mathcal{L}(\psi_n) \|_{H^1(M, \Sigma M)} \to 0
\]

as \( n \to \infty \).

From \((11)\), we have

\[
2\mathcal{L}(\psi_n) - \langle d\mathcal{L}(\psi_n), \psi_n \rangle = \int_M (H_\psi(x, \psi_n), \psi_n) dx - 2\int_M H(x, \psi_n) dx \to 2c \quad (12)
\]
as \( n \to \infty \).

Set \( \omega_n = \frac{\varphi_n}{\| D \varphi_n \|_2} \), which imply \( \| \omega_n \|_{1,2} \leq C \). By the \( H^1(M, \Sigma M) \) is a reflexive space and Sobolev embedding theorem, there exist \( \omega \) such that

\[
\begin{align*}
\omega_n &\to \omega \quad \text{in} \ H^1(M, \Sigma M) \\
\omega_n &\to \omega \quad \text{in} \ L^p(M, \Sigma M) \\
\omega_n(x) &\to \omega(x) \quad \text{a.e.} \ x \in M
\end{align*}
\]

(13)

where \( 1 \leq p < 2^* \).

On the one hand, if \( \| \omega(x) \|_2 \equiv 0 \), then \( \omega(x) = 0 \) a.e. on \( M \).

Define \( I(t) : [0, 1] \to \mathbb{R} \) such that

\[
I(t_n \psi_n) = \max_{t \in [0, 1]} \mathcal{L}(t \psi_n).
\]

(14)

If, for \( n \in \mathbb{N} \), \( t_n \) is defined by (14) is not unique, we choose any one of those values.

For any \( m > 0 \), set \( \varpi_n(x) = \sqrt{2m} \omega_n(x) \), by \( \omega_n(x) \to \omega(x) = 0 \), a.e. \( x \in M \) and \((H_1)\), it is easy to see that

\[
\lim_{n \to \infty} \int_M H(x, \varpi_n(x))dx = \int_M \lim_{n \to \infty} H(x, \varpi_n(x))dx = 0.
\]

(15)

Therefore, for enough \( n \), according to \( \| \psi_n \|_{1,2} \to \infty \), (14) and (15), we have

\[
\mathcal{L}(t_n \psi_n) \geq \mathcal{L}(\varpi_n(x)) = 2m - \int_M H(x, \varpi_n(x))dx \geq m.
\]

The above argument shows that \( \mathcal{L}(t_n \psi_n) \to \infty \) as \( n \to \infty \).

Since \( \mathcal{L}(0) = 0 \) and \( \mathcal{L}(\psi_n) \to c \), we have \( 0 < t_n < 1 \). Hence for enough \( n \), we conclude that

\[
d\mathcal{L}(t_n \psi_n), t_n \psi_n) = t_n \frac{d}{dt} |_{t=t_n} \mathcal{L}(t \psi_n) = 0.
\]

It implies that

\[
\int_M |D(t_n \psi_n)|^2 dx = \int_M \langle H(x, t_n \psi_n), t_n \psi_n \rangle dx.
\]

(16)

According to \( 0 < t_n < 1 \) and \((H_3)\), there exist \( \theta \geq 1 \) such that \( \theta h(x, \psi_n) \geq h(x, t_n \psi_n) \). Plugging (10) into the following inequality, we obtain

\[
2\mathcal{L}(\psi_n) - \langle d\mathcal{L}(\psi_n), \psi_n \rangle = \int_M h(x, \psi_n) \geq \frac{1}{\theta} h(x, t_n \psi_n) dx
\]

\[
= \frac{1}{\theta} \int_M \langle H(x, t_n \psi_n), t_n \psi_n \rangle dx - 2 \int_M H(x, t_n \psi_n))
\]

\[
= \frac{2}{\theta} \mathcal{L}(t_n \psi_n) \to \infty
\]
as \( n \to \infty \). This is a contraction to (12).

On the other hand, if \( \| \omega(x) \|_2 \neq 0 \), it follows from (H2) and Fatou’s Lemma that
\[
\int_M \frac{\langle H_\omega(x, \psi_n), \psi_n \rangle}{|\psi_n|^2} |\omega_n|^2 \, dx \to \infty
\]
as \( n \to \infty \). But from (11), we get
\[
\int_M |D\psi_n|^2 \, dx - \int_M \langle h(x, \psi_n), \psi_n \rangle \, dx = \langle d\mathfrak{L}(\psi_n), \psi_n \rangle = o(1)
\]
as \( n \to \infty \). This is
\[
1 - o(1) = \int_M \frac{\langle h(x, \psi_n), \psi_n \rangle}{|D\psi_n|^2} \, dx = \int_M \frac{\langle h(x, \psi_n), \psi_n \rangle}{|\psi_n|^2} |\omega_n|^2 \, dx.
\]
Thus we get a contradiction. The proof is complete, \( \mathfrak{L} \) satisfies the Cerami condition with respect to \( H^1(M, \Sigma M) \).

4 Proofs of Theorem 1

To obtain the theorem 1, we recall Fountain theorem for semi-definite functionals, see [22] for the detailed exposition.

Let \( X \) be a Banach space with basis \( \{e_j\}_{j=1}^\infty \), i.e. \( X = \text{span} \{e_j\}_{j=1}^\infty \). We set
\[
Y_k := \bigoplus_{j=1}^k \mathbb{R}e_j, \quad Z_k := \bigoplus_{j=k}^\infty \mathbb{R}e_j,
\]
We then have \( X = Y_k \bigoplus Z_k \).

**Theorem 2.** (Fountain theorem) Let \( J \in C^1(X, \mathbb{R}) \) is an even functional, i.e. \( J(-u) = J(u) \) for all \( u \in X \). \( J \) satisfies the Cerami condition. If for every \( k \in \mathbb{N} \), there exists \( r_k > 0 \) such that
\[
(A_1) : a_k := \inf_{u \in Z_k, \|u\| = r_k} J(u) \to \infty, \quad k \to \infty;
\]
\[
(A_2) : b_k := \sup_{u \in Y_k, \|u\| = r_k} J(u) \leq 0
\]
then \( J \) has an unbounded sequence of critical values.

We shall apply the fountain theorem for the functional \( \mathfrak{L} \) on \( H^1(M, \Sigma M) \). First of all, we prove:

**Lemma 4.** Define \( \beta_k = \sup\{\| \psi \|_2 : \psi \in Z_k, \| \psi \|_{1,2} = 1 \} \). We then have \( \beta_k \to 0 \) as \( k \to \infty \).
Proof: By the definition of $\beta_k$, for each $j$ there exists $\psi_j \in Z_k$ such that $\| \psi_j \|_{1,2} = 1$ and $\frac{1}{2} \beta_k < \| \psi_j \|_2$. According to the compactness of the embedding, we may assume (after taking a subsequence if necessary) that $\psi_j \rightharpoonup \psi$ weakly in $H^1(M, \Sigma M)$ and $\psi_j \to \psi$ strongly in $L^p(M, \Sigma M)$ for some $\psi \in H^1(M, \Sigma M)$, where $1 \leq p < 2^*$. Therefore,

$$1 = \| \psi \|_{1,2}^2 = \| D\psi \|_2^2 + \| \psi \|_2^2 = (1 + \lambda_k^2) \| \psi_k \|_2^2$$

where $\lambda_k$ is the eigenvalue of $\psi_k$, and $\lambda_k \to \infty$ as $k \to \infty$. Obviously, we have $\| \psi \|_2 = 0$ and $\beta_k \to 0$ as $k \to \infty$.

**Lemma 5.** There exists $\rho_k > r_k > 0$ such that

(A1): $a_k := \inf_{\| \psi \|_{1,2} = r_k} \mathcal{L}(\psi) \to \infty, k \to \infty$;

(A2): $b_k := \sup_{\| \psi \|_{1,2} = \rho_k} \mathcal{L}(\psi) \leq 0$

Proof: (i) Let $\psi \in Z_k$ with $\| \psi \|_{1,2} = r_k$. Then by (H1) and Lemma 4 we have

$$\mathcal{L}(\psi) = \frac{1}{2} \| D\psi \|_2^2 - \int_M H(x, \psi) dx$$

$$\geq \frac{1}{2} \| \psi \|_{1,2}^2 - \frac{1}{2} \| \psi \|_2^2 - C \| \psi \|_q^q - C$$

$$\geq \frac{1}{2} r_k^2 - \frac{1}{2} r_k^2 \beta_k^2 - Cr_k \beta_k^q - C$$

(17)

By Lemma 3 we have $\beta_k^q \leq \frac{1}{4}$ for enough $k$, we obtain from (17) that

$$\mathcal{L}(\psi) \geq \frac{1}{4} r_k^2 - Cr_k \beta_k^q - C$$

(18)

As in the proof of [19, Lemma 3.6], choosing $r_k = (2pC\beta_k^q)^{\frac{1}{p-q}}$, (18) imply that

$$\mathcal{L}(\psi) \geq \frac{1}{4} r_k^2 - Cr_k \beta_k^q - C \geq (\frac{1}{4} - \frac{1}{2p})(2pC\beta_k^q)^{\frac{1}{p-q}} \to \infty$$

as $k \to \infty$ and the condition (A1) is satisfied.

(ii) Since $\text{dim}(Y_k) < \infty$, the $\| \cdot \|_2$ and $\| \cdot \|_{1,2}$-norms are equivalent on $Y_k$, for any $\psi \in Y_k$, there exist $C_k > \frac{1}{2}$ such that

$$\frac{1}{2} \int_M | D\psi | dx \leq \frac{1}{2} \| \psi \|_{1,2} \leq C_k \| \psi \|_2^2.$$  

(19)

By (H2), these exist $R_k > 0$ such that

$$H(x, \psi) \geq 2C_k | \psi |^2,$$

(20)

for any $(x, \psi) \in Y_k$ with $| \psi | \geq R_k$. 


Choosing $M_k = \max\{|H(x, \psi)|: |\psi| \leq R_k, x \in M\}$, we obtain
\[
H(x, \psi) \geq -M_k \geq 2C_k |\psi|^2 - 2C_k R_k^2 - M_k
\tag{21}
\]
for any $|\psi| \leq R_k$.

Using (20) and (21) we have
\[
H(x, \psi) \geq 2C_k |\psi|^2 - M_k
\tag{22}
\]
where $M_k = 2C_k R_k^2 + M_k > 0$. Hence by (19) and (22), we obtain
\[
L(\psi) = \frac{1}{2} \int_M |D\psi| \, dx - \int_M H(x, \psi) \, dx \leq -C_k \|\psi\|_2^2 + M_k |M|.
\]
Therefore, we have $b_k \leq 0$ for enough $\rho_k > 0$ where $\rho_k > r_k$, the condition (A2) is satisfied.

**Proof of Theorem**

According to Lemma 3, $L$ satisfy the Cerami condition. Lemma 4 and Lemma 5 imply that the condition (A1) and (A2) is satisfied. By Fountain theorem there exists critical points $\psi_n \in H^1(M, \Sigma M)$ of $L$ such that $L(\psi_n) \to \infty$ as $n \to \infty$. Since $D^2 \psi_n = D(D\psi_n) = H(\psi, \psi_n)$, where $H$ is $\alpha$ Hölder continuous in the direction of the base, where $0 < \alpha < 1$. By the Interior Schauder estimates in [2], we have $D\psi_n \in C^{1}(M, \Sigma M)$. Using the Interior Schauder estimates again, we obtain those weak solutions $\psi_n$ are in fact $C^2(M, \Sigma M)$ and classical solutions to problem (1).

**Founding**
The corresponding author: Xu Yang was supported by the NSFC (grant no.11801499) of China.

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