Spin Supercurrent in the Canted Antiferromagnetic Phase

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(Dated: May 5, 2014)

The spin and layer (pseudospin) degrees of freedom are entangled coherently in the canted antiferromagnetic phase of the bilayer quantum Hall system at the filling factor $\nu = 2$. There emerges a complex Goldstone mode describing such a combined degree of freedom. In the zero tunneling-interaction limit ($\Delta_{\text{SAS}} \to 0$), its phase field provokes a supercurrent carrying both spin and charge within each layer. The Hall resistance is predicted to become anomalous precisely as in the $\nu = 1$ bilayer system in the counterflow and drag experiments. Furthermore, it is shown that the total current flowing in the bilayer system is a supercurrent carrying solely spins in the counterflow geometry. It is intriguing that all these phenomena occur only in imbalanced bilayer systems.

PACS numbers: 73.43.-f, 11.30.Qc, 73.43.Qt, 64.70.Tg

I. INTRODUCTION

Physics of the bilayer quantum Hall (QH) system is enormously rich owing to the intralayer and interlayer phase coherence controlled by the interplay between the spin and the layer (pseudospin) degrees of freedom\(^1\,^2\). At the filling factor $\nu = 1$ there arises a unique phase, the spin-ferromagnet and pseudospin-ferromagnet phase, which has well been studied both theoretically and experimentally. One of the most intriguing phenomena is the Josephson-like tunneling between the two layers predicted in Refs.\(^3\,^4\), whose first experimental indication was obtained in Ref.\(^6\). Other examples are the anomalous behavior of the Hall resistance reported in counterflow and drag experiments\(^7\,^8\). They are triggered by the supercurrent within each layer\(^10\). Quite recently, careful experiments\(^11\) were performed to explore the condition for the tunneling current to be dissipationless. These phenomena are driven by the Goldstone mode describing an interlayer phase coherence. It exhibits the linear dispersion relation in the zero tunneling-interaction limit ($\Delta_{\text{SAS}} \to 0$).

On the other hand, at $\nu = 2$ the bilayer QH system has three phases, the spin-ferromagnet and pseudospin-singlet phase, the spin-singlet and pseudospin ferromagnet phase, and a canted antiferromagnetic phase\(^12\,^14\) (abridged as the CAF phase), depending on the relative strength between the Zee-man energy $\Delta_Z$ and the tunneling energy $\Delta_{\text{SAS}}$. The pattern of the symmetry breaking is SU(4)→U(1)⊗SU(2)⊗SU(2), associated with which there appear four complex Goldstone modes\(^19\). A part of them has been studied in Refs.\(^17\,^18\). We have recently analyzed the full details of these Goldstone modes in each phase\(^19\). The CAF phase is the most interesting, where the spins are canted coherently and making antiferromagnetic correlations between the two layers. Moreover, one of the Goldstone modes becomes gapless and has a linear dispersion relation\(^19\) as $\Delta_{\text{SAS}} \to 0$. It is an urgent and intriguing problem what kind of phase coherence this Goldstone mode develops.

In this paper, we show that it is the entangled spin-pseudospin phase coherence, and we explore associated phase coherent phenomena. We employ the Grassmannian formalism\(^18\), where the basic field is the Grassmannian field consisting of two complex-projective (CP\(^3\)) fields. The CP\(^3\) field emerges when composite bosons undergo Bose-Einstein condensation\(^1\). The formalism provides us with a clear physical picture of the spin-pseudospin phase coherence in the CAF phase. Furthermore, it enables us to analyze nonperturbative phase coherent phenomena, where the phase field $\vartheta(x)$ is essentially classical and may become very large. We show that the supercurrent flows within the layer when there is in-homogeneity in $\vartheta(x)$. This is precisely the same as in the $\nu = 1$ bilayer QH system. Indeed, the supercurrent leads to the same formula\(^10\) of the anomalous Hall resistivity for the counterflow and drag geometries as the one at $\nu = 1$. What is remarkable is that the total current flowing the bilayer system is a supercurrent carrying solely spins and not charges in the counterflow geometry. We note that the supercurrent flows both in the balanced and imbalanced systems at $\nu = 1$ but only in imbalanced systems at $\nu = 2$.

II. GRASSMANNIAN FIELD AND ENTANGLLED SPIN-PSEUDOSPIN PHASE COHERENCE

In the bilayer system an electron has two types of indices, the spin index ($\uparrow, \downarrow$) and the layer index (f,b). They can be incorporated into 4 types of isospin index $\alpha = \uparrow, \downarrow, f, b$, where the electron field $\psi_\alpha(x)$ has four components, and the bilayer system possesses the underlying algebra SU(4) with the subalgebra SU(2)⊗SU(2). We denote the three generators of the SU(2) by $\tau^a_{\text{spin}}$, and those of SU(2) by $\tau^a_{\text{spin}}$. There are remaining nine generators $\tau^a_{\text{spin}}\tau^b_{\text{spin}}$, which are the generators of the R-spin operators. Their explicit forms are given in Appendix D in Ref.\(^1\).

All the physical operators required for the description of the system are constructed as the bilinear combinations of $\psi(x)$ and $\psi^\dagger(x)$. They are 16 density operators $\rho(x) = \psi^\dagger(x)\psi(x)$, $S_\alpha(x) = \frac{1}{2}\psi^\dagger(x)\tau^a_{\text{spin}}\psi(x)$, $P_\alpha(x) = \frac{1}{2}\psi^\dagger(x)\tau^a_{\text{spin}}\psi(x)$, and $R_{ab}(x) = \frac{1}{2}\psi^\dagger(x)\tau^a_{\text{spin}}\tau^b_{\text{spin}}\psi(x)$, where $S_\alpha$ describes the total spin, $2P_z$ measures the electron-
density difference between the two layers. The operator $R_{ab}$ transforms as a spin under SU$_{\text{spin}}(2)$ and as a pseudospin under SU$_{\text{ppin}}(2)$. It is $R_{ab}$ that plays the key role in the entangled spin-pseudospin phase coherence in the CAF phase.

The kinetic Hamiltonian is quenched, since the kinetic energy is common to all states in the lowest Landau level (LLL). The Coulomb Hamiltonian is decomposed into the SU(4)-invariant term $H_C^2$ and the SU(4)-noninvariant term $H_C^-$. The additional potential terms are the Zeeman, tunneling, and bias terms, $H_{2D} = -\int d^2x (\Delta_2 S_z + \Delta_{\text{ppin}} P_x + e V_{\text{bias}} P_z)$, where $V_{\text{bias}}$ is the bias voltage which controls the density imbalance between the two layers. The total Hamiltonian is $H = H_C^2 + H_C^- + H_{2D}$.

We project the density operators to the LLL. What are obtained experimentally are the classical densities, which are expectation values such as $\rho^a(x) = \langle \hat{S}_a \rho(x) \hat{S}_a \rangle$, where $\langle \hat{S} \rangle$ represents a generic state in the LLL. We may set $\rho^a(x) = \rho_0$, $S^a_{\text{cap}}(x) = \rho_0 \rho_s(x)$, $P^a_{\text{cap}}(x) = \rho_0 \rho_s(x)$, and $R^a_{\text{cap}}(x) = \rho_0 \rho_s(x)$ for the study of Goldstone modes, where $\rho_0 = \rho_0/v$ is the density of states. Taking the nontrivial lowest order terms in the derivative expansion, we obtain the SU(4) effective Hamiltonian density\(^2\):

$$H_{\text{eff}} = J_{s}^0 \left( \sum (\partial_k S_a)^2 + (\partial_k P_a)^2 + (\partial_k R_{ab})^2 \right)$$

$$+ 2 J_s^- \left( \sum (\partial_k S_a)^2 + (\partial_k P_a)^2 + (\partial_k R_{ab})^2 \right)$$

$$+ \rho_0 \left[ \epsilon_{\text{cap}} (P_a) - 2 e \sqrt{\sum (S_a)^2 + (R_{ab})^2} \right] - (\Delta_2 S_z + \Delta_{\text{ppin}} P_x + \Delta_{\text{bias}} P_z),$$

where $J_s^- = \frac{1}{2} (J_s - J_s^0)$ with $J_s$ and $J_s^0$ the intralayer and interlayer stiffness, $\epsilon_{\text{cap}}$ the capacitance energy, $\epsilon_s$ the exchange Coulomb energy due to $H_C^-$. Their explicit formulas are given in Appendix A in Ref.\(^1\). This effective Hamiltonian is valid at $\nu = 1, 2, 3$.

The ground state is obtained by minimizing the effective Hamiltonian\(^1\) for homogeneous configurations of the classical densities. The order parameters are the classical densities for the ground state. They are explicitly given in Ref.\(^2\) for the $\nu = 2$ system. In the limit $\Delta_{\text{ppin}} \to 0$, we read

$$S^0_z = 1 - |\sigma_0|, \quad P^0_z = \sigma_0, \quad R^0_{xx} = \text{sgn}(\sigma_0) R^0_{yy},$$

$$R^0_{yy} = -\sqrt{|\sigma_0| (1 - |\sigma_0|)},$$

and all others being zero. Here, $\sigma_0 = (\rho_0^b - \rho_0^d)/(\rho_0^b + \rho_0^d)$ is the imbalance parameter with $\rho_0^b$ being the electron density in the front (back) layer. Both the spin and pseudospin are polarized into the $z$-axis in this limit.

We have analyzed the excitations around the classical ground state\(^1\). There emerge four complex Goldstone modes associated with the spontaneous symmetry breaking SU(4)→U(1)⊗SU(2)⊗SU(2). When $H_C^- = 0$ and $\Delta_2 = \Delta_{\text{ppin}} = \Delta_{\text{bias}} = 0$, the SU(4) symmetry is exact and all of them are gapless, but they get gapped by these interactions.

We are interested in the limit $\Delta_{\text{ppin}} \to 0$ since we expect the enhancement of the interlayer phase coherence just as in the $\nu = 1$ system. We have already shown that there exists one gapless Goldstone mode with a linear dispersion relation in a perturbation theory\(^\text{[19]}\). In this paper we employ the Grassmannian formalism\(^\text{[18]}\) to make the physical picture of this Goldstone mode and its phase coherence clearer, and to construct a nonperturbative theory in terms of the density difference field $\sigma(x)$ and its conjugate phase field $\varphi(x)$. The Grassmannian field $Z(x)$ consists of two CP$^3$ fields $n_1(x)$ and $n_2(x)$ at $\nu = 2$, since there are two electrons per one Landau site. Due to the Pauli exclusion principle they should be orthogonal one to another.

Hence, we require $n_1^\dagger (x) \cdot n_2 (x) = \delta_{ij}$ with $i = 1, 2$. Using a set of two CP$^3$ fields subject to this normalization condition we introduce a $4 \times 4$ matrix field, the Grassmannian field given by $Z(x) = (n_1, n_2)$ obeying $Z^\dagger Z = 1$.

The dimensionless SU(4) isospin densities are given by

$$S_a (x) = \frac{1}{2} \text{Tr} [Z^\dagger \sigma_a^\text{spin} Z] = \frac{1}{2} \sum_{i=1}^2 n_i^\dagger \sigma_a^\text{spin} n_i,$$

$$P_a (x) = \frac{1}{2} \text{Tr} [Z^\dagger \tau_a^\text{spin} Z] = \frac{1}{2} \sum_{i=1}^2 n_i^\dagger \tau_a^\text{spin} n_i,$$

$$R_{ab} (x) = \frac{1}{2} \text{Tr} [Z^\dagger \tau_a^\text{ppin} \tau_b^\text{ppin} Z] = \frac{1}{2} \sum_{i=1}^2 n_i^\dagger \tau_a^\text{ppin} \tau_b^\text{ppin} n_i,$$

where $n_i$ consists of the basis $n_i (x) = (n_1^\uparrow, n_1^\downarrow, n_2^\uparrow, n_2^\downarrow)^\dagger$. It is a straightforward task to carry out the perturbative analysis of the effective Hamiltonian\(^1\) in terms of the Grassmannian field and obtain the same results as given in Ref.\(^\text{[19]}\).

We concentrate solely on the gapless mode in the limit $\Delta_{\text{ppin}} \to 0$. We parametrize the CP$^3$ fields as

$$n_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 0 \\ -e^{i\varphi(x)/2} \sqrt{\sigma(x)} \\ e^{-i\varphi(x)/2} \sqrt{1 - \sigma(x)} \\ 0 \end{pmatrix},$$

for $\sigma(x) > 0$, and

$$n_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad n_2 = \begin{pmatrix} e^{i\varphi(x)/2} \sqrt{1 + \sigma(x)} \\ 0 \\ 0 \\ e^{-i\varphi(x)/2} \sqrt{-\sigma(x)} \end{pmatrix},$$

for $\sigma(x) < 0$. The isospin density fields are expressed in terms of $\sigma(x)$ and $\varphi(x)$.

$$S_z (x) = 1 - |\sigma(x)|, \quad P_z (x) = \sigma(x),$$

$$R_{yy} (x) = \text{sgn}(\sigma_0) R_{xx} (x) = -\sqrt{|\sigma(x)| (1 - |\sigma(x)|)} \cos \varphi(x),$$

$$R_{xz} (x) = -\text{sgn}(\sigma_0) R_{yz} (x) = -\sqrt{|\sigma(x)| (1 - |\sigma(x)|)} \sin \varphi(x),$$

with all others being zero. The ground-state expectation values $\langle \sigma(x) \rangle = \sigma_0$, $\langle \varphi(x) \rangle = 0$, with which the order parameters\(^3\) are reproduced from\(^4\). It is notable that the fluctuations of the phase field $\varphi(x)$ affect both spin and pseudospin components of the $R$-spin. This is very different from
the spin wave in the monolayer QH system or the pseudospin wave in the bilayer QH system at $\nu = 1$. Hence we call it the entangled spin-pseudospin phase field $\vartheta(x)$.

By substituting (6) into (1), apart from irrelevant constant terms the resulting effective Hamiltonian is

$$\mathcal{H}_{\text{eff}} = \frac{J_0}{2} (\nabla \vartheta)^2 + \frac{J_\sigma}{2} (\nabla \sigma)^2 + \rho_0 \epsilon_{\text{cap}}^{-1} (\sigma - \sigma_0)^2,$$

where $J_\sigma = 4 J_s (\sigma_0 [1/(\sigma_0 + 1)] - 1)$, $J_\vartheta = 4 J_s^d [\sigma_0 (1 - \sigma_0)]$, and $\epsilon_{\text{cap}}^{-1} = 4 (\epsilon_D - \epsilon_X)$ is the capacitance parameter at $\nu = 1$. The effective Hamiltonian is correct up to $O(\Delta^2_{\text{SAS}})$ as $\Delta_{\text{SAS}} \to 0$.

When we require the equal-time commutation relation,

$$\rho_0 \left[ \sigma(x), \vartheta(y) \right] = i \delta(x - y),$$

the Hamiltonian (7) is second quantized, and it has the linear dispersion relation,

$$E_k = |k| \sqrt{\frac{2 J_\sigma}{\rho_0} \left( \frac{2 J_\vartheta}{\rho_0} k^2 + 2 \epsilon_{\text{cap}}^{-1} \right)},$$

This agrees with Eq.(136) of Ref.[19]. It should be emphasized that the effective Hamiltonian (7) is valid in all orders of the phase field $\vartheta(x)$. It may be regarded as a classical Hamiltonian as well, where (8) should be replaced with the corresponding Poisson bracket.

The effective Hamiltonian (7) for $\vartheta(x)$ and $\sigma(x)$ reminds us of the one that governs the Josephson-like effect at $\nu = 1$. The main difference is the absence of the tunneling term, as implies that there exists no Josephson-like tunneling. Nevertheless, the supercurrent is present within the layer, which is our main issue.

By using the Hamiltonian (7) and the commutation relation (8), we obtain the equations of motion,

$$\hbar \partial_t \vartheta(x) = \frac{2 J_\sigma}{\rho_0} \nabla^2 \sigma(x) - 2 \epsilon_{\text{cap}}^{-1} (\sigma(x) - \sigma_0),$$

$$\hbar \partial_t \sigma(x) = - \frac{2 J_\vartheta}{\rho_0} \nabla^2 \partial_t \vartheta(x).$$

### III. ANOMALOUS HALL RESISTANCE AND SPIN SUPERCURRENT

We now study the electric supercurrent carried by the gapless mode $\vartheta(x)$. The electron densities are $\rho_e^{(b)} = - e \rho_0 (1 \pm \mathcal{P}_z) / 2 = - e \rho_0 (1 \pm \sigma(x)) / 2$ on each layer. Taking the time derivative and using (11) we find

$$\partial_t \rho_e^{(b)} = \frac{\epsilon J_\vartheta}{\hbar} \nabla^2 \partial_t \vartheta(x).$$

The time derivative of the charge is associated with the current via the continuity equation, $\partial_t \rho_e^{(b)} = \partial_t J_i^{(b)}$. We thus identify $J_i^{(b)} = \pm J_i^{(\text{hos})}(x) + \text{constant}$, where

$$J_i^{(\text{hos})}(x) \equiv \frac{\epsilon J_\vartheta}{\hbar} \partial_t \vartheta(x).$$

Consequently, the current $J_i^{(\text{hos})}(x)$ flows when there exists inhomogeneity in the phase $\vartheta(x)$. It is a supercurrent because the coherent mode exhibits a linear dispersion relation. It is intriguing that the current does not flow in the balanced system since $J_\vartheta = 0$ at $\sigma_0 = 0$.

Let us inject the current $J_\vartheta$ into the $x$ direction of the bilayer sample, and assume the system to be homogeneous in the $y$ direction (Fig.1). It creates the electric field $E_y^{(b)}$ so that the Hall current flows into the $x$-direction. A bilayer system consists of the two layers and the volume between them. The Coulomb energy in the volume is minimized by the condition $E_y^{(b)} = E_y^{(a)}$. We thus impose $E_y^{(a)} = E_y^{(b)} = E_y$. The current is the sum of the Hall current and the supercurrent,

$$J_x^{(a)}(x) = \frac{\nu}{R_K} \rho_0 E_y + J_x^{(\text{hos})}, \quad J_x^{(b)}(x) = \frac{\nu}{R_K} \rho_0 E_y - J_x^{(\text{hos})},$$

with $R_K = 2 \pi \hbar / e^2$ the von Klitzing constant. We obtain the standard Hall resistance when $J_x^{(\text{hos})} = 0$. Namely, the emergence of the supercurrent ($J_x^{(\text{hos})} \neq 0$) is detected if the Hall resistance becomes anomalous.

We apply these formulas to analyze the counterflow and drag experiments since they occur without tunneling. In the counterflow experiment, the current $J_\vartheta$ is injected to the front layer and extracted from the back layer at the same edge. Since there is no tunneling we have $J_x^{(b)} = - J_x^{(a)} = - J_\vartheta$. Hence, it follows from (14) that $E_y = 0$, or

$$R_{xy}^e J_x^{(e)} = 0, \quad R_{xy}^b J_x^{(b)} = 0.$$
All the input current is carried by the supercurrent, $J_{\text{in}}^{\text{bs}} = J_m$. It generates such an inhomogeneous phase field that
\[ \vartheta(x) = (\hbar/eJ_0)J_m x. \]

On the other hand, in the drag experiment, since the interlayer coherent tunneling is absent, no current flows on the back layer, or $J_x^b = 0$. Hence, it follows from (14) that $J_m = J_x^{t} = (\nu/RK)E_y$, or
\[ R_{xy}^{f} = \frac{E_y}{J_x^{t}} = \frac{RK}{\nu} = \frac{1}{2}RK \quad \text{at} \quad \nu = 2, \quad (16) \]

A part of the input current is carried by the supercurrent, $J_{\text{in}}^{\text{bs}} = \frac{1}{2}(1 - \sigma_0)J_m$.

The standard Hall resistance is given by $R_{xy}^{f} = \frac{2}{\nu}RK = RK$ at $\nu = 2$. We thus predict the anomalous Hall resistance (15) and (16) in the CAF phase at $\nu = 2$ by carrying out similar experiments (7,9) due to Kellogg et al. and Tutuc et al. in imbalanced configuration ($\sigma_0 \neq 0$).

The phase field $\vartheta(x)$ describes the entangled spin-pseudospin coherence according to the basic formula (3) in the CAF phase. The spin density in each layer is defined by $\rho_{\text{spin}}^{\alpha}(x) \equiv s_{\alpha}x_{\alpha}^{1/2}f_{\alpha}$, where $s_{\alpha} = \frac{1}{2}\hbar$ for $\alpha = f \uparrow$, $b \uparrow$ and $s_{\alpha} = -\frac{1}{2}\hbar$ for $\alpha = f \downarrow$, $b \downarrow$. We note the relation
\[
\begin{pmatrix}
\rho_{\uparrow\uparrow}(x) \\
\rho_{\uparrow\downarrow}(x) \\
\rho_{\downarrow\uparrow}(x) \\
\rho_{\downarrow\downarrow}(x)
\end{pmatrix} = \frac{1}{4}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
2S_z(x) \\
2P_z(x) \\
2R_{zz}(x)
\end{pmatrix},
\]

Up to $O((\sigma - \sigma_0)^2)$, we obtain $S_z = 1 - |\sigma(x)|$, and
\[
\partial_x \rho_{\text{spin}}^{\text{spin}} = \rho_{\text{spin}}^{\text{spin}}(x) = \frac{J_0}{4}[1 + \text{sgn}(\sigma_0)]\partial_x^2\vartheta(x), \quad (18)
\]
\[
\partial_x \rho_{\text{spin}}^{\text{spin}} = \rho_{\text{spin}}^{\text{spin}}(x) = -\frac{J_0}{4}[1 - \text{sgn}(\sigma_0)]\partial_x^2\vartheta(x). \quad (19)
\]

The time derivative of the spin is associated with the spin current via the continuity equation, $\partial_x \rho_{\text{spin}}^{\text{spin}}(x) = \partial_x J_{\text{in}}^{\text{bs}}(x)$ for each $\alpha$. We thus identify
\[
J_{b\uparrow}^{\text{spin}}(x) = J_{t\uparrow}^{\text{spin}}(x) = \frac{J_0}{2} \partial_x \vartheta(x), \quad \text{for } \sigma_0 > 0, \quad (20)
\]
\[
J_{b\downarrow}^{\text{spin}}(x) = J_{t\downarrow}^{\text{spin}}(x) = -\frac{J_0}{2} \partial_x \vartheta(x), \quad \text{for } \sigma_0 < 0. \quad (21)
\]

The spin current $J_{\text{in}}^{\text{spin}}(x)$ flows along the $x$-axis, when there exists an inhomogeneous phase difference $\vartheta(x)$.

In the counterflow experiment, the total charge current along the $x$-axis is zero, $J_x^{f}(x) + J_x^{b}(x) = 0$. Consequently, the input current generates a pure spin current,
\[
J_x^{\text{spin}} = J_{t\uparrow}^{\text{spin}} + J_{t\downarrow}^{\text{spin}} + J_{b\uparrow}^{\text{spin}} + J_{b\downarrow}^{\text{spin}} = \text{sgn}(\sigma_0) \frac{\hbar}{e} J_m. \quad (22)
\]

This current is dissipationless since the dispersion relation is linear. It is appropriate to call it a spin supercurrent. It is intriguing that the spin current flows in the opposite directions for $\sigma_0 > 0$ and $\sigma_0 < 0$, as illustrated in Fig[1] A comment is in order: The spin current only flows within the sample, since spins are scattered in the resistor $R$ and spin directions become random outside the sample.

We have explored the entangled spin-pseudospin phase coherence in the CAF phase, governed by the Goldstone mode $\vartheta(x)$ describing the $R$-spin according to the formula (6). We have predicted anomalous Hall resistivity in the counterflow and drag experiments in the imbalanced regime ($\sigma_0 \neq 0$) at $\nu = 2$. In particular, there flows a spin supercurrent in the counterflow geometry.

This research was supported in part by JSPS Research Fellowships for Young Scientists, and a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan (No. 21540254).

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