POSITIVITY OF RIEMANN–ROCH POLYNOMIALS
AND TODD CLASSES OF HYPERKÄHLER
MANIFOLDS

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Abstract. For a hyperkähler manifold \( X \) of dimension \( 2n \), Huybrechts showed that there are constants \( a_0, a_2, \ldots, a_{2n} \) such that

\[
\chi(L) = \sum_{i=0}^{n} \frac{a_{2i}}{(2i)!} q_X c_i(L)^i
\]

for any line bundle \( L \) on \( X \), where \( q_X \) is the Beauville–Bogomolov–Fujiki quadratic form of \( X \). Here the polynomial \( \sum_{i=0}^{n} \frac{a_{2i}}{(2i)!} q^i \) is called the Riemann–Roch polynomial of \( X \).

In this paper, we show that all coefficients of the Riemann–Roch polynomial of \( X \) are positive. This confirms a conjecture proposed by Cao and the author, which implies Kawamata’s effective non-vanishing conjecture for projective hyperkähler manifolds. It also confirms a question of Riess on strict monotonicity of Riemann–Roch polynomials.

In order to estimate the coefficients of the Riemann–Roch polynomial, we produce a Lefschetz-type decomposition of \( \text{td}^{1/2}(X) \), the root of the Todd genus of \( X \), via the Rozansky–Witten theory following the ideas of Hitchin, Sawon, and Nieper-Wißkirchen.

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1. INTRODUCTION

Throughout this paper, we work over the complex number field \( \mathbb{C} \).

A compact Kähler manifold \( X \) is called a hyperkähler manifold or an irreducible holomorphic symplectic manifold if \( X \) is simply connected and \( H^0(X, \Omega^2) \) is spanned by an everywhere non-degenerate 2-form. Hyperkähler manifolds are higher-dimensional analogues of K3 surfaces and appear to be a very important class of manifolds with \( c_1 = 0 \). Their rich geometry attracts much attention from different areas of mathematics. The only known examples are (up to deformations): Hilbert schemes of points on K3 surfaces, generalized Kummer varieties (due to Beauville’s construction \([1]\)), and 2 examples in dimensions 6 and 10 constructed by O’Grady \([21, 22]\).

The main goal of this paper is to study general properties of the Riemann–Roch polynomial and Todd classes of a hyperkähler manifold.

1.1. Positivity of Riemann–Roch polynomials. For a hyperkähler manifold \( X \) of dimension \( 2n \), Huybrechts \([11]\) showed that there are constants \( a_0, a_2, \ldots, a_{2n} \) such that

\[
\chi(L) = \sum_{i=0}^{n} \frac{a_{2i}}{(2i)!} q_X (c_1(L))^i
\]

for any line bundle \( L \) on \( X \), where \( q_X \) is the Beauville–Bogomolov–Fujiki quadratic form of \( X \) (see Section \([22]\)). Here the polynomial \( RR_X(q) = \sum_{i=0}^{n} \frac{a_{2i}}{(2i)!} q^i \) is called the Riemann–Roch polynomial of \( X \). Note that \( RR_X \) is a deformation invariant of \( X \). To study the behavior of line bundles on hyperkähler manifolds, it is crucial to have a good understanding of Riemann–Roch polynomials. In \([4]\), Cao and the author conjectured that the coefficients of the Riemann–Roch polynomial are
all non-negative for any projective hyperkähler manifold, and proved it up to dimension 6. The main theorem of this paper is the following.

**Theorem 1.1.** Let $X$ be a hyperkähler manifold. Then all coefficients of the Riemann–Roch polynomial $\text{RR}_X(q)$ are positive.

In fact, in Corollary 5.2 we will have a more precise estimate on the lower bounds of the coefficients of $\text{RR}_X$. We remark that Nieper-Wißkirchen [19] gave a closed formula for the coefficients $a_{2k}$ in terms of Chern numbers of $X$ by the Rozansky–Witten theory, but the expression is quite complicated and not sufficient to determine the positivity of coefficients.

**Example 1.2.** The Riemann–Roch polynomials of known hyperkähler manifolds are as the following:

1. If $X$ is a hyperkähler manifold of dimension $2n$ deformation equivalent to the Hilbert scheme of $n$ points on a K3 surface or O’Grady’s 10-dimensional example, then $\text{RR}_X(q) = \binom{q^{2+n+1}}{n}$ by [5, Lemma 5.1] and [23, Theorem 2];
2. If $X$ is a hyperkähler manifold of dimension $2n$ deformation equivalent to a generalized Kummer variety or O’Grady’s 6-dimensional example, then $\text{RR}_X(q) = (n + 1)\binom{q/2+n}{n}$ by [3, Lemma 5.2] and [23, Theorem 2].

From the known examples, we can observe that $\text{RR}_X$ might satisfy more properties than positivity, so it is natural to raise up the following conjecture (the first one is a question asked by Ortiz).

**Conjecture 1.3.** Let $X$ be a hyperkähler manifold.

1. The sequence of coefficients of $\text{RR}_X(q)$ is log concave.
2. All roots of $\text{RR}_X(q)$ are negative real numbers.
3. More wildly, all roots of $\text{RR}_X(q)$ are negative even integers forming an arithmetic sequence.

As applications of Theorem 1.1, we give an affirmative answer to the conjecture of Cao and the author [4] which leads to a solution of Kawamata’s effective non-vanishing conjecture for projective hyperkähler manifolds (Corollary 5.3) and also an affirmative answer to a question of Riess [24] on the strict monotonicity of Riemann–Roch polynomials (Corollary 5.4).

1.2. **A Lefschetz-type decomposition of $\text{td}^{1/2}(X)$ via the Rozansky–Witten theory.** To prove Theorem 1.1 we need to have a good understanding of the Todd genus of a hyperkähler manifold. In fact, as observed by Hitchin and Sawon [9] and Nieper-Wißkirchen [19], the root of the Todd genus $\text{td}^{1/2}(X)$ is a more interesting object, especially from the point of view of the Rozansky–Witten theory. Following their ideas, we use the Rozansky–Witten theory to produce a Lefschetz-type decomposition of $\text{td}^{1/2}(X)$. 

Theorem 1.4 (=Proposition 4.2+Theorem 4.5). Let \( X \) be a hyperkähler manifold of dimension \( 2n \) and fix a non-zero \( \sigma \in H^0(X, \Omega^2_X) \). Consider 
\[
\lambda_{\sigma} = \frac{24n \int \exp(\sigma + \overline{\sigma})}{\int c_2(X) \exp(\sigma + \overline{\sigma})}.
\]
For \( 0 \leq k \leq n/2 \), denote 
\[
\text{tp}_{2k} := \sum_{i=0}^{k} \frac{(n - 2k + 1)! \Lambda_{\sigma/4}(\text{td}_i^{1/2} \wedge (\sigma \overline{\sigma})^{k-i})}{(-\lambda_{\sigma})^{k-i}(k-i)!(n-k-i+1)!} \in H^{4k}(X).
\]
Then \( \text{tp}_{2k} \) is \((\sigma + \overline{\sigma})\)-primitive for any \( 0 \leq k \leq n/2 \). Furthermore, for any \( 0 \leq k \leq n \), 
\[
\text{td}_{2k}^{1/2} = \sum_{i=0}^{\min\{k,n-k\}} \frac{(n - k - i)!}{\lambda_{\sigma}^{k-i}(k-i)!(n-2i)!} \text{tp}_{2i} \wedge (\sigma \overline{\sigma})^{k-i}.
\]

Applying the Hodge–Riemann bilinear relation to the decomposition in Theorem 1.4 will give a good estimate to \( \int \text{td}(X) \exp(\sigma + \overline{\sigma}) \), which proves Theorem 1.1. Also this decomposition can recover known results due to Hitchin and Sawon [9] and Nieper-Wißkirchen [19] (see Corollary 4.6). Meanwhile, this result might be also interesting for its own sake to help us to study the cohomological structure of hyperkähler manifolds.

We remark that the idea of using the Hodge–Riemann bilinear relation to prove Theorem 1.1 originates from [4], where we used a Lefschetz-type decomposition from [8] for \( c_2(X) \) to show that Theorem 1.1 holds in dimension 6. The decomposition there is given by the projection to the Verbitsky component, that is, the subalgebra \( SH^2(X) \subset H^*(X) \) generated by \( H^2(X) \). However, this method only works for \( c_2(X) \), and is not applicable to higher dimensions, as this decomposition is too coarse and we can not control the orthogonal complements in \( SH^2(X)^\perp \).

In order to prove the decomposition in Theorem 1.4, the key ingredient is to show the following result.

Corollary 1.5 (=Corollary 3.19). Let \( X \) be a hyperkähler manifold and fix a non-zero \( \sigma \in H^0(X, \Omega^2_X) \). Consider 
\[
\Lambda_{\sigma/4}(\text{td}_i^{1/2}) = \frac{1}{\lambda_{\sigma}} \text{td}_i^{1/2} \wedge \overline{\sigma}.
\]

See Section 2.6 for the definition of \( \Lambda_{\sigma/4} \). This result is proved using the Rozansky–Witten theory and the wheeling theorem, following the ideas of Hitchin and Sawon [9] and Nieper-Wißkirchen [19]. The key is to show a formula comparing the \( \Lambda_{\sigma/4} \)-action on Rozansky–Witten classes and the differential operator action on Jacobi diagrams (Theorem 3.16). Such a formula was originally observed by Nieper-Wißkirchen in his thesis [18].

Finally, it is worth-mentioning that during the proof, we get the following by-product. It can be viewed as a counterpart of the result
\[ \int \frac{\text{td}^{1/2}}{2}(X) > 0 \text{ in } [9], \text{ and it might have further applications to the topological structure of hyperkähler manifolds.} \]

**Corollary 1.6** (=Corollary 5.5). *Let X be a hyperkähler manifold of dimension \( 2n > 2 \). Then \( \int \frac{\text{td}^{1/2}}{2}(X) < 1 \).*

This paper is organized as the following. In Section 2, we prepare necessary background knowledge. In Section 3, we briefly recall the Rozansky–Witten theory and prove Theorem 3.16 and Corollary 1.5. In Section 4, we give a Lefschetz-type decomposition of \( \text{td}^{1/2} \) (Theorem 1.4). In Section 5, we study the positivity of the Riemann–Roch polynomials, prove Theorem 1.1 and give various applications.

## 2. Preliminaries

In this section, we collect basic knowledge on hyperkähler manifolds. The readers may refer to [11].

### 2.1. Complex structures.

Let \( X \) be a hyperkähler manifold with a non-zero \( \sigma \in H^0(X, \Omega^2_X) \) and a Kähler form \( \omega \). Then there are 3 complex structures \( I, J, K \) on \( X \) satisfying \( K = IJ = -JI \) and a hyperkähler metric \( g \) compatible with all of them with corresponding Kähler forms \( \omega = \omega_I, \omega_J, \omega_K \). Up to a scalar, we may assume that \( \sigma = \omega_J + \sqrt{-1} \omega_K \).

### 2.2. Beauville–Bogomolov–Fujiki form and Riemann–Roch polynomial.

Beauville [1], Bogomolov [2], and Fujiki [6] proved that there exists a quadratic form \( q_X : H^2(X, \mathbb{R}) \rightarrow \mathbb{R} \) and a constant \( c_X \in \mathbb{Q}^+ \) such that for all \( \alpha \in H^2(X, \mathbb{R}) \),

\[ \int \alpha^{2n} = c_X \cdot q_X(\alpha)^n. \]

The above equation determines \( c_X \) and \( q_X \) uniquely if assuming:

1. \( q_X \) is a primitive integral quadratic form on \( H^2(X, \mathbb{Z}) \);
2. \( q_X(\sigma + \overline{\sigma}) > 0 \) for \( 0 \neq \sigma \in H^{2,0}(X) \).

Here \( q_X \) and \( c_X \) are called the Beauville–Bogomolov–Fujiki form and the Fujiki constant of \( X \) respectively.

Recall the following important result by Fujiki [3] (see also [7, Corollary 23.17] for a generalization).

**Theorem 2.1** ([3], [7] Corollary 23.17). *Let \( X \) be a hyperkähler manifold of dimension \( 2n \). Assume that \( \alpha \in H^{4j}(X, \mathbb{R}) \) is of type \( (2j, 2j) \) on all small deformations of \( X \). Then there exists a constant \( C(\alpha) \in \mathbb{R} \) depending only on \( \alpha \) such that*

\[ \int \alpha \beta^{2n-2j} = C(\alpha) \cdot q_X(\beta)^{n-j} \]

*for all \( \beta \in H^2(X, \mathbb{R}) \).*
A direct application of this result (cf. [11, 1.11]) is that, for a line bundle \(L\) on \(X\), the Hirzebruch–Riemann–Roch formula gives

\[
\chi(X, L) = \sum_{i=0}^{n} \frac{1}{(2i)!} \int \text{td}_{2n-2i}(X)(c_1(L))^{2i} = \sum_{i=0}^{n} \frac{a_{2i}}{(2i)!} q_X(c_1(L))^i,
\]

where

\[
a_{2i} = C(\text{td}_{2n-2i}(X)).
\]

The polynomial \(RR_X(q) := \sum_{i=0}^{n} \frac{a_{2i}}{(2i)!} q^i\) is called the Riemann–Roch polynomial of \(X\).

### 2.3. Characteristic values.

For a hyperkähler manifold, the characteristic value is defined by Nieper-Wißkirchen, which is a quadratic form proportional to \(q_X\). This quadratic form is more convenient than \(q_X\) when playing with Rozansky–Witten classes and Riemann–Roch polynomials.

**Definition 2.2** ([19, Definition 17]). Let \(X\) be a hyperkähler manifold. For any \(\alpha \in H^2(X, \mathbb{R})\), Nieper-Wißkirchen defined the characteristic value of \(\alpha\),

\[
\lambda(\alpha) := \begin{cases} 
\frac{24n \int \exp(\alpha)}{\int c_2(X) \exp(\alpha)} & \text{if well-defined;} \\
0 & \text{otherwise.}
\end{cases}
\]

For simplicity, we often denote \(\lambda_{\sigma} := \lambda(\sigma + \sigma)\).

**Proposition 2.3** (cf. [19, Proposition 10]). \(\lambda(\alpha)\) is a positive constant multiple of \(q_X(\alpha)\), more precisely,

\[
\lambda(\alpha) = \frac{12c_X}{(2n-1)c_2(X)} q_X(\alpha).
\]

Note that to study \(RR_X\), we may always view it as a polynomial in terms of \(\lambda\). Here we remark that this multiple is positive (i.e., \(C(c_2(X)) > 0\)) by Yau’s solution to Calabi’s conjecture [30] (cf. [20, Proposition 3.11]).

Recall that a line bundle \(L\) on a projective manifold \(X\) is said to be nef if \(L \cdot C \geq 0\) for any curve \(C \subset X\), moreover, it is said to be big if \(L_{\text{dim}X} > 0\). We have the following easy lemma.

**Lemma 2.4** (cf. [11, 1.10]). Let \(X\) be a hyperkähler manifold and fix a non-zero \(\sigma \in H^0(X, \Omega_X^2)\). Let \(L\) be a line bundle on \(X\).

1. If \(X\) is projective and \(L\) is nef and big, then \(q_X(c_1(L)) > 0\) and \(\lambda(L) > 0\).
2. \(\lambda_{\sigma} = \lambda(\sigma + \sigma) > 0\).
2.4. Todd genus and Todd classes. Let $X$ be a hyperkähler manifold. It is well-known that all its odd Chern classes vanish. The Todd genus of $X$ (see [19, (4.13)]) can be defined by

$$
\text{td}(X) = \exp \left( -2 \sum_{k=1}^{\infty} b_{2k}(2k)! \text{ch}_{2k}(X) \right),
$$

where $\text{ch}_{2k}(X)$ are the Chern characters of $X$ and $b_{2k}$ are the modified Bernoulli numbers defined by

$$
\sum_{k=0}^{\infty} b_{2k} x^{2k} = \frac{1}{2} \ln \frac{\sinh(x/2)}{x/2}.
$$

The square root of the Todd genus of $X$ is defined by

$$
\text{td}^{1/2}(X) = \exp \left( -\sum_{k=1}^{\infty} b_{2k}(2k)! \text{ch}_{2k}(X) \right)
$$

which satisfies $(\text{td}^{1/2}(X))^2 = \text{td}(X)$. We will use $\text{td}^{1/2}_{2k} = \text{td}^{1/2}_{2k}(X)$ to denote the $2k$-th term of $\text{td}^{1/2}(X)$. For example, $\text{td}^{1/2}_0 = 1$, $\text{td}^{1/2}_2 = \frac{1}{2} \text{td}_2 = \frac{1}{24} c_2(X)$, $\text{td}^{1/2}_4 = \frac{1}{5760} (7c_2^2(X) - 4c_4(X))$.

Here we remark that, for hyperkähler manifolds, rational Chern classes are determined by rational Pontrjagin classes (cf. [20, Proposition 1.13]), hence rational Chern classes (and hence Todd classes) are topological invariants of $X$. In particular, rational Chern classes are independent of complex structures.

2.5. Some linear algebra on symplectic forms. Let $k$ be a field of characteristic zero, $V$ a $k$-vector space, and $A$ a $k$-algebra.

An element $\sigma \in \bigwedge^2 V^*$ is called a symplectic form on $V$ if it defines a non-degenerate bilinear form on $V$. Note that if a vector space $V$ admits a symplectic form $\sigma$, then its dimension is even, say $2n$. In this case we can always choose a symplectic basis $e_1, \ldots, e_{2n}$ of $V$ such that $\sigma = \sum_{i=1}^{n} \vartheta^{2i-1} \wedge \vartheta^{2i}$, where $\vartheta^1, \ldots, \vartheta^{2n}$ is the corresponding dual basis of $V^*$.

**Definition 2.5.** Let $V$ be a $k$-vector space of dimension $2n$ admitting a symplectic form $\sigma$. The **contraction** by $\sigma$ is the map $\delta : \bigwedge V^* \otimes A \to \bigwedge V^* \otimes A$ defined by

$$
\delta((\alpha_1 \wedge \cdots \wedge \alpha_l) \otimes a) = \sum_{r=1}^{n} \left( \sum_{1 \leq s < t \leq l} (-1)^{s+t-1} (\alpha_s(e_{2r-1}) \alpha_t(e_{2r}) - \alpha_s(e_{2r}) \alpha_t(e_{2r-1})) \cdot \alpha_1 \wedge \cdots \wedge \widehat{\alpha_s} \wedge \cdots \wedge \widehat{\alpha_t} \wedge \cdots \wedge \alpha_l \right) \otimes a
$$

for $\alpha_1, \ldots, \alpha_l \in V^*$ and $a \in A$. Here $e_1, \ldots, e_{2n}$ is a symplectic basis of $V$. 
Note that we can regard $\sigma$ as an operator $\sigma : \bigwedge V^* \otimes A \to \bigwedge V^* \otimes A$ by taking wedge product with $\sigma$, and consider the operator $\Pi : \bigwedge V^* \otimes A \to \bigwedge V^* \otimes A$ acting on $\bigwedge^p V^* \otimes A$ by multiplying with $p - n$.

**Proposition 2.6.** Keep the above setting. Then $(\sigma, \delta, \Pi)$ gives an $\mathfrak{sl}_2$-action on $\bigwedge V^* \otimes A$. Namely,

$$[\sigma, \delta] = \Pi, \quad [\Pi, \sigma] = 2\sigma, \quad [\Pi, \delta] = -2\delta.$$

**Proof.** The proof is standard. To avoid tedious computations, we illustrate by the following example: it is easy to see that for any $a \in A$,

$$\delta(\sigma \otimes a) = \delta \left( \sum_{i=1}^{n} \varphi^{2i-1} \wedge \varphi^{2i} \otimes a \right) = n \otimes a,$$

hence $[\sigma, \delta](1 \otimes a) = -\delta(\sigma \otimes a) = -n \otimes a = \Pi(1 \otimes a)$. \hfill $\square$

### 2.6. An $\mathfrak{sl}_2$-action on the cohomology of a hyperkähler manifold

The cohomology of a hyperkähler manifold has been studied by many authors, see for example [3, 16, 29]. In particular, there is a natural $\mathfrak{so}(4,1)$-action on the cohomology of a hyperkähler manifold by [28]. In this paper, we mainly focus on a special $\mathfrak{sl}_2$-action induced by $\sigma$, which has been considered by Fujiki [3], Huybrechts [10], and Nieper-Wißkirchen [18].

Let $X$ be a hyperkähler manifold of dimension $2n$ and fix a non-zero $\sigma \in H^0(X, \Omega_X^2)$. After a rescaling we may assume that $\sigma = \omega_J + \sqrt{-1}\omega_K$ as in Section 2.1. For any $0 \leq p, q \leq 2n$, let $L_\sigma : H^q(X, \Omega_X^p) \to H^q(X, \Omega_X^{p+2})$ be the Lefschetz operator giving by the cup-product with $\sigma$. Define $\Lambda_{\sigma/4} = \ast^{-1} \circ L_{\sigma/4} \circ \ast$ where $\ast$ is the Hodge operator associated to the Kähler metric $g$ compatible with the hyperkähler structure of $X$, and define $\Pi : H^q(X, \Omega_X^p) \to H^q(X, \Omega_X^p)$ to be the map multiplying by $(p - n)$. Then we have

$$[L_\sigma, \Lambda_{\sigma/4}] = \Pi, \quad [\Pi, L_\sigma] = 2L_\sigma, \quad [\Pi, \Lambda_{\sigma/4}] = -2\Lambda_{\sigma/4},$$

and hence $(L_\sigma, \Lambda_{\sigma/4}, \Pi)$ gives an $\mathfrak{sl}_2$-action on $H^*(X, \Omega_X^*)$ (cf. [10] Proof of Theorem 6.3]).

**Remark 2.7.** There is a natural local interpretation of this $\mathfrak{sl}_2$-action as the following: fix a point $x \in X$, consider $V = T_{x,X}$ the holomorphic tangent space and $A = \bigwedge \Omega_{x,X}$, note that $\sigma_x$ is a symplectic form on $V$, then we can consider operators $(\sigma_x, \delta_x, \Pi_x)$ acting on $\bigwedge V^* \otimes A$ as in Definition 2.5 and Proposition 2.6, which, on the level of cohomology, induce exactly $(L_\sigma, \Lambda_{\sigma/4}, \Pi)$ acting on $H^*(X, \Omega_X^*)$.

**Definition 2.8** (cf. [6]). A class $\alpha \in H^*(X, \Omega_X^k)$ is called $\sigma$-primitive if $\Lambda_{\sigma/4}(\alpha) = 0$, which is equivalent to $L_\sigma^{p-k+1}(\alpha) = 0$.

The following lemma is standard.

**Lemma 2.9.** $[L_\sigma, L_{\sigma}] = [\Lambda_{\sigma/4}, L_{\sigma}] = 0$. 

Proof. The first one is trivial. Let us consider the second one. We may assume that 
\[ \sigma = \omega_j + \sqrt{-1}\omega_K \] as in Section 2.1. Then by definition,
\[ L_{\sigma} = L_{\omega_j} - \sqrt{-1}L_{\omega_K}, \quad \Lambda_{\sigma/4} = \frac{1}{4}(\Lambda_{\omega_j} - \sqrt{-1}\Lambda_{\omega_K}). \]

Then the conclusion follows immediately from [28, (2.1)]. □

The following lemma is standard by the representation theory of \( \mathfrak{sl}_2 \), see for example [13, Corollary 1.2.28].

Lemma 2.10. For \( \alpha \in H^*(X, \Omega^k_X) \) and \( m \geq 1 \),
\[ [L_{\sigma}^m, \Lambda_{\sigma/4}](\alpha) = m(k - n + m - 1)L_{\sigma}^{m-1}(\alpha). \]

In particular, if moreover \( \alpha \) is \( \sigma \)-primitive, then
\[ \Lambda_{\sigma/4}L_{\sigma}^m(\alpha) = m(n + 1 - k - m)L_{\sigma}^{m-1}(\alpha). \]

3. The Rozansky–Witten theory

For a hyperkähler manifold \( X \) with a non-zero \( \sigma \in H^0(X, \Omega^k_X) \), the Rozansky–Witten theory associates to every Jacobi diagram \( \Gamma \) to a cohomology class \( \text{RW}_\sigma(\Gamma) \), which is due to Rozansky and Witten [25] and later developed by Kapranov [14]. Later Hitchin and Sawon [9] and Nieper-Wißkirchen [19] discovered that this is a powerful tool to study the characteristic classes of hyperkähler manifolds.

The main goal of this section is to apply the Rozansky–Witten theory to prove Corollary 3.19, which calculates \( \Lambda_{\sigma/4}(\text{td}^{1/2}_{2k}) \) for a hyperkähler manifold. In order to explain the proof, we will briefly recall basic knowledge of the Rozansky–Witten theory, the readers may refer to [26, 18, 19, 20] for details. Most of the contents in this section are from [19], while Proposition 3.12 and Theorem 3.16 are originally claimed by Nieper-Wißkirchen in [18], and we provide a self-contained proof for the reader’s convenience.

3.1. The graph homology space. A graph is a collection of vertices connected by edges, where every edge connects 2 vertices. A flag or a half-edge is an edge together with an adjacent vertex. So every edge consists of exactly 2 flags, and every flag belongs to exactly 1 vertex. Note that an edge or a vertex can be identified with the set of flags belonging to it. A vertex is called univalent, if there is only 1 flag belonging to it, and it is called trivalent, if there are exactly 3 flags belonging to it. A graph is called vertex-oriented if, for every vertex, a cyclic ordering of its flags is fixed.

Definition 3.1 (Jacobi diagram). A Jacobi diagram is a vertex-oriented graph with only univalent and trivalent vertices. A trivalent Jacobi diagram is a Jacobi diagram with no univalent vertices. The degree of a Jacobi diagram is the number of its vertices.
When we draw a Jacobi diagram as a planar graph, we want the counter-clockwise ordering of the flags at each trivalent vertex in the drawing to be the same as the given cyclic ordering.

**Example 3.2.** (1) The empty graph is a Jacobi diagram, which is denoted by 1.

(2) The Jacobi diagram consisting of 2 univalent vertices connecting by 1 edge is denoted by $\ell$, and called a *strut*.

(3) The Jacobi diagram $\Theta$ consisting of 2 trivalent vertices connecting by 3 edges is denoted by $\Theta$.

(4) For each positive integer $k$, the $2k$-wheel $w_{2k}$ is a Jacobi diagram defined to be a closed path with $2k$ vertices and $2k$ edges, while every vertex has a third edge outside the closed path. So it contains $2k$ trivalent vertices and $2k$ univalent vertices. For example, the 8-wheel $w_8$ looks like $\circledast$.

**Definition 3.3 (Graph homology space).** The space $\mathcal{B}$ is defined to be the $\mathbb{Q}$-vector space spanned by all Jacobi diagrams modulo the IHX relation and the anti-symmetry (AS) relation (see [27, 19] for definitions). The space $\mathcal{B}'$ is defined to be the subspace of $\mathcal{B}$ spanned by all Jacobi diagrams not containing $\ell$ as a component. The space $\mathcal{B}^t$ is defined to be the subspace of $\mathcal{B}$ spanned by all trivalent Jacobi diagrams. These spaces are graded by degrees, and bi-graded by the numbers of trivalent and univalent vertices. We denote $\mathcal{B}_{k,l}$ to be the homogeneous part of $\mathcal{B}$ generated by Jacobi diagrams with $k$ trivalent and $l$ univalent vertices. The completion of $\mathcal{B}$ (resp. $\mathcal{B}'$, $\mathcal{B}^t$) with respect to the grading is denoted by $\hat{\mathcal{B}}$ (resp. $\hat{\mathcal{B}}'$, $\hat{\mathcal{B}}^t$).

There are 2 natural operations on the graph homology spaces.

**Definition 3.4 (Disjoint union).** The disjoint union of Jacobi diagrams induces a bilinear map

$$\hat{\mathcal{B}} \times \hat{\mathcal{B}} \to \hat{\mathcal{B}} : (\gamma, \gamma') \mapsto \gamma \gamma' := \gamma \cup \gamma'.$$

By identifying $1 \in \mathbb{Q}$ with $1 \in \hat{\mathcal{B}}$, this gives a natural graded $\mathbb{Q}$-algebra structure of $\hat{\mathcal{B}}$.

**Definition 3.5 (Differential operator).** There is a differential operator $\partial : \hat{\mathcal{B}}' \to \hat{\mathcal{B}}'$ defined by

$$\partial \Gamma = \sum_{\{u,v\} \subseteq U} \Gamma/\{u,v\}$$

for every Jacobi diagram $\Gamma$. Here $U$ is the set of univalent vertices of $\Gamma$, and $\Gamma/\{u,v\}$ is the Jacobi diagram obtained by removing vertices $\{u,v\}$ and gluing 2 edges belonging to $u,v$ into a new edge. Here $\Gamma/\{u,v\}$ admits a natural orientation from $\Gamma$ as trivalent vertices remain unchanged. In other words, the action of $\partial$ on a Jacobi diagram
means to glue 2 of its univalent vertices in all possible ways. Note that $\partial : \hat{B}' \rightarrow \hat{B}'$ is a $\hat{B}'$-linear map.

**Example 3.6.** $\partial w_2 = \Theta$.

**Definition 3.7 (Wheeling element).** The wheeling element $\Omega \in \hat{B}$ is defined via the expression

$$\Omega = \exp \left( \sum_{k=1}^{\infty} b_{2k} w_{2k} \right)$$

using the graded $\mathbb{Q}$-algebra structure of $\hat{B}$, where $b_{2k}$ are the modified Bernoulli numbers as in Section 2.4. We may write $\Omega = \sum_{k=0}^{\infty} \Omega_{2k}$ where $\Omega_{2k}$ is the homogeneous component of degree $4k$ of $\Omega$. For example, $\Omega_0 = 1$, $\Omega_2 = \frac{1}{48} w_2$.

The wheeling element $\Omega$ has an important property called the wheeling theorem (see [27]) by the knot theory. The method of combining the wheeling theorem with Rozansky–Witten classes to deal with characteristic classes of hyperkähler manifolds was discovered by Hitchin and Sawon [9] and later generalized by Nieper-Wißkirchen [19]. As observed by Nieper-Wißkirchen, all we need is the following special case.

**Theorem 3.8 ([27, Lemma 6.2], [19, Theorem 3.1]).** As elements in $\hat{B}$, $\partial \Omega = \frac{\Theta}{48} \Omega$.

An elementary proof can be found in [19].

3.2. **Rozansky–Witten classes in general setting.** Let $k$ be a field of characteristic zero, $V$ a finite-dimensional $k$-vector space, $A = \bigoplus_{i=0}^{\infty} A_i$ a skew-commutative $\mathbb{Z}$-graded $k$-algebra, and $\sigma$ a symplectic form on $V$. We will apply this general setting later to the case that $V = T_{X,x}$ the holomorphic tangent space and $A = \bigwedge \overline{T}_{X,x}$, where $x \in X$ is a point on a hyperkähler manifold $X$.

For every Jacobi diagram $\Gamma$ with $k$ trivalent and $l$ univalent vertices and every $\alpha \in \text{Sym}^3 V \otimes A_1$, we define an element

$$\text{RW}_{\sigma,\alpha}(\Gamma) \in \bigwedge^l V^* \otimes A_k$$

as the following (see [14], [19, Section 4.1]).

Denote $T$ to be the set of trivalent vertices of $\Gamma$, denote $U$ to be the set of univalent vertices of $\Gamma$, and denote $F$ to be the set of flags of $\Gamma$. Recall that an edge is identified with 2 flags belonging to it, and a vertex is identified with the set of its flags. We can label the set $F = \{1, 2, \ldots, 3k + l - 1, 3k + l\}$ such that the set of edges are just $E = \{\{1, 2\}, \ldots, \{3k + l - 1, 3k + l\}\}$. Identifying vertices with sets of flags, we may write the set of univalent vertices as $U = \{u_1, \ldots, u_l\} \subset F$, and write the set of trivalent vertices as

$$T = \{\{t_1, t_2, t_3\}, \ldots, \{t_{3k-2}, t_{3k-1}, t_{3k}\}\} \subset 2^F,$$
where the ordering \( \{ t_{3i-2}, t_{3i-1}, t_{3i} \} \) coincides with the orientation of \( \Gamma \) (which is a given cyclic ordering for each trivalent vertex). Note that we have the relation

\[
\bigcup_{t \in T} t \cup \bigcup_{u \in U} u = \bigcup_{e \in E} e = F.
\]

Divide \( U = U' \cup U'' \) where \( U' = \{ u'_1, \ldots, u'_{l_1} \} \) consists of univalent vertices connected to trivalent vertices, and \( U'' = \{ u''_1, \ldots, u''_{2l_2} \} \) consists of univalent vertices contained in some component \( \ell \) of \( \Gamma \). Here \( l = l_1 + 2l_2 \) and \( \Gamma \) has exactly \( l_2 \) copies of \( \ell \) as connected components.

We may assume that for \( 1 \leq j \leq l_2 \), \( (u''_{2j-1}, u''_{2j}) = (3k + l_1 + 2j - 1, 3k + l_1 + 2j) \), that is, in the above ordering, the last \( l_2 \) edges correspond to the components \( \ell^{l_2} \subset \Gamma \). Note that for \( 1 \leq i \leq l_1 \), we have \( 1 \leq u'_i \leq 3k + l_1 \), so we may further assume that each \( u'_i \) is even, that is, the flag belonging to it takes the second position in the ordering on the corresponding edge, which is just \( \{ u'_1 - 1, u'_1 \} \).

We may choose the labelling of \( F \) properly such that

\[
\bigwedge_{1 \leq j \leq 3k + l_1, j \neq u'_{1}, \ldots, u'_{l_1}}^3 f_j = f_{t_1} \wedge \cdots \wedge f_{t_{3k}}.
\]

in \( \wedge (k^{3k}) \) where \( \{ f_{t_j} \}_{j=1}^{3k} \) is a basis of \( k^{3k} \), and this condition is called the compatibility with the orientation of \( \Gamma \). See Figure 1 for an example of \( \w_2 \cup \ell \) with \( k = 2, l = 4 \).

**Remark 3.9.**

1. If we ignore the compatibility of orientation, then the Rozansky–Witten invariants are only defined up to a sign.

2. The compatibility condition (3.1) we state here is different from the one in [19, (3.3)] up to a sign \((-1)^{l_1(l_1-1)/2}\) because we ignore the contribution from univalent vertices. In fact, the referee reminded the author that in [19, (4.2)], each isomorphism \( V \otimes A_1 \to A_1 \otimes V \) introduces a sign change, and so the definition in this paper is indeed the same as that in [19].

3. If we glue \( u'_s \) and \( u'_t \) in \( U' \) (\( 1 \leq u'_s < u'_t \leq 3k + l_1 \)), then we get \( \Gamma_{s,t} := \Gamma / \{ u'_s, u'_t \} \) as in Definition 3.5. By the assumption, the opposite flags corresponding to the edges containing \( u'_s, u'_t \) are \( u'_s - 1, u'_t - 1 \) respectively. Note that the set of flags \( F_{s,t} \) (resp. the set of univalent vertices \( U_{s,t} \)) of \( \Gamma_{s,t} \) is \( F \) (resp. \( U \))
removing \( u'_s, u'_t \), and the set of edges \( E_{s,t} \) of \( \Gamma_{s,t} \) is \( E \) removing \( \{u'_s - 1, u'_s\}, \{u'_t - 1, u'_t\} \) and adding \( \{u'_s - 1, u'_t - 1\} \) as a new edge after the edge \( \{3k + l_1 - 1, 3k + l_1\} \). Then the ordering of \( F_{s,t} \) induced from \( E_{s,t} \) satisfies the compatibility with the orientation of \( \Gamma_{s,t} \) up to a sign \((-1)^{s+t-1} \). This is because from Definition 3.10.

With the above notation, for every Jacobi diagram \( s \), we define the element

\[
\Phi_\sigma(s) := \prod_{j=1}^{(3k+t)/2} \sigma(v_{2j-1}, v_{2j}) \cdot (\alpha_1 \wedge \cdots \wedge \alpha_t) \otimes (a_1 \cdot \cdots \cdot a_k).
\]

Roughly speaking, \( s \) assigns to every flag an element in \( V \), and \( \Phi_\sigma \) contracts every two elements on the same edge by \( \sigma \) in a way compatible with the orientation of \( \Gamma \). This in fact extends to a linear map

\[
\Phi_\sigma : (\text{Sym}^3 V \otimes A_1)^{\otimes |T|} \otimes \text{End}(V)^{\otimes |U|} \to \bigwedge^1 V^* \otimes A_k.
\]

This definition can be extended to \( \Phi_\gamma \) for any \( \mathbb{Q} \)-linear combinations of Jacobi diagrams with \( k \) trivalent and \( l \) univalent vertices \( \gamma \) by linearity.

**Definition 3.10.** With the above notation, for every Jacobi diagram \( \Gamma \) with \( k \) trivalent and \( l \) univalent vertices and every \( \alpha \in \text{Sym}^3 V \otimes A_1 \), we define the element

\[
\text{RW}_{\sigma,\alpha}(\Gamma) := \Phi_\Gamma(\alpha^{\otimes k} \otimes (\text{id}_V)^{\otimes l}) \in \bigwedge^l V^* \otimes A_k.
\]
This definition can be extended to $\text{RW}_{\sigma,\alpha}(\gamma) \in \bigwedge V^* \otimes A$ for any $\mathbb{Q}$-linear combinations of Jacobi diagrams $\gamma$ by linearity.

**Example 3.11** (cf. [19, Proposition 3, (4.5)]). Let us compute $\Phi^{w_{2k}}$. We fix the labeling of flags as the following: the $2k$ univalent vertices are labelled as $U = \{2, 4, 6, \ldots, 4k\}$ counter-clockwisely, the $2k$ trivalent vertices are labelled as $T = \{\{1, 4k + 1, 8k\}, \{3, 4k + 3, 4k + 2\}, \{5, 4k + 5, 4k + 4\}, \ldots, \{4k - 1, 8k - 1, 8k - 2\}\}.$

Here $\{2i - 1, 2i\}$ is the edge from a trivalent vertex to a univalent vertex for $1 \leq i \leq 2k$. Then the set of edges is just $E = \{\{1, 2\}, \ldots, \{8k - 1, 8k\}\}.$

See Figure 3. Note that this given labeling is compatible with the orientation of $w_{2k}$ up to a sign $(-1)$, as

$$f_1 \wedge f_3 \wedge \cdots \wedge f_{4k-1} \wedge f_{4k+1} \wedge \cdots \wedge f_{8k}$$

$$= - (f_1 \wedge f_{4k+1} \wedge f_{8k}) \wedge \cdots \wedge (f_{4k-1} \wedge f_{8k-1} \wedge f_{8k-2})$$

in $\bigwedge (k^\otimes 6k)$ where $\{f_i\}$ is a basis of $k^\otimes 6k$ (cf. [26, Page 36]). One can compare this labelling with the labelling of $w_2$ in Figure 1 to feel the difference.

Then for an element in $(\text{Sym}^3 V \otimes A_1)^\otimes 2k \otimes \text{End}(V)^\otimes 2k$ of the form

$$s = \bigotimes_{i=1}^{2k} (w_i^3 \otimes a_i) \otimes \bigotimes_{j=1}^{2k} (w'_j \otimes \alpha_j),$$

where $w_i \in V$, $a_i \in A_1$, $w'_j \in V$, and $\alpha_j \in V^*$ for each $1 \leq i, j \leq 2k$, we have

$$\Phi^{w_{2k}}(s) = - \prod_{i=1}^{2k} \left( \sigma(w_i, w_{i+1}) \sigma(w_i, w'_j) \right) \cdot \bigwedge_{i=1}^{2k} \alpha_i \otimes \prod_{i=1}^{2k} a_i.$$
Properties of the map $\Phi^\Gamma$ are summarized in [19, Proposition 3]. Here we introduce a property originally observed by Nieper-Wißkirchen [18, Proposition 3.12].

**Proposition 3.12** (cf. [18 (85)]). Keep the above notation. Fix a Jacobi diagram $\Gamma$ with $k$ trivalent and $l$ univalent vertices, suppose moreover that $[\Gamma] \in B'$. Then for any $\beta \in (\text{Sym}^3 V \otimes A_1)^{\otimes k}$,

$$\Phi^\partial(\beta \otimes (\text{id}_V)^{\otimes (l-2)}) = \delta(\Phi^\Gamma(\beta \otimes (\text{id}_V)^{\otimes l})).$$

In particular, $\text{RW}_{\sigma,\alpha}(\partial \Gamma) = \delta(\text{RW}_{\sigma,\alpha}(\Gamma))$ for any $\alpha \in \text{Sym}^3 V \otimes A_1$. Here $\delta$ is the contraction map in Definition 2.3.

**Proof.** As in the construction, we may write the set of flags $F$ of $\Gamma$ by $\{1, 2, \ldots, 3k + l - 1, 3k + l\}$ such that the set of edges $E$ are just $\{\{1, 2\}, \ldots, \{3k + l - 1, 3k + l\}\}$. Identifying vertices with sets of flags, we may write the set of univalent vertices $U$ as $\{u_1, \ldots, u_l\} \subset F$, and write the set of trivalent vertices $T$ as

$$\{\{t_1, t_2, t_3\}, \ldots, \{t_{3i-2}, t_{3i-1}, t_{3i}\}\} \subset 2^E,$$

where the ordering $\{t_{3i-2}, t_{3i-1}, t_{3i}\}$ coincides with the orientation of $\Gamma$. Note that $U = U'$ and $l = l'$ by the assumption that $[\Gamma] \in B'$. Without loss of generality, we may assume that for $1 \leq i \leq l$, $(t_{3i-2}, u_i) = (2i - 1, 2i)$, that is, in the above ordering, the $i$-th univalent vertex is connected to the $i$-th trivalent vertex via the $i$-th edge.

Without loss of generality, we may assume that the element $\beta \in (\text{Sym}^3 V \otimes A_1)^{\otimes k}$ is of the form

$$\beta = \bigotimes_{i=1}^k (w_i^3 \otimes a_i)$$

where $v_{3i-1} = v_{3i-2} = v_{3i} = w_i \in V$ and $a_i \in A_1$ for each $i$. Choose a symplectic basis $e_1, \ldots, e_{2n}$ of $V$ such that

$$\sigma = \sum_{i=1}^n \vartheta^2i - 1 \wedge \vartheta^{2i},$$

where $\vartheta^1, \ldots, \vartheta^{2n}$ is the corresponding dual basis of $V^*$. Then $\text{id}_V = \sum_{m=1}^{2n} e_m \otimes \vartheta^m$ via $\text{End}(V) \simeq V \otimes V^*$. Then

$$\Phi^\Gamma(\beta \otimes (\text{id}_V)^{\otimes l})$$

$$= \Phi^\Gamma \left( \beta \otimes \left( \sum_{m=1}^{2n} e_m \otimes \vartheta^m \right)^{\otimes l} \right)$$

$$= \sum_{1 \leq m_1, \ldots, m_l \leq 2n} \left( \prod_{i=1}^l \sigma(v_{2i-1}, e_{m_i}) \cdot \prod_{i'=l+1}^{(3k+l)/2} \sigma(v_{2i'-1}, v_{2i'}) \cdot (\vartheta^{m_1} \wedge \cdots \wedge \vartheta^{m_l}) \otimes (a_1 \cdot \cdots \cdot a_k) \right).$$
\[= \bigwedge_{i=1}^{l} \left( \sum_{m=1}^{2n} \sigma(v_{2i-1}, e_m) \vartheta^m \right) \otimes \left( \prod_{i'=l+1}^{(3k+l)/2} \sigma(v_{2i'-1}, v_{2i'}) \cdot (a_1 \cdots a_k) \right) \]

\[= \bigwedge_{i=1}^{l} \left( \sum_{m=1}^{2n} \sigma(v_{2i-1}, e_m) \vartheta^m \right) \otimes N. \]

Here we denoted \( N = \prod_{i'=l+1}^{(3k+l)/2} \sigma(v_{2i'-1}, v_{2i'}) \cdot (a_1 \cdots a_k) \). Then

\[\delta(\Phi^\Gamma(\beta \otimes (id_V)^{\otimes l}))\]

\[= \sum_{r=1}^{n} \sum_{s \leq r \leq t} \left( \sum_{1 \leq m \leq 2n} \sum_{i \neq s, t} \lambda_{1 \leq i \leq l} \sum_{m=1}^{2n} \sigma(v_{2i-1}, e_m) \vartheta^m \right) \otimes N. \]

Here we used the fact that

\[\sigma(v_{2s-1}, v_{2t-1}) = \sum_{r=1}^{n} \left( \sigma(v_{2s-1}, e_{2r-1}) \sigma(v_{2t-1}, e_{2r}) - \sigma(v_{2s-1}, e_{2r}) \sigma(v_{2t-1}, e_{2r-1}) \right). \]

On the other hand, denote \( \Gamma_{s,t} = \Gamma/\{u_s, u_t\} \) for \( \{u_s, u_t\} \subset U \), note that

\[\partial \Gamma = \sum_{s \leq t} \Gamma_{s,t}.\]

Here as the above notation, \( (u_s, u_t) = (2s, 2t) \). Now we compute \( \Phi^{\Gamma_{s,t}}(\beta \otimes (id_V)^{\otimes (l-2)}) \). Note that the set of flags (resp. univalent vertices) of \( \Gamma_{s,t} \) is \( F \) (resp. \( U \)) removing \( 2s, 2t \), the set of edges of \( \Gamma_{s,t} \) is \( E \) removing \( \{2s-1, 2s\}, \{2t-1, 2t\} \) and adding \( \{2s-1, 2t-1\} \) as a new edge to the end. So we may compute

\[\Phi^{\Gamma_{s,t}}(\beta \otimes (id_V)^{\otimes (l-2)}) = \Phi^{\Gamma_{s,t}} \left( \beta \otimes \left( \sum_{m=1}^{2n} \epsilon_{m} \otimes \vartheta^m \right)^{\otimes (l-2)} \right) \]

\[= \sum_{1 \leq m_i \leq 2n, \ \sum_{i \neq s, t} \lambda_{1 \leq i \leq l} \sum_{m=1}^{2n} \sigma(v_{2i-1}, e_m) \cdot \prod_{i'=l+1}^{(3k+l)/2} \sigma(v_{2i'-1}, v_{2i'}) \cdot (a_1 \cdots a_k) \right) \otimes N. \]

Here the sign \( (-1)^{s+t-1} \) comes from the compatibility of the orientation of \( \Gamma_{s,t} \) (Remark 3.9(3)).

To conclude, we get

\[\Phi^{\partial \Gamma}(\beta \otimes (id_V)^{\otimes (l-2)}) = \sum_{1 \leq s \leq t \leq l} \Phi^{\Gamma_{s,t}}(\beta \otimes (id_V)^{\otimes (l-2)}) \]
= \delta\left(\Phi^\Gamma(\beta \otimes (\text{id}_V)^{\otimes l})\right)

by comparing the above computations.

3.3. **Rozansky–Witten classes of hyperkähler manifolds.** Let \(X\) be a hyperkähler manifold and fix a non-zero \(\sigma \in H^0(X, \Omega^2_X)\). Denote \(A^k(X, E)\) to be the space of \((0,k)\)-forms with values in a holomorphic vector bundle \(E\), and set \(A^{l,k}(X) := A^k(X, \Omega^l_X)\). Denote \(\alpha_X\) to be a Dolbeault representative of the Atiyah class of \(X\). Recall that according to Kapranov [14], \(\alpha_X \in A^1(X, \text{Sym}^3 T_X)\) when we identify \(T_X\) with \(\Omega_X\) via \(\sigma\).

**Definition 3.13** ([19, Definition 14]). For a Jacobi diagram \(\Gamma\) with \(k\) trivalent and \(l\) univalent vertices, we define \(\text{Rozansky–Witten class} \ \text{RW}_\sigma(\Gamma) \in H^k(X, \Omega^l_X)\) to be the Dolbeault cohomology class of the \((\bar{\partial})\)-closed \((l,k)\)-form

\[
\left(\frac{\sqrt{-1}}{2\pi}\right)^k \cdot (x \mapsto \text{Rozansky–Witten class} \ \text{RW}_\sigma(\gamma_x)(\Gamma)) \in A^{l,k}(X).
\]

This definition can be extended to \(\text{Rozansky–Witten class} \ \text{RW}_\sigma(\gamma)\) for any \(\mathbb{Q}\)-linear combinations of Jacobi diagrams \(\gamma\) by linearity.

It was proved by Kapranov [14] that \(\text{Rozansky–Witten class} \ \text{RW}_\sigma(\Gamma) = \text{Rozansky–Witten class} \ \text{RW}_\sigma(\Gamma')\) if \([\Gamma] = [\Gamma'] \in \hat{B}\). So we have a linear map \(\text{Rozansky–Witten class} \ \text{RW}_\sigma : \hat{B} \to H^*(X, \Omega^*_X)\) which maps elements of \(\hat{B}_{k,l}\) into \(H^k(X, \Omega^l_X)\). The values of \(\text{Rozansky–Witten class} \ \text{RW}_\sigma\) are called the **Rozansky–Witten classes** of \(X\). In fact, this map preserves \(\mathbb{Q}\)-algebra structures.

**Proposition 3.14** ([19, Proposition 6]). \(\text{Rozansky–Witten class} \ \text{RW}_\sigma : \hat{B} \to H^*(X, \Omega^*_X)\) is a morphism of \(\mathbb{Q}\)-algebras.

For special Jacobi diagrams, we have the following corresponding Rozansky–Witten classes.

**Proposition 3.15** ([19]). Let \(X\) be a hyperkähler manifold and fix a non-zero \(\sigma \in H^0(X, \Omega^2_X)\). Then

1. \(\text{Rozansky–Witten class} \ \text{RW}_\sigma(\ell) = 2\sigma\);
2. \(\text{Rozansky–Witten class} \ \text{RW}_\sigma(\Theta) = \frac{48}{\lambda_\sigma}\sigma\), where \(\lambda_\sigma = \lambda(\sigma + \overline{\sigma}) > 0\) as in Definition 2.2;
3. \(\text{Rozansky–Witten class} \ \text{RW}_\sigma(\omega_{2k}) = -(2k)!\text{ch}_{2k}(X)\);
4. \(\text{Rozansky–Witten class} \ \text{RW}_\sigma(\Omega) = \text{td}^{1/2}(X)\).

**Proof.** (1)–(3) follow from [19] (4.11), (4.16), (4.12) (cf. Remark 3.9(2)). (4) follows from (3) and Proposition 3.14 (cf. [19] (4.14)). □

3.4. **Conclusions.** As a direct application of Proposition 3.12 and Remark 2.7, we have the following theorem originally observed by Nieper-Wiśkirchen [18] (85).

**Theorem 3.16** ([Nieper-Wiśkirchen [18] (85)]). Let \(X\) be a hyperkähler manifold and fix a non-zero \(\sigma \in H^0(X, \Omega^2_X)\). Then for any \(\gamma \in \hat{B}'\),

\[
\text{Rozansky–Witten class} \ \text{RW}_\sigma(\partial \gamma) = \Lambda_{\sigma/4}(\text{Rozansky–Witten class} \ \text{RW}_\sigma(\gamma)).
\]
Example 3.17. Let $X$ be a K3 surface and fix a non-zero $\sigma \in H^0(X, \Omega^2_X)$. We may assume that $\int \sigma \sigma = 1$. Then we have $c_2(X) = 24\sigma \sigma$ and $\lambda_\sigma = \lambda(\sigma + \overline{\sigma}) = 1$. Consider $\gamma = w_2$. Then
$$\text{RW}_\sigma(\partial w_2) = \text{RW}_\sigma(\Theta) = \frac{48}{\lambda_\sigma} \sigma.$$ On the other hand,
$$\Lambda_{\sigma/4}(\text{RW}_\sigma(w_2)) = \Lambda_{\sigma/4}(2c_2(X)) = \Lambda_{\sigma/4}(48\sigma \sigma) = 48\sigma.$$ Hence
$$\text{RW}_\sigma(\partial w_2) = \Lambda_{\sigma/4}(\text{RW}_\sigma(w_2)).$$

Remark 3.18. Here we make a historical remark on Theorem 3.16. In [18, (85)], Nieper-Wißkirchen claimed a formula as in Theorem 3.16 with a different sign. The reason is that his definition of $\delta$ differs from ours (Definition 2.5) by a sign. In fact, Definition 2.5 coincides with [18, Definition 29], but differs from [18, Remark 30] by a sign. So to clarify which sign is the right choice, we decide to include a detailed proof of Proposition 3.12 and Theorem 3.16 in this paper following our sign conventions. In any sense these two results should be credited to Nieper-Wißkirchen who observed this interesting correspondence.

Combining Theorem 3.16 with Theorem 3.8, we get the following consequence, which is crucial in the study of $\text{td}^{1/2}(X)$.

Corollary 3.19. Let $X$ be a hyperkähler manifold and fix a non-zero $\sigma \in H^0(X, \Omega^2_X)$. Then for any integer $k \geq 1$,
$$\Lambda_{\sigma/4}(\text{td}^{1/2}_{2k}) = \frac{1}{\lambda_\sigma} \text{td}^{1/2}_{2k-2} \wedge \sigma.$$ 

Proof. By Proposition 3.15(4), $\text{td}^{1/2}_{2k} = \text{RW}_\sigma(\Omega_{2k})$. By Theorem 3.8 we have $\partial \Omega_{2k} = \frac{\Theta}{48} \Omega_{2k-2}$ by taking the homogenous parts of degree $4k - 2$. So by Theorem 3.16
$$\Lambda_{\sigma/4}(\text{td}^{1/2}_{2k}) = \Lambda_{\sigma/4}(\text{RW}_\sigma(\Omega_{2k})) = \text{RW}_\sigma(\partial \Omega_{2k})$$
$$= \text{RW}_\sigma\left( \frac{\Theta}{48} \Omega_{2k-2} \right) = \text{RW}_\sigma\left( \frac{\Theta}{48} \right) \text{RW}_\sigma(\Omega_{2k-2})$$
$$= \frac{1}{\lambda_\sigma} \text{td}^{1/2}_{2k-2} \wedge \sigma.$$ Here we applied Proposition 3.14 and Proposition 3.15(2). 

Corollary 3.19 can be also written as the following with respect to Kähler forms.

Corollary 3.20. Let $X$ be a hyperkähler manifold with a Kähler form $\omega$. Then for any integer $k \geq 1$,
$$\Lambda_{\omega}(\text{td}^{1/2}_{2k}) = \frac{1}{\lambda(\omega)} \text{td}^{1/2}_{2k-2} \wedge \omega.$$
Proof. Consider \( \sigma = \omega_J + \sqrt{-1}\omega_K \in H^0(X, \Omega^2_X) \) as in Section 2.1. Then Corollary 3.19 says that
\[
\frac{1}{4}(\Lambda_{\omega_J} - \sqrt{-1}\Lambda_{\omega_K}) (\text{td}^{1/2}_{2k}) = \frac{1}{\lambda_{\sigma}} \text{td}^{1/2}_{2k-2} \wedge (\omega_J - \sqrt{-1}\omega_K).
\]
As \( \text{td}^{1/2}_{2k} \) and \( \text{td}^{1/2}_{2k-2} \) are real, we get
\[
\Lambda_{\omega_J}(\text{td}^{1/2}_{2k}) = 4 \frac{1}{\lambda_{\sigma}} \text{td}^{1/2}_{2k-2} \wedge \omega_J = \frac{1}{\lambda(\omega_J)} \text{td}^{1/2}_{2k-2} \wedge \omega_J.
\]
Here we used the fact that \( \lambda(\omega_J) = \lambda(\frac{1}{2}(\sigma + \sigma)) = \frac{1}{4}\lambda_{\sigma} \). Hence the conclusion also holds by replacing \( \omega_J \) with \( \omega \) according to the hyperkähler structure. \( \square \)

**Remark 3.21.** (1) As \( \text{td}^{1/2}_{2k} \) and \( \text{td}^{1/2}_{2k-2} \) are real, by conjugation, Corollary 3.19 also implies that
\[
\Lambda_{\sigma/4}(\text{td}^{1/2}_{2k}) = \frac{1}{\lambda_{\sigma}} \text{td}^{1/2}_{2k-2} \wedge \sigma,
\]
where \( \Lambda_{\sigma/4} \) can be defined similarly as \( \Lambda_{\sigma/4} \) as in Section 2.6. So Corollaries 3.19 and 3.20 show that \( \text{td}^{1/2}_{2k} \) and \( \text{td}^{1/2}_{2k-2} \) coincide after taking proper Lefschetz operations, as shown by the following diagram:

\[
\begin{array}{ccc}
H^k(X, \Omega^{k-2}_X) & \xrightarrow{\lambda} & H^{2k-2}(X, \Omega^{k-2}_X) \\
\text{td}^{1/2}_{2k-2} & \xrightarrow{\lambda_{\sigma/4}} & \text{td}^{1/2}_{2k} \\
L_{\omega} & \xrightarrow{\lambda_{\sigma/4}} & L_{\omega} \\
\end{array}
\]

(2) Note that by the Hodge theory, \( L_{\omega} \) is injective on \( H^{2k}(X, \Omega^k_X) \) for \( k \leq \frac{n}{2} \) and \( \Lambda_{\omega} \) is injective on \( H^{2k}(X, \Omega^k_X) \) for \( k > \frac{n}{2} \). So by Corollary 3.20 \( \text{td}^{1/2}_{2k} \) carries all information of \( \{\text{td}^{1/2}_{2k} \mid 0 \leq k \leq n\} \).

4. A LEFSCHETZ-TYPE DECOMPOSITION OF \( \text{td}^{1/2}(X) \)

In this section, we apply Corollary 3.19 to give a Lefschetz-type decomposition of \( \text{td}^{1/2}(X) \). Note that we can also apply Corollary 3.20 to give a similar Lefschetz-type decomposition of \( \text{td}^{1/2}(X) \) (see Remark 4.3), but the former one is easier to handle in computations.

First, we find several natural primitive elements given by linear combinations of \( \text{td}^{1/2}_{2k} \).
Definition 4.1. Let $X$ be a hyperkähler manifold of dimension $2n$ and fix a non-zero $\sigma \in H^0(X, \Omega_X^2)$. Consider $\lambda_\sigma = \lambda(\sigma + \overline{\sigma}) > 0$ as in Definition 2.2. For $0 \leq k \leq n/2$, denote

$$tp_{2k} := \sum_{i=0}^{k} \frac{(n - 2k + 1)!td_{2i}^{1/2} \wedge (\sigma \overline{\sigma})^{k-i}}{(-\lambda_\sigma)^{k-i}(k-i)!(n-k-i+1)!} \in H^{4k}(X).$$

Note that it is of type $(2k, 2k)$ and fix a non-zero $\sigma$ and $\overline{\sigma}$. As $tp_{2k}$ is a real class, $tp_{k}$ is primitive. Here recall that $\lambda_\sigma$ is a non-zero scalar.

The following proposition shows that $tp_{2k}$ is indeed primitive in several senses.

Proposition 4.2. Let $X$ be a hyperkähler manifold of dimension $2n$ and fix a non-zero $\sigma \in H^0(X, \Omega_X^2)$. Then for any $0 \leq k \leq n/2$, $tp_{2k}$ is both $\sigma$-primitive and $(\sigma + \overline{\sigma})$-primitive.

Proof. To show that $tp_{2k}$ is $\sigma$-primitive, we need to check that $\Lambda_{\sigma/4}(tp_{2k}) = 0$. This can be checked directly by using Corollary 3.19. In fact, by Corollary 3.19, Lemmas 2.9 and 2.10.

$$\Lambda_{\sigma/4}(tp_{2k})/(n - 2k + 1)! = \sum_{i=0}^{k} \frac{\Lambda_{\sigma/4}(td_{2i}^{1/2} \wedge (\sigma \overline{\sigma})^{k-i})}{(-\lambda_\sigma)^{k-i}(k-i)!(n-k-i+1)!} = \sum_{i=0}^{k} \frac{\Lambda_{\sigma/4}L_{\sigma}^{k-i}(td_{2i}^{1/2}) \wedge (\sigma \overline{\sigma})^{k-i}}{(-\lambda_\sigma)^{k-i}(k-i)!(n-k-i+1)!} = \sum_{i=0}^{k} \frac{L_{\sigma}^{k-i}\Lambda_{\sigma/4}(td_{2i}^{1/2}) \wedge (\sigma \overline{\sigma})^{k-i}}{(-\lambda_\sigma)^{k-i}(k-i)!(n-k-i+1)!} = \frac{(-\lambda_\sigma)^{k-i}(k-i)!(n-k-i+1)!}{(-\lambda_\sigma)^{k-i}(k-i)!(n-k-i+1)!} = 0.$$

Hence $tp_{2k}$ is $\sigma$-primitive, in other words,

$$tp_{2k} \wedge \sigma^{n-2k+1} = L_{\sigma}^{n-2k+1}(tp_{2k}) = 0.$$

As $tp_{2k}$ is a real class, $tp_{2k} \wedge (\sigma + \overline{\sigma})^{2n-4k+1} = 0$ by complex conjugation. Hence it follows that $tp_{2k} \wedge (\sigma + \overline{\sigma})^{2n-4k+1} = 0$, that is, $tp_{2k}$ is $(\sigma + \overline{\sigma})$-primitive. Here recall that $\sigma + \overline{\sigma}$ is a Kähler form. □
By Proposition 4.2, we can show that for any Kähler form $\omega$ and for $0 \leq k \leq n/2$,
\[
\sum_{i=0}^{k} \frac{(2n - 4k + 2)!((n - k - i + 1)!td^{1/2}_{2i} \wedge \omega^{2k-2i})}{(-\lambda(\omega))^{k-i}(k-i)!((n - 2k + 1)!(2n - 2k - 2i + 2)!} \in H^{4k}(X)
\]
is $\omega$-primitive.

(1) We may assume that $\omega$ is $(4,0) + (2,2) + (0,4)$ with respect to the Kähler form $(\sigma + \bar{\sigma})$, so components of $tp_{2k} = td^{1/2}_{2k} - \frac{\sigma\bar{\sigma}}{n\lambda_\sigma}$ are of types $(4,0)$ and $(0,4)$. By Corollary 3.20, $\Lambda_1(\omega^2) + (2\omega_l) = 0$ and $\Lambda_1(\omega^2) = 0$. This implies that $\Lambda_1(\omega^2) = 0$. In particular, $tp_{2k}$ is both $\omega_l$-primitive and $\omega_k$-primitive. One may wonder whether $tp_{2k}$ is $\omega_l$-primitive, but unfortunately this is not true if $n > 1$ even for $k = 1$. Recall that by definition,
\[
\sum_{i=0}^{k} \frac{(2n - 4k + 2)!((n - k - i + 1)!td^{1/2}_{2i} \wedge \omega^{2k-2i})}{(-\lambda(\omega))^{k-i}(k-i)!((n - 2k + 1)!(2n - 2k - 2i + 2)!} \in H^{4k}(X)
\]
is $\omega$-primitive.

(2) Keep the notation in Section 2.3 and fix $0 \leq k \leq n/2$. Then Proposition 4.2 shows that $A_{\sigma/4}(tp_{2k}) = 0$ and $A_{\omega_l}(tp_{2k}) = 0$. This implies that $A_{\omega_k}(tp_{2k}) = 0$. In particular, $tp_{2k}$ is both $\omega_l$-primitive and $\omega_k$-primitive. One may wonder whether $tp_{2k}$ is $\omega_l$-primitive, but unfortunately this is not true if $n > 1$ even for $k = 1$. Recall that by definition,
\[
\sum_{i=0}^{k} \frac{(2n - 4k + 2)!((n - k - i + 1)!td^{1/2}_{2i} \wedge \omega^{2k-2i})}{(-\lambda(\omega))^{k-i}(k-i)!((n - 2k + 1)!(2n - 2k - 2i + 2)!} \in H^{4k}(X)
\]
is $\omega$-primitive.

Remark 4.3. (1) Similar to Proposition 4.2 by applying Corollary 3.20, we can show that for any Kähler form $\omega$ and for $0 \leq k \leq n/2$,
\[
\sum_{i=0}^{k} \frac{(2n - 4k + 2)!((n - k - i + 1)!td^{1/2}_{2i} \wedge \omega^{2k-2i})}{(-\lambda(\omega))^{k-i}(k-i)!((n - 2k + 1)!(2n - 2k - 2i + 2)!} \in H^{4k}(X)
\]
is $\omega$-primitive.

(2) Keep the notation in Section 2.3 and fix $0 \leq k \leq n/2$. Then Proposition 4.2 shows that $A_{\sigma/4}(tp_{2k}) = 0$ and $A_{\omega_l}(tp_{2k}) = 0$. This implies that $A_{\omega_k}(tp_{2k}) = 0$. In particular, $tp_{2k}$ is both $\omega_l$-primitive and $\omega_k$-primitive. One may wonder whether $tp_{2k}$ is $\omega_l$-primitive, but unfortunately this is not true if $n > 1$ even for $k = 1$. Recall that by definition,
\[
\sum_{i=0}^{k} \frac{(2n - 4k + 2)!((n - k - i + 1)!td^{1/2}_{2i} \wedge \omega^{2k-2i})}{(-\lambda(\omega))^{k-i}(k-i)!((n - 2k + 1)!(2n - 2k - 2i + 2)!} \in H^{4k}(X)
\]
is $\omega$-primitive.

For computation we adopt notation in [28] or [7] Section 24.2, where the Lefschetz operators of $\omega_j, \omega_l, \omega_k$ are denoted by $L_j, L_l, L_k, A_j, A_l, A_k$. By Corollary 3.20, $\Lambda_1(\omega^2) = \frac{1}{\lambda(\omega_1)}\omega_1$. On the other hand,
\[
\Lambda_1(\omega_j) = L_j\Lambda_1(\omega_j) + K_{jl}(\omega_l) + L_k\Lambda_1(\omega_k) - K_{kl}(\omega_l)
\]
$= 4\omega_j$.

Here we used the facts that $\Lambda_1(\omega_1) = \Lambda_1(\omega_l) = 0$ and $K_{jl}(\omega_l) = K_{kl}(\omega_l) = -2\omega_l$ (see [7] Proposition 24.2). Note that $\lambda(\omega_1) = \lambda(\omega_l) = \frac{1}{4}\lambda(\sigma)$. So we conclude that $\Lambda_1(tp_{2k}) = \frac{n-1}{n\lambda(\omega_1)}\omega_1 \neq 0$.

From the primitivity of $tp_{2k}$, we get the following.

Corollary 4.4. Let $X$ be a hyperkähler manifold of dimension $2n$ and fix a non-zero $\sigma \in H^0(X, \Omega_X)$). Consider two integers $0 \leq k, k' \leq n/2$ such that $k \neq k'$. Then
\[
\begin{enumerate}
\item \[ \int tp_{2k}tp_{2k'}(\sigma\bar{\sigma})^{n-k-k'} = 0; \text{ in particular, } \int tp_{2k'}(\sigma\bar{\sigma})^{n-k'} = 0 \text{ if } k' \neq 0. \]
\item \[ \int (tp_{2k})^2(\sigma\bar{\sigma})^{n-2k} \geq 0; \text{ moreover, the equality holds if and only if } tp_{2k} = 0. \]
\end{enumerate}
\]

Proof. (1) We may assume that $k > k'$, then $n-k-k' \geq n-2k+1$. By Proposition 4.2, $tp_{2k} \wedge \sigma^{n-k-k'} = 0$. Hence the conclusion is clear.

(2) By Proposition 4.2, $tp_{2k}$ is $(\sigma + \bar{\sigma})$-primitive. Note that $\sigma\bar{\sigma}$ is of type $(4,0) + (2,2) + (0,4)$ with respect to the Kähler form $(\sigma + \bar{\sigma})$, so components of $tp_{2k}$ in the Hodge decomposition with respect to the Kähler form $(\sigma + \bar{\sigma})$ are of types $(2k - 2m, 2k + 2m)$ for $-k \leq m \leq k$. On the other hand, $tp_{2k}$ is real, hence by the Hodge–Riemann bilinear
In summary, for any relation (13 Proposition 3.3.15),
\[ \int (t\sigma)^2 (\sigma + \sigma)^{2n-4k} \geq 0, \]
where the equality holds if and only if \( t\sigma = 0 \). This is equivalent to the conclusion by degree reason.

The following is the main theorem of this section, which gives a Lefschetz-type decomposition of \( t\sigma^{1/2} \) in terms of \( t\sigma \). One special and important phenomenon is that the coefficients in this decomposition are all positive.

**Theorem 4.5.** Let \( X \) be a hyperkähler manifold of dimension \( 2n \) and fix a non-zero \( \sigma \in H^0(X, \Omega^2_X) \). Consider \( \lambda_\sigma = \lambda(\sigma + \sigma) > 0 \) as in Definition 2.2.

1. For \( k \leq n/2 \),
\[
(1) \quad \text{For } k \leq n/2, \quad \text{td}_{2k}^{1/2} = \sum_{i=0}^{k} \frac{(n - 2k + i)!}{\lambda_\sigma^i!(n - 2k + 2i)!} t\sigma^{2k-2i} \wedge (\sigma \sigma)^i \\
= \sum_{i=0}^{k} \frac{(n - k - i)!}{\lambda_\sigma^{k-i}(k - i)!(n - 2i)!} t\sigma^{2i} \wedge (\sigma \sigma)^{k-i}.
\]

2. For \( k > n/2 \),
\[
(2) \quad \text{For } k > n/2, \quad \text{td}_{2k}^{1/2} = \sum_{i=0}^{n-k} \frac{(n - k - i)!}{\lambda_\sigma^{k-i}(k - i)!(n - 2i)!} t\sigma^{2i} \wedge (\sigma \sigma)^{k-i}.
\]

In summary, for any \( 0 \leq k \leq n \),
\[
\text{td}_{2k}^{1/2} = \sum_{i=0}^{\min\{k, n-k\}} \frac{(n - k - i)!}{\lambda_\sigma^{k-i}(k - i)!(n - 2i)!} t\sigma^{2i} \wedge (\sigma \sigma)^{k-i}.
\]

**Proof.** (1) This can be checked directly as the following.

\[
\begin{align*}
\sum_{i=0}^{k} \frac{(n - 2k + i)!}{\lambda_\sigma^i!(n - 2k + 2i)!} t\sigma^{2k-2i} \wedge (\sigma \sigma)^i \\
= \sum_{i=0}^{k} \frac{(n - 2k + i)!}{\lambda_\sigma^i!(n - 2k + 2i)!} \left( \sum_{j=0}^{k-i} \frac{(n - 2k + 2i + 1)!}{(n - 2k + i)} \text{td}_{2j}^{1/2} \wedge (\sigma \sigma)^{k-i-j} \wedge (\sigma \sigma)^i \right) \\
= \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{(-1)^{k-i-j}(n - 2k + 2i + 1)(n - 2k + i)!}{\lambda_\sigma^{k-i-j}(k - i - j)!(n - k + i + j + 1)!} \text{td}_{2j}^{1/2} \wedge (\sigma \sigma)^{k-j} \\
= \sum_{j=0}^{k} \frac{(-1)^{k-j} \text{td}_{2j}^{1/2} \wedge (\sigma \sigma)^{k-j}}{\lambda_\sigma^{k-j}} \left( \sum_{i=0}^{k-j} \frac{(n - 2k + 2i + 1)(n - 2k + i)!}{i!(k - i - j)!(n - k + i + j + 1)!} \right) \\
= \text{td}_{2k}^{1/2}.
\end{align*}
\]
Here in the last step, we applied Lemma 2.11 in the appendix.

(2) Note that by Corollary 3.19 and Lemma 2.9

\[ \lambda_{\sigma/4}^{2k-n}(td_{2k}^{1/2}) = \frac{1}{\lambda_{\sigma/4}^{2k-n}}td_{2n-2k}^{1/2} \wedge \sigma^{2k-n} \]

and

\[ \lambda_{\sigma/4}^{2k-n}\left( \sum_{i=0}^{n-k} \frac{(n-k-i)!}{\lambda_{\sigma}^{k-i}(k-i)!(n-2i)!}tp_{2i} \wedge (\sigma\bar{\sigma})^{k-i} \right) \]

\[ = \sum_{i=0}^{n-k} \frac{(n-k-i)!}{\lambda_{\sigma}^{k-i}(k-i)!(n-2i)!} \lambda_{\sigma/4}^{2k-n} L_{\sigma}^{n-k-i}(tp_{2i}) \wedge (\sigma\bar{\sigma})^{k-i} \]

\[ = \frac{n-k}{k-i} \left( \frac{(k-i)^2}{n-k-i} \right) ! L_{\sigma}^{n-k-i}(tp_{2i}) \wedge (\sigma\bar{\sigma})^{k-i} \]

\[ = \frac{n-k}{k-i} \left( \frac{(k-i)!}{n-k-i} \right) ! \lambda_{\sigma}^{k-i}(k-i)!(n-2i)! \]

\[ = \frac{1}{\lambda_{\sigma}^{2k-n}}td_{2n-2k}^{1/2} \wedge \sigma^{2k-n}. \]

Here for the second equality, we applied Lemma 2.10 repeatedly \((2k-n)\) times; for the last one, we applied (1). So the conclusion follows immediately as \(\lambda_{\sigma/4}^{2k-n} : H^{2k}(X, \Omega_{X}^{2k}) \rightarrow H^{2k}(X, \Omega_{X}^{2n-2k})\) is an isomorphism by standard representation theory of \(\mathfrak{sl}_2\).

As a direct application of this decomposition, we recover an important result of Nieper-Wißkirchen [19] generalizing Hitchin and Sawon [19]. It was used by Huybrechts [12] to prove finiteness results for hyperkähler manifolds.

**Corollary 4.6** ([19 (5.17)]). Let \(X\) be a hyperkähler manifold. Then for any \(\alpha \in H^2(X)\),

\[ \int td^{1/2}(X) \exp(\alpha) = (1 + \lambda(\alpha))^n \int td^{1/2}(X). \]

**Proof.** By Theorem 2.1, it suffices to prove the result for \(\alpha = \sigma + \bar{\sigma}\). By Theorem 1.5(1) and Corollary 4.4, for any \(0 \leq k \leq n\),

\[ \int td_{2k}^{1/2}(\sigma\bar{\sigma})^{n-k} = \int \frac{(n-k)!}{\lambda_{\sigma}^k n!}(\sigma\bar{\sigma})^n \]

as the integrals on components other than \(tp_0\) vanish. In particular,

\[ \int td_{2k}^{1/2}(\sigma\bar{\sigma})^{n-k} = \frac{(n-k)!}{\lambda_{\sigma}^k n!}(\sigma\bar{\sigma})^n. \]

Hence

\[ \int td_{2k}^{1/2}(\sigma\bar{\sigma})^{n-k} = \frac{(n-k)!}{\lambda_{\sigma}^k n!}(\sigma\bar{\sigma})^n. \]

\[ \int td_{2n}^{1/2}(\sigma\bar{\sigma})^{n} = \frac{1}{\lambda_{\sigma}^n (n!)^2} \int td_{2n}^{1/2}(X) \]

\[ = \frac{(n-k)!}{\lambda_{\sigma}^k n!}(\sigma\bar{\sigma})^n. \]
This concludes the proof. □

5. Positivity of Riemann–Roch polynomials and applications

In this section, we study the positivity of the Riemann–Roch polynomials and its applications.

5.1. Positivity of Riemann–Roch polynomials. The following theorem is a more precise version of Theorem 4.1

Theorem 5.1. Let \( X \) be a hyperkähler manifold of dimension 2n and fix a non-zero \( \sigma \in H^0(X, \Omega^2_X) \). Consider \( \lambda_\sigma = \lambda(\sigma + \overline{\sigma}) > 0 \) as in Definition 2.2. Then for any \( 0 \leq m \leq n \),

\[
\int \text{td}_{2m} \exp(\sigma + \overline{\sigma}) \geq \left( \frac{2n - m + 1}{m} \right) \lambda_{\sigma}^{n-m} \int \text{td}^{1/2}(X).
\]

Moreover, the inequality is strict for \( m > 1 \) and \( n > 1 \).

Proof. Note that by definition, \( \text{td}_{2m} = \sum_{k=0}^{m} \text{td}_{2k}^1 \text{td}_{2m-2k}^1 \). Hence by Theorem 4.5 and Corollary 4.4,

\[
\int \text{td}_{2m}(\sigma \overline{\sigma})^{n-m} = \sum_{k=0}^{m} \sum_{i=0}^{\min(k,m-k)} \frac{(n-k-i)!}{\lambda_{\sigma}^{k-i}(k-i)! (n-2i)!} (\text{tp}_{2i}(\sigma \overline{\sigma})^{k-i}) \cdot \left( \sum_{i=0}^{\min(m-k,n-m+k)} \frac{(n-m+k-i)!}{\lambda_{\sigma}^{m-k-i}(m-k-i)! (n-2i)!} (\text{tp}_{2i}(\sigma \overline{\sigma})^{m-k-i}) \right) (\sigma \overline{\sigma})^{n-m}
\]

\[
= \sum_{k=0}^{m/2} \sum_{i=0}^{m-i} \frac{(n-k-i)! (n-m+k-i)!}{\lambda_{\sigma}^{m-2i}(k-i)! (m-k-i)! (n-2i)!^2} \int (\text{tp}_{2i})^2 (\sigma \overline{\sigma})^{n-2i}
\]

\[
= \sum_{i=0}^{m/2} \frac{(n-m)!^2}{\lambda_{\sigma}^{m-2i} (n-2i)!^2} \left( \frac{2n-2i-m+1}{m-2i} \right) \int (\text{tp}_{2i})^2 (\sigma \overline{\sigma})^{n-2i}
\]

\[
\geq \frac{(n-m)!^2}{\lambda_{\sigma}^{m} n!^2} \left( \frac{2n-m+1}{m} \right) \int (\sigma \overline{\sigma})^{n}.
\]
Here in the last three steps we applied Lemma [A.2], Corollary 4.4(2), and Equality (4.1). This proves the desired inequality.

If the equality holds for some $m > 1$, then by Corollary 4.4(2), $t p_2 = 0$. Then Theorem 4.5 implies that $t d^{1/2} (X)$ is proportional to $\sigma \sigma$, which is absurd if $n > 1$, as $t d^{1/2}$ does not depend on the complex structure of $X$.

Finally we remark that, from the above expression, if one could get a better estimate for $\int (t p_2) (\sigma \sigma) n - 2$ for $i > 0$, then we can get a better estimate for $\int t d^{2m} (\sigma \sigma)^{n - m}$.

\[ P_{\lambda (\alpha)} := \int t d(X) \exp(\alpha) - \sum_{m=0}^{n} \binom{2n - m + 1}{m} \lambda (\alpha)^{n - m} \int t d^{1/2} (X) \]

is a polynomial in terms of $\lambda (\alpha)$ of degree $n - 2$ with positive coefficients.

**Proof.** By Theorem 2.1, the coefficient of $\lambda (\alpha)^{n - m}$ in $P_{\lambda (\alpha)}$ is just

\[ b_{n-m} = \frac{1}{\lambda \sigma} \int t d_{2m} \exp(\sigma + \sigma) - \binom{2n - m + 1}{m} \int t d^{1/2} (X). \]

If $m > 1$, then $b_{n-m} > 0$ by Theorem 5.1. If $m = 0$, then $b_n = 0$ by Equality (4.1). If $m = 1$, then by the definition of $\lambda \sigma$,

\[
\begin{align*}
    b_{n-1} &= \frac{1}{\lambda \sigma} \int t d_2 \exp(\sigma + \sigma) - 2n \int t d^{1/2} (X) \\
    &= \frac{1}{\lambda \sigma} \int 2 \sigma_2 (X) \exp(\sigma + \sigma) - 2n \frac{1}{\lambda \sigma} \int \exp(\sigma + \sigma) = 0.
\end{align*}
\]

Hence $P_{\lambda (\alpha)}$ is a polynomial in terms of $\lambda (\alpha)$ of degree $n - 2$ with positive coefficients.

**Proof of Theorem 1.1.** This follows from Proposition 2.3 and Corollary 5.2.\qed

**5.2. Kawamata’s effective non-vanishing conjecture and Riess’s question.** Recall that a special version of Kawamata’s effective non-vanishing conjecture predicts that, if $L$ is a nef and big line bundle on a projective manifold $X$ with $c_1 (X) = 0$, then $h^0 (X, L) > 0$. In [1] we studied this conjecture and proposed a stronger version for projective hyperkähler manifolds ([1, Conjecture 3.6]), which is actually equivalent to Theorem 1.1 for projective hyperkähler manifolds. So by Theorem 1.1 we get the following corollary.

**Corollary 5.3 ([1, Conjecture 3.6]).** Let $X$ be a projective hyperkähler manifold of dimension $2n$ and $L$ a nef and big line bundle on $X$. Then
(1) $h^0(X, L) \geq n + 2$;
(2) $\int td_{2n-2}(X) \cdot L^i > 0$ for all $0 \leq i \leq n$.

Proof. (2) directly follows from Theorem 5.1 and Theorem 2.1. For (1), by the Kawamata–Viehweg vanishing theorem ([15]), Theorem 1.1, and Proposition 2.4,
$$h^0(X, L) = \chi(L) > \chi(O_X) = n + 1.$$ Here recall that the constant term of $RR_X$ is just $\chi(O_X)$.

As a related topic, Riess [24] studied the base loci of linear systems of line bundles on hyperkähler manifolds and naturally raised up the question whether the Riemann–Roch polynomial $RR_X(q)|_{q>0}$ is strictly monotonic. Theorem 1.1 answers her question affirmatively.

Corollary 5.4 (Riess’s question). Let $X$ be a hyperkähler manifold. Then the Riemann–Roch polynomial $RR_X(q)$ is strictly monotonic for $q > 0$.

5.3. An upper bound of $\int td^{1/2}(X)$. As an application of Theorem 5.1, we can give an upper bound for the value $\int td^{1/2}(X)$.

Corollary 5.5. Let $X$ be a hyperkähler manifold of dimension $2n > 2$. Then $\int td^{1/2}(X) < 1$. Equivalently, let $g$ be a hyperkähler metric on $X$ compatible with the hyperkähler structure on $X$, then
$$||R||^{2n} < (192\pi^2 n^n (\text{vol} X)^{n-1},$$ where $||R||$ is the $L_2$-norm of the curvature tensor of $g$.

Proof. In Theorem 5.1, taking $m = n$, we get
$$\int td^{1/2}(X) < \frac{1}{n+1} \int td_n = \frac{1}{n+1} \chi(O_X) = 1.$$ The second statement follows directly from [9, Theorem 5].

Example 5.6. (1) For a K3 surface $S$, $\int td^{1/2}(S) = c_2(S)/24 = 1$.
(2) If $X$ is the Hilbert scheme of $n$ points on a K3 surface, then $\int td^{1/2}(X) = \frac{(n+3)^n}{4n!}$ by Sawon [26] Proposition 19].
(3) If $X$ is a generalized Kummer variety of dimension $2n$, then $\int td^{1/2}(X) = \frac{(n+1)^n}{4n!}$ by Sawon [26] Proposition 21].
(4) As all Chern numbers of O’Grady’s examples are known due to [17] (6-dimensional case) and [23] (10-dimensional case), we can compute $\int td^{1/2}(X)$ for these examples. In fact, Belmans informed the author that he computed that $\int td^{1/2}(X) = \frac{2}{3}$ or $\frac{4}{3}$ for O’Grady’s 6-dimensional example and 10-dimensional example respectively, which coincides with the value for a generalized Kummer variety of dimension 6, or the Hilbert scheme of 5 points on a K3 surface respectively. Then we realized that all the above numbers can be obtained directly by Example 1.2 and the following Lemma 5.7.
Lemma 5.7. Let $X$ be a hyperkähler manifold of dimension $2n$ and suppose that the first two leading terms of $\text{RR}_X(q)$ are $Aq^n$ and $Bq^{n-1}$, then $\int \text{td}^{1/2}(X) = \left(\frac{2n}{(2n)!A^{-1}}\right)^n$. In particular, if $X$ and $Y$ are two hyperkähler manifolds with the same Riemann–Roch polynomial $\text{RR}_X(q) = \text{RR}_Y(q)$, then $\int \text{td}^{1/2}(X) = \int \text{td}^{1/2}(Y)$.

Proof. Write $\text{RR}_X(q) = Aq^n + Bq^{n-1} + \text{(lower terms)}$. Recall that by [23, Proof of Lemma 3], $c_X = (2n)!A$ and $\lambda(\sigma + \overline{\sigma}) = \frac{2nA}{B}q_X(\sigma + \overline{\sigma})$, where $c_X$ is the Fujiki constant and $\lambda$ is in Definition 2.2. Then by (4.1),

$$
\int \text{td}^{1/2}(X) = \frac{1}{\lambda(\sigma + \overline{\sigma})(n!)^2} \int (\sigma \overline{\sigma})^n = \frac{1}{\lambda(\sigma + \overline{\sigma})(2n)!} \int (\sigma + \overline{\sigma})^{2n} = \frac{c_Xq_X(\sigma + \overline{\sigma})^n}{\lambda(\sigma + \overline{\sigma})(2n)!} = \frac{B^n}{(2n)^n A^{n-1}}.
$$

The examples suggest that $\int \text{td}^{1/2}(X)$ might get very small as $n$ getting large, so it is natural to ask whether there is a better upper bound for $\int \text{td}^{1/2}(X)$ of exponential order $c < 0$ in terms of $\dim X$.

Recall that for a hyperkähler manifold $X$ of dimension $2n$, its Chern numbers are given by integrals of the form $\int c_{2k_1}c_{2k_2}\ldots c_{2k_m}$ for non-negative integers $k_1,\ldots,k_m$ satisfying $\sum_{i=1}^m k_i = n$. As observed by Sawon [26] and Nieper-Wißkirchen [18, Appendix B] (see also [17] and [23]), all known Chern numbers of hyperkähler manifolds are positive. So it is natural to propose the following conjecture, which is a question by Nieper-Wißkirchen [18, Appendix B].

Conjecture 5.8. Let $X$ be a hyperkähler manifold of dimension $2n$. Then all Chern numbers $\int c_{2k_1}c_{2k_2}\ldots c_{2k_m}$ for non-negative integers $k_1,\ldots,k_m$ satisfying $\sum_{i=1}^m k_i = n$ are positive integers.

If this conjecture is true, then it reflects very special geometry of hyperkähler manifolds. For example, it predicts that the topological Euler characteristic of any hyperkähler manifold is positive as a special case, which is unfortunately unknown even in dimension 4. One can expect that the methods in this paper might give some partial solutions to this conjecture.

Appendix A. Some combinatorial identities

In this appendix, we prove two combinatorial identities.

Lemma A.1. Given non-negative integers $n, k, j$ satisfying $n/2 \geq k \geq j$, we have

$$
\sum_{i=0}^{k-j} \frac{(-1)^i(n-2k+2i+1)(n-2k+i)!}{i!(k-i-j)!(n-k+i-j+1)!} = \begin{cases} 
0 & \text{if } k > j; \\
1 & \text{if } k = j.
\end{cases}
$$
Proof. The case when $k = j$ is trivial. Suppose that $k > j$. The desired equality is equivalent to
\[
\sum_{i=0}^{k-j} (-1)^i (n - 2k + 2i + 1) \binom{n - 2k + i}{i} \binom{n - 2j + 1}{k - i - j} = 0.
\]

Note that
\[
\sum_{i=0}^{k-j} (-1)^i (n - 2k + 2i + 1) \binom{n - 2k + i}{i} \binom{n - 2j + 1}{k - i - j}
\]
\[= \sum_{i=0}^{k-j} (-1)^i ((n - 2k + i + 1) + i) \binom{n - 2k + i}{i} \binom{n - 2j + 1}{k - i - j}
\]
\[= \sum_{i=0}^{k-j} (-1)^i (n - 2k + 1) \binom{n - 2k + i + 1}{i} \binom{n - 2j + 1}{k - i - j}
\]
\[+ \sum_{i=1}^{k-j} (-1)^i (n - 2k + 1) \binom{n - 2k + i}{i-1} \binom{n - 2j + 1}{k - i - j}.
\]

To conclude the proof, we claim that
\[
\sum_{i=0}^{k-j} (-1)^i \binom{n - 2k + i + 1}{i} \binom{n - 2j + 1}{k - i - j}
\]
\[= \sum_{i=1}^{k-j} (-1)^{i-1} \binom{n - 2k + i}{i-1} \binom{n - 2j + 1}{k - i - j}
\]
\[= \binom{2k - 2j - 1}{k - j}.
\]

In fact, the first item is the coefficient of $x^{k-j}$ in the generating function
\[
(1 + x)^{(n-2k+2)} \cdot (1 + x)^{n-2j+1} = (1 + x)^{2k-2j-1},
\]
so it equals to $\binom{2k-2j-1}{k-j}$; meanwhile, the second item is the coefficient of $x^{k-j-1}$ in the generating function
\[
(1 + x)^{(n-2k+2)} \cdot (1 + x)^{n-2j+1} = (1 + x)^{2k-2j-1},
\]
so it equals to $\binom{2k-2j-1}{k-j-1}$. \hfill \Box

Lemma A.2. Given non-negative integers $n, m$ satisfying $n \geq m$, we have
\[
\sum_{k=0}^{m} \frac{(n-k)!(n-m+k)!}{k!(m-k)!} = (n-m)! \binom{2n-m+1}{m}.
\]

Furthermore, if $i$ is an integer satisfying $m \geq 2i$, then
\[
\sum_{k=1}^{m-i} \frac{(n-k-i)!(n-m+k-i)!}{(k-i)!(m-k-i)!} = (n-m)! \binom{2n-2i-m+1}{m-2i}.
\]
Proof. Consider
\[
\frac{1}{(n-m)^2} \sum_{k=0}^{m} \frac{(n-k)!(n-m+k)!}{k!(m-k)!} = \sum_{k=0}^{m} \binom{n-k}{m-k} \binom{n-m+k}{k}.
\]
This is exactly the coefficient of $x^m$ in the generating function
\[
(1-x)^{-(n-m+1)} \cdot (1-x)^{-(n-m+1)} = (1-x)^{-(2n-2m+2)},
\]
which is just \( \binom{2n-m+1}{m} \). The second equality follows from the first one by considering $n-2i$ and $m-2i$, and changing the range of $k$ to $[0, m-2i]$. □

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References

[1] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Differential Geom. 18 (1983), no. 4, 755–782.
[2] F.A. Bogomolov, Hamiltonian Kählerian manifolds, Dokl. Akad. Nauk SSSR 243 (1978), no. 5, 1101–1104.
[3] M. Britze, M.A. Nieper, Hirzebruch–Riemann–Roch formulae on irreducible symplectic Kähler manifolds, arXiv:math/0101062v1.
[4] Y. Cao, C. Jiang, Remarks on Kawamata’s effective non-vanishing conjecture for manifolds with trivial first Chern classes, Math. Z. 296 (2020), 615–637.
[5] G. Ellingsrud, L. Göttsche, M. Lehn, On the cobordism class of the Hilbert scheme of a surface, J. Algebraic Geom. 10 (2001), no. 1, 81–100.
[6] A. Fujiki, On the de Rham cohomology group of a compact Kähler symplectic manifold, Algebraic geometry, Sendai, 1985, 105–165, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.
[7] M. Gross, D. Huybrechts, D. Joyce, Calabi–Yau manifolds and related geometries, Lectures from the Summer School held in Nordfjordeid, June 2001. Universitext. Springer-Verlag, Berlin, 2003.
[8] D. Guan, On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four, Math. Res. Lett. 8 (2001), no. 5-6, 663–669.
[9] N. Hitchin, J. Sawon, Curvature and characteristic numbers of hyper-Kähler manifolds, Duke Math. J. 106 (2001), no. 3, 599–615.
[10] D. Huybrechts, Compact hyperkähler manifolds, Habilitationsschrift Essen (1997), 65 pages.
[11] D. Huybrechts, Compact hyper-Kähler manifolds: basic results, Invent. Math. 135 (1999), no. 1, 63–113; Erratum: “Compact hyper-Kähler manifolds: basic results”, Invent. Math. 152 (2003), no. 2, 209–212.
[12] D. Huybrechts, Finiteness results for compact hyperkähler manifolds, J. Reine Angew. Math. 558 (2003), 15–22.
[13] D. Huybrechts, *Complex geometry. An introduction*. Universitext. Springer-Verlag, Berlin, 2005. xii+309 pp.
[14] M. Kapranov, *Rozansky–Witten invariants via Atiyah classes*, Compositio Math. 115 (1999), no. 1, 71–113.
[15] Y. Kawamata, *A generalization of Kodaira–Ramanujam’s vanishing theorem*. Math. Ann. 261 (1982), no. 1, 43–46.
[16] E. Looijenga, V.A. Lunts, *A Lie algebra attached to a projective variety*, Invent. Math. 129 (1997), no. 2, 361–412.
[17] G. Mongardi, A. Rapagnetta, G. Saccà, *The Hodge diamond of O’Grady’s 6-dimensional example*, Compos. Math. 154 (2018), no. 5, 984–1013.
[18] M.A. Nieper-Wißkirchen, *Characteristic classes and Rozansky–Witten invariants of compact hyperkähler manifolds*, Ph.D Thesis, Köln 2002.
[19] M.A. Nieper, *Hirzebruch–Riemann–Roch formulae on irreducible symplectic Kähler manifolds*, J. Algebraic Geom. 12 (2003), no. 4, 715–739.
[20] M.A. Nieper-Wißkirchen, *Chern numbers and Rozansky–Witten invariants of compact hyper-Kähler manifolds*, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.
[21] K.G. O’Grady, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. 512 (1999), 49–117.
[22] K.G. O’Grady, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. 12 (2003), no. 3, 435–505.
[23] Á.D.R. Ortiz, *Riemann–Roch polynomials of the known hyperkähler manifolds*, with an appendix by Yalong Cao and Chen Jiang, arXiv:2006.09307v2.
[24] U. Riess, *Base divisors of big and nef line bundles on irreducible symplectic varieties*, arXiv:1807.05192v1, appear in Ann. Inst. Fourier (Grenoble).
[25] L. Rozansky, E. Witten, *Hyper-Kähler geometry and invariants of three-manifolds*, Selecta Math. (N.S.) 3 (1997), no. 3, 401–458.
[26] J. Sawon, *Rozansky–Witten invariants of hyperkähler manifolds*, Ph.D. thesis, University of Cambridge, arXiv:math/0404360v1, October 1999.
[27] D.P. Thurston, *Wheeling: A diagrammatic analogue of the Dafoe isomorphism*, Ph.D. thesis, University of California at Berkeley, arXiv:math/0006083v1, Spring 2000.
[28] M. Verbitsky, *Action of the Lie algebra of SO(5) on the cohomology of a hyper-Kähler manifold*, Funktsional. Anal. i Prilozhen. 24 (1990), no. 3, 70–71; translation in Funct. Anal. Appl. 24 (1990), no. 3, 229–230 (1991).
[29] M. Verbitsky, *Cohomology of compact hyper-Kähler manifolds and its applications*, Geom. Funct. Anal. 6 (1996), no. 4, 601–611.
[30] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411.

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