Vacancy-like dressed states in topological waveguide QED

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We identify a class of dressed atom-photon states forming at the same energy of the atom at any coupling strength. As a hallmark, their photonic component is an eigenstate of the bare photonic bath with a vacancy in place of the atom. The picture accommodates waveguide-QED phenomena where atoms behave as perfect mirrors, connecting in particular dressed bound states (BS) in the continuum or BIC with geometrically-confined photonic modes. When applied to photonic lattices, the framework provides a general criterion to predict dressed BS in lattices with topological properties by putting them in one-to-one correspondence with photonic BS. New classes of dressed BS are thus predicted in the photonic Creutz-ladder and Haldane models. In the latter case, states with non-zero local photon flux occur, where an atom is dressed by a photon orbiting around it.

Atom-photon dressed states are a basic concept of quantum electrodynamics (QED) [1, 2]. A dressed bound state (BS), in particular, features a photonic cloud localized close to the atom, which can be pictured as the attempted emission of a photon that is yet reflected back exciting the atom again, instead of flying away as in free space. Dramatic departure from spontaneous decay thus occurs, such as vacuum Rabi oscillations [2, 3] or population trapping [4–7].

The interest in in-gap dressed BS, in particular, has thrived in the last few years [8–22], prompting their experimental detection in various setups such as circuit QED [20] and cold atoms coupled to photonic crystal waveguides (PCW) [23] or optical lattices [24, 25]. A major appeal of dressed BS is their ability to mediate dispersive many-body Hamiltonians [9, 11, 17, 26–30]. Unlike cavity-QED schemes, these feature short-range, tunable inter-atomic couplings with promising applications in quantum technologies and many-body physics.

In this work, we focus on a class of dressed states, which can be both bound and unbound, that we dub “Vacancy-like Dressed States” (VDS) for reasons that will become clear shortly. Their definition is simple: a VDS is a single-photon dressed state having exactly the same energy as the bare atom, irrespective of the coupling strength (under the rotating wave approximation). Familiar instances of dressed states, such as those arising in the Jaynes-Cummings model [2] and most of the in-gap BS studied so far [11], are not VDS. While it might appear strange that the dressed-state energy can be insensitive to the coupling strength, in fact eigenstates with an analogous property recur in several fields such as quantum biology [31] and dark states in atomic physics [32].

As will be shown, the hallmark of VDS is that their photonic component is an eigenstate of the (bare) photonic bath with a vacancy on the atomic position (hence the name). Intuitively, the atom imposes a pointlike hard-wall boundary condition on the field and is then dressed by one of the resulting photonic eigenstates. This allows to embrace and re-interpret waveguide-QED phenomena [33–35] where atoms behave as perfect mirrors [36–42], in particular spotlighting the link between dressed BS in the continuum (BIC) [43–54] and photonic confined modes [55]. When applied to photonic lattices, VDS prove especially fruitful to establish a general criterion for occurrence of topological dressed BS, so far predicted and experimentally observed only in the photonic Su-Schrieffer-Heeger (SSH) model [28, 56]. Guided by this, new classes of topological dressed BS are predicted in the photonic Creutz-ladder and Haldane models, highlighting potential applications and exotic properties such as persistent single-photon fluxes dressing the atom.

Vacancy-like dressed states.—Consider a general Hamiltonian model [see Fig. 1(a)] describing a two-level (pseudo) atom with frequency $\omega_0$ weakly coupled to a structured photonic bath $B$ (field), the latter being an unspecified network of coupled bosonic modes (“cavities”). The Hamiltonian reads

$$H = \omega_0 \sigma_+ \sigma_- + H_B + g (b^\dagger \sigma_- + b \sigma_+), \quad (1)$$

with

$$H_B = \sum_i \omega_i b^\dagger_i b_i + \sum_{i \neq j} J_{ij} b^\dagger_i b_j \quad (2)$$

the bath free Hamiltonian ($J_{ji} = J_{ij}^\ast$). Here, $b_i$ are bosonic ladder operators on $B$ fulfilling $[b_i, b^\dagger_j] = \delta_{ij}$, while $\sigma_\pm$ are usual pseudo-spin ladder operators of the atom. The atom is locally coupled to cavity $i = v$ (henceforth, at times referred to as the atom’s “position”). The overall ground state is $|G\rangle = |g\rangle |\text{vac}\rangle$ with $|g\rangle$ the atom’s ground state and $|\text{vac}\rangle$ the field vacuum. The single-excitation subspace is spanned by $|e\rangle = \sigma_+ |G\rangle$ (atom excited and no photons) and $\{ |i\rangle \}$ with $|i\rangle = b^\dagger_i |G\rangle$ single-photons states.

A vacancy-like dressed state (VDS) $|\Psi\rangle$ is defined by

$$|\Psi\rangle = \varepsilon |e\rangle + |\psi\rangle, \quad (3)$$
single-photon eigenstates of the bare photonic bath with

where, explicitly,

by restricting the sum to

where

can be seen by projecting (4) onto

ψ

|v⟩

is coupled to eigenstates of

H_{B_v}

(among which

|ψ⟩

and the atom excited state

|e⟩.

If

|ψ⟩

has the same energy as

|e⟩,

a VDS exists. When

B

has bands, both in-band VDS (d) and in-gap VDS (e) can occur.

fulfilling

\[ H |\Psi⟩ = ω_0 |\Psi⟩, \]

where

|ψ⟩ = \sum_i ψ_i |i⟩

is the (unnormalized) photonic wavefunction.

VDS are single-photon dressed states with a node on the atom: \( ψ_v = 0 \) (the converse holds as well). This can be seen by projecting (4) onto \(|e⟩\), yielding \( ω_0 ε + g ψ_v = ω_0 ε \), hence \( ψ_v = 0 \). Accordingly, if \( B_v \) is the photonic bath obtained from \( B \) by replacing cavity \( v \) with a vacancy [see Fig. 1(b)], the photon dressing the atom fully lives in \( B_v \). It is then easily shown [57] that \(|ψ⟩\) is an eigenstate of \( B_v \), again with energy \( ω_0 \),

\[ H_{B_v} |ψ⟩ = ω_0 |ψ⟩, \]

where \( H_{B_v} \) (free Hamiltonian of \( B_v \)) is obtained from (2) by restricting the sum to \( i, j \neq v \). Also [57],

\[ g ε + \langle e |H_B|ψ⟩ = 0, \]

where, explicitly, \( \langle e |H_B|ψ⟩ = \sum_{i\neq v} J_{v,i} ψ_i \). Conversely, given \(|ψ⟩\) fulfilling (5), the superposition of \(|e⟩\) and \(|ψ⟩\) defined by (6) is a VDS.

Thus a one-to-one mapping exists between VDS and single-photon eigenstates of the bare photonic bath with a vacancy in place of the atom (note that \(|ψ⟩\) is not an eigenstate of \( H_B \)): Searching for VDS in fact reduces to searching for photonic normal modes in the presence of a vacancy.

We point out that, for each \(|ψ⟩\) fulfilling (5), the existence of the VDS is guaranteed regardless of the coupling strength \( g \) and bath structure. This is easily seen from the star-like structure of \( H \) in the single-excitation sector [see Fig. 1(c)]. Owing to the Λ-configuration with vertexes \(|e⟩\), \(|e⟩\) and \(|ψ⟩\), it is clear that – when \(|e⟩\) and \(|ψ⟩\) have the same energy \( ω_0 \) – there always exists a superposition \(|Φ⟩\) of them which, through destructive interference, is uncoupled from all other states (in formal analogy with, e.g., dark states [32]).

In general, \(|Φ⟩\) can be normalizable (i.e., a dressed BS in/out of the continuum) or not (i.e., unbound). Also, degeneracies can occur. When \(|Φ⟩\) is bound, condition \( \langle Φ |Φ⟩ = 1 \) and (6) can be used to express it in the form

\[ |Φ⟩ = \cos θ |e⟩ + e^{iv} \sin θ |ψ⟩, \]

where

\[ θ = \arctan |η|, \quad φ = \arg η \text{ with } η = -\frac{g}{|e| H_B |ψ⟩}. \]

(|ψ⟩ fulfills \( \langle ψ |ψ⟩ = 1 \) and (5)).

The extension of VDS to many atoms is straightforward [57].

Two cavities.—The simplest VDS occurs when \( B \) is a pair of cavities \((v \text{ and } 1)\) coupled with strength \(-J\) [see Fig. 2(a)]. When \( ω_0 = ω_1 \), (5) has the only solution \( |ψ⟩ = |1⟩ \) yielding the VDS defined by \( θ = g/J, \quad φ = 0 \). In all other dressed states the photon can be found at \( v \). Instead, for \( ω_0 \neq ω_1 \), (5) has no solution and no VDS arises. Also, no VDS exists for \( J = 0 \) (usual Jaynes-Cummings model [2]).

Atom as a mirror.—When \( B \) is a 1D waveguide, where a vacancy is equivalent to a perfect mirror, VDS formalize the known mirror-like behavior of atoms [33].

Let \( B \) be an infinite waveguide (discretized for the sake of argument) [see Fig. 2(b)] with \( ω_0 \) well within the photonic band [see Fig. 1(d)]. Then \( B_v \) is the waveguide with a perfect mirror on the atom’s location, i.e., \( B_v = B_v^L \cup B_v^R \) with \( B_v^L \) (\( B_v^R \)) the semi-infinite waveguide on the left (right) of \( v \). Clearly, the eigenstates of \( H_{B_v} \) are a continuum of sinusoidal, unbound, stationary waves with a node on \( v \), each living either in \( B_v^R \) or \( B_v^L \) [see Fig. 2(b)]. The pair at energy \( ω = ω_0 \) fulfill Eq. (5), thus two VDS exist. Each is a scattering state describing a left- or right-incoming photon of frequency \( ω_0 \) fully reflected back from the atom, a major effect in waveguide QED [36–38] (see Ref. [57] for details). The one above is an instance of unbound VDS.

When the waveguide is semi-infinite [see Fig. 2(c)], \( B_v^L \) turns into a perfect cavity the related eigenstates being now discrete and bound, each corresponding to a cavity protected mode with wavevector \( k_m = m π/d \) (\( m = 1, 2, \ldots \)) and frequency \( ω_m \). A bound VDS will thus arise when an \( m \) exists such that \( ω_m = ω_0 \), i.e., \( k_0 = m π \) (with \( k_0 \) defined by \( ω_0 = ω_0 \)). Since the system is gapless, this VDS is a dressed BS in the continuum (BIC). Its explicit form is obtained from that of a textbook cavity mode by a direct application of Eqs. (7)-(8) [57]. A VDS for two atoms in an infinite waveguide is found likewise [57]. We thus retrieve a class of dressed BIC (or quantum BIC) [43–54]: the VDS framework ex-
explicitly connects these quantum BIC to geometrically-confined photonic modes (corresponding respectively to $|\Psi\rangle$ and $|\psi\rangle$).

**In-gap dressed bound states.**—Henceforth, we focus on translationally-invariant photonic lattices $B$, thus subject to periodic boundary conditions (BCs), and bound VDS. Let us first recall general known properties. The spectrum of $H_B$ comprises continuous bands of unbound modes, separated by bandgaps. In the one-excitation sector, the total Hamiltonian $H$ [cf. Eq. (1)] will feature the same bands as $H_B$ [6] each band corresponding to a continuum of unbound dressed states. In addition, in-gap dressed BS – at most one per bandgap – generally occur [6, 57, 58]. In particular, when $\omega_0$ lies within a bandgap, and $g$ is much smaller than the bandgap width $\Delta_{\text{gap}}$, a dressed BS occurs in the same bandgap, which reduces to $|\psi\rangle$ for $g \rightarrow 0$ [cf. Eq. (1)]. For $g \ll \Delta_{\text{gap}}$, when many atoms are present, these dressed BS mediate decoherence-free atom-atom interactions described by an effective Hamiltonian $H_{\text{eff}}$, whose inter-atomic potential inherits just the same spatial profile as the BS photonic component [11, 27].

Let us explore VDS in this scenario. In general, replacing a site of the bare lattice with a vacancy seeds in-gap photonic BS, at most one within each finite (i.e., internal) bandgap [57, 58]. Then tuning the atom on resonance [see Fig. 1(e)] with one such state, say $|\psi\rangle$, the corresponding VDS $|\Psi\rangle$ [cf. Eq. (7)] will form (and this will then be the only dressed BS in the bandgap where $|\psi\rangle$ occurs). The ensuing many-atom effective Hamiltonian (for $g \ll \Delta_{\text{gap}}$) reads $H_{\text{eff}} = \sum_{\nu \nu'} K_{\nu \nu'} \sigma_{\nu} \sigma_{\nu'} + H.c.$, where the inter-atomic potential is given by [57]

$$K_{\nu \nu'} = -\frac{g^2}{2 \langle \nu | H_B | \nu' \rangle} \langle \psi_{\nu'} | \psi_{\nu} \rangle$$

(9)

with $|\psi_{\nu'}\rangle$ the photonic BS arising when atom $\nu$ is replaced by a vacancy (in absence of all other atoms) and $\psi_\nu$ its amplitude at the position of atom $\nu$. Thus the spatial shape of the potential is just the $\psi$ wavefunction, while its strength depends on how tightly connected is $|\psi_{\nu'}\rangle$ to site $\nu'$ this being measured by $\langle \nu | H_B | \nu' \rangle$ [cf. Eq. (6)].

In passing, we note that, for $g \ll \Delta_{\text{gap}}$, the VDS wavefunction is stable against an imperfect setting of condition $\omega_0 = \omega_\psi$ with $H_{\text{eff}} |\psi\rangle = \omega_\psi |\psi\rangle$ [57].

A major advantage of VDS is that they bridge dressed BS to topological condensed matter/photonics [59, 60]: Topological classifications of translationally invariant lattices – such as the ten Altland-Zirnbauer (AZ) classes [61, 62] – allow to predict if a vacancy will seed topologically-protected in-gap photonic BS [63, 64]. These thus become criteria to predict in-gap dressed BS and ensuing dispersive effective Hamiltonians (once atoms are tuned on resonance with $|\psi\rangle$). Moreover, the resulting dressed BS will inherit properties analogous to their photonic counterparts such as protection against disorder and circulating chiral currents. Very recently, disorder resilience was observed both theoretically [28] and experimentally [56] in the SSH model. Our framework establishes that it is a general expected property of atoms coupled to any lattice with the right symmetries to admit topological vacancy-induced BS.

We note that, in 1D, if $R \cdot d$ open BCs [59] of $B$ occur under open BCs (i.e., $B$ without a full cell) then a vacancy-induced BS always exists and can be inferred from the edge states (here $R = 1, 2, \ldots$ is the interaction range of the lattice and $d$ the number of sites per cell). This is based on a theorem proven in Ref. [57].

Three instances of lattices with topological properties follow.

**SSH model.**—The photonic SSH model [65–68] is the simplest 1D topological lattice [see Fig. 2(d)]. The unit cell has two cavities, $a$ and $b$, both of frequency $\omega_\nu$, coupled with strength $J(1-\delta)$, where $|\delta| \leq 1$, while the inter-cell coupling is $J(1+\delta)$. The total number of cavities is $2N$ (even) with $N$ the number of cells. The $H_B$ spectrum has two bands separated by a bandgap, centered at $\omega_{\text{mid}} = \omega_c$, of width $\Delta_{\text{gap}} = 4|\delta|J$. In this simple instance, $B_v$ is just an open SSH chain with an odd number of sites $2N-1$: this is well-known to exhibit (see, e.g., Refs. [69, 70]) a single in-gap edge state $|\psi\rangle$ of energy $\omega_c$ with non-zero amplitude only on sites of given parity. If $v = a$, $|\psi\rangle$ is localized [see Fig. 2(d)] close to the edge of $B_v$ on the right (left) of $v$ for $\delta > 0$ ($\delta < 0$) (right and left are swapped if $v = B$). Thus, for $\omega_0 = \omega_c$,
a corresponding VDS arises with a strongly asymmetric shape ("chiral BS" [28, 56]), which is worked out from the known form of $|\psi\rangle$ [71] via a direct application of (7) [57]. Note that this dispenses with the resolvent method [1, 7], by which this state was first found very recently [28]. Also note that, for $\delta = -1$, $B$ reduces to uncoupled pairs of cavities (dimers), linking this VDS to that for two cavities in Fig. 2(a).

Creutz-ladder (CL) model.—Another 1D lattice with topological properties is the photonic CL model [72], a circuit-QED implementation of which was recently put forward [73]. The cell has again two cavities $a$ and $b$ each of frequency $\omega_c$ [see Fig. 2(e)] with vertical (diagonal) coupling strength $-2mJ$ ($J$, where $|m| \leq 1$, and upper (lower) horizontal strength $Je^{-i\alpha}$ ($Je^{i\alpha}$). The bandgap is centered at $\omega_{mid} = \omega_c - 2m \cos \alpha J$, its width $\Delta \omega_{gap}$ being the smallest of the four quantities $4\delta_e J$ and $2(\delta_e+\delta_h \pm 2 \cos \alpha)J$ with $\delta_e = |m| \pm 1$. In particular, $\Delta \omega_{gap} = 0$ for $m = \pm 1$.

Using methods in Refs. [74, 75] combined with the aforementioned theorem for 1D lattices [57], we find that, when $\Delta \omega_{gap} > 0$, $B_v$ admits a BS of energy $\omega_{mid}$. This reads (we place the atom on site $a$ of cell $n = 1$ and assume $N \gg 1$)

$$\psi_{an} = \frac{1}{2} \sqrt{1-m^2} (e^{i\alpha}m^{-n+2}+e^{-i\alpha}m^{N-n}) \text{ (sites } a), \quad (10)$$

while $\psi_{bn}$ (sites $b$) is the same but $e^{\pm i\alpha} \rightarrow -1$ (here, $n = 2, \ldots, N$; observe that cells on the left of $v$ are labeled by $N, N-1, \ldots$). An analogous BS occurs for $v = b$ [57]. When $\omega_0 = \omega_{mid}$, a corresponding VDS is seeded being defined by [cf. Eqs. (7)-(8)]

$$\eta = g/(2J)e^{\pm \frac{\pi}{2}} \sin^{-2} \sqrt{1-m^2} \text{.}$$

Note that, unlike SSH, $|\psi_{an}| = |\psi_{bn}|$. Remarkably, for $\alpha = \pm \pi/2$, a topological phase occurs [72], ensuring that the above pair of edge states — hence BS (10) and the associated VDS — are topologically protected.

In contrast to SSH, here no chirality manifests in the photon probability distribution since $|\psi_{j,n}|$ (for $j = a, b$) is mirror-symmetrical around $v$. The same holds for $\psi_{bn}$ (phase included). Yet, $\psi_{an} \sim e^{\pm i\alpha}$ on the right of $v$ while $\psi_{an} \sim e^{-i\alpha}$ on the left. Thus, in the Creutz model, BS possess a chirality of phase (instead of modulus as in the SSH model). This is inherited by the corresponding VDS and thus by the following associated $H_{eff}$. Plugging $|\psi\rangle$ into (9) yields $K_{m,n}^{(aa)} = \frac{g^2}{2m} e^{\pm i\alpha}m^{-n}m^{-n'}$ for two atoms sitting at cells $n$ and $n'$ both on sites $a$, while $K_{m,n}^{(bb)}$ and $K_{m,n}^{(ab)}$ are obtained from $K_{m,n}^{(aa)}$ by replacing $\alpha$ with $-\alpha$ and $\pi$, respectively. This in particular allows to implement spin Hamiltonians with complex couplings [30] (e.g., placing all atoms on sites $a$), whose phase can be tuned via parameter $\alpha$ [see Fig. 2(e)]. Moreover, for $\alpha = \pm \pi/2$, $H_{eff}$ is topologically protected.

Haldane model.—The Haldane model is a prototypical 2D topological lattice [76], the first proposed to observe anomalous quantum Hall effect (QHE), whose photonic version [77] is considered next. Its honeycomb lattice [see Fig. 2(f)] features a unit cell with two cavities $(a$ and $b)$ of frequencies $\omega_c \pm mJ$. Nearest-neighbour (next-nearest-neighbour) cavity-cavity couplings are $J$ ($J'$) with $J' = Jte^{i\phi}$. The bandgap, centered at $\omega_{mid} = \omega_c - 3t \cos \phi J$, has width $\Delta \omega_{gap} = |m| - 3\sqrt{3}t |\sin \phi|$. When $|m| < 3\sqrt{3}t |\sin \phi|$ the model features two topological phases [named I and II in Fig. 3(a)], witnessed under open BCs by a continuum of in-gap edge modes close to the lattice boundaries. These modes carry a stationary chiral current (as in the usual QHE [78]).

It can be shown [57] that a vacancy seeds an in-gap BS only within regions I-II. In particular, for $\phi = \pm \pi/2$ and $m = 0$, the BS has energy $\omega_c$ (bandgap center) and is topologically-protected [63]. Also, similarly to edge modes in open BCs, the BS features a chiral current density (CD) circulating around $v$. A corresponding VDS thus arises for $\omega_0 = \omega_c$ whose photonic component inherits analogous properties [see numerical instance in Fig. 3(b)]. We thus get that the the atom is dressed by a persistent single-photon current orbiting around it, a phenomenon with no 1D analogue.

We note that the $\phi$-$m/t$ plane [cf. Fig. 3(a)] contains a whole set of points outside regions I-II having the same $\Delta \omega_{gap}$ as Fig. 3(b), where however (see above) no BS of $B_v$ occurs. Yet, in each point, for $\omega_0 = \omega_{mid}$ and $g$ small enough, an in-gap dressed BS (which is not a VDS) still arises. This also features a circulating CD, which is yet orders of magnitude weaker than the VDS in Fig. 3(b) [57].

Conclusions.—To sum up, we studied a class of dressed states, dubbed vacancy-like dressed states (VDS), forming at the same energy as the atom. These are in one-to-one correspondence with normal modes of the bare photonic bath with a vacancy replacing the atom: if one among the latter modes has frequency matching the atom’s then a VDS is seeded. Waveguide-QED phenomena where atoms behave as mirrors are naturally interpreted in terms of VDS, based on which we explicitly
linked dressed BIC to purely photonic bound modes. For photonic lattices, VDS in fact provide a general criterion to find dressed bound states (BS), and associated many-body Hamiltonians, inheriting topological properties of the bare photonic lattice. This was used to predict new classes of topological dressed BS in the photonic Creutz-ladder (CL) and Haldane models. Either of these exhibits chiral properties. In the CL model, BS show phase chirality (as opposed to modulus chirality in the SSH model). Haldane-model VDS instead feature a chiral single-photon current encircling the atom.

We expect several other classes of dressed BS can be unveiled by an analogous approach. From a more general perspective, our work suggests a new beneficial link between quantum optics in structured baths and areas such as photonic BIC [55] and topological photonics/condensed matter [59, 79].

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[1] C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, and P. Thickstun, Atom-photon interactions: basic processes and applications (Wiley Online Library, 1992, 2004).
[2] Serge Haroche and Jean-Michel Raimond, Exploring the Quantum: Atoms, Cavities, and Photons (Oxford University Press, 2006).
[3] J. M. Raimond, M. Brune, and S. Haroche, “Manipulating quantum entanglement with atoms and photons in a cavity,” Reviews of Modern Physics 73, 565–582 (2001).
[4] V. P. Bykov, “Spontaneous Emission From a Medium With a Band Spectrum.” Sov. J. Quantum Electron. 4, 861–871 (1975).
[5] Sajeev John and Tran Quang, “Spontaneous emission near the edge of a photonic band gap,” Physical Review A 50, 1764 (1994).
[6] A. G. Kofman, G. Kurizki, and B. Sherman, “Spontaneous and induced atomic decay in photonic band structures,” Journal of Modern Optics 41, 353–384 (1994).
[7] P. Lambropoulos, Georgios M. Nikolopoulos, Torben R. Nielsen, and Seven Bay, “Fundamental quantum optics in structured reservoirs,” Reports on Progress in Physics 63, 455–503 (2000).
[8] Paolo Lonigo, Peter Schmitteckert, and Kurt Busch, “Few-photon transport in low-dimensional systems: Interaction-induced radiation trapping,” Physical Review Letters 104, 023602 (2010).
[9] Ephraim Shahmoon and Gershon Kurizki, “Nonradiative interaction and entanglement between distant atoms,” Phys. Rev. A 87, 033831 (2013).
[10] F. Lombardo, F. Ciccarello, and G. M. Palma, “Photon localization versus population trapping in a coupled-cavity array,” Physical Review A 89, 053826 (2014).
[11] J. S. Douglas, H. Habibian, C. L. Hung, A. V. Gorshkov, H. J. Kimble, and D. E. Chang, “Quantum many-body models with cold atoms coupled to photonic crystals,” Nature Photonics 9, 326–331 (2015).
[12] Giuseppe Calajó, Francesco Ciccarello, Derrick Chang, and Peter Rabl, “Atom-field dressed states in slow-light waveguide QED,” Physical Review A 93, 033833 (2016).
[13] Tao Shi, Ying-Hai Wu, A. González-Tudela, and J. I. Cirac, “Bound States in Boson Impurity Models,” Physical Review X 6, 021027 (2016).
[14] A. González-Tudela and J. I. Cirac, “Quantum Emitters in Two-Dimensional Structured Reservoirs in the Nonperturbative Regime,” Physical Review Letters 119, 143602 (2017), 1705.06673.
[15] A. González-Tudela and J. I. Cirac, “Markovian and non-Markovian dynamics of quantum emitters coupled to two-dimensional structured reservoirs,” Physical Review A 96, 043811 (2017).
[16] Yanbing Liu and Andrew A. Houck, “Quantum electrodynamics near a photonic bandgap,” Nature Physics 13, 48–52 (2017).
[17] A. González-Tudela and J. I. Cirac, “Exotic quantum dynamics and purely long-range coherent interactions in Dirac conelike baths,” Physical Review A 97, 043831 (2018).
[18] A. González-Tudela and J. I. Cirac, “Non-Markovian Quantum Optics with Three-Dimensional State-Dependent Optical Lattices,” Quantum 2, 97 (2018).
[19] Alejandro González-Tudela and Fernando Galve,
“Anisotropic Quantum Emitter Interactions in Two-Dimensional Photonic-Crystal Baths,” ACS Photonics (2019).

[20] Neeraja M. Sundaresan, Rex Lundgren, Guanyu Zhu, Alexey V. Gorshkov, and Andrew A. Houck, “Interacting Qubit-Photon Bound States with Superconducting Circuits,” Physical Review X 9, 011021 (2019).

[21] E. Sánchez-Burillo, L. Martín-Moreno, J. J. García-Ripoll, and D. Zueco, “Single Photons by Quenching the Vacuum,” Physical Review Letters 123, 013601 (2019).

[22] Juan Román-Roche, Eduardo Sánchez-Burillo, and David Zueco, “Bound states in ultrastrong waveguide QED,” (2020), arXiv:2001.07643.

[23] Jonathan D. Hood, Akhisa Goban, Ana Añesjo-García, Mingwu Lu, Su-Peng Yu, E. Chang, and H. J. Kimble, “Atom–atom interactions around the band edge of a photonic crystal waveguide,” Proceedings of the National Academy of Sciences 113, 10507–10512 (2016).

[24] Ludwig Kriinner, Michael Stewart, Arturo Pazmiño, Joohnyuk Kwon, and Dominik Schneble, “Spontaneous emission of matter waves from a tunable open quantum system,” Nature 559, 589–592 (2018).

[25] Michael Stewart, Joohnyuk Kwon, Alfonso Lamuza, and Dominik Schneble, “Fractional Decay of Matter-Wave Quantum Emitters in a Synthetic Bandgap Material,” (2020), arXiv:2003.02816.

[26] Alejandro González-Tudela, C. L. Hung, Darrick E. Chang, J. Ignacio Cirac, and H. J. Kimble, “Subwavelength vacuum latitudes and atom-atom interactions in two-dimensional photonic crystals,” Nature Photonics 9, 320–325 (2015).

[27] T. Shi, Y-H Wu, A González-Tudela, and J I Cirac, “Effective many-body hamiltonians of qubit-photon bound states,” New Journal of Physics 20, 105005 (2018).

[28] M. Bello, G. Platero, J. I. Cirac, and A. González-Tudela, “Unconventional quantum optics in topological waveguide QED,” Science Advances 5, eaaw0297 (2019).

[29] Inaki García-Ecaino, Alejandro González-Tudela, and Jorge Bravo-Abad, “Quantum electrodynamics near photonic weyl points,” arXiv preprint arXiv:1903.07513 (2019).

[30] Eduardo Sánchez-Burillo, Chao Wan, David Zueco, and Alejandro González-Tudela, “Chiral quantum optics in photonic sawtooth lattices,” Phys. Rev. Research 2, 023003 (2020).

[31] F. Caruso, A. W. Chin, A. Datta, S. F. Huelga, and M. B. Plenio, “Highly efficient energy excitation transfer in light-harvesting complexes: The fundamental role of noise-assisted transport,” The Journal of Chemical Physics 131, 105106 (2009).

[32] Peter Lambropoulos and David Petrosyan, Fundamentals of Quantum Optics and Quantum Information (Springer-Verlag Berlin, Heidelberg, 2007) pp. 1–325.

[33] Dibyendu Roy, C. M. Wilson, and Ofer Firstenberg, “Colloquium: Strongly interacting photons in one-dimensional continuum,” Reviews of Modern Physics 89, 21001 (2017).

[34] Zeyang Liao, Xiaodong Zeng, Hyunchul Nha, and M. Suhail Zubairy, “Photon transport in a one-dimensional nanophotonic waveguide QED system,” Physica Scripta 91, 63004 (2016).

[35] Xi Gu, Anton Frisk Kockum, Adam Miranowicz, Yu Xi Liu, and Franco Nori, “Microwave photonics with superconducting quantum circuits,” Physics Reports 718-719, 1–102 (2017).

[36] J. T. Shen and Shanhu Fan, “Coherent photon transport from spontaneous emission in one-dimensional waveguides,” Optics Letters 30, 2001 (2005).

[37] Darrick E. Chang, Anders S. Sørensen, Eugene A. Demler, and Mikhail D. Lukin, “A single-photon transistor using nanoscale surface plasmons,” Nature Physics 3, 807–812 (2007).

[38] Lan Zhou, Z. R. Gong, Yu Xi Liu, C. P. Sun, and Franco Nori, “Controllable scattering of a single photon inside a one-dimensional resonator waveguide,” Physical Review Letters 101, 100501 (2008).

[39] Lan Zhou, H. Dong, Yu-xi Liu, C. P. Sun, and Franco Nori, “Quantum supercavity with atomic mirrors,” Phys. Rev. A 78, 063827 (2008).

[40] F. Ciccarello, D. E. Browne, L. C. Kwek, H. Schomerus, M. Zarcone, and S. Bose, “Quasideterministic realization of a universal quantum gate in a single scattering process,” Phys. Rev. A 85, 050305(R) (2012).

[41] D. E. Chang, L. Jiang, A. V. Gorshkov, and H. J. Kimble, “Cavity QED with atomic mirrors,” New Journal of Physics 14, 63003 (2012).

[42] Mohammad Mirhosseini, Eunjong Kim, Xueyue Zhang, Alp Sipahigil, Paul B. Dieterle, Andrew J. Keller, Ana Añesjo-García, Darrick E. Chang, and Oskar Painter, “Cavity quantum electrodynamics with atom-like mirrors,” Nature (2019).

[43] Gonzalo Ordonez, Kyungsun Na, and Sungyun Kim, “Bound states in the continuum in quantum-dot pairs,” Physical Review A 73, 22113 (2006).

[44] S. Tanaka, S. Garmon, G. Ordonez, and T. Petrosky, “Electron trapping in a one-dimensional semiconductor quantum wire with multiple impurities,” Physical Review B 76, 153308 (2007).

[45] S. Longhi, “Bound states in the continuum in a single-level Fano-Anderson model,” European Physical Journal B 57, 45–51 (2007).

[46] Tommaso Tufarelli, Francesco Ciccarello, and M. S. Kim, “Dynamics of spontaneous emission in a single-end photonic waveguide,” Physical Review A 87, 13820 (2013).

[47] C. González-Ballestero, F. J. García-Vidal, and Esteban Moreno, “Non-Markovian effects in waveguide-mediated entanglement,” New Journal of Physics 15, 73015 (2013).

[48] Tommaso Tufarelli, M. S. Kim, and Francesco Ciccarello, “Non-Markovianity of a quantum emitter in front of a mirror,” Physical Review A 90, 12113 (2014).

[49] E. S. Redchenko and V. I. Yudson, “Decay of metastable excited states of two qubits in a waveguide,” Physical Review A 90, 63829 (2014).

[50] I. C. Hoi, A. F. Kockum, L. Tornberg, A. Pourkabirian, G. Johansson, P. Delsing, and C. M. Wilson, “Probing the quantum vacuum with an artificial atom in front of a mirror,” Nature Physics 11, 1045–1049 (2015).

[51] Paolo Facchi, M. S. Kim, Saverio Pascacio, Francesco V. Pepe, Domenico Pomarico, and Tommaso Tufarelli, “Bound states and entanglement generation in waveguide quantum electrodynamics,” Physical Review A 94, 43839 (2016).

[52] Giuseppe Calajó, Yao-Lung L. Fang, Harold U. Baranger, and Francesco Ciccarello, “Exciting a Bound State in the Continuum through Multiphoton Scattering Plus Delayed Quantum Feedback,” Physical Review Letters 122, 073601 (2019).
González-Tudela, “Qubit-photon corner states in all dimensions,” Physical Review Research 2, 023082 (2020).

Stefano Longhi, “Photonic simulation of giant atom decay,” Opt. Lett. 45, 3017–3020 (2020).

Chia Wei Hsu, Bo Zhen, A. Douglas Stone, John D. Joannopoulos, and Marin Soljacic, “Bound states in the continuum,” Nature Reviews Materials 1, 16048 (2016).

Eunjong Kim, Xueyue Zhang, Vinicius S. Ferreira, Jash Banker, Joseph K. Iverson, Alp Sipahigil, Miguel Bello, Alejandro Gonzalez-Tudela, Mohammad Mirhosseini, and Oskar Painter, “Quantum electrodynamics in a topological waveguide,” (2020), arXiv:2005.03802.

See Supplemental Material at xxx for technical details.

Eleftherios N Economou, Green’s functions in quantum physics Vol. 7 (Springer Science & Business Media, 2006).

M. Z. Hasan and C. L. Kane, “Colloquium: Topological insulators,” Rev. Mod. Phys. 82, 3045–3067 (2010).

I. Carusotto, D. Gerace, H. E. Tureci, S. De Liberato, C. Ciuti, and A. Imamolu, “Fermionized Photons in an Array of Driven Dissipative Nonlinear Cavities,” Physical Review Letters 103, 33601 (2009).

Andreas P. Schnyder, Shimeji Ryu, Akira Furusaki, and Andreas W W Ludwig, “Classification of topological insulators and superconductors in three spatial dimensions,” Phys. Rev. B 78, 195125 (2008).

Alexei Kitaev, Vladimir Lebedev, and Mikhail Feigel’man, “Periodic table for topological insulators and superconductors,” in AIP Conf. Proc., Vol. 1134 (AIP, 2009) pp. 22–30.

Jeffrey C Y Teo and C L Kane, “Topological defects and gapless modes in insulators and superconductors,” Phys. Rev. B 82, 115120 (2010).

Ching-Kai Chiu, Jeffrey C. Y. Teo, Andreas P. Schnyder, and Shimeji Ryu, “Classification of topological quantum matter with symmetries,” Rev. Mod. Phys. 88, 035005 (2016).

W. P. Su, J. R. Schrieffer, and A. J. Heeger, “Solitons in polyacetylene,” Phys. Rev. Lett. 42, 1698 (1979).

W. P. Su, J. R. Schrieffer, and A. J. Heeger, “Soliton excitations in polyacetylene,” Phys. Rev. B 22, 2099–2111 (1980).

Guilherme M.A. Almeida, Francesco Ciccarello, Tony J.G. Apollaro, and Andre M.C. Souza, “Quantum-state transfer in staggered coupled-cavity arrays,” Physical Review A 93, 032310 (2016).

Stefano Longhi, Gian Luca Giorgi, and Roberta Zambrini, “Landau–zener topological quantum state transfer,” Advanced Quantum Technologies 2, 1800090 (2019).

Byeong Chun Shin, “A formula for Eigenpairs of certain symmetric tridiagonal matrices,” Bulletin of the Australian Mathematical Society 55, 249–254 (1997).

J. Sirker, M. Maiti, N. P. Konstantinidis, and N. Sedlmayr, “Boundary fidelity and entanglement in the symmetry protected topological phase of the SSH model,” Journal of Statistical Mechanics: Theory and Experiment 2014 (2014).

Francesco Ciccarello, “Resonant atom-field interaction in large-size coupled-cavity arrays,” Phys. Rev. A 83, 043802 (2011).

Michael Creutz, “End states, ladder compounds, and domain-wall fermions,” Phys. Rev. Lett. 83, 2636 (1999).

Hadiseh Aalaeian, Chung Wai Sandbo Chang, Mehran Vahdani Moghaddam, Christopher M. Wil- son, Enrique Solano, and Enrique Rico, “Creating lattice gauge potentials in circuit qed: The bosonic creutz ladder,” Phys. Rev. A 99, 053834 (2019).

Abhijeet Alase, Emilio Cobanera, Gerardo Ortiz, and Lorenza Viola, “Generalization of bloch’s theorem for arbitrary boundary conditions: Theory,” Phys. Rev. B 96, 195133 (2017).

Emilio Cobanera, Abhijeet Alase, Gerardo Ortiz, and Lorenza Viola, “Generalization of bloch’s theorem for arbitrary boundary conditions: Interfaces and topological surface band structure,” Phys. Rev. B 98, 245423 (2018).

F. D.M. Haldane, “Model for a quantum hall effect without landau levels: Condensed-matter realization of the “parity anomaly”,” Phys. Rev. Lett. 61, 2015 (1988).

Marco Polini, Francisco Guinea, Maciej Lewenstein, Hari Manoharan, and Vittorio Pellegrini, “Artificial honeycomb lattices for electrons, atoms and photons,” Nature nanotechnology 8, 625 (2013).

K. v. Klitzing, G. Dorda, and M. Pepper, “New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance,” Phys. Rev. Lett. 45, 494 (1980).

Tomoki Ozawa, Hannah M. Price, Alberto Amo, Nathan Goldman, Mohammad Hafezi, Ling Lu, Mikael C. Rechtsman, David Schuster, Jonathan Simon, Oded Zilberberg, and Iacopo Carusotto,”Topological photonics,” Rev. Mod. Phys. 91, 015006 (2019).
This Supplemental Material presents technical proofs of some properties and theorems discussed in the main text. We note that most of Section SM4 deals with essentially known material, which is yet not easily found in explicit form in the literature. This is used to formulate a general necessary and sufficient condition for an in-gap dressed BS to exist.
SM1. PROOF OF EQS. (5) AND (6)

We decompose Hamiltonian $H_B$ as

$$H_B = H_v + H_{B_v} + V_{v-B_v}, \quad (S1)$$

with $H_v = \omega_v b_v^\dagger b_v$ the free Hamiltonian of cavity $v$ and

$$V_{v-B_v} = b_v^\dagger \sum_{i \neq v} J_{v,i} b_i + \text{H.c.} \quad (S2)$$

the coupling Hamiltonian between cavity $v$ and $B_v$ (bath with vacancy). Now note that

$$H |\psi\rangle = H_{B_v} |\psi\rangle + \sum_{i \neq v} J_{v,i} |\psi_i\rangle |v\rangle \quad ,$$

where we used decomposition (S1) and (see main text) $\psi_v = \langle v |\psi\rangle = 0$. On the other hand, using (3), Eq. (4) can be rearranged as

$$H |\psi\rangle = \omega_0 |\psi\rangle - \varepsilon (H - \omega_0) |e\rangle = \omega_0 |\psi\rangle - g\varepsilon |v\rangle . \quad (S4)$$

Upon comparison with (S3), we get

$$H_{B_v} |\psi\rangle + \sum_{i \neq v} J_{v,i} |\psi_i\rangle |v\rangle = \omega_0 |\psi\rangle - g\varepsilon |v\rangle . \quad (S5)$$

Projecting either side onto $|i \neq v\rangle$, we just end up with (5) in matrix form, hence (5) holds true. Projecting instead (S5) onto $|v\rangle$ we get Eq. (6).

SM2. MANY-ATOM VACANCY-LIKE DRESSED STATES

The generalization of Hamiltonian (1) to $N_a$ atoms indexed by $\nu = 1, ..., N_a$ with the $\nu$th atom coupled to cavity $v$ reads

$$H = \omega_0 \sum_{\nu=1}^{N_a} \sigma_{\nu+} \sigma_{\nu-} + H_B + g \sum_{\nu=1}^{N_a} (b_v^\dagger \sigma_{\nu-} + \text{H.c.}) . \quad (S6)$$

The single-excitation subspace is spanned by $\{|e_\nu\rangle\},\{|i\rangle\}$ with $|e_\nu\rangle = \sigma_{\nu+}|G\rangle$. A vacancy-like dressed state (VDS) is defined as

$$|\Psi\rangle = \sum_{\nu=1}^{N_a} \varepsilon_\nu |e_\nu\rangle + |\psi\rangle \quad (S7)$$

with $H |\Psi\rangle = \omega_0 |\Psi\rangle$. Projecting the latter equation onto $|\nu\rangle$ yields $\omega_0 \varepsilon_\nu + g \psi_\nu = \omega_0 \varepsilon_\nu$, hence $\psi_\nu = 0$ for any $\nu$.

Let $B_v$ be now the bare bath with cavities $\nu = 1, ..., N_a$ replaced by vacancies, whose free Hamiltonian $H_{B_v}$ is obtained from (2) by restricting the sum to $i,j \neq \nu$ (i.e., all indexes $i,j$ different from $\nu = 1, ..., N_a$). To show that $H_{B_v} |\psi\rangle = \omega_0 |\psi\rangle$, we decompose $H_B$ as

$$H_B = H_v + H_{B_v} + V_{v-B_v}, \quad (S8)$$

with $H_v = \sum_{\nu=1}^{N_a} \omega_\nu b_v^\dagger b_\nu$ and

$$V_{v-B_v} = \sum_{\nu=1}^{N_a} \sum_{i \in B_v} J_{\nu,i} b_v^\dagger b_i + \text{H.c.,} \quad (S9)$$

where the second sum runs over all cavities of $B_v$. 

Now note that

\[ H |\psi\rangle = H_{B_v} |\psi\rangle + \sum_{\nu=1}^{N_a} \sum_{i \in B_v} J_{\nu,i} |\psi^\nu_i\rangle, \quad (S10) \]

where we used decomposition (S8) and \( \psi_\nu = 0 \) for any \( \nu \). On the other hand, using (S7), \( H |\Psi\rangle = \omega_0 |\Psi\rangle \) can be rearranged as

\[ H |\psi\rangle = \omega_0 |\psi\rangle - \sum_{\nu=1}^{N_a} \varepsilon_\nu (H - \omega_0) |\psi^\nu\rangle = \omega_0 |\psi\rangle - g \sum_{\nu=1}^{N_a} \varepsilon_\nu |\nu\rangle. \]

Upon comparison with (S10), we get

\[ H_{B_v} |\psi\rangle + \sum_{\nu=1}^{N_a} \sum_{i \in B_v} J_{\nu,i} |\psi^\nu_i\rangle = \omega_0 |\psi\rangle - g \sum_{\nu=1}^{N_a} \varepsilon_\nu |\nu\rangle. \]

Finally, projecting either side onto \(|i\rangle\) with \( i \in B_v \), we just end up with \( H_{B_v} |\psi\rangle = \omega_0 |\psi\rangle \) in matrix form, completing the proof.

Projecting onto \( \nu = 1, ..., N_a \) instead yields

\[ g \varepsilon_\nu + \sum_{i \in B_v} J_{\nu,i} \psi^\nu_i = 0, \quad (S11)\]

which can be used to express the atomic amplitudes in terms of \(|\psi\rangle\).

**SM3. ATOM AS A MIRROR**

**SM3.1. Perfect reflection of a resonant photon**

Consider the model in the main text [cf. Eq. (1)] in the case that \( B \) is a discrete, infinite waveguide described by the usual tight-binding Hamiltonian (cavities are indexed by integer \( n \), which for the present lattice coincides with the cell index)

\[ H_B = \omega_c \sum_{n=-\infty}^{\infty} b_n^\dagger b_n - J \sum_{n=-\infty}^{\infty} (b_n^\dagger b_{n+1} + \text{H.c.}). \quad (S12) \]

The system can be thought as a homogeneous coupled-cavity array with \( \omega_c \), the frequency of each cavity and \( -J \) the cavity-cavity coupling strength. The waveguide spectrum is \( \omega_k = \omega_c - 2J \cos k \), with \( k \in [-\pi, \pi] \) the wavevector whose corresponding group velocity is \( v_k = 2J \sin k \).

Setting \( \nu = 0 \) (atom coupled to cavity \( n = 0 \)), \( B_v \) is the union of the semi-infinite arrays \( B_v^L \) and \( B_v^R \), respectively defined by \( n \in [-\infty, -1] \) and \( n = 1, 2, ..., \) as sketched in Fig. 2(b) of the main text. The normal modes of \( B_v \) are thus

\[ b_k = \sum_{n=-\infty}^{-1} \sin(kn) b_n, \quad b'_k = \sum_{n=1}^{\infty} \sin(kn) b_n \quad (0 \leq k \leq \pi) \quad (S13)\]

respectively corresponding to \( B_v^L \) and \( B_v^R \) (any normal mode of \( B_v^{L(R)} \) is trivially also a normal mode of \( B_v \) because \( B_v^R \) and \( B_v^L \) are uncoupled). The corresponding normal frequencies are \( \omega_k = \omega_c - 2J \cos k \) and \( \omega'_k = \omega_c - 2J \cos k \). Henceforth, we focus on \( B_v^L \) (an analogous reasoning will apply to \( B_v^R \)).

The single-photon eigenstates of \( H_{B_v} \) corresponding to modes \( b_k \) are \(|\psi_k\rangle\) with energy \( \omega_k \) and wavefunction \( \langle n |\psi_k\rangle = \sin(kn) \) for \( n \leq -1 \) and \( 0 \) otherwise. Hence, a VDS [cf. Eq. (3)] occurs for \(|\psi\rangle = |\psi_{k0}\rangle\), where \( \omega_{k0} = \omega_0 \). Noting that we can write \( \langle n |\psi\rangle \propto e^{ikn} + e^{-ikn} \) with \( r = -1 \), we see that the VDS is a scattering state describing a left-incoming photon fully reflected back from the atom as if this were a perfect mirror. Condition (6) in this case simply reads \( g \varepsilon - J \sin k = 0 \), hence \( \varepsilon = J \sin k/g \propto v_k/g \), matching known results obtained via scattering theory (see e.g. Ref. [1]).
SM3.2. Dressed bound states (BS) in the continuum (BIC): one atom in a semi-infinite waveguide

The previous infinite waveguide is now replaced by a semi-infinite waveguide made out of cavities \( n = 1, 2, ..., \) hence in each sum of Hamiltonian (S12) \( n \) now starts from 1 (equivalently, one can think of the right half of an infinite waveguide with a perfect mirror on site \( n = 0 \)). Placing the atom at site \( d \) (thus \( v = d \)), \( B_v \) [see Fig.2(c) of the main text] is the union of the finite array \( n = 1, 2, ..., d - 1 \) (\( B_v^L \)) and the semi-infinite lattice \( n = d + 1, d + 2, ... \) (dubbed \( B_v^R \)). The normal frequencies of \( B_v^L \) are the same as in the infinite-waveguide case with normal modes \( b_k^L \) [cf. Eq. (S13)] now displaced by the amount \( v \). These are at the same time a (continuous) subset of normal frequencies and normal modes of \( B_v \) (since \( B_v^L \) and \( B_v^R \) are disjoint). The remaining frequencies and normal modes of \( B_v \) are those of \( B_v^R \) (discrete). These are \( \omega_{km} = \omega_c - 2J \cos k_m \), with \( k_m = m\pi/d \) and \( m = 1, 2, ..., d \), and

\[
b_{km} = \sqrt{\frac{2}{d}} \sum_{n=1}^{d-1} \sin(k_m n) b_n .
\]

The corresponding single-photon bound eigenstates are \( |\psi_{km}\rangle \), with \( \langle n |\psi_{km}\rangle = \sqrt{2/d} \sin(k_m n) \) for \( 1 \leq n \leq d - 1 \) and \( \langle n |\psi_{km}\rangle = 0 \) for \( n \geq d \), and energies \( \omega_{km} \). A VDS (3) arises when one of these energies resonates with the atom, i.e., there exists a value of \( m \) such that \( \omega_{km} = \omega_0 \). By defining \( k_0 \) such that \( \omega_{k=k_0} = \omega_0 \), the VDS condition can be expressed in terms of wavevectors simply as \( k_m = k_0 \), that is

\[
k_0 d = m\pi .
\]

Thus, if \( m \) fulfills (S15), we set \( |\psi\rangle = |\psi_{km}\rangle \). Since \( |\psi\rangle \) is normalized, the corresponding VDS [cf. Eq. (7)] is a BS in the continuum or BIC (the atom frequency lies within the photonic band). To get the mixing angle \( \theta \) [cf. Eq. (8) where \( \varphi = 0 \) in this case], we use that

\[
\langle d-1 |\psi\rangle = \psi_{d-1} = \sqrt{\frac{2}{d}} \sin[k_m(d-1)] = \sqrt{\frac{2}{d}} (-1)^{m+1} \sin k_m = \sqrt{\frac{2}{d}} (-1)^{m+1} \frac{v_0}{2J} ,
\]

where we set \( v_0 = v_{km} \) (recall that \( v_k = 2J \sin k \)). Hence, \( \theta = (-1)^{m+1} \arctan[\sqrt{2d (g/v_0)}] \), which once plugged into (7) yields the dressed BIC [2–4]

\[
|\Psi\rangle = \frac{1}{\sqrt{1 + \frac{\Gamma \tau}{2}}} \left( |e\rangle + (-1)^{m+1} \sqrt{\frac{2\Gamma}{v_0}} \sum_{n=1}^{d-1} \sin(k_m n) |n\rangle \right) ,
\]

where we introduced the decay rate \( \Gamma = 2g^2/v_0 \) and time delay \( \tau = 2d/v_0 \).

SM3.3. Dressed BIC: two atoms in an infinite waveguide

A two-atom dressed BIC closely related to the previous one occurs in a waveguide, this time infinite. Let atoms 1 and 2 be coupled to cavities \( n = 0 \) and \( n = d \), respectively. The setup is obtained from that in Fig.2(c) of the main text by adding cavities \( n = -\infty, ..., 0 \) to the waveguide with cavity \( n = 0 \) \((n = d) \) coupled to atom 1 \((2) \). Domain \( B_v \) (see Section SM2) is now the union of two semi-infinite waveguides (comprising sites \( n < 0 \) and \( n > d \), respectively) plus the same cavity as in the previous section (i.e., the finite set of sites \( n = 1, 2, ..., d \). As in the previous case, modes (S14) and \( \omega_{km} \) are thus bound normal modes and normal frequencies of \( B_v \). A two-atom bound VDS (S7) will thus arise with \( |\psi\rangle \propto |\psi_{km}\rangle \) provided that \( m \) fulfills condition (S15) [note that in (S7) \( |\psi\rangle \) is not normalized]. Analogously to (S16),

\[
\langle 1 |\psi_{km}\rangle = \sqrt{\frac{2}{d}} \frac{v_0}{2J} , \langle d-1 |\psi_{km}\rangle = \sqrt{\frac{2}{d}} (-1)^{m+1} \frac{v_0}{2J} .
\]

Thus, using Eq. (S11),

\[
\varepsilon_1 = \frac{J}{g} \langle 1 |\psi_{km}\rangle = \sqrt{\frac{2}{d}} \frac{v_0}{2g} , \varepsilon_2 = \frac{J}{g} \langle d-1 |\psi_{km}\rangle = \sqrt{\frac{2}{d}} (-1)^{m+1} \frac{v_0}{2g} ,
\]

entailing \( \varepsilon_1 = (-1)^{m+1} \varepsilon_2 \). Therefore, through Eq. (S7), we get that the (unnormalized) VDS corresponding to \( |\psi_{km}\rangle \) is

\[
|\Psi\rangle = \frac{v_0}{\sqrt{g}} |\Phi^\pm\rangle + |\psi_{km}\rangle ,
\]
where $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle \pm |e_2\rangle)$ and the plus (minus) sign holds for odd (even) $m$. Upon normalization, we end up with

$$|\Psi\rangle = \frac{1}{\sqrt{1 + \Gamma^2}} \left( |\Phi^+\rangle + \sqrt{\frac{\Gamma}{\nu}} \sum_{n=1}^{d-1} \sin(k_m n) |n\rangle \right),$$

matching the known expression of the two-atom dressed BIC obtained with other methods (see e.g. Refs. [4, 5]).

**SM4. VACANCY IN A TRANSLATIONALLY-INvariant LATTICE: UNICITY OF BS AND GENERAL CONDITION FOR HAVING A VDS**

Here, we will show that in a translationally-invariant lattice with a single vacancy (i.e., $B_v$ with $B$ a translationally-invariant lattice) there is at most one non-degenerate bound state (BS) in each internal bandgap.

Consider a generic translationally-invariant $D$-dimensional lattice ($D = 1, 2, 3$) with finite-range interactions and $d$ sites in each unit cell. A vacancy on site $\alpha_v$ in the cell $n_v$ breaks translational invariance transforming the lattice Hamiltonian as $H_B \rightarrow H_{B_v} = H_B + H_1$ with the perturbation Hamiltonian defined by

$$H_1 = \epsilon |n_v, \alpha_v\rangle \langle n_v, \alpha_v| \quad \text{for} \quad \epsilon \to \infty. \tag{S20}$$

Consider the Green functions of the unperturbed and perturbed Hamiltonians, respectively defined as $G_B(z) = (z - H_B)^{-1}$ and $G_{B_v}(z) = (z - H_{B_v})^{-1}$, which fulfill [6]

$$G_{B_v}(z) = G_B(z) + G_B(z)T(z)G_B(z) \tag{S21}$$

with

$$T(z) = H_1 \frac{1}{1 - G_B(z)H_1} = |n_v, \alpha_v\rangle \langle n_v, \alpha_v| \frac{1/\epsilon - \langle G_B(z)\rangle_v}{1/\epsilon - \langle G_B(z)\rangle_v} \tag{S22}$$

with $\langle \ldots \rangle_v = \langle n_v, \alpha_v| \ldots |n_v, \alpha_v\rangle$.

The bound states of $H_{B_v}$ correspond to the poles $z = \omega_p$ of $G_{B_v}(z)$, which are the solutions of the equation

$$\langle G_B(\omega_p)\rangle_v = 0. \tag{S23}$$

This cannot be satisfied by $\omega_p$ inside a band of $H_B$ since within each band $\langle G_B(\omega)\rangle_v$ has a non-zero imaginary part (this being proportional to the density of states [6]).

For $z = \omega \in \mathbb{R}$ outside of bands of $H_B$, $G_B$ is a Hermitian operator, analytical function of $z$, and $\langle G'_B(\omega)\rangle_v = -(\omega - H_B)^{-2}v = -\langle G_B(\omega)^2\rangle_v \leq 0$ ($G'$ is the derivative with respect to $z$). This shows that, in each given bandgap (including the outermost semi-infinite bandgaps), $\langle G_B(\omega)\rangle_v$ is a monotonic function of $\omega$. Therefore, there is at most one solution of Eq. (S23) within each energy interval where $\langle G_B(\omega)\rangle_v$ is a continuous function of $\omega$, i.e., within each finite and semi-infinite bandgap.

However, notice that $\lim_{\omega \to \pm \infty} G_B(\omega) = 0$, hence there is no finite value of $\omega$ satisfying Eq. (S23) in the two external semi-infinite bandgaps (above and below the uppermost and lowermost bands). Therefore Eq. (S23) can only be fulfilled within an internal finite bandgap.

Finally, we derive the degeneracy $\delta_p$ of the BS of $H_{B_v}$ for a given solution $\omega_p$. The projector onto the eigenspace of $H_{B_v}$ corresponding to eigenvalue $\omega_p$ is the residue of $G_{B_v}(z)$ at the pole $z = \omega_p$. By plugging the expansion

$$\langle G_B(z)\rangle_v = 1/\epsilon + \langle G'_B(\omega_p)\rangle_v (z - \omega_p) + o(z - \omega_p)$$

into Eq. (S21) and using Eq. (S22) one gets

$$\delta_p = \text{Tr} \{ \text{Res}_{\omega_p} [G_{B_v}] \} = -\text{Tr} \left\{ \frac{G_B(\omega_p) |n_v, \alpha_v\rangle \langle n_v, \alpha_v| G_B(\omega_p)}{\langle G'_B(\omega_p)\rangle_v} \right\} = -\frac{\langle G'_B(\omega_p)\rangle_v}{\langle G_B(\omega_p)\rangle_v} = 1. \tag{S24}$$

Thus each solution of Eq. (S23) corresponds to a non-degenerate BS.

Combining the above with the main text discussion about VDS, it turns out that a general necessary and sufficient condition for an in-gap VDS to arise is

$$\langle G_B(\omega_0)\rangle_v = 0 \tag{S25}$$

with $\omega_0$ the atomic transition frequency.
SM5. STABILITY OF THE VDS AGAINST DETUNING

Let $B$ be a translationally-invariant lattice and assume that $B$ admits an in-gap BS $|\psi\rangle$ of energy $\omega_\psi$. Then, if the atomic frequency is tuned such that $\omega_0 = \omega_\psi$, a VDS exists with energy $\omega_\psi = \omega_0 = \omega_\psi$. Here, assuming weak coupling, we ask how sensitive is the VDS to the condition $\omega_0 = \omega_\psi$.

We thus introduce a small detuning between the atom and BS $|\psi\rangle$, $\Delta = \omega_0 - \omega_\psi$, which corresponds to a perturbation of the total Hamiltonian according to $H \rightarrow H + \Delta \sigma_+ \sigma_-$. The unperturbed Hamiltonian can be written as

$$H = \omega_\psi \langle \Psi \rangle |\Psi\rangle + \sum_m \omega_m |\Psi_m\rangle \langle \Psi_m|,$$

where $|\Psi\rangle$ is the VDS [cf. Eq. (7)] and $|\Psi_m\rangle$ all the other single-photon dressed states. Note that $|\omega_m - \omega_\psi| \geq \frac{1}{2} \Delta \omega_{\text{gap}}$. Since $|\Psi\rangle$ is non-degenerate, we can apply standard second-order perturbation theory. Accordingly, the corrected dressed state $|\Psi_\Delta\rangle$ (such that $|\Psi\rangle_{\Delta=0} = |\Psi\rangle$ with $|\Psi\rangle$ the ideal VDS) is given by [cf. Eq. (7)]

$$|\Psi_\Delta\rangle = |\Psi\rangle + \Delta \cos \theta \sum_m \frac{\langle \Psi_m | e \rangle}{\omega_\psi - \omega_m} |\Psi_m\rangle + \Delta^2 \cos \theta \sum_m \frac{\langle \Psi_m | e \rangle}{\omega_\psi - \omega_m} \left( \sum_m \frac{\langle \Psi_m | e \rangle^2}{\omega_\psi - \omega_m} - \cos^2 \theta \right) |\Psi_m\rangle,$$

and the corrected energy by

$$\omega_\Delta = \omega_\psi + \Delta |\cos \theta|^2 + \Delta^2 |\cos \theta|^2 \sum_m \frac{|\langle e | \Psi_m \rangle|^2}{\omega_\psi - \omega_m}.$$ (S28)

On the other hand, to first order in $g$ (we are assuming weak coupling), the unperturbed eigenstates $|\Psi_m\rangle$ can be expressed as

$$|\Psi_m\rangle = |\beta_m\rangle + g \frac{\langle v | \beta_m \rangle}{\omega_\psi - \omega_m} |e\rangle,$$

with $|\beta_m\rangle$ single-photon eigenstates of $H_B$ such that $H_B |\beta_m\rangle = \omega_m |\beta_m\rangle$ and $V = g (b_+ \sigma_+ + b_- \sigma_-)$ [cf. Eq. (1) in the main text]. Thus, so long as both $g$ and $\Delta$ are small compared to the bandgap width $\Delta \omega_{\text{gap}}$, recalling that $|\omega_m - \omega_\psi| \geq \frac{1}{2} \Delta \omega_{\text{gap}}$, up to first order the VDS wavefunction is insensitive to the detuning $\Delta$, only acquiring a small energy shift $\Delta |\cos \theta|^2$ [cf. Eq. (S28)].

SM6. MANY-ATOM EFFECTIVE HAMILTONIANS MEDIATED BY IN-GAP VDS

SM6.1. Bound VDS in terms of the $H_B$’s normal modes

Let $\{|k\rangle\}$ be the single-photon eigenstates of $H_B$ such that $H_B |k\rangle = \omega_k |k\rangle$, where $k$ in the present subsection generically labels the $B$’s normal modes. These states can be used as a basis to expand $|\psi\rangle$ [cf. Eq. (3)] as $|\psi\rangle = \sum_k \psi_k |k\rangle$. Then Eq. (4) is equivalent to the coupled equations

$$\omega_0 \varepsilon + \sum_k g \langle v | k \rangle \psi_k = \omega_0 \varepsilon, \quad \omega_k \psi_k + g \langle k | v \rangle \varepsilon = \omega_0 \psi_k,$$

(S30)

Solving the latter equation for $\psi_k$ we get

$$\psi_k = \frac{g \langle k | v \rangle}{\omega_0 - \omega_k} \varepsilon.$$ (S31)

When the VDS (3) is bound, the normalization condition $|\varepsilon|^2 + \sum_k |\psi_k|^2 = 1$ must hold. Using (S31) and solving for $\varepsilon$ yields

$$\varepsilon = \frac{1}{\sqrt{1 + g^2 \sum_k \frac{|\langle k | v \rangle|^2}{(\omega_0 - \omega_k)^2}}}.$$ (S32)
Here, we assumed $\varepsilon \equiv |\varepsilon|$ (always possible by attaching to $|\Psi\rangle$ a suitable global phase factor). Replacing in (S31) we end up with

$$\tilde{\psi}_k = \frac{g \langle k| v \rangle}{\omega_0 - \omega_k} \frac{1}{\sqrt{1 + g^2 \sum_k \frac{|\langle k| v \rangle|^2}{(\omega_0 - \omega_k)^2}}}.$$  \hfill (S33)

Here, we added the tilde to recall that (S33) is unnormalized, thus different from that appearing in (7). The two are related as $\tilde{\psi}_k = e^{i\phi} \sin \theta \psi_k$.

Eqs. (S32) and (S33) express a bound VDS as an explicit function of $g_k = g \langle k| v \rangle$, i.e., the coupling strength between the atom and mode $k$ of $B$ (note that it is $B$, not $B_c$). We point out that, for a general dressed BS which is not necessarily also a VDS, this functional dependence is implicit in that in Eqs. (S32)-(S33) $\omega_0$ is replaced by the dressed-state energy which is a priori unknown when the dressed BS is not a VDS.

For $g = 0$ (zero coupling), Eqs. (S32) and (S33) yield $\varepsilon = 1$ and $\tilde{\psi}_k = 0$ as expected ($|\Psi\rangle = |\varepsilon\rangle$). The next order of approximation is

$$\varepsilon \simeq 1, \quad \tilde{\psi}_k \simeq \frac{g \langle k| v \rangle}{\omega_0 - \omega_k},$$  \hfill (S34)

(note that normalization is ensured to leading order). In terms of basis $\{|i\}\}$ of $B$ (real-space representation), using that $\tilde{\psi}_i = \sum_k \tilde{\psi}_k \langle i|k \rangle$, we get

$$\tilde{\psi}_i = g \sum_k \frac{\langle k| v \rangle \langle i|k \rangle}{\omega_0 - \omega_k},$$  \hfill (S35)

which we recall that holds in the weak-coupling limit.

### SM6.2. Photonic lattice: many-atom effective Hamiltonian

Let now $B$ be a translationally-invariant photonic lattice, whose unit cell has $d$ cavities. Its free Hamiltonian is written in terms of normal modes as

$$H_B = \sum_{\mu, \mathbf{k}} \omega_{\mu, \mathbf{k}} \beta_{\mu, \mathbf{k}}^\dagger \beta_{\mu, \mathbf{k}}$$  \hfill (S36)

with $\mu$ a band index and $\mathbf{k}$ now standing for a (generally three-dimensional) wave vector. Denoting by $n$ the cell index, let $\mathbf{x}_{n, \alpha}$ with $\alpha = 1, \ldots, d$ be the (generally three-dimensional) position of the $\alpha$th cavity in cell $n$. When applied to the present lattice, Eq. (S35) thus specifically reads

$$\tilde{\psi}_{n, \alpha} = g \sum_{\mu, \mathbf{k}} \frac{\langle \mu| \mathbf{k} \rangle \langle n_{\alpha}, \alpha \rangle \langle n, \alpha|\mu|\mathbf{k} \rangle}{\omega_0 - \omega_{\mu, \mathbf{k}}}$$  \hfill (S37)

(the atom is coupled to the $\alpha$th cavity of cell $n_{\alpha}$). Here, $|\mu, \mathbf{k}\rangle = \beta_{\mu, \mathbf{k}}^\dagger |\text{vac}\rangle$, which due to translational invariance has the real-space Bloch-form form $|\mu| = \sum_{n, \alpha} c_{\mu, \mathbf{k}, \alpha} e^{i \mathbf{k} \cdot \mathbf{x}_{n, \alpha}} |n, \alpha\rangle$. Therefore,

$$\tilde{\psi}_{n, \alpha} = g \sum_{\mu, \mathbf{k}} \frac{c_{\mu, \mathbf{k}, \alpha}^* c_{\mu, \mathbf{k}, \alpha} e^{i \mathbf{k} \cdot (\mathbf{x}_{n, \alpha} - \mathbf{x}_{\alpha})}}{\omega_0 - \omega_{\mu, \mathbf{k}}}$$  \hfill (S38)

where we set $\mathbf{x}_{\alpha} = \mathbf{x}_{n_{\alpha}, \alpha}$, and (in view of the many-atom generalization) added superscript $\nu$.

Consider next $N_n$ identical atoms indexed by $\nu = 1, \ldots, N_n$ with the $\nu$th atom coupled to cavity $(n_{\nu}, \alpha_{\nu})$. The interaction Hamiltonian then reads

$$V = g \sum_{\nu} \frac{b^\dagger_{n_{\nu}, \alpha_{\nu}} \sigma_{\nu-} + \text{H.c.}}{2} = \sum_{\nu} \sum_{\mu, \mathbf{k}} \left( g_{\mu, \mathbf{k}}^\nu \beta_{\mu, \mathbf{k}}^\dagger + \text{H.c.} \right)$$ \hfill (S39)

with

$$g_{\mu, \mathbf{k}}^\nu = g c_{\mu, \mathbf{k}, \alpha_{\nu}}^* e^{-i \mathbf{k} \cdot \mathbf{x}_{\nu}},$$  \hfill (S40)
where we set $x_\nu = x_{n_\nu,\alpha_\nu}$.

Let the atomic transition frequency $\omega_0$ lie well within a bangap of $H_B$. Thus, if $g$ is much smaller than the bandgap width $\Delta \omega_{\text{gap}}$, the atoms are far-detuned from all lattice modes $\beta_{\mu,k}$. Then it can be shown in various ways [7, 8] that the photonic degrees of freedom can be adiabatically eliminated giving rise to the effective decoherence-free atom-atom interaction Hamiltonian

$$H_{\text{eff}} = \sum_{\nu \nu'} K_{\nu \nu'} \sigma_{\nu'} + \sigma_{\nu} + \text{H.c.} \tag{S41}$$

with the second-order coupling strengths given by

$$K_{\nu \nu'} = \frac{1}{2} \sum_{\mu,k} \frac{g^2_{\mu,k} \delta_{\nu'} (\nu)}{\omega_0 - \omega_{\mu,k}} = \frac{g^2}{2} \sum_{\mu,k} \frac{C_{\mu,k,\alpha} \theta_{\mu,k,\alpha} e^{i k (x_\nu - x_{\nu'})}}{\omega_0 - \omega_{\mu,k}}, \tag{S42}$$

where in the last identity we used (S40). Upon comparison with (S38), we thus end up with

$$K_{\nu \nu'} = \frac{1}{2} g \psi_{n_\nu,\alpha_\nu} = \frac{1}{2} g \sin \theta(e^{i \phi} \psi (\nu')_{n_\nu,\alpha_\nu} \tag{S43}$$

where in the last identities we introduced $|\psi\rangle$ (normalized) using (7) and added a subscript $\nu'$ to the angles (8) since these depend on the position of the $\nu$th atom. Finally, in order to ensure consistency with Eqs. (7)-(8) (where $\theta$ and $\varphi$ are at all orders in $g$), we must approximate $\theta = \arctan |\eta| \approx |\theta|$. Thus, recalling that $\varphi = \arctan \eta$, we get $\sin \theta e^{i \varphi} \approx \theta e^{i \varphi} = \eta$ so as to end up with

$$K_{\nu \nu'} = \frac{1}{2} g \varphi (\nu') \psi (\nu')_{n_\nu,\alpha_\nu} \tag{S44}$$

where for brevity we set $\nu \equiv (n_\nu,\alpha_\nu)$.

**SM7. VDS IN THE PHOTONIC SSH MODEL**

When $v = 1$ (atom coupled to cavity $a$ in cell $n = 1$) and for $\delta > 0$, the photonic wavefunction is non-zero only on even sites (i.e., cavities $b$) and reads (see, e.g., Ref. [9])

$$\psi_{2n} = \frac{2 \sqrt{\delta}}{1 + \delta} \left( \frac{\delta - 1}{\delta + 1} \right)^{N-n} \tag{S45}$$

with $n = 2, \ldots, N$. For $\delta < 0$, this must be mirror-reflected around $v = 1$ making the simultaneous replacement $\delta \rightarrow -\delta$. Plugging this into (8) directly yields the dressed BS (7) with $\theta = \arctan [g/(2J\sqrt{\delta})] \approx \theta$ and $\varphi = 0$.

**SM8. THEOREM FOR 1D LATTICES: BS OF $B_v$ FROM EDGE STATES UNDER OPEN BCS**

Consider a 1D photonic lattice $B$. Note that the lattice under open BCs is obtained from the translationally-invariant lattice $B$ by removing an entire cell (instead of a single cavity as in the definition of $B_v$). Here, we derive a condition allowing to deduce both the existence and wavefunction of the photonic BS of $B_v$ from in-gap edge states under open BCs (if any).

Let $B$ have $N$ unit cells, each with $d$ cavities. Then the most general free Hamiltonian of the lattice can be written as

$$H^N_\lambda = \sum_{n=1}^{N} \sum_{\alpha,\alpha' = 1}^{d} b_{n,\alpha}^\dagger b_{n,\alpha'} + \sum_{r=1}^{R} \sum_{\alpha = 1}^{d} \sum_{\alpha' = 1}^{d} (b_{n,\alpha}^\dagger J^r_{\alpha\alpha'}, b_{n+r,\alpha'} + \text{H.c.}) \tag{S46}$$

Here, the $d \times d$ Hermitian matrix $h_{\alpha\alpha'}$ specifies the intra-cell cavity couplings (off-diagonal entries) and on-site cavity frequencies (diagonal entries), while the (generally non-symmetric) $d \times d$ matrix $J^r_{\alpha\alpha'}$ contains all the inter-cell cavity-cavity couplings with range $r = 1, 2, \ldots, R$ (for nearest-neighbor cells, $r = 1$; $R$ is the maximum range). Here, we conveniently introduced the notation $H^N_\lambda$, where the superscript is the number of cells while the subscript $\lambda = P, O, v$.
specifies the BCs with \( P \) standing for periodic BCs (translationally-invariant lattice \( B \)), \( O \) for the lattice subject to open BCs (\( B \) without an entire cell) and \( v \) for lattice \( B_v \). Thus, to connect with the main-text notation, \( H_B = \mathcal{H}_P^N \), \( H_B = \mathcal{H}_O^N \).

We note that, if \( R > 1 \) one can always redefine the lattice unit cell such that it contains \( Rd \) cavities. Accordingly, without loss of generality, henceforth we focus on nearest-neighbor inter-cell couplings and thus set \( R = 1 \) (it is understood that, if \( R > 1 \), \( d \) must be intended as \( Rd \)). It is convenient to introduce a vector-matrix formalism allowing to rewrite (S46) as

\[
H^N = \sum_{n=1}^{N} \varphi_n^\dagger \cdot h \cdot \varphi_n + \sum_{n=1}^{N} (\varphi_n^\dagger \cdot J \cdot \varphi_{n+1} + \text{H.c.})
\]  

(S52)

with \( h_0 \) and \( h_1 \) the matrices corresponding to rates \( h_{ij} \) and \( J_{ij} \), respectively, and where \( \varphi_n = (b_{n,1} \ldots b_{n,d})^T \).

Consider now a vacancy on site \( \alpha_v \) of cell \( n = 0 \). It is convenient to define \( \tilde{\varphi}_0 = (I_d - P) \cdot \varphi_0 \), where \( P \) is the \( d \)-dimensional projector on the vacancy site (inside the unit cell); thus \( \tilde{\varphi}_0 \) is simply \( \varphi_0 \) with the \( \alpha_v \)th component set to zero. The free Hamiltonian of \( B_v \), with \( B \) having \( N + 1 \) cells, can thus be written as

\[
H_v^{(N+1)} = H_O^N + \tilde{\varphi}_{N+1}^\dagger \cdot h \cdot \tilde{\varphi}_{N+1} + \tilde{V},
\]  

(S53)

where \( \tilde{V} = \tilde{\varphi}_{N+1}^\dagger \cdot (J \cdot \varphi_1 + J^\dagger \cdot \varphi_N + \text{H.c.}) \) (S54)

is the coupling Hamiltonian between all cavities of cell \( n = 0 \) but \( \alpha_v \) (vacancy site) and lattice cells \( n = 1, \ldots, N \) (i.e., \( B \) under open BCs).

Assume now that \( B \) under open BCs (cells \( n = 1, \ldots, N \)) admits \( N_e \) degenerate edge states of energy \( \omega_e \), which we call \( |\mathcal{E}^e_s\rangle = \sum_{n=1}^{N} \varphi_n^\dagger \cdot \mathcal{E}^e_n |\text{vac}\rangle \) with \( s = 1, \ldots, N_e \) (here \( \mathcal{E}^e_n \) is a \( d \)-dimensional row vector). Being these localized, \( \omega_e \) lies within a bandgap (note that \( B \), \( B_B \) under open BCs and \( B_v \) share the same bands and bandgaps). Consider now a linear combination of these edge states, \( |\psi\rangle = \sum_{s=1}^{N_e} \gamma_s |\mathcal{E}^e_s\rangle \). The condition in order for \( |\psi\rangle \) to be an eigenstate of \( H_v^{(N+1)} \) with eigenvalue \( \omega_e \) is

\[
\tilde{V} |\psi\rangle = 0,
\]  

(S55)

where we used that \( H_O^N |\mathcal{E}^e_s\rangle = \omega_e |\mathcal{E}^e_s\rangle \) and \( \langle 0|b_{0,\alpha}|\mathcal{E}^e_s\rangle = 0 \) for \( \alpha = 1, \ldots, d \).

Eq. (S55) is a linear system of \( d - 1 \) equations in the \( N_e \) unknowns \( \{\gamma_s\} \). This has \( N_e - d + 1 \) non-trivial solutions. Hence, if \( N_e = d \) there is only one non-trivial solution, which is a bound state of \( B_v \). This completes the proof.

**SM9. VDS IN THE PHOTONIC CREUTZ-LADDER MODEL**

In this section, we consider \( B \) to be a photonic Creutz-ladder model and show that, when the bandgap is open, \( B_v \) admits a photonic BS (hence a corresponding VDS occurs). This task is carried out by applying the theorem in the last section by first deriving the edge states of \( B \) under open BCs through the methods introduced in Refs. [10, 11].

**SM9.1. Edge States of \( B \) under open BCs**

The free Hamiltonian of the Creutz model with open BCs is

\[
\mathcal{H}_O^{(N)} = -2mJ \sum_{n=1}^{N} (a_n^\dagger b_n + \text{H.c.}) + J \sum_{n=1}^{N-1} \left[ e^{i\alpha} a_n^\dagger b_{n+1} + e^{-i\alpha} b_n^\dagger b_{n+1} + a_n^\dagger b_{n+1} + b_n^\dagger a_{n+1} + \text{H.c.} \right],
\]  

(S56)

where \( a_n \) and \( b_n \) are ladder operators corresponding to cavities \( a \) and \( b \) of cell \( n \) [see Fig. 2(e)] and where we set \( \omega_e = 0 \) (which does not affect the calculation). Column vector \( \varphi_n \) and matrices \( h_{0,1} \) in Eq. (S47) in this case thus read

\[
\varphi_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad h_0 = -2Jm \sigma_x, \quad h_1 = J \begin{pmatrix} e^{i\alpha} & 1 \\ 1 & e^{-i\alpha} \end{pmatrix}
\]  

(S57)
Consider first the generalized Bloch Hamiltonian [10, 11]
\[ H(z) = h_0 + z h_1 + \cdots + e^{i \alpha n} \],
(S62)
with subscript \( v = a, b \) indicating whether the atom is coupled to cavity \( a \) or \( b \).

Let \( H_\infty \) be the Hamiltonian obtained from (S51) by replacing each sum with \( \sum_{n=\infty}^\infty \).
Then a generic eigenstate of \( H_\infty \) can be written as
\[ |\chi_\pm\rangle = \sum_{n=-\infty}^{\infty} z_\pm^n \varphi_n^1, u(z_\pm) |\text{vac}\rangle, \]
where \( u(z_\pm) \) is the (two-dimensional) eigenvector of \( H(z) \) with eigenvalue \( E \).

Next, to work out the eigenstates of \( H_\infty^{(N)} \), we make the ansatz
\[ |\mathcal{E}\rangle = \mathbb{P}^N (c_+ |\chi_+\rangle + c_- |\chi_-\rangle), \]
(S57)
where \( \mathbb{P}^N = \sum_{n=1}^{N} a_n^\dagger |\text{vac}\rangle \langle a_n + b_n^\dagger |\text{vac}\rangle \langle b_n \) is the projector on all the sites that belong to the lattice. In order to be an eigenstate of \( H_\infty^{(N)} \), \( |\mathcal{E}\rangle \) must satisfy the eigenvalue equation in particular on the lattice boundary. This corresponds to enforcing the conditions
\[ \sum_{i=\pm} c_i \langle \text{vac} | a_n (H_\infty^{(N)} - E \mathbb{1}) | \chi_i \rangle = 0, \sum_{i=\pm} c_i \langle \text{vac} | b_n (H_\infty^{(N)} - E \mathbb{1}) | \chi_i \rangle = 0 \quad \text{with} \quad n = 1, N, \]
(S58)
which is a linear system of four equations in the unknowns \( c_\pm \) (each equation corresponds to site \( a \) or \( b \) of cell \( n = 1 \) or \( n = N \)). The solutions for \( z_\pm \) are those values of \( z_\pm \) such that system (S58) admits non-trivial solutions. Based on Eq. (S56), note that if \( |z_\pm| \neq 1 \) then the corresponding eigenstate is localized close to one of the two lattice edges, the left edge if \( |z_\pm| < 1 \), the right one if \( |z_\pm| > 1 \) (left and right edge correspond to cells \( n = 1/n = N \), respectively).

After some manipulations, in the thermodynamical limit \( N \to \infty \), we end up with [recall Eq. (S55)]
\[ E = 2m \cos \alpha J, \quad z_\pm = m^{\pm1}. \]
(S59)
Plugging these into (S57) [recall Eq. (S56)], we get the pair of edge states
\[ |\mathcal{E}^L\rangle = \sqrt{1/m^2} \sum_{n=1}^{N} m^{n-1} (e^{i \hat{\mathcal{Z}} a_n^\dagger - e^{-i \hat{\mathcal{Z}} b_n^\dagger}) |\text{vac}\rangle, \quad |\mathcal{E}^R\rangle = \sqrt{1/m^2} \sum_{n=1}^{N} m^{N-n} (e^{-i \hat{\mathcal{Z}} a_n^\dagger - e^{i \hat{\mathcal{Z}} b_n^\dagger}) |\text{vac}\rangle, \]
(S60)
with subscript \( L \) (\( R \)) indicating whether the state is localized close to the left (right) edge (the wavefunction modulus decays from right to left in the case of \( \mathcal{E}^R \) and from left to right for \( \mathcal{E}^L \)).

**SM9.2. Bound state of \( B_v \)**

In light of the theorem in Section SM8, the knowledge of edge states (S60) ensure the existence of a BS of \( B_v \). To work out the corresponding wavefunction, we just impose (S50) using (S60), obtaining
\[ |\psi_{v=a}\rangle = \frac{1}{\sqrt{2}} \left( |\mathcal{E}^R\rangle + e^{-i \alpha} |\mathcal{E}^L\rangle \right) = \frac{\sqrt{1 - m^2}}{2} e^{-i \hat{\mathcal{Z}}} \sum_{n=2}^{N} \left[ (e^{i \alpha} m^{n-2} + e^{-i \alpha} m^{N-n}) a_n^\dagger - (m^{n-2} + m^{N-n}) b_n^\dagger \right], \]
(S61)
\[ |\psi_{v=b}\rangle = \frac{1}{\sqrt{2}} \left( |\mathcal{E}^R\rangle + e^{i \alpha} |\mathcal{E}^L\rangle \right) = \frac{\sqrt{1 - m^2}}{2} e^{i \hat{\mathcal{Z}}} \sum_{n=2}^{N} \left[ (m^{n-2} + m^{N-n}) a_n^\dagger - (e^{-i \alpha} m^{n-2} + e^{i \alpha} m^{N-n}) b_n^\dagger \right], \]
(S62)
with subscript \( v = a, b \) indicating whether the atom is coupled to cavity \( a \) or \( b \).
SM10. HALDANE MODEL

SM10.1. Existence of a VDS

Here we will prove that a BS of $B_v$ in the Haldane model occurs if and only if the model lies within regions I and II of the parameters space [see Fig. 3(a) in the main text]. Thus, a VDS can occur only within the same regions.

Recall from Section SM4 that a BS of $B_v$ exists iff $\langle G_B(z) \rangle_v$ admits a root $z = \omega_p$ within the bandgap. Since $\langle G_B(z) \rangle_v$ is analytic and monotonic, this occurs iff $\langle G_B(z) \rangle_v$ takes opposite signs at the bandgap edges.

In the reciprocal space, the Haldane model can be expressed as

$$H_B = \int \sum_{\omega} d^3k \frac{\varphi_k^\dagger \mathcal{H}_k \varphi_k}{2} = \frac{1}{2\pi} \int \sum_{\omega} d^3k e^{-i\omega x} \varphi_x,$$

where $\varphi_k$ is the annihilation operator of cavity $a$ (b) on the unit cell specified by vector $x$, $u_1 = (0,1)^T$, $u_2 = (-\sqrt{3}/2,-1/2)^T$, $u_3 = (\sqrt{3}/2,-1/2)^T$ and $d_i = u_i - u_{i-1}$.

For $\Delta \omega_{\text{gap}} \ll J$, the low-energy physics is well described by the 2D-Dirac Hamiltonian

$$h_\pm(\kappa) \sim \omega_{\text{mid}} I \pm [M_\pm c^2 \sigma_z + c(-\kappa_x, \sigma_x \pm \kappa_y, \sigma_y)]$$

which yields

$$\langle G_B(z) \rangle_v \sim -\frac{\pi c}{z'} \sum_{\pm} (z' \pm M_\pm c^2 \langle \sigma_z \rangle_v) \ln \left[ 1 + \frac{c^2 z'}{M_\pm^2 c^2 - z'^2} \right] \quad \text{with } z' = z - \omega_{\text{mid}} \in \mathbb{R},$$

with $v = a$ or $v = b$, in which cases $\langle \sigma_z \rangle_v = 1$ and $\langle \sigma_z \rangle_v = -1$, respectively.

Without loss of generality, we assume $m > 0$ and $0 < \phi < \pi$ such that $|M_+| > |M_-|$ and the bandgap edges are at $z' = \pm M_- c^2$. In regions I and II [see Fig. 3(a) in the main text], $|m| < 3 \sqrt{3} |t| \sin \phi$, hence $M_- > 0$ so that the upper and lower bandgap edges lie at $M_- c^2$ and $-M_- c^2$, respectively. Thereby, the Green function takes the following values on the bandgap edges

lower edge: $\lim_{z' \rightarrow (-M_- c^2)^-} \langle G_B(z') \rangle_v \sim \left\{ \begin{array}{ll} +\infty & \text{for } v = a \\ -\pi \ln \left[ 1 + \frac{(\pi a)^2}{(M_+^2 - M_-^2) c^2} \right] (M_- + M_+) & \text{for } v = b \end{array} \right.$

upper edge: $\lim_{z' \rightarrow (+M_- c^2)^-} \langle G_B(z') \rangle_v \sim \left\{ \begin{array}{ll} -\pi \ln \left[ 1 + \frac{(\pi a)^2}{(M_+^2 - M_-^2) c^2} \right] (M_- + M_+) & \text{for } v = a \\ -\infty & \text{for } v = b \end{array} \right.$

Therefore, no matter whether the atom sits on $a$ or $b$, $\langle G_B(z') \rangle_v$ changes sign across the bandgap, entailing $(G_B(z'))_v = 0$ for some $\omega_p$ within the gap.

On the other hand, for $|m| > 3 \sqrt{3} |t| \sin \phi$ (i.e., outside regions I-II), $M_- < 0$ and the upper (lower) band-gap edge occurs at $z' = -M_- c^2$ ($z' = M_- c^2$). Then

lower edge: $\lim_{z' \rightarrow (+M_- c^2)^-} \langle G_0(z') \rangle_v \sim \left\{ \begin{array}{ll} -\pi \ln \left[ 1 + \frac{(\pi a)^2}{(M_+^2 - M_-^2) c^2} \right] (M_- + M_+) & \text{for } v = a \\ +\infty & \text{for } v = b \end{array} \right.$

upper edge: $\lim_{z' \rightarrow (-M_- c^2)^-} \langle G_0(z') \rangle_v \sim \left\{ \begin{array}{ll} -\infty & \text{for } v = a \\ +\pi \ln \left[ 1 + \frac{(\pi a)^2}{(M_+^2 - M_-^2) c^2} \right] (M_- + M_+) & \text{for } v = b \end{array} \right.$

Thus, contrary to the previous case, $\langle G_0(z') \rangle_v$ does not change sign across the bandgap wherein it cannot vanish.
SM10.2. Additional remarks on topological protection of the VDS for $\phi = \pm \pi/2$ and $m = 0$.

As mentioned in the main text, a VDS inherits its topological features (if any) from the BS of $B_v$, i.e., the BS induced by a (zero-dimensional) vacancy.

In the Haldane model, a topologically-protected BS (TPBS) around a vacancy is guaranteed only for $\phi = \pm \pi/2$ and $m = 0$. Indeed, according to the topological classification in Ref. [12], a zero-dimensional defect in a 2D model may seed around it a TPBS only when the model lies in a suitable Atland-Zirnbauer class [13]. Classes are identified by occurrence of time-reversal, particle-hole and chiral symmetry, or the absence thereof. The Haldane model generally belongs to class A, the class of models lacking any of the above symmetries (although some symmetry can occur on special points of the parameter space as discussed shortly). According to Ref. [12], class-A models may feature one-dimensional topologically-protected states. In the Haldane model, these states are the well known chiral edge states, which appear in the topological phases I and II of Fig. 3(a) in the main text. Note that, according to the classification of Ref. [12], models within class A do not admit zero-dimensional TPBS. Yet, it turns out that, for the special values $\phi = \pm \pi/2$ and $m = 0$, the Haldane model does possess particle-hole symmetry, so that the model falls within a different class, namely class D. Two-dimensional class-D models may have zero-dimensional TPBS, whenever a suitable $Z_2$ topological invariant [14] acquires a non-vanishing value. This is indeed the case in the instance of Fig. 3(b) [white dot of Fig. 3(a)]. The existence of these TPBS in the Haldane model is indeed analytically proven in Ref. [15], where moreover their topological protection is numerically confirmed. Note that a BS may still exist within the whole phases I and II [see Fig. 3(a) in the main text]: however, due to lack of particle-hole symmetry their topological protection is not guaranteed except on the special point $\phi = \pm \pi/2$ and $m = 0$ [15].

SM10.3. Dressed bound states that are not VDS

Fig. 3(b) shows the photon density current (CD) of the topologically protected VDS for $m = 0$, $\pi/2$ and $t = 0.1$. In this case, the CD is highly picked, reaching a maximum value of 0.04.

As already mentioned in the main text, there are points of the phase diagram of Fig. 3(a) outside the regions I and II, where the gap $\Delta \omega_{\text{gap}}$ coincides with the value $\Delta_0$ assumed at $\phi = \pm \pi/2$ and $m/t = 0$, and all other parameters unchanged. If one takes $\omega = \omega_{\text{mid}}$ and $g$ small enough, an in-gap dressed BS arises (which is not a VDS) in which the localisation of the photon probability density is quantitatively similar to the one in the topologically protected VDS of Fig. 3(b).

This dressed BS may still display a CD pinned around $v$, whose magnitude, however, is several orders of magnitude smaller than in Fig. 3(b). For example, the set of dressed BS occurring for $m = \Delta_0 - 3\sqrt{3}|\sin \phi|$, $\phi \in [-\pi, \pi]$, $t = 0.1$ and $g = 0.01$, for which $\Delta \omega_{\text{gap}} = \Delta_0$, we numerically observe a maximum CD that is at least six order of magnitudes smaller than the VDS in Fig. 3(b). This is based on exact numerical diagonalization of the Hamiltonian using a lattice of $30 \times 30$ unit cells and the CD formula [16]

$$j(x_j) = \sum_k (x_j - x_k) \Im \left( \langle \Psi | x_j \rangle \langle x_j | H | x_k \rangle \langle x_k | \Psi \rangle \right)$$

(S70)

where $x_j$ is the position of the $j$-th site on the lattice.

[S1] J. T. Shen and S. Fan, Coherent photon transport from spontaneous emission in one-dimensional waveguides, Optics Letters 30, 2001 (2005).
[S2] S. Longhi, Bound states in the continuum in a single-level Fano-Anderson model, European Physical Journal B 57, 45 (2007).
[S3] T. Tufarelli, F. Ciccarello, and M. S. Kim, Dynamics of spontaneous emission in a single-end photonic waveguide, Physical Review A 87, 13820 (2013).
[S4] G. Calajó, Y.-L. L. Fang, H. U. Baranger, and F. Ciccarello, Exciting a Bound State in the Continuum through Multi-photon Scattering Plus Delayed Quantum Feedback, Physical Review Letters 122, 073601 (2019).
[S5] C. Gonzalez-Ballestero, F. J. Garcia-Vidal, and E. Moreno, Non-Markovian effects in waveguide-mediated entanglement, New Journal of Physics 15, 73015 (2013).
[S6] E. N. Economou, Green’s Functions in Quantum Physics, Springer Series in Solid-State Sciences, Vol. 7 (Springer Berlin Heidelberg, Berlin, Heidelberg, 2006).
[S7] C. Cohen-Tannoudji, J. Dupont-Roc, G. Gryenberg, and P. Thickstun, Atom-photon interactions: basic processes and applications (Wiley Online Library, 1992, 2004).