The Bismut-Elworthy-Li formula for jump-diffusions and applications to Monte Carlo methods in finance

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Abstract

We extend the Bismut-Elworthy-Li formula to non-degenerate jump diffusions and “payoff” functions depending on the process at multiple future times. In the spirit of Fournié et al [14] and Davis and Johansson [10] this can improve Monte Carlo numerics for stochastic volatility models with jumps. To this end one needs so-called Malliavin weights and we give explicit formulae valid in presence of jumps: (a) In a non-degenerate situation, the extended BEL formula represents possible Malliavin weights as Itô integrals with explicit integrands; (b) in a hypoelliptic setting we review work of Arnaudon and Thalmaier [1] and also find explicit weights, now involving the Malliavin covariance matrix, but still straight-forward to implement. (This is in contrast to recent work by Forster, Lükebohmert and Teichmann where weights are constructed as anticipating Skorohod integrals.) We give some financial examples covered by (b) but note that most practical cases of poor Monte Carlo performance, Digital Cliquet contracts for instance, can be dealt with by the extended BEL formula and hence without any reliance on Malliavin calculus at all. We then discuss some of the approximations, often ignored in the literature, needed to justify the use of the Malliavin weights in the context of standard jump diffusion models. Finally, as all this is meant to improve numerics, we give some numerical results with focus on Cliquets under the Heston model with jumps.
1 Introduction

Modern arbitrage theory reduces the pricing of (non-American) options to the computation of an expectation under a risk neutral measure. It is common practice to assume that the risk neutral measure is induced by a parametric family of jump diffusions which can then be calibrated to liquid option prices. We can therefore assume that all expectations are with respect to a fixed pricing measure. A typical option on some underlying \((S_t)\) then has (undiscounted) price

\[
E[f(S_{T_1}, S_{T_2}, \ldots, S_{T_n})] \equiv E[f(S)].
\]

For hedging and risk-management purposes it is crucial to understand the dependence on \(S_0\) and other model parameters. Computing

\[
\Delta = \frac{\partial}{\partial S_0} E[f(S)] = E \left[ \nabla f(S) \frac{\partial S}{\partial S_0} \right]
\]

via finite differences can present computational challenges in Monte Carlo; just think of an at-the-money digital option near expiration. Broadie and Glasserman [7] showed that this problem is overcome by

\[
\frac{\partial}{\partial S_0} E[f(S)] = E[f(S) \pi]
\]

where \(\pi\) is the logarithmic derivative of the joint density of the random vector \(S\). On the other hand, the random weight \(\pi\) adds noise itself and it is important to localise: for instance by using (1) for an irregular, but compactly support and bounded, \(f\) and the usual finite difference technique for \(f - \tilde{f}\), assumed to be nice (\(C^1\) will usually suffice).

In two seminal papers, Fourniè et al [14] and [15] use Malliavin calculus to compute \(\pi\) when no explicit transition density is known. They work with non-degenerate (or: elliptic) continuous diffusions but also cover some hypoelliptic situations. As is well known, elliptic results can be obtained by the Bismut-Elworthy-Li formula (Elworthy and Li [11], Bismut [6]) and there are, in fact, other ways to obtain such results without Malliavin calculus: we mention in particular the idea of Thalmaier [27] of differentiation at the level of local martingales which was employed by Gobet and Munos [18] in the present context. The point was that in many cases of practical interest, at least in absence of jumps, one does not need Malliavin calculus. (Specialists will note that Malliavin techniques are more flexible in the sense that different perturbations of Brownian motion yield different weights and there is an apriori interest to pick weights with small variance. In reality, it is hard to justify much effort in this direction as the potential gains are negligible to the improvements obtained by localisation.)

Over the last decade it has become clear that pure diffusion models are unable to fit the short-dated smile and jumps have been included to models to rectify this situation; Cont and Tankov [9] and Gatheral [16] provide
two excellent accounts. The question has arisen as to how the above ideas can be adapted to models based on jump diffusion processes and we shall propose a quite simple solution to this along the ideas of Elworthy-Li bypassing both classical Malliavin techniques and its extensions to Lévy processes that have been used in this financial Monte Carlo context. We note that a similar extension of the BEL formula, slightly less general than ours, was used recently by Priola and Zabczyk [22] to establish Liouville theorems for non-local operators.

Let us briefly mention that in some cases a random weight \( \pi \) can be constructed by conditioning arguments. Consider for instance the trivial example \( X_t = z + B_t + N_t \), where \( B \) is a standard Brownian motion and \( N \) a Poisson process. Conditional on \( N_t \), any function of \( X_t \) is a (different) function of \( z + B_t \), a pure diffusion with no jumps, and since the associated random weight \( \pi \) is universal (i.e. do not depend on the particular payoff function) this also solves the problem for the jump diffusion \( X \). This kind of reasoning leads immediately to the class of “separable” jump diffusions, considered in Davis and Johansson [10] via Malliavin calculus for simple Lévy processes. We shall omit a detailed discussion since a refined, iterated conditioning argument can be used assuming only finite activity of the jumps (and without assuming separability in the sense of [10]). To this end, we quickly recall the BEL for continuous diffusions (see Section 3 for notation and assumptions)

\[
\frac{\partial}{\partial z_j} \mathbb{E}[f(x_T^z)] = \mathbb{E} \left[ f(x_T^z) \int_0^T a(t) \left( \frac{\partial x_t^z}{\partial z_j} \right)^T \right] dW_t
\]

where \( \int_0^T a(t) dt = 1 \) and \( x_0 = z \). Let \( 0 < S < T \) be deterministic. The standard choice \( a \equiv 1/T \) gives a weight, say \( \pi_{0,T} \). Another weight (of higher variance) comes from \( a \equiv 1/S \) on \([0,S]\), 0 otherwise, and we call it \( \pi_{0,S} \). One can also condition on \( x_S \) and apply the BEL formula over the time interval \([S,T]\), this yields another weight \( \pi_{S,T} \) for the derivative of \( \mathbb{E}[f(x_T)|x_S] \) w.r.t. \( x_S \). We leave it to the reader to check that, combined with the chain-rule, \( \left( \frac{\partial}{\partial z_j} \right) = \left( \frac{\partial x_S}{\partial z_j} \right) \frac{\partial}{\partial x_S} \), the weight \( \pi_{0,T} \) can be assembled from \( \pi_{0,S} \) and \( \pi_{S,T} \). In other words, instead of applying BEL on \([0,T]\) one can apply it on \([0,S]\) and \([S,T]\). While we did not assume jumps in this discussion, it is clear that a cadlag discontinuity of \( x \) at time \( S \) does not pose a problem. This extends to any number of intervals and if we are dealing with a finite activity jump diffusion conditioning will reduce the problem to the one just discussed. The flaw with this sort of reasoning is that it makes fundamental use of a property which is completely irrelevant for the result to hold true: finite activity of jumps. In the general case, i.e. beyond finite activity, not only does the preceding argument break down, but jumps arise from a genuine stochastic integral w.r.t. a compensated Poisson random measure and any conditioning on jumps must fail.
On the other hand, by maintaining a finite activity assumption on the jumps and some conditions on linkage operators, the ellipticity condition has been relaxed to hypoellipticity by Forster, Lütkebohmert and Teichmann [13]. Unfortunately, their ‘linkage’ condition on the jump vector fields excludes many examples of financial interest\(^1\). The main contribution of [13], in our view, is to establish new conditions for integrability of the inverse of the Malliavin covariance matrix \(C\) in presence of jumps. Recent progress in this direction was also made by Takeuchi [26] who manages to bypass Norris’ lemma, which, in a sense is the bottleneck of the arguments in [13]. Thus, noting that criteria for integrability of \(C^{-1}\) are available in the literature, and can also be checked by hand in many examples, we show that suitable integrability of \(C^{-1}\) allows to extend a recent result by Arnaudon and Thalmaier [1] and we so obtain non-anticipating Malliavin weights for hypoelliptic diffusions with jumps of possibly infinite activity also allowing for the ‘linkage’ condition in [13] to be relaxed.

It is worthwhile to ponder for a moment which financial examples really benefit from BEL / Fourni et al type formulae. The standard hypoelliptic example in finance is an Asian option but computation of Greeks with (intelligently chosen) finite difference perform rather well. In fact, most jump diffusion models used in practice have an essentially elliptic diffusion part and also quasi-closed form expressions for European option prices and the usual Greeks, typically by Fourier methods i.e. by low-dimensional integration. Thus, the focus should really be on instruments without (quasi-)closed form prices for which finite difference methods perform poorly. In fact, there is a very popular family of such contracts in equity markets, namely digital cliquets, and the numerical difficulties for risk management are well-known to practitioners. Surprisingly perhaps, there seems to be no result in the literature that applies to computing sensitivities of digital cliquets under, for instance, the Heston-model with jumps: the separability conditions of [10] are far too stringent, the relevant statement in [13], Proposition 1 to be precise, still contains the (here unnecessary) linkage condition which is not satisfied\(^3\) nor do Heston-type models satisfy the strong \(C^\infty\) assumptions of [13]\(^4\).

This paper is organised as follows. We prove that the Bismut-Elworthy-Li formula holds for a generic non-degenerate time-inhomogeneous Markovian jump-diffusion; \(\pi\) is given explicitly as a stochastic integral involv-

\(^1\)Indeed it is easy to see that this condition fails in the case where the jumps in the stock are log normal as in the Merton model (or any example in which the Lévy measure has full support).

\(^2\)The gap between elliptic and what one has in some real examples is subject of Section 5 of this paper.

\(^3\)One could re-run the Malliavin calculus arguments of [13] in the elliptic setting to get rid of this condition or, in fact, make rigorous the iterated conditioning argument outlined above.

\(^4\)This is just to say, that approximation arguments similar to those discussed in Section 5 of this paper would be needed.
ing the flow and the right-inverse of the diffusion matrix, just as in the classical Bismut-Elworthy-Li formula (which is recovered in the absence of jumps). A similar presentation is given for the second derivative. In section 4 we demonstrate how Malliavin calculus may be used with an appropriate choice of perturbation to provide explicit weights in a hypoelliptic setting. In section 5, as a case study, we show how to represent the spot sensitivity\(^5\) in the Heston model with jumps (also known as SVJ) and the Matytsin double jump model (an extension of SVJ and also known as SVJJ). Both models are described in detail in Gatheral [16] and are popular in the industry because of their quasi-closed form solutions for European options in terms of Fourier-transforms, which we use for numerical benchmarks for some simulations in the last chapter. An honest application of the BEL formula\(^6\) to these (and many other practically relevant) examples requires approximation argument which, in our view, have been neglected in the literature.

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### 2 Preliminaries

We collect some background material from Gikhman and Skorohod [17]. Our focus is on the strong solution of

\[
x_t^z = z + \int_0^t Z(s, x_{s-}^z) ds + \int_0^t X(s, x_{s-}^z) dW_s + \int_0^t \int_{E} Y(t, x_{s-}^z, y)(\mu - \nu)(dy, ds)
\]

where \(W_t \equiv (W^1_t, \ldots, W^m_t)\) is an \(\mathbb{R}^m\)-valued Brownian motion on some probability space \((\Omega, \mathcal{F}_t, \mathbb{P})\) and \(\mu\) is a \((\Omega, \mathcal{F}_t, \mathbb{P})\)-Poisson random measure on \(E \times [0, \infty)\) for some topological space \(E\) such that \(\nu\), the compensator of \(\mu\), is of the form \(G(dy) dt\) for some \(\sigma\)-finite measure \(G\). The vector fields \(Z(t, x)\) and \(Y(t, x, y) \in \mathbb{R}^d\), \(X(t, x) \in L(\mathbb{R}^m, \mathbb{R}^d)\) for all \(t \in [0, T]\), \(x \in \mathbb{R}^d\) and \(y \in E\). We will always assume at least the following conditions which guarantee the existence and uniqueness of a solution to the SDE (see Gikhman and Skorohod [17])

1. For all \(x \in \mathbb{R}^d\) and \(t \in [0, T]\)

\[
|Z(t, x)|^2 + |X(t, x)|^2 + \int_E |Y(t, x, z)|^2 G(dz) \leq C(1 + |x|^2)
\]
2. For all $x, z \in \mathbb{R}^d$ and $t \in [0, T]$

$$|Z(t, x) - Z(t, z)|^2 + |X(t, x) - X(t, z)|^2$$

$$+ \int_E |Y(t, x, y) - Y(t, z, y)|^2 G(dz) \leq C|x - z|^2$$

Throughout we fix the option expiry time $T > 0$ and consider a payoff $f: \mathbb{R}^d \to \mathbb{R}$, and we frequently work with the process $x_{t'}^{t, z}$ for $t' < t$ defined as the solution to the SDE

$$x_{t'}^{t, z} = z + \int_t^{t'} Z(s, x_{s'}^{t, z}) ds + \int_t^{t'} X(s, x_{s'}^{t, z}) dW_s$$

$$+ \int_0^t \int_E Y(s, x_{s'}^{t, z}, y)(\mu - \nu)(dy, ds).$$

We will write $f \in C^k_b(\mathbb{R}^j)$ to mean that the function $f: \mathbb{R}^j \to \mathbb{R}$ is $k$-times continuously differentiable with $f$ and all its derivatives up to order $k$ uniformly bounded. Our method of proof will rely on ensuring that the function $u(t, z) : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ given by $u(t, z) = E[f(x_T^{t, z})]$ satisfies the backward equation of Kolmogorov.

$$G_t u(t, z) + \frac{\partial u}{\partial t}(t, z) = 0.$$
The functions $Z(.,.), \nabla Z(.,.), \nabla^2 Z(.,.), X(.,.), \nabla X(.,.), \nabla^2 X(.,.)$
and $\int_E |Y(., y)|^2 G(dy), \int_E |\nabla Y(., y)|^2 G(dy), \int_E |\nabla^2 Y(., y)|^2 G(dy)$
are all continuous on $[0, T] \times \mathbb{R}^d$ (6)

$$\sup_{(t,x)\in[0,T] \times \mathbb{R}^d} \left( \int_E (|Y(t, x, z)|^k + |\nabla Y(t, x, z)|^k) G(dy) \right) < \infty$$ (7)

for $k = 2, 3, 4$. Then if $f \in C^2_b(\mathbb{R}^d)$ the function $u(t, z) = \mathbb{E}[f(x^z_t)]$ is such that $u \in C^{1,2}_b([0, T] \times \mathbb{R}^d)$ and satisfies

$$\frac{\partial u}{\partial t}(t, z) + G_t u = 0$$

with boundary condition $\lim_{t \to T} u(t, z) = f(z)$ and where $G_t$ is given by (4).

### 3 The Bismut-Elworthy-Li formula for jump-diffusions

The argument of Elworthy and Li [11] extends in a straight-forward way to jump-diffusions.

**Theorem 2.** Fix some $T > 0$ and consider $x^z_t \equiv x^{0,z}_t$ the solution to SDE (2) on the interval $[0, T]$ and suppose that the conditions of Theorem 1 are satisfied. Further assume that the diffusion matrix $X(t, x)$ has a right inverse $R(t, x)$, and satisfies the following uniform ellipticity condition

$$y^T X(t, x) X^T(t, x) y \geq \epsilon |y|^2$$

for every $t \in [0, T], x, y \in \mathbb{R}^d$ and some $\epsilon > 0$. Then, if $a \in L^2[0, T]$ is any deterministic function which satisfies

$$\int_0^T a(t) dt = 1$$

and $f \in C^2_b(\mathbb{R}^d)$ the following is true for all $1 \leq k \leq d$

$$\frac{\partial}{\partial z^k} \mathbb{E}[f(x^z_T)] = \mathbb{E} \left[ f(x^z_T) \int_0^T a(t) \left( R(t, x^z_{t-}) \frac{\partial x^z_t}{\partial z^k} \right) dW_t \right].$$ (8)

Moreover, if we consider $0 < T_1 \leq \ldots \leq T_n \leq T$ and a function of the form

$f(x^z_{T_1}, \ldots, x^z_{T_n})$, where $f \in C^2_b(\mathbb{R}^d \times \ldots \times \mathbb{R}^d)$ and let $a \in L^2[0, T]$ be a deterministic function satisfying

$$\int_0^{T_1} a(t) dt = 1.$$
Then, for all $1 \leq k \leq d$, the following is true

\[
\frac{\partial}{\partial z_k} \mathbb{E}[f(x_{T_1}^z, \ldots, x_{T_n}^z)] = \mathbb{E} \left[ f(x_{T_1}^z, \ldots, x_{T_n}^z) \int_0^{T_1} a(t) \left( R(t, x_{T_1}^z) \frac{\partial x_{T_1}^z}{\partial z_k} \right)^T dW_t \right].
\]

(9)

**Remark 1.** In the absence of jumps and with $a(t) = T^{-1}$ on $[0, T]$ we recover the classical Bismut-Elworthy-Li formula.

**Remark 2.** It is easy to see that the uniform ellipticity condition gives rise to the fact that

\[
\int_0^T a(s) \left( R(s, x_{s-}^z) \frac{\partial x_{s-}^z}{\partial z_k} \right)^T dW_s
\]

is a martingale on $[0, T]$. To see this take $t = s$, $x = x_{s-}$ and $y = R(s, x_{s-}) \frac{\partial x_{s-}^z}{\partial z_k}$ and observe that

\[
\left| R(s, x_{s-}) \frac{\partial x_{s-}^z}{\partial z_k} \right|^2 \leq \epsilon^{-1} \left| \frac{\partial x_{s-}^z}{\partial z_k} \right|^2 \text{ a.s.}
\]

Consequently,

\[
\mathbb{E} \left[ \int_0^T a(t)^2 \left| R(t, x_{T_1}^z) \frac{\partial x_{T_1}^z}{\partial z_k} \right|^2 dt \right] \leq \epsilon^{-1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \frac{\partial x_{s-}^z}{\partial z_k} \right|^2 \right] \int_0^T a(t)^2 dt < \infty.
\]

**Proof.** For $t < T$ we apply Itô’s formula to the function

\[
u(t, z) = \mathbb{E}[f(x_t^z)] = P_{T-t} f(z)
\]

for $t < T$. Since

\[
\mathcal{G}_t u + \frac{\partial u}{\partial t}(t, x) = 0
\]

the $\mathcal{G}$-term vanishes, leaving only a constant and the two martingale terms. Letting $t \to T$ we find

\[
f(x_T^z) = u(0, x) + \int_0^T (\nabla u(s, x_{s-}^z) X(s, x_{s-}^z))^T dW_s
\]

\[
+ \int_0^T \int_E (u(s, x_{s-}^z + Y(s, x_{s-}^z, y)) - u(s, x_{s-}^z))(\mu - \nu)(ds, dy)
\]

(10)

The integral featuring above with respect to $\mu - \nu$ is a discontinuous $L^2$-martingale which is orthogonal to the martingale $\int a(s) \left( R(s, x_{s-}) \frac{\partial x_{s-}^z}{\partial z_k} \right)^T dW_s$. 

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Multiplying by \( \int_0^T a(s) \left( R(s, x_{s-}^z) \frac{\partial x_{s-}^z}{\partial z_k} \right)^T \) \( dW_s \) and using Itô’s isometry gives the result
\[
\mathbb{E} \left[ f(x_T^z) \int_0^T a(s) \left( R(s, x_{s-}^z) \frac{\partial x_{s-}^z}{\partial z_k} \right)^T \right] = \mathbb{E} \left[ \int_0^T a(s) \nabla u(s, x_{s-}^z) \frac{\partial x_{s-}^z}{\partial z_k} \right] ds
= \int_0^T a(s) \mathbb{E} \left[ \nabla u(s, x_{s-}^z) \frac{\partial x_{s-}^z}{\partial z_k} \right] ds
= \int_0^T a(s) \frac{\partial}{\partial z_k} \mathbb{E}[u(s, x_{s-}^z)] ds
= \int_0^T a(s) \frac{\partial}{\partial z_k} \mathbb{E}[f(x_T^z)] ds
= \frac{\partial}{\partial z_k} \mathbb{E}[f(x_T^z)]
\] (11)

We justify the progression from the second to third line by a routine argument based on the boundedness of \( \nabla u \) and \( \nabla^2 u \) and the definition of \( \frac{\partial x_{s-}^z}{\partial z_k} \) as the \( L^2 \)-limit (as \( h \to 0 \)) of the random variables \( h^{-1}(x_i^{z+he_k} - x_i^z) \) for fixed \( t \). Also, we justify the third to fourth line in (11) by the observation that \( u(t, x_i^z) \to u(s, x_{s-}^z) \) almost surely as \( t \uparrow s \) and so bounded convergence gives \( \mathbb{E}[u(t, x_i^z)] \to \mathbb{E}[u(s, x_{s-}^z)] \). But for each \( t \in [0, T] \) we have \( \mathbb{E}[u(t, x_i^z)] = \mathbb{E}[\mathbb{E}[f(x_T^z)]_{\mathcal{F}_{t}}] = \mathbb{E}[f(x_T^z)] \), so \( \mathbb{E}[u(s, x_{s-}^z)] = \mathbb{E}[f(x_T^z)] \).

For the final part, we note that the function \( g : \mathbb{R}^d \to \mathbb{R} \) defined by \( g(x) = \mathbb{E}[f(x, x_{T_{T_n}} t, \ldots, x_{T_{T_n}} t)] \) has the property that \( g \in C^0_b(\mathbb{R}^d) \) and, moreover, by the Markov property
\[
g(x_{T_n}^z) = \mathbb{E}[f(x_{T_n}^z, \ldots, x_{T_n}^z)|\sigma(x_{T_n}^z)] = \mathbb{E}[f(x_{T_n}^z, \ldots, x_{T_n}^z)|\mathcal{F}_{T_n}] \text{ a.s.}
\]
Consequently, by (8) we have
\[
\frac{\partial}{\partial z_k} \mathbb{E}[f(x_{T_n}^z, \ldots, x_{T_n}^z)] = \frac{\partial}{\partial z_k} \mathbb{E}[g(x_{T_n}^z)]
= \mathbb{E} \left[ g(x_{T_n}^z) \int_0^{T_n} a(t) \left( R(t, x_{t-}^z) \frac{\partial x_{t-}^z}{\partial z_k} \right)^T \right] dW_t,
\]
which concludes the proof.

**Remark 3.** Under stronger condition on the vector fields (see Theorem (2-28) of Bichteler, Jacod and Gravereaux [5]) we can ensure the existence of a density \( p_T(z, y) \) for the random variable \( x_T^z \) with \( p_T \in C^1(\mathbb{R}^d \times \mathbb{R}^d) \). It is then possible to relax the regularity restrictions on \( f \) so that we need only make a measurability assumption on \( f \).

**Remark 4.** The result (9) extends the result of Section 3.2 Fourniè et al [14] to jump-diffusions. We notice that the form of the results do not correspond exactly,
their weight is represented by

\[
\pi = \int_0^T a(t) \left( R(t, x_i^-) \frac{\partial x_i^-}{\partial z_k} \right)^T dW_t,
\]

with \( a \in L^2[0, T] \) satisfying \( \int_0^T a(t) = 1 \) for all \( 1 \leq i \leq n \), and our weight \( \tilde{\pi} \) is a particular case of this when \( a = 0 \) on \([T_1, T]\). However, it is clear that if \( a \neq 0 \) on \([T_1, T]\) then

\[
\text{Var}(\tilde{\pi}) \leq \text{Var}(\pi).
\]

Since the efficiency of Monte Carlo is optimised by the choice of the minimal variance weight we would always choose \( a \equiv 0 \) on \([T_1, T]\) and hence there is no conceivable practical advantage to representing the weight by \( \pi \).

We may adapt this approach to deal with higher order derivatives as well.

**Theorem 3.** Suppose that \( XX^T \) is uniformly elliptic and further assume that the conditions on the vector fields are strengthened so that the following conditions are satisfied. For every \( t \in [0, T] \) and \( y \in E \)

\[
Z(t, \cdot), X(t, \cdot) \text{ and } Y(t, \cdot, y) \in C^\infty_b(\mathbb{R}^d). \tag{12}
\]

For every \( l \in \mathbb{N} \cup \{0\}, \nabla^l Z(\cdot, \cdot), \nabla^l X(\cdot, \cdot) \) and \( \int_E |\nabla^l Y(\cdot, \cdot, y)|^2 G(dy) \) are continuous on \([0, T] \times \mathbb{R}^d \) \( \tag{13} \)

For \( r \in \mathbb{N} \) with \( r \geq 2 \) and \( l = 1, 2 \)

\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \left( \int_E (|Y(t, x, y)|^r + |\nabla^l Y(t, x, y)|^r) G(dy) \right) < \infty \tag{14}
\]

Then, if \( f \in C^3_b(\mathbb{R}^d) \) and, for each \( t \in [0, T] \), \( R(t, \cdot) \in C^1_b(\mathbb{R}^d) \) (where the bounds on \( R(t, \cdot) \) and \( \nabla R(t, \cdot) \) hold uniformly in \( t \in [0, T] \)), the following formula holds for all \( 1 \leq j, k \leq d \)

\[
\frac{\partial^2}{\partial z_j \partial z_k} \mathbb{E}[f(x_T^-)] = \frac{4}{T^2} \mathbb{E} \left[ f(x_T^-) \int_{T/2}^T \left( R(t, x_i^-) \frac{\partial x_i^-}{\partial z_j} \right)^T dW_t \int_0^{T/2} \left( R(t, x_i^-) \frac{\partial x_i^-}{\partial z_k} \right)^T dW_t \right] + \frac{2}{T^2} \mathbb{E} \left[ f(x_T^-) \int_0^{T/2} \left( \nabla R(t, x_i^-) \frac{\partial x_i^-}{\partial z_j} \right)^T dW_t \right] + \frac{2}{T^2} \mathbb{E} \left[ f(x_T^-) \int_0^{T/2} \left( \frac{\partial^2 x_i^-}{\partial z_j \partial z_k} \right)^T dW_t \right].
\]

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Moreover, if we consider \(0 < T_1 \leq \ldots \leq T_n \leq T\) and a function of the form \(f(x_{T_1}, \ldots, x_{T_n})\), where \(f \in C^3_b(\mathbb{R}^d \times \ldots \times \mathbb{R}^d)\). Then, the above result remains true when we replace \(f(x_{T_1}, x_{T_2}, \ldots, x_{T_n})\) and \(T\) by \(T_1\) in the above formula.

**Remark 5.** The conditions on the vector fields are stronger than needed, but we state them in their current form for simplicity.

**Proof.** Define the function \(w : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\) by \(w(x, y) = \nabla f(x) y\). Then it is easy to verify that the function \(p : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\) defined by \(p(t, z, y) = \mathbb{E}\left[w\left(x_t, z, \frac{\partial x_t}{\partial z}\right)\right]\) satisfies the backward equation associated to the generator of the \(\mathbb{R}^d \times \mathbb{R}^d\)-valued diffusion \((x_t, z_t, \frac{\partial x_t}{\partial z}, \frac{\partial z_t}{\partial x})_T\). The argument now proceeds as before; applying Itô’s formula to \(p(t, x_t, z_t, \frac{\partial x_t}{\partial z})\), letting \(t \to T\) and then multiplying by \(\int_0^T R(t, x_t, z_t, \frac{\partial x_t}{\partial z}) T dW_t\) and taking expectations allows the argument to be concluded as in Theorem 2.3 of Elworthy and Li [11].

## 4 Relaxing the Ellipticity Criterion

For simplicity we now assume that the vector fields are time homogeneous. We denote by \(U\) the \(L(\mathbb{R}^d, \mathbb{R}^d)\)-valued process given by \(U_t = \nabla z x_t^z\) and denote its inverse, when it exists, by \(V_t\). We define the Malliavin covariance matrix

\[
C_t(z) = \int_0^t \langle (V_s - X(z^s_\cdot))(V_s - X(z^s_\cdot))^T \rangle ds
\]

and make the following a standing assumption.

**Assumption 1.** For fixed \(T > 0\), \(C_T\) is invertible a.s. and moreover \(|C_T^{-1}| \in L^p\) for all \(p \geq 1\).

This assumption is known to be true in certain cases, for instance it holds in the diffusion case under Hörmander conditions on the vector fields (see Nualart [21]), and more recently it has been shown to hold in the jump diffusion case for finite intensity jumps under uniform Hörmander condition (see Forster, Lütkebohmert and Teichmann [13]). For more general jump processes the problem is more involved but ideas in this setting have been developed in Cass [8] and Takeuchi [26].

We now prove an extension of Theorem 3.2 of Arnaudon and Thalmaier [1] which allows us to give an explicit representation of the weight in terms of an adapted \(\mathbb{R}^d\)-valued process. Note that the result of Forster, Lütkebohmert and Teichmann [13] where the weight is given in the form of a anticipating Skorokhod integral may be converted into sum of integrals of adapted processes using the expansion formula (1.49) in Nualart [21].
A representation of this type is more desirable from the point of view of simulation. First we recall some concepts from Malliavin calculus. Let \( a \) be an \( L(R^d, R^m) \)-valued previsible process such that for \( T > 0 \) fixed
\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T |a_s h|^2 ds \right) \right] < \infty, \ h \in R^d \text{ locally at } 0
\] (15)
and define a new probability measure by
\[
Z_h^T = \frac{dP}{dP} \bigg|_{\mathcal{F}_T} = \exp \left( -\int_0^T <a_s h, dW_s> - \frac{1}{2} \int_0^T |a_s h|^2 ds \right),
\]
and let \( Z_t = \mathbb{E}[Z_h^T | \mathcal{F}_t] \) for \( 0 \leq t \leq T \). Introduce a perturbed Brownian motion \( dW^h_t = dW_t + a_t hdt \) and denote by \( x^h_t, C^h_t(z) \) the processes corresponding to \( x^t \) and \( C_t(z) \) when the underlying Brownian motion is replaced by \( W^h_t \). The crucial ingredient to the following result is the observation that the change of measure described above has no effect on the Poisson random measure.

**Theorem 4.** Suppose Assumption 1 is in force along with the following conditions on the vector fields
\[
Z(.) \in C^\infty_b(R^d, R^d), \ X(.) \in C^\infty_b(R^d, L(R^m, R^d)), \ Y(., y) \in C^\infty_b(R^d, R^d)
\]
and \( \sup_{x \in R^d} \int_{E} |\nabla_x Y(x, y)|^2 G(dy) < \infty \) for all \( n \in N \).

Further assume
\[
\sup_{x \in R^d} \sup_{y \in E} |(I + \nabla_x Y(x, y))^{-1}| < \infty. \tag{16}
\]
Then, for any \( f \in C^1_c(R^d) \) and \( j \in \{1, 2, \ldots, d\} \) we have
\[
\frac{\partial}{\partial z_j} \mathbb{E}[f(x^h_T)] = \mathbb{E} \left[ f(x^h_T) \left( \int_0^T V_s X(x_t-)_dW_s \right)^T C^{-1}_T(z) e_j + \sum_{k=1}^d \left( C^{-1}_T(z) \left( \frac{\partial}{\partial h_k} \bigg|_{h=0} C^h_T(z) \right) C^{-1}_T(z) \right) k,j \right].
\]

**Remark 6.** Condition (16) is there to ensure both the existence of \( V = U^{-1} \) and that \( V \in L^p \) for all \( p \geq 1 \). In practice this can often be relaxed in favour of some less stringent condition (see Example 2 below).

**Proof.** The fact that \( \mu \) remains a Poisson random measure with compensator \( \nu \) under \( P^h \) follows from Theorems (3.15) and (3.34) of Jacod [20]. We then observe, since \( x \) is a strong solution to the SDE (2), that
\[
\sum_{k=1}^d \left. \frac{\partial}{\partial h_k} \right|_{h=0} \mathbb{E}[f(x^h_T) Z^h_T (C^h_T(z)^{-1})_{k,j}] = 0. \tag{17}
\]
Choosing the perturbation

$$a^n_{s-} = V_{s-} X(x_{s-}) 1_{\{s \leq \tau_n\}}$$

with an increasing sequence of previsible stopping times $(\tau_n)$ chosen such that $a^n$ satisfies condition (15) and such that $\tau_n \uparrow T$. An elementary application of Itô’s formula can be used to show

$$\frac{\partial}{\partial h_k}|_{h=0} x_T = U_T \int_0^{\tau_n} V_s - X(x_s)(V_s - X(x_s)) dse_k = U_T C_{\tau_n} e_k.$$

Using this we may expand (16) to get

$$E[f(x_T^2)U_T C_{\tau_n} C_T^{-1} e_j] = E \left[ f(x_T^2) \left( \int_0^{\tau_n} V_t - X(x_t) dW_t \right)^T C_T^{-1}(z) e_j \right. \left. - \left( \sum_{k=1}^d \frac{\partial}{\partial h_k}|_{h=0} (C_T^{-1}) e_k \right)^T e_j \right].$$

We let $n \to \infty$ and expand the second term on the right hand side to give the stated result.

**Example 1.** (Bachelier with jumps, Asian options) We assume

$$dS_t = \sigma dW_t + dN_t$$

$$dA_t = S_t - dt,$$

with some Poisson process $N$ of finite rate. The Malliavin covariance matrix has the particularly simple form

$$C_T = \sigma^2 \begin{pmatrix} T & -T^2/2 & T^3/3 \\ -T^2/2 & -T^3/2 & T^3/3 \end{pmatrix}$$

and so the second term on the right hand side of the formula in Theorem 4 drops out leaving us with

$$\frac{\partial}{\partial S_0} E[f(S_T, A_T)] = \frac{6}{\sigma T} E \left[ \left( \frac{1}{T} \int_0^T W_t dt - \frac{1}{3} W_T \right) f(S_T, A_T) \right].$$

**Example 2.** (Exponential Lévy, Asian options) Consider the following model for the evolution of a stock price

$$dS_t = \beta S_t dt + \sigma S_t dW_t + \int_{y \geq -1} y S_t \nu(dy, dt)$$

$$dA_t = S_t dt,$$
with $A_0 = 0$, and where $W$ is a Brownian motion and $\mu$ a Poisson random measure with compensator $\nu(dy, dt) = G(dy)dt$. Make the assumptions that for all $p \geq 1$ and arbitrary $\delta > 0$

$$\int_{-\delta}^{-1} (1 + y)^{-p} G(dy) < \infty \quad \text{and} \quad \int_{y \geq 1} (1 + y)^p G(dy) < \infty.$$  \hfill (18)

Condition (16) is not satisfied in this example, however it is easy to show by truncating the jumps at some arbitrary level that the theorem may be applied. Assumption (18) may then be invoked to guarantee the resulting formula remains valid in the limit as the truncation parameter goes to zero. We notice also in this case that the vector fields are not bounded and similar approximation results are needed, details on how this type of argument can be made rigorous are given in the next section but we omit them here for the purpose of clear exposition. The Malliavin covariance matrix may be computed

$$C_T = \int_0^T \begin{pmatrix} -\sigma^2 S_0^2 & -\sigma^2 S_0 A_t \\ -\sigma^2 S_0 A_t & \sigma^2 A_t^2 \end{pmatrix} dt = \begin{pmatrix} \sigma^2 S_0^2 T & -\sigma^2 S_0 \int_0^T A_t dt \\ -\sigma^2 S_0 \int_0^T A_t dt & \sigma^2 \int_0^T A_t^2 dt \end{pmatrix}.$$  

It is easy to show that

$$\frac{\partial}{\partial h_1} \bigg|_{h=0} S_t^h = \sigma^2 S_0 S_t t \quad \frac{\partial}{\partial h_2} \bigg|_{h=0} S_t^h = -\sigma^2 S_t \int_0^t A_s ds$$

and then

$$\frac{\partial}{\partial h_i} \bigg|_{h=0} A_t^h = \begin{cases} \sigma^2 S_0 \int_0^t sS_{s-} ds & \text{if } i = 1 \\ \sigma^2 \left( \int_0^t A_s^2 ds - A_t \int_0^t A_s ds \right) & \text{if } i = 2 \end{cases}.$$  

We notice that $\det C_t = \sigma^4 S_0^2 \left( t \int_0^t A_t^2 dt - \left( \int_0^t A_t dt \right)^2 \right)$, and

$$C_T^{-1} = (\det C_T)^{-1} \begin{pmatrix} \sigma^2 \int_0^T A_t^2 dt & \sigma^2 S_0 \int_0^T A_t dt \\ \sigma^2 S_0 \int_0^T A_t dt & \sigma^2 S_0^2 T \end{pmatrix}.$$  

We must show that $C_T^{-1} \in L^p$ for all $p \geq 1$. To see this it suffices to check that

$$\mathbb{P}(\det C_T \leq \epsilon) \quad \text{is } o(\epsilon^p) \quad \text{as } \epsilon \to 0.$$  

To this end we note that for any $0 < \delta < 1$

$$\mathbb{P}(\det C_T \leq \epsilon) \leq \mathbb{P}(\det C_T \leq \epsilon, \inf_{0 \leq t \leq T} S_t > \delta, \sup_{0 \leq t \leq T} S_t < \delta^{-1}) + \mathbb{P}\left( \inf_{0 \leq t \leq T} S_t \leq \delta \right) + \mathbb{P}\left( \sup_{0 \leq t \leq T} S_t \geq \delta^{-1} \right).$$  

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We now show that on the set \( B := \{ \inf_{0 \leq t \leq T} S_t > \delta, \sup_{0 \leq t \leq T} S_t < \delta^{-1} \} \) we have, for \( \delta = \delta(\varepsilon) \) appropriately chosen, \( \det C_T > \varepsilon \). To see this note that on \( B \) we have \( \delta t \leq A_t(\omega) \leq \delta^{-1} t \) for all \( t \in [0, T] \), and so we define
\[
A = \{ f : [0, T] \to \mathbb{R}, \text{ such that } \delta t \leq f(t) \leq \delta^{-1} t \text{ for all } t \in [0, T] \}.
\]
Then by examining the form of the determinant we have
\[
\det C_T \geq \sigma^4 S_0^2 T^2 \inf_{f \in A} \text{var} f(U)
\]
where \( U \sim \text{Uniform}[0, T] \). We may bound the left hand side from below by Chebyshev’s inequality, so that for any \( a > 0 \)
\[
\text{var} f(U) \geq a^2 \mathbb{P}(|f(U) - \mathbb{E}[f(U)]| \geq a)
\]
\[
\geq a^2 \mathbb{P}(f(U) \leq -a + \mathbb{E}[f(U)])
\]
\[
\geq a^2 \mathbb{P}\left( f(U) \leq -a + \frac{\delta T}{2} \right)
\]
and taking \( a = \frac{\delta T}{16} \) gives \( \text{var} f(U) \geq \frac{\delta^4 T^2}{256} \mathbb{P}(f(U) \leq \frac{\delta T}{4}) \). Since \( f(t) \leq \delta^{-1} t \) in \([0, T]\) we have \( \mathbb{P}(f(U) \leq \frac{\delta T}{4}) \geq \mathbb{P}(U \leq \frac{\delta T}{4}) \) which gives
\[
\text{var} f(U) \geq \frac{\delta^4 T^3}{64}
\]
and so \( \det C_T \geq \frac{\delta^4 T^3 \sigma^4 S_0^2}{64} := C \delta^4 \). Choosing \( \delta = (C^{-1} \varepsilon)^{1/4} \) we see that \( \det C_T \geq \varepsilon \) on \( B \). It therefore suffices to show that
\[
\mathbb{P}\left( \inf_{0 \leq t \leq T} S_t \leq \varepsilon \right) \quad \text{and} \quad \mathbb{P}\left( \sup_{0 \leq t \leq T} S_t > \varepsilon^{-1} \right) \quad \text{are } o(\varepsilon^p) \text{ as } \varepsilon \to 0
\]
for every \( p \geq 1 \). We show this for the infimum, the supremum being a simple modification of this argument. To this end we write \( S_t = S_0 e^{X_t} \) where \( X_t \) is a Lévy process with triplet \((\sigma^2, \mu, \tilde{G})\) with
\[
\tilde{\mu} = \mu - \frac{1}{2} \sigma^2 - \int_{|y| \geq 1} y G(dy), \quad \tilde{G}(A) = G(\{ e^y - 1 : x \in A \}) \text{ for } A \in \mathcal{B}(A).
\]
It is easy to verify using the definition of \( \tilde{G} \) that and assumptions (18) that \( \int_{|x| \geq 1} e^{ux} \tilde{G}(dx) < \infty \) for all \( u \in \mathbb{R} \) and from Theorem 25.17 of Sato [25] this means that \( \mathbb{E}[e^{uX_1}] < \infty \) and, moreover, \( \mathbb{E}[e^{uhX_1}] = e^{t \Psi(u)} \) where
\[
\Psi(u) = \frac{1}{2} \sigma^2 u^2 + \int_{\mathbb{R}} \left( e^{ux} - 1 - ux \mathbb{1}_{[-1,1]}(x) \right) \tilde{G}(dx) + \tilde{\mu} u.
\]
Next, by choosing \( a > 0 \) sufficiently large we may ensure that \( \mathbb{P} \left( \sup_{0 \leq t \leq T} X_t > a \right) \leq 1/2 \). Then,

\[
\mathbb{P} \left( \inf_{0 \leq t \leq T} X_t \leq -2a \right) = \mathbb{P} \left( \inf_{0 \leq t \leq T} X_t \leq -2a, X_T \leq -2a \right) + \mathbb{P} \left( \inf_{0 \leq t \leq T} X_t \leq -2a, X_T > -2a \right) \\
\leq \mathbb{P} (X_T \leq -2a) + \mathbb{P} \left( \inf_{0 \leq t \leq T} X_t \leq -2a \right),
\]

and for the second term in the preceding inequality we may use the strong Markov property at the stopping time \( \zeta = \inf \{ t \geq 0 : X_t \leq -2a \} \) to give \( \mathbb{P} (\inf_{0 \leq t \leq T} X_t \leq -2a, X_T > -2a) \leq 1/2 \mathbb{P} (\inf_{0 \leq t \leq T} X_t \leq -2a) \), and so for any \( p \geq 1 \) we have

\[
\mathbb{P} (\inf_{0 \leq t \leq T} X_t \leq -2a) \leq 2^{p-1} \mathbb{P} (X_T \leq -a) \leq 2e^{-pa} e^{T\Psi(-p)}.
\]

Finally we finish by noting that for \( \epsilon \) sufficiently small

\[
\mathbb{P} (\inf_{0 \leq t \leq T} \epsilon_S \leq \epsilon) = \mathbb{P} \left( \inf_{0 \leq t \leq T} X_t \leq \log \left( \frac{\epsilon}{S_0} \right) \right) \leq 2e^{T\Psi(-p)} \mathbb{P} \left( \inf_{0 \leq t \leq T} \epsilon_S \leq \epsilon \right).
\]

Using these facts and the previous theorem we have a random variable \( \pi = \pi_1 + \pi_2 \), where

\[
\pi_1 = \frac{1}{\sigma S_0} \left( \frac{W_T t^0 A^2 dt - \int_0^T A_t dW_t - \int_0^T A_t dt}{T \int_0^T A_t dt - \left( \int_0^T A_t dt \right)^2} \right)
\]

and \( \pi_2 = \pi_{2,1} + \pi_{2,2} \) with

\[
\pi_{2,1} = \frac{S_0^2 \sigma^6}{(\text{det}C_T)^2} \left( -2 \int_0^T A_t^2 dt \int_0^T A_t dt \int_0^T \frac{\partial}{\partial h_1} \bigg|_{h=0} A_t^h dt \\
+ \left( \int_0^T A_t dt \right)^2 \int_0^T \frac{\partial}{\partial h_1} \bigg|_{h=0} (A_t^h)^2 dt \right)
\]

and

\[
\pi_{2,2} = \frac{S_0^3 \sigma^6}{(\text{det}C_T)^2} \left( - \left( \int_0^T A_t dt \right)^2 \int_0^T \frac{\partial}{\partial h_2} \bigg|_{h=0} A_t^h dt \\
+ T \left( \int_0^T A_t dt \int_0^T \frac{\partial}{\partial h_2} \bigg|_{h=0} (A_t^h)^2 dt - \int_0^T A_t^2 dt \int_0^T \frac{\partial}{\partial h_2} \bigg|_{h=0} A_t^h dt \right) \right)
\]

such that

\[
\frac{\partial}{\partial S_0} \mathbb{E} [f(S_T, A_T)] = \mathbb{E} [f(S_T, A_T) \pi].
\]

Numerical implementation of these results shows a good degree of accuracy comparable to that achieved by finite difference Monte Carlo in the case of a European call.
5 Examples

We show how the formula derived in the previous section should be implemented to obtain appropriate representations. We will find that the restrictions imposed by Theorems 2 and 3 on the vector fields are often too stringent and that we have to get round this problem by localisation.

5.1 Stochastic volatility models with jumps

We will consider a volatility process \( \sigma_t \) described by the Heston model

\[
\frac{d\sigma_t^2}{\sigma_t^2} = \kappa(\theta - \sigma_t^2)dt + \eta \sigma_t dW_t.
\]  

(19)

We will need the following lemma

Lemma 1. For any parameter choice with \( 2\kappa\theta > \eta^2 \) and for every finite \( T > 0 \)

\[
\sup_{0 \leq t \leq T} \mathbb{E}[\sigma_t^{-2}] < \infty.
\]

Proof. We let \( Y_t \) be the squared \( \delta \)-dimensional Bessel process defined as the unique strong solution to the SDE

\[
Y_t = \sigma_0^2 + \delta t + 2 \int_0^t \sqrt{Y_s} dW_s.
\]

For the choice \( \delta = \frac{4\kappa\theta}{\eta^2} \) we can relate \( Y_t \) and \( \sigma_t^2 \) by the time change (see Going-Jaeschke and Yor [19])

\[
\sigma_t^2 = e^{-\kappa t} \left( \frac{\eta^2}{4\kappa} (e^{\kappa t} - 1) \right).
\]

If we let \( f(t) = \frac{\eta^2}{4\kappa} (e^{\kappa t} - 1) \) and \( T^* = f(T) < \infty \) then as \( t \) takes values in \([0, T]\) so \( f(t) \) ranges over \([0, T^*]\). Consequently,

\[
\sup_{0 \leq t \leq T} \mathbb{E}[\sigma_t^{-2}] = \sup_{0 \leq t \leq T} \mathbb{E}[(e^{-\kappa t} Y(f(t)))^{-1}] \leq e^{\kappa T^*} \sup_{0 \leq t \leq T^*} \mathbb{E}[Y_t^{-1}] .
\]

(20)

Next we notice from the expression for the Laplace transform of \( Y_t \) (Revuz and Yor [23], page 422)

\[
\mathbb{E}[Y_t^{-1}] = \int_0^\infty \mathbb{E}[e^{-\lambda Y_t}] d\lambda = \int_0^\infty (1 + 2\lambda t)^{-\delta/2} \exp \left( \frac{-\lambda \sigma_0^2}{1 + 2\lambda t} \right) d\lambda.
\]

From this and the fact that \( \delta > 2 \) we have, for any \( \epsilon > 0 \),

\[
\sup_{0 \leq t \leq T^*} \mathbb{E}[Y_t^{-1}] \leq \int_0^\infty (1 + 2\lambda t)^{-\delta/2} d\lambda \leq (2\epsilon)^{-1}.
\]

So the proof will be complete if we can show

\[
\limsup_{t \to 0} \int_0^\infty (1 + 2\lambda t)^{-\delta/2} \exp \left( \frac{-\lambda \sigma_0^2}{1 + 2\lambda t} \right) d\lambda < \infty.
\]
By using the substitution $\xi = 1 - (1 + 2\lambda t)^{-1}$ and writing $z = t^{-1}, y = \frac{\sigma_0^2}{2}$ we need to examine the behaviour of

$$\frac{z}{2} \int_0^1 (1 - \xi)^{\delta/2 - 2} e^{-\sigma_0^2 z \xi/2} d\xi = \sigma_0^{-2} \int_0^1 (1 - \xi)^{\delta/2 - 2} ye^{-y \xi} d\xi$$

as $y \to \infty$, and it suffices the check that the expression on the right hand side is bounded for large $y$. To show this, first suppose $\frac{\delta}{2} - 2 \geq 0$ then we trivially have

$$\sigma_0^{-2} \int_0^1 (1 - \xi)^{\delta/2 - 2} ye^{-y \xi} d\xi \leq \sigma_0^{-2} (1 - e^{-y}) \leq \sigma_0^{-2}.$$

Next, suppose $\frac{\delta}{2} - 2 < 0$, then by making the substitution $w = (1 - \xi)y$ and noticing that for $y > 1$

$$\int_0^1 (1 - \xi)^{\delta/2 - 2} ye^{-y \xi} d\xi = ye^{-y} \left( e^{1 - \int_0^1 w^{\delta/2 - 2} dw + \int_1^y e^w dw } \right),$$

we see that the right hand side may be bounded uniformly in $y$ since $\frac{\delta}{2} - 2 > -1$.

It will be convenient to think of the process $\sigma_t$ instead, so writing $X_t = \log S_t$ to represent the evolution of the logarithm of the stock price the system can be described by the vector SDE

$$\begin{pmatrix} X_t \\ \sigma_t \end{pmatrix} = \begin{pmatrix} x \\ \sigma_0 \end{pmatrix} + \int_0^t \left( \begin{array}{c} r - \frac{1}{2} \sigma_s^2 \\ \rho \sigma_s \end{array} \right) ds + \int_0^t \left( \begin{array}{c} \sqrt{1 - \rho^2} \sigma_s \\ \rho \sigma_s \\ \mu - \nu \end{array} \right) (dy, ds).$$

We shall call this model SVJ. Before the next theorem we introduce the notation $C^k_c(\mathbb{R}^d)$ to indicate the set of real-valued, $k$-times differentiable, compactly supported functions with domain $\mathbb{R}^d$. We then define $I(\mathbb{R}^d)$ to be the collection of indicator functions of the form $1_{(a,b)}, 1_{(a,b]}$, $1_{[a,b)}$, or $1_{[a,b]}$ for some $|a| < |b| < \infty$ and finally a class of real-valued functions on $\mathbb{R}^d$, $J(\mathbb{R}^d)$, by

$$J(\mathbb{R}^d) = \left\{ f : f = \sum_{i=1}^n a_i f_i, \ a_i \in \mathbb{R}, \ n \in \mathbb{N}, \ f_i \in C^k_c(\mathbb{R}^d) \cup I(\mathbb{R}^d) \right\}.$$
Lemma 2. For every \( y \in \mathbb{R}, T > 0 \) and under the assumption \( 2\kappa \theta > \eta \) the following is true
\[
\lim_{\epsilon \downarrow 0} \sup_{x \in \mathbb{R}} \mathbb{P}(X_T^x \in (y - \epsilon, y + \epsilon)) = 0.
\]

Proof. We write \( J_t = \int_0^t \int \rho \mu(dy, ds) \) and observe that the distribution of \( X_T^x \) conditional on \( J_T \) and \( \{W_t: 0 \leq t \leq T\} \) is Gaussian. Indeed we have
\[
X_T^x | J_T, \{W_t: 0 \leq t \leq T\} \sim N(x + \alpha, \beta^2),
\]
where
\[
\alpha = \int_0^T \left(r - \frac{1}{2}\sigma_t^2\right) dt + \rho \int_0^T \sigma_t dW_t + J_T,
\]
\[
\beta^2 = (1 - \rho^2) \int_0^T \sigma_t^2 dt.
\]
This gives
\[
\sup_{x \in \mathbb{R}} \mathbb{P}(X_T^x \in (y - \epsilon, y + \epsilon)) = \sup_{x \in \mathbb{R}} \mathbb{E}[\mathbb{1}_{\{X_T^x \in (y - \epsilon, y + \epsilon)\}} | J_T, \{W_t: 0 \leq t \leq T\}]
\]
\[
= \sup_{x \in \mathbb{R}} \mathbb{E} \left[ \int_{y-\epsilon}^{y+\epsilon} \frac{1}{\sqrt{2\pi\beta}} \exp \left( -\frac{(z - \alpha - x)^2}{2\beta^2} \right) dz \right] 
\leq C \mathbb{E}[\beta^{-1}]
\]
for some constant \( C < \infty \). The proof will be complete if we can show \( \mathbb{E}[\beta^{-1}] < \infty \), but this is true since the Cauchy-Schwarz inequality gives
\[
\beta^{-1} \leq (T \sqrt{1 - \rho^2})^{-1} \left( \int_0^T \sigma_t^{-2} dt \right)^{1/2} \quad \text{a.s.}
\]
Then, from the previous lemma,
\[
\mathbb{E} \left[ \left( \int_0^T \sigma_t^{-2} dt \right)^{1/2} \right] \leq \mathbb{E} \left[ \int_0^T \sigma_t^{-2} dt \right]^{1/2}
\]
\[
\leq (T \sup_{0 \leq t \leq T} \mathbb{E}[\sigma_t^{-2}])^{1/2} < \infty.
\]

An application of the extended Bismut-Elworthy-Li formula will give the following result.

Theorem 5. Suppose that the parameters of the SVJ model satisfy \( 2\kappa \theta > \eta^2 \) and \( f \in \mathcal{F}(\mathbb{R}) \) then, provided \(|\rho| < 1\), the following is true
\[
\frac{\partial}{\partial S_0} \mathbb{E}[f(S_T)] = \mathbb{E} \left[ f(S_T) \int_0^T \frac{1}{TS_0 \sqrt{1 - \rho^2} \sigma_s} dZ_s \right].
\]
Remark 7. For the purposes of Monte Carlo applications one would make use of the localised Malliavin technique described in Fournié et al [14], and it is clear that the class of functions \( J(\mathbb{R}) \) is sufficiently rich for this purpose. In particular, it enables us to deal with digital payoffs and European call and put option payoffs.

Proof. Step 1 We assume that \( f \in C^2_c(\mathbb{R}) \subset C^2(\mathbb{R}) \) and note that this implies \( f \in C^2(\mathbb{R}) \) and we let \( D \subset \mathbb{R} \) be some arbitrary compact subset with \( x \in D \).

It suffices to derive a representation for \( X_T \) for \( f \in J(\mathbb{R}) \), the conclusion for \( S_T \) will then follow by applying the result for \( X_T \) to the function \( f \circ \exp \in J(\mathbb{R}) \), and changing the variable of differentiation to \( S_0 \).

Step 2 We construct an approximating sequence of SDEs with solution \( X^N_T \) such that \( X^N_T \to X \) a.s. and such that the extended Bismut-Elworthy-Li formula can be applied for each \( X^N_T \). To this end we define for \( N \geq 2 \)

\[
\begin{aligned}
(X^N_T) &= \left( \frac{x}{\sigma_0} \right) + \int_0^t \left( \frac{r - h^N(\sigma^N_s)}{\sigma^N_s} \right) ds + \int_0^t \left( \sqrt{1 - \rho^2} p^N(\sigma^N_s) \right) dZ_s \\
&\quad + \int_0^t \left( \frac{\mu}{\sigma^N_s} \right) dW_s + \int_0^t \int E \left( \frac{y_0}{\mu - \nu} \right) (\mu - \nu)(dy, ds)
\end{aligned}
\]

(21)

where the functions \( h^N, g^N, p^N \in C^2(\mathbb{R}) \) are such that

\[
h^N(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } |x| \leq N \\ 0 & \text{if } |x| \geq N + 1 \end{cases}
\]

\[
g^N(x) = \begin{cases} \left( \frac{\theta}{2} - \frac{\eta^2}{8} \right) \frac{1}{x} & \text{if } x \geq \frac{1}{N} \\ 0 & \text{if } x \leq \frac{1}{N} \end{cases}
\]

and

\[
p^N(x) = \begin{cases} \frac{x}{N^2} & \text{if } x \leq 0 \\ x & \text{if } x \geq \frac{1}{N} \end{cases}
\]

where, \( \xi = \frac{1}{2} (\frac{\theta}{2} - 1) \) and, as in Lemma 1, \( \delta = \frac{4 \alpha_0 \beta}{\eta^2} \). Moreover, for each \( N \), \( h^N(x) \leq \frac{1}{2}x^2 \) for all \( x \in \mathbb{R} \), \( g^N(x) \leq \left( \frac{\theta}{2} - \frac{\eta^2}{8} \right) \frac{1}{x} \) for all \( x \in [0, \infty) \) and \( \frac{1}{2N^2} \vee x \leq p^N(x) \leq 1 \) for all \( x \in [0, \frac{1}{N}] \) (similar approximating sequences for the volatility have been discussed in Ewald [12]). Next, we define the stopping times

\[
\tau_N = \inf \left\{ t \geq 0 : \sigma_t \leq \frac{1}{N} \right\}, \quad \zeta_N = \inf \{ t \geq 0 : \sigma_t \geq N \}.
\]

Then, it is well known that for \( \eta^2 < 2\alpha \theta \) the volatility never hits zero so we have \( \tau_N \to \infty \) a.s. as \( N \to \infty \), and since the solution to (19) is non-explosive we also have \( \zeta_N \to \infty \) a.s. as \( N \to \infty \). Consequently, for each
$t \in [0, T], X^N_t = X_t$ a.s. on the set $\{\tau_N > t, \zeta_N > t\}$ and so $X^N_t \to X_t$ a.s. as $N \to \infty$.

Step 3 We confirm that the extended Bismut-Elworthy-Li formula applies for each $N$ to deduce

$$\frac{\partial}{\partial x} \mathbb{E}[f(X^N_T)] = \mathbb{E} \left[ f(X^N_T) \frac{1}{T\sqrt{1 - \rho^2 \sigma^N_\rho}} dZ_s \right].$$

The vector fields driving the SDE defining $X^N$ satisfy the conditions of Theorem 2 so we need only verify that the process

$$K^N_t := \int_0^t R(s, x_s) \frac{\partial x^N_s}{\partial s_1} dW_s = \int_0^t \frac{1}{T\sqrt{1 - \rho^2 \sigma^N_\rho}} dZ_s$$

is a martingale for all $N$. But this is immediate from the fact that the integrand is bounded (by $2N^2/\sqrt{1 - \rho^2}$).

Step 4 Next we check that

$$\frac{\partial}{\partial x} \mathbb{E}^x[f(X_T)] = \lim_{N \to \infty} \frac{\partial}{\partial x} \mathbb{E}^x[f(X^N_T)]$$

To do this we define the sequence of functions $\phi^N : D \to \mathbb{R}$ by $\phi^N : x \mapsto \mathbb{E}^x[f(X^N_T)]$. We know that each $\phi^N$ is differentiable, and it is clear by bounded convergence that $\phi^N(x) \to \phi(x) := \mathbb{E}^x[f(X_T)]$ for every $x \in D$.

We now confirm that $\phi$ is differentiable with $\phi'(x) = \lim_{N \to \infty} \phi'^N(x) = \mathbb{E}[f'(X_T)]$. Since, for every $N$, $\frac{\partial X^N_T}{\partial x} \equiv 1$ this will be achieved if we can show

$$\lim_{N \to \infty} \mathbb{E}[f'(X^N_T)] = \mathbb{E}[f'(X_T)] \quad (22)$$

and the convergence is uniform over $x \in D$. To do this we first show that $X^N_t \to X_T$ in $L^1$ uniformly in $x \in D$, but since each term $\mathbb{E}[|X_T - X^N_T|]$ is independent of $x$ it suffices the show that $X^N_T \to X_T$ in $L^1$, since any convergence will then immediately be uniform in $x$. Before we do this we note that a straightforward application of the comparison theorem (page 269 Rogers and Williams [24]) tells us for each $t \in [0, T]$ that $y_t \leq \sigma_t$ a.s. where $y_t$ is the Ornstein-Uhlenbeck process solving the SDE

$$dy_t = -\frac{k}{2} y_t dt + \frac{\eta}{2} dW_t. \quad (23)$$

We may also use the proof of the comparison theorem combined with the fact that $\sigma > 0$ a.s. to show that $\sigma_t \leq \sigma_t$ a.s. for each $t \in [0, T]$. Then, we use the Cauchy-Schwarz and Burkholder-Davis-Gundy inequalities together with the fact that $\left(\sum_{i=1}^k x_i\right)^2 \leq k \sum_{i=1}^k x_i^2$ to show that the family $\{X^N_t : N \geq 2\}$ is bounded in $L^2$. We end up with

$$\mathbb{E}[(X^N_T)^2] \leq 4x^2 + \int_0^T (8T \rho^2 + 8T \mathbb{E}[\rho^N(\sigma^N_\rho)^2] + 4\mathbb{E}[\rho^N(\sigma^N_\rho)^2]) dt$$
Since \( h^N(x) \leq x^2/2, p^N(x) \leq 1 + |x|, \sup_{0 \leq t \leq T} E[(\sigma_t^N)^4] \leq \sup_{0 \leq t \leq T} E[\sigma_t^4] + \sup_{0 \leq t \leq T} E[y_t^4] < \infty \) we conclude
\[
E[(X^N_T)^2] \leq 4x^2 + 8r^2T^2 + 2T^2 \sup_{0 \leq t \leq T} E[(\sigma_t^N)^4] + 8T(1 + \sup_{0 \leq t \leq T} E[(\sigma_t^N)^2])
\]
and the right hand side of the inequality may be bounded uniformly in \( N \), and consequently \( X^N_T \rightarrow X_T \) in \( L^1 \) uniformly in \( x \). Finally, we verify (22) by noting that
\[
E[f'(X^N_T) - f'(X_T)] \leq \E[|f'(X^N_T) - f'(X_T)|1_{(|X^N_T - X_T| \leq \varepsilon)}]
+ \E[|f'(X^N_T) - f'(X_T)|1_{(|X^N_T - X_T| > \varepsilon)}]
\]
The first term on the right converging to zero uniformly in \( x \) by the uniform continuity of \( f' \) and the second term likewise by the convergence in probability (from Chebyshev’s inequality) of \( X^N_T \) to \( X_T \) uniformly for \( x \in D \) and the boundedness of \( f' \).

Step 5 We now establish
\[
\mathbb{P}\left[\tau_N \leq T\right] = \mathbb{P}\left[\sup_{0 \leq t \leq T} \sigma_t^2 \leq \frac{1}{N}\right] \leq \mathbb{P}\left[\sup_{0 \leq t \leq T} Y \left(\frac{\nu^2}{4N} (e^{\kappa t} - 1)\right) \leq \frac{e^{\kappa T}}{N}\right]
\leq \mathbb{P}\left[\inf_{0 \leq t \leq \infty} Y_t \leq \frac{e^{\kappa T}}{N}\right] = \frac{e^{2\kappa T}}{N^{2\kappa}}.
\]
The last line following from the observation that the scale function for a \( \delta \)-dimensional squared Bessel process is \( s(x) = -x^{-2\delta} \) (see page 286 of Rogers and Williams [24]). We now observe by the Itô-isometry that
\[
E[|K^N_T - K_T|^2] = \frac{1}{T(1 - \rho^2)} \mathbb{E}\left[\int_0^T \left(\frac{1}{\sigma_s} - \frac{1}{p^N(\sigma_s)}\right)^2 ds\right] \leq \frac{2}{T(1 - \rho^2)} \mathbb{E}\left[\int_0^T \left(\frac{1}{\sigma_s} - \frac{1}{p^N(\sigma_s)}\right)^2 + \left(\frac{1}{p^N(\sigma_s)} - \frac{1}{p^N(\sigma_s)}\right)^2 ds\right].
\]
Using the three facts $p^N(x) \geq x$ for all $x$, $p^N(\sigma_i^N) = p^N(\sigma_i)$ for $t < \tau_N$ and $p^N(x) \geq \frac{1}{2N^T}$ for all $x$, we see that

\[
\mathbb{E}[|K^N_T - K_T|^2] \leq \frac{4\sup_{0 \leq t \leq T} \mathbb{E}[|\sigma_t^{-2}|]}{(1 - \rho^2)} + \frac{2}{T(1 - \rho^2)} \mathbb{E}\left[\int_{\tau_N}^T \left(\frac{1}{p^N(\sigma_s)} - \frac{1}{p^N(\sigma_s^N)}\right)^2 dt \mathbb{1}_{\{\tau_N < T\}}\right]
\]

\[
\leq \frac{4}{1 - \rho^2} \left(\sup_{0 \leq t \leq T} \mathbb{E}[|\sigma_t^{-2}|] + 4N^{2\epsilon} \mathbb{P}(\tau_N < T)\right)
\]

\[
\leq \frac{4}{1 - \rho^2} \left(\sup_{0 \leq t \leq T} \mathbb{E}[|\sigma_t^{-2}|] + 4\epsilon^2 e^{2T}\right) < \infty,
\]

and the fact that $K^N_T \to K_T$ in $L^1$ is immediate.

**Step 6** We now relax the regularity conditions on $f$ in two stages. Firstly, we extend to $f \in C_0(\mathbb{R})$. To do this we notice that we can identify a sequence of functions $f_n \in C^\infty(\mathbb{R})$ with $f_n \to f$ uniformly and boundedly as $n \to \infty$. The extension is then immediate since bounded convergence implies $\mathbb{E}[f_n(X_T)] \to \mathbb{E}[f(X_T)]$ and, for any compact subset $H \subset \mathbb{R}$, we have

\[
\sup_{x \in H} \left|\frac{\partial}{\partial x} \mathbb{E}[f_n(X^{x}_T)] - \mathbb{E}[f(X_T)K_T]\right| \leq \mathbb{E}[(f_n(X^{x}_T) - f(X^{x}_T))^2]^{1/2} \leq \mathbb{E}[K^2_T]^{1/2} \mathbb{E}[(f_n(X^{x}_T) - f(X^{x}_T))^2]^{1/2}
\]

(25)

The convergence of the right hand side to zero being immediate from the fact that $f_n \to f$ uniformly, and that $X^{x}_T = x + S$ for some random variable $S$ independent of $x$. Secondly, we extend to indicator functions of the form $f = \mathbb{1}_{[a, b]}$ (the extension to indicators of open and half-open intervals being similar). To do this we note that we can construct an approximating sequence $f_n \in C_0(\mathbb{R})$ having the properties that $f_n \to f$ pointwise and, for any neighbourhoods $B_a$ and $B_b$ of $a$ and $b$ respectively, $f_n - f = 0$ on $L := \mathbb{R} \cap B_a^c \cap B_b^c$ for $n$ sufficiently large. We can now repeat the argument of the previous paragraph to obtain (25). To show that the right hand side of (25) can be made arbitrarily small we let $\delta > 0$ and fix some $\epsilon > 0$ chosen such that

\[
\sup_{x \in H} \mathbb{P}(X^{x}_T \in (a - \epsilon, a + \epsilon)) < \frac{\delta}{4\mathbb{E}[K^2_T]^{1/2}}
\]

and

\[
\sup_{x \in H} \mathbb{P}(X^{x}_T \in (b - \epsilon, b + \epsilon)) < \frac{\delta}{4\mathbb{E}[K^2_T]^{1/2}}
\]

as we may by Lemma 2. With $L = \mathbb{R} \cap (a - \epsilon, a + \epsilon)^c \cap (b - \epsilon, b + \epsilon)^c$ we may then choose $N$ such that for all $n \geq N$ we have $\sup_{y \in L} |f_n(y) - f(y)| = 0$
and we can bound the right hand side of (26) by
\[ 2E[K_{T}^{2}]^{1/2} \sup_{x \in H} (P(X_{T}^{2} \in (a - \epsilon, a + \epsilon)) + P(X_{T}^{2} \in (b - \epsilon, b + \epsilon))) < \delta. \]

Since \( \delta \) was arbitrary this completes the result. Since it is clear that (24) is stable under taking finite linear combinations the extension to the class \( \mathcal{J}(\mathbb{R}) \) is immediate. The result for \( S_{T} \) follows as described in Step 1.

**Remark 8.** By the same argument and under the same conditions as the last theorem we can also obtain
\[ \frac{\partial}{\partial S_{0}} E[f(S_{T_{1}}, \ldots, S_{T_{n}})] = E \left[ f(S_{T_{1}}, \ldots, S_{T_{n}}) \int_{0}^{T_{1}} \frac{1}{T_{0} \sqrt{1 - \rho^{2} \sigma^{2}}} dZ_{s} \right], \]
for any \( n \in \mathbb{N} \) and \( 0 < T_{1} \leq T_{2} \leq \ldots \leq T_{n} \leq T \).

**Remark 9.** We may apply Theorem 3 together with a similar approximation procedure described above to deduce the representation for the gamma
\[ \frac{\partial^{2}}{\partial S_{0}^{2}} E[f(S_{T})] = \frac{4}{(1 - \rho^{2})T^{2}S_{0}} E \left[ \left( \int_{T/2}^{T} \frac{1}{\sigma_{t}} dZ_{t} \int_{0}^{T/2} \frac{1}{\sigma_{t}} dZ_{t} - \frac{T \sqrt{1 - \rho^{2} \sigma^{2}}}{4} \int_{0}^{T} \frac{1}{\sigma_{t}} dZ_{s} \right) f(S_{T}) \right], \]
\[ = E \left[ \frac{4}{(1 - \rho^{2})T^{2}S_{0}} \left( \int_{T/2}^{T} \frac{1}{\sigma_{t}} dZ_{t} \int_{0}^{T/2} \frac{1}{\sigma_{t}} dZ_{t} \right) f(S_{T}) \right] - \frac{1}{S_{0}} \frac{\partial}{\partial S_{0}} E[f(S_{T})] \]
for \( f \in \mathcal{J}(\mathbb{R}) \). Where, as above, we have initially used Theorem 3 for \( X_{T} \) and deduced the result for \( S_{T} \) by applying it to the function \( f \circ \exp \) and using the observation that
\[ \frac{\partial^{2}}{\partial x^{2}} = S_{0} \frac{\partial}{\partial x} + S_{0}^{2} \frac{\partial^{2}}{\partial s^{2}}. \]

### 5.2 Stochastic volatility with jumps in the volatility - the Matytsin model

We consider how these ideas may be extended to the model of Matytsin where the volatility evolves according to the Heston model with the exception that there are jumps which occur in the stock and volatility simultaneously, the volatility jumps being of positive deterministic size. This volatility process is written as
\[ d\sigma_{t}^{2} = \kappa(\theta - \sigma_{t}^{2}) dt + \eta \sigma_{t} dW_{t} + \gamma dJ_{t}, \]
where \( J_{t} \) is a Poisson process. With \( X_{t} \) given as in the the SVJ model and \( \sigma_{t}^{2} \) as above the pair \( (X_{t}, \sigma_{t}^{2}) \) describes the Matytsin double jump model ( or
Applying Itô’s formula we can express the system \((X_t, \sigma_t)\) in terms of our previous notation by the SDEs

\[
\begin{align*}
(X_t, \sigma_t) &= (x, \sigma_0) + \int_0^t \left( \frac{\sigma_0^2 - \sigma_s^2}{\sigma_s} \right) ds + \frac{r - \frac{1}{2} \sigma_s^2}{\sigma_s} \sum_{s} \int_0^t \left( \frac{\sigma_s^2 - \gamma - \sigma_{s-}}{2} \right) ds \\
&+ \int_0^t \left( \rho \sigma_s \right) dZ_s + \int_0^t \left( \sqrt{1 - \rho^2} \sigma_s \right) dW_s \\
&+ \int_0^t \int E \left( \frac{\gamma - \sigma_s}{\sigma_{s-}} \right) (\mu - \nu)(dy, ds)
\end{align*}
\]

where \(E = \mathbb{R}, \gamma\) is the constant jump size in the volatility and \(\mu\) is a Poisson random measure with mean measure \(\nu(dy, dt) = \lambda G(dy) dt = \lambda p(y) dy dt\) where here \(p(y)\) is the density of the jumps in \(X\).

**Theorem 6.** Suppose that the parameters in the SVJJ model satisfy \(2\kappa\theta > \eta^2\) and \(f \in \mathcal{J}(\mathbb{R})\). Then, provided \(|\rho| < 1\), the following is true

\[
\frac{\partial}{\partial S_0} \mathbb{E}[f(S_T)] = \mathbb{E} \left[ f(S_T) \int_0^T \frac{1}{S_0 T \sqrt{1 - \rho^2 \sigma_{s-}}} dZ_s \right]
\]

**Proof.** The proof may be completed by following the steps of the previous theorem. The approximating system used is

\[
\begin{align*}
(X^N_t, \sigma^N_t) &= (x, \sigma_0) + \int_0^t \left( \frac{\sigma_0^2 - \sigma_s^2}{\sigma_s} \right) ds + \frac{r - h^N(\sigma^N_{s-})}{\sigma_s} \sum_{s} \int_0^t \left( \frac{\sigma_s^2 - \gamma - \sigma_{s-}^N}{2} \right) ds \\
&+ \int_0^t \left( \rho \sigma_s \right) dZ_s + \int_0^t \left( \sqrt{1 - \rho^2} \sigma_s \right) dW_s \\
&+ \int_0^t \int E \left( \frac{\gamma - \sigma_s}{\sigma_{s-}} \right) (\mu - \nu)(dy, ds)
\end{align*}
\]

and then, by the same argument as before, \(\sigma^N_t \to \sigma_t\) almost surely. Denoting \(\hat{\sigma}\) to be the solution of the usual continuous Heston process with the same parameters and its approximating process by \(\hat{\sigma}^N\), and using the fact that the jumps in the volatility in Matytsin are non-negative we can apply the comparison theorem in between jumps to give the relation

\[
y_t \leq \hat{\sigma}^N_t \leq \sigma^N_t \leq \sigma_t
\]

a.s. for every \(t \in [0, T]\), where \(y_t\) is as in (23). Consequently, using \((x + y)^p \leq 2^{p-1}(x^p + y^p)\) we can deduce

\[
\begin{align*}
\mathbb{E} \left[ \sup_{0 \leq t \leq T} (\sigma^N_t)^4 \right] &\leq 8 \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sigma_t^4 \right] + 8 \mathbb{E} \left[ \sup_{0 \leq t \leq T} y_t^4 \right] < \infty
\end{align*}
\]

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and so $\sigma_N^t \to \sigma_t$ in $L^2$ for every $t \in [0,T]$ by dominated convergence. The remainder of the argument follows as before, only the elementary observation (which follows from (26)) that $\sup_{0 \leq t \leq T} \mathbb{E}[(\sigma_t^{-2} - \tilde{\sigma}_t^{-2})^2]$ is needed to recycle the estimates already established for the process $\tilde{\sigma}$ to give new estimates on the Matytsin volatility $\sigma$. The extension from $C^2_b(\mathbb{R})$ to $\mathcal{J}(\mathbb{R})$ proceeds in the same way as before after Lemma 2 has been verified with the Matytsin volatility, which follows from an elementary adaptation of the argument given.

Remark 10. Under the same assumptions of Remark 8 we may again derive an representation for the gamma for the SVJJ model analogous to the one for SVJ.

6 Numerical Results

We implement the results for the SVJJ model firstly in the case of a European call option (payoff $(S_T - K)_+$) with $T = 1$ and $S_0 = 100$ and strike $K = 100$, secondly for a double digital payoff of the form $1_{[K_1, K_2]}(S_T)$ again with $T = 1, S_0 = 100$ and $K_1 = 100, K_2 = 110$. Finally, we implement for the delta of a digital Cliquet option with payoff profile $1_{[K_1^*, K_2^*]}(S_T - S_T^1)$, where $T = 1, T_1 = 0.5, K_1^* = 5$ and $K_2^* = 10$. The model parameters we use are $r = 0, \rho = -0.7, \gamma = 0.4, \sigma_0^2 = 0.1, \lambda = 1.0, \theta = 0.08, \kappa = 4.0, \eta = 0.6, \eta = 0$, and we assume that the jumps in $\log S$ are distributed normally with mean $-0.1$ and standard deviation 0.1.
Figure 1: Gamma for a European call option with parameters as above

Figure 2: Delta for a double digital option with parameters as above
Figure 3: Delta for a digital Cliquet option with parameters as above

References

[1] Arnaudon M., Thalmaier A. The Differentiation of Hypoelliptic Diffusion Semigroups Preprint

[2] Bass R.F. Stochastic differential equations with jumps Probability Surveys, Vol. 1 (2004) 1-19

[3] Bavouzet M.P., Messauod M. Computation of Greeks using Malliavin’s Calculus in jump type market models. Preprint (2005)

[4] Benhamou E. Smart Monte Carlo: various tricks using Malliavin calculus Quant. Finance 2 (2002), no. 5, 329-336. 91B28

[5] Bichteler K., Gravereaux, J-B., Jacod J. Malliavin Calculus for Processes with Jumps Gordon and Breach Science Publications 1987

[6] Bismut J.M. Large deviation and Malliavin calculus in Progress in Mathematics, Vol 45, Birkhäuser, Boston-Basel-Stuttgart, 1984

[7] Broadie M., Glasserman P. Estimating security price derivatives using simulation Manag. Sci. 42, 269-285 (1996)

[8] Cass T.R Smoothness of density for solutions to stochastic differential equations with jumps Preprint (2006)
[9] Cont R., Tankov P. Financial modelling with jump processes Chapman and Hall CRC Press 2003

[10] Davis M.H.A, Johansson M.P, Malliavin Monte Carlo Greeks for Jumps Diffusions Preprint (2004)

[11] Elworthy K.D., Li X-M., Formulae for the Derivatives of Heat Semigroups Journal of Functional Analysis, 125; 252-286 (1994)

[12] Ewald, C-O A Note on the Malliavin Differentiability of the Heston Model available from www.recercat.net/bitstream/2072/1024/880.pdf (2005)

[13] Forster B., Lütkebohmert E., Teichmann J., Calculation of the Greeks for Jump-Diffusions Preprint (2005)

[14] Fournié E., Lasry J.M., Lebuchoux J., Lions P.L. Applications of Malliavin Calculus to Monte Carlo Methods in Finance Finance and Stochastics, 3(4), 391 -412 (1999)

[15] Fournié E., Lasry J.M., Lebuchoux J., Lions P.L. Applications of Malliavin Calculus to Monte Carlo Methods in Finance II Finance and Stochastics, 5(2), 201-236 (2001)

[16] Gatheral J. The volatility surface - a practitioner’s guide Wiley 2006

[17] Gikhman, I.I., Skorokhod A.V. Stochastic Differential Equations Springer-Verlag 1972

[18] Gobet E., Munos R. Sensitivity analysis using Itô-Malliavin calculus and martingales and applications to stochastic optimal control SIAM J. Control Optim. 43, no. 5, 1676-1713, (2005)

[19] Göing-Jaeschke A, Yor M. A Survey and Some Generalizations of Bessel Processes ETH Zurich (1999)

[20] Jacod J., Calcul stochastique et problèmes des martingales Lecture Notes in Mathematics, 714, Springer Berlin 1979

[21] Nualart, D. The Malliavin Calculus and Related Topics Springer-Verlag, New York 1995

[22] Priola E., Zabczyk J., Liouville theorems for non-local operators J. Funct. Anal., 216 (2004) no.2, 455-490

[23] Revuz D., Yor M. Continuous Martingales and Brownian Motion (second edition) Springer-Verlag 1999

[24] Rogers L.C.G., Williams D. Diffusion Markov Processes and Martingales (Volume 2) Cambridge University Press 2000
[25] Sato K-I. *Levy processes and infinitely divisible distributions* Cambridge University Press 1999

[26] Takeuchi A. *The Malliavin Calculus for SDE with jumps and the partially hypoelliptic problem* Osaka J. Math. 39 (2002)

[27] Thalmaier A. *On the differentiation of heat semigroups and Poisson integrals* Stochastics Stochastic Rep. 61 (1997), no. 3-4, 297-321