On the fine structure and hierarchy of gradient catastrophes for multidimensional homogeneous Euler equation

B.G.Konopelchenko¹ and G.Ortenzi ² *

1 Dipartimento di Matematica e Fisica “Ennio De Giorgi”, Università del Salento, 73100 Lecce, Italy
2 Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, via Cozzi 55, 20126 Milano, Italy
2INFN, Sezione di Milano-Bicocca, Piazza della Scienza 3, 20126 Milano, Italy

October 11, 2022

Abstract

Blow-ups of derivatives and gradient catastrophes for the n-dimensional homogeneous Euler equation are discussed. It is shown that, in the case of generic initial data, the blow-ups exhibit a fine structure in accordance of the admissible ranks of certain matrix generated by the initial data. Blow-ups form a hierarchy composed by n + 1 levels with the strongest singularity of derivatives given by \( \partial u_i / \partial x_k \sim |\delta x|^{-(n+1)/(n+2)} \) along certain critical directions. It is demonstrated that in the multi-dimensional case there are certain bounded linear superposition of blow-up derivatives. Particular results for the potential motion are presented too. Hodograph equations are basic tools of the analysis.

1 Introduction

Multidimensional quasi-linear partial differential equations and systems have been studied intensively during last decades. The homogeneous n-dimensional Euler equation

\[
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = 0, \quad \mathbf{u} : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]  

(1.1)

is one of the most important representative of this class of equations. It is the basic equation in the theory of continuous media in the case when effects of dissipation, pressure etc. are negligible (see e.g. [14, 13, 20]). Equation (1.1) arises in various branches of physics from hydrodynamics to astrophysics [14, 13, 20, 21, 17].

A remarkable feature of the homogeneous Euler equation (HEE) is that it is solvable by the multidimensional version of the classical hodograph method [21, 17, 4, 5, 6]. Namely, solutions of the HEE are provided by the vector hodograph equation

\[
x_i - u_i t - f_i(u) = 0, \quad i = 1, \ldots, n
\]  

(1.2)

where \( f_i(u) \) are the local inverse of the initial data \( u_i(x,0) \) for equation (1.1). The hodograph equation (1.2) is also a powerful tool for the analysis of the singularities of solutions for HEE, in particular, of blow-ups of derivatives and gradient catastrophes [21, 17, 4, 11, 10]. It was demonstrated that features and properties of blow-ups and gradient catastrophes (GC) for the multidimensional HEE are quite different from those in the text-book one-dimensional examples [14, 20]. Existence or nonexistence of blow-ups in different dimensions, boundedness of certain linear superpositions of derivatives, the case of potential flows have been discussed briefly recently in paper [10].

In this paper the detailed analysis of the blow-ups and GC for HEE (1.1) is presented. It is shown that the blow ups of derivatives, which occur on the hypersurface \( \Gamma \) defined by the equation

\[
\det(M(t, u)) = 0, \quad \text{where} \quad M_{il} = \delta_{il} + \frac{\partial f_i}{\partial u_l}, \quad i, l = 1, \ldots, n,
\]  

(1.3)

form a hierarchy. Degree of singularities of derivatives and dimensions of subspaces in \( \Gamma \) at which they occur are different for different members (levels) of such hierarchy.

Blow-up of the first level have a fine structure due to the different admissible ranks \( r_1 \) of the degenerate matrix \( M \). It is shown that there are \( r_1 \)-dimensional subspaces in the spaces of variation \( \delta u_k \) and \( \delta x_i \) such that the corresponding derivatives \( \partial u_k / \partial x_i \) are bound. The rest of the derivatives blow-up on \( \Gamma \) as

\[
\frac{\partial u_k}{\partial x_i} \sim \epsilon^{-1/2}, \quad \epsilon \rightarrow 0,
\]  

(1.4)

*Corresponding author. E-mail: giovanni.ortenzi@unimib.it, Phone: +39(0)264485725

arXiv:2210.03939v1 [nlin.SI]  8 Oct 2022
where \( \epsilon \) is the distance from the catastrophe point in the most singular direction. Admissible values from the rank \( r_1 \) and the dimensions \( \text{dim} \Gamma_{r_1} \) of the corresponding subspaces in \( \Gamma \) are calculated for generic functions \( f(u) \), i.e., for generic initial data \( u(x, 0) \). For instance, for dimension \( n = 3 \), the admissible ranks \( r_1 \) are \( r_1 = 2, 1 \) and correspondingly \( \text{dim} \Gamma_{r_1=2} = 3 \) and \( \text{dim} \Gamma_{r_1=1} = 0 \), while for \( n = 8 \), one has \( r_1 = 7, 6, 5 \) and \( \text{dim} \Gamma_{r_1} = 8, 5, 0 \). On the other side, in the particular case of potential flows, \( \text{dim} \Gamma_{r_1} \) assumes others values. This case is analysed in the paper too.

It is shown that in the generic case the hierarchy of blow-ups has \( n + 1 \) levels. On the \( m \)-th level there are also the subsections of smaller dimensions where superpositions of derivatives are bounded while the blowing-up derivatives behave as

\[
\frac{\partial u_k}{\partial x_k} \sim \epsilon^{-m/(m+1)}, \quad \epsilon \to 0.
\]

So, the most singular behavior of derivatives for HEE in the generic case is given by

\[
\frac{\partial u_k}{\partial x_k} \sim \epsilon^{-(n+1)/(n+2)}, \quad \epsilon \to 0.
\]

Non-generic and Poincaré cases for initial data are discussed too. Characteristic features of blow-ups of derivatives for the two- and three-dimensional HEE are considered in details. Some concrete cases are analyzed. The results presented in this paper confirm an expected great difference between the properties of the one-dimensional and multi-dimensional homogeneous Euler equations.

The paper is organized as follows. In Section 2 blow-ups of derivatives of the first level are analyzed. Admissible ranks of the matrix \( M(t, u) \) and corresponding \( \text{dim} \Gamma_{r_1} \) are calculated in Section 3. Non-generic and Poincaré cases are considered in Section 4. The situation with potential motion is discussed in Section 5. Blow-ups of higher level and hierarchy are described in Section 6 for the maximal rank case and in Section 7 for the lower rank case. Two-dimensional and three-dimensional cases are described in Sections 8 and 9 respectively. An explicit 2D example has been carried out in Section 10.

### 2 Generic blow-ups of derivatives: first level

The hodograph equations (1.2) will be our basic tool in the study of blow-ups of derivatives and gradient catastrophes for the HEE (1.1).

Differentiating (1.2) with respect to \( x_k \), one obtains [6, 11, 10]

\[
\sum_{l=1}^{n} M_{il} \frac{\partial u_l}{\partial x_k} = \delta_{ik}, \quad \text{with} \quad M_{il} = t \delta_{il} + \frac{\partial f_i}{\partial u_l}, \quad i, k, l = 1, \ldots, n,
\]

where \( \delta_{ik} \) is the Kronecker symbol. This relation implies that the derivatives \( \frac{\partial u_j}{\partial x_k} \) and also \( \frac{\partial u_j}{\partial t} \) become unbounded at the hypersurface \( \Gamma \) defined by the equation

\[
\text{det}(M(t, u)) = 0.
\]

The degeneracy of the matrix \( M \) is the central point in the analysis of blow-ups. General properties of the hypersurface (2.2) and some of their implications have been discussed in [10].

For deeper study of blow-ups one needs to consider the infinitesimal version of the hodograph mapping (1.2) around a point \( u_0 \) on the blow-up hypersurface \( \Gamma \) (2.2) at fixed time \( t \), i.e., with the expansion

\[
\delta x_i = t \delta u_i + \sum_k \frac{\partial f_i}{\partial u_k}(u_0) \delta u_k + \frac{1}{2} \sum_{k,l} \frac{\partial^2 f_i}{\partial u_k \partial u_l}(u_0) \delta u_k \delta u_l + \frac{1}{6} \sum_{k,l,m} \frac{\partial^3 f_i}{\partial u_k \partial u_l \partial u_m}(u_0) \delta u_k \delta u_l \delta u_m + \ldots
\]

\[
= \sum_k M_{ik}(u_0) \delta u_k + \frac{1}{2} \sum_{k,l} \frac{\partial^2 f_i}{\partial u_k \partial u_l}(u_0) \delta u_k \delta u_l + \frac{1}{6} \sum_{k,l,m} \frac{\partial^3 f_i}{\partial u_k \partial u_l \partial u_m}(u_0) \delta u_k \delta u_l \delta u_m + \ldots, \quad I = 1, \ldots, n.
\]

For variable \( t \) one has the expansions (2.3) with the substitution \( \delta x \to \delta(x - u_0 t) \) in the l.h.s. after the subtraction of the contribution from the infinitesimal Galilean transformation.

An important feature of the multi-dimensional case is the invariance of HEE (1.1) under the group \( SO(n) \) of rotations in \( \mathbb{R}^n \). It is easy to see that in order to ensure that the variables \( (u_1, \ldots, u_n) \) are the component of the \( n \)-dimensional vector \( u \) it is sufficient to require that the functions \( f(u) = (f_1(u), \ldots, f_n(u)) \) in (1.2) are transformed as the component of a vector. In such a case \( M_{ik} \) are components of a tensor and the hodograph equations (1.2), condition (2.2) and (2.3) are invariant to the group of rotations \( SO(n) \) too. Consequently, it would be preferable to formulate the basic properties of blow-ups in a covariant form.

Another important peculiarity of the multi-dimensional HEE is that the terms linear in \( \delta u \) do not disappear in the expansion (2.3) in contrast to the one-dimensional case. The coefficients \( M_{ik}(u_0) \) do not vanish on \( \Gamma \), however they are very special. They are elements of the degenerate \( n \times n \) matrix \( M \), which is a key element for the subsequent analysis.
Let us assume that the rank of the matrix $M$ at the point $u_0 \in \Gamma$ is equal to $r_1$. It means that (see e.g. [7]) there exist $n - r_1$ vectors $R^{(\alpha)}(u_0)$ and $L^{(\alpha)}(u_0)$ with $\alpha = 1, \ldots, n - r_1$ such that

$$\sum_{k=1}^{n} M_{ik}(u_0) R^{(\alpha)}_k = 0, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, n - r_1, \tag{2.4}$$

and

$$\sum_{i=1}^{n} L^{(\alpha)}_i M_{ik}(u_0) = 0, \quad k = 1, \ldots, n, \quad \alpha = 1, \ldots, n - r_1. \tag{2.5}$$

It is well known (see e.g. [7]) that the requirements that the matrix $M$ has rank $r_1$ and that there are $n - r_1$ linearly independent vectors $R^{(\alpha)}$ and $L^{(\alpha)}$ satisfying the conditions (2.4) and (2.5) are equivalent. An advantage of conditions (2.4) and (2.5) is that they are invariant under the rotation in $\mathbb{R}^n$.

The properties (2.4) and (2.5) of the matrix $M(u)$ have immediate consequences for the expansion (2.3). First, the relations (2.4) imply that for the variations $\delta u$ of the form

$$\delta_k u_k = \sum_{\alpha=1}^{n-r_1} R^{(\alpha)}_k \delta a_\alpha, \tag{2.6}$$

where $\delta a_\alpha$ are arbitrary infinitesimals, one has

$$\sum_{k=1}^{n} M_{ik}(u) \delta_k u_k = 0, \quad i = 1, \ldots, n. \tag{2.7}$$

This fact suggests us to introduce $r_1$ linearly independent vectors $\tilde{R}^{(\beta)}$, $\beta = 1, \ldots, r_1$ complementary to the vectors $R^{(\alpha)}$, $\alpha = 1, \ldots, n - r_1$ such that arbitrary variation $\delta u$ can be decomposed as

$$\delta u = \sum_{\alpha=1}^{n-r_1} R^{(\alpha)} \delta a_\alpha + \sum_{\beta=1}^{r_1} \tilde{R}^{(\beta)} \delta b_\beta, \tag{2.8}$$

where $\delta b_\beta$ are arbitrary infinitesimals. Substituting (2.8) into (2.3) one gets

$$\delta x = \sum_{\beta=1}^{r_1} Q^{\beta}_{(0 \beta)} \delta b_\beta + \sum_{\beta_1 = 1}^{r_1} Q^{\beta_1 \beta_2}_{(0 \beta)} \delta b_\beta_1 \delta b_\beta_2 + \sum_{\alpha_1, \alpha_2=1}^{n-r_1} Q^{\alpha_1 \alpha_2}_{(20)} \delta a_{\alpha_1} \delta a_{\alpha_2} + \sum_{\alpha=1}^{n-r_1} \sum_{\beta=1}^{r_1} Q^{\alpha \beta}_{(1 \beta)} \delta a_\alpha \delta b_\beta + \ldots, \tag{2.9}$$

where

$$Q^{\beta}_{(0 \beta)} = \frac{1}{2} \sum_{k,\ell=1}^{n} \frac{\partial^2 f_k}{\partial u_k \partial u_\ell} (u_0) R^{(\alpha_1)}_k R^{(\alpha_2)}_\ell, \tag{2.10}$$

$$Q^{\beta_1 \beta_2}_{(0 \beta)} = \frac{1}{2} \sum_{k,\ell=1}^{n} \frac{\partial^2 f_k}{\partial u_k \partial u_\ell} (u_0) \tilde{R}^{(\beta_1)}_k \tilde{R}^{(\beta_2)}_\ell, \tag{2.11}$$

$$Q^{\alpha_1 \alpha_2}_{(20)} = \frac{1}{2} \sum_{k,\ell=1}^{n} \frac{\partial^2 f_k}{\partial u_k \partial u_\ell} (u_0) R^{(\alpha_1 \alpha_2)}_k R^{(\alpha_2 \alpha_1)}_\ell, \tag{2.12}$$

$$Q^{\alpha \beta}_{(1 \beta)} = \frac{1}{2} \sum_{k,\ell=1}^{n} \frac{\partial^2 f_k}{\partial u_k \partial u_\ell} (u_0) R^{(\alpha \beta)}_k \tilde{R}^{(\beta)}_\ell, \tag{2.13}$$

$$Q^{\gamma}_{(s \gamma)} = \frac{1}{(r+s)!} \sum_{k_1,\ldots,k_r} \frac{\partial^r \partial^s f_k}{\partial u_{k_1} \ldots \partial u_{k_r} \partial u_1 \ldots \partial u_{r_1}} (u_0) R^{(\alpha_1 \alpha_2) \cdots R^{(\alpha_2 \alpha_1)}_r R^{(\beta_1 \beta_2) \cdots R^{(\beta_2 \beta_1)}_r \cdots R^{(\beta r_1 \beta)}_1 \cdots R^{(\beta r_1 \beta)}_1 \cdots R^{(\beta r_1 \beta)}_1 \cdots R^{(\beta r_1 \beta)}_1}, \quad r + s \geq 2$$

$$\gamma = 1, \ldots, n - r_1, \quad \beta, \beta_1, \beta_2, \cdots = 1, \ldots, r_1.$$

One observes that the variables $\delta a_\alpha$ and $\delta b_\beta$ behave differently at the blow-up points.

Further, the existence of the vectors $L^{(\alpha)}$, $\alpha = 1, \ldots, n - r_1$ with the property (2.5) suggests to introduce new variables

$$\delta y_\alpha = L^{(\alpha)} \cdot \delta x, \quad \delta y_\gamma = \tilde{L}^{(\gamma)} \cdot \delta x, \quad \alpha = 1, \ldots, n - r_1, \quad \gamma = 1, \ldots, r_1 \tag{2.11}$$

where $(L^{(1)} \ldots L^{(n-r_1)}, \tilde{L}^{(1)} \ldots \tilde{L}^{(r_1)})$ form a basis of $\mathbb{R}^n$. 

\[ \text{Page 3} \]
Note that the transformation from the variables $\delta x_i$ to the variables $\delta y_a, \delta \gamma$, is invertible. Taking the scalar product of (2.9) with $L^{(a)}$ and $\tilde{L}^{(\gamma)}$, one obtains respectively the expansions

$$
\delta y_a = \sum_{\beta_1, \beta_2 = 1}^{r_1} Q^{(\alpha)}_{(02)}(\beta_1, \beta_2) \delta b_{1\beta_1} \delta b_{2\beta_2} + \sum_{\alpha_1, \alpha_2 = 1}^{n-r_1} Q^{(\alpha)}_{(20)}(\alpha_1, \alpha_2) \delta a_{\alpha_1} \delta a_{\alpha_2} + \sum_{\beta = 1}^{n-r_1} Q^{(\alpha)}_{(11)}(\alpha) \delta a_{\alpha} \delta b_{\beta} + \ldots ,
$$

(2.12)

and

$$
\delta \gamma = \sum_{\beta = 1}^{r_1} Q^{(\gamma)}_{(01)}(\beta) \delta b_{\beta} + \sum_{\beta_1, \beta_2 = 1}^{r_1} Q^{(\gamma)}_{(02)}(\beta_1, \beta_2) \delta b_{1\beta_1} \delta b_{2\beta_2} + \sum_{\alpha_1, \alpha_2 = 1}^{n-r_1} Q^{(\gamma)}_{(20)}(\alpha_1, \alpha_2) \delta a_{\alpha_1} \delta a_{\alpha_2} +
$$

$$
+ \sum_{\alpha = 1}^{n-r_1} \sum_{\beta = 1}^{r_1} Q^{(\gamma)}_{(11)}(\beta) \delta a_{\alpha} \delta b_{\beta} + \ldots ,
$$

(2.13)

where the coefficients in (2.12) and (2.13) are defined as

$$
Q^{(\alpha)}_{(x)}(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) \equiv L^{(\alpha)} \cdot Q_{(x)}^{(\alpha)}(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s),
$$

$$
\tilde{Q}^{(\gamma)}_{(x)}(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) \equiv \tilde{L}^{(\gamma)} \cdot Q_{(x)}^{(\gamma)}(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s).
$$

(2.14)

It is noted that the coefficients in these expansions depend on the point $u_0 \in \Gamma$ and generically $Q^{(\alpha)}_{(01)} = \tilde{L}^{(\alpha)} \cdot B^{(\beta)} \neq 0$.

It was briefly mentioned in paper [10] that the relations (2.9) and (2.12) are essential for the analysis of blow-ups of derivatives for HEE (1.1). As we will see, the expansions (2.9), (2.12) and (2.13) completely define the structures and characteristic properties of the blow-up of the first and higher levels as well as the corresponding hierarchy of blow-ups.

Blow-ups of derivatives of the first level correspond to the situation when all families of bilinear forms in (2.9), (2.12) and (2.13) and all their possible linear superpositions do not vanish. Let us denote the $(n - r_1)$-dimensional subspace of variation of $u$ around the point $u_0$ given by the formula (2.6) as $H^+_1(u_0)$ and as $H^-_1(u_0)$ its complement spanned by the vectors $\tilde{R}^{(\delta)}$

$$
\delta \cdot u = \sum_{\beta = 1}^{r_1} \tilde{R}^{(\delta)}(\beta) \delta b_{\beta}.
$$

(2.15)

Due to the invertibility of the transformations (2.11), the generic variation of $x$ around the point $x_0$ can also be represented in the form

$$
\delta x = \delta_+ x + \delta_- x
$$

(2.16)

where

$$
\delta_+ x = \sum_{\alpha = 1}^{n-r_1} P_{\alpha} \delta y_{\alpha},
$$

$$
\delta_- x = \sum_{\beta = 1}^{r_1} \tilde{P}_{\beta} \delta \gamma_{\beta}
$$

(2.17)

and

$$
\sum_{\alpha = 1}^{n-r_1} P_{\alpha} L^{(\alpha)}_k + \sum_{\beta = 1}^{r_1} \tilde{P}_{\beta} \tilde{L}^{(\beta)}_k = \delta_{ik}.
$$

(2.18)

We will denote the subspace of variations of $x_i$ given by the first of (2.17) as $H^+_{(1)0}$ and that given by the second of (2.17) as $H^-_{(1)0}$.

Now, it is easy to see that the expansions (2.12) and (2.13) define the behavior of derivations with the variations of $\delta u$ and $\delta x$ belonging to different subspaces. Indeed, the expansion (2.13) implies that if

$$
\delta u \in H^+_1(u_0), \quad \delta x \in H^-_{(1)0},
$$

(2.19)

one has

$$
\delta \gamma \sim O(1),
$$

(2.20)

and, consequently,

$$
\frac{\partial b_{\beta}}{\partial \gamma} \sim O(1).
$$

(2.21)

Analogously, if

$$
\delta u \in H^+_1(u_0), \quad \delta x \in H^+_{(1)0},
$$

(2.22)

the expansions (2.12) and (2.13) imply that

$$
\delta a_{\alpha} \sim |\delta x|^{1/2}
$$

(2.23)
and so
\[ \frac{\partial u_i}{\partial x_k} \sim |\delta x|^{-1/2}, \quad |\delta x| \to 0. \]  

Finally if
\[ \delta u \in H_{(1)u_0}^+ \quad \delta x \in H_{(1)x_0}^+ \]
one has
\[ \delta b_{\beta} \sim |\delta y|^{1/2}, \]
and, hence,
\[ \frac{\partial b_{\beta}}{\partial y_{\alpha}} \sim |\delta y|^{-1/2}, \quad |\delta y| \to 0. \]  

It is noted that
\[ \frac{\partial}{\partial y_{\alpha}} = P_{\alpha} \cdot \frac{\partial}{\partial x} = \sum_{i=1}^{n} P_{\alpha i} \frac{\partial}{\partial x_i}, \quad \alpha = 1, \ldots, n - r_1 \]  

and
\[ \frac{\partial}{\partial y_{\gamma}} = \tilde{P}_{\gamma} \cdot \frac{\partial}{\partial x} = \sum_{i=1}^{n} \tilde{P}_{\gamma i} \frac{\partial}{\partial x_i}, \quad \gamma = 1, \ldots, r_1 \]
where the matrices $P_{\alpha i}$ and $\tilde{P}_{\gamma i}$ are defined by (2.17). Summarizing one has the following table for the behavior of derivatives $\frac{\partial u_i}{\partial x_k}$ as $\delta x \to 0$

| $\delta x$ | $H_{(1)u_0}^+$ | $H_{(1)x_0}^+$ |
|-----------|----------------|----------------|
| $H_{(1)u_0}^+$ | $\propto$ | $\propto$ |
| $H_{(1)x_0}^+$ | $\propto$ | $O(1)$ |

It is noted that the derivatives $\frac{\partial u_i}{\partial x_k}$ do not blow-up in $r_1$-dimensional subspace of variations $\delta u, \delta x$. The rank $r_1$ of the matrix $M(u_0)$ and dimensions of corresponding subspaces may vary from point to point on the blow-up hypersurface $\Gamma$. Subspaces $H_{(1)u_0}^+, H_{(1)u_0}^-, H_{(1)x_0}^+, H_{(1)x_0}^-$ are elements of the orbits generated by the group $SO(n)$ of rotations. The dimensions of all elements of corresponding orbits are, obviously, the same.

3 On the admissible ranks of the matrix $M(u_0)$

The matrix $M(u)$, playing the central role in the whole construction, has a rather special form (2.1). It is parametrized by $n$ functions $f_i(u)$, $i = 1, \ldots, n$ of $n$ variables $u = u_1, \ldots, u_n$. Functions $f_i$ are essentially local inverse to the initial data $u(x, t = 0)$ for the equation (1.1).

In the analysis of the admissible constraints for matrix $M$ one should distinguish two cases. First, usually considered case, corresponds to the so-called generic initial data $u_0$ and, consequently, to the generic functions $f(u)$. In such a case the elements of the matrix $M$ depend on $n + 1$ variables $t, u_1, \ldots, u_n$ for the generic functions $f_1, \ldots, f_n$.

The second case corresponds to the situation when one considers not the fixed initial data but a family of initial data. In this case one can view the function $f_i$, as depending not only on $n$ variables $u_1, \ldots, u_n$, but also on a certain number (possibly infinite) of parameters $\lambda_i, i > 0$. For the discussion of such situation in the one-dimensional case see e.g. [10]. As we will see, the situation with possible ranks of the matrix $M$ is quite different in these two cases.

Let us begin with the generic case. The matrix $M$ depends on $n + 1$ variables $t, u$. The requirement that $M(u_0)$ has the rank $r_1$ imposes $(n - r_1)^2$ constraints of order $(r_1 + 1)$ (see e.g. [2]).

Hence, the dimension of the subspace at which the matrix $M(u_0)$ has rank $r_1$ is given by
\[ \dim \Gamma_{r_1} = n + 1 - (n - r_1)^2. \]  

Since $\dim \Gamma_{r_1}$ cannot be negative, in the generic case, one has the following constraint on the rank $r_1$
\[ (n - r_1)^2 \leq n + 1, \]
\[ \left( n - \sqrt{n + 1} \right) \leq r_1 < n, \]  

i.e.
Table 1: Admissible ranks $r_1$ for the singular matrix $M$ until dimension 11 and the related values of geometric objects characterizing the catastrophe.

| space dimension $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | ... |
|---------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| admissible ranks $r_1$ | 0   | 1   | 2.1 | 3.2 | 4.3 | 5.4 | 6.5 | 7.6 | 8.7 | 9.8 | 10.9 | 8.7 |
| $\dim \Gamma_{r_1}(u,t)$ | 1   | 2   | 3.0 | 4.1 | 5.2 | 6.3 | 7.4 | 8.5 | 9.6 | 10.7 | 11.8 | 3.7 |
| $\dim R^+_{u_0} = \dim R^+_{\mathcal{X}_0}$ | 1   | 1   | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | ... |
| $\dim R^-_{u_0} = \dim R^-_{\mathcal{X}_0}$ | 0   | 1   | 2.1 | 3.2 | 4.3 | 5.4 | 6.5 | 7.6 | 8.7 | 9.8 | 10.9 | 8.7 |

where $[x]$ indicates the first integer greater or equal to $x$ or, equivalently

$$n - k \leq r_1 < n, \quad \text{if} \quad k^2 - 1 \leq n \leq (k + 1)^2 - 2 \quad \text{with} \quad k = 1, 2, 3, \ldots$$

So, the admissible ranks of the degenerate matrix $M(u_0)$, in general, cannot assume all values $r_1 = 0, 1, 2, \ldots, n - 1$. It is easy to show that the admissible ranks for the one-dimensional case $n = 1$ is $r_1 = 0$, for $n = 2$ one has $r_1 = 1$ and for $n = 3, 4, 5, 6, 7$ the rank $r_1$ may assume values $n - 1, n - 2, n - 3, n - 4$ and so on for $n = 8, 9, \ldots, 14$ admissible ranks are equal to $n - 1, n - 2, n - 3, n - 4$ and so on for $n = 15, 16, \ldots, 23$ the rank $r_1$ can be $n - 1, n - 2, n - 3, n - 4$ and so on following formulas (3.3) or (3.4). See table 1 for explicit rank values for smaller dimensions. So, in the generic case the matrix $M(u_0)$ may have the lowest rank $r_1 = 1$ only in the two- and three-dimensional spaces. In two-dimensional case $\dim \Gamma_1 = 2$ and the space $\Gamma_1$ coincides with the singular hypersurface given by $\det M(t, u_1, u_2) = 0$.

In the three-dimensional case, the matrix $M$ has the rank $r_1 = 2$ at the generic points of the three-dimensional singular hypersurface $\det M(t, u_1, u_2, u_3) = 0$. The matrix $M$ has rank $r_1 = 1$ at a point on the singular hypersurface defined by the four equations that, after eliminating $t$ in the generic case when $\frac{\partial f_i}{\partial u_j} \neq 0$, can be reduced to the three conditions

$$
\begin{align*}
\frac{\partial f_1}{\partial u_2} \frac{\partial f_2}{\partial u_3} \frac{\partial f_3}{\partial u_1} - \frac{\partial f_1}{\partial u_1} \frac{\partial f_2}{\partial u_2} \frac{\partial f_3}{\partial u_3} &= 0, \\
\frac{\partial f_1}{\partial u_1} \frac{\partial f_2}{\partial u_2} - \frac{\partial f_1}{\partial u_3} \frac{\partial f_2}{\partial u_1} + \frac{\partial f_2}{\partial u_3} \frac{\partial f_3}{\partial u_1} &+ \frac{\partial f_3}{\partial u_2} \frac{\partial f_1}{\partial u_3} \left( \frac{\partial f_1}{\partial u_1} \right)^2 - \frac{\partial f_1}{\partial u_3} \left( \frac{\partial f_1}{\partial u_2} \right)^2 &= 0, \\
\frac{\partial f_1}{\partial u_3} - \frac{\partial f_3}{\partial u_1} \left( \frac{\partial f_1}{\partial u_2} \right)^2 - \frac{\partial f_1}{\partial u_1} \left( \frac{\partial f_3}{\partial u_2} \right)^2 - \frac{\partial f_1}{\partial u_1} \left( \frac{\partial f_3}{\partial u_2} \right)^2 &= 0.
\end{align*}
$$

For the dimensions $n = 4, 5, 6, 7$, the dimension of $\Gamma_{r_1}$ may assume the values $n$ and $n - 3$ while for $n = 8, 9, \ldots, 14$, the possible dimensions of the subspaces $\Gamma_{r_1}$ are $n, n - 3$ and $n - 8$ and so on as resumed in table 1.

In correspondence of the admissible ranks of the matrix $M(u_0)$, the possible dimension $n - r_1$ of the subspaces $R^+_{u_0}$ and $R^-_{u_0}$ are equal to 1 for $n = 2$; they are equal to 1 and 2 for $n = 3, 4, 5, 6, 7$; they are equal to 1, 2 and 3 for $n = 8, 9, \ldots, 14$ and so on as resumed in tables 1.

As it is shown in table 1 in correspondence of the admissible ranks of the matrix $M(u_0)$ the possible dimensions of the subspaces $R^+_{u_0}$ and $R^-_{u_0}$ where the derivatives blow-up, remain pretty small with increasing dimension $n$ while subspaces with bounded derivatives become larger. The Table 3 in the Appendix gives the formulas for generic dimensions $n$.

So, in the case of generic initial data and functions $f_i$ the structure of the subspaces with blow-ups of derivatives at first level is rather nontrivial and exhibits a great difference with the one-dimensional case.

## 4 Non-generic and Poincaré cases

Generic cases typically attract main attention in the study of singularities. However, an analysis of particular classes of initial data or their families are of interest too.
We begin with the three-dimensional case of rank $r_1 = 1$ situation. Within the generic case approach the equations \((4.3)\) for generic functions $f_1$, $f_2$, and $f_3$ define generically a point on the blow-up hypersurface.

Let us change the viewpoint: namely, let us consider the equations \((4.3)\) as the system of equations to define functions $f_1$, $f_2$, and $f_3$. For such solutions $f_1$, $f_2$, and $f_3$, the matrix $M(u_0)$ has rank one on the whole blow-up hypersurface $\Gamma_{r_1}$. So, for such functions $f_1$, $f_2$, and $f_3$ and corresponding initial data $u(x, t = 0)$ the singularity subspaces $R_{u_0}^+$ and $R_{u_0}^-$ have the dimensions 2 on the whole blow-up hypersurface.

In the $n$-dimensional case all $(n - r_1)^2$ constraints with $\dim \Gamma_{r_1} \geq 0$, treated in a similar way after the elimination of $t$, represent themselves the system of $(n - r_1)^2 - 1$ nonlinear PDEs which define those functions $f_1, \ldots, f_n$ of corresponding initial data and admissible ranks $r_1$ for which one has $\dim R_{u_0}^+ = \dim R_{u_0}^- = n - r_1$ on the whole blow-up hypersurface $\Gamma$ instead of the subspaces of the dimensions $\dim \Gamma_{r_1}(u, t) = n + 1 - (n - r_1)^2$ in the generic case. For example, in the 4-dimensional case, the matrix $M(u_0)$ has rank $r_1 = 2$ and $\dim \Gamma_{r_1}(u, t) = 2$ on the whole blow-up hypersurface $\Gamma_{r_1}$ for the functions $f_1, f_2, f_3, f_4$ defined by the system of equations analogue to \((3.5)\), i.e.

\[-f_{4,2}^2 + (f_{1,2}f_{4,1} - f_{1,1}f_{4,2} - f_{3,3}f_{4,2} + f_{3,2}f_{4,3})t^2 + f_{1,2}f_{3,3}f_{4,1} - f_{1,1}f_{3,3}f_{4,2} + f_{1,3}f_{3,1}f_{4,2} - f_{3,1}f_{3,2}f_{4,1} - f_{1,2}f_{1,4} + f_{1,1}f_{3,2}f_{4,3} + f_{1,1}f_{3,2}f_{4,3} = 0,\]

\[f_{3,2}^2 + (f_{1,2}f_{3,1} + f_{1,1}f_{3,2} - f_{3,3}f_{4,3} + f_{3,2}f_{4,4})t + f_{1,4}f_{3,3}f_{4,1} + f_{1,3}f_{3,3}f_{4,2} + f_{1,1}f_{3,4}f_{4,2} - f_{3,1}f_{3,4}f_{4,2} - f_{1,2}f_{3,4}f_{4,4} + f_{1,3}f_{3,4}f_{4,4} = 0,\]

\[f_{4,1}^2 + (f_{2,2}f_{3,1} + f_{3,3}f_{4,1} - f_{2,1}f_{4,2} - f_{3,1}f_{4,3})t + f_{3,1}f_{3,3}f_{4,1} + f_{2,3}f_{3,3}f_{4,1} + f_{3,3}f_{3,1}f_{4,2} - f_{2,3}f_{3,3}f_{4,2} - f_{3,1}f_{3,3}f_{4,2} + f_{3,1}f_{3,3}f_{4,2} = 0,\]

\[-f_{3,1}^2 + (f_{2,2}f_{3,1} - f_{3,3}f_{4,1} + f_{2,1}f_{3,3} + f_{3,1}f_{4,1})t + f_{2,2}f_{3,3}f_{4,1} + f_{2,3}f_{3,3}f_{4,1} + f_{2,1}f_{3,1}f_{4,2} - f_{2,1}f_{3,3}f_{4,2} - f_{2,1}f_{3,3}f_{4,2} - f_{2,1}f_{3,3}f_{4,2} = 0,\]

where $f_{i,j} = \frac{\partial f_i}{\partial u_j}$. In the case with $\dim \Gamma_{r_1} = n + 1 - (n - r_1)^2 < 0$ the number of corresponding PDEs exceeds the number of functions $f_1, \ldots, f_n$. So, one may have only rather specific functions $f_1, \ldots, f_n$ and corresponding initial data.

An opposite approach consists in the analysis of not concrete, even generic initial data $f$, but the whole families of them. This approach has been suggested by H. Poincaré in 1879 [15] and asserts that “one has to study not only a single situation (even generic one) but the whole family of close situations in order to get a complete and deep understanding of certain phenomena.” Such type of approach is typical in general theory of singularities of functions and mappings (see e.g. [1] [14]). Recently it has been used in the study of singularities of parabolic type mapping in hydrodynamic type [9].

In our case it means that one should consider a large family of initial data for HEE and corresponding functions $f$. In fact, one can view them as parameterized by a large (or possibly infinite) number of parameters $\lambda_i$. In such a case the situation is pretty simple. Indeed, in this case an effective number $K$ of independent variables $t, u_1, u_2, \ldots, u_n, \lambda_1, \lambda_2, \ldots$ is larger enough or infinite. For the matrix $M$ of rank $r_1$ one has

\[\dim \Gamma_{r_1} = K - (n - r_1)^2.\]

So, there exists always a sufficiently large $K$ such that the condition $\dim \Gamma_{r_1} \geq 0$ or

\[\left[ n - \sqrt{K} \right] < r_1 < n \quad \text{for} \quad K < n^2\]

\[0 \leq r_1 < n \quad \text{for} \quad K \geq n^2\]

is satisfied.

In particular, for families of the initial data for HEE \((1.1)\) with infinite $K$, all ranks $r_1 = n - 1, n - 2, \ldots, 0$ and correspondingly all dimensions $\dim R_{u_0}^+ = \dim R_{u_0}^- = 1, \ldots, n$ are realizable at any dimension $n$.

## 5 Potential motion

Another nongeneric, but important case corresponds to potential flows. Outside the blow-up hypersurface it holds $\frac{\partial u_i}{\partial x_k} = -(M^{-1})_{ik}$, $i, k = 1, \ldots, n$. Hence, for potential flows with $u = \nabla \phi$ one has

\[(M^{-1})_{ik} = \frac{\partial^2 \phi}{\partial x_i \partial x_k}, \quad i, k = 1, \ldots, n\]

and consequently the matrix $M$ is symmetric. Then the definition \((2.1)\) implies that

\[\frac{\partial f_i}{\partial u_k} = \frac{\partial f_k}{\partial u_i}, \quad i, k = 1, \ldots, n,\]

\[\frac{\partial^2 f_i}{\partial u_k \partial u_l} = \frac{\partial^2 f_k}{\partial u_l \partial u_i}, \quad i, k, l = 1, \ldots, n.\]
and, hence,
\[ f_i = \frac{\partial W}{\partial u_i}, \quad i = 1, \ldots, n. \] (5.3)

So, for potential flows, the hodograph equations [1.2] assume the form of the \( n \)-dimensional gradient mappings
\[ x_i = \frac{\partial W}{\partial u_i}, \quad i = 1, \ldots, n. \] (5.4)

with
\[ W = t + \frac{1}{2} |\mathbf{u}|^2 + \tilde{W}(\mathbf{u}). \] (5.5)

The hodograph equations are also the critical points of the family of functions
\[ \Phi(x, u, t) = -x \cdot u + \frac{t}{2} |\mathbf{u}|^2 + \tilde{W}(\mathbf{u}), \] (5.6)

while the elements of the matrix \( M \) [5.1] are given by
\[ M_{ik} = \frac{\partial^2 W}{\partial u_i \partial u_k}, \quad i, k = 1, \ldots, n. \] (5.7)

Some consequences of such representation have been discussed in [10].

For the potential flows, the blow-up hypersurface is given by the Monge-Ampère type equation
\[ \det \left( \frac{\partial^2 W}{\partial u_i \partial u_k} \right) = 0, \] (5.8)

and expansion [2.3] assumes the form
\[ \delta x_i = \sum_{k=1}^{n} \frac{\partial^2 W}{\partial u_i \partial u_k} (u_0) \delta u_k + \frac{1}{2} \sum_{k, l=1}^{n} \frac{\partial^2 W}{\partial u_i \partial u_k \partial u_l} (u_0) \delta u_k \delta u_l + \cdots, \quad i = 1, \ldots, n. \] (5.9)

In the potential case the matrix \( M \) is symmetric. So, its rank \( r_1 \) is equal to the number of nonzero eigenvalues. So, the dimensions of the subspaces \( R_{u_0}^1 \) and \( R_{u_0}^2 \) coincide with the number of zero eigenvalues of \( M \).

It is noted that for a non-symmetric matrix \( M \) the situation is different. Namely, the number of vectors \( R^{(n)} \) of type [2.4] not necessarily coincides with number of zero eigenvalues. As a simple example let us consider the case
\[ M(u_0) = \begin{pmatrix} 0 & f(u_0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (5.10)

In the generic case, the matrix rank is 1. The zero eigenvalue has algebraic multiplicity 3, but only 2 as geometrical multiplicity being the dimension of the eigenspace related to zero eigenvalues generated by the vectors \((1, 0, 0)\) and \((0, g(u_0), -f(u_0))\).

Another feature of the potential case is that the number of constraints for the matrix \( M \) in the form [5.7] is smaller than \((n - r_1)^2\). It is easy to see that for \( n = 3 \) and \( r_1 = 1 \) the first equation [3.5] is automatically satisfied in the potential case when \( \frac{\partial f}{\partial u_1} = \frac{\partial^2 W}{\partial u_1 \partial u_2} \). So, for three-dimensional potential flows, the matrix \( M \) has rank \( r_1 = 1 \) on the curve \( R_{u_0}^{(3)} \) defined by two equations
\[ \left( \frac{\partial^2 W}{\partial u_1 \partial u_2} - \frac{\partial^2 W}{\partial u_2 \partial u_1} \right) \frac{\partial^2 W}{\partial u_1 \partial u_2} + \frac{\partial^2 W}{\partial u_1 \partial u_2} + \left( \frac{\partial^2 W}{\partial u_1 \partial u_2} \right)^2 = 0, \] (5.11)
\[ \left( \frac{\partial^2 W}{\partial u_2 \partial u_3} - \frac{\partial^2 W}{\partial u_3 \partial u_2} \right) \frac{\partial^2 W}{\partial u_2 \partial u_3} + \frac{\partial^2 W}{\partial u_2 \partial u_3} + \left( \frac{\partial^2 W}{\partial u_2 \partial u_3} \right)^2 = 0. \] (5.12)

In general, for potential flows in \( n \) dimensions, the requirement that the matrix \( M(u_0) \) has rank \( r_1 \) imposes \( n - r_1 \) constraints. This fact can be proved in different ways. One of them is to consider the characteristic polynomial \( P_{M(u_0)}(\lambda) \), that is
\[ P_{M(u_0)}(\lambda) = \lambda^n + m_{n-1}(u_0)\lambda^{n-1} + \cdots + m_{n-r_1}(u_0)\lambda^{n-r_1}. \] (5.13)
Table 2: Admissible ranks \( r_1 \) for symmetric singular matrix \( M \) in generic dimensions and the related values of geometric objects characterizing the catastrophe.

| Space dimension | \( n \) |
|-----------------|-------|
| Admissible ranks \( r_1 \) for symmetric singular \( M \) | \( n - 1, n - 2, \ldots, 0 \) |
| \( \dim \Gamma_{r_1}(u,t) = r_1 + 1 \) | \( n, n - 1, \ldots, 2, 1 \) |
| \( \dim R^+_u = \dim R^+_u = n - r_1 \) | \( 1, 2, \ldots, n \) |

Table 2: Admissible ranks \( r_1 \) for the singular symmetric matrix \( M \) in generic dimensions and the related values of geometric objects characterizing the catastrophe.

that is equivalent to the \( n - r_1 \) constraints

\[
m_{n-r_1-1}(u_0) = m_{n-r_1-2}(u_0) = \cdots = m_1(u_0) = m_0(u_0) = 0. \tag{5.14}
\]

So, the dimension of the subspace \( \Gamma_{r_1} \) where the matrix \( M(u_0) \) has rank \( r_1 \) is equal to

\[
\dim \Gamma_{r_1}(u_0) = n + 1 - (n - r_1) = r_1 + 1. \tag{5.15}
\]

Since it is always positive for potential motion all ranks \( r_1 = n - 1, n - 2, \ldots, 1 \) are admissible. The simplification of the table \( \Gamma \) in the potential case is presented at the table 2.

## 6 Blow-ups of second and higher levels and their hierarchy: maximal rank blow-up

Blow-ups and gradient catastrophes of the second, third and higher levels, occur when come bilinear, trilinear, etc. terms in the expansions (2.12) and (2.13) vanish.

In the one-dimensional case one has only the second level due to the constraint \( \frac{\partial^2 f(u)}{\partial u^2} = 0 \) \[13, 20\], where \( f \) is the local inverse of the initial datum, i.e. the scalar analogue of \( F(u) \) in (1.2). In one-dimensional Poincaré case studied in \[8\], there is an infinite hierarchy of blow-ups for which \( m - 1 \)th level corresponds to the constraints \( \frac{\partial^2 f(u)}{\partial u^2} = \frac{\partial^3 f(u)}{\partial u^3} = \cdots = \frac{\partial^n f(u)}{\partial u^n} = 0 \).

For the multi-dimensional HEE (1.1) the situation is more structured.

Let us start with the simplest case of maximal rank \( r_1 = n - 1 \). The corresponding expansions (2.12) and (2.13) are of the form

\[
\delta y_1 = Q^{(11)}(\delta a_1)^2 + \sum_{\beta_1, \beta_2 = 1}^{n-1} Q^{(11)}(\beta_1, \beta_2) \delta b_{\beta_1} \delta b_{\beta_2} + \sum_{\beta_1 = 1}^{n-1} Q^{(11)}(\beta_1) \delta a_1 \delta b_{\beta_1} + Q^{(111)}(\delta a_1)^3 + \sum_{\beta_1 = 1}^{n-1} Q^{(11)}(\beta_1) \delta b_{\beta_1} + \cdots, \tag{6.1}
\]

and

\[
\delta y_1 = \sum_{\beta_1 = 1}^{n-1} Q^{(11)}(\delta a_1)^2 + \sum_{\beta_1 = 1}^{n-1} Q^{(11)}(\beta_1) \delta a_1 \delta b_{\beta_1} + \sum_{\beta_1 = 1}^{n-1} Q^{(11)}(\beta_1) \delta a_1 \delta b_{\beta_1} + \cdots, \tag{6.2}
\]

In the subspace \( H^{(1)}_{(1)}u_0 \) the expansions (6.1) and (6.2) are simplified to

\[
\delta y_1 = Q^{(11)}(\delta a_1)^2 + O((\delta a_1)^3), \quad \delta y_1 = Q^{(11)}(\delta a_1)^2 + O((\delta a_1)^3), \quad \gamma = 1, \ldots, n - 1, \tag{6.3}
\]
while in the subspace $H_{(1)u_0}^-$ one has

$$
\delta y_1 = \sum_{\beta_1, \beta_2 = 1}^{r_1} Q_{(02)}^{(1)\beta_1 \beta_2} \delta b_{\beta_1} \delta b_{\beta_2} + O(|\delta b|^3),
$$

$$
\delta \tilde{y}_\gamma = \sum_{\beta = 1}^{r_1} \tilde{Q}_{(01)}^{(\gamma)\beta} \delta b_{\beta} + \sum_{\beta_1, \beta_2 = 1}^{r_1} \tilde{Q}_{(02)}^{(\gamma)\beta_1 \beta_2} \delta b_{\beta_1} \delta b_{\beta_2} + O(|\delta b|^3), \quad \gamma = 1, \ldots, n - 1.
$$

(6.4)

According to the table (1) we are interested in the behavior of derivatives only in three blow-up sectors. This means that the second equation in (6.3) is not involved in the following considerations.

Blow-ups of the second level correspond to the situation when some quadratic or bilinear terms in the expansions and first of (6.4) vanish.

The simplest case with the weakest constraint corresponds to the condition

$$
Q_{(20)}^{(1)11} = 0.
$$

(6.5)

In such a case

$$
\delta a_1 \sim (\delta y_1)^{1/3},
$$

(6.6)

and hence in the subsectors $\delta u \in H_{(1)u_0}^+, \delta x \in H_{(1)x_0}^+,$

$$
\frac{\partial a_1}{\partial y_1} \sim (\delta y_1)^{-2/3}, \quad |\delta y_1| \to 0.
$$

(6.7)

Due to the constraint (6.5) such type of blow-up occurs on the $(n-1)$-dimensional subsurface of the blow-up surface (2.2).

Next we analyse the second expansion in (6.3). If there exist $m(2)$ linearly independent vectors of dimension $(n-1)$ $\tilde{L}_{(2)}^{(\delta)}, \delta = 1, \ldots, m(2)$ such that

$$
\sum_{\gamma = 1}^{n-1} \tilde{L}_{(2)\gamma}^{(\delta)} \tilde{Q}_{(20)}^{(\gamma)11} = 0, \quad \delta = 1, \ldots, m(2),
$$

(6.8)

then $n-1$ formulas (6.3) are equivalent to the following

$$
\delta \tilde{y}_\delta^{(2)} \equiv \sum_{\gamma = 1}^{n-1} \tilde{L}^{(\delta)}_{(2)\gamma} \delta \tilde{y}_\gamma = \sum_{\gamma = 1}^{n-1} \tilde{L}^{(\delta)}_{(2)\gamma} \tilde{Q}_{(30)}^{(\gamma)111} (\delta a_1)^3 + O((\delta a_1)^4), \quad \delta = 1, \ldots, m(2),
$$

$$
\delta \tilde{y}_\gamma \equiv \tilde{Q}_{(30)}^{(\gamma)111} (\delta a_1)^2 + O((\delta a_1)^3), \quad \gamma = m(2) + 1, \ldots, n. \tag{6.9}
$$

So, in the subspace $H_{(2)x_0} \subset H_{(1)x_0}^+$ of the variations of the form $\delta \tilde{y}_\delta^{(2)}$ one generically has

$$
\delta a_1 \sim |\delta \tilde{y}_\delta^{(2)}|^{1/3}
$$

(6.10)

and, hence, in the subsector $\delta u \in H_{(1)u_0}^+$ and $\delta x \in H_{(2)x_0}^+$ the derivative blows-up still as

$$
\frac{\partial a_1}{\partial \tilde{y}_\delta^{(2)}} \sim |\delta \tilde{y}_\delta^{(2)}|^{-2/3}, \quad |\delta \tilde{y}_\delta^{(2)}| \to 0.
$$

(6.11)

The dimension of the corresponding subspace of $\Gamma$ is equal to $n - m(2)$.

Finally, let us consider the first expansion (6.4). The term bilinear in $b_\beta$ vanishes if the matrix $Q_{(02)}^{(1)\beta_1 \beta_2}$ is degenerate. Indeed, if the rank $r_2$ of this matrix is smaller than $n - 1$, there are $n-1-r_2$ vectors $R_{(2)}^{(\delta)}$ of dimension $(n-1)$ such that

$$
\sum_{\beta_2 = 1}^{n-1} Q_{(02)}^{(1)\beta_1 \beta_2} R_{(2)\beta_2}^{(\delta)} = 0, \quad \delta = 1, \ldots, n - 1 - r_2.
$$

(6.12)

Hence, in the complementary subspace $H_{(2)u_0}^- \subset H_{(1)u_0}^-$ spanned by $n - 1 - r_2$ vectors $R_{(2)\beta}^{(\delta)}$ with coordinates $\delta b_\delta^{(2)}$ defined by

$$
\delta b_\delta = \sum_{\beta = 1}^{n-1-r_2} R_{(2)\beta}^{(\delta)} \delta b_\delta^{(2)}, \quad \beta = 1, \ldots, n - 1, \tag{6.13}
$$

first expansion (6.4) assumes the form

$$
\delta y_1 = \sum_{\delta_1, \delta_2, \delta_3 = 1}^{n-1-r_2} \sum_{\beta_1, \beta_2, \beta_3 = 1}^{n-1} Q_{(03)}^{(1)\beta_1 \beta_2 \beta_3} R_{(2)\beta_1}^{(\delta_1)} R_{(2)\beta_2}^{(\delta_2)} R_{(2)\beta_3}^{(\delta_3)} \delta b_\delta^{(2)} \delta b_\delta^{(2)} \delta b_\delta^{(2)} + O(|\delta b^{(2)}|^4).
$$

(6.14)
So, in the subspace $H_{(2)u_0}$

$$\delta y^{(3)} \sim (\delta y_1)^{1/3}$$

(6.15)

and, hence,

$$\frac{\partial \delta y^{(2)}}{\partial y_1} \sim (\delta y_1)^{-2/3}, \quad |\delta y_1| \to 0.$$ 

(6.16)

Since, the requirement that the symmetric matrix $Q^{(1)\beta_1\beta_2}_{(0)2}$ has rank $r_2$ is equivalent to $n - 1 - r_2$ constraints the dimension of the subspace $H_{(2)u_0}$ is \(n - (n - 1 - r_2) = r_2 + 1\).

Thus, at the second level, the derivatives blow-up as

$$\left| \frac{\partial \delta u}{\partial x} \right| \sim |\delta x|^{-2/3}, \quad |\delta x| \to 0.$$ 

(6.17)

If the blow-up time $t_i$ is positive one has a gradient catastrophe for initial data.

Blows-up of third level occur when the third orders terms in expansions (6.3), (6.9) and (6.14) vanish. The weakest constraints on the functions $f$ are

$$Q_{(3)0}^{(1)111} = \sum_{i=1}^{n} \sum_{l=1, l \neq i}^{n} \frac{\partial^2 f_i}{\partial y_{i1} \partial y_{il}} (u_0) L_i^{(1)} R_{i1}^{(1)} R_{i2}^{(1)} = 0,$$

$$Q_{(3)0}^{(3)111} = \sum_{i=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 f_i}{\partial y_{i1} \partial y_{il}^2} (u_0) L_i^{(1)} R_{i1}^{(1)} R_{i2}^{(1)} R_{i3}^{(1)} = 0.$$ 

(6.18)

Consequently, in this case

$$\delta a_1 \sim |\delta y_1|^{1/4},$$

(6.19)

and the derivative $\frac{\partial a_1}{\partial y_1}$ blow up in sectors $\delta u \in H_{(1)u_0}^+, \delta x \in H_{(1)x_0}^+$ as

$$\frac{\partial a_1}{\partial y_1} \sim |\delta y_1|^{-3/4}, \quad |\delta y_1| \to 0.$$ 

(6.20)

The dimension of the corresponding subspace of the hypersurface $\Gamma$ is equal to $n - 2$.

In analysis of expansions (6.9) and (6.14) one proceeds in a way similar to that of the second level. Namely, if there exists $m_{(3)}$ linearly independent vectors of dimension $m_{(2)}$ denoted by $L_{(3)}^{(\xi)}$, $\xi = 1, \ldots, m_{(3)}$ such that

$$\sum_{\xi=1}^{m_{(2)}} L_{(3)}^{(\xi)} \left( \sum_{\gamma=1}^{n-1} L_{(2)}^{(\gamma)} \delta y_\gamma \right) = 0, \quad \xi = 1, \ldots, m_{(3)}$$ 

(6.21)

then the $m_{(2)}$ expansions of the first line in (6.9) are equivalent to the following

$$\delta y_3^{(3)} \equiv \sum_{\xi=1}^{m_{(2)}} L_{(3)}^{(\xi)} \delta y_3^{(2)} = \sum_{\xi=1}^{m_{(2)}} L_{(3)}^{(\xi)} \left( \sum_{\gamma=1}^{n-1} L_{(2)}^{(\gamma)} Q^{(\gamma)1111}_{(0)} \delta a_1 \right) (\delta a_1)^4 + O((\delta a_1)^5), \quad \xi = 1, \ldots, m_{(3)}$$ 

(6.22)

$$\delta y_3^{(2)} \equiv \sum_{\gamma=1}^{n-1} L_{(2)}^{(\gamma)} Q^{(\gamma)1111}_{(0)} \delta a_1 (\delta a_1)^3 + O((\delta a_1)^5), \quad \xi = m_{(3)} + 1, \ldots, m_{(2)}.$$

Thus, in the subspace $H_{(3)x_0}^+ \subset H_{(2)x_0}$ of the variations $\delta y_3^{(3)}$ one generically has

$$\delta a_1 \sim |\delta y_3^{(3)}|^{1/4},$$

(6.23)

and, hence, in the subsector $\delta u \in H_{(1)u_0}^+, \delta x \in H_{(3)x_0}^+$ the derivatives blow-up as

$$\frac{\partial \delta a_1}{\partial y_3^{(3)}} \sim |\delta y_3^{(3)}|^{-3/4}, \quad |\delta y_3^{(3)}| \to 0.$$ 

(6.24)

The dimension of the corresponding subspace of $\Gamma$ is $n - 1 - m_{(2)} - m_{(3)}$.

Now let us consider the expansion (6.14) and assume that there are $n_{(3)} = n - 1 - r_2 - r_3 > 0$ linearly independent vectors of dimension $n - 1 - r_2$ denoted by $R_{(3)}^{(\sigma)}$, $\sigma = 1, \ldots, n_{(3)}$ such that

$$\sum_{\delta_1=1}^{n_{(3)}} \sum_{\delta_2=1}^{n_{(3)}} \sum_{\delta_3=1}^{n_{(3)}} Q^{(\delta_1)\delta_2\delta_3}_{(0)3} R^{(\delta_1)}_{(2)\delta_1} R^{(\delta_2)}_{(2)\delta_2} R^{(\delta_3)}_{(2)\delta_3} R_{(3)\delta_3}^{(\sigma)} = 0, \quad \sigma = 1, \ldots, n_{(3)}, \quad \delta_1, \delta_2, \delta_3 = 1, \ldots, n - 1 - r_2.$$ 

(6.25)
In such a case \((6.14)\) becomes
\[
\delta y_1 \sim |\delta b^{(3)}_\sigma|^4
\]  
where the variations \(\delta b^{(3)}_\sigma\) are defined by
\[
\delta b^{(3)}_\sigma = \sum_{\sigma=1}^{n(3)} P^{(\sigma)}_{3,4} \delta b^{(3)}_\sigma, \quad \delta = 1, \ldots, n - 1 - r_2.
\]  
(6.27)

So, it would be
\[
\delta b^{(3)}_\sigma \sim |\delta y_1|^{1/4},
\]  
(6.28)

and
\[
\frac{\partial b^{(3)}_\sigma}{\partial y_1} \sim |\delta y_1|^{-3/4}, \quad |\delta y_1| \to 0,
\]  
(6.29)

if the dimension of the corresponding subspace in \(\Gamma\) equal to \(r_2 + 1 - \frac{1}{2} n(3) (n - 1 - r_2) (n - r_2)\) is nonnegative.

Thus, the characteristic behavior of the blow-up derivations on the third level in the corresponding subsections is
\[
\frac{\partial u_i}{\partial x_k} \sim |\delta x|^{-n+1}, \quad |\delta x| \to 0, \quad m = 1, 2, 3, \ldots.
\]  
(6.30)

An analysis of highest level is straightforward, but cumbersome. However, it is not difficult to show, considering for example the first expansion in \((6.3)\), that in the generic case and for the rank \(r_1 = n - 1\) on the \(m\)-th level the derivatives in corresponding subsectors slow up as
\[
\frac{\partial u_i}{\partial x_k} \sim |\delta x|^{-n+1}, \quad |\delta x| \to 0.
\]  
(6.31)

For the subspace \(H^+_{(1)u_0}\) of the corresponding subspaces of the hypersurface \(\Gamma\) is equal to \(n + 1 - m\).

In the case of generic functions \(f(u)\) (i.e. generic initial data), the hierarchy of blow-ups for the \(n\)-dimensional HEE is finite. It has \(n + 1\) levels and the most singular behavior of derivatives is given by
\[
\frac{\partial u_i}{\partial x_k} \sim |\delta x|^{-n+1}, \quad |\delta x| \to 0.
\]  
(6.32)

It should be noted that the structure of the blow-ups hierarchy depends on the approach which one follow in the description of the blow-ups. In generic case considered above the hierarchy is finite with the most singular behavior given by \((6.32)\). In contrast, in the Poincaré approach discussed in Section 4 all the constraints for the functions \(f\) can be satisfied at any \(n\) and \(m\). So, the corresponding hierarchy of blow-ups is infinite. For the one-dimensional case see \([8]\).

### 7 Blow-ups of second and higher levels and their hierarchy: lower ranks blow-ups

For lower ranks \(r_1 < n - 1\) the structure of higher level blow-ups is more complicated. The expansions \((7.12)\) and \((7.13)\), i.e. the analogue of \((6.3)\) and \((6.4)\) studied in previous section 6, are of the form in the subspace \(H^+_{(1)u_0}\)
\[
\delta y_\alpha = \sum_{\alpha_1, \alpha_2 = 1}^{n-r_1} Q^{(\alpha)}_{(20)} \delta a_{\alpha_1} \delta a_{\alpha_2} + \sum_{\alpha_1, \alpha_2, \alpha_3 = 1}^{n-r_2} Q^{(\alpha)}_{(30)} \delta a_{\alpha_1} \delta a_{\alpha_2} \delta a_{\alpha_3} + \ldots, \quad \alpha = 1, \ldots, n - r_1,
\]  
(7.1)

and in the subspace \(H^+_{(1)u_0}\)
\[
\delta y_\gamma = \sum_{\gamma_1, \gamma_2 = 1}^{n-r_1} \hat{Q}^{(\gamma)}_{(20)} \delta a_{\gamma_1} \delta a_{\gamma_2} + \sum_{\gamma_1, \gamma_2, \gamma_3 = 1}^{n-r_2} \hat{Q}^{(\gamma)}_{(30)} \delta a_{\gamma_1} \delta a_{\gamma_2} \delta a_{\gamma_3} + \ldots, \quad \gamma = 1, \ldots, r_1,
\]  
(7.2)

Again we are interested only in expansions \((7.1)\) and the first of \((7.2)\). One observes that these expansions are rather similar in their form, only the numbers of equations and independent variations are varying.
Let us start with the first expansion in (7.1). These \( n - r_1 \) of them instead of one in \( r_1 = n - 1 \) case. Blow-up of the second level occurs when some of bilinear term vanish. The simplest case is
\[
Q_{(20)}^{(\alpha_1 \alpha_2)} = 0, \quad \alpha = 1, \ldots, l < n - r_1, \quad \alpha_1, \alpha_2 = 1, \ldots, n - r_1. \tag{7.3}
\]
These conditions are equivalent to \( l \frac{(n-r_1+1)(n-r_1)}{2} \) constraints and the dimension of the corresponding subspace of \( \Gamma \) is
\[
\dim \Gamma_{r_1}(u_0) = l \frac{(n-r_1+1)(n-r_1)}{2}. \tag{7.4}
\]
For \( r_1 = n - 2 \) and \( l = 1 \) it is equal to \( n - 6 \); for \( l = 2 \) it is \( n - 9 \) and so on.

Similar constraints and dimensions one has if there exist \( l \) vectors \( L_{(2)}^{(\delta)} \), \( \delta = 1, \ldots, l \) of dimension \( n - r_1 \) such that
\[
\sum_{\alpha=1}^{n-r_1} L_{(2)}^{(\delta)} Q_{(20)}^{(\alpha_1 \alpha_2)} = 0, \quad \delta = 1, \ldots, l, \quad \alpha_1, \alpha_2 = 1, \ldots, n - r_1. \tag{7.5}
\]
In this case for variables defined as \( z_\delta \) defined as
\[
\delta z_\delta = \sum_{\alpha=1}^{n-r_1} L_{(2)}^{(\delta)} \delta y_\alpha, \quad \delta = 1, \ldots, l, \tag{7.6}
\]
one has
\[
\delta z_\delta = \sum_{\alpha=1}^{n-r_1} L_{(2)}^{(\delta)} \left( \sum_{\alpha_1, \alpha_2, \alpha_3=1}^{n-r_1} Q_{(20)}^{(\alpha_1 \alpha_2 \alpha_3)} \right) \delta a_{\alpha_1} \delta a_{\alpha_2} \delta a_{\alpha_3} + \ldots, \quad \delta = 1, \ldots, l. \tag{7.7}
\]

The number of constraints is smaller and the dimensions of the subspaces of \( \Gamma \) is larger if the symmetric matrices \( Q_{(20)}^{(\alpha_1 \alpha_2)} \) or \( \sum_{r=1}^{n-r_1} L_{(2)}^{(\delta)} Q_{(20)}^{(\alpha_1 \alpha_2)} \) have rank \( r_2 \) instead of zero rank (conditions (7.3) and (7.5) with \( l = 1 \)). In this case there are \( n - r_1 - r_2 \) vectors \( L_{(2)}^{(\sigma)} \), \( \sigma = 1, \ldots, n - r_1 - r_2 \) such that
\[
\sum_{\sigma=1}^{n-r_1-r_2} Q_{(20)}^{(\alpha_1 \alpha_2)} R_{(2)}^{(\sigma)} = 0, \quad \alpha_1 = 1, \ldots, n - r_1. \tag{7.8}
\]
and for the subspace \( H_{(2)}^{(\alpha_1 \alpha_2)} \) of variations given by
\[
\delta a_\alpha = \sum_{\sigma=1}^{n-r_1-r_2} R_{(2)}^{(\sigma)} \delta a_\sigma^{(2)}, \quad \alpha = 1, \ldots, n - r_1. \tag{7.9}
\]
one has
\[
\delta y_\delta = \sum_{\sigma_1, \sigma_2, \sigma_3=1}^{n-r_1-r_2} \left( \sum_{\sigma_1, \sigma_2, \sigma_3=1}^{n-r_1-r_2} Q_{(20)}^{(\alpha_1 \alpha_2 \alpha_3)} R_{(2)}^{(\sigma_1 \sigma_2 \sigma_3)} \right) \delta a_{\sigma_1}^{(2)} \delta a_{\sigma_2}^{(2)} \delta a_{\sigma_3}^{(2)} + \ldots. \tag{7.10}
\]
The number of such constraints now is equal to \( n - r_1 - r_2 \) and the dimensions of the corresponding subspaces are
\[
\dim \Gamma_{r_1}(u_0) = r_1 + r_2 - (n - r_1)^2. \tag{7.11}
\]
So, in the subspace \( H_{(2)}^{(\alpha_1 \alpha_2)} \) one has
\[
\delta a_\sigma^{(2)} \sim \delta(y_\delta)^{1/3}, \tag{7.12}
\]
and hence
\[
\frac{\partial a_\sigma^{(2)}}{\partial y_\delta} \sim |\delta y_\delta|^{-2/3}, \quad |\delta y_\delta| \rightarrow 0, \tag{7.13}
\]
on the subspaces of the blow-up hypersurface \( \Gamma \) of dimension \( r_1 + r_2 - (n - r_1)^2 \).

Similar calculations can be done also for the second expansion in (7.1) and the first of (7.2). In all case the characteristic behavior of derivatives is of the type (7.13). Dimensions of the corresponding subspaces of the blow-up hypersurface \( \Gamma \) rapidly decrease.

Analysis of third and higher levels and the whole hierarchy is straightforward, but cumbersome. The hierarchy of blow-ups is finite in the generic case, while for special non generic initial data corresponding to \( f(u) \), the structure can be quite different. For instance, the derivatives \( \frac{\partial f(y_\delta)}{\partial a_\alpha^{(n-1)}}(u_0) \) of certain order may vanish themselves, leading to low-ups in some subspaces of \( R_{(\alpha_1 \ldots \alpha_n)} \).
8 Two dimensional case

Two and three dimensional cases are of the most interest in physics. Here we will consider them in detail.

It was shown in [10] that in even dimension the real hypersurface \( \Gamma \) may not exists. In what follows we assume that there is real blow-up hypersurface \( \Gamma \) in two dimensions.

At \( n = 2 \) the only admissible configuration has \( \operatorname{rank} M(u_0) = 1 \), \( \dim \Gamma_{r_1} = 2 \) and \( \dim H^+_{u_0} = \dim H^-_{u_0} = 1 \). Denoting the vectors appearing in the formulas \( (2.4), (2.5), \) and \( (2.11) \) by \( R, \tilde{R}, L, \) one has the following decomposition \( (2.8) \) and formulas \( (2.11) \):

\[
\delta u = \delta_+ u + \delta_- u = R \delta a + \tilde{R} \delta b, \\
\delta y_1 = L \cdot \delta x, \\
\delta \tilde{y}_1 = L \cdot \delta \bar{x},
\]

and

\[
\delta x = \delta_+ x + \delta_- x = P \delta y_1 + \tilde{P} \delta \tilde{y}_1,
\]

where \( P \) and \( \tilde{P} \) are defined in \( (2.18) \). The expansions \( (2.12), (2.13) \) are of the form

\[
\delta y_1 = Q_{(11)}(\delta a)^2 + Q_{(12)}(\delta a \delta b) + Q_{(22)}(\delta b)^2 + \ldots, \\
\delta \tilde{y}_1 = Q_{(11)}(\delta b) + Q_{(20)}(\delta a)^2 + Q_{(12)}(\delta a \delta b) + Q_{(02)}(\delta b)^2 + \ldots,
\]

where the coefficients are defined as in \( (2.10) \) and \( (2.14) \), i.e.

\[
Q_{(11)} = \sum_{i,j=1}^{2} M_{ij}(u_0) \tilde{L}_i \tilde{R}_j, \\
Q_{(rs)} = \frac{1}{(r+s)!} \sum_{i_1,j_1, \ldots, i_r,j_r, k_1, \ldots, k_s=1}^{2} \frac{\partial^{r+s} f_i}{\partial u_{i_1} \ldots \partial u_{j_r} \partial u_{k_1} \ldots \partial u_{k_s}} (u_0) L_{i_1} R_{j_1} \ldots L_{i_r} R_{j_r} \tilde{R}_{k_1} \ldots \tilde{R}_{k_s}, \\
\tilde{Q}_{(rs)} = \frac{1}{(r+s)!} \sum_{i_1,j_1, \ldots, i_r,j_r, k_1, \ldots, k_s=1}^{2} \frac{\partial^{r+s} f_i}{\partial u_{i_1} \ldots \partial u_{j_r} \partial u_{k_1} \ldots \partial u_{k_s}} (\tilde{u}_0) \tilde{L}_{i_1} R_{j_1} \ldots L_{i_r} R_{j_r} \tilde{R}_{k_1} \ldots \tilde{R}_{k_s}.
\]

The expansions \( (8.3) \) imply that at \( \delta y_1 \sim \delta \tilde{y}_1 \sim \epsilon, \epsilon \to 0 \) one has

\[
\frac{\partial a}{\partial y_1} \sim \epsilon^{-1/2}, \quad \frac{\partial b}{\partial y_1} \sim \epsilon^{-1/2}, \quad \frac{\partial a}{\partial \tilde{y}_1} \sim \epsilon^{-1/2}, \quad \frac{\partial b}{\partial \tilde{y}_1} \sim \epsilon^{-1/2},
\]

while

\[
\frac{\partial b}{\partial y_1} \sim O(1).
\]

Note that

\[
\delta b = R^\perp \cdot \delta u,
\]

where \( R^\perp \) is a vector orthogonal to \( R \) such that \( R^\perp \cdot \tilde{R} \neq 0 \) and (see \( (2.17) \))

\[
\frac{\partial}{\partial y_1} = \tilde{P}_1 \cdot \frac{\partial}{\partial x_1} + \tilde{P}_2 \cdot \frac{\partial}{\partial x_2}.
\]

So, there are particular directions in the two-dimensional spaces of variations of \( \delta u \) and \( \delta x \) defined by the vector \( R^\perp \) and \( \tilde{P} \) such that

\[
\left( \tilde{P}_1 \cdot \frac{\partial}{\partial x_1} + \tilde{P}_2 \cdot \frac{\partial}{\partial x_2} \right) (R^\perp \cdot u) \sim O(1).
\]

The fact that certain combinations of blow-up derivatives are bound is a principal novelty of the 2D HEE in comparison with the one-dimensional case given by the Burgers-Hopf equation.

The absence of such special directions of boundedness would correspond to a very specific initial data or, equivalently, for functions \( f(u_0) = (f_1(u_0), f_2(u_0)) \) for which all elements of the matrix \( M \) vanish, i.e. \( M_{ij}(u_0) = 0, i, j = 1, 2 \). It is easy to see that in such a case \( \frac{\partial f_i}{\partial u_j}(u_0) = \frac{\partial f_j}{\partial u_i}(u_0) = 0 \) and the two-dimensional HEE decomposes in \( u_0 \) into two disconnected Burgers-Hopf equations.

Blow-up at the second level occurs if

\[
Q_{20} = \frac{1}{2} \sum_{i,j,k=1}^{2} \frac{\partial^2 f_i}{\partial u_j \partial u_k}(u_0) L_i R_j R_k = 0,
\]

(8.10)
and \( Q_{20} \neq 0 \). It happens on a curve defined by the equation \( Q_{20}(u_{10}, u_{20}) \equiv Q_{20}(u_0) = 0 \) and
\[
\frac{\partial \alpha}{\partial y_1} \sim \epsilon^{-2/3}, \quad \epsilon \to 0
\]
for the points \( u_0 \) on this curve.

Instead, if
\[
\hat{Q}_{(20)}(u_0) = \frac{1}{2} \sum_{i,j,k=1}^{2} \frac{\partial^2 f_i}{\partial u_j \partial u_k}(u_0) \hat{L}_i R_j R_k = 0 ,
\]
then
\[
\frac{\partial \alpha}{\partial y_1} \bigg|_{\delta b = 0} \sim \epsilon^{-2/3}, \quad \epsilon \to 0 ,
\]
on the line defined by the equation
\[
\hat{Q}_{(20)}(u_0) = \hat{Q}_{(20)}(u_{10}, u_{20}) = 0 .
\]
If both conditions \( Q_{10}, Q_{20} \) are satisfied then one has blow-ups\( (8.11), (8.13) \) at the point defined by the equations \( Q_{20}(u_0) = \hat{Q}_{(20)}(u_0) = 0 \).

Then in the case when \( Q_{(02)} = 0 \) one has
\[
\frac{\partial b}{\partial y_1} \sim \epsilon^{-2/3} , \quad \epsilon \to 0 ,
\]
on the curve.

Third level of blow-up is characterized by two conditions
\[
Q_{(20)}(u_{10}, u_{20}) = Q_{(20)}(u_{10}, u_{20}) = 0 ,
\]
which generically define a point on the blow-up hypersurface \( \Gamma \). At this point
\[
\frac{\partial \alpha}{\partial y_1} \sim \epsilon^{-3/4}, \quad \epsilon \to 0 .
\]

Similarly, if
\[
\hat{Q}_{(20)}(u_{10}, u_{20}) = \hat{Q}_{(20)}(u_{10}, u_{20}) = 0 ,
\]
one has
\[
\frac{\partial \alpha}{\partial y_1} \sim \epsilon^{-3/4}, \quad \epsilon \to 0 .
\]
at the point defined by equation \( 8.18 \).

Finally, if
\[
Q_{(02)}(u_{10}, u_{20}) = Q_{(03)}(u_{10}, u_{20}) = 0 ,
\]
one has
\[
\frac{\partial b}{\partial y_1} \sim \epsilon^{-3/4}, \quad \epsilon \to 0 .
\]
at a point on the hypersurface \( \Gamma \).

So, in the generic case the hierarchy of blow-ups for the two-dimensional HEE has only three levels and the most singular blow-up is given by \( \epsilon^{-3/4}, \epsilon \to 0 \).

For potential motion one has the same results. For the Poincaré case the corresponding hierarchy is infinite.

### 9 Three dimensional case

In the dimension three the blow-up surface \( \Gamma \) always has at least one real branch \( [10] \) and in the generic case the admissible ranks \( r_1 \) of the matrix \( M(u_0) \) are 2 and 1.

For \( r_1 = 2 \) one has \( \dim \Gamma_{r_1=2} = 3 \) and \( \dim H^+_{u_0} = \dim H^-_{u_0} = 1 \) or \( \dim H^+_{u_0} = \dim H^-_{u_0} = 1 \). Expansions \( 2.12 \) and \( 2.13 \) in this case are of the form
\[
\delta y_1 = \sum_{\beta_1, \beta_2 = 1}^{2} Q_{(02)}^{(1)\beta_1\beta_2} \delta b_{\beta_1} \delta b_{\beta_2} + Q_{(20)}^{(1)11} (\delta a_1)^2 + \sum_{\beta=1}^{2} Q_{(11)}^{(1)\beta} \delta a_{1} \delta b_{\beta} + \ldots
\]
and
\[
\delta y_\gamma = \sum_{\beta=1}^{2} Q_{(01)}^{(\gamma)\beta} \delta b_{\beta} + \sum_{\beta_1, \beta_2 = 1}^{2} Q_{(02)}^{(\gamma)\beta_1\beta_2} \delta b_{\beta_1} \delta b_{\beta_2} + Q_{(20)}^{(\gamma)11} (\delta a_1)^2 + \sum_{\beta=1}^{2} Q_{(11)}^{(\gamma)\beta} \delta a_{1} \delta b_{\beta} + \ldots , \quad \gamma = 1, 2 .
\]
The decomposition formulae (2.8) and (2.16) assume the form
\[ \delta u = R(1) \delta a_1 + \tilde{R}(1) \delta b_1 + R(2) \delta b_2, \]
\[ \delta x = P(1) \delta y_1 + \tilde{P}(1) \delta \tilde{y}_1 + P(2) \delta \tilde{y}_2. \]

(9.3)

The subspaces \( H_{u_0}^- \) and \( H_{x_0}^- \) are now the planes spanned by the pairs of vectors \( \tilde{R}(1), \tilde{R}(2) \) and \( \tilde{P}(1), \tilde{P}(2) \), respectively (see fig. 1). On the first level, the derivatives \( \frac{\partial a_1}{\partial y_1}, \frac{\partial a_1}{\partial \tilde{y}_1}, \gamma = 1, 2 \) and \( \frac{\partial b_1}{\partial y_1}, \gamma = 1, 2 \) blow-up as \( \epsilon^{-1/2} \) while the derivatives \( \frac{\partial b_2}{\partial \tilde{y}_1} \).

remain bounded.

Equivalently,
\[ \tilde{P}_\gamma \cdot \nabla_x (\tilde{R}_\beta^+ u) \sim O(1), \quad \beta, \gamma = 1, 2, \]

(9.4)

where the vectors \( \tilde{R}_\beta^+ \) are generically defined by the relations
\[ R_{(1)} \cdot R_{(1)}^+ = 0, \quad R_{(1)}^+ \cdot \tilde{R}_{(2)} = 0, \]
\[ R_{(2)} \cdot R_{(1)}^+ = 0, \quad R_{(2)}^+ \cdot \tilde{R}_{(1)} = 0. \]

(9.5)

The four combinations (9.4) of blow-up derivatives \( \frac{\partial a_1}{\partial y_1} \) are bound on the whole three-dimensional hypersurface \( \Gamma \).

Blow-ups of the second level occur when (see (9.2))
\[ Q_{(20)}^{(1)11} = 0, \quad Q_{(20)}^{(1)111} \neq 0. \]

(9.6)

In this case
\[ \frac{\partial a_1}{\partial y_1} \sim \epsilon^{-2/3}, \quad \epsilon \to 0, \]

(9.7)

and it happens on the two-dimensional subspace of \( \Gamma \).

If there is a two-dimensional vector \( \tilde{L}_{(2)}^+ \) (\( \gamma \)) such that (see (6.8))
\[ \sum_{\gamma=1}^2 \tilde{L}_{(2)\gamma}^+ Q_{(20)}^{(\gamma)111} = 0, \]

(9.8)

then also
\[ \frac{\partial a_1}{\partial \tilde{y}_1} \mid_{\epsilon=0} \sim \epsilon^{-2/3}, \quad \epsilon \to 0, \]

(9.9)

where \( \tilde{y}_1 \equiv \sum_{\gamma=1}^2 \tilde{L}_{(2)\gamma} \tilde{y}_\gamma \). Blow-up (9.9) happens on a certain two-dimensional subspace of \( \Gamma \).

Similarly, if the matrix \( Q_{(20)}^{(1)11\beta_1\beta_2} \) has rank one and
\[ \sum_{\beta_2=1}^2 Q_{(20)}^{(1)11\beta_1\beta_2} R_{(2)\beta_2} = 0, \]

(9.10)
one has
\[ \frac{\partial \delta b_1}{\partial y_1} \sim \epsilon^{-2/3}, \quad \epsilon \to 0, \] (9.11)
where \( \delta b_\beta = R_1^{(1)} b_1^{(2)} \delta b_1^{(2)} \), \( \beta = 1, 2 \).

Third level of blow-up corresponds to the constraints
\[ Q_{(20)}^{(1)111} = Q_{(30)}^{(1)111} = 0, \quad Q_{(40)}^{(1)1111} \neq 0. \] (9.12)
In case (9.2) one has a similar expansion. It occurs on a curve on \( \Gamma \) and derivatives blow up as
\[ \frac{\partial a_1}{\partial y_1} \sim \epsilon^{-3/4} \quad \epsilon \to 0. \] (9.13)
On the last, fourth level for which in particular
\[ Q_{(1)1}^{(111)} = Q_{(30)}^{(1)111} = Q_{(40)}^{(1)1111} = 0, \] (9.14)
and which occur generically at a single point on \( \Gamma \) the derivatives blow-up as
\[ \frac{\partial a_1}{\partial y_1} \sim \epsilon^{-4/5} \quad \epsilon \to 0. \] (9.15)

This is the most singular blow-up of derivatives which occurs in the generic case for three-dimensional HEE.

One has similar results for potential motion and \( r_1 = 2 \).

It is noted that the matrix \( M \) may have rank 2 on the whole three-dimensional blow-up hypersurface \( \Gamma \) and, consequently, some constraints of the type (9.6), (9.12), and (9.14) are admissible in the generic case.

In contrast, the matrix \( M \) may have rank 1 generically at a point \( u_0 \) in \( \Gamma \) (see table 1). Consequently, in the generic case and \( r_1 = 1 \) only blow-ups of the first level are realizable.

The subspaces \( H^- \) are one-dimensional and one has the straight lines (see fig 2) instead of planes present in in figure 1.

![Figure 2: Subspaces \( H^- \) and \( H_- \) in 3D case when rank \( M=1 \). If both \( \delta u \) and \( \delta x \) are in this subspace then the spatial \( u \) derivative remains bounded also at the catastrophe point.](image)

All derivative at such point \( u_0 \) blow-up as \( \epsilon^{-1/2}, \epsilon \to 0 \) and only one is bound, namely
\[ \frac{\partial b_1}{\partial y_1} \sim O(1). \] (9.16)
Equivalently, one has
\[ \sum_{i=1}^{3} \delta P_1 \cdot \nabla x \left( R^\perp \cdot u \right) = O(1), \] (9.17)
where the vector \( R^\perp \) is defined by the conditions
\[ R^\perp \cdot R^{(\alpha)} = 0, \quad \alpha = 1, 2. \] (9.18)
For potential motion the situation of rank $M = 1$ is realizable on the two-dimensional subspace of hypersurface $\Gamma$ (see Table 2). So, the blow-ups of the second and third level, in principle, may occur. The expansions (7.1) and (7.2) for $n = 3$, $r_1 = 1$ have the form

$$\delta y_\alpha = \sum_{\alpha_1, \alpha_2 = 1}^{2} Q_{(a)20}^{(\alpha)1a_1a_2} \delta a_{\alpha_1} \delta a_{\alpha_2} + \sum_{\alpha_1, \alpha_2, \alpha_3 = 1}^{2} Q_{(b)30}^{(\alpha)1a_1a_2a_3} \delta a_{\alpha_1} \delta a_{\alpha_2} \delta a_{\alpha_3} + \ldots, \quad \alpha = 1, 2,$$

(9.19)

$$\delta \tilde{y} = \sum_{\alpha_1, \alpha_2 = 1}^{2} \tilde{Q}_{(a)20}^{(1)1a_1a_2} \delta a_{\alpha_1} \delta a_{\alpha_2} + \sum_{\alpha_1, \alpha_2, \alpha_3 = 1}^{2} \tilde{Q}_{(b)30}^{(1)1a_1a_2a_3} \delta a_{\alpha_1} \delta a_{\alpha_2} \delta a_{\alpha_3} + \ldots,$$

and

$$\delta y_\alpha = Q_{(a)02}^{(\alpha)11} (\delta b_1)^2 + Q_{(a)03}^{(\alpha)111} (\delta b_1)^3 + \ldots, \quad \alpha = 1, 2,$$

$$\delta \tilde{y} = \tilde{Q}_{(a)01}^{(1)1} \delta b_1 + \tilde{Q}_{(a)02}^{(1)11} (\delta b_1)^2 + \ldots.$$  

(9.20)

Blow-up of the second level happens if the matrix $\tilde{Q}_{(a)20}^{(1)1a_1a_2}$ is degenerate with (see (7.8) for $\delta \tilde{y}$ case)

$$\sum_{\alpha_2 = 1}^{2} \tilde{Q}_{(a)20}^{(1)1a_1a_2} \tilde{Q}_{(2)20}^{(1)} = 0,$$

(9.21)

and for the subspace $H_{(2)u_0}^+$ of variations given by

$$\delta a_{\alpha} = S_{(2)0}^{(1)} \delta a_{\alpha}^{(2)}, \quad \alpha = 1, \ldots, 2,$$

(9.22)

one has

$$\delta \tilde{y} = \sum_{\alpha_1, \alpha_2, \alpha_3 = 1}^{2} Q_{(a)30}^{(\alpha)1\alpha_1\alpha_2\alpha_3} S_{(2)20}^{(1)1} S_{(2)20}^{(1)1} S_{(2)20}^{(1)1} \left(\delta a_{\alpha_1}^{(2)}\right)^3 + O \left(\left(\delta a_{\alpha_1}^{(2)}\right)^4\right).$$

(9.23)

If the term of the fourth order in (9.23) does not vanish, then

$$\frac{\partial a_{\alpha}^{(2)}}{\partial \tilde{y}_1} \sim \epsilon^{-2/3}, \quad \epsilon \rightarrow 0$$

(9.24)

at some curve on the hypersurface $\Gamma$. If the term of the fourth order in (9.23) is equal to zero, then

$$\frac{\partial a_{\alpha}^{(2)}}{\partial \tilde{y}_1} \sim \epsilon^{-3/4}, \quad \epsilon \rightarrow 0$$

(9.25)

at the point $u_0$ on $\Gamma$.

Under certain constraints in the coefficients in the expansions (9.19) and (9.20), one gets similar behavior of the derivatives $a_{\alpha}$ and $b_1$ along suitable directions in $(y_1, y_2)$ plane.

10 A two-dimensional example: the first catastrophe time

In the multidimensional case, the definition of the catastrophe time is the same as in the one-dimensional case, e.g. the smallest positive blow-up time. The crucial difference is that this definition do not coincides with the annihilation of all the second order terms in (2.3) development (as we have shown in the previous sections, also some linear terms survive at blow-up time). The study of the various catastrophe behaviors requires a separate study and here we limit ourselves to a two-dimensional example.

In the space $\mathbb{R}^2$ with coordinates $x, y$ let us consider the following initial data of the velocity $u = (u, v)$

$$u(x, 0) = \frac{1}{1 + x^2 + y^2}, \quad v(x, 0) = \frac{1}{1 + x^2 + 2y^2}.$$ (10.1)

Such initial datum cannot be inverted globally and we select here the invertibility region where the first blow-up (catastrophe) happens. We obtain the hodograph relations

$$x - ut = \sqrt{\frac{3}{2u} - \frac{1}{2v} - 1}, \quad y - vt = \sqrt{\frac{1}{2v} - \frac{1}{2u}}.$$ (10.2)

18
The corresponding matrix $M$ is

$$M = \begin{pmatrix}
t - \frac{3}{2u^2} \sqrt{\frac{uv}{u^2 - 2u + 4uv}} & \frac{1}{2u^2} \sqrt{\frac{uv}{u^2 + 2u - 4uv}} \\
t - \frac{1}{2u^2} \sqrt{\frac{uv}{u^2 + 2u - 4uv}} & \frac{3}{2u^3} \frac{v(u - v)}{u}
\end{pmatrix} \quad (10.3)$$

and the condition $\det(M) = 0$ gives two curves $t = t(u,v)$ where the generic blow-up points are located. Analogously to the one-dimensional case the lowest positive minimum of such curves [4][11][10] give the catastrophe time. In this case we obtain (using here and in the following the software Mathematica)

$$u_c = 0.870058, \quad v_c = 0.73282, \quad t_c = 0.846079, \quad x_c = 0.940412, \quad y_c = 0.94808, \quad (10.4)$$

and the comparison of such results with the numerical evolution of the initial data is given in figure 3.

![Figure 3](image-url)

Figure 3: Evolution of the initial data 10.1 for the homogeneous Euler equation. The panels a and c are the plots of the initial data for $u$ and $v$ respectively. The panels b and d are the plots of $u$ and $v$ respectively at the critical time. The black point indicates the catastrophe point 10.4.

As we have proven in the previous sections, the main difference with the one-dimensional case is the derivative behavior at the catastrophe point. In particular, there is a direction along which the derivative does not blow up also at the catastrophe point. In two dimensional case such direction is identified by the eigenvector of $M$ related to nonzero eigenvalue. In particular

$$M(t_c, u_c) = \begin{pmatrix}
-4.00403 & 2.27893 \\
1.00669 & -0.572967
\end{pmatrix} \quad (10.5)$$

The behavior of the solution near to the catastrophe point is given by

$$\begin{pmatrix}
\delta x \\
\delta y
\end{pmatrix} = \begin{pmatrix}
-4.00403 & 2.27893 \\
1.00669 & -0.572967
\end{pmatrix} \begin{pmatrix}
\delta u \\
\delta v
\end{pmatrix} + O(|\delta u|^2). \quad (10.6)$$
The eigenvalues of $M$ are 0 and $\lambda \equiv -4.57699$ and the related right eigenvectors are

|                  | eigenvalue 0 | eigenvalue $\lambda \equiv -4.57699$ |
|------------------|--------------|---------------------------------------|
| right eigenvector $R$  | $\begin{pmatrix} 0.494652 \\ 0.869091 \end{pmatrix}$ | $\begin{pmatrix} 0.969818 \\ -0.243831 \end{pmatrix}$ |
| left eigenvector $L$   | $\begin{pmatrix} 0.253075 \\ 1.00659 \end{pmatrix}$ | $\begin{pmatrix} 0.902041 \\ -0.513406 \end{pmatrix}$ |

(10.7)

The theory developed previously indicates that $L$ gives the direction perpendicular to the catastrophe front and $W$ the tangent one (note that in 2D case $L \cdot W = 0$). Such condition is depicted in figure (4). Using the vector $L$, we obtain

\[ \delta y_1 = 0.253075 \delta x + 1.00659 \delta y = O(|\delta u|^2). \]

(10.8)

The successive terms of the expansion of (10.2) near to the catastrophe point at fixed $t = t_c$ are

\[ \delta x = -4.00403 \delta u + 2.27893 \delta v - 52.0038 \delta u^2 + 54.1089 \delta u \delta v - 15.822 \delta v^2 + O(|\delta u|^3), \]

\[ \delta y = 1.00669 \delta u - 0.572967 \delta v - 2.70162 \delta u^2 + 4.35455 \delta u \delta v - 1.1327 \delta v^2 + O(|\delta u|^3). \]

(10.9)

If we define $\delta a = L \cdot \delta x$ and $\delta b = V \cdot \delta x$ that the quadratic term in $a$ disappears

\[ \delta y_1 = -19.5167 \delta b^2 + O(|\delta a|^3), \]

(10.10)
as it is expected.

11 Conclusions

We have studied the possible generic and non-generic behaviors of blow-ups and catastrophes for homogeneous Euler equations (1.1). Such PDE can be considered as prototypical for the study of many nonlinear behaviors being the multidimensional version of the classical Burgers-Hopf equation. We have shown that the presence of more than one spatial dimension introduces new phenomena such as the possibility of initial data without blow-ups for both positive and negative times and the existence of directions of finite derivatives at the blow-up times. The study of singularities of the hodographic mapping (1.2) leads to many qualitatively different cases depending on the rank of the matrix $M$ defined in
The matrix $M$ encodes the initial data and the restriction on its admissible ranks translates on the classification of the possible blow-ups for different initial data.

Mimicking what has been done in one dimension, the possible future directions of this line of research are substantially two. The first one is the construction of weak solutions after the catastrophe time. The understanding of the $n$-dimensional analogue of the Rankine Hugoniot conditions is an open problem and also the recent literature on the complete Euler equation is large (see e.g. [18] and references therein as an example). Here we simply stress that the classical approach based on selected conserved quantities along the shock cannot be applied straightforward here because the generic function $f(u)$ is not automatically, as in the one-dimensional case, a conserved density. The simplest way manage this problem is to extend the HEE equation to the well known pressureless isentropic Euler equations (see e.g. [21, 3, 2])

$$u_t + u \cdot \nabla u = 0, \quad \rho_t + \nabla \cdot (\rho u) = 0,$$

which admits the classical conservation laws of mass, momentum and energy. The relation of such shocks with the vorticity generation for full Euler equations is a subject of the recent study [12].

The second direction is the role of dispersion in the post-catastrophe evolution. Contrarily to the complete Euler equations, HEE generically does not admit incompressible solutions because $\nabla \cdot u = \text{tr} (M^{-1})$ does not remain zero even if it starts from zero. A natural question is to compare, at least numerically, the HEE solution and the complete Euler one for small values of $\nabla \cdot u$.

**Acknowledgments**

The authors are grateful to E. A. Kuznetsov, F. Magri and M. Pedroni for useful and clarifying discussions. This project thanks the support of the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant No. 778010 IPaDEGAN. We also gratefully acknowledge the auspices of the GNFM Section of INdAM under which part of this work was carried out and the financial support of the project MMNLP (Mathematical Methods in Non Linear Physics) of the INFN.

**Appendix**

The generic results for the admissible ranks $r_1$ of $M$ are resumed in the following table 3.

| space dimension $n$ | $k^2 - 1 \leq n \leq (k + 1)^2 - 2, \quad k = 1, 2, 3, \ldots$ |
|---------------------|---------------------------------------------------------------|
| admissible ranks $r_1$ of $M$ | $n - \sqrt{n + 1} \leq r_1 < n$ |
| $\dim \Gamma_{r_1}(u, t) = n + 1 - (n - r_1)^2$ | $n - 1, n - 2, \ldots, n - k + 1, n - k$ |
| $\dim R_{u_0}^+ = \dim R_{x_0}^+ = n - r_1$ | $n, n - 3, \ldots, n + 1 - (k - 1)^2, n + 1 - k^2$ |
| $1, 2, \ldots, k$ |

Table 3: Admissible ranks $r_1$ for the singular matrix $M$ in generic dimensions and the related values of geometric objects characterizing the catastrophe.

**References**

[1] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko *Singularities of Differentiable Maps* Boston, MA: Birkhäuser (2009)

[2] S. Bianchini and S. Daneri *On the sticky particle solutions to the multi-dimensional pressureless Euler equations.* [arXiv:2004.06557](arXiv:2004.06557)
[3] A. Bressan and T. Nguyen *Non-existence and non-uniqueness for multidimensional sticky particle systems* Kinetic and Related Models 7(2): 205-218 (2014)

[4] S. G. Chefranov *An exact statistical closed description of vortex turbulence and of the diffusion of an impurity in a compressible medium* Sov. Phys. Dokl. 36(4) 286-289 (1991)

[5] D.B. Fairlie, *Equations of Hydrodynamic Type* DTP/93/31(1993)

[6] D.B.Fairlie & A.N.Leznov *General solutions of the Monge-Ampère equation in n-dimensional space* J. Geom. Phys. 16(6) 385-390 (1995)

[7] I. M. Gelfand *Lectures on linear algebra* Ed. Dover (1989)

[8] Y. Kodama and B. G. Konopelchenko *Singular sector of the Burgers-Hopf hierarchy and deformations of hyperelliptic curves* J. Phys. A: Math. Gen. 35 L489–L500 (2002)

[9] B. G. Konopelchenko and G. Ortenzi *On the plane into plane mappings of hydrodynamic type. Parabolic case* Rev. Math. Phys. vol 32 (2020)

[10] B. G. Konopelchenko and G. Ortenzi *Homogeneous Euler equation: blow-ups, gradient catastrophes and singularity of mappings* J. Phys. A: Math. Theor. 55, 035203 (2022)

[11] E. A. Kuznetsov *Towards a sufficient criterion for collapse in 3D Euler equations* Physica D 184 266-275 (2003)

[12] E. A. Kuznetsov and E. A. Mikhailov *Slipping flows and their breaking* Annals of Physics in press; [arXiv:2207.10621](https://arxiv.org/abs/2207.10621) (2022)

[13] H. Lamb *Hydrodynamics* Cambridge: Cambridge University Press (1993)

[14] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon press (1987)

[15] H. Poincaré, *Sur les propriétés des fonctions définies par les équations aux différences partielles : 1re thèse*. Gautiers-Villars, Paris (1879)

[16] E. Sernesi *Geometria 1* Ed. Bollati Boringhieri (1997)

[17] S. F. Shandarin and Ya. B. Zeldovich *The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium* Rev. Mod. Phys. 61 185-220 (1989)

[18] T. Buckmaster, S. Shkoller and V. Vicol *Formation of Shocks for 2D Isentropic Compressible Euler* Comm. Pure App. Math. 1–56 (2020)

[19] H. Whitney, *On singularities of mappings of Euclidean spaces. I. Maps of the plane into the plane* Ann. of Math. 62(3) 374–410 (1955)

[20] G. B. Whitham *Linear and Nonlinear Waves* John Wiley & Sons, New York, N.Y., USA (1999).

[21] Y. B. Zel’dovich *Gravitational instability: An approximate theory for large density perturbations*. Astron. and Astrophys., 5 84-89 (1970)