The pseudo-perturbative shifted - \( \ell \) expansion technique PSLET is shown applicable in the non-Hermitian \( \mathcal{PT} \) - symmetric context. The construction of bound states for several \( \mathcal{PT} \) - symmetric potentials is presented, with special attention paid to \( V(r) = ir^3 - \alpha \sqrt{ir} \) oscillators.
I. INTRODUCTION

In their recent studies Dorey, Dunning and Tateo (DDT) [1] have considered the manifestly non-Hermitian Schrödinger equation, in $\hbar = 2m = 1$ units,

$$\left[ -\frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} - \alpha \sqrt{i r + i r^3} \right] \psi_{k,\ell}(r) = E_{k,\ell} \psi_{k,\ell}(x).$$

They have rigorously proved that the spectrum $E_{k,\ell}$ is real and discrete in the domain of the sufficiently large angular momenta,

$$\ell > \max \left[ \frac{1}{4}(2\alpha - 7), -\frac{1}{2} \right] \equiv \ell_{DDT}(\alpha).$$

This inspired our subsequent study of this model [2] where we have shown that in the strong coupling regime with $\ell \gg 1$, the low lying DDT bound states may be very well approximated by the harmonic oscillators. At the same time we have noticed that the quality of such an asymptotic approximation may deteriorate quite significantly with both the increase of excitation $k$ and/or with the decrease of $\ell$.

Such a situation is, obviously, challenging. Firstly, our study [2] revealed that the manifest non-Hermiticity of the models of the type (1) leads to the reliable leading order approximation only after we select our harmonic oscillator approximant as lying very far from the real axis (i.e., from the Hermitian regime). Such a recipe is, apparently, deeply incompatible with a smooth modification of the traditional zero-order approximants occurring in current Hermitian $1/\ell$ recipes (cf. a small sample of some references in [3]). At the same time, the smallness of $1/\ell$ still supports the feeling that the similar perturbation techniques should prove efficient after their appropriate modification.

This observation offered a sufficiently strong motivation for our continued interest in the complex, non-Hermitian model (1) which may be understood as a characteristic representative of a very broad class of the so called pseudo-Hermitian models with real spectra, the analyses of which became very popular in the recent literature [4] - [18]. Within this class, the strong-coupling version of DDT oscillators (1) with $\alpha \gg 1$ forms a particularly suitable
testing ground as it combines the necessary reality of its spectrum with the smallness of the inverse quantity $1/\ell$. Moreover, the phenomenologically appealing non-Hermitian models like (1) are rarely solvable in closed form so that the presence of a ”universal” small parameter $1/\ell \ll 1$ offers one of not too many ways towards their systematic approximate solution.

In our present paper II we intend to discuss such a possibility in more detail.

II. FRAMEWORK

The first stages of interest in the non-Hermitian oscillators (1) date back to an old paper by Caliceti et al [5]. It studied the imaginary cubic problem in the context of perturbation theory and, more than twenty years ago, it offered the first rigorous explanation why the spectrum in such a model may be real and discrete. In the literature, this result has been quoted as a mathematical curiosity [6] and only many years later, its possible relevance in physics re-emerged and has been emphasized [7]. This initiated an extensive discussion which resulted in the proposal of the so called $\mathcal{PT}$ symmetric quantum mechanics by Bender et al [8].

The key idea of the new formalism lies in the empirical observation that the (phenomenologically desirable) existence of the real spectrum need not necessarily be attributed to the Hermiticity of the Hamiltonian. The current Hermiticity assumption $H = H^\dagger$ is replaced by the mere $\mathcal{PT}$ symmetry $H = H^\dagger \equiv \mathcal{PT}H\mathcal{PT}$. Here, $\mathcal{P}$ denotes the parity ($\mathcal{P}x\mathcal{P} = -x$) while the anti-linear operator $\mathcal{T}$ mimics the time reflection ($\mathcal{T}i\mathcal{T} = -i$). It is easy to verify that example (1) exhibits such a type of symmetry [2] and may serve as an elementary illustration of the latter extension of quantum mechanics.

The Bender’s and Boettcher’s conjecture that $H = H^\dagger$ may imply $\text{Im} \ E = 0$ is fragile. The extent as well as limitations of its validity are most easily analyzed in the language of linear algebra using the biorthogonal bases [4,9] and/or exactly solvable Hamiltonians [10]. Nevertheless, the relevance of many unsolvable oscillators originates from their applicability
in physics [11] and field theory [12]. In such a setting it is necessary to develop and test also some efficient approximation methods. New and intensive studies employed the ideas of the strong-coupling expansions [13] as well as the complex version of WKB [14], Hill determinants and Fourier transformation [15], functional analysis [16], variational and truncation techniques [17], and linear programming [18].

In what follows we intend to use the method based on the smallness of the inverse angular momentum parameter $1/\ell$. Various versions [19] of such an approach are available for Hermitian models where the combination of the central repulsive core $\ell(\ell + 1)/r^2$ with a confining (i.e., asymptotically growing) interaction $V(r)$ forms a practical effective potential $V_p(r)$ which possesses a pronounced minimum. Near such a minimum the shape of the potential is naturally fitted by the elementary and solvable harmonic oscillator well. Corrections can be then evaluated in an unambiguous and systematic manner [20].

As long as we intend to move to the complex plane, the leading-order approximation may become non-unique. One finds several different complex and/or real minima of $V_p(r)$ even in our oversimplified examples (1) [2]. For all the similar non-Hermitian Hamiltonians, even the most sophisticated forms of the perturbation expansions in the powers of our small parameter $1/\ell$ lose their intuitive background and deserve careful new tests, therefore.

Some of the related reopened questions will be clarified by our forthcoming considerations inspired, basically, by the pseudo-perturbative shifted$-\ell$ expansion technique (PSLET) in its form designed for the standard Hermitian Hamiltonians and described, say, by Mustafa and Odeh in ref. [20]. In this recipe, the shifts of $\ell$ were admitted as suitable optional auxiliary parameters while the use of the prefix ”pseudo-” just indicated that $\ell$ itself is an artificial, kinematical parameter rather than a genuine dynamical coupling.

III. $\mathcal{PT}$-SYMmetric PSLET RECIPE

As we already mentioned, one of the first $\mathcal{PT}$-symmetric models with an immediate impact on physics has been the Buslaev’s and Grecchi’s quartic anharmonic oscillator [7]
described by the radial Schrödinger equation in $d$–dimensional space,

$$\left[-\frac{d^2}{dr^2} + \frac{\ell_d (\ell_d + 1)}{r^2} + V(r)\right] \chi_{k,\ell}(r) = E_{k,\ell} \chi_{k,\ell}(r). \quad (3)$$

In this model they shifted the coordinate axis to the complex plane, $r = t - ic$ with a constant $\Im(r) = -c < 0$ and variable $\Re(r) = t \in (-\infty, \infty)$. They also required that $\chi_{k,\ell}(r) \in L_2(-\infty, \infty)$ at all the partial waves $\ell_d = \ell + (d - 3)/2$ and dimensions $d > 2$.

This example may find the various sophisticated generalizations some of which will be also mentioned in due course in what follows. For example, a $t$–dependent shift $c = c(t)$ may be needed both for the exactly solvable Coulombic model of ref.[21] and for all the more general and purely numerically tractable potentials $V(r) \sim -(ir)^N$ of ref.[8] with the positive exponents $N > 3$. Fortunately, the transition to $c = c(t)$ remains particularly elementary, being mediated by the mere change of the variable in eq.(3) within this class (c.f., e.g., ref.[21] for an explicit illustration). Another remark might concern the assumption that the wave functions are square integrable. For the most elementary $\mathcal{PT}$– symmetric Hamiltonians this assumption seems very natural but in certain more sophisticated models its use may require a more careful analysis as presented, e.g. in refs. [4] and [16].

The practical experience with the Hermitian version of the pseudo-perturbation shifted - $\ell$ expansion technique of Mustafa and Odeh [20] may serve as a key inspiration of an appropriate complexified new PSLET recipe. Firstly we notice a formal equivalence between the assumed smallness of our parameter $1/\ell_d \approx 0$ and of its (arbitrarily) shifted form. Thus we introduce a new symbol $\bar{\ell} = \ell_d - \beta$ and, simultaneously, move and rescale our coordinates $r \rightarrow x$ to

$$x = \bar{\ell}^{1/2}(r - r_0). \quad (4)$$

Here $r_0$ is an arbitrary point, with its particular value to be determined later. Equation (3) thus becomes

$$\left\{-\bar{\ell} \frac{d^2}{dx^2} + \frac{\bar{\ell} + (2\beta + 1)}{r_0^2} \frac{\bar{\ell} + \beta (\beta + 1)}{[x/(r_0 \bar{\ell}^{1/2}) + 1]^2} + \frac{\bar{\ell}^2}{Q} V(x(r))\right\} \Psi_{k,\ell}(x(r)) = E_{k,\ell}\Psi_{k,\ell}(x(r)), \quad (5)$$
where $Q$ is a constant that scales the potential $V$ at large-$\ell_d$ limit and is set, for any specific choice of $\ell_d$ and $k$, equal to $\bar{l}^2$ at the end of calculation. Expansions about this point, $x = 0$ (i.e. $r = r_0$), yield

$$\frac{1}{r_0^2 \left[ x/(r_0\bar{l}/2) + 1 \right]^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{r_0^{n+2}} x^n \bar{l}^{-n/2}, \quad (6)$$

$$\frac{\bar{l}^2}{Q} V(x(r)) = \sum_{n=0}^{\infty} \left( \frac{d^n V(r_0)}{dr_0^n} \right) \frac{x^n}{n! Q} \bar{l}^{-(n-4)/2}. \quad (7)$$

It is also convenient to expand $E_{k,\ell}$ as

$$E_{k,\ell} = \sum_{n=-2}^{\infty} E_{k,\ell}^{(n)} \bar{l}^{-n}. \quad (8)$$

Of course one may consider also energy coefficient of half-entire power of $\bar{l}$ in (8) but all these coefficients vanish (c.f., e.g., ref.[19,20]). Equation (5), therefore, reads

$$\{ \begin{array}{c} -\frac{d^2}{dx^2} + \sum_{n=0}^{\infty} B_n x^n \bar{l}^{-(n-2)/2} + (2 \beta + 1) \sum_{n=0}^{\infty} T_n x^n \bar{l}^{-n/2} \\ + \beta (\beta + 1) \sum_{n=0}^{\infty} T_n x^n \bar{l}^{-(n+2)/2} \end{array} \} \Psi_{k,\ell}(x)$$

$$= \left\{ \sum_{n=-2}^{\infty} E_{k,\ell}^{(n)} \bar{l}^{-(n+1)} \right\} \Psi_{k,\ell}(x) \quad (9)$$

where

$$B_n = T_n + \left( \frac{d^n V(r_0)}{dr_0^n} \right) \frac{1}{n! Q}; \quad T_n = (-1)^n \frac{(n+1)}{r_0^{n+2}}. \quad (10)$$

Equation (9) is to be compared with the non-Hermitian $\mathcal{PT}$ - symmetrized perturbed harmonic oscillator in the one dimensional Schrödinger equation

$$\left[ -\frac{d^2}{dy^2} + \frac{\omega^2}{4} (y - ic)^2 + \varepsilon_0 + P(y - ic) \right] \Phi_k(y) = \lambda_k \Phi_k(y), \quad (11)$$

where $P(y - ic)$ is a complexified perturbation - like term and $\varepsilon_0$ is obviously a constant. Such a comparison implies
\[ \varepsilon_0 = B_0 \bar{l} + (2\beta + 1)T_0 + \beta(\beta + 1)T_0/\bar{l} \]

\[ \lambda_k = \varepsilon_0 + (2k + 1)\frac{\omega}{2} + \sum_{n=0}^{\infty} \lambda_k^{(n)} \bar{l}^{-(n+1)} \]

\[ = B_0 \bar{l} + [(2\beta + 1)T_0 + (2k + 1)\frac{\omega}{2}] + \frac{1}{\bar{l}} \left[ \beta(\beta + 1)T_0 + \lambda_k^{(0)} \right] + \sum_{n=2}^{\infty} \lambda_k^{(n-1)} \bar{l}^{-n} \]

\[ = E_{k,\ell}^{(-2)} \bar{l} + E_{k,\ell}^{(-1)} + \sum_{n=1}^{\infty} E_{k,\ell}^{(n-1)} \bar{l}^{-n} \quad (12) \]

The first two dominant terms are obvious

\[ E_{k,\ell}^{(-2)} = \frac{1}{r_0^2} + \frac{V(r_0)}{Q}, \quad (13) \]

\[ E_{k,\ell}^{(-1)} = \frac{(2\beta + 1)}{r_0^2} + (2k + 1)\frac{\omega}{2}, \quad (14) \]

and with appropriate rearrangements we obtain

\[ E_{k,\ell}^{(0)} = \frac{\beta(\beta + 1)}{r_0^2} + \lambda_k^{(0)}, \quad (15) \]

\[ E_{k,\ell}^{(n)} = \lambda_k^{(n)}; \quad n \geq 1. \quad (16) \]

Here \( r_0 \) is chosen to minimize \( E_{k,\ell}^{(-2)} \), i.e.

\[ \frac{dE_{k,\ell}^{(-2)}}{dr_0} = 0 \quad \text{and} \quad \frac{d^2E_{k,\ell}^{(-2)}}{dr_0^2} > 0. \quad (17) \]

Equation (13) in turn gives, with \( (\ell_d - \beta)^2 = Q \),

\[ |\ell_d - \beta| = \sqrt{\frac{r_0^3V'(r_0)}{2}}. \quad (18) \]

Consequently, \( B_0 \bar{l} \equiv \bar{l} E_{k,\ell}^{(-2)} \) adds a constant to the energy eigenvalues and \( B_1 = 0 \).

The next leading correction to the energy series, \( \bar{l} E_{k,\ell}^{(-1)} \), consists of a constant term
and the exact eigenvalues of the unperturbed one-dimensional harmonic oscillator potential $\omega^2 x^2 / 4 (= B_2 x^2)$, where

$$0 < \omega = \omega^{(\pm)} = \pm \frac{2}{r_0^2} \Omega ; \quad \Omega = \sqrt{3 + \frac{r_0 V''(r_0)}{V'(r_0)}},$$

(19)

Evidently, equation (19) implies that $r_0$ is either pure real, $\omega = \omega^{(+)0}$. Next, the shifting parameter $\beta$ is determined by choosing $\bar{l} E^{(-1)}_{k,\ell} = 0$. That is

$$\beta = \beta^{(\pm)} = -\frac{1}{2} [1 \pm (2 k + 1) \Omega],$$

(20)

where $\beta = \beta^{(+)0}$ for $r_0$ pure real and $\beta = \beta^{(-)0}$ for $r_0$ pure imaginary. Then equation (9) reduces to

$$\left[ -\frac{d^2}{dx^2} + \sum_{n=0}^{\infty} v^{(n)} \bar{l}^{n-2} \right] \Psi_{k,\ell}(x) = \left[ \sum_{n=-1}^{\infty} E^{(n)}_{k,\ell} \bar{l}^{-(n+1)} \right] \Psi_{k,\ell}(x).$$

(21)

with

$$v^{(0)}(x) = B_2 x^2 + (2 \beta + 1) T_0,$$

(22)

and for $n \geq 1$

$$v^{(n)}(x) = B_{n+2} x^{n+2} + (2 \beta + 1) T_n x^n + \beta (\beta + 1) T_{n-2} x^{n-2}.$$

(23)

Equation (21) upon setting the wave functions

$$\Psi_{k,\ell}(x) = F_{k,\ell}(x) \exp(U_{k,\ell}(x))$$

readily transforms into the Riccati-type equation:
\[ F_{k,\ell}(x) \left[ -\left( U''_{k,\ell}(x) + U'_{k,\ell}(x) U'_{k,\ell}(x) \right) + \sum_{n=0}^{\infty} v^{(n)}(x) \bar{l}^{-n/2} \right. \\
- \left. \sum_{n=0}^{\infty} E_{k,\ell}^{(n-1)} \bar{l}^{-n} \right] - 2F'_{k,\ell}(x) U'_{k,\ell}(x) - F''_{k,\ell}(x) = 0, \]

where the primes denote derivatives with respect to \( x \). It is evident that this equation admits solutions (c.f., e.g., ref.[20]) of the form

\[ U'_{k,\ell}(x) = \sum_{n=0}^{\infty} U^{(n)}_{k}(x) \bar{l}^{-n/2} + \sum_{n=0}^{\infty} G^{(n)}_{k}(x) \bar{l}^{-(n+1)/2}, \]

\[ F_{k,\ell}(x) = x^k + \sum_{n=0}^{\infty} \sum_{p=0}^{k-1} d^{(n)}_{p,k} x^p \bar{l}^{-n/2}, \]

with

\[ U^{(n)}_{k}(x) = \sum_{m=0}^{n+1} D_{m,n,k} x^{2m-1}; \quad D_{0,n,k} = 0, \]

\[ G^{(n)}_{k}(x) = \sum_{m=0}^{n+1} C_{m,n,k} x^{2m}. \]

Obviously, equating the coefficients of the same powers of \( \bar{l} \) and \( x \) (for each \( k \)), respectively, one can calculate the energy eigenvalues and eigenfunctions (following the uniqueness of power series representation) from the knowledge of \( C_{m,n,k}, D_{m,n,k}, \) and \( a^{(n)}_{p,k} \) in a hierarchical manner.

In order to test the performance of our strategy, let us first apply it to the two trivial, exactly solvable \( \mathcal{PT} \)-symmetric examples.
IV. AN ELEMENTARY ILLUSTRATION OF THE RECIPE

A. $\mathcal{PT}$–symmetric Coulomb

Using the potential $V(r) = iA/r$ (where $A$ is a real coupling constant) in the above $\mathcal{PT}$–symmetric PSLET setting, one reveals the leading-order energy approximation

$$\bar{l}^2 E_{k,\ell}^{(-2)} = \frac{\bar{l}^2}{r_0^2} + \frac{iA}{r_0}.$$  \hfill (24)

The unique minimum at $r_0 = 2\bar{l}/A$ occurs in the upper-half of the complex plane. In this case $\Omega = 1$, $\beta = \beta^{(-)} = k$, the leading energy term reads

$$\bar{l}^2 E_{k,\ell}^{(-2)} = \frac{A^2}{2r_0} = \frac{A^2}{4(k - \ell_d)^2},$$  \hfill (25)

and higher-order corrections vanish identically. Therefore, the total energy is

$$E_{n,\ell} = \frac{A^2}{4(n - 2\ell - 1)^2}, \quad n = 1, 2, 3, ...$$  \hfill (26)

where $n = k + \ell + 1$ is the principle quantum number. Evidently, the degeneracy associated with ordinary (Hermitian) Coulomb energies $E_n = -A^2/(2n)^2$ is now lifted upon the complexification of, say, the dielectric constant embedded in $A$. Moreover, the phenomenon of flown away states at $k = \ell_d$ emerge, of course if they do exist at all (i.e. the probability of finding such states is presumably zero, the proof of which is already beyond our current methodical proposal). Therefore, for each $k$–state there is an $\ell_d$–state to fly away.

Next, let us replace the central-like repulsive/attractive core through the transformation

$$\ell_d(\ell_d + 1) \to \alpha_o^2 - 1/4,$$  i.e. $\ell_d = -1/2 + q|\alpha_o|$, with $q = \pm 1$ denoting quasi-parity, and recast (25) as

$$E_{k,q} = \frac{A^2}{(2k + 1 - 2q|\alpha_o|)^2}.$$  \hfill (27)

Which is indeed the exact result obtained by Znojil and Lévai [21]. Equation (27) implies that even-quasi-parity, $q = +1$, states with $k = |\alpha_o| - 1/2$ fly away and disappear from the spectrum. Nevertheless, quasi-parity-oscillations are now manifested by energy levels
crossings. That is, a state-$k$ with even-quasi-parity crosses with a state-$k'$ with odd-quasi-parity when $|\alpha_o| = (k - k')/2$. However, when $\alpha_o = 0$ the central-like core becomes attractive and the corresponding states cease to perform quasi-parity-oscillations. For more details on the result (27) the reader may refer to Znojil and Lévai [21].

**B. $\mathcal{PT}$--symmetric harmonic oscillator $V(r) = r^2$**

For this potential the leading energy term reads

$$\bar{l}^2 E_{k,\ell}^{(-2)} = \frac{\bar{l}^2}{r_0^2} + r_0^2, \quad (28)$$

and supports four eligible minima (all satisfy our conditions in (17)) obtained through $r_0^4 = \bar{l}^2$ as $r_0 = \pm i |\bar{l}|^{1/2}$ and $r_0 = \pm |\bar{l}|^{1/2}$. In this case $\Omega = 2$,

$$\beta = \beta^{(+)} = -(2k + 3/2) \quad (29)$$

for $r_0 = \pm |\bar{l}|^{1/2}$, and

$$\beta = \beta^{(-)} = (2k + 1/2) \quad (30)$$

for $r_0 = \pm i |\bar{l}|^{1/2}$. Whilst the former (29) yields

$$\bar{l}^2 E_{k,\ell}^{(-2)} = 2r_0^2 = 4k + 2\ell_d + 3, \quad (31)$$

the latter (30) yields

$$\bar{l}^2 E_{k,\ell}^{(-2)} = 2r_0^2 = 4k + 1 - 2\ell_d. \quad (32)$$

In both cases $\beta = \beta^{(\pm)}$ the higher-order corrections vanish identically. Yet, one would combine (31) and (32) by the superscript $(\pm)$ and cast

$$E_{k,\ell}^{(\pm)} = 4k + 2 \pm (2\ell_d + 1). \quad (33)$$

Therefore, the $\mathcal{PT}$--symmetric oscillator is exactly solvable, by our recipe, and its spectrum, non-equidistant in general, exhibits some unusual features (cf. [22] for more
details). However, it should be noted that for the one-dimensional oscillator (where \( \ell_d = -1 \), and 0, even and odd parity, respectively) equation (33) implies (I) \( E^{(+)}/2 = 2k + 1/2 \), \( E^{(-)}/2 = 2k + 3/2 \), for \( \ell_d = -1 \), and (II) \( E^{(+)}/2 = 2k + 3/2 \), \( E^{(-)}/2 = 2k + 1/2 \), for \( \ell_d = 0 \). Which can be combined together to form the exact well known result

\[
E_N = 2N + 1; \quad N = 0, 1, 2, ....
\]

with a new, redefined quantum number \( N \).

V. APPLICATION: \( \mathcal{P} \mathcal{T} \)-SYMMETRIC DDT OSCILLATORS

In our \( \mathcal{P} \mathcal{T} \)-symmetric Schrödinger equation (1) with the practical effective potential

\[
V_p(r) = \frac{\ell(\ell + 1)}{r^2} - \alpha \sqrt{ir + i r^3}
\]

the general solutions themselves are analytic functions of \( r \) (c.f., e.g., [6]). We may construct them in the complex plane which is cut, say, from the origin upwards. This means that \( r = \xi \exp(i \varphi) \) with the length \( \xi \in (0, \infty) \) and with the span of the angle \( \varphi \in (-3\pi/2, \pi/2) \). Compact accounts of the related mathematics may be found in Bender et al [8].

Let us proceed with our \( \mathcal{P} \mathcal{T} \)-PSLET and search for the minimum/minima of our leading energy term for the DDT-oscillators (1)

\[
\hat{l}^2 E_{k,\ell}^{(-2)} = \frac{\ell^2}{r_0^2} - \alpha \sqrt{ir_0} + i r_0^3.
\]

Evidently, condition (17) yields

\[
r_0^5 + i \left[ \frac{1}{6} \alpha r_0^2 \sqrt{ir_0} + \frac{2}{3} \ell^2 \right] = 0.
\]

Obviously, a closed form solution for this equation is hard to find (if it exists at all) and one has to appeal to numerical techniques to solve for \( r_0 \).

\textit{A priori}, it is convenient to do some elementary analyzes, in the vicinity of the extremes of \( \alpha \) (mandated by condition (2)), and distinguish between the two different domains of \( \alpha \). For this purpose let us denote \( (2\ell^2)/3 = G^2 \), re-scale \( r_0 = -i|G^{2/5}| \rho \) and abbreviate
\[ \alpha = \delta \sqrt{6 \ G^2} \text{ with } 0 \lesssim \delta \lesssim 1. \] This gives the following new algebraic transparent form of our implicit definition of the minimum/minima in (37),

\[ 1 - Z = \delta \sqrt{\frac{Z}{6}} ; \quad Z = \rho^5. \quad (38) \]

In the weak-coupling domain, vanishing \( \delta \approx 0 \), equation (38) becomes trivial \((1 - Z = 0)\). It is easy to verify that (36), with \( \delta \approx 0 \), has a unique absolute minimum at

\[ Z = 1 \implies \rho = 1 \implies r_0 = -i|G^{2/5}|, \quad \delta = 0. \quad (39) \]

In the strong-coupling regime, \( \delta \approx 1 \), equation (38) yields \( Z = 2/3 \).

At this point, one may choose to work with \( \beta = 0 \) ( i.e. \( E^{(-1)}_{k,\ell} \neq 0 \)) and obtain the leading (zeroth)-order approximation

\[ \bar{l}^2 E^{(-2)}_{k,\ell} = \left( \frac{G^6}{Z^2} \right)^{1/5} \left[ 5Z - \frac{15}{2} \right], \quad (40) \]

and, with

\[ \frac{\omega^2}{4} = B_2 = \frac{1}{r_0^4} \left( \frac{5}{2} \right) [1 + Z], \quad r_0 = -iG^{2/5}Z^{1/5}, \]

the first-order correction

\[ \bar{l}E^{(-1)}_{k,\ell} = \sqrt{\frac{3}{2}} G \left[ \frac{1}{r_0^4} + \frac{(2k + 1)\omega}{2} \right] \]

\[ = \left( \frac{G}{Z^2} \right)^{1/5} \left[ \sqrt{\frac{15}{4} (1 + Z)} (2k + 1) - \sqrt{\frac{3}{2}} \right]. \quad (41) \]

Consequently, the energy series (8) reads, up to the first-order correction

\[ E_{k,\ell} = \frac{1}{Z^{2/5}} \left[ \frac{G^6}{2} (5Z - \frac{15}{2}) + G^{1/5} \left( \sqrt{\frac{15}{4} (1 + Z)} (2k + 1) - \sqrt{\frac{3}{2}} \right) \right]. \quad (42) \]

Nevertheless, one may choose to work with \( \beta = \beta^{(-)} \neq 0 \) ( i.e. \( E^{(-1)}_{k,\ell} = 0 \)) and obtain

\[ \beta = \beta^{(-)} = -\frac{1}{2} \left[ 1 - (2k + 1)\sqrt{5(1 + Z)/2} \right]. \quad (43) \]

Thus the zeroth-order approximation yields
\[ \ell^2 E_{k,\ell}^{(-2)} = \left( \frac{G_s^6}{Z^2} \right)^{1/5} \left[ 5Z - \frac{15}{2} \right]; \ G_s = \sqrt{\frac{2}{3}} \left[ \ell_d + \frac{1}{2} - (2k + 1) \sqrt{\frac{5}{8}} (1 + Z) \right]. \quad (44) \]

In Figure I we plot the energies of (44) vs \( \ell \in (-5, 5) \) at different values of \( Z = Z(\delta) \in (2/3, 1) \). Obviously, our results show that even with \( \ell < \ell_{DDT}(\alpha) \) the spectrum remains real and discrete. Moreover, once we replace \( \ell(\ell + 1) \rightarrow \alpha_o^2 - 1/4 \), i.e. \( \ell \rightarrow -1/2 + q|\alpha_o| \) with \( q = \pm 1 \) denoting quasi-parity, quasi-parity-oscillations are manifested by the unavoidable energy levels crossings, see Figure II.

Table I shows that our results from Eqs.(42) and (44) compare satisfactorily with those obtained by Znojil, Gemperle and Mustafa [2], via direct variable representation (DVR). We may mention that even in the domain of not too large \( \ell \), the difference between the exact and approximate energies remain small, of order of \( \approx 0.05\% \) from Eq.(42), with \( \beta = 0 \), and \( \approx 0.2\% \) from Eq.(44), with \( \beta = \beta^{(-)} \neq 0 \), for the ground state. Such a prediction should not mislead us in connection with the related convergence/divergence of our energy series (8), which is in fact the genuine test of our present \( \mathcal{PT} \)-symmetric PSLET formulae. The energy series (8) with \( \beta = \beta^{(-)} \neq 0 \) converge more rapidly than it does with \( \beta = 0 \). Nevertheless, our leading energy term remains the benchmark for testing the reality and discreteness of the energy spectrum.

In Table II we compare our results ( using \( \beta = \beta^{(-)} \neq 0 \), hereinafter, numerically solve for \( r_0 \) and following the procedure of section III) for (1) with \( \alpha = 0 \) using the first ten-terms of (8) and the corresponding Padé approximant, again with those from the DVR approach. They are in almost exact accord. Hereby, we may emphasize that the digital-precision enhances for larger \( \bar{l} \) ( smaller \( 1/\bar{l} \) ) values, where the energy series (8) and the related Padé approximants stabilize more rapidly.

Extending the recipe of our test beyond the weak-coupling regime \( \delta \approx 0 \) (i.e. \( \alpha \approx 0 \)) we show, in table III, the energy dependence on the non-vanishing \( \alpha \). Evidently, the digital-precision of our \( \mathcal{PT} \)- PSLET recipe reappears to be \( \bar{l} \)-dependent and almost \( \alpha \)- independent. In table IV we witness that the leading energy approximation inherits a substantial amount of the total energy documenting, on the computational and practical methodical side, the
usefulness of our pseudo-perturbation recipe beyond its promise of being quite handy. Yet, a broad range of $\alpha$ is considered including the domain of negative values, safely protected against any possible spontaneous $\mathcal{PT}$-symmetry breaking.

VI. SUMMARY

Our purpose was to find a suitable perturbation series like expansion of the bound states. We have started from the observation that the physical consistency of the model (1) (i.e., the reality of its spectrum) is characterized by the presence of a strongly repulsive/attractive core in the potential $V_p(r)$. This is slightly counterintuitive since the phenomenologically useful values of $\ell$ are usually small and only the first few lowest angular momenta are relevant in the Hermitian Schrödinger equations with central symmetry.

Only exceptionally, very high partial waves are really needed for phenomenological purposes (say, in nuclear physics [23]). The strong repulsion is required there, first of all, due to its significant phenomenological relevance and in spite of the formal difficulties.

Fortunately, efficient $\ell \gg 1$ approximation techniques already exist for the latter particular realistic Hermitian models. They have been developed by many authors (cf., e.g., their concise review in [24]). Their thorough and critical tests are amply available but similar studies were still missing in the non-Hermitian context.

Our present purpose was to fill the gap at least partially. We have paid thorough attention to the first few open problems related, e.g., to the possible complex deformation of the axis of coordinates. Our thorough study of a few particular $\mathcal{PT}$ symmetric examples revealed that the transition to the non-Hermitian models is unexpectedly smooth. We did not encounter any serious difficulties, in spite of many apparent obstacles as mentioned in Section II (e.g., an enormous ambiguity of the choice of the most suitable zero-order approximation).

In this way, our present study confirmed that the angular momentum (or dimension) parameter $\ell$ in the “strongly spiked” domain where $|\ell| \gg 1$ offers its formal re-interpretation and introduction of an artificial perturbation-like parameter $1/\ell$ which may serve as a guide
to the interpretation of many effective potentials as a suitably chosen solvable (harmonic oscillator) zero-order approximation followed by the systematically constructed corrections which prove obtainable quite easily.

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FIGURE CAPTIONS

**Figure 1.** Cubic oscillator (35) eigenenergies \((-E)\) in (44) vs \(\ell\) for \(k = 0\) and \(|\ell| < 5\) at different values of \(Z = Z(\delta) \in (2/3, 1)\).

**Figure 2.** Cubic oscillator (35) eigenenergies \((-E)\) in (44) vs \(|\alpha_o|\) for \(k = 0\), \(\ell = -1/2 + q|\alpha_o|\), and different values of \(Z = Z(\delta) \in (2/3, 1)\) at even- and odd-quasi-parities.
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TABLE I. Comparison of the energy levels for model (1) (with $\alpha = 0$). The benchmark, numerically exact DVR values are cited from Znojil, Gemperle and Mustafa [2].

| $\ell$ | $k$ | DVR          | Eq.(42) | Eq.(44) |
|-------|-----|--------------|---------|---------|
| 5     | 0   | -11.52191    | -11.517 | -11.542 |
|       | 1   | -4.56482     | -4.260  | -4.900  |
|       | 2   | 1.87017      | 2.997   | -0.109  |
| 10    | 0   | -28.76552    | -28.762 | -28.776 |
|       | 1   | -20.59867    | -20.426 | -20.756 |
|       | 2   | -12.70640    | -12.090 | -13.230 |
|       | 3   | -5.11663     | -3.754  | -6.380  |
|       | 4   | 2.14032      | 4.58    | -0.727  |
| 20    | 0   | -68.72646    | -68.724 | -68.733 |
|       | 1   | -59.24706    | -59.149 | -59.330 |
|       | 2   | -49.91773    | -49.574 | -50.171 |
|       | 3   | -40.74589    | -39.998 | -41.283 |
|       | 4   | -31.73951    | -30.423 | -32.705 |
|       | 5   | -22.90712    | -20.847 | -24.489 |
|       | 6   | -14.25769    | -11.272 | -16.713 |
|       | 7   | -5.80054     | -1.696  | -9.512  |
|       | 8   | 2.45491      | 7.879   | -3.172  |
| 50    | 0   | -211.13555   | -211.134| -211.138|
|       | 1   | -199.68009   | -199.633| -199.718|
|       | 2   | -188.29459   | -188.132| -188.406|
|       | 3   | -176.98040   | -176.631| -177.206|
|       | 4   | -165.73889   | -165.129| -166.123|
|       | 5   | -154.57149   | -153.628| -155.161|
|   |       |       |       |
|---|-------|-------|-------|
| 6 | -143.47967 | -142.127 | -144.328 |
| 7 | -132.46494 | -130.626 | -133.628 |
| 8 | -121.52886 | -119.124 | -123.069 |
TABLE II. Same as table I with $\mathcal{PT}$–PSLET results from the first ten-terms of (8) and the corresponding Padé approximant.

| $k$ | $\ell$ | DVR        | $\mathcal{PT}$–PSLET | Padé        | $\bar{l} \approx$ |
|-----|--------|------------|-----------------------|-------------|------------------|
| 0   | 5      | -11.52191  | -11.52191             | -11.52191336| 4.4              |
|     | 10     | -28.76552  | -28.76552             | -28.76552178| 9.4              |
|     | 20     | -68.72646  | -68.72646             | -68.72645928| 19.4             |
|     | 50     | -211.13555 | -211.135548           | -211.13554785| 49.4             |
| 1   | 5      | -4.56482   | -4.565                | -4.564813   | 2.2              |
|     | 10     | -20.59867  | -20.598669            | -20.59866911| 7.2              |
|     | 20     | -59.24706  | -59.2470556           | -59.24705572| 17.2             |
|     | 50     | -199.68009 | -199.68008657         | -199.6800865747| 47.2             |
| 2   | 10     | -12.70640  | -12.7065              | -12.7964017 | 4.9              |
|     | 20     | -49.91773  | -49.917727            | -49.917727248| 14.9             |
|     | 50     | -188.29459 | -188.294591           | -188.29459127507| 44.9             |
| 3   | 10     | -5.11663   | -9.388                | -5.1166     | 2.7              |
|     | 20     | -40.74589  | -40.74589             | -40.74589026| 12.7             |
|     | 50     | -176.98040 | -176.98039968         | -176.98039968| 42.7             |
TABLE III. Comparison of the energies for model (1) with $\ell = 10$ and different values of $\alpha$.

| $\alpha$ | $k$ | DVR       | $\mathcal{PT}$–PSLET | Padé             |
|----------|-----|-----------|-----------------------|------------------|
| 20       | 0   | -58.62190 | -58.6219026           | -58.621902667    |
|          | 1   | -49.83626 | -49.836258            | -49.8362579      |
|          | 2   | -41.29014 | -41.2905              | -41.29014        |
| 10       | 0   | -43.85223 | -43.85223324          | -43.852233235    |
|          | 1   | -35.40717 | -35.407174            | -35.40717408     |
|          | 2   | -27.23003 | -27.23010             | -27.23003        |
| 0        | 0   | -28.76552 | -28.76552             | -28.76551777     |
|          | 1   | -20.59867 | -20.59867             | -20.598669108    |
|          | 2   | -12.70640 | -12.70653             | -12.7064017      |
| -10      | 0   | -13.35529 | -13.3552878           | -13.355287796    |
|          | 1   | -5.40717  | -5.40717              | -5.4071736       |
|          | 2   | 2.27617   | 2.08                  | 2.27617          |
| -20      | 0   | 2.38131   | 2.381306              | 2.38130557       |
|          | 1   | 10.16462  | 10.1646               | 10.1646191       |
|          | 2   | 17.70339  | ...                  | ...              |
TABLE IV. Energies for model (1) with $\ell = 0$ and different values of $\alpha, d, k$.

| $d$ | $\alpha$ | $k$ | Leading term | $\mathcal{PT}$−PSLET | Padé | $\tilde{l}$ ≈ |
|-----|---------|-----|--------------|----------------------|-------|----------|
| 3   | -5      | 0   | 3.07         | 2.903                | 2.881 | 1        |
|     |         | 1   | -1.65        | -1.58                | -1.579391 | 3.2 |
|     |         | 2   | -7.962       | -7.652               | -7.65505 | 5.4 |
| -10 | 0       |     | 8.23         | 7.82                 | 7.820 | 1.3 |
|     | 1       |     | 3.98         | 3.803                | 3.80992 | 3.5 |
|     | 2       |     | -1.87        | -1.80                | -1.798903 | 5.7 |
| -15 | 0       |     | 13.95        | 13.38                | 13.38419 | 1.6 |
|     | 1       |     | 9.95         | 9.52                 | 9.542833 | 3.8 |
|     | 2       |     | 4.45         | 4.27                 | 4.273163 | 6.0 |
| 1   | 10      | 0   | -12.46       | -12.471              | -12.471 | 1.4 |
|     |         | 1   | -20.65       | -20.304              | -20.3040 | 3.5 |
|     |         | 2   | -29.00       | 028.341              | -28.34038 | 5.7 |
| 5   | 0       |     | -8.12        | -8.082               | -8.082 | 1.5 |
|     | 1       |     | -15.38       | -15.058              | -15.05795 | 3.6 |
|     | 2       |     | -23.15       | -22.569              | -22.56820 | 5.9 |
| -5  | 0       |     | 1.57         | 1.5393               | 1.5393 | 1.8 |
|     | 1       |     | -4.16        | -4.055               | -4.055465 | 4.0 |
|     | 2       |     | -10.94       | -10.63               | -10.6341055 | 6.3 |
| -15 | 0       |     | 12.99        | 12.754               | 12.75391 | 2.2 |
|     | 1       |     | 8.06         | 7.84                 | 7.84006 | 4.6 |
|     | 2       |     | 1.99         | 1.93                 | 1.9269341 | 6.8 |
Figure 1

Z=0.67
Z=0.70
Z=0.75
Z=0.80
Z=0.85
Z=0.90
Z=0.95
Z=1.00
Figure 2

Z = 0.67, q = -1
Z = 0.70, q = -1
Z = 0.75, q = -1
Z = 0.80, q = -1
Z = 0.67, q = +1
Z = 0.70, q = +1
Z = 0.85, q = -1
Z = 0.75, q = +1
Z = 0.90, q = -1
Z = 0.80, q = +1
Z = 0.95, q = -1
Z = 0.85, q = +1
Z = 1.00, q = -1
Z = 0.90, q = +1
Z = 0.95, q = +1
Z = 1.00, q = +1

0.00  4.00  8.00  12.00  16.00