Probabilistic Systems with Hidden State and Unobservable Transitions

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Abstract. We consider probabilistic systems with hidden state and unobservable transitions, an extension of Hidden Markov Models (HMMs) that in particular admits unobservable $\varepsilon$-transitions (also called null transitions), allowing state changes of which the observer is unaware. Due to the presence of $\varepsilon$-loops this additional feature complicates the theory and requires to carefully set up the corresponding probability space and random variables. In particular we present an algorithm for determining the most probable explanation given an observation (a generalization of the Viterbi algorithm for HMMs) and a method for parameter learning that adapts the probabilities of a given model based on an observation (a generalization of the Baum-Welch algorithm). The latter algorithm guarantees that the given observation has a higher (or equal) probability after adjustment of the parameters and its correctness can be derived directly from the so-called EM algorithm.

1 Introduction

There are many practical applications that involve the observation of a probabilistic system with hidden state, where the aim is to infer properties about the state of the system only from the observations that are available.

In particular we are motivated by the following scenario: imagine a building equipped with sensors that are triggered when a person walks past. However, these sensors might produce both false positives (nobody walked past, but the sensor sends a signal) and false negatives (somebody was present, but did not trigger the sensor). This can be modelled by a probabilistic transition system which has both observable symbols and $\varepsilon$-transitions (also referred to as null transitions), corresponding to false negatives. Now assume that there are three rooms, the bedroom (B), the corridor (C) and the kitchen (K), all of them connected through C and equipped with sensors. Sensors B, K trigger, but not C. However, in order to reach the kitchen from the bedroom, the person should have passed the corridor! Hence our analysis should tell us that the most likely explanation for the observation is indeed the sequence B, C, K.

While here this reasoning is straightforward, it may become increasingly more complex with additional missing sensor data and multiple possible paths.

In order to make matters more concrete, consider the following system depicting our motivational example. The start state is $s_0$ and from each state we
label the transitions with symbols and probabilities. For instance, from $s_0$ there is a probability of 0.1 of going to C with an unobservable $\varepsilon$-transition.

Now we detect the observation: b,k. What happened? In fact there are several possible paths, the most probable being $s_0$, B, C, K (first transition b, second one $\varepsilon$ and the third k) whose probability is

$$\delta(s_0)(b, B) \cdot \delta(B)(\varepsilon, C) \cdot \delta(C)(k, K) = 0.4 \cdot 0.3 \cdot 0.5 = 0.06$$

The question is how to efficiently determine the most likely path and its probability.

A second issue is how to learn the probabilities that label the transitions. Assume the basic structure of the system is known, in particular the number of states, but the parameters, i.e., the probabilities are not. Now we observe the system and want to estimate its parameters.

Of course, such systems have been extensively studied under the name of Hidden Markov Models (HMMs) \[11,13\]. Unobservable transitions, also known as null transitions or $\varepsilon$-transitions, i.e. HMMs that may change state without the observer being aware of it, have been proposed in the literature, especially in the context of speech recognition \[2,8\]. However, to the best of our knowledge, there is no theory of systems that allows for $\varepsilon$-loops. Related work only treats specific HMMs, where either loops can not appear altogether (so-called left-to-right HMMs) \[10\] or $\varepsilon$-loops are forbidden \[3,2\]. Also \[8\] does not allow $\varepsilon$-loops in the context of learning. It does describe an algorithm to eliminate $\varepsilon$-transitions, which can increase the size of the model and alters the system such that probabilities of $\varepsilon$-transitions can not be learned directly.

On the other hand $\varepsilon$-loops can occur naturally in applications, e.g., in the application described above that we have in mind, and there are no easy work-arounds. We can only speculate why the generalization has not been made, but observe that for systems with $\varepsilon$-loops the theory (in particular for parameter learning) is more complex since the Baum-Welch algorithm has to be fundamentally adapted to deal with this scenario. In particular the usual forward-backward algorithm \[11\] for parameter estimation can not be used directly, but has to be generalized.

Given an observation, we propose to compute the conditional expectation of passing each transition (remember that the state and so the transitions are hidden) and this is turned into a probability function (by normalizing), which is the new parameter estimate. The guarantee provided by this estimate is that the observation sequence becomes more probable given the new parameters. This result can be derived from the so-called expectation-maximization (EM) algorithm that gives us a general framework for such results.

Our contributions are:
- We rephrase the theory behind HMMs with $\varepsilon$-transitions, being precise about the used probability space and random variables.
- We extend the theory to systems with $\varepsilon$-transitions, a very natural extension for such systems and indispensable for our application, which complicates the
formalization and the algorithms. In particular, we have to handle $\varepsilon$-loops which we deal with by setting up fixpoint equations.

We spell out explicitly why parameter learning based on the EM algorithm works in this setting.

While HMMs have been known for some time, we feel that, due to the current large interest in learning approaches (e.g., machine learning or the L* algorithm [1]), it makes sense to revive the theory and close existing gaps in the literature.

2 Preliminaries

Probability theory. We recapitulate the basics of discrete probability theory.

A probability space $(\Omega, P)$ consists of a (countable) sample space $\Omega$ and a probability function $P : \Omega \rightarrow [0, 1]$ such that $\sum_{\omega \in \Omega} P(\omega) = 1$. Given a set $\Omega$ we denote by $\mathcal{D}(\Omega)$ the set of all probability functions on $\Omega$.

A random variable for a probability space $(\Omega, P)$ is a function $X : \Omega \rightarrow V$. We assume a special random variable $Z : \Omega \rightarrow \Omega$, which is the identity. For $v \in V$ we denote by $P(X = v) = \sum_{\omega \in \Omega} X(\omega) = v P(\omega)$ the probability that $X$ has value $v$.

Given two random variables $X_i : \Omega \rightarrow V_i$, $i = 1, 2$, the conditional probability that $X_1$ takes value $v_1$, under the condition that $X_2$ takes value $v_2$, is

$$P(X_1 = v_1 \mid X_2 = v_2) = \frac{P(X_1 = v_1 \land X_2 = v_2)}{P(X_2 = v_2)},$$

provided that $P(X_2 = v_2) > 0$.

For a random variable $X : \Omega \rightarrow \mathbb{R}$ (where we might enrich the real numbers with $-\infty$), we define its expectation as $E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$. Given another random variable $Y : \Omega \rightarrow V$, conditional expectation is defined, for $v \in V$, as

$$E[X \mid Y = v] = \sum_{\omega \in \Omega} X(\omega) \cdot P(Z = \omega \mid Y = v).$$

Hidden Markov Models. We are working with HMMs with $\varepsilon$-transitions (also called null transitions in the literature [28]). In particular we put observations on the transitions, rather than on the states [11]. This is a standard variant for HMMs and is for instance done in [28,6]. Since we use $\varepsilon$-transitions, we need not assume an initial probability function, but instead later fix a start state $s_0$.

Definition 2.1 (HMM with $\varepsilon$-transitions). An HMM with $\varepsilon$-transitions is a three-tuple $(S, \Sigma, \delta)$, consisting of a finite state space $S$, an alphabet $\Sigma$ and a transition function $\delta : S \rightarrow \mathcal{D}((\Sigma \cup \{\varepsilon\}) \times S)$. The set of transitions is defined as $T_\delta = \{(s, a, s') \mid s, s' \in S, a \in \Sigma \cup \{\varepsilon\}, \delta(s)(a, s') > 0\}$. In addition $T_\delta^\varepsilon = T_\delta \cap ((\{s\} \times (\Sigma \cup \{\varepsilon\}) \times S)$.}

3 Probability Space and Random Variables

The probability space of observation sequences contains alternating sequences of states and observation symbols (or $\varepsilon$) and is dependent on $n$, denoting the
number of observable symbols (from $\Sigma$). We fix a start state $s_0 \in S$ and restrict the possible sequences to those where the second to last element is contained in $\Sigma$ and is hence observable. This is needed to make sure that the probabilities in fact sum up to 1.

$$\Omega^n_{s_0} = s_0((\varepsilon S)^* \Sigma S)^n$$

The probability for an element from the probability space $\tilde{z} \in \Omega^n_{s_0}$ can be calculated by multiplying the corresponding transition probabilities. Whenever $\tilde{z} = s_0a_1s_1a_2 \ldots a_ms_m \in \Omega^n_{s_0}$ where $s_i \in S$ and $a_i \in \Sigma \cup \{\varepsilon\}$ we define:

$$P^n_{s_0}(\tilde{z}) = \prod_{i=1}^{m} \delta(s_{i-1})(a_i, s_i).$$

Furthermore $P^0_{s_0}(s_0) = 1.$

Note that due to the presence of $\varepsilon$-transitions we have to take care to set up a probability space where the probabilities add up to 1. An alternative could be to use the solution of [8] and to distinguish a final state, which is however inconvenient for some applications. We continue by showing that the probability space is well-defined under some mild conditions. These conditions have the additional benefit that the fixpoint equations become contractive (after a number of iterations) and hence have unique solutions (for more details see Appendix D).

**Proposition 3.1.** Assume that for each state $s \in S$ there is an outgoing path of non-zero probability that contains a symbol in $\Sigma$. Then the probability space is well-defined, in particular $\sum_{\tilde{z} \in \Omega^n_{s_0}} P^n_{s_0}(\tilde{z}) = 1.$

We will in the following assume that the requirement of Prop. 3.1 holds. Otherwise there might be states that can never reach an observation symbol, for which the probability of all outgoing paths is 0.

Given this probability space, we define some required random variables.

| Random Variables | Description |
|------------------|-------------|
| $Z : \Omega^n_{s_0} \to \Omega^n_{s_0}$ | Identity on $\Omega^n_{s_0}$ |
| $Y : \Omega^n_{s_0} \to \Sigma^n$ | Projection to observable symbols (removal of $\varepsilon$’s and states) |
| $L : \Omega^n_{s_0} \to S$ | Last state of a given observation sequence |
| $X_t : \Omega^n_{s_0} \to \mathbb{N}_0$ | Number of times a transition $t = (s, a, s')$ occurs in a sequence |

We omit the indices $n, s_0$ if they are clear from the context: if we write $P(\tilde{z})$ or $P(Z = \tilde{z})$ we work in the probability space $\Omega^n_{s_0}$ and mean the probability function $P^n_{s_0}$, where $n = |Y(\tilde{z})|$ and $s_0$ is the first element of $\tilde{z}$. And if we write $P_{s_0}(Y = \tilde{y})$, the value $n$ is understood to be $|\tilde{y}|$. We do the same for expectations.

## 4 Finding the Best Explanation for an Observation

As a warmup we will describe a method for finding the best explanation, given an observation sequence. More concretely, an observation sequence $\tilde{y} \in \Sigma^*$ is given and it is our aim to compute the most probable sequence of states and its probability. For standard HMMs there is a well-known algorithm for this task: the Viterbi algorithm [12911]. Instead of enumerating all paths and checking which one is most probable, it uses intermediate results, by computing step-by-step the most probable path ending at a given state $s$, for each prefix of the observation sequence $\tilde{y}$.
We now adapt the Viterbi algorithm, taking \( \varepsilon \)-transitions into account. While in the standard case it is straightforward to obtain the likeliest path in the case of a single observation symbol, in our case the path might have taken an arbitrary number of \( \varepsilon \)-transitions in between. Remember that the probability space is set up in such a way that the last transition in every sequence that we consider is always observable, which is no restriction, since there is always some explanation with maximal probability that satisfies this condition.

**Proposition 4.1 (Maximal probability for one observation).** Let \((S, \Sigma, \delta)\) be an HMM and \(a \in \Sigma\) be an observation. With \(E_{s_0, s}^a\), for \(s_0, s \in S\) we denote the probability for the most likely path in \(\Omega_s^1\), starting in state \(s_0\) and ending in state \(s\), where \(a\) is the observation. Then we have:

\[
E_{s_0, s}^a = \max_{\tilde{z} \in \Omega_{s_0}^1} P(Z = \tilde{z} \land Y = a) = \max \left( \delta(s_0)(a, s), \max_{s' \in S} \delta(s_0)(\varepsilon, s') \cdot E_{s', s}^a \right)
\]

The equation of Prop. 4.1 has a unique fixpoint due to the requirement that from every state there is a path of non-zero probability that contains an observation. In order to compute \(E_{s_0, s}^a\), one could hence perform fixpoint iteration or use an external solver. In fact, the computation is simplified in this case since among the paths with the highest probability there is always one that does not contain duplicate states (apart from the final state \(s\)). By equipping the computation with an extra parameter \(S_0 \subseteq S\) (the set of states that can still be visited), we can easily ensure termination, even in the presence of \(\varepsilon\)-loops, and the equation becomes the following, where \(E_{s_0, s}^a = E_{s_0, s}^a(S)\).

\[
E_{s_0, s}(S_0) = \max \left( \delta(s_0)(a, s), \max_{s' \in S_0 \setminus \{s_0\}} \delta(s_0)(\varepsilon, s') \cdot E_{s', s}^a(S_0 \setminus \{s_0\}) \right)
\]

We can now address the task of computing the maximal probability for a longer sequence of observations. For this purpose, we extend the established Viterbi algorithm [12]. Here, the probability for the likeliest path that results in a given observation is computed inductively and is based on Prop. 4.1.

**Proposition 4.2 (Maximal probability for observation sequence).** Let \((S, \Sigma, \delta)\) be an HMM and let \(\tilde{y} = a_1 \ldots a_n = \tilde{y}_1 a_n\) be an observation sequence. Then \(V_{s_0, s}^\tilde{y}\) denotes the maximum probability of observing \(\tilde{y}\) and ending in state \(s\), more formally

\[
V_{s_0, s}^\tilde{y} = \max_{\tilde{z} \in \Omega_{s_0}^n} P(Z = \tilde{z} \land Y = \tilde{y})
\]

For \(n = 0\) we have \(V_{s_0, s}^\varepsilon = 1\) if \(s = s_0\) and \(0\) otherwise. For \(n > 0\):

\[
V_{s_0, s}^\tilde{y} = \max_{s' \in S} V_{s_0, s'}^\tilde{y}_1 \cdot E_{s', s}^a
\]

In order to obtain the best explanation starting at \(s_0\), regardless of its final state, we still have to take the maximum \(\max_{s \in S} V_{s_0, s}^\tilde{y}\). If we are instead interested in the conditional probability, i.e., \(\max_{\tilde{z} \in \Omega_{s_0}^n} P(Z = \tilde{z} \mid Y = \tilde{y})\), it can be obtained from this maximum by dividing by \(P_{s_0}(Y = \tilde{y})\).
Since the computation of the most likely path is almost identical to the computation of the highest probability, we elaborate on this only in the appendix.

5 Parameter Learning

We now discuss a method for determining the system parameters. We assume that the structure of the system and initial probabilities are given, and those probabilities have to be adjusted through observing output sequences. This core problem for HMMs is traditionally solved by the Baum-Welch algorithm [3], which is based on the forward-backward algorithm, but because of \( \varepsilon \)-transitions and in particular \( \varepsilon \)-loops, it is necessary to develop a different approach.

5.1 Conditional Expectation of the Number of Transition Traversals

To adjust the probabilities, we have to solve the following subtask: Given an HMM with initial state \( s_0 \), an observation sequence \( \bar{y} \) and a transition \( t \), determine the expected value of the number of traversals of \( t \), when observing sequence \( \bar{y} \), starting from \( s_0 \). For each state, we determine these values for all outgoing transitions and normalize them to obtain probabilities. This gives us new parameters and we later discuss the guarantees that this approach provides.

If there are no \( \varepsilon \)-loops, it is sufficient to compute the probability of crossing a given transition \( t \) while reading the \( i \)-th symbol of the observation sequence and to sum up over all \( i \). This is done with the forward-backward algorithm, determining the probability of reaching the source state of \( t \), multiplied with the probability of \( t \) and the probability of reading the remaining observation sequence from the target state. In the present setup, this has to be adapted, since we may cross \( t \) several times while reading the \( i \)-th symbol.

We want to determine \( E_{s_0}[X_t | Y = \bar{y}] \) or, equivalently, \( E_{s_0}[X_t | Y = \bar{y}] \cdot P_{s_0}(Y = \bar{y}) \). This is defined if \( P_{s_0}(Y = \bar{y}) > 0 \), which we assume since the sequence \( \bar{y} \) has actually been observed. Note that due to the nature of our probability space, \( \varepsilon \)-transitions that might be traversed after the last observation do not count. We compute the conditional expectation by setting up a suitable fixpoint equation.

**Proposition 5.1.** Fix an HMM and an observation sequence \( \bar{y} = a_1 \ldots a_n = a_1 \bar{y}_1 \). Let \( t = (s, a, s') \) and define

\[
C_{s_0, t}^{\bar{y}} = E_{s_0}[X_t | Y = \bar{y}] \cdot P_{s_0}(Y = \bar{y}).
\]

Then \( C_{s_0, t}^{\varepsilon} = 0 \) and the following fixpoint equation holds: whenever \( a \in \Sigma \)

\[
C_{s_0, t}^{\bar{y}} = \sum_{s_1 \in S} \delta(s_0)(a_1, s_1) \cdot C_{s_1, t}^{\bar{y}_1} + \sum_{s_1 \in S} \delta(s_0)\varepsilon(s_1) \cdot C_{s_1, t}^{\bar{y}} + \begin{cases} [s_0 = s \land a_1 = a] \cdot \delta(s)(a, s') \cdot P_{s'}(Y = \bar{y}_1) \\
&\text{and whenever } a = \varepsilon \text{ the last summand has to be replaced by } [s_0 = s] \cdot \delta(s)(\varepsilon, s') \cdot P_{s'}(Y = \bar{y}). \end{cases}
\]

We use the convention that \([b] = 1\) if \( b \) holds and \([b] = 0\) otherwise.
Since the equations are contractive after some iterations (cf. Appendix 10), they have a unique fixpoint, which can be approximated by (Kleene) iteration or computed via a solver. For this we have to be able to determine $P_{s_0}(Y = \tilde{y})$ for $\tilde{y} = a_1 \tilde{y}_1$, which can be done with a similar fixpoint equation (adapt the proof of Prop. 5.1 to the case where $X_t$ is the constant 1-function): $P_{s_0}(Y = \varepsilon) = 1$ and otherwise:

$$P_{s_0}(Y = \tilde{y}) = \sum_{s_1 \in S} \delta(s_0)(a_1, s_1) \cdot P_{s_1}(Y = \tilde{y}_1) + \sum_{s_1 \in S} \delta(s_0)(\varepsilon, s_1) \cdot P_{s_1}(Y = \tilde{y}).$$

Example 5.2. Given the following HMM on the right where the states and transitions are known, but the probabilities have to be adjusted by observing the system. The three transitions are named $t_1 = (s_0, \varepsilon, s_0), t_2 = (s_0, \alpha, s_0), t_3 = (s_0, \beta, s_0)$ and the observation sequence is $\tilde{y} = \alpha$. Then:

$$C_{s_0, t_1}^{\tilde{y}} = \delta(s_0)(\alpha, s_0) \cdot C_{s_0, t_1}^{\varepsilon} + \delta(s_0)(\varepsilon, s_0) \cdot C_{s_0, t_1}^{\tilde{y}} + 1 \cdot \delta(s_0)(\varepsilon, s_0) \cdot P_{s_0}(Y = \tilde{y})$$

\[= C_{s_0, t_1}^{\tilde{y}} = \frac{1}{2} = E_{s_0}[X_{t_1} \mid Y = \tilde{y}] \cdot P_{s_0}(Y = \tilde{y}) \Rightarrow E_{s_0}[X_{t_1} \mid Y = \tilde{y}] = 1 \]

Similarly, we compute $E_{s_0}[X_{t_2} \mid Y = \tilde{y}] = 1$ and $E_{s_0}[X_{t_3} \mid Y = \tilde{y}] = 0$. The adjusted and normalized probability parameters are:

$$\delta(s_0)(\varepsilon, s_0) = \frac{1}{2} \quad \delta(s_0)(\alpha, s_0) = \frac{1}{2} \quad \delta(s_0)(\beta, s_0) = 0$$

In practice one will of course make longer or multiple observations before adjusting the parameters.

5.2 Using the EM Algorithm

We will now introduce the so-called Expectation Maximization (EM) Algorithm [5], which is commonly used to derive the Baum-Welch algorithm [11] for parameter estimation. It explains how to suitably adjust (probabilistic) parameters of a system in such a way that the likelihood of observing the given output of the system increases. We assume that the higher the probability for observed sequences, the closer the parameters are to their actual values. This procedure is divided into two phases: the Expectation and Maximization phase.

Fix an HMM with known (graph structure) and unknown parameters (transition probabilities). The unknown parameters, denoted by $\theta$, can be learned by observing the system. We will use $\theta$ in conditional probabilities or expectations to clarify the parameter dependency. E.g., $\delta(t \mid \theta)$ with $t = (s, a, s')$ stands for $\delta(s)(a, s')$ under the parameter setting $\theta$.

The algorithm works iteratively in two phases. $\theta^l$ always denotes our current best guess of the probabilistic parameters, $\theta$ denotes the new parameters that we wish to learn and improve iteratively given an observation sequence $\tilde{y}$. In the first
phase we calculate $Q(\theta \mid \theta^t)$ denoting the expected value of the log likelihood function for $\theta$ with respect to the current conditional probability of $Z$ given an observation and the current estimates of the parameter $\theta^t$. More concretely:

$$Q(\theta \mid \theta^t) = E_{Z \mid \theta^t}[\log P(Y, Z \mid \theta)],$$

which denotes the expectation of the random variable $\tilde{z} \mapsto \log P(Y = \tilde{y}, Z = \tilde{z} \mid \theta)$ in an updated probability space where the probability function is $P'(\tilde{z}) = P(Z = \tilde{z} \mid Y = \tilde{y}, \theta^t)$. Here it is understood that $\log 0 = -\infty$ and $0 \cdot (-\infty) = 0$.

After the first phase follows the Maximization phase, where $\theta^{t+1}$ is determined as $\arg \max_\theta Q(\theta \mid \theta^t)$ and the algorithm subsequently starts again with phase one. This happens iteratively until $\theta^{t+1} = \theta^t$ or the improvements are below some threshold. In general we will converge to a local optimum, finding the global optimum is typically infeasible. The guarantee of the EM algorithm is that $P(Y = \tilde{y} \mid \theta) > P(Y = \tilde{y} \mid \theta^t)$ whenever $Q(\theta \mid \theta^t) > Q(\theta^t \mid \theta^t)$.

**Theorem 5.3.** In our setting it holds that

$$Q(\theta \mid \theta^t) = \sum_{s \in S} \sum_{(s,a,s') \in T_3} \log \delta(t \mid \theta) \cdot E_{s_0}[X_t \mid Y = \tilde{y}, \theta^t].$$

The value $Q(\theta \mid \theta^t)$ is maximal when the parameters $\theta$ are as follows: for every transition $t$ we set $\delta(t \mid \theta)$ proportional to $E_{s_0}[X_t \mid Y = \tilde{y}, \theta^t]$.

Note that there might be states where all outgoing transitions have conditional expectation zero, i.e., such a state can not be reached via the observation sequence. In this case we keep the previous parameters. If we adhere to this, we can always guarantee that the requirement of Lemma 3.1 is maintained, since if an outgoing transition of a state has conditional expectation greater than zero, there must be a path of non-zero probability to an observation.

6 Conclusion

In this paper, we considered HMMs that admit unobservable $\varepsilon$-transitions. We presented algorithms for determining the most probable explanation (i.e. a sequence of hidden states) given an observation and a method for parameter learning. For this, we generalized the Viterbi and the Baum-Welch algorithm to consider $\varepsilon$-transitions (including $\varepsilon$-loops) and provided the respective proofs of their soundness. By allowing state changes of which the observer is unaware we can model false negatives, i.e. actions that have taken place but have not been observed by a sensor. This extends the applicability of HMMs as a modeling technique to the domain of sensor-based systems, which always have to consider the probability of sensor errors. For example, we now have the methods to compare observations made by sensors with the computed most likely explanation. When these two drift further apart over time, we can conclude that the real-world system is subject to parameter drift or degrading sensor quality. Furthermore, we plan to use the HMMs to clean data sets by replacing observations with their most probable explanation. Parameter learning will be needed to learn and adapt the model parameters based on recorded observations.
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A Proofs for Section 3 (Probability Space and Random Variables)

Proposition 3.1 Assume that for each state \( s \in S \) there is an outgoing path of non-zero probability that contains a symbol in \( \Sigma \). Then the probability space is well-defined, in particular \( \sum_{\tilde{z} \in \Omega_{s_0}} P^n_{s_0}(\tilde{z}) = 1 \).

Proof. Given a sequence \( \tilde{z} = s_0s_1s_2\ldots s_ns_{m} = s_0s_1s_2 \) we observe that

\[
P^n_{s_0}(\tilde{z}) = \delta(s_0)(a_1, s_1) \cdot P^n_{s_1}(\zeta_1) \quad \text{if} \quad a_1 = \varepsilon
\]

\[
P^n_{s_0}(\tilde{z}) = \delta(s_0)(a_1, s_1) \cdot P^{n-1}_{s_1}(\zeta_1) \quad \text{if} \quad a_1 \in \Sigma
\]

We abbreviate \( S^n_{s_0} = \sum_{\tilde{z} \in \Omega_{s_0}} P^n_{s_0}(\tilde{z}) \). Since \( \Omega^n_{s_0} = \{ s_0 \} \), it is easy to see that \( S^n_{s_0} = 1 \) and if \( n \geq 1 \):

\[
S^n_{s_0} = \sum_{\tilde{z} \in \Omega^n_{s_0}} P^n_{s_0}(\tilde{z}) = \sum_{s_1 \in S} \sum_{s_1 \in \Omega^n_{s_1}} \delta(s_0)(\varepsilon, s_1) \cdot P^n_{s_1}(\zeta_1) + \\
\sum_{a_1 \in \Sigma, s_1 \in S} \sum_{\tilde{z}_1 \in \Omega^{n-1}_{s_1}} \delta(s_0)(a_1, s_1) \cdot P^{n-1}_{s_1}(\zeta_1)
\]

\[
= \sum_{s_1 \in S} \delta(s_0)(\varepsilon, s_1) \cdot \sum_{\tilde{z}_1 \in \Omega^n_{s_1}} P^n_{s_1}(\zeta_1) + \\
\sum_{a_1 \in \Sigma, s_1 \in S} \delta(s_0)(a_1, s_1) \cdot \sum_{\tilde{z}_1 \in \Omega^{n-1}_{s_1}} P^{n-1}_{s_1}(\zeta_1)
\]

\[
= \sum_{s_1 \in S} \delta(s_0)(\varepsilon, s_1) \cdot S^n_{s_1} + \sum_{a_1 \in \Sigma, s_1 \in S} \delta(s_0)(a_1, s_1) \cdot S^{n-1}_{s_1}
\]

Since the probabilities of all outgoing transitions of a state sum up to 1, we observe that \( S^n_{s} = 1 \) for all \( n, s \in S \) is a solution to this system of fixpoint equations. Since we assume that each state will eventually reach an observation symbol in \( \Sigma \) with non-zero probability, the corresponding fixpoint function is contractive after some iterations, since the probability to stay with index \( n \) is strictly less than 1 after at most \( |S| \) steps (for more details see Appendix D).

This implies that the fixpoint is unique and hence the statement of the proposition follows. \( \square \)

B Proofs for Section 4 (Finding the Best Explanation for an Observation)

Proposition 4.1 (Maximal probability for one observation). Let \((S, \Sigma, \delta)\) be an HMM and \( a \in \Sigma \) be an observation. With \( \mathcal{E}_{s_0,s}^a \), for \( s_0, s \in S \) we denote the probability for the most likely path in \( \Omega^n_{s_0} \), starting in state \( s_0 \) and ending in state \( s \), where \( a \) is the observation. Then we have:

\[
\mathcal{E}_{s_0,s}^a = \max_{\tilde{z} \in \Omega^n_{s_0}} P(Z = \tilde{z} \land Y = a) = \max_{z \in S} \left( \delta(s_0)(a, s), \max_{s' \in S} \delta(s_0)(\varepsilon, s') \cdot \mathcal{E}_{s',s}^a \right)
\]
Proof.

\[ E_{a_1, s} = \max_{\tilde{z} \in D_{s_0}^{n_0}} P(\tilde{z} = \tilde{z} \wedge Y = a) = \max_{\tilde{z} \in S^{n_0}} P(\tilde{z} = \tilde{z} \wedge Y = a) \]

\[ = \max_{\tilde{z} \in S^{n_0}} P(\tilde{z} = s_0 \varepsilon s \wedge Y = a), \delta(s_0)(a, s) \]

\[ \delta(s_0)(s', \varepsilon) \cdot P(\tilde{z} = \tilde{z} \wedge Y = a) \]

\[ = \max_{s' \in S} \delta(s_0)(s, s') \cdot \delta(s_0)(\varepsilon, s') \cdot P(\tilde{z} = \tilde{z} \wedge Y = a) \]

\[ = \max_{s' \in S} \delta(s_0)(a, s') \cdot E_{s', s} \]

\[ \sqrt{\text{Proposition 4.2 (Maximal probability for observation sequence). Let } (S, \Sigma, \delta) \text{ be an HMM and let } \tilde{y} = a_1 \ldots a_n = \tilde{y}_1 a_n \text{ be an observation sequence. Then } V_{s_0, s}^\tilde{y} \text{ denotes the maximum probability of observing } \tilde{y} \text{ and ending in state } s, \text{ more formally} \]

\[ V_{s_0, s}^\tilde{y} = \max_{\tilde{z} \in D_{s_0}^{n_0}} P(\tilde{z} = \tilde{y} \wedge Y = \tilde{y}) \]

For \( n = 0 \) we have \( V_{s_0, s}^\tilde{y} = 1 \) if \( s = s_0 \) and 0 otherwise. For \( n > 0 \):

\[ V_{s_0, s}^\tilde{y} = \max_{s' \in S} V_{s_0, s'}^{\tilde{y}_1} \cdot E_{s', s}^{a_n} \]

Proof.

\[ V_{s_0, s}^\tilde{y} = V_{s_0, s}^{a_1 \ldots a_n} \]

\[ = \max_{\tilde{z} \in D_{s_0}^{n_0}} P(Z = \tilde{z} \wedge Y = a_1 \ldots a_n) \]

\[ = \max_{s' \in S} \max_{\tilde{z}_1 \in D_{s_0}^{n_1} \tilde{z}_2 \in D_{s_0}^{n_2} \ldots \tilde{z}_n} P(Z = \tilde{z}_1 \wedge Y = a_1 \ldots a_{n-1}) \cdot P(Z = \tilde{z}_2 \wedge Y = a_n) \]

\[ = \max_{s' \in S} \max_{\tilde{z}_1 \in D_{s_0}^{n_1} \tilde{z}_2 \in D_{s_0}^{n_2} \ldots \tilde{z}_n} P(Z = \tilde{z}_1 \wedge Y = a_1 \ldots a_{n-1}) \cdot \delta(s_0)(a_1, s') \cdot \delta(s_0)(\varepsilon, s') \cdot P(\tilde{z}_2 = \tilde{z}_2 \wedge Y = a_n) \]

\[ = \max_{s' \in S} V_{s_0, s'}^{a_1 \ldots a_{n-1}} \cdot E_{s', s}^{a_n} \]

\[ \boxrule 11
In order to obtain the most likely path resulting in a given observation, we make use of the calculated highest probabilities. We do not only expect a sequence of states as explanation, but also the intermediate symbols or $\varepsilon$'s used when transitioning from state to state. This is necessary because a possible $\varepsilon$-transition can implicitly occur in an observation, creating an ambiguity problem when working out which exact transitions were taken at what time in the state sequence.

By unravelling the fixpoint equation of Prop. 4.1, we obtain the following construction, where $\mathcal{E}\text{Path}^a_{s_0,s} \in \Omega^l_0$ denotes the likeliest state sequence for a path starting in state $s_0$ and ending in state $s$ that produces the observation $a$ in its last transition. This is feasible whenever $\mathcal{E}^a_{s_0,s} > 0$.

$$\mathcal{E}\text{Path}^a_{s_0,s} = \begin{cases} s_0 a s & \text{if } \mathcal{E}^y_{s_0,s} = \delta(s_0)(a,s) \\ s_0 \varepsilon \mathcal{E}\text{Path}^a_{s',s} & \text{otherwise, with } s' = \operatorname{arg\ max}_{s' \in S} \delta(s_0)(\varepsilon,s') \cdot \mathcal{E}^y_{s',s} \end{cases}$$

Note that in the following the operator $\circ$ denotes concatenation and the tail function $T$ removes the first element of a given input sequence and returns the rest.

**Proposition B.1 (Likeliest state sequence for observation sequence).**

*Given an HMM $(S, \Sigma, \delta)$ and an observation sequence $\tilde{y} = a_1 \ldots a_n$. Whenever $V^\tilde{y}_{s_0,s} > 0$, the term $\psi_{s_0,s}^\tilde{y}$ denotes the likeliest state sequence starting in state $s_0$, ending in $s$, that explains $\tilde{y}$, in particular

$$\psi_{s_0,s}^\tilde{y} = \operatorname{arg\ max}_{\tilde{z} \in \Omega^l_{s_0}} P(Z = \tilde{z} \land Y = \tilde{y}).$$

It holds that $\psi_{s_0,s}^\varepsilon = s_0$. Furthermore, whenever $\tilde{y} = \tilde{y}_1 a_n$:

$$\psi_{s_0,s}^\tilde{y} = \psi_{s_0,s'}^\tilde{y}_1 \circ T(\mathcal{E}\text{Path}^a_{s',s}),$$

where $s' = \operatorname{arg\ max}_{s' \in S} V^\tilde{y}_1_{s_0,s'} \cdot \mathcal{E}^a_{s',s}$.

**Proof.**

$$\psi_{s_0,s}^\tilde{y} = \psi_{s_0,s}^{a_1 \ldots a_n} = \operatorname{arg\ max}_{\tilde{z} \in \Omega^l_{s_0}} P(Z = \tilde{z} \land Y = a_1 \ldots a_n)$$

$$= \operatorname{arg\ max}_{\tilde{z}_1 \in \Omega^l_{s_0} \land (\tilde{z}_1)=s} P(Z = \tilde{z}_1 \land Y = a_1 \ldots a_{n-1}) \circ T(\operatorname{arg\ max}_{\tilde{z}_2 \in \Omega^l_{s'} \land (\tilde{z}_2)=s} P(Z = \tilde{z}_2 \land Y = a_n))$$

$$= \psi_{s_0,s}^{a_1 \ldots a_{n-1}} \circ T(\mathcal{E}\text{Path}^a_{s',s}) = \psi_{s_0,s'}^\tilde{y}_1 \circ T(\mathcal{E}\text{Path}^a_{s',s})$$

where $s' = \operatorname{arg\ max}_{s' \in S} V^\tilde{y}_1_{s_0,s'} \cdot \mathcal{E}^a_{s',s}$ is the state where the maximum is reached.

$\square$
C  Proofs for Section 5 (Parameter Learning)

Proposition 5.1  Fix an HMM and an observation sequence \( \tilde{y} = a_1 \ldots a_n = a_1 \tilde{y}_1 \). Let \( t = (s, a, s') \) and define

\[
C^{\tilde{y}}_{s_0, t} = E_{s_0}[X_t \mid Y = \tilde{y}] \cdot P_{s_0}(Y = \tilde{y}).
\]

Then \( C^{\tilde{y}}_{s_0, t} = 0 \) and the following fixpoint equation holds: whenever \( a \in \Sigma \)

\[
C^{\tilde{y}}_{s_0, t} = \sum_{s_1 \in S} \delta(s_0)(a_1, s_1) \cdot C^{\tilde{y}}_{s_1, t} + \sum_{s_1 \in S} \delta(s_0)(\varepsilon, s_1) \cdot C^{\tilde{y}}_{s_1, t} +
\]

\[
\{s_0 = s \land a_1 = a \} \cdot \delta(s)(a, s') \cdot P_{s'}(Y = \tilde{y}_1)
\]

and whenever \( a = \varepsilon \) the last summand has to be replaced by \( \{s_0 = s\} \cdot \delta(s)(\varepsilon, s') \cdot P_{s'}(Y = \tilde{y}) \). We use the convention that \( [b] = 1 \) if \( b \) holds and \( [b] = 0 \) otherwise.

Proof. Note that by assumption \( P_{s_0}(Y = \tilde{y}) > 0 \). We compute

\[
C^{\tilde{y}}_{s_0, t} = E_{s_0}[X_t \mid Y = \tilde{y}] \cdot P_{s_0}(Y = \tilde{y})
\]

\[
= \sum_{\tilde{z} \in \Omega^n_0} X_{t}(\tilde{z}) \cdot P(Z = \tilde{z} \mid Y = \tilde{y}) \cdot P_{s_0}(Y = \tilde{y})
\]

\[
= \sum_{\tilde{z} \in \Omega^n_0} X_{t}(\tilde{z}) \cdot P(Z = \tilde{z} \land Y = \tilde{y})
\]

\[
= \sum_{\tilde{z} \in \Omega^n_0} X_{t}(\tilde{z}) \cdot P(Z = \tilde{z})
\]

\[
= \sum_{s_1 \in S} \sum_{\tilde{z}_1 \in \Omega^n_{s_1}} X_{t}(s_0 a_1 \tilde{z}_1) \cdot P(Z = s_0 a_1 \tilde{z}_1) +
\]

\[
\sum_{s_1 \in S} \sum_{\tilde{z}_1 \in \Omega^n_{s_1}} X_{t}(s_0 \varepsilon \tilde{z}_1) \cdot P(Z = s_0 \varepsilon \tilde{z}_1)
\]

\[
= \sum_{s_1 \in S} \sum_{\tilde{z}_1 \in \Omega^n_{s_1}} \left( X_{t}(\tilde{z}_1) + [s_0 = s \land a_1 = a \land s_1 = s'] \right) \cdot
\]

\[
\cdot \delta(s_0)(a_1, s_1) \cdot P(Z = \tilde{z}_1) +
\]

\[
\sum_{s_1 \in S} \sum_{\tilde{z}_1 \in \Omega^n_{s_1}} X_{t}(\tilde{z}_1) \cdot \delta(s_0)(\varepsilon, s_1) \cdot P(Z = \tilde{z}_1)
\]

\[
= \sum_{s_1 \in S} \delta(s_0)(a_1, s_1) \cdot \sum_{\tilde{z}_1 \in \Omega^n_{s_1}} X_{t}(\tilde{z}_1) \cdot P(Z = \tilde{z}_1) +
\]

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\[
\sum_{s_1 \in S} \delta(s_0)(\varepsilon, s_1) \cdot \sum_{\tilde{z}_1 \in \Omega^n_{s_1}} X_t(\tilde{z}_1) \cdot P(Z = \tilde{z}_1) + \\
\sum_{s_1 \in S} [s_0 = s \land a_1 = a \land s_1 = s'] \cdot \delta(s_0)(a_1, s_1) \cdot \sum_{\tilde{z}_1 \in \Omega^n_{s_1}} P(Z = \tilde{z}_1) + \\
= \sum_{s_1 \in S} \delta(s_0)(a_1, s_1) \cdot C_{s_1,t}^{y_1} + \sum_{s_1 \in S} \delta(s_0)(\varepsilon, s_1) \cdot C_{s_1,t}^{\tilde{y}} + \\
[s_0 = s \land a = a_1] \cdot \delta(s)(a, s') \cdot P_s(Y = \tilde{y}_1).
\]

\[\Box\]

**Theorem 5.3.** In our setting it holds that
\[
Q(\theta \mid \theta^t) = \sum_{s \in S} \sum_{t = (s, a, s') \in T_y} \log \delta(t \mid \theta) \cdot E_{s_0}[X_t \mid Y = \tilde{y}, \theta^t].
\]

The value \(Q(\theta \mid \theta^t)\) is maximal when the parameters \(\theta\) are as follows: for every transition \(t\) we set \(\delta(t \mid \theta)\) proportional to \(E_{s_0}[X_t \mid Y = \tilde{y}, \theta^t]\).

**Proof.** In this proof we write \(t_i(\tilde{z})\) for the \(i\)-th transition of \(\tilde{z}\), i.e., if \(\tilde{z} = s_0a_1s_1a_2 \ldots a_ms_m\), then \(t_i(\tilde{z}) = (s_{i-1}, a_i, s_i)\). Note that \(1 \leq i \leq (|z| - 1)/2\). Hence
\[
P(Z = \tilde{z} \mid \theta) = \prod_{i=1}^{(|z| - 1)/2} \delta(t_i(\tilde{z}) \mid \theta).
\]

Furthermore:
\[
Q(\theta \mid \theta^t) = E_{Z \mid Y, \theta^t}[\log P(Y, Z \mid \theta)]
\]
\[
= \sum_{\tilde{z} \in \Omega^n_{s_0}} \log P(Y = \tilde{y}, Z = \tilde{z} \mid \theta) \cdot P(Z = \tilde{z} \mid Y = \tilde{y}, \theta^t)
\]
\[
= \sum_{\tilde{z} \in \Omega^n_{s_0}} \log P(Z = \tilde{z} \mid \theta) \cdot P(Z = \tilde{z} \mid Y = \tilde{y}, \theta^t)
\]
\[
= \sum_{\tilde{z} \in \Omega^n_{s_0}} \log \prod_{i=1}^{(\frac{|z| - 1}{2})} \delta(t_i(\tilde{z}) \mid \theta) \cdot P(Z = \tilde{z} \mid Y = \tilde{y}, \theta^t)
\]
\[
= \sum_{\tilde{z} \in \Omega^n_{s_0}} \sum_{Y(\tilde{z}) = \tilde{y}} \log \delta(t_i(\tilde{z}) \mid \theta) \cdot P(Z = \tilde{z} \mid Y = \tilde{y}, \theta^t)
\]
\[
= \sum_{t \in T_y} \sum_{\tilde{z} \in \Omega^n_{s_0}} X_t(\tilde{z}) \cdot \log \delta(t \mid \theta) \cdot P(Z = \tilde{z} \mid Y = \tilde{y}, \theta^t)
\]
\[
= \sum_{t \in T_y} \sum_{\tilde{z} \in \Omega^n_{s_0}} X_t(\tilde{z}) \cdot P(Z = \tilde{z} \mid Y = \tilde{y}, \theta^t)
\]
\[
= \sum_{t \in T_y} \log \delta(t \mid \theta) \cdot \sum_{\tilde{z} \in \Omega^n_{s_0}} X_t(\tilde{z}) \cdot P(Z = \tilde{z} \mid Y = \tilde{y}, \theta^t)
\]
\[
E_{s_0}[X_t \mid Y = \tilde{y}, \theta^t]
\]

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= \sum_{s \in S} \sum_{t \in T} \log \delta(t \mid \theta) \cdot E_{\theta_0}[X_t \mid Y = \tilde{y}, \theta^t]$

The third-last equality is given by the following computation, where $A \subseteq B^*$ and $a_i$ denotes the $i$-th symbol of $a \in B^*$. In this case nested sums can be rewritten as follows, where $\#_b(a)$ stands for the number of occurrences of $b$ in $a$:

$$
\sum_{a \in A} |a| \sum_{i=1} T_a \cdot S_a = \sum_{a \in A} S_a \cdot \sum_{i=1} |a| T_a \cdot \sum_{b \in B} T_b \cdot \sum_{a_i = b} \#_i(a) \cdot T_a \cdot S_a
$$

The last value in the computation above can be maximized for each $s \in S$ independently and we obtain:

$$
\sum_{t \in T} \sum_{i=1} \log \delta(t \mid \theta) \cdot E_{\theta_0}[X_t \mid Y = \tilde{y}, \theta^t] = \sum_i (\log p_i) \cdot a_i
$$

This value is maximal if $p_i = \frac{a_i}{\sum_{i=1} a_i}$, which concludes the proof. This is a consequence of Gibbs’ inequality, which says that, given a probability function $p : I \to [0, 1]$ with $I$ finite, then for every other probability function $q : I \to [0, 1]$, we have that

$$
\sum_{i=1} a_i \cdot \log p_i \geq \sum_{i=1} a_i \cdot \log q_i.
$$

It is also related to the fact that Kullback-Leibler divergence is always non-negative. Since the result is easy to derive we prove it in Lemma C.1.

**Lemma C.1.** Let $a_i \geq 0$, $i \in \{1, \ldots, m\}$, be fixed. Let $p_i \in [0, 1]$ be unknown values such that $\sum_{i=1} p_i = 1$. Then

$$
\sum_{i=1}^m a_i \cdot \log p_i
$$

is maximal if $p_i = \frac{a_i}{\sum_{i=1} a_i}$.

**Proof.** We can assume that $\log = \ln$, since logarithms differ only by a constant factor. Furthermore we can see that in order to achieve the maximal value, $p_i$ must be strictly larger than 0 whenever $a_i > 0$ and $p_i = 0$ otherwise. (Remember the convention that $0 \cdot \log 0 = 0 \cdot (-\infty) = 0$.) Hence we can assume without loss of generality that $a_i > 0$ for all $i$.

Since $\sum_{i=1} p_i = 1$ we can replace $p_m$ by $1 - \sum_{i=1}^{m-1} p_i$ and obtain

$$
\sum_{i=1}^{m-1} a_i \cdot \ln p_i + a_m \cdot \ln \left(1 - \sum_{i=1}^{m-1} p_i \right)
$$

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Remembering that $\frac{d}{dx} \ln x = \frac{1}{x}$, we now compute the partial derivatives with respect to $p_j$ where $j \neq m$.

$$\frac{\partial}{\partial p_j} \left( \sum_{i=1}^{m-1} a_i \cdot \ln p_i + a_m \cdot \ln \left( 1 - \sum_{i=1}^{m-1} p_i \right) \right)$$

$$= \frac{a_j}{p_j} + a_m \cdot \frac{1}{1 - \sum_{i=1}^{m-1} p_i} \cdot (-1)$$

$$= \frac{a_j}{p_j} - \frac{a_m}{p_m} = \frac{a_j \cdot p_m - p_j \cdot a_m}{p_j \cdot p_m}$$

This equals 0 if $a_j \cdot p_m - p_j \cdot a_m = 0$. We sum up over all indices $j$ and get

$$0 = \sum_{j=1}^{m} (a_j \cdot p_m - p_j \cdot a_m) = p_m \sum_{j=1}^{m} a_j - a_m \sum_{j=1}^{m} p_j = p_m \sum_{j=1}^{m} a_j - a_m \sum_{j=1}^{m} p_j$$

This implies

$$p_m = \frac{a_m}{\sum_j a_j}$$

and by substitution, we obtain an analogous formula for all $p_j$. We can check that all conditions $a_j \cdot p_m - p_j \cdot a_m = 0$ are satisfied.

The maximum must be reached in the point where all derivatives are zero, from which the statement follows.

\[ \Box \]

### D Contractivity

The claims on contractivity made in the paper deserve further elaboration. We first define the notion of a contractive function.

**Definition D.1 (Contractive function).** Let $\mathbb{R}^W$ be the set of all functions from a set $W$ to $\mathbb{R}$. We use the supremum (or maximum) distance and define $d_{\text{sup}}(g_1, g_2) = \sup_{W \subseteq W} |g_1(W) - g_2(W)|$ for $g_1, g_2 : W \to \mathbb{R}$.

A function $F : \mathbb{R}^W \to \mathbb{R}^W$ is contractive whenever for all $g_1, g_2 \in \mathbb{R}^W$ it holds that $d_{\text{sup}}(F(g_1), F(g_2)) \leq q \cdot d_{\text{sup}}(g_1, g_2)$ for some $0 \leq q < 1$.

We say that $F$ is contractive after $k$ iterations if $F^k$ is contractive.

It is well-known from the Banach fixpoint theorem that contractive functions over complete metric spaces have unique fixpoints and $\mathbb{R}^W$ with the sup-metric is complete. Furthermore any sequence $(g_i)_{i \in \mathbb{N}}$ with $g_{i+1} = F(g_i)$ converges to this fixpoint. Now, every fixpoint of $F$ is a fixpoint of $F^k$ and vice versa. The latter direction holds, since fixpoint iteration for $F$ from a fixpoint $x$ of $F^k$ (with $F^k(x) = x$) clearly converges again to $x$. Hence a function $F$ that is contractive after $k$ iterations also has a unique fixpoint and enjoys the same convergence property (although with a potentially slower convergence rate).

We now argue why the fixpoint functions that we consider are contractive after a certain number of iterations.
The fixpoint equation systems set up in Sct. 3 (proof of Prop. 3.1) and Sct. 5.1 are over a set $W$ of variables of the form $W_\tilde{y}^s$, where $s \in S$ and $\tilde{y} \in \Sigma^*$ is the suffix of a given word $\tilde{y} \in \Sigma^*$ with $n = |\tilde{y}|$. (Or alternatively the variables are of the form $S^n_s$, see the proof of Prop. 3.1 leading to an analogous argument.) We define $o(W_\tilde{y}^s) = \tilde{y}$. Note that $W$ is finite, since the state space $S$ is finite.

The corresponding fixpoint function is a monotone function $F: \mathbb{R}^W \to \mathbb{R}^W$ where, for $g: W \to \mathbb{R}$:

$$F(g)(W) = \sum_{W' \in W} p_{W,W'} \cdot g(W') + D_W,$$

where $p_{W,W'} \in [0,1]$ such that for each $W \in W$ we have $\sum_{W' \in W} p_{W,W'} \leq 1$ and $D_W$ is a non-negative constant. Furthermore $p_{W,W'} > 0$ implies $|o(W)| \geq |o(W')|$. In addition we can assume that $o(W) = \varepsilon$ implies $p_{W,W'} = 0$ (and hence $F(g)(W) = D_W$ is a constant).

Such functions are clearly non-expansive (which means that the contractivity requirement holds for $q = 1$) but not necessarily contractive.

However, we know that the probabilities $p_{W,W'}$ are transition probabilities and the length of the observed sequence decreases if one takes a transition that is labelled with an observable symbol. Due to the requirement of Prop. 3.1 we know that each state has a path of non-zero probability that contains such an observation. For each state $s \in S$ we consider the minimum length of such a path and we take the maximum over all these minimums and obtain $k$. Then we know that the fixpoint equation associated with $F^k$ is of the same form as for $F$ above (see (1)) and additionally for each $W \in W$

- there exists $W' \in W$ with $p_{W,W'} > 0$ and $|o(W)| > |o(W')|$ (if we take a transition with the next label to observe, reducing the length of the observation sequence) or
- $\sum_{W' \in W} p_{W,W'} < 1$ (if a state has an outgoing transition with a label that does not match the next observation).

This means that either the second condition holds after at most $n \cdot k$ iterations or we reach the last observation of $\tilde{y}$ on a path of non-zero probability of length at most $n \cdot k$. The latter means that the term for $F^{n-k}(g)(W)$ contains – multiplied with a non-zero probability – a variable $W'$ with $o(W') = \varepsilon$, for which $F(g)(W')$ is constant. That is, after $m+1$ iterations the corresponding fixpoint equation (1) satisfies $q_W := \sum_{W' \in W} p_{W,W'} < 1$ for each $W \in W$. Then we have, given $g_1, g_2: W \to \mathbb{R}$:

$$d_{\sup}(F^{m+1}(g_1), F^{m+1}(g_2)) = \max_{W \in W} \left| \left( \sum_{W' \in W} p_{W,W'} \cdot g_1(W') + D_W \right) - \left( \sum_{W' \in W} p_{W,W'} \cdot g_2(W') + D_W \right) \right|$$

$$\leq \max_{W \in W} \sum_{W' \in W} p_{W,W'} \cdot |g_1(W') - g_2(W')|$$

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\[
\begin{align*}
\leq & \max_{W \in \mathcal{W}} \left( \sum_{W' \in \mathcal{W}} p_{W,W'} \right) \cdot \max_{W' \in \mathcal{W}} |g_1(W') - g_2(W')| \\
\leq & \left( \max_{W \in \mathcal{W}} q_w \right) \cdot \max_{W' \in \mathcal{W}} |g_1(W') - g_2(W')| = q \cdot d_{\text{sup}}(g_1, g_2)
\end{align*}
\]

The first inequality uses the fact that \( |\sum_{i \in I} a_i| \leq \sum_{i \in I} |a_i| \), while the second inequality holds since \( \sum_{i \in I} p_i \cdot a_i \leq (\sum_{i \in I} p_i) \cdot \max_{i \in I} a_i \).

Since \( q < 1 \) we have contractivity after \( m + 1 \) iterations.

In fact, the arguments are similar to the setting of absorbing Markov chains [7], where the absorption property is used to guarantee unique solutions.