A special exact spherically symmetric solution in f(T) gravity theories

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1 Introduction

Common consensus in the scientific circle is that the characterization of the gravitational field, powered by Einstein General Relativity (GR), is inaccurate at scales of magnitude of the Planck’s length. The spacetime frame of such characterization should be clarified by a quantum regime. In the opposite extreme of the physical phenomena,
GR also faces a fascinating problem linked to the late cosmic speed-up stage of the universe. According to the previous reasons, and for other defects, GR has been the topic of many modifications which have attempted to supply a more satisfactory qualification of the gravitational field in the above aforementioned extreme regimes. Among the most important modified gravitational theories is the one called “$f(T)$ gravity”, which is a theory constructed in a spacetime having absolute parallelism (cf., [1–8]).

Many of $f(T)$ gravity theories have been analyzed in [9–13]. It has been suggested that $f(T)$ gravity theory is not dynamically synonymous with the teleparallel equivalent of GR Lagrangian through conformal transformation [14]. Many observational restrictions have been studied [15–18]. Large-scale structure in $f(T)$ gravity theory has been analyzed [19]. Perturbations in the area of cosmology in $f(T)$ gravity have been demonstrated [20–30]. Birkhoff’s theorem, in $f(T)$ gravity has been studied [4,5,31]. Stationary, spherical symmetry solutions have been derived for $f(T)$ theories [32]. Relativistic stars and the cosmic expansion have been investigated [33,34].

$f(T)$ gravity theories have engaged many concerns: It has been indicated that the Lagrangian and the equations of motion are not invariant under local Lorentz transformations [35]. The reasons for such phenomena has been explained in [36].

The equations of motion of $f(T)$ theories differ from those of $f(R)$ theories [37–54], because they are of the second order instead of the fourth order as in $f(R)$ theories. Such property has been believed as an indicator which shows that the theory might be of much interest. The non locality of such theories indicate that $f(T)$ seems to comprise more degrees of freedom.

The target of this study is to find an analytic vacuum spherically symmetric solution, within the framework of higher-torsion theories, i.e., for $f(T)$ gravitational theory. In Sect. 2, a brief review is presented of the covariant formalism for the gravitational energy-momentum, described by the pair $(\vartheta^\alpha, \Gamma_\alpha^{\beta\gamma})$. In Sect. 3, a brief survey of the $f(T)$ gravitational theory is provided. A non-diagonal spherically symmetric tetrad field, with four unknown functions of the radial coordinate $r$, is presented. The application of such tetrad field to the equations of motion of $f(T)$ is demonstrated in Sect. 4. Also, in Sect. 4, an analytic vacuum spherically symmetric solution with one constant of integration is obtained. In Sect. 5, we calculate the energy associated with this solution in order to understand the physical meaning of the constant. This calculation uses traditional computation employing the Riemannian connection: $\Gamma_\alpha^{\beta}$. The final section is devoted to discussions of these results.

**Notation** By convention, we denote the exterior vector product by $\wedge$, the interior of a vector product $\xi$ and a p-form $\Psi$ by $\xi \lrcorner \Psi$. The vector basis dual to the frame 1-forms $\vartheta^\alpha$ is denoted by $e_\alpha$ and they satisfy $e_\alpha \lrcorner \vartheta^\beta = \delta_\alpha^\beta$. Using local coordinates $x^i$, we have $\vartheta^\alpha = h^\alpha_\beta dx^\beta$ and $e_\alpha = h_\alpha^\beta \partial_i$ where $h^\alpha_\beta$ and $h_\alpha^\beta$ are the covariant and contravariant components of the tetrad field. We define the volume 4-form by

$$\eta \stackrel{\text{def}}{=} \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3. \quad (1)$$

Furthermore, with the help of the interior product we define

$$\eta_\alpha \stackrel{\text{def}}{=} e_\alpha \lrcorner \eta = \frac{1}{3!} \epsilon_{\alpha\beta\gamma\delta} \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta = \ast \vartheta_\alpha, \wedge$$

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1 where $\epsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\epsilon_{0123} = 1$.

$$\eta_{\alpha\beta} \overset{\text{def.}}{=} e_\beta \eta_\alpha = \frac{1}{2!} \epsilon_{\alpha\beta\gamma\delta} \partial^\gamma \wedge \partial^\delta = \ast (\partial_\alpha \wedge \partial_\beta),$$

$$\eta_{\alpha\beta\gamma} \overset{\text{def.}}{=} e_\gamma \eta_{\alpha\beta} = \frac{1}{1!} \epsilon_{\alpha\beta\gamma\delta} \partial^\delta = \ast (\partial_\alpha \wedge \partial_\beta \wedge \partial_\gamma),$$

which are bases for 3-, 2- and 1-forms respectively. Finally,

$$\eta_{\alpha\beta\mu\nu} \overset{\text{def.}}{=} e_\nu \eta_{\alpha\beta\mu} = e_\nu e_\mu e_\beta e_\alpha \eta = \ast (\partial_\alpha \wedge \partial_\beta \wedge \partial_\mu \wedge \partial_\nu),$$

is the Levi-Civita tensor density. The $\eta$-forms satisfy the useful identities:

$$\partial^\beta \wedge \eta_\alpha = \delta_\alpha^\beta \eta, \quad \partial^\beta \wedge \eta_{\mu\nu} = \delta_\nu^\beta \eta_\mu - \delta_\mu^\beta \eta_\nu,$$

$$\partial^\beta \wedge \eta_{\alpha\mu\nu} = \delta_\nu^\beta \eta_{\alpha\mu} + \delta_\mu^\beta \eta_{\alpha\nu} + \delta_\alpha^\beta \eta_{\mu\nu},$$

$$\partial^\beta \wedge \eta_{\alpha\gamma\mu\nu} = \delta_\nu^\beta \eta_{\alpha\gamma\mu} - \delta_\mu^\beta \eta_{\alpha\gamma\nu} + \delta_\gamma^\beta \eta_{\alpha\mu\nu} - \delta_\alpha^\beta \eta_{\gamma\mu\nu}. \quad (2)$$

2 Brief review of teleparallel gravity

Teleparallel geometry can be viewed as a gauge theory of translation [55–65]. The coframe $\theta^\alpha$ plays the role of the gauge translational potential of the gravitational field. GR can be reformulated as a teleparallel theory. Geometrically, teleparallel gravity can be considered as a special case of the metric-affine gravity (MAG) in which $\theta^\alpha$ and the local Lorentz connection are subject to the distant parallelism constraint $R_{\alpha\beta} = 0$ (cf., [66–75]). In this geometry the torsion 2-form

$$T^\alpha = D\theta^\alpha = d\theta^\alpha + \Gamma^{\alpha}_{\beta\lambda} \wedge \theta^\beta = \frac{1}{2} T_{\mu\nu}^\alpha \partial^\mu \wedge \partial^\nu = \frac{1}{2} T_{ij}^\alpha dx^i \wedge dx^j, \quad (3)$$

arises as the gravitational gauge field strength, $\Gamma^{\alpha}_{\beta\lambda}$ being the Weitzenböck 1-form connection, $d$ in front of the coframe $\theta^\alpha$ the exterior derivative and $D$ the covariant derivative associated with $\Gamma^{\alpha}_{\beta\lambda}$. The torsion form $T^\alpha$ can be decomposed into three irreducible pieces [57,76]: the tensor, the trace and the axial trace given respectively by

$$T^\alpha \overset{\text{(1)}}{=} T^\alpha - T^\alpha \overset{\text{(2)}}{=} - T^\alpha - T^\alpha \overset{\text{(3)}}{=} T^\alpha, \quad \text{with}$$

$$T^\alpha \overset{\text{(2)}}{=} \frac{1}{3} \theta^\alpha \wedge T, \quad \text{where} \quad T = (e_\beta \wedge T^\beta), e_\alpha \wedge T = T_{\mu\alpha}^\mu, \quad \text{vector trace of torsion} \quad (4)$$

$$T^\alpha \overset{\text{(3)}}{=} \frac{1}{3} e_\alpha \wedge P, \quad \text{with} \quad P = (\theta^\beta \wedge T^\beta), e_\alpha \wedge P = T^{\mu\nu\lambda} \eta_{\mu\nu\lambda\alpha}, \quad \text{axial trace of torsion.} \quad (4)$$

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1 Refer to Hodge dual operator.

‡ Refers to Hodge dual operator.
The Lagrangian of the teleparallel equivalent of GR has the form [76]\(^2\)

\[ V(\vartheta^\alpha, \Gamma_\beta^\alpha) = -\frac{1}{2\kappa} T^\alpha \wedge \ast \left( (1) T_\alpha - 2(2) T_\alpha - \frac{1}{2}(3) T_\alpha \right), \]  

(5)

where \(\kappa = 8\pi G/c^3\), \(G\) is the Newtonian constant, \(c\) is the speed of light and the metric \(g_{\alpha\beta}\) is assumed to be flat Minkowski metric \(g_{\alpha\beta} = O_{\alpha\beta} = \text{diag}(+1, -1, -1, -1)\), that is used to raise and lower local frame (Greek) indices. In accordance with the general Lagrange-Noether scheme [2,63], one can derive from Eq. (5) the translational momentum 2-form \(H_\alpha\) and the canonical energy-momentum 3-form \(E_\alpha\) are given respectively by:

\[ H_\alpha = -\frac{\partial V}{\partial T^\alpha} = \frac{1}{\kappa} \ast \left( (1) T_\alpha - 2(2) T_\alpha - \frac{1}{2}(3) T_\alpha \right), \]  

(6)

\[ E_\alpha \overset{\text{def.}}{=} \frac{\partial V}{\partial \vartheta_\alpha} = e_\alpha] V + (e_\alpha] T^\beta) \wedge H_\beta. \]  

(7)

Due to geometric identities [82], the Lagrangian of Eq. (5) can be rewritten as

\[ V = -\frac{1}{2} T^\alpha \wedge H_\alpha. \]  

(8)

The variation of the total action with respect to the coframe gives the equations of motion in the form:

\[ DH_\alpha - E_\alpha = \Sigma_\alpha, \text{ where } \Sigma_\alpha \overset{\text{def.}}{=} \frac{\delta L_{\text{matter}}}{\delta \vartheta^\alpha}, \]  

(9)

is the canonical energy-momentum current 3-form which is considered as the source of matter. The presence of the connection form \(\Gamma_\beta^\alpha\) in Eq. (3) plays an important regularizing role due to the following:

First: The theory becomes explicitly covariant under the local Lorentz transformations of the coframe, i.e. the special Lagrangian given by Eq. (5) is invariant under the change of variables

\[ \vartheta^\alpha = L^\alpha_\beta \vartheta^\beta, \quad \Gamma'^\beta_\alpha = \left(L^{-1}\right)^\mu_\alpha \Gamma^\mu_\mu \Gamma_\mu^\beta L^\beta_\nu + L^\beta_\gamma d\left(L^{-1}\right)^\gamma_\alpha. \]  

(10)

When \(\Gamma_{\alpha}^{\beta} = 0\), which is the tetrad gravity, the Lagrangian (5) is only quasi-invariant, i.e., it changes by a total divergence.

Second: \(\Gamma_{\alpha}^{\beta}\) plays an important role in the teleparallel framework. This role represents the inertial effects which arise from the choice of the reference system [83]. The contributions of this inertial in many cases lead to a non-physical results for

\(^2\) The effect of adding the non-Riemannian parity odd pseudoscalar curvature to the Hilbert–Einstein–Cartan scalar curvature was studied by many authors (cf., [77–81] and references therein).
the total energy of the system. Therefore, the role of the teleparallel connection is to separate the inertial contribution from the truly gravitational one. Since the teleparallel curvature is zero, the connection is a “pure gauge”, that is

\[ \Gamma_\alpha^\beta \overset{\text{def.}}{=} \left( \Lambda^{-1} \right)^\beta_\gamma d \left( \Lambda^\gamma_\alpha \right). \] (11)

The Weitzenböck connection form \( \Gamma_\alpha^\beta \) always has the form of Eq. (11).

The translational momentum of the Lagrangian (5) has the form [76]

\[ \tilde{H}_\alpha = \frac{1}{2k} \tilde{\Gamma}_\beta^\gamma \wedge \eta_\alpha^\beta, \] where \( \Gamma_\alpha^\beta \overset{\text{def.}}{=} \Gamma_\alpha^\beta - K_\alpha^\beta, \] (12)

with \( \Gamma_\alpha^\beta \) is the purely Riemannian connection form and \( K^{\mu\nu} \) is the contortion 1-form which is related to the torsion through the relation

\[ T^\alpha \overset{\text{def.}}{=} K_\alpha^\beta \wedge \partial^\beta. \] (13)

3 Brief review of f(T)

In a spacetime having absolute parallelism the parallel vector field \( h_{\mu}^i \) [84] defines the nonsymmetric affine connection:

\[ \Gamma^i_{jk} \overset{\text{def.}}{=} h_{\mu}^i h_{jk}^\mu, \] (14)

where \( h_{\mu i, j} = \partial_j h_{\mu i} \). The curvature tensor of \( \Gamma^i_{jk} \) vanishes identically. The metric tensor \( g_{ij} \) is defined by

\[ g_{ij} \overset{\text{def.}}{=} O_{\mu\nu} h_{i}^\mu h_{j}^\nu. \] (15)

Define the torsion components by:

\[ T^i_{jk} \overset{\text{def.}}{=} \Gamma^i_{kj} - \Gamma^i_{jk} = h_{\mu}^i \left( \partial_j h_{\mu k} - \partial_k h_{\mu j} \right), \]

and the contortion components by:

\[ K^{ij}_{\ k} \overset{\text{def.}}{=} -\frac{1}{2} \left( T^{i}_{jk} - T^{j}_{ik} - T^{ij}_{\ k} \right). \] (16)

\[ \text{3 We use the Latin indices } i, j, \ldots \text{ for local holonomic spacetime coordinates and the Greek indices } \alpha, \beta, \ldots \text{ for the (co)frame components.} \]
Here the contortion equals the difference between Weitzenböck and Levi-Civita connections, i.e., \( K^i_{jk} = \Gamma^i_{jk} - \{^i_{jk}\} \).

The tensor \( S_i^j{}^k \) is defined as

\[
S_i^j{}^k \defby \frac{1}{2} \left( K^j_{ik} + \delta^j_i T^a_{ak} - \delta^k_i T^a_{aj} \right). 
\]  

(17)

The torsion scalar is defined as

\[
T \defby T^i{}_{jk} S_i^j{}^k. 
\]  

(18)

As was done in the development of the \( f(R) \) theory [37–54], one can define the action of \( f(T) \) theory as

\[
\mathcal{L}(h^\mu{}^i, \Phi_A) = \int d^4x h \left[ \frac{1}{16\pi} f(T) + \mathcal{L}_{\text{Matter}}(\Phi_A) \right], 
\]

where \( h = \sqrt{-g} = \text{det}(h^\mu{}^i) \).

(19)

We have assumed that the units such that \( G = c = 1 \). The variables \( \Phi_A \) are the matter fields.

Considering the Eq. (19) as a function of the field variables \( h^\mu{}^i, \Phi_A \), and equating the variation of the function with respect to the tetrad field \( h^\mu{}^i \) to zero, one can obtain the following equation of motion [9]:

\[
S_i^a{}^j T_a{}^\nu f(T)_{TT} + \left[ h^{-1} h^\mu{}^i \delta_b \left( h h^a{}^\mu S_a{}^{bij} - T^a{}_{bi} S_a{}^{jb} \right) - T^a{}_{bi} S_a{}^{jb} \right] f(T)_T - \frac{1}{4} \delta_i^j f(T) = 4\pi T_i^j, 
\]

where \( T_a = \frac{\partial T}{\partial x^a}, f(T)_T = \frac{\partial f(T)}{\partial T}, f(T)_{TT} = \frac{\partial^2 f(T)}{\partial T^2} \) and \( T^\nu_\mu \) is the energy momentum tensor. In this work we are interested in the vacuum case of \( f(T) \) theory, i.e., \( T_i^j = 0 \).

4 Spherically symmetric solution in f(T) gravity theory

Assume that the manifold possesses a stationary and spherical symmetry with local Lorentz transformations the tetrad field \( (h^\mu{}^a) \) has the form:

\[
(h^\mu{}^a) = \begin{pmatrix}
A(r) & B(r) & 0 & 0 \\
C(r) \sin \theta \cos \phi & D(r) \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
C(r) \sin \theta \sin \phi & D(r) \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
C(r) \cos \theta & D(r) \cos \theta & -r \sin \theta & 0
\end{pmatrix} 
\]

(21)

where \( A(r), B(r), C(r) \) and \( D(r) \) are four unknown functions of the radial coordinate, \( r \).
Using Eqs. (19) and (21), one can obtain \( h = \det(h^a_i) = r^2 \sin \theta (AD - BC)^4 \): By the use of Eqs. (16) and (17), we obtain the torsion scalar and its derivatives in terms of \( r \) as:

\[
T(r) = -\frac{4 (rA'[(D - 1)A - BC] + rCC' - \frac{1}{2}[(D - 1)A - C(B - 1)][(D - 1)A - C(B + 1)])}{r^2(AD - BC)^2},
\]

where \( A' = \frac{\partial A(r)}{\partial r}, B' = \frac{\partial B(r)}{\partial r}, C' = \frac{\partial C(r)}{\partial r} \) and \( D' = \frac{\partial D(r)}{\partial r} \).

\[
T'(r) = \frac{\partial T(r)}{\partial r} = -\frac{4}{r^3(AD - BC)^3} \left( r^2(AD - BC)[[(D - 1)A - BC]A'' + CC''] \right.
\]

\[
- r^2[(D - 1)DA - BC(D + 1)]A^2 + 4rA' \left[ rC'(B(D - 2)A - C(B^2 + 2D)) \right.
\]

\[
- r[AD' - CB'][(D - 2)A - BC] - D[(D - 1)A^2 - ABC + C^2] \right) + r^2(AD + BC)C^2
\]

\[
- 2rC' \left[ rACD' - rC^2B' - B \right]
\]

\[
+ (AD - BC)(D - 1)A - (B - 1)C][(D - 1)A - (B + 1)C].
\]

(22)

The equations of motion (20) can be rewritten in the form

\[
4\pi T_0^0 = -\frac{f_{TT}T'[(D - 1)A^2 - ABC + C^2]}{r^2(AD - BC)^2}
\]

\[
- \frac{f_T}{r^2(AD - BC)^3} \left( rA'\{A(D - 1)(DA - 2BC) + C^2(B^2 - D)\} \right.
\]

\[
- rC'(A^2B + BC^2 - 2ACD) + r(AD' - CB')(A^2 - C^2)
\]

\[
+ (AD - BC)((D - 1)A^2 - ABC + C^2)] + \frac{f}{4}
\]

(23)

\[
4\pi T_1^0 = \frac{4f_{TT}T'}{r(AD - BC)^2} - \frac{f_T[AB - DC](DA' - CB' + AD' - BC')}{r(AD - BC)^3},
\]

(24)

\[
4\pi T_1^1 = \frac{f_T[C^2 - ABC + (D - 1)A^2 + 2rCC' + r[(D - 2)A - BC]A']}{r^2(AD - BC)^2} + \frac{f}{4}
\]

(25)

\[
4\pi T_2^2 = 4\pi T_3^3 = \frac{f_{TT}T'}{2r(AD - BC)^2}
\]

\[
+ \frac{f_T}{2r^2(AD - BC)^3} \left( r^2(AD - BC)[AA'' - CC''] - r^2BCA^2 \right.
\]

\[
- rA' \left[ rC'(BA + CD) - rA(AD' - CB') + A^2D(-2D + 3) \right.
\]

\[
- 4BCA(1 - D) - (D + 2B^2)C^2 \right)
\]

(4) We will denote \( A, B, \ldots \) instead of \( A(r), A(r), \ldots \)
\[ r^2ADC^2 - rC\left(rACD' + rC^2B' + A^2B - 4ACD - 3BC^2\right) + r(AD' - CB')\left(A^2 - C^2\right) - (AD - BC)\left[(D - 1)A - (B - 1)C\right][(D - 1)A - (B + 1)C] + \frac{f}{4}. \quad (26) \]

From Eqs. (22)–(26), it is clear that \( AD \neq BC \).

To find an exact solution of the Eqs. (23)–(26), we impose the following constraints:

\[ T' = 0, \]
\[ DA' - CB' + AD' - BC' = 0, \]
\[ C^2 - ABC + (D - 1)A^2 + 2rCC' + r[(D - 2)A - BC]A' = 0, \]
\[ rA'[A(D - 1)(DA - 2BC) + C^2(B^2 - D)] - rC'(A^2B + BC^2 - 2ACD) + r(AD' - CB')(A^2 - C^2) + (AD - BC)[(D - 1)A^2 - ABC + C^2] = 0. \quad (27) \]

As a consequence of Eq. (27), we get

\[ \left(r^2(AD - BC)[AA'' - CC''] - r^2BCA^2\right) - rA'\left[rC'(BA + CD) - rA(AD' - CB') + A^2D(-2D + 3) - 4BCA(1 - D) - (D + 2B^2)C^2\right] + r^2ADC^2 - rC'\left(rACD' + rC^2B' + A^2B - 4ACD - 3BC^2\right) + r(AD' - CB')(A^2 - C^2) - (AD - BC)[(D - 1)A - (B - 1)C][(D - 1)A - (B + 1)C], \quad (28) \]

which does not represent any new constraint. Equations (27) are four differential equations in four unknown functions \( A, B, C \) and \( D \). The only solution of these equations is the following

\[ A = 1 - \frac{c_1}{r}, \quad B = \frac{c_1}{r(1 - \frac{c_1}{r})}, \quad C = c_1/r, \quad D = \frac{1 - \frac{c_1}{r}}{1 - \frac{2c_1}{r}}, \quad (29) \]

where \( c_1 \) is a constant of integration. Substituting from (29) into Eq. (22) we get a vanishing value of the scalar torsion which satisfy the second equation of Eq. (22). Therefore, solution (29) is an exact vacuum solution of Eqs. (23)–(26), provided that

\[ f(0) = 0, \quad f_T(0) \neq 0, \quad f_{TT} \neq 0. \quad (30) \]
To understand the physical meaning of the constant of integration appearing in solution (29), \(c_1\), we are going to discuss the physics related to this solution by calculating the energy associated with the tetrad field (21) after using solution (29).

5 Total energy

In this section, we use the solution (29) to calculate the total energy using the translational momentum given by Eq. (12). The coframe of this solution, i.e., \(\theta^\alpha = (h^\delta_i) dx^i\), has the form

\[
\begin{align*}
\theta^0 &= Bdr + Adt, \quad \theta^1 = \sin \theta \cos \phi [Ddr + Cdt] + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi, \\
\theta^2 &= \sin \theta \sin \phi [Ddr + Cdt] + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi, \\
\theta^3 &= \cos \theta [Ddr + Cdt] - r \sin \theta d\theta.
\end{align*}
\] (31)

The coframe (31) has the following non-vanishing components of the Riemannian connection

\[
\begin{align*}
\tilde{\Gamma}^0_1 &= \tfrac{[Bdr + Adt]A' - [Ddr + Cdt]C'}{AD - CB} \sin \theta \cos \phi + C[\sin \theta \sin \phi d\phi - \cos \theta \cos \phi d\theta], \\
\tilde{\Gamma}^1_2 &= \tfrac{[Bdr + Adt]A' - [Ddr + Cdt]C'}{AD - CB} \sin \theta \sin \phi - C[\sin \theta \cos \phi d\phi + \cos \theta \sin \phi d\theta], \\
\tilde{\Gamma}^2_3 &= \tfrac{[Bdr + Adt]A' - [Ddr + Cdt]C'}{AD - CB} \cos \theta + C \sin \theta d\theta, \\
\tilde{\Gamma}^3_1 &= \tfrac{[AD - BC - A]}{AD - CB} \sin^2 \theta d\phi, \\
\tilde{\Gamma}^3_2 &= \tfrac{[AD - BC - A]}{AD - CB} \sin \theta \cos \phi \cos \theta d\phi + \sin \phi d\theta.
\end{align*}
\] (32)

The non-vanishing components of the superpotential 2-form are thus

\[
\begin{align*}
\tilde{H}_0 &= \frac{c_1}{4\pi} \sin \theta (d\theta \wedge d\phi), \quad \tilde{H}_1 = -\frac{c_1}{4\pi} \sin^2 \theta \cos \phi (d\theta \wedge d\phi), \\
\tilde{H}_2 &= -\frac{c_1}{4\pi} \sin \theta \cos \phi (d\theta \wedge d\phi), \quad \tilde{H}_3 = -\frac{c_1}{4\pi} \sin \theta \cos \phi (d\theta \wedge d\phi).
\end{align*}
\] (33)

Computing the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface \(\partial S = \{r = R, \theta, \phi\}\), we obtain

\[
\tilde{E} = \int_{\partial S} \tilde{H}_0 = c_1,
\] (34)
and the spatial momentum

$$\tilde{\rho}_a = \int_{\partial S} \tilde{H}_a = 0, \quad \alpha = 1, 2, 3.$$  \hfill (35)

Equation (34) shows in a clear way that the constant of integration $c_1$ is related to the gravitational mass $M$ [85].

6 Main results and discussion

The $f(T)$ gravitational theory has been considered in the vacuum case. The equations of motion have been applied to a non-diagonal spherically symmetric tetrad field having four unknown functions of the radial coordinate. Four nonlinear differential equations are obtained. To solve these differential equations, we have imposed four constraints (in four unknown functions). The only solution that is compatible with these constraints contains one constant of integration. Therefore, an exact vacuum spherically symmetric solution to the field equations of $f(T)$ gravitational theory has been derived. This solution has a vanishing scalar torsion and is satisfying the equations of motion of $f(T)$, provided that Eq. (30) holds. To understand the physical meaning of the constant of integration, we calculate its associated energy. It has been shown that it is related to the gravitational mass.

To understand the construction of the derived solution, let us rewrite it in the following form:

$$(h^\mu_i) = (\Lambda^\mu_v)_1 (\Lambda^v_\lambda)_2 (h^\lambda_i)_d,$$

where

$$\begin{align*}
(\Lambda^\mu_v)_1 &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\
0 & \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\
0 & \cos \theta & -\sin \theta & 0
\end{pmatrix}, \\
(\Lambda^\mu_v)_2 &= \begin{pmatrix}
\frac{1 - M}{r} & \frac{M}{r(1 - \frac{2M}{r})} & 0 & 0 \\
\frac{M}{r(1 - \frac{2M}{r})} & \frac{1 - M}{r} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \\
(h^\mu_i)_d &= \begin{pmatrix}
\sqrt{1 - \frac{2M}{r}} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & r & 0 \\
0 & 0 & 0 & r \sin \theta
\end{pmatrix}.
\end{align*}$$  \hfill (36)
The matrices (36) show that the exact solution consists of a diagonal solution given by \((h^\mu_i)_d\) and two local Lorentz transformations, i.e., \((\Lambda^\mu_i)_1\) and \((\Lambda^\mu_i)_2\). The tetrad \((\Lambda^\mu_i)_1(h^\nu_i)_d\) has been applied to \(f(T)\) theory [86,87] and no exact solution has been obtained but only an asymptotic one, that represents the Schwarzschild-Ads and traversable wormhole solution. The extra local Lorentz transformation \((\Lambda^\mu_i)_2\) plays a key role in adjusting the solution to be exact to the \(f(T)\) gravitational theory.

Maluf et al. [88] have used a tetrad that can be decomposed into two local Lorentz transformations and a diagonal tetrad. This tetrad has given the Schwarzschild space-time and it has been shown that the energy content related to this tetrad is vanishing. Maluf et al. [88] have discussed this result as a freely falling test body in this space-time. However, Obukhov et al. [76] have shown that there is inertia that contributes to the gravitational mass. This result produces the vanishing energy. Here, in this study, we have obtained a solution similar to that studied by Maluf et al. [88] and Lucas et al. [76]. Our’s solution is similar in the sense that it consists of two local Lorentz transformations and a diagonal tetrad. There is no inertia contributing to the physics as shown by Eq. (34).

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