Stable lattices in modular Galois representations
and Hida deformation

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Abstract
In this paper, we discuss the variation of the numbers of the isomor-
phic classes of stable lattices when the weight and the level vary in a
Hida deformation by using the Kubota-Leopoldt $p$-adic $L$-function. Then
in Corollary 1.7, we give a sufficient condition for the numbers of the
isomorphic classes of stable lattices in Hida deformation to be infinite.

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1 Introduction

Fix a prime $p \geq 3$. We denote by $\mathbb{Q}(\mu_{p^{\infty}})$ the extension of the field of rational
numbers $\mathbb{Q}$ obtained by adjoining all $p$-power roots of unity. We fix a complex
embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and a $p$-adic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ of an algebraic closure
$\overline{\mathbb{Q}}$ throughout the paper, where $\mathbb{C}$ is the field of complex numbers and $\overline{\mathbb{Q}}_p$
an algebraic closure of the field $\mathbb{Q}_p$ of $p$-adic numbers. We fix a topological
generator $u$ of $1 + p\mathbb{Z}_p$ throughout the paper. We denote by $\mathbb{Q}_{\infty}$ the cyclotomic
$\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let

$$\chi_{\text{cyc}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \text{Gal}(\mathbb{Q}(\mu_{p^{\infty}})/\mathbb{Q}) \cong \mathbb{Z}_p^\times$$

be the $p$-adic cyclotomic character. Thus $\chi_{\text{cyc}}$ is decomposed into the product
$\chi_{\text{cyc}} = \kappa_{\text{cyc}} \omega$ where

$$\kappa_{\text{cyc}} : \text{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q}) \cong \text{Gal}(\mathbb{Q}(\mu_{p^{\infty}})/\mathbb{Q}(\mu_p)) \xrightarrow{\chi_{\text{cyc}}} 1 + p\mathbb{Z}_p$$

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is the canonical character and
\[ \omega : \text{Gal}(\mathbb{Q}(\mu_p) / \mathbb{Q}) \cong \text{Gal}(\mathbb{Q}(\mu_{p^n}) / \mathbb{Q}_\infty)^{\chi_{\infty}} \mu_{p-1} \]
the Teichmüller character. Let \( \mathcal{O} \subset \mathbb{Q}_p \) be a commutative ring which is finite flat over \( \mathbb{Z}_p \) and let \( \psi \) be a Dirichlet character modulo \( M \). We denote by \( S_k(\Gamma_0(M), \psi, \mathcal{O}) \) the space of cusp forms of weight \( k \), level \( M \), Neben character \( \psi \) and Fourier coefficients in \( \mathcal{O} \). We also denote by the same symbol \( \psi \) the corresponding character of \( \text{Gal}(\mathbb{Q}(\mu_M)/\mathbb{Q}) \cong (\mathbb{Z}/M)^\times \). For a group \( \Delta_M \) which is isomorphic to \( \text{Gal}(\mathbb{Q}(\mu_M)/\mathbb{Q}) \), a \( \mathbb{Z}_p \)-module \( M \) which has a \( \mathbb{Z}_p \)-linear action of \( \Delta_M \) and a character \( \varepsilon \) of \( \Delta_M \), we denote by \( \mathcal{M} = M \otimes_{\mathbb{Z}_p} [\Delta_M] Z_p[e] \).

In 1976, Ribet [16] proved the converse of Herbrand’s theorem as follows:

**Theorem 1.1** (Ribet). Let \( k \) be an even integer satisfying \( 2 \leq k \leq p - 3 \) and \( B_k \) the \( k \)-th Bernoulli number. We denote by \( \text{Cl}(\mathbb{Q}(\mu_p))[p^\infty] \) the \( p \)-part of the ideal class group of \( \mathbb{Q}(\mu_p) \) on which the Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts by functoriality. Suppose \( p \) divides \( B_k \). Then \( \text{Cl}(\mathbb{Q}(\mu_p))[p^\infty]^{\omega^{1-k}} \neq 0 \).

The method of Ribet’s proof is to construct a normalized Hecke eigen cusp form \( f = \sum_{n=1}^{\infty} a(n, f)q^n \in S_2(\Gamma_0(p), \chi) \) which is congruent to Eisenstein series by the condition \( p \) divides \( B_k \). Then by using the Galois representation \( \rho_f \) attached to \( f \) due to Deligne and Shimura, Ribet constructed an unramified \( p \)-extension of \( \mathbb{Q}(\mu_p) \) by using a canonical stable lattice (see Proposition 1.2 below) of \( \rho_f \). By extending Ribet’s method, Mazur-Wiles [12] and Wiles [24] proved the Iwasawa main conjecture for \( \mathbb{Q} \) and for totally real fields.

The key lemma Ribet used, which is called “Ribet’s lemma”, is the following proposition:

**Proposition 1.2** (Ribet’s lemma). Let \( (\mathcal{O}, \varpi, \mathcal{O}/(\varpi)) \) be the ring of integers of a finite extension of \( \mathbb{Q}_p \), where \( \varpi \) is an uniformizer of \( \mathcal{O} \). Let \( K = \text{Frac}(\mathcal{O}) \) be the fraction field of \( \mathcal{O} \) and \( V \) a 2-dimensional \( K \)-vector space. For a given \( p \)-adic representation \( \rho : G \to \text{Aut}_K(V) \) of a compact group \( G \), let \( \bar{\rho}^{ss} \) be the semi-simplification of the mod \( \varpi \) representation (see Section 2.1 below). Suppose \( \rho \) is irreducible and \( \bar{\rho}^{ss} \cong \psi_1 \oplus \psi_2 \), where \( \psi_1, \psi_2 : G \to (\mathcal{O}/(\varpi))^{\times} \) are characters. Then there exists a stable lattice \( T \subset V \) for which \( \bar{\rho}_T \) is the form \( \left( \begin{array}{cc} \psi_1 & * \\ 0 & \psi_2 \end{array} \right) \) but is not semi-simple.

Let \( f \) be a normalized Hecke eigen cusp form and \( \rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(K) \) the continuous irreducible representation attached to \( f \), where \( K \) is the field \( \mathbb{Q}_p(\{a(n, f)\}_{n \geq 1}) \). We denote by \( \mathcal{L}(\rho_f) \) the set of the isomorphic classes of stable lattices of \( \rho_f \). Since \( \rho_f \) is irreducible, \( \sharp \mathcal{L}(\rho_f) \) is finite (see (5) of Proposition 2.2 below). The author wants to determine \( \sharp \mathcal{L}(\rho_f) \) for a given \( f \). For example the known result is obtained by Greenberg and Monsky for the Ramanujan’s cusp form \( \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in S_{12}(\text{SL}_2(\mathbb{Z})) \) and \( p = 691 \):

**Proposition 1.3** (Greenberg, Monsky). Let \( \rho_\Delta : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_{691}) \) be the 691-adic representation attached to \( \Delta \). Then \( \sharp \mathcal{L}(\rho_\Delta) = 2 \).
Theorem 1.5. Suppose $X$ is a primitive Hecke eigen cusp form. Now we are going to determine denoted by $S$ generated over $\Lambda$ for all $S$ modulo $Np$. If $\zeta \in \mu_{p^r}(r \geq 0)$ is a $p^r$-th root of unity, we denote by $\chi_\zeta$ the Dirichlet character as follows:

$$\chi_\zeta : (\mathbb{Z}/p^{r+1}\mathbb{Z})^{\times} \to \overline{\mathbb{Q}}_p^{\times}, \text{ } \text{ } u \text{ } \text{ } \text{ } \text{mod} \text{ } p^{r+1} \rightarrow \zeta.$$ 

Now let $I$ be an integrally closed local domain which is finite flat over $\Lambda_\chi = \mathbb{Z}_p[\chi][[X]]$ and $\mathfrak{X}_I$ the set of homomorphisms defined as follows:

$$\mathfrak{X}_I = \{ \varphi : I \to \overline{\mathbb{Q}}_p \mid \varphi(1 + X) = \zeta_\varphi u^{k_\varphi - 2}, (k_\varphi, \zeta_\varphi) \in \mathbb{Z}_{\geq 2} \times \mu_{p^\infty} \}.$$ 

Let $\mathcal{F} = \sum_{n=1}^\infty c(n, \mathcal{F})q^n \in \mathbb{Z}[q]$ be an $I$-adic cusp form (resp. $I$-adic normalized Hecke eigen cusp form) with character $\chi$. That is,

$$f_\varphi := \sum_{n=1}^\infty \varphi(c(n, \mathcal{F}))q^n \in S_{k_\varphi} \Gamma_1(Np^\infty), \chi_{\zeta_\varphi} \chi_{\omega^{1-k_\varphi}, \varphi(I)}$$

is a $p$-ordinary cusp form (resp. $p$-ordinary normalized Hecke eigen cusp form) for all $\varphi \in \mathfrak{X}_I$, where $\zeta_\varphi$ is a primitive $p^r$-th root of unity. We denote by $S^{\text{ord}}(\chi, I)$ the space of $I$-adic forms with character $\chi$. Let $T(\chi, \Lambda_\chi)$ the ring generated over $\Lambda_\chi$ by all Hecke operators $T(l)$ for all primes $l$. Then $T(\chi, I) = T(\chi, \Lambda_\chi) \otimes_{\Lambda_\chi} I$ acts on the space $S^{\text{ord}}(\chi, I)$.

Let $\mathcal{F}$ be an $I$-adic normalized Hecke eigen cusp form and $\text{Frac}(I)$ the field of fraction of $I$. Hida [8] proved that there is a continuous representation

$$\rho_\mathcal{F} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\text{Frac}(I))$$

such that for any $\varphi \in \mathfrak{X}_I$, the residual representation $\rho_\mathcal{F}(\text{Ker} \varphi)$ (see Definition 2.8) is isomorphic to $\rho_{f_\varphi}$.

From now on throughout the paper, we denote by $\phi$ the Euler function and we fix a positive integer $N$ prime to $p$. Let $\chi$ be a primitive Dirichlet character modulo $NP$. Let $l = \mathbb{Z}$ be the same as above with $m$ the maximal ideal of $l$. We denote by $S^{\text{ord}}(\chi, l)$, $T(\chi, l)$ the same as above. Let $\mathcal{F}$ be an $I$-adic normalized Hecke eigen cusp form. Now we are going to determine $\mathcal{L}(\mu_{f_\varphi})$ when $\varphi$ varies in $\mathfrak{X}_I$. Our result is the following theorem:

Theorem 1.5. Suppose $p \nmid \phi(N)$ and $\rho_\mathcal{F}(m) \cong \psi_1 \oplus \psi_2$ such that $\psi_1$ (resp. $\psi_2$) is unramified (resp. ramified) at $p$. Assume the following condition:

(D) There exist Dirichlet characters $\chi_1, \chi_2$ with relative prime conductors such that $\chi_1 \chi_2 = \chi$, $\chi_1 \neq \chi_2 \omega$ and $\chi_i = \psi_i (i = 1, 2)$.

We enlarge $I$ such that $l$ is also finite flat over $\Lambda_{\chi_1 \chi_2}$. Then we have the following statements:
boundedness of 

Then we have the following statements:

We denote by \( \hat{\varphi} \)

Let the assumptions and the notations be as in Theorem 1.5.

Corollary 1.6.

Theorem 1.5 will be proved at the end of Section 3.2. Now we discuss the

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(1) There exists an integer \( r \in \mathbb{Z}_{\geq 0} \) such that

is bounded when \( \varphi \) varies in \( X_{i}^{(r)} \), where \( \text{rank}_{\Lambda_{\chi}} \mathbb{I} \) is the rank of the \( \Lambda_{\chi} \)-module \( \mathbb{I} \).
(2) For each integer \( k \geq 2 \), \( \sharp L(\varphi) \) is bounded when \( \varphi \) varies in \( X_{1,k} \).

(3) Suppose that \( \mathcal{I} \) is isomorphic to \( \mathcal{O}[[X]] \) with \( \mathcal{O} \) the ring of integers of a finite extension of \( \mathbb{Q}_p \). Then there exists an integer \( r' \in \mathbb{Z}_{\geq 0} \) such that \( \sharp L(\varphi) \) is constant when \( \varphi \) varies in \( X_{1,\mathcal{I}} \).

(4) Assume the condition (R) or both of the conditions (C) and (P). For each \( \zeta \in \mu_{p^\infty} \), \( \sharp L(\varphi) \) is unbounded when \( \varphi \) varies in \( X_{1,\mathcal{I}}(\zeta) \) if and only if \( L_p(1 - s, \chi\zeta\chi_1^{-1}\chi_2\omega) \) has a zero in \( \mathbb{Z}_p \).

Corollary 1.6 will be proved in Section 3.3. Let \( \mathcal{L}(\varphi) \) be the set of the isomorphic classes of stable lattices of Hida deformation \( \rho_F \). Now we give a result of \( \sharp L(\varphi) \) answering Question 4.5 1 of [15].

**Corollary 1.7.** Let the assumptions and the notations be as in Theorem 1.5. Assume the conditions (D), (C) and (P). Further assume the following condition

(F) There exists a stable lattice \( \mathcal{T} \) which is free over \( \mathcal{I} \).

Suppose that there exists a \( \zeta \in \mu_{p^\infty} \) such that \( L_p(1 - s, \chi\zeta\chi_1^{-1}\chi_2\omega) \) has a zero in \( \mathbb{Z}_p \). Then \( \sharp L(\varphi) = \infty \).

Corollary 1.7 will be proved in Section 3.4.

**Remark 1.8.** Mazur-Wiles [13, §9], Tilouine [21, Theorem 4.4] and Mazur-Tilouine [14, §2, Corollary 6] give a list of cases where the condition (F) is known to be true.

**Outline.** The outline of this paper is as follows. In Section 2, we recall known results concerning the Bruhat-Tits tree of \( GL_2 \), Hida deformation and Kubota-Leopoldt \( p \)-adic \( L \)-function. These will be used frequently in Section 3. In Section 3, first we determine the number of isomorphic classes of stable lattices in a given \( p \)-adic representation by using the Bellaïche-Chenevier reducibility ideal \( I(\rho) \). Then we give the proof of the main results. In Section 4, we give two examples of Hida deformations associated to an \( \mathcal{I} \)-adic normalized Hecke eigen cusp form \( \mathcal{F} \in S^{\text{ord}}(\omega^{k_0-1}, \mathcal{I}) \) when \( (p, k_0) = (691, 12) \) and \( (547, 486) \).

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## 2 The Bruhat-Tits Tree, Hida deformation and \( p \)-adic \( L \)-function

### 2.1 The lattices and the Bruhat-Tits Tree

Let \( A \) be a commutative integral domain with field of fractions \( K = \text{Frac}(A) \) and \( V \) a \( n \)-dimensional \( K \)-vector space. We say an \( A \)-submodule \( T \) of \( V \) is a lattice of \( V \) if there exist two free \( A \)-submodules \( L_1, L_2 \) of \( V \) such that \( L_1 \subseteq T \subseteq L_2 \) and \( \text{rank}_A L_1 = n \). If \( A \) is Noetherian, we have that an \( A \)-submodule \( T \) of \( V \) is a lattice of \( V \) if and only if \( T \) is finitely generated and \( T \otimes_A K = V \) (see [6, VI.5].
I. 4.1, Corollary to Proposition 1). Now we assume \( A = \mathcal{O} \) which is the ring of integers of a finite extension field of \( \mathbb{Q}_p \) with a fixed uniformizer \( \varpi \) of \( \mathcal{O} \). For a given \( p \)-adic representation

\[
\rho : G \to \text{Aut}_K(V)
\]
of a compact group \( G \), we say that \( T \) is a \( G \)-stable lattice of \( V \) if \( T \) is a lattice and \( \rho(G)T = T \). This means that \( T \) is also an \( \mathcal{O}[G] \)-module. Since \( G \) is compact, there exists a \( G \)-stable lattice in \( V \) (see [18] pp.1-2). We denote by \( \rho_T \) the representation

\[
\rho_T : G \to \text{Aut}_\mathcal{O}(T) \cong \text{GL}_n(\mathcal{O})
\]
and by \( \bar{\rho}_T \mod \varpi \) the representation \( \rho_T \mod \varpi \) as follows:

\[
\rho_T \mod \varpi : G \to \text{Aut}_\mathcal{O}(T) \mod \varpi \to \text{Aut}_\mathcal{O}((\varpi))(T/\varpi T) \cong \text{GL}_n(\mathcal{O}/(\varpi)).
\]

For stable lattices \( T \) and \( T' \), the representation \( \bar{\rho}_T, \bar{\rho}_{T'} \), can be non-isomorphic to each other. However the semi-simplification \( \bar{\rho}_T^{\text{ss}} \) of \( \bar{\rho}_T \) is isomorphic to \( \bar{\rho}_{T'}^{\text{ss}} \), by the Brauer-Nesbitt theorem. We denote by \( \bar{\rho}_T^{\text{ss}} \) the semi-simplification of \( \bar{\rho}_T \).

Now following [17] and [4], we introduce the graph structure of lattices, which will be used to prove Proposition 3.4. From now on to the end of this section we assume \( n = 2 \).

For a lattice \( T \) of \( V \), we denote by \( [T] = \{ xT \mid x \in K^\times \} \) the equivalence class up to homotheties. Let \( \mathcal{X} \) be the set of all \( [T] \) where \( T \) is a lattice. We say that a point \( x' \) in \( \mathcal{X} \) is a neighbor of a point \( x \in \mathcal{X} \) if \( x' \neq x \) and there are lattices \( T, T' \) of \( V \) such that \( x = [T], x' = [T'] \) and \( \varpi T \subset T' \subset T \). In this way one defines a combinatorial graph structure on \( \mathcal{X} \).

**Theorem 2.1** ([17, Chapter II, Theorem 1]). The graph \( \mathcal{X} \) is a tree.

Now we recall some basic notions on the tree \( \mathcal{X} \). Let \( x, x' \in \mathcal{X} \). A path without backtracking from \( x \) to \( x' \), which is denoted by \( \text{Path}_{x,x'} \), is a sequence \( x = x_0, x_1, \ldots, x_n = x' \) of points in \( \mathcal{X} \) such that \( x_i \) is a neighbor of \( x_{i+1} \) and \( x_i \neq x_j \) if \( i \neq j \). We define the integer \( n = d(x, x') \geq 0 \) to be the distance between \( x \) and \( x' \). Let \( x = [T] \) and we fix a positive integer \( n \), then there is a natural bijection between the set of the points \( x' \) in \( \mathcal{X} \) such that \( d(x, x') = n \) and the set of lattices \( \varpi^n T \subset T' \subset T \) such that \( T/T' \cong \mathcal{O}/(\varpi)^n \) as an \( \mathcal{O} \)-module.

In a tree, we define the segment \([x, x']\) as

\[
[x, x'] = \begin{cases} 
\{ x \} & (x = x') \\
\text{Path}_{x,x'}(x \neq x') & \end{cases}
\]

A subset \( C \) of \( \mathcal{X} \) is called a convex if for every \( x, x' \in C \), the segment \([x, x'] \subset C \).

We denote by \( \mathcal{X}(\rho) \) the set of \( \mathcal{X} \) that are fixed by \( \rho(G) \). We summarize some results on \( \mathcal{X}(\rho) \):

**Proposition 2.2.**

1. \( \mathcal{X}(\rho) \) is a convex ([4, §3.1]).

2. If \( x \in \mathcal{X}(\rho) \), then \( x \) has no neighbor in \( \mathcal{X}(\rho) \) if and only if \( \overline{\rho}_x \) is irreducible ([4, Proposition 11 (d)-(i)]).
If $x \in X(\rho)$, then $x$ has exactly one neighbor in $X(\rho)$ if and only if $\overline{\rho}_x$ is reducible but indecomposable ([4, Proposition 11 (d)-(ii)]).

(4) If $x \in X(\rho)$, then $x$ has exactly two neighbors in $X(\rho)$ if and only if $\overline{\rho}_x$ is decomposed into two distinct characters ([4, Proposition 11 (d)-(iii)]).

(5) $\rho$ is irreducible if and only if $X(\rho)$ is bounded ([4, Lemme 10]).

(6) Assume that $\rho$ is irreducible and $\rho \approx \chi \otimes \psi_1 \otimes \psi_2$ of characters $\psi_1, \psi_2 : G \to (\mathbb{O}/(\varpi))^\times$ with $\psi_1 \neq \psi_2$. Then $X(\rho)$ is a segment.

The assertion (6) easily follows from the assertions (1), (4) and (5) (cf. [2] the arguments before §1.3 in page 7).

2.2 $\mathbb{I}$-adic forms and Galois representations

In this section, we review some fundamental results on $\mathbb{I}$-adic cusp forms and their Galois representations. For more detail on this theory, the reader can refer to Chapter 7 of [9].

Recall that $\mathbb{I}$ is an integrally closed local domain which is finite flat over $\Lambda_\chi = \mathbb{Z}_p[[X]]$, where $\chi$ is a primitive Dirichlet character modulo $N_p$. We denote by $X_{\Lambda_\chi}$ and $X_\mathbb{I}$ the sets of homomorphisms defined as follows:

$$X_{\Lambda_\chi} = \left\{ \nu_{k,\zeta} : \Lambda_\chi \to \overline{\mathbb{Q}}_p \mid \nu_{k,\zeta}(1 + X) = \zeta u^{k-2}, (k, \zeta) \in \mathbb{Z}_{\geq 2} \times \mu_{p^n} \right\},$$

$$X_\mathbb{I} = \left\{ \varphi : \mathbb{I} \to \overline{\mathbb{Q}}_p \mid \varphi|_{\Lambda_\chi} = \nu_{k,\zeta} u^{k-2}, \nu_{k,\zeta} \in X_{\Lambda_\chi} \right\}.$$

If $\zeta \in \mu_{p^r} (r \geq 0)$ is a primitive $p^r$-th root of unity, let

$$\chi_\zeta : (\mathbb{Z}/p^{r+1}\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p, u \mod p^{r+1} \mapsto \zeta$$

be the character associated to $\zeta$.

**Definition 2.3.** We call $\mathcal{F} = \sum_{n=1}^{\infty} c(n, \mathcal{F})q^n \in \mathbb{I}[[q]]$ an $\mathbb{I}$-adic form (resp. $\mathbb{I}$-adic normalized Hecke eigen cusp form) with Dirichlet character $\chi$ if for each $\varphi \in X_\mathbb{I}$ with $\varphi|_{\Lambda_\chi} = \nu_{k,\zeta,\zeta}$ such that $\zeta_{\varphi}$ is a primitive $p^r$-th root of unity,

$$f_{\varphi} := \sum_{n=1}^{\infty} \varphi(c(n, \mathcal{F}))q^n \in S_{k_{\varphi}}(\Gamma_0(Np^{r+1}), \chi_{\varphi} \chi_{\omega}^{1-k_{\varphi}}, \varphi(1))$$

is the $q$-expansion of a $p$-ordinary cusp form (resp. $p$-ordinary normalized Hecke eigen cusp form).

We also denote by $\varphi(\mathcal{F})$ the above cusp form $f_{\varphi}$. Recall that $S^{\text{ord}}(\chi, \mathbb{I})$ is the space of $\mathbb{I}$-adic forms with Dirichlet character $\chi$.

**Theorem 2.4** (Hida [9, §7.4, Theorem 7]). Let $\zeta \in \mu_{p^r} (r \geq 0)$ be a primitive $p^r$-th root of unity and $\chi_\zeta$ the character associated to $\zeta$. Let

$$f \in S_k(\Gamma_0(Np^{r+1}), \chi_{\zeta} \chi_{\omega}^{1-k}, \overline{\mathbb{Q}})$$

be
be a $p$-ordinary normalized Hecke eigen cusp form of weight $k \geq 1$. Then there exist an integrally closed local domain $I$ which is finite flat over $\Lambda$, an $I$-adic normalized Hecke eigen cusp form $F \in S^{\text{ord}}(\chi, I)$ and a $\varphi \in \mathfrak{X}_I$ such that $\varphi(F) = f$.

**Definition 2.5.** A Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\text{Frac}(I))$ is continuous if there exists a lattice $T \subset \text{Frac}(I)^{\oplus 2}$ which is stable under $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action such that $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}_I(T)$ is continuous with respect to the topology of $T$ defined by $m$ the maximal ideal of $I$.

Hida associates a continuous Galois representation over $\text{Frac}(I)$ to an $I$-adic normalized Hecke eigen cusp form $F$ as follows:

**Theorem 2.6** (Hida [8, Theorem 2.1]). There exists a continuous irreducible representation $\rho_F : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\text{Frac}(I))$ with the following properties:

1. $\rho_F$ is unramified outside $Np$.
2. For the geometric Frobenius element $\text{Frob}_l$ at $l \nmid Np$, we have:
   \[
   \text{tr} \rho_F(\text{Frob}_l) = c(l, \mathfrak{F}),
   \]
   \[
   \det \rho_F(\text{Frob}_l) = \chi(l)(u(1+X)^{s_l}) \mod P,
   \]
   where $d = \omega(d)(1+p)^{*l}$ under the isomorphism $\mathbb{Z}_p^* \cong \mu_{p-1} \times (1+p\mathbb{Z}_p)$.

We have the following local property due to Mazur and Wiles:

**Theorem 2.7** (Wiles [23, Theorem 2.2.2]). With the same notations as above, the restriction of $\rho_F$ to the decomposition group $D_p = \text{Gal}((\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is given up to equivalence by

\[
\rho_F \mid D_p \sim \begin{pmatrix} \varepsilon_1 & 0 \\ * & \varepsilon_2 \end{pmatrix}
\]

with $\varepsilon_1$ unramified and $\varepsilon_1(\text{Frob}_p) = c(p, \mathfrak{F})$.

**Definition 2.8.** For a prime ideal $P$ of $I$, a Galois representation $\rho_F(P) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\text{Frac}(I/P))$ is called a residual representation of $\rho_F$ modulo $P$ if $\rho_F(P)$ is semi-simple, continuous under the $\mathfrak{m}$-adic topology of $\text{Frac}(I/P)$ and satisfies the following properties:

1. $\rho_F(P)$ is unramified outside $Np$.
2. For the geometric Frobenius element $\text{Frob}_l$ at $l \nmid Np$,
   \[
   \text{tr} \rho_F(P)(\text{Frob}_l) = c(l, \mathfrak{F}) \mod P,
   \]
   \[
   \det \rho_F(P)(\text{Frob}_l) = \chi(l)(u(1+X)^{s_l}) \mod P.
   \]

Although $\rho_F$ may not have $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice which is isomorphic to $I^{\oplus 2}$, the following fact is well-known (see [9, §7.5, Corollary 1] for example).

**Proposition 2.9.** For every prime ideal $P$, the residual representation $\rho_F(P)$ exists and is unique up to isomorphism over an algebraic closure of $\text{Frac}(I/P)$.
2.3 Kubota-Leopoldt $p$-adic $L$-function

Now we recall some facts about the Kubota-Leopoldt $p$-adic $L$-function. Let $\psi$ be an arbitrary Dirichlet character. Kubota-Leopoldt (see [10, 3, Theorem 2]) showed that there exists a $p$-adic continuous function $L_p(s, \psi)$ for $s \in \mathbb{Z}_p - \{1\}$ (also continuous at $s = 1$ if $\psi$ is non-trivial) with the following interpolate property for $k \geq 1$:

$$L_p(1 - k, \psi) = (1 - \psi \omega^{-k}(p)p^{k-1})L(1 - k, \psi \omega^{-k}).$$

Set

$$H_\psi(X) = \begin{cases} \psi(u)(1 + X) - 1 & \text{if } \psi \text{ factors through } \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}), \\ 1 & \text{otherwise.} \end{cases}$$

Iwasawa [10, §6] showed that there exists a unique power series $G_\psi(X) \in \mathbb{Z}_p[[X]]$ such that

(i) $L_p(1 - s, \psi) = G_\psi(u^s - 1)/H_\psi(u^s - 1),$

(ii) if $\rho$ factors through $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, then $G_{\psi \rho}(X) = G_\psi(\rho(u)(1 + X) - 1)$.

We define

$$\hat{G}_\psi(X) = G_{\psi \omega}(u^2(1 + X) - 1),$$

$$\hat{H}_\psi(X) = H_{\psi \omega}(u^2(1 + X) - 1)$$

for later reference.

3 Proof of Theorem 1.5 and its corollaries

3.1 Calculation of $\mathfrak{zL}(\rho)$ by means of $I(\rho)$

For a given $p$-adic representation $\rho$, recall that $\mathcal{L}(\rho)$ is the set of the isomorphic classes of stable lattices of $\rho$. First we determine $\mathfrak{zL}(\rho)$ in this section. The following lemma is proved by Bellaïche-Chenevier [3].

Lemma 3.1 (Bellaïche-Chenevier [3, Lemme 1]). Let $(A, m, k)$ be a complete local domain such that $\text{char}(k) \neq 2$, where $m$ is the maximal ideal of $A$ and $k$ the residue field $A/m$. Let $\rho : G \to \text{GL}_2(\text{Frac}(A))$ be a linear representation of a group $G$ satisfying $\text{tr}\rho(G) \subset A$ and $\text{tr}\rho$ mod $m = \psi_1 + \psi_2, \psi_1 \neq \psi_2$, where $\psi_1, \psi_2 : G \to k^\times$ are characters. Let $g_0 \in G$ be an element satisfying $\psi_1(g_0) \neq \psi_2(g_0)$ and $\lambda_1, \lambda_2 \in A$ the roots of the characteristic polynomial of $\rho(g_0)$. Choose a basis $\{e_1, e_2\}$ of the representation $\rho$ such that $\rho(g_0)e_i = \lambda_i e_i$ ($i = 1, 2$). Write $\rho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$ for any $g \in G$.

Let $I \subseteq A$ be an ideal such that there exist two characters $\vartheta_1, \vartheta_2 : G \to (A/I)^\times$ such that

$$\text{tr}\rho(g) \mod I = \vartheta_1(g) + \vartheta_2(g)$$

for any $g \in G$. Assume $\vartheta_1 \mod m = \psi_1, \vartheta_2 \mod m = \psi_2$ without loss of generality. Then for any $g, g' \in G$, we have $a(g), d(g) \in A, a(g) \mod I = \vartheta_1(g), d(g) \mod I = \vartheta_2(g)$, and $b(g)c(g') \in I$. 

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Remark 3.2. If char($k$) = 2, the statement holds assuming an extra condition on the determinate (cf. [3, Lemme 1]).

Definition 3.3. Let $(A, m, k)$ be a complete local domain such that char($k$) ≠ 2, where $m$ is the maximal ideal of $A$ and $k$ the residue field. Let $\rho : G \rightarrow \text{GL}_2(\text{Frac}(A))$ be a linear representation of a group $G$ satisfying $\text{tr} \rho(G) \subset A$ and $\text{tr} \rho \mod m = \psi_1 + \psi_2, \psi_1 \neq \psi_2$, where $\psi_1, \psi_2 : G \rightarrow k^\times$ are characters. For any $g \in G$, write $\rho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$ with respect to the basis taken as in Lemma 3.1. We define $I(\rho)$ the ideal of $A$ which is generated by $b(g)c(g')$ for all $g, g' \in G$.

The ideal $I(\rho)$ is well-defined by Lemma 3.1. Under the above preparation, we are ready to determine $\sharp L(\rho)$ of a $p$-adic representation $\rho$.

Proposition 3.4. Let $(\mathcal{O}, \varpi, \mathcal{O}/(\varpi))$ be the ring of integers of a finite extension of $\mathbb{Q}_p$, where $\varpi$ is a fixed uniformizer of $\mathcal{O}$. Let $V$ be a vector space of dimension 2 over $K = \text{Frac}(\mathcal{O})$ and $\rho : G \rightarrow \text{Aut}_K(V)$ a continuous irreducible representation of a compact group $G$.

Assume that

$$\text{tr} \rho \mod \varpi = \psi_1 + \psi_2, \psi_1 \neq \psi_2,$$

where $\psi_1, \psi_2 : G \rightarrow (\mathcal{O}/\varpi)^\times$ are characters. Then we have

$$\text{ord}_\varpi I(\rho) + 1 = \sharp X(\rho) = \sharp L(\rho).$$

This proposition is a special case of Bellaïche-Graffieux [5, Théorème 4.1.3] (see also the remark immediately after it), but we give the proof here for self-containing.

Proof. We first show $\text{ord}_\varpi I(\rho) + 1 = \sharp X(\rho)$. Fix a $g_0 \in G$ such that $\psi_1(g_0) \neq \psi_2(g_0)$. The characteristic polynomial of $\rho(g_0)$

$$X^2 - \text{tr} \rho(g_0)X + \det \rho(g_0)$$

has roots $\lambda_1 \neq \lambda_2$ in $A$ such that $\lambda_i \mod \varpi = \psi_i(g_0) (i = 1, 2)$ by Hensel’s lemma. Choose a basis $\{e_1, e_2\}$ of the representation $\rho$ such that $\rho(g_0)e_i = \lambda_i e_i$ $(i = 1, 2)$. Write $\rho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$ for all $g \in G$. Let $B$ be the module of $\mathcal{O}$ generated by $b(g)$ for all $g \in G$. Since $\rho$ is irreducible, we have $B \neq 0$. Since $\rho$ is continuous and $G$ is compact, there exists a stable lattice. This implies $B = (\varpi)^r$ for an integer $r$.

If we replace $\rho$ by $\begin{pmatrix} 1 & 0 \\ 0 & \varpi^r \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & \varpi^r \end{pmatrix}^{-1}$ and we denote by the same symbol $\rho$ for this new representation. Then we have the following properties for the new $\rho$:

1. $\rho$ takes values in $\text{GL}_2(\mathcal{O})$ by Lemma 3.1.

2. $\rho(g_0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

3. For any $g \in G$, write $\rho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$. There exists an $h \in G$ such that $\varpi \nmid b(h)$. 

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We denote by the same symbol $B$ (resp. $C$) the ideal of $\mathcal{O}$ which is generated by new $b(g)$ (resp. new $c(g)$) for all $g \in G$. Since $BC = I(\rho)$ by Lemma 3.1 and $B = \mathcal{O}$ by (1), we must have $C = I(\rho) = (\varpi)^n$ for a positive integer $n$. This also means that we have chosen a stable lattice $T$ such that 

$$
\rho = \rho_T : G \to \text{GL}_2(\mathcal{O})
$$

and $T/\varpi T$ is not semi-simple. By reduction mod $(\varpi)^i$ ($i = 1, 2, \ldots, n$), we obtain the $G$-stable lattices $T_1, \ldots, T_n$ such that $[T_i] \neq [T_j]$ if $i \neq j$. Then $n + 1 = \text{ord}_{\varpi} I(\rho) + 1 \leq \sharp X(\rho)$.

Let $\sharp X(\rho) = m + 1$. We have $X(\rho)$ is a segment $[x, x_m]$ by (6) of Proposition 2.2. Let $T, T_m$ be the representatives of $x, x_m$ such that $T_m \subset T$ and $T/T_m \cong \mathcal{O}/(\varpi)^m$ as an $\mathcal{O}$-module. Hence there exists a basis of $T$ such that $\rho_T : G \to \text{GL}_2(\mathcal{O}), g \mapsto \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$ satisfies $\varpi^m | c(g)$ for any $g \in G$. Then

$$
a \bmod \varpi^m : G \to (\mathcal{O}/(\varpi)^m)\times, g \mapsto a(g) \bmod \varpi^m
$$

and

$$
d \bmod \varpi^m : G \to (\mathcal{O}/(\varpi)^m)\times, g \mapsto d(g) \bmod \varpi^m
$$

are two characters. Thus $I(\rho) \subset (\varpi)^m$ by Lemma 3.1 and $\sharp X(\rho) \leq \text{ord}_{\varpi} I(\rho) + 1$.

Next we prove $\sharp X(\rho) = \sharp L(\rho)$. Suppose $\sharp X(\rho) = n + 1$. Since $X(\rho)$ is a segment, by Section 2.1 there exist $T_0, T_1, \ldots, T_n$ the representatives of the points in $X(\rho)$ such that

(i) $[T_i]$ is a neighbor of $[T_{i-1}]$ and $T_0/T_i \cong \mathcal{O}/(\varpi)^i$ as an $\mathcal{O}$-module for $i = 1, \ldots, n$.

(ii) $T_0, T_n$ are mod $\varpi$ not semi-simple lattices and the others are not.

Thus it is sufficient to show that for $i \neq j$, $T_i$ and $T_j$ are non-isomorphic to each other as $\mathcal{O}[G]$-modules.

1. Suppose we have $f : T_0 \to T_n$ as $\mathcal{O}[G]$-modules. Then $\varpi T_n \subset f(T_1) \subset T_n$ since $[T_1]$ is a neighbor of $[T_0]$. Since $T_n$ is a mod $\varpi$ not semi-simple lattice, we have $f(T_1) = \varpi T_n$, by (3) of Proposition 2.2. Hence

$$
T_1/\varpi T_0 \cong \varpi T_n - 1/\varpi T_n \cong \mathcal{O}/(\varpi) [\psi_1] \ (\text{resp. } \mathcal{O}/(\varpi) [\psi_2])
$$

as an $\mathcal{O}[G]$-module. Thus there is no mod $\varpi$ not semi-simple stable lattice $T$ such that $T/\varpi T$ has a submodule which is isomorphic to $\mathcal{O}/(\varpi) [\psi_1]$ (resp. $\mathcal{O}/(\varpi) [\psi_2]$). This contradicts to the Ribet’s lemma (Proposition 1.2).

2. Suppose we have $f : T_i \to T_j$ as $\mathcal{O}[G]$-modules for some $0 < i < j < n$. Since $T_0$ is a mod $\varpi$ not semi-simple lattice and $T_0, T_n$ are non-isomorphic as $\mathcal{O}[G]$-modules, we have $f(\varpi T_0) = \varpi T_j$. Hence

$$
\mathcal{O}/(\varpi)^i \cong T_0/T_i \cong T_0/\varpi^{i-1} T_j
$$

as an $\mathcal{O}$-module. This implies $d([T_0], [T_j]) = i$. On the other hand, $[T_0]$ is an edge of the segment $X(\rho)$, there exists an unique point $y \in X(\rho)$ such that $d([T_0], y) = i$. This contradicts to $i \neq j$. 

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Now we give an example by using Proposition 3.4 to determine $\#L(\rho_f)$, where $\rho_f$ is the Galois representation attached to a normalized Hecke eigen cusp form $f$. Let $\Delta \in S_{12}(\text{SL}_2(\mathbb{Z}))$ be the Ramanujan’s cusp form, whose $q$-expansion is

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n$$

the Galois representation attached to $\Delta$.

Proposition 3.5. The ideal $I(\rho_\Delta) \subset \mathbb{Z}_p$ defined as in Definition 3.3 is the minimal ideal such that there exists an integer $a \in \mathbb{Z}$ such that for any prime $l \neq p$,

$$\tau(l) \equiv l^a + l^{11-a} \mod I(\rho_\Delta).$$

Proof. We denote by $\mathbb{Q}_{[p,\infty]}$ be the maximal Galois extension of $\mathbb{Q}$ which is unramified outside $\{p, \infty\}$ and by $G_{[p,\infty]} = \text{Gal}(\mathbb{Q}_{[p,\infty]}/\mathbb{Q})$. Since $\rho_\Delta$ is unramified outside $\{p, \infty\}$, $\rho_\Delta$ must factor through $G_{[p,\infty]}$.

Let $\vartheta_1, \vartheta_2 : G_{[p,\infty]} \to (\mathbb{Z}_p/I(\rho_\Delta))^\times$ be the character such that

$$\text{tr}_p \mod I(\rho_\Delta) = \vartheta_1 + \vartheta_2.$$

Since $\rho_\Delta$ is unramified outside $p$, $\vartheta_1$ and $\vartheta_2$ must factor through $\text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$ by class field theory. Thus $\vartheta_1$ and $\vartheta_2$ must be power of mod $I(\rho_\Delta)$ $p$-adic cyclotomic character $\chi_{cyc}$ mod $I(\rho_\Delta)$. For the geometric Frobenius element $\text{Frob}_l$ with prime $l \neq p$, we have $\chi_{cyc}(\text{Frob}_l) = l$ and $\det(\text{Frob}_l) = l^{11}$. Thus the proposition follows by the Chebotarev’s density theorem.

Serre and Swinnerton-Dyer showed that $\bar{\rho}_\Delta$ is reducible if and only if $p = 2, 3, 5, 7$ and 691 (see [19, Corollary to Theorem 4]). [20] also showed the congruence mod $p^n$ for $p = 3, 5, 7$ and 691 (see [20], page 77 for $p = 691$, Theorem 4 for $p = 5, 7$ and the table after Theorem 6 for $p = 3$). Then combined with our arguments, we have the following table for odd primes.

| $p$  | 3 | 5 | 7 | 691 |
|------|---|---|---|-----|
| $\#L(\rho_\Delta)$ | 7 | 4 | 2 | 2 |

3.2 The relation between $I(\rho_\bar{\mathcal{F}})$ and the Iwasawa power series

Let us take an $I$-adic normalized Hecke eigen cusp form $\bar{\mathcal{F}}$. Let the notations and the assumptions be as in Theorem 1.5. We denote by $\gamma$ a topological generator of $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ such that $\kappa_{cyc}(\gamma) = u$ and by $\kappa$ the universal cyclotomic character as follows:

$$\kappa : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \xrightarrow{\sim} 1 + p\mathbb{Z}_p \hookrightarrow \Lambda_\kappa^\vee,$$
where \(1 + p\mathbb{Z}_p \rightarrow \Lambda \) is the homomorphism defined by sending \(u \) to \(1 + X\).

Write \(\eta = \det \rho_\varphi\) for short. By Theorem 2.7, we have

\[
\rho_\varphi |_{D_\varphi} \sim \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix},
\]

with \(\varepsilon_1\) unramified. Recall that \(\rho_\varphi(\mathfrak{m}) \cong \chi_1 \oplus \chi_2\). Then for any \(g \in D_\mathfrak{p}\), \(\{\chi_1(g), \chi_2(g)\}\) are the set of the roots of the mod \(\mathfrak{m}\) characteristic polynomial of \(\rho_\varphi(g)\): \(X^2 - \text{tr}\rho_\varphi(g)X + \det\rho_\varphi(g) \mod \mathfrak{m}\), hence they must be coincide. Thus \(\chi_1 = \chi_1|_{D_\mathfrak{p}}\) and \(\chi_2 = \chi_2|_{D_\mathfrak{p}}\) under the assumption that \(\chi_1\) (resp. \(\chi_2\)) is unramified (resp. ramified). We denote by \(I_\mathfrak{p}\) the inertia group of \(\mathfrak{p}\) and we choose a \(g_0 \in I_\mathfrak{p}\) such that \(\chi_1(g_0) \neq \chi_2(g_0)\).

Let \(\{e_1, e_2\}\) be a basis of \(\text{Frac}(\mathbb{I}^{\otimes 2})\) such that

\[
\rho_\varphi(g_0) = \begin{pmatrix} 1 & 0 \\ 0 & e_2(g_0) \end{pmatrix}, \rho_\varphi |_{D_{\mathfrak{p}}} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}.
\]

Write \(\rho_\varphi(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}\) for any \(g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\). We have \(a(g), d(g)\) and \(b(g)c(g') \in \mathbb{I}\) for any \(g, g' \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) by Lemma 3.1. Recall that \(I(\rho_\varphi)\) is the ideal of \(I\) generated by \(b(g)c(g')\) for all \(g, g' \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\). Since \(\rho_\varphi(\mathfrak{m})\) is reducible, we have \(I(\rho_\varphi) \subset \mathfrak{m}\) by Lemma 3.1.

**Lemma 3.6.** Let us take the basis of \(\text{Frac}(\mathbb{I})^{\otimes 2}\) to be the same as the beginning of this section. For any \(\varphi \in X_1\), let \(\varpi_\varphi\) be a fixed uniformizer of \(\varphi(\mathbb{I})\). Then

\[
\text{ord}_{\varpi_\varphi}(\varphi(I(\rho_\varphi))) + 1 = t\mathcal{L}(\rho_{\mathfrak{p}_\varphi}).
\]

**Proof.** For any \(\varphi \in X_1\), we denote by \(\mathcal{P} = \text{Ker} \varphi\) and by \(\mathbb{I}_\mathcal{P}\) the localization of \(\mathbb{I}\) at \(\mathcal{P}\). Then \(\mathbb{I}_\mathcal{P}\) is a discrete valuation ring with \(\theta\) a fixed uniformizer of \(\mathcal{P}\). For the Galois representation

\[
\rho_{\mathcal{P}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{\otimes \mathfrak{p}} \rightarrow \text{GL}_2(\text{Frac}(\mathbb{I}_\mathcal{P})),
\]

let \(B\) be the \(\mathcal{P}\)-submodule of \(\text{Frac}(\mathbb{I}_\mathcal{P})\) generated by \(b(g)\) for all \(g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\). Since \(\rho_{\mathcal{P}}\) is irreducible, \(B \neq (0)\). Since \(\rho_{\mathcal{P}}\) is continuous, by Definition 2.5 there exists a lattice \(T \subset \text{Frac}(\mathbb{I})^{\otimes 2}\) which is stable under \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-action. Then \(T^\mathcal{P} = T \otimes \mathbb{I}_\mathcal{P}\) is a stable lattice of \(\rho_{\mathcal{P}}\) and \(\text{Im} \rho_{\mathcal{P}}\) is bounded. This implies \(B = (\theta)^n\) for an integer \(n\).

We replace \(\rho_{\mathcal{P}}\) by \(\begin{pmatrix} 1 & 0 \\ 0 & \theta^n \end{pmatrix}^{-1} \rho_{\mathcal{P}} \begin{pmatrix} 1 & 0 \\ 0 & \theta^n \end{pmatrix}\) and we denote by the same symbol \(\rho_{\mathcal{P}}\) for this new Galois representation. Then \(\text{Im} \rho_{\mathcal{P}} \subset \text{GL}_2(\mathbb{I}_\mathcal{P})\) for new \(\rho_{\mathcal{P}}\). We also denote by the same symbol \(a(g), b(g), c(g), d(g)\) such that \(\rho_{\mathcal{P}}(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}\) for all \(g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\).

We denote by \(\rho_{\varphi}\) the Galois representation

\[
\rho_{\varphi} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{\otimes \mathfrak{p}} \rightarrow \text{GL}_2(\varphi(\mathbb{I}_\mathcal{P})), g \mapsto \begin{pmatrix} a_{\varphi}(g) & b_{\varphi}(g) \\ c_{\varphi}(g) & d_{\varphi}(g) \end{pmatrix}
\]

and by \(\rho_{f_{\varphi}}\) the Galois representation associated to \(f_{\varphi}\). Since \(\text{tr}\rho_{\varphi} (\text{Frob}_l) = \text{tr}\rho_{f_{\varphi}} (\text{Frob}_l)\) and \(\det\rho_{\varphi} (\text{Frob}_l) = \det\rho_{f_{\varphi}} (\text{Frob}_l)\) for all primes \(l \nmid N\), we have
conjugation. Then we have the following statements:

Let us first show $\rho \cong \rho_f$. Since $a_\varphi(g) = \varphi(a(g)), b_\varphi(g) = \varphi(b(g)c(g))$ and $a_\varphi(g_0) \neq b_\varphi(g_0) \mod \varphi$, we have $I(\rho_\varphi) = \varphi(I(\rho))$ by the definition of $I(\rho_\varphi)$. Thus the statement follows from Proposition 3.4.

Lemma 3.7. Let us take the basis of $\text{Frac}(\mathcal{L}) \otimes \mathbb{Z}_p$ to be the same as the beginning of this section and let $\eta_1 = \chi_1, \eta_2 = \chi_2\kappa_{\text{cyc}}$. Let $J$ be the ideal of $I$ generated by $tr_{\rho}(g) - \eta_1(g) - \eta_2(g)$ for all $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $J' \text{ the ideal generated by } a(g) - \eta_1(g)$ for all $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then we have the following statements:

1. $I(\rho_\varphi) = J = J'$.

2. Suppose $N = 1$. Then $\rho_\varphi \mod m \cong \overline{T} \oplus \overline{T}_\kappa$ where $\overline{1}$ is the trivial character.

Proof. We first show $J = J'$. Since $tr_{\rho}(g) \equiv \eta_1 + \eta_2 \mod J, a(g) \equiv \eta_1(g) \mod J$ or $a(g) \equiv \eta_2(g) \mod J$ for all $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by Lemma 3.1. Since the character $a \mod m = \overline{T}$ is unramified at $p$, we have $a(g) \equiv \eta_1(g) \mod J$ for all $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This implies $J \subset J'$. We also have $J \subset J'$ since $tr_{\rho}(g) - \eta_1(g) - \eta_2(g) = (a(g) - \eta_1(g)) + (a(g^{-1}) - \eta_1(g^{-1}))\eta \eta_2(g) \in J'$.

Now we prove $I(\rho_\varphi) \subset J = J'$. Let $K$ be the abelian extension of $\mathbb{Q}$ corresponding to

$$\text{Ker}[a \mod I(\rho_\varphi) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow (\mathbb{Z}/I(\rho_\varphi))^\mathbb{Q}, g \mapsto a(g) \mod I(\rho_\varphi)].$$

We denote by $\tilde{a} : \text{Gal}(K/\mathbb{Q}) \rightarrow (\mathbb{Z}/I(\rho_\varphi))^\mathbb{Q}$ the induced homomorphism. For all $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we denote by $\overline{g}$ the image of $g$ under $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Gal}(K/\mathbb{Q})$. Write $a(g) = \eta_1(g)(1 + m(g))$ where $m(g) \in \mathfrak{m}$. Then $\tilde{a}((\overline{g})) = \eta_1(g) \mod I(\rho_\varphi) - (1 + m(g)) \mod I(\rho_\varphi)$. Note that $a \mod I(\rho_\varphi)$ is unramified outside $N$ by the equation (1), hence $K$ is a subfield of $\mathbb{Q}_{N_\mathfrak{p}}$ by class field theory. On the other hand, the kernel of the map $\mathbb{Z}/I(\rho_\varphi)^\mathbb{Q} \rightarrow (\mathbb{Z}/\mathfrak{m})^\mathbb{Q}$ is a pro-$\mathfrak{p}$ group, thus $(1 + m(g)) \mod I(\rho_\varphi)$ must be trivial under the assumption $p \not| \phi(N)$. This implies $\eta_1(g) \equiv a(g) \mod I(\rho_\varphi)$, hence $J' \subset I(\rho_\varphi)$. Specially when $N = 1$, we have that $a \mod I(\rho_\varphi)$ is an unramified character. Thus $a \mod I(\rho_\varphi)$ is trivial by class field theory.

Lemma 3.7 tells us that $I(\rho_\varphi)$ is a closed ideal in $I$ under the $\mathfrak{m}$-adic topology.

Proposition 3.8. Let us take the basis of $\text{Frac}(\mathcal{L}) \otimes \mathbb{Z}_p$ to be the same as at the beginning of this section. Let $L_\infty, L_\infty(Np)$ be the maximal unramified abelian $p$-extension of $\mathbb{Q}(\mu_{Np\infty})$ and the maximal abelian $p$-extension unramified outside $Np$ of $\mathbb{Q}(\mu_{Np\infty})$. We denote by $X_\infty = \text{Gal}(L_\infty/\mathbb{Q}(\mu_{Np\infty}))$ and by $Y_\infty = \text{Gal}(L_\infty(Np)/\mathbb{Q}(\mu_{Np\infty}))$ on which $\Delta_{Np} = \text{Gal}(\mathbb{Q}(\mu_{Np\infty})/\mathbb{Q}_\infty)$ acts by conjugation. Then we have the following statements:

1. $\hat{G}_{\chi_1^{-1}\chi_2}(X)_I \subset I(\rho_\varphi)$.

2. Suppose the $A_{\chi_1^{-1}\chi_2}$-modules $X_\infty^{\chi_1^{-1}\chi_2}$ and $Y_\infty^{\chi_1^{-1}\chi_2}$ are cyclic. Then $I(\rho_\varphi)$ is principal.
Proof. Recall that \( \kappa \) is the character \( \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong 1 + p\mathbb{Z}_p \rightarrow A^\times_\kappa \). Let \( \eta_1 = \chi_1 \) and \( \eta_2 = \chi_2 \kappa_{cyc} \kappa \). We have that \( \eta_1(g) \equiv a(g) \mod I(\rho_\mathfrak{p}) \) and \( \eta_2(g) \equiv d(g) \mod I(\rho_\mathfrak{p}) \) for all \( g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) by Lemma 3.1 and Lemma 3.7. We prove the proposition by using Wiles’ construction (cf. [24, Section 6]) of an unramified extension \( N_\infty \) of \( \mathbb{Q}(\mu_{N_\infty}) \).

Let \( B \) (resp. \( C \)) be an \( \mathfrak{I} \)-submodule of \( \text{Frac}(\mathfrak{I}) \) generated by \( b(g) \) (resp. \( c(g) \)) for all \( g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Since \( c(g)B \) and \( b(g)C \) are ideals of \( \mathfrak{I} \) for all \( g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) by Lemma 3.1, \( B \) and \( C \) are finitely generated. We denote by \( b \) the function

\[
b : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow B, \ g \mapsto b(g)
\]

and we endow \( B \) with the \( \mathfrak{m} \)-adic topology.

Claim 1. \( b \) is continuous.

Proof. Since \( \rho_\mathfrak{p} \) is continuous, by Definition 2.5 there exists a lattice \( \mathcal{T} \subset \text{Frac}(\mathfrak{I})^{\oplus 2} \) which is stable under \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-action such that \( \rho_\mathfrak{p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}_\mathbb{I}(\mathcal{T}) \) is continuous with respect to the \( \mathfrak{m} \)-adic topology of \( \text{Aut}_\mathbb{I}(\mathcal{T}) \). We denote by \( \mathcal{T}_i = \text{Frac}(\mathfrak{I})_{\mathfrak{I}i} \) and by \( \mathcal{T}_i = \mathcal{T} \cap \mathcal{T}_i \) (\( i = 1, 2 \)). Then \( \rho_\mathfrak{p}(\mathcal{T}_i) \subset \mathcal{T} \). For any \( x_1 \in \mathcal{T}_1 \) and \( y_2 \in \mathcal{T}_2 \), we have

\[
\rho_\mathfrak{p} (g)(x_1) = a(g)x_1 + c(g)y_2,
\]

\[
\rho_\mathfrak{p} (g)(y_2) = b(g)y_2.
\]

Since \( a(g) \in \mathfrak{I} \) by Lemma 3.1, \( a(g)x_1 \in \mathcal{T} \cap \mathcal{T}_1 = \mathcal{T}_1 \) and \( c(g)y_2 = \rho_\mathfrak{p} (g)(x_1) - a(g)x_1 \in \mathcal{T} \cap \mathcal{T}_2 = \mathcal{T}_2 \). We also have \( b(g)y_2 \in \mathcal{T}_1 \) by the same argument. This implies that \( \mathcal{T}_1 \oplus \mathcal{T}_2 \) is also a stable lattice of \( \text{Frac}(\mathfrak{I})^{\oplus 2} \).

We replace \( \mathcal{T} \) with \( \mathcal{T}_1 \oplus \mathcal{T}_2 \). The representation \( \rho_\mathfrak{p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}_\mathbb{I}(\mathcal{T}) \) is also continuous by the Artin-Rees lemma. We may regard \( B \) as an \( \mathfrak{I} \)-submodule of \( \text{Hom}_\mathbb{I}(\mathcal{T}_2, \mathcal{T}_1) \) via the injective homomorphism as follows:

\[
B \hookrightarrow \text{Hom}_\mathbb{I}(\mathcal{T}_2, \mathcal{T}_1), \ b(g) \mapsto b(g)(y_2) = b(g) \cdot y_2
\]

for all \( y_2 \in \mathcal{T}_2 \). Then \( b \) is the following map:

\[
\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \overset{b}{\rightarrow} \text{Aut}_\mathbb{I}(\mathcal{T}) \rightarrow \text{Hom}_\mathbb{I}(\mathcal{T}_2, \mathcal{T}_1).
\]

The homomorphism \( \text{Aut}_\mathbb{I}(\mathcal{T}) \rightarrow \text{Hom}_\mathbb{I}(\mathcal{T}_2, \mathcal{T}_1) \) is continuous under the \( \mathfrak{m} \)-adic topology, hence \( b \) is continuous.

Define \( \overline{b} \) the following homomorphism:

\[
\overline{b} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{N_\infty^\infty})) \overset{b}{\rightarrow} B \hookrightarrow B/I(\rho_\mathfrak{p})B.
\]

Let \( N_\infty^\infty \) be the abelian extension of \( \mathbb{Q}(\mu_{N_\infty^\infty}) \) corresponding to \( \text{Ker} \overline{b} \) and we denote by the same symbol \( \overline{b} \)

\[
\overline{b} : G = \text{Gal}(N_\infty^\infty/\mathbb{Q}(\mu_{N_\infty^\infty})) \hookrightarrow B/I(\rho_\mathfrak{p})B.
\]

For any \( h \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) and \( g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{N_\infty^\infty})) \), a matrix calculation shows that

\[
\overline{b}(hgh^{-1}) = \eta_1 \eta_2^{-1}(h) \overline{b}(g).
\]
Let $\tilde{\gamma}$ be a topological generator of $\text{Gal}(\mathbb{Q}(\mu_{Np^{\infty}})/\mathbb{Q}(\mu_{Np}))$ which is sent to $\gamma$ under the canonical isomorphism $\text{Gal}(\mathbb{Q}(\mu_{Np^{\infty}})/\mathbb{Q}(\mu_{Np})) \to \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. The above arguments tell us that $\overline{b}(G)$ is a $\Lambda_{\chi_1\chi_2^{-1}}$-module under the surjection

$$Z_p[[\text{Gal}(\mathbb{Q}(\mu_{Np^{\infty}})/\mathbb{Q})]] \xrightarrow{\eta_2^{-1}} \Lambda_{\chi_1\chi_2^{-1}}, \ u^{-1}\tilde{\gamma}^{-1} \mapsto 1 + X$$

and $\text{Gal}(\mathbb{Q}(\mu_{Np^{\infty}})/\mathbb{Q}_\infty)$ acts on $\overline{b}(G)$ via $\chi_1\chi_2^{-1}$.

**Claim 2.** The canonical homomorphism $\overline{b}(G) \otimes_{\Lambda_{\chi_1\chi_2^{-1}}} \mathbb{I} \to B/I(\rho,\mathcal{F})B$ is an isomorphism.

**Proof.** The injectivity follows from the assumption that $\mathbb{I}$ is flat over $\Lambda_{\chi_1\chi_2^{-1}}$, by applying the base extension $\otimes_{\Lambda_{\chi_1\chi_2^{-1}}} \mathbb{I}$ to the equation (2). For any $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, consider the commutator $[g, g] \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{Np^{\infty}}))$ we have

$$\overline{b}([g, g]) = \frac{\lambda - 1}{\lambda} \eta_2(g)^{-1} \tilde{\sigma}(g),$$

where $\lambda = \varepsilon_2(g_0)$. Since $\lambda \neq 1 \mod m$, we have $\overline{b}([g, g]) \otimes \eta_2(g) \frac{\lambda}{\lambda - 1} \in \overline{b}(G) \otimes_{\Lambda_{\chi_1\chi_2^{-1}}} \mathbb{I}$. This completes the proof of Claim 2. \qed

$N_\infty/\mathbb{Q}(\mu_{Np^{\infty}})$ is unramified at $p$ by the equation (1). Since the conductor of $\chi_1\chi_2^{-1}$ is $Np$ under the condition (D), $N_\infty/\mathbb{Q}(\mu_{Np^{\infty}})$ is also unramified at the primes dividing $N$ by class field theory (see the proof of [24, Lemma 6.1]). Thus $N_\infty/\mathbb{Q}(\mu_{Np^{\infty}})$ is everywhere unramified.

We fix the Iwasawa-Serre isomorphism as follows:

$$Z_p[\chi_1\chi_2^{-1}[[\text{Gal}(\mathbb{Q}(\mu_{Np^{\infty}})/\mathbb{Q}(\mu_{Np}))]] \xrightarrow{\tilde{\gamma}^{-1}} \Lambda_{\chi_1\chi_2^{-1}}, \tilde{\gamma} \mapsto u^{-1}(1 + X)^{-1}. \quad (3)$$

Then we have the following $\mathbb{I}$-homomorphisms:

$$X_{\chi_1\chi_2^{-1}} \otimes_{\Lambda_{\chi_1\chi_2^{-1}}} \mathbb{I} \to \overline{b}(G) \otimes_{\Lambda_{\chi_1\chi_2^{-1}}} \mathbb{I} \xrightarrow{\sim} B/I(\rho,\mathcal{F})B. \quad (4)$$

By taking the Fitting ideal, we have the inclusion relation as follows:

$$\text{Fitt}_{\Lambda_{\chi_1\chi_2^{-1}}}(X_{\chi_1\chi_2^{-1}}) \mathbb{I} = \text{Fitt}_2(X_{\chi_1\chi_2^{-1}} \otimes_{\Lambda_{\chi_1\chi_2^{-1}}} \mathbb{I}) \subset \text{Fitt}_2(B/I(\rho,\mathcal{F})B) \subset I(\rho,\mathcal{F}).$$

By the Iwasawa main conjecture (Theorem of Mazur-Wiles) we have

$$\text{Fitt}_{\Lambda_{\chi_1\chi_2^{-1}}}(X_{\chi_1\chi_2^{-1}}) = G_{\chi_1^{-1}\chi_2}(u^2(1 + X) - 1)\Lambda_{\chi_1\chi_2^{-1}} = \tilde{G}_{\chi_1^{-1}\chi_2}(X)\Lambda_{\chi_1\chi_2^{-1}}.$$ 

Thus $\tilde{G}_{\chi_1^{-1}\chi_2}(X) \mathbb{I} \subset I(\rho,\mathcal{F})$. This completes the proof of (1) of the proposition.

Similarly, we denote by $M_\infty(Np)$ the abelian extension of $\mathbb{Q}(\mu_{Np^{\infty}})$ corresponding to

$$\text{Ker}(\tilde{\sigma} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{Np^{\infty}})) \to C/I(\rho,\mathcal{F})C, \ g \mapsto \tilde{\sigma}(g))$$

and by $H = \text{Gal}(M_\infty(Np)/\mathbb{Q}(\mu_{Np^{\infty}}))$. Then $\tilde{\sigma}(H)$ is a $\Lambda_{\chi_1^{-1}\chi_2}$-module under the surjection
Lemma 3.7. Thus the canonical isomorphism $B(T) \otimes_{A_{\chi_1^{-1} \chi_2}^{\infty}} \mathbb{I} \to C/I(\rho_\mathcal{F})C$ induced by $\pi$ is an isomorphism by the same arguments as in Claim 2. Hence we have the surjective homomorphism as follows:

$$Y_{\infty}^{\chi_1^{-1} \chi_2} \otimes_{A_{\chi_1^{-1} \chi_2}^{\infty}} \mathbb{I} \twoheadrightarrow C/I(\rho_\mathcal{F})C.$$  

(5)

Note that in the equation (5), we endowed $Y_{\infty}^{\chi_1^{-1} \chi_2}$ with the $A_{\chi_1^{-1} \chi_2}^{\infty}$-module structure under the isomorphism as follows:

$$\mathbb{Z}_p[\mathbb{I}][[\text{Gal}(\mathbb{Q}(\mu_{N_p^\infty})/\mathbb{Q})]]^{\hat{\eta}_2-1} \otimes_{A_{\chi_1^{-1} \chi_2}^{\infty}} \mathbb{I} \twoheadrightarrow \mathbb{Z}_p[\mathbb{I}][[\text{Gal}(\mathbb{Q}(\mu_{N_p^\infty})/\mathbb{Q}(\mu_{N_p^\infty}))]] \cong A_{\chi_1^{-1} \chi_2}^{\infty}, \quad u^{-1}\gamma \mapsto u(1 + X).$$

By the equations (4) and (5), there exists a $g_B \in X_\infty$ (resp. $g_C \in Y_\infty$) such that $B/I(\rho_\mathcal{F})B$ (resp. $C/I(\rho_\mathcal{F})C$) is generated by $\rho(g_B)$ (resp. $\rho(g_C)$). By Nakayama’s lemma, $B$ (resp. $C$) is generated by $b(g_B)$ (resp. $c(g_C)$) over $\mathbb{I}$. This implies $I(\rho_\mathcal{F}) = BC = (b(g_B), c(g_C))$.

Define the Eisenstein ideal $I(\chi, \mathbb{I})$ the ideal of $T(\chi, \mathbb{I})$ which is generated by $T(l) - 1 - \eta(\text{Frob}_p)$ for all primes $l \neq p$ and $T(p) - 1$.

**Corollary 3.9.** Let the assumptions and the notations be as in Theorem 1.5. Assume the condition (R). We have $I(\rho_\mathcal{F}) = \hat{G}_\chi(X) \mathbb{I}$.

**Proof.** Since $T(\chi, \mathbb{I}) = T(\chi, \Lambda_\chi) \otimes_{A_\chi} \mathbb{I}$, $T(\chi, \mathbb{I})$ is isomorphic to $\mathbb{I}$ by the assumption (R). Since $N = 1$, $I(\rho_\mathcal{F})$ is generated by $c(l, \mathcal{F}) - 1 - \eta(\text{Frob}_l)$ for all primes $l \neq p$ by Lemma 3.7 and the Chebotarev density theorem. We also have $c(p, \mathcal{F}) - 1 = \varepsilon_1(\text{Frob}_p) - 1 = a(\text{Frob}_p) - 1 \in I(\rho_\mathcal{F})$ by Theorem 2.7 and Lemma 3.7. Thus the canonical isomorphism $\mathbb{I} \to T(\chi, \mathbb{I})$ sends $I(\rho_\mathcal{F})$ to the Eisenstein ideal $I(\chi, \mathbb{I})$. On the other hand, the canonical homomorphism

$$\mathbb{I} \to \hat{G}_\chi(X) \mathbb{I} \to T(\chi, \mathbb{I})/ (I(\chi, \mathbb{I}), \hat{G}_\chi(X))$$

is an isomorphism by [24, Theorem 4.1]. This implies $I(\rho_\mathcal{F}) \subset \hat{G}_\chi(X) \mathbb{I}$. Hence they must be coincide by Proposition 3.8.

The next corollary is obviously deduced from (2) of Proposition 3.8.

**Corollary 3.10.** Let the assumptions and the notations be as in Theorem 1.5. Assume the conditions (C) and (P). We have $I(\rho_\mathcal{F}) = \hat{G}_\chi(X) \mathbb{I}$.

Now we prove Theorem 1.5.

**Proof of Theorem 1.5.** For any $\phi \in X_1$, let $\omega_\phi$ be a fixed uniformizer of $\phi(\mathbb{I})$. Then

$$\text{ord}_{\omega_\phi}(\phi(I(\rho_\mathcal{F}))) \leq \text{ord}_{\omega_\phi}(\hat{G}_{\chi_1^{-1} \chi_2}(\zeta_\phi u^{k_\phi-2} - 1))$$

by (1) of Proposition 3.8. Since $\chi_1 \neq \chi_2$, the character $\chi_{\zeta_\phi}^{-1} \chi_1^{-1} \chi_2$ does not factor through $\text{Gal}((\mathbb{Q}_\infty)/\mathbb{Q})$. Thus

$$L_p(1 - k_{\phi}, \chi_{\zeta_\phi}^{-1} \chi_1^{-1} \chi_2) = \hat{G}_{\chi_1^{-1} \chi_2}(\zeta_\phi u^{k_\phi-2} - 1)$$
by Section 2.3. Combine Proposition 3.4 and Lemma 3.6 we have
\[ \text{ord}_p (\varphi (I (\rho \varphi))) + 1 = \mathcal{L} (\rho f_\varphi), \]
then
\[ \mathcal{L} (\rho f_\varphi) \leq \text{ord}_p (L_p (1 - k\varphi, \chi \zeta_1 \chi_1^{-1} \chi \omega)) + 1. \]

If we assume the condition (R) or both of the conditions (C) and (P), we have
\[ I (\rho \varphi) = \tilde{G}_{\chi_1 \chi_2} (X) I \]
by Corollary 3.9 and Corollary 3.10. Then
\[ \mathcal{L} (\rho f_\varphi) = \text{ord}_p (L_p (1 - k\varphi, \chi \zeta_1 \chi_1^{-1} \chi \omega)) + 1. \]
Specially when (R) satisfied, \( \chi_1 = 1 \) and \( \chi_2 = \chi \) by Lemma 3.7. This completes the proof of Theorem 1.5.

3.3 Discussion of the variation of \( \mathcal{L} (\rho f_\varphi) \) by means of \( L_p \)

We use the following lemma to prove Corollary 1.6.

Lemma 3.11. Let \( \mathcal{O} \) be the ring of integers of a finite extension of \( \mathbb{Q}_p \) and \( F(X) \in \mathcal{O}[X] \) a distinguished polynomial. Then there exists an integer \( r \in \mathbb{Z}_{\geq 0} \) such that for any \( (k, \zeta) \in \mathbb{Z}_{\geq 0} \times (\mu_{p \infty} \setminus \mu_{p r}), \)
\[ \text{ord}_p (F (\zeta u^k - 1)) = \frac{\deg F(X)}{(p - 1) p^{r - 1}}, \]
where \( \zeta \) is a primitive \( p^r \)-th root of unity.

Proof. Decompose
\[ F(X) = \prod_{i=1}^{n} (X - \alpha_i) \]
and choose an integer \( r \geq 0 \) such that \( \text{ord}_p (\alpha_i) > \frac{1}{(p - 1) p^{r - 1}} \) for any \( \alpha_i \). Then for any \( (k, \zeta) \in \mathbb{Z}_{\geq 0} \times (\mu_{p \infty} \setminus \mu_{p r}), \) we have
\[ \text{ord}_p (\zeta u^k - 1 - \alpha_i) = \text{ord}_p (\zeta (\exp (k \cdot \log(u)) - 1) + (\zeta - 1) - \alpha_i) \]
\[ = \frac{1}{(p - 1) p^{r - 1}}, \]
where \( \exp \) and \( \log \) are the \( p \)-adic exponential and logarithm functions. Thus
\[ \text{ord}_p (F (\zeta u^k - 1)) = \sum_{i=1}^{n} \text{ord}_p (\zeta u^k - 1 - \alpha_i) = \frac{\deg F(X)}{(p - 1) p^{r - 1}}. \]

Let us return to the proof of Corollary 1.6.
Proof of Corollary 1.6. For (1) it is sufficient to show that there exists an \( r \in \mathbb{Z}_{\geq 0} \) such that for any \( \varphi \in \mathbb{X}^{(r)}_1 \),

\[
\text{ord}_{\varphi}(L_p(1 - k_{\varphi}, \chi_{\varphi} \chi_1^{-1} \chi_2 \omega)) \leq \text{rank}_{\mathbb{A}} I \cdot \deg \hat{G}^{*}_{\chi_1^{-1} \chi_2}(X).
\]

Since \( \hat{G}^{*}_{\chi_1^{-1} \chi_2}(X) \) is not divisible by a uniformizer of \( \mathbb{Z}_p[\chi_1^{-1} \chi_2] \) by the Ferrero-Washington’s theorem \([7]\), the Weierstrass preparation theorem enables one to decompose

\[
\hat{G}^{*}_{\chi_1^{-1} \chi_2}(X) = \hat{G}^{*}_{\chi_1^{-1} \chi_2}(X)U(X),
\]

where \( \hat{G}^{*}_{\chi_1^{-1} \chi_2}(X) \) is a distinguished polynomial and \( U(X) \) a unit in \( \mathbb{A}_{\chi_1^{-1} \chi_2} \). We apply Lemma 3.11 to \( \hat{G}^{*}_{\chi_1^{-1} \chi_2}(X) \). Then there exists an \( r \in \mathbb{Z}_{\geq 0} \) such that for any \( \varphi \in \mathbb{X}^{(r)}_1 \),

\[
\text{ord}_{\varphi}(\varphi(\hat{G}^{*}_{\chi_1^{-1} \chi_2}(X))) = \frac{\deg \hat{G}^{*}_{\chi_1^{-1} \chi_2}(X)}{(p - 1)^{r^{\phi} - 1}},
\]

where \( \varphi |_{\mathbb{A}} = \nu_{\mathbb{A}, \varphi} \) and \( \zeta_{\varphi} \) is a primitive \( p^{\phi} \)-th root of unity such that \( r_{\varphi} > r \).

Let us take a \( \varphi \in \mathbb{X}_1^{(r)} \). For the extension of the discrete valuation rings \( \varphi(\mathbb{I}) \supset \mathbb{Z}_p[\chi][\zeta_\varphi] \), since \( [\text{Frac}(\varphi(\mathbb{I})) : \text{Frac}(\mathbb{Z}_p[\chi][\zeta_\varphi])] \leq \text{rank}_{\mathbb{A}} I \), so is the ramification index \( e_{\varphi} \). Since \( r_{\varphi} > 0 \), the ramification index in the extension \( \mathbb{Z}_p[\chi][\zeta_\varphi] \supset \mathbb{Z}_p \) is \( (p - 1)^{r^{\phi} - 1} \). Then by the equation (6), we have

\[
\text{ord}_{\varphi}(L_p(1 - k_{\varphi}, \chi_{\varphi} \chi_1^{-1} \chi_2 \omega)) = \text{ord}_{\varphi}(\varphi(\hat{G}^{*}_{\chi_1^{-1} \chi_2}(X)))
= e_{\varphi}(p - 1)^{r^{\phi} - 1} \frac{\deg \hat{G}^{*}_{\chi_1^{-1} \chi_2}(X)}{(p - 1)^{r^{\phi} - 1}}
= e_{\varphi} \deg \hat{G}^{*}_{\chi_1^{-1} \chi_2}(X)
\leq \text{rank}_{\mathbb{A}} I \cdot \deg \hat{G}^{*}_{\chi_1^{-1} \chi_2}(X).
\]

This completes the proof of (1) of Corollary 1.6 and (2) is easily deduced from (1).

Now we assume that \( I \) is isomorphic to \( \mathcal{O}[[X]] \) with \( \mathcal{O} \) the ring of integers of a finite extension \( K \) of \( \mathbb{Q}_p \). We choose a uniformizer \( \varpi \) of \( \mathcal{O} \). Let \( f_1(X), \ldots, f_m(X) \) be the generators of \( I(\rho \varphi) \). For each \( i = 1, \ldots, m \), decompose

\[
f_i(X) = \varpi^\mu_i P_i(X)U_i(X),
\]

where \( P_i(X) \) is a distinguished polynomial and \( U_i(X) \) a unit in \( \mathcal{O}[[X]] \). Let

\[
F(X) = \prod_{i=1}^m P_i(X).
\]

We apply Lemma 3.11 to \( F(X) \). Then there exists an \( r_1 \in \mathbb{Z}_{\geq 0} \) such that for any \( \varphi \in \mathbb{X}^{(r_1)}_1 \),

\[
\text{ord}_{\varphi}(f_i(X)) = \mu_i \text{ord}_{\varphi} \varpi + \frac{\deg P_i(X)}{(p - 1)^{r^{\phi} - 1}}.
\]

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Then the argument above also holds, i.e. there exists a constant \( b \) by the equation (8). Combine the equation (9) and (11), we have that \( K_{(p \cdot 2)} \) for each 1\( \leq s \leq 1.5 \). We fix a \( \mathcal{Q} \) index of coincides, so are the ramification index. Hence the ramification index of \( \mathcal{K} \) is isomorphic to \( \mathcal{O}[\mathcal{X}] \), the residue degree in \( \mathcal{K}_{(p \cdot 2)} \) \( \mathcal{K} \). If \( r_2 = 0 \), we may enlarge \( \mathcal{O} \) to \( \mathcal{O}' = \mathcal{O}[\mathcal{Q}] \) since \( \mathcal{O}[\mathcal{Q}] = \mathcal{O}'[\mathcal{Q}] \) for \( \mathcal{P} \in \mathcal{X}^{(p \cdot 2)} \). Then the argument above also holds, i.e. there exists a constant \( e \) such that the ramification index of \( \mathcal{K}_{(p \cdot 2)} \) over \( \mathcal{Q}_p \) is \( e(p - 1)p^r - 1 \). Note that \( e \) is the ramification index of \( \mathcal{K}_{(p \cdot 2)} / \mathcal{Q}_p \) if \( r_2 = 0 \).

Since \( \mathcal{G}_{\chi_1, \chi_2}^{1, \mathcal{X}}(\mathcal{Y}) \subseteq \mathcal{I}(\mathcal{Q}) \) and \( \omega \mid \mathcal{G}_{\chi_1, \chi_2}^{1, \mathcal{X}}(\mathcal{X}) \), we have that

\[
\mathcal{Z} = \{ i = 1, \ldots, m \mid \mu_i = 0 \}
\]

is nonempty. Let

\[
l = \min \{ \deg \mathcal{P}_i(\mathcal{X}) \mid i \in \mathcal{Z} \}
\]

and let us take an \( r_3 \in \mathbb{Z}_{\geq 0} \) such that for any \( i \notin \mathcal{Z} \),

\[
(p - 1)p^r - 1 \mu_i \ord_p \omega + \deg \mathcal{P}_i(\mathcal{X}) \geq l. \tag{8}
\]

Let \( r' = \max \{ r_1, r_2, r_3 \} \) and let us take a \( \varphi \in \mathcal{X}^{(p \cdot 3)} \). Then we have

\[
\sharp \mathcal{L}(\mathcal{Q}_p) = \min \{ \ord_{\varphi}(\mathcal{P}_i(\mathcal{X})) \mid 1 \leq i \leq n \} + 1. \tag{9}
\]

Since the ramification index of \( \mathcal{K}_{(p \cdot 2)} \) over \( \mathcal{Q}_p \) is \( e(p - 1)p^r - 1 \), we have

\[
\ord_{\varphi}(\mathcal{P}_i(\mathcal{X})) = e(p - 1)p^r - 1 \mu_i \ord_p \omega + e \cdot \deg \mathcal{P}_i(\mathcal{X}) \tag{10}
\]

for each \( 1 \leq i \leq n \) by the equation (7). Thus

\[
\min \{ \ord_{\varphi}(\mathcal{P}_i(\mathcal{X})) \mid 1 \leq i \leq n \} = el \tag{11}
\]

by the equation (8). Combine the equation (9) and (11), we have that \( \sharp \mathcal{L}(\mathcal{Q}_p) = el + 1 \) is constant. This completes the proof of (3) of Corollary 1.6.

Now we assume the condition (R) or both of the conditions (C) and (P). We have \( \sharp \mathcal{L}(\mathcal{Q}_p) = \ord_{\varphi}(\mathcal{L}_p(1 - k_\varphi, \chi_\varphi, \chi_1^{-1} \chi_2 \omega)) + 1 \) by (2) and (3) of Theorem 1.5. We fix a \( \zeta \in \mu_{p \cdot \infty} \). First we assume that \( \mathcal{L}_p(1 - s, \chi_\varphi, \chi_1^{-1} \chi_2 \omega) \) has a zero \( s_0 \in \mathbb{Z}_p \). Let \( \{ k_n \} \) be the sequence defined as follows:

(i) \( k_n = s_0 + p^i \) if \( s_0 \in \mathbb{Z} \),

(ii) \( k_n = \sum_{i=0}^{n} a_ip^i \) if \( s_0 = \sum_{i=0}^{\infty} a_ip^i \) such that \( 0 \leq a_i \leq p - 1 \) and \( s_0 \notin \mathbb{Z} \).

Then \( \sharp \mathcal{L}(\mathcal{Q}_p) \) is unbounded when \( k_\varphi \) runs over the sequence \( \{ k_n \} \).

Now we suppose that \( \mathcal{L}_p(1 - k_\varphi, \chi_\varphi, \chi_1^{-1} \chi_2 \omega) \) has no zero in \( \mathbb{Z}_p \) and we prove that \( \ord_{\varphi}(\mathcal{L}_p(1 - k_\varphi, \chi_\varphi, \chi_1^{-1} \chi_2 \omega)) \) is bounded by contradiction. Suppose that
\(\text{ord}_\pi \left( L_p \left( 1 - k\pi, \chi\zeta^{-1}_1 \chi_2 \omega \right) \right) \) is unbounded. Then there exists a sequence \(\{k_n\} \) such that \(k_n \geq 2\) and
\[
\lim_{n \to \infty} L_p \left( 1 - k_n, \chi\zeta^{-1}_1 \chi_2 \omega \right) = 0.
\]
Since \(\mathbb{Z}_p\) is compact and \(L_p\) is a continuous function, \(L_p \left( s, \chi\zeta^{-1}_1 \chi_2 \omega \right)\) must have zero in \(\mathbb{Z}_p\) which contradicts to our assumption. Hence \(\mathcal{L}(\rho_{f_\varphi})\) is bounded. This completes the proof of (4) of Corollary 1.6.

### 3.4 Proof of Corollary 1.7

We denote by \(F\) the residue field \(\mathbb{Z}/m\). The following lemma is a generalization of the arguments in [11, Appendix I] for more general settings.

**Lemma 3.12.** Let the assumptions and the notations be as in Theorem 1.5. Assume the conditions (D), (C), (P) and (F). Let \(\mathcal{T}\) be a stable lattice which is free over \(I\). Then \(\mathcal{T} \otimes I\varphi(I)\) is a mod \(\varpi\) semi-simple lattice for any \(\varphi \in \mathfrak{X}_i\).

**Proof.** We have \(I \left( \rho \varphi \right) = \hat{G}_{\chi^{-1}_1 \chi_2}(X)\mathbb{Z}\) under the conditions (D), (C) and (P) by Corollary 3.10. Let us take a stable lattice \(\mathcal{T} \cong \mathbb{I}^{\oplus 2}\) and we consider the following representation:
\[
\rho = \rho_{\varphi, \mathcal{T}} : \text{Gal} \left( \overline{\mathbb{Q}}/\mathbb{Q} \right) \to \text{GL}_2(I).
\]
Write \(\mathcal{L} = \hat{G}_{\chi^{-1}_1 \chi_2}(X)\mathbb{Z}\) for short. The condition (P) enables us to define \(\text{Frac}(I/\mathcal{L})\) and \(\text{Frac}(I/\mathcal{L})\) is of characteristic zero by the Ferrero-Washington theorem. We denote by \(\rho \bmod \mathcal{L}\) the representation as follows:
\[
\rho \bmod \mathcal{L} : \text{Gal} \left( \overline{\mathbb{Q}}/\mathbb{Q} \right) \xrightarrow{\rho} \text{GL}_2(I) \xrightarrow{\bmod \mathcal{L}} \text{GL}_2(I/\mathcal{L}).
\]

Since \(\text{tr}_\rho \rho \bmod \mathcal{L}\) is the sum of two characters, we have \(\rho \bmod \mathcal{L}\) is reducible by the Brauer-Nesbitt theorem. Let \(\{v_1, v_2\}\) be a basis corresponding to \(\rho \bmod \mathcal{L}\) such that \((I/\mathcal{L})v_1\) is stable under \(\rho \bmod \mathcal{L}\). Let \(\tilde{v}_1 \in \mathcal{T}\) be a lift of \(v_1\) \((i = 1, 2)\).

Since \(I\) is complete under the \(m\)-adic topology, \(\mathcal{T}\) is generated by \(\tilde{v}_1\) and \(\tilde{v}_2\) over \(I\). Since \(\bigcap_n \mathbb{Z}^n = \{0\}\), we have \(\mathcal{T} = I\tilde{v}_1 \oplus I\tilde{v}_2\). Thus \(\mathcal{T}' = I\tilde{v}_1 \oplus I\tilde{v}_2\) is also a stable \(I\)-free lattice and \(\mathcal{T}/\mathcal{T}' \cong \mathbb{I}/\hat{G}_{\chi^{-1}_1 \chi_2}(X)\mathbb{Z}\).

For any \(\varphi \in \mathfrak{X}_i\), we denote by \(T, T'\) the lattices \(\mathcal{T} \otimes I\varphi(I)\) and \(\mathcal{T}' \otimes I\varphi(I)\). Let \(\mathcal{L}(\rho_{f_\varphi}) = n + 1\). Since \(I \left( \rho_{\varphi} \right) = \hat{G}_{\chi^{-1}_1 \chi_2}(X)\mathbb{Z}\), we have
\[
T/T' = \frac{\mathcal{T} \otimes I\varphi(I)}{\mathcal{T}' \otimes I\varphi(I)} \cong (\mathcal{T}/\mathcal{T}') \otimes I\varphi(I) \cong \left( \mathbb{I}/\hat{G}_{\chi^{-1}_1 \chi_2}(X)\mathbb{Z} \right) \otimes I\varphi(I) \cong \varphi(I)/\varpi^n.
\]

Thus \(d([T], [T']) = n\). Since \(\mathcal{L}(\rho_{f_\varphi})\) is a segment by (6) of Proposition 2.2 and Proposition 3.4, \([T]\) has exactly one neighbor in \(\mathcal{L}(\rho_{f_\varphi})\). Thus \(T\) is a mod \(\varpi\) semi-simple lattice by (3) of Proposition 2.2.

\(\square\)

Under the above preparation, we return to the proof of Corollary 1.7.

**Proof of Corollary 1.7.** Let us take an \(m \in \mathbb{Z}_{>0}\). Let \(\zeta \in \mu_{p^\infty}\) such that \(L_p \left( 1 - s, \chi\zeta^{-1}_1 \chi\omega \right)\) has a zero in \(\mathbb{Z}_p\). Then Corollary 1.6 (4) tells us that there exists a \(\varphi \in \mathfrak{X}_{\delta\zeta}\) such that \(\mathcal{L}(\rho_{f_\varphi}) = n + 1 > m\).
Now we fix such $\varphi$ and we denote by $\mathcal{P} = \ker \varphi$. Let $\mathcal{T}$ be the stable lattice which satisfies the condition (F). We denote by $T = \mathcal{T} \otimes \varphi(1)$ and let

$$\pi : \mathcal{T} \twoheadrightarrow \mathcal{T} \otimes \varphi(\mathbb{I}) = T$$

be the reduction map. We have $T$ is a mod $\varpi$ not semi-simple by Lemma 3.12.

Let

$$\mathcal{L}(\rho_{f_\varphi}) = \{ [T], [T_1], \ldots, [T_n] \}$$

such that for any $1 \leq i \leq n$, $T/T_i \cong \varphi(\mathbb{I})/(\varpi^i)\varphi(\mathbb{I})$ as a $\varphi(\mathbb{I})$-module. We denote by $\mathcal{T}_i = \pi^{-1}(T_i)$. Since $\mathcal{P}T \subset \mathcal{T}_i \subset \mathcal{T}$, $\mathcal{T}_i$ is a lattice. By the definition of $\mathcal{T}_i$ we have $\mathcal{T}_i$ is stable under the Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Thus we obtain stable $\mathbb{I}$-lattices

$$\mathcal{T} \supset \mathcal{T}_1 \supset \ldots \supset \mathcal{T}_n.$$

For $i \neq j$, if there exists an $\mathbb{I} \langle \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rangle$-isomorphism $\Xi : \mathcal{T}_i \cong \mathcal{T}_j$, then $\Xi$ induces a $\varphi(\mathbb{I}) \langle \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rangle$-isomorphism

$$\mathcal{T}_i \cong \mathcal{T}_j, \nu \otimes 1 \mapsto \Xi(\nu) \otimes 1$$

in $\mathcal{T} \otimes \varphi(\mathbb{I})$. For $i \neq j$, $\mathcal{T}_i$ and $\mathcal{T}_j$ are non-isomorphic to each other by Proposition 3.4. This contradicts to our assumption. Hence $\mathcal{T}_i$ and $\mathcal{T}_j$ are non-isomorphic to each other and $\sharp \mathcal{L}(\rho_{\varphi}) \geq n + 1 > m$. This completes the proof of Corollary 1.7.

Remark 3.13. By Corollary 3.9, we also have $I(\rho_{\varphi}) = \hat{G}_{\chi_1, \chi_2}(X) \mathbb{I}$ is a prime ideal under the conditions (D), (R) and (P). Thus Corollary 1.7 is also satisfied if we assume the conditions (D), (R), (P) and (F).

4 Examples

Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ and $f \in S_2(k, \Gamma_0(M), \varepsilon, \mathcal{O})$ a newform. Assume that the eigenvalue $a(p, f)$ of $f$ for the Hecke operator $T_p$ is a $p$-adic unit. We define $f^* \in S_2(k, \Gamma_0(Mp), \varepsilon, \mathcal{O})$ by $f^* = f(q) - \beta f(q^p)$, where $\beta$ is the unique root of $x^2 - a(p, f)x + \psi(p)p^{k-1}$ with $p$-adic absolute $|\beta| < 1$. We call this $f^*$ the $p$-stabilized newform associated to $f$.

Let $(p, k_0)$ be the irregular pair such that $p \mid B_{k_0}$. We give two examples as follows:

1. $(p, k_0) = (691, 12)$. Let $\Delta \in S_{12}(\text{SL}_2(\mathbb{Z}))$ be the Ramanujan’s cuspform. Since $\dim S_{12}(\text{SL}_2(\mathbb{Z})) = 1$, there exists an unique $\Lambda$-adic normalized Hecke eigen cusp form $\mathcal{F} \in S_{\text{ord}}(\omega^{11}, \Lambda)$ such that

$$\mathcal{F}(u^{10} - 1) = \Delta^*$$

and $T(\omega^{11}, \Lambda)$ is isomorphic to $\Lambda$ (see [9, §7.6]), where $\Lambda = \mathbb{Z}_p[[X]]$. Hence $I(\rho_{\varphi}) = \hat{G}_{\omega^{11}}(X) \mathbb{I}$. By Corollary 3.9. The ideal generated by the Iwasawa power series $\hat{G}_{\omega^{11}}(X)$ is equal to $\langle X - a_{\omega^{11}} \rangle$ with $a_{\omega^{11}} \in p\mathbb{Z}_p \setminus p^2\mathbb{Z}_p$ which is calculated by Iwasawa-Sims (see [22, §1]). Then we have the following statements:
L(ρfϕ) is unbounded when ϕ varies in $X_{Λ,1}$ by (4) of Corollary 1.6.

(ii) $L(ρfϕ) = 2$ is constant when ϕ varies in $X_{Λ}^{(0)}$ by (1) and (3) of Corollary 1.6.

(iii) For each $k ≥ 2$, $L(ρfϕ)$ is bounded with maximum value $\text{ord}_p(L_p(1-k, χω)) + 1$ when ϕ varies in $X_{Λ,k}$ by (i) and (ii).

(iv) Since $I = Λ$ is a regular local ring, for a stable Λ-lattice $T$, we have that $T^{**} = \text{Hom}_Λ(\text{Hom}_Λ(T, Λ), Λ)$ is a Λ-free lattice. Hence the condition (F) is satisfied and we have $L(ρF) = ∞$ by Corollary 1.7.

**Remark 4.1.** [11, Appendix II] tells us that for the irregular pair $(p, k_0)$ with $p < 10^7$ and $k_0 < 8000$ such that $p | B_0$, $T(χ, Λ)$ is isomorphic to Λ except for $(p, k_0) = (547, 486)$. Hence we can apply Theorem 1.5 (2) for these pairs.

2. $(p, k_0) = (547, 486)$. By [11, Appendix II], there is a conjugate pair of $p$-stabilized newforms of weight 486 with the required Eisenstein congruence condition mod 547 and the corresponding Hida Hecke algebra $T(ω^{485}, Λ)$ is finite flat of rank two over Λ. We denote by $f_{486}, f_{′486}$ the corresponding newforms.

Let $F$ (resp. $F′$) be the I-adic normalized Hecke eigen cusp form associated to $f_{486}$ (resp. $f_{′486}$). Note that I is an integral closure of a quotient of $T(ω^{484}, Λ)$ by a minimum prime ideal of Λ by the proof of [9, §7.4, Theorem 7]. Hence Frac(I) is a quadratic extension of Frac(Λ). The ideal generated by the Iwasawa power series $\left(\hat{G}_{ω^{485}}(X)\right)$ is equal to $(X - a_{ω^{485}})$ with $a_{ω^{485}} ∈ p\mathbb{Z}_p \setminus p^2\mathbb{Z}_p$ which is calculated by Iwasawa-Sims (see also [22, §1]). Then $L(ρ_{f_2}) ∈ \{2, 3\}$ when ϕ varies in $X_{I}^{(0)}$ by (1) of Corollary 1.6. The condition (C) satisfied for $F$ (this is because the Vandiver’s conjecture is true for $p = 547$), thus $I(ρ_{F′})$ is a principal ideal which is generated by a factor of $X - a_{ω^{485}}$ in I by Corollary 3.10. The same holds for $F′$.

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