A physical eigenstate for Loop Quantum Gravity

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The absence of driving experimental tests is one of the main challenges for quantum gravity theories. In Loop Quantum Gravity (LQG) the quantisation of General Relativity leads to precise predictions for the eigenvalues of geometrical observables like volume and area, up to the value of the only free parameter in the theory, the Barbero-Immirzi (BI) parameter. With the help of the eigenvalues equation for the area operator, LQG successfully derives the Bekenstein-Hawking entropy of large black holes with isolated horizon, fixing in this limit the BI parameter as $\gamma \approx 0.274$. Any further empirical test should give this same value. In the present paper we first show that a black hole with angular momentum $\hbar$ and Planck mass is eigenstate of the area operator provided that $\gamma = \sqrt{\frac{3}{4}} \approx 1.05 \times 0.274$. As the black hole is extremal, there is no Hawking radiation and the horizon is isolated. We then show, with error $< 0.1\%$, that such a black hole can be formed in the head-on scattering of two parallel magnetic dipoles carried by Standard Model neutrinos in the mass state $m_2$ (assuming $m_1 = 0$). Finally, we use the obtained BI parameter to numerically compute the entropy of isolated horizons with areas ranging up to $160\ell_P^2$, by counting the number of micro-states associated to a given area. For area bins of $0.5\ell_P^2$, the resulting entropy has a leading term $S \approx 0.249A$, in agreement to the Bekenstein-Hawking entropy.

I. INTRODUCTION

As well known, the quantisation of gravity suffers, among others difficulties, from the non-convergence of its perturbative expansions, related to the absence of an adimensional coupling constant, contrary to what happens in other gauge theories [1]. This has lead to the development of non-perturbative approaches, from which Loop Quantum Gravity (LQG) is probably the most complete from a theoretical viewpoint [2][3]. Another difficulty is related to the absence of empirical facts that could drive the postulation of quantisation rules. In spite of that, LQG has successfully builded consistent conjugate operators and their commutation relations, up to a free parameter that fixes a particular quantum representation, the Barbero-Immirzi (BI) parameter. Once this parameter is determined by any experiment, additional tests can rule out the theory, which in this way is falsifiable. The task is to find at least two independent tests in the realm of a so weak interaction and so small scales. Surprisingly enough, the theory has been confronted to the derivation of the horizon entropy of large black holes [1], explaining the linear relation between entropy and horizon area and fixing the BI parameter as $\gamma \approx 0.274$ in order to have the expected slope of $1/4$. This is done by counting the number of spin networks - the quantum states in the space of loops - that generate a given horizon area. In this count the horizon is assumed isolated as usually done for thermodynamic systems.

The main goal of this paper is to provide an independent determination of the BI parameter. For that, we will initially identify a physical eigenstate of the LQG area operator, constituted by a black hole of angular momentum $\hbar$ and Planck mass $m_P$ with 99.9% precision [6]. The BI parameter determined in this way differs 5% from the approximate value derived from the entropy of large black holes. As a consistency test, we perform an exact counting of microstates associated to small horizon areas running up to $160\ell_P^2$. The entropy $S(A)$ shows a linear leading term which slope - depending on the adopted area bins - differ by less than 1% from the Bekenstein-Hawking recipe.

II. A PLANCK SCALE BLACK HOLE

The simplest quantum black hole one can probably think is formed in the scattering of two identical, repulsively interacting particles, at a centre-of-mass energy of the order of the Planck scale [7]. If the particles, for instance, have spin 1/2 and carry parallel magnetic moments, the repulsion between the dipoles can lead to the formation of a Kerr black hole with angular momentum...
The formed horizon is stable only in the extremal case, when the surface gravity is zero and there is no Hawking radiation. In this case mass, angular momentum and horizon radius are related by $a^2 = r_H^2 = J$, where $a = J/M$. This leads to $M = m_P$ and $r_H = l_P$. On the other hand, in the extremal limit the horizon area is reduced to $A = 4\pi(r_H^2 + a^2) = 8\pi J$. If we write the angular momentum as the identity

$$J = \frac{\sqrt{3}}{3} \sum_{i=1}^{2} \sqrt{j_i(j_i + 1)},$$

with $j_i = 1/2$ for each of the two dipoles, the horizon area can be written as

$$A = 8\pi\gamma l_P^2 \sum_{i=1}^{4} \sqrt{j_i(j_i + 1)},$$

where $\gamma = \sqrt{3}/6$. This is the eigenvalues equation for the area operator of LQG [12], that fixes in this way the BI parameter. It can be interpreted as a horizon pierced by four spin network lines of colour 1/2 or, equivalently, crossed by two lines, with two punctures per line [6].

Nevertheless, we still should find in Nature an actual physical system with the above features. It is noteworthy that it can indeed be formed in the head-on scattering of parallel neutrinos in a suitable mass state. Dirac neutrinos carry the smallest magnetic moment among the known particles of Standard Model, provided they have mass. Their magnetic dipole originates from vacuum fluctuations and its value involves the weak coupling constant, the fine structure constant and the masses of leptons and gauge bosons. At 1-loop approximation and neglecting lepton masses as compared to gauge boson lepton and gauge bosons. At 1-loop approximation and constant, the fine structure constant and the masses of the known particles of Standard Model, provided they neutrinos carry the smallest magnetic moment among.

III. THE HORIZON ENTROPY

The entropy of a black hole of horizon area $A$ can be found by counting the number $N$ of spin network configurations that satisfy the eigenvalues equation of the area operator [16],

$$A = 8\pi\gamma l_P^2 \sum_{i=1}^{n} \sqrt{j_i(j_i + 1)},$$

where $n$ is the number of points on the horizon pierced by spin network lines of colours $j_i \in \mathbb{Z}_+/2$. Furthermore, the condition of horizon isolation imposes to the punctures a set of second labels $m_i$ that must satisfy the “projection constraint” [17, 18]

$$\sum_{i=1}^{n} m_i = 0,$$

with

$$m_i \in \{-j_i, -j_i-1, \ldots, -j_i-1, j_i\}.$$

In the limit of large horizon areas, condition [5] can be analytically solved up to terms that vanish in the limit $A \rightarrow \infty$, leading to [19]

$$S = \ln N = \frac{\tilde{\gamma}}{4\gamma} A,$$

where $\tilde{\gamma} \approx 0.274$ is the root of

$$1 = \sum_{k=1}^{\infty} (k + 1) \exp \left(-\pi\tilde{\gamma}\sqrt{k(k + 2)}\right).$$

Eq. (8) fits the Bekenstein-Hawking entropy for $\gamma = \tilde{\gamma}$. It is also possible to show that (in the same limit of large areas) the projection constraint [6] does not affect the leading term [8].

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2. For a recent study of the classical Kerr solution in real Ashtekar variables, see [5]. Quantum black holes are discussed e.g. in [9, 10].

3. Here the association between spins and spin network lines is just formal. A more than formal relation is conjectured in [13, 14].

4. The gravitational radius of an extremal Kerr black hole of event horizon radius $r_H = l_P$. which leads to

$$M \approx \frac{9e^2G^2m_P^2}{1024\pi^4}. (4)$$

If $M = m_P$, we have the Kerr solution described above, i.e. a physical eigenstate of the area operator. The no-hair conjecture assures that the magnetic dipoles are not observable from outside when the horizon is formed. Although we do not know the absolute values of the neutrinos masses, flavor oscillation measurements give with precision the gaps between the squared masses. If we assume normal ordering and set the smallest mass $m_1 = 0$, the lightest massive state is $m_2 \approx 8.66 \times 10^{-3}$ eV [15]. From [4] we then have $M \approx 1.001 m_P$. Reversing the argument, we would have an exact eigenstate if $m_2 \approx 8.654$ eV [6]. The difference to the measured value has the same order of 2-loops corrections to Eq. (3).
For small black holes the entropy can be exactly evaluated with the help of appropriate generating functions \cite{18} or by a direct computation of all permitted micro-states for a given area \cite{20}. Although the former allows generate larger areas in shorter times, we have followed the latter procedure for simplicity, which has allowed us to obtain the number of states for areas running up to 160 \( l_p^2 \). The entropy was then found as \( S = \ln N \). The computation consisted in the following algorithm:

(i) fix a value for \( A_0 \) (in units of \( l_p^2 \));

(ii) calculate the maximum number \( n_{\text{max}} \) of punctures for which condition (5) is satisfied, given by the integer part of \( A_0/(4\pi \sqrt{3}) \);

(iii) generate all vectors \( |j_i,m_i\rangle \) of length \( n \leq n_{\text{max}} \) for which \( A \in (A_0 - \delta A, A_0 + \delta A) \) for a chosen semi-interval \( \delta A \), excluding permutations of equals \( j_i \);

(iv) for each allowed vector, find all combinations \( \{j_i,m_i\} \) satisfying the projection constraint (6);

(v) vary \( A_0 \) from \( A_{\text{min}} \) to \( A_{\text{max}} \) with a chosen step.

The prohibition of permutations with equals \( j_i \) comes from the indistinguishability of punctures with equal labels \( (j,m) \). Therefore, a given vector \( |j_i\rangle \) of length \( n \) with \( n_s \) elements \( j_s \) will have multiplicity

\[
\frac{n!}{\prod_s (n_s!)}.
\]

If we do not impose the projection constraint, each vector will also have an additional multiplicity

\[
\prod_s (2j_s + 1)^{n_s}.
\]

We fixed \( \delta A = 0.5l_p^2 \) as in \cite{20} and, to avoid superposition of intervals, varied \( A_0 \) in steps of \( 2\delta A \). Without imposing the projection constraint, and taking \( \gamma = 0.274 \), we reproduced the finds of \cite{20}, i.e. a linear relation between \( S \) and \( A \) with slope 0.2504, in excellent agreement with the large area approximation. The resulting curve (in orange) is shown in Fig. 1. When we impose the projection constraint and still take \( \gamma = 0.274 \), the oscillatory behaviour found in \cite{18} \cite{20} is evidenced. The fitting of a straight line gives a slope 0.254, in contrast to the result 0.237 reported in \cite{20}. Actually, the computation with the projection constraint is sensitive to the adopted semi-interval \( \delta A \) and to the step of variation of \( A_0 \). With \( \gamma = \sqrt{3}/6 \) and without including the projection constraint, a linear relation is recovered with slope 0.238, in agreement to the analytic approximation \cite{18}. When we include the projection constraint the oscillations reappear and the fitting of a straight line gives a slope 0.243. The corresponding curve (in purple) is also shown in Fig. 1. One can see that the oscillations are attenuated for larger areas, as expected in the thermodynamic limit. Note as well that, with or without the projection constraint, the slopes for \( \gamma = 0.274 \) are 1.05 times higher than for \( \gamma = \sqrt{3}/6 \), in accordance to the large areas expression \cite{20} and to the ratio between these values of the BI parameter.

We redone the analysis for \( \gamma = \sqrt{3}/6 \) varying the semiinterval \( \delta A \) in steps of \( 10^{-4}l_p^2 \), for \( A_0 \) running from 50 \( l_p^2 \) to 150 \( l_p^2 \), and we have found the correlations shown in Fig. 1 between the \( S \times A \) slope and the adopted area bin. Without the projection constraint (red line), the slope grows for smaller bins, approximating a maximum around \( \delta A = 0.02l_p^2 \), for which the angular coefficient is 0.246. When the projection constraint is imposed (blue line), the slope approaches a maximum around \( \delta A = 0.25l_p^2 \). At this point the angular coefficient is 0.249, less than 1% below the Bekenstein-Hawking value. The correlations found may possibly be better understood with the help of the counting presented in \cite{18}, based on number theory and combinatorial methods. Nevertheless, let us comment that for large area bins the fine grain structure of micro-states distribution is lost even for the smallest areas, which could explain the low values obtained for the slope with \( \delta A \sim 0.5l_p^2 \), that coincide with the large areas approximation \cite{18} when the projection constraint is not considered. On the other hand, for small area bins the entropy oscillations become pronounced even for the largest areas, allowing in this way the fitting of straight lines with lower inclinations. The competition between these two effects, that are stronger when the projection constraint is imposed, may explain the maximum around \( \delta A \sim 0.25l_p^2 \) and the oscillatory pattern observed. The correspondent results for \( \gamma = 0.274 \) are also shown in Fig. 1. As expected, the curves are shifted by a factor of \( \approx 1.05 \), with a maximum slope of \( \approx 0.263 \).

IV. CONCLUDING REMARKS

The eigenstate presented here has implications that transcend a signature of quantum gravity and the deter-
mination of the BI parameter. Some assumptions were made and finding $m_2$ with 99.9% precision is a consistency test for these assumptions. In the gravity sector we have assumed that quantum gravity corrections (e.g. to the black holes horizon area) are negligible above the Planck length. We have also taken for granted that the scattered magnetic dipoles are no more observed from outside after the horizon formation, leading to a Kerr solution characterised only by $J$ and $M$, as postulated by the no-hair conjecture. Furthermore, the value found for the BI parameter agrees with the Ghosh-Mitra count of states [17], in opposition to the Dogamala-Lewandowski original count [21]. These two counts lead to the same $S \times A$ relation in the large area limit, but with different values for $\tilde{\gamma}$ in Eq. (8) [19]. Nevertheless, the most surprising implication is perhaps the corroboration of some relevant assumptions made in the neutrinos sector. Expression (3) for the neutrinos magnetic moment is only valid if they are Dirac neutrinos, because Majorana neutrinos do not carry magnetic moments. In its derivation it is also assumed the minimal extension of the Standard Model needed to accommodate massive neutrinos, with the addition of right-handed singlets [3]. Finally, the assumption of normal ordering of the mass states, with $m_1 = 0$, was necessary. As we see, the empirical validation of quantum gravity may shed light on other, apparently uncorrelated open questions.

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[1] R. P. Woodard, arXiv:0907.4238 [gr-qc].
[2] A. Ashtekar and J. Lewandowski, Class. Quantum Grav. 21, R53 (2004).
[3] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, 2008).
[4] G. A. Mena Marugán and S. Carneiro, Phys. Rev. D65, 087303 (2002); S. Carneiro, Int. J. Mod. Phys. D12, 1669 (2003).
[5] C. Giunti and A. Studenikin, Rev. Mod. Phys. 87, 531 (2015), Eqs. (4.1)-(4.6).
[6] S. Carneiro, arXiv:1811.08731 [gr-qc].
[7] X. Calmet, B. Carr and E. Winstanley, *Quantum Black Holes* (Springer, 2014).
[8] R. Gambini, E. Mato, J. Olmedo and J. Pullin, Class. Quant. Grav. 36, 125009 (2019).
[9] R. Gambini, J. Olmedo and J. Pullin, Class. Quantum Grav. 31, 095009 (2014).
[10] A. Ashtekar, J. Olmedo and P. Singh, Phys. Rev. Lett. 121, 241301 (2018).
[11] V. P. Frolov and A. Zelnikov, *Introduction to Black Hole Physics* (Oxford University Press, 2011).
[12] C. Rovelli and L. Smolin, Nucl. Phys. B442, 593 (1995), Erratum: Nucl. Phys. B456, 753 (1995).
[13] K. Krasnov, Class. Quantum Grav. 16, L15 (1999).
[14] M. Bojowald, gr-qc/0008054.
[15] I. Esteban et al., JHEP 1701, 087 (2017), Table 1.
[16] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, Phys. Rev. Lett. 80, 904 (1998).
[17] A. Ghosh and P. Mitra, Phys. Lett. B616, 114 (2005).
[18] I. Agulló et al., Phys. Rev. Lett. 100, 211301 (2008); Phys. Rev. D82, 084029 (2010).
[19] K. Meissner, Class. Quantum Grav. 21, 5245 (2004).
[20] A. Corichi, E. F. Borja and J. Díaz-Polo, Class. Quantum Grav. 24, 243 (2007).
[21] M. Dogamala and J. Lewandowski, Class. Quant. Grav. 21, 5233 (2004).