DYNAMICS OF PIEZOELECTRIC BEAMS WITH MAGNETIC EFFECTS AND DELAY TERM

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Abstract. In this paper, we consider a piezoelectric beams system with magnetic effects and delay term. We study its long-time behavior through the associated dynamical system. We prove that the system is gradient and asymptotically smooth, which as a consequence, implies the existence of a global attractor, which is characterized as unstable manifold of the set of stationary solutions. We also get the quasi-stability of the system by establishing a stabilizability estimate and therefore obtain the finite fractal dimension of the global attractor.

1. Introduction. In recent years, we have seen a large number of published works on piezoelectric materials [49, 56, 4, 16, 28, 27, 5]. Piezoelectric materials represent a class of intelligent materials capable of generating electrical energy from mechanical deformations and vice versa, which originate from direct and indirect piezoelectric effects [20]. In terms of applications, piezoelectric materials have become increasingly useful to society and realizing this, the industry has established strategies to deploy such materials in many diverse sectors of human activity, seeking to utilize the maximum of the mechanical energy produced in the machine’s operations, movements of the human body and environmental sources such as waves, winds and other [8].

To achieve the benefits from piezoelectric materials, a lot of investment in technology is needed. In addition, it is need to study more efficient mathematical models which do not neglect important physical phenomena present in modeling. A recent example of this is the magnetic effects in mathematical models of a single piezoelectric beam, which until recently were overlooked because these are very small compared to mechanical effects. However, recent studies [35, 36, 40] showed that

2020 Mathematics Subject Classification. Primary: 35B40, 35B41, 35L53; Secondary: 74K10.
Key words and phrases. Piezoelectric beams, Time delay, Magnetic effects, Quasi-stable system, Global attractor, Exponential attractors.
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magnetic effects due to Maxwell’s equations [33, 34] with one of the essential electric boundary conditions prescribed on the electrodes are capable of significantly modifying the control of such materials. According to Tiersten [52], in modeling piezoelectric beams, there are mainly three approaches to include electric and magnetic effects, i.e.: (i) Electrostatic electric field: In this approach the magnetic effects are ignored, i.e., \( B = \bar{D} = i_b = \sigma_b = 0 \) and Maxwell’s equations become \( \nabla \cdot D = 0 \) and \( \nabla \times E = 0 \). In this case, the Poincaré’s Theorem [14] guarantees the existence of a scalar potential such that \( E = -\nabla \phi \) and \( \phi \) is determined up to a constant. (ii) Quasi-static electric field: In this approach, magnetic effects are also ignored. However, it is allowed that \( B \) and \( \bar{D} \) are non-zero, but \( i_b \) and \( \sigma_b \) are null. Here Maxwell’s equations become \( \nabla \cdot D = 0 \), \( \nabla \cdot B = 0 \), \( \bar{D} = -\nabla \times E \) and \( \bar{D} = \frac{1}{\mu}(\nabla \times B) \). From equation \( \nabla \cdot B = 0 \) we have that there exists a magnetic potential vector \( A \) such that Poincaré’s Theorem \( B = \nabla \times A \). From \( \bar{D} = -\nabla \times E \) and \( B = \nabla \times A \) we have \( \nabla \times (A + E) = 0 \). This implies that there is a scalar electric potential \( \phi \) such that \( E = -A - \nabla \phi \), since \( \text{curl}(\text{grad} \ \phi) = 0 \) of any scalar function \( \phi \). In addition, for \( A, A \ll \phi \) we can consider simplification \( A = \bar{A} = 0 \). Note that \( D \) is nonzero. (iii) Fully dynamic electric field: Here, unlike the quasi-static assumption \( A \) and \( \bar{A} \) are not assumed zero. Furthermore, depending on the type of material, body charge density \( \sigma_b \) and body current density \( i_b \) may also be non-zero.

The mathematical models of a single piezoelectric beam were modeled based on the electrostatic hypothesis due to Maxwell equations [33, 34], which neglects the dynamic interactions between electromagnetism. Therefore, the electrostatic or quasi-static model given by

\[
\rho v_{tt} - \alpha_1 v_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \quad (1.1)
\]

\[
v(0, t) = \alpha_1 v_x(L, t) + \delta v_t(L, t) = 0, \quad t \in \mathbb{R}^+, \quad (1.2)
\]

is exponentially stable (see Tebou [50] and Haraux [19]).

On the other hand, considering the magnetic effects, Morris and Özer [35] used a variational approach to derive the differential equations and boundary conditions that model a single piezoelectric beam with magnetic effects given by

\[
\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \quad (1.3)
\]

\[
\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \quad (1.4)
\]

\[
v(0, t) = p(0, t) = \alpha v_x(L, t) - \gamma \beta p_v(L, t) = 0, \quad t \in \mathbb{R}^+, \quad (1.5)
\]

\[
\beta p_x(L, t) - \gamma \beta v_x(L, t) + \frac{V(t)}{k} = 0, \quad t \in \mathbb{R}^+, \quad (1.6)
\]

where \( \rho, \alpha, \gamma, \mu, \beta \) and \( V(t) = p(t, L, t) \) denote the mass density per unit volume, elastic rigidity, piezoelectric coefficient, magnetic permeability, water resistance coefficient of the beam and the prescribed voltage on electrodes of beam, respectively. In [36] the authors proved that for almost all system parameters, the piezoelectric beam can be strongly stabilized, but it is not exponentially stabilizable in the energy space. In [40] Özer shows that the uncontrolled system is exactly observable in a space larger than the energy space and by using a \( B^* \)-type feedback controller, explicit polynomial decay estimates are obtained for more regular initial data.

Recently, Ramos et al. [44], considered the system of piezoelectric beams with magnetic effects given by

\[
\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + \delta v_t = 0, \quad \text{in} \quad (0, L) \times (0, T), \quad (1.7)
\]

\[
\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0, \quad \text{in} \quad (0, L) \times (0, T), \quad (1.8)
\]
They proved using the energy method, that damping of the friction type \( \delta v_t \), acting in the mechanical equation, is strong enough to exponentially stabilize the energy of the system. In [45] the authors proved the system

\[
\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} = 0 \quad \text{in} \quad (0, L) \times (0, T),
\]

\[
\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0 \quad \text{in} \quad (0, L) \times (0, T),
\]

with boundary conditions

\[
v(0, t) = \alpha v_x(L, t) - \gamma \beta p_x(L, t) = 0, \quad 0 < t < T, \tag{1.13}
\]

\[
p(0, t) = \beta p_x(L, t) - \gamma \beta v_x(L, t) = 0, \quad 0 < t < T, \tag{1.14}
\]

exponentially stable independent of any relation between the coefficients using terms of feedback at the boundary and consequently prove their equivalence with the exact observability at the boundary.

Many strategies are used for the purpose of controlling piezoelectric vibrations [43, 29, 42, 54]. As a control strategy we can mention the time-delayed feedback control that can be used to improve the stability of the system [21, 53]; however, we must be careful when using time-delayed feedback control in hyperbolic systems. The study of the delay effect in the stabilization of hyperbolic systems has been made in recent years and many researchers have shown the so called destabilizing effect. In many cases it was shown that time delays can be a source of instability, same with an arbitrary small delay. That is to say, the time delay may destabilize an evolution equation which is uniformly asymptotically stable in the absence of delay unless some control terms have been used. The first contributions in that direction appear with the papers due to Datko et al. [10, 12, 13, 11]. In particular, Datko [10] considered the equation given by

\[
u_{tt} - u_{xx} + 2u_t(x, t - \tau) = 0, \quad (x, t) \in (0, 1) \times (0, +\infty). \tag{1.15}
\]

He showed that the time delay (represented by \( t - \tau \) for \( \tau > 0 \)) in the damping given by velocity term can destabilize the system. The same result was obtained by Datko et al. [11] for time delay acting on boundary control and the authors showed that the well-behaved hyperbolic system turns in a chaotic system from which they concluded that delay becomes a source of instability. In particular, Datko [12] presented examples of hyperbolic equations which change for an unstable regime by small time delays in the boundary feedback control. On the other hand, in order to stabilize a hyperbolic system with time delay terms, it is necessary to add control terms. In that direction, Nicaise and Pignotti [38] (see also the same authors in [37]) considered the following system

\[
u_{tt} - \Delta u = 0, \quad x \in \Omega, \ t > 0, \tag{1.16}
\]

\[
u(x, t) = 0, \quad x \in \Gamma_0, \ t > 0, \tag{1.17}
\]

\[
\frac{\partial u}{\partial n}(x, t) = \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau), \quad x \in \Gamma_1, \ t > 0, \tag{1.18}
\]

where \( u \) denotes wave propagations in a bounded domain region \( \Omega \subset \mathbb{R}^n \) with a smooth boundary \( \Gamma \) which is divided in two closed and disjoint parts \( \Gamma_0 \) and \( \Gamma_1 \), i.e., \( \Gamma = \Gamma_0 \cup \Gamma_1 \) and \( \Gamma_0 \cap \Gamma_1 = \emptyset \). Moreover, \( \nu \) denotes the outer unit normal
vector \( \frac{\partial u}{\partial \nu} \) is the normal derivative, \( \mu_1 \) and \( \mu_2 \) are two real numbers and \((u_0, u_1, g_0)\) denotes the initial data. The authors prove that under the assumption
\[
\mu_2 < \mu_1, \tag{1.19}
\]
the total energy of solutions is exponentially stable. However, if \( \mu_2 > \mu_1 \), the system \((1.16)-(1.18)\) is unstable.

In this direction we highlight the recent work by Ramos et al. [46] in which the authors consider the system \((1.7)-(1.10)\) with a delay term in the internal state feedback is given by
\[
\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + \xi_1 v_t + \xi_2 v_t(x, t - \tau) = 0 \quad \text{in} \quad (0, L) \times (0, T), \tag{1.20}
\]
\[
\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} = 0 \quad \text{in} \quad (0, L) \times (0, T), \tag{1.21}
\]
where \( \xi_2 v_t(x, t - \tau) \) with \( \xi_2 > 0 \) represents the time delay on the vertical displacement and \( \tau > 0 \) represents the respective retardation time. They proved the existence and uniqueness of the solutions, the exponential stability of the solutions from \((1.20)-(1.21)\) under a constraint in coefficients \( \xi_1 \) and \( \xi_2 \) by using an energy-based approach.

A common point in all the works cited is that they do not consider the presence of external forces or the action of nonlinear source terms or delay. In this article, we consider the following nonlinear piezoelectric beams system with delay term
\[
\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + f_1(v, p) + v_t = h_1, \tag{1.22}
\]
\[
\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} + f_2(v, p) + \mu_1 p_t + \mu_2 p_t(x, t - \tau) = h_2, \tag{1.23}
\]
where \((x, t) \in (0, L) \times (0, T)\), the functions \( f_1(v, p) \) and \( f_2(v, p) \) are nonlinear source terms, \( h_1 \) and \( h_2 \) represent external forces, whereas \( v_t \) and \( p_t \) denote damping in displacement and magnetic current, respectively.

Remark 1.1. As in the study of von Karman’s evolution [22], the non-linearities \( f_i(v, p) \) \((i = 1, 2)\) are given in an abstract way so that they are able to represent a wide range of problems. In addition, they must meet assumptions inherent to the structure of the problem studied, in order to guarantee the limitations dictated by an abstract framework.

This system is subjected to the following initial conditions
\[
v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in (0, L), \tag{1.24}
\]
\[
p(x, 0) = p_0(x), \quad p_t(x, 0) = p_1(x), \quad x \in (0, L), \tag{1.25}
\]
and boundary conditions
\[
v(0, t) = v_x(L, t) = p(0, t) = p_x(L, t) = 0, \quad t \geq 0. \tag{1.26}
\]
For the interaction between nonlinear source terms \( G_1(v, p) := f_1(v, p) + v_t \) and \( G_2(v, p) := f_2(v, p) + \mu_1 p_t + \mu_2 p_t(x, t - \tau) \) produce a stable dissipation in the system, we adopt the condition
\[
\tau \mu_2 \leq \xi \leq \tau (2\mu_1 - \mu_2), \tag{1.27}
\]
with \( \mu_2 < \mu_1 \) and \( \xi > 0 \). Otherwise, the system is unstable (cf. Lemma 2.3). The same condition applies, if the delay term was acting in the equation \((1.22)\).

The issues related to well-posedness of the model, including existence and uniqueness of solutions and qualitative properties of solutions are of fundamental importance within the study of mathematical models that arise in the context of the concrete applications of piezoelectric systems. On the other hand, of particular
interest are the issues related to the long-term behavior of solutions. This property of evolution of the system cannot be tested empirically, so any a priori information about the long-term asymptotic behavior of the system can be used to predict the final response of the dynamic behavior of the solutions.

Daqaq et al.\cite{9} presented an overview of the nonlinearities of piezoelectric materials, which shows the topicality of the theme and the need to consider nonlinear terms such as $f_i(v, p)$ ($i = 1, 2$) in the mathematical models of piezoelectric beams. In addition, as nonlinearities have become an important feature for the use of piezoceramics, many researchers have introduced nonlinear piezoelectric coupling in their work and have thus shown that energy capture can be more or less efficient\cite{23, 24, 25, 2}.

The interesting thing about studying nonlinear piezoelectric models is that many physical phenomena like hysteresis\cite{26}, saturation\cite{17, 47, 32} and others, are present only in non-linear systems. In fact, no physical system is strictly linear, linear constraints are applied only to very small amplitude vibrations\cite{31}. Therefore, in order to study and understand with precision the dynamic behavior of structural systems under general loading conditions, it is essential to consider more general source terms, due to their intrinsic mathematical properties.

We believe that this work is the first to study the dynamics of attractors in the system (1.22)–(1.26) and our main results refer to the existence of global and exponential attractors. The present results essentially follow the methodology of\cite{22}, we show the system is gradient system and asymptotic smoothness, and prove the existence of a global attractor, which is characterized as unstable manifold of the set of stationary solutions. In addition, we establish a stabilizability inequality to get the quasi-stability of the system and therefore obtain the finite fractal dimension of the global attractor and exponential attractors.

Outline of the paper. The work is divided as follows: In Section 2, we introduce some assumptions and well-posedness results. In Section 3, we prove the existence of global attractors with finite fractal dimension (Theorem 4.5). Finally, in Section 5, we prove smoothness properties (Theorem 5.1) and the existence of a generalized fractal exponential attractor (Theorem 5.2).

2. Well-Posedness. In order to obtain the well-posedness of the problem (1.22)–(1.26), consider the following change of variable as found in\cite{39, 48},

$$z(x, y, t) = p_t(x, t - \tau y) \quad \text{in} \quad (0, L) \times (0, 1) \times (0, T),$$

so we have readily

$$\tau z_t(x, y, t) + z_y(x, y, t) = 0 \quad \text{in} \quad (0, L) \times (0, 1) \times (0, T).$$

Therefore, the system (1.22)–(1.26) takes the following form

$$\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + f_1(v, p) + v_t = h_1 \quad \text{in} \quad (0, L) \times (0, T),$$

$$\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} + f_2(v, p) + \mu_1 p_t + \mu_2 z(x, 1, t) = h_2 \quad \text{in} \quad (0, L) \times (0, T),$$

$$\tau z_t(x, y, t) + z_y(x, y, t) = 0 \quad \text{in} \quad (0, L) \times (0, 1) \times (0, T),$$

with initial conditions

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad p(x, 0) = p_0(x), \quad p_t(x, 0) = p_1(x), \quad x \in (0, L),$$

$$z(x, y, 0) = f_0(x, -\tau y), \quad (x, y) \in (0, L) \times (0, 1),$$

$$f(x, y, 0) = g_0(x, y, -\tau y), \quad (x, y) \in (0, L) \times (0, 1),$$

$$f(x, y, t) = g_1(x, y, t), \quad (x, y) \in (0, L) \times (0, 1).$$
and boundary conditions
\[ v(0, t) = v_x(L, t) = p(0, t) = p_x(L, t) = 0, \quad t \geq 0. \] (2.8)

The existence and uniqueness of solutions of our problem will be given by using nonlinear semigroup theory. We introduce the dependent variables \( v' = v_t \) and \( p' = p_t \). Then, problem (2.3)–(2.8) is equivalent to the Cauchy problem
\[
\begin{cases}
\frac{d}{dt} U(t) + AU(t) = F(U(t)), & t > 0, \\
U(0) = U_0 \in H,
\end{cases}
\] (2.9)

where \( U_0 = (v_0, p_0, v_1, p_1, f_0(\cdot, -\tau)) \in H \), the operator \( A : D(A) \subset H \to H \) and the forcing function \( F : H \to H \) are defined by
\[
AU = \begin{pmatrix}
-v' \\
-p'
\end{pmatrix} =
\begin{pmatrix}
\frac{\rho}{\gamma} v_x + \frac{\beta}{\gamma} p_{xx} + \frac{1}{\gamma} v' \\
\frac{-\beta}{\gamma} v_{xx} + \frac{\beta}{\gamma} p_{xx} + \frac{\mu_2}{\gamma} p' + \frac{\mu_2}{\gamma} \zeta(1)
\end{pmatrix},
F(U) = \begin{pmatrix}
0 \\
0
\end{pmatrix} + \begin{pmatrix}
\frac{1}{\gamma} [h_1 - f_1(v, p)] \\
\frac{1}{\gamma} [h_2 - f_2(v, p)]
\end{pmatrix}.
\] (2.10)

The domain of \( A \) is given by
\[ D(A) := \left\{ U = (v, p, v', p', z) \in H : v_x(L) = p_x(L) = 0 \text{ and } p' = z(0) \text{ in } (0, L) \right\}, \]
where
\[ H := (H^2(0, L) \cap H^1_0(0, L))^2 \times (H^1_0(0, L))^2 \times L^2(0, 1; H^1_0(0, L)). \] (2.11)

The energy space is given by
\[ H := (H^1_0(0, L))^2 \times (L^2(0, L))^2 \times L^2((0, L) \times (0, 1)). \] (2.12)

We consider \( \xi \) a positive constant satisfying
\[ \tau \mu_2 \leq \xi \leq \tau (2 \mu_1 - \mu_2). \] (2.13)

We define in \( H \) the following inner product and norm
\[
(U, \bar{U})_H = \rho(v', \hat{v}') + \mu(p', \hat{p}') + \alpha_1(v_x, \hat{v}_x) + \beta(\gamma v_x - p_x, \gamma \hat{v}_x - \hat{p}_x)
+ \xi \int_0^L \int_0^1 z(x, y) \zeta(x, y) dy dx,
\] (2.14)

and
\[
\|U\|_H^2 = \rho \|v'\|_2^2 + \mu \|p'\|_2^2 + \alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2
+ \xi \int_0^L \int_0^1 z^2(x, y) dy dx,
\] (2.15)

for any \( U = (v, p, v', p', z) \) and \( \bar{U} = (\hat{v}, \hat{p}, \hat{v}', \hat{p}', \hat{z}) \) in \( H \), where \((\cdot, \cdot)\) and \(\|\cdot\|_2\) are inner product and norm in \( L^2(0, L) \), respectively.

We also observe that there exist constants \( d_1, d_2 > 0 \) such that
\[
d_1(\|v_x\|_2^2 + \|p_x\|_2^2) \leq \alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2 \leq d_2(\|v_x\|_2^2 + \|p_x\|_2^2).
\] (2.16)

Then the Poincaré’s inequality implies that
\[
\|v\|_2^2 + \|p\|_2^2 \leq d_0(\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2),
\] (2.17)

where \( d_0 = (\lambda_0 d_1)^{-1} \) and \( \lambda_0 > 0 \) is the Poincaré’s constant.
2.1. **Existence and uniqueness.** The question of the existence and uniqueness of the solution of problem (2.9) will be considered in this subsection. Firstly, let us remember the following concepts:

- A function $U : [0, T) \to \mathcal{H}$, with $T > 0$, is a **strong solution** of (2.9), if $U$ is continuous on $[0, T)$, continuously differentiable on $(0, T)$, with $U(t) \in D(A)$ for all $t \in (0, T)$ and satisfies (2.9) on $[0, T)$ almost everywhere.
- A function $U \in C((0, T), \mathcal{H})$, $T > 0$, satisfying the integral equation

$$U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)}F(U(s))ds, \quad t \in [0, T), \quad (2.18)$$

is called a **mild solution** of initial value problem (2.9).

In order to obtain well-posedness, consider the following assumptions on $f_i$ and $h_i$ for $i = 1, 2$.

- **(A1):** $h_i \in L^2(0, L)$.
- **(A2):** There is a $C^2$-function $F : \mathbb{R}^2 \to \mathbb{R}$ and constants $d, m_F \geq 0$, $C_F > 0$ and $r \geq 1$ satisfying

$$\nabla F = (f_1, f_2), \quad (2.19)$$

$$F(v, p) \geq -d(|v|^2 + |p|^2) - m_F, \quad \forall v, p \in \mathbb{R}, \quad (2.20)$$

$$0 \leq d < \frac{1}{2d_0}, \quad (2.21)$$

and

$$|\nabla f_i(v, p)| \leq C_F(1 + |v|^{r-1} + |p|^{r-1}), \quad \forall v, p \in \mathbb{R}. \quad (2.22)$$

By successive integration, the previous inequality produces a positive constant $C_F$ such that

$$F(v, p) \leq C_F(1 + |v|^{r+1} + |p|^{r+1}), \quad \forall v, p \in \mathbb{R}, \quad (2.23)$$

and we suppose in addition that

$$\nabla F(v, p) \cdot (v, p) - F(v, p) \geq -d(|v|^2 + |p|^2) - m_F, \quad \forall v, p \in \mathbb{R}. \quad (2.24)$$

**Lemma 2.1.** Suppose that $\mu_2 \leq \mu_1$, then the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ defined in (2.10) is maximal monotone in $\mathcal{H}$.

**Proof.** For all $U = (v, p, v', p', z) \in D(A)$, we get

$$(AU, U)_{\mathcal{H}} = ||v'||^2_2 + \left(\mu_1 - \frac{\xi}{2\tau}\right)||p'||^2_2 + \mu_2(z(1), p') + \frac{\xi}{2\tau}||z(1)||^2_2. \quad (2.25)$$

Therefore, the Young’s inequality yields

$$(AU, U)_{\mathcal{H}} \geq ||v'||^2_2 + \left(\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau}\right)||p'||^2_2 + \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)||z(1)||^2_2. \quad (2.26)$$

From (2.13), we have

$$\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau} \geq 0 \quad \text{and} \quad \frac{\xi}{2\tau} - \frac{\mu_2}{2} \geq 0,$$

which implies

$$(AU, U)_{\mathcal{H}} \geq 0.$$
Therefore $\mathcal{A}$ is monotone. In order to prove that $\mathcal{A}$ is maximal monotone, we need to prove $R(\mathcal{A} + I) = \mathcal{H}$. Let $U^* = (g_1, g_2, g_3, g_4, g_5) \in \mathcal{H}$. To achieve this, we first show that there exists $U = (v, p, v', p', z) \in D(\mathcal{A})$ such that $(\mathcal{A} + I)U = U^*$, that is,

\begin{align}
-v' + v &= g_1, \\
-p' + p &= g_2, \\
-\frac{\alpha}{\rho} v_{xx} + \frac{\gamma \beta}{\rho} p_{xx} + \frac{1}{\rho} v' + v' &= g_3, \\
-\frac{\gamma \beta}{\mu} v_{xx} + \frac{\beta}{\mu} p_{xx} + \frac{\mu_1}{\mu} p' + \frac{\mu_2}{\mu} z(1) + p' &= g_4, \\
\frac{1}{\tau} z_y + z &= g_5.
\end{align}

(2.25) - (2.29)

Following the ideas in [39, 48], we can obtain a solution to the equations (2.25)-(2.29). Then, we can infer that $\mathcal{A}$ is maximal monotone. The proof is complete.

**Lemma 2.2.** Suppose that (A1) and (A2) hold, then $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ defined in (2.10) is locally Lipschitz continuous operator.

**Proof.** Let $U = (v, p, v', p'), \tilde{U} = (\tilde{v}, \tilde{p}, \tilde{v}', \tilde{p}') \in \mathcal{H}$ such that $\|U\|_\mathcal{H}, \|\tilde{U}\|_\mathcal{H} \leq R$ where $R > 0$. By definition of $\mathcal{F}$ and norm in $\mathcal{H}$, we have

\begin{align}
\|\mathcal{F}(U) - \mathcal{F}(\tilde{U})\|_\mathcal{H}^2 &= \frac{1}{\rho} \int_0^L |f_1(v, p) - f_1(\tilde{v}, \tilde{p})|^2 dx \\
&\quad + \frac{1}{\mu} \int_0^L |f_2(v, p) - f_2(\tilde{v}, \tilde{p})|^2 dx.
\end{align}

(2.30)

Using the assumption (2.22) yields

\begin{align}
|f_i(v, p) - f_i(\tilde{v}, \tilde{p})|^2 &= |\nabla f_i(\theta(v, p) + (1 - \theta)(\tilde{v}, \tilde{p})) - (\tilde{v}, \tilde{p})|^2 \\
&\leq C \left(|v|^{r-1} + |\tilde{v}|^{r-1} + |p|^{r-1} + |\tilde{p}|^{r-1} + 1\right)^2 (|v - \tilde{v}|^2 + |p - \tilde{p}|^2),
\end{align}

(2.31)

for some $0 \leq \theta \leq 1$. It follows from (2.31) that there exists a constant $C_R > 0$ such that

\begin{align}
\int_0^L |f_i(v, p) - f_i(\tilde{v}, \tilde{p})|^2 dx \leq C_R \|U - \tilde{U}\|_\mathcal{H}^2, \quad i = 1, 2.
\end{align}

(2.32)

By substituting (2.32) into (2.30), we conclude that there exists $C'_R > 0$ such that

\begin{align}
\|\mathcal{F}(U) - \mathcal{F}(\tilde{U})\|_\mathcal{H} \leq C'_R \|U - \tilde{U}\|_\mathcal{H}.
\end{align}

This proves that $\mathcal{F}$ is locally Lipschitz continuous.

It is opportune now to define the functional energy $E(t)$ of a solution $U = (v, p, v_t, p_t, z)$ by the expression

\begin{align}
E(t) = E(t) + \int_0^L F(v, p) dx - \int_0^L (h_1 v + h_2 p) dx,
\end{align}

where $E(t) = \frac{1}{2} \|U(t)\|_\mathcal{H}^2$.

**Lemma 2.3.** Suppose that $\mu_2 \leq \mu_1$, then energy functional $E(t)$ is non-increasing, more precisely, for any strong solution $U = (v, p, v_t, p_t, z)$ of (2.9), we have

\begin{align}
\frac{d}{dt} E(t) \leq -\|v_t\|^2 - \left(\mu_1 - \mu_2 - \frac{\xi}{2\tau}\right) \|p_t\|^2 - \left(\frac{\xi}{2\tau} - \mu_2 \right) \|z(1)\|^2 \leq 0, \quad \forall t \geq 0.
\end{align}

(2.34)
Moreover, there exist constants $\chi_0, C_F > 0$ such that
\[
\mathcal{E}(t) \geq \chi_0 \|U(t)\|_H^2 - C_F, \quad \forall t \geq 0, \quad (2.35)
\]
and
\[
\mathcal{E}(t) \leq C_F \left(\|U(t)\|_{H^1}^2 + 1\right), \quad \forall t \geq 0. \quad (2.36)
\]

**Proof.** Multiplying (2.3) by $v_t$ and (2.4) by $p_t$, integrating by parts over $[0, L]$ and applying Young's inequality, yields
\[
\frac{d}{dt} \left\{ \frac{1}{2} \|v_t\|^2 + \frac{\mu}{2} \|p_t\|^2 + \frac{\alpha_1}{2} \|v_x\|^2 + \frac{\beta}{2} \|v - p_x\|^2 \right\}
+ \frac{d}{dt} \left\{ \int_0^L F(v, p) dx - \int_0^L (h_1 v + h_2 p) dx \right\}
\leq -\|v_t\|^2 - \left( \mu_1 - \frac{\mu_2}{2} \right) \|p_t\|^2 + \frac{\mu_2}{2} \|z(1)\|^2. \quad (2.37)
\]

Multiplying (2.5) by $\xi z$ and integrating in $[0, L] \times [0, 1]$, we have
\[
\frac{\xi}{2} \frac{d}{dt} \int_0^L \int_0^1 z^2(x, y, t) dy dx = -\frac{\xi}{2\tau} \int_0^L \int_0^1 \frac{\partial}{\partial y} z^2(x, y, t) dy dx
\]
\[
= \frac{\xi}{2\tau} \int_0^L \left( z^2(x, 0, t) - z^2(x, 1, t) \right) dx. \quad (2.38)
\]

Combining (2.37) and (2.38), we get (2.34).

From (2.20) and (2.17), we have
\[
\int_0^L F(v, p) dx \geq -d(\|v\|_H^2 + \|p\|_H^2) - Lm_F
\]
\[
\geq -dd_0(\alpha_1 \|v_x\|^2 + \beta \|v - p_x\|^2)
\]
\[
\geq -dd_0 \|U(t)\|_H^2 - Lm_F. \quad (2.39)
\]

Then, we obtain
\[
\mathcal{E}(t) \geq \left( \frac{1}{2} - dd_0 \right) \|U(t)\|_H^2 - Lm_F - \int_0^L (h_1 v + h_2 p) dx.
\]

In light of (2.21), setting
\[
\chi_0 = \frac{1}{4} \left( 1 - 2dd_0 \right) > 0, \quad (2.40)
\]
and using the estimate
\[
\int_0^L (h_1 v + h_2 p) dx \leq \frac{\chi_0}{d_0} (\|v\|_H^2 + \|p\|_H^2)
+ \frac{d_0}{4\chi_0} (\|h_1\|_H^2 + \|h_2\|_H^2), \quad (2.41)
\]
we obtain the inequality in (2.35) with
\[
C_F = Lm_F + \frac{\chi_0}{4d_0} (\|h_1\|_H^2 + \|h_2\|_H^2).
\]

Finally, using (2.23) and (2.16), we deduce that (2.36) holds. The proof is therefore complete. \hfill \square

**Theorem 2.4 (Local and Global Solution).** Suppose that $\mu_2 \leq \mu_1$ and assumptions (A1) and (A2) hold. Then:
(i): If \( U_0 \in \mathcal{H} \), then there exists \( T_{\text{max}} > 0 \) such that (2.9) has a unique mild solution \( U : [0, T_{\text{max}}) \to \mathcal{H} \). In addition, if \( U_0 \in D(A) \), then the mild solution is strong solution.

(ii): The solution \( U(t) \) is globally bounded in \( \mathcal{H} \) and thus \( T_{\text{max}} = +\infty \).

(iii): If \( U_1 \) and \( U_2 \) are two mild solutions of problem (2.9), then there exists a positive constant \( C_0 = C(U_1(0), U_2(0)) \) such that, for any \( T > 0 \)

\[
\|U_1(t) - U_2(t)\|_{\mathcal{H}} \leq e^{C_0 t}\|U_1(0) - U_2(0)\|_{\mathcal{H}}, \quad \forall t \in [0, T).
\]  

(2.42)

Proof. (i) The result follows from Lemmas 2.1, 2.2 and of [41, Chap. 6, Theorems 1.4 and 1.5].

(ii) From (2.34), we have

\[ \mathcal{E}(t) \leq \mathcal{E}(0), \quad \forall t \geq 0, \]

and combining with (2.35) yields

\[ \|U(t)\|_{\mathcal{H}}^2 \leq \chi_0^{-1}(\mathcal{E}(0) + C_F), \quad \forall t \geq 0, \]

which implies that \( T_{\text{max}} = +\infty \).

(iii) Since \( U_1 \) and \( U_2 \) are mild solutions of (2.9), we have

\[
\|U_1(t) - U_2(t)\|_{\mathcal{H}} = \left\| e^{At}(U_1(0) - U_2(0)) - \int_0^t e^{A(t-s)}(F(U_1(s)) - F(U_2(s)))ds \right\|_{\mathcal{H}}.
\]

Since \( e^{At} \) is a contraction semigroup, we have

\[
\|U_1(t) - U_2(t)\|_{\mathcal{H}} \leq \|U_1(0) - U_2(0)\|_{\mathcal{H}} + \int_0^t \|F(U_1(s)) - F(U_2(s))\|_{\mathcal{H}} ds. \quad (2.45)
\]

From Lemma 2.2 and (2.44), there exists a constant \( C > 0 \) such that, for any \( T > 0 \)

\[
\|U_1(t) - U_2(t)\|_{\mathcal{H}} \leq \|U_1(0) - U_2(0)\|_{\mathcal{H}} + C \int_0^t \|U_1(s) - U_2(s)\|_{\mathcal{H}} ds, \quad \forall t \in [0, T).
\]

Applying the Gronwall’s inequality we get (2.42). This completes the proof of Theorem 2.4.

Remark 2.5. It is worth emphasizing that being \( D(A) \) dense in \( \mathcal{H} \), then for every \( U_0 \in \mathcal{H} \) and its respective mild solution \( U : [0, \infty) \to \mathcal{H} \), it is possible to obtain a sequence \( \{U_0^n\} \) in \( D(A) \) with \( U_0^n \to U_0 \) and a sequence \( U^n \in C([0, +\infty); \mathcal{H}) \) where \( U^n \) is a strong solution of

\[
\begin{cases}
\frac{d}{dt} U^n(t) + AU^n = F(U^n), & t > 0, \\
U^n(0) = U_0^n,
\end{cases}
\]

with

\[
U^n \to U \quad \text{in} \quad C([0, T]; \mathcal{H}), \quad \forall T > 0. \quad (2.46)
\]

This means that the regularity for the solutions obtained in Theorem 2.4 is sufficient to justify the calculations that will be performed in this work.
3. Some concepts and results related to dynamical systems. To facilitate the reading, we shall introduce some concepts and results related to dynamical systems (see [1, 6, 7, 22, 18, 55, 51]). In particular, the reference [22] is more relevant for the rest of the paper. A dynamical system is a pair \((H, S(t))\) where \(H\) is a Banach space and \(S(t)\) is a strongly continuous semigroup in \(H\). A compact set \(A \subset H\) is a called a global attractor for \((H, S(t))\) if \(A\) is an invariant set, that is, \(S(t)A = A\), for all \(t \geq 0\) and \(A\) is uniformly attracting, that is, for every bounded set \(B \subset H\), we have
\[
\lim_{t \to +\infty} d_H(S(t)B, A) = 0,
\]
where \(d_H\) denotes the Hausdorff semi-distance.

A dynamical system \((H, S(t))\) is called dissipative if it has an absorbing set, that is, a bounded set \(B \subset H\) such that for any bounded set \(D \subset H\) there exists \(t_D > 0\) with
\[
S(t)D \subset B, \quad \forall t \geq t_D.
\]

A dynamical system is called asymptotically smooth if, for any bounded set \(D \subset H\) forward invariant (i.e. \(S(t)D \subseteq D\)) there exists a compact set \(K \subset \overline{D}\) that uniformly attracts \(D\).

A dynamical system \((H, S(t))\) is called gradient, if there exists a strict Lyapunov function on \(H\), that is, there exists a continuous function \(\Phi : H \to \mathbb{R}\) such that \(t \mapsto \Phi(S(t)U)\) is non-increasing for any \(U \in H\), and if \(\Phi(S(t)U) = \Phi(U)\) for all \(t > 0\) and some \(U \in H\), then \(U\) is a stationary point of \((H, S(t))\).

**Theorem 3.1.** [22, Corollary 7.5.7] Let \((H, S(t))\) be a asymptotically smooth gradient system on a Banach space \(H\), with the corresponding Lyapunov functional denoted by \(\Phi\). Suppose that
\[
\Phi(U) \to \infty \quad \text{if and only if} \quad \|U\|_H \to \infty,
\]
and that the set of stationary points \(\mathcal{N}\) is bounded. Then the system \((H, S(t))\) possesses a compact global attractor which coincides with the unstable manifold \(A = M^u_{\mathcal{N}}(\mathcal{N})\).

The fractal dimension of a compact set \(M\) in \(H\) is defined by
\[
\dim^H_f M := \lim_{\varepsilon \to 0} \sup \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)}, \tag{3.1}
\]
where \(n(M, \varepsilon)\) is the minimal number of closed balls of radius \(\varepsilon\) which covers \(M\).

An compact set \(A_{\exp} \subset H\) is called a fractal exponential attractor for \((H, S(t))\) if
\begin{itemize}
  \item \(A_{\exp}\) is a positively invariant set, that is, \(S(t)A_{\exp} \subset A_{\exp}\) for all \(t \geq 0\),
  \item \(A_{\exp}\) has finite fractal dimension in \(H\),
  \item \(A_{\exp}\) attracts bounded sets of \(H\) at an exponential rate, that is, for any bounded set \(D \subset H\) there exist \(t_D, C_D, \gamma_D > 0\) such that
    \[
    d_H(S(t)D, A_{\exp}) \leq C_D e^{-\gamma_D(t-t_D)}, \quad \forall t \geq t_D.
    \]
\end{itemize}

If \(A_{\exp}\) has finite fractal dimension in some extended space \(\tilde{H} \supseteq H\), then \(A_{\exp}\) is called a generalized exponential fractal attractor.

Let \(X, Y\) and \(Z\) be reflexive Banach spaces with \(X\) compactly embedded in \(Y\). We consider the space \(H = X \times Y \times Z\) and the dynamical system \((H, S(t))\) given by
\[
S(t)U_0 = (u(t), u(t), \theta(t)), \quad U_0 = (u(0), u_0(0), \theta(0)) \in H, \tag{3.2}
\]
where the function $u$ and $v$ has the regularity

$$u \in C([0, \infty); X) \cap C^1([0, \infty); Y), \quad \theta \in C([0, \infty); Z).$$

The dynamical system $(H, S(t))$ is called quasi-stable on a set $B \subset H$ if there exist a compact seminorm $\eta_X$ on the space $X$ and nonnegative scalar functions $a$, $b$ and $c$ on $[0, \infty)$ such that

- $a, b, c$ are locally bounded on $[0, \infty)$;
- $b \in L^1(0, \infty)$ possesses the property
  $$\lim_{t \to \infty} b(t) = 0;$$
- for every $U_1, U_2 \in B$ and $t \geq 0$ the following relations
  $$\|S(t)U_1 - S(t)U_2\|_H^2 \leq a(t)\|U_1 - U_2\|_H^2,$$
  and
  $$\|S(t)U_1 - S(t)U_2\|_H^2 \leq b(t)\|U_1 - U_2\|_H^2 + c(t) \sup_{0 \leq s \leq t} [\eta_X(u^1(s) - u^2(s))]^2$$
  hold. Here we denote $S(t)U_i = (u^i(t), u^i(t), \theta^i(t)), i = 1, 2.$

**Theorem 3.2.** [22, Proposition 7.9.4] Let $(H, S(t))$ be a dynamical system satisfying (3.2). Assume that $(H, S(t))$ is quasi-stable over any bounded invariant set $B \subset H$. Then, $(H, S(t))$ is asymptotically smooth.

**Theorem 3.3.** [22, Theorem 7.9.6] Suppose $(H, S(t))$ be a dynamical system satisfying (3.2). Assume that $(H, S(t))$ possesses a compact global attractor $A$ and is quasi-stable on $A$. Then the fractal dimension of $A$ is finite.

**Theorem 3.4.** [22, Theorem 7.9.9] Suppose $(H, S(t))$ be a dynamical system satisfying (3.2). Assume that $(H, S(t))$ is dissipative and quasi-stable on some bounded absorbing set $B$. We also assume that there exists an extended space $\tilde{H} \supseteq H$ such that

$$\|S(t_1)y - S(t_2)y\|_H \leq C_{B,T}|t_1 - t_2|^\gamma, \quad \forall t_1, t_2 \in [0, T], y \in B.$$ 

where $C_{B,T} > 0$ and $\gamma \in [0, 1)$ are constants. Then the dynamical system possesses a generalized exponential attractor $A_{exp} \subset \tilde{H}$ whose dimension is finite in the space $\tilde{H}$.

**Theorem 3.5.** [22, Theorem 7.9.8] Suppose $(H, S(t))$ be a dynamical system satisfying (3.2). Assume that the dynamical system possesses a compact global attractor $A$ and is quasi-stable on the attractor $A$. Moreover, we assume that (3.5) holds with the function $c(t)$ possessing the property $c_\infty = \sup_{t \in [0, \infty)} c(t) < \infty$. Then any full trajectory $(u(t), u(t), \theta(t))$ in the global attractor has the following regularity properties,

$$u \in L^\infty(\mathbb{R}; X) \cap C(\mathbb{R}; Y), \quad u_{tt} \in L^\infty(\mathbb{R}; Y), \quad \theta_t \in L^\infty(\mathbb{R}; Z).$$

Moreover, there exists $R > 0$ such that

$$\|u(t)\|_X^2 + \|u_{tt}(t)\|_Y^2 + \|\theta_t(t)\|_Z^2 \leq R^2, \quad \forall t \in \mathbb{R},$$

where $R$ depends on the constant $c_\infty$, on the seminorm $\mu_X$, and also on the embedding properties of $X$ into $Y$. 

4. Existence of global attractor. In this section, we will study the existence of global attractor for the dynamic system generated by the problem (1.22)–(1.26), we will often refer to the inequality \( \mu_2 \leq \mu_1 \) associated with (2.13). We observe that the well-posedness of problem (1.22)–(1.26) ensures that the solution operator \( S(t) : \mathcal{H} \to \mathcal{H} \) defined by
\[
S(t)(v_0, p_0, v_1, p_1, f_0) = (u(t), p(t), v_i(t), p_i(t), z(t)), \quad t \geq 0,
\]
is a strongly continuous semigroup in \( \mathcal{H} \). Therefore, we will often refer to the inequality \( \mu_2 \leq \mu_1 \) associated with (2.13). We observe that the well-posedness of problem (1.22)–(1.26) ensures that the solution operator \( S(t) : \mathcal{H} \to \mathcal{H} \) defined by
\[
S(t)(v_0, p_0, v_1, p_1, f_0) = (u(t), p(t), v_i(t), p_i(t), z(t)), \quad t \geq 0,
\]
is a strongly continuous semigroup in \( \mathcal{H} \). Thus, the pair \( (\mathcal{H}, S(t)) \) constitutes a dynamical system that will describe the long-time behavior of problem (1.22)–(1.26).

**Lemma 4.1.** If \( \mu_2 \leq \mu_1 \), then the dynamical system \( (\mathcal{H}, S(t)) \) is gradient, that is, there exists a strict Lyapunov function \( \Phi \) defined in \( \mathcal{H} \). In addition,
\[
\Phi(U) \to \infty \quad \text{if and only if} \quad \|U\|_\mathcal{H} \to \infty.
\]

**Proof.** Let us consider the functional energy defined in (2.33) as the Lyapunov function, that is, \( \Phi \equiv E \). Thus, given \( U_0 = (v_0, p_0, v_1, p_1, f_0) \in \mathcal{H} \), it follows from (2.34) that
\[
\frac{d}{dt} \Phi(S(t)U_0) \leq -\|v\|^2 - \left( \mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau} \right) \|p_1\|^2 - \left( \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \|z(1)\|^2 \leq 0,
\]
and hence, \( \Phi(S(t)U_0) \) is nonincreasing. Now let us suppose \( \Phi(S(t)U_0) = \Phi(U_0) \) for all \( t \geq 0 \). Since
\[
\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} > 0 \quad \text{and} \quad \frac{\xi}{2\tau} - \frac{\mu_2}{2} > 0,
\]
then (4.3) implies
\[
v_i(x, t) = 0, \quad p_i(x, t) = 0, \quad \text{a.e. in} \ (0, L), \ \forall t \geq 0.
\]
In view of (2.1), we deduce
\[
z(x, y, t) = 0, \quad \text{a.e. in} \ (0, L), \ \forall y \in (0, 1), \ \forall t \geq 0.
\]
It follows from (4.4) and (4.5) that
\[
v(t) = v_0, \quad p(t) = p_0 \quad \text{and} \quad z(t) = 0 \quad \forall t \geq 0.
\]
Then we can obtain that \( U(t) = S(t)U_0 = (v_0, p_0, 0, 0, 0) \) for all \( t \geq 0 \), that is, \( U_0 \) is a stationary point of \( (\mathcal{H}, S(t)) \), thus proving that \( \Phi \) is a strict Lyapunov function of \( (\mathcal{H}, S(t)) \) and therefore, the dynamical system is gradient. Using (2.35) and (2.36) we conclude that (4.2) holds.

**Lemma 4.2.** The set of stationary points \( \mathcal{N} \) of the dynamical system \( (\mathcal{H}, S(t)) \) is bounded.

**Proof.** Based on the Lemma 4.1, the set \( \mathcal{N} \) is given by
\[
\mathcal{N} = \left\{ U = (v, p, 0, 0, 0) \in \mathcal{H} : \mathcal{A}U = \mathcal{F}(U) \right\}.
\]
Therefore, \( v \) and \( p \) must satisfy
\[
-\alpha v_{xx} + \gamma \beta p_{xx} + f_1(v, p) = h_1 \quad \text{in} \ (0, L),
\]
\[
-\beta p_{xx} + \gamma \beta v_{xx} + f_2(v, p) = h_2 \quad \text{in} \ (0, L).
\]
Multiplying (4.7) by \( v \) and (4.8) by \( p \), integrating over \( (0, L) \) and adding the results, yields
\[
\alpha_1 \|v_x\|^2 + \beta \|v_x - p_x\|^2 = -\int_0^L \nabla F(v, p) \cdot (v, p) dx + \int_0^L (h_1 v + h_2 p) dx.
\]

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Then the Young’s inequality yields

\[ - \int_0^L \nabla F(v, p) \cdot (v, p) \, dx \leq 2d_0(\alpha_1\|v_x\|^2 + \beta\|v_x - p_x\|^2) + 2LM_F, \]

and therefore, in light of (2.40),

\[ 4\chi_0(\alpha_1\|v_x\|^2 + \beta\|v_x - p_x\|^2) \leq 2LM_F + \int_0^L (h_1v + h_2p) \, dx. \]

Hence, using the estimate (2.41), we conclude that \( N \) is bounded. \( \square \)

**Lemma 4.3.** Suppose that \( \mu_2 \leq \mu_1 \) and assumptions (A1) and (A2) hold. For every set bounded \( B \subset H \), there are positive constants \( \eta, \vartheta \) and \( C_B \), with \( C_B \) depending on \( B \), such that

\[ \|S(t)U_1 - S(t)U_2\|_H^2 \leq \eta e^{-\eta t}\|U_1 - U_2\|_H^2 + C_B \int_0^t e^{-(t-s)}(\|v(s)\|^2 + \|p(s)\|^2) \, ds, \quad t \geq 0, \]  

(4.10)

for any \( U_i = (v_i, p_i, v'_i, p'_i, f_i) \in B \), where \( S(t)U_i = (v^i(t), p^i(t), v'_i(t), p'_i(t), z^i(t)) \) is mild solution of (2.9) to \( i = 1, 2 \), \( v = v^1 - v^2 \), \( p = p^1 - p^2 \) and \( z = z^1 - z^2 \).

**Proof.** The differences \( v = v^1 - v^2 \), \( p = p^1 - p^2 \) and \( z = z^1 - z^2 \) solves the problem

\[ \mu_1 v_{tt} - \alpha_1 v_{xx} + \gamma_2 p_{xx} + f_1(v^1, p^1) - f_1(v^2, p^2) + v_t = 0, \]

(4.11)

\[ \mu_2 p_{tt} - \beta_1 p_{xx} + \gamma_2 v_{xx} + f_2(v^1, p^1) - f_2(v^2, p^2) + \mu_1 v_t + \mu_2 z(x, 1, t) = 0, \]

(4.12)

\[ \tau z_t(x, y, t) + z_y(x, y, t) = 0, \]

(4.13)

with boundary condition

\[ v(0) = p(0) = v_x(L) = p_x(L) = 0, \]

and initial condition

\[ (v(0), p(0), v_t(0), p_t(0), z(0)) = U^1 - U^2. \]

(4.14)

**Step 1.** We claim that there exists a constant \( K_B > 0 \) such that

\[ \frac{d}{dt} E(t) \leq -\frac{1}{2} \|v_t\|^2 - \frac{1}{2} \left( \mu_1 - \mu_2 - \frac{\xi}{2\tau} \right) \|p_t\|^2 - \left( \frac{\xi}{2\tau} - \mu_2 \right) \|z(1)\|^2 + K_B(\|v\|^2 + \|p\|^2). \]

(4.15)

Multiplying (4.11) by \( v_t \), (4.12) by \( p_t \) and integrating in \([0, L]\), and then (4.13) by \( \xi z \) and integrating in \([0, L] \times [0, 1]\) and adding the results, we obtain

\[ \frac{d}{dt} E(t) = -\|v_t\|^2 - \mu_1 \|p_t\|^2 - \frac{\mu_2}{2} \int_0^L z(1)p_t \, dx - \int_0^L (F_1(v, p)v_t + F_2(v, p)p_t) \, dx \]

\[ + \frac{\xi}{2\tau} (\|p_t\|^2 - \|z(1)\|^2). \]

where

\[ F_i(v, p) = f_i(v^i, p^i) - f_i(v^2, p^2). \]

Then the Young’s inequality yields

\[ \frac{d}{dt} E(t) \leq -\|v_t\|^2 - \left( \mu_1 - \mu_2 - \frac{\xi}{2\tau} \right) \|p_t\|^2 - \left( \frac{\xi}{2\tau} - \mu_2 \right) \|z(1)\|^2 - \int_0^L (F_1(v, p)v_t + F_2(v, p)p_t) \, dx. \]

(4.16)
Using (2.22), Hölder’s and Young’s inequalities, we obtain

\[
\int_0^L F_1(v, p)v_t dx \\
\leq C_f \int_0^L (|v^1|^r - 1 + |v^2|^r - 1 + |p^1|^r - 1 + |p^2|^r - 1)(|v| + |p|)|v_t| dx \\
\leq C_f (1 + \|v^1\|^r_{L^r} + \|v^2\|^r_{L^r} + \|p^1\|^r_{L^r} + \|p^2\|^r_{L^r}) (\|v\|_{L^2} + \|p\|_{L^2}) \|v_t\|_2 \\
\leq K_B(\|v\|_{L^2} + \|p\|_{L^2}) \|v_t\|_2
\]

(4.17)

Similarly to (4.17) and (4.18), we deduce that

\[
\int_0^L F_2(v, p)v_t dx \\
\leq K_B(\|v\|_{L^2} + \|p\|_{L^2}) \|v_t\|_2 + \frac{1}{2} \|v_t\|^2.
\]

Analogously, we have

\[
\int_0^L F_2(v, p)p_t dx \\
\leq K_B(\|v\|_{L^2} + \|p\|_{L^2}) \|p_t\|_2 \\
\leq K_B(\|v\|^2_{L^2} + \|p\|^2_{L^2}) + \frac{1}{2} \left( \mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau} \right) \|p_t\|^2.
\]

(4.18)

Substituting the estimates (4.17) and (4.18) in (4.16), we see that (4.15) is obtained.

**Step 2.** Now, we define the functional \( \mathcal{K} \) by

\[
\mathcal{K}(t) = \rho \int_0^L v v_t dx + \mu \int_0^L p p_t dx.
\]

Then, \( \mathcal{K} \) satisfies

\[
\frac{d}{dt} \mathcal{K}(t) \leq \left( \rho + \frac{C}{d_0} \right) \|v_t\|^2 + \left( \mu + \frac{C}{d_0} \right) \|p_t\|^2 - \frac{\alpha_1}{2} \|v_x\|^2 - \frac{\beta}{2} \|v_x - p_x\|^2 \\
+ \frac{C}{d_0} \|z(0)\|^2 + K_B(\|v\|^2_{L^2} + \|p\|^2_{L^2}),
\]

(4.19)

for some constant \( C > 0 \). Indeed, taking the derivation of \( \mathcal{K} \), we can get

\[
\frac{d}{dt} \mathcal{K}(t) = \rho\|v_t\|^2 + \mu\|p_t\|^2 - \alpha_1\|v_x\|^2 - \beta\|v_x - p_x\|^2 \\
- \int_0^L v v dx - \mu_1 \int_0^L p v dx - \mu_2 \int_0^L z(1)p dx \\
- \int_0^L (F_1(v, p)v + F_2(v, p)p) dx.
\]

(4.20)

Similarly to (4.17) and (4.18), we deduce that

\[
\int_0^L F_1(v, p)v dx \leq K_B(\|v\|^2_{L^2} + \|p\|^2_{L^2}),
\]

and

\[
\int_0^L F_2(v, p)p dx \leq K_B(\|v\|^2_{L^2} + \|p\|^2_{L^2}).
\]
Now, using (2.17) and Young’s inequalities, we have

\[
- \int_0^L v_t v dx - \mu_1 \int_0^L p_t p dx - \mu_2 \int_0^L z(1) p dx \leq \|v_t\|_2^2 + \mu_1 \|p_t\|_2^2 + \mu_2 \|z(1)\|_2^2
\]
\[
\leq \frac{d_2}{2} (\|v\|_2^2 + \|p\|_2^2) + \frac{C}{d_0} (\|v_t\|_2^2 + \|p_t\|_2^2 + \|z(1)\|_2^2)
\]
\[
\leq \frac{1}{2} (\alpha_1 \|v_x\|_2^2 + \beta \|\gamma v_x - p_x\|_2^2) + \frac{C}{d_0} (\|v_t\|_2^2 + \|p_t\|_2^2 + \|z(1)\|_2^2).
\]

Combining these estimates with (4.20), we obtain (4.19).

**Step 3.** We define the functional \(\mathcal{P}\) by

\[
\mathcal{P}(t) = \tau \int_0^L \int_0^1 e^{-2\tau y} z^2 dydx.
\] (4.21)

Deriving \(\mathcal{P}\) with respect to \(t\) and using the equation (2.5), we have

\[
d\frac{d}{dt}\mathcal{P}(t) = 2\tau \int_0^L \int_0^1 e^{-2\tau y} z z_\tau dydx = -2\tau \int_0^L \int_0^1 e^{-2\tau y} z z_\tau dydx
\]
\[
= - \int_0^L \int_0^1 e^{-2\tau y} \frac{\partial}{\partial y} z^2 dydx = - \int_0^L \left[ e^{-2\tau y} z^2 \right]_{y=0}^{y=1} dx
\]
\[
- 2\tau \int_0^L \int_0^1 e^{-2\tau y} z^2 dx
\]
\[
= \|p_t\|_2^2 - e^{-2\tau} \|z(1)\|_2^2 - 2\tau \tau \int_0^L \int_0^1 e^{-2\tau y} z^2 dydx.
\] (4.22)

**Step 4.** Now, we define the functional \(\mathcal{L}\),

\[
\mathcal{L}(t) = N E(t) + K(t) + M \mathcal{P}(t),
\]

where \(N\) and \(M\) are positive constants to be determined later.

It is not difficult to check that there exists a constant \(C_0 > 0\) such that

\[
|\mathcal{L}(t) - N E(t)| \leq C_0 E(t), \quad \forall t \geq 0.
\]

Therefore, for \(N\) large enough, we obtain positive constants \(C_1\) and \(C_2\) such that

\[
C_1 E(t) \leq \mathcal{L}(t) \leq C_2 E(t), \quad \forall t \geq 0.
\] (4.23)

**Step 5.** Deriving \(\mathcal{L}\) and using (4.15), (4.19) and (4.22), we have

\[
\frac{d}{dt} \mathcal{L}(t) \leq \left[ \frac{N}{2} - \rho - \frac{C}{d_0} \right] \|v_t\|_2^2 - \left[ \frac{N}{2} \left( \mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau} \right) - \mu - \frac{C}{d_0} + M \right] \|p_t\|_2^2
\]
\[
- \left[ N \left( \mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau} \right) - \frac{C}{d_0} + M \right] \|z(1)\|_2^2 - \frac{\alpha_1}{2} \|v_x\|_2^2 - \frac{\beta}{2} \|\gamma v_x - p_x\|_2^2
\]
\[
- 2 M \tau e^{-2\tau} \int_0^L \int_0^1 z^2 dydx + K_B(N + 1)(\|v\|_{2\tau}^2 + \|p\|_{2\tau}^2).
\]

Now, we choose \(N > 0\) large enough such that

\[
\frac{N}{2} - \rho - \frac{C}{d_0} > 0.
\]
Finally, choosing \( M > 0 \) large enough, we conclude that exists constants \( N_0, K_B > 0 \) such that
\[
\frac{d}{dt} \mathcal{L}(t) \leq -N_0 E(t) + K_B (\|v\|_{2r}^2 + \|p\|_{2r}^2).
\]  
Combining (4.24) and (4.23), we have
\[
\mathcal{L}(t) \leq e^{-\frac{N_0}{C_2} t} \mathcal{L}(0) + K_B \int_0^t e^{-\frac{N_0}{C_2} (t-s)} (\|v(s)\|_{2r}^2 + \|p(s)\|_{2r}^2) ds.
\]
Using (4.23) again, we obtain
\[
E(t) \leq \vartheta e^{-\eta t} E(0) + C_B \int_0^t e^{-\eta (t-s)} (\|v(s)\|_{2r}^2 + \|p(s)\|_{2r}^2) ds,
\]
with
\[
\vartheta = \frac{C_2}{C_1}, \quad \eta = \frac{N_0}{C_2}, \quad C_B = \frac{K_B}{C_1}.
\]
This implies the estimate (4.10). The proof is complete.

**Lemma 4.4.** Suppose that \( \mu_2 \leq \mu_1 \) and (A1)-(A2) hold. Then the dynamical system \((\mathcal{H}, S(t))\) is quasi-stable on any bounded positively invariant subset of \(\mathcal{H}\).

**Proof.** Let \( B \subset \mathcal{H} \) be a bounded set positively invariant to \((\mathcal{H}, S(t))\) and consider \( U_1, U_2 \in B \). As already mentioned, we denote to \( i = 1, 2 \)
\[
S(t)U_1 = (v^i(t), p^i(t), v_i^i(t), p_i^i(t), z^i(t)), \quad (v, p) = (v^1 - v^2, p^1 - p^2).
\]
From Theorem 2.4 (iii), we obtain \( a(t) = e^{C_0 t} > 0 \) which is locally bounded in \([0, \infty)\). We also consider the seminorm \( \eta_X(\cdot) \) in \( X = H^1_1(0, L) \times H^1_1(0, L) \) given by
\[
\eta_X(v, p) = \sqrt{\|v\|_{2r}^2 + \|p\|_{2r}^2},
\]
which is compact in \( X \), since the embedding \( H^1_1(0, L) \hookrightarrow L^2(0, L) \) is compact. It follows from Lemma 4.3 that
\[
\|S(t)U_1 - S(t)U_2\|_H^2 \leq b(t) \|U_1 - U_2\|_H^2 + c(t) \sup_{0 \leq s \leq t} [\eta_X(v(s), p(s))]^2,
\]
where
\[
b(t) = \vartheta e^{-\eta t} \quad \text{and} \quad c(t) = C_B \int_0^t e^{-\eta (t-s)} ds, \quad t \geq 0.
\]
Thus we have \( b(t) \in L^1(\mathbb{R}^+) \), with \( \lim_{t \to \infty} b(t) = 0 \) and \( c(t) \leq \frac{C_0}{\eta} < \infty \). Hence, (3.5) holds. Therefore \((\mathcal{H}, S(t))\) is quasi-stable on any bounded positively invariant set. The proof is complete.

As a consequence of the previous lemmas, we obtain the following theorem, which is the main result of this section.

**Theorem 4.5.** Suppose that \( \mu_2 \leq \mu_1 \) and (A1)-(A2) hold. Then the dynamical system \((\mathcal{H}, S(t))\) possesses a unique compact global attractor \( A \subset \mathcal{H} \), with finite fractal dimension. Moreover, the global attractor \( A \) is characterized by
\[
A = M^{\mu_1}_+(\mathcal{N}),
\]
where \( \mathcal{N} \) is the set of stationary point of \((\mathcal{H}, S(t))\) and \( A = M^{\mu_1}_+(\mathcal{N}) \) is the unstable manifold of \( \mathcal{N} \).
Proof. Since \((\mathcal{H}, S(t))\) is a asymptotically smooth gradient system with Lyapunov functional satisfying (4.2) and the set of its stationary points \(\mathcal{N}\) is bounded in \(\mathcal{H}\), the existence of a compact global attractor \(\mathcal{A} = \mathcal{M}^\infty_\omega(\mathcal{N})\) is established by Theorem 3.1. The proof is complete. \(\square\)

5. Regularity and exponential attractors. In this section, regularities properties of trajectories and the existence of a generalized fractal exponential attractor are analyzed.

**Theorem 5.1.** Suppose that \(\mu_2 \leq \mu_1\) and assumptions (A1)-(A2) hold. Then any full trajectory 
\[
(v(t), p(t), v_\varepsilon(t), p_\varepsilon(t), z(t)) \quad \text{in} \quad \mathcal{A}
\]
has further regularity 
\[
(v_\varepsilon, p_\varepsilon, v_\varepsilon(t), p_\varepsilon(t), z(t)) \in L^\infty(\mathbb{R}, \mathcal{H}).
\]  
Moreover, there exists \(R > 0\) such that 
\[
\|v(t, p(t))\|_{H^1_x(0, L)}^2 + \|v_\varepsilon(t, p_\varepsilon(t))\|_{L^2_x(0, L)}^2 + \|z(t)\|_{L^2_x((0, L) \times (0, 1))}^2 \leq R,
\]
for all \(t \in \mathbb{R}\).

**Proof.** Since we have shown that \((\mathcal{H}, S(t))\) is quasi-stable on the global attractor \(\mathfrak{A}\) with \(c_\infty = \sup_{t \in \mathbb{R}} c(t) < \infty\), then the regularity properties (5.2) and (5.3) follows by [22, Theorem 7.9.8]. The proof is complete. \(\square\)

**Theorem 5.2.** Suppose that \(\mu_2 \leq \mu_1\) and (A1)-(A2) hold. Then the dynamical system \((\mathcal{H}, S(t))\) possesses a generalized exponential attractor representing \(\mathcal{A}_\exp \subset \mathcal{H}\) with finite dimension in the extended space 
\[
\mathcal{H}_{-\delta} := (H^{-1}_x(0, 1))^2 \times (L^2(0, 1))^2 \times L^2((0, 1) \times (0, 1)),
\]
where \(H^{-1}_x(0, L)\) denotes the dual space of \(H^1_x(0, L)\). In addition, from the interpolation theorem, for all \(0 < \delta < 1\) there exists a generalized fractal exponential attractor whose fractal dimension is finite in the extended space \(\mathcal{H}_{-\delta}\), where 
\[
\mathcal{H}_0 := \mathcal{H}, \quad \text{and} \quad \mathcal{H} \subset \mathcal{H}_{-\delta} \subset \mathcal{H}_{-\delta}.
\]

**Proof.** Let \(\Phi\) be the functional of Lyapunov considered in Lemma 4.1, let us take 
\[
\mathfrak{B} := \{U: \Phi(U) \leq R\}.
\]
It is clear that for \(R\) large enough, the set \(\mathfrak{B}\) is absorbing and positively invariant, thus \((\mathcal{H}, S(t))\) is quasi-stable on \(\mathfrak{B}\). In another hand, for strong solutions \(U(t)\) with initial data \(U_0 \in \mathfrak{B}\), from (2.9) and the positive invariance of \(\mathfrak{B}\), we get \(C_{\mathfrak{B}, T} > 0\) such that for any \(0 \leq t \leq T\), 
\[
\|U(t)\|_{\mathcal{H}_{-\delta}} \leq \|AU(t)\|_{\mathcal{H}} + \|\mathcal{F}(U(t))\|_{\mathcal{H}} \leq C_{\mathfrak{B}, T}.
\]
Consequently, 
\[
\|S(t_1)U_0 - S(t_2)U_0\|_{\mathcal{H}_{-\delta}} \leq \int_{t_1}^{t_2} \|U_\varepsilon(s)\|_{\mathcal{H}_{-\delta}} ds \leq C_{\mathfrak{B}}|t_1 - t_2|, \quad 0 \leq t_1 \leq t_2 \leq T.
\]
Therefore, the application \(t \to S(t)U_0\) is Hölder continuous on space extending \(\mathcal{H}_{-\delta}\) with exponent \(\delta = 1\) for every \(U_0 \in \mathfrak{B}\). Thus, based on [22, Theorem 7.9.9] the system \((\mathcal{H}, S(t))\) possesses a generalized exponential attractor with finite fractal dimension in generalized space \(\mathcal{H}_{-\delta}\).

Using an analogous argument to that found in [3, 30] we can show the existence of exponential attractor with finite fractal dimension in the generalized space \(\mathcal{H}_{-\delta}\) with \(\delta \in (0, 1)\), thus concluding the proof of the Theorem 5.2. \(\square\)
Acknowledgments. We would like to thank the anonymous referees for carefully reading our manuscript and for giving constructive comments which improved the quality of this paper.

A. J. A. Ramos thanks the CNPq for financial support through the projects “Asymptotic stabilization and numerical treatment for carbon nanotube” (CNPq Grant 310729/2019-0).

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Received February 2020; revised December 2020.

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