LP Formulations of Two-Player Zero-Sum Stochastic Bayesian games

Nabiha Nasir Orpa · Lichun Li

Abstract This paper studies two-player zero-sum stochastic Bayesian games where each player has its own dynamic state that is unknown to the other player. Using typical techniques, we provide the recursive formulas and the sufficient statistics in both the primal game and its dual games. It’s also shown that with a specific initial parameter, the optimal strategy of one player in a dual game is also the optimal strategy of the player in the primal game. We, then, construct linear programs to compute the optimal strategies in both the primal game and the dual games and the special initial parameters in the dual games. The main results are demonstrated in a security problem of underwater sensor networks.

Keywords Game theory · Stochastic · LP · Zero-sum · Dual game

1 Introduction

Because of the multi-agent nature, game theory has great potential in solving or explaining economic, social, and engineering problems. Game theory has been used in addressing AdWord problems [2], enhancing the security of Los Angeles airport [12], advising in presidential election and nuclear disarmament [10], explaining and anticipating disease spreading [3], and many other problems. One common property of these problems is that the individuals or agents in the problems have their own private information not shared with the others. For example, didders in AdWord problems may not reveal its budget to the other bidders.

If one or more agents in a game don’t have complete information about the game, we call the game a game with incomplete information, which was first introduced in [1]. In this case, a player in the game makes its strategy according to its observations like the other players’ actions and/or its own payoff. Two-player zero-sum games with incomplete information are special cases of games with incomplete information, and the focus of this paper.

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This paper studies two player zero-sum stochastic games with incomplete information on both sides, which are also called stochastic Bayesian games. In these games, both players have their own types (private information) which will change stage by stage, and the payoffs depend on both players’ types. We assume that both players can observe the actions of each other, and the one-stage payoffs won’t be revealed until the end of the game.

Compared with the previous work in [1,3,13,15,19], our model considers the stochastic property of both players’ types and the incomplete information on both sides. [1,3,7,10,13-22] considered repeated games with incomplete information in which every player’s type is fixed at the beginning of the game, while [13,15,16,19] studied games with incomplete information from one side.

Our main results in Section 3 and 4 are closely related to [15]. In [15], every player’s type is fixed, and a public stochastic state is considered. Compared with [15], our model doesn’t have the public stochastic state, and the players’ types change stage by stage. Considering the model differences, it is necessary to develop the corresponding results though the techniques are typical and frequently used in the prior work [3,13,15,18]. We develop a recursive formula in primal game to compute the game value and show that the pair of believes of both players’ types is a sufficient statistic of the players. This sufficient statistic depends on both players’ strategies, and hence neither player has full access to it. To address this problem, we study dual games of the two player zero-sum stochastic Bayesian game. The relationships between the game values of the primal game and the dual games are provided. Moreover, we show that with some special initial parameters, the optimal strategy of one player in the corresponding dual game is also its optimal strategy in the primal game, and this optimal strategy only relies on the belief of its own type and the realized vector payoff of the other player’s type. In other words, the belief of its own type and the realized vector payoff of the other player’s type is the player’s sufficient statistic in the dual game. Different from the believes of both players’ types, the sufficient statistic in the dual game is fully accessible by the player.

We, then, further propose LP formulations to compute the special initial parameters in the dual games, and the optimal strategies in both the primal game and the dual games, which is closely related to [21]. Previous work in [21] introduced the sequence form to develop an LP formulation to compute the optimal strategies in primal games. Compared with [21], we use the recursive formula of vector payoff to develop the LP formulation for the primal game. Moreover, we extend the results to dual games, and figure out how to compute the special initial parameters in the dual game, which was not discussed in [21]. The main results are demonstrated in a jamming problem in underwater acoustic sensor networks.

The rest of the paper is organized as follows. Section 2 describes the game model. Primal game value and its recursive formula are computed in section 3. Section 4 describes the dual game and its recursive form. The strategies of players both in primal and dual game are computed with linear programming in section 5. In section 6, we apply the computed optimal strategies of players in an underwater acoustic sensor network jamming problem and show that for specific initial vector payoff, the optimal strategy of the dual game is also the optimal strategy in the primal game.
Let \( \mathbb{R}^n \) denote the n-dimensional real space, and \( \mathcal{K} \) be a finite set. The cardinality of \( \mathcal{K} \) and the set of probability distribution over \( \mathcal{K} \) is denoted by \( |\mathcal{K}| \) and \( \Delta(\mathcal{K}) \), respectively. A two player zero sum stochastic game with incomplete information on both sides is specified by the nine-tuple \((\mathcal{K}, \mathcal{L}, \mathcal{A}, \mathcal{B}, p, q, P, Q, G)\), where

- \( \mathcal{K} \) and \( \mathcal{L} \) are non-empty finite sets, called player 1 and 2’s state sets, respectively.
- \( \mathcal{A} \) and \( \mathcal{B} \) are non-empty finite sets, called player 1 and 2’s action sets, respectively where \( a_t \in \mathcal{A} \) and \( b_t \in \mathcal{B} \) are actions of player 1 and 2 in stage \( t \), respectively.
- \( G_{k,l} \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{B}|} \) is the payoff matrix given player 1’s state \( k \in \mathcal{K} \) and player 2’s state \( l \in \mathcal{L} \). The element \( G_{k,l}(a,b) \) is player 1’s one stage payoff or player 2’s one stage cost if the state of player 1 and 2 are \( k \) and \( l \), respectively and the current action of player 1 and 2 are \( a \) and \( b \), respectively.
- \( p \in \Delta(\mathcal{K}) \) and \( q \in \Delta(\mathcal{L}) \) are the initial probability on \( \mathcal{K} \) and \( \mathcal{L} \) respectively.
- \( P \in \mathbb{R}^{|\mathcal{K}| \times |\mathcal{K}|} \) and \( Q \in \mathbb{R}^{|\mathcal{L}| \times |\mathcal{L}|} \) are player 1 and 2’s transition matrices, respectively. \( P_{a_t,b_t}(k_{t-1},k_{t+1}) \) is a conditional probability of the next state of player 1, \( k_{t+1} \), given the current actions of both players and current state of player 1. Similarly, \( Q_{a_t,b_t}(l_{t-1},l_{t+1}) \) is a conditional probability of the next state of player 2, \( l_{t+1} \), given the current actions of both players and current state of player 2.

The nine-tuple \((\mathcal{K}, \mathcal{L}, \mathcal{A}, \mathcal{B}, p, q, P, Q, G)\) is the common knowledge of the two players. At stage \( t = 1 \), the initial state of player 1 and player 2 are chosen independently by nature according to the initial probability \( p \) and \( q \), respectively. At stage \( t \geq 2 \), the state of player 1 and 2 are chosen according to the transition probability \( P_{a_{t-1},b_{t-1}}(k_{t-1},k_{t}) \) and \( Q_{a_{t-1},b_{t-1}}(l_{t-1},l_{t}) \), respectively. Each player does not know the state of the opponent player. Once the initial state is chosen, at stage \( t = 1, 2, ..., N \), each player simultaneously chooses its action which is observed by both players. This is a perfect monitoring and perfect recall game, i.e. every player can observe the current actions of both players, and record all history actions of both players. At stage \( t \), \( G_{k_t,l_t}(a_t,b_t) \) is the one stage payoff of player 1, and the one stage cost of player 2. None of the players can observe the one stage payoff, and the total payoff is revealed to both players at the end of the game.

At the beginning of stage \( t \), the available information of player 1 and 2 is denoted by \( \mathcal{I}_t = \{k_1, a_1, b_1, ..., k_{t-1}, a_{t-1}, b_{t-1}, k_t\} \) and \( \mathcal{J}_t = \{l_1, a_1, b_1, ..., l_{t-1}, a_{t-1}, b_{t-1}, l_t\} \), respectively. The behavioral strategy of player 1 and 2 at stage \( t \) are \( \sigma_t \) and \( \tau_t \), respectively, where \( \sigma_t : \mathcal{I}_t \mapsto \Delta(\mathcal{A}) \) and \( \tau_t : \mathcal{J}_t \mapsto \Delta(\mathcal{B}) \). \( \sigma_t \) and \( \tau_t \) are the probability distributions over player 1’s actions \( a_t \) and player 2’s action \( b_t \) at stage \( t \), respectively. \( \Sigma \) and \( \mathcal{T} \) are the sets of behavior strategies of player 1 and 2, respectively. Strategy of player 1, \( \sigma \in \Sigma \), is a sequence of \( \sigma_t \) and strategy of player 2, \( \tau \in \mathcal{T} \), is a sequence of \( \tau_t \). The payoff with initial probabilities \( p, q \) and strategies \( \sigma, \tau \) of an \( N \)-stage \( \lambda \) discounted game with \( \lambda \in (0, 1) \) is defined as

\[
\gamma_{N,\lambda}(p,q,\sigma,\tau) = E_{\sigma,q,\sigma,\tau} \left( \sum_{t=1}^{N} \lambda^{t-1} G_{k_t,l_t}(a_t,b_t) \right)
\]

Here, \( \lambda \in (0, 1) \) if \( N \) is infinite.
An $N$-stage $\lambda$ discounted game $\Gamma_{N,\lambda}(p, q)$ is defined as two player zero-sum stochastic bayesian game equipped with initial probability $p$ and $q$, strategy spaces $\Sigma$ and $\mathcal{T}$ and payoff function $\gamma_{N,\lambda}(p, q, \sigma, \tau)$. If $\lambda = 1$ and $N$ is finite then it represents $N$-stage game payoff. If $\lambda < 1$ and $N$ is infinite then it is a discounted game. If $\lambda < 1$ and $N$ is finite then it is a truncated discounted game. Here, we exclude the case when $N = \infty$ and $\lambda = 1$. In $\Gamma_{N,\lambda}(p, q)$, player 1 wants to maximize the payoff and player 2 wants to minimize it. Therefore, player 1 has a security level $\underline{v}_{N}(p, q)$, which is also called the maxmin value of the game, defined as

$$\underline{v}_{N,\lambda}(p, q) = \max_{\sigma \in \Sigma} \min_{\tau \in \mathcal{T}} \gamma_{N,\lambda}(p, q, \sigma, \tau)$$

A strategy $\sigma^* \in \Sigma$ which can ensure player 1's security level, i.e. $\underline{v}_{N,\lambda}(p, q) = \min_{\tau \in \mathcal{T}} \gamma_{N,\lambda}(p, q, \sigma^*, \tau)$, is called the security strategy of player 1.

Similarly, player 2's security level $\overline{v}_{N,\lambda}(p, q)$, which is also called the minmax value of the game, is defined as

$$\overline{v}_{N,\lambda}(p, q) = \min_{\tau \in \mathcal{T}} \max_{\sigma \in \Sigma} \gamma_{N,\lambda}(p, q, \sigma, \tau)$$

The security strategy of player 2, $\tau^* \in \mathcal{T}$ guarantees player 2's security level.

When $\underline{v}_{N,\lambda}(p, q) = \overline{v}_{N,\lambda}(p, q)$, we can say the game has a value denoted by $v_{N,\lambda}(p, q) = \underline{v}_{N,\lambda}(p, q) = \overline{v}_{N,\lambda}(p, q)$. According to Theorem 3.2 in [9], this game has a value if $N$ is finite or $\lambda \in (0, 1)$.

### 3 Primal games and the related results

The $N$-stage $\lambda$-discounted game is also called the primal game in this paper. This section introduces some properties of the game value and both players’ security strategies in the primal game. Typical techniques are used to prove the lemmas, and the proofs are in the Appendix.

In the primal game, there are two essential variables $p_t(k_t) = P_r(k_t|I_t)$ and $q_t(l_t) = P_r(l_t|J_t)$, which are called the believes of $k_t$ and $l_t$, respectively. It is easy to see that $p_1 = p$ and $q_1 = q$. It is straightforward to show that

$$p_{t+1}(k') = \sum_k P_{a,b}(k,k') \frac{p_t(k)X(a,k)}{\bar{x}_{p_t,X}(a)} \overline{p}_{a,X}(k') \quad (1)$$

$$q_{t+1}(l') = \sum_l Q_{a,b}(l,l') \frac{q_t(l)Y(b,l)}{\bar{y}_{q_t,Y}(b)} \overline{q}_{a,Y}(l') \quad (2)$$

where $X(\cdot, k) \in \Delta(A)$ and $Y(\cdot, l) \in \Delta(B)$ are player 1 and 2’s strategies at stage $t$ given state $k$ and $l$, respectively. $\bar{x}_{p_t,X}(a) = \sum_k p_t(k)X(a,k)$, and $\bar{y}_{q_t,Y}(b) = \sum_l q_t(l)Y(b,l)$. Later, we will show that these two variables, together with stage $t$ if $N$ is finite, is a sufficient statistic of the primal game. To prove this result, we first show the following lemma.

**Lemma 1** In primal game $\Gamma_{N,\lambda}(p, q)$, the game value $v_{N,\lambda}(p, q)$ is concave in $p$ and convex in $q$. 
For the simplicity of the mathematical expression of Lemma 2, we define
\[
\bar{G}(p, q, X, Y) = \sum_{k,l} p(k)q(l) \sum_{a,b} X(a, k)G_{k,l}(a, b)Y(b, l)
\]
\[
\bar{T}_{p,q,X,Y}(v) = \bar{G}(p, q, X, Y) + \lambda \sum_{a\in A} \sum_{b\in B} \bar{x}_{p,X}(a)p_{p,X}(v) + \bar{y}_{q,Y}(b)q_{q,Y}(v)
\]

**Lemma 2** Consider a primal game \( \Gamma_{N,\lambda}(p, q) \). Let \( p_t, q_t \) be the believes at stage \( t = 1, \ldots, N \) with \( n = N + 1 - t \) stages to go. The game value of the primal game satisfies the following recursive formula
\[
v_{n,\lambda}(p_t, q_t) = \max_{X \in \Delta(A)} \min_{Y \in \Delta(B)} \bar{T}_{p_t,q_t,X,Y}(v_{n-1,\lambda}) \tag{3}
\]
\[
= \min_{Y \in \Delta(B)} \max_{X \in \Delta(A)} \bar{T}_{p_t,q_t,X,Y}(v_{n-1,\lambda}) \tag{4}
\]

The optimal solution \( X^* \) to equation (3) and the optimal solution \( Y^* \) to equation (4) are player 1 and 2’s security strategies at stage \( t \), respectively.

Moreover, player 1 and 2’s security strategies only depend on \( p_t \) and \( q_t \), together with \( t \) if \( N \) is finite.

The problem of the sufficient statistic \( p_t \) and \( q_t \) is that no player has full access to it. Equation (1) and (2) indicate that the belief of player i’s state depends on player i’s strategy, for \( i = 1, 2 \). While each player has its own strategy, it has no access to the other player’s strategy, and hence can only compute either \( p_t \) or \( q_t \).

To explore the full accessible sufficient statistic, we study the dual games of the primal game in the next section.

### 4 Dual game and the related results

This section introduces two dual games of the primal game \( \Gamma_{N,\lambda}(p, q) \), explains how the game values and the strategies of the dual games are related to that of the primal game, and finally provides the recursive formula and a sufficient statistic of the two dual games.

The dual games are rooted from the Fenchel’s conjugate of the game value of the primal game \( \Gamma_{N,\lambda}(p, q) \) [4][18]. The Fenchel’s conjugate of \( v_{N,\lambda}(p, q) \) regarding \( p \) is the game value of type 1 dual game, and the Fenchel’s conjugate of \( v_{N,\lambda}(p, q) \) regarding \( q \) is the game value of type 2 dual game. Next, we will introduce the two types of dual games.

Type 1 dual game can be specified by the nine-tuple \((K, L, A, B, \mu, q, P, Q, G)\), where \( K, L, A, B, q, P, Q, G \) are defined the same as in the primal game and \( \mu \in \mathbb{R}^{|K|} \) is the initial realized vector payoff over player 1’s state. Type 1 dual game \( \tilde{\Gamma}_{N,\lambda}(\mu, q) \) is played similarly as in the primal game \( \Gamma_{N,\lambda}(p, q) \) except that the initial state of player 1 is chosen by itself rather than the nature. If \( p \) is player 1’s strategy to choose its initial state, the payoff is
\[
\tilde{\gamma}_{N,\lambda}(\mu, q, p, \sigma, \tau) = E_{p,q,\sigma,\tau} \left( \mu(k_1) + \sum_{t=1}^{N} \lambda^{t-1}G_{k_t,l_t}(a_t, b_t) \right)
\]
Similarly, type 2 dual game is specified by the nine-tuple \((K, L, A, B, p, ν, P, Q, G)\), where \(K, L, A, B, p, P, Q, G\) are defined the same as in the primal game and \(ν ∈ \mathbb{R}^{|L|}\) is the initial realized vector payoff over player 2’s state. Type 2 dual game \(\tilde{Γ}^2_{N, λ}(p, ν)\) is played similarly as in the primal game \(Γ^N_{N, λ}(p, q)\) except that the initial state of player 2 is chosen by itself rather than the nature. If \(q\) is player 2’s strategy to choose its initial state, then the payoff function is

\[
\tilde{γ}^2_{N, λ}(p, ν, q, σ, τ) = E_{p, q, σ, τ} \left( ν(l_1) + \sum_{t=1}^{N} \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) \right)
\]

In both dual games, player 1 wants to maximize the payoff and player 2 wants to minimize it. The game values of type 1 and 2 dual games are denoted by \(w^1_{N, λ}(μ, q)\) and \(w^2_{N, λ}(p, ν)\), respectively. All the propositions and lemmas in this section can be derived using typical techniques, and the proofs are in the Appendix.

We are interested in how the game values of the dual games are related to the game value of the primal game.

**Proposition 1** Let \(v_{n, λ}(p, q)\) be the game value of the primal game \(Γ^N_{N, λ}(p, q)\), and \(w^1_{n, λ}(μ, q)\) and \(w^2_{n, λ}(p, ν)\) be the game values of type 1 and 2 dual games, respectively. We have

\[
v_{n, λ}(p, q) = \min_μ w^1_{n, λ}(μ, q) - p^T μ \tag{5}
\]

\[
v_{n, λ}(p, q) = \max_ν w^2_{n, λ}(p, ν) - q^T ν \tag{6}
\]

Moreover, player 2’s optimal strategy in type 1 dual game \(\tilde{Γ}^1_{n, λ}(μ^*, q)\) is also its optimal strategy in primal game \(Γ^N_{N, λ}(p, q)\), where \(μ^*\) is the optimal solution to equation \((5)\).

Similarly, player 1’s optimal strategy in type 2 dual game \(\tilde{Γ}^2_{n, λ}(p, ν^*)\) is also its optimal strategy in primal game \(Γ^N_{N, λ}(p, q)\), where \(ν^*\) is the optimal solution to equation \((6)\).

In the primal game \(Γ^N_{N, λ}(p, q)\), define

\[
α_1(σ) = \min_τ E_{p, q, σ, τ} \left( \sum_{t=1}^{N} \lambda^{t-1} G_{k_t, l_t}(a_t, b_t)|l_1 = l \right)
\]

\[
β_k(τ) = \max_σ E_{p, q, σ, τ} \left( \sum_{t=1}^{N} \lambda^{t-1} G_{k_t, l_t}(a_t, b_t)|k_1 = k \right)
\]

Let \(α(σ) = (α_1(σ))|_σ \in L\) and \(β(τ) = (β_k(τ))|_k \in K\). From the proof of Proposition 1, we also show the following corollary.

**Corollary 1** An optimal solution to equation \((5)\) and equation \((6)\) is

\[
μ^* = -β(τ^*) \tag{7}
\]

\[
ν^* = -α(σ^*) \tag{8}
\]

where \(σ^*\) and \(τ^*\) are player 1 and 2’s security strategies in the primal game \(Γ^N_{N, λ}(p, q)\), respectively.

Moreover, \(w^1_{n, λ}(μ^*, q) = w^2_{n, λ}(p, ν^*) = 0\), where \(μ^*\) and \(ν^*\) is given in equation \((7)\) and \((8)\), respectively.
Proposition 2 The values of type 1 and 2 dual games satisfy

\[ w^{1}_{n,\lambda}(\mu, q) = \max_{p} p^T \mu + v_{n,\lambda}(p, q) \] (9)

\[ w^{2}_{n,\lambda}(p, \nu) = \min_{q} q^T \nu + v_{n,\lambda}(p, q) \] (10)

Proof For type 1 dual game,

\[ w^{1}_{n,\lambda}(\mu, q) = \max_{p} \min_{\sigma} \gamma^{1}_{n,\lambda}(\mu, q, p, \sigma, \tau) \]

\[ = \max_{p} \min_{\sigma} \max_{\tau} E_{p}(\mu(k)) + \sum_{t=1}^{N} \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) \]

By similar technique we can prove equation (10) for type 2 dual game. \(\square\)

Proposition 3 The value of type 1 dual game satisfies the following recursive formula:

\[ w^{1}_{n,\lambda}(\mu, q) = \min_{Y} \min_{\beta_{a\in A, b\in B}} \max_{\Pi} \sum_{a, k} \Pi(a, k) \left[ \mu(k) + \sum_{l, b} G_{k, l}(a, b)Y(b, l)q(l) + \lambda \sum_{b} \tilde{q}_{l, Y}(b) \left( w^{1}_{n-1,\lambda}(\beta_{a, b}, \tilde{q}_{l, Y}) - \sum_{k'} \Pr_{a, b}(k, k') \beta_{a, b}(k') \right) \right] \]

where \( \Pi(a, k) = Pr(a \cap k) \). The value of type 2 dual game satisfies the following recursive formula:

\[ w^{2}_{n,\lambda}(p, \nu) = \max_{X} \max_{\alpha_{a\in A, l\in \Lambda}} \min_{\Psi} \sum_{b, l} \Psi(b, l) \left[ \nu(l) + \sum_{k, a} G_{k, l}(a, b)X(b, l)p(k) + \lambda \sum_{a} \tilde{X}_{a}(a) \left( w^{2}_{n-1,\lambda}(p_{a, X}, \alpha_{a, b}) - \sum_{l'} Q_{a, b}(l, l')\alpha_{a, b}(l') \right) \right] \]

where \( \Psi(b, l) = Pr(b \cap l) \).
Based on Proposition 3, we see that besides the belief, there is another important variable, \( \mu \) in type 1 dual game, and \( \nu \) in type 2 dual game. We call them the realized vector payoffs on state \( k \) and \( l \), respectively, and define them in the following way.

**Definition 1** Consider a type 1 dual game \( \tilde{\Gamma}^{1}_{N,\lambda}(\mu, q) \). Let \( \mu^{1}_{1} = \mu \) be the initial realized vector payoff, and \( \beta^{*} \) be the optimal solution to the problem

\[
 w^{1}_{n,\lambda}(\mu, q) = \min_{Y} \min_{\beta \in A, \lambda \in \pi} \max_{\mu, q, Y, \beta, \lambda} \left( w^{1}_{n-1,\lambda} \right)
\]

where \( n = N + 1 - t \) for stage \( t \). The realized vector payoff over state \( k \) at stage \( t+1 \) is defined as \( \mu^{1}_{t+1} = \beta^{*}_{n,b} \).

Similarly, consider a type 2 dual game \( \tilde{\Gamma}^{2}_{N,\lambda}(p, \nu) \). Let \( \nu^{1}_{1} = \nu \) be the initial realized vector payoff, and \( \alpha^{*} \) be the optimal solution to the problem

\[
 w^{2}_{n,\lambda}(p, \nu) = \max_{X} \max_{\alpha \in A, \lambda \in \psi} \min_{p, \nu, X, \alpha, \lambda} \left( w^{2}_{n-1,\lambda} \right)
\]

where \( n = N + 1 - t \) for stage \( t \). The realized vector payoff over state \( l \) at stage \( t+1 \) is defined as \( \nu^{1}_{t+1} = \alpha^{*}_{n,b} \).

In the next corollary, we show that \( p_{t}, \nu_{t} \) and \( \mu_{t}, q_{t} \) is player 1 and player 2’s sufficient statistic in type 2 and type 1 dual game, respectively. The proof is in the Appendix.

**Corollary 2** Player 1’s security strategy at stage \( t \) in type 2 dual game \( \tilde{\Gamma}^{2}_{N,\lambda}(p, \nu) \) only depends on \( p_{t} \) and \( \nu_{t} \) if \( N \) is finite. Player 2’s security strategy at stage \( t \) in type 1 dual game \( \tilde{\Gamma}^{1}_{N,\lambda}(\mu, q) \) only depends on \( \mu_{t} \) and \( q_{t} \), together with \( t \) if \( N \) is finite.

From Proposition 3, Corollary 1, and Corollary 2, we build a path of looking for the fully accessible sufficient statistics, and the security strategies of both players in the primal game. Let’s assume \( \mu^{*} \) and \( \nu^{*} \) defined as in equation (7) and (8) are found. Player 1 figures out its security strategy in type 2 dual game \( \tilde{\Gamma}^{2}_{N,\lambda}(p, \nu^{*}) \). Notice that according to Corollary 2, player 1’s security strategy only depends on \( p_{t} \) and \( \nu_{t} \) which are fully accessible by player 1. This security strategy in type 2 dual game is then used in the primal game. According to Proposition 1, this strategy is also the security strategy in the primal game. Player 2 can follow the similar steps to derive a security strategy in the primal game.

The problem is how to compute the \( \mu^{*} \) and \( \nu^{*} \) defined as in equation (7) and (8). The next section will answer this question.

### 5 LP’s for both primal and dual games

Section 3 and 4 provide descriptive results about the game value and the security strategies in primal games and dual games. This section provides prescriptive results for both the primal and dual games. In [21], LP formulation was provided to compute the optimal strategies in primal games but they did not introduce it for dual games. To be more specific, this section gives the LP formulations to compute the game values and security strategies of both players in primal and dual games. Moreover, LP formulation to compute the optimal solution to problem (5)
and \( \alpha \) are proposed.

We define the realization plan of player 1, \( R_{J_1}(a_t) \) as

\[
R_{J_1}(a_t) = p(k_1) \prod_{s=1}^{t-1} P_{a_s, b_s}(k_s, k_{s+1}) \prod_{s=1}^{t} \sigma^s_t(I_s)
\]

with \( R_{J_1}(a_0) = p(k_1) \), and \( P_{a_0, b_0}(k_0, k_1) = 1 \). It is straightforward to show that

\[
R_{J_1}(a_t) = P_{a_{t-1}, b_{t-1}}(k_{t-1}, k_t) \sigma^t_{t-1}(I_t) R_{J_{t-1}}(a_{t-1})
\]  

(14)

Similarly, the realization plan of player 2 is defined as,

\[
S_{J_2}(b_t) = q(l_1) \prod_{k=1}^{t-1} Q_{a_k, b_k}(l_k, l_{k+1}) \prod_{k=1}^{t} \tau^k_s(J_k)
\]

with \( S_{J_2}(b_0) = q(l_1) \) and \( Q_{a_0, b_0}(l_0, l_1) = 1 \). We have

\[
S_{J_2}(b_t) = Q_{a_{t-1}, b_{t-1}}(l_{t-1}, l_t) \tau^t_t(J_t) S_{J_{t-1}}(b_{t-1})
\]  

(15)

To build the LP formulation to compute the optimal solution to problem (5) and (6), we introduce the weighted payoff \( U_{J_1}(\sigma, \tau) \) and \( Z_{J_1}(\sigma, \tau) \) of player 1 and 2 as follows.

\[
U_{J_1}(\sigma, \tau) = \sum_{k_1, \ldots, k_t} R_{J_{t-1}}(a_{t-1}) P_{a_{t-1}, b_{t-1}}(k_{t-1}, k_t)
\]

\[
E\left( \sum_{s=1}^{N} \lambda^{s-1} G_{k_s, l_s}(a_s, b_s) | k_1, \ldots, k_t, J_t \right)
\]

\[
Z_{J_1}(\sigma, \tau) = \sum_{l_1, \ldots, l_t} S_{J_{t-1}}(b_{t-1}) Q_{a_{t-1}, b_{t-1}}(l_{t-1}, l_t)
\]

\[
E\left( \sum_{s=1}^{N} \lambda^{s-1} G_{k_s, l_s}(a_s, b_s) | l_1, \ldots, l_t, I_t \right)
\]

Based on the definition of \( U \) and \( Z \), we find that \( \alpha(\sigma^*) \) and \( \beta_k(\tau^*) \) are related to \( U \) and \( Z \) in the following way.

\[
\alpha(\sigma^*) = \min_{\tau} U_{J_1}(\sigma^*, \tau), \text{ where } J_1 = \{l\}
\]  

(16)

\[
\beta_k(\tau^*) = \max_{\sigma} Z_{J_1}(\sigma, \tau^*), \text{ where } I_1 = \{k\}
\]  

(17)

To build the LP formulation to compute \( \alpha(\sigma^*) \) and \( \beta_k(\tau^*) \) and hence \( \mu^* \) and \( \nu^* \) in Corollary 4, we first provide recursive formulas for \( U_{J_1}(\sigma, \tau) \) and \( Z_{J_1}(\sigma, \tau) \) in Lemma 3 and recursive formulas for \( \alpha(\sigma^*) \) and \( \beta_k(\tau^*) \) in Corollary 4 and then set up LP’s for \( \min_{\tau} U_{J_1}(\sigma, \tau) \) and \( \max_{\sigma} Z_{J_1}(\sigma, \tau) \) in Lemma 3 and finally \( \alpha(\sigma^*) \) and \( \beta_k(\tau^*) \) in Theorem 1.
Lemma 3 The weighted payoffs $U_{J_t}(\sigma, \tau)$ and $Z_{J_t}(\sigma, \tau)$ satisfy the following recursive formulas.

$$U_{J_{t+1}}(\sigma, \tau) = \sum_{a_t \in \mathcal{A}} \left( \sum_{b_t \in \mathcal{B}} R_{J_t}(a_t) \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) + \sum_{a_t} \sum_{l_t=1} Q_{a_t, b_t}(l_t, l_{t+1}) \right) \tau^b_t(J_t) ; \text{ with } U_{J_{N+1}}(\sigma, \tau) = 0$$

$$Z_{J_{t+1}}(\sigma, \tau) = \sum_{a_t \in \mathcal{A}} \left( \sum_{b_t \in \mathcal{B}} \sum_{l_t=1} S_{J_t}(b_t) \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) + \sum_{b_t} \sum_{k_t=1} P_{a_t, b_t}(k_t, k_{t+1}) \right) \sigma^a_t(J_t) ; \text{ with } Z_{J_{N+1}}(\sigma, \tau) = 0$$

Proof According to the definition, we have

$$U_{J_{t+1}}(\sigma, \tau) = \sum_{k_{t+1}, \ldots, k_{t+1}} R_{J_t}(a_t) P_{a_t, b_t}(k_t, k_{t+1}) E \left( \sum_{s=t+1}^N \lambda^{s-1} G_{k_s, l_s}(a_s, b_s)|k_1, \ldots, k_{t+1}, J_{t+1} \right)$$

$$Z_{J_{t+1}}(\sigma, \tau) = \sum_{k_{t+1}, \ldots, k_{t+1}} S_{J_t}(b_t) P_{a_t, b_t}(k_t, k_{t+1}) E \left( \sum_{s=t+1}^N \lambda^{s-1} G_{k_s, l_s}(a_s, b_s)|k_1, \ldots, k_{t+1}, J_{t+1} \right)$$

**Term 1:**

$$= \sum_{k_{t+1}, \ldots, k_{t+1}} R_{J_t}(a_t) P_{a_t, b_t}(k_t, k_{t+1}) \left( \sum_{a_t, b_t} \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) \right) \Pr(a_t, b_t|k_1, \ldots, k_t, J_t)$$

$$= \sum_{a_t, b_t} \sum_{k_{t+1}, \ldots, k_{t+1}} R_{J_t}(a_t) \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) \sigma^a_t(J_t) \tau^b_t(J_t)$$

**Term 2:**

$$= \sum_{k_{t+1}, \ldots, k_{t+1}} R_{J_t}(a_t) P_{a_t, b_t}(k_t, k_{t+1}) \left( \sum_{k_{t+1}, \ldots, k_{t+1}} \sum_{a_t, b_t} P_{a_t, b_t}(k_t, k_{t+1}) Q_{a_t, b_t}(l_t, l_{t+1}) \right) \sigma^a_t(J_t) \tau^b_t(J_t)$$

$$= \sum_{l_{t+1}, a_t, b_t} \left( \sum_{k_{t+1}, \ldots, k_{t+1}} R_{J_t}(a_t) P_{a_t, b_t}(k_t, k_{t+1}) E \left( \sum_{s=t+1}^N \lambda^{s-1} G_{k_s, l_s}(a_s, b_s)|k_1, \ldots, k_{t+1} \right) \right) \tau^b_t(J_t)$$
\[ J_{t+1})Q_{a_t, b_t} (l_t, l_{t+1}) \tau^b_t (J_t) \]
\[ = \sum_{a_t, b_t} \sum_{l_t, l_{t+1}} Q_{a_t, b_t} (l_t, l_{t+1}) U_{J_{t+1}} \tau^b_t (J_t) \]

So, equation (18) is proved. Following the similar method, we can prove the recursive formula of \( Z^*_I (\sigma, \tau) \).

Define player 1 and 2's optimal weighted payoffs \( U^*_J (\sigma) \) and \( Z^*_I (\tau) \) as

\[
U^*_J (\sigma) = \min \sum_{a_t, b_t} \sum_{l_t, l_{t+1}} R_{J_t} (a_t) \lambda^{l_t-1} G_{k_t, l_t} (a_t, b_t) \\
+ \sum_{a_t, b_t} \sum_{l_t, l_{t+1}} Q_{a_t, b_t} (l_t, l_{t+1}) U^*_{J_{t+1}} (\sigma) \tau^b_t (J_t) \\
\]

with \( U^*_J (\sigma) = 0 \).

\[
Z^*_I (\tau) = \max \sum_{a_t} \sum_{b_t} \sum_{l_t, l_{t+1}} S_{J_t} (b_t) \lambda^{l_t-1} G_{k_t, l_t} (a_t, b_t) \\
+ \sum_{a_t} \sum_{b_t} \sum_{k_{t+1}} P_{a_t, b_t} (k_t, k_{t+1}) Z^*_{I_{t+1}} (\tau) \sigma^b_t (I_t); \\
\]

with \( Z^*_I (\tau) = 0 \).

The optimal weighted payoff \( U^*_J (\sigma) \) and \( Z^*_I (\tau) \) can be computed by LP. The LP formulations are given in Lemma 4.

**Lemma 4** For any \( t = 1, \ldots, N \), \( U^*_J (\sigma) \) and \( Z^*_I (\tau) \) can be computed by the following LPs.

\[
U^*_J (\sigma) = \max \sum_{a_t} \sum_{b_t} \sum_{l_t, l_{t+1}} R_{J_t} (a_t) \lambda^{l_t-1} G_{k_t, l_t} (a_t, b_t) \\
+ \sum_{a_t} \sum_{l_{t+1}} Q_{a_t, b_t} (l_t, l_{t+1}) U^*_{J_{t+1}} (\sigma) \tau^b_t (J_t) \\
(l_t, l_{t+1}) U_{J_{t+1}} \geq U_{J_t}; \forall s = t, \ldots, N; \forall b_s; \forall J_t \subset J_s,
\]
with $U_{J_{N+1}} = 0, \forall J_{N+1}$.

$$Z_{I_{s}}(\tau) = \min_{Z} Z_{I_{s}}$$

s.t.

$$\sum_{l_{i_1}...l_{i_s}} S_{J_{s}}(b_s) \lambda^{s-1} G_{k_{i_s},l_{i_s}}(a_{i_s}, b_{i_s}) + \sum_{k_{s+1}} P_{a_{i_s},b_{i_s}} (k_{s+1}, k_{s+1}) Z_{I_{s+1}} \leq Z_{I_{s}}, \forall I_{s} \subset I_{s}$$ \label{20}

with $Z_{I_{N+1}} = 0, \forall I_{N+1}$.

Proof Let us define two optimization problems $P1$ and $P2$.

$$P1 : \bar{U}_{J_{t}}(\sigma) = \max_{U_{J_{t}}}$$

s.t.

$$\sum_{k_{t_1}...k_{t_s}} R_{I_{s}}(a_{t_s}) \lambda^{s-1} G_{k_{t_s},l_{t_s}}(a_{t_s}, b_{t_s}) + \sum_{l_{t+1}} Q_{a_{t_s},b_{t_s}} (l_{t_s}, l_{t+1}) \bar{U}_{J_{t+1}}(\sigma) \geq U_{J_{t}}, \forall s = t, ...N$$ \label{21}

We know,

$$U_{J_{t}}^{*}(\sigma) = \min_{\tau} \sum_{b_{t}} \left( \sum_{a_{t}} \sum_{k_{t_1}...k_{t_s}} R_{I_{s}}(a_{t_s}) \lambda^{s-1} G_{k_{t_s},l_{t_s}}(a_{t_s}, b_{t_s}) + \sum_{l_{t+1}} Q_{a_{t_s},b_{t_s}} (l_{t_s}, l_{t+1}) \bar{U}_{J_{t+1}}(\sigma) \right) \tau^{b_{t}}_{t}(J_{t})$$

s.t. $\bar{U}_{J_{t}}^{*}(\sigma)$ = max $U_{J_{t}}$

With $\tau^{b_{t}}_{t}(J_{t}) \geq 0, \forall b_{t}$

The dual of this LP is,

$$P2 : U_{J_{t}}^{*}(\sigma) = \max_{\bar{U}_{J_{t}}}$$

s.t.

$$\sum_{k_{t_1}...k_{t_s}} R_{I_{s}}(a_{t_s}) \lambda^{s-1} G_{k_{t_s},l_{t_s}}(a_{t_s}, b_{t_s}) + \sum_{l_{t+1}} Q_{a_{t_s},b_{t_s}} (l_{t_s}, l_{t+1}) U_{J_{t+1}}^{*}(\sigma) \geq U_{J_{t}}, \forall b_{t}$$

For $t = N$,

$$\bar{U}_{J_{N}}(\sigma) = \max_{U} U_{J_{N}}$$

s.t.

$$\sum_{k_{N_1}...k_{N_s}} R_{I_{s}}(a_{N_s}) \lambda^{s-1} G_{k_{N_s},l_{N_s}}(a_{N_s}, b_{N_s}) \geq U_{J_{N}}(\sigma), \forall b_{N}$$

$$U_{J_{N}}^{*}(\sigma) = \max_{U} U_{J_{N}}$$

s.t.

$$\sum_{k_{N_1}...k_{N_s}} R_{I_{s}}(a_{N_s}) \lambda^{s-1} G_{k_{N_s},l_{N_s}}(a_{N_s}, b_{N_s}) \geq U_{J_{N}}(\sigma), \forall b_{N}$$
From this we can see, for $t = N$, $U_{J_t}(\sigma) = U_{J_t}^*(\sigma)$. Let us assume, $U_{J_{t+1}} = \bar{U}_{J_{t+1}}$.

$P3 : U_{J_{t+1}}^*(\sigma) = \max_{\bar{U}} U_{J_{t+1}}$

\[
\begin{align*}
    &\text{s.t. } \sum_{k_1\ldots k_t \at I_t} \sum_{a_t} R_{I_t}(a_t)^{\lambda-1} G_{k_t, I_t}(a_t, b_t) + \sum_{l_{t+1} \at I_t} \sum_{a_t} Q_{a_t, b_t}(l_t, l_{t+1}) U_{J_{t+1}} \\
    &\hspace{5cm} \geq U_{J_t}; \quad \forall s = (t + 1) \ldots N, \forall b_s, J_s \supset J_{t+1}
\end{align*}
\]

Let $\bar{U}_{J_1}, \bar{U}_{J_2}, \ldots, \bar{U}_{J_N}$ be the optimal solution of $P1$. $U_{J_t}^*$ be the optimal solution of $P2$. $\bar{U}_{J_{t+1}}^*, \ldots, \bar{U}_{J_N}^*$ be the optimal solution of $P3$. Here, $U^*$ is feasible in $P1$. So, $\bar{U}_{J_t} \geq U_{J_t}^*$. $\bar{U}_{J_{t+1}} \ldots \bar{U}_{J_N}^*$ is feasible in $P3$. So, $U_{J_{t+1}}^* \geq \bar{U}_{J_{t+1}}$. For $s = t$, we can write equation (21) as,

\[
\begin{align*}
    &\sum_{k_1\ldots k_t \at I_t} \sum_{a_t} R_{I_t}(a_t)^{\lambda-1} G_{k_t, I_t}(a_t, b_t) + \sum_{l_{t+1} \at I_t} \sum_{a_t} Q_{a_t, b_t}(l_t, l_{t+1}) U_{J_{t+1}}(\sigma) \\
    &\geq U_{J_t}(\sigma); \quad \forall t = 1, \ldots, N, \forall b_s, J_s \supset J_{t+1}
\end{align*}
\]

So, $\bar{U}_{J_t}$ is feasible in $P2$. Then $U_{J_t}^* \geq \bar{U}_{J_t}$. Therefore, $\bar{U}_{J_t} = U_{J_t}^*$, which completes the proof. Similarly we can prove equation (22).

Based on the LP in Lemma[3] we further develop the LP’s to compute $U_{J_t}^*(\sigma^*)$ and $Z_{I_t}^*(\tau^*)$ by considering the game value and the security strategies of the primal game in Theorem[1]

**Theorem 1** Consider a primal game $\Gamma_{n, \lambda}(p, q)$. Its game value satisfies

\[
\begin{align*}
    v_{n, \lambda}(p, q) = \max_R \max_U \sum_l q(l) U_{J_1}; \quad \text{where } J_1 = \{l\} \quad (22) \\
    &\text{s.t. } \sum_{k_1\ldots k_t \at I_t} \sum_{a_t} R_{I_t}(a_t)^{\lambda-1} G_{k_t, I_t}(a_t, b_t) + \sum_{l_{t+1} \at I_t} \sum_{a_t} Q_{a_t, b_t}(l_t, l_{t+1}) \\
    &\hspace{5cm} U_{J_{t+1}} \geq U_{J_t}; \quad \forall t = 1, \ldots, n, \forall b_t, \forall J_{t+1} \supset J_t \quad (23) \\
    &\sum_{a_t} R_{I_t}(a_t) = P_{a_{t-1}, b_{t-1}}(k_{t-1}, k_t) R_{I_{t-1}}(a_{t-1}); \quad \forall t = 1, \ldots, n, \forall I_t \quad (24) \\
    &U_{J_{n+1}} = 0; \quad \forall J_{n+1} \quad (25) \\
    &R_{I_t}(a_t) \geq 0; \quad \forall I_t \quad (26)
\end{align*}
\]

The optimal strategy of player 1 is,

\[
\begin{align*}
    \sigma_{t}^{*a_t}(I_t) = \frac{R_{I_t}(a_t)}{P_{a_{t-1}, b_{t-1}}(k_{t-1}, k_t) R_{I_{t-1}}^*(a_{t-1})} \quad (27) \\
    a_t(\sigma^*) \equiv U_{J_t}^* \quad (28)
\end{align*}
\]
where $R^*$ and $U^*$ are the optimal solution to (22). Similarly, we also have

$$ v_{n,\lambda}(p,q) = \min_S \min_Z \sum_k p(k)Z_{k^t}; \text{ where } I_1 = \{ k \} $$

s.t. $\sum_{l_1,\ldots,l_t} S_{\beta_i}(b_t) \lambda^{l_{t-1}} G_{k_t, l_t} (a_t, b_t) + \sum_{k_{t+1}} P_{a_t, b_t}(k_t, k_{t+1})$

$$ Z_{k_{t+1}} \leq Z_{k_t}; \forall t = 1, \ldots, n, \forall \lambda_t, \forall I_{t+1} \supset I_t $$

$$ \sum_{b_{t_t}} S_{\beta_i}(b_t) = Q_{a_{t-1}, b_{t-1}} (l_{t-1}, l_t) S_{\beta_i-1}(b_{t-1}); \forall t = 1, \forall \beta_i $$

$$ Z_{k_{t+1}} = 0; \forall I_{n+1} $$

$$ S_{\beta_i}(b_t) \geq 0; \forall \beta_i $$

The optimal strategy of player 2 is,

$$ \tau^*_{l_t}(b_t) = \frac{S_{\beta_i}(b_t)}{Q_{a_{t-1}, b_{t-1}} (l_{t-1}, l_t) S_{\beta_i-1}(b_{t-1})} $$

$$ \beta_t(\tau^*) = Z_{I_1} $$

where $Z^*$ and $S^*$ are the optimal solution to (47).

Proof According to the discussion in the second paragraph on page 248 in [21], the realization probabilities can serve as strategic variables of a player. We have,

$$ v_{n,\lambda}(p,q) = \max \min_{\sigma} E \left( \sum_{t=1}^n \lambda^{t-1} G_{k_t, l_t} (a_t, b_t) \right) $$

$$ = \max_R \sum_t q(t) \min_{\sigma} E \left( \sum_{t=1}^n \lambda^{t-1} G_{k_t, l_t} (a_t, b_t) | t \right) $$

$$ = \max_R \sum_t q(t) U^*_{\beta_t}(\sigma) $$

$$ = \max_R \max_{\mu} \sum_{l_t} q(l) U_{\beta_i}, \text{ where } J_1 = \{ l \} $$

s.t. equation (23 - 26)

Let $R^*$ and $U^*$ be the optimal solution. Equation (14) implies that the optimal strategy satisfies equation (27), and equation (10) implies equation (28). Following the same steps, equation (17) can be shown. \( \square \)

The size of the linear program (22) is polynomial with respect to the size of $|K|, |L|, |A|, |B|$ and exponential with respect to the length of the stage $n$. We first examine the size of variables. When $t = 1$ the number of scalar variables for $R^*_t$ is $|K||A|$. When $t = 2$ the number of scalar variables for $R^*_t$ is $|K||A||B||K||A|$. So, when $t = n$ the number of scalar variables for $R^*_t$ is $|K|([|A||B||K|]|A|).$ The number of scalar variables for $U^*_t$ is $|L|([|A||B||L|]|A|).$ Therefore, the number of scalar variables has the order of $O((|A|^n|B|^n(|K|^n + |L|^n))).$ For the constraints, constraint set (23) has $|B||C|$ inequalities when $t = 1$, $|B||C||A||B||L|$ inequalities when $t = 2$, and $|B||C|(|A||B||C|)^{n-1}$ inequalities when $t = n$. In constraint set (24), the
The optimal strategy of player 1 in type 2 dual game is, where the security strategy of player 2 at the same time. Based on the relation between the security level and the security strategy of player 1 in type 2 dual game, and the security level and the security strategy of player 2 in type 1 dual game, can further develop the LPs to compute the security level and the security strategy for order of \( n, \lambda \) set (25) and (26), respectively. Therefore, the number of the constraints is in the order of \( O(|A|^n|B|(|K| + |L|)^n) \). The computational complexity of LP (47) is the same as that of LP (22-26).

Theorem 1 provides one LP to compute \( \alpha_l(\sigma^*) \) (and hence \( \mu^* \)), the security level of player 1, and the security strategy of player 1 at the same time, and another LP to compute \( \beta_l(\tau^*) \) (and hence \( \nu^* \)), the security level of player 2, and the security strategy of player 2 at the same time. Based on the relation between the game values of the primal game and the two dual games in Proposition 2, we can further develop the LPs to compute the security level and the security strategy of player 1 in type 2 dual game, and the security level and the security strategy of player 2 in type 1 dual game.

**Proposition 4** The game value of type 2 dual game \( \tilde{\Gamma}^{\alpha\lambda}_{n,\lambda}(p, \nu) \) satisfies,

\[
\begin{align*}
    w_{n,\lambda}^2(p, \nu) &= \max \max \max U_0 \\
    & \text{s.t. } \nu(l) + U_{\mathcal{J}_t} \geq U_0, \quad \forall l, \text{ where } \mathcal{J}_1 = \{l\} \\
    & \quad \sum_{k_t} \sum_{a_t} R_{\mathcal{I}_t}(a_t) \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) + \sum_{l_{t+1}} Q_{a_t, b_t}(l_t, l_{t+1}) \\
    & \quad U_{\mathcal{J}_{t+1}} \geq U_{\mathcal{J}_t}; \quad \forall t = 1, \ldots, n, \forall b_t, \forall \mathcal{J}_{t+1} \supset \mathcal{J}_t \\
    & \quad \sum_{a_t} R_{\mathcal{I}_t}(a_t) = P_{a_{t-1}, b_{t-1}}(k_{t-1}, k_t) R_{\mathcal{I}_{t-1}}(a_{t-1}); \quad \forall t = 1, \ldots, n; \forall \mathcal{I}_t \\
    & \quad U_{\mathcal{J}_{N+1}} = 0; \quad \forall \mathcal{J}_{N+1} \\
    & \quad R_{\mathcal{I}_t}(a_t) \geq 0; \quad \forall \mathcal{I}_t 
\end{align*}
\]

The optimal strategy of player 1 in type 2 dual game is,

\[
\begin{align*}
    \sigma^*_n\lambda(I_t) &= \frac{R_{\mathcal{I}_t}^*(a_t)}{P_{a_{t-1}, b_{t-1}}(k_{t-1}, k_t) R_{\mathcal{I}_{t-1}}^*(a_{t-1})} 
\end{align*}
\]

where \( R^* \) and \( U^* \) is the optimal solution to problem (32). Similarly, the game value of type 1 dual game \( \Gamma^{\alpha\lambda}_{n,\lambda}(\mu, q) \) satisfies,

\[
\begin{align*}
    w_{n,\lambda}^1(\mu, q) &= \min \min \min Z_0 \\
    & \text{s.t. } \mu(k) + Z_{\mathcal{I}_t} \leq Z_0, \quad \forall k, \text{ where } \mathcal{I}_1 = \{k\} \\
    & \quad \sum_{l_t, \ldots, l_{t-1}} \sum_{b_t} S_{\mathcal{I}_{t-1}}(b_t) \lambda^{t-1} G_{l_{t-1}, l_t}(a_t, b_t) + \sum_{k_{t+1}} P_{a_t, b_t}(k_t, k_{t+1}) \\
    & \quad Z_{\mathcal{I}_{t+1}} \leq Z_{\mathcal{I}_t}; \quad \forall t = 1, \ldots, n; \forall a_t, \forall \mathcal{I}_{t+1} \supset \mathcal{I}_t \\
    & \quad \sum_{b_t} S_{\mathcal{I}_{t-1}}(b_t) = Q_{a_{t-1}, b_{t-1}}(l_{t-1}, l_t) S_{\mathcal{I}_{t-1}}^*(b_{t-1}); \quad \forall t = 1, \ldots, n; \forall \mathcal{I}_t \\
    & \quad Z_{\mathcal{I}_{N+1}} = 0; \quad \forall \mathcal{I}_{N+1} \\
    & \quad S_{\mathcal{I}_{t-1}}(b_t) \geq 0; \quad \forall \mathcal{I}_t 
\end{align*}
\]
The optimal strategy of player 2 is,
\[ \tau^*_t(J_t) = \frac{S^*_t(b_t)}{Q_{a_{t-1},b_{t-1}}(l_{t-1},l_t)S^*_{J_{t-1}}(b_{t-1})} \]
where \( S^* \) and \( Z^* \) is the optimal solution to problem (33).

Proof

By using theorem 1 we can write the above equation as,
\[ w_{n,\lambda}^2(p, \nu) = \min_{q} q^T \nu + v_{n,\lambda}(p, q) \]
Given \( U, R \) s.t. equation (34) - (37), which are linear constraints. Hence \( R \) is in a compact convex set. Given \( U, q \) s.t. equation (38) and (39), which are linear. Hence \( q \) is in a compact convex set. Given \( R, F_1(q, R, U) \) is linear w.r.t. \( q \) and \( R \). Then we can write,
\[ \max_R \min_q \sum_l q(l)(\nu(l) + U_{J_t}) \]
\[ \text{s.t. equation (35) to (39)} \]
Table 1 The total channel capacity matrix (Payoff matrix), given sensors' type $k$ and jammer's type $l$

| $k$ | l (0.5 km) | l (2 km) |
|-----|------------|----------|
| 1 ([1 1] km) | 108.89 | 113.78 |
|       | 108.89 | 113.78 |
| 2 ([1 5] km) | 11.48 | 107.38 |
|       | 99.04 | 20.15 |
| 3 ([5 5] km) | 1.64 | 13.75 |
|       | 1.64 | 13.75 |

We have,

$$\min \quad \sum_{l} q(l)(\nu(l) + U_{J_1})$$

s.t.

$$\sum_{l} q(l) = 1$$

$$q(l) \geq 0; \quad \forall l$$

The dual problem of this is,

$$\max_{U_0} U_0$$

s.t.

$$\nu(l) + U_{J_1} \geq U_0, \quad \forall l$$

So the proposition is proved. By using the same method we can prove the linear program (33).

The level of computational complexity of the LP for dual game is same as the primal game LP.

6 Case study

Jamming problems in underwater acoustic sensor networks can be modeled as a strategic game by game theoretic approach. [20] represented jamming in an underwater sensor network as a Bayesian two-player zero-sum one-shot game and evaluated how the nodes’ position (state) effect the equilibrium and [7] extended it to a repeated Bayesian game with uncertainties on both sides. In this paper we have used the same network model and formulated it as a two-player zero-sum stochastic game where both players are partially informed. The network has four sensors ($s_1, s_2, s_3, s_4$) and one jammer. The goal of the sensors is to transmit data to a sink node by using a shared spectrum at [10, 40] KHz which is divided into two channels, $B_1 = [10, 25]$ kHz and $B_2 = [25, 40]$ kHz whereas the jammer wants to block this data transfer. Each channel can be used by one sensor so that at a time only 2 sensors can transmit data. The distance between the sensor and sink node can be 1 km or 5 km. Let, the distances from sink node to $s_1, s_2, s_3$ and $s_4$ are 1 km, 5 km, 1 km and 5 km respectively. The jammer distance from the sink node can be 0.5 km or 2 km. Let us consider sensors as player 1 and the jammer
Table 2  P (transition matrix for sensor’s state)

|   | 1   | 2   |
|---|-----|-----|
| 1 | 0.8 | 0.1 |
|   | 0.1 | 0.5 |
|   | 0.2 | 0.1 |
| 2 | 0.2 | 0.6 |
|   | 0.5 | 0.5 |
|   | 0   | 1   |
|   | 0   | 0.1 |

Table 3  Q (transition matrix for jammer’s state)

|   | 1   | 2   |
|---|-----|-----|
| 1 | 1   | 0   |
|   | 0.2 | 0.8 |
|   | 0.5 | 0.5 |
|   | 0.1 | 0.9 |
| 2 | 0.6 | 0.4 |
|   | 0.7 | 0.3 |
|   | 0.5 | 0.5 |

Table 4  Optimal security strategy of the sensors in the 2-stage primal stochastic game given history information

|   | 1   | 2   |
|---|-----|-----|
| 11 | 1   | 0   |
| 12 | 1   | 1   |
| 21 | 0   | 1   |
| 22 | 1   | 0   |

Table 5  Optimal security strategy of the jammer in the 2-stage primal stochastic game given history information

|   | 1   | 2   |
|---|-----|-----|
| 11 | 1   | 0   |
| 12 | 1   | 1   |
| 21 | 0   | 1   |
| 22 | 1   | 0   |
as player 2. The possible states for player 1 are \([1,1], [1,5] \) and \([5,5] \) and states for player 2 are \([0.5] \) and \([2] \). For sensors, we are denoting \([1,5] \) and \([5,1] \) as the same state. The initial state probabilities of player 1 and 2 are \(p = [0.5 \ 0.3 \ 0.2] \) and \(q = [0.5 \ 0.5] \) respectively. The sensors coordinate with each other to use the channels so that they can maximize the data transmission. Channel 1 is more effective for the sensors which are far away and channel 2 is better for the sensors close by. Though the sensors and the jammer do not know each others position, the jammer can observe whether a channel is used by a far away sensor or a close by sensor. For each time period, the jammer can only generate noises in one channel and sensors are able to detect it. The jammer tries to minimize the throughput of the channels. If the sensor state is \([1,1] \), the active sensors are \(s_1 \) and \(s_3 \). The feasible actions are \(s_1 \) using channel 1 while \(s_3 \) using channel 2 (action 1), and \(s_1 \) using channel 2 while \(s_3 \) using channel 1 (action 2). If the sensor state is \([5,5] \), the active sensors are \(s_2 \) and \(s_4 \). The feasible actions are \(s_2 \) using channel 1 while \(s_4 \) using channel 2 (action 1), and \(s_2 \) using channel 2 while \(s_4 \) uses channel 1 (action 2). If the sensor state is \([1,5] \), the feasible actions are faraway sensor using channel 1 while nearby sensor using channel 2 (action 1), and faraway sensor using channel 2 while nearby sensor using channel 1 (action 2). Similarly, the jammers actions are blocking channel 1 which is action 1 or channel 2 which is action 2. The payoff matrix is in Table 1. As we are modeling this network as a stochastic game there will be transition matrices for both player’s state. Table 2 and 3 show the transition matrices of sensors’ and jammer’s state, respectively.

Let the game be played for two stages. The linear program of the sensors is given in Theorem 1 from which we can compute the security level and the security strategy of sensors. The computed security level of sensors is 167.26 bit/s and the optimal security strategy is shown in Table 4. Similarly, for jammer we computed the security level and optimal strategy by using LP 47 and equation 30. The security level of jammer is 167.26 and the security strategy is in Table 5.

The security levels of both players are the same. Hence, this is the game value. We, then, used the security strategies of both players in the jamming game. For each experiment, we run the game for 1000 times, and the experiment was did for 60 times. The total throughput in the jamming game varies from 163.5 bits/s to 171.4 bits/s with an average capacity to be 167.45 which is close to the security level we got from LP of sensors and jammer.

According to equation 28 and 31 we computed the value of a specific initial realized vector payoff of sensor’s state in type 1 dual game and initial realized vector payoff of jammer’s state in type 2 dual game. We had \(\mu^* = \begin{bmatrix} 219.28 \\ 136.16 \\ 83.83 \end{bmatrix} \) and \(\nu^* = \begin{bmatrix} 151.24 \\ 183.27 \end{bmatrix} \). By using this initial realized vector payoff, we computed the LPs of dual game 52 and 53. In both cases the value of the game is zero which satisfies corollary 1. The optimal strategy of sensors in type 2 dual game is in Table 6 and the optimal strategy of jammer in type 1 dual game is in Table 7.

We can see for this specific initial realized vector payoff, the security strategy of sensors and jammer in the dual games is the same as in primal game which satisfies Proposition 1.
Table 6 Optimal security strategy of the sensors in the 2-stage Dual stochastic game given history information

| I(t) | 1    | 2    | 3    | 1111 | 1112 |
|------|------|------|------|------|------|
| \sigma^*_t(I(t))(a = 1) | 1 | 0 | 0 | 0 | 0 |
| \sigma^*_t(I(t))(a = 2) | 0 | 1 | 0 | 1 | 1 |
| J(t) | 1113 | 1121 | 1122 | 2211 | 2212 |
| \sigma^*_t(J(t))(a = 1) | 0 | 0 | 0 | 0 | 0.423 |
| \sigma^*_t(J(t))(a = 2) | 1 | 1 | 1 | 1 | 0.577 |
| J(t) | 2222 | 3111 | 3112 | 3113 | 3121 |
| \sigma^*_t(J(t))(a = 1) | 0.451 | 0 | 0.490 | 0 | 0 |
| \sigma^*_t(J(t))(a = 2) | 0.549 | 1 | 0.510 | 1 | 1 |
| J(t) | 3122 | 3123 |
| \sigma^*_t(J(t))(a = 1) | 0.265 | 0 |
| \sigma^*_t(J(t))(a = 2) | 0.735 | 1 |

Table 7 Optimal security strategy of the jammer in the 2-stage dual stochastic game given history information

| J(t) | 1    | 2    | 1111 | 1121 |
|------|------|------|------|------|
| \tau^*_t(J(t))(b = 1) | 0 | 1 | 0 | 0 |
| \tau^*_t(J(t))(b = 2) | 1 | 0 | 0 | 1 |
| J(t) | 1112 | 1211 | 1212 | 1221 |
| \tau^*_t(J(t))(b = 1) | 0.56 | 0 | 0 | 0.31 |
| \tau^*_t(J(t))(b = 2) | 0.44 | 0 | 0 | 0.69 |
| J(t) | 1222 | 2111 | 2112 | 2211 |
| \tau^*_t(J(t))(b = 1) | 1 | 0.068 | 1 | 0.068 |
| \tau^*_t(J(t))(b = 2) | 0 | 0.932 | 0 | 0.932 |
| J(t) | 2212 |
| \tau^*_t(J(t))(b = 1) | 1 |
| \tau^*_t(J(t))(b = 2) | 0 |

7 Conclusion

This paper studies two-player zero-sum stochastic Bayesian games, computes the characteristics of primal and dual game, and shows the sufficient statistics in dual game are fully accessible. It has also formulated LPs for both primal and dual game from which we can get the game values and optimal strategies of both players. We are interested in extending this work to infinite horizon cases and check how the optimal strategies change for infinite time.

Appendix

Lemma 5

\[ v_{n-1, \lambda}(q, p) = \alpha v_{n-1, \lambda}(p, q) \]
\[ v_{n-1, \lambda}(p, q) = \alpha v_{n-1, \lambda}(p, q) \]
Proof For $n = 1$,

$$v_{n,\alpha}(\alpha p, q) = \max_X \min_Y \left( \sum_{k,l} \alpha p(k) q(l) \sum_{a,b} X(a,k) G_{k,l}(a,b) Y(b,l) \right)$$

$$= \max_X \min_Y \left( \sum_{k,l} p(k) q(l) \sum_{a,b} X(a,k) G_{k,l}(a,b) Y(b,l) \right)$$

$$= \alpha v_{n,\alpha}(p, q)$$

Suppose, for $n$, $v_{n,\alpha}(\alpha p, q) = \alpha v_{n,\alpha}(p, q)$. For $n + 1$,

$$v_{n+1,\alpha}(\alpha p, q) = \max_X \min_Y \left( \sum_{k,l} \alpha p(k) q(l) \sum_{a,b} X(a,k) G_{k,l}(a,b) Y(b,l) + \sum_{a,b} \sum_k \alpha p(k) X(a,k) Y(b,l) \right)$$

$$= \max_X \min_Y \left( \sum_{k,l} p(k) q(l) \sum_{a,b} X(a,k) G_{k,l}(a,b) Y(b,l) + \sum_{a,b} \sum_k p(k) X(a,k) Y(b,l) \right)$$

$$= \alpha v_{n+1,\alpha}(p, q)$$

So, the proof is complete. □

Lemma 6 $F(\Pi, Y)$ is concave in $\Pi$, where $F(\Pi, Y)$ is defined as equation (48).

Proof According to Lemma 5, $v_{n-1,\alpha}(\alpha p, q) = \alpha v_{n-1,\alpha}(p, q)$. Then we can write,

$$F(\Pi, Y) = \max_\Pi \min_Y \left[ \sum_{k,a} \Pi(a,k) \mu(k) + \sum_{k,l,a,b} \Pi(a,k) q(l) G_{k,l}(a,b) Y(b,l) + \lambda \sum_{a,b,k} \Pi(a,k) \eta_{a,b}(k,k') \left( \sum_k \sum_\alpha p(a,k) \eta_{a,b}(k,k') \Pi(a,k) \right) \right]$$

$$= \max_\Pi \min_Y \left[ \sum_{k,a} \Pi(a,k) \mu(k) + \sum_{k,l,a,b} \Pi(a,k) q(l) G_{k,l}(a,b) Y(b,l) + \lambda \sum_{a,b,k} \Pi(a,k) \eta_{a,b}(k,k') \Pi(a,k) \right]$$

$$= \max_\Pi \min_Y \left[ \sum_{k,a} \Pi(a,k) \mu(k) + \sum_{k,l,a,b} \Pi(a,k) q(l) G_{k,l}(a,b) Y(b,l) + \lambda \sum_{a,b,k} \Pi(a,k) \eta_{a,b}(k,k') \Pi(a,k) \right]$$

From this equation we can see that $F(\Pi, Y)$ is linear in $\Pi$. So $F(\Pi, Y)$ is both concave and convex in $\Pi$. The proof is complete. □
7.1 Proof of Lemma 1

Proof Let, initial probability, \( p = \alpha_w p_w \) where \( W \in \{1, 2\} \). Here, \( p, p_w \in \Delta K \), \( \alpha_w \in \Delta W \) and \( \sigma^w \) is optimal for player 1 in game \( \Gamma_n,\lambda(p_w, q) \). In initial state \( k \), player 1 uses \( \sigma^w(k, \cdot) \) with probability \( \alpha_w \frac{p_w(k)}{p(k)} \). Then the payoff is,

\[
\gamma_{n,\lambda}(\delta_k, \delta_l, \sigma(k), \tau(l)) = E_{\sigma(k), \tau(l)} \left( \sum_{t=1}^{n} \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) | k, l \right) \\
= E_w \left[ E_{\sigma^w(k), \tau(l)} \left( \sum_{t=1}^{n} \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) | k, l, w \right) \right] \\
= \sum_w Pr(w | k, l) \left[ E_{\sigma^w(k), \tau(l)} \left( \sum_{t=1}^{n} \lambda^{t-1} G_{k_t, l_t}(a_t, b_t) | k, l, w \right) \right] \\
= \sum_w Pr(w | k) \gamma_{n,\lambda}(\delta_k, \delta_l, \sigma^w(k), \tau_l) \\
= \sum_w \alpha_w \frac{p_w(k)}{p(k)} \gamma_{n,\lambda}(\delta_k, \delta_l, \sigma^w(k), \tau_l) \\
= \sum_{k,l} p(k)q(l) \gamma_{n,\lambda}(\delta_k, \delta_l, \sigma(k), \tau(l)) \\
= \sum_{k,l} p(k)q(l) \sum_w \alpha_w \frac{p_w(k)}{p(k)} \gamma_{n,\lambda}(\delta_k, \delta_l, \sigma^w(k), \tau_l) \\
= \sum_{k,l} q(l) \sum_w \alpha_w p_w(k) \gamma_{n,\lambda}(\delta_k, \delta_l, \sigma^w(k), \tau_l) \\
= \sum_w \alpha_w \gamma_{n,\lambda}(p_w, q, \sigma^w, \tau) \\
\]

This is true for any \( \tau \). Let, \( \tau^* \) is the optimal strategy in \( \Gamma_n,\lambda(p, q) \). So,

\[
\sum_w \alpha_w \gamma_{n,\lambda}(p_w, q, \sigma^w, \tau^*) \geq \sum_w \alpha_w v_{n,\lambda}(p_w, q) \\
\Rightarrow \gamma_{n,\lambda}(p, q, \sigma, \tau^*) \geq \sum_w \alpha_w v_{n,\lambda}(p_w, q) \\
\Rightarrow v_{n,\lambda}(p, q) \geq \sum_w \alpha_w v_{n,\lambda}(p_w, q); \quad [As \; v_{n,\lambda}(p, q) \geq \gamma_{n,\lambda}(p, q, \sigma, \tau^*)] 
\]

From this we can say that \( v_{n,\lambda}(p, q) \) is concave on \( p \). By following the similar method me can prove that \( v_{n,\lambda}(p, q) \) is convex on \( q \). \( \square \)
7.2 Proof of Lemma [2]

Proof The payoff of the primal game can be written as,

\[
\gamma_{n,\lambda}(p, q, \sigma, \tau) = \tilde{G}(p, q, X, Y) + E_{p, q, \sigma, \tau} \left( \sum_{t'=2}^{n} \lambda^{t'-1} G_{k, l, i, r}(a_{t'}, b_{t'}) \right)
\]

\[
E_{p, q, \sigma, \tau} \left( \sum_{t'=2}^{n} \lambda^{t'-1} G_{k, l, i, r}(a_{t'}, b_{t'}) \right)
\]

\[
= E \left[ \sum_{k_2, l_2, a_1, b_1} E \left( \sum_{t'=2}^{n} \lambda^{t'-1} G_{k, l, i, r}(a_{t'}, b_{t'}) | k_2, l_2, a_1, b_1 \right) Pr(k_2, l_2, a_1, b_1) \right]
\]

\[
= \sum_{k_2, l_2, a_1, b_1} E \left( \sum_{t'=2}^{n} \lambda^{t'-1} G_{k, l, i, r}(a_{t'}, b_{t'}) | k_2, l_2, a_1, b_1 \right) \sum_{k_1, l_1} Pr(k_2, l_2, a_1, b_1, k_1, l_1) \sum_{k_1, l_1} Pr(k_1, l_1)
\]

\[
= \sum_{k_2, l_2, a_1, b_1} E \left( \sum_{t'=2}^{n} \lambda^{t'-1} G_{k, l, i, r}(a_{t'}, b_{t'}) | k_2, l_2, a_1, b_1 \right) \sum_{k_1, l_1} Pr(k_2, l_2, a_1, b_1, k_1, l_1) \sum_{k_1, l_1} Pr(k_1, l_1) \sum_{l_1} Pr(l_2 | a_1, b_1, l_1) Pr(b_1, l_1)
\]

\[
= \sum_{k_2, l_2, a_1, b_1} E \left( \sum_{t'=2}^{n} \lambda^{t'-1} G_{k, l, i, r}(a_{t'}, b_{t'}) | k_2, l_2, a_1, b_1 \right) \sum_{k_1, l_1} Pr(k_2, l_2, a_1, b_1, k_1, l_1) \sum_{k_1, l_1} Pr(k_1, l_1) \sum_{l_1} Pr(l_2 | a_1, b_1, l_1) Pr(b_1, l_1) \sum_{l_1} Pr(b_1, l_1)
\]

\[
= \sum_{a_1, b_1, k_2, l_2, a_1, b_1} \sum_{k_1, l_1} Pr(k_2 | a_1, b_1, k_1, l_1) \sum_{k_1, l_1} Pr(l_2 | a_1, b_1, l_1) \sum_{k_1, l_1} Pr(b_1, l_1) \sum_{l_1} Pr(l_2 | a_1, b_1, l_1) Pr(b_1, l_1)
\]

\[
E \left( \sum_{t=1}^{n-1} \lambda^{t} G_{k, l, i, r}(a_{t}, b_{t}) | k_2, l_2, a_1, b_1 \right) \sum_{k_1, l_1} Pr(a_1, k_1) \sum_{l_1} Pr(l_1) \sum_{l_1} Pr(b_1, l_1) \sum_{l_1} Pr(l_1)
\]

\[
= \lambda \sum_{a_1, b_1} \gamma_{n-1, \lambda}(p^+, q^+, \sigma, \tau) \sum_{k_1, l_1} Pr(a_1 | k_1) \sum_{l_1} Pr(l_1) \sum_{l_1} Pr(b_1, l_1)
\]

\[
= \lambda \sum_{a_1, b_1} \left[ \sum_{k_1} Pr(a_1 | k_1) Pr(k_1) \right] \left[ \sum_{l_1} Pr(b_1 | l_1) Pr(l_1) \right] \gamma_{n-1, \lambda}(p^+, q^+, \sigma, \tau)
\]

So we can write,
\[ \gamma_{n,\lambda}(p, q, \sigma, \tau) = \bar{G}(p, q, X, Y) + \lambda \sum_{a_1, b_1} x_{p, X}(a_1) \bar{y}_{q, Y}(b_1) \gamma_{n-1,\lambda}(p^{+}, q^{+}, \sigma, \tau) \]

Let us define an auxiliary zero-sum game \( \hat{f}_{n,\lambda}(p, q) \) where the payoff function for player 1 is defined by,

\[ f_{n,\lambda}^{p,q} = \bar{G}(p, q, X, Y) + \lambda \sum_{a \in A, b \in B} \bar{x}_{p, X}(a) \bar{y}_{q, Y}(b) v_{n-1,\lambda}(p^{+}, q^{+}) \]

By following the similar proof as [13] we can say, \( f_{n,\lambda}^{p,q} \) is concave in \( X \) and convex in \( Y \). Given \( X \), \( \bar{G}(p, q, X, Y) \) is linear w.r.t. \( Y \) and given \( Y \), it is linear w.r.t. \( X \).

Let, \( X = \Theta X^\prime + (1 - \Theta) X'' \). As \( v_{n-1,\lambda} \) is concave in \( p \),

\[ \bar{y}_q Y(b)v_{n-1,\lambda}(p^{+}, q^{+}) \geq \Theta v_{n-1,\lambda}(\bar{x}_{p, X'}(a)\bar{y}_{q, Y}(b)p^{+}_{X'}, q^{+}) + (1 - \Theta) \]

As \( v_{n-1,\lambda}(\alpha p, q) = \alpha v_{n-1,\lambda}(p, q) \), we can write the above equation as,

\[ \sum_{a, b} \bar{x}_{p, X}(a) \bar{y}_{q, Y}(b)v_{n-1,\lambda}(p^{+}, q^{+}) \geq \Theta \sum_{a, b} \bar{x}_{p, X'}(a) \bar{y}_{q, Y}(b)v_{n-1,\lambda}(p^{+}_{X'}, q^{+}) + (1 - \Theta) \]

\[ \sum_{a, b} \bar{x}_{p, X''}(a) \bar{y}_{q, Y}(b)v_{n-1,\lambda}(p^{+}, q^{+}) \]

\[ \Rightarrow \bar{G}(p, q, X, Y) + \sum_{a, b} \bar{x}_{p, X}(a) \bar{y}_{q, Y}(b)v_{n-1,\lambda}(p^{+}, q^{+}) \geq \Theta \left( \bar{G}(p, q, X', Y) + \sum_{a, b} \bar{x}_{p, X'}(a) \bar{y}_{q, Y}(b)v_{n-1,\lambda}(p^{+}_{X'}, q^{+}) \right) \]

\[ \Rightarrow f_{n,\lambda}^{p,q}(X, Y) \geq \Theta f_{n,\lambda}^{p,q}(X', Y) + (1 - \Theta) f_{n,\lambda}^{p,q}(X'', Y) \]

From this we can say \( f_{n,\lambda}^{p,q}(X, Y) \) is concave in \( X \). By using the similar steps we can prove \( f_{n,\lambda}^{p,q}(X, Y) \) is convex in \( Y \). So, according to [17], \( \bar{f}_{n,\lambda}(p, q) \) has a value. Let, the value is \( f_{n,\lambda}(p, q) \). In \( \hat{f}_{n,\lambda}(p, q) \) at stage 1, \( X^* \) and \( Y^* \) is optimal for player 1 and 2 respectively. We will now prove that player 1 can guarantee \( f_{n,\lambda}(p, q) \) in \( \hat{f}_{n,\lambda}(p, q) \). Let, \( \sigma \in \Sigma \) be as follows: At stage 1, play \( X^* \). From stage 2 to \( n \), play the optimal strategy \( \sigma_n \) of stage 2 to \( n \) in \( \hat{f}_{n,\lambda}(p, q) \). Let, \( \tau \in T \) and denote by \( Y \) in \( \Delta(B) \) the mixed action played by \( \tau \) at stage 1 and for each \( (a, b) \in A \times B \), by \( \tau_{a, b} \) the strategy played by player 2 at stage 2 to \( n \) if \( (a, b) \) is played at stage 1. Then

\[ \gamma_{n,\lambda}(p, q, \sigma, \tau) = \bar{G}(p, q, X^*, Y) + \lambda \sum_{a_1, b_1} x_{p, X'}(a_1) \bar{y}_{q, Y}(b_1) \gamma_{n-1,\lambda}(p^{+}_{X'}, q^{+}, \sigma, \tau) \quad (40) \]

\[ f_{n,\lambda}^{p,q}(X^*, Y) = \bar{G}(p, q, X^*, Y) + \lambda \sum_{a, b} x_{p, X'}(a) \bar{y}_{q, Y}(b)v_{n-1,\lambda}(p^{+}_{X'}, q^{+}) \quad (41) \]
For 1 stage game,

\[ \gamma_{n-1, \lambda}(p_X^+, q^+, \sigma_a, \tau) \geq v_{n-1, \lambda}(p_X^+, q^+) \quad (42) \]

From equation (40), (41), (42) we can write, \( \gamma_{n, \lambda}(p, q, \sigma, \tau) \geq f_{p,q}^{n, \lambda}(X^*, Y) \) and we know \( f_{p,q}^{n, \lambda}(X^*, Y) \geq f_{n, \lambda}(p, q) \).

\[ \therefore \gamma_{n, \lambda}(p, q, \sigma, \tau) \geq f_{p,q}^{n, \lambda}(X^*, Y) \geq f_{n, \lambda}(p, q) \quad (43) \]

\[ \therefore v_n(p, q) \geq f_n(p, q); \quad \text{[This is true for any } \tau] \]

By using the similar method we can prove \( v_n(p, q) \leq f_n(p, q) \). That means, \( v_n(p, q) = f_n(p, q) \).

7.3 Proof of Proposition [1]

Proof For type 1: Let, \( \bar{\sigma}^* \) and \( \bar{\tau}^* \) be the optimal strategy of player 1 and 2 respectively in type 1 dual game.

\[ \bar{\gamma}_N, \lambda(\mu, q, p, \sigma, \bar{\tau}^*) \leq \bar{\gamma}_N, \lambda(\mu, q, \bar{\sigma}^*, \bar{\tau}^*) \]

\[ \Rightarrow \sum_{k_1} \mu(k_1) Pr(k_1) + \gamma_N, \lambda(p, q, \sigma, \bar{\tau}^*) \leq w^{1}_{N, \lambda}(\mu, q) \]

\[ \Rightarrow \sum_{k_1} \mu(k_1) Pr(k_1) + \gamma_N, \lambda(p, q, \sigma, \bar{\tau}^*) \leq w^{1}_{N, \lambda}(\mu, q) \]

\[ \Rightarrow p^T \mu + \gamma_N, \lambda(p, q, \sigma, \bar{\tau}^*) \leq w^{1}_{N, \lambda}(\mu, q) \]

\[ \Rightarrow \gamma_N, \lambda(p, q, \sigma^*, \bar{\tau}^*) \leq w^{1}_{N, \lambda}(\mu, q) - p^T \mu \quad (43) \]

\[ \Rightarrow v_N, \lambda(p, q) \leq w^{1}_{N, \lambda}(\mu, q) - p^T \mu; \quad \text{[for any } p, q \text{ and } \mu] \]

\[ v_N, \lambda(p, q) \leq w^{1}_{N, \lambda}(\mu, q) - p^T \mu \quad (44) \]
Let, $\tau^*$ be the optimal strategy of player 2 in primal game $\Gamma_{N,\lambda}(p, q)$. Let us define,

$$\beta(\tau^*) = \max_{\sigma_k} E_{\sigma, \tau^*, q}(\sum_{t=1}^{N} \lambda^{t-1} G_{k_t, i_t}(a_t, b_t)|k_1 = k)$$

$$p^T \beta(\tau^*) = \sum_{k_1} Pr(k_1) \max_{\sigma_k} E_{\sigma, \tau^*, q}(\sum_{t=1}^{N} \lambda^{t-1} G_{k_t, i_t}(a_t, b_t)|k_1 = k)$$

$$= \max_{\sigma} \sum_{k_1} Pr(k_1) E_{\sigma, \tau^*, q}(\sum_{t=1}^{N} \lambda^{t-1} G_{k_t, i_t}(a_t, b_t)|k_1 = k)$$

$$= \max_{\sigma} E_p \left[ E_{\sigma, \tau^*, q} \left( \sum_{t=1}^{N} \lambda^{t-1} G_{k_t, i_t}(a_t, b_t) \right) \right]$$

$$= \max_{\sigma} E_{p, \sigma, \tau^*, q} \left( \sum_{t=1}^{N} \lambda^{t-1} G_{k_t, i_t}(a_t, b_t) \right)$$

$$= \nu_{N,\lambda}(p, q)$$

In dual game $\hat{\Gamma}^1(-\beta(\tau^*), q)$,

$$\hat{\gamma}_{N,\lambda}(p, q, \sigma, \tau^*) = E_{\sigma} \left( -\beta(\tau^*)(k_1) + E_{\sigma, \tau^*, p, q}(\sum_{t=1}^{N} \lambda^{t-1} G_{k_t, i_t}(a_t, b_t)) \right)$$

$$= \sum_{k_1} -\beta(\tau^*)(k_1) Pr(k_1) + \gamma_{N,\lambda}(p, q, \sigma, \tau^*)$$

$$= p^T [-\beta_k(\tau^*)] + \gamma_{N,\lambda}(p, q, \sigma, \tau^*)$$

$$= -\nu_{N,\lambda}(p, q) + \gamma_{N,\lambda}(p, q, \sigma, \tau^*)$$

$$\gamma_{N,\lambda}(p, q, \sigma^*, \tau^*) \geq \gamma_{N,\lambda}(p, q, \sigma, \tau^*)$$

$$\Rightarrow \nu_{N,\lambda}(p, q) \geq \gamma_{N,\lambda}(p, q, \sigma, \tau^*)$$

So, we can say,

$$\hat{\gamma}_{N,\lambda}(p, q, \sigma, \tau^*) \leq 0; \text{[For any } \sigma, p, q]$$

$$\hat{\gamma}_{N,\lambda}(p, q, \sigma^*, \tau^*) \leq 0$$

$$\Rightarrow w_{N,\lambda}(-\beta(\tau^*), q) \leq 0$$

$$\Rightarrow w_{N,\lambda}(-\beta(\tau^*), q) \leq \nu_{N,\lambda}(p, q) + p^T (-\beta(\tau^*))$$

$$\Rightarrow \nu_{N,\lambda}(p, q) \geq w_{N,\lambda}(-\beta(\tau^*), q) - [p^T (-\beta(\tau^*))]$$

$$\Rightarrow \nu_{N,\lambda}(p, q) \geq w_{N,\lambda}(\mu, q) - [p^T \mu]$$

So we can say for some specific $\mu = -\beta(\tau^*)$ in type 1 dual game,

$$\nu_{N,\lambda}(p, q) = w_{N,\lambda}(\mu, q) - p^T \mu$$

(46)

In $\hat{\Gamma}^1_{N,\lambda}(\mu, q)$ the optimal strategy of player 2 is $\hat{\tau}^*$. So from equation (43) we get,

$$\gamma_{n,\lambda}(p, q, \sigma^*, \hat{\tau}^*) \leq w_{n,\lambda}(\mu, q) - p^T \mu; \ \forall \mu$$

$$\gamma_{n,\lambda}(p, q, \sigma^*, \hat{\tau}^*) \geq \nu_{n,\lambda}(p, q)$$
If $\mu = -\beta(\tau^*)$ we can write equation (47) as,

$$\gamma_{n,\lambda}(p, q, \sigma^*, \tilde{\tau}^*) \leq v_{n,\lambda}(p, q)$$

$$\therefore \gamma_{n,\lambda}(p, q, \sigma^*, \tilde{\tau}^*) = v_{n,\lambda}(p, q)$$

$\therefore \tilde{\tau}^*$ is also the optimal in $\Gamma_{n,\lambda}(p, q)$. Similarly, we can prove the equation for type 2 dual game. According to equation (45) $p^T \beta(\tau^*) = v_{N,\lambda}(p, q)$. Then for $\mu = -\beta(\tau^*)$ we can write equation (46) as,

$$v_{n,\lambda}(p, q) = w_{n,\lambda}^1(\mu, q) - p^T \mu$$

$$\Rightarrow v_{n,\lambda}(p, q) = w_{n,\lambda}^1(\mu, q) + v_{n,\lambda}(p, q)$$

$$\Rightarrow w_{n,\lambda}^1(\mu^*, q) = 0$$

Similarly, we can prove $w_{n,\lambda}^2(p, \nu^*) = 0$.

7.4 Proof of Proposition 3

Proof According to equation (49),

$$w_{n,\lambda}^1(\mu, q) = \max_p p^T \mu + v_{n,\lambda}(p, q)$$

$$= \max_p p^T \mu + \min_X \left[ \tilde{G}(p, q, X, Y) + \lambda \sum_{a,b} \tilde{x}_{p,X}(a)\tilde{y}_{q,Y}(b)v_{n-1,\lambda}(p^+, q^+) \right]$$

$$= \max_p p^T \mu + \min_X \left[ \sum_{k,l} p(k)q(l) \sum_{a,b} X(a,k)G_{k,l}(a,b)Y(b,l) + \lambda \sum_{a,b} \sum_k p(k)X(a,k) \right]$$

$$= \max_p \max X \min Y \left[ \sum_k P_{r,k}(k) + \sum_{k,l,a,b} Pr(k)q(l)Pr(a|k)G_{k,l}(a,b)Y(b,l) \right]$$

$$= \max_p \max \min \left[ \sum_k II(a,k)G_{k,l}(a,b)Y(b,l) + \sum_{a,b,k} Pr(k)Pr(a|k) \tilde{y}_{q,Y}(b)v_{n-1,\lambda}(\sum_k P_{a,b,k}(k)\tilde{y}_{q,Y}(b)) \right]$$

$$= \max \min \left[ \sum_k II(a,k) \mu(k) + \sum_{k,l,a,b} II(a,k)q(l)G_{k,l}(a,b)Y(b,l) + \sum_{a,b,k} II(a,k) \tilde{y}_{q,Y}(b)v_{n-1,\lambda}(\sum_k P_{a,b,k}(k)\tilde{y}_{q,Y}(b)) \right]$$

$$= \max \min H \left[ \sum_k II(a,k) \mu(k) + \sum_{k,l,a,b} II(a,k)q(l)G_{k,l}(a,b)Y(b,l) + \sum_{a,b,k} II(a,k) \tilde{y}_{q,Y}(b)v_{n-1,\lambda}(\sum_k P_{a,b,k}(k)\tilde{y}_{q,Y}(b)) \right]$$

$$= \max \min F(H, Y)$$

(47)
According to Lemma 6 we can write,
\[
F(\Pi, Y) = \max_{\Pi} \min_{Y} \left[ \sum_{k,a} \Pi(a, k) \mu(k) + \sum_{k,l,a,b} \Pi(a, k) q(l) G_{k,l}(a, b) Y(b, l) + \lambda \sum_{a,b,k} \Pi(a, k) \bar{y}_{q,Y}(b) v_{n-1,\lambda} \left( \sum_{k} P_{a,b}(k, k') \frac{\Pi(a, k)}{\sum_{k} \Pi(a, k')} q^+ \right) \right]
\]
\[
= \max_{\Pi} \min_{Y} \left[ \sum_{k,a} \Pi(a, k) \mu(k) + \sum_{k,l,a,b} \Pi(a, k) q(l) G_{k,l}(a, b) Y(b, l) + \lambda \sum_{a,b,k} \Pi(a, k) \bar{y}_{q,Y}(b) v_{n-1,\lambda} \left( \sum_{k} P_{a,b}(k, k') \Pi(a, k), q^+ \right) \right]
\]
\[
= \max_{\Pi} \min_{Y} \left[ \sum_{k,a} \Pi(a, k) \mu(k) + \sum_{k,l,a,b} \Pi(a, k) q(l) G_{k,l}(a, b) Y(b, l) + \lambda \sum_{a,b,k} \Pi(a, k) \bar{y}_{q,Y}(b) v_{n-1,\lambda} \left( \sum_{k} P_{a,b}(k, k'), q^+ \right) \right]
\]
\[
F(\Pi, Y) \text{ is linear in } Y \text{ and concave in } \Pi. \text{ According to Sion’s theorem } [17],
\]
\[
w_{n,\lambda}^1(\mu, q) = \max_{Y} \min_{\Pi} F(\Pi, Y) = \min_{\Pi} \max_{Y} F(\Pi, Y)
\]
\[
w_{n,\lambda}^1(\mu, q)
\]
\[
= \min_{Y} \max_{\Pi} \sum_{k,a} \Pi(a, k) \left[ \mu(k) + \sum_{l,b} q(l) G_{k,l}(a, b) Y(b, l) + \lambda \sum_{b} \bar{y}_{q,Y}(b) v_{n-1,\lambda} (p^+ + q^+) \right]
\]
\[
= \min_{Y} \max_{\Pi} \sum_{k,a} \Pi(a, k) \left[ \mu(k) + \sum_{l,b} q(l) G_{k,l}(a, b) Y(b, l) + \lambda \sum_{b} \bar{y}_{q,Y}(b) \min_{\beta} \left( w_{n-1,\lambda}^1(\beta, q) - p^+ \beta \right) \right]
\]
\[
= \min_{Y} \max_{\Pi} \beta \left[ \sum_{k,a} \Pi(a, k) \mu(k) + \sum_{k,a} \Pi(a, k) \sum_{l,b} q(l) G_{k,l}(a, b) Y(b, l) + \lambda \sum_{k,a} \Pi(a, k) \sum_{b} \bar{y}_{q,Y}(b) \sum_{k,k'} P_{a,b}(k, k') \frac{p(k)X(a, k)}{\sum_{k} p(k)X(a, k)} \beta(k') \right]
\]
\[
= \min_{Y} \max_{\Pi} \beta \left[ \sum_{k,a} \Pi(a, k) \mu(k) + \sum_{k,a} \Pi(a, k) \sum_{l,b} q(l) G_{k,l}(a, b) Y(b, l) + \lambda \sum_{k,a} \Pi(a, k) \sum_{b} \bar{y}_{q,Y}(b) \sum_{k,k'} P_{a,b}(k, k') \frac{\Pi(a, k)}{\sum_{k} \Pi(a, k)} \beta(k') \right]
\]
\[
= \min_{Y} \max_{\Pi} \beta \left[ \sum_{k,a} \Pi(a, k) \left[ \mu(k) + \sum_{l,b} q(l) G_{k,l}(a, b) Y(b, l) + \lambda \sum_{b} \bar{y}_{q,Y}(b) \left( w_{n-1,\lambda}^1(\beta, q) - \sum_{a,k'} P_{a,b}(k, k') \beta(k') \right) \right] \right]
\]
\[
= \min_{Y} \max_{\beta} \Phi(\Pi, \beta)
\]
\( w_{n-1,\lambda}(\beta, q) \) is convex in \( \beta \). To prove that we need to show,

\[
w_{n-1,\lambda}(\beta, q) \leq tw_{n-1,\lambda}(\beta_1, q) + (1-t)tw_{n-1,\lambda}(\beta_2, q)
\]

where, \( \beta = t\beta_1 + (1-t)\beta_2 \) and \( t \in [0, 1] \)

**L.H.S** = \( w_{n-1,\lambda}(\beta, q) \)

\[
= w_{n-1,\lambda}(t\beta_1 + (1-t)\beta_2, q) \\
= \max_p p^T (t\beta_1 + (1-t)\beta_2) + v_{n-1,\lambda}(p, q) \\
= \max_p \left[ t[p^T \beta_1 + v_{n-1,\lambda}(p, q)] + (1-t)[p^T \beta_2 + v_{n-1,\lambda}(p, q)] \right]
\]

**R.H.S** = \( tw_{n-1,\lambda}(\beta_1, q) + (1-t)tw_{n-1,\lambda}(\beta_2, q) \)

\[
= \max_p t[p^T \beta_1 + v_{n-1,\lambda}(p, q)] + \max_p (1-t)[p^T \beta_2 + v_{n-1,\lambda}(p, q)]
\]

We know,

\[
\max_p \left[ t[p^T \beta_1 + v_{n-1,\lambda}(p, q)] + (1-t)[p^T \beta_2 + v_{n-1,\lambda}(p, q)] \right] \leq \max_p t[p^T \beta_1 + v_{n-1,\lambda}(p, q)] + \max_p (1-t)[p^T \beta_2 + v_{n-1,\lambda}(p, q)]
\]

\[
\therefore w_{n-1,\lambda}(\beta, q) \leq tw_{n-1,\lambda}(\beta_1, q) + (1-t)tw_{n-1,\lambda}(\beta_2, q)
\]

According to Sion’s theorem \[17\],

\[
w_{n-1,\lambda}(\mu, q) = \min_Y \min_{\beta} \max_{\Phi} \Phi(H, \beta)
\]

By using similar method we can find the recursive formula for the game value of type 2 dual game. \( \square \)

### 7.5 Proof of Corollary 2

**Proof** At stage 1 in \( N \)-stage game, player 2 knows its initial state \( q \) and initial realized vector \( \mu \). Using these information it can compute it’s optimal strategy \( \pi_1^* \) for stage 1 by solving the optimal problem in equation \[11\]. Player 2 chooses it’s stage 1 action according to the optimal strategy. If player 1 chooses some action
for stage 1 then the stage 1 payoff of this game is,

\[
E_{\mu,q,\sigma,\tau}\left(\mu(k_1) + G_{k_1,l_1}(a_1, b_1)\right)
\]

\[
= \sum_{k_1} \mu(k_1) Pr(k_1) + \sum_{k_1,l_1,a_1,b_1} G_{k_1,l_1}(a_1, b_1) Pr(k_1, l_1, a_1, b_1)
\]

\[
= \sum_{k_1} \mu(k_1) Pr(k_1) + \sum_{k_1,l_1,a_1,b_1} G_{k_1,l_1}(a_1, b_1) Pr(a_1 | k_1) Pr(b_1 | l_1) Pr(k_1) Pr(l_1)
\]

\[
= \sum_{k,a} II(a, k) \mu(k) + \sum_{k,l,a,b} G_{k,l}(a,b) II(a, k) Pr(b | l) Pr(l)
\]

\[
= \sum_{k,a} II(a, k) \left(\mu(k) + \sum_{l,b} G_{k,l}(a,b) Y(b,l)q(l)\right)
\]

After stage 1, player 2 plays the optimal strategy \(\tau^*\) for \(N-1\) stage game with initial parameter \(q^*\) and \(\beta_{a,b}^*\). When player 1 chooses some action and player 2 chooses action according to its optimal strategy then payoff of the game \(\Gamma_{N-1,\lambda}(\beta_{a,b}^*, q^*)\) is,

\[
E_{p^*,q^*,\sigma,\tau^*}\left(\beta_{a,b}^*(k') + \sum_{t=1}^{N-1} \lambda^{t-1} G_{k_1,l_1}(a_t, b_t)\right)
\]

If player 2 uses the optimal strategy to take action for stage 2 : \(N\) in the \(N\) stage game then the primal game payoff of stage 2 : \(N\) will be,

\[
E\left(E_{p^*,q^*,\sigma,\tau^*}\left(\sum_{t=2}^{N} \lambda^{t-1} G_{k_1,l_1}(a_t, b_t)\right)|a_1, b_1\right)
\]

\[
= E\left(\lambda E\left(\beta_{a,b}^*(k') + \sum_{t=1}^{N-1} \lambda^{t-1} G_{k_1,l_1}(a_t, b_t) - \beta_{a,b}^*(k')\right)|a_1, b_1\right)
\]

\[
= \lambda \sum_{a_2,b_1} E\left(\beta_{a,b}^*(k') + \sum_{t=1}^{N-1} \lambda^{t-1} G_{k_1,l_1}(a_t, b_t) - \beta_{a,b}^*(k')\right) Pr(a_1, b_1)
\]

\[
= \lambda \sum_{a_2,b_1} \left[ E\left(\beta_{a,b}^*(k') + \sum_{t=1}^{N-1} \lambda^{t-1} G_{k_1,l_1}(a_t, b_t) - E\left(\beta_{a,b}^*(k')\right)\right) Pr(a_1, b_1)\right]
\]

\[
= \lambda \sum_{a_2,b_1} Pr(a_1, b_1) \left[ w_{n-1,\lambda}(\beta_{a,b}^*, q^*) - \sum_{k'} \beta_{a,b}^*(k') Pr(k') \right]
\]

\[
= \lambda \sum_{a,b,k,l} Pr(a \cap k) Pr(b | l) Pr(l) \left[ w_{n-1,\lambda}(\beta_{a,b}^*, q^*) - \sum_{k'} \beta_{a,b}^*(k') P_{a,b}(k, k') \right]
\]

\[
= \sum_{a,k} II(a,k) \sum_{b} \bar{y}_{q,Y}(b) \left[ w_{n-1,\lambda}(\beta_{a,b}, q^*) - \sum_{k'} P_{a,b}(k, k') \beta_{a,b}(k') \right]
\]

From the maximization of the total payoff of stage 1 and 2 : \(N\) we can say, in the worst case player 2 can achieve the game value \(w_{n,\lambda}(\mu, q)\). Similarly, we can prove this for player 1. \(\Box\)
References

1. R. J. Aumann, M. Maschler, and R. E. Stearns. *Repeated games with incomplete information*, MIT press, 1995.
2. D. Charles, D. Chakraborty, M. Chickering, N. R. Devanur, and L. Wang. Budget smoothing for internet ad auctions: a game theoretic approach. In *Proceedings of the fourteenth ACM conference on Electronic commerce*, pages 163–180. ACM, 2013.
3. B. De Meyer. Repeated games, duality and the central limit theorem. *Mathematics of Operations Research*, 21(1):237–251, 1996.
4. B. De Meyer and D. Rosenberg. cav u and the dual game. *Mathematics of Operations Research*, 24(3):619–626, 1999.
5. C. Eksin, J. S. Shamma, and J. S. Weitz. Disease dynamics in a stochastic network game: a little empathy goes a long way in averting outbreaks. *Scientific reports*, 7:44122, 2017.
6. T. Feddersen and A. Sandroni. A theory of participation in elections. *American Economic Review*, 96(4):1271–1282, 2006.
7. L. Li, C. Langbort, and J. Shamma. An lp approach for solving two-player zero-sum repeated bayesian games. *IEEE Transactions on Automatic Control*, 64(9):3716–3731, Sep. 2019.
8. L. Li and J. S. Shamma. Efficient strategy computation in zero-sum asymmetric information repeated games. *IEEE Transactions on Automatic Control*, 2020.
9. J.-F. Mertens, S. Sorin, S. Zamir, et al. Repeated games, part a: Background material. Technical report, Université catholique de Louvain, Center for Operations Research and , 1994.
10. J.-F. Mertens and S. Zamir. The value of two-person zero-sum repeated games with lack of information on both sides. *International Journal of Game Theory*, 1(1):39–64, 1971.
11. R. B. Myerson. Comments on games with incomplete information played by bayesian players, iii harsanyi’s games with incomplete information. *Management Science*, 50(12 supplement):1818–1824, 2004.
12. J. Pita, M. Jain, F. Ordóñez, C. Portway, M. Tambe, C. Western, P. Paruchuri, and S. Kraus. Using game theory for los angeles airport security. *AI magazine*, 30(1):43–43, 2009.
13. J. Renault. The value of markov chain games with lack of information on one side. *Mathematics of Operations Research*, 31(3):490–512, 2006.
14. J. Renault. The value of markov chain games with lack of information on one side. *Mathematics of Operations Research*, 31(3):490 – 512, 2006.
15. D. Rosenberg. Duality and markovian strategies. *International Journal of Game Theory*, 27(4):577–597, Dec 1998.
16. D. Rosenberg, E. Solan, and N. Vieille. Stochastic games with a single controller and incomplete information. *SIAM journal on control and optimization*, 43(1):86–110, 2004.
17. M. Sion et al. On general minimax theorems. *Pacific Journal of mathematics*, 8(1):171–176, 1958.
18. S. Sorin. *A first course on zero-sum repeated games*, volume 37. Springer Science & Business Media, 2002.
19. S. Sorin. Stochastic games with incomplete information. In *Stochastic Games and applications*, pages 375–395. Springer, 2003.
20. V. Vadorn, M. Scalabrin, A. V. Gugleimi, and L. Badia. Jamming in underwater sensor networks as a bayesian zero-sum game with position uncertainty. In *2015 IEEE Global Communications Conference (GLOBECOM)*, pages 1–6, Dec 2015.
21. B. Von Stengel. Efficient computation of behavior strategies. *Games and Economic Behavior*, 14(2):220–246, 1996.
22. S. Zamir. Repeated games of incomplete information: Zero-sum. *Handbook of Game Theory with Economic Applications*, 1:109–154, 1992.