On the degree of strong approximation of almost periodic functions in the Stepanov sense

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Abstract

Considering the class of almost periodic functions in the Stepanov sense we extend and generalize the results of the first author [11] as well as the results of L. Leindler [7] and P. Chandra [4, 5].

Key words: Degree of strong approximation, Almost periodic functions, Strong approximation, Special sequences.

2000 Mathematics Subject Classification: 42A24, 41A25.

1 Introduction

Let $S^p$ ($1 < p \leq \infty$) be the class of all almost periodic functions in the Stepanov sense ($1 < p < \infty$) or uniformly almost periodic ($p = \infty$) with the norm

$$
\|f\|_{S^p} := \begin{cases} 
\left( \sup_{u} \left\{ \frac{1}{\pi} \int_{u}^{u+\pi} |f(t)|^p \, dt \right\} \right)^{1/p} & \text{when } 1 < p < \infty, \\
\sup_{u} |f(u)| & \text{when } p = \infty.
\end{cases}
$$

Suppose that the Fourier series of $f \in S^p$ has the form

$$
Sf(x) = \sum_{\nu=-\infty}^{\infty} A_{\nu}(f) e^{i\lambda_{\nu}x}, \quad \text{where} \quad A_{\nu}(f) = \lim_{L \to \infty} \frac{1}{L} \int_0^L f(t) e^{-i\lambda_{\nu}t} \, dt,
$$

with the partial sums

$$
S_{\gamma_k}f(x) = \sum_{|\lambda_{\nu}| \leq \gamma_k} A_{\nu}(f) e^{i\lambda_{\nu}x}
$$
and that $0 = \lambda_0 < \lambda_\nu < \lambda_{\nu+1}$ if $\nu \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $\lim_{\nu \to \infty} \lambda_\nu = \infty$, $\lambda_{-\nu} = -\lambda_\nu$, $|A_\nu| + |A_{-\nu}| > 0$. Let $\Omega_{\alpha,p}$, with some fixed positive $\alpha$, be the set of functions of class $S^p$ whose Fourier exponents satisfy the condition

$$\lambda_{\nu+1} - \lambda_\nu \geq \alpha \quad (\nu \in \mathbb{N}).$$

In case $f \in \Omega_{\alpha,p}$,

$$S_{\lambda_\nu} f(x) = \int_0^\infty \{f(x + t) + f(x - t)\} \Psi_{\lambda_{\nu},\lambda_{\nu+\alpha}}(t) \, dt,$$

where

$$\Psi_{\lambda,\eta}(t) = \frac{2 \sin \frac{(n-\lambda)t}{2} \sin \frac{(n+\lambda)t}{2}}{\pi (\eta - \lambda) t^2} \quad (0 < \lambda < \eta, \ |t| > 0).$$

Let $A := (a_{nk}) \ (k, n = 0, 1, \ldots)$ be a lower triangular infinite matrix of real numbers satisfying the following condition:

$$a_{nk} \geq 0 \quad (k, n = 0, 1, \ldots), \ a_{nk} = 0 \quad (k > n) \quad \text{and} \quad \sum_{k=0}^{n} a_{nk} = 1. \quad (1)$$

Let us consider the strong mean

$$H_{\alpha,\gamma}^q f(x) = \left\{ \sum_{k=0}^{n} a_{n,k} |S_{\gamma_k} f(x) - f(x)|^q \right\}^{1/q} \quad (q > 0). \quad (2)$$

As measures of approximation by the quantity (2), we use the best approximation of $f$ by entire functions $g_\sigma$ of exponential type $\sigma$ bounded on the real axis, shortly $g_\sigma \in B_\sigma$ and the moduli of continuity of $f$ defined by the formulas

$$E_{\sigma}(f)_{S^p} = \inf_{g_\sigma} \|f - g_\sigma\|_{S^p},$$

$$\omega f(\delta)_{S^p} = \sup_{|t| \leq \delta} \|f(\cdot + t) - f(\cdot)\|_{S^p}$$

and

$$w_\sigma f(\delta)_p := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_\sigma(t)|^p \, dt \right\}^{1/p} \quad \text{with} \ 1 < p < \infty,$$

where $\varphi_\sigma(t) := f(x + t) + f(x - t) - 2f(x)$, respectively.

A sequence $c := (c_n)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c \in RBVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_n - c_{n+1}| \leq K(c) c_m \quad (3)$$

for all natural numbers $m$, where $K(c)$ is a constant depending only on $c$. 

2
A sequence \( c := (c_n) \) of nonnegative numbers will be called the Head Bounded Variation Sequence, or briefly \( c \in HBVS \), if it has the property

\[
\sum_{k=0}^{m-1} |c_n - c_{n+1}| \leq K(c) c_m \tag{4}
\]

for all natural numbers \( m \), or only for all \( m \leq N \) if the sequence \( c \) has only finite nonzero terms and the last nonzero terms is \( c_N \).

Therefore we assume that the sequence \( (K(\alpha_n))_{n=0}^{\infty} \) is bounded, that is, that there exists a constant \( K \) such that

\[
0 \leq K(\alpha_n) \leq K
\]

holds for all \( n \), where \( K(\alpha_n) \) denote the sequence of constants appearing in the inequalities \( \text{(4)} \) or \( \text{(4)} \) for the sequence \( \alpha_n := (a_{nk})_{k=0}^{\infty} \). Now we can give the conditions to be used later on. We assume that for all \( n \) and \( 0 \leq m \leq n \)

\[
\sum_{k=m}^{\infty} |a_{nk} - a_{nk+1}| \leq K a_{nm} \tag{5}
\]

and

\[
\sum_{k=0}^{m-1} |a_{nk} - a_{nk+1}| \leq K a_{nm} \tag{6}
\]

hold if \( \alpha_n := (a_{nk})_{k=0}^{\infty} \) belongs to \( RBVS \) or \( HBVS \), respectively.

The \( C \)-norm of the deviation \( \left| \sum_{k=0}^{n} a_{n,k} [S_kf(x) - f(x)] \right| \), with the partial sums \( S_kf \) of classical trigonometric Fourier series, was estimated by P. Chandra \[4\] \[5\] for monotonic sequences \( (a_{nk}) \) and by L. Leindler \[7\] for the sequences of bounded variation. These results were generalized by W. Lenski \[11\] who considered the strong means \( H^q_{n,A} \), also in classical case, and the functions belonging to the \( L^p \). In present paper we shall considered the almost periodic functions from the Stepanov class giving similarly estimations for the strong means \( H^q_{n,A} \) in individual points and in norms.

We shall write \( I_1 \ll I_2 \) if there exists a positive constant \( C \) such that \( I_1 \leq CI_2 \).

2 Main results

Let us consider a function \( w_x \) of modulus of continuity type on the interval \([0, +\infty)\), i.e. a nondecreasing continuous function having the following properties: \( w_x(0) = 0 \), \( w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2) \) for any \( \delta_1, \delta_2 \geq 0 \) with \( x \) such that the set

\[
\Omega_{\alpha,p}(w_x) = \left\{ f \in \Omega_{\alpha,p} : \left[ \frac{1}{\delta} \int_0^\delta |\varphi_x(t) - \varphi_x(t \pm \gamma)|^p \ dt \right]^{1/p} \ll w_x(\gamma) \right. \\
\left. \text{and } w_x f(\delta)_p \ll w_x(\delta), \text{ where } \gamma, \delta > 0 \right\}
\]
is nonempty. It is clear that $\Omega_{\alpha,p}(w_x) \subseteq \Omega_{\alpha,p'}(w_x)$, for $p' \leq p < \infty$.

Our main results are the following:

**Theorem 1** Let (1) and (6) hold. Suppose $w_x$ is such that

$$\left\{ u^\frac{p}{q} \int_0^\infty \frac{(w_x(t))^p}{t^{1+\frac{p}{q}}} \, dt \right\}^{\frac{1}{p}} = O(uH_x(u)) \quad \text{as} \quad u \to 0^+, \quad (7)$$

where $H_x(u) \geq 0$, $1 < p \leq q$ and

$$\int_0^t H_x(u) \, du = O(tH_x(t)) \quad \text{as} \quad t \to 0^+. \quad (8)$$

If $f \in \Omega_{\alpha,p}(w_x)$, then

$$H_{n,A,\gamma}^{q,q}f(x) = O \left( a_{nn}H_x(a_{nn}) + \left\{ \sum_{k=0}^{n} a_{n,k} \left( E_{ak/2}^{f} (S_{Sf})^{q} \right)^{1/q} \right\} \right), \quad (9)$$

where $q$ is such that $1 < q (q-1)^{-1} \leq p \leq q$.

**Theorem 2** Let (1), (5), (7) and (8) hold. If $f \in \Omega_{\alpha,p}(w_x)$, then

$$H_{n,A,\gamma}^{q,q}f(x) = O \left( a_{nn}H_x(a_{nn}) + \left\{ \sum_{k=0}^{n} a_{n,k} \left( E_{ak/2}^{f} (S_{Sf})^{q} \right)^{1/q} \right\} \right), \quad (10)$$

where $q$ is such that $1 < q (q-1)^{-1} \leq p \leq q$.

Consequently, we can immediately derive the results on norm approximation.

**Theorem 3** Let (1) and (6) hold. Suppose $\omega f(\cdot)_{S^{\bar{p}}}$ is such that

$$\left\{ u^\frac{p}{q} \int_0^\infty \frac{\omega f(t)_{S^{\bar{p}}}^p}{t^{1+\frac{p}{q}}} \, dt \right\}^{\frac{1}{p}} = O(uH(u)) \quad \text{as} \quad u \to 0^+ \quad (11)$$

holds, with $1 < p \leq q \leq \bar{p}$, where additionally $H \geq 0$ instead of $H_x$ satisfies the condition (5). If $f \in \Omega_{\alpha,\bar{p}}$, then

$$\| H_{n,A,\gamma}^{q,q}f(\cdot) \|_{S^{\bar{p}}} = O(a_{nn}H_x(a_{nn})), \quad (q', q'] = 1 < q (q-1)^{-1} \leq p \leq q.$$

\[4\]
Theorem 4 Let (1) and (5) hold. Suppose \( \omega f(\cdot) \) \( \tilde{S}_p \) is such that (11) holds, with \( 1 < p \leq q \leq \tilde{p} \), where additionally \( H(\geq 0) \) instead of \( H_x \) satisfies the condition (8). If \( f \in \Omega_{\alpha, \tilde{p}} \), then\[
abla H^q_{n,A,\gamma} f(\cdot) \nabla = O(a_n H_x(a_n)),
\]
with \( q' \in (0, q] \), where \( q \) is such that \( 1 < q(q-1)^{-1} \leq p \leq q \).

Remark 1 Analyzing our proofs and dividing the integral in the formula\[
\left\{ \sum_{k=0}^{n} a_{n,k} \left| \int_{0}^{\infty} \varphi_x(t) \Psi_{k+N}(t) \, dt \right|^{q/2} \right\}^{1/q}
\]
into parts with \( \frac{1}{n+1} \) instead of \( a_{n,n} \) or \( a_{n,0} \) we can obtain the next series of theorems analogously as in [11].

3 Lemmas

To prove our theorems we need the following lemmas.

Lemma 1 [11] If (7) and (5) hold, then\[
\int_{0}^{u} \frac{w_x f(t)}{t} \, dt = O(u H_x(u)) \quad (u \to 0_+).
\]

Lemma 2 [16, Theorem 5.20 II, Ch. XII] Suppose that \( 1 < q(q-1)^{-1} \leq p \leq q \) and \( \xi = \frac{1}{p} + \frac{1}{q} - 1 \). If \( |x^{-\xi} g(t)| \in L^p \), then\[
\left\{ \frac{|a_0(g)|^q}{2} + \sum_{k=0}^{\infty} (|a_k(g)|^q + |b_k(g)|^q) \right\}^{1/2} \ll \left\{ \int_{-\infty}^{\pi} |x^{-\xi} g(t)|^p \, dt \right\}^{1/p}.
\]

4 Proofs of the Results

4.1 Proof of Theorem 1

In the proof we will use the following function \( \Phi_x f(\delta, \nu) = \frac{1}{\pi} \int_{\nu}^{\nu+\delta} \varphi_x(u) \, du \), with \( \delta = \delta_n = \frac{\pi}{n+1} \) and its estimate from [10] Lemma 1, p.218\[
|\Phi_x f(\xi_1, \xi_2)| \leq w_x(\xi_1) + w_x(\xi_2)
\]
for \( f \in \Omega_{\alpha,p}(w_x) \) and any \( \xi_1, \xi_2 > 0 \).

Since, for \( n = 0 \) our estimate is evident we consider \( n > 0 \), only.
Denote by $S_k^* f$ the sums of the form

$$S_{\frac{ak}{2}} f (x) = \sum_{|\lambda_\nu| \leq \frac{ak}{2}} A_\nu (f) e^{i\lambda_\nu x}$$

such that the interval $\left( \frac{ak}{2}, \frac{a(k+1)}{2} \right)$ does not contain any $\lambda_\nu$. Applying Lemma 1.10.2 of [9] we easily verify that

$$S_k^* f (x) - f (x) = \int_0^\infty \varphi_x (t) \Psi_k (t) \, dt,$$

where $\varphi_x (t) := f (x + t) + f (x - t) - 2 f (x)$ and $\Psi_k (t) = \Psi_{\frac{ak}{2}, \frac{a(k+1)}{2}} (t)$, i.e.

$$\Psi_k (t) = \frac{4 \sin \frac{\alpha t}{4} \sin \frac{\alpha (2k+1) t}{4}}{\alpha \pi t^2}$$

(see also [3], p.41). Evidently, if the interval $\left( \frac{ak}{2}, \frac{a(k+1)}{2} \right)$ contains a Fourier exponent $\lambda_\nu$, then

$$S_{\frac{ak}{2}} f (x) = S_{\frac{ak}{2} + 1} f (x) - (A_\nu (f) e^{i\lambda_\nu x} + A_{-\nu} (f) e^{-i\lambda_\nu x}) .$$

Since (see [11] p.78 and [21] p. 7)

$$\left\{ \sum_{\nu = -\infty}^{\infty} |A_\nu (f)|^q \right\}^{1/q} \leq \| f \|_{B^p} \quad \text{and} \quad \| f \|_{B^p} \leq \| f \|_{S^p} ,$$

where $\| \cdot \|_{B^p}$, with $p \geq 1$, is the Besicovitch norm, so we have

$$|A_{\pm \nu} (f)| = |A_{\pm \nu} (f - g_{\alpha \mu/2})| \leq \| f - g_{\alpha \mu/2} \|_{S^p} = E_{\alpha \mu/2} (f)_{S^p} ,$$

for some $g_{\alpha \mu/2} \in B_{\alpha \mu/2}$, with $\alpha k/2 < \alpha \mu/2 < \lambda_\nu$. Therefore, the deviation

$$\left\{ \sum_{k=0}^{n} a_{n,k} \left| S_{\frac{ak}{2}} f (x) - f (x) \right|^q \right\}^{1/q}$$

can be estimated from above by

$$\left\{ \sum_{k=0}^{n} a_{n,k} \left| \int_{0}^{\infty} \varphi_x (t) \Psi_{k+\kappa} (t) \, dt \right|^q \right\}^{1/q} + \left\{ \sum_{k=0}^{n} a_{n,k} \left( E_{\alpha \mu/2} (f)_{S^p} \right)^q \right\}^{1/q} ,$$

where $\kappa$ equals 0 or 1. Applying the Minkowski inequality we obtain

$$\left\{ \sum_{k=0}^{n} a_{n,k} \left| \int_{0}^{\infty} \varphi_x (t) \Psi_{k+\kappa} (t) \, dt \right|^q \right\}^{1/q}$$
\[
\begin{align*}
&= \left\{ \sum_{k=0}^{n} a_{n,k} \left| \frac{\varphi_x(t)}{t} \right|_{a_n,n}^\alpha + \left| \varphi_x(t) \Psi_{k+\kappa}(t) \right| dt \right\}^{1/q} \\
&\leq \left\{ \sum_{k=0}^{n} a_{n,k} |I_1(k)|^q \right\}^{1/q} + \left\{ \sum_{k=0}^{n} a_{n,k} |I_2(k)|^q \right\}^{1/q} + \left\{ \sum_{k=0}^{n} a_{n,k} |I_3(k)|^q \right\}^{1/q}.
\end{align*}
\]

By (1), integrating by parts, we obtain
\[
\left\{ \sum_{k=0}^{n} a_{n,k} |I_1(k)|^q \right\}^{1/q} \leq \left\{ \sum_{k=0}^{n} a_{n,k} \left| \frac{4e^{2\pi a_n n}}{\varphi_x(t)} \right|_{a_n,n}^{2\pi a_n n} \varphi_x(t) \sin \frac{\alpha t}{t^2} \sin \frac{\alpha t}{4} (2k + 2\kappa + 1) dt \right\}^{1/q}.
\]

\[
= \frac{1}{\pi} \int_0^{2\pi a_n n} \left| \varphi_x(t) \right| dt = \frac{1}{\pi} \int_0^{2\pi a_n n} \left( \frac{d}{dt} \left| \varphi_x(s) \right| ds \right) dt
\]

\[
= \frac{1}{\pi} \left[ \frac{t}{\pi} \left| \varphi_x(s) \right| ds \right]_{t=0}^{t=2\pi a_n n} + \frac{1}{\pi} \int_0^{\frac{1}{2} \varphi_x(t)} \left( \frac{t}{\pi} \left| \varphi_x(s) \right| ds \right) dt
\]

\[
= \frac{1}{\pi} \int_0^{2\pi a_n n} \left( \frac{2\pi}{\varphi_x(t)} \right)_1 + \frac{2\pi a_n n}{\pi} \int_0^{2\pi a_n n} \left( \frac{1}{\lambda} \right) w_x f(t) dt
\]

\[
\ll w_x f(a_n,n)_1 + \int_0^{\frac{1}{2} \varphi_x(t)} \frac{1}{\lambda} w_x f(t) dt
\]

\[
= \frac{1}{\pi} \int_0^{2\pi a_n n} \left( \frac{2\pi}{\varphi_x(t)} \right)_1 + \frac{2\pi a_n n}{\pi} \int_0^{2\pi a_n n} \left( \frac{1}{\lambda} \right) w_x f(t) dt.
\]

\[
\ll w_x f(a_n,n)_1 + \int_0^{2\pi a_n n} \frac{1}{\lambda} w_x f(t) dt.
\]

It is clear that \( w_x f(\delta)_1 / \delta \) is nondecreasing with respect to \( \delta > 0 \) and \( w_x f(\delta)_1 \leq w_x f(\delta)_p \) for \( p \geq 1 \). Using these properties we have
\[
\left\{ \sum_{k=0}^{n} a_{n,k} |I_1(k)|^q \right\}^{1/q} \ll a_{n,n} \int_0^{\frac{1}{2} \varphi_x(t)} \frac{w_x f(t)}{t^2} dt + \int_0^{\frac{1}{2} \varphi_x(t)} \frac{1}{\lambda} w_x f(t) dt.
\]
\[ \ll \left\{ a_{n,n} \int_{a_{n,n}}^{(w_x f'(t))} \right\}^{1/p} + \int_{0}^{1} w_x f'(t) \, dt. \]

Since \( f \in \Omega_{n,p}(w_x) \) and \( \mathbf{8} \) holds, Lemma 1 and \( \mathbf{7} \) give

\[ \left\{ \sum_{k=0}^{n} a_{n,k} |I_1(k)|^q \right\}^{1/q} = O \left( a_{nn} H_x (a_{nn}) \right). \]

If \( \mathbf{6} \) holds, then

\[ a_{n,\mu} - a_{n,m} \leq |a_{n,\mu} - a_{n,m}| \leq \sum_{k=\mu}^{m-1} |a_{n,k} - a_{n,k+1}| \leq K a_{n,m} \]

for any \( m \geq \mu \geq 0 \). Hence we have

\[ a_{n,\mu} \leq (K + 1) a_{n,m}. \quad (16) \]

From this, we get

\[ \left\{ \sum_{k=0}^{n} a_{n,k} |I_2(k)|^q \right\}^{1/q} \leq \{(K + 1) a_{nn} \}^{1/q} \left\{ \sum_{k=0}^{n} \frac{4}{\alpha \pi} \left\| \int_{2 \pi / a_{n,n}}^{2 \pi} \frac{\varphi_x(t) \sin \alpha t}{t^2} \sin \frac{\alpha t}{4} (2k + 2) \, dt \right\| \right\}^{1/q} \]

\[ \ll \frac{8}{\alpha^2} (a_{nn})^{1/4} \left\{ \sum_{k=0}^{n} \frac{\alpha}{2 \pi} \left\| \int_{2 \pi / a_{n,n}}^{2 \pi} \frac{\varphi_x(t) \sin \alpha t}{t^2} \sin \frac{\alpha t}{4} (2k + 2) \cos \frac{\alpha k t}{2} \, dt \right\| \right\}^{1/q} \]

\[ \ll \frac{8}{\alpha^2} (a_{nn})^{1/4} \left\{ \sum_{k=0}^{n} \frac{\alpha}{2 \pi} \left\| \int_{2 \pi / a_{n,n}}^{2 \pi} \frac{\varphi_x(t) \sin \alpha t}{t^2} \cos \frac{\alpha t}{4} (2k + 1) \sin \frac{\alpha k t}{2} \, dt \right\| \right\}^{1/q}. \]

Using inequality \( \mathbf{13} \), we have

\[ \left\{ \sum_{k=0}^{n} a_{n,k} |I_2(k)|^q \right\}^{1/q} \ll (a_{nn})^{1/4} \left\{ \int_{2 \pi / a_{n,n}}^{2 \pi} |\varphi_x(t)|^p \, dt \right\}^{1/p}. \]

Integrating by parts, we obtain

\[ \left\{ \sum_{k=0}^{n} a_{n,k} |I_2(k)|^q \right\}^{1/q} = (a_{nn})^{1/4} \left\{ \int_{0}^{t} \left[ \frac{1}{t^{1+\frac{p}{4}}} \int_{0}^{t} |\varphi_x(s)|^p \, ds \right]_{t=2 \pi / a_{n,n}}^{t} \right\}^{1/p}. \]
\[
\left( 1 + \frac{p}{q} \right) \frac{\alpha \pi}{4} \int_{\frac{2\pi}{\alpha}} \frac{1}{t^{2+\frac{q}{p}}} \left( \int_0^t |\varphi_x (t)|^p \, ds \right) \, dt \right] \frac{1}{p} \\
= (a_{nn})^{\frac{1}{q}} \left\{ \left[ \frac{1}{t^{2+\frac{q}{p}}} \left( w_x f (t) \right)^p \right] \int_{\frac{2\pi}{\alpha}} + \left( 1 + \frac{p}{q} \right) \frac{\alpha \pi}{4} \int_{\frac{2\pi}{\alpha}} \frac{1}{t^{1+\frac{q}{p}}} \left( w_x f (t) \right)^p \, dt \right\} \frac{1}{p} \\
\leq (a_{nn})^{\frac{1}{q}} \left\{ \left( w_x (\pi) \right)^p + \int_{a_{nn}} \frac{\alpha \pi}{4} \frac{1}{t^{1+\frac{q}{p}}} (w_x (t))^p \, dt \right\} \frac{1}{p}
\]

Since \( f \in \Omega_{\alpha,p} (w_x)\), (7) gives

\[
\left\{ \sum_{k=0}^n a_{n,k} |I_2(k)|^q \right\}^{1/q} \leq (a_{nn})^{\frac{1}{q}} \left\{ \left( w_x (\pi) \right)^p + \int_{a_{nn}} \frac{\alpha \pi}{4} \frac{1}{t^{1+\frac{q}{p}}} (w_x (t))^p \, dt \right\} \frac{1}{p}
\]

\[
\leq \left\{ \sum_{k=0}^n a_{n,k} \left[ \sum_{\mu=1}^\infty \frac{2\pi}{\alpha \mu} (\mu+1) \int_{\frac{2\pi}{\alpha \mu}} |\varphi_x (t) - \Phi_x f (\delta_k, t)| \Psi_{k+\kappa} (t) \, dt \right] \right\}^{q} \frac{1}{q}
\]

\[
= \left\{ \sum_{k=0}^n a_{n,k} |I_2(k)|^q \right\}^{1/q} + \left\{ \sum_{k=0}^n a_{n,k} |I_3(k)|^q \right\}^{1/q}
\]

and
\[
|I_3(k)| \leq \frac{4}{\alpha \pi} \sum_{\mu=1}^{\frac{2\pi}{\alpha \mu}} \int_{\frac{2\pi}{\alpha \mu}} |\varphi_x (t) - \Phi_x f (\delta_k, t)| t^{-2} \, dt
\]

\[
9
\]
\[ \leq \frac{4}{\alpha \pi} \sum_{\mu = 1}^{\infty} \frac{2\pi}{\mu} \left( \frac{\alpha}{\delta_k} \right) \int_{0}^{\delta_k} |\varphi_x(t) - \varphi_x(t + u)| \, du \] 

\[ = \frac{4}{\alpha \pi} \frac{\delta_k}{\delta_k} \sum_{\mu = 1}^{\infty} \left\{ \int_{t}^{t+u} |\varphi_x(t) - \varphi_x(t + u)| \, dt \right\} \] 

\[ = \frac{4}{\alpha \pi} \frac{\delta_k}{\delta_k} \sum_{\mu = 1}^{\infty} \left\{ \left[ \frac{1}{t^2} \int_{0}^{t} |\varphi_x(s) - \varphi_x(s + u)| \, ds \right] t = \frac{2\pi}{\mu} \right\} \] 

\[ + 2 \int_{t}^{t+u} \left[ \frac{1}{t^3} \int_{0}^{t} |\varphi_x(s) - \varphi_x(s + u)| \, ds \right] \, du \] 

\[ \ll \left| \frac{1}{\delta_k} \int_{0}^{\infty} \sum_{\mu = 1}^{\infty} \left\{ \frac{1}{[\alpha/\mu]^2} \int_{0}^{t} |\varphi_x(s) - \varphi_x(s + u)| \, ds \right\} \right| \] 

\[ + \frac{1}{\delta_k} \sum_{\mu = 1}^{\infty} \left\{ \left[ \frac{1}{t^3} \int_{0}^{t} |\varphi_x(s) - \varphi_x(s + u)| \, ds \right] \right\} \] 

Since \( f \in \Omega_{\alpha,p}(w_x) \), thus for any \( x \)

\[ \lim_{\zeta \to \infty} \frac{1}{\zeta^2} \int_{0}^{\zeta} |\varphi_x(s) - \varphi_x(s + u)| \, ds \leq \lim_{\zeta \to \infty} \frac{1}{\zeta^2} w_x(u) \\
\leq \lim_{\zeta \to \infty} \frac{1}{\zeta} w_x(\delta_k) \] 

and therefore

\[ |I_{31}(k)| \leq \frac{1}{\delta_k} \int_{0}^{\delta_k} \left\{ \frac{\alpha}{2\pi} \int_{0}^{2\pi/\alpha} |\varphi_x(s) - \varphi_x(s + u)| \, ds \right\} \] 

\[ + \frac{1}{\delta_k} \int_{0}^{\delta_k} w_x(u) \, du \sum_{\mu = 1}^{\infty} \left\{ \frac{2\pi}{\mu} \right\} \]
Since
\[
\int_0^{\delta_k} w_x(u) \, du + w_x(\delta_k) \sum_{\mu=1}^{\infty} \frac{1}{2\pi \mu^2} \ll w_x(\delta_k).
\]
Next, we will estimate the term \(|I_{32}(k)|\). So,
\[
I_{32}(k) = \frac{2}{\alpha \pi} \sum_{\mu=1}^{\infty} \int_{\frac{2\pi \mu}{\alpha}}^{2\pi (\mu+1)\mu} \frac{\Phi_x f(\delta_k, t) \, dt}{t^2} \left( -\cos \frac{\alpha t (k+\kappa)}{2} + \frac{\cos \frac{\alpha t (k+\kappa+1)}{2}}{2} \right) dt
\]
\[
= \frac{2}{\alpha \pi} \sum_{\mu=1}^{\infty} \frac{2\pi (\mu+1)}{\Phi \mu} \left( -\cos \frac{\alpha t (k+\kappa)}{2} + \frac{\cos \frac{\alpha t (k+\kappa+1)}{2}}{2} \right) \left( \cos \frac{\alpha t (k+\kappa)}{2} + \frac{\cos \frac{\alpha t (k+\kappa+1)}{2}}{2} \right) dt
\]
\[
= I_{321}(k) + I_{322}(k)
\]
Since \(f \in \Omega, (w_x)\), thus for any \(x\) (using (23))
\[
\lim_{\zeta \to \infty} \left| \frac{\Phi_x f(\delta_k, 2\pi \zeta)}{2\pi \zeta^2 k} \right| \left( -\cos \frac{\pi \zeta (k+\kappa)}{\alpha (k+\kappa)} + \frac{\cos \frac{\pi \zeta (k+\kappa+1)}{\alpha (k+\kappa+1)}}{2} \right) \left( \cos \frac{\pi \zeta (k+\kappa)}{\alpha (k+\kappa)} + \frac{\cos \frac{\pi \zeta (k+\kappa+1)}{\alpha (k+\kappa+1)}}{2} \right)
\]
\[
\leq \lim_{\zeta \to \infty} \frac{w_x(\delta_k) + w_x\left(\frac{2\pi \zeta}{\alpha}\right)}{2\pi \zeta^2 k} \ll \lim_{\zeta \to \infty} \frac{w_x(\delta_k) + \zeta w_x\left(\frac{2\pi \zeta}{\alpha}\right)}{\zeta^2 k} \ll w_x(\pi) \lim_{\zeta \to \infty} \frac{1 + \zeta}{\zeta^2} = 0,
\]
and therefore
\[
I_{321}(k) = \frac{2}{\alpha \pi} \sum_{\mu=1}^{\infty} \left[ \frac{\Phi_x f(\delta_k, \frac{2\pi}{\alpha} (\mu+1))}{2\pi \zeta^2 (\mu+1)^2} \left( -\cos \frac{\pi (\mu+1) (k+\kappa)}{\alpha (k+\kappa)} \right) + \frac{\cos \frac{\pi (\mu+1) (k+\kappa+1)}{\alpha (k+\kappa+1)}}{2} \right]
\]
\[
- \Phi_x f(\delta_k, \frac{2\pi}{\alpha} \mu) \left( -\cos \frac{\pi \mu (k+\kappa)}{\alpha (k+\kappa)} + \frac{\cos \frac{\pi \mu (k+\kappa+1)}{\alpha (k+\kappa+1)}}{2} \right)
\]
\[
= -\frac{2}{\alpha \pi} \frac{\Phi_x f(\delta_k, 2\pi/\alpha)}{[2\pi/\alpha]^2} \left( -\frac{(-1)^{(k+\kappa)}}{\alpha (k+\kappa)} + \frac{(-1)^{(k+\kappa+1)}}{\alpha (k+\kappa+1)} \right)
\]
\[
= -\frac{1}{\pi^3} \Phi_x f(\delta_k, 2\pi/\alpha) (-1)^{(k+\kappa+1)} \left( \frac{1}{k+\kappa+1} + \frac{1}{k+\kappa} \right).
\]
Using (14), we get

\[ |I_{321} (k)| \ll \frac{1}{\pi^3} \frac{2}{k + 1} |\Phi_x f (\delta_k, 2\pi/\alpha)| \leq \frac{2}{\pi^3} \frac{1}{(k + 1)} (w_x (\delta_k) + w_x (2\pi/\alpha)) . \]

Similarly

\[ I_{322} (k) = \frac{2}{\alpha \pi} \sum_{\mu=1}^{\infty} \int_{\frac{\alpha}{\delta k}}^{\frac{\alpha}{2\mu} \alpha} \left( \frac{\Phi_x f (\delta_k, t)}{t^2} - \frac{2\Phi_x f (\delta_k, t)}{t^3} \right) \cdot \left( \cos \frac{\alpha (k + \mu)}{2} - \cos \frac{\alpha (k + \mu + 1)}{2} \right) dt \]

and

\[ |I_{322} (k)| \ll \frac{8}{\alpha^2} \frac{1}{(k + 1) \pi} \sum_{\mu=1}^{\infty} \left[ \int_{\frac{\alpha}{\delta k}}^{\frac{\alpha}{2\mu} \alpha} \frac{|\varphi_x (t + \delta_k) - \varphi_x (t)|}{\delta_k t^2} dt \right] \]

\[ + \frac{2}{\alpha^2} \frac{1}{(k + 1) \pi} \sum_{\mu=1}^{\infty} \int_{\frac{\alpha}{\delta k}}^{\frac{\alpha}{2\mu} \alpha} \frac{|\Phi_x f (\delta_k, t)|}{t^3} dt \]

\[ \leq \frac{8}{\alpha^2} \frac{1}{(k + 1) \pi} \sum_{\mu=1}^{\infty} \frac{2\alpha^2 (\mu + 1)}{2\mu \alpha} \int_{\frac{\alpha}{\delta k}}^{\frac{\alpha}{2\mu} \alpha} \frac{|\varphi_x (t + \delta_k) - \varphi_x (t)|}{t^2} dt \]

\[ + \frac{16}{\alpha^2} \frac{1}{(k + 1) \pi} \sum_{\mu=1}^{\infty} \frac{\alpha^2 (\mu + 1)}{2\mu \alpha} \int_{\frac{\alpha}{\delta k}}^{\frac{\alpha}{2\mu} \alpha} w_x (\delta_k) + w_x (t) \frac{1}{t^3} dt \]

\[ \ll \frac{1}{(k + 1) \delta_k} w_x (\delta_k) + \frac{1}{k + 1} \sum_{\mu=1}^{\infty} \left[ \left( w_x (\delta_k) + w_x \left( \frac{2\pi (\mu + 1)}{\alpha} \right) \right) \frac{\alpha^2}{4\pi^2 \mu^3} \right] \]

\[ \ll w_x (\delta_k) + \frac{1}{k + 1} \left( w_x (\delta_k) \sum_{\mu=1}^{\infty} \frac{1}{\mu^3} + \sum_{\mu=1}^{\infty} w_x \left( \frac{2\pi (\mu + 1)}{\alpha} \right) \frac{1}{\mu^3} \right) \]

\[ \ll w_x (\delta_k) + \frac{1}{k + 1} \left( w_x (\delta_k) + w_x \left( \frac{4\pi}{\alpha} \right) \sum_{\mu=1}^{\infty} \frac{\mu + 1}{\mu^3} \right) \]

\[ \ll w_x (\delta_k) + \frac{1}{k + 1} \left( w_x (\delta_k) + w_x \left( \frac{4\pi}{\alpha} \right) \right) . \]

Therefore

\[ |I_3 (k)| \ll w_x (\delta_k) + \frac{1}{k + 1} \left( w_x (\delta_k) + w_x \left( \frac{2\pi}{\alpha} \right) + w_x \left( \frac{4\pi}{\alpha} \right) \right) \]
and thus
\[
\left\{ \sum_{k=0}^{n} a_{n,k} |I_3(k)|^q \right\}^{1/q} \ll \left\{ \sum_{k=0}^{n} a_{n,k} \left( w_x \left( \frac{\pi}{k+1} \right) + \frac{1}{k+1} w_x \left( \frac{\pi}{\alpha_k} \right) \right)^q \right\}^{1/q}
\]
\[
\ll \left\{ \sum_{k=0}^{n} a_{n,k} \left( w_x \left( \frac{\pi}{k+1} \right) \right)^q \right\}^{1/q}.
\]

From (16) we obtain
\[
\sum_{k=0}^{n} a_{n,k} \left( w_x \left( \frac{\pi}{k+1} \right) \right)^q \leq \left\{ \sum_{k=0}^{n} a_{n,k} \left( w_x \left( \frac{\pi}{k+1} \right) \right)^q \right\}^{1/q}.
\]

Using (1), (16) and the monotonicity of the function \( w_x \), from (7) and (12), we get
\[
\sum_{k=0}^{n} a_{n,k} \left( w_x \left( \frac{\pi}{k+1} \right) \right)^q \leq (K+1) \sum_{k=0}^{n} a_{n,k} \left( w_x \left( \frac{\pi}{k+1} \right) \right)^q + \sum_{k=0}^{n} a_{n,k} \left( w_x \left( \pi (K+1) a_{n,n} \right) \right)^q.
\]

Summing up we obtain that (9) is proved and the proof is complete.
4.2 Proof of Theorem 2

Under the notation of the before proof we can write

\[
\left\{ \sum_{k=0}^{n} a_{n,k} \left| \phi_{x} (t) \Psi_{k+\kappa} (t) dt \right|^q \right\}^{1/q} = \left\{ \sum_{k=0}^{n} a_{n,k} \left( \int_{0}^{\frac{2\pi}{k}} + \int_{\frac{2\pi}{k}}^{\frac{2\pi}{k+\kappa}} + \int_{\frac{2\pi}{k+\kappa}}^{\frac{2\pi}{2\pi}} \phi_{x} (t) \Psi_{k+\kappa} (t) dt \right) \right\}^{q \cdot \frac{1}{q}} \leq \left\{ \sum_{k=0}^{n} a_{n,k} \left( J_{1}(k) \right) \right\}^{1/q} + \left\{ \sum_{k=0}^{n} a_{n,k} \left( J_{2}(k) \right) \right\}^{1/q} + \left\{ \sum_{k=0}^{n} a_{n,k} \left( J_{3}(k) \right) \right\}^{1/q},
\]

using the Minkowski inequality. Applying the property of the class \textit{RBVS} instead of the property of \textit{HBVS} our proof will be similar to the proof of Theorem 1.

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