Intrinsic periodicity of turbulence

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Abstract

Quasi-Periodic Behavior (QPB) is a common feature of both low and high Reynolds number flows. We carefully assess three-dimensional flows driven by a steady forcing to reveal the robustness of QPB. A close examination of the flow allows the formulation of a simple three-equation model which reproduces QPB with temporal behavior similar to predator-prey dynamics. We find that non-local triad interactions are necessary to maintain QPB. Bifurcation analyses illustrate that the model can reproduce several critical features of the Navier-Stokes dynamics, such as deterministic chaos with QPB and sudden relaminarization of transient chaos. These findings suggest that the model is not specific to the studied flow but is of more general interest in investigating turbulence and its transition in different flow geometries.

INTRODUCTION

List of previous research

Both laminar and turbulent flows display spontaneous periodic or quasi-periodic features. An illustrative example is the vortex shedding behind a cylinder. For low values of the Reynolds number (Re), i.e., when the effects of inertia compared to viscous diffusion are moderate, the so-called von-Kármán vortex street behind a cylinder is perfectly periodic. When the flow becomes fully turbulent at higher Re, this periodicity is still present, even though the stochastic nature of the turbulent fluctuations prevents the system from being perfectly periodic. This close-to-periodic motion, embedded in chaotic or turbulent fluctuations, is what we will call Quasi-Periodic Behavior (QPB). Understanding such QPB is at the heart of the present investigation.

Another important example of QPB is the temporal behavior of turbulent channel flow, where quasi-periodic bursts and streaks govern the dynamics [1–3]. The simplified descriptions of this phenomenon are in general specific to the case of channel flow, e.g. [4–10]. Furthermore, it was realized that there is QPB in flows without obstacles or solid walls. Indeed, even in cubic boxes with periodic boundary conditions and steady forcing, QPB is observed [11–13], which suggests that the dynamics are an intrinsic feature of the Navier-Stokes equations and not a consequence of a specific flow geometry. Such robust, stable periodic behavior is of particular interest given the amount of recent research dedicated to
the identification of unstable periodic orbits embedded in turbulent flows \cite{14-21}. Such stud-
ies show that periodic solutions of turbulence possess vital statistics or dynamical properties
of turbulence.

In this investigation, we quest after the simplest possible model that can reproduce QPB while retaining a close link with the governing equations. Such an approach has been tried for different systems in the literature \cite{21-27}. The bursty nature of the statistics and the observed time-delay between different flow structures \cite{12} hint at a predator-prey type behavior as described by the Lotka-Volterra equations \cite{28} as recently mentioned in Ref. \cite{29}. However, we show that QPB needs the interactions of three “species”, so one more term than present in the Lotka-Volterra system.

The temporal behavior of fully turbulent three-dimensional flows is hard to address in detail since many different length scales interact to produce fluctuations at a wide range of time scales. The present investigation disentangles a flow simpler than the fully turbulent flow but complicated enough to retain its QPB. This approach is motivated by the recent observation of a non-trivial QPB in a confined cylindrical flow between two counter-rotating disks (the so-called von Kármán flow) \cite{30} and that QPB is also observed at \( \text{Re} \gg 1 \) \cite{31}.

To simplify the analysis, we will consider three-dimensional flows driven by a steady forcing in a periodic domain. First, we report a temporally periodic but non-trivial state at low Reynolds numbers. This laminar state shares some key features with the high Reynolds number case. We dissect this periodic flow and assess Fourier-mode interactions in detail to show three types of modes and three types of interactions among them. We propose a minimal model consisting of three ordinary differential equations from these observations. The model exhibits QPB, somewhat similar to predator-prey dynamics. Let us now detail our approach.

**Quasi Periodic Behavior**

We conduct Direct Numerical Simulation (DNS) of the Navier-Stokes equation driven by a steady forcing \cite{11-13},

\[
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f},
\]

\[
\mathbf{f} = (-f_0 \sin x \cos y, f_0 \cos x \sin y, 0),
\]
FIG. 1. (a,c) Visualizations of vortical structures and (b,d) parametric time series of the energy input rate $P(t)$ and the energy dissipation rate $\epsilon(t)$ in (a,b) low Reynolds number periodic flow and (c,d) higher Reynolds number turbulent flow. Visualizations show isosurfaces of (a) $\sqrt{|\omega|^2}$ in white and (c) $\sqrt{|\omega|^2}$ in cyan and low-pass filtered $\sqrt{|\omega|^2}$ in red. Low-pass filtered quantity is computed from $u^<(x) \equiv \int dr \, G(r/\sigma)u(x+r)$, where $G(r/\sigma)$ is Gaussian function and we set $\sigma = \sqrt{2}|k_f| = 2$. Parametric time series show (a) original and (d) phase-averaged $P(t)$-$\epsilon(t)$. Here, $\langle \cdot \rangle_{\text{phase}}$ denotes the phase-averaging procedure. Dark to light colors represent the time evolution, and the gap between two consecutive dots denotes $5T$.

with $\nabla \cdot u = 0$. Here, $u$, $p$, and $f$ are the velocity, pressure, and forcing field, respectively. The kinematic viscosity $\nu$ is the control parameter. The forcing amplitude $f_0$ is set to unity. We employ the pseudo-spectral method in a $(2\pi)^3$ periodic box [32] (See the supplemental material for further detail of the DNS). We define the Reynolds number and the characteristic timescale as

$$Re \equiv \sqrt{f_0/|k_f|^{3/2}\nu} \quad \text{and} \quad T \equiv 1/\sqrt{|k_f|f_0} = 0.840,$$

respectively. For a specific value of $Re = 5.83$, we obtain a perfectly time-periodic three-dimensional flow, visualized in Fig. 1(a). We distinguish four large-scale columnar vortices, associated with the steady body force $f$, and anti-parallel smaller vortices, generated by vortex stretching. Parametric time series of the energy input rate $P \equiv \langle f \cdot u \rangle$ where $\langle \cdot \rangle$ denotes the spatial average and the energy dissipation rate $\epsilon \equiv \nu \langle |\omega|^2 \rangle$ where $\omega = \nabla \times u$ in Fig. 1(b) shows an anti-clockwise cycle, suggesting a time-delayed causality between $P$ and $\epsilon$.

We extract similar dynamics in turbulent flow [Fig. 1(c,d)] at $Re = 29.7$ [33]. The isosurfaces of the enstrophy $|\omega|^2$ consist of complex structures [Fig. 1(c)]. However, by applying a low-pass filter to the velocity field, it is observed that the principal structures are
FIG. 2. (a) Time series of energy $E(t)$ and energy dissipation rate $\epsilon(t)$ computed from all modes (solid line) and the forcing plus six primary energetic modes (with subscript $X + Y$, shown in dashed line). (b) Schematic of three different scales: “forcing”, “primary”, and “secondary”. We visualize vorticity distributions of typical Fourier modes in each scale simultaneously. Forcing scale is $k_f = (1, 1, 0)$. In the primary scale, we visualize $k = (2, 0, 2)$ and $(0, 2, 2)$ modes. The secondary scale with $k = (3, 1, 0)$ and $(2, 2, 2)$. Triads denote energy transfer between three Fourier modes in different scales satisfying the triad-interaction conditions.

the forcing-induced columnar vortices and perpendicular rolls, very similar to those observed in the periodic flow. We apply a phase average to the complex time series of $P(t)$ and $\epsilon(t)$ conditioned on the peaks of $P(t)$ in order to extract smooth time-delayed oscillations shown in Fig. 1 (d) (See the supplemental material for the detailed procedure). These results suggest that QPB of turbulent flow driven by the steady body force is robust regardless of the Reynolds number \[34\].
THREE-EQUATION MODEL

Construction of the model

Our objective is to construct the simplest possible model capable of reproducing this QPB while retaining a close connection with the structure of the Navier-Stokes equations. For this, we recall that in a Fourier-representation the individual modes $q_i$ are governed by the discrete Navier-Stokes dynamics \cite{35,36}

$$[\partial_t + \nu_i]q_i = \sum_{j,m} A_{ijm}q_j q_m + f_i,$$

where $\nu_i \equiv \nu |k_i|^2$, $f_i$ is the forcing applied to mode $i$, and $A_{ijm}$ are the coupling constants resulting from the advection and pressure terms. The nonlinear term associated with triad interactions rapidly yields an overwhelming complexity when the number of retained modes increases. Therefore, if the origin of the QPB is common to both the periodic and the turbulent flows, it seems sensible to analyze the periodic flow with less dynamically active modes. However, even this seemingly simple flow is constituted by a considerable number of interacting scales. We spectrally investigate the periodic flow to find that only seven Fourier modes are responsible for 98\% of its energy, as shown in Fig. 2 (a) (See the supplemental material for further detail of energetic modes). In the following, $X \in \mathbb{R}$ denotes the characteristic velocity of the forced mode, and $Y \in \mathbb{R}$ corresponds to that of the other six modes.

Writing explicitly Eq. (4) for a general system of seven modes yields already very complicated nonlinear interactions. However, a close inspection of these modes shows that they can be expressed by a single triad interaction of the forced mode $X$ and two of the six other modes $Y$ by invoking symmetry. Thus, Eq. (4) can be rewritten as the equations of $X$ and $Y$ as,

$$\begin{align*}
d_t X &= -AY^2 - \nu K_X^2 X + F, \\
d_t Y &= +AXY - \nu K_Y^2 Y,
\end{align*}$$

(5)

with a coefficient $A > 0$, typical wavenumbers $K_{\alpha} \in \mathbb{R}$ with $\alpha \in \{X,Y\}$, and a steady force $F > 0$. The first term on the RHS represents nonlinear coupling and conserves energy. We performed an extensive parameter scan, but do not observe periodic behavior in the two-equation model \cite{5} for $\nu \neq 0$ and $F \neq 0$. Retaining only this simple triad interaction
between the forcing and most energetic modes is not enough to reproduce QPB.

The additional ingredient for QPB is a small-scale representative and its associated triad interaction terms. This additional feature can be interpreted as a minimal representation of the energy cascade. Indeed, in Fig. 2 (a), it is observed that while the energy is almost entirely contained in the modes constituting X and Y, a visible bump in the energy dissipation rate $\epsilon$ is omitted in these modes. Thus, we add a mode $Z \in \mathbb{R}$ associated with the remaining dynamically active modes of the flow to obtain a refined three-equation model represented by a schematic diagram in Fig. 2 (b). The model is defined by

$$\begin{cases}
\frac{d}{dt} X = -A_1 Y^2 + A_3 YZ - \nu K_X^2 X + F, \\
\frac{d}{dt} Y = +A_1 XY - A_2 Z^2 + A_4 XZ - \nu K_Y^2 Y, \\
\frac{d}{dt} Z = +A_2 YZ - (A_3 + A_4) XY - \nu K_Z^2 Z.
\end{cases}$$

where $A_1, A_2 > 0$ and $A_3, A_4 \in \mathbb{R}$ are triad coefficients which retain the Navier-Stokes structure $[4]$. We choose the signs and the values of the triad coefficients so that the detailed balance holds in the energy transfer between the three scales. The signs of $A_1$ and $A_2$ are defined so that energy cascades towards small scales: from X to Y and Y to Z. The additional triad with coefficients $A_3$ and $A_4$ represents non-local interactions. The full model is not a conservative system since we introduce the viscous damping term. We also note that the model does not satisfy Liouville’s theorem in the $\nu \to 0$ limit.

**Determination of the parameters**

We then fit six out of eight model constants in (6) by comparing them to the DNS of the periodic flow. To do so, we use the energy equation associated with Eq. (6),

$$\begin{cases}
\frac{d}{dt} E_X = T_X - \epsilon_X + P, \\
\frac{d}{dt} E_Y = T_Y - \epsilon_Y, \\
\frac{d}{dt} E_Z = T_Z - \epsilon_Z,
\end{cases}$$

where energy $E_\alpha \equiv \alpha^2/2$ with $\alpha \in \{X,Y,Z\}$, energy transfer terms $T_X \equiv -A_1 XY^2 + A_3 XYZ$, $T_Y \equiv A_1 XY^2 - A_2 YZ^2 + A_4 XYZ$, $T_Z \equiv A_2 YZ^2 - (A_3 + A_4) XY Z$, energy dissipation rate $\epsilon_\alpha \equiv 2\nu K_\alpha^2 E_\alpha$, and energy input rate $P \equiv FX$. The quantities $K_\alpha^2$ and $F$ are determined directly by comparison with their time-averaged values in the DNS. $A_1$ and $A_2$
FIG. 3. Time series of fluctuating energy of the forcing scale $E_{X-X_0} \equiv (X - F \frac{Re}{K_X^2})^2/2$ and energy of residual scales $E_{Y+Z} \equiv Y^2/2 + Z^2/2$ in the model (6). Parameters are $A_1 = 0.4$, $A_2 = 4$, $A_3 = 0.5$, $A_4 = -1$, $F = 0.7$, $K_X^2 = 2$, $K_Y^2 = 5$, $K_Z^2 = 15$, and $Re = 15$. Inset: similar plot by DNS of the periodic flow where $E_{X-X_0} \equiv \left\langle |\mathbf{u}_X - f/2\nu|k_f|^2\right\rangle^2/2$ and $E_{Y+Z} \equiv \left\langle |\mathbf{u}_Y|^2\right\rangle/2 + \left\langle |\mathbf{u}_Z|^2\right\rangle/2$. Time is normalized by $T$.

are adjusted to yield the observed energy flux, neglecting the non-local triad (Full description of the parameter fitting can be found in the supplemental material). The free parameters of the model are then $A_3$ and $A_4$, and the control parameter is $Re \equiv 1/\nu$. We numerically integrate the model using a fourth-order Runge-Kutta scheme with $\Delta t = 0.01$ starting from random initial conditions. We find that no periodic fluctuations are observed for the system without the non-local interactions: $A_3 = A_4 = 0$. Conversely, periodic behavior is observed for a wide range of values when $A_3, A_4 \neq 0$, showing that the non-local triad interactions are responsible for the periodic behavior of the model.

Figure 3 compares the periodic solution obtained by the model and the DNS. Here, we set the free parameters $A_3 = 0.5$ and $A_4 = -1$, which allows us to reproduce the intrinsic periodicity as in the DNS. We compute two quantities. One is $E_{X-X_0}$, that is, the fluctuating energy of the forcing mode around the laminar base flow $X_0 \equiv F \frac{Re}{K_X^2}$. The other quantity is the energy of the rest of the modes. We compare them to the corresponding quantities in the DNS of the periodic flow (the base flow is $\mathbf{u}_0 \equiv f/2\nu|k_f|^2$). The two quantities show qualitatively the same behavior in terms of amplitude and phase difference. We stress that this periodic solution is independent of the initial condition, unlike the standard two-species Lotka-Volterra equations, which is a favorable feature for fluid turbulence.
FIG. 4. Bifurcation diagrams of the model (6) for (a) $A_3 = 0.5$ and $A_4 = -1$ and (b) $A_3 = 0.5$ and $A_4 = -0.63$. Other parameters are same as Fig. 3. We plot local extrema of $Y$ while varying $Re$ systematically. (a) Yellow vertical line corresponds to $Re = 15$ shown in Fig. 3. Inset: close-up of the bifurcation diagram in the range shown by the red rectangle in the main plot. Blue shaded region denotes $Re \in [15.673, 15.675]$. (b) Blue and yellow data correspond to descending and ascending $Re$. Inset: an example of time series of the model at $Re = 23.9$, denoted by the pink vertical line in the main plot. The time series exhibits a sudden relaminarization from the chaotic to the steady state.

RESULTS

The system (6) retains several main features of turbulent flows. First of all, the $A_1$ and $A_2$ triads represent two steps of the Richardson-Kolmogorov energy cascade. The viscous terms show how dissipation acts on modes of different scales. For our purposes, the essential feature of the model is the non-local coupling between different scales represented by $A_3$ and $A_4$. We understand that this coupling is responsible for the predator-prey-like periodic behavior in the here considered flow.

We have thus succeeded in reproducing the fully periodic behavior observed in the low
Reynolds number DNS. Another important question is whether the model can also reproduce QPB when the dynamics are not fully periodic anymore, as observed in turbulent QPB. First, we need to illustrate that the model can yield deterministic chaos. The bifurcation diagrams in Fig. 4 confirm this and show the existence of chaos. For example, there are narrow windows of chaos for higher values of Re in Fig. 4 (a) (with the same parameters as in Fig. 3) and we find a critical Reynolds number $Re_{cr} \in [15.673, 15.675]$. In these chaotic states, the behavior of the system remains close to periodic. The same model does, therefore, reproduce not only the purely periodic fluctuations but also chaotic QPB for smaller values of $\nu$, as observed in the DNS.

A natural question is now how universal the model is in reproducing the features of different flows. Indeed, the system (6) has a general form associated with the discrete representation of the Navier-Stokes equations (4), and it is not clear from the outset whether it can describe features from pipe flow, channel flow, and Kolmogorov flow, or that it is restricted to the forcing we consider in this study. The behavior of these flows is different, particularly their transition properties, some of them being subcritical, others supercritical. If the present model is universal, it should exhibit QPB and different bifurcation scenarios for different parameter set-ups. We plot another example of the bifurcation diagram in Fig. 4 (b) to reveal that this is indeed the case. Whereas Fig. 4 (a) shows supercritical bifurcation from a periodic solution to chaos with QPB, Fig. 4 (b) shows different bifurcation dynamics with a hysteresis while we only modified one of the free parameters $A_4$ from $-1$ to $-0.63$. The solution exhibits a subcritical transition around $Re \approx 24$ and turns into a steady-state for lower Reynolds numbers, and eventually merges to the trivial solution with $Y = 0$. We identify a transient chaos regime in a narrow range of Re near this subcritical bifurcation, as shown in the inset of Fig. 4 (b). These sudden transitions from chaotic to steady states remind us of the sudden relaminarization observed in a linearly forced turbulence [37] and pipe flow [38]. Note that the current steady forcing (2) in the DNS does not permit such transition; however, the somewhat similar steady Kolmogorov forcing exhibits sudden relaminarization [39].

From the present investigation, the minimum requirement to reproduce periodic behavior while keeping the structure of the Navier-Stokes equation is the non-local interaction of three distinct types of modes. Combining with the other retained characteristics such as forcing, damping, and local interactions, the present model reproduces QPB and features such as an
energy cascade, deterministic chaos, and sudden relaminarization. We, therefore, hope that the present minimal model of turbulence can guide us, like the Lorenz model has done for general chaotic dynamics, to a better understanding of turbulence.

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FIG. 5. Time series of energy input rate $P(t)$ and energy dissipation rate $\epsilon(t)$ of the turbulent flow at $Re = 29.7$.

DIRECT NUMERICAL SIMULATIONS

We use an in-house parallelized code [32] to conduct DNS. It employs the pseudo-spectral method with a 2/3 dealiasing rule for spatial discretization and the Adams-Bashforth scheme in the time domain. The initial condition is generated in Fourier space, following Rogallo’s method [40].

We perform two DNS in $(2\pi)^3$ triply periodic box: three-dimensional periodic and turbulent flows. The periodic flow is obtained in a box of $(64)^3$ resolution by adjusting $\nu = 0.102$. It is equivalent to the Reynolds number (3) of $Re = 5.83$. We employ $(128)^3$ box to obtain turbulent flow at $\nu = 0.02$ ($Re = 29.7$). For the statistical phase averaging procedure, we use $t \in [1.28 \times 10^3, 1.04 \times 10^4]$ time series to guarantee statistics to converge. Note that we discard the transient part from the analysis. This time series is equivalent to approximately $1.08 \times 10^4T$ with the characteristic time scale $T$ defined in Eq. (3) being 0.840.

DETAILED PROCEDURE OF THE PHASE AVERAGING

Figure 5 shows the temporal evolutions of energy input rate $P(t) \equiv \langle f \cdot u \rangle$ and $\epsilon(t) \equiv \nu \langle |\omega|^2 \rangle$ in the turbulent flow at $Re = 29.7$. To extract an intrinsic periodicity in the chaotic fluctuations, we phase average the complex time series. First, we pick up the local maxima of $P(t)$ (as in Fig. 6 (a)) with the following two criteria:
FIG. 6. Detailed procedure of the phase-averaging. (a) Time series of $P(t)$ with its local maxima denoted by dots. The pink horizontal line corresponds to the threshold of the magnitude $\langle P \rangle_t + \sigma(P)$. (b) Overlapped segments of the time series of $P(t)$ based on its local maxima. Note that peaks are normalized to unity. (c) Phase averaged time series of $P(t)$ and $\epsilon(t)$. Shaded region represents $\pm \sigma(f)$ where $f(t)$ is $P(t)$ or $\epsilon(t)$. The pink vertical dashed line indicates the peak of phase-averaged $\epsilon(t)$.

1. The magnitude of the local peak must be above $\langle P \rangle_t + \sigma(P)$ where $\langle \cdot \rangle_t$ and $\sigma(\cdot)$ denote time average and standard deviation of the time series, respectively.

2. The time increment between two consecutive local peaks must be larger than $\tau_{\text{max}}/2$ where $\tau_{\text{max}}$ corresponds to the second peak of autocorrelation function of $P$ (note that the first peak is at $\tau = 0$).

Criterion 1 corresponds to the horizontal pink line in Fig. 6 (a). Second, segments of time series of $P(t)$ centered at local maxima are overlapped [Fig. 6 (b)]. At the same time, we normalize them by their peak values. Then, we compute the average over them to obtain the phase averaged time series [Fig. 6 (c)].

Concerning $\epsilon(t)$, we again normalize by their local maxima, but we use the time instant corresponding to the peak of $P(t)$ to normalize time. Thereby, we can evaluate the relative time delay between the two quantities, as shown in the pink vertical dash line in Fig. 6 (c).

In Fig. 1 (d), we plot a parametric plot of them. Although we do not observe a closed-loop in Fig. 1 (d), possibly because of the time series are not long enough, Fig. 6 (c) suggests that these two phase-averaged quantities exhibit QPB.
FIG. 7. Schematic of forcing plus six primary energetic Fourier modes in the periodic flow. Visualizations show vorticity distributions of each Fourier mode at the same instance. Triangles denote the possible combinations of triad interactions. On the top, we compare the isosurface of $\sqrt{|\omega|^2}$ computed from the forcing plus six primary energetic modes and total $\sqrt{|\omega|^2}$.

PRIMARY ENERGETIC MODES OF THE PERIODIC FLOW

Figure 7 represents one forcing mode plus six primary energetic Fourier modes with visualizations of vorticity distributions of each mode. Triangles indicate combinations of different modes where energy transfer via triad interactions is possible considering wave vectors but ignoring their phase. Figure 7 also compares visualizations of vortical structures of the sum of these seven primary energetic modes, and that of all modes and similar structures (large columnar vortices and small vortices in anti-parallel alignment) are observed.

We plot the energy of each of these seven modes in Fig. 8. The forcing mode is dominant since the flow is at a low Reynolds number.
FIG. 8. Time series of forcing plus six primary energetic modes. Total energy and energy corresponding to the sum of the seven modes are denoted by $E$ and $E_{X+Y}$. Energy for the specific Fourier mode $(k_x, k_y, k_z)$ is denoted by $E_{k_x k_y k_z}$.

FIG. 9. Time series of (a) forcing coefficient $F(t)$ and (b) scale coefficients $K_{\alpha}^2(t)$. Both (a) and (b) are computed in the DNS of the three-dimensional periodic flow.

DETAILED PROCEDURE OF THE PARAMETER FITTING

Figure 9(a) shows time evolution of the forcing coefficient evaluated by

$$F = \frac{P}{\sqrt{2E_X}} = \frac{\langle f \cdot u \rangle}{\sqrt{\langle |u_X|^2 \rangle}}.$$  \hspace{1cm} (8)
FIG. 10. Time series of (a) energy transfer terms $T_\alpha$ and (b) transfer coefficients $A_i$. $T_\alpha$ is indirectly evaluated by adapting (7) to DNS, and $A_i$ is computed by (12, 13). Both (a) and (b) are computed in the DNS of the three-dimensional periodic flow.

For the value of the forcing, we find $F \approx 1/\sqrt{2}$, since we set $f_0 = 1$ in (2) so that we can evaluate that $P = \sqrt{E_X}$. However, we observe a periodic decrease of $F$ associated with a phase modulation of $u_X$.

We also compute scale factors

$$K^2_\alpha(t) = \frac{\epsilon_\alpha(t)}{2\nu E_\alpha(t)},$$

(9)

to plot temporal evolution of them in Fig. 9 (b), where $K^2_X(t)$ is almost constant since it corresponds to only one mode, $k_f$. On the other hand, $K^2_Y(t)$ and $K^2_Z(t)$ exhibit fluctuations due to competition of different Fourier modes in these scales of modes. We take their time average as the model parameters; $K^2_\alpha = \langle K^2_\alpha(t) \rangle_t$.

For triad coefficients $A_1$ to $A_4$, there are more degrees of freedom than the number of equations, thus we cannot determine all of them. In order to obtain rough estimates of $A_1$ and $A_2$, we compute the average energy transfer rate from $X$ to $Y$ and $Y$ to $Z$ without the non-local interactions. This idea corresponds to considering the energy equation,

$$\begin{cases}
    \frac{d}{dt}E_X = T_X - \epsilon_X + P, \\
    \frac{d}{dt}E_Y = T_Y - \epsilon_Y, \\
    \frac{d}{dt}E_Z = T_Z - \epsilon_Z,
\end{cases}$$

(10)
with modified energy transfer terms,

\[
\begin{align*}
    T_X &= -A_1 XY^2 + A_3 XYZ \approx -A_1 XY^2, \\
    T_Y &= +A_1 XY^2 - A_2 YZ^2 + A_4 XYZ \approx +A_1 XY^2 - A_2 YZ^2, \\
    T_Z &= +A_2 YZ^2 - (A_3 + A_4) XYZ \approx +A_2 YZ^2,
\end{align*}
\]  

(11)

with \( A_3 = A_4 = 0 \). We compute \( T_X, T_Y, \) and \( T_Z \) by (10) to show them in Fig. 10 (a). It shows that \( T_X < 0 \), thus \( X \) is transferring energy to smaller scales on average. The fact that \( T_Y, T_Z > 0 \) means that these smaller scales receive energy from the larger scales. We then evaluate their time-dependent coefficients \( A_1(t) \) and \( A_2(t) \) by comparing (10) and (11),

\[
\begin{align*}
    A_1(t) &\approx -\frac{T_X}{XY^2} = -\frac{1}{2\sqrt{2} \sqrt{E_X E_Y}}, \\
    A_2(t) &\approx \frac{T_Z}{YZ^2} = \frac{1}{2\sqrt{2} \sqrt{E_Y E_Z}},
\end{align*}
\]  

(12)  (13)

again, under the assumption that \( A_3 = A_4 = 0 \). The result is shown in Fig. 10 (b) and we take their time averages to determine \( A_1 \) and \( A_2 \). Noise in these quantities comes from the post-processing evaluation of the time derivative of energy.

We summarize our estimation of the model constants from the DNS of three-dimensional periodic flow which we use in the manuscript:

\[
A_1 = 0.4, \ A_2 = 4, \ F = 0.7, \ K_X^2 = 2, \ K_Y^2 = 5, \ K_Z^2 = 15. \quad (14)
\]

This estimate is incomplete as we have not determined \( A_3 \) and \( A_4 \). These constants have been chosen subsequently to reproduce the similar periodic fluctuations observed in the DNS.