Propagation of Fronts and Information in Dispersive Media

Shi-Yao Zhu\textsuperscript{1,2}, Ya-Ping Yang\textsuperscript{1,2}, Li-Gang Wang\textsuperscript{2}, Nian-Hua Lin\textsuperscript{2,3}, and M. Suhail Zubairy\textsuperscript{4}

\textsuperscript{1}Department of Physics, Tongji University, Shanghai 200092, China
\textsuperscript{2}Department of Physics, Hong Kong Baptist University, Kowloon Tong, Hong Kong
\textsuperscript{3}Department of Physics, Nanchang University, Nanchang 330047, China
\textsuperscript{4}Institute for Quantum Studies and Department of Physics, Texas A\&M University, College Station, TX 77843

(Dated: January 31, 2022)

PACS numbers: 42.25.Bs, 42.50.Lc, 42.70.Qs

We present a general proof based on Kramers-Kronig relations that, in a normal or anomalous dispersive linear medium, any (discontinuity/nonanalytic disturbance) in an electromagnetic pulse can not propagate faster than the phase velocity, $c$. Consequently the information carried by the discontinuity (nonanalytical disturbance) can not be transmitted superluminally.

According to the Einstein’s theory of special relativity, the speed of any moving object can not exceed the speed of light in vacuum $c$. However it is well-known that the group velocity of a light pulse $v_g$ can exceed $c$ in an anomalously dispersive medium.\footnote{\textsuperscript{1}, \textsuperscript{10}, \textsuperscript{11}, \textsuperscript{12}, \textsuperscript{13}} This interesting effect is a result of interference of different frequency components of the light pulse.\footnote{\textsuperscript{2}, \textsuperscript{3}} The superluminal phenomenon disappears when the pulse loses the coherence.\footnote{\textsuperscript{4}}

Sommerfeld and Brillouin observed in 1914 that a superluminal group velocity does not violate causality.\footnote{\textsuperscript{5}, \textsuperscript{6}, \textsuperscript{7}} They observed that the front velocity, (the velocity of a sharp nonanalytic discontinuity in a light pulse), should be used as the signal velocity at which information is transmitted and this velocity does not exceed $c$. The analysis of Sommerfeld and Brillouin is based on the propagation of an electromagnetic wave of the form

$$f(t) = \begin{cases} 0 & (t < 0) \\ \sin(2\pi t/\tau) & (t > 0) \end{cases}$$

through an anomalously dispersive medium with strong absorption, characterized by a Lorentzian susceptibility that is proportional to $\omega^{-2}$. The sharp begining of the light wave (1) corresponds to the signal. The debate concerning the information velocity in media still remains due to progress in experiments. For example, a recent experiment by Wang et al. reports superluminal propagation in a gain medium with susceptibility proportional to $\omega^{-1}$.\footnote{\textsuperscript{8}} It is therefore a matter of great interest to give a general proof about the causal nature of the propagation of classical information (carried by the front or discontinuities) with a velocity less than or equal to $c$.

We note that the peak superluminal propagation of a light pulse in a dielectric medium, when the spectrum of the incident pulse is in the anomalous dispersion frequency range of the medium, can be derived from Maxwell’s electromagnetic theory. The superluminal propagation is therefore a classical phenomenon and the question whether the information can be transmitted with a velocity faster than $c$ needs to be addressed classically. Any practical pulse must have a beginning (a starting point of a non-equilibrium process)\footnote{\textsuperscript{1}, \textsuperscript{11}, \textsuperscript{12}}. Furthermore, any nonanalytical disturbance (a discontinuity of the field or its first or higher order derivatives)\footnote{\textsuperscript{4}} carries classical information.

In this letter, we prove, based only on Kramers-Kronig relations and the Maxwell equations, that any nonanalytical disturbance (any discontinuity including the front) in pulses propagates at the phase velocity in a linear medium. It should be emphasized that our general and rigorous proof has no requirement on the form of incident field and the particularity of the medium.

We consider the propagation of an electromagnetic pulse through a medium occupying the space from $z = 0$ to an arbitrary $z > 0$ whose response to the electric field of the light pulse is linear. After passing through the medium, the light field at the position $z$ and at time $t$ can be written as

$$E_m(z,t) = \int_{-\infty}^{+\infty} dt_1 E_0(0,t_1) G(t_1 - t + z/c; z),$$

where

$$G(\xi;z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{i\omega\xi} e^{i\omega\sigma(\omega)z/c}$$

has the property of the retarded Green function. Here $E_0(0,t_1)$ is the input field and $\sigma(\omega) = n(\omega) - 1 + i\kappa(\omega)$ with $n(\omega)$ and $\kappa(\omega)$ being the real and imaginary parts of the complex refractive index, respectively. Since $n(\omega)$ and $\kappa(\omega)$ satisfy the Kramers-Kronig relations,\footnote{\textsuperscript{12}} it can be shown that $\sigma(\omega)$ is an analytic function in the upper half plane and consequently $\sigma(\omega)$ can be expanded into the form: $\sigma(\omega) = \sum_{n=1}^{\infty} A_n/\omega^n$ for $|\omega| \to \infty$, where $A_n$ are the expansion coefficients.

For the pulse propagation through vacuum, $\sigma(\omega) = 0$. Therefore the output field at $z$ is $E_n(z,t) = E_0(0,t - z/c)$, i.e., the light pulse propagates with velocity $c$. 

For the pulse propagation through a medium, the function $G(\xi; z)$ as given by Eq. (3) can be rewritten as

$$G(\xi; z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{i\omega \xi} \left\{ e^{i\omega \sigma(\omega)/c} - e^{i\frac{\Delta_1}{c}} e^{i\omega \sigma(\omega)/c} + e^{i\frac{\Delta_1}{c}} \right\}$$

$$= e^{i\frac{\Delta_1}{c}} \delta(\xi) + e^{i\frac{\Delta_1}{c}} J(\xi; z)$$

(4)

where

$$J(\xi; z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{i\omega \xi} \left[ e^{-i\Delta_1} e^{i\omega \sigma(\omega)/c} - 1 \right].$$

(5)

We then have

$$E_m(z, t) = e^{i\Delta_1} E_0(z, t - z/c) + e^{i\Delta_1} \tilde{E}_m(z, t),$$

(6)

where

$$\tilde{E}_m(z, t) = \int_{-\infty}^{t-z/c} dt_1 E_0(0, t_1) J(t_1 - t + z/c; z).$$

(7)

From Eq. (6), we see that the output field has two parts: the first part is an instant response which leads to a time delay $(z/c)$ for the discontinuities in the field, and the second part is the retarded response from the medium. In the following, we will prove that the second term is a continuous function. The discontinuity in the field is therefore determined by the first term, thus proving that the discontinuity always propagates with the phase velocity.

As the integral function in Eq. (3) is analytic in the upper half plane, $J(\xi; z)$ is equal to 0 for $\xi > 0$ since the integral function $\left[ e^{-i\frac{\Delta_1}{c}} e^{i\omega \sigma(\omega)/c} - 1 \right] \to 0$ when $|\omega| \to \infty$.

When $\xi = 0$, we have

$$J(\xi; z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \left[ e^{-i\Delta_1} e^{i\omega \sigma(\omega)/c} - 1 \right]$$

$$= \frac{1}{2\pi} \lim_{R \to \infty} \int_C d\omega \left[ e^{-i\Delta_1} e^{i\omega \sigma(\omega)/c} - 1 \right]$$

$$= \frac{1}{2\pi} \lim_{R \to \infty} \int_C d\omega \sum_{n=1}^{\infty} \frac{B_n}{\omega^n}$$

$$= -\frac{i}{2} B_1,$$

(8)

where the integrations in the second and third lines are along the open semicircle $C$ under the condition $R \to \infty$ (see Fig. 1(a)). From the first to second lines, we have used that the integration along the closed path composed of the real axis and the semicircle $C$ (with $R \to \infty$) is zero, because the integral, $\exp[-i(\xi/c + i\omega \sigma(\omega)/c)] - 1$, is analytical. In the third line we expanded $\exp[-i(\xi/c + i\omega \sigma(\omega)/c)] - 1$ for $|\omega| \to \infty$, into $\sum_{n=1}^{\infty} (B_n/\omega^n)$, where the coefficients $\{B_n\}$ are related to the coefficients $\{A_n+1\}$.

![FIG. 1: (a) The integral path of Eq. (3); (b) The integral path of Eq. (4).](image)

For $\xi < 0$, as shown in Fig. 1(b), we have

$$J(\xi; z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{i\omega \xi} \left[ e^{-i\frac{\Delta_1}{c}} e^{i\omega \sigma(\omega)/c} - 1 \right]$$

$$= \frac{1}{2\pi} \oint_{C'} d\omega e^{i\omega \xi} \left[ e^{-i\frac{\Delta_1}{c}} e^{i\omega \sigma(\omega)/c} - 1 \right]$$

$$= -\frac{1}{2\pi} \left[ \oint + \oint + \cdots \right]$$

$$\left[ e^{-i\frac{\Delta_1}{c}} e^{i\omega \sigma(\omega)/c} - 1 \right] e^{i\omega \xi} d\omega$$

$$= -\frac{1}{2\pi} \oint_r \left[ e^{-i\frac{\Delta_1}{c}} e^{i\omega \sigma(\omega)/c} - 1 \right] e^{i\omega \xi} d\omega$$

$$\times e^{i\omega \xi} r e^{i\varphi} d\varphi,$$

(9)

where the step leading to second line is because the function $\left[ e^{-i\frac{\Delta_1}{c}} e^{i\omega \sigma(\omega)/c} - 1 \right]$ tends to zero when $|\omega| \to \infty$; in the second line, the integral is along the closed path $C'$ (see Fig. 1(b)); in the third line the integrals are along the finite numbers of neighborhoods of the isolated singular points and tangent lines, and these integrals are finite; In the fourth line we describe a dashed circle of a radius $r$ which embraces all these finite singularities and tangent lines as shown in Fig. 1(b); and in the fifth line, we let $\omega = r \times e^{i\varphi}$. Therefore, we have the inequality

$$|J(\xi; z)| \leq \frac{1}{2\pi} \int_0^{2\pi} M \times r d\varphi = r \times M$$

where $M$ is the maximum value of the function $\left[ e^{-i\frac{\Delta_1}{c}} e^{i\omega \sigma(\omega)/c} - 1 \right] e^{i\omega \xi} e^{i\varphi}$ on the dashed circle of
radius \( r \). This means that \( J(\xi; z) \) is bounded. Because the function \( J(\xi, z) \) is bounded, the second term in Eq. (10) is continuous. A discontinuity in the input field amplitude at time \( t_a \) will occur in the output field after passing through the medium only at the time equal to \( t_a + z/c \). This is the arrival time of information carried by the discontinuity after passing through the medium, which propagates with the phase velocity \( c \).

The intensity discontinuity might disappear under some specific cases due to the second term. The reason is that although \(|a|^2 - |b|^2 \neq 0\), \(|a|^2 + |b|^2 \) may be equal to zero for complex numbers \( a, b, c \). Once such a discontinuity disappears, there is no intensity discontinuity at any time, and the information is lost. It should be pointed out that there is no requirement for the form of the incident field in the above proof and particularity of the medium except the Kramers-Kronig relations.

Next we consider the propagation of a nonanalytic disturbance in the derivatives of the field amplitude. Suppose the 0th to the \((n - 1)\)th order derivatives of the field are continuous functions, while the \(n\)th derivative has a discontinuity. How does this kind of nonanalytical disturbance propagate? From Eq. (4), we obtain

\[
\frac{\partial^n E_m(z, t)}{\partial t^n} = e^{i\frac{z_A}{c}} \frac{\partial^n E_0(z, t - z/c)}{\partial t^n} + e^{i\frac{z_A}{c}} \frac{\partial^n E_m(z, t)}{\partial t^n}.
\]

The second term is

\[
\frac{\partial^n E_m(z, t)}{\partial t^n} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 E_0(0, t_1) \int_{-\infty}^{\infty} d\omega (-i\omega)^n e^{i\omega\xi} \left[ e^{-i\frac{z_A}{c}} e^{i\omega(\omega z/c - 1)} \right]
\]

where the integral \( \int_{-\infty}^{\infty} d\omega (-i\omega)^n e^{i\omega\xi} \left[ e^{-i\frac{z_A}{c}} e^{i\omega(\omega z/c - 1)} \right] \) is equal to \((-i)^n \int_{-\infty}^{\infty} d\omega e^{i\omega\xi} (\omega \cdots (\omega(e^{-i\frac{z_A}{c}} e^{i\omega(\omega z/c - 1)} - 1) - B_1 \cdots - B_n) + 2\pi \sum_{j=1}^{n} (-i)^j B_j \frac{\partial^{(n-j)} E_0(z, t - z/c)}{\partial t^{(n-j)}} \). Therefore, we finally obtain

\[
\frac{\partial^n E_m(z, t)}{\partial t^n} = e^{i\frac{z_A}{c}} \sum_{j=0}^{n} (-i)^j B_j \frac{\partial^{(n-j)} E_0(z, t - z/c)}{\partial t^{(n-j)}} \]

\[
+ e^{i\frac{z_A}{c}} \frac{1}{2\pi} \int_{-\infty}^{\infty} dt_1 E(0, t_1) J_n'(t_1 - t + z/c, z).
\]

We can conclude that the discontinuity in the \(n\)th order derivative propagates with the phase velocity \( c \). This discontinuity might also be washed out under certain cases. Once this discontinuity disappears, there is no discontinuity at any other time (the information is lost). Any initial discontinuity in the \(n\)th order derivative (including the 0-th order) never leads to any new nonanalytical disturbances during the propagation. Therefore there is one to one correspondence between the disturbances (information) in the output field and the input field.

As a numerical example, the propagation time of the discontinuities in the first order derivative are calculated, see Fig. 2. There are two nonanalytical disturbances in the first order derivative of the input field at \( t_1 = 0 \) and \( t_2 = 0.2 \mu s \), and other parts of the input field are analytic. After passing through the gain medium of length \( L = 6 cm \), these two discontinuities occur at time \( t_{1,2}' = t_{1,2} + L/c \) given by the phase velocity, but not by the group velocity. Please note that the initial intensity is a parabolic line in the interval \((0, 0.2)\mu s\). The information carried by the discontinuities propagate at the phase velocity \( c \).

In the second example, we calculate the propagation of the nonanalytical disturbances in the second derivative (see Fig.3). There are two such discontinuities at time \( t_1 = -4.8 \mu s \) and \( t_2 = 4.8 \mu s \) (see Fig.3). The analytical disturbances in the 2nd order derivative of the output field after passing through the gain medium arrive at \( t_{1,2}' = t_{1,2} + L/c \) (same as passing through vacuum). That
The intensities and their 2nd order derivatives of the output field after passing through the gain medium (solid lines) and the same distance vacuum (dashed lines). The input field is $E(0, t) = A[1 + \cos(\pi t/2.4)]$ when $-7.2 < t < -4.8$ and $4.8 < t < 7.2$, $E(0, t) = A[1.5+0.5 \cos(\pi t/2.4)]$ when $-4.8 < t < 4.8$, and otherwise $E(0, t) = 0$. The parameter of the medium is the same as WKD’s experiment [8].

In the two examples of numerical calculation, we used a gain medium of the type considered in the experiment [8] with anomalous dispersion. Here we would like to note that the Kramers-Kronig relations is applicable to the gain medium. To our knowledge there is no dielectric medium that do not obey the Kramers-Kronig relations.

For a practical medium, the background refraction index of the medium always exists. In this case, we can write $n(\omega) = n_0 + \tilde{n}(\omega)$ ( $n_0$ is the background refractive index and is larger than unity, and $\tilde{n}(\omega) \to 0$ as $|\omega| \to \infty$). In this case, we can again show that the propagation of any nonanalytical disturbance is at the speed of $c/n_0$ (the phase velocity).

We have proved that the information carried by the nonanalytical disturbances in the amplitude or in its any order derivatives (including the front of a pulse) propagates with the phase velocity, but not the group velocity. Our proof is based only on Maxwell’s electromagnetic theory and the Kramers-Kronig relations without any condition on whether the medium is normal or anomalous. We believe this is why Einstein only considered the phase velocity in his relativity theory.

The authors gratefully acknowledge the support from RGC and CRGC from Hong Kong Government, National Science Foundation of China, and the Air Force Research Laboratories (Rome, New York).
