On the high rank $\pi/3$ and $2\pi/3$-congruent number elliptic curves

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Abstract

In this article, we try to find high rank elliptic curves in the family $E_{n,\theta}$ defined over $\mathbb{Q}$ by the equation $y^2 = x^3 + 2snx - (r^2 - s^2)n^2x$, where $0 < \theta < \pi$, $\cos(\theta) = s/r$ is rational with $0 \leq |s| < r$ and $\gcd(r, s) = 1$. These elliptic curves are related to the $\theta$-congruent number problem as a generalization of the classical congruent number problem. We consider two special cases $\theta = \pi/3$ and $\theta = 2\pi/3$. Then by searching in a certain known family of $\theta$-congruent numbers and using Mestre-Nagao sum as a sieving tool, we find some square free integers $n$ such that $E_{n,\theta}(\mathbb{Q})$ has Mordell-Weil rank up to 7 in the first case and 6 in the second case.

1 Introduction

Constructing high rank elliptic curves is one of the major problems concerned the elliptic curves. Dujella [6] collected a list of known high rank elliptic curves with prescribed torsion groups. The largest known rank, found by Elkies [9] in 2006, is 28. Several authors studied this problem for elliptic curves with certain properties. For instance, we cite [6, 16] for the curves with given torsion groups, [11, 21] for the curves $x^3 + y^3 = k$ related to the so-called taxicab problem, [7] for the curves $y^2 = (ax + 1)(bx + 1)(cx + 1)(dx + 1)$ induced by Diophantine quadruples $\{a, b, c, d\}$, [14] for the curves $y^2 = x^3 + dx$, [8, 20] for the classical congruent number elliptic curves $y^2 = x^3 - n^2x$.

In this paper we treat with special cases of a family of elliptic curves which are closely related to the $\theta$-congruent numbers as an extension of the classical congruent numbers. Let $0 < \theta < \pi$ and $\cos(\theta) = s/r$ be a rational number with $0 \leq |s| < r$ and $\gcd(r, s) = 1$. A positive integer $n$ is called a $\theta$-congruent number if there exists a triangle with rational sides and area equal to $na_\theta$, where $a_\theta = \sqrt{r^2 - s^2}$. Note that for $\theta = \pi/2$, a $\theta$-congruent number is the ordinary congruent number. It is easy to see that if a positive integer $n$ is $\theta$-congruent, then so is $nt^2$, for any positive integer $t$. Throughout this paper, we assume $n$ is a square free positive integer and concentrate on finding $\theta$-congruent number.
elliptic curves with high Mordell-Weil rank for two special cases \( \theta = \pi/3 \) and \( 2\pi/3 \).

In Section 2, we recall some known results about \( \theta \)-congruent number elliptic curves; in particular, a criterion for a square free positive integer to be \( \theta \)-congruent number, a result on which our work hinges. In Section 3, we describe briefly the Mestre-Nagao sum and Birch and Swinnerton-Dyer conjecture on any elliptic curves defined on \( \mathbb{Q} \). In section 4, we describe our strategy for searching the high rank \( \theta \)-congruent elliptic curves in two cases \( \theta = \pi/3 \) and \( \theta = 2\pi/3 \) and then collect the main results of our works, which includes elliptic curves \( E_{n,\theta} \) with high Mordell-Weil (algebraic) rank \( r_{\theta}(n) \) in these cases. By an analytic methods, Yoshida \[24\] proved that \( r_{\pi/3}(6) = 1, r_{39}(\pi/3)(39) = 2 \) and \( r_{2\pi/3}(5) = 1, r_{2\pi/3}(14) = 2 \). These integers, indeed, are the smallest ones by moderate Mordell-Weil rank. Our searching leads to finding square free integers \( n \) such that \( 3 \leq r_{\pi/3}(n) \leq 7 \) and \( 3 \leq r_{2\pi/3}(n) \leq 6 \).

In our computations we use the Pari/Gp software \[2\], William Stein’s SAGE software \[27\] and Cremona’s MWrank program \[4\], which use the method of descent via 2-isogeny for computing the Mordell-Weil rank of the elliptic curves.

2 \( \theta \)-congruent numbers elliptic curves

The problem of determining \( \theta \)-congruent numbers is related to the problem of finding a non-2-torsion points on the family of elliptic curves

\[ E_{n,\theta} : y^2 = x^3 + 2snx - (r^2 - s^2)n^2x, \]

called \( \theta \)-congruent number elliptic curves, where \( r \) and \( s \) are as in the previous section. This family introduced and studied by Fujiwara \[11\], for the first time, and some authors in various point of views. For any \( n \) and \( \theta \) with \( 0 < \theta < \pi \), let \( E_{n,\theta}(\mathbb{Q}) \) be the group of rational points on \( E_{n,\theta} \). Fujiwara \[12\] studied the torsion groups of the curves \( E_{n,\theta} \). Hibinio and Kan \[13\], using a criterion of Birch, considering modular parameterizations, and studying Heegner points on some modular curves, constructed some families of prime \( \pi/3 \) and \( 2\pi/3 \)-congruent numbers. The most important results on \( E_{n,\theta} \) was proved by Yoshida \[24, 25, 26\]. In \[24\], he constructed new families of \( \pi/3 \) and \( 2\pi/3 \)-congruent numbers using 2-descent methods, Heegner points, and Waldesporger’s results on modular forms of half-integral weight. He also conjectured that:

1) \( n \) is \( \pi/3 \)-congruent number if \( n \equiv 6, 10, 11, 13, 17, 18, 21, 22 \) or \( 23 \) (mod 24);
2) \( n \) is \( 2\pi/3 \)-congruent number if \( n \equiv 5, 9, 10, 15, 17, 19, 21, 22 \) or \( 23 \) (mod 24).

Using ternary quadratic forms, Yoshida \[24\] proved a theorem analogous to the Tunnell’s theorem \[28\] for the classical \( \pi/2 \)-congruent number problem. He also constructed new families of \( \pi/3 \) and \( 2\pi/3 \)-congruent numbers with two and three prime factors.

The curve \( E_{n,\pi/2} \) is the well known congruent number elliptic curve defined by \( y^2 = x^3 - n^2x \). Finding high rank curves in this family is due to Rogers \[20, 21\] and co-work of the present authors with Dujella \[8\] in which reference,
there is a list of congruent number elliptic curves with \( r_{\pi/2}^2(n) \leq 7 \). In particular, it is shown that the integers \( n = 5, 34, 1254, 29297, 48272239 \), are the smallest \( n \) with \( r_{\pi/2}^2(n) = 1, 2, 3, 4, 5 \), respectively. The smallest known integer \( n \) with \( r_{\pi/2}^2(n) = 6 \) is \( n = 661719866 \), however, its minimality is not proved yet. The largest known value for \( r_{\pi/2}^2(n) \) is 7 with \( n = 797507543735 \), which is found by Rogers [21]. There is no other known congruent number \( n \) for which the Mordell-Weil rank of \( E_{n,\pi/2} \) is equal to 7.

It is known [15] that \( n \) is a congruent number if and only if \( r_{\pi/2}^2(n) > 0 \) for the congruent number elliptic curve \( E_{n,\pi/2} \). A similar result holds for \( \theta \)-congruent numbers.

**Theorem 1.** (Fujiwara [11]) Let \( n \) be any square free positive integer and consider the elliptic curve \( E_{n,\theta} \) as above. Then we have:

(i) \( n \) is a \( \theta \)-congruent number if and only if there exists a non-2-torsion point in \( E_{n,\theta}(\mathbb{Q}) \);

(ii) If \( n \neq 1, 2, 3, 6 \), then \( n \) is a \( \theta \)-congruent number if and only if \( r_{\theta}^2(n) > 0 \).

Kan [14] proved the following result which gives a family of \( \theta \)-congruent numbers. This result is an efficient tool in our work.

**Lemma 2.** A square free positive integer \( n \) is a \( \theta \)-congruent number if and only if \( n \) is the square free part of

\[
pq(p + q)(2rq + p(r - s)),
\]

for some positive integers \( p, q \) with gcd\((p, q) = 1\).

### 3 Mestre-Nagao sum and analytic rank

We recall the Mestre-Nagao sum [17, 18, 19] for elliptic curves. Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and \( p \) be any prime. There is both theoretical and experimental evidence to suggest that elliptic curves of high ranks have the property that \( N_p \), the number of elements in \( E(\mathbb{F}_p) \), is large for finitely many primes \( p \).

Let \( N \) be a positive integer and let \( P_N \) be the set of all primes less than \( N \). Mestre-Nagao sum is defined by

\[
S(N, E) = \sum_{p \in P_N} \left(1 - \frac{p-1}{N_p}\right) \log p = \sum_{p \in P_N} \frac{-a_p + 2}{N_p} \log p,
\]

which can be computed for any elliptic curve. It is experimentally known [18, 19] to expect that high rank curves have large values \( S(N, E) \). We cite [3] for a heuristic argument which links this concept to the famous Birch and Swinnerton-Dyer conjecture which is simply stated as follows.
Conjecture 3. Let $E$ be an elliptic curve over $\mathbb{Q}$. Let $L(E, s)$ be the Hasse-Weil $L$-function of $E$ and denote by $r^g$ the Mordell-Weil rank of $E(\mathbb{Q})$. Then the Taylor expansion of $L(E, s)$ about $s = 1$ has the form

$$L(E, s) = c(s - 1)^{r^a} + \text{higher order terms},$$

with $c \neq 0$ and $r^a = r^g$.

The integer $r^a$ is called the analytic rank of elliptic curve $E$, which is the order of $L(E_{n, \theta}, s)$ at $s = 1$. For an elliptic curve $E_{n, \theta}$, denote $r^a$ by $r^a_{n}(n)$. There are some algorithms [5] to compute the analytic rank of elliptic curves. In SAGE software [27], there are three functions to compute the analytic rank of elliptic curves with small coefficients. We shall use the following function of SAGE in our computations:

$$\text{lcalc.analytic_rank}(E)$$

4 Our searching strategy and the main results

Now we attempt to find high rank elliptic curves $E_{n, \theta}$ when $\theta = \pi/3$ and $2\pi/3$. We divide our attempting into two steps depending on the range of the square free positive integers $n$.

**Step (I) $n \leq 5 \times 10^6$.** First of all, using the $s$-option of MWrank program, we compute $s_{\theta}(n)$, the Selmer rank of $E_{n, \theta}$ for all 3039633 square free positive integers in this range. It is easily checked that $r^a_{\theta}(n) \leq s_{\theta}(n)$. For more details on Selmer groups of elliptic curves and their ranks we cite [23]. Table 1 distributes these square free integers through the various values of $s_{\theta}(n)$ in two cases $\theta = \pi/3$ and $2\pi/3$. Using MWrank and considering the Birch and Swinnerton-Dyer conjecture, we find the smallest $n$’s with $r^a_{\pi/3}(n) = 3, 4, 5$ and $r^a_{2\pi/3}(n) = 3, 4$.

| $s_{\theta}(n)$ | 0   | 1   | 2   | 3   | 4   | $\geq 6$ | Total          |
|-----------------|-----|-----|-----|-----|-----|---------|----------------|
| $\theta = \pi/3$ | 783043 | 1401045 | 734390 | 166158 | 5045 | 52       | 3039633        |
| $\theta = 2\pi/3$ | 760511 | 1374165 | 751192 | 144641 | 9038 | 86       | 3039633        |

Table 1: Distribution of square free integers less than $5 \times 10^6$ through the various values of $s_{\theta}(n)$ in two cases $\theta = \pi/3$ and $2\pi/3$.

**Step (II) $n > 5 \times 10^6$.** In this step, we search for $n$’s with $r^a_{\pi/3}(n) \geq 6$ and $r^a_{2\pi/3}(n) \geq 5$. We consider all different square free $\theta$-congruent numbers $n$ of the form (11) in Lemma 2 where positive integers $p$ and $q$ satisfy in the following conditions:

$$1 < p, q \leq 10^4, \quad \gcd(p, q) = 1, \quad w(n) \geq 4,$$

where $w(n)$ is the number of odd prime factors of $n$. Then we get a list of different $n$’s with more than $7 \times 10^6$ elements for each of the cases $\theta = \pi/3$ and
\( \theta = 2\pi/3 \). Applying to Mestre-Nagao sum and using the \( s \)-option of MWrank, we reduce the length of this list. In fact, we choose \( n \)'s for which

\[
S(10^3, E_{n, \theta}) > 15, \quad S(10^4, E_{n, \theta}) > 20, \quad S(10^5, E_{n, \theta}) > 40,
\]

where \( s_{\pi/3}(n) > 5 \), and \( s_{2\pi/3}(n) > 4 \). These computations are done by the Pari/Gp software [2]. After computing the value of \( r_{\theta}^g(n) \) by MWrank for these candidates, we can find \( n \)'s with \( r_{\pi/3}^g(n) = 6,7 \) for \( \theta = \pi/3 \) and \( n \)'s with \( r_{2\pi/3}^g(n) = 5,6 \) for \( \theta = 2\pi/3 \).

In the following subsections, we collect all \( n \)'s with \( 3 \leq r_{\pi/3}^g(n) \leq 7 \) and \( 3 \leq r_{2\pi/3}^g(n) \leq 6 \). In each case, using MWrank, we find a minimal generating set for the Mordell-Weil groups. To improve the generators, we used the LLL-algorithm to find those generators with smaller heights.

4.1 The case \( \theta = \pi/3 \)

**Rank 3**: The integers 407 and 646 are the two smallest integers among 116158 integers \( n \) less than \( 5 \times 10^6 \) with \( s_{\pi/3}(n) = 3 \). We have \( r_{\pi/3}^g(407) = r_{\pi/3}^a(407) = 1 \), however, for \( n = 646 \) these ranks are both 3 and the generators of \( E_{646, \pi/3} : y^2 = x^3 + 1292x^2 - 1251948x \) are:

\[
P_1 = [-722, 34656], \quad P_2 = [6137, 521645], \quad P_3 = [-1216, 40432].
\]

**Rank 4**: There are 63 integers \( n \) less than 172081 with \( s_{\pi/3}(n) = 4 \). For 29 cases we have \( 0 \leq r_{\pi/3}^g(n) \leq 4 \) and the others satisfy \( 2 \leq r_{\pi/3}^g(n) \leq 4 \). Using SAGE, one can find that \( r_{\pi/3}^a(n) = 0 \) for the former 29 integers, and \( r_{\pi/3}^a(n) = 2 \) for the latter group. So by assuming Birch and Swinnerton-Dyer Conjecture, the smallest positive integer with \( r_{\pi/3}^g(n) = 4 \) is 172081 whose related curve

\[
E_{172081, \pi/3} : y^2 = x^3 + 443492x^2 - 88835611683x
\]

has the generators:

\[
P_1 = [-608141, -61627202], \quad P_2 = [-58621, -78669382],
\]

\[
P_3 = [-440076, -143244738], \quad P_4 = [224175, 92987790].
\]

**Rank 5**: An easy computation shows that 221746 is the smallest among 52 integers \( n \) with \( s_{\pi/3}(n) = 5 \). By MWrank, one can see that \( r_{\pi/3}^g(221746) = 5 \), and the related elliptic curve

\[
E_{221746, \pi/3} : y^2 = x^3 + 443492x^2 - 147513865548x
\]

has the following generators:
P1 = \([345450, 207822720]\),
P2 = \([-15792, 49357896]\),
P3 = \([994896, 113003604]\),
P4 = \([-13254, -45063600]\),
P5 = \([-386575, -255989965]\).

**Rank 6:** By part (II) of our searching technique, we can get finitely many \(n\) with \(s_{\pi/3}(n) = 6\) and \(n > 5 \times 10^6\). Using MWrank, we can find nine \(n\)'s with \(r^g_{\pi/3}(n) = 6\) the smallest of which is \(n = 11229594411\) and the related curve is of the form

\[E_{11229594411, \pi/3} : y^2 = x^3 + 22459188822x^2 - 378311371906687310763x\]

whose generators are:

P1 = \([904103532759/25, -992069570757491352/125]\),
P2 = \([154173188897/16, 2090318638263775025/64]\),
P3 = \([265444083202036/2025, 463687440736982658134/91125]\),
P4 = \([719501508201/64, 40873417425022581/512]\),
P5 = \([13006760076899761/269361, 169318158533140000267498/139798359]\),
P6 = \([50286669020153449/278784, 1189609671289659453790795/147197952]\).

Note that there are also some \(n\)’s (even smaller than 11229594411) with \(s_{\pi/3}(n) = 6\), however, MWrank cannot give the exact values of \(r^g_{\pi/3}(n)\). The other 8 square free numbers are as:

\[167514827545, 198606002595, 2713148227665, 3302971161265, 3492293850595, 6634009064865, 4058213000419, 45563303263450.\]

**Rank 7:** We can find only one \(n\) with \(r^g_{\pi/3}(n) = 7\). This is \(n = 36580346586\) and the corresponding curve is

\[E_{36580346586, \pi/3} : y^2 = x^3 + 731606929172x^2 - 40143652410936286454188x\]

with the generators:

P1 = \([433764757524, 2124566767469092628]\),
P2 = \([191274050073, -1698645579158165609]\),
P3 = \([-933533874904423/2025, 3644281, -570541658990431976790514695/6956932429]\),
P4 = \([1994920524369/4, -4277466996084516865/8]\),
P5 = \([21388826856027602/6239, 568939954835494296233212/389017]\),
P6 = \([78676918101433554, 80088, 21982407380563692008852160/22665187]\),
P7 = \([-562236028164373765342/540237049, 361716521043536625559445197360/12556729729907]\).

Also we can find three integers \(n = 2185135410173, 27441232583014\) and 1892439367910454 with \(s_{\pi/3}(n) = 7\) while, using MWrank gives only the bound \(1 \leq r^g_{\pi/3}(n) \leq 7\) for all of them.

**4.2 The case \(\theta = 2\pi/3\)**

**Rank 3:** There is no any positive square free integer less than \(n = 221\) for which \(r^g_{2\pi/3}(n) = r^a_{\pi/3}(n) = s_{2\pi/3}(n) = 3\). So, we get the curve

\[E_{221,2\pi/3} : y^2 = x^3 - 442x^2 - 146523x\]

with the generators:
$P_1 = [-204, 1734],
P_2 = [-169, 2704],
P_3 = [4131, -249696]$.  

**Rank 4**: The smallest $n$ with $r_{2\pi/3}(n) = r_{\pi/3}(n) = s_{2\pi/3}(n) = 4$ is 12710. There are only two integers, $n = 4718$ and 6398, less than 12710 with $s_{2\pi/3}(n) = 4$, but for these integers we have $r_{2\pi/3}(n) = r_{\pi/3}(n) = 0$. Hence we have the curve  

$$E_{12710, 2\pi/3} : y^2 = x^3 - 25420x^2 - 484632300x$$

with the generators:  

$P_1 = [-310, 384400],
P_2 = [-9920, -1153200],
P_3 = [48050, 5381600],
P_4 = [76880, 16337000]$.  

**Rank 5**: By part (II), we get finitely many $n$’s with $r_{2\pi/3}(n) = 5$ and $n > 5 \times 10^6$, the smallest of which is $n = 16470069$. The corresponding curve  

$$E_{16470069, 2\pi/3} : y^2 = x^3 - 32940138x^2 - 813789518594283x$$

has the generators:  

$P_1 = [-3115959/4, -198146948769/8],
P_2 = [-16255958103/1024, -813789518594283/32768],
P_3 = [118172745075/1849, -21701053829180880/79507],
P_4 = [174895662711/3481, -10850525914590440/205379],
P_5 = [18013358979/361, -27582052686448/6859]$.  

Note that there are finitely many $n$’s less than 16470069 with $s_{2\pi/3}(n) = 5$, but MWrank can not calculate the exact values of $r_{2\pi/3}(n)$.  

**Rank 6**: We found 29 positive integers $n$ with $r_{2\pi/3}(n) = 6$ such that $n = 4562490669$ is the smallest of them, which gives the curve  

$$E_{4562490669, 2\pi/3} : y^2 = x^3 - 912498132x^2 - 624489630677617068x$$

with the generators:  

$P_1 = [1372171206, 29309567690616],
P_2 = [24303608784, 3714988879700280],
P_3 = [1677715326, -32359028622624],
P_4 = [3653049873, -183588193835865],
P_5 = [27273656667348/2571353, 18789, 39342486732689875284/2571353],
P_6 = [36967427406/25, 2217080599939296/125]$.  

The other 28 square free numbers are as:  

456249066, 764046170, 902472906, 5062245006, 9667090290, 11801899970, 1996987310, 2024772006, 23819590518, 24080567966, 30834423438, 30960775454, 5818159120,  

64256704710, 9870770590, 106366908126, 148280772900, 181684390314, 292826163690, 309090045354, 33515184000, 685374515826, 713465075246, 685374515826, 713465075246, 860842004286, 1185986591790, 1248260820170, 1185986591790, 1248260820170.  

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Note that we can find two $n$’s with $s_9(n) = 7$, but by MWrank one can see that $1 \leq r_{2\pi/3}^9(n) \leq 7$. These integers are $n = 162552566$ and $45010115083565$. Also, for $n = 2118002187593054$, we have $s_9(n) = 8$ but MWrank gives only the bound $1 \leq r_{2\pi/3}^9(n) \leq 8$.

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