Spectrum of the harmonic oscillator in a general noncommutative phase space

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Abstract

Harmonic oscillator, in 2-dimensional noncommutative phase space with non-vanishing momentum-momentum commutators, is studied using an algebraic approach. The corresponding eigenvalue problem is solved and discussed.

Keywords: spectrum, harmonic oscillator, noncommutative phase space.

In recent years, there is an increasing interest in the application of noncommutative (NC) geometry to physical problems \cite{1} in solid-state and particle physics \cite{2}, mainly motivated by the idea of a strong connection of noncommutativity with field and string theories. Besides, the evidence coming from the latter and other approaches to the issues of quantum gravity suggests that attempts to unify gravity and quantum mechanics could ultimately lead to a non-commutative geometry of spacetime. The phase space of ordinary quantum mechanics is a well-known example of noncommuting space \cite{3}. The momenta of a system in the presence of a magnetic field are noncommuting operators as well. Since the noncommutativity between spatial and time coordinates may lead to some problems with unitarity and causality, usually only spatial noncommutativity is considered. Besides, so far quantum theory on the NC space has been extensively studied, the main approach is based on the Weyl-Moyal correspondence which amounts to replacing the usual product by a $\star$-product in the NC space. Therefore, deformation quantization has special significance in the study of physical systems on the NC space. Moreover, the problem of quantum mechanics on NC spaces can be understood in the framework of deformation quantization \cite{5}-\cite{8}. In the same vein, some works on harmonic oscillators (ho) in the NC space from the point of view of deformation quantization have been reported in \cite{9}-\cite{16} and references therein. In this paper, we consider different representations...
of a harmonic oscillator in a general full noncommutative phase space with both the spatial and momentum coordinates being noncommutative. Indeed, noncommutativity between momenta arises naturally as a consequence of noncommutativity between coordinates, as momenta are defined to be the partial derivatives of the action with respect to the noncommutative coordinates. This work continues the investigations pursued in [9], [10] and [11] devoted to the study of a quantum exactly solvable $D$-dimensional NC oscillator with quasi-harmonic behavior. We intend to extend previous results presenting a similar analysis to the quantum version of the two-dimensional NC system with non-vanishing momentum components. For additional details in the motivation, see [9]. The physical model resembles to the Landau problem in NC quantum mechanics extensively studied in the literature. See [12] and [13] and references therein for more details. Broadly put, it is worth noticing that the description of a system of a two-dimensional harmonic oscillator in a full NC phase space is equivalent to that of the model of a two-dimensional ho in a constant magnetic field in some NC space.

Consider a 2D general NC phase space. The coordinates of position and momentum, $x = (x^1, x^2)$ and $p = (p^1, p^2)$, modeling the classical system of a two-dimensional ho maps into their respective quantum operators $\hat{x}$ and $\hat{p}$ giving rise to the Hamiltonian operator 

$$\hat{H} = \frac{1}{2} \left( \hat{p}_\mu \hat{p}^\mu + \hat{x}_\mu \hat{x}^\mu \right)$$ (1)

with commutation relations

$$[\hat{x}^\mu, \hat{p}^\nu] = i\hbar \delta^{\mu\nu}, \quad [\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}, \quad [\hat{p}^\mu, \hat{p}^\nu] = i\bar{\Theta}^{\mu\nu}, \quad \mu, \nu = 1, 2$$ (2)

where $\Theta^{\mu\nu}$ and $\bar{\Theta}^{\mu\nu}$ are skew-symmetric tensors carrying the dimensions of (length)$^2$ and (momentum)$^2$, respectively. The effective Planck constant is given by

$$\hbar_{eff} = h \left( 1 + \frac{\Theta^{\mu\nu} \bar{\Theta}^{\mu\nu}}{4D\hbar^2} \right),$$ (3)

where $D = 2$ is the dimension of the NC space. One can readily check that one can rewrite the operators $\hat{x}^\mu$ and $\hat{p}^\nu$ as

$$\hat{p}^\mu = \pi^\mu + \frac{1}{2\hbar} \bar{\Theta}^{\mu\nu} \pi^\nu, \quad \hat{x}^\mu = \hat{q}^\mu - \frac{1}{2\hbar} \Theta^{\mu\nu} \pi^\nu$$ (4)

in terms of $\pi^\mu$ and $\hat{q}^\nu$ that obey the standard Weyl-Heisenberg algebra

$$[\hat{q}^\mu, \pi^\nu] = i\hbar \delta^{\mu\nu}; \quad [\hat{q}^\mu, \hat{q}^\nu] = 0 = [\pi^\mu, \pi^\nu].$$ (5)
In the deformation quantization theory of a classical system in the noncommutative space, one treats \((x, p)\) and their functions as classical quantities, but replaces the ordinary product between these functions by the following generalized \(*\)–product

\[
* = *_{\text{eff}} * \Theta * \bar{\Theta}
\]

\[
* = \exp \left[ \frac{i}{\hbar_{\text{eff}}} \left( \partial_{x} \mu \partial_{p} \mu - \partial_{p} \mu \partial_{x} \mu \right) + \frac{i \Theta_{\mu \nu}}{2} \partial_{x} \mu \partial_{x} \nu + \frac{i \bar{\Theta}_{\mu \nu}}{2} \partial_{p} \mu \partial_{p} \nu \right].
\]  

(6)

The variables \(x^\mu, p^\mu\) on the NC phase space satisfy the following commutation relations similar to (7)

\[
[x^\mu, p^\nu]_* = i \hbar_{\text{eff}} \delta^{\mu \nu}, \quad [x^\mu, x^\nu]_* = i \Theta_{\mu \nu}, \quad [p^\mu, p^\nu]_* = i \bar{\Theta}_{\mu \nu}
\]

\(
\mu, \nu = 1, 2
\)

(7)

with the following uncertainty relations

\[
\Delta x^1 \Delta x^2 \geq \Theta \quad \Delta p^1 \Delta p^2 \geq \bar{\Theta}
\]

\[
\Delta x^1 \Delta p^1 \geq \frac{\hbar_{\text{eff}}}{2} \quad \Delta x^2 \Delta p^2 \geq \frac{\hbar_{\text{eff}}}{2}.
\]

(8)

The first two uncertainty relations show that measurements of positions and momenta in both directions \(x^1\) and \(x^2\) are not independent. Taking into account the fact that \(\Theta\) and \(\bar{\Theta}\) have dimensions of \((\text{length})^2\) and \((\text{momentum})^2\) respectively, then \(\sqrt{\Theta}\) and \(\sqrt{\bar{\Theta}}\) define fundamental scales of length and momentum which characterize the minimum uncertainties possible to achieve in measuring these quantities. One expects these fundamental scales to be related to the scale of the underlying field theory (possible the string scale), and thus to appear as small corrections at the low-energy level or quantum mechanics. Commonly, the time evolution function for a time-independent Hamiltonian \(H\) of a system is denoted by the \(*\)–exponential function denoted here by \(e_{*}^{(\mu)}\):

\[
e_{*}^{(\mu)} := \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{t}{i \hbar_{\text{eff}}} \right)^n H * H * \cdots * H,
\]

(9)

which is the solution of the following time-dependent Schrödinger equation

\[
i \hbar_{\text{eff}} \frac{d}{dx} e_{*}^{(\mu)} = H(x, p) * e_{*}^{(\mu)}
\]

\[
= H \left( x^\mu + \frac{i \hbar_{\text{eff}}}{2} \partial_{p} \mu, \frac{i \Theta_{\mu \nu}}{2} \partial_{x} \mu \partial_{x} \nu, p^\nu - \frac{i \hbar_{\text{eff}}}{2} \partial_{p} \nu \partial_{x} \nu + \frac{i \bar{\Theta}_{\mu \sigma}}{2} \partial_{x} \mu \partial_{x} \sigma \right) e_{*}^{(\mu)}.
\]

(10)

There corresponds the generalized \(*\)–eigenvalue time-independent Schrödinger equation:

\[
H * \mathcal{W}_n = \mathcal{W}_n * H = \mathcal{E}_n \mathcal{W}_n
\]

(11)
where \( \mathcal{W}_n \) and \( \mathcal{E}_n \) stand for the Wigner function and the corresponding energy eigenvalue of the system. The Fourier-Dirichlet expansion for the time-evolution function defined as

\[
\frac{\hbar}{e^{i\hbar t}} = \sum_{n=0}^{\infty} e^{-i\mathcal{E}_n t} \mathcal{W}_n
\]

links the Wigner function to the \(*\)-exponential function. Provided the above, the operators on a NC Hilbert space can be represented by the functions on a NC phase space, where the operator product is replaced by relevant star-product. The algebra of functions of such noncommuting coordinates can be replaced by the algebra of functions on ordinary spacetime, equipped with a NC star-product. So, considering the transformations (4) and leaving out the operator symbol \( \hat{\cdot} \), we arrive at \((q, \pi)\) phase space and the commutation relation change into (5), with the star-product defined in the following way.

**Definition 0.1** Let \( C^\infty(\mathbb{R}^4) \) be the space of smooth functions \( f : \mathbb{R}^4 \to \mathbb{C} \). For \( f, g \in C^\infty(\mathbb{R}^4) \), the formal star product is defined by

\[
f \ast g = f \exp \left[ \frac{i\hbar}{2} \frac{\partial \mu}{\partial q^1} \frac{\partial \nu}{\partial q^2} \right] g.
\]

Here the smooth functions \( f \) and \( g \) depend on the real variables \( q^1, q^2, \pi^1 \) and \( \pi^2 \), and

\[
\frac{\partial}{\partial q^1} \frac{\partial \mu}{\partial \pi^1} \frac{\partial \nu}{\partial \pi^2} = \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right) \left( \begin{array}{c} \frac{\partial \mu}{\partial q^1} \\ \frac{\partial \mu}{\partial q^2} \\ \frac{\partial \mu}{\partial \pi^1} \\ \frac{\partial \mu}{\partial \pi^2} \end{array} \right)
\]

Therefore, the star product \( f \ast g \) represents a deformation of the classical product \( fg \). This deformation depends on the Planck constant \( \hbar \). In term of physics, the difference \( f \ast g - fg \) describes quantum fluctuation depending on \( \hbar \). For the present case,

\[
q^\mu \ast \pi^\nu - q^\mu \pi^\nu = \frac{i\hbar}{2} \delta^{\mu \nu}, \quad \pi^\nu \ast q^\mu - \pi^\nu q^\mu = -\frac{i\hbar}{2} \delta^{\mu \nu}.
\]

Hence

\[
[q^\mu, \pi^\nu]_\ast = i\hbar \delta^{\mu \nu}.
\]

Building now, in the standard manner, the creation and annihilation operators of the system as

\[
a_l = \frac{q^l + i\pi^l}{\sqrt{2}} \quad \tilde{a}_l = \frac{q^l - i\pi^l}{\sqrt{2}} \quad l = 1, 2
\]
and using the polar coordinates such that
\[ q^l = \rho \cos \varphi^l, \quad \pi^l = \rho \sin \varphi^l, \] (17)
we solve the right and left eigenvalue equations
\[
\begin{align*}
& a_l \ast f_{mn} = \sqrt{m \hbar} f_{m-1,n} \\
& \bar{a}_l \ast f_{mn} = \sqrt{(m+1) \hbar} f_{m+1,n} \\
& f_{mn} \ast a_l = \sqrt{(n+1) \hbar} f_{m,n+1} \\
& f_{mn} \ast \bar{a}_l = \sqrt{n \hbar} f_{m,n-1}
\end{align*}
\] (18)
to find the eigenfunctions \( f_{mn} \) as
\[
f_{mn} \equiv 2(-1)^m \sqrt{\frac{m!}{n!}} e^{(n-m)\varphi^l} \left( \frac{2\rho^2}{\hbar} \right)^{\frac{n-m}{2}} L_{m-n}^{n-m} \left( \frac{2\rho^2}{\hbar} \right) e^{-\frac{\rho^2}{2\hbar}}, \quad m, n \in \mathbb{N}
\] (19)
with
\[
f_{00} = 2e^{-\rho^2/\hbar}. \] (20)

\( L_{m-n}^{n-m} \) are the generalized Laguerre polynomials defined for \( n = 0, 1, 2, \ldots, \alpha > 1 \), by
\[
L_{n}^{\alpha}(x) = \frac{1}{n!} e^{x-x^\alpha} \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!}.
\] (21)

Then the states defined by \( b^{(4)}_{mn} = f_{m_1n_1} f_{m_2n_2} \), where \( m = (m_1, m_2), n = (n_1, n_2), m_1, m_2, n_1, n_2 \in \mathbb{N} \), exactly solve the right and left eigenvalue problems of the Hamiltonian \( H_0 = \sum_{l=1}^{2} \bar{a}_l a_l \) as
\[
\begin{align*}
H_0 \ast b^{(4)}_{mn} &= \hbar (|m| + 1) b^{(4)}_{mn} \quad \text{and} \quad b^{(4)}_{mn} \ast H_0 = \hbar (|n| + 1) b^{(4)}_{mn}
\end{align*}
\] (22)
where \( |m| = m_1 + m_2 \).

Now, consider the Hamiltonian (11) and use the relation (5) to re-express it with the help of variables \( q \) and \( \pi \) as follows:
\[
H = H_0 + H_L + H_q(\bar{\Theta}) + H_{\pi}(\Theta)
\] (23)
where
\[
\begin{align*}
H_0 &= \frac{1}{2} \left( (q^1)^2 + (q^2)^2 + (\pi^1)^2 + (\pi^2)^2 \right) \\
H_L &= -\frac{\Theta + \bar{\Theta}}{2\hbar} \bar{q} \wedge \bar{\pi} \quad \bar{q} \wedge \bar{\pi} = q^1 \pi_2 - q^2 \pi_1
\end{align*}
\] (25)
and
\[
\begin{align*}
H_q(\bar{\Theta}) &= \frac{\bar{\Theta}^2}{8\hbar^2} \left( (q^1)^2 + (q^2)^2 \right) \\
H_{\pi}(\Theta) &= \frac{\Theta^2}{8\hbar^2} \left( (\pi^1)^2 + (\pi^2)^2 \right).
\end{align*}
\] (26)
It is a matter of computation to verify that the Hamiltonians $H_0$ and $H_L$ commute. Indeed, for the Hamiltonians $H_L$ and $H_I = H_q(\bar{\Theta}) + H_\pi(\Theta)$. Therefore, the Hamiltonians of family \{H_0, H_L\}, (respectively \{H_L, H_I\}) can be simultaneously measured.

**Proposition 0.1** In the case $\Theta = -\bar{\Theta}$, the Hamiltonian $H$ can be expressed as

$$H = \left(1 + \frac{\Theta^2}{4\hbar^2}\right)H_0$$

and the states $b_{mn}^{(4)}$ solve the right and left eigenvalue problems of $H$ as

$$H \ast b_{mn}^{(4)} = \mathcal{E}_{m0}^{R} b_{mn}^{(4)} \quad \mathcal{E}_{m0}^{R} = \hbar \left(1 + \frac{\Theta^2}{4\hbar^2}\right)(|m| + 1)$$

where $m = (m_1, m_2)$, $n = (n_1, n_2)$ $m_1, m_2, n_1, n_2 \in \mathbb{N}$, $|m| = m_1 + m_2$.

In the general $(q, \pi)$-representation, the problem to solve is equivalent to that of the two-dimensional Landau problem in a symmetric gauge on a noncommutative space. Indeed, the Hamiltonian $H$ can be re-transcribed as

$$H = \frac{\alpha^2}{2}(q_1^2 + q_2^2) + \frac{\beta^2}{2}(\pi_1^2 + \pi_2^2) - \frac{\gamma}{\hbar} \vec{q} \wedge \vec{\pi} =: H_0^2 + H_L$$

where

$$\alpha^2 = 1 + \frac{\Theta^2}{4\hbar^2}, \quad \beta^2 = 1 + \frac{\Theta^2}{4\hbar^2}, \quad \gamma = \frac{\Theta + \bar{\Theta}}{2\hbar}$$

Remark that the Hamiltonian terms $H_0^2$ and $H_L$ commute. Therefore, the eigenvectors of \{H_0^2, H_L\} are automatically eigenvectors of $H$. As matter of convenience, to solve the Schrödinger eigen-equation, let us choose the polar coordinates

$$q_1 = \rho \cos \varphi \quad q_2 = \rho \sin \varphi$$

and assume the variable separability to write

$$\tilde{f}(\rho, \varphi) = \xi(\rho)e^{ik\varphi}, \quad k = 0, \pm 1, \pm 2, \cdots$$

Then, from the static Schrödinger equation on NC space, $H \ast \tilde{f}(\rho, \varphi) = \mathcal{E} \tilde{f}(\rho, \varphi)$, we deduce the radial equation as follows:

$$\left[ - \frac{\hbar^2 \beta^2}{2} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho}\right) + \frac{\alpha^2}{2} \rho^2 - \gamma \hbar k \right] \xi(\rho, \varphi) = \mathcal{E} \xi(\rho, \varphi)$$
yielding the spectrum of $H$ under the form
\[ \mathcal{E} = \hbar \frac{\alpha^2}{\beta^2} (n + 1) - \hbar \gamma k, \quad n = 0, 1, 2, \cdots \] (35)
with
\[ \xi(\rho, \varphi) \propto e^{-\frac{\alpha}{\hbar \beta} \rho^2} H_n \left( \frac{\alpha}{\hbar \beta} \rho^2 \right). \] (36)

The last term of the energy spectrum $\mathcal{E}$ falls down when $\gamma = 0$, i.e. $\Theta = -\bar{\Theta}$. In this case, $\alpha^2 = \beta^2$ and we recover the discrete spectrum of the usual two-dimensional harmonic oscillator as expected. The results obtained here can be reduced to specific expressions reported in the literature [9] for particular cases. Besides, the formalism displayed in this work permits to avoid the appearance of infinite degeneracy of states observed when $\hbar^2_{\text{eff}} - \Theta \bar{\Theta} = 0$ in [12] where the phase space is divided into two phases based on the following conditions on the deformation parameters:

- Phase I for $\hbar^2_{\text{eff}} - \Theta \bar{\Theta} > 0$
- Phase II for $\hbar^2_{\text{eff}} - \Theta \bar{\Theta} < 0$.

Finally, let us mention that the direct computation of the energy spectrum from the relation (23) instead of (30) introduces an unexpected feature, i.e. the energy spectrum depends on the phase space variables as it should not be with respect to the study performed in [13]. Such a pathology is generated by the phase space variable dependence of the commutator
\[ [H_0, H_I]_* = \frac{i}{4\hbar} \Theta^2 - \bar{\Theta}^2 (q^1 \pi^1 + q^2 \pi^2). \] (37)
This could explain why previous investigations (see [9], [14] and [15] and references therein) were restricted to the cases $\Theta = \pm \bar{\Theta}$.

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