A viscous blast-wave model for relativistic heavy-ion collisions

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Using a viscosity-based survival scale for geometrical perturbations formed in the early stages of relativistic heavy-ion collisions, we model the radial flow velocity during freeze-out. Subsequently, we employ the Cooper-Frye freeze-out prescription, with first-order viscous corrections to the distribution function, to obtain the transverse momentum distribution of particle yields and flow harmonics. For initial eccentricities, we use the results of Monte Carlo Glauber model. We fix the blast-wave model parameters by fitting the transverse momentum spectra of identified particles at the Large Hadron Collider (LHC) and demonstrate that this leads to a fairly good agreement with transverse momentum distribution of elliptic and triangular flow for various centralities. Within this viscous blast-wave model, we estimate the shear viscosity to entropy density ratio $\eta/s \simeq 0.24$ at the LHC.

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I. INTRODUCTION

High-energy heavy-ion collision experiments at the Relativistic Heavy Ion Collider (RHIC) [1, 2] and the Large Hadron Collider (LHC) [3–5] have conclusively established the formation of a strongly interacting Quark-Gluon Plasma (QGP). The QGP formed in these collisions exhibit strong collective behaviour, and therefore can be studied within the framework of relativistic hydrodynamics. The hydrodynamical analyses of the flow data suggests that the QGP behaves like a nearly perfect fluid with an extremely small shear viscosity to entropy density ratio $\eta/s$ [6, 7]. Local pressure gradients due to deformations and inhomogeneities in the initial stages of the collision results in anisotropic fluid velocity. These anisotropies subsequently translate into flow harmonics describing the momentum asymmetry of produced particles [8, 9]. The $\eta/s$ of the fluid governs the conversion efficiency, which results in a suppression of elliptic flow and higher order flow coefficients [10–23].

Apart from hydrodynamics, the observables pertaining to collective behaviour of QGP can also be studied using the so-called blast-wave model. Using a simple functional form for the phase-space density at kinetic freezeout, Schnedermann et. al. [24] approximated hydrodynamical results with boost-invariant longitudinal flow [25]. They used this blast-wave model to successfully fit the transverse momentum spectra with only two parameters: a kinetic temperature, and a radial flow strength. However, this model was only valid for central collisions at midrapidity. In order to make it applicable for non-central collisions, Huovinen et. al. [26] generalized this model to account for the anisotropies in the transverse flow profile by introducing an additional parameter. This new parameter controlled the difference between the strength of the flow in and out of the reaction plane. This lead to a fairly good fit with the measured elliptical flow as a function of transverse momentum. However, the STAR Collaboration achieved better fits when they generalized the model even further by introducing a fourth parameter to account for the anisotropic shape of the source in coordinate space [27]. Teaney made the first attempt to estimate the effect of viscosity on elliptic flow using a variant of the blast-wave model model [28]. However, the centrality dependence of the fit parameters left the model with little predictive power.

In this paper we generalize the blast-wave model to include viscous effects by employing a viscosity-based survival scale for geometrical anisotropies, formed in the early stages of relativistic heavy-ion collisions, in the parametrization of the radial flow velocity. The present model has five parameters, including $\eta/s$, which has to be fitted only for one centrality. In the Cooper-Frye freeze-out prescription for particle production, we consider the first-order viscous corrections to the distribution function [28]. In essence, we provide a model which incorporates the important features of viscous hydrodynamic evolution but does not require to do the actual evolution. We use this viscous blast-wave model to obtain the transverse momentum distribution of particle yields and flow harmonics for LHC. The blast-wave model parameters are fixed by fitting the transverse momentum spectra of identified particles. Subsequently, we show that this leads to fairly good agreement with transverse momentum distribution of elliptic and triangular flow for various centralities as well as centrality distribution for integrated flow. We estimate the shear viscosity to entropy density ratio $\eta/s \simeq 0.24$ at the LHC, within the present model.

II. BLAST WAVE MODEL

The blast-wave model has been used extensively to fit experimental data and it provides good description of spectra and elliptic flow observed in relativistic heavy-ion collisions [26–30]. The previously used blast-wave mod-
els employ a simple parametrization for the flow velocity of boost invariant ideal hydrodynamics. The most important feature is the parametrization of the transverse velocity which is assumed to increase linearly with respect to the radius. This parametrization is found to be in agreement with hydro results [31, 32]. This essentially leads to an exponential expansion of the fireball in the transverse direction, hence the term blast-wave. Apart from boost invariance, the model also assumes rotational invariance. In the following, we quickly outline the key features of the blast-wave model.

In order to consider a boost invariant framework, it is easier to work in the Milne co-ordinate system where,

\[
\begin{align*}
\tau &= \sqrt{t^2 - z^2}, \\
\eta_s &= \tanh^{-1}(z/t), \\
r &= \sqrt{x^2 + y^2}, \\
\varphi &= \text{atan2}(y, x).
\end{align*}
\]

The metric tensor for this co-ordinate system is \(g_{\mu\nu} = \text{diag}(1, -\tau^2, -1, -r^2)\). Boost invariance and rotational invariance implies \(u^\varphi = u^\tau = 0\), whereas linearly rising transverse velocity flow profile leads to \(u^r \sim r\). The blast-wave model further assumes that the particle freeze-out happens at a proper time \(\tau_f\) having a constant temperature \(T_f\) and uniform matter distribution, in the transverse plane. In summary, the hydrodynamic fields are parametrized as [28]

\[
\begin{align*}
T &= T_f \Theta(R - r), \\
u^r &= u_0 \frac{r}{R} \Theta(R - r), \\
u^\tau &= u^\tau = 0, \\
u^\varphi &= \sqrt{1 + (u^r)^2},
\end{align*}
\]

where \(R\) is the transverse radius of the fireball at freeze-out. The expression for \(u^\tau\) is obtained by requiring that the fluid four-velocity satisfy the condition \(u^\mu u_\mu = 1\).

The hadron spectra can be obtained using the Cooper-Frye prescription for particle production [33]

\[
\frac{dN}{d^2 p_T dy} = \frac{1}{(2\pi)^3} \int p_\mu d\Sigma^\mu f(x, p),
\]

where \(d\Sigma^\mu\) is the oriented freeze-out hyper-surface and \(f(x, p)\) is the phase-space distribution function of the particles at freeze-out. The distribution function can be written in terms of the equilibrium and non-equilibrium parts, \(f = f_0 + \delta f\). The equilibrium distribution function is given by

\[
f_0 = \frac{1}{\exp(u_\mu p^\mu/T) + a},
\]

where \(a = +1\) for baryons and \(a = -1\) for mesons.

For small deviations from equilibrium, i.e., \(\delta f \ll f_0\), we use the Grad’s 14-moment approximation for the non-equilibrium part [34, 35]

\[
\delta f = \frac{f_0 \delta f_0}{2(\epsilon + P)/T^2} p^\alpha p^\beta \pi_{\alpha\beta},
\]

where \(\delta f_0 = 1 - af_0\) and \(\pi_{\alpha\beta}\) is the shear stress tensor. Approximating the shear stress tensor with its first-order relativistic Navier-Stokes expression, \(\pi_{\alpha\beta} = 2\eta \nabla_{(\alpha u_\beta)}\), the expression for the 14-moment approximation reduces to

\[
\delta f_1 = \frac{f_0 \delta f_0}{T^3} \left( \frac{n}{s} \right) p^\alpha p^\beta \nabla_{(\alpha u_\beta)}.
\]

Here \(\eta\) is the coefficient of shear viscosity, \(s = (\epsilon + P)/T\) is the entropy density and the angular brackets denote traceless symmetric projection orthogonal to the fluid four-velocity [36]. The form of \(p^\alpha p^\beta \nabla_{(\alpha u_\beta)}\) in the case of blast-wave model is calculated in Appendix A.

The anisotropic flow is defined as

\[
v_n(p_T) = \frac{1}{\pi} \int_0^{\pi} d\phi \cos[n(\phi - \Psi_n)] \frac{dN}{dy_p d^3 p_T d\phi},
\]

where \(\Psi_n\) is the \(n\)-th harmonic event-plane angle. In the present case, we do not consider event-by-event fluctuations and therefore \(\Psi_n = 0\). Up to first order in viscosity [28],

\[
v_n(p_T) = v_n^{(0)}(p_T) \left( 1 - \frac{d \phi}{dy_p d^3 p_T d\phi} \right) \
+ \frac{d \phi}{dy_p d^3 p_T d\phi} \frac{dN^{(1)}}{d^3 p_T d\phi} \left( \frac{d \phi}{dy_p d^3 p_T d\phi} \right),
\]

where the superscript ‘(0)’ denotes quantities calculated using the ideal distribution function, Eq. (10), and ‘(1)’ denotes those obtained using the first-order viscous correction, Eq. (12).

### III. VISCOUS BLAST-WAVE MODEL

The definition of the participant anisotropies, \(\varepsilon_n\), via the Fourier expansion for a single-particle distribution is

\[
f(\varphi) = \frac{1}{2\pi} \left[ 1 + 2 \sum_{n=1}^{\infty} \varepsilon_n \cos[n(\varphi - \psi_n)] \right],
\]

where \(\psi_n\) are the angles between the \(x\) axis and the major axis of the participant distribution. The geometrical anisotropies in the initial particle distribution, \(\varepsilon_n\), eventually converts to anisotropies in the radial fluid velocity,

\[
u^r = u_0 \frac{r}{R} \left[ 1 + 2 \sum_{n=1}^{\infty} u_n \cos[n(\varphi - \psi_n)] \right].
\]
In the following, we determine the conversion efficiency of the initial eccentricity to anisotropy in the radial fluid velocity, $u_n/\varepsilon_n$.

Using the well known dispersion relation for sound in a viscous medium \cite{35},
\begin{equation}
\omega = c_s k + ik^2 \frac{1}{T} \left( \frac{2 \eta}{3 s} \right),
\end{equation}
the authors of Ref. \cite{37} introduced a viscosity-based survival scale which all structures formed by point like perturbations should attain at freeze-out. In the above equation, $\eta$ is the coefficients of shear viscosity and $c_s$ is the speed of sound in the medium. In the present work, we ignore the contribution due to bulk viscosity. Using a plane-wave Fourier ansatz, $\exp(i\omega t - ikx)$, we observe that the amplitudes of the stress tensor harmonics with momentum $k$ are attenuated by a factor
\begin{equation}
\delta T^{\mu\nu}(t, k) = \exp \left[ - \left( \frac{2 \eta}{3 s} \right) \frac{k^2 t}{T} \right] \delta T^{\mu\nu}(0, k),
\end{equation}
where we have suppressed the oscillatory pre-factor. We note that the presence of momentum squared in the exponent leads to enhanced effect of viscosity for the higher harmonics. We expect the same qualitative behaviour for the radial flow velocity as will be explained in the following.

First and foremost, we notice that each harmonics is essentially a damped oscillator with wave-vector $k$. Moreover, throughout the evolution the harmonics form standing waves on the fireball circumference, as shown in Fig. 1, whose amplitude is progressively damped due the viscous effects. Therefore, the fireball circumference is an integer multiple of the wavelength with wave-vector $k$, i.e.,
\begin{equation}
2\pi R = n \frac{2\pi}{k},
\end{equation}
where $R$ is the transverse radius of the fireball at freeze-out. Hence, at the freeze-out time $t_f$, the wave amplitude reaction is given by
\begin{equation}
\frac{\delta T^{\mu\nu}|_{t=t_f}}{\delta T^{\mu\nu}|_{t=0}} = \exp \left[ -n^2 \left( \frac{2 \eta}{3 s} \right) \frac{t_f}{R^2 T_f} \right],
\end{equation}
where $T_f$ is the freeze-out temperature.

In absence of viscosity, the initial geometrical perturbations in the fluid will result in the development of radial flow velocity and the conversion efficiency will remain the same for all harmonics. In the case of a viscous medium, however, the conversion efficiency of the initial geometrical perturbation to radial fluid velocity must be proportional to the wave amplitude reaction,
\begin{equation}
\frac{u_n}{\varepsilon_n} = \alpha_0 \exp \left[ -n^2 \left( \frac{2 \eta}{3 s} \right) \frac{t_f}{R^2 T_f} \right],
\end{equation}
where $\alpha_0$ is the constant of proportionality. The sudden stopping of the damped oscillator at the freeze-out time may lead to certain phases due to the oscillatory pre-factor. These phases can, in general, lead to secondary peaks in the power spectrum of higher harmonics. However, as no secondary peaks has been observed in the spectrum of relativistic heavy-ion collisions, we will continue to ignore these phase factors.

We emphasize that the acoustic damping should be applied to the hydrodynamic variables, such as the moments of the flow velocity, $u_n$, rather then to the final state observables such as $v_n$, as done in Ref. \cite{38}. In Ref. \cite{38}, Shuryak and Zahed (S-Z) proposed that the ratio of the initial eccentricity $\varepsilon_n$ to the final $p_T$-integrated $v_n$ should be proportional to the wave amplitude reaction, i.e., the r.h.s. of Eq. (21) should be equal to $u_n/\varepsilon_n$. However, $\varepsilon_n$ is the eccentricity in the configuration space whereas $v_n$ is the momentum anisotropy of the particles after freeze-out. The momentum space asymmetries are not hydrodynamic variables and are only affected indirectly via damping. On the other hand, the acoustic damping should be applicable to hydrodynamic variables and it should only capture the viscous effects of the hydrodynamic evolution. This assumption also misses the additional effect of viscosity at freeze-out using the Cooper-Frye formula. Moreover, it does not provide the opportunity to study the $p_T$ dependence of anisotropic flow and hence to estimate the viscosity of the expanding medium.

\section{IV. INITIAL CONDITIONS}

In this section, we set-up the initial conditions of the collisions in order to reduce the number of free parameters in the blast-wave model. To this end, we evaluate the parameters corresponding to the initial geometry of the collisions. We also approximate the subsequent transverse expansion of the fireball by using the radial velocity parametrization in the blast-wave model.

We consider the collision of two identical nuclei with mass number $A$. The radius of each nucleus is given by $R_0 = 1.25A^{1/3}$ fm and the impact parameter is $b$, as shown in Fig. 2. The shaded region in Fig. 2 is the overlap zone of the colliding nuclei. We draw a circle of radius $r_0$, with its centre coinciding with that of the overlap zone, in such a way that the boundary of the
Overlap zone is equally divided in four parts, i.e., $123 = 345 = \beta = 781$; see Fig. 2. This approximates $\varepsilon_2$ as the second harmonics of initial geometrical fluctuations, analogous to the $n = 2$ case as shown in Fig. 1. The radius $r_0$ is therefore the initial transverse radius of the expanding fireball and is given by

$$r_0 = \frac{1}{2} \left( b^2 - 2bR_0 \sqrt{2 + \frac{b}{R_0} + 4R_0^2} \right)^{1/2}, \quad (22)$$

which reduces to $r_0 = R_0$ for head-on collisions ($b = 0$). Since $\varepsilon_2$ is the most prominent eccentricity for non-central collisions, all other geometrical eccentricities are treated as boundary perturbations to this circle.

The subsequent transverse expansion of the fireball is obtained by employing the radial velocity parametrization of the blast-wave model. Using the perturbation-free expression for the transverse velocity, Eq. (6), we get

$$u' \equiv \frac{dr}{d\tau} = u_0 \frac{r}{R} \Rightarrow \int_{r_0}^{R} \frac{dr}{r} = \int_{0}^{\tau_f} u_0 \frac{d\tau}{R}. \quad (23)$$

After performing the straightforward integration, we obtain a transcendental equation for the freeze-out radius, $R$,

$$R = r_0 \exp \left( \frac{u_0 \tau_f}{R_0} \right), \quad (24)$$

which can be solved for $R$ given the isotropic expansion velocity $u_0$ and the freeze-out time $\tau_f$. In the following, using Bjornen’s scaling solution for one-dimensional boost-invariant expansion, we obtain an expression to determine the freeze-out times for non-central collisions once it is fixed for the central one.

For the ideal hydrodynamic evolution of relativistic fluid, in the one-dimensional boost-invariant scenario, the evolution of the energy density follows $\epsilon \propto \tau^{-4/3}$. Assuming the initial thermalization time and the final freeze-out energy density (i.e., the freeze-out temperature) to be same for all collisions, we get

$$\tau_f = \tau_{f0} \left( \frac{\epsilon_i}{\epsilon_{i0}} \right)^{3/4}. \quad (25)$$

Here $\tau_{f0}$ is the freeze-out time for most central collisions, which has to be fixed by fitting the corresponding transverse momentum spectra. The freeze-out times for other centralities can then be obtained using the above equation and therefore they are not free parameters. The ratio $\epsilon_i/\epsilon_{i0}$ is the initial central energy density scaled by its corresponding value in most central collisions.

Figure 3 shows $\epsilon_i/\epsilon_{i0}$ for various centralities obtained using the Glauber model calculations [39, 40] in the case of Pb–Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV at the LHC. We observe that $\epsilon_i/\epsilon_{i0}$ (and hence the initial temperature) decreases for non-central collisions compared to central ones. Therefore according to Eq. (25), freeze-out happens earlier in peripheral collisions which is also reflected in Fig. 3 for $\tau_f/\tau_{f0}$. The inset of Fig. 3 shows the centrality dependence of the freeze-out radius of the fireball $R$, obtained using Eq. (24). We find a rather large transverse radius of the fireball at freeze-out. Finally, the parameters that we need to fix within the viscous-blend wave model to fit the spectra are the freeze-out temperature $T_f$, the freeze-out time for central collision $\tau_{f0}$ and the unperturbed maximum radial flow velocity $u_0$. An interplay of the coefficient of proportionality $a_0$ in Eq. (21) and $\eta/s$ will be crucial to reproduce the flow harmonics.
V. RESULTS AND DISCUSSIONS

In this section, we show our results for Pb+Pb collisions at \( \sqrt{s_{NN}} = 2.76 \) TeV and compare it with experimental data measured at LHC by the ALICE and ATLAS collaborations.

Figure 4 shows the transverse momentum distribution of pions, kaons, and protons spectra for 0 − 5% and 20 − 30% central Pb+Pb collisions at \( \sqrt{s_{NN}} = 2.76 \) TeV measured at LHC by the ALICE collaboration (symbols) and calculated using the viscous blast-wave model (lines). The results are obtained using root mean square values of eccentricities \( \varepsilon_2 \) and \( \varepsilon_3 \) in a Monte-Carlo Glauber model, with a shear viscosity to entropy density ratio \( \eta/s = 0.24 \). We observe that the spectra for \( \pi^+ \) and \( K^+ \) from the viscous blast-wave model are in good overall agreement with the experimental data for a freeze-out temperature of 120 MeV. On the other hand, within the viscous blast-wave model, the proton yield for a freeze-out temperature of 120 MeV is severely underestimated. To obtain an overall fair agreement with the experimental data, the freeze-out temperature for protons is considered to be 135 MeV. The freeze-out time for 0 − 5% most central collision was found to be 8 fm. For other centralities, the freeze-out time was obtained by using Eq. (25) and results shown in Fig. 3.

Figure 5 shows our results for the \( v_n(p_T) \), in comparison with the ATLAS data [4] for various centralities. We find overall fair agreement with the data for elliptic (\( v_2 \)) and triangular (\( v_3 \)) flow, all centralities. This is achieved by choosing a single fixed value \( \eta/s = 0.24 \) to obtain the required suppression (relative to ideal blast-wave results) of \( v_2 \) and \( v_3 \), for all centralities. Apart from \( \eta/s \), flow also depends on the constant of proportionality \( \alpha_0 \) appearing in Eq. (21). It controls the conversion efficiency of the initial eccentricity to final fluid velocity. We consider the initial eccentricity \( \varepsilon_n \) to be the rms values of the eccentricities obtained in the MC-Glauber model, as given in Ref. [42]. A large value of \( \alpha_0 \) means larger conversion of eccentricity leading to increased flow velocity and hence higher \( v_2 \) and \( v_3 \). On the other hand, as is well known, an increase in \( \eta/s \) leads to suppression of \( v_2 \) and \( v_3 \). Large (small) value of \( \alpha_0 \) can be compensated by choosing a higher (lower) value of \( \eta/s \) up to a certain extent. However, beyond a certain range of \( \eta/s \), the relative behaviour of \( v_2 \) and \( v_3 \) is destroyed.

In order to match the elliptic and triangular flow data, we find the most suitable parameter values for \( \alpha_0 = 0.4 \) and \( \eta/s = 0.24 \). Moreover, we find that \( v_n \) is insensitive to \( a \) for transverse velocity of the form \( v^r \sim (r/R)^a \). This may be attributed to the fact that the exponent \( a \) controls the rate of isotropic transverse expansion. The anisotropic flow originates from the initial eccentricity which translates into final flow. Therefore \( v_n \) is sensitive to \( u_n \), which depends on \( \alpha_0 \) and \( \eta/s \), as is apparent from Eq. (21). On the other hand, it should be noted that the slope of the particle spectra is sensitive to the exponent \( a \) and could be tuned to get a better fit with the experimental data. However, in the present work, we are interested in the anisotropic flow and therefore we set \( a = 1 \) and do not fit it to match the particle spectra.

In Fig. 6(a), \( p_T \)-integrated values of \( v_2 \) and \( v_3 \) obtained from the viscous blast-wave model are compared with ALICE data [3], as a function of centrality. We observe that using the same constant \( \eta/s = 0.24 \) for all central-
FIG. 6: (Color online) (a): Centrality dependence of the $p_T$ integrated anisotropic flow coefficients $v_n$ of charged hadrons in Pb+Pb collisions at $√s_{NN} = 2.76$ TeV calculated in the viscous blast-wave model (lines) with $η/s = 0.24$, as compared to ALICE data [3] (symbols). (b): Centrality dependence of the ratio $v_n/ε_n$ in the viscous blast-wave model where $ε_n$ is the rms values of the eccentricities obtained in the MC-Glauber model [42]. In panels (a) and (b), we show results for $n = 2$ and $3$. (c): Centrality dependence of the ratio $v_2/v_3$ for ALICE data (symbols), using the viscous blast-wave model (solid line) and from the Shuryak-Zahed estimate (dashed line).

FIGURE 6(b) shows the conversion efficiency of the initial spatial anisotropy into the final momentum anisotropy. A linear relation between $v_n$ and $ε_n$, as observed in some hydrodynamic calculations [23], is not obtained within the viscous blast-wave model presented here. On the other hand, in view of the non-linear nature of the hydrodynamic equations a linear relation between $v_n$ and $ε_n$ is not obvious. Indeed other calculations [43] as well as a recent analysis of the LHC data [44] also result in a similar centrality dependence as obtained in our analysis. In Fig. 6(c), we show the ratio $v_2/v_3$ as a function of centrality for the ALICE data (symbols), the present viscous blast-wave model (blue solid line) and the estimate due to Shuryak and Zahed (red dashed line). We see that the viscous blast-wave model provides a better agreement with ALICE data compared to the S-Z estimate. However, we should keep in mind that the freeze-out parameters used in the S-Z estimate is the same as that of the viscous blast-wave fit values.

VI. CONCLUSIONS AND OUTLOOK

In this paper we have generalized the blast-wave model to include viscous effects by employing a viscosity-based survival scale for geometrical anisotropies formed in the early stages of relativistic heavy-ion collisions. This viscous damping is introduced in the parametrization of the radial flow velocity. The viscous blast wave model presented here involved five parameters, including $η/s$, which has to be fitted for only one centrality. This model therefore incorporates the important features of viscous hydrodynamic evolution but does not require to do the actual evolution. We have used this viscous blast-wave model to obtain the transverse momentum distribution of particle yields and anisotropic flow harmonics for LHC. The blast-wave model parameters were fixed by fitting the transverse momentum spectra of identified particles. We demonstrated that a fairly good agreement was achieved for transverse momentum distribution of elliptic and triangular flow for various centralities as well as centrality distribution for the integrated flow. Within the present model, we estimated the shear viscosity to entropy density ratio $η/s \simeq 0.24$ at the LHC.

One of the drawbacks of the present model is that we have employed root mean squared eccentricity over number of events, which is analogous to “single shot” hydrodynamic evolution. On the plus side, the present model could also be implemented on an event-by-event basis. Another difficulty that we encountered was that in order to fit the proton spectra, we had to consider a different freeze-out temperature (135 MeV) compared to that for pions and Kaons (120 MeV). This problem could be addressed by parametrizing the transverse velocity in the form $v^r \sim (r/R)^a$ and fitting $a$ for different particle species, separately. We leave these problems for a future work.

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Appendix A: Viscous stress tensor

In this Appendix, we calculate the viscous tensor $\nabla \alpha \beta$ and therefore obtain the viscous corrections to the distribution function at freeze-out. We work in Milne co-ordinate system, Eqs. (1)-(4) with the metric tensor $g_{\mu \nu} = \text{diag}(1, -τ^2, -1, -τ^2)$. Therefore, the inverse metric tensor is $g^{\mu \nu} = \text{diag}(1, 1/τ^2, -1, -1/τ^2)$, its determinant $g$ is $\sqrt{-g} = τ$ and the non-vanishing Christoffel symbols are $\Gamma^r_{r r} = τ$, $\Gamma^r_{rr} = 1/τ$, $\Gamma^r_{ϕϕ} = -r$, and
Using the parametrization of the fluid velocity given in Eqs. (6)-(8), we get

\[ \Delta \nu = 0, \quad \Delta \varphi = -\frac{1}{r}, \quad \Delta \tau = -1 - (u_r)^2, \tag{A1} \]
\[ \partial_r u^r = \frac{u^r}{r}, \quad \partial_u u^r = -2u_0r \sum_{n=1}^\infty a_n \sin[n(\varphi - \psi_n)]. \tag{A2} \]

where \( \Delta \nu \equiv g_{\mu\nu} - u^\mu u^\nu \) is the projection operator orthogonal to the fluid four-velocity.

To fix the time derivatives of the fluid velocity, we assume that if the particles are freezing-out, they are free streaming, which means that \( Du^\mu = 0 \). Here \( D \equiv u^\mu d_\mu \) is the co-moving derivative and \( d_\mu \) is the covariant derivative. With this prescription, we have

\[ \partial_r u^\varphi = 0, \]
\[ \partial_r u^r = -v \partial_r u^r - \frac{(u_r)^2}{ru^r}, \]
\[ \partial_r u^\tau = v \partial_r u^\tau - \frac{(u^\tau)^3}{r(u_r)^2} \tag{A3} \]

where \( v = u^r/u^r \) is the radial velocity. The expansion scalar is given by

\[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} u^\mu) = \frac{u^r}{r} + \frac{u^\varphi}{r} + \partial_\varphi u^r + \partial_r u^\varphi + \partial_\tau u^r, \]
\[ = \frac{u^r}{r} + 2\frac{u^\varphi}{r} - \frac{(u_r)^3}{r(u_r)^2}. \tag{A4} \]

Assuming boost invariance, the spatial components of the viscous tensor are given by

\[ r \nabla^\nu u^\nu = -\frac{r}{2} \partial_\nu u^\nu - \frac{1}{2r} \partial_r u^r - \frac{r}{2} u^\nu \partial_r u^\nu - r \frac{u^\nu}{2} \partial_r u^\nu \]
\[ - \frac{1}{3} \Delta \nabla^\nu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} u^\mu) \]
\[ = \frac{u_0}{R} \sum_{n=1}^\infty a_n \sin[n(\varphi - \psi_n)], \tag{A5} \]

where we have used the fact that \( (u_r)^2 = 1 + (u_r)^2 \).

\[ \tau^2 \nabla^\nu (u_\nu u^\mu) = -\frac{u^r}{\tau} + \frac{1}{3} \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} u^\mu) \]
\[ = \frac{1}{3} \left[ \frac{2u^r}{\tau} - 2\frac{u^\varphi}{\tau} - \frac{(u_r)^3}{r(u_r)^2} \right], \tag{A8} \]

\[ \nabla^\nu (u_\nu u^\mu) = \nabla^\nu (u_\nu u^\nu) = 0. \tag{A9} \]

To obtain the temporal components of the viscous stress energy tensor, we use the Landau frame condition, \( \nabla (u_\nu u^\beta)_{\nu\beta} = 0 \).

\begin{align*}
\nabla^\nu (u_\nu u^\nu) &= v \nabla^\nu (u_\nu u^\nu), \\
\nabla^\nu (u_\nu u^\beta)_{\nu\beta} &= 0, \\
\nabla^\nu (u_\nu u^\nu) &= v \nabla^\nu (u_\nu u^\nu). \tag{A10} \\
\end{align*}

Using Eqs. (A6), (A7), (A8) and (A14)

\[ g_{\mu\nu} \nabla^\mu u^\nu = \nabla^\nu u^\nu - \tau \nabla^\nu (u_\nu u^\mu) - \nabla^\nu (u_\nu u^\nu) + \tau^{-2} \nabla^\nu (u_\nu u^\nu) \]
\[ = \frac{1}{3} \left[ \frac{(u^r)^2}{\tau^2} - \frac{(u^\varphi)^3}{r(u_r)^2} + \frac{(u_r)^5}{r(u_r)^2} \right] \tag{A14} \]

Next, in order to verify our algebra, we confirm that the viscous stress tensor is traceless, i.e., \( \nabla^\mu u^\nu = 0 \). Using Eqs. (A6), (A7), (A8) and (A14)

\[ g_{\mu\nu} \nabla^\mu u^\nu = \nabla^\nu (u_\nu u^\nu) - \tau^2 \nabla^\nu (u_\nu u^\nu) - \nabla^\nu (u_\nu u^\nu) - \tau^2 \nabla^\nu (u_\nu u^\nu) \]
\[ = \frac{1}{3} \left[ \frac{(u^r)^2}{\tau^2} - \frac{(u^\varphi)^3}{r(u_r)^2} + \frac{(u_r)^5}{r(u_r)^2} \right] \tag{A14} \]

For a particle at the space-time point \( (\tau, \eta, r, \varphi) \) with the four momentum \( p^\mu = (E, p^r, p^\varphi, p^\tau) = (m_T \cosh y, m_T \cos \varphi, m_T \sin \varphi, m_T \sinh y) \), we get

\[ p^\tau = m_T \cosh(y - \eta_s) \quad \Rightarrow \quad p_T = m_T \cosh(y - \eta_s), \]
\[ \tau p^\tau = m_T \sinh(y - \eta_s) \quad \Rightarrow \quad p_\eta = -\tau m_T \sinh(y - \eta_s), \]
\[ p^\varphi = p_T \cos(\varphi - \varphi_p) \quad \Rightarrow \quad p_\varphi = -p_T \cos(\varphi - \varphi_p), \]
\[ p^r = p_T \sin(\varphi - \varphi_p) \quad \Rightarrow \quad p_\varphi = -p_T \sin(\varphi - \varphi_p). \tag{A16} \]

The oriented freeze-out hyper-surface is \( d\Sigma_\mu = (\tau d\eta_s, r d\tau, r d\varphi, 0, 0, 0) \), and therefore the integration measure is given by

\[ p^\mu d\Sigma_\mu = m_T \cosh(y - \eta_s) \tau d\eta_s, r d\varphi. \tag{A17} \]
The viscous correction to the equilibrium distribution function is proportional to

\[ p_\mu p_\nu \nabla^{(\mu} u^{\nu)} = p_r^2 \nabla^{(r} u^{r)} + p_{\eta s} \nabla^{(\eta \nu} u^{s)} + p_\varphi^2 \nabla^{(\varphi\nu} u^{\varphi)} + 2p_r p_\varphi \nabla^{(r} u^{\varphi)} + 2p_r p_\varphi \nabla^{(r} u^{\nu)} + 2p_r p_\varphi \nabla^{(r} u^{\varphi)} \]

(A18)

The final form of \( f = f_0 + \delta f \) obtained from Eq. (12) using the above equation is required to evaluate the spectra given by

\[ \frac{d^2 N}{d^2 p_T dy} = \frac{1}{(2\pi)^3} \int_0^R d\tau \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} d\eta_s m_T \cosh(y - \eta_s)f. \]

(A19)

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