NONEXISTENCE RESULT FOR A SEMILINEAR ELLIPTIC PROBLEM

SALVADOR LÓPEZ-MARTÍNEZ AND ALEXIS MOLINO

Abstract. In this paper we prove the nonexistence of nontrivial solution to
\[
\begin{cases}
-\Delta u = f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
being \( \Omega \subset \mathbb{R}^N (N \geq 2) \) a bounded domain with boundary smooth and \( f \) Lipschitz with non-positive primitive.

1. Introduction

Problems of partial differential equations are extensively studied at present, mainly motivated by their applications in the fields of physics, biology and engineering among others. One of the simplest model of a nonlinear elliptic differential equation is the following
\[
(P) \quad \begin{cases}
-\Delta u = f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
being \( \Omega \subset \mathbb{R}^N \) a bounded domain with boundary of class \( C^{1,1} \), \( N \geq 2 \) and \( f : \mathbb{R} \to \mathbb{R} \) an \( L \)-Lipschitz function.

Along this note, a classical solution to \( (P) \) (solution from now on) will be a function \( u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega}) \), for some \( \alpha \in (0,1) \), satisfying \( (P) \) pointwise. Observe that, by regularity results, every weak solution is a solution to this problem (see e.g. [Struwe(2008)]).

On the other hand, solutions to \( (P) \) must satisfy the well known Pohozaev identity [Pohozaev(1965)]:
\[
\frac{1}{2} \int_{\partial \Omega} |\nabla u(x)|^2 \cdot \nu(x) dx + \frac{N-2}{2} \int_{\Omega} |\nabla u(x)|^2 dx = N \int_{\Omega} F(u(x)) dx,
\]
where \( F(s) = \int_0^s f(t)dt \) and \( \nu \) denotes the unit outward normal to \( \partial \Omega \) vector. This equality is very useful for proving nonexistence of solutions. In fact, if \( \Omega \) is starshaped with respect to 0 (i.e., \( x \cdot \nu(x) \geq 0 \) on \( \partial \Omega \)), there is no nontrivial solution when either \( f(s) = \lambda |s|^{p-2}s \) for \( \lambda > 0 \) and \( p \geq 2^* \) (supercritical case), or
\[
F(s) \leq 0, \quad \forall s \in \mathbb{R},
\]
(non-positive primitive case). Observe that condition (1) together with the continuity of \( f \) imply that \( f(0) = 0 \). This means that \( u \equiv 0 \) is always a solution in any of the two mentioned cases. In this context, it is natural to ask whether there is a solution different from the trivial one if the domain is not starshaped. In this line, the supercritical case has been extensively studied and positive solutions have been found for all \( \lambda > 0 \) and for all \( p \geq 2^* \) when the domain is an annulus.
\{x \in \mathbb{R}^N : 0 < a < |x| < b\} \] (see the seminal paper [Kazdan and Warner(1975)] and references therein). Thus, existence in supercritical case depends on a special geometric property of the domain \(\Omega\).

In contrast, much less is known about problem \((P)\) in the non-positive primitive case. Analogously to the supercritical case, it is reasonable to expect that there are bounded domains \(\Omega\) and \(L-\)Lipschitz functions \(f : \Omega \to \mathbb{R}\) satisfying (1) for which problem \((P)\) admits a nontrivial solution. In fact, in [Ricceri(2018)] it is conjectured the existence of \(\lambda > 0\) such that, if \(\Omega\) is an annulus and \(f(u) = -\lambda \sin u\), then problem \((P)\) has at least one non-zero solution. However, the literature concerning the non-positive primitive case provides only nonexistence results. Indeed, it has been proved in [Ricceri(2008)] that, for any smooth bounded domain \(\Omega\), zero is the only solution if \(L < 3\lambda_1\), where \(\lambda_1\) is the first eigenvalue for the Laplacian operator with zero Dirichlet boundary conditions. Next, in [Fan(2009a)] (see also [Fan(2009b)]), it is shown the nonexistence of nontrivial solution in the limit case \(L = 3\lambda_1\). Later, in [Fan and Ricceri(2010)] the authors showed the nonexistence of radially symmetric solutions when \(\Omega\) is an annulus.

The aim of this paper is to prove that, apart from the trivial solution, there is no solution to problem \((P)\) provided (1) is satisfied. Here, no additional hypotheses on \(\Omega\) nor \(f\) are imposed. This exposes the unexpected fact that, in contrast to the supercritical case, there is no geometric assumption on \(\Omega\) that gives a nontrivial solution.

2. MAIN RESULT

**Theorem 2.1.** Trivial solution is the unique solution to problem \((P)\) provided (1) holds.

**Proof.** Clearly, 0 is a solution. We argue by contradiction and assume that there exists a nontrivial solution \(v\) to \((P)\). First of all, notice that \(-v\) is a solution to

\[
\begin{cases}
-\Delta u = -f(-u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Since the function \(-f(-s)\) is under the hypotheses of the theorem, there is no loss of generality in assuming that \(v_\infty := \max_{x \in \Omega} v(x) > 0\). On the other hand, since the value of \(f(s)\) for \(s > v_\infty\) is irrelevant, we can also assume that \(\lim_{s \to +\infty} f(s) = -\infty\).

We claim now that, in fact, there is a positive solution \(u > 0\) to problem \((P)\). Indeed, observe that there exists \(s_0 > v_\infty\) such that \(f(s_0) < 0\). Then, \(u_0 \equiv s_0\) is a bounded supersolution to \((P)\). We establish now the classical iterative scheme: let us consider for all \(n \in \mathbb{N}\) the solution \(u_n\) to problem

\[
\begin{cases}
-\Delta u_n + Lu_n = f(u_{n-1}) + Lu_{n-1} & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Note that the function \(s \mapsto f(s) + Ls\) is non-decreasing for all \(s \in \mathbb{R}\). Then, we easily deduce that the sequence \(\{u_n\}\) is non-increasing. Moreover, again the monotonicity of \(f(s) + Ls\) together with the fact that \(u_0 > v\) lead to \(u_n \geq v\) for all \(n\). Hence, there exists a solution \(u\) to problem \((P)\) as a limit of \(u_n\), so \(u \geq v\) (in particular, \(u \neq 0\)). On the other hand, since 0 is also a solution, then \(u_n \geq 0\) for all \(n\), so necessarily \(u \geq 0\). Now the strong maximum principle implies that \(u > 0\), and the claim is true.
Let us denote $u_\infty := \max_{x \in \Omega} u > 0$. We will show now, following the arguments in [Ambrosetti and Hess (1980), Lemma 6.2], that $f(u_\infty) > 0$. Indeed, arguing by contradiction, assume that $f(u_\infty) \leq 0$. Then,

$$-\Delta u_\infty + Lu_\infty \geq f(u_\infty) + Lu_\infty \quad \text{in } \Omega. \quad (2)$$

Moreover, we have proved that

$$-\Delta u + Lu = f(u) + Lu \quad \text{in } \Omega, \quad (3)$$

Subtracting (3) from (2), and using that $f(s) + Ls$ is non-decreasing, we obtain

$$-\Delta(u_\infty - u) + L(u_\infty - u) \geq f(u_\infty) + Lu_\infty - f(u) - Lu \geq 0 \quad \text{in } \Omega. \quad (4)$$

Since $u_\infty > u$ on $\partial \Omega$, the strong maximum principle implies that $u_\infty > u$ in $\Omega$, which is a contradiction.

Thus, the fact that $f(u_\infty) > 0$ implies that there are $s_1, s_2 > 0$ such that $s_1 < u_\infty < s_2$ and $f(s) > 0$ for $s \in (s_1, s_2)$. Moreover, since $F(s) \leq 0$ and $\lim_{s \to +\infty} f(s) = -\infty$, we can choose respectively $s_1$ and $s_2$ such that $f(s_1) = f(s_2) = 0$. Further, we can assume that $F(s_2) < 0$ since, otherwise (i.e., if $F(s_2) = 0$), we can modify $f$ to another $L$-Lipschitz function $f^*$ such that $f(s) > f^*(s) > 0$ for $s \in (u_\infty, s_2)$ and $f = f^*$ elsewhere. In this way, $u$ is still a solution to $(P)$, but now $F(s_2) < 0$.

Now we will find a family of supersolutions to $(P)$ which will lead to a contradiction by comparison with $u$. For this purpose, we follow the original reasoning in [Clement and Sweers (1987)], which in principle is performed for $f \in C^1(\mathbb{R})$. Here we adapt the proof to our setting and check that it also works for Lipschitz functions $f$.

Indeed, consider the following initial value problem

$$\begin{cases}
-w''(r) = f(w(r)), & \forall r > 0, \\
w(0) = s_2, \\
w'(0) = -\sqrt{-F(s_2)}.
\end{cases} \quad (5)$$

Since $f$ is Lipschitz there is a unique solution $w \in C^2([0, +\infty))$. Multiplying the equation by $w'(r)$ and integrating, we obtain

$$(w'(r))^2 = -F(s_2) + 2 \int_{w(r)}^{s_2} f(s) ds$$

$$= F(s_2) - 2F(w(r)), \quad (4)$$

Thus, we get that

$$\begin{aligned}
(w'(r))^2 &> 0 \quad \text{for } w(r) \in [s_1, s_2].
\end{aligned} \quad (5)$$

Now, since $w(0) = s_2$ and $w'(0) < 0$, we deduce easily that $w(r) \in (s_1, s_2)$ for all $r > 0$ small enough. We claim now that there exists $r_0 > 0$ such that $w(r_0) = s_1$. Indeed, assume by contradiction that $w(r) > s_1$ for all $r > 0$. Then, by (5) we have that $w$ is decreasing in $(0, +\infty)$. Hence, there exists $s_3 \in [s_1, s_2)$ such that $\lim_{r \to +\infty} w(s) = s_3$. But this is impossible as $w''(r) = -f(w(r)) < 0$ for all $r > 0$, i.e., $w$ is concave.

In consequence, since $w(r_0) = s_1$ and $w'(r_0) < 0$, we deduce that $\inf_{r > 0} w(r) < s_1$. Moreover, it is easy to show that $\inf_{r > 0} w(r) > 0$. Indeed, assuming otherwise, there exists a sequence $\{r_n\} \subset [0, +\infty)$ such that $\lim_{n \to +\infty} w(r_n) = 0$. Then, for $n$ large enough, we deduce from (4) that $(w'(r_n))^2 < \frac{F(s_2)}{2} < 0$, a contradiction.
Thus, we have proved that

\[ 0 < \inf w < s_1. \]

Next, we define

\[ W(r) = \begin{cases} s_2, & r \in (-\infty, 0], \\ \min \{w(r), s_2\}, & r \in (0, \infty). \end{cases} \]

Since we can assume that \( f(s) < 0 \) for \( s > s_2 \), it follows that \( w \) is convex if \( w(r) > s_2 \). This implies that, if \( w(r_2) = s_2 \) for some \( r_2 > 0 \), then \( W(r) = s_2 \) for all \( r \geq r_2 \). Otherwise, \( w(r) < s_2 \) for all \( r > 0 \), so \( W(r) = w(r) \) for all \( r > 0 \).

For every \( t \in \mathbb{R} \), consider the family of parametric functions \( W_t(x) = W(x_1 - t) \), for all \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \), note that \( W_t(x) = s_2 \) for \( t \) large enough. The fact that \( W \) is a bounded Lipschitz function implies that \( W_t \) is a bounded Lipschitz function too. Moreover, it is easy to see that \( -\Delta W_t \geq f(W_t) \) in \( \Omega \) (in the weak sense). In addition, due to (6), \( W_t > 0 \) on \( \partial \Omega \). Therefore, the strong maximum principle implies that \( u < W_t \) in \( \Omega \). In consequence,

\[ u(x) \leq \inf_{t \in \mathbb{R}} W_t(x) = \inf_{r > 0} w(r) < s_1, \quad \forall x \in \Omega, \]

which is a contradiction with the fact that \( u_\infty \in (s_1, s_2) \).

\[ \Box \]

ACNOWLEDGEMENTS

Research supported by MINECO-FEDER grant MTM2015-68210-P (first and second authors) and Programa de Contratos Predoctorales del Plan Propio de la Universidad de Granada (first author).

REFERENCES

[Ambrosetti and Hess(1980)] Ambrosetti, A. and Hess, P. Positive solutions of asymptotically linear elliptic eigenvalue problems. J. Math. Anal. Appl., 73(2):411–422, 1980. ISSN 0022-247X.

[Clément and Sweers(1987)] Clément, P. and Sweers, G. Existence and multiplicity results for a semilinear elliptic eigenvalue problem. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 14(1):97–121, 1987. ISSN 0391-173X.

[Fan(2009a)] Fan, X. A remark on Ricceri’s conjecture for a class of nonlinear eigenvalue problems. Journal of Mathematical Analysis and Applications, 349(2):436 – 442, 2009a. ISSN 0022-247X.

[Fan(2009b)] Fan, X. On Ricceri’s conjecture for a class of nonlinear eigenvalue problems. Appl. Math. Lett., 22(9):1386–1389, 2009b. ISSN 0893-9659.

[Fan and Ricceri(2010)] Fan, X. and Ricceri, B. On the Dirichlet problem involving non-linearities with non-positive primitive: a problem and a remark. Applicable Analysis, 89(2):189–192, 2010.

[Kazdan and Warner(1975)] Kazdan, J. L. and Warner, F. W. Remarks on some quasilinear elliptic equations. Comm. Pure Appl. Math., 28(5):567–597, 1975. ISSN 0010-3640.

[Pohožaev(1965)] Pohožaev, S. I. On the eigenfunctions of the equation \( \Delta u + \lambda f(u) = 0 \). Dokl. Akad. Nauk SSSR, 165:36–39, 1965. ISSN 0002-3264.

[Ricceri(2008)] Ricceri, B. A remark on a class of nonlinear eigenvalue problems. Nonlinear Analysis: Theory, Methods & Applications, 69(9):2964 – 2967, 2008. ISSN 0362-546X.

[Ricceri(2018)] Ricceri, B. Four Conjectures in Nonlinear Analysis, pages 681–710. Springer International Publishing, Cham, 2018. ISBN 978-3-319-89815-5.

[Struwe(2008)] Struwe, M. Variational methods, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, fourth edition, 2008. ISBN 978-3-540-74012-4. Applications to nonlinear partial differential equations and Hamiltonian systems.
