Stability estimates for the fault inverse problem

Faouzi Triki$^1$ and Darko Volkov$^{2,3}$

$^1$ Laboratoire Jean Kuntzmann, UMR CNRS 5224, Université Grenoble-Alpes, 700 Avenue Centrale, 38401 Saint-Martin-d’Hères, France
$^2$ Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, United States of America
E-mail: faouzi.triki@univ-grenoble-alpes.fr and darko@wpi.edu

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Abstract
We show in this paper stability estimates for the fault inverse problem. In this problem, faults are assumed to be planar open surfaces in a half-space elastic medium with known Lamé coefficients. A traction-free condition is imposed on the boundary of the half space. Displacement fields present jumps across faults, called slips, while traction derivatives are continuous. It was proved in Volkov et al (2017 Inverse Problems 33 055018) that if the displacement field is known on an open set on the boundary of the half space, then the fault and the slip are uniquely determined. In this present paper, we study the stability of this uniqueness result with regard to the coefficients of the equation of the plane containing the fault. If the slip field is known, we state and prove a Lipschitz stability result. In the more interesting case where the slip field is unknown, we state and prove another Lipschitz stability result under the additional assumption, which is still physically relevant, that the slip field is one-directional.

Keywords: stability estimates, Lamé system, fault inverse problem, higher order jump formulas for convolution integrals with the elasticity Green’s tensor

1. Introduction

Mapping the structure of Earth’s crust in ever finer details has always captured the interest of geophysicists. Seismic and displacements data are collected by sensors and then processed using partial differential equations (PDE) models and inverse problem formulations. Typical models for the Earth’s crust involve linear elasticity equations: this is because displacements...
and deformations are very small compared to the thickness of the crust. Moreover, if local phenomena such as earthquakes or active subduction zones are studied, a half-space formulation is adequate [3, 10, 12]. With the advent of ultra accurate satellite-based measurements of surface displacements (2–5 mm resolution) the study of so called ‘slow earthquakes’ [5, 7, 9, 11, 16] has recently attracted a lot of attention. Most authors first set a profile for the interface between tectonic plates (also called faults) derived from seismicity or gravimetry as in [14] and then use a linear inverse algorithm for determining slip fields on faults. A popular algorithm is the one explained in Tarantola’s textbook [17]. In addition to recovering these slip fields from surface displacement measurements, some authors have sought to simultaneously recover some geometric features of the fault, such as the dip angle [3, 10]. However, until recently, there was no formal mathematical proof that the simultaneous recovery of the (piecewise linear) geometry of the fault and the slip was at all possible. This was achieved in [21]. From there the second author and Sandmuniene have derived a deterministic and a stochastic fault reconstruction algorithm [20] and estimated convergence rates to the solution of the inverse problem. In [20], these convergence rates still depend on the intrinsic stability of the underlying inverse problem. Although numerical investigations hinted at a possible Lipschitz type stability, these stability estimates were still unknown at the time of writing [20], and they are the subject of this present study. In general, a uniqueness statement for solving an inverse problem is not of great practical use without a stability result. From a pragmatic and computational point of view, mathematical objects can only be computed approximately and real life field data is always tinted by measurement errors, so one would not want these errors to grow exponentially in inversion algorithms.

A literature review of the field of inverse problems will show that stability results are notoriously difficult to derive and prove. The major difficulty in proving such results is that solutions to inverse problems are not explicitly formulated. There are a few papers on stability estimates for the recovery of cracks in materials, which is the analog of faults in Earth’s crust. In an earlier paper, Friedman and Vogelius [8] showed a stability result for the recovery of linear cracks. In that paper the governing equation was the two dimensional conductivity and outer boundary conditions were prescribed to adequate values. In [1], Alessandrini et al proved a general log log stability estimate for the Hausdorff distance between two $C^{1,1}$ domains where in each domain there is a solution to the same conductivity equation with same Neumann condition and the stability estimate is in the $L^2$ distance between the corresponding Dirichlet outputs on one part of the boundary. In [2], Ammari and the first author were able to improve this log log estimate based on the assumptions that Dirichlet data is available for a whole range of frequencies and boundaries are a priori known to be open real analytic curves. In [4], Beretta et al were also able to derive and prove an interesting Lipschitz stability result. Their result pertains to two dimensional linear elasticity in bounded domains with cracks. In the case of linear cracks they were able to derive Lipschitz continuity of the Hausdorff distance between cracks in terms of overdetermined boundary data. Here we need to explain that the case of faults which pertains to our research project is drastically different since we can not impose boundary conditions, we are simply passively measuring displacements on one part of the boundary of an infinite domain while an unknown slip field on an unknown fault is the forcing term of our governing equations.

This paper is organized as follows. In section 2 we introduce the PDE for the forward fault problem and we recall the uniqueness statement for the inverse fault problem proved in [21]. Section 3 contains our first stability result and its proof. It relies on the implicit function theorem. Indeed, using the Green function for the forward PDE, solutions can be represented by convolution with the slip on the fault. We thus define a fault to surface operator. At fixed slip, in effect, this introduces a $C^1$ function $\phi$ from the set of geometry parameters $m$ in $\mathbb{R}^3$ defining
the plane containing the fault to the space of surface measurements. We know from [21] that \( \phi \) is injective. Thus, by the inverse function theorem, if \( \nabla \phi(m) \) has full rank, \( \phi^{-1} \) is \( C^\infty \) in a neighborhood of \( \phi(m) \) and is therefore Lipschitz continuous. The crux of the proof is in proving that \( \nabla \phi(m) \) has full rank. This is established by an argument by contradiction. If \( \nabla \phi(m) \) does not have full rank, then using relations on jumps for the Green function of the forward problem (and of its derivatives), we can derive a PDE for the slip field \( h \) on the fault. Finally, we prove that this PDE can only have the trivial solution, completing the proof. It turns out that although the PDE on \( h \) is a relatively simple transport equation in most cases, in the particular case of horizontal faults a much more complicated system of PDE for \( h \) must be solved. In section 4, we assume that a fixed but unknown slip \( h_0 \) is occurring on a plane with geometry parameter \( m_0 \). Thus in this case it is not possible to evaluate the difference \( \| \phi(m) - \phi(m_0) \| \).

Instead, we use a linear operator \( A_m \) mapping any slip \( \phi \) to surface measurements \( A_m \phi \) and we may minimize \( \inf A_m \phi - A_m h_0 \| \) where the inf is taken over all possible slips. This quantity is proven to be bounded below by a constant times \( |m - m_0| \) under the additional assumption that \( h_0 \) is one-directional (that direction is not known for the inverse problem), or that \( h_0 \) is a gradient. Finally, the rather technical formulas for the jumps of integrals containing a convolution of the elasticity Green tensor with a vector field density are shown in the appendix. Although a related formula is standard in solid mechanics, we have not found formulas for the jumps of the first and second derivatives in the literature. This is probably due to the fact that they are not directly related to a physical problem and that they may be too intricate to prove without the use of symbolic computation software.

2. Mathematical model and uniqueness result

2.1. Forward problem

Using the standard rectangular coordinates \( x = (x_1, x_2, x_3) \) of \( \mathbb{R}^3 \), we define \( \mathbb{R}^3^- \) to be the open half-space \( x_3 < 0 \). Let \( \partial_i \) denote the derivative in the \( i \)th coordinate. In this paper we consider the case of linear, homogeneous, isotropic elasticity; the two Lamé constants \( \lambda \) and \( \mu \) will be two positive constants. For a vector field \( u = (u_1, u_2, u_3) \) the stress and strain tensors will be denoted as follows:

\[
\sigma_{ij}(u) = \lambda \text{div} u \delta_{ij} + \mu (\partial_i u_j + \partial_j u_i),
\]

\[
\epsilon_{ij}(u) = \frac{1}{2} (\partial_i u_j + \partial_j u_i),
\]

and the stress vector in the direction \( e \in \mathbb{R}^3 \) will be denoted by

\[
T_e u = \sigma(u) e.
\]

Let \( \Gamma \) be a Lipschitz open surface which is strictly included in \( \mathbb{R}^3^- \), with a normal vector \( n \). We define the jump \( [v] \) of a vector field \( v \) across \( \Gamma \) to be

\[
[v](x) = \lim_{h \to 0^+} v(x + hn) - v(x - hn),
\]

for \( x \) in \( \Gamma \), if this limit exists. Let \( u \) be the displacement field solving

\[
\mu \Delta u + (\lambda + \mu) \nabla \text{div} u = 0 \text{ in } \mathbb{R}^3^- \setminus \Gamma, \tag{2.1}
\]

\[
T_e u = 0 \text{ on the surface } x_3 = 0, \tag{2.2}
\]

\[
T_n u \text{ is continuous across } \Gamma, \tag{2.3}
\]
that is, \( u = g \) is a given jump across \( \Gamma \),
\[
[u] = g
\]
where \( e_3 \) is the vector \((0,0,1)\). For vector fields \( v, w \) in \( \mathbb{R}^3 \setminus \Gamma \) whose gradient is square integrable we introduce the bilinear product
\[
B(v, w) = \int_{\mathbb{R}^3 \setminus \Gamma} \lambda \, \text{tr}(\nabla v) \, \text{tr}(\nabla w) + 2\mu \, \text{tr} (e(v)e(w)),
\]
where \( \text{tr} \) is the trace. Let \(|x|\) be the Euclidean norm of \( x \). In [21], we defined the functional space \( \mathcal{V} \) of vector fields \( v \) defined in \( \mathbb{R}^3 \setminus \Gamma \) such that \( \nabla v \) and \( \frac{v}{(1 + |x|^2)^{1/2}} \) are in \( L^2(\mathbb{R}^3 \setminus \Gamma) \), and we proved that the following four norms are equivalent on \( \mathcal{V} \):
\[
\|v\|_1 = \left( \int_{\mathbb{R}^3 \setminus \Gamma} |\nabla v|^2 \right)^{1/2} + \left( \int_{\mathbb{R}^3 \setminus \Gamma} \frac{|v|^2}{1 + |x|^2} \right)^{1/2},
\]
\[
\|v\|_2 = \left( \int_{\mathbb{R}^3 \setminus \Gamma} |\nabla v|^2 \right)^{1/2},
\]
\[
\|v\|_3 = \left( \int_{\mathbb{R}^3 \setminus \Gamma} |e(v)|^2 \right)^{1/2},
\]
\[
\|v\|_4 = B(v, v)^{1/2}.
\]
Let \( D \) be a bounded domain with a Lipschitz boundary \( \partial D \) containing \( \Gamma \). We define the Sobolev space \( \dot{H}^2(\Gamma)^2 \) to be the set of restrictions to \( \Gamma \) of tangential fields in \( H^2(\partial D)^2 \) supported in \( \Gamma \). We proved in [21] the following theorem.

**Theorem 2.1.** Let \( g \) be in \( \dot{H}^2(\Gamma)^2 \). The problem (2.1)–(2.4) has a unique solution in \( \mathcal{V} \). In addition the solution \( u \) satisfies the decay conditions (2.5).

In this paper we will only consider forcing terms \( g \) which are tangential to \( \Gamma \). Physically, this suggests that the fault \( \Gamma \) is not opening or starting to self intersect: only slip is allowed. We recall that if \( g \) is continuous, the support of \( g \), \( \text{supp} \, g \), is equal to the closure of the set of points in \( \Gamma \) where \( g \) is nonzero; in general \( \text{supp} \, g \) is defined in the sense of distributions.

### 2.2. Fault inverse problem

Can we determine both \( g \) and \( \Gamma \) from the data \( u \) given only on the plane \( x_3 = 0 \)? The following theorem in [21] asserts that this is possible if the data is known on a relatively open set of the plane \( x_3 = 0 \).

**Theorem 2.2.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be two bounded open surfaces, with smooth boundary, such that each of them is included in a rectangle strictly contained in \( \mathbb{R}^3 \). For \( i \) in \( \{1, 2\} \), assume that \( u^i \) solves (2.1)–(2.5) for \( \Gamma_i \) in place of \( \Gamma \) and \( g^i \), a tangential field in \( H^1(\Gamma)^2 \), in place of \( g \). Assume that \( g^i \) has full support in \( \Gamma_i \), that is, \( \text{supp} \, g^i = \overline{\Gamma_i} \). Let \( V \) be a nonempty open subset in \( \{x_3 = 0\} \). If \( u^1 \) and \( u^2 \) are equal in \( V \), then \( \Gamma_1 = \Gamma_2 \) and \( g^1 = g^2 \).

The solution \( u \) to problem (2.1)–(2.4) can also be written out as the convolution on \( \Gamma \)
\[
\int_{\Gamma} H(x, y, n) g(y) \, d\sigma(y),
\]
where \( H \) is the Green tensor associated to the system (2.1)–(2.5), and \( n \) is the normal to \( \Gamma \). The practical determination of this adequate half-space Green tensor \( H \) was first studied in [15] and later, more rigorously, in [18]. In particular, \( H \) satisfies the decay conditions
\[
H(x, y, n) = O(|x|^{-2}), \quad \nabla_x H(x, y, n) = O(|x|^{-3}), \quad |x| \to \infty,
\]
uniformly in \( y \) and in \( n \), as long as \( y \) remains in a bounded subset of \( \mathbb{R}^3 \). Due to formula (2.6) we can define a continuous mapping \( M \) from tangential fields \( g \) in \( H^1_0(\Gamma)^2 \) to surface displacement fields \( u(x_1, x_2, 0) \) in \( L^2(\mathcal{V}) \) where \( u \) and \( g \) are related by (2.1)–(2.5). Theorem 2.2 asserts that this mapping is injective, so an inverse operator can be defined. It is well known, however, that such an operator \( M \) is compact, therefore its inverse is unbounded. It is thus clear that any stable numerical method for reconstructing \( g \) from \( u(x_1, x_2, 0) \) will have to use some regularization process. Our goal in this paper is to analyze the stability properties of the fault inverse problem with regard to the plane containing \( \Gamma \), first in section 3 as the slip on the fault is fixed, and then in section 4 in the case of unknown slips.

3. Lipschitz stability of the fault geometry for a fixed slip

3.1. Preliminary results

Let \( G(x, y, n) \) be the free space elasticity Green function such that for any open Lipschitz surface \( \Gamma \) in \( \mathbb{R}^3 \) and any displacement field \( u' \) satisfying
\[
\mu \Delta u' + (\lambda + \mu) \nabla \text{div} u' = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Gamma,
\]
\[
T_n u' \quad \text{is continuous across} \quad \Gamma,
\]
\[
[u'] = g \quad \text{is a given jump across} \quad \Gamma,
\]
\[
\begin{aligned}
\nabla u'(x) &= O \left( \frac{1}{|x|^2} \right), \\
\nabla u'(x) &= O \left( \frac{1}{|x|^3} \right), \quad \text{uniformly as} \quad |x| \to \infty,
\end{aligned}
\]
we have
\[
u'(x) = \int_{\Gamma} G(x, y, n) g(y) \, d\sigma(y),
\]
for all \( x \) in \( \mathbb{R}^3 \setminus \Gamma \). The existence of \( G \) has been known for a long time for smooth surfaces and later it was proved that formula (3.1) is also valid if \( \Gamma \) is just Lipschitz regular, [6]. An explicit formula for \( G \) can be easily found from Kelvin’s tensor \( K \), where \( K \) itself is given by
\[
K_y(x, y) = \frac{1}{8\pi \mu(\lambda + 2\mu)} \left( (\lambda + \mu) \partial_n r \partial_n r + (\lambda + 3\mu) \delta_{ij} \right) \frac{1}{r},
\]
and \( r = |x - y| \). Now if \( v \) is any fixed vector, define
\[
G(x, y, v) = \left( T_{v(\cdot)} K(x, y) \right)'.
\]
The difference \( (H - G)(x, y, v) \) is smooth for all \( x \) and \( y \) in \( \mathbb{R}^3 \) and \( v \) in \( \mathbb{R}^3 \); this is true by construction of \( H \), [18]. In particular
\[
\int_{\Gamma} (H - G)(x, y, n) g(y) \, d\sigma(y)
\]
is a smooth function of \( x \) in all of \( \mathbb{R}^3 \). It follows that any jump formula across \( \Gamma \) for 
\[
\int_{\Gamma} G(x, y, n) g(y) \, d\sigma(y)
\]
and its derivatives will also hold for 
\[
\int_{\Gamma} H(x, y, n) g(y) \, d\sigma(y).
\]
Evidently, it is more convenient to derive jump formulas for 
\[
\int_{\Gamma} G(x, y, n) g(y) \, d\sigma(y)
\]
since it is much simpler to write \( G \) in closed form rather than \( H \). In this paper, the following jump formulas will be needed.

**Lemma 3.1.** Let \( \Gamma \) be an open surface in \( \mathbb{R}^3 \) included in the plane \( x_3 = 0 \). Let \( g \) be a three dimensional vector field on \( \Gamma \) with regularity \( C_\infty^\infty(\Gamma) \). Then the following jump formulas across \( \Gamma \) hold:

\[
\begin{align*}
&\int_{\Gamma} G(x, y, e_3) g(y) \, dy_1 dy_2 = g(x), \\
&\int_{\Gamma} (\partial_3 G)(x, y, e_3) g(y) \, dy_1 dy_2 = -\partial_i g(x), \\
&\int_{\Gamma} G(x, y, e_1) g(y) \, dy_1 dy_2 = \frac{\lambda}{\lambda + 2\mu} g_1(x) e_3 + g_3(x) e_1, \\
&\int_{\Gamma} (\partial_3 G)(x, y, e_3) g(y) \, dy_1 dy_2 = \frac{\lambda}{\lambda + 2\mu} (\text{div } r g_\Gamma)(x) e_3 + \nabla g_3(x), \\
&\partial_3 \int_{\Gamma} G(x, y, e_3) g(y) \, dy_1 dy_2 = -\frac{\lambda}{\lambda + 2\mu} (\text{div } r g_\Gamma)(x) e_3 - \nabla g_3(x),
\end{align*}
\]

where \( g = (g_1, g_2, g_3) \), \( g_\Gamma = (g_1, g_2, 0) \), \( \text{div } r g_\Gamma = \partial_1 g_1 + \partial_2 g_2 \), and \( \nabla g \cdot e_3 = (\partial_1 g_3, \partial_2 g_3, 0) \). For the normal derivative of (3.7) we have the jump formula:

\[
\begin{align*}
\partial_3 \int_{\Gamma} (\partial_3 G)(x, y, e_3) g(y) \, dy_1 dy_2 &= \left( \frac{3\lambda + 4\mu}{\lambda + 2\mu} \partial_1^2 g_1 + \partial_2^2 g_1 + 2 \frac{\lambda + \mu}{\lambda + 2\mu} \partial_1 \partial_2 g_2 \right) e_1 \\
&+ \left( \frac{3\lambda + 4\mu}{\lambda + 2\mu} \partial_2^2 g_2 + \partial_3^2 g_2 + 2 \frac{\lambda + \mu}{\lambda + 2\mu} \partial_1 \partial_3 g_3 \right) e_2 \\
&- \frac{\lambda}{\lambda + 2\mu} (\partial_1^2 + \partial_2^2) g_3 e_3.
\end{align*}
\]

Finally, we give a jump formula for the normal derivative of (3.6):

\[
\begin{align*}
\partial_3 \int_{\Gamma} G(x, y, e_1) g(y) \, dy_1 dy_2 &= \left( \frac{3\lambda + 4\mu}{\lambda + 2\mu} \partial_1 g_1 + \partial_2 g_2 \right) e_1 + \left( \frac{\lambda}{\lambda + 2\mu} \partial_1 g_1 + \partial_2 g_2 \right) e_2 \\
&- \frac{\lambda}{\lambda + 2\mu} \partial_3 g_3 e_3.
\end{align*}
\]

**Proof.** Formula (3.4) is well known, however, formulas (3.5)–(3.10) are not readily found in the literature. It is therefore worth providing a proof, which can be found in the appendix. \( \square \)

**Lemma 3.2.** Let \( g \) be a tangential vector field on \( \Gamma \) with regularity \( H^1_0(\Gamma) \). Then the jump formulas (3.4) and (3.6) still hold in the \( H^1_0(\Gamma) \) norm, while the jump formulas (3.5), (3.7), (3.8) and (3.10) hold as continuous linear operations from \( H^1_0(\Gamma) \) to \( L^2(\Gamma) \). The jump
formula \((3.9)\) holds as a continuous linear operator from \(H^1_0(\Gamma)\) to \(H^{-1}(\Gamma)\).

The same jump formulas also hold with \(H\) in place of \(G\).

**Proof.** This is clear since \(C_c^\infty(\Gamma)\) is dense in \(H^1_0(\Gamma)\) and \((H - G)(x, y, n)\) is smooth for all \(x\) and \(y\) in \(\mathbb{R}^3\).

**Lemma 3.3.** Let \(\Omega\) be an open bounded subset in \(\mathbb{R}^2\). Let \(f\) be in \(C_c^\infty(\mathbb{R}^2)\) such that \(f^{-1}(\{0\}) \cap \Omega\) has zero measure. Let \(\tau\) be a nonzero vector in \(\mathbb{R}^2\). Assume that \(u\) is in \(H^1_0(\Omega)\) and satisfies in \(\Omega\) the partial differential equation

\[\partial_\tau(fu) + \alpha u = 0,\]  

(3.11)

where \(\alpha\) is a constant in \(\mathbb{R}\). Then \(u\) is zero.

**Proof.** We first assume that \(\alpha \neq 0\). We note that for any function \(g\) in \(C_c^\infty(\Omega)\) the divergence theorem implies that

\[0 = \int_\Omega \text{div}(g\tau) = \int_\Omega \nabla g \cdot \tau,\]  

(3.12)

which can be extended by density to any \(g\) in \(H^1_0(\Omega)\). Let \(f_n\) be a sequence in \(C_c^\infty(\Omega)\) which converges to \(f\) in the \(H^1\) norm. By formula (3.12),

\[0 = \int_\Omega \nabla (f_n fu^2) \cdot \tau = \int_\Omega f_n u \nabla (fu) \cdot \tau + fu \nabla (f_n u) \cdot \tau.\]  

(3.13)

Next we want to prove that

\[\lim_{n \to \infty} \int_\Omega f_n u \nabla (fu) \cdot \tau = \lim_{n \to \infty} \int_\Omega fu \nabla (f_n u) = \int_\Omega f^+ u \nabla (fu).\]  

(3.14)

The first limit in (3.14) is clear. We observe that

\[f_n u \nabla (fu) - fu \nabla (f_n u) = u^2 (f \nabla f_n - f_n \nabla f).\]

Since \(\Omega\) is two dimensional and \(u\) is \(H^1_0(\Omega)\), we can assert by the Sobolev embeddings that \(u^2\) is in \(L^2(\Omega)\). \(f \nabla f_n - f_n \nabla f\) converges to \(f^+ \nabla f^+ + f^- \nabla f^-\) in \(L^2(\Omega)\), which is zero, so the second limit in (3.14) is proved. Going back to (3.12), we have now shown that

\[\int_\Omega f^+ u \nabla (fu) \cdot \tau = 0.\]

We now multiply equation (3.11) by \(f^+ u\), we integrate over \(\Omega\), and we use that \(\alpha\) is nonzero to find that \(\int_{\Omega} f^+ u^2\) is zero. Similarly, \(\int_{\Omega} f^- u^2\) is zero. As \(f^{-1}(\{0\}) \cap \Omega\) has measure zero, this shows that \(u\) is zero.

We now consider the case where \(\alpha\) is zero. After a linear change of variables, we may assume that \(\tau\) is the base vector \(e_1\). Let \(A\) be a constant such that \(\Omega\) is included in the box \(-A \leq x_1, x_2 \leq A\). We first note that for any function \(g\) in \(C_c^\infty((-A, A) \times (-A, A))\) for any \(-A \leq x_1, x_2 \leq A \):
\[ |g(x_1, x_2)| \leq \left( \int_{-A}^{A} |\partial_{x_1} g(x_1, x_2)|^2 dx_1 \right)^{1/2} (2A)^{1/2}, \]

Thus
\[
\int_{-A}^{A} \int_{-A}^{A} |g(x_1, x_2)|^2 dx_1 dx_2 \leq 4A^2 \int_{-A}^{A} \int_{-A}^{A} |\partial_{x_1} g(x_1, x_2)|^2 dx_1 dx_2,
\]

and this estimate can be extended to all \( g \) in \( H^1_0((-A, A) \times (-A, A)) \). It thus follows that \( fu \) in zero in \((-A, A) \times (-A, A)\), thus \( u \) is zero in \( \Omega \).

### 3.2. Lipschitz stability theorem for a fixed slip

Let \( R \) be a closed rectangle in the plane \( x_3 = 0 \). Let \( B \) be a set of triplets \((a, b, d)\) such that the set
\[
\Gamma_{a,b,d} = \{(x_1, x_2, ax_1 + bx_2 + d) : (x_1, x_2) \in R\}
\]
is included in the half-space \( x_3 < 0 \). When appropriate, we will use the short hand notation \( m = \{a, b, d\} \). We assume that \( B \) is a closed and bounded subset of \( \mathbb{R}^3 \). It follows that
\[
\inf \{ \text{dist}(\Gamma_m, \{x \in \mathbb{R}^3 : x_3 = 0\}) : m \in B \} > 0,
\]
where \( \text{dist} \) is the distance between subsets of \( \mathbb{R}^3 \). We set \( \mathbf{n} = (-a, -b, 1)/\sqrt{1 + a^2 + b^2} \) to be the normal vector on \( \Gamma_m \) and \( \sigma = \sqrt{1 + a^2 + b^2} \) the surface element. Let \( H^1_0(R)^2 \) be the space of vector fields \( g = (g_1, g_2) \) on \( R \) with \( H^1_0 \) regularity. Define \( g_m \) the tangential vector field on \( \Gamma_m \), \( g_m = (g_1, g_2, ag_1 + bg_2) \). Define the operator
\[
A_m : H^1_0(R)^2 \to L^2(V)^3
\]
\[
g \to \int_R H(x, y_1, y_2, ay_1 + by_2 + d, \mathbf{n})g_m(y_1, y_2)\sigma dy_1 dy_2.
\]

It is clear that \( A_m \) is linear, continuous, and compact. Note that due to theorem 2.2, \( A_m \) is injective. In the remainder of this section we fix a nonzero \( h \) in \( H^1_0(R) \), and we define a nonlinear function
\[
\phi : B \to L^2(V)^3
\]
\[
\phi(m) = \int_R H(x, y_1, y_2, ay_1 + by_2 + d, \mathbf{n})h_m(y_1, y_2)\sigma dy_1 dy_2,
\]
where \( m = \{a, b, d\} \). Due to the regularity of the Green tensor \( H(x, y, n) \), it is clear that \( \phi \) is real analytic in \( m \). Now due to theorem 2.2, \( \phi \) is injective. We now want to prove that the inverse of \( \phi \) defined on \( \phi(B) \) and valued in \( B \) is Lipschitz continuous. This will be achieved by showing that we can apply the inverse function theorem.

**Theorem 3.1.** Fix a nonzero \( h \) in \( H^1_0(R)^2 \) and define the function \( \phi \) from \( B \) to \( L^2(V)^3 \) by (3.17). Assume that:

(i) \( B \) does not contain any triplet in the form \((0, 0, d)\), in other words no horizontal profiles are allowed for the faults.
or

(ii). \( h \) is one-directional.

or

(iii). \( h \) is in \( H^2_0(\mathbb{R})^2 \).

There is a positive constant \( C \) such that

\[
C|m - m'| \leq \|\phi(m) - \phi(m')\|_{L^2(V)},
\]

for all \( m \) and \( m' \) in \( B \).

**Remark 3.1.** The constant \( C \) in estimate (3.19) depends on the fixed slip \( h \) and on the compact set of values of the geometry parameters \( B \), but is otherwise independent of the choice of \( m \) and \( m' \).

**Proof.** Fix \( m \) in \( B \). Our first task is to evaluate \( \nabla \phi(m) \). We first note that \( n \sigma \) simplifies to \((-a, -b, 1)\). We recall that \( H(x, y, n) \) is linear in \( n \). By the chain rule, for \( y = ay_1 + by_2 + d \),

\[
\frac{\partial}{\partial a} H(x, y, n) = \frac{\partial y_3}{\partial a} (\partial_n H)(x, y, n) - H(x, y, e_1) = y_1 (\partial_n H)(x, y, n) - H(x, y, e_1).
\]

Similarly,

\[
\frac{\partial}{\partial b} H(x, y, n) = y_2 (\partial_n H)(x, y, n) - H(x, y, e_2),
\]

and

\[
\frac{\partial}{\partial d} H(x, y, n) = (\partial_n H)(x, y, n).
\]

Arguing by contradiction, assume that for some \( m \) in \( B \), \( \nabla \phi(m) \) does not have full rank. Then there is a nonzero vector \((\gamma_1, \gamma_2, \gamma_3)\) in \( \mathbb{R}^3 \) such that

\[
\gamma_1 \frac{\partial}{\partial a} \phi(m) + \gamma_2 \frac{\partial}{\partial b} \phi(m) + \gamma_3 \frac{\partial}{\partial d} \phi(m) = 0.
\]

Set \( f(y_1, y_2) = \gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 \). We note that \((\gamma_1 \frac{\partial}{\partial w_a} + \gamma_2 \frac{\partial}{\partial w_b} + \gamma_3 \frac{\partial}{\partial w_d})h_m = \nabla f \cdot h_m e_3 \). Relation (3.20) can be expressed as

\[
\int_R \left[ H(x, y_1, y_2, ay_1 + by_2 + d, n) \nabla f \cdot h_m(y_1, y_2)e_3 \sigma dy_1 dy_2 + \int_R (\partial_n H)(x, y_1, y_2, ay_1 + by_2 + d, n) h_m(y_1, y_2)f(y_1, y_2) \sigma dy_1 dy_2 
\right.

\[
- \int_R H(x, y_1, y_2, ay_1 + by_2 + d, n) f(y_1, y_2) h_m(y_1, y_2) dy_1 dy_2 = 0,
\]

for all \( x \) in \( V \). Set \( u(x) \) to be the left-hand side of (3.21) where \( x \) has been extended to \( \mathbb{R}^3 \setminus \Gamma_m \). We now proceed to prove that \( u \) is zero in \( \mathbb{R}^3 \setminus \Gamma_m \). First, it is clear that \( u \) satisfies
the elasticity equations in $\mathbb{R}^3 - \Gamma_m$ since the scalar differential operators $\partial_y$ and $\partial_x$ commute. Next, due to (3.21), $w$ is zero on $V$. By construction of the Green tensor $H$, for any $x$ on the plane $x_3 = 0$, any $y$ in $\mathbb{R}^3$, and any fixed vector $p$ in $\mathbb{R}^3$,

$$T_{e_i}(x)H(x, y, p) = 0.$$ 

We can thus take a $\partial_y$ derivative and commute the matrix differential operator $T_{e_i}(x)$ with the scalar differential operator $\partial_y$, to obtain

$$T_{e_i}(x)\partial_y H(x, y, p) = 0.$$ 

It follows that $T_{e_i}w$ is also zero in $V$ and a Cauchy–Kowaleski type argument as in the proof of theorem 2.2, which was given in [21], shows that $w$ must be zero everywhere in $\mathbb{R}^3 - \Gamma_m$. In particular the jump of $w$ across $\Gamma_m$ must also be zero. Recall the definition of $G$ given by (3.3). We note that for any vector $v$ in $\mathbb{R}^3$, $H(x, y, v) - G(x, y, v)$ is smooth for all $x$ and $y$ in $\mathbb{R}^3$, see [18]. Therefore, the jump across $\Gamma_m$ of

$$\int_R G(x, y_1, y_2, ay_1 + by_2 + d, n)\nabla f \cdot h(y_1, y_2)e_3\sigma dy_1dy_2$$

$$\int_R (\partial_y G)(x, y_1, y_2, ay_1 + by_2 + d, n)h_m(y_1, y_2)f(y_1, y_2)\sigma dy_1dy_2$$

$$- \int_R G(x, y_1, y_2, ay_1 + by_2 + d, \nabla f)h_m(y_1, y_2)\sigma dy_1dy_2. \tag{3.22}$$

is also zero. Let us write

$$e_3 = \alpha n + \tau,$$

where $\tau$ is parallel to $\Gamma_m$. We now use the fact that the free space Green function is rotation invariant. After a change of coordinates by rotation, we can assume that $\Gamma_m$ is horizontal and $\tau = \beta e_3$ (for the sake of lighter notations, the new coordinates will be named in the same way as the old coordinates). In the new coordinates we note that $h_m, e_3 = 0$, and we simply write $h$ in place of $h_m$. The expression (3.22) can be written out as

$$\int_R G(x, y_1, y_2, \tilde{d}, e_3)(\nabla \tilde{f} \cdot h)((\alpha e_3 + \beta e_1)\sigma dy_1dy_2$$

$$+ \alpha \int_R \partial_y G(x, y_1, y_2, \tilde{d}, e_3)h\tilde{f}\sigma dy_1dy_2$$

$$+ \beta \int_R \partial_y G(x, y_1, y_2, \tilde{d}, e_3)h\tilde{f}\sigma dy_1dy_2$$

$$- \int_R G(x, y_1, y_2, \tilde{d}, \nabla \tilde{f})h\sigma dy_1dy_2,$$

where $\tilde{f}$ is a nonzero affine function, and in the new coordinates $\Gamma_m$ is contained in the plane $y_3 = \tilde{d}$. This must also have a zero jump across $R + \tilde{d}$. We now proceed to write down the expression for that jump thanks to lemma 3.2 and formulas (3.4)–(3.7) to find
\[
(\nabla \tilde{f} \cdot h)(\alpha e_3 + \beta e_1)\sigma \\
+ \alpha \sigma \frac{\lambda}{\lambda + 2\mu} \left( \text{div}(\tilde{f}h) \right)e_3 \\
- \beta \sigma \partial_1(\tilde{f}h) \\
- \delta_3 \tilde{f}h \\
- \frac{\lambda}{\lambda + 2\mu} h \cdot \nabla \tilde{f}e_3 = 0.
\]

As \(\sigma \alpha = 1\), this simplifies along \(e_3\) to
\[
\frac{\lambda}{\lambda + 2\mu} \tilde{f} \text{div} h + \nabla \tilde{f} \cdot h = 0. \quad (3.23)
\]

The remaining terms lead to the equation
\[
\beta \sigma (\nabla \tilde{f} \cdot h) e_1 - \beta \sigma \partial_1(\tilde{f}h) - \partial_3 \tilde{f}h = 0,
\]
that is, to the system
\[
- \beta \sigma \partial_1(\tilde{f}h_2) - \partial_3 \tilde{f}h_2 = 0 \\
- \beta \sigma \partial_1(\tilde{f}h_1) + \beta \sigma (\partial_2 \tilde{f}h_1 + \partial_3 \tilde{f}h_2) - \partial_3 \tilde{f}h_1 = 0. \quad (3.24)
\]

Assume that condition (i) in the statement of theorem 3.1 holds. Then \(\Gamma_m\) is not horizontal, thus \(\beta \neq 0\). Note that \(\nabla \tilde{f}\) is a constant vector. Then we can use the first line of equation (3.24) in conjunction to lemma 3.3 to find that \(h_2 = 0\). Then due to the second line of (3.24) and lemma 3.3, \(h_1 = 0\). Thus we showed that \(h\) is zero in \(H^1_0(\mathbb{R}^2)\): contradiction.

If condition (ii) in the statement of theorem 3.1 holds, we set \(h = uV\), where \(u\) is a scalar function and \(V\) is a fixed vector and equation (3.23) simplifies to
\[
\frac{\lambda}{\lambda + 2\mu} V \cdot \nabla (\tilde{f}u) + \frac{2\mu}{\lambda + 2\mu} (V \cdot \nabla \tilde{f}) u,
\]
so equation (3.23) in conjunction to Lemma 3.3 can be used to show that \(u\) is zero.

Now, assume that \(\Gamma_m\) is horizontal and that condition (iii) holds. In that case \(\beta = 0\) and equation (3.24) is void. We also note that here \(\alpha = \sigma = 1\) and that equation (3.23) is still valid. We use that the jump of the \(\partial_3\), derivative across \(\Gamma_m\) of
\[
\int_{\Gamma_m} G(x, y_1, y_2, d, e_3)(\nabla f \cdot h) e_3 dy_1 dy_2 \\
+ \int_{\Gamma_m} \partial_3 G(x, y_1, y_2, d, e_3) hf dy_1 dy_2 \\
- \int_{\Gamma_m} G(x, y_1, y_2, d, \nabla f) h dy_1 dy_2
\]

is zero. To complete the proof we apply a change of coordinates by rotation about \(e_3\) such that \(\nabla f\) becomes parallel to \(e_1\) in the new coordinates. By homogeneity, we can then assume that \(f(x_1, x_2) = x_1 + \gamma_3\).
We now apply formula (3.8)–(3.10) to the (zero) \( \partial_n \) jump of (3.25) to obtain the following equation in the direction of \( e_1 \)

\[
-\partial_1 (\partial_1 (f h_1)) + \frac{3\lambda + 4\mu}{\lambda + 2\mu} \partial_1^2 (f h_1) + \partial_1^2 (f h_1) + 2 \frac{\lambda + \mu}{\lambda + 2\mu} \partial_1 \partial_2 (f h_2) - \partial f \left( \frac{3\lambda + 4\mu}{\lambda + 2\mu} \partial_1 h_1 + \partial_2 h_2 \right) = 0.
\]  

(3.26)

We then eliminate \( h_2 \) in (3.26). This is done by using (3.23) and observing that as \( \partial f = 1 \),

\[
\partial_1 \partial_2 (f h_2) = -\frac{2\mu}{\lambda} \partial_1 h_1,
\]

so (3.26) reduces to, as \( 1 + 4\frac{\mu}{\lambda + 2\mu} + \frac{3\lambda + 4\mu}{\lambda + 2\mu} = 4 + \frac{2\mu}{\lambda} \),

\[
\partial_1^2 (f h_1) + \partial_2^2 (f h_1) + \partial_1 h_1 (-4 - \frac{2\mu}{\lambda}) - \partial_2 h_2 = 0.
\]  

(3.27)

We multiply by \( f \), use again (3.23) and simplify to obtain

\[
f^2 \Delta h_1 + f \partial_1 h_1 (-3 - \frac{2\mu}{\lambda}) + (1 + \frac{2\mu}{\lambda}) f h_1 = 0.
\]  

(3.28)

Note that this is not an elliptic PDE as \( f \) may be equal to zero in \( \Gamma \). To show that \( h_1 \) is zero, fix \( \epsilon > 0 \), let \( \Gamma^+ = \{ x \in \Gamma : f(x) > \epsilon \} \) and \( \Gamma^- = \{ x \in \Gamma : f(x) < -\epsilon \} \). As \( h \) is in \( H_0^2(\Gamma) \), since \( \Gamma^+ \) is the intersection of \( \Gamma \) and a half plane, if it is nonempty, the Cauchy–Kowaleski Theorem can be applied to (3.28) to claim that \( h_1 \) is zero in \( \Gamma^+ \). We carry out the same argument on \( \Gamma^- \). Finally we let \( \epsilon \) tend to zero: this proves that \( h_1 \) is zero in \( \Gamma \). From there we claim that \( h_2 \) is also zero by recalling (3.23) and applying lemma 3.3.

We have thus proved that for all \( m \) in \( B, \nabla \phi(m) \) has full rank. We now include the set \( B \) in a subset \( B' \) of \( \mathbb{R}^3 \) such that \( B' \) is open and property (3.15) still holds for \( B' \). As for every \( m \) in \( B' \), \( \nabla \phi(m) \) has full rank, by the inverse function theorem \( \phi \) defines a \( C^1 \) diffeomorphism from an open neighborhood \( U_m \) to its image by \( \phi \) on \( L^2(V) \). Thus, there is a positive constant \( C_m \) such that for all \( m' \) and \( m'' \) in \( U_m \):

\[
C_m |m' - m''| \leq \| \phi(m') - \phi(m'') \|_{L^2(V)}.
\]

Arguing by contradiction, assume that estimate (3.19) fails to be true. Then there are two sequences \( m'_n \) and \( m''_n \) in \( B \) such that \( m'_n \neq m''_n \) and \( \| \phi(m'_n) - \phi(m''_n) \|_{L^2(V)} \) tends to zero. As \( B \) is compact, we may assume after extracting subsequences that \( m'_n \) converges to \( \tilde{m} \) and \( m''_n \) converges to \( \tilde{m} \). Since \( \phi \) is continuous and injective we must have \( \tilde{m} = \tilde{m} \). But for all \( n \) large enough \( m'_n \) and \( m''_n \) must be in the open neighborhood \( U_{\tilde{m}} \); contradiction.

\[ \square \]

4. Second stability theorem: the case of unknown slips

In applications the slip on \( \Gamma \) is unknown, therefore this slip cannot be used to minimize \( \| \phi(m) - \phi(m_0) \|_{L^2(V)} \) for \( m \) over \( B \) as in (3.19) to find the geometry \( m_0 \). Instead, one has to minimize \( |A_m h - A_m h_0|_{L^2(V)} \) over all geometries \( m \) and all slips \( h \). The unique minimum is zero and only achieved for \( m = m_0 \) and \( h = h_0 \) according to theorem 2.2. To obtain Lipschitz stability in \( \| m - m_0 \| \) we need to add an additional assumption on \( h_0 \). A possible additional
assumption is to require that $h_0$ be one-directional. Physically, this means that the slip on the fault $\Gamma$ occurs in only one direction. Interestingly, this condition already appeared in another theoretical study of destabilization modes of faults, [19], as discussed in section 2.2.

Recall the definition (3.16) of the operator $A_m$. We will need the following lemma.

**Lemma 4.1.** Let $P_m$ be the orthogonal projection onto $\overline{R(A_m)}$ in $L^2(V)$. Fix $m_0$ in $B$. Then there is a constant $C$ depending on $m_0$ but not on $m$ such that

$$\|P_m - P_{m_0}\| \leq C|m - m_0|,$$  \hspace{1cm} (4.1)

for all $m$ in $B$.

**Proof.** We first note that the closure $\overline{R(A_m)}$ of the range of $A_m$ in $L^2(V)$ is equal to $\overline{R(A_mA_m^*)}$: this is true because the nullspace $N(A_m^*)$ is equal to $N(A_mA_m^*)$ and we can then take the orthogonals of each of this subspace. If $m$ tends to $m_0$, it is clear $A_mA_m^*$ is norm convergent to $A_mA_m^*$, and that $\|A_mA_m^* - A_mA_m^*\| = O(|m - m_0|)$. Let $C$ be the circle in the complex plane centered at the origin with radius $\|A_mA_m^*\| + 1$. The orthogonal projection on the image of $A_mA_m^*$ can be represented by the contour integral as follows (see [13]):

$$P_m = \frac{1}{2\pi i} \int_C (\zeta I - A_mA_m^*)^{-1} d\zeta,$$

for all $m$ large enough, and where $I$ is the identity operator in $L^2(V)$. This leads to (4.1). \hfill \Box

**Theorem 4.1.** Fix a nonzero $h_0$ in $H^1_0(R)$ and $m_0$ in $B$. Assume that $h_0$ satisfies one of the two following additional assumptions:

(i). $h_0$ is one-directional, that is, $h_0$ is parallel to a fixed tangential vector.

(ii). $h_0$ is the gradient of a function $\varphi$ in $H^2(\Gamma)$.

Then there exists a positive constant $C$ depending on $m_0$ and $h_0$ but not on $m$ such that

$$\inf_{h \in H^1_0(R)} \|A_mh - A_mh_0\|_{L^2(V)} \geq C|m - m_0|,$$  \hspace{1cm} (4.2)

for all $m$ in $B$.

**Proof.** Since $I - P_m$ is an orthogonal projection,

$$\|A_mh - A_mh_0\|_{L^2(V)}^2 \geq \|(I - P_m)(A_mh - A_mh_0)\|_{L^2(V)}^2.$$  

Since $P_m$ is the orthogonal projection on $\overline{R(A_m)}$, $P_mA_mh = A_mh$, and we obtain

$$\|A_mh - A_mh_0\|_{L^2(V)}^2 \geq \|(I - P_m)(A_mh_0)\|_{L^2(V)}^2 = \|P_mA_mh_0 - A_mh_0\|_{L^2(V)}^2.$$  

Arguing by contradiction, assume that there is a sequence $m_n$ in $B$ converging to $m_0$ such that

$$\|P_mA_mh_0 - A_mh_0\|_{L^2(V)} = o(|m_n - m_0|).$$  \hspace{1cm} (4.3)

It clearly follows that

$$\|(I - P_m_0)(A_{m_n} - A_{m_0})h_0\|_{L^2(V)} = o(|m_n - m_0|).$$  

As
\[\|(P_{m_n} - P_m)(A_{m_n} - A_{m_0})h_0\|_{L^2(V)} = o(|m_n - m_0|),\]
we may write
\[\|(I - P_{m_n})(A_{m_n} - A_{m_0})h_0\|_{L^2(V)} = o(|m_n - m_0|).\]
Equivalently,
\[\|(I - P_{m_n})(A_{m_n} - A_{m_0})h_0\|_{L^2(V)} = o(1).\]
(4.4)

As \(\frac{m_n - m_0}{|m_n - m_0|}\) is a sequence on the unit sphere of \(\mathbb{R}^3\), after possibly extracting a subsequence we may assume that it converges to some \(q\) with \(|q| = 1\). Taking the limit as \(n \to \infty\) in (4.4) we find
\[(I - P_{m_n})\partial_q A_{m_n} h_0 = 0,
\]
thus, there is a \(g_0\) in \(H^1_0(\mathbb{R})\) such that
\[\partial_q A_{m_n} h_0 = A_{m_n} g_0.\]
(4.5)

We then set \(q = (\gamma_1, \gamma_2, \gamma_3)\) as in the proof of theorem 3.1. Given the form (3.16) of the operator \(A_m\) for \(m\) in \(B\), \(A_m g_0\) can be extended to a vector field on \(\mathbb{R}^3 - \Gamma_m\) satisfying equations (2.1)–(2.5) with \(g_0\) in place of \(g\) and \(\Gamma_m\) in place of \(\Gamma\). In particular, the normal jump of that extended vector field across \(\Gamma_m\) is zero. The same argument as in the proof of theorem 3.1 can then be carried out to show that \(h_0\) must satisfy, due to (4.5), a partial differential equation on \(\Gamma_m\). Due to the \(A_m g_0\) term on the right-hand side this equation will be unhelpful along any direction which is tangential to \(\Gamma_m\). However we obtain the same homogeneous equation in the normal direction which we write here for \(h_0\):
\[\lambda \frac{\partial}{\partial \lambda} f \text{div} h_0 + \nabla f \cdot h_0 = 0,\]
(4.6)

where this equation was written in a rotated coordinate system such that \(\Gamma_m\) is parallel to the new \(x_1, x_2\) plane, \(h_0\) depends only on the new coordinates \(x_1, x_2\) and \(f\) is a nonzero affine function whose coefficients depend linearly on \(\gamma_1, \gamma_2, \gamma_3\).

If assumption (i) on \(h_0\) holds then we can apply lemma 3.3 to claim that \(h_0\) is zero: contradiction.

If assumption (ii) on \(h_0\) holds then \(\varphi\) satisfies the partial differential equation
\[\lambda \partial_\lambda f \Delta \varphi + \nabla f \cdot \nabla \varphi = 0.\]
(4.7)

Let \(\text{sgn}_0\) be the sign function defined on \(\mathbb{R}\) by: \(\text{sgn}_0(t) = -1\) if \(t < 0\), \(\text{sgn}_0(0) = 0\) and \(\text{sgn}_0(t) = 1\) if \(t > 0\). Multiplying (4.7) by \((1 + \frac{\lambda}{2\mu})|f|^{\frac{\lambda}{\mu}} \text{sgn}_0(f)\), we obtain
\[\text{div} \left(|f|^{1 + \frac{\lambda}{\mu}} \nabla \varphi\right) = 0.\]
As by assumption \( h_0 = \nabla \varphi \) is in \( H^1_0(\Gamma) \), multiplying by \( \varphi \) and applying Green’s theorem leads to
\[
\int_{\Gamma} |f|^{1 + \frac{2i}{\pi}} |\nabla \varphi|^2 = 0. \tag{4.8}
\]

Since \( f \) is affine, it vanishes on a set with low dimensionality. We then deduce from the identity (4.8) that \( \varphi \) is zero: contradiction. \( \Box \)

5. Conclusion

In this paper we studied the well-posedness of the fault inverse problem. We derived stability estimates for determining the plane containing the fault. We proved that if the slip field is known, this determination is Lipschitz stable. In the more realistic case where the slip field is unknown, we showed another Lipschitz stability result under the additional assumption, which seems physically relevant, that the slip field is one-directional. The proofs of the results presented in this paper are nonconstructive and thus provide no insight on how the stability constants depend on the physics and on the geometry of the problem. This will be the subject of forthcoming work.

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Appendix. Proof of lemma 3.1

In this section we need to use the following notations for three by three matrices \( M \). Let \( M_1, M_2, M_3 \) be the three column vectors of \( M \). Let \( (M_1, M_2, M_3) \) denote the vector \( M \). When necessary, we will also use the notation
\[
\begin{pmatrix}
M_1 \\
M_2 \\
M_3
\end{pmatrix}.
\]

To show formula (3.5), we observe that if \( x \) is not in \( \Gamma \), given that \( g \) is in \( C^\infty_c(\Gamma) \), we can use integration by parts to write
\[
\int_{\Gamma} (\partial_{y_1} G)(x, y, n)g(y) \, dy_1dy_2 = -\int_{\Gamma} G(x, y, n)(\partial_{y_1} g)(y) \, dy_1dy_2,
\]
and then we can apply formula (3.4). We are not aware of formulas (3.6) and (3.7) appearing anywhere in the literature, so we believe that a full proof is called for. By a Taylor expansion,
\[
g(y_1, y_2) = g(0, 0) + \partial_{y_1} g(0, 0)y_1 + \partial_{y_2} g(0, 0)y_2 + O(\rho^2), \tag{A.1}
\]
where \( \rho = \sqrt{y_1^2 + y_2^2} \). Let \( \epsilon > 0 \) be small enough so that the circle in the plane \( x_3 = 0 \) centered at the origin and with radius \( \epsilon \) is strictly included in \( \Gamma \). A long calculation (which we performed...
thanks to the use of a symbolic computation software), leads to the following expression for 
\( G(x, y, e_1) \) where we only indicate twice the odd \( x_3 \) terms for \( x_1 = x_2 = 0 \): setting

\[
A = \frac{1}{8} \frac{\lambda + 3\mu}{\pi \mu (\lambda + 2\mu)}, \quad B = \frac{1}{8} \frac{\lambda + \mu}{\pi \mu (\lambda + 2\mu)}.
\]

\( G(x, y, e_1) \) is the product of \( (\rho^2 + x^2_3)^{-5/2} \) and the matrix whose columns are

\[
\begin{pmatrix}
0, 0, 2 \left((\rho^2 + x^2_3)(A - B)\lambda - 2B\mu (x^2_3 - 2y_1^2 + y^2_2)\right) x_3,

0, 0, 12B y_1 y_2 x_3,

2x_3 \left((A - B)x^2_3 + \rho^2 A + B(5y_1^2 - y_2^2)\right) \mu, 12B y_1 y_2 x_3, 0.
\end{pmatrix}
\]

We note that for \( x_3 > 0 \)

\[
\int_0^\epsilon \frac{x^3_3 \rho \, d\rho}{(x^2_3 + \rho^2)^{5/2}} = \frac{1}{3} \left(\frac{\epsilon^2 + x^2_3}{\epsilon^2 + x^2_3}\right)^{3/2} - x^3_3,
\]

thus

\[
\lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\epsilon \frac{x^3_3 \rho \, d\rho \, d\theta}{(x^2_3 + \rho^2)^{5/2}} = \frac{2\pi}{3}, \tag{A.2}
\]

Similarly

\[
\lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\epsilon \frac{x^3_3 \rho^3 \cos^2 \theta \, d\rho \, d\theta}{(x^2_3 + \rho^2)^{5/2}} = \frac{2\pi}{3}, \tag{A.3}
\]

and

\[
\lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\epsilon \frac{x^3_3 \rho^3 \sin^2 \theta \, d\rho \, d\theta}{(x^2_3 + \rho^2)^{5/2}} = \frac{2\pi}{3}, \tag{A.4}
\]

while by symmetry

\[
\lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\epsilon \frac{x^3_3 \rho^3 \sin \theta \cos \theta \, d\rho \, d\theta}{(x^2_3 + \rho^2)^{5/2}} = 0. \tag{A.5}
\]

Thus integrating the matrix \( G(x, y, e_1) \) times \( g(0, 0) \) over the disk in the \( x_1 - x_2 \) plane with radius \( \epsilon \) centered at the origin for \( x_3 > 0 \) and taking the limit as \( x_3 \) approaches zero we find

\[
\begin{pmatrix}
g_3(0, 0) \\
0 \\
\frac{\lambda}{\lambda + 2\mu} g_1(0, 0)
\end{pmatrix}.
\]

Given that

\[
\lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\epsilon \frac{x^4_3 \rho \, d\rho \, d\theta}{(x^2_3 + \rho^2)^{5/2}} = \lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\epsilon \frac{x^3_3 \rho^4 \, d\rho \, d\theta}{(x^2_3 + \rho^2)^{5/2}} = 0, \tag{A.6}
\]

using Taylor’s expansion (A.1), formula (3.6) is proved.

To prove (3.7), we perform another calculation aided by the use of symbolic computation software. We now write the column vectors of the odd part of \( \partial_\gamma G(x, y, e_3) \) multiplied by
two, where the symmetry is taken in \( x_3 \) and \( x_1 \) and \( x_2 \) are set to zero. They are equal to the product of

\[
\frac{1}{(x_3^2 + \rho^2)^{3/2}}
\]

and the three column vectors

\[
\begin{align*}
(0, 0, -6((A + 5B)x_3^2 + \rho^2(A - 5B)x_3y_1)),
(0, 0, -6((A + 5B)x_3^2 + \rho^2(A - 5B)x_3y_2)),
(-6y_3x_3((\rho^2 + x_3^2)(A - B) + 2B(-3\rho^2 + 2x_3^2)\mu), -6y_2x_3((\rho^2 + x_3^2)(A - B) + 2B(-3\rho^2 + 2x_3^2)\mu), 0).
\end{align*}
\]

Clearly, by symmetry,

\[
\int_0^{2\pi} \int_0^\infty y_jx_3\rho^3 \, d\rho \, d\theta = \int_0^{2\pi} \int_0^\infty y_jx_3\rho^3 \, d\rho \, d\theta = 0,
\]

for \( j = 1 \) or \( 2 \). Thus there will be no contribution from \( g(0, 0) \). Similarly, the cross terms

\[
\int_0^{2\pi} \int_0^\infty y_jy_3x_3\rho^3 \, d\rho \, d\theta = \int_0^{2\pi} \int_0^\infty y_jy_3x_3\rho^3 \, d\rho \, d\theta = 0,
\]

are zero if \( j \neq k \), where \( j \) and \( k \) are in \( \{1, 2\} \). Now a calculation will show that the following limits hold:

\[
\lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\infty \frac{x_3^3 \rho^5 \cos^2 \theta \, d\rho \, d\theta}{(x_3^2 + \rho^2)^{3/2}} = \lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\infty \frac{x_3^3 \rho^5 \sin^2 \theta \, d\rho \, d\theta}{(x_3^2 + \rho^2)^{3/2}} = \frac{8\pi}{15},
\]

\[
\lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\infty \frac{x_3^3 \rho^5 \cos^2 \theta \, d\rho \, d\theta}{(x_3^2 + \rho^2)^{3/2}} = \lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\infty \frac{x_3^3 \rho^5 \sin^2 \theta \, d\rho \, d\theta}{(x_3^2 + \rho^2)^{3/2}} = \frac{2\pi}{15}.
\]

We then combine (A.7), (A.8) and (A.11) with Taylor formula (A.1) to find a contribution of \( \frac{\lambda}{\lambda + 2\rho} \) (\( \partial_3, g(0, 0) + \partial_{y_3} g(0, 0) \)) in the direction of \( e_3 \), and \( \nabla_F g(x, 0) \) in the \( e_1, e_2 \) plane. Higher order terms will not contribute since

\[
\lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\infty \frac{x_3^3 \rho^6 \, d\rho}{(x_3^2 + \rho^2)^{7/2}} = \lim_{x_3 \to 0^+} \int_0^{2\pi} \int_0^\infty \frac{x_3^3 \rho^4 \, d\rho}{(x_3^2 + \rho^2)^{7/2}} = 0,
\]

and formula (3.7) is proved.

Formula (3.8) is derived likewise.

To prove formula (3.9) we need a higher order Taylor formula. We write

\[
\begin{align*}
g(y_1, y_2) &= g(0, 0) + \partial_{y_1} g(0, 0)y_1 + \partial_{y_2} g(0, 0)y_2 \quad + \frac{1}{2} \partial^2_{y_1} g(0, 0)y_1^2 + \frac{1}{2} \partial^2_{y_2} g(0, 0)y_2^2 + \partial_{y_1} \partial_{y_2} g(0, 0)y_1y_2 + O(\rho^3).
\end{align*}
\]

We perform another calculation aided by the use of symbolic computation software to find expressions for \( \partial_{x_1}, \partial_{x_2}, G(x, y, e_3) \). For the sake of brevity we only give a proof in the case where \( x_3 = 0 \), so we only need the first two columns. We only indicate twice the odd \( x_3 \) terms for \( x_1 = x_2 = 0 \). They are the product of \( \frac{1}{(\rho^2 + x_3^2)^{3/2}} \) and a matrix whose first column is
\[
\begin{pmatrix}
6\mu(2x_3^4 - (41y_1^2 + y_2^2)x_3^2 + 3\rho^2(9y_1^2 - y_2^2))x_3B + 6(3\rho^4 + \rho^2x_3^2 - 2x_3^4)\mu x_3A \\
-60y_1y_2(-3\rho^2 + 4x_3^2)Bx_3\mu \\
0
\end{pmatrix},
\]
and whose second column is
\[
\begin{pmatrix}
-60y_1y_2(-3\rho^2 + 4x_3^2)Bx_3\mu \\
6\mu(2x_3^4 - (41y_1^2 + y_2^2)x_3^2 + 3\rho^2(9y_1^2 - y_2^2))x_3B + 6(3\rho^4 + \rho^2x_3^2 - 2x_3^4)\mu x_3A \\
0
\end{pmatrix}.
\]
Further calculations show that if we multiply these two columns by \(\frac{\rho}{(\rho^2 + \rho_1^2)^\frac{3}{2}}\) integrate in \(\rho\) from 0 to \(\epsilon\) and then take the limit as \(x_3\) tends to zero, we find zero. We also find zero as we multiply by \(\frac{\rho^2\cos\theta}{(\rho^2 + \rho_1^2)^\frac{3}{2}}\), \(\frac{\rho^2\sin\theta}{(\rho^2 + \rho_1^2)^\frac{3}{2}}\), or by \(\frac{\rho^2}{(\rho^2 + \rho_1^2)^\frac{3}{2}}\), if \(p \geq 4\). We only find nonzero contributions (which are independent of \(\epsilon > 0\)) for the second derivatives of \(g\): they are found by multiplying by \(\frac{\rho^2\cos\theta}{(\rho^2 + \rho_1^2)^\frac{3}{2}}\), where \(p, q\) are in \(\{0, 1, 2\}\) with \(p + q = 2\). More precisely, for the \(\partial_1^2\) derivative, we find, for the first column:
\[
\begin{pmatrix}
6\frac{\lambda + 8\mu}{\lambda + 2\mu} & 0 & 0
\end{pmatrix},
\]
for the second column:
\[
(0, 2, 0).
\]
For the \(\partial_2^2\) derivative, we find, for the first column:
\[
(2, 0, 0),
\]
for the second column:
\[
\begin{pmatrix}
0 & 6\frac{\lambda + 8\mu}{\lambda + 2\mu} & 0
\end{pmatrix}.
\]
For the \(\partial_1\partial_2\) derivative, we find, for the first column:
\[
\begin{pmatrix}
0 & 2\frac{\lambda + 2\mu}{\lambda + 2\mu} & 0
\end{pmatrix},
\]
for the second column:
\[
\begin{pmatrix}
2\frac{\lambda + 2\mu}{\lambda + 2\mu} & 0 & 0
\end{pmatrix}.
\]
Finally, we arrive at jump formula (3.9) thanks to a linear combination.

The same proof technique is used for deriving formula (3.10). For the sake of brevity, here too we assume \(g_3 = 0\). This time a Taylor expansion of order 1 is sufficient. The first two columns of \(\partial_{x_i}G(x, y, e_1)\), where we only indicate twice the odd \(x_i\) terms for \(x_1 = x_2 = 0\), are the product of\(\frac{1}{(x_i + \rho_1)^{\frac{3}{2}}}\) and
\[
\begin{pmatrix}
6x_3y_1\left(\rho^2 + x_3^2)(A-B)\lambda + \mu\left(2A(\rho^2 + x_3^2) - 2B(2x_3^2 - 3y_1^2 + 2y_2^2)\right)\right) \\
6x_3y_2\left(\rho^2 + x_3^2)(A-B)\lambda - 2B\mu(x_3^2 - 4y_1^2 + y_2^2)\right) \\
0
\end{pmatrix}.
\]
and
\[
\begin{pmatrix}
6\mu x_3 y_2 \left( (A - B)y_2^2 + (A + 9B)y_1^2 + x_3(A - B) \right)

6\mu x_3 y_1 \left( x_3^2(A - B) + (A - B)y_1^2 + y_2^2(A + 9B) \right)

0
\end{pmatrix}.
\]
(A.15)

We then multiply (A.14) and (A.15) by \( \frac{\rho y_i}{(\rho^2 + x^2)^{7/2}} \), \( i = 1, 2 \) and integrate in \( \rho \) from 0 to \( \epsilon \), and \( \theta \) from 0 to \( 2\pi \) to only find nonzero contributions (which are independent of \( \epsilon > 0 \)) for the first derivatives of \( g \). After simplification, we obtain formula (3.10).

**ORCID iDs**

Faouzi Triki [https://orcid.org/0000-0002-7181-6299](https://orcid.org/0000-0002-7181-6299)

Darko Volkov [https://orcid.org/0000-0001-9203-9583](https://orcid.org/0000-0001-9203-9583)

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