TEICHMÜLLER SPACE OF CIRCLE DIFFEOMORPHISMS WITH HÖLDER CONTINUOUS DERIVATIVE

KATSUHIKO MATSUZAKI

Abstract. Based on the quasiconformal theory of the universal Teichmüller space, we introduce the Teichmüller space of diffeomorphisms of the circle with α-Hölder continuous derivatives as a subspace of the universal Teichmüller space. We characterize such a diffeomorphism quantitatively in terms of the complex dilatation of its quasiconformal extension and the Schwarzian derivative given by the Bers embedding. Then we provide a complex Banach manifold structure for it and prove that its topology coincides with the one induced by local $C^{1+\alpha}$-topology at the base point.

1. Introduction

Parametrization of orientation-preserving diffeomorphisms of the unit circle $\mathbb{S}$ can be studied in the framework of the theory of Teichmüller spaces. In this case, we utilize the universal Teichmüller space $T$, which is identified with the group $QS$ of quasisymmetric automorphisms of $\mathbb{S}$ modulo post-composition of Möbius transformations $\text{Möb}(\mathbb{S})$. Here a quasisymmetric automorphism of $\mathbb{S}$ is the boundary extension of a quasiconformal automorphism of the unit disk $\mathbb{D}$. The Teichmüller space of an arbitrary hyperbolic Riemann surface can be understood as the fixed point locus of the corresponding Fuchsian group acting on $T$. If we replace the group invariance with certain regularity conditions for quasisymmetric automorphisms, we can also embed the Teichmüller space of such a family of circle automorphisms in the universal Teichmüller space $T = \text{Möb}(\mathbb{S}) \setminus QS$.

In this paper, we formulate the Teichmüller space $T_0^\alpha = \text{Möb}(\mathbb{S}) \setminus \text{Diff}^{1+\alpha}(\mathbb{S})$ of circle diffeomorphisms with Hölder continuous derivatives of exponent $\alpha \in (0, 1)$. We provide a complex Banach manifold structure for $T_0^\alpha$ and prove basic properties of this space. The arguments for $T_0^\alpha$ are modeled on those for the universal Teichmüller space $T$ and certain refinements are imported from the theory for the little subspace $T_0 = \text{Möb}(\mathbb{S}) \setminus \text{Sym}$. Here the subgroup $\text{Sym} \subset QS$ consists of symmetric automorphisms $\mathbb{S}$ which are the boundary extension of asymptotically conformal automorphisms of $\mathbb{D}$ whose complex dilatations vanish at the boundary $\mathbb{S}$. It contains all circle diffeomorphisms and hence $\text{Diff}^{1+\alpha}(\mathbb{S}) \subset$...
We will survey necessary results on the universal Teichmüller space $T$ in Section 2 and on the little subspace $T_0$ in Section 3.

We first characterize circle diffeomorphisms with Hölder continuous derivatives in terms of their quasiconformal extension to $\mathbb{D}$. This originates in the work of asymptotically conformal maps by Carleson \[15\]. Later, Gardiner and Sullivan \[24\] developed the theory of symmetric automorphisms of $S$ using previous results on quasiconformal extension and Schwarzian derivatives of univalent functions due to Becker and Pommerenke \[12\]. We will refine these results quantitatively concerning the decay order of the corresponding maps vanishing at the boundary.

We verify in Section 4 that if the complex dilatation $\mu$ of an asymptotically conformal automorphism of $D$ decays in the order of $O((1 - |z|)^\alpha)$, then the Schwarzian derivative of the developing map of the projective structure on the exterior disk $D^*$ determined by $\mu$ decays exactly in the same order $\alpha$ with respect to the hyperbolic metric. This is carried out by dividing the support of $\mu$ suitably into annular regions and estimating the pre-Schwarzian derivative of the composition of conformal homeomorphisms. A different and qualitative proof was previously obtained by Dyn’kin \[18\], but we have to prepare more precise estimates in terms of a weighted supremum norm of $\mu$ (Theorem 4.1 and Corollary 4.7).

In Section 5, we mainly consider one-dimensional properties of circle diffeomorphisms with Hölder continuous derivatives. We first provide a topology for $\text{Diff}^{1+\alpha}(S)$ and see that it is a topological group (Proposition 5.2). The topology is defined in a neighborhood of the identity map by $C^{1+\alpha}$ convergence and then distributed to every point by the right translation of the group. For the characterization of an element of $\text{Diff}^{1+\alpha}(S)$, a result of Carleson \[15\] plays an important role, which gives a connection between the Hölder continuity of the derivative and the quasisymmetric quotient of $g$. We review his theorem and supply necessary claims for our arguments.

A fundamental result is that if the complex dilatation of an asymptotically conformal automorphism of $D$ decays in the order of $O((1 - |z|)^\alpha)$, then the regularity of its boundary extension $g$ to $S$ is exactly $C^{1+\alpha}$. Carleson gave that it is at least $C^{1+\alpha/2}$. This problem was investigated further by Anderson and Hinkkanen \[7\] among others, and settled qualitatively by Dyn’kin \[18\] and Anderson, Cantón and Fernández \[6\]. Combined with the aforementioned results, this can be summarized as follows (Theorem 6.7).

**Proposition 1.1.** Let $\alpha$ be a constant with $0 < \alpha < 1$. The following conditions are equivalent for $g \in \text{QS}$:

1. $g$ is a diffeomorphism of $S$ with Hölder continuous derivative of exponent $\alpha$;
2. $g$ extends continuously to a quasiconformal automorphism of $D$ whose complex dilatation $\mu(z)$ decays in the order of $O((1 - |z|)^\alpha)$ as $z \in D$ tends to the boundary;
3. the Schwarzian derivative $\varphi(z)$ of the conformal homeomorphism of $D^*$ determined by $g$ decays in the order of $O((|z| - 1)^{-2+\alpha})$ as $z \in D^*$ tends to the boundary.

In Section 6, we will give a more improved statement than Proposition 1.1 with a different proof which is necessary for the arguments of Teichmüller spaces (Theorem 6.9). Our
strategy is to represent a circle diffeomorphism $g$ by conformal welding, which is originally
due to Anderson, Becker and Lesley [5]. For the argument in this method, we need to
know that an asymptotically conformal automorphism $f$ of $\mathbb{D}$ and its inverse mapping
$f^{-1}$ have the complex dilatations of order $O((1 - |z|)^{\alpha})$ at the same time. For this
purpose, we extend the consequence of the Mori theorem to a quasiconformal automorphism
$f$ of $\mathbb{D}$ with complex dilatation of order $O((1 - |z|)^{\alpha})$. The result is that $1 - |f(z)|$ is
comparable to $1 - |z|$ without the power of the maximal dilatation $K(f)$ (Theorem 6.4). This
guarantees that the complex dilatation of $f^{-1}$ is also of order $O((1 - |z|)^{\alpha})$.

**Theorem 1.2.** Let $f$ be a quasiconformal automorphism of $\mathbb{D}$ with $f(0) = 0$ whose
complex dilatation $\mu(z)$ satisfies $|\mu(z)| \leq \ell(1 - |z|)^{\alpha}$ almost every $z \in \mathbb{D}$ for some $\ell \geq 0$.
Then there is a constant $A \geq 1$ depending only on $K(f), \alpha$ and $\ell$ such that

$$\frac{1}{A}(1 - |z|) \leq 1 - |f^\mu(z)| \leq A(1 - |z|)$$

for every $z \in \mathbb{D}$.

For Beltrami coefficients and Schwarzian derivatives as above, we prepare the following
spaces: $\text{Bel}_{0}^{\alpha}(\mathbb{D})$ is the space of Beltrami coefficients $\mu$ on $\mathbb{D}$ with finite norm $\|\mu\|_{\infty,\alpha} = \rho_{\mathbb{D}}^{\alpha}(z) \text{ess.sup} |\mu(z)|$; and $B_{0}^{\alpha}(\mathbb{D}^*)$ is the Banach space of holomorphic functions $\varphi$ on $\mathbb{D}^*$
with finite norm $\|\varphi\|_{\infty,\alpha} = \rho_{\mathbb{D}^*}^{-2 + \alpha}(z) \text{sup} |\varphi(z)|$. Here $\rho$ stands for the hyperbolic density
of each space. Then Proposition 1.1 implies that the Teichmüller projection $\pi$, the Bers
projection $\Phi$ and the Bers embedding $\beta$ for the universal Teichmüller space $T$ also work
for our spaces by restriction of the original mappings:

$$\begin{align*}
\text{Bel}_{0}^{\alpha}(\mathbb{D}) \xrightarrow{\pi} T_{0}^{\alpha} = \text{Möb}(\mathbb{S}) \setminus \text{Diff}^{1+\alpha}(\mathbb{S}) \xrightarrow{\beta} \beta(T) \cap B_{0}^{\alpha}(\mathbb{D}^*)
\end{align*}$$

The topology on $T_{0}^{\alpha}$ is induced from $\text{Bel}_{0}^{\alpha}(\mathbb{D})$ by $\pi$. If we regard $T_{0}^{\alpha}$ as the subgroup of
$\text{Diff}^{1+\alpha}(\mathbb{S})$ consisting of normalized elements, then we can also provide it with the right
uniform topology of $\text{Diff}^{1+\alpha}(\mathbb{S})$, which is generated by the right translations of local $C^{1+\alpha}$-
topology at the identity. In Section 7 we prove that these topologies on $T_{0}^{\alpha}$ are the same
(Theorem 1.3).

**Theorem 1.3.** The quotient topology on $T_{0}^{\alpha}$ induced by $\pi : \text{Bel}_{0}^{\alpha}(\mathbb{D}) \to T_{0}^{\alpha}$
c coinides with the $\text{Diff}^{1+\alpha}$-topology and in particular $T_{0}^{\alpha}$ is a topological group.

The complex structure on $T_{0}^{\alpha}$ is given by showing that the Bers embedding $\beta$ as above
is homeomorphic onto its image. Moreover, we want to find that the base point change
map (right translation) of $T_{0}^{\alpha}$ is compatible with this complex structure. To this end,
we will prove that the Bers projection $\Phi$ is a holomorphic submersion. To see that $\Phi$ is
continuous, we use an integral representation of the Schwarzian derivative $\Phi(\mu)$, which
is originally due to Astala and Zinsmeister [8]. Then a careful estimate of this integral
taking the dependence of constants into account yields the assertion. The holomorphy
is a consequence from the continuity in our situation. To see that \( \Phi \) is a submersion,
we construct a local holomorphic section of \( \Phi \). For the universal Teichmüller space (and
Teichmüller spaces of Riemann surfaces), this was proved by Bers [13]. We adapt this
argument to our situation for showing the continuity with respect to the topology in our
spaces. In Section 7, we will prove the following.

**Theorem 1.4.** The Bers projection \( \Phi : \text{Bel}_0^\alpha (\mathbb{D}) \to B_0^\alpha (\mathbb{D}^*) \) is a holomorphic submersion onto its image. This implies that the Bers embedding \( \beta : T_0^\alpha \to \beta (T) \cap B_0^\alpha (\mathbb{D}^*) \) is a homeomorphism. With this complex structure of \( T_0^\alpha \) identified with a domain of the complex Banach space \( B_0^\alpha (\mathbb{D}^*) \), every base point change map of \( T_0^\alpha \) is a biholomorphic automorphism of \( T_0^\alpha \).

A motivation of this work is to apply the Bers embedding of the Teichmüller space \( T_0^\alpha \) to
the studies of a rigidity of Diff\( ^{1+\alpha} (\mathbb{S}) \)-representation of a Möbius group and a conjugation
problem of a subgroup of Diff\( ^{1+\alpha} (\mathbb{S}) \) to a Möbius group. These arguments are developed
in the continuation [33] of the present work. A survey of our project can be found in [31].

2. The universal Teichmüller space

In this section, we define the universal Teichmüller space in terms of the group of qua-
sisymmetric automorphisms of the circle and then introduce a topological and a complex
structure on this space by using the quasiconformal theory; the Beltrami equation and
the Schwarzian derivative. Basic results can be found in Lehto [27].

We denote the group of all quasiconformal automorphisms of the unit disk \( \mathbb{D} \) by \( \text{QC}(\mathbb{D}) \).
Each quasiconformal automorphism \( f \in \text{QC}(\mathbb{D}) \) extends continuously to the boundary \( \mathbb{S} \) as a homeomorphism. Then we have a homomorphism \( q : \text{QC}(\mathbb{D}) \to \text{Homeo}(\mathbb{S}) \) into the group of self-homeomorphisms of the unit circle \( \mathbb{S} \). An orientation-preserving self-
homeomorphism \( g \) of \( \mathbb{S} \) is called *quasisymmetric* if \( g \in \text{Im} q \). We denote the group \( \text{Im} q \) of all quasisymmetric automorphisms of \( \mathbb{D} \) by \( \text{QS} \). Let \( \text{Möb}(\mathbb{D}) \subset \text{QC}(\mathbb{D}) \) denotes the subgroup of all conformal automorphisms of \( \mathbb{D} \), which are Möbius transformations of \( \mathbb{D} \).

We define \( \text{Möb}(\mathbb{S}) = q(\text{Möb}(\mathbb{D})) \subset \text{QS} \).

**Definition.** The *universal Teichmüller space* \( T \) is defined to be the set of the cosets \( \text{Möb}(\mathbb{S}) \setminus \text{QS} \). We denote the coset of \( g \in \text{QS} \) by \( [g] \).

The *Beltrami coefficient* \( \mu \) on a domain \( D \subset \widehat{\mathbb{C}} \) is a measurable function with supremum norm \( ||\mu||_\infty \) less than 1. We denote the set of all Beltrami coefficients on \( D \) by

\[ \text{Bel}(D) = \{ \mu \in L^\infty (D) \mid ||\mu||_\infty < 1 \} . \]

Every quasiconformal homeomorphism \( f : D \to D' \) has partial derivatives \( \partial f \) and \( \bar{\partial} f \) in the distribution sense and the ratio \( \mu_f (z) = \bar{\partial} f (z)/\partial f (z) \) called the *complex dilatation* is a Beltrami coefficient on \( D \). The *maximal dilatation* of \( f \) is defined by

\[ K(f) = \frac{1 + ||\mu_f||_\infty}{1 - ||\mu_f||_\infty} . \]
Given \( K \geq 1 \), we call \( f \) a \( K \)-quasiconformal if \( K(f) \leq K \). The \textit{measurable Riemann mapping theorem} due to Ahlfors and Bers [3] asserts that a Beltrami coefficient determines uniquely a quasiconformal homeomorphism up to post-composition of conformal homeomorphisms.

Applying this theorem to quasiconformal automorphisms of \( \mathbb{D} \), we see that \( \text{Bel}(\mathbb{D}) \) can be identified with the set of the cosets \( \text{M"{o}b}(\mathbb{D}) \setminus \text{QC}(\mathbb{D}) \). Then the boundary extension \( e : \text{QC}(\mathbb{D}) \rightarrow \text{QS} \) induces a surjective map \( \pi : \text{Bel}(\mathbb{D}) \rightarrow T \) by taking the quotient of \( \text{M"{o}b}(\mathbb{D}) = \text{M"{o}b}(\mathbb{S}) \). This is called the \textit{Teichm"{u}ller projection}. The topology of the universal Teichm"{u}ller space \( T \) is given as the quotient topology of the unit ball \( \text{Bel}(\mathbb{D}) \) of the Banach space \( L^\infty(\mathbb{D}) \) by the projection \( \pi \) so that \( \pi \) is continuous.

There is a global continuous section for the Teichmüller projection \( \pi : \text{Bel}(\mathbb{D}) \rightarrow T \). This is defined by giving a canonical quasiconformal extension \( e \) be used to obtain such a section. Douady and Earle [17] introduced another extension for any \( \phi \) by \( \pi \)

The measurable Riemann mapping theorem implies that, for every \( \nu \in \text{Bel}(\mathbb{D}) \), there is a unique normalized quasiconformal automorphism \( f \in \text{QC}(\mathbb{D}) \) whose complex dilatation coincides with \( \nu \). Here the \textit{normalization} is given by fixing three boundary points 1, \( i \) and \(-1\) on \( \mathbb{S} \). We denote this normalized quasiconformal automorphism by \( f^\nu \). The subgroup of \( \text{QC}(\mathbb{D}) \) consisting of all normalized elements is defined as \( \text{QC}_e(\mathbb{D}) \). This also defines the normalized elements of \( \text{QS} \), which constitute the subgroup \( \text{QS}_e = q(\text{QC}_e(\mathbb{D})) \).

Applying this normalization, we can define a group structure on \( \text{Bel}(\mathbb{D}) \) and \( T \) as follows. For any \( \nu_1, \nu_2 \in \text{Bel}(\mathbb{D}) \), set \( \nu_1 \ast \nu_2 \) to be the complex dilatation of the composition \( f^{\nu_1} \circ f^{\nu_2} \). Then \( \text{Bel}(\mathbb{D}) \) has a group structure with this operation \( \ast \). In other words, by the identification of \( \text{Bel}(\mathbb{D}) \) with \( \text{QC}_e(\mathbb{D}) \), we regard \( \text{Bel}(\mathbb{D}) \) as a subgroup of \( \text{QC}(\mathbb{D}) \). We denote the inverse element of \( \nu \in \text{Bel}(\mathbb{D}) \) by \( \nu^{-1} \), which is the complex dilatation of \( (f_\nu)^{-1} \). The chain rule of partial differentials yields a formula

\[
\nu_1 \ast \nu_2^{-1}(\zeta) = \frac{\nu_1(z) - \nu_2(z)}{1 - \nu_2(z)\nu_1(z)} \frac{\partial f^{\nu_2}(z)}{\partial f^{\nu_2}(2)} (\zeta = f^{\nu_2}(z)).
\]

For the base point \([\text{id}]\) of \( T \), the inverse image of the Teichmüller projection

\[
\pi^{-1}([\text{id}]) = \{ \nu \in \text{Bel}(\mathbb{D}) \mid q(f^\nu) = \text{id} \}
\]

is a normal subgroup of \( \text{Bel}(\mathbb{D}) \) since \( q : \text{QC}(\mathbb{D}) \rightarrow \text{QS} \) is a homomorphism. Having \( T = \text{Bel}(\mathbb{D})/\pi^{-1}([\text{id}]) \), we see that \( T \) has a group structure with the operation \( \ast \) defined by \( \pi(\nu_1) \ast \pi(\nu_2) = \pi(\nu_1 \ast \nu_2) \). Then \( \pi : \text{Bel}(\mathbb{D}) \rightarrow T \) is a surjective homomorphism with
\(\pi^{-1}([\text{id}])\) its kernel. If we identify \(T\) with \(Q_{S^*}\), we may regard \(T\) as a subgroup of \(Q\) and the projection \(\pi\) as the restriction of \(q\) to \(Q_{S^*}(\mathbb{D})\).

Each \(\nu \in \text{Bel}(\mathbb{D})\) induces the right translation \(r_\nu : \text{Bel}(\mathbb{D}) \to \text{Bel}(\mathbb{D})\) by \(\mu \mapsto \mu * \nu^{-1}\). The projection under \(\pi\) yields a well-defined map \(R_{\pi(\nu)} : T \to T\) by

\[
\pi(\mu) \mapsto \pi(\mu * \nu^{-1}) = \pi(\mu) * \pi(\nu)^{-1}.
\]

In this way, for every point \(\tau \in T\), we have the base point change \(R_\tau : T \to T\) sending \(\tau\) to \([\text{id}]\). By the above formula, we see that \(r_\nu\) and \((r_\nu)^{-1} = (r_\nu)^{-1}\) are continuous, and hence \(r_\nu\) is a homeomorphism onto \(\text{Bel}(\mathbb{D})\). Since \(R_{\pi(\nu)} = \pi \circ r_\nu \circ s_{\text{DE}}\) and \(R_{\pi(\nu)}^{-1} = \pi \circ (r_\nu)^{-1} \circ s_{\text{DE}}\) are continuous, the base point change of \(T\) is also a homeomorphism onto \(T\). In addition, we see the following.

**Proposition 2.2.** The Bers projection \(\Phi : \text{Bel}(\mathbb{D}) \to T\) is an open map.

**Proof.** For an arbitrary open subset \(U \subset \text{Bel}(\mathbb{D})\), we have

\[
\pi^{-1}(\pi(U)) = \bigcup_{\nu \in \pi^{-1}([\text{id}])} r_\nu(U).
\]

This proves that \(\pi\) is an open map. \(\Box\)

The universal Teichmüller space \(T\) has a complex structure modeled on a certain complex Banach space. This is seen as follows. For \(\mu \in \text{Bel}(\mathbb{D})\), we extend \(\mu(z)\) to \(\hat{\mathbb{C}}\) by setting \(\mu(z) \equiv 0\) for \(z \in \mathbb{D}^* = \hat{\mathbb{C}} - \overline{\mathbb{D}}\). By the measurable Riemann mapping theorem, there exists a unique quasiconformal automorphism \(f_\mu\) of \(\hat{\mathbb{C}}\) up to post-composition of Möbius transformations whose complex dilatation coincides with the extended Beltrami coefficient \(\mu\). Take the Schwarzian derivative \(S_f(z)\) of the conformal homeomorphism \(f(z) = f_\mu|_{\mathbb{D}^*}(z)\) on \(\mathbb{D}^*\). The ambiguity of \(f_\mu\) by Möbius transformations is killed by taking the Schwarzian derivative because \(S_{h \circ f}(z) = S_f(z)\) for every \(h \in \text{Möb}(\hat{\mathbb{C}})\).

We define the Banach space of holomorphic functions on \(\mathbb{D}^*\) with finite hyperbolic supremum norm by

\[
B(\mathbb{D}^*) = \{ \varphi \in \text{Hol}(\mathbb{D}^*) \mid \|\varphi\|_\infty = \sup_{z \in \mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| < \infty \},
\]

where \(\rho_{\mathbb{D}^*}(z) = 2/(|z|^2 - 1)\) is the hyperbolic density on \(\mathbb{D}^*\). The Nehari-Kraus theorem says that \(\|\varphi\|_\infty \leq 3/2\) for the Schwarzian derivative \(\varphi(z) = S_f(z)\) of any conformal homeomorphism \(f\) of \(\mathbb{D}^*\). Hence we have a map \(\Phi : \text{Bel}(\mathbb{D}) \to B(\mathbb{D}^*)\) by the correspondence of \(\mu \in \text{Bel}(\mathbb{D})\) to \(S_{f_\mu}|_{\mathbb{D}^*}\), which is called the Bers projection.

Concerning the Teichmüller projection \(\pi : \text{Bel}(\mathbb{D}) \to T\) and the Bers projection \(\Phi : \text{Bel}(\mathbb{D}) \to B(\mathbb{D}^*)\), it can be proved that \(\pi(\mu_1) = \pi(\mu_2)\) if and only if \(\Phi(\mu_1) = \Phi(\mu_2)\). Therefore we have a well-defined injection \(\beta : T \to B(\mathbb{D}^*)\) that satisfies \(\beta \circ \pi = \Phi\). This is called the Bers embedding of the universal Teichmüller space \(T\).

**Proposition 2.2.** The Bers projection \(\Phi : \text{Bel}(\mathbb{D}) \to B(\mathbb{D}^*)\) is continuous.
Proof. For arbitrary two points \( \mu, \nu \in \text{Bel}(\mathbb{D}) \), we apply the right translation \( r_{\nu} \) to \( \mu \). On the quasidisk \( f_{\nu}(\mathbb{D}^*) \), we use an estimate of the Schwarzian derivative of the conformal homeomorphism \( f_{\mu} \circ f_{\nu}^{-1} \) in terms of \( \| r_{\nu}(\mu) \|_\infty \). See [27, Theorem II.3.2]. Then we have
\[
\| \Phi(\mu) - \Phi(\nu) \|_\infty \leq 3 \| r_{\nu}(\mu) \|_\infty \leq \frac{3 \| \mu - \nu \|_\infty}{1 - \| \nu \|_\infty \| \mu \|_\infty},
\]
which implies that \( \Phi \) is continuous. \( \square \)

In fact, the Bers projection \( \Phi \) is holomorphic. Once we have \( \Phi \) is continuous, then the holomorphy is a consequence from the point-wise holomorphic dependance of the normalized solution \( f_{\mu}(z) \) of the Beltrami equation for \( \mu \), which comes from the arguments for the measurable Riemann mapping theorem due to Ahlfors and Bers [3]. Moreover, the following result was proved by Bers [13].

**Theorem 2.3.** The Bers projection \( \Phi : \text{Bel}(\mathbb{D}) \to B(\mathbb{D}^*) \) is a holomorphic submersion.

The condition for \( \Phi \) to be a holomorphic submersion is equivalent to the existence of a local holomorphic section for \( \Phi \) at every \( \varphi \in \Phi(\text{Bel}(\mathbb{D})) \) sending \( \varphi \) to an arbitrary point of \( \Phi^{-1}(\varphi) \). See Nag [34, Section 1.6] concerning holomorphic submersion on a domain of a Banach space. This implies that \( \Phi \) is an open map and in particular the image \( \Phi(\text{Bel}(\mathbb{D})) \) in \( B(\mathbb{D}^*) \) is open (hence it is a bounded domain).

Since both \( \pi \) and \( \Phi \) are continuous and open, the Bers embedding \( \beta = \Phi \circ \pi^{-1} : T \to B(\mathbb{D}^*) \) is a homeomorphism onto the image \( \beta(T) = \Phi(\text{Bel}(\mathbb{D})) \). By identifying \( T \) with a bounded domain \( \beta(T) \subset B(\mathbb{D}^*) \), we provide a complex structure for \( T \). Then the base point change \( R_{\tau} \) for every \( \tau \in T \) is a biholomorphic automorphism of \( T \). Indeed, for an arbitrary point \( \varphi \in \beta(T) \), take a holomorphic local section \( \eta \) of \( \Phi \). Also, take \( \nu \in \text{Bel}(\mathbb{D}) \) such that \( \pi(\nu) = \tau \). Represent \( R_{\tau} \) at \( \beta^{-1}(\varphi) \) by
\[
R_{\tau} = \beta^{-1} \circ \Phi \circ r_{\nu} \circ \eta \circ \beta.
\]
Since \( \Phi \circ r_{\nu} \circ \eta \) is holomorphic, this shows that \( R_{\tau} \) is holomorphic. By \( R_{\tau}^{-1} = R_{\tau-1} \), \( R_{\tau}^{-1} \) is also holomorphic, namely, \( R_{\tau} \) is biholomorphic.

3. Symmetric homeomorphisms and the little Teichmüller space

A quasisymmetric homeomorphism was originally introduced as a function on \( \mathbb{R} \) that has quasiconformal extension to the upper half-plane \( \mathbb{H} \). It can be characterized by the quasisymmetric quotient defined as follows.

**Definition.** An increasing homeomorphism \( h : \mathbb{R} \to \mathbb{R} \) is called a quasisymmetric function if there exists a constant \( M \geq 1 \) such that
\[
\frac{1}{M} \leq \frac{h(x + t) - h(x)}{h(x) - h(x - t)} \leq M
\]
holds for every \( x \in \mathbb{R} \) and for every \( t > 0 \). The ratio in the mid-term is called the quasisymmetric quotient of \( h \) and is denoted by \( m_h(x,t) \).
For an orientation-preserving self-homeomorphism $g : \mathbb{S} \to \mathbb{S}$, we can take its lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$ with $u \circ \tilde{g} = g \circ u$ for the universal cover $u : \mathbb{R} \to \mathbb{S}$ given by $u(x) = e^{2\pi ix}$. It is an increasing homeomorphism of $\mathbb{R}$ satisfying $\tilde{g}(x + 1) = \tilde{g}(x) + 1$. Conversely, for an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(x + 1) = h(x) + 1$, we can take its projection $\tilde{h} : \mathbb{S} \to \mathbb{S}$ with $u \circ \tilde{h} = \tilde{h} \circ u$.

It is known that $g \in \text{QS}$ is a quasisymmetric automorphism of $\mathbb{S}$ if and only if its lift $\tilde{g}$ is a quasisymmetric function on $\mathbb{R}$ (see [30, Theorem 4.4]). To see that $\tilde{g}$ is quasisymmetric, it is enough to check the quasisymmetric quotient $m_\tilde{g}(x, t)$ for $0 \leq x < 1$ and $0 < t < 1/2$ (see [30, Proposition 4.5]). For each $g \in \text{QS}$, we introduce the quasisymmetry constant of $g$ by

$$M(g) = \sup_{0 \leq x < 1, 0 < t < 1/2} m_\tilde{g}(x, t)^{\pm 1}.$$ 

This defines a topology on $\text{QS}$. More precisely, $g_n \in \text{QS}$ converge $g \in \text{QS}$ if $M(g_n \circ g^{-1}) \to 1$ as $n \to \infty$. Then the relative topology on $\text{QS}_* \subset \text{QS}$ coincide with the Teichmüller topology on $T \cong \text{QS}_*$, which is the quotient topology under $\pi : \text{Bel}(\mathbb{D}) \to T$. See Lehto [27, Theorem III.3.1]

We consider a special class of quasisymmetric functions on $\mathbb{R}$ whose quasisymmetric quotient is uniformly tends to 1 as $t \to 0$. We also consider the corresponding quasisymmetric automorphisms of $\mathbb{S}$.

**Definition.** A quasisymmetric function $h : \mathbb{R} \to \mathbb{R}$ is called symmetric if there exists a non-negative increasing function $\varepsilon(t)$ for $t > 0$ with $\lim_{t \to 0} \varepsilon(t) = 0$ such that

$$(1 + \varepsilon(t))^{-1} \leq m_h(x, t) \leq 1 + \varepsilon(t)$$

for all $x \in \mathbb{R}$. We call $\varepsilon(t)$ a gauge function for symmetry. A quasisymmetric automorphism $g \in \text{QS}$ is called symmetric if its lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is a symmetric function. We denote the subset of all symmetric automorphisms of $\mathbb{S}$ by $\text{Sym}$.

As the corresponding concept for quasiconformal maps, there are asymptotically conformal automorphisms whose complex dilatations vanish at the boundary. We will review the relation of two maps, especially certain quantitative estimate of the complex dilatation of the quasiconformal extension in terms of the quasisymmetric quotient, which is originally due to Carleson [15].

For a quasisymmetric function $h : \mathbb{R} \to \mathbb{R}$, set

$$\alpha(x, y) = \int_0^1 h(x + ty)dt; \quad \beta(x, y) = \int_0^1 h(x - ty)dt$$

and define

$$F(z) = \frac{1}{2} \{\alpha(x, y) + \beta(x, y)\} + i\{\alpha(x, y) - \beta(x, y)\}$$

for $z = x + iy \in \mathbb{H}$. Beurling and Ahlfors [13] proved that $F$ is a quasiconformal automorphism of $\mathbb{H}$ with an estimate of the maximal dilatation of $F$ in terms of the quasisymmetry constant $M \geq 1$. We call this the Beurling-Ahlfors extension of $h$. Concerning the Beurling-Ahlfors extension of symmetric functions, the following result is crucial, which
was proved in [15, Lemma 3] and improved slightly by giving explicit computation for involved constants in [30].

**Theorem 3.1.** Let $h : \mathbb{R} \to \mathbb{R}$ be a symmetric function such that $m_h(x,t)^{\pm 1} \leq 1 + \varepsilon(t)$ for a gauge function $\varepsilon(t)$. Let $F$ be the Beurling-Ahlfors extension of $h$, which is a quasiconformal automorphism of $\mathbb{H}$. Then the complex dilatation $\mu_F$ of $F$ satisfies $|\mu_F(z)| \leq 4\varepsilon(y)$ for every $z = x + iy \in \mathbb{H}$.

In particular, this theorem shows that a symmetric function $h : \mathbb{R} \to \mathbb{R}$ extends continuously to a quasiconformal automorphism $F : \mathbb{H} \to \mathbb{H}$ with $F(\infty) = \infty$ whose complex dilatation $\mu_F(z)$ tends to 0 as $y \to 0$ uniformly on $x \in \mathbb{R}$.

Conversely, such a quasiconformal automorphism $F$ of $\mathbb{H}$ extends to a symmetric function on $\mathbb{R}$. Carleson [15, Lemma 2] proved this fact giving the order of a gauge function for symmetry. We will reprove this result in the following form with a more explicit estimate for the gauge function. This estimate is useful in later arguments.

**Theorem 3.2.** If a $K$-quasiconformal automorphism $F : \mathbb{H} \to \mathbb{H}$ with $F(\infty) = \infty$ satisfies $|\mu_F(z)| \leq \tilde{\varepsilon}(y)$ uniformly on $x \in \mathbb{R}$ for a function $\tilde{\varepsilon}(y)$ with $\tilde{\varepsilon}(y) \to 0$ as $y \to 0$, then its boundary extension $h : \mathbb{R} \to \mathbb{R}$ is a symmetric function whose quasisymmetric quotient satisfies $m_h(x,t)^{\pm 1} \leq 1 + \varepsilon(t)$ for a gauge function $\varepsilon(t)$ with

$$\varepsilon(t) \leq c\tilde{\varepsilon}(\sqrt{t}) + R\sqrt{t} \quad (0 < t < 1/2),$$

where $c = c(K) > 0$ is a constant depending only on $K \geq 1$ and $R > 0$ is an absolute constant.

**Proof.** For each $t \in (0,1/2)$, define a Beltrami coefficient $\mu_t(z)$ by letting $\mu_t(z) = \mu_F(z)$ on $\{z \in \mathbb{H} \mid y > \sqrt{t}\}$ and $\mu_t(z) \equiv 0$ elsewhere. Let $F_t$ be the quasiconformal automorphism of $\mathbb{H}$ with complex dilatation $\mu_t$ and with $F_t(\infty) = \infty$, and $h_t$ the quasiconformal automorphism of $\mathbb{H}$ such that $F = h_t \circ F_t$. Since $h_t = F \circ F_t^{-1}$, the complex dilatation of $h_t$ satisfies

$$|\mu_{h_t}(F_t(z))| = \frac{|\mu_F(z) - \mu_t(z)|}{|1 - \mu_t(z)\mu_F(z)|} \leq \frac{\tilde{\varepsilon}(\sqrt{t})}{1 - \|\mu_F\|_{\infty}}.$$

In particular, there is a constant $c' > 0$ depending only on $K$ such that the maximal dilatation of $h_t$ is estimated as $K(h_t) \leq 1 + c'\tilde{\varepsilon}(\sqrt{t})$.

By reflection with respect to $\mathbb{R}$, we may assume that $F_t$ is a quasiconformal automorphism of $\mathbb{C}$. The restriction of $F_t$ to the strip domain $\{z \in \mathbb{C} \mid |y| < \sqrt{t}\}$ is conformal. For each $x \in \mathbb{R}$, consider the ball of radius $\sqrt{t}$ with center $x$ and apply the Koebe distortion theorem (Proposition 3.3 below) to the conformal map $F_t$ on this disk. Then, we have

$$\frac{|F'_t(x)|t}{(1 + t/\sqrt{t})^2} \leq F_t(x + t) - F_t(x) \leq \frac{|F'_t(x)|t}{(1 - t/\sqrt{t})^2}.$$
The middle term can be replaced with \( F_t(x) - F_t(x-t) \). This leads us to the following estimate for the quasisymmetric quotient \( m_{F_t}(x,t) \) of \( F_t \):

\[
\frac{(1 - \sqrt{t})^2}{(1 + \sqrt{t})^2} \leq m_{F_t}(x,t) = \frac{F_t(x + t) - F_t(x)}{F_t(x) - F_t(x-t)} \leq \frac{(1 + \sqrt{t})^2}{(1 - \sqrt{t})^2}.
\]

In particular, there is an absolute constant \( R' > 0 \) such that \( m_{F_t}(x,t)^{\pm 1} \leq 1 + R' \sqrt{t} \) for \( 0 < t < 1/2 \).

Next, we apply the quasiconformal automorphism \( h_t \) to the points \( F_t(x-t), F_t(x) \) and \( F_t(x+t) \), which are mapped to \( h(x-t), h(x) \) and \( h(x+t) \) respectively. Note that the quasisymmetric quotients can be given by the conformal moduli as follows:

\[
m_{F_t}(x,t) = \lambda(\text{mod} \, \mathbb{H}(F_t(x-t), F_t(x), F_t(x+t), \infty));
m_h(x,t) = \lambda(\text{mod} \, \mathbb{H}(h(x-t), h(x), h(x+t), \infty)).
\]

Here, \( \text{mod} \, Q(x_1, x_2, x_3, x_4) \in (0, \infty) \) stands for the conformal modulus of a quadrilateral \( Q \) with positively ordered four vertices \( x_1, x_2, x_3, x_4 \in \partial Q \). Also, \( \lambda : (0, \infty) \to (0, \infty) \) is the distortion function, which transforms conformal moduli to quasisymmetric quotients. See [27, Section I.2.4] and [25, Section II.6].

On the other hand, the ratio of the conformal moduli are bounded by the maximal dilatation \( K(h_t) \leq 1 + c' \tilde{\varepsilon}(\sqrt{t}) \):

\[
\frac{1}{K(h_t)} \leq \frac{\text{mod} \, \mathbb{H}(h(x-t), h(x), h(x+t), \infty)}{\text{mod} \, \mathbb{H}(F_t(x-t), F_t(x), F_t(x+t), \infty)} \leq K(h_t).
\]

Plugging the quasisymmetric quotients in this inequality gives

\[
m_h(x,t) = \lambda(\text{mod} \, \mathbb{H}(h(x-t), h(x), h(x+t), \infty)) \leq \lambda(K(h_t) \text{mod} \, \mathbb{H}(F_t(x-t), F_t(x), F_t(x+t), \infty)) = \lambda(K(h_t)\lambda^{-1}(m_{F_t}(x,t))) \leq \lambda((1 + c' \tilde{\varepsilon}(\sqrt{t}))\lambda^{-1}(1 + R' \sqrt{t}))
\]

for all \( x \in \mathbb{R} \) and \( t \in (0, 1/2) \). An estimate for \( m_h(x,t)^{-1} \) is similarly obtained. Since \( \lambda \) is continuous with \( \lambda(1) = 1 \) and differentiable at 1 with a non-vanishing derivative (see [4]), we see that the last term can be represented as \( 1 + \varepsilon(t) \) for a gauge function \( \varepsilon(t) \) as in the statement of the theorem.

We state the Koebe distortion theorem as follows, which includes the one-quarter theorem. See [36, Theorem 1.3] for example.

**Proposition 3.3.** A conformal homeomorphism \( f \) of \( \mathbb{D} \) into \( \mathbb{C} \) satisfies

\[
|f'(0)| \frac{|z|}{(1 + |z|)^2} \leq |f(z) - f(0)| \leq |f'(0)| \frac{|z|}{(1 - |z|)^2}.
\]

\[
|f'(0)| \frac{1 - |z|^2}{(1 + |z|)^2} \leq |f'(z)| \leq |f'(0)| \frac{1 + |z|}{(1 - |z|)^3}.
\]
for every \( z \in \mathbb{D} \). The first inequality in the former line in particular shows that the image \( f(\mathbb{D}) \) contains a disk of center at \( f(0) \) and radius \(|f'(0)|/4\).

Using the Beurling-Ahlfors extension, we can also define a quasiconformal extension of a quasisymmetric automorphism \( g \) of \( S \) to \( \mathbb{D} \). Actually, for the lift \( \tilde{g} : \mathbb{R} \to \mathbb{R} \) of \( g \) under the universal cover \( u : \mathbb{R} \to S \), we take the Beurling-Ahlfors extension \( F : \mathbb{H} \to \mathbb{H} \) of \( \tilde{g} \). Here we also use the extension of \( u \) to the holomorphic universal cover \( u : \mathbb{H} \to \mathbb{D} - \{0\} \) defined by \( u(z) = e^{2\pi i z} \). Projecting down \( F \) to a quasiconformal automorphism of \( \mathbb{D} - \{0\} \) by the holomorphic universal cover \( u \) and filling the puncture 0, we obtain a quasiconformal automorphism \( f \) of \( \mathbb{D} \). By this correspondence \( g \mapsto f \), we have a map

\[
e_{BA} : QS \to QC(\mathbb{D}),
\]

which satisfies \( q \circ e_{BA} = \text{id}|_{QS} \).

Differently from the Douady-Earle extension \( e_{DE} \), the Beurling-Ahlfors extension \( e_{BA} \) does not have conformal naturality. Accordingly, it does not descend to a section \( T \to \text{Bel}(\mathbb{D}) \) naturally. In order to define a section, we have to use the normalized quasisymmetric automorphism \( g \in QS \) as a representative of an element \([g] \in T\). From this \( g \), we make the quasiconformal automorphism \( f \) of \( \mathbb{D} \) as above and then take its complex dilatation \( \mu_f \). By this correspondence \([g] \mapsto \mu_f\), we have a map \( s_* : T \to \text{Bel}(\mathbb{D}) \), which is a section for the Teichmüller projection \( \pi : \text{Bel}(\mathbb{D}) \to T \). It can be also proved that \( s_* \) is continuous.

We say that a quasiconformal automorphism \( f \in QC(\mathbb{D}) \) is asymptotically conformal if the complex dilatation \( \mu_f(z) \) vanishes at the boundary \( S \). This means that

\[
\lim_{t \to 1} \text{ess.sup} \{ |\mu_f(z)| \mid |z| > t \} = 0.
\]

We denote the subset of \( QC(\mathbb{D}) \) consisting of all asymptotically conformal automorphisms by \( AC(\mathbb{D}) \). Theorem 3.1 implies that the restriction of \( e_{BA} \) to \( \text{Sym} \) gives

\[
e_{BA} : \text{Sym} \to AC(\mathbb{D}).
\]

On the other hand, Theorem 3.2 implies that the restriction of \( q \) to \( AC(\mathbb{D}) \) gives

\[
q : AC(\mathbb{D}) \to \text{Sym}.
\]

Note that, for a given point \( z_0 \) in \( \mathbb{D} \), there is a quasiconformal automorphism \( \phi \) with \( \phi(z_0) = 0 \) and \( q(\phi) = \text{id}|_S \) such that whose complex dilatation vanishes outside some compact subset in \( \mathbb{D} \). This arranges any asymptotically conformal automorphism of \( \mathbb{D} \) to be a quasiconformal automorphism fixing 0 without changing the property of vanishing at the boundary.

By the above two claims, we have the following result attributed to Fehlmann [22] in Gardiner and Sullivan [24].

**Corollary 3.4.** A quasisymmetric automorphism \( g \in QS \) is symmetric if and only if \( g \) extends continuously to a quasiconformal automorphism \( f \in QC(\mathbb{D}) \) whose complex dilatation \( \mu_f \) vanishes at the boundary \( S \).
By the chain rule of complex dilatations, the composition of asymptotically conformal automorphisms of $D$ is also asymptotically conformal. Hence $AC(D)$ is a subgroup of $QC(D)$. Accordingly, Corollary 3.4 shows that $Sym$ is a subgroup of $QS$. Moreover, it was proved in [24] that $Sym$ is the characteristic topological subgroup of the partial topological group $QS$ for which the neighborhood base is given at id by using the quasisymmetry constant and is distributed at every point $g \in QS$ by the right translation.

In the rest of this section, we review the Teichmüller space of symmetric automorphisms, which is already well-known in the theory of asymptotic Teichmüller spaces. This will be a prototype of our construction of the Teichmüller space of circle diffeomorphisms.

**Definition.** The little subspace $T_0$ of the universal Teichmüller space $T = \text{Mob}(S) \setminus QS$ (or the Teichmüller space of symmetric automorphisms) is defined to be $T_0 = \text{Mob}(S) \setminus Sym$.

We define the subset $\text{Bel}_0(D)$ of $\text{Bel}(D)$ consisting of all Beltrami coefficients vanishing at the boundary. Since $	ext{Mob}(D) \setminus AC(D)$ can be identified with $\text{Bel}_0(D)$, Corollary 3.4 implies that the image of $\text{Bel}_0(D)$ under the Teichmüller projection $\pi : \text{Bel}(D) \rightarrow T$ is $T_0$. This also implies that its Bers embedding $\beta(T_0)$ coincides with $\Phi(\text{Bel}_0(D))$ for the Bers projection $\Phi : \text{Bel}(D) \rightarrow B(D^*)$. Under the group structure $*$ of $\text{Bel}(D)$, $\text{Bel}_0(D)$ is a subgroup. Correspondingly, $T_0$ is a subgroup of $(T, *)$. Actually $T_0 \subset T$ is a topological subgroup since $T_0$ is identified with $\text{Sym}_* = \text{Sym} \cap QS_*$ and $\text{Sym}$ is a topological subgroup.

It is proved by Earle, Markovic and Saric [20] that the Douady-Earle extension $\epsilon_{DE}(g)$ of a symmetric automorphism $g \in \text{Sym}$ is asymptotically conformal; $\epsilon_{DE} : \text{Sym} \rightarrow AC(D)$ is a section for $\epsilon : AC(D) \rightarrow \text{Sym}$. Hence the conformally natural section $s_{DE} : T \rightarrow \text{Bel}(D)$ sends $T_0$ into $\text{Bel}_0(D)$. Note that $\text{Bel}_0(D)$ is the unit ball of the Banach subspace $L^\infty_0(D) \subset L^\infty(D)$ consisting of bounded measurable functions vanishing at the boundary: $\text{Bel}_0(D) = \text{Bel}(D) \cap L^\infty_0(D)$. In particular, $\text{Bel}_0(D)$ is contractible. Therefore, $T_0$ is also contractible as a topological subspace of $T$.

To consider the complex structure of $T_0$, we introduce the Banach subspace $B_0(D^*)$ of $B(D^*)$ as follows:

$$B_0(D^*) = \{ \varphi \in B(D^*) \mid \lim_{|z| \rightarrow 1} \rho_{D^*}^2(z)|\varphi(z)| = 0 \}.$$  

An element in $B_0(D^*)$ is also called vanishing at the boundary. The following theorem is in Gardiner and Sullivan [24], which was essentially proved by Becker and Pommerenke [12].

**Theorem 3.5.** For the Bers projection $\Phi : \text{Bel}(D) \rightarrow B(D^*)$, it holds that

$$\Phi(\text{Bel}_0(D)) = \beta(T) \cap B_0(D^*).$$

By this theorem, we have $\beta(T_0) = \beta(T) \cap B_0(D^*)$. Hence $T_0$ is identified with a bounded domain of the complex Banach space $B_0(D^*)$. 
4. The decay order of Schwarzian and pre-Schwarzian derivatives

We focus on the decay order of a Beltrami coefficient $\mu \in \text{Bel}_0(\mathbb{D})$ vanishing at the boundary $\mathbb{S}$. Define

$$\kappa_\mu(t) = \text{ess.sup}_{1-t \leq |\zeta| < 1} |\mu(\zeta)|| (0 < t \leq 1)$$

for $\mu \in \text{Bel}_0(\mathbb{D})$, which satisfies $\kappa_\mu(t) \to 0$ as $t \to 0$. Let $\alpha$ be a fixed constant with $0 < \alpha < 1$. For a Beltrami coefficient $\mu \in \text{Bel}_0(\mathbb{D})$, we define a new norm by

$$\|\mu\|_{\infty, \alpha} = \text{ess.sup}_{\zeta \in \mathbb{D}} \rho_\alpha(\zeta)|\mu(\zeta)|.$$ 

Clearly $\|\mu\|_{\infty, \alpha} < \infty$ if and only if $\kappa_\mu(t) = O(t^\alpha)$ $(t \to 0)$.

**Definition.** Let $\alpha$ be a constant with $0 < \alpha < 1$. The space of Beltrami coefficients $\mu \in \text{Bel}(\mathbb{D})$ with $\|\mu\|_{\infty, \alpha} < \infty$ is denoted by $\text{Bel}_\alpha^0(\mathbb{D})$.

As in the definition of the Bers projection, we extend a Beltrami coefficient $\mu \in \text{Bel}_\alpha^0(\mathbb{D})$ to $\hat{\mathbb{C}}$ by setting $\mu(z) \equiv 0$ for $z \in \mathbb{D}^*$ and take a quasiconformal automorphism $f_\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ having the complex dilatation $\mu$. Then $f_\mu|_{\mathbb{D}^*}$ is a conformal homeomorphism (univalent function). Hereafter, we always give the following normalization for $f_\mu$:

$$f_\mu(\infty) = \infty; \quad \lim_{z \to \infty} (f_\mu)'(z) = 1.$$ 

Equivalently, the Laurent expansion of $f_\mu$ at $\infty$ is

$$f_\mu(z) = z + b_0 + \frac{b_1}{z} + \cdots.$$ 

We consider its pre-Schwarzian derivative and Schwarzian derivative on $\mathbb{D}^*$ defined respectively as follows:

$$T_{f_\mu|_{\mathbb{D}^*}}(z) = \frac{f_\mu''(z)}{f_\mu'(z)};$$

$$S_{f_\mu|_{\mathbb{D}^*}}(z) = (T_{f_\mu|_{\mathbb{D}^*}})'(z) - \frac{1}{2} \{T_{f_\mu|_{\mathbb{D}^*}}(z)\}^2.$$ 

It has been shown in Becker and Pommerenke [12] that the condition $\mu \in \text{Bel}_0(\mathbb{D})$ is equivalent to each of the conditions

$$\lim_{|z| \to 1} \rho_{\mathbb{D}^*}^{-1}(z)|T_{f_\mu|_{\mathbb{D}^*}}(z)| = 0; \quad \lim_{|z| \to 1} \rho_{\mathbb{D}^*}^{-2}(z)|S_{f_\mu|_{\mathbb{D}^*}}(z)| = 0.$$ 

To estimate their decay order quantitatively in terms of $\kappa_\mu(t)$, we set

$$\beta_\mu(t) = \sup_{1<|z|\leq 1+t} (|z| - 1)|T_{f_\mu|_{\mathbb{D}^*}}(z)|,$$

$$\sigma_\mu(t) = \sup_{1<|z|\leq 1+t} (|z| - 1)^2|S_{f_\mu|_{\mathbb{D}^*}}(z)| \quad (0 < t < \infty).$$
It was proved by Becker [11, Theorem 2] that
\[ \beta_\mu(t^{1+\varepsilon}) \leq 3\{\kappa_\mu(t) + t^\varepsilon\}, \quad \sigma_\mu(t^{1+\varepsilon}) \leq \frac{3}{2}\{\kappa_\mu(t) + t^{2\varepsilon}\} \quad (0 < t \leq 1) \]
for any \( \varepsilon > 0 \).

We will improve these estimates regarding the power of \( t \) for the case where \( \kappa_\mu(t) = O(t^\alpha) \). In this case, the elimination of the constant \( \varepsilon \) was done by Dyn’kin [18]. Our improvement can be stated as follows.

**Theorem 4.1.** For every \( \alpha \in (0, 1) \), there is a constant \( C = C(\alpha) > 0 \) depending only on \( \alpha \) such that
\[ \rho_{D^1}(z)|T_{f_\mu}|_{D^*}(z)| \leq C\|\mu\|_{\infty, \alpha}(|z| - 1)^\alpha \]
for every \( \mu \in \text{Bel}^0_\alpha(D) \) and for every \( z \in D^* \). Equivalently,
\[ \beta_\mu(t) \leq C\|\mu\|_{\infty, \alpha} \frac{2t^\alpha}{t + 2} \]
for every \( t > 0 \).

We decompose a Beltrami coefficient \( \mu \in \text{Bel}^0_\alpha(D) \) suitably into a finite number of those whose supports are in mutually disjoint annular domains of \( D \). Then a computation of the pre-Schwarzian derivative of the composition of the corresponding conformal homeomorphisms establishes the estimate. These steps are given in the following two lemmata.

**Lemma 4.2.** For every \( \alpha \in (0, 1) \), there is a constant \( \lambda \) with \( 0 < \lambda < 1 \) depending only on \( \alpha \) such that, if a sequence \( \{s_n\}_{n=0}^{\infty} \) of positive numbers satisfies a recurrence relation
\[ \left( \frac{1}{1 + s_{n-1}} \right)s_n^\alpha = \lambda^n \]
for every \( n \geq 1 \) and \( s_0 = 1 \), then \( \{s_n\} \) is increasing and diverges to \( +\infty \).

**Proof.** The recurrence relation is equivalent to
\[ s_n = \lambda^{\frac{1}{\alpha}}(1 + s_{n-1})^{1/\alpha} \]
for every \( n \geq 1 \) and \( s_0 = 1 \). For comparison with this formula, we consider another recurrence relation
\[ s'_n = \lambda^{\frac{2}{\alpha}}s'_{n-1}^{\frac{1}{\alpha}} \]
for every \( n \geq 2 \) by giving the initial value \( s'_1 = s' = (2\lambda)^{1/\alpha} \). It is easy to see that \( s_n \geq s'_n \) for every \( n \geq 1 \) and hence \( \lim_{n \to \infty} s'_n = +\infty \) implies \( \lim_{n \to \infty} s_n = +\infty \). Also, if \( \{s'_n\} \) is increasing then so is \( \{s_n\} \).

Set \( b_n = s'_{n+1}/s'_n \). Then we have
\[ b_n = \lambda^{\frac{1}{\alpha}}(b_{n-1})^{\frac{1}{\alpha}} \]
for every \( n \geq 2 \) and
\[ b_1 = \frac{s'_2}{s'_1} = \frac{\lambda^{\frac{2}{\alpha}}(2\lambda)^{1/\alpha}}{(2\lambda)^{1/\alpha}}. \]
Taking the logarithm yields

$$\log b_n = \frac{1}{\alpha} \log b_{n-1} + \frac{1}{\alpha} \log \lambda$$

with

$$\log b_1 = \left( \frac{1}{\alpha^2} + \frac{1}{\alpha} \right) \log \lambda + \left( \frac{1}{\alpha^2} - \frac{1}{\alpha} \right) \log 2.$$ 

This shows that if

$$\log b_1 > -\log \lambda_{n1} - \alpha,$$

then \( \log b_n \) are positive and uniformly bounded away from 0 for all \( n \geq 1 \). By choosing \( \lambda < 1 \) sufficiently close to 1, we have such a situation. For instance, \( \lambda \) can be chosen so that \( \lambda > (1/2)^{(1-\alpha)^2/(1+\alpha+\alpha^2)} \). This proves that \( \{s'_n\} \) is increasing and diverges to \( +\infty \). \( \Box \)

**Lemma 4.3.** For a finite sequence of real numbers

$$1 = r_{-1} > r_0 > r_1 > \cdots > r_N > r_{N+1} = 0,$$

let \( A_n = \{ r_n > |\zeta| \geq r_{n+1} \} \) be an annulus (or a disk) in \( \mathbb{D} \) for each \( n = -1, 0, \ldots, N \). For \( \mu \in \text{Bel}(\mathbb{D}) \), set

$$\mu_n(\zeta) = \begin{cases} \mu(\zeta) & (\zeta \in A_n) \\ 0 & (\zeta \in \hat{\mathbb{C}} - A_n), \end{cases}$$

and \( k_n = \| \mu_n \|_{\infty} \). Then the pre-Schwarzian derivative of \( f_\mu|_{\mathbb{D}^*} \) satisfies

$$|T_{f_\mu|_{\mathbb{D}^*}}(z)| \leq 12 \sum_{n=-1}^{N} \frac{k_n r_n}{|z|^2 - r_n^2}$$

for every \( z \in \mathbb{D}^* \).

**Proof.** First, we take a quasiconformal automorphism \( f_N = f_{\mu_N} \) of \( \mathbb{C} \) (namely, an automorphism of \( \hat{\mathbb{C}} \) fixing \( \infty \)) with the complex dilatation \( \mu_N \) and consider the push-forward \( \tilde{\mu}_{N-1} = (f_N)_* \mu_{N-1} \) of \( \mu_{N-1} \) by \( f_N \). Here, the push-forward \( f_* \mu \) of \( \mu \in \text{Bel}(D) \) by a conformal homeomorphism \( f \) of a domain \( D \) is defined in general by

$$(f_* \mu)(z) = \mu(f^{-1}(z)) \frac{(f^{-1})'(z)}{(f^{-1})'(f(z))} \quad (z \in f(D)).$$

Next, we take a quasiconformal automorphism \( f_{N-1} = f_{\tilde{\mu}_{N-1}} \) of \( \mathbb{C} \) and the push-forward \( \tilde{\mu}_{N-2} = (f_{N-1} \circ f_N)_* \mu_{N-2} \) similarly. Inductively, for each \( n \geq 0 \), let \( f_n = f_{\tilde{\mu}_n} \) be a quasiconformal automorphism of \( \mathbb{C} \) with the complex dilatation \( \tilde{\mu}_n \) and let

$$\tilde{\mu}_{n-1} = (f_n \circ \cdots \circ f_N)_* \mu_{n-1}$$

be the push-forward of \( \mu_{n-1} \) by \( f_n \circ \cdots \circ f_N \). Finally, choose a quasiconformal automorphism \( f_{-1} = f_{\tilde{\mu}_{-1}} \) of \( \mathbb{C} \) so that \( f_{-1} \circ \cdots \circ f_N \) coincides with \( f_\mu \).
By the chain rule of pre-Schwarzian derivatives, we see that
\[
T_{f_n|\hat{D}}(z) = T_{f_N}(z) + T_{f_{N-1}}(f_N(z)) + \cdots + T_{f_{-1}}(f_0 \circ \cdots \circ f_N)(z)
\]
\[
= T_{f_N}(z) + \sum_{n=1}^{N-1} T_{f_n}(f_{n+1} \circ \cdots \circ f_N)(f_{n+1} \circ \cdots \circ f_N)'(z)
\]
for \( z \in \hat{D}^* \).

Here we use the following estimates for pre-Schwarzian derivative. For any conformal homeomorphism \( f \) of \( \hat{D}^* \) with \( f(\infty) = \infty \), it was shown by Avhadiev \[9\] (cf. Sugawa \[38, Theorem 4.2.3\]) that
\[
\rho_{\hat{D}^*}^{-1}(z)|T_f(z)| \leq \frac{|z|^2 - 1}{2}|z T_f(z)| \leq 3 \quad (|z| > 1).
\]

In addition, if \( f \) extends to a quasiconformal automorphism of \( \hat{C} \) of complex dilatation \( \mu \) with \( \|\mu\|_{\infty} \leq k \), then the majorant principle due to Lehto \[27, Section II.3.5\] yields that
\[
|T_f(z)| \leq 3k\rho_{\hat{D}^*}(z).
\]

On the other hand, for any simply connected domain \( \Omega_* \subset \hat{C} \) containing \( \infty \) and for any conformal homeomorphism \( f \) of \( \Omega_* \) with \( f(\infty) = \infty \), we see that
\[
|T_f(\omega)| \leq 6\rho_{\Omega_*}(\omega) \quad \text{for} \quad \omega \in \Omega_*.
\]

Again if this extends to a quasiconformal automorphism of \( \hat{C} \) with \( \|\mu\|_{\infty} \leq k \) then
\[
|T_f(\omega)| \leq 6k\rho_{\Omega_*}(\omega).
\]

The conformal homeomorphism \( f_N \) of the disk \( \Omega_N = \{|z| > r_N\} \cup \{\infty\} \) into \( \hat{C} \) with \( f_N(\infty) = \infty \) satisfies
\[
|T_{f_N}(z)| \leq \frac{6k_Nr_N}{|z|^2 - r_N^2}.
\]

The conformal homeomorphism \( f_n \) of the quasidisk \( \Omega'_n \) into \( \hat{C} \) with \( f_n(\infty) = \infty \) for \(-1 \leq n \leq N-1\), where \( \Omega'_n \) is the image of the disk \( \{|z| > r_n\} \cup \{\infty\} \) under \( f_{n+1} \circ \cdots \circ f_N \), satisfies
\[
|T_{f_n}(\omega)| \leq 6k_n\rho_{\Omega'_n}(\omega)
\]
for every \( \omega \in \Omega'_n \) in terms of the hyperbolic density \( \rho_{\Omega'_n}(\omega) \) of \( \Omega'_n \). Hence, by replacing \( \omega \) with \( f_{n+1} \circ \cdots \circ f_N(z) \), we have
\[
|T_{f_n}(f_{n+1} \circ \cdots \circ f_N(z))(f_{n+1} \circ \cdots \circ f_N)'(z)|
\]
\[
\leq 6k_n\rho_{\Omega'_n}(f_{n+1} \circ \cdots \circ f_N(z))(f_{n+1} \circ \cdots \circ f_N)'(z) = \frac{12k_nr_n}{|z|^2 - r_n^2}.
\]

This gives the desired inequality
\[
|T_{f_n|\hat{D}}(z)| \leq 12 \sum_{n=1}^{N} \frac{k_nr_n}{|z|^2 - r_n^2}
\]
for every \( z \in \hat{D}^* \). \( \square \)
Proof of Theorem 4.1. Suppose that $\mu \in \text{Bel}_0^a(\D)$ is given such that $\ell = \|\mu\|_{\infty,a} < \infty$. Then we have

$$\kappa_\mu(t) = \sup_{1-t \leq |\zeta| < 1} |\mu(\zeta)| \leq \ell^t \alpha \quad (0 < t \leq 1).$$

Fixing $\tau \in \D^*$, we will estimate $\rho_{\D^*}^{-1}(\tau)\left|T_{f_n|\D^*}(\zeta)\right|$ in terms of $\ell$. Set $\tau = |\zeta| - 1 \in (0, \infty)$.

We choose $t_0 = \tau$ and inductively define a sequence $\{t_n\}_{n \geq 1}$ of positive numbers by a recurrence relation

$$\frac{\tau}{\tau + t_{n-1}} \cdot \ell t_n^\alpha = \lambda^n \cdot \ell^{\tau^\alpha}$$

for some constant $\lambda$ with $0 < \lambda < 1$. If we set $s_n = t_n/\tau$, this is equivalent to

$$\left(\frac{1}{1 + s_{n-1}}\right) s_n^\alpha = \lambda^n$$

with the initial condition $s_0 = 1$. Then by Lemma 4.2, we can find the constant $\lambda = \lambda(\alpha)$ so that the sequence $\{s_n\}$ and hence $\{t_n\}$ are increasing and diverge to $+\infty$. In particular, there is the smallest positive integer $N$ such that $t_{N+1} \geq 1$.

By using the positive numbers $\{t_n\}_{n=0}^N$, we set $r_n = 1 - t_n$. Also, set $r_{-1} = 1$ and $r_{N+1} = 0$. Then, as in Lemma 4.3, we divide $\D$ into the annuli (or the disk)

$$A_n = \{r_n > |\zeta| > r_{n+1}\} \quad (n = -1, 0, \ldots, N)$$

and define $k_n = \|\mu_n\|_{\infty}$ for $\mu_n = \mu \cdot 1_{A_n}$. Since $\kappa_\mu(t) \leq \ell^t$, we see that $k_n \leq \ell t_{n+1}^\alpha$. Note that for $n = N$ this is valid as $\|\mu\|_{\infty} \leq \ell \leq \ell t_{n+1}^\alpha$. Now the application of Lemma 4.3 yields

$$\rho_{\D^*}^{-1}(\tau)\left|T_{f_n|\D^*}(\zeta)\right| \leq 6(|z|^2 - 1) \sum_{n=1}^N \frac{k_n r_n}{|z|^2 - r_n^2} \leq 6 \sum_{n=-1}^N \frac{\tau}{\tau + t_n} \cdot \ell t_{n+1}^\alpha.$$  

Here the recurrence relation for $\{t_n\}$ gives that the last sum is taken for $\lambda^{n+1} \cdot \ell^{\tau^\alpha}$. Thus we have

$$\rho_{\D^*}^{-1}(\tau)\left|T_{f_n|\D^*}(\zeta)\right| \leq \frac{6 \ell}{1 - \lambda^{\tau^\alpha}},$$

where $\lambda$ is depending only on $\alpha$. By taking $C = 6/(1 - \lambda)$, we obtain the assertion. \hfill $\Box$

Next, we consider the relation between $T_f$ and $S_f$ for a conformal homeomorphism $f$ of $\D^*$. It is known that there is some absolute constant $A > 0$ such that

$$\rho_{\D^*}^{-2}(\zeta)\left|S_f(z)\right| \leq A \rho_{\D^*}^{-1}(\tau)\left|T_f(z)\right| \quad (z \in \D^*).$$

See Becker [10, Lemma 6.1]. This in particular implies the following.

Proposition 4.4. If $\beta_\mu(t) = O(t^{\alpha})$ then $\sigma_\mu(t) = O(t^{\alpha}) \ (t \to 0)$.

Remark. Lemmata 4.2 and 4.3 can be easily modified to be suitable for the estimate of Schwarzian derivatives. Hence inequalities

$$\rho_{\D^*}^{-2}(\zeta)\left|S_{f_n|\D^*}(\zeta)\right| \leq C'\|\mu\|_{\infty,a}(|z| - 1)^\alpha; \quad \sigma_\mu(t) \leq C'\|\mu\|_{\infty,a} \frac{4t^\alpha}{(t + 2)^2}.$$
for some $C' = C'(\alpha) > 0$ can be also derived directly from these modifications in the same way as in the proof of Theorem 4.1.

Finally we will show that $\sigma_{\mu}(t) = O(t^\alpha)$ implies $\kappa_{\mu}(t) = O(t^\alpha)$ $(t \to 0)$. This is a consequence of the next lemma, which we can be found in Becker [10, Theorem 5.4].

**Lemma 4.5.** Suppose that $\varphi = S_f$ belongs to $B_0(\mathbb{D}^*)$, where $f$ is a conformal homeomorphism of $\mathbb{D}^*$ having quasiconformal extension to $\hat{\mathbb{C}}$. Set

$$F(z) = f(z^*) - \frac{(z^* - z)f'(z^*)}{1 + (z^* - z)f''(z^*)/(2f'(z^*))}$$

for $z \in \mathbb{D}$, where $z^* = 1/\bar{z}$ is the reflection of $z$ with respect to $\mathbb{S}$. Then there is some $t > 0$ such that $f$ extends to a quasiconformal automorphism of $\hat{\mathbb{C}}$ that coincides with $F$ on the annulus \{ $1 - t < |z| < 1$ \} having the complex dilatation

$$\mu_F(z) = \frac{\partial F(z)}{\partial F(z')} = -2\rho_D^{-2}(z^*)(zz^*)^2\varphi(z^*).$$

In particular, if $\varphi$ satisfies $\|\varphi\|_{\infty, \alpha} < \infty$, then there is some $\mu \in \text{Bel}_0(\mathbb{D})$ such that $S_{f_{\mu|_{\mathbb{D}^*}}} = \varphi$.

Theorem 4.1, Proposition 4.4 and Lemma 4.3 conclude the equivalence of all the conditions above. Note that the condition $\pi(\mu) = \pi(\mu')$ for $\mu, \mu' \in \text{Bel}(\mathbb{D})$ is equivalent to that $f_{\mu|_{\mathbb{D}^*}} = f_{\mu'|_{\mathbb{D}^*}}$.

**Theorem 4.6.** The following conditions are equivalent for $\mu \in \text{Bel}(\mathbb{D})$ and $\alpha \in (0, 1)$:

1. $\|\mu'\|_{\infty, \alpha} < \infty$ for some $\mu' \in \text{Bel}(\mathbb{D})$ with $\pi(\mu) = \pi(\mu')$;
2. $\beta_{\mu}(t) = O(t^\alpha)$ $(t \to 0)$;
3. $\sigma_{\mu}(t) = O(t^\alpha)$ $(t \to 0)$ or equivalently $\sup_{z \in \mathbb{D}^*} \rho_D^{-2+\alpha}(z)|S_{f_{\mu|_{\mathbb{D}^*}}}(z)| < \infty$.

The above results can be also proved when we exchange the role of $\mathbb{D}$ and $\mathbb{D}^*$. We will briefly mention this fact. For any Beltrami coefficient $\mu \in \text{Bel}(\mathbb{D})$, we define its reflection by

$$\mu^*(z) = \overline{\mu(z^*)}(zz^*)^2 \in \text{Bel}(\mathbb{D}^*).$$

This coincides with the complex dilatation of the reflection of $f^\mu : \mathbb{D} \to \mathbb{D}$ with respect to $\mathbb{S}$. If $\mu \in \text{Bel}_0(\mathbb{D})$ and $\|\mu\|_{\infty, \alpha} = \ell$, then $\mu^*$ satisfies

$$|\mu^*(z)| = |\mu(z^*)| \leq \ell \left( \frac{|z|^2 - 1}{2|z|^2} \right)^\alpha \leq \ell(|z| - 1)^\alpha \quad (z \in \mathbb{D}^*);$$

$$\kappa_{\mu^*}(t) = \sup_{1 < |z| \leq 1 + t} |\mu^*(z)| \leq \ell t^\alpha \quad (0 < t < \infty).$$

We define the function $\beta$ for $\mu^* \in \text{Bel}(\mathbb{D}^*)$ similarly. Extend $\mu^*$ to $\hat{\mathbb{C}}$ by setting $\mu^*(\zeta) \equiv 0$ for $\zeta \in \mathbb{D}$ and take a quasiconformal automorphism $f_{\mu^*} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ having the complex
dilatation \( \mu^* \) with \( f_{\mu^*}(\infty) = \infty \). Then, for the pre-Schwarzian derivative \( T_{f_{\mu^*}}(\zeta) = f''_{\mu^*}(\zeta)/f'_{\mu^*}(\zeta) \) on \( \mathbb{D} \), we define
\[
\tilde{\beta}_{\mu^*}(t) = \sup_{1 - t \leq |\zeta| < 1} (1 - |\zeta|)|T_{f_{\mu^*}}(\zeta)|.
\]

We can modify Lemma 4.3 appropriately by using the corresponding estimates of pre-Schwarzian derivatives on \( \mathbb{D} \) and any simply connected domain \( \Omega \subset \mathbb{C} \):
\[
|T_f(\zeta)| \leq 3\rho_D(\zeta) \quad (\zeta \in \mathbb{D}); \quad |T_f(\omega)| \leq 4\rho_D(\omega) \quad (\omega \in \Omega).
\]

Concerning the relation between \( T_f \) and \( S_f \) for a conformal homeomorphism \( f \) of \( \mathbb{D} \), there is some absolute constant \( A' > 0 \) such that
\[
\rho_D^{-2}(\zeta)|S_f(\zeta)| \leq A'|\rho_D^{-1}(\zeta)|T_f(\zeta)| \quad (\zeta \in \mathbb{D}).
\]
See [11, pp.117–119] and [38, Sections 4.2 and 5.3]. Thus the corresponding statement to Proposition 4.4 holds true also in this case. Moreover, the interior version of Lemma 4.5 is given in [11, Theorem 3].

Therefore, the corresponding statements to Theorems 4.1 and 4.6 are also valid in this case, in particular, we record the following claim as a corollary for later use.

**Corollary 4.7.** For every \( \alpha \in (0, 1) \), there is a constant \( C' = C'(\alpha) > 0 \) depending only on \( \alpha \) such that \( \tilde{\beta}_{\mu^*}(t) \leq C'||\mu||_{\infty, \alpha}t^\alpha \) for every \( \mu \in \text{Bel}_D^0(\mathbb{D}) \) and for every \( t \in (0, 1] \).

### 5. Hölder continuity of derivatives and quasisymmetric quotients

We define a class of orientation-preserving diffeomorphisms of the circle with Hölder continuous derivatives, which is of importance in our theory of Teichmüller spaces. In this section, we investigate the topology of the space of such circle diffeomorphisms. In particular, we relate this topology to the quasisymmetric quotients and the dilatations of their quasiconformal extensions.

**Definition.** An orientation-preserving diffeomorphism \( g \) of \( \mathbb{S} \) belongs to a class \( \text{Diff}^{1+\alpha}(\mathbb{S}) \) for some \( \alpha \) \((0 < \alpha < 1)\) if its derivative \( g' \) is \( \alpha \)-Hölder continuous. This means that the lift \( \tilde{g} : \mathbb{R} \to \mathbb{R} \) of \( g \) satisfies
\[
|\tilde{g}'(x) - \tilde{g}'(y)| \leq c|x - y|^{\alpha} \quad (x, y \in \mathbb{R})
\]
for some \( c \geq 0 \).

We provide the right uniform topology for \( \text{Diff}^{1+\alpha}(\mathbb{S}) \). This is induced by the \( C^{1+\alpha} \) constant, which measures the difference of an element \( g \in \text{Diff}^{1+\alpha}(\mathbb{S}) \) from the identity as follows:
\[
p_{1+\alpha}(g) = \sup_{0 \leq x < 1} |\tilde{g}(x) - \tilde{g}(0) - x| + \sup_{0 \leq x < 1} |\tilde{g}'(x) - 1| + c_\alpha(g),
\]
where
\[
c_\alpha(g) = \sup_{0 < |x - y| \leq 1/2} \frac{|\tilde{g}'(x) - \tilde{g}'(y)|}{|x - y|^{\alpha}}.
\]
Then we define that $g_n$ converge to $g$ in $\text{Diff}^{1+\alpha}(\mathbb{S})$ if $p_{1+\alpha}(g_n \circ g^{-1}) \to 0$ as $n \to \infty$.

**Remark.** The right uniform topology on $\text{Diff}^{1+\alpha}(\mathbb{S})$ as above is different from the $C^{1+\alpha}$-topology given in Herman [25].

We first verify that the neighborhood base at $id \in \text{Diff}^{1+\alpha}(\mathbb{S})$ is compatible with the group structure. In other words, $\text{Diff}^{1+\alpha}(\mathbb{S})$ is a partial topological group in the sense of Gardiner and Sullivan [24].

**Proposition 5.1.** The $C^{1+\alpha}$ constant $p_{1+\alpha}$ satisfies the following:

1. If $p_{1+\alpha}(g_n) \to 0$ and $p_{1+\alpha}(h_n) \to 0$ as $n \to \infty$ then $p_{1+\alpha}(g_n \circ h_n) \to 0$;
2. If $p_{1+\alpha}(g_n) \to 0$ as $n \to \infty$ then $p_{1+\alpha}(g_n^{-1}) \to 0$.

**Proof.** (1) It is obvious that $g_n \circ h_n \to id$ and $(g_n \circ h_n)'(x) = g'_n(h_n(x))h'_n(x) \to 1$ uniformly. Concerning the convergence of $c_\alpha$, we have

$$|(g_n \circ h_n)'(x) - (g_n \circ h_n)'(y)|$$

$$\leq \frac{|g'_n(h_n(x))h'_n(x) - g'_n(h_n(y))h'_n(y)|}{|x - y|^\alpha} + \frac{|g'_n(h_n(y))h'_n(x) - g'_n(h_n(y))h'_n(y)|}{|x - y|^\alpha}$$

$$\leq \frac{c_\alpha(g_n)|h_n(x) - h_n(y)|^\alpha|h'_n(x)|}{|x - y|^\alpha} + \frac{|g'_n(h_n(y))|c_\alpha(h_n)|x - y|^\alpha}{|x - y|^\alpha}.$$ 

Since $c_\alpha(g_n), c_\alpha(h_n) \to 0$ and $g'_n(x), h'_n(x) \to 1$ uniformly, we see that $c_\alpha(g_n \circ h_n) \to 0$ as $n \to \infty$.

(2) It is obvious that $g_n^{-1} \to id$ and $(g_n^{-1})'(x) = 1/(g'_n(g_n^{-1}(x))) \to 1$ uniformly. Concerning the convergence of $c_\alpha$, we have

$$|(g_n^{-1})'(x) - (g_n^{-1})'(y)| = \frac{|g'_n(g_n^{-1}(x)) - g'_n(g_n^{-1}(y))|}{|x - y|^\alpha}$$

$$\leq \frac{c_\alpha(g_n)|g_n^{-1}(x) - g_n^{-1}(y)|^\alpha}{|x - y|^\alpha}.$$ 

Since $c_\alpha(g_n) \to 0$ and $g'_n(x), (g_n^{-1})'(x) \to 1$ uniformly, we see that $c_\alpha(g_n^{-1}) \to 0$ as $n \to \infty$. 

Actually, we see more: $\text{Diff}^{1+\alpha}(\mathbb{S})$ is a topological group.

**Proposition 5.2.** With respect to the right uniform topology, $\text{Diff}^{1+\alpha}(\mathbb{S})$ is a topological group.

**Proof.** According to [24] Lemma 1.1], we have only to show that the adjoint map is continuous at $id$; if $p_{1+\alpha}(g_n) \to 0$ as $n \to \infty$ then $p_{1+\alpha}(h \circ g_n \circ h^{-1}) \to 0$ for every $h \in \text{Diff}^{1+\alpha}(\mathbb{S})$. We have that $h \circ g_n \circ h^{-1} \to id$ and

$$(h \circ g_n \circ h^{-1})'(x) = \frac{h'(g_n \circ h^{-1}(x))}{h'(h^{-1}(x))} g'_n(h^{-1}(x)) \to 1.$$
uniformly. Also,
\[
\left| (h \circ g_n \circ h^{-1})'(x) - (h \circ g_n \circ h^{-1})'(y) \right| \leq \frac{h'(g_n \circ h^{-1}(x)) - h'(g_n \circ h^{-1}(y))}{h'(h^{-1}(x)) - h'(h^{-1}(y))} \cdot |x - y|^{-\alpha},
\]
which is uniformly asymptotic to
\[
\frac{|g'_n(h^{-1}(x)) - g'_n(h^{-1}(y))|}{|x - y|^{\alpha}} \leq c_\alpha(g_n)|h^{-1}(x) - h^{-1}(y)|^{\alpha}.\]
Since \( c_\alpha(g_n) \to 0 \), we see that \( c_\alpha(h \circ g_n \circ h^{-1}) \to 0 \) as \( n \to \infty \).

Since every circle diffeomorphism is symmetric, \( \text{Diff}^{1+\alpha}(\mathbb{S}) \) is a subgroup of \( \text{Sym} \). We will characterize an element \( g \) of \( \text{Diff}^{1+\alpha}(\mathbb{S}) \) in terms of the quasisymmetric quotient of \( g \). This is due to Carleson [15, Lemma 5]. See also Gardiner and Sullivan [24, Section 9]. The following statement and a detailed proof can be found in [30, Theorem 7.1] and its corollary.

**Theorem 5.3.** Fix \( \alpha \in (0, 1) \). Suppose that there is some \( b \geq 0 \) such that the lift \( \tilde{g} \) of \( g \in \text{Sym} \) satisfies
\[
(1 + bt^\alpha)^{-1} \leq m_{\tilde{g}}(x, t) \leq 1 + bt^\alpha
\]
for every \( x \in [0, 1) \) and every \( t \in (0, 1/2) \). Then \( g \) belongs to \( \text{Diff}^{1+\alpha}(\mathbb{S}) \) and \( c_\alpha(g) \) depends only on \( b \) and tends to 0 uniformly as \( b \to 0 \). Moreover, \( \tilde{g}'(x) \) is uniformly bounded from above and away from 0 by constants depending only on \( b \) with \( \alpha \) fixed, which tend to 1 as \( b \to 0 \).

Conversely, every element \( g \in \text{Diff}^{1+\alpha}(\mathbb{S}) \) (\( 0 < \alpha < 1 \)) belongs to \( \text{Sym} \) with a gauge function for symmetry of order \( O(t^\alpha) \). More precisely, we have the following.

**Proposition 5.4.** For \( g \in \text{Diff}^{1+\alpha}(\mathbb{S}) \), there is a constant \( b \geq 0 \) such that
\[
(1 + bt^\alpha)^{-1} \leq m_g(x, t) \leq 1 + bt^\alpha
\]
for every \( x \in [0, 1) \) and every \( t \in (0, 1/2) \), where \( b \) can be taken depending only on \( c_\alpha(g) \) when \( c_\alpha(g) \leq 1 \) and tends to 0 uniformly as \( c_\alpha(g) \to 0 \).

For the proof, we need a simple claim.

**Proposition 5.5.** Every \( g \in \text{Diff}^{1+\alpha}(\mathbb{S}) \) satisfies
\[
1 - c_\alpha(g) < 1 - c_\alpha(g)(1/2)^\alpha \leq \tilde{g}'(x) \leq 1 + c_\alpha(g)(1/2)^\alpha < 1 + c_\alpha(g).
\]

**Proof.** Since \( \int_0^1 \tilde{g}'(x)dx = 1 \), there exists some \( x_0 \in [0, 1] \) such that \( \tilde{g}'(x_0) \geq 1 \). Also, there exists some \( x'_0 \in [0, 1] \) such that \( \tilde{g}'(x'_0) \leq 1 \). The Hölder continuity of \( \tilde{g}' \) implies that
\[
|\tilde{g}'(x) - \tilde{g}'(x_0)| \leq c_\alpha(g)|x - x_0|^\alpha \leq c_\alpha(g)(1/2)^\alpha
\]
for every \( x \in \mathbb{R} \) with \( |x - x_0| \leq 1/2 \), and the same is true for \( x'_0 \). Then using the periodicity \( \tilde{g}'(x + 1) = \tilde{g}'(x) \), we have the assertion. \( \square \)
Proof of Proposition 5.4. The mean value theorem says that there are $\xi_+$ and $\xi_-$ such that

$$g(x + t) - g(x) = tg'(\xi_+), \quad (x < \xi_+ < x + t)$$
$$g(x) - g(x - t) = tg'(\xi_+), \quad (x - t < \xi_- < x).$$

This gives

$$m_{\tilde{g}}(x, t) = 1 + \frac{\tilde{g}'(\xi_+) - \tilde{g}'(\xi_-)}{\tilde{g}'(\xi_-)}; \quad m_{\tilde{g}}(x, t)^{-1} = 1 + \frac{\tilde{g}'(\xi_-) - \tilde{g}'(\xi_+)}{\tilde{g}'(\xi_+)}.$$

Here we have

$$|\tilde{g}'(\xi_+) - \tilde{g}'(\xi_-)| \leq c_\alpha(g)|\xi_+ - \xi_-|^\alpha \leq c_\alpha(g)(2t)^\alpha$$

by the Hölder continuity of $\tilde{g}'$. Proposition 5.4 gives the lower estimate of $\tilde{g}'$. Moreover, since $g$ is diffeomorphic, there is some $c_0 > 0$ depending on $g$ such that $\tilde{g}'(x) \geq c_0$. Therefore

$$m_{\tilde{g}}(x, t)^{\pm 1} \leq 1 + \frac{2^\alpha c_\alpha(g)}{\max\{1 - c_\alpha(g)(1/2)^\alpha, c_0\}} t^\alpha.$$ 

We set the coefficient of $t^\alpha$ by $b$. If $c_\alpha(g) \leq 1$, then $1 - c_\alpha(g)(1/2)^\alpha > 0$ and $b$ depends only on $c_\alpha(g)$.

Now we see that $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$ if and only if $m_{\tilde{g}}(x, t)^{\pm 1} = 1 + O(t^\alpha)$ ($t \to 0$). We set

$$b_\alpha(g) = \sup_{0 \leq x < 1, 0 < t < 1/2} \max_{\xi = \pm 1} \frac{m_{\tilde{g}}(x, t)^\xi - 1}{t^\alpha}.$$ 

The quasisymmetry constant satisfies $M(g) \leq 1 + b_\alpha(g)$.

Corollary 5.6. For a sequence $\{g_n\} \subset \text{Diff}^{1+\alpha}(\mathbb{S})$, $c_\alpha(g_n) \to 0$ if and only if $b_\alpha(g_n) \to 0$ as $n \to \infty$. Moreover, under the extra assumption that each $g_n \in \text{Diff}^{1+\alpha}(\mathbb{S})$ is normalized so that it fixes the three points on $\mathbb{S}$ ($g_n \in QS\times$), $p_{1+\alpha}(g_n) \to 0$ if and only if $b_\alpha(g_n) \to 0$ or $c_\alpha(g_n) \to 0$ as $n \to \infty$.

Proof. The first statement directly follows from Theorem 5.3 and Proposition 5.4. For the second statement, we have only to show that $b_\alpha(g_n) \to 0$ or $c_\alpha(g_n) \to 0$ implies $p_{1+\alpha}(g_n) \to 0$ under the normalization. Theorem 5.3 or Proposition 5.5 verifies that $\tilde{g}'(x)$ converge to 1 uniformly. Also, since $M(g_n) \leq 1 + b_\alpha(g_n) \to 1$ and $g_n$ are normalized, $g_n$ converge to id uniformly. Hence we have $p_{1+\alpha}(g_n) \to 0$.

Finally in this section, we prepare to investigate $\text{Diff}^{1+\alpha}(\mathbb{S})$ by quasiconformal extension to $\mathbb{D}$. This will complete in the next section. Recall that, since $\text{Diff}^{1+\alpha}(\mathbb{S}) \subset \text{Sym}$, there is a quasiconformal extension that is asymptotically conformal. We look at the decay order of its complex dilatation close to the boundary.

Theorem 5.7. For every $g \in \text{Diff}^{1+\alpha}(\mathbb{S})$, there exists a quasiconformal extension $f \in AC(\mathbb{D})$ of $g$ whose complex dilatation $\mu$ belongs to $\text{Bel}_0^\alpha(\mathbb{D})$. Here $\|\mu\|_{\infty, \alpha}$ tends to 0 uniformly as $b_\alpha(g) \to 0$ or $c_\alpha(g) \to 0$. 

Proof. By Proposition 5.4 the lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$ of $g$ satisfies $m_{\tilde{g}}(x, t)^{\pm t} \leq 1 + bt^\alpha$ for $b = b_\alpha(g) \geq 0$. Then, by Theorem 3.1 the complex dilatation $\mu_F(z)$ of the Beurling-Ahlfors extension $F(z)$ satisfies $|\mu_F(z)| \leq 4by^\alpha$ for every $z = x + iy \in \mathbb{H}$. The projection $f : \mathbb{D} - \{0\} \to \mathbb{D} - \{0\}$ of $F$ under the holomorphic universal covering $u : \mathbb{H} \to \mathbb{D} - \{0\}$ is defined to be $e_{BA}(g)$ after filling 0.

The complex dilatation $\mu$ of $f = e_{BA}(g)$ satisfies
\[ |\mu(\zeta)| = |\mu_F(z)| = |\mu_F((\log \zeta)/(2\pi i))|.\]
Since $\text{Im}\{(\log \zeta)/(2\pi i)\} = -\log |\zeta|/(2\pi)$, the condition $|\mu_F(z)| \leq 4by^\alpha$ yields
\[ |\mu(\zeta)| \leq \frac{4b}{(2\pi)^\alpha} \{ -\log |\zeta| \}^\alpha.\]
Since $-\log |\zeta|$ is comparable to $1 - |\zeta|$ near $|\zeta| = 1$, we can find a continuous increasing function $d : [0, 1) \to [1, \infty)$ with $\lim_{t \to 0} d(t) = 1$ such that
\[ |\mu(\zeta)| \leq \frac{4b}{(2\pi)^\alpha} d(||\mu||_\infty)(1 - |\zeta|)^\alpha\]
for every $\zeta \in \mathbb{D}$. Moreover, if $c_\alpha(g) \to 0$, then $b_\alpha(g) \to 0$ by Proposition 5.4 and hence $M(g) \to 1$ which implies $||\mu||_\infty \to 0$ (see [27, Theorem I.5.2]). Therefore we see that $||\mu||_{\infty, \alpha} \to 0$ as $b_\alpha(g) \to 0$ or $c_\alpha(g) \to 0$. \qed

6. QUASICONFORMAL CHARACTERIZATION OF CIRCLE DIFFEOMORPHISMS

We will establish the relationship among the following three indices: the exponent of Hölder continuity of the derivative of a circle diffeomorphism $g$; the decay order of the complex dilatation of quasiconformal extension of $g$; and the decay order of the Schwarzian derivative of the corresponding conformal homeomorphism. We have seen the equivalence on the last two quantities (Theorem 4.6) and the implication of the second one from the first (Theorem 5.7).

The new ingredient is the converse of the statement of Theorem 5.7. In Theorem 5.2 and Corollary 3.4, we have seen that an asymptotically conformal automorphism $f \in \text{AC}(\mathbb{D})$ extends to a symmetric automorphism $g \in \text{Sym}$ and provided a certain estimate of the gauge function for symmetry in terms of the decay order of $\mu_f$. Note that the order of the gauge function is reduced to $\alpha/2$ from decay order $\alpha$ of $\mu_f$ according to Theorem 3.2. Also in the course of transforming the situation from $\mathbb{H}$ to $\mathbb{D}$, we need certain normalization on $g \in \text{Sym}$ to obtain some quantitative estimate. The order of the gauge function and the Hölder continuity of the derivative are related to each other as in Theorem 5.3 and Proposition 5.4.

Lemma 6.1. For an $K$-quasiconformal automorphism $f$ of $\mathbb{D}$ with complex dilatation $\mu \in \text{Bel}_0^\alpha(\mathbb{D})$, its boundary extension $g$ belongs to $\text{Diff}^{1+\alpha/2}(S)$. In addition, under the normalization such as $f(0) = 0$ or $g \in \text{QS}_*$, the derivative of $g$ is uniformly bounded from
above and away from 0. More precisely, there is a constant $D = D(\alpha, K, \ell) \geq 1$ depending only on $\alpha$, $K$ and $\ell$ with $\|\mu_f\|_{\infty, \alpha} \leq \ell$ such that
\[
\frac{1}{D} \leq g'(x) \leq D
\]
for every $x \in \mathbb{R}$.

**Proof.** Suppose that $f \in AC(\mathbb{D})$ fixes 0. In this case, $f$ lifts to the quasiconformal automorphism $F$ of $\mathbb{H}$ under the holomorphic universal covering $u : \mathbb{H} \rightarrow \mathbb{D} - \{0\}$. As in the proof of Theorem 5.7, the complex dilatation of $F$ satisfies $|\mu_F(z)| \leq (2\pi)^\alpha \ell y^\alpha$ ($z = x + iy \in \mathbb{H}$). Then Theorem 3.2 is applied for $\tilde{\zeta}(y) = (2\pi)^\alpha \ell y^\alpha$ to verify that the quasisymmetric quotient of $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$, which is the boundary extension of $F$ as well as the lift of $g$, satisfies
\[
m_{\tilde{g}}(x, t)^{\pm 1} \leq 1 + c\ell t^{\alpha/2} + R t^{1/2} \leq 1 + b t^{\alpha/2}
\]
for every $x \in [0, 1)$ and every $t \in (0, 1/2)$, where $b = b(K, \ell) > 0$ is a constant depending only on $K = K(F)$ and $\ell$. Then Theorem 5.3 asserts that $g$ belongs to $\text{Diff}^{1+\alpha/2}(\mathbb{S})$. Moreover, the derivative $g'(x)$ is estimated in terms of $\alpha$ and $b$ by the same theorem.

For a general $f$ not necessarily fixing 0, we take $\phi \in \text{Möb}(\mathbb{D})$ such that $\phi \circ f(0) = 0$. The complex dilatation of $\phi \circ f$ is the same as that of $f$. Then we can apply the previous argument to $\phi \circ f$; we obtain that $\phi \circ g \in \text{Diff}^{1+\alpha/2}(\mathbb{S})$ where $\phi \in \text{Möb}(\mathbb{S})$ denotes the boundary extension of $\phi \in \text{Möb}(\mathbb{D})$ by the same symbol. This in particular shows that $g$ itself belongs to $\text{Diff}^{1+\alpha/2}(\mathbb{S})$. Moreover, if $g$ is normalized, Proposition 6.2 below shows that $|f(0)| \leq r$ for some $r = r(K) \in [0, 1)$. Then $\phi$ satisfies
\[
\frac{1 - r}{1 + r} \leq |\phi'(z)| \leq \frac{1 + r}{1 - r}.
\]
From the uniform boundedness of $(\phi \circ g)'$ by the previous argument, we also see that $g'$ is uniformly bounded from above and away from 0.

In this section, we will compare again the condition $f(0) = 0$ with our normalization fixing 1, $i$ and $-1$ for $f = f^\mu \in QC(\mathbb{D})$. The following proposition ensures that they have little difference as we have seen as above.

**Proposition 6.2.** There is a constant $r = r(K) \in [0, 1)$ depending only on $K$ such that every $K$-quasiconformal automorphism $f \in QC(\mathbb{D})$ fixing 1, $i$ and $-1$ satisfies $|f(0)| \leq r$.

**Proof.** The distortion theorem for cross-ratio due to Teichmüller (see [1] Section III.D]) implies that for any four points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$, the hyperbolic distance between the cross-ratios $[z_1, z_2, z_3, z_4]$ and $[f(z_1), f(z_2), f(z_3), f(z_4)]$ in $\mathbb{C} - \{0, 1\}$ is bounded by $\log K$. Take $z_1 = 0$ and $z_2 = \infty$. If we choose two distinct points from $\{1, i, -1\}$ for $z_3$ and $z_4$, we see that $f(0)$ cannot be close to $\mathbb{S}$ except some neighborhoods of $z_3$ and $z_4$ within a distance depending only on $K$. By considering all such choices from $\{1, i, -1\}$, we obtain the assertion. \qed
The full converse of Theorem 5.7 should be a statement that if the complex dilatation \( \mu_f \) of \( f \in AC(\mathbb{D}) \) is in \( Bel_0^\alpha(\mathbb{D}) \) then the boundary extension \( g \) of \( f \) belongs to \( Diff^{1+\alpha}(S) \) for the same \( \alpha \). We will improve the weaker consequence \( g \in Diff^{1+\alpha/2}(S) \) in Lemma 6.1 into this result. The other consequence of the lemma itself is also necessary for a certain estimate of the \( C^{1+\alpha} \)-constant.

We need some distortion estimates of quasiconformal automorphisms of \( \mathbb{D} \), which are variants of the Mori theorem. The first one is its direct consequence.

**Proposition 6.3.** Let \( f \) be a \( K \)-quasiconformal automorphism of \( \mathbb{D} \) with \( f(0) = 0 \). Then
\[
\frac{1}{16}(1 - |z|)K \leq 1 - |f(z)| \leq 16(1 - |z|)^{1/K}
\]
is satisfied for every \( z \in \mathbb{D} \).

**Proof.** The Mori theorem (see [1, Section III.C] and [28, Theorem II.3.2]) says that
\[
|f(w) - f(z)| \leq 16|w - z|^{1/K}
\]
for any \( w \) and \( z \) in \( \overline{\mathbb{D}} \). We choose \( w = z/|z| \in S \) for every \( z \in \mathbb{D} \). Then \( 1 - |f(z)| \leq |f(w) - f(z)| \) yields the inequality. Considering \( f^{-1} \), we obtain the other inequality. \( \square \)

We can remove the power \( 1/K \) in Proposition 6.3 if the complex dilatation is in our class \( Bel_0^\alpha(\mathbb{D}) \). The following result verifies this, which will be crucial in our arguments.

**Theorem 6.4.** Let \( f^\mu \) be a normalized \( K \)-quasiconformal automorphism of \( \mathbb{D} \) with \( \mu \in Bel_0^\alpha(\mathbb{D}) \) and \( \|\mu\|_{\infty,\alpha} \leq \ell \). Then there is a constant \( A = A(\alpha, K, \ell) \geq 1 \) depending only on \( \alpha, K \) and \( \ell \) such that
\[
\frac{1}{A}(1 - |z|) \leq 1 - |f^\mu(z)| \leq A(1 - |z|)
\]
for every \( z \in \mathbb{D} \).

**Proof.** For the moment, we prove the inequality for \( f \in AC(\mathbb{D}) \) with \( f(0) = 0 \) whose complex dilatation \( \mu \) satisfies the same assumption as in the statement. Set
\[
t_0 = \min\{(2\ell)^{-2/\alpha}, 1/4\}.
\]
It is easy to show the inequality for \( z \in \mathbb{D} \) with \( 1 - |z| \geq t_0 \). Indeed, using Proposition 6.3 we have
\[
\frac{iK}{16}(1 - |z|) \leq \frac{iK}{16} \leq \frac{1}{16}(1 - |z|)^K \leq 1 - |f(z)| \leq 1 \leq t_0^{-1}(1 - |z|).
\]
Thus we may assume that \( 1 - |z| < t_0 \) hereafter.

Set \( t = 1 - |z| < t_0 \) for a given point \( z \in \mathbb{D} \). Define a Beltrami coefficient \( \mu_t(\zeta) \) by setting \( \mu_t(\zeta) = \mu(\zeta) \) on \( \{\zeta \in \mathbb{D} \mid |\zeta| \leq 1 - \sqrt{t}\} \) and \( \mu_t(\zeta) \equiv 0 \) elsewhere. Let \( f_t \) be the quasiconformal automorphism of \( \mathbb{D} \) with the complex dilatation \( \mu_t \) and with \( f_t(0) = 0 \). Let \( h_t \) be the quasiconformal automorphism of \( \mathbb{D} \) such that \( f = h_t \circ f_t \). Since
annulus \{ |z| < 1, \rho |z| > 1 \} for every \( w \in \mathbb{D} \), we see that \( |\mu_h(w)| \leq \ell t^{\alpha/2} < 1/2 \) for \( w \in \mathbb{D} \), which implies that the maximal dilatation \( K_t \) of \( h_t \) satisfies

\[
\frac{1}{K_t} \geq \frac{1 - \ell t^{\alpha/2}}{1 + \ell t^{\alpha/2}} \geq 1 - 2\ell t^{\alpha/2}; \quad K_t \leq \frac{1 + \ell t^{\alpha/2}}{1 - \ell t^{\alpha/2}} \leq 1 + 4\ell t^{\alpha/2}.
\]

First we apply a distortion theorem to the conformal homeomorphism \( f_t(\zeta) \) restricted to \( |\zeta| > 1 - \sqrt{t} \). Actually, we may assume that \( f_t \) is a conformal homeomorphism of an annulus \( \{ 1 - \sqrt{t} < |\zeta| < 1/(1 - \sqrt{t}) \} \) by the reflection principle. On the other hand, \( f_t \) is also an \( K \)-quasiconformal automorphism of \( \mathbb{D} \) whose complex dilatation satisfies \( \|\mu_{f_t}\|_{\infty,\alpha} \leq \ell \) independent of \( t \). Then we see from Lemma [3.3] that there is a constant \( D = D(\alpha, K, \ell) \geq 1 \) independent of \( t \) such that the derivative \( f_t' \) satisfies \( D^{-1} \leq |f_t'(\zeta)| \leq D \) for every \( \zeta \in \mathbb{S} \).

Now the Koebe distortion theorem (Proposition [3.3]) in the disk \( \Delta(\xi, \sqrt{t}) \) of radius \( \sqrt{t} \) and center \( \xi = z/|z| \) yields a upper estimate

\[
1 - |f_t(z)| \leq |f_t(z) - f_t(\xi)| \leq (|f_t'(\xi)|\sqrt{t}) \frac{t/\sqrt{t}}{(1 - t/\sqrt{t})^2} \leq 4Dt
\]

by using \( t < 1/4 \). A lower estimated is more complicated. Proposition [3.3] shows that

\[
|f_t'(z)| \geq |f_t'(\xi)| \frac{1 - t/\sqrt{t}}{(1 + t/\sqrt{t})^3} \geq \frac{4}{27D}
\]

with \( t < 1/4 \). Consider the reflection \( z^* \) of \( z \) with respect to \( \mathbb{S} \). The Koebe distortion theorem applied after sending \( z \) to \( \xi \) by a conformal automorphism of the disk \( \Delta(\xi, \sqrt{t}) \) (see [36 Corollary 1.1.3]) gives that

\[
|f_t(z) - f_t(z^*)| \geq (1 - (t/\sqrt{t})^2)(|f_t'(z)|\sqrt{t}) \cdot \frac{2(t/\sqrt{t})}{4(1 - (t/\sqrt{t})^2)} \geq \frac{2t}{27D}.
\]

Since \( f_t(z^*) \) is the reflection of \( f_t(z) \) with respect to \( \mathbb{S} \), \( 1 - |f_t(z)| \) is nearly a half of \( |f_t(z) - f_t(z^*)| \) if it is small, for example, \( 1 - |f_t(z)| \geq 9|f_t(z) - f_t(z^*)|/20 \) if \( 1 - |f_t(z)| \leq 2/11 \). This in particular shows that

\[
1 - |f_t(z)| \geq \frac{9}{20} \cdot \frac{2t}{27D} = \frac{t}{30D}.
\]

Next we apply Proposition [5.3] to the quasiconformal automorphism \( h_t \) of \( \mathbb{D} \). It implies that

\[
1 - |h_t(w)| \leq 16(1 - |w|)^{1/K_t} \leq 16(1 - |w|)^{1 - 2\ell t^{\alpha/2}};
\]

\[
1 - |h_t(w)| \geq \frac{1}{16}(1 - |w|)^{K_t} \geq \frac{1}{16}(1 - |w|)^{1 + 4\ell t^{\alpha/2}}
\]

for every \( w \in \mathbb{D} \). Then by setting \( w = f_t(z) \) we have

\[
\frac{1}{16}\{t/(30D)\}^{1 + 4\ell t^{\alpha/2}} \leq 1 - |f(z)| \leq 16\{4Dt\}^{1 - 2\ell t^{\alpha/2}}.
\]
Dividing this inequality by $t = 1 - |z|$ and taking the logarithm, we obtain
\[
- \log(480D) + 4t^{\alpha/2} \log(t/(30D)) \leq \log \left( \frac{1 - |f(z)|}{1 - |z|} \right) \leq \log(64D) - 2t^{\alpha/2} \log(4Dt).
\]
This is bounded from above and below and hence $(1 - |f(z)|)/(1 - |z|)$ is also bounded from above and away from 0. Thus we can find a constant $A' = A'(\alpha, K, \ell) \geq 1$ such that
\[
\frac{1}{A'} (1 - |z|) \leq 1 - |f(z)| \leq A'(1 - |z|)
\]
for the case of $1 - |z| < t_0$ as well as for the previous case $1 - |z| \geq t_0$.

Now we consider the normalized quasiconformal automorphism $f^\mu \in QC(\mathbb{D})$. Proposition 6.2 says that there is a constant $\mu$ such that $|f^\mu(0)| \leq r$. Take a Möbius transformation $\phi \in \text{Möb}(\mathbb{D})$ such that $\phi \circ f^\mu(0) = 0$. Then $f = \phi \circ f^\mu$ satisfies the desired inequality. On the other hand, $|f^\mu(0)| \leq r$ implies that
\[
\frac{1 - r}{1 + r} \leq |\phi'(z)| \leq \frac{1 + r}{1 - r}.
\]
Since
\[
\{ \min_{z \in \mathbb{D}} |\phi'(z)| \} (1 - |f(z)|) \leq 1 - |f^\mu(z)| \leq \{ \max_{z \in \mathbb{D}} |\phi'(z)| \} (1 - |f(z)|),
\]
we can choose $A = A'(1 + r)/(1 - r)$ for the required inequality, which depends only on $\alpha, K$ and $\ell$.

We have several consequences from this lemma.

**Proposition 6.5.** For any $\mu$ and $\nu$ in $\text{Bel}_0^\alpha(\mathbb{D})$, the composition $\mu \ast \nu^{-1}$ also belongs to $\text{Bel}_0^\alpha(\mathbb{D})$. Hence $\text{Bel}_0^\alpha(\mathbb{D})$ is a subgroup of $\text{Bel}(\mathbb{D})$.

**Proof.** We apply Theorem 6.4 to $\zeta = f^\nu(z)$ in the formula
\[
\mu \ast \nu^{-1}(\zeta) = \frac{\mu(z) - \nu(z)}{1 - \nu(z)\mu(z)} \cdot \frac{\partial f^\nu(z)}{\partial \nu(z)}.
\]
Then $\rho^\alpha_\nu(\zeta) \leq (2A)^\alpha \rho^\alpha_\nu(z)$, from which we have
\[
\|\mu \ast \nu^{-1}\|_{\infty, \alpha} \leq \frac{(2A)^\alpha}{1 - \|\mu\|_{\infty} \|\nu\|_{\infty}} \|\mu - \nu\|_{\infty, \alpha}.
\]
The statement follows from this inequality.

**Corollary 6.6.** If $\nu \in \text{Bel}_0^\alpha(\mathbb{D})$ then $\nu^{-1} \in \text{Bel}_0^\alpha(\mathbb{D})$. More precisely, every $\nu \in \text{Bel}_0^\alpha(\mathbb{D})$ with $\|\nu\|_{\infty, \alpha} \leq \ell$ and $\|\nu\|_{\infty} \leq k < 1$ satisfies $\|\nu^{-1}\|_{\infty, \alpha} \leq \tilde{A}\|\nu\|_{\infty, \alpha}$ for a constant $\tilde{A} = \tilde{A}(\alpha, k, \ell) \geq 1$.

**Proof.** As a special case of the above inequality by setting $\mu = 0$, we have
\[
\|\nu^{-1}\|_{\infty, \alpha} \leq (2A)^\alpha \|\nu\|_{\infty, \alpha}.
\]
Then setting $\tilde{A} = (2A)^\alpha$ gives the statement since $A$ is depending only on $\alpha, k$ and $\ell$ by Theorem 6.4.
Now we state our main result of this section including the solution to the problem on the converse of Theorem 5.7. We use the following notation hereafter.

**Definition.** For a bounded holomorphic function \( \varphi \in B(D^*) \), we define a new norm by
\[
\| \varphi \|_{\infty, \alpha} = \sup_{z \in D^*} \rho_{D^*}^{2+\alpha}(z) |\varphi(z)|.
\]
The Banach space of bounded holomorphic functions with respect to this norm is given by
\[
B^\alpha_0(D^*) = \{ \varphi \in B(D^*) \mid \| \varphi \|_{\infty, \alpha} < \infty \} \subset B_0(D^*).
\]

**Theorem 6.7.** Let \( \alpha \) be a constant with \( 0 < \alpha < 1 \). For a quasisymmetric automorphism \( g \in QS \), the following conditions are equivalent:

1. \( g \) belongs to \( \text{Diff}^{1+\alpha}(S) \);
2. there is \( \mu \in \text{Bel}_0^\alpha(D) \) such that \( \pi(\mu) = [g] \in T \);
3. \( \beta([g]) \in \beta(T) \) is in \( B^\alpha_0(D^*) \).

**Proof.** The implication \( (1) \Rightarrow (2) \) is a reformulation of the statement of Theorem 5.7. This is essentially due to Carleson [15]. The equivalence \( (2) \iff (3) \) has been reviewed in Theorem 4.6, where contributions to this equivalence are gathered. Note that \( (1) \Rightarrow (3) \) was also proved in Tam and Wan [39] by using harmonic extension of diffeomorphisms of \( S \). On the other hand, the converse \( (2) \Rightarrow (1) \) was given in Dyn’kin [18] by his results on pseudoanalytic extension of differentiable functions, and independently in Anderson, Cantón and Fernández [6] relying on a certain approximation theorem of quasiconformal maps on the disk by polynomials. Theorem 6.9 below proves \( (2) \Rightarrow (1) \) in complex analytic methods and provides necessary results for our theorems on the Teichmüller space. \( \square \)

For the proof of Theorem 6.9 and later purpose, we prepare the following proposition. Actually, we will use this for both \( \mu \in \text{Bel}_0^\alpha(D) \) and its reflection \( \mu^* \). According to the difference of assumptions which we will impose on them, we claim both cases separately.

**Proposition 6.8.** (1) Let \( f \) be a conformal homeomorphism of \( D^* \) with \( f(\infty) = \infty \) and \( \lim_{z \to \infty} f'(z) = 1 \) whose quasiconformal extension to \( D \) has the complex dilatation \( \mu \) in \( \text{Bel}_0^\alpha(D) \) with \( \| \mu \|_{\infty, \alpha} \leq \ell \). Then there is a constant \( B = B(\alpha, \ell) \geq 1 \) such that
\[
\frac{1}{B} \leq |f'(z)| \leq B
\]
for every \( z \in D^* \).

(2) Let \( f \) be a conformal homeomorphism of \( D \) with \( e^{-s} \leq |f'(0)| \leq e^s \) whose quasiconformal extension to \( D^* \) has the complex dilatation \( \mu^* \) for \( \mu \in \text{Bel}_0^\alpha(D) \) with \( \| \mu \|_{\infty, \alpha} \leq \ell \). Then there is a constant \( B' = B'(\alpha, \ell, s) \geq 1 \) such that
\[
\frac{1}{B'} \leq |f'(z)| \leq B'
\]
for every \( z \in D \).
Hence, by taking $B = \exp((0, 1))$, the integral is bounded by

$$\int_{\xi}^{(1+t)\xi} \left| \frac{d}{dz} \log f'(z) \right| dz \leq L \int_0^t \frac{2t^{\alpha-1}}{t+2} dt.$$  

This integral is bounded by $L t^{\alpha}/\alpha$, which implies that $\log f'$ extends continuously to $S$ (see [37, Theorem 4.1]). Moreover, by taking the limit as $t \to \infty$, we have

$$|\log f'(z)| \leq \frac{L}{\alpha} + 2L \int_1^\infty t^{\alpha-2} dt = \frac{L}{\alpha} + \frac{2L}{1-\alpha}$$

for every $\xi \in S$. Then the maximal principle yields $|\log f'(z)| \leq 2L/((1-\alpha))$ for $z \in \mathbb{D}^\ast$. Hence, by taking $B = \exp((2L/(1-\alpha)))$, we obtain the assertion.

(2) By Corollary 4.7, there is a constant $L' = L'(\alpha, \ell) \geq 0$ such that $\bar{\beta}_{\mu^*}(t) \leq L't^\alpha$. Since

$$\left| \frac{f''(z)}{f'(z)} \right| = \frac{\bar{\beta}_{\mu^*}(t)}{t}$$

for $t = 1 - |z|$, the integration on the radial segment connecting $(1-t)\xi$ and $\xi$ for any $\xi \in S$ gives

$$\int_{(1-t)\xi}^{\xi} \left| \frac{d}{dz} \log f'(z) \right| dz \leq L' \int_0^t t^{\alpha-1} dt = \frac{L't^\alpha}{\alpha}.$$  

Similarly to the above, $\log f'$ extends continuously to $S$. By taking $t = 1$, we have

$$|\log f'(\xi) - \log f'(0)| \leq \frac{L'}{\alpha}$$

for every $\xi \in S$. Then the maximal principle yields $|\log f'(z) - \log f'(0)| \leq L'/\alpha$ for $z \in \mathbb{D}$. Since $-s \leq \log |f'(0)| \leq s$, we have $|\log |f'(z)|| \leq L'/\alpha + s$, and hence by taking $B = \exp(L'/\alpha + s)$, we obtain the assertion. \[\square\]

Theorem 6.9. If $\mu \in \text{Bel}^0_0(\mathbb{D})$ then $g \in QS$ with $\pi(\mu) = [g]$ belongs to $\text{Diff}^{1+\alpha}(S)$. Moreover, if $g$ is normalized ($g \in QS_*$), then $p_{1+\alpha}(g)$ tends to 0 uniformly as $\|\mu\|_{\infty, \alpha} \to 0$.

Proof. We represent $g \in QS$ by conformal welding. We may assume that $g$ is normalized because being in $\text{Diff}^{1+\alpha}(S)$ is preserved by post-composition of an element of $\text{Möb}(S)$. Then $g$ extends to the normalized quasiconformal automorphism $f^\mu$ of $\mathbb{D}$ whose complex dilatation is $\mu$. The quasiconformal automorphism of $\hat{\mathbb{C}}$ extended by the reflection of $f^\mu$ with respect to $S$ is also denoted by $f^\mu$. Let $f_\mu$ be a normalized quasiconformal automorphism of $\hat{\mathbb{C}}$ whose complex dilatation is $\mu$ on $\mathbb{D}$ and 0 on $\mathbb{D}^\ast$, which satisfies $f_\mu(\infty) = \infty$ and $\lim_{z \to \infty} f_\mu'(z) = 1$. Define the quasiconformal automorphism $f_\mu \circ (f^\mu)^{-1}$
of \( \widehat{C} \) by \( f \), which is conformal on \( D \) with \( f(D) = f_{\mu}(D) \) and whose complex dilatation on \( D^* \) is \( (\mu^*)^{-1} \), the inverse of the reflection of \( \mu \). Note that \( (\mu^*)^{-1} = (\mu^{-1})^* \), where \( \mu^{-1} \) belongs to \( \text{Bel}^{1}_{1}(D) \) and \( ||\mu^{-1}||_{\infty,\alpha} \) can be estimated in terms of \( ||\mu||_{\infty,\alpha} \) by Corollary 6.6.

We have \( g = f^{-1} \circ f_{\mu} \) on \( S \).

We will estimate the modulus of continuity of the derivative of \( g : S \to S \) at \( e^{2\pi i} z \in S \) in terms of \( \beta_{\mu}(t) \) and \( \beta_{(\mu^*)^{-1}}(t) \). This is based on the argument by Anderson, Becker and Lesley [5]. By Theorem 4.1, we see that \( \beta_{\mu}(t) \leq L \) for some constant \( L \geq 0 \) tending to 0 uniformly as \( ||\mu||_{\infty,\alpha} \to 0 \). By Corollary 4.7, we also have \( \beta_{(\mu^*)^{-1}}(t) \leq L' \) for a constant \( L' \geq 0 \) with the same property as \( L \); if \( ||\mu||_{\infty,\alpha} \to 0 \), then \( ||\mu^{-1}||_{\infty,\alpha} \to 0 \) and hence \( L' \to 0 \) uniformly.

Now we consider the derivative of the lift \( \tilde{g} : R \to R \) at \( x \in R \) represented by

\[
\tilde{g}'(x) = \lim_{s \to 0} \left| \frac{g(e^{2\pi i(x+s)}) - g(e^{2\pi i}))}{e^{2\pi i(x+s)} - e^{2\pi i}} \right| = |g'(e^{2\pi i})|,
\]

where \( g'(e^{2\pi i}) \) is the directional derivative along the tangent of \( S \) at \( e^{2\pi i} z \). We see that \( g \) is continuously differentiable and

\[
g'(e^{2\pi i}) = (f_{\mu})'(e^{2\pi i})/g'(e^{2\pi i})).
\]

Indeed, as in the proof of Proposition 6.8 if \( ||\mu||_{\infty,\alpha} < \infty \), then \( (f_{\mu})'(z) \) \( (z \in \mathbb{D}^*) \) has a non-vanishing continuous extension to \( S = \partial \mathbb{D}^* \). This is also true for \( f'(z) \) \( (z \in \mathbb{D}) \).

Since \( g \) is normalized, Lemma 6.1 asserts that \( \tilde{g}'(x) \leq D \) for a constant \( D \geq 1 \) uniformly bounded when \( ||\mu||_{\infty,\alpha} \to 0 \).

The modulus of continuity of \( \tilde{g}'(x) \) is defined by

\[
I(t; \tilde{g}') = \sup_{|x-y| \leq t} |\tilde{g}'(x) - \tilde{g}'(y)|
\]

for \( t \in (0, 1/2] \). Then we have

\[
c_{\alpha}(g) = \sup_{0 < t \leq 1/2} \frac{I(t; \tilde{g}')}{{t^{\alpha}}}.
\]

The mean value theorem gives \( |\tilde{g}(x) - \tilde{g}(y)| \leq D|x - y| \), and if \( \tilde{g}'(x) \geq \tilde{g}'(y) > 0 \), then

\[
|\tilde{g}'(x) - \tilde{g}'(y)| \leq D \left| 1 - \frac{\tilde{g}'(y)}{\tilde{g}'(x)} \right| \leq D \left| \log \frac{\tilde{g}'(y)}{\tilde{g}'(x)} \right|.
\]

This yields \( I(t; \tilde{g}') \leq D I(t; \log \tilde{g}') \). The case where \( \tilde{g}'(y) \geq \tilde{g}'(x) > 0 \) deduces the same estimate. Moreover,

\[
I(t; \log \tilde{g}') \leq I(t; \log |f_{\mu})'(e^{2\pi i} \cdot)|) + I(t; \log |f'(g(e^{2\pi i} \cdot))|)
\]

\[
\leq I(t; \log |(f_{\mu})'(e^{2\pi i} \cdot))|) + I(t; \log |f'(e^{2\pi i} \cdot))|).
\]

Here, we note that \( \log |f'_{\mu})'(z)| = T_{f_{\mu})}'(z) \) and \( \log |f'(z)| = T_{f_{\mu})}(z) \). Taking a path of integration including the circular arc \( \gamma \) connecting \( e^{2\pi i} (1 + t) \) and \( e^{2\pi i y} (1 + t) \) in \( \mathbb{D}^* \).
Lemma 6.12. For every $e^{2\pi i x}, e^{2\pi i y} \in \mathbb{S}$ with $|x - y| \leq t$, we have
\[
|\log |(f_\mu)'(e^{2\pi i x})| - \log |(f_\mu)'(e^{2\pi i y})|| \leq |\log (f_\mu)'(e^{2\pi i x}) - \log (f_\mu)'(e^{2\pi i y})|
\]
\[
\leq \int_{e^{2\pi i x}}^{e^{2\pi i (1+t)}} |T_{f_\mu}dz| + \int |T_{f_\mu}dz| + \int_{e^{2\pi i y}}^{e^{2\pi i y(1+t)}} |T_{f_\mu}dz|
\]
\[
\leq 2 \int_0^t \beta_\mu(t) \cdot dt + 2\pi(1 + t)\beta_\mu(t).
\]
This implies that, if $\beta_\mu(t) \leq Lt^\alpha$, then
\[
I(t; \log |(f_\mu)'(e^{2\pi i x})|) \leq (2/\alpha + 3\pi)Lt^\alpha
\]
for $t \in (0, 1/2]$. The same holds for $f'$ and we have
\[
I(Dt; \log |f'(e^{2\pi i x})|) \leq (2/\alpha + 3\pi)L'D^\alpha t^\alpha.
\]
Hence $I(t; \tilde{\gamma}' = O(t^\alpha)$, which means that $g$ belongs to $\text{Diff}_{1+\alpha}^1(\mathbb{S})$ by definition.

Under the normalization $g \in \text{QS}_s$, we have seen that $D$ is uniformly bounded as $\|\mu\|_{\infty, \alpha} \to 0$. Since $L, L' \to 0$ as $\|\mu\|_{\infty, \alpha} \to 0$, this shows that $I(t; \tilde{\gamma}' / t^\alpha)$ tends to 0, which means that $c_\alpha(g) \to 0$. We also have $g \to \text{id}$ and $\tilde{\gamma}' \to 1$ by Proposition 5.5. Thus $p_{1+\alpha}(g) \to 0$ uniformly as $\|\mu\|_{\infty, \alpha} \to 0$.

Condition (2) of Theorem 6.7 says that there exists some Beltrami coefficient $\mu \in \text{Bel}_{1}^0(\mathbb{D})$ whose Teichmüller projection $\pi(\mu)$ coincides with $[g]$ for a given $g \in \text{Diff}_{1+\alpha}^1(\mathbb{S})$. Alternatively, this means that $g \in \text{Diff}_{1+\alpha}^1(\mathbb{S})$ has some quasiconformal extension to $\mathbb{D}$ whose complex dilatation belongs to $\text{Bel}_{1}^0(\mathbb{D})$. We will show here that the Douady-Earle extension actually gives such an extension provided that Theorem 6.7 is known.

**Theorem 6.10.** For every $g \in \text{Diff}_{1+\alpha}^1(\mathbb{S})$, the image $s_{DE}(\mu)$ under the conformally natural section belongs to $\text{Bel}_{1+\alpha}^0(\mathbb{D})$.

Let $\sigma : \text{Bel}(\mathbb{D}) \to \text{Bel}(\mathbb{D})$ be defined by the correspondence of $\mu$ to $s_{DE}(\pi(\mu))$ for the conformally natural section $s_{DE}$. We call this the *conformally natural transformation* on $\text{Bel}(\mathbb{D})$. A crucial result on this transformation is the following result, which was proved by Cui [16, Theorem 1].

**Lemma 6.11.** Let $\bar{\mu} = (\sigma(\mu^{-1}))^{-1}$ for any $\mu \in \text{Bel}(\mathbb{D})$. Then
\[
|\bar{\mu}(w)|^2 \leq C(1 - |w|^2)^2 \int_{\mathbb{D}} \frac{|\mu(z)|^2}{|1 - \bar{w}z|^4}dxdy
\]
for every $w \in \mathbb{D}$, where $C_1 = C_1(k) > 0$ is a constant depending only on $k$ with $\|\mu\|_{\infty} \leq k$.

We also need the following claim, which can be found in Zhu [22, Lemma 3.10].

**Lemma 6.12.** If $\mu \in \text{Bel}_{1}^0(\mathbb{D}) (0 < \alpha < 1)$, then
\[
\int_{\mathbb{D}} \frac{|\mu(z)|^2}{|1 - \bar{w}z|^4}dxdy \leq C_2(1 - |w|^2)^{2\alpha - 2}
\]
for every \( w \in \mathbb{D} \), where \( C_2 = C_2(\tilde{k}) > 0 \) is a constant depending only on \( \tilde{k} \) with \( \|\mu\|_{\infty,\alpha} \leq \tilde{k} \).

Proof of Theorem 6.10. For \( g \in \text{Diff}^{1+\alpha}(\mathbb{S}) \), we choose \( \nu \in \text{Bel}^\alpha_0(\mathbb{D}) \) such that \( \pi(\nu) = [g] \) by Theorem 6.7. Then \( \nu^{-1} \) also belongs to \( \text{Bel}^\alpha_0(\mathbb{D}) \) by Corollary 6.6. For \( \mu = \nu^{-1} \), we apply Lemmata 6.11 and 6.12 to obtain that \( \tilde{\mu} = (\sigma(\mu^{-1}))^{-1} \) belongs to \( \text{Bel}^\alpha_0(\mathbb{D}) \). Again by Corollary 6.6 this shows that \( \sigma(\nu) = \sigma(\mu^{-1}) \in \text{Bel}^\alpha_0(\mathbb{D}) \). Since \( \sigma(\nu) = s_{DE}(\pi(\nu)) = [g] \), we have the assertion. □

We can also show that the restriction of the conformally natural transformation \( \sigma \) to \( \text{Bel}^\alpha_0(\mathbb{D}) \) is continuous with respect to the topology induced by the norm \( \| \cdot \|_{\infty,\alpha} \). To see this, we use the relation between the norm \( \| \cdot \|_{\infty,\alpha} \) and the right uniform topology on \( \text{Diff}^{1+\alpha}(\mathbb{S}) \), which will be shown in Theorem 7.9 in the next section. The detailed proof appears in [32].

7. The Teichmüller space of circle diffeomorphisms

Now we are ready to realize the Teichmüller space of circle diffeomorphisms with Hölder continuous derivatives as a subspace of the universal Teichmüller space. Then we will give some application of using the structure of this space at the end of this section.

Definition. For a constant \( \alpha \) with \( 0 < \alpha < 1 \), the Teichmüller space of circle diffeomorphisms with Hölder continuous derivatives is defined by

\[
T^\alpha_0 = \text{Möb}(\mathbb{S}) \setminus \text{Diff}^{1+\alpha}(\mathbb{S})
\]

Theorem 6.7 implies that the Teichmüller projection \( \pi : \text{Bel}(\mathbb{D}) \to T \) gives

\[
\pi(\text{Bel}^\alpha_0(\mathbb{D})) = T^\alpha_0,
\]

and the Bers embedding \( \beta : T \to B(\mathbb{D}^*) \) gives

\[
\beta(T^\alpha_0) = \beta(T) \cap B^\alpha_0(\mathbb{D}^*),
\]

which coincides with \( \Phi(\text{Bel}^\alpha_0(\mathbb{D})) \) for the Bers projection \( \Phi : \text{Bel}(\mathbb{D}) \to B(\mathbb{D}^*) \). Here, we see that \( \beta(T) \cap B^\alpha_0(\mathbb{D}^*) \) is an open subset of the Banach space \( B^\alpha_0(\mathbb{D}^*) \). Indeed, this follows from the fact that \( \beta(T) \) is open in \( B(\mathbb{D}^*) \) and the norm inequality \( \|\varphi\|_\infty \leq \|\varphi\|_{\infty,\alpha} \) for \( \varphi \in B^\alpha_0(\mathbb{D}^*) \).

We restrict \( \pi, \Phi \) and \( \beta \) to the spaces as above and consider continuity and openness of these maps. We provided \( T^\alpha_0 \) with the quotient topology from \( \text{Bel}^\alpha_0(\mathbb{D}) \) by \( \pi \), which is so defined that \( \pi \) is continuous. Then, from the facts listed in the proof below, we are able to prove the following.

Theorem 7.1. The Bers embedding \( \beta : T^\alpha_0 \to \beta(T) \cap B^\alpha_0(\mathbb{D}^*) \) is a homeomorphism. Hence \( T^\alpha_0 \) is equipped with the complex structure modeled on the complex Banach space \( B^\alpha_0(\mathbb{D}^*) \).

Proof. For the proof of this theorem, it suffices to show the following claims:

1. \( \pi : \text{Bel}^\alpha_0(\mathbb{D}) \to T^\alpha_0 \) is open;
Lemma 7.3. \( \Phi : \text{Bel}_0^\alpha (\mathbb{D}) \rightarrow \beta(T) \cap B_0^\alpha (\mathbb{D}^*) \) is continuous;
(3) \( \Phi : \text{Bel}_0^\alpha (\mathbb{D}) \rightarrow \beta(T) \cap B_0^\alpha (\mathbb{D}^*) \) has a continuous local section.
These are proved in Lemma 7.3, Lemma 7.4 and Lemma 7.6 below, respectively. \( \square \)

We begin with showing a basic fact on the group \( \text{Bel}_0^\alpha (\mathbb{D}) \).

**Proposition 7.2.** The right translation \( r_\nu : \text{Bel}_0^\alpha (\mathbb{D}) \rightarrow \text{Bel}_0^\alpha (\mathbb{D}) \) for \( \nu \in \text{Bel}_0^\alpha (\mathbb{D}) \) defined by \( \mu \mapsto \mu * \nu^{-1} \) is a homeomorphism with respect to \( \| \cdot \|_{\infty,\alpha} \).

**Proof.** We have the following inequality for \( \zeta = f^\nu(z) \):
\[
|r_\nu(\mu_1)(\zeta) - r_\nu(\mu_2)(\zeta)| = |\mu_1 * \nu^{-1}(\zeta) - \mu_2 * \nu^{-1}(\zeta)|
\leq \frac{|\mu_1(z) - \nu(z)|}{1 - \nu(z)|\mu_1(z)|} \frac{|\mu_2(z) - \nu(z)|}{1 - \nu(z)|\mu_2(z)|}
= \frac{|\mu_1(z) - \mu_2(z)|}{|1 - \nu(z)|} \frac{|\mu_1(z)|}{|1 - \nu(z)|} \frac{|\mu_2(z)|}{|1 - \nu(z)|}.
\]
Applying Theorem 6.4 to \( f^\nu \), we have \( \rho_0^\alpha (\zeta) \leq (2A)\rho_0^\alpha (z) \) for some \( A > 0 \). Hence
\[
\|r_\nu(\mu_1) - r_\nu(\mu_2)\|_{\infty,\alpha} \leq \frac{(2A)^\alpha}{(1 - \|\mu_1\|_{\infty}^2)(1 - \|\mu_2\|_{\infty}^2)\|\mu_1 - \mu_2\|_{\infty,\alpha}}.
\]
This shows that \( r_\nu \) is continuous. Since \( (r_\nu)^{-1} = r_{\nu^{-1}} \), we have the assertion. \( \square \)

To see that \( \pi \) is open, we follow the same argument as in the original case (Proposition 2.1).

**Lemma 7.3.** The Teichmüller projection \( \pi : \text{Bel}_0^\alpha (\mathbb{D}) \rightarrow T_0^\alpha \) is an open map.

**Proof.** Take an open subset \( U \subset \text{Bel}_0^\alpha (\mathbb{D}) \). To see that \( \pi(U) \) is open, we consider
\[
\pi^{-1}(\pi(U)) = \bigcup_{\nu \in \text{Ker } \pi \cap \text{Bel}_0^\alpha (\mathbb{D})} r_\nu(U).
\]
Here \( r_\nu(U) \) is open by Proposition 7.2. Then \( \pi^{-1}(\pi(U)) \) is also open in \( \text{Bel}_0^\alpha (\mathbb{D}) \) and hence \( \pi(U) \) is open. \( \square \)

Note that the right translation \( r_\nu \) for \( \nu \in \text{Bel}_0^\alpha (\mathbb{D}) \) projects down to the base point change \( R_{\pi(\nu)} : T_0^\alpha \rightarrow T_0^\alpha \) and then the openness of \( \pi \) guarantees that \( R_{\pi(\nu)} \) is homeomorphic. This will be discussed later again.

The continuity of \( \Phi : \text{Bel}_0^\alpha (\mathbb{D}) \rightarrow \beta(T) \cap B_0^\alpha (\mathbb{D}^*) \) can be proved as a special case of the following assertion. In contrast to the original case (Proposition 2.2), we need to introduce here a certain representation of Schwarzian derivatives by Beltrami coefficients and gather their estimates we have obtained, which will be a crucial step for proving Theorem 7.1.
Lemma 7.4. Let \( \nu \in \text{Bel}_{0}^{\nu}(\mathbb{D}) \) possibly with \( \alpha' \neq \alpha \). Then every \( \mu \in \text{Bel}(\mathbb{D}) \) satisfies
\[
\| \Phi(\mu) - \Phi(\nu) \|_{\infty, \alpha} \leq C \| \mu - \nu \|_{\infty, \alpha},
\]
where \( C = C(\nu, \alpha, k) > 0 \) is a constant depending only on \( \nu, \alpha \) and \( k \) with \( \| \mu \|_{\infty} \leq k \).

The dependence on \( \nu \) is actually given by \( \alpha' \), \( \| \nu \|_{\infty} \) and \( \| \nu \|_{\infty, \alpha'} \). The right side term is assumed to be \( \infty \) when \( \mu - \nu \notin \text{Bel}_{0}^{\alpha}(\mathbb{D}) \).

For the proof, we use the following integral representation of Schwarzian derivatives by Yanagishita [11, Lemma 3.1, Proposition 3.2], which is obtained by generalizing the arguments in Astala and Zinsmeister [8].

Proposition 7.5. For Beltrami coefficients \( \mu \) and \( \nu \) in \( \text{Bel}(\mathbb{D}) \), let \( f_{\mu} \) and \( f_{\nu} \) be the quasiconformal automorphisms of \( \mathbb{C} \) that are conformal on \( \mathbb{D}^{*} \). Set \( \Omega = f_{\nu}(\mathbb{D}) \) and \( \Omega^{*} = f_{\nu}(\mathbb{D}^{*}) \). Then
\[
\left| S_{f_{\mu} \circ f_{\nu}^{-1}}(\zeta) \right| \leq \frac{6\rho_{\Omega^{*}}(\zeta)}{\sqrt{\pi}} \left( \int_{\Omega} \frac{|\mu(f_{\nu}^{-1}(w)) - \nu(f_{\nu}^{-1}(w))|^{2}}{(1 - |\mu(f_{\nu}^{-1}(w))|^{2})(1 - |\nu(f_{\nu}^{-1}(w))|^{2}) |w - \zeta|^{2}} \, dw \right)^{1/2}
\]
holds for every \( \zeta \in \Omega^{*} \).

To consider the norm \( \| \Phi(\mu) \|_{\infty, \alpha} \) of the Schwarzian derivative \( S_{f_{\mu} \circ f_{\nu}^{-1}} \), we need an estimate of the derivative of the conformal homeomorphism \( f_{\mu} \) of \( \mathbb{D}^{*} \) defined by \( \mu \in \text{Bel}_{0}^{\alpha}(\mathbb{D}) \). We use Proposition 6.8 for this purpose.

Proof of Lemma 7.4. By the definition of the norm,
\[
|\mu(z) - \nu(z)| \leq \rho_{\mathbb{D}}^{-\alpha}(z) \| \mu - \nu \|_{\infty, \alpha}
\]
for every \( z \in \mathbb{D} \). By Theorem 6.4 there is a constant \( a = a(\alpha, \nu) \geq 1 \) such that \( \rho_{\mathbb{D}}^{-\alpha}(z) \leq a \rho_{\mathbb{D}}^{-\alpha}(f_{\nu}(z)) \).

We set \( f = f_{\nu} \circ (f_{\nu}^{-1}) \). This is a conformal homeomorphism of \( \mathbb{D} \) extending to a quasiconformal automorphism of \( \mathbb{C} \) whose complex dilatation on \( \mathbb{D}^{*} \) coincides with \( (\nu)^{-1} \). We can choose \( f_{\nu} \) so that \( f(0) = 0 \) keeping the normalization \( f_{\nu}(\infty) = \infty \) and \( \lim_{z \to \infty} f'_{\nu}(z) = 1 \). Note that \( f(\mathbb{D}) = f_{\nu}(\mathbb{D}) \).

By the normalization of \( f_{\nu} \) appealing to the Schwarz lemma and the Koebe one-quarter theorem (Proposition 6.4) on \( \mathbb{D}^{*} \), we see that \( f_{\nu}(\mathbb{D}) \) is not strictly contained in \( \mathbb{D} \) but contained in the disk \( \{ |z| < 4 \} \). Hence, there is some \( x_{1} \in \mathbb{S} \) such that \( 1 \leq |f_{\nu}(x_{1})| \leq 4 \).

On the other hand, Proposition 6.2 says that there is some \( r \in [0, 1) \) depending only on \( \| \nu^{-1} \|_{\infty} = \| \nu \|_{\infty} \) such that \( |(f_{\nu}^{-1})'(0)| \leq r \). Take \( z \in \mathbb{D} \) with \( |z| = (1 + r)/2 \) arbitrarily and consider the cross-ratio \( [(f_{\nu})^{-1}(0), x_{1}, \infty, z] \). By the distortion theorem for cross-ratio due to Teichmüller (see [11, Section III.D]), the hyperbolic distance on \( \mathbb{C} - \{0, 1\} \) between \( [(f_{\nu})^{-1}(0), x_{1}, \infty, z] \) and
\[
[f_{\nu}((f_{\nu})^{-1}(0)), f_{\nu}(x_{1}), f_{\nu}(\infty), f_{\nu}(z)] = [0, f_{\nu}(x_{1}), \infty, f_{\nu}(z)]
\]
is bounded by \( \log K \), where \( K = (1 + \| \nu \|_{\infty})/(1 - \| \nu \|_{\infty}) \). This implies that there is a constant \( \rho = \rho(|\nu|_{\infty}) > 0 \) such that \( |f_{\nu}(z)| \geq \rho \) for \( |z| = (1 + r)/2 \) and hence \( f(\mathbb{D}) = f_{\nu}(\mathbb{D}) \) contains the disk of center at 0 and radius \( \rho \).
By the Schwarz lemma applied to the conformal homeomorphism \( f \) of \( \mathbb{D} \), we see that there is a constant \( s = s(\rho) > 0 \) depending only on \( \rho \) and hence on \( \nu \) such that \( e^{-s} \leq |f'(0)| \leq 4 \). It follows from Proposition 6.8 that there is a constant \( B = B(\nu) > 0 \) such that \( |f'(z)| \geq 1/B \) for every \( z \in \mathbb{D} \). Hence there is a constant \( b = b(\nu, \alpha) \geq 1 \) such that \( \rho^{-\alpha}_\mathbb{D}(f^\nu(z)) \leq b\rho^{-\alpha}_\mathbb{D}(f^\nu(z)) \).

For \( w = f^\nu(z) \in \Omega \), the above inequalities yield
\[
|\mu(f^\nu_1(w)) - \nu(f^\nu_1(w))| \leq ab\rho^{-\alpha}_\mathbb{D}(w)\|\mu - \nu\|_{\infty, \alpha}.
\]

By substituting this inequality to the integral in Proposition 7.5, we will estimate
\[
\left( \int_{\Omega} \frac{\rho^{-2\alpha}_\Omega(w)}{|w - \zeta|^4} \, dudv \right)^{1/2}.
\]

We can find a similar estimate in Nag [34, Section 3.4] and we follow this. Let \( \eta_\Omega(w) \) be the euclidean distance from \( w \in \Omega \) to \( \partial \Omega \) and \( \eta_\Omega^*(\zeta) \) the euclidean distance from \( \zeta \in \Omega^* \) to \( \partial \Omega \). As a consequence from the Koebe one-quarter theorem (Proposition 3.3), we see that both \( \rho_\Omega(w)\eta_\Omega(w) \) and \( \rho_\Omega^*(\zeta)\eta_\Omega^*(\zeta) \) are bounded below by \( 1/2 \). We have
\[
|\mu(f^\nu_1(w)) - \nu(f^\nu_1(w))| \leq 4\rho^{-\alpha}_\mathbb{D}(w) \leq 4|w - \zeta|^{2\alpha}
\]
for every \( w \in \Omega \) and every \( \zeta \in \Omega^* \). Hence the integral can be estimated as
\[
\int_{\Omega} \frac{\rho^{-2\alpha}_\Omega(w)}{|w - \zeta|^4} \, dudv \leq 4 \int_{\Omega} \frac{dudv}{|w - \zeta|^{4-2\alpha}} \leq 4 \int_{|w - \zeta| \geq \eta_\Omega^*(\zeta)} \frac{dudv}{|w - \zeta|^{4-2\alpha}} \leq 8\pi \frac{1}{2 - 2\alpha} \eta_\Omega^*(\zeta)^{2-2\alpha} \leq 16\pi \frac{1}{1 - \alpha} \rho_\Omega^*(\zeta)^{2-2\alpha}.
\]

Plugging this estimate in the inequality of Proposition 7.5, we have
\[
\rho^{-2\alpha}_\Gamma^\nu(\zeta)|S_{f^\nu f^\nu_1}|_{\Gamma^\nu}^* (\zeta) \leq \frac{24ab\|\mu - \nu\|_{\infty, \alpha}}{\sqrt{(1 - \alpha)(1 - \|\mu\|_\infty^2)(1 - \|\nu\|_\infty^2)}} \rho^{-\alpha}_{\Gamma^\nu}(\zeta).
\]

For \( \zeta = f^\nu(z) \) with \( z \in \mathbb{D}^* \), the left side term equals to
\[
\rho^{-2\alpha}_\Gamma^\nu(z)|S_{f^\nu f^\nu_1}|_{\Gamma^\nu}^* (z) - |S_{f^\nu}|_{\Gamma^\nu}^* (z)|.
\]

For the right side term, we apply Proposition 6.8 again to the quasiconformal automorphism \( f^\nu \) of \( \hat{\mathbb{C}} \) which is conformal on \( \mathbb{D}^* \). Then there is a constant \( b' = b'(\nu, \alpha) \geq 1 \) such that \( \rho^{-\alpha}_\Gamma^\nu(f^\nu(z)) \leq b'\rho^{-\alpha}_\Gamma^\nu(z) \). Therefore the above inequality turns out to be
\[
\rho^{-2\alpha}_\Gamma^\nu(z)|S_{f^\nu}|_{\Gamma^\nu}^* (z) - |S_{f^\nu}|_{\Gamma^\nu}^* (z)| \leq \frac{24ab\|\mu - \nu\|_{\infty, \alpha}}{\sqrt{(1 - \alpha)(1 - \|\mu\|_\infty^2)(1 - \|\nu\|_\infty^2)}} \rho^{-\alpha}_\Gamma^\nu(z).
\]
This implies that
\[ \|\Phi(\mu) - \Phi(\nu)\|_{\infty,a} \leq \frac{24abc}{\sqrt{(1 - \alpha)(1 - \|\mu\|_{\infty})(1 - \|\nu\|_{\infty})}} \|\mu - \nu\|_{\infty,a}. \]

We can choose the multiplier of the right side term as the constant \(C\).

The existence of a local continuous section for \(\Phi : \text{Bel}_0^\alpha(\mathbb{D}) \to \beta(T) \cap B_0^\alpha(\mathbb{D}^*)\) is verified by using a local continuous section for the original Bers projection \(\Phi\) defined by quasiconformal reflection due to Ahlfors [2]. This was improved later by Earle and Nag [21].

**Lemma 7.6.** The Teichmüller projection \(\Phi : \text{Bel}_0^\alpha(\mathbb{D}) \to \beta(T) \cap B_0^\alpha(\mathbb{D}^*)\) has a continuous local section.

**Proof.** For each \(f \in \beta(T) \cap B_0^\alpha(\mathbb{D}^*)\), take \(\nu \in \text{Bel}_0^6(\mathbb{D})\) such that \(\Phi(\nu) = S_{f_\nu}|_{\mathbb{D}} = \psi\) and that \(f_{\nu}|_{\mathbb{D}}\) is a diffeomorphism. The quasiconformal reflection \(\lambda : f_{\nu}(\mathbb{D}) \to f_{\nu}(\mathbb{D}^*)\) with respect to the quasicircle \(f_{\nu}(S)\) is defined by \(\lambda(\zeta) = f_{\nu}(f_{\nu}^{-1}(\zeta)^*)\), where \(z^*\) denotes the reflection of \(z\) with respect to \(S\).

We follow the arguments in [27 Section II.4.2] and [23 Section 14.4]. We have a constant \(\varepsilon = \varepsilon(k) > 0\) depending only on \(k\) with \(\|\nu\|_{\infty} \leq k\) such that if \(\varphi \in B(\mathbb{D}^*)\) satisfies \(\|\varphi\|_{\infty} < \varepsilon\) then there is a quasiconformal automorphism \(f^*\) of \(\hat{\mathbb{C}}\) conformal on \(f_{\nu}(\mathbb{D}^*)\) such that \(S_{\hat{f}^*|_{\mathbb{D}^*}} = \psi + \varphi\) (see also [27 Theorem III.4.2]). In this case, the Beltrami coefficient \(\mu_{\hat{f}}\) of \(\hat{f}\) is given by
\[
\mu_{\hat{f}}(\zeta) = \frac{S_{\hat{f}}(\lambda(\zeta))(\zeta - \lambda(\zeta))^2\bar{\partial}\lambda(\zeta)}{2 + S_{\hat{f}}(\lambda(\zeta))(\zeta - \lambda(\zeta))^2\bar{\partial}\lambda(\zeta)}
\]

for \(\zeta = f_{\nu}(z) \in f_{\nu}(\mathbb{D})\). Here
\[
|((\zeta - \lambda(\zeta))^2\bar{\partial}\lambda(\zeta))| \leq c\rho_{f_{\nu}(\mathbb{D}^*)}^{-2}(\lambda(\zeta)); \quad |((\zeta - \lambda(\zeta))^2\bar{\partial}\lambda(\zeta))| \leq c\rho_{f_{\nu}(\mathbb{D}^*)}^{-2}(\lambda(\zeta))
\]
for some constant \(c = c(k) > 0\). Then, by replacing the \(\varepsilon > 0\) so that \(\varepsilon \leq 1/c\), we have
\[
|\mu_{\hat{f}}(f_{\nu}(z))| \leq c|\varphi(z^*)||f_{\nu}'(z^*)|^{-2}\rho_{f_{\nu}(\mathbb{D}^*)}^{-2}(f_{\nu}(z^*))
\]
\[
= \frac{1}{\varepsilon} |\varphi(z^*)|\rho_{\mathbb{D}^*}^{-2}(z^*) < 1
\]
for \(z \in \mathbb{D}\).

Now we take \(\varphi \in B_0^0(\mathbb{D}^*)\) such that \(\|\varphi\|_{\infty} \leq \|\varphi\|_{\infty,a} < \varepsilon\). Then we can apply the above argument and obtain
\[
|\mu_{\hat{f}}(f_{\nu}(z))| \leq \frac{1}{\varepsilon} |\varphi(z^*)|\rho_{\mathbb{D}^*}^{-2}(z^*) \leq \frac{\|\varphi\|_{\infty,a}}{\varepsilon} \rho_{\mathbb{D}^*}^{-2}(z^*) \leq |z|^{-2\alpha} \rho_{\mathbb{D}^*}^{-\alpha}(z).
\]
This in particular shows that \(\mu_{\hat{f}} \circ f_{\nu} \in \text{Bel}_0^\alpha(\mathbb{D})\).
Denoting the complex dilatation of $\hat{f} \circ f_\nu$ by $\mu_\varphi$, we will show that $\mu_\varphi \in \text{Bel}_0^\alpha(\mathbb{D})$. A formula of the complex dilatation of composed quasiconformal maps gives

$$\mu_\varphi(z) = \frac{e^{-2i\theta} \mu_\hat{f}(f_\nu(z)) + \nu(z)}{1 + e^{-2i\theta} \mu_\hat{f}(f_\nu(z))\nu(z)}$$

where $\theta = \arg \partial f_\nu(z)$. Thus we have

$$|\mu_\varphi(z)| \leq \frac{|\mu_\hat{f} \circ f_\nu(z)| + |\nu(z)|}{1 - \|\mu_\hat{f}\|\|\nu\|_\infty}.$$ 

Since both $\mu_\hat{f} \circ f_\nu$ and $\nu$ belong to $\text{Bel}_0^\alpha(\mathbb{D})$, so does $\mu_\varphi$. Since $\Phi(\mu_\varphi) = S_{f_\circ f_\nu|_{\mathbb{D}^*}} = \psi + \varphi$, we have a local section of $\Phi$ at $\psi$ by the correspondence $\varphi \mapsto \mu_\varphi$. By the above formula of $\mu_\hat{f}$ in terms of $\varphi$, we see that this local section is continuous at $\psi$. \hfill \qed

Now we have obtained the continuity of the Bers projection $\Phi$ and its local section restricted to $\text{Bel}_0^\alpha(\mathbb{D})$ and $\beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ with respect to the norm $\| \cdot \|_{\infty, \alpha}$. Note that the local section at $\psi \in \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ can be chosen so that $\psi$ is sent to an arbitrary point in the fiber $\Phi^{-1}(\psi)$ by post-composition of the right translation map $r_\lambda$ for $\lambda \in \ker \pi$.

These maps are given by the same form of the original ones for $\text{Bel}(\mathbb{D})$ and $B(\mathbb{D}^*)$. Moreover, we know that these maps are holomorphic on $\text{Bel}(\mathbb{D})$ and $B(\mathbb{D}^*)$ with respect to the norm $\| \cdot \|_\infty$ (Theorem 2.3). Once we are in this situation, to see that the new maps are actually holomorphic is a matter of general argument. Indeed, $\Phi$ and its local section are holomorphic as mappings to $\mathbb{C}$ if we fix the complex variable $z$ of functions $\varphi(z) \in B_0^\alpha(\mathbb{D}^*)$ and $\mu(z) \in \text{Bel}_0^\alpha(\mathbb{D})$. Then the norm inequality $\| \cdot \|_\infty \leq \| \cdot \|_{\infty, \alpha}$ and the continuity under $\| \cdot \|_{\infty, \alpha}$ justify the claim. See Earle [19] Lemma 3.4 and Lehto [27] Lemma V.5.1.

**Corollary 7.7.** The Bers projection $\Phi : \text{Bel}_0^\alpha(\mathbb{D}) \to \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ is a holomorphic submersion.

Moreover, we have seen in Proposition 7.2 that the right translation and hence the base point change map are homeomorphic. By the same reasoning as above, we also see that they are actually biholomorphic.

**Corollary 7.8.** The right translation $r_\nu : \text{Bel}_0^\alpha(\mathbb{D}) \to \text{Bel}_0^\alpha(\mathbb{D})$ for $\nu \in \text{Bel}_0^\alpha(\mathbb{D})$ and the base point change map $R_\tau : T^\alpha_0 \to T^\alpha_0$ for $\tau \in T^\alpha_0$ are biholomorphic.

**Remark.** The Teichmüller space $T^0_\alpha$ is equipped with the Kobayashi metric as an invariant metric of a complex manifold. By Yanagishita [10] Theorem 1.1 generalizing the result of Hu, Jiang and Wang [26], we see that the Kobayashi distance on $T^0_\alpha$ coincides with the restriction of the Teichmüller distance on $T$ to $T^0_\alpha$, and hence the infinitesimal Kobayashi metric on each tangent space of $T^0_\alpha$ coincides with its restriction of the infinitesimal Teichmüller metric on the tangent space of $T$. 
Finally in this section, we investigate the topology on $T_0^\alpha$, which has been defined to be the quotient topology induced from the norm $\| \cdot \|_{\infty, \alpha}$ on $\text{Bel}_0^\alpha(\mathbb{D})$ by the Teichmüller projection $\pi$. However, since $\text{Diff}^{1+\alpha}(\mathbb{S})$ is equipped with the right uniform topology, we can also introduce another topology on $T_0^\alpha = \text{Möb}(\mathbb{S}) \setminus \text{Diff}^{1+\alpha}(\mathbb{S})$. Namely, it is the relative topology under the identification of $T_0^\alpha$ with the subgroup $\text{Diff}_1^{1+\alpha}(\mathbb{S}) \subset \text{Diff}^{1+\alpha}(\mathbb{S})$ of all normalized elements. We call this the $\text{Diff}^{1+\alpha}$-topology on $T_0^\alpha$. Concerning the relation between these two topologies on $T_0^\alpha$, we have the following.

**Theorem 7.9.** The $\text{Diff}^{1+\alpha}$-topology on $T_0^\alpha$ coincides with the quotient topology induced from $\text{Bel}_0^\alpha(\mathbb{D})$.

**Proof.** Suppose that $[g_n] \to [g]$ in $T_0^\alpha$ for $g_n, g \in \text{Diff}_1^{1+\alpha}(\mathbb{S})$. Then there are $\mu_n$ and $\mu$ in $\text{Bel}_0^\alpha(\mathbb{D})$ with $\pi(\mu_n) = [g_n]$ and $\pi(\mu) = [g]$ such that $\mu_n \to \mu$ with respect to $\| \cdot \|_{\infty, \alpha}$. Since the right translation $r_\mu$ is a homeomorphism of $\text{Bel}_0^\alpha(\mathbb{D})$ by Proposition 7.2, the condition $\mu_n \to \mu$ is equivalent to the condition $r_\mu(\mu_n) = \mu_n \ast \mu^{-1} \to 0$ in $\text{Bel}_0^\alpha(\mathbb{D})$. Then by Theorem 6.9 the normalized representatives $\gamma_n \in \text{Diff}_1^{1+\alpha}(\mathbb{S})$ with $[\gamma_n] = \pi(r_\mu(\mu_n))$ satisfy $p_{1+\alpha}(\gamma_n) \to 0$ as $n \to \infty$. This means that $\gamma_n = g_n \circ g^{-1}$ converge to id in $\text{Diff}_1^{1+\alpha}(\mathbb{S})$. Hence $[g_n]$ converge to $[g]$ in the $\text{Diff}^{1+\alpha}$-topology.

Conversely, suppose that $[g_n] \to [g]$ in the $\text{Diff}^{1+\alpha}$-topology for $g_n, g \in \text{Diff}_1^{1+\alpha}(\mathbb{S})$. Then $\gamma_n = g_n \circ g^{-1}$ converge to id, that is, $p_{1+\alpha}(\gamma_n) \to 0$ as $n \to \infty$. In particular, $c_\alpha(\gamma_n) \to 0$. Then by Theorem 5.7 we have quasiconformal extensions $f_n : \mathbb{D} \to \mathbb{D}$ of $\gamma_n$ whose complex dilatations $\nu_n$ satisfy $\| \nu_n \|_{\infty, \alpha} \to 0$. Hence $[\gamma_n] = [g_n] \ast [g]^{-1} \to [\text{id}]$ in $T_0^\alpha$ and thus $[g_n] \to [g]$ in $T_0^\alpha$ by the continuity of the base point change map $R_{[g]^{-1}}$. 

Combined with Proposition 5.2 this implies the following.

**Corollary 7.10.** $(T_0^\alpha, \ast)$ is a topological group.

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Department of Mathematics, School of Education, Waseda University, Shinjuku, Tokyo 169-8050, Japan

E-mail address: matsuzak@waseda.jp