Non-Fermi Liquid Behaviour of Electrons
in the Half-filled Honeycomb Lattice
(A renormalization group approach)

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Abstract

A system of electrons in the two-dimensional honeycomb lattice with Coulomb interactions is described by a renormalizable quantum field theory similar but not equal to QED$_3$. Renormalization group techniques are used to investigate the infrared behavior of the system that flows to a fixed point with non-Fermi liquid characteristics. There are anomalous dimensions in the fermionic observables, no quasiparticle pole, and anomalous screening of the Coulomb interaction. These results are robust as the Fermi level is not changed by the interaction. The system resembles in the infrared the one-dimensional Luttinger liquid.

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1 Introduction

This paper addresses the problem of interacting electrons in a band of a two-dimensional crystal. This problem uncovers all the macroscopic properties of the materials of which the maximum interest is centered at the present on the high-$T_c$ superconductors, in particular, on the nature of their normal state. To fix ideas and some notation let us review the highlights of the problem. Start with a free theory of, say, electrons in a metal, i. e., a free Fermi gas whose ground state is known and whose low energy elementary excitations are electrons and holes obeying a free dispersion relation. Then switch on an interaction among the electrons and ask the question of what the new ground state is and what is the nature of the low energy excitations of the full interacting theory. This program has been successfully addressed by Landau in three spatial dimensions $d$ where it leads to the famous theory of Landau’s Fermi liquid [1] and the concept of quasiparticles. Things are different in $d = 1$ where an exact solution can be found and the behavior of the system is always of the Luttinger type[2].

The case $d = 2$ is physically very interesting both in practice (most of the high-$T_c$ materials are organized in two dimensional layers and experimental data are available) and under a theoretical point of view. Despite the great effort devoted to it, no exact solution is known at the present and the situation remains controversial[3]. The ultimate origin of the trouble lies on the very complex nature of the many-body vacuum and the corresponding difficulties of doing perturbations around the Fermi surface.

The theory of the Fermi liquid and the perturbations of it that lead to the BCS and charge density wave instabilities have recently been rephrased as effective field theories [4, 5, 6] to which renormalization group techniques can be applied [7]. The main idea of this approach is that, although we are dealing with problems that naturally have an ultraviolet cutoff $\rho_0$ (the width of the band), we are only interested in the physics at a scale much lower than it, namely, a very narrow strip above and below the Fermi surface. By integrating out in the path integral all the momenta above a certain cutoff $\rho \leq \rho_0$, we end up with an effective field theory whose couplings are cutoff dependent. The renormalization group transformations are rescalings of $\rho$. In this way one can visualize the Fermi liquid as a fixed point of this RG transformations (it is gapless) that flows towards a BCS state when a weak attraction is introduced between the quasiparticles.

When comparing the effective field theories that arise in this context with

\footnote{One of the clearest examples of the mentioned difficulties is the “infrared catastrophe” [3] discovered by Anderson that is attracting a major interest in relation with the study of the high $T_c$ superconductivity.}
standard quantum field theory, we encounter two main differences. The first one is the nonrelativistic nature of the many body physics where there is a parameter, the Fermi velocity, whose typical value is much smaller than the speed of light. This prevents the use of the general theorems concerning renormalizability. The more severe difference lies in the above mentioned nature of the many body vacuum. When treating the interacting system as a perturbation of the “free” system constituted by the fixed point, perturbations are to be described around a Fermi surface in $d = 3$, a Fermi line in $d = 2$, and Fermi points in $d = 1$. We are then faced with couplings that are functions of the shape of the Fermi surface which is also modified by the interaction. Only in the $d = 1$ case we deal, as in quantum field theory, with a true vacuum around which one can expand and get a bunch of coupling constants.

In ref. [8], we developed a model for the electronic structure of free electrons in a two-dimensional honeycomb lattice. This lattice constitutes the basic building block of periodic systems made of identical atoms with planar, threefold coordination. That is the case of graphite, built of carbon atoms. The rapidly growing family of fullerene compounds [9, 10] can be thought of as closed, curved surfaces derived from the planar honeycomb lattice. These compounds show a variety of unexpected features, the most striking of them being the superconductivity of doped crystals of C$_{60}$ molecules [11, 12]. Moreover, this superconductivity can be considered a form of high-$T_c$ superconductivity, as the number of electrons available to form Cooper pairs is unusually low.

The main feature that makes the honeycomb lattice the object of our attention in the present context is that its Fermi level at half-filling consists of exactly two Fermi points. This peculiarity opens the possibility of performing a complete analysis of the problems mentioned above in analogy with the $d = 1$ case. In particular it makes of this lattice an ideal laboratory for investigating departures from Fermi liquid behavior in $d = 2$ which have shown very elusive in more conventional approaches. The major interest of this investigation relies on the fact, made clear by now from the experiments, that the normal state of high-$T_c$ materials is not of Fermi liquid type. Besides, it can be directly applied to the physics of the fullerenes that, as mentioned above, are two-dimensional graphitic structures at half-filling [13].

The paper is organized as follows. In section 2 we review results on the free electrons on the honeycomb lattice that lead to the low energy effective action used as a starting point of our analysis. We emphasize the point that, unlike what usually occurs in many body physics, our effective action constitutes a genuine quantum field theory with a single, well-defined, vacuum state. Next the free theory is completed with the addition of the Coulomb interaction. In
the field theory, this is mediated by an electromagnetic field whose propagator reproduces, in the nonrelativistic limit, the usual four-fermion interaction considered in the many-body approach. We end up with a standard local quantum field theory. A discussion is done on the differences to be expected by the use of the relativistic versus the instantaneous electromagnetic field propagator illustrated by the computation of the density of states in both cases. The results announce an infrared instability which is clarified in section 4 with the help of the renormalization group.

Section 3 is devoted to the study of the renormalizability of the model. It is shown that, despite its nonrelativistic nature and due to its gauge invariance, it is renormalizable. The Fermi velocity and the electron wave function get renormalized while the electric charge does not.

Sections 4 and 5 contain the analysis of the RG flow of the model and its physical implications. The first section is devoted to the nonrelativistic limit of the theory. We see that the perturbative expansion has an effective coupling given by the ratio of the square of the electric charge and the Fermi velocity. The RG shows that this coupling decreases to zero in the infrared although there is no fixed point in the nonrelativistic regime. We then investigate the possibility of having a natural length scale in the theory and make some comments on the system away from half-filling. Finally we find an anomalous dimension for the electron wave function at two-loop order that points towards a non-Fermi liquid behavior of the system.

The previous analysis is completed in section 5 where we study the relativistic regime which is the relevant regime in the infrared. We find a fixed point of the RG flow obtained when the Fermi velocity equals the speed of light. Next we compare our model with standard RG analysis of the Fermi liquid theory and find significant differences both from the technical and from the physical point of view. The most significant one is the anomalous dimension found for the electron Green function. We discuss the non renormalization of the electric charge and establish the non-Fermi liquid nature of our fixed point.

We set our main conclusions and a summary of the more relevant points of the article in section 6.

2 Many-body theory of the 2D layer

In this section we first review the one-particle electronic states in the two-dimensional honeycomb lattice, which serve afterwards as a starting point for the many-body problem. The present one-particle description has been worked out with complete detail in ref. [8], regarding the conduction band of
a two-dimensional graphite layer. The honeycomb lattice has two different atoms per primitive cell, so that two degrees of freedom arise in a variational computation of the energy eigenstates, corresponding to the respective orbitals for the two atoms. In the tight-binding approximation the problem reduces to the diagonalization of the one-particle hamiltonian

\[ H = -t \sum_{\langle i,j \rangle} a_i^+ a_j \]  

where the sum is over pairs of nearest neighbors atoms \( i, j \) on the lattice and \( a_i, a_j^+ \) are canonically anticommuting operators

\[ \{ a_i, a_j \} = \{ a_i^+, a_j^+ \} = 0 \quad \{ a_i, a_j^+ \} = \delta_{ij} \]  

The states which are simultaneously eigenvectors of the hamiltonian and of the lattice generators have to be of the form

\[ \Psi = \sum_{i} c_i e^{i \mathbf{k} \cdot \mathbf{r}_i} a_i^+ |O\rangle + \sum_{\circ} c_\circ e^{i \mathbf{k} \cdot \mathbf{r}_i} a_i^+ |O\rangle \]  

with coefficients \( c_\bullet \) and \( c_\circ \) for black and blank points, respectively, as depicted in figure 1. Obviously, under the action of (1) black points are mapped into blank points and viceversa. We have, indeed,

\[ H \Psi = -t \sum_{i} c_\circ e^{i \mathbf{k} \cdot \mathbf{r}_j} a_i^+ |O\rangle - t \sum_{\circ} c_\bullet e^{i \mathbf{k} \cdot \mathbf{r}_j} a_i^+ |O\rangle \]

\[ = -t \sum_{j} e^{i \mathbf{k} \cdot \mathbf{u}_j} \sum_{i} c_i e^{i \mathbf{k} \cdot \mathbf{r}_i} a_i^+ |O\rangle - t \sum_{\circ} e^{i \mathbf{k} \cdot \mathbf{v}_j} \sum_{\circ} c_\circ e^{i \mathbf{k} \cdot \mathbf{r}_i} a_i^+ |O\rangle \]

where \( \{ \mathbf{u}_j \} \) is a triad of vectors connecting a \( \bullet \) atom with its nearest neighbors, and \( \{ \mathbf{v}_j \} \) the triad made of their respective opposites (see figure 1). Therefore, the state (3) is an eigenvector of \( H \) provided that the coefficients \( c_\bullet \) and \( c_\circ \) are solutions of the eigenvalue problem

\[ \begin{pmatrix} 0 & -t \sum_j e^{i \mathbf{k} \cdot \mathbf{u}_j} \\ -t \sum_j e^{i \mathbf{k} \cdot \mathbf{v}_j} & 0 \end{pmatrix} \begin{pmatrix} c_\bullet \\ c_\circ \end{pmatrix} = E(k) \begin{pmatrix} c_\bullet \\ c_\circ \end{pmatrix} \]  

The parameter \( t \) measures the hopping between nearest neighbors and provides the natural energy scale in the one-particle description. The diagonalization of (5) gives the band of levels

\[ E(k) = \pm t \sqrt{1 + 4 \cos^2 \frac{\sqrt{3}}{2} ak_x + 4 \cos \frac{\sqrt{3}}{2} ak_x \cos \frac{3}{2} ak_y} \]
the parameter $a$ being the lattice spacing. We will be interested in the consideration of this band at half-filling, which is the pertinent instance for the carbon-based materials and, in particular, graphite layers and undoped fullerenes. The dispersion relation (5) has the amazing property that the Fermi level at $E(\mathbf{k}) = 0$ is only reached by six isolated points in momentum space (see figure 2). These are the six corners of the first Brillouin zone and, due to the periodicity of the dual lattice, they correspond to only two independent states. The existence of a finite number of Fermi points is quite unusual in more than one spatial dimension. It has important consequences since it makes possible to encode the low-lying excitations about the Fermi level into a simple field theory. When considering an arbitrarily large sample, it is appropriate to amplify the region about any of the two independent Fermi points by taking the limit $a \rightarrow 0$. In either case we obtain for the operator $\mathcal{H}$ in equation (5)

$$\mathcal{H} = \frac{3}{2} ta \mathbf{\sigma} \cdot \delta \mathbf{k} + O \left( (a \delta k)^2 \right)$$

(7)

where $\delta \mathbf{k}$ is the displacement about the Fermi point and $\sigma_1, \sigma_2$ stand for the two Pauli matrices. The conclusion is that, in the long-wavelength limit $a \rightarrow 0$, the one-particle (energy) eigenstates of the electrons in the honeycomb lattice are given by the field theory of two massless Dirac spinors (one for each independent Fermi point) in two spatial dimensions. This is all about the free theory.

We now turn to introduce the electronic interaction by applying the method of second quantization. We remark that, proceeding in this way with the above set of one-particle states, we end up with a formalism which falls more into the framework of quantum field theory than into that of quantum statistical theory. The difference is subtle but significant. In quantum statistical theory the ground state of the system is, in general, a condensate with an extended Fermi surface (in more than one spatial dimension) and the elementary excitations (quasiparticles) are only stable right at the Fermi level. The second quantized theory built from the dispersion relation (5) has, however, a definite vacuum in the sense of quantum field theory. Its excitations are similar to the particle-hole excitations of the Dirac sea. We may consider separately, for instance, each of the two Dirac spinors and work out the many-body electron propagator of the free theory

$$G^{(0)}(t, r; t', r') = -i \langle T \Psi_I(t, r) \overline{\Psi}_I(t', r') \rangle$$

(8)

We denote by $\Psi_I$ the electron operator $\Psi$ in the interaction representation while $\overline{\Psi}_I = \Psi^\dagger_I \sigma_3$. In terms of the respective modes $\psi_{k,+}, \psi_{k,-}$ for unoccupied

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4Similar dispersion relations can be found in higher dimensions. The best known of them is the one derived from the 3D diamond structure with one state per site.\(^4\)
and occupied levels we have

\[ \Psi_I(t, r) = \sum_k e^{ik \cdot r} e^{-i\omega |k| t} \psi_{k,+} + \sum_k e^{ik \cdot r} e^{i\omega |k| t} \psi_{k,-} \tag{9} \]

where \( v = 3ta/2 \) is the Fermi velocity and the operators \( a_{k,\pm} \) satisfy anticommutation relations

\[ \{a_{k,\pm}, a_{k',\pm}\} = \delta(k - k') \tag{10} \]

Taking the statistical average with the system at half-filling, we get for the Fourier transform of the propagator

\[ G^{(0)}(\omega, k) = \frac{i(-\gamma_0 + v \gamma \cdot k)}{-\omega^2 + v^2 k^2 - i\epsilon} \tag{11} \]

\{\gamma_0, \gamma\} is a standard set of \( \gamma\)-matrices satisfying

\[ \{\gamma_0, \gamma_0\} = -2 \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij} \quad \{\gamma_0, \gamma_i\} = 0 \tag{12} \]

and related to the above quoted Pauli matrices by \( \sigma_3 = -i\gamma_0, \sigma_1 = i\sigma_3\gamma_1, \sigma_2 = i\sigma_3\gamma_2 \). The above expression coincides precisely with the Feynman propagator for a massless spinor in \( 2 + 1 \) dimensions, which confirms our assertion that the system at half-filling is equivalent to the Dirac sea in quantum field theory. We must notice however the presence of the Fermi velocity \( v \) in front of the spatial part of the scalar product, which is a feature of the many-body theory.

In the nonrelativistic approximation, the introduction of the electronic interaction leads, in first instance, to the second quantized hamiltonian

\[ H_{\text{Coulomb}} = \frac{3}{2}ta \int d^2r \overline{\Psi(r)} \gamma \cdot \nabla \Psi(r) \]

\[ + \frac{e^2}{2} \int d^2r_1 \int d^2r_2 \frac{\overline{\Psi(r_1)} \sigma_3 \Psi(r_1) \overline{\Psi(r_2)} \sigma_3 \Psi(r_2)}{4\pi |r_1 - r_2|} \tag{13} \]

We attach to this hamiltonian the name of “Coulomb” since it describes the instantaneous interaction between electric charges. The model given in terms of the Coulomb interaction is, however, a highly nonlocal field theory, what makes very awkward its investigation from a formal point of view. Both for the sake of studying the properties of the quantum theory as well as for computational purposes, a complete description of the interaction with the electromagnetic field \( A_\mu \) is more desirable. One of the points that we support

\[ ^5 \text{From now on we replace } \delta k \text{ by the wavevector } k \text{ with origin at the Fermi point.} \]
in what follows is that, actually, the model of interacting electrons is sensitive to retardation effects of the electromagnetic propagation. The reason for such unconventional behaviour is the massless character of the spinor $\Psi$. This property is not accidental in the description of the two-dimensional layer, as long as the expansion (7) gives only one marginal operator dictating the long-distance behaviour of the free theory.

We propose, therefore, a quantum field theory description based on the second quantized Hamiltonian

$$H = \frac{3}{2} \tau a \int d^2r \bar{\Psi}(r) \gamma \cdot \nabla \Psi(r) - e \int d^2r \, j_\mu A^\mu$$

(14)

The interaction of the electromagnetic field $A_\mu$ and the electrons in the layer is described in the standard fashion, by coupling to the conserved current

$$j_\mu \sim \left( i \bar{\Psi} \gamma_0 \Psi, i \bar{\Psi} \gamma^\nu \Psi \right)$$

(15)

This poses some technical problems since the electromagnetic field propagates in three-dimensional space while we want the dynamics of the electrons to be confined to the two-dimensional layer. Although this may not be achieved in general, it turns out to be possible by specializing to the Feynman gauge$[15]$, which enforces the constraint

$$\nabla_\mu A^\mu = 0$$

(16)

and has also the property of placing the $A_\mu$ field in the same direction as the electronic current. In this gauge we have the propagator

$$\langle TA_\mu(t, \mathbf{r})A_\nu(t', \mathbf{r}') \rangle = -i \delta_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}}{-\omega^2 + k^2 - i\epsilon} e^{-i\omega(t-t')}$$

(17)

and the coupling to the $2 + 1$ dimensional current is perfectly consistent. In the computation of electronic properties we have just to take care of placing the points $\mathbf{r}, \mathbf{r}'$ on the plane of the two-dimensional layer. With this description we certainly get more than we would need in a nonrelativistic theory, but we can recover at any time the nonrelativistic limit of all quantities by expanding in powers of $v/c$.

It is important to stress that we have to expect, in general, different answers in the computation of a given observable taking (13) or (14) as starting point. The dimensionality of the system is low enough so that the massless condition of the spinor reflects in the appearance of infrared divergences in perturbation theory. Thus, although we should not expect any discrepancy

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6Unless otherwise stated, we work henceforth in units $\hbar = c = 1$. 

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in the computation of local quantities (like cutoff dependent quantities) in
the nonrelativistic model (13) and in the model given by (14) in the limit
\(v/c \to 0\), the response of nonlocal quantities to the interaction turns out to
be quite different in the infrared regime. From a technical point of view, per-
turbation theory loses predictive power because of the mentioned infrared
instabilities, and one has to resort to more sophisticated methods in order to
obtain information from the quantum theory.

Let us illustrate the above fact with the computation of the quantum
corrections to the density of states \(n(\omega)\) near the Fermi level \(\omega = 0\). This
observable is given by

\[
\frac{d^2k Tr \left[ G(\omega, \mathbf{k}) \sigma_3 \right]}{2(2\pi)^2} \int \frac{d^2k}{k^2 + k_z^2 - i\epsilon}
\]

In the theory with the instantaneous Coulomb interaction, the density of
states is not modified to first order in perturbation theory. The instantaneous
interaction can be obtained from (17), for instance, by considering formally
its expression for \(c \to \infty\). In this limit we have, computing always at points
\(\mathbf{r}, \mathbf{r}'\) in the two-dimensional layer

\[
\langle TA_{\mu}(t, \mathbf{r})A_{\nu}(t', \mathbf{r}') \rangle \approx -i\delta_{\mu\nu} \int \frac{d^2k d\omega}{(2\pi)^3} \int \frac{dk_z}{2\pi} E^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} e^{-i\omega(t-t')}
\]

Then, it becomes obvious that this kind of interaction cannot modify the
\(\omega\) dependence of the propagator \(G\), to the first perturbative order (see also
the discussion at the end of section 4). However, as mentioned above, the
propagator (14) cannot be safely used in the computation of nonlocal quan-
tities. This remark is appropriate to the case of the density of states, whose
determination in the quantum field theory given by (14) goes as follows.

The inverse propagator may be decomposed, as usual, into a free part
and the electron self-energy \(\Sigma(\omega, \mathbf{k})\)

\[
\frac{1}{G} = \frac{1}{G^{(0)}} - \Sigma
\]

\(\Sigma(\omega, \mathbf{k})\) has a perturbative expansion in terms of one-particle-irreducible
diagrams[1], which are in general ultraviolet divergent. Since our model can
be treated as a genuine quantum field theory we expect these divergences
to be local and susceptible of being absorbed into \(1/G^{(0)}\). For the moment
we are only interested in finite corrections and will leave the issue of renor-
malization to the next section. To first order in perturbation theory, the
relevant diagram is shown in figure 3 and gives (in the limit $v/c \to 0$) the finite contribution to $\Sigma$

$$\Sigma_R(\omega, k) = i \frac{e^2}{16 \pi^2} v \gamma \cdot k \int_0^1 dx \frac{\sqrt{1-x}}{(1 - x + v^2 x)^2} \log \left\{ v^2 k^2 x - \omega^2 x (1 - x) \right\} + O(e^4)$$

(21)

When inserted in equation (18) the above expression gives the first quantum corrections to the density of levels

$$n(\omega) = \frac{2 \pi^2 |\omega|}{v^2} \left( 1 + \frac{1}{32 \pi} v \log \omega^2 + \frac{1}{8 \pi^2} v^2 \int_0^1 dx \frac{\sqrt{1-x}}{(1 - x + v^2 x)^2} \log x \right.$$\n
$$+ \frac{1}{16 \pi^2} v^2 \int_0^1 dx \frac{\sqrt{1-x}}{(1 - x + v^2 x)^2} \frac{1}{x} + O(e^4) \right)$$

(22)

It is already appreciated that the perturbative expansion of $n(\omega)$ is not well-defined, since the last integral in (22) is clearly divergent. Even if we could manage to regularize in some way this infrared divergence, we still would be left with a more serious problem. At the point near which we want to measure the density of levels, i.e. near $\omega = 0$, the first perturbative correction dominates over the zeroth order term, and one can presume that higher order terms can produce increasingly large powers of $\log \omega$. Perturbation theory is clearly not reliable in the infrared regime, but the kind of infrared instability described points at new physics near the Fermi level.

3 Renormalization

Henceforth we will pay attention to the local quantum field theory (14), which offers the possibility of analyzing the infrared behaviour by application of renormalization group methods. We study first in this section the renormalization properties of the model. Though the many-body theory of the 2D layer has a natural ultraviolet cutoff, we insist in absorbing ultraviolet divergent contributions into bare parameters of the theory since this is a way of extracting relevant information about the scaling properties in the infrared[16, 17]. As we have mentioned before, the renormalizability of a quantum many-body theory is, in general, a nontrivial issue, since the couplings in the effective theory may depend on the point chosen on the Fermi surface. In our case things become much simpler as our low-energy effective theory is a genuine quantum field theory. Regarding the usual considerations for renormalizability in field theory, the only condition that we are missing is relativistic invariance since the Fermi velocity of the electrons $v = 3ta/2$ does not match the speed of light $c$ that appears in the dispersion relation of
photons. For this reason it is still necessary to check that the ultraviolet divergences of the theory can be absorbed into a redefinition of the scale of the fields and of the parameters in the Hamiltonian. We present here the analysis at the one-loop level, and conclude with a compelling argument showing that the model is renormalizable to all orders in perturbation theory.

The conditions which make the theory renormalizable can be stated more clearly in the Lagrangian formalism. Corresponding to the Hamiltonian (14), we have the action

$$S = \int dtd^2r \overline{\Psi}(-\gamma_0 \partial_0 + v \gamma \cdot \nabla)\Psi - ie \int dtd^2r \overline{\Psi}(-\gamma_0 A_0 + v \gamma \cdot A)\Psi$$  \hspace{1cm} (23)$$

In the quantum theory the coefficients of the terms in the action have to be adjusted so as to cancel the ultraviolet divergences. We should start therefore with the bare action

$$S_{\text{bare}} = Z_{\text{kin}} \int dtd^2r \overline{\Psi}(-\gamma_0 \partial_0 + Z_v v \gamma \cdot \nabla)\Psi - Z_{\text{int}} ie \int dtd^2r \overline{\Psi}(-\gamma_0 A_0 + Z_v v \gamma \cdot A)\Psi$$  \hspace{1cm} (24)$$

The renormalization coefficient $Z_{\text{kin}}$ may be determined from the electron self-energy $\Sigma(\omega, k)$, as well as the renormalization coefficient for the Fermi velocity $Z_v$. The coefficient $Z_{\text{int}}$ is determined from the renormalization of the time component of the interaction vertex. It turns out therefore that the spatial components have necessarily to be renormalized by the product $Z_{\text{int}}Z_v$. This is a nontrivial check of the renormalizability of the theory, which arises in the absence of relativistic invariance. Provided that it is fulfilled, the renormalized theory may be made finite in terms of renormalized quantities $t_R, \epsilon_R, \Psi_R, A^\mu_R$

$$v_{\text{bare}} = Z_v v_R$$

$$e_{\text{bare}} = Z_v e_R$$

$$\Psi_{\text{bare}} = Z^{1/2}_\Psi \Psi_R$$

$$A^\mu_{\text{bare}} = Z^{1/2}_{A^\mu} A^\mu_R$$  \hspace{1cm} (25)$$

In the previous section we already quoted the one-loop renormalized contribution to the electron self-energy $\Sigma(\omega, k)$. We deal here with the ultraviolet divergent part of this object, regularizing it by working in analytical continuation to dimension $d = 3 - \epsilon$[15]. From the diagram in figure 3 we have

$$\frac{1}{G^{(0)}} - \Sigma$$
\begin{align*}
\Gamma_0 &= -Z_0 i(\gamma_0 \omega + Z_e v \gamma \cdot k) - i \frac{e^2}{8\pi^2} \gamma_0 \omega \left(1 - 2v^2\right) \int_0^1 dx \frac{\sqrt{1 - x}}{1 - x + v^2 x} \frac{1}{\varepsilon} \\
&\quad - i \frac{e^2}{8\pi^2} v \gamma \cdot k \int_0^1 dx \frac{\sqrt{1 - x}}{(1 - x + v^2 x)^2} \frac{1}{\varepsilon} + \text{finite terms} + O(e^4)
\end{align*}

In order to ensure the finiteness of the electron propagator we take the renormalization coefficients

\begin{align*}
Z_\Psi &= 1 + \frac{e^2}{8\pi^2} \left(1 - 2v^2\right) \int_0^1 dx \frac{\sqrt{1 - x}}{1 - x + v^2 x} \frac{1}{\varepsilon} + O(e^4) \\
Z_v &= 1 - \frac{e^2}{8\pi^2} \int_0^1 dx \frac{\sqrt{1 - x}}{(1 - x + v^2 x)^2} \frac{1}{\varepsilon} \\
&\quad - \frac{e^2}{8\pi^2} \left(1 - 2v^2\right) \int_0^1 dx \frac{\sqrt{1 - x}}{1 - x + v^2 x} \frac{1}{\varepsilon} + O(e^4)
\end{align*}

These coefficients have a more complicated structure than those of a relativistic theory, as long as \(v\) does not match the speed of light—which, in our units, means that \(v \neq 1\). \(Z_v\) has, for instance, an infinite power series expansion in \(v\)

\begin{align*}
Z_v &= 1 - \frac{1}{16\pi^2} e^2 \left\{ \pi \frac{1}{v} F \left(\frac{1}{2}; 1; \frac{3}{2}; v^2\right) - 2\pi v \left(1 - 2v^2\right) F \left(\frac{3}{2}; \frac{3}{2}; \frac{3}{2}; v^2\right) \\
&\quad + 4 \left(1 - 2v^2\right) F \left(1, 1; \frac{1}{2}; v^2\right) - 4F \left(1, 2; \frac{3}{2}; v^2\right) \right\} \frac{1}{\varepsilon} + O(e^4)
\end{align*}

We will later use this expression in the consideration of the nonrelativistic approximation \(v \to 0\).

Now we come to the renormalization of the interaction term in the action \(\mathcal{L}\). The first quantum corrections to the vertex are given by the diagram in figure 4, and we denote them symbolically by \(\Gamma_\mu\). The divergent contribution to the time component of the vertex is actually different than that to the spatial components. We find that

\begin{align*}
\Gamma_0 &= i \frac{e^3}{16\pi^2} \left(1 - 2v^2\right) \int_0^1 dx \frac{1 - x}{\sqrt{x} (v^2 + (1 - v^2)x)} \\
&\quad - i \frac{e^3}{8\pi^2} v^2 (1 - 2v^2) \int_0^1 dx \frac{1 - x}{\sqrt{x} (v^2 + (1 - v^2)x)^2} \frac{1}{\varepsilon} \\
&\quad + \text{finite terms} + O(e^5) \\
\Gamma &= -i \frac{e^3}{16\pi^2} v \gamma \int_0^1 dx \frac{1 - x}{\sqrt{x} (v^2 + (1 - v^2)x)} \frac{1}{\varepsilon} + \text{finite terms} + O(\varepsilon^3)
\end{align*}

We may think of \(\Gamma_\mu\) as a correction to the bare vertex in \(\mathcal{L}\). We have then

\begin{align*}
Z_{\text{int}} &= 1 - \frac{e^2}{16\pi^2} \left(1 - 2v^2\right) \int_0^1 dx \frac{1 - x}{\sqrt{x} (v^2 + (1 - v^2)x)} \frac{1}{\varepsilon}
\end{align*}
\[ e^2 \frac{v^2}{8\pi^2} (1 - 2v^2) \int_0^1 dx \frac{1 - x}{\sqrt{x} (v^2 + (1 - v^2)x)^2} \frac{1}{\varepsilon} + O(e^4) \quad (32) \]

\[ Z_{\text{int}} Z_v = 1 - \frac{e^2}{16\pi^2} \int_0^1 dx \frac{1 - x}{\sqrt{x} (v^2 + (1 - v^2)x)^2} \frac{1}{\varepsilon} + O(e^4) \quad (33) \]

By taking the quotient between (33) and (32), we get an independent determination of the renormalization coefficient \( Z_v \)

\[ Z_v = 1 - \frac{e^2}{16\pi^2} \int_0^1 dx \frac{1 - x}{\sqrt{x} (v^2 + (1 - v^2)x)^2} \frac{1}{\varepsilon} + \frac{e^2}{16\pi^2} \int_0^1 dx \frac{1 - x}{\sqrt{x} (v^2 + (1 - v^2)x)^2} \left( 1 - 2v^2 - \frac{2v^2(1 - 2v^2)}{v^2 + (1 - v^2)x} \right) \frac{1}{\varepsilon} + O(e^4) \]

\[ = 1 - \frac{e^2}{16\pi^2} \left\{ \pi \frac{1}{v} (1 - 2v^2) F \left( \frac{1}{2}, 1; 1; v^2 \right) - \frac{8}{3} v^2 (1 - 2v^2) F \left( 2, 2; \frac{5}{2}; v^2 \right) + 2\pi v F \left( \frac{1}{2}, 3; 1; v^2 \right) - 8v^2 F \left( 1, 2; 3; v^2 \right) \right\} \frac{1}{\varepsilon} + O(e^4) \quad (34) \]

It is not obvious that this expression coincides with the alternative form (29), though it has to for the theory to be renormalizable. In fact, after using the recursion relations which hold for hypergeometric functions[18] on can show that the two functions in (34) and (29) are identical. We then have the nice situation of dealing with a nonrelativistic quantum field theory in which the Fermi velocity of the electrons is renormalized exactly in the same way in the different terms of the action.

This precise coincidence is not casual. One is led to suspect that it has to be consequence of some symmetry operating in the theory, which turns out to be nothing but gauge invariance. Actually, this should automatically lead in the quantum theory to the same renormalization of the kinetic part and the interaction term in the action, \( Z_{\text{kin}} = Z_{\text{int}} \). It is not difficult to check that this is satisfied in our model, to the perturbative order in which we are working. From expressions (27) and (32) we have

\[ Z_{\text{kin}} = 1 + \frac{e^2}{8\pi^2} (1 - 2v^2) \left\{ 2F \left( 1, 1; \frac{1}{2}; v^2 \right) - \pi v F \left( \frac{3}{2}, \frac{3}{2}; \frac{3}{2}; v^2 \right) \right\} \frac{1}{\varepsilon} + O(e^4) \quad (35) \]

\[ Z_{\text{int}} = 1 - \frac{e^2}{16\pi^2} (1 - 2v^2) \left\{ -\frac{\pi}{v} F \left( \frac{1}{2}, \frac{1}{2}; 1; v^2 \right) - \frac{8}{3} v^2 F \left( 2, 2; \frac{5}{2}; v^2 \right) \right. \]

\[ + \frac{\pi}{v} F \left( \frac{1}{2}, 3; 1; v^2 \right) + 4F \left( 1, 2; \frac{3}{2}; v^2 \right) \right\} \frac{1}{\varepsilon} + O(e^4) \quad (36) \]

which become identical after use again of the relations among hypergeometric functions.
The gauge symmetry

\[ \Psi \rightarrow e^{ie\theta(t, r)} \Psi \]
\[ A_\mu \rightarrow A_\mu + \partial_\mu \theta(t, r) \]  

(37)

is manifest at the classical level. Provided that it holds for the bare action \( (24) \), it ensures that \( Z_{\text{kin}} = Z_{\text{int}} \). It is easily seen, by taking a gauge parameter \( \theta \) independent of \( t \), that it also enforces the same renormalization of the Fermi velocity \( v \) in the different terms of the action. As long as one is able to preserve the gauge invariance in the quantum theory, this turns out to guarantee the renormalizability to all orders of the perturbative expansion.

The rest of the renormalization coefficients in \( (24) \) can be obtained from the following remark. The fact that \( Z_\Psi \) has to equal \( Z_{\text{int}} \) to all orders in perturbation theory implies, as in standard quantum electrodynamics, that \( Z_\varepsilon Z_A^{1/2} = 1 \). The renormalization coefficient of the electromagnetic field \( Z_A^{1/2} \) is determined from the polarization tensor, whose first perturbative contribution shown diagrammatically in figure 5 is a finite quantity

\[ \Pi_{\mu\nu}(k) = \frac{1}{2\pi v^2} (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) |k| \int_0^1 dx \sqrt{x(1-x)} \]  

(38)

We conclude, therefore, that

\[ Z_A^{1/2} = Z_\varepsilon^{-1} = 1 + O(e^4) \]  

(39)

remarking the absence of renormalization of the electric charge to first order in perturbation theory.

4 Scaling of the electronic interaction (I). Non-relativistic regime.

In this section we seek the connection of our model to the real physics of electrons in a two-dimensional layer. A single graphite sheet or any fullerene aggregate of sufficiently large size is the appropriate physical instance to look at. The first question concerns the magnitude of the parameters pertinent to the real layer. A standard value for the hopping parameter in graphite is \( t \approx 2.2eV \). For the nearest neighbor separation in the lattice we can take \( a \approx 0.14 \cdot 10^{-9}m \). From these values and restoring for a moment the dependence on \( c \) and \( \hbar \), the Fermi velocity in our model turns out to be

\[ v = \frac{3ta}{(2\hbar)} \approx 1.5 \cdot 10^{-3}c \]

This is certainly much smaller than the speed of light, for which to extract relevant quantities to the electron physics in the
layer we will have to take the limit \( v/c \to 0 \) and keep the leading order in previous formulas.

For a clearer description of our approach to renormalization group we will quote the renormalization coefficients in this section as functions of the ultraviolet momentum cutoff \( \Lambda \) rather than in terms of the dimensional regularization poles \( 1/\varepsilon \). At the one-loop level this just amounts to replace \( 1/\varepsilon \) by \( \log \Lambda \) in all formulas. The renormalization coefficient of the Fermi velocity \( Z_v \) becomes, for instance, in the nonrelativistic limit

\[
Z_v = 1 - \frac{1}{16\pi} \frac{e^2}{v} \log \Lambda + O \left( \frac{e^2}{v^2} \right) \quad (40)
\]

It is not difficult to see that at each level of the perturbative expansion the leading order in the nonrelativistic approximation enters with a power \((e^2/v)^n\). We define, therefore, a consistent perturbative expansion in the nonrelativistic limit by taking \( v \to 0 \) (or, equivalently, \( c \to \infty \)) and \( e^2/v = \text{const.} \). This latter quantity becomes then the effective strength of the electronic interaction. In this limit we obtain from (35)

\[
Z_\Psi = 1 + O \left( \frac{e^2}{v^2} \right) \quad (41)
\]

so that we do not find wavefunction renormalization at the one-loop level in the nonrelativistic limit. We recall also the former result (39)

\[
Z^{1/2}_A = Z^{-1}_e = 1 + O \left( e^4 \right) \quad (42)
\]

We have a nonrelativistic quantum theory with no renormalization of the electric charge, but in which the effective strength of the interaction \( e^2/(4\pi v) \) is renormalized. We rely on the original interpretation of renormalization adopted by Wilson[19] to determine the scale dependence of the effective interaction. Since we have shown that the model is renormalizable, we may impose the independence of the renormalized theory on the momentum cutoff \( \Lambda \). By means of renormalization group transformations we are able therefore to relate bare theories at different scales (different cutoff) which correspond to the same renormalized theory[18, 17]. The essential information is encoded in the renormalization group equation, which for the electron propagator, for instance, states that under a change in the scale of the cutoff \( \Lambda \)

\[
\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta_v(v, e^2) \frac{\partial}{\partial v} + \beta_e(v, e^2) \frac{\partial}{\partial e} - \gamma(v, e^2) \right) G(\omega, k; \Lambda; v, e^2) = 0 \quad (43)
\]
where

\[ \beta_v(v, e^2) = \Lambda \frac{\partial Z_v}{\partial \Lambda} v_R \]  \hspace{1cm} (44)  
\[ \beta_e(v, e^2) = \Lambda \frac{\partial Z_e}{\partial \Lambda} e_R \]  \hspace{1cm} (45)  
\[ \gamma(v, e^2) = \Lambda \frac{\partial}{\partial \Lambda} \log Z_\Psi \]  \hspace{1cm} (46)  

The well-known solution to this equation takes the form

\[ G(\omega, k, \rho \Lambda; v, e^2) = \exp \left\{ \int_\rho^\rho' \frac{d\rho'}{\rho'} \gamma \right\} G(\omega, k, \Lambda; v_{\text{eff}}(\rho), e_{\text{eff}}^2(\rho)) \]  \hspace{1cm} (47)  

where the effective parameters are given by

\[ \rho \frac{\partial}{\partial \rho} v_{\text{eff}}(\rho) = -\beta_v(v_{\text{eff}}, e_{\text{eff}}^2) \]  
\[ \rho \frac{\partial}{\partial \rho} e_{\text{eff}}^2(\rho) = -\beta_e(v_{\text{eff}}, e_{\text{eff}}^2) \]  \hspace{1cm} (48)  

Increasing the scale \( \rho \) of the cutoff is equivalent to measure the observables of the theory at large distance scale. From the one-loop order results of the previous section we see that \( e_{\text{eff}} \) is constant at this level, while \( v_{\text{eff}} \) is not. We expect, therefore, the effective coupling of the electronic interaction \( e^2/(4\pi v) \) to have a nontrivial renormalization group flow in the infrared regime. Actually, from equations (40) and (44)

\[ \rho \frac{\partial}{\partial \rho} v_{\text{eff}} v_R = \frac{1}{16\pi} \frac{e^2}{v_{\text{eff}}^2} + O \left( \frac{e^4}{v_{\text{eff}}^2} \right) \]  \hspace{1cm} (49)  

so that the asymptotic behaviour of the coupling is

\[ \frac{1}{4\pi} \frac{e^2}{v_{\text{eff}}} (\Lambda/\Lambda_0) = \frac{1}{4\pi} \frac{e^2}{v_R^2} + \frac{1}{8\pi} \frac{e^2}{v_R} \log \frac{\Lambda}{\Lambda_0} \left( \frac{1}{\Lambda_0} \right)^{-1/2} \]  \hspace{1cm} (50)  

We have to bear in mind that, when integrating equation (49), we are relying heavily on perturbation theory. If we recall our original estimate of the Fermi velocity \( v \), we see that the coupling \( e^2/(4\pi v) \) of the nonrelativistic theory is not a small parameter but, in fact, larger than one. Nevertheless, we know that when dealing with a renormalized quantum theory the statement that a coupling like \( e^2/(4\pi v) \) is small or large applies to a particular energy scale. We then have to be very precise in establishing the regime in which the integration of the renormalization group equation becomes consistent. Let us remark that the solution (50) of the renormalization group
equation is equivalent to performing the sum of leading logarithm terms of the perturbative expansion and curing, therefore, the problem of the infrared instability of the density of states addressed at the end of section 2. Taking the limit \( \Lambda \to \infty \) in the bare theory corresponds to measure at large distances (or small energies with respect to the Fermi level) and in this regime perturbation theory becomes more and more accurate. This is another instance of a physical system which has strong coupling at certain energies and an asymptotic regime at which perturbation theory becomes reliable, similar to what is believed to be true (as supported by experiment) in the quantum field theory of the strong interactions. We are providing the example of a nonrelativistic theory in which this also happens, exchanging the ultraviolet regime in the case of the strong interactions by the infrared regime here.

As for the practical application of the flow (50), we remark that the nonrelativistic theory appears to be formally incomplete, since it does not give any hint about what the value of \( v_R \) has to be. This is related to the unlimited growth of the Fermi velocity, and to the fact that, in the nonrelativistic approximation, there is no signal of a renormalization group fixed point. Later we will see that in the complete quantum field theory the fixed point is given by \( v_{\text{eff}} = c \), but the flow in the graphite layer may not be necessarily controlled by this value of the renormalized coupling. We believe, anyhow, that the observation of the phenomenon of scaling in the asymptotic region should be possible in a single graphite sheet.

Regarding the possible existence of a natural infrared cutoff, it can be shown that in our model the Coulomb interaction is not screened, in the usual metallic manner, by quantum effects. This is due to the vanishing of the density of states at the Fermi level. Thus, a finite screening length is only generated by giving to the system an excess of charge, as a consequence of doping. In order to study this effect, we may place the system away from half-filling by introducing a chemical potential \( \mu \) for the conserved charge

\[
\frac{n}{V} = - \lim_{t' \to t+0} \text{Tr} \langle T \Psi(t,0) \Psi^\dagger(t',0) \rangle
\]

The correct expression of the free electron propagator at finite \( \mu \) is

\[
G^{(0)}(\omega, k_{\mu \neq 0}) = i \frac{-\gamma_0 \omega + v \gamma \cdot k}{-\omega^2 + v^2 k^2} + i \epsilon - \pi \frac{-\gamma_0 \omega + v \gamma \cdot k}{v^2 k^2} \theta(\mu - v |k|) \delta(\omega - v |k|)
\]

Obviously, the \( \mu \)-dependent part cannot change the ultraviolet behaviour of the loop-integrals and the renormalization coefficients of the quantum field theory are not modified by the nonvanishing chemical potential. This may lead, however, to finite corrections. The fermion loop diagram is now
nonvanishing, opposite to what happened before at $\mu = 0$. The computation of (51) gives

$$\frac{n}{V} = \frac{1}{(2\pi)^2} \int d^2 k \, d\omega \frac{\omega}{v|k|} \theta(\mu - v|k|) \delta(\omega - v|k|) = \frac{1}{4\pi \frac{\mu^2}{v^2}}$$

(53)

which, after multiplying by 2 (the two independent Fermi points) gives the correct number of particles. Regarding the polarization tensor, a nonvanishing value of $\Pi_{00}(\omega = 0, k \to 0)$ signals the appearance of a photon effective mass. At small $\mu$ one finds actually the estimate for the inverse of the screening length

$$\Pi_{00}(0, k \to 0) \approx \frac{e^2 \mu}{2\pi v^2}$$

(54)

which gives a measure of the screening effects away from half-filling.

We now turn, finally, to the issue of whether the electron observables may get anomalous dimensions in the nonrelativistic limit. We remark that the absence of one-loop wavefunction renormalization in the limit $v \to 0$ is just an accident. The signal of wavefunction renormalization is given by an ultraviolet divergent self-energy contribution of the form

$$\Sigma(\omega, k) \approx \frac{e^2}{v} \gamma_0 \omega + \ldots$$

(55)

To establish the degree of divergence of the local contributions one can safely implement the nonrelativistic limit in the photon propagator, which leads to the expression (19). The $\omega$-dependence in the vertex diagram in figure 3 can be absorbed then by a simple change of variables in the loop integral, so that the divergent part of the diagram cannot bear any dependence on it. At the two-loop level, the same independence of the local contributions on $\omega$ holds except for the vertex diagrams in figure 6. In these diagrams the loop integrals overlap in such a way that the previous argument does not apply. By using the nonrelativistic approximation (19) and after a very lengthy calculation, one obtains from the sum of the two-loop diagrams

$$Z_\Psi = 1 - \frac{c_1}{16\pi^2 \frac{v^2}{e^4}} \log \Lambda + O(e^6)$$

(56)

where $c_1$ is a nonvanishing quantity, $c_1 \approx 5.49 \cdot 10^{-2}$. We see therefore that the electron observables acquire anomalous dimensions in the infrared regime. According to equation (47), the electron propagator transforms under a change of scale $\rho$ by the factor

$$\exp\left\{-\frac{c_1}{16\pi^2} \frac{e^4}{\rho^2 \frac{v^2}{v_{eff}}}(\rho')\right\}$$

(57)
We have therefore an electronic system which deviates slightly from Fermi liquid behaviour. In particular, we have seen that the vanishing of the density of states at the Fermi level suppresses the usual screening of the Coulomb interaction. This property is maintained in the quantum theory, at least at the one-loop level, since the Fermi level is not shifted by quantum corrections. Also a gap does not open at the Fermi points to the one-loop order (as it would be if an effective mass for the spinor had been generated in perturbation theory). The system is not an insulator, though it does not exhibit usual metallic properties either.

5 Scaling of the electronic interaction (II).
Non-Fermi liquid fixed point.

In the previous section we have mentioned that the nonrelativistic theory is formally incomplete since it does not know of any bound in the effective Fermi velocity \( v_{\text{eff}} \) of electrons. As this parameter grows steadily at large distances the nonrelativistic approximation becomes less accurate, up to a point in which the power series expansion in \( v/c \) loses its meaning. In this relativistic regime, we have to use the full expression of the beta function \( \beta_v(v, e^2) \) to analyze the behaviour of \( v_{\text{eff}} \). In particular, we are interested in knowing if a possible fixed point of the renormalization group flow exists, characterized by the condition

\[
\beta_v(v, e^2) = 0 \tag{58}
\]

By quoting (29) in terms of elementary functions, this equation reads, at the one-loop level,

\[
\frac{1}{v} \left(1 - 2v^2 + 4v^4\right) \arccos v + \frac{1 - 4v^2}{1 - v^2} = 0 \tag{59}
\]

For positive \( v \), the only solution to (59) is \( v = 1 \), which in our units means \( v \) equal to \( c \). Thus, the full quantum field theory has a built-in upper bound of the renormalization group flow, and the speed of light turns out to be the fixed point for the Fermi velocity of electrons.

In fact, it is not difficult to check that the one-loop beta function has a simple zero at \( v = 1 \), and to compute the flow in the neighborhood of the fixed point. The renormalization group equation for \( v_{\text{eff}} \) becomes

\[
\rho \frac{\partial}{\partial \rho} v_{\text{eff}} \approx \frac{2 e^2}{5 \pi^2} (1 - v_{\text{eff}}) \tag{60}
\]

\(^7\)This is, obviously, a statement regarding the limit value of the left hand side of (59) as \( v \to 1 \).
The asymptotic behaviour of the solution is, taking for \( v_R \) the fixed point value,

\[
\log(1 - v_{\text{eff}}) \approx -\frac{2}{5} \frac{e^2}{\pi^2} \log \left( \Lambda / \Lambda_0 \right)
\]

(61)

In figure 7 we show the plot of this scaling behaviour, which is practically indistinguishable in the range considered from that obtained with the non-relativistic approximation.

Recalling the vanishing of the electric charge beta function \( \beta_e(v, e^2) \) at the one-loop level, we may assert that to this perturbative order we have identified an infrared fixed point given by \( v_{\text{eff}} = c \) and arbitrary (though small) value of \( e_{\text{eff}} \). The one-loop result is not satisfactory, however, since the independence of \( e_{\text{eff}} \) on the flow may be destabilized by higher-loop effects. In the renormalization group approach to Fermi liquid theory, the consideration of the one-loop graph in figure 5 is enough since it encodes, via the renormalization group, information of the series of ladder diagrams. In the usual analysis of the many-body theory with an extended Fermi surface these are which dominate in the infrared regime. Because of the existence of Fermi points in the honeycomb lattice at half-filling, though, there is no argument here to conclude the relevance of one set of diagrams against the rest. In our quantum field theory description it seems that no diagram can be discarded a priori. It is also for this reason that there are diagrams which are irrelevant in Fermi liquid theory but give rise in our model to wavefunction renormalization and anomalous dimensions in the infrared regime.

Regarding the renormalization of the electric charge, we have analyzed the polarization tensor \( \Pi_{\mu\nu} \) to second order in perturbation theory. As stated above, the renormalization of the photon field \( A_\mu \) is directly related by gauge invariance to the renormalization of the charge \( e \). Computing in the framework of dimensional regularization we have not found any divergent contribution of order \( e^4 \) to \( \Pi_{\mu\nu} \). The reason is the same operating in the one-loop polarization tensor. A naïve dimensional analysis of the diagram shows that it should diverge linearly in momentum space. In the dimensional regularization scheme this leads to the appearance of gamma functions at half-integer values of the argument, instead of poles at \( d \to 3 \). We believe that this scheme is the best suited for the regularization of the gauge theory —it certainly is not going to break the gauge invariance in any event. If, however, one insists on understanding the regularization in terms of a cutoff dependence, it can be checked that this does not lead either to a renormalization of the scale of the field \( A_\mu \). One would have to look for divergent contributions of \( \log \Lambda \) type accompanying the gauge invariant expression (38) for \( \Pi_{\mu\nu} \). These are not present at the two-loop order and one may conjecture, following the above dimensional argument, that they do not appear to any
order in perturbation theory.

At or near the fixed point, the two-dimensional electron liquid we are describing is definitely not a Fermi liquid. The main reason for this assertion is the anomalous dimension acquired by the electron field Ψ, which may be obtained from (27) evaluated now at \( v = 1 \). At the fixed point the behaviour of the Green function under a change of scale is

\[
G(\omega, k)\big|_{\Lambda} \equiv \Lambda^\gamma \Phi(\omega, k)
\]

or, equivalently, after introducing the precise value of \( \gamma \)

\[
G(\rho \omega, \rho k) = \rho^{-1+e^2/(12\pi^2)} G(\omega, k)
\]

We arrive in this way to solve the problem of the proliferation of \( \log \omega \) terms in the perturbative computation of the density of states. As is well-known, the renormalization group approach succeeds in summing up all these increasingly divergent contributions, in the form of an anomalous dependence on the electron frequency. We have found that this sum is consistently achieved in the relativistic regime, where the fixed point lies and the electron field has a simple, anomalous scaling behaviour.

6 Summary and conclusions

We have studied a system of electrons in the two-dimensional honeycomb lattice interacting via a Coulomb potential. The fact that the Fermi surface of this lattice at half-filling consists of two isolated points renders the problem similar to the \( d = 1 \) case and allows the formulation of the low energy effective theory as a standard quantum field theory. We have been able to investigate the infrared behavior of the system by using renormalization group techniques and have seen that, indeed, it resembles more that of one-dimensional electrons than that of a two-dimensional Fermi liquid.

Before entering into the analysis of the phenomenological consequences of our work, there are a few points that we would like to emphasize.

Let us first comment on the use made through the paper of the electromagnetic field propagator. This use is unconventional both from the condensed matter point of view where the photon is mostly absent, and from the quantum field theory since, as we will strength now, we are not dealing with standard massless QED.

Expression (17) shows the usual three-dimensional photon of QED. This is what we use in our relativistic computations. The constraint that, in the electronic interactions, the two points lay on the lattice plane, allows us to integrate out the perpendicular dimension. The resulting expression read
as a propagator on the plane certainly differs from the standard propagator in QED$_3$. In particular its infrared behavior is quite different, it goes like $1/r$ instead of the standard $\log r$. This is the origin of, for instance, the finiteness of the photon self-energy up to the two-loop level and, hence, of the nonrenormalization of the electric charge.

Let us now comment on the comparison between the instantaneous versus the retarded propagator. The effect of retardation on the infrared behavior of nonlocal quantities has been exemplified with the computation of the quantum corrections to the density of states at the end of section 2. In the theory given by the Coulomb interaction (13), no one-loop corrections were found to the free value of the density of states. In the theory with the retarded interaction our RG analysis shows that, despite the infrared divergence found at the one-loop level, the approach is consistent in the sense that the perturbative expansion of the nonrelativistic approximation does not have to make sense in the infrared when the Fermi velocity is still far from the fixed-point value. Hence, we must resort to the full relativistic theory when computing the infrared behavior of nonlocal observables. There, the RG allows to interpret the one-loop infrared instability shown in section 2 as the manifestation, in perturbation theory, of the anomalous dimension.

We have then found that our original system flows in the infrared towards a fixed point of the renormalization group that is a relativistic quantum field theory with anomalous dimension of the fermionic propagator.

Our analysis has important consequences in condensed matter physics. One of the more interesting is the anomalous screening of the Coulomb interactions of the model. In a Fermi liquid, formation of virtual electron-hole pairs (i.e., quantum corrections to the photon self-energy) generates a new scale, the screening length. This is the basis of the absence of long-range interactions in the ultralocal approach to interacting electrons (Hubbard model). What we have seen is that, due to the vanishing of the density of states at the Fermi level and to the absence of renormalization of the photon self-energy, there is no metallic screening in our materials. The robustness of this result stands on the non-renormalization of the chemical potential (the Fermi level remains a set of two points after the corrections due to the interactions). Moreover we have seen that, although the low energy density of states goes to zero, no finite gap appears in the spectrum of charge excitations. Our results rule out the validity of descriptions in terms of on-site interactions only. Screening effects are not strong enough to reduce the couplings to simple on-site terms.

The one-particle properties are very different from those of a conventional Fermi system. The wavefunction renormalization leads to the disappearance of the quasiparticle peak at low energies. Thus, no effective description in
terms of independent, and coherent, quasiparticles is possible. The situation resembles the 1D Luttinger liquid, where any amount of electron-electron interaction suffices to wipe out the quasiparticle pole, leading to incoherent one-particle excitations.

Finally, the existence of nontrivial scaling laws implies that anomalous exponents should show up in a variety of properties, like the specific heat, susceptibility, etc. The possibility of having an experimental test of these results remains open.

To conclude this summary and to point out possible future work, we would like to remark the interest of investigating the connections of our work with the infrared catastrophe mentioned in the introduction as well as with other unconventional Fermi liquid behavior discussed in the literature [21]. On this respect, this work can be seen as a rigorous realization of the ideas settled in [22].

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References

[1] A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics (Dover, New York, 1975). E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics, Vol. 2 (Pergamon Press, Oxford, 1980).

[2] F. D. M. Haldane, J. Phys C14 (1981) 2585. P. W. Anderson, Phys. Rev. Lett. 64 (1990) 1839; 65 2306; 66 3226.

[3] For a recent review on the infrared catastrophe see the talk of P. W. Anderson in “The Physics and the Mathematical Physics of the Hubbard Model”, Nato A.R.W., San Sebastian, October 1993.

[4] For a theoretical physicist’s vision of the problem see J. Polchinski, “Effective Field Theory and the Fermi Surface”, hep-th-9210046.

[5] G. Benfatto and G. Gallavotti, J. Stat. Phys. 59 (1990) 541; Phys. Rev. B42 (1990) 9967.

[6] R. Shankar, Physica A177 (1991) 530. For an extensive review see R. Shankar, “Renormalization Group Approach to Interacting Fermions”, Rev. Mod. Phys., to appear.

[7] K.G. Wilson, Phys. Rev. B4 (1971) 3174, 3184.

[8] J. González, F. Guinea, and M. A. H. Vozmediano, Nucl. Phys. B406 (1993) 771.

[9] H. W. Kroto, J. R. Heath, S. C. O’Brien, R. F. Curl and R. E. Smalley, Nature 318 (1985) 162.

[10] W. Krätschmer, L. D. Lamb, K. Fostiropoulos and D. R. Huffman, Nature 347 (1990) 354.

[11] A. F. Hebard, M. J. Rosseinsky, R. C. Haddon, D. W. Murphy, S. H. Glarum, T. T. M. Palstra, A. P. Ramirez, and A. R. Kortan, Nature 350 (1991) 600. M. J. Rosseinsky et al., Phys. Rev. Lett. 66 (1991) 2830. K. Holczer, O. Klein, S.-M. Huang, R. B. Kaner, K.-J. Fu, R. L. Whetten and F. Diederich, Science 252 (1991) 1154.

[12] For a review on the properties of fullerenes, see K. Holczer and R. L. Whetten, Carbon 30 (1992) 1261.
[13] J. González, F. Guinea and M. A. H. Vozmediano, “Electrostatic Screening in Fullerene Molecules”, preprint cond-mat/9303007 (to be published in Mod. Phys. Lett. B).

[14] S. Sorella and E. Tosatti, Europhys. Lett. 19 (1992) 699, G. Santoro, M. Airoldi, S. Sorella and E. Tosatti, Phys. Rev. B 47 (1993) 16216.

[15] P. Ramond, Field Theory. A Modern Primer. (Benjamin/Cummings, London, 1981).

[16] J. Zinn-Justin, Lectures delivered at the Cargèse Summer School 1973 (unpublished). K. Symanzik, in Particles, Quantum Fields and Statistical Mechanics (edited by M. Alexanian and A. Zepeda) (Springer Verlag, Berlin, 1975).

[17] D. J. Amit, Field Theory, Renormalization Group and Critical Phenomena, Chap. 8 (McGraw-Hill, New York, 1978).

[18] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, p. 1044 (Academic Press, New York, 1980).

[19] K. G. Wilson, Phys. Rev. Lett. 28 (1972) 548.

[20] S. A. Chin, Ann. Phys. 108 (1977) 301.

[21] X. G. Wen, Phys. Rev. B42 (1990) 6623.

[22] C. M. Varma et al., Phys. Rev. Lett. 63 (1989) 1996.
Figure Captions

Figure 1: The planar honeycomb lattice.
Figure 2: Representation in \((E, k)\) space of the lower branch of the electronic dispersion relation (in units \(t = a = 1\)). The cusps appear at the six corners of the first Brillouin zone.
Figure 3: One-loop contribution to the electron self-energy.
Figure 4: First quantum corrections to the interaction vertex.
Figure 5: One-loop polarization tensor.
Figure 6: Two-loop diagrams contributing to wavefunction renormalization.
Figure 7: Plot of the effective Fermi velocity \(v_{\text{eff}}\) (in units \(c = 1\)) versus length scale \(L/L_0\).
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