Casimir Energies due to Matter Fields in $T^2$ and $T^2/Z_2$ Compactifications

Masato Ito *

Department of Physics, Nagoya University, Nagoya, JAPAN 464-8602

Abstract

We calculate the Casimir energies due to matter fields with various boundary conditions along two compact directions in $T^2$ compactification. We discuss whether the Casimir energies generate attractive or repulsive forces. On the theories with extra dimensions, the Casimir energy plays a crucial role in the mechanism for stabilizing the size of extra dimensions. Finally we argue a procedure of the application to $Z_2$ orbifold.

*E-mail address: mito@eken.phys.nagoya-u.ac.jp
1 Introduction

Motivated by Kaluza [1] and Klein [2], it is expected that the phenomenology of low energy physics should be explained by extra spatial dimensions. Based on the Kaluza-Klein theory it is natural to consider that the radii of the compactified extra dimensions should be Planck size. After the Kaluza-Klein theory, there had been many works in the framework of theories with extra dimensions [3, 4, 5, 6]. Relying on the assumption that the gravity only feels large extra compact dimensions at sub-millimeter range, a remarkable model of Ref. [7] provided a breakthrough of hierarchy problem.

Recent developments are based on the idea that ordinary matter fields could be confined to a three-brane world embedded in the higher dimensional world. The assumption of such models is that the three-brane can be identified with the fixed plane via orbifold of extra dimensional manifold. At the beginning of warped braneworld scenario, Randall and Sundrum had proposed a new suggestion to the hierarchy problem by using separated three-branes on $S^1/Z_2$ orbifold embedded in $AdS_5$ [8]. Furthermore, it was shown that the localization of the gravity occurs on the positive tension three-brane and usual four-dimensional Newton law can be recovered at distance which is much larger than radius of $AdS_5$ [9]. Thus the orbifold of compact spaces may be very fascinating idea in the framework of braneworld scenarios.

Interestingly, an operation of the orbifold is very useful to construct phenomenological models because unwanted matter fields can be projected out, and the operation enables for the models to be chiral. In order that the model can be phenomenologically viable, it is possibility that wanted matter fields can be assigned to appropriate charges under orbifold symmetry. Furthermore it is widely discussed that the orbifold brings on the supersymmetry breaking or the breaking of gauge group [10, 11]. The idea of the orbifold had been developed by string theory in nontrivial background. Recent phenomenological successful models via the orbifold are not necessary based on string theory. However it is expected that these remarkable models can provide several solutions to triplet-doublet splitting, proton decay, neutrino models and so on.

On the theories with extra dimensions, an important issue arises. The stabilization mechanism for the size of the extra spatial dimensions has not been discovered yet. As a clue to the issue, the Casimir energy plays a crucial role in radius stabilization
of models with compactified extra dimensions [12, 13]. The fate of compactified extra dimensions depends on the Casimir force generated by the Casimir energy. Attractive force leads to shrink of the compactified manifold, on the other hand, repulsive force leads to inflation of the compactified manifold. Accordingly it is considered that radius stabilization can be realized by balance between the attractive and the repulsive force.

In this paper, we calculate the Casimir energy $V$ for a massless scalar field in $M^4 \times T^2$, where $M^4$ is ordinary four-dimensional Minkowski space and $T^2$ is two-torus. It is assumed that the fifth and the sixth dimensions are compactified. Note that the Casimir energy depends on the topology and boundary conditions of compactified manifold, and we consider that the boundary condition is periodic or anti-periodic along each compactified direction. We focus on the Casimir energies with respect to the following four cases, $V^{(P,P)}$, $V^{(P,A)}$, $V^{(A,P)}$ and $V^{(A,A)}$, where $P$ denotes periodic boundary condition and $A$ does anti-periodic boundary condition. In the round bracket of $V$, the former letter and the latter letter represents the boundary condition for the fifth direction and for the sixth direction, respectively. The evaluation of $V^{(P,P)}$ has been performed in Ref. [13], accordingly, we will evaluate the remaining three cases. By examining the sign of the Casimir energy, we will investigate whether the Casimir force is attractive or repulsive. Finally we discuss an application to $Z_2$ orbifold. Since we are interested in the Casimir energies due to the matter fields, in the present paper we neglect the contributions of the various localized terms on the orbifold fixed planes in addition to a bulk cosmological constant. We are going to provide the formulas of the Casimir energies due to the matters fields when considering the radius stabilization of a certain model.

The Casimir energy due to a scalar field with the Kaluza-Klein modes can be given by

$$V = \frac{1}{2} \sum_{m,n} \int \frac{d^4k}{(2\pi)^4} \log \left( k^2 + M_{m,n}^2 \right)$$

$$= -\frac{1}{2} \sum_{m,n} \left. \frac{\partial}{\partial s} \right|_{s=0} \int \frac{d^4k}{(2\pi)^4} \left( k^2 + M_{m,n}^2 \right)^{-s}$$

$$= -\frac{1}{32\pi^2} \left. \frac{\partial}{\partial s} \right|_{s=-2} \frac{1}{s(s+1)} \sum_{m,n} \left( M_{m,n}^2 \right)^{-s},$$

(1)

where $m, n \in \mathbb{Z}$. Note that $M_{m,n}^2$ corresponds to the Kaluza-Klein spectrum which is
determined by boundary conditions. The evaluation of the double sum in (1) is explicitly performed by using the techniques of \(\zeta\)-function regularization [14, 15, 16, 17, 18]. When we calculate the Casimir energy of matter field with various boundary conditions on \(T^2\), we use the above formula. Applying our results to a certain phenomenological models, we multiply (1) by the number of degrees of freedom of scalar field in the model, while for fermion we need to multiply by \(-1\).

The paper is organized as follows. In section 2 we describe the detailed calculations of the Casimir energies due to a massless scalar field with boundary conditions along two directions of \(T^2\). The Casimir energies can be written in terms of the area of the torus and the ratio of two radii, and we study the Casimir forces. Finally we shall briefly discuss the case of \(Z_2\) orbifold. In section 3 we mention a summary with respect to the results obtained in the paper. In Appendix A we show the detailed evaluations of the double sums including in calculations of the Casimir energies. In Appendix B we derive the relation between \(\zeta'(-2n)\) and \(\zeta(2n+1)\) by calculating the Casimir energy on \(S^1\) compactification. Because we encounter the first derivative of zeta function when evaluating the Casimir energy. In final part, a procedure of the evaluation of \(\zeta'(0)\) is described.

## 2 Calculation of the Casimir energies

We consider \(T^2\) compactification in the six-dimensional theory, where the two compact directions are mutually perpendicular. It is assumed that the compact radius of the fifth dimension is \(R_1\) and the sixth dimension is \(R_2\). We begin to calculate the Casimir energy of a massless scalar field with periodic boundary conditions along two directions in \(T^2\). The double sum of the corresponding Kaluza-Klein spectrum in (1) can be expressed as

\[
\sum_{m,n} \left( \frac{1}{R_1^2} n^2 + \frac{1}{R_2^2} m^2 \right)^{-s} = R_1^{2s} \left\{ 2\zeta(2s) + 2\sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \left( \frac{R_1}{R_2} \right)^{1-2s} \zeta(2s-1) \right. \\
+ \left. \frac{8\pi^s}{\Gamma(s)} \left( \frac{R_1}{R_2} \right)^{1-s} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{n}{m} \right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}} \left( 2\pi \frac{R_1}{R_2} mn \right) \right\}, \tag{2}
\]
where the prime denotes the exclusion of a zero mode. Here $K_s(x)$ is the modified Bessel function and $\zeta(x)$ is the Riemann zeta function. In the Appendix A the evaluation of the double sum is explicitly performed. Plugging into (1) leads to the Casimir energy

$$V^{(P,P)} = -\frac{1}{64\pi^2R_1^4} \left\{ \frac{3}{\pi^4} \zeta(5) + \frac{8\pi}{945} \left( \frac{R_1}{R_2} \right)^5 + \frac{16}{\pi^2} \left( \frac{R_1}{R_2} \right)^{\frac{5}{2}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{m}{n} \right)^{\frac{5}{2}} K_{-\frac{5}{2}} \left( 2\pi \frac{R_1}{R_2} mn \right) \right\},$$

where $(P,P)$ denotes periodic boundary conditions along two compact directions in $T^2$, the former letter corresponds to the fifth direction and the latter letter does the sixth direction. Performing the calculation of the derivative with respect to $s$ in (1), we used the following particular values:

$$\zeta'(-4) = \frac{3}{4\pi^4} \zeta(5), \quad \Gamma'(-2) = \frac{\pi}{\Gamma(-2)^2} = -2, \quad \Gamma \left( \frac{5}{2} \right) = -\frac{8}{15} \sqrt{\pi},$$

$$\zeta(-5) = -\frac{15}{4\pi^6} \zeta(6) = \frac{1}{252}.$$  

(4)

In the Appendix B we represented the evaluation of $\zeta'(-2n)$, where $n$ is nonnegative integer. By using the following form of the modified Bessel function :

$$K_{\frac{5}{2}}(z) = \sqrt{\frac{\pi}{2}} \left( z^{-\frac{5}{2}} + 3z^{-\frac{3}{2}} + 3z^{-\frac{1}{2}} \right) e^{-z},$$

we can obtain

$$V^{(P,P)} = -\frac{1}{64\pi^2R_1^4} \left\{ \frac{3}{\pi^4} \zeta(5) + \frac{8\pi}{945} \left( \frac{R_1}{R_2} \right)^5 + \frac{16}{\pi^2} \left( \frac{R_1}{R_2} \right)^{\frac{5}{2}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{m}{n} \right)^{\frac{5}{2}} K_{-\frac{5}{2}} \left( 2\pi \frac{R_1}{R_2} mn \right) \right\},$$

where we defined

$$L_3(\tau) \equiv \sum_{m=1}^{\infty} m^2 Li_3(q^m) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{\coth(\pi \tau n)}{n^3 \sinh^2(\pi \tau n)},$$

$$L_4(\tau) \equiv \sum_{m=1}^{\infty} m Li_4(q^m) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^4 \sinh^2(\pi \tau n)},$$

$$L_5(\tau) \equiv \sum_{m=1}^{\infty} Li_5(q^m) = \sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi \tau n} - 1)},$$

$$q \equiv e^{-2\pi \tau},$$

(9)
where $\tau = R_1/R_2$ and $Li_k(x) = \sum_{n=1}^{\infty} x^n/n^k$ is the polylogarithm function. In (7), (8) and (9), the infinite sums of geometric sequence over $m$ are performed.

The toroidal compactification has modular symmetry, namely, it is always possible to redefine the values of $R_1$ and $R_2$. Therefore the area of the torus $A$ and the ratio of two radii $\tau$ have physical meaning. The ordinary modular symmetry can be specified by the three parameters, two radii $R_1$, $R_2$ and relative angle $\theta$ between two directions of compactification. In terminology of string theory, the area $A$ corresponds to the Kähler moduli and the modular parameter $(R_1/R_2)e^{i\theta}$ corresponds to the complex moduli. The modular invariance of the Casimir energy can be demonstrated by the Poisson resummation technique described in [19]. In Ref. [13], it was shown that the Casimir energy $V^{(P,P)}$ has extreme points at the two self-dual points ($\theta = \pi/2, 2\pi/3$ and $R_1 = R_2$) of modular symmetry. In the present paper it is assumed that $\theta = \pi/2$.

From (6), the Casimir energy can be written in terms of $A = 4\pi^2 R_1 R_2$ and $\tau = R_1/R_2$:

$$V^{(P,P)} = -\frac{1}{4A^2} \left\{ \frac{8\pi^3}{945} \tau^3 + 8L_3(\tau) + \frac{12L_4(\tau)}{\tau} \pi + \frac{3\zeta(5) + 6L_5(\tau)}{\pi^2} \frac{1}{\tau^2} \right\}. \quad (11)$$

Note that since the Casimir energy $V^{(P,P)}$ is negative for arbitrary $\tau$, the Casimir force due to a scalar field with periodic boundary conditions along two directions is attractive. For instance, adopting $\tau = 1$ ($R_1 = R_2$) to be maximal symmetry, the value of $V^{(P,P)}$ is approximately given by

$$V^{(P,P)} \simeq -0.1502385/A^2, \quad (12)$$

where the area is fixed. Thus it turns out to be attractive force.

Next we calculate the Casimir energy of a massless scalar field with periodic boundary condition in the fifth direction and anti-periodic boundary condition in the sixth direction. The double sum of the corresponding Kaluza-Klein spectrum in (1) can be expressed as

$$\sum_{m,n} \left( \frac{1}{R_1^2 n^2} + \frac{1}{R_2^2} \left( m + \frac{1}{2} \right)^2 \right)^{-s}$$

$$= R_1^{2s} \left\{ 2\sqrt{\pi} \frac{\Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s)} \frac{R_1}{R_2} \right\}^{1-2s} \zeta \left( 2s - 1, \frac{1}{2} \right)$$
\[ V^{(P,A)} = -\frac{1}{64\pi^2 R_1^4} \left\{ -\frac{31\pi}{3780} \left( \frac{R_1}{R_2} \right)^5 + \frac{16}{\pi^2} \left( \frac{R_1}{R_2} \right)^5 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left( m + \frac{1}{2} \right)^2 n^{-\frac{3}{2}} K_{s-\frac{1}{2}} \left( 2\pi \frac{R_1}{R_2} \left( m + \frac{1}{2} \right) n \right) \right\}, \]

where \( \zeta(s,\nu) \) is the Hurwitz's zeta function. In the Appendix A we evaluated the above double sum which corresponds to the case of \( \alpha \to 0 \) in (41). From (1), we can obtain

\[ V^{(P,A)} = -\frac{1}{64\pi^2 R_1^4} \left\{ -\frac{31\pi}{3780} \left( \frac{R_1}{R_2} \right)^5 + \frac{16}{\pi^2} \left( \frac{R_1}{R_2} \right)^5 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left( m + \frac{1}{2} \right)^2 n^{-\frac{3}{2}} K_{s-\frac{1}{2}} \left( 2\pi \frac{R_1}{R_2} \left( m + \frac{1}{2} \right) n \right) \right\}, \]

where we used \( \zeta(s,1/2) = (2^s - 1)\zeta(s) \) and \((P,A)\) denotes periodic boundary condition in the fifth direction and anti-periodic boundary condition in the sixth direction. Using (5), the Casimir energy can be expressed as

\[ V^{(P,A)} = -\frac{1}{64\pi^2 R_1^4} \left\{ -\frac{31\pi}{3780} \left( \frac{R_1}{R_2} \right)^5 + \frac{8\bar{L}_3(\tau)}{\pi^2} \left( \frac{R_1}{R_2} \right)^2 + \frac{12\bar{L}_4(\tau)}{\pi^3} \frac{R_1}{R_2} + \frac{6\bar{L}_5(\tau)}{\pi^4} \right\}, \]

where we defined

\[ \bar{L}_3(\tau) \equiv \sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right)^2 Li_3 \left( q^{m+\frac{1}{2}} \right) = \frac{1}{8} \sum_{n=1}^{\infty} \frac{\cosh^2 \pi \tau n + 1}{n^3 \sinh^3 \pi \tau n}, \]

\[ \bar{L}_4(\tau) \equiv \sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right) Li_4 \left( q^{m+\frac{1}{2}} \right) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{\coth \pi \tau n}{n^4 \sinh \pi \tau n}, \]

\[ \bar{L}_5(\tau) \equiv \sum_{m=0}^{\infty} Li_5 \left( q^{m+\frac{1}{2}} \right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^5 \sinh \pi \tau n}. \]

Using the area of torus \( \mathcal{A} \) and the ratio of two radii \( \tau \), (15) is given by

\[ V^{(P,A)} = -\frac{1}{4\mathcal{A}^2} \left\{ -\frac{31\pi^3}{3780} \tau^3 + 8\bar{L}_3(\tau) + \frac{12\bar{L}_4(\tau)}{\pi} \frac{1}{\tau} + \frac{6\bar{L}_5(\tau)}{\pi^2} \frac{1}{\tau^2} \right\}. \]

In the bracket of the above equation, the first term contributes to the repulsive force and the remaining terms have the contributions of the attractive forces. The value of the Casimir energy for \( \tau = 1 \), when the area is fixed, is approximately given by

\[ V^{(P,A)} \approx +0.0139727/\mathcal{A}^2, \]

consequently, repulsive force arises. In the case of \( \tau \ll 1 \), since the last term in (19) is dominant, the attractive force is generated. By choosing the appropriate value of \( \tau \), vanishing \( V^{(P,A)} \) can be realized by balance between attractive and repulsive.
Successively we calculate the Casimir energy of a massless scalar field with anti-periodic boundary condition in the fifth direction and periodic boundary condition in the sixth direction. By making an exchange of $\tau \leftrightarrow 1/\tau$ ($R_1 \leftrightarrow R_2$) in (19), we can obtain

$$V^{(A,P)} = -\frac{1}{4A^2} \left\{ -\frac{31\pi^3}{3780} \frac{1}{\tau^3} + 8\tilde{L}_3(\tau^{-1}) + \frac{12\tilde{L}_4(\tau^{-1})}{\pi} \tau + \frac{6\tilde{L}_5(\tau^{-1})\tau^2}{\pi^2} \right\} .$$

(21)

As a matter of course, $V^{(P,A)}$ is equal to $V^{(A,P)}$ for $\tau = 1$. In the case of $\tau \ll 1$, since the first term is dominant, it is repulsive force.

Finally we will calculate the Casimir energy of a massless scalar field with anti-periodic boundary conditions in two directions. The double sum of the corresponding Kaluza-Klein modes in (1) can be given by

$$\sum_{m,n} \left( \frac{1}{R_1^2} \left( n + \frac{1}{2} \right)^2 + \frac{1}{R_2^2} \left( m + \frac{1}{2} \right)^2 \right)^{-s}$$

$$= R_1^{2s} \left\{ 2\sqrt{\pi} \frac{\Gamma \left( s - \frac{1}{2} \right)}{\Gamma(s)} \left( \frac{R_1}{R_2} \right)^{1-2s} \zeta \left( 2s - 1, \frac{1}{2} \right) \right.$$  

$$+ \frac{8\pi s}{\Gamma(s)} \left( \frac{R_1}{R_2} \right)^{1-s} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left( m + \frac{1}{2} \right)^{-s} n^{s-\frac{1}{2}} (-1)^n K_{s-\frac{1}{2}} \left( 2\pi \frac{R_1}{R_2} \left( m + \frac{1}{2} \right) n \right) \right\} .$$

(22)

The evaluation of the above double sum corresponds to the case of $\alpha = \beta = 1/2$ in the Appendix A. Plugging into (1), we have

$$V^{(A,A)} = -\frac{1}{64\pi^2 R_1^5} \left\{ -\frac{31\pi}{3780} \left( \frac{R_1}{R_2} \right)^5 + \frac{8\tilde{L}_3(\tau)}{\tau^2} \left( \frac{R_1}{R_2} \right)^2 + \frac{12\tilde{L}_4(\tau) R_1}{\pi^3 R_2} + \frac{6\tilde{L}_5(\tau)}{\pi^4} \right\} ,$$

(23)

where we defined

$$\tilde{L}_3(\tau) \equiv \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n}{n^3} \left( m + \frac{1}{2} \right)^2 e^{-2\pi\tau(m+1/2)} = -\tilde{L}_3(\tau) + \frac{1}{4} \tilde{L}_3(2\tau) ,$$

(24)

$$\tilde{L}_4(\tau) \equiv \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n}{n^4} \left( m + \frac{1}{2} \right) e^{-2\pi\tau(m+1/2)} = -\tilde{L}_4(\tau) + \frac{1}{8} \tilde{L}_4(2\tau) ,$$

(25)

$$\tilde{L}_5(\tau) \equiv \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n}{n^5} e^{-2\pi\tau(m+1/2)} = -\tilde{L}_5(\tau) + \frac{1}{16} \tilde{L}_5(2\tau) .$$

(26)

Here we rewrote the above sums by using (16), (17) and (18). The Casimir energy with anti-periodic boundary conditions in two directions can be rewritten in terms of
\[ V^{(A,A)} = -\frac{1}{4A^2} \left\{ -\frac{31\pi^3}{3780}\tau^3 + 8\hat{L}_3(\tau) + \frac{12\hat{L}_4(\tau)}{\pi} \frac{1}{\tau} + \frac{6\hat{L}_5(\tau)}{\pi^2} \frac{1}{\tau^2} \right\}. \]  

(27)

Since \( \tilde{L}_3, \tilde{L}_4 \) and \( \tilde{L}_5 \) are decreasing for \( \tau \), \( \hat{L}_3, \hat{L}_4 \) and \( \hat{L}_5 \) are always negative. Namely \( V^{(A,A)} \) is positive for arbitrary \( \tau \). Therefore it generates repulsive force. For instance, in the case of \( \tau = 1 \) when \( A \) is fixed, the value of \( V^{(A,A)} \) is approximately

\[ V^{(A,A)} \approx +0.1126133/A^2, \]  

(28)

thus it is repulsive force.

Consequently, \( V^{(P,P)} \) and \( V^{(A,A)} \) can generate the attractive force and repulsive force, respectively. Whether \( V^{(P,A)} \) and \( V^{(A,P)} \) generate attractive or repulsive forces depends on the value of \( \tau \). We calculated the Casimir energies of a single massless scalar field in the present paper, while for fermion field we need to multiply by \( -1 \).

Taking into account of the number of degrees of freedom in matter fields with various boundary conditions, the contributions of the matter fields are given by the sum of (11), (19), (21) and (27). Thus the total Casimir energy due to the matter fields depends on model-building. Moreover we need to add the contribution of a bulk cosmological constant:

\[ \int d^6x \sqrt{G} \Lambda = \int d^4x \Lambda A, \]

where \( \Lambda \) is a bulk cosmological constant which corresponds to the vacuum energy. Accordingly it is possible for the total Casimir energy to have minimum for \( A \) and \( \tau \). However, the stability of the system depends on the sign of \( \Lambda \) which leads to the familiar cosmological constant problem. In the present paper we do not mention the problem. Setting to a certain supersymmetric model, the preservation of supersymmetry via toroidal compactification guarantees vanishing Casimir energy [13].

3 Application to \( Z^2 \) orbifold

We shall discuss a procedure of the application to \( T^2/Z_2 \) orbifold. The matter fields with boundary conditions of \( (P,P) \), \( (P,A) \), \( (A,P) \) and \( (A,A) \) can be decomposed by \( Z_2 \) orbifold. The wave function of each matter field can be separated into the cosine function and the sine function by performing the operation of the orbifold.
Equivalently, this implies $Z_2$ projections of the following three transformations

$$r_1 : (x_5, x_6) \mapsto (x_5 + 2\pi R_1, x_6),$$  \hspace{1cm} (29)

$$r_2 : (x_5, x_6) \mapsto (x_5, x_6 + 2\pi R_2),$$  \hspace{1cm} (30)

$$r_3 : (x_5, x_6) \mapsto (-x_5, -x_6),$$ \hspace{1cm} (31)

where $x_5$ and $x_6$ denotes the fifth coordinate and the sixth coordinate, respectively. The $Z_2$ parity of matter fields via these transformations can be specified by $(r_1, r_2, r_3)$, where $r_1, r_2, r_3 = \pm 1$ under the orbifold symmetry. Namely this means that +1 corresponds to the even state and -1 odd state. The illustration of the decompositions via $Z_2$ projection can be represented as follows

$$(P, P) : \exp\left(i\frac{x_5}{R_1}m + i\frac{x_6}{R_2}n\right) \mapsto \begin{cases} \cos\left(\frac{x_5}{R_1}m + \frac{x_6}{R_2}n\right) \\ \sin\left(\frac{x_5}{R_1}m + \frac{x_6}{R_2}n\right) \end{cases},$$ \hspace{1cm} (32)

$$(P, A) : \exp\left(i\frac{x_5}{R_1}m + i\frac{x_6}{R_2}\left(n + \frac{1}{2}\right)\right) \mapsto \begin{cases} \cos\left(\frac{x_5}{R_1}m + \frac{x_6}{R_2}\left(n + \frac{1}{2}\right)\right) \\ \sin\left(\frac{x_5}{R_1}m + \frac{x_6}{R_2}\left(n + \frac{1}{2}\right)\right) \end{cases},$$ \hspace{1cm} (33)

$$(A, P) : \exp\left(i\frac{x_5}{R_1}\left(m + \frac{1}{2}\right) + i\frac{x_6}{R_2}n\right) \mapsto \begin{cases} \cos\left(\frac{x_5}{R_1}\left(m + \frac{1}{2}\right) + \frac{x_6}{R_2}n\right) \\ \sin\left(\frac{x_5}{R_1}\left(m + \frac{1}{2}\right) + \frac{x_6}{R_2}n\right) \end{cases},$$ \hspace{1cm} (34)

$$(A, A) : \exp\left(i\frac{x_5}{R_1}\left(m + \frac{1}{2}\right) + i\frac{x_6}{R_2}\left(n + \frac{1}{2}\right)\right) \mapsto \begin{cases} \cos\left(\frac{x_5}{R_1}\left(m + \frac{1}{2}\right) + \frac{x_6}{R_2}\left(n + \frac{1}{2}\right)\right) \\ \sin\left(\frac{x_5}{R_1}\left(m + \frac{1}{2}\right) + \frac{x_6}{R_2}\left(n + \frac{1}{2}\right)\right) \end{cases},$$ \hspace{1cm} (35)

where the symbol $\mapsto$ denotes the operation of $Z_2$ orbifold. Here we omitted the normalization factors and the four-dimensional parts in matter fields. Under the orbifold symmetry, note that $(+1,+1,+1)$ and $(+1,+1,-1)$ states result from $(P, P)$, $(+1,-1,+1)$ and $(+1,-1,-1)$ from $(P, A)$, $(-1,+1,+1)$ and $(-1,+1,-1)$ from $(A, P)$, $(-1,-1,+1)$ and $(-1,-1,-1)$ from $(A, A)$. Thus the $Z_2$ orbifold can produce the eight states. Furthermore there are four orbifold fixed planes on $T^2/Z_2$ at $(x_5, x_6) = (0, 0), (\pi R_1, 0), (0, \pi R_2), (\pi R_1, \pi R_2)$. There exist extra contributions of the localized terms on these fixed planes, for example, kinetic terms, mass terms and interaction.
terms between bulk and brane as well as brane tension. Since the Kaluza-Klein spectrum is modified by these brane-localized terms, these effects must be considered. For example, in five-dimensional $S^1/Z_2$ model, the Casimir energy including brane-localized kinetic terms had been calculated [13]. Consequently the total Casimir energy consists of the matter fields, a bulk cosmological constant as well as brane-localized terms. It is considered that the Casimir energy including all contributions is very complicated form, and minimum problem of the Casimir energy is closely related to the cosmological constant problem. We are going to investigate the points elsewhere.

Adopting a certain model on $T^2/Z_2$ orbifold, the Casimir energies due to the matter fields will be calculated by taking account of the number of the degrees of freedom and $(r_1, r_2, r_3)$ parity assignments given in matter content of the model when neglecting the effects of branes. When performing the calculation, we need to make the replacements $R_1 \rightarrow R_1/2$ and $R_2 \rightarrow R_2/2$, simultaneously, multiply $1/2 \times 1/2$ factor via half modes over $m, n$. Therefore we can obtain the following forms

\[
V^{(++\pm)} = \frac{1}{4} V^{(P,P)} \left( \frac{R_1}{2}, \frac{R_2}{2} \right)
\]

\[
V^{(+-\pm)} = \frac{1}{4} V^{(P,A)} \left( \frac{R_1}{2}, \frac{R_2}{2} \right)
\]

\[
V^{(-+\pm)} = \frac{1}{4} V^{(A,P)} \left( \frac{R_1}{2}, \frac{R_2}{2} \right)
\]

\[
V^{(--\pm)} = \frac{1}{4} V^{(A,A)} \left( \frac{R_1}{2}, \frac{R_2}{2} \right)
\]

When considering the radius stabilization of the concrete model on $T^2/Z_2$ orbifold, it is important to use results obtained here. The concrete model is beyond the scope of this paper and we do not mention it here.

4 Summary and Discussion

We calculated the Casimir energies due to a massless scalar field with various boundary conditions on $T^2$ compactification, assuming that two compact directions are perpendicular each other. The Casimir energies can be explicitly represented in terms of the area of torus and the ratio of two radii. Consequently, it was shown that the case of $(P, P)$ boundary condition is attractive force and the case of $(A, A)$ boundary condition
is repulsive force. For \((P,A)\) and \((A,P)\), whether attractive or repulsive depends on the ratio of two radii. For fermion field, opposite force works.

When calculating the Casimir energies of the matter fields on a certain \(T^2/Z_2\) orbifold model, we need to multiply our results to the number of degrees of freedom assigning eight kinds of \(Z_2\) parity \((\pm 1, \pm 1, \pm 1)\) of matter content. Thus the contributions of the matter fields in radius stabilization of the model will be explicitly evaluated. Furthermore there are contributions of a bulk cosmological constant and brane-localized terms (kinetic term, mass term, interaction terms and brane tension) on the orbifold fixed planes. Since Kaluza-Klein spectrum is modified by these brane-localized terms, it can be considered that the total Casimir energy is very complicated form. That the total Casimir energy has a minimum point for the area of the torus and the ratio of two radii is related to the cosmological constant problem. We are going to describe it elsewhere. In the present paper we could provide the formulas of the Casimir energies with various boundary conditions when considering the radius stabilization in the model.

**Appendix A: Evaluation of double sums**

Calculating the Casimir energy for field with various boundary conditions on toroidal compactification, we encounter the double summation of infinite series [14].

When we compute the Casimir energy of a massless scalar field with periodic boundary conditions along two compact directions in \(T^2\), we must evaluate the following double sum

\[
I(a; s) = \sum_{m,n}' \left[ n^2 + a^2 m^2 \right]^{-s},
\]

where the prime denotes \((m, n) \neq (0, 0)\). The double sum can be decomposed as follows

\[
\sum_n' n^{-2s} + \sum_m' \sum_n \left[ n^2 + a^2 m^2 \right]^{-s}.
\]

The first term can be written in terms of the Riemann zeta function. The second term can be rewritten by using the gamma function what is called Mellin transformation. Consequently, we obtain [13]

\[
I(a; s) = 2\zeta(2s) + \sum_m' \sum_n \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} e^{-\left(n^2 + a^2 m^2\right)t}
\]
\[ \begin{aligned}
&= 2\zeta(2s) + \sum_m \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty dt \: t^{s-\frac{3}{2}} e^{-a^2 m^2 t} \left( 1 + 2 \sum_{n=1}^\infty e^{-\frac{x^2}{t} n^2} \right) \\
&= 2\zeta(2s) + 2\sqrt{\pi} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} |a|^{1-2s} \zeta(2s - 1) \\
&\quad + \frac{8\pi^s}{\Gamma(s)} |a|^{\frac{1}{2}-s} \sum_m \sum_n \left( \frac{n}{m} \right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}} \left( 2\pi |a| mn \right),
\end{aligned} \]

where \( K_s(x) \) is the modified Bessel function, we used the Poisson re-summation formula in the second line.

Next we shall evaluate the double sum with non-periodic boundary conditions as follows

\[ I(a; \alpha, \beta; s) = \sum_{m,n} \left[ (n + \alpha)^2 + a^2 (m + \beta)^2 \right]^{-s}, \]

where \( 0 < \alpha, \beta < 1 \). Following the similar procedures in (39), we obtain

\[ I(a; \alpha, \beta; s) \]
\[ = \sum_{m,n} \frac{1}{\Gamma(s)} \int_0^\infty dt \: t^{s-1} e^{-((n+\alpha)^2+a^2(m+\beta)^2)t} \\
= \frac{\sqrt{\pi}}{\Gamma(s)} \sum_{m=-\infty}^\infty \int_0^\infty t^{s-\frac{3}{2}} e^{-a^2 (m+\beta)^2 t} \left( 1 + 2 \sum_{n=1}^\infty \cos(2\pi n\alpha) e^{-\frac{\pi^2}{4} n^2} \right) \\
= \sqrt{\pi} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} |a|^{1-2s} \left( \zeta(2s - 1, \beta) + \zeta(2s - 1, 1 - \beta) \right) \\
&\quad + \frac{4\pi^s}{\Gamma(s)} |a|^{\frac{1}{2}-s} \sum_{m=0}^\infty \sum_{n=1}^\infty n^{s-\frac{1}{2}} \cos(2\pi n\alpha) \left\{ (m + \beta)^{-s+\frac{1}{2}} K_{s-\frac{1}{2}} \left( 2\pi |a|(m + \beta)n \right) \\
&\quad + (m + 1 - \beta)^{-s+\frac{1}{2}} K_{s-\frac{1}{2}} \left( 2\pi |a|(m + 1 - \beta)n \right) \right\}, \]

where

\[ \zeta(s, \nu) = \sum_{n=0}^\infty \frac{1}{(n + \nu)^s} \]

is the Hurwitz’s zeta function.

**Appendix B: Evaluation of \( \zeta'(-2n) \)**

We derive the relation between \( \zeta'(-2n) \) and \( \zeta(2n+1) \) by calculating the Casimir energy for \( R^{2n} \times S^1 \), where \( n \) is nonnegative integer. We consider a massless scalar field with
periodic boundary condition on the $S^1$ compactification with radius $L$. The Casimir energy $E$ is given by

$$E = \frac{1}{2} \sum_{m=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} \log \left( k^2 + \frac{m^2}{L^2} \right)$$

Next we can rewrite the first line of (43) as follows

$$E = \frac{1}{2} \sum_{m=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} \int_0^\infty ds \frac{1}{s} e^{-\left( k^2 + \frac{m^2}{L^2} \right)s}$$

where we used the Poisson re-summation formula in the third line. Since (43) is equal to (44), we obtain

$$\zeta'(-2n) = \frac{(-1)^n}{2\pi^{2n+1}} \Gamma(n+1) \Gamma \left( n + \frac{1}{2} \right) \zeta(2n+1)$$

Below we tabulate the particular values of $\zeta'(-2n)$, for instance, for $n = 1, 2, 3, 4$.

$$\zeta'(-2) = -\frac{1}{4\pi^2} \zeta(3) \sim -0.03048,$$

$$\zeta'(-4) = \frac{3}{4\pi^4} \zeta(5) \sim 0.0080,$$

$$\zeta'(-6) = -\frac{45}{8\pi^6} \zeta(7) \sim -0.00592,$$

$$\zeta'(-8) = \frac{315}{4\pi^8} \zeta(9) \sim 0.00835.$$
Next, for your information, we can obtain a specific value of $n = 0$ by using the following identity [20]

$$\zeta(1 - z) = 2^{1-z}\pi^{-z}\cos\frac{z\pi}{2}\Gamma(z)\zeta(z). \quad (47)$$

Performing the logarithmic derivative with respect to $z$, we get

$$\frac{\zeta'(1 - z)}{\zeta(1 - z)} = \log(2\pi) + \frac{\pi}{2} \tan\frac{z\pi}{2} - \psi(z) - \frac{\zeta'(z)}{\zeta(z)}, \quad (48)$$

where $\psi(z)$ is the polygamma function. Taking the limit of $z \to 1$, we have

$$\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi) - \psi(1) + \lim_{z \to 1} \left( -\frac{\zeta'(z)}{\zeta(z)} + \frac{\pi}{2} \tan\frac{z\pi}{2} \right) = \log(2\pi). \quad (49)$$

Here we used $\psi(1) = -\gamma$ and $\zeta(z) = 1/(z - 1) + \gamma + \mathcal{O}(|z - 1|)$ for $z \sim 1$, where $\gamma$ is the Euler constant. Therefore we have

$$\zeta'(0) = \zeta(0) \log(2\pi) = -\frac{1}{2} \log(2\pi). \quad (50)$$

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