ON THE ORDERS OF THE NON-FRATTINI ELEMENTS
OF A FINITE GROUP

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Abstract. Let $G$ be a finite group and let $p_1, \ldots, p_n$ be distinct primes. If $G$ contains an element of order $p_1 \cdot \cdots \cdot p_n$, then there is an element in $G$ which is not contained in the Frattini subgroup of $G$ and whose order is divisible by $p_1 \cdot \cdots \cdot p_n$.

1. Introduction

Let $G$ be a finite group. We say that $g \in G$ is a non-Frattini element of $G$ if $g \notin \text{Frat}(G)$, where Frat($G$) denotes the Frattini subgroup of $G$. Let $d(G)$ be the smallest cardinality of a generating set of $G$. In [6] L. G. Kovács, J. Neubüser and B. H. Neumann proved that the set $G \setminus \text{Frat}(G)$ of the non-Frattini elements of $G$ coincides with the set $\kappa_{d(G)+1}(G)$ of those elements $g$ of $G$ that are not omissible from some family of $d(G)+1$ generators of $G$. In the same paper they noticed that if a prime $p$ divides $|G|$, then $p$ divides the order of some element of $G \setminus \text{Frat}(G) = \kappa_{d(G)+1}(G)$. This follows immediately from the fact that if a prime $p$ divides $|G|$, then $p$ divides also $|G/\text{Frat}(G)|$ [3, III Satz 3.8]. In other words the set $\pi(G)$ of the prime divisors of $|G|$ can be deduced from the knowledge of the non-Frattini elements of $G$ or, equivalently, looking at the elements which appear in the minimal generating sets of cardinality $d(G)+1$. One can ask whether some more elaborated information about the arithmetical properties of the orders of the elements of $G$ can be recovered from $G \setminus \text{Frat}(G)$. Looking for results in this direction, we consider in this note the prime graph of $G$.

If $G$ is a finite group, its prime graph $\Gamma(G)$ is defined as follows: its vertices are the primes dividing the order of $G$ and two vertices $p, q$ are joined by an edge if $p \cdot q$ divides the order of some element in $G$. It has been introduced by Gruenberg and Kegel in the 1970s and studied extensively in recent years (see for examples [5], [8], [9]). We consider now the subgraph $\tilde{\Gamma}(G)$ of $\Gamma(G)$ (called the non-Frattini prime graph) defined by saying that two vertices $p$ and $q$ are joined by an edge if and only if $p \cdot q$ divides the order of some element of $G \setminus \text{Frat}(G)$. Our main result is the following.

Theorem 1. Let $G$ be a finite group. Then the prime graph $\Gamma(G)$ and the non-Frattini prime graph $\tilde{\Gamma}(G)$ coincide.

Notice that, although $\pi(G) = \pi(G/\text{Frat}(G))$, it is not in general true that $\Gamma(G) = \Gamma(G/\text{Frat}(G))$. Consider for example $G = \langle a, b \mid a^3 = 1, b^4 = 1, a^b = a^{-1} \rangle$. We have $|G| = 12$ and $|ab^2| = 6$, so $\Gamma(G)$ is the complete graph on the two vertices 2 and 3. However Frat($G$) = $\langle b^2 \rangle$ and $G/\text{Frat}(G) \cong \text{Sym}(3)$, hence $\Gamma(G/\text{Frat}(G))$ consists of two isolated vertices.
Clearly if \( p \cdot q \) divides the order of some elements of \( G \), then \( G \) contains an element of order \( p \cdot q \). This is no more true for \( G \setminus \text{Frat}(G) \). For example all the elements of order \( p \cdot q \) in a cyclic group \( G \) of order \( p^2 \cdot q^2 \) are contained in \( \text{Frat}(G) \).

A question that is not easy to be answered is, in the case when \( G \setminus \text{Frat} G \) contains an element of order divisible by \( p \cdot q \), whether this element could be chosen so that \( p \) and \( q \) are the unique prime divisors of its order. We prove that this is true for soluble groups and we provide a reduction to this question to a problem concerning the finite nonabelian simple groups and their representations.

2. Proofs and remarks

Given an element \( g \) in a finite group \( G \), we will denote by \( \pi(g) \) the set of the prime divisors of \( |g| \). Theorem 1 is a consequence of the following stronger result.

**Theorem 2.** Let \( G \) be a finite group and suppose that \( \pi = \{p_1, \ldots, p_n\} \) is a subset of the set \( \pi(G) \) of the prime divisors of \( |G| \). Set \( \pi^* = \pi \) if \( G \) is soluble, \( \pi^* = \pi \cup \{2\} \) otherwise. If \( G \) contains an element \( g \) of order \( p_1 \cdots p_n \), then there exists an element \( \gamma \) in \( G \setminus \text{Frat}(G) \) such that \( \pi \subseteq \pi(\gamma) \subseteq \pi^* \).

**Proof.** Choose \( g \in G \) with \( |g| = p_1 \cdots p_n \). We may assume \( g \in \text{Frat}(G) \), otherwise we have done. Let \( N \) be a minimal normal subgroup of \( G \). Consider the element \( \bar{g} = gN \) of the factor group \( \bar{G} = G/N \) and let \( M/N = \text{Frat}(G/N) \). If \( |\bar{g}| = |g| \), then by induction there exists \( x \in G \) such that \( \bar{x} = xN \notin M/N \) and \( \pi \subseteq \pi(\bar{x}) \subseteq \pi^* \).

We may choose \( x \) such that \( \bar{\pi}(x) = \bar{\pi}(\bar{x}) \). Since \( x \notin M \) and, by [4, 5.2.13 (iii)], \( N \text{Frat}(G) \leq M \), we have that \( x \notin \text{Frat}(G) \) and we are done. So we may assume that there exists \( i \in \{1, \ldots, n\} \) with \( 1 \neq g^{p_i} \in N \). If \( N \notin \text{Frat}(G) \), then \( N \cap \text{Frat}(G) = 1 \), hence \( g^{p_i} \notin \text{Frat}(G) \), a contradiction.

We remain with the case in which for every minimal normal subgroup \( N \) of \( G \), we have that \( N \) is contained in \( \text{Frat}(G) \) and \( (g) \cap N \neq 1 \). Since \( \text{Frat}(G) \) is nilpotent, we deduce that \( \pi(\text{Frat}(G)) \subseteq \pi \). On the other hand we must have that \( \pi \subseteq \pi(\text{Frat}(G)) \), otherwise \( g \notin \text{Frat}(G) \). So \( \pi(\text{Frat}(G)) = \pi \) and \( \text{Frat}(G) = P_1 \times \cdots \times P_n \), where for each \( i \in \{1, \ldots, n\} \), \( P_i \) is a \( p_i \)-group. Let \( i \in \{1, \ldots, n\} \) and \( N \) a minimal normal subgroup of \( G \) with \( N \leq P_i \). Assume \( N \neq P_i \). In this case, choose \( x \in P_i \setminus N \), and take \( y = xp_i^k \). The element \( \bar{y} = yN \) of the factor group \( \bar{G}/N \) has order divisible by \( p_1 \cdots p_n \) and, as in the first paragraph of this proof, this allows us to conclude that \( G \setminus \text{Frat}(G) \) contains an element of order divisible by \( p_1 \cdots p_n \). So we may assume \( N = P_i \). In particular \( P_i \) is an irreducible \( G \)-module, for every \( i \in \{1, \ldots, n\} \).

First assume that \( G/\text{Frat}(G) \) contains an abelian minimal normal subgroup \( M/\text{Frat}(G) \). There exists a prime \( p \) such that \( M/\text{Frat}(G) \) is a \( p \)-group. If \( p \notin \pi \), then \( \text{Frat}(G) \) is a normal \( \pi \)-Hall subgroup of \( M \) and therefore, by the Schur-Zassenhaus Theorem, \( \text{Frat}(G) \) has a complement, say \( K \), in \( M \) and all these complements are conjugate in \( M \). By the Frattini’s Argument, \( G = \text{Frat}(G)N_G(K) \), hence \( G = N_G(K) \), so \( K \) is a nontrivial normal subgroup of \( G \). However all the minimal normal subgroups of \( G \) are contained in \( \text{Frat}(G) \) and \( \text{Frat}(G) \cap K = 1 \), a contradiction. So \( p = p_i \) for some \( i \in \{1, \ldots, n\} \). Let \( Q_i = \prod_{j \neq i} P_j \) and let \( T_i \) be a Sylow \( p_i \)-subgroup of \( M \). Again by the Frattini Argument, \( G = M \cdot N_G(T_i) = Q_i \cdot N_G(T_i) = Q_i \cdot \text{Frat}(G)N_G(T_i) \), so \( N_G(T_i) = G \) and \( M = Q_i \times T_i \).

Take \( x \in T_i \setminus P_i \) and consider \( \gamma = xq^{p_i} \). Since \( q^{p_i} \in Q_i \), we have \( |\gamma| = |x||q^{p_i}| \) and consequently \( \pi(\gamma) = \pi \).
Finally assume that \( M/\text{Frat}(G) \) is a nonabelian minimal normal subgroup of \( G/\text{Frat}(G) \) and let \( x = z\frac{Frat}{G} \) be an element of \( M/\text{Frat}(G) \) of order 2. We may assume \( |z| = 2^c \) for some positive integers \( c > 0 \). Let \( I \) be the subset of \( i \in \{1, \ldots, n\} \) consisting of the indices \( i \) such that \( p_i \) is odd. If \( i \in I \) and \( M \leq C_G(P_i) \), let \( x_i \) be an arbitrarily chosen nontrivial element of \( P_i \). Assume that \( i \in I \) and that \( M \not\leq C_G(P_i) \). In this case \( M/\text{Frat}(G) \) is isomorphic to a subgroup of \( \text{GL}(F_i) \). It can be easily seen that if \( y \in \text{GL}(P_i) \) has order 2, then either 1 is an eigenvalue of \( y \) or \( y \) is the scalar multiplication by -1. Since \( Z(M/\text{Frat}(G)) = 1 \), we deduce that \( z \) fixes a non-trivial element of \( x_i \) of \( P_i \). Now let \( x = \prod_{i \in I} x_i \) and take \( \gamma = z x \). Since \( |\gamma| = 2^c \prod_{i \in I} p_i \), we conclude \( \pi(\gamma) = \pi^* \).

Given a pair \((p, q)\) of distinct prime divisors of the order of a finite group \( G \), let
\[
\Omega_{p\cdot q}(G) = \{ g \in G \mid p \cdot q \text{ divides } |g| \}, \quad \Omega_{p\cdot q}^*(G) = \Omega_{p\cdot q}(G) \setminus \text{Frat}(G).
\]
Moreover, denote by \( \Omega_{p\cdot q}^{**}(G) \) the set of the elements \( g \in \Omega_{p\cdot q}^*(G) \) whose order is not divisible by any prime different from \( p \) and \( q \). One can ask the following question.

**Question 1.** Is it true that if \( \Omega_{p\cdot q}(G) \neq \emptyset \), then \( \Omega_{p\cdot q}^{**}(G) \neq \emptyset \) ?

By Theorem 2, Question 1 has an affirmative answer if \( 2 \in \{p, q\} \).

**Definition 3.** Let \( S \) be a finite nonabelian simple group, \( P \) a faithful irreducible \( S \)-module of \( p \)-power order and \( Q \) a faithful irreducible \( S \)-module of \( q \)-power order. We say that \((S, P, Q)\) is a \((p, q)\)-Frattini triple if the following conditions are satisfied:

1. \( \Omega_{p\cdot q}^*(S) = \emptyset \);
2. \( H^2(S, P) \neq 0 \);
3. \( H^2(S, Q) \neq 0 \);
4. \( C_P(s) = 0 \) for every nontrivial element \( s \) of \( S \) with \( q \)-power order;
5. \( C_Q(s) = 0 \) for every nontrivial element \( s \) of \( S \) with \( p \)-power order.

Notice that if \((S, P, Q)\) is a \((p, q)\)-Frattini triple then we may construct a Frattini extension \( G \) of \( P \times Q \) by \( S \).

**Lemma 4.** If \((S, P, Q)\) is a \((p, q)\)-Frattini triple and \( G \) is a Frattini extension \( G \) of \( P \times Q \) by \( S \), then \( \Omega_{p\cdot q}^{**}(G) = \emptyset \).

**Proof.** Let \( F = \text{Frat}(G) = P \times Q \). Assume that \( g \) is an element of \( G \) with \( |g| = p^a \cdot q^b \), for some positive integers \( a \) and \( b \). We can write \( g = xy \) where \( x \) has order \( p^a \), \( y \) has order \( q^b \) and \( x \) and \( y \) commute. By (1), \( S \cong G/F \) does not contain elements of order \( p \cdot q \), hence either \( x \in P \) or \( y \in Q \). It is not restrictive to assume \( x \in P \). Then \( y \) is a \( q \)-element of \( C_G(x) \), i.e. \( C_P(yF) \neq 0 \). By (4), this implies \( y \in Q \) and consequently \( g = xy \in F \).

**Proposition 5.** Let \( p \) and \( q \) be two odd primes. Assume that \( G \) is a finite group of minimal order with respect to the property that \( \Omega_{p\cdot q}(G) \neq \emptyset \) but \( \Omega_{p\cdot q}^*(G) = \emptyset \). Then there exists a \((p, q)\)-Frattini triple \((S, P, Q)\) such that \( G \) is a Frattini extension \( G \) of \( P \times Q \) by \( S \).

**Proof.** Let \( X = G/\text{Frat}(G) \). As in the proof of Theorem 2, \( \text{Frat}(G) = P \times Q \), where \( P \) and \( Q \) are, respectively, the Sylow \( p \)-subgroup and the Sylow \( q \)-subgroup of \( \text{Frat}(G) \). Moreover \( P \) and \( Q \) are the unique minimal normal subgroups of \( G \). Now let \( C = C_G(P) \). Clearly \( Q \leq C \). Let \( x \in C \) such that \( |x| \) is a nontrivial \( q \)-power and let \( 1 \neq y \in P \). The order of \( xy \) is divisible by \( p \cdot q \) so \( xy \in \text{Frat}(G) \) i.e.
x \in Q. Hence Q is a normal q-Sylow subgroup of C and therefore, by the Schur-Zassenhaus Theorem, Q has a complement, say K, in C and all these complements are conjugate in C. By the Frattini’s Argument, G = QG(K) = Frat(G)NG(K), hence G = N_G(K) and C = Q \times K. In particular K \leq C_G(Q), and, repeating the same argument as before, we deduce that P is a normal Sylow subgroup of K. Again by the Schur-Zassenhaus Theorem, P is complemented in K: so we have C = (P \times Q) \rtimes T = Frat(G) \rtimes T for a suitable subgroup T whose order is coprime with p \cdot q. By the Frattini’s Argument, G = Frat(G)NG(T) hence G = N_G(T) so T is normal in G. However all the minimal normal subgroups of G are contained in Frat(G) = P \times Q and Frat(G) \cap T = 1. So it must be T = 1, i.e. C_G(P) = Frat(G). With a similar argument we deduce that C_G(Q) = Frat(G) so P and Q are faithful nontrivial irreducible X-module, setting X = G/Frat(G).

By [7, 5.2.13 (iii)], Frat(G/P) = Frat(G)/P \cong Q, so G/P is a non-split extension of Q by X and consequently H^2(X, Q) \neq 0 and, similarly, H^2(X, P) \neq 0. Let Y be a non-trivial normal subgroup of X. Since C_P(Y) is X-invariant and P is an irreducible and faithful X-module, it must be C_P(Y) = 0, hence, by [2 Corollary 3.12 (2)], if p would not divide |S|, then H^2(X, P) = 0, so p (and similarly q) must divide |Y|. Let T be a Sylow p-subgroup of G. If there exists t \in T \setminus Frat commutes with a non-trivial element y in Q, then ty \in G \setminus Frat(G) and |ty| = |t||y| = p^a q for some a \in \mathbb{N}, against our assumption. But then T/P \cong T Frat(G)/Frat(G) is a fixed-point-free group of automorphisms of Q, and therefore, by [7, 10.5.5], T/Frat(G) is a cyclic group. Similarly, a Sylow q-subgroup of G/Frat(G) is cyclic. Now let S = soc X. Since p divides the order of every minimal normal subgroup of X, S = S_1 \times \cdots \times S_t, where, for each 1 \leq i \leq t, S_i is a simple group whose order is a multiple of p. However a Sylow p-subgroup of S is cyclic, hence we must have t = 1, i.e. S is a simple group. Moreover by [2 Lemma 5.2], S is nonabelian. Let now U be a Sylow p-subgroup of X. Since U is cyclic, if U \not\leq S, then U \cap S \leq Frat(U).

By a theorem of Tate (see for instance [3 p. 431]), S would be p-nilpotent, a contradiction. Hence U \leq S and therefore, by [2 Lemma 3.6], the restriction map H^2(X, P) \to H^2(S, P) is an injection. So in particular H^2(S, P) \neq 0 and similarly H^2(S, Q) \neq 0. This implies that there exist an irreducible S-submodule P^* of P and an irreducible S-submodule Q^* of Q such that H^2(S, P^*) \neq 0 and H^2(S, Q^*) \neq 0. We have that (S, P^*, Q^*) is a (p, q)-Frattini triple.

Let S_{p,q} be the set of the nonabelian simple groups S for which there exist P and Q such that (S, P, Q) is a (p, q)-Frattini triple. From Proposition [5] and its proof, the following can be easily deduced.

**Theorem 6.** Let p and q be two distinct odd primes. Assume that G is a finite group with \Omega_{p,q}(G) \neq \emptyset. If no composition factor of G is in S_{p,q}, then \Omega_{p,q}(G) \neq \emptyset.

It seems a difficult problem to determine whether S_{p,q} is non empty. The following remark can give some help in dealing with this question. Following [1], we say that a subset \{g_1, …, g_d\} of a finite group G invariably generates G if \{g_1^{x_1}, …, g_d^{x_d}\} generates G for every choice of x_i \in G.

**Proposition 7.** Assume that S \in S_{p,q}. If no proper subgroup of S is isomorphic to a group in S_{p,q}, then there exist x and y in S such that:

1. \langle x \rangle is a Sylow p-subgroup of S;
2. \langle y \rangle is a Sylow q-subgroup of S;
(3) \(\{x, y\}\) invariably generates \(S\).

Proof. Let \((S, P, Q)\) be a \((p, q)\)-Frattini triple and let \(X\) and \(Y\) be, respectively, a Sylow \(p\)-subgroup and a Sylow \(q\)-subgroup of \(S\). The subgroup \(X\) is a fixed-point-free group of automorphisms of \(Q\), and therefore, by [7, 10.5.5], \(X\) is a cyclic group. Similarly \(Y\) is a cyclic group. Let \(X = \langle x \rangle\) and \(Y = \langle y \rangle\). Assume, by contradiction, that \(\{x, y\}\) does not invariably generate \(S\). Then there exist \(s, t \in S\) such that \(H = \langle x^s, y^t \rangle\) is a proper subgroup of \(S\). Since \(H\) contains both a Sylow \(p\)-subgroup and a Sylow \(q\)-subgroup of \(G\), by [2, Lemma 3.6] the restriction maps \(H^2(S, P) \to H^2(H, P)\) and \(H^2(S, Q) \to H^2(H, Q)\) are injective. Hence \(H^2(H, P) \neq 0\) and \(H^2(H, Q) \neq 0\). Arguing as in the proof of Proposition 5 we deduce that \(T = \text{soc} H\) is a finite nonabelian simple group and \(T \in S_{p,q}\), against our assumption.

\(\square\)

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