Graph cohomologies and rational homotopy type of configuration spaces

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We compare the cohomology complex defined by Baranovsky and Sazdanović, that is the $E_1$ page of a spectral sequence converging to the homology of the configuration space depending on a graph, with the rational model for the configuration space given by Kriz and Totaro. In particular we generalize the rational model to any graph and to an algebra over any field. We show that, in the case of configuration spaces of point on a even dimensional manifold, the dual of the Baranovsky and Sazdanović’s complex is quasi equivalent to this generalized version of the Kriz’s model.

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1 Introduction

In 2006, in the study of the chromatic polynomial, Eastwood and Huggett define a generalized configuration space of points in a manifold, depending of a graph \[6\]. Let \(M\) is a manifold and \(\Gamma\) a graph with set of vertices \(V = \{v_1, \ldots, v_n\}\) and set of edges \(E(\Gamma)\). The configuration space of \(n\) points in \(M\), \(\text{Conf}(M, \Gamma)\) is defined as

\[
\text{Conf}(M, \Gamma) = \{(x_1, \ldots, x_n) \in M^n; x_i \neq x_j \text{ if } e_{i,j} \in E(\Gamma)\}.
\]

When \(\Gamma\) is a complete graph, \(\text{Conf}(M, \Gamma)\) coincide with the classical configuration space of \(n\) points in a manifold:

\[
\text{Conf}(M, n) = \{(x_1, \ldots, x_n) \in M^n; x_i \neq x_j \text{ for } i \neq j\}.
\]

In 2012 Baranovsky and Sazdanović \[2\] define a graph cohomology complex-inspired by the one defined by Helme-Guizon and Rong in \[8\]. We denote this complex by \(C_{BS}\). They prove the existence a spectral sequence with \(E_1\) page given \(C_{BS}\) converging to the homology of the Eastwood and Huggett’s configuration space \[2\]. The spectral sequence for the case where the graph is the complete graph was given in 1991 by Bendersky and Gitler \[3\].

The cohomology and the rational homotopy type of \(\text{Conf}(M, n)\) has been studied before. If the manifold is \(\mathbb{R}^d\), the results about the cohomology of \(\text{Conf}(\mathbb{R}^d, n)\) is due to Arnold \[1\] in the case \(r = 2\) and Cohen \[5\] for \(r \geq 3\). In 1994 Fulton and MacPherson \[7\] constructed a model for the rational homotopy type of \(\text{Conf}(X, n)\), where \(X\) is a non singular, compact, complex variety. This model depends on the cohomology ring \(H^*(X, \mathbb{Q})\), the orientation and the Chern classes. The same year, Kriz in \[12\] described a differential graded algebra \(E[n]\) that is a rational model for \(\text{Conf}(X, n)\) and that is independent from the Chern classes. \(E[n]\) was described in the same time in the work by Totaro \[17\] and it appeared to be isomorphic to the \(E_2\) page of the Larey spectral sequence of the inclusion \(\text{Conf}(X, n) \hookrightarrow X^n\).

The algebra \(E[n]\) will be here discussed in details in \textit{Theorem 3.1}. Later, Lambecht and Stanley studied the rational models for configurations spaces where \(X\) is a simply connected closed manifold. They showed that a simply connected closed manifold always admits a Poincaré duality model \(A\) \[14\].

In 2004 they described the case \(k = 2\), a configuration space of 2 points in a manifold \[13\] and they defined a model for its rational homotopy type. In 2008 they presented a potential model for the general case \[15\]. This commutative graded algebra is denoted by \(G_A(n)\). They conjectured that if \(X\) is a simply connected \(m\)-manifold, \(G_A(n)\) is a rational model for \(\text{Conf}(X, n)\). In 2019 Idrissi \[10\] proved the conjecture true for the real homotopy type and manifolds of dimension at least 4. Campos and Willwacher few years before \[4\] constructed a real model for the configuration space of points in a manifold.

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In this article, we compare the graph cohomology complex $C_{BS}(\Gamma)$ defined by Baranovsky and Sazdanović in [2] and described in Section 2, with the model for the rational homotopy type given by Kriz and Totaro denoted by $E[n]$. In the second and third sections we give the definition of these two cohomology complexes. In the following section, we describe Frobenius algebras and give some technical results that will be later used. In Section 5 we define the dual of the complex $(C_{BS}(\Gamma), \partial)$ that we will call $(C_{BS}(\Gamma)^*, \delta)$. The complex $(C_{BS}(\Gamma), \partial)$ is the $E_1$ page of a spectral sequence converging to the relative cohomology $H^*(M^n, Z_{\Gamma}, R)$, and if the space $M$ is a compact oriented manifold of dimension $m$ the cohomology is isomorphic to the homology $H_{mn-*}(\text{Conf}(M, \Gamma), R)$. On the other hand, the cohomology of the complex $E[n]$ is the cohomology of the configuration space. By Remark 2.8, if $M$ is a compact Kähler manifold and the coefficient ring is $R = \mathbb{Q}$, the spectral sequence degenerates at page $E_2$. In this case the two complexes are quasi equivalent. In the following sections we prove that there is in general a quasi equivalence between $C_{BS}(\Gamma)^*$ and a generalized version of $E[n]$, called $R(\Gamma, A)$. In the definition of this generalised complex, a ring $\Delta [G_{a,b}] / \sim$ is involved. This is the exterior algebra over the generator corresponding to the edges in the graph $\Gamma$ quotient by a relation, that we call the generalised Arnold relation. We denote this ring by $R(\Gamma)$. Section 6 describes it for a complete graph $K_n$, and in this case the relation is the usual Arnold relation. We will call it $R(K_n)$. In the following section we define $R(\Gamma, A)$ for an even dimensional manifold. It depends on a graph $\Gamma$ not necessarily complete, and a Frobenius algebra $A$ over any field. In the case of an even dimensional formal manifold and a complete graph $\Gamma$, $R(\Gamma, A)$ coincide with the CDGA that Idrissi [10] proves to be a real model for $\text{Conf}(M, n)$. Section 8 contains the main theorem of the chapter.

**Theorem.** Let $S \subseteq \Gamma$. The map

$$F : C_{BS}(\Gamma)^* \to R(\Gamma, A)$$

$$F(G_S \otimes x) = [G_S \otimes x]$$

is a quasi equivalence.

In [16] Thomas and Felix prove that the $mn$ suspension of the $E_2$ term of the Bendersky-Gitler spectral sequence is isomorphic to the $E_2$ term of the Cohen and Taylor spectral sequence of which the Kriz’s model is a special case. Our theorem presents an alternative proof and generalization of this result. In the last section we discuss the chain complex $C_{BS}(\Gamma)^*/I(\Gamma)$, where $I(\Gamma)$ is the ideal generated by the generalised Arnold relation and we show that it is isomorphic to $R(\Gamma, A)$. 

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2 Baranovsky-Sazdanović’s graph cohomology

This section presents the graph complex defined by Baranovsky and Sazdanović in [2]. Their definition is inspired by the work by Helme-Guizon and Rong [8], whose construction develops from the cohomology theory defined by M. Khovanov in [11]. There he associates to each link a family of cohomology groups whose Euler characteristic is the Jones polynomial of the link. Helme-Guizon’s and Rong’s graph cohomology expands the Khovanov’s definition associating to each graph, graded cohomology groups whose Euler characteristic is the chromatic polynomial of the graph. Baranovsky and Sazdanović in [2] prove that there is a spectral sequence that relates the graph cohomology defined by Helme-Guizon and Rong with the cohomology of configuration spaces, verifying a conjecture posed by Khovanov.

We give here the definition of the graph cohomology complex. We refer to [2] for the definitions and the notation with the exception of the notation of the complex that we will call $C_{BS}(\Gamma)$. Then, we state some results relating the complex to the homology of configuration spaces.

Let $A$ be a graded commutative algebra over a commutative ring $R$, and assume that $A$ is a projective $R$-module. Let $\Gamma$ be a finite graph, $V = V(\Gamma)$ be the set of vertices and $E(\Gamma)$ the set of edges. We choose an order on the vertices. This gives an orientation on every edge $\alpha$ in $E(\Gamma)$, if $\alpha$ connects the vertices $i$ and $j$ and $i \leq j$, $\alpha : i \to j$. For any subset $S$ of $E(\Gamma)$, we denote by $[\Gamma : S]$ the subgraph that has as vertices the same vertices of $\Gamma$ and as edges the edges in $S$, we denote by $l(S)$ the number of connected components of $[\Gamma : S]$.

Definition 2.1 ([2]). Let $\Lambda$ be an exterior algebra over $R$ with generators $e_\alpha$, $\alpha \in E(\Gamma)$, and $e_S$ be the exterior product of $e_\alpha$, $\alpha \in S$, ordered with the lexicographic order of the pair $(i,j)$ where $\alpha : i \to j$.

The bigraded complex, that we will here denote by $C_{BS}(\Gamma)$, is defined as

$$C_{BS}(\Gamma) = \Lambda \otimes A^\otimes n / e_\alpha \otimes (a[i] - a[j]),$$

the algebra $\Lambda \otimes A^\otimes n$ quotient by the relation $e_\alpha \otimes (a[i] - a[j])$, where $a \in A$, $\alpha : i \to j \in E(\Gamma)$ and $a[i]$ denotes the element $1 \otimes a \otimes 1 \otimes \cdots \in A^\otimes n$. The complex has a bigrading given by the sum of the grading of the elements $e_\alpha$ of bidegree $(0,1)$ and the elements $1 \otimes a_1 \otimes \cdots \otimes a_n$ with bidegree $(\sum_{i=1}^n \deg_A a_i, 0)$, so the degree of $e_S \otimes a_1 \otimes \cdots \otimes a_n$ in $C_{BS}(\Gamma)$ is $(\sum_{i=1}^n \deg_A a_i, |S|)$. 

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The differential of degree \((0, 1)\) is given by the exterior product

\[
\partial = \sum_{\alpha \in E(\Gamma)} e_\alpha
\]

**Remark 2.2.** The assumption of \(A\) be a projective \(R\)-module is used in the proof of the convergence of the spectral sequence. We refer to [2] for the definition of the spectral sequence and the proof.

**Remark 2.3.** The complex \(C_{BS}(\Gamma)\) is isomorphic to

\[
\bigoplus_{S \subseteq E(\Gamma)} e_S \otimes A^{l(S)}.
\]

That is for every \(n \in \mathbb{N}\)

\[
C^{n}_{BS}(\Gamma) = \bigoplus_{S \subseteq E(\Gamma), |S| = n} e_S \otimes A^{l(S)}.
\]

For \(S \subseteq E(\Gamma)\), each term \(a_i\) of the element \(e_S \otimes a_1 \otimes \cdots \otimes a_{l(S)} \in A^{l(S)}\), corresponds to a component in \([\Gamma : S]\). In the case \(S = \emptyset\), the components of \([\Gamma : \emptyset]\) are the vertices in \(\Gamma\). We can construct a map

\[
\phi : (\Lambda \otimes A^{\otimes n} / \sim) \to \bigoplus_{S \subseteq E(\Gamma)} e_S \otimes A^{l(S)}
\]

such that if \(\alpha\) is an edge in \(E(\Gamma)\), \(\alpha : i \to j\), then

\[
\phi(e_\alpha \otimes a_1 \otimes \cdots \otimes a_n) = e_\alpha \otimes a_1 \otimes \cdots \otimes (-1)^{l(\alpha)} a_i a_j \otimes \cdots \otimes a_{l(\alpha)}.
\]

The terms \(a_i\) and \(a_j\) are multiplied with a sign that is the Kozul sign given by the permutation in the tensor product that brings \(a_j\) close to \(a_i\), here \(l(\alpha) = n - 1\). The inverse is given by

\[
\phi^{-1}(e_\alpha \otimes a_1 \otimes \cdots \otimes (-1)^{l(\alpha)} a_i a_j \otimes \cdots \otimes a_{n-1}) = e_\alpha \otimes a_1 \otimes \cdots \otimes (-1)^{l(\alpha)} a_i a_j \otimes \cdots \otimes 1 \otimes \cdots \otimes a_{n-1}
\]

where 1 is in the position \(j\). Then, it is enough to notice that

\[
e_\alpha \otimes a_1 \otimes \cdots \otimes (-1)^{l(\alpha)} a_i a_j \otimes \cdots \otimes 1 \otimes \cdots \otimes a_{n-1}
\]

is in the same equivalence class of \(e_\alpha \otimes a_1 \otimes \cdots \otimes a_n\), since for \(a, b \in A\),

\[
a \otimes b = (a \otimes 1)(1 \otimes b) \sim (1 \otimes a)(1 \otimes b) = (1 \otimes ab)
\]

and

\[
(a \otimes 1)(1 \otimes b) \sim (a \otimes 1)(b \otimes 1) = (ab \otimes 1).
\]
Remark 2.4. The differential \( \partial : C^k_{BS} \to C^{k+1}_{BS} \) induced by \( \Phi \) on \( C^*_{BS}(\Gamma) = \bigoplus_{S \subseteq E(\Gamma)} e_S \otimes A^{l(S)} \) is
\[
\partial(e_S \otimes a_1 \otimes \cdots \otimes a_{l(S)}) = \sum_{\alpha \in E(\Gamma), l(S \cup \alpha) = l(S)} e_\alpha e_S \otimes a_1 \otimes \cdots \otimes a_{l(S)} + \sum_{\alpha \in E(\Gamma), l(S \cup \alpha) = l(S) - 1} (-1)^{\tau} e_\alpha e_S \otimes a_1 \otimes \cdots \otimes a_i \cdot a_j \otimes \cdots \otimes a_{l(S)}.
\]

The first term of the sum represents the case where the edge \( \alpha \) connects two vertices of the edges in \( S \) that are in the same component. Therefore, the number of components of \( [\Gamma : S] \) and \( [\Gamma : S \cup \alpha] \) are the same, so \( l(S) = l(S \cup \alpha) \). The second term of the sum refers to the case where the edge \( \alpha \) connects two different components, so \( l(S \cup \alpha) = l(S) - 1 \). Suppose that \( \alpha : i \to j \), and that \( a_k \) is the term corresponding to the component containing \( i \), and \( a_k \) to the component containing \( j \). Then \( \tau \) is the Kozul sign given by the permutation in the tensor product that moves \( a_k \) to the position immediate to the right of \( a_k \).

Example 2.5. Let \( \Gamma \) be \( K_3 \), the complete graph with 3 vertices. The order of the vertices induces an order on the edges given by the lexicographic order \( E(K_3) = \{e_{0,1}, e_{0,2}, e_{1,2}\} \) and an orientation on the edges.

![Figure 1: The graph \( K_3 \).](image)

The chain \( C_{BS}(\Gamma) \) in this case is the following
\[
A^\otimes 3 \to A^\otimes 2 \oplus A^\otimes 2 \oplus A^\otimes 2 \to A \oplus A \oplus A \to A
\]

The chain groups are given by
\[
C_{BS}(\Gamma)^n = \bigoplus_{|S|=n} A^{\otimes l(S)}
\]
where \( S \) is a subset of \( E(\Gamma) \), and \( l(S) \) is the number of components of \( [\Gamma : S] \). The picture shows the components of \( [\Gamma : S] \) with increasing cardinality of
$S$ and the differential that adds every time an edge in $[\Gamma : S]$, connecting its components. Let $a_0 \otimes a_1 \otimes a_2 \in C^0_{BS}$, then
\[
\partial(a_0 \otimes a_1 \otimes a_2) = e_{0,1} \otimes a_0 a_1 \otimes a_2 + e_{1,2} \otimes a_0 \otimes a_1 a_2 + (-1)^{|a_1||a_2|} e_{0,2} \otimes a_0 a_2 \otimes a_1
\]
Notice that the fact that $\partial^2 = 0$ is provided by the sign coming from the graded commutativity of $A$. For example,
\[
\partial e_{0,1}((-1)^{|a_1||a_2|} e_{0,2} \otimes a_0 a_2 \otimes a_1) = (-1)^{|a_1||a_2|} e_{0,1} e_{0,2} \otimes a_0 a_2 a_1 = e_{0,1} e_{0,2} \otimes a_0 a_1 a_2
\]
and
\[
\partial e_{0,2}(e_{0,1} \otimes a_0 a_1 \otimes a_2) = e_{0,2} e_{0,1} \otimes a_0 a_1 a_2 = -e_{0,1} e_{0,2} \otimes a_0 a_1 a_2.
\]
All the terms in $\partial^2$ given by adding an edge $e_i$ and then $e_j$ cancel with the terms given by adding the edges in opposite order, so $\partial^2 = 0$.

As anticipated, the complex $C_{BS}$ is related to the homology of configuration spaces depending on a graph, as defined by Eastwood and Hugget [6].

Let $M$ be simplicial complex and $\Gamma$ a graph as defined in the first section. Let $\alpha : i \to j$ be an edge in $E(\Gamma)$, $Z_\alpha$ be the diagonal of the Cartesian product $M^n$ corresponding to the edge $\alpha$,
\[
Z_\alpha = \{(m_1, \ldots, m_n) \in M^n; m_i = m_j\}
\]
and

\[ Z_\Gamma = \bigcup_{\alpha \in E(\Gamma)} Z_\alpha. \]

We define the graph configuration space of \( M \) dependent on \( \Gamma \) to be

\[ \text{Conf}(M, \Gamma) = M^n \setminus Z_\Gamma. \]

If \( M \) is a manifold the definition corresponds to the generalized configuration space depending on a graph studied by Eastwood and Huggett in [6].

Baranovsky and Sazdanović in [2] prove that \( C_{BS} \) is the \( E_1 \) page of a spectral sequence converging to the cohomology of such configuration space. This confirms a conjecture by Khovanov that there is a spectral sequence between the graph homology defined by L.Helme-Guizon and Y. Rong and the work by Eastwood and Huggett.

**Theorem 2.6** ([2]). Assume that the cohomology algebra \( A = H^*(M, R) \) is a projective \( R \)-module and that \( \Gamma \) has no loops or multiple edges. There exist a spectral sequence with \( E_1 \) term isomorphic to \( C_{BS} \) which converges to the relative cohomology \( H^*(M^n, Z_\Gamma; R) \).

**Remark 2.7** (Remark 4 [2]). When \( M \) is a compact \( R \)-oriented manifold of dimension \( m \), the relative cohomology groups \( H^*(M^n, Z_\Gamma; R) \) are isomorphic to the homology groups \( H_{nm-\ast}(\text{Conf}(M, \Gamma); R) \) by Lefschetz duality.

Moreover in the case where \( M \) is a Kähler manifold the following result holds.

**Remark 2.8** ([2]). If \( M \) is a compact Kähler manifold and the coefficient ring \( R \) is the rationals \( \mathbb{Q} \) the spectral sequence degenerates at page \( E_2 \).

### 3 The Kriz model

In this section we describe the rational model for the configuration space of points in a complex projective variety defined by Kriz in [12].

Let \( X \) be a smooth projective variety over \( \mathbb{C} \) and \( \text{Conf}(X, n) \) be the ordered configuration space of \( n \) points in a space \( X \),

\[ \text{Conf}(X, n) = X^n \setminus \bigcup_{i \neq j} \Delta_{i,j} \]

\( \Delta_{i,j} = \{(x_1, \ldots , x_n) \in X^n; x_i = x_j\} \). For \( a, b \in \{1, \ldots , n\}, a \neq b, \) let \( p^a_* : H^*(X) \to H^*(X^n) \) the pullback of the projection \( p_a : X^n \to X \) to the \( a \)-th coordinate and let \( p^a_{a,b} : H^*(X^2) \to H^*(X^n) \) the pullback of the projection \( p_{a,b} : X^n \to X^2 \). Let \( \Delta \in H^*(X^2) \) be the class of the diagonal.
Theorem 3.1. Let $X$ be a complex projective variety of complex dimension $m$. Then the space $\text{Conf}(X, n)$ has a model $E(n)$ that is isomorphic to 

$$H^*(X^n, \mathbb{Q})[G_{a,b}]$$

where $G_{a,b}$ are generators of degree $2m - 1$, $a, b \in \{1, \ldots, n\}$, $a \neq b$ modulo the relations

- $G_{a,b} = G_{b,a}$
- $p_a^*(x)G_{a,b} = p_b^*(x)G_{a,b}$, $x \in H^*(X)$
- $G_{a,b}G_{b,c} + G_{b,c}G_{c,a} + G_{c,a}G_{a,b} = 0$

The differential is given by $d(G_{a,b}) = p_{a,b}^*\Delta$.

Remark 3.2. The third relation $G_{a,b}G_{b,c} + G_{b,c}G_{c,a} + G_{c,a}G_{a,b} = 0$ is known in the literature as Arnold relation.

The definition of this graded algebra presents some similarities with the graded complex defined in Section 2: the structure of the exterior algebra with generators $G_{a,b}$ and the first two relations. However, the differential in $C_{BS}(\Gamma)$ ”adds edges” while the one in $E[n]$ ”removes edges”. Therefore, we would like to relate the dual of the graded complex $C_{BS}(\Gamma)$ with the DGA $E(n)$.

Moreover, the complex $C_{BS}(\Gamma)$ makes perfect sense in positive characteristic, so that we will also consider the following situation. Let $k$ be a ground ring, which could typically be $\mathbb{Z}$, $\mathbb{Q}$, or a prime field $\mathbb{F}_p$. Assume that $A = H^*(X, k)$ is an algebra over $k$ which is free as a $k$-module. We extend the definition given by Kriz to this case by defining $E[n]$ as $A[G_{a,b}]/\sim$, where the relations are given by exactly the same three formulas as in the theorem above. It will be convenient to extend the definition further to the case where $A$ is a Frobenius algebra. To do this, we have to give a definition of $\Delta$ in this case, we do that in the next section.

4 Structures of tensor powers of Frobenius algebras

We will consider a graded version of Frobenius algebras. To be precise about how we understand that term in this chapter:

Definition 4.1. A graded commutative Frobenius algebra $A$ over a commutative ground ring $k$ is a graded commutative ring, free and finite over $k = A^0$ as a module, together with a perfect, graded symmetrical pairing

$$\langle -, - \rangle : A \otimes A \to k,$$
such that 
\[ \langle ab, c \rangle = \langle a, bc \rangle. \]

**Remark 4.2.** Main example: Let \( X \) be a compact, connected \( k \)-orientated manifold such that each cohomology group \( H^i(X; k) \) is a free \( k \)-module. The cohomology ring \( H^*(X, k) \) is a graded commutative Frobenius algebra over \( k \). In this case, the pairing has degree \(-\dim(X)\).

If \( A \) is a graded Frobenius algebra, so is \( A \otimes A \). The multiplication is given by the usual tensor product of DGAs, involving the Koszul sign 
\[
(a \otimes b)(c \otimes d) = (-1)^{|b||c|}ac \otimes bd,
\]
and the pairing is given by 
\[
\langle a \otimes b, c \otimes d \rangle_2 = (-1)^{|b||c|}\langle a, c \rangle\langle b, d \rangle
\]
where \(|-|\) stands for the degree of the an element in the graded algebra.

We can construct the dual \( A^* = \text{hom}(A, k) \) and we have an isomorphism of vector spaces given by 
\[
k : A \cong A^*
\]
\[
a \mapsto k(a)(-) = \langle -, a \rangle
\]
\( A \) is equipped with a multiplication \( m : A \otimes A \to A \) and \( A^* \) a dual map given by \( m^* : A^* \to (A \otimes A)^* \cong A^* \otimes A^* \). Therefore we have a map \( \mu^* : A \to A \otimes A \) defined by composing the map \( k \) that gives the isomorphism with the dual:
\[
\mu^* : A \xrightarrow{k} A^* \xrightarrow{m^*} A^* \otimes A^* \xrightarrow{k^{-1} \otimes k^{-1}} A \otimes A.
\]
Alternatively, \( \mu^* \) is defined by that 
\[
\langle x \otimes y, \mu^*(a) \rangle_2 = \langle xy, a \rangle
\]
We see from this definition that \( \mu^* : A \to A \otimes A \) is an \( A \otimes A^{op} \) module map, since 
\[
\begin{align*}
\langle x \otimes y, (a \otimes 1)\mu^*(b)(1 \otimes c) \rangle_2 &= (-1)^{|a|+|x||y|+|b||c|}\langle (a \otimes 1)(x \otimes y)(1 \otimes c), \mu^*(b) \rangle_2 \\
&= (-1)^{|a|+|x||y|+|b||c|}\langle axyc, b \rangle \\
&= \langle xy, abc \rangle.
\end{align*}
\]

We define \( \Delta \in A \otimes A \) by the property 
\[
\langle a \otimes b, \Delta \rangle_2 = \langle ab, 1 \rangle.
\]

**Remark 4.3.** In the case \( A = H^*(M, k) \) as considered above, \( A \otimes A \cong H^*(M \times M, k) \), and \( \Delta \) corresponds under this isomorphism to the Poincaré dual of the homology class of the diagonal \( M \subset M \times M \).
Lemma 4.4. The class $\Delta$ satisfies that $(a \otimes b)\Delta = \mu^*(ab)$. In particular, $\mu^* : A \to A \otimes A$ is given by $\mu^*(a) = (a \otimes 1)\Delta$.

Proof. Because the paring $\langle -, - \rangle_2$ is perfect, it suffices to prove that for any $x, y \in A$, we have that $\langle x \otimes y, \mu^*(ab) \rangle_2 = \langle x \otimes y, (a \otimes b)\Delta \rangle_2$. We do the computation

\[
\langle x \otimes y, (a \otimes b)\Delta \rangle_2 = \langle (x \otimes y)(a \otimes b), \Delta \rangle_2
= \langle xyab, 1 \rangle
= \langle xy, ab \rangle
= \langle x \otimes y, \mu^*(ab) \rangle_2.
\]

Remark 4.5. $\Delta$ has the property that $(1 \otimes a)\Delta = (a \otimes 1)\Delta$, $a \in A$.

We introduce some notation. Let $S$ be a subset of the set of edges $E(\Gamma)$. Each $S$ determines a partition of the set of vertices so we have a map

\[\Phi : E(\Gamma) \to \mathcal{P}(\Gamma)\]

where $\mathcal{P}(\Gamma)$ is the set of all partitions of $V(\Gamma)$ and $\mathcal{E}(E)$ the set of subsets of $E(\Gamma)$. The sets $\mathcal{E}(\Gamma)$ form a partially ordered set by reversed inclusion, and $\mathcal{P}(\Gamma)$ form partially ordered sets by refinement. The map $\Phi$ is order preserving.

Note that the number $l(S)$ introduced in Definition 2 corresponds to the number of sets in the partition $P = \Phi(S)$, that is the cardinality of $\Phi(S)$.

There is a contravariant functor $\Psi$ from $\mathcal{P}(\Gamma)$ to graded algebras given by

\[\Psi(P) = A \otimes |P|\]

The dual of the canonical surjective map $\Psi(P \to V(\Gamma)) : A^\otimes n \to A^\otimes |P|$ is a canonical injective map

\[A^\otimes |P| \cong (A^\otimes |P|)^* \hookrightarrow (A^\otimes n)^* \cong A^\otimes n.\]

For any partition $P$ we consider the image $\Delta_P \in A^\otimes n$ of $1 \in A^\otimes |P|$. Using lemma 4.4 inductively, we see that multiplying with this element is dual to the multiplication map in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
A^\otimes |P| & \xrightarrow{\Delta_P} & A^\otimes n \\
\downarrow{\kappa^\otimes |P|} & & \downarrow{\kappa^\otimes n} \\
(A^\otimes |P|)^* & \xrightarrow{\Psi(P \to V(\Gamma))^*} & (A^\otimes n)^* \\
\end{array}
\]
This element is invariant under any permutation in $S_n$ preserving $P$. If $Q$ is a refinement of $P$, there is similarly a relative element $\Delta_{Q,P} \in A^{\otimes |Q|}$ such that the following diagram commutes:

\[
\begin{CD}
A^{\otimes |Q|} @>\Delta_{Q,P}>> A^{\otimes |P|} \\
@V\Delta_Q VV @VV\Delta_P V \\
A^{\otimes n} @. A^{\otimes n}
\end{CD}
\]

Each algebra $A^{\otimes |P|}$ is a module over $A^{\otimes n}$, and multiplication by $\Delta_{P,Q}$ is a map of $A^{\otimes n}$-modules.

5 The dual graded complex

Using the notation of the previous section, we can re-write $C_{BS}(\Gamma)$ as the graded chain complex

\[
C_{BS}(\Gamma) = \bigoplus_{S \subseteq E(\Gamma)} e_S \otimes A^{\otimes |S|} = \bigoplus_{P \in P} \bigoplus_{S, \phi(S) = P} e_S \otimes A^{\otimes |P|}).
\]

The differential is given by

\[
\partial = \sum_{e \in E(\Gamma)} \partial_e
\]

\[
\partial_e(e_S \otimes x) = (-1)^\tau e_{S \cup \{e\}} \otimes \Psi(\Phi(S \cup \{e\})).
\]

The sign $(-1)^\tau$ is determined by the number $\tau$ of edges in $S$ that precede $e$ in the chosen ordering of the edges.

We note that as a graded vector space

\[
C_{BS}(\Gamma) = \Lambda(e_\alpha) \otimes (A^{\otimes n})/\sim
\]

where $\sim$ indicates the relation $e_\alpha \otimes (a[i] - a[j])$, $a \in A$, $\alpha : i \to j \in E(\Gamma)$ and $a[i] \otimes a[j]$ denotes the element $1^{\otimes i-1} \otimes a \otimes 1^{\otimes n-i} \in A^{\otimes n}$, described in the first section in Definition 2.1. This relation corresponds to the second relation of the definition of the DGA defined by Kriz, since $p_\alpha(x) = 1 \otimes \cdots \otimes x \otimes \cdots 1 \in A^{\otimes n}$ where $x$ is the $a$-th component of the tensor product.

We want to describe the dual graded chain complex

\[
C_{BS}^*(\Gamma) = \left( \bigoplus_{S \subseteq E(\Gamma)} e_S \otimes A^{\otimes |S|} \right)^*.
\]

We will denote by $G_\alpha$ the basis for the dual exterior algebra over $e_\alpha$, for the edge $\alpha : i \to j$. We can so write the dual graded chain complex

\[
C_{BS}^*(\Gamma) = \left( \bigoplus_{S \subseteq E(\Gamma)} e_S \otimes A^{\otimes |S|} \right)^* = \bigoplus_{S \subseteq E(\Gamma)} (e_S)^* \otimes (A^*)^{\otimes |S|} = \bigoplus_{S \subseteq E(\Gamma)} G_S \otimes A^{\otimes |S|},
\]
where $G_S$ denotes the product of all the $G_{ij}$ where $\alpha : i \to j$ is an edge in $S$.

The dual of the differential $\partial$, that we denote by $\delta$, acts by removing edges in the graph and therefore increasing the number of components. Let $G_S$ be the product of all the $G_{ij}$ where $\alpha : i \to j$ is an edge in $S$,

$$\delta(G_S) = \sum_{i<j} (-1)^\nu \delta_{i,j}(G_S) = \sum_{i<j} (-1)^\nu G_{S \setminus \alpha_{i,j}}$$

where $\nu$ is the number corresponding to the position of the edge $\alpha_{i,j}$ in the ascending order. We have

$$\delta(G_S \otimes a_1 \otimes \cdots \otimes a_l(S)) = \sum_{i<j} (-1)^\nu \delta_{i,j}(G_S \otimes a_1 \otimes \cdots \otimes a_l(S))$$

and

$$\delta_{i,j}(G_S \otimes a_1 \otimes \cdots \otimes a_l(S)) = (-1)^\tau G_{S \setminus \alpha} \otimes (\Delta_{S, S \setminus \alpha} \cdot a_1 \otimes \cdots \otimes a_l(S))$$

in the case $\alpha : i \to j$ is an edge belonging to $S$ and $l(S \setminus \alpha) = l(S) - 1$, and $\tau$ is the Kozul sign given by moving the factor in $\mu(a)$ in the $j$-th position. While

$$\delta_{i,j}(G_S \otimes a_1 \otimes \cdots \otimes a_l(S)) = G_{S \setminus \alpha} \otimes a_1 \otimes \cdots \otimes a_l(S)$$

in the case $\alpha : i \to j$ is an edge belonging to $S$ and $l(S \setminus \alpha) = l(S)$. Finally,

$$\delta_{i,j}(G_S \otimes a_1 \otimes \cdots \otimes a_l(S)) = 0$$

if $\alpha$ does not belong to $S$.

**Remark 5.1.** We discuss here the grading of the dual complex. Let $S \subseteq E(\Gamma)$. We assign to an element $G_S \otimes a_1 \otimes \cdots \otimes a_l(S)$ in $C_{BS}(\Gamma)^*$ the grading

$$(m - 1)r_{ext} - r_{int} + \sum_i |a_i|,$$

where $m$ is the dimension of the manifold. $r_{ext}$ is the number of external edges, that are the edges that, if removed, disconnect components, $r_{int}$ the number of internal edges, that are the edges that do not disconnect components if removed and $|a_i|$ is the degree of the element $a_i$ in $A$. The differential has degree $1$ since

$$\delta(G_S \otimes a_1 \otimes \cdots \otimes a_l(S)) = \begin{cases} 0 & \text{if } \alpha \notin S \\ \sum_{\alpha \in E(\Gamma)} G_{S \setminus \alpha} \otimes \Delta_{S,S \setminus \alpha} \cdot a_1 \otimes \cdots \otimes a_l(S), & \text{if } \alpha \text{ disconnects } S \\ \sum_{\alpha \in E(\Gamma)} G_{S \setminus \alpha} \otimes a_1 \otimes \cdots \otimes a_l(S), & \text{if } \alpha \text{ non disconnects } S \end{cases}$$

and $\Delta_{S,S \setminus \alpha}$ has degree $m$. If $S$ is a forest, the grading of $C_{BS}(\Gamma)^*$ and the DGA $R(\Gamma, A)$ that we define later, coincide.
6 The ring $R(K_n)$

In this section we want to study the ring defined by the exterior algebra $\Lambda[G_{a,b}]$, where $G_{a,b}$ are edges in a complete graph with $n$ vertices $K_n$, quotient by the relations introduced by Kriz in Theorem 3.1

Let $K_n$ be a complete graph with $n$ vertices and $\Lambda[G_{a,b}]$ be the exterior algebra with generators $G_{a,b}$ given corresponding to the edges in $K_n$. We define

$$R(K_n) = \Lambda[G_{a,b}]/\sim$$

where $\sim$ is the Arnold relation $G_{i,j}G_{j,k} + G_{j,k}G_{k,i} + G_{k,i}G_{i,j}$. We call $I(K_n)$ the ideal generated by this relation, in order to simplify the notation this will be denoted also by $I$.

The following lemmas characterize the ideal $I$. We denote by $G_\Gamma$ the product of the generators corresponding to edges in $\Gamma$.

Lemma 6.1. Denote by $v = (v_1, \ldots, v_k)$, $k \geq 3$ a set of vertices in $K_n$ and denote by $s(v)$ the product $s(v) = G_{v_1,v_2} \cdot \ldots \cdot G_{v_{i-1},v_i} \cdot G_{v_i,v_{i+1}} \cdot \ldots \cdot G_{v_{k-2},v_{k-1}} \cdot G_{v_{k-1},v_{k}}$ where $G_{v_i,v_{i+1}}$ is the generator in the exterior algebra corresponding to the edge $\alpha : v_i \rightarrow v_{i+1}$, so $s(v)$ is the product of the generators corresponding to edges of a cycle of length $k$. Let $J$ be the ideal generated by the elements $s(v)$ with $k \geq 3$. Let $I$ be the ideal generated by $\delta(s(v))$ for every $v$ with $k \geq 3$. Then $J$ is contained in $I$ and $I$ is generated by $\delta(s(v_1,v_2,v_3)) = G_{v_1,v_2}G_{v_2,v_3} + G_{v_2,v_3}G_{v_3,v_1} + G_{v_3,v_1}G_{v_1,v_2}$.

Proof. We first show that $J$ is contained in $I$.

$$G_{v_1,v_2} \cdot \delta(s(v)) = G_{v_1,v_2} \cdot (G_{v_2,v_3} \cdot \ldots \cdot G_{v_{i-1},v_i} \cdot G_{v_{i+1},v_{i+2}} \cdot \ldots \cdot G_{v_k,v_1})$$

$$- G_{v_1,v_2} \cdot (G_{v_1,v_2} \cdot G_{v_3,v_4} \cdot \ldots \cdot G_{v_{i-1},v_i} \cdot G_{v_{i+1},v_{i+2}} \cdot \ldots \cdot G_{v_k,v_1})$$

$$+ \ldots$$

$$= G_{v_1,v_2} \cdot G_{v_2,v_3} \cdot \ldots \cdot G_{v_1,v_{i+1}} \cdot G_{v_{i+1},v_{i+2}} \ldots \cdot G_{v_k,v_1} = s(v).$$

Now we want to show that $I$ is generated by $\delta(s(v_1,v_2,v_3)) = G_{v_1,v_2}G_{v_2,v_3} + G_{v_2,v_3}G_{v_3,v_1} + G_{v_3,v_1}G_{v_1,v_2}$. Let $I_k$ the ideal generated by $\delta(s(v))$ where $v = (v_1, \ldots, v_l)$, $l \leq k$. $I = \cup I_k$, we want to show by induction that $I_k = I_3$, $k \geq 3$ where $I_3$ is generated by $\delta(s(v_1, v_2, v_3))$. It is obviously true for $k = 3$. Suppose it true for $k - 1$, we want to show that for every $s(v_1, \ldots, v_k)$, $\delta(s(v_1, \ldots, v_k)) \in I_{k-1}$. Consider $X = \delta(s(v_k, v_1, v_2))\delta(G_{v_2,v_3} \cdot \ldots \cdot G_{v_{k-1},v_k}) \in I_3 = I_{k-1}$, we can expand the expression

$$X = (G_{v_1,v_2} \cdot G_{v_2,v_k} - G_{v_k,v_1} \cdot G_{v_2,v_k} + G_{v_k,v_1} \cdot G_{v_1,v_2}) \cdot \left( \sum_{2 \leq j \leq k-1} (-1)^j G_{v_2,v_3} \cdot \ldots \cdot G_{v_{j-1},v_{j+1}} \cdot \ldots \cdot G_{v_{k-1},v_k} \right)$$

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\[ = (G_{v_1,v_2}) \cdot \left( \sum_{2 \leq j \leq k-1} (-1)^j G_{v_k,v_2} \cdot G_{v_2,v_3} \cdot \ldots \cdot \hat{G}_{v_j,v_{j+1}} \cdot \ldots \cdot G_{v_{k-1},v_k} \right) \]
\[ - (G_{v_k,v_1}) \cdot \left( \sum_{2 \leq j \leq k-1} (-1)^j G_{v_k,v_2} \cdot G_{v_2,v_3} \cdot \ldots \cdot \hat{G}_{v_j,v_{j+1}} \cdot \ldots \cdot G_{v_{k-1},v_k} \right) \]
\[ + \sum_{2 \leq j \leq k-1} (-1)^j G_{v_k,v_1} \cdot G_{v_1,v_2} \cdot G_{v_2,v_3} \cdot \ldots \cdot \hat{G}_{v_j,v_{j+1}} \cdot \ldots \cdot G_{v_{k-1},v_k} \]
\[ = (G_{v_1,v_2}) \cdot (-\delta(s(v_k,v_2,\ldots,v_{k-1})) + G_{v_2,v_3} \cdot \ldots \cdot G_{v_j,v_{j+1}} \cdot \ldots \cdot G_{v_{k-1},v_k}) \]
\[ - (G_{v_k,v_1}) \cdot (-\delta(s(v_k,v_2,\ldots,v_{k-1})) + G_{v_2,v_3} \cdot \ldots \cdot G_{v_j,v_{j+1}} \cdot \ldots \cdot G_{v_{k-1},v_k}) \]
\[ + \delta(s(v_k,v_1,\ldots,v_{k-1})) - G_{v_1,v_2} \cdot G_{v_2,v_3} \cdot \ldots \cdot G_{v_j,v_{j+1}} + G_{v_1,v_1} \cdot G_{v_2,v_3} \cdot \ldots \cdot G_{v_j,v_{j+1}} \]
\[ = -G_{v_1,v_2} \cdot \delta(s(v_k,v_2,\ldots,v_{k-1})) + G_{v_1,v_1} \delta(s(v_k,v_2,\ldots,v_{k-1})) + \delta(s(v_k,v_1,\ldots,v_{k-1})). \]

Note that the third equality comes from the fact that
\[ \delta(s(v_k,v_2,\ldots,v_{k-1})) = \sum_{2 \leq j \leq k} (-1)^j G_{v_k,v_2} \cdot \ldots \cdot G_{v_j,v_{j+1}} \cdot \ldots \cdot G_{v_{k-1},v_k} \]
that equals the first term in the sum in the expression apart from the missing term \( G_{v_2,v_3} \cdot \ldots \cdot G_{v_j,v_{j+1}} \cdot \ldots \cdot G_{v_{k-1},v_k} \).

Now, \( \delta(s(v_k,v_1,\ldots,v_{k-1})) = (-1)^{k-1} \delta(s(v_1,v_2,\ldots,v_k)), \) so we can write
\[ \delta(s(v_1,v_2,\ldots,v_k)) = -G_{v_1,v_2} \cdot \delta(s(v_2,v_3,\ldots,v_k)) + G_{v_1,v_1} \delta(s(v_2,v_3,\ldots,v_k)) - X. \]

We can conclude that \( \delta(s(v_1,v_2,\ldots,v_k)) \in I_{k-1} = I_3. \) This end the proof by induction, so \( I_k = I_3 \) for all \( k \) and so \( I = I_3. \)

\[ \textbf{Corollary 6.2. If the graph } K_n \text{ contains a cycle, then } G_{\Gamma} \in I. \]

We conclude that every element in \( R_n(\Gamma) \) where \( \Gamma = K_n \) can be written as a linear combination of the classes \( G_{\Gamma'} \) where \( \Gamma' \) are graphs which do not contain any cycles. Such a graph is a disjoint union of trees, that is, it is a forest. However, these classes are not linearly independent in \( R_n(\Gamma) \). Let \( F \) denote a forest in \( \Gamma \). We can rewrite the complex \( R_n(\Gamma) \) as
\[ R_n(\Gamma) = \Lambda[G_{a,b}] / \sim = \bigoplus_{T \subseteq F} \mathbb{Z}[T] / \sim \]
where \( \mathbb{Z}[T] \) is the free group generated by the trees. We have from a result by Vassilev \[18\] that \( \mathbb{Z}[T] = \mathbb{Z}^{(n-1)!} \).
The generalised DGA

We want to extend the definition of \( E[n] \) to a graded algebra dependent by any graph \( \Gamma \) and defined on a Frobenius algebra over any ring. In order to do so, we need to modify the ideal \( I(K_n) \) and introduce the following definition:

**Definition 7.1.** Let \( \Gamma \) be a graph, and \((v_1, \ldots, v_k) \) \( k \geq 3 \) a set of vertices in \( \Gamma \). We call a cycle \( w \) a subset of the set of edges of \( \Gamma \) of the form \( \{(v_1, v_2), \ldots, (v_i, v_{i+1}), \ldots, (v_k, v_1)\} \). Let \( \Lambda[G_{a,b}] \) be the exterior algebra with generators \( G_{a,b} \) corresponding to the edges \( (a, b) \) in \( \Gamma \). We denote by \( G_w \) the product \( G_{v_1, v_2} \cdot \ldots \cdot G_{v_i, v_{i+1}} \cdot \ldots \cdot G_{v_k, v_1} \). We define

\[
R(\Gamma) = \Lambda[G_{a,b}] / \sim
\]

where \( \sim \) are the relations

- \( G_{a,b} = G_{b,a} \)
- \( \delta(G_w) = \sum_i (-1)^i G_{v_1, v_2} \cdot \ldots \cdot G_{v_i, v_{i+1}} \cdot \ldots \cdot G_{v_k, v_1} = 0 \)

for all the cycles \( w \) in \( \Gamma \). We call generalised Arnold relations the second set of relations and \( I(\Gamma) \) the ideal generated by them.

**Remark 7.2.** Note that by the results in the previous section, if \( \Gamma = K_n \) then \( I(\Gamma) = I(K_n) \).

**Lemma 7.3.** If \( \Gamma \) contains a cycle then \( G_v \in I(\Gamma) \).

**Proof.** Let \( \Gamma \) be the cycle with edges \( \{(v_1, v_2), \ldots, (v_i, v_{i+1}), \ldots, (v_k, v_1)\} \)

\[
G_{v_1, v_2} \cdot d(G_v) = G_{v_1, v_2} \cdot (G_{v_2, v_3} \cdot \ldots \cdot G_{v_i, v_{i+1}} \cdot G_{v_{i+1, v_{i+2}} \cdot \ldots \cdot G_{v_k, v_1}}) - G_{v_1, v_2} \cdot (G_{v_1, v_2} \cdot G_{v_3, v_4} \cdot \ldots \cdot G_{v_i, v_{i+1}} \cdot G_{v_{i+1, v_{i+2}} \cdot \ldots \cdot G_{v_k, v_1}}) + \ldots
\]

\[
= G_{v_1, v_2} \cdot G_{v_2, v_3} \cdot \ldots \cdot G_{v_i, v_{i+1}} \cdot G_{v_{i+1, v_{i+2}} \cdot \ldots \cdot G_{v_k, v_1}} = G_v.
\]

**Corollary 7.4.** If \( \Gamma \) contains a cycle then \( G_\Gamma \in I(\Gamma) \).

We can conclude that the elements in \( R_n(\Gamma) \) are linear combinations of forests. We now define the generalized complex.

**Definition 7.5.** Let \( M \) a compact, connected \( k \)-orientated manifold of even dimension \( m \), \( A = H^*(M, k) \) be a Frobenius algebra, where \( k \) is the ground
ring. Let $\Gamma$ be a graph with $n$ edges and $k$ cycles $w_j, j = 0, \ldots, k$. We define the differential graded algebra

$$R(\Gamma, A) = \Lambda[G_{a,b}] \otimes A^\otimes n / \sim$$

where $G_{a,b}$ are generators of degree $m - 1$, $(a, b) \in E(\Gamma)$, and $\sim$ are the relations

- $G_{a,b} = G_{b,a}$
- $p^*_a(x)G_{a,b} = p^*_b(x)G_{a,b}, x \in H^*(X)$
- $\delta(G_{w_j}) = \sum (-1)^i G_{v_{1,j},v_{2,j}} \cdot \ldots \cdot G_{v_{i,j},v_{i+1,j}} \cdot \ldots \cdot G_{v_{h,j},v_{1,j}} = 0, \text{ for all } j = 0, \ldots, k$

The differential is given by

$$d(G_{a,b}) = p^*_{a,b} \Delta$$

here $\Delta$ is the class of the diagonal as described in Section 4 and $p^*_{a,b}$ the pull back of the projection defined in Section 3.

8 A quasi equivalence

Let $M$ be an even dimensional, compact, connected $k$-orientated manifold of dimension $m$, $A = H^*(M, k)$ a graded commutative Frobenius algebra and $\Gamma$ be any graph. Let $(C^*_{BS}(\Gamma), \delta)$ be the dual complex defined in Section 5 as

$$C^*_{BS}(\Gamma) = \bigoplus_{S \subseteq E(\Gamma)} G_S \otimes A^\otimes l(S).$$

We consider the generalized complex given in Definition 7.5

$$R(\Gamma, A) = \Lambda[G_{a,b}] \otimes A^\otimes n / \sim$$

where $\sim$ are the relations introduced in Definition 7.5 and $\Lambda[G_{a,b}]$ is the exterior algebra with generators given by the edges in $\Gamma$. We want to show that there is a quasi equivalence between $(C^*_{BS}(\Gamma), \delta)$ and $(R(\Gamma, A), d)$.

Remark 8.1. The differential in $C^*_{BS}(\Gamma)$ can be written as

$$\delta = \delta_{\text{int}} + \delta_{\text{ext}}$$

where $\delta_{\text{int}}$ is the differential that removes internal edges, meaning edges such that if removed they don’t disconnect components, and $\delta_{\text{ext}}$ is the differential that removes external edges, that are the edges that if removed they disconnect components. By Lemma 7.3 we have that $R(\Gamma)$ is given by linear combination of forests and therefore $d = \delta_{\text{ext}}$. 

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**Definition 8.2.** Let $S \subseteq E(\Gamma)$. We define the following map of graded groups:

$$F : C_{BS}^*(\Gamma) \to R(\Gamma, A)$$

$$F(G_S \otimes x) = [G_S \otimes x]_{\sim}.$$ 

In order to simplify the notation we will write $G_S \otimes x$ instead of $[G_S \otimes x]_{\sim}$.

**Lemma 8.3.** The map $F$ is compatible with the differential.

**Proof.** Let $G_{\Gamma'} \otimes x \in C_{BS}^*(\Gamma)$, where $\Gamma'$ is a subgraph of $\Gamma$ and we denote by $G_{\Gamma'}$ the product of the generators corresponding to the edges in a graph $\Gamma'$. We want to check the commutativity of the following diagram.

$$\begin{array}{ccc}
C_{BS}^*(\Gamma) & \xrightarrow{F} & R(\Gamma, A) \\
\delta \downarrow & & \downarrow d \\
C_{BS}^*(\Gamma) & \xrightarrow{F} & R(\Gamma, A)
\end{array}$$

We consider first the case where $\Gamma'$ does not contain any cycle. If $\Gamma'$ does not contain any cycle,

$$d \circ F(G_{\Gamma'} \otimes x) = d(G_{\Gamma'} \otimes x).$$

On the other hand by Remark 8.1

$$F \circ \delta(G_{\Gamma'} \otimes x) = F \circ \delta_{ext}(G_{\Gamma'} \otimes x) = d(G_{\Gamma'} \otimes x).$$

Now, suppose that $\Gamma'$ contains a cycle, that we denote by $S$, then

$$d \circ F(G_{\Gamma'} \otimes x) = d(0) = 0,$$

by definition of $F$. To prove the commutativity of the diagram we want to show that $F \circ \delta(G_{\Gamma'} \otimes x) = 0$. By the previous remark,

$$\delta(G_{\Gamma'} \otimes x) = \delta_{int} + \delta_{ext}(G_{\Gamma'} \otimes x),$$

the first summand is given by $\delta_{int}(G_{\Gamma'} \otimes x) = \delta(G_{S \cup S'})G_{\Gamma' / S \cup S'} \otimes x$, where $S'$ is the graph given by the internal edges in $\Gamma'$ that are not in $S$. The differential $\delta_{int}$ doesn’t change the number of components and so it doesn’t act on $x \in A^{\otimes l(\Gamma)}$. Now,

$$\delta(G_{S \cup S'}) = \delta(G_{S})G_{S'} + G_{S}\delta(G_{S'}) \in I(\Gamma)$$

by Lemma 7.3 because $G_{S}$ and $\delta(G_{S})$ belongs to $I(\Gamma)$, so $F \circ \delta_{int}(G_{\Gamma'} \otimes x) = 0$. The second summand is

$$\delta_{ext}(G_{\Gamma'} \otimes x) = \delta_{ext}(G_{\Gamma' / S \cup S'})G_SG_{S'} \otimes x'.$$
where $x' \in A^{\otimes l(\Gamma')}$. The term belongs to the ideal $I(\Gamma)$ since $G_S \in I(\Gamma)$ and so
\[
F \circ \delta_{\text{ext}}(G_{\Gamma'} \otimes x) = 0.
\]

**Theorem 8.4.** The map $F$ is a quasi equivalence.

**Proof.** We want to introduce two filtrations on $C_{BS}^*(\Gamma)$ and on $R(A, \Gamma)$, and prove that $F$ is compatible with them and that it induces a quasi equivalence on the filtration quotients.

Let $\Gamma$ be a graph with $n$ vertices, $S$ be a subset of the set of edges $E(\Gamma)$. $S$ determines a partition of the set of vertices, so we have a map $\Phi : E \to \mathcal{P}$ where $\mathcal{P}$ is the set of all partitions of $V(\Gamma)$ and $E$ the set of subsets of $E(\Gamma)$. As noted in Section 5 we can rewrite the complex $C_{BS}^*(\Gamma)$ as
\[
C_{BS}^*(\Gamma) = \bigoplus_{P \in \mathcal{P}} \bigoplus_{S, \phi(S) = P} G_S \otimes A^{\otimes |P|}
\]
where $|P|$ is the number of classes in the partition $P$.

There is a filtration of $C_{BS}^*(\Gamma)(A)^*$ given by
\[
F_k = \bigoplus_{P \in \mathcal{P}, |P| \geq k} \bigoplus_{S, \phi(S) = P} G_S \otimes A^{\otimes |P|}.
\]

$F_k$ is a subcomplex of $C_{BS}^*(\Gamma)^*$ since the differential $\delta$ acts by removing edges and so increasing the number of components. So
\[
F_n \subseteq \cdots \subseteq F_k \subseteq F_{k-1} \subseteq \cdots \subseteq C_{BS}^*(\Gamma)^*
\]
and $|V(\Gamma)| = n$.

Similarly we have a filtration on $R(A, \Gamma)$ in terms of partitions. Since
\[
R(\Gamma) = \bigoplus_{P \in \mathcal{P}} \bigoplus_{S, \phi(S) = P} G_S \big/ \bigoplus_{S, \phi(S) = P} G_S \cap I(\Gamma)
\]
we can define
\[
F'_k = \bigoplus_{P \in \mathcal{P}, |P| \geq k} \bigoplus_{S, \phi(S) = P} G_S \big/ \bigoplus_{S, \phi(S) = P} G_S \cap I(\Gamma) \otimes A^{\otimes |P|}
\]
as before $F'_k$ is a subcomplex of $R(A, \Gamma)$ since the differential $d$ acts by removing edges and so increasing the number of components. So
\[
F'_n \subseteq \cdots \subseteq F'_k \subseteq F'_{k-1} \subseteq \cdots \subseteq R(A, \Gamma)
\]
and $|V(\Gamma)| = n$. We want to show that $F$ is compatible with the filtrations, that is $F(F_k) \subseteq F'_k$. This is clearly true since $F(G_{\Gamma} \otimes x) = G_{\Gamma} \otimes x$ is $\Gamma$ if a forest and 0 otherwise.
There are two short exact sequences given by inclusion and quotient map
\[ \mathcal{F}_{k-1} \rightarrow \mathcal{F}_k \rightarrow \mathcal{F}_{k-1}/\mathcal{F}_k \]
and
\[ \mathcal{F}'_{k-1} \rightarrow \mathcal{F}'_k \rightarrow \mathcal{F}'_{k-1}/\mathcal{F}'_k. \]
The last step of the proof consists in showing that for every \( k \),
\[ F : \mathcal{F}_{k-1}/\mathcal{F}_k \rightarrow \mathcal{F}'_{k-1}/\mathcal{F}'_k \]
is a quasi equivalence and then use the long exact sequences in homology induced from the short exact sequences to prove the result. Now,
\[ \mathcal{F}_{k-1}/\mathcal{F}_k = \bigoplus_{P \in \mathcal{P}, |P| = k-1} \left( \bigoplus_{S, \phi(S) = P} G_S \otimes A^{\otimes |P|} \right) \]
is determined by the partitions with exactly \( k-1 \) classes.

Let \( S \) be the maximal subset of \( E(\Gamma) \) with respect to the inclusion that determines a partition of the set of vertices \( P = \{P_1, \ldots, P_l\} \) and let \( \Gamma_i^S, 0 \leq i \leq l \), be the connected subgraph of \( S \) corresponding to the element \( P_i \) in the partition. By Lemma 8.5 we can rewrite \( \mathcal{F}_{k-1}/\mathcal{F}_k \) as
\[ \mathcal{F}_{k-1}/\mathcal{F}_k = \bigoplus_{P \in \mathcal{P}, |P| = k-1} \left( \bigotimes_{1 \leq i \leq k-1} C_{\text{con}}(\Gamma_i^S) \right) \otimes A^{\otimes |P|}. \]
We define the chain complex \( C_{\text{con}}(\Gamma) \) for connected graphs with \( h \) vertices, and for every \( 0 \leq i \leq h \), \( C_{\text{con}}(\Gamma)^i \) is the free abelian group generated by all connected subgraphs of \( \Gamma \) with \( i \) edges. Let \( S \) be a connected subgraph of \( \Gamma \), the differential is given by
\[ d_{\text{con}}(S) = \sum_{e \in E(\Gamma)} (-1)\nu(S \setminus e) \]
where \( \nu \) is the position of edge \( e \in E(\Gamma) \) in ascending order. If \( S \setminus e \) is not connected \( d_{\text{con}}(S) = 0. \)

\[ \mathcal{F}'_{k-1}/\mathcal{F}'_k = \bigoplus_{P \in \mathcal{P}', |P| = k-1} \left( \bigotimes_{1 \leq i \leq k-1} \bigoplus_{j} C_{\text{con}}(\Gamma_{i,j}) \right) \otimes A^{\otimes |P|} \]
where now \( \Gamma_{i,j} \subset \Gamma_i \) are spanning trees. In particular, we have that \( C_{\text{con}}(\Gamma') = C_{\text{con}}(\Gamma)/I(\Gamma) \). By the K"unneth formula, the problem reduces to checking if
\[ q : C_{\text{con}}(\Gamma) \rightarrow C_{\text{con}}(\Gamma)/I(\Gamma) \]
is a quasi equivalence. Here by \( I(\Gamma) \) we mean the subgroup given by \( \alpha(I(\Gamma)) \) and \( \alpha \) is the isomorphism defined in Lemma 8.8. By Lemma 8.8 the homology of \( C_{\text{con}}(\Gamma) \) is concentrated in degree \( n-1 \). Now, \( C_{\text{con}}(\Gamma)/I(\Gamma) \) is a
complex concentrated in dimension \(n - 1\) by Remark 8.6 that is the chain group generated by the trees and
\[
(C_{\text{con}}(\Gamma)/I(\Gamma))^{n-1} = C_{\text{con}}(\Gamma)^{n-1}/d_{\text{con}}(C_{\text{con}}(\Gamma)^n)
\]
because by Lemma 8.7 \(d_{\text{con}}(C_{\text{con}}(\Gamma)^n) = I(\Gamma)\). Since \(C_{\text{con}}^{m-2} = 0\), we have
\[
H_{n-1}(C_{\text{con}}(\Gamma)) = C_{\text{con}}(\Gamma)^{n-1}/I(\Gamma) = H_{n-1}(C_{\text{con}}(\Gamma)/I(\Gamma))
\]
and
\[
H_i(C_{\text{con}}(\Gamma)) = H_i(C_{\text{con}}(\Gamma)/I(\Gamma)) = 0
\]
for \(i \neq n - 1\). Then
\[
H_s(C_{\text{con}}(\Gamma)) = H_s(C_{\text{con}}(\Gamma)/I(\Gamma)),
\]
so \(q : C_{\text{con}}(\Gamma) \to C_{\text{con}}(\Gamma)/I(\Gamma)\) is a quasi equivalence.

Finally we consider the long exact sequences in homology
\[
\begin{array}{ccccccc}
H_{i+1}(\mathcal{F}_{k-1}/\mathcal{F}_k) & \longrightarrow & H_i(\mathcal{F}_k) & \longrightarrow & H_i(\mathcal{F}_{k-1}) & \longrightarrow & H_i(\mathcal{F}_{k-1}/\mathcal{F}_k) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
H_{i+1}(\mathcal{F}'_{k-1}/\mathcal{F}'_k) & \longrightarrow & H_i(\mathcal{F}'_k) & \longrightarrow & H_i(\mathcal{F}'_{k-1}) & \longrightarrow & H_i(\mathcal{F}'_{k-1}/\mathcal{F}'_k)
\end{array}
\]

Since \(\mathcal{F}_n = \mathcal{F}'_n\), we have that \(H_i(\mathcal{F}_n) = H_i(\mathcal{F}'_n)\) for every \(i \geq 0\). We can then use the Five Lemma and induction on \(k\) with initial step given by \(H_i(\mathcal{F}_n) = H_i(\mathcal{F}'_n)\).

\[
\begin{array}{ccccccc}
H_i(\mathcal{F}_{n-1}/\mathcal{F}_n) & \longrightarrow & H_{i-1}(\mathcal{F}_n) & \longrightarrow & H_{i-1}(\mathcal{F}_{n-1}/\mathcal{F}_n) & \longrightarrow & H_{i-2}(\mathcal{F}_n) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
H_i(\mathcal{F}'_{n-1}/\mathcal{F}'_n) & \longrightarrow & H_{i-1}(\mathcal{F}'_n) & \longrightarrow & H_{i-1}(\mathcal{F}'_{n-1}/\mathcal{F}'_n) & \longrightarrow & H_{i-2}(\mathcal{F}'_n)
\end{array}
\]

Therefore \(H_i(\mathcal{F}_k) \cong H_i(\mathcal{F}'_k)\) for every \(k\) and \(i\). This concludes the proof that \(F\) is a quasi equivalence.

\[\square\]

**Lemma 8.5.** Let \(S\) be the maximal subset of \(E(\Gamma)\) with respect to the inclusion that determines the partition of the set of vertices \(P = \{P_1, \ldots, P_l\}\). Let \(\Gamma_i^S\), \(0 \leq i \leq l\), be the connected subgraph of \(S\) corresponding to the element \(P_i\) in the partition. The map

\[
\alpha : \bigoplus_{P, |P|=k-1, T, \phi(T)=P} G_T \otimes A^{|P|} \rightarrow \bigoplus_{P \in \mathcal{P}, |P|=k-1} \left( \bigotimes_{1 \leq i \leq k-1} C_{\text{con}}(\Gamma_i^S) \otimes A^{|P|} \right)
\]
defined for every \(T \subset S\) by

\[
\alpha(G_T \otimes a_1 \otimes \cdots \otimes a_{|T|}) = \Gamma_T^T \otimes \cdots \otimes \Gamma_i^T \otimes \cdots \otimes \Gamma_{k-1}^T \otimes a_1 \otimes \cdots \otimes a_{|T|}
\]
is a graded group isomorphism.
Proof. Let \( P \) be a partition with \( k - 1 \) classes and consider the maximal subset \( S \subseteq E(\Gamma) \), such that \( \Phi(S) = P \). Consider a class \( P_i \) corresponding to a connected subgraph of \( S, \Gamma_i \). Since the tensor product \( A^{\otimes |P|} \) is not affected by the differential, we can reduce to building a map

\[
\alpha : \bigoplus_{T, \phi(T) = P} G_T \to \bigotimes_{1 \leq i \leq k-1} C_{\text{con}}(\Gamma_i)
\]

where \( G_T \) is the element of \( \Lambda[G_S] \) the exterior algebra with generators given by the edges in \( S \). Note that if \( T \) be a subgraph of \( S \), it can be written as \( T = \Gamma_1^T \cup \cdots \cup \Gamma_{k-1}^T \), where \( \Gamma_i^T \) are connected subgraphs of \( \Gamma_i \). The map \( \alpha \) is a group isomorphism, since there is a bijection between elements of the base of \( \bigoplus_{T, \phi(T) = P} G_T \) and elements \( \Gamma_1^T \otimes \cdots \otimes \Gamma_i^T \otimes \cdots \otimes \Gamma_{k-1}^T \) of the base of \( \bigotimes_{1 \leq i \leq k-1} C_{\text{con}}(\Gamma_i) \). Moreover, the map preserve the grading since the degree in \( \bigoplus_{T, \phi(T) = P} G_T \) is given by the number of edges in \( S \) as well as the degree in \( \bigotimes_{1 \leq i \leq k-1} C_{\text{con}}(\Gamma_i) \) by the definition of the complex \( C_{\text{con}}(\Gamma) \).

Remark 8.6. As a consequence of Lemma 7.3 we have that \( \alpha(I(\Gamma)) \cap C_{\text{con}} \) contains the connected graphs with cycles, so the connected graphs that are not spanning trees. Therefore, \( C_{\text{con}}^i / I(\Gamma) \) is trivial for all \( i \neq n-1 \).

Lemma 8.7. Let \( (C_{\text{con}}(\Gamma), d_{\text{con}}) \) be the chain complex defined in the proof, \( \Gamma \) a connected graph with \( n \) vertices, then \( d_{\text{con}}(C_{\text{con}}^n) = I(\Gamma) \cap C_{\text{con}}^{n-1} \).

Proof. \( C_{\text{con}}^n \) is the free group generated by connected subgraph \( S \) of \( \Gamma \) with \( n \) edges. That is the image under the map \( \alpha \) of the algebra

\[
\bigoplus_{S \subseteq E(\Gamma), |S| = n} G_S.
\]

Since \( \Gamma \) has \( n \) vertices, \( S \) must contain a cycle. We call \( C \) the cycle, and the exterior algebra \( G_S \) is given by the product of the edges in \( C \) and the product of the rest of the edges, that we call \( S' \). Now \( d_{\text{con}}(C) = \sum_{e} d_{\text{con}, e}(C)S' \), since removing one edge in \( S' \) will give a non connected graph. By Lemma 8.7 the ideal generated by \( d_{\text{con}}(C) \) is \( \alpha(I(\Gamma)) \cap C_{\text{con}}^{n-1} \).

Lemma 8.8. Let \( \Gamma \) be a graph with \( n \) vertices. The homology of the complex \( C_{\text{con}}(\Gamma) \) is concentrated in degree \( n - 1 \).

Proof. Let \( \Gamma \) be a connected graph with \( n \) vertices and let \( e \) be an edge in \( E(\Gamma) \). We order the edges in \( \Gamma \) so that \( e \) is the last edge. We denote by \( \Gamma \setminus e \) the graph obtained from \( \Gamma \) by deleting the edge \( e \) and \( \Gamma / e \) the graph obtained from \( \Gamma \) by contracting the edge \( e \). We want to prove that we have a short exact sequence

\[
0 \to C_{\text{con}}(\Gamma \setminus e) \xrightarrow{\alpha} C_{\text{con}}(\Gamma) \xrightarrow{\beta} C_{\text{con}}(\Gamma / e) \to 0.
\]
The map $\alpha$ is the inclusion of subgraphs and is injective since $\ker(\alpha)$ is given by graphs that are mapped to 0, but these are the disconnected graph that are 0 also in $C^i_{\text{con}}(\Gamma \setminus e)$, so $\ker(\alpha) = 0$. $\beta$ is the contraction of the edge $e$ and it is surjective since every element in $C^i_{\text{con}}(\Gamma/e)$ is the image of an element in $C^i_{\text{con}}(\Gamma)$, and if a graph is disconnected in $C^i_{\text{con}}(\Gamma/e)$ so it is in $C^i_{\text{con}}(\Gamma)$.

Now we want to show that $\alpha$ and $\beta$ are chain maps, so that the squares in the following diagram commute.

The right and left square are clearly commutative. Consider the second square, let $S \in C^i_{\text{con}}(\Gamma \setminus e)$ then

$$\alpha(d_{\text{con}}(S)) = \alpha\left( \sum_{l \in E(S)} (-1)^{\nu_l} S \setminus l \right) = \sum_{l \in E(S)} (-1)^{\nu_l} S \setminus l$$

and

$$d_{\text{con}}(\alpha(S)) = d_{\text{con}}(S) = \sum_{l \in E(S)} (-1)^{\nu_l} S \setminus l.$$ 

Consider now the third square, let $S \in C^i_{\text{con}}(\Gamma)$,

$$\beta(d_{\text{con}}(S)) = \beta\left( \sum_{l \in E(S)} (-1)^{\nu_l} S \setminus l \right) = \sum_{l \in E(S \setminus e)} (-1)^{\nu_l} (S \setminus l)/e$$

and

$$d_{\text{con}}(\beta(S)) = d_{\text{con}}(S/e) = \sum_{l \in E(S \setminus e)} (-1)^{\nu_l} S/e \setminus l,$$

since we ordered the edges in $\Gamma$ so that $e$ is the last edge. Reordering the edges of $\Gamma$ commutes with the differential so considering the chain where the edge $e$ is the last edge in $\Gamma$ doesn’t affect the computation of the homology.

There are long exact sequences in homology and we proceed by induction.

$$H_{i-1}(C^i_{\text{con}}(\Gamma/e)) \rightarrow H_{i-1}(C^i_{\text{con}}(\Gamma \setminus e)) \rightarrow H_{i-1}(C^i_{\text{con}}(\Gamma)) \rightarrow H_{i-2}(C^i_{\text{con}}(\Gamma/e)) \rightarrow H_{i-2}(C^i_{\text{con}}(\Gamma \setminus e))$$

From the long exact sequence follows that if the homology of $C^i_{\text{con}}(\Gamma \setminus e)$ is concentrated in degree $n-1$ and the homology of $C^i_{\text{con}}(\Gamma/e)$ is concentrated in degree $n-2$, that are the degrees represented by trees, then the homology of $C^i_{\text{con}}(\Gamma)$ is concentrated in degree $n-1$. We prove by induction on the number of edges in $\Gamma$ that the homology of $C^i_{\text{con}}(\Gamma)$ is concentrated
in degree $n - 1$, where $n$ is the number of vertices in $\Gamma$. Let $\Gamma$ be a connected graph with one edge $e$ and two vertices, then $\Gamma \setminus e$ is a disconnected graph and so $C_{\text{con}}(\Gamma \setminus e)$ is trivial. $\Gamma / e$ is a graph with one vertex and no edges, the complex $C_{\text{con}}(\Gamma / e)$ is concentrated in degree 0. $H_*(C_{\text{con}}(\Gamma / e))$ is concentrated in degree 0 that is $n - 2$ and so $H_*(C_{\text{con}}(\Gamma))$ is concentrated in degree $1 = n - 1$. Now suppose by induction that the statement is true for any graph $\Gamma$ with $|E(\Gamma)| < k$. We want to prove it for $|E(\Gamma)| = k$. Then $\Gamma \setminus e$ is either disconnected or it is a connected graph with $k - 1$ edges and $n$ vertices. In the first case the homology is trivial, in the second case the homology $H_*(C_{\text{con}}(\Gamma \setminus e))$ is concentrated in degree $n - 1$ by inductive hypothesis. $\Gamma / e$ is a graph with $k - 1$ edges and $n - 1$ vertices, it can have a loop or a multiple edge, by Lemma 8.10 the homology $H_*(C_{\text{con}}(\Gamma / e))$ is either trivial or concentrated in degree $n - 2$.

Remark 8.9. This lemma provides an alternative proof of the result given by Vassilev in [18] regarding the homology of the complex of connected subgraphs of the complete graph. The proof could be already present in the literature but we are not aware of any references.

Lemma 8.10. Let $\Gamma$ be a graph. If $\Gamma$ contains a loop, then the homology groups $H_i(C_{\text{con}}(\Gamma))$ are trivial, and replacing a multiple edge by a single edge doesn’t change the homology.

Proof. Let $\Gamma$ be a graph with a loop $l$, then $\Gamma \setminus l$ and $\Gamma / l$ are the same graph and so $H(C_{\text{con}}(\Gamma \setminus l)) = H(\Gamma / l))$. By the long exact sequence we have that $H(C_{\text{con}}(\Gamma)) = 0$. Let now $\Gamma$ have a multiple edge $e$. Then $\Gamma \setminus e$ is a graph with single edges and $\Gamma / l$ is a graph with a loop, and so $H(C_{\text{con}}(\Gamma / e)) = 0$. By the long exact sequence we have that $H(C_{\text{con}}(\Gamma \setminus e)) = H(\Gamma / l))$.

Remark 8.11. The idea in the proof of Lemma 8.10 is the same as the proof of the similar statement in Corollary 3.2 in [9].

9 The chain $C_{BS}^*(\Gamma) / I(\Gamma)$

In this chapter we want to discuss the complex $C_{BS}^*(\Gamma) / I(\Gamma)$ and show that it is isomorphic to $R(A, \Gamma)$.

The DGA $R(\Lambda, A)$ is defined as $\Lambda [G_{a,b}] \otimes A \otimes n$ quotient by the relations

- $G_{a,b} = G_{b,a}$
- $p_a^*(x) G_{a,b} = p_b^*(x) G_{a,b}, x \in H^*(\Lambda)$
- $\delta(w_i) = 0$

where $w_i$ is a cycle in $\Gamma$, and with differential

$$d(G_{a,b}) = p_{a,b}^* \Delta.$$
Let \( \tilde{R}(A, \Gamma) \) be \( \Lambda[G_{a,b}] \otimes A^{\otimes n} \) quotient only by the first two relations, with the same differential \( d \) and let \( (C_{BS}^*(\Gamma), \delta) \) be the complex defined in Section 5.

**Remark 9.1.** The complexes \( (C_{BS}^*(\Gamma), \delta) \) and \( \tilde{R}(A, \Gamma) \) are the same as vector spaces and they differ only for the differentials. The differential of the first complex is defined as

\[
\delta(G_S \otimes a_1 \otimes \cdots \otimes a_{l(S)}) = \begin{cases} 
0 & \text{if } \alpha \notin S \\
\sum_{\alpha \in E(\Gamma)} G_{S \setminus \alpha} p_{i,j}(\Delta) \otimes a_1 \otimes \cdots \otimes a_n, & \text{if } \alpha \text{ disconnects } S \\
\sum_{\alpha \in E(\Gamma)} G_{S \setminus \alpha} \otimes a_1 \otimes \cdots \otimes a_n, & \text{if } \alpha \text{ non disconnects } S
\end{cases}
\]

Here \( \alpha : i \to j \) and \( p_{i,j}(\Delta) \) is the pullback of the projection. Note that \( G_{S \setminus \alpha} p_{i,j}(\Delta) \otimes a_1 \otimes \cdots \otimes a_n = G_{S \setminus \alpha} \Delta_{S, S \setminus \alpha} \otimes a_1 \otimes \cdots \otimes a_{l(S)} \), due to the relation in the definition of \( C_{BS}^*(\Gamma) \). The differential multiplies by the diagonal class only when removing the edge disconnects components. On the other hand, the differential in \( \tilde{R}(\Gamma, A) \), given by

\[
d(G_{a,b}) = p_{a,b}^* \Delta,
\]

always multiplies by the diagonal class. The differentials \( \delta \) and \( d \) can be therefore written as \( \delta = \delta_{\text{int}} + \delta_{\text{ext}} \) and \( d = d_{\text{int}} + d_{\text{ext}} \) (see Remark 8.1), and \( d_{\text{ext}} = \delta_{\text{ext}} \). Removing an edge in a cycle leaves the number of components unchanged, while removing edges in a forest disconnects components, so from the point where \( S \) is a spanning forest the two complexes are the same.

We now prove that there is an isomorphism of chain complexes between \( C_{BS}^*(\Gamma)/I(\Gamma) \) and \( R(A, \Gamma) \).

**Theorem 9.2.** There is an isomorphism of chain complexes,

\[
\text{id} : (C_{BS}^*(\Gamma), \delta)/I(\Gamma) \to (R(A, \Gamma), d)
\]

**Proof.** Consider the identity map \( \text{id} : C_{BS}^*(\Gamma)/I(\Gamma) \to R(A, \Gamma) \). By Remark 9.1 the complexes \( C_{BS}^*(\Gamma) \) and \( \tilde{R}(A, \Gamma) \) differ only from the internal differential since the two differentials agree in the case where the subgraph \( S \) of the set of vertices \( E(\Gamma) \) is a forest. By Lemma 7.3 the third relation assures that elements in the complexes corresponding to graphs \( S \) with a component containing cycles are zero. Therefore \( C_{BS}^*(\Gamma)/I(\Gamma) \) and \( R(A, \Gamma) \) agree as vector spaces, as the two differentials \( \delta = d \), since \( d_{\text{int}} = \delta_{\text{int}} = 0 \). \( \square \)
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