Unitary transformations can be distinguished locally

Xiang-Fa Zhou, Yong-Sheng Zhang and Guang-Can Guo

Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei, Anhui 230026, People’s Republic of China

We show that in principle, N-partite unitary transformations can be perfectly discriminated under local measurement and classical communication (LOCC) despite of their nonlocal properties. Based on this result, some related topics, including the construction of the appropriate quantum circuit together with the extension to general completely positive trace preserving operations, are discussed.

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Superposition plays the central role in quantum mechanics. The quantum nonorthogonality and entanglement due to superposition, which show many counter-intuitive behaviors compared with those in classical world, have drawn much attention in the past two decades. Quantum nonorthogonality put many constraints on physically accessible manipulations on input states. It is well-known that two nonorthogonal pure state cannot be perfectly discriminated [1]. On the other hand, quantum nonlocality due to entanglement, which was first brought into attention by Einstein, Podolsky, and Rosen (EPR) in 1935 [2], is also one of the most interesting and important parts in quantum information science. Today, quantum entanglement has been viewed as a significant resource for quantum information processing, and currently the behavior of entanglement in quantum information science is still under investigation.

Although perfect identification of nonorthogonal quantum states are impossible in quantum world, when we refer to quantum operations, thing becomes very different. It was proved that two unitary operations can be perfectly discriminated after applying the unitary gate a finite number of times in parallel [3, 4]. On the other hand, the nonlocality of unitary transformation has been extensively studied because of its fundamental importance during the construction of universal quantum circuit [5]. For example, it has been shown that a sequence of a nonlocal gate (e.g., Control-not gate or Control-phase gate) and single-qubit rotations can be used to construct any desired transformations. Also nonlocal gate can be classified and simulate each other under specific conditions [6, 7]. Based on these results, one natural problem arises - what is the influence of the nonlocality of quantum operation on the discrimination.

In this work, we consider to discriminate two unitary transformations with local methods. Compared with its counterpart, i.e., local identification of quantum states, which is often considered for orthogonal states [8, 9], we find that any two unitary transformations can be perfectly identified locally despite of their nonlocal properties.

Before concentrating on the specific topics, let us make a few remarks about the difference between the discrimination of quantum states and of quantum operations. Generally to identify a quantum state, one should make a measurement on the given state followed by an estimation based on the measurement results. Such process usually collapses the input states which thus cannot be used any more. However, thing becomes different when we refer to quantum operations. The reason lies in the fact that quantum operations never collapse, and in principle it can be repeated any times if we need. What’s more, when unitary operations are considered, by exchanging the input and output ports of the whole setup, we can obtain the reverse transformations. Actually, these facts make the discrimination of quantum operations very different from that of quantum states.

Generally the strategy of operation identification is formulated as this: we employ a quantum circuit $f(U)$ which is made up of the selected operation $U$ on the suitable input state $\rho_{s,a}$, where $s(a)$ denotes the circuit system (auxiliary system). If only local methods are required, $\rho_{s,a}$ must also be separable. To obtain the maximal distinguishability, the overlap of the output states should be as small as possible for different quantum operations. Fig. 1 shows the sketch of the identification process under local operation and classical communication (LOCC). When global operation are permitted, both of the circuit and the input state can be constructed to realize a perfect discrimination for unitary transformations [3, 4]. However, when only local operations and resources are permitted, thing becomes not so obvious. To simplify our consideration, in the following, we mainly focus on bipartite system.

Let us begin with some simple observations. Here we mainly concentrate on unitary operations, one can check that some of the discussions are also suitable for general quantum operations. As we have mentioned above, to realize perfect identification, one need to find a suitable input state such that the corresponding output states are orthogonal to each other for different selected operations. Assume that we want to discriminate two unitary operations $U$ and

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*Electronic address: x Zhon@mail.ustc.edu.cn
†Electronic address: yshzhang@ustc.edu.cn
FIG. 1: Illustration of the identification of unitary transformations under local operation and classical communication. Alice and Bob input a locally implemented state $\rho_{AA'B'B'}$ to a quantum circuit $f(U)$ followed by local measurement operations. The measurement results are transmitted through classical channels to realize perfect discrimination.

V. By inputting a locally implemented quantum state $\rho_{AA'B'B'} = \sum_i \lambda_i \rho_{AA}^i \otimes \rho_{BB}^i$, we have that the two output states $\rho_U = (U \otimes I_{AB'})\rho_{AA'B'B'}(U^\dagger \otimes I_{AB'})$ and $\rho_V = (V \otimes I_{AB'})\rho_{AA'B'B'}(V^\dagger \otimes I_{AB'})$ should be orthogonal to each other. Now consider the spectral decompositions of $\rho_{AA}^i = \sum_j r_j^i |r_j^i\rangle\langle r_j^i|$, and $\rho_{BB}^i = \sum_k s_k^i |s_k^i\rangle\langle s_k^i|$. The requirement of $\rho_U \perp \rho_V$ is equivalent to $(U \otimes I_{AB'})|r_j^i\rangle_{AA}|s_k^i\rangle_{BB} \perp (V \otimes I_{AB'})|r_j^i\rangle_{AA}|s_k^i\rangle_{BB}$ for any $i, j, k, i'$. This observation shows in general, a pure input state $|r\rangle_{AA'}|s\rangle_{BB'}$ is enough to perfectly discriminate two unitary operations if they can. Moreover, since two orthogonal pure states can be locally identified [8, 9], hence in this case $U$ and $V$ can also be discriminated with local methods.

Consider two unitary transformations $U_{AB}$ and $V_{AB}$ with zero overlap in trace norm, i.e., $\text{Tr}(V_{AB}^\dagger U_{AB}) = 0$. Then by preparing the following locally maximal entangled state as the input

$$|\phi\rangle_{AA'B'B'} = |\phi\rangle_{AA'} \otimes |\phi\rangle_{BB'},$$

where $|\phi\rangle_{AA'} = \sum_i |i\rangle_A|i\rangle_{A'}$ (or $|\phi\rangle_{BB'} = \sum_i |i\rangle_B|i\rangle_{B'}$) is a nonnormalized entangled state between the system $A$ and the corresponding local environment $A'$ (or $B$ and $B'$). From the following equation

$$\langle \phi| V_{AB}^\dagger U_{AB} \otimes I |\phi \rangle = \text{Tr}(V_{AB}^\dagger U_{AB}) = 0,$$

one immediately obtain that the two output states $U_{AB} \otimes I |\phi\rangle_{AA'B'B'}$ and $V_{AB} \otimes I |\phi\rangle_{AA'B'B'}$ are orthogonal to each other, hence can be locally discriminated perfectly. Equations (1) can be viewed as the extension of Jamiolkowski isomorphism in local case [10]. The input state $|\phi\rangle_{AA'B'B'}$ is universal for any two operations $U$ and $V$ satisfying $\text{Tr}(V_{AB}^\dagger U_{AB}) = 0$. Actually, given $U$ and $V$, if global input states are permitted, one can always choose a suitable pure input state in the composite system of only $A$ and $B$, namely, the auxiliary system can be neglected in this case [3, 4]. However, if only local resources are required, in order to achieve the maximal distinguishability of the output states, an entangled state between the system and the environment seems to be required unless the global optimal pure state is separable.

In the above case, perfect identification can be realized in a single run for both global and local methods. In the more general cases, one needs to run the selected gate $N$ times ($N$ is finite). The optimal $N$ has been found for global discrimination of $U$ and $V$, which asserts that if the minimal arclength $\delta$ spread by the eigenvalue of $(U \otimes V)^{\otimes N}$ in the circle $|z| = 1$ is not less than $\pi$, then a perfect discrimination scheme is allowed. Now assume $U^{\otimes N} = U_1^{\otimes N} \otimes U_2^{\otimes N}$ to be local operation, with $\delta_1$ and $\delta_2$ being the minimal arclengths of $U_1^{\otimes N}$ and $U_2^{\otimes N}$ respectively. Then perfect global discrimination can be implemented by inputting an entangled state if $\delta_1 + \delta_2 \geq \pi$. However, if only local input states (e.g., $|r\rangle|s\rangle$) are allowed, since

$$\langle r| U_1^{\otimes N} |r\rangle \langle s| U_2^{\otimes N} |s\rangle = 0 \iff \langle r| U_1^{\otimes N} |r\rangle = 0 \text{ or } \langle s| U_2^{\otimes N} |s\rangle = 0,$$

this indicates that to distinguish $U$ and $V$ locally, at least one of the two arclength $\delta_1$ and $\delta_2$ must be not less than $\pi$. Therefore, generally in the local case the optimal running times $N$ of the selected operation should be greater than that of the global case.

As a special example, consider the following control unitary transformation $U_{AB}^\dagger = P_1 \otimes I + P_2 \otimes u$, where $P_i P_j = \delta_{ij} P_i$ and $\sum_i P_i = I$, $I$ is the identity operation, and $u$ is a local unitary manipulations. The eigenvalues $r_i$ of $U_{AB}^\dagger$ belong to the set $\{1, b_1, b_2, ...\}$ with $b_1$ and $b_2$ being the eigenvalues and eigenvectors of $u^{\otimes N}$ separately. If only local input state $\rho_A \otimes \rho_B = \text{Tr}(|\psi_{AA}'\rangle\langle \psi_{AA}'| \otimes |\psi_{BB}'\rangle\langle \psi_{BB}'|)$ is permitted, then

$$\text{Tr}[(U_{AB}^\dagger)^{\otimes N}(\rho_A \otimes \rho_B)] = x + (1 - x) \sum_i b_i |b_i\rangle\langle b_i|.$$
where $x = \text{Tr}(P_i \rho_A) \geq 0$ and can be chosen arbitrarily by input appropriate $\rho_A$. In order to make the right-hand-side of Eq. (1) to be zero, one can easily obtain that the minimal angular spread of $\{1, b_1, b_2, \ldots \}$ should be not less than $\pi$. Therefore, in this case the minimal $N$ required equals to that of global case. Similarly, suppose $U^1V = (P_1 \otimes I + P_2 \otimes V \cdot (U_0 \otimes U_2)$ and $U^2 \neq I$ or $U^2 \neq u^1$. If $U^2 \neq u^1$, then by inputting appropriate state $|\psi_A|\psi_B$ with $|\psi_A \rangle$ lying in the support of $P_1$, $U^1V$ is equivalent to the local transformation $(uu_2)|\psi_B \rangle$, hence can be perfectly identified.

In the above discussions, we have considered to discriminate several special kinds of unitary transformations. They all can be perfectly identified and the optimal quantum circuit and input state can be easily obtained. In the following, we mainly focus on the most general case. Although we cannot present the optimal quantum circuit and input state, we prove that, in principle, any two unitary operations $U$ and $V$ can be perfectly identified locally.

Following [10], we call a 2-qudit gate $U_{AB}$ to be primitive if $U_{AB}$ maps a separable state to another separable state; otherwise, $U_{AB}$ is imprimitive. Generally, a primitive gate $U_{AB}$ can be expressed as the product of 1-qudit gate up to a swap operation $P$, namely, $U_{AB} = U_A \otimes U_B$ or $U_{AB} = U_A \otimes U_B \cdot P$ with $P\langle \alpha | \beta \rangle_B = \langle \beta | \alpha \rangle_B$. For simplicity, in the following, we use $H$ to denote the set of all 2-qubit gates of the form $U_A \otimes U_B$. Under these assumptions, we then introduce the following lemma.

Lemma 1. $H$ together with an imprimitive gate $Q$ can generate the unitary group $U(d^2)$.

A detailed proof of this lemma can be found in [10], which is used to study the university of quantum gate. This lemma indicates that if $Q$ and all local unitary transformations are permitted, we can then construct $H' = HQQ^{-1}$. By choosing suitable sequence of $H$ and $H'$, we can obtain any desired elements in $U(d^2)$. The length of the sequence is finite, therefore it is only need to run the imprimitive gate a finite number of times.

Based on this lemma, we now prove the main theorem of this work.

Theorem 1. Any two unitary transformation $U_{AB}$ and $V_{AB}$ can be perfectly identified with local methods.

Proof: Following our former discussions, we obtain that if both $U_{AB}$ and $V_{AB}$ are primitive, then they can be perfectly discriminated locally.

Now assume that only one of the two unitary gate is primitive. Without loss of generality, we suppose $V_{AB}$ to be imprimitive. According to the lemma, we obtain that there exists a quantum circuit $f(V_{AB})$ made up of the elements in $H$ and $H' = V_{AB}HV_{AB}^+ \subseteq$ such that $f(V_{AB}) \in (HH')^\circ$ is some control unitary transformation. On the other hand, since $U_{AB}$ is primitive, which means $H' = U_{AB}HU_{AB}^+ = H$, one immediately obtain that $f(U_{AB})$ is also primitive. Because $f(U_{AB}) \neq f(V_{AB})$, we have that the two unitary operations can be locally identified.

If $U_{AB}$ and $V_{AB}$ are both imprimitive, Following the lemma, we obtain that there is a quantum circuit such that $f(U_{AB}) = (L_{A})^i_{12} \otimes L_{B}^{i1} \delta_{i1} \delta_{i2} i + \delta_{i2} \delta_{i1} (or (L_{B}^{i1})_{ij}$. (2) If $f(V_{AB})$ is primitive, then perfect local discrimination can be realized. Otherwise, both $f(U_{AB})$ and $f(V_{AB})$ are imprimitive. Since $f(U_{AB})^\dag f(U_{AB})$ with $A = \text{diag}(\sigma_x, I_{(d-2)}) \otimes I$, $I \otimes \text{diag}(\sigma_y, I_{(d-2)})$, and $\text{diag}(\sigma_y, I_{(d-2)}) \otimes I$, or $I \otimes \text{diag}(\sigma_y, I_{(d-2)})$. One can easily check that if the similar result occurs for $V_{AB}$, then $f(V_{AB})$ can be expressed as $f(V_{AB}) = e^{ixL_{A}^{i1} \otimes L_{B}^{i2}}$ for some $x \in \mathbb{R}$. Therefore the whole question can be divided into the following two parts:

i) If $f(V_{AB}) \neq e^{ixL_{A}^{i1} \otimes L_{B}^{i2}}$ for any $x \in \mathbb{R}$, then by employing the transformation $Af(\cdot)A^\dag f(\cdot)$, we can obtain an identity operation for $U_{AB}$. Because $Af(V_{AB})A^\dag f(V_{AB}) \neq I$, the two operations thus are locally distinguishable.

ii) If $f(V_{AB}) = e^{ixL_{A}^{i1} \otimes L_{B}^{i2}}$, then when $x \neq 1$, $f(U_{AB})$ and $f(V_{AB})$ can be reduced to $e^{ixL_{A}^{i1} \otimes I}$ and $e^{ixL_{A}^{i1} \otimes I}$ by inputting a product state $|\phi \rangle \psi$ with $|\psi \rangle$ being an eigenvector of $L_{A}^{i1}$, which, therefore, can be perfectly identified locally by running the circuit a finite number of times in parallel. Otherwise we have $f(U_{AB}) = f(V_{AB})$. Since $e^{ixL_{A}^{i1} \otimes L_{B}^{i2}}$ is imprimitive, it can be used to construct the desired operator $U_{AB}$. Thus the original problem is reduced to the locally identification of the identity operation and $U_{AB}^\dag V_{AB}$, which can be implemented perfectly.

This completes the proof.

The above theorem shows that in principle, to realize a perfect local identification, we only need to run the selected unitary operation a finite number of times. Although we have assumed that the two subsystems $A$ and $B$ have equal dimensions, one can easily obtain that the same result holds even if $A$ and $B$ have different dimensions. For example, if $\text{dim}H_A < \text{dim}H_B$, then by introducing another subsystem $A_1$ in Alice’s side such that $\text{dim}H_A + \text{dim}H_{A_1} = \text{dim}H_B$, we can obtain two extended unitary transformations $U \oplus I_{A_1}$ and $V \oplus I_{A_1}$, which thus can be identified with the methods described above.

It should be mentioned that the ancillary subsystem $A_1$ usually plays nontrivial role during the discussion of operation discrimination [11]. In practice, given two different operations $\langle \xi_1, \xi_2 \rangle$ acting on the same Hilbert space $A$, it is always possible to prepare a larger system $A'$ such that $A' = A \oplus A_1$. Therefore, the original problem can be reduced to the discrimination of the two newly defined operations $\langle \xi_1 \oplus I_{A_1}, \xi_2 \oplus I_{A_1} \rangle$. For instance, in the global discrimination of two unitary operations $\{U, V\}$, the minimal running times usually reads $N = \lceil \frac{1}{2} \rceil$. However, when
identity operators by preparing suitable pure input state in the total Hilbert's space. However, if the ancillary level [2] is concerned, then perfect identification can be implemented by preparing suitable pure input state in the total Hilbert space.

From the practical viewpoint, it will be valuable if one can provide an optimal circuit to implement such kind of identification operation. Generally, it is not easy to do this. Here, to simplify our consideration, we take two-qubit gates as an example.

For any two-qubit unitary transformation $U$, it has the following canonical decomposition

$$U = (U_1 \otimes U_2)e^{i(h_x \sigma_x \otimes \sigma_x + h_y \sigma_y \otimes \sigma_y + h_z \sigma_z \otimes \sigma_z)}(U_3 \otimes U_4),$$

where $\sigma_x, \sigma_y, \sigma_z$ are the usual Pauli matrices, $U_i$ are local single-qubit gate and $\pi/4 \geq h_x \geq h_y \geq |h_z|$. Benefitting from the nice decomposition, one need not to reverse the whole setup because $U^\dagger$ can be constructed from $U$ directly. Now suppose we have two unitary operations $U$ and $V$. After applying the selected gate at most 2 times, we can transform one of them, e.g., $U$, into $f(U) = e^{ih_y \sigma_y \otimes \sigma_y}$ if $f(V) = e^{ih_y \sigma_y \otimes \sigma_y}$ for some $h_y \in \mathbb{R}$, we can employ the manipulation $g(\cdot) = AF(\cdot)A^\dagger f(\cdot)$ ($A = \sigma_3 \otimes I, I \sigma_3 \otimes I, I \otimes \sigma_y, or I \otimes \sigma_z$), where $A$ can be selected to meet the requirement, i.e., to reduce the original $U$ and $V$ to $I$ and $g(V)$ respectively. Similarly, by running $g(V)$ at most 4 times, we can then obtain two local unitary transformations $U'$ and $V'$. One can easily check that by choosing suitable single-qubit gates, $U'$ and $V'$ can always be different. Therefore, after repeating the selected gate at most 20 times, we reduce the original problem to the discrimination of two local gates, which can be perfectly implemented with the method we described in the former context.

The same question can also be investigated in multi-partite case. To answer this problem, we should introduce the generalized version of the primitive gates. We call $U_{12...N}$ is $\{[i_1,...,i_s],..., [j_1,...,j_t]\}$-primitive if $U_{12...N}$ together with all single-qudit gates can generate the group $U_N = U_{1,2,...,N}$ is primitive. Following the same routine in 10, we can obtain that a $\{[i_1,...,i_s],..., [j_1,...,j_t]\}$-primitive gate can be expressed as $U_{i_1,...,i_s} \otimes \cdots \otimes U_{j_1,...,j_t} \cdot P_{[i_1,...,i_s]|[j_1,...,j_t]}$, where $P_{[i_1,...,i_s]|[j_1,...,j_t]}$ is permutation operator which preserves the structure of the partition $\{[i_1,...,i_s],..., [j_1,...,j_t]\}$. For example, if $U_{12345}$ is $\{1,2,3,4,5\}$-primitive, then $P_{[1,2]|3,4,5} = P_{12,34} \otimes I_5$ or $I_{12345}$, where $P_{12,34}$ is the swap operation between Hilbert spaces $H_1 \otimes H_2$ and $H_3 \otimes H_4$; if $U_{12345}$ is $\{1,2,3,4,5\}$-primitive, then $P_{1,2,3,4,5}$ can be any element in the permutation group $S_5$.

We take 3-partite unitary transformations as an instance. According to the above discussion, if one of the two 3-partite unitary transformations $U_{ABC}$ and $V_{ABC}$ is $\{A,B,C\}$-primitive, then perfect local identification can be realized. If both of the two selected transformations are primitive, then there exists a sequence $f(U_{ABC}) = (H^tH) \cdots (H^tH)$ with $H^t = U_{ABC}H_{ABC}^t$, such that $f(U) = e^{iL_1^t \sigma_1 \otimes \sigma_1 + L_2^t \sigma_2 \otimes \sigma_2 + L_3^t \sigma_3 \otimes \sigma_3}$.

Following the discussion of bipartite case, we conclude that $U_{ABC}$ and $V_{ABC}$ can be locally discriminated. Finally, if $U_{ABC}$ is $\{A,B,C\}$-primitive with $V_{ABC}$ being $\{A,B,C\}$-primitive, then there exists a circuit such that $f(U_{ABC}) = (U_{A_1} \otimes P_{B_1} + U_{A_2} \otimes P_{B_2}) \otimes U_{C}$ and $f(V_{ABC}) = (V_{A_1} \otimes P_{B_1} + V_{A_2} \otimes P_{B_2}) \otimes U_{C}$, where $(U_{A_1} \otimes P_{B_1} + U_{A_2} \otimes P_{B_2})$ is some control-unitary transformation. Since $U_{A_1} \neq V_{A_1}$ or $U_{A_2} \neq V_{A_2}$, by choosing suitable input state, the original problem can be reduced to the discrimination of two different local unitary manipulations, hence can be realized perfectly.

The above discussion can be extended to $N$-partite case, and we have that it is always possible to discriminate two unitary operations locally, although in general, we need to run the unknown operation many times. Interestingly, unlike the previous results for quantum states, where “the hidden entanglement” plays a very important role, it seems that the nonlocality of unitary transformations does not affect the distinguishability much (in this work, it only changes the total run times $N$). We can also generalize this result to the case of $M$ unitary transformations. To discriminate the unknown operation from others, we could perform $M - 1$ tests; after each test, one of the $M$ operations can be ruled out. Therefore perfect local identification can be realized after a finite number of runs of the unknown gate.

One can also consider the same problem for nonunitary transformations. For general completely positive trace preserving operations $\xi_1$ and $\xi_2$, the reverse transformations donot always exist unless they are unitary. Moreover, the output states usually are mixed even if we employ a pure input state, and $\xi_1, \xi_2$ may contain common Kraus operators. To realize perfect identification operation, these components should have no contribution to the output states. Thus totally solve this problem seems to be quite complicated.

To summarize, we have shown that besides global operations, multi-unitary unitary transformations can also be discriminated perfectly with local methods. Nonlocal schemes together with entangled input states usually can improve the efficiency of the identification, i.e., we can run the unknown operation less times to realize perfect discrimination. However, it does not affect the distinguishability of the whole problem. In principle, by running the
secretly chosen operations a finite number of times, we can also realize perfect identification under LOCC. From the practical viewpoint, one need to provide an optimal methods to implement the discrimination operations. Our investigation indicates that this question has a close relation to the exact universality of unitary evolution and the optimal quantum circuit in d-level system \[\mathbb{I}\].

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**APPENDIX**

We now present a simple proof about the exact universality of \(N\)-partite unitary transformations. The method used here are mainly based on ref. \[\mathbb{II}\]. First we introduce the following lemma.

**Lemma 2.** Let \(G\) be a compact Lie group. If \(H_1, \ldots, H_k\) are closed connected subgroups and they generate a dense subgroup of \(G\), then in fact they generate \(G\).

Suppose \(U\) is a \(N\)-partite unitary map, and we also use \(H\) to denote all 1-qudit gates \(V_1 \otimes \ldots \otimes V_N\). We introduce the subgroup \(H_1 = UHU^{-1}\). Now consider the \(n\)-fold products \(\Sigma = \Sigma_1 \ldots \Sigma_n\) with \(\Sigma = H_i H\). One can find that when \(n \to \infty\), \(\Sigma∞\) is a subgroup of all \(N\)-partite unitary transformations \(U(d^N)\), hence we have \(H \subseteq \Sigma∞ \subseteq U(d^N)\).

Assume \(h, r, g\) are the corresponding Lie algebras of the group \(H, \Sigma∞, U(d^N)\) separately. Consider the representation of \(K = SU(d) \otimes \ldots \otimes SU(d)\) on the Lie algebra \(g\)

\[
\pi^{S_1 \ldots S_N}(\xi) = (S_1 \otimes \ldots \otimes S_N)\xi(S_1 \otimes \ldots \otimes S_N)^{-1}, \quad \xi \in g.
\]

Since \(K\) is a compact Lie group, \(\pi\) can be decomposed as a direct sum of irreducible representations of \(K\). Therefore, we obtain the following decomposition of \(g\)

\[
g = \bigoplus_{j=0}^{N} \bigoplus_{k=1}^{n_j} i^{N+1-j} P_{[\alpha_i^{j,k}, \ldots, \alpha_j^{j,k}]} \tag{A-1}
\]

with

\[
P_0 = \mathbb{R} I \otimes \ldots \otimes I, \tag{A-2}
\]

\[
P_{[\alpha_i^{j,k}]} = I_1 \otimes \ldots \otimes SU(d)_{\alpha_i^{j,k}} \otimes \ldots \otimes SU(d)_{\alpha_j^{j,k}} \otimes \ldots, \tag{A-3}
\]

where \(i^2 = -1\), \(SU(d)\) is the Lie algebra of \(SU(d)\), and \(\alpha_i^{j,k}\) are indices selected from the set \([1, \ldots, N]\).

Similarly, because \(H \subseteq \Sigma∞\), \(r\) can also be decomposed into the direct sum of a finite number of terms on the right-hand-side of Eq. \(A-1\)

\[
r = \bigoplus_{j=0}^{N} \bigoplus_{k=1}^{n_j} c_{jk} P_{[\alpha_i^{j,k}, \ldots, \alpha_j^{j,k}]}, \quad \text{and} \quad c_{jk} \in \{\pm 1, \pm i\}. \tag{A-4}
\]

We call two indices \(\alpha_i^{j,k}\) and \(\alpha_j^{j',k'}\) to be connected if there exists a subset \(C = [\alpha_i^{j,k}, \ldots, \alpha_j^{j,k}]\) such that \(\alpha_i^{j,k} \in C\) and \(\alpha_j^{j',k'} \in C\). Thus the connectedness of indices lead to the following decomposition of \([1, \ldots, N]\)

\[
[\ldots, \alpha_i^{j_1, k_1}, \ldots] \oplus [\ldots, \alpha_j^{j_2, k_2}, \ldots] \oplus \ldots \tag{A-5}
\]

On the other hand, since \(r\) is a Lie algebra, one can immediately obtained that \(r\) is the Lie algebra of the compact Lie group \(U_r = U[\ldots, \alpha_i^{j_1, k_1}, \ldots] \otimes U[\ldots, \alpha_j^{j_2, k_2}, \ldots] \otimes \ldots\). According to Lemma 2, we obtain that \(\Sigma∞ = U_r\), hence there exist some \(p\) such that \(\Sigma^p = U_r\).

After we have obtained the group \(U_r\), we can now define the new \(n\)-fold products as \(\Sigma^p = \Sigma_1 \ldots \Sigma_1\) with \(\Sigma_1 = (U_r U_r U_r^+)\). Repeat the above discussions, we have that \(U\) together with all 1-qudit gates can generate the following unitary group

\[
U_\beta = U[\ldots, \alpha_i^{j_1, k_1}, \ldots] \otimes U[\ldots, \alpha_j^{j_2, k_2}, \ldots] \otimes \ldots \tag{A-6}
\]
with $U_3 U_3^\dagger = U_3$. Therefore, $U$ is $\{[\ldots, \beta_{1, k_1}, \ldots], [\ldots, \beta_{2, k_2}, \ldots], \ldots\}$-primitive. Moreover, $U$ normalize $U_3$. Following the similar discussion in ref. [10], we have that $U$ can be expressed as $U = U_\beta \cdot P_\beta$ for some $U_\beta \in U_\beta$, where $P_\beta$ is the corresponding permutation operator of the Hilbert spaces $H_L \{\ldots, \beta_{j, k}, \ldots\}$ which have the same dimension.

For example, if $U_{12345}$ is $\{[1, 2], [3, 4], 5\}$-primitive, then $P_{\{[1, 2], [3, 4], 5\}} = P_{12,34} \otimes I_5$ or $I_{12345}$, where $P_{12,34}$ is the swap operation between Hilbert spaces $H_1 \otimes H_2$ and $H_3 \otimes H_4$; if $U_{12345}$ is $\{1, 2, 3, 4, 5\}$-primitive, then $P_{\{1, 2, 3, 4, 5\}}$ can be any element in the permutation group $S_5$.

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