Mass and spin coevolution during the alignment of a black hole in a warped accretion disc

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Accepted 2009 July 20. Received 2009 July 17. In original form 2009 March 16

\section{INTRODUCTION}

Astrophysical black holes (BHs) are Kerr black holes fully characterized by their mass $M_{BH}$ and spin $J_{BH}$, customarily expressed in terms of the dimensionless spin parameter $a$ ($\leq 1$), and unit vector $\hat{J}_{BH}$:

$$J_{BH} = a \frac{GM_{BH}^2}{c} \hat{J}_{BH}. \quad (1)$$

The spin and mass of BHs residing in galaxy nuclei do not remain constant, close to their birth values, but change sizably through cosmic time, in response to major accretion events. In current cosmological scenarios for the evolution of galaxies, repeated interactions among gas-rich halos play a key role not only in shaping galaxies, but also in triggering quasar activity \cite{White & Rees 1978, Di Matteo et al. 2005}. Massive gaseous nuclear discs that form in the aftermath of major galaxy mergers \cite{Mihos & Hernquist 1996, Mayer et al. 2007} may provide enough fuel to feed, on sub-parsec scales, the BH through a Keplerian accretion disc.

If these episodes repeat recursively and/or at random phases \cite{King & Pringle 2006}, the BH spin $J_{BH}$ is expected, initially, to be misaligned relative to the direction of the angular momentum of the disc $J_{disc, out}$, at its unperturbed, outer edge $R_{out}$. In this configuration, the gas elements inside the disc undergo Lense-Thirring precession \cite{Wilkins 1972}. In the fluid, the action of viscosity onto the differentially precessing disc ensures that the inner portion of the accretion disc aligns (or anti-aligns) its orbital angular momentum with the BH spin $J_{BH}$, out to a transition radius $R_{warp}$ beyond which the disc remains aligned to the outer disc, as first shown by \cite{Bardeen & Petterson 1975} \cite{Armitage & Natarajan 1999, Nelson & Papaloizou 2000, Fragile & Anninos 2005, Fragile et al. 2007}. Warping of the inner disc at distance $R$ from the BH is communicated through the fluid elements on a timescale $t_{BP}(R)$ related to the vertical shear viscosity of the accretion disc. Therefore, the inner regions of the disc align (or counter-align if the disc is counter-rotating) with the BH spin on the scale $t_{BP}(R_{warp})$ when the viscous time for vertical propagation of disturbances equals the Lense-Thirring precession time. On a longer timescale, the joint evolution...
We consider a BH with spin \( J_{BH} \) surrounded by a geometrically thin, standard Shakura-Sunyaev \( \alpha \)-disc (e.g. Shakura & Syunyaev 1973; Frank et al. 2002). The \( \alpha \)-disc is initially misaligned relative to \( J_{BH} \), i.e. the angular momentum unit vector of the disc at the outer edge is \( J_{disc, out} \neq J_{BH} \); the relative inclination angle between the two unit vectors is \( \theta_{out} \).

Following Pringle (1992), we assume that the accretion disc has a high viscosity (\( \alpha \gg H/R \), where \( H \) is the disc vertical scale height) so that perturbations propagate diffusively. We introduce two viscosity parameters, \( \nu_1 \) and \( \nu_2 \); \( \nu_1 \) is the standard radial shear viscosity while \( \nu_2 \) is the vertical share viscosity associated to the diffusion of vertical warps through the disc, due to Lense-Thirring precession. For \( \nu_1 \) we adopt the \( \alpha \) prescription

\[
\nu_1 = \alpha H c_s
\]

where \( c_s \) is the sound speed inside the accretion disc. It is still poorly understood which is the relation between the radial and the vertical viscosity: in particular, if \( \nu_1 \sim \nu_2 \) or \( \nu_1 \ll \nu_2 \). In order to simplify our discussion, we refer to the recent analysis of Lodato & Pringle (2007), and for \( \nu_2 \) we take:

\[
\nu_2 = \frac{f_{c_2}}{2a^2}
\]

where \( f_{c_2} \) (given in Table 1) is a coefficient determined in numerical simulations that accounts for non-linear effects.

The disc model is defined after specifying five free parameters (subscript 0 will be introduced to indicate initial values when mass and spin evolution is considered):

1. The BH mass, \( M_{BH} \); we explore a mass range between \( 10^5 M_\odot < M_{BH} < 10^7 M_\odot \). For the BH mass we introduce the dimensionless parameter \( M_0 \) as \( M_{BH} = M_0 \times 10^6 M_\odot \).
2. The spin modulus, in terms of the dimensionless spin parameter \( a \), which varies between 0 \( \leq a \leq 0.95 \). We do not use the theoretical limit \( a = 1 \) because, if accretion is driven by magneto-rotational instabilities in a relativistic MHD disc, the final equilibrium spin due to continuous accretion is \( a \approx 0.95 \) (Gammie et al. 2004).
3. The relative inclination angle \( \theta_{out} \), between the spin vector \( J_{BH} \) and the orbital angular momentum vector at the outer edge of the accretion disc, \( J_{disc, out} \). This angle varies isotropically from 0 to \( \pi \). In the following, however, we will confine this interval to \( 0 \sim \pi/6 \) in order to satisfy the used approximations.
4. The viscosity parameter \( \alpha \) which is assumed to vary between \( 10^{-2} \lesssim \alpha \lesssim 10^{-1} \) to bracket uncertainties (King et al. 2007). For our purposes we selected values of \( \alpha \) according to Lodato & Pringle (2007), as in Table 1. In this study, \( \alpha \) is considered as a constant inside the disc.
5. The accretion rate onto the BH, \( \dot{M} \), is expressed in terms of the Eddington ratio \( f_{\dot{Edd}} = L/\dot{L}_{Edd} \) and of the accretion efficiency \( \eta \) (where \( \dot{L}_{Edd} \) is the Eddington luminosity: \( M = f_{\dot{Edd}} \dot{L}_{Edd}/(\eta c^2) \)). We consider values of \( f_{\dot{Edd}} \) in the interval \( 10^{-4} < f_{\dot{Edd}} < 1 \) and compute \( \eta \) as a function of the BH spin modulus.

If the disc, warped in its innermost parts, is described to first order by the Shakura-Sunyaev \( \alpha \)-model, both \( \nu_1 \) and

| \( \alpha \) | \( f_{c_2} \) |
|------|------|
| 0.18 | 1.00 |
| 0.15 | 0.85 |
| 0.09 | 0.60 |
| 0.05 | 0.38 |

Table 1. Table of the coefficients \( \alpha \) and \( f_{c_2} \).
Following standard Shakura-Sunyaev disc solutions for external regions of an accretion disc (Frank et al. 2002), we have $\beta = 3/4$ and

$$A_{v_1} = 9.14 \times 10^6 \alpha_{0.1}^{4/5} \alpha_0^{2/5} (\frac{f_{\text{Edd}}}{\eta_{0.1}})^{3/10} \text{ cm}^{5/4} \text{s}^{-1}$$

(5)

$$A_{v_2} = \left(\frac{v_2}{v_1}\right) A_{v_1} = 50 f_{v_2} \alpha_{0.1}^{2/5} A_{v_1}.$$

In equation $A_{v_1}$, $\alpha_{0.1}$ and $\eta_{0.1}$ are the $\alpha$ coefficient and the BH radiative efficiency in unit of 0.1. $f_{v_2}$ is tabulated in Table 1 (Lodato & Pringle 2007).

3 WARPED ACCRETION DISC

3.1 The angular momentum content of discs: extended versus truncated discs

The dynamics of a fluid element in a misaligned disc around a spinning BH is given by the combination of three different motions: the Keplerian rotation around the BH; the radial drift, due to radial shear viscosity, and finally the Lense-Thirring precession, due to the gravitomagnetic field $\mathbf{H}_g$ generated by $\mathbf{J}_{\text{BH}}$ (see, e.g., Weinberg 1972; Thorne et al. 1986). In response to Lense-Thirring induced precession, viscous stresses in the disc acts rapidly to produce in the vicinity of the BH an axysymmetric configuration whereby adjacent fluid elements rotates in the equatorial plane of the spinning BH. The disc thus warps and the warp disturbance propagates diffusely (Papaloizou & Pringle 1983) in the disc.

As the Bardeen-Petterson effect modifies the inclination of the orbital plane of consecutive infinitesimal rings, then the warped profile of the accretion disc can be described by the specific angular momentum density, $L$, expressed as

$$L = L\hat{\bf i} = \Sigma \Omega_{\text{K}} R^2 \hat{\bf i}$$

(6)

where $\hat{\bf i}(R)$ is a unit vector indicating the local direction of the orbital angular momentum, $L$ is the modulus, $\Sigma$ is the surface density of the disc and $\Omega_{\text{K}}$ is the local Keplerian angular velocity. The angle describing the tilted disc is defined as

$$\theta(R) = \cos^{-1}(\hat{\bf i}(R) \cdot \mathbf{J}_{\text{BH}}),$$

(7)

so that $\hat{\bf i}(R)$ carries information of the warped structure of the accretion disc. The angular momentum of the accretion disc within radius $R$ is given by

$$\mathbf{J}_{\text{disc}}(R) = \int_{R_{\text{ISO}}}^{R} 2\pi x \mathbf{L}(x) \, dx$$

(8)

where the integration domain extends from the innermost stable orbit $R_{\text{ISO}}$ out to $R$. In order to calculate the total disc angular momentum we define an outermost radius, $R_{\text{out}}$. For an extended disc with $R_{\text{out}} \to \infty$, the disc angular momentum $\mathbf{J}_{\text{disc}}$ always dominates over $\mathbf{J}_{\text{BH}}$.

Real discs are likely to be truncated by their own self-gravity that becomes important at distances where the disc mass $M_{\text{disc}}(R) \sim (H/R) M_{\text{BH}}$ (see, e.g., Pringle 1981; Frank et al. 2002; Lodato 2007). Outside the truncation radius, gas can be either turned into stars or expelled by winds from stars which do form (Levin 2007, King & Pringle 2007). Thus, we are led to define a disc outer edge as the distance where the Toomre parameter for stability, $Q = \kappa_{c_\text{iso}} / (\pi G \Sigma)$ (where $\kappa^2 = R (dV^2/dR) + 4V^2$), becomes less than unity, and the cooling timescale of the clumping gas is less than its dynamical timescale. When the Toomre parameter drops toward unity, the disc becomes unstable on a timescale $\lambda = c_\text{s}^2 / (G \Sigma)$ (Polyachenko et al. 1997; Levine et al. 2008) for a nearly Keplerian, Shakura-Sunyaev $\alpha$-disc, this scale is much smaller than the disc radial dimension, and the cooling time of the associated perturbation is less or of the same order of its orbital period. Then, as long as the accretion disc can be described as a Shakura-Sunyaev disc $\alpha$, the external radius can be defined from the condition $Q(R_{\text{out}}) = 1$, so that

$$R_{\text{out}} = 1.21 \times 10^5 \alpha_{0.1}^{28/45} \alpha_0^{52/45} \frac{(f_{\text{Edd}})}{\eta_{0.1}}^{-22/45} \frac{M}{M_{\text{BH}}} R_\text{S},$$

(9)

where $R_\text{S} = 2GM_{\text{BH}}/c^2$ is the Schwarzschild radius. At the outer edge of the disc, $L(R_{\text{out}}) = J_{\text{disc, out}}$, and $\theta(R_{\text{out}}) = \theta_{\text{out}}$.

Definitions (6) and (8) for $L$ and $J_{\text{disc}}(R)$ hold for any disc profile. At first order, we can neglect details about the warped disc structure around $R_{\text{warp}}$ assuming $\hat{\bf i} \approx (0, 0, 1)$, and estimate the modulus of the orbital angular momentum within radius $R$, $J_{\text{disc}}(R)$, using Shakura-Sunyaev solutions for a flat disc. In this approximation, the surface density is $\Sigma_{\text{out}} \approx M / (3\pi R_L)$ (see, e.g., Pringle 1981; Frank et al. 2002)

$$L(R) \approx \frac{M}{3\pi R_L} \sqrt{GM_{\text{BH}}R}.$$ (10)

Using equations (8) and (10), and expression (4) for $v_1$, in the case of $\beta = 3/4$, the modulus of the disc angular momentum within $R$ reads:

$$J_{\text{disc}}(R) = \frac{8}{21} \frac{M \sqrt{GM_{\text{BH}}}}{A_{v_1}} R^{7/4}.$$ (11)

If expression (11) is estimated at the outer radius (9), the resulting dimensionless ratio between the disc and BH angular momenta is

$$\frac{J_{\text{disc}}(R_{\text{out}})}{J_{\text{BH}}} = 7.3 \alpha_{0.1}^{13/45} \alpha_0^{-37/45} \left(\frac{f_{\text{Edd}}}{\eta_{0.1}}\right)^{-7/45} a^{-1}.$$ (12)

3.2 Timescales and warp radius

The time-dependent evolution of the disc is described by the continuity equation

$$R \frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial R} (v_R \Sigma R) = 0,$$

(13)

where $v_R$ is the radial component of the velocity vector, and by the equation of conservation of angular momentum.

\footnote{This condition is fulfilled only for $(M_{\text{BH}}/f_{\text{Edd}})/(\alpha_{0.1} \eta_{0.1}) \gtrsim 4.3$. If this condition is not satisfied the gas temperature drops below $\sim 10^5$ K, in the external region of the disc where $Q$ is still greater than unity. The change in the opacity likely modifies the structure of the outer disc, and we can not explicitly use (13). In this paper we assume that the outer region is sufficiently extended to provide matter and angular momentum to the inner regions and use self-consistently the Shakura-Sunyaev model to describe the disc in regions where the gravitomagnetic interaction takes place.}
In presence of a gravitomagnetic field, for a geometrically thin disc characterized by the two viscosities $\nu_1$ and $\nu_2$, the equation reads \cite{Pringle1992}:
\[
\frac{\partial \mathbf{L}}{\partial t} = -\frac{1}{R} \frac{\partial}{\partial R} (R L_{\text{fr}}) + \frac{1}{R} \frac{\partial}{\partial R} \left( \nu_1 \frac{R^3}{d} \Omega \mathbf{I} \right) + \frac{1}{2} \frac{\partial}{\partial R} \left( \nu_2 R L_{\text{fr}} \frac{\partial \mathbf{I}}{\partial R} \right) + \frac{2G J_{\text{BH}} \times \mathbf{L}}{R^3}.
\]
(14)

The last term is the Lense-Thirring precession term and the associated angular velocity is
\[
\Omega_{\text{LT}}(R) = \frac{2G J_{\text{BH}}}{c^2 R^3}.
\]
(15)

The time-dependent equation \cite{14} describes the radial drift of matter and the diffusion of warping disturbances across the high-viscosity disc. This equation introduces several key scales:

(i) The viscous/accretion timescale for radial drift, related to the angular momentum transport parallel to $J_{\text{disc, out}}$, $\tau_{\text{acc}}(R)$. It can be seen as the time it takes for a fluid element at $R$ to accrete onto the BH (see, e.g., \cite{Pringle1981}). Considering equation \cite{14}, the balance between the advection term and the viscous term proportional to $\nu_1$ (both on the right side of equation \cite{14}) leads to an estimate of the accretion time:
\[
\tau_{\text{acc}}(R) \sim \frac{R^2}{\nu_1}.
\]
(16)

According to equation \cite{16}, we can introduce the disc consumption timescale $t_{\text{disc}}$, a concept useful when considering transient, truncated discs, as the accretion timescale at the outer radius:
\[
t_{\text{disc}} \sim \tau_{\text{acc}}(R_{\text{out}}) = 1.71 \times 10^6 \alpha_0^{-1/45} \frac{M_6^{11/45}}{\left(\frac{f_{\text{Edd}}}{\eta_{0.1}}\right)^{41/45}} \text{yr}.
\]
(17)

(ii) The timescale for warp propagation, related to the radial diffusion of gravitomagnetic perturbations that transport the component of the disc angular momentum lying in the plane of the disc; this scale is inferred from equation \cite{14} considering the term proportional to $\nu_2$,
\[
\tau_{\text{BP}}(R) \sim \frac{R^2}{\nu_2} \sim \frac{\nu_1}{\nu_2} \tau_{\text{acc}}(R).
\]
(18)

The physical interpretation of this timescale has been recently investigated by solving numerically equation \cite{14} for a thin disc starting at $t = 0$ with a flat disc misaligned relative to the fixed BH spin, $t \approx \tau_{\text{BP}}$ indicates the time it takes for the radial diffusion of the warp to reach radius $R$; on longer timescale, the disc approaches a steady warped state.

(iii) The characteristic extension of the warp $R_{\text{warp}}$, defined as the distance at which the Bardeen-Petterson timescale $\tau_{\text{BP}}(R)$ equals the Lense-Thirring precession timescale $\Omega_{\text{LT}}$:
\[
R_{\text{warp}} = \frac{4G J_{\text{BH}}}{\nu_2 c^2}.
\]
(19)

For power-law viscosity model, equations \cite{1}, \cite{5} and \cite{19} give
\[
R_{\text{warp}} = 476 \alpha_0^{-24/35} M_6^{47/35} \left(\frac{f_{\text{Edd}}}{\eta_{0.1}}\right)^{-6/35} a^{7/4} R_s.
\]
(20)

The warp radius represents the dividing between the outer region for $R \gg R_{\text{warp}}$, where the disc keeps its original inclination, given by $J_{\text{disc, out}}$, and the inner region for $R \ll R_{\text{warp}}$, where the disc aligns (or anti-aligns) its orbital angular momentum with the BH spin, $\mathbf{I} \parallel J_{\text{BH}}$. The warp radius fixes also the magnitude of the relevant Bardeen-Petterson timescale, which reads
\[
t_{\text{BP}}(R_{\text{warp}}) = 33.5 \alpha_0^{-34/35} f_{\nu_2}^{-12/7} M_6^{17/35} \times \left(\frac{f_{\text{Edd}}}{\eta_{0.1}}\right)^{-18/35} a^{5/7} \text{yr}.
\]
(21)

If we define the function
\[
\psi(R) = \frac{d\hat{\mathbf{I}}}{dR}
\]
(22)

and $R_{\text{BP}}$ the radius where the disc is maximally deformed
\[
\Psi \equiv \psi(R_{\text{BP}}) = \max (\psi),
\]
(23)

we expect that:
\[
R_{\text{BP}} = n_{\text{BP}} R_{\text{warp}}
\]
(24)

with $n_{\text{BP}}$ of order unity. $R_{\text{BP}}$ has two important properties: first, if it is the radius where the disc is maximally warped, i.e., where the difusive propagation of vertical perturbations is more significant; second, it provides a reliable estimate of the distance from the BH where the gravitomagnetic interaction is stronger. From equation \cite{14} this interaction is proportional to $\left(\mathbf{L} \times J_{\text{BH}}\right)/R^2$: this term vanishes in the inner part of the disc ($R \ll R_{\text{BP}}$) since the Bardeen-Petterson effect aligns $\mathbf{L}$ with $J_{\text{BH}}$, and also in the outer regions ($R \gg R_{\text{BP}}$), due to the rapid decline with $R$. Accordingly, the region near $R_{\text{BP}}$ (or equivalently $R_{\text{warp}}$) is the only one significantly misaligned with $J_{\text{BH}}$.

### 3.3 Analytical solutions

In this Section we summarise the properties of the steady warped disc structure used to compute the joint evolution of the disc and the BH.

Following previous studies we assume that the viscosity profiles are power-laws with exponent $\beta$, as in equation \cite{1}, and explore two possible cases. In the first, we formally extend the Shakura-Sunyaev solution everywhere in the disc, i.e. $\nu_1 \propto R^\beta$ with $\beta = 3/4$, and $\nu_2$ given by equation \cite{3} (Martin et al. 2007). In the second case, we assume the viscosities to remain approximately constant everywhere in the disc (Scheuer & Feiler 1996). In order to compare the two models (Martin et al. 2007, cfr.), we impose the continuity of the viscosities at $R_{\text{BP}}$ where the gravitomagnetic torque is most important.

Before solving equations \cite{13} and \cite{14} we introduce two appropriate reference frames. The first is the inertial reference frame $Oxyz$ referred to the outer disc; we can always rotate it so that its $z'$ axis is parallel to the direction of $J_{\text{disc, out}}$. The second reference frame is the not-inertial frame $O'x'y'z'$ referred to the BH spin, which is always centered to the BH and whose $z'$ axis is parallel to the black hole time varying spin $J_{\text{BH}}$. If we use the adiabatic approximation, then frame $O'x'y'z'$ can be approximated, with time $t$, as a sequence of frames, one for every quasi-stationary state of the system. The shape of the warped accretion disc is studied in the
O'x'y'z' frames and the cartesian components of any vector $v$ are there indicated as $v_x', v_y', v_z'$. Oxyz is the natural frame to study the temporal evolution of the BH spin and here the Cartesian components of the previous vector are denoted as $v_x, v_y, v_z$.

For a stationary state, continuity equation (13) can be easily intergrated introducing the accretion rate $\dot{M}$ as constant of integration:

$$R\dot{\Sigma}v_R = -\frac{\dot{M}}{2\pi}$$

while the projection of equation (14) along $i$ reads:

$$\left(\frac{3}{2}\nu_1 \frac{dL}{dR} - \frac{\dot{M} \sqrt{GM_{BH}}}{4\pi \sqrt{R}} \right) + \frac{1}{2} R \nu_2 L \left| \frac{d\hat{l}}{dR} \right|^2 = 0$$

In the small deformation approximation (Scheuer & Feiler 1996) the warp is gradual and we can neglect the non-linear term, proportional to $\partial A/\partial R$). Using the boundary condition $\Sigma(R_{ISO}) = 0$, the integral of (26) is

$$L(R) = \frac{\dot{M}}{3\pi \nu_1} \sqrt{GM_{BH} R} \left( 1 - \sqrt{\frac{R_{ISO}}{R}} \right).$$

This means that, in this approximation scheme, the modulus of the angular momentum density for a warped accretion disc far from the horizon is the same as for a flat disc, equation (10).

Following Scheuer & Feiler (1996) we study the disc profile of the steady disc introducing the complex variable $W' = \hat{l}_x' + i \hat{l}_y'$ and considering the case $\theta_{out} < \pi/2$. Using power-law viscosities according to (14), analytic solutions of equation (14), in the small deformation approximation have been found by Martin et al. (2007):

$$W_{PL} = B \left( \frac{R}{R_{warp}} \right)^{-\frac{\beta}{2}} \times K_{1/2(1+\beta)}\left( \frac{\sqrt{2}(1-i)}{(1+\beta)} \left( \frac{R}{R_{warp}} \right)^{-\frac{1+\beta}{2}} \right)$$

where $B$ is a complex constant of integration, depending on the boundary condition at the external edge, the subscript "PL" is a reminder of the power-law viscosities and $K_{1/2(1+\beta)}$ is the modified Bessel function of order $1/(2(1+\beta))$. In the particular case where we consider constant viscosities, i.e. $\beta = 0$, the solution can be written as

$$W_C = A \exp \left( -\sqrt{2} (1-i) \left( \frac{R}{R_{warp}} \right)^{-\frac{\beta}{2}} \right)$$

Figure 1. $\psi$ as function of $R/R_{warp}$, for $W_C$ (solid line) and $W_{PL}$ (dashed line) for different inclination angles: black lines correspond to $\theta_{out} = \pi/3$, blue lines to $\pi/30$, red lines to $\pi/300$. The vertical black dotted line represents $R_{BP}/R_{warp} = 0.42$, the Bardeen-Petterson radius for constant viscosity profiles. The parameters set for the BH and the disc is given by $M_{BH,0} = 10^6 M_{\odot}$, $\alpha = 0.5$, $f_{Edd} = 0.1$, $\alpha = 0.09$.

Figure 2. $\chi_{\beta=0}(R)$ (solid line) and $\chi_{\beta=3/4}$ (dashed lines) as functions of $R/R_{warp}$ for different inclination angles. Black lines correspond to $\theta_{out} = \pi/3$, blue ones to $\pi/30$, red ones to $\pi/300$; the vertical black dotted line represents $R_{BP}/R_{warp} = 0.42$, the Bardeen-Petterson radius for constant viscosity profiles. The parameters set for the BH and the disc is given by $M_{BH,0} = 10^6 M_{\odot}$, $\alpha = 0.5$, $f_{Edd} = 0.1$, $\alpha = 0.09$. In Figure 1 we plot the modulus of the gradient of $\hat{l}_i$, $\psi(R)$, which is a local measure of the deformation degree of the disc, for a particular set of parameters and for three different angles, $\theta_{out} = \pi/3, \pi/30, \pi/300$. The shape of $\psi$ is similar for the two different disc profiles and for all the angles; there is a well defined maximum near $R_{warp}$, where we
expect the disc to be more deformed. At radii smaller than $R_{\text{warp}}$ and far from $R_{\text{warp}}$ the disc is almost flat (note that the graph is logarithmic in both axes). For the constant viscosity (power-law) profile the peak is at $R_{\text{BP}} \approx 0.42R_{\text{warp}}$ ($R_{\text{BP}} \approx 0.38R_{\text{warp}}$). In Figure 1 we also see that a constant viscosity disc is less warped (since the maximum deformation is the smaller) than the power law viscosity disc. The ratio between the maximum deformations in the power law vs constant viscosity is roughly a factor 2, and it does not depend on the inclination angle (except for a scale factor, approximately equal to the ratio between the corresponding angles).

3.4 Validity of the approximation

We calculated the warped disc profile under the small deformation approximation. We neglected second order terms in equation (26) and found an analytic solution for $L$; in order to verify the consistence of this approximation, we define $\chi_{\beta}$ as the ratio between the neglected term and the first term into the round brackets of (26), assuming to have a Keplerian disc with power-law viscosity profile with exponent $\beta$, like in equation (4). Considering equations (27) for $L$, we have $dL/dR \approx (1/2 + \beta)(L/R)$ and then $\chi_{\beta}$ reads

$$\chi_{\beta}(R) = \frac{2}{3(\beta + 1)} \frac{\nu^2}{\nu_1} \left[ \frac{d}{dR} \right]^2 R^2. \tag{30}$$

Once we know the explicit solutions, the consistence of this approximation can be tested a posteriori calculating $\chi_{\beta}$: the approximation is well satisfied if $\chi_{\beta} \ll 1$. From equation (30), $\chi_{\beta}$ can be expressed also as a function of $R/R_{\text{warp}}$ and $\psi$:

$$\chi_{\beta} = \frac{2}{3(\beta + 1)} \frac{\nu^2}{\nu_1} R_{\text{warp}}^2 \left( \frac{R}{R_{\text{warp}}} \right)^2 \psi^2 \left( \frac{R}{R_{\text{warp}}} \right). \tag{31}$$

Figure 2 shows the function $\chi_{\beta=0}$ for constant viscosity profiles (dashed lines) and the function $\chi_{\beta=3/4}$ for power-law viscosity profiles (solid lines), for the same parameters as in Figure 1.

The function $\chi_{\beta}$ exhibits a maximum, $(\chi_{\beta})_{\text{max}}$, around $R_{\text{warp}}$. Far from $R_{\text{warp}}$ the accuracy of the approximation increases, albeit slowly. The function $\chi_{\beta}$ is most sensitive to the inclination angle, as expected (notice that Figure 2 uses logarithmic axes).

In Figure 3 we test the validity of the small deformation approximation plotting, in the BH mass versus $\theta_{\text{out}}$ plane, the color coded values of $(\chi_{\beta})_{\text{max}}$ for different values of the viscosity parameter $\alpha$ (Table 1), using the constant viscosity profile model (we fix $f_{\text{Edd}} = 0.1$ and $a = 0.9$). White zones represent the regions where $(\chi_{\beta=0})_{\text{max}} > 1$, i.e. where the small deformation approximation becomes invalid. $(\chi_{\beta=-0})_{\text{max}}$ shows mainly a strong dependence on inclination angle $\theta_{\text{out}}$, but also a weaker dependence on the BH mass which reveals that the small deformation approximation is less accurate for $M_{\text{BH}} \geq 10^6M_\odot$ and increasing BH mass. Comparing different $\alpha$ values, the approximation is better satisfied for large viscosities parameters (i.e. $\alpha = 0.18$). We repeated the analysis for the power-law viscosity model that shows no significant differences in the parameters dependence.

In Figure 4, using the same colours conventions, we explored $(\chi_{\beta})_{\text{max}}$ in the $\theta_{\text{out}}$ versus $a$ (left panels), and $f_{\text{Edd}}$ versus $\theta_{\text{out}}$ (right panels) planes, once we have fixed the viscosity parameter $(\alpha = 0.09)$, the BH mass $(M_{\text{BH}} = 10^6M_\odot)$, and $f_{\text{Edd}} = 0.1$ for the left panels and $a = 0.9$ for the right panels. For both constant $(\beta = 0)$ and power-law $(\beta = 3/4)$ models the relative inclination angle is again the leading parameter gauging the goodness of the fit as the approximation depends very weakly on $a$ and $f_{\text{Edd}}$.

4 BLACK HOLE EVOLUTION

4.1 Basic equations

In this section we explore the equations for the BH evolution. The BH is accreting and its mass increases, from an initial value $M_{\text{BH,0}}$, according to

$$\frac{dM_{\text{BH}}}{dt} = \frac{M E(R_{\text{ISO}})}{c^2} \tag{32}$$

where $E(R_{\text{ISO}})$ is the energy per unit mass of a test particle at the innermost stable orbit. $E(R_{\text{ISO}})/c^2 = 1 - \eta(a)$ is related to the efficiency $\eta(a)$ that depends only on the spin parameter $\eta(a)$ (Bardeen 1970; Bardeen et al. 1972). Equation (32) introduces a natural timescale for BH mass growth, known as Salpeter time $t_s$:

$$t_s = 4.5 \times 10^5 \frac{\eta}{f_{\text{Edd}}} \frac{M_{\text{BH}}}{1 - \eta} \text{ yr.} \tag{33}$$

As argued by Rees (1978) and shown by Thorne et al. (1986), there is a coupling between the BH spin and the angular momentum of the disc. Even though the disc is much less massive than the BH, the moving fluid elements perturb the Kerr metric and interact with the BH spin, causing spin precession, and if viscous dissipation is present, alignment. For an infinitesimal ring of inviscid matter with total angular momentum $J_{\text{ring}}$, the BH spin precesses, following the equation

$$\frac{dJ_{\text{BH}}}{dt} = \frac{2G M J_{\text{ring}}}{c^2 R^3} \times J_{\text{BH}}, \tag{34}$$

with a precession frequency

$$\Omega_{\text{precession}} = \Omega_{\text{LT}} \frac{J_{\text{ring}}}{J_{\text{BH}}} \tag{35}$$

Equation (34) can be extended to the case of an accretion disc to yield:

$$\frac{dJ_{\text{BH}}}{dt} = M\Lambda(R_{\text{ISO}}) \frac{\dot{M}(R_{\text{ISO}})}{c^2} + \frac{4\pi G}{c^2} \int_{\text{disc}} \frac{L(R) \times J_{\text{BH}}}{R^2} dR. \tag{36}$$

The first contribution is due to accretion of matter at $R_{\text{ISO}}$ where $\Lambda(R_{\text{ISO}})$ indicates the orbital angular momentum per unit mass carried by matter at ISO; the Bardeen-Petterson effect ensures that the direction of $\dot{J}(R_{\text{ISO}})$ is parallel or anti-parallel to $J_{\text{BH}}$, so that the accretion modifies only the spin modulus. As shown by Bardeen (1970), a variation of mass $\Delta M_{\text{BH}} = \sqrt{M_{\text{BH,0}}}$ is necessary to pass from a Schwarzschild BH ($a = 0$) to an extreme Kerr BH ($a = 1$), while spin flip of $\pi$, due only to accretion on an initially extreme Kerr BH, needs $\Delta M_{\text{BH}} = 3M_{\text{BH,0}}$. So, the spin accretion timescale for the spin modulus is of the same order of the mass accretion timescale $t_s$. The second term in equation (36) describes the gravitomagnetic interaction.
Mass and spin coevolution during the alignment of a BH

\[ (\chi_\beta = 0)_{\text{max}} \]

\[ \alpha = 0.05 \]

\[ \alpha = 0.09 \]

\[ \alpha = 0.15 \]

\[ \alpha = 0.18 \]

\[ M_{\text{BH}} / 10^6 \, M_\odot \]

\[ M_{\text{BH}} / 10^6 \, M_\odot \]

\[ \theta_{\text{out}} \]

\[ \theta_{\text{out}} \]

\[ \theta_{\text{out}} \]

\[ \theta_{\text{out}} \]

Figure 3. Color coded plot of \((\chi_\beta)_{\text{max}}\) in the \(\theta_{\text{out}}\) versus \(M_{\text{BH}}\) plane, for four different \(\alpha\) parameters: \(\alpha = 0.05\) top left panel, \(\alpha = 0.09\) top right panel, \(\alpha = 0.15\) bottom left panel, \(\alpha = 0.18\) bottom right panel. The disc has constant viscosity profiles, i.e. \(\beta = 0\). The accretion rate is \(f_{\text{Edd}} = 0.1\) and the spin parameter is \(a = 0.9\).

between the rotating viscous disc and the BH spin vector. This term modifies only the spin direction of the BH in order to conserve the total angular momentum of the system. Under the working hypothesis that the disc is continually fed by matter carrying the same angular momentum (see Section 6 for a critical discussion), the BH aligns its spin \(J_{\text{BH}}\) in the direction of \(J_{\text{disc, out}}\). Alignment implies that \(\theta_{\text{out}}(t) = \cos^{-1}(\hat{J}_{\text{BH}}(t) \cdot \hat{J}_{\text{disc, out}})\) goes to 0 with time. Figure 5 shows the function \(I\) defined as the modulus of the integral kernel of equation (36)

\[ I(R) = \frac{4\pi G L(R)J_{\text{BH}} \sin[\theta(R)]}{R^2} \]

as a function of \(R/R_{\text{warp}}\), for different value of \(\theta_{\text{out}} = \pi/3, \pi/30, \pi/300\), where \(\theta(R)\) is computed along the profile of the steady warped disc of equation (25) and (29). The function \(I\), similarly to \(\psi\) (defined in eq. [22]), peaks near \(R_{\text{warp}}\). Contrary to \(\psi\), power law viscosity profiles have lower peaks, compared with constant viscosity profiles. This figure indicates also that the BH-disc gravitomagnetic interaction is spread over a relatively small region of the disc around the warp radius; the characteristic spreading length, which is slightly larger for constant viscosity profiles, is usually of a few warp radii.
4.2 Alignment time

In this Paragraph we want to give simple estimations for the alignment and the precession timescales, starting from equation (36).

Assuming BH mass and spin modulus variations due to accretion to be small compared with gravitomagnetic effects during the alignment, we neglect the term proportional to $\Lambda(R_{ISOL})$ in (36); if BH spin aligns and precess, left hand side of (36) can be estimated introducing a characteristic gravitomagnetic timescale $\tau_{gm}$ as

$$\left| \frac{dJ_{BH}}{dt} \right| \sim \frac{\Delta J_{BH}}{\tau_{gm}} \sim \frac{J_{BH} \sin \theta_{out,0}}{\tau_{gm}}.$$  

and the integral on the right hand side as

$$\left| 4\pi G \int_{\text{disc}} \frac{L(R) \times J_{BH}}{R^2} dR \right| \sim \frac{4\pi G \; L(R_{warp}) \; J_{BH} \sin \theta_{out}}{c^2 \; R_{warp}}.$$  

since the bulk of the gravitomagnetic interaction occurs around $R_{warp}$. Equating these two expressions and using equation (27) for the specific angular momentum density modulus, we obtain

$$\tau_{gm} \sim \frac{3 \; c \; \nu_1(R_{warp}) \; \sqrt{\frac{R_{warp}}{R_s}}}{4 \; GM} \sqrt{\frac{R_{warp}}{R_s}}.$$  

Using equation (19) and (11) which imply $M \sqrt{G M_{BH}/\nu_1(R_{warp})} \approx (21/8) \; J_{\text{disc}}(R_{warp}) \; R_{warp}^{-5/2}$, the gravitomagnetic scale $\tau_{gm}$ can be written in terms of the
Bardeen-Petterson warp timescale (eq. [18]):

$$\tau_{\text{gm}} \sim \frac{4\sqrt{2}}{\gamma} \frac{J_{\text{BH}}}{J_{\text{disc}}(R_{\text{warp}})} \tau_{\text{BP}}(R_{\text{warp}})$$

(39)

and also in term of the accretion timescale (eq. [16])

$$\tau_{\text{gm}} \sim \frac{4\sqrt{2}}{\gamma} \frac{\nu_{1}}{\nu_{2}} \frac{J_{\text{BH}}}{J_{\text{disc}}(R_{\text{warp}})} \tau_{\text{acc}}(R_{\text{warp}})$$

(40)

where $J_{\text{disc}}(R_{\text{warp}})$ is the disc angular momentum modulus within the warp radius, estimated by [11]. Finally, considering equation [11] and [16] for $\tau_{\text{acc}}$ together with the expression for the spin modulus and Schwarzschild radius, $\tau_{\text{gm}}$ of expression [40] can be rearranged as

$$\tau_{\text{gm}} \sim \frac{3}{2} \frac{a}{\nu_{2}} \frac{M_{\text{BH}}}{M} \sqrt{\frac{R_{S}}{R_{\text{warp}}}} \frac{R_{S}}{M_{\text{BH}}^2}.$$

(41)

Since the disc carries very little angular momentum at the warp radius, from equation [39] $\tau_{\text{gm}} \gg \tau_{\text{BP}}$, always. The gravitomagnetic BH-disc interaction causes BH spin precession and alignment at the same time, and then introduces two scales related with $\tau_{\text{gm}}$, the precession and the alignment timescales, $t_{\text{precc}}$ and $t_{\text{al}}$ respectively. We separate their relative importance following Martin et al. (2007) results, and define the parameter $\mu$, so that

$$t_{\text{al}} = \frac{\tau_{\text{gm}}}{\cos \mu}, \quad t_{\text{precc}} = \frac{\tau_{\text{gm}}}{\sin \mu}.$$

(42)

The exact value of $\mu$ depends on the viscosity profile, and can be estimated either analytically (Martin et al., 2007), or numerically as in this paper. Initially, we assume alignment and precession to have the same timescale, $\cos \mu = \sin \mu = \sqrt{2}/2$

according to [Scheuer & Feiler (1996)]. Substituting expressions [9] for the viscosities, [20] for the warp radius, and [38] for $\tau_{\text{gm}}$ in [12], the alignment time reads

$$t_{\text{al}} = 1.13 \times 10^5 \frac{\alpha^{58/35}}{\alpha_{0.1}^{58/35}} f_{\nu_{2}}^{-5/7} M_{6}^{-2/35} \left(\frac{f_{\text{Edd}}}{\gamma_{0.1}}\right)^{-32/35} a^{5/7} \text{yr.}$$

(43)

The timescale $t_{\text{al}}$ increases with $a$, indicating that a rapidly rotating Kerr BH offers some resistance before changing its direction. Interestingly, the alignment timescale does not depend on the initial inclination $\theta_{\text{out,0}}$ since a more inclined configuration implies more pronounced disc deformations and stronger mutual gravitomagnetic interactions (as also shown in Figure 2 and 3). $t_{\text{al}}$ has a weak dependence on the BH mass and scales nearly as $M^{-5/7}$: a higher accretion rate implies a higher angular momentum density $L(R)$ and thus a stronger gravitomagnetic coupling. We notice also that, apart from numerical factors of order unity, this timescale is consistent with the alignment scales found by [Scheuer & Feiler (1996); Natarajan & Pringle (1998); Natarajan & Armitage (1999); Martin et al. (2007)].

4.3 The adiabatic approximation

In Section 3.2 and 4.1 we described the equations governing the evolution of a warped accretion disc around a fixed BH, and the evolution of an accreting Kerr BH in gravitomagnetic interaction with its accretion disc. The BH and the accretion disc evolve contemporary and their evolution is coupled, so that we can solve simultaneously equations [13] and [14] for a Keplerian disc and [42] and [36] for the accreting and precessing BH.

In this paper, we solve these coupled equations using the *adiabatic* approximation that separates the rapid temporal evolution of the warped disc from the longer temporal evolution of the BH. Equations are integrated starting from given initial conditions: at $t = 0$ the BH spin $\hat{\alpha}$ is inclined with respect to the BH and disc timescales, as functions of $M_{\text{BH}}$ and $f_{\text{Edd}}$, for two selected values of the viscosity and spin parameter: $\alpha = 0.15$ and $\alpha = 0.9$. In Figure 6 and Figure 7, we draw in the $M_{\text{BH}}$-$f_{\text{Edd}}$ plane lines of constant $t_{\text{BP}}(R_{w})/t_{\text{al}}$ and $t_{\text{al}}/t_{\text{al}}$ ratios. The comparison between the different timescales lead to the following hierarchy of timescales:

$$t_{\text{BP}}(R_{w}) \ll t_{\text{al}} \ll t_{s}.$$  

(44)

Then, in the adiabatic approximation, the disc transits through a sequence of warped states over the shortest timescale $t_{\text{BP}}(R_{w})$, while, on the longer timescale $t_{\text{al}}$, the BH aligns its spin to $J_{\text{out,0}}$ and modifies a little its spin modulus and mass due to accretion. Considering one of these disc quasi-steady states, initially at time $t$, after a time gap $\delta t \sim t_{\text{BP}}(R_{w})$ the BH mass and spin $J_{\text{BH}}$ are updated according to

$$\begin{align*}
M_{\text{BH}}(t + t_{\text{BP}}(R_{w})) & = M_{\text{BH}}(t) + \delta M_{\text{BH}} \\
J_{\text{BH}}(t + t_{\text{BP}}(R_{w})) & = J_{\text{BH}}(t) + \delta J_{\text{BH}}
\end{align*}$$

(45)
For the BH mass variation state at \( t \) and these variations produce a new quasi-stationary warped interaction and changes only the spin direction. After the interaction, \( \delta J \) where

\[
\delta J = \frac{J(R_{\text{ISO}})}{t_{\text{BP}}(R_{\text{ISO}})}
\]

For the spin variation, we need to integrate equation (36) from \( t \) to \( t + t_{\text{BP}}(R_{\text{BP}}) \):

\[
\delta M_{\text{BH}} \approx \dot{M} \left( \frac{E(R_{\text{ISO}})}{c^2} \right) t_{\text{BP}}(R_{\text{BP}})
\]

where \( R_{\text{ISO}} \) is the last innermost stable orbit associated with the current value of \( a(t) \). For the spin variation, we need to integrate equation (36) that includes the two different and coupled contributions due to accretion and gravitomagnetic interaction; if \( \delta M_{\text{BH}} \) and \( \delta J \) are small on the timescale \( t_{\text{BP}}(R_{\text{BP}}) \), to first order the two contributions decouple and they can be integrated separately:

\[
(\delta J_{\text{BH,acc}}) \approx M \Lambda(R_{\text{ISO}}) t_{\text{BP}}(R_{\text{BP}})
\]

\[
(\delta J_{\text{BH,gm}}) \approx \frac{4\pi G}{c^2} t_{\text{BP}}(R_{\text{BP}}) \int_{\text{disc}} L(R,t) \times J_{\text{BH}}(t) \frac{R^2}{dR}
\]

where \( (\delta J_{\text{BH,acc}}) \) is due to accretion and changes only the spin modulus while \( (\delta J_{\text{BH,gm}}) \) is due to gravitomagnetic interaction and changes only the spin direction. After the interval \( t_{\text{BP}}(R_{\text{BP}}) \), the angular momentum of \( J_{\text{BH}} \) are updated according to this rule

\[
J_{\text{BH}}(t + t_{\text{BP}}(R_{\text{BP}})) = \left( J_{\text{BH}}(t) + (\delta J_{\text{BH}})_{gm} \right)
\times \frac{J_{\text{BH}}(t) + (\delta J_{\text{BH}})_{acc}}{J_{\text{BH}}(t)}
\]

This procedure can be repeated iteratively on a timescale \( t_{\text{al}} \) to study the coupled evolution of \( L(R,t), J_{\text{BH}} \) and \( M_{\text{BH}} \) during the alignment process.

5 SPIN ALIGNMENT

5.1 Set up

In this Section we study the coupled evolution of the BH and warped accretion disc using the approximation scheme described in the previous Section, in order to infer the evolution of \( M_{\text{BH}} \) and \( J_{\text{BH}} \) as a function of time, in response to the gravitomagnetic interaction and matter accretion.

At \( t = 0 \) the outer disc, extending up to a radius \( R_{\text{out}} \), defines the fixed reference frame \( Oxyz \). In this frame the external edge of the disc lies in the \( x, y \) plane and the orbital angular momentum at \( R_{\text{out}} \) is

\[
(L_x(R_{\text{out}}), L_y(R_{\text{out}}), L_z(R_{\text{out}})) = L(R_{\text{out}})(0,0,1)
\]

while the BH spin is initially inclined of \( \theta_{\text{out},0} \) with respect to the \( z \) axis:

\[
(J_{\text{BH},x}, J_{\text{BH},y}, J_{\text{BH},z}) = J_{\text{BH}}(\sin \theta_{\text{out},0}, 0, \cos \theta_{\text{out},0}).
\]

If at \( t \neq 0 \) we know the components of \( J_{\text{BH}} \) in the fixed reference frame \( Oxyz \) there is always a rotated reference frame \( Ox'y'z' \) where \( J_{\text{BH}} \) is along the new \( z' \) axis (see also the discussion about reference frames of Section 3.3). The two reference frames are related by a rotation \( \mathcal{R} \), which depends only on the components \( J_{\text{BH},x}, J_{\text{BH},y}, J_{\text{BH},z} \) of \( J_{\text{BH}}(t) \) in the fixed reference frame. If \( \mathcal{R}_{ij} \) is the matrix associated with this rotation, we can easily find the components of \( J_{\text{BH}}(t) \) and \( L(R_{\text{out}}, t) \) in the rotated frame:

\[
J_{\text{BH},i}(t) = \mathcal{R}_{ij} J_{\text{BH},j}(t) \Rightarrow J_{\text{BH}}(t) = (0,0,J_{\text{BH}}(t))
\]

\[
L'_i(R_{\text{out}}, t) = \mathcal{R}_{ij} L_j(R_{\text{out}}, t).
\]

As shown by Scheuer & Feiler (1996) for the constant viscosity profile and Martin et al. (2007) for the power-law viscosity profile, in this special rotated frame of reference it is possible to calculate analytically the expression of the
Figure 8. Results for precession and alignment processes. Black lines refer to our result while red lines refer to results published by Martin et al. (2007); solid lines (dashed lines) refer to constant (power-law) viscosity profile. Top left panel represents temporal evolution of relative inclination angle $\theta_{\text{out}}$ while top right shows evolution of $J_{\text{BH,x}}/J_{\text{BH}}$ against $J_{\text{BH,y}}/J_{\text{BH}}$, both for an initial BH with $M_{\text{BH,0}} = 10^6 M_\odot$, $a_0 = 0.5$ and an accretion disc with $f_{\text{Edd}} = 0.1$ and $\alpha = 0.09$, with $\theta_{\text{out},0} = \pi/6$. Blue dashed line represents the evolution of the spin components for a pure precession motion around $\hat{J}_{\text{disc},\text{out}}||\hat{z}$. In bottom left (right) panel we represent evolution of $J_{\text{BH,x}}/J_{\text{BH}}$ for an initial relative inclination angle $\theta_{\text{out},0} = \pi/30$ ($\theta_{\text{out},0} = \pi/3$), for an initial BH with $M_{\text{BH,0}} = 10^6 M_\odot$, $a_0 = 0.5$ and an accretion disc with $f_{\text{Edd}} = 0.1$ and $\alpha = 0.09$.

5.2 Results

We computed, within the adiabatic approximation, the joint evolution of the BH mass and spin during the process of alignment under the assumption that matter is corotating with the BH. We iterated equations (46) and (49), from the initial conditions (50) and (51), recording the updated values of $M_{\text{BH}}$, $a$ and of the relative inclination angle $\theta_{\text{out}}$ every snapshot of time $\delta t \sim t_{\text{BP}}(R_{\text{warp}})$. We initially choose a spinning BH with $M_{\text{BH}} = 10^6 M_\odot$ and $a = 0.5$, and an ac-
Figure 9. Coupled evolution of the relative inclination angle $\theta_{\text{out}}$, BH mass $M_{\text{BH}}$ and spin parameter $a$. Solid (dashed) lines refer to constant (power-law) viscosity profile. Black lines to $f_{\text{Edd}} = 1$, red lines to $f_{\text{Edd}} = 0.1$, blue lines to $f_{\text{Edd}} = 0.01$. Dotted horizontal lines which appear in top panels represent angles $\theta_{\text{out}}/\theta_{\text{out,0}} = 10^{-1}, 10^{-2}, 10^{-3}$. Initial configuration: $M_{\text{BH},0} = 10^6 M_\odot$, $a_0 = 0.5$, $f_{\text{Edd}} = 0.1$ and $\alpha = 0.09$, with $\theta_{\text{out,0}} = \pi/6$.

Figure 9 shows the evolution of $\theta_{\text{out}}$ and $a$ as functions of time and of the increasing BH mass, for an initial BH with $M_{\text{BH},0} = 10^6 M_\odot$, $\theta_{\text{out,0}} = \pi/6$ and spin parameter $a_0 = 0.5$, and for $f_{\text{Edd}} = 1, 0.1$, and 0.01. Both constant and power-law viscosity profiles are explored, always with viscosity parameter $\alpha = 0.09$. Alignment is a process that shows a strong
dependence on the accretion rate: for the constant (power-

law) viscosity model the time necessary to reduce the rela-
tion inclination angle by a factor 100 varies from 3.0 \times 10^5\,\text{yr} (5.3 \times 10^5\,\text{yr}) for $f_{\text{Edd}} = 1$ to 1.86 \times 10^5\,\text{yr} (3.63 \times 10^5\,\text{yr}) for $f_{\text{Edd}} = 0.01$. During this alignment time, the BH has in-
creased its mass by a small fraction, between 0.74\% (1.30\%) for $f_{\text{Edd}} = 1$ and 0.46\% (0.89\%) for $f_{\text{Edd}} = 0.01$. The spin parameter $a$ increases due to accretion, but only by a small amount, between 3.13\% (5.47\%) for $f_{\text{Edd}} = 1$ to 1.96\% (3.79\%) for $f_{\text{Edd}} = 0.01$. In Table 2 we summarize the results of Figure 9.

5.3 Exploring the parameters space

Here we explore more systematically how the fractional in-
creases of $M_{\text{BH}}$ and $a$, and the alignment time vary with initial mass $M_{\text{BH,0}}$, spin $a_0$, $f_{\text{Edd}}$ and $a$, for both constant and power-law viscosity profiles, fixing $\theta_{\text{out,0}} = \pi/6$. The evolution is followed until $\theta_{\text{out}}$ has decreased by a factor 100; we define as $\Delta t_{\theta_0\rightarrow\theta_0/100}$ the corresponding "alignment" time, computed self-consistently. We also infer from the numerical model the relative growths of BH mass $\Delta M_{\text{BH}}/M_{\text{BH,0}}$ and spin parameter $\Delta a/a_0$ during $\Delta t_{\theta_0\rightarrow\theta_0/100}$.

Figure 10 and Figure 11 show the weak dependence of the alignment time $\Delta t_{\theta_0\rightarrow\theta_0/100}$ on the initial BH mass $M_{\text{BH,0}}$, and of the relative mass and spin parameter increases, for eight different sets of the other parameters. Comparing numerical scaling factors for $M_0$ in $\Delta t_{\theta_0\rightarrow\theta_0/100}$ with that of expression 43, we notice again a good agreement, in particular for $f_{\text{Edd}}$ not too close to the Eddington limit and $M_{\text{BH,0}} \lesssim 10^8M_\odot$.

By contrast, the alignment process is more sensitive on $f_{\text{Edd}}$, $a_0$ and $a$. Color-coded maps of $\Delta t_{\theta_0\rightarrow\theta_0/100}$ (Figure 12), of $\Delta M/M_{\text{BH,0}}$ (Figure 13) and $\Delta a/a_0$ (Figure 14) are constructed in the $a_0$ versus $f_{\text{Edd}}$ plane, varying the coefficient $\alpha$ and the viscosity law inside the accretion disc. In Figure 12 we infer the interval of the alignment time $\Delta t_{\theta_0\rightarrow\theta_0/100}$ (as inferred from the numerical model) of interest for the study of BH evolution. The alignment time can vary by many orders of magnitude from $\sim 10^5\,\text{yr}$ to $\sim 10^{10}\,\text{yr}$, and it reveals strong dependencies both on the accretion rate and on the initial spin parameter. In addition, smaller viscosities ($\alpha = 0.09$) gives shorter timescales compared to higher viscosities ($\alpha = 0.18$). A simple-

| VP | $f_{\text{Edd}}$ | $\theta_{\text{out,0}}/\theta_{\text{out,0}}$ | $\Delta t$ (10^6\,\text{yr}) | $\Delta t/t_{\text{al}}$ | $\Delta M_{\text{BH}}/M_{\text{BH,0}}$ (in units of 10^{-2}) | $\Delta a/a_0$ (in units of 10^{-2}) |
|-----|------------------|---------------------------------|-----------------|----------------|---------------------------------|----------------|
|     | 1                |                                 |                 |               |                                 |                |
|     | $10^{-1}$        | 0.15                            | 2.2             | 0.67         | 1.57                            |                |
|     | $10^{-2}$        | 0.30                            | 4.4             | 0.74         | 3.13                            |                |
|     | $10^{-3}$        | 0.45                            | 6.6             | 1.11         | 4.69                            |                |
|     | 0.1              |                                 |                 |               |                                 |                |
|     | $10^{-1}$        | 1.12                            | 2.0             | 0.29         | 1.24                            |                |
|     | $10^{-2}$        | 2.33                            | 4.2             | 0.58         | 2.46                            |                |
|     | $10^{-3}$        | 3.52                            | 6.3             | 0.87         | 3.68                            |                |
|     | 0.01             |                                 |                 |               |                                 |                |
|     | $10^{-1}$        | 9.28                            | 2.0             | 0.23         | 0.99                            |                |
|     | $10^{-2}$        | 18.6                            | 4.1             | 0.46         | 1.96                            |                |
|     | $10^{-3}$        | 27.9                            | 6.1             | 0.68         | 2.94                            |                |
|     | 0.1              |                                 |                 |               |                                 |                |
|     | $10^{-1}$        | 2.18                            | 3.9             | 0.54         | 2.30                            |                |
|     | $10^{-2}$        | 4.39                            | 7.9             | 1.08         | 4.57                            |                |
|     | $10^{-3}$        | 6.64                            | 11.9            | 1.64         | 6.84                            |                |
|     | 0.01             |                                 |                 |               |                                 |                |
|     | $10^{-1}$        | 18.1                            | 3.9             | 0.45         | 1.91                            |                |
|     | $10^{-2}$        | 36.3                            | 7.9             | 0.89         | 3.79                            |                |
|     | $10^{-3}$        | 54.7                            | 11.9            | 1.35         | 5.67                            |                |

Table 2. Summary of our parameters and results for the co-rotating case; we consider viscosity coefficient $\alpha = 0.09$ and initial inclination angle $\theta_{\text{out,0}} = \pi/6$, both for constant (C) and power-law (PL) viscosity profiles (VP). The initial BH has $M_{\text{BH,0}} = 10^6M_\odot$ and $a_0 = 0.5$. Accretion rate $f_{\text{Edd}}$ varies over three orders of magnitude and we record times needed to decrease the relative inclination angle of a factor 10, 100 or 1000, comparing it with estimated alignment timescale, equation 43; we also report mass and spin relative variations.

Mass and spin coevolution during the alignment of a BH

The spin parameter $a$ and the viscosity law inside the accretion disc. In Figure 12 we infer the interval of the alignment time $\Delta t_{\theta_0\rightarrow\theta_0/100}$ (as inferred from the numerical model) of interest for the study of BH evolution. The alignment time can vary by many orders of magnitude from $\sim 10^5\,\text{yr}$ to $\sim 10^{10}\,\text{yr}$, and it reveals strong dependencies both on the accretion rate and on the initial spin parameter. In addition, smaller viscosities ($\alpha = 0.09$) gives shorter timescales compared to higher viscosities ($\alpha = 0.18$). A simple-com
comparison between alignment times $\Delta t_{\theta_0}$ for different initial spin parameters, but identical $f_{\text{Edd}}$, reveals that the scaling factors for $a$ and $\theta_0$, $f_0$ in equation (43) are in good agreement with numerical results.

Figure 13 shows that the relative amount of mass accreted during the alignment process is small, compared to the initial BH mass. It varies between $\sim 10^{-9}$ and $\sim 10^{-2}$ for the constant viscosity profile, and between $\sim 2.5 \times 10^{-3}$ and $\sim 3 \times 10^{-2}$ for the power-law viscosity profile. Even if the accretion rate varies over four orders of magnitude, there are no comparable variations for the relative BH mass growth, $\Delta M_{\text{BH}}$, which increases significantly during the alignment for initially counter-aligned BHs, but the particles at its innermost stable orbit carry the outer regions of the accretion disc, if this disc is regularly and coherently fed. Due to the Bardeen-Petterson effect, we expect the innermost part of the disc (approximately within $R_{\text{warp}}$) to orbit in a plane which is perpendicular to $J_{\text{BH}}$, with orbital angular momentum density $\mathbf{L}$ counter-aligned with respect to the BH spin. In this BH-disc configuration, one of the major changes is in the radius of the innermost stable orbit, which increases due to the asymmetry seeded in the geodetic motion of particles in Kerr metrics. As a consequence, the energy and the orbital angular momentum of particles at $R_{\text{ISO}}$ increase, while the BH radiative efficiency decreases (see, i.e., Wilkins [1972] Bardeen et al. [1972]. The Bardeen-Petterson timescale $\tau_{\text{BP}}$ and the warp radius $\tau_{\text{d}}$ have the same values as in the co-rotating case. Since $\tau_{\text{BP}} \ll \tau_{\text{d}}$, the adiabatic approximation holds again, but the small deformation approximation [7] has a limited validity, requiring $\theta_{\text{out}} \sim \pi$. In order to remain consistent with the approximation scheme, we trace the alignment process, from $\pi$ to $(\pi - \pi/6)$, only.

In the counter-rotating case and small deformation approximation (i.e. $\theta_{\text{out}} \sim \pi$), the disc profile can be solved analytically. We choose a reference frame $O''x''y''z''$ where $\mathbf{J}_{\text{BH}} = (0, 0, -1)$ and we solved equation (14) for $i$ in it. For constant viscosities, the function $W''(R/R_{\text{warp}}) = T'' + i''_{\psi}$

\[ T'' + i''_{\psi} \]

5.4 Counter-rotating case

In this Section, we investigate the counter-rotating configuration for a BH and its misaligned accretion disc, for initial values of $\theta_{\text{out},0}$ close to $\pi$.

As shown by Scheuer & Feiler [1996] and Martin et al. [2007], on the timescale $t_{\text{al}}$ the BH spin again aligns with the outer regions of the accretion disc, if this disc is regularly and coherently fed. Due to the Bardeen-Petterson effect, we expect the innermost part of the disc (approximately within $R_{\text{warp}}$) to orbit in a plane which is perpendicular to $J_{\text{BH}}$, with orbital angular momentum density $\mathbf{L}$ counter-aligned with respect to the BH spin. The Bardeen-Petterson timescale $\tau_{\text{BP}}$ and the warp radius $\tau_{\text{d}}$ have the same values as in the co-rotating case. Since $\tau_{\text{BP}} \ll \tau_{\text{d}}$, the adiabatic approximation holds again, but the small deformation approximation [7] has a limited validity, requiring $\theta_{\text{out}} \sim \pi$. In order to remain consistent with the approximation scheme, we trace the alignment process, from $\pi$ to $(\pi - \pi/6)$, only.

In the counter-rotating case and small deformation approximation (i.e. $\theta_{\text{out}} \sim \pi$), the disc profile can be solved analytically. We choose a reference frame $O''x''y''z''$ where $\mathbf{J}_{\text{BH}} = (0, 0, -1)$ and we solved equation (14) for $i$ in it. For constant viscosities, the function $W''(R/R_{\text{warp}}) = T'' + i''_{\psi}$

\[ T'' + i''_{\psi} \]
\[ \Delta t_{\theta_0 \rightarrow \theta_0}/100 \left( f_{\text{Edd}}, a_0 \right) \]

Describing the warp is

\[ W_{C,\text{cut}}'' = C \exp \left( -\sqrt{2} \left( 1 + i \right) \left( \frac{R}{R_{\text{warp}}} \right)^{-\frac{1}{2}} \right) \]  

(55)

while for a power-law viscosity profile

\[ W_{\text{PL, cut}}'' = D \left( \frac{R}{R_{\text{warp}}} \right)^{-\frac{1}{4}} \times K_{1/2(1+\beta)} \left( \frac{\sqrt{2}(1+i)}{(1+\beta)} \left( \frac{R}{R_{\text{warp}}} \right)^{-\frac{1+2\beta}{2}} \right) \]  

(56)

We then apply the adiabatic approximation to study the coupled evolutions of the system BH-disc. The jointed evolutions of \( \theta_{\text{out}} \), \( M_{\text{BH}} \) and \( a \) are presented in Figure 15 and in Table 3 for different accretion rates and viscosity profiles.

The shorter timescales in the counter-rotating configuration stem from the dependence of the alignment timescale on the spin modulus, \( t_{\text{al}} \propto a^{5/7} \). Counter-rotating matter carries larger and opposite angular momentum, reducing the spin modulus and the alignment timescale in the process.

6 DISCUSSION AND CONCLUSIONS

In this paper, we followed the joint evolution of the mass \( M_{\text{BH}} \) and spin \( J_{\text{BH}} \) of a BH inside a geometrically thin, ex-
Figure 13. Color coded map of the relative increase of mass during the alignment time (defined as the time necessary for the relative inclination angle to go from $\theta_{\text{out},0} = \pi/6$ to $\theta_{\text{out}} = \pi/600$) for a BH of $M_{\text{BH},0} = 10^6 M_\odot$, as a function of the accretion rate expressed through the Eddington factor $f_{\text{Edd}}$ and of the initial BH spin parameter $a_0$. The colour scale represents $\Delta M_{\text{BH}}/M_{\text{BH},0}$. Top (bottom) panels refer to the constant (power-law) viscosity profile. Left (right) panels refer $\alpha = 0.09$ ($\alpha = 0.18$).

The BH spin is initially misaligned with the angular momentum of the disc in its outer regions. On the short Bardeen-Petterson timescale, the disc responds to the Lense-Thirring precession, imposed by the BH spin, and propagates a warp that is maximum around $R_{\text{warp}}$; within this radius matter orbits around the BH in a plane which is perpendicular to the BH spin. According to angular momentum conservation, the warped disc interacts with the BH spin and, on the longer alignment timescale, the BH aligns its spin to $\hat{J}_{\text{disc,out}}$. In its outer regions, the disc is assumed to be fed by matter that flows along a plane that keeps its coherence in direction $\hat{J}_{\text{disc},\text{out}}$ for a sufficiently long timescale to allow for the gravitomagnetic interaction to complete BH-disc alignment. While doing so the BH is accreting matter and angular momentum from the inner portion of the disc, which is aligned or anti-aligned to the BH spin. Given the mismatch between the timescale for warp propagation and the alignment time [Scheuer & Feiler 1996; Natarajan & Pringle 1998], we devised a method that enabled us to follow, in the small-deformation approximation, the co-evolution of the BH mass and spin in a self-consistent manner, carrying out a large survey of the parameter space and a critical review of the used approximations.

It is found that, considering an initial small relative inclination angle ($\theta_{\text{out},0} \lesssim \pi/6$, small deformation approximation), matter in the inner part of the accretion disc has orbital angular momentum density parallel to $J_{\text{BH}}$. The gravitomagnetic interaction of the BH with this warped accretion
Figure 14. Color coded map of the relative increase of spin parameter during the alignment time (defined as the time necessary for the relative inclination angle to go from $\theta_{\text{out},0} = \pi/6$ to $\theta_{\text{out}} = \pi/600$) for a BH of $M_{\text{BH},0} = 10^6 M_\odot$, as a function of the accretion rate expressed through the Eddington factor $f_{\text{Edd}}$ and of the initial BH spin parameter $a_0$. The colour scale represents $\Delta a/a_0$. Top (bottom) panels refer to the constant (power-law) viscosity profile. Left (right) panels refer $\alpha = 0.09$ ($\alpha = 0.18$).

disc and their coupled evolution bring the BH into alignment with the outer regions of the disc, i.e. $\theta_{\text{out}}(t) \rightarrow 0$. The timescale $t_{\text{al}}$ of equation (43) gives a good estimate of the BH-disc alignment time for an $\epsilon$-folding reduction of the angle of misalignment, in very good agreement with numerical results. For a maximally rotating Kerr BH accreting at the Eddington rate, $t_{\text{al}} \sim 10^{5-6}$ yr, depending on the viscosity parameter $\alpha$ and on the viscosity profile model, in agreement with early findings by Natarajan & Pringle (1998). On the other hand, environments where the accretion rate is extremely low imply longer alignment timescales, as $t_{\text{al}} \propto M^{-32/35}$. In the explored BH mass range, the alignment time displays a weak dependence on $M_{\text{BH},0}$: fixed all the other parameters, alignment of a $10^7 M_\odot$ BH occurs, on average, at the same pace of a $10^5 M_\odot$ BH. The BH mass and spin modulus increase during alignment, but their fractional increases are modest. After surveying a wide parameters space, we find that $0.1\% \lesssim \Delta M_{\text{BH}}/M_{\text{BH},0} \lesssim 3\%$ while the spin parameter increases by $0.5\% \lesssim \Delta a/a_0 \lesssim 20\%$.

Starting with an almost anti-parallel BH-disc configuration ($\theta_{\text{out},0} \approx \pi$), the orbital angular momentum density of the inner part of the disc is initially counter-aligned with respect to the BH spin. Nevertheless, the BH still tends to reduce the degree of misalignment (i.e. $\theta_{\text{out}}(t)$ decreases), because of the nature of the gravitomagnetic interaction (see also Scheuer & Peifer 1996; Martin et al. 2007). The accretion of matter with opposite angular momentum at $R_{\text{ISO}}$ decreases $J_{\text{BH}}$ and $a$ with higher rates, compared with their
Figure 15. Coupled evolution of the relative inclination angle \( (\pi - \theta_{\text{out}}) \), BH mass \( M_{\text{BH}} \) and spin parameter \( a \) for a counter-rotating disc. Solid (dashed) lines refer to constant (power-law) viscosity profile. Black lines to \( f_{\text{Edd}} = 1 \), red lines to \( f_{\text{Edd}} = 0.1 \), blue lines to \( f_{\text{Edd}} = 0 \). Dotted horizontal lines which appear in top panels represent angles \( \theta_{\text{out}}/\theta_{\text{out},0} = 10^{0}, 10^{2}, 10^{3} \). Initial configuration: \( M_{\text{BH},0} = 10^{6}M_{\odot}, a_{0} = 0.5, f_{\text{Edd}} = 0.1 \) and \( \alpha = 0.09 \), with \( \pi - \theta_{\text{out},0} = \pi/600 \).

It is still poorly known whether a spinning BH in an active galactic nucleus is fed through a disc that maintains its angular momentum direction stable over a Salpeter timescale \( t_{S} \). Two opposite, still plausible scenarios, have been proposed and discussed. Natarajan & Pringle (1998) speculated that the stability of jets in radio-loud AGNs requires a long-lived phase of stable accretion capable to maintain spatial coherence, i.e. a fixed direction of \( \mathbf{J}_{\text{disc-out}} \), for a time as long as \( 10^{8} \) yr. By contrast, King & Pringle (2006, 2007); King et al. (2008) speculated recently that AGN activity, triggered by gas-rich major mergers, is chaotic in nature even within a single merger event, i.e. is occurring through a sequence of uncorrelated short-lived accretion episodes. In their picture the corresponding discs, truncated by their own-self gravity, continuously change their inclination and feed the BH on their consumption timescale. Under these circumstances the BH spin modulus is seen to either increase or decrease at random clustering around small average values \( a \sim 0.1 - 0.3 \). This model would simultaneously explain the relatively low radiative efficiency of the quasar population as inferred from the background light (e.g., Merloni 2004; Merloni & Heinz 2008), and the possibility of growing
BH as massive as $10^6 M_\odot$ from small BH seeds already at redshift $z \sim 6$ (King & Pringle 2006).

Isolated discs, truncated by their own self-gravity, carry a well defined disc angular momentum and are accreted by the BH on a finite timescale. Starting with a misaligned BH-disc configuration, the BH spin changes direction significantly only if (i) the alignment time is shorter than the disc consumption time, $t_{al} < t_{\text{disc}}$; and (ii) the magnitude of the disc angular momentum is comparable to the BH spin magnitude, i.e. $J_{\text{disc}} \gtrsim J_{\text{BH}}$. The first condition is verified for the whole parameter range explored in this paper. The estimate $J_{\text{disc}} \gtrsim J_{\text{BH}}$ depends instead sensitively upon $R_{\text{out}}$. Equation (12) establishes that isolated discs around large BHs truncate at $R_{\text{out}}$ such that $J_{\text{disc}} < J_{\text{BH}}$. Condition (ii) is satisfied for BH masses $\lesssim 3 \times 10^7 M_\odot$. We note here that since we model our discs using the Shakura-Sunyaev solution for Kramer’s opacity, we cannot rigourously estimate $R_{\text{out}}$, and $J_{\text{disc}}$, for BHs with mass $\lesssim 10^4 - 10^5 M_\odot$. An extension of disc solutions to different, self-consistent opacities is non trivial (Huré et al. 1994a[b]) and we postpone a detailed analysis to future work.

BHs with masses $\lesssim 3 \times 10^7$ align efficiently in discs truncated by their own self-gravity, implying alignment also in the case of stochastically fed AGN? where not only a fluctuates with time, but also the direction of the BH spin continually changes due to the rapidity of the alignment process. By contrast, rapidly spinning ($a \sim 1$) heavier BHs with $M_{\text{BH}} \gtrsim 10^6 M_\odot$ have truncated discs that carry little angular momentum compared with $J_{\text{BH}}$. In this case alignment is uneffective and the orientation of the BH spin is not influenced significantly by the surrounding short-lived disc.

In light of these findings, the vector $J_{\text{BH}}$ appears to carry precious information on the orientation of the plane through which the BH has been fed, and on whether accretion has been long-lived and coherent or short-lived and random.

The method developed in the paper is sufficiently versatile that it will be implemented in numerical simulations describing the process of pairing of dual BHs in circumnuclear discs during their on-fly accretion (Dotti et al., in preparation) to improve upon the speculation (Bogdanovic et al. 2007) that, in gas-rich galaxy mergers, binary BHs have time to align their spin orthogonally to their orbital plane, as discussed in Escala et al. (2005); Dotti et al. (2006); Mayer et al. (2007); Dotti et al. (2007, 2009); Colpi & Dotti (2009). The spin-orbit configuration is relevant to study the impact of BH recoils, that occur after two BHs have coalesced (see, e.g., Pretorius 2007). Detection of gravitational waves, emitted by coalescing BHs, with the Laser Interferometer Space Antenna (LISA) (Bender et al. 1994; Hils & Bender 1999) will be able to constrain the moduli and the directions of the coalescing BHs spins (Vecchio 2004; Lang & Hughes 2006).

Therefore the alignment of $J_{\text{BH}}$ around $J_{\text{tot}}$ (close to $J_{\text{BH}}$) is expected to be unimportant.

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### Table 3. Summary of our parameters and results for the counter-rotating case; we consider viscosity coefficient $\alpha = 0.09$ and initial inclination angle $\theta_{\text{out,0}} = \pi(1 - 1/6000)$, both for constant (C) and power-law (PL) viscosity profiles (VP). The initial BH has $M_{\text{BH,0}} = 10^6 M_\odot$ and $a_0 = 0.5$. Accretion rate $f_{\text{Edd}}$ varies over three orders of magnitude and we record times needed to $(\pi - \theta_{\text{out,0}})$ of a factor 10, 100 or 1000; we also report mass and spin relative variations.

| VP   | $f_{\text{Edd}}$ | $\Delta t$ (10^6yr) | $\Delta M_{\text{BH}}/M_{\text{BH,0}}$ (in units of 10^{-2}) | $\Delta a/a_0$ (in units of 10^{-2}) |
|------|------------------|---------------------|----------------------------------------------------------|---------------------------------|
| C    | 1                | 10                  | 0.40                                                     | -3.99                          |
|      | 10^2             | 10^2                | 0.17                                                     | -7.81                          |
|      | 10^3             | 10^3                | 0.25                                                     | -11.5                          |
| 0.1  | 10^2             | 10^2                | 0.66                                                     | -3.12                          |
|      | 10^3             | 10^3                | 1.31                                                     | -6.15                          |
|      | 0.01             | 10^2                | 5.29                                                     | -4.92                          |
|      | 10^3             | 10^3                | 1.96                                                     | -9.11                          |
| PL   | 0.1              | 10^2                | 1.21                                                     | -5.68                          |
|      | 10^3             | 10^3                | 2.38                                                     | -11.0                          |
|      | 0.01             | 10^2                | 3.52                                                     | -16.2                          |
|      | 10^3             | 10^3                | 10.1                                                     | -13.6                          |

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3 Here we are not considering accretion events involving a disc with a very small amount of mass, i.e. below $(H/R)M_{\text{BH}}$. This light accretion disc has an outer radius much smaller than $H$ and thus carries an angular momentum $J_{\text{disc}} \ll J_{\text{BH}}$. As a consequence, the disc has a very short consumption timescale, and the alignment process is active for a very short period of time.
7 ACKNOWLEDGMENTS

We wish to thank Vittorio Gorini, Sergio Cacciatori, Alberto Sesana, Bernadetta Devecchi, Oliver Piattella and Luca Rizzi for useful discussions and suggestions.

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APPENDIX A: EXPLICIT EXPRESSIONS OF THE GRAVITOMAGNETIC TORQUE

Constant viscosity: For the constant viscosity model, the disc profile is described by (29) and the equation (53) becomes

$$\delta(J_{\text{BH},x} + iJ_{\text{BH},y})_{\text{GM}} = (1 - i) \frac{2\sqrt{2}A}{3} \sqrt{GMm_{\text{BH}}} \frac{GMJ_{\text{BH}}}{A_{\nu}c^{2}R_{\text{warp}}^{3/2}} t_{\text{BP}}(R_{\text{warp}}) \times \exp \left( -\sqrt{2}(1 - i) \left( \frac{R}{R_{\text{warp}}} \right)^{-1/2} \right) \left| \frac{R_{\text{out}}}{R_{\text{ISO}}} \right| .$$ (A1)

Power-law viscosity: For the power-law case (4) with exponent $\beta$, the disc profile is given by $W_{\text{PL}}$, defined as (28). In this case, equation (53) was integrated by Martin et al. (2007):

$$\delta(J_{\text{BH},x} + iJ_{\text{BH},y})_{\text{GM}} = -i \frac{SGMJ_{\text{BH}}}{3(1 + \beta)} A_{\nu} c^{2} \times B \left( \frac{\sqrt{2}}{1 + \beta} (1 - i) - \frac{\sqrt{2 + \beta}}{1 + \beta} \right)$$

$$\times t_{\text{BP}}(R_{\text{warp}}) \int_{z_{\text{in}}}^{z_{\text{out}}} z^{2(1 + \beta)/3(1 + \beta)} K \frac{1}{\sqrt{\delta_{\text{warp}}}} (z) dz$$

where $z$ is a new complex variable, defined as

$$z = \frac{\sqrt{2}}{1 + \beta}(1 - i) \left( \frac{R}{R_{\text{warp}}} \right)^{-1/2} .$$ (A3)
Assuming that
\[ \int_{z_{in}}^{z_{out}} z^{2(\frac{3}{2} + \beta)} K_{\frac{1}{2}(3 + \beta)} (z) \, dz \approx \]
\[ \int_{0}^{(1-i)\infty} z^{2(\frac{3}{2} + \beta)} K_{\frac{1}{2}(3 + \beta)} (z) \, dz = 2^{-\frac{1}{2(1 + \beta)}} \]  
\( (A4) \)
we can rewrite the infinitesimal gravitomagnetic spin variation as
\[ \frac{\delta (J_{BH,x} + iJ_{BH,y})_{gm}}{J_B H} = i -\frac{1}{2(1 + \beta)} \frac{t_{BP}(R_{warp})}{T_{PL}} \]  
\( (A5) \)
where
\[ T_{PL}^{-1} = \frac{4GM\sqrt{GM_R}}{3 \Lambda_{r_1} c^2} B \left( \frac{\sqrt{2}}{1 + \beta} \right) \frac{2^{\frac{3}{2} + 1}}{\frac{1}{2}(3 + \beta)} \]
\times R_{warp}^{-\left(\beta + \frac{1}{2}\right)} 2^{\frac{3}{2} + 3} \Gamma \left( \frac{1 + 2\beta}{2(1 + \beta)} \right). \]  
\( (A6) \)

Martin et al. (2007) estimate the alignment timescale as
\[ t_{al,M} = \frac{T_{PL}}{\cos \left( \frac{\pi}{2(1 + \beta)} \right)}. \]  
\( (A7) \)