Some remarks on $n$-uninorms in IF-sets

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Abstract: Uninorms and nullnorms are well-known monoidal and monotone operations on the unit interval. Akella [2007] proposed their generalization to $n$-uninorms. Really, we get both, proper uninorms as well as proper nullnorms as special cases of 2-uninorms. Moreover, proper uninorms as well as proper nullnorms can be characterized as 2-uninorms with some special types of 2-neutral elements. In the present paper, we discuss a classification of 2-uninorms from another point of view as it was done by Akella in 2007 and 2009. Then, we look at 2-uninorms in IF-sets and point out some differences between 2-uninorms on the unit interval and 2-uninorms in IF-sets.

Keywords: IF-sets, Nullnorm, Uninorm, $n$-Uninorm.

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1 Introduction

Uninorms were introduced by Yager and Rybalov [21] as a generalization of both $t$-norms and $t$-conorms (for details on $t$-norms and their duals, $t$-conorms, see, e.g., [13, 17]). Since that time, researchers study properties of several distinguished families of uninorms. In [16], Karaçal and Mesiar introduced uninorms in bounded lattices. In [5], Bodjanova and Kalina constructed uninorms in bounded lattices with arbitrarily given underlying $t$-norm and $t$-conorm.

Another generalization of $t$-norms and $t$-conorms, called $t$-operators, was introduced by Mas et al. In [18, 19], Mas et al. studied $t$-operators on finite chains. In 2001, Calvo et al. [6] introduced nullnorms when trying to solve Frank’s functional equation [11] where one of the operations in the equation was a uninorm. Afterwards, Mas et al. [20] showed that nullnorms
and \(t\)-operators coincide in the unit interval. Karaçal et al. [15] introduced nullnorms in bounded lattices.

Akella [1, 2] introduced 2- and \(n\)-uninorms in the unit interval and gave a characterization of these operations. In this paper, we characterize 2-uninorms (or more general on \(n\)-uninorms) from the point of view of their two-neutral elements. Particularly, we split the system of all 2-uninorms into 9 (not necessarily disjoint) subclasses. Afterwards, we point out some differences in the structure of 2- (and \(n\)-) uninorms in IF sets.

Intuitionistic fuzzy sets (also, IF-sets), introduced by Atanassov, are a special type of lattice-valued fuzzy sets, introduced by Goguen [12]. Important milestones in the theory of IF-sets, besides the monograph by Atanassov [3], are the papers by Deschrijver [7, 8], and Deschrijver and Kerre [9]. In [7] Deschrijver has shown that there exist \(t\)-norms which are not representable as a pair of a \(t\)-norm and a \(t\)-conorm. In [8] the author has shown that there exist uninorms in IF-sets which are neither conjunctive nor disjunctive. In [9], Deschrijver and Kerre have shown that the theory of IF-sets is equivalent to the theory of interval-valued sets.

A further development of uninorms in IF-sets (or, equivalently, in interval-valued sets) is the paper by Kalina and Král’ [14], where the authors have shown that for arbitrary pair \((a, e)\) of incomparable elements of interval-valued sets there exists a uninorm having \(a\) as the annihilator and \(e\) as the neutral element.

## 2 Basic definitions and some known facts

An IF-set [3] can be represented as a special case of \(L\)-fuzzy set [12], where \(L\) is a bounded lattice. Membership grades of an IF-set are elements \((x_1, x_2) \in [0, 1]^2\) such that \(x_1 + x_2 \leq 1\). The set of all IF-membership grades will be denoted by \(L^*\). For arbitrary \((x_1, x_2), (y_1, y_2) \in L^*\) the following holds

\[
(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq x_2 \land y_1 \geq y_2.
\]

Thus, the least and the greatest elements of \(L^*\) are \(0 = (0, 1)\), \(1 = (1, 0)\), respectively. We will write these values in bold letters to distinguish them from the real numbers 0 and 1.

Following the notation introduced in [4], we will write \(x \parallel y\) if \(x, y \in L^*\) are incomparable. For \(x \in L^*\) we denote \(\Vert x = \{z \in L^*; z \parallel x\}\).

**Definition 1** ([21]). An associative, commutative and monotone operation \(U : [0, 1]^2 \to [0, 1]\) is said to be a uninorm if it has a neutral element \(e \in [0, 1]\)

A uninorm \(U\) has an annihilator \(a = U(0, 1)\), where \(a \in \{0, 1\}\).

**Definition 2** ([10]). A uninorm \(U\) is said to be conjunctive if \(U(0, 1) = 0\), and \(U\) is called disjunctive if \(U(0, 1) = 1\).

**Lemma 1** ([10]). A uninorm \(U\) is a \(t\)-norm whenever its neutral element is \(e = 1\). In that case the annihilator of \(U\) is \(a = 0\).

\(U\) is a \(t\)-conorm whenever its neutral element is \(e = 0\). In that case the annihilator of \(U\) is \(a = 1\).
Lemma 2 ([10]). Let \( U \) be a uninorm, \( e \in ]0, 1[ \) be its neutral element. Then
\[
T_U(x, y) = \frac{U(ex, ey)}{e}, \quad S_U(x, y) = \frac{U(e + (1 - e)x, e + (1 - e)y) - e}{1 - e},
\]
are a \( t \)-norm and a \( t \)-conorm, respectively.

The operations \( T_U \) and \( S_U \) from Lemma 2 are called the underlying \( t \)-norm and the underlying \( t \)-conorm, respectively.

Definition 3 ([6]). An associative, commutative and monotone operation \( V : [0, 1]^2 \to [0, 1] \) is said to be a nullnorm if there exists an element \( a \in [0, 1] \) such that
\[
\begin{align*}
& (1b) \quad V(0, x) = x \text{ for all } x \in [0, a], \\
& (2b) \quad V(1, x) = x \text{ for all } x \in [a, 1].
\end{align*}
\]

Lemma 3 ([6]). Let \( V \) be a nullnorm and \( a \in [0, 1] \) be such that
\[
\begin{align*}
& (1b) \quad V(0, x) = x \text{ for all } x \in [0, a], \\
& (2b) \quad V(1, x) = x \text{ for all } x \in [a, 1].
\end{align*}
\]
Then \( a \) is the annihilator of \( V \).

Similarly like for uninorm \( U \), also for nullnorm \( V \) there exist its underlying \( t \)-norm \( V_T \) and \( t \)-conorm \( V_S \) given by, respectively,
\[
V_T(x, y) = \frac{V(a + (1 - a)x, a + (1 - a)y) - a}{1 - a}, \quad V_S(x, y) = \frac{V(ax, ay)}{a}
\]
for \( a \in ]0, 1[ \).

Definition 4 ([1]). Let \( F : [0, 1]^2 \to [0, 1] \) be a commutative operation. Then \( \{e_1, e_2\}_z \) is called a 2-neutral element of \( F \) if \( F(e_1, x) = x \) for all \( x \in [0, z] \) and \( V(e_2, x) = x \) for all \( x \in [z, 1] \), where \( 0 < z < 1 \) and \( e_1 \in [0, z] \), \( e_2 \in [z, 1] \).

Definition 5 ([1]). Let \( F : [0, 1]^2 \to [0, 1] \) be a monotone, commutative and associative operation that has a 2-neutral element \( \{e_1, e_2\}_z \).

Lemma 4 ([1]). Let \( F \) be a 2-uninorm whose 2-neutral element is \( \{e_1, e_2\}_z \). Then
\[
U_1(x, y) = \frac{F(zx, zy)}{z}, \quad U_2(x, y) = \frac{F(z + (1 - z)x, z + (1 - z)y) - z}{1 - z}
\]
are uninorms whose neutral elements are \( \tilde{e}_1 = \frac{z_1}{z} \) and \( \tilde{e}_2 = \frac{z - z}{1 + z} \), respectively.

Definition 6 ([1]). Let \( F : [0, 1]^2 \to [0, 1] \) be a commutative operation and \( 0 = z_0 < z_1 < z_2 < \cdots < z_{n-1} < z_n = 1 \). Then \( \{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, z_{n-1})} \) is called an \( n \)-neutral element of \( F \) if for all \( i \in \{1, 2, \ldots, n\} \) we have \( e_i \in [z_{i-1}, z_i] \).

Definition 7 ([1]). An associative, commutative and monotone operation \( F : [0, 1]^2 \to [0, 1] \) will be called \( n \)-uninorm if it has an \( n \)-neutral element \( \{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, z_{n-1})} \).
3 Characterization and classes of 2-uninorms and a generalization to \( n \)-uninorms

Let us consider proper uninorms and proper nullnorms as 1-uninorms. Then we adopt the following definition:

**Definition 8.** Let \( F_n \) be an \( n \)-uninorm for \( n > 1 \). We say that \( F_n \) is a proper \( n \)-uninorm if \( F_n \) is not an \((n - 1)\)-uninorm.

For a proper 2-uninorm \( F \), the operations \( U_1 \) and \( U_2 \) given by equality (1), will be called the lower and the upper underlying uninorm, respectively.

Let us a look at 2-neutral elements. For a given \( 0 < z < 1 \), there are 9 possibilities how to set a 2-neutral element \( \{e_1, e_2\}_z \). Namely,

\[
\begin{align*}
  e_1 & = 0, & e_2 & = z, \\
  e_1 & \in [0, z[, & e_2 & \in ]z, 1[, \\
  z, & \quad & 1.
\end{align*}
\]

As a corollary to Lemma 1 we get the following

**Corollary 1.** Let \( F \) be a 2-uninorm whose 2-neutral element is \( \{e_1, e_2\}_z \) for \( z \in ]0, 1[ \). Set \( U_1(x, y) = \frac{F(x,y)}{z} \) and \( U_2(x, y) = \frac{F(x+(1-z)x,y+(1-z)y)}{1-z} \).

Then

(a) \( U_1 \) is a \( t \)-norm if \( e_1 = z \), \( U_2 \) is a \( t \)-norm if \( e_2 = 1 \);

(b) \( U_1 \) is a proper uninorm if \( e_1 \in ]0, z[ \), \( U_2 \) is a proper uninorm if \( e_2 \in ]z, 1[ \);

(c) \( U_1 \) is a \( t \)-conorm if \( e_1 = 0 \), \( U_2 \) is a \( t \)-conorm if \( e_2 = z \).

Let us check all 9 possibilities of setting a 2-neutral element.

**Lemma 5.** Let \( F \) be a 2-uninorm whose 2-neutral element is \( \{z, 1\}_z \) \((\{0, z\}_z)\) for \( 0 < z < 1 \). Then \( F \) is a \( t \)-norm \((t \)-conorm\) that is the ordinal sum of two \( t \)-norms \( F = (\langle T_1, 0, z \rangle, \langle T_2, z, 1 \rangle) \) \((of two \ t \)-conorms \( F = (\langle S_1, 0, z \rangle, \langle S_2, z, 1 \rangle))\).

**Proof.** We will prove only the \( t \)-norm case.

As the first step, let us prove that \( F(0, 1) = 0 \). Since \( \{z, 1\}_z \) is the 2-neutral element of \( F \), we have by associativity

\[
F(0, 1) = F(F(0, z), 1) = F(0, F(z, 1)) = F(0, z) = 0.
\]

Monotonicity of \( F \) implies that 0 is the annihilator of \( F \).

As the second step, we prove that 1 is the neutral element of \( F \). Since we know that 1 is the partial neutral element of \( F \) in the interval \([z, 1]\). Let \( x \in [0, z] \).

\[
F(x, 1) = F(F(x, z), 1) = F(x, F(z, 1)) = F(x, z) = x.
\]

The proof is completed. \( \square \)
Lemma 6. Let $F$ be a 2-uninorm whose 2-neutral element is $\{z\}_z (\{e_1, z\}_z, \{z, e_2\}_z)$ for $0 < z < 1$ and $0 < e_1 < z, z < e_2 < 1$. Then $F$ is a uninorm whose neutral element is $z$ (the underlying $t$-conorm $S$ is the ordinal sum of two $t$-conorms $S = (\langle S_1, e_1, z \rangle, \langle S_2, z, 1 \rangle)$, $e_2$ and the underlying $t$-norm $T$ is the ordinal sum of two $t$-norms $T = (\langle T_1, 0, z \rangle, \langle T_2, z, e_2 \rangle)$).

Proof. In the case that $\{z\}_z$, we have that $z$ is a partial neutral element in the interval $[0, z]$ as well as in the interval $[z, 1]$, i.e., $z$ is the neutral element of $F$. Hence, directly by Definition 1 we get that $F$ is a uninorm with the neutral element $z$.

In the case that $\{e_1, z\}_z$ is the 2-neutral element of $F$, we get applying Lemma 5 to the interval $[e_1, 1]$ that $S = (\langle S_1, e_1, z \rangle, \langle S_2, z, 1 \rangle)$ is a $t$-conorm which is the underlying operation of $F$. The rest of the proof is due to Definition 1.

Dually we could prove the case when $\{z, e_2\}_z$ is the 2-neutral element of $F$. □

Lemma 7. Let $F$ be a 2-uninorm whose 2-neutral element is $\{0, 1\}_z$ for $0 < z < 1$. Then, $F$ is a nullnorm and $z$ is its annihilator.

Proof. The fact that $F$ is a nullnorm with the annihilator $z$ is directly due to Definition 3. □

The remaining three cases lead to proper 2-uninorms.

Lemma 8. Let $F$ be a 2-uninorm whose 2-neutral element is $\{e_1, e_2\}_z$ for $0 < e_1 < z < e_2 < 1$. Then $F$ is a proper 2-uninorm.

We omit the proof of this lemma since the assertion is obvious.

Lemma 9. Let $F$ be a 2-uninorm whose 2-neutral element is $\{e, 1\}_z$ for $0 < e < z < 1$. Then $F$ is a proper 2-uninorm whose upper underlying uninorm is reduced to a $t$-norm.

Proof. The fact that the upper underlying uninorm is reduced to a $t$-norm is due to Lemma 5. The rest of the proof is obvious. □

Lemma 10. Let $F$ be a 2-uninorm whose 2-neutral element is $\{0, e\}_z$ for $0 < z < e < 1$. Then $F$ is a proper 2-uninorm whose lower underlying uninorm is reduced to a $t$-conorm.

The assertion of Lemma 10 is a dual case of Lemma 9. That is why the proof is omitted.

Generalizing Lemma 6, we get the following

Proposition 1. For $n \geq 2$, let $F$ be a proper $n$-uninorm where $\{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, z_{n-1})}$ is its $n$-neutral element. Then there exists $1 \leq i \leq n$ such that $z_{i-1} < e_i < z_i$ and moreover, $F$ is an $(n + 1)$-uninorm whose $(n + 1)$-neutral element is $\{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, z_i, z_{i+1}, \ldots, z_{n-1})}$.

Proof. We have to prove two items for $n \geq 2$:

1) There exists $i$ such that $z_{i-1} < e_i < z_i$,

2) $\{e_1, e_2, \ldots, e_n\}_{(z_1, z_2, \ldots, z_i, z_{i+1}, \ldots, z_{n-1})}$ is an $(n + 1)$-neutral element of $F$. 

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To prove item 1), it is enough to realize that, for \( n \geq 2 \), if there were no \( i \) such that \( z_{i-1} < e_i < z_i \), the operation \( F \) would have diagonal blocks either \((T_1, S_1, T_2, S_2, \ldots)\) or \((S_1, T_1, S_2, T_2, \ldots)\), where \( T_1, T_2 \) are \( t \)-norms, and \( S_1, S_2 \) are \( t \)-conorms. In each of these two cases the \( n \)-neutral element could be reduced to the \((n - 1)\)-neutral element, since in the first case \( e_1 = e_2 \) and in the second case \( e_2 = e_3 \) either \( \{e_2, \ldots, e_n\}(z_2, \ldots, e_i, \ldots, z_{n-1}) \) or \( \{e_1, \ldots, e_n\}(z_1, \ldots, e_i, \ldots, z_{n-1}) \), respectively. This proves the item 1) for \( n \geq 3 \). For \( n = 2 \) the statement is due to Lemmas 8, 9 and 10.

Item 2) is a direct consequence of item 1). □

### 4 2-uninorms on IF-sets

\((L^*, \leq_{L^*})\) is a bounded lattice with incomparable elements. The incomparability of some elements will be crucial in our considerations.

**Example 1.** On the bounded lattice \((L^*, \wedge, \vee, 0, 1)\), \(T_\wedge(z_1, z_2) = z_1 \wedge z_2\) is the greatest \( t \)-norm. When we choose an arbitrary element \( x \notin \{0, 1\}\), \(T_\wedge\) can be considered as the ordinal sum \( t \)-norm \(((T_\wedge, 0, x), (T_\wedge, x, 1))\).

On the other hand, since \( \|x\| \neq \emptyset \), we can define

\[
\hat{T}_\wedge(z_1, z_2) = \begin{cases} 
  z_1 \wedge z_2 & \text{for } (z_1, z_2) \in ([0, x] \cup [x, 1])^2, \\
  z_2 & \text{for } z_1 \in \|x\|, z_2 \in [0, x] \\
  z_1 & \text{for } z_1 \in [0, x], z_1 \in \|x\|, \\
  x & \text{otherwise.}
\end{cases}
\]

(2)

Hence, \(\hat{T}_\wedge\) is not a \( t \)-norm, but \(\{x, 1\}_x\) is a 2-neutral element of \(\hat{T}_\wedge\). This means that \(\hat{T}_\wedge\) is a proper 2-uninorm.

**Example 2.** Let \( x \notin \{0, 1\} \) be an element in \( L^*. \)

\[
\hat{U}(z_1, z_2) = \begin{cases} 
  z_1 \wedge z_2 & \text{for } (z_1, z_2) \in [0, x]^2, \\
  z_1 & \text{for } z_1 \in [0, x] \text{ and } z_2 \notin [0, x], \\
  & \text{and for } z_1 \in [x, 1] \text{ and } z_2 \in \|x\|, \\
  z_2 & \text{for } z_2 \in [0, x] \text{ and } z_1 \notin [0, x], \\
  & \text{and for } z_2 \in [x, 1] \text{ and } z_1 \in \|x\|, \\
  z_1 \vee z_2 & \text{for } (z_1, z_2) \in [x, 1]^2, \\
  x & \text{otherwise.}
\end{cases}
\]

(3)

The operation \(\hat{U}\) is restricted to \([0, x] \cup [x, 1]\), if \(\hat{U}\) has no neutral element on the whole \( L^* \), i.e., it is not a uninorm. On the other hand, \(\{x\}_x\) is a 2-neutral element, hence \(\hat{U}\) is a proper 2-uninorm.
Example 3. Let \( x \notin \{0, 1\} \) be an element in \( L^* \).

\[
V(z_1, z_2) = \begin{cases} 
  z_1 \lor z_2 & \text{for } (z_1, z_2) \in [0, x]^2, \\
  z_1 \land z_2 & \text{for } (z_1, z_2) \in [x, 1]^2, \\
  x & \text{otherwise.}
\end{cases}
\]

(4)

\( V \) is a nullnorm whose annihilator is \( x \). In this case, if we are looking for a modification \( \tilde{V} \) of \( V \) in such a way that \( \tilde{V} \) is reduced to \( [0, x] \cup [x, 1] \), but \( \tilde{V} \) is not a nullnorm, we will not succeed. Really, we have that \( V(1, 0) = x \) and hence also \( \tilde{V}(1, 0) = x \) and this implies that \( x \) is the annihilator of \( \tilde{V} \).

Remark 1. Dually to the operation \( \hat{T}_α \) introduced by (2), we can define on \( L^* \) an operation \( \hat{S}_α \) starting from the t-conorm \( S_α(z_1, z_2) = z_1 \lor z_2 \) and an element \( x \notin \{0, 1\} \). This means that, unlike the situation with the operations in the unit interval, an arbitrary form of the 2-neutral element, except of the case when \( \{0, 1\} \), is the 2-neutral element, may lead to proper 2-uninorms.

As a corollary to the above considerations in Examples 1 – 3, we get the following proposition.

Proposition 2. For arbitrary \( n \geq 2 \) there exists a proper \( n \)-uninorm \( F : L^* \times L^* \to L^* \) such that \( F \) has no \((n + 1)\)-neutral element, i.e., \( F \) is not an \((n + 1)\)-uninorm.

Proof. It is enough to modify the construction in Example 1. For arbitrary \( n \geq 2 \), let us choose \( 0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{n-1} < \zeta_n = 1 \) and we define an operation \( \hat{T} \) by

\[
\hat{T}(z_1, z_2) = \begin{cases} 
  z_1 & \text{for } z_1 \in [\zeta_{i-1}, \zeta_i], i \in \{1, 2, \ldots, n-1\} \text{ and } z_2 \geq \zeta_i, \\
  z_2 & \text{for } z_2 \in [\zeta_{i-1}, \zeta_i], i \in \{1, 2, \ldots, n-1\} \text{ and } z_1 \geq \zeta_i, \\
  \zeta_{i-1} & \text{for } i \in \{1, 2, \ldots, n\} \text{ and } (z_1, z_2) \in [\zeta_{i-1}, \zeta_i]^2, \\
  \text{or } (z_1, z_2) \in [\zeta_{i-1}, 1]^2 \text{ and } z_1 \|_{\zeta_i} \text{ or } z_2 \|_{\zeta_i}.
\end{cases}
\]

We get that \( \{\zeta_1, \zeta_2, \ldots, \zeta_n\}(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}) \) is the \( n \)-neutral element of \( \hat{T} \) and there exists no \((n + 1)\)-neutral element of \( \hat{T} \).

5 Conclusions

In this paper, we have discussed 2-uninorms in the unit interval and in the \( L^* \) lattice of IF-membership grades. We have shown that there are substantial differences between 2-uninorms in the unit interval and 2-uninorms in the \( L^* \) lattice. The results on 2-uninorms we have generalized to \( n \)-uninorms.

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