An Optimal Policy for Dynamic Assortment Planning Under Uncapacitated Multinomial Logit Models

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Abstract

We study the dynamic assortment planning problem, where for each arriving customer, the seller offers an assortment of substitutable products and customer makes the purchase among offered products according to an uncapacitated multinomial logit (MNL) model. Since all the utility parameters of MNL are unknown, the seller needs to simultaneously learn customers’ choice behavior and make dynamic decisions on assortments based on the current knowledge. The goal of the seller is to maximize the expected revenue, or equivalently, to minimize the expected regret. Although dynamic assortment planning problem has received an increasing attention in revenue management, most existing policies require the estimation of mean utility for each product and the final regret usually involves the number of products N. The optimal regret of the dynamic assortment planning problem under the most basic and popular choice model—MNL model is still open. By carefully analyzing a revenue potential function, we develop a trisection based policy combined with adaptive confidence bound construction, which achieves an item-independent regret bound of $O\left(\sqrt{T}\right)$, where T is the length of selling horizon. We further establish the matching lower bound result to show the optimality of our policy. There are two major advantages of the proposed policy. First, the regret of all our policies has no dependence on N. Second, our policies are almost assumption free: there is no assumption on mean utility nor any “separability” condition on the expected revenues for different assortments. Our result also extends the unimodal bandit literature.

Keywords: dynamic assortment optimization, multinomial logit choice model, trisection algorithm, regret analysis.

1 Introduction

Assortment planning has a wide range of applications in retailing and online advertising. Given a large number of substitutable products, the assortment planning problem refers to the selection of a subset of products (a.k.a., an assortment) offering to a customer such that the expected revenue is maximized. To model customers’ choice behavior when facing a set of offered products, discrete

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choice models have been widely used, which capture demand for each product as a function of the entire assortment. One of the most popular discrete choice models is the \textit{multinomial logit model (MNL)}, which is naturally resulted from the random utility theory where a customer’s preference of a product is represented by the mean utility of the product with a random factor \cite{18}. In many scenarios, customers’ choice behavior (e.g., mean utilities of products) may not be given as \textit{a priori} and cannot be easily estimated well due to the insufficiency of historical data (e.g., fast fashion sale or online advertising). To address this challenge, dynamic assortment planning that simultaneously learns choice behavior and makes decisions on the assortment has received a lot of attentions \cite{7,19,21,2,3}. More specifically, in a dynamic assortment planning problem, the seller offers an assortment to each arriving customer in a finite time horizon of length $T$. The goal of the seller is to maximize the cumulative expected revenue over $T$ periods, or equivalently, to minimize the \textit{regret}, which is defined as the gap between the expected revenue generated by the policy and the oracle expected revenue when the mean utility for each product is known as \textit{a priori}.

Despite a lot of research in the area of dynamic assortment planning under various choice models (see Section 2), the optimal policy for the most fundamental uncapacitated MNL model still remains open in the literature. A natural idea to tackle this problem is to conduct some form of maximum likelihood estimation (MLE) of mean utilities of different products on-the-fly, and then select the assortment that maximizes the expected revenue based on the current estimate of mean utilities. However, when the number of products $N$ is large as compared to the horizon length $T$, accurate estimation of mean utilities is extremely difficult, if not impossible, without additional assumptions. In terms of regret analysis, this approach usually incurs a regret that is polynomial in $N$, which is sub-optimal according to our lower bound result (i.e., $\Omega(\sqrt{T})$). Therefore, the following question naturally arises: can we design dynamic assortment policies without explicit estimation of mean utilities and achieve the optimal regret that is independent of $N$?

In this paper, we provide affirmative answers to this question under the most fundamental and popular uncapacitated multinominal logit model. As mentioned above, the estimation of utility parameters will be inaccurate when $N$ is large and thus existing methods based on maximum likelihood estimation cannot be directly used. We design several new techniques to address the challenge. Under an MNL model, we leverage the structure of the optimal assortment in static problems and convert the problem into a \textit{dynamic optimization} of a carefully designed \textit{potential function}. In particular, the seminal result by \cite{22,14,16} shows that the optimal assortment belongs to the set of revenue-ordered assortments. More precisely, assuming that $N$ products are revenue-ordered with the revenues $r_1 \geq r_2 \geq \ldots \geq r_N$, then the optimal assortment must belong to the set $\{\emptyset, \{1\}, \{1,2\}, \ldots, \{1,\ldots,N\}\}$. Therefore, it suffices to only consider the following level sets of products: for each cutoff parameter $\theta \geq 0$, we define the \textit{level set} to be the products whose revenue is greater than or equal to $\theta$. Further, motivated by \cite{19}, we can define the \textit{potential function} $F(\theta)$ to be the expected revenue when this level set is offered as an assortment.

To construct our policy, we first establish a set of important properties of the potential function $F(\theta)$, including 1) we show that the fixed point of $F(\theta)$ is the maximizer $\theta^*$ and leads to the optimal assortment; 2) we set up a reference line and comparing $F(\theta)$ with the reference line to decide whether $F$ is increasing or decreasing locally at $\theta$. Based on these properties, we propose a trisection search policy that dynamically searches the maximizer $\theta^*$ of the potential function and achieves an optimal regret up to logarithmic factors in $T$. Then we further develop an approach with adaptive confidence levels to remove the logarithmic factor in $T$. The matching lower bound result has also been established, which shows the optimality of the proposed policy. By exploring
the structure of the potential function, we no longer need to estimate \( N \) parameters of mean utilities; instead, we only estimate the expected revenue of level sets at a few cutoff points. Before we present an overview of our technical result in Sec. 1.1, we briefly highlight two important advantages of the proposed policies.

1. First, the regrets of our policies have no dependence on the number of products \( N \). This property makes our result more favorable for scenarios when a large number of potential items are available, e.g., online sales or online advertisement. And a key message behind this result is that by exploring the structure of the problem, the explicit estimation of utility parameters could be avoided in dynamic assortment planning.

2. Second, our policy is almost assumption-free: we only require the revenue for each product is upper bounded by a constant and the knowledge of total selling horizon \( T \), which is usually available in practice. We have no assumption on the mean utilities (e.g., the assumption that the no-purchase is the most frequent outcome as in \([2, 3]\)). This relaxation of assumptions is possible because we do not attempt to estimate individual mean utilities in our algorithms. Moreover, we do not have any “separation condition” on the expected revenue between a pair of candidate assortments, which has been assumed in the existing literature \([19, 21]\).

1.1 Our results and techniques

The main contribution of this paper is an optimal characterization of the worst-case regret for dynamic assortment planning under the MNL model. More specifically, we have the following informal statement of the main results in this paper.

**Theorem 1** (informal). There exists a policy whose worst-case regret over \( T \) time periods is upper bounded by \( C_1 \sqrt{T} \) for some universal constant \( C_1 > 0 \); furthermore, there exists another universal constant \( C_2 > 0 \) such that no policy can achieve a worst-case regret smaller than \( C_2 \sqrt{T} \).

To enable such an \( N \)-independent regret, we provide a refined analysis of a certain unimodal revenue potential function first studied in \([20]\) and consider a trisection algorithm on revenue levels, extending some ideas in unimodal bandits on either discrete or continuous arm domains \([25, 12, 1]\). An important challenge in our problem is that the revenue potential function (defined in Eq. (5)) does not satisfy convexity or local Lipschitz growth, and therefore previous results on unimodal bandits cannot be directly applied (see the related work section 2 for details). Moreover, it is a simple exercise that mere unimodality in multi-armed bandits cannot lead to regret smaller than \( \sqrt{NT} \), because the worst-case constructions in the classical lower bound in multi-armed bandits are based on unimodal arms (see, e.g., \([5, 6]\)).

To overcome these difficulties, we establish additional properties of the revenue potential function which are different from classical convexity or Lipschitz growth properties. In particular, we prove connections between the potential function and the straight line \( F(\theta) = \theta \), which is then used as guidelines in our update rules of trisection. Also, because the potential function behaves differently on \( F(\theta) \leq \theta \) and \( F(\theta) \geq \theta \), our trisection algorithm is asymmetric in the treatments of the two trisection mid-points, which is in contrast to previous trisection based methods for unimodal bandits \([25, 12]\) that treat both trisection mid-points symmetrically.

We also remark that the trisection search policy leads to a regret \( O(\sqrt{T \log T}) \), where the optimal regret should be \( \Theta(\sqrt{T}) \). The removal of additional \( \log T \) terms in dynamic assortment selection and unimodal bandit problems is quite non-trivial, which requires new technical development.
In fact, most previous results on dynamic assortment selection [20, 2, 3] and unimodal/convex bandits [25, 12, 1] have additional log $T$ terms in regret upper bounds. The removal of this log($T$) term is achieved by using confidence bounds with adaptively chosen confidence levels corresponding to different amounts of data collected. At a higher level, our strategy shares a similar spirit to the MOSS (Minimax Optimal Strategy in the Stochastic case) algorithm for multi-armed bandits [4]. On the other hand, the analysis is quite different from the analysis of the MOSS algorithm, involving new concentration inequalities and induction arguments tailored specifically to our model and proposed policy.

The rest of the paper is organized as follows. Sec. 2 discusses the related work from both revenue management and bandit learning fields. We introduce the model and notations in Sec. 3. We further define the revenue potential function and investigate its properties in Sec. 4. The policy and regret analysis will be provided in Sec. 5 and the lower bound results are developed in Sec. 7. In Sec. 8, we provide some simulation studies to illustrate the performance of the proposed policies and conclusion and discussions will be followed in Sec. 9. Some technical proofs will be relegated to the supplementary material.

2 Related work

There are two lines of related work — dynamic assortment planning and unimodal bandits. We will provide a brief review of both fields and highlight some closely related work.

2.1 Dynamic assortment planning

Static assortment planning with known choice behavior has been an active research area since the seminal work by [23, 17]. When the customer makes the choice according to the MNL model, [22, 14] prove the the optimal assortment will belong to revenue-ordered assortments (see Lemma 1 in Sec. 4). An alternative proof is provided in [16]. This important structural result enables efficient computation of static assortment planning under the MNL model, which reduces the number of candidate assortments from $2^N$ to $N$ and will also be used in our policy development.

Motivated by the large-scale online retailing, researchers start to relax the assumption on prior knowledge of customers’ choice behavior. The question of dynamic optimization of assortments has received increasing attention in both the machine learning and operations management society [7, 19, 21, 2, 3], where the mean utilities of products are unknown and have to be learnt on the fly. Motivated by fast-fashion retailing, the work by [7] was the first to study dynamic assortment planning problem, which assumes that the demand for product is independent of each other. The work [19] and [21] incorporate choice models of MNL into dynamic assortment planning and formulate the problem into a online regret minimization problem.

The work [19] is closely related to our paper, which analyzes the same revenue potential function and proposes a golden ratio search algorithm based on the unimodal property of the potential function. However, only using the unimodal property leads a regret bound involving log($N$) [19], which is not $N$-independent. Moreover, the golden ratio search algorithm imposes a strong “separability assumption” (see Proposition 8 in [19]), which assumes a constant gap between the expected revenues of any pair of candidate assortments, which may fail when the number of items $N$ is large. In this work we relax the gap assumption and also remove the additional log $N$ dependency by a more refined analysis of properties of the revenue potential function.
Table 1: Summary of the state-of-the-art worst-case regrets for dynamic assortment planning under uncapacitated MNL and capacitated MNL, where $T$ and $N$ denote the length of the horizon and the number of products, respectively. We also provide the reference for each result, either the theorem number (when the result is first derived in this paper) or the reference. Here, the tilde-$O$ notation $\tilde{O}$ is used as a variant of the standard big-$O$ notation but hides logarithmic factors.

| Worst-case Regret | uncapacitated MNL | capacitated MNL ($K \leq N/4$) |
|-------------------|-------------------|-------------------------------|
| Upper bound       | $O(\sqrt{T})$    | $O(\sqrt{NT} + N)$            |
|                   | (Theorem 3)       | [2, 3]                        |
| Lower bound       | $\Omega(\sqrt{T})$ | $\Omega(\sqrt{NT})$          |
|                   | (Theorem 4)       | [8]                           |

Our paper is also closely related to recent work [2] and [3]. These work develop variants of UCB and Thompson sampling type methods for capacitated MNL assortment models, where the size of each assortment is not allowed to exceed a pre-specified parameter $K$. Here the capacity limit $K$ is usually much smaller than $N$. For the capacitated MNL model, the paper [8] further establishes a lower bound result, which shows an $\Omega(\sqrt{NT})$ regret lower bound exists provided that $K \leq N/4$. By comparing this result with our result described in Theorem 1, it is interesting to see that the regret behavior in capacitated and uncapacitated MNL models is significantly different (see Table 1). While the dependence on $N$ in regret is unavoidable in the capacitated case, this paper shows that it can be got rid of in the uncapacitated case. We remove this dependence on $N$ by designing a novel policy that does not explicitly estimate utility parameters.

In addition to MNL models, there are some recent work studying dynamic assortment under more complicated choice models, such as nested logit models [10] and contextual MNL models [24, 9]. We also note that to highlight our key idea and focus on the balance between information collection and revenue maximization, we study stylized dynamic assortment planning problems following the existing literature [19, 21, 2, 3], which ignore operational considerations such as price decisions and inventory replenishment.

2.2 Unimodal bandits

Another relevant line of research is unimodal bandit [25, 12, 1, 13], in which discrete or continuous multi-armed bandit problems are considered with additional unimodality constraints on the means of the arms. Apart from unimodality, additional structures such as “inverse Lipschitz continuity” (e.g., $|\mu(i) - \mu(j)| \geq L|i - j|$ for some constant $L$, where $\mu(i)$ denotes the mean reward of the $i$-th arm) or convexity are imposed to ensure smaller regret compared to unstructured multi-armed bandits. However, both conditions fail to hold for the revenue potential function arising from uncapacitated MNL-based assortment planning problems. In addition, under the gap-free setting where an $O(\sqrt{T})$ regret is to be expected, most previous works have additional $\log T$ terms in their regret upper bounds (except for the work of [13] which introduces additional strong regularity conditions on the underlying functions). In [11], a more general problem of optimizing piecewise-constant function is considered, without assuming a unimodal structure of the function. Consequently, a weaker $\tilde{O}(T^{2/3})$ regret is derived.
3 Model specification

Let \( \mathcal{N} \) be a finite set of all products/items with \(|\mathcal{N}| = N\), and each item \( i \in \mathcal{N} \) is associated with a revenue parameter \( r_i > 0 \) and a utility parameter (a.k.a., preference parameter) \( v_i \geq 0 \). Throughout the paper we conveniently label all items in \( \mathcal{N} \) as \( 1, 2, \cdots, N \). The revenue parameters \( r_1, \cdots, r_N \) are known to the retailer, who has full knowledge of each items’ price/cost; while the utility parameters \( v_1, \cdots, v_N \) are unknown. Let \( \mathcal{S} = 2^{\mathcal{N}} \) be the set of all possible assortments. At every time time \( t \), a retailer picks an assortment \( S_t \in \mathcal{S} \) (\( S_t \neq \emptyset \)), and observes a purchasing action \( i_t \in S_t \cup \{0\} \), where \( i_t = 0 \) means no purchase occurs at time \( t \). If a purchasing action is made (i.e., \( i_t \neq 0 \)), the corresponding revenue \( r_{i_t} \) is collected. It is worthy noting that since items are substitutable, a typical setting of assortment planning usually restricts each purchase to be a single item.

The distribution of \( i_t \) is modeled by the following multinomial-logit (MNL) model:

\[
\Pr[i_t = j] = \begin{cases} 
  \frac{v_j}{1 + \sum_{i \in S_t} v_i} & j \in S_t; \\
  \frac{1}{1 + \sum_{i \in S_t} v_i} & j = 0.
\end{cases} 
\]

Define also \( R(S_t) \) as the expected revenue by supplementing \( S_t \) to a customer; more specifically,

\[
R(S_t) := \sum_{j \in S_t} \Pr[i_t = j] \cdot r_j = \frac{\sum_{j \in S_t} r_j v_j}{1 + \sum_{j \in S_t} v_j}.
\]

For normalization purposes the utility parameter for the “no-purchase” action is assumed to be \( v_0 = 1 \). Apart from that, the rest of the preference parameters \( \{v_i\}_{i=1}^N \) are unknown to the retailer and have to be either explicitly or implicitly learnt from customers’ purchasing actions \( \{i_t\}_{t=1}^T \).

The retailer’s objective is to maximize the expected revenue over the \( T \) time periods. Such an objective is equivalent to the “regret minimization”, in which the retailer’s assortment sequence is compared against the optimal assortment. More specifically, the goal of the retailer is to design a policy \( \pi \) that generates \( \{S_t\}_{t=1}^T \) to minimize the following cumulative regret:

\[
\text{Reg}(\{S_t\}_{t=1}^T) := \sum_{t=1}^T R(S_t) - \mathbb{E}^\pi [R(S_t)] \quad \text{where } S^* \in \arg\max_{S \in \mathcal{S}} R(S).
\]

Here, \( R(S_t) = \mathbb{E}[r_{i_t}|S_t] \) is the expected revenue the retailer collects on assortment \( S_t \). For notational convenience we define \( r_0 = 0 \) corresponding to the “no-purchase” action.

Finally, throughout this paper we only make the following standard assumption on the revenue parameters (see, e.g., Theorem 1 in [2]):

(A1) \( r_\infty := \max_{i \in \mathcal{N}} r_i \leq 1 \).

We note that upper bound on the maximum revenue is assumed to be one without loss of generality, since one can always normalize the revenues.

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1From random utility theory, we have \( v_i = \exp(u_i) \), where \( u_i \) is the underlying mean utility. For the ease of presentation, we will call \( v_i \) the “utility parameter” since we only use \( v_i \) throughout this paper.
4 The revenue potential function and its properties

The set $S$ consists of $2^N$ different assortments, which poses a significant challenge on both regret minimization (treating each assortment in $S$ independently results in exponentially large regret) and computation (as it is intractable to enumerate all assortments in $S$). To address the challenge, we can reduce the number of candidate assortments in $S$ by constraining such assortment selections to “level sets”. In particular, for a given real number $\theta \geq 0$, define the $\theta$-level set to be

$$L_\theta(N) := \{ i \in N : r_i \geq \theta \}$$

as all items whose revenues are not smaller than $\theta$. For notational simplicity, we will use $L_\theta$ (omitting $N$ in the parenthesis) when the context is clear. Further, let

$$P := \{ L_\theta(N) : \theta \geq 0 \} \subseteq S$$

be the class of all candidate assortments in $S$ that can be expressed as level sets. It is easy to verify that $|P| \leq N$, which is significantly smaller than $|S| = 2^N$.

It is well-known that the optimal expected revenue for the static assortment optimization problem will remain the same when reducing the candidate assortments from $S$ to $P$. More precisely, the following lemma is a classical result in revenue management [22, 14, 16], which shows the optimal expected revenue can be achieved by only considering the restricted level set class $P$ under the MNL model.

**Lemma 1** ([22, 14, 16]). Under the MNL model, there exists a subset $S^* \subseteq N$ such that $R(S^*) = \max_{S \subseteq N} R(S) = \max_{P \subseteq S} R(S)$.

In other words, Lemma 1 suggests that it suffices to consider “level-set” type assortments $L_\theta$ and to find $\theta \in [0, 1]$ that gives rise to the largest $R(L_\theta)$.

This motivates the following “potential” function, which takes a revenue threshold $\theta$ as input and outputs the expected revenue of its corresponding level set assortments:

$$The revenue potential function: \quad F(\theta) := R(L_\theta), \quad \theta \in [0, 1].$$

Intuitively, $F(\theta)$ is the expected revenue obtained by providing the assortment consisting of all items whose revenues exceed or are equal to $\theta$. The potential function plays a central role in the development of our dynamic trisection search algorithm and item-independent regret bounds. Similar idea of studying the expected revenue of revenue-ordered items was also considered in [20]. But we will derive a more comprehensive list of properties of the potential function $F$ to facilitate our algorithmic development and analysis. The derived properties in this section could also be potentially useful for solving other assortment planning problems under the MNL.

Because item revenues $r_i$ are discrete, $F$ is a piecewise-constant function as illustrated in the left picture in Fig. 1, where $S = \{ s_1, \cdots, s_m \}$ are the changing points of $F$. More specifically, we have the following proposition and its verification is easy from the definition and the discretized nature of $F$.

**Proposition 1.** There exist $c_0, \cdots, c_m \geq 0$ satisfying $c_i \neq c_{i+1}$ for all $i = 0, \cdots, m - 1$, and $S = \{ s_1, \cdots, s_m \} \subseteq \{ r_i \}_{i=1}^N$, such that

$$F(\theta) = c_0 \cdot I[\theta \leq s_1] + \sum_{i=1}^{m-1} c_i \cdot I[s_i < \theta \leq s_{i+1}] + c_m \cdot I[\theta > s_m],$$

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where \( c_m = 0 \).

Define \( F^* := \max_{0 \leq i \leq m} c_i = \sup_{\theta \geq 0} F(\theta) \) as the maximum value of \( F \). By Lemma 1, we have the following corollary saying that \( F^* \) equals the expected revenue of the optimal assortment.

**Corollary 1.** \( F^* = R(S^*) \).

We further establish some more refined structural properties of \( F \). For notational simplicity, let \( F(x^+) := \lim_{y \to x^+} F(y) \) and \( F(x^-) := \lim_{y \to x^-} F(y) \).

**Lemma 2.** There exists \( \theta^* > 0 \) such that \( \theta^* = F(\theta^*) = F^* \).

**Lemma 3.** For any \( \theta \geq \theta^* \), \( F(\theta) \leq \theta \) and \( F(\theta) \geq F(\theta^+) \).

**Lemma 4.** For any \( \theta \leq \theta^* \), \( F(\theta) \geq \theta \) and \( F(\theta) \leq F(\theta^-) \).

The proofs of the above lemmas are given in the supplementary material. Lemmas 2, 3 and 4 provide a complete picture of the structure of the potential function \( F \), and most importantly the relationship between \( F \) and the central straight line \( F(\theta) = \theta \), as depicted in the right picture of Fig. 1. In particular, \( F \) intersects with the \( y = x \) line at \( \theta^* \) that attains the maximum function value \( F^* \), and monotonically decreases as one moves away from \( \theta^* \), meaning that \( F \) is uni-modal. Furthermore, Lemmas 3 and 4 show that (1) \( F \) is left-continuous; (2) \( F^* \) lies below the \( y = x \) line to the right of \( \theta^* \) and above the \( y = x \) line to the left of \( \theta^* \). This helps us judge the positioning of a particular revenue level \( \theta \) by simply comparing the expected revenue of \( R(\mathcal{L}_\theta) \) with \( \theta \) itself, motivating an asymmetric trisection algorithm which we describe in the next section.

## 5 Trisection and regret analysis

We propose an algorithm based on trisections of the potential function \( F \) in order to locate level \( \theta^* \) at which the maximum expected revenue \( F^* = F(\theta^*) \) is attained. Our algorithm avoids explicitly estimating individual items’ mean utilities \( \{v_i\}_{i=1}^N \), and subsequently yields a regret independent of the number of items \( N \). We first give a simplified algorithm (pseudo-code description in Algorithm
\textbf{Input:} revenue parameters $r_1, \ldots, r_n \in [0, 1]$, time horizon $T$
\textbf{Output:} sequence of assortment selections $S_1, S_2, \ldots, S_T \subseteq \mathcal{N}$

1 Initialization: $a_0 = 0$, $b_0 = 1$;
2 for $\tau = 0, 1, \cdots$ do
3 \hspace{1em} $x_\tau = \frac{2}{3} a_\tau + \frac{1}{3} b_\tau$, $y_\tau = \frac{1}{3} a_\tau + \frac{2}{3} b_\tau$; \hspace{1em} $\triangleright$ \textit{trisection}
4 \hspace{1em} $\ell_0(x_\tau) = \ell_0(y_\tau) = 0$, $u_0(x_\tau) = u_0(y_\tau) = 1$; \hspace{1em} $\triangleright$ \textit{initialization of confidence intervals}
5 \hspace{1em} $\rho_0(x_\tau) = \rho_0(y_\tau) = 0$; \hspace{1em} $\triangleright$ \textit{initialization of accumulated rewards}
6 for $t = 1$ to $16[(y_\tau - x_\tau)^{-2} \ln(T)]^\dagger$ do
7 \hspace{2em} if $\ell_{t-1}(y_\tau) \leq y_\tau \leq u_{t-1}(y_\tau)$ then $\rho_t(y_\tau), \ell_t(y_\tau), u_t(y_\tau) \leftarrow \text{EXPLORE}(y_\tau, t, 1/T^2)$;
8 \hspace{2em} else $\rho_t(y_\tau), \ell_t(y_\tau), u_t(y_\tau) \leftarrow \rho_{t-1}(y_\tau), \ell_{t-1}(y_\tau), u_{t-1}(y_\tau)$;
9 \hspace{2em} Exploit the left endpoint $a_\tau$: pick assortment $S = \mathcal{L}_{a_\tau}$;
10 end \hspace{1em} $\triangleright$ \textit{Update trisection parameters}
11 if $u_t(y_\tau) < y_\tau$ then $a_{\tau+1} = a_\tau$, $b_{\tau+1} = y_\tau$;
12 else $a_{\tau+1} = x_\tau$, $b_{\tau+1} = b_\tau$;
13 end
\vspace{1em}
\hspace{1em} $\dagger$\textit{Stop whenever the maximum number of iterations $T$ is reached.}

\textbf{Algorithm 1:} The trisection algorithm.

\begin{algorithm}
\textbf{Input:} revenue level $\theta$, time $t$, confidence level $\delta$
\textbf{Output:} accumulated revenue $\rho_t(\theta)$, confidence intervals $\ell_t(\theta)$ and $u_t(\theta)$
1 Pick assortment $S = \mathcal{L}_\theta(\mathcal{N})$ and observe purchasing action $j \in S \cup \{0\}$;
2 Update accumulated reward: $\rho_t(\theta) = \rho_{t-1}(\theta) + r_j$; \hspace{1em} $\triangleright$ $r_0 := 0$
3 Update confidence intervals: $[\ell_t(\theta), u_t(\theta)] = \frac{\rho_t(\theta)}{t} \pm \sqrt{\frac{\log(1/\delta)}{2t}}$.
\end{algorithm}

\textbf{Algorithm 2:} \textsc{Explore} Subroutine: exploring a certain revenue level $\theta$

1) with an additional $O(\sqrt{\log T})$ term in the regret upper bound and outline its proofs. We further show how the additional logarithmic dependency on $T$ can be removed by using more advanced techniques.

To assist with readability, below we list notations used in the algorithm description together with their meanings:

- $a_\tau$ and $b_\tau$: left and right boundaries that contain $\theta^*$; it is guaranteed that $a_\tau \leq \theta^* \leq b_\tau$ with high probability, and the regret incurred on failure events is strictly controlled;
- $x_\tau$ and $y_\tau$: trisection points; $x_\tau$ is closer to $a_\tau$ and $y_\tau$ is closer to $b_\tau$;
- $\ell_t(y_\tau)$ and $u_t(y_\tau)$: lower and upper confidence bounds for $F(y_\tau)$ established at iteration $t$; it is guaranteed that $\ell_t(y_\tau) \leq F(y_\tau) \leq u_t(y_\tau)$ with high probability, and the regret incurred on failure events is strictly controlled;
- $\rho_t(y_\tau)$: accumulated reward by exploring level set $\mathcal{L}_{y_\tau}$ up to iteration $t$. 
With these notations in place, we provide a detailed description of Algorithm 1 to facilitate the understanding. The algorithm operates in epochs (outer iterations) \( \tau = 1, 2, \ldots \) until a total of \( T \) assortment selections are made. The objective of each outer iteration \( \tau \) is to find the relative position between trisection points \( (x_\tau, y_\tau) \) and the “reference” location \( \theta^* \), after which the algorithm either moves \( a_\tau \) to \( x_\tau \) or \( b_\tau \) to \( y_\tau \), effectively shrinking the length of the interval \([a_\tau, b_\tau]\) that contains \( \theta^* \) to its two thirds. Furthermore, to avoid a large cumulative regret, level set corresponding to the left endpoint \( a_\tau \) is exploited in each time period within the epoch \( \tau \) to offset potentially large regret incurred by exploring \( y_\tau \).

In Step 7 and 8 of Algorithm 1, lower and upper confidence bounds \([\ell_t(y_\tau), u_t(y_\tau)]\) for \( F(y_\tau) \) are constructed using concentration inequalities (e.g. Hoeffding’s inequality [15]). These confidence bounds are updated until the relationship between \( y_\tau \) and \( F(y_\tau) \) is clear, or a pre-specified number of inner iterations for outer iteration \( \tau \) has been reached (set to \( n_\tau := \lceil 16(y_\tau - x_\tau)^{-2} \ln(T^2) \rceil \) in Step 6). Algorithm 2 gives detailed descriptions on how such confidence intervals are built, based on repeated exploration of level set \( L_{y_\tau} \).

After sufficiently many explorations of \( L_{y_\tau} \), a decision is made on whether to advance the left boundary (i.e., \( a_{\tau+1} \leftarrow x_\tau \)) or the right boundary (i.e., \( b_{\tau+1} \leftarrow y_\tau \)). Below we give high-level intuitions on how such decisions are made, with rigorous justifications presented later as part of the proof of the main regret theorem for Algorithm 1.

1. If there is sufficient evidence that \( F(y_\tau) < y_\tau \) (e.g., \( u_t(y_\tau) < y_\tau \)), then \( y_\tau \) must be to the right of \( \theta^* \) (i.e., \( y_\tau \geq \theta^* \)) due to Lemma 3. Therefore, we will shrink the value of right boundary by setting \( b_{\tau+1} \leftarrow y_\tau \).

2. On the other hand, when \( u_t(y_\tau) \geq y_\tau \), we can conclude that \( x_\tau \) must be to the left of \( \theta^* \) (i.e., \( x_\tau \leq \theta^* \)). We show this by contradiction. Assuming that \( x_\tau > \theta^* \), since \( y_\tau \) is always greater than \( x_\tau \) (and thus \( y_\tau > \theta^* \)) and the gap between \( y_\tau \) and \( F(y_\tau) \) is at least \( y_\tau - x_\tau \), the gap will be detected by the confidence bounds and thus we will have \( u_t(y_\tau) < y_\tau \) with high probability. This leads to a contradiction. Since \( x_\tau \) is to the left of \( \theta^* \), we should increase the value of the left boundary by setting \( a_{\tau+1} \leftarrow x_\tau \).

The following theorem is our main upper bound result for the (worst-case) regret incurred by Algorithm 1.

**Theorem 2.** There exists a universal constant \( C_1 > 0 \) such that for all parameters \( \{v_i\}_{i=1}^N \) and \( \{r_i\}_{i=1}^N \) satisfying \( r_i \in [0, 1] \), the regret incurred by Algorithm 1 satisfies

\[
\text{Reg}\left(\{S_t\}_{t=1}^T\right) = \mathbb{E} \sum_{t=1}^T R(S^*) - R(S_t) \leq C_1 \sqrt{T \ln T}.
\] (7)

### 5.1 Proof sketch

In the rest of the section we sketch key steps and lemmas towards the proof of Theorem 2. The proofs of technical lemmas are provided in the supplementary material. We first state a simple lemma showing that the confidence bound \( \ell_t(y_\tau) \) and \( u_t(y_\tau) \) constructed in Algorithm 1 contains \( F(y_\tau) \) with high probability.

\(^2\)By Lemma 3, we have \( y_\tau - F(y_\tau) \geq y_\tau - F(x_\tau) \geq y_\tau - x_\tau \)
Combining Eqs. (9) and (10) we conclude that Reg at each outer iteration \( \tau \) times, we have subsequently invoking Lemma 6 and using summation of geometric series we have that (with high probability) the shrinkage of \( a_\tau \) or \( b_\tau \) are “consistent”; i.e., \( \theta^* \) is always contained in \([a_\tau, b_\tau]\). Its proof is based on the intuitive two-case analysis discussed before Theorem 2 and will be provided in the supplementary material.

**Lemma 6.** With probability \( 1 - O(T^{-1}) \), \( a_\tau \leq \theta^* \leq b_\tau \) for all \( \tau = 1, 2, \cdots, \tau_0 \), where \( \tau_0 \) is the last outer iteration of Algorithm 1.

Using Lemmas 5 and 6, we are able to prove the following lemma that upper bounds the regret incurred at each outer iteration \( \tau \) using the distance between the trisection points \( x_\tau \) and \( y_\tau \).

**Lemma 7.** For \( \tau = 0, 1, \cdots \) let \( \mathcal{T}(\tau) \) denote the set of all indices of inner iterations at outer iteration \( \tau \). Conditioned on the success events in Lemmas 5 and 6, it holds that

\[
\mathbb{E} \sum_{t \in \mathcal{T}(\tau)} R(S^*) - R(S_t) \leq \varepsilon^{-1}_\tau \log T. \tag{8}
\]

We are now ready to prove Theorem 2.

**Proof.** Recall the definition that \( \varepsilon_\tau = y_\tau - x_\tau \) for outer iterations \( \tau = 0, 1, \cdots \). Because after each outer iteration we either set \( b_{\tau+1} = y_\tau \) or \( a_{\tau+1} = x_\tau \), it is easy to verify that \( \varepsilon_\tau = (2/3) \cdot \varepsilon_{\tau-1} \).

Subsequently, invoking Lemma 6 and using summation of geometric series we have

\[
\mathbb{E} \sum_{t=1}^T R(S^*) - R(S_t) \leq \sum_{\tau=0}^{\tau_0} \varepsilon^{-1}_\tau \log T \leq \varepsilon^{-1}_{\tau_0} \log T, \tag{9}
\]

where \( \tau_0 \) is the total number of outer iterations executed by Algorithm 1. On the other hand, because at each outer iteration \( \tau \) the revenue level \( a_\tau \) is exploited for exactly \( n_\tau = 16[(y_\tau - x_\tau)^{-2} \ln(T^2)] \) times, we have

\[
T \geq n_{\tau_0} \geq \varepsilon^{-2}_{\tau_0} \log T. \tag{10}
\]

Combining Eqs. (9) and (10) we conclude that \( \text{Reg}((S_t)_{t=1}^T) \leq \sqrt{T \log T}. \)

\( \square \)

### 6 Improved regret with adaptive confidence levels

In this section we consider a variant of Algorithm 1 that achieves an improved regret of \( O(\sqrt{T}) \). The key idea is to use an adaptive allocation of confidence levels, by allowing larger failure probability as more data are collected. This is because later failures result in smaller accumulated regret. Such a strategy is motivated by the MOSS algorithm [4] for multi-armed bandits. However, our analysis is quite different from [4], involving new concentration inequalities and induction arguments tailored specifically to our model and proposed policy.

We start with a new uniform concentration inequality for adaptively chosen confidence levels.

**Lemma 8.** Let \( X_1, \cdots, X_L \) be i.i.d. random variables with mean \( \mu \) and satisfy \( a \leq X_i \leq b \) almost surely for all \( \ell \in [L] \). For any \( \delta \in (0, 1] \), it holds that

\[
\Pr \left[ \forall \ell \in [L], \left| \frac{1}{\ell} \sum_{i=1}^{\ell} X_i - \mu \right| \leq \sqrt{\frac{2(b-a)^2 \ln(8/(\delta \ell))}{\ell}} \right] \geq 1 - L\delta. \tag{11}
\]
The proof of Lemma 8 is placed in the supplementary material, based on a careful doubling argument with Hoeffding’s maximal inequality ([15], re-phrased in Lemma 16). Compared to the classical Hoeffding’s inequality (Lemma 15) with the union bound, one notable difference is the increasing “failure probability” as $\ell$ increases (effectively $\ell \delta$ in $\sqrt{\frac{2 \ln(8/(\delta \ell))(b-a)^2}{\ell}}$ instead of $\delta$). This allows the confidence intervals to be much shorter for large $\ell$.

With Lemma 8, we are ready to describe the variant of Algorithm 1, which attains the tight regret bound. Most steps in Algorithms 1 and 2 remain unchanged, and the changes are summarized below:

- Step 3 in Algorithm 2 is replaced with
  $$[\ell_t(\theta), u_t(\theta)] = \frac{\rho_t(\theta)}{t} \pm \sqrt{\frac{2 \ln \left( \frac{8}{\delta t} \right)}{t}}. \quad (12)$$

- Step 7 in Algorithm 1 is replaced with $\text{EXPLORE}(y_\tau, t, 1/T)$; correspondingly, the number of inner iterations is changed to $n_\tau = 8[(y_\tau - x_\tau)^{-2} \ln(8T(y_\tau - x_\tau)^2)]$.

The first change for improving the regret is the way how confidence intervals $[\ell_t(\theta), u_t(\theta)]$ of $F(\theta)$ is constructed. Instead of using fixed confidence level $1/T^2$ as in the baseline policy, in the revised policy varying confidence levels are employed, with “effective” failure probabilities increase as the algorithm collects more data.

We also remark that similar confidence parameter choices were also adopted in [4] to remove additional $\log(T)$ factors in multi-armed bandit problems.

The following theorem shows that the algorithm variant presented above achieves an asymptotic regret of $O(\sqrt{T})$, considerably improving Theorem 2 with an $O(T \log T)$ regret bound. Its proof is rather technical and involves careful analysis of failure events at each outer iteration $\tau$ of the trisection algorithm. To highlight the main idea behind the proof, we provide a sketch of the proof in Sec. 6.1 and defer the entire proof of Theorem 3 to the supplement.

**Theorem 3.** There exists a universal constant $C_1 > 0$ such that for all parameters $\{v_i\}_{i=1}^N$ and $\{r_i\}_{i=1}^N$ satisfying $r_i \in [0, 1]$, the regret incurred by the variant of Algorithm 1 described above satisfies

$$\text{Regret}(\{S_t\}_{t=1}^T) = \mathbb{E} \sum_{t=1}^T R(S^*) - R(S_t) \leq C_1 \sqrt{T}. \quad (13)$$

**6.1 Proof sketch**

We sketch key steps and lemmas towards the proof of Theorem 2. The proofs of technical lemmas are provided in the supplementary material. We first define some notations. Let $\tau = 0, 1, \ldots$ be the number of outer iterations in Algorithm 1, $\varepsilon_\tau = (y_\tau - x_\tau)$ be the distance between the two trisection points at outer iteration $\tau$, and $n_\tau = 8[\varepsilon_\tau^{-2} \ln(8T\varepsilon_\tau^2)]$ be the pre-specified number of inner iterations. Recall also that $\theta^* = F(\theta^*) = F^*$ is the optimal revenue value suggested by Lemma 2.

Define the following three disjoint events that partition the entire probabilistic space:

- Event $E_1(\tau)$: $\theta^* < a_\tau < b_\tau$;


- Event $E_2(\tau)$: $a_\tau \leq \theta^* \leq b_\tau$;
- Event $E_3(\tau)$: $a_\tau < b_\tau < \theta^*$.

Let $\tau_0 \in \mathbb{N}$ be the last outer iteration in Algorithm 1. Let also $T(\tau) \subseteq [T]$ be the indices of inner iterations in outer iteration $\tau$, satisfying $|T(\tau)| \leq 2n_\tau$ almost surely. For $\omega \in \{1, 2, 3\}$, $\tau \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}^+$, define

$$
\psi^\omega_\tau(\alpha, \beta) := \mathbb{E}\left[\sum_{\tau' = \tau}^{\tau_0} \sum_{\tau' \in T(\tau')} R(S^\omega) - R(S_\tau)|\mathcal{E}_\omega(\tau), |a_\tau - \theta^*| = \alpha, |F(a_\tau) - a_\tau| = \beta\right].
$$

Intuitively, $\psi^\omega_\tau(\alpha, \beta)$ is the expected regret Algorithm 1 incurs for outer iterations $\tau, \tau + 1, \ldots, \tau_0$, conditioned on the event $\mathcal{E}_\omega(\tau)$ and other boundary conditions at the left margin $a_\tau$.

The following three lemmas are the central steps in our proof, which establish recurrence relationships among $\psi^\omega_\tau(\alpha, \beta)$, for $\omega \in \{1, 2, 3\}$. The proofs are technically involved and, as we have mentioned, deferred to the supplementary material. To simplify notations, we write $a_n \leq b_n$ or $b_n \geq a_n$ if there exists a universal constant $C > 0$ such that $|a_n| \leq C|b_n|$ for all $n \in \mathbb{N}$.

**Lemma 9** (Regret in Case 1). $\psi^1_\tau(\alpha, \beta) \leq \beta T \sum_{\tau' = \tau}^{\tau_0} \sup_{\Delta > \varepsilon_\tau} \Delta T \exp\{-n_\tau \Delta^2\} + O(\varepsilon_\tau^{-1} \log(T\varepsilon^2_\tau))$.

**Lemma 10** (Regret in Case 2). $\psi^2_\tau(\alpha, \beta) \leq O(\varepsilon_\tau^{-1} \log(T\varepsilon^2_\tau)) + \psi^2_{\tau + 1}(\alpha'_2, \beta'_2) + \psi^3_{\tau + 1}(\alpha'_3, \beta'_3) \cdot O(\log(T\varepsilon^2_\tau)/(T\varepsilon^2_\tau)) + \sup_{\Delta > \varepsilon_\tau} \psi^1_{\tau + 1}(\alpha'_1, \beta'_1(\Delta)) \exp\{-n_\tau \Delta^2\}$ for parameters $\alpha'_1, \beta'_1(\Delta), \alpha'_2, \beta'_2, \alpha'_3, \beta'_3$ that satisfy $\beta'_1(\Delta) \leq \Delta$ and $\alpha'_3 \leq 3\varepsilon_\tau$.

**Lemma 11** (Regret in Case 3). $\psi^3_\tau(\alpha, \beta) \leq \alpha T$.

We are now ready to complete the proof of Theorem 3 by combining Lemmas 10, 9 and 11.

**Proof.** We first get a cleaning expression of $\psi^1_\tau(\alpha, \beta)$ using Lemma 9. First note that $\Delta \mapsto \Delta \exp\{-n_\tau \Delta^2\}$ attains its maximum on $\Delta > 0$ at $\Delta = \sqrt{1/2n_\tau}$. Also note that $n_\tau = [8\varepsilon^{-2}_{\tau} \ln(8T\varepsilon^2_{\tau})]$ and therefore $\sqrt{1/2n_\tau} \leq \varepsilon_{\tau}$. Subsequently,

$$
\sum_{\tau' = \tau}^{\tau_0} \sup_{\Delta > \varepsilon_\tau} \Delta T \exp\{-n_\tau \Delta^2\} \leq \sum_{\tau' = \tau}^{\tau_0} \varepsilon_\tau T \exp\{-n_\tau \varepsilon^2_\tau\} \leq \sum_{\tau' = \tau}^{\tau_0} \varepsilon_\tau T \exp\{-\ln(T\varepsilon^2_\tau)\} \leq \sum_{\tau' = \tau}^{\tau_0} \varepsilon^{-1}_\tau = O(\varepsilon^{-1}_{\tau_0}),
$$

where the last asymptotic holds because $\{\varepsilon_{\tau}\}$ forms a geometric series. Subsequently,

$$
\psi^1_\tau(\alpha, \beta) \leq \beta T + \sum_{\tau' = \tau}^{\tau_0} O(\varepsilon^{-1}_{\tau'} \log(T\varepsilon^2_{\tau'})).
$$

It remains the bound the summation term on the right-hand side of the above inequality. Denote $s_{\tau'} = \varepsilon^{-1}_{\tau'} \ln(T\varepsilon^2_{\tau'}) = \rho^{-\tau'} \ln(T\rho^{2\tau'})$, where $\rho = 2/3$. We then have $s_{\tau'} = \rho^{-\tau_0-\tau'}[1 + \ln \rho^{-2(\tau_0-\tau')}]s_{\tau_0} \leq 2(\tau_0 - \tau' + 1)\rho^{-\tau_0-\tau'} \ln(1/\rho)$ for all $\tau' \leq \tau_0$. Subsequently,

$$
\sum_{\tau' = \tau}^{\tau_0} s_{\tau'} \leq \sum_{\tau' = 0}^{\tau_0} 2(\tau_0 - \tau' + 1)\rho^{-\tau_0-\tau'} \ln(1/\rho) \cdot s_{\tau_0} \leq O(1) \cdot s_{\tau_0}.
$$

13
Therefore,
\[ \psi^1(\alpha, \beta) \leq \beta T + O(\varepsilon_{r_0}^{-1} \log(T \varepsilon_{r_0}^2)). \] (18)

We are now ready to derive the final regret upper bound by analyzing \( \psi^2(\alpha, \beta) \), because the event \( \mathcal{E}_2(0) \) always holds since \( 0 \leq \theta^* \leq 1 \). Applying Lemma 10 with Lemma 11 and Eq. (18), we have for all \( \tau \in \{0, 1, \cdots, \tau_0\} \) that
\[
\begin{align*}
\psi^2(\alpha, \beta) &\leq \psi^2_{r+1}(\alpha_2', \beta_2') + O(\varepsilon_r^{-1} \log(T \varepsilon_r^2)) + O(\varepsilon_T T) \cdot \frac{\ln(T \varepsilon_T^2)}{T \varepsilon_T^2} \\
&\quad + \sup_{\Delta > \varepsilon_r} (\Delta T + O(\varepsilon_{r_0}^{-1} \log(T \varepsilon_{r_0}^2))) \exp\{-n_r \Delta^2\} \\
&\leq \psi^2_{r+1}(\alpha_2, \beta_2) + O(\varepsilon_r^{-1} \log(T \varepsilon_r^2)) + \sup_{\Delta > \varepsilon_r} \Delta T \exp\{-n_r \Delta^2\} \\
&\quad + O(\varepsilon_{r_0}^{-1} \log(T \varepsilon_{r_0}^2)) \cdot \exp\{-n_r \varepsilon_r^2\}. \\
\end{align*}
\] (19)

Using the same analysis as in Eq. (15), we know \( \sup_{\Delta > \varepsilon_r} \Delta T \exp\{-n_r \Delta^2\} \leq O(\varepsilon_r^{-1}) \) and \( \exp\{-n_r \varepsilon_r^2\} \leq 1/(T \varepsilon_r^2) \). Subsequently, summing all terms \( \tau = 0, 1, \cdots, \tau_0 \) together we have
\[
\psi^2_0(\alpha, \beta) \leq \sum_{\tau=0}^{\tau_0} O(\varepsilon_r^{-1} \log(T \varepsilon_r^2)) + O(\varepsilon_{r_0}^{-1} \log(T \varepsilon_{r_0}^2)) \cdot \frac{1}{T \varepsilon_r^2} \\
\leq \varepsilon_{r_0}^{-1} \log(T \varepsilon_{r_0}^2) \cdot (1 + 1/(T \varepsilon_{r_0}^2)). \\
\] (20)

Finally, note that \( n_{r_0} \geq \varepsilon_{r_0}^{-2} \) and \( n_{r_0} \leq T \), implying that \( \varepsilon_{r_0} \geq \sqrt{1/T} \). Plugging the lower bound on \( \varepsilon_{r_0} \) into the above inequality we have \( \psi^2_0(\alpha, \beta) \leq \sqrt{T} \), which completes the proof of Theorem 3. \( \square \)

7 Lower bound

We prove the following theorem showing that no policy can achieve an accumulated regret smaller than \( \Omega(\sqrt{T}) \) in the worst case.

**Theorem 4.** Let \( N \) and \( T \) be the number of items and the time horizon that can be arbitrary. There exists revenue parameters \( r_1, \cdots, r_N \in [0, 1] \) such that for any policy \( \pi \),
\[
\sup_{v_1, \cdots, v_N \geq 0} \text{Reg}(\{S_t\}_{t=1}^T) \geq \frac{\sqrt{T}}{384}. \\
\] (21)

Theorem 4 shows that our regret upper bounds in Theorems 2 and 3 are tight up to \( \sqrt{\log T} \) or \( \sqrt{\log \log T} \) factors and numerical constants. We conjecture (in Sec. 9) that the additional \( \sqrt{\log \log T} \) term can also be removed, leading to upper and lower bounds that match up to universal constants.

7.1 Proof sketch of Theorem 4

We next give a sketch of the proof of Theorem 4. Due to space constraints, we only present an outline of the proof and defer proofs of all technical lemmas to the supplement.
We first describe the underlying parameter values on which our lower bound proof is built. Fix revenue parameters \( \{r_i\}_{i=1}^{N} \) as \( r_1 = 1, r_2 = 1/2 \) and \( r_3 = \cdots = r_N = 0 \), which are known a priori. We then consider two constructions of the unknown utility parameters \( \{v_i\}_{i=1}^{N} \):

\[
\begin{align*}
P_0 &: \quad v_1 = 1 - 1/4\sqrt{T}, \quad v_2 = 1, \quad v_3 = \cdots = v_N = 0; \\
P_1 &: \quad v_1 = 1 + 1/4\sqrt{T}, \quad v_2 = 1, \quad v_3 = \cdots = v_N = 0.
\end{align*}
\]

We note that \( P_0 \) and \( P_1 \) also give the probability distributions that characterize the customer random purchasing actions; and thus we will use \( P_j[A] \) to denote the probability of event \( A \) under the utility parameters specified by \( P_j \) for \( j \in \{0,1\} \).

The first lemma shows that there does not exist estimators that can identify \( P_0 \) from \( P_1 \) with high probability with only \( T \) observations of random purchasing actions. Its proof involves careful calculation of the Kullback-Leibler (KL) divergence between the two hypothesized distributions and subsequent application of Le Cam’s lemma to the testing question between \( P_0 \) and \( P_1 \).

**Lemma 12.** For any estimator \( \hat{\psi} \in \{0,1\} \) whose inputs are \( T \) random purchasing actions \( i_1, \cdots, i_T \), it holds that \( \max_{j \in \{0,1\}} P_j[\hat{\psi} \neq j] \geq 1/3 \).

On the other hand, the following lemma shows that, if the policy \( \pi \) can achieve a small regret under both \( P_0 \) and \( P_1 \), then one can construct an estimator based on \( \pi \) such that with large probability the estimator can distinguish between \( P_0 \) and \( P_1 \) from observed customers’ purchasing actions.

**Lemma 13.** Suppose a policy \( \pi \) satisfies \( \text{Regret}(\{S_t\}_{t=1}^{T}) < \sqrt{T}/384 \) for both \( P_0 \) and \( P_1 \). Then there exists an estimator \( \hat{\psi} \in \{0,1\} \) such that \( P_j[\hat{\psi} \neq j] \leq 1/4 \) for both \( j = 0 \) and \( j = 1 \).

Lemma 13 is proved by explicitly constructing a classifier (tester) \( \hat{\psi} \) from any sequence of low regret. In particular, for any assortment sequence \( \{S_t\}_{t=1}^{T} \), we construct \( \hat{\psi} \) as \( \hat{\psi} = 0 \) if \( \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}[1 \in S_t, 2 \notin S_t] \geq 1/2 \) and \( \hat{\psi} = 1 \) otherwise. Using Markov’s inequality and the construction of \( \{r_i, v_i\} \), it can be shown that if \( \text{Regret}(\{S_t\}_{t=1}^{T}) > \sqrt{T}/384 \) then \( \hat{\psi} \) is a good tester with small testing error.

Detailed calculations and the complete proof is deferred to the supplement.

Combining Lemmas 12 and 13 we proved our lower bound result in Theorem 4.

### 8 Simulation results

We present numerical results of our proposed trisection (and its improved variant) algorithm and compare their performance with several competitors on synthetic data.

**Experimental setup.** We generate each of the revenue parameters \( \{r_i\}_{i=1}^{N} \) independently and identically from the uniform distribution on \([.4,.5]\). For the preference parameters \( \{v_i\}_{i=1}^{N} \), they are generated independently and identically from the uniform distribution on \([10/N, 20/N]\), where \( N \) is the total number of items available.

To motivate our parameter setting, consider the following three types of assortments: the “single assortment” \( S = \{i\} \) for some \( i \in N \), the “full assortment” \( S = \{1, 2, \cdots, N\} \), and the “appropriate” assortment \( S = \{i \in N : r_i \geq 0.42\} \). For the single assortment \( S = \{i\} \), because the preference parameter for each item is rather small (\( v_i \leq 20/N \)), no single assortment can produce an expected revenue exceeding \( 0.5 \times (20/N)/(1 + 20/N) = 10/(20 + N) \). For the full assortment
Table 2: Average (mean) and worst-case (max) regret of our trisection (TRISEC.) and adaptive trisection (ADAP-TRISEC.) algorithms and their competitors on synthetic data. $N$ is the number of items and $T$ is the time horizon.

| $(N, T)$ | UCB mean | UCB max | THOMPSON mean | THOMPSON max | GRS mean | GRS max | TRISEC. mean | TRISEC. max | ADAP-TRISEC. mean | ADAP-TRISEC. max |
|----------|-----------|---------|---------------|-------------|---------|---------|-------------|-------------|------------------|------------------|
| (100,500) | 34.9 | 38.1 | 1.28 | 2.97 | 10.9 | 22.4 | 7.68 | 7.68 | 1.99 | 1.99 |
| (250,500) | 54.3 | 56.2 | 2.81 | 4.95 | 7.93 | 34.2 | 7.57 | 7.57 | 2.23 | 2.23 |
| (500,500) | 73.4 | 75.5 | 4.90 | 4.95 | 7.02 | 43.4 | 7.34 | 7.43 | 2.23 | 2.23 |
| (1000,500) | 90.3 | 93.5 | 8.17 | 10.7 | 5.34 | 45.1 | 7.44 | 7.44 | 2.25 | 2.25 |
| (100,1000) | 73.1 | 78.2 | 1.36 | 2.79 | 139.9 | 175.0 | 8.69 | 8.69 | 3.90 | 3.90 |
| (250,1000) | 113.7 | 119.3 | 3.36 | 5.17 | 90.1 | 110.1 | 8.69 | 8.69 | 4.13 | 4.14 |
| (500,1000) | 136.8 | 140.3 | 5.65 | 7.64 | 65.7 | 113.9 | 9.38 | 9.38 | 3.80 | 3.80 |
| (1000,1000) | 160.8 | 165.4 | 9.31 | 12.4 | 8.43 | 22.8 | 9.77 | 9.77 | 3.97 | 3.97 |

$S = \{1, 2, \cdots, N\}$, because $\sum_{i=1}^N r_i v_i \stackrel{P}{\longrightarrow} 0.45 \times 15/N \times N = 6.75$ and $\sum_{i=1}^N v_i \stackrel{P}{\longrightarrow} 15$ by the law of large numbers, the expected revenue of $S$ is around $6.75/(1 + 5) = 0.422$. Finally, for the “appropriate” assortment $S = \{i \in \mathcal{N}: r_i \geq 0.42\}$, we have $\sum_{i \in S} r_i v_i \stackrel{P}{\longrightarrow} 0.46 \times 15/N \times 0.8N = 5.52$ and $\sum_{i \in S} v_i \stackrel{P}{\longrightarrow} 15/N \times 0.8N = 12$. Therefore, the expected revenue of $S$ is around $5.52/(1 + 12) = 0.425 > 0.422$. The above discussion shows that a revenue threshold $r^* \in (0.4, 0.5)$ is mandatory to extract a portion of the items $\{i \in \mathcal{N}: r_i \geq r^*\}$ that attain the optimal expected revenue, which is highly non-trivial for a dynamic assortment selection algorithm to identify.

**Comparative methods.** Our trisection algorithm with $O(\sqrt{T \log T})$ regret is denoted as TRISEC, and its improved adaptive variant (with regret $O(\sqrt{T})$) is denoted as ADAP-TRISEC. The other methods we compare against include the Upper Confidence Bound algorithm of [2] (denoted as UCB), the Thompson sampling algorithm of [3] (denoted as THOMPSON), and the Golden Ratio Search algorithm of [19] (denoted as GRS). Note that both UCB and THOMPSON proposed in [2, 3] were initially designed for the capacitated MNL model, in which the number of items each assortment contains is restricted to be at most $K < N$. In our experiments, we operate both the UCB and THOMPSON algorithms under the uncapacitated setting, simply by removing the constraint set when performing each assortment optimization.

Most hyper-parameters (such as constants in confidence bounds) are set directly using the theoretical values. One exception is our improved adaptive trisection algorithm (ADAP-TRISEC), in which we replace the $\sqrt{\frac{2\ln(8/\delta)}{\epsilon}}$ confidence interval configuration with $\sqrt{\frac{0.1\ln(8/\delta)}{\epsilon}}$. We observe that a smaller constant value leads to better empirical performance. Another is the GRS algorithm: in [19] the number of exploration iterations is set to $34 \ln(2N)/\beta^2$ where $\beta = \min_{j \neq j'} |R(\mathcal{L}_T) - R(\mathcal{L}_{T,j'})|$, which is inappropriate for our “gap-free” synthetical setting in which $\beta = 0$. Instead, we use the common choice of $\sqrt{T}$ exploration iterations in typical gap-independent bandit problems for GRS.

**Results.** In Table 2 we report the mean and maximum regret from 20 independent runs of each algorithm on our synthetic data, with different settings of $N$ (number of items) and $T$ (time hori-
zon length). We observe that as the number of items ($N$) becomes large, our algorithms (TRISEC and ADAP-TRISEC) achieve smaller mean and maximum regret compared to their competitors, and ADAP-TRISEC consistently outperforms TRISEC in all settings. Unlike UCB and THOMPSON whose regret depend polynomial on $N$, our TRISEC and ADAP-TRISEC algorithms have no dependency on $N$ and hence their regret does not increase with $N$. Moreover, the separate exploration and exploitation structure in GRS makes its performance somewhat unstable, which leads to a larger gap between mean and maximum regrets.

9 Conclusion and future directions

In this paper we consider the dynamic assortment planning problem under uncapacitated MNL models and derive an optimal regret bound, which is independent of $N$.

There are a few interesting future work. In this paper, we assume that the time horizon length $T$ is known. It is interesting to design “horizon-free” algorithms which adapt to the time horizon $T$. Moreover, the uncapacitated MNL can be viewed as a capacitated MNL with the capacity upper bound $K = N$. It is known from [2] and [8] that the optimal regret is $\Theta(\sqrt{NT})$ when $K \leq N/4$ and from this paper that the optimal regret is $\Theta(\sqrt{T})$ when $K = N$. It is interesting to investigate the phase transition from $\Theta(\sqrt{NT})$ to $\Theta(\sqrt{T})$. Finally, another direction is to investigate “instance-optimal” regret bounds whose regret depends explicitly on the problem parameters $\{r_i\}_{i=1}^n$, $\{v_i\}_{i=1}^n$ and matching corresponding (instance-dependent) minimax lower bounds in which $\{v_i\}_{i=1}^n$ are known up to permutations. Such instance-optimal regret might potentially depend on “revenue gaps” $\Delta_i = R(S^*) - R(L_{r_i})$, where $S^*$ is the optimal assortment and $r_i$ is the revenue parameter of the item with the $i$th largest revenue.

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Supplementary Material for: An Optimal Policy for Dynamic Assortment Planning Under Uncapacitated Multinomial Logit Models

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This supplementary material provides detailed proofs for technical lemmas whose proofs are omitted in the main text.

A Proof of technical lemmas in Sec. 4

A.1 Proof of Lemma 2

Let \( s < s' \) be the two endpoints such that \( F(s^+) = F(s') = F^* \) (if there are multiple such \( s, s' \) pairs, pick any one of them). We will prove that \( s < F^* \leq s' \), which then implies Lemma 2.

We first prove \( s < F^* \). Assume by contradiction that \( F^* \leq s \). Clearly \( s \neq 0 \) because \( F^* > 0 \).

By definition of \( F \) and \( F^* \), we have

\[
F^* = F(s') = \frac{\sum_{r_i \geq s'} r_i v_i}{1 + \sum_{r_i \geq s'} v_i} \implies \sum_{r_i \geq s'} (r_i - F^*) v_i = F^*. \tag{S1}
\]

Because \( F^* \leq s \), adding we have that

\[
\sum_{r_i \geq s'} (r_i - F^*) v_i \geq F^* \implies F(s) \geq F^*. \tag{S2}
\]

This contradicts with the fact that \( F(s) \neq F(s^+) \) and that \( F^* \) is the maximum value of \( F \).

We next prove \( F^* \leq s' \). Assume by contradiction that \( F^* > s' \). Removing all items corresponding to \( r_i = s' \) in Eq. (S1), we have

\[
\sum_{r_i > s'} (r_i - F^*) v_i \geq F^* \implies F(s^+) \geq F^*. \tag{S3}
\]

This contradicts with the fact that \( F(s^+) \neq F(s') \) and that \( F^* \) is the maximum value of \( F \).

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A.2 Proof of Lemma 3

Because $F(\theta^*) = \theta^* = F^*$ and $F^*$ is the maximum value of $F$, we have $F(\theta) \leq \theta$ for all $\theta \geq \theta^*$. In addition, for any $\theta \geq \theta^*$, by definition of $F$ we have

$$F(\theta) - F(\theta^+) = R(\{i \in \mathcal{N} : r_i \geq \theta\}) - R(\{i \in \mathcal{N} : r_i > \theta\}) \quad (S4)$$

$$= \frac{\sum_{r_i \geq \theta} r_i v_i}{1 + \sum_{r_i \geq \theta} v_i} - \frac{\sum_{r_i > \theta} r_i v_i}{1 + \sum_{r_i > \theta} v_i} \quad (S5)$$

$$= \frac{(1 + \sum_{r_i \geq \theta} v_i)(\sum_{r_i \geq \theta} r_i v_i) - (1 + \sum_{r_i > \theta} v_i)(\sum_{r_i > \theta} r_i v_i)}{(1 + \sum_{r_i \geq \theta} v_i)(1 + \sum_{r_i > \theta} v_i)} \quad (S6)$$

$$= \frac{(1 + \sum_{r_i > \theta} v_i)(\sum_{r_i = \theta} r_i v_i) - (\sum_{r_i = \theta} v_i)(\sum_{r_i > \theta} r_i v_i)}{(1 + \sum_{r_i > \theta} v_i)(1 + \sum_{r_i > \theta} v_i)} \quad (S7)$$

$$= \frac{\sum_{r_i = \theta} r_i v_i}{1 + \sum_{r_i > \theta} v_i} \left[ \theta - F(\theta^+) \right]. \quad (S8)$$

Because $\theta \geq F(\theta)$ holds for all $\theta \geq \theta^*$, we conclude that $\theta \geq F(\theta^+) \geq F(\theta^+)$ also holds for all $\theta \geq \theta^*$. Subsequently, the right-hand side of Eq. (S8) is non-negative and therefore $F(\theta) \geq F(\theta^+)$.

A.3 Proof of Lemma 4

If $F(\theta) \equiv F^*$ for all $\theta \leq \theta^*$ then the lemma clearly holds. In the rest of the proof we shall assume that there is at least one jumping point strictly smaller than $\theta^*$. Formally, we let $0 < s_1 < s_2 < \cdots < s_t < \theta^*$ be all jumping points that are strictly smaller than $\theta^*$. To prove Lemma 4, it suffices to show that $F(s_j) \geq s_j$ and $F(s_j) \geq F(s_j^+)$ for all $j = 1, \cdots, t$.

We use induction to establish the above claims. The base case is $j = t$. Because $F^*$ is the maximum value of $F$, we conclude that $F(s_t) \leq F^* = F(s_t^+)$. In addition, because $s_t \leq \theta^* = F^* = F(s_t^+)$, invoking Eq. (S8) we have that $F(s_t) \leq F(s_t^+)$. The base case is then proved.

We next prove the claim for $s_j$, assuming it holds for $s_{j+1}$ by induction. By inductive hypothesis, $F(s_{j+1}) \geq s_{j+1} \geq s_j$. Also, $F(s_j^+) = F(s_{j+1})$ because there is no jump points between $s_j$ and $s_{j+1}$, and subsequently $F(s_j^+) \geq s_j$. Invoking Eq. (S8) we proved $F(s_j) \leq F(s_j^+)$. To prove $F(s_j) \geq s_j$, define $\gamma_j := (\sum_{r_i = s_j} v_i)/(1 + \sum_{r_i > s_j} v_i)$. It is clear that $0 \leq \gamma_j \leq 1$. By Eq. (S8), we have

$$F(s_j) - s_j = F(s_j) - F(s_j^+) + F(s_j^+) - s_j \quad (S9)$$

$$= \gamma_j \left[ s_j - F(s_j^+) \right] + F(s_j^+) - s_j \quad (S10)$$

$$= (1 - \gamma_j) \left[ F(s_j^+) - s_j \right]. \quad (S11)$$

As we have already proved $F(s_j^+) \geq s_j$, the right-hand side of the above inequality is non-negative and therefore $F(s_j) \geq s_j$. 
B Proof of technical lemmas in Sec. 5

B.1 Proof of Lemma 5

Let $\delta = 1/T^2$ be the confidence parameter in Algorithm 2. By Hoeffding’s inequality (Lemma 15) and the fact that $0 \leq F(\theta) \leq 1$ for all $\theta$, we have

$$
\Pr \left[ F(\theta) \notin [\ell_t(\theta), u_t(\theta)] \right] = \Pr \left[ \frac{\ell_t(\theta)}{t} - F(\theta) > \sqrt{\frac{\ln(1/\delta)}{2t}} \right] \leq 2 \exp \left\{ -2t \cdot \frac{\ln(1/\delta)}{2t} \right\} \leq 2 \delta = 2/T^2. \tag{S13}
$$

Subsequently, by union bound the probability of $F(\theta) \notin [\ell_t(\theta), u_t(\theta)]$ for at least one $t$ is at most $O(T^{-1})$.

B.2 Proof of Lemma 6

We use induction to prove this lemma. We also conditioned on the fact that $\ell_t(x_\tau) \leq F(x_\tau) \leq u_t(x_\tau)$ and $\ell_t(y_\tau) \leq F(y_\tau) \leq u_t(y_\tau)$ for all $t$ and $\tau$, which happens with probability at least $1 - O(T^{-1})$ by Lemma 5.

We first prove the lemma for the base case of $\tau = 0$. According to the initialization step in Algorithm 1, we have $a_0 = 0$ and $b_0 = 1$. On the other hand, for any $\theta \geq 0$ it holds that $0 \leq F(\theta) \leq F^* \leq 1$. Therefore, $0 \leq \theta^* \leq 1$ and hence $a_\tau \leq \theta^* \leq b_\tau$ for $\tau = 0$.

We next prove the lemma for outer iteration $\tau$, assuming the lemma holds for outer iteration $\tau - 1$ (i.e., $a_{\tau-1} \leq r^* \leq b_{\tau-1}$). According to the trisection parameter update step in Algorithm 1, the proof can be divided into two cases:

**Case 1:** $u_t(y_{\tau-1}) < y_{\tau-1}$. Because $\ell_t(y_{\tau-1}) \leq F(y_{\tau-1}) \leq u_t(y_{\tau-1})$ always holds, we conclude in this case that $F(y_{\tau-1}) < y_{\tau-1}$. Invoking Lemma 4 we conclude that $b_\tau = y_{\tau-1} > \theta^*$. On the other hand, by inductive hypothesis $a_\tau = a_{\tau-1} \leq \theta^*$. Therefore, $a_\tau \leq r^* \leq b_\tau$.

**Case 2:** $u_t(y_{\tau-1}) \geq y_{\tau-1}$. In this case, the revenue level $y_{\tau-1}$ must be explored at every inner iteration in Algorithm 1 at outer iteration $\tau - 1$, because $u_t(y_{\tau-1})$ is a non-increasing function of $t$. Denote $\varepsilon_\tau = y_\tau - x_\tau$ and $n_\tau = 16\varepsilon_\tau^{-2} \ln(T^2)$ as the number of inner iterations in outer iteration $\tau$. Subsequently, the length of the confidence intervals on $y_{\tau-1}$ at the end of all inner iterations can be upper bounded by

$$
|u_t(y_{\tau-1}) - \ell_t(y_{\tau-1})| \leq 2 \sqrt{\ln(T^2) / n_\tau} \leq \frac{1}{2} \varepsilon_\tau^{-1}. \tag{S14}
$$

Invoking Lemma 5 we then have

$$
F(y_{\tau-1}) \geq \ell_t(y_{\tau-1}) \geq u_t(y_{\tau-1}) - \frac{y_{\tau-1} - x_{\tau-1}}{2} \geq y_{\tau-1} - \frac{y_{\tau-1} - x_{\tau-1}}{2}. \tag{S15}
$$

We now establish that $x_{\tau-1} \leq \theta^*$, which implies $a_\tau \leq \theta^* \leq b_\tau$ because $a_\tau = x_{\tau-1}$ and $b_\tau = y_{\tau-1} > \theta^*$ by the inductive hypothesis. Assume by contradiction that $x_{\tau-1} > \theta^*$. By Lemma 3, $F(x_{\tau-1}) \leq x_{\tau-1}$ and $F(x_{\tau-1}) \geq F(y_{\tau-1})$. Subsequently,

$$
F(y_{\tau-1}) \leq x_{\tau-1} = y_{\tau-1} - (y_{\tau-1} - x_{\tau-1}) < y_{\tau-1} - \frac{y_{\tau-1} - x_{\tau-1}}{2}, \tag{S16}
$$
which contradicts Eq. (S15).

### B.3 Proof of Lemma 7

This lemma upper bounds the expected regret incurred at each outer iteration \( \tau \), conditioned on the success events in Lemmas 5 and 6.

We analyze the regret incurred at outer iteration \( \tau \) from exploration of \( y_\tau \) and exploitation of \( a_\tau \) separately.

1. **Regret from exploring \( y_\tau \):** suppose the level set \( \mathcal{L}_{y_\tau}(N) \) is explored for \( m_\tau \leq n_\tau \) times at outer iteration \( \tau \). Then we have \( u_{m_\tau}(y_\tau) \geq y_\tau \). In addition, by Lemma 5 and widths in the constructed confidence bands \( \ell_{m_\tau}(y_\tau) \) and \( u_{m_\tau}(y_\tau) \), we have with probability \( 1 - O(T^{-1}) \) that \( \ell_{m_\tau}(y_\tau) \leq F(y_\tau) \leq u_{m_\tau}(y_\tau) \) and \( |u_{m_\tau}(y_\tau) - \ell_{m_\tau}(y_\tau)| \leq 2\sqrt{\frac{\ln(T^2)}{2m_\tau}} \). Subsequently,

\[
F(y_\tau) \leq \ell_{m_\tau}(y_\tau) \leq u_{m_\tau}(y_\tau) - 2\sqrt{\frac{\ln(T^2)}{2m_\tau}} \geq y_\tau - 2\sqrt{\frac{\ln T}{m_\tau}}.
\]  

(S17)

Note also that \( y_\tau \geq a_\tau \geq \theta^*-3\varepsilon_\tau = F^*-3\varepsilon_\tau \); we have

\[
F^* - F(y_\tau) \leq 3\varepsilon_\tau + 2\sqrt{\frac{\ln T}{m_\tau}}.
\]

(S18)

By Lemma 2, \( F^* = R(S^*) \) and therefore the right-hand side of the above inequality is an upper bound on the regret incurred by exploring revenue level \( y_\tau \) (corresponding to the assortment selection \( \mathcal{L}_{y_\tau} \)) once. As the exploration is carried out for \( m_\tau \) times, the total regret for all exploration steps at revenue level \( x_\tau \) can be upper bounded by

\[
m_\tau \left[ 3\varepsilon_\tau + 2\sqrt{\frac{\ln T}{m_\tau}} \right] \leq 3n_\tau \varepsilon_\tau + \sqrt{4m_\tau \ln T} \leq 3n_\tau \varepsilon_\tau + \sqrt{4n_\tau \ln T} \leq \varepsilon_\tau^{-1} \log T.
\]  

(S19)

Here the last inequality holds because \( n_\tau \leq 16\varepsilon_\tau^{-2} \ln(T^2) \).

2. **Regret from exploiting \( a_\tau \):** by Lemma 6, \( a_\tau \leq \theta^* \), and therefore \( F(a_\tau) \geq a_\tau \). In addition, \( a_\tau \geq \theta^* - 3\varepsilon_\tau \) by the definition of \( \varepsilon_\tau \). Subsequently,

\[
F(a_\tau) \geq a_\tau \geq \theta^* - 3\varepsilon_\tau = F^* - 3\varepsilon_\tau.
\]

(S20)

Re-organizing terms on both sides of the above inequality and noting that \( F^* = F(S^*) \), we have

\[
F(S^*) - F(a_\tau) \leq 3\varepsilon_\tau.
\]

(S21)

Therefore, the regret for each exploitation of revenue level \( a_\tau \) (corresponding to the assortment selection \( \mathcal{L}_{a_\tau} \)) can be upper bounded by \( \varepsilon_\tau \). Because the revenue level \( a_\tau \) is exploited for \( n_\tau \) times and \( n_\tau \leq 16\varepsilon_\tau^{-2} \ln(T^2) \), the total regret of exploitation of \( a_\tau \) at outer iteration \( \tau \) can be upper bounded by

\[
n_\tau \cdot 3\varepsilon_\tau \leq \varepsilon_\tau^{-1} \log T.
\]

(S22)
C Proof of technical lemmas in Sec. 6

C.1 Proof of Lemma 8

Without loss of generality we assume $X_1, \ldots, X_L \in [0, 1]$ almost surely, while the general case of $X_1, \ldots, X_L \in [a, b]$ can be dealt with by a simple re-scaling argument. Denote $k := \lfloor \log_2 L \rfloor$. For each $\ell \in \{1, 2, 4, \cdots, 2^k\}$, by standard Hoeffding's inequality (Lemma 15), we have

$$\Pr \left[ \frac{1}{\ell} \sum_{i=1}^{\ell} X_i - \mu \leq \sqrt{\frac{\ln[8/(\delta\ell)]}{2\ell}} \right] \geq 1 - \frac{\delta\ell}{4}.$$  

Subsequently, by union bound and the fact that $1 + 2 + 4 + \cdots + 2^k \leq 2^{k+1} \leq 2L$, we have

$$\Pr \left[ \forall \ell = 1, 2, 4, \cdots, 2^k, \frac{1}{\ell} \sum_{i=1}^{\ell} X_i - \mu \leq \sqrt{\frac{\ln[8/(\delta\ell)]}{2\ell}} \right] \geq 1 - \frac{\delta L}{2}. \quad (S23)$$

Next consider any $\ell \in \{1, 2, 4, \cdots, 2^k\}$. By Hoeffding’s maximal inequality (Lemma 16), we have

$$\Pr \left[ \forall i \leq \min\{\ell, n - \ell\}, |X_{\ell+1} + \cdots + X_{\ell+i} - i \cdot \mu| \leq \sqrt{\frac{\ell}{2} \ln[8/(\delta\ell)]} \right] \geq 1 - \frac{\delta\ell}{4}. \quad (S24)$$

Again using union bound over all $\ell = 1, 2, 4, \cdots, 2^k$ and the fact that $1 + 2 + 4 + \cdots + 2^k \leq 2^{k+1} \leq 2L$, we have

$$\Pr \left[ \forall \ell = 1, 2, \cdots, 2^k, i \leq \min\{\ell, n - \ell\}, |X_{\ell+1} + \cdots + X_{\ell+i} - i \cdot \mu| \leq \sqrt{\frac{\ell}{2} \ln[8/(\delta\ell)]} \right] \geq 1 - \frac{\delta L}{2}. \quad (S24)$$

Combining Eqs. (S23,S24), we have with probability $1-\delta L$ uniformly over all $\ell = 1, 2, 4, \cdots, 2^k$ and $i \leq \min\{\ell, n - \ell\}$ that

$$|X_1 + \cdots + X_\ell + X_{\ell+1} + \cdots + X_{\ell+i} - (\ell + i)\mu| \leq \sqrt{2\ell \ln[8/(\delta\ell)]}.$$  

Dividing both sides of the above inequality by $(\ell + i)$ we complete the proof of Lemma 8.

C.2 Proof of Lemma 10

First analyze the expected regret incurred at outer iteration $\tau$. by exploiting the left end-point $a_\tau$ (corresponding to assortment $L_{a_\tau}$) for $n_\tau$ iterations. Also, because $a_\tau \leq \theta^* \leq b_\tau$ conditioned on $E_2(\tau)$, by Lemmas 2 and 4 we have $F(a_\tau) \geq a_\tau \geq \theta^* - |b_\tau - a_\tau| = F(\theta^*) - |b_\tau - a_\tau| \geq R(S^*) - 3\varepsilon_\tau$. Subsequently,

$$\text{Regret by exploiting } L_{a_\tau} : \quad \leq 3\varepsilon_\tau \cdot n_\tau \leq \varepsilon_\tau^{-1} \log(T\varepsilon_\tau^2). \quad (S25)$$

Next we analyze the expected regret incurred at outer iteration $\tau$ by exploring the right trisection point $y_\tau$ (corresponding to assortment $L_{y_\tau}$). This is done by a case analysis. If $y_\tau \leq \theta^*$, then the regret incurred by exploiting $L_{y_\tau}$ at outer iteration $\tau$ is again upper bounded (up to numerical
constants) by \( \varepsilon_\tau^{-1} \log(T \varepsilon_\tau^2) \), similar to Eq. (S25). Otherwise, for the case of \( y_\tau > \theta^* \), define \( \Delta_\tau := y_\tau - F(y_\tau) \). By Lemma 3, we know \( \Delta_\tau \geq 0 \), and also by Lemma 2, each exploration of \( L_{y_\tau} \) incurs a regret of no more than \( \Delta_\tau \). Let \( m_\tau \) be the number of times \( L_{y_\tau} \) is explored at outer iteration \( \tau \). By definition of the stopping rule in Algorithm 1, we have

\[
\Pr[m_\tau \geq \ell] \leq \Pr \left[ \frac{\rho_\ell}{\ell} + \sqrt{\frac{2 \ln(8T/\ell)}{\ell}} \geq y_\tau \right] = \Pr \left[ \frac{\rho_\ell}{\ell} - F(y_\tau) \geq \Delta_\tau - \sqrt{\frac{2 \ln(8T/\ell)}{\ell}} \right].
\]

(S26)

Because \( \rho_\ell \) is a sum of \( \ell \) i.i.d. random variables with mean \( F(y_\tau) \) and values in \([0, 1] \) almost surely, applying Hoeffding’s inequality (Lemma 15) we have

\[
\Pr[m_\tau \geq \ell] \leq \exp \left\{ -2 \left( \sqrt{\ell} \Delta_\tau - \sqrt{2 \ln(8T/\ell)} \right)^2 \right\}
\]

\[
\leq \begin{cases} 
1, & \text{if } \Delta_\tau \leq \sqrt{8 \ln(8T/\ell)/\ell}; \\
\exp\{-\ell \Delta_\tau^2/2\}, & \text{otherwise}
\end{cases}
\]

Subsequently,

**Regret by exploring** \( L_{y_\tau} \):

\[
\Delta_\tau \Pr[m_\tau = \ell] \leq \sum_{\ell=1}^{n_\tau} \Delta_\tau \Pr[m_\tau \geq \ell]
\]

\[
\leq \sum_{\ell=1}^{\ell_0-1} \sqrt{\frac{\ln(\ell/\ell_0)}{\ell_0}} + \sum_{\ell=\ell_0}^{n_\tau} \Delta_\tau \exp\{-\ell \Delta_\tau^2/2\}
\]

(S27)

\[
\leq \sqrt{\ell_0 \ln(\ell/\ell_0)} + \sup_{\Delta > \sqrt{8 \ln(8T/\ell_0)/\ell_0}} \Delta \cdot \sum_{\ell=\ell_0}^{\infty} \exp\{-\ell \Delta^2/2\}
\]

(S28)

\[
\leq \sqrt{\ell_0 \ln(\ell/\ell_0)} + \sup_{\Delta > \sqrt{8 \ln(8T/\ell_0)/\ell_0}} \frac{\Delta \exp\{-\ell_0 \Delta^2/2\}}{1 - \exp\{-\Delta^2/2\}}
\]

\[
\leq \sqrt{\ell_0 \ln(\ell/\ell_0)} + \sup_{\Delta > \sqrt{8 \ln(8T/\ell_0)/\ell_0}} \frac{\Delta \exp\{-\Delta^2/2\}}{1 - \exp\{-\Delta^2/2\}}
\]

\[
\leq \sqrt{\ell_0 \ln(\ell/\ell_0)} + \sqrt{\frac{8 \ln(8T/\ell_0)}{\ell_0}} \cdot \frac{1}{1 - \exp\{-4 \ln(8T/\ell_0)\}}
\]

(S29)

\[
\leq \sqrt{\ell_0 \ln(\ell/\ell_0)}.
\]

(S30)

Here in Eq. (S27), \( \ell_0 \) is the smallest positive integer not exceeding \( n_\tau \) such that \( \Delta_\tau > \sqrt{8 \ln(8T/\ell_0)/\ell_0} \). (If \( \Delta_\tau \leq \sqrt{8 \ln(8T/\ell_0)/\ell_0} \) holds for all \( 1 \leq \ell_0 \leq n_\tau \), then the second term in Eq. (S27) is 0 and one can conveniently set \( \ell_0 = n_\tau + 1 \) in this case.) Eq. (S28) holds because

\[
\sum_{\ell=1}^{\ell_0} \sqrt{\frac{\ln(\ell/\ell_0)}{\ell}} \leq \sum_{j=1}^{\lfloor \log_2 \ell_0 \rfloor} 2^j \sqrt{\frac{\ln(2^j/2^j)}{2^j}} = \sum_{j=1}^{\lfloor \log_2 \ell_0 \rfloor} \sqrt{2^j \ln(2^j/2^j)} \leq \sqrt{\ell_0 \ln(\ell/\ell_0)};
\]
Eq. (S29) holds because $\Delta \rightarrow \Delta e^{-\Delta^2/2}/(1 - e^{-\Delta^2/2})$ is monotonically decreasing on $\Delta > 0$. Finally, because $t_0 \leq n_\tau$ and $n_\tau \leq \varepsilon_\tau^{-2} \log(T\varepsilon_\tau^2) \geq \varepsilon_\tau^{-2}$, we have

$$\text{Regret by exploring } \mathcal{L}_{y_\tau} \leq \sqrt{n_\tau \ln(T/n_\tau)} \leq \varepsilon_\tau^{-1} \log(T\varepsilon_\tau^2).$$  \hspace{1cm} (S31)

Finally, we consider regret incurred at later outer iterations $\tau' = \tau + 1, \ldots, \tau_0$. This is done by another case analysis on the relative location of $\theta^*$ with respect to $a_{\tau+1}$ and $b_{\tau+1}$:

- $\mathcal{E}_2(\tau + 1): a_{\tau+1} \leq \theta^* \leq b_{\tau+1}$: the additional regret is upper bounded by $\psi_{\tau+1}^2(\alpha'_1, \beta'_1)$ for some values of $\alpha'_1, \beta'_1$ that are not important;

- $\mathcal{E}_1(\tau + 1): \theta^* < a_{\tau+1} < b_{\tau+1}$: the additional regret is upper bounded by $\psi_{\tau+1}^1(\alpha'_2, \beta'_2)$ with $\beta'_2 \leq \Delta_\tau = y_\tau - F(y_\tau)$ and the value of $\alpha'_2$ not important;

- $\mathcal{E}_3(\tau + 1): a_{\tau+1} < b_{\tau+1} < \theta^*$: the additional regret is upper bounded by $\psi_{\tau+1}^3(\alpha'_3, \beta'_3)$ with $\alpha'_3 \leq 3\varepsilon_\tau$ and the value of $\beta'_3$ not important.

It remains to upper bound the probability the latter two cases above occur. $\mathcal{E}_1(\tau + 1)$ occurs if for all inner iterations $t \in \mathcal{T}(\tau)$, the exploration step fails to detect $F(y_\tau)$ below $y_\tau$, meaning that $\rho_\tau^2 + \sqrt{\frac{2 \ln(8T/\ell)}{\ell}} > y_\tau$ for all $\ell \in \{1, \ldots, n_\tau\}$. Also note that because $\theta^* < a_{\tau+1} = x_\tau = y_\tau - \varepsilon_\tau$, by Lemma 3 we know that $\Delta_\tau = y_\tau - F(y_\tau) \geq \varepsilon_\tau$. Using Hoeffding’s inequality, we have

$$\Pr[\mathcal{E}_1(\tau + 1)] \leq \Pr \left[ \forall \ell, \frac{\rho_\tau}{\ell} - F(y_\tau) > \Delta_\tau - \sqrt{\frac{2 \ln(8T/\ell)}{\ell}} \right] \leq \Pr \left[ \frac{\rho_\tau}{n_\tau} - F(y_\tau) > \Delta_\tau - \sqrt{\frac{2 \ln(8T/\ell)}{n_\tau}} \right] \leq \exp \left\{ -2 \left( \sqrt{n_\tau \Delta_\tau} - \sqrt{2 \ln(8T/n_\tau)} \right)^2 \right\} \leq \exp \left\{ -n_\tau \Delta_\tau^2 \right\}. \hspace{1cm} (S32)$$

Here Eq. (S32) holds because $\sqrt{n_\tau \Delta_\tau} \geq \sqrt{n_\tau \varepsilon_\tau} \geq \sqrt{8 \ln(8T\varepsilon_\tau^2)} \geq 2\sqrt{2 \ln(8T/n_\tau)}$ by the choice of $n_\tau$.

The $\mathcal{E}_3(\tau + 1)$ event occurs if the exploration step in Algorithm 1 falsely detects $y_\tau > F(y_\tau)$ at some stage $\ell \in \{1, \ldots, n_\tau\}$, meaning that $\rho_\tau^2 + \sqrt{\frac{2 \ln(8T/\ell)}{\ell}} < y_\tau$. Note that because $b_{\tau+1} = y_\tau < \theta^*$, by Lemma 4, we know $F(y_\tau) \geq \varepsilon_\tau$. By Lemma 8,

$$\Pr[\mathcal{E}_3(\tau + 1)] = \Pr \left[ \exists \ell, \left| \frac{\rho_\tau}{\ell} - F(y_\tau) \right| > \sqrt{2 \ln(8T/\ell)} \right] \leq \Pr \left[ \exists \ell, \frac{\rho_\tau}{\ell} - F(y_\tau) > \sqrt{\frac{2 \ln(8T/\ell)}{\ell}} \right] \leq \frac{n_\tau}{T} \leq \frac{\ln(T\varepsilon_\tau^2)}{T\varepsilon_\tau^2}. \hspace{1cm} (S33)$$

Combining all regret parts we complete the proof of Lemma 10.
C.3 Proof of Lemma 9

The regret for all outer iterations after $\tau$ (conditioned on $E_2(\tau): \theta^* < a_\tau < b_\tau$) consists of two parts: the regret from exploiting $L_{y_\tau'}$ for $\tau' \geq \tau$, and the regret from exploring $L_{a_\tau'}$.

For any $\tau' \in \{\tau, \tau + 1, \cdots, \tau_0\}$, the expected regret from exploiting $L_{y_\tau'}$ can always be upper bounded by $O(\varepsilon_{\tau'}^{-1}\log(T\varepsilon_{\tau'}^2))$ by the same analysis in the proof of Lemma 10 (more specifically the array of inequalities leading to Eqs. (S28) and (S30)), regardless of the values of $\alpha$ and $\beta$. This corresponds to the $\sum_{\tau' = \tau}^{\tau_0} O(\varepsilon_{\tau'}^{-1}\log(T\varepsilon_{\tau'}^2))$ term in Lemma 9.

We next upper bound the expected regret incurred by exploring $L_{a_\tau'}$ for all $\tau' = \tau, \tau + 1, \cdots, \tau_0$. Because $a_\tau - F(a_\tau) = \beta$ by the definition of $\psi_1(\alpha, \beta)$, the expected regret incurred by exploring $L_{a_\tau}$, $\tau' \in \{\tau, \tau + 1, \cdots, \tau_0\}$ is at most $\beta T$ assuming $a_\tau = a_{\tau + 1} = \cdots = a_{\tau_0}$. It then remains to bound the additional regret incurred by the movements of $a_\tau$ in subsequent outer iterations.

Let $\mathcal{W} = \{\tau'_1, \tau'_2, \cdots, \tau'_\ell\}$ be outer iterations at which the update rule $a_{\tau + 1} \leftarrow x_{\tau'}$ is applied. We then have the following observations:

1. Each $\tau' \in \mathcal{W}$ would incur an additional regret upper bounded by $\Delta_{\tau'} T$, where $\Delta_{\tau'} = y_{\tau'} - F(y_{\tau'}) \geq \varepsilon_{\tau'}$;

2. For each $\tau' \in \{\tau, \tau + 1, \cdots, \tau_0\}$, the probability update $a_{\tau' + 1} \leftarrow x_{\tau'}$ is applied is at most $\exp\{-n_{\tau'} \Delta_{\tau'}\}$, using the same analysis in the proof of Lemma 10 (more specifically the array of inequalities leading to Eq. (S32)).

Summarizing the above observations, by the law of total expectation the expected regret from exploring $L_{a_\tau}$ at subsequent iterations $\tau' \geq \tau$ can be upper bounded by $\beta T + \sum_{\tau' = \tau}^{\tau_0} \sup_{\Delta > \varepsilon_{\tau'}} \Delta T \exp\{-n_{\tau'} \Delta^2\}$.

C.4 Proof of Lemma 11

Because $a_\tau = \theta^* - \alpha < \theta^*$, by Lemma 4 we have $F(a_\tau) \geq a_\tau = \theta^* - \alpha = F(\theta^*) - \alpha$. Subsequently, $F(S^*) - F(a_\tau) \leq \alpha$ thanks to Lemma 2. Also note that conditioned on $E_3(\tau)$, the revenue levels explored or exploited at each time epoch $t \in T(\tau')$, $\tau \leq \tau' \leq \tau_0$ are sandwiched between $a_\tau$ and $\theta^*$, and therefore $R(S^*) - R(S_t) \leq \alpha$. Hence, $\psi_3(\alpha, \beta) \leq \alpha \cdot \mathbb{E} \sum_{\tau' = \tau}^{\tau_0} |T(\tau')| \leq \alpha T$.

D Proofs of technical lemmas in Sec. 7

D.1 Proof of Lemma 12

We first state a lemma that upper bounds the KL divergence under $P_0$ and $P_1$ for arbitrary assortment selections $S \in \mathcal{S}$.

**Lemma 14.** For any $S \in \mathcal{S}$ let $P_0(S)$ and $P_1(S)$ be the distribution of the purchasing action under $P_0$ and $P_1$, respectively. Then $\text{KL}(P_0(S)\|P_1(S)) \leq 1/18T$.

**Proof of Lemma 14.** If $1 \notin S$ then $P_0(S) \equiv P_1(S)$ and therefore $\text{KL}(P_0(S)\|P_1(S)) = 0$. In addition, because $v_i = r_i = 0$ for all $i \geq 3$, the items apart from 1 and 2 in $S$ do not affect the distribution of the purchasing action under both $P_0$ and $P_1$. Therefore, it suffices to compute $\text{KL}(P_0(\{1\})\|P_1(\{1\}))$ and $\text{KL}(P_0(\{1, 2\})\|P_1(\{1, 2\}))$.

Before delving into detailed calculations, we first state a simple proposition bounding the KL divergence between two categorical distributions. It is simple to verify.
Proposition 2. Let \( P \) and \( Q \) be two categorical distributions on \( J \) items, with parameters \( p_1, \ldots, p_J \) and \( q_1, \ldots, q_J \) respectively. Denote also \( \varepsilon_j := p_j - q_j \). Then \( \text{KL}(P||Q) \leq \sum_{j=1}^{J} \varepsilon_j^2/q_j \).

We first consider \( \text{KL}(P_0(\{1\})||P_1(\{1\})) \). By definition, \( P_0(i = 1|\{1\}) \leq 1 - 2/124\sqrt{T} \) and \( P_1[i = 2|\{2\}] \leq 1/2 + 1/24\sqrt{T} \). Also, \( \min_{i=0,1} \{ P_1(i|\{1\}) \} \geq 1/3 \). Subsequently,

\[
\text{KL}(P_0(\{1\})||P_1(\{1\})) \leq 2 \times \frac{1/144T}{1/3} \leq \frac{1}{24T} \leq \frac{1}{18T}. \tag{S34}
\]

We next consider \( \text{KL}(P_0(\{1,2\})||P_1(\{1,2\})) \). Note that \( P_0(i = 0|\{1,2\}) > P_1(i = 0|\{1,2\}) \), \( P_0(i = 1|\{1,2\}) < P_1(i = 1|\{1,2\}) \) and \( P_0(i = 2|\{1,2\}) > P_1(i = 2|\{1,2\}) \). Also, \( P_0(i = 1|\{1,2\}) \leq 1/3 - 1/48\sqrt{T} \), \( P_1(i = 1|\{1,2\}) \geq 1/3 + 1/48\sqrt{T} \) and \( \min_{0 \leq i \leq 2} \{ P_1(i|\{1,2\}) \} \geq 1/4 \). Subsequently,

\[
\text{KL}(P_0(\{1,2\})||P_1(\{1,2\})) \leq 3 \times \frac{1/576T}{1/4} \leq \frac{1}{48T} \leq \frac{1}{18T}. \tag{S35}
\]

The lemma is thus proved.

We are now ready to prove Lemma 12.

Proof of Lemma 12. Denote \( \|P - Q\|_{\text{TV}} := 2 \sup_A |P(A) - Q(A)| \) as the total variation norm between \( P \) and \( Q \), and let \( P_0^{\otimes T}, P_1^{\otimes T} \) denote the distribution of \( \{ i_t | S_t \}_{t=1}^T \) parameterized by \( P_0 \) and \( P_1 \). By Pinsker’s inequality and the conditional independence of \( i_t \) conditioned on \( S_t \), we have

\[
\|P_0^{\otimes T} - P_1^{\otimes T}\|_{\text{TV}} \leq \sqrt{2\text{KL}(P_0^{\otimes T}||P_1^{\otimes T})} \leq \sup_{S^{(1)}, \ldots, S_t} \left[ \sum_{t=1}^{T} \text{KL}(P_0(S_t)||P_1(S_t)) \right] \leq \sqrt{2T} \cdot \sqrt{KL(P_0(S)||P_1(S))} \leq \sqrt{2T} \cdot \sqrt{1/18T} \leq 1/3. \tag{S36}
\]

Using Le Cam’s inequality we have

\[
\inf_{\psi} \max_{j=0,1} P_j \left[ \hat{\psi} \neq j \right] \geq \frac{1}{2} \left( 1 - \|P_0^{\otimes T} - P_1^{\otimes T}\|_{\text{TV}} \right) \geq \frac{1}{3}. \tag{S37}
\]

D.2 Proof of Lemma 13

Denote \( \varphi_0 := 1/T \cdot \sum_{t=1}^{T} \mathbb{I}[1 \in S_t, 2 \notin S_t] \), \( \varphi_1 := 1/T \cdot \sum_{t=1}^{T} \mathbb{I}[1, 2 \in S_t] \), \( \varphi_2 := 1/T \cdot \sum_{t=1}^{T} \mathbb{I}[2 \in S_t, 1 \notin S_t] \) and \( \bar{\varphi} := 1/T \cdot \sum_{t=1}^{T} \mathbb{I}[1, 2 \notin S_t] \). Because the four events partition the entire probability space, we have \( \varphi_0 + \varphi_1 + \varphi_2 + \bar{\varphi} = 1 \). In addition, it is easy to verify that \( S^* = \{1\} \) under \( P_0 \) and under \( P_1 \). Subsequently,

\[
\frac{\text{Reg}_\pi(T)}{T} \leq \frac{\varphi_0}{12\sqrt{T}} + \frac{\varphi_2 + \bar{\varphi}}{24} \quad \text{under } P_0;
\]

\[
\frac{\text{Reg}_\pi(T)}{T} \leq \frac{\varphi_1}{48\sqrt{T}} + \frac{\varphi_2 + \bar{\varphi}}{6} \quad \text{under } P_1.
\]
Using Markov’s inequality and the fact that \( \text{Reg}_P(T) \leq \sqrt{T}/384 \) under both \( P_0 \) and \( P_1 \), we have
\[
P_0 \left[ \frac{\varphi_0}{12\sqrt{T}} + \frac{\varphi_2 + \varphi}{24} > \frac{1}{96 \sqrt{T}} \right] \leq \frac{1}{4} \quad \text{and} \quad P_1 \left[ \frac{\varphi_1}{48 \sqrt{T}} + \frac{\varphi_2 + \varphi}{6} > \frac{1}{96 \sqrt{T}} \right] \leq \frac{1}{4}. \quad (S39)
\]
Subsequently, because \( \varphi_0 + \varphi_1 + \varphi_2 + \varphi = 1 \), we know that \( \varphi_0 > 1/2 \) with probability \( \geq 2/3 \) under \( P_0 \) and \( \varphi_0 < 1/2 \) with probability \( \geq 2/3 \) under \( P_1 \). Define \( \hat{\psi} \) as
\[
\hat{\psi} := \begin{cases} 
0 & \text{if } \varphi_0 \geq 1/2; \\
1 & \text{if } \varphi_0 < 1/2.
\end{cases} \quad (S40)
\]
The estimator \( \hat{\psi} \) then satisfies Lemma 13 by the above argument.

## E Concentration inequalities

The following lemma is the celebrated Hoeffding’s inequality [15].

**Lemma 15.** Suppose \( X_1, \cdots, X_n \) are i.i.d. random variables with mean \( \mu \) and satisfy \( a \leq X_i \leq b \) almost surely for all \( i \in [n] \). Then for any \( t > 0 \),
\[
\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right| > t \right] \leq 2 \exp \left\{ -\frac{2nt^2}{(b-a)^2} \right\}. \quad (S41)
\]

The following lemma is the Hoeffding’s maximal inequality, also by [15].

**Lemma 16.** Let \( X_1, \cdots, X_n \) be i.i.d. random variables with mean \( \mu \) and satisfy \( a \leq X_i \leq b \) almost surely for all \( i \in [n] \). Then for any \( t > 0 \),
\[
\Pr \left[ \forall i \in [n], X_1 + \cdots + X_i \geq i \cdot \mu + t \right] \leq \exp \left\{ -\frac{2t^2}{n(b-a)^2} \right\}. \quad (S42)
\]