On Stability Of The Crystal Universe Models

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Abstract

We generalize the Goldberger-Wise mechanism and study the stability of the Crystal Universe models. We show that the model can be stabilized, however for configurations of Crystal Universe in the absence of fine-tuning, brane crystals are not equidistant, \textit{i.e.} a ”$-+$” pair is far away from adjacent ”$-+$” pair, except for the fixed points of the orbifold, which differs from the assumptions taken in the literature.

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Recently there has been considerable interest in studying the brane universe models. Randall and Sundrum (RS) [1] have proposed a high dimensional picture to solve the gauge hierarchy problem between the Planck scale and the electroweak scale. In RS’s scenario there are two 3-branes with opposite tension which sit at the fixed points of an $S^1/Z_2$ orbifold with $AdS_5$ bulk geometry, the gravity is shown to be localized on the brane of positive tension (”+” brane) and the exponential warp factor in the spacetime metric generates a scale hierarchy on the brane of negative tension (”−” brane). If the standard model (SM) fields reside on the ”−” brane, this leads to the resolution of the gauge hierarchy problem, however even though it is plausible in string theory, it is still questionable whether we can live on ”−” brane. There are many variants of RS model proposed recently to avoid this problem. In Ref. [2], Lykken and Randall [2] proposed a multi-brane model, i.e. a ”++−” brane configuration in which the SM fields reside on intermediate ”+” brane and a warp factor accounts for scale hierarchy. Kogan et al. in Ref. [3] (see also [4] and [5]) considered ”+−−+” multi-brane model and the Crystal Universe model (see also [6], [7], [8], [9], [10]). These models provide a way of having the visible sector on a ”+” brane with an hierarchical warp factor and interestingly predict gravity be different from we expect not only at small scale but also at ultralarge scale. Furthermore it has been argued in [11] that this class of models give rise to many new phenomena such as neutrino mixing, Dark Matter which can be tested experimentally.

In this paper we study the issue of stability of the crystal universe models. Following Goldberger and Wise (GW) [12] we introduce a bulk scalar field into the models, then minimize the potential generated by the bulk scalar with quartic interaction localized on two 3-branes. We will show that the brane crystals is not equidistant in the absence of fine-tuning and generally a ”−+” pair is far away from adjacent ”−+” pair.

To begin with, we consider a Crystal Universe Model shown in Fig.1 which consists of $n$ array of parallel 3-branes with ”+” brane every other ”−” brane in a $AdS_5$ space with negative cosmological constant $\Lambda$. The fifth dimension $y$ has orbifold geometry $S^1/Z_2$. The $n+1$ array of parallel 3-branes are located at $y_0 = 0$, $y_1$, $y_2$ $...$ $y_n$, where $y_0 = 0$ and $y_n$ are orbifold fixed points. The action for this configuration is

$$S = \int d^4x \int dy \sqrt{G} \left( 2M^3 R - \Lambda \right) - \sum_{i=1}^{n+1} \int_{y=y_{i-1}}^{y_{i}} d^4x \sqrt{g^{(i-1)}}, \quad (1)$$

where $i = 0, 1, 2, ... n$, $g^{(i-1)}_{\mu\nu}$ are the induced metric on the branes, $V_{i-1}$ are their tensions and $M$ is the 5D fundamental scale. The 5D metric ansatz that respects 4D Poincaré invariance is given by

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \quad (2)$$

here the warp function $\sigma(y)$ is essentially a conformal factor that rescales the 4D component
of the metric. Substituting (2) into the Einstein equations we have that:

\[
(\sigma^\prime)^2 = k^2, \quad \sigma'' = \sum_{i-1} \frac{V_{i-1}}{12M^3} \delta(y - y_{i-1}),
\]

(3)

where \( k = \sqrt{-\Lambda/24M^3} \) is effectively the bulk curvature in the region between the two adjacent brane crystals. There are two solution to equations (3):

(i). \( \sigma_i(y) = (-1)^{i+1}ky + \sum_{j=0}^{i-1}2(-1)^{j+1}ky_j \),

(4)

and

(ii). \( \sigma_i(y) = (-1)^iky + \sum_{j=0}^{i-1}2(-1)^jk y_j \),

(5)

here \( \sigma_i(y) \) are warp factors between the \((i - 1)\)th brane and \(i\)th brane. For solution (i) the " + " brane sits on the fixed point \( y_0 = 0 \) and the corresponding brane tensions are \( V_{i-1} = (-1)^i\Lambda/k \); for solution (ii) on the fixed point is the " − " brane and \( V_{i-1} = (-1)^i\Lambda/k \).

To study the stability of Crystal Universe, we introduce and couple a bulk scalar field to the brane crystals. This technique is a generalization of the GW mechanism, however the calculation involved in this paper will be much more complicated than that in [12]. For a bulk scalar field with mass \( m \),

\[
S_{\text{Bulk}} = \frac{1}{2} \int d^4x \int dy \sqrt{G} (G^{AB} \partial_A \Phi \partial_B \Phi - m^2 \Phi^2),
\]

(6)

where \( G_{AB} \) is the 5D metric given in (2) with \( \sigma(y) \) given in (4) and (5). And the boundary potentials of scalar field are

\[
S_{i-1} = - \int d^4x \lambda_{i-1} \sqrt{g^{(i-1)}} \left( \Phi^2 - v_{i-1}^2 \right)^2,
\]

(7)

where \( v_{i-1} \) are the vacuum expectation values of bulk scalar field in the \((i - 1)\)th brane, \( \lambda_{i-1} \) are coupling contants. We are interested in those configurations of the bulk scalar where the boundary potentials are minimised. This essentially amounts to negligible dynamics of \( \Phi \) along the direction tangential to any of the 3-branes. This assumption is reasonable because we focus on the stability of the 3-branes system at the moment and do not study phenomenological consequence of possible coupling of the bulk scalar field \( \Phi \) to matter fields living on the branes. It, therefore, suffices to concentrate on equation of motion of \( \Phi \) only in \( y \) direction, which is

\[
\partial_y^2 \Phi - 4\sigma_i'(y)\partial_y \Phi - m^2 \Phi = 0,
\]

(8)

where \( \sigma_i'(y) = d\sigma_i(y)/dy \). Solution of this equation is

\[
\Phi(y) = \exp (2\sigma_i'(y)y)[A_i \exp (\sigma_i'(y)\nu_i y) + B_i \exp (-\sigma_i'(y)\nu_i y)].
\]

(9)
In Eq.(9) \( \nu_i = \sqrt{4 + m^2/\sigma^2(y)} \) is independent of \( y \) (which we denote by \( \nu \) in the following discussion), however \( A_i \) and \( B_i \) vary in the range of \( y_{i-1} < y < y_i \).

To determine the coefficients \( A_i \) and \( B_i \), we require \( \Phi \) minimize the boundary potential. We firstly consider the case (i) where the “+” brane is at \( y_0 = 0 \), then discuss the case (ii).

For case (i):

\[
A_i = \left( \frac{v_{i-1} - v_i R_i^{2-\nu}}{1 - R_i^{-2\nu}} \right) Y_{i-1}^{2+\nu},
\]

\[
B_i = \left( \frac{-v_{i-1} R_i^{-2\nu} + v_i R_i^{2-\nu}}{1 - R_i^{-2\nu}} \right) Y_{i-1}^{2-\nu},
\]

where \( i = 2j - 1 \) with \( j = 1, 2, 3... \)

and

\[
A_i = \left( \frac{v_{i-1} - v_i R_i^{\nu-2}}{1 - R_i^{-2\nu}} \right) Y_{i-1}^{-(2+\nu)},
\]

\[
B_i = \left( \frac{-v_{i-1} R_i^{2\nu} + v_i R_i^{\nu-2}}{1 - R_i^{-2\nu}} \right) Y_{i-1}^{-\nu-2},
\]

where \( i = 2j \) with \( j = 1, 2, 3... \)

In Eqs.(10-13), \( Y_i \) and \( R_i \) are defined as : \( Y_i = \exp (-k y_i) \), \( R_1 = \exp (-k y_1) \equiv Y_1 \), and \( R_i = \exp [-k(y_i - y_{i-1})] \equiv Y_i/Y_{i-1} \) (\( i \neq 1 \)).

Substituting \( A_i, B_i \) in Eqs.(10-13) and \( \Phi(y) \) in Eq.(9) into the action (6) and integrating out \( y \) give rise to a 4D effective potential \( V(R_i, v_i) \),

\[
k^{-1} V(R_i, v_i) = f_1(R_1, v_0, v_1) + \frac{R_i^4}{R_2^4} f_2(R_2, v_2, v_1) + \frac{R_i^4}{R_2^4} f_3(R_3, v_2, v_3) + \frac{R_i^4}{R_4^4} f_4(R_4, v_4, v_3) + ... + \frac{R_i^4 R_{2j}^4 ... R_{2j-3}^4}{R_{2j}^4 R_{2j-2}^4} [f_{2j-1}(R_{2j-1}, v_{2j-2}, v_{2j-1}) + ...],
\]

where \( f \) is defined as

\[
f(R, u, v) = \frac{(\nu + 2)(R^\nu u - R^{2\nu})^2 + (\nu - 2)(u - R^{\nu+2}v)^2}{1 - R^{2\nu}}.
\]

The effective potential in (14) is an iterated function with many variables, however it can be shown that a minimum exists in certain conditions. Defining that \( r_{(2j-2,2j-1)} \equiv v_{2j-2}/v_{2j-1}, r_{(2j,2j-1)} \equiv v_{2j}/v_{2j-1} \), we obtain that

\[
\hat{V}_{2j-1} \equiv V_{2j-1} + \frac{R_{2j-1}^4}{R_{2j}^4} \left( \frac{v_{2j+1}}{v_{2j-1}} \right)^2 \hat{V}_{2j+1}
\]
\[ \begin{align*}
&= f_{2j-1}(R_{2j-1}, r_{(2j-2,2j-1)}, 1) + \frac{R_{2j-1}^4}{R_{2j}^4} f_{2j}(R_{2j}, r_{(2j,2j-1)}, 1) \\
&+ \frac{R_{2j-1}^4}{R_{2j}^4} (\frac{v_{2j+1}}{v_{2j-1}})^2 \tilde{V}_{2j+1}. \quad (16)
\end{align*} \]

Substituting (15) into (16), \( \tilde{V}_{2j-1} \) can be rewritten as

\[ \tilde{V}_{2j-1} = \left[ \frac{(\nu + 2)(R_{2j-1}^\nu - R_{2j-1}^2) - R_{2j-1}^2}{1 - R_{2j-1}^{2\nu}} \right] + (17) \]

\[ \frac{R_{2j-1}^4}{R_{2j}^4} \left[ (\nu + 2)(R_{2j}^2 - R_{2j-1}^2) + (\nu - 2)(R_{2j-1}^2 - R_{2j-1}^{2\nu}) \right] + \tilde{V}_{2j+1}(\frac{v_{2j+1}}{v_{2j-1}})^2. \quad (18) \]

From (17), one can see that for arbitrary positive values of \( \nu \), \( \tilde{V}_{2j-1} \) grows as \( R_{2j-1} \to 1 \) or as \( R_{2j} \to 1 \) as long as \( v_{2j-1} \neq v_{2j-2} \) and \( v_{2j-1} \neq v_{2j} \). These two limits correspond to the \((2j-1)\)th " - " brane approaching the " + " brane at \( y = y_{2j-2} \) and at \( y = y_{2j} \), respectively, and in these limits \( \tilde{V}_{2j-1} \) is singular, i.e., \( \tilde{V}_{2j-1}(R_{2j-1}, R_{2j}) \sim (1 - R_{2j-1}^{2\nu})^{-1} > 0 \) as \( R_{2j-1} \to 1 \) and \( \tilde{V}_{2j-1}(R_{2j-1}, R_{2j}) \sim (1 - R_{2j}^{2\nu})^{-1} > 0 \) as \( R_{2j} \to 1 \). This implies that the \((2j-1)\)th " - " brane experiences repulsive forces exerted on it by the " + " brane of its either side and consequently the numbers of the branes can not be reduced. We note that when \( v_{2j-1} = v_{2j-2} \) and/or \( v_{2j-1} = v_{2j} \), the leading singularity in \( \tilde{V}_{2j-1} \) is removed and the subleading terms in \( \tilde{V}_{2j-1} \) is attractive. In this case, therefore, the less brane crystals will be more stable.

From now on, we assume that \( v_i \) takes different numerical values in the different branes. To obtain the values of \( R_{2j-1} \) and \( R_{2j} \), we minimize the effective potential \( \tilde{V}_{2j-1} \). For \( R_{2j} \) it satisfies the following equation:

\[ r_{(2j,2j-1)}^\pm(R_{2j}) = \frac{\nu R_{2j}^{2+\nu} \left[ (2 \pm \sqrt{Q_{2j}}) (R_{2j}^{2\nu} - 1) + \nu (1 + R_{2j}^{2\nu}) \right]}{2 \left( \nu^2 R_{2j}^{2\nu} + 2 (R_{2j}^{2\nu} - 1)^2 + \nu (R_{2j}^{4\nu} - 1) \right)}, \quad (19) \]

where

\[ Q_{2j} = \nu^2 - 4 + 4\tilde{V}_{2j+1}(\frac{v_{2j+1}}{v_{2j-1}})^2 \left[ \frac{(\nu + 2)R_{2j}^{2\nu} + (\nu^2 - 4)R_{2j}^{2\nu} + 2 - \nu}{\nu^2 R_{2j}^{4\nu + 4}} \right]. \quad (20) \]

Since \( R_{2j} \to 0 \) or \( 1, \tilde{V}_{2j-1} \to \infty \) which one can see from (17), this extremum represents the minima of \( \tilde{V}_{2j-1} \) in the \( R_{2j} \)-direction. To see whether simultaneous minima in the \( R_{2j-1} \)-direction exist, we extremize \( \tilde{V}_{2j-1} \) with respect to \( R_{2j-1} \) and get

\[ r_{(2j-2,2j-1)}^\pm(R_{2j-1}) = \frac{R_{2j-1}^{2-\nu}}{2 \nu} \left[ 2 (1 - R_{2j-1}^{2\nu}) + \nu (1 + R_{2j-1}^{2\nu}) \pm R_{2j-1}^{2+\nu}(1 - R_{2j-1}^{2\nu})\sqrt{Q_{2j-1}} \right], \quad (21) \].
Note that for \( r_{(2j-2,2j-1)} = r_{(2j-2,2j-1)}^+ \), \( \partial^2 \tilde{V}_{2j-1} / \partial R_{2j-1}^2 < 0 \), which corresponds to a sequence of saddle points. While for \( r_{(2j-2,2j-1)} = r_{(2j-2,2j-1)}^- \), \( \partial^2 \tilde{V}_{2j-1} / \partial R_{2j-1}^2 > 0 \) and this extremum represents the minima of \( \tilde{V}_{2j-1} \) in the \( R_{2j-1} \)-direction. Thus in the parameter space where \( r_{(2j,2j-1)}^+ \) and \( r_{(2j-2,2j-1)}^- \) co-exist, the absolute minima of \( \tilde{V}_{2j-1} \) exists. Therefore, the minima of \( V(R_i, r_i) \) exists, i.e. Crystal Universe can be stabilized.

Having shown the possibility of stabilizing the Crystal Universe models, we discuss and analyze the configuration of the brane crystals when they are stabilized. Following GW\[12\], we will also limit ourselves to the regime where \( \epsilon \) is small, \( \epsilon \equiv \nu - 2 \approx \frac{m^2}{4k^2} \ll 1 \). For a stabilized Crystal Universe model, from (21) we have

\[
\frac{f_{2j}(R_{2j}, r_{(2j,2j-1)}, 1)}{R_{2j}^4} + \frac{\tilde{V}_{2j+1}}{R_{2j}} \left( \frac{v_{2j+1}}{v_{2j-1}} \right)^2 \leq \epsilon. \tag{23}
\]

Note that the two terms on the left-handed side of eq.(22) are positive, we have seperately

\[
\frac{f_{2j}(R_{2j}, r_{(2j,2j-1)}, 1)}{R_{2j}^4} \leq \epsilon, \tag{24}
\]

\[
\frac{\tilde{V}_{2j+1}}{R_{2j}} \left( \frac{v_{2j+1}}{v_{2j-1}} \right)^2 \leq \epsilon. \tag{25}
\]

In Fig.2 we plot the allowed region of \( R_{2j} \) based on (23), from which one can see that for \( \epsilon = 0.01 \) \( R_{2j} \) varies from 0.85 to 1, which corresponds to the \((2j-1)th \) ” – ” brane very close to the \(2jth \) ” + ” brane of its right side. With \( R_{2j} \) in this range we have \( r_{(2j,2j-1)} \approx 1 \), i.e. \( v_{2j} \approx v_{2j-1} \).

From eq. (24) we get

\[
\left( \frac{v_{2j+1}}{v_{2j-1}} \right)^2 \left( \frac{v_{2j+1}}{v_{2j-1}} \right)^2 \left( \frac{v_{2j+3}}{v_{2j+1}} \right)^2 \left( \frac{v_{2j+3}}{v_{2j+1}} \right)^2 \tilde{V}_{2j+3} \leq \epsilon. \tag{26}
\]

Thus

\[
\left( \frac{v_{2j+1}}{v_{2j-1}} \right)^2 \frac{\tilde{V}_{2j+1}}{R_{2j}^4} \leq \epsilon, \tag{27}
\]

\[
\frac{R_{2j}^4}{R_{2j}^4 R_{2j+2}^4} \left( \frac{v_{2j+1}}{v_{2j-1}} \right)^2 \left( \frac{v_{2j+3}}{v_{2j+1}} \right)^2 \tilde{V}_{2j+3} \leq \epsilon. \tag{28}
\]
Given that $v_{2j-1} \approx v_{2j}$, we have from (26)
\[
\frac{f_{2j+1}(R_{2j+1}, r_{(2j,2j+1)}, 1)}{r_{(2j,2j+1)}^2} \leq \epsilon R_{2j}^4,
\]
(29)
\[
\frac{1}{r_{(2j,2j+1)}^2} \frac{R_{2j+1}^4}{R_{2j}^2 R_{2j+2}^2} f_{2j+2}(R_{2j+2}, r_{(2j+2,2j+1)}, 1) \leq \epsilon.
\]
(30)

In Fig.3 we plot the allowed range of $R_{2j+1}$ from which we see that with $R_{2j}$ in the range of $0.85 \sim 1$, $R_{2j+1}$ varies from 0 to $0.8 \sim 1$.

Now we consider an additional constraint on $R_{2j+1}$ from (27). Combining Eqs.(26), (27) and (29) we obtain
\[
R_{2j+1}^4/r_{(2j,2j+1)}^2 \leq R_{2j+2}^4,
\]
(31)
which we plot in Fig.4 for the allowed range of $R_{2j+1}$.

From Figs. 3 and 4 one can see that a Crystal Universe can be stabilized for a large parameter space of $R_{2j+1}$, however $R_{2j}$ is required to be very close to 1.

Similarly we can discuss the parameter space for solution (ii) and our results show to have a stabilized Crystal Universe $R_{2j+1}$ approaches 1, and correspondingly $r_{(2j+1,2j)} \approx 1$. However we should point out that the absolute value of $R_1$ can not be fixed and it depends mostly on the $r_{(1,0)}$.

In summary, in this paper by explicit calculation we show that Crystal Universe can be stabilized by introducing a bulk scalar field to brane system. Our results differ from the assumptions taken in the literature for the discussion of the Crystal Universe. For example, in Refs.[3, 11], they have assumed that the branes are equidistant.

We should point out that Choudhury et al. [13] studied the stability of the ” + – + ” brane configuration and find that the ” – ” brane chooses to stay close to the visible ” + ” brane. Taking $n = 2$, we recover the results of Ref.[13]. For $n = 1$ we agree with GW’s results. So our results apply for general Crystal Universe models.

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Figure 1: Crystal Universe is made up \( n + 1 \) array of \( '+' \) and \( '-' \) branes with lattice spacing \((y_1 - y_0), \,(y_2 - y_1) \ldots \,(y_n - y_{n-1})\) and bulk curvature \(k\). The fifth dimension \(y\) has orbifold geometry \(S^1/Z_2\), \(y_0 = 0\) and \(y_n\) are orbifold fixed points.

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Figure 2: The allowed parameter space of $R_{2j}$. The y-axis is $g_{2j} \equiv \frac{f_{2j}}{R_{2j}}$ i.e. the left-handed side of eq.(23); the x-axis is $R_{2j}$. The figure on the right-handed side is an amplification of the left in the range of $R_{2j} \approx 0.85 \sim 1$.

Figure 3: The allowed parameter space of $R_{2j}$. The y-axis is $g_{2j} \equiv \frac{f_{2j}}{R_{2j}}$ i.e. the left-handed side of eq.(28); the x-axis is $R_{2j+1}$.

Figure 4: The allowed parameter space of $R_{2j}$. The y-axis is $h_{2j} \equiv \frac{R_{2j+1}}{r_{(2j,2j+1)}}$ i.e. the left-handed side of eq.(30); the x-axis is $R_{2j+1}$.
Figure 5: Illustration of a Crystal Universe model with " + " brane at orbifold fixed point $y_0 = 0$. The up figure ends with " + " brane and the down with " − " brane.

Figure 6: Illustration of a Crystal Universe model with " − " brane at orbifold fixed point $y_0 = 0$. The up figure ends with " + " brane and the down with " − " brane.