EXISTENCE OF A POSITIVE SOLUTION TO A NONLINEAR SCALAR FIELD EQUATION WITH ZERO MASS AT INFINITY

MÓNICA CLAPP AND LILIANE A. MAIA

ABSTRACT. We establish the existence of a positive solution to the problem
\[-\Delta u + V(x)u = f(u), \quad u \in D^{1,2}(\mathbb{R}^N),\]
for $N \geq 3$, when the nonlinearity $f$ is subcritical at infinity and supercritical near the origin, and the potential $V$ vanishes at infinity. Our result includes situations in which the problem does not have a ground state. Then, under a suitable decay assumption on the potential, we show that the problem has a positive bound state.

Key words: scalar field equations; zero mass; superlinear; double-power nonlinearity; positive solution; variational methods.
MSC2010: 35Q55 (35B09, 35J20).

1. Introduction

This paper is concerned with the existence of a positive solution to the problem
\[
\left\{
\begin{array}{ll}
-\Delta u + V(x)u = f(u), \\
u \in D^{1,2}(\mathbb{R}^N),
\end{array}
\right.
\]
for $N \geq 3$, where the nonlinearity $f$ is subcritical at infinity and supercritical near the origin, and the potential $V$ vanishes at infinity. Our precise assumptions on $V$ and $f$ are stated below.

In their groundbreaking paper [10], Berestycki and Lions considered the case where $V \equiv \lambda$ is constant and $f$ has superlinear growth. They showed that, if $f$ is subcritical, the problem $(\wp_V)$ has a solution for $\lambda > 0$ and it does not have a solution for $\lambda < 0$. They also studied the limiting case $V \equiv 0$, which they called the zero mass case. They showed that, if $f$ is subcritical at infinity and supercritical near the origin, the problem
\[
(\wp_0) \\
-\Delta u = f(u), \quad u \in D^{1,2}(\mathbb{R}^N),
\]
has a ground state solution $\omega$, which is positive, radially symmetric and decreasing in the radial direction.

The motivation for studying this type of equations came from some problems in particle physics, related to the nonabelian gauge theory which underlies strong interaction, called quantum chromodynamics or QCD. Their solutions give rise to some special solutions of the pure Yang-Mills equations via 't Hooft’s Ansatz; see [17].

Date: May 13, 2018.

M. Clapp was supported by CONACYT grant 237661 (Mexico) and UNAM-DGAPA-PAPIIT grant IN104315 (Mexico). L. Maia was supported by CNPq/PQ 308173/2014-7 (Brazil) and PROEX/CAPES (Brazil).
For a radial potential \( V(|x|) \), Badiale and Rolando established the existence of a positive radial solution to the problem \((pV)\) in [4]. On the other hand, under suitable hypotheses, but without assuming any symmetries on \( V \), Benci, Grisanti and Micheletti showed in [7] that the problem \((pV)\) has a positive least energy solution if \( V(x) \leq 0 \) for all \( x \in \mathbb{R}^N \) and \( V(x) < 0 \) on a set of positive measure. They also showed that, if \( V(x) \geq 0 \) for all \( x \in \mathbb{R}^N \) and \( V(x) > 0 \) on a set of positive measure, this problem does not have a ground state solution, i.e., the corresponding variational functional does not attain its (least energy) mountain pass value. Other related results may be found in [2, 8, 9, 16].

The result that we present in this paper includes the existence of a positive bound state for positive or sign changing potentials which decay to 0 at infinity with a suitable velocity. More precisely, we assume that \( V \) and \( f \) have the following properties:

\begin{enumerate}[(V1)]
  \item \( V \in L^{N/2}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \) for some \( r > N/2 \), and \( \int_{\mathbb{R}^N} |V|^2 \leq S^{N/2} \), where \( V^- := \min\{0,V\} \) and \( S \) is the best constant for the embedding \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \) with \( 2^* := \frac{2N}{N-2} \).
  \item There are constants \( A_0 > 0 \) and \( \kappa > \max\{2, N-2\} \) such that \( V(x) \leq A_0(1 + |x|)^{-\kappa} \) for all \( x \in \mathbb{R}^N \).
  \item \( f \in C^1[0,\infty) \), and there are constants \( A_1 > 0 \) and \( 2 < p < 2^* < q \) such that, for \( m = -1,0,1 \),
    \[ |f^{(m)}(s)| \leq \begin{cases} 
      A_1 |s|^{p-(m+1)} & \text{if } |s| \geq 1, \\
      A_1 |s|^{q-(m+1)} & \text{if } |s| \leq 1,
    \end{cases} \]
    where \( f^{(-1)} := F \), \( f^{(0)} := f \), \( f^{(1)} := f' \), and \( F(s) := \int_0^s f(t)dt \).
  \item There is a constant \( \theta > 2 \) such that \( 0 \leq \theta F(s) \leq f(s)s < f'(s)s^2 \) for all \( s > 0 \).
  \item The function \( g(s) := \frac{sf(s)}{f'(s)} \) is a decreasing function of \( s > 0 \) and \( \lim_{s \to \infty} g(s) < 2^* - 1 < \lim_{s \to 0} g(s) \).
\end{enumerate}

Our main result is the following one.

**Theorem 1.1.** Assume that (V1)-(V2) and (f1)-(f3) hold true. Then the problem \((pV)\) has a positive solution.

It is easy to see that the model nonlinearity

\[ f(s) := \frac{s^{q-1}}{1 + s^{q-p}} \]

satisfies the assumptions (f1)-(f3).

We point out that assumptions (V1), (f1) and (f2) are quite natural and have been also considered in previous works, in particular, in [7]. Assumption (f3) guarantees that the limit problem \((p0)\) has a unique positive solution. This fact, together with some fine estimates, which involve assumption (V2), allows us to show the existence of a positive bound state for the problem \((pV)\) when the ground state is not attained.

The positive mass case, in which the potential \( V \) tends to a positive constant at infinity, has been widely investigated. A brief account may be found in [13], where a result, similar to Theorem 1.1, was obtained for subcritical nonlinearities. On
the other hand, except for the case of the critical pure power nonlinearity, only few results are known for the zero mass case.

There are several delicate issues in dealing with the zero mass case. Already the variational formulation requires some care, because the energy space $D^{1,2}(\mathbb{R}^N)$ is only embedded in $L^{2^*}(\mathbb{R}^N)$. The growth assumptions $(f1)$ on the nonlinearity, however, provide the basic interpolation and boundedness conditions that allow to establish the differentiability of the variational functional and to study its compactness properties. Benci and Fortunato, in [6], expressed these conditions in the framework of Orlicz spaces, which was also used and further developed in [2–4, 7–9, 16]. The crucial facts, for our purposes, are stated in Proposition 3.1 below.

Another sensitive issue is the lack of compactness. In the positive mass case, a fundamental tool for dealing with it, is Lions’ vanishing lemma, whose proof relies deeply on the fact that the sequences involved are bounded in $H^1(\mathbb{R}^N)$. Once again, assumption $(f1)$ allowed us to obtain a suitable version of this result for sequences which are only bounded in $D^{1,2}(\mathbb{R}^N)$ (see Lemma 3.5). This new version of Lions’ vanishing lemma plays a crucial role in the proof of the splitting lemma (Lemma 3.9) which describes the lack of compactness of the variational functional. When the ground state is not attained, due to the uniqueness of the positive solution to limit problem $(\wp_0)$, the splitting lemma provides an open interval of values at which the energy functional satisfies the Palais-Smale condition.

We give a topological argument to establish the existence of a critical value in this interval. This argument requires, on the one hand, some fine estimates which are based on the precise asymptotic decay for the solutions of the limit problem $(\wp_0)$, obtained recently by Vétois in [18], and on a suitable deformation lemma for $C^1$-manifolds that was proved by Bonnet in [11].

This paper is organized as follows: In Section 2 we collect the information that we need about the solutions of the limit problem. Section 3 is devoted to the study of the variational problem and, specially, of the compactness properties of the variational functional. In Section 4 we derive the estimates that we need, and we prove our main result.

2. The limit problem

We define $f(u) := -f(-u)$ for $u < 0$. Then, $f \in C^1(\mathbb{R})$ and it is an odd function. Note that, if $u$ is a positive solution of the problem $(\wp_V)$ for this new function, it is also a solution of $(\wp_V)$ for the original function $f$. Hereafter, $f$ will denote this extension.

We consider the Hilbert space $D^{1,2}(\mathbb{R}^N) := \{ u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N, \mathbb{R}^N) \}$ with its standard scalar product and norm

$$\langle u, v \rangle := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v, \quad \| u \| := \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^{1/2}.$$

In this section we collect the information that we need on the positive solutions to the limit problem $(\wp_0)$.

Since $f \in C^1(\mathbb{R})$ and $f$ satisfies $(f1)$, a classical result of Berestycki and Lions establishes the existence of a ground state solution $\omega \in C^2(\mathbb{R}^N)$ to the problem $(\wp_0)$, which is positive, radially symmetric and decreasing in the radial direction; see Theorem 4 in [10].
Observe that assumption (f1) implies that $|f(s)| \leq A_1 |s|^{2^*-1}$ and $|f'(s)| \leq A_1 |s|^{2^*-2}$. Note also that assumption (f2) yields that $f(s) > 0$ if $s > 0$. Therefore, a recent result of Vétois implies that every positive solution $u$ to $(\Phi_0)$ satisfies the decay estimates
\begin{equation}
A_2 (1 + |x|)^{-(N-2)} \leq |u(x)| \leq A_3 (1 + |x|)^{-(N-2)} \frac{\|\nabla u(x)\|}{|\nabla u(x)|} \leq A_3 (1 + |x|)^{-(N-1)},
\end{equation}
for some positive constants $A_2$ and $A_3$ and, moreover, $u$ is radially symmetric and strictly radially decreasing about some point $x_0 \in \mathbb{R}^N$; see Theorem 1.1 and Corollary 1.2 in [18].

Concerning uniqueness, Erbe and Tang showed that, if $f$ also satisfies (f3), then the problem $(\Phi_0)$ has a unique fast decaying radial solution, up to translations, where fast decaying means that $u$ is positive and there is a constant $c \in (0, \infty)$ such that $\lim_{|x| \to \infty} |x|^{N-2} u(|x|) = c$; see Theorem 2 in [14] and the remark in the paragraph following it.

We summarize these results in the following statement.

**Proposition 2.1.** Under the assumptions (f1)-(f3), the limit problem $(\Phi_0)$ has a unique positive solution $\omega$, up to translations. Moreover, $\omega \in C^2(\mathbb{R}^N)$, it is radially symmetric and strictly decreasing in the radial direction, and it satisfies the decay estimates (2.1).

## 3. The variational setting

For $u, v \in D^{1,2}(\mathbb{R}^N)$ we set
\begin{equation}
\langle u, v \rangle := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv, \quad \|u\|^2_V := \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2).
\end{equation}
By assumption (V1), these expressions are well defined and, using the Sobolev inequality, we conclude that $\|\cdot\|_V$ is a norm in $D^{1,2}(\mathbb{R}^N)$ which is equivalent to the standard one.

Let $2 < p < 2^* < q$. The following proposition, combined with assumption (f1), provides the interpolation and boundedness properties that are needed to obtain a good variational problem.

**Proposition 3.1.** Let $\alpha, \beta > 0$ and $h \in C^0(\mathbb{R})$. Assume that $\frac{2}{\beta} \leq \frac{2}{\alpha}, \beta \leq q$, and there exists $M > 0$ such that
\[|h(s)| \leq M \min\{|s|^{\alpha}, |s|^{\beta}\} \quad \text{for every } s \in \mathbb{R}.
\]
Then, for every $t \in \left[\frac{q}{\beta}, \frac{2}{\alpha}\right]$, the map $D^{1,2}(\mathbb{R}^N) \to L^t(\mathbb{R}^N)$ given by $u \mapsto h(u)$ is well defined, continuous and bounded.

**Proof.** The decomposition $u = u_1 \chi_{\Omega_u} + u_1 \chi_{\mathbb{R}^N \setminus \Omega_u}$, where $\Omega_u := \{x \in \mathbb{R}^N : |u(x)| > 1\}$, gives a continuous embedding of $L^{2^*}(\mathbb{R}^N)$ and, hence, of $D^{1,2}(\mathbb{R}^N)$, into the Orlicz space
\[L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N) := \{u : u = u_1 + u_2 \text{ with } u_1 \in L^p(\mathbb{R}^N), u_2 \in L^q(\mathbb{R}^N)\},
\]
whose norm is defined by
\[|u|_{L^p + L^q} := \inf\{|u_1|_p + |u_2|_q : u = u_1 + u_2, \ u_1 \in L^p(\mathbb{R}^N), u_2 \in L^q(\mathbb{R}^N)\}.
\]
Therefore, our claim is a special case of Proposition 3.5 in [3].

□
Let $F(u) := \int_0^u f(s) \, ds$. Assumption (f1) implies that $|F(s)| \leq A_1 |s|^{2^*}$ and $|f(s)| \leq A_1 |s|^{2^* - 1}$. Therefore, the functionals $\Phi, \Psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ given by
\[ \Phi(u) := \int_{\mathbb{R}^N} F(u), \quad \Psi(u) := \int_{\mathbb{R}^N} f(u)u \]
are well defined. Using Proposition 3.1 it is easy to show that $\Phi$ is of class $C^2$ and $\Psi$ is of class $C^1$; see Lemma 2.6 in [9] or Proposition 3.8 in [3]. Hence, the functional $I_V : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ given by
\[ I_V(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \int_{\mathbb{R}^N} F(u), \]
is of class $C^2$, with derivative
\[ I_V'(u)v = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) - \int_{\mathbb{R}^N} f(u)v, \quad u, v \in D^{1,2}(\mathbb{R}^N), \]
and the functional $J_V : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by
\[ J_V(u) := I_V'(u)u = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \int_{\mathbb{R}^N} f(u)u, \]
is of class $C^1$.

The solutions to the problem (\psi_V) are the critical points of the functional $I_V$. The nontrivial ones lie on the set
\[ \mathcal{N}_V := \{ u \in D^{1,2}(\mathbb{R}^N) : u \neq 0, \ J_V(u) = 0 \}. \]
We define
\[ c_V := \inf_{u \in \mathcal{N}_V} I_V(u), \]
and we write $I_0, J_0, N_0$ and $c_0$ for the previous expressions with $V = 0$.

The proofs of the next two lemmas use well known arguments. We include them for the sake of completeness. Hereafter $C$ will denote a positive constant, not necessarily the same one.

**Lemma 3.2.**  
(a) There exists $\varrho > 0$ such that $\|u\|_V \geq \varrho$ for every $u \in \mathcal{N}_V$.  
(b) $\mathcal{N}_V$ is a closed $C^1$-submanifold of $D^{1,2}(\mathbb{R}^N)$ and a natural constraint for the functional $I_V$.  
(c) $c_V > 0$.  
(d) If $u \in \mathcal{N}_V$, the function $t \mapsto I_V(tu)$ is strictly increasing in $[0, 1)$ and strictly decreasing in $(1, \infty)$. In particular,
\[ I_V(u) = \max_{t > 0} I_V(tu). \]

**Proof.** (a): Assumption (f1) implies that $|f(s)s| \leq A_1 |s|^{2^*}$. So, using Sobolev’s inequality, we get
\[ J_V(u) \geq \|u\|_V^{2^*} - C \int_{\mathbb{R}^N} \|u\|^{2^*} - C \|u\|_V^{2^*} \quad \forall u \in D^{1,2}(\mathbb{R}^N). \]
As $2^* > 2$, there exists $\varrho > 0$ such that $J_V(u) > 0$ if $0 < \|u\|_V \leq \varrho$. This proves (a).  
(b): It follows from (a) that $\mathcal{N}_V$ is closed in $D^{1,2}(\mathbb{R}^N)$. Moreover, assumption (f2) yields
\[ J_V'(u)u = 2\|u\|_V^2 - \int_{\mathbb{R}^N} f'(u)u^2 - \int_{\mathbb{R}^N} f(u)u = \int_{\mathbb{R}^N} [f(u) - f'(u)u] u < 0 \]
for every $u \in \mathcal{N}_V$. This implies that $0$ is a regular value of the restriction of $J_V$ to $D^{1,2}(\mathbb{R}^N) \setminus \{0\}$, which is of class $C^1$. Hence, $\mathcal{N}_V$ is a $C^1$-submanifold of $D^{1,2}(\mathbb{R}^N)$ and a natural constraint for $I_V$.

(c): Let $u \in \mathcal{N}_V$. From hypothesis $(f2)$ and statement (a), we obtain that

$$I_V(u) = I_V(u) - \frac{1}{\theta} I'_V(u) u = \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u\|_V^2 + \int_{\mathbb{R}^N} \left( \frac{1}{\theta} f(u) - F(u) \right) u$$

$$\geq \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u\|_V^2 \geq \left( \frac{1}{2} - \frac{1}{\theta} \right) \theta^2.$$

Hence, $c_V > 0$.

(d): Let $u \in \mathcal{N}_V$. Then,

$$\frac{d}{dt} I_V(tu) = \frac{1}{t} I_V(tu) = t \|u\|_V^2 - t \int_{\mathbb{R}^N} f(tu) u = t \int_{\mathbb{R}^N} \left( \frac{f(u)}{t} - \frac{f(tu)}{tu} \right) u$$

$$= t \left[ \int_{u > 0} \left( \frac{f(u)}{u} - \frac{f(tu)}{tu} \right) u^2 + \int_{u < 0} \left( \frac{f(u)}{u} - \frac{f(tu)}{tu} \right) u^2 \right].$$

Property $(f2)$ implies that $\frac{f(s)}{s}$ is strictly increasing for $s > 0$ and strictly decreasing for $s < 0$. Therefore $\frac{d}{dt} I_V(tu) > 0$ if $t \in (0, 1)$ and $\frac{d}{dt} I_V(tu) < 0$ if $t \in (1, \infty)$. This proves (d).

**Lemma 3.3.** If $u$ is a solution of $(\wp_V)$ with $I_V(u) \in [c_V, 2c_V]$, then $u$ does not change sign.

**Proof.** If $u$ is a solution of $(\wp_V)$ then $0 = I'_V(u) u^\pm = J_V(u^\pm)$, where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. Thus, if $u^+ \neq 0$ and $u^- \neq 0$, then $u^\pm \in \mathcal{N}_V$ and

$$I_V(u) = I_V(u^+) + I_V(u^-) \geq 2c_V,$$

contradicting our assumption. \qed

**Lemma 3.4.** The limit problem $(\wp_0)$ does not have a solution $u$ with $I_0(u) \in (c_0, 2c_0)$.

**Proof.** If $u$ is a solution of $(\wp_0)$ such that $I_0(u) \in [c_0, 2c_0]$ then, by Lemma 3.3, $u$ does not change sign. So, by Proposition 2.1, we have that $u = \pm \omega$, up to a translation. Hence, $I_0(u) = c_0$. \qed

The following version of Lions’ vanishing lemma plays a crucial role in the proof of Lemma 3.6 and of the splitting lemma (Lemma 3.9). Its proof was inspired by that of Lemma 2 in [1]. We write $B_R(y) := \{x \in \mathbb{R}^N : |x - y| < R\}$.

**Lemma 3.5.** If $(u_k)$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and there exists $R > 0$ such that

$$\lim_{k \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_k|^2 \right) = 0,$$

then $\lim_{k \to \infty} \int_{\mathbb{R}^N} f(u_k) u_k = 0$.

**Proof.** Fix $\varepsilon \in (0, 1)$ and set $\eta := \frac{\varepsilon}{2} > 1$. For each $k$, consider the function $w_k := \left\{ \begin{array}{ll} |u_k| & \text{if } |u_k| \geq \varepsilon, \\ \varepsilon^{-(\eta - 1)} |u_k|^{\eta} & \text{if } |u_k| \leq \varepsilon. \end{array} \right.$
Observe that
\[
|\nabla w_k|^2 = \eta^2 \varepsilon^{-(\eta-1)} |u_k|^2 |\nabla u_k|^2 \leq \eta^2 |\nabla u_k|^2 \quad \text{if } |u_k| \leq \varepsilon,
\]
and
\[
|w_k|^2 \leq \varepsilon^{-2} |u_k|^2 |\nabla u_k|^2 \leq |u_k|^2 \quad \text{if } |u_k| \leq \varepsilon,
\]
\[
|w_k|^2 \leq |u_k|^{2-2}\varepsilon |u_k|^2 \leq \varepsilon^{-2} |u_k|^2 \quad \text{if } |u_k| \geq \varepsilon.
\]
Using these inequalities we obtain that
\[
|w_k|^2 \leq |u_k|^2, \quad |w_k|^2 \leq \varepsilon^{-2} |u_k|^2, \quad |\nabla w_k|^2 \leq \eta^2 |\nabla u_k|^2.
\]
Therefore, as \((u_k)\) is bounded in \(D^{1,2}(\mathbb{R}^N)\), we have that
\[
\|w_k\|_{H^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\nabla w_k|^2 + |w_k|^2 \leq \int_{\mathbb{R}^N} \eta^2 |\nabla u_k|^2 + \int_{\mathbb{R}^N} \varepsilon^{-2} |u_k|^2 \leq C,
\]
i.e., \((w_k)\) is bounded in \(H^1(\mathbb{R}^N)\). Moreover,
\[
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |w_k|^2 = 0.
\]
It follows from Lions’ vanishing lemma [19, Lemma 1.21] that
\[
w_k \to 0 \quad \text{in } L^s(\mathbb{R}^N) \quad \text{for each } 2 < s < 2^*.
\]
Now, using \((f1)\), we obtain
\[
\left| \int_{\mathbb{R}^N} f(u_k)u_k \right| \leq A_1 \left( \int_{|u_k| \geq 1} |u_k|^p + \int_{|u_k| \leq 1} |u_k|^q \right)
\leq A_1 \left( \int_{|u_k| \geq \varepsilon} |u_k|^p + \int_{\varepsilon \leq |u_k| \leq 1} |u_k|^q + \int_{|u_k| \leq \varepsilon} |u_k|^q \right)
\leq 2A_1 \int_{|u_k| \geq \varepsilon} |u_k|^p + A_1 \int_{|u_k| \leq \varepsilon} |u_k|^q
\leq 2A_1 \int_{|u_k| \geq \varepsilon} |w_k|^p + A_1 \int_{|u_k| \leq \varepsilon} |u_k|^q - 2\varepsilon |u_k|^{2^*}
\leq 2A_1 \int_{\mathbb{R}^N} |w_k|^p + A_1 \varepsilon^{2-2^*} \int_{\mathbb{R}^N} |u_k|^{2^*}.
\]
As \((u_k)\) is bounded in \(D^{1,2}(\mathbb{R}^N)\) and \(w_k \to 0\) in \(L^p(\mathbb{R}^N)\), we conclude that
\[
\left| \int_{\mathbb{R}^N} f(u_k)u_k \right| \leq C\varepsilon^{2-2^*}.
\]
Since \(\varepsilon \in (0, 1)\) was arbitrarily chosen, the statement is proved. \(\square\)

We write \(\nabla I_V(u)\) and \(\nabla J_V(u)\) for the gradients of \(I_V\) and \(J_V\) at \(u\) with respect to the scalar product (3.1).

**Lemma 3.6.** Let \((u_k)\) be a sequence in \(D^{1,2}(\mathbb{R}^N)\) such that \(I_V(u_k) \to d > 0\) and \(J_V(u_k) \to 0\). Then there exist \(a_1 > a_0 > 0\) such that, after passing to a subsequence,
\[
a_0 \leq \|u_k\|_V \leq a_1, \quad a_0 \leq \|\nabla J_V(u_k)\|_V \leq a_1, \quad |J_V'(u_k)u_k| \geq a_0.
\]
Hence, there exists a subset $\Lambda$ of positive measure such that

$$d + o(1) = I_V(u_k) \leq I_V(u_k) + \int_{\mathbb{R}^N} F(u_k) = \frac{1}{2} \|u_k\|^2_V,$$

and

$$d + o(1) = I_V(u_k) - \frac{1}{\theta} J_V(u_k)$$

$$= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_k\|^2_V + \int_{\mathbb{R}^N} \left[\frac{1}{\theta} f(u_k) u_k - F(u_k)\right] \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_k\|^2_V.$$

Hence, $(u_k)$ is bounded and bounded away from 0 in $D^{1,2}(\mathbb{R}^N)$.

By assumption $(f1)$, for any $v \in D^{1,2}(\mathbb{R}^N)$, we have that

$$\left|\int_{\mathbb{R}^N} [f'(u_k) u_k + f(u_k)] v\right| \leq C \int_{\mathbb{R}^N} |u_k|^{2^* - 1} |v| \leq C |u_k|^{2^* - 1} |v|_{2^*},$$

$$\leq C \||u_k||^{2^* - 1} \|v\| \leq C \|v\|_V.$$

Therefore,

$$|\langle \nabla J_V(u_k), v \rangle| = \left|2 \langle u_k, v \rangle_V - \int_{\mathbb{R}^N} [f'(u_k) u_k + f(u_k)] v\right| \leq C \|v\|_V.$$

This implies that $(\nabla J_V(u_k))$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Hence, after passing to a subsequence, we have that $|J'_V(u_k)u_k| \to a \geq 0$. Next, we show that $a > 0$.

As $J_V(u_k) \to 0$ we have that

$$0 < a^2_0 \leq \|u_k\|^2_V = \int_{\mathbb{R}^N} f(u_k) u_k + o(1).$$

So, by Lemma 3.5, there exist $\delta > 0$ and a sequence $(y_k)$ in $\mathbb{R}^N$ such that

$$\int_{B_1(y_k)} |u_k|^2 = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_k|^2 > \delta.$$ 

Set $\tilde{u}_k := u_k(\cdot + y_k)$. Replacing $(\tilde{u}_k)$ by a subsequence, we have that $\tilde{u}_k \rightharpoonup u$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $\tilde{u}_k \to u$ in $L^2_{\text{loc}}(\mathbb{R}^N)$. The inequality (3.2) implies that $u \neq 0$. Hence, there exists a subset $\Lambda$ of positive measure such that $u(x) \neq 0$ for every $x \in \Lambda$. Assumption $(f2)$ implies that $f'(s)s^2 - f(s)s > 0$ if $s \neq 0$. So, using Fatou’s lemma, we conclude that

$$a = \lim_{k \to \infty} |J'_V(u_k)u_k| = \lim_{k \to \infty} \left|2\|u_k\|^2_V - \int_{\mathbb{R}^N} (f'(u_k)u_k^2 + f(u_k)u_k)\right|$$

$$= \lim_{k \to \infty} \int_{\mathbb{R}^N} (f'(u_k)u_k^2 - f(u_k)u_k) = \lim_{k \to \infty} \int_{\mathbb{R}^N} (f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k)$$

$$\geq \liminf_{k \to \infty} \int_{\Lambda} (f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k) \geq \int_{\Lambda} (f'(u)u^2 - f(u)u) > 0.$$

This proves that $a > 0$ and, hence that $(J'_V(u_k)u_k)$ is bounded away from 0 in $\mathbb{R}$.

It follows that $\|\nabla J_V(u_k)\|_V \geq a_{\delta} > 0$. This finishes the proof. \hfill $\Box$

For $\sigma \in \mathbb{R}$, we set $\mathcal{M}_\sigma := J^{-1}_V(\sigma)$ if $\sigma \neq 0$ and $\mathcal{M}_0 := \mathcal{N}_V$. If $|\sigma|$ is small enough and $u \in \mathcal{M}_\sigma$, we write $\nabla_{\mathcal{M}_\sigma} I_V(u)$ for the orthogonal projection of $\nabla I_V(u)$ onto the tangent space to $\mathcal{M}_\sigma$ at $u$.

Recall that a sequence $(u_k)$ in $D^{1,2}(\mathbb{R}^N)$ is said to be a $(PS)_d$-sequence for $I_V$ if $I_V(u_k) \to d$ and $\nabla I_V(u_k) \to 0$. 

$\textbf{Proof.}$ From assumption $(f2)$ we get that

$$d + o(1) = I_V(u_k) \leq I_V(u_k) + \int_{\mathbb{R}^N} F(u_k) = \frac{1}{2} \|u_k\|^2_V,$$
Lemma 3.7. Let $\sigma_k \in \mathbb{R}$ and $u_k \in \mathcal{M}_{\sigma_k}$ be such that $\sigma_k \to 0$, $I_V(u_k) \to d > 0$ and $\nabla_{\mathcal{M}_{\sigma_k}} I_V(u_k) \to 0$. Then $(u_k)$ is a $(PS)_d$-sequence for $I_V$.

Proof. Let $t_k \in \mathbb{R}$ be such that
\begin{equation}
\nabla I_V(u_k) = \nabla_{\mathcal{M}_{\sigma_k}} I_V(u_k) + t_k \nabla J_V(u_k).
\end{equation}
Taking the scalar product with $\nabla \psi$, we get that
\begin{equation}
\sigma_k = J_V(u_k) = I_V(u_k) u_k = \left( \nabla_{\mathcal{M}_{\sigma_k}} I_V(u_k), u_k \right)_V + t_k J_V(u_k) u_k.
\end{equation}
By Lemma 3.6 we have that $(u_k)$ is bounded and $(J_V(u_k))$ is bounded away from 0. So, as $\sigma_k \to 0$ and $\nabla_{\mathcal{M}_{\sigma_k}} I_V(u_k) \to 0$, we conclude that $t_k \to 0$. Moreover, as $(\nabla J_V(u_k))$ is bounded in $D^{1,2}(\mathbb{R}^N)$, from equation (3.3) we get that $\nabla I_V(u_k) \to 0$, as claimed. \hfill \Box

Lemma 3.8. If $u_k \to u$ weakly in $D^{1,2}(\mathbb{R}^N)$, the following statements hold true:
(a) $\|u_k\|_V^2 = \|u_k - u\|^2 + \|u\|^2 + o(1)$.
(b) $\int_{\mathbb{R}^N} |f(u_k) - f(u)| |\varphi| = o(1)$ for every $\varphi \in C_0^\infty(\mathbb{R}^N)$.
(c) $\int_{\mathbb{R}^N} F(u_k) = \int_{\mathbb{R}^N} F(u_k - u) + \int_{\mathbb{R}^N} F(u) + o(1)$.
(d) $f(u_k) - f(u_k - u) \to f(u)$ in $(D^{1,2}(\mathbb{R}^N))'$.

Proof. (a): Set $v_k := u_k - u$. Assumption (V1) states that $V \in L^{N/2}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r > N/2$. As $\eta := \frac{2r}{N} < 2^*$, we have that $v_k \to 0$ in $L^2_{loc}(\mathbb{R}^N)$. Given $\varepsilon > 0$, we fix $R > 0$ such that $\int_{\mathbb{R}^N \setminus B_R(0)} |V|^N \leq \varepsilon N/2$. Then,
\begin{equation}
\int_{\mathbb{R}^N} |V| v_k^2 = \int_{B_R(0)} |V| v_k^2 + \int_{\mathbb{R}^N \setminus B_R(0)} |V| v_k^2 \leq |V|_{L^2(B_R(0))} |v_k|^2 + \|v_k\|^2 + o(1)
\end{equation}
for $k$ large enough. It follows that
\begin{equation}
\|u_k\|_V^2 = \|u_k - u\|^2 + \|u\|^2 + o(1) = \|u_k - u\|^2 + \|u\|^2 + o(1).
\end{equation}
(b): Let $s, t \in \mathbb{R}$. By the mean value theorem, there exists $\zeta \in (0,1)$ such that
\begin{equation}
|f(s + t) - f(s)| = |f'(s + \zeta t)| |t| \leq A_1 \min\{|s + \zeta t|^{p-2}, |s + \zeta t|^{q-2}\} |t|
\leq A_1 \min\{|s|^{p-2}, |s|^{q-2}\} |t|
= h(|s| + |t|) |t|,
\end{equation}
where $h(s) := A_1 \min\{|s|^{p-2}, |s|^{q-2}\}$. Applying Proposition 3.1 to this function, we get that $\{h(|u| + |u_k - u|)\}$ is bounded in $L^p/(p-2)(\mathbb{R}^N)$. So, as $u_k \to u$ in $L^p_{loc}(\mathbb{R}^N)$, we conclude that
\begin{equation}
\int_{\mathbb{R}^N} |f(u_k) - f(u)| |\varphi| \leq \int_{\mathbb{R}^N} h(|u| + |u_k - u|) |u_k - u| |\varphi|
\leq h(|u| + |u_k - u|) |\varphi| \left( \int_{\text{supp}(\varphi)} |u_k - u|^p \right)^{1/p} = o(1).
\end{equation}
(c): Arguing as in (b), we have that
\begin{equation}
|F(s + t) - F(s)| \leq H(|s| + |t|) |t|
\end{equation}
for all $s, t \in \mathbb{R}$, where $H(s) := A_1 \min\{|s|^{p-1}, |s|^{q-1}\}$. Let $\varepsilon > 0$ and set $v_k := u_k - u$. Then, noting that $|F(s)| \leq A_1 |s|^{2^*}$ and using Proposition 3.1, we may choose $R > 1$ such that

$$
\int_{|x| > R} |F(u_k) - F(v_k) - F(u)| \leq \int_{|x| > R} |F(u_k) - F(v_k)| + \int_{|x| > R} |F(u)|
$$

$$
\leq \int_{|x| > R} H(|v_k| + |u|) |u| + A_1 \int_{|x| > R} |u|^{2^*}
$$

$$
\leq |H(|v_k| + |u|)|_{2^*/(2^* - 1)} \left( \int_{|x| > R} |u|^{2^*} \right)^{1/2^*} + A_1 \int_{|x| > R} |u|^{2^*}
$$

$$
\leq C \left( \int_{|x| > R} |u|^{2^*} \right)^{1/2^*} + A_1 \int_{|x| > R} |u|^{2^*} < \varepsilon.
$$

On the other hand, as $u_k \to u$ in $L^p_{loc}(\mathbb{R}^N)$, we have that

$$
\int_{|x| \leq R} |F(u_k) - F(v_k) - F(u)|
$$

$$
\leq \int_{|x| \leq R} |F(u_k) - F(u)| + \int_{|x| \leq R} |F(v_k)| + \int_{|x| \leq 1} |F(v_k)|
$$

$$
\leq \int_{|x| \leq R} H(|u_k| + |u|) |v_k| + A_1 \int_{|x| \leq R} |v_k|^p + A_1 \int_{|x| \leq 1} |v_k|^q
$$

$$
\leq |H(|u_k| + |u|)|_{p/(p-1)} \left( \int_{|x| \leq R} |v_k|^p \right)^{1/p} + C \int_{|x| \leq R} |v_k|^p
$$

$$
\leq C \left( \int_{|x| \leq R} |v_k|^p \right)^{1/p} + C \int_{|x| \leq R} |v_k|^p < \varepsilon
$$

if $k$ is large enough. This proves (c).

(d): Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ and $R > 0$. Set $h(s) := A_1 \min\{|s|^{p-2}, |s|^{q-2}\}$. From (3.4) and Proposition 3.1 we get that

$$
\int_{|x| > R} |f(u_k) - f(u_k - u)| \varphi \leq \int_{|x| > R} h(|u_k| + |u|) |u| |\varphi|
$$

$$
\leq |h(|u_k| + |u|)|_{2^*/(2^* - 2)} \left( \int_{|x| > R} |u|^{2^*} \right)^{1/2^*} |\varphi|_{2^*} \leq C \left( \int_{|x| > R} |u|^{2^*} \right)^{1/2^*} \|\varphi\|.
$$

Moreover, as $|f(u)| \leq A_1 |u|^{2^*-1}$, we have that

$$
\int_{|x| > R} |f(u)| |\varphi| \leq C \left( \int_{|x| > R} |u|^{2^*} \right)^{(2^*-1)/2^*} \|\varphi\|.
$$

Thus, given $\varepsilon > 0$, we may choose $R > 0$ large enough so that

$$
\int_{|x| > R} |f(u_k) - f(u_k - u) - f(u)| |\varphi| \leq \varepsilon \|\varphi\|.
$$
Next, we fix \( \delta \in (0, 1) \) such that \( \eta := 2^* \delta \in (p, 2^*) \) and \( \nu := \frac{\eta}{\eta - 1 - \delta} \in \left[ \frac{q}{q-2}, \frac{p}{p-2} \right] \).

As \( u_k \to u \) strongly in \( L^p_{\text{loc}}(\mathbb{R}^N) \), from (3.4) and Proposition 3.1 we get that
\[
\int_{|x| \leq R} |f(u_k) - f(u)| |\varphi| \leq |h(|u| + |u_k - u|)| \nu \left( \int_{|x| \leq R} |u_k - u|^\nu \right)^{1/\nu} \|\varphi\|_{2^*} \leq \varepsilon \|\varphi\|
\]
for \( k \) large enough and, similarly,
\[
\int_{|x| \leq R} |f(u_k - u)| |\varphi| \leq |h(|u_k - u|)| \nu \left( \int_{|x| \leq R} |u_k - u|^\nu \right)^{1/\nu} \|\varphi\|_{2^*} \leq \varepsilon \|\varphi\|.
\]
Therefore,
\[
\left| \int_{\mathbb{R}^N} (f(u_k) - f(u_k - u) - f(u)) \varphi \right| \leq \varepsilon \|\varphi\| \quad \text{for } k \text{ large enough.}
\]
This proves the claim. \( \square \)

The following lemma is stated in [7] but its proof contains a gap. This can be fixed with the help of Lemma 3.5. We give the details.

**Lemma 3.9** (Splitting lemma). Let \( (u_k) \) be a bounded \((PS)_{\delta}\)-sequence for \( I_V \). Then, after replacing \((u_k)\) by a subsequence, there exists a solution \( u \) of problem \((p\nu)\), a number \( m \in \mathbb{N} \cup \{0\} \), \( m \) nontrivial solutions \( w_1, \ldots, w_m \) to the limit problem \((p0)\) and \( m \) sequences of points \((y_{j,k}) \in \mathbb{R}^N, 1 \leq j \leq m, \) satisfying
(i) \( |y_{j,k}| \to \infty, \) and \( |y_{j,k} - y_{i,k}| \to \infty \) if \( i \neq j, \)
(ii) \( u_k - \sum_{i=1}^m w_i (\cdot - y_{i,k}) \to u \) in \( D^{1,2}(\mathbb{R}^N), \)
(iii) \( d = I_V(u) + \sum_{i=1}^m I_0(w_i). \)

**Proof.** Passing to a subsequence, we have that \( u_k \rightharpoonup u \) weakly in \( D^{1,2}(\mathbb{R}^N) \). It follows from Lemma 3.8 that
\[
o(1) = I'_V(u_k)\varphi = \langle u_k, \varphi \rangle_V - \int_{\mathbb{R}^N} f(u_k)\varphi \\
= \langle u, \varphi \rangle_V - \int_{\mathbb{R}^N} f(u)\varphi + o(1) = I'_V(u)\varphi + o(1)
\]
for every \( \varphi \in C_0^\infty(\mathbb{R}^N) \). Hence, \( u \) solves \((p\nu)\). Set \( u_{1,k} := u_k - u \). Then, \( u_{1,k} \to 0 \) weakly in \( D^{1,2}(\mathbb{R}^N) \). Moreover, Lemma 3.8 implies that
\[
I_V(u_k) = I_0(u_{1,k}) + I_V(u) + o(1),
\]
\[
o(1) = I'_V(u_k) = I'_0(u_{1,k}) + o(1) \quad \text{in } \left(D^{1,2}(\mathbb{R}^N)\right)' .
\]
If \( u_{1,k} \to 0 \) strongly in \( D^{1,2}(\mathbb{R}^N) \), the proof is finished. So assume it does not. Then, as \( I'_V(u_{1,k})u_{1,k} \to 0 \), after passing to a subsequence, we have that
\[
0 < C \leq \|u_{1,k}\|_V^2 = \int_{\mathbb{R}^N} f(u_{1,k})u_{1,k} + o(1).
\]
So, by Lemma 3.5, there exist \( \delta > 0 \) and a sequence \((y_{1,k})\) in \( \mathbb{R}^N \) such that
\[
\int_{B_1(y_{1,k})} |u_{1,k}|^2 = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_{1,k}|^2 > \delta.
\]
Set \( v_k := u_{1,k}(\cdot + y_{1,k}) \). Passing to a subsequence, we have that \( v_k \to w_1 \) weakly in \( D^{1,2}(\mathbb{R}^N) \) and \( v_k \to 0 \) in \( L^{2}_{\text{loc}}(\mathbb{R}^N) \). The inequality (3.6) implies that \( w_1 \neq 0 \)
and, as \( u_{1,k} \to 0 \) weakly in \( D^{1,2}(\mathbb{R}^N) \), we conclude that \( |y_{1,k}| \to \infty \). Next, we show that \( w_1 \) is a solution to the limit problem \((\psi_0)\). Let \( \varphi \in C_c^\infty(\mathbb{R}^N) \) and set \( \varphi_k := \varphi(\cdot - y_{1,k}) \). Using Lemma 3.8 and performing a change of variable, we obtain
\[
I_0'(w_1)\varphi + o(1) = I_0'(v_k)\varphi = I_0'(u_{1,k})\varphi_k = o(1).
\]
This proves that \( w_1 \) solves the problem \((\psi_0)\). Moreover, Lemma 3.8 implies that
\[
I_0(v_k) = I_0(v_k - w_1) + I_0(w_1) + o(1),
\]
\[
o(1) = I'_V(v_k) = I'_V(v_k - w_1) + o(1) \quad \text{in} \quad (D^{1,2}(\mathbb{R}^N))'.
\]
Set \( u_{2,k} := u_{1,k} - w_1(\cdot - y_{1,k}) = u_k - u - w_1(\cdot - y_{1,k}) \). Then, \( u_{2,k} \to 0 \) weakly in \( D^{1,2}(\mathbb{R}^N) \) and, after a change of variable, from the identities (3.5) and (3.7) we obtain that
\[
I_0(u_{2,k}) = I_0(u_{1,k}) - I_0(w_1) + o(1) = I_V(u_k) - I_V(u) - I_0(w_1) + o(1),
\]
\[
I'_0(u_{2,k}) = I'_0(u_{1,k}) + o(1) = I'_V(u_k) + o(1) = o(1) \quad \text{in} \quad (D^{1,2}(\mathbb{R}^N))'.
\]
If \( u_{2,k} \to 0 \) strongly in \( D^{1,2}(\mathbb{R}^N) \), the proof is finished. If not, we repeat the argument. After a finite number of steps, we will arrive to a sequence \( (u_{m+1,k}) \) which converges strongly to 0 in \( D^{1,2}(\mathbb{R}^N) \). This finishes the proof. \( \square \)

**Corollary 3.10 (Compactness).** If \( c_V \) is not attained by \( I_V \) on \( N_V \), then the following statements hold true.

(a) \( c_V \geq c_0 \).

(b) If \( \sigma_k \in \mathbb{R} \) and \( u_k \in M_{\sigma_k} \) are such that \( \sigma_k \to 0 \), \( I_V(u_k) \to d \in (c_0, 2c_0) \) and \( \nabla M_{\sigma_k} I_V(u_k) \to 0 \), then \( (u_k) \) contains a convergent subsequence.

**Proof.** (a): Let \( (u_k) \) be a minimizing sequence for \( I_V \) on \( N_V \). By Ekeland’s variational principle and Lemma 3.6, we may assume that \( (u_k) \) is a bounded \((PS)_{c_V}\)-sequence for \( I_V \). As \( c_V \) is not attained, the splitting lemma implies that \( c_V \geq c_0 \).

(b): By Lemmas 3.6 and Lemma 3.7 we have that \( (u_k) \) is a bounded \((PS)_d\)-sequence for \( I_V \). Arguing by contradiction, assume that \( (u_k) \) does not contain a convergent subsequence. Then, the splitting lemma yields a solution \( w \) of the limit problem \((\psi_0)\) with \( d = I_0(w) \), contradicting Lemma 3.4. This proves our claim. \( \square \)

4. Existence of a positive solution

The proof of our main result requires some delicate estimates. The following lemma will help us obtain them.

**Lemma 4.1.**

(a) If \( y_0, y \in \mathbb{R}^N, y_0 \neq y, \) and \( \alpha \) and \( \beta \) are positive constants such that \( \alpha + \beta > N \), then there exists \( C_1 = C_1(\alpha, \beta, |y - y_0|) > 0 \) such that
\[
\int_{\mathbb{R}^N} \frac{dx}{(1 + |x - R y_0|)^\alpha (1 + |x - R y|)^\beta} \leq C_1 R^{-\mu}
\]
for all \( R \geq 1 \), where \( \mu := \min\{\alpha, \beta, \alpha + \beta - N\} \).

(b) If \( y_0, y \in \mathbb{R}^N \setminus \{0\}, \) and \( \kappa \) and \( \gamma \) are positive constants such that \( \kappa + 2\gamma > N \), then there exists \( C_2 = C_2(\kappa, \gamma, |y_0|, |y|) > 0 \) such that
\[
\int_{\mathbb{R}^N} \frac{dx}{(1 + |x|)^\kappa (1 + |x - R y_0|)^\gamma (1 + |x - R y|)^\gamma} \leq C_2 R^{-\tau},
\]
for all \( R \geq 1 \), where \( \tau := \min\{\kappa, 2\gamma, \kappa + 2\gamma - N\} \).
Similarly, as claimed. □

(a): After a suitable translation, we may assume that \( y = -y_0 \). Let \( 2\rho := |y - y_0| > 0 \). In the following, \( C \) will denote different positive constants which depend on \( \alpha, \beta \) and \( \rho \). If \( |x - R y_0| \leq \rho R \), then \( |x - R y| \geq \rho R \). Hence

\[
\int_{B_{\rho R}(R y_0)} \frac{dx}{(1 + |x - R y_0|)^\alpha (1 + |x - R y|)^\beta} = \int_{B_{\rho R}(R y_0)} \frac{dx}{(1 + |x - R y_0|)^\alpha (\rho R)^\beta} = C R^{-\beta} \int_{B_{\rho R}(0)} \frac{dx}{(1 + |x|)^\beta} \leq C (R^{-\beta} + R^\alpha) \leq C R^{-\mu}.
\]

Similarly,

\[
\int_{B_{\rho R}(R y_0)} \frac{dx}{(1 + |x - R y_0|)^\alpha (1 + |x - R y|)^\beta} \leq C (R^{-\alpha} + R^\alpha) \leq C R^{-\mu}.
\]

Let

\[
H^+ := \{ z \in \mathbb{R}^N : |z - R y| \geq |z - R y_0| \},
\]

\[
H^- := \{ z \in \mathbb{R}^N : |z - R y| \leq |z - R y_0| \}.
\]

Setting \( x = R z \) we obtain

\[
\int_{H^+ \setminus B_{\rho R}(R y_0)} \frac{dx}{(1 + |x - R y_0|)^\alpha (1 + |x - R y|)^\beta} \leq \int_{H^+ \setminus B_{\rho R}(R y_0)} \frac{dx}{(1 + |x - R y_0|)^\alpha + \beta} \leq \int_{H^- \setminus B_{\rho R}(y_0)} \frac{R^N dx}{|R^2 - y_0|^{\alpha + \beta}} = C R^{N-(\alpha + \beta)} \leq C R^{-\mu}.
\]

Similarly,

\[
\int_{H^- \setminus B_{\rho R}(R y)} \frac{dx}{(1 + |x - R y_0|)^\alpha (1 + |x - R y|)^\beta} \leq C R^{-\mu}.
\]

Since \( \mathbb{R}^N \setminus \{ B_{\rho R}(R y_0) \cup B_{\rho R}(R y) \} = [H^+ \setminus B_{\rho R}(R y_0)] \cup [H^- \setminus B_{\rho R}(R y)] \), the previous estimates yield (a).

(b): From Hölder’s inequality and inequality (a) we obtain

\[
\int_{\mathbb{R}^N} \frac{dx}{(1 + |x|)^\alpha (1 + |x - R y_0|)^\gamma (1 + |x - R y|)^\gamma} \leq \left( \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|)^\alpha (1 + |x - R y_0|)^{2\gamma}} \right)^{1/2} \left( \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|)^\alpha (1 + |x - R y|)^{2\gamma}} \right)^{1/2} \leq C_2 R^{-\tau},
\]

as claimed. □

Let \( \omega \) be the positive radial ground state of the limit problem \((\psi_0)\). Fix \( y_0 \in \mathbb{R}^N \) with \( |y_0| = 1 \), and let \( B_2(y_0) := \{ x \in \mathbb{R}^N : |x - y_0| \leq 2 \} \). For \( R \geq 1 \) and each \( y \in \partial B_2(y_0) \), we define

\[
\omega^R_{0,y} := \omega(-R y_0), \quad \omega^R_{y,y} := \omega(-R y),
\]

and we set

\[
\varepsilon_R := \int_{\mathbb{R}^N} f(\omega^R_{0,y}) \omega^R_{y,y} = \int_{\mathbb{R}^N} f(\omega(x - R y_0)) \omega(x - R y) \, dx.
\]

As before, \( C \) will denote a positive constant, not necessarily the same one.
Lemma 4.2. There exists a constant $C_3 > 0$ such that
\[ \varepsilon_R \leq C_3 R^{-(N-2)} \]
for all $y \in \partial B_2(y_0)$ and all $R \geq 1$.

Proof. By assumption $(f1)$, we have that $|f(s)| \leq A_1 |s|^{2^*-1}$. On the other hand, from the estimates (2.1) and Lemma 4.1(a) we obtain
\[ \int_{R_n} (\omega_0^R)^{2^*-1} \omega_y^R \leq C \int_{R_n} (1 + |x - R_{y_0}|)^{N+2}(1 + |x - R_y|)^{N-2} \]
\[ \leq C R^{-(N-2)}. \]
Therefore,
\[ \int_{R_n} f(\omega_0^R) \omega_y^R \leq A_1 \int_{R_n} (\omega_0^R)^{2^*-1} \omega_y^R \leq C R^{-(N-2)} \]
for all $y \in \partial B_2(y_0)$ and all $R \geq 1$, as claimed. \hfill \Box

Lemma 4.3. There exists a constant $C_4 > 0$ such that
\[ \int_{R_n} f(s\omega_0^R) t \omega_y^R \geq C_4 R^{-(N-2)} \]
for all $s, t \geq \frac{1}{2}, y \in \partial B_2(y_0)$ and $R \geq 1$.

Proof. Note that, if $|x| < 1$, then, for every $y \in \partial B_2(y_0)$ and $R \geq 1$,
\[ 1 + |x - R(y - y_0)| < 1 + |x| + R |y - y_0| < 4R. \]
Assumption $(f_3)$ implies that $\frac{f(s)}{s}$ is strictly increasing for $s > 0$. Hence, so is $f$. Performing a change of variable and using the estimate (2.1) we obtain
\[ \int_{R_n} f(s\omega_0^R) t \omega_y^R \geq t \int_{R_n} f \left( \frac{1}{2} \omega_0^R \right) \omega_y^R \geq \frac{1}{2} \int_{B_1(R_{y_0})} f \left( \frac{1}{2} \omega_0^R \right) \omega_y^R \]
\[ \geq \frac{1}{4} \min_{x \in B_1(0)} f \left( \frac{1}{2} \omega(x) \right) \int_{B_1(0)} \omega(x - R(y - y_0)) dx \]
\[ \geq C \int_{B_1(0)} (1 + |x - R(y - y_0)|)^{-(N-2)} dx \leq CR^{-(N-2)} \]
for all $s, t \geq \frac{1}{2}, y \in \partial B_2(y_0)$ and $R \geq 1$, as claimed. \hfill \Box

Note that Lemmas 4.2 and 4.3 yield
\[ C_4 R^{-(N-2)} \leq \varepsilon_R := \int_{R_n} f(\omega_0^R) \omega_y^R \leq C_3 R^{-(N-2)} \]
for all $y \in \partial B_2(y_0)$ and $R \geq 1$.

Lemma 4.4. For each $b > 1$ there is a constant $C_b > 0$, such that
\[ \left| \int_{R_n} (sf(\omega_0^R) - f(s\omega_0^R)) \omega_y^R \right| \leq C_b |s - 1| \varepsilon_R \]
for all $s \in [0, b], y \in \partial B_2(y_0)$ and $R \geq 1$. 
Proof. Fix \( t \in \mathbb{R} \) and set \( g(s) := sf(t) - f(st) \). By the mean value theorem, there exists \( \zeta \) between 1 and \( s \) such that

\[
|sf(t) - f(st)| = |g(s) - g(1)| = |f(t) - f'(\zeta)t||s - 1|
\]

\[
\leq (|f(t)| + |f'(\zeta)t|)|s - 1|
\]

\[
\leq A_1 \left( |t|^{2^{*} - 1} + |\zeta|^{2^{*} - 2} |t|^{2^{*} - 1} \right) |s - 1|
\]

\[
\leq A_1 (1 + B^{2^{*} - 2}) |t|^{2^{*} - 1} |s - 1| \quad \forall s \in [0, b],
\]

where the second-to-last inequality follows from assumption (f1). So, from the inequalities (4.1) and (4.2) we obtain that

\[
\int_{\mathbb{R}^N} (sf(\omega_0^R) - f(s\omega_0^R)) \omega_y^R \leq C |s - 1| \int_{\mathbb{R}^N} (\omega_0^R)^{2^{*} - 1} \omega_y^R
\]

\[
\leq C |s - 1| R^{-(N-2)} \leq C |s - 1| \varepsilon_R
\]

for all \( s \in [0, b] \), \( y \in \partial B_2(0) \) and \( R \geq 1 \), as claimed.

**Lemma 4.5.** There exists \( \tau > N - 2 \) such that

\[
\int_{\mathbb{R}^N} V^+ (\omega_0^R + \omega_y^R)^2 \leq CR^{-\tau}
\]

for every \( y \in \partial B_2(y_0) \) and \( R \geq 1 \).

**Proof.** From assumption (V2), the estimates (2.1) and Lemma 4.1(b), we immediately obtain that

\[
\int_{\mathbb{R}^N} V^+ (\omega_0^R + \omega_y^R)^2 = \int_{\mathbb{R}^N} V^+ (\omega_0^R)^2 + 2 \int_{\mathbb{R}^N} V^+ \omega_0^R \omega_y^R + \int_{\mathbb{R}^N} V^+ (\omega_y^R)^2
\]

\[
\leq CR^{-\tau},
\]

with \( \tau := \min\{\kappa, 2(N - 2), \kappa + N - 4\} > N - 2 \).

For each \( R \geq 1 \), \( y \in \partial B_2(y_0) \) and \( \lambda \in [0, 1] \), we define

\[
\omega_{\lambda,y}^R := \lambda \omega_0^R + (1 - \lambda)\omega_y^R.
\]

**Lemma 4.6.** For each \( R \geq 1 \), \( y \in \partial B_2(y_0) \) and \( \lambda \in [0, 1] \), there exists a unique \( T_{\lambda,y}^R > 0 \) such that

\[
T_{\lambda,y}^R \omega_{\lambda,y}^R \in \mathcal{N}_V.
\]

Moreover, there exist \( R_0 \geq 1 \) and \( T_0 > 2 \) such that \( T_{\lambda,y}^R \in (0, T_0) \) for all \( R \geq R_0 \), \( y \in \partial B_2(y_0) \) and \( \lambda \in [0, 1] \), and \( T_{\lambda,y}^R \) is a continuous function of the variables \( \lambda, y \) and \( R \).

**Proof.** The proof is the same as that of Lemma 3.2 in [13], with the obvious changes.

**Lemma 4.7.** For \( \lambda = \frac{1}{2} \) we have that \( T_{\lambda,y}^R \to 2 \) as \( R \to \infty \) uniformly in \( y \in \partial B_2(y_0) \).
Hence, there exists $s, t > 0$ such that $f$ is increasing for $0 < s < t$. The inequality is clearly satisfied if $s = t > 0$ as $R \to \infty$, uniformly in $y \in \partial B_1(\eta_0)$. Hence, for $\lambda = \frac{\nu}{2}$, we get that

$$J_V(2 z_{\lambda, y}) = J_V(\omega_0^R + \omega_1^R) = J_0(\omega_0^R + \omega_1^R) + \int_{\mathbb{R}^N} V(\omega_0^R + \omega_1^R) \, \omega_y^R = o_R(1).$$

This yields the claim. \hfill $\Box$

The proof of the next result follows that of Lemma 2.1 in [15].

**Lemma 4.8.** For each $a > 0$ there exists $C_a \geq 0$ such that

$$F(s + t) - F(s) - F(t) - f(s)t - f(t)s \geq -C_a (st)^{1+\frac{\nu}{2}}$$

for all $s, t \in [0, a]$ and $\nu \in (0, q - 2)$.

**Proof.** The inequality is clearly satisfied if $s = 0$ or $t = 0$. Assumption $(f_3)$ implies that $f$ is increasing for $s > 0$. Therefore,

$$F(s + t) - F(s) = \int_s^{s+t} f(\zeta) \, d\zeta \geq f(s) t$$

for all $s, t > 0$. Moreover, by $(f_2)$, we have that $f(s) = o(|s|^{1+\nu})$ for any $\nu \in (0, q-2)$. Hence, there exists $M_a > 0$ such that $f(s) \leq M_a s^{1+\nu}$ for all $s \in [0, a]$. It follows that

$$F(s + t) - F(s) - F(t) - f(s)t - f(t)s \geq -F(t) - f(t)s \geq -M_a \int_0^t \zeta^{1+\nu} \, d\zeta - M_a st^{1+\nu} = -M_a \left( \frac{t^{2+\nu}}{2+\nu} + st^{1+\nu} \right)$$
for all \( s, t \in [0, a] \). So, if \( t \leq s \), we get that
\[
F(s + t) - F(s) - F(t) - f(s)t - f(t)s \geq -\frac{3}{2}M_{a}(st)^{1+\frac{\nu}{2}}.
\]
As this expression is symmetric in \( s \) and \( t \), it holds true also when \( s \leq t \), and the proof is complete. \( \square \)

With the previous lemmas on hand, we now prove the following estimate.

**Proposition 4.9.** There exists \( R_1 \geq 1 \) and, for each \( R > R_1 \), a number \( \eta_R > 0 \) such that
\[
I_V(T_{\lambda, y}^{R, R}) \leq 2c_0 - \eta_R
\]
for all \( \lambda \in [0, 1] \) and all \( y \in \partial B_2(y_0) \).

**Proof.** To simplify the notation, let us set
\[
s := T_{\lambda, y}^{R, R}, \quad t := T_{\lambda, y}(1 - \lambda), \quad \omega_0 := \omega_0^R, \quad \omega_y := \omega_y^R.
\]
Then \( s, t \in [0, T_0] \) if \( R \geq R_0 \), with \( R_0 \geq 1 \) and \( T_0 > 2 \) as in Lemma 4.6, and
\[
I_V(T_{\lambda, y}^{R, R}) = I_V(s\omega_0 + t\omega_y)
\]
\[
= \frac{s^2}{2} \int_{\mathbb{R}^N} |\nabla \omega_0|^2 + \frac{s^2}{2} \int_{\mathbb{R}^N} V\omega_0^2 + \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \omega_y|^2 + \frac{t^2}{2} \int_{\mathbb{R}^N} V\omega_y^2
\]
\[
+ st \int_{\mathbb{R}^N} \nabla \omega_0 \cdot \nabla \omega_y + st \int_{\mathbb{R}^N} V\omega_0\omega_y - \int_{\mathbb{R}^N} F(s\omega_0 + t\omega_y)
\]
(4.3)
\[
= \frac{s^2}{2} \int_{\mathbb{R}^N} |\nabla \omega_0|^2 - \int_{\mathbb{R}^N} F(s\omega_0) + \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \omega_y|^2 - \int_{\mathbb{R}^N} F(t\omega_y)
\]
(4.4)
\[
+ st \int_{\mathbb{R}^N} \nabla \omega_0 \cdot \nabla \omega_y
\]
(4.5)
\[
- \int_{\mathbb{R}^N} [F(s\omega_0 + t\omega_y) - F(s\omega_0) - F(t\omega_y) - f(s\omega_0)t\omega_y - f(t\omega_y)s\omega_0]
\]
(4.6)
\[
- \int_{\mathbb{R}^N} f(s\omega_0)t\omega_y - \int_{\mathbb{R}^N} f(t\omega_y)s\omega_0
\]
(4.7)
Next, we estimate each of the numbered lines. As \( \omega_0 \) and \( \omega_y \) are ground states of the limit problem \( \langle \omega_0 \rangle \), Lemma 3.2(d) yields
\[
(4.3) = I_0(s\omega_0) + I_0(t\omega_y) \leq I_0(\omega_0) + I_0(\omega_y) = 2c_0.
\]
From Lemma 4.5 and estimates (4.2) we get that
\[
(4.4) \leq CR^{-\tau} = o(\varepsilon_R).
\]
Lemma 4.8 with \( \nu \in \left( \frac{2}{N-2}, q - 2 \right) \) and Lemma 4.1(a) with \( \alpha = \beta = (1 + \frac{\nu}{2})(N - 2) \), imply that, for some \( \mu > N - 2 \),
\[
(4.6) = - \int_{\mathbb{R}^N} [F(s\omega_0 + t\omega_y) - F(s\omega_0) - F(t\omega_y) - f(s\omega_0)t\omega_y - f(t\omega_y)s\omega_0]
\]
\[
\leq C |st|^{1+\frac{\nu}{2}} \int_{\mathbb{R}^N} (\omega_0\omega_y)^{1+\frac{\nu}{2}} \leq CR^{-\mu} = o(\varepsilon_R).
\]
We write the sum of the remaining terms as
\[
\frac{t}{2} \int_{\mathbb{R}^N} |s f(\omega_0) - f(s\omega_0)|\omega_y + \frac{s}{2} \int_{\mathbb{R}^N} |t f(\omega_y) - f(t\omega_y)|\omega_0
\]
\[
- \frac{1}{2} \int_{\mathbb{R}^N} f(s\omega_0) t\omega_y - \frac{1}{2} \int_{\mathbb{R}^N} f(t\omega_y) s\omega_0.
\]
By Lemma 4.4 there exists a constant \(C > 0\) such that
\[
\frac{t}{2} \int_{\mathbb{R}^N} |s f(\omega_0) - f(s\omega_0)|\omega_y + \frac{s}{2} \int_{\mathbb{R}^N} |t f(\omega_y) - f(t\omega_y)|\omega_0 \leq C(|s - 1| + |t - 1|) \varepsilon_R,
\]
for all \(s, t \in [0, T_0]\), \(y \in \partial B_2(y_0)\) and \(R \geq R_0\). Moreover, Lemma 4.3, yields a constant \(C_0 > 0\) such that
\[
\frac{1}{2} \int_{\mathbb{R}^N} f(s\omega_0) t\omega_y + \frac{1}{2} \int_{\mathbb{R}^N} f(t\omega_y) s\omega_0 \geq C_0 \varepsilon_R
\]
for all \(s, t \geq \frac{1}{2}, y \in \partial B_2(y_0)\) and \(R \geq R_0\). By Lemma 4.7, if \(\lambda = \frac{1}{2}\), then \(s, t \to 1\) as \(R \to \infty\). Therefore, there exist \(R_1 \geq R_0\) and \(\delta \in (0, \frac{1}{2})\) such that
\[
(4.5) + (4.7) \leq - \frac{C_0}{2} \varepsilon_R
\]
for all \(\lambda \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta], y \in \partial B_2(y_0)\) and \(R \geq R_1\). Summing up, we have shown that
\[
(4.8) \quad I_V(s\omega_0 + t\omega_y) \leq 2c_0 - \frac{C_0}{2} \varepsilon_R + o(\varepsilon_R)
\]
for all \(\lambda \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta], y \in \partial B_2(y_0), R \geq R_1\).

On the other hand, by Lemma 3.2(d), there exists \(\gamma \in (0, c_0)\) such that
\[
(4.3) = I_0(s\omega_0) + I_0(t\omega_y) \leq 2c_0 - \gamma
\]
for all \(\lambda \in [0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1], y \in \partial B_2(y_0)\) and \(R\) sufficiently large. Since
\[
(4.4) + \cdots + (4.7) \leq O(\varepsilon_R),
\]
we conclude that
\[
(4.9) \quad I_V(s\omega_0 + t\omega_y) \leq 2c_0 - 2\gamma + O(\varepsilon_R)
\]
for all \(\lambda \in [0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1], y \in \partial B_2(y_0)\) and \(R\) sufficiently large.

Inequalities (4.8) and (4.9), together, yield the statement of the proposition. \(\square\)

**Lemma 4.10.** For any \(\delta > 0\), there exists \(R_2 > 0\) such that
\[
I_V(T^{R}_{\lambda, y} \cdot z^R_{\lambda, y}) < c_0 + \delta
\]
for \(\lambda = 1\) and every \(y \in \partial B_2(y_0)\) and \(R \geq R_2\). In particular, \(c_V \leq c_0\).

**Proof.** By Lemma 4.6, \(T^{R}_{\lambda, y}\) is bounded uniformly in \(\lambda, y\) and \(R\). So, from Lemmas 3.2(d) and 4.5, we obtain that
\[
I_V(T^{R}_{1, y} \cdot z^R_{1, y}) = I_0(T^{R}_{1, y} \omega^R_{1, y}) + (T^{R}_{1, y})^2 \int_{\mathbb{R}^N} V(\omega^R_{y})^2 \leq c_0 + o_R(1),
\]
where \(o_R(1) \to 0\) as \(R \to \infty\), uniformly in \(y \in \partial B_2(y_0)\), and the claim is proved. \(\square\)

For \(c \in \mathbb{R}\), set
\[
I^c_V := \{u \in D^{1,2}(\mathbb{R}^N) : I_V(u) \leq c\}.
\]
Lemma 4.11 (Deformation). If $c_V$ is not attained by $I_V$ on $\mathcal{N}_V$, then $c_V = c_0$. If, moreover, $I_V$ does not have a critical value in $(c_0, 2c_0)$ then, for any given $\delta, \eta \in (0, \frac{c_0}{4})$, there exists a continuous function

$$\pi : \mathcal{N}_V \cap I_V^{2c_0-\eta} \to \mathcal{N}_V \cap I_V^{c_0+\delta}$$

such that $\pi(u) = u$ for all $u \in \mathcal{N}_V \cap I_V^{c_0+\delta}$.

Proof. If $c_V$ is not attained, Corollary 3.10(a) and Lemma 4.10 imply that $c_V = c_0$.

Recall that $\mathcal{N}_V$ is a $C^1$-manifold. From Lemma 3.6 and Corollary 3.10(b) we have that the following statement is true: If $\sigma_k \in \mathbb{R}$ and $u_k \in \mathcal{M}_{\sigma_k}$ are such that $\sigma_k \to 0$, $I_V(u_k) \to d \in (c_0, 2c_0)$ and, either $\nabla_{\mathcal{M}_{\sigma_k}} I_V(u_k) \to 0$ or $\nabla J_V(u_k) \to 0$, then $(u_k)$ contains a convergent subsequence. (In fact, Lemma 3.6 says that $(\nabla J_V(u_k))$ must be bounded away from 0.) This allows us to apply Theorem 2.5 in [11] to conclude that there exists $\hat{\varepsilon} > 0$ such that, for each $\epsilon \in (0, \hat{\varepsilon})$, there exists a homeomorphism $\phi : \mathcal{N}_V \to \mathcal{N}_V$ such that

1. $\phi(u) = u$ if $I_V(u) \notin [d - \epsilon, d + \epsilon]$,
2. $I_V(\phi(u)) \leq I_V(u)$ for all $u \in \mathcal{N}_V$,
3. $I_V(\phi(u)) \leq d - \epsilon$ for all $u \in \mathcal{N}_V$ with $I_V(u) \leq d + \epsilon$.

Our claim follows easily from this fact. □

Let $b : L^2(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ be a barycenter map, i.e., a continuous map such that

$$b(u(-y)) = b(u) + y \quad \text{and} \quad b(u \circ \Theta^{-1}) = \Theta(b(u))$$

for all $u \in L^2(\mathbb{R}^N) \setminus \{0\}$ and $y \in \mathbb{R}^N$, and every linear isometry $\Theta$ of $\mathbb{R}^N$; see [5, 12]. Note that $b(u) = 0$ if $u$ is radial.

Lemma 4.12. If $c_V$ is not attained by $I_V$ on $\mathcal{N}_V$, then there exists $\delta > 0$ such that

$$b(u) \neq 0 \quad \forall u \in \mathcal{N}_V \cap I_V^{c_0+\delta}.$$

Proof. The proof is the same as that of Lemma 3.11 in [13]. □

Proof of Theorem 1.1. If $c_V$ is attained by $I_V$ at some $u \in \mathcal{N}_V$, then $u$ is a nontrivial solution of problem $(\psi_V)$. So assume that $c_V$ is not attained. Then, by Lemma 4.11, $c_V = c_0$. We will show that $I_V$ has a critical value in $(c_0, 2c_0)$.

Lemma 4.12 allows us to choose $\delta \in (0, \frac{c_0}{4})$ such that

$$b(u) \neq 0 \quad \forall u \in \mathcal{N}_V \cap I_V^{c_0+\delta}$$

and, by Proposition 4.9 and Lemma 4.10, we may choose $R \geq 1$ and $\eta \in (0, \frac{c_0}{4})$ such that

$$I_V(T_{\lambda, y}^R z_{\lambda, y}^R) \leq \begin{cases} 2c_0 - \eta & \text{for all } \lambda \in [0, 1] \text{ and all } y \in \partial B_2(y_0), \\ c_0 + \delta & \text{for } \lambda = 1 \text{ and all } y \in \partial B_2(y_0). \end{cases}$$

Define $\iota : B_2(y_0) \to \mathcal{N}_V \cap I_V^{2c_0-\eta}$ by

$$\iota((1 - \lambda)y_0 + \lambda y) := T_{\lambda, y}^R z_{\lambda, y}^R, \quad \text{with } \lambda \in [0, 1], \ y \in \partial B_2(y_0).$$

Arguing by contradiction, assume that $I_V$ does not have a critical value in $(c_0, 2c_0)$. Then, by Lemma 4.11, there exists a continuous function

$$\pi : \mathcal{N}_V \cap I_V^{2c_0-\eta} \to \mathcal{N}_V \cap I_V^{c_0+\delta}.$$
such that $\pi(u) = u$ for all $u \in \mathcal{N}_V \cap I_{V}^{\delta}$. The function $\psi : B_{2}(y_{0}) \to \partial B_{2}(y_{0})$ given by

$$\psi(x) := 2 \frac{(b \circ \pi \circ \iota)(x)}{|(b \circ \pi \circ \iota)(x)|}$$

is well defined and continuous, and $\psi(y) = y$ for every $y \in \partial B_{2}(y_{0})$. This is a contradiction. Therefore, $I_V$ must have a critical point $u \in \mathcal{N}_V$ with $I_V(u) \in (c_0, 2c_0)$.

By Lemma 3.3, $u$ does not change sign and, since $f$ is odd, $-u$ is also a solution of $(\varphi_V)$. This proves that problem $(\varphi_V)$ has a positive solution. $\square$

REFERENCES

[1] Azzollini, Antonio; Benci, Vieri; D’Aprile, Teresa; Fortunato, Donato: Existence of static solutions of the semilinear Maxwell equations. Ric. Mat. 55 (2006), no. 2, 283–297.

[2] Azzollini, A.; Pomponio, A.: Compactness results and applications to some ”zero mass” elliptic problems. Nonlinear Anal. 69 (2008), no. 10, 3559–3576.

[3] Badiale, Marino; Pisani, Lorenzo; Rolando, Sergio: Sum of weighted Lebesgue spaces and nonlinear elliptic equations. NoDEA Nonlinear Differential Equations Appl. 18 (2011), no. 4, 369–405.

[4] Badiale, Marino; Rolando, Sergio: Elliptic problems with singular potential and double-power nonlinearity. Mediterr. J. Math. 2 (2005), no. 4, 417–436.

[5] Bartsch, Thomas; Weth, Tobias: Three nodal solutions of singularly perturbed elliptic equations on domains without topology. Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), no. 3, 259–281.

[6] Cerami, Giovanna; Passaseo, Donato: The effect of concentrating potentials in some singularly perturbed problems. Calc. Var. Partial Differential Equations 17 (2003), no. 3, 257–281.

[7] Clapp, Mónica; Maia, Liliane A.: A positive bound state for an asymptotically linear or superlinear Schrödinger equation. J. Differential Equations 260 (2016), no. 4, 3173–3192.

[8] Erbe, Lynn; Tang, Moxun: Structure of positive radial solutions of semilinear elliptic equations. J. Differential Equations 133 (1997), no. 2, 179–202.

[9] Vétois, Jérôme: A priori estimates and application to the symmetry of solutions for critical p-Laplace equations. J. Differential Equations 260 (2016), no. 1, 149–161.
[19] Willem, Michel: Minimax theorems. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, 04510 Coyoacán, CDMX, Mexico.
E-mail address: monica.clapp@im.unam.mx

Departamento de Matemática, UNB, 70910-900 Brasília, Brazil.
E-mail address: lilimaia@unb.br