Hidden $Sp(2s + 1)$- or $SO(2s + 1)$-symmetry and new exactly solvable models in ultracold atomic systems

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Abstract

The high-spin ultracold atomic models with a special form of contact interactions, i.e. the scattering lengths in the total spin-2, 4, \ldots channels are equal, but may be different from that in the spin-0 channel, are studied. Those models have either $U(1) \otimes Sp(2s + 1)$-symmetry for the fermions or $U(1) \otimes SO(2s + 1)$-symmetry for the bosons, and the generators are found to be magnetic multipole operators. Based on the symmetry analysis, a class of exactly solvable models is proposed and solved via the Bethe ansatz. The ground states and excitations for repulsive fermions are also discussed.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Recently, the study of cold atoms with high spin has aroused much attention in the fields of atomic, molecular, optical and condensed matter physics. Due to the spin exchange interactions, many interesting spin ordered states arise and the phase diagrams of these systems are very rich. For instance, in the spin-1 spinor Bose–Einstein condensations, the bosons are found to form pairs and the pairs condense even in the repulsive regime [1–3]. In experiments, by using atom cooling and trapping techniques, one can prepare the high-spin cold atomic systems, such as $^7$Li, $^{23}$Na, $^{87}$Rb with hyperfine spin 1 [4–10]; $^{53}$Cr with hyperfine spin 3/2 [11] and $^{40}$K, $^{173}$Yb, $^{43}$Ca, $^{87}$Sr, $^{133}$Cs with higher ones [12–17]. Using Feshbach resonance [18, 19] and confinement induced resonance [20] techniques, the interactions among the atoms can be manipulated. In theoretical approaches, the low-energy effective models of the dilute ultracold atomic systems are the quantum gas with contact interactions, and the spin exchanging interactions should also be considered for systems with internal degrees of freedom [21, 22].
Symmetry analysis plays a very important role in studying quantum many-body systems. Physical properties such as the ground state manifold and order parameters are closely related to the symmetry of a system [23–25]. The analysis of the symmetry can give some hints for suitable approximation and to study the physics such as phase diagrams in the framework of mean-field theory. In cold atomic systems with delta function interactions, the symmetric or anti-symmetric properties of the identical particles restrict the forms of spin exchange interactions. Effective spin exchanging interactions only take place in the channels with a symmetric spatial wavefunction. Such properties may make the systems have intrinsic symmetries in the spin sector. For example, in the spin-3/2 system, the $SO(5)$ symmetry is found [26].

Strong quantum fluctuations and correlations make the physics of a one-dimensional (1D) system quite different from the ones of higher dimensions. Exact solution is a good starting point to study these systems, since it can give conclusive results. The 1D atomic $SU(2s + 1)$ symmetric quantum gases, with $s$ the spin of the particles, are integrable either for Bose–Fermi mixtures with equal masses of each species or for pure fermionic or bosonic gases [27–37]. In these models, the spin exchange interactions are not considered. However, they usually cannot be neglected in experiments, and many novel ordered states are induced by the spin exchange. Motivated by this consideration, we proposed a $SO(3)$ integrable spin-1 bosonic model [38] and a $Sp(4)$ integrable spin-3/2 fermionic model [39], where the spin exchange interactions are considered.

To find 1D delta function interacting cold atomic integrable models with spin exchange interactions, spin chain models are important reference points. It is easy to find the corresponding integrable spin chain from an integrable cold atomic model with delta function. However, it is very hard to find integrable atomic models from integrable spin chains, e.g. the Takhtajan–Babujian model with spin larger than 1.

The $Sp(2s + 1)$ and $SO(2s + 1)$-symmetric models are studied widely in the discussion of pairing problems and the long-range interaction integrable models [40–45], and spin chain integrable models with such symmetry and the related algebraic methods are discussed in [46, 47]. Kennedy and Batchelor give the integrable $Sp(2s + 1)$ and $SO(2s + 1)$ models with spin exchange interactions in forms of projection operators [48, 49], and these models have corresponding integrable cold atomic gas models, which are discussed here.

In this paper, we discuss the symmetry and integrability of some diluted cold atomic models with contact spin exchange interaction. For a special interaction form of atoms with hyperfine spin $s$, the fermionic system with half-odd spin is found to have $U(1) \otimes Sp(2s + 1)$ symmetry while the bosonic system with integer spin is found to have $U(1) \otimes SO(2s + 1)$ symmetry. The generators of the corresponding algebra are constructed by the magnetic multipole operators. Based on the symmetry analysis, we propose a new class of exactly solvable atomic models in one dimension.

2. Model and symmetry

For the delta function interaction models of dilute cold atomic gases with hyperfine spin $s$, the spin exchange interaction between two particles $i$ and $j$ is usually written as spin projection operators $\hat{P}_{ij}^{l}$ in different channels with total spin-$l$ ($l = 0, 1, 2, \ldots, 2s$). Nontrivial scattering processes occur only in the even $l$ channels because of the symmetry or anti-symmetry behavior of the wavefunctions. To study the behavior of such systems away from
the $SU(2s+1)$ symmetry point, we consider a simple case, i.e. all the scattering lengths of nonzero $l$ channels are the same. The Hamiltonian reads
\begin{equation}
\hat{H} = -\sum_{i=1}^{N} \nabla_{r_{ij}}^{2} + \sum_{ij \neq j} \left[ c_{1} \hat{P}_{ij}^{0} + c_{2} \sum_{l=2,4,\ldots} \hat{P}_{ij}^{l} \right] \vec{\delta}(r_{i} - r_{j}).
\end{equation}

Here, $N$ is the number of atoms, $r_{i}$ is the position of the $i$th atom, $c_{1}$ is the interaction strength in the spin-0 channel and $c_{2}$ is the one in the other channels.

The two-body scattering in the system (1) is quite interesting. There are two kinds of scattering processes in the spin sector. One is $|s, m; s, m'\rangle\langle s, m' ; s, m| + h.c.$ provided by two-particle permutation as shown in figure 1(a). Here, $m$ and $m'$ are the spins along the $z$ direction, and $m, m' = s, s - 1, \ldots, -s$. In this process, the particle numbers $\hat{N}_{m}$ with different $m$ are invariant. As a consequence, the total spin $\hat{S}$ and total particle number $\hat{N}$ are also conserved. The other scattering process is $|s, m; s, -m\rangle\langle s, -m' ; s, -m'| + h.c.$ provided by the projector operator $\hat{P}_{ij}^{0}$ as shown in figure 1(b). In this process, two particles with opposite spins scatter into another pair, and the absolute value $|m'|$ can be unequal to $|m|$. Obviously, this process does not affect the total particle number $\hat{N}$, but it destroys the invariability of $\hat{N}_{m}$. Therefore, the particle number $\hat{N}_{m}$ is no longer conserved. Nevertheless, after careful consideration, we find that $\hat{J}_{m} = \hat{N}_{m} - \hat{N}_{-m}$ with $m = s, s-1, \ldots$ and $m \geq 0$ are still invariant. The invariance of $\hat{J}_{m}$ means that the total spin $\hat{S} = \sum_{m} m \hat{J}_{m}$ is still a good quantum number. Besides, some of the magnetic multipole operators are also invariant. The multipole operators are observable physical quantities and can be defined in the form of irreducible tensors:
\begin{equation}
\hat{T}_{m}^{l} = \sqrt{(l+m)!(l-m)!/(2l)!} \left( \prod_{j=1}^{l-1} \sum_{m_{j}} \hat{s}_{m_{j}} \right) \hat{s}_{m-l},
\end{equation}
\begin{equation}
m = -l, -l+1, \ldots, l, l = 1, 2, \ldots, 2s.
\end{equation}

Here, $\hat{s}_{-1} = (\hat{s}_{x} - i\hat{s}_{y})/\sqrt{2}$, $\hat{s}_{0} = \sqrt{2}\hat{s}_{z}$, $\hat{s}_{1} = -(\hat{s}_{x} + i\hat{s}_{y})/\sqrt{2}$, $\hat{s}_{\alpha} (\alpha = x, y, z)$ are the spin operators of one particle, $m_{j} = -1, 0, 1$, and the sum $\Sigma'$ means $|m - \sum_{j} m_{j}| < l - j$ for any $j$. The total multipole operators for $N$ particles are $\hat{M}_{m}^{l} = \sum_{j=1}^{N} \hat{T}_{m}^{l}$, where $\hat{M}_{m}^{l}$ is the $l$-rank multipole operator of the $i$th particle. It can be proved that the multipole operators with odd rank are commutative with the Hamiltonian
\begin{equation}
\left[ \hat{V}, \hat{M}_{m}^{l} \right] = 0,
\end{equation}
where $\hat{V} = \sum_{i \neq j} \left( c_{1} \hat{P}_{ij}^{0} + c_{2} \sum_{l=2,4,\ldots} \hat{P}_{ij}^{l} \right)$, and thus are the conserved quantities of the system (1). This can be understood from the two-body scattering processes. If we only consider...
the process of permutation, all multipole operators are commutative with the spin part of the Hamiltonian, since the exchanges of spins have no effect on the magnetic properties. However, the scattering processes of $\hat{P}_{ab}^0$ make the magnetic quadrupole change.

The odd rank multipole operators can be used to construct the generators $Y$ of $Sp(2s + 1)$ (half odd $s$) and $SO(s, s + 1)$ (integer $s$) algebras

$$\hat{Y}_m = i\hat{T}_m^I$$ (odd $I$). (4)

For the odd rank-$l$ multipole operators are commutative with the spin part of Hamiltonian (3), the corresponding symmetries hold for the system (1). The algebras $so(s, s + 1)$ and $so(2s + 1)$ have the same complex extensions $so(2s + 1, \mathbb{C})$, so that if a model possesses $SO(s, s + 1)$ symmetry, it must have $SO(2s + 1)$ symmetry. There is also a $U(1)$ symmetry for the coordinate part; then, the system (1) has $U(1) \otimes Sp(2s + 1)$ symmetry for the fermionic case and $U(1) \otimes SO(2s + 1)$ symmetry for the bosonic case.

There are three homomorphisms $Sp(2) \simeq SU(2)$, $SO(3) \simeq SU(2)$ and $Sp(4) \simeq SO(5)$. For the case $s = 1/2$, only one channel $\hat{P}_{ab}^0$ is involved; the model in the spin sector has $SU(2)$ ($Sp(2)$) symmetry. For the cases $s = 1$ and $3/2$, two channels $\hat{P}_{ab}^0$ and $\hat{P}_{ab}^2$ are involved. When $s = 1$, the model has $SU(2)$ ($SO(3)$) symmetry, and when $s = 3/2$ the system has $SO(5)$ ($Sp(4)$) symmetry, which are consistent with the results obtained in [26].

When $c_1 = c_2$, the symmetry of the system (1) in the spin sector degenerates into the $SU(2s + 1)$ one, where all the interaction strengths in different channels are the same. The interaction of the spin part is the spin permutation operator up to a constant. The permutation operator acting on the symmetric wavefunctions gives eigenvalue 1, and the one acting on the anti-symmetric wavefunctions gives $-1$. Thus, the effective interaction is just the contact interaction and all magnetic multipoles are conserved. This can also be explained from the view that only the permutation operators are involved in the scattering process. In the form of multipole operators, the $(2s + 1)^2 - 1$ generators of the $SU(2s + 1)$ group read $\hat{Y}_{jm} = i^{m}(\hat{T}_{m} + \hat{T}_{-m})/2$ and $\hat{\psi}_{jm} = i^{m+1}(\hat{T}_{m} - \hat{T}_{-m})/2$.

3. Exactly solvable models

In one dimension, it is well known that at the $SU(2s + 1)$ symmetry point, the model is integrable. As we showed in the spin-1 [38] and spin-3/2 [39] cases, there is indeed another integrable point. For the $Sp(2s + 1)$ or $SO(2s + 1)$-invariant Hamiltonian (1), we construct the following exactly solvable model by constricting the parameters $c_1$ and $c_2$:

$$\hat{H}_{int} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{N} \hat{V}_{ij} \delta(x_i - x_j),$$ (5)

$$\hat{V}_{ij} = (-1)^{2s+1} \left[ s + \frac{1}{2} - (-1)^{2s} \right] c \hat{P}_{ij}^0 + c \sum_{l=2, 4, 6, \ldots} \hat{P}_{ij}^l.$$ (6)

With the standard coordinate Bethe ansatz method, the wavefunction of the system (5) is assumed as

$$\Psi_E = \sum_{Q, \mathcal{P}} \Theta(Q) A_{m_1, \ldots, m_N} (Q, \mathcal{P}) e^{i \sum_{k_1, \ldots, k_N} k_{Q_{i, k}} x_{Q_{i, k}}},$$ (7)

Here, $m_i$ is the spin component along the $z$-direction of the $i$th particle, $m_i = s, s-1, \ldots, -s$ and $k_i (i = 1, 2, \ldots, N)$ are the quasi-momenta carried by the particles. $Q$ and $\mathcal{P}$ are all $N!$ permutations of $\{1, 2, \ldots, N\}$, and $Q_i (\mathcal{P})$ is the $i$th number of the permutation $Q (\mathcal{P})$. $\Theta(Q) = \prod_{i=1}^{N} \Theta(x_{Q_{i}} - x_{Q_{i+1}})$ is a continuous multiplication of the step function $\theta(x)$. When
x \geq 0, \theta(x) = 1, \text{ and otherwise } \theta(x) = 0. \text{ Thus, } the \text{ function } \Theta \text{ divides the coordinate space into } N! \text{ intervals.}

The two-particle scattering occurs at the interface of two adjacent coordinate intervals \( Q' = \{Q_1, Q_2, \ldots, Q_{\xi-1}, Q_{\xi}, Q_{\xi+1}, Q_{\xi+2}, \ldots, Q_N\} \) and \( Q'' = \{P_1, P_2, P_{\xi-1}, P_{\xi}, P_{\xi+1}, P_{\xi+2}, \ldots, P_N\} \). 

\( \hat{A}(Q, P) = \hat{\xi}_{s_{1},s_{1+1}}(k_{v_{s+1}} - k_{v_{s}}\hat{A}(Q', P'), \tag{8} \)

\( Q_{\xi} = a, Q_{\xi+1} = b \) and \( \hat{A} \) is the vector denotation of superposition coefficients \( A_{m_{1}, \ldots, m_{n}} \). In the system (5), the wavefunction should be continuous and the first-order derivative of the wavefunction with respect to coordinates should be discontinuous. Solving the Schrödinger equation and using the symmetry or antisymmetry condition, we can obtain the scattering matrix. For the \( S_{\mathrm{P}}(2s+1) \)-invariant fermionic model, the two-body scattering matrix is

\[
\hat{S}_{ab}^{(s)}(\lambda) = \sum_{k=1}^{s} \hat{P}_{ab}^{2k-1} + \sum_{k=1}^{s} \frac{\lambda - ic}{\lambda + ic} \hat{P}_{ab}^{2k} + \frac{\lambda - (s + 1)ic}{\lambda + (s + 1)ic} \hat{P}_{ab}^{-0}.
\]

\[
\hat{S}_{ab}^{(s)}(\lambda) = \sum_{k=1}^{s} \hat{P}_{ab}^{2k-1} + \sum_{k=1}^{s} \frac{\lambda - ic}{\lambda + ic} \hat{P}_{ab}^{2k} + \frac{\lambda - (s + 1)ic}{\lambda + (s + 1)ic} \hat{P}_{ab}^{-0}.
\]

The scattering matrices (10) and (11) are different. In order to prove the integrability of the bosonic and fermionic models uniformly, we introduce the \( R \)-matrix for these two kinds of symmetries by the following mapping:

\[
\hat{R}_{ab}^{(s)}(\lambda) = \begin{cases} 
-\hat{P}_{ab} \hat{S}_{ab}^{(s)}(\lambda) & \text{(half odd)} \\
\hat{a}(\lambda) - \hat{b}(\lambda) \hat{P}_{ab} \hat{S}_{ab}^{(s)}(-\lambda) & \text{(integer odd)}
\end{cases}
\]

where \( \hat{b}(\lambda) = ic/(\lambda + ic) \) and \( \hat{a}(\lambda) = \lambda/(\lambda + ic) \). With this mapping, the explicit form of the \( R \)-matrix is

\[
\hat{R}_{ab}^{(s)}(\lambda) = \hat{b}(\lambda) \hat{I} + \hat{a}(\lambda) \hat{P}_{ab} - (2s + 1) \epsilon^{(s)}(\lambda) \hat{P}_{ab}^{0}.
\]

Here, \( \hat{I} \) is a unitary operator and \( \epsilon^{(s)}(\lambda) \) is a scalar function depending on \( s \):

\[
\epsilon^{(s)}(\lambda) = (-1)^{2s} \hat{b}(\lambda) \hat{a}(\lambda) [s + 1/2 + (-1)^{2s+1}].
\]

The last term in (13), \(- (2s + 1) \epsilon^{(s)}(\lambda) \hat{P}_{ab}^{0}\), is a representation of the Temperley–Lieb algebra and the \( R \)-matrix (13) satisfies the Yang–Baxter equation [44, 45, 50, 51]

\[
\hat{R}_{ab}(\lambda) \hat{R}_{bc}(\mu + \lambda) \hat{R}_{ab}(\mu) = \hat{R}_{bc}(\mu) \hat{R}_{ab}(\lambda + \mu) \hat{R}_{bc}(\lambda).
\]

In the proof, the following relations have been used:

\[
\hat{P}_{ab}^{0} \hat{P}_{bc} = (2s + 1) \hat{P}_{ab}^{0} \hat{P}_{ac}, \quad \hat{P}_{ab} = \sum_{k=0}^{2s} (-1)^{2s-k} \hat{P}_{ab}^{k}.
\]

Since there are two invariant mappings, \( \hat{R}(\lambda) \mapsto f(\lambda) \hat{R}(\lambda) \) and \( \hat{R}(\lambda) \mapsto \hat{R}(-\lambda) \), for the Yang–Baxter equation (15) [52], Hamiltonian (5) is integrable.

The \( R \)-matrix defined in equation (13) only has two sets of solutions of the Yang–Baxter equation (15). One set is \( \epsilon^{(s)} = 0 \) where the system has \( SU(2s + 1) \) symmetry. In this case, there are no effective spin exchange interactions. The other set is equation (14). The system has

\[
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\]
$Sp(2s+1)$ symmetry for half-odd $s$ and $SO(2s+1)$ symmetry for integer $s$. The corresponding integrable spin chains are Kennedy–Batchelor models in [48, 49]. When $s = 1/2$, the $Sp(2)$-invariant integrable model is discussed in [29, 30]; when $s = 1$, the $SO(3)$-invariant integrable model is discussed in [38] and when $s = 3/2$, the $Sp(4)$-symmetry integrable model is discussed in [39].

The integrable model (5) has one tunable interacting parameter $c$. For the $Sp(2s+1)$ fermionic model, the interaction is repulsive when $c > 0$ and is attractive when $c < 0$. For the $SO(2s+1)$ bosonic model, the interaction in the spin-0 channel is attractive and that in other channels is repulsive when $c > 0$, while the interaction in the spin-0 channel is repulsive and is attractive in other channels when $c < 0$. To obtain the exact energy spectrum of the system, we need to determine all the values of quasi-momenta $k_j$. This can be done by solving the eigenvalue problem given by the periodic boundary condition, in which we can obtain the Bethe ansatz equations.

### 4. Exact solutions

For the integrable systems with high symmetry, the exact solutions are usually obtained by using the nested algebraic Bethe ansatz method. The Bethe ansatz equations of integrable quantum gas models are composed of the ones of the coordinate part, i.e. $U(1)$ symmetry, and the ones given by the spin part. The spin sector usually has nesting integrable symmetries for high spin models. Using the method suggested in [53, 54], we can obtain the Bethe ansatz equations for the $Sp(2s + 1)$ model ($C_n$ type algebra, $n = s + 1$) and the $SO(2s + 1)$ model.

For the $Sp(2s+1)$ ($s > 1/2$) case, there are $s + 3/2$ sets of coupled equations. When $s > 3/2$, the equations are

$$e^{ik_j l} = \prod_{j=1}^{M^{(i)}} \lambda_j - \lambda_j^{(0)} + i\frac{s}{2}, \quad j = 1, 2, \ldots, N, \quad (17)$$

$$\prod_{i=1}^{M^{(j+1)}} \frac{\lambda_j^{(j+1)} - \lambda_j^{(j+1)} + i\frac{s}{2}}{\lambda_j^{(j)} - \lambda_j^{(j)} - i\frac{s}{2}} \prod_{i=1}^{M^{(j-1)}} \frac{\lambda_j^{(j-1)} - \lambda_j^{(j-1)} - i\frac{s}{2}}{\lambda_j^{(j)} - \lambda_j^{(j)} + i\frac{s}{2}} = \prod_{j \neq j'}^{M^{(i)}} \frac{\lambda_j - \lambda_{j'} + ic}{\lambda_j - \lambda_{j'} - ic},$$

$$l = s, s - 1, \ldots, s/2, \quad j = 1, 2, \ldots, M^{(i)}, \quad (18)$$

$$\prod_{j=1}^{M^{(j+2)}} \frac{\lambda_j^{(j+2)} - \lambda_j^{(j+2)} + ic}{\lambda_j^{(j+2)} - \lambda_j^{(j+2)} - ic} \prod_{j=1}^{M^{(j-2)}} \frac{\lambda_j^{(j-2)} - \lambda_j^{(j-2)} + ic}{\lambda_j^{(j-2)} - \lambda_j^{(j-2)} - ic} = \prod_{j \neq j'}^{M^{(i)}} \frac{\lambda_j - \lambda_{j'} + ic}{\lambda_j - \lambda_{j'} - ic}, \quad j = 1, 2, \ldots, M^{(i)}, \quad (19)$$

$$\prod_{j=1}^{M^{(j+2)}} \frac{\lambda_j^{(j+2)} - \lambda_j^{(j+2)} + ic}{\lambda_j^{(j+2)} - \lambda_j^{(j+2)} - ic} \prod_{j=1}^{M^{(j-2)}} \frac{\lambda_j^{(j-2)} - \lambda_j^{(j-2)} + ic}{\lambda_j^{(j-2)} - \lambda_j^{(j-2)} - ic} = \prod_{j \neq j'}^{M^{(i)}} \frac{\lambda_j - \lambda_{j'} + 2ic}{\lambda_j - \lambda_{j'} - 2ic}, \quad j = 1, 2, \ldots, M^{(i)}. \quad (20)$$

Here, $M^{(i)}$ is the number of rapidity $\lambda^{(i)}$, $M^{(2s+1)} = N$, $\lambda^{(s+1)} = k_j$ and $L$ is the length of the system. When $s = 3/2$, the Bethe ansatz equations reduce to the ones obtained in [39]. When $s = 1/2$, the system (5) reduces to the $Sp(2)$-invariant spin-1/2 Fermi gas, and the Bethe ansatz equations are as given in [29].

For the $SO(2s + 1)$ bosons, the Bethe ansatz equations have $s + 1$ sets, and when $s > 1$ they are

$$e^{ik_j l} = \prod_{j=1}^{N} \frac{k_j - k_j + ic}{k_j - k_j - ic} \prod_{j=1}^{M^{(i)}} \frac{k_j - \lambda_j^{(0)} - i\frac{s}{2}}{k_j - \lambda_j^{(0)} + i\frac{s}{2}}, \quad j = 1, 2, \ldots, N. \quad (21)$$

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The total spin is of the thermodynamic properties of the system can be obtained. If the temperature tends to zero, only the rapidities of the atoms are frozen, i.e. $k_j$. Since the string distributions are symmetric around the real axis, the total momentum is

$$E = \sum_{j=1}^{N} k_j^2, \quad K = \sum_{j=1}^{N} k_j. \quad (24)$$

The total spin is $S = sN - \sum_{i} M_i^{(1)}$.

Obviously, the Bethe ansatz equations of the present system are different from the $SU(2s + 1)$ ones. The physical properties can be obtained from the solutions of the Bethe ansatz equations. For example, solutions of the $SO(3)$-invariant spin-1 bosonic model show that there are bound states in the regimes of $c > 0$ and $c < 0$ [38], so there always exist attractive interactions in some scattering channels.

5. Repulsive fermions

For the repulsive fermionic models, detailed analysis of the Bethe ansatz equations shows that all quasi-momenta $k$ are real, which means there are no charge bound states, and the spin rapidities $\lambda^{(l)}$ form strings. In the thermodynamic limit, the string solutions read

$$\lambda^{(l)}_{n,z,j} = \lambda^{(l)}_{n,z} + (n + 1 - 2j)i\epsilon/2, \quad j = 1, 2, \ldots, n, \quad 3/2 \leq l \leq s, \quad (25)$$

$$\lambda^{(l/2)}_{n,z,j} = \lambda^{(l/2)}_{n,z} + (n + 1 - 2)j\epsilon, \quad j = 1, 2, \ldots, n. \quad (26)$$

Here, $\lambda^{(l)}_{n,z}$ denotes the real parts of the $n$-string rapidities, $z = 1, 2, \ldots, M_n^{(k)}$, and $M_n^{(k)}$ is the number of $n$-strings for $\lambda^{(l)}$. Based on the above string hypothesis, the finite temperature thermodynamic properties of the system can be obtained. If the temperature tends to zero, only the real rapidities and 2-strings for $\lambda^{(l)} (3/2 \leq l \leq s)$ are left in the ground state. When the momenta of atoms are frozen, i.e. $k_j = 0$, the models reduce to be the integrable $Sp(2s + 1)$ spin chains [48]. So, only the real rapidities and the 2-strings of $\lambda^{(l)} (3/2 \leq l \leq s)$ are left in the spin chains at the ground state. This result coincides with the discussion by Martins et al [54, 55].

Substituting these solutions into the Bethe ansatz equations and taking the thermodynamic limit, we obtain the coupled integral equations. Solving these equations, we obtain the numbers of the $i$-string $\lambda^{(l)}$ analytically as

$$M_{1}^{(l)} = \frac{l - 1/2}{s + 1/2} N, \quad M_{2}^{(l)} = \frac{s - l + 1}{s + 1/2} N \quad (l > 3/2), \quad M_{1}^{(l/2)} = N/2. \quad (27)$$

Thus, the numbers of $\lambda^{(l)}$ are $M^{(l)} = N(l > 3/2), M^{(l/2)} = N/2$, and the conserved quantities $J_0 = 0$ in the ground state. The total spin is zero, so that the ground state is a spin singlet state. Since the string distributions are symmetric around the real axis, the total momentum $K$
of the ground state is also zero. The dressed energy of charge rapidities \( k \) in the ground state satisfies the equations

\[
\epsilon(k) = k^2 - \mu + \hat{D}^{(s+1)} \ast \epsilon^-(k), \quad \epsilon^{(l)}(k) = \hat{D}_l^{(l)} \ast \epsilon^-(k), \quad l = s, s-1, \ldots, 1/2.
\] (28)

Here, \( \mu \) is the chemical potential, \( \ast \) is an integral operation defined by \( \hat{w} \ast y(x) = \int w(x-x')y(x')dx' \) and \( \epsilon^-(k) \) is the negative part of the dressed energy \( \epsilon^-(k) = \epsilon(k), |k| < Q \).

\( Q \) is the Fermi point which is determined by the particle density \( n = N/L \). The kernels \( D(k) \) of integral operators \( \hat{D} \) in equations (28) are

\[
D(k) = \hat{a}_{1/2} \ast D_1^{(1)}(k) + \hat{a}_1 \ast D_2^{(1)}(k),
\] (29)

\[
D_1^{(l)}(k) = \frac{1}{(2s+1)c} \sin \left( \frac{2l-1}{2s+1} \pi \right),
\] (30)

\[
D_2^{(l)}(k) = \hat{D}_l^{(s-l+3/2)} \ast D_1^{(1)}(k), \quad 3/2 \leq l \leq s,
\] (31)

\[
D_1^{(1/2)}(k) = \frac{1}{(2s+3)c \cosh[\pi k/(c(s+3/2))]}.
\] (32)

where \( a_i(x) = tc/[\pi(x^2 + tc^2)] \). The dressed energies for models with spin-5/2 and -9/2 when \( c = 1 \) and \( n = 1 \) are shown in figure 2.

The physical properties of such 1D systems are controlled by the parameter \( \gamma = c/n \) [27]. When \( \gamma \rightarrow \infty \), we obtain the density of charge rapidities, energy and the Fermi point in the strong repulsive limit as

\[
\rho(k) = \frac{1}{2\pi} \quad (|k| \leq Q), \quad \rho(k) = 0 \quad (|k| > Q), \quad E = \frac{1}{3\pi}Q^3, \quad Q = n\pi.
\] (33)
When $\gamma \to 0$, the system degenerates into the free fermions and we have
\[
\rho(k) = \frac{2s + 1}{2\pi} \quad (|k| \leq Q), \quad \rho(k) = 0 \quad (|k| > Q), \quad E = \frac{2s + 1}{3\pi}Q^3, \quad Q = \frac{n\pi}{2s + 1}. \tag{34}
\]

Adding holes and strings into the ground state, we can get the excited state of Hamiltonian (5). For the charge sector, the addition of a hole with quasi-momentum $|k| < Q$ and string with quasi-momentum $|k| > 0$ will give us the charge–hole excitation. The excitation spectra of spinor excitations are more interesting in these models. In this case, holes could be added into the strings already existing in the ground state, while strings could be added for the other ones. For these excitations are collective, these changes will affect the numbers of other strings. Such effects are represented in the restriction of the Bethe ansatz equations. Assuming that $m^{(i)}_n$ are the numbers of holes for $n$-strings $\lambda^{(i)}$ added, $m^{(i)}_n'$ are the numbers of high strings and $\Delta M^{(i)}_n$ are the number changing of corresponding strings, then the restriction is
\[
\Delta M^{(i)}_n = -\frac{s - l + 1}{2} \delta_{n,2} m^{(i)}_n - \delta_{n,2} m^{(i)}_n'
\]
\[
+ \sum_{j=1}^{l} \frac{s - j + 1}{2s + 1} \sum_{n'=1}^{2} \left[ (2l - 1)C_{n,n'}^{(i)} - (s + l)C_{n,n'}^{(i)} \right] m^{(i)}_n
\]
\[
+ \sum_{j=\frac{l}{2}}^{l} \frac{s - l + 1}{2s + 1} \sum_{n'=1}^{2} \left[ (2j - 1)C_{n,n'}^{(i)} - (s + j)C_{n,n'}^{(i)} \right] m^{(i)}_n', \quad n = 1, 2, \tag{35}
\]
\[
\Delta M^{(1/2)}_n = -\sum_{j=\frac{l}{2}}^{l} \frac{s - j + 1}{2} m^{(i)}_n - m^{(1/2)}_n. \tag{36}
\]

Here, $m^{(i)}_n$ are the total numbers of the strings added. For $l \neq 1/2$, $m^{(1/2)}_n = \sum_{j=1}^{l} m^{(1/2)}_j$, while for $l = 1/2$, $m^{(1/2)}_n = \sum_{j=1}^{l} m^{(1/2)}_j$. The matrices $C^1$ and $C^2$ are
\[
C^1 = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}, \quad C^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{37}
\]

From the framework of the Bethe ansatz method, the numbers of strings should be integer; thus, only proper choices of $m$ give the low-energy excitations. The possible choices can be found from equations (35) and (36).

The strong correlation of one-dimensional systems makes the spin excitation split into spinons with fractional spins. In the language of the Bethe ansatz method, the quasi-particles are holes and strings. From equations (35) and (36), we learn that the spin carried by the holes and strings added is
\[
S = \sum_{j=\frac{l}{2}}^{l} \left[ \frac{1}{2} (s + l)(s - l + 1)m^{(i)}_j + \sum_{j=1}^{l} (2 - j)m^{(i)}_j \right]
+ \frac{1}{2} \left( s + \frac{1}{2} \right)^2 m^{(1/2)}_j + \sum_{j=\frac{l}{2}+1}^{l} (1 - j)m^{(1/2)}_j. \tag{38}
\]

The holes added for the real $\lambda^{(i)}$ strings of $l \geq 3/2$ do not affect the spin, and they are always neutral, while the other holes carry spins. Table 1 gives some of the possible low-energy excitations of spin-5/2 repulsive fermions. The spins of the excitations are listed in the last
Table 1. Some possible excitations ($s = 5/2$).

| No | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|
| $m_{1/2}$ | 3 | 2 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $m_{3/2}$ | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 4 | 0 | 1 | 3 | 0 | 2 | 4 | 6 |
| $m_{5/2}$ | 0 | 1 | 3 | 0 | 0 | 2 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $m_{1/2}$ | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 3 | 2 | 1 | 0 | 0 | 0 |
| $m_{1}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 |
| spin | 0 | 7 | 0 | 0 | 4 | 5 | 4 | 5 | 7 | 8 | 9 | 10 | 9 | 11 | 12 | 12 | 13 | 14 | 15 |

Figure 3. The dispersion relation of the excited state listed in table 1. Here $c = 1$ and $n = 1$.

row of the table. The momenta and energies carried by the holes and strings can also be calculated based on the above discussion. The spectra for the excitations in table 1 are shown in figure 3. A 2-string hole of $\lambda^{(5/2)}$ carries the lowest excited energy. The single hole carries spin-$5/2$.

The low temperature specific heat is determined by single charge (hole) excitations at the Fermi surface. For the repulsive $Sp(2s + 1)$ fermions here, the low temperature specific heat is given by

$$C = \frac{\pi}{3} TL \left( \frac{1}{v_F} + \sum_{l=1/2}^{s} \frac{1}{v_1^{(l)}} + \sum_{l=3/2}^{s} \frac{1}{v_2^{(l)}} \right) ,$$

(39)

where $v_s$ are the velocities of the holes given by $v(\lambda) = \epsilon'(\lambda)/2\pi \rho(\lambda), v_F$ is the Fermi velocity contributed by the charge (hole) of $k = Q$ and $v_n^{(l)}$ is the spinwave velocity contributed by the hole of $\lambda_n^{(l)}$ at $\lambda_n^{(l)} \to \infty$. The low temperature specific heat of the $SU(2s + 1)$ model has a similar expression with equation (39). It is well known that the spinwave velocities of the $SU(2s + 1)$ model are equal to each other [56]. However, they are different in the $Sp(2s + 1)$ models here. When $\gamma \gg 1$, we can find that the Fermi velocity and spinwave velocities are

$$v_F \approx 2n\pi, \quad v_1^{(l)} \approx \frac{2n^2\pi^3}{3c(s/2 + 1/4)}, \quad v_2^{(l)} \approx \frac{2n^2\pi^3}{3c(s + 3/2)}, \quad s > l > \frac{1}{2}.$$

(40)
$v_{1}^{1/2}$ and $v_{2}^{1/2}$ are lower than the others. This indicates that the low temperature behavior of the repulsive fermions here is different from the $SU(2S + 1)$ repulsive fermions and they have different low energy effective models.

6. Conclusion

In conclusion, we find that there is a hidden symmetry of the high-spin cold atomic systems with a special interaction form away from the $SU(2S + 1)$ symmetry point. Based on the symmetry analysis, a new class of integrable models for cold atoms with arbitrary spin is proposed. The ground state and the excitations of repulsive fermions are also discussed briefly.

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