Skew braces from Rota–Baxter operators: a cohomological characterisation and some examples

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Received: 31 January 2022 / Accepted: 6 May 2022 / Published online: 13 June 2022
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Abstract

Rota–Baxter operators for groups were recently introduced by Guo, Lang, and Sheng. Bardakov and Gubarev showed that with each Rota–Baxter operator, one can associate a skew brace. Skew braces on a group $G$ can be characterised in terms of certain gamma functions from $G$ to its automorphism group $\text{Aut}(G)$ that are defined by a functional equation. For the skew braces obtained from a Rota–Baxter operator, the corresponding gamma functions take values in the inner automorphism group $\text{Inn}(G)$ of $G$. In this paper, we give a characterisation of the gamma functions on a group $G$, with values in $\text{Inn}(G)$, that come from a Rota–Baxter operator, in terms of the vanishing of a certain element in a suitable second cohomology group. Exploiting this characterisation, we are able to exhibit examples of skew braces whose corresponding gamma functions take values in the inner automorphism group, but cannot be obtained from a Rota–Baxter operator. For gamma functions that can be obtained from a Rota–Baxter operators, we show how to get the latter from the former, exploiting the knowledge that a suitable central group extension splits.

Keywords Endomorphisms · Skew braces · Rota–Baxter operators · Group cohomology · Central extensions · Liftings

Mathematics Subject Classification 17B38 · 20J06 · 20E22 · 20D15 · 08A35

Both authors are members of INdAM—GNSAGA. The first author gratefully acknowledges support from the Department of Mathematics of the University of Trento. The second author gratefully acknowledges support from the Department of Mathematics of the University of Pisa.
1 Introduction

Rota–Baxter operators for various kinds of algebras have been studied by several authors since G. Baxter introduced them for commutative algebras [2] in 1960.

Recently, L. Guo, H. Lang, and Y. Sheng introduced Rota–Baxter operators for groups [11]. These were studied further by V. G. Bardakov and V. Gubarev [3, 4]. In particular, in [4], it is showed how to associate a skew brace with a Rota–Baxter operator.

Recall that a skew (left) brace, defined in [13], is a triple $(G, \cdot, \circ)$, where $(G, \cdot)$ and $(G, \circ)$ are groups and the two operations are related by the identity

$$ g \circ (h \cdot k) = (g \circ h) \cdot g^{-1} \cdot (g \circ k) $$

for all $g, h, k \in G$; here the inverse refers to the operation “$\cdot$”.

The interplay between set-theoretic solutions of the Yang-Baxter equation of Mathematical Physics, (skew) braces, regular subgroups, and Hopf-Galois structures has spawned a considerable body of the literature in recent years; see, for example, [6, 8, 12, 13, 15, 16].

Given a group $(G, \cdot)$, the group operations “$\circ$” such that $(G, \cdot, \circ)$ is a skew brace can be characterised in the form

$$ g \circ h = g \cdot \gamma(g) h, \quad (1.1) $$

where $\gamma : G \to \text{Aut}(G)$ is a function, which we write as an exponent, satisfying the identity

$$ \gamma(g \cdot \gamma(g) h) = \gamma(g) \gamma(h) $$

for all $g, h \in G$. We call such a function a gamma function. These functions are usually referred to as $\gamma$ in the literature of skew braces. (See, for instance, [7, 9] for this context.)

A Rota–Baxter operator on the group $(G, \cdot)$ is a function $B : G \to G$ that satisfies the identity

$$ B(g \cdot B(g) \cdot h \cdot B(g)^{-1}) = B(g) \cdot B(h) $$

for all $g, h \in G$.

Consider the morphism from the group $G$ onto the group of its inner automorphisms

$$ i : G \to \text{Inn}(G) $$

$$ g \mapsto (x \mapsto g \cdot x \cdot g^{-1}). $$

whose kernel is the centre $Z(G)$ of $G$. If $B$ is a Rota–Baxter operator on the group $G$, then the function

$$ \gamma(g) = i(B(g)) $$

is immediately seen to be a gamma function on $G$, with values in the inner automorphisms group $\text{Inn}(G)$. This function yields the skew brace $G(B)$ introduced in [4], where

$$ g \circ h = g \cdot i(B(g)) h = g \cdot B(g) \cdot h \cdot B(g)^{-1} $$

for all $g, h \in G$.

We address here the question whether the converse holds.
Question 1.1 Let $\gamma$ be a gamma function on the group $(G, \cdot)$ that takes values in $\text{Inn}(G)$.

Can the skew brace $(G, \cdot, \circ)$, where $goh = g \cdot \gamma(h)$ for all $g, h \in G$, be obtained from a Rota–Baxter operator $B : G \rightarrow G$?

Equivalently, does there exist a Rota–Baxter operator $B$ on $G$ such that

$$\gamma(g) = \iota(B(g))$$

holds for all $g \in G$?

In this case, we say that the gamma function comes from a Rota–Baxter operator.

Remark 1.2 It might be noted that it is not difficult to find skew braces for which the image of the corresponding gamma function is not contained in the group of inner automorphisms.

For instance, this is the case when $(G, \cdot, \circ)$ is a non-trivial skew brace with $(G, \cdot)$ abelian (that is, $(G, \cdot, \circ)$ is a non-trivial brace; see [15]).

In Sect. 7, we exhibit a class of examples where $(G, \cdot)$ is non-abelian.

Question 1.1 has a positive answer when $G$ is a centreless group. In fact, in this case, the map $\iota$ is an isomorphism, so if $\gamma : G \rightarrow \text{Inn}(G)$ is a gamma function, then the composition of $\gamma$ with $\iota^{-1}$ yields a Rota–Baxter operator $B$ for which (1.2) holds. (Compare with [4, Proposition 3.13].)

In this paper, we show that the question has a negative answer in general, by exhibiting two counterexamples, which are finite $p$-groups, for $p$ an odd prime.

Our examples rely on a cohomological characterisation of the gamma functions that come from a Rota–Baxter operators, in terms of the vanishing of a certain cocycle in a suitable second cohomology group over a trivial module. Our characterisation is given in Theorem 3.2 and depends in turn on a well-known cohomological characterisation (Proposition 3.1) of the morphisms from a group $U$ to a quotient of another group $V$ by an abelian normal subgroup that can be lifted to a morphism $U \rightarrow V$.

In Sect. 2, we recall the standard connection between group extensions and the second cohomology group. In Sect. 4, we show that if a gamma function comes from a Rota–Baxter operator, then our cohomological characterisation allows us to reconstruct the latter from the former, on the basis of the knowledge that a certain central group extension splits.

Our first example, given in Sect. 5, has order $p^5$, and it is a simplified version of an example of [10]. With the cohomological setting in place, the proof that this group provides a negative answer to Question 1.1 reduces to the elementary fact that the Heisenberg group of order $p^3$ does not split over its centre.

In Sect. 6, we study a family of gamma functions on the Heisenberg group of order $p^3$ and show, once more via the connection with the splitting of certain extensions, that precisely one of them provides another, smaller counterexample to Question 1.1. For the other gamma functions in this family, we apply the methods of Sect. 4 to find explicitly the corresponding Rota–Baxter operators.

2 Group extensions and the second cohomology group

We recall here the connection between group extensions and a suitable second cohomology group; see [5, Chapter IV, Sect.3]. [17, Sect. 5.1].
Let $G$ be a group, let $Q$ be a $G$-module, where the action is written as left exponent, and let $E$ be an extension of $G$ by $Q$, that is, a group $E$ together with an exact sequence of groups

$$1 \to Q \to E \to G \to 1$$

such that the action by conjugation of $G \cong E/Q$ on $Q$ coincides with the original one. (Here, we identify $Q$ with its image in $E$.) Let $s : G \to E$ be a section, that is, a set-theoretic map $G \to E$ such that $s(x) = x$ for all $x \in G$. Then, there exists a function $\theta : G \times G \to Q$ such that for all $a, b \in G$,

$$s(a)s(b) = \theta(a, b)s(ab).$$

(2.2)

AssOCIATIvITY now yields that $\theta : G \times G \to Q$ is a 2-cocycle: for all $a, b, c \in G$,

$$^a\theta(b, c)\theta(a, bc) = \theta(ab, c)\theta(a, b).$$

Conversely, if $\theta : G \times G \to Q$ is a 2-cocycle, then one may endow the set $E = Q \times G$ with the group operation

$$(q_1, a_1)(q_2, a_2) = (q_1, a_1q_2)\theta(a_1, a_2), a_1a_2)$$

(2.3)

and obtain an exact sequence (2.1), with the natural injection on the first component and projection on the second one, for which (2.2) holds with $s(a) = (1, a)$ for all $a \in G$.

Putting all these facts together, we have obtained most of the following standard result.

**Proposition 2.1** Let $G$ be a group, and let $Q$ be a $G$-module.

1. Given an extension $E$ of $G$ by $Q$ as in (2.1), and a section $s$, Eq. (2.2) defines a 2-cocycle $\theta$, whose class in $H^2(G, Q)$ does not depend on the particular section we have chosen.

2. If $\theta : G \times G \to Q$ is a 2-cocycle, then there exists an exact sequence (2.1) and a section $s$ such that Eq. (2.2) holds.

3. The following are equivalent:
   (a) The extension $E$ of $G$ by $Q$ splits, that is, there exists a section which is a morphism.
   (b) The cocycle $\theta$ yields the trivial class in $H^2(G, Q)$.

**Remark 2.2** We will apply this result only for trivial modules. In particular, if $Q$ is a trivial $G$ module, then the corresponding group extension $E$ is central, meaning that (the image of) $Q$ is contained in the centre of $E$.

Moreover, note that if a central extension $E$ of $G$ by $Q$ splits, then $E \cong G \times Q$.

## 3 Rota–Baxter operators and cohomology

Proposition 3.1 below is well known; it follows, for instance, from the diagrammatic approach of [5, Chapter IV, Sect. 3, Exercise 1(a)], and Proposition 2.1. In the following, we sketch a direct, elementary argument.

Let $U$ and $V$ be groups, let $A$ be an abelian, normal subgroup of $V$, and let $\psi : U \to V/A$ be a morphism. Note that $A$ is a $V/A$-module under conjugation, and then a $U$-module via $\psi$; we denote this action by a left exponent.
Let $C : U \to V$ be a lifting of $\psi$, that is, a set-theoretic map such that $\psi(u) = C(u)A$ for all $u \in U$. Since $\psi$ is a morphism, we have for all $u_1, u_2 \in U$,
\begin{equation}
C(u_1)C(u_2) = \kappa(u_1, u_2)C(u_1u_2),
\end{equation}
for a suitable function $\kappa : U \times U \to A$. Expanding
\[ C((u_1u_2)u_3) = C(u_1(u_2u_3)), \]
one sees that $\kappa : U \times U \to A$ is a 2-cocycle. Moreover, it is immediate to see that if we set, for all $u \in U$,
\[ C'(u) = \sigma(u)C(u), \]
for a function $\sigma : U \to A$, then the cocycle $\kappa'$ associated with $C'$ is given by
\[ \kappa'(u_1, u_2) = \kappa(u_1, u_2)\sigma(u_1)(\sigma(u_2))\sigma(u_1u_2)^{-1}, \]
so it differs from $\kappa$ by a 2-coboundary. From this discussion, we obtain the following well-known result.

**Proposition 3.1** Let $U$ and $V$ be groups, let $A$ be an abelian, normal subgroup of $V$, and let $\psi : U \to V/A$ be a morphism. Let $C : U \to V$ be a lifting of $\psi$, and let $\kappa$ be the map $U \times U \to A$ defined by (3.1). Then, the following hold:

1. $\kappa$ is a 2-cocycle, whose class in $H^2(U, A)$ does not depend on the choice of the lifting $C$.
2. The following are equivalent:
   
   (a) There exists a morphism $\varphi : U \to V$ which is a lifting of $\psi$.
   (b) The class of $\kappa$ in $H^2(U, A)$ is trivial.
3. Two morphisms $\varphi_1, \varphi_2$ are liftings of the same $\psi$ if and only if there exists a 1-cocycle $\zeta : U \to A$ such that, for all $u \in U$,
   \begin{equation}
   \varphi_2(u) = \zeta(u)\varphi_1(u).
   \end{equation}

Let now $\gamma$ be a gamma function on a group $(G, \cdot)$, whose values are inner automorphisms and write $Z(G)$ for the centre of $(G, \cdot)$. For all $g, h \in G$, let
\[ g \circ h = g \cdot \gamma(g) h. \]
Then, $(G, \circ)$ is a group and $\gamma : (G, \circ) \to \text{Inn}(G)$ is a morphism. Composing $\gamma$ with the natural isomorphism $\text{Inn}(G) \to (G, \cdot)/Z(G)$, we obtain a morphism $\psi : (G, \circ) \to (G, \cdot)/Z(G)$.

Note that a lifting of $\psi$ to a morphism $B : (G, \circ) \to (G, \cdot)$ is precisely a Rota–Baxter operator on $G$ from which $\gamma$ comes, as for such a $B$,
\[ B(g \cdot \gamma(g)) h = B(g \circ h) = B(g) \cdot B(h) \]
for all $g, h \in G$. Thus, we can specialise Proposition 3.1 to the following result, where $Z(G)$ is a trivial $(G, \circ)$-module.
Theorem 3.2  Let $\gamma$ be a gamma function on the group $G$, which takes values in $\Inn(G)$. Let $C : G \to G$ be a function such that $\gamma(g) = \iota(C(g))$ for all $g \in G$, and let $\kappa : G \times G \to Z(G)$ be defined by
\[ C(g) \cdot C(h) = \kappa(g, h) \cdot C(g \cdot h) \] (3.3)
for all $g, h \in G$. Then, the following hold:

1. $\kappa$ is a 2-cocycle, whose cohomology class in
   \[ H^2((G, \circ), Z(G)) \]
does not depend on the choice of $C$.
2. The following are equivalent:
   (a) The gamma function $\gamma$ comes from a Rota–Baxter operator.
   (b) The cohomology class of $\kappa$ in $H^2((G, \circ), Z(G))$ is trivial.
3. Two Rota–Baxter operators $B_1, B_2$ yield the same gamma function if and only if there exists a morphism $\xi : (G, \circ) \to Z(G)$ such that
   \[ B_2(g) = \xi(g) \cdot B_1(g) \]
   for all $g \in G$.

Note that since $Z(G)$ is a trivial $(G, \circ)$-module, the morphisms $\xi$ are precisely the 1-cocycles.

4 Reconstructing the Rota–Baxter operator

With the same notation of the previous section, we now show how one can explicitly find the Rota–Baxter operator associated with the gamma function, when the cohomology class of the corresponding cocycle is trivial.

Let $(G, \cdot)$ be a group with centre $Z(G)$, let $\gamma(g) = \iota(C(g))$ be a gamma function on $G$ with values in $\Inn(G)$, corresponding to an operation “$\circ$”, and let $\kappa$ be the cocycle obtained as in (3.3).

Assume that the central extension
\[ 1 \to Z(G) \to E \to (G, \circ) \to 1 \]
associated with $\kappa$ splits, that is, that there is a section $s' : (G, \circ) \to E$ which is a morphism. The section $s'$ yields the trivial cocycle, and to write $\kappa$, associated with the section $s(g) = (1, g)$, as a coboundary, consider the map $\sigma : G \to Z(G)$ such that $s'(g) = \sigma(g)s(g)$. We immediately deduce that for all $g, h \in G$,
\[ \kappa(g, h) = \sigma(g)^{-1} \cdot \sigma(h)^{-1} \cdot \sigma(g \cdot h). \]

If we now define $B(g) = \sigma(g) \cdot C(g)$ for all $g \in G$, then by an immediate application of Theorem 3.2, $B$ is the Rota–Baxter operator on $G$ from which $\gamma$ comes.
5 An example of order $p^5$

This is a simplified version of [10, Example 5.4].

Let $p$ be an odd prime, and let $H$ be the Heisenberg group of order $p^3$:

$$H = \langle u, v, k : u^p, v^p, k^p, [u, v] = k, [u, k], [v, k] \rangle;$$

every element of $H$ can be written uniquely as

$$u^i v^j k^q,$$

with $0 \leq i, j, q < p$.

Let $(G, \cdot) = S \times H$, where $S = \langle x, y \rangle$ is elementary abelian of order $p^2$, so that $G$ has order $p^5$. Write $Z(G) = Z(G, \cdot)$ and $K = \langle k \rangle = Z(H) \leq Z(G)$. Since $H/K = \langle uK, vK \rangle$ is elementary abelian of order $p^2$, the assignment

$$\psi(xK) = uK, \quad \psi(yK) = vK, \quad \psi(hK) = K$$

for all $h \in H$ uniquely defines an endomorphism $\psi$ of $G/K$, whose kernel and image are $H/K$.

Let $C : G \to G$ be any function such that

$$C(g)K = \psi(gK) \quad (5.1)$$

for all $g \in G$. (Such a function is denoted by $\psi^1$ in [10].) Note that we do not assume $C$ to be constant on the cosets of $K$. Also, note that $C(G) \subseteq H$ and $C(H) \subseteq K$.

It was proved in [10], and it is immediate to see, that the function $\gamma : G \to \text{Inn}(G)$ defined by $\gamma(g) = \iota(C(g))$ depends only on $\psi$, and not on the particular choice of $C$. It was also proved in [10, Theorem 3.2] that $\gamma$ is a gamma function, which thus defines a skew brace $(G, \cdot, \circ)$, where “$-$” is the original group operation on $G$ and “$\circ$” is the group operation of (1.1), namely

$$a \circ b = a \cdot C(a) \cdot b \cdot C(a)^{-1}$$

for all $a, b \in G$. (Actually, gamma functions are not mentioned explicitly in [10], but they are used implicitly via (1.1).)

Note that $[a, b] = 1$ for all $a \in S$ and $b \in C(G) \subseteq H$; therefore, the operations “$-$” and “$\circ$” coincide on $S$, which is thus a subgroup of $(G, \circ)$.

We now compute the cohomology class of the 2-cocycle $\kappa$ associated with $\gamma$, as per Sect. 3. To compute this class, we are free to choose any of the functions $C$ satisfying (5.1). Our choice is the map $C : G \to G$ defined by

$$C(x^i y^j c) = u^i v^j \in H$$

for all $0 \leq i, j < p$ and $c \in H$. Let us compute the relevant 2-cocycle. We have, for all $0 \leq i, j, m, n < p$ and $c, d \in H$,

$$C((x^i y^j c) \cdot C(x^i y^j c) \cdot (x^m y^n d) \cdot C(x^i y^j c)^{-1}) = C(x^{i+m} y^{i+n} e) = u^{i+m} v^{i+n},$$

for some $e \in H$. On the other hand,
So the relevant 2-cocycle here is

\[ \kappa(x^i y^j c, x^m y^n d) = k^{-jm}, \]  

(5.2)

with image in \( K = Z(H) \leq Z(G) = S \times K \).

The skew brace \((G, \cdot, \circ)\) provides a negative answer to Question 1.1. This will follow from Theorem 3.2 and the following lemma.

**Lemma 5.1** The cocycle \( \kappa \) of (5.2) yields a non-trivial class in the 2-cohomology group \( H^2((G, \circ), Z(G)) \).

We give two proofs of this lemma. Both rely on the connection between central extensions and a suitable second cohomology group, as recalled in Sect. 2.

**First proof of Lemma 5.1** Consider \( Z(G) = S \times K \) as a trivial module for \((G, \circ)\) and \( K \) as a trivial module for \( S \).

As the actions are trivial, the pair of morphisms

\[ \iota : H \hookrightarrow G, \quad \varphi : S \times K \rightarrow K, \]

the inclusion and the projection, respectively, yields a morphism

\[ H^2((G, \circ), Z(G)) \rightarrow H^2((S, \circ), K); \]

see, for example, [5, Chapter III, Sect. 8]. Explicitly, the cohomology class of the 2-cocycle \( \kappa \) is mapped to the cohomology class of the 2-cocycle \( \kappa' \) defined by

\[ \kappa'(x^i y^j, x^m y^n) = k^{-jm} \]  

(5.3)

for all \( 0 \leq i, j, m, n < p \).

The surjective morphism

\[ \pi : H \rightarrow S \]

\[ u^i v^j c \mapsto x^i y^j, \]

with \( 0 \leq i, j < p \) and \( c \in K \), has kernel \( K \), so it yields the central extension

\[ 1 \rightarrow K \hookrightarrow H \xrightarrow{\pi} S \rightarrow 1. \]

Now, patently, the Heisenberg group \( H \) does not split over its centre \( K \).

We now show that the cohomology class associated with this central extension is that of \( \kappa' \). According to Proposition 2.1, this will yield that \( \kappa' \) is non-trivial in cohomology. It will follow that the same holds for \( \kappa \), thereby concluding the proof of the lemma.

Consider the following section of \( \kappa' \):
where \(0 \leq i, j < p\). As

\[
\begin{align*}
s(x'y^j)s(x^m y^n) &= u^i v^j u^m v^n = u^{i+m} v^{i+j+n} [u, v]^{-jm} = u^{i+m} v^{i+j+n} k^{-jm}
\end{align*}
\]

and

\[
\begin{align*}
s(x'y^j)(x^m y^n) &= s(x^{i+m} y^{i+j+n}) = u^{i+m} v^{i+j+n}
\end{align*}
\]

for all \(0 \leq i, m, n < p\), the cocycle we are looking for is

\[
(x'y^j, x^m y^n) \mapsto k^{-jm},
\]

as claimed. \(\square\)

We now give an alternative proof of Lemma 5.1, which is elementary, in that it avoids the use of the above map between cohomology groups, replacing it with a group-theoretic argument.

**Second proof of Lemma 5.1** Let \(E = (S \times K) \times (G, \circ), \) and consider the central extension

\[
\begin{align*}
1 & \rightarrow S \times K \rightarrow E \rightarrow (G, \circ) \rightarrow 1
\end{align*}
\]

(5.4)

defined by \(\kappa\) and the standard section \(s(g) = (1, g)\). Thus, the operation on \(E\) is given by (2.3), with \(\theta = \kappa\); it follows from this and formula (5.2) for \(\kappa\) that

\[
\begin{align*}
[(1, x), (1, y)] &= (1, x)(1, y)((1, y)(1, x))^{-1} \\
&= (1, xy)(\kappa(x, y), 1)((\kappa(y, x), 1)(1, yx))^{-1} \\
&= (1, xy)(1, yx)^{-1}(\kappa(y, x), 1)^{-1} \\
&= (\kappa(y, x)^{-1}, 1) = (k, 1),
\end{align*}
\]

so that \((1 \times K, 1)\) is contained in the derived subgroup \([E, E]\) of \(E\).

Assume by way of contradiction that the sequence (5.4) splits, and let \(T\) be a complement to \(S \times K\) in \(E\). Then, \(M = (S \times 1, 1)T\) is a maximal subgroup of the finite \(p\)-group \(E\), which does not contain \((1 \times K, 1)\), a non-trivial subgroup of \([E, E]\).

This contradiction shows that (5.4) does not split, so that, by Proposition 2.1, \(\kappa\) is non-trivial in \(H^2((G, \circ), Z(G))\). \(\square\)

### 6 Examples of order \(p^3\)

Consider again the Heisenberg group of order \(p^3\), where \(p\) is an odd prime:

\[
(H, \cdot) = \langle u, v, k : u^p, v^p, k^p, [u, v] = k, [u, k], [v, k] \rangle.
\]

Write, as usual, \(Z(H)\) for the centre of \((H, \cdot)\).

For all \(a \in Z/pZ\), consider the function \(\gamma(g) = \iota(g^a)\), which is showed in [10, Proposition 5.6] to be a gamma function on \(H\). The associated circle operation is

\[
g \circ h = g \cdot \iota(g^a) h = g \cdot h \cdot [g, h]^a
\]
for all \( g, h \in H \). Note that powers with respect to \( \circ \) (in particular, inverses) coincide with the corresponding powers in \( H \). Moreover, the commutator in \((H, \circ)\) of \( g \) and \( h \) is given by

\[
[g, h]_\circ = gohg^{-1}oh^{-1} = (g \cdot h \cdot [g, h]^a) \circ (g^{-1} \cdot h^{-1} \cdot [g^{-1}, h^{-1}]^a) \\
= g \cdot h \cdot [g, h]^a \cdot g^{-1} \cdot h^{-1} \cdot [g^{-1}, h^{-1}]^a \cdot [g \cdot h, g^{-1} \cdot h^{-1}]^a \\
= [g, h]^{1+2a} \cdot [h \cdot g \cdot h]^a = [g, h]^{1+2a}.
\]

In particular, \((H, \circ)\) is abelian precisely when \( \alpha = -1/2 \), where the fraction is taken in \( \mathbb{Z}/p\mathbb{Z} \). (Note that this situation is a particular case of the Baer correspondence [1], which is in turn an approximation of the Lazard correspondence and the Baker-Campbell-Hausdorff formulas [14, Chapters 9 and 10].)

We now show the following result.

**Proposition 6.1**

1. When \( \alpha = -1/2 \), the gamma function \( \gamma \) does not come from a Rota–Baxter operator.
2. When \( \alpha \neq -1/2 \), the gamma function \( \gamma \) comes from the Rota–Baxter operator

\[
B(u^i \cdot v^j \cdot k^r) = u^{ia} \cdot v^{ja} \cdot k^{r \cdot (r - i \alpha)(1 + 2\alpha)^{-1}}, \tag{6.1}
\]

for \( 0 \leq i, j, r < p \).

When \( \alpha = -1/2 \), we have thus obtained another example giving a negative answer to Question 1.1.

**Proof** Take \( C : H \to H \) to be \( C(g) = g^a \). We have

\[
C(g \circ h) = C(g \cdot h \cdot [g, h]^a) = g^a \cdot h^a \cdot [h, g]^{(\alpha a)} \cdot [g, h]^{a^2} \\
= C(g) \cdot C(h) \cdot [g, h]^{e^a a^2}.
\]

The critical cocycle \( \kappa : H \times H \to Z(H) \) is thus

\[
\kappa(g, h) = [g, h]^{-e^a a^2}
\]

for all \( g, h \in H \). Consider, as in Sect. 2, the standard central extension defined by \( \kappa \):

\[
1 \to Z(H) \to Z(H) \times H \to (H, \circ) \to 1, \tag{6.2}
\]

where the operation on \( E = Z(H) \times H \) is given by

\[
(a, g)(b, h) = (a \cdot b \cdot \kappa(g, h), g \circ h)
\]

for all \( a, b \in Z(H) \) and \( g, h \in H \), with the standard section \( s : H \to E \) given by \( s(g) = (1, g) \) for all \( g \in H \).

Let us compute the commutator of two elements \((1, g), (1, h)\) in \( E \). We have
\[(1, g), (1, h) = (1, g)(1, h)((1, h)(1, g))^{-1}
= (\kappa(g, h), g_0 h)(\kappa(h, g), h_0 g)^{-1}
= (\kappa(g, h), g_0 h)(\kappa(h, g)^{-1}, (h_0 g)^{-1})
= (\kappa(g, h)^2, [g, h]_a)
= ([g, h]^{-a(a+1)}, [g, h]^{1+2a}),\]

where we have exploited
\[\kappa((g_0 h), (h_0 g)^{-1}) = \kappa(g \cdot h, (h \cdot g)^{-1}) = 1.\]

It is immediate to see that \((Z(H), Z(H)) \leq Z(E).\) (The following arguments yield that they actually coincide.) Therefore, the commutator subgroup \([E, E]\) is of order \(p\), generated by
\[[(1, u), (1, v)] = (k^{-a(a+1)}, k^{1+2a}). \tag{6.3}\]

When \(a = -1/2\), we have seen above that the group \((H, \circ)\) is abelian, so \((6.2)\) does not split, as otherwise \(E \cong Z(H) \times (H, \circ)\) would be abelian, whereas \((6.3)\) shows that \([E, E]\) is non-trivial.

Consider now the case \(a \neq -1/2\). The previous argument yields that both \((H, \circ)\) and \(S = \langle (1, u), (1, v) \rangle\) are Heisenberg groups of order \(p^3\). The former is generated by \(u, v\), with \(\bar{k} = [u, v]_c = k^{1+2a}\), and the latter is generated by \((1, u), (1, v)\), with \([(1, u), (1, v)] = (k^{-a(a+1)}, k^{1+2a})\). An isomorphism between \((H, \circ)\) and \(S\) is clearly given by
\[s' : (u^i \circ v^j \circ \tilde{k}^q) \mapsto (1, u)^i (1, v)^j (k^{-a(a+1)}, k^{1+2a})^q\]
\[= (k^{-\frac{2a+4}{2}}(2q+i)), u^i \circ v^j \circ \tilde{k}^q),\]

with \(0 \leq i, j, q < p\). This implies immediately that \(s'\) is a section which splits the extension \((6.2)\).

We now employ the methods of Sect. 4 to compute the Rota–Baxter operator \(B\) from which \(\gamma\) comes. As
\[s'(u^i \circ v^j \circ \tilde{k}^q) = (k^{-\frac{2a+4}{2}}(2q+i)), u^i \circ v^j \circ \tilde{k}^q),\]
\[= (k^{-\frac{2a+4}{2}}(2q+i)), 1)(1, u^i \circ v^j \circ \tilde{k}^q),\]
\[= (k^{-\frac{2a+4}{2}}(2q+i)), 1)s(u^i \circ v^j \circ \tilde{k}^q),\]

we have
\[B(g) = \sigma(g) \cdot C(g),\]

where
\[\sigma(u^i \circ v^j \circ \tilde{k}^q) = k^{-\frac{2a+4}{2}}(2q+i)}\]

and
\[C(u^i \circ v^j \circ \tilde{k}^q) = (u^i \cdot v^j \cdot k^{i+a+q(1+2a)}a = u^a \cdot v^a \cdot k^{a+2a} + (\frac{2a}{2a})\tilde{k}^{q+ij}).\]
Explicitly,
\[
B(u^i \circ v^j \tilde{k}^q) = k^{-\frac{q^2 + q}{2}}(2q + ij) \cdot u^{ai} \cdot v^{ja} \cdot k^{q(a + 2a^2) + i(j + \frac{q}{2})} \\
= u^{ia} \cdot v^{ja} \cdot k^{qa^2}.
\]

The expression (6.1) for $B$ in terms of the operation $\cdot$ is then easily obtained (see Remark 6.2 below).

We can double-check that $B$ is a Rota–Baxter operator:
\[
B(u^i \circ v^j \tilde{k}^q \circ u^m \circ v^n \tilde{k}^r) = B(u^{i+m} \circ v^{j+n} \tilde{k}^{r-jm}) \\
= u^{(i+m)a} \cdot v^{(j+n)a} \cdot k^{(q+r-jm)a^2}
\]
and
\[
B(u^i \circ v^j \tilde{k}^q) \cdot B(u^m \circ v^n \tilde{k}^r) = u^{ia} \cdot v^{ja} \cdot k^{qa^2} \cdot u^{ma} \cdot v^{na} \cdot k^{ra^2} \\
= u^{ia+ma} \cdot v^{ja+na} \cdot k^{-jma^2} \cdot k^{ra^2+ra^2},
\]
so the assertion follows.

**Remark 6.2** For $\alpha \neq -1/2$, it is easy to switch from a writing of the kind $u^m \cdot v^n \cdot k^r$ to one of the kind $u^i \circ v^j \tilde{k}^q$. Indeed,
\[
u^i \circ v^j \tilde{k}^q = u^i \cdot v^j \cdot k^{ija} \cdot k^{(1+2a)} = u^i \cdot v^j \cdot k^{ija+q(1+2a)},
\]
so that
\[
u^i \cdot v^j \cdot k^r = u^i \circ v^j \tilde{k}^q,
\]
where $q = (r - ij)(1 + 2a)^{-1}$.

## 7 A class of gamma functions whose values are not all inner

Let $A$ and $B$ be two groups, and let $\psi : A \to \text{Aut}(B)$ be a morphism such that $\psi(A) \nsubseteq \text{Inn}(B)$. Write the action of $\psi(A)$ on $B$ as left exponent and consider two group structures on the set $G = A \times B$:

1. $(G, \cdot)$ is the direct product.
2. $(G, \circ)$ is the semidirect product $A \ltimes_{\psi} B$.

Then, $(G, \cdot, \circ)$ is a skew brace; see [13, Example 1.4]. Here, the gamma function is given by
\[
\gamma(a, b) = (a, b)^{-1} \cdot ((a, b) \circ (a', b')) \\
= (a, b)^{-1} \cdot (a \cdot a', b \cdot \psi(a) b') \\
= (a', \psi(a) b'),
\]
for all $a, a' \in A$ and $b, b' \in B$. Clearly $\gamma(a, b) \in \text{Inn}(G, \cdot)$ if and only if $\psi(a) \in \text{Inn}(B)$. Moreover, $(G, \cdot)$ is non-abelian if one of $A$ and $B$ is non-abelian.
To specify a concrete example, let $A = \langle a \rangle$ be the cyclic group of order 4, and let $B = D_8 = \langle r, s : r^4 = s^2 = 1, rsr = r^{-1} \rangle$ be the dihedral group of order 8. If $\psi(a)$ maps $r$ in $r$ and $s$ in $rs$, then $\psi(a) \not\in \operatorname{Inn}(B)$, so that $\gamma(a, b) \not\in \operatorname{Inn}(A \times B, \cdot)$ for all $b \in B$.

As another example, take $V$ to be an elementary abelian group of order $p^2$ and $H = \langle h_0 \rangle$ to be a cyclic group of order a prime $q$, with $q | p - 1$. Choose two elements $\mu$ and $\nu$ of order $q$ in $\operatorname{Aut}(V) \cong \operatorname{GL}(2, p)$ such that $\mu$ is scalar while $\nu$ is not, so that $\mu$ and $\nu$ commute, but they are not conjugate in $\operatorname{GL}(2, p)$. Define $B$ to be the semidirect product of $V$ by $H$ via $\Phi(h_0) = \mu$. Now let $A = \langle a \rangle$ be another group of order $q$, and take in the construction above $\psi(a)(h, v) = (h, \nu(v))$ for all $h \in H$ and $v \in V$.

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