Bäcklund Transformations
of MKdV and Painlevé Equations

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Abstract

For $N \geq 3$ there are $S_N$ and $D_N$ actions on the space of solutions of the first non-trivial equation in the $SL(N)$ MKdV hierarchy, generalizing the two $Z_2$ actions on the space of solutions of the standard MKdV equation. These actions survive scaling reduction, and give rise to transformation groups for certain (systems of) ODEs, including the second, fourth and fifth Painlevé equations.

Given a solution $j$ of the MKdV equation

$$j_t = j_{xxx} - \frac{3}{2} j^2 j_x$$

(1)

we can construct new solutions, $-j$ and $j - \frac{2}{q}$, where $q$ satisfies

$$q_x + qj = 1$$

$$q_t + q(j_{xx} - \frac{1}{2}j^3) = (j_x - \frac{1}{2}j^2).$$

(2)

Equations (2) constitute a strong auto-Bäcklund transformation for the MKdV equation, distinct from the usual one given in the literature (see for example [1], Chapter 8, Exercise 2), and discovered, I believe, in the context of Painlevé analysis [2]. If we choose the integration constant arising in the solution of (2) appropriately, the square of this transformation is the identity; but when combined with the $j \to -j$ transformation it can,
generically, be used to generate an infinite number of solutions from a particular one (solutions of MKdV that are periodic under the action of the combined symmetry are discussed in [3]).

Unlike the standard auto-Bäcklund transformation for MKdV, the transformations \( j \to -j \), \( j \to j - \frac{2}{q} \) do not contain dimensionful parameters, and hence survive scaling reduction.\(^1\) Setting \( j = -2(3t)^{-\frac{1}{3}} J(w) \) where \( w = -x(3t)^{-\frac{4}{3}} \), (1) reduces to the second Painlevé equation (PII)

\[
J'' = 2J^3 + Jw + \alpha \tag{3}
\]

where a prime denotes differentiation with respect to \( w \) and \( \alpha \) is an integration constant. Setting \( q = (3t)^{-\frac{1}{3}} Q(w) \) in (2), we can solve for \( Q \) and thus we have two explicit transformations for (3),

\[
J \to -J, \quad \alpha \to -\alpha
\]

\[
J \to J + \frac{\frac{1}{2} - \alpha}{J' - J^2 - \frac{1}{3}w}, \quad \alpha \to 1 - \alpha. \tag{4}
\]

These transformations, which both square to the identity, generate the well-known transformation group of PII (see [5], [6] and references therein). In solving for \( Q \) I have assumed \( \alpha \neq \frac{1}{2} \); for \( \alpha = \frac{1}{2} \) there is a one parameter family of solutions of PII given by the solutions of \( J' = J^2 + \frac{1}{3}w \) (which can be solved in terms of Airy functions).

The purpose of this note is to present generalizations of the above transformations for the \( SL(N) \) MKdV equations, \( N \geq 3 \). The above transformations for the standard MKdV equation actually extend to the MKdV hierarchy, and similarly in the \( SL(N) \) case the relevant transformations extend to the hierarchy. But for clarity we will focus just on the lowest nontrivial equation in the hierarchy. On scaling reduction all the transformations become explicit. The (lowest nontrivial) \( SL(3) \) and \( SL(4) \) MKdV equations reduce, respectively, to the PIV and PV equations (the first of these facts originated, I believe, in [7]; the second is, I believe, new); we recover the transformation groups investigated by Okamoto for PIV [6] (see also [8]) and PV [9].

The \( SL(N) \) MKdV hierarchy describes evolutions of \( N \) fields \( j_i, i = 1, \ldots, N \), with \( \sum_{i=1}^{N} j_i = 0 \). Writing \( \Sigma = \frac{1}{N} \sum_{i=1}^{N} j_i^2 \), the lowest nontrivial evolution in the hierarchy is

\[
\partial_t j_i = \partial_x \left[ \sum_{r=1}^{N-1} \left( 1 - \frac{2r}{N} \right) \partial_x j_{i + r \text{mod} \ N} + j_i^2 - \Sigma \right], \quad i = 1, \ldots, N \tag{5}
\]

or, equivalently,

\[
\partial_t (j_i - j_{i+1}) = \partial_x [\partial_x (j_i + j_{i+1}) + j_i^2 - j_{i+1}^2], \quad i = 1, \ldots, N - 1. \tag{5'}
\]

\(^1\) When the dimensionful parameter in the standard transformation is set to zero, the transformation becomes trivial. The standard auto-Bäcklund transformation for KdV [1] remains non-trivial even when the dimensionful parameter is set to zero, and this was used in [4], along with the Miura map, to obtain a different derivation of the transformation group of PII from the one we are about to see.
A simple way to obtain (5) is by reduction of the $SL(N)$ self-dual Yang-Mills equations with an ansatz given in the appendix. (5) has one obvious symmetry group:

**Prop.1:** $D_N$ Invariance of (5).

Equations (5) are invariant under the $D_N$ action generated by

\[
A : j_i \rightarrow j_{i+1} \mod N, x \rightarrow x, t \rightarrow t
\]

\[
B : j_i \rightarrow j_{N+1-i}, x \rightarrow -x, t \rightarrow -t
\]

(6)

which satisfy $A^N = B^2 = I, ABAB = I$.

The other symmetry group is less obvious:

**Prop.2:** $S_N$ Invariance of (5).

There is an $S_N$ action on solutions of (5); the action of the fundamental transpositions $T_i = (i \; i+1)$, $i = 1, ..., N-1$, is given by

\[
\begin{align*}
  j_i &\rightarrow j_i + q_i^{-1} \\
  j_{i+1} &\rightarrow j_{i+1} - q_i^{-1} \\
  j_r &\rightarrow j_r, \quad r \neq i, i+1
\end{align*}
\]

(7)

where $q_i$ satisfies

\[
\begin{align*}
  q_{ix} + (j_i - j_{i+1})q_i + 1 &= 0 \\
  q_{it} + [(j_i + j_{i+1})x + j_i^2 - j_{i+1}^2]q_i + (j_i + j_{i+1}) &= 0.
\end{align*}
\]

(8)

For a complete understanding of the origin of these transformations, and why the group generated by the transformations $T_i$ is $S_N$, the reader is referred to [10]. The basic argument however is quite simple: consider the $N$th order differential operator $L = (\partial + j_N)\cdots(\partial + j_2)(\partial + j_1)$, and choose a basis $\{\psi_1, ..., \psi_N\}$ for the kernel of $L$ such that $\{\psi_1, ..., \psi_i\}$ is a basis for the kernel of $(\partial + j_i)\cdots(\partial + j_2)(\partial + j_1)$ for each $i, i = 1, ..., N$. It is easy to check that switching $\psi_i$ and $\psi_{i+1}$ induces the change in the $j_r$ given by (7), with $q_i$ satisfying the first equation in (8); the second equation in (8) is then deduced directly from (5).

There are relations between the $D_N$ and $S_N$ generators; two obvious ones are

\[
\begin{align*}
  AT_i &= T_{i-1}A, \quad i = 2, 3, ..., N-1 \\
  BT_i &= T_{N-i}B, \quad i = 1, 2, ..., N-1.
\end{align*}
\]

(9)

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2 Equations (8) determine $q_i$ up to a parameter. The precise action of $T_i$ is determined by picking a suitable boundary condition satisfied by the solution $j_1, ..., j_N$ on which we wish to act, and requiring the $T_i$ to preserve this condition. For example, for the scaling reduction of (5) we will shortly consider, we fix this parameter by requiring that $q_i$ should also have a well-defined scaling behavior.
It is natural to define a transformation $T_N$ by $T_N \equiv A^{-1}T_{N-1}A$; this satisfies $AT_1 = T_NA$ and $BT_N = T_NB$. The explicit action of $T_N$ is

$$
\begin{align*}
    j_N & \rightarrow j_N + q_N^{-1} \\
    j_1 & \rightarrow j_1 - q_N^{-1} \\
    j_r & \rightarrow j_r, \quad r \neq N, 1
\end{align*}
$$

(10)

where $q_N$ satisfies

$$
\begin{align*}
    q_{Nx} + (j_N - j_1)q_N + 1 &= 0 \\
    q_{Nt} + [(j_N + j_1)x + j_N^2 - j_1^2]q_N + (j_N + j_1) &= 0.
\end{align*}
$$

(11)

It must be emphasized that $T_N$ is not a pure $S_N$ transformation, and should not be confused with the fundamental transposition $(1\,N)$ in $S_N$, which generically changes all the $j_i$. Having introduced $T_N$, it is clear that the transformation group for $(5)$ is a semi-direct product of the group generated by $T_1, .., T_N$ with the group $D_N$.

We now consider the scaling reduction of $(5)$. Writing $j_i = t^{-\frac{1}{2}}J_i(w)$ where $w = t^{-\frac{1}{2}}x$, we find that we can at once integrate each equation of $(5)$ to obtain the reduced system

$$
-\frac{1}{2}wJ_i + \alpha_i = \sum_{r=1}^{N-1} \left(1 - \frac{2r}{N}\right) J_i^{\prime}+ r(\mod N) + J_i^2 - S, \quad i = 1, .., N.
$$

(12)

Here $S = \frac{1}{N} \sum_{r=1}^{N} J_r^2$, a prime denotes differentiation with respect to $w$, the $\alpha_i, i = 1, .., N$, are constants satisfying $\sum_{i=1}^{N} \alpha_i = 0$, and $\sum_{r=1}^{N} J_i = 0$. Because of the square roots in the reduction formulae, $(12)$ displays a residual scale invariance $w \rightarrow -w, J_i \rightarrow -J_i$. This can be eliminated by setting $J_i(w) = w^{-1}K_i(z)$, where $z = w^2$, to obtain the system

$$
(\alpha_i - \frac{1}{2}K_i)z = \sum_{r=1}^{N-1} \left(1 - \frac{2r}{N}\right) \left(2zK_i^{\prime}+ r(\mod N) - K_i^{\prime}+ r(\mod N)\right) + K_i^2 - T, \quad i = 1, .., N.
$$

(13)

Here a dot denotes differentiation with respect to $z$, $T = \frac{1}{N} \sum_{r=1}^{N} K_i^2$, and $\sum_{r=1}^{N} K_i = 0$. We could, of course, have obtained $(13)$ directly from $(5)$ by substituting $j_i = x^{-1}K_i(z)$ where $z = t^{-1}x^2$, but if we do this it is somewhat harder to see the integrations that can be done.

Under scaling reduction (i.e. setting $q_i = xQ_i(z)$) we find we can solve $(8)$ for $q_i$; we can thus write down both the $D_N$ and $S_N$ actions explicitly:

$\text{Prop.1'}$: $D_N$ Invariance of $(13)$.

Equations $(13)$ are invariant under the $D_N$ action generated by

$$
\begin{align*}
    A & : K_i \rightarrow K_{i+1(\mod N)}, z \rightarrow z, \alpha_i \rightarrow \alpha_{i+1(\mod N)} \\
    B & : K_i \rightarrow -K_{N+1-i}, z \rightarrow -z, \alpha_i \rightarrow -\alpha_{N+1-i}
\end{align*}
$$

(14)

Similar considerations give rise to the extra invariances of equation $(3)$ under $J \rightarrow \lambda J$, $w \rightarrow \lambda^2w$ with $\lambda^3 = 1$. 

which satisfy \( A^N = B^2 = I \), \( ABAB = I \).

**Prop. 2′**: \( S_N \) Invariance of (13).

There is an \( S_N \) action on solutions of (13); the action of the fundamental transformations \( T_i = (i,i+1) \), \( i = 1, \ldots, N-1 \), is given by

\[
K_i \rightarrow K_i + \frac{z(\alpha_{i+1} - \alpha_i - \frac{1}{2})}{K_i + K_{i+1} + \frac{1}{2}z} \]

\[
K_{i+1} \rightarrow K_{i+1} - \frac{z(\alpha_{i+1} - \alpha_i - \frac{1}{2})}{K_i + K_{i+1} + \frac{1}{2}z} \]

\[
K_r \rightarrow K_r, \quad r \neq i,i+1
\]

\[
\alpha_i \rightarrow \alpha_{i+1} - \frac{1}{2}
\]

\[
\alpha_{i+1} \rightarrow \alpha_i + \frac{1}{2}
\]

\[
\alpha_r \rightarrow \alpha_r, \quad r \neq i,i+1.
\]

The transformations \( T_i \) can easily be checked using the reduced form of (5)' (equivalent to (13)):

\[2z(\dot{K}_i + \dot{K}_{i+1}) = (K_{i+1} + K_i)(K_{i+1} - K_i + 1) + \frac{1}{2}z(K_{i+1} - K_i) + z(\alpha_i - \alpha_{i+1}), \quad i = 1, \ldots, N-1.\]

(13)′

For comparison with [6],[9] it is useful to define \( \beta_i \equiv N^{-1}(2\alpha_i - i + \frac{1}{2}(N+1)) \); the action of \( T_i \), \( i = 1, \ldots, N-1 \), on the \( \beta_r \) is \( \beta_i \rightarrow \beta_{i+1} \), \( \beta_{i+1} \rightarrow \beta_i \), and \( \beta_r \rightarrow \beta_r \) for \( r \neq i,i+1 \). The action of \( T_N \) is \( \beta_1 \rightarrow \beta_N + 1 \), \( \beta_N \rightarrow \beta_1 - 1 \), and \( \beta_r \rightarrow \beta_r \) for \( r \neq 1, N \). The action of \( A \) is \( \beta_r \rightarrow \beta_{r+1} + \frac{1}{N} \) for \( r \neq N \), and \( \beta_N \rightarrow \beta_1 - \frac{N-1}{N} \), and the action of \( B \) is \( \beta_r \rightarrow -\beta_{N+1-r} \). The transformation \( AT_1T_2\ldots T_{N-1} \) acts as a “parallel transformation”, mapping \( \vec{\beta} \) to \( \vec{\beta} + \frac{1}{N}(1,1,\ldots,1,1-N) \).

We now relate the above systems for \( N = 3,4 \) to PIV and PV, and discuss the relevant transformation groups. The following results are elementary to establish.

**Prop. 3**

The general solution of (13) for \( N = 3 \) is

\[
K_3 = \frac{M + z}{2}
\]

\[
K_2 - K_1 = \frac{2z\dot{M}}{M} - 1 + \frac{z(1 + 2\alpha_1 - 2\alpha_2)}{M}
\]

(16)

where \( M(z) \) solves the equation

\[
\ddot{M} = \frac{\dot{M}^2}{2M} + \frac{3\dot{M}^3}{32z^2} + \frac{3M^2}{8z} + \left( \frac{3}{4} - \frac{2\alpha_3}{z} - \frac{1}{z^2} \right) \frac{3M}{8} - \frac{(\frac{1}{2} + \alpha_1 - \alpha_2)^2}{2M}.
\]

(17)

\( M(z) \) solves (17) if and only if \( J(p) \) defined by \( M(z) = 2pJ(p) \), where \( p = (3z/4)^{\frac{1}{2}} \), satisfies PIV:

\[
\frac{d^2J}{dp^2} = \frac{1}{2J} \left( \frac{dJ}{dp} \right)^2 + \frac{3}{2}J^3 + 4pJ^2 + 2(p^2 - 2\alpha_3)J - \frac{2}{J} \left( \frac{1 + 2\alpha_1 - 2\alpha_2}{3} \right)^2.
\]

(18)
The transformation group for (17) is a semi-direct product of the group generated by $T_1, T_2, T_3$, which Okamoto [6] calls $s_1, s_2, \tilde{s}$, with the $D_N$ group generated by $A, B$. Okamoto writes $l$ for $A^{-1}$ (and $l$ for $T_2 T_1 A^{-1}$), and instead of $B$ uses $x = AB$ (all this can easily be checked; Okamoto’s coefficients $v_1, v_2, v_3$ are the coefficients $\beta_1, \beta_2, \beta_3$ introduced above). The transformation group for (18) is just that for (17) supplemented with the extra symmetry $J \rightarrow -J, p \rightarrow -p$, which Okamoto denotes $\psi$. Explicit formulae for all the transformations can easily be written.

**Prop. 4**

The general solution of (13) for $N = 4$ is

$$K_1 = \frac{z \dot{V}}{V(V - 1)} + \frac{z V + 1}{4 V - 1} - \frac{V + 1 + 2(V - 1)(\alpha - \alpha_1)}{4V}$$

$$K_2 = -\frac{z \dot{V}}{V(V - 1)} + \frac{z V + 1}{4 V - 1} + \frac{V + 1 + 2(V - 1)(\alpha - \alpha_1)}{4V}$$

$$K_3 = \frac{z \dot{V}}{(V - 1)} - \frac{z V + 1}{4 V - 1} - \frac{V + 1 + 2(V - 1)(\beta - \beta_4)}{4}$$

where $V(z)$ solves PV:

$$\ddot{V} = \left(\frac{1}{2V} + \frac{1}{V - 1}\right) \dot{V}^2 - \frac{\dot{V}}{V} - \frac{V(V + 1)}{8(V - 1)} + \frac{(\alpha + \alpha_2)V}{2z}$$

$$+ \frac{(V - 1)^2}{32z^2} \left(1 + 2\alpha_3 - 2\alpha_4\right)^2 V - \frac{(1 + 2\alpha_1 - 2\alpha_2)^2}{V}.$$  

(20)

Note that for $N \neq 4$ the order of the system (13) is $N - 1$, but for $N = 4$ it is 2. The form of PV in (20) is brought to the standard form of Okamoto [9] by rescaling $z$. Having done this, it is straightforward to check that the coefficients $\beta_1, ..., \beta_4$ are exactly the coefficients $v_1, ..., v_4$ of Okamoto. The relationship between the transformations we have introduced and those of [9] is as follows. First we note that since $V$ is determined by $K_1 + K_2$, the transformations $T_1, T_3$ leave $V$ unchanged. These are $\pi'_1$ and $\pi_1$ in [9], respectively. $s_1, s_2, s_3, s_0$ in [9] are $T_1, T_2, T_3, T_4$ respectively, and $l$ is $T_3 T_2 T_1 A^{-1}$. Finally $x$ (or $\pi_2$) of [9] is $BA^2$, and $w'$ of [9] is $T_1 T_3 B$.

**Discussion**

It is pleasing that we have obtained the results of [6] and [9] in a unified and extended framework; we see the rather complicated transformation groups for PIV and PV have a fairly simple origin in the symmetries of (5). It is to be expected that useful applications will be found for the system (13) for $N \geq 5$, and for the systems obtained by scaling reduction of higher equations in the MKdV hierarchies (all of these systems possess the Painlevé property). For example, in generalization of the results of [11], we should expect the scaling reductions of equations in the $SL(N)$ MKdV hierarchy to arise as the “string equations” for certain matrix models.
One thing missing from this paper is an explanation of the origin of the transformation groups for PIII [12] and PVI [13]. From Okamoto’s work on these systems one might guess that they arise as scaling reductions of an MKdV equation associated with the Lie algebras $B_2$ and $D_4$ respectively [14]. The lowest nontrivial flow in the $B_2$ MKdV hierarchy (which describes the evolution of two fields $j_1, j_2$) can be computed, and has a consistent scaling reduction; each of the resulting pair of equations can be integrated once, as above, but the remaining system is a fourth order system with two arbitrary constants. Remarkably, this system can actually be written as a single fourth order ODE. But so far I have been unable to establish any connection between this equation and PIII. From [5] it is clear that PIII and PVI have to be discussed in tandem with other systems, so it might not be surprising if they arose naturally embedded in some larger system; but currently the existence of a relation between the $B_2$ MKdV and PIII remains conjecture.

Another possibility for the origin of the transformation groups of PIII and PVI is that these equations might arise as scaling reductions of some other bihamiltonian integrable system in 1 + 1 dimensions (it is known that PIII and PVI arise as reductions of certain 1 + 1 dimensional systems - see [15], p.343 for references - but not as scaling reductions). It can be shown that group actions which survive scaling reduction exist on the spaces of solutions of other bihamiltonian systems. Indeed the reader can check that PIV arises as a scaling reduction of the system

$$
\begin{align*}
  j_t &= (j_x + j^2 - 2jj_x) \\
  \tilde{j}_t &= (-\tilde{j}_x - \tilde{j}^2 + 2j\tilde{j})_x
\end{align*}
$$

(21)

which is intimately related to the nonlinear Schrödinger equation (see [16]). Simple Bäcklund transformations of (21), such as $j \rightarrow \tilde{j}, \tilde{j} \rightarrow j, x \rightarrow x, t \rightarrow -t$ generate at least some of the transformations for PIV (compare [17] where some of the transformations for PIV were obtained directly from Bäcklund transformations of NLS).

As a final comment, I note that a new derivation of all the Painlevé equations has recently been given by Mason and Woodhouse, who examined certain symmetry reductions of the $SL(2)$ self-dual Yang-Mills equations [18]. It would be very interesting to see if the transformation groups had an explanation from this viewpoint as well.

Acknowledgements
I thank P.Aspinwall and C.Johnson for discussions. This work was supported by the U.S.Department of Energy under grant #DE-FG02-90ER40542.

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Appendix: Derivation of (5)

The self-dual Yang-Mills equations can be written \( F_{\bar{x}\bar{t}} = 0, F_{\bar{t}x} = F_{\bar{t}\bar{x}}, F_{xt} = 0 \), where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \) (\( \mu, \nu \in \{ x, t, \bar{x}, \bar{t} \} \)) and the \( A_\mu \) are the “potentials” i.e. Lie-algebra valued functions of \( x, t, \bar{x}, \bar{t} \). Equations (5), up to a rescaling of \( t \), are obtained from the \( SL(N) \) self-dual Yang-Mills equations with an ansatz:

\[
\begin{align*}
(A_{\bar{x}})_{ij} &= \delta_{i,j+(N-1)} \\
(A_{\bar{t}})_{ij} &= f \delta_{i,j+(N-1)} - \delta_{i,j+(N-2)} \\
(A_x)_{ij} &= j_i \delta_{i,j} + \delta_{i,j-1} \\
(A_t)_{ij} &= A_i \delta_{i,j} + B_i \delta_{i,j-1} - \delta_{i,j-2}
\end{align*}
\]

Here the \( A_i \) (\( i = 1, .., N \)), \( B_i \) (\( i = 1, .., N - 1 \)), \( j_i \) (\( i = 1, .., N \)) and \( f \) are functions of \( x, t \) alone, with \( \sum_{i=1}^{N} j_i = 0 \) and \( \sum_{i=1}^{N} A_i = 0 \).