Normal and Rectifying Curves in Pseudo-Galilean Space $G^1_3$ and Their Characterizations

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Abstract
We defined normal and rectifying curves in Pseudo-Galilean Space $G^1_3$. Also we obtained some characterizations of this curves in $G^1_3$.

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1 Introduction
In the Euclidean space $E^3$, the notion of rectifying curves was introduced by B.Y. Chen in [4]. By definition, a regular unit speed space curve $\alpha(s)$ is called a rectifying curve, if its position vector always lies its rectifying plane $\{t, b\}$, spanned by the tangent and the binormal vector field. This subject have been studied by many researcher. The curves are studied from different way in [4,5,6,7].

A Galilean space may be considered as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics. [10]

The Pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature $(0,0,+,-)$). The absolute of the Pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where $w$ is the ideal (absolute) plane, $f$ is line in $w$ and $I$ is the fixed hyperbolic involution of points of $f$. [2]. Differential geometry of the Pseudo - Galilean space $G^1_3$ has been largely developed in [1,2,3,8,9].

In the Pseudo-Galilean Space $G^1_3$, to each regular unit speed curve $r: I \rightarrow G^1_3$, $I \subset \mathbb{R}$, it is possible to associate three mutually orthogonal unit vector fields. $t, n$ and $b$, called respectively the tangent, the principal normal and the binormal vector field. The planes spanned by the vector fields $\{t, n\}$, $\{t, b\}$ and $\{n, b\}$ are defined as the osculating plane, the rectifying plane and the normal plane, respectively.
In this paper, we study the normal and rectifying curves in the Pseudo-Galilean Space $G^3_1$. By using similar method as in [4] we show that there is some characterizations of normal and rectifying curves.

2 Preliminaries

Let $r$ be a spatial curve given first by

$$r(t) = (x(t), y(t), z(t)), \quad (2.1)$$

where $x(t), y(t), z(t) \in C^3$ (the set of three-times continuously differentiable functions) and $t$ run through a real interval $[2]$.

**Definition 1** A curve $r$ given by (2.1) is admissible if

$$\dot{x}(t) \neq 0. \quad (2.2)$$

Then the curve $r$ can be given by

$$r(x) = (x, y(x), z(x)) \quad (2.3)$$

and we assume in addition that, in $[2]$

$$y''^2(x) - z''^2(x) \neq 0. \quad (2.4)$$

**Definition 2** For an admissible curve given by (2.1) the parameter of arc length is defined by

$$ds = |\dot{x}(t)dt| = |dx|. \quad (2.5)$$

For simplicity we assume $dx = ds$ and $x = s$ as the arc length of the curve $r$. From now on, we will denote the derivation by $s$ by upper prime $' [2]$. The vector $t(s) = r'(s)$ is called the tangential unit vector of an admissible curve $r$ in a point $P(s)$. Further, we define the so called osculating plane of $r$ spanned by the vectors $r'(s)$ and $r''(s)$ in the same point. The absolute point of the osculating plane is

$$H(0 : 0 : y''(s) : z''(s)). \quad (2.6)$$

We have assumed in (2.4) that $H$ is not lightlike. $H$ is a point at infinity of a line which direction vector is $r''(s)$. Then the unit vector

$$u(s) = \frac{r''(s)}{\sqrt{|y''^2(s) - z''^2(s)|}} \quad (2.7)$$

is called the principal normal vector of the curve $r$ in the point $P$.

Now the vector
\[ b(s) = \frac{(0, \varepsilon z''(s), \varepsilon y''(s))}{\sqrt{|y''^2(s) - z''^2(s)|}} \] (2.8)

is orthogonal in pseudo-Galilean sense to the osculating plane and we call it the binormal vector of the given curve in the point \( P \). Here \( \varepsilon = +1 \) or \(-1\) is chosen by the criterion \( \det(t, n, b) = 1 \). That means

\[ |y''^2(s) - z''^2(s)| = \varepsilon (y''^2(s) - z''^2(s)). \] (2.9)

By the above construction the following can be summarized [2].

**Definition 3** In each point of an admissible curve in \( G^1_3 \) the associated orthonormal (in pseudo-Galilean sense) trihedron \( \{t(s), n(s), b(s)\} \) can be defined. This trihedron is called pseudo-Galilean Frenet trihedron [2].

If a curve is parametrized by the arc length i.e. given by (2.3), then the tangent vector is non-isotropic and has the form of

\[ t(s) = r'(s) = (1, y'(s), z'(s)). \] (2.10)

Now we have

\[ t'(s) = r''(s) = (0, y''(s), z''(s)). \] (2.11)

According to the classical analogy we write (2.7) in the form

\[ r''(s) = \kappa(s)n(s), \] (2.12)

and so the curvature of an admissible curve \( r \) can be defined as follows

\[ \kappa(s) = \sqrt{|y''^2(s) - z''^2(s)|}. \] (2.13)

**Remark 4** In [2] for the pseudo-Galilean Frenet trihedron of an admissible curve \( r \) given by (2.3) the following derivative Frenet formulas are true.

\[
\begin{align*}
t'(s) & = \kappa(s)n(s) \\
n'(s) & = \tau(s)b(s) \\
b'(s) & = \tau(s)n(s)
\end{align*}
\] (2.14)

where \( t(s) \) is a spacelike, \( n(s) \) is a spacelike and \( b(s) \) is a timelike vektor, \( \kappa(s) \) is the pseudo-Galilean curvature given by (2.13) and \( \tau(s) \) is the pseudo-Galilean torsion of \( r \) defined by

\[ \tau(s) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)}. \] (2.15)
The formula (2.15) can be written as
\[ \tau(s) = \frac{\det(r'(s), r''(s), r'''(s))}{\kappa^2(s)}. \] (2.16)

3 Normal and Rectifying Curves in Pseudo-Galilean Space \( G^1_3 \).

**Definition 5** Let \( r \) be an admissible curve in 3-dimensional Pseudo-Galilean Space \( G^1_3 \). If the position vector of \( r \) always lies in its normal plane, then it is called normal curve in \( G^1_3 \).

By this definition, for a curve in \( G^1_3 \), the position vector of \( r \) satisfies
\[ r(s) = \xi(s)n(s) + \eta(s)b(s), \] (3.1)
where \( \xi(s) \) and \( \eta(s) \) are differentiable functions.

**Theorem 6** Let \( r \) be an admissible curve in \( G^1_3 \), with \( \kappa, \tau \in \mathbb{R} \). Then \( r \) is a normal curve if and only if the principal normal and binormal components of the position vector are respectively given by
\[ < r, n > = (c_1 + c_2 s)e^{-\tau s} + (c_3 + c_4 s)e^{\tau s} + \frac{\kappa}{\tau^2} \] (3.2)
and
\[ < r, b > = (c_1 + c_2 s)e^{-\tau s} - (c_3 + c_4 s)e^{\tau s} \] (3.3)
where \( c_1, c_2, c_3, c_4 \in \mathbb{R} \).

**Proof.** Let us assume that \( r \) is a normal curve in \( G^1_3 \), then from Definition 1 we have
\[ r(s) = \xi(s)n(s) + \eta(s)b(s). \] (3.4)
Differentiating this with respect to \( s \), we have
\[ r'(s) = \xi'(s)n(s) + \eta'(s)b(s) + \xi(s)n'(s) + \eta(s)b'(s). \] (3.5)
By using the Frenet equation (2.14), we write
\[ t = \xi' n + \eta' b + \xi \tau b + \eta \tau n. \] (3.6)
Again differentiating this with respect to \( s \) and by using the Frenet equation (2.14), we get
\[ \kappa n = (\xi' + \eta \tau)' + \tau(\xi \tau + \eta') n + \tau(\xi' + \eta \tau) + (\xi \tau + \eta') b \] (3.7)
From equation (3.7), we obtain the differential equation system.
\[\begin{align*}
\xi'' + 2\tau \eta' + \tau^2 \xi &= \kappa \\
\eta'' + 2\tau \xi' + \tau^2 \eta &= 0.
\end{align*}\]  
(3.8)

By solving this system, we obtain

\[\xi(s) = (c_1 + c_2 s)e^{-\tau s} + (c_3 + c_4 s)e^{\tau s} + \frac{\kappa}{\tau^2}, \quad c_1, c_2, c_3, c_4 \in \mathbb{R}\]  
(3.9)

and

\[\eta(s) = (c_1 + c_2 s)e^{-\tau s} - (c_3 + c_4 s)e^{\tau s}, \quad c_1, c_2, c_3, c_4 \in \mathbb{R}\]  
(3.10)

which completes the proof.

**Definition 7** Let \( r \) be an admissible curve in 3-dimensional Pseudo-Galilean Space \( G^1_3 \). If the position vector of \( r \) always lies in its rectifying plane, then it is called rectifying curve in \( G^1_3 \).

By this definition, for a curve in \( G^1_3 \), the position vector of \( r \) satisfies

\[r(s) = \lambda(s)t(s) + \mu(s)b(s),\]  
(3.11)

where \( \lambda(s) \) and \( \mu(s) \) are some differentiable functions.

**Theorem 8** Let \( r \) be a rectifying curve in \( G^1_3 \), with curvature \( \kappa > 0 \), \( < t, t > = 1 \), \( < n, n > = 1 \), \( < b, b > = \varepsilon \), \( \varepsilon = \pm 1 \). Then the following statements hold:

(i) The distance function \( \rho = ||r|| \) satisfies

\[\rho^2 = |< r, r >| = |s^2 + 2m_1 s + m_1^2 + \varepsilon n_1^2|\]

for some \( m_1 \in \mathbb{R}, n_1 \in \mathbb{R} - \{0\} \).

(ii) The tangential component of the position vector of \( r \) is given by \( < r, t > = s + m_1 \), where \( m_1 \in \mathbb{R} \).

(iii) The normal component \( r^N \) of the position vector of the curve has a constant length and the distance function \( \rho \) is non-constant.

(iv) The torsion \( \tau(s) \neq 0 \) and binormal component of the position vector of the curve is constant, i.e. \( < r, b > \) is constant.

**Proof.** Let us assume that \( r \) is a rectifying curve in \( G^1_3 \). Then from Definition 3, we can write the position vector of \( r \) by

\[r(s) = \lambda(s)t(s) + \mu(s)b(s),\]  
(3.12)

where \( \lambda(s) \) and \( \mu(s) \) are some differentiable functions of the invariant parameters.

(i) Differentiating the equation (3.12) with respect to \( s \) and considering the Frenet equations (2.14), we get

\[\begin{align*}
\lambda'(s) &= 1 \\
\lambda(s)\kappa(s) + \mu(s)\tau(s) &= 0 \\
\mu'(s) &= 0.
\end{align*}\]  
(3.13)
Thus, we obtain

\[
\lambda(s) = s + m_1, \quad m_1 \in \mathbb{R}
\]
\[
\mu(s) = n_1, \quad n_1 \in \mathbb{R}
\]
\[
\mu(s) \tau(s) = -\lambda(s) \kappa(s) \neq 0,
\]

and hence \( \mu(s) = n \neq 0, \quad \tau(s) \neq 0 \). From the equation (3.12), we easily find that

\[
\rho^2 = |< r, r >| = |s^2 + 2m_1s + m_1^2 + \varepsilon n_1^2|, \quad \varepsilon = \mp 1
\]

(ii) If we consider equation (3.12), we get

\[
< r, t > = \lambda(s)
\]

which means that the tangential component of the position vector of \( r \) is given by

\[
< r, t > = s + m_1, \quad m_1 \in \mathbb{R}.
\]

(iii) From the equation (3.12), it follows that the normal component \( r^N \) of the position vector \( r \) is given by

\[
r^N = \mu b.
\]

Therefore,

\[
|| r^N || = |\mu| = |n_1| \neq 0.
\]

Thus we proved statement (iii).

(iv) If we consider equation (3.12), we easily get

\[
< r, b > = \varepsilon \mu = \text{const.}, \quad \varepsilon = \mp 1
\]

and since \( \tau(s) \neq 0 \), the statement (iv) is proved.

Conversely, suppose that statement (i) or statement (ii) holds. Then we have

\[
< r, t > = s + m_1, \quad m_1 \in \mathbb{R}.
\]

Differentiating equation (3.21) with respect to \( s \), we obtain

\[
\kappa < r, n > = 0.
\]

Since \( \kappa > 0 \), it follows that

\[
< r, n > = 0
\]

which means that \( r \) is a rectifying curve.

Next, suppose that statement (iii) holds. Let us can write

\[
r(s) = l(s)t(s) + r^N, \quad l(s) \in \mathbb{R}.
\]

Then we easily obtain that

\[
< r^N, r^N >= C = \text{const.} = < r, r > - < r, t >^2.
\]
If we differentiate equation (3.25) with respect to $s$, we get

$$< r, t > = [1 + \kappa < r, n >]. \tag{3.26}$$

Since $\rho \neq \text{const.}$, we have

$$< r, t > \neq 0. \tag{3.27}$$

Moreover, since $\kappa > 0$ and from (3.26) we obtain

$$< r, n > = 0, \tag{3.28}$$

that is $r$ is rectifying curve.

Finally, if the statement (iv) holds, then from the Frenet equations (2.14), we get

$$< r, n > = 0, \tag{3.29}$$

which means that $r$ is rectifying curve.

**Theorem 9** Let $r$ be a curve in $G^1_3$. Then the curve $r$ is a rectifying curve if and only if there holds

$$\frac{\tau(s)}{\kappa(s)} = as + b \tag{3.30}$$

where $a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$.

Proof. Let us first suppose that the curve $r(s)$ is rectifying. From the equations (3.13) and (3.14) we easily find that

$$\frac{\tau(s)}{\kappa(s)} = as + b \tag{3.31}$$

where $a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$.

Conversely, let us suppose that $\frac{\tau(s)}{\kappa(s)} = as + b$, $a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$. Then we may choose

$$a = \frac{1}{n_1}, \quad b = \frac{m_1}{n_1}, \tag{3.32}$$

where $n_1 \in \mathbb{R} - \{0\}$, $m_1 \in \mathbb{R}$.

Thus we have

$$\frac{\tau(s)}{\kappa(s)} = \frac{s + m_1}{n_1}. \tag{3.33}$$

If we consider the Frenet equations (2.14), we easily find that

$$\frac{d}{ds}[r(s) - (s + m_1)t(s) - n_1b(s)] = 0 \tag{3.34}$$

which means that $r$ is a rectifying curve.

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