Maximization of the volume of zonotopes

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Abstract. We study properties of projections of the unit cube \( I^n = [0,1]^n \) onto \( k \)-dimensional subspaces. We prove that the surface area of such a projection is at least \( 2^k \) and that the minimum is attained only for projections onto coordinate subspaces. We give a first-order necessary condition for a \( k \)-dimensional subspace \( H_k \subset \mathbb{R}^n \) to be a (local) maximizer of the problem of finding the maximum possible volume of such projections. Using this condition, we show that the squared lengths \( |v|^2 \) and \( |u|^2 \) of the projections of two vectors of the standard basis onto a maximizer satisfy the inequality \( |v|^2 \geq (\sqrt{2} - 1)|u|^2 \). For these purposes, we study the properties of the exterior powers of the projection operator and give several equivalent properties (in terms of exterior algebra) for a set of vectors to be a projection of an orthonormal basis.

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1. Introduction

A zonotope is the Minkowski sum of several segments in \( \mathbb{R}^k \). Every zonotope can be represented (up to an affine transformation) as a projection of a higher-dimensional cube \( I^n = [0,1]^n \). For formal definitions and explanations, we refer the reader to the next sections. To get more information about zonotopes, we refer the reader to Zong’s book [21, Chapter 2] and to [18].

This paper deals with properties of \( k \)-dimensional subspaces \( H_k \) on which the (local) maximum of the volume of a projection of \( I^n \) is attained. Several different bounds for the maximum of the following function

\[
F(H_k) = \text{vol}(I^n|H_k),
\]

where \( H_k \) is a \( k \)-dimensional subspace of \( \mathbb{R}^n \) and \( I^n|H_k \) denotes the projection of \( I^n \) onto \( H_k \), are known. For example, G. D. Chakerian and P. Filliman [7] prove

\[
\text{vol}(I_n|H_k) \leq \sqrt{n! / (n-k)!k!} \quad \text{and} \quad \text{vol}(I_n|H_k) \leq \frac{\omega_{k-1}^k}{\omega_k^k} \left( \frac{n}{k} \right)^{k/2},
\]

where \( \omega_i \) is the volume of the \( i \)-dimensional Euclidean unit ball. The right-hand side inequality is asymptotically tight, as was shown in [7], the left-hand side inequality is tight in the cases \( k = 1,2,n-2,n-1 \). The tight upper and lower bounds in the limit case for the volume of a zonotope in a specific position (even for so called \( L_p \)-zonoids) were obtained in [14, Theorem 2].

Among different upper bounds, all maximizers of (1.1) were described in the cases \( k = 1,2,n-2,n-1 \) and \( k = 3,n = 6 \) in [7]. It appears that projections of the standard basis vectors onto \( H_k \) have the same length whenever \( H_k \) is a maximizer of (1.1) for all cases mentioned above, which gives a rather reasonable conjecture.

Conjecture 1.1. The maximum volume of a projection of the cube \( I^n = [0,1]^n \) on a subspace is attained when the projections of all edges of the cube have the same length.
We reformulate the problem (1.1) to avoid working in the Grassmannian of $k$-dimensional linear subspaces of $\mathbb{R}^n$. We will consider $I^n|H_k$ as a zonotope in $\mathbb{R}^k$ given by $\sum_{i=1}^n[0, v_i]$, where the $v_i$ are the projections of an orthonormal basis of $\mathbb{R}^n$ onto $\mathbb{R}^k$. We refer the reader to Lemma 2.3 and Section 2 for the explanation of this approach. Then, in Section 3, we explain why and how small perturbations of the vectors $\{v_1, \ldots, v_n\}$ can be treated as the small rotations of the corresponding $H_k$. This allows us to obtain in Lemma 3.3 a first-order necessary condition for $H_k$ to be a (local) maximizer in (1.1).

In fact, we separate the problem in two steps. First, we perturb the set of the initial vectors $S = \{v_1, \ldots, v_n\}$ to a new one $S' = \{v'_1, \ldots, v'_n\}$, where $v'_i = v_i + t_i x_i$ for a sufficiently small scalar $t_i$ and $x_i \in \mathbb{R}^k$. Second, we introduce in the rather obvious Theorem 2.5 an affine transformation that maps $S'$ to a new "projection" of the unit cube $I^n$ and calculate in Corollary 6.1 the gradient of the determinant of the matrix of this affine transformation as a function on $(t_1, \ldots, t_n)$ at $(0, \ldots, 0)$ for fixed vectors $x_1, \ldots, x_n$. We show in Lemma 3.5 and Lemma 3.3 that the differentiability of this determinant implies the differentiability of the volume of the zonotope $\sum_{i=1}^n[0, v'_i]$ as function of $(t_1, \ldots, t_n)$ at $(t_1, \ldots, t_n) = (0, \ldots, 0)$ for a local maximizer $S$.

Using Lemma 3.3, we give a partial answer for Conjecture 1.1 in Section 7.

**Theorem 1.2.** Let $H_k$ be such that the maximum in (1.1) is attained. Let $v_i$ be a projection of the standard basis vector $e_i$ onto $H_k$, for $i \in [n]$. Then for any $i, j \in [n]$ the inequality holds

$$|v_i|^2 \geq (\sqrt{2} - 1)|v_j|^2.$$  

As a consequence of Theorem 1.2 and McMullen’s symmetric formula ([15]), which is

$$(1.2) \quad \text{vol}(I^n|H_k) = \text{vol}(I^n|H_k^\perp),$$

where $H_k^\perp$ is the orthogonal complement of $H_k$, we prove.

**Corollary 1.3.** Fix $q \in \mathbb{N}$. Let $H_{n-q}$ be a maximizer of (1.1) for $k = n - q, n \geq q$. By $M_n$ and $m_n$ we denote the maximum and the minimum length of the projections of the standard basis’s vectors of $\mathbb{R}^n$ onto $H_{n-q}$, respectively. Then

$$\frac{m_n}{M_n} \to 1 \quad \text{as} \quad n \to \infty.$$

Moreover, we calculate in Lemma 6.1 the explicit formula for the first-order approximation of the above-mentioned determinant (and, consequently, for the volume of $\sum_{i=1}^n[0, v'_i]$ near local maximizer of (1.1)). For this purpose we use exterior algebra for studying properties of the exterior powers of the projection operators in Section 5. It seems to be the proper language for our type of problems. The relevant information on exterior algebra is included in Section 4 for completeness. Among other results on projection operators, we prove

**Corollary 1.4.** Vectors $(v_i)_1^n \subset \mathbb{R}^k$ are the projections of some orthonormal basis of $\mathbb{R}^n$ onto $\mathbb{R}^k$ if and only if all their cross products $[v_{i_1}, \ldots, v_{i_{k-1}}]$ for all $(k - 1)$-tuples $\{i_1, \ldots, i_{k-1}\} \subset ([n])_{k-1}$ are projections of some orthonormal basis of $\mathbb{R}^{(k-1)}$ onto $\mathbb{R}^k$.

As a simple application of our method, we provide a new proof of a generalization of the inequality

$$\text{vol}(I^n|H_k) \geq \text{vol} I^k$$

([5], Inequality (C))

for the surface area of zonotopes in Section 6.

**Theorem 1.5.** Let $k \geq 2$. For every $k$-dimensional subspace $H_k \subset \mathbb{R}^n$, the following inequality holds

$$\text{vol} \partial(I^n|H_k) \geq \text{vol} \partial(I^k) = 2k,$$
where \( \text{vol} \partial K \) is the surface area of a convex compact set \( K \). The equality is attained iff \( H_k \) is a coordinate subspace.

This result coincides with the assertion of Corollary 1 in [11] for \( j = n - 1 \), which is given in different notation.

**Remark 1.6.** The inequality \( \text{vol}(I^n|H_k) \geq \text{vol}I^k = 1 \) is an elementary consequence of the well-known Vaaler’s theorem \( \text{vol}([-1, 1]^n \cap H_k) \geq \text{vol}([-1, 1]^k) = 2^k \) (see [20], Corollary on the second page). In general, the trivial inequality between the volume of the projection onto \( H_k \) and the volume of the central section of \( I^n \) by the affine \( k \)-plane parallel to \( H_k \) gives the lower bounds to the maximum in (1.1), as the latter volume is well-studied, e.g. see [1], [2] and [3].

In the last Section 8, we discuss some open problems connected to our results.

### 2. Definitions and Preliminaries

We use \( I^n \) to denote an \( n \)-dimensional cube \( \{ x : 0 \leq x[i] \leq 1 \} \) in \( \mathbb{R}^n \). Here and throughout the paper \( x[i] \) stands for \( i \)-th coordinate of a vector \( x \). As usual, \( \{ e_i \}_n \) is the standard orthonormal basis of \( \mathbb{R}^n \). We use \( \langle p, x \rangle \) to denote the value of a linear functional \( p \) at a vector \( x \).

Throughout the paper \( H_k \) will be a \( k \)-dimensional subspace of \( \mathbb{R}^n \), \( P \) will be the orthogonal projection onto \( H_k \) from \( \mathbb{R}^n \). For a convex body \( K \subset \mathbb{R}^n \) and a \( k \)-dimensional subspace \( H_k \) of \( \mathbb{R}^n \) we denote by \( K \cap H_k \) and \( K|H_k \) the section of \( K \) by \( H_k \) and the orthogonal projection of \( K \) onto \( H_k \), respectively. For a \( k \)-dimensional subspace \( H_k \) of \( \mathbb{R}^n \) and a convex body \( K \subset H_k \) we denote by \( \text{vol} K \) and \( \text{vol} \partial K \) the \( k \)-dimensional volume of \( K \) and the surface area of \( K \), respectively. We consider only \( k \geq 2 \).

For a positive integer \( n \), we refer to the set \( \{ 1, 2, \ldots, n \} \) as \( [n] \). The set of all \( \ell \)-element subsets (or \( \ell \)-tuple) of a set \( M \subset [n] \) is denoted by \( \binom{M}{\ell} \).

For two \( \ell \)-tuples \( I \in \binom{[a]}{\ell} \), \( J \in \binom{[b]}{\ell} \), we will use \( M_{I,J} \) to denote the determinant of the proper minor of the \( a \times b \) matrix \( M \). For the sake of convenience, we will write \( M_I \) whenever \( I = J \). We use \( M_{i,j} \) to denote the entry in the \( i \)-th row and the \( j \)-th column of a matrix \( M \).

Recall that a **zonotope** in a vector space is a Minkowski sum of several line segments, i.e. \( \sum_{i=1}^n [a_i, b_i] \), where \( \sum \) stands for the Minkowski sum in a vector space and \( [a, b] \) means the line segment between points \( a \) and \( b \). Since \( \sum_{i=1}^n [a_i, b_i] = \sum_{i=1}^n [0, b_i-a_i] + \sum_{i=1}^n a_i \), we always put \( a_i = 0 \), for \( i \in [n] \), i.e. we consider only zonotopes \( \sum_{i=1}^n [0, b_i] \).

**Definition 2.1.** We will say that the set \( S = \{ w_1, \ldots, w_n \} \) of \( n \) vectors of \( \mathbb{R}^k \) generates

1. a zonotope \( \sum_{i \in [n]} [0, w_i] \). We use \( I^n|S \) to denote this zonotope.
2. an operator \( \sum_{i \in [n]} w_i \otimes w_i \). We use \( A_S \) to denote this operator and the matrix of this operator in the standard basis.
3. a subspace \( H_k^S \subset \mathbb{R}^n \) which is the linear hull of the rows of the \( k \times n \) matrix \( (w_1, \ldots, w_n) \).
4. a projection operator \( P^S \) from \( \mathbb{R}^n \) onto \( H_k^S \). We use the same notation \( P^S \) for the matrix of this operator in the standard basis.

First of all, it is convenient to put (by an affine transformation) a zonotope in a “standard” position. It is known that any zonotope \( \sum_{i=1}^n [0, b_i] \subset \mathbb{R}^k \) with a non-empty interior is an affine image of the projection of the unit cube \( I^n = \sum_{i=1}^n [0, e_i] \) onto a proper \( k \)-dimensional subspace.

Let \( v_i = P e_i \). Since \( I_n = \sum_{i=1}^n [0, e_i] \), we have that

\[
(2.1) \quad I^n|H_k = P \left( \sum_{i=1}^n [0, e_i] \right) = \sum_{i=1}^n [0, v_i] = \sum_{i=1}^n [0, v_i].
\]
This means that a projection of $I^n$ is determined by the set of $n$ vectors $(v_i)_i^n$, which are the projections of the orthonormal basis. To deal with those vectors we introduce several definitions following [12].

**Definition 2.2.** We will say that some vectors $(w_i)_i^n \subset H$ give a unit decomposition in a vector space $H$ if

$$(2.2) \quad \left( \sum_{i=1}^n w_i \otimes w_i \right)_{|H} = I_H,$$

where $I_H$ is the identity operator in $H$ and $A|_H$ is the restriction of an operator $A$ onto $H$.

In the following Lemma we understand $\mathbb{R}^k \subset \mathbb{R}^n$ as the subspace of vectors, whose last $n-k$ coordinates are zero. For convenience, we will consider $(w_i)_i^n \subset \mathbb{R}^k \subset \mathbb{R}^n$ to be $k$-dimensional vectors.

**Lemma 2.3.** The following assertions are equivalent:

1. the vectors $(w_i)_i^n \subset \mathbb{R}^k$ give a unit decomposition in $\mathbb{R}^k$;
2. there exists an orthonormal basis $\{f_1, \ldots, f_n\}$ of $\mathbb{R}^n$ such that $w_i$ is the orthogonal projection of $f_i$ onto $\mathbb{R}^k$, for any $i \in [n]$;
3. $\text{Lin}\{w_1, \ldots, w_n\} = \mathbb{R}^k$ and the Gram matrix $\Gamma$ of vectors $\{w_1, \ldots, w_n\} \subset \mathbb{R}^k$ is the matrix of a projection operator from $\mathbb{R}^n$ onto the linear hull of the rows the matrix $M = (w_1, \ldots, w_n)$.
4. the $k \times n$ matrix $M = (w_1, \ldots, w_n)$ is a sub-matrix of an orthogonal matrix of order $n$.

This Lemma was proven in [12, Lemma 2.1].

**Definition 2.4.** A set of $n$ vectors in $\mathbb{R}^k$ which span $\mathbb{R}^k$ is called an $(n,k)$-frame. If the vectors of an $(n,k)$-frame $S$ give a unit decomposition, we say that $S$ is an $(n,k)$-uframe. We use $\Omega(n,k)$ to denote the set of all $(n,k)$-uframes.

The simple Lemma 2.3 from linear algebra allows us to identify vectors $(v_i)_i^n \subset H_k$ with a set of vectors in $\mathbb{R}^k$, which give a unit decomposition. To be more precise, there is an isometric isomorphism between any $k$-dimensional subspace $H_k \subset \mathbb{R}^n$ and $\mathbb{R}^k$. Thus, the projections $I^n|H_k$ correspond to $I^n|S$, where $S$ is the $(n,k)$-uframe corresponding to this isometry. Vice versa, given an $(n,k)$-uframe $S = \{w_1, \ldots, w_n\}$ the assertion (3) of Lemma 2.3 gives the proper $H_k$, and this is exactly $H_k^S$ defined in Definition 2.1 (it is a $k$-dimensional space in case $S$ is an $(n,k)$-frame).

Summarizing these observations, we have that the problem to find and to study maximizers of

$$(2.3) \quad F(S) = \text{vol}(I^n|S), \quad \text{where} \quad S \in \Omega(n,k)$$

is equivalent to that of (1.1).

As will be shown below, it is much more convenient (in some sense) to speak about a transformation of the set of vectors $(v_i)_i^n \subset \mathbb{R}^k$ and to write the first-order necessary condition of the maximum in problem (1.1) in this setup, than to consider $H_k$ as a point in the Grassmannian of $k$-dimensional subspaces of $\mathbb{R}^n$.

For an $(n,k)$-frame $S = \{w_1, \ldots, w_n\}$, by definition put

$$B_S = A_S^{-\frac{1}{2}}.$$

The operator $B_S$ is well-defined as the condition $\text{Lin} \ S = \mathbb{R}^k$ implies that $A_S$ is a strictly positive operator.

Clearly, $B_S$ maps any $(n,k)$-frame $S$ to the $(n,k)$-uframe. This means that $B_S$ transforms any zonotope $I^n|S$ to a zonotope generated by an $(n,k)$-uframe, which is the image of the orthogonal transformation of $I^n|H_k$ for a proper $k$-dimensional $H_k$. As this simple observation plays a crucial role in our studies, we provide a proof.
**Theorem 2.5.** The \((n, k)\)-frame \(B_S S = \{B_S v_1, \ldots, B_S v_n\}\) is an \((n, k)\)-uframe for any \((n, k)\)-frame \(S\). Also, the determinant of \(A_S\) coincides with the determinant of the Gram matrix of the vectors of \(S\) whenever \(n = k\).

**Proof.**
The first claim is trivial
\[
\sum_{i=1}^{n} B_S v_i \otimes B_S v_i = B_S \left( \sum_{i=1}^{n} v_i \otimes v_i \right) B_S^T = B_S A_S B_S = I_k.
\]

The second claim is the straightforward consequence of the first one. Indeed, the linear transformation by the matrix \(M\), which is the inverse to the matrix \((v_1, \ldots, v_k)\), maps vectors of \(S\) to the vectors of standard basis in \(\mathbb{R}^k\), which give a unit decomposition in \(\mathbb{R}^k\). This implies \(\det(v_1, \ldots, v_k) = 1/\det B\) and, therefore, \(\det A_S = (\det(v_1, \cdots, v_k))^2\). The latter is exactly the determinant of the Gram matrix of \(\{v_1, \cdots, v_k\}\).

\(\Box\)

Our approach in choice of the standard position for zonotopes is not new. Recall that a finite Borel measure \(\mu\) on the sphere \(\mathbb{S}^{k-1}\) is called *isotropic* if
\[
\int_{\mathbb{S}^{k-1}} ((v,a))^2 \mu(dv) = |a|^2 \text{ for any } a \in \mathbb{R}^k.
\]

Directly from the definition, \(S = \{v_1, \ldots, v_n\}\) is an \((n, k)\)-uframe iff the discrete measure
\[
\mu(x) = \sum_{|v_i| \neq 0} |v_i|^2 \delta_{v_i}(x)
\]
is an isotropic measure. The notion of an isotropic measures is often used in various problems in asymptotic convex analysis (see [4] for more details) and, particularly, in problems on zonotopes, e.g. in [11].

3. Perturbations of the frames

Lemma 2.3 allows us to describe all substitutions \(w_i \to w'_i\), \(i \in [n]\), which preserves the operator \(A_S\) for an \((n, k)\)-frame \(S = \{w_1, \ldots, w_n\}\). But we need a more suitable geometric description. So, let \(S = \{w_1, \ldots, w_n\}\) be an \((n, k)\)-frame and \(\ell \in [k], I = \{i_1, \ldots, i_\ell\} \subseteq \binom{[n]}{\ell}\).

Consider a substitution \(w_i \to w'_i\), for \(i \in I\), and \(w'_i = w_i\), for \(i \notin I\), which preserves \(A_S\), and denote by \(S' = \{w'_1, \ldots, w'_n\}\) the new \((n, k)\)-frame.

In this notation, we get.

**Lemma 3.1.** The substitution preserves \(A_S\) (i.e. \(A_S = A_{S'}\)) iff there exists an orthogonal matrix \(U\) of rank \(\ell\) such that
\[
(w'_1, \ldots, w'_n) = (w_{i_1}, \ldots, w_{i_\ell}) U.
\]

Additionally, in case \(S\) is an \((n, k)\)-uframe, let \(\{f_1, \ldots, f_n\}\) be any orthonormal basis of \(\mathbb{R}^n\) given by the assertion (2) of Lemma 2.3. Then the substitution preserves \(A_S\) iff the vectors of \(S'\) are the projection of an orthonormal basis \(\{f'_1, \ldots, f'_n\}\) of \(\mathbb{R}^n\), which is obtained from \(\{f_1, \ldots, f_n\}\) by an orthogonal transformation of \(\{f_i\}_{i \in I}\) in their linear hull.

**Proof.**
The identity \(A_{S'} = A_S\) holds if and only if
\[
\sum_{i \in I} w_i \otimes w_i = \sum_{i \in I} w'_i \otimes w'_i.
\]
Writing this in a matrix form, we get another equivalent statement that the Gram matrices of the rows of the matrices \((w'_1, \ldots, w'_n)\) and \((w_{i_1}, \ldots, w_{i_\ell})\) are the same. The latter
is equivalent to the existence of an isometry of $\mathbb{R}^t$, which maps the rows of the first matrix to the rows of the second. This isometry defines an orthogonal matrix $U$, which satisfies the assumptions of the lemma.

In case of an $(n, k)$-uframe, extending $U$ as the orthogonal transformation of $\text{Lin}(f_i)_{i \in I}$, we obtain the proper $\{f'_1, \ldots, f'_n\}$.

\[ \square \]

As well as $A_S$, this substitution preserves $\sum_{i \in I} w_i \otimes w_i$, and, therefore, it preserves $A_Q$ and $\det A_Q$ for any set of indices $Q \supset I$. In particular, by Theorem 2.5, $\det A_Q$ is the determinant of the Gram matrix of the corresponding vectors whenever $|Q| = k$. This means that our substitution preserves the absolute value of the determinant of the $k$ vectors with indices from such $Q$.

Also, we can reformulate the second claim of Lemma 3.1 in the following equivalent way: Given a block-matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that the rows of $(A \ B)$ are orthonormal and the columns of $(A \ C)$ are orthonormal, then $D$ can be chosen in such a way that $M$ will be an orthogonal matrix.

Again, we can transform any $(n, k)$-frame $S$ to an $(n, k)$-uframe using $B_S$. This implies that we can write a necessary and sufficient condition for the problem (2.3).

**Lemma 3.2.** The maximum in (2.3) is attained on an $(n, k)$-uframe $S = \{v_1, \ldots, v_n\}$ iff the following inequality holds for an arbitrary $(n, k)$-frame $S'$

\begin{equation}
\frac{\text{vol}(I^n | S')}{\text{vol}(I^n | S)} \leq \sqrt{\det A_S'}.
\end{equation}

**Proof.**

As mentioned above, $B_S S'$ is an $(n, k)$-uframe, and clearly, $\text{vol}(I^n | B_S S') = \det B_S \text{ vol}(I^n | S')$. The $(n, k)$-uframe $S$ is a maximizer iff $\text{vol}(I^n | B_S S') \leq \text{vol}(I^n | S)$ for an arbitrary $(n, k)$-frame $S'$. Using these observations and the definition of $B_S$, we have

\[ \frac{\text{vol}(I^n | S')}{\text{vol}(I^n | S)} = \frac{1}{\det B_S'} \frac{\text{vol}(I^n | B_S S')}{\text{vol}(I^n | S)} \leq \frac{1}{\det B_S'} = \sqrt{\det A_S'}. \]

\[ \square \]

A few words about local extrema of (1.1) and of (2.3). A nice geometric way of endowing a Grassmann manifold with a metric is to use a largest principal angle between two $k$-dimensional subspaces (and this is exactly the Hausdorff distance between the unit circles in these subspaces). Also, we may consider the natural metric on the Grassmanian of $k$-dimensional subspaces of $\mathbb{R}^n$ (the Grassmanian is a homogeneous manifold). As shown in Neretin’s paper [16], this metric is the square root of the sum of squares of all principal angles. Therefore, the two metrics are equivalent. We can consider a set of all $(n, k)$-frames as a open set of $\mathbb{R}^{nk}$ equipped with the metric of $\mathbb{R}^{nk}$. Clearly, $\det B_S$ is a continuous function of $S$ in this metric. Hence, $B_S S$ is a continuous function on the set of all $(n, k)$-frames, and one can see that a small enough neighbourhood in $\Omega(n, k)$ of an $(n, k)$-uframe $S$ is the image of the proper neighbourhood of $S$ in the set of all $(n, k)$-frames under the mapping $B_S$. Any element $H_k$ of the Grassmanian can be obtained as a proper $H^S_k$ (not in a unique way), and the Hausdorff distance depends smoothly on $S$. Combining this with the previous observation, we get that a small enough neighbourhood of an $(n, k)$-uframe $S$ is homeomorphic to the proper neighbourhood of $H^S_k$ in the Grassmanian, and the latter covers all of the Grassmanian. But this means that the local extrema are the same for the both problems.

Developing a first-order condition for the problem (2.3), we prove in Section 6.
Lemma 3.3. Let $H_k$ be such that the a local maximum in (1.1) is attained. Let $v_i$ be a projection of the vector $e_i$ onto $H_k$, for $i \in [n]$. Then for any vectors $\{x_1, \ldots, x_n\} \subset H_k$ and scalars $\{t_1, \ldots, t_n\} \in \mathbb{R}$, the function $f(t_1, \ldots, t_n) = \text{vol} \left( \sum_{i=1}^{n} [0, v_i + t_i x_i] \right)$ is differentiable at the origin and the following identity holds

$$\frac{f(t_1, \ldots, t_n)}{\text{vol}(I^n|H_k)} = 1 + \sum_{i=1}^{n} t_i \langle x_i, v_i \rangle + o \left( \sqrt{t_1^2 + \cdots + t_n^2} \right).$$

In order to get the first-order necessary condition for (2.3), we will slightly move the vectors $v_i$ of the local maximizer $S = \{v_1, \ldots, v_n\}$. More precisely, let $S'$ be an $(n, k)$-frame obtained from $S$ by substitution $v_i \rightarrow v_i + tx$, where $t \in \mathbb{R}, x \in \mathbb{R}^k$. We consider det $A_{S'}$ and $\text{vol}(I^n|S')$ as a function of $t$ for fixed $S, x, i$. Then, in case of differentiability, we can write $\text{vol}(I^n|S') = \text{vol}(I^n|S) + tK_1(v_i, x) + o(t)$; det $A_{S'} = \det A_S + tK_2(v_i, x) + o(t)$, where $K_1(v_i, x), K_2(v_i, x)$ are some coefficients depending only on $v_i \in S$ and the vector $x$. It is not quite clear why the volume of $I^n|S'$ (or of $I^n|B_S S'$) as a function of $t$ is a differentiable function at $t = 0$, but we show that this is the case for a local maximizer of (2.3) (Moreover, the volume of $I^n|S'$ as a function of $t$ is not a differentiable function at $t = 0$ for an arbitrary $(n, k)$-uframe $S$).

The rather obvious Lemma 3.2 helps us to deal with differentiability. It separates an algebraic question of the differentiability of the determinant of $A_{S'}$ (this is quite clear as $A_S = I_k$) from a geometric question of the differentiability of the volume $I^n|S'$ at $t = 0$. By the above arguments, the inequality in Lemma 3.2 is a necessary condition of a local maximum for (2.3).

Despite this, we will use an explicit algebraic formula (4.1) for the volume of $I^n|S'$. The main idea in the proof of Lemma 3.3 is the following.

1) We show that $\sqrt{\det A_{S'}}$ is a smooth function in some neighbourhood of $t = 0$. After some exterior algebra preliminaries, we prove the following in section 6.

Lemma 3.4. Let $S$ be an $(n, k)$-uframe and $S'$ be an $(n, k)$-frame obtained from $S$ by substitution $v_i \rightarrow v_i + tx$, where $t \in \mathbb{R}, x \in \mathbb{R}^k$. Then

$$\sqrt{\det A_{S'}} = 1 + t \langle v_i, x \rangle + o(t).$$

That is, $\sqrt{\det A_{S'}}$ is a differentiable function of $t$ at $t = 0$, and the derivative equals $\langle v_i, x \rangle$.

2) We show that the volume of $I^n|S'$ is a convex function on $t$. This, the smoothness of $\sqrt{\det A_{S'}}$, and Lemma 3.2 imply that $\text{vol}(I^n|S')$ as a function of $t$ must be smooth at $t = 0$. In section 6, we prove.

Lemma 3.5. Let $S = \{v_1, \ldots, v_n\}$ be a (local) maximizer of (2.3) and $S'$ be an $(n, k)$-frame obtained from $S$ by substitution $v_i \rightarrow v_i + tx$, where $t \in \mathbb{R}, x \in \mathbb{R}^k$. Then

$$\frac{\text{vol}(I^n|S')}{\text{vol}(I^n|S)} = 1 + t \langle v_i, x \rangle + o(t).$$

That is, $\frac{\text{vol}(I^n|S')}{\text{vol}(I^n|S)}$ is a differentiable function of $t$ at $t = 0$ for a (local) maximizer and the derivative equals $\langle v_i, x \rangle$.

Remark 3.6. The same approach works in different extremal problems on volumes of projections or sections of convex bodies. For example, in a problem of finding the maximal volume of the projection of the $n$-dimensional cross-polytope (i.e. $\text{co}\{\pm e_1, \ldots, \pm e_n\}$) onto a $k$-dimensional subspace or in a problem of finding the minimal volume of the cross-section of the $n$-dimensional cube. The only problem is to show that the corresponding volume is a convex function of $t$ (the volume of the body "associated" with $S'$ in our notation). In the case of the maximal volume of the projections of the cross-polytope, it is quite obvious. Since in this case, the volume is the sum of the volumes of the simplices of some subdivision, and the volume of simplex is a convex function of a shift of one of its vertices.
Also, one can notice that $\text{tr} A_S = \sum_{i=1}^{n} |v_i|^2$ for any $(n,k)$-frame $S$. Hence $\text{tr} A_S = k$ for an $(n,k)$-uframe $S$. Let $\Omega(n,k)$ denote a class of $(n,k)$-frames $S = \{v_1, \ldots, v_n\}$ such that $\sum_{i=1}^{n} |v_i|^2 = k (= \text{tr} A_S)$. The same arguments as in Lemma 3.2 imply.

**Corollary 3.7.**

$$\max_{S \in \Omega'(n,k)} \text{vol}(I^n|S) = \max_{S \in \Omega(n,k)} \text{vol}(I^n|S).$$

*That is, in problem (2.3) it is enough to fix the sum of squared length of vectors.*

**Proof.**

Since $\Omega(n,k) \subset \Omega'(n,k)$, it is enough to show that for any $S' \in \Omega'(n,k), S' \notin \Omega(n,k)$ there exists an $(n,k)$-uframe $S$ such that $\text{vol}(I^n|S') < \text{vol}(I^n|S)$. We put $S = B_{S'}$. Then, by Lemma 3.2, it is enough to prove that $\det A_{S'} < 1$ in our case.

Considering $A_{S'}$ in the basis of its eigenvectors (in which it is a diagonal operator) and using the inequality of arithmetic and geometric means, we obtain

$$\det A_{S'} \leq \left( \frac{\text{tr} A_{S'}}{k} \right)^k = 1,$$

where equality is attained iff all eigenvalues of $A_{S'}$ are ones. This means, the equality is attained iff $S' \in \Omega(n,k)$. This completes the proof. □

**Remark 3.8.** Corollary 3.7 was proven in [7] (see Theorem 1) with the same idea but different notation. Also, the same argument works for the volumes of the projections and sections mentioned in Remark 3.6.

### 4. Exterior algebra

In [19], the following Shepard’s formula for the volume of a zonotope $\sum_{i=1}^{n} [0, b_i] \subset \mathbb{R}^k$ was proven

$$(4.1) \quad \text{vol} \left( \sum_{i=1}^{n} [0, b_i] \right) = \sum_{\{i_1, \ldots, i_k\} \subseteq \binom{[n]}{k}} |\text{det}\{b_{i_1}, \ldots, b_{i_k}\}|,$$

which gives a nice tool to work with the volume of zonotopes.

To deal with the determinants in (4.1) or in Lemma 3.4 we need machinery from exterior algebra. For this purpose we need to recall some definitions from exterior algebra (for more details see [9]).

Let $H$ be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. We define the vector space $\Lambda^\ell(H)$ as the space of the multilinear skew-symmetric functions on $\ell$ vectors of $H$ with the natural linear structure. The vectors of $\Lambda^\ell(H)$ are called $\ell$-forms. As we consider only linear spaces with inner product, we assume that a space $H$ and its dual $H^*$ coincide. This allows us to simplify the following definitions.

**Remark 4.1.** We consider only vector spaces with a fixed inner product, and, by this, we identify a space and its dual, and the spaces of $\ell$-vectors and $\ell$-forms. For the sake of completeness and clarity, all needed definitions from exterior algebra are given here in a brief and non-canonical way. We assume that the equivalence of our definitions to the usual one is quite obvious. As a proper introduction to multilinear algebra we refer the reader to Greub’s book [9].

For $\ell$ vectors $\{x_1, \ldots, x_\ell\} \subset H$ we define an $\ell$-form $x_1 \wedge \cdots \wedge x_\ell$ by its evaluation on vectors $\{y_1, \ldots, y_\ell\} \subset H$ given by

$$x_1 \wedge \cdots \wedge x_\ell(y_1, \ldots, y_\ell) = \text{det} M, \quad \text{where} \quad M_{ij} = \langle x_i, y_j \rangle, \quad i, j \in [\ell].$$
By the properties of the determinant \( x_1 \wedge \cdots \wedge x_\ell \) is a multilinear skew-symmetric function on \( \ell \) vectors of \( H \). For the sake of convenience, given \( n \) vectors \( (x_i)_1^n \) we denote by \( x_L \) the \( \ell \)-form \( x_{i_1} \wedge \cdots \wedge x_{i_\ell} \) for \( \ell \)-tuple \( L = \{i_1, \ldots, i_\ell \} \in ([n]_\ell) \).

As usual, we choose the set of \( \ell \)-forms \( e_{i_1} \wedge \cdots \wedge e_{i_\ell} \), where \( \{i_1, \ldots, i_\ell \} \in ([n]_\ell) \), to be a basis of \( \Lambda^\ell (\mathbb{R}^n) \). We use the lexicographical order on \( \ell \)-tuples \( \{i_1, \ldots, i_\ell \} \in ([n]_\ell) \) to denote a number of \( e_{i_1} \wedge \cdots \wedge e_{i_\ell} \) in this basis.

Recall that for a vector space \( H \), an \( \ell \)-form \( w \in \Lambda^\ell (H) \) is said to be decomposable if it can be represented in the form \( x_1 \wedge \cdots \wedge x_\ell \) for \( (x_i)_1^n \subset H \). The main point is that every \( \ell \)-form is a linear combination of some decomposable \( \ell \)-forms (e.g. of the forms from the standard basis), but not all \( \ell \)-forms are decomposable, e.g. \( e_1 \wedge e_2 + e_3 \wedge e_4 \in \Lambda^2 (\mathbb{R}^4) \). A line \( l \subset \Lambda^\ell (\mathbb{R}^n) \) has a decomposable directional vector iff \( l \) is "generated" by some \( \ell \)-dimensional subspace \( H_\ell \subset \mathbb{R}^n \).

Since the standard basis \( \{e_I, I \in ([n]_\ell)\} \) of \( \Lambda^\ell (\mathbb{R}^n) \) consists of decomposable vectors, it is enough to define a linear operator (or the inner product) on \( \Lambda^\ell (\mathbb{R}^n) \) only for decomposable \( \ell \)-forms, by linearity.

Recall that the exterior \( \ell \)-power of an operator \( A \) on \( \mathbb{R}^n \) is a linear operator on \( \Lambda^\ell (\mathbb{R}^n) \), which is defined on decomposable forms by

\[
\wedge^\ell A (x_1 \wedge \cdots \wedge x_\ell) = Ax_1 \wedge \cdots \wedge Ax_\ell.
\]

Recall that the inner product on two decomposable \( \ell \)-forms \( a = a_1 \wedge \cdots \wedge a_\ell \) and \( b = b_1 \wedge \cdots \wedge b_\ell \) is defined by

\[
\langle a, b \rangle = \langle a_1 \wedge \cdots \wedge a_\ell, b_1 \wedge \cdots \wedge b_\ell \rangle = a_1 \wedge \cdots \wedge a_\ell (b_1, \ldots, b_\ell).
\]

In this way, once we have fixed the inner product on \( \mathbb{R}^n \), we fix the inner product on \( \Lambda^\ell (\mathbb{R}^n) \). Then for \( \ell \in [n] \) we can define a special isometry, the so called Hodge star operator; \( \star : \Lambda^\ell (\mathbb{R}^n) \to \Lambda^{n-\ell} (\mathbb{R}^n) \) by the equation

\[
a \wedge \star (b) = \langle a, b \rangle e_1 \wedge \cdots \wedge e_n,
\]

where \( a, b \) are \( \ell \)-forms.

Using this one can show the following. For a given \( a = a_1 \wedge \cdots \wedge a_\ell \in \Lambda^\ell (\mathbb{R}^n) \) and \( b = b_1 \wedge \cdots \wedge b_{n-\ell} \in \Lambda^{n-\ell} (\mathbb{R}^n) \), we get

\[
\langle a, \star (b) \rangle = (-1)^\ell \det \{a_1, \ldots, a_\ell, b_1, \ldots, b_{n-\ell}\}.
\]

We use \( \wedge^\ell H_k \) to denote the linear hull of forms \( x_1 \wedge \cdots \wedge x_\ell \) in \( \Lambda^\ell (\mathbb{R}^n) \) such that \( \{x_1, \ldots, x_\ell\} \subset H_k \) for a \( k \)-dimensional subspace \( H_k \subset \mathbb{R}^n \). Since \( H_k \) inherits the inner product on \( \mathbb{R}^n \), the space \( \wedge^\ell H_k \) inherits the inner product of \( \Lambda^\ell (\mathbb{R}^n) \), and thus can be identified with the space \( \Lambda^\ell (H_k) \) in the tautological way. This allows us to use the Hodge star operator for the spaces \( \Lambda^\ell (H_k) \) and \( \Lambda^{k-\ell} (H_k) \) for \( \ell \in [k] \). Also, the identity (4.2) can be rewritten in the following form. For a given \( a = a_1 \wedge \cdots \wedge a_\ell \in \Lambda^\ell (H_k) \) and \( b = b_1 \wedge \cdots \wedge b_{k-\ell} \in \Lambda^{k-\ell} (H_k) \), we get

\[
\langle a, \star (b) \rangle = (-1)^\ell \det \{a_1, \ldots, a_\ell, b_1, \ldots, b_{k-\ell}\}.
\]

where we understand the determinant as the determinant of \( k \)-vectors in a \( k \)-dimensional space \( H_k \).

The next Lemma is a straightforward consequence of the definitions. We show that the \( \wedge \) product and \( \otimes \) product commute for the exterior powers of operators in our case. Usually, definitions of the outer product involve some kind of (anti-)symmetrization, which can give an additional constant when we exchange the \( \wedge \) product and \( \otimes \) product. For example, we get \(-1\) to the proper power for the skew tensor product of graded algebras \( \Lambda (E^*) \) and \( \Lambda (E) \) (see [9], Section 6.17) in the general case. To avoid misunderstandings we prefer to give a proof of the following statement.
Lemma 4.2. Let $A = \sum_{i=1}^{t} v_i \otimes v_i$ for some integer $t$ and vectors $(v_i)_1^{t} \subset \mathbb{R}^n$. Then 
\[ \Lambda^\ell A = \sum_{L \in \binom{[n]}{\ell}} v_L \otimes v_L. \]

Proof.
We can assume that $\ell \leq \dim \text{Lin}\{v_1, \ldots, v_t\} \leq t$, otherwise $\Lambda^\ell A = 0 = \sum_{L \in \binom{[n]}{\ell}} v_L \otimes v_L$. It suffices to show the identity only for the decomposable $\ell$-forms. Fix $\{x_1, \ldots, x_\ell\} \subset \mathbb{R}^n$. We have

\[ \Lambda^\ell A(x_1 \wedge \cdots \wedge x_\ell) = Ax_1 \wedge \cdots \wedge Ax_\ell = \sum_{i=1}^{t} (v_i, x_1) v_i \wedge \cdots \wedge \sum_{i=1}^{t} (v_i, x_\ell) v_i. \]

By linearity and by skew-symmetry, we expand the last identity and the coefficient at $v_L = v_i \wedge \cdots \wedge v_i$ is

\[ \sum_{\sigma} \text{sgn} \sigma \prod_{j=1}^{\ell} (v_{\sigma(j)}, x_{\sigma(j)}), \]

where is the summation is taken over all permutation of $[\ell]$. This is the determinant of the matrix $M_{ij} = \langle v_i, x_j \rangle$, where $i, j \in [\ell]$. By the definition of the inner product, this determinant is just $\langle v_L, (x_1 \wedge \cdots \wedge x_\ell) \rangle$, i.e.,

\[ \Lambda^\ell A(x_i \wedge \cdots \wedge x_i) = \sum_{L \in \binom{[n]}{\ell}} \langle v_L, (x_1 \wedge \cdots \wedge x_\ell) \rangle v_L = \sum_{L \in \binom{[n]}{\ell}} v_L \otimes v_L(x_1 \wedge \cdots \wedge x_\ell). \]

This completes the proof. □

Considering $\Lambda^{k-1} H_k$ to be $\Lambda^{k-1}(H_k)$, we can identify $H_k$ with the $\wedge^{k-1} H_k$ by the Hodge star operator. Then we can define the cross product of $k - 1$ vectors $\{x_1, \ldots, x_{k-1}\}$ by

\[ [x_1, \ldots, x_{k-1}] = \star (x_1 \wedge \cdots \wedge x_{k-1}). \]

Or, in other words, by linearity of the determinant, the cross product $x = [x_1, \ldots, x_{k-1}]$ of $k - 1$ vectors $\{x_1, \ldots, x_{k-1}\}$ in a $k$-dimensional space $H_k$ with the fixed inner product $\langle \cdot, \cdot \rangle$ is the vector defined by

\[ \langle x, y \rangle = \det(x_1, \ldots, x_{k-1}, y) \quad \text{for all} \quad y \in H_k. \]

5. Properties of the exterior power of a projection operator

The main observation is that $P$ is actually the Gram matrix of vectors $\{v_1, \ldots, v_n\} \subset H_k$. Since $\langle Pe_i, e_j \rangle = \langle P^2 e_i, e_j \rangle = \langle Pe_i, Pe_j \rangle = \langle v_i, v_j \rangle$, we have that for a fixed $\ell \in [k]$ and $\ell$-tuple $L \subset \binom{[n]}{\ell}$ the proper sub-matrix of $P$ is the Gram matrix of vectors $(v_i)_{i \in L}$ and the determinant of this Gram matrix is $P_L$. It is well known that for $\ell$ vectors $(w_i)_1^\ell \subset \mathbb{R}^\ell$ their squared determinant is equal to the determinant of their Gram matrix. Thus, we can rewrite formula (4.1) for the set $I^n|H_k$ in the following way

\[ \text{vol}(I^n|H_k) = \sum_{I \in \binom{[n]}{\ell}} \sqrt{P_I}. \]

By the same arguments, if $\text{Lin}\{v_i | i \in M\} = d$ for a set $M \subset [n]$, then we have

\[ \text{vol}\left( \sum_{i \in M} [0, v_i] \right) = \sum_{I \in \binom{M}{d}} \sqrt{P_I}. \]

(5.1)
Lemma 5.1. Let $P$ be the orthogonal projection from $\mathbb{R}^n$ onto a $k$-dimensional subspace $H_k$. Then for $\wedge^\ell P$, where $1 \leq \ell \leq k$, we have

1. $\wedge^\ell P$ is an $\binom{n}{\ell} \times \binom{n}{\ell}$ matrix such that

\begin{equation}
\wedge^\ell \begin{bmatrix} P_{I,J} \end{bmatrix} = P_{I,J} \quad \text{for} \quad I, J \in \left( \binom{[n]}{\ell} \right).
\end{equation}

2. $\wedge^\ell P : \Lambda^\ell (\mathbb{R}^n) \to \Lambda^\ell (\mathbb{R}^n)$ is the orthogonal projection onto $\wedge^\ell H_k$.

Proof.

1) By definition $\wedge^\ell P$ is an $\binom{n}{\ell} \times \binom{n}{\ell}$ matrix. There is nothing to prove. The identity (5.2) is the consequence of the definition of the scalar product in $\Lambda^\ell (\mathbb{R}^n)$.

2) By the definitions, for any decomposable $\ell$-form $x_1 \wedge \cdots \wedge x_\ell$, we have

$$
\wedge^\ell P(x_1 \wedge \cdots \wedge x_\ell) = P x_1 \wedge \cdots \wedge P x_\ell \in \wedge^\ell H_k.
$$

By linearity, we have that $\wedge^\ell P x \in \wedge^\ell H_k$ for an arbitrary $\ell$-form $x$. For $x \in \wedge^\ell H_k$, we see that $\wedge^\ell P x = x$. Thus, we showed that $\text{Im} \; \wedge^\ell P = \wedge^\ell H_k$ and $(\wedge^\ell P)^2 = \wedge^\ell P$. This completes the proof. \hfill \square

In the following Lemma, we understand $\mathbb{R}^k \subset \mathbb{R}^n$ as the subspace of vectors, whose last $n-k$ coordinates are zero. This embedding generates natural embeddings $\Lambda^\ell (\mathbb{R}^k) \subset \Lambda^\ell (\mathbb{R}^n)$.

Theorem 5.2. The following assertions are equivalent for $\{v_1, \ldots, v_n\} \subset \mathbb{R}^k$:

1. there exists an orthonormal basis $\{f_1, \ldots, f_n\}$ of $\mathbb{R}^n$ such that $v_i$ is the orthogonal projection of $f_i$ onto $\mathbb{R}^k$, for any $i \in [n]$;

2. $P_k = \sum i v_i \otimes v_i$, where $P_k$ is the projector from $\mathbb{R}^n$ onto $\mathbb{R}^k$

3. for a fixed $\ell \in [k-1]$ there exists an orthonormal basis $\{f_L\}$ of $\Lambda^\ell (\mathbb{R}^n)$ such that $v_L$ is the orthogonal projection of $f_L$ onto $\Lambda^\ell (\mathbb{R}^k)$, for all $L \in \binom{[n]}{\ell}$.

4. for a fixed $\ell \in [k-1]$ the following identity is true

$$
\Lambda^\ell P_k = \sum_{L \in \binom{[n]}{\ell}} v_L \otimes v_L
$$

where $\Lambda^\ell P_k$ is the projector from $\Lambda^\ell (\mathbb{R}^n)$ onto $\Lambda^\ell (\mathbb{R}^k)$.

Proof.

By the equivalence (1) $\iff$ (2) in Lemma 2.3 and since $I_k$ is the restriction of $P_k$ onto $\mathbb{R}^k$, we have that (1) $\iff$ (2).

Identifying $f_i$ and $e_i$ for $i \in [n]$, we identify $\mathbb{R}^k$ with some subspace $H_k \subset \mathbb{R}^n$, $\Lambda^\ell (\mathbb{R}^k) \subset \Lambda^\ell (\mathbb{R}^n)$ with $\wedge^\ell (H_k) \subset \Lambda^\ell (\mathbb{R}^n)$, and $P_k$ with $P$.

Lemma 5.1 says that $\Lambda^\ell P_k$ is exactly $\Lambda^\ell P$. Thus, identifying $f_i$ and $e_i$ for $i \in [n]$, we identify $\Lambda^\ell P_k$ with $\Lambda^\ell P$. Again, by the equivalence (1) $\iff$ (2) in Lemma 2.3, we get that (3) $\iff$ (4).

By the assertion (2) in Lemma 5.1, we have that (2) $\Rightarrow$ (4).

Hence, we must show that the implication (2) $\iff$ (4) holds to complete the proof.

Let $\{v_1, \ldots, v_n\} \subset H_k$ such that $\wedge^\ell P = \sum_{L \in \binom{[n]}{\ell}} v_L \otimes v_L$ for a fixed $\ell \in [k-1]$. Let us prove that $P = \sum_{i=1}^n v_i \otimes v_i$ to complete the proof. Assume the contrary, i.e. $P \neq \sum_{i=1}^n v_i \otimes v_i$, and define $A = \sum_{i=1}^n v_i \otimes v_i$. By Lemma 4.2, we have that $\wedge^\ell A = \wedge^\ell A$. Since the restriction of $A$ on $H_k$ is a positive-semidefinite operator, the restriction of $A$ has an orthonormal basis of eigenvectors in $H_k$. Let us denote these eigenvectors by $x_1, \ldots, x_k$ and the corresponding eigenvalues by
\[\lambda_1, \ldots, \lambda_k.\] Since \(\{x_1, \ldots, x_k\} \subset H_k\) and \(\wedge^\ell P = \wedge^\ell A\) is the projection onto \(\wedge^\ell H_k\), we have that
\[0 \neq x_{i_1} \wedge \cdots \wedge x_{i_\ell} = P x_{i_1} \wedge \cdots \wedge P x_{i_\ell} = \wedge^\ell P (x_{i_1} \wedge \cdots \wedge x_{i_\ell}) =
\]
\[\wedge^\ell A (x_{i_1} \wedge \cdots \wedge x_{i_\ell}) = Ax_{i_1} \wedge \cdots \wedge Ax_{i_\ell} = \left(\prod_{j=1}^{\ell} \lambda_{i_j}\right) \cdot x_{i_1} \wedge \cdots \wedge x_{i_\ell}\]
for any \(\ell\)-tuple \(\{i_1, \ldots, i_\ell\} \subset \binom{[n]}{\ell}\). Therefore, \(\prod_{j=1}^{\ell} \lambda_{i_j} = 1\) for any \(\ell\)-tuple \(\{i_1, \ldots, i_\ell\} \subset \binom{[n]}{\ell}\).

Obviously, since \(0 < \ell < k\), it implies that all eigenvalues are one. Hence, the restriction of \(A\) onto \(H_k\) is the identity in \(H_k\). This means that \(A = P\).

\[\square\]

The implication (2) \(\iff\) (4) of Theorem 5.2 still holds for \(\ell = k\). By the same arguments as in the proof, we have that (4) implies \(\det \left(\sum_{1 \leq i \leq n} v_i \otimes v_i\right)\) for any \(\ell\)-tuple \(\{i_1, \ldots, i_\ell\} \subset \binom{[n]}{\ell}\).

\[\text{Corollary 5.3.}\] The following assertions are equivalent for \(\{v_1, \ldots, v_n\} \subset \mathbb{R}^k \subset \mathbb{R}^n:\]
\[(1)\] \(\det \left(\sum_{1 \leq i \leq n} v_i \otimes v_i\right)\) for \(v_i \in \mathbb{R}^k\).
\[(2)\] the following identity is true
\[\Lambda^k P_k = \sum_{L \in \binom{[n]}{\ell}} v_L \otimes v_L.\]

\[\text{Proof of Corollary 1.4}\]
By the definition of the cross product, Corollary 1.4 is the equivalence (1) \(\iff\) (3) of Theorem 5.2.

\[\square\]

Now we are ready to calculate some determinants.

By the identity \((\wedge^\ell P)^2 = \wedge^\ell P\), we have
\[P_{\{I,J\}} = \sum_{L \in \binom{[n]}{\ell}} P_{\{I,L\}} P_{\{L,J\}},\]
for a fixed \(\ell \in [k]\) and two \(\ell\)-tuples \(I, J \subset \binom{[n]}{\ell}\). Particularly,
\[P_T = P_{\{I,I\}} = \sum_{L \in \binom{[n]}{\ell}} P^2_{\{I,L\}}.\]

But Theorem 5.2 gives us another identity, which connects the squared \(\ell\)-dimensional volume (that is \(P_T\)) with the sum of squared the \(k\)-dimensional volumes.

\[\text{Lemma 5.4.}\] Let \(\{v_1, \ldots, v_n\}\) give a unit decomposition in \(H_k\). Then the following identity holds
\[P_T = \sum_{Q \in \binom{[n]}{\ell} : |Q \cap T| = 0} P_{T,Q} = \sum_{T \in \binom{[n]}{\ell} |I \subset T} P_T,\]
for a fixed \(\ell \in [k]\) and an \(\ell\)-tuple \(I \in \binom{[n]}{\ell}\).

\[\text{Proof.}\]
Now we identify \(\wedge^p H_k\) with \(\Lambda^p(\Lambda^k H_k)\) for a fixed integer \(p \in [k]\). Then the Hodge star operator maps a \(p\)-form \(\nu \in \Lambda^p(H_k)\) to a \((k - p)\)-form \(\nu^* \in \Lambda^{k-p}(H_k)\). By Theorem 5.2, we know that the \((k - \ell)\)-forms \(v_Q\), where \(Q \in \binom{[n]}{k-\ell}\), give us a unit decomposition in \(\Lambda^{(k-\ell)} H_k = \Lambda^{(k-\ell)}(H_k)\).

Using the Hodge star operator, we get that the \(\ell\)-forms \(\nu^*(v_Q)\), where \(Q \in \binom{[n]}{k-\ell}\), give a unit
decomposition in $\wedge^\ell H_k \equiv \Lambda^\ell(H_k)$. By the definition of the inner product of two $\ell$-forms, we have $P_I = \langle v_I, v_I \rangle$. Now we can expand this using properties of the $\ell$-forms $\star(v_Q)$, where $Q \in \binom{[n]}{k-\ell}$:

\begin{equation}
(5.3) \quad P_I = \langle v_I, v_I \rangle = \sum_{Q \in \binom{[n]}{k-\ell}} \langle v_I, \star(v_Q) \rangle \langle \star(v_Q), v_I \rangle.
\end{equation}

By the identity (4.3), $\langle v_I, \star(v_Q) \rangle$ is the determinant of the $k$ vectors $(v_i)_{i \in I}, (v_q)_{q \in Q}$ of $H_k$. Thus, $\langle v_I, \star(v_Q) \rangle = 0$ whenever $I \cap Q \neq \emptyset$, and $\langle v_I, \star(v_Q) \rangle^2 = P_{I \cup Q}$ whenever $I \cap Q = \emptyset$. Combining these formulas with 5.3, we obtain

$$P_I = \sum_{Q \in \binom{[n]}{k-\ell}, Q \cap I = \emptyset} \langle v_I, \star(v_Q) \rangle \langle \star(v_Q), v_I \rangle = \sum_{Q \in \binom{[n]}{k-\ell}, |Q \cap I| = 0} P_{I \cup Q}.$$

\[\square\]

6. PROOFS OF LEMMA 3.3 AND THEOREM 1.5

**Proof of Lemma 3.4**

First of all, by the Cauchy-Binet formula, we have

\begin{equation}
(6.1) \quad \det \left( \sum_1^n v_i \otimes v_i \right) = \sum_{I \in \binom{[n]}{k}} P_I^S.
\end{equation}

Given a substitution $v_i \to v_i + tx$ we change a minor $P_I^S$ iff $i \in I$. By the properties of minors of $P^S$ and the definition of the Hodge star operator, we have that $P_I^S = \langle v_i, \star(v_{I \setminus i}) \rangle^2$ if $i \in I$. After the substitution, we get

$$\langle v_i + tx, \star(v_{I \setminus i}) \rangle^2 = \langle v_i, \star(v_{I \setminus i}) \rangle^2 + 2t \langle v_i, \star(v_{I \setminus i}) \rangle \langle \star(v_{I \setminus i}), x \rangle + o(t).$$

Hence,

$$\det A_{S'} = \sum_{I \in \binom{[n]}{k}} P_I^S + \sum_{I \in \binom{[n]}{k}, i \in I} P_I^S + 2t \sum_{I \in \binom{[n]}{k}, i \in I} \langle v_i, \star(v_{I \setminus i}) \rangle \langle \star(v_{I \setminus i}), x \rangle + o(t).$$

If $i \in J$ for $J \in \binom{[n]}{k-1}$, then we have $\langle v_i, \star(v_J) \rangle = 0$. Therefore, we can rewrite the last identity

$$\det A_{S'} = \sum_{I \in \binom{[n]}{k}} P_I^S + 2t \sum_{J \in \binom{[n]}{k-1}} \langle v_i, \star(v_J) \rangle \langle \star(v_J), x \rangle + o(t).$$

By Corollary 1.4, we know that the vectors $(\star(v_J))_{J \in \binom{[n]}{k-1}}$ give a unit decomposition in $H_k$. Therefore,

$$\sum_{J \in \binom{[n]}{k-1}} \langle v_i, \star(v_J) \rangle \langle \star(v_J), x \rangle = \langle v_i, x \rangle.$$

Using this and by (6.1), we obtain

$$\sqrt{\det A_{S'}} = \sqrt{\det A_S + 2t \langle v_i, x \rangle + o(t)} = 1 + t \langle v_i, x \rangle + o(t).$$

\[\square\]

For the sake of convenience, we denote by $d_S(\{M\})$ the determinant of $\{v_i, i \in M\}$ for an $(n, k)$-frame $S$ and a set $M$ of indexes from $[n]$ of cardinality $k$ (the indexes may repeat). Given $k$ vectors in $\mathbb{R}^k$ we can compute their determinant. By linearity and the identity $P_I^S = (d_S(\{I\}))^2$, one can see that
Corollary 6.1. For an arbitrary \((n,k)\)-uframe \(S\) the following identity holds

\[
\sqrt{\det A_{S'}} = 1 + \sum_{i=1}^{n} t_i(x_i, v_i) + o\left(\sqrt{t_1^2 + \ldots + t_n^2}\right),
\]

where \(S'\) is obtained from \(S\) by the substitution \(v_i \rightarrow v_i + t_i x_i, \ i \in [n]\).

Proof of Lemma 3.5
Given a substitution \(v_i \rightarrow v_i + tx\), we change \(d_S(I)\) for \(I \in \binom{n}{l}\) in (4.1) iff \(i \in I\). Since the determinant is a linear function of each vector, this means that \(|d_{S'}(I)|\) as function of \(t\) (even as function of \(x' = tx\)) is a convex function of \(t\) (or even of \(x' = tx\)) is a convex function. Using this and Lemma 3.2, we obtain that whenever \(S\) is a local maximizer of (2.3), we have that \(\text{vol}(I^n|S')/\text{vol}(I^n|S)\) is a convex function and is bounded from above by the differentiable function \(\sqrt{\det A_{S'}}\), moreover, their values coincide at \(t = 0\). It is a basic fact from sub-differential calculus that the function \(\text{vol}(I^n|S')/\text{vol}(I^n|S)\) is therefore a differentiable function of \(t\) at \(t = 0\) and its derivative coincides with the derivative of \(\sqrt{\det A_{S'}}\) at \(t = 0\). This along with Lemma 3.4 completes the proof.

Remark 6.2. In fact, in Lemma 3.4 we prove that \(\sqrt{\det A_{S'}}\) as a function of \(x' = tx \in \mathbb{R}^k\) is a differentiable function at \(x' = 0\), and in Lemma 3.5 the function \(\text{vol}(I^n|S')/\text{vol}(I^n|S)\) is differentiable at \(x' = 0\) as well.

Corollary 6.3. Let \(S = \{v_1, \ldots, v_n\} \) be a local maximizer of (2.3). Then the vectors of \(S\) are in general position in \(\mathbb{R}^k\), i.e. \(d_S(I) \neq 0\) for each \(I \in \binom{n}{k}\).

Proof. Assume the contrary, that there is a \(k\)-tuple \(J\) such that \(d_S(I) = 0\). As the rank of the vectors of \(S\) is \(k\), this implies that there is an \(I = \{i_1, \ldots, i_k\} \in \binom{n}{k}\) such that the vectors \(v_{i_1, \ldots, v_{i_k}}\) are linearly independent and the vectors \(\{v_1, \ldots, v_{i_k-1}, v_{i_k}\}\) are linearly dependent (i.e. \(d_S(I) = 0\)). Taking \(x \neq 0\) in the orthogonal complement of \(\text{Lin}\{v_1, \ldots, v_{i_k-1}\}\) and obtaining \(S'\) by a substitution \(v_{i_k} \rightarrow v_{i_k} + tx\), we get that \(\text{vol}(I^n|S')/\text{vol}(I^n|S)\) as well as the absolute value of \(d_S(I)\) is not differentiable at \(t = 0\). This contradicts Lemma 3.5.

\[\square\]

Proof of Lemma 3.3.
As explained in Section 3, we may prove the theorem for a local maximizer \(S = \{v_1, \ldots, v_n\}\) of (2.3). Let \(S'\) be a \((n,k)\)-frame obtained from \(S\) by substitution \(v_i \rightarrow v_i + t_i x_i, \ i \in [n]\).

The function \(|d_{S'}(I)|\) as function of \((t_1, \ldots, t_n)\) is the absolute value of a polynomial of \((t_1, \ldots, t_n)\). Hence, it is differentiable at the point \((t_1, \ldots, t_n)\) whenever \(d_{S'}(\{I\}) \neq 0\). By Corollary 6.3, we have \(d_S(I) \neq 0\) for all \(k\)-tuples. Therefore the function

\[
\frac{\text{vol}(I^n|S')}{\text{vol}(I^n|S)} = \frac{\sum_{I \in \binom{n}{k}} |d_{S'}(\{I\})|}{\text{vol}(I^n|S)}
\]

as function of \((t_1, \ldots, t_n)\) is differentiable at the origin.

Since the determinant is a linear function of each row, we conclude that the coefficient at \(t_i\) in the linear approximation of \(\text{vol}(I^n|S')/\text{vol}(I^n|S)\) coincides with that in Corollary 6.1, i.e.

\[
\frac{\text{vol}(I^n|S')}{\text{vol}(I^n|S)} = 1 + \sum_{i=1}^{n} t_i(x_i, v_i) + o\left(\sqrt{t_1^2 + \ldots + t_n^2}\right).
\]

\[\square\]

Proof of Theorem 1.5.
We say that a set \(M \subset [n]\) is a maximal set if the set \(V_M = \{v_i | i \in M\}\) is maximal (in the sense of inclusion) such that the dimension of the linear hull of \(V_M\) is \(k - 1\), i.e.

\[
\dim\text{Lin} V_M = k - 1 \quad \text{and} \quad \dim\text{Lin}\{V_M, v_n\} = k.
\]
for any $m \in [n] \setminus M$.

Since the support function of the Minkowski sum of convex sets is the sum of their support functions, each facet of $I^n|H_k$ is a $(k-1)$-dimensional zonotope, which is a translate of $\sum_{i \in M \subset [n]} [0, v_i]$ for a maximal set $M$. Clearly, for a fixed $L \in {n \choose k-1}$ such that $\dim V_L = k-1$ (or $P_L \neq 0$) the set $\sum_{i \in L \subset [n]} [0, v_i]$ is the summand of a unique zonotope $\sum_{i \in M \subset [n]} [0, v_i]$ for some maximal set $M$, i.e. calculating the surface area measure of $I^n|H_k$ we count the volume of $\sum_{i \in L \subset [n]} [0, v_i]$ exactly twice. By this and by formula (5.1), we have

$$\text{vol } \partial(I^n|H_k) = 2 \sum_{L \in {n \choose k-1}} \sqrt{P_L}.$$  

Clearly, $0 \leq P_L \leq 1$ for any $L \in {n \choose k-1}$. Therefore,

$$\text{vol } \partial(I^n|H_k) = 2 \sum_{L \in {n \choose k-1}} \sqrt{P_L} \geq 2 \sum_{L \in {n \choose k-1}} P_L = 2 \text{tr}(\wedge^{k-1} P) = 2k = \text{vol } (I^k).$$

Where $\text{tr}(\wedge^{k-1} P) = k$ since $\wedge^{k-1} P$ is the orthogonal projection onto the $k$-dimensional space $\wedge^{k-1} H_k$.

It is easy to understand that equality is attained iff each $P_L$ either equals zero or one. The latter means that $P$ is the projection onto a coordinate subspace.

\[\square\]

\textbf{Remark 6.4.} Since the trace of the exterior $\ell$-power of an operator $A$ is, up to sign, the proper coefficient of the characteristic polynomial of $A$, one can prove Theorem 1.5 without using exterior algebra. But we believe that our method reveals some hidden geometric properties of the volume of zonotopes.

\section{Proof of Theorem 1.2}

In order to prove Theorem 1.2, we need to understand some geometric properties of the determinants $d_S(\{J\}), J \in {n \choose k}$. For a fixed $i \in [n]$, all $n \choose k-1$ values $d_S(i, I), I \in {n \choose k-1}$ can be considered as a vector in $\Lambda^{k-1}(\mathbb{R}^n)$. By definition let $d_S(i)$ be the vector in $\Lambda^{k-1}(\mathbb{R}^n)$ such that its $i$th coordinate in the standard basis of $\Lambda^{k-1}(\mathbb{R}^n)$ is $d_S(i, I)$.

The idea of the proof is the following. We show that the vectors $\{d_S(1), \ldots, d_S(n)\} \subset \Lambda^{k-1}(\mathbb{R}^n)$ are isometric to $\{v_1, \ldots, v_n\}$. Then, using the first-order necessary condition, we understand the geometric sense of the signs of the coordinates of $d_S(i)$. Finally, we use Lemma 3.1 to rotate vectors by $\pi/4$ (i.e. $v_i \rightarrow \cos(\pi/4)v_i - \sin(\pi/4)v_j, v_j \rightarrow \sin(\pi/4)v_i + \cos(\pi/4)v_j$), and after some simple calculations we get the inequality of Theorem 1.2.

Let $S = \{v_1, \ldots, v_n\}$ be an arbitrary $(n, k)$-uframe. Recall that we can go from an $(n, k)$-uframe to the proper $k$-dimensional subspace $H_k^S \subset \mathbb{R}^n$, and that the vectors of $S$ map to the projections of the standard basis onto $H_k^S$. Some geometric properties of the vectors $d_S(i)$ are described in the following lemma.

\textbf{Lemma 7.1.} Let $S = \{v_1, \ldots, v_n\}$ be an arbitrary $(n, k)$-uframe. Then the vectors $d_S(i), i \in [n]$ lie in $\Lambda^{k-1}(H_k^S) \subset \Lambda^{k-1}(\mathbb{R}^n)$. Moreover, they give a unit decomposition in $\Lambda^{k-1}(H_k^S)$ and the following identity holds

\begin{equation}
\langle d_S(i), d_S(j) \rangle = \langle v_i, v_j \rangle,
\end{equation}

for all $i, j \in [n]$.

\textbf{Proof.}\n
Let $a_{ij}, j \in [k]$ be the rows of the $k \times n$ matrix $M^S = (v_1, \ldots, v_n)$. By the definition of $H_k^S$, we know that $a_{ij} \in H_k^S, j \in [k]$ and that they form an orthonormal system in $H_k^S$. Hence, the $(k-1)$-forms $(a_{jk}|^k)_{j=1}^k$ are an orthonormal basis of $\Lambda^{k-1}(H_k^S)$. Consider the $(k-1)$-forms
\[ b_i = \sum_{j=1}^{k} (-1)^{i+j} v_i[j] \cdot a_{[k] \setminus j}, \text{ where } i \in [n]. \]
Then, by the definition of the inner product on \( \Lambda^{k-1}(\mathbb{R}^n) \) and the Laplace expansion of the determinant, we have
\[
 b_i[I] = (b_i, e_I) = \sum_{j=1}^{k} (-1)^{i+j} v_i[j] a_{[k] \setminus j}, e_I = \sum_{j=1}^{k} (-1)^{i+j} v_i[j] M^S_{[k] \setminus j, I} = M^S_{i, I}, d_s(i) = d_S(i)[I],
\]
for \( I \in \binom{[k]}{k-1}, i \notin I \). We get \( b_i[I] = d_S(i)[I] = 0 \) if \( i \in I \). Thus, \( d_S(i) = b_i \in \Lambda^{k-1}(H^S_k) \), and, clearly, the identity (7.1) holds. (Also, the identity (7.1) is a straightforward consequence of Corollary 1.4). 
\[ \square \]

As one can see, we use only the properties of vectors that give a unit decomposition in Lemma 7.1. But the necessary condition in Lemma 3.3 provides us with some restrictions for local maximizers of problem (2.3).

Let an \((n, k)\)-uframe \( S \) be a local maximizer of (1.1). Then we define a \textit{sign-function} \( \sigma(i, I) \) by
\[
\sigma_S(i, I) = \begin{cases} 
1/\text{vol}(I^n|S), & \text{if } d(\{i, I\}) > 0; \\
-1/\text{vol}(I^n|S), & \text{if } d(\{i, I\}) < 0; \\
0, & \text{if } d(\{i, I\}) = 0,
\end{cases}
\]
for \( i \in [n] \) and \( I \in \binom{[n]}{k-1} \).

In the same way as with the vectors \( d_S(i), i \in [n] \), we identify \( \sigma_S(i) \) with a vector in \( \Lambda^{k-1}(\mathbb{R}^n) \) such that its \( I \)th coordinate in the standard basis of \( \Lambda^{k-1}(\mathbb{R}^n) \) is \( \sigma_S(i, I) \). As a direct consequence of Lemma 3.3, we obtain.

**Lemma 7.2.** Let \( S = \{v_1, \ldots, v_n\} \) be a local maximizer of (2.3). Then
\[
\wedge^{k-1} P^S(\sigma_S(i)) = d_S(i) \quad \text{and} \quad \langle \sigma_S(i), d_S(j) \rangle = \langle v_i, v_j \rangle, \quad i, j \in [n].
\]

**Proof.**
By Lemma 7.1, we have that the vectors \( (d_S(i))_{i=1}^n \) give a unit decomposition in \( \wedge^{k-1} H^S_k \). Therefore, by Theorem 5.2, it is enough to show that the right-hand side of (7.2) is true.

Fix \( i, j \in [n] \). Let \( S' \) be an \((n, k)\)-frame obtained from \( S \) by the substitution \( v_i \rightarrow v_i + tv_j \). By Corollary 6.3, we have that \( d_S(\{I\}) \neq 0 \) for every \( k \)-tuple \( I \). Therefore, \( d_S(\{I\}) \neq 0 \) and \( d_S(\{I\}) \) has the same sign as \( d_S(\{I\}) \) for all \( k \)-tuples \( I \) and for a small enough \( t \). Using the substitution \( v_i \rightarrow v_i + tv_j \), we only change the determinants of type \( d_S(\{i, J\}) \), where \( i \notin J \subseteq \binom{[n]}{k-1} \). Thus, by the properties of absolute value (for a small enough \( t \)), we get
\[
\frac{\text{vol}(I^n|S')}{\text{vol}(I^n|S)} = \frac{\sum_{I \in \binom{[n]}{k-1}} |d_S(\{I\})|}{\text{vol}(I^n|S)} = 1 + t \sum_{J \in \binom{[n]}{k-1}, i \notin J} \sigma_S(i, J)d_S(\{j, J\}).
\]
Since \( \sigma_S(i, J) = 0 \) whenever \( i \in J \), we have that the coefficient at \( t \) in the previous formula is
\[
\sum_{J \in \binom{[n]}{k-1}} \sigma_S(i, J)d_S(\{j, J\}).
\]
But this is the inner product \( \langle \sigma_S(i), d_S(j) \rangle \) of the \((k-1)\)-forms \( \sigma_S(i) \) and \( d_S(j) \) written in the standard basis of \( \Lambda^{k-1}(\mathbb{R}^n) \). By Lemma 3.3, we have that \( \langle \sigma_S(i), d_S(j) \rangle = \langle v_i, v_j \rangle \). 
\[ \square \]

**Proof of Theorem 1.2.**
We prove the theorem for a maximizer \( S = \{v_1, \ldots, v_n\} \) of (2.3), and we fix \( i \) and \( j \) in \([n]\).
We assume that $|v_i|^2 < |v_j|^2$, otherwise it is nothing to prove. Using Lemma 3.1, we have that the substitution $v_i \to \cos(\pi/4)v_i - \sin(\pi/4)v_j, v_j \to \sin(\pi/4)v_i + \cos(\pi/4)v_j$ preserves $A_S$ and the absolute value of $d_S(\{I\})$ for all $I \subset (n)_k$. Hence, we show that

$$\sqrt{2} \sum_{J \in (n)_k, i,j \notin J} (|d_S(\{i, J\})| - d_S(\{j, J\})) + |d_S(\{i, J\}) + d_S(\{j, J\}))| \leq$$

$$\sum_{J \in (n)_k, i,j \notin J} (|d_S(\{i, J\})| + |d_S(\{j, J\}))|.$$  

By the identity $|a+b| + |a-b| = 2 \max\{|a|, |b|\}$, we obtain that each summand in the left-hand side is at least $2 \max\{|d_S(\{i, J\})|, |d_S(\{i, J\})|\}$, and, consequently, is at least $2 |d_S(\{j, J\})|$. Hence, we show that

$$(\sqrt{2} - 1) \sum_{J \in (n)_k, i,j \notin J} |d_S(\{j, J\})| \leq \sum_{J \in (n)_k, i,j \notin J} |d_S(\{i, J\})|.$$  

For all $(k-1)$-tuples $I$ such that $i \in I$ and $j \notin I$ there is one-to-one correspondence with the set of all $(k-1)$-tuples $J$ such that $i \notin J$ and $j \in J$ given by $I \to (I \setminus \{i\}) \cup \{j\}$. In this case, $|d_S(\{j, I\})| = |d_S(\{i, \emptyset \} \cup \{i\})|$. Adding all such the determinants to the last inequality, we obtain

$$\sqrt{2} - 1 \sum_{J \in (n)_k, i \notin J} |d_S(\{i, J\})| \geq (\sqrt{2} - 1) \sum_{J \in (n)_k, i,j \notin J} |d_S(\{j, J\})| + \sum_{J \in (n)_k, \emptyset \notin J} |d_S(\{j, J\})| \geq$$

$$(\sqrt{2} - 1) \left( \sum_{J \in (n)_k, i,j \notin J} |d_S(\{j, J\})| + \sum_{J \in (n)_k, \emptyset \notin J} |d_S(\{j, J\})| \right) = (\sqrt{2} - 1) \sum_{J \in (n)_k, \emptyset \notin J} |d_S(\{j, J\})|.$$  

Finally, the sum in the left-hand side is exactly $\text{vol}(P^n|S)\langle \sigma_S(i), d_S(i) \rangle$, and by Lemma 7.2, it is equal to $\text{vol}(P^n|S)|v_i|^2$. Similarly, we have $(\sqrt{2} - 1) \text{vol}(P^n|S)|v_j|^2$ in the right-hand side of the last inequality. Dividing by $\text{vol}(P^n|S)$, we obtain $|v_i|^2 \geq (\sqrt{2} - 1)|v_j|^2$.

\[\square\]

**Remark 7.3.** By Shepard’s formula 4.1 and properties of the determinant, the sum

$$\frac{1}{|v_i|} \sum_{J \in (n)_k, i \notin J} |d_S(\{i, J\})| = \frac{1}{|v_i|} \sum_{J \in (n)_k} |d_S(\{i, J\})|$$

is equal to the volume of the projection of the zonotope $P^n|H_k$ onto the orthogonal complement of the line $L_{\{v_i\}}$ in $H_k$.

As mentioned in the Introduction, Corollary 1.3 is a consequence of Theorem 1.2 and McMullen’s symmetric formula (1.2).

**Remark 7.4.** McMullen’s formula is the consequence of Shepard’s formula (4.1) and Lagrange’s identity $|U_j| = |U_{[n]\setminus J}|$ for an orthogonal matrix $U$ of rank $n$ and any $J \subset [n]$. In terms of properties of a projection operator, Lagrange’s identity is expressed by $\star(U^H P x) = (U^H P) P^\perp \star x$, where $P$ is the projection onto a $k$-dimensional subspace $H_k$ and $P^\perp$ is the projection onto its orthogonal complement.

**Proof of Corollary 1.3.**

In the notation of the corollary, let $H^k_q$ be the orthogonal complement of $H_{n-q}$ in $\mathbb{R}^n$. Let $v_i$ and $v_i'$ be the projections of the vector $e_i$ onto $H_{n-q}$ and $H^k_q$, respectively. Clearly, $|v_i|^2 + |v_i'|^2 = 1$.  

By Theorem 1.2, we conclude that \(|v_i'|^2\) is at most \(1/(\sqrt{2} - 1)\) and at least \((\sqrt{2} - 1)\) of the average squared length of the projections of the standard basis, which is \(q/n\). Therefore,

\[
1 \geq \frac{m_n}{M_n} \geq \frac{1 - \frac{1}{\sqrt{2} - 1} \frac{q}{n}}{1 - (\sqrt{2} - 1)\frac{q}{n}},
\]

which tends to 1 as \(n\) tends to infinity.

□

8. FURTHER DISCUSSION AND OPEN QUESTIONS

The line \(\wedge^k H_k \subset \Lambda^k(\mathbb{R}^n)\) has a decomposable directional vector and all decomposable vectors in \(\Lambda^k(\mathbb{R}^n)\) are generated by a proper \(k\)-dimensional subspace. Combining this with (5.1) and the properties of \(\wedge^k P\), we obtain

\[
\max \text{vol}(I^n|H_k) = \max \|x\|_1,
\]

where in the left-hand side we take the maximum over all \(k\)-dimensional subspaces of \(\mathbb{R}^n\) and in the right-hand side we take the maximum over all unit decomposable \(k\)-forms, and \(\|x\|_1\) means \(\ell_1\)-norm of the vector \(x\). In case \(k = 1, n - 1\) all \(k\)-forms are decomposable, and this equality gives an exact answer \(\sqrt{n} = \sqrt{\dim \Lambda^1(\mathbb{R}^n)}\).

And again, by identity (5.1), it is enough to understand what the diagonal elements of \(\wedge^k P\) are. We described all possible diagonals of \(P = \wedge^1 P\) in [12], Lemma 2.5, and this description is the straightforward consequence of the well-known Horn’s theorem [10], which describes all possible vectors \((c_1, \ldots, c_n)\) that can be the main diagonal of a Hermitian matrix with a prescribed vector of eigenvalues \((\lambda_1, \ldots, \lambda_n)\).

We ask what is a generalisation of this result for the exterior powers of positive-semidefinite operators.

By the symmetry arguments, \(H_k\) is a local maximizer whenever the vectors \(\{\pm v_1, \ldots, \pm v_n\}\) are distinct vertices of a symmetric platonic solid. We conjecture that those are the global maximizer for corresponding \(n\) and \(k\). Also, we think that the \(k\)-dimensional permutohedron is the maximizer in (2.3).

Finally, it is interesting to describe all continuous transformation of \(I^n|S\) which change the volume in a monotone way. Can we transform any given configuration \(S\) to the \(\{e_1, \ldots, e_k, 0, \ldots, 0\}\) decreasing the volume in a monotone way? Can we transform in such a way to get a local maximum?

If \(k = 1\) the answer is clear. We can answer in case \(k = 2\) for the last question, but the questions are open in general.

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