DIVISION FORMULAS ON PROJECTIVE VARIETIES

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ABSTRACT. We introduce a division formula on a possibly singular projective subvariety $X$ of complex projective space $\mathbb{P}^N$, which, e.g., provides explicit representations of solutions to various polynomial division problems on the affine part of $X$. In particular we consider a global effective version of the Briançon-Skoda-Huneke theorem.

1. INTRODUCTION

In this paper we construct a division-interpolation integral formula for polynomial ideals on an algebraic subvariety $V$ of $\mathbb{C}^N$. Such division formulas, for smooth $V$, have been used by several authors, see, e.g., [11], [10] and [9]. There are two main novelties in our approach. Our formula is the restriction to $V$ of a formula on the closure $X$ of $V$ in $\mathbb{P}^N$; this makes it possible to get sharper estimates. In case $X = \mathbb{P}^n$ we precisely get back the formula introduced in [4]. In this paper we also allow non-smooth varieties.

Our main interest is a global effective Briançon-Skoda-Huneke type polynomial division result on $V$ that was proved in [8], generalizing theorems of Ein-Lazarsfeld, [13], and Hickel, [17], to non-smooth $V$. Our integral formula does not in any substantial way contribute to the proof of the existence of the polynomials but provides an integral representation of the them. In particular, applied to Nullstellensatz data we get a representation of a solution with a polynomial degree that is not too far from the optimal one, cf., Remark 1.2 below.

Let $X$ be the closure of $V$ in $\mathbb{P}^N$, let $J_X$ denote the associated homogeneous ideal in the graded ring $S = \mathbb{C}[z_0, \ldots, z_N]$, and let $S(\ell)$ denote the module $S$ but where all degrees are shifted by $\ell$. Let

$$0 \to S_M \overset{a_M}{\to} \cdots \overset{a_1}{\to} S_0$$

be a minimal graded free resolution of $S/J_X$; here $S_k = S(-d_k^1) \oplus \cdots \oplus S(-d_k^N)$ and the mappings $a_k = (a_{ij}^k)$ are matrices of homogeneous forms in $\mathbb{C}^{N+1}$ with

$$\deg a_{ij}^k = d_k^j - d_k^i - 1.$$

Since $J_X$ has pure dimension it follows from [11 Corollary 20.14] that $M \leq N$, see also [8, Section 2.7]. Let

$$\kappa_0 = \max_{k \leq M} d_k^i.$$

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Given polynomials \( F_1, \ldots, F_m \) on \( V \) of degrees at most \( d \), let \( f_j \) be the corresponding \( d \)-homogeneous forms on \( \mathbb{C}^{N+1} \). That is, if \( z = (z_0, \ldots, z_N) \) and \( z' = (z_1, \ldots, z_N) \), then \( f_j(z) = z_0^d F(z'/z_0) \). As usual we can consider \( f_j \) as sections of the restriction to \( X \) of the line bundle \( \mathcal{O}(d) \) over \( \mathbb{P}^N \). Let \( Z_f \) be the zero set of \( f_j \) on \( X \).

We now present our main result in this paper. Assume that \( \rho \) is an integer that is larger than or equal to \( d \min(m, n+1) + \kappa_0 - N \), and let \( \phi \) be a section of \( \mathcal{O}(\rho) \) over \( X \). We construct a division-interpolation formula

\[
\phi(z) = f(z) \cdot \int_X A(\zeta, z) \phi(\zeta) + \int_X B(\zeta, z) \wedge R_f(\zeta) \phi(\zeta),
\]

where, for fixed \( z \), the first integral exists as a principal value at \( X_{\text{sing}} \cup Z_f \), \( B(\cdot, z) \) is a smooth form, and \( R_f \) is a residue current on \( X \) with support on \( Z_f \), so that the second “integral” is the action of \( R_f \) on \( \pm B \phi \). Moreover, both \( A \) and \( B \) are holomorphic in \( z \). For the precise formula, see Theorem 4.2 below. Assume that \( \Phi \) is a polynomial of degree at most \( \rho \) and let \( \phi \) be its \( \rho \)-homogenization. If the current \( R_f \phi \) vanishes, i.e., \( \phi \) annihilates \( R_f \), it follows that

\[
Q_j(z') = \int_X A_j(\zeta, z') \phi(\zeta)
\]

are polynomials such that \( \deg F_j Q_j \leq \rho \) and \( F_1 Q_1 + \cdots + F_m Q_m = \Phi \) on \( V \).

Let us now describe our main application of this formula. Let \( J_f \) be the coherent analytic ideal sheaf over \( X \) generated by \( f_j \). Furthermore, let \( c_\infty \) be the maximal codimension of the so-called distinguished varieties of the sheaf \( J_f \), in the sense of Fulton-MacPherson, that are contained in

\[
X_\infty := X \setminus V;
\]

see, e.g., [8, Section 5]. If there are no distinguished varieties contained in \( X_\infty \), then we interpret \( c_\infty \) as \( -\infty \). We always have

\[
c_\infty \leq \mu := \min(m, n).
\]

Let \( |F|^2 = |F_1|^2 + \cdots + |F_M|^2 \).

**Theorem 1.1.** Assume that \( V \) is a reduced \( n \)-dimensional algebraic subvariety of \( \mathbb{C}^N \).

(i) There exists a number \( \mu_0 \) such that if \( F_1, \ldots, F_m \) are polynomials of degree \( \leq d \) and \( \Phi \) is a polynomial such that

\[
|\Phi| \leq C|F|^d + \mu_0
\]

locally on \( V \), then, with the appropriate choice of \( \rho \), (1.3) are polynomials on \( V \) such that

\[
\Phi = F_1 Q_1 + \cdots + F_m Q_m
\]

on \( V \) and

\[
\deg (F_j Q_j) \leq \max \left( \deg \Phi + (\mu + \mu_0) d^\infty \deg X, d \min(m, n+1) + \kappa_0 - N \right).
\]
(ii) Assume that $V$ is smooth. There is a number $\mu'$ such that if $F_1, \ldots, F_m$ are polynomials of degree $\leq d$ and $\Phi$ is a polynomial such that
\begin{equation}
|\Phi| \leq C|F|^{\mu'}
\end{equation}
locally on $V$, then \eqref{1.4} gives polynomials $Q_j$ such that \eqref{1.7} holds on $V$ and
\begin{equation}
\deg(F_j Q_j) \leq \max\left(\deg \Phi + \mu' d, \deg X + \mu', d \min(m, n+1) + \kappa_0 - N\right).
\end{equation}
If $X$ is smooth one can take $\mu' = 0$.

Recall that the regularity of $X$ is defined as the regularity of the module $J_X$ which in turn is $1 + \max_{k \leq M} d_k$, see \cite[Ch. 4]{15}. Notice that
\begin{equation}
\kappa_0 - N \leq \reg X - 1.
\end{equation}

Theorem 1.1 in \cite{8} states that there are polynomials $Q_j$ as in Theorem 1.1 with the slightly sharper degree bound $d \min(m, n+1) + \reg X - \min(m, n+1)$ rather than $d \min(m, n+1) + \kappa_0 - N$ in the last entry in \eqref{1.8} and \eqref{1.10}.

A key step in the proof of Theorem 1.1 (i) in \cite{8} is the following result:
(* There is a number $\mu_0$, only depending on $V$, such that the following holds: If $F_j$ are polynomials of degree at most $d$ and $\Phi$ is a polynomial such that \eqref{1.6} holds, $\rho \geq (\deg \Phi + (\mu + \mu_0) d \sup\deg X, \rho \geq \deg \Phi$, and $\phi$ is the $\rho$-homogenization of $\Phi$, then $R^j \phi = 0$.
If we choose
\begin{equation}
\rho = \max\left(\deg \Phi + (\mu + \mu_0) d \sup\deg X, d \min(m, n+1) + \kappa_0 - N\right),
\end{equation}
we can represent $\phi$ by formula \eqref{1.3}, and since furthermore then $R^j \phi = 0$ in view of (*) it follows that the polynomials in \eqref{1.4} satisfy \eqref{1.7} and \eqref{1.8}. Thus we get Theorem 1.1 (i) above.

In the same way, if the hypotheses in Theorem 1.1 (ii) are fulfilled, it follows from \cite{8} that $R^j \phi = 0$ if $\phi$ is the $\rho$-homogenization where $\rho \geq \deg \Phi + (\mu + \mu_0) d \sup\deg X + \mu'$. Again we thus get the integral \eqref{1.4} representation of polynomials $Q_j$ satisfying \eqref{1.7} and \eqref{1.10}.

Remark 1.2. If we apply Theorem 1.1 to Nullstellensatz data, i.e., $F_j$ with no common zero on $V$ and $\Phi = 1$, then we get polynomials $Q_j$ by \eqref{1.4} such that $\sum F_j Q_j = 1$ on $V$ and
\begin{equation}
\deg(F_j Q_j) \leq \max\left((\mu + \mu_0) d \sup\deg X, d \min(m, n+1) + \kappa_0 - N\right).
\end{equation}
It was proved by Jelonek, \cite{19}, that one can find a solution such that
\begin{equation}
\deg(F_j Q_j) \leq c_m d \deg X,
\end{equation}
where $c_m = 1$ if $m \leq n$ and $c_m = 2$ otherwise, and this result is essentially optimal. In general it is clearly much sharper than what we get; however, if $c_\infty < \mu$ and $d$ is large enough, then our estimate can compete.

For a further discussion of Theorem 1.1 and a comparison with the results in \cite{13} and \cite{17}, see \cite{8}.

\footnote{If $m \geq n+1$, then $\reg X - \min(m, n+1) \leq \kappa_0 - N$; when $m \leq n$ at least the proof in \cite{8} gives an estimate that is not less sharp than $\kappa_0 - N$.}

\footnote{The optimal result for case $V = \mathbb{C}^n$ was proved by Kollár, \cite{20}, for $d \geq 3$; see \cite{21} and \cite{19} for $d = 2$.}
Remark 1.3. One can consider Theorem 1.1 in [8] as an effective global version of the Briançon-Skoda-Huneke theorem:

\[ \psi \text{ of } L \rightarrow F \text{ are any holomorphic functions at } x, t \geq 1, \]  
\[ |\Phi| \leq C \left| F \right|^{\mu_0 + \ell} \text{ in a neighborhood of } x, \]  
then \( \Phi \) belongs to the local ideal \( (F_j)_x \) at \( x \).

If \( x \) is a smooth point, then one can take \( \mu_0 = 0 \); this is the classical Briançon-Skoda theorem, [12]. The general case was proved by Huneke, [14], by purely algebraic methods. An analytic proof appeared in [5].

Remark 1.4. In view of the Briançon-Skoda-Huneke theorem, the hypothesis (1.6) implies that \( \Phi \) is in the ideal \( (F_j) \) at each point, and hence (1.7) holds for some polynomials. However, membership in \( (F_j) \) only implies a representation (1.7) with a degree estimate like \( \deg \Phi + d(2^n) \) rather than \( \deg \Phi + d^n \).

2. Integral Representation on \( \mathbb{P}^N \)

We first recall from [4] how one can generate representation formulas for holomorphic sections of a vector bundle \( F \rightarrow \mathbb{P}^N \). The construction is an adaptation to \( \mathbb{P}^N \) of the idea introduced in [11]; see also [16]. Let \( F_z \) denote the pull-back of \( F \) to \( \mathbb{P}^N \times \mathbb{P}^N \) under the natural projection \( \mathbb{P}^N \times \mathbb{P}^N \rightarrow \mathbb{P}^N \) and define \( F_\zeta \) analogously. Let \( \delta_\eta \) denote contraction with the sheaf of currents on \( \mathbb{P}^N \times \mathbb{P}^N \),

\[ \eta = 2\pi i \sum_{0}^{N} z_i \frac{\partial}{\partial \zeta_i} \]

over \( \mathbb{P}^N \times \mathbb{P}^N \). We thus have the mapping

\[ \delta_\eta : C_{\ell+1,q}(\mathcal{O}_\zeta(1) \otimes \mathcal{O}_z(j)) \rightarrow C_{\ell,q}(\mathcal{O}_\zeta(k-1) \otimes \mathcal{O}_z(j+1)), \]

where \( C_{\ell,q}(\mathcal{O}_\zeta(k) \otimes \mathcal{O}_z(j)) \) denotes the sheaf of currents of bidegree \( (\ell, q) \) in \( \zeta \) and \( (0, 0) \) in \( z \) that take values in \( \mathcal{O}_\zeta(k) \otimes \mathcal{O}_z(j) \). In this paper we only deal with forms and currents with respect to \( \zeta \) and always consider \( z \) as a parameter. Given a vector bundle \( L \rightarrow \mathbb{P}^N \times \mathbb{P}^N \), let

\[ \mathcal{L}^{\nu}(L) = \bigoplus_j C_{j,j+\nu}(\mathcal{O}_\zeta(j) \otimes \mathcal{O}_z(-j) \otimes L). \]

If \( \nabla_\eta = \delta_\eta - \bar{\partial} \), where \( \bar{\partial} = \bar{\partial}_\zeta \), then \( \nabla_\eta : \mathcal{L}^{\nu}(L) \rightarrow \mathcal{L}^{\nu+1}(L) \). Furthermore, \( \nabla_\eta \) is a anti-derivation, and \( \nabla_\eta^2 = 0 \).

A weight with respect to \( F \rightarrow \mathbb{P}^N \) and a point \( z \in \mathbb{P}^N \) is a section \( g \) of \( \mathcal{L}^{0}(\text{Hom}(F_\zeta,F_z)) \) such that \( \nabla_\eta g = 0 \), \( g \) is smooth for \( \zeta \) close to \( z \), and \( g_{0,0} \) is the identity endomorphism on \( F \). The following basic formula appeared as Proposition 4.1 in [4].

Proposition 2.1. Let \( g \) be a weight with respect to \( F \rightarrow \mathbb{P}^N \) and \( z \), and assume that \( \psi \) is a holomorphic section of \( F \otimes \mathcal{O}(-N) \). We then have the representation formula

\[ \psi(z) = \int_{\mathbb{P}^N} g_{N,N} \psi. \]
Recall that an \( \ell \)-homogeneous form \( \xi \) on (an open subset of) \( \mathbb{C}^{N+1} \setminus \{0\} \) is \textit{projective}, i.e., the pullback under the natural projection \( \pi: \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N \) of a form on (an open subset of) \( \mathbb{P}^N \), if and only if \( \delta_z \xi = \delta \bar{z} \xi = 0 \), where \( \delta_z \) is interior multiplication by

\[
\sum_{j=0}^{N} z_j \frac{\partial}{\partial z_j},
\]

and \( \delta \bar{z} \) is its conjugate.

\textit{Example 2.2.} For fixed \( z \),

\[
\alpha = \alpha_{0,0} + \alpha_{1,1} = \frac{z \cdot \bar{\zeta} - \bar{\partial}_\zeta \cdot d\zeta}{2\pi i |\zeta|^2}
\]

is a well-defined smooth form in \( L^0(\text{Hom}(\mathcal{O}_z(-1),\mathcal{O}_z(1))) \), such that

\[
(2.2) \quad \nabla_\eta \alpha = 0,
\]

and \( \alpha_{0,0} \) is equal to \( I_{\mathcal{O}(1)} \) at \( z \). Thus \( \alpha \) is a weight with respect to \( \mathcal{O}(1) \) and \( z \).

For a given point \( z \in \mathbb{P}^N \),

\[
b = \frac{1}{2\pi i} \frac{|\zeta|^2 \bar{\zeta} \cdot d\zeta - (\bar{\zeta} \cdot \zeta) \bar{\zeta} \cdot d\zeta}{|\zeta|^2 |\zeta|^2 - |\zeta \cdot z|^2}
\]

is the \( \mathcal{O}_z(1) \otimes \mathcal{O}_z(-1) \)-valued \((1,0)\)-form in \( d\zeta \) on \( X \setminus \{z\} \) of minimal norm such that \( \eta \cdot b = \delta_\eta b = 1 \). Let

\[
B = \frac{b}{\nabla_\eta b} = b + b \wedge \bar{\partial} b + b \wedge (\bar{\partial} b)^2 + \cdots + b \wedge (\bar{\partial} b)^{N-1};
\]

here \( b \wedge (\bar{\partial} b)^{k-1} \) is an \( \mathcal{O}_z(k) \otimes \mathcal{O}_z(-k) \)-valued \((k,k-1)\)-form that is \( \mathcal{O}(1/d(\zeta,z)^{2k-1}) \), where \( d(\zeta,z) \) is the distance between \( \zeta \) and \( z \) on \( \mathbb{P}^N \). In particular, \( B \) is locally integrable at \( z \) and can thus be considered as a current on \( X \). Let \([z] \) denote the \( \mathcal{O}_z(N) \otimes \mathcal{O}_z(-N) \)-valued \((N,N)\)-current point evaluation at \( z \), i.e.,

\[
\int_{\mathbb{P}^N} [z] \xi = \xi(z)
\]

for each smooth \((0,0)\)-form \( \xi \) with values in \( \mathcal{O}(-N) \). Then, see, e.g., [4, formula (4.1)],

\[
(2.3) \quad \nabla_\eta B = 1 - [z]
\]

in the current sense.

For further reference we recall from [4] how \( (2.1) \) follows from \( (2.3) \): Since \( g \) is \( \nabla_\eta \)-closed, \( \nabla_\eta (g \wedge B) = g \wedge (1 - [z]) \), and by identifying components of bidegree \((N,N)\) therefore

\[
-\bar{\partial}(g \wedge B)_{N,N-1} = g_{N,N} - g_{0,0} [z] = g_{N,N} - I_{F_1}[z].
\]

Now \( (2.1) \) follows after multiplication by \( \phi \) and an application of Stokes’ theorem.
2.1. Division-interpolation formulas on $\mathbb{P}^N$. Assume that

$$0 \to E_M \xrightarrow{f_M} \ldots \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0$$

is a generically exact complex of Hermitian vector bundles over $\mathbb{P}^N$ and let $Z$ be the projective variety where (2.4) is not pointwise exact. In [3] and [7], were introduced currents

$$U = U_1 + \ldots + U_N + U_{\min(M,N+1)}, \quad R = R_1 + \ldots + R_{\min(M,N+1)}$$

associated to (2.4) with the following properties: The current $U$ is smooth outside $Z$, $U_k$ are $(0, k-1)$-currents that take values in $\text{Hom}(E_0, E_k)$, and $R_k$ are $(0, k)$-currents with support on $Z$, taking values in $\text{Hom}(E_0, E_k)$. Moreover, they satisfy the relations

$$f_1 U_1 = I_{E_0}, \quad f_{k+1} U_{k+1} - \bar{\partial} U_k = -R_k, \quad k \geq 1,$$

which can be compactly written

$$\nabla f U = I_{E_0} - R$$

if $\nabla f = f - \bar{\partial} = f_1 + f_2 + \cdots f_N - \bar{\partial}$. We have the corresponding complex of locally free sheaves

$$0 \to \mathcal{O}(E_{-1}) \xrightarrow{f_{-1}} \mathcal{O}(E_0) \xrightarrow{f_1} \mathcal{O}(E_1) \xrightarrow{f_2} \mathcal{O}(E_2) \to \cdots \to \mathcal{O}(E_k) \xrightarrow{f_{k+1}} \mathcal{O}(E_0).$$

In this paper we will only consider $E_k$ that are direct sums of line bundles $\mathcal{O}(\nu)$. Let $E_k$ be disjoint trivial line bundles over $\mathbb{P}^N$ with basis elements $e_{k,j}$, and let

$$E_k = (E_k^1 \otimes \mathcal{O}(-d_k^1)) \oplus \cdots \oplus (E_k^{r_k} \otimes \mathcal{O}(-d_k^{r_k})).$$

Then

$$f_k = \sum_{ij} f_k^{ij} e_{k-1,i} \otimes e_{k,j}^{*}$$

are matrices of homogeneous forms; here $e_{k,j}^{*}$ are the dual basis elements, and

$$\deg f_k^{ij} = d_k^{i-1} - d_k^j.$$

We equip $E_k$ with the natural Hermitian metric, i.e., such that

$$|\xi(z)|^2_{E_k} = \sum_{j=1}^{r_k} |\xi_j(z)|^2 |z|^{2d_k^j},$$

if $\xi = (\xi_1, \ldots, \xi_{r_k})$.

If $\psi$ is a holomorphic section of $\mathcal{O}(E_0)$ that annihilates $R$, i.e., the current $R \psi$ vanishes, then $\psi$ is in the sheaf $\mathcal{F} = \text{Im} f_1$, see [7, Proposition 2.3]. In order to represent the membership by an integral formula we will introduce a weight $g$ that contains $f_1(z)$ as a factor and apply Proposition 2.1. To this end we introduced in [3] a generalization of so-called Hefer forms, inspired by [3] and [7], to the case with non-trivial vector bundles.

**Definition 1.** We say that $H = (H_k^f)$ is a Hefer morphism for the complex $E_\bullet$ in (2.4) if $H_k^f$ are smooth sections of

$$\mathcal{L}^{-k+\ell}(\text{Hom}(E_{z,k}, E_{z,\ell}))$$

that are holomorphic in $z$, $H_k^f = 0$ for $k < \ell$, the term $(H_0^f)_{0,0}$ of bidegree $(0,0)$ is the identity $I_{E_0}$ on the diagonal $\Delta$, and

$$\nabla_\eta H_k^f = H_k^f \nabla f_k - f_{\ell+1}(z) H_k^{\ell+1},$$

with $\nabla_\eta = \text{grad} \mathcal{F}$ on $\mathbb{P}^N$.
where \( f_k \) stands for \( f_k(\zeta) \).

Notice that we do not require \( H \) to be holomorphic in \( \zeta \). Assume that \( H \) is a Hefer morphism for \( E_* \) and let \( U \) and \( R \) be the associated currents. We can then form the currents \( HU = \sum_j H^1_j U_j \) (notice the superscript 1 here) and \( HR = \sum_j H^0_j R_j \). To be precise with the signs one has to introduce a superbundle structure on \( E = \oplus E_k \); then for instance \( f \) is mapping of even order since it maps \( E_k \rightarrow E_{k-1} \) (and therefore \( f \) anti-commutes with odd order forms) whereas, e.g., \( H \) is even since \( H^\ell_k \) is a form of degree \( k-\ell \) (mod 2) that takes values in \( \text{Hom} (E_\ell, E_k) \), giving another factor \( k-\ell \) (mod 2). See [3, Section 5] for details.

Let \( f \) be a non-trivial section that vanishes on \( Z \). If \( \Re \lambda >> 0 \), then
\[
U^\lambda := |f_1|^2 U, \quad R^\lambda := 1 - |f_1|^2 + \bar{\partial}|f_1|^2 \wedge U
\]
are smooth forms. Moreover there admit current-valued analytic continuations to \( \Re \lambda > -\epsilon \), and the values at \( \lambda = 0 \) are \( U \) and \( R \), respectively, see, e.g., [7]. It is readily checked that
\[
\nabla f U^\lambda = I_{E_0} - R^\lambda,
\]
and from the proof of Proposition 4.2 in [4] it follows that
\[
g^\lambda := f_1(z) HU^\lambda + HR^\lambda
\]
is a weight (as long as \( \Re \lambda >> 0 \)) if \( H \) is a Hefer morphism for the complex \( E_* \). From Proposition 2.1 applied to \( g^\lambda \wedge g \), and letting \( \lambda = 0 \), we get the following interpolation-division formula.

**Proposition 2.3** ([4], Proposition 4.2). Assume that \( H \) is a Hefer morphism for the complex \( E_* \). If \( \psi \) is a holomorphic section of \( F \otimes E_0 \otimes \mathcal{O}(-N) \) and \( g \) is a weight with respect to \( F \rightarrow \mathbb{P}^N \), then we have the representation
\[
(2.10) \quad \psi(z) = f_1(z) \int_{\mathbb{P}^N} (HU^\lambda \wedge g)_{N,N} \psi + \int_{\mathbb{P}^N} (HR^\lambda \wedge g)_{N,N} \psi, \quad z \in \mathbb{P}^N.
\]

If \( R\psi = 0 \) we thus have the explicit holomorphic solution
\[
q(z) = \int_{\mathbb{P}^N} (HU^\lambda \wedge g)_{N,N} \psi
\]
to \( f_1 q = \psi \). Moreover, if \( \psi \) a priori is only defined in a neighborhood of \( Z \) and \( g \) depends holomorphically on \( z \), then the second integral in (2.10) is well-defined and provides a global holomorphic section \( \tilde{\psi} \) such that \( \tilde{\psi} - \psi \) belongs to the ideal sheaf generated by \( f_1 \). This is the reason for the notion division-interpolation formula. For a more precise interpolation result, see Proposition 3.1 below.

**Remark 2.4.** In [7] occur more general currents \( U^\ell_k \) and \( R^\ell_k \) taking values in \( \text{Hom} (E_\ell, E_k) \). With the same proof as below we get the more general formula
\[
\psi(z) = f_{\ell+1}(z) \int_{\mathbb{P}^N} (HU^\ell \wedge g)_{N,N} \psi + \int_{\mathbb{P}^N} (HR^\ell \wedge g)_{N,N} \psi + \int_{\mathbb{P}^N} (HU^{\ell-1} \wedge g)_{N,N} f_\ell \psi
\]
for holomorphic sections of $F \otimes E_\ell \otimes O(-N + \ell)$; here $HR^\ell = \sum_j H_j^\ell R^\ell_j$ and $HU^\ell = \sum_j H_j^{\ell+1} U^\ell_j$. If $f_\ell \psi = 0$ and $R^\ell \psi = 0$ we thus get an explicit holomorphic solution to $f_{\ell+1} q = \psi$.

2.2. A choice of Hefer morphism. Assume now that $E_\bullet$ is a complex with $E_k$ of the form (2.7) and choose $\kappa$ such that $\kappa \geq d_i^k$ for all $i,k$. We can then construct a Hefer morphism for the complex $E_\bullet \otimes O(\kappa)$. Notice that the currents $U$ and $R$ that are associated to $E_\bullet$ are also the associated currents to $E_\bullet \otimes O(\kappa)$. We thus obtain a division formula for sections $\psi$ of $E_0 \otimes O(\kappa - N)$.

From the complex $E_\bullet$ we form a complex $E'_\bullet$ of trivial bundles over $\mathbb{C}^{n+1}$ in the following way: Let $E'_k := E_1^k \oplus \cdots \oplus E_r^k k$ and take the mappings $F_k$ that are formally just the matrices $f_k$ of homogeneous forms. Let $\delta_{w-z}$ denote interior multiplication with $2\pi i \sum_0^N (w_j - z_j) \frac{\partial}{\partial w_j}$ in $\mathbb{C}_w^{N+1} \times \mathbb{C}_{z}^{N+1}$.

**Proposition 2.5.** There exist $(k-\ell,0)$-form-valued mappings

$$h_k^\ell = \sum_{ij} (h_k^\ell)_{ij} e_\ell e_i^* : \mathbb{C}_w^{n+1} \times \mathbb{C}_z^{n+1} \to \text{Hom} (E'_k, E'_\ell),$$

such that $h_k^\ell = 0$ for $k<\ell$, $h_\ell^\ell = I_{E'_\ell}$, and

$$-\delta_{w-z} h_k^\ell = h_k^{\ell-1} F_k(w) - F_{\ell+1}(z) h_k^{\ell+1},$$

and the coefficients in the form $(h_k^\ell)_{ij}$ are homogeneous polynomials of degree $d_k^1 - d_k^\ell - (k-\ell)$.

In [3, Section 4] there is an explicit formula that provides $h_k^\ell$ such that (2.11) holds. The components of $h_k^\ell$ of the desired degrees then must satisfy (2.11) as well. One can also check that the forms given by the formula actually have the desired degrees. Notice that

$$\gamma_j = d\zeta_j - \frac{\zeta \cdot d\zeta}{|\zeta|^2} \zeta_j, \quad j = 1, \ldots, N,$$

are projective forms such that

$$\nabla_\eta \gamma_j = 2\pi i (z_j - \alpha \zeta_j),$$

where $\alpha$ is the form in Example 2.2.

If $h(w,z)$ is a homogeneous form in $\mathbb{C}_w^{N+1} \times \mathbb{C}_z^{N+1}$ with differentials $dw$ and polynomial coefficients, we let $\tau^* h$ be the projective form obtained by replacing $w$ by $\alpha \zeta$ and $dw_j$ by $\gamma_j$. We then have

$$\nabla_\eta \tau^* h = \tau^* (\delta_{w-z} h),$$

in light of (2.12) and (2.2).

---

3The initial minus sign here is because we have $\delta_{w-z}$ rather than $\delta_{z-w}$ as in [3].
Proposition 2.6 ([4], Proposition 4.4). Assume that \( \kappa \geq d_k^j \) for all \( k \) and \( j \). Then

\[(2.14) \quad (H_E^E)^\ell_k = \sum_{ij} (r^* h_k^\ell)_{ij} \wedge \alpha - d_k^j e_{\ell,i} \otimes e_k^j \]

is a Hefer morphism for the complex \( E_\bullet \otimes \mathcal{O}(\kappa) \).

For degree reasons it is enough that \( \kappa \geq d_k^j \) for \( k \leq \min(M, N+1) \).

3. Integral Representation on \( X \)

Let \( i: X \to \mathbb{P}^N \) be a projective subvariety of pure dimension \( n \), let \( p := N - n \), and let \( \mathcal{J}_X \subset \mathcal{O}^{\mathbb{P}^N} \) be the radical sheaf. From the free resolution (1.1) of \( S/\mathcal{J}_X \) (of minimal length \( M \leq N \)) we can form the locally free sheaf complex \( \mathcal{O}(E_\bullet) \) as in (2.6), defined by

\[(3.1) \quad E_k = (E_1^k \otimes \mathcal{O}(-d_1^k)) \oplus \cdots \oplus (E_r^k \otimes \mathcal{O}(-d_r^k)) ,
\]

where \( E_k^1 \) are disjoint trivial line bundles, and the mappings \( a_j \) are formally the same mappings as in (1.1). It turns out, see, e.g., [7], that (3.1) is actually a (locally free) resolution of the sheaf \( \mathcal{O}^{\mathbb{P}^N}/\mathcal{J}_X \). Let \( R \) and \( U \) be the associated currents as above and recall that \( R = R_p + R_{p+1} + \cdots \).

In view of

Notice that

\[ \Omega := \delta_\xi \left( d\zeta_0 \wedge \cdots \wedge d\zeta_{N+1} \right) = \sum (-1)^j \zeta_j d\zeta_0 \wedge \cdots \wedge d\zeta_j \wedge \cdots \wedge d\zeta_N \]

is a non-vanishing section of the trivial bundle over \( \mathbb{P}^N \) realized as an \((N,0)\)-form with values in \( \mathcal{O}(N + 1) \). It follows from [6, Section 3] that there is a unique current \( \omega = \omega_0 + \cdots + \omega_k \), where \( \omega_k \) has bidegree \((n,k)\) and take values in \( E_k \otimes \mathcal{O}(N + 1) \), such that

\[(3.2) \quad i_* \omega = \Omega \wedge R, \quad i_* \omega_\ell = \Omega \wedge R_{p+\ell}.
\]

The form \( \omega \), in [6] it is called a structure form for \( X \), is smooth on \( X_{reg} \) and if \( \xi \) is a smooth form then

\[ \int_X \omega \wedge \xi \]

exists as a principal value at \( X_{sing} \); actually \( \omega \) is almost semi-meromorphic in the sense introduced in [6].

For any smooth form \( \xi \) on \( \mathbb{P}^N \) there is a unique form \( \vartheta(\xi) \) such that

\[(3.3) \quad \vartheta(\xi) \wedge \Omega = \xi_{N,*}.
\]

From (3.2) we have that

\[(3.4) \quad \xi_{N,*} \wedge R = \vartheta(\xi) \wedge \Omega \wedge R = i_* (\vartheta(\xi) \wedge \omega). \]

From Proposition 2.6 we have a Hefer morphism \( H^E_\kappa \) for the complex \( E_\bullet \otimes \mathcal{O}(\kappa_0) \).

Proposition 3.1. Let \( \ell \) be any integer and assume that \( g \) is a smooth weight on \( \mathbb{P}^N \) with respect to \( \mathcal{O}(\ell - \kappa_0 + N) \) and \( z \in X \). For holomorphic sections \( \phi \) of \( \mathcal{O}(\ell) \) over \( X \) we have the representation

\[ \phi(z) = \int_X \vartheta(g \wedge H^E_\kappa) \wedge \omega \phi. \]
Proof. In view of (2.9) and (2.14),
\[ g^\lambda = a_1(z)H^E_{\kappa_0}U^\lambda + H^E_{\kappa_0}R^\lambda \]
is a smooth weight in \( \mathbb{P}^N \) with respect to \( O(\kappa_0) \) and \( z \) if \( \text{Re} \lambda >> 0 \). Since \( a_1(z) = 0 \),
\[ g^\lambda = H^E_{\kappa_1}R^\lambda. \]

Let \( \Phi_0 \) be a smooth global section of \( O(z) \) that is equal to \( \phi \) on \( X \) and holomorphic in a neighborhood (in \( \mathbb{P}^N \)) of \( z \). Since \( B \) is smooth outside \( z \),
\[ \nabla_n B = 1 \] there and \( \partial \Phi_0 = 0 \) in a neighborhood (in \( \mathbb{P}^N \)) of \( z \),
\[ \Phi := \Phi_0 - \bar{\partial} \Phi_0 \wedge B \]
is a smooth \( \nabla_n \)-closed section of \( L^0(O(\ell - \kappa_0 + N)) \) on \( \mathbb{P}^N \). Let \( \tilde{g} \) be a smooth weight with respect to \( O(\ell - \kappa_0 + N) \) in \( \mathbb{P}^N \). Then
\[ \nabla_n (g^\lambda \wedge \tilde{g} \wedge \Phi \wedge B) = g^\lambda \wedge \tilde{g} \wedge \Phi (1 - [z]) = g^\lambda \wedge \tilde{g} \wedge \Phi - (g^\lambda \wedge \tilde{g} \wedge \Phi)_{0,0} \wedge [z]. \]

Notice that \( (g^\lambda \wedge \tilde{g} \wedge \Phi)_{0,0} \) is equal to \( \phi(z) \) at the point \( z \). By Stokes’ theorem we thus find, cf., the discussion preceding Section 2.1, that
\[ \phi(z) = \int_{\mathbb{P}^N} g^\lambda \wedge \tilde{g} \wedge \Phi. \]

Taking \( \lambda = 0 \), we get
\[ \phi(z) = \int_{\mathbb{P}^N} H^E_{\kappa_0}R \wedge \tilde{g} \wedge \Phi = \int_X \partial (\tilde{g} \wedge H^E_{\kappa_0}) \wedge \omega \phi, \]
cf., (3.4), since the \( i^* \Phi = \phi \). \( \square \)

Example 3.2. If \( \ell \geq \kappa_0 - N \), then we can take \( g = \alpha^{\ell - \kappa_0 + N} \) in Proposition 3.1. Since this weight is holomorphic for all \( z \in \mathbb{P}^N \), the integral then provides a holomorphic extension to \( \mathbb{P}^N \) of \( \phi \). Thus we get an “explicit” proof of the surjectivity of the natural restriction mapping
\[ \Gamma(\mathbb{P}^N, O(\ell)) \to \Gamma(X, O(\ell)) \]
for \( \ell \geq \kappa_0 - N \).

4. Division formulas on \( X \)

Let \( f_1, \ldots, f_m \) be our sections of \( O(d) \) on \( X \) from Section 1.4 and let \( J_f \) be the associated ideal sheaf on \( X \). If \( X \) is smooth we can find a resolution of \( O^X/J_f \) over \( X \) and obtain a residue current whose annihilator is precisely \( J_f \). We can then form a division-interpolation formula like (1.3), if \( \rho \) is big enough, such that the remainder term vanishes as soon as \( \phi \) is in \( J_f \). However, our main objective is to find an explicit representation of the solutions in Theorem 1.1 and to this end we use the Koszul complex generated by the \( f_j \). In this way we get a residue current that is explicitly defined and, moreover, perfectly adapted to the theorem. For the case with different degrees of \( f_j \), see [4].

Let \( E' \) be a trivial rank \( m \) bundle with basis elements \( e_1, \ldots, e_m \), let \( E := E' \otimes O(-d) \), and let \( e^*_i \) be the dual basis elements for \( (E')^* \) so that

\[ \text{In the intermediate sections } f_j \text{ stand for mappings in a complex in Sections 2.1 and 2.2, we hope this will not cause any confusion.} \]
Lemma 4.1. If $\delta_f$ denotes contraction with $f$. If we let $E_k := \mathcal{O}(-dk) \otimes \Lambda^k E'$ we thus have a complex like (2.4) and the associated currents

$$0 \to \mathcal{O}(-md) \otimes \Lambda^m E' \overset{\delta_f}{\to} \cdots \overset{\delta_f}{\to} \mathcal{O}(-2d) \otimes \Lambda^2 E' \overset{\delta_f}{\to} \mathcal{O}(-d) \otimes E' \overset{\delta_f}{\to} \mathbb{C} \to 0,$$

where $\delta_f$ denotes contraction with $f$. Let us describe them in more detail. In $\mathbb{P}^N \setminus Z_f$ we define the section

$$\sigma = \sum_{j=1}^m \overline{f_j(z)} e_j / |f(z)|^2$$

of $\mathcal{O}(d) \otimes E$. If $\bar{f} \cdot e = \sum \overline{f_j} e_j$ and $d\bar{f} \cdot e = \sum df_j \wedge e_j$, then

$$\sigma \wedge (\bar{\partial} \sigma)^k = \frac{\bar{f} \cdot e \wedge (d\bar{f} \cdot e)^{k-1}}{|f|^{2k}}.$$

It turns out, cf., Section 2.1 and, e.g., [4],

$$U^{f, \lambda} := |f|^{2\lambda} \sum_{k=1}^m \overline{f} \cdot e \wedge (d\overline{f} \cdot e)^{k-1} / |f|^{2k}$$

and

$$R^{f, \lambda} := 1 - |f|^{2\lambda} + \bar{\partial}|f|^{2\lambda} \Lambda \sum_{k=1}^m \overline{f} \cdot e \wedge (d\overline{f} \cdot e)^{k-1} / |f|^{2k}.$$

From Section 2.2 we know that there is a Hefer morphism $H^{f}_{\kappa'}$ for the complex (4.1) with $\kappa' := d \min(m, N + 1)$, and as noticed in Section 2.1

$$g^{f, \lambda}_{\kappa'} := f(z) \cdot H^{f}_{\kappa'} U^{f, \lambda} + H^{f}_{\kappa'} R^{f, \lambda}$$

is then a smooth global weight with respect to $\mathcal{O}(\kappa')$ on $\mathbb{P}^N$ when $\text{Re} \lambda >> 0$. For an explicit choice of Hefer form, see the end of this section.

If now $\phi$ is a section of $\mathcal{O}(\rho)$ over $X$ and $g$ is a weight on $\mathbb{P}^N$ with respect to $\mathcal{O}(\rho - \kappa' - \kappa_0 + N)$ it follows from Proposition 3.1 that

$$\phi(z) = \int_X \partial (g^{f, \lambda}_{\kappa'} \wedge g \wedge H^{E}_{\kappa_0}) \wedge \omega \phi.$$

Let

$$\kappa := d \min(m, n + 1).$$

Since terms of $g^{f, \lambda}_{\kappa'}$ of bidegree $(k, k)$, $k > n$, vanish on $X$, one sees that $g^{f, \lambda}_{\kappa'} = \alpha \kappa' - \kappa \wedge g^{f, \lambda}_{\kappa'}$ on $X$ where $g^{f, \lambda}_{\kappa'}$ is smooth on $X$.

Lemma 4.1. If $\tilde{g}$ is a weight with respect to $\mathcal{O}(\rho - \kappa - \kappa_0 + N)$, then we have the representation

$$\phi(z) = \int_X \partial (g^{f, \lambda}_{\kappa'} \wedge \tilde{g} \wedge H^{E}_{\kappa_0}) \wedge \omega \phi, \quad z \in X.$$
***Proof.*** Fix $z \in X$. Let $\chi(t)$ be a cutoff function that is 1 when $t > 3/4$ and 0 when $t < 1/4$, let

$$\chi_\delta := \chi\left(\frac{|\zeta \cdot z|^2}{|z|^2}|\zeta|^2\delta\right),$$

and

$$g_\delta := \chi_\delta - \partial\chi_\delta \wedge B.$$ 

Since $g_\delta$ vanishes in a neighborhood of the hyperplane $\zeta \cdot z = 0$, $\alpha^{-r} \wedge g_\delta$ is a smooth weight with respect to $O(-r)$ and $z$. Thus we can choose $g = \chi_\delta \alpha^{-r} \wedge \tilde{g}$ in (4.2), where $r = \kappa' - \kappa$ and get

$$\phi(z) = \int_X A(\zeta, z) \phi(\zeta) + \int_X B(\zeta, z) \wedge R(\zeta) \phi(\zeta),$$

where

$$A(\zeta, z) := \vartheta(\alpha^L \wedge H^E_{\kappa_0}) \wedge U^f \wedge \omega,$$

$$B(\zeta, z) := \vartheta(\alpha^L \wedge H^E_{\kappa_0})$$

and

$$R := R^f \wedge \omega.$$
From above it is clear that $A(\zeta, z)$ is holomorphic in $z \in X$ and that, for fixed $z$, is the product of the principal value current $U^f \wedge \omega$ and a smooth form. Moreover, $B(\zeta, z)$ is smooth and holomorphic in $z \in X$. If $R^f \wedge \omega \phi = 0$, then the second term in (4.7) vanishes and so we get a solution $q$ to $f \cdot q = \phi$.

**Remark 4.3.** If $X$ is smooth, then $\omega$ is smooth as well. If in addition $\text{codim } Z_f = m$, then $R^f \wedge \omega \phi = 0$ if and only if $\phi$ is in the sheaf $\mathcal{F}_f$, see [7]. Moreover, $R^f = R^f_m$ coincides with the classical Coleff-Herrera product $\overline{\partial} f^m \wedge \cdots \wedge \overline{\partial} f^1$ on $X$, cf., [2] p. 112. Since $\omega$ is smooth, thus $R^f \wedge \omega = R^f_m \wedge \omega$.

We can express $H_f^\kappa$ somewhat more explicitly. Let $\tilde{h}_j(w, z)$ be $(1, 0)$-forms in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ of polynomial degrees $d - 1$ such that

$$
\delta w - z \tilde{h}_j = f_j(w) - f_j(z)
$$

and let $h_j = \tau^* \tilde{h}_j$. Let

$$
h := h_1 \wedge e_1 + \cdots + h_m \wedge e_m.
$$

If $\delta_h$ is interior multiplication with $h$ and $(\delta_h)^k := (\delta_h)^k/k!$, then

$$
(H_f^\kappa)_k^\ell = \alpha^{n - dk}(\delta_h)^k - \ell,
$$

see [4] Section 5]. Recall that $(H_f^\kappa)_k^\ell$ occurs in (4.7) with $\ell = 1$ in the first integral and with $\ell = 0$ in the second one.

### 4.1. Explicit representation of the solutions in Theorem 1.1

Assume that all the hypotheses in Theorem 1.1 (i) hold, let $f_j$ be the $d$-homogenizations of $F_j$, let

$$
\rho = \max(\text{deg } + (\mu + \mu_0)d^\infty \text{deg } X, d \min(n, m + 1) + \kappa_0 - N),
$$

and let $\phi$ be the $\rho$-homogenization of $\Phi$, cf., Section 1. It follows from the proof of Theorem 1.1 in [8] Section 6] that $R^f \wedge \omega \phi = 0$ so that the second term in (4.7) vanishes. In view of the discussion succeeding Theorem 1.1 above, thus the dehomogenization of (4.8)

$$
q = \sum_{k=1}^{\min(m, n+1)} \int_X \partial \left[ \alpha^{\rho - \kappa_0 + N - dk} \wedge (\delta_h)^k \cdot e \wedge (d \tilde{f} \cdot e)^{k-1} \wedge H_{E_{\kappa_0}}^E \right] \wedge \omega \phi
$$

is a tuple of polynomials $Q_j$ such that (1.7) and (1.8) hold. There is an analogous representation of the solution in Theorem 1.1 (ii).

The integral in (1.8) is defined as a principal value at $Z_f \cup X_{\text{sing}}$. For instance let $h$ be a tuple (of holomorphic sections of line bundles on $X$) that vanishes on $Z_f \cup X_{\text{sing}}$ and let $\chi(t)$ be (a smooth approximand of) the characteristic function for the interval $[1, \infty)$. If we multiply the integrand in (1.8) by $\chi(|h|^2/\delta)$ then the integral exists in the ordinary sense, and the principal value is obtained by letting $\delta \to 0$. One can just as well multiply
the integrand by $|h|^{2\lambda}$ for $\text{Re} \lambda >> 0$. The principal value is then obtained by analytic continuation to $\lambda = 0$.

**Remark 4.4.** If $m \leq n$ and $|\psi| \leq |f|^{\min(m,n)}$, then $U^J \psi$ is integrable on $X_{reg}$ so (4.8) is a convergent integral locally on $X_{reg}$. In general, however, $U^J \psi$ may be a distribution of higher order than zero, and then the integral in (4.8) must be regarded as a principal value even on $X_{reg}$.

5. **THE CASE WHEN $X$ IS A (REDUCED) HYPERSURFACE**

In this section we illustrate the results in the special case when $X$ is a reduced hypersurface in $\mathbb{P}^{n+1}$. We thus assume that $X = \{a = 0\}$ where $a$ is a section of $\mathcal{O}(\kappa_0)$ in $\mathbb{P}^{n+1}$, i.e., $a = a(\zeta_0, \ldots, \zeta_{n+1})$ is a $\kappa_0$-homogeneous polynomial in $\mathbb{C}^{n+2}$, and $da \neq 0$ on (the pull-back to $\mathbb{C}^{n+2}$ of) $X$. We first discuss the general representation formula in Proposition 3.1 in this case. It can of course be obtained from this proposition but we find it instructive to derive it directly from the general representation formula (Proposition 2.1) on $\mathbb{P}^{n+1}$.

Notice that $\|a\| := |a|/|\zeta|^\kappa_0$ is the natural pointwise norm of $a$ considered as a section of $\mathcal{O}(\kappa_0)$. It is well-known that $\partial B(a)$ is a current-valued analytic continuation to $\text{Re} \lambda > -\epsilon$, and that the value at $\lambda = 0$ is $\partial(1/a)$, i.e., $\partial$ applied to the principal value current $1/a$.

**Remark 5.1.** We have the locally free sheaf resolution

$$0 \rightarrow \mathcal{O}(\kappa_0) \rightarrow \mathcal{O}(0)$$

of $\mathcal{O}^n/J_X$, and it is readily checked that the associated residue current $R^E$ is the current $\partial B(1/a)$ times a homomorphism (from $\mathcal{O}(0)$ to $\mathcal{O}(\kappa_0)$) that has odd order. However since we will derive the representation formula on $X$ directly we do not have to bother about these sign problems. Notice that the number $\kappa_0$ here is consistent with the definition in the general case.

Let $h^\alpha(z, w)$ be a $(1,0)$-form such that $\delta_{w-z} h^\alpha = a(z) - a(w)$ on $\mathbb{C}^{n+2} \times \mathbb{C}^{n+2}$, and let us write $h^\alpha(z, \alpha \zeta)$ for $\tau^* h$, cf. Section 2.2. If $\text{Re} \lambda > 0$, then

$$g^\lambda := \alpha^\kappa_0 - \nabla_\eta(h^\alpha(z, \alpha \zeta)\|a\|^{2\lambda}/a)$$

is a weight in $\mathbb{P}^{n+1}$ with respect to $\mathcal{O}(\kappa_0)$ and $z$. If $\phi$ is a holomorphic section of $\mathcal{O}(\ell)$ and $g$ is a weight with respect to $\mathcal{O}(\ell - \kappa_0 + n + 1)$, then we have from Proposition 2.1 the representation

$$\phi(z) = \int_{\mathbb{P}^{n+1}} g^\lambda \wedge g\phi.$$

Since

$$-\nabla_\eta h^\alpha(z, \alpha \zeta) = a(z) - a(\alpha \zeta) = a(z) - \alpha^\kappa_0 a(\zeta),$$

we have that

$$g^\lambda = (1 - \|a\|^{2\lambda})\alpha^\kappa_0 + \frac{a(z)}{a(\zeta)}\|a\|^{2\lambda} + h^\alpha(z, \alpha \zeta)\wedge \partial\|a\|^{2\lambda}/a.$$
vanishes when \( \lambda = 0 \) by the dominated convergence theorem. Let us now assume that \( z \in X \) so that \( a(z) = 0 \). We then have

\[
\phi(z) = \int_{\mathbb{P}^{n+1}} g \wedge h^0(z, \alpha \zeta) \wedge \overline{\partial} \frac{1}{a} \phi, \quad z \in X.
\]

Arguing as in the proof of Proposition 3.1 it is enough to assume that \( \phi \) a priori is defined on \( X \).

We want to write the right hand side in (5.2) as a principal value integral over \( X \). As before, let

\[
\Omega := \delta_\zeta d\zeta,
\]

where

\[
d\zeta := d\zeta_0 \wedge \ldots \wedge d\zeta_{n+1}.
\]

Recall, cf., (3.2), that the form \( \omega \) on \( X \) is defined by the equality\(^6\)

\[
i_\ast \omega = \overline{\partial}(1/a) \wedge \Omega.
\]

We shall give an explicit representation of \( \omega \). Let

\[
\partial_j a = \frac{\partial a}{\partial \zeta_j}, \quad j = 0, \ldots, n+1, \quad |\partial a|^2 = |\partial_0 a|^2 + \cdots + |\partial_{n+1} a|^2,
\]

let \( \delta_A \) denote interior multiplication by

\[
2\pi i \frac{1}{|\partial a|^2} \sum_{j=0}^{n+1} \partial_j a \partial_j,
\]

and define

\[
(5.4) \quad \omega' := \delta_A \Omega = \delta_A \delta_\zeta d\zeta.
\]

**Lemma 5.2.** For any test form \( \xi \) we have that

\[
(5.5) \quad \int_{\mathbb{P}^{n+1}} \xi \wedge \overline{\partial} \frac{1}{a} \wedge \Omega = \int_X \xi \wedge \omega'.
\]

In view of (5.3) we thus have that

\[
\omega = i \ast \omega'.
\]

**Proof.** Notice that

\[
Da := da - \kappa_0 \overline{\zeta} \cdot \frac{d\zeta}{|\zeta|^2} a
\]

is the Chern connection on \( \mathcal{O}(\kappa_0) \) acting on \( a \). One can verify directly that \( Da \) is a projective form, since by the \( \kappa_0 \)-homogeneity of \( a \), \( \partial_0 a \zeta_0 + \cdots + \partial_{n+1} a \zeta_{n+1} = \kappa_0 a \). In any case, \( \delta_\zeta(Da) = 0 \) so we have

\[
Da \wedge \omega' = Da \wedge \delta_A \delta_\zeta d\zeta = \delta_\zeta(Da \wedge \delta_A d\zeta) = \delta_\zeta(\delta_A Da \wedge d\zeta) = 2\pi i \left( 1 - \frac{\kappa_0}{|\partial a|^2} \frac{\kappa_0 a}{|\zeta|^2} \right) \Omega.
\]

In particular,

\[
Da \wedge \omega' = 2\pi i \Omega
\]

---

\(^6\)Since \( R^E \) has even degree this is consistent with (3.2), cf. Remark 3.1.
on $X$. By the Poincaré-Lelong lemma,

$$\bar{\partial} \frac{1}{a} \wedge Da = 2\pi i [X],$$

and therefore

$$2\pi i \bar{\partial} \frac{1}{a} \wedge \Omega = \bar{\partial} \frac{1}{a} \wedge Da \wedge \omega' = 2\pi i [X] \wedge \omega'$$

which is the same as (5.4) $\Box$.

If $\xi$ has bidegree $(n+1,n)$, then by (5.4),

$$\delta_A \xi = \delta_A (\vartheta(\xi) \wedge \Omega) = (-1)^n \vartheta(\xi) \wedge \omega'.$$

Moreover,

$$\vartheta(\xi) \wedge \Omega \wedge \bar{\partial} \frac{1}{a} = (-1)^{n+1} \vartheta(\xi) \wedge \bar{\partial} \frac{1}{a} \wedge \Omega.$$

From (5.2) and Lemma 5.2 we thus get

**Proposition 5.3.** If $\phi$ is a holomorphic section of $O(\ell)$ and $g$ is a weight with respect to $O(\ell - \kappa_0 + n + 1)$, then we have the representation

(5.6) $\phi(z) = (-1)^{n+1} \int_X \vartheta(g \wedge h^a(z, \alpha \zeta)) \wedge \omega' \phi = - \int_X \delta_A(h^a \wedge g) \phi$.

Given the situation in Section 4.1, the dehomogenization of

(5.7) $q = \sum_{k=1}^{\min(m,n+1)} (-1)^{n+1}$

$$\int_{\alpha=0} \vartheta \left[ \alpha^{p-\kappa_0+n+1-dk} \wedge (\delta_h)_{k-1} \frac{\tilde{f} \cdot e \wedge (d \tilde{f} \cdot e)^{k-1}}{|\tilde{f}|^{2k}} \wedge h^a(z, \alpha \zeta) \right] \wedge \omega' \phi.$$

is thus a tuple of polynomials $Q_j$ such that (1.7) and (1.8) hold on $X$.

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