Dirac Equation in Standard Cosmological Models

M. Sharif *
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus Lahore-54590, PAKISTAN.

Abstract

The time equation associated to the Dirac Equation (DE) is studied for the radiation-dominated Friedmann-Robertson-Walker (FRW) universe. The results are analysed for small and large values of time. We also incorporate the corrections of the paper studied by Zecca [1] for the matter-dominated FRW universe.

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1 Introduction

It is of cosmological interest to consider a universe of incoherent radiation. The primordial radiation played a dominant role in the early universe for the times smaller than the times of re-combination. Zecca [1] has discussed the time equation associated to the DE in the FRW spacetime in the case of Standard Cosmology. He confined his paper to the study of matter-dominated cosmological model only. This paper is dedicated to a further study of the time equation associated to the DE of a radiation-dominated cosmological model. Also, we will incorporate the corrections in the expansion terms of Eqs.(14) and (17) in the Zecca’s paper [1].

It is well known that the DE can be written in General Relativity in terms of covariant derivatives and generalised Pauli matrices. In the context of the Newmann-Penrose formalism [2], the DE can further be expressed in terms of directional derivatives and spin coefficients. The DE, formulated by the spinorial formalism of Newmann and Penrose [3], is given by

$$\nabla_{AA'} P^A + \mu_\alpha \bar{Q}_{A'} = 0,$$

$$\nabla_{AA'} Q^A + \mu_\alpha \bar{P}_{A'} = 0,$$

(1)

where $\nabla_{AA'}$ are the covariant spinor derivatives and $\mu_\alpha \sqrt{2}$ is the mass of the particle [4]. If we correspond $\phi \leftrightarrow (P, Q), \psi \leftrightarrow (U, V)$, we can define the spinor

$$J^{AA'}(\phi, \psi) = P^A \bar{U}^{A'} + V^A \bar{Q}^{A'}$$

(2)

This is divergence free, i.e., $\nabla_{AA'} J^{AA'}$ $\equiv$ $\nabla_\alpha J^\alpha = 0$ due to Eq.(1). We can define inner product between the solutions of the DE by setting [5]

$$\langle \phi, \psi \rangle = \int_{\Sigma} J_\alpha(\phi, \psi)(-g_\Sigma(x))^{1/2} n^\alpha d\Sigma$$

(3)

$$= \int_{t=t_0} d^3x(-g_{t_0})^{1/2} \sigma^t_{AA'} J^{AA'}(\phi, \psi)$$

(4)

$$= \frac{1}{2} \int_{t=t_0} d^3x(-g_{t_0})^{1/2}(P^0 U^0 + P^1 U^1 + V^0 \bar{Q}^0 + V^1 \bar{Q}^1).$$

(5)
The Eq.(3) is independent of the spacelike Cauchy hypersurface $\Sigma$ of volume element $d\Sigma$, $n^\alpha$ is a future directed unit vector orthogonal to $d\Sigma$, $\sigma'_{AA'}$ is the generalised Pauli matrix and has the form $\sigma'_{AA'} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The separation of the angular part in Eq.(1) is performed by a standard separation method which gives the Teukolski-like equation for spin(1/2) field admitting explicit solution [6]. The surviving coupled equations in the time and radial variables are separated by writing the unknown wave function in terms of a known particular solution. These equations can be expressed in the conformal time parameter $\tau$ and in the spatial parameter $s$ defined by

$$\tau(t) = \int_0^t \frac{dt'}{R(t')}, \quad s(r) = \int_0^r \frac{dr'}{\sqrt{1 - a r'^2}}, \quad (a = 0, \pm 1). \quad (6)$$

The complete spinorial solution of Eq.(1) has been found and is of the form

$$P^i = \frac{s^{(i+1)}_l}{2 R(t)} H(r,t) \left\{ (-1)^i A_{lk}(s) T_k(\tau) + \left[ \frac{2\lambda}{r} A_{lk}(s) - A_{lk}'(s) \right] \int_0^\tau d\tau T_k \right\},$$

$$Q_i = -\frac{s^{(i+2)}_l}{2 R(t)} H(r,t) \left\{ A_{lk}(s) T_k(\tau) + (-1)^i \left[ \frac{2\lambda}{r} A_{lk}(s) - A_{lk}'(s) \right] \int_0^\tau d\tau T_k \right\}, \quad (7)$$

where $i = 0, 1$. The angular functions appearing in Eq.(7) are of the form $S^{(j)}_{lm} = S^{(j)}_{lm}(\theta, \phi)'s(j = 1, 2)$, where $S^{(j)}_{lm}(\theta, \phi)$ are solutions of an eigen-value problem originated by the solution of the angular part of Eq.(1). The $A_{lk}(s)'s$ are the solutions of the angular and radial equations, respectively whose explicit expressions can be found in [7]; $\lambda, k$ are separation constants such that $\lambda^2 = (l + 1)^2$, $l = 0, 1, 2, ...$ for $m = 0$; $\lambda^2 = \frac{(l + 1/2)^2}{2}, l = |m|$, $|m| + 1, |m| + 2, ...$ for $m = 0, \pm 1, \pm 2, \pm 3, ...$. The angular functions are assumed to satisfy the normalization condition

$$\int d\Omega S^{(j)}_{lm}(\theta, \phi) S^{(j)}_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}. \quad (8)$$

The function $H$ is connected to a particular integral of Eq.(1) and has the form

$$H(r,t) = \frac{1}{R^{l/2}(t)} \left( 1 + \sqrt{1 - ar^2} \right)^\lambda \exp[i \mu_\ast \sqrt{2t}], \quad (a = 0, \pm 1). \quad (9)$$
The function $T_k(\tau)$ is a solution of the separated time equation in the conformal time parameter $\tau$ and satisfies [7] the equation

$$T'' + 2\sqrt{2} \mu_* R(\tau) T' + (2\sqrt{2} \mu_* R'(\tau) + k^2) T = 0,$$

(10)

where prime denotes differentiation with respect to $\tau$. The solution of Eq.(10) depends on the dynamical evolution of the cosmological background and this satisfies a property similar to the Wronskian property of the scalar field time equation in FRW spacetime [8,9]. By using a first formal integration of Eq.(10), one can easily show that any solution $T_k(\tau)$ of Eq.(10) satisfies the relation

$$|T_k(\tau)|^2 + k^2 \left| \int_0^\tau T_k(\hat{\tau}) d\hat{\tau} \right|^2 = \text{constant} = 1.$$

(11)

This normalization is a necessary requirement of a second quantization of the Dirac field. It is noted that Eq.(10) can be solved in particular simple cases such as that of static universe or of the neutrino case. However, it seems difficult to give the solution of Eq.(10) for a general cosmological evolution.

The layout of this paper is as follows. In section two, we analyse the matter-dominated FRW universe to incorporate the corrections studied by Zecca [1]. In the next section we shall extend this procedure to study the solutions for the radiation-filled cosmological model. Section four is devoted to discussion of the results.

### 2 Solutions in the Matter-Dominated Cosmological Model

It is well known that the FRW spacetime whose metric

$$ds^2 = dt^2 - R^2(t) \left[ \frac{dr^2}{1-ar^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (a = 0, \pm 1)$$

(12)

is the base of the standard model [10,11]. This is the natural context for the formulation of the Einstein Field Equation for a general isotropic and homogeneous universe. We shall first consider the matter-dominated universe
for the purpose of correction in [1]. For matter-dominated universe, \( R(t) \) in parametric form is given as

\[
R(t) = E t^{2/3}, \quad (a = 0),
\]
\[
R(t) = A (\cosh \psi - 1), \quad t = B (\sinh \psi), \quad \psi \geq 0, \quad (a = -1),
\]
\[
R(t) = C (1 - \cos \theta), \quad t = D (\theta - \sin \theta), \quad 0 \leq \theta \leq 2\pi, \quad (a = 1).
\]  

(13)

In terms of the conformal time parameter \( \tau \) defined in Eq.(6) we obtain

\[
R(\tau) = E^3 \tau^2 / 9, \quad (a = 0),
\]
\[
R(\tau) = A [\cosh(\tau A / B) - 1], \quad (a = -1),
\]
\[
R(\tau) = C [1 - \cos(\tau C / D)], \quad (a = 1).
\]  

(14)

If we use Eq.(14) in Eq.(10) they all give a linear differential equation with finite regular point of the form

\[
T'' + \left( \sum_{n=0}^{\infty} p_n \tau^n \right) T' + \left( \sum_{n=0}^{\infty} q_n \tau^n \right) T = 0.
\]  

(15)

By using a standard method [12], we can set

\[
T = \sum_{n=0}^{\infty} c_n \tau^n,
\]  

(16)

where the coefficients \( c_n \) can be found by the recurrence relation

\[
(n + 2)(n + 1)c_{n+2} + \sum_{j=0}^{\infty} p_j (n + 1 - j) c_{n+1-j} + \sum_{j=0}^{\infty} q_j c_{n-j} = 0, \quad n = 0, 1, 2, ...
\]  

(17)

We shall always take \( c_1 = 0 \) due to our interest only in the time integral.

For the flat model \( a = 0 \), the recurrence relation (14) in [1] will become

\[
T = c_0 \left\{ 1 - \frac{\tau^2}{2} k^2 - \frac{\tau^3}{9} 2\alpha E^3 / 9 + \frac{\tau^4}{4!} k^4 + \frac{\tau^5}{5!} 14\alpha k^2 E^3 / 9 \\
+ \frac{\tau^6}{6!} [40(\alpha E^3 / 9)^2 - k^6] - \frac{\tau^7}{7!} 14\alpha E^3 k^4 / 9 + \frac{\tau^8}{8!} [k^8 - 628k^2(\alpha E^3 / 9)^2] \\
+ \frac{\tau^9}{9!} [100\alpha E^3 k^6 / 9 - 2240(\alpha E^3 / 9)^3] + \ldots \right\},
\]  

(18)

where \( \alpha = 2\sqrt{2} \epsilon \mu_* \).
For the open model \( a = -1 \), the recurrence relation (17) in [1] will take the form

\[
T = c_0 \left\{ 1 - \frac{\tau^2}{3} k^2 - \frac{\tau}{3} \alpha A^3 / B^2 + \frac{\tau}{4} \alpha A^5 / B^4 \right\} + \frac{\tau}{6} \left[ 10(\alpha A^3 / B^2)^2 - k^2 \right] + \frac{\tau}{7} (16\alpha A^5 k^2 / B^4 - 22\alpha A^3 k^4 / B^2 - \alpha A^7 / B^6)
\]

By combining Eqs.(10),(13) and (14) one has \( T \sim R^{-1} \sim t^{-2/3} \sim \tau^{-2} \) for \( \tau \gg 1 \). Thus by using Eqs.(7),(9) and (18), the time dependence of \( H \) will take the form

\[
t^{-1/3} \exp[i \mu \sqrt{2t}], \quad (t \gg 1, a = 0).
\]

3 Solutions in Radiation-Dominated Cosmological Model

In this section we shall extend the above procedure to solve Eq.(10) for radiation-filled FRW universe model. For radiation-dominated FRW spacetimes, the scale factor \( R \) is given as

\[
R(t) = E t^{1/2}, \quad (a = 0),
R(t) = A \sinh \psi, \quad t = B(\cosh \psi - 1), \quad \psi \geq 0, \quad (a = -1),
R(t) = C \sin \theta, \quad t = D(1 - \cos \theta), \quad 0 \leq \theta \leq 2\pi, \quad (a = 1).
\]

We can write these values of scale factor in terms of the conformal time parameter \( \tau \) as

\[
R(\tau) = E^2 \tau / 2, \quad (a = 0),
R(\tau) = A \sinh(\tau A / B), \quad (a = -1),
R(\tau) = C \sin(\tau C / D), \quad (a = 1).
\]

(i) The Flat Model \( a=0 \)

By using Eqs.(10), (15) and the 1st equation in (22), we can have

\[
\begin{align*}
p_0 &= 0, \quad p_1 = \alpha E^2 / 2, \quad p_n = 0, \quad n \neq 1, \\
q_0 &= \alpha E^2 / 2 + k^2, \quad q_n = 0, \quad n \geq 1.
\end{align*}
\]
With the help of Eq.(23) in the recurrence relation (17), we can find all values of \( c \). Replacing these values of the \( c \)'s in Eq.(16), we get

\[
T = c_0 \{ 1 - \frac{\tau^2}{2} (\alpha E^2/2 + k^2) + \frac{\tau^4}{4!} (3\alpha E^2/2 + k^2)(\alpha E^2/2 + k^2) \\
- \frac{\tau^6}{6!} (5\alpha E^2/2 + k^2)(3\alpha E^2/2 + k^2)(\alpha E^2/2 + k^2) \\
+ \frac{\tau^8}{8!} (7\alpha E^2/2 + k^2)(5\alpha E^2/2 + k^2)(3\alpha E^2/2 + k^2)(\alpha E^2/2 + k^2) + ... \}. 
\]  
(24)

(ii) The Open Model \( a=-1 \)

From Eqs.(10), (15) and the second in Eq.(22) with the use of the series expansions of \( \cosh x, \sinh x \), we can make the identifications

\[
p_0 = 0, \quad p_{2n-1} = \frac{\alpha A}{(2n-1)!} (A/B)^{2n-1}, \quad n \geq 1, \\
q_0 = \alpha A^2/B + k^2, \quad q_{2n} = \frac{\alpha A^2}{B(2n)!} (A/B)^{2n}, \quad n \geq 1. \tag{25}
\]

Making use of these values in the recurrence relation (17) to find \( c \)'s and then Eq.(16) yields

\[
T = c_0 \{ 1 - \frac{\tau^2}{2} (\alpha A^2/B + k^2) + \frac{\tau^4}{4!} [(3\alpha A^2/B + k^2)(\alpha A^2/B + k^2) \\
- \alpha (A^2/B)(A/B)^2 - \frac{\tau^6}{6!} [(5\alpha A^2/B + k^2)((3\alpha A^2/B + k^2) \\
(\alpha A^2/B + k^2) - \alpha (A^2/B)(A/B)^2 - 10\alpha (A^2/B)(A/B)^2 \\
(\alpha A^2/B + k^2) + \alpha (A^2/B)(A/B)^4] + \frac{\tau^8}{8!}[(7\alpha A^2/B + k^2) \\
(5\alpha A^2/B + k^2) (\alpha A^2/B + k^2) \\
- \alpha (A^2/B)(A/B)^2)] - 10\alpha (A^2/B)(A/B)^2 (\alpha A^2/B + k^2) \\
+ \alpha (A^2/B)(A/B)^4] - 35\alpha (A^2/B)(A/B)^2 ((3\alpha A^2/B + k^2) \\
(\alpha A^2/B + k^2) - \alpha (A^2/B)(A/B)^2) + 3\alpha (A^2/B)(A/B)^4 \\
(\alpha A^2/B + k^2) - \alpha (A^2/B)(A/B)^6) + ... \}. \tag{26}
\]

(iii) The Closed Model \( a=1 \)

The closed model can be obtained from the open model \( (a = -1) \) by replacing \( A \) by \( \i C \) and \( B \) by \( -D \). Using the normalization Eq.(11), we can have \( c_0 = 1 \) in every case.
4 Discussion of the Results

We see from Eqs. (24) and (26) that when \( \tau \) tends to zero \( T(\tau) \) approaches to 1. This implies that the complete spinorial solution (7) becomes independent of the conformal time parameter \( \tau \) for \( \tau \to 0 \). When we combine Eqs. (7), (9), (21), (22), (24) and (26), we obtain the same behaviour in every case for \( t << 1 \).

Consider the case of a cosmological model with \( a = 0 \). For large \( t \), \( R \simeq t^{1/2} \) hence \( R \simeq \tau \) so that the asymptotic behaviour of the solutions of Eq. (10) are now approximated by \( T \simeq \tau^{-1} \simeq t^{-1/2} \) so that \( H \) has the time dependence

\[
t^{-1/4} \exp[i \mu \sqrt{2t}], \quad (t \gg 1, a = 0)
\]

Similarly, for the open (\( a = -1 \)) radiation-dominated universe we have \( R \simeq t \) and hence \( R \simeq \exp(\tau A/B) \) for large \( t \). Thus the asymptotic behaviour of the solutions of Eq. (10) is given by \( T \simeq \exp(-\tau A/B) \simeq t^{-1} \) and for \( H \) the time dependence is of the form

\[
t^{-1/2} \exp[i \mu \sqrt{2t}], \quad (t \gg 1, a = -1)
\]

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References

[1] Zecca, A.: Nuovo Cimento B113(1998)915.
[2] Montaldi, E. and Zecca, A.: Int. J. Theor. Phys. 37(1998)995.
[3] Newman, E. and Penrose, R.: J. Math. Phys. 3(1962)566.
[4] Chandrasekhar, S.: The Mathematical Theory of Black Holes (Oxford University Press, 1983).
[5] Hawking, S.W. and Ellis, G.F.R.: *The Large Scale Structure of Spacetime* (Cambridge University Press, 1973).

[6] Montaldi, E. and Zecca, A.: Int. J. Theor. Phys. 33(1994)1053.

[7] Zecca, A.: J. Math. Phys. 37(1996)874.

[8] Birrell, N.D. and Davies, P.C.W.: *Quantum Fields in Curved Spacetime* (Cambridge University Press, 1982).

[9] Zecca, A.: Int. J. Theor. Phys. 36(1997)1387.

[10] Misner, C.W., Thorne, K.S. and Wheeler, J.A.: *Gravitation* (W.H. Freeman San Francisco, 1973).

[11] Peebles, P.J.E.: *Principles of Physical Cosmology* (Princeton University Press, 1993);
    Kolb, E.W. and Turner, M.S.: *The Early Universe* (Addison Wesley, 1990).

[12] Arfken, G.B. and Weber, H.J.: *Mathematical Methods for Physicists* (Academic Press New York, 1995).