The INTERNODES method for the treatment of non-conforming multipatch geometries in Isogeometric Analysis

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Abstract

In this paper we apply the INTERNODES method to solve second order elliptic problems discretized by Isogeometric Analysis methods on non-conforming multiple patches in 2D and 3D geometries. INTERNODES is an interpolation-based method that, on each interface of the configuration, exploits two independent interpolation operators to enforce the continuity of the traces and of the normal derivatives. INTERNODES supports non-conformity on NURBS spaces as well as on geometries. We specify how to set up the interpolation matrices on non-conforming interfaces, how to enforce the continuity of the normal derivatives and we give special attention to implementation aspects. The numerical results show that INTERNODES exhibits optimal convergence rate with respect to the mesh size of the NURBS spaces and that it is robust with respect to jumping coefficients.

Key words. Isogeometric Analysis, Multi-patch Geometries, Domain Decomposition Methods, Non-conforming Interfaces, Internodes, Elliptic Problems

1 Introduction

Nowadays Isogeometric Analysis (IgA) represents one of the most popular methods for numerical simulations. Its paradigm consists in expanding the Partial Differential Equations (PDE) solution with respect to the basis functions of the same type of the ones (either B-splines or NURBS) used to describe the geometry of the computational domain generated by CAD software.

Often, real-life problems are defined on complex geometries that usually consist of several patches, moreover these patches can feature non-conformity. By non-conformity we intend discretization non-conformity, that is when different and totally unrelated NURBS spaces (not necessarily the refinement one of the other) are considered inside adjacent patches, but also geometrical non-conformity, that is when two adjacent patches share a subset of a face of their, or their common boundaries don’t match exactly (i.e. they are not watertight).

In the last years, the treatment of multipatch geometries has been investigated in several papers, far from be exhaustive we mention [9, 11, 27, 35, 36, 32, 13, 31, 23, 22]. In this paper we propose to apply the INTERNODES method to solve elliptic problems within the IgA framework.

INTERNODES (INTERpolation for NOncconforming DEcompositionS) [18] is a general purpose method to deal with non-conforming discretizations of PDEs in 2D and 3D geometries split into non-overlapping subdomains (in fact IgA patches play the role of non-overlapping subdomains in the domain decomposition context [11, 35]). The method was proposed in [18] to solve elliptic PDEs by Finite Element Methods (FEM) and Spectral Element Methods (SEM) on two non-conforming subdomains, then its theoretical analysis, as well as its extension to decompositions with more than two subdomains, has been carried out in [22, 23, 25].

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INTERNODES has been successfully applied to solve Navier-Stokes equations ([20][21]) and multi-physics problems like the Stokes-Darcy coupling to simulate the filtration of fluid in porous domains [24][25] and the Fluid-Structure Interaction problem [20][21].

It has been proved that, inside the FEM-SEM context, INTERNODES exhibits optimal accuracy with respect to the $H^1$-broken norm, i.e., the error between the global INTERNODES solution and the exact one behaves like the best approximation error inside the subdomains.

Inside the subregions (or patches) of the decomposition we discretize the PDE (here we have implemented the Galerkin formulation of IgA) by using NURBS spaces that are totally unrelated one each other.

To enforce the continuity of the traces and the equilibration of the fluxes across the interfaces between two adjacent patches, INTERNODES exploits two independent interpolation operators: one for transferring the Dirichlet trace of the solution, the other for the normal derivatives. Like mortar methods, INTERNODES tags the opposite sides of an interface either master or slave: the continuity of the traces is enforced on the slave side of the interface (more precisely the Dirichlet trace is interpolated from the master side to the slave one), while the equilibration of the fluxes is enforced on the master side of the same interface (the normal derivative is interpolated in a suitable way from the slave side to the master one).

In this paper we apply the interpolation at the Greville abscissae of the knot vectors [36][17][1], nevertheless other choices are possible (see, e.g. [1][33]), and when a face of one patch is adjacent to at least two other patches (as in Figures 7 and 8), we use a sort of partition of unity matrix to weigh different contributions coming from contiguous patches (see Sect. 8.1).

To interpolate correctly the normal derivatives we need to assemble local interface mass matrices, but differently than in mortar methods, no cross-mass matrix involving basis functions living on the two opposite sides of the interface and no ad hoc numerical quadrature ([5]) are required by INTERNODES to build the inter-grid operators.

We assume that the patches are watertight, but non-watertight patches could be considered as well. Possible strategies to deal with non-watertight patches are based on the projection of the interpolation nodes from one face to the adjacent one, or on the application of Radial Basis Functions (see, e.g., [19][20] in the FEM context and [32] in the IgA context).

To solve the problem at the algebraic level, the degrees of freedom internal to the patches can be eliminated and the Schur complement system associated with the degrees of freedom on the master skeleton can be solved by Krylov methods (e.g., Bi-CGStab or GMRES), as typical in domain decomposition methods of sub-structuring type ([38][41][13]).

The numerical results of Sections 6 and 10 show that INTERNODES applied to IgA discretizations exhibits optimal accuracy versus the mesh size $h$ for both 2D and 3D geometries and it is robust with respect to jumping coefficients.

This is the first paper that joins INTERNODES and IgA and a lot of questions remain open: the analysis of the convergence rate in the IgA framework, the efficient solution of the Schur complement system and the design of suitable preconditioners, the formulation of the method on surfaces in 3D, its application to contact mechanics problems and, last but not least, the extension of the method to deal with multi-physics problems. Even though these are indeed challenging tasks, the authors of this paper have no reason to think that they cannot be accomplished within INTERNODES, being the theoretical setting presented herein clear and the results promising. Compared to the mortar method, the removal of the necessity of inter-grid quadrature is one of the most attractive feature of INTERNODES. The authors believe that it alone can be a sufficient reason to further develop INTERNODES in the IgA framework.

The paper is organized as follows. In Sect. 2 we formulate the transmission problem; in Sect. 3 we present INTERNODES for two patches; in Sect. 4 we recall the definition of the NURBS basis functions, we define the interpolation operators at the Greville nodes and we specify how to interpolate the normal derivatives at the interface. In Sect. 5 we give the algebraic formulation of the method on two patches, while in Sect. 8 we present INTERNODES on more general configurations with $M > 2$ patches. Finally, in Sect. 9 we provide the algorithms to implement INTERNODES and solve the Schur complement system with respect to degrees of freedom on the master skeleton. The numerical results are shown in Sect. 6 and 10.
2 Problem setting

Let $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$, be an open domain with Lipschitz boundary $\partial \Omega$ $f \in L^2(\Omega)$, $\alpha \in L^\infty(\Omega)$ and $g \in H^{1/2}(\partial \Omega)$ be given functions. We look for the solution $u$ of the self-adjoint second order elliptic problem

$$
\begin{aligned}
-\Delta u + \alpha u &= f \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial \Omega.
\end{aligned}
$$

(1)

For sake of simplicity, in the first part of the paper we deal with this simple problem, then starting from Sect. 7 onwards we extend the method to more general elliptic operators.

We denote by $\tilde{g}$ a lifting of the Dirichlet datum $g$, i.e. any function $\tilde{g} \in H^1(\Omega)$ such that $\tilde{g}|_{\partial \Omega} = g$.

The weak form of problem (1) reads: find $u \in H^1(\Omega)$ with $(u - g) \in H_0^1(\Omega)$ such that

$$
a(u, v) = (f, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega),
$$

(2)

where $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + \alpha uv \, d\Omega$.

Under the assumption that $\alpha \geq 0$ a.e. in $\Omega$, problem (2) admits a unique solution (see, e.g., [37]) that is stable w.r.t. the data $f$ and $g$.

2.1 The transmission problem for two subdomains

We define a non-overlapping decomposition of $\Omega$ into two subdomains $\Omega_1$ and $\Omega_2$ with Lipschitz boundary, such that

$$
\overline{\Omega} = \overline{\Omega}^{(1)} \cup \overline{\Omega}^{(2)}, \quad \Omega^{(1)} \cap \Omega^{(2)} = \emptyset,
$$

while $\Gamma_{12} = \overline{\Omega}^{(1)} \cap \overline{\Omega}^{(2)}$ is the common interface that we assume to be of class $C^{1,1}$ (see [26, Def. 1.2.1.2]) to allow the normal derivative of $u$ on it to be well defined.

Then, for $k = 1, 2$ we define: $\partial \Omega_D^{(k)} = \partial \Omega^{(k)} \cap \partial \Omega_D$. Let $u^{(k)}$ be the restriction of $u$ to $\Omega^{(k)}$, then $u^{(1)}$ and $u^{(2)}$ are the solutions of the transmission problem (see [14, Ch. VII, Sect. 4])

$$
\begin{aligned}
-\Delta u^{(k)} + \alpha u^{(k)} &= f \quad \text{in } \Omega^{(k)}, \quad k = 1, 2, \\
u^{(k)} &= g \quad \text{on } \partial \Omega_D^{(k)}, \quad k = 1, 2, \\
u^{(1)} &= u^{(2)} \quad \text{on } \Gamma_{12}, \\
\frac{\partial u^{(1)}}{\partial \mathbf{n}_1} + \frac{\partial u^{(2)}}{\partial \mathbf{n}_2} &= 0 \quad \text{on } \Gamma_{12},
\end{aligned}
$$

(3)

where $\mathbf{n}_k$ is the outward unit normal vector to $\partial \Omega^{(k)}$ (on $\Gamma_{12}$ it holds $\mathbf{n}_1 = -\mathbf{n}_2$).

For $k = 1, 2$, we define the functional spaces

$$
V^{(k)} = \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \partial \Omega_D^{(k)} \right\}, \quad V_0^{(k)} = H_0^1(\Omega^{(k)}),
$$

$$
\Lambda = \left\{ \lambda \in H^{1/2}(\Gamma_{12}) \mid \exists v \in H^1(\Omega) \text{ such that } v|_{\Gamma_{12}} = \lambda \right\},
$$

noticing that $\Lambda = H_0^{1/2}(\Gamma_{12})$ if $\Gamma_{12} \cap \partial \Omega \neq \emptyset$.

We denote by $a^{(k)}(u, v) = \int_{\Omega^{(k)}} \nabla u \nabla v + \alpha uv \, d\Omega$ the restriction of the bilinear form $a(\cdot, \cdot)$ to $\Omega^{(k)}$ and we set $\tilde{g}^{(k)} = \tilde{g}|_{\Omega^{(k)}}$.

The weak form of the transmission problem reads (see [38]): for $k = 1, 2$ look for $u^{(k)} \in H^1(\Omega^{(k)})$ with $(u^{(k)} - \tilde{g}^{(k)}) \in V^{(k)}$ such that

$$
\begin{aligned}
&\quad \begin{cases}
 a^{(k)}(u^{(k)}, v^{(k)}) = F^{(k)}(v^{(k)}) & \forall v^{(k)} \in V_0^{(k)} \\
u^{(1)} = u^{(2)} & \text{on } \Gamma_{12} \\
\sum_{k=1,2} a^{(k)}(u^{(k)}, L^{(k)}\eta) = \sum_{k=1,2} F^{(k)}(L^{(k)}\eta) & \forall \eta \in \Lambda
\end{cases}
\end{aligned}
$$

(4)
where
\[
\mathcal{F}^{(k)}(v^{(k)}) = (f, v^{(k)})_{L^2(\Omega^{(k)})}, \quad \forall v^{(k)} \in V^{(k)},
\] (5)

while
\[
\mathcal{L}^{(k)} : \Lambda \to V^{(k)}, \quad \text{s.t.} \quad (\mathcal{L}^{(k)} \eta)_{|\Gamma} = \eta \quad \forall \eta \in \Lambda
\] (6)
denotes any possible linear and continuous \textit{lifting operator} from \(\Gamma_{12}\) to \(\Omega^{(k)}\).

**Remark 2.1** Denoting by \(\langle \cdot, \cdot \rangle\) the duality pairing between \(\Lambda\) and its dual space \(\Lambda'\), the distributional form of the interface condition (3) reads
\[
\langle \frac{\partial u^{(1)}}{\partial n_1} + \frac{\partial u^{(2)}}{\partial n_2}, \eta \rangle = 0 \quad \forall \eta \in \Lambda
\] (7)
and it is equivalent to (4).3.

### 3 Formulation of INTERNODES

Bearing in mind the Isogeometric Analysis framework, the two subdomains \(\Omega^{(1)}\) and \(\Omega^{(2)}\) introduced in the previous section play the role of two disjoint patches of a suitable multipatch decomposition of \(\Omega\).

For \(k = 1, 2\), let \(N_h^{(k)}\) be two finite dimensional spaces arising from Isogeometric Analysis discretization that can be totally unrelated one each other and set
\[
V_h^{(k)} = N_h^{(k)} \cap V^{(k)}, \quad V_{0,h}^{(k)} = N_h^{(k)} \cap V_{0}^{(k)}. \quad \text{(8)}
\]
Then we denote by \(u_h^{(k)} \in N_h^{(k)}\) the approximation of \(u^{(k)}\) we are looking for.

Let us denote by \(\Gamma_1\) and \(\Gamma_2\) the two sides of \(\Gamma_{12}\) as part of the boundary of either \(\Omega^{(1)}\) or \(\Omega^{(2)}\) (see Fig. 1), and by \(Y_h^{(k)}\) the space of the trace on \(\Gamma_k\) of the functions of \(N_h^{(k)}\), for \(k = 1, 2\) (see Fig. 1).

Even if \(\Gamma_1\) and \(\Gamma_2\) may represent the same geometric curve (when \(d = 2\)) or surface (when \(d = 3\)), we distinguish them to underline on which side of the interface we are working.

The choice of non-conforming discretizations in \(\Omega^{(1)}\) and \(\Omega^{(2)}\) implies that the trace spaces \(Y_{h_1}^{(1)}\) and \(Y_{h_2}^{(2)}\) are non-conforming. In such a case, to enforce the continuity of the trace (i.e., the interface condition (4)2 and the equilibration of normal derivatives (i.e., the interface condition (7) or, equivalently, (4)3), we introduce two independent interpolation operators: the first one, named \(\Pi_{21}\), is designed to interpolate the trace of \(u_{h_1}^{(1)}\) from \(\Gamma_1\) to \(\Gamma_2\), while the second one, named \(\tilde{\Pi}_{12}\), is used to interpolate in a suitable way the normal derivative \(\frac{\partial u_{h_2}^{(2)}}{\partial n_2}\) from \(\Gamma_2\) to \(\Gamma_1\) (see Fig. 1).

![Figure 1: The interface \(\Gamma_{12} = \partial\Omega^{(1)} \cap \partial\Omega^{(2)}\) and the two sides \(\Gamma_1\) and \(\Gamma_2\) for a multipatch geometry when \(d = 2\). The core idea of INTERNODES: \(\Pi_{21}\) interpolates the trace from \(\Gamma_1 \subset \partial\Omega^{(1)}\) to \(\Gamma_2 \subset \partial\Omega^{(2)}\), \(\tilde{\Pi}_{12}\) interpolates the normal derivative from \(\Gamma_2\) to \(\Gamma_1\).](image-url)
We give here the basic idea of the INTERNODES method when it is applied to the weak transmission problem (9), and we postpone the rigorous description of the method to the next sections, after defining the interpolation operators and after explaining how to transfer the normal derivative across the interface.

The INTERNODES method applied to (9) reads as follows. For \( k = 1, 2 \), let \( \tilde{g}^{(k)}_{h_k} \in \mathcal{N}_{h_k}^{(k)} \) be a suitable approximation of \( g^{(k)} \). Then, for \( k = 1, 2 \), we look for \( u^{(k)}_{h_k} \in \mathcal{N}_{h_k}^{(k)} \) such that \( (u^{(k)}_{h_k} - g^{(k)}_{h_k}) \in V^{(k)} \) and

\[
\begin{align*}
& a^{(k)}(u^{(k)}_{h_k}, v^{(k)}_{h_k}) = \mathcal{F}(k)(v^{(k)}_{h_k}) & \forall v^{(k)}_{h_k} \in V^{(k)}_{0, h_k}, \quad k = 1, 2 \\
& u^{(1)}_{h_2} = \Pi_{21} u^{(1)}_{h_1} & \text{on } \Gamma_2 \\
& \left( \frac{\partial u^{(1)}_{h_2}}{\partial n_1} + \Pi_{12} \frac{\partial u^{(2)}_{h_2}}{\partial n_2}, \eta^{(1)}_{h_1} \right) = 0 & \forall \eta^{(1)}_{h_1} \in Y^{(1)}_{h_1}.
\end{align*}
\]

(9)

Following the terminology typical of mortar methods, the interface condition \( \eta^{(1)}_{h_2} \) characterizes the role of the interfaces \( \Gamma_1 \) and \( \Gamma_2 \). Since the trace on \( \Gamma_2 \) depends on the trace on \( \Gamma_1 \), the interface \( \Gamma_1 \) is named master, while \( \Gamma_2 \) is named slave.

**Remark 3.1 (Analysis of INTERNODES)** The INTERNODES method has been analyzed in \([22]\) in the Finite Element framework. More precisely, if quasi-uniform and affine triangulations are considered inside each subdomain and Lagrange interpolation is applied to enforce the interface conditions, it has been proved \([22]\) that INTERNODES yields a solution that is unique, stable, and convergent with an optimal rate of convergence (i.e., that of the best approximation error in every subdomain).

Two interpolation operators are needed to guarantee the optimal convergence rate of the method with respect to the discretization parameters. As matter of fact, it is well known that using a single interpolation operator (jointly with its transpose) instead of two different operators is not optimal. The approach using a single interpolation operator is also known as point-wise approach, see \([\text{3, 2]}\) and \([\text{IN, Sect. 6}]\).

The same arguments used in \([22]\) can be used in the Isogeometric Analysis framework too, to prove the existence, the uniqueness, and the stability of the solution of problem (9).

A convergence theorem, establishing the error bound for the INTERNODES method with respect to the mesh size \( h = \max_k h_k \) in the framework of Isogeometric Analysis is an open problem at the moment of writing the present paper. Nevertheless, the numerical results provided in the next Sections show that INTERNODES exhibits optimal accuracy versus the mesh size \( h \) for both 2D and 3D geometries.

**Remark 3.2** INTERNODES could be applied to the strong form \((7)\) of the transmission problem. In this case it is sufficient to replace \([9,1] \) with the discrete counterpart of \([9,1]\).

### 4 Discretization by Isogeometric Analysis

Let \( Z = \{0 = \zeta_0, \zeta_1, \ldots, \zeta_{n-1}, \zeta_{n-1} \} \) be the set of \((n_{cl} + 1)\) distinct knot values in the one-dimensional patch \([0,1]\). Given the integer \( p \geq 1 \), let

\[
\Xi = \{\xi_1, \xi_2, \ldots, \xi_q\} = \left\{ \underbrace{\zeta_0, \ldots, \zeta_0, \zeta_1, \ldots, \zeta_1, \ldots, \zeta_{n_{cl}-1}, \ldots, \zeta_{n_{cl}-1}}_{m_{\xi}} \right\}
\]

be a \( p \)-open knot vector, whose internal knots are repeated at most \( p \) times, their multiplicity being denoted \( m_{\xi} \). If \( q \) is the cardinality of \( \Xi \), we consider the number \( n = n(\Xi) = q - p - 1 \).

Starting from the knot vector \( \Xi \) we define the \( n(\Xi) \) uni-variate B-spline functions of degree \( p \) and of global regularity \( C^{p-\max\{m_{\xi}\}} \) in the patch \([0,1]\) by means of the Cox-de Boor recursion formula as follows \((8)\). For \( i = 1, \ldots, n(\Xi) \), set

\[
\hat{B}_{i,0}(\hat{x}) = \begin{cases} 
1 & \text{if } \xi_i \leq \hat{x} < \xi_{i+1} \\
0 & \text{otherwise,}
\end{cases}
\]

(11)
and, for $\ell = 1, \ldots, p$ and $i = 1, \ldots, q - \ell - 1$, set
\[ \tilde{B}_{i, \ell}(\hat{x}) = \frac{\hat{x} - \xi_i}{\xi_{i+\ell} - \xi_i} \tilde{B}_{i, \ell-1}(\hat{x}) + \frac{\xi_{i+\ell+1} - \hat{x}}{\xi_{i+\ell+1} - \xi_{i+1}} \tilde{B}_{i+1, \ell-1}(\hat{x}). \] (12)

The $d$-times tensor product of the set $Z$ induces a Cartesian grid in the parameter domain $\hat{\Omega} = [0,1]^d$. Then we exploit the tensor product rule for the construction of multivariate B-splines functions:
\[ \tilde{B}_{i_1, \ldots, i_d, p_1, \ldots, p_d}(\hat{x}_1, \ldots, \hat{x}_d) = \tilde{B}_{i_1, p_1}(\hat{x}_1) \cdots \tilde{B}_{i_d, p_d}(\hat{x}_d). \] (13)

We assume for sake of simplicity that the knots $\xi_i$ are equally spaced (i.e. the resulting knot vector is uniform) along all the parameter directions, and we define the mesh size $h = 1/n_{el}$. We also assume that the multiplicities of the internal knots are all equal to 1, thus the resulting B-spline functions belong to $C^{p-1}$. Finally we assume that the knots vectors $\Xi_1, \ldots, \Xi_d$ and the polynomial degrees $p_1, \ldots, p_d$ are the same along any direction of the parameter domain, bearing in mind that what we are going to formulate applies as well to more general situations for which either different knot vectors (uniform or non-uniform) or different polynomial degrees or different global regularities are considered along the directions of the parameter domain. With these assumptions, the number of uni-variate basis functions along each direction is equal to $n = n_{el} + p$ and, by tensor product means, the number of multivariate basis functions is $N = n^d$.

B-splines are the building blocks for the parametrization of geometries $\Omega \subset \mathbb{R}^d$ of interest. Given a set of $N$ so-called control points $P_i \in \mathbb{R}^d$, the geometrical map $F : [0,1]^d \rightarrow \Omega$ defined as
\[ F(\hat{x}_1, \ldots, \hat{x}_d) = \sum_{i=1}^{N} P_i \tilde{B}_{i, p}(\hat{x}_1, \ldots, \hat{x}_d) \] (14)
is a parametrization of $\Omega$, its shape being governed by the control points. This can be seen as the starting point of the techniques typically adopted by the CAD community for the representation of geometries.

Even though they can be used to parametrize a wide variety of shapes, B-splines do not allow to exactly represent objects such as conic sections and many others typical of the engineering design. To overcome this drawback the CAD community and hence Isogeometric Analysis exploits NURBS (Non-Uniform Rational B-Splines).

Given a set of positive weights $\{w_1, w_2, \ldots, w_N\}$ associated with the control points $P_i$, multivariate NURBS basis functions are obtained as follows. When $d = 2$, for any $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in \hat{\Omega} = [0,1]^2$, we define the multivariate NURBS basis function
\[ \tilde{\varphi}_{(i_1,i_2),p}(\tilde{x}) = \frac{\tilde{B}_{i_1,p}(\tilde{x}_1)\tilde{B}_{i_2,p}(\tilde{x}_2) w_{i_1,i_2}}{\sum_{j_1,j_2=1}^{n} \tilde{B}_{j_1,p}(\tilde{x}_1)\tilde{B}_{j_2,p}(\tilde{x}_2) w_{j_1,j_2}}, \quad i_1, i_2 = 1, \ldots, n, \] (15)
then we set $i = (i_2 - 1)n + i_1$ and $\tilde{\varphi}_{i,p}(\tilde{x}) = \tilde{\varphi}_{(i_1,i_2),p}(\tilde{x})$.

Similarly, when $d = 3$, for any $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \hat{\Omega} = [0,1]^3$, we define
\[ \tilde{\varphi}_{(i_1,i_2,i_3),p}(\tilde{x}) = \frac{\tilde{B}_{i_1,p}(\tilde{x}_1)\tilde{B}_{i_2,p}(\tilde{x}_2)\tilde{B}_{i_3,p}(\tilde{x}_3) w_{i_1,i_2,i_3}}{\sum_{j_1,j_2,j_3=1}^{n} \tilde{B}_{j_1,p}(\tilde{x}_1)\tilde{B}_{j_2,p}(\tilde{x}_2)\tilde{B}_{j_3,p}(\tilde{x}_3) w_{j_1,j_2,j_3}}, \quad i_1, i_2, i_3 = 1, \ldots, n, \] (16)
then we set $i = (i_3 - 1)n^2 + (i_2 - 1)n + i_1$ and $\tilde{\varphi}_{i,p}(\tilde{x}) = \tilde{\varphi}_{(i_1,i_2,i_3),p}(\tilde{x})$.

Notice that, by the definition of the knot vectors, the basis functions associated with the vertices of $\hat{\Omega}$ are interpolatory.

Then we denote by
\[ \hat{N}_h = \hat{N}_h(\Xi, p) = \text{span}\{\tilde{\varphi}_{i,p}(\tilde{x}) : i = 1, \ldots, N\} \] (17)
*For the sake of simplicity, we index the B-spline basis functions with a uni-variate index $i = 1, \ldots, N$ instead of a more precise multi-index $i = (i_1, \ldots, i_d)$. The same simplification holds for $p$.\]
the space spanned by the multivariate NURBS basis functions (15) (or (16)) on the parameter domain \( \hat{\Omega} \). The sub-index \( h \) is an abridged notation that expresses the dependence of the space on both the number of elements \( n_d \) induced by the knot vector \( \Xi \) and the polynomial degree \( p \) of the B-spline.

NURBS are in fact piecewise rational B-splines and inherit the global continuity in the patch by the B-spline \( B_{i,p} \). The index \( p \) used in the definition of \( \hat{\varphi}_{i,p} \) represents the polynomial degree of the originating B-splines, but it is evident that \( \hat{\varphi}_{i,p} \) are not piecewise polynomials.

For a deeper analysis of NURBS basis functions and their practical use in CAD frameworks, we refer to [36].

Even if in [11] it is shown that the isoparametric paradigm can be relaxed (i.e. by using a NURBS space for the parametrization of \( \Omega \) via \( F \) and an unrelated B-spline space for the discretization of the PDE), in this paper we will follow this concept and we will use the same NURBS space for both the parametrization of the subdomain and for the discrete space.

Since the discretizations in the two patches \( \Omega^{(1)} \) and \( \Omega^{(2)} \) are independent one each-other, for any \( k = 1, 2 \), we consider \( p \)-open multivariate knot vectors \( \Xi^{(k)} \) (with \( \Xi^{(1)} \) and \( \Xi^{(2)} \) independent one each other) and polynomial degrees \( p^{(k)} \) (again with \( p^{(1)} \) and \( p^{(2)} \) independent one each other). Again, the cardinality of the associated NURBS spaces are \( n^{(k)} = n_d^{(k)} + p^{(k)} \) along each direction so that the global cardinality of the multivariate NURBS spaces are \( N^{(k)} = (n^{(k)})^d \).

Given a set of real positive weights in each patch, we define the (parameter-)multivariate NURBS spaces

\[
\mathcal{N}^{(k)}_{h_k} = \mathcal{N}^{(k)}_{h_k}(\Xi^{(k)}, p^{(k)}) = \text{span}\{ \varphi_{i,p^{(k)}}^{(k)}(\hat{x}), \ i = 1, \ldots, N^{(k)} \}, \quad k = 1, 2,
\]

where the sub-index \( h_k \) is an abridged notation that expresses the dependence of the space on both the number of elements \( n_d^{(k)} \) induced by the knot vector \( \Xi^{(k)} \) and the polynomial degree \( p^{(k)} \) of the B-splines.

We assume that each physical patch \( \Omega^{(k)} \) is given through a NURBS transformation \( F \) of the parameter domain \( \hat{\Omega} \). Such transformation, denoted by \( F^{(k)} : \hat{\Omega} \rightarrow \Omega^{(k)} \), is defined by a set of control points \( P_i^{(k)} \in \mathbb{R}^d \) for \( i = 1, \ldots, N^{(k)} \), thus every point \( x \in \Omega^{(k)} \) is given by

\[
x = F^{(k)}(\hat{x}) = \sum_{i=1}^{N^{(k)}} P_i^{(k)} \varphi_{i,p^{(k)}}^{(k)}(\hat{x}).
\]

Throughout the paper we assume that the parameter and the physical spaces coincide and that the mappings \( F^{(k)} \) are invertible, of class \( C^1 \) and their inverses are of class \( C^1 \). After setting \( \hat{\Gamma}_k = (F^{(k)})^{-1}(\Gamma_k) \), we define the space of the traces on \( \hat{\Gamma}_k \)

\[
\hat{\mathcal{N}}^{(k)}_{h_k} = \left\{ \hat{\lambda} = \hat{\varphi}_{i,p^{(k)}}^{(k)} | \hat{\varphi} \in \mathcal{N}^{(k)}_{h_k} \right\},
\]

whose dimension is \( n^{(k)}_{\hat{\Gamma}} = (n^{(k)})^d - 1 \).

The basis functions \( \hat{\mu}_j^{(k)} \) (with \( j = 1, \ldots, n^{(k)}_{\hat{\Gamma}} \)) of \( \hat{\mathcal{N}}^{(k)}_{h_k} \) are defined starting from those of \( \mathcal{N}^{(k)}_{h_k} \), more precisely they are the restriction to \( \hat{\Gamma}_k \) of those basis functions of \( \mathcal{N}^{(k)}_{h_k} \) that are not identically null on \( \hat{\Gamma}_k \). Thus, for any basis function \( \hat{\mu}_j^{(k)} \) of \( \hat{\mathcal{N}}^{(k)}_{h_k} \) there exists a unique basis function \( \hat{\varphi}_{i,j}^{(k)} \in \hat{\mathcal{N}}^{(k)}_{h_k} \), such that

\[
\hat{\mu}_j^{(k)} = (\hat{\varphi}_{i,j}^{(k)})|_{\hat{\Gamma}_k}.
\]

We define the NURBS function space over the physical domain \( \Omega^{(k)} \) as the push-forward of the NURBS function space over \( \hat{\Omega} \) through \( F \):

\[
\mathcal{N}^{(k)}_{h_k} = \mathcal{N}^{(k)}_{h_k}(\Xi^{(k)}, p^{(k)}) = \text{span}\left\{ \varphi_{i,p^{(k)}}^{(k)}(x) = \varphi_{i,p^{(k)}}^{(k)}((F^{(k)})^{-1}(x)), \ i = 1, \ldots, N^{(k)} \right\}.
\]

From now on, for sake of clearness, we denote the basis functions of \( \mathcal{N}^{(k)}_{h_k} \) by \( \varphi_i^{(k)} \) (instead of \( \varphi_{i,p^{(k)}}^{(k)} \)).
Starting from [21], we define the finite dimensional spaces $V^{(k)}_{h_k}$ and $V^{(k)}_{0,h_k}$ as in [8] and the trace space

$$Y^{(k)}_{h_k} = \text{span}\{\mu_j^{(k)}(x) = \hat{\mu}_j^{(k)}((F^{(k)})^{-1}(x)), \ j = 1, \ldots, n^{(k)}_\Gamma\}.$$ 

Relations [26] suggest how to define the discrete counterpart of the lifting operators $L^{(k)}$ invoked in the weak transmission problem [4].

For $k = 1, 2$ we define the linear and continuous discrete lifting operator $L^{(k)} : Y^{(k)}_{h_k} \rightarrow N^{(k)}_{h_k}$ such that $L^{(k)} \mu_j^{(k)} = \varphi_{ij}^{(k)}$ for any basis function $\mu_j^{(k)} \in Y^{(k)}_{h_k}$, i.e. the lifting of $\mu_j^{(k)}$ is the NURBS basis function $\varphi_{ij}^{(k)}$ whose trace on $\Gamma_k$ is $\mu_j^{(k)}$.

### 4.1 Interpolation operators

Contrary to mortar methods that are based on $L^2$—projection operators, INTERNODES takes advantage of two interpolation operators to exchange information across the interface $\Gamma$.

Starting from the $p$—open knot vector $\Xi = \{\xi_i\}_{i=1}^q$ (with $q = n + p + 1$) in the parameter domain $[0, 1]$, the Greville abscissae (also known as averaged knot vector [36 Ch. 9]) are defined by

$$\xi_{i,G} = \frac{1}{p} \sum_{j=i+1}^{i+p} \xi_j, \quad i = 1, \ldots, n. \tag{22}$$

The assumption that the knot vector $\Xi$ is $p$—open implies that $\xi_{1,G} = \xi_1 = 0$ and $\xi_{n,G} = \xi_n = 1$. The Greville abscissae interpolation is proved to be stable up to degree 3 ([1]), while there are examples of instability for degrees higher than 19 on particular non-uniform meshes (more precisely, meshes with geometric refinement [1 33]).

For $k = 1, 2$, we define $\xi_{i,G}^{(k)} \in [0, 1]$ as in (22) and, by tensor product, we build the Greville nodes

$$\hat{x}_{i,G}^{(k)} = (\xi_{i_1,G}^{(k)}, \ldots, \xi_{i_d,G}^{(k)}) \in \hat{\Omega}, \quad \text{for } i_1, \ldots, i_d \in \{1, \ldots, n^{(k)}\}. \tag{23}$$

Finally the points

$$x_{i,G}^{(k)} = F^{(k)}(\hat{x}_{i,G}^{(k)}) \tag{24}$$

are the images of the Greville nodes in the physical patch $\Omega^{(k)}$. In fact, only the Greville nodes laying on $\Gamma_k$ will be used during the interpolation process, these points are denoted by $x_{i,G}^{(k)}$, while $\hat{x}_{i,G}^{(k)} = (F^{(k)})^{-1}(x_{i,G}^{(k)})$.

We make the following assumptions.

**Assumptions 4.1** In this paper we assume that the two maps $F^{(1)}$ and $F^{(2)}$ defining the geometries of the two patches are watertight, this implies that $\Gamma_1$ and $\Gamma_2$ will coincide from the geometrical point of view, i.e.,

describe either the same curve in $\mathbb{R}^2$ or the same surface in $\mathbb{R}^3$.

Given $\lambda^{(1)} \in Y^{(1)}_{h_1}$ and $\lambda^{(2)} \in Y^{(2)}_{h_2}$, we define

$$\Pi_{21} : Y^{(1)}_{h_1} \rightarrow Y^{(2)}_{h_2}, \quad \text{and} \quad \Pi_{12} : Y^{(2)}_{h_2} \rightarrow Y^{(1)}_{h_1}$$

by the interpolation conditions at the Greville nodes on $\Gamma_2$ and $\Gamma_1$, respectively, i.e.,

$$\begin{align*}
(\Pi_{21} \lambda^{(1)})(x_{i,G}^{(2)}) &= \lambda^{(1)}(x_{i,G}^{(2)}), & \text{for } i = 1, \ldots, n^{(2)}_\Gamma, \\
(\Pi_{12} \lambda^{(2)})(x_{i,G}^{(1)}) &= \lambda^{(2)}(x_{i,G}^{(1)}), & \text{for } i = 1, \ldots, n^{(1)}_\Gamma. \tag{25}
\end{align*}$$

These interpolation operators are particular instances of those analyzed, e.g., in [17 36 1].
Let us suppose that $\lambda^{(1)} \in Y_{h_1}^{(1)}$ is known and we want to compute the function
\[ Y_{h_2}^{(2)} \ni \psi = \Pi_{21} \lambda^{(1)}, \] (26)
we proceed as follows. First of all, for $k, \ell \in \{1, 2\}$ we define the matrices $G_{k\ell}$ (with $n^{(k)}_i$ rows and $n^{(\ell)}_i$ columns) by the relations
\[ (G_{k\ell})_{ij} = \hat{\mu}^{(\ell)}_{j}(\tilde{X}_{i,G}^{(k)}), \] (27)
i.e., we evaluate the basis functions $\hat{\mu}^{(\ell)}_{j}$ of the trace space $\tilde{Y}_{h_1}^{(\ell)}$ (for $\ell = 1, 2$) at the Greville nodes of both $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$. Notice that, in view of Assumptions 4.1, the entries of the matrices $G_{k\ell}$ with $k \neq \ell$ are well defined, moreover due to the modal (and not nodal) nature of the basis functions, the matrices $G_{k\ell}$ are not diagonal, even when $k = \ell$.

The matrix $G_{k\ell}$ is non-singular (see [36, Ch. 9.2]).

Then, we denote by $\lambda^{(1)}_j$ (for $j = 1, \ldots, n^{(1)}_i$) the known coefficients of the expansion of $\lambda^{(1)}$ with respect to the basis function $\mu^{(1)}_j$ of $Y_{h_1}^{(1)}$ and by $\psi_j$ the unknown coefficients of the expansion of $\psi$ with respect to the basis functions of $Y_{h_2}^{(2)}$, i.e.
\[ \lambda^{(1)}(x) = \sum_{j=1}^{n^{(1)}_i} \lambda^{(1)}_j \mu^{(1)}_j(x), \quad \psi(x) = \sum_{j=1}^{n^{(2)}_i} \psi_j \mu^{(2)}_j(x). \] (28)

From now on, the expansion of a function of $Y_{h_1}^{(k)}$ with respect to the basis functions $\mu^{(k)}_j$ is named primal and the associated coefficients are named primal coefficients.

The interpolation conditions $\psi(\vec{x}_{i,G}^{(1)}) = \lambda^{(1)}(\vec{x}_{i,G}^{(1)})$ (25) read:
\[ \sum_{j=1}^{n^{(2)}_i} \psi_j \hat{\mu}^{(2)}_{j}(\vec{X}_{i,G}^{(1)}) = \sum_{j=1}^{n^{(1)}_i} \lambda^{(1)}_j \hat{\mu}^{(1)}_{j}(\vec{X}_{i,G}^{(1)}), \quad i = 1, \ldots, n^{(2)}_\Gamma, \] (29)
where we have used the identities $\mu^{(\ell)}_j(\vec{X}^{(k)}_{i,G}) = \hat{\mu}^{(\ell)}_{j}(\vec{X}_{i,G}^{(k)})$ for $\ell, k = 1, 2$.

Denoting by $\lambda^{(1)}$ (\psi, resp.) the array whose components are the values $\lambda^{(1)}_j$ (\psi_j, resp.), (29) becomes
\[ G_{22} \psi = G_{21} \lambda^{(1)}. \] (30)

In conclusion, given $\lambda^{(1)}$, we compute $\psi$ by
\[ \psi = P_{21} \lambda^{(1)}, \quad \text{with} \quad P_{21} = G_{22}^{-1} G_{21}. \] (31)

The matrix $P_{21}$ (with $n^{(1)}_i$ rows and $n^{(2)}_i$ columns) implements the interpolation operator $\Pi_{21}$.

Proceeding in a similar way, we define the matrix
\[ P_{12} = G_{12}^{-1} G_{12} \] (32)
that is the algebraic counterpart of the interpolation operator $\Pi_{12}$.

**Remark 4.1** Notice that $\Pi_{21} u_{h_1}^{(1)}$ in $\tilde{\Gamma}_2$ stands for $\Pi_{21}(u_{h_1}^{(1)})|\tilde{\Gamma}_2$.

**Remark 4.2** If either the parametrizations of the two physical patches mismatch or the patches are not watertight, a projection is required, otherwise Radial Basis Functions can be considered, see, e.g., [19, 20] in the FEM context and [32] in the IgA context.
Figure 2: The lifting \( \mathcal{L}^{(2)} \mu_j^{(2)} \). At left, \( \mu_j^{(2)} \) (the purple function defined on \( \Gamma_2 \)) is identically null on \( \partial \Gamma_2 \) and \( \mathcal{L}^{(2)} \mu_j^{(2)} \) is identically null on \( G_2 \) (\( G \) is drawn in blue); at right, \( \mu_j^{(2)} \) is not identically null on \( \partial \Gamma_2 \) and \( \mathcal{L}^{(2)} \mu_j^{(2)} \) is not identically null on \( G_2 \).

4.2 Transferring the normal derivatives

We start by underlying that the interpolation matrix \( P_{21} \) (\( P_{12} \), resp.) defined in (31) ((32), resp.) applies to the primal coefficients of the functions belonging to \( Y_h^{(1)} \) (\( Y_h^{(2)} \), resp.). Despite this, the normal derivative \( \frac{\partial u_k^{(k)}}{\partial n_k} \) is a functional belonging to the space \( (Y_h^{(k)})' \) dual of \( Y_h^{(k)} \), thus we have to understand how to apply the interpolation to the normal derivatives and we have to explain which is the meaning of \( \tilde{\Pi}_{12} \frac{\partial u_2^{(2)}}{\partial n_2} \) as it appears in the interface condition (3).

For \( k = 1, 2 \) and for any basis function \( \mu_i^{(k)} \) of \( Y_h^{(k)} \) we define the real values

\[
r_i^{(k)} = \left( \frac{\partial u_h^{(k)}}{\partial n_k}, \mu_i^{(k)} \right) = \int_{\Gamma_k} \frac{\partial u_h^{(k)}}{\partial n_k} \mu_i^{(k)} \, d\Gamma, \quad i = 1, \ldots, n^{(k)}. \tag{33}
\]

When dealing with the weak form of the transmission problem, to compute \( r_i^{(k)} \) with a small effort, we can work as follows. Let \( u_h^{(k)} \) be any function in \( \mathcal{N}_h^{(k)} \) such that

\[
a^{(k)}(u_h^{(k)}, v_h^{(k)}) = \mathcal{F}^{(k)}(v_h^{(k)}) + \int_{\partial \Omega^{(k)}} \frac{\partial u_h^{(k)}}{\partial n_k} v_h^{(k)} \, d\Gamma \quad \forall v_h^{(k)} \in \mathcal{N}_h^{(k)}, \tag{34}
\]

and let us set \( G_k = \partial \Omega^{(k)} \setminus \Gamma_k \). Then the values \( r_i^{(k)} \) read also

\[
r_i^{(k)} = a^{(k)}(u_h^{(k)}, \mathcal{L}^{(k)} \mu_i^{(k)}) - \mathcal{F}^{(k)}(\mathcal{L}^{(k)} \mu_i^{(k)}) - \int_{G_k} \frac{\partial u_h^{(k)}}{\partial n_k} \mathcal{L}^{(k)} \mu_i^{(k)} \, d\Gamma. \tag{35}
\]

The algebraic implementation of (35) is in fact a matrix-vector product between the stiffness matrix and the array of the degrees of freedom associated with the interface (see (42)–(44)). Obviously, in the case that the strong form of the transmission problems is preferred instead of the Galerkin one, the formula (33) must be considered instead of (35).

The presence of the last term in (35) is justified as follows. We notice that when \( \mu_j^{(k)} \) is not identically null on \( \partial \Gamma_k \), then \( \mathcal{L}^{(k)} \mu_j^{(k)} \) is not identically null on the set \( G_k \) (see Fig. 2).

For \( k = 1, 2 \) we denote by \( r_{\Gamma_k} \), the array whose entries are the real values \( r_i^{(k)} \), for \( i = 1, \ldots, n_1^{(k)} \) and, following the nomenclature used in linear algebra, \( r_{\Gamma_k} \) is named residual vector.
The values $r_i^{(k)}$ are not the coefficients of the primal expansion of $\frac{\partial u_{h_i}^{(k)}}{\partial n_k}$, so we cannot apply the interpolation matrix $P_{\Gamma_i}$ to the array $r_{\Gamma_i}$. Rather they are the coefficients of $\frac{\partial u_{h_i}^{(k)}}{\partial n_k}$ with respect to the basis $\{\Phi_j^{(k)}\}_{j=1}^{n_i^{(k)}}$ in $(Y_{h_i}^{(k)})'$ that is dual to $\{\mu_j^{(k)}\}_{j=1}^{n_i^{(k)}}$ (see, e.g., [22, 4]), i.e. satisfying

$$\langle \Phi_j^{(k)}, \mu_i^{(k)} \rangle = \int_{\Gamma_k} \Phi_j^{(k)} \mu_i^{(k)} d\Gamma = \delta_{ij}, \quad i, j = 1, \ldots, n_i^{(k)}$$

(where $\delta_{ij}$ is the Kronecker delta), and it holds

$$\frac{\partial u_{h_i}^{(k)}}{\partial n_k} = \sum_{j=1}^{n_i^{(k)}} r_j^{(k)} \Phi_j^{(k)}.$$  \hfill (36)

Nevertheless, $Y_{h_i}^{(k)}$ and $(Y_{h_i}^{(k)})'$ are the same (finite dimensional) algebraic space ([4]) and we denote by $J_k$ the canonical isomorphism between $Y_{h_i}^{(k)}$ and its dual $(Y_{h_i}^{(k)})'$.

To transfer the normal derivative $\frac{\partial u_{h_i}^{(k)}}{\partial n_k}$ from $\Gamma_2$ to $\Gamma_1$, we define the operator $\tilde{\Pi}_{12} : (Y_{h_2}^{(2)})' \to (Y_{h_1}^{(1)})'$ such that $\tilde{\Pi}_{12} = J_1 \Pi_{12} J_2^{-1}$, i.e.

$$\tilde{\Pi}_{12} : (Y_{h_2}^{(2)})' \xrightarrow{J_2} Y_{h_2}^{(2)} \xrightarrow{\Pi_{12}} Y_{h_1}^{(1)} \xrightarrow{J_1} (Y_{h_1}^{(1)})'.$$

The interface mass matrix $M_{\Gamma_i}$ on $\Gamma^{(k)}$, whose entries are

$$(M_{\Gamma_i})_{ij} = \langle \mu_j^{(k)}, \mu_i^{(k)} \rangle_{L^2(\Gamma_k)}, \quad i, j = 1, \ldots, n_i^{(k)},$$  \hfill (37)

is the matrix corresponding to the isomorphism $J_k$.

Assume that $u_{h_2}^{(2)}$ is known, the computation of $\tilde{\Pi}_{12} \frac{\partial u_{h_2}^{(2)}}{\partial n_2}$ is carried out as follows:

1. compute the entries of the array $r_{\Gamma_2}$ by (35);

2. compute the array $z_{\Gamma_2} = M_{\Gamma_2}^{-1} r_{\Gamma_2}$, the entries of $z_{\Gamma_2}$ are in fact the coefficients of the expansion (named $z_{h_2}^{(2)}$) of $\frac{\partial u_{h_2}^{(2)}}{\partial n_2}$ with respect to the primal basis $\mu_j^{(2)}$ of $Y_{h_2}^{(2)}$ (see, e.g., [22, 4]);

3. compute the array $s_{\Gamma_1} = P_{12} z_{\Gamma_2}$, the entries of $s_{\Gamma_1}$ are the primal coefficients of the function $s = \Pi_{12} z_{h_2}^{(2)} \in Y_{h_1}^{(1)}$;

4. compute the array $\tilde{r}_{\Gamma_1} = M_{\Gamma_1} s_{\Gamma_1}$, i.e., come back to the dual expansion.

The entries of the array

$$\tilde{r}_{\Gamma_1} = M_{\Gamma_1} P_{12} M_{\Gamma_2}^{-1} r_{\Gamma_2}$$  \hfill (38)

are the coefficients of the expansion of $\tilde{\Pi}_{12} \frac{\partial u_{h_2}^{(2)}}{\partial n_2}$ with respect to the dual basis $\{\Phi_j^{(1)}\}$, i.e.,

$$\tilde{\Pi}_{12} \frac{\partial u_{h_2}^{(2)}}{\partial n_2} = \sum_{j=1}^{n_i^{(1)}} \tilde{r}_j^{(1)} \Phi_j^{(1)}, \quad \text{and} \quad \langle \tilde{\Pi}_{12} \frac{\partial u_{h_2}^{(2)}}{\partial n_2}, \mu_i^{(1)} \rangle = \tilde{r}_i.$$  \hfill (39)

In conclusion, in view of (36) and (39), the algebraic counterpart of the interface condition ([9], 3) reads

$$r_{\Gamma_1} + M_{\Gamma_1} P_{12} M_{\Gamma_2}^{-1} r_{\Gamma_2} = 0.$$  \hfill (40)
5 The algebraic form of INTERNODES

For \( k = 1, 2 \) we define the following sets of indices:

- \( \mathcal{I}_{\Omega}^{(k)} = \{1, \ldots, N^{(k)}\} \);
- \( \mathcal{I}_k \) the subset of the indices of \( \mathcal{I}_{\Omega}^{(k)} \) associated with the basis functions of \( \Lambda_h^{(k)} \) that are identically null on \( \partial \Omega^{(k)} \);
- \( \mathcal{I}_{\Gamma_k} \) the subset of the indices of \( \mathcal{I}_{\Omega}^{(k)} \) associated with the basis functions of \( \Lambda_h^{(k)} \) that are not identically null on \( \Gamma_k \) (even if \( \Gamma_k = \Gamma_k^e \), the bar over \( \Gamma_k \) stresses the fact that we are taking into account for all the basis functions that are not identically null on \( \Gamma_k \));
- \( \mathcal{I}_{\partial \Omega_k} \) the subset of the indices of \( \mathcal{I}_{\Omega}^{(k)} \) associated with the Dirichlet degrees of freedom;
- \( \mathcal{I}_{\partial \Gamma_k} = \mathcal{I}_{\Gamma_k} \setminus \mathcal{I}_{\Gamma_k} \).

We define the local stiffness matrices \( A^{(k)} \) whose entries are

\[
A_{ij}^{(k)} = a^{(k)}(\varphi_j, \varphi_i), \quad i, j \in \mathcal{I}_{\Omega}^{(k)},
\]

then let

\[
A^{(k,k)} = A^{(k)}(\mathcal{I}_k, \mathcal{I}_k)
\]

be the submatrix of \( A^{(k)} \) obtained by taking both rows and columns of \( A^{(k)} \) whose indices belong to \( \mathcal{I}_k \). Similarly, we define the submatrices \( A^{(k,\Gamma_k)} = A^{(k)}(\mathcal{I}_k, \mathcal{I}_{\Gamma_k}), A^{(\Gamma_k, \Gamma_k)} = A^{(k)}(\mathcal{I}_{\Gamma_k}, \mathcal{I}_{\Gamma_k}) \), \( A^{(\Gamma_k, \Gamma_k)} = A^{(k)}(\mathcal{I}_{\Gamma_k}, \mathcal{I}_k), A^{(\Gamma_k, \partial \Omega_k)} = A^{(k)}(\mathcal{I}_{\Gamma_k}, \mathcal{I}_{\partial \Omega_k}), A^{(\Gamma_k, \partial \Gamma_k)} = A^{(k)}(\mathcal{I}_{\Gamma_k}, \mathcal{I}_{\partial \Gamma_k}) \), and so on.

Moreover, we define the array \( f^{(k)} \) whose entries are

\[
f_i^{(k)} = f_k(\varphi_i^{(k)}), \quad i \in \mathcal{I}_{\Omega}^{(k)},
\]

the array \( u^{(k)} \) of the degrees of freedom in \( \mathcal{I}_k \), and using the same notation as above, the subarrays

\[
f_0^{(k)} = f^{(k)}(\mathcal{I}_k), \quad f_k = f^{(k)}(\mathcal{I}_{\Gamma_k}), \quad f_{\Gamma_k} = f^{(k)}(\mathcal{I}_{\Gamma_k}),
\]
\[
u_0^{(k)} = u^{(k)}(\mathcal{I}_k), \quad v_k = u^{(k)}(\mathcal{I}_{\Gamma_k}), \quad v_{\Gamma_k} = u^{(k)}(\mathcal{I}_{\Gamma_k}).
\]

Finally, we denote by \( g_k \) and \( g_{\partial \Gamma_k} \) the arrays of all the Dirichlet degrees of freedom associated with \( \partial \Omega^{(k)} \) and \( \partial \Gamma_k \), respectively.

To evaluate the last integral of (35) we define the matrix \( C^{(k)} \) whose non-null entries are

\[
C_{ij}^{(k)} = -\int_{G_k} \frac{\partial \varphi_j^{(k)}}{\partial n_k} \varphi_i^{(k)}, \quad \text{for } i \in \mathcal{I}_{\partial \Gamma_k}, \quad j \in \mathcal{I}_{\Omega}^{(k)}
\]

and, as done for the stiffness matrix \( A^{(k)} \), we set \( C^{(\Gamma_k, \Gamma_k)} = C^{(k)}(\mathcal{I}_{\Gamma_k}, \mathcal{I}_{\Gamma_k}), C^{(\Gamma_k, \Gamma_k)} = C^{(k)}(\mathcal{I}_{\Gamma_k}, \mathcal{I}_k), C^{(\Gamma_k, \partial \Omega_k)} = C^{(k)}(\mathcal{I}_{\Gamma_k}, \mathcal{I}_{\partial \Omega_k}), \) and so on.

The integrals in (42) can be easily computed by exploiting the definition of the NURBS basis functions, moreover the rows of \( C^{(k)} \) associated with all the degrees of freedom not belonging to \( \mathcal{I}_{\partial \Gamma_k} \) are null. Thus the computation of \( C^{(k)} \) is very cheap.

Then we define

\[
\hat{A}^{(\Gamma_k, X)} = A^{(\Gamma_k, X)} + C^{(\Gamma_k, X)} \quad \text{and} \quad \hat{A}^{(\Gamma_k, X)} = A^{(\Gamma_k, X)} + C^{(\Gamma_k, X)},
\]

where \( X \in \{\Omega_k, k, \Gamma_k, \bar{\Gamma}_k, D_k\} \), so that the algebraic implementation of (35) reads, for \( k = 1, 2 \),

\[
r_{\Gamma_k} = \hat{A}^{(\Gamma_k, \Omega_k)} u^{(k)} - f_{\Gamma_k},
\]

(44)
By defining the two intergrid matrices

\[ Q_{21} = P_{21}, \quad Q_{12} = M_{I1} P_{12} M_{I2}^{-1}, \]

the algebraic counterpart of \([31, 40]\) read (see \([31, 40]\))

\[ u_{\Gamma_2} = Q_{21} u_{\Gamma_1}, \quad r_{\Gamma_1} + Q_{12} r_{\Gamma_2} = 0. \]  

By introducing the following submatrices:

\[ Q_{21}(\Gamma_2, \Gamma_1) = Q_{21}(I_{\Gamma_2}, I_{\Gamma_1}), \quad Q_{21}(\Gamma_2, \partial \Gamma_1) = Q_{21}(I_{\Gamma_2}, I_{\partial \Gamma_1}), \quad Q_{12}(\Gamma_1, \Gamma_2) = Q_{12}(I_{\Gamma_1}, I_{\Gamma_2}), \]

and by using \([46]\), the algebraic form of \([9]\) reads

\[
\begin{bmatrix}
    A^{(1,1)} & 0 \\
    0 & A^{(2,2)}
\end{bmatrix}
\begin{bmatrix}
    A^{(1,1)} & A^{(1,\Gamma_1)} \\
    A^{(2,\Gamma_2)} & A^{(2,\Gamma_1)}
\end{bmatrix}
\begin{bmatrix}
    u^{(1)}_0 \\
    u^{(2)}_0
\end{bmatrix}
+ \begin{bmatrix}
    u^{(1)}_{\Gamma_1} \\
    u^{(2)}_{\Gamma_1}
\end{bmatrix}
= \begin{bmatrix}
    f^{(1)}_0 \\
    f^{(2)}_0 + Q_{12}^{(\Gamma_1, \Gamma_2)} f_{\Gamma_2}
\end{bmatrix}
\]

where the array

\[ G = \begin{bmatrix}
    G_1 \\
    G_2
\end{bmatrix}
\begin{bmatrix}
    A^{(1,\Gamma_1)D_1} g_1 \\
    A^{(2,\Gamma_2)} g_2 + A^{(2,\Gamma_2)} Q_{21}^{(\Gamma_2, \partial \Gamma_1)} g_{\partial \Gamma_1}, \\
    A^{(1,\Gamma_1)D_1} g_1 + Q_{12}^{(\Gamma_1, \Gamma_2)} (A^{(\Gamma_2,D_2)} g_2 + A^{(\Gamma_2,D_2)} Q_{21}^{(\Gamma_2, \partial \Gamma_1)} g_{\partial \Gamma_1})
\end{bmatrix} \]

is non null only when non-homogeneous Dirichlet conditions are given on \( \partial \Omega_D \) and implements the lifting of the Dirichlet datum.

Finally the degrees of freedom in \( \Omega^{(1)} \) are given by \( u^{(1)} = [u^{(1)}_0, u^{(1)}_{\Gamma_1}, g_1] \) while the one in \( \Omega^{(2)} \) are given by \( u^{(2)} = [u^{(2)}_0, Q_{21}^{(\Gamma_2, \Gamma_1)} u^{(1)}_{\Gamma_1}, g_2] \).

The presence of the terms \( A^{(\Gamma_2, \Gamma_1)} Q_{21}^{(\Gamma_2, \partial \Gamma_1)} g_{\partial \Gamma_1}, \) and \( A^{(\Gamma_2, \Gamma_1)} Q_{21}^{(\Gamma_2, \partial \Gamma_1)} g_{\partial \Gamma_1} \) in the last two rows of \([48]\) is motivated by the fact that the trace of \( u^{(2)}_{\partial \Gamma_2} \) on the interface \( \Gamma_2 \) is the interpolation through \( \Pi_{21} \) of the trace of \( u^{(1)}_{\partial \Gamma_2} \) on \( \Gamma_1 \).

System \([47]\) represents the algebraic form of INTERNODES implemented in practice. By taking \( Q_{12} = Q_{21} = I \) we recover the algebraic system associated with classical conforming domain decomposition (see, e.g., \([35, 41]\)).

Notice that, even though the residuals are defined up to the boundary of \( \Gamma_k \), the algebraic counterpart of condition \([33]\) is imposed only on the degrees of freedom internal to \( \Gamma_1 \). In this way the number of equations and the number of unknowns in \([47]\) do coincide.

### 5.1 An iterative method to solve \([47]\)

An alternative way to solve system \([47]\) consists in eliminating the variables \( u^{(1)}_0 \) and \( u^{(2)}_0 \) and in solving the Schur complement system \([33, 35]\)

\[ S u_{\Gamma_1} = b, \]

where

\[
\begin{align*}
S &= S_{\Gamma_1} + Q_{12}^{(\Gamma_1, \Gamma_2)} S_{\Gamma_2} Q_{21}^{(\Gamma_2, \Gamma_1)}, \\
S_{\Gamma_1} &= \hat{A}^{(\Gamma_1, \Gamma_1)} - \hat{A}^{(\Gamma_1, 1)} (A^{(1,1)})^{-1} A^{(1, \Gamma_1)}, \\
b_{\Gamma_1} &= f_{\Gamma_1} - \hat{A}^{(\Gamma_1, 1)} (A^{(1,1)})^{-1} f^{(1)}_0 - G_1, \\
S_{\Gamma_2} &= \hat{A}^{(\Gamma_2, \Gamma_2)} - \hat{A}^{(\Gamma_2, 2)} (A^{(2,2)})^{-1} A^{(2, \Gamma_2)}, \\
b_{\Gamma_2} &= f_{\Gamma_2} - \hat{A}^{(\Gamma_2, 2)} (A^{(2,2)})^{-1} f^{(2)}_0 - G_2.
\end{align*}
\]
balanced & master refined & slave refined \\
\hline
n_{el}^{(1)} & n_{el}^{(2)} & n_{el}^{(1)} & n_{el}^{(2)} & n_{el}^{(1)} & n_{el}^{(2)} \\
8 & 9 & 16 & 9 & 8 & 17 \\
16 & 17 & 32 & 17 & 16 & 33 \\
32 & 33 & 64 & 33 & 32 & 65 \\
64 & 65 & 128 & 65 & 64 & 129 \\
\hline

Figure 3: Two-patches test case. At left, the computational domain split into two patches. At right, the three discretization sets used in the simulations.

\[ Q_{12}^{(1,2)} S_{1} Q_{21}^{(2,1)} \] is not the transpose of \( Q_{21} \), even if the differential operator is symmetric, the Schur complement system \( S \) is not. Moreover, the matrix \( Q_{12}^{(1,2)} S_{1} Q_{21}^{(2,1)} \) is not a good candidate to play the role of preconditioner since it may be singular.

To this aim, it is sufficient to compute and store a suitable factorization of the matrices \( A^{(k,k)} \) and dispose of a function that implements the action of \( S \) on a given array \( \mathbf{\lambda} \) whose entries are the degrees of freedom associated with \( \Gamma_1 \). We will describe this approach in Sect. 9 for general multipatch configurations.

Once \( \mathbf{u}_{\Gamma_1} \) is known, the variables \( \mathbf{u}_{0}^{(1)} \) and \( \mathbf{u}_{0}^{(2)} \) are recovered by solving the local subsystems

\[
A^{(1,1)} \mathbf{u}_{0}^{(1)} = \mathbf{f}_0^{(1)} - G_1 - A^{(1,\Gamma_1)} \mathbf{u}_{\Gamma_1},
\]

\[
A^{(2,2)} \mathbf{u}_{0}^{(2)} = \mathbf{f}_0^{(2)} - G_2 - A^{(2,\Gamma_2)} Q_{21}^{(2,1)} \mathbf{u}_{\Gamma_1}.
\]

Finally, \( \mathbf{u}_{\Gamma_2} \) is recovered by assembling \( \mathbf{u}_{\Gamma_1} \) and \( \mathbf{g}_{\partial \Omega_1} \), and the numerical solution on \( \Gamma_2 \) is reconstructed by the interpolation formula \( \mathbf{u}_{\Gamma_2} = Q_{21} \mathbf{u}_{\Gamma_1} \).

6 Numerical results for 2 patches

Let us consider the differential problem \( 1 \) in \( \Omega = \{ (x, y) \in \mathbb{R}^2 : x \geq 0, \ y \geq 0, \ 1 \leq x^2 + y^2 \leq 4 \} \) with \( \alpha = 0 \), and \( f \) and \( g \) such that the exact solution is \( u(x, y) = \sin(1.5\pi x) \sin(3\pi y) \).

The computational domain \( \Omega \) is split into the patches \( \Omega^{(1)} = \{ (x, y) \in \Omega : x^2 + y^2 \leq (1.5)^2 \} \) and \( \Omega^{(2)} = \{ (x, y) \in \Omega : x^2 + y^2 \geq (1.5)^2 \} \) (see Fig. 3) and let

\[
u_h = \begin{cases} 
  u_{h_1}^{(1)} & \text{in } \Omega^{(1)} \\
  u_{h_2}^{(2)} & \text{in } \Omega^{(2)} 
\end{cases}
\]  

(53)

denote the numerical solution computed with INTERNODES.

We consider three (all non-conforming) discretization sets. In the first set, named balanced the number of elements in the two patches is quite the same, we fix \( n_{el}^{(2)} = n_{el}^{(1)} + 1 \) (recalling the notations introduced in Sect. 4, \( n_{el}^{(2)} \) is the number of elements along each direction of the parameter domain).

In the second set, named master refined the number of elements in the patch \( \Omega^{(1)} \) (whose interface \( \Gamma_1 \) plays the role of master interface) is about twice the number of the elements in \( \Omega^{(2)} \), more precisely we fix \( n_{el}^{(1)} = 2(n_{el}^{(2)} - 1) \).

In the third set, named slave refined the number of elements in the patch \( \Omega^{(2)} \) (whose interface \( \Gamma_2 \) plays the role of slave interface) is about twice the number of the elements in \( \Omega^{(1)} \), more precisely we fix \( n_{el}^{(2)} = 2n_{el}^{(1)} + 1 \). In the Table at the left of Fig. 3 we show the values of \( n_{el}^{(1)} \) and \( n_{el}^{(2)} \) we have considered.
Figure 4: Two-patches test case. The broken-norm error \( \| u_h - u \|_* \) versus the mesh size. At left the error for the balanced configuration, in the middle the error for the master refined configuration, at right the error for the slave refined configuration.

\[
\| u_h - u \|_* = \left( \sum_k \frac{\| u_h - u \|_{H^1(\Omega^{(k)})}^2}{\| u \|_{H^1(\Omega^{(k)})}^2} \right)^{1/2}
\]

with respect to the maximum mesh size \( h = \max_k h_k \), where \( h_k = 1/n_{el}^{(k)} \), for the three discretization sets. In all the cases, the local polynomial degrees \( p^{(1)} \) and \( p^{(2)} \) are equal and we set \( p = p^{(1)} = p^{(2)} \).

The convergence of INTERNODES is optimal versus the mesh-size \( h \), in the sense that the broken-norm errors behave like \( h^p \) when \( h \to 0 \), exactly as the error in \( H^1 \)-norm of the Galerkin Isogeometric methods (see, e.g., [12, Thm. 3.4 and Cor. 4.16]).

In Fig. 5 we show the broken-norm errors versus the polynomial degree \( p^{(1)} \) in the patch \( \Omega^{(1)} \). At left the error for the balanced configuration with \( n_{el}^{(1)} = 20 \) and \( n_{el}^{(2)} = 19 \), at right the error for the slave refined configuration with \( n_{el}^{(1)} = 20 \) and \( n_{el}^{(2)} = 41 \).

In Fig. 6, we present three qualitative pictures of the two-domain problem.
7 More general second order elliptic PDEs

INTERNODES methods can be applied to solve general elliptic second order PDEs, where the differential operator is

\[ Lu = -\nabla \cdot (\nu \nabla u) + b \cdot \nabla u + \alpha u, \quad (55) \]

with \( \nu \in L^\infty(\Omega) \) such that there exists \( \nu_0 > 0 \) and \( \nu \geq \nu_0 \) a.e. in \( \Omega \); \( b = (b_1, \ldots, b_d) \), with \( b_i \in L^\infty(\Omega) \); \( \alpha \in L^\infty(\Omega) \) with \( \alpha \geq 0 \) a.e. in \( \Omega \).

Given \( f \in L^2(\Omega) \) and \( g \in H^{1/2}(\partial \Omega) \), and under the assumption that \( \alpha - \frac{1}{2} \nabla \cdot b \geq 0 \) a.e. in \( \Omega \), the problem to find \( u \in H^1(\Omega) \) such that

\[
\begin{aligned}
Lu &= f \quad \text{in } \Omega \\
u_1 \frac{\partial u^{(1)}}{\partial n_1} + \nu_2 \frac{\partial u^{(2)}}{\partial n_2} &= 0 \quad \text{on } \Gamma_{12},
\end{aligned}
\]

admits a unique solution that is table w.r.t. the data \( f \) and \( g \).

In such a case, while the interface condition enforcing the continuity of the traces across \( \Gamma_{12} \) does not change, the interface conditions involving the normal derivatives must be replaced by

\[ \nu_k \frac{\partial u^{(1)}}{\partial n_1} + \nu_k \frac{\partial u^{(2)}}{\partial n_2} = 0 \quad \text{on } \Gamma_{12}, \quad (57) \]

where \( \nu_k = \nu_{|\Omega^{(k)}} \).

When Neumann boundary conditions are assigned on a subset \( \partial \Omega_N \) of the boundary \( \partial \Omega \), the definition of the set \( G_k \) (that is used to define the real values \( r_i^{(k)} \), see \( \ref{eq:35} \)) becomes \( G_k = \partial \Omega^{(k)} \setminus (\Gamma_k \cup \partial \Omega_N) \), (see \ref{eq:25} formula (44)).

8 INTERNODES for decompositions with \( M \geq 2 \) patches

Let now \( \Omega^{(k)} \), with \( k = 1, \ldots, M \), denote a family of disjoint patches of \( \Omega \subset \mathbb{R}^d \), with \( d = 2, 3 \), s.t. \( \cup_k \Omega^{(k)} = \Omega \). Let us suppose that each \( \Omega^{(k)} \) has Lipschitz boundary \( \partial \Omega^{(k)} \) (for \( k = 1, \ldots, M \)).

Let \( \Gamma_k = \partial \Omega^{(k)} \setminus \partial \Omega \) be the part of the boundary of \( \Omega^{(k)} \) internal to \( \Omega \), and

\[ \Gamma_{kl} = \Gamma_{lk} = \partial \Omega^{(k)} \cap \partial \Omega^{(l)} \]

be the interface between the two subdomains \( \Omega^{(k)} \) and \( \Omega^{(l)} \). Intersections having null measure in the topology of \( \mathbb{R}^{d-1} \) are considered empty. Finally, let \( L \) be the differential operator introduced in \( \ref{eq:55} \).

Figure 6: Two-patches test case. The numerical solution. At right, the zoom around the interface is shown.
The physical domain $\Omega \subset \mathbb{R}^2$ split in 3 patches and the faces $\gamma_{k,\alpha}$. The interfaces $\Gamma_{k\ell}$ are:

- $\Gamma_{12} = \gamma_{1,1} \cap \gamma_{2,2}$,
- $\Gamma_{13} = \gamma_{1,1} \cap \gamma_{3,2}$,
- $\Gamma_{23} = \gamma_{2,1} \cap \gamma_{3,1}$.

At right, a possible choice of the master/slave faces is shown.

The multidomain formulation of problem (1) reads: look for $u^{(k)}$ for $k = 1, \ldots, M$ such that:

\[
\begin{align*}
Lu^{(k)} &= f & \text{in } \Omega^{(k)}, \quad k = 1, \ldots, M \\
u_k \frac{\partial u^{(k)}}{\partial n_k} + \nu_\ell \frac{\partial u^{(\ell)}}{\partial n_\ell} &= 0 & \text{on } \Gamma_{k\ell} \neq \emptyset,
\end{align*}
\]

(58)

We split the internal boundary $\Gamma_k$ of $\partial \Omega^{(k)}$ in faces and we denote by $\gamma_{k,\alpha}$ the $\alpha$th face of $\Gamma_k$ (see Fig. 7 and Fig. 8), the first sub-index $k$ identifies the domain, while the second one $\alpha$ is the index of the face of $\Gamma_k$. As for the case of two subdomains, we assume that each interface $\gamma_{k,\ell}$ is sufficiently regular (i.e. of class $C^{1,1}$) to allow the conormal derivative of $u_k$ on $\gamma_{k,\ell}$ to be well defined.

**Remark 8.1** We assume that $\gamma_{k,\alpha}$ includes its boundary.

For example, in the multipatch configuration of Fig. 7 we have $\Gamma_1 = \gamma_{1,1}$, $\Gamma_2 = \gamma_{2,1} \cup \gamma_{2,2}$ and $\Gamma_3 = \gamma_{3,1} \cup \gamma_{3,2}$.

Moreover, for any face $\gamma_{k,\alpha}$ we define the set

\[
A_{k,\alpha} = \{ (\ell, \beta) : \gamma_{\ell,\beta} \cap \gamma_{k,\alpha} \neq \emptyset \}
\]

(59)
of the faces (of the other patches) that are adjacent to $\gamma_{k,\alpha}$. In the multipatch configuration of Fig. 7 we have $A_{1,1} = \{ (2, 2), (3, 2) \}$, $A_{2,2} = \{ (1, 1) \}$, $A_{2,1} = \{ (3, 1) \}$ and so on.

Between $\gamma_{k,\alpha}$ and $\gamma_{\ell,\beta}$, one is tagged as master and the other as slave and we define the master skeleton

\[
\Gamma = \bigcup_{(k,\alpha)} \gamma_{k,\alpha} \text{ with } \gamma_{k,\alpha} \text{ master.}
\]

(60)

In the mortar community $\Gamma$ is named mortar interface.

A-priori there is no constraint in tagging a face as either master or slave. In the example of Fig. 7 right, we could tag as master the face $\gamma_{1,1}$ (in which case $\gamma_{2,2}$ and $\gamma_{3,2}$ will be both slave), or other way around.

In the patch $\Omega^{(k)}$ we define a NURBS space $N_h^{(k)}$ as defined in (21) and the corresponding finite dimensional spaces $V_h^{(k)}$ (see (8)) that are totally independent of the discretizations inside the adjacent patches. For each face $\gamma_{k,\alpha} \subset \Gamma_k$ we define the trace space

\[
Y_{h_k}^{(k,\alpha)} = \{ \lambda = v|_{\gamma_{k,\alpha}} : v \in N_h^{(k)} \}
\]

whose dimension is denoted by $n_{(k,\alpha)}$. 

17
and \( x \) 

8.1 Interpolation operators

In Sect. 8.2 we will precise how to generalize formula (33) for the computation of normal derivatives. The INTERNODES method for \( M > 2 \) patches reads as follows. For \( k = 1, \ldots, M \), let \( \tilde{g}^{(k)} \in \mathcal{N}^{(k)} \) be a suitable approximation of \( g^{(k)} \), we look for \( u_h^{(k)} \in \mathcal{N}^{(k)}_{h_k} \) such that \( (u_h^{(k)} - \tilde{g}^{(k)}) \in V_h^{(k)} \) and

\[
\begin{align*}
\alpha^{(k)}(u_h^{(k)}, v_h^{(k)}) &= \mathcal{F}^{(k)}(v_h^{(k)}) \\
\text{for any } (\ell, \beta) : \gamma_{\ell, \beta} \text{ is slave} &
\sum_{(k, \alpha) \in A_{\ell, \beta}} \Pi_{(\ell, \beta)(k, \alpha)} u_h^{(k)} \\
\text{for any } (k, \alpha) : \gamma_{k, \alpha} \text{ is master} &
\sum_{(\ell, \beta) \in A_{k, \alpha}} \Pi_{(k, \alpha)(\ell, \beta)} \left( \nu_{\ell} \frac{\partial u_h^{(k)}}{\partial \nu_{\ell}} \right), \eta = 0 \\
\forall \eta &\in Y_h^{(k, \alpha)} 
\end{align*}
\]

(61)

where:

- \( \Pi_{(\ell, \beta)(k, \alpha)} \) and \( \Pi_{(k, \alpha)(\ell, \beta)} \) are the interpolation operators used to transfer information from one side to the other of \( \gamma_{k, \alpha} \cap \gamma_{\ell, \beta} \neq \emptyset \), more precisely, \( \Pi_{(\ell, \beta)(k, \alpha)} \) moves from \( \gamma_{\ell, \beta} \) to \( \gamma_{k, \alpha} \), while \( \Pi_{(k, \alpha)(\ell, \beta)} \) moves from \( \gamma_{k, \alpha} \) to \( \gamma_{\ell, \beta} \);
- \( \mathcal{J}_{k, \alpha} \) (resp. \( \mathcal{J}_{\ell, \beta} \)) denotes the canonical isomorphism between \( Y_{h_k}^{(k, \alpha)} \) (resp. \( Y_{h_k}^{(\ell, \beta)} \)) and its dual space;
- \( \Pi_{(k, \alpha)(\ell, \beta)} = \mathcal{J}_{k, \alpha} \Pi_{(k, \alpha)(\ell, \beta)} \mathcal{J}_{\ell, \beta}^{-1} \).

When the cardinality of \( A_{k, \alpha} \) is equal to one, i.e. there is only one face adjacent to \( \gamma_{k, \alpha} \), then the summations in (61) disappear and we recover the interface conditions (9). In such a case the definition of the interpolation operators is as in Sect. 4.1.

When instead the cardinality of \( A_{k, \alpha} \) is greater than one, that is the face \( \gamma_{k, \alpha} \) interfaces with at least two adjacent faces (as, e.g. for the face \( \gamma_{k, \alpha} \) in Fig. 9), and we want to interpolate from \( \bigcup_{(\ell, \beta) \in A_{k, \alpha}} \gamma_{\ell, \beta} \) to \( \gamma_{k, \alpha} \), then we have to slightly modify the definition of the interpolation operators (see Sect. 8.1). Finally, in Sect. 8.2 we will precise how to generalize formula (33) for the computation of normal derivatives.

### 8.1 Interpolation operators

Let \( x_{i,G}^{(k, \alpha)} \), for \( i = 1, \ldots, n^{(k, \alpha)} \), be the Greville nodes associated with the patch \( \Omega^{(k)} \) that belong to \( \gamma_{k, \alpha} \) and \( \tilde{x}_{i,G}^{(k, \alpha)} = (F^{(k)})^{-1}(x_{i,G}^{(k, \alpha)}) \) their counter-image in the parameter domain. For any face \( \gamma_{k, \alpha} \) and for any
of the indices of the Greville nodes associated with the domain $\Omega$ coming from two contiguous faces adjacent to

$$\gamma$$

following interpolation conditions for $i$

$$A$$

At right, the black dots are the Greville nodes $x_{i,G}^{(k,\alpha)} \in \gamma_{k,\alpha}$. The indices of the nodes surrounded by blue rings belong to $G_{(k,\alpha)(\ell_1,\beta_1)}$, while the indices of the nodes marked with a red cross belong to $G_{(k,\alpha)(\ell_2,\beta_2)}$ (see formula (62)). There is one Greville node that belongs to both the sets $G_{(k,\alpha)(\ell_1,\beta_1)}$ and $G_{(k,\alpha)(\ell_2,\beta_2)}$, the corresponding value $(U_{k,\alpha})_i$ is equal to 1/2. For all the other Greville nodes, $(U_{k,\alpha})_i$ is equal to 1

$$(\ell, \beta) \in A_{k,\alpha}$$

we define the set

$$G_{(k,\alpha)(\ell,\beta)} = \{i = 1, \ldots, n^{(k,\alpha)} | F^{(\ell)}(x_{i,G}^{(k,\alpha)}) \in \gamma_{\ell,\beta}\}$$

(62)

of the indices of the Greville nodes associated with the domain $\Omega^{(k)}$ belonging to $\gamma_{k,\alpha}$ that lay on $\gamma_{\ell,\beta}$ too (see Fig. 9 right).

Notice that $G_{(k,\alpha)(\ell,\beta)}$ and $G_{(\ell,\beta)(k,\alpha)}$ denote two different sets.

Finally, we define a sort of partition of unity function, with the aim of interpolating correctly the data coming from two contiguous faces adjacent to $\gamma_{k,\alpha}$.

For any $i = 1, \ldots, n^{(k,\alpha)}$, we set

$$(U_{k,\alpha})_i = 1/\text{card}\{(\ell, \beta) \in A_{k,\alpha} | i \in G_{(k,\alpha)(\ell,\beta)}\},$$

(63)

that is $(U_{k,\alpha})_i$ is the inverse of the number of faces adjacent to $\gamma_{k,\alpha}$ which the Greville node $x_{i,G}^{(k,\alpha)}$ lays on (up to the boundary). Here, card$A$ denotes the cardinality of the set $A$.

Given $\lambda \in Y_{h_{(\ell,\beta)}}$, we define the interpolation operator $\Pi_{(k,\alpha)(\ell,\beta)} : Y_{h_{(\ell,\beta)}} \rightarrow Y_{h_{k}}$ by imposing the following interpolation conditions for $i = 1, \ldots, n^{(k,\alpha)}$;

$$\Pi_{(k,\alpha)(\ell,\beta)}(\lambda)(x_{i,G}^{(k,\alpha)}) = \begin{cases} (U_{k,\alpha})_i \lambda(x_{i,G}^{(k,\alpha)}) & \text{if } i \in G_{(k,\alpha)(\ell,\beta)}, \\ 0 & \text{if } i \notin G_{(k,\alpha)(\ell,\beta)} \end{cases}$$

(64)

Notice that $\Pi_{(k,\alpha)(\ell,\beta)}(\lambda)$ is defined on the whole face $\gamma_{k,\alpha}$, even when $\gamma_{\ell,\beta} \supsetneq \gamma_{k,\alpha}$.

Let us consider the multipatch geometry shown in the top left picture of Fig. 10, we interpolate a trace function from $\gamma_{2,2} \cup \gamma_{3,2}$ to $\gamma_{1,1}$. In the bottom pictures of Fig. 10, we show how the interpolation operators $\Pi_{(1,1)(2,2)}$ (from $\gamma_{2,2}$ to $\gamma_{1,1}$) and $\Pi_{(1,1)(3,2)}$ (from $\gamma_{3,2}$ to $\gamma_{1,1}$) work. The point whose coordinates are $(x, y) = (1, 1)$ is a Greville point for $\Omega^{(1)}$ belonging to $\gamma_{1,1}$ and it lays on both $\gamma_{2,2}$ and $\gamma_{3,2}$ too (the faces include their boundary), thus the corresponding weight defined in (63) is equal to 1/2. Notice that $\Pi_{(1,1)(2,2)}(\lambda)$ takes null value at the Greville nodes of $\gamma_{1,1}$ not laying on $\gamma_{2,2}$. Analogous considerations hold for $\Pi_{(1,1)(3,2)}(\lambda)$.

The sum $\Pi_{(1,1)(2,2)}(\lambda^{(2,2)}) + \Pi_{(1,1)(3,2)}(\lambda^{(3,2)})$ interpolates the piece-wise function $\lambda$ such that $\lambda_{|\gamma_{2,2}} = \lambda^{(2,2)}$ and $\lambda_{|\gamma_{3,2}} = \lambda^{(3,2)}$ (see the right picture of Fig. 10).
Figure 10: Example of how the interpolation operators \( \Pi_{(k, \alpha)(\ell, \beta)} \) work. The black dots represent the interpolated values at the Greville nodes.
Let $\mu_{j}^{(k,\alpha)}$, with $j = 1, \ldots, n^{(k,\alpha)}$, be the NURBS basis functions of the trace space $Y_{h_{k}}^{(k,\alpha)}$ and $\hat{\mu}_{j}^{(k,\alpha)}((F^{(k)})^{-1}(x)) = \mu_{j}^{(k,\alpha)}(x)$. By setting

\begin{align}
(G_{(k,\alpha)(k,\alpha)})_{ij} &= \hat{\mu}_{j}^{(k,\alpha)}(\hat{x}_{i,G}), & i, j = 1, \ldots, n^{(k,\alpha)}, \\
(G_{(k,\alpha)(\ell,\beta)})_{ij} &= \begin{cases} 
\hat{\mu}_{j}^{(\ell,\beta)}(\hat{x}_{i,G}) & \text{if } i \in G_{(k,\alpha)(\ell,\beta)}, \\
0 & \text{if } i \notin G_{(k,\alpha)(\ell,\beta)}
\end{cases}, & i = 1, \ldots, n^{(k,\alpha)}, \quad j = 1, \ldots, n^{(\ell,\beta)}
\end{align}

(65)

the matrix associated with $\Pi_{(k,\alpha)(\ell,\beta)}$ is

$$P_{(k,\alpha)(\ell,\beta)} = G_{(k,\alpha)(k,\alpha)}^{-1} \text{diag}(U_{k,\alpha}) G_{(k,\alpha)(\ell,\beta)}.$$  (66)

Similarly, we define the interpolation operator $\Pi_{(\ell,\beta)(k,\alpha)} : Y_{h_{k}}^{(k,\alpha)} \to Y_{h_{\ell}}^{(\ell,\beta)}$ and the corresponding matrix

$$P_{(\ell,\beta)(k,\alpha)} = G_{(\ell,\beta)(\ell,\beta)}^{-1} \text{diag}(U_{\ell,\beta}) G_{(\ell,\beta)(k,\alpha)}.$$  (67)

Denoting by $\lambda_{k,\alpha}$ ($\lambda_{\ell,\beta}$, resp.) the array whose entries are the degrees of freedom of the function $u_{h_{k}}^{(k)}$ ($u_{h_{\ell}}^{(\ell)}$, resp.) associated with the face $\gamma_{k,\alpha}$ ($\gamma_{\ell,\beta}$, resp.), the algebraic implementation of the interface conditions (61) reads:

for any $(\ell, \beta) : \gamma_{\ell,\beta}$ is slave, \[ \lambda_{\ell,\beta} = \sum_{(k,\alpha) \in \mathcal{A}_{\ell,\beta}} P_{(\ell,\beta)(k,\alpha)} \lambda_{k,\alpha}. \]  (68)

### 8.2 Transferring normal derivatives

For each face (either slave or master) $\gamma_{k,\alpha} \subset \Gamma_{k} = \partial \Omega^{(k)} \setminus \partial \Omega$, we define the set $G_{k,\alpha} = \partial \Omega^{(k)} \setminus \gamma_{k,\alpha}$ and the real values (similar to (33))

$$r_{i}^{(k,\alpha)} = \int_{\gamma_{k,\alpha}} \nu_{k} \frac{\partial u_{h_{k}}^{(k)}}{\partial n_{k}} \mathcal{L}^{(k)} \mu_{i}^{(k,\alpha)} \, d\Gamma, \quad i = 1, \ldots, n^{(k,\alpha)}. \tag{69}$$

As done for the configuration with only two patches, we can compute $r_{i}^{(k,\alpha)}$ by exploiting the weak form of the differential equations inside the patches, i.e.

$$r_{i}^{(k,\alpha)} = a^{(k)}(u_{h_{k}}^{(k)}, \mathcal{L}^{(k)} \mu_{i}^{(k,\alpha)}) - F^{(k)}(\mathcal{L}^{(k)} \mu_{i}^{(k,\alpha)}) - \int_{G_{k,\alpha}} \nu_{k} \frac{\partial u_{h_{k}}^{(k)}}{\partial n_{k}} \mathcal{L}^{(k)} \mu_{i}^{(k,\alpha)} \, d\Gamma, \quad i = 1, \ldots, n^{(k,\alpha)}. \tag{70}$$

Following the notations of Sect. 3 for any $k = 1, \ldots, M$ and for any face $\gamma_{k,\alpha} \subset \Gamma_{k}$, we define the sets $\mathcal{I}_{\gamma_{k,\alpha}}$, $\mathcal{I}_{\gamma_{k,\alpha}}$, and $\mathcal{I}_{\partial \gamma_{k,\alpha}}$. To evaluate the last integral of (70) we define the matrix $C^{(k,\alpha)}$ (of size $N^{(k)} \times N^{(k)}$) whose non-null entries are

$$C_{ij}^{(k,\alpha)} = -\int_{G_{k,\alpha}} \nu_{k} \frac{\partial \varphi_{i}^{(k)}}{\partial n_{k}} \varphi_{j}^{(k)} \, d\Gamma, \quad \text{for } i \in \mathcal{I}_{\partial \gamma_{k,\alpha}}, \quad j \in \mathcal{I}_{\mathcal{I}_{\mathcal{I}_{\mathcal{I}_{\gamma_{k,\alpha}}}}}.$$  (71)

Then we set

$$A^{(\gamma_{k,\alpha}, \overline{\gamma}_{k})} = A^{(\gamma_{k,\alpha}, \overline{\gamma}_{k})} + C^{(\gamma_{k,\alpha}, \overline{\gamma}_{k})},$$  (72)

and $f_{k,\alpha} = f(\mathcal{I}_{\gamma_{k,\alpha}})$, and we compute

$$r_{k,\alpha} = A^{(\gamma_{k,\alpha}, \overline{\gamma}_{k})} u_{h_{k}}^{(k)} - f_{k,\alpha},$$

that will contain the values $r_{i}^{(k,\alpha)}$ defined in (70).
Finally, for any face $\gamma_{k,\alpha}$ we define the mass matrix

$$(M_{k,\alpha})_{ij} = (\mu_j^{(k)}, \mu_i^{(k)})_{L^2(\gamma_{k,\alpha})}, \quad i, j = 1, \ldots, n^{(k,\alpha)}. \quad (73)$$

The algebraic implementation of the interface conditions $[61]_3$ reads:

for any $(k, \alpha): \gamma_{k,\alpha}$ is master,

$$r_{k,\alpha} + \sum_{(t,\beta) \in A_{k,\alpha}} M_{k,\alpha} P_{(k,\alpha)_{(t,\beta)}} M_{(t,\beta)}^{-1} r_{t,\beta} = 0. \quad (74)$$

9 The iterative algorithm to solve (61).

We extend to each patch with index $k = 1, \ldots, M$ the notations on the matrices and on the arrays introduced in Sect. 5 and we split the degrees of freedom (the unknown coefficients of each $u_{h_k}^{(k)}$ with respect to the NURBS basis functions) in:

1. $u_0^{(k)}$: the degrees of freedom internal to $\Omega^{(k)}$,
2. $g^{(k)}$: the degrees of freedom associated with the Dirichlet boundary $\partial\Omega_D^{(k)}$, if it is not empty,
3. $u_\Gamma$: the (not replicated) degrees of freedom associated with the master skeleton $\Gamma$ defined in $[60]$ deprived of the degrees of freedom associated with $\Gamma \cap \partial\Omega_D^{(k)}$.

Notice that only the degrees of freedom associated with the vertices of the patches are interpolatory and they are the only degrees of freedom which we must be careful to not replicate inside $u_\Gamma$. In this way we automatically enforce the continuity of the solution at such interpolatory points.

For example, when we consider a decomposition like that sketched in Fig. 11 the vertex shared by all the four patches belongs to four master faces (the pink ones), but only one occurrence of it must be considered in the array $u_\Gamma$.

Notice that in the decompositions depicted in Fig. 7 the vertex of $\gamma_{3,1}$ (internal to $\Omega$) lays on the face $\gamma_{1,1}$, but it is not a degree of freedom for the patch $\Omega^{(1)}$ (the solely interpolation degrees of freedom of $\Omega^{(1)}$ are at the vertices of the patch itself).

If we eliminate the internal degrees of freedom $u_0^{(k)}$, we obtain the Schur complement system with respect to $u_\Gamma$:

$$Su_\Gamma = b \quad (75)$$

and we can solve it by a Krylov method (e.g. Bi-CGStab, GMRES and so on), for which it is sufficient to provide an algorithm (see Algorithm 3) that, given an array $\lambda$ of the same size of $u_\Gamma$, computes $\psi = S\lambda$.

Since the array $u_\Gamma$ does not contain the degrees of freedom associated with $\Gamma \cap \partial\Omega_D^{(k)}$, but at the same time the interpolation matrices work on the degrees of freedom associated with the faces up to their boundary, for practical purposes it is convenient to extend $u_\Gamma$ to an array $\bar{u}_\Gamma$ that includes also the null degrees of freedom associated with $\Gamma \cap \partial\Omega_D^{(k)}$.

We denote by $n_\Gamma$ the size of $u_\Gamma$, by $\bar{n}_\Gamma$ the size of $\bar{u}_\Gamma$ and we define the restriction matrix $R_D$ of size $\bar{n}_\Gamma \times n_\Gamma$ (whose entries are 0 or 1) that, with any array $\bar{u}$ defined on $\Gamma$, associates its restriction to $\Gamma \setminus \partial\Omega$ such that $u_\Gamma = R_D \bar{u}_\Gamma$, and the prolongation (or extension-by-zero) matrix of size $n_\Gamma \times \bar{n}_\Gamma$ such that $\bar{u}_\Gamma = R_D^T u_\Gamma$.

Similarly, for any couple $(k, \alpha)$, we define the restriction matrix $R_{k,\alpha}$ of size $n^{(k,\alpha)} \times \bar{n}_\Gamma$ that implements the restriction of $u_\Gamma$ from $\Gamma$ to the degrees of freedom associated with $\gamma_{k,\alpha}$ (up to the boundary), i.e. $u_{k,\alpha} = R_{k,\alpha} \bar{u}_\Gamma$. Consequently, $R_{k,\alpha}^T$ implements the prolongation of $u_{k,\alpha}$ to $\bar{u}_\Gamma$.

In Fig. 11 and 12 we sketch a decomposition with 4 patches, we identify master (pink) and slave (green) faces and we explain how the interpolation operators act.

Algorithm 1 contains the instructions to initialize INTERNODES. Algorithm 2 computes the right hand side $b$ of (75), while Algorithm 3 implements the matrix vector product $\psi = S\lambda$ that can be used at each iteration of the iterative method called to solve (75). Once that $u_\Gamma$ has been computed, we can recover the solution $u^{(k)}$ in every patch by applying Algorithm 4.
Figure 11: At left, the global layout for a situation with 4 patches, the circles denote the degrees of freedom. The pink color denotes the master faces, while the green color denotes the slave ones. At right, the degrees of freedom on the master skeleton (the pink cross) and the correspondence between the master skeleton $\Gamma$ and the master faces $\gamma_{k,\alpha}$

Figure 12: At left, the interpolation of the trace from the master faces to the slave faces, $P$ stands for the interpolation operator $\Pi_{(k,\alpha)(\ell,\beta)}$, “drc” stands for Dirichlet condition. At right, the interpolation of the normal derivatives from slave faces to master faces, $MPM$ stands for the interpolation operator $\tilde{\Pi}_{(\ell,\beta)(k,\alpha)}$. “C” identifies the degrees of freedom on which the correction matrix $C^{(k)}$ defined in [71] acts
Algorithm 1 Initialization of \textsc{Internodes}

for all patch $k = 1,\ldots,M$ do

\begin{itemize}
  \item build the local stiffness matrices $A^{(k)}$
  \item build the arrays $f^{(k)}$, $g^{(k)}$
  \item build the matrices $C^{(k)}$ (see (71)) and $\hat{A}^{(\gamma_k,\alpha_k)}$ (see (72))
  \item build the Greville nodes in $\Omega^k$
\end{itemize}

for all face $\alpha$ of $\Gamma_k$ do

\begin{itemize}
  \item build the local interface mass matrices $M^{(k,\alpha)}$ (formula (73))
  \item for all face $(\ell,\beta) \in A^{(k,\alpha)}$ do
    \begin{itemize}
      \item build the interpolation matrices $P^{(k,\alpha)}(\ell,\beta)$ (formulas (66))
    \end{itemize}
\end{itemize}

end for

end for

end for

Algorithm 2 Computation of the right hand side $b$ of (75)

\% Distribute the Dirichlet dof

for all patch $k = 1,\ldots,M$ do

\begin{itemize}
  \item $t^{(k)} = 0$ (array of the dof associated with $\partial\Omega^{(k)}$)
  \item $t^{(k)}|_{\partial\Omega^{(k)}} = g|_{\partial\Omega^{(k)}}$
\end{itemize}

end for

\% Interpolate from master faces to slave faces

for all patch $\ell = 1,\ldots,M$ do

\begin{itemize}
  \item for all slave face $\beta$ of $\Gamma_\ell$ do
    \begin{itemize}
      \item $t^{(\ell)}|_{(\ell,\beta)} = \sum_{(k,\alpha) \in A^{(\ell,\beta)}} P^{(\ell,\beta)(k,\alpha)} t^{(k)}|_{(k,\alpha)}$
    \end{itemize}
\end{itemize}

end for

\% Solve local independent subproblems and compute $r_k,\alpha$ face by face

for all patch $k = 1,\ldots,M$ do

\begin{itemize}
  \item solve $A^{(k)} u^{(k)}_0 = f^{(k)}_0 - A^{(k,\partial\Omega^{(k)})} t^{(k)}$
  \item assemble $u^{(k)} = [u^{(k)}_0, t^{(k)}]$
  \item for all face $\alpha$ of $\Gamma_k$ do
    \begin{itemize}
      \item $r^{(k,\alpha)} = A^{(\gamma_k,\alpha_k)} u^{(k)} - f^{(k,\alpha)}$
    \end{itemize}
  \end{itemize}

end for

\% Interpolate the derivatives from slave to master faces and assemble from local faces to global master skeleton

$b = 0$

for all patch $k = 1,\ldots,M$ do

\begin{itemize}
  \item for all master face $\alpha$ of $\Gamma_k$ do
    \begin{itemize}
      \item $\bar{b} = \bar{b} + R^T_{(k,\alpha)} \left( r^{(k,\alpha)} + \sum_{(\ell,\beta) \in A^{(k,\alpha)}} M^{(k,\alpha)} P^{(k,\alpha)(\ell,\beta)} M^{-1}_{(\ell,\beta)} r^{(\ell,\beta)} \right)$
    \end{itemize}
  \end{itemize}

end for

end for

% Restrict $\bar{b}$ to $\Gamma \setminus \partial \Omega$

$b = R_D \bar{b}$
Algorithm 3 Given $\lambda$, computation of $\psi = S\lambda$. This is the matrix-vector product needed to solve (75) by Krylov methods

% Expand $\lambda$ from $\Gamma \setminus \partial\Omega$ to $\Gamma$
$\tilde{\lambda} = R_D^T \lambda$
% Distribute the trace from the global master skeleton $\Gamma$ to the local master faces
for all patch $k = 1, \ldots, M$ do
  $t^{(k)}(0)$ (array of the dof associated with $\partial\Omega^{(k)})$
  for all master face $\alpha$ of $\Gamma_k$ do
    $t^{(k)}|_{(k,\alpha)} = R_{(k,\alpha)} \tilde{\lambda}$
  end for
end for
% Interpolate from master faces to slave faces
for all patch $\ell = 1, \ldots, M$ do
  for all slave face $\beta$ of $\Gamma_\ell$ do
    $t^{(\ell)}|_{(\ell,\beta)} = \sum_{(k,\alpha) \in A_{(\ell,\beta)}} P_{(\ell,\beta)}^{(k,\alpha)} t^{(k)}|_{(k,\alpha)}$
  end for
end for
% Solve local independent subproblems and compute $r_{k,\alpha}$ face by face
for all patch $k = 1, \ldots, M$ do
  solve $A^{(k,k)} u^{(k)}_0 = -A^{(k,\partial\Omega^{(k)})} t^{(k)}$
  assemble $u^{(k)} = [u^{(k)}_0, t^{(k)}]$
  for all face $\alpha$ of $\Gamma_k$ do
    compute $r_{k,\alpha} = \bar{A}^{(k,\alpha)} u^{(k)}$
  end for
end for
$\tilde{\psi} = 0$ (same size as $\tilde{\lambda}$)
% Interpolate the derivatives from slave to master faces and assemble from local faces to global master skeleton
for all patch $k = 1, \ldots, M$ do
  for all master face $\alpha$ of $\Gamma_k$ do
    $\bar{\psi} = \psi + R_D^T (r_{k,\alpha} + \sum_{(\ell,\beta) \in A_{(k,\alpha)}} M_{(k,\alpha)} P_{(k,\alpha)(\ell,\beta)} M_{(\ell,\beta)}^{-1} r_{\ell,\beta})$
  end for
end for
% Restrict $\psi$ to $\Gamma \setminus \partial\Omega$
$\psi = R_D \psi$

10 Numerical results for $M > 2$ patches

10.1 2D geometry

Let us consider again the differential problem in $\Omega = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$ with $\alpha = 0$, and $f$ and $g$ such that the exact solution is $u(x,y) = \sin(1.5\pi x) \sin(3\pi y)$. Now we split $\Omega$ in 7 patches and we tag the master/slave interfaces as shown in the left picture of Fig. We fix the polynomial degree equal to $p = 2, \ldots, 5$ in all the patches, while the number of elements inside the patches is defined as in the following table, with $\pi = 4, 8, 16, 32$:
Algorithm 4 Given $u_\Gamma$, computation of the local solutions $u^{(k)}$, for $k = 1, \ldots, M$

% Expand $u_\Gamma$ from $\Gamma \setminus \partial \Omega$ to $\Gamma$
$\tilde{u}_\Gamma = R_T^\Gamma u_\Gamma$

% Distribute the Dirichlet dof and the trace from $\Gamma$ to the master faces $\gamma_{k,\alpha}$
\nfor all patch $k = 1, \ldots, M$ do
    $t^{(k)} = 0$ (array of the dof associated with $\partial \Omega^{(k)}$)
    for all master face $\alpha$ of $\Gamma_k$ do
        $t^{(k)}|_{(k,\alpha)} = R_{(k,\alpha)} \tilde{u}_\Gamma$
    end for
    $t^{(k)}|_{\partial \Omega^{(k)}} = g|_{\partial \Omega^{(k)}}$
end for

% Interpolate from master faces to slave faces
\nfor all patch $\ell = 1, \ldots, M$ do
    for all slave face $\beta$ of $\Gamma_\ell$ do
        $t^{(\ell)}|_{(\ell,\beta)} = \sum_{(k,\alpha) \in A_{(\ell,\beta)}} P_{(\ell,\beta),(k,\alpha)} t^{(k)}|_{(k,\alpha)}$
    end for
end for

% Solve local independent subproblems
\nfor all patch $k = 1, \ldots, M$ do
    solve $A^{(k,k)} u^{(k)}_0 = f^{(k)} - A^{(k,\partial \Omega^{(k)})} t^{(k)}$
    assemble $u^{(k)} = [u^{(k)}_0, t^{(k)}]$
end for

| patch       | number of elements               |
|-------------|----------------------------------|
| $\Omega_1$  | $(\pi + 2) \times \pi$          |
| $\Omega_2$ and $\Omega_5$ | $\pi \times \pi$          |
| $\Omega_3$ and $\Omega_4$ | $(\pi + 1) \times (\pi + 1)$ |
| $\Omega_6$  | $\pi \times 3\pi$               |
| $\Omega_7$  | $(\pi + 2) \times \pi$          |

In the right picture of Fig. 13 the broken-norm errors \[54\] are shown versus $h = \max_k h_k$ for $p = 2, \ldots, 5$. As in the case of two subdomains, INTERNODES exhibits optimal convergence order with respect to the mesh size $h$.

10.2 Jumping coefficients, the Kellogg’s test case

We solve the elliptic problem $-\nabla \cdot (\nu \nabla u) = 0$ in $\Omega = (-1, 1)^2$ with Dirichlet boundary conditions on $\partial \Omega$ and piece-wise constant coefficient $\nu$ such that the exact solution is the so-called Kellogg’s function (see, e.g., \[34, 23, 22\]). This is a very challenging problem whose solution features low regularity.

The Kellogg’s solution can be written in terms of the polar coordinates $r$ and $\theta$ as $u(r, \theta) = r^\gamma \mu(\theta)$, where $\gamma \in (0, 2)$ is a given parameter, while $\mu(\theta)$ is a $2\pi$-periodic continuous function defined like follows:

$$
\mu(\theta) = \begin{cases} 
\cos((\pi/2 - \sigma)\gamma) \cos((\theta - \pi/2 + \rho)\gamma) & 0 \leq \theta \leq \pi/2 \\
\cos(\rho\gamma) \cos((\theta - \pi + \sigma)\gamma) & \pi/2 \leq \theta \leq \pi \\
\cos(\sigma\gamma) \cos((\theta - \pi - \rho)\gamma) & \pi \leq \theta \leq 3\pi/2 \\
\cos((\pi/2 - \rho)\gamma) \cos((\theta - 3\pi/2 - \sigma)\gamma) & 3\pi/2 \leq \theta \leq 2\pi.
\end{cases}
$$

(76)

The parameters $\sigma$, $\rho$, $\gamma$ and the coefficient $R$ (that is involved in the definition of $\alpha$) must satisfy the following
broken-norm error

\[ \begin{cases} R = -\tan((\pi/2 - \sigma)\gamma) \cot(\rho\gamma) \\ \frac{1}{R} = -\tan(\rho\gamma) \cot(\sigma\gamma) \\ R = -\tan(\sigma\gamma) \cot((\pi/2 - \rho)\gamma) \\ 0 < \gamma < 2 \\ \max\{0, \pi\gamma - \pi\} < 2\gamma\rho < \min\{\gamma\pi, \pi\} \\ \max\{0, \pi - \gamma\pi\} < -2\gamma\sigma < \min\{\pi, 2\pi - \gamma\pi\}. \end{cases} \] (77)

We set \( \nu = R > 0 \) in the first and the third quadrants, and \( \nu = 1 \) in the second and in the fourth ones.

The case \( \gamma = 1 \) is trivial since the solution is a plane. When \( \gamma \neq 1 \), then \( u \in H^{1+\gamma-\varepsilon}(\Omega) \) for any \( \varepsilon > 0 \), in particular the solution features low regularity at the origin and on the axes.

We look for the approximation of the Kellogg’s solution by applying INTERNODES to the 4-subdomains decomposition induced by the discontinuity of \( \nu \).

For \( \bar{n} \in \{5, 10, 20, 25\} \) we set \( n_{el}^{(1)} = n_{el}^{(3)} = 2\bar{n} + 1 \) and \( n_{el}^{(2)} = n_{el}^{(4)} = \bar{n} - 1 \), so that the discretizations on the two opposite sides of any interface are non-conforming. To analyze the errors we take the same polynomial degree \( p \) along each direction and in all patches.

In the left picture of Fig. 14 the multipatch configuration is shown; in the middle picture of the same Figure the numerical solution corresponding to \( \gamma = 0.6 \) is plotted, it is computed by setting the polynomial degree \( p = 2 \) in each patch, \( n_{el}^{(1)} = n_{el}^{(3)} = 11 \) and \( n_{el}^{(2)} = n_{el}^{(4)} = 4 \).

The broken-norm errors (54) versus the mesh size \( h = \max_k h_k \) are shown in the right picture of Fig. 14.

We have considered two different values for \( \gamma \): first \( \gamma = 0.6 \) and \( R \approx 3.85 \), so that the corresponding Kellogg’s solution belongs to \( H^{1.6-\varepsilon}(\Omega) \) (for any \( \varepsilon > 0 \)), after \( \gamma = 1.8 \) and \( R \approx 2.5 \cdot 10^{-2} \), so that the corresponding Kellogg’s solution belongs to \( H^{2.8-\varepsilon}(\Omega) \).

The broken-norm errors behave like \( h^{\min(s-1,p)} \) when \( h \to 0 \), where \( s = 1 + \gamma - \varepsilon \) is the Sobolev regularity of the Kellogg’s solution. We conclude that the INTERNODES solution is converging to the exact one when \( h \to 0 \) with the best possible convergence rate dictated by the regularity of the Kellogg’s solution.

### 10.3 3D geometry

Let us consider the differential problem (1) with \( \alpha = 1 \) in the domain \( \Omega \subset \mathbb{R}^3 \) represented in Fig. 8. The functions \( f \) and \( g \) are such that the exact solution is \( u(x, y, z) = \sin(\pi x) \sin(\pi y) \cos(2\pi z) \).

The domain \( \Omega \) is split into four patches like in Fig. 8, the multipatch configuration is geometrically non-conforming. The master skeleton is \( \Gamma = \gamma_{1,1} \cup \gamma_{3,1} \cup \gamma_{3,3} \cup \gamma_{4,3} \) (see the caption of Fig. 15 for the numbering of the faces).
Figure 14: Kellogg’s test case. At left, the geometry for the Kellogg’s test case. At middle, the numerical solution with $\gamma = 0.6$. At right the errors versus the discretization parameter $h_1 = 1/n_{el}^{(1)}$.

Figure 15: 3D-ring test case. At left, the broken-norm errors versus the discretization parameter $h$, $p$ is the same in all patches. At right, the plot of the numerical solution in the patches $\Omega^{(1)} \cup \Omega^{(2)} \cup \Omega^{(4)}$, computed with: $p^{(1)} = 4, n_{el}^{(1)} = 3, p^{(2)} = 3, n_{el}^{(2)} = 4, p^{(3)} = 4, n_{el}^{(3)} = 4, p^{(4)} = 3, n_{el}^{(4)} = 3$. We have removed the patch $\Omega^{(3)}$ to have a look on the solution inside $\Omega$.

For $\bar{n} = 2, \ldots, 8$ we set $n_{el}^{(1)} = n_{el}^{(4)} = \bar{n}$ and $n_{el}^{(2)} = n_{el}^{(3)} = \bar{n} + 1$. The discretizations on the two sides of any interface are totally unrelated and non-conforming.

To analyze the behaviour of the broken-norm error with respect to the mesh size, we have considered the same polynomial degree $p$ along each direction and in all patches.

In the left picture of Fig. 15 we show the broken-norm error (54) versus the mesh size $h = \max_k h_k$. We observe that $\|u - u_h\|_* \sim O(h^p)$ when $h \to 0$ and we conclude that the INTERNODES solution is converging to the exact one when $h \to 0$ with the best possible convergence rate dictated by the NURBS-discretization inside the patches (see, e.g., [12, Thm. 3.4 and Cor. 4.16]).

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