\(\kappa\)-deformed Spacetime From Twist

Jong-Geon Bu, Hyeong-Chan Kim, Youngone Lee, Chang Hyon Vac, and Jae Hyung Yee

Department of Physics, Yonsei University, Seoul, Korea.

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We twist the Hopf algebra of \(i\mathfrak{gl}(n, R)\) to obtain the \(\kappa\)-deformed spacetime coordinates. Coproducts of the twisted Hopf algebras are explicitly given. The \(\kappa\)-deformed spacetime obtained this way satisfies the same commutation relation as that of the conventional \(\kappa\)-Minkowski spacetime, but its Hopf algebra structure is different from the well known \(\kappa\)-deformed Poincaré algebra in that it has larger symmetry algebra than the \(\kappa\)-Minkowski case. There are some physical models which consider this symmetry \([42, 43, 44]\). Incidentally, we obtain the canonical (\(\theta\)-deformed) non-commutative spacetime from canonically twisted \(i\mathfrak{gl}(n, R)\) Hopf algebra.

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I. INTRODUCTION

There have been extensive efforts to understand the gravity and quantum physics in a unified viewpoint. These led to the developments in many new directions of research in theoretical physics and mathematics. To accommodate the quantum aspect of spacetime, spacetime non-commutativity has been studied intensively \([1, 2, 3, 4, 5]\). The majority of the research in this direction focused on two types of noncommutative spacetimes \([6, 7]\), i.e., the canonical noncommutative spacetime satisfying,

\[
[x^\mu, x^\nu] = i\theta^{\mu\nu},
\]

where \(\theta^{\mu\nu}\) is a constant antisymmetric matrix, and the commutation relation for the \(\kappa\)-Minkowski spacetime,

\[
[x^\mu, x^\nu] = \frac{i}{\kappa}(a^\mu x^\nu - a^\nu x^\mu),
\]

where \(\kappa\) is a parameter of mass dimension. We call this \(\kappa\)-non-commutativity as time-like if \(a^\mu a_\mu < 0\), space-like if \(a^\mu a_\mu > 0\), and light-cone \(\kappa\)-commutation relation if \(a^\mu a_\mu = 0\). In this paper, we consider time-like noncommutativity (i.e. for \(a^\mu = (1, 0, 0, 0)\)).

The quantum field theories on canonical non-commutative spacetime have many interesting features. By the Weyl-Moyal correspondence, the theory can be thought as a theory on commutative spacetime with Moyal product of the field variables \([8]\). The theories break the classical symmetries (for example, Poincaré symmetry), and may not satisfy unitarity, locality, and some other properties of the corresponding commutative quantum field theory depending on the structure of \(\theta^{\mu\nu}\). The attempts to cure these pathologies are still under progress \([9, 10, 11, 12, 13]\).

Extensive studies have been devoted to the quantum group theory developed in the course of constructing the solutions of the Quantum Yang-Baxter equations. A branch of quantum group theory, deformation theory, was led by Drinfeld \([14]\) and Jimbo \([15]\). Especially, Drinfeld \([14]\) discovered a method of finding one parameter solutions of the Quantum Yang-Baxter equation from simple Lie algebra \(\mathfrak{g}\), that is, he found one parameter family of solutions of Hopf algebras \(U_q(\mathfrak{g})\) deformed from Hopf algebra of the universal enveloping algebra \(U(\mathfrak{g})\) \([16]\).

Recently, following Oeckl \([17]\), Chaichian et.al. \([18]\) and Wess \([19]\) proposed a new kind of symmetry group which is deformed from the classical Poincaré group. Especially, Chaichian et.al. \([18]\) use the twist deformation of quantum group theory to interpret the symmetry of the canonical noncommutative field theory as twisted Poincaré symmetry. There have been some attempts to apply the idea to the field theory with different symmetries, for example, to theories with the \(\Theta\)-Poincaré symmetry \([20]\), conformal symmetry \([21, 22]\), super conformal symmetry \([23]\), super

*Electronic address: bjgeon@yonsei.ac.kr
†Electronic address: hckim@phya.yonsei.ac.kr
‡Electronic address: youngone@phya.yonsei.ac.kr
§Electronic address: Shoutpeace@yonsei.ac.kr
¶Electronic address: jhyee@phya.yonsei.ac.kr
symmetry \cite{24, 25}, Galilean symmetry \cite{26}, Galileo Schrödinger symmetry \cite{27}, translational symmetry of $R^d$ \cite{17}, gauge symmetry \cite{28, 29}, diffeomorphic symmetry \cite{30, 31, 32}, and fuzzy diffeomorphism \cite{33}. The virtue of these twists is that the irreducible representation of the twisted group does not change from that of the original untwisted group, and moreover, the Casimir operators remain the same. And there have been some studies in constructing consistent quantization formalism of field theory with this twisted symmetry group \cite{34, 35, 36, 37}.

In the same reasoning, if one finds a twist that gives $\kappa$-deformed commutation relation (especially time-like $\kappa$-deformed noncommutativity) between coordinates from Poincaré Hopf algebra, it would be very useful in constructing quantum field theory in $\kappa$-deformed spacetime since we can use the irreducible representations of Poincaré algebra for the $\kappa$-noncommutative quantum field theory. There has been some attempts to obtain the $\kappa$-commutation relation of the coordinate system, Eq. (4), from twisting the Poincaré Hopf algebra \cite{35, 39, 40}. In \cite{38}, Lukierski et.al. have argued that one can only get a light-cone $\kappa$-deformation from the Poincaré Hopf algebra. Hence, as far as we know, there is no twist of the Poincaré algebra which gives a time-like $\kappa$-deformed coordinate commutation relation.

This is our motivation to seek for the twist that gives $\kappa$-deformed commutation relation from different Hopf algebra which is larger than the Poincaré algebra. The paper is organized as follows. In section III we briefly review the Hopf algebra and twist deformation. We present two types of abelian twist elements, leading to the two twisted spacetime coordinate systems ($\kappa$-deformed and $\theta$-deformed spacetime) in section IV. In section IV we discuss some aspects of the symmetry algebra and $\kappa$-deformed spacetime induced from twisting $igl(n, R)$ with some physical examples.

II. BRIEF SUMMARY OF TWISTING HOPF ALGEBRA

For any Lie algebra $g$, we have a unique universal enveloping algebra $U(g)$ which preserves the central property of the Lie algebra (Lie commutator relations) in terms of unital associative algebra \cite{41}. This $U(g)$ becomes a Hopf algebra if it is endowed with a co-algebra structure. For $Y \in U(g)$, $U(g)$ becomes a Hopf algebra if we define

$$\Delta : U(g) \rightarrow U(g) \otimes U(g), \quad \Delta Y = Y \otimes 1 + 1 \otimes Y,$$

$$\epsilon(Y) = 0, \quad S(Y) = -Y,$$

(3)

where $\Delta Y$ is a coproduct of $Y$, $\epsilon(Y)$ is a counit, and $S(Y)$ is a coinverse (antipode) of $Y$. In other words, the set $\{U(g), \cdot, \Delta, \epsilon, S\}$ constitutes a Hopf algebra.

$Y$ acts on the module algebra $A$ and on the tensor algebra of $A$, and the action satisfies the relation (hereafter we use Sweedler’s notation $\Delta Y = \sum Y(1) \otimes Y(2)$ \cite{41})

$$Y \triangleright (\phi \cdot \psi) = \sum (Y(1) \triangleright \phi) \cdot (Y(2) \triangleright \psi),$$

(4)

where $\phi, \psi \in A$, the symbol $\cdot$ is a multiplication in the algebra $A$, and the symbol $\triangleright$ denotes the action of the Lie generators $Y \in U(g)$ on the module algebra $A$.

We have a new (twisted) Hopf algebra $\{U_F(g), \cdot, \Delta_F, \epsilon_F, S_F\}$ from the original $\{U(g), \cdot, \Delta, \epsilon, S\}$ if there exists a twist element $F \in U(g) \otimes U(g)$, which satisfies the relations

$$(F \otimes 1) \cdot (\Delta \otimes \text{id})F = (1 \otimes F) \cdot (\text{id} \otimes \Delta)F,$$

(5)

$$(\epsilon \otimes \text{id})F = 1 = (\text{id} \otimes \epsilon)F.$$

(6)

This relations are called counital 2-cocycle condition. The relation between the two Hopf algebras, $U_F(g)$ and $U(g)$, is

$$\Delta_F Y = F \cdot \Delta Y \cdot F^{-1}, \quad \epsilon_F(Y) = \epsilon(Y),$$

$$S_F(Y) = u \cdot S(Y) \cdot u^{-1}, \quad u = \sum F(1) \cdot S(F(2)).$$

(7)

If $A$ is an algebra on which $U(g)$ acts covariantly in the sense of Eq. (4), then

$$\phi \ast \psi = \cdot [F^{-1} \triangleright (\phi \otimes \psi)],$$

(8)

for all $\phi, \psi \in A$, defines a new associative algebra $A_F$. In constructing new Hopf algebra and getting a twisted module algebra, Eq. (5) is crucial for the associativity of the twisted module algebra.

This construction of twisted Hopf algebra has great advantages when it is applied to physical problems whose symmetry group and the irreducible representations are known, since we can use the same irreducible representations and Casimir operators in the twisted theory.
III. \(\kappa\)-DEFORMED COMMUTATION RELATIONS FROM TWISTING \(i\text{gl}(n, R)\)

In this paper we focus on twisting the Hopf algebra of \(i\text{gl}(n, R)\) as a symmetry algebra. In this section we present two abelian twists which result in two non-commutative coordinate systems, Eq (11) and Eq (2).

We use the commutation relation of the Lie algebra, \(g = i\text{gl}(n, R)\),

\[
\begin{align*}
[P_\mu, P_\nu] &= 0 & [M_\mu^\nu, P_\sigma] &= i\delta_\mu^\sigma \cdot P_\nu \\
[M_\mu^\nu, M_\lambda^\rho] &= i(\delta_\mu^\lambda \cdot M_\nu^\rho - \delta_\nu^\rho \cdot M_\mu^\lambda).
\end{align*}
\]

(9)

where \(P_\mu\) can be interpreted as generators of the translation in the \(x^\mu\)-direction and \(M_\mu^\nu\) can be generators of rotations, dilations and contractions.

In coordinate space, the generators are represented as

\[
P_\mu \rightarrow -i\partial_\mu, \quad M_\mu^\nu = -ix^\mu\partial^\nu.
\]

(10)

(note that generators \(M_\mu^\nu\) are different from those of the Poincaré algebra.)

A. \(\kappa\) deformed non-commutativity

The \(\kappa\)-deformation is generated by the twist element \(F_\kappa\),

\[
F_\kappa = \exp \left[ \frac{i}{2\kappa} (E \otimes D - D \otimes E) \right],
\]

(11)

where

\[
D = h_\mu^\nu M_\nu^\mu, \quad E = n_\mu P_\mu.
\]

(12)

\(h_\mu^\nu\) and \(n_\mu\) are defined as \(h_\mu^\nu = \delta_\mu^\nu - \delta_\mu^0 \delta_0^\nu, (n_\mu) = (1, 0, 0, 0)\), that is, \(D = \sum M_\mu^\nu, E = P_0\). These generators \((E, D)\) commute with each other,

\[
[E, D] = -i n_\nu h_\mu^\nu P_\mu = 0.
\]

(13)

Hence we confirm that \(F_\kappa\) is an abelian twist element.

With this twist element Eq. (11), we twist the Hopf algebra of \(U(g = i\text{gl}(n, R))\) to get \(U_\kappa(g)\) as in section II. From the fact that \([E \otimes D, D \otimes E] = 0\), we can rewrite \(F_\kappa\) as

\[
F_\kappa = \exp \left[ \frac{i}{2\kappa} (E \otimes D) \right] \exp \left[ -\frac{i}{2\kappa} (D \otimes E) \right],
\]

(14)

which greatly simplifies the calculation of co-product of twisted Hopf algebra. From the commutation relations (9) we have

\[
[D, M_\mu^\nu] = i(\Omega M^\mu_{\rho\sigma} \mu^\nu, \quad \Omega^\mu_{\rho\sigma} = \delta_\rho^\mu h_\nu^\sigma - h_\rho^\mu \delta_\nu^\sigma \equiv (\mathbb{1} \otimes h - h \otimes \mathbb{1})^{\mu\nu}_{\rho\sigma}.
\]

(15)

From \(h^2 = h\), we get the relation

\[
\Omega^3 = \Omega.
\]

(16)

With the commutation relations (9), we obtain the explicit forms of the coproduct \(\Delta(Y)\),

\[
\begin{align*}
\Delta_\kappa(P)_{\mu}^{\nu} &= \left. \left\{ e^{h \otimes E/(2\kappa)}(P \otimes 1) + e^{-E/(2\kappa) \otimes h} (1 \otimes P) \right\} \right|_{\mu}^{\nu}, \\
\Delta_\kappa(M)_{\mu}^{\nu} &= \left. e^{\Omega \otimes E/(2\kappa)}(M \otimes 1) + e^{-\Omega \otimes E/(2\kappa)} (1 \otimes M) \\
&\quad + \frac{1}{2\kappa} \left. [e^{h \otimes E/(2\kappa)}(P \otimes D) - e^{-E/(2\kappa) \otimes h}(D \otimes P)] \right|_{\mu}^{\nu}.
\end{align*}
\]

(17)
In this calculation, we use the well-known operator relation, \( \text{Ade}^B C = \sum_{n=0}^{\infty} (\text{Ad}B)^n C \), with \( (\text{Ad}B)C = [B, C] \), and the relation

\[
\Delta_\kappa(Y) = \text{Ade}^{-\frac{i}{2\kappa}(E \otimes D \otimes D \otimes E)} \Delta(Y) \\
= e^{\frac{i}{2\kappa}(E \otimes D \otimes D \otimes E)} \Delta(Y) e^{-\frac{i}{2\kappa}(E \otimes D \otimes D \otimes E)} \\
= e^{-\frac{i}{2\kappa}(E \otimes D)} e^{-\frac{i}{2\kappa}(D \otimes E) Y \otimes 1} e^{\frac{i}{2\kappa}(D \otimes E)} e^{-\frac{i}{2\kappa}(E \otimes D)} \\
+ e^{-\frac{i}{2\kappa}(D \otimes E)} e^{\frac{i}{2\kappa}(E \otimes D)} (1 \otimes Y) e^{-\frac{i}{2\kappa}(E \otimes D)} e^{\frac{i}{2\kappa}(D \otimes E)} \\
= \text{Ade}^{-\frac{i}{2\kappa}(E \otimes D)} \text{Ade}^{-\frac{i}{2\kappa}(D \otimes E)} (Y \otimes 1) + \text{Ade}^{-\frac{i}{2\kappa}(D \otimes E)} \text{Ade}^{-\frac{i}{2\kappa}(E \otimes D)} (1 \otimes Y). \tag{18}
\]

Note that, from \( h^2 = h \) and \( \Omega^3 = \Omega \), we have relations

\[
e^{\frac{\Omega \otimes E}{(2\kappa)}} = 1 \otimes 1 + h \otimes \left( e^{E/2\kappa} - 1 \right), \tag{19}
\]

\[
e^{\frac{\Omega \otimes E}{(2\kappa)}} = 1 \otimes 1 + \Omega \otimes \sinh \left( \frac{E}{2\kappa} \right) + \Omega^2 \otimes \left[ \cosh \left( \frac{E}{2\kappa} \right) - 1 \right]. \tag{20}
\]

The algebra acts on the spacetime coordinates \( x^\mu \) with commutative multiplication:

\[
m(f(x) \otimes g(x)) := f(x)g(x). \tag{21}
\]

When twisting \( U(P) \), one has to redefine the multiplication as in Eq. (3), while retaining the action of the generators of the Hopf algebra on the coordinates as in (11):

\[
m_\kappa(f(x) \otimes g(x)) := f(x) \ast g(x) = m \left[ \mathcal{F}_\kappa^{-1}(f(x) \otimes g(x)) \right]. \tag{22}
\]

It is represented as:

\[
(f \ast g)(x) := \exp \left[ \frac{i}{2\kappa} \left( \frac{\partial}{\partial x_0} y^k \frac{\partial}{\partial y_k} - x^k \frac{\partial}{\partial x_k} \frac{\partial}{\partial y_0} \right) \right] f(x)g(y) \bigg|_{x=y}. \tag{23}
\]

Since \( P_\alpha = -i\partial_\alpha \) in this representation, the commutation relations between spacetime coordinates are deduced from this \( \ast \)-product:

\[
x^\mu \ast x^\nu = \left[ e^{\frac{i}{2\kappa}(\partial_0 \otimes x^k \partial_k - x^k \partial_0 \otimes \partial_k)} \right] (x^\mu \otimes x^\nu) \\
= x^\mu \cdot x^\nu + \frac{i}{2\kappa} (\delta_\mu^0 \delta^\nu_k x^k - x^k \delta_\mu^k \delta^\nu_0), \\
\implies [x^0, x^k] = \frac{i}{\kappa} x^k, \quad [x_i, x_j] = 0, \tag{24}
\]

which corresponds to those of the time-like \( \kappa \)-deformed spacetime.

The case of the tachyonic \( (\alpha_p \kappa^\mu = 1) \) and light-cone \( (\alpha_p \kappa^\mu = -1) \) \( \kappa \)-deformation is obtained in Lukierski et.al.’s work [3]. It should be noted that the twisted Hopf algebra in this section is different from that of the conventional \( \kappa \)-Minkowski algebra, which is a deformed Poincaré algebra, in that it has different co-algebra structure from a bigger symmetry.

### B. \( \theta \)-deformed non-commutativity

Since affine algebra contains Poincaré algebra, there is also a twist which has same form as that of the canonical non-commutativity case [18]. We use the same twist element given by

\[
\mathcal{F}_\theta = \exp \left( \frac{i}{2} \theta^{\alpha \beta} P_\alpha \otimes P_\beta \right). \tag{25}
\]

This twist element satisfies the 2-cocycle condition, Eq. (5). Following the same procedures as in subsection (III A) we get the twisted Hopf algebra given by the coproducts,

\[
\Delta_\theta(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu, \tag{26}
\]

\[
\Delta_\theta(M_\mu^\nu) = M_\mu^\nu \otimes 1 + 1 \otimes M_\mu^\nu + \frac{1}{2} \theta^{\alpha \beta} \cdot [\delta_\alpha^\mu P_\nu \otimes P_\beta + \delta_\beta^\nu P_\alpha \otimes P_\nu].
\]
Here also note that generators \( M^\mu_\nu \) are different from those of the Poincaré algebra.

Similarly as in the \( \kappa \)-deformed case, we have used

\[
\Delta_\theta(Y) = e^{\frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta} (Y \otimes 1 + 1 \otimes Y) e^{-\frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta} = \text{Ad} \exp \left( \frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta \right) \Delta(Y),
\]

and

\[
[\text{Ad}(P_\alpha \otimes P_\beta)]^n (M^\mu_\nu \otimes 1) = -i \delta^n_1 \cdot \delta^\mu_\alpha (P_\nu \otimes P_\beta),
\]

\[
[\text{Ad}(P_\alpha \otimes P_\beta)]^n (1 \otimes M^\mu_\nu) = -i \delta^n_1 \cdot \delta^\mu_\beta (P_\alpha \otimes P_\nu),
\]

for \( n \geq 1 \).

When \( \phi, \psi \in A_\theta \) are the functions of the same spacetime coordinate \( x^\mu \), the product \(*\) becomes the well known Moyal product. Since \( P_\alpha = -i \partial_\alpha \) in this representation, the commutation relations between spacetime coordinates are

\[
x^\mu * x^\nu = \cdot \left[ \exp \left( \frac{i}{2} \theta^{\alpha\beta} \partial_\alpha \otimes \partial_\beta \right) \right] (x^\mu \otimes x^\nu)
\]

\[
= x^\mu \cdot x^\nu + \frac{i}{2} \theta^{\mu\nu},
\]

which leads to the commutation relation

\[
[x^\mu, x^\nu]_* = i \theta^{\mu\nu}.
\]

This is the same commutation relation of coordinates as those in the canonical noncommutative spacetime. This twist is different from the conventional twist \([19]\), in that the relevant group is different. The twist in the work of Wess \([19]\) and Chaichian, et al. \([18]\) is that of the Poincaré Hopf algebra. Since we use bigger symmetry algebra, \( i-sl(n, R) \), than the Poincaré algebra, \( iso(n, R) \), only antisymmetric part of our twisted coproduct of generators \( M^\mu_\nu \) corresponds to those of \([19]\) and \([18]\). We have more components (coproducts of the symmetric part of the generators \( M^\mu_\nu \)) in the coproduct sector.

\[\text{IV. DISCUSSION}\]

In this paper we obtained time-like \( \kappa \)-deformed commutation relation by twisting the Hopf algebra of \( i-sl(n, R) \) not by twisting the Poincaré Hopf algebra. To understand why it is difficult to obtain time-like \( \kappa \)-deformed commutation relation by twisting the Poincaré Hopf algebra, \( U(\mathcal{P}) \), it is instinctive to try a element \( F \in U(\mathcal{P}) \otimes U(\mathcal{P}) \) as a twist element.

For \( F = 1 + r + O(r^2) \), \( r = r_1 \otimes r_2 \) for all \( r_1, r_2 \in U(\mathcal{P}) \)

\[
x^\mu * x^\nu = \cdot [F \triangleright (x^\mu \otimes x^\nu)]
\]

\[
= x^\mu \cdot x^\nu + r_1(x^\mu) \cdot r_2(x^\nu) + \cdots
\]

From the action of \( r_1, r_2 = (P_\rho, L_{\mu\nu}) \), where \( P_\rho \rightarrow -i \partial_\rho \), \( L_{\mu\nu} \rightarrow -i (x_\mu \partial_\nu - x_\nu \partial_\mu) \), to the coordinates, we infer the ansatz of the classical r-matrix in order to obtain the first order relation of the \( \kappa \)-deformed commutation relation, Eq. \([2]\), as

\[
r = r^{\rho\mu\nu} P_\rho \wedge L_{\mu\nu},
\]

where \( r^{\rho\mu\nu} \) is a constant.

In order to this element \( F \) to be a twist, the above classical r-matrix have to satisfy ‘classical Yang-Baxter equation’, i.e. the relation,

\[
[r, r] = 0
\]

Actually one can show that the above form of classical r-matrix in Eq.\([32]\) cannot satisfy classical Yang-Baxter equation, in general, except in a very special combination of \( P_\rho, L_{\mu\nu} \), i.e. \([P_\rho, L_{\mu\nu}] = 0 \), which is the crucial condition
for the twist to satisfy a 2-cocycle condition. For that special case, Lukierski and Woronowicz give the ‘abelian’ twist element from classical r-matrix \[39\]. Although there have been many studies on field theories in the \(\kappa\)-Minkowski spacetime, the attempts to twist the Poincaré group to get the \(\kappa\)-deformed coordinate spacetime have succeeded only in the light-cone \(\kappa\)-deformation \[38, 39\].

Hence, in order to obtain a twist which gives time-like \(\kappa\)-deformation we need a bigger symmetry algebra than the Poincaré algebra. We have obtain the \(\kappa\)-deformed non-commutativity at the cost of bigger symmetry. Our choice is the Hopf algebra of affine group \(IGL(n, R)\) and we have successfully twisted \(U(igl(n, R))\) in two different ways corresponding to the \(\kappa\)-deformed and the \(\theta\)-deformed non-commutativity.

Incidentally, since there are two abelian subalgebras in \(igl(n, R)\), we derive two the non-commutativity corresponding to the \(\kappa\)-deformed and the \(\theta\)-deformed case in section III. Is there a twist which transform the \(\theta\)-noncommutativity to the \(\kappa\)-noncommutativity? Since the inverses of \(F_\theta\) and \(F_\kappa\) also satisfy the counital 2-cocycle condition, we may think of the maps \(F_\theta \circ F_\kappa^{-1}\) and \(F_\kappa \circ F_\theta^{-1}\) as mapping between the two different non-commutative spaces as shown in the following figure:

\[
\begin{array}{c}
U(g) \\
| \quad \theta \quad | \\
F_\theta \quad \kappa \quad F_\theta \circ F_\kappa^{-1} \quad \kappa \\
U_\theta(g) \quad \kappa \quad U_\kappa(g).
\end{array}
\]

However, they \((F_\theta \circ F_\kappa^{-1}\) and \(F_\kappa \circ F_\theta^{-1}\) do not satisfy the counital 2-cocycle condition in general. Hence the composition map of them can not be a twist. We can not regard this relation between the two twists as a kind of symmetry (while it can be a kind of symmetry in the context of quasi-Hopf algebra).

The algebra \(igl(n, R)\) we twisted in this paper can be interpreted as a physical symmetry algebra in two ways.

One interpretation is to regard the \(igl(n, R)\) as the generalization of \(O(1, 3)\) \[42, 43, 44\]. In those works, they consider the generalization of the Einstein-Hilbert action. That is, from the theory of the local symmetry group

\[Diff(M) \times O(1, 3),\]  

(34)

to the theory of local symmetry group

\[Diff(M) \times GL(4).\]  

(35)

The present method will be useful when these kinds of non-commutativity are applied to those theories with translational symmetry.

Another possible interpretation is to regard \(igl(n, R)\) as the algebra of a subgroup of diffeomorphism group. Since the Minkowski spacetime is a solution of Einstein-Hilbert action, we obtain by twist the \(\kappa\)-Minkowski spacetime and the canonical non-commutative spacetime using \(F_\kappa\) and \(F_\theta\), respectively. We may summarize this relations as

\[
\text{local symmetry group} \supset \text{Diff}(M) \times O(1, 3) \\
\quad \supset \cdots \times (GL(4, R) \times O(1, 3)) \ltimes T. \quad (36)
\]

In this sense, we are in the same direction as in the work of Aschieri et.al. \[31, 32\]. From the above relation, we realize that the canonical twist in this interpretation has different origin from the twists in earlier studies \[18, 19\].

We twist the subgroup of the \(\text{Diff}(M)\), while in earlier studies (for example, Chaichian et.al. \[18\]) one twists the Hopf algebra, \(o(1, 3)\), i.e., the algebra of the Poincaré group,

\[P \supset O(1, 3) \ltimes T,\]  

(37)

or more general symmetry group (for example, conformal symmetry by Matlock \[21\], etc). The twisted \(\kappa\)-deformation gives the same coordinate non-commutativity, but is distinguished from the \(\kappa\)-deformed Minkowski algebra in that the co-algebra structures are different.

The blackholes in \(\kappa\)-deformed spacetime can be regarded as an example of the application of the result of this paper. Since field theories in \(\kappa\)-deformed spacetime are the same as field theories with a \(\kappa\)-moyal product as in Eq. \[23\] in commutative spacetime, the deformed Einstein equations will be the ones in which the products between the metric and its derivatives are changed to the \(\kappa\)-moyal product. Among the solutions of the Einstein equations in commutative
spacetime, a static solution is also a solution in a $\kappa$-deformed spacetime. The Einstein equations are turned into the form:

$$R_{\mu\nu}(g(x)) = 0 \rightarrow R^*_{\mu\nu}(g(x)) = 0. \quad (38)$$

Since a static solution of the left hand side of Eq. (38) has no time dependence, $\kappa$-moyal products in the right hand side are the same as normal products between them, it would automatically satisfy the deformed Einstein equations. The $\kappa$-deformed spacetime here denotes the module space of the twist induced algebra. It should be distinguished from the $\kappa$-deformed spacetime which is a module space of the well-known $\kappa$-deformed algebra.

Though a static solution is also a solution in $\kappa$-deformed spacetime, it has different dynamics. The time dependance changes the dynamics through $\kappa$-moyal products. That is, the perturbation equation of the static solution will be different:

$$\delta R^*_{\mu\nu}(g(x)) = 0 \neq \delta R_{\mu\nu}(g(x)) = 0. \quad (39)$$

Hence to tell the stability of the static solution in $\kappa$-deformed spacetime, careful analysis of the deformed perturbation equation Eq. (39) is expected. The stability analysis of these static solutions in $\kappa$-deformed spacetime is under investigation.

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