Cache-aided Interference Management using Hypercube Combinatorial Design with Reduced Subpacketizations and Order Optimal Sum-Degrees of Freedom

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Abstract

We consider a cache-aided interference network which consists of a library of $N$ files, $K_T$ transmitters and $K_R$ receivers (users), each equipped with a local cache of size $M_T$ and $M_R$ files respectively, and connected via a discrete-time additive white Gaussian noise (AWGN) channel. Each receiver requests an arbitrary file from the library. The objective is to design a cache placement without knowing the receivers' requests and a communication scheme such that the sum Degrees of Freedom (sum-DoF) of the delivery is maximized. This network model with one-shot transmission was firstly investigated by Naderializadeh et al., who proposed a scheme that achieves a one-shot sum-DoF of $\min\{\frac{M_T K_T + K_R M_R}{N}, K_R\}$, which is optimal within a constant of 2. One of the biggest limitations of this scheme is the requirement of high subpacketization level. This paper attempts to design new algorithms to reduce the file subpacketization in such a network without hurting the sum-DoF. In particular, we propose a new approach for both prefetching and linearly coded delivery based on a combinatorial design called hypercube. The proposed approach reduces the subpacketization exponentially in terms of $K_R M / N$ and achieves the identical one-shot sum DoF when $\frac{M_T K_T + K_R M_R}{N} \leq K_R$.

Index Terms

Interference management, subpacketization reduction, hypercube cache design, Degree-of-Freedom (DoF)

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I. Introduction

Wireless traffic has grown dramatically in recent years due to the increasing mobile data demand, mainly due to video delivery services \cite{2}. One promising approach to handle this traffic bottleneck is to exploit local cache memories at end user devices or network edge nodes (e.g., small cell base stations) to pre-store part of the contents (e.g., movies) which might be requested in the near future. With the help of these cache nodes, the system can serve users with a much higher rate and lower latency \cite{3}–\cite{14}. Among all schemes based on caching approaches, coded caching, introduced in \cite{3}, has attracted significant attentions. In particular, Maddah-Ali and Niesen considered a \textit{shared-link network} and studied the problem of minimizing the worst-case traffic load or \textit{transmission rate}. It was shown that prefetching packets of the library files in a uniform manner during the placement phase, and employing coded scheme based on linear index code during the delivery phase, is sufficient to provide optimal rate under uncoded cache placement \cite{4}, \cite{15}, \cite{16}. Later, the idea of coded caching was extended to many other network topologies including Device-to-Device (D2D) caching networks \cite{5}, multi-server caching networks \cite{17} and combination caching networks \cite{18}–\cite{22}, where the channels between the transmitters and receivers are either wireline channel or noiseless broadcast channel.

The concept of coded caching was also extended to the wireless channels with the consideration of interference \cite{6}–\cite{13}. For example, in \cite{6}, the authors considered a three-user interference channel where only transmitters are equipped with cache memories (no cache memories at the receivers) and showed that via a specific cache prefetching strategy, an efficient delivery scheme can be designed by exploiting the gains based on interference cancellation and interference alignment. In \cite{9}, the additive Gaussian channel in a broadcast setting with cache-aided receivers was studied. Later, the study was extended to the case where both transmitters and receivers are equipped with cache memories \cite{7}, \cite{8}, \cite{12}, \cite{13}. Moreover, cache-aided fog radio access network was also investigated in \cite{10}, \cite{14}.

As shown in the above works, in most of the network models, the remarkable multiplicative gain of coded caching in terms of network aggregate cache memory has been established in the asymptotic regime when the number of packets per file, denoted by $F$, scales to infinity. It has been shown that in most of the cases, to achieve the desired caching gain, $F$ has to increase exponentially as a function of the number of nodes in the network. The finite length analysis of
coded caching for the shared-link network was initiated in [23] in which the authors proposed to encode the data only across a small subset of the total $K$ users in the system to obtain reduced subpacketization level at the cost of a reduced coded caching gain. Significant efforts have been made to reduce the subpacketization levels in shared-link caching networks such as placement delivery array (PDA) [24], resolvable design [25] and hypergraph based design [26].

The finite length analysis of coded caching in other network topologies other than shared-link and MIMO broadcast channel is very limited. In [13], the authors considered a MISO broadcast channel with $L$ transmitting antennas and showed that a reduced subpacketization can be achieved. In addition, they extended the achievable scheme to the cache-aided interference networks. In [27], we considered a D2D caching network over noiseless broadcast channel model and introduced a combinatorial design called hypercube, and the corresponding placement and coded delivery schemes with a substantially lower subpacketization level while still achieving order optimal throughput.

In this paper, we consider the general wireless interference network with cache memories equipped at both the transmitter and receiver sides. In particular, we consider a wireless interference network with $K_T$ transmitters and $K_R$ receivers, each equipped with a local cache memory of size $M_T$ and $M_R$ files, from a library of $N$ files. We restrict the communication scheme to one-shot linear schemes due to its practicality. This network model was first considered by Naderializadeh, Maddah-Ali and Avestimehr (NMA) in [7]. Interestingly, in this work, we will show that our previously introduced hypercube based combinatorial approach, which was designed for D2D caching networks with noiseless broadcast channels, can be extended to cache-aided interference networks in a non-straightforward way such that the subpacketizations can be significantly reduced.

Our main contribution in this paper is two-fold. First, based on the hypercube cache placement introduced in [27], we designed a cache placement scheme at both transmitters and receivers, and proposed a linear one-shot delivery scheme by exploiting zero-forcing opportunities via transmitter collaboration and cache-induced interference cancellation opportunities at receivers side. The proposed scheme achieves an order-wise subpacketization level reduction compared to that achieved in [7]. Second, when $\frac{K_T M_T + K_R M_R}{N} \leq K_R$, the proposed scheme achieves a one-shot sum-DoF of $\frac{K_T M_T + K_R M_R}{N}$, which is within a factor of 2 to the optimum as shown by
More importantly and surprisingly, it achieves the same sum-DoF as in [7]. This implies that there is no loss in terms of one-shot sum-DoF by using the proposed scheme while requiring a much less file subpacketization. In the rest of the paper, we will refer the scheme in [7] as NMA scheme.

Notation Convention: We use calligraphic symbols to denote sets and $|\cdot|$ to represent the cardinality of a set or the length of a vector or the norm of a random variable; $\mathbb{Z}^+$ denotes the positive integer set and $\mathbb{C}$ denote the set of all complex numbers. “$a \mod b$” denotes the module operation of $a$ modulo $b$; For some $m, n \in \mathbb{Z}^+$ and $m \leq n$, let $[n] \triangleq \{0, 1, \cdots, n-1\}$ and $[m : n] \triangleq \{m, m+1, \cdots, n-1, n\}$.

II. NETWORK MODEL AND PROBLEM FORMULATION

A. General Problem Formulation

Consider a wireless interference network, as illustrated in Fig. 1, which consists of $K_T$ transmitters and $K_R$ receivers, denoted by $\{\text{Tx}_i : i \in [K_T]\}$ and $\{\text{Rx}_j : j \in [K_R]\}$, respectively. The system contains a library of $N$ files denoted by $\{\mathcal{W}_n : n \in [N]\}$, where the file $\mathcal{W}_n$ contains $F$ packets $\mathcal{W}_n \triangleq \{w_{n,p} : p \in [F]\}$ with size of $L$ bits each, i.e., $w_{n,p} \in \mathbb{F}_2^L$. Transmitters and receivers are equipped with cache memories to store part of the file library. In particular, each transmitter and receiver are equipped with a local cache of size $M_T$ and $M_R$ files, respectively. The communication channel between transmitters and receivers is modeled as discrete-time additive white Gaussian noise (AWGN) channel, which can be written as

$$Y_j(t) = \sum_{i=0}^{K_T-1} h_{ji} S_i(t) + N_j(t),$$

where $t$ is the index of the time slot, $S_i(t) \in \mathbb{C}$ is the complex transmit signal of $\text{Tx}_i$ at time slot $t$, satisfying the power constraint $\mathbb{E}[|S_i(t)|^2] \leq P$. $Y_j(t)$ is the received signal of $\text{Rx}_j$ and $N_j(t) \sim \mathcal{CN}(0,1)$ is the complex additive white Gaussian noise at receiver $\text{Rx}_j$. Moreover, $h_{ji} \in \mathbb{C}$ denotes the complex channel gain from $\text{Tx}_i$ to $\text{Rx}_j$, which is assumed to keep unchanged

1 Note that when $K_T M_T + K_R M_R > K_R$, using the similar argument presented in [7], the order optimal sum-DoF of $K_R$ is also achievable using our proposed approach. However, it is not straightforward to compare the subpacketizations. Hence, we do not consider this case in this paper.

2 In this paper, we let $L$ be a designed variable and equals to $|\mathcal{W}_n|/F$.

3 We will ignore the index of $t$ when it does not cause confusion.
during the entire transmission process and is known to all transmitters and receivers. The system operates in two phases: the prefetching phase and the delivery phase as described in [7]. In the prefetching phase, each transmitter and receiver can store up to $M_T F$ and $M_R F$ arbitrary packets from the file library, respectively. This phase is done without the prior knowledge of the receivers’ future requests. In the following delivery phase, each receiver $Rx_j$ randomly requests a file $W_{d_j}, d_j \in [N]$ from the library. These requests are represented by a demand vector denoted as $d \triangleq [d_0, d_1, \ldots, d_{K_R-1}]$. For a specific demand vector, since the receivers have already cached some packets of their requested files, the transmitters only need to deliver the remaining packets to those receivers. The task in this phase is to design the an efficient transmission procedure based on the cache placement in the prefetching phase so that the receivers’ demands can be satisfied. In order to guarantee that any possible demands can be satisfied, we require that the entire file library is cached among all transmitters, i.e., $K_T M_T \geq N$.

For each cached packet $w_{n,p} \in \mathbb{F}_2^L$, the transmitter performs random Gaussian coding $\psi : \mathbb{F}_2^L \mapsto \mathbb{C}^\hat{L}$ to obtain the coded packet $\hat{w}_{n,p} \triangleq \psi(w_{n,p})$ which consists of $\hat{L}$ complex symbols. Assume that the communication will take place in $H$ blocks, each of which consists of $\hat{L}$ time slots. In addition, we allow only one-shot linear transmission schemes in each block $m \in [1 : H]$ to deliver a set of requested (coded) packets $P_m$ to a subset of the receivers, denoted by $\mathcal{R}_m$. That is, each transmitter $Tx_i, i \in [K_T]$ will send a linearly coded message

$$s_i^m = \sum_{(n,p) : w_{n,p} \in C_i \cap P_m} \alpha_{i,n,p}^m \hat{w}_{n,p},$$

(2)
where $C^T_i$ denotes the cached contents of Tx$_i$ and $\alpha_{m,n,p}^i$ is the linear combination coefficients used by Tx$_i$ at the $m$-th block. Accordingly, the received signal of the intended receivers Rx$_j$, $j \in R_m$ in the $m$-th block is
\[
y^m_j = \sum_{i=0}^{K_T-1} h_{ji} s^m_i + n^m_j,
\]
where $n^m_j \in \mathbb{C}^L$ is the random noise at Rx$_j$ in block $m$. Each receiver will utilize its cached contents, consisting of packets stored in the prefetching phase, to subtract some of the interference caused by undesired packets. In particular, each receiver will perform a linear combination $L^m_j(.)$ if possible in block $m$ to recover its requested packets from all received signals as follows
\[
L^m_j(y^m_j, \hat{C}_j^R) = \hat{w}_{d,j,p} + n^m_j,
\]
where $\hat{w}_{d,j,p} \in P_m$ is the desired coded packet of Rx$_j$ and $\hat{C}_j^R$ denotes the Gaussian coded version of the packets cached by Rx$_j$.

The one-shot linear sum-DoF is defined as the maximum achievable one-shot linear sum-DoF for the worst-case demands under a given caching realization [7], i.e.,
\[
\text{DoF}^*_{L,\text{sum}}(\{C^T_i\}_{i=0}^{K_T-1}, \{C^R_j\}_{j=0}^{K_R-1}) = \sup_{H, \{P_m\}_{m=1}^H} \inf_{d} \left| \bigcup_{m=1}^H P_m \right| \frac{H}{H}.
\]

The one-shot linear sum-DoF of the network is correspondingly defined as the maximum achievable one-shot linear sum-DoF over all possible caching realizations, i.e.,
\[
\text{DoF}^*_{L,\text{sum}}(N, M_T, M_R, K_T, K_R) = \sup_{\{C^T_i\}_{i=0}^{K_T-1}, \{C^R_j\}_{j=0}^{K_R-1}} \text{DoF}^*_{L,\text{sum}}(\{C^T_i\}_{i=0}^{K_T-1}, \{C^R_j\}_{j=0}^{K_R-1}),
\]
in which the cached contents of all transmitter and receivers satisfy the memory constraints, i.e., $|C^T_i| \leq M_T F, \forall i \in [K_T]$ and $|C^R_j| \leq M_R F, \forall j \in [K_R]$.

B. Combinatorial Cache Placement Design

In this paper, the combinatorial cache placement design based on hypercube, proposed in [27], [28] to reduce the subpacketization level in wireless D2D networks is adopted in the prefetching phase. The hypercube cache placement has a nice geometric interpretation: each packet of the file can be represented by a lattice point in a high-dimensional hypercube and the cached content
of each D2D node is represented by a hyperplane in that hypercube (see Fig. 2). Based on the
hypercube cache placement and the corresponding communication scheme, order-optimal rate
can be achieved with exponentially less number of packets compared to the Ji-Caire-Molisch
(JCM) scheme \[5\]. It turns out that by a non-trivial extension, the hypercube scheme can also
significantly reduce the required subpacketizations in cache-aided interference networks. The
details of hypercube cache placement \[27\], \[28\] is described as follows.

1) Hypercube cache placement design for wireless D2D caching networks: Consider a wire-
less D2D network consisting of a library of \( N \) files, each with \( F \) packets, and \( K \) users, each of
which is equipped with a local cache memory of size \( M \) files, or equivalently, \( MF \) packets. The
caching parameter, defined as \( t \triangleq \frac{KM}{N} \in \mathbb{Z}^+ \), represents the average number of times that each file
is cached among all users. In the hypercube cache placement, each file \( W_n \) is split into \( \left( \frac{N}{M} \right)^t \) sub-
files\(^4\)(assuming that \( \frac{N}{M} \) and \( t \) are both integers), i.e., \( W_n = \{ W_{n, (\ell_0, \ell_1, \cdots, \ell_{t-1})} : \ell_j \in \left[ \frac{N}{M} \right], j \in [t] \} \).

It can be seen that each subfile of a file \( W_n \) is uniquely marked by a \emph{t}-tuple \( (\ell_0, \ell_1, \cdots, \ell_{t-1}) \)
where \( \ell_j, j \in [t] \) represents the index of the lattice point along the \( j \)-th dimension. In the
prefetching phase, each user \( u \in [K] \) caches a set of subfiles \( \{ W_{n, (\ell_0, \ell_1, \cdots, \ell_{t-1})} : \forall n \in [N] \} \),
where \( \ell_j = u \mod \frac{N}{M} \), for \( j = \lfloor u / \left( \frac{N}{M} \right) \rfloor \), and \( \ell_i \in \left[ \frac{N}{M} \right] \) for any \( i \neq j \). As a result, each
user will cache \( (\frac{N}{M})^{t-1} \) subfiles from each file \( W_n \). It can be verified that the total number of
subfiles cached by any user is equal to \( N (\frac{N}{M})^{t-1} = \frac{N(M/M)^t}{N/M} = N \frac{EM}{N/M} = MF \), satisfying the
memory constraint. The hypercube cache placement has a nice geometric interpretation. Under
the hypercube file splitting method, each subfile will represent a lattice point with coordinate
\( (\ell_0, \ell_1, \cdots, \ell_{t-1}) \) in a \emph{t}-dimensional hypercube, and \( \frac{N}{M} \in \mathbb{Z}^+ \) is the number of lattice points
along each dimension. We will further illustrate the details of the hypercube cache placement
via the following example.

\textbf{Example 1: (Hypercube Cache Placement)} Consider a set of \( K = 9 \) users labelled as
\( \{0, 1, 2, \cdots, 8\} \) and a set of \( N = 9 \) files \( \{ W_n, n \in [9] \} \). Each user has a cache memory of
size \( M = 3 \) files. We first partition the users into \( t \triangleq \frac{KM}{N} = 3 \) disjoint groups denoted as
\( \mathcal{U}_0 = \{0, 1, 2\}, \mathcal{U}_1 = \{3, 4, 5\} \) and \( \mathcal{U}_2 = \{6, 7, 8\} \). Each file \( W_n \) is split into \( (\frac{N}{M})^t = 3^3 = 27 \)
subfiles, i.e., \( W_n = \{ W_{n, (\ell_0, \ell_1, \ell_2)} : \ell_0, \ell_1, \ell_2 \in [3]\} \), each of which can be represented by a unique

\footnote{In the prefetching phase, each file is split into multiple smaller files and each of these smaller subfiles is then spread
across the user caches. We use ‘subfile’ to refer to these smaller subfiles. In the delivery phase, in order to perform interference
cancellation, each subfile needs to be further split into multiple even smaller ones. We use ‘packet’ to refer to such smaller files
resulting from splitting the subfiles.}
Fig. 2. A 3-dimensional example of the hypercube cache placement. Each subfile is represented by a unique lattice point in the 3-dimensional hypercube (cube). Each of the 9 users caches a set of packets represented by plane of lattice points. As a result, each user caches $9 \times 9 = 81$ subfiles in total.

lattice point in the 3-dimensional cube (see Fig. 2). As a result, each lattice point will represent a set of $N = 9$ subfiles, each from a distinct file. For the cache placement, each user caches all subfiles represented by a plane of lattice points of the cube. For example, user $u_0 = 2$, $u_1 = 4$ and $u_2 = 8$ will cache subfiles represented by the green, red and blue planes respectively in Fig. 2. We can see that the set of subfiles $\{W_{n,(2,1,2)} : \forall n \in [9]\}$ represented by the lattice point $(2, 1, 2)$, which is the intersection of the three orthogonal planes of different colors, is cached exclusively by users $u_0$, $u_1$ and $u_2$. Similarly, each subfile is cached by three distinct users. △

2) Hypercube cache placement design for cache-aided interference networks: Different from the D2D setting in [27], in cache-aided interference networks, we have a set of explicit transmitters and receivers instead of D2D users. However, the hypercube approach can still be applied to design the cache placement in this case illustrated as follows.

*File Splitting:* let $D_T \triangleq \frac{N}{M_T} \in \mathbb{Z}^+$ and $D_R \triangleq \frac{N}{M_R} \in \mathbb{Z}^+$ denote the number of transmitters and receivers on each edge of the hypercube associated with the transmitters’ cache and receivers’ cache respectively. For the set of $K_T = D_T t_T$ transmitters $\{\text{Tx}_k : k \in [K_T]\}$, we denote the $t_T \triangleq \frac{K_T M_T}{N}$ dimensions of the transmitters as $U_i^T = \{k : \lfloor \frac{k}{D_T} \rfloor = i\}, i \in [t_T]$. Similarly, for

5 Since we apply the hypercube cache placement at both the transmitters’ and receivers’ sides, there are two hypercubes associated with the cache-aided interference network, including the *transmitter hypercube* which is a $t_T$-dimensional hypercube with each edge containing $N/M_T$ lattice points (transmitters), and the *receiver hypercube* which is a $t_R$-dimensional hypercube with each edge containing $N/M_R$ lattice points (receivers).

6 The superscript 'T' means 'Transmitter'. Readers should not confuse this with the transpose operator.
the set of $K_R = D_R t_R$ receivers $\{Rx_k : k \in [K_R]\}$, we denote the $t_R \triangleq \frac{K_R m R}{N}$ dimensions of the receivers as $U^R_t = \{k : \left\lfloor \frac{k}{D_R} \right\rfloor = j\}, j \in [t_R]$. It can be seen that $|U^R_t| = D_T, \forall i \in [t_T]$ and $|U^R_i| = D_R, \forall i \in [t_R]$, i.e., for both the transmitter and the receiver hypercubes, all distinct dimensions (edges) contain the same number of lattice points.

With this file splitting, the prefetching phase is then described as follows.

**Prefetching Phase:** The hypercube cache placement is employed at both the transmitters’ and receivers’ sides. That is, each file $W_n$ is split into $D_T t_T D_R t_R = (\frac{N}{M_T}) t_T (\frac{N}{M_R}) t_R$ disjoint equal-size subfiles, denoted by

$$W_n = \{W_{n,T,R}\}_{T \in \ell^0 U^T_t \otimes U^T_t \otimes \cdots \otimes U^T_{t-1}, \ \ R \in \ell^0 U^R_t \otimes U^R_t \otimes \cdots \otimes U^R_{t-1}}$$

in which the definition of the operator $\otimes$ is as follows. For $m \in \mathbb{Z}^+$ sets $A_0, A_1, \ldots, A_{m-1}$, we define $A_0 \otimes A_1 \otimes \cdots \otimes A_{m-1}$ as the set of all un-ordered elements in $A_0 \times A_1 \times \cdots \times A_{m-1}$, where $\times$ denotes the Cartesian product. We use $\{\cdot\}$ to convert the $m$-tuple $A$ to a set. For example, for a tuple $(1, 2, 3)$, we have $(1, 2, 3) = \{1, 2, 3\}$. Hence, $A_0 \otimes A_1 \otimes \cdots \otimes A_{m-1} \triangleq \{A : A \in A_0 \times A_1 \times \cdots \times A_{m-1}\}$. The subfile $W_{n,T,R}$ is exclusively cached by a set of transmitters in $T$ and a set of receivers in $R$. Under this file splitting strategy, each transmitter $Tx_i$ caches a set of subfiles $\{W_{n,T,R} : \forall T : i \in T, \forall R, \forall n \in [N]\}$ and each receiver $Rx_j$ caches a set of subfiles $\{W_{n,T,R} : \forall T, \forall R : j \in R, \forall n \in [N]\}$. As a result, the number of subfiles cached by $Tx_i, i \in [K_T]$ is equal to $ND_T t_T-1 D_R t_R$ and hence the number of packets cached by $Tx_i, i \in [K_T]$ is equal to

$$ND_T t_T-1 D_R t_R \frac{F}{D_T t_T D_R t_R} = M_T F \text{ packets,} \tag{8}$$

where $\frac{F}{D_T t_T D_R t_R}$ is the number of packets of each subfile (note that in the following delivery phase, each subfile needs to be further split into multiple packets). Similarly, the number of subfiles cached by $Rx_j, j \in [K_R]$ is equal to $ND_T t_T D_R t_R-1$ and hence the number of packets cached by $Rx_j, \forall j \in [K_R]$ is equal to

$$ND_T t_T D_R t_R-1 \frac{F}{D_T t_T D_R t_R} = M_R F \text{ packets,} \tag{9}$$

which also satisfies the memory constraint. The application of the hypercube cache placement method to cache-aided interference networks is illustrated via the following example.
Example 2: Consider a wireless network with $K_T = 4$ transmitters and $K_R = 4$ receivers. Each transmitter and receiver is equipped with a cache memory of size $M_T = 2$ and $M_R = 2$ files, respectively. The file library contains $N = 4$ files denoted by $\mathcal{W}_0 = A, \mathcal{W}_1 = B, \mathcal{W}_2 = C$ and $\mathcal{W}_3 = D$. Hence, we have the parameters $D_T = \frac{N}{M_T} = 2, D_R = \frac{N}{M_R} = 2, t_T = \frac{K_T}{D_T} = 2$ and $t_R = \frac{K_R}{D_R} = 2$.

In the prefetching phase, each file $\mathcal{W}_n$ is split into $D_T t_T D_R t_R = 2^2 \times 2^2 = 16$ subfiles $\{\mathcal{W}_n, T, R\}$ of equal sizes for any $T \in \{(0, 2), (0, 3), (1, 2), (1, 3)\}$ and $R \in \{(0, 2), (0, 3), (1, 2), (1, 3)\}$. Each subfile is then cached by the two transmitters in $T$ and the two receivers in $R$, respectively.

For example, file $A$ is split into 16 subfiles:

$A_{02,02}, A_{02,03}, A_{02,12}, A_{02,13},$  
$A_{03,02}, A_{03,03}, A_{03,12}, A_{03,13},$  
$A_{12,02}, A_{12,03}, A_{12,12}, A_{12,13},$  
$A_{13,02}, A_{13,03}, A_{13,12}, A_{13,13},$

where for example, $A_{02,02}$ is cached by transmitters $T_{x0}$ and $T_{x2}$ as well as receivers $R_{x0}$ and $R_{x2}$. The same file splitting is done for files $B, C$ and $D$. It can be seen that each transmitter caches 8 subfiles of each file. Since each subfile contains $\frac{F}{16}$ packets, the total number of packets cached by each transmitter is $4 \times 8 \times \frac{F}{16} = 2F$, which satisfies the memory constraint of the transmitters. Similarly, the memory constraint of the receivers is also satisfied.  

III. MAIN RESULTS

The main results on the one-shot linear sum-DoF using the hypercube cache placement approach are presented in this section. Note that for the case where $K_R < \frac{K_T M_T + K_R M_R}{N}$, it was shown in [7] that the one-shot linear sum-DoF of $K_R$ is always achievable by utilizing only a fraction of the Tx/Rx cache memories. Hence, for simplicity, we focus on the case where $K_R \geq \frac{K_T M_T + K_R M_R}{N}$.

Theorem 1: For a $K_T \times K_R$ wireless interference network with a library of $N$ files, each consisting of $F$ packets, and with transmitter and receiver cache sizes of $M_T F$ and $M_R F$ packets,

\footnote{With a slight abuse of notation, we write $A_{\{0,2\},(0,2)}$ as $A_{02,02}$ for simplicity and the same for other symbols.}
respectively, given the hypercube cache placement approach employed in the prefetching phase, and for any $\delta \triangleq \frac{t_T}{t_R} \in \mathbb{Z}^+$, $D_R = \frac{K_R}{t_R} \geq \delta + 1$, where $t_T \in [K_T], t_R \in [K_R], D_R \in \mathbb{Z}^+$, the one-shot linear sum-DoF of $\frac{K_T M_T + K_R M_R}{N}$ is achievable when $K_R \geq \frac{K_T M_T + K_R M_R}{N}$ with

$$F = \left( \frac{N}{M_T} \right)^{t_T} \left( \frac{N}{M_R} \right)^{t_R} \left( D_R - 2 \right) \left( \frac{D_R - 1}{\delta} - 1 \right) \frac{(\delta!)^{t_R}}{\delta} (t_R - 1)!$$

(10)

**Proof:** The achievability of Theorem 1 is proved by the general achievable scheme described in Section IV-C which focuses on the case $K_R \geq \frac{M_T K_T + M_R K_R}{N}$. The converse results follows directly from [7] which will not be presented in this paper.

The implications of Theorem 1 are two-folds, which includes the optimality of the achievable one-shot linear DoF and the reduced subpacketization level. Note that if either $t_T$ or $t_R$ is not an integer, or both of them are not integers, we can still achieve the sum-DoF of $t_T + t_R$ for any values of $t_T$ and $t_R$ using the the memory-sharing in [3] which will be briefly introduced later.

1) **Sum-DoF Optimality:** As shown in [7], when $K_R \geq \frac{K_T M_T + K_R M_R}{N}$, the optimal one-shot linear sum-DoF of the interference network studied in this paper, $\text{DoF}_{L,\text{sum}}^*$, over any possible cache placement strategies and caching realizations, is bounded by

$$\frac{K_T M_T + K_R M_R}{N} \leq \text{DoF}_{L,\text{sum}}^* \leq \frac{2(K_T M_T + K_R M_R)}{N},$$

(11)

which indicates that when $\frac{K_T M_T + K_R M_R}{N} \leq K_R$, the achievable one-shot linear sum-DoF under the hypercube cache placement is equal to the achievable one-shot linear DoF in [7] and is within a factor of 2 to the optimal one-shot linear sum-DoF of the network. This result indeed shows that the DoF of $\frac{M_T K_T + M_R K_R}{N}$ can be achieved by different cache placement methods, which provides the potential to reduce the total number of packets required.

2) **Subpacketization Level Reduction:** Under the hypercube cache placement strategy, the number of packets per file, i.e., $F$, required for implementing the interference cancellation in the delivery phase is significantly reduced compared to the NMA scheme. In particular, the NMA scheme requires to split each file into $\left( \frac{K_T}{t_T} \right) \left( \frac{K_R}{t_R} \right)$ subfiles in the prefetching phase and further split each subfile into $\frac{t_R!}{[K_R-(t_T+1)]!}$ packets in the delivery phase. However, if we employ the hypercube cache placement strategy, each file is going to be split into $\left( \frac{N}{M_T} \right)^{t_T} \left( \frac{N}{M_R} \right)^{t_R}$ subfiles in the prefetching phase, and is further split into $\frac{D_R-2}{\delta-1} \left( \frac{D_R-1}{\delta} \right)^{t_R-1} \frac{(\delta!)^{t_R}}{\delta} (t_R - 1)!$ packets in the delivery phase. In Section IV-D, we will show that for any system parameter, the hypercube
scheme requires less number of packets than the NMA scheme and the gain of subpacketization can be unbounded with the increase of the cache sizes of transmitters and receivers. Together with the sum-DoF optimality, the hypercube based scheme can achieve the same one-shot linear DoF as in [7] while requiring a significantly smaller $F$.

3) Non-integer Caching Parameters: When the caching parameters $t_T = \frac{K_T M_T}{N}$ and $t_R = \frac{K_R M_R}{N}$ are not integers, we can still achieve the one-shot linear sum-DoF of $t_T + t_R$ using the memory-sharing method of [3]. More specifically, we can split the Tx/Rx memories and files proportionally so that for each of the new partitions, our proposed scheme can be applied for the updated parameters $t_T$ and $t_R$ which are integers. That is, for each new partition of memories and files, it can be treated as a new interference network with updated Tx/Rx cache memory sizes $M'_T, M'_R$, file size $L'$ and the corresponding caching parameters $t'_T = \frac{K_T M'_T}{N} \in \mathbb{Z}^+, t'_R = \frac{K_R M'_R}{N} \in \mathbb{Z}^+$, where the proposed scheme can be directly applied.

IV. Achievable Delivery Scheme

A. An Example

We first present the achievable delivery scheme under the hypercube cache placement via the following example.

Example 3: (Achievable Delivery Scheme) We consider the same network setting as in Example 2. Let receiver Rx$_j$ request the file $W_{d_j}$. Without loss of generality, we assume that $W_{d_0} = A, W_{d_1} = B, W_{d_2} = C$ and $W_{d_3} = D$. In the prefetching phase, each receiver has already cached 8 subfiles of its requested file. Therefore, the transmitters only need to deliver the $16 - 8 = 8$ remaining subfiles to each receiver. In particular, the following 32 subfiles need to be delivered to the receivers:

$$A_{02,12}, A_{03,12}, A_{12,12}, A_{13,12}, A_{02,13}, A_{03,13}, A_{12,13}, A_{13,13} \} \text{ to Rx}_0,$$

$$B_{02,02}, B_{03,02}, B_{12,02}, B_{13,02}, B_{02,03}, B_{03,03}, B_{12,03}, B_{13,03} \} \text{ to Rx}_1,$$
\[ C_{02,03}, C_{03,03}, C_{12,03}, C_{13,03}, C_{02,13}, C_{03,13}, C_{12,13}, C_{13,13} \] to Rx\(_2\),
\[ D_{02,02}, D_{03,02}, D_{12,02}, D_{13,02}, D_{02,12}, D_{03,12}, D_{12,12}, D_{13,12} \] to Rx\(_3\).

Note that in the hypercube-based delivery scheme, each subfile needs to be further split into
\[ \binom{D_R-2}{\delta-1} \binom{D_R-1}{\delta} \frac{(\delta t_R-1)!}{\delta} (t_R-1)! \] packets. In this example, since \( \delta = \frac{t_T}{t_R} = 1, D_T = t_T = D_R = t_R = 2, \delta = 1 \), we have
\[ \binom{D_R-2}{\delta-1} \binom{D_R-1}{\delta} \frac{(\delta t_R-1)!}{\delta} (t_R-1)! = \binom{0}{0} \binom{1}{1} (2-1)! = 1, \]
which implies that no further file splitting is needed and thus 32 packets will be delivered.

We now show how the above 32 packets can be grouped in 8 subsets, each of which contains 4 packets, such that the packets within the same subset can be delivered simultaneously to the receivers without interference. Fig. 3 shows how the 32 packets to be delivered are grouped and transmitted. In each communication step, \( t_T + t_R = 4 \) packets are delivered to the receivers simultaneously, and the interference among different users can be effectively eliminated by choosing proper linear combination coefficients at the \( t_T + t_R = 4 \) transmitters. For example, in step 1 of Fig. 3, four packets \( A_{02,12}, B_{13,03}, C_{12,13} \) and \( D_{03,02} \) are delivered to receivers Rx\(_0\), Rx\(_1\), Rx\(_2\) and Rx\(_3\) respectively. We write the transmitted signals \( S_i, i \in [4] \) of each transmitter \( \text{Tx}_i \) as a linear combination of a subset of these four packets as follows:

\[ S_0 = h_{32} \hat{A}_{02,12} - h_{13} \hat{D}_{03,02}, \]
\[ S_1 = h_{23} \hat{B}_{13,03} - h_{02} \hat{C}_{12,13}, \]
\[ S_2 = h_{01} \hat{C}_{12,13} - h_{30} \hat{A}_{02,12}, \]
\[ S_3 = h_{10} \hat{D}_{03,02} - h_{21} \hat{B}_{13,03}, \]

where for each packet \( W_{n,T,R}, \hat{W}_{n,T,R} \) denotes its physical layer coded version. As a result, due to the careful choice of the linear coefficients, some interferences are canceled over the air by zero forcing (e.g., \( \hat{C}_{12,13} \) is canceled at Rx\(_0\)). The corresponding received signals by Rx\(_0\), Rx\(_1\),
Rx_2 and Rx_3 after zero forcing are given by

\[
Y_0 = (h_{32}h_{00} - h_{30}h_{12})\hat{A}_{02,12} + (h_{23}h_{01} - h_{21}h_{03})\hat{B}_{13,03} + (h_{10}h_{03} - h_{13}h_{00})\hat{D}_{03,02} + N_0,
\]

\[
Y_1 = (h_{23}h_{11} - h_{21}h_{13})\hat{B}_{13,03} + (h_{32}h_{10} - h_{30}h_{12})\hat{A}_{02,12} + (h_{02}h_{11} - h_{01}h_{12})\hat{C}_{12,13} + N_1,
\]

\[
Y_2 = (h_{01}h_{22} - h_{02}h_{21})\hat{C}_{12,13} + (h_{32}h_{20} - h_{30}h_{22})\hat{A}_{02,12} + (h_{10}h_{23} - h_{13}h_{20})\hat{D}_{03,02} + N_2,
\]

\[
Y_3 = (h_{10}h_{33} - h_{13}h_{30})\hat{D}_{03,02} + (h_{23}h_{31} - h_{21}h_{33})\hat{B}_{13,03} + (h_{01}h_{32} - h_{02}h_{31})\hat{C}_{12,13} + N_3,
\]

where \( N_i, i \in [4] \) represents the Gaussian noise.

We can see that receiver Rx_0 can cancel the interference caused by \( B_{13,03} \) and \( D_{03,02} \) since these two packets have already been cached by Rx_0 and the desired packet \( A_{02,12} \) can be successfully decoded by subtracting the undesired but prefetched packets. Similarly, Rx_1, Rx_2 and Rx_3 can also cancel the interference caused by undesired packets by utilizing their cached contents. Therefore, all the interference including inter-user interference and interference caused by cached packets can be eliminated so that all receivers can decode their desired packets. It can be verified that there exists such linear combinations and all receivers can decode their desired packets in all remaining 7 communication steps. Hence, the 32 packets, each consisting of \( \frac{|W_n|}{16} \) bits, can be delivered to the receivers in 8 communication steps, each containing \( F_{16} = 1 \) resource block. As a result, a sum-DoF of \( \frac{32}{8} = 4 = K_T M_T + K_R M_R \) can be achieved. Hence, the proposed file subpacketization, cache placement, precoding and scheduling strategy in the delivery phase allow transmitters to collaboratively zero-force some of the outgoing interference and allow receivers to cancel the leftover interference using cached contents for any receivers’ demands. △

B. Hypercube Permutation

Before we proceed to the description of the general achievable scheme, we introduce two definitions of special permutations on a given set of points, i.e., the hypercube permutation and circular hypercube permutation, which are essential to the description of the general delivery phase.

Definition 1: (Hypercube Permutation) Given a set of \( D \times t \) points, denoted by \( \mathcal{Q} \), i.e., \( |\mathcal{Q}| = Dt \), we label each of these points by a unique number \( u_{i,j} \in [Dt] \), where \( i \in [t], j \in [D] \). Assume that these points are partitioned into \( t \) disjoint groups, which we refer to as dimensions. Each dimension consists of \( D \) points, denoted by \( \mathcal{U}_i = \{u_{i,j} : \left\lfloor \frac{u_{i,j}}{D} \right\rfloor = i, j = 0, 1, \cdots, D - 1\}, i \in [t] \).
Define a hypercube permutation of the set $Q$, denoted by $\pi^{HC}$, as such a permutation of the $Dt$ points that satisfies the following condition: For any set of points $U_i, i \in [t]$, the positions in the permutation (denoted by $pos(\cdot)$, meaning that $pos(u) = i$ if $\pi(i) = u$) of any two of them, $u_{i,j_1}$ and $u_{i,j_2}$ ($j_1 \neq j_2$), should satisfy $|pos(u_{i,j_1}) - pos(u_{i,j_2})| = kt, 1 \leq k \leq D - 1, k \in \mathbb{Z}^+$ and $j_1, j_2 \in [D]$. \hfill \triangleleft

Definition 2: (Circular Hypercube Permutation) A circular permutation of a set $Q$ is a way of arranging the elements of $Q$ around a fixed table. Denote the set of circular permutations of $Q$ as $\Pi^{circ}_Q$. For example, if $Q = \{1, 2, 3\}$, then $\Pi^{circ}_Q = \{[1 \ 2 \ 3], [1 \ 3 \ 2]\}$. A circular hypercube permutation of a set $Q$ is a way of arranging the elements of $Q$ around a fixed table, and meanwhile, the corresponding arrangement should be a hypercube permutation. \hfill \triangleleft

We illustrate the concept of hypercube permutation and circular hypercube permutation via the following example.

Example 4: For $Q = \{0, 1, 2, 3\}$ with $t = 2$ dimensions and $D = 2$ points in each dimension,
i.e., \( \mathcal{U}_0 = \{0, 1\} \), \( \mathcal{U}_1 = \{2, 3\} \), we have
\[
\Pi_{Q}^{HCB} = \{ [0 2 1 3], [0 3 1 2], [1 2 0 3], [1 3 0 2], \\
[2 1 3 0], [2 0 3 1], [3 1 2 0], [3 0 2 1] \}.
\]

It is clear that, for any two points within one dimension, \( 0, 1 \in \mathcal{U}_0 \) or \( 2, 3 \in \mathcal{U}_1 \), we have \( |\text{pos}(0) - \text{pos}(1)| = |\text{pos}(2) - \text{pos}(3)| = 2 \), which satisfies the condition \( |\text{pos}(u_{i,j_1}) - \text{pos}(u_{i,j_2})| = t \) (note that \( k = 1 \)). Furthermore, we have \( \Pi_{Q}^{HCB, circ} = \{ [0 2 1 3], [0 3 1 2] \} \). △

**Lemma 1:** For a set of points (users) \( Q \) of dimension \( t \) and \( D \) points (users) in each dimension, denote the set of all hypercube permutations as \( \Pi_{Q}^{HCB} \), then \( |\Pi_{Q}^{HCB}| = (D!)^t \). The set of circular hypercube permutations of \( Q \), denoted by \( \Pi_{Q}^{HCB, circ} \), has size \( |\Pi_{Q}^{HCB, circ}| = \frac{(D!)^t (t-1)!}{D} \). □

**Proof:** See Appendix A.

### C. General Achievable Scheme

In this section, we present the general achievable scheme which is formally described in Algorithm 1. Recall that \( T = \frac{K_T M_T}{N} \) and \( R = \frac{K_R M_R}{N} \), and we assume \( T, R \in \mathbb{Z}^+ \), \( M_T K_T + M_R K_R \leq K_R \). In this paper, we focus on the case \( \delta \triangleq \frac{t_T}{t_R} \in \mathbb{Z}^+ \), implying that \( t_T \geq 2 \).

The corresponding prefetching and delivery phases are described as follows.

1) **Prefetching Phase:** The hypercube cache placement is employed at both the transmitters’ and receivers’ sides in the prefetching phase. Refer to Section II-B2 for detailed descriptions.

2) **Delivery Phase:** In the delivery phase, the receivers’ demand vector \( \mathbf{d} = [d_0, d_1, \ldots, d_{K_R-1}] \) is revealed, i.e., each receiver \( \text{Rx}_j, j \in [K_R] \) requests a file \( \mathcal{W}_{d_j} \). Since some subfiles of the requested file have already been cached by the receiver in the prefetching phase, the transmitters only need to send those subfiles which have not been cached by \( \text{Rx}_j \), i.e., \( \{\mathcal{W}_{d_j, T}, \forall T, \forall R : j \notin \mathcal{R} \} \).

Following a similar methodology of [7], we need to further split the set of subfiles to be delivered to the receivers into packets so that they can be scheduled in subsets of size \( t_T + t_R \) and delivered to the receivers simultaneously without interference. In particular, for any packet in the subset of \( t_T + t_R \) packets, it is requested by one particular receiver and can be cancelled by another \( t_R \) receivers by utilizing their cached packets. Also, the transmitters can collaborate to zero-force the the interference to another \( t_T - 1 \) unintended receivers. We describe how to do such a further splitting based on the hypercube cache placement in the following.
Algorithm 1 General Hypercube-based Achievable Scheme

**Prefetching Phase:**

1: for $i = 0, 1, \ldots, K_T - 1$ do
2:   Group Tx$_i$ into the transmitter dimension $U_j^T$, where $j = \lfloor \frac{i}{D_T} \rfloor$.
3: end for
4: for $i = 0, 1, \ldots, K_R - 1$ do
5:   Group Rx$_i$ into the receiver label set $U_j^R$, where $j = \lfloor \frac{i}{D_R} \rfloor$.
6: end for
7: for $n = 0, 1, \ldots, N - 1$ do
8:   Split $W_n$ into $(\frac{N}{M_T})^T (\frac{N}{M_R})^R$ disjoint equal-size subfiles:
9:     $W_n = \{W_{n,T,R} : i \in T\}$ for all $n \in [N]$.
10: end for
11: for $i = 0, 1, \ldots, K_T - 1$ do
12:   Tx$_i$ caches $\{W_{n,T,R} : i \in T\}$ for all $n \in [N]$.
13: end for
14: for $j = 0, 1, \ldots, K_R - 1$ do
15:   Rx$_j$ caches $\{W_{n,T,R} : j \in R\}$ for all $n \in [N]$.
16: end for

**Delivery Phase:**

17: for $j = 0, 1, \ldots, K_R - 1$ do
18:   for $T \in U_0^T \otimes U_1^T \otimes \cdots \otimes U_{T-1}^T$ do
19:     for $R \in U_0^R \otimes U_1^R \otimes \cdots \otimes U_{R-1}^R$ do
20:       Split the subfile $W_{d_j,T,R}$ into $(\frac{D_R-2}{D-1}) (\frac{D_R-1}{D-1})^{t_R-1} (\frac{1}{D})^{t_R-1} (t_R - 1)!$ disjoint packets of equal-sizes:
21:         $\{W_{d_j,T,R,\pi} : \pi = \{1 : t_R\}, \pi = \{1 : t_R+1 : t_T+t_R\} \}$
22:         $\pi \in \Pi_{Q''}^{U_T}, \pi(0) = j, \pi(t_R) = \frac{1}{t_R}$
23:         $\{\pi(1), \pi(2), \ldots, \pi(t_R-1)\} \in R \setminus \{\pi(1) : t_R-1\}$
24:       end for
25:   end for
26: end for
27: for $T \in U_0^T \otimes U_1^T \otimes \cdots \otimes U_{T-1}^T$ do
28:   for $R \in U_0^R \otimes U_1^R \otimes \cdots \otimes U_{R-1}^R$ do
29:     for $\pi \in \Pi_{HCB}^{U_T}$ do
30:       Each transmitter sends a linear combination (Lemma $3$) of the coded packets:
31:         $S_i = L_{i,T,\pi} \left( \{\hat{W}_{d(t,\ell)}(\ell,\pi(\ell+t_R)), \pi(\ell+t_R+1:t_T+t_R+t_T-1) : \ell \in [t_T+t_R], i \in T(\ell)\} \right)$
32:     end for
33: end for
34: end for
For any $j \in [K_R]$, \( \mathcal{T} = \{(\tau_0, \tau_1, \ldots, \tau_{t_T-1})\} \) with \((\tau_0, \tau_1, \ldots, \tau_{t_T-1}) \in \mathcal{U}_0^T \times \mathcal{U}_1^T \times \cdots \times \mathcal{U}_{t_T-1}^T\), and \( \mathcal{R} = \{(r_0, r_1, \ldots, r_{t_T-1})\} \) with \((r_0, r_1, \ldots, r_{t_T-1}) \in \mathcal{U}_R^0 \times \mathcal{U}_R^1 \times \cdots \times \mathcal{U}_R^t\), \( t \geq 0 \), \( (\tau_{t_T}) \in \mathcal{U}_{t_T}^T \), \( \tau_{t_T} \in \mathcal{U}_{t_T+1}^T \), \( (r_{t_T}) \in \mathcal{U}_R^{t_T} \), \( \mathcal{U}_R^{t_T} \subseteq \{0,1, \ldots, t_T\} \), \( |\mathcal{R}| = t_R \), we split \( \mathcal{W}_{d_j, \mathcal{T}, \mathcal{R}} \) into \( (D_R-2)(D_R-1)^{t_R-1} = (t_R-1)! \) disjoint packets of equal-sizes denoted by

\[
\mathcal{W}_{d_j, \mathcal{T}, \mathcal{R}, i} = \mathcal{W}_{d_j, \mathcal{T}, \mathcal{R}, i}, \quad (13)
\]

where \( \mathcal{Q} \in \mathcal{U}_0 \delta_{t_R} \times \cdots \times \mathcal{U}_\delta \delta_{t_R} \times \cdots \times \mathcal{U}_{t_T-1} \delta_{t_R} \) and the notations are defined as follows. For a set \( S \), \( \Gamma_{S,s} \) is defined as a set whose elements are all subsets of \( S \) of size \( s \), i.e., \( \Gamma_{S,s} = \{A : A \subseteq S, |A| = s\} \), \( s = 1, 2, \ldots, \, |S| \). For example, for \( S = \{0, 1, 2\} \), we have \( \Gamma_{S,2} = \{\{0, 1\}, \{1, 2\}, \{0, 2\}\} \). For a set \( \mathcal{Q} \) whose elements are sets, \( \mathcal{Q}^U \) denotes the union of the elements in \( \mathcal{Q} \). For example, if \( \mathcal{Q} = \{\{0, 1\}, \{2, 3\}\} \), we have \( \mathcal{Q}^U = \{0, 1\} \cup \{2, 3\} = \{0, 1, 2, 3\} \).

Moreover, for a set \( S \), and a hypercube permutation \( \pi \in \Pi_S \), \( \delta \) integers \( i, j \), where \( i \leq j \), \( \pi[i : j] \) is defined as \( \pi[i : j] = \pi\{ (i \oplus |S|) \, \pi, (i \oplus |S|) \, (j - i) \} \), in which for two integers \( m, n, m \oplus |S| \) is defined as

\[
m \oplus |S| = n + (m + n - 1 \mod |S|). \quad (14)
\]

After such a further splitting, for a specific set of \( t_T + t_R \) receivers and a corresponding hypercube permutation \( \pi \), the packet \( \mathcal{W}_{d_j, \mathcal{T}, \mathcal{R}, \pi} \), which is desired by \( Rx_j \), can be cancelled at receivers in \( \pi \) by utilizing their individual cached contents and can be zero-forced at receivers in \( \bar{\pi} \) through the collaboration of some transmitters. Lemma 2 shows how this further splitting is done. For a set \( \mathcal{T} = \{\tau_0, \tau_1, \ldots, \tau_{t_T-1}\} \) whose elements are from the \( t_T \) different transmitter dimensions, i.e., \( \tau_i \in \mathcal{U}_i^T, i \in [t_T] \), we define the corresponding sets \( \mathcal{T}(\ell) \triangleq \{\tau_0^{(\ell)}, \tau_1^{(\ell)}, \cdots, \tau^{(\ell)}_{t_T-1}\} \), \( \ell \in [t_T + t_R] \), where \( \mathcal{T}(0) = \mathcal{T} \), \( \ell \), \( \tau_0^{(0)} = \tau_i, \forall i \in [t_T] \) and

- When \( 1 \leq \ell \leq t_T \),

\[
\tau^{(\ell)}_i = \begin{cases} 
\tau_0^{(0)} + 1 \mod D_T & 0 \leq i \leq \ell - 1, \\
\tau_0^{(0)} & \ell \leq i \leq t_T - 1.
\end{cases}
\quad (15)
\]

Here we have implicitly assumed that \( D_R - 2 \geq \delta - 1 \), i.e., \( D_R \geq \delta + 1 \).
When $t_T + 1 \leq \ell \leq t_T + t_R - 1$,

$$
\tau_i^{(\ell)} = \begin{cases} 
\tau_i^{(0)} & 0 \leq i \leq \ell - t_T - 1, \\
\tau_i^{(0)} + 1 \mod D_T & \ell - t_T \leq i \leq \ell - t_T - 1.
\end{cases}
$$

(16)

Lemma 2: Based on the hypercube cache placement, for any receivers’ demand vector $d$, the set of packets needed to be sent to the receivers can be grouped into disjoint subsets of size $t_T + t_R$ as

$$
\bigcup_{T \in U_{\delta+1}^T} \times \left\{ \mathcal{W}_{d_{\tau(\ell)}, T(\ell), \pi[\ell + 1; \ell + t_R], \pi[\ell + t_R + 1; \ell + t_R + t_T - 1]} : \ell \in [t_T + t_R] \right\},
$$

(17)

Proof: See Appendix B.

Given the grouping method of the packets in Lemma 2, we will have $D_T^{t_T} (D_R)^{t_R} (\delta + 1)! (t_R - 1)! / \delta + 1$ steps of communications. More specifically, the term $D_T^{t_T}$ corresponds to the number of possible choices of $T$, $(D_R)^{t_R}$ corresponds to the number of choices of $R$. We also have $|\Pi_{U_{\delta}}^{HC_B, circ}| = (\delta + 1)! (t_R - 1)! / \delta + 1$ which is a direct result of Lemma 1, i.e., the number of different hypercube permutations of the set $R_U$ partitioned into $t = t_R$ dimensions and $D = \delta + 1$ points in each dimension. In each of these communication steps, specific sets $T$ and $R$ and a hypercube permutation are fixed, and each transmitter $Tx_i, i \in T(\ell)$ transmits a linear combination of the coded packets, i.e.,

$$
S_i = \mathcal{L}_{i, T, \pi} \left( \left\{ \hat{W}_{d_{\tau(\ell)}, T(\ell), \pi[\ell + 1; \ell + t_R], \pi[\ell + t_R + 1; \ell + t_R + t_T - 1]} : \ell \in [t_T + t_R], i \in T(\ell) \right\} \right),
$$

(18)

in which for any packet $W_{d_j, \tau, \pi, \hat{\cdot}}, \hat{W}_{d_j, \tau, \pi, \hat{\cdot}}$ denotes its coded version, and $\mathcal{L}_{i, T, \pi}(\cdot)$ represents the linear combination that $Tx_i$ chooses to transmit set of packets in Eq. (18).

The following lemma shows the existence of the linear combination coefficients.

Lemma 3: For any subset of $t_T$ transmitters $T \in U_{\delta+1}^T \times U_1^T \times \cdots \times U_{t_T-1}^T$, any set of $t_T + t_R$ receivers $R_U$ for which $R \in \Gamma_U^T \times \cdots \times \Gamma_U^{t_T-1}$, and any circular hypercube permutation $\pi \in \Pi_{R_U}^{HC_B, circ}$, there exists a choice of the linear combinations $\left\{ \mathcal{L}_{i, T, \pi}(\cdot) \right\}_{i=1}^{K_{\delta+1}}$ in Eq.
such that the set of $t_T + t_R$ packets in
\[
\mathcal{W}_{d_{\pi(\ell)}, \mathcal{T}(\ell), \pi[\ell+1: \ell+t_R], \pi[\ell+t_R+1: \ell+t_T-1]} : \ell \in [t_T + t_R]
\]
(19)
can be delivered simultaneously without interference by the transmitters in $\bigcup_{\ell \in [t_T+t_R]} \mathcal{T}(\ell)$ to the receivers in $\mathcal{R}^U$.

**Proof:** The proof of Lemma 3 follows exactly the same steps given in [7]. To show the existence of such linear combinations, we require the linear coefficients to be designed such that for any receiver in $\mathcal{R}^U$, its desired packets must be received with non-zero coefficients, and the undesired subfiles which cannot be cancelled by utilizing its cached content, must be zero-forced. Then we can show the existence of such linear combinations simply by observing the fact that the number of variables (coefficients) equals the number of equations (received signal requirements). The details of proof are omitted here. ■

**D. Subpacketization Complexity Analysis**

In this section, we provide a comprehensive performance comparison between the proposed hypercube-based scheme and the NMA scheme.

Each file in the library is split into $(N/M_T)^{t_T}(N/M_R)^{t_R}$ subfiles in the hypercube-based scheme while the NMA needs to partition each file into $(K_T/t_T)(K_R/t_R)$ subfiles. In the delivery phase, to implement interference cancellation, each requested subfile is further split into
\[
\Delta_{\text{HC}}(K_T, M_T, K_R, M_R, N) \triangleq (D_R - 2) \left( D_R - 1 \right)^{t_T-1} \left( \frac{\delta !}{\delta} \right)^{t_R-1} (t_T-1)!
\]
(20)
packets in the proposed hypercube-based scheme and
\[
\Delta_{\text{NMA}}(K_T, M_T, K_R, M_R, N) \triangleq t_T! \left( K_R - t_R - 1 \right) \left( \frac{t_T-1}{t_T-1} \right) (t_T-1)!
\]
(21)
packets in the NMA scheme. Since each receiver $\text{Rx}_j, j \in [K_R]$ has already cached a fraction of $M_R/N$ of the subfiles for each file in the prefetching phase, these pre-stored subfiles do not need to be further split into packets. To measure the subpacketization complexity, we only count the number of packets that a specific file needs to be split into and ignores the effect of the pre-stored subfiles. In particular, in the NMA scheme, each requested file is split into $(K_T/t_T)(K_R/t_R)$ subfiles and $(K_T/t_T)(K_R-1/t_R-1)$ of them are pre-stored contents, implying that only the remaining
Fig. 4. The multiplicative gap $G$ between the hypercube scheme and the NMA scheme. The comparison is done under the setting $t_T = t_R = t$, $N/M_T = N/M_R = d$, which implies $K_T = K_R = dt$. It can be seen that: (a) For a fixed $d$, $G$ decreases quickly as $t$ increases and approaches zero as $t$ goes to infinity, and (b) For a fixed $t$, $G$ converges to some specific non-zero value as $d$ goes to infinity.

\[
\begin{align*}
(K_T) \left( \binom{K_R}{t_T} - \binom{K_R-1}{t_R-1} \right) &= \binom{K_T}{t_R} \binom{K_R-1}{t_R-1} \\
F_{\text{HCB}}(K_T, M_T, K_R, M_R, N) &= D_T^{t_T} - D_R^{t_R} = D_T^{t_T} D_R^{t_R-1} = D_T^{t_T} D_R^{t_R} - D_T^{t_T} D_R^{t_R-1} = D_T^{t_T} D_R^{t_R-1} (D_R - 1) \\
F_{\text{NMA}}(K_T, M_T, K_R, M_R, N) &= \left( \binom{K_T}{t_T} \binom{K_R-1}{t_R} \right) \Delta_{\text{HCB}}(K_T, M_T, K_R, M_R, N).
\end{align*}
\]

Since the comparison of subpacketization levels is always done under the same set of system parameters $K_T, M_T, K_R, M_R, N$, we ignore these parameters in the expressions of $\Delta_{\text{HCB}}(\cdot), \Delta_{\text{NMA}}(\cdot), F_{\text{HCB}}(\cdot)$ and $F_{\text{NMA}}(\cdot)$ for ease of notation. To compare the subpacketization level between our scheme and the NMA scheme, we define the multiplicative gap of the subpacketization levels between these two schemes as follows.

**Definition 3:** (Multiplicative Gap of Subpacketization Levels) For a set of system parameters $K_T, M_T, K_R, M_R, N$, the multiplicative gap between the hypercube-based scheme and the NMA

\[
\begin{align*}
F_{\text{HCB}}(K_T, M_T, K_R, M_R, N) &= D_T^{t_T} - D_R^{t_R} = D_T^{t_T} D_R^{t_R-1} = D_T^{t_T} D_R^{t_R} - D_T^{t_T} D_R^{t_R-1} = D_T^{t_T} D_R^{t_R-1} (D_R - 1) \\
F_{\text{NMA}}(K_T, M_T, K_R, M_R, N) &= \left( \binom{K_T}{t_T} \binom{K_R-1}{t_R} \right) \Delta_{\text{HCB}}(K_T, M_T, K_R, M_R, N).
\end{align*}
\]
scheme, denoted by $G$, is defined as
\[
G(K_T, M_T, K_R, M_R, N) \triangleq \frac{F_{\text{HCB}}(K_T, M_T, K_R, M_R, N)}{F_{\text{NMA}}(K_T, M_T, K_R, M_R, N)}.
\] (24)

For ease of notation, we use $G \triangleq \frac{F_{\text{HCB}}}{F_{\text{NMA}}} = \frac{D_T^t T D_R^t R^{-1}(D_R^{-1})_R}{(K_T^t)_{K_T}^t R^{-1} t R}$ alternatively.

We next show that for any system parameters, the hypercube scheme has a strictly less subpacketization level than that of the NMA scheme.

**Theorem 2:** For any system parameters $K_T, K_R, M_T, M_R$ and $N$ satisfying $t_T = \frac{K_T M_T}{N} \in \mathbb{Z}^+$, $t_R = \frac{K_R M_R}{N} \in \mathbb{Z}^+$, $D_T = \frac{K_T}{t_T} \in \mathbb{Z}^+$, $D_R = \frac{K_R}{t_R} \in \mathbb{Z}^+$ and $\delta \triangleq \frac{t_T}{t_R} \in \mathbb{Z}^+$, $D_R \geq \delta + 1$, the multiplicative gap $G$ is strictly less than 1. Moreover, under the setting $t_T = \delta t_R = \delta t$, $D_T = D_R = d$, we have $\lim_{t \to \infty} G(d, t, \delta) = 0$. More specifically, for fixed $d, \delta$ and large enough $t$, we have
\[
G(d, t, \delta) \leq \frac{C_0 e^{-(\delta+1)t}}{C_1^{(\delta-1)t}}
\] (25)
in which the constants are $C_0 = \frac{2\pi \sqrt{3(d-2)}e^{\frac{1}{2} \pi^2 \frac{\delta}{(d-\delta-1)!}}}{{(d-\delta-1)!}}$, $C_1 = \frac{(d-1)!}{(d-\delta-1)!}$. For fixed $t, \delta$, it holds that
\[
\lim_{d \to \infty} G(d, t, \delta) = \frac{(t-1)!(\delta t)!}{\delta t^{(2\delta+1)t-1}}.
\] (26)

**Proof:** See Appendix C.

One important implication of Theorem 2 is the subpacketization level reduction of the hypercube scheme over the NMA scheme, which yields a significant advantage since it holds for any possible system parameters while preserving the same one-shot linear sum-DoF gain. Fig. 4 shows the multiplicative gain $G(d, t, 1)$ under logarithmic scale for the case when $\delta \triangleq \frac{t_T}{t_R} = 1, t_T = t_R = t$ and $D_R = D_R = d$. It can be seen that the gap decreases exponentially as $t$ increases and goes to zero as $t$ goes to infinity (see Fig. 4(a)) and $G$ converges to some specific value as $d$ goes to infinity (see Fig. 4(b)).

V. DISCUSSIONS

In this section, we will first provide two possible extensions of the proposed scheme, which are cache-aided Device-to-Device (D2D) interference networks and wireless coded distributed computing networks. Second, we will discuss the connection between the proposed scheme and
the scheme in [13].

A. Extension to cache-aided D2D Interference Networks and Wireless Distributed Computing Systems

In the settings of a typical cache-aided D2D interference networks, all the nodes (or devices) are expected to have homogeneous cache memory sizes. The proposed hypercube-based scheme can be directly extended to such D2D interference networks to achieve an order-optimal one-shot linear sum-DoF while maintaining the promised subpacketization levels compared to the direct translation of the NMA scheme. There are multiple approaches to apply the hypercube-based approach to cache-aided D2D interference networks. In the following, we will illustrate one example of such applications. We consider a D2D interference network with a library of $N$ files and $K$ nodes, each equipped with a cache memory of size $M$ files. We assume $K$ is even and $t = KM/N \leq K/2$. We partition the network into two groups with equal number of devices, i.e., each group has $K/2$ devices. Let $t' = \frac{KM}{2N} \in \mathbb{Z}^+$. In the prefetching phase, in each group, we perform the hypercube cache placement such that the two groups have identical cache placement. The delivery phase has two steps, in the first step, one group of nodes will performance as transmitters and the other group will perform as receivers. Note that since $K_T = K_R = K/2$, the proposed delivery scheme based on the hypercube cache placement can be directly used. The achievable sum-DoF is $t = t_T + t_R = KM/N$. In the first phase, the requests from one group of receivers can be served. In the second step, we exchange the groups of transmitters and receivers such that the other group can be served with the same achievable sum-DoF. Therefore, the total achievable sum-DoF is given by $t = KM/N$.

Moreover, due to the similarity between the cache-enabled D2D interference network and the Coded Distributed Computing (CDC, [29]), the hypercube cache placement can be directly applied to the wireless CDC interference networks. This system model has been considered under the assumption of full-duplex transmissions in [30]. From the wireless D2D caching network example, it can be seen that the proposed hypercube-based scheme can be applied in a more practical half-duplex transmission settings. For example, the hypercube cache placement scheme can be employed in the file assignment phase in the CDC networks. Then we use the same delivery scheme as in the wireless D2D caching networks to achieve an order optimal communication-computation trade-off.
B. Connection to and Difference from [13]

The result of [13] shows that adding multiple \(L\) transmit antennas can reduce the supacketization level approximately to its \(L\)-th root compared to the shared-link coded caching scheme. It turns out that this scheme can be extended to the cache-aided \(K_T \times K_R\) interference networks to achieve the same sum-DoF as the hypercube-based scheme proposed in this paper and achieves a smaller subpacketization level. One of the major limitation of the scheme proposed in [13] is the “asymmetric” cache placement at the transmitters and receivers, which means that the cache placement schemes used at the transmitters and receivers are completely different. This results in the fact that the scheme proposed in [13] cannot be directly applied in either wireless cache-aided D2D interference networks or wireless CDC interference networks where each node is both a transmitter and a receiver.

VI. Conclusion

In this paper, we considered the cache-aided interference management problem where the transmitters and receivers are equipped with cache memories of certain sizes to pre-store parts of the contents. We adopt a new cache placement method called hypercube at both the transmitters’ and receivers’ sides. Based on the hypercube cache placement, we proposed a corresponding delivery scheme where the one-shot linear DoF of \(\min \left\{ \frac{M_T K_T + M_R K_R}{N}, K_R \right\} \) is achievable with exponentially less subpacketizations compared to the well-known NMA scheme. More specifically, via the design of the cache placement and the communication scheme, a set of \(\frac{M_T K_T + M_R K_R}{N}\) packets can be delivered to the receivers simultaneously and interference-free, which is a joint effect of the zero-forcing (collaboration of transmitters via cache placement design at the transmitters’ side) and cache cancellation (neutralization of known interference via the cache placement design at the receivers’ side). The result shows that our proposed scheme can achieve exactly the same DoF performance as the NMA scheme while requiring significantly lower supacketization level.

APPENDIX A

Proof of Lemma 1

First, we show that given a set of \(|Q| = Dt\) points (users) with \(t\) dimensions and \(D\) points in each dimension, the number of different hypercube permutations is equal to \(\left| \prod_{Q}^{HC} \right| = (D!)^t(t)!\).
According to Definition 1, for a hypercube permutation $\pi^\text{HCB}$, the users belonging to the same dimension $U_i$ can only appear in positions $p_{i,1}, p_{i,2}, \ldots, p_{i,D-1}$ such that $p_{i,j} \mod t = C_i$, $\forall j \in [0 : D - 1]$, where $C_i$ is a constant in terms of $j$ and $C_i \in [0 : t - 1]$. For two different dimensions $U_{i_1}$ and $U_{i_2}$, the corresponding modulo residues $C_{i_1} \neq C_{i_2}$ if $i_1 \neq i_2$. As a result, $\{C_0, C_1, \ldots, C_{t-1}\} = \{0, 1, \ldots, t - 1\}$. Thus, given a group of users $U_i$ and a prescribed modulo residue $C_i$, there are $D!$ ways to arrange these users to the corresponding set of positions $\{p_{i,j} : p_{i,j} \mod t = C_i, j \in [0 : D - 1]\}$. Since we have $t$ such user groups (dimensions), according to the multiplication principle, there are $(D!)^t$ ways to arrange all the users $Q$ to the positions $\{p_{i,j} : p_{i,j} \mod t = C_i, j \in [0 : D - 1], i = 0, 1, \ldots, t - 1\}$ under a prescribed modulo residue assignment. Since there are $t!$ different ways to assign the modulo residues $C_0, C_1, \ldots, C_{t-1}$ to the $t$ user groups, we conclude that $|\Pi^\text{HCB}_Q| = (D!)^t(t)!$.

Now, for any $\pi \in \Pi^\text{HCB}_Q$, it is easy to see that there are $Dt - 1$ other permutations in $\Pi^\text{HCB}_Q$ which are resulted from circularly shifting the elements of $\pi$. Since circular shifting is not allowed in the circular permutation, we have

$$|\Pi^\text{HCB, circ}_Q| = \frac{|\Pi^\text{HCB}_Q|}{Dt} = \frac{(D!)^t(t-1)!}{D}, \quad (27)$$

which completes the proof of Lemma 1.

**APPENDIX B**

**PROOF OF LEMMA 2**

The proof of Lemma 2 can be completed by verifying the following two conditions: (1) For a specific receiver $Rx_j$, the number of packets it receives in the delivery phase equals the number of packets which are desired but have not been cached by $Rx_j$; (2) The number of packets received by all $K_R$ receivers equals the number of packets desired by them.

Each set in the union of Eq. (17) is composed of $t_T + t_R$ packets. The number of such sets is equal to

$$D_T^{t_T} \left( \frac{D_R}{\delta + 1} \right)^{t_T} \frac{((\delta + 1)!)^{t_R} (t_R - 1)!}{\delta + 1}. \quad (28)$$

Therefore, the total number of packets in Eq. (17) is equal to

$$D_T^{t_T} \left( \frac{D_R}{\delta + 1} \right)^{t_T} \frac{((\delta + 1)!)^{t_R} (t_R - 1)!}{\delta + 1} (t_T + t_R) = \frac{D_T^{t_T} K_R (D_R - 1)! (D_R!)^{t_R - 1} (t_R - 1)!}{((D_R - \delta - 1)!)^{t_R}}, \quad (29)$$
where we used the fact that $\delta = \frac{t_T}{t_R}$ and $t_R = \frac{K_R}{D_T}$.

On the other hand, $Rx_j, j \in [K_R - 1]$ has cached $D_T^{t_T} D_R^{t_R-1}$ subfiles in the prefetching phase, so the number of subfiles $Rx_j$ needs is equal to $D_T^{t_T} D_R^{t_R-1} (D_R - 1)$. Since in the delivery phase, each desired subfile is further split into $(\frac{D_R - 2}{\delta - 1}) (\frac{D_R - 1}{\delta})^{t_R-1} (t_R - 1)!$ packets, the total number of packets needed by $Rx_j$ is equal to

$$D_T^{t_T} D_R^{t_R-1} (D_R - 1) \left( \frac{D_R - 2}{\delta - 1} \right) \left( \frac{D_R - 1}{\delta} \right)^{t_R-1} (\delta! )^{t_R} (t_R - 1)!$$

$$= D_T^{t_T} (D_R - 1)! (D_R!)^{t_R-1} (t_R - 1)! ((D_R - \delta - 1)!)^{t_R}.$$ (30)

Therefore, the total number of packets needed by all $K_R$ receivers is equal to

$$K_R D_T^{t_T} (D_R - 1)! (D_R!)^{t_R-1} (t_R - 1)! ((D_R - \delta - 1)!)^{t_R},$$ (31)

which equals the total number of packets in Eq. [29], implying that the set of packets needed by the receivers can be grouped into subsets of size $t_T + t_R$, verifying the second condition. Moreover, the number of packets received by $Rx_j$ in the delivery phase is equal to

$$D_T^{t_T} \left( \frac{D_R - 1}{\delta} \right)^{t_R-1} \left( \frac{(\delta + 1)!}{\delta + 1} \right)^{t_R} (t_R - 1)!$$

$$= D_T^{t_T} (D_R - 1)! (D_R!)^{t_R-1} (t_R - 1)! ((D_R - \delta - 1)!)^{t_R},$$ (32)

which equals the number of packets calculated in Eq. (30), verifying the first condition. As a result, the proof of Lemma 2 is complete.

**APPENDIX C**

**PROOF OF THEOREM 2**

We will first show that for any system parameters $K_T, K_R, M_T, M_R$ and $N$, which satisfy $K_T = D_T t_T, K_R = D_R t_R$ and $\delta = \frac{t_T}{t_R} \in \mathbb{Z}^+$, we have 1) $D_T^{t_T} < \left( \frac{K_T}{t_T} \right)$; 2) $D_R^{t_R-1} (D_R - 1) < \left( \frac{K_R t_T}{t_R} \right)$; and 3) $\Delta_{HC} < \Delta_{NMA}$. As a result, we obtain $G < 1$.

We first prove that $D_T^{t_T} < \left( \frac{K_T}{t_T} \right)$. For ease of notation, we denote $D_T$ as $d$ and $t_T$ as $t$ for the time being. We have

$$\frac{D_T^{t_T}}{\left( \frac{K_T}{t_T} \right)} = \frac{d^t}{(\frac{d}{t})} = \frac{d^t t!}{dt(dt-1)(dt-2) \cdots (dt-(t-1))} = \frac{t}{t} \left( \frac{t-1}{t-\frac{1}{d}} \right) \cdots \left( \frac{t-(t-1)}{t-\frac{1}{d}} \right).$$ (33)

Since we have assumed that $d \geq \delta + 1 \geq 2$ where $\delta \geq 1$, it can be seen that $t - i \leq t - \frac{i}{d}, \forall i \in \mathbb{Z}^+$. Therefore, we obtain $G < 1$.
We then have

\[ \frac{D_R^{t_R-1}(D_R - 1)}{\binom{K_{t_R}}{t_R}} = \frac{d^{t-1}(d-1)}{\binom{dt-1}{t}} \]

which is a direct result from Eq. (33) and in (a) we used the identity \( \binom{dt}{t} = \frac{dt}{d} \binom{dt-1}{t} \).

Last we prove that \( \Delta_{HCB} < \Delta_{NMA} \). Denote \( t_R \) as \( t \) and \( D_R \) as \( d \), we have \( t_T = \delta t_R = \delta t \).

Thus, \( \Delta_{HCB} \) and \( \Delta_{NMA} \) can be simplified as

\[ \Delta_{HCB} = \left( \frac{d - 2}{\delta - 1} \right) \binom{d - 1}{\delta}^{-1} (\delta!)^{\frac{t}{\delta}} (t - 1)! = \frac{(d - 1)!}{(\delta - 1)!} (t - 1)! \]

\[ \Delta_{NMA} = \binom{dt - t - 1}{\delta t - 1} (\delta t - 1)! = \frac{(dt - t - 1)!}{(\delta - 1)!} \]

Therefore, we have

\[ \frac{\Delta_{NMA}}{\Delta_{HCB}} = \frac{(d - \delta - 1)!}{(d - 1)!} \frac{(d - 1)!}{(\delta - 1)!} = \frac{\prod_{i=0}^{\delta t - 1} ((d - 1)t - i)}{\prod_{i=0}^{\delta t - 1} (d - 1 - i)} = \lambda_0 \lambda_1 \cdots \lambda_{t-1}, \]

in which the parameter \( \lambda_k \) is defined as

\[ \lambda_k \triangleq \frac{\prod_{i=k}^{(k+1)\delta - 1} ((d - 1)t - i)}{\prod_{i=0}^{\delta t - 1} (d - 1 - i)}, \quad \forall k \in \mathbb{Z}, \]

Note that \( \lambda_0 > \lambda_1 > \cdots > \lambda_{t-1} \). Next we show that \( \lambda_{t-1} \geq 1 \). From Eq. (38), we have

\[ \lambda_{t-1} = \frac{\prod_{i=0}^{\delta t - 1} ((d - 1)t - i)}{\prod_{i=0}^{\delta t - 1} (d - 1 - i)} = \prod_{i=0}^{\delta t - 1} \left( \frac{t - (\delta - i)(t - 1)}{d - 1 - i} \right) \]

\[ \geq \prod_{i=0}^{\delta t - 1} \left( \frac{t - (\delta - i)(t - 1)}{d - 1 - i} \right) = \prod_{i=0}^{\delta t - 1} \left( \frac{t - (t - 1)}{d - 1 - i} \right) = 1, \]

where in (a) we used the assumption that \( d \geq \delta + 1 \). Hence, we obtain that \( \lambda_{t-1} \geq 1 \). Since \( \lambda_0 > \lambda_1 > \cdots > \lambda_{t-1} \geq 1 \), we have \( \frac{\Delta_{NMA}}{\Delta_{HCB}} = \lambda_0 \lambda_1 \cdots \lambda_{t-1} > 1 \), implying \( \Delta_{HCB} < \Delta_{NMA} \).

Combining the above results, we conclude that the multiplicative gap is strictly less than 1 for 

\([t - 1]\), implying that each individual term in Eq. (33) is less than 1. As a result, the product is less than 1, implying \( D_T^{t_R} < \binom{K_{t_R}}{t_R} \).

We next prove that \( D_R^{t_R-1}(D_R - 1) < \binom{K_{t_R}}{t_R} \). Denote \( D_R \) as \( d \) and \( t_R \) as \( t \) for the time being. We then have

\[ \frac{D_R^{t_R-1}(D_R - 1)}{\binom{K_{t_R}}{t_R}} = \frac{d^{t-1}(d-1)}{\binom{dt-1}{t}} = \frac{d^t}{(\frac{dt}{d})}\text{=} \frac{dt}{d} \frac{dt}{d} \text{=} \frac{dt}{d} \frac{dt}{d} \text{=} \frac{dt}{d} \frac{dt}{d} \text{=} \frac{dt}{d} \frac{dt}{d} \text{=} \frac{dt}{d} \]

which is a direct result from Eq. (33) and in (a) we used the identity \( \binom{dt}{t} = \frac{dt}{d} \binom{dt-1}{t} \).
any valid system parameters, i.e.,

\[ G = \frac{D_T^{t_T}}{t_T} \cdot \frac{D_R^{t_R} (D_R - 1)}{t_R} \cdot \frac{\Delta_{HCB}}{\Delta_{NMA}} < 1. \]  

(40)

This proof also indicates that the hypercube based scheme requires less number of subfiles per file in the prefetching phase and and less number of packets per subfile in the delivery phase than the NMA scheme.

Next we prove the asymptotic results in Theorem 2. We set \( D_T = D_R = d \) and \( t_R = t, t_T = \delta t_R = \delta t \).

First we show that \( G(d, t, \delta) \leq C_0 e^{-(\delta+1)t}C_1^{-(\delta-1)t} \) for fixed \( \delta \) and \( d \), and large enough \( t \), in which \( C_0, C_1 \) (specified later) are constants independent of \( t \). As a result, we have \( \lim_{t \to \infty} G(d, t, \delta) = 0 \). For sufficiently large \( t \), we have

\[
G(d, t, \delta) = \frac{d^{\delta t} d^{t-1} (1 + (d-1)\Delta_{HCB})}{(\delta t) \left( \frac{d}{t} \right)^{t-1} (1 + (d-1)\Delta_{NMA})} \\
\leq \frac{(d-1)d^{(\delta+1)t-1} \Delta_{HCB} + o(d^{(\delta+1)t-1} \Delta_{HCB})}{(d-1)\left( \frac{d}{t} \right)^{t-1} \Delta_{NMA} + o(\left( \frac{d}{t} \right)^{t-1} \Delta_{NMA})} \\
= \frac{d^{(\delta+1)t-1} \left( \frac{d}{t} \right)^{t-1} (\delta!)^{t-1} (\delta - 1)! (t-1)!}{\left( \frac{d}{t} \right)^{t-1} (\delta t - 1)! t!} \\
\approx 2\pi \sqrt{\delta (\delta - 1)!} \left( \frac{d - 2}{\delta - 1} \right) e^{-(\delta+1)t + \frac{\delta}{2} + \frac{\delta}{2 \pi} t - \delta t + 1} C_1^{t-1} \\
= C_0 e^{-(\delta+1)t} C_1^{-(\delta-1)t} t^{-\delta t + 1} \left( \frac{C_1}{t} \right)^{t-1} \\
\leq C_0 e^{-(\delta+1)t} C_1^{-(\delta-1)t},
\]

(41)

in which the constants are \( C_0 = \frac{2\pi \sqrt{\delta (d-2)!} e^{\frac{\delta}{2} + \frac{\delta}{2 \pi} t}}{(d-\delta-1)!} \), \( C_1 = \frac{(d-1)!}{(d-\delta-1)!} \). In (a), we used the standard “order” notation: given two functions \( f(n) \) and \( g(n) \), we say \( f(n) = o(g(n)) \) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \). In (b) we used Stirling’s approximation \( n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \).
Next we prove that \( G(d, t, \delta) \rightarrow \frac{(t-1)! (\delta t)!}{\delta t t^{2 \delta t - 1}} \) as \( d \rightarrow \infty \) for fixed values of \( \delta \) and \( t \). We have

\[
\lim_{d \to \infty} G(d, t, \delta) = \lim_{d \to \infty} \frac{d^{\delta t} d^{-1} (1 + (d-1) \Delta_{\text{HC}})}{(\delta d t) (\delta t - 1) (1 + (d-1) \Delta_{\text{NM}})} \frac{(d-1) d^{\delta t-1} \Delta_{\text{HC}} + o \left( d^{(\delta t - 1)} \Delta_{\text{HC}} \right)}{(d-1) (\delta d t) (\delta t - 1) \Delta_{\text{NM}} + o \left( (\delta d t) (\delta t - 1) \Delta_{\text{NM}} \right)}
\]

\[
= \lim_{d \to \infty} \frac{d^{\delta t-1} \Delta_{\text{HC}}}{(\delta d t) (\delta t - 1) \Delta_{\text{NM}}}
\]

\[
= \lim_{d \to \infty} \frac{d^{\delta t-1} (d-1) (\delta d t) (\delta t - 1) (\delta t - 1)! (t-1)!}{(\delta d t) (\delta t - 1) (\delta t - 1)!}
\]

\[
\overset{(a)}{=} \lim_{d \to \infty} \frac{(t-1)! (\delta t)!}{\delta t t^{2 \delta t - 1}} \frac{d^{\delta t-1} (d-1) \delta t (d t - 1) (d t - 1) \delta t - 1}{d^{\delta t-1} (d t - 1) \delta t - 1}
\]

\[
= \frac{(t-1)! (\delta t)!}{\delta t t^{2 \delta t - 1}},
\]

(42)

where in (a), we used the fact that \( \binom{n}{m} \approx \frac{n^m}{m!} \) for large enough \( n \) and some \( m \ll n \). This completes the proof of Theorem [2].

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