QUASI-LIE BIALGEBRAS OF LOOPS IN QUASI-SURFACES

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Abstract. We discuss natural operations on loops in a quasi-surface and show that these operations define a structure of a quasi-Lie bialgebra in the module generated by the set of free homotopy classes of non-contractible loops.

Keywords: surfaces, quasi-surfaces, loops, brackets, cobrackets

1. Introduction

The module generated by free homotopy classes of loops in an oriented surface carries a natural Lie bracket introduced by W. Goldman [Go1], [Go2] and closely related to the Poisson brackets on the moduli spaces of the surface, see [AB], [Wo], [FR]. Goldman’s Lie bracket is complemented by a natural Lie cobracket so that together they form a Lie bialgebra, see [Tu1]. Here we study similar operations on loops in more general topological spaces called quasi-surfaces. This leads us to an algebraic notion of a quasi-Lie algebra, a dual notion of a quasi-Lie coalgebra, and a self-dual notion of a quasi-Lie bialgebra. We show that our operations on loops in quasi-surfaces satisfy the axioms of a quasi-Lie bialgebra.

The idea behind the study of quasi-surfaces is that a topological space containing an oriented surface with boundary \( \Sigma \) - which is separated from the rest of the space by segments in \( \partial \Sigma \) - must inherit certain features of \( \Sigma \). Formally, a quasi-surface \( X \) is obtained by gluing \( \Sigma \) to an arbitrary topological space \( Y \) along a mapping of several disjoint subsegments of \( \partial \Sigma \), called the gates, to \( Y \). The space \( X \) splits as a union of the “surface core” \( \Sigma \) and the “singular part” \( Y \) which meet at the gates. Considering loops in \( X \) and their intersections in \( \Sigma \), we obtain a bracket in the module \( M = M(X) \) freely generated by the set of free homotopy classes of loops in \( X \), see [Tu2]. This bracket is skew-symmetric but may not satisfy the Jacobi identity. Here we compute its Jacobianator in terms of operations on loops associated with the gates. Similarly, considering self-intersections of loops, we obtain a skew-symmetric cobracket in \( M \). We compute its co-Jacobianator and coboundary in terms of the gates. These computations show that our operations on loops induce a quasi-Lie bialgebra structure in the quotient of the module \( M \) by the homotopy class of contractible loops in \( X \). For loops in the surface core \( \Sigma \) of \( X \), we recover the standard Lie bialgebra of loops in \( \Sigma \).

A major source of quasi-surfaces are finite families of disjoint segments in (ordinary) surfaces. Consider such a family \( C \) in an oriented surface \( \Sigma \). Assume that \( C \cap \partial \Sigma = \partial C \neq \emptyset \) and that \( C \) splits \( \Sigma \) into two surfaces meeting at \( C \). Taking one of them as the surface core, the other one as the singular part, and the components of \( C \) as the gates, we turn \( \Sigma \) into a quasi-surface. Our operations on loops yield then a quasi-Lie bialgebra structure in \( M(\Sigma) \) depending on \( C \). Further examples of quasi-surfaces may be produced by collapsing several segments in \( \partial \Sigma \) into a single point which plays the role of the singular part.
Other known operations on loops in surfaces, see [KK1], [KK2], [MT], can also be generalized to quasi-surfaces. The author plans to discuss these generalizations elsewhere.

The paper starts with the definitions of quasi-Lie algebras, coalgebras, and bialgebras (Section 2). Then we discuss gate operations on loops in topological spaces (Section 3) and formulate our main theorems (Section 4). The rest of the paper is devoted to the proof of these theorems.

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2. Brackets, cobrackets, and bi-endomorphisms

2.1. Brackets. Throughout the paper we fix a commutative ring \( R \). By a module we mean an \( R \)-module. For a module \( M \) and an integer \( m \geq 1 \), we let \( M^m \) be the direct product of \( m \) copies of \( M \). An \( m \)-bracket in \( M \) is a map \( M^m \to M \) which is linear in all \( m \) variables. A bracket in \( M \) is fully symmetric if it is invariant under all permutations of the variables. A bracket \( \mu : M^m \to M \) is cyclically symmetric if \( \mu(x_1, ..., x_m) = \mu(x_2, ..., x_m, x_1) \) for all \( x_1, ..., x_m \in M \). Given a 2-bracket \( \mu : M^2 \to M \), its transpose \( \mu^t \) is the 2-bracket in \( M \) defined by \( \mu^t(x, y) = \mu(y, x) \) for all \( x, y \in M \). The Jacobiator \( J_\mu \) of \( \mu \) is the cyclically symmetric 3-bracket in \( M \) defined by

\[
J_\mu(x, y, z) = \mu(\mu(x, y), z) + \mu(\mu(y, z), x) + \mu(\mu(z, x), y)
\]

for all \( x, y, z \in M \). Note the identity

\[
J_{\mu^t}(x, y, z) = \mu(z, \mu(y, x)) + \mu(x, \mu(z, y)) + \mu(y, \mu(x, z)).
\]

A 2-bracket \( \mu \) in \( M \) is symmetric (respectively, skew-symmetric) if \( \mu^t = \mu \) (respectively, \( \mu^t = -\mu \)). In both cases \( J_{\mu^t} = J_\mu \).

2.2. Quasi-Lie algebras. A quasi-Lie algebra is a module \( M \) endowed with a skew-symmetric 2-bracket \([-\cdot, -\cdot]\) and a cyclically symmetric 3-bracket \([-\cdot, -\cdot, -\cdot]\) such that the Jacobiator of \([-\cdot, -\cdot]\) is obtained by antisymmetrization of \([-\cdot, -\cdot, -\cdot]\), i.e.,

\[
J_{[-\cdot, -\cdot]}(x, y, z) = [x, y, z] - [z, y, x]
\]

for all \( x, y, z \in M \). Such a pair \([[-\cdot, -\cdot], [-\cdot, -\cdot, -\cdot]]\) is called a quasi-Lie pair of brackets in \( M \). We recover the usual Lie algebras when \([-\cdot, -\cdot, -\cdot] = 0 \).

The following lemma is our main tool producing quasi-Lie algebras.

**Lemma 2.1.** For any module \( M \) and any bilinear form \( M^2 \to M, (x, y) \mapsto x \bullet y \), the 2-bracket \([x, y] = x \bullet y - y \bullet x\) and the 3-bracket

\[
[x, y, z] = J_*^t(x, y, z) + J_*^t(x, y, z)
\]

form a quasi-Lie pair of brackets in \( M \).

**Proof.** That the 2-bracket \([-\cdot, -\cdot]\) is skew-symmetric and the 3-bracket \([-\cdot, -\cdot, -\cdot]\) is cyclically symmetric is clear. Direct computations show that

\[
[x, y, z] = [x \bullet y - y \bullet x, z] = (x \bullet y) \bullet z - (y \bullet x) \bullet z - z \bullet (x \bullet y) + z \bullet (y \bullet x),
\]

\[
[y, z, x] = [y \bullet z - z \bullet y, x] = (y \bullet z) \bullet x - (z \bullet y) \bullet x - x \bullet (y \bullet z) + x \bullet (z \bullet y),
\]

\[
[z, x, y] = [z \bullet x - x \bullet z, y] = (z \bullet x) \bullet y - (x \bullet z) \bullet y - y \bullet (z \bullet x) + y \bullet (x \bullet z).
\]
Summing up, we get
\[ J_{[-,-]}(x, y, z) = J_{\bullet}(x, y, z) - J_{\bullet}(z, y, x) - J_{\bullet}(z, y, x) + J_{\bullet}(x, y, z) = [x, y, z] - [z, y, x]. \]

2.3. Remarks. 1. Given a quasi-Lie pair of brackets \([\cdot, \cdot], [-, -], [-, -] = u\) in a module \(M\) and a fully symmetric 3-bracket \(v\) in \(M\), the pair \([\cdot, \cdot], [-, -], [-, -] = u\) is also a quasi-Lie pair of brackets in \(M\). For example, taking in Lemma 2.1
\[ u(x, y, z) = J_{\bullet}(x, y, z) + J_{\bullet}(z, y, x) \]
we deduce that the 2-bracket \([x, y] = x \bullet y - y \bullet x\) and the 3-bracket carrying any triple \((x, y, z) \in M^3\) to
\[ (x \bullet y) \bullet z - x \bullet (y \bullet z) + (y \bullet z) \bullet x - y \bullet (z \bullet x) + (z \bullet x) \bullet y - z \bullet (x \bullet y) \]
form a quasi-Lie pair of brackets in \(M\). This shows that quasi-Lie algebras naturally arise in the study of non-associative multiplications.

2. A quasi-Lie pair of brackets in a module \(M\) gives rise to a fully symmetric 3-bracket \(s : M^3 \to M\) by
\[ s(x, y, z) = [x, y, z] + [z, y, x] = 2[x, y, z] - J_{[-,-]}(x, y, z) \]
for any \(x, y, z \in M\). If 2 is invertible in \(R\), then we can recover the 3-bracket \([-, -, -]\) from \([- , - , -]\) and \(s\). This yields a bijective correspondence between quasi-Lie pairs of brackets in \(M\) and pairs (a skew-symmetric 2-bracket in \(M\), a fully symmetric 3-bracket in \(M\)).

2.4. Cobrackets and quasi-Lie coalgebras. To discuss the dual notions of cobrackets and quasi-Lie coalgebras we use the language of tensor products. For a module \(M\) and an integer \(m \geq 1\), we let \(M^\otimes m\) be the tensor product over \(R\) of \(m\) copies of \(M\). It is clear that \(m\)-brackets in \(M\) bijectively correspond to linear maps \(M^\otimes m \to M\). An \(m\)-cobracket in \(M\) is a linear map \(M \to M^\otimes m\). A cobracket \(\nu : M \to M^\otimes m\) is cyclically symmetric if \(\nu = Q_m \circ \nu\) where \(Q_m\) is the linear automorphism of \(M^\otimes m\) defined by \(Q_m(x \otimes y) = y \otimes x\) for all \(x \in M, y \in M^\otimes (m-1)\). For a 2-cobracket \(\nu : M \to M^\otimes 2\), we denote by \(\nu^2\) the 3-cobracket \((\nu \otimes \text{id}_M) \circ \nu : M \to M^\otimes 3\).

The co-Jacobitator \(j_\nu\) of \(\nu\) is the cyclically symmetric 3-cobracket
\[ j_\nu = (I + Q + Q^2) \circ \nu^2 : M \to M^\otimes 3 \]
where \(I = \text{id}_{M^\otimes 3}\) and \(Q = Q_3\). A 2-cobracket \(\nu\) is symmetric (respectively, skew-symmetric) if \(P \circ \nu = \nu\) (respectively, \(P \circ \nu = -\nu\)) where \(P = Q_2\) is the permutation automorphism of \(M^\otimes 2\) defined by \(P(x \otimes y) = y \otimes x\) for all \(x, y \in M\).

A quasi-Lie coalgebra is a module \(M\) carrying a skew-symmetric 2-cobracket \(\nu\) and a cyclically symmetric 3-cobracket \(\gamma : M \to M^\otimes 3\) such that
\[ j_\nu = E \circ \gamma : M \to M^\otimes 3 \]
where \(E\) is the endomorphism of \(M^\otimes 3\) defined by
\[ E(x \otimes y \otimes z) = x \otimes y \otimes z - z \otimes y \otimes x \]
for all \(x, y, z \in M\). The pair \((\nu, \gamma)\) is called then a quasi-Lie pair of cobrackets in \(M\). We recover the usual Lie coalgebras for \(\gamma = 0\).
2.5. Bi-endomorphisms and quasi-Lie bialgebras. By a bi-endomorphism of a module \( M \) we mean a linear endomorphism of the module \( M^{\otimes 2} \). For instance, the permutation automorphism \( P \) of \( M^{\otimes 2} \), see Section 2.2 and \( \overline{P} = \text{id}_{M^{\otimes 2}} - P \) are bi-endomorphisms of \( M \). We say that a bi-endomorphism \( \zeta \) of \( M \) is equivariant if \( P\zeta = \zeta P \). Examples: \( P \) and \( \overline{P} \) are equivariant; for each bi-endomorphism \( \zeta \) of \( M \), the bi-endomorphism \( \zeta^{\text{eq}} = \zeta + P\zeta P \) is equivariant. We say that a bi-endomorphism \( \zeta \) of \( M \) is skew-symmetric if \( P\zeta = \zeta P = -\zeta \). Examples: \( \overline{P} \) is skew-symmetric; for each bi-endomorphism \( \zeta \) of \( M \), the composition \( \delta(\zeta) = \overline{P}\zeta\overline{P} \) is skew-symmetric. Note that

\[
\delta(\zeta^{\text{eq}}) = 2 \delta(\zeta).
\]

A bracket \([-,-]\) and a 2-cobracket \( \nu \) in \( M \) determine a bi-endomorphism \( \partial\nu \) of \( M \), the coboundary of \( \nu \). Note that any \( z \in M \) yields a linear map \( a_z : M^{\otimes 2} \to M^{\otimes 2} \) carrying \( x \otimes y \) to \([z,x] \otimes y + x \otimes [z,y]\) for any \( x, y \in M \). By definition,

\[
\partial\nu(x \otimes y) = \nu([x,y]) - ad_x(\nu(y)) + ad_y(\nu(x)) \in M^{\otimes 2}.
\]

If \([-,-]\) and \( \nu \) are skew-symmetric, then so is \( \partial\nu \). We have \( \partial\nu \circ P = -\partial\nu \) because

\[
\partial\nu(y \otimes x) = \nu([y,x]) - ad_y(\nu(x)) + ad_x(\nu(y))
\]

\[
= -\nu([x,y]) + ad_x(\nu(y)) - ad_y(\nu(x)) = -\partial\nu(x \otimes y)
\]

for any \( x, y \in M \). Also \( P \circ \partial\nu = -\partial\nu \) because

\[
P(\partial\nu(x \otimes y)) = P(\nu([x,y]) - ad_x(\nu(y)) + ad_y(\nu(x)))
\]

\[
= -\nu([x,y]) + ad_x(\nu(y)) - ad_y(\nu(x)) = -\partial\nu(x \otimes y).
\]

We define a quasi-Lie bialgebra to be a module endowed with a quasi-Lie pair of brackets \([-,-],[-,-,-])\), a quasi-Lie pair of cobrackets \((\nu,\gamma)\), and an equivariant bi-endomorphism \( \zeta \) such that \( \partial\nu = \delta(\zeta) \). We leave it to the reader to verify that this notion of a quasi-Lie bialgebra is self-dual. For \([-,-,-] = 0, \gamma = 0, \zeta = 0 \), we recover the usual Lie bialgebras.

3. Gates and loops

We define gates in topological spaces and derive from the gates certain algebraic operations on homotopy classes of loops.

3.1. Preliminaries. A loop in a topological space \( X \) is a continuous map \( a : S^1 \to X \) where the circle \( S^1 = \{ p \in \mathbb{C} \mid |p| = 1 \} \) is oriented counterclockwise. The point \( a(1) \in X \) is the base point of \( a \). For \( p \in S^1 \), we let \( a_p : S^1 \to X \) be the loop obtained as the composition of \( a \) with the rotation \( S^1 \to S^1 \) carrying \( 1 \in S^1 \) to \( p \).

Two loops \( a, b : S^1 \to X \) are freely homotopic if there is a continuous map \( F : S^1 \times [0,1] \to X \) such that \( F(p,0) = a(p) \) and \( F(p,1) = b(p) \) for all \( p \in S^1 \). Such a map \( F \) is a homotopy between \( a \) and \( b \).

We let \( L(X) \) denote the set of free homotopy classes of loops in \( X \) and let \( M(X) \) be the free module with basis \( L(X) \). For a loop \( a \) in \( X \), we let \( \langle a \rangle \in L(X) \subset M(X) \) be the free homotopy class of \( a \). Set \( \langle a \rangle_0 = \langle a \rangle \in M(X) \) if \( a \) is non-contractible and \( \langle a \rangle_0 = 0 \in M(X) \) if \( a \) is contractible.
3.2. Gates. A cylinder neighborhood of a subset $C$ of a topological space $X$ is a pair consisting of a closed set $U \subset X$ with $C \subset \text{Int}(U)$ and a homeomorphism $U \approx C \times [-1, 1]$ carrying $\text{Int}(U)$ onto $C \times (-1, 1)$ and carrying each point $c \in C$ to $(c, 0)$. A gate in $X$ is a closed path-connected subspace $C \subset X$ endowed with a cylinder neighborhood and such that all loops in $C$ are contractible in $X$. We will identify the cylinder neighborhood in question with $C \times [-1, 1]$ via the given homeomorphism.

For a gate $C \subset X$, consider the map $H : X \to S^1$ carrying the complement of $C \times (-1, 1)$ in $X$ to $-1 \in S^1$ and carrying $C \times \{t\}$ to $\exp(\pi it) \in S^1$ for all $t \in [-1, 1]$. We say that a loop $a : S^1 \to X$ is transversal to $C$ if the map $Ha : S^1 \to S^1$ is transversal to $1 \in S^1$. Then the set $a^{-1}(C) = (Ha)^{-1}(1)$ is finite. For each $p \in a^{-1}(C)$, we define the crossing sign $\varepsilon_p(a)$: if at $p$ the loop $a$ goes from $C \times [-1, 0)$ to $C \times (0, 1]$ then $\varepsilon_p(a) = +1$, otherwise, $\varepsilon_p(a) = -1$. The integer

$$a \cdot C = \sum_{p \in a^{-1}(C)} \varepsilon_p(a)$$

is the algebraic intersection number of $a$ and $C$. Note that the formula $a \mapsto a \cdot C$ defines a homomorphism $H_1(X) \to \mathbb{Z}$. An example of a gate is provided by a simply connected proper codimension 1 submanifold $C$ of a manifold together with a suitable homeomorphism of a closed neighborhood of $C$ onto $C \times [-1, 1]$. In this case, $a \cdot C$ is the usual intersection number of a loop $a$ with $C$.

3.3. Gate brackets. We derive from a gate $C \subset X$ a sequence of brackets $\{\mu^n_C\}_{m \geq 1}$ in the module $M = M(X)$. Fix a point $\star \in C$. For a loop $a$ in $X$ with $a(1) \in C$ we let $\tilde{a}$ be the loop based in $\star$ and obtained from $a$ by conjugation along a path in $C$ from $\star$ to $a(1)$. The loop $\tilde{a}$ represents an element of $\pi_1(X, \star)$ independent of the choice of the latter path. Consider now $m \geq 1$ loops $a_1, \ldots, a_m$ in $X$ transversal to $C$. For any $i = 1, \ldots, m$ and $p_i \in a_i^{-1}(C) \subset S^1$, consider the reparametrization $a_{i,p_i} = (a_i)_{p_i}$ of $a_i$ based at $a_i(p_i) \in C$, see Section 3.1. Consider the associated loop $\tilde{a}_{i,p_i}$ based at $\star$. Set

$$\mu^n_C(a_1, \ldots, a_m) = \sum_{p_1 \in a_1^{-1}(C), \ldots, p_m \in a_m^{-1}(C)} \prod_{i=1}^m \varepsilon_{p_i}(a_i) \prod_{i=1}^m \tilde{a}_{i,p_i} \in M.$$ (3.3.1)

Lemma 3.1. $\mu^n_C(a_1, \ldots, a_m)$ depends only on the free homotopy classes of the loops $a_1, \ldots, a_m$.

Proof. For any pair of freely homotopic loops in $X$ transversal to $C$, there is a free homotopy between these loops which splits as a product of several homotopies of the following two types (and inverse homotopies): (i) deformations in the class of loops transversal to $C$ and (ii) deformations pushing a branch of the loop lying in $X \setminus C$ across $C$ and creating two new crossings of the loop with $C$. It is clear that homotopies of $a_1, \ldots, a_m$ of type (i) do not change $\mu^n_C(a_1, \ldots, a_m)$. A homotopy of $a_k$ of type (ii) with $k \in \{1, \ldots, m\}$ creates two crossings $p, p'$ such that $\varepsilon_p(a_k) = -\varepsilon_{p'}(a_k)$ and $\tilde{a}_{k,p} = \tilde{a}_{k,p'}$. As a consequence, the terms of the sum (3.3.1) arising from $p = p$ and $p_k = p'$ cancel each other, so that the sum is preserved. It is clear that $\mu^n_C(a_1, \ldots, a_m)$ does not depend on the choice of the point $\star \in C$. \hfill $\square$

Using the product structure in the cylinder neighborhood of $C$, we easily observe that each loop in $X$ is freely homotopic to a loop transversal to $C$. Therefore
Lemma 3.1 yields a map

\[(L(X))^m \to M, \quad (\langle a_1 \rangle, \ldots, \langle a_m \rangle) \mapsto \mu_C^m(a_1, \ldots, a_m).\]

This map extends uniquely to an \(m\)-bracket \(\mu_C^m\) in \(M\). Since the loops \(\prod_{i=1}^m \tilde{a}_{i,p_i}\) and \((\prod_{i=2}^m \tilde{a}_{i,p_i})\tilde{a}_{1,p_1}\) are freely homotopic, this bracket is cyclically symmetric for all \(m\). Clearly, \(\mu_C^1(a) = (a \cdot C)\langle a \rangle\) for any loop \(a\) in \(X\) transversal to \(C\).

### 3.4. Gate cobrackets

For a gate \(C\) in a topological space \(X\) and for any integer \(m \geq 1\), we define an \(m\)-cobracket in the module \(M = M(X)\). The idea is to consider all splittings of a loop \(a : S^1 \to X\) as a product of \(m\) paths with endpoints in \(C\) and to sum up the associated elements of \(M^\otimes m\). To this end, for any distinct points \(p_1, p_2 \in S^1\), we consider the path in \(X\) obtained by restricting \(a\) to the arc in \(S^1\) which starts in \(p_1\) and goes counterclockwise to \(p_2\). If \(a(p_1), a(p_2) \in C\), then the product of this path with a path from \(a(p_2)\) to \(a(p_1)\) in \(C\) is a loop in \(X\) based at \(a(p_1)\). This loop is denoted \(a_{p_1,p_2}\). Since \(C\) is path-connected and all loops in \(C\) are contractible in \(X\), the loop \(a_{p_1,p_2}\) is well defined up to homotopy. An \(m\)-sequence of the loop \(a\) is a sequence of \(m\) distinct points \(p_1, \ldots, p_m \in a^{-1}(C)\) such that moving along \(S^1\) counterclockwise, we meet \(p_1, \ldots, p_m\) in this cyclic order. The set of all \(m\)-sequences of \(a\) is denoted \(S_m(a)\). It is nonempty if and only if \(m \leq \text{card}(a^{-1}(C))\).

If \(a\) is transversal to \(C\), then the set \(S_m(a)\) is finite and we set

\[
(3.4.1) \quad \gamma_{C,m}(a) = \sum_{(p_1,\ldots,p_m) \in S_m(a)} \prod_{i=1}^m \varepsilon_{p_i}(a) \langle a_{p_i,p_{i+1}} \rangle_0 \in M^\otimes m
\]

where \(p_{m+1} = p_1\). Since all cyclic permutations of \(m\)-sequences of \(a\) are \(m\)-sequences of \(a\), the vector \(\gamma_{C,m}(a)\) is invariant under cyclic permutations of \(M^\otimes m\). In particular, \(\gamma_{C,1}(a) = (a \cdot C)\langle a \rangle_0\) and

\[
\gamma_{C,2}(a) = \sum_{(p_1,p_2) \in S_2(a)} \varepsilon_{p_1}(a) \varepsilon_{p_2}(a) \langle a_{p_1,p_2} \rangle_0 \otimes \langle a_{p_2,p_1} \rangle_0
\]

where \(S_2(a)\) is the set of all ordered pairs of distinct elements of \(a^{-1}(C) \subset S^1\).

**Lemma 3.2.** For all \(m \geq 1\), Formula (3.4.1) defines a map \(L(X) \to M^\otimes m\).

**Proof.** We need only to prove that if two loops \(a, a'\) transversal to \(C\) are freely homotopic in \(X\), then \(\gamma_{C,m}(a) = \gamma_{C,m}(a')\). As in the proof of Lemma 3.1, it is enough to consider a homotopy pushing a branch of the loop across \(C\) and creating two transversal crossings with opposite crossing signs. The contributions to \(\gamma_{C,m}\) of the \(m\)-sequences containing neither of these two points are the same before and after the deformation. The contributions to \(\gamma_{C,m}\) of the \(m\)-sequences containing exactly one of these new crossings cancel each other. The \(m\)-sequences containing both new crossings contribute zero to \(\gamma_{C,m}\) because at least one of the corresponding loops \(a_{p_i,p_{i+1}}\) is contractible in \(X\). This implies our claim. \(\square\)

The map \(L(X) \to M^\otimes m\) from Lemma 3.2 extends uniquely to a linear map \(\gamma_{C,m} : M \to M^\otimes m\). This map is an \(m\)-cobracket in \(M\).

### 3.5. Gate bi-endomorphisms

We derive from any gate \(C \subset X\) a bi-endomorphism \(\zeta_C\) of the module \(M = M(X)\). Note that for any loops \(a, b : S^1 \to X\) transversal to \(C\) and for any points \(p \in a^{-1}(C), q \in b^{-1}(C)\), we can multiply the loops \(a_p, b_q\) connecting their base points \(a(p), b(q) \in C\) by a path in \(C\). The product
loop is denoted \( a_p \circ C \ b_q \). Its free homotopy class \( \langle a_p \circ C \ b_q \rangle \) does not depend on the choice of the path in \( C \) connecting the base points. Set

\[
|a|_C = \text{card}(a^{-1}(C))
\]

and

\[
(3.5.1) \quad \zeta(a, b) = |a|_C (b \cdot C) \langle a \rangle_0 \otimes \langle b \rangle_0 + 2 \sum_{(p_1, p_2) \in S_2(a), q \in b^{-1}(C)} \varepsilon_{p_1}(a) \varepsilon_{p_2}(a) \varepsilon_q(b) \langle a_{p_1, p_2} \rangle_0 \otimes \langle a_{p_2, p_1} \circ C \ b_q \rangle_0.
\]

Here \( a_{p_1, p_2}, a_{p_2, p_1} \) are the loops as in Section 3.4 based respectively at \( p_1, p_2 \).

**Lemma 3.3.** \( \zeta(a, b) \in M \otimes^2 \) is invariant under free homotopies of \( a \) and \( b \) in \( X \).

**Proof.** We first show that \( \zeta(a, b) = \zeta(a, b') \) for any loop \( b' \) transversal to \( C \) and freely homotopic to \( b \). As in Lemma 3.2, it is enough to show that \( \zeta(a, b) \) is preserved under any deformation \( b \mapsto b' \) of \( b \) pushing a branch across \( C \) and creating two crossing points \( q_+, q_- \) so that \( (b')^{-1}(C) = b^{-1}(C) \cup \{q_+, q_-\} \) and \( \varepsilon_{q_\pm}(b') = \pm 1 \). Then for any pair \( (p_1, p_2) \in S_2(a) \), the contributions of the triples \( p_1, p_2, q_+ \) and \( p_1, p_2, q_- \) to \( \zeta(a, b') \) are opposite and cancel each other. Also, \( b' \cdot C = b \cdot C \) and \( \langle b' \rangle_0 = \langle b \rangle_0 \). Therefore \( \zeta(a, b') = \zeta(a, b) \).

We next show that \( \zeta(a, b) = \zeta(a', b) \) for any loop \( a' \) transversal to \( C \) and freely homotopic to \( a \). As above, it suffices to consider the case where \( a' \) is obtained by pushing a branch of \( a \) across \( C \) and creating two crossing points \( p_+, p_- \) so that \( (a')^{-1}(C) = a^{-1}(C) \cup \{p_+, p_-\} \) and \( \varepsilon_{p_\pm}(a') = \pm 1 \). Let \( s \) be the sum over \( p_1, p_2, q \) appearing in (3.5.1). Let \( s' \) be the similar sum with \( a \) replaced by \( a' \). For any \( p \in a^{-1}(C), q \in b^{-1}(C) \), the contributions of the triples \( p_+, p, q \) and \( p_-, p, q \) to \( s \) cancel each other. The same holds for the triples \( p, p_+, q \) and \( p, p_-, q \). To compute the contributions of the triples \( p_+, p_-, q \) and \( p_-, p_+, q \) to \( s' \), we distinguish two cases. In the first case, the deformation \( a \mapsto a' \) pushes a branch of \( a \) “up”, that is from \( C \times [-1, 0] \) to \( C \times [0, 1] \). Then the loop \( a'_{p_+, p_-} \) is contractible and the loop \( a'_{p_-, p_+} \) is freely homotopic to \( a \). So, the triples \( p_+, p_-, q \) and \( p_-, p_+, q \) contribute to \( s' \) respectively \( 0 \) and \( -\varepsilon_q(b) \langle a \rangle_0 \otimes \langle b \rangle_0 \). In the second case, the deformation \( a \mapsto a' \) pushes a branch of \( a \) “down”, that is from \( C \times [0, 1] \) to \( C \times [-1, 0] \). Then the loop \( a'_{p_+, p_-} \) is freely homotopic to \( a \), the loop \( a'_{p_-, p_+} \) is contractible, and the triples \( p_+, p_-, q \) and \( p_-, p_+, q \) contribute to \( s' \) respectively \( -\varepsilon_q(b) \langle a \rangle_0 \otimes \langle b \rangle_0 \) and \( 0 \). In both cases,

\[
s' = s - \sum_{q \in b^{-1}(C)} \varepsilon_q(b) \langle a \rangle_0 \otimes \langle b \rangle_0 = s - (b \cdot C) \langle a \rangle_0 \otimes \langle b \rangle_0.
\]

Multiplying this equality by 2 and adding to the obvious formula

\[
|a'|_C (b \cdot C) \langle a' \rangle_0 \otimes \langle b \rangle_0 = (|a|_C + 2) (b \cdot C) \langle a \rangle_0 \otimes \langle b \rangle_0
\]

we obtain that \( \zeta(a', b) = \zeta(a, b) \). \( \square \)

By Lemma 3.3, the formula \( (a, b) \mapsto \zeta(a, b) \) defines a map \( L(X) \times L(X) \to M \otimes^2 \). We let \( \zeta_C : M \otimes^2 \to M \otimes^2 \) be the linear extension of this map.

4. **Quasi-surfaces and main theorems**

We recall quasi-surfaces from [11] and state our main theorems.
4.1. **Quasi-surfaces.** By a surface we mean a smooth oriented 2-dimensional manifold with boundary. A *quasi-surface* is a path-connected topological space $X$ obtained by gluing a surface $\Sigma$ to a topological space $Y$ along a continuous map $f : \alpha \to Y$ where $\alpha \subset \partial \Sigma$ is a union of a finite number ($\geq 1$) of disjoint closed segments in $\partial \Sigma$. Note that then $Y \subset X$ and $X \setminus Y = \Sigma \setminus \alpha$. We fix a closed neighborhood of $\alpha$ in $\Sigma$ and identify it with $\alpha \times [-1, 1]$ so that

$$\alpha = \alpha \times \{-1\} \quad \text{and} \quad \partial \Sigma \cap (\alpha \times [-1, 1]) = \alpha \cup (\partial \alpha \times [-1, 1]).$$

The surface

$$\Sigma' = \Sigma \setminus (\alpha \times \{-1\}) \subset \Sigma \setminus \alpha \subset X$$

is a copy of $\Sigma$ embedded in $X$. We provide $\Sigma'$ with the orientation induced from that of $\Sigma$. We call $Y$ the *singular part* of $X$, call $\Sigma$ the *surface core* of $X$, and call $\Sigma'$ the *reduced surface core* of $X$.

Set $\pi_0 = \pi_0(\alpha)$. For $k \in \pi_0$, we let $\alpha_k$ be the corresponding segment component of $\alpha \times \{0\} \subset \partial \Sigma' \subset X$. It is clear that $\alpha_k$ is a gate of $X$ in the sense of Section 3.2. The gates $\{\alpha_k\}_{k \in \pi_0}$ separate $\Sigma' \subset X$ from the rest of $X$. For any loop $a : S^1 \to X$ and $k \in \pi_0$, we set $a \cap \alpha_k = a(S^1) \cap \alpha_k$.

We say that a loop $a : S^1 \to X$ is *generic* if (i) all branches of $a$ in $\Sigma'$ are smooth immersions meeting $\partial \Sigma'$ transversely at a finite set of points which all lie in the interior of the gates, and (ii) all self-intersections of $a$ in $\Sigma'$ are double transversal intersections which all lie in $\text{Int}(\Sigma') = \Sigma' \setminus \partial \Sigma'$. A generic loop $a$ traverses any point of a gate $\alpha_k$ at most once. So, the restriction of the map $a : S^1 \to X$ to $a^{-1}(\alpha_k) \subset S^1$ is a bijection onto the set $a \cap \alpha_k$. In this context, we adjust notation of Section 3.3 and systematically use the letter $p$ for points in $a \cap \alpha_k$ rather than for their preimages under $a$. Accordingly, the crossing sign $\varepsilon_p(a)$ at $p \in a \cap \alpha_k$ is $+1$ if $a$ goes at $p$ from $X \setminus \Sigma'$ to $\text{Int}(\Sigma')$ and is $-1$ otherwise.

More generally, a finite family of loops in $X$ is *generic* if these loops are generic and all their intersections in $\Sigma'$ are double transversal intersections in $\text{Int}(\Sigma')$. In particular, these loops do not meet at the gates. Using cylinder neighborhoods of the gates, it is easy to see that any finite family of loops in $X$ can be transformed into a generic family by a small deformation.

We keep the objects $X, \Sigma, \Sigma', \alpha, \pi_0$ for the rest of the paper.

4.2. **The brackets.** We recall the 2-bracket in the module $M = M(X)$ introduced in [Tu2]. For a loop $a : S^1 \to X$ and a point $r \in X$ traversed by $a$ exactly once, we let $a_r$ be the loop which starts at $r$ and goes along $a$ until coming back to $r$. For any loops $a, b$ in $X$ set

$$a \cap b = a(S^1) \cap b(S^1) \cap \Sigma'.$$

If $a, b$ is a generic pair of loops then the set $a \cap b \subset \text{Int}(\Sigma')$ is finite and each point $r \in a \cap b$ is traversed by $a$ and $b$ only once so that we can consider the loops $a_r, b_r$ based at $r$. Set $\varepsilon_r(a, b) = 1$ if the tangent vectors of $a$ and $b$ at $r$ form a positive basis in the tangent space of $\Sigma'$ at $r$ and set $\varepsilon_r(a, b) = -1$ otherwise. The sum $\sum_{r \in a \cap b} \varepsilon_r(a, b) = 1(a_r, b_r) \in M$ may be viewed as the "homotopy intersection" of $a, b$. Generally speaking, this sum depends on the choice of $a, b$ in their free homotopy classes. We therefore add further terms involving the gates.

We start with terminology. By a *gate orientation* of $X$ we mean an orientation of all gates $\{\alpha_k\}_{k \in \pi_0}$ of $X$. Gate orientations of $X$ canonically correspond to orientations of the 1-manifold $\alpha \subset \partial \Sigma$. Pick a gate orientation $\omega$ of $X$. For $k \in \pi_0$, set $\varepsilon(\omega, k) = +1$ if the $\omega$-orientation of $\alpha_k$ is compatible with the orientation of $\Sigma'$, i.e.,
if the pair (a ω-positive tangent vector of αk ⊂ ∂Σ', a vector directed inside Σ') is positively oriented in Σ'. Otherwise, set ε(ω, k) = −1. For points p, q ∈ αk, we write p <ω q if p ≠ q and the ω-orientation of αk leads from p to q.

Any generic pair of loops a, b in X determines a finite set of triples

\[ T(a, b) = \{(k, p, q) \mid k \in \pi_0, p \in a \cap \alpha_k, q \in b \cap \alpha_k \} \]

These triples are called chords of the pair (a, b). For any such chord (k, p, q) we have p ≠ q since generic loops do not meet at the gates. Also, we can multiply the loops ap, bq based at p, q using an arbitrary path connecting p, q in αk. The product loop determines an element of L(X) denoted ⟨ap, bq⟩. Clearly, if (k, p, q) is a chord of (a, b) then (k, q, p) is a chord of (b, a) and ⟨ap, bq⟩ = ⟨bq, ap⟩. Finally, set

\[ T_ω(a, b) = \{(k, p, q) \in T(a, b) \mid q <ω p \} \subset T(a, b) \]

**Lemma 4.1.** (i) There is a unique 2-bracket [−, −]X,ω in the module M = M(X) such that for any generic pair of loops a, b in X, we have

\[ \langle a, b \rangle |X,ω = \sum_{r \in a \cap b} \varepsilon_r(a, b) ⟨a_r, b_r⟩ + \sum_{(k, p, q) \in T_ω(a, b)} \varepsilon(ω, k) \varepsilon_p(a) \varepsilon_q(b) ⟨a_p, b_q⟩. \]

(ii) The skew-symmetric bracket [−, −]X in M defined by

\[ [x, y]_X = [x, y]_{X,ω} - [y, x]_{X,ω} \]

for all x, y ∈ M does not depend on the choice of ω.

Claim (i) is Lemma 4.1 of [1u2] and Claim (ii) is Theorem 4.2 of [1u2]. Both brackets [−, −]X,ω and [−, −]X generalize Goldman’s bracket ([Go1], [Go2]): the value of [−, −]X,ω (respectively, [−, −]X) on any pair of free homotopy classes of loops in Σ ⊂ X is equal to their Goldman’s bracket (respectively, twice this bracket). For all ω, the bracket 2[−, −]X,ω can be computed from [−, −]X and the gate 2-brackets of X, see [1u2], Remark 4.5.1.

We now compute the Jacobian of [−, −]X. By Section 3.3 each gate αk of X determines cyclically symmetric brackets \( \mu_k = \mu_{α_k} \) in M. The sum

\[ \mu^m = \sum_{k \in \pi_0} \mu_k : M^m \to M \]

is a cyclically symmetric bracket in M called the total gate m-bracket of X.

**Theorem 4.2.** The module M = M(X) endowed with the 2-bracket [−, −] = [−, −]X and the total gate 3-bracket μ = μ3 is a quasi-Lie algebra.

Theorem 4.2 may be rephrased by saying that

\[ J_{[−, −]X} (x, y, z) = \mu(x, y, z) - \mu(z, y, x) \]

for any x, y, z ∈ M. Theorem 4.2 is proved in Section 5.

### 4.3. The cobrackets

We recall from [1u2] the intersection cobracket in M = M(X). Consider a generic loop a in X. Denote by #a the set of self-intersections of a in Σ'. This set is finite and lies in Int(Σ'). The loop a crosses each point r ∈ #a twice; we let v1, v2 be the tangent vectors of a at r numerated so that the pair (v1, v2) is positively oriented. For i = 1, 2, let ai be the loop starting in r and going along a in the direction of the vector vi until the first return to r. Up to parametrization, a = a1a2 is the product of the loops a1, a2 based at r. As in Section 3.4 for any distinct points p1, p2 ∈ a ∩ αk with k ∈ π0 we have the loop
Lemma 4.3. (i) There is a unique 2-cobracket \( \nu_{X,\omega} \) in the module \( M = M(X) \) such that for any generic loop \( a \) in \( X \), we have

\[
\nu_{X,\omega}(\langle a \rangle) = \sum_{r \in \#a} (\langle a_r^1 \rangle_0 \otimes \langle a_r^2 \rangle_0 - \langle a_r^2 \rangle_0 \otimes \langle a_r^1 \rangle_0) + \sum_{(k,p_1,p_2) \in T_\omega(a)} \varepsilon(\omega,k) \varepsilon_{p_1}(a) \varepsilon_{p_2}(a) \langle a_{p_2,p_1} \rangle_0 \otimes \langle a_{p_1,p_2} \rangle_0.
\]

(ii) Let \( P \) be the linear automorphism of \( M \otimes M \) carrying \( x \otimes y \) to \( y \otimes x \) for all \( x, y \in M \). The skew-symmetric cobracket

\[ \nu_X = \nu_{X,\omega} - P \nu_{X,\omega} : M \to M \otimes M \]

does not depend on the choice of \( \omega \).

Claim (i) is Lemma 5.1 of [Tu2] and Claim (ii) is Theorem 5.2 of [Tu2]. For loops in \( \Sigma' \), the cobracket \( \nu_{X,\omega} \) coincides with the cobracket introduced in [Tu1] and the cobracket \( \nu_X \) is twice the one in [Tu1]. For all \( \omega \), the cobracket \( 2 \nu_{X,\omega} \) can be recovered from \( \nu_X \) and the gate 2-cobrackets of \( X \).

We now compute the co-Jacobiator of \( \nu_X \). By Section 3 each gate \( \alpha_k \) of \( X \) determines cyclically symmetric cobrackets \( \{ \gamma_{\alpha_k,m} \}_{m \geq 1} \) in \( M \). The map

\[ \gamma^m = \sum_{k \in \pi_0} \gamma_{\alpha_k,m} : M \to M^\otimes m \]

is a cyclically symmetric \( m \)-cobracket in \( M \) called the total gate \( m \)-cobracket of \( X \). Though we shall not need it, note that \( \gamma^1 = 0 \).

Theorem 4.4. The module \( M = M(X) \) endowed with the 2-cobracket \( \nu = \nu_X \) and the total gate 3-cobracket \( \gamma = \gamma^3 \) is a quasi-Lie coalgebra.

Theorem 4.4 is proved in Section 6. Next, we compute the coboundary \( \partial \nu \) of \( \nu = \nu_X \) via the gates (at least in the case where \( 1/2 \in R \)). By Section 3.3 each gate \( \alpha_k \) of \( X \) yields a bi-endomorphism \( \zeta_{\alpha_k} \) of \( M \). The map

\[ \zeta = \sum_{k \in \pi_0} \zeta_{\alpha_k} : M^\otimes 2 \to M^\otimes 2 \]

is called the total gate bi-endomorphism of the module \( M = M(X) \).

Theorem 4.5. Let \( e \in L(X) \subset M \) be the homotopy class of contractible loops. Then

\[ \delta(\zeta) = 2 \partial \nu \mod(Re \otimes M + M \otimes Re). \]

Theorem 4.5 is proved in Section 7. This theorem suggests to consider the quotient module \( M_\nu = M/Re \). We can represent \( e \in M \) by an embedded circle in a small disc in \( \text{Int}(\Sigma') \). This easily implies that

\[ \mu(Re \times M \times M) = \mu(M \times Re \times M) = \mu(M \times M \times Re) = 0, \]

\[ [Re,M] = [M,Re] = 0, \nu(Re) = 0, \gamma(Re) = 0, \zeta(Re \times M) = \zeta(M \times Re) = 0. \]
Denoting the projection $M \to M_0$ by $\psi$, we deduce the existence and uniqueness of maps $[-,-], \mu_\circ, \nu_\circ, \gamma_\circ, \zeta_\circ$ such that the following five diagrams commute:

$$
\begin{align*}
M \times M & \xrightarrow{[-,-]} M, & M \times M \times M & \xrightarrow{\mu} M, \\
\psi \times \psi & \downarrow & \psi & \downarrow \\
M_0 \times M_0 & \xrightarrow{[-,-]_0} M_0 & M_0 \times M_0 \times M_0 & \xrightarrow{\mu_0} M_0 \\
M & \xrightarrow{\nu} M_0^{\otimes 2}, & M & \xrightarrow{\gamma} M_0^{\otimes 3}, & M^{\otimes 2} & \xrightarrow{\zeta} M_0^{\otimes 2}.
\end{align*}
$$

Theorems 4.2, 4.4, 4.5 and Formula (2.5.1) imply the following:

**Corollary 4.6.** If $1/2 \in R$, then the module $M_0$ with brackets $[-,-]_0, \mu_0, \nu_0, \gamma_0, \zeta_0$, and bi-endomorphism $(1/4)(\zeta_0)^q$ is a quasi-Lie bialgebra.

**4.4. Remark.** M. Chas [Ch] proved that the Lie bialgebras of loops on (ordinary) surfaces are involutive. It would be interesting to extend this result to the quasi-Lie bialgebra of Corollary 4.6.

5. **Proof of Theorem 1.2**

5.1. **Preliminaries.** A finite family of loops in the quasi-surface $X$ is said to be **simple** if these loops meet the gates of $X$ transversely and have no crossings or self-crossings in the reduced surface core $\Sigma' \subset X$. In particular, these loops do not meet at the gates. A simple family of loops is necessarily generic.

**Lemma 5.1.** Any finite family of loops in $X$ can be deformed in $X$ into a simple family of loops.

**Proof.** Consider first a single loop in $X$. Since $X$ is path-connected and contains a gate, we can deform our loop into a generic loop $a$ which meets a gate. If the set $#a$ of double points of $a$ in $\text{Int}(\Sigma')$ is empty, then we are done. Otherwise, pick a point $r \in #a$. Starting at $r = r_0$ and moving along $a$ we meet several double points $r_1, \ldots, r_n \in #a$ with $n \geq 0$ and then come to a point $p \in a \cap \alpha_k$ in a certain gate $\alpha_k$. The segment of $a$ running from $r_n$ to $p$ is embedded in $\Sigma'$ and meets $#a$ only at $r_n$. Denote this segment by $s$ and let $t$ be the branch of $a$ transversal to $s$ at $r_n$. Push $t$ towards $s$ keeping $t$ transversal to $s$ and eventually push $t$ across $\alpha_k$ at $p$. This deformation of $a$ increases $\text{card}(a \cap \alpha_k)$ by 2 and decreases $\text{card}(#a)$ by 1. Continuing by induction, we deform $a$ into a generic loop without self-intersections in $\Sigma'$. If the original family of loops contains two or more loops, then we first deform it into a generic family of loops which all meet some gates. Then, as above, pushing branches at crossings and self-crossings across the gates, we obtain a simple family of loops.

5.2. **Proof of the theorem.** Fix an arbitrary gate orientation $\omega$ of $X$ and let $\overline{\gamma}$ be the gate orientation of $X$ opposite to $\omega$ on all gates. To shorten our formulas, we set $x \bullet y = [x,y]|_{X,\omega}$ for any $x, y \in M = M(X)$. If $x, y \in L(X) \subset M$ are represented
by a generic pair of loops $a, b$, then
\begin{equation}
(5.2.1)
\quad x \bullet y = \sum_{r \in \Gamma(a,b)} \varepsilon(r) (a,b) (a_r b_r) + \sum_{(k,p,q) \in T_{\omega}(a,b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b) \langle a_p b_q \rangle.
\end{equation}

Note that the inclusion $(k,p,q) \in T_{\omega}(a,b)$ holds if and only if $(k, q, p) \in T_{\omega}(b, a)$. This and the obvious identities
\[ (a_r b_r) = (b_r a_r), \quad \varepsilon(r, a, b) = -\varepsilon(r, b, a), \quad \varepsilon(\omega, k) = -\varepsilon(\omega, k) \]

imply that
\begin{equation}
(5.2.2)
\quad [x, y]_{X, \omega} = -[y, x]_{X, \omega} = -y \bullet x.
\end{equation}

By the bilinearity of the brackets, the same holds for all $x, y \in M$. For arbitrary $x, y, z \in M$, set
\begin{equation}
(5.2.3)
\quad u_\omega(x, y, z) = (x \bullet y) \bullet z + (y \bullet z) \bullet x + (z \bullet x) \bullet y.
\end{equation}

By $(5.2.2)$,
\begin{equation}
(5.2.4)
\quad u_\omega(x, y, z) = [[x, y]_{X, \omega}, z]_{X, \omega} + [[y, z]_{X, \omega}, x]_{X, \omega} + [[z, x]_{X, \omega}, y]_{X, \omega} = z \bullet (y \bullet x) + x \bullet (z \bullet y) + y \bullet (x \bullet z).
\end{equation}

We now apply Lemma $2.1$ to the bilinear form $M^2 \to M, (x, y) \mapsto x \bullet y$. The 2-bracket $[-, -]$ in this lemma is the bracket $[-, -]_{X}$. Formulas $(5.2.3)$ and $(5.2.4)$ show that the 3-bracket $[-, - , -]$ in Lemma $2.1$ is equal to $u_\omega + u_\omega$. By $(2.2.1)$, to prove the theorem it suffices to show that for all $x, y, z \in M$, we have
\begin{equation}
(5.2.5)
\quad [x, y, z] - [z, y, x] = \mu^3(x, y, z) - \mu^3(z, y, x).
\end{equation}

Since both sides of $(5.2.5)$ are linear in $x, y, z$, it suffices to handle the case $x, y, z \in L(X) \subset M$. By Lemma $5.1$ we can represent the triple $x, y, z$ by a simple triple of loops $a, b, c$ in $X$. Then Formula $(5.2.1)$ simplifies to
\begin{equation}
(5.2.6)
\quad x \bullet y = \sum_{(k,p,q) \in T_{\omega}(a,b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b) \langle a_p b_q \rangle \in M
\end{equation}

where $a_p, b_q$ are loops reparametrizing $a, b$ and based respectively in $p, q \in \alpha_k$. Here $a_p b_q = a_p(b_q)^{\ell - 1}$ for a path $\ell = \ell(p, q)$ in the gate $\alpha_k$ from $p$ to $q$. We deform the loop $a_p b_q$ by slightly pushing its subpaths $\ell$ and $\ell^{-1}$ into $X \setminus \Sigma'$. (The endpoints $p, q$ of these subpaths are pushed into $X \setminus \Sigma$ along $a, b$, respectively.) The resulting loop is denoted by $a \circ_{p,q} b$ or by $b \circ_{p,q} a$. Thus,
\begin{equation}
(5.2.7)
\quad x \bullet y = \sum_{(k,p,q) \in T_{\omega}(a,b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b) \langle a \circ_{p,q} b \rangle.
\end{equation}

Note that the loop $a \circ_{p,q} b$ is simple; moreover, the pair of loops $a \circ_{p,q} b$, $c$ is simple. Applying $(5.2.6)$ to this pair, we get
\[ \langle a \circ_{p,q} b \rangle \bullet z = \sum_{(l,s,t) \in T_{\omega}(a \circ_{p,q} b, c)} \varepsilon(\omega, l) \varepsilon_s(a \circ_{p,q} b) \varepsilon_t(c) \langle (a \circ_{p,q} b) \times c_t \rangle. \]

For each $l \in \pi_0$, the set $\langle a \circ_{p,q} b \rangle \cap \alpha_l$ is a disjoint union of the sets $a \cap \alpha_l$ and $b \cap \alpha_l$. Therefore for every triple $(k, p, q) \in T_{\omega}(a, b)$, we have
\[ T_{\omega}(a \circ_{p,q} b, c) = T_{\omega}(a, c) \cup T_{\omega}(b, c) \]

and
\[ \langle a \circ_{p,q} b \rangle \bullet z = \lambda_{\omega}(k, p, q) + \rho_{\omega}(k, p, q) \]
where

\[(5.2.8)\]
\[\lambda_\omega(k, p, q) = \sum_{(l, s, t) \in T_\omega(a, c)} \varepsilon(\omega, l) \varepsilon_s(a) \varepsilon_t(c) \langle (a \circ_{p, q} b) s c_t \rangle \in M,\]

\[(5.2.9)\]
\[\rho_\omega(k, p, q) = \sum_{(l, s, t) \in T_\omega(b, c)} \varepsilon(\omega, l) \varepsilon_s(b) \varepsilon_t(c) \langle (a \circ_{p, q} b) s c_t \rangle \in M.\]

Combining with \[(5.2.7)\] we obtain

\[(5.2.10)\]
\[(x \bullet y) \bullet z = \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k) \varepsilon_p(a) \varepsilon_q(b) \langle \lambda_\omega(k, p, q) + \rho_\omega(k, p, q) \rangle.\]

We now compute the right-hand sides of Formulas \[(5.2.8), (5.2.10)\]. The free homotopy class \[\langle (a \circ_{p, q} b) s c_t \rangle\] in \[(5.2.8)\] is represented by the loop \[b \circ_{q, p} a_{s, t} \circ c\] obtained by grafting \[b\] and \[c\] to \[a\] using a path in \[\alpha_k\] from \[p \in a \cap \alpha_k\] to \[q \in b \cap \alpha_k\] and a path in \[\alpha_l\] from \[s \in a \cap \alpha_l\] to \[t \in c \cap \alpha_l\]. So,

\[(5.2.11)\]
\[\lambda_\omega(k, p, q) = \sum_{(l, s, t) \in T_\omega(a, c)} \varepsilon(\omega, l) \varepsilon_s(a) \varepsilon_t(c) \langle b \circ_{q, p} a_{s, t} \circ c \rangle.\]

To give a more precise description of the loop \[b \circ_{q, p} a_{s, t} \circ c\], we distinguish two classes of triples \[(l, s, t) \in T_\omega(a, c)\] determined by whether or not \[s = p\].

Class 1: \[s \neq p\] so that the points \[p, q, s, t\] are distinct (possibly, \[l = k\]). Then the loop \[b \circ_{q, p} a_{s, t} \circ c\] goes along \[\alpha_k\] from \[p\] to \[q\], along the full loop \[b\] from \[q\] to \[q\], along \[\alpha_k\] from \[q\] to \[p\], along \[a\] from \[p\] to \[s\], along \[\alpha_l\] from \[s\] to \[t\], along the full loop \[c\] from \[t\] to \[t\], along \[\alpha_l\] from \[t\] to \[s\], and finally along \[a\] from \[s\] to \[p\].

Class 2: \[s = p\] so that \[l = k\] and \[p, q, t\] are \([k\) are three distinct points. If \[\varepsilon_p(a) = +1\], then the loop \[b \circ_{q, p} a_{s, t} \circ c\] goes along \[\alpha_k\] from \[p\] to \[q\], along the full loop \[b\] from \[q\] to \[q\], along \[\alpha_k\] from \[q\] to \[p\], along the full loop \[c\] from \[t\] to \[t\], along \[\alpha_k\] from \[t\] to \[p\], and finally along the full loop \[a\] from \[p\] to \[p\]. If \[\varepsilon_p(a) = -1\], then the loop \[b \circ_{q, p} a_{s, t} \circ c\] goes along \[\alpha_k\] from \[p\] to \[q\], along the full loop \[b\] from \[q\] to \[q\], along \[\alpha_k\] from \[q\] to \[p\], along the full loop \[a\] from \[p\] to \[p\], along \[\alpha_l\] from \[p\] to \[t\], along the full loop \[c\] from \[t\] to \[t\], and finally along \[\alpha_k\] back to \[p\].

Similarly, the free homotopy class \[\langle (a \circ_{p, q} b) s c_t \rangle\] in \[(5.2.9)\] is represented by the loop \[a \circ_{p, q} b_{s, t} \circ c\] obtained by grafting \[a\] and \[c\] to \[b\] via a path in \[\alpha_k\] from \[p \in a \cap \alpha_k\] to \[q \in b \cap \alpha_k\] and a path in \[\alpha_l\] from \[s \in a \cap \alpha_l\] to \[t \in c \cap \alpha_l\]. A precise description of this loop involves three classes of triples \[(l, s, t) \in T_\omega(b, c)\] determined by whether or not \[s = q\]; we leave the details to the reader. Thus,

\[(5.2.12)\]
\[\rho_\omega(k, p, q) = \sum_{(l, s, t) \in T_\omega(b, c)} \varepsilon(\omega, l) \varepsilon_s(b) \varepsilon_t(c) \langle a \circ_{p, q} b_{s, t} \circ c \rangle.\]

Substituting these expansions of \(\lambda_\omega, \rho_\omega\) in \[(5.2.10)\], we obtain an expansion of \((x \bullet y) \bullet z\) as a sum. We call the summands determined by pairs \((k, p, q) \in T_\omega(a, b)\), \((l, s, t) \in T_\omega(a, c)\) with \(s \neq p\) the left 4-terms. The summands determined by pairs \((k, p, q) \in T_\omega(a, b)\), \((l, s, t) \in T_\omega(a, c)\) with \(s = p\) and \(l = k\) are called the left 3-terms. Similarly, the summands of \((x \bullet y) \bullet z\) determined by pairs \((k, p, q) \in T_\omega(a, b)\), \((l, s, t) \in T_\omega(b, c)\) with \(s \neq q\) are called the right 4-terms. The summands of \((x \bullet y) \bullet z\) determined by pairs \((k, p, q) \in T_\omega(a, b)\), \((l, s, t) \in T_\omega(b, c)\) with \(s = q\) and \(l = k\) are called the right 3-terms. Thus, \((x \bullet y) \bullet z\) is a sum of (left and right) 3-terms and 4-terms. Further, using \[(5.2.20)\], we expand \(u_\omega(x, y, z)\) as a sum of 3-terms and 4-terms. Combining with a parallel expansion of \(u_\omega(x, y, z)\) we get an expansion
of \( [x, y, z] = u_\omega(x, y, z) + u_\epsilon(x, y, z) \) as a sum of 3-terms and 4-terms. The total contribution of the 3-terms (respectively, 4-terms) to \( [x, y, z] \) is denoted by \( P_3(a, b, c) \) (respectively, \( P_4(a, b, c) \)). Thus,
\[
[x, y, z] = P_3(a, b, c) + P_4(a, b, c).
\]

To proceed, we simplify our notation. Note that each point \( p, q, s, t \) in a 4-term or a 3-term is traversed by exactly one of the loops \( a, b, c \). We will write \( \varepsilon_p, \varepsilon_q, \varepsilon_s, \varepsilon_t \) for the corresponding signs \( \pm 1 \). For example, \( \varepsilon_p = \varepsilon_p(a), \varepsilon_q = \varepsilon_q(b) \), etc. Also, set \( \varepsilon(\omega, k, l) = \varepsilon(\omega, k) \varepsilon(\omega, l) \). In this notation, the contribution of the left 4-terms to \((x \bullet y) \bullet z\) is equal to
\[
\lambda^{a,b,c}_{\omega} = \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle b \circ_{p,q} a_{s,t} \circ c \rangle.
\]
The contribution of the right 4-terms to \((x \bullet y) \bullet z\) is equal to
\[
\rho^{a,b,c}_{\omega} = \sum_{(k, p, q) \in T_\omega(a, b)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle a \circ_{p,q} b_{s,t} \circ c \rangle.
\]
Applying to the formula for \( \lambda^{a,b,c}_{\omega} \) the permutations
\[
a \mapsto b \mapsto c \mapsto a, \quad k \mapsto l \mapsto k, \quad p \mapsto s \mapsto q \mapsto t \mapsto p
\]
we get
\[
\lambda^{b,c,a}_{\omega} = \sum_{(l, s, t) \in T_\omega(b, c)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle c \circ_{t,s} b_{q,p} \circ a \rangle.
\]
Set
\[
\Delta^{a,b,c}_{\omega} = \rho^{a,b,c}_{\omega} + \lambda^{b,c,a}_{\omega} \quad \text{and} \quad \Delta a,b,c = \Delta^{a,b,c}_{\omega} + \Delta^{b,c,a}_{\omega}.
\]
The summands in the expansions of \( \rho^{a,b,c}_{\omega} \) and \( \lambda^{b,c,a}_{\omega} \) are given by the same formula as follows from the equality
\[
(\varepsilon_p, \varepsilon_q, \varepsilon_s, \varepsilon_t) = (\varepsilon_{p,s}, \varepsilon_{q,t}, \varepsilon_{s,p}, \varepsilon_{t,q})
\]
for \( q \neq s \). The summation in these two expansions goes over complementary sets of triples \((k, p, q)\) as the inclusion \((k, q, p) \in T_\omega(b, a)\) holds if and only if \((k, p, q) \in T(a, b) \setminus T_\omega(a, b)\). Therefore
\[
\Delta^{a,b,c}_{\omega} = \sum_{(k, p, q) \in T(a, b)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle a \circ_{p,q} b_{s,t} \circ c \rangle.
\]
Since \( \varepsilon(\omega, k, l) = \varepsilon(\omega, k) \), we deduce that
\[
\Delta^{a,b,c}_{\omega} = \sum_{(k, p, q) \in T(a, b)} \varepsilon(\omega, k, l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle a \circ_{p,q} b_{s,t} \circ c \rangle.
\]
Applying to this formula the label permutations
\[
a \mapsto c \mapsto a, \quad k \mapsto l \mapsto k, \quad p \mapsto t \mapsto p, \quad q \mapsto s \mapsto q
\]
we get
\[ \Delta^{c,b,a} = \sum_{(l,t,s) \in T(c,b)} \varepsilon(\omega,k,l) \varepsilon_p \varepsilon_q \varepsilon_s \varepsilon_t \langle c \circ_{t,s} b_{q,p} \circ a \rangle. \]
Comparing with the expression for \( \Delta^{a,b,c} \) above and using (5.2.13), we get
\[ \Delta^{a,b,c} = \Delta^{c,b,a}. \]
It is clear that the total contribution of the 4-terms to \( u_\omega(x,y,z) \) is equal to
\[ \lambda_\omega^{a,b,c} + \rho_\omega^{a,b,c} + \lambda_\omega^{b,c,a} + \rho_\omega^{b,c,a} + \lambda_\omega^{c,a,b} + \rho_\omega^{c,a,b} = \Delta^{a,b,c} + \Delta^{b,c,a} + \Delta^{c,a,b}. \]
Therefore
\[ P_4(a,b,c) = \Delta^{a,b,c} + \Delta^{b,c,a} + \Delta^{c,a,b}. \]
Formula (5.2.14) implies that \( P_4(a,b,c) = P_4(c,b,a) \). Therefore all 4-terms cancel out in \([x,y,z] - [z,y,x] \) and
\[ [x,y,z] - [z,y,x] = P_3(a,b,c) - P_3(c,b,a). \]
It remains to check that
\[ P_3(a,b,c) - P_3(c,b,a) = \mu^3(x,y,z) - \mu^3(z,y,x). \]
We will use the function \( \eta \) on the set \( \{ \pm 1 \} = \{-1,1\} \) defined by \( \eta(1) = 1 \) and \( \eta(-1) = 0 \). For any triple \( \varepsilon, \varepsilon', \varepsilon'' \in \{ \pm 1 \} \) set
\[ |\varepsilon, \varepsilon', \varepsilon''| = \varepsilon\varepsilon'\eta(\varepsilon'') + \varepsilon\varepsilon''\eta(\varepsilon') + \varepsilon'\varepsilon''\eta(\varepsilon) - \varepsilon\varepsilon'\varepsilon'' \in \mathbb{Z}. \]
Clearly, the integer \( |\varepsilon, \varepsilon', \varepsilon''| \) is invariant under permutations of \( \varepsilon, \varepsilon', \varepsilon'' \).
Observe that each 3-term in \( P_3(a,b,c) \) is associated with a certain \( k = l \in \pi_0 \) and a triple \( p \rightarrow a \cap \alpha_l, q \rightarrow b \cap \alpha_l, t \rightarrow c \cap \alpha_l \). Set
\[ P_{3,k}(a,b,c) = \sum_{p,q,t} |a \ b \ c \ p \ q \ t| = |\varepsilon_p, \varepsilon_q, \varepsilon_t| (\langle a_pb_q c_t \rangle + \langle c_t b_q a_p \rangle) - |\varepsilon_p \varepsilon_q \varepsilon_t| (c_t b_q a_p) \in M \]
where \( a_pb_q c_t \) is the product of the loops \( a_p, b_q, c_t \) formed by connecting their base points \( p,q,t \) by arbitrary paths in \( \alpha_k \). The loop \( c_t b_q a_p \) is defined similarly.
Let \( P_{3,k}(a,b,c) \) be the sum of the 3-terms in \([x,y,z] \) associated with \( k \in \pi_0 \). We prove below that
\[ P_{3,k}(a,b,c) = \sum_{p,q,t} |a \ b \ c \ p \ q \ t|. \]
Here and below \( p,q,t \) run respectively over the sets \( a \cap \alpha_l, b \cap \alpha_l, c \cap \alpha_l \). Observe that (5.2.19) implies (5.2.16). Indeed, permuting \( a \leftrightarrow c, p \leftrightarrow t \) in (5.2.19) we get
\[ P_{3,k}(c,b,a) = \sum_{p,q,t} |c \ b \ a \ p \ q \ t| + |\varepsilon_p \varepsilon_q \varepsilon_t| (c_t b_q a_p) - |\varepsilon_p \varepsilon_q \varepsilon_t| (a_p b_q c_t) \]
\[ = P_{3,k}(a,b,c) + \mu^3_k(z,y,x) - \mu^3_k(x,y,z). \]
Therefore
\[ P_{3,k}(a,b,c) - P_{3,k}(c,b,a) = \mu^3_k(x,y,z) - \mu^3_k(z,y,x). \]
Summing up over all \( k \in \pi_0 \) and using the obvious equality
\[ P_3(a,b,c) = \sum_{k \in \pi_0} P_{3,k}(a,b,c) \]
we get (5.2.16).
We now prove (5.2.19). Observe first that the formulas above simplify for the left 3-terms: \( \varepsilon(\omega, k, l) = \varepsilon(\omega, k) \varepsilon(\omega, l) = +1 \) as \( k = l \) and \( \varepsilon \varepsilon_p = +1 \) as \( s = p \). Thus, the contribution to \( (x \circ y) \circ z \) of the left 3-terms with the given \( k \) is equal to

\[
\mathcal{L}_{\omega,k}^{a,b,c} = \sum_{q < \omega, p, t < \omega} \varepsilon_q \varepsilon_t (b \circ_{q, p, t} a_p \circ c).
\]

The sum here runs over all \( p \in a \cap k, q \in b \cap l, t \in c \cap k \) satisfying the indicated inequalities which reformulate the conditions \( (k, p, q) \in T_{\omega}(a, b) \) and \( (l, s, t) \in T_{\omega}(a, c) \). The description of the loop \( b \circ_{q, p, t} a_p \circ c \) above shows that it is freely homotopic to \( a_p b_q c_t \) if \( \varepsilon_p = 1 \) and to \( c_t b_q a_p \) if \( \varepsilon_p = -1 \). Thus,

\[
\langle b \circ_{q, p, t} a_p \circ c \rangle = \eta(\varepsilon_p) \langle a_p b_q c_t \rangle + \eta(\varepsilon_p) - \varepsilon_p \rangle \langle c_t b_q a_p \rangle
\]

and therefore

\[
(5.2.20) \quad \mathcal{L}_{\omega,k}^{a,b,c} = \sum_{q < \omega, p, t < \omega} \varepsilon_q \varepsilon_t \langle \eta(\varepsilon_p) \langle a_p b_q c_t \rangle + (\eta(\varepsilon_p) - \varepsilon_p) \langle c_t b_q a_p \rangle \rangle.
\]

Cyclically permuting \( a, b, c \) and \( p, q, t \), we get

\[
(5.2.21) \quad \mathcal{L}_{\omega,k}^{b,c,a} = \sum_{q < \omega, p, t < \omega} \varepsilon_t \varepsilon_p \langle \eta(\varepsilon_q) \langle a_p b_q c_t \rangle + (\eta(\varepsilon_q) - \varepsilon_q) \langle c_t b_q a_p \rangle \rangle,
\]

\[
(5.2.22) \quad \mathcal{L}_{\omega,k}^{c,a,b} = \sum_{p < \omega, q, t < \omega} \varepsilon_q \varepsilon_p \langle \eta(\varepsilon_t) \langle a_p b_q c_t \rangle + (\eta(\varepsilon_t) - \varepsilon_t) \langle c_t b_q a_p \rangle \rangle.
\]

Similarly, the contribution to \( (x \circ y) \circ z \) of the right 3-terms with the given \( k \) is equal to

\[
\mathcal{P}_{\omega,k}^{a,b,c} = \sum_{t < \omega, q < \omega} \varepsilon_q \varepsilon_t \langle a_p \circ_{q, p, t} b_{q, t} \circ c \rangle
\]

where the conditions on \( p, q, t \) reformulate the inclusions \( (k, p, q) \in T_{\omega}(a, b) \) and \( (l, s, t) \in T_{\omega}(b, c) \). Since

\[
\langle a_p \circ_{q, p, t} b_{q, t} \circ c \rangle = \eta(\varepsilon_q) \langle c_t b_q a_p \rangle + (\eta(\varepsilon_q) - \varepsilon_q) \langle a_p b_q c_t \rangle
\]

we have

\[
(5.2.23) \quad \mathcal{P}_{\omega,k}^{a,b,c} = \sum_{t < \omega, q < \omega} \varepsilon_q \varepsilon_t \langle \eta(\varepsilon_q) \langle c_t b_q a_p \rangle + (\eta(\varepsilon_q) - \varepsilon_q) \langle a_p b_q c_t \rangle \rangle.
\]

Cyclically permuting \( a, b, c \) and \( p, q, t \), we get

\[
(5.2.24) \quad \mathcal{P}_{\omega,k}^{b,c,a} = \sum_{p < \omega, t < \omega} \varepsilon_t \varepsilon_q \langle \eta(\varepsilon_q) \langle c_t b_q a_p \rangle + (\eta(\varepsilon_q) - \varepsilon_q) \langle a_p b_q c_t \rangle \rangle,
\]

\[
(5.2.25) \quad \mathcal{P}_{\omega,k}^{c,a,b} = \sum_{q < \omega, p, t < \omega} \varepsilon_t \varepsilon_q \langle \eta(\varepsilon_p) \langle c_t b_q a_p \rangle + (\eta(\varepsilon_p) - \varepsilon_p) \rangle \langle a_p b_q c_t \rangle \rangle.
\]

The contribution of the 3-terms (with given \( k \)) to \( u_\omega(x, y, z) \) is the sum of the expressions \( 5.2.20 \)–\( 5.2.25 \). Then \( F_\omega(a, b, c) \) is the sum of these six expressions and of similar expressions obtained by replacing \( \omega \) with \( \overline{\omega} \). Under this replacement, the only change on the right-hand sides of \( 5.2.20 \)–\( 5.2.25 \) concerns the summation domain. For example, the summation domain in \( 5.2.20 \) changes from the set of triples \( p, q, t \) such that \( q < \omega, p, t < \omega \) to the set of triples \( p, q, t \) such that \( q < \overline{\omega}, p, t < \overline{\omega} \). The latter conditions may be rewritten as \( p < q, p < q, t \). Thus,

\[
(5.2.26) \quad \mathcal{L}_{\omega,k}^{a,b,c} = \sum_{p < \omega, q, p < \omega} \varepsilon_q \varepsilon_t \langle \eta(\varepsilon_p) \langle a_p b_q c_t \rangle + (\eta(\varepsilon_p) - \varepsilon_p) \rangle \langle c_t b_q a_p \rangle \rangle.
\]
this chord.

We claim that the sum of these twelve summands is equal to $a$.

In Case (b), only the\(\sum_{t<\omega}X\) equal to $P$ and pick their\(\sum_{t<\omega}X\) may be non-zero and their sum is easily computed to be equal to $\sum_{t<\omega}X$. This leaves us with three cases: (a) $t<\omega p$; (b) $p<\omega t<\omega q$, and (c) $q<\omega t$. In Case (a), only the $(p,q,t)$-summands of $L_{\omega,k}$, $P_{\omega,k}$, $L_{\omega,k}$ may be non-zero and their sum is

\[
\phi t \epsilon p (\epsilon(\omega) \phi p b q c t) + \epsilon t \epsilon p (\epsilon(\omega) - \epsilon q) (c t b q a p) \]

\[
+ \epsilon p \epsilon q (\epsilon(\omega) \epsilon t) (a p b q c t) + \epsilon p \epsilon q (\epsilon(\omega) - \epsilon t) (c t b q a p) \]

\[
+ \epsilon t \epsilon q (\epsilon(\omega) \epsilon t) (c t b q a p) + \epsilon t \epsilon q (\epsilon(\omega) - \epsilon t) (a p b q c t) = \begin{vmatrix} a & b & c \\ p & q & t \end{vmatrix}.
\]

In Case (b), only the $(p,q,t)$-summands of $L_{\omega,k}$, $P_{\omega,k}$, $L_{\omega,k}$ may be non-zero and their sum is easily computed to be equal to $\begin{vmatrix} a & b & c \\ p & q & t \end{vmatrix}$. In Case (c), only the $(p,q,t)$-summands of $L_{\omega,k}$, $P_{\omega,k}$, $L_{\omega,k}$ may be non-zero and again their sum is equal to $\begin{vmatrix} a & b & c \\ p & q & t \end{vmatrix}$. This proves the claim above and completes the proof of the theorem.

6. Proof of Theorem 3.4

6.1. Conventions. Set $\nu = \nu X : M \to M \otimes M$ and fix the gate orientation $\omega$ of $X$ such that $\epsilon(\omega, k) = 1$ for all $k \in \pi_0$. For points $p, q$ of a gate, we write $p < q$ if $p <\omega q$, i.e., if the $\omega$-positive direction on this gate leads from $p$ to $q$. Recall that a loop in $X$ is simple if it is transversal to the gates and has no self-intersections in $\Sigma' \subset X$. By Lemma 6.1, any loop in $X$ is freely homotopic to a simple loop. By Lemma 4.3 for a simple loop $a$ in $X$, we have

\[
\nu(a) = \sum_{(k,p_1,p_2) \in T_\omega(a)} \epsilon_{p_1} \epsilon_{p_2} (\langle a_{p_2,p_1} \rangle_0 \otimes \langle a_{p_1,p_2} \rangle_0 - \langle a_{p_1,p_2} \rangle_0 \otimes \langle a_{p_2,p_1} \rangle_0)
\]

where $\epsilon_p = \epsilon_p(a)$ for any point $p$ of a gate traversed by $a$. We say that the summand of (6.1.1) determined by the chord $(k,p_1,p_2) \in T_\omega(a)$ is obtained by splitting $a$ at this chord.
6.2. Proof of the theorem. It is enough to prove that

\[(I + Q + Q^2) \circ \nu^2(\langle a \rangle) = E \circ \gamma^\ast(\langle a \rangle)\]

for any simple loop \(a\) in \(X\). We compute \(\nu(\langle a \rangle)\) via (6.2.1). Note that the loops \(a_{p_1,p_2}, a_{p_2,p_1}\) in this formula are not transversal to \(\alpha_k\) as they contain subpaths connecting \(p_1,p_2 \in a \cap \alpha_k\) in \(\alpha_k\). We deform both loops pushing these subpaths into \(X \setminus \Sigma'\) while sliding their endpoints \(p_1,p_2\) into \(X \setminus \Sigma'\) along \(a(S^1)\). This gives simple loops \(a'_{p_1,p_2}, a'_{p_2,p_1}\) in \(X\) whose intersections with the gates are among the intersections of \(a\) with the gates. Therefore all chords of these two loops are among the chords of \(a\). Since \(\langle a'_{p_1,p_2} \rangle_0 = \langle a_{p_1,p_2} \rangle_0\) and \(\langle a'_{p_2,p_1} \rangle_0 = \langle a_{p_2,p_1} \rangle_0\), we have

\[(6.2.2) \quad \nu(\langle a \rangle) = \sum_{(k,p_1,p_2) \in T_\omega(a)} \varepsilon_{p_1} \varepsilon_{p_2} (\langle a'_{p_2,p_1} \rangle_0 \otimes \langle a'_{p_1,p_2} \rangle_0 - \langle a'_{p_1,p_2} \rangle_0 \otimes \langle a'_{p_2,p_1} \rangle_0).\]

Note that for a chord \((k,p_1,p_2) \in T_\omega(a)\), the point \(p_1 \in a \cap \alpha_k\) is traversed by the loop \(a'_{p_1,p_2}\) if \(\varepsilon_{p_1} = +1\) and by the loop \(a'_{p_2,p_1}\) if \(\varepsilon_{p_1} = -1\). The point \(p_2 \in a \cap \alpha_k\) is traversed by \(a'_{p_2,p_1}\) if \(\varepsilon_{p_2} = +1\) and by \(a'_{p_1,p_2}\) if \(\varepsilon_{p_2} = -1\).

It is clear that \(\nu^2(\langle a \rangle)\) is a sum of expressions obtained by splitting \(a\) along a chord and then splitting one of the resulting two loops (deformed as in the previous paragraph) along a chord. The summation runs over ordered pairs of chords of \(a\) arising in this way. We call such pairs splitting pairs. We will show that two chords of \(a\) are unlinked if (i) they have distinct endpoints and (ii) going along the loop \(a\) we meet consecutively both endpoints of one chord and then both endpoints of the other chord. It is clear that the chords in a splitting pair are either unlinked or have at least one common endpoint.

We claim that the sum of the terms of \(\nu^2(\langle a \rangle)\) derived from unlinked pairs of chords is annihilated by \(I + Q + Q^2\). Indeed, pick any unlinked pair of chords \((k,p_1,p_2),(l,u_1,u_2)\) of \(a\) (possibly, \(k = l\)). Then the pair \((l,u_1,u_2),(k,p_1,p_2)\) also is unlinked. We will show that the terms of \((I + Q + Q^2)\nu^2(\langle a \rangle)\) derived from these two pairs are opposite to each other. This will imply our claim. Consider first the case where the loop \(a\) traverses the points \(p_1,p_2,u_1,u_2\) in the cyclic order \(p_2,p_1,u_2,u_1\). Let \(b,c,d,e\) be respectively the paths in \(X\) going along \(a\) from \(p_2\) to \(p_1\), from \(p_1\) to \(u_2\), from \(u_2\) to \(u_1\), and from \(u_1\) to \(p_2\). Let \(f\) be the path going from \(p_1\) to \(p_2\) along \(\alpha_k\) and let \(g\) be the path going from \(u_1\) to \(u_2\) along \(\alpha_l\). The term of \(\nu(\langle a \rangle)\) obtained by splitting \(a\) at \((k,p_1,p_2)\) is equal to

\[(6.2.3) \quad \varepsilon_{p_1} \varepsilon_{p_2} (\langle b f \rangle_0 \otimes \langle c d e f^{-1} \rangle_0 - \langle c d e f^{-1} \rangle_0 \otimes \langle b f \rangle_0).\]

Consider the loop \(v = e f^{-1} c g^{-1}\). Splitting the left tensor factors in (6.2.3) at \((l,u_1,u_2)\) we get

\[(6.2.4) \quad \varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{u_1} \varepsilon_{u_2} (0 + \langle v \rangle_0 \otimes \langle d g \rangle_0 \otimes \langle b f \rangle_0 - \langle d g \rangle_0 \otimes \langle v \rangle_0 \otimes \langle b f \rangle_0)\]

where the zero term reflects the fact that \((l,u_1,u_2)\) is not a chord of the loop \(b f\).

Similarly, the summand of \(\nu^2(\langle a \rangle)\) arising by splitting \(a\) first at \((l,u_1,u_2)\) and then at \((k,p_1,p_2)\) is equal to

\[(6.2.5) \quad \varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{u_1} \varepsilon_{u_2} (\langle v \rangle_0 \otimes \langle b f \rangle_0 \otimes \langle d g \rangle_0 - \langle b f \rangle_0 \otimes \langle v \rangle_0 \otimes \langle d g \rangle_0)\]

Applying \(I + Q + Q^2\) to (6.2.4) and (6.2.5), we obtain opposite values. The cases where the points \(p_1,p_2,u_1,u_2\) are traversed by \(a\) in other possible cyclic orders \((p_2,p_1,u_1,u_2), (p_1,p_2,u_1,u_2),\) and \((p_1,p_2,u_2,u_1)\) are treated similarly.
We check next that a splitting pair of chords consisting of the same chord \((k, p_1, p_2)\) of \(a\) taken twice contributes zero to \(\nu^2(\langle a \rangle)\). If \(\varepsilon_{p_1} = \varepsilon_{p_2}\) then one of the points \(p_1, p_2\) is traversed by \(a'_{p_1, p_2}\) and the other one by \(a'_{p_2, p_1}\), so that \((k, p_1, p_2)\) is not a chord of these loops. Then the pair \((k, p_1, p_2), (k, p_1, p_2)\) is not a splitting pair of \(a\) and does not contribute to \(\nu^2(\langle a \rangle)\). If \(\varepsilon_{p_1} = -\varepsilon_{p_2}\) then both points \(p_1, p_2\) are traversed by the same loop, either \(a'_{p_1, p_2}\) or \(a'_{p_2, p_1}\). Splitting that loop at the chord \((k, p_1, p_2)\), we obtain two loops one of which lies in a neighborhood of \(\alpha_k\) and is contractible in \(X\). So, the corresponding term of \(\nu^2(\langle a \rangle)\) is equal to zero.

It remains to consider splitting pairs \((k, p_1, p_2), (l, u_1, u_2)\) consisting of chords of \(a\) with one common endpoint. Then \(k = l\) and our chords have 3 distinct endpoints. For any three distinct points of \(a \cap \alpha_k\) we compute the contribution to \(\nu^2(\langle a \rangle)\) of all associated splitting pairs. Label our points \(p_1, p_2, p_3\) so that \(p_1 \prec_\omega p_2 \prec_\omega p_3\). The splitting pairs in question are ordered pairs of distinct chords from the list \((k, p_1, p_2), (k, p_2, p_3), (k, p_1, p_3)\). Recall the function \(\eta\) on the set \(\{+1, -1\}\) defined by \(\eta(+1) = 1, \eta(-1) = 0\). In the following computations,

\[
x = \langle a'_{p_1, p_2} \rangle_0 \in M, \ y = \langle a'_{p_2, p_3} \rangle_0 \in M, \ z = \langle a'_{p_3, p_1} \rangle_0 \in M.
\]

Consider first the case where the loop \(a\) traverses the points \(p_1, p_2, p_3\) in the cyclic order \(p_1, p_2, p_3\). In this notation, the contribution of the chord \((k, p_1, p_2)\) to \(\nu(\langle a \rangle)\) is equal to

\[
\varepsilon_{p_1} \varepsilon_{p_2} \langle \langle a'_{p_1, p_2} \rangle_0 \otimes x - x \otimes \langle a'_{p_2, p_1} \rangle_0 \rangle.
\]

Clearly, the loop \(a'_{p_1, p_2}\) representing \(x\) does not traverse the point \(p_3\) and does not have \((k, p_2, p_3)\) as a chord. If \(\varepsilon_{p_2} = -1\), then the loop \(a'_{p_2, p_3}\) does not traverse \(p_2\) and does not have \((k, p_2, p_3)\) as a chord. Therefore the contribution of the splitting pair \((k, p_1, p_2), (k, p_2, p_3)\) to \(\nu^2(\langle a \rangle)\) is equal to

\[
\eta(\varepsilon_{p_2}) \varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{p_3} (z \otimes y - y \otimes z) \otimes x = \eta(\varepsilon_{p_2}) \varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{p_3} (z \otimes y - y \otimes z) \otimes x.
\]

Similarly, the splitting pair \((k, p_1, p_2), (k, p_1, p_3)\) contributes to \(\nu^2(\langle a \rangle)\) the term

\[
(1 - \eta(\varepsilon_{p_1})) \varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{p_3} (z \otimes y - y \otimes z) \otimes x.
\]

So, the splitting pairs starting with \((k, p_1, p_2)\) contribute to \(\nu^2(\langle a \rangle)\) the expression

\[
(\eta(\varepsilon_{p_1}) + \eta(\varepsilon_{p_2}) - 1) \varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{p_3} (z \otimes y \otimes x - y \otimes z \otimes x).
\]

Similarly, the contributions to \(\nu^2(\langle a \rangle)\) of the splitting pairs with first term \((k, p_2, p_3)\) and the second term \((k, p_1, p_2)\) or \((k, p_1, p_3)\) sum up to

\[
(\eta(\varepsilon_{p_2}) + \eta(\varepsilon_{p_3}) - 1) \varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{p_3} (z \otimes x \otimes y - x \otimes z \otimes y)
\]

and the contributions of the splitting pairs with first chord \((k, p_1, p_3)\) sum up to

\[
(\eta(\varepsilon_{p_1}) + 1 - \eta(\varepsilon_{p_3})) \varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{p_3} (x \otimes y \otimes z - y \otimes z \otimes x).
\]

Applying \(I + Q + Q^2\) to the sum of these three expressions we get

\[
(6.2.6) \quad \varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{p_3} (I + Q + Q^2)(x \otimes y \otimes z - z \otimes y \otimes x).
\]

By definition, \(\gamma^3(\langle a \rangle) \in M_{g \geq 3}^\otimes\) is a sum whose terms are numerated by triples of distinct points of \(S^1\) carried by \(a : S^1 \to X\) to the same gate. The term associated with the triple \(p_1, p_2, p_3\) above is

\[
\varepsilon_{p_1} \varepsilon_{p_2} \varepsilon_{p_3} (x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y).
\]
Applying the operator $E$ (defined by (2.4.1)) to this expression we again get (6.2.6). Thus, the triple $p_1, p_2, p_3$ contributes to both sides of (6.2.1). Similar computations show that this is also true when the points $p_1, p_2, p_3$ are traversed by $a$ in the cyclic order $p_3, p_2, p_1$. Summing up over all $k \in \pi_0$ and all triples $p_1, p_2, p_3 \in a \cap \alpha_k$ with $p_1 < p_2 < p_3$, we get (6.2.1).

7. **Proof of Theorem 4.5**

We start by computing $\partial \nu(x \otimes y)$ for arbitrary $x, y \in L(X) \subset M$. Recall that

$$\partial \nu(x \otimes y) = \nu([x, y]) - ad_x(\nu(y)) + ad_y(\nu(x)).$$

Since all terms on the right-hand side involve two of our (co)bracket operations, they expand as sums of terms determined by ordered pairs of chords. To make this expansion explicit, we use notation of Section 6.1 and represent $x, y$ by simple loops $a, b$ which do not meet in $\Sigma$. For any chord $(k, p, q) \in T(a, b)$, consider the simple loop $a \circ_{p, q} b$ in $X$ obtained by deformation of the loop $a_p b_q$ as in the proof of Theorem 4.2 in Section 5.2. Formula (5.2.7) and the equality $\langle a \circ_{p, q} b \rangle = b \circ_{q, p} a$ imply that

$$[x, y] = x \cdot y - y \cdot x = \sum_{(k, p, q) \in T(a, b)} \epsilon_p \epsilon_q \langle a \circ_{p, q} b \rangle - \sum_{(k, p, q) \in T(b, a)} \epsilon_p \epsilon_q \langle b \circ_{p, q} a \rangle$$

$$= \sum_{(k, p, q) \in T(a, b)} \delta(p, q) \epsilon_p \epsilon_q \langle a \circ_{p, q} b \rangle$$

where $\epsilon_p = \epsilon_p(a)$, $\epsilon_q = \epsilon_q(b)$, and

$$\delta(p, q) = \begin{cases} +1 & \text{if } q < p, \\ -1 & \text{if } p < q. \end{cases}$$

Therefore

$$\nu([x, y]) = \sum_{(k, p, q) \in T(a, b)} \delta(p, q) \epsilon_p \epsilon_q \nu(\langle a \circ_{p, q} b \rangle).$$

By (6.1.1), $\nu(\langle a \circ_{p, q} b \rangle)$ is a sum of terms numerated by the chords of $a \circ_{p, q} b$. Thus, $\nu([x, y])$ is a sum of terms numerated by pairs $(a, b)$, a chord of the loop $a \circ_{p, q} b$. We call a chord $(k, p, q)$ of the pair $(a, b)$ *positive* if $p < q$. For all $l \in \pi_0$, we have

$$(7.0.1) \quad (a \circ_{p, q} b) \cap \alpha_l = (a \cap \alpha_l) \cup (b \cap \alpha_l).$$

Therefore a chord of the loop $a \circ_{p, q} b$ is either a chord of $a$, or a chord of $b$, or a positive chord of the pair $(a, b)$, or a positive chord of the pair $(b, a)$. Next we use Formula (6.2.2) to compute $\nu(x)$. Since the crossings of the gates with the loops $a'_{p_1, p_2}, a'_{p_2, p_3}$ appearing in (6.2.2) are among the crossings of the gates with $a$, the brackets $[y, \langle a'_{p_1, p_2} \rangle]$ and $[y, \langle a'_{p_2, p_3} \rangle]$ are sums of expressions numerated by chords of the pair $(b, a)$. So, $ad_y(\nu(x))$ is a sum of terms numerated by pairs $(a \circ b, a)$, a chord of $(b, a))$. Similarly, $ad_x(\nu(y))$ is a sum of terms numerated by pairs $(a \circ b, a)$, a chord of $(a, b))$. We conclude that there are five types of ordered pairs of chords which may contribute to $\partial \nu(x \otimes y)$:

(i) (a chord of $(a, b)$, a chord of $a$);

(ii) (a chord of $(a, b)$, a chord of $b$);

(iii) (a chord of $(a, b)$, a positive chord of $(a, b)$ or of $(b, a)$);

(iv) (a chord of $a$, a chord of $(b, a)$).
(v) (a chord of \( b \), a chord of \( (a,b) \)).

Here pairs of types (i)–(iii) contribute to \( \nu([x,y]) \) and pairs of types (iv), (v) contribute respectively to \( \text{ad}_y(\nu(x)) \), \( \text{ad}_x(\nu(y)) \). In the sequel, the contribution of an ordered pair of chords \( s \) to \( \partial \nu(x \otimes y) \) is denoted by \( |s| \).

Pick now a set \( S \subset \bigcup \alpha_k \) such that there is an ordered pair of chords in the list (i)–(v) whose set of endpoints is equal to \( S \). We call such a pair of chords an \( S \)-pair. The existence of an \( S \)-pair implies that \( 2 \leq \text{card}(S) \leq 4 \) and \( S \) meets both \( a(S^1) \) and \( b(S^1) \). Set \( |S| = \sum_s |s| \) where \( s \) runs over all \( S \)-pairs.

Claim: \( \text{card}(S) = 4 \Rightarrow |S| = 0 \). We first consider the case where \( S \) meets both \( a(S^1) \) and \( b(S^1) \) in two points. Then all \( S \)-pairs have type (iii) and \( S \) consists of four distinct points

\[
p \in a \cap \alpha_k, \quad q \in b \cap \alpha_k, \quad p' \in a \cap \alpha_l, \quad q' \in b \cap \alpha_l
\]

with \( k, l \in \pi_0 \) (possibly, \( k = l \)). We let \( c_{p',q}' \) be the loop going from \( p' \) to \( q' \) along \( a \circ_{p,q} b \) and then back to \( p' \) along the gate \( \alpha_l \). Viewed up to homotopy, this loop goes along \( a \) from \( p' \) to \( p \), then along \( \alpha_k \) to \( q \), then along \( b \) to \( q' \), and then along \( \alpha_l \) back to \( p' \). Similarly, we let \( c_{q',p}' \) be the loop going from \( q' \) to \( p' \) along \( a \circ_{p,q} b \) and then back to \( q' \) along \( \alpha_l \). Viewed up to homotopy, this loop goes along \( b \) from \( q' \) to \( q \), then along \( \alpha_k \) to \( p \), then along \( a \) to \( p' \), and then along \( \alpha_l \) back to \( q' \).

Note that there is just one \( S \)-pair, \( s \), with the first term \( (k,p,q) \). Namely,

\[
|s| = \delta(p,q) \delta(p',q') \varepsilon_p \varepsilon_q \varepsilon_{p'} \varepsilon_{q'} (c_{p',q}'_0) \otimes (c_{q',p}'_0) - (c_{p,q}'_0) \otimes (c_{q',p}'_0)
\]

where \( \varepsilon_p = \varepsilon_p(a) \), \( \varepsilon_{p'} = \varepsilon_{p'}(a) \), \( \varepsilon_q = \varepsilon_q(b) \), \( \varepsilon_{q'} = \varepsilon_{q'}(b) \). We rewrite this as

\[
|s| = \delta(p,q) \delta(p',q') \varepsilon_p \varepsilon_q \varepsilon_{p'} \varepsilon_{q'} \mathcal{T}((c_{p',q}'_0) \otimes (c_{q',p}'_0)).
\]

Similarly, there is a unique \( S \)-pair \( t \) with the first term \( (l,p',q') \) and

\[
|t| = \delta(p,q) \delta(p',q') \varepsilon_p \varepsilon_q \varepsilon_{p'} \varepsilon_{q'} \mathcal{T}((c_{p,q}'_0) \otimes (c_{q,p}'_0)).
\]

The obvious equalities \( c_{p,q}'_0 = (c_{p,q}'_0)_0 \) and \( c_{q,p}'_0 = (c_{q,p}'_0)_0 \) imply that \( |s| = -|t| \). If \( k \neq l \), then there are no other \( S \)-pairs and \( |S| = |s| + |t| = 0 \). If \( k = l \), then there are two more \( S \)-pairs \( s' \) and \( t' \) with the first terms respectively \( (k,p,q) \) and \( (k,p',q) \). The argument above (with \( p \) replaced by \( p' \) and vice versa) shows that \( |s'| = -|t'| \). So, \( |S| = |s| + |t| + |s'| + |t'| = 0 \).

Suppose now that our 4-element set \( S \) meets \( a(S^1) \) in three points and meets \( b(S^1) \) in one point. The existence of an \( S \)-pair implies that \( S \) consists of points

\[
p \in a \cap \alpha_k, \quad q \in b \cap \alpha_k, \quad p' \in a \cap \alpha_l
\]

for some \( k, l \in \pi_0 \). Assume first that \( k \neq l \) and choose \( p', p'' \) so that \( p' < p'' \). Then there are only two \( S \)-pairs of chords:

\[
(7.0.2) \quad s = ((k,p,q),(l,p',q')) \quad \text{and} \quad t = ((l,p',p''),(k,q,p))
\]

of types respectively (i) and (iv). If the points \( p,p',p'' \) are traversed by \( a \) in this cyclic order, then

\[
|s| = \delta(p,q) \varepsilon_p \varepsilon_q \varepsilon_{p'} \varepsilon_{p''} \mathcal{T}((a_{p',q'})_0) \otimes (a_{p',p''})_0).
\]
If the points \( p, p', p'' \) are traversed by \( a \) in the opposite cyclic order, then
\[
|s| = \delta(p, q) \varepsilon_p \varepsilon_q \varepsilon_{p'} \overline{F}((a_{p', p})_0 \otimes (a_{p', p'})_0 b_q)_0.
\]

At the same time, the contribution of the chord \((l, p', p'')\) of \( a \) to \( \nu(x) \) is equal to
\[
\varepsilon_{p'} \varepsilon_{p''} \overline{F}((a_{p', p'})_0 \otimes (a_{p', p''})_0).
\]

As a consequence, \(|t| = -|s|\) where the minus is due to the minus in the formula \( \delta(q, p) = -\delta(p, q) \). Thus, \(|S| = |s| + |t| = 0\). If \( k = l \), then we label the three points of the set \( S \cap a(S^1) \) by \( p, p', p'' \) so that \( p < p' < p'' \). Here, besides the \( S \)-pairs \( s, t \) as in (7.0.2), we have four more \( S \)-pairs
\[
\begin{align*}
& s' = ((k, p', q), (l, p, p''))), \\
& t' = ((l, p, p''), (k, q, p'))), \\
& s'' = ((k, p'', q), (l, p, p')), \\
& t'' = ((l, p, p'), (k, q, p'')).
\end{align*}
\]

The same computations as above yield
\[
|s| + |t| = |s'| + |t'| = |s''| + |t''| = 0.
\]

Hence,
\[
|S| = |s| + |t| = |s'| + |t'| = |s''| + |t''| = 0.
\]

The case where \( S \) meets \( a(S^1) \) in one point and meets \( b(S^1) \) in three points is treated similarly. This completes the proof of our claim \( \text{card}(S) = 4 \Rightarrow |S| = 0 \).

Consider now the case \( \text{card}(S) = 2 \). Then \( S \) consists of the endpoints of a chord \((k, p, q)\) of \((a, b)\). Suppose first that \( p < q \). Then the only \( S \)-pair is \( s = ((k, p, q), (k, p, q)) \). The chord \((k, p, q)\) contributes to \([x, y]\) the expression
\[
\delta(p, q) \varepsilon_p \varepsilon_q \langle a \circ_{p, q} b \rangle = -\varepsilon_p \varepsilon_q \langle a \circ_{p, q} b \rangle.
\]

The same chord \((k, p, q)\) is a chord of the loop \( a \circ_{p, q} b \) and splitting \( a \circ_{p, q} b \) at it we obtain two loops. If \( \varepsilon_p = -\varepsilon_q \), then one of them is contractible and therefore \(|S| = |s| = 0\). If \( \varepsilon_p = \varepsilon_q \), then these two loops are freely homotopic to \( a, b \).

Analyzing separately the cases \( \varepsilon_p = \varepsilon_q = +1 \) and \( \varepsilon_p = \varepsilon_q = -1 \) we obtain that
\[
|S| = |s| = \varepsilon_p \overline{F}(a)_0 \otimes (b)_0 = \varepsilon_p \overline{F}(x \otimes y) \mod(Re \otimes M + M \otimes Re).
\]

Thus, modulo \( Re \otimes M + M \otimes Re \), we have
\[
|S| = \begin{cases} 0 & \text{if } \varepsilon_p = -\varepsilon_q, \\ \varepsilon_p \overline{F}(x \otimes y) & \text{if } \varepsilon_p = \varepsilon_q. \end{cases}
\]

In the case \( q < p \) there is only one \( S \)-pair \((k, p, q), (k, q, p))\) and similar computations again give (7.0.3).

Finally, assume that \( \text{card}(S) = 3 \), i.e., that \( S \) consists of the endpoints of two distinct chords sharing one endpoint. Then \( S \) is contained in a single gate \( \alpha_k \) and meets one of the sets \( a(S^1), b(S^1) \) in two points and the other set in one point. Suppose first that \( S = \{p, p', q\} \) for some \( p, p' \in a \cap \alpha_k, q \in b \cap \alpha_k \) with \( p < p' \). Consider the loops \( u = a_{p, p'}, v' = a_{p', p}, v = b_q \) based in \( p, p', q \) respectively. We view the gate \( \alpha_k \) as a “big” base point of these loops; this allows us to form the products \( uu', uv', \) etc. For example, in this notation \( \langle a \circ_{p, q} b \rangle = \langle uu'v \rangle \) and \( \langle a \circ_{p', q} b \rangle = \langle u'uv \rangle \). Set
\[
U = \langle u \rangle \otimes \langle u'v \rangle = \langle u \rangle \otimes \langle vu' \rangle \in M \otimes M
\]

and
\[
U' = \langle u' \rangle \otimes \langle uv \rangle = \langle u' \rangle \otimes \langle vu \rangle \in M \otimes M.
\]
We claim that modulo $Re \otimes M + M \otimes Re$ we have

$|S| = \varepsilon_p \varepsilon_{p'} \varepsilon_q \mathcal{P}(U + U').$  

(7.0.4)

To see it, we list all $S$-pairs of chords. Pairs of type (i):

$s = ((k, p, q), (k, p, p'))$ and $s' = ((k, p', q), (k, p, p')).$

Pairs of types (ii) and (v): none. Pairs of type (iii):

$t = \begin{cases} 
((k, p, q), (k, p', q)) & \text{if } p' < q, \\
((k, p, q), (k, q, p')) & \text{if } q < p',
\end{cases}

$\text{and}$

$t' = \begin{cases} 
((k, p', q), (k, p, q)) & \text{if } p < q, \\
((k, p', q), (k, q, p)) & \text{if } q < p.
\end{cases}$

Pairs of type (iv):

$w = ((k, p, p'), (k, q, p))$ and $w' = ((k, p, p'), (k, q, p'))$. We compute the contributions of these six $S$-pairs to $\partial \nu(x \otimes y)$. All the computations to follow proceed modulo $Re \otimes M + M \otimes Re$. As we know, the contribution of the chord $(k, p, q)$ to $[x, y]$ is equal to

$|s| = -\delta(p, q) \varepsilon_p \varepsilon_{p'} \varepsilon_q \times \begin{cases} 
\mathcal{P}(U') & \text{if } \varepsilon_p = -1, \\
\mathcal{P}(U) & \text{if } \varepsilon_p = +1.
\end{cases}$

Similarly, the chord $(k, p', q)$ contributes $\delta(p', q) \varepsilon_{p'} \varepsilon_q \langle a \circ_{p, q} b \rangle$ to $[x, y]$. Applying $\nu$ and focusing on the contribution of the chord $(k, p, p')$, we get

$|s'| = \delta(p', q) \varepsilon_p \varepsilon_{p'} \varepsilon_q \times \begin{cases} 
\mathcal{P}(U) & \text{if } \varepsilon_{p'} = -1, \\
\mathcal{P}(U') & \text{if } \varepsilon_{p'} = +1.
\end{cases}$

Next, applying $\nu$ to (7.0.5) and focusing on the contribution of the chord $(k, p', q)$ if $p' < q$ and the chord $(k, q, p')$ if $q < p'$, we get

$|t| = \delta(p, q) \delta(p', q) \varepsilon_p \varepsilon_{p'} \varepsilon_q \times \begin{cases} 
\mathcal{P}(U) & \text{if } \varepsilon_q = -1, \\
\mathcal{P}(U') & \text{if } \varepsilon_q = +1.
\end{cases}$

This computation of $|t|$ does not use the assumption $p < p'$; therefore exchanging $p, U$ with $p', U'$, respectively, we get

$|t'| = \delta(p, q) \delta(p', q) \varepsilon_p \varepsilon_{p'} \varepsilon_q \times \begin{cases} 
\mathcal{P}(U) & \text{if } \varepsilon_q = -1, \\
\mathcal{P}(U') & \text{if } \varepsilon_q = +1.
\end{cases}$

By definition, the contribution of the chord $(k, p, p')$ to $\nu(x)$ is equal to

$(7.0.6) \quad \varepsilon_p \varepsilon_{p'} (\langle u' \rangle_0 \otimes \langle u \rangle_0 - \langle u \rangle_0 \otimes \langle u' \rangle_0)$. $

Applying $ad_y$ and focusing on the contribution of the chord $(k, q, p)$, we get

$|w| = -\delta(p, q) \varepsilon_p \varepsilon_{p'} \varepsilon_q \times \begin{cases} 
\mathcal{P}(U) & \text{if } \varepsilon_p = -1, \\
\mathcal{P}(U') & \text{if } \varepsilon_p = +1.
\end{cases}$

Similarly, the chord $(k, p', q)$ contributes

$\delta(p', q) \varepsilon_{p'} \varepsilon_q \langle a \circ_{p, q} b \rangle$.
where we use the identity $\delta(q, p) = -\delta(p, q)$. Applying $\alpha d_y$ to (7.0.6) and focusing on the contribution of the chord $(k, q, p')$, we similarly get

$$|w'| = \delta(p', q) \varepsilon_p \varepsilon_{p'} \varepsilon_q \times \begin{cases} \overline{\mathcal{P}}(U') & \text{if } \varepsilon_{p'} = -1, \\
\mathcal{P}(U) & \text{if } \varepsilon_{p'} = +1. \end{cases}$$

From these formulas we deduce that

(7.0.7) \quad |s| + |w| = -\delta(p, q) \varepsilon_p \varepsilon_{p'} \varepsilon_q \overline{\mathcal{P}}(U + U'),

(7.0.8) \quad |s'| + |w'| = \delta(p', q) \varepsilon_p \varepsilon_{p'} \varepsilon_q \overline{\mathcal{P}}(U + U'),

(7.0.9) \quad |t| + |t'| = \delta(p, q) \delta(p', q) \varepsilon_p \varepsilon_{p'} \varepsilon_q \overline{\mathcal{P}}(U + U').

Note that

$$-\delta(p, q) + \delta(p', q) + \delta(p, q) \delta(p', q) = 1$$

as either $q < p'$ and then $\delta(p', q) = 1$ or $p < p' < q$ and then $\delta(p, q) = -1$. Therefore adding up the equalities (7.0.7) - (7.0.9) we obtain (7.0.4).

It remains to consider the case where $S = \{p, q, q'\}$ for some $p \in a \cap \alpha_k, q, q' \in b \cap \alpha_k$ with $q < q'$. Consider the loops $u = a_p, v = b_{q, q'}, v' = b_{q', q}$ based respectively at $p, q, q'$. As above, we view the gate $\alpha_k$ as a “big” base point of these loops. Set

$$V = \langle \iota \rangle \otimes \langle \iota' \rangle = \langle \iota \rangle \otimes \langle \iota' \rangle \iota \iota' \rangle = M \otimes M$$

and

$$V' = \langle \iota' \rangle \otimes \langle \iota u \rangle = \langle \iota' \rangle \otimes \langle \iota' \rangle \iota \iota' \rangle = M \otimes M.$$

Then modulo $Re \otimes M + M \otimes Re$ we have

(7.0.10) \quad |S| = -\varepsilon_p \varepsilon_q \varepsilon_{q'} \overline{\mathcal{P}}(V + V').

This follows from the observation that the equality $\partial \nu(x \otimes y) = -\partial \nu(y \otimes x)$ holds term-wise so that the contribution of $S$ to $\partial \nu(x \otimes y)$ is obtained by negation from the contribution of $S$ to $\partial \nu(y \otimes x)$. By (7.0.4), the latter is equal to $\varepsilon_p \varepsilon_q \varepsilon_{q'} \overline{\mathcal{P}}(V + V')$. This implies (7.0.10).

Let $\sigma_0$ be the total contribution to $\partial \nu(x \otimes y)$ of the 2-element sets $S$ consisting of the endpoints of a chord of $(a, b)$. Summing up the equalities (7.0.3) over all such $S$ we obtain that

$$\sigma_0 = \sum_{k \in \pi_0} (n_k^+ - n_k^-) \overline{\mathcal{P}}(x \otimes y)$$

where $n_k^\pm$ is the number of pairs $(p \in a \cap \alpha_k, q \in b \cap \alpha_k)$ with $\varepsilon_p(a) = \varepsilon_q(b) = \pm$. In the notation of Section 3.3 applied to $C = \alpha_k$ we have

$$n_k^+ - n_k^- = \frac{|a|_{\alpha_k} (b \cdot \alpha_k) + |b|_{\alpha_k} (a \cdot \alpha_k)}{2}.$$

Therefore

$$2\sigma_0 = \sum_{k \in \pi_0} (|a|_{\alpha_k} (b \cdot \alpha_k) + |b|_{\alpha_k} (a \cdot \alpha_k)) \overline{\mathcal{P}}(x \otimes y) = \sigma_0' - \sigma_0''$$

where

$$\sigma_0' = \sum_{k \in \pi_0} |a|_{\alpha_k} (b \cdot \alpha_k) \overline{\mathcal{P}}(x \otimes y) \quad \text{and} \quad \sigma_0'' = \sum_{k \in \pi_0} |b|_{\alpha_k} (a \cdot \alpha_k) \overline{\mathcal{P}}(y \otimes x).$$
Let $\sigma_1$ be the total contribution to $\partial \nu (x \otimes y)$ of all sets $S$ consisting of two points of $a \cap \alpha_k$ and a point of $b \cap \alpha_k$ for some $k \in \pi_0$. Summing up the equalities (7.0.4) over all such $S,k$, we obtain that (modulo $Re \otimes M + M \otimes Re$)

$$\sigma_1 = \sum_{k \in \pi_0} \sum_{p_1,p_2 \in a \cap \alpha_k \atop p_1 \neq p_2 \atop q \in \beta \cap \alpha_k} \varepsilon_{p_1}(a)\varepsilon_{p_2}(a)\varepsilon_q(b) \mathcal{T}(\langle a_{p_1,p_2} \rangle \otimes \langle a_{p_2,p_1} \circ a_q \rangle).$$

Similarly, Formula (7.0.10) implies that the total contribution to $\partial \nu (x \otimes y)$ of all sets $S$ consisting of a point of $a \cap \alpha_k$ and two points of $b \cap \alpha_k$ for some $k$ is equal (modulo $Re \otimes M + M \otimes Re$) to $-\sigma_2$ where

$$\sigma_2 = \sum_{k \in \pi_0} \sum_{p \in a \cap \alpha_k} \sum_{q_1,q_2 \in b \cap \alpha_k \atop q_1 \neq q_2} \varepsilon_p(a)\varepsilon_{q_1}(b)\varepsilon_{q_2}(b) \mathcal{T}(\langle b_{q_1,q_2} \rangle \otimes \langle b_{q_2,q_1} \circ a_p \rangle).$$

Since $|S| = 0$ for all other $S$, we have

$$\partial \nu (x \otimes y) = \sigma_0 + \sigma_1 - \sigma_2 \mod (Re \otimes M + M \otimes Re).$$

Therefore

$$2 \partial \nu (x \otimes y) = (\sigma'_0 + 2\sigma_1) - (\sigma''_0 + 2\sigma_2) \mod (Re \otimes M + M \otimes Re).$$

In terms of the bi-endomorphism $\zeta$ of $M$, we have

$$\sigma'_0 + 2\sigma_1 = \mathcal{T}\zeta (x \otimes y) \quad \text{and} \quad \sigma''_0 + 2\sigma_2 = \mathcal{T}\zeta (y \otimes x).$$

Thus, modulo $Re \otimes M + M \otimes Re$, we have

$$2 \partial \nu (x \otimes y) = \mathcal{T}\zeta (x \otimes y - y \otimes x) = \mathcal{T}\zeta \mathcal{T}\zeta (x \otimes y) = \delta(\zeta)(x \otimes y).$$

So, $\delta(\zeta) = 2 \partial \nu$ modulo $Re \otimes M + M \otimes Re$.

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