

**Abstract.** In this paper, we consider nonlocal Schrödinger equations with certain potentials $V$ given by an integro-differential operator $L_K$ as follows:

$$L_K u + V u = f \text{ in } \mathbb{R}^n$$

where $V \in \text{RH}^q$ for $q > \frac{n}{2s}$ and $0 < s < 1$. We denote the solution of the above equation by $S_V f := u$, which is called the inverse of the nonlocal Schrödinger operator $L_K + V$ with potential $V$; that is, $S_V = (L_K + V)^{-1}$. Then we obtain a weak Harnack inequality of weak subsolutions of the nonlocal equation

$$\begin{cases}
L_K u + V u = 0 & \text{in } \Omega, \\
u = g & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

where $g \in H^s(\mathbb{R}^n)$ and $\Omega$ is a bounded open domain in $\mathbb{R}^n$ with Lipschitz boundary, and also get an improved decay of a fundamental solution $e^V$ for $L_K + V$. Moreover, we obtain $L^p$ and $L^p - L^q$ mapping properties of the inverse $S_V$ of the nonlocal Schrödinger operator $L_K + V$.

**Contents**

1. Introduction
2. Preliminaries
3. The fractional auxiliary function $m_V(x)$
4. A weak Harnack inequality
5. Weak solutions and Caccioppoli estimate for nonlocal Schrödinger operators $L_V$
6. $L^p$ and $L^p - L^q$ mapping properties of the inverse of the nonlocal Schrödinger operator
7. Appendix
8. References

1. **Introduction**

Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ with Lipschitz boundary. Then we introduce integro-differential operators of form

$$L_K u(x) = \frac{1}{2} \text{ p.v. } \int_{\mathbb{R}^n} \mu(u, x, y)K(y) \, dy, \ x \in \Omega,$$

where $V \in \text{RH}^q$ for $q > \frac{n}{2s}$ and $0 < s < 1$. We denote the solution of the above equation by $S_V f := u$, which is called the inverse of the nonlocal Schrödinger operator $L_K + V$ with potential $V$; that is, $S_V = (L_K + V)^{-1}$. Then we obtain a weak Harnack inequality of weak subsolutions of the nonlocal equation

$$\begin{cases}
L_K u + V u = 0 & \text{in } \Omega, \\
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\end{cases}$$

where $g \in H^s(\mathbb{R}^n)$ and $\Omega$ is a bounded open domain in $\mathbb{R}^n$ with Lipschitz boundary, and also get an improved decay of a fundamental solution $e^V$ for $L_K + V$. Moreover, we obtain $L^p$ and $L^p - L^q$ mapping properties of the inverse $S_V$ of the nonlocal Schrödinger operator $L_K + V$.

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Theorem 1.1. Let for any \(\phi\) \(K\) interested in \(L\) functions \(v\) (1.4) for any \(x,y\) weak solution is a Schrödinger operator with nonnegative potential \(V\) with potential \(V\), such that \(L_K = (-\Delta)^s\) is the fractional Laplacian and it is well-known that \(\lim_{s \to 1^-} (-\Delta)^s u = -\Delta u\) for any function \(u\) in the Schwartz space \(S(\mathbb{R}^n)\).

We focus our attention on the nonlocal Schrödinger operator \(L_V = L_K + V\) with potential \(V\); as a matter of fact, we consider the nonlocal Schrödinger equation with potential \(V\) given by

\[
\begin{cases}
L_V u = 0 & \text{in } \Omega, \\
u = g & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

where \(K \in \mathcal{K}\) and \(V\) is a nonnegative potential with \(V \in L^1_{\text{loc}}(\mathbb{R}^n)\). Then we are interested in \(L^p\)-estimates and \(L^p-L^q\) estimates for the inverse \(S_V\) of the nonlocal Schrödinger operator with nonnegative potential \(V\) to be given in Section 6.

Let \(\Omega\) be a bounded open domain in \(\mathbb{R}^n\) with Lipschitz boundary and let \(K \in \mathcal{K}\). Let \(X(\Omega)\) be the linear function space of all real-valued Lebesgue measurable functions \(v\) on \(\mathbb{R}^n\) such that \(v|_{\Omega} \in L^2(\Omega)\) and

\[
\iint_{\mathbb{R}^{2n}\setminus(\Omega^c \times \Omega^c)} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dx \, dy < \infty.
\]

Set \(X_0(\Omega) = \{v \in X(\Omega) : v = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}\). For \(g \in H^s(\mathbb{R}^n)\), we consider the convex set of \(H^s(\mathbb{R}^n)\) given by \(X_0(\Omega) = \{v \in H^s(\mathbb{R}^n) : g - v \in X_0(\Omega)\}\).

Let \(V \in RH^q\) for \(q > \frac{n}{2s}\) and \(s \in (0,1)\). Then we say that a function \(u \in X_0(\Omega)\) is a weak solution of the nonlocal equation (1.4), if it satisfies the weak formulation

\[
\iint_{\mathbb{R}^{2n}\setminus(\Omega^c \times \Omega^c)} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y) \, dx \, dy + \langle Vu, \varphi \rangle_{L^2(\Omega)} = 0
\]

for any \(\varphi \in X_0(\Omega)\).

We now state our main theorems as follows.

Theorem 1.1. Let \(V \in RH^q\) for \(q > \frac{n}{2s}\) with \(s \in (0,1)\) and \(n \ge 2\). Then there are constants \(\varepsilon, C > 0\) depending only on \(n, s\), and \(s\) such that

\[
0 \le \varepsilon(x-y) \le C \frac{1}{\Xi(\varepsilon + 1 + \frac{1}{2} |x-y| \, m_V(x))^{\frac{2s}{n+2s}}} |x-y|^{n-2s}
\]

for any \(x,y \in \mathbb{R}^n\), where \(\Xi(x) = \sum_{k=0}^\infty x^k/(k!)^{1+s}, d_0 > 0\) is the constant given in Lemma 3.2 and \(m_V\) is the fractional auxiliary function to be given in Section 3.
The main step in proving Theorem 1.1 is to obtain an improved version of the weak Harnack inequality for weak solutions of the equation (1.4) as follows. To get this, certain type of Caccioppoli estimates for weak solutions of the equation (1.4) to be obtained in Section 5 will play an important role.

**Theorem 1.2.** Let \( s \in (0, 1) \) and \( x_0 \in \Omega \). If \( u \) is a nonnegative weak solution of the equation (1.4) in \( \Omega \), then there are universal constants \( \varepsilon, C > 0 \) depending only on \( n, s, \lambda, \Lambda \) such that

\[
\sup_{B_{\frac{R}{2}}(x_0)} u \leq \frac{C}{\Xi(1 + R \ell^2 V(x_0))^{\frac{1}{n}} \left( \frac{1}{R^n} \int_{B_R(x_0)} u^2(y) \, dy \right)^\frac{1}{2}},
\]

for any \( R \in (0, d(x_0, \partial \Omega)) \), where \( \Xi(x) = \sum_{k=0}^\infty x^{k+1}/(k!)^{\frac{1}{2}+s} \), \( d_0 > 0 \) is the constant in Lemma 3.2 and \( \ell^2 \) is the fractional auxiliary function to be given in Section 3.

In the next, we get mapping properties of \( L_K \circ S_V \) from \( L^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \) for \( p \in [1, q] \), whenever \( V \in \mathcal{R}^n \) for \( q > \frac{n}{2s} \) with \( s \in (0, 1) \) and \( n \geq 2 \).

**Theorem 1.3.** If \( V \in \mathcal{R}^n \) is nonnegative for \( q > \frac{n}{2s} \) with \( s \in (0, 1) \) and \( n \geq 2 \), then there is a universal constant \( C = C(n, s, q) > 0 \) such that

\[
\| (L_K \circ S_V) f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}
\]

for any \( p \) with \( 1 \leq p \leq q \), where \( S_V = (L_K + V)^{-1} \).

For any \( p, q \) with \( 1 \leq p < q < \infty \), \( s \in (0, 1) \), \( \theta \in [0, n] \), and \( n \geq 2 \), we write

\[
\Gamma_\theta = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in (0, 1) \times (0, 1) : \frac{1}{p} - \frac{1}{q} = \frac{\theta}{n}, p \leq q \right\}
\]

and \( W = M^{2s-\theta} \).

Denote by \( \Delta_\theta = \{(p, q) \in (1, \infty) \times (1, \infty) : (1/p, 1/q) \in \Gamma_\theta\} \). Let us introduce the weak \( L^p(\mathbb{R}^n) \) space which is denoted by \( L^{p, \infty}(\mathbb{R}^n) \). For \( 0 < p < \infty \), the space \( L^{p, \infty}(\mathbb{R}^n) \) is the class of all real-valued Lebesgue measurable functions \( g \) on \( \mathbb{R}^n \) such that

\[
\|g\|_{L^{p, \infty}(\mathbb{R}^n)} := \sup_{\gamma > 0} \gamma \omega_\theta(\gamma)^{1/p} < \infty
\]

where \( \omega_\theta(\gamma) = |\{y \in \mathbb{R}^n : |g(y)| > \gamma\}| \) for \( \gamma > 0 \). In fact, it is a quasi-normed linear space for \( 0 < p < \infty \). Then we obtain the mapping properties of \( M_W \circ S_V \) in the following theorem, where \( M_W \) is the multiplication operator, i.e. \( M_Wf = Wf \).

**Theorem 1.4.** Let \( V \in \mathcal{R}^n \) be nonnegative for \( \tau > \frac{n}{2s} \) with \( s \in (0, 1) \) and \( n \geq 2 \).

(a) If \( (p, q) \in (\bigcup_{\theta \in [0, 2s]} \Delta_\theta) \cup \{(\infty, \infty)\} \), then we have that

\[
\| (M_W \circ S_V) f \|_{L^p(\mathbb{R}^n)} \leq C_a \| f \|_{L^p(\mathbb{R}^n)}
\]

with a constant \( C_a = C_a(n, s, \lambda, \tau) > 0 \).

(b) If \( p = 1 \), then there is a universal constant \( C_b = C_b(n, s, \lambda, \tau) > 0 \) such that

\[
\| (M_W \circ S_V) f \|_{L^{1, \infty}(\mathbb{R}^n)} \leq C_b \| f \|_{L^1(\mathbb{R}^n)}
\]

for any \( q \in [1, \frac{n}{n-2s}] \).

(c) If \( (p, q) \in \Delta_{2s} \), then there is a universal constant \( C_c = C_c(n, s, \lambda, \tau) > 0 \) such that

\[
\| (M_W \circ S_V) f \|_{L^{p, \infty}(\mathbb{R}^n)} \leq C_c \| f \|_{L^p(\mathbb{R}^n)}
\]

for any \( q \in (\frac{n}{n-2s}, \infty) \).
The paper is organized as follows. In Section 2, we define several function spaces and give the fractional Poincaré inequality which was proved in [BBM, MS]. In Section 3, we introduce the fractional auxiliary function \( m_V(x) \) related with certain potential \( V \in RH^q \) for \( q > \frac{n}{2} \) and \( 0 < s < 1 \), and deduce a nonlocal version of the Fefferman-Phong inequality [F] associated with \( V \) and \( m_V \). In Section 4, we also get a weak Harnack inequality for nonnegative weak subsolutions of the equation

\[
\begin{aligned}
L_K u &= 0 \quad \text{in } \Omega, \\
u &= g \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

where \( g \in H^s(\mathbb{R}^n) \) and \( \Omega \) is a bounded open domain in \( \mathbb{R}^n \) with Lipschitz boundary. In Section 5, we furnish a relation between the weak solutions (weak subsolutions and weak supersolutions) of the nonlocal Schrödinger equation and the minimizers (subminimizers and superminimizers) of its energy functional, respectively, and also get a Caccioppoli estimate for weak solutions of the equation. In Section 6, we get an explicit improved upper bound of \( \epsilon_V \). Finally, we obtain \( L^p \)-estimate and \( L^p - L^q \) estimates for the inverse of the nonlocal Schrödinger operator \( L_V \) with potential \( V \) in Section 7.

2. Preliminaries

Denote by \( \mathcal{F}^n \) the family of all real-valued Lebesgue measurable functions on \( \mathbb{R}^n \). Let \( \Omega \) be a bounded open domain in \( \mathbb{R}^n \) with Lipschitz boundary and let \( K \in \mathcal{K} \). Let \( X(\Omega) \) be the linear function space of all Lebesgue measurable functions \( v \in \mathcal{F}^n \) such that \( v|_\Omega \in L^2(\Omega) \) and

\[
\int\int_{\mathbb{R}^n_+^2} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dx \, dy < \infty
\]

where we denote by \( \mathbb{R}^{2n}_+ := \mathbb{R}^{2n} \setminus (D^c \times D^c) \) for a set \( D \subset \mathbb{R}^n \). We also set

\[
X_0(\Omega) = \{ v \in X(\Omega) : v = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}
\]
Since $C^0_0(\Omega) \subset X_0(\Omega)$, we see that $X(\Omega)$ and $X_0(\Omega)$ are not empty. Then we see that $(X(\Omega), \| \cdot \|_{X(\Omega)})$ is a normed space, where the norm $\| \cdot \|_{X(\Omega)}$ is defined by

\[
\|v\|_{X(\Omega)} := \|v\|_{L^2(\Omega)} + \left( \int_{\mathbb{R}^n_+} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2} < \infty, \quad v \in X(\Omega).
\]

For $p \geq 1$, let $W^{s,p}(\Omega)$ be the usual fractional Sobolev spaces with the norm

\[
\|v\|_{W^{s,p}(\Omega)} := \|v\|_{L^p(\Omega)} + \|v\|_{W^{s,p}(\Omega)} < \infty
\]

where the seminorm $\| \cdot \|_{W^{s,p}(\Omega)}$ is defined by

\[
[v]_{W^{s,p}(\Omega)} = \left( \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{1/p}.
\]

Denote by $H^s(\Omega) = W^{s,2}(\Omega)$ and let $H^s_0(\mathbb{R}^n)$ be the class of all functions in $H^s(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$.

By [SV], there exists a constant $c > 1$ depending only on $n, \lambda, s$ and $\Omega$ such that

\[
\|v\|_{X_0(\Omega)} \leq \|v\|_{X(\Omega)} \leq c \|v\|_{X_0(\Omega)}
\]

for any $v \in X_0(\Omega)$, where

\[
\|v\|_{X_0(\Omega)} := \left( \int_{\mathbb{R}^n_+} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2}.
\]

Thus $\| \cdot \|_{X_0(\Omega)}$ is a norm on $X_0(\Omega)$ equivalent to (2.4). Moreover it is known [SV] that $(X_0(\Omega), \| \cdot \|_{X_0(\Omega)})$ is a Hilbert space with inner product

\[
\langle u, v \rangle_{X_0(\Omega)} := \int_{\mathbb{R}^n_+} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy.
\]

**Lemma 2.1.** Let $s \in (0, 1)$ and $h > 0$. If $K \in K_0$ and $u \in X_0(\Omega)$, then we have the following properties: for any $x \in \mathbb{R}^n$ and $\varrho \in (0, h)$,

\[
\begin{align*}
(i) & \quad \varrho^{-2} \int_{|x-y| < \varrho} |x-y|^2 K(x-y) \, dy + \int_{|x-y| \geq \varrho} K(x-y) \, dy \leq \Theta_{n,s} \varrho^{-2s}, \\
(ii) & \quad \frac{1}{\lambda c_{n,s}} \int_{\Omega \times \Omega} |u(x) - u(y)|^2 K(x-y) \, dx \, dy \leq \frac{1}{\lambda c_{n,s}} \int_{\Omega \times \Omega} |u(x) - u(y)|^2 K(x-y) \, dx \, dy
\end{align*}
\]

where $\Theta_{n,s} = \frac{\omega_n \varrho}{s}$ and $\omega_n$ denotes the surface measure of the unit sphere $S^{n-1}$.

**Proof.** Refer to [FK] for (i). Also the proof of (ii) is very straightforward. \hfill $\square$

Next we give the fractional Poincaré inequality, which was proved in [BM, MS].

**Proposition 2.2.** Let $n \geq 1$, $p \geq 1$, $s \in (0, 1)$ and $sp < n$. Then there is a universal constant $c_{n,p} > 0$ depending only on $n, p$ such that

\[
\|u - u_B\|_{L^p(B)}^p \leq c_{n,p}(1-s)|B|^{\frac{p}{n}} \left( \frac{1}{(n-sp)^{p-1}} \|u\|_{W^{s,p}(B)}^p \right)
\]

for any ball $B \subset \mathbb{R}^n$. 
3. The fractional auxiliary function $\mathfrak{m}_V(x)$

A locally integrable function in $\mathbb{R}^n$ that takes values in $[0, \infty)$ almost everywhere is called a weight. For $p \geq 1$ and a weight $w \in L^{loc}_p(\mathbb{R}^n)$, let $L^p_w(\Omega)$ be the weighted $L^p$ class of all real-valued measurable functions $g$ on $\mathbb{R}^n$ satisfying

$$\|g\|_{L^p_w(\Omega)} := \left( \int_{\Omega} |g(y)|^p w(y) \, dy \right)^{\frac{1}{p}} < \infty.$$  

Consider a class of weights, so-called the Muckenhoupt $A_p$-class, satisfying the following conditions $[\text{St}]:$ Let $1 \leq p < \infty$. Then we say that a weight $w \in L^{loc}_p(\mathbb{R}^n)$ satisfies the $A_p$-condition (and we denote by $w \in A_p$), if there is a universal constant $C > 0$ such that

$$\left( \frac{1}{|B|} \int_B w(y) \, dy \right) \left( \frac{1}{|B|} \int_B w(y)^{-\frac{1}{p-1}} \, dy \right)^{p-1} \leq C \quad \text{for } 1 \leq p < \infty$$

for all balls $B$ in $\mathbb{R}^n$ (here, the second $L^{\frac{1}{p-1}}$-average of $w^{-1}$ on $B$ must be replaced by $\|w^{-1}\|_{L^{\infty}(B)}$, when $p = 1$). The smallest constant $C$ in (3.1) is called the $A_p$-norm of $w$ and denoted by $[w]_{A_p}$. If $w \in A_\infty := \cup_{p \geq 1} A_p$, then $w \in A_p$ for some $p \geq 1$. In this case, it is well-known that $w$ satisfies a reverse Hölder’s inequality with exponent $q = 1 + \eta > 1$; that is, there are universal constants $C > 0$ and $\eta > 0$ depending only on $n, p$ and $[w]_{A_p}$ such that

$$\left( \frac{1}{|B|} \int_B w(y)^q \, dy \right)^{\frac{1}{q}} \leq C \left( \frac{1}{|B|} \int_B w(y) \, dy \right)$$

for all balls $B$ in $\mathbb{R}^n$. Let RH$^q$ be the class of all weights $w$ satisfying (3.2) for some $q > 1$, and let RH$^\infty$ be the class of all weights $w$ satisfying that there is a universal constant $C > 0$ such that

$$\|w\|_{L^{\infty}(B)} \leq C \left( \frac{1}{|B|} \int_B w(y) \, dy \right)$$

for all balls $B$ in $\mathbb{R}^n$. Thus it is obvious that if $w \in A_\infty$ then there is some $q > 1$ such that $w \in \text{RH}^q$, and also RH$^\infty$ is a subclass of any RH$^q$. Moreover, if $w \in \text{RH}^q$ for $q > 1$, then it is well-known that $w \in A_\infty$, which is equivalent to the following condition; there are some $\alpha_0, \beta_0 \in (0,1)$ such that

$$|\{x \in B : w(x) \geq \alpha_0 w_B\}| \geq \beta_0 |B|$$

for all balls $B$ in $\mathbb{R}^n$, where $w_B = \frac{1}{|B|} \int_B w(y) \, dy$ denotes the average of $w$ on $B$.

Throughout this paper, we shall assume that $V \in \text{RH}^q$ for some $q > \frac{n}{2s}$ with $s \in (0,1)$ and $n \geq 2$. We will consider the auxiliary function to be used in measuring an efficient growth of such weight function $V$, which was introduced by Shen $[\text{S}]$: as a matter of fact, we are considering the nonlocal adaptation of such auxiliary function. For $r \in (0, \infty)$, $s \in (0,1)$ and $x \in \mathbb{R}^n$, we set

$$\mathcal{G}^s_V(x, r) = \frac{1}{r^{n-2s}} \int_{B_r(x)} V(z) \, dz.$$  

In this case, using Hölder’s inequality, it is quite easy to check that

$$\frac{\mathcal{G}^s_V(x, r)}{\mathcal{G}^s_V(x, R)} \leq c_0 \left( \frac{R}{r} \right)^{\frac{s}{2} - 2s}$$
with a universal constant $c_0 > 0$, for any $x \in \mathbb{R}^n$, $s \in (0, 1)$ and $0 < r < R < \infty$. The assumption $q > \frac{n}{2s}$ and (3.5) imply that

$$\lim_{r \to 0} \mathcal{G}^s_V(x, r) = 0 \quad \text{and} \quad \lim_{r \to \infty} \mathcal{G}^s_V(x, r) = \infty$$

for any $x \in \mathbb{R}^n$ and $s \in (0, 1)$. For $s \in (0, 1)$, $x \in \mathbb{R}^n$ and $V \in \text{RH}^q$ with $q > \frac{n}{2s}$, we define

$$\mathcal{G}^s_V(x, \rho) = \sup \{\rho > 0 : \mathcal{G}^s_V(x, \rho) \leq 1\}.$$  

From (3.6), it is trivial that $0 < \mathcal{G}^s_V(x) < \infty$ for all $x \in \mathbb{R}^n$. If $\mathcal{G}^s_V(x) = 1/\rho$, then we see that

$$\mathcal{G}^s_V(x, \rho) = 1.$$  

Moreover, it follows from (3.7) that

$$\mathcal{G}^s_V(x, \rho) \sim 1 \quad \text{if and only if} \quad \mathcal{G}^s_V(x) \sim \frac{1}{\rho}.$$  

In addition, we mention a well-known fact that the measure $V(z)\, dz$ satisfies the following doubling condition; that is, there is a universal constant $c_1 > 0$ such that

$$\int_{B_{2r}(x)} V(z)\, dz \leq c_1 \int_{B_r(x)} V(z)\, dz,$$

provided that $V \in \text{RH}^q$ for $q > 1$.

For the fractional auxiliary function $\mathcal{G}^s_V(x)$, we have the following inequalities whose proof is a nonlocal adaptation of that in $\text{S}$.

**Lemma 3.1.** For $s \in (0, 1)$, there are universal constants $C_0, d_0 > 0$ such that

(a) $\mathcal{G}^s_V(x) \sim \mathcal{G}^s_V(y)$ if $|x - y| \leq \frac{C_0}{\mathcal{G}^s_V(x)}$,

(b) $\mathcal{G}^s_V(y) \leq C_0 \left[1 + |x - y| \mathcal{G}^s_V(x)\right]^{d_0} \mathcal{G}^s_V(x)$ for any $x, y \in \mathbb{R}^n$,

(c) $\mathcal{G}^s_V(y) \geq \frac{\mathcal{G}^s_V(x)}{C_0 \left[1 + |x - y| \mathcal{G}^s_V(x)\right]^{\frac{d_0}{q-n}}} \mathcal{G}^s_V(x)$ for any $x, y \in \mathbb{R}^n$.

**Proof.** (a) Assume that $|x - y| \leq C_0 \rho$ where $\mathcal{G}^s_V(x) = 1/\rho$. Since the measure $V\, dz$ has the doubling condition (3.10), by (3.8) we see that

$$\mathcal{G}^s_V(y, \rho) \sim \mathcal{G}^s_V(x, \rho) = 1.$$  

Thus it follows from (3.9) that $\mathcal{G}^s_V(y) \sim \mathcal{G}^s_V(x)$.

(b) Assume that $|x - y| \sim 2^k \rho$ for $k \in \mathbb{N}$. For $\rho_1 \in (0, \rho)$, we choose some $j \in \mathbb{N}$ so that $2^j \rho_1 \sim 2^k \rho$. Then by (3.5) and (3.10) we have that

$$\int_{B_{2^j \rho_1}(y)} V(z)\, dz \leq c_0 \int_{B_{2^j \rho_1}(y)} \int_{B_{2^j \rho_1}(y)} V(z)\, dz \leq C \int_{B_{2^j \rho_1}(y)} V(z)\, dz \leq C \int_{B_{2^k \rho}(y)} V(z)\, dz \leq C \int_{B_{2^k \rho}(y)} V(z)\, dz \leq C \int_{B_{2^k \rho}(y)} V(z)\, dz = c_1^k C 2^{j[n/q-n]} \rho^{n-2s}.$$
Thus this leads us to obtain that
\[ \mathcal{E}_V^*(y, \rho_1) \leq c_1^k C 2^{[n/q-n]} \left( \frac{\rho}{\rho_1} \right)^{n-2s} \leq C [2^n q^{-n} c_1]^k \left( \frac{\rho}{\rho_1} \right)^{n/q - 2s}. \]
If we choose some large number \( C_1 > 0 \) so that
\[ \mathcal{E}_V^*(y, \rho_1) \leq a_0 \quad \text{for} \quad \rho_1 \geq C_1^{-k} \rho, \]
then by (3.7) we conclude that
\[ \frac{1}{m_V(y)} \geq C_1^{-k} \rho. \]
Hence it follows from (3.11) and (3.12) that
\[ m_V(y) \leq C_1^k m_V(x) \leq C_0 (1 + |x - y| m_V(x))^{d_0} m_V(x) \]
where \( d_0 = \log_2(C_1) \).
(c) Without loss of generality, we may assume that \( |x - y| \geq 1/m_V(y) \); for, otherwise it can be shown by (a). From (b), we have that
\[ m_V(x) \leq C_0 (1 + |x - y| m_V(y))^{d_0} m_V(y) \leq C_0 |x - y|^{d_0} |m_V(y)|^{d_0 + 1}. \]
Therefore we obtain that
\[ m_V(y) \geq \frac{|m_V(x)|^{\frac{d_0}{d_0 + 1}}}{C_0 |x - y|^{\frac{d_0}{d_0 + 1}}} \geq \frac{m_V(x)}{C_0 (1 + |x - y| m_V(x))^{\frac{d_0}{d_0 + 1}}}. \]
Hence we complete the proof. \( \square \)

In the following lemma, we get a nonlocal version of Fefferman-Phong inequality [F] related with the potential \( V \) and the fractional auxiliary function \( m_V \).

**Lemma 3.2.** Let \( n \geq 1, s \in (0, 1) \) and \( 2s < n \). If \( u \in H^s_c(\mathbb{R}^n) \), then there exists a universal constant \( C_0 = C_0(n, s) > 0 \) such that
\[ \int_{\mathbb{R}^n} |u(x)|^2 [m_V(x)]^{2s} \, dx \leq C_0 \left( ||u||_{H^s(\mathbb{R}^n)}^2 + ||u||_{L^2(\mathbb{R}^n)}^2 \right). \]

**Proof.** We take any \( z \in \mathbb{R}^n \). Let \( B = B_{r}(z) \) and set \( m_V(z) = 1/\eta \) for \( \eta > 0 \). By (3.8), we then observe that
\[ V_B = \frac{1}{|B_1| \eta^{2s}}. \]
By Proposition 2.2, we have that
\[ \frac{1}{|B_1| \eta^{2s}} \geq d_{n,s} \int_{B \times B} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \geq \frac{d_{n,s}}{|B_1| \eta^{n+2s}} \int_{B \times B} |u(x) - u(y)|^2 \, dx \, dy \]
where \( d_{n,s} = \frac{n - 2s}{2s \eta (1 - 2s)} \). Also we easily obtain the following equality
\[ \int_B |u(x)|^2 V(x) \, dx = \frac{c_n}{\eta^n} \int_{B \times B} |u(y)|^2 V(y) \, dx \, dy \]
where \( c_n = 1/|B_1| \). Since \( V \in A_\infty \), by (3.3) there are universal constants \( a_0, b_0 \in (0, 1) \) not depending on \( B \) such that
\[ \left| \{ x \in B : V(x) > a_0 V_B \} \right| \geq b_0 |B|. \]
By (3.13) and (3.16), adding up (3.14) and (3.15) yields that
\[
\int\int_{B \times B} \frac{|u(x) - u(y)|^2}{|x - y|^{n + 2s}} \, dx \, dy + \int_B |u(y)|^2 V(y) \, dy \\
\geq \frac{d_{n,s} \wedge c_n}{2 \eta^n} \int B \times B \left( \frac{\alpha_0}{|B_1| \eta^{2s}} \wedge V(y) \right) |u(x)|^2 \, dx \, dy \\
\geq \frac{\alpha_0 \beta_0 (d_{n,s} \wedge c_n)}{2 \eta^{2s}} \int_B |u(x)|^2 \, dx.
\]

Thus by (3.17) and (a) of Lemma 3.1 we have that
\[
\int_B |u(x)|^2 [m_V(x)]^{n + 2s} \, dx \\
\leq C \left( \int\int_{B \times B} \frac{|u(x) - u(y)|^2}{|x - y|^{n + 2s}} [m_V(x)]^n \, dx \, dy + \int_B |u(x)|^2 V(x) [m_V(x)]^n \, dx \right),
\]
where the constant \( C = C_{n,s} \) is given by
\[
C := \frac{2 \epsilon}{\alpha_0 \beta_0 (d_{n,s} \wedge c_n)}.
\]

Applying (a) of Lemma 3.1 again, by (3.18) we obtain that
\[
\int_B |u(x)|^2 [m_V(x)]^{n + 2s} \, dx \\
\leq C \left( \int\int_{B \times B} \frac{|u(x) - u(y)|^2}{|x - y|^{n + 2s}} [m_V(x)]^n \, dx \, dy + \int_B |u(x)|^2 V(x) [m_V(x)]^n \, dx \right).
\]

Since \( B \times B = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - z| \leq |y - z| \leq \eta \} \), integrating both sides of the above inequality in \( z \) over \( \mathbb{R}^n \) and changing the order of integrations yield that
\[
\int_{\mathbb{R}^n} |u(x)|^2 [m_V(x)]^{n + 2s} \left( \int_{B_{\rho}(x)} \, dz \right) \, dx \\
\leq C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n + 2s}} [m_V(x)]^n \left( \int_{|z - x| \leq |z - y| < \rho} \, dz \right) \, dy \\
+ C \int_{\mathbb{R}^n} |u(x)|^2 V(x) [m_V(x)]^n \left( \int_{B_{\rho}(x)} \, dz \right),
\]
where \( \rho = 1/m_V(x) \). Here we note that
\[
\int_{|z - x| \leq |z - y| < \rho} \, dz \leq \int_{B_{\rho}(x)} \, dz = \frac{|B_1|}{|m_V(x)|^n}.
\]

Therefore we complete the proof by using the fact that \( u \) is supported in \( \Omega \). \( \square \)

**Lemma 3.3.** For \( V \in RH^q \) with \( q > \frac{n}{2s} \) and \( s \in (0, 1) \), there are some universal constants \( d_1 > 0 \) and \( C = C(n, s) > 0 \) such that
\[
\frac{1}{R^{n - 2s}} \int_{B_R(x)} V(y) \, dy \leq C [R m_V(x)]^{d_1} \quad \text{whenever} \quad R m_V(x) \geq 1.
\]

**Proof.** Set \( r = 1/m_V(x) \). If \( R m_V(x) \geq 1 \), then we may write \( 2^{k-1}r \leq R < 2^k r \) for \( k \in \mathbb{N} \). By (3.8) and the doubling condition (3.10), we have that
\[
\int_{B_R(x)} V(y) \, dy \leq c_1^k \int_{B_{r^k}(x)} V(y) \, dy = c_1^k r^{n - 2s}.
\]
Thus we conclude that
\[
\frac{1}{R^{n-2s}} \int_{B_R(x)} V(y) \, dy \leq c_1^k \left( \frac{r}{R} \right)^{n-2s} \leq 2^{n-2s}(c_1 2^{2s-n})^k \leq C \left[ R \text{m}_V(x) \right]^{d_1},
\]
where \( d_1 = \log_2 c_1 + 2s - n \). Hence we are done. \( \square \)

4. A weak Harnack inequality

In this section, we obtain a weak Harnack inequality of the weak solution to the following nonlocal elliptic boundary value problem

\[
\begin{align*}
L_K u &= 0 \quad \text{in } \Omega, \\
u &= g \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{align*}
\]

where \( g \in H^s(\mathbb{R}^n) \). In what follows, we consider a bilinear form by

\[
\langle u, v \rangle_K = \iint_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(v(x) - v(y)) K(x - y) \, dx \, dy \quad \text{for } u, v \in X(\Omega).
\]

For given \( g \in H^s(\mathbb{R}^n) \), we consider the convex subsets of \( H^s(\mathbb{R}^n) \) by

\[
\begin{align*}
X^+_g(\Omega) &= \{ v \in H^s(\mathbb{R}^n) : (g - v)^\pm \in X_0(\Omega) \}, \\
X^-_g(\Omega) &:= X^+_g(\Omega) \cap X^-_g(\Omega) = \{ v \in H^s(\mathbb{R}^n) : g - v \in X_0(\Omega) \}.
\end{align*}
\]

The weak formulation of the equation (4.1) is as follows; if \( u \in X^-_g(\Omega) \) is a weak solution of the equation (4.1), then it satisfies that

\[
\langle u, \varphi \rangle_K \leq 0 \quad \text{for every nonnegative } \varphi \in X_0(\Omega).
\]

Moreover, we observe that the weak solution \( u \) is the minimizer of the energy functional

\[
\mathcal{E}(v) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v(x) - v(y)|^2 K(x - y) \, dx \, dy, \quad v \in X^-_g(\Omega).
\]

**Definition 4.1.** A function \( u \in X^-_g(\Omega) \) is said to be a weak subsolution (weak supersolution) of the equation (4.1), if it satisfies that

\[
\langle u, \varphi \rangle_K \leq (\geq) 0 \quad \text{for every nonnegative } \varphi \in X_0(\Omega).
\]

Also a function \( u \) is a weak solution of the equation (4.1), if it is both a weak subsolution and a weak supersolution. So any weak solution \( u \) of (4.1) must be in \( X^-_g(\Omega) \) and satisfy (4.4).

In the next, we consider the definition of subminimizer and superminimizer of the functional in (4.5) to get better understanding about weak subsolutions and supersolutions.

**Definition 4.2.** Let \( g \in H^s(\mathbb{R}^n) \). (a) A function \( u \in X^-_g(\Omega) \) is said to be a subminimizer of the functional (4.5) over \( X^-_g(\Omega) \), if it satisfies that

\[
\mathcal{E}(u) \leq \mathcal{E}(u + \varphi) \quad \text{for all nonpositive } \varphi \in X_0(\Omega).
\]

A function \( u \in X^+_g(\Omega) \) is said to be a superminimizer of the functional (4.5) over \( X^+_g(\Omega) \), if it satisfies that

\[
\mathcal{E}(u) \leq \mathcal{E}(u + \varphi) \quad \text{for all nonnegative } \varphi \in X_0(\Omega).
\]
(b) A function $u$ is said to be a minimizer of the functional (4.5) over $X_0(\Omega)$, if it is both a subminimizer and a superminimizer. So any minimizer $u$ must be in $X_0(\Omega)$ and satisfies that $\mathcal{E}(u) \leq \mathcal{E}(u + \varphi)$ for all $\varphi \in X_0(\Omega)$.

**Theorem 4.3.** If $s \in (0,1)$, then there is a unique minimizer of the functional (4.5). Moreover, a function $u \in X_0^+(\Omega)(X_0^+(\Omega))$ is a subminimizer (superminimizer) of the functional (4.5) over $X_0^-(\Omega)(X_0^+(\Omega))$ if and only if it is a weak subsolution (weak supersolution) of the equation (4.1). In particular, a function $u \in X_0^+(\Omega)$ is a minimizer of the functional (4.5) if and only if it is a weak solution of the equation (4.1).

**Proof.** Applying standard way of calculus of variations, we proceed with our proof. Take any minimizing sequence $\{u_k\} \subset X_0(\Omega)$. By the fractional Sobolev inequality [DPV], we can take a subsequence $\{u_{k_j}\} \subset X_0(\Omega)$ which converges strongly to $u$ in $L^2(\Omega)$. Then there is a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ converging a.e. in $\Omega$ to $u \in X_0(\Omega)$. Thus it follows from Fatou’s lemma that the energy functional $\mathcal{E}(\cdot)$ is weakly lower semicontinuous on $X_0(\Omega)$. This implies that $u$ is an actual minimizer of (4.5). Also the uniqueness of the minimizer easily follows from the strict convexity of the functional (4.5).

Next, we prove the equivalence only for the weak supersolution case, because the other cases can be shown in a similar way. First, if $u \in X_0^+(\Omega)$, then we note that

$$\mathcal{E}(u + \varphi) - \mathcal{E}(u) = 2\langle u, \varphi \rangle_{X(\Omega)} + \|\varphi\|^2_{X_0(\Omega)}$$

for all nonnegative $\varphi \in X_0(\Omega)$. Thus this implies that a weak supersolution $u \in X_0^+(\Omega)$ of the equation (4.1) is a superminimizer of the functional (4.5) over $X_0^+(\Omega)$. On the other hand, we assume that $u \in X_0^+(\Omega)$ is a superminimizer of the functional (4.5). Then by (4.7) we see that

$$2\langle u, \varphi \rangle_{X(\Omega)} + \|\varphi\|^2_{X_0(\Omega)} \geq 0$$

for all nonnegative $\varphi \in X_0(\Omega)$. Since $\varepsilon \varphi \in X_0(\Omega)$ and it is nonnegative for any $\varepsilon > 0$ and $\varphi \in X_0(\Omega)$, we have that

$$2\langle u, \varphi \rangle_{X(\Omega)} + \varepsilon \|\varphi\|^2_{X_0(\Omega)} \geq 0 \quad \text{for any } \varepsilon > 0.$$ 

Taking $\varepsilon \to 0$, we conclude that $\langle u, \varphi \rangle_{X(\Omega)} \geq 0$ for any nonnegative $\varphi \in X_0(\Omega)$. Thus $u$ is a weak supersolution of the equation (4.1). Therefore we complete the proof. \qed

As in [DKP2], we consider the nonlocal tail $T(f; x_0, R)$ of a function $f$ in the open ball $B_R(x_0) \subset \Omega$ with center $x_0 \in \mathbb{R}^n$ and radius $R > 0$, which plays a crucial role in the regularity results of the nonlocal equations unlike the local equations. For $f \in X_0(\Omega)$ and $r > 0$, we define

$$T(f; x_0, r) = r^{2s} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|f(y)|}{|y - x_0|^{n+2s}} \, dy.$$ 

We note that this nonlocal tail is well-defined because $X_0(\Omega)$ is compactly imbedded in $L^2(\Omega)$ by the fractional Sobolev inequality on $X(\Omega)$ [DPV] for a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary.

Next, we mention some results on local boundedness and nonlocal tail properties of weak subsolutions of the equation (4.1) obtained in [DKP1] [DKP2].
Theorem 4.4. Let \( u \in X^-_g(\Omega) \) be a weak subsolution of the equation (4.1) where \( g \in H^s(\mathbb{R}^n) \), and let \( s \in (0, 1) \) and \( B_r(x_0) \subset \Omega \). Then there is a constant \( c_1 > 0 \) depending only on \( n, s, \lambda \) and \( \Lambda \) such that
\[
\sup_{B_{r/2}(x_0)} u \leq \delta T(u^+; x_0, r/2) + c_1 \delta^{-\frac{n}{2s}} \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u^+(y)|^2 \, dy \right)^{1/2}
\]
for any \( \delta \in (0, 1] \). Moreover, if \( u \geq 0 \) in \( B_R(x_0) \subset \Omega \) with \( 0 < r < R \), then there is a constant \( c_2 > 0 \) depending only on \( n, s, \lambda \) and \( \Lambda \) such that
\[
T(u^+; x_0, r) \leq c_2 \sup_{B_r(x_0)} u + c_2 \left( \frac{r}{R} \right)^{2s} T(u^-; x_0, R).
\]

Next we shall prove a weak Harnack inequality of nonnegative weak subsolutions of the equation (4.1) by using Theorem 4.4. Interestingly, this estimate no longer depends on the nonlocal tail term of the weak solution, but the proof is quite simple.

Proposition 4.5. Let \( u \in X_g(\Omega) \) be a nonnegative weak subsolution of the equation (1.6) where \( g \in H^s(\mathbb{R}^n) \), and let \( s \in (0, 1) \) and \( B_r(x_0) \subset \Omega \). Then there is a constant \( C > 0 \) depending only on \( n, s, \lambda \) and \( \Lambda \) such that
\[
\sup_{B_{r/2}(x_0)} u \leq C \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u^2(y) \, dy \right)^{1/2}.
\]

Proof. We take some \( \delta \in (0, 1] \) so that \( 1 - \delta c_2 > 0 \) and choose some \( R > r \) with \( B_R(x_0) \subset \Omega \). Then by Theorem 4.4 we have that
\[
\sup_{B_{r/2}(x_0)} u \leq \delta c_2 \sup_{B_{r/2}(x_0)} u + \delta c_2 \left( \frac{r}{R} \right)^{2s} T(u^-; x_0, R) + c_1 \delta^{-\frac{n}{2s}} \left( \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u^2(y) \, dy \right)^{1/2}.
\]
Since \( T(u^-; x_0, R) = 0 \), we can obtain the required result by taking
\[
C = c_1 \delta^{-\frac{n}{2s}} \frac{1}{1 - \delta c_2}.
\]
Hence we complete the proof. 

5. WEAK SOLUTIONS AND CACCIOPPOLI ESTIMATE FOR NONLOCAL SCHRÖDINGER OPERATORS \( \mathbf{L}_V \)

In this section, we give a relation between weak solutions (weak subsolutions and weak supersolutions) for the nonlocal Schrödinger equation and minimizers (subminimizers and superminimizers) of its energy functional, respectively. Also, we obtain a certain type of Caccioppoli estimate for nonnegative weak subsolutions of the nonlocal Schrödinger equation.

Definition 5.1. Let \( g \in H^s(\mathbb{R}^n) \) and \( V \in RH^q \) for \( q > \frac{n}{2s} \) and \( s \in (0, 1) \). Then we say that a function \( u \in X_g(\Omega) \) is a weak solution of the nonlocal equation
\[
\mathbf{L}_V u = 0 \quad \text{in } \Omega,
\]
\[
u = g \quad \text{in } \mathbb{R}^n \setminus \Omega,
\]
Moreover, any bounded sequence in $E$ of the energy functional $E$ universal constant $C > 0$ we need a compactness theorem $Y(\Omega) \hookrightarrow V(\Omega)$. In order to obtain the existence of the minimizer of the functional $\langle u, \varphi \rangle_K + \langle Vu, \varphi \rangle_{L^2(\mathbb{R}^n)} = 0$ for any $\varphi \in X_0(\Omega)$.

In fact, it turns out that the weak solution of the equation (5.1) is the minimizer of the energy functional
\begin{equation}
E_V(v) = E(v) + \|v\|^2_{L^2(\mathbb{R}^n)}, \ v \in Y_g(\Omega) := X_g(\Omega) \cap L^2_{V}(\mathbb{R}^n),
\end{equation}
where $g \in H^s(\mathbb{R}^n)$. We consider function spaces $Y_g^+(\Omega)$ and $Y_g^-(\Omega)$ defined by
\[ Y_g^\pm(\Omega) = \{v \in Y_g(\Omega) : (g - v)^\pm \in X_0(\Omega)\}. \]
Then we see that $Y_g(\Omega) = Y_g^+(\Omega) \cap Y_g^-(\Omega)$. When $u = g = 0$ in $\mathbb{R}^n \setminus \Omega$, we easily see that $Y_0(\Omega) = X_0(\Omega) \cap L^2_V(\Omega)$ and $Y_0(\Omega)$ is a Hilbert space with the inner product defined by $\langle u, v \rangle_{Y_0(\Omega)} = \langle u, v \rangle_{X_0(\Omega)} + \langle Vu, v \rangle_{L^2(\Omega)}$. Moreover, we see that $Y_0(\Omega) = X_0(\Omega)$ and they are norm-equivalent (refer to [CK]).

As in Section 4, we define weak subsolutions and weak supersolutions of the nonlocal equation (5.1) in the following definition.

**Definition 5.2.** Let $g \in H^s(\mathbb{R}^n)$. A function $u \in Y_g^-(\Omega) (Y_g^+(\Omega))$ is said to be a weak subsolution (weak supersolution) of the equation (5.1), if it satisfies that
\begin{equation}
\langle u, \varphi \rangle_K + \langle Vu, \varphi \rangle_{L^2(\mathbb{R}^n)} \leq (\geq) 0
\end{equation}
for every nonnegative $\varphi \in X_0(\Omega)$. Also a function $u$ is a weak solution of the equation (5.1), if it is both a weak subsolution and a weak supersolution. So any weak solution $u$ of (5.1) must be in $Y_g(\Omega)$ and satisfy (5.2).

In the next, we furnish the definition of subminimizer and superminimizer of the functional (5.3) to understand well weak subsolutions and supersolutions of the nonlocal Schrödinger equation (5.1).

**Definition 5.3.** Let $g \in H^s(\mathbb{R}^n)$. (a) A function $u \in Y_g^-(\Omega)$ is said to be a subminimizer of the functional (5.3) over $Y_g^-(\Omega)$, if it satisfies that
\[ E_V(u) \leq E_V(u + \varphi) \quad \text{for all nonpositive } \varphi \in X_0(\Omega). \]
Also, a function $u \in Y_g^+(\Omega)$ is said to be a superminimizer of the functional (5.3) over $Y_g^+(\Omega)$, if it satisfies that
\[ E_V(u) \leq E_V(u + \varphi) \quad \text{for all nonnegative } \varphi \in X_0(\Omega). \]
(b) A function $u$ is said to be a minimizer of the functional (5.3) over $Y_g(\Omega)$, if it is both a subminimizer and a superminimizer. So any minimizer $u$ must be in $Y_g(\Omega)$ and satisfy $E_V(u) \leq E_V(u + \varphi)$ for all $\varphi \in X_0(\Omega)$.

Let $Y(\Omega)$ be the normed subspace of $X(\Omega)$ which is endowed with the norm
\[ \|u\|_{Y(\Omega)} := \sqrt{\|u\|^2_{X(\Omega)} + \|u\|^2_{L^2(\Omega)}} < \infty, \ u \in Y(\Omega). \]
In order to obtain the existence of the minimizer of the functional $E_V$ on $Y_g(\Omega)$, we need a compactness theorem $Y(\Omega) \hookrightarrow L^2(\Omega)$ as follows.

**Theorem 5.4.** Let $n \geq 1$, $s \in (0, 1)$ and $2s < n$. If $u \in Y(\Omega)$, then there exists a universal constant $C > 0$ depending on $n, s$ and $\lambda$ such that
\[ \|u\|_{L^2(\Omega)} \leq C \|u\|_{Y(\Omega)}. \]
Moreover, any bounded sequence in $Y(\Omega)$ is precompact in $L^2(\Omega)$. 
Proof. Since \( \|u\|_{H^r(\Omega)} \leq c(\lambda)\|u\|_{X(\Omega)} \leq c(\lambda)\|u\|_{Y(\Omega)} \) for any \( u \in Y(\Omega) \), it follows from the fractional Sobolev inequality [DPV] that
\[
\|u\|_{L^2(\Omega)} \leq C\|u\|_{H^r(\Omega)} \leq C\|u\|_{Y(\Omega)}.
\]
Thus the precompactness in \( L^2(\Omega) \) can be obtained by weak compactness. \( \square \)

**Lemma 5.5.** If \( s \in (0, 1) \), then there is a unique minimizer of the functional (5.3). Moreover, a function \( u \in Y_g^- (\Omega) (Y_g^+ (\Omega)) \) is a subminimizer (superminimizer) of the functional (5.3) over \( Y_g^- (\Omega) (Y_g^+ (\Omega)) \) if and only if it is a weak subsolution (weak supersolution) of the equation (5.1). In particular, a function \( u \in Y_g(\Omega) \) is a minimizer of the functional (5.3) if and only if it is a weak solution of the nonlocal equation (5.1).

Proof. We proceed with our proof as in Lemma 4.4. Take any minimizing sequence \( \{u_k\} \subset Y_g(\Omega) \). By applying Theorem 5.4, we can take a subsequence \( \{u_{k_j}\} \subset \{u_k\} \) such that
\[ u_{k_j} \to u \quad \text{in} \quad L^2(\Omega) \]
as \( j \to \infty \). So there exist a subsequence \( \{u_{k_{j_i}}\} \) of \( \{u_{k_j}\} \) which converges a.e. in \( \Omega \) to \( u \in Y_g(\Omega) \). Thus, by applying Fatou’s lemma, we can show that the energy functional \( \mathcal{E}_V(\cdot) \) is weakly semicontinuous in \( Y_g(\Omega) \). This implies that \( u \) is a minimizer of (5.3). The uniqueness of the minimizer also follows from the strict convexity of the functional (5.3).

Next, we show the equivalency only for the weak supersolution case, because the other case can be done in a similar way. First, if \( u \in Y_g^+(\Omega) \), then we observe that
\[
\mathcal{E}_V(u + \varphi) - \mathcal{E}_V(u) = 2\langle u, \varphi \rangle_K + 2\langle Vu, \varphi \rangle_{L^2(\mathbb{R}^n)} + \|\varphi\|_{X_0(\Omega)}^2 + \|\varphi\|_{L^2_0(\Omega)}^2
\]
for all nonnegative \( \varphi \in X_0(\Omega) \). This implies that a weak supersolution \( u \in Y_g^+(\Omega) \) of the equation (5.1) is a superminimizer of the functional (5.3) over \( Y_g^+(\Omega) \).

On the other hand, we suppose that \( u \in Y_g^+(\Omega) \) is a superminimizer of the functional (5.3). Then it follows from (5.5) that
\[
2\langle u, \varphi \rangle_K + 2\langle Vu, \varphi \rangle_{L^2(\mathbb{R}^n)} + \|\varphi\|_{X_0(\Omega)}^2 + \|\varphi\|_{L^2_0(\Omega)}^2 \geq 0
\]
for all nonnegative \( \varphi \in X_0(\Omega) \). Since \( \varepsilon \varphi \in X_0(\Omega) \) and it is nonnegative for any \( \varepsilon > 0 \) and \( \varphi \in X_0(\Omega) \), we obtain that
\[
2\langle u, \varphi \rangle_K + 2\langle Vu, \varphi \rangle_{L^2(\mathbb{R}^n)} + \varepsilon\|\varphi\|_{X_0(\Omega)}^2 + \varepsilon\|\varphi\|_{L^2_0(\Omega)}^2 \geq 0
\]
for any \( \varepsilon > 0 \). Taking \( \varepsilon \to \infty \), we can conclude that \( \langle u, \varphi \rangle_K + \langle Vu, \varphi \rangle_{L^2(\mathbb{R}^n)} \geq 0 \) for any nonnegative \( \varphi \in X_0(\Omega) \). Hence \( u \) is a weak supersolution of the equation (5.1). Therefore we are done. \( \square \)

**Lemma 5.6.** If \( \alpha, \beta \in \mathbb{R} \) and \( a, b \geq 0 \), then we have the equality
\[
(\beta - \alpha)(b^2\beta - a^2\alpha) = (b\beta - a\alpha)^2 - \alpha\beta(b - a)^2.
\]

Proof. By simple calculation, we have that
\[
(\beta - \alpha)(b^2\beta - a^2\alpha) = b^2\beta^2 - 2ab\alpha\beta + a^2\alpha^2 + 2ab\alpha\beta - b^2\alpha\beta - a^2\alpha\beta
= (b\beta - a\alpha)^2 - \alpha\beta(b - a)^2.
\]
Hence we are done. \( \square \)
Next we will prove the following type of Caccioppoli estimate for weak solutions of the equation (5.1).

**Lemma 5.7.** Let $s \in (0, 1)$ and $x_0 \in \Omega$. Suppose that $u$ is a nonnegative weak subsolution of the nonlocal equation (5.1) in $\Omega$. Then there is a constant $C > 0$ depending only on $n, s, \lambda$ and $\Lambda$ such that

$$
\|u\|_{L^2(B_{R_*}(x_0))}^2 + \lambda c_{n,s} \|\phi u\|_{H^s(\mathbb{R}^n)}^2 \leq \frac{20 \Theta_{n,s} + C}{(R-r)^{2s}} \left(\frac{R}{R-r}\right)^n \|u\|_{L^2(B_R(x_0))}^2
$$

for any $r \in (0, d(x_0, \partial \Omega)/2)$ and any $R \in (r, 2r]$, where $\Theta_{n,s}$ is the constant in Lemma 2.3 and $\phi$ is the function defined by

$$
\phi(x) = \phi_{r,R_*,x_0}(x) := \left(\frac{R_* - |x - x_0|}{R_* - r}\right) \vee 0 \wedge 1,
$$

where $r < R_*(R + r)/2 < R$.

**Proof.** We use $\varphi(x) = \phi^2(x)u(x)$ as a testing function in (5.2). Then we note that

$$
\langle u, \varphi \rangle_{X(\Omega)} + \langle Vu, \varphi \rangle_{L^2(\Omega)} = \langle u, \varphi \rangle_K + \langle Vu, \varphi \rangle_{L^2(\mathbb{R}^n)} \leq 0.
$$

Applying Lemma 5.6, we obtain that

$$
\langle u, \varphi \rangle_{X(\Omega)} = \int_{\mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x - y) \, dx \, dy
$$

$$
= \int_{B_{R_*}^2(x_0)} (u(x) - u(y))^2 K(x - y) \, dx \, dy
$$

$$
+ \int_{\mathbb{R}^n \setminus B_{R_*}^2(x_0)} (\varphi(x)u(x) - \varphi(y)u(y))^2 K(x - y) \, dx \, dy
$$

$$
- \int_{\mathbb{R}^n \setminus B_{R_*}^2(x_0)} (\varphi(x) - \varphi(y))^2 u(x)u(y)K(x - y) \, dx \, dy
$$

where $B_{R_*}^2(x_0) := B_r(x_0) \times B_r(x_0)$. Thus by (5.6) we have that

$$
\int_{B_{R_*}(x_0)} Vu(x)u(x) \, dx + \int_{\mathbb{R}^n} (\varphi(x)u(x) - \varphi(y)u(y))^2 K(x - y) \, dx \, dy
$$

$$
\leq \int_{\mathbb{R}^n \setminus B_{R_*}^2(x_0)} (\varphi(x) - \varphi(y))^2 u(x)u(y)K(x - y) \, dx \, dy := I.
$$

From the property of $\phi$, we see that

$$
\sup_{x,y \in \mathbb{R}^n} \frac{(\phi(x) - \phi(y))^2}{|x - y|^2} \leq \left(\frac{1}{R_* - r}\right)^2 \leq \frac{4}{(R-r)^2}.
$$

Since we have the estimate

$$
|y - x| \geq |y - x_0| - |x - x_0| \geq \frac{(R - r)|y - x_0|}{2R} \text{ for any } (x, y) \in B_{R_*}(x_0) \times B_{R_*}(x_0),
$$
it follows from Lemma 2.1, (5.7), (5.8) and Cauchy’s inequality that

\[ \mathcal{I} = \frac{1}{2} \int \int_{B_R^c(x_0) \setminus B_{r_1}^c(x_0)} (\phi(x) - \phi(y))^2 (u^2(x) + u^2(y)) K(x-y) \, dx \, dy \\
+ \frac{1}{2} \int \int_{B_R(x_0) \times B_{r_1}^c(x_0)} \phi^2(x) u(x) u(y) K(x-y) \, dx \, dy \\
\leq \int \int_{B_R(x_0)} (\phi(x) - \phi(y))^2 u^2(x) K(x-y) \, dx \, dy \\
+ \frac{1}{2} \int \int_{B_R(x_0)} \phi^2(x) u(x) \left( \int_{B_{r_1}^c(x_0)} u(y) K(x-y) \, dy \right) \, dx \\
\leq \frac{4 \Theta_{n,s}}{(R-r)^{2s}} \|u\|^2_{L^2(B_R(x_0))} \\
+ \Lambda c_{n,s} \left( \frac{2R}{R-r} \right)^{n+2s} \|u\|_{L^1(B_R(x_0))} \int_{B_{r_1}^c(x_0)} \frac{|u(y)|}{|y-x_0|^{n+2s}} \, dy. \]

Since \( \mathcal{T}(u^-; x_0, R) = 0 \), it follows from Theorem 4.4 and Proposition 4.5 that

\[ \mathcal{I} - \frac{4 \Theta_{n,s}}{(R-r)^{2s}} \|u\|^2_{L^2(B_R(x_0))} \leq \frac{2^{n+2s} \Lambda c_{n,s}}{(R-R)^{2s}} \|u\|^2_{L^2(B_R(x_0))} \left( \frac{R}{R-2} \right)^{-2s} \mathcal{T}(u; x_0, R/2) \]

\[ \leq \frac{2^{n+4} c_2 \Lambda c_{n,s}}{(R-r)^{2s}} \|u\|^2_{L^2(B_R(x_0))} \sup_{B_{R/2}(x_0)} u \]

\[ \leq \frac{C}{(R-r)^{2s}} \left( \frac{R}{R-r} \right)^n |B_R(x_0)| \frac{1}{\|B_R(x_0)\|} \int_{B_R(x_0)} u^2(y) \, dy \]

\[ = \frac{C}{(R-r)^{2s}} \left( \frac{R}{R-r} \right)^n \|u\|^2_{L^2(B_R(x_0))}. \]

Thus, by (5.7), we obtain that

\[ \int_{B_r(x_0)} V(x)|u(x)|^2 \, dx + \lambda c_{n,s} \int_{\mathbb{R}^n_{+2s}} \frac{|\phi(x)u(x) - \phi(y)u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \\
\leq \mathcal{I} \leq \frac{4 \Theta_{n,s} + C}{(R-r)^{2s}} \left( \frac{R}{R-r} \right)^n \|u\|^2_{L^2(B_R(x_0))}. \]

Using \( \varphi(x) = \phi_R^2(x)u(x) \) as a testing function where \( \phi_0 = \phi_{R^*} \cdot x_0 \) for \( r < R^* < (3R + r)/4 < R \), via the same way as the above we arrive at the estimate

\[ \int_{B_R(x_0)} V(x)|u(x)|^2 \, dx + \lambda c_{n,s} \int_{\mathbb{R}^n_{+2s}} \frac{|\phi_0(x)u(x) - \phi(y)u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \\
\leq \mathcal{I} \leq \frac{16 \Theta_{n,s} + C}{(R-r)^{2s}} \left( \frac{R}{R-r} \right)^n \|u\|^2_{L^2(B_R(x_0))}. \]

Combining this with (5.10), the required estimate can be achieved. \( \square \)

**[Proof of Theorem 1.2]** Let \( s \in (0, 1) \) and \( x_0 \in \Omega \). Given \( k \in \mathbb{N} \), let \( \ell_i = \frac{3}{4} + \frac{1}{4k} \) and \( R_i = \ell_i R \) for \( i = 1, \cdots, k + 1 \), and also let \( R_i^* = R_i + \frac{1}{8k} R \) for \( i = 1, \cdots, k \).
For $i = 1, \cdots, k$, let us denote by
\[
\phi_i(x) = \left( \frac{R_i^- - |x - x_0|}{R_i^- - R_i^+} \vee 0 \right) \wedge 1
\]
with $R_i < R_i^+ = \frac{R_i + R_{i+1}^-}{2} < R_{i+1}$. Fix any $i = 1, \cdots, k$ and any $R \in (0, d(x_0, \partial \Omega))$. Applying Lemma 3.2 with $\phi_i u$, it follows from Lemma 5.7 that
\[
\int_{B_{R_i}(x_0)} |u(y)|^2 \left[ m V(y) \right]^{2s} dy \leq C_{n,s} \left( \|u\|^2_{L^2(B_{R_i^+}(x_0))} + \|\phi_i u\|^2_{H^s(\mathbb{R}^n)} \right) \tag{5.11}
\]
\[
\leq \frac{C k^{n+2s}}{R^{2s}} \|u\|^2_{L^2(B_{R_{i+1}}(x_0))},
\]
because $R_{i+1} - R_i = \frac{1}{4k} R$ and $\frac{R_{i+1}^- - R_i^+}{R_{i+1}^- - R_i^-} \leq 4k$. From (c) of Lemma 3.1 and (5.11), we obtain that
\[
\|u\|^2_{L^2(B_{R_i}(x_0))} \leq \frac{C k^{n+2s}(1 + R m V(x_0)) \frac{2^{kd_0}}{|\mathbb{R}^n|^k}}{R^{2s} [m V(x_0)]^{2s}} \|u\|^2_{L^2(B_{R_{i+1}}(x_0))}.
\]
Continuing this process $k$-times from $i = 1$ to $i = k$ yields that
\[
\|u\|^2_{L^2(B_{R_k}(x_0))} \leq \frac{C k^{k(n+2s)}(1 + R m V(x_0)) \frac{2^{skd_0}}{|\mathbb{R}^n|^k}}{R^{2sk} [m V(x_0)]^{2sk}} \|u\|^2_{L^2(B_{R_0}(x_0))}.
\]
Applying Proposition 4.5, we can derive from the above inequality that
\[
\sup_{B_{\frac{R_k}{2}}(x_0)} u \leq \frac{C^\frac{k}{2} k^{(\frac{n}{2} + s)k} (1 + R m V(x_0)) \frac{2^{k^2d_0}}{|\mathbb{R}^n|^k}}{R^{sk} [m V(x_0)]^{sk}} \left( \frac{1}{R^n} \int_{B_{R_k}(x_0)} u^2(y) dy \right)^{\frac{1}{2}}. \tag{5.12}
\]
From the well-known Stirling’s formula $k^k \sim e^k k!/(2\pi k)^{-1/2}$ as $k \to \infty$, we see that there is a constant $c_0 > 0$ such that $k^k \leq c_0 e^{k^2 k}$ for any $k \in \mathbb{N}$. Combining (5.12) with Proposition 4.5 yields that
\[
\frac{\sup_{B_{\frac{R_k}{2}}(x_0)} u}{\left( \frac{1}{R^n} \int_{B_{R_k}(x_0)} u^2(y) dy \right)^{\frac{1}{2}}} \leq \frac{C^\frac{k}{2} k^{(\frac{n}{2} + s)k} \left( 1 + R m V(x_0) \right) \frac{2^{k^2d_0}}{|\mathbb{R}^n|^k}}{(1 + R m V(x_0)) \frac{2^{skd_0}}{|\mathbb{R}^n|^k}} \quad \text{for all } k \in \mathbb{N}. \tag{5.13}
\]
Multiplying (5.13) by $(1 + R m V(x_0)) \frac{2^{kd_0}}{|\mathbb{R}^n|^k} e^{k^2} / (k!)^{\frac{n}{2} + s}$ and adding up on $k \in \mathbb{N}$ where the constant $\varepsilon > 0$ is chosen so small that $\varepsilon e^{\frac{n}{2} + s} C^{1/2} < 1$, we obtain that
\[
\Xi \left( \varepsilon \left( 1 + R m V(x_0) \right) \frac{2^{kd_0}}{|\mathbb{R}^n|^k} \right) \left( \sup_{B_{\frac{R_k}{2}}(x_0)} u \right) = \left( \sup_{B_{\frac{R_k}{2}}(x_0)} u \right) \sum_{k=0}^{\infty} \left( \varepsilon \left( 1 + R m V(x_0) \right) \frac{2^{kd_0}}{(k!)^{\frac{n}{2} + s}} \right)^k \leq c_0 \sum_{k=0}^{\infty} \left( \varepsilon e^{\frac{n}{2} + s} C^{1/2} \right)^k \left( \frac{1}{R^n} \int_{B_{R_k}(x_0)} u^2(y) dy \right)^{\frac{1}{2}} \leq C \left( \frac{1}{R^n} \int_{B_{R_k}(x_0)} u^2(y) dy \right)^{\frac{1}{2}}.
\]
Hence we conclude that
\[
\sup_{B_{\frac{1}{2}}(x_0)} u \leq \frac{C}{\Xi(\epsilon(1 + R m_V(x_0)))^{\frac{n}{n+1}}} \left( \frac{1}{R^n} \int_{B_R(x_0)} u^2(y) dy \right)^{\frac{1}{2}}.
\]
□

[Proof of Theorem 1.1] Let \( \mathbf{e}_V \) be a fundamental solution for the operator \( L_V = L_K + V \). By Theorem 1.1 [CK], we have that
\[
0 \leq \mathbf{e}_V(x - y) \leq \frac{C}{|x - y|^{n-2s}}.
\]
Take any \( x \in \mathbb{R}^n \). Since \( \text{supp}(\delta_y) = \{y\} \), we see that \( u(z) := \mathbf{e}_V(z - y) \) satisfies the nonlocal equation (5.1) on \( B_{R}(x) \) where \( R = |x - y|/2 \). Applying Theorem 1.2 to \( u(z) \), we obtain that
\[
\mathbf{e}_V(x - y) \leq \sup_{B_{\frac{1}{2}}(x)} |u| \leq \frac{C}{\Xi(\epsilon(1 + R m_V(x)))^{\frac{n}{n+1}}} \left( \frac{1}{R^n} \int_{B_R(x)} |u(z)|^2 dz \right)^{\frac{1}{2}}.
\]
Since \( R = |x - y|/2 \), we see that \( |z - y| \geq |x - y| - |z - x| \geq |x - y|/2 \) for any \( z \in B_R(x) \), and thus we have that
\[
\mathbf{e}_V(x - y) \leq \frac{C}{\Xi(\epsilon(1 + \frac{1}{2}|x - y| m_V(x)))^{\frac{n}{n+1}}} |x - y|^{n-2s}.
\]
Hence we are done.
□

Corollary 5.8. Let \( V \in RH^q \) be a nonnegative potential for \( q > \frac{n}{2s} \) with \( s \in (0, 1) \) and \( n \geq 2 \). Then for any \( N > 0 \) there exists a constant \( C_N > 0 \) possibly depending on \( n, \lambda, s \) such that
\[
0 \leq \mathbf{e}_V(x - y) \leq \frac{C_N}{(1 + |x - y| m_V(x))^N} |x - y|^{n-2s} \quad \text{for } x, y \in \mathbb{R}^n.
\]
Proof. For any \( N \in (0, \infty) \), it is easy to check that
\[
C_N = \sup_{t > 0} \frac{t^N}{\Xi(t)} < \infty.
\]
Hence the required estimate immediately follows from Theorem 1.1.
□

6. \( L^p \) and \( L^p - L^q \) mapping properties of the inverse of the nonlocal Schrödinger operator

In this section, we consider the nonhomogeneous nonlocal Schrödinger equation with potential \( V \) given by
\[
L_K u + V u = f \quad \text{in } \mathbb{R}^n,
\]
where \( V \in RH^q \) is nonnegative for \( q > \frac{n}{2s} \) with \( s \in (0, 1) \) and \( n \geq 2 \). Then we see that the function
\[
u(x) = \int_{\mathbb{R}^n} \mathbf{e}_V(x - y) f(y) dy
\]
is a solution of the equation (6.1). We denote the solution by \( S_V f(x) := u(x) \), and so we may write \( S_V = (L_K + V)^{-1} \). We call \( S_V \) the inverse of the nonlocal Schrödinger operator with nonnegative potentials \( V \).
[Proof of Theorem 1.3.] Set \( r = 1/m_V(x) \). Then by (6.2) we may write

\[
S_V f(x) = \int_{B_r(x)} e_V(x - y) f(y) dy + \int_{\mathbb{R}^n \setminus B_r(x)} e_V(x - y) f(y) dy
\]

\[
:= S^1_V f(x) + S^2_V f(x).
\]

By (5.14) and Hölder’s inequality, we have that

\[
|S^1_V f(x)| \leq C r^{2s - \frac{n}{q}} \left( \int_{B_r(x)} |f(y)|^q dy \right)^{\frac{1}{q}}.
\]

Then it follows from (6.4) and changing the order of integrations that

\[
\|V(S^1_V f)\|^q_{L^q(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} \left( \int_{B_{\frac{1}{m_V(x)}}(x)} |f(y)|^q dy \right) \frac{V^q(x)}{m_{2q}^q(x)} dx
\]

\[
= C \int_{\mathbb{R}^n} |f(y)|^q \left( \int_{B_{\frac{1}{m_V(y)}}(y)} \frac{V^q(x)}{m_{2q}^q(x)} dx \right) dy.
\]

By Lemma 3.1 and Lemma 3.3, we obtain that

\[
\int_{B_{\frac{1}{m_V(y)}}(y)} \frac{V^q(x)}{m_{2q}^q(x)} dx \leq C \frac{1}{m_{2q}^q(y)} \int_{B_{\frac{1}{m_V(y)}}(y)} V(x) dx \leq C
\]

\[
\leq C.
\]

From (3.2), (6.5) and (6.6), we have that

\[
\|V(S^1_V f)\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)} \quad \text{for} \quad q > \frac{n}{2s}.
\]

Using Hölder’s inequality as in (6.4) and applying (3.2), Lemma 3.1, Lemma 3.3 and changing the order of integrations, we obtain that

\[
\|V(S^1_V f)\|_{L^1(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} |f(y)| \left( \int_{B_{\frac{1}{m_V(y)}}(y)} \frac{V(x)}{|x - y|^{n-2s}} dx \right) dy
\]

\[
\leq C \int_{\mathbb{R}^n} |f(y)| \frac{1}{m_{2s}^n(y)} \left( m_V^n(y) \int_{B_{\frac{1}{m_V(y)}}(y)} V^q(x) dx \right)^{\frac{1}{q}} dy
\]

\[
\leq C \int_{\mathbb{R}^n} |f(y)| \left( m_V^{n-2s}(y) \int_{B_{\frac{1}{m_V(y)}}(y)} V(x) dx \right) dy
\]

\[
\leq C \|f\|_{L^1(\mathbb{R}^n)}.
\]

From standard interpolation argument between the estimates (6.7) and (6.8), we have that

\[
\| (M_V \circ S^1_V) f \|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for any} \quad p \text{ with} \quad 1 \leq p \leq q.
\]
To deal with $S_V^2 f(x)$, we note that Corollary 5.8 and Hölder’s inequality yield
\[ |S_V^2 f(x)| \leq C_N \int_{B(x)} \frac{|f(y)|}{1 + |x-y|m_V(x))^N |x-y|^{n-2s}} dy \]
\[ \leq C_N r^{2s(1-1/p)} \left( \int_{B(x)} \frac{|f(y)|^p}{(1 + |x-y|m_V(x))^N |x-y|^{n-2s}} dy \right)^{\frac{1}{p}} \]
for $1 \leq p \leq q$ and $r = 1/m_V(x)$, provided that $N > 2s$. Thus we have that
\[ \|V(S_V^2 f)\|_{L^p(\mathbb{R}^n)} \]
\[ \leq C_N \int_{\mathbb{R}^n} |f(y)|^p \left( \int_{B^c(x)} m_V^{2s(1-p)}(x) V^p(x) dx \right) dy. \] (6.10)

If we set $N_1 = \frac{-2s(p-1)d_0}{d_0+1}$ and $S_k = B_{\frac{1}{m_V(x)}}(y) \setminus B_{\frac{1}{m_V(x)}(y)}$ for each $k \in \mathbb{N}$, then it follows from (c) of Lemma 3.1, (3.2), (3.10) and Lemma 3.3 that
\[
\int_{B^c(x)} m_V^{2s(1-p)}(x) V^p(x) dx \]
\[ \leq \int_{B^c(x)} m_V^{2s(p-1)}(y)(1 + |x-y|m_V(y))^{N_1} |x-y|^{n-2s} \]
\[ \leq \frac{C}{m_V^{2s(p)}(y)} \sum_{k=1}^{\infty} \frac{2^{2s(k-1)}}{(1 + 2^{k-1})N_1} \left[ \frac{1}{B_{\frac{1}{m_V(y)}}(y)} \int_{S_k} V^p(x) dx \right] \]
\[ \leq \frac{C}{m_V^{2s(p)}(y)} \sum_{k=1}^{\infty} \frac{2^{2s(k-1)c_1}}{(1 + 2^{k-1})N_1} \left[ \frac{1}{B_{\frac{1}{m_V(y)}}(y)} \int_{S_k} V(x) dx \right]^p \]
\[ \leq C \sum_{k=1}^{\infty} 2^{-(N_1-2s-log_2 c_1)} \left( m_V^{2s}(y) \int_{B_{\frac{1}{m_V(y)}}(y)} V(x) dx \right)^p \]
\[ \leq C, \quad \text{provided that } N > 2(q-1) \text{ is chosen sufficiently large.} \] (6.11)

From (6.10) and (6.11), we thus conclude that
\[ \|(M_V \circ S_V^2 f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for any } p \text{ with } 1 \leq p \leq q. \] (6.12)

From the definition of $S_V$, we observe that
\[ (L_K \circ S_V)f = f - (M_V \circ S_V)f \]
for $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq q$. Therefore the required result can easily be obtained from (6.9) and (6.12). \qed

[Proof of Theorem 1.4.] (a) For $\theta \in [0, n]$, let $\mathfrak{M}_\theta g$ be the fractional maximal operator defined by
\[ \mathfrak{M}_\theta g(x) = \sup_{B(x)} \frac{1}{|B|^{1-\frac{n}{\theta}}} \int_B |g(y)| dy, \]
where the supremum is taken over every ball $B$ containing $x$. Then it is well-known in standard harmonic analysis \[\text{St}\] that there is a constant $C = C(n, p, q) > 0$ such that

\[(6.13) \quad \|\mathcal{M}_\theta g\|_{L^q(\mathbb{R}^n)} \leq C \|g\|_{L^p(\mathbb{R}^n)}\]

for any $p, q$ with $1 < p \leq q < \infty$ and $\theta = n \left(\frac{1}{p} - \frac{1}{q}\right)$. Thus the proof of (a) of Theorem 1.4 can easily derived from the following lemma.

**Lemma 6.1.** Let $s \in (0, 1)$, $n \geq 2$ and $\theta \in [0, 2s)$, and let $V \in RH^\tau$ be nonnegative for $\tau > \frac{n}{2s}$. Then there is a constant $C = C(n, s, \lambda, \theta) > 0$ such that

\[|(M_W \circ S_V) f(x)| \leq C \mathcal{M}_\theta f(x)\]

for any $x \in \mathbb{R}^n$.

**Proof.** Take any $x \in \mathbb{R}^n$ and set $m_V(x) = 1/\rho$ for $V \in RH^\tau$ with $\tau > \frac{n}{2s}$ and $s \in (0, 1)$. If $0 \leq \theta < 2s$, then it follows from (6.2) and Corollary 5.8 that

\[|(M_W \circ S_V) f(x)| = |m_V^{2s-\theta}(x) S_V f(x)| \leq C m_V^{2s-\theta}(x) \int_{\mathbb{R}^n} \frac{|f(y)|}{(1 + |x - y| m_V(x))^{2(2s-\theta)} |x - y|^{n-2s} dy}
\]

\[\leq C \sum_{k=-\infty}^{\infty} A_k^\theta(x) \rho^{2s-\theta} (1 + 2^{k-1}) \left(\frac{(2k-1) n}{2s} - 2n\right) \left(\frac{1}{(2k \rho)^{2s-\theta}} \int_{B_{2k,\rho}(x)} |f(y)| dy\right)
\]

\[\leq C \mathcal{M}_\theta f(x) \sum_{k=-\infty}^{\infty} \text{log} \left(\frac{(2k)^{2s-\theta}}{(1 + 2^{k-1}) (2k-1) n} \right) \leq C \mathcal{M}_\theta f(x),
\]

where $A_k^\theta(x) = B_{2^{k+1},\rho}(x) \setminus B_{2k,\rho}(x)$. Hence we are done. \[\square\]

(b) & (c) We note that

\[(6.14) \quad \frac{1}{|B|^{1-\frac{n}{\theta}}} \int_B |g(y)| dy = \left|\frac{|B_1(0)|^{\theta-1}}{|B|/|B_1(0)|^{1-\frac{n}{\theta}}} \right| \int_{|y-x| < (|B|/|B_1(0)|)^{1/n}} |g(y)| dy
\]

\[\leq |B_1(0)|^{\theta-1} \int_{\mathbb{R}^n} \frac{|g(y)|}{|x - y|^{n-\theta} dy}
\]

for any ball $B \subset \mathbb{R}^n$ with center $x \in \mathbb{R}^n$. This implies that the fractional maximal operator $\mathcal{M}_\theta$ is dominated by the Riesz potential $I_\theta$, i.e.

\[(6.15) \quad \mathcal{M}_\theta g(x) \leq C_{n, \theta} I_\theta |g|(x) = C_{n, \theta} |g| * I_\theta(x)
\]

where $C_{n, \theta} > 0$ is certain constant depending only on $n$ and $\theta \in [0, n)$ and $I_\theta$ is the Riesz kernel given by $I_\theta(y) = |y|^{-n+\theta}$. Then it is easy to check that

\[(6.16) \quad \|I_\theta\|_{L^\infty(\mathbb{R}^n)} < \infty \text{ for any } \theta \in [0, n).
\]
Also, it is well-known [St] that there is a universal constant \( C(n, s) > 0 \) such that
\[
\|I_Qg\|_{L^{s,\infty}(\mathbb{R}^n)} \leq C(n, s)\|g\|_{L^1(\mathbb{R}^n)}
\]
for any \( q \in (1, \frac{n}{n-2s}) \). Thus we can easily derive (b) from (6.15) and (6.17). Finally, we can easily obtain (c) from (6.16) and Proposition 7.1 below, because
\[
1 - \frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{s} = \frac{2s}{n}
\]
when \( r = \frac{n}{n-2s} \). Hence we complete the proof. \( \square \)

7. Appendix

In order to obtain the mapping properties of \( M_W \circ S_V \) on the boundary of the trapezoidal area in Figure 1, we need the following estimates whose proof is self-contained.

**Proposition 7.1.** If \( p \in [1, \infty) \) and \( q, r \in (1, \infty) \) satisfy that
\[
\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r},
\]
then there is a constant \( C = C(p, q, r) > 0 \) such that
\[
\|g \ast h\|_{L^{s,\infty}(\mathbb{R}^n)} \leq C \|h\|_{L^{r,\infty}(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}
\]
for any \( g \in L^p(\mathbb{R}^n) \) and \( h \in L^{r,\infty}(\mathbb{R}^n) \). Moreover, \( C = O((r - 1)^{-p'/q}) \) as \( r \to 1^- \).

**Proof.** For \( N > 0 \) and \( h \in L^{r,\infty}(\mathbb{R}^n) \), we denote by
\[
L_N = \{ y \in \mathbb{R}^n : |h(y)| \leq N \}
\qquad \text{and} \quad
U_N = \{ y \in \mathbb{R}^n : |h(y)| > N \}.
\]
If we set \( h_1 = h \mathbb{1}_{L_N} \) and \( h_2 = h \mathbb{1}_{U_N} \), then we have that
\[
\begin{align*}
\omega_{h_1}(\gamma) &= [\omega_h(\gamma) - \omega_h(N)]\mathbb{1}_{(0,N)}(\gamma), \\
\omega_{h_2}(\gamma) &= \omega_h(N)\mathbb{1}_{(0,N)}(\gamma) + \omega_h(\gamma)\mathbb{1}_{(N,\infty)}(\gamma)
\end{align*}
\]
for any \( \gamma > 0 \).

It is easy to check that
\[
\omega_{g \ast h}(\gamma) \leq \omega_{g \ast h_1}(\gamma/2) + \omega_{g \ast h_2}(\gamma/2)
\]
for any \( \gamma > 0 \).

From (7.1), we see that \( 1 < r < p' \) where \( p' \) is the dual exponent of \( p \). If \( p' < \infty \), then by simple calculation and (7.2) we have that
\[
\int_{\mathbb{R}^n} |h_1(y)|^{p'} \, dy = p' \int_0^N \gamma^{p'-1} \omega_{h_1}(\gamma) \, d\gamma
\]
\[
= p' \int_0^N \gamma^{p'-1} [\omega_h(\gamma) - \omega_h(N)] \, d\gamma
\]
\[
\leq p' \int_0^N \gamma^{p'-1} \|h\|_{L^{r,\infty}(\mathbb{R}^n)}^{r-1} \, d\gamma - p' \int_0^N \gamma^{p'-1} \omega_h(N) \, d\gamma
\]
\[
= \frac{p'N^{p'-r}}{p' - r} \|h\|_{L^{r,\infty}(\mathbb{R}^n)}^{r-1} - N^{p'} \omega_h(N) \leq \frac{p'N^{p'-r}}{p' - r} \|h\|_{L^{r,\infty}(\mathbb{R}^n)}^{r-1},
\]
and so it follows from Hölder’s inequality and (7.4) that
\[
\|g \ast h_1\|_{L^{s,\infty}(\mathbb{R}^n)} \leq \|g\|_{L^p(\mathbb{R}^n)} \|h_1\|_{L^{r,\infty}(\mathbb{R}^n)}
\]
\[
\leq \|g\|_{L^p(\mathbb{R}^n)} \left( \frac{p'N^{p'-r}}{p' - r} \|h\|_{L^{r,\infty}(\mathbb{R}^n)}^{r-1} \right)^{1/p'}.
\]
If \( p' = \infty \), then by applying Hölder’s inequality again we obtain that
\[
\|g * h_1\|_{L^\infty(\mathbb{R}^n)} \leq \|g\|_{L^p(\mathbb{R}^n)} N.
\]

Fix any \( \gamma > 0 \) and choose some \( N \) so that
\[
\frac{\gamma}{2\|g\|_{L^p(\mathbb{R}^n)}} = \begin{cases} 
\left( \frac{p' N^{p'-r}}{p'-r} \gamma/\|h\|_{L^r,\infty(\mathbb{R}^n)} \right)^{1/p'} & \text{if } p' < \infty, \\
N & \text{if } p' = \infty.
\end{cases}
\]

On the other hand, by simple calculation and (7.2), we have that
\[
\int_{\mathbb{R}^n} |h_2(y)| \, dy = \int_0^\infty \omega_{h_2}(\gamma) \, d\gamma = \int_0^N \omega_h(N) \, d\gamma + \int_N^\infty \omega_h(\gamma) \, d\gamma
\]
\[
\leq N \omega_h(N) + \int_N^\infty \gamma^{-r} \|h\|_{L^r,\infty(\mathbb{R}^n)} \, d\gamma
\]
\[
\leq N^{1-r} \|h\|_{L^r,\infty(\mathbb{R}^n)} + \frac{N^{1-r}}{r-1} \|h\|_{L^r,\infty(\mathbb{R}^n)}
\]
\[
= \frac{rN^{1-r}}{r-1} \|h\|_{L^r,\infty(\mathbb{R}^n)},
\]
and thus it follows from Young’s inequality and (7.8) that
\[
\|g * h_2\|_{L^p(\mathbb{R}^n)} \leq \|g\|_{L^p(\mathbb{R}^n)} \|h_2\|_{L^1(\mathbb{R}^n)} \leq \frac{rN^{1-r}}{r-1} \|h\|_{L^r,\infty(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.
\]

By (7.6), (7.7), (7.9) and Chebychev’s inequality, we conclude that
\[
\omega_{g*h}(\gamma) \leq \omega_{g*h_2}(\gamma/2) \leq \frac{2p}{\gamma^p} \|g * h_2\|_{L^p(\mathbb{R}^n)}^p
\]
\[
\leq \frac{2p}{\gamma^p} \left( \frac{rN^{1-r}}{r-1} \|h\|_{L^r,\infty(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} \right)^p
\]
\[
= \frac{C^g_h}{\gamma^q} \|h\|_{L^r,\infty(\mathbb{R}^n)}^q \|g\|_{L^p(\mathbb{R}^n)}^q.
\]
Therefore the required result can be obtained from (7.10). \( \square \)

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