Controlling the consensus state in networks with symmetries

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Abstract—We study networks described by linear dynamics in the presence of symmetries of the pair \((A, B)\), which induce a partition of the network nodes in orbital clusters. An invariant group consensus subspace can be defined, in which the nodes in the same cluster evolve along the same trajectory in time. Our analysis can be extended to the more general case of equitable clusters. We prove that the network dynamics is uncontrollable in directions orthogonal to the group consensus subspace. We focus on cases for which the dynamics is controllable in directions parallel to this subspace and, under appropriate conditions, we design optimal controllers that drive the group consensus dynamics towards a desired state.

I. INTRODUCTION

The number of real-world systems modeled as complex networks is ever increasing, and ranges from natural [1], [2], technological, [3], [4] and social systems [5], [6] to epidemic spreading [7]. The ultimate goal of being able to arbitrarily affect the behavior of these systems has spurred researchers across different scientific communities to investigate the controllability properties of linear complex networks [8], [9]. In this framework, several works [10], [11] have revisited the classical tools of structural controllability [12] from the viewpoint that in order to control complex networks, controllability must be guaranteed by a proper selection of the set of nodes (the driver set) in which control signals are injected. If the selection of the driver nodes ensures structural controllability, then the network will also be controllable in Kalman’s sense for all possible edge weights but for a set of Lebesgue measure zero. Among the combinations of edge weights inside this set, there are those that induce the emergence of symmetries [13], [14] or equitable partitions in the network graph. In the presence of symmetries, there exist permutations of the network nodes that leave the graph unchanged, and the sets of nodes that permute among themselves induce a partition of the network in clusters. On the other hand, an equitable partition [15] clusters the network nodes such that the sum of the incoming edges in any node of the same cluster from nodes in any cluster is the same. While symmetries and equitable partitions cause loss of controllability [16], they also induce the emergence of group consensus [17], [18], i.e., solutions in which the state of each node in the same cluster is the same.

In this work we focus on networks with symmetries or equitable partitions and we show that loss of controllability and emergence of group consensus are different sides of the same coin, which is due to the presence of invariant subspaces that are smaller than the entire network state space. While these subspaces allow group consensus solutions to emerge, we also show that they encompass the network controllable subspace. Altogether, our results show that the best case scenario is that one can control the group consensus solution. This can be done by designing controllers on a reduced network, whose nodes correspond to clusters of nodes of the original network, yielding a substantial computational advantage in the control design. Our theoretical analysis is supported by an illustrative numerical example.

II. MATHEMATICAL PRELIMINARIES

We denote by \(\mathcal{G}(V, E)\) an undirected graph with \(V = \{v_i, \ i = 1, \ldots, N\}\), the set of \(N\) nodes, and \(E \subseteq V \times V\), the set of edges defining the interconnections among the nodes. The symmetric binary matrix \(A \in \mathbb{R}^{N \times N}\) is the adjacency matrix of the graph, that is, a matrix whose elements are \(A_{ij} = A_{ji} = 1\) if \((i, j) \in E\) and \(A_{ij} = A_{ji} = 0\) otherwise. A permutation \(\pi(V) = \tilde{V}\) is an automorphism (or symmetry) of \(\mathcal{G}\) if (i) \(\forall i \in V\), i.e., \(\pi\) does not add or remove nodes, and (ii) \((i, j) \in E\), then \((\pi(i), \pi(j)) \in \tilde{E}\). The set of automorphisms of a graph with adjacency matrix \(A\), with the operation composition, is the automorphism group which we will denote by \(\text{aut}(\mathcal{G}(A))\). Any permutation of this group can be represented by a permutation matrix \(P\) that commutes with \(A\), i.e., such that \(PA = AP\). The set of all automorphisms in the group will only permute certain subsets of nodes (the orbits or clusters) among each other. For any two nodes in the same orbit there exists a permutation that maps them into each other. Moreover, we call coarsest orbital partition the partition of the nodes corresponding to the orbits of the automorphism group.

Given a partition \(\Pi\) of the set \(V\) of the network nodes \(V\) into \(s\) subsets \(\{S_1, S_2, \ldots, S_s\}\), such that \(\bigcup_{i=1}^{s} S_i = V\), \(S_i \cap S_j = \emptyset\) for \(i \neq j\), we can introduce the \(N \times s\) indicator matrix \(E_{\Pi}^\text{in}\), such that \(E_{\Pi}^\text{in}_{ij} = 1\) if node \(i\) belongs to \(S_j\) and \(E_{\Pi}^\text{in}_{ij} = 0\) otherwise.

III. NETWORK DYNAMICS

We consider a linear dynamical network described by

\[
\dot{x} = Ax + Bu.
\]

where \(x \in \mathcal{X} = \mathbb{R}^N\) is the vector stacking the states of the \(N\) network nodes and \(u\) is the vector stacking the \(M\) input signals.
injected in the network. Consistently, the $N \times N$ symmetric matrix $A$ defines the network topology, while the $N \times M$ matrix $B$ describes the way in which the $M$ input signals affect the network dynamics. Namely, if the $j$-th input is injected in the $i$-th node then $B_{ij} = 1$, while $B_{ij} = 0$ otherwise.

IV. CONTROLLABILITY PROPERTIES OF NETWORKS WITH SYMMETRIES

In this section, we will show how the presence of symmetries in the controlled network $\Gamma$ affects its controllability.

Lemma 1. The subset of automorphisms of $G(A)$ given by the set of matrices $P := \{ P_i : P_i A = A P_i \text{ and } P_i B = B \}$ forms a subgroup of $\text{aut}(G(A))$.

Proof. For the set $\mathcal{P}$ to be a subgroup, the following four properties must be true:

(i) $P_i (P_i P_j) = (P_i P_j) P_i \forall (P_i, P_j) \in \mathcal{P}$;
(ii) $P_i \in \mathcal{P}$ is non singular $\forall i$;
(iii) $I \in \mathcal{P}$;
(iv) given any two matrices $P_i, P_j \in \mathcal{P}$, then $P_i P_j \in \mathcal{P}$.

Proving that the matrices in $\mathcal{P}$ satisfy property (i) and (ii) is trivial as (i) is true for any three square matrices with the same dimensions $(P_i, P_j, P_k) \in \mathcal{P}$ regardless of whether these are, or are not, in $\mathcal{P}$, while (ii) is true as permutation matrices are non singular. Moreover, (iii) holds as $IA = AI = A$, and $IB = B$. Moreover, property (iv) is proved as

$$(P_i P_j) A = P_i (P_j A) = P_i (AP_j) = AP_i P_j = A (P_i P_j)$$

which proves that $P_i P_j A = AP_i P_j \forall (P_i, P_j) \in \mathcal{P}$. Then, finally, the proof is completed by noting that, as from our hypotheses $P_i B = P_j B = B \forall (P_i, P_j) \in \mathcal{P}$, it follows that $P_i P_j B = P_i B = B$. \hfill \square

We will denote as $\text{aut}(G(A, B))$ the group represented by the permutation matrices $P$ such that $PA - AP = 0$ and $PB - B = 0$. Similarly to $\text{aut}(G(A))$, $\text{aut}(G(A, B))$ partitions the set of network nodes into orbits, where an orbit is a subset of symmetric nodes. Hence, we can define the coarsest orbital partition $\Pi_{or}$ into clusters corresponding to the orbits of the automorphism group $\text{aut}(G(A, B))$, $C_1, C_2, \ldots, C_K$, such that $\cup_{i=1}^{K} C_i = V$, and $C_i \cap C_j = 0$ for $i \neq j$. We will rely on the indicator matrix $E^{\Pi_{or}}$ to keep track of the orbit to which each node belongs.

Lemma 2. Each orbit of the coarsest partition $\Pi_{or}$, induced by $\text{aut}(G(A, B))$ is a subset of an orbit of the coarsest partition induced by $\text{aut}(G(A))$.

Proof. The thesis follows from the observation that if two (or more) nodes are permuted by a permutation matrix $P$ in $\text{aut}(G(A, B))$ and thus belong to the same orbit, then they also belong to the same orbit of the coarsest orbital partition induced by $\text{aut}(G(A))$, as the same matrix $P$ also belongs to $\text{aut}(G(A))$. \hfill \square

Theorem 1. If there exists a permutation matrix $P \neq I$ such that $PA - AP = 0$ and $PB - B = 0$, then

(i) the set of states $X_{or} := \{ x : x_i = x_l \forall i, l \in C_j, \forall j \} \subset X$, is an invariant subspace of the matrix $A$, i.e., $\forall x \in X_{or}, Ax \in X_{or}$;
(ii) if $x_i = x_j$ then $\dot{x}_i = \dot{x}_j \forall (i, j) \in C_j$ and for all $j$.

Proof. Let us start by showing that if there exists a permutation matrix $P$ such that $PA = AP$ and $PB = B$, then the network state $x$ and the permuted state vector $y := Px$ share the same dynamics. Indeed, by left multiplying both sides of eq. (1) by $P$ we get

$$P \dot{x} = PAx + PBu.$$  

Then, as $PA = AP$ and $PB = B$, we get

$$\dot{y} = Ay + Bu.$$  

Now, as there always exists a permutation matrix $P \in \text{aut}(G(A, B))$ that maps into each other any two nodes belonging to the same clusters [19], this proves statement (ii), i.e., that nodes in the same clusters share the same dynamics, and thus that if $x_i = x_j$ for all $i$ and $j$ in the same cluster, then also $\dot{x}_i = \dot{x}_j$. Moreover, this also means that the subspace made of all the points of the state-space such that $x_i = x_j$ for all $(i, j)$ in the same cluster and for each of the $K$ clusters is $A$-invariant (statement (i)). \hfill \square

Theorem 1 establishes the existence of the group consensus subspace $X_{or}$ for network $\Gamma$. Hence, to tackle consensus control problems, it is useful to perform a transformation that allows us to separate the dynamics along the subspace $X_{or}$ from that orthogonal to the subspace $X_{or}$ itself. This task is accomplished by the so called Irreducible Representation (IRR) of the symmetry group through a transformation in a new coordinate system [17]. This is a state transformation $z_{or} = T_{or} x$ where the transformation matrix

$$T_{or} = \left[ \begin{array}{c|c} T^\parallel & T^\perp \end{array} \right] \in \mathbb{R}^{N \times N}$$

is orthogonal, and the elements of the block $T^\parallel \in \mathbb{R}^{K \times N}$ are such that

$$T^\parallel_{ij} = \sqrt{|C_i|}^{-1}$$

if node $j$ is in cluster $i$ and 0 otherwise. The $K$ rows of the matrix $T^\parallel$ are thus a basis of the group consensus subspace $X_{or}$. The rows of the matrix $T^\perp \in \mathbb{R}^{(N-K) \times N}$, which complete the transformation, are thus a basis of the orthogonal complement to the group consensus subspace. Consistently, we have that the dynamic matrix $\tilde{A} = T_{or} A T_{or}^\perp$ has the following structure:

$$\tilde{A} = T_{or} A T_{or}^\perp = \left[ \begin{array}{c|c} A^\parallel & 0 \\ \hline 0 & A^\perp \end{array} \right].$$

From eq. (3), we see that the IRR decouples motion along the consensus subspace from that orthogonal to the group consensus subspace. In this new coordinate system, the dynamics of network $\Gamma$ can be rewritten as

$$\dot{z}_{or} = \tilde{A} z_{or} + \tilde{B} u,$$

and

$$\tilde{B} = T_{or} B = \left[ \begin{array}{c} \tilde{B}_1 \\ \tilde{B}_2 \end{array} \right].$$
Indeed, the pair \((A\|, B\|)\), which we will denote as the quotient pair, determines the controllability properties of the dynamics along the subspace \(\mathcal{X}_{or}\) and thus our ability to control the consensus state, while the pair \((A_⊥, B_⊥)\) determines our ability to stabilize such solution. We are interested in studying the controllability properties of the two pairs \((A\|, B\|)\) and \((A_⊥, B_⊥)\). Before doing so, we will present a few more details on this representation. First of all, let us point out that the block \(T_{ij}\) of the matrix \(T\) is such that \(T_{ij} = E_{or}^i\), where \(E_{or} \in \mathbb{R}^{N \times K}\) is the indicator matrix corresponding to the coarsest partition \(\Pi_{or}\). Consistently, the state of the quotient network, the network associated to pair \((A\|, B\|)\), can be computed as

\[ z_{or}^\parallel = E_{or}^\parallel x \in \mathbb{R}^K \]

and thus, we have that \(A\| = E_{or}^\parallel A E_{or}\) and \(B\| = E_{or}^\parallel B\).

**Remark 1.** Note that the quotient network associated to the coarsest orbital partition does not encompass symmetries, i.e., the only permutation matrix \(P\) such that \(PA\| - A\|P = 0\) and \(PB\| - B\| = 0\) is the identity matrix.

Now, we are ready to give the following theorem.

**Theorem 2.** If there exists a matrix \(P \neq I\) such that \(PA = AP\) and \(PB = B\), then \(\mathcal{X}_{or}\), the invariant subspace of the matrix \(A\) associated to the cluster consensus solution, encompasses the controllable subspace.

**Proof.** To prove the statement we must show that if \(PB = B\), this subspace encompasses the range of \(B\). Indeed, if \(PB = B\), as left-multiplying a vector by the matrix \(P\) only permutes the elements associated to nodes of the same cluster, \(B\) is such that \(b_{il} = b_{jl}\) for all \(l\) and for all \(i, j\) in the same cluster. Hence, all the columns of \(B\) and thus its range, are encompassed in the \(A\)-invariant subspace defined by the clusters (see Theorem 1). As the controllable subspace is defined as the smallest \(A\)-invariant subspace encompassing the range of \(B\), the thesis follows. \(\square\)

**Corollary 1.** \(B_⊥ = 0_{(N-K) \times M}\).

**Proof.** The statement is a direct consequence of the statement of Theorem 2 and of the definition of \(B_⊥\). \(\square\)

V. CONTROLLABILITY PROPERTIES OF NETWORKS WITH EQUITABLE PARTITIONS

In this section we extend the results of section IV to the case in which the network clusters correspond to an equitable partition.

**Definition 1.** Given a graph \(\mathcal{G}\), a partition of the nodes \(V(\mathcal{G})\) in \(K\) clusters \(C_1, C_2, \ldots, C_K\) is equitable if

\[ \sum_{p \in C_i} A_{lp} = d_{ip} \quad \forall l \in C_i; \]

(5)

We will denote such a partition by \(\Pi_{eq}\) and the corresponding indicator matrix by \(E_{eq}\).

Let us now extend the definition of equitable partition to the graph induced by the pair \((A, B)\).

**Definition 2.** A partition \(\Pi_{eq}\) of the node set \(V(\mathcal{G}(A, B))\) of the graph \(\mathcal{G}(A, B)\) induced by the pair \((A, B)\) is equitable if and only if

1) \(\sum_{k \in C_i} A_{lk} = d_{il} \quad \forall l \in C_i;\)
2) \(B_{lp} = d_{lp} \quad \forall l \in C_i, \quad \forall p = 1, \ldots, M.\)

We denote by \(\bar{E}\) the indicator matrix corresponding to \(\Pi_{eq}\).

Note that all the orbital partitions of a graph \(\mathcal{G}(A, B)\) are equitable but the converse is not true [19], [20]. An example of an equitable partition that is not orbital is shown in Fig as the equitable partition has two clusters \(C_1\) and \(C_2\), with its nodes colored in red and yellow respectively, while the coarsest orbital partition defines three clusters \(\{1, 2, 3, 4\}\), \(\{5, 6, 7, 8\}\) and \(\{9, 10\}\). Also, all the clusters of the orbital partition \(\Pi_{or}\) are subsets of the clusters of the equitable partition \(\Pi_{eq}\). Now, we are ready to give the following theorem.

**Theorem 3.** Let \(\mathcal{G}(A, B)\) be the graph induced by the pair \((A, B)\) and \(\Pi_{eq}\) be an equitable partition of the nodes of \(\mathcal{G}(A, B)\) with indicator matrix \(\bar{E}\). Then,

a) \(\Pi_{eq}\) is equitable if and only if the column space of \(\bar{E}\) is \(A\)-invariant;

b) the column space of \(\bar{E}\) encompasses the controllable subspace.

**Proof.** In proving a) we start from the definition of \(A\)-invariance, that is, the column space of \(\bar{E}\) is \(A\)-invariant if and only if there exists a matrix \(Q\) such that \(A \bar{E} = \bar{E}Q\) \(\subseteq \mathbb{R}^{K \times M}\). Then, we show that if \(\Pi_{eq}\) is equitable, then \(Q = (\bar{E}^T \bar{E})^{-1} \bar{E}^T A \bar{E}\). To do so, we need to prove that

\[ A \bar{E} = \bar{E} (\bar{E}^T \bar{E})^{-1} \bar{E}^T A \bar{E} \]

which can be easily done by left multiplying both terms of this expression by \(\bar{E}^T\), yielding

\[ \bar{E}^T A \bar{E} = \bar{E}^T (\bar{E}^T \bar{E})^{-1} \bar{E}^T A \bar{E} \]

which implies that \(\bar{E}^T A \bar{E} = \bar{E}^T A \bar{E}\) thus proving statement a). To prove b) note that, as \(B_{lp} = d_{lp} \quad \forall l \in C_i, \quad \forall p = 1, \ldots, M\) the range of \(B\) is encompassed in the \(A\)-invariant subspace generated by the columns of \(\bar{E}\) and as the controllable subspace is defined as the smallest \(A\)-invariant subspace encompassing the range of \(B\), b) is proved. \(\square\)

**Definition 3.** The coarsest equitable partition \(\varphi_{eq}\) of the graph \(G(A,B)\) is the equitable partition of the graph \(G(A,B)\) with the minimum number \(K\) of clusters. We denote by \(E_{\varphi}\) the corresponding indicator matrix.

Let us write the transformation matrix as done in Section III

\[ T_{eq} = \begin{bmatrix} E_{eq}^T \hline T_\perp \end{bmatrix} \]

(6)

with the rows of \(E_{\varphi}^i = \text{span}\{E_{\varphi}^{i(1)}, E_{\varphi}^{i(2)}, \ldots, E_{\varphi}^{i(K)}\}\) where \(E_{\varphi}^{i(j)}\) is the \(i\)-th column of \(E_{\varphi}\), and \(T_\perp\) is an \((N-K) \times N\) matrix whose rows span the orthogonal complement to the column space of \(E\). Then, we can give the following two Corollaries to Theorem 3.
Moreover, statement b) of Theorem 3 implies the existence of \( E \) which admits solution \( T_{eq}x = \mathbb{R}^N \) the transformed network dynamics is

\[
\dot{z}_{eq} = \hat{A}z_{eq} + \hat{B}u
\]

where the matrices

\[
\hat{A} = \begin{bmatrix} A_\parallel & 0 \\ 0 & A_\perp \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_\parallel \\ 0 \end{bmatrix},
\]

with the dimensions of each block being defined by that of the matrix \( A_\parallel := E_c^\dagger AE_c \in \mathbb{R}^{K \times K} \). Moreover, if the pair \((A_\parallel, B_\parallel)\) is controllable, then the transformation \( T_{eq} \) is a controllability transformation.

**Proof.** Note that, as by definition of the matrix \( T_{eq} \) in eq. (6), \( A_\parallel \) is the quotient network, and thus the first \( K \) state variables capture the dynamics along the column space of \( E_c \). Hence, from Theorem 3 a), which states that the column space of \( E_c \) is \( A \)-invariant, we can prove the existence of the \( 0 \) block in \( \hat{A} \). Moreover, statement b) of Theorem 3 implies the existence of the \( 0 \) block in \( \hat{B} \), as the dynamics orthogonal to the column space of \( E_c \) are uncontrollable.

**VI. CONTROLLING GROUP CONSENSUS**

In Sections IV and V, we have established some controllability limitations of networks with symmetries and equitable partitions. Here, we show how to operate within these limitations so to control group consensus.

**Corollary 3.** Consider a graph \( G(A,B) \) with coarsest equitable partition \( \varphi_c \). If the pair \((A_\parallel, B_\parallel)\) is controllable, then for any cost function \( J(u(t)) \) the optimal control problem

\[
\min_u \int_0^{t_f} J(u(t))dt \quad \text{s.t.} \quad \dot{x} = Ax + Bu \\
(8a)
\]

admits solution \( u(t)^* := \arg\min_{u(t)} \int_0^{t_f} J(u(t))dt \) if and only if \( x_{f,i} = z_{f,i}^\parallel \) for all \( i \in C_1 \) and for all \( l \). Moreover, if \( x_{f,i} = z_{f,i}^\parallel \), then \( u^* = u^{**} \), where \( u^{**} \) is the solution of the following optimal control problem

\[
\min_u \int_0^{t_f} J(u(t))dt \quad \text{s.t.} \quad \dot{z}_{eq} = A_\parallel z_{eq} + B_\parallel u \\
(9a)
\]

which implies that \( z_{f}^\parallel = 0 \). This is ensured independently of \( u \) as \( z_{f}^\parallel(0) = 0 \) and as from Theorem 3 we know that \( z_{f}^\parallel \) are the state variables of the non-controllable subsystem of the pair \((A,B)\). Finally, from eq. (7) we know that left-multiplying eq. (6a) by \( T_{eq} \) yields the set of equations

\[
\dot{z}_{eq} = A_\parallel z_{eq} + B_\parallel u
\]

which captures completely the dynamics in eq. (8b) independently of \( u \). Hence, problem (8) and the reduced order problem in (9) share the same decision variables, cost function, and constraints which implies that \( u^* = u^{**} \).

**Remark 2.** Note that as orbital partitions are also equitable, Corollary 3 also holds for networks with symmetries.

**Remark 3.** Corollary 3 provides an approach to control the consensus solution. Note however that this solution is not stabilizable neither in the case of symmetries nor in that of equitable partitions, as the dynamics orthogonal to the group consensus subspace are uncontrollable (see Theorems 2 and 3). However, in both cases, the transformations in eqs. 3 and 6 allow to study the stability of the group consensus solution by computing the eigenvalues of the block \( A_\perp \) of the matrices \( \hat{A} \) in eq. (3) and \( \hat{A} \) in eq. (7) respectively. Note that the block \( A_\perp \) of the matrix \( \hat{A} \) of the irreducible representation in eq. (3) is itself block-diagonal, with each block representing the dynamics orthogonal to the consensus subspace of single or intertwined clusters [17]. Hence, in the case of symmetries, analysis of the eigenvalues of each one of the diagonal subblocks of \( A_\perp \) in eq. (3) provides information about which clusters will asymptotically reach consensus (and which ones will not).

**Remark 4.** Note that Corollary 3 provides an approach to design an input to control group consensus. A viable alternative is to solve

\[
\min_u \int_0^{t_f} J(u(t))dt
\]
with $E_{\phi}$ being the indicator matrix of an equitable partition $C_1, C_2, \ldots, C_K$ of the network nodes, and

$$\begin{bmatrix} y_i \\ \|C_i \end{bmatrix}$$

being the consensus value for all the nodes of the cluster $C_i$.

VII. NUMERICAL EXAMPLE

We consider the $N = 10$ node network in Fig. 1. The reader familiar with structural controllability theory [12], [22], will note that this network is structurally controllable as there is a cycle encompassing all of its nodes, which are also all accessible from the control signals. Hence, one could expect this network to be controllable also in Kalman’s sense. However, if the edge weights are selected as in Fig. 1, then an equitable partition $\phi_{eq}$ clusters the network nodes in $K = 2$ clusters, $C_1 \cup C_2 = V$ and $C_1 = \{1, 2, 3, 4\}$ and $C_2 = V \setminus C_1$. The corresponding indicator matrix is

$$E_{\phi}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \quad (12)$$

Consistently with Corollary 2 performing the state transformation $z_{eq} = T_{eq}x$, with the matrix $T_{eq}$ selected according to eq. (6) we obtain that $B_{\perp \phi} = 0$. Moreover, we have that

$$A_{\parallel} = \begin{bmatrix} -10 & 3 \\ 2 & -8 \end{bmatrix}, B_{\parallel} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (13)$$

and the reader may easily check that the pair $(A_{\parallel}, B_{\parallel})$ is controllable. Hence, we can exploit the results in Section VI to control group consensus. Indeed, to steer the network towards the group consensus state $[1_{1 \times 4} \ 2_{1 \times 6}]^T$ with minimum energy, from Corollary 3 instead of solving

$$\begin{bmatrix} x(0) = 0 \\ x(1) = [1_{1 \times 4} \ 2_{1 \times 6}]^T \end{bmatrix}$$

we can solve

$$\begin{bmatrix} \|z_{eq} \| = A_{\parallel} \|z_{eq} \| + B_{\parallel}u \\ \|z_{eq}(0) = 0_{2 \times 1} \\ \|z_{eq}(1) = [1 \ 2]^T \end{bmatrix}$$

s.t.

$$\begin{bmatrix} \min_{u} \frac{1}{2} \int_{0}^{1} u(t)^T u(t) dt \\ \dot{x} = Ax + Bu \end{bmatrix}$$

$$y = E_{\phi}^T x$$

$$x(0) = 0$$

$$y(t_f) = y_f.$$ 

s.t.

$$\begin{bmatrix} x(0) = 0 \\ x(1) = [1_{1 \times 4} \ 2_{1 \times 6}]^T \end{bmatrix}$$

$$(11b) \quad (11c) \quad (11d) \quad (11e)$$

$$= A_{\parallel} \|z_{eq} \| + B_{\parallel}u$$

$$\|z_{eq}(0) = 0_{2 \times 1}$$

$$\|z_{eq}(1) = [1 \ 2]^T$$

that is, the optimal control input is a linear combination of the two eigenmodes corresponding to the two clusters of the partition $\phi_{eq}$ of $G(A, B)$.

Thus $u^{**}$ can be used to control the original network whose graph is depicted in Figure 1. Note that the optimal control
input (17), that is shown in Figure 2, is able to steer nodes in $C_1$ to 1 and nodes in $C_2$ to 2 at $t_f = 1$, as shown in Figure 3.

VIII. CONCLUSIONS
Motivated by the observation that symmetries and equitable partitions induce both loss of controllability and the emergence of group consensus, in this work we studied the controllability properties of networks endowed of symmetries or equitable partitions. We found that in either case, controllability is lost in directions orthogonal to the group consensus subspace, but we can still control the consensus state either if the network initial condition belongs to the group consensus subspace, or if the subsystem of the dynamics orthogonal to this subspace is asymptotically stable. Moreover, we showed that when the network controllable subspace coincides with the group consensus subspace, we can control consensus by designing control strategies on a lower-dimensional network, the quotient network, thus reducing the computational burden. We demonstrated our theoretical analysis through a representative numerical example.

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