MARSTRAND-TYPE PROJECTION THEOREMS FOR LINEAR PROJECTIONS AND IN NORMED SPACES

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Abstract. We establish Marstrand-type as well as Besicovich-Federer-type projection theorems for closest-point projections onto hyperplanes in the normed space $\mathbb{R}^n$. In particular, we prove that if a norm on $\mathbb{R}^n$ is $C^{1,1}$-regular, then the analogues of the well-known statements from the Euclidean setting hold. On the other hand, we construct an example of a $C^1$-regular norm in $\mathbb{R}^2$ for which Marstrand-type theorems fail. These results are obtained by comparison arguments.

1. Introduction

This paper is concerned with the behavior of Hausdorff measure and Hausdorff dimension under projections along linear foliations of $\mathbb{R}^n$ and in finite dimensional normed spaces. Let $A \subseteq \mathbb{R}^2$ be a Borel set. By $\dim A$ denote its Hausdorff dimension and by $\mathcal{H}^s$ its Hausdorff $s$-measure where $s > 0$. For every angle $\theta \in [0, \pi)$ consider the orthogonal projection $P_{\mathbb{R}^2}^\theta : \mathbb{R}^2 \to L_\theta$ of $\mathbb{R}^2$ onto the line $L_\theta = \{r(\cos \theta, \sin \theta) : r \in \mathbb{R}\} \subset \mathbb{R}^2$. From the facts that $L_\theta$ is a set of dimension 1 and that the projection $P_{\mathbb{R}^2}^\theta$ is a 1-Lipschitz mapping, one easily deduces that $\dim P_{L_\theta}^\theta A \leq \min\{1, \dim A\}$ for all $\theta \in [0, \pi)$. In 1954, Marstrand [22] proved that given a Borel set $A \subseteq \mathbb{R}^2$, the orthogonal projection of $A$ onto the line $L_\theta$ is a set of Hausdorff dimension

$$\dim P_{L_\theta}^\theta A = \min\{1, \dim A\}$$

for $\mathcal{H}^1$-a.e $\theta \in [0, \pi)$, i.e., given a Borel set $A$, there exists an $\mathcal{H}^1$-zero set $E \subset [0, \pi)$ such that $\dim P_{L_\theta}^\theta A = \min\{1, \dim A\}$ for all angles $\theta \in [0, \pi) \setminus E$. This theorem marked the start of a long sequence of results in the same spirit. They are known as Marstrand-type projection theorems and we summarize some of them in Theorem 1.1 below. In order to formally make sense of Theorem 1.1 recall the following definitions. For positive integers $m < n$ we denote by $G(n, m)$ the Grassmannian manifold, i.e. the family of $m$-dimensional linear subspaces ($m$-planes) of $\mathbb{R}^n$. For every $m$-plane $V \subseteq G(n, m)$, let $P_V^\theta : \mathbb{R}^n \to V$ be the orthogonal projection of $\mathbb{R}^n$ onto $V$. We will refer to the set $\{P_V^\theta : V \in G(n, m)\}$ as the family of orthogonal projections (onto $m$-planes). Notice that the Grassmannian $G(n, m)$ is equipped with a natural measure $\sigma_{n,m}$ which is induced by the action of $O(n)$ on $G(n, m)$ and the invariant Haar measure on $O(n)$; see [25, Chapter 3]. Moreover, since $G(n, m)$ carries a (smooth) manifold structure the notion of Hausdorff dimension of subsets of $G(n, n - 1)$ is well-defined.

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The following Theorem is a summary of results due to Marstrand 22, Kaufman 19, Mattila 24, Falconer 11, and Peres-Schlag 29.

**Theorem 1.1.** For each $m$-plane $V \in G(n,m)$ denote by $P^\mathbb{E}_V : \mathbb{R}^n \rightarrow V$ the orthogonal projection of $\mathbb{R}^n$ onto $V$. Then, for all Borel sets $A \subseteq \mathbb{R}^n$, the following hold:

1. If $\dim A \leq m$, then
   a) $\dim P^\mathbb{E}_V A = \dim A$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$,
   b) For $0 < \alpha \leq \dim A$, $\dim \{V \in G(n,m) : \dim(P^\mathbb{E}_V A) < \alpha\} \leq (n - m - 1)m + \alpha$.

2. If $\dim A > m$, then
   a) $\mathcal{H}^m P^\mathbb{E}_V A > 0$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$,
   b) $\dim \{V \in G(n,m) : \mathcal{H}^m(P^\mathbb{E}_V A) = 0\} \leq (n - m)m + m - \dim A$.

3. If $\dim A > 2m$, then
   a) $P^\mathbb{E}_V A \subset \mathbb{R}^m$ has non-empty interior for $\sigma_{n,m}$-a.e. $V \in G(n,m)$,
   b) $\dim \{V \in G(n,m) : \text{the interior of } P^\mathbb{E}_V A \text{ is empty} \} \leq (n - m)m - \dim A + 2m$

Many of the above statements are proven to be sharp; see e.g. 20 [11, 12]. Similar problems have been studied in various settings such as the Heisenberg groups 2, 1, 15 and Riemannian surfaces 4, 5, 21. Moreover, for an overview on the numerous works on the topic of projection theorems, we recommend the textbooks 25, 10, 27 as well as the survey articles 26, 23.

Another important projection theorem with a rather different flavor relates the size of sets under projections to their rectifiability properties. Recall that a subset $A$ of a metric space $(X,d)$ is called $m$-rectifiable if there exist a collection of at most countably many Lipschitz mappings $f_i : \mathbb{R}^m \rightarrow X$ such that $\mathcal{H}^m (A \setminus \bigcup_i f_i(\mathbb{R}^m)) = 0$. On the other hand, a set $E \subseteq \mathbb{R}^n$ is called purely $m$-unrectifiable, if $\mathcal{H}^m (E \cap A) = 0$ for every $m$-rectifiable set $A \subseteq \mathbb{R}^n$. The following theorem is due to Besicovitch 7 and Federer 14; see also 25.

**Theorem 1.2.** An $\mathcal{H}^m$-measurable set $A \subseteq \mathbb{R}^n$ with $\mathcal{H}^m(A) < \infty$ is purely $m$-unrectifiable if and only if $\mathcal{H}^m(P^\mathbb{E}_V A) = 0$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$.

Theorem 1.2 has been generalized to other settings such as families of transversal projections in metric spaces 16 and families of projections in the Heisenberg group 15.

In this paper, we establish versions of the above Theorems for families of linear and surjective projections and families of closest-point projections with respect to sufficiently regular norms on $\mathbb{R}^n$. These results improve parts of the results in 3 jointly achieved with Balogh.

We call a family of mappings $\{P_V : V \in G(n,m)\}$ a family of linear and surjective projections (onto $m$-planes), if for every $V \in G(n,m)$, $P_V : \mathbb{R}^n \rightarrow V$ is a linear and surjective mapping. Notice that the family of orthogonal projections $\{P^\mathbb{E}_V : V \in G(n,m)\}$ is a family of linear and surjective projections. Moreover, every linear and surjective projection $P_V : \mathbb{R}^n \rightarrow V$ is a Lipschitz mapping. Hence it is a natural question whether Marstrand-type projection theorems generalize to families of linear and surjective projections.
Many families of linear and surjective projections \( \{P_V : V \in (n,m)\} \) are given in terms of linear foliations. Namely, if for \( V \in G(n,m) \), we have \( P_V(P_V x) = P_V x \) for all \( x \in \mathbb{R}^n \), then there exists an \((n-m)\)-plane \( W \in G(n,n-m) \) with \( V \cap W = \{0\} \) such that for all \( x \in \mathbb{R}^n \), \( P_V x = a \), where \( x = a + w \), \( a \in V \) and \( w \in W \). The affine \( m \)-planes \( a+W \) with \( a \in V \) are fibers of the foliation of \( \mathbb{R}^n \) induced by \( V \) and \( W \), and it follows that \( \ker P_V = W \). It is straightforward to see that there exist families of linear and surjective projections for which Marstrand-type theorems must fail. Namely, consider \( V_0 \in G(n,m) \) and \( W_0 \in G(n,n-1) \) with \( V_0 \cap W_0 = \{0\} \). Let \( U \) a small open neighbourhood of \( V_0 \) such that \( V \cap W_0 = \{0\} \) for all \( V \in U \). Now, for each \( V \in U \) define \( P_V \) to be the projection onto \( V \) along the fibers \( a+W_0 \), i.e., \( P_V x = a \) where \( x = a + w \), \( a \in V \) and \( w \in W_0 \). Then, whenever the measure or dimension of a Borel set \( A \) is decreased under \( P_{V_0} \), then the same is true for all \( V \in U \). Hence, Marstrand-type results must fail.

On the other hand, we shall prove that given a family \( \{P_V : V \in G(n,m)\} \) of linear and surjective projections, if for every \( V_0 \in G(n,m) \) we can control the size of the set of \( m \)-planes \( V \in G(n,m) \) for which \( P_V \) and \( P_{V_0} \) are projections along the same foliation, then Marstrand-type as well as Besicovitch-Federer-type theorems hold for this family. Define the mapping \( \mathcal{G} : G(n,m) \to G(n,m) \) associated with the family \( \{P_V : V \in G(n,m)\} \) by

\[
\mathcal{G}(V) = (\ker P_V)^\perp.
\]

This notation allows us to state the following analog of classical Marstrand-type projection theorems for families of linear and surjective projections.

**Theorem A.** Let \( \{P_V : V \in G(n,m)\} \) be a family of linear and surjective projections whose associate mapping \( \mathcal{G} \) is dimension non-decreasing and maps \( \sigma_{n,m} \)-positive sets to \( \sigma_{n,m} \)-positive sets. Then, the following hold for all Borel sets \( A \subseteq \mathbb{R}^n \).

1. If \( \dim A \leq m \), then
   a) \( \dim P_V A = \dim A \) for \( \sigma_{n,m} \)-a.e. \( V \in G(n,m) \),
   b) For \( 0 < \alpha \leq \dim A \),
      \[
      \dim \{V \in G(n,m) : \dim (P_V A) < \alpha \} \leq (n-m-1)m + \alpha.
      \]
2. If \( \dim A > m \), then
   a) \( \mathcal{H}^m(P_V A) > 0 \) for \( \sigma_{n,m} \)-a.e. \( V \in G(n,m) \),
   b) \( \dim \{V \in G(n,m) : \mathcal{H}^m(P_V A) = 0\} \leq (n-m)m + m - \dim A \).
3. If \( \dim A > 2m \), then
   a) \( P_V A \subseteq V \simeq \mathbb{R}^m \) has non-empty interior for \( \sigma_{n,m} \)-a.e. \( V \in G(n,m) \),
   b) \( \dim \{V \in G(n,m) : (P_V A)^\circ \neq \emptyset\} \leq (n-m)m + 2m - \dim A \).

By the same methods we also obtain a Besicovitch-Federer-type projection theorem.

**Theorem B.** Let \( \{P_V : V \in G(n,m)\} \) be a family of linear and surjective projections such that for all \( E \subseteq G(n,m) \), \( \sigma_{n,m}(\mathcal{G}^{-1}(E)) = 0 \) if and only if \( \sigma_{n,m}(E) = 0 \). Then, an \( \mathcal{H}^m \)-measurable set \( A \subseteq \mathbb{R}^n \) with \( \mathcal{H}^m(A) < \infty \) is purely \( m \)-unrectifiable if and only if \( \mathcal{H}^m(P_V(A)) = 0 \) for \( \sigma_{n,m} \)-a.e. \( V \in G(n,m) \).
Although Theorems \([\text{A}]\) and \([\text{B}]\) are interesting in their own right (see Section 4) we are particularly interested in applying them to the setting of normed spaces. Let \(\| \cdot \|\) be a strictly convex norm on \(\mathbb{R}^n\), i.e., a norm whose unit sphere \(S_{n-1}^n = \{ x \in \mathbb{R}^n : \| x \| = 1 \}\) is the boundary of a strictly convex set. Then for every \(x \in \mathbb{R}^n\) and every \(m\)-plane \(V \in G(n,m)\), there exists a unique point \(q \in V\) that realizes the distance between \(x\) and \(V\) with respect to \(\| \cdot \|\), i.e., \(\| x - q \| = \text{dist}_{\| \cdot \|}(x, V) := \inf\{ \| x - y \| : y \in V \}\). We call the mapping \(P_{Vx}^n : \mathbb{R}^n \to V\) given by \(P_{Vy}x = q\), where \(\| x - q \| = \text{dist}_{\| \cdot \|}(x, V)\), the closest-point projection with respect to \(\| \cdot \|\) onto \(V\). Obviously, in case that \(\| \cdot \|\) is the standard Euclidean norm \(| \cdot |\) on \(\mathbb{R}^n\) we have \(P_{Vx}^n = P_{V}^1\), for all \(V \in G(n,m)\). We denote the unit sphere with respect to \(\| \cdot \|\) in \(\mathbb{R}^n\) by \(S^{n-1}\).

If \(m = n - 1\) and \(\| \cdot \|\) is a strictly convex norm on \(\mathbb{R}^n\), then one can check that \(\{ P_{Vx}^n : V \in G(n,m) \}\) is a family of linear and surjective projections (see Section 3). If in addition, \(\| \cdot \|\) is assumed to be \(C^1\)-regular (i.e. continuously differentiable outside of \(\{0\}\)), then at every point \(x\) in the hypersurface \(S_{n-1}^n\), the unit outward normal \(G(x) \in S^{n-1}\) in well-defined. This yields a mapping \(G : S_{n-1}^n \to S^{n-1}\) that we call the Gauss map of \(\| \cdot \|\). As we will show in Lemma 3.1, the mapping \(G\) associated with the family \(\{ P_{Vx}^n : V \in G(n,n-1) \}\) of linear and surjective projections can be expressed in terms of the inverse of \(G\). This will allow us to prove the following results for families of projections onto hyperplanes.

**Theorem C.** Let \(\| \cdot \|\) be a strictly convex \(C^1\)-regular norm on \(\mathbb{R}^n\). If the Gauss map \(G\) is dimension non-increasing and maps \(\mathcal{H}^{n-1}\)-zero sets to \(\mathcal{H}^{n-1}\)-zero sets, then the following hold for all Borel sets \(A \subseteq \mathbb{R}^n\).

1. If \(\dim A \leq n - 1\), then
   a) \(\dim (P_{wA}^1) = \dim A\) for \(\mathcal{H}^{n-1}\)-a.e. \(w \in S^{n-1}\),
   b) For \(0 < \alpha \leq \dim A\), \(\dim \{ w \in S^{n-1} : \dim (P_{wA}) < \alpha \} \leq \alpha\).

2. If \(\dim A > n - 1\), then
   a) \(\mathcal{H}^{n-1}(P_{wA}) > 0\) for \(\mathcal{H}^{n-1}\)-a.e. \(w \in S^{n-1}\),
   b) \(\dim \{ w \in S^{n-1} : \mathcal{H}^{n-1}(P_{wA}) = 0 \} \leq 2(n-1) - \dim A\).

**Theorem D.** Let \(\| \cdot \|\) be a strictly convex \(C^1\)-regular norm on \(\mathbb{R}^n\) such that for all \(E \in G(n,n-1)\), \(\sigma_{n,n-1}(G(E)) = 0\) if and only if \(\sigma_{n,n-1}(E) = 0\). Then, an \(\mathcal{H}^{n-1}\)-measurable set \(A \subseteq \mathbb{R}^n\) with \(\mathcal{H}^{n-1}(A) < \infty\) is purely \(m\)-unrectifiable if and only if \(\mathcal{H}^{n-1}(P_{V \perp}^1(A)) = 0\) for \(\sigma_{n,n-1}\)-a.e. \(V \in G(n,n-1)\).

Note that if \(\| \cdot \|\) is \(C^{1,1}\)-regular, then the Gauss map \(G\) (which essentially is the gradient of the norm) is locally Lipschitz and hence the requirements of Theorem \([\text{C}]\) are satisfied. Thus the following corollary is a direct consequence of Theorem \([\text{C}]\).

**Corollary E.** If \(\| \cdot \|\) is a strictly convex \(C^{1,1}\)-regular norm on \(\mathbb{R}^n\), then the conclusions of Theorem \([\text{C}]\) hold for the projections \(P_{wA}^1 : \mathbb{R}^n \to w \perp, w \in S^{n-1}\).
Exploiting the arguments from the proof of Theorem A allows the construction of a $C^1$-regular norm on $\mathbb{R}^2$ for which Theorem C fails.

**Theorem F.** There exists a $C^1$-regular norm $\| \cdot \|$ on $\mathbb{R}^2$ and a Borel set $A \subset \mathbb{R}^2$ with $\dim A \leq 1$ such that $\mathcal{H}^1(\{w \in S^{n-1} : \dim(P_w \| A) < \dim A\}) > 0$. Thus, in particular, Conclusion 1 of Theorem C fails for $\| \cdot \|$. Analogously, there exists a $C^1$-regular norm on $\mathbb{R}^2$ for which Conclusion 2 of Theorem C fails.

Theorem F shows that Marstrand-type projection theorems do not trivially hold for families of projections induced by a norm unless the norm is induced by a scalar product; see Section 6. Thereby, it underlines the importance of Theorem C and in some sense shows the sharpness of Theorem C. Comparing Corollary E with Theorem C raises the question whether or not Marstrand-type theorems hold for $C^{1,\delta}$-regular norms in $\mathbb{R}^2$ (i.e. derivatives of $\| \cdot \|$ of first order are locally $\delta$-Hölder). Surprisingly, the answer to this question is related to the study of the structure of exceptional sets for Euclidean projections. We will address this relation in Section 6.

The paper is organized as follows. In Section 2, we prove Theorems A and B. In Section 3, we prove Theorems C and D by applying Theorems A and B respectively. Section 4 is for Propositions and Examples underlining the independent interest of Theorems A and B. In Section 5, we explicitly construct a $C^1$-regular norm in $\mathbb{R}^2$ for which Theorem C fails and thereby prove Theorem F. Section 6 is for final remarks.

### 2. Linear projections

In this section we prove Theorem A. Consider a family of linear and surjective projections $\{P_V : V \in G(n,m)\}$. Recall that we defined the mapping $\mathcal{G} : G(n,m) \rightarrow G(n,m)$ by

$$\mathcal{G}(V) = (\operatorname{Ker} P_V)^\perp.$$

The following two lemmas will be used to compare the images of a Borel set under $P_V$ and $P_{\mathcal{G}(V)}$.

**Lemma 2.1.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^d$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be linear mappings with $\operatorname{Ker} f = \operatorname{Ker} g$. Then, there exists a bijective linear mapping $h : f(\mathbb{R}^n) \rightarrow g(\mathbb{R}^n)$ such that for all $x \in \mathbb{R}^n$, $h(f(x)) = g(x)$.

**Proof.** In case $U := \operatorname{Ker} f = \operatorname{Ker} g$ equals $\mathbb{R}^n$ or $\{0\}$, the Lemma is trivial. Therefore, we may assume without loss of generality that $0 < k := \dim(U) < n$. Let $u_1, \ldots, u_k$ be a basis of $U$ and extend it to a basis $u_1, \ldots, u_k, w_1, \ldots, w_{n-k}$ of $\mathbb{R}^n$. Then, $f(w_1), \ldots, f(w_{n-k})$ is a basis of $f(\mathbb{R}^n)$ and $g(w_1), \ldots, g(w_{n-k})$ is a basis of $g(\mathbb{R}^n)$. Define a linear mapping $h : f(\mathbb{R}^n) \rightarrow g(\mathbb{R}^n)$ by setting $h(f(w_j)) = g(w_j)$ for all $j = 1, \ldots, n-k$. Then, $h$ is a bijection and for every $x \in \mathbb{R}^n$, $h(f(x)) = g(x)$.

The following Lemma is a trivial consequence of Lemma 2.1. It can be considered the key ingredient in the proofs of Theorem A and Theorem B.

**Lemma 2.2.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^d$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be linear mappings with $\operatorname{Ker} f = \operatorname{Ker} g$ and $A \subset \mathbb{R}^n$ a Borel set. Then, $\dim f(A) = \dim g(A)$, and $\mathcal{H}^m(f(A)) = 0$ if and only if $\mathcal{H}^m(g(A)) = 0$.
Proof of Theorem A. Let $A \subseteq \mathbb{R}^n$ be a Borel set and $0 < \alpha \leq \dim(A) \leq m$. We know that 1.a) and 1.b) of Theorem A hold for \{P_{\mathcal{F}}^E : V \in V\}, that is,

\begin{align*}
(2) & \quad \sigma_{n,m}(\{W \in G(n,m) : \dim P_{\mathcal{F}}^E(A) < \alpha\}) = 0 \\
(3) & \quad \dim \{W \in G(n,m) : \dim P_{\mathcal{F}}^E(A) < \alpha\} \leq \alpha.
\end{align*}

By Lemma 2.2 with $f = P_V$ and $g = P_{\mathcal{F}(V)}^E$, it follows that, for all $V \in G(n,m)$,

\begin{align*}
(4) & \quad \dim P_V(A) = \dim P_{\mathcal{F}(V)}^E(A),
\end{align*}

On the other hand, notice that

\begin{align*}
\sigma_{n,m}(\mathcal{F}\{V \in G(n,m) : \dim P_{\mathcal{F}(V)}^E(A) < \alpha\}) &= \sigma_{n,m}(\{\mathcal{F}(V) \in G(n,m) : \dim P_{\mathcal{F}(V)}^E(A) < \alpha\}) \\
&\leq \sigma_{n,m}(\{W \in G(n,m) : \dim P_{\mathcal{F}}^E(A) < \alpha\})
\end{align*}

Thus, by (4), (5) and the fact that $\mathcal{F}$ does not map $\sigma_{n,m}$-positive sets to $\sigma_{n,m}$-zero sets it follows that $\sigma_{n,m}(\{V \in G(n,m) : \dim P_V(A) < \alpha\}) = 0$. This proves 1.a).

Furthermore, combining (3) and (4) with the fact that $\mathcal{F}$ is dimension non-decreasing, yields

\begin{align*}
\dim \{V \in G(n,m) : \dim P_V(A) < \alpha\} &= \dim \{V \in G(n,m) : \dim P_{\mathcal{F}(V)}^E(A) < \alpha\} \\
&\leq \dim \mathcal{F}\{V \in G(n,m) : \dim P_{\mathcal{F}(V)}^E(A) < \alpha\} \\
&= \dim \{\mathcal{F}(V) \in G(n,m) : \dim P_{\mathcal{F}(V)}^E(A) < \alpha\} \\
&\leq \dim \{W \in G(n,m) : \dim P_{\mathcal{F}}^E(A) < \alpha\} \leq \alpha.
\end{align*}

This proves 1.b). The proofs of 2 and 3 are analogous. \qed

Proof of Theorem B. Let $A \subseteq \mathbb{R}^n$ be $\mathcal{H}^m$-measurable and $\mathcal{H}^m(A) < \infty$. Let

\begin{align*}
E &= \{V \in G(n,m) : \mathcal{H}^m(P_V(A)) > 0\} \\
F &= \{V \in G(n,m) : \mathcal{H}^m(P_{\mathcal{F}}^E(A)) > 0\}
\end{align*}

Recall that by Lemma 2.2 with $f = P_V$ and $g = P_{\mathcal{F}}^E$ we have $\mathcal{H}^m(P_V(A)) = 0$ if and only if $\mathcal{H}^m(P_{\mathcal{F}}^E(A)) = 0$. Thus, it follows that $E$ equals the preimage $g^{-1}(F)$. Hence Theorem B follows from Theorem 1.2. \qed

Remark 2.3. The above proof reveals that the conditions for Theorem B can be slightly weakened. Namely, the following condition on $\mathcal{F}$ suffices for the conclusion of Theorem B to hold: For every $\mathcal{H}^m$-measurable set $A \subseteq \mathbb{R}^n$ with $\mathcal{H}^m(A) < \infty$, the set $E_A := \{V \in G(n,m) : \mathcal{H}^m(P_V(A)) > 0\}$ is a $\sigma_{n,m}$-zero set if and only if $g^{-1}(F_A)$ is a $\sigma_{n,m}$-zero set.
3. Codimension-one projections in normed spaces

In this Section, we consider closest-point projections onto hyperplanes of \( \mathbb{R}^n \) (i.e. \( m = n - 1 \)) that are induced by a norm.

Recall that for a strictly convex norm \( \| \cdot \| \) for every linear subspace \( V \in G(n, m) \) the closest-point projection \( P^\|_V : \mathbb{R}^n \to V \) given by \( \| P^\|_V x - x \| = \text{dist}_{\| \cdot \|}(x, V) \). \( x \in \mathbb{R}^n \), is well-defined. Notice, that for every point \( x \in \mathbb{R}^n \) the point \( P^\|_V x \) can be characterized as the unique point in the intersection \( S^{n-1}_{\| \cdot \|}(x, r) \cap V \), where \( r = \text{dist}_{\| \cdot \|}(V, x) \). Now, in addition, assume that \( \| \cdot \| \) is \( C^1 \)-regular. Then, the unit sphere \( S^{n-1}_{\| \cdot \|} \) with respect to \( \| \cdot \| \) is a compact \( C^1 \)-hypersurface of \( \mathbb{R}^n \) that admits an unit outward normal \( G(x) \in \mathbb{S}^{n-1} \) at every point \( x \in S^{n-1}_{\| \cdot \|} \). We call the map \( G : S^{n-1}_{\| \cdot \|} \to \mathbb{S}^{n-1}, x \mapsto G(x) \) the Gauss map of \( \| \cdot \| \). Notice that by the assumption of \( C^1 \)-regularity of \( \| \cdot \| \), \( G \) is continuous. Moreover, it has the following properties.

**Lemma 3.1.** Let \( \| \cdot \| \) be a strictly convex \( C^1 \)-regular norm on \( \mathbb{R}^n \). Then the Gauss map \( G : S^{n-1}_{\| \cdot \|} \to \mathbb{S}^{n-1} \) is a homeomorphism, \( G(-v) = -G(v) \) and \( \langle v, G(v) \rangle \neq 0 \) for all \( v \in \mathbb{S}^{n-1} \).

**Proof.** Injectivity of \( G \) follows immediately from strict convexity. To see this, assume that \( G \) is not injective, thus, there exist two points \( v, w \in S^{n-1}_{\| \cdot \|}, v \neq w \), with \( G(v) = G(w) \). Hence for the tangent planes we have \( T_vS^{n-1}_{\| \cdot \|} = T_wS^{n-1}_{\| \cdot \|} =: H \). Assume without loss of generality that \( w \) lies on the same side of \( H \) (if not, replace \( w \) by \( -w \)). In case that \( v + H = w + H \), strict convexity implies that \( v = w \) which contradicts the choice of \( v \) and \( w \). Consider the case when \( v + H \neq w + H \). Then, \( H, v + H \) and \( w + H \) are three parallel hyperplanes in \( \mathbb{R}^n \). Moreover, by the assumption that \( v \) and \( w \) lie on the same side of \( H \), \( H \) is not the middle one of these three hyperplanes. Assume that \( v + H \) is the middle one (the other case is analogous). Since \( S^{n-1}_{\| \cdot \|} \setminus \{v\} \) is a continuum containing \( w \) and \( -w \), the affine plane \( v + H \) must intersect \( S^{n-1}_{\| \cdot \|} \) in more than one point. This contradicts strict convexity. Hence, it follows that \( G \) is injective.

Now, consider a direction \( u \in \mathbb{S}^{n-1} \) and let \( V \) be its orthogonal complement. Since \( S^{n-1}_{\| \cdot \|} \) is compact, the set \( \{ t > 0 : S^{n-1}_{\| \cdot \|} \cap (tu + V) \neq \emptyset \} \) has a maximum \( t_0 > 0 \). Thus, \( H := t_0u + V \) is the (affine) tangent plane of \( S^{n-1}_{\| \cdot \|} \) at the point \( x \) where \( S^{n-1}_{\| \cdot \|} \) intersects \( L_u = \{ tu : t \in \mathbb{R} \} \). Moreover, since \( V \) was chosen to be orthogonal to \( u \), it follows that \( G(x) = u \). Hence, \( G \) is surjective.

Finally, notice that by antipodal symmetry of \( \| \cdot \| \), that is \( \| v \| = \| -v \| \) for all \( v \in \mathbb{S}^{n-1} \), it follows that \( G(-v) = -G(v) \) for all \( v \in \mathbb{S}^{n-1} \).

We will prove Theorem [C] by applying Theorem [A]. Therefore, the following lemma is essential.

**Lemma 3.2.** For a strictly convex \( C^1 \)-norm \( \| \cdot \| \), \( \{ P^\|_V : V \in G(n, n - 1) \} \) is a family of linear and surjective projections. Moreover, for all \( V \in G(n, n - 1) \),

\[
\mathcal{G}(V) = (G^{-1}(w))^\perp,
\]

where \( w = w(V) \in \mathbb{S}^{n-1} \) orthogonal to \( V \).
Proof. Let \( V \in G(n, n-1) \). First, recall that for all \( x \in \mathbb{R}^n \setminus V \), \( P^{|\cdot|}_V(x) \) is the unique point in the intersection \( S^{|\cdot|}_V \cap V \), where \( r = \text{dist}_{|\cdot|}(x, V) \). Therefore, \( V \) must be the tangent plane of \( S^{|\cdot|}_V \) at \( P^{|\cdot|}_V(x) \); see Figure 1.

![Figure 1. Gauss map and projections.](image)

However, this implies that the unit outward normal of \( S^{|\cdot|}_V \) at \( P^{|\cdot|}_V(x) \) is orthogonal to \( V \), or, equivalently (see Figure 1), that \( G(u) \perp V \), where \( u = P^{|\cdot|}_V(x) - x \parallel P^{|\cdot|}_V(x) - x \parallel \).

Let \( w = w(V) \in S^{n-1} \) be a direction that is orthogonal to \( V \), then for some \( \lambda \in \{-1, 1\} \), we have \( G(u) = \lambda w \). Using the fact that \( G \) is invertible and antipodally symmetric yields \( u = \lambda G^{-1}(w) \).

Thus, for every \( x \in \mathbb{R}^n \), the projection direction \( P_V x - x \) is collinear with \( u = G^{-1}(w) \) and \( u \) does not depend on \( x \) but only on \( V \). Moreover, by Lemma 3.1 \( u = G^{-1}(w) \) is not contained in \( V \). Hence, \( P^{|\cdot|}_V(x) \) is the unique intersection point of the affine line \( x + L_u \) with the \( m \)-plane \( V \) (recall that \( L_u := \{rv : r \in \mathbb{R}\} \) for all \( v \in \mathbb{R}^n \setminus \{0\} \)). This proves that \( P^{|\cdot|}_V : \mathbb{R}^n \to V \) is a linear and surjective mapping. Moreover, \( (P^{|\cdot|}_V)^{-1}(\{0\}) = L_u \), and thus, \( \mathcal{G}(V) = u^\perp = (G^{-1}(w))^\perp \).

Notice that in order to prove Theorem C it suffices to check that the map \( \mathcal{G} \) associated with the family of closest-point projections with respect to \( |\cdot| \) is dimension non-increasing and maps \( \sigma_{n,n-1} \)-positive sets to \( \sigma_{n,n-1} \)-positive sets. The main ingredient for this will be Lemma 3.2. Lemma 3.2 states that the associated mapping \( \mathcal{G} : G(n, n-1) \to G(n, n-1) \) basically equals the inverse Gauss map \( G^{-1} : S^{n-1} \to S^{n-1} \), once we identify hyperplanes \( V \in G(n, n-1) \) by the outward normals \( \{w, -w\} \subset S^{n-1} \). However, by our assumptions on \( G \), the inverse Gauss map \( G^{-1} \) has all the desired properties. In the below proof we carry out the details of this strategy.

**Proof of Theorem C.** Let \( F \subseteq G(n, n-1) \) measurable. We will show that \( \dim(\mathcal{G}(F)) \geq \dim F \) and thereby establish that \( \mathcal{G} \) is dimension non-decreasing. Recall from the introduction that the
notion of $\mathcal{H}^s$-zero sets on $G(n, n-1)$ can be understood in terms of smooth chart maps for the Grassmannian manifold $G(n, n-1)$. From this fact, one easily deduces that a set $A \subset G(n, n-1)$ is an $\mathcal{H}^s$-zero set in $G(n, n-1)$ if and only if $\{v \in S^{n-1} : v^\perp \in A\}$ is an $\mathcal{H}^s$-zero set in $S^{n-1}$. Moreover, as a consequence of this equivalence, $\dim A = \dim \{v \in S^{n-1} : v^\perp \in A\}$. Thus, for our set $F$ it follows that

\begin{equation}
\dim \mathcal{G}(F) = \dim \{v \in S^{n-1} : v^\perp \in \mathcal{G}(F)\}.
\end{equation}

(6)

Recall that any norm on $\mathbb{R}^n$ is bi-Lipschitz equivalent to the Euclidean norm. In particular, so is our norm $\|\cdot\|$. This is equivalent to the fact that the mapping $S^{n-1} \to S^{n-1}_{\|\cdot\|}$, $v \mapsto \frac{v}{\|v\|}$ is bi-Lipschitz equivalent (with respect to the Euclidean norm $|\cdot|$). Hence, for all sets $A \subset S^{n-1}$, it follows that $\mathcal{H}^s(A) = 0$ if and only if $\mathcal{H}^s(\{\frac{v}{\|v\|} : v \in A\}) = 0$, for all $s > 0$. Therefore, in particular,

\begin{equation}
\dim A = \dim \{\frac{v}{\|v\|} : v \in S^{n-1} : v^\perp \in A\}.
\end{equation}

(7)

Combining this equality with \(6\) yields

\begin{equation}
\dim \mathcal{G}(F) = \dim \{u \in S^{n-1}_{\|\cdot\|} : u^\perp \in \mathcal{G}(F)\}.
\end{equation}

(8)

The condition that $u^\perp \in \mathcal{G}(F)$ in \(7\) is equivalent to the existence of a hyperplane $V \in F$ for which $u^\perp = \mathcal{G}(V)$. However, by Lemma 3.2 the equality $u^\perp = \mathcal{G}(F)$ is equivalent to the equality $u = G^{-1}(w)$ where $w \in S^{n-1}$ with $w^\perp = V$. Plugging this into \(7\) yields

\begin{equation}
\dim \mathcal{G}(F) = \dim \{G^{-1}(w) \in S^{n-1} : w^\perp \in F\}
\end{equation}

(9)

\begin{equation}
= \dim(G^{-1}\{w \in S^{n-1} : w^\perp \in F\}).
\end{equation}

(10)

By Lemma 3.1 $G$ is a homeomorphism and by our assumption it is dimension non-increasing. Thus, $G^{-1}$ is dimension non-decreasing homeomorphism. Hence, from \(8\) and the argument above \(6\), it follows that

\begin{equation}
\dim \mathcal{G}(F) \geq \dim \{w \in S^{n-1} : w^\perp \in F\} = \dim F.
\end{equation}

This proves that $\mathcal{G}$ is dimension non-decreasing.

Now we prove that $\mathcal{G}$ maps $\sigma_{n,n-1}$-positive sets to $\sigma_{n,n-1}$-positive sets. Let $F \subset G(n, n-1)$ be measurable. It follows from the definition of $\sigma_{n,n-1}$ that

\begin{equation}
\sigma_{n,n-1}(F) = \mathcal{H}^{n-1}(\{v \in S^1 : v^\perp \in F\})
\end{equation}

(11)

\begin{equation}
\sigma_{n,n-1}(\mathcal{G}(F)) = \mathcal{H}^{n-1}(\{v \in S^1 : v^\perp \in \mathcal{G}(F)\})
\end{equation}

(12)

Then, by the arguments given above equations \(11\) and \(12\), we may conclude

\begin{equation}
\sigma_{n,n-1}(\mathcal{G}(F)) = \mathcal{H}^{n-1}(\{u \in S^{n-1}_{\|\cdot\|} : u^\perp \in \mathcal{G}(F)\})
\end{equation}

(13)

\begin{equation}
= \mathcal{H}^{n-1}(G^{-1}\{w \in S^{n-1} : w^\perp \in F\}).
\end{equation}

(14)

Recall that $G$ is a homeomorphism that maps $\mathcal{H}^{n-1}$-zero sets to $\mathcal{H}^{n-1}$-zero sets. Hence, in case $\sigma_{n,n-1}(F) > 0$ it follows that

\begin{equation}
\sigma_{n,n-1}(\mathcal{G}(F)) = \mathcal{H}^{n-1}(G^{-1}\{w \in S^{n-1} : w^\perp \in F\}) > 0.
\end{equation}

□
Proof of Theorem C. Given the above proof of Theorem C, in order to prove Theorem D it suffices to check that $\mathcal{G}$ maps $\sigma_{n,n-1}$-zero sets to $\sigma_{n,n-1}$-zero sets. Let $F \subset G(n, n-1)$ be a $\sigma_{n,n-1}$-zero set. Since zero sets are measurable, by (11), it follows that

\begin{equation}
\sigma_{n,n-1}(\mathcal{G}(F)) = B^{-1}(G^{-1}\{w \in S^{n-1} : w^\perp \in F\}).
\end{equation}

Recall that $G$ is a homeomorphism and that by assumption the preimages of zero sets under $G^{-1}$ are zero sets. Thus, by (12), it follows that $\sigma_{n,n-1}(\mathcal{G}(F)) = 0$. □

4. LINEAR PROJECTIONS THAT ARE NOT INDUCED BY A NORM

In this section we emphasize the independent interest of Theorems A and B. Namely, we will show that there exist many families of linear and surjective projections onto hyperplanes satisfying the conditions of Theorem A that cannot be induced by a norm. First of all, notice that Theorems A and B apply in all codimensions (i.e. for all $1 \leq m < n$) while Theorems C and D only apply for codimension 1 (i.e. $m = n - 1$). Indeed, projections induced by a norm are in general not linear if the codimension is larger that 1; see Section 6. In the sequel of this section, we will show that also for codimension 1 there are many natural families of linear and surjective projections that are not induced by a norm.

Given a mapping $\mathcal{G} : G(n, m) \to G(n, m)$ we may define a family of linear and surjective projections

\begin{equation}
P_V(x) = P_{\mathcal{G}(V)}(x),
\end{equation}

$V \in G(n, m)$. Then, the associated mapping (1) for this family of projections $\{P_V : V \in G(n, m)\}$ is the given mapping $\mathcal{G}$. Thus, if $\mathcal{G}$ is dimension non-decreasing and does not map $\sigma_{n,m}$-positive sets to $\sigma_{n,m}$-zero sets, then Theorem A applies to the family $\{P_V : V \in G(n, m)\}$. Notice that in order for a mapping $\mathcal{G} : G(n, n-1) \to G(n, n-1)$ to satisfy these conditions, properties such as continuity or injectivity are not required. However, for families of linear and surjective projections that are induced by a strictly convex $C^1$-norm it is known that $\mathcal{G}$ is given by the inverse Gauss map $G^{-1}$. Recall from Lemma 3.1 that $G^{-1}$ is known to be a homeomorphism in this setting. Therefore, we may conclude the following proposition.

**Proposition 4.1.** Every dimension non-decreasing mapping $\mathcal{G} : G(n, n-1) \to G(n, n-1)$ that does not map $\sigma_{n,n-1}$-positive sets to $\sigma_{n,n-1}$-zero sets and fails to be continuous and injective induces a family of linear and surjective projections that satisfies Theorem A and is not given by a strictly convex norm on $\mathbb{R}^n$.

Moreover, as the following Lemma shows, any mapping $\mathcal{G} : G(n, n-1) \to G(n, n-1)$ that is given in terms of the inverse Gauss map of a strictly convex $C^1$-norm possesses at least two fixed points.

**Lemma 4.2.** There exist two vectors $v, w \in S^{n-1}_{\|\cdot\|}$, $v \notin \{w, -w\}$, such that $G(v) = \frac{v}{\|v\|}$ and $G(w) = \frac{w}{\|w\|}$.

**Proof.** Let $v_0 \in S^{n-1}_{\|\cdot\|}$ be a point that maximizes the Euclidean distance to the origin among all $v \in S^{n-1}_{\|\cdot\|}$. Let $\gamma : (-\epsilon, \epsilon) \to S^{n-1}$ be a $C^1$-curve for which $\gamma(0) = v_0$. Thus, $\gamma(0) \in T_{v_0} S^{n-1}_{\|\cdot\|}$. 
Moreover, by choice of $v_0$ and the product rule for derivations, it follows that

$$0 = \frac{d}{dt} \langle \gamma(t), \gamma(t) \rangle |_{t=0} = 2 \langle \dot{\gamma}(0), \gamma(0) \rangle.$$

Since $G(v_0)$ is orthogonal to $T_{v_0}S^{n-1}_{||.||}$ it follows that $G(v_0) = \pm \frac{v_0}{|v_0|}$. Since $G(v_0)$ points outward of $S^{n-1}_{||.||}$ at $v_0$, hence $G(v_0) = \frac{v_0}{|v_0|}$. Analogously, one proceeds for a point $w_0$ that minimizes the Euclidean distance to the origin among all $w \in S^{n-1}_{||.||}$. Then, unless $||.||$ equals the Euclidean norm $|.|$, we have $v_0 \neq w_0$. Notice that for $||.|| = |.|$ the lemma is trivially true. □

Lemma 4.2 immediately implies the following proposition.

**Proposition 4.3.** Every dimension non-decreasing mapping $G : G(n, n-1) \rightarrow G(n, n-1)$ that does not map $\sigma_{n,n-1}$-positive sets to $\sigma_{n,n-1}$-zero sets and fails to have two fixed points, by (13) induces a family of linear and surjective projections that satisfies Theorem [A] and is not given by a strictly convex norm on $\mathbb{R}^n$.

Propositions 4.1 and 4.3 allow the construction of many families of linear and surjective projections that are not induced by a norm and for which Theorem [A] holds. In particular, it is easy to explicitly define and illustrate such families in $\mathbb{R}^2$. Consider the following simple example. For every line $L \in G(2,1)$, let $\alpha(L) \in (0, \pi)$ be some angle and define $h(L) \in G(2,1)$ to be the line that makes a counter-clockwise angle $\alpha(L)$ with $L$. By definition of $h$, for every $L \in G(2,1)$ and every $x \in \mathbb{R}^2$, there exist unique unique $x_L \in L$ and $x_h(L) \in h(L)$ such that $x = x_L + x_h(L)$. Define $P_L : \mathbb{R}^2 \rightarrow L$ by $P_L x = x_L$; see Figure 2. Notice that then $G(L) = (h(L))^\perp$. Hence a line $L \in G(2,1)$ is a fixed point of $G$ if and only if $\alpha(L) = \frac{\pi}{2}$. Therefore, if $\alpha(L) \neq \frac{\pi}{2}$ for all $L \in G(2,1)$, by Proposition 4.3, the family $\{P_L : L \in (2,1)\}$ is not induced by a norm. In particular, if $\alpha$ is constant and not equal to $\frac{\pi}{2}$, then the family $\{P_L : L \in (2,1)\}$ is not induced by a norm and trivially satisfies Theorem [A].

![Figure 2. The projections $P_L : \mathbb{R}^2 \rightarrow L$ induced by $h : G(2,1) \rightarrow (0, \pi)$.](image)

5. A norm for which Marstrand-type theorems fail

It is easy to construct families of linear and surjective projections for which Marstrand-type projection theorem fails. Similar examples are obtained from norms for which the Gauss map is not defined or multivalued for some points; see [3, Figures 4 and 6]. This raises the natural question,
whether there exists a $C^1$-regular norm on $\mathbb{R}^n$ for which Marstrand-type theorems fail for projections onto hyperplanes. In this section, we will construct such a norm on $\mathbb{R}^2$ and thereby prove Theorem F.

The following lemmas will be used in the proof of Theorem F.

**Lemma 5.1.** For $0 < d < 2$, there exists a Borel set $A \subset \mathbb{R}^n$ of dimension $\dim A = d$ whose exceptional set $E = \{w \in S^1 : \dim P_{w \perp} A < \min\{\dim A, 1\}\}$ for the family of orthogonal projections is a set of dimension $\dim E = d$.

**Proof.** Let $0 < d < 1$. As established in [20] $(0 < d < 1)$, there exists a compact set $A \subset \mathbb{R}^2$ of dimension $d$ such that the exceptional set $E = \{w \in S^1 : \dim(P_{w \perp} A) < d\}$ is a set of dimension $\dim(E) = d$. Moreover, by [19] $E$ is a Borel set and by Marstrand’s theorem it follows that $\mathcal{H}^1(E) = 0$. Let $1 \leq d < 2$, then by [11], there exists a compact set $A \subset \mathbb{R}^2$ of dimension $d$ such that the exceptional set $E = \{w \in S^1 : \mathcal{H}^1(P_{w \perp} A) = 0\}$ is a set of dimension $\dim(E) = 2 - d > 0$. Again, this set $E$ is a Borel set and by Marstrand’s theorem it follows that $\mathcal{H}^1(E) = 0$. \qed

**Lemma 5.2.** Let $\|\cdot\|$ be a strictly convex $C^1$-regular norm on $\mathbb{R}^2$. Consider closest-point projections $P_{w \perp} : \mathbb{R}^n \to w \perp$, $w \in S^{n-1}$ and the Gauss map $G : S^1_{\perp \cdot} \to S^1$. Let $0 < d < 1$ (resp. $1 \leq d < 2$) and let $A \subset \mathbb{R}^n$ and $E \subset S^1$ be the sets from Lemma 5.1. Let $E' = \{u \in S^1_{\perp \cdot} : \frac{u}{\|u\|} \in E\}$. Then, whenever $\mathcal{H}^1(G(E')) > 0$, Conclusion 1 (resp. Conclusion 2) of Theorem C fails for $\|\cdot\|$.

The proof of Lemma 5.2 is very similar to the proofs of Theorem A and Theorem C.

**Proof.** Consider the case when $0 < d < 1$. By Lemma 2.2 (applied as in the proof of Theorem A) and Lemma 3.2 we have

$$\mathcal{H}^1(\{v \in S^1 : \dim P_{w \perp} A < \dim A\}) = \mathcal{H}^1(\{w \in S^1 : \dim P_{(G^{-1}(w)) \perp} A < \dim A\})$$

$$= \mathcal{H}^1(\{G(u) \in S^1 : \dim P_{u \perp} A < \dim A\})$$

$$= \mathcal{H}^1(G(\{u \in S^1_{\perp \cdot} : \dim P_{u \perp} A < \dim A\}))$$

$$= \mathcal{H}^1(G(E')) > 0.$$ (14)

Hence, Conclusions 1.a) and 1.b) of Theorem C fail. The case when $1 \leq d < 2$ is analogous. Then, Conclusions 2.a and 2.b of Theorem C fail. \qed

The following two lemmas outsource some technicalities from the proof of Theorem F.

**Lemma 5.3.** Consider an interval $I \subset \mathbb{R}$ and two continuous curves $\alpha : I \to \mathbb{R}^m$ and $\beta : I \to \mathbb{R}^n$. Suppose that there exists a constant $M > 0$ for which

$$|\beta(s) - \beta(s')| \leq M|\alpha(s) - \alpha(s')|,$$

for all $s, s' \in I$. Then, for all Borel sets $F \subseteq [0, 1]$ and for all $t > 0$,

$$\mathcal{H}^d(\beta(F)) \leq (2M)^t \mathcal{H}^d(\alpha(F)).$$ (16)

In particular, if follows that if $\mathcal{H}^1(\beta(F)) > 0$, then $\mathcal{H}^1(\alpha(F)) > 0$. 

**Proof.**
We prove Lemma \textbf{[5.3]} by applying a straightforward covering argument based on the definition of the Hausdorff measure.

\textbf{Proof.} Let \( t > 0 \) and \( F \subseteq I \) a Borel set. In the case when \( \mathcal{H}^d(\alpha(F)) = \infty \), \cite{16} holds trivially. Therefore, we assume that \( \mathcal{H}^d(\alpha(F)) = c \) where \( 0 \leq c < \infty \). Let \( \delta > 0 \). Then, there exists an open covering \( A := \{A_i\}_{i=1}^N \) of \( \alpha(F) \) where \( N \in \mathbb{N} \cup \{\infty\} \) for which \( \text{diam} \ A_i \leq \delta \), for all \( i = 1, \ldots, N \) and \( \sum_{i=1}^N \text{diam} \ A_i \leq c + \delta \). Without loss of generality, assume that \( A_i \cap \alpha(F) \neq \emptyset \) for all \( i = 1, \ldots, N \). Let \( s_i \in I \) such that \( \alpha(s_i) \in A_i \cap \alpha(F) \). Then, by \cite{15}, the family of closed balls \( B_i \) with center \( \beta(s_i) \) and radius \( M \text{diam} \ A_i \) covers \( \beta(F) \) and \( \text{diam} \ B_i = 2M \text{diam} \ A_i \leq 2M\delta \) for all \( i = 1, \ldots, N \). This yields

\[ \mathcal{H}^d_{2M\delta}(\beta(F)) \leq \sum_{i=1}^N (\text{diam} \ B_i)^d \leq (2M)^d \sum_{i=1}^N (\text{diam} \ A_i)^d \leq (2M)^d(c + \delta), \]

and hence \( \mathcal{H}^d(\beta(I)) \leq (2M)^d c \). \( \square \)

The following lemma is an application of Lemma \textbf{[5.3]}

\textbf{Lemma 5.4.} Let \( b \in (0, \infty] \) and let \( f, g : [0, b] \to [0, \infty) \) be two strictly increasing functions. Define \( h(t) := f(t)g(t) \) for all \( t \in [0, b] \). Then, for all Borel sets \( F \subseteq [0, b] \), if \( \mathcal{H}^1(f(F)) > 0 \), then \( \mathcal{H}^1(h(F)) > 0 \).

\textbf{Proof.} Let \( F \subseteq [0, b] \) be a Borel set with \( \mathcal{H}^1(f(F)) > 0 \). Then, by sub-additivity of \( \mathcal{H}^1 \) and the fact that \( f \) is increasing, there exists a number \( n \in \mathbb{N} \) with \( n > \frac{1}{t} \), such that for \( F_n := F \cap \left[ \frac{1}{n}, b \right] \), we have \( \mathcal{H}^1(f(F_n)) > 0 \). For \( s < s' \in \left[ \frac{1}{n}, b \right] \), we have

\[ h(s') - h(s) = f(s')g(s') - f(s)g(s) \geq (f(s') - f(s))g(s') \geq g(h) \frac{f(s)}{g(h)} > 0. \]

Applying Lemma \textbf{[5.3]} for \( \alpha = f : \left[ \frac{1}{n}, b \right] \to [0, \infty) \), \( \beta = h : \left[ \frac{1}{n}, b \right] \to [0, \infty) \), and \( M = \frac{1}{g(h)} \), yields \( \mathcal{H}^1(h(F)) \geq \mathcal{H}^1(h(F_n)) > 0 \). \( \square \)

Our strategy for the proof of Theorem \textbf{[7]} goes as follows. For \( 0 < d < 1 \) consider the Borel set \( A \subseteq \mathbb{R}^2 \) from Lemma \textbf{[5.1]} and its exceptional set \( E = \{ v \in S^1 : \dim(P_{v \perp} A) < \dim A \} \). We construct the norm \( \| \cdot \| \) such that the Gauss map for \( \| \cdot \| \) blows up the exceptional set \( E \) to a set of positive \( \mathcal{H}^1 \)-measure. Thus, by Lemma \textbf{[5.2]} Conclusion 1 of Theorem \textbf{[C]} fails for \( \| \cdot \| \). The construction of such a norm \( \| \cdot \| \) roughly goes as follows. Identify \( S^1 \) with the interval \( [0, 2\pi] \). This identification will be denoted by \( \alpha^{-1} : S^1 \to [0, 2\pi] \). We consider a suitable subset \( K \subseteq \alpha^{-1}(E) \) and construct a strictly increasing and continuous function \( f \) that blows up the set \( K \) to a set of positive length. Then, the integral \( F \) of \( f \) will be strictly convex and \( C^1 \). Now, we roll the graph of \( F \) back up with \( \alpha \) (resp. its extension \( h \)); see Figure \textbf{[3]} Thus, the image \( \Gamma \) of the graph of \( F \) will be a piece of the boundary of a strictly convex set which defines a norm \( \| \cdot \| \) on \( \mathbb{R}^2 \), see Figure \textbf{[4]} We will show that the Gauss map of this norm restricted to \( \Gamma \), will still behave like the function \( f \) in terms of its measure theoretic properties. (The case where \( 1 \leq d < 2 \) is analogous.)

\textbf{Proof of Theorem \textbf{[7]}.} Let \( 0 < d < 1 \) and consider the Borel set \( A \subseteq \mathbb{R}^2 \) from Lemma \textbf{[5.1]} and its exceptional set \( E = \{ v \in S^1 : \dim(P_{v \perp} A) < \dim A \} \). Consider the parameterization \( \alpha : [0, 2\pi) \to S^1 \)
given by $\alpha(t) := (\cos(t), \sin(t))$. Since $\alpha$ is locally bi-Lipschitz, it follows that $\dim(\alpha^{-1}(E)) = d$. Let $0 < s < d$. Then, by definition of the Hausdorff dimension, $\mathcal{H}^s(\alpha^{-1}(E)) = \infty$. Therefore, by [25, Theorem 8.13], there exists a compact set

$$K \subset \alpha^{-1}(E) = \{t \in [0, 1] : \dim P_{\alpha(t)}^e(A) < \dim A\}. \tag{17}$$

with $0 < \mathcal{H}^s(K) < \infty$. We assume without loss of generality that $K \subset [0, 1]$.

Now, define $f : [0, 1] \to [0, 1]$ by

$$f(t) := \frac{1}{2} \left( \frac{1}{\mathcal{H}^s(K)} \mathcal{H}^s([0, t] \cap K) + t \right). \tag{18}$$

Notice that $t \mapsto \mathcal{H}^s([0, t])$ is non-decreasing and continuous. (In case $K$ is the triadic Cantor set the function then $t \mapsto \mathcal{H}^s([0, t] \cap K$ is the triadic Cantor staircase function). Thus, $f$ is a strictly increasing homeomorphism. Furthermore, since $K$ is compact, $[0, 1] \setminus K$ consists of countably many (relatively) open intervals in $[0, 1]$. On each interval in $[0, 1] \setminus K$, $f$ is linear with slope $\frac{1}{2}$. Hence, $\mathcal{H}^1(f([0, 1] \setminus K)) = \frac{1}{2}$ and it follows that $\mathcal{H}^1(f(K)) = \frac{1}{2} > 0$.

Define the mapping $F : [0, 1] \to [0, 1]$ by

$$F(u) := \frac{1}{4} \int_0^u f(t) dt.$$ 

Then, $F : [0, 1] \to [0, 1]$ is an injective and strictly convex $C^1$-mapping with $F(1) \leq \frac{1}{4}$. Define $S \subset \mathbb{R}^2$ by $S := \{r \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} : t \in [0, 1], \ r \geq 0\}$. Moreover, we define the mapping $h : [0, 1] \times [0, 1] \to S$ by $h(x, y) := (1 - y) \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix}$, and the curve $\gamma : [0, 1] \to S$ by $\gamma(t) := h(t, F(t))$. Thus, the curve $\gamma$ parameterizes the arc $h(\text{Graph}(F))$; see Figure 3.

![Figure 3. Construction of the curve $\gamma$ from $F : [0, 1] \to [0, \frac{1}{4}]$.](image-url)

Observe that for all $t \in [0, 1]$,

$$\alpha(t) = \frac{\gamma(t)}{|\gamma(t)|}.$$
Moreover, $\gamma$ is a regular $C^1$-curve and $\dot{\gamma}$ is given by

$$
\dot{\gamma}(t) = \begin{pmatrix}
-(1 - F(t)) \sin(t) & -\cos(t) \\
(1 - F(t)) \cos(t) & -\sin(t)
\end{pmatrix} \begin{pmatrix}
1 \\
\tfrac{1}{4} f(t)
\end{pmatrix}
$$

$$
= (1 - F(t)) \begin{pmatrix}
\cos(t + \frac{\pi}{2}) & -\sin(t + \frac{\pi}{2}) \\
\sin(t + \frac{\pi}{2}) & \cos(t + \frac{\pi}{2})
\end{pmatrix} \begin{pmatrix}
1 \\
\tfrac{1}{4(1 - F(t))} f(t)
\end{pmatrix}.
$$

Notice that since $0 \leq F(t) \leq \frac{1}{4}$, it follows that $4(1 - F(t)) \geq 3$ and

$$
\frac{1}{4(1 - F(t))} \leq \frac{1}{3}.
$$

Consider the curve $\beta : [0, 1] \to S^1$, defined by $\beta(t) := \frac{\dot{\gamma}(t)}{||\dot{\gamma}(t)||}$. As we will establish later, $\beta$ has the following properties:

(P1) $\beta : [0, 1] \to S^1$ is an injective curve that travels in $S^1$ in counterclockwise direction from $\beta(0) = (\tfrac{1}{2}, 0)$ to $\beta(1)$ where $\beta(1) = (\cos(s), \sin(s))$, with $s \in (\tfrac{\pi}{2}, \pi)$.

(P2) $\mathcal{H}^1(\beta(K)) > 0$.

Denote the image of $[0, 1]$ under $\gamma$ by $\Gamma$. From our bounds for the values of $\beta$ at $t = 0$ and $t = 1$ from property (P1), it follows that we can extend the union $\Gamma \cup (-\Gamma)$ to the image of a closed $C^1$-curve $\bar{\Gamma}$, by gluing arcs $R$ and $-R$ to $\Gamma$ and $-\Gamma$, such that the tangential directions at the gluing points agree, as illustrated in Figure 4.

Recall that by property (P1), $\beta$ is injective. Thus, $\bar{\Gamma}$ is a simply closed curve that bounds a strictly convex, antipodally symmetric subset of $\mathbb{R}^2$ with non-empty interior. Hence, $\bar{\Gamma}$ defines a norm $\| \cdot \|$.
on \( \mathbb{R}^2 \) by setting \( S^1_{11} := \tilde{\Gamma} \). Moreover, since \( \beta(t) \) is tangential to \( \tilde{\Gamma} \) at \( \gamma(t) \in \tilde{\Gamma} \) for \( t \in [0,1] \), the Gauss map \( G : S^1_{11} \to S^1 \) of the norm \( \| \cdot \| \) in such points is given by

\[
G(\gamma(t)) = R_{\frac{\pi}{2}} \beta(t),
\]

where \( R_{\frac{\pi}{2}} \) denotes the counterclockwise rotation about the angle \( \frac{\pi}{2} \). Recall that by property (P2), \( \mathcal{H}^1(\beta(K)) > 0 \). Thus, (22) implies that \( \mathcal{H}^1(G(\gamma(t))) > 0 \), where \( G \) denotes the Gauss map of the arc \( \Gamma \) parameterized by \( \gamma \). This proves Theorem \( \Box \) given properties (P1) and (P2) for \( \beta \).

Thus, we are left to prove that \( \beta \) actually does satisfy properties (P1) and (P2). Let us begin by defining shorter notations for the objects appearing in \( \Box \). For \( t \in [0,1] \), we write

\[
M(t) := \begin{pmatrix} \cos(t + \frac{\pi}{2}) & -\sin(t + \frac{\pi}{2}) \\ \sin(t + \frac{\pi}{2}) & \cos(t + \frac{\pi}{2}) \end{pmatrix}
\]

and

\[
v(t) := \left( \frac{1}{4(1-F(t))} f(t) \right).
\]

Hence, \( M(t) \in O(2) \), \( v(t) \in \{(1) \times [0, \frac{1}{3}]\} \subset \mathbb{R}^2 \) (see (21)) and \( \dot{\gamma}(t) = (1-F(t)) M(t) v(t) \), for all \( t \in [0,1] \). Set \( w(t) := \frac{v(t)}{|v(t)|} \) for \( t \in [0,1] \). Then, by (20), and the fact that \( M(t) \in O(2) \) for all \( t \in [0,1] \), it follows that \( \beta(t) = M(t) w(t) \).

Recall that the functions \( f : [0,1] \to [0,1] \) as well as \( F : [0,1] \to [0, \frac{1}{3}] \) are strictly increasing. Thus, in particular, \( t \mapsto \frac{1}{4(1-F(t))} \) is strictly increasing. Also, recall that \( \mathcal{H}^1(f(K)) > 0 \). Hence, the mapping \( \psi : [0,1] \to [0, \frac{1}{3}] \), defined by

\[
\psi(t) := \frac{1}{4(1-F(t))} f(t)
\]

is strictly increasing as well. Moreover, Lemma \( \Box \) implies that \( \mathcal{H}^1(\psi(K)) > 0 \).

Note that \( \mathbb{R} \to \{(1) \times \mathbb{R} \} \subset \mathbb{R}^2, x \mapsto (\frac{1}{2}) \) is an isometric embedding (i.e. a 1-bi-Lipschitz mapping) and \( v(t) = \begin{pmatrix} 1 \\ \psi(t) \end{pmatrix} \). Therefore, \( v : [0,1] \to \{(1) \times [0, \frac{1}{3}]\} \) is injective with \( v(0) = (\frac{1}{0}) \) and \( v(1) = \left( 1/(4(1-F(1))) \right) \), and \( \mathcal{H}^1(v(K)) > 0 \).

Recall that \( w(t) = \frac{v(t)}{|v(t)|} \), for \( t \in [0,1] \). Thus, \( w : [0,1] \to S^1 \) is an injective curve that travels from \( w(0) = v(0) = (\frac{1}{0}) \) to \( w(1) \), see Figure \( \Box \).

For \( t \in [0,1] \), denote by \( \theta(t) \in [0,2\pi) \) the counterclockwise angle from the \( x \)-axis to \( w(t) \), thus

\[
w(t) = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}.
\]
Recall that $v(1) = \left( \frac{1}{4(1 - F(1)))} \right)$ and notice that

$$\frac{1}{1/(4(1 - F(1)))} = 4(1 - F(1)) \geq 3 > \frac{\cos(\frac{s}{2} - 1)}{\sin(\frac{s}{2} - 1)}.$$  

Therefore, it follows that $w(1) = v(1) = \left( \frac{\cos(\theta(1))}{\sin(\theta(1))} \right)$ with $\theta(1) \in (0, \frac{\pi}{2} - 1)$. Moreover, from the fact that $(\{1\} \times [0, \frac{1}{3}]) \to S^1$, $x \mapsto \frac{x}{||x||}$ is a bi-Lipschitz mapping, it follows that $H^1(w(K)) > 0$.

Now, consider the curve $\beta : [0, 1] \to S^1$, $t \mapsto M(t)w(t)$. The matrix $M(t)$ is the matrix of the counterclockwise rotation about the angle $t + \frac{\pi}{2}$.

Thus, it follows that

$$\beta(t) = \left( \frac{\cos(t + \frac{\pi}{2} + \theta(t))}{\sin(t + \frac{\pi}{2} + \theta(t))} \right).$$

This makes $\beta : [0, 1] \to S^1$ an injective curve that travels in $S^1$ in counterclockwise direction from $\beta(0) = (\frac{1}{1})$ to $\beta(1) = \left( \frac{\cos(s)}{\sin(s)} \right)$, where $s := 1 + \frac{\pi}{2} + \theta(1)$ and thus $s \in (1 + \frac{\pi}{2}, \frac{\pi}{2})$; see Figure 6. This proves property (P1).
Moreover, it follows from (23) and (24) that $|\beta(t) - \beta(t')| \geq |w(t) - w(t')|$, for all $t, t' \in [0,1]$. Thus, by Lemma 5.3 and the fact that $H^1(w(K)) > 0$, it follows that $H^1(\beta(K)) > 0$. This proves property (P2).

\[\Box\]

6. Final Remarks

6.1. Codimension greater than 1. As pointed out in the introduction, for every strictly convex norm $\|\cdot\|$ on $\mathbb{R}^n$ and for every $1 \leq m < n$, the family $P_v^{\|\cdot\|} : \mathbb{R}^n \to \mathbb{R}^n$, $V \in G(n,m)$ of closest-point projections with respect to $\|\cdot\|$ is well-defined. Nevertheless, Theorem \[A\] only covers the case when $m = n - 1$. We strongly believe that a statement similar to Theorem \[1,1\] holds for general codimension, i.e., for all $1 \leq m < n$. However our methods do not allow a proof yet. One can check that, in general, for strictly convex norms (even if they have a good differentiable regularity) projections onto plane of codimension greater than one ($m < n - 1$) are not linear mappings and therefore Theorem \[A\] is not applicable. For example, a simple calculation (see [17] Section 5.5) shows that projections onto lines induced by the $L_p$-norm on $\mathbb{R}^n$ for $n \geq 3$ are linear mappings if and only if $p = 2$ (recall that the $L_2$ norm on $\mathbb{R}^n$ is the standard Euclidean norm).

On the other hand, as we shall prove now, in case that a norm $\|\cdot\|$ on $\mathbb{R}^n$ is induced by an inner product space then all Euclidean projection theorems stated in the introduction hold for the family $\{P_v : V \in G(n,m)\}$ for all $1 \leq m < n$. For this, denote the Euclidean inner product (the scalar product) in $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ which is an orthonormal basis with respect to $\langle \cdot, \cdot \rangle$. Moreover, let $\langle \cdot, \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^n$ and $b_1, \ldots, b_n$ an orthonormal basis of $\mathbb{R}^n$ with respect to $\langle \cdot, \cdot, \cdot \rangle$. Then, the linear mapping $\Psi : (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \to (\mathbb{R}^n, \langle \cdot, \cdot, \cdot \rangle)$ defined by $\Psi(b_i) = e_i$ for all $i = 1, \ldots, n$, is an isometry in the sense that $\langle x, y \rangle = \langle \Psi(x), \Psi(y) \rangle$ for all $x, y \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $V \in G(n,m)$, then by definition of $P_v^{\|\cdot\|}$, we have $\|x - P_v^{\|\cdot\|}(x)\| = \text{dist}_V(x, V)$. Since $\Psi$ is an isometry, this implies that $|\Psi(x) - \Psi(P_v^{\|\cdot\|}(x))| = \text{dist}_V(\Psi(x), \Psi(V))$, and hence, by definition of the Euclidean projection, $P_{\Psi(V)}(\Psi(x)) = \Psi(P_v^{\|\cdot\|}(x))$. Hence, it follows that

$$P_v^{\|\cdot\|}(x) = \psi^{-1} \circ P_{\Psi(V)}^{\Psi} \circ \Psi(x),$$

for all $x \in \mathbb{R}^n$ and $V \in G(n,m)$. Therefore, in particular, the projection $P_v^{\|\cdot\|} : \mathbb{R}^n \to V$ is linear and surjective for all $V \in G(n,m)$. Moreover, the mapping $\mathcal{G}$ associated with the family $P_v^{\|\cdot\|} : \mathbb{R}^n \to \mathbb{R}^n$, $V \in G(n,m)$ is given by $\Psi$. Since, $\Psi$ is a linear bijection, $\mathcal{G} : G(n,m) \to G(n,m)$ is a smooth diffeomorphism of manifolds and thus preserves zero-sets and Hausdorff dimension. Therefore, Theorem \[A\] and Theorem \[E\] apply.

6.2. Possibility for a generalization of Theorem \[F\]. The main reason why we cannot state Theorem \[F\] in any greater generality is lack of knowledge about the structure of exceptional sets for orthogonal projection. Notice that the Gauss map $G : S_1^{\|\cdot\|} \to S^1$ of the norm $\|\cdot\|$ constructed in the proof of Theorem \[F\] might turn out to be a $\delta$-Hölder mapping for some $\delta > 0$ depending on the geometry of $K$. This would then imply that there exists a $C^{1,\delta}$-regular norm for which Conclusions 1 and 2 of Theorem \[A\] fail. For example, if $K$ happened to be the triadic cantor set, the mapping $f : [0,1] \to [0,1]$ defined in [18] and therefore also the Gauss map $G : S_1^{\|\cdot\|} \to S^1$
would be $\frac{\log(2)}{\log(3)}$-Hölder mappings. The question about the geometry of the exceptional sets is in general open. In particular, we do not know, whether a set like the triadic Cantor set appears as a subset of such exceptional sets. For a more detailed account on the study of the structure of exceptional sets for orthogonal projections we refer to the works [18, 13, 28, 8] and references therein. Furthermore, we do not know whether Theorem F generalizes to families of projections $P_V : \mathbb{R}^n \to V$, onto $(n-1)$-planes $V \in G(n, n-1)$. The main obstacle is that we do not have a suitable analog of the function $f$ given in equation (18) if $n \neq 2$. Notice that it is of great importance for the construction of $f$ that the continuity of $t \mapsto \mathcal{H}^s([0,t] \cap K)$ is independent of the structure of $K$. However, the tentative higher-dimensional analog of this is not true. For example, if $K \subset [0,1]^2$ contains an isolated line segment parallel to an axis then the mapping $(t,r) \mapsto \mathcal{H}^1(([0,t] \times [0,r]) \cap K)$ is not continuous.

It is an interesting question whether Theorem D holds for the $C^1$-regular norms constructed in the proof of Theorem F. In order to approach this question, we suggest to study the rectifiability properties of the set sets $A$ of Lemma 5.1. As pointed out in the proof of Lemma 5.1 the construction of these sets are due to [20]. They are based on the number theoretic considerations in [9].

6.3. Projection theorems via differentiable transversality. Peres and Schlag [29] establish a very general projection theorem for families of (abstract) projections from compact metric spaces to Euclidean space. Their result states that if a sufficiently regular family of projections satisfies a certain transversality condition, then this yields bounds for the Sobolev dimension of the push-forward (by the projections) of certain measures. All the classical Marstrand-type projection theorems for orthogonal projection in $\mathbb{R}^n$ can be deduced as corollaries from their result; see [29, Section 6] and [27, Section 18.3]. Moreover, Hovila et. al. [16] has proven that if a family of abstract projections satisfies transversality with sufficiently good transversality constants , then this yields a Besicovitch-Federer-type projection theorem for this family of projections. This makes differentiable transversality a very powerful method in establishing projection theorems in various settings. In particular, the works [15] (Heisenberg groups) and [5] (Riemannian surfaces of constant curvature) are based on Peres and Schlag’s notion of transversality.

In fact, one can check that if a strictly convex norm $\| \cdot \|$ on $\mathbb{R}^n$ is $C^{2,\delta}$-regular for some $\delta > 0$ then the induced family of closest-point projections satisfies differentiable transversality. Also, the better the regularity of the norm, the better the transversality constant. This is worked out in detail in [17]. Notice that the transversality constants affect the bounds for the size of the exceptional sets for Marstrand-type theorems deduced from transversality. Therefore, whenever $\| \cdot \|$ fails to be $C^\infty$-regular the Marstrand-type theorems that can be obtained by establishing differentiable transversality are worse than Theorem C. On the other hand, the fact that families of projections induced by a sufficiently regular norm are transversal to some extend can be considered a result of interest independent of projection theorems. Note that for example, families of closest-point projections in infinity dimensional Banach spaces fail to be transversal [6].
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