On the Complexity of Learning Description Logic Ontologies*

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Abstract. Ontologies are a popular way of representing domain knowledge, in particular, knowledge in domains related to life sciences. (Semi-)automating the process of building an ontology has attracted researchers from different communities into a field called “Ontology Learning”. We provide a formal specification of the exact and the probably approximately correct learning models from computational learning theory. Then, we recall from the literature complexity results for learning lightweight description logic (DL) ontologies in these models. Finally, we highlight other approaches proposed in the literature for learning DL ontologies.

Keywords: Description Logic · Exact Learning · Complexity Theory

1 Introduction

Ontologies have been used to build concept and role hierarchies mapping and integrating the vocabulary of data sources, to model definitional sentences in a domain of interest, to support the inference of facts in knowledge graphs, among others. However, modelling an ontology that captures in a precise and clear way the relevant knowledge of a domain can be quite time-consuming. To imagine this, consider the task of writing a text on a particular topic. The writer needs to select the right words, think about their meaning, delineate the scope, and the essential information she or he wants to convey. The knowledge of the writer needs to be clearly represented in a language, using the vocabulary and the constructs available in it. In this way, building an ontology can be seen as a similar process but often there is the additional challenge that the ontology needs to capture the knowledge of a domain in which the ‘writer’—an ontology engineer—is not familiar with. The information in an ontology may need to be validated by a domain expert. Because building and maintaining ontologies are demanding tasks, several researchers have worked on developing theoretical results and practical tools for supporting this process [33,36].

Here we consider two learning models that were applied to model the process of building an ontology. The first is the exact learning model [2]. In the exact learning model, a learner attempts to communicate with a teacher in order to identify an abstract target concept. When instantiating the exact learning model

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to capture the process of building an ontology, one can consider that the teacher is a domain expert who knows the domain but cannot easily formulate it as an ontology \[27\]. On the other side, the role of the learner is played by an ontology engineer. The abstract target that the ontology engineer wants to identify is an ontology that reflects the knowledge of the domain expert (the teacher).

Although the teacher in the exact learning model is often described as a human (potentially a domain expert), it can also be a batch of examples \[7\], an artificial neural network \[51,52\], a Tsetlin machine \[24\] or any another formalism that can be used to simulate the teacher. The most studied communication protocol is based on *membership* and *equivalence* queries. A membership query gives to the learner the ability to formulate an example and ask for its classification (“does X hold in the domain?”). This mode of learning is called *active learning*. In an equivalence query, the learner asks whether a certain hypothesis is equivalent to the target. There are various results in the literature showing that the combination of these two types of queries allows the learner to correctly identify the target concept in polynomial time, with hardness results for the case in which one of the two queries is disallowed \[2\[1\][2][25][40]\].

The second model we study is the classical probably approximately correct (PAC) learning model \[46\]. In the PAC model, a learner receives classified examples according to a probability distribution. Then, the learner attempts to build a hypothesis that is consistent with the examples. This mode of learning is called *passive learning* because, in contrast with the active learning mode, the learner has no control of which examples are going to be classified. One can instantiate this model to the problem of learning ontologies by considering that the ontology engineer starts attempting to collect information about the domain at random (instead of interacting with the domain expert). One can also consider the case in which, in addition to the search at random, the ontology engineer can pose membership queries to the expert. Equivalence queries are not considered in the PAC model because of a general result showing that learners that can pose equivalence queries to learn a certain target are also able to accomplish this task within the PAC model \[2\] (the combination is not interesting because one problem setting is ‘easier’ than the other, more details in Subsection 3.3).

Having these models in mind, we first present the syntax and the semantics of the ontology language \(\mathcal{ELH}\) \[10,13\]. This is a prototypical lightweight ontology language based on description logic (DL). Then, we formalise the exact and the PAC learning models using notions from the theory of computation. We recall from the literature complexity results for learning lightweight DL ontologies in these models and provide intuitions about these results. Finally, we point out other approaches that have been applied for learning DL ontologies.

## 2 Description Logic

We introduce \(\mathcal{ELH}\) \[10,13\], a classical lightweight DL which features existential quantification \((\exists)\) and conjunction \((\sqcap)\). Let \(N_C\), \(N_R\), and \(N_I\) be countably infinite and disjoint sets of *concept* and *role* names. An \(\mathcal{ELH}\) ontology (or *TBox*) is a
finite set of role inclusions (RIs) \( r \sqsubseteq s \) with \( r, s \in \mathbb{N}_R \) and concept inclusions (CIs) \( C \sqsubseteq D \) with \( C, D \in \mathcal{EL} \) concept expressions built according to the rule

\[
C, D ::= A \| \top \| C \cap D \| \exists r.C
\]

with \( A \in \mathbb{N}_C \) and \( r \in \mathbb{N}_R \). An \( \mathcal{ELH} \) TBox is a finite set of RIs and CIs \( C \sqsubseteq D \) with \( C, D \) being \( \mathcal{EL} \) concept expressions. We denote by \( \mathcal{ELH}_{lhs} \) and \( \mathcal{ELH}_{rhs} \) the fragments of \( \mathcal{ELH} \) that allow only a concept name on the right-hand side and on the left-hand side, respectively. That is, \( \mathcal{ELH}_{lhs} \) is the language that allows complex \( \mathcal{EL} \) expressions on the left-hand side and the same idea applies for \( \mathcal{ELH}_{rhs} \). An \( \mathcal{EL} \) TBox is an \( \mathcal{ELH} \) TBox that does not have RIs. We may write \( C \equiv D \) as a short hand for having both \( C \sqsubseteq D \) and \( D \sqsubseteq C \). An assertion is an expression of the form \( r(a, b) \) or of the form \( A(a) \), where \( A \in \mathbb{N}_C \), \( r \in \mathbb{N}_R \), and \( a, b \in \mathbb{N}_I \). An ABox is a finite set of assertions. An \( \mathcal{ELH} \) instance query (IQ) is of the form \( C(a) \) or \( r(a, b) \) with \( C \) an \( \mathcal{EL} \) concept expression, \( r \in \mathbb{N}_R \), and \( a, b \in \mathbb{N}_I \).

We now present the usual semantics of \( \mathcal{ELH} \), which is based on interpretations. An interpretation \( I \) is a pair \((\Delta^I, \cdot^I)\) where \( \Delta^I \) is a non-empty set, called the domain of \( I \), and \( \cdot^I \) is a function mapping each \( A \in \mathbb{N}_C \) to a subset \( A^I \) of \( \Delta^I \), each \( r \in \mathbb{N}_R \) to a subset \( r^I \) of \( \Delta^I \times \Delta^I \), and each \( a \in \mathbb{N}_I \) to an element in \( \Delta^I \). The function \( \cdot^I \) extends to arbitrary \( \mathcal{EL} \) concept expressions as follows:

\[
\begin{align*}
(\top)^I &= \Delta^I \\
(C \cap D)^I &= C^I \cap D^I \\
(\exists r.C)^I &= \{ d \in \Delta^I \mid \exists e \in C^I \text{ such that } (d, e) \in r^I \}
\end{align*}
\]

The interpretation \( I \) satisfies an RI \( r \sqsubseteq s \) iff \( R^I \subseteq s^I \). It satisfies a CI \( C \sqsubseteq D \) iff \( C^I \subseteq D^I \). It satisfies a TBox \( T \) iff \( I \) satisfies all RIs and CIs in \( T \). We write \( I \models \alpha \) if \( I \) satisfies an RI, a CI, or a TBox \( \alpha \). A TBox \( T \) entails a CI \( \alpha \) (or a TBox \( T' \)), written \( T \models \alpha \) (or \( T \models T' \)), iff all interpretations satisfying \( T \) also satisfy \( \alpha \) (or \( T' \)). Two TBoxes \( T \) and \( T' \) are equivalent, written \( T \equiv T' \), iff \( T \models T' \) and \( T' \models T \). These notions can be adapted as expected for defining satisfaction and entailment of assertions, ABoxes, and IQs \[10\]. Given a TBox \( T \), the signature \( \Sigma_T \) of \( T \) is the set of concept and role names occurring in it.

**Example 1.** Although \( \mathcal{ELH} \) is a very simple language, it is already useful to represent certain kinds of static knowledge. One can express in \( \mathcal{ELH} \) that ‘Penicillamine nephropathy is a renal disease’ with the CI

\[
\text{PenicillamineNephropathy} \sqsubseteq \text{RenalDisease}
\]

The definitional sentence ‘Penicillamine nephropathy is a renal disease of the kidney structure caused by penicillamine’ can be expressed as:

\[
\text{PenicillamineNephropathy} \equiv \text{RenalDisease} \cap \text{RenalStructure}
\]

\[1\] This example follows modelling guidelines and terms found in the ontology from the medical domain SNOMED CT \[45\].
An important advantage of representing domain knowledge in an ontology is that ambiguities found in natural language can be removed. For example, one can distinguish when ‘is’ should mean a subset relation (represented syntactically with ‘⊆’ from when it means an equivalence (represented with the symbol ‘≡’).

In the next section, we introduce notions from the theory of computation that are relevant to define learnability and complexity classes in the exact and PAC learning models.

3 The Complexity of Learning

Complexity classes are defined in terms of a model of computation, a type of problem, and bounds on the resources (usually time and memory) needed to solve a problem [44]. In this section, we formally define complexity classes for learning problems (Subsection 3.3). Before that, we define a general model of computation that represents the communication of a learner and a teacher via queries (Subsection 3.1). This model can be specialised for learning problems in the exact and the PAC learning models. For the exact learning model, we assume that the learner can pose membership and equivalence queries. In the PAC learning model, the learner can pose sampling queries. We describe the queries in detail in Subsection 3.2.

3.1 Model of Computation

The main advantage of defining the model of computation is that this opens the possibility of analysing the learning phenomenon in light of the theory of computation. Our model of computation for learning problems is based on learning systems [50]. Learning systems are formulated using the notion of a pair \((L, T)\) of multitape Turing machines (MTM), one for the learner, \(L\), and one for the teacher, \(T\). There are four kinds of tape:

- \(L, T\) share a read-only input tape;
- \(L, T\) share a read-write communication tape;
- \(T\) has a read-only tape, called oracle tape, not accessed by \(L\); and
- \(L\) has a write-only output tape, not accessed by \(T\).

Intuitively, the computation of the two MTMs represents the interaction of the learner and the teacher via queries, where posing a query to the teacher means writing down the input of the query in the communication tape and entering the corresponding query state. Then the teacher computes the answer, writes it in the communication tape (if the computation of the answer terminates), and enters the corresponding answer state. The learner reads the answer in the communication tape and continues its computation. This process may continue forever or halt when the learner writes its final hypothesis in the output tape.
and enters the final state. It is assumed that the teacher never enters the final state (only the learner can enter the final state).

In the definition of a learning system \((L, T)\), we consider that \(L\) is a deterministic MTM (DMTM) with three tapes (input, communication, and output tapes) and that the set of states contains special elements called *query states*, one for each type of query. The teacher \(T\) is a non-deterministic MTM (NMTM) with three tapes (input, communication, and oracle tapes) and the set of states contains special elements called *answer states*, one for each type of query. We describe the types of queries for the exact and PAC learning models in Subsection 3.2.

A DMTM with \(k\) tapes can be defined as a tuple \(M = (Q, \Sigma, \Theta, q_0, q_f)\) where: \(Q\) is a finite set of states; \(\Sigma\) is a finite alphabet containing the blank symbol \(\bot\); \(\Theta: (Q \setminus \{q_f\}) \times \Sigma^k \rightarrow Q \times \Sigma^k \times \{l, r\}^k\) is the transition function; and \(\{q_0, q_f\} \subseteq Q\) are the initial and final states. The expression \(\Theta(q, a_1, \ldots, a_k) = (q', b_1, \ldots, b_k, D_1, \ldots, D_k)\), with \(D_i \in \{l, r\}\), means that if \(M\) is in state \(q\) and heads 1 through \(k\) are reading the symbols \(a_1\) through \(a_k\) (resp.) then \(M\) goes to state \(q'\), the symbols \(b_1, \ldots, b_k\) are written in tapes 1, \ldots, \(k\) (resp.) and each head \(i\) moves to the direction corresponding to \(D_i\). An NMTM is defined in the same way as a DMTM except that \(\Theta: (Q \setminus \{q_f\}) \times \Sigma^k \rightarrow \mathcal{P}(Q \times \Sigma^k \times \{l, r\}^k)\). That is, \(\Theta(q, a_1, \ldots, a_k)\) is now a set of expressions of the form \((q', b_1, \ldots, b_k, D_1, \ldots, D_k)\).

A configuration of a MTM with \(k\) tapes is a \(k\)-tuple \((w_1 q w_1', \ldots, w_k q w_k')\) with \(w_i, w'_i \in \Sigma^*\) and \(q \in Q\), meaning that the tape \(i\) contains the word \(w_i w'_i\), the machine is in state \(q\) and the head is on the position of the left-most symbol of \(w_i\). The notion of *successive configurations* is defined as expected in terms of the transition relations of \(L\) and \(T\). Whenever \(L\) enters a query state, the transition relation of \(T\) is used to define a successive configuration (it may not be unique due to non-determinism of \(T\)) and whenever \(T\) enters again in an answer state then the transition relation of \(L\) defines the (unique) successive configuration \((L\) resumes its execution). A *computation* of \((L, T)\) on an input word \(w_0\) is a tree whose paths are sequences of successive configurations \(\alpha_0, \alpha_1, \ldots\), where \(\alpha_0 = q_0 w_0\) is the initial configuration for the input \(w_0 \in (\Sigma \setminus \{\bot\})^*\) and \(q_0\) is the initial state of \(L\). The branches of the tree correspond to the different possibilities for \(T\) to move from one state to another.

The model of computation that we presented can be generalised to the case in which there are multiple learners and multiple teachers. For our purposes, it suffices to consider only one learner and one teacher. In the following, we explain how the computational model we described can be tailored to the exact and the PAC learning models, as well as some variants of these models.

### 3.2 Learning Frameworks and Queries

To define the learnability in the exact and PAC models (Subsection 3.3), we use the notion of a *learning framework* and three types of queries (membership and equivalence queries for the exact learning model and sample queries for the PAC learning model). A learning framework \(\mathfrak{F}\) is a triple \((\mathcal{E}, \mathcal{L}, \mu)\) where

- \(\mathcal{E}\) is a set of examples,
L is a set of concept representations called hypothesis space, and \( \mu \) is a function that maps each element of \( L \) to a set of examples in \( \mathcal{E} \).

We call target a fixed but arbitrary element of \( L \) that the learner wants to acquire. A hypothesis is an element of \( L \) that represents the ‘idea’ of the learner about the target. This element is often updated during the computation of a learning system \((L, T)\) until \( L \) reaches its final state (if ever). Given a target \( t \in L \), we say that an example \( e \) is positive for \( t \) if \( e \in \mu(t) \), and negative otherwise.

Given a hypothesis \( h \) and a target \( t \) in \( L \) and an example \( e \in \mathcal{E} \), we say that \( e \) is a counterexample for \( t \) and \( h \) if \( e \in \mu(t) \oplus \mu(h) \) (where \( \oplus \) denotes the symmetric difference). We may omit ‘for \( t \)’ and ‘for \( t \) and \( h \)’ if this is clear from the context.

Remark 1. Given a DL \( \mathcal{L} \), we denote by \( \mathcal{F}(\mathcal{L}) \) the learning framework \((\mathcal{E}, L, \mu)\) where \( \mathcal{E} \) is the set of CIs and RIs that can be formulated in \( \mathcal{L} \) (using symbols from \( NC \) and \( NR \)), \( L \) is the set of all \( L \) TBoxes, and

\[
\mu(T) = \{ \alpha \mid T \models \alpha \}, \text{ with } \alpha \text{ a CI or an RI in } \mathcal{L}.
\]

This setting is called learning from entailments. In the learning framework \( \mathcal{F}(\mathcal{E}LHRHS) = (\mathcal{E}, L, \mu) \) we have that \( T = \{ A \sqsubseteq \exists r.A \in L \} \) and, for all \( n \in \mathbb{N} \), the CI \( A \sqsubseteq \exists r^n.A \) is in \( \mu(T) \), where \( \exists r^{n+1}.A := \exists r.\exists r^n.A \) and \( \exists r^1.A := \exists r.A \).

One could define a more general notion of a learning framework, where the hypothesis space for the hypothesis of the learner differs from the hypothesis space that contains the target. Also, the mapping function \( \mu \) could be adapted to represent fuzzy sets of examples. We keep the version introduced above because it is general enough for our purposes and covers classical problems in the literature \([2,46]\). We now describe in detail the queries that the learner can pose and how the teacher answers these queries. Consider a learning framework \( \mathcal{F} = (\mathcal{E}, L, \mu) \), and a learning system \((L, T)\) with a fixed but arbitrary target \( t \in L \) in the oracle tape.

- A **membership query** happens whenever \( L \) writes an example \( e \) in the communication tape and enters the membership query state. In this case, \( T \) resumes the execution and (if the computation terminates) writes ‘yes’ in the communication tape if \( e \in \mu(t) \), otherwise, it writes ‘no’ (assume such answers can be formulated using symbols from the alphabets of \( L \) and \( T \)).
- An **equivalence query** happens whenever \( L \) writes a hypothesis \( h \in L \) in the communication tape and enters the equivalence query state. The teacher \( T \) resumes its execution and (if the computation terminates) writes some \( e \in \mu(t) \oplus \mu(h) \) in the communication tape, or ‘yes’ if \( \mu(t) = \mu(h) \).
- A **sample query** happens whenever \( L \) enters the sample query state. In this case, the teacher \( T \) resumes its execution and (if the computation terminates) writes some \((e, \ell_t(e))\) in the communication tape, where the choice of \( e \in \mathcal{E} \) is according to a fixed but arbitrary probability distribution on \( \mathcal{E} \) (unknown to the learner) and \( \ell_t(e) = 1 \), if \( e \in \mu(t) \), and 0 otherwise.

\[ \text{In Machine Learning, a concept representation is a way of representing a set of examples. This differs from the notion of a concept in DL.} \]
We write \((L_F, T_F(t))\) to indicate that \(t\) is in the oracle tape and queries/answers are as just described for a learning framework \(F\) (we may omit the subscript \(F\) if this is clear from the context). For some learning frameworks and some types of queries, it can be assumed that the computation of answers by the teacher always terminates independently of which \(t \in \mathcal{L}\) happens to be in the oracle tape. One example is when the \(\mu\) function encodes the entailment relation and the entailment problem of the logic represented in \(L\) is decidable (e.g., entailment in \(E\mathcal{LH}\) is decidable in polynomial time [9]). However, if the entailment problem is undecidable this assumption cannot be made independently of the content of the oracle tape (e.g., entailment in first-order logic).

Even if there is a teacher that always terminates depending on the content of the oracle tape, naturally, one cannot assume that all of them will terminate. So we define the following notion. Let \(T(t)\) be a teacher with \(t \in \mathcal{L}\) in the oracle tape. We say that \(T(t)\) is terminating for membership queries if for every possible membership query (within a learning framework) the teacher \(T(t)\) always terminates the computation of the answer. This notion can be easily adapted for other types of queries.

The multiple ways of choosing \(e \in \mu(t) \oplus \mu(h)\) in an equivalence query and an example \(e \in \mathcal{E}\) in a sample query is captured by the non-determinism of \(T\) (see Subsection 3.1). For representing sample queries, one can consider the special case in which the NMTM is a multitape probabilistic Turing machine [44]. We may write \(T_D\) to indicate that, whenever a sample query is posed by \(L\), we have that \(T\) chooses an example according to the (same) probability distribution \(D\), with the events of drawing examples being mutually independent (see e.g. [43] for more details on sample queries and [36] for a presentation using this notation).

### 3.3 Learnability and Complexity Classes

We are ready to define the notion of learnability and complexity classes for learning problems. We write \(Y \in (L, T(t))(X)\) if there is a finite computation of the learning system \((L, T(t))\) with \(X\) in the input tape, \(t\) in the oracle tape, and the content written by the learner in the output tape, \(L\), is \(Y\). We first define learnability for the exact learning model.

Let \(\mathcal{F} = (\mathcal{E}, \mathcal{L}, \mu)\) be a learning framework. Assume that the learner can pose membership and equivalence queries and these are truthfully replied by the teacher, as described in Subsection 3.2. We say that \(\mathcal{F}\) is exactly learnable if there is a learner \(L\) such that, for every \(t \in \mathcal{L}\), there is a terminating \(T(t)\) (for membership and equivalence queries). Moreover,

- every learning system \((L, T'(t))\) with a terminating \(T'(t)\) halts and every \(h \in (L, T'(t))(\Sigma_t) \cap \mathcal{L}\) satisfies \(\mu(h) = \mu(t)\), where \(\Sigma_t\) is the signature of \(t\).

If the number of steps made by \(L\) in each path of the computation tree is always bounded by a polynomial \(p(|t|, |e|)\), where \(t \in \mathcal{L}\) is the target and \(e \in \mathcal{E}\) is the largest counterexample written so far in the communication tape by \(T'(t)\) (in the corresponding path), then \(\mathcal{F}\) is exactly learnable in polynomial time.
We denote by $E_{\text{L}}(MQ, EQ)$ and $E_{\text{LP}}(MQ, EQ)$ the classes of all learning frameworks that are, respectively, exactly learnable and exactly learnable in polynomial time with membership and equivalence queries. One can easily adapt this notation to the case in which the learner is allowed to make an exponential number of steps, denoted $E_{\text{LExp}}(MQ, EQ)$, or to the case in which the learner can only pose one type of query. For representing this, we simply drop $MQ$ or $EQ$ from the class name (e.g., $E_{\text{LP}}(EQ)$ is the class of all learning frameworks that are exactly learnable in polynomial time with only equivalence queries). One can also consider other types of queries, such as subset and superset queries [2], or queries that take into account the history of previous queries [38]. It follows from these definitions that $E_{\text{LP}}(MQ, EQ) \subseteq E_{\text{LExp}}(MQ, EQ) \subseteq E_{\text{L}}(MQ, EQ)$.

We now define learnability in the PAC model. Let $\mathcal{F} = (E, L, \mu)$ be a learning framework. Assume that the learner can pose sample queries and these are replied by the teacher as in Subsection 3.2. The goal is to build a hypothesis such that ‘with high probability there is not much difference between the hypothesis and the target’. A parameter $\epsilon$ quantifies the error of the hypothesis w.r.t. the target (how different they are). Another parameter $\delta$ is used to quantify the confidence of meeting the error requirement (whether this has high probability). Both parameters are real numbers ranging between 0 and 1. Formally, we say that $\mathcal{F}$ is PAC learnable if there is a function $f : (0, 1)^2 \to \mathbb{N}$ and a learner $L$ such that, for every $(\epsilon, \delta) \in (0, 1)^2$, every probability distribution $D$ on $E$, and every target $t \in L$, there is a terminating $T_D(t)$ (for sample queries). Moreover,

- every $(L, T'(t)_\mathcal{D})$ with a terminating $T'(t)_\mathcal{D}$ halts after $L$ poses $m \geq f(\epsilon, \delta)$ samples queries and, with probability at least $(1 - \delta)$ (over the choice of sets of $m$ examples), $h \in (L, T'(t)_\mathcal{D})(\Sigma_t) \cap L$ satisfies $D(\mu(h) \oplus \mu(t)) \leq \epsilon$.

If the number of steps made by $L$ in each path of the computation tree is always bounded by a polynomial function $p(|t|, |e|, 1/\epsilon, 1/\delta)$, where $e$ is the largest example written in the communication tape by $T'(t)_\mathcal{D}$ (in the corresponding path), then $\mathcal{F}$ is PAC learnable in polynomial time. We can easily extend these notions to the case in which the learner can also pose membership queries (with a terminating teacher for both sample and membership queries). We denote by $P_{\text{L}}$ and $P_{\text{LP}}(SQ)$ the classes of all learning frameworks that are, respectively, PAC learnable and PAC learnable in polynomial time with sample queries. Also, we write $P_{\text{LP}}(MQ, SQ)$ for the case the learner can also pose membership queries.

Remark 2. There is an important difference between the polynomial bound for the exact and the PAC learning models. In the exact model, $e$ is the largest counterexample written so far by the teacher in the path of computation, while in the PAC model $e$ is the largest example written by the teacher (at any point of the path). The more strict requirement of the exact model is to avoid a loophole in the definition [3]. Whenever the hypothesis of the learner is not equivalent, the teacher needs to provide a counterexample. Since this depends on both the target and the hypothesis, there could be a case in which the learner spends an exponential amount of time (in the size of the target) to discover a hypothesis that would force the teacher to provide an exponential counterexample. Then
the learner would have spent a polynomial amount of time in the size of the largest counterexample but not in the size of the largest example given so far. This requirement is not necessary in the PAC model because the teacher does not need to provide an example that depends on the hypothesis of the learner, so there is no way the learner can ‘force’ the teacher to return a large example.

Theorem 1 states that positive results for the exact learning model with only equivalence queries are transferable to the PAC model and this also holds if both models allow membership queries.

**Theorem 1.** The following holds [3]:

- \( \text{EL}(\text{EQ}) \subseteq \text{P}(\text{SQ}) \);
- \( \text{ELP}(\text{EQ}) \subseteq \text{P}(\text{SQ}) \);
- \( \text{ELP}(\text{MQ}, \text{EQ}) \subseteq \text{P}(\text{MQ}, \text{SQ}) \).

The intuition for Theorem 1 is that the learner can pose sample queries instead of equivalence queries. By posing sample queries, the learner can obtain a set of classified examples, drawn according to a fixed but arbitrary probability distribution. If the current hypothesis of the learner misclassifies one of the examples of this set then the learner has found a counterexample. So it can proceed as if it had posed an equivalence query and the teacher had returned the counterexample. Otherwise, it is shown in the proof of the theorem that if the sample is large enough then any hypothesis consistent with the sample satisfies the criteria for PAC learnability.

For presentation purposes, we have presented only time complexity classes for the exact and the PAC learning models. One can also consider classes that capture other ways of measuring the resources used by the learner and/or the teacher [6]. For example, one can measure the number and size of queries posed by the learner. In this way, query complexity classes could also be defined [5, 27].

## 4 Learning DL Ontologies

We provide some examples and intuitions about the notions presented so far (Subsection 4.1). Then, in Subsection 4.2 we recall results on learning DL ontologies in the exact and PAC learning models.

### 4.1 An Example

To illustrate the ideas for learning DL ontologies in the exact and the PAC learning models, we start by considering the problem of exactly learning an ontology in a toy language that allows only concept inclusions of the form \( A \sqsubseteq B \) with \( A, B \in \mathcal{N}_C \).

Consider the learning framework \( \mathcal{F}_{\text{toy}} = (\mathcal{L}, \mathcal{E}, \mu) \) with \( \mathcal{L} \) and \( \mathcal{E} \) being the set of all TBoxes and the set of all CIs that can be formulated in the toy language, respectively. The \( \mu \) function maps TBoxes \( \mathcal{T} \) in \( \mathcal{L} \) to CIs in \( \mathcal{E} \) entailed by \( \mathcal{T} \).
Suppose the target \( T \in L \) is \( \{ A \sqsubseteq B, B \sqsubseteq C \} \) and let \( (L, T(T)) \) be a learning system such that on the input \( \Sigma_T = \{ A, B, C \} \) (the signature of \( T \)) returns \( \mathcal{H} \equiv T \). In symbols, \( \mathcal{H} \in (L, T(T))(\Sigma_T) \). Clearly, for all \( T' \in L \), there is a terminating \( T(T') \) for membership, equivalence, and sample queries.

Figure 1 illustrates part of a computation of \( (L, T(T)) \) on the input \( \Sigma_T = \{ A, B, C \} \) where \( L \) poses the membership query \( A \sqsubseteq B \in \mu(T) \) and receives ‘yes’ as an answer. A simple strategy for \( L \) is to formulate all CIs within \( \Sigma_T \) and pose membership queries with each such CI, one at a time. The CI is added to \( \mathcal{H} \) if, and only if, the answer is ‘yes’. With this strategy, the hypothesis \( \mathcal{H} \) computed by the learner is \( \{ A \sqsubseteq B, A \sqsubseteq C, B \sqsubseteq C \} \). At most \( |\Sigma_T|^2 \) membership queries are needed. Thus, \( \mathfrak{S}_{\text{toy}} \in \text{ElP}(MQ) \).

**Adding Conjunctions**

Now, consider an extension of the toy language that allows conjunctions of concept names in CIs. We denote the underlying learning framework as \( \mathfrak{S}_{\text{toy}} \). In this case, the strategy of posing membership queries for each possible CI formulated within the signature \( \Sigma_T \) of a target \( T \) still terminates (since \( \Sigma_T \) is finite). However, it does not terminate in polynomial time in \( |\Sigma_T| \) because with conjunctions one can formulate an exponential number of CIs.

In the following, we provide a simple argument showing that there is no strategy that guarantees polynomial time learnability with only membership queries. In other words, \( \mathfrak{S}_{\text{toy}} \not\in \text{ElP}(MQ) \).

The main idea is to define a superpolynomial set \( S \) of TBoxes in this extension of the toy language and show that any membership query can distinguish at most polynomially many elements of \( S \). Let \( \Sigma = \{ A_1, \ldots, A_n, \bar{A}_1, \ldots, \bar{A}_n, M \} \). For any sequence \( \sigma = \sigma^1 \ldots \sigma^n \) with \( \sigma^i \in \{ A_i, \bar{A}_i \} \) the expression \( \sigma \sqsubseteq M \) stands for the CI \((\sigma^1 \sqcap \ldots \sqcap \sigma^n \sqsubseteq M)\). For every such sequence \( \sigma \) (of which there are \( 2^n \) many), consider the TBox \( T_\sigma \) defined as:

\[
T_\sigma = \{ \sigma \sqsubseteq M \} \cup T_0 \quad \text{with} \quad T_0 = \{ A_i \sqcap \bar{A}_i \sqsubseteq M \mid 1 \leq i \leq n \}
\]

The CI \( \sigma \sqsubseteq M \) represents a unique binary sequence for each \( T_\sigma \), ‘marked’ by the concept name \( M \). The CIs in \( T_0 \) are shared by all \( T_\sigma \) in \( S \).
Lemma 1. For any CI $\alpha$ in the extended toy language over $\Sigma$ either:

- for every $T_\sigma \in S$, we have $T_\sigma \models \alpha$; or
- $T_\sigma \models \alpha$, for at most one $T_\sigma \in S$.

Proof. Suppose there is $T_\sigma \in S$ such that $T_\sigma \models \alpha$ (otherwise we are done).
Assume the CI $\alpha$ is $C \sqsubseteq D$. If $D$ is $M$ then either $\alpha$ is a tautology or there is no
$T_\sigma \in S$ such that $T_\sigma \models \alpha$. In both cases, we have that $T_\sigma \models \alpha$ for at most one
$T_\sigma \in S$. Then, we can assume that $D$ is $M$. Regarding $C$ (the concept on the
left side of the CI $\alpha$), we make a case distinction:

- there is $1 \leq i \leq n$ such that $A_i, \overline{A_i}$ are conjuncts in $C$. In this case, by
definition of $T_0$, we have that $T_\sigma \models \alpha$, for every $T_\sigma \in S$.
- there is no $1 \leq i \leq n$ such that $A_i, \overline{A_i}$ are conjuncts in $C$. This means that
$T_0 \not\models \alpha$. If $\alpha$ is of the form $\sigma \sqsubseteq M$ then there is exactly one $T_\sigma \in S$
such that $T_\sigma \models \alpha$. Otherwise, there is no $T_\sigma \in S$ such that $T_\sigma \models \alpha$. So, $T_\sigma \models \alpha$,
for at most one $T_\sigma \in S$.

Since any membership query can eliminate only polynomially many elements
from $S$ (in our case at most one), the learner cannot distinguish between the
remaining elements from our initial superpolynomial set $S$ in polynomial time.
Thus, $\mathcal{F}^{\text{toy}} \not\in \text{ELP}(\text{MQ})$.

This language can be easily translated into propositional Horn. It is known
that propositional Horn expressions are exactly learnable in polynomial time if
equivalence queries are also allowed \[1,20\]. That is, $\mathcal{F}^{\text{toy}} \in \text{ELP}(\text{MQ}, \text{EQ})$.

Adding Existentials We discuss here a further extension the toy language that
also allows existential quantification. This language coincides with $\mathcal{EL}$, defined in
Section 2. Our first observation is that in $\mathcal{EL}$ there is an infinite number of CIs
that can be formulated with a finite signature $\Sigma_T$ of a target $T$. This happens
because existential quantifiers can be nested in concept expressions. Moreover,
due to cyclic references between concepts in an $\mathcal{EL}$ TBox, an infinite number
of CIs can be entailed by a (finite) TBox (see Remark 1). This means that
the strategy of posing membership queries for each possible CI formulated with
the signature $\Sigma_T$ of a target $T$ does not terminate in this case. If equivalence
queries are allowed then one can still enumerate all TBoxes of size $n$ that can
be formulated with $\Sigma_T$ (up to logical equivalence) and ask equivalence queries
with such TBoxes, one by one. Then one can increase $n$ until it reaches the
size of $T$ (which is finite). This strategy is guaranteed to terminate, although
not in polynomial time. In the next subsection, we discuss further results for
$\mathcal{EL}$ extended with role inclusions (that is, $\mathcal{ELH}$) and its fragments $\mathcal{ELH}_{lhs}$ and
$\mathcal{ELH}_{rhs}$, introduced in Section 2.
4.2 Complexity Results

We now recall from the literature polynomial time complexity results for learning DL ontologies in the exact and the PAC learning models. Figure 2 illustrates some of these results (some results and complexity classes have been omitted to simplify the presentation). Dashed lines are for the classes associated with the PAC learning model. In what follows, we give an overview of the complexity results and provide additional explanations for the complexity classes.

Fig. 2. Learning Frameworks and Complexity Classes

Konev et al. (2018) have shown that $\mathcal{ELH}$ (in fact already $\mathcal{EL}$) TBoxes are not exactly learnable from entailments in polynomial time while $\mathcal{ELH}_{lhs}$ and $\mathcal{ELH}_{rhs}$ are polynomially learnable [27]. In symbols, $\mathcal{F}(\mathcal{ELH}) \not\in \mathcal{ElP}(MQ, EQ)$ but $\mathcal{F}(\mathcal{ELH}_{lhs}), \mathcal{F}(\mathcal{ELH}_{rhs}) \in \mathcal{ElP}(MQ, EQ)$. Similar results also hold for a variant of this problem setting where the examples are pairs of the form $(A, q)$ (instead of being CIs and RIs), where $A$ is an ABox and $q$ is an $(\mathcal{ELH})$ IQ [28]. In this setting, $(A, q)$ is a positive example for $\mathcal{T}$ iff $(\mathcal{T}, A) \models q$. We denote these learning frameworks with $\mathcal{F}(L, IQ)$, where $L$ is the DL. In both problem settings, if the return of an equivalence query is ‘yes’ then this means that the hypothesis of the learner and the target are logically equivalent. Recently, it has been shown that if the ABox is fixed and one only aims at preserving IQ results w.r.t. the fixed ABox (not logical equivalence between the hypothesis and the target) then there is a polynomial time algorithm for $\mathcal{ELH}$ terminologies [37]. We denote this learning framework by $\mathcal{F}(\mathcal{ELH}, A, IQ)$ where $A$ is the fixed ABox. The intuition for why the problem is ‘easier’ in this case is because, since the ABox is fixed, the possible counterexamples the teacher can give are constrained. The fixed
ABox setting avoids the difficult scenario described in the hardness proof for the learning framework $\mathcal{F}(\mathcal{ELH}, \mathcal{IQ})$, where the teacher can give counterexamples of the form $(A_\sigma, A(a))$, with $\sigma = \sigma^1, \ldots, \sigma^n$, for $\sigma^i \in \{A, \overline{A}\}$, and $A_\sigma$ an ABox of the form $\{\sigma^1(a), \ldots, \sigma^n(a)\}$.

By Theorem 1, positive results in the exact learning model are transferable to the PAC model extended with membership queries. We point out that the complexity class $\text{PlP}(\text{SQ})$ is not contained in $\text{ElP}(\text{MQ}, \text{EQ})$. This has been discovered by Blum in 1994 [14]. He constructed an artificial counterexample to prove the result and the argument relies on cryptographic assumptions. Another (artificial) counterexample appears in the work by Ozaki et al. (2020) [37]. The argument in this case does not rely on cryptographic assumptions. Apart from these carefully constructed learning frameworks, in many cases, learning frameworks in $\text{PlP}(\text{SQ})$ are also in $\text{ElP}(\text{MQ}, \text{EQ})$.

We now explain why $\mathcal{F}(\mathcal{ELH})$ and $\mathcal{F}(\mathcal{ELH}, \mathcal{IQ})$ appear in $\text{ELExp}(\text{MQ}, \text{EQ})$. In fact, they are already in $\text{ELExp}(\text{EQ})$. The reason is that, as explained at the end of Subsection 4.1, since the learning system receives the (finite) signature $\Sigma_T$ of the target $\mathcal{T}$ as input, it can enumerate all TBoxes (up to logical equivalence) of a certain size and ask whether any of them is equivalent to $\mathcal{T}$, one by one, increasing this size until a TBox equivalent to $\mathcal{T}$ is found. This naive procedure clearly requires an exponential number of steps in the size of $\mathcal{T}$. The same holds for other DL languages more expressive than $\mathcal{ELH}$, as long as TBoxes can also be enumerated in this way. An exponential (but non-trivial) algorithm for $\mathcal{EL}$ terminologies and its implementation is provided by Duarte et al. (2018) [17].

It remains to explain the $\mathcal{F}(\text{DL-Lite}_R)$ case. DL-Lite$_R$ is a member of a well-known family of DLs [8]. What we would like to explain is that, for some ontology languages, such as DL-Lite$_R$, the number of RIs and CIs that can be formulated within the (finite) signature $\Sigma_T$ of a target $\mathcal{T}$ is polynomial in the size of $\Sigma_T$. Since $\Sigma_T$ is given as input to the learning system, this means that the learner can identify the target with only membership queries (see toy example in Subsection 4.1), and moreover, in polynomial time in the size of $\Sigma_T$. Therefore, $\mathcal{F}(\text{DL-Lite}_R)$ belongs to $\text{ElP}(\text{MQ})$. Learning an equivalent DL-Lite$_R$ TBox with only equivalence queries is also easy. This happens because there are only polynomially many counterexamples that can be given. The learner can start by posing an equivalence query with an empty hypothesis. Then, the teacher is obliged to return a positive counterexample (unless the target is equivalent to the empty hypothesis and we are done). All the learner needs to do is to add this positive counterexample to its hypothesis and then proceed by posing another equivalence query. After polynomially many equivalence queries, the learner will terminate with an equivalent hypothesis. So $\mathcal{F}(\text{DL-Lite}_R) \in \text{ElP}(\text{EQ})$ also holds.

5 Related Work

We now highlight some other approaches from the literature for learning DL ontologies, when the focus is on finding how terms of an ontology should relate to each other using the expressivity of the ontology language at hand. These
approaches are mainly based on association rule mining, formal concept analysis, inductive logic programming, and neural networks [36].

Formal concept analysis [23] has been applied to mine $\mathcal{EL}$ CIs [15] (see also [11,12,41]). In this setting, a learner receives a finite interpretation $I$ as input and attempts to build a finite ontology $T$ such that, for all $\mathcal{EL}$ CIs formulated in a finite signature, $T \models C \subseteq D$ if, and only if, $I \models C \subseteq D$. This ontology, called base, should also satisfy certain minimality conditions. It is known that, given a finite interpretation $I$, a finite base (expressed within a finite signature) always exists for the $\mathcal{EL}$ ontology language. However, this may not be the case for other ontology languages. The main difficulty in applying formal concept analysis for building ontologies is that, as originally proposed, it cannot build CIs one may expect to hold when the data is (even just slightly) incorrect. If a certain CI holds in practice but there is an element that violates it in the interpretation then the CI will not be included in the base. One could argue that this method is then useful to find such errors but the application for mining CIs has this issue.

When there is a certain threshold for tolerating errors, then association rule mining offers an interesting solution. This method is based on the measures of support and confidence [1]. The support is a metric for measuring statistical significance, while confidence measures the ‘strength’ of a rule, in this case, expressed as a CI in an ontology language. Many authors have already employed this method for building DL ontologies [19,48,49] (see also [42]) and for finding relational rules in knowledge graphs [22]. The usual approach is to fix the depth of the CIs in order to restrict the search space.

There is a vast literature on algorithms and techniques for learning DL concepts based on inductive logic programming [18,21,26,29–32] (see [34] for learning logical rules also based in inductive logic programming). One of the most well known tools for supporting the construction of DL concepts is the DL-Learner [29]. In this approach, the learner receives as input examples of assertions classified as positive and negative and the goal is to construct a DL concept expression that ‘fits’ the classified examples.

Deep learning has also been applied for learning DL ontologies [39]. In the mentioned work, the authors use definitional sentences labelled with their DL translation to train a recurrent neural network (see also [35] for more work on definitional sentences in a DL context). It is an interesting approach that deals extremely well with data variability. The main difficulties pointed out by the authors are how to find large amounts of classified examples to train the neural network (the authors have trained it using synthetic data) and how to capture the semantics of the ontology. The neural network could capture the syntax, for example, map the word ‘and’ to the logical operator $\sqcap$. However, as reported by the authors, the method does not really capture the semantics of the sentences and how they relate to each other. There is an extensive literature on learning assertions using neural networks [16,53] but not many works on building DL ontologies with complex concept expressions.
6 Conclusion

We have presented a formalisation of the exact and the PAC learning models and defined learning complexity classes. This opens the possibility of investigating other questions such as the problem of deciding whether a learning framework is PAC or exactly learnable. Some authors have already investigated this problem for the PAC model, with a different formalisation of PAC learnability \[47\].

An interesting application of exact learning algorithms is to verify neural networks, as in the already mentioned works by Weiss et al. \[51,52\]. These works are based on Angluin’s exact learning algorithm for learning regular languages represented by deterministic finite automata with membership and equivalence queries (an abstraction of the automata is used to find counterexamples). One of the goals of this strategy is to find adversarial inputs: examples neither present in the training nor in the test set which were misclassified by a neural network \[51,52\]. It would be interesting to investigate whether algorithms for exactly learning ontologies can also be applied to verify if a neural network captures certain rules.

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