KÄHLER NON-COLLAPSING, EIGENVALUES AND THE CALABI FLOW

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ABSTRACT. We first prove a new compactness theorem of Kähler metrics, which confirms a prediction in \cite{19}. Then we establish several eigenvalue estimates along the Calabi flow. Combining the compactness theorem and these eigenvalue estimates, we generalize the method developed for the Kähler-Ricci flow in \cite{23} to obtain several new small energy theorems of the Calabi flow.

1. INTRODUCTION

In 1982, E. Calabi \cite{4} introduced extremal Kähler (extK) metrics in a fixed cohomology class of Kähler metrics. They are critical points of the Calabi energy which is the $L^2$-norm of the scalar curvature. The Kähler-Einstein metrics and which are more general, the constant scalar curvature Kähler (cscK) metrics are both extremal Kähler metrics. In the same paper, Calabi also introduced a decreasing flow of the Calabi energy, which is now well-known as the Calabi flow. The Calabi flow is expected to be an effective tool to find cscK metrics in a Kähler class. In this paper, we shall prove a compactness theorem of Kähler metrics with its applications on the existence of extK/cscK metrics, and discuss the long time behavior of Calabi flow.

1.1. Kähler non-collapsing. In order to study the Calabi flow in the frame of geometric analysis, an important step is to establish a compactness theorem of Kähler metrics under suitable geometric conditions. Comparing with the compactness properties of Riemannian metrics, the set of Kähler metrics within a fixed cohomological class has more rigidity. The elliptic equations of Riemannian metrics could be written down as function equations of Kähler potentials. This point of view is different from the compactness of Riemannian metrics.

The Cheeger-Gromov theorem \cite{10} \cite{38} states that the set of Riemannian metrics with uniform curvature, diameter upper bound and the volume lower bound has $C^{1,\alpha}$-compactness up to diffeomorphisms. There are two points of view to treat the compactness of the set of Kähler metrics. One is to apply the compactness theorem of the corresponding Riemannian metrics. However, the complex structure under the diffeomorphisms might "jump" in the limit (see Tian \cite{57}). The other is to consider the point-wise convergence of the Kähler forms and their Kähler potentials (see Ruan \cite{50}). There are examples that a sequence of Riemannian metrics converges up to diffeomorphisms, but the corresponding Kähler metrics collapse in some Zariski open set in the point-wise sense. In particular, within the same Kähler class this phenomenon was called Kähler collapsing in Chen \cite{19}. The $C^{1,\alpha}$-compactness of the set of the Kähler metrics was proved in Chen-He \cite{15} under uniform Ricci bound from both sides and the uniform $L^\infty$-bound of the Kähler potential, and recently in Székelyhidi \cite{56} under the conditions that the Riemannian curvature is uniformly

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bounded and the $K$-energy is proper (which is called $I$-proper in our paper in order to distinguish different notions of properness. See Definition 2.2 in Section 2.1).

In [19] Chen initiated the problem that what kinds of geometric conditions are required to avoid the Kähler collapsing? He showed that the properness of the $K$-energy in a Kähler class, the Kähler metrics in a bounded geodesic ball in the space of Kähler potentials, the uniform bound of Ricci curvature and the uniform diameter are sufficient. Furthermore, he predicted that the upper bound of the Ricci curvature could be dropped. In general, when the Ricci curvature only has uniform lower bound, the sequence of Riemannian metrics could collapse to a lower dimensional manifold (see Cheeger-Colding [11][12][13][9]). Together with the uniform injectivity radius lower bound and the uniform volume upper bound, Anderson-Cheeger [1] proved the $C^\alpha$-compactness of the subset of Riemannian metrics. We have the following Kähler noncollapsing theoreom, which relaxes the conditions of Chen’s theorem on the two-side bounds of the Ricci curvature.

**Theorem 1.1.** Let $(M, \Omega)$ be an $n$-dimensional compact Kähler manifold for which the $K$-energy is $I$-proper in the Kähler class $\Omega$. If $S$ is the set of Kähler metrics in $\Omega$ satisfying the following properties:

- the $K$-energy is bounded;
- the $L^p$-norm of the scalar curvature is bounded for some $p > n$;
- the Ricci curvature is bounded from below;
- the Sobolev constant is bounded;

then $S$ is compact in $C^{1,\alpha}$-topology of the space of the Kähler metrics for some $\alpha \in (0, 1)$. In particular, Kähler collapsing does not occur.

In order to prove this theorem, we essentially use the equation of the scalar curvature $S$ of the Kähler metric $g_\varphi$, which reads

$$\begin{cases}
\frac{\det(g_\varphi)}{\det(g)} = e^P,

\triangle_{g_\varphi} P = g_\varphi^{ij} R_{ij}(\omega) - S(g_\varphi).
\end{cases}$$

This is a second order elliptic system. The first equation is a complex Monge-Ampère equation with a pseudo-differential term $P$. The main difficulty of proving this compactness theorem is how to derive the a priori estimates of the second equation which is a complex linearized Monge-Ampère equation. Aiming to solve this problem, we first introduce a decomposition method, then apply the De Giorgi-Nash-Moser iteration with the Sobolev constant. The second condition in the theorem suggests us that there might exist analytic singularities for the critical exponent $p = n$. A nature way to apply the compactness theorem is to find a nice path decreasing the energy, such as the minimizing sequence from the variational direct method, the continuity path from the continuity method (Aubin-Yau path) and the geometric flows including the Calabi flow, the pseudo-Calabi flow, the Kähler-Ricci flow, etc. In this paper, we focus on the discussion of the Calabi flow. Actually, there are large classes of Fano surfaces, where the Sobolev constant is bounded along the Calabi flow (see the end of Section 5).

The $I$-properness of the $K$-energy (cf. Definition 2.2) was introduced by Tian in [58] and Tian proved that the $I$-properness of the $K$-energy in the first Chern class is equivalent to the existence of Kähler-Einstein metrics when the underlying Kähler manifold has no non-zero holomorphic vector fields. The properness condition is a "coercive" condition in the frame of variational theory. However, regarding to the existence of cscK/extK metrics, the accurate positive function appearing in the "coercive" condition is still not
known clearly. Tian [59] conjectured that in a general Kähler class the $I$-properness of the $K$-energy is equivalent to the existence of cscK metrics.

Theorem 1.1 has direct applications to the Calabi flow. Since the $K$-energy is decreasing, the first condition appearing in Theorem 1.1 is satisfied automatically along the Calabi flow. On the other hand, according to the functional inequality $\nu(\varphi) - \nu(0) \leq \sqrt{C/\alpha(\varphi)} \cdot d(0, \varphi)$ of Chen [19], the uniform bound of the $K$-energy could be replaced by the uniform bounds of the Calabi energy and the geodesic distance.

1.2. The Calabi flow on the level of Kähler potentials. The Calabi flow is a fourth order flow of Kähler potentials

$$\frac{\partial}{\partial t} \varphi = S(\varphi) - \bar{S},$$

where $\bar{S}$ is the average of the scalar curvature of $\omega$. On Riemann surfaces, the long time existence and the convergence of the Calabi flow have been proved by Chrusical [27] using the Bondi mass estimate. Chen [14] gave a new geometric analysis proof which is based on the compactness properties of the conformal metrics with bounded Calabi energy and area. In [55] Struwe gave a proof by the integral method without using the maximum principle.

On Kähler manifolds of higher dimensions, since the Calabi flow is a fourth order flow, so far for now, restricted to the difficulties from PDE, not much progress has been made. However, due to the intrinsic geometric property of the Calabi flow, Chen conjectured that the Calabi flow always has long time existence with any smooth initial Kähler metrics. Furthermore the general conjectural picture of the convergence of the Calabi flow was outlined in Donaldson [31]. Assuming the existence of the critical metrics, the stability problem studies the asymptotic behavior of the Calabi flow near the critical metrics, in particular, the exponential convergence of the Calabi flow. The first stability theorem of Calabi flow was proved by Chen-He [15] near a cscK metric. By using a different method, Huang and the second author [41] proved a stability theorem when the initial Kähler metric is near an extK metrics. Other stability theorems were proved when perturbing the complex structure in Chen-Sun [24] and in Tosatti-Weinkove [61] when the first Chern class is less than or equals to zero. There are also several different kinds of notions of weak solution of the Calabi flow are constructed by Berman [3] in the sense of current in the first Chern class, approximated by the balancing flow in Fine [33] and by the De Giorgi’s notion of minimizing movement in Streets [53][54].

In this paper, the main problem we concerned is the convergence of the Calabi flow without assuming the existence of cscK/extK metrics. The first main technique result of the Calabi flow in this paper is the following theorem, which assumes that the initial Kähler potential is bounded and the Calabi energy is sufficiently small.

**Theorem 1.2.** Let $(M, \omega)$ be an compact Kähler manifold with vanishing Futaki invariant. For any $\lambda, \Lambda > 0$, there is a constant $\epsilon = \epsilon(\lambda, \Lambda, \omega)$ such that for any metric $\omega_\varphi \in [\omega]$ satisfying

$$\omega_\varphi \geq \lambda \omega, \quad |\varphi|_{C^{1,\alpha}} \leq \Lambda, \quad Ca(\omega_\varphi) \leq \epsilon,$$

the Calabi flow with the initial metric $\omega_\varphi$ exists for all time and converges exponentially fast to a cscK metric.

An analogous result to Theorem 1.2 also holds for extK metrics (cf. Theorem 5.2). We accumulate the discussion above and obtain the corollary of Theorem 1.1 as the following:

**Corollary 1.3.** Let $(M, \omega)$ be an n-dimensional compact Kähler manifold for which the $K$-energy is $I$-proper in the Kähler class $\Omega$. If the $L^p$-norm of the scalar curvature for
some $p > n$, the Sobolev constant and the lower bound of Ricci curvature are uniformly bounded along the flow, then the Calabi flow converges exponentially fast to a cscK metric.

Combining Theorem 1.1 with Theorem 1.2 we also have the following result using the $K$ energy:

**Theorem 1.4.** Let $(M, \omega)$ be a $n$-dimensional compact Kähler manifold for which the $K$-energy is $I$-proper. For any constants $\lambda, \Lambda, K > 0$ and $p > n$, there is a constant $\epsilon = \epsilon(\lambda, \Lambda, K, p, \omega)$ such that if there exists a metric $\omega_0 \in [\omega]$ satisfying the following conditions

$\text{Ric}(\omega_0) \geq -\lambda \omega_0, \quad \|S(\omega_0)\|_{L^p(\omega_0)} \leq \Lambda, \quad C_S(\omega_0) \leq K, \quad \nu_\omega(\omega_0) \leq \inf_{\omega' \in \Omega} \nu_\omega(\omega') + \epsilon,$

then the Calabi flow with the initial metric $\omega_0$ exists for all time and converges exponentially fast to a cscK metric.

We sketch the proof of Theorem 1.4. By Theorem 1.1 we obtain a $C^{3,\alpha}$ Kähler potential with small $K$-energy. Then a contradiction argument implies that the resulting smooth metric from the Calabi flow has small Calabi energy. This together with Theorem 1.2 implies Theorem 1.4. Analogous results also hold for modified Calabi flow and extK metrics (cf. Theorem 5.4).

1.3. The Calabi flow on the level of Kähler metrics. According to the rich geometries of the Calabi flow, it could be formulated from different point of views. The equation (1.2) of Kähler potentials can be written as the evolution equation of Kähler metrics

$$\frac{\partial}{\partial t} \omega_\varphi = \sqrt{-1} \partial \bar{\partial} S(\varphi).$$

In the following, we study the Calabi flow of the Kähler metrics (1.4). Unlike the Kähler Ricci flow or the mean curvature flow, we have no maximum principle and it is difficult to obtain the explicit lifespan of the curvature tensor. For this reason, we introduce the following definition.

**Definition 1.5.** Given any constants $\tau, \Lambda > 0$. A solution $\omega_t (t \in [0, T]) (T \geq \tau)$ of Calabi flow is called

- **type I(\tau, \Lambda), if**

$$|Rm|(t) \leq \Lambda, \quad \forall t \in [0, \tau],$$

- **type II(\tau, \Lambda), if**

$$|Rm|(t) + |\partial \bar{\partial} S| \leq \Lambda, \quad \forall t \in [0, \tau].$$

Starting from a type $(\tau, \Lambda)$ Calabi flow with small Calabi energy, we show the flow has long time solution and converges to a cscK metric.

**Theorem 1.6.** Let $(M, \omega)$ be an $n$-dimensional compact Kähler manifold with no nonzero holomorphic vector fields. For any $\tau, \Lambda, K, \delta > 0$, there is a constant $\epsilon = \epsilon(\tau, \Lambda, K, \delta, n, \omega) > 0$ such that if the solution $\omega_t$ of the Calabi flow with any initial metric $\omega_0 \in [\omega]$ satisfies the following properties:

(a) the Calabi flow $\omega_t$ is of type II(\tau, \Lambda);

(b) $C_S(\omega_0) \leq K, \quad \mu_1(\omega_0) \geq \delta, \quad C\alpha(\omega_0) \leq \epsilon,$

then the Calabi flow $\omega_t$ exists for all the time and converges exponentially fast to a cscK metric.
The proof of Theorem 1.6 essentially depends on the estimates of the eigenvalue of the linearized operator of the scalar curvature. We show that under some curvature assumptions the eigenvalue will decay slowly along the Calabi flow, which guarantees the exponential decay of the Calabi energy. On the other hand, the exponential decay of the Calabi energy will force the curvature tensor bounded along the flow, and the curvature tensor bound will help to improve the eigenvalue estimates. Repeating this argument, we can show the Calabi flow exists for all time and converges. The idea of this proof is developed from [23], and here we need to overcome several difficulties since the linearized operator of the scalar curvature is a fourth order operator and there is no maximum principle along the flow.

When \( M \) admits non-zero holomorphic vector fields, under the holomorphic action the limit of the orbit of the complex structure might contain larger holomorphic group. The geometric stability condition (see Definition 4.9) which is called pre-stable avoids this phenomenon. It is also used in Chen-Li-Wang [23] and Phong-Sturm [48].

**Theorem 1.7.** Let \( (\mathcal{M}, \omega) \) be a \( n \)-dimensional compact Kähler manifold with vanishing Futaki invariant. Assume that \( M \) is pre-stable. For any \( \tau, \Lambda, K > 0 \), there is a constant \( \epsilon = \epsilon(\tau, \Lambda, K, n, \omega) > 0 \) such that if the solution \( \omega_t \) of the Calabi flow with any initial metric \( \omega_0 \in [\omega] \) satisfies the following properties:

(a) the Calabi flow \( \omega_t \) is of type \( I(\tau, \Lambda) \);
(b) \[
C_S(\omega_0) \leq K, \quad C(\omega_0) \leq \epsilon,
\]
then the Calabi flow \( \omega_t \) exists for all the time and converges exponentially fast to a cscK metric.

### 1.4. The organization

The organization of this paper is as follows: In Section 2, we will discuss the energy functionals and prove the Kähler non-collapsing theorem. In Section 3, we recall some basic facts on the Calabi flow and the modified Calabi flow. In Section 4, we will estimate the eigenvalue in various cases and prove the exponential decay of the Calabi energy in a short time interval, which is used to obtain uniform bounds on the curvature. In Section 5, we will use the results in previous sections to show Theorem 1.2, Theorem 1.4 and Corollary 1.3. In Section 6, we finish the proof of Theorem 1.6 and Theorem 1.7. Moreover, we will give some conditions on the initial metric which implies the type \( (\tau, \Lambda) \) Calabi flow solution.

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### 2. A Kähler non-collapsing theorem

In this section, we prove that there is no Kähler collapsing by replacing the uniform bound of the Ricci curvature by its lower bound and the \( L^p \) bound of the scalar curvature.

#### 2.1. Energy functionals

Let \( (M, \omega) \) be a compact Kähler manifold with a Kähler metric \( \omega \). The space of Kähler potentials is denoted by

\[
\mathcal{H}(M, \omega) = \{ \phi \in C^\infty(M, \mathbb{R}) \mid \omega + \sqrt{-1} \partial \bar{\partial} \phi > 0 \}.
\]
Recall that Aubin functions $I$ and $J$ are defined on $\mathcal{H}(M, \omega)$ by
\[
I(\varphi) = \frac{1}{V} \int_M \varphi(\omega^n - \omega^n_\varphi) = \sqrt{-1} \sum_{i=0}^{n-1} \int_M \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i}_\varphi,
\]
\[
J(\varphi) = \frac{\sqrt{-1}}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i}_\varphi.
\]

Note that the functionals $I$ and $J$ satisfy the inequalities
\[
\frac{1}{n+1} I \leq J \leq \frac{n}{n+1} I.
\]

In Ding [29], the Lagrangian functional of the Monge-Ampère operator is
\[
D_\omega(\varphi) = \frac{1}{V} \int_M \varphi \omega^n - J(\varphi).
\]

The derivative of $D_\omega$ along a general path $\varphi_t \in \mathcal{H}(M, \omega)$ is given by
\[
\frac{d}{dt} D_\omega(\varphi_t) = \frac{1}{V} \int_M \dot{\varphi}_t \omega^n _{\varphi_t}.
\]

We compute the explicit formula of $D_\omega(\varphi)$ as the following
\[
D_\omega(\varphi) = \frac{1}{V} \sum_{i=0}^{n} \frac{n!}{(i+1)!(n-i)!} \int_M \varphi^i \omega^n \wedge (\sqrt{-1} \partial \bar{\partial} \varphi)^i
\]
\[
= \frac{1}{V} \int_M \varphi \omega^n - \sqrt{-1} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i}_\varphi.
\]

Let $\mathcal{H}_0(M, \omega)$ be the subspace of $\mathcal{H}(M, \omega)$ with the normalization condition
\[
\mathcal{H}_0(M, \omega) := \{ \varphi \in \mathcal{H}(M, \omega) | D_\omega(\varphi) = 0 \}.
\]

It is obvious that the Calabi flow stays in $\mathcal{H}_0(M, \omega)$ once it starts from a Kähler potential in $\mathcal{H}_0(M, \omega)$. It is proved in Chen [18] that any two Kähler potentials in $\mathcal{H}(M, \omega)$ are connected by a $C^{1,1}$ geodesic. The length of the $C^{1,1}$ geodesic from 0 to $\varphi$ is given by
\[
d_\omega(\varphi) = \int_0^1 \sqrt{\int_M \dot{\varphi}^2 \omega^n } \, dt.
\]

It satisfies the following inequality
\[
d_\omega(\varphi) \geq V^{-\frac{1}{4}} \max \left\{ \int_{\varphi > 0} \varphi \omega^n, - \int_{\varphi < 0} \varphi \omega^n \right\}.
\]

Recall the explicit formula of Mabuchi’s $K$-energy (cf. [17] [59])
\[
u_\omega(\varphi) = E_\omega(\varphi) + SD_\omega(\varphi) + j_\omega(\varphi).
\]

The advantage of this formula is that it is well-defined for all bounded Kähler metrics which may be degenerate. The first and the third terms of (2.3) will be discussed in the following. We call the first functional the entropy of Kähler metrics
\[
E_\omega(\varphi) = \int_M \log \frac{\omega^n _\varphi}{\omega^n} \omega^n _\varphi.
\]
The entropy \( E_\omega(\varphi) \) is uniformly bounded below, since
\[
\log \frac{\omega^n}{\omega^n} \geq -e^{-1}.
\]
Applying Tian’s \( \alpha \)-invariant and the Jensen inequality, we obtain a lower bound of the entropy.

**Lemma 2.1.** There are uniform constants \( \delta = \delta(\omega), C = C(\omega) > 0 \) such that for any \( \varphi \in \mathcal{H}(M, \omega) \) we have
\[
(2.5) \quad E_\omega(\varphi) \geq \delta I_\omega(\varphi) - C.
\]

The third term of (2.3) is the \( j \)-functional which is given by
\[
(2.6) \quad j_\omega(\varphi) = \frac{1}{V} \sum_{i=0}^{n-1} \frac{n!}{(i + 1)! (n - i - 1)!} \int_M \varphi \text{Ric}(\omega) \wedge \omega^{n-1-i} \wedge (\sqrt{-1} \partial \bar{\partial} \varphi)^i.
\]

Along a path \( \varphi_t \in \mathcal{H}(M, \omega) \), we have
\[
\frac{d}{dt} j_\omega(\varphi_t) = -\frac{n}{V} \int_M \dot{\varphi}_t \text{Ric}(\omega) \wedge \omega^{n-1}_t.
\]

Under the topological condition that
\[
(2.7) \quad C_1(M) < 0 \quad \text{and} \quad -n \frac{C_1(M)}{[\omega]^{n-1}} \frac{[\omega]}{[\omega]} + (n - 1) C_1(M) > 0,
\]
the \( j \)-flow which is the gradient flow of the \( j \)-functional converges and thus the \( j \)-functional has lower bound (see Chen [17], Song-Weinkove [52]). Combining this result with (2.3), we get that along the Calabi flow the entropy \( E \) has upper bound. When \( C_1(M) < 0 \) and \( M \) admits a cscK metric, it is observed by J. Streets [53] that along the Calabi flow the lower bound of \( j \) is controlled by the geodesic distance. Since the geodesic distance to a cscK metric (which is stable along the Calabi flow) is decreasing, the functional \( j \) is bounded below. Thus along the Calabi flow the entropy \( E \) also has upper bound. In general, in order to obtain the upper bound of the entropy, we require the “coercive” condition on the \( K \)-energy. The properness of \( K \)-energy is introduced by Tian to prove the existence of Kähler-Einstein metrics on Fano manifolds in [58]. The space of Kähler metrics equipped with \( L^2 \)-metric introduced by Donaldson [30], Mabuchi [46] and Semmes [51] is an infinite-dimensional, nonpositive curved, symmetric space, its geodesic distance function is a natural positive function. Chen defined another properness of the \( K \)-energy regarding to the entropy in [19] and the geodesic distance \( d \) in [17]. We list them as the following.

**Definition 2.2.** Let \( \rho(t) \) be a nonnegative, non-decreasing functions satisfying \( \lim_{t \to \infty} \rho(t) = \infty \). The \( K \)-energy is called to be:
- **I-proper on** \( \mathcal{H}(M, \omega) \), if \( \nu_\omega(\varphi) \geq \rho(I_\omega(\varphi)) \) for all \( \varphi \in \mathcal{H}(M, \omega) \);
- **E-proper on** \( \mathcal{H}(M, \omega) \), if \( \nu_\omega(\varphi) \geq \rho(E_\omega(\varphi)) \) for all \( \varphi \in \mathcal{H}(M, \omega) \);
- **d-proper on** \( \mathcal{H}(M, \omega) \), if \( \nu_\omega(\varphi) \geq \rho(d_\omega(\varphi)) \) for all \( \varphi \in \mathcal{H}(M, \omega) \).

By Lemma 2.2 we see that \( E \)-properness implies \( I \)-properness. The following result says that the converse is also true in our situation.

**Lemma 2.3.** Fix a constant \( C \). Let \( \mathcal{H}_C \) be the set of functions \( \varphi \in \mathcal{H}(M, \omega) \) which satisfies the inequality
\[
(2.8) \quad \text{osc} \ (\varphi) \leq I_\omega(\varphi) + C.
\]
Then the \( K \)-energy is \( E \)-proper on \( \mathcal{H}_C \) if and only if the \( K \)-energy is \( I \)-proper on \( \mathcal{H}_C \).
Proof. That $E$-properness implies $I$-properness follows from Lemma 2.1. It suffices to show the converse. Suppose that the $K$-energy is $I$-proper on $\mathcal{H}_C$. From (2.8), $\int_M \varphi \omega^n$ is bounded by $I_\omega(\varphi) + C$. So from (2.2) the function $D_\omega(\varphi)$ satisfies

$$|D_\omega(\varphi)| \leq C_1(I_\omega(\varphi) + 1), \quad \forall \varphi \in \mathcal{H}_C$$

for some constant $C_1$. On the other hand, each term of (2.6) can be written as a linear combination of the following expressions

$$\int_M \varphi \text{Ric}(\omega) \wedge \omega^{n-1}, \quad \int_M \varphi \text{Ric}(\omega) \wedge \omega^{n-k} \wedge \omega^k_\varphi \wedge \sqrt{-1}\partial\bar{\partial}\varphi, \quad k = 0, \cdots, n - 2.$$  

Since $\varphi \in \mathcal{H}_C$, the first term of (2.9) can be controlled by $I_\omega(\varphi)$. The rest terms of (2.9) can be estimated as follows:

$$\left| \int_M \varphi \text{Ric}(\omega) \wedge \omega^{n-k} \wedge \omega_\varphi^k \wedge \sqrt{-1}\partial\bar{\partial}\varphi \right| \leq C_2 \int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^{n-k} \wedge \omega_\varphi^k \leq C_3 I_\omega(\varphi),$$

where we used $\text{Ric}(\omega) \leq C_2 \omega$ in the first inequality. Combining the above estimates with (2.6), we have

$$|j_\omega(\varphi)| \leq C_4 I_\omega(\varphi).$$

Thus, by (2.5) we have

$$E_\omega(\varphi) = \nu_\omega(\varphi) - S D_\omega(\varphi) - j_\omega(\varphi) \leq \nu_\omega(\varphi) + C_5 I_\omega(\varphi).$$

Here all $C_i$ are positive constants which depends only on $C$, $n$ and $\omega$. Therefore, if the $K$-energy is $I$-proper and is bounded from above, the functional $E_\omega$ is also bounded. Thus, the $K$-energy is $E$-proper. The lemma is proved.

In the case of extremal Kähler metrics, we can define the modified $K$-energy by (3.5) for the $K$-invariant Kähler potentials (see Section 3 for the details). As in Definition 2.2, we can define the $E$ and $I$ properness for the modified $K$-energy. After slightly modification, we can prove the following result as Lemma 2.3

**Lemma 2.4.** Fix a constant $C$. Let $\mathcal{H}_C^K$ be the set of $K$-invariant functions $\varphi \in \mathcal{H}(M, \omega)$ which satisfies the inequality

$$\text{osc}(\varphi) \leq I_\omega(\varphi) + C. \quad (2.10)$$

Then the modified $K$-energy is $E$-proper on $\mathcal{H}_C^K$ if and only if the modified $K$-energy is $I$-proper on $\mathcal{H}_C^K$.

**Proof.** It suffices to show the sufficiency part. By the expression of the modified $K$-energy,

$$\tilde{E}_\omega(\varphi) = \nu_\omega(\varphi) + \int_0^1 dt \int_M \frac{\partial \varphi_t}{\partial t} \theta_{\text{Re}V}(\varphi_t) \omega^n_{\varphi_t},$$

where $\varphi_t \in \mathcal{H}_C^K$ is a path from 0 to $\varphi$. Following the argument of Lemma 2.3, we only need to show

$$\left| \int_0^1 dt \int_M \frac{\partial \varphi_t}{\partial t} \theta_{\text{Re}V}(\varphi_t) \omega^n_{\varphi_t} \right| \leq C' I_\omega(\varphi) \quad (2.11)$$
for some constant $C' = C'(n, \omega) > 0$. Choosing the path $\varphi_t = t \varphi (0 \leq t \leq 1)$, we calculate the left hand side of (2.11)
\[
\int_0^1 dt \int_M \frac{\partial \varphi_t}{\partial t} \theta_{Re V} (\varphi_t) \omega^n_{\varphi_t} = \int_0^1 dt \int_M \varphi (\theta_{Re V} + t X (\varphi)) \left[ (1 - t) \omega_g + t \omega_\varphi \right]^n.
\]
The right hand side of (2.12) is a linear combination of the following expressions:
\[
\int_M \varphi \theta_{Re V} \omega_i g \land \omega^n_{\varphi} - i \varphi, \quad \int_M \varphi X (\varphi) \omega_i g \land \omega^n_{\varphi} - i \varphi, \quad i = 0, \cdots, n.
\]
Since $\varphi$ satisfies the condition (2.10) and $X (\varphi)$ is bounded (cf. \[36\] \[65\]), all the expressions in (2.13) are bounded by $I_{\omega} (\varphi)$. Thus, (2.11) is proved and the lemma holds.

2.2. Linear estimate.

Lemma 2.5. (Sobolev inequality, Hebey \[40\]) Let $(M, g)$ be an $m$-dimensional complete Riemannian manifold with Ricci curvature bounded below and positive injective radius. For any $\epsilon > 0$ and $1 \leq q < m$, there exists constant $A(q, \epsilon)$ such that for any $f \in W^{1, q} (M)$,
\[
\left( \frac{1}{V} \int_M |f|^p dV_g \right)^{\frac{1}{p}} \leq (K (m, p) + \epsilon) \left( \frac{1}{V} \int_M |\nabla f|^q dV_g \right)^{\frac{1}{q}} + A(q, \epsilon) \left( \frac{1}{V} \int_M |f|^q dV_g \right)^{\frac{1}{q}},
\]
where the constant $p$ is defined by $\frac{1}{p} + \frac{1}{m} = \frac{1}{q}$.

In the following, we call $C_S = \max \{ (K (m, p) + \epsilon), A(q, \epsilon) \}$ the Sobolev constant.

Lemma 2.6. (Poincaré inequality, Li-Yau \[44\]) Let $(M, g)$ be an $m$-dimensional complete Riemannian manifold with Ricci curvature bounded below and bounded diameter, then for any $f \in W^{1, q} (M)$ we have
\[
\frac{1}{V} \int_M (f - \frac{1}{V} \int_M f dV_g)^2 dV_g \leq \frac{1}{\lambda_1 V} \int_M |\nabla f|^2 dV_g.
\]

In the following, we call $C_p = \frac{1}{\lambda_1 V}$ the Poincaré constant. The following result says the Poincaré constant is bounded by the Sobolev constants and the lower bound of the Ricci curvature.

Lemma 2.7. Let $C$ be a set of Kähler metrics in the Kähler class $\Omega$ such that the Kähler metrics in $C$ have uniform Sobolev constant and uniform lower bound of the Ricci curvature. Then the Kähler metrics in $C$ have the uniform Poincaré constant.

Proof. Since the Sobolev constant is bounded, by Carron \[8\] there are two positive constants $r_0$ and $\kappa$ such that
\[
\text{Vol}(B(p, r)) \geq \kappa r^{2n}, \quad \forall r \in (0, r_0).
\]
Since the volume of $\omega$ is fixed, the diameter is uniformly bounded from above. This together with the lower boundedness of Ricci curvature implies that the first eigenvalue of the Laplacian operator is uniformly bounded from below (cf. Li-Yau \[44\]). Thus, the Poincaré constant is uniformly bounded.

Now we consider the following linear equation on a compact Kähler manifold $(M, \omega_\varphi)$ of complex dimension $n$:
\[
\Delta \varphi = f.
\]
Since the metrics $\omega_\varphi$ are in the same Kähler class, we assume the volume of $\omega_\varphi$ is $1$. The zero order estimate follows from the De Giorgi-Nash-Moser iteration.

**Proposition 2.8.** (Global boundedness) If $v$ is a $W^{1,2}$ sub-solution (respectively super-solution) of (2.14) in $M$, moreover, if $f \in L^{2^*}$ with $p > 2n$, then there is a constant $p^* = \frac{2np}{2n+p}$ and $C$ depending on the Sobolev constant $C_S$ and the first eigenvalue $\lambda_1$ with respect to $\omega_\varphi$ such that

\begin{equation}
(2.15) \quad \sup_M \left( v - \int_M v \omega_\varphi^n \right) \leq C \left\| f \right\|_{p^*} \quad \text{resp.} \quad \sup_M \left( v - \int_M v \omega_\varphi^n \right) \leq C \left\| f \right\|_{p^*}.
\end{equation}

**Proof.** Let $\tilde{v} = v - \int_M v \omega_\varphi^n$. For any constant $k$ we denote by $u = (\tilde{v} - k)_+$ the positive part of $\tilde{v} - k$. Set $A(k) = \{ x \in M | \tilde{v}(x) > k \}$. Multiplying (2.14) with $u$ on both sides and integrating by parts, we have the inequality

\begin{equation}
(2.16) \quad \int_M |\nabla u|^2 \omega_\varphi^n \leq - \int_M u f \omega_\varphi^n.
\end{equation}

By the Hölder’s inequality, the right hand side is bounded by

\[ \|u\|_{2^*} \cdot \|f\|_{p^*} \cdot |A(k)|^r, \]

where $\| \cdot \|_p$ denotes the $L^p$ norm with respect to the metric $\omega_\varphi$ and $|A(k)|$ denotes the volume of $A(k)$ with respect to $\omega_\varphi$. Here $m = 2n, 2^* = \frac{2m}{m-2}, p^* = \frac{mp}{m+p}$ and $r = \frac{1}{r} - \frac{1}{p}$. The Sobolev inequality implies

\[ \|u\|_{2^*} \leq C_1 (\|\nabla u\|_2^2 + \|u\|_2^2). \]

Here all constants $C_i$ in the proof depend on $C_S(\omega_\varphi)$. Since $\|u\|_2 \leq C_2 \|u\|_{2^*} |A(k)|^\frac{1}{r}$, we obtain

\begin{equation}
(2.17) \quad \|u\|_{2^*} \leq C_3 (\|u\|_{2^*} \cdot \|f\|_{p^*} \cdot |A(k)|^r + \|u\|_2^2 \cdot |A(k)|^\frac{1}{r}).
\end{equation}

We will choose a constant $k_0$ later such that for any $k \geq k_0$ we have $|A(k)|^\frac{1}{r} \leq \frac{1}{2C_5}$. Using the Poincaré inequality, we have

\begin{equation}
(2.18) \quad \|\tilde{v}\|_2^2 \leq C_4 \|\nabla \tilde{v}\|_2^2.
\end{equation}

By multiplying the equation (2.14) with $\tilde{v}$ and applying the Hölder inequality, we have the inequality $\|\nabla \tilde{v}\|_2^2 \leq \|\tilde{v}\|_2 \|f\|_2$. Thus we have

\[ \|\tilde{v}\|_2 \leq C_4 \|f\|_2 \leq C_5 \|f\|_{p^*}. \]

Since $k_0^2 |A(k_0)| \leq \|\tilde{v}\|_2^2$, we choose $k_0^2 = C_5 \|f\|_{p^*} (2C_3)^\frac{1}{r}$ so that we have $|A(k_0)|^\frac{1}{r} = \frac{1}{2C_5}$. Combining this with (2.16), we obtain for any $k \geq k_0$,

\[ \|u\|_{2^*} \leq C_6 \|f\|_{p^*} \cdot |A(k)|^r. \]

Now we choose $h > k \geq k_0$ so that we have the inequality

\[ \|u\|_{2^*} \geq C_7 (h - k) \cdot |A(h)|^\frac{1}{r}. \]

Thus by the iteration lemma (see [37]), we have $A(k_0 + d) = 0$ and $d = C_8 \|f\|_{p^*}$. Thus, we have

\[ \tilde{v} \leq k_0 + d \leq C_9 \|f\|_{p^*}. \]

The proposition is proved. \qed

**Remark 2.9.** Since $p^* = \frac{2np}{2n+p}$, we have $n < p^* < \frac{4}{2}$. 


2.3. Log volume ratio.

**Lemma 2.10.** The log volume ratio $h_\omega(\varphi) := \log \frac{\omega^n}{\omega_{\varphi}}$ satisfies the equality

$$|h_\omega(\varphi)| \leq E_\omega(\varphi) + C(n, p, C_S(\omega_\varphi)) \left( \osc_M \varphi + \|S(\omega_\varphi)\|_{L^p(\omega_\varphi)} \right),$$

where $p > n$.

**Proof.** For brevity, we write $h$ for $h_\omega(\varphi)$. We compute the equation of the scalar curvature,

$$\triangle_\varphi h = g^i_\varphi R_{i\bar{j}}(\omega) - S(\omega_\varphi).$$

Since the background metric $\omega$ is smooth, we have

$$\inf_M \text{Ric} \cdot \omega \leq \text{Ric}(\omega) \leq \sup_M \text{Ric} \cdot \omega.$$

Letting $R_- = - \inf_M \text{Ric}$, $R_+ = \sup_M \text{Ric}$ be two positive constants, we have

$$-R_- \cdot (n - \triangle_\varphi \varphi) - S(\omega_\varphi) \leq \triangle_\varphi h \leq R_+ \cdot (n - \triangle_\varphi \varphi) - S(\omega_\varphi).$$

Thus we have the inequality

$$\triangle_\varphi (h - R_- \cdot \varphi) \geq -nR_- - S(\omega_\varphi),$$

and

$$\triangle_\varphi (h + R_+ \cdot \varphi) \leq nR_+ - S(\omega_\varphi).$$

By Proposition 2.8 and the inequalities (2.19)-(2.20), we have

$$h - E_\omega(\varphi) \leq C \left( R_- (\varphi - \int_M \varphi \omega^n_\varphi) + \|S(\omega_\varphi)\|_{L^p(\omega_\varphi)} \right)$$

and

$$h - E_\omega(\varphi) \geq -C \left( R_+ (\varphi - \int_M \varphi \omega^n_\varphi) + \|S(\omega_\varphi)\|_{L^p(\omega_\varphi)} \right).$$

Thus, the lemma is proved. □

**Remark 2.11.** The lower bound of $h$ is obtained by Chen-Tian [16] under the Ricci curvature lower bound. If $\text{Ric}_\varphi$ is bounded from below for some constant $L$, i.e.

$$\text{Ric}_\varphi \geq -\Lambda \omega_\varphi,$$

then there exists a constant $C(\Lambda + \frac{1}{n} \sup_M S(\omega))$ such that

$$\inf_M h_\omega(\varphi) \geq -4 \left( L \sup_M (\varphi - \int_M \varphi \omega^n_\varphi) + C \right) e^{2(1+E_\omega(\varphi))}.$$

2.4. Second order estimate. The following result is a standard application of Chern-Lu inequality:

**Lemma 2.12.** There is a constant $C > 0$ depending on $\inf_M \text{Ric}(\omega_\varphi)$, $\sup_{i,j} R_{i\bar{j}\bar{j}}(\omega)$, $\osc_M (\varphi)$ and $\sup_M h_\omega(\varphi)$ such that

$$\frac{1}{C} \omega \leq \omega_\varphi \leq C \omega.$$
Proof. As in [64][45], we have the Chern-Lu inequality:

\[
\Delta \phi \log \text{tr}_{\omega^\phi} \omega = \frac{\Delta_\phi (\text{tr}_{\omega^\phi} \omega)}{\text{tr}_{\omega^\phi} \omega} - \frac{g^{ki}_\phi \cdot \partial_k g^{ij}_\phi g_{ij} \cdot \partial_l g^{pq}_\phi g_{pq}}{(\text{tr}_{\omega^\phi} \omega)^2} \geq \frac{R^{ij}_\phi \cdot R_{ij} \cdot \text{tr}_{\omega^\phi} \omega}{\text{tr}_{\omega^\phi} \omega} \geq \inf_M \text{Ric}(\omega^\phi) - \sup_M R_{i\bar{i}j} (\omega) \cdot \text{tr}_{\omega^\phi} \omega.
\]

Let \( \lambda = \sup_M R_{i\bar{i}j} (\omega) + 1 \). The function \( u := \log \text{tr}_{\omega^\phi} \omega - \lambda \phi \) satisfies the inequality

\[
\Delta \phi u \geq (\inf_M \text{Ric}(\omega^\phi) - \lambda n) + (\lambda - \sup_M R_{i\bar{i}j} (\omega)) \cdot \text{tr}_{\omega^\phi} \omega.
\]

By the maximum principle, we have

\[
\text{tr}_{\omega^\phi} \omega \leq C(\inf_M \text{Ric}(\omega^\phi), \sup_{i,j} R_{i\bar{i}j} (\omega), \text{osc}_M (\phi)).
\]

This implies that \( \omega^\phi \geq \frac{1}{C'} \omega \). On the other hand, using the identity \( \omega_n^\phi = e^h \omega_n \) we have

\[
\omega^\phi \leq C' \omega
\]

for some positive constant \( C' = C' (\inf_M \text{Ric}(\omega^\phi), \sup_{i,j} R_{i\bar{i}j} (\omega), \sup_M h, \text{osc}_M (\phi)) \). The lemma is proved.

\[\square\]

2.5. Zero order estimate.

Lemma 2.13. For any Kähler potential \( \phi \in \mathcal{H}(M, \omega) \), we have

\[
\text{osc} (\phi) \leq I(\omega, \omega^\phi) + C(C_S(\omega^\phi), C_S(\omega)).
\]

Proof. Since \( n + \Delta \phi > 0 \), by Proposition 2.8 we have the upper bound of \( \phi \),

\[
\sup_M \phi - \int_M \phi \omega^n \leq C(C_S(\omega)).
\]

On the other hand, using the inequality \( n - \Delta \phi \phi > 0 \) we have the lower bound

\[
\inf_M \phi - \int_M \phi \omega^n \geq -C(\omega^\phi).
\]

The inequality (2.21) follows directly from (2.22) and (2.23).

\[\square\]

Combining Lemma 2.13 with Lemma 2.3 (or Lemma 2.4), we have

Lemma 2.14. Suppose that the (modified) K-energy is I-proper. If the (modified) K-energy \( \nu_\omega (\phi) \) (or \( \tilde{\nu}_\omega (\phi) \)) and \( C_\phi (\omega) \) are bounded, then \( \text{osc} (\phi) \) and \( E_\omega (\phi) \) are bounded.
2.6. **Proof of Theorem 1.1**

**Proof of Theorem 1.1** We consider the equation

\[ \omega_n^\varphi = e^{h_\omega(\varphi)} \omega_n. \]  

(2.24)

Under the assumptions of Theorem 1.1 by Lemma 2.14 \( \text{osc}(\varphi) \) and \( E_{\omega}(\varphi) \) are bounded and by Lemma 2.10 \( h_\omega(\varphi) \) is bounded. So \( \varphi \) has \( C^\alpha \) bound by Kolodziej [42]. Moreover by Lemma 2.12, we obtain the second order estimates. Now since the metric \( \omega_\varphi \) and \( \omega \) are equivalent, we apply weak Harnack inequality and the local boundedness estimate to (2.19) and (2.20) respectively (see Section 5.2 in Calamai-Zheng [6]), we obtain that \( h \) has \( C^\alpha \) bound. So the \( C^{2,\alpha} \) estimate of \( \varphi \) follows from the Schauder estimate of complex Monge-Ampère equation (see Wang [62]). Now we run the bootstrap argument. Considering (2.24), the right hand side belong to \( L^p \) for some \( p \geq n \). Since the coefficient is \( C^\alpha \), we could apply the \( L^p \) theory and obtain that \( h \) belongs to \( W^{2,p} \). Consequently, by the Sobolev imbedding theorem \( h \) is \( C^{1,\alpha} \), then \( \varphi \) is \( C^{3,\alpha} \) by linearizing (2.24) and applying the Schauder estimate of the Laplacian equation. The theorem is proved.

\[ \square \]

The following version of Theorem 1.1 is used to the modified Calabi flow in the remainder sections.

**Theorem 2.15.** Let \((M, \Omega)\) be a compact Kähler manifold for which the modified K-energy is \( I \)-proper in the Kähler class \( \Omega \). If \( S \) is the set of K-invariant Kähler metrics in \( \Omega \) satisfying the following properties:

- the modified K-energy is bounded;
- the \( L^p \)-norm of the scalar curvature is bounded for some \( p > n \);
- the Sobolev constant is bounded;
- the Ricci curvature is bounded from below;

then \( S \) is compact in \( C^{1,\alpha} \)-topology of the space of the Kähler metrics for some \( \alpha \in (0, 1) \). In particular, Kähler collapsing does not occur.

3. **Calabi Flow**

In this section, we will recall some basic facts of Calabi flow and modified Calabi flow, which will be used in next sections. Let \((M, \omega)\) be a compact Kähler manifold with a Kähler metric \( \omega \). A family of Kähler potentials \( \varphi(t) \in \mathcal{H}(M, \omega)(t \in [0, T]) \) is called a solution of Calabi flow, if it satisfies the equation

\[ \frac{\partial}{\partial t} \varphi(t) = S(\varphi(t)) - \underline{S}, \]

where \( S(\varphi(t)) \) is the scalar curvature of the metric \( \omega_\varphi(t) \). In the level of Kähler metrics, the equation of Calabi flow can be written as

\[ \frac{\partial}{\partial t} g_{ij} = S_{ij}, \]

where \( g_{ij} \) denotes the metric tensor of the Kähler form \( \omega_\varphi(t) \). Along the Calabi flow, the evolution equation of the curvature tensor is given by

\[ \frac{\partial}{\partial t} Rm = -\nabla \nabla \nabla \nabla S = -\Delta^2 Rm + \nabla^2 Rm \ast Rm + \nabla Rm \ast \nabla Rm, \]

(3.1)

where the operator \( \ast \) denotes some contractions of tensors.
Next, we introduce the modified Calabi flow. Let $\text{Aut}(M)$ be the holomorphic group and $\text{Aut}_0(M)$ the identity component of the holomorphic group. According to Fujiki [34], $\text{Aut}_0(M)$ has a unique meromorphic subgroup $G$ which is a linear algebraic group and the quotient $\text{Aut}_0(M)/G$ is a complex torus. Furthermore, the Chevalley decomposition shows that $G$ is the semidirect product of reductive subgroup $H$ and the unipotent radical $R_u$. Moreover, $H$ is a complexification of a maximal compact subgroup $K$ of $G/R_u$. Let $\mathfrak{h}_0(M), \mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebra of $\text{Aut}_0(M), G$ and $H$ respectively.

Let $\mathcal{M}_{\text{inv}}$ contains the metric in Kähler class $\Omega$ whose isometric group is a maximal compact subgroup $K$ of $\text{Aut}_0(M)$. We call these metrics the $K$-invariant Kähler metrics. Each metric $\omega \in \mathcal{M}_{\text{inv}}$ defines a holomorphic vector field

$$V_\omega = g^{-j}\frac{\partial \text{pr}(\omega)}{\partial \bar{z}^j} \frac{\partial}{\partial z^j},$$

where $\text{pr}$ is the projection from the space of complex valued functions to the space of the Hamiltonian functions of the holomorphic vector fields. Due to Calabi [5] we have the isometric group of an extremal metric is a maximal compact subgroup of $\text{Aut}_0(M)$ and the associated $V$ is a holomorphic vector field. Hence, all extremal metrics stay in $\mathcal{M}_{\text{inv}}$. Futaki-Mabuchi [35] showed that $\text{Re} V$ lies in the center of $\mathfrak{h}$ and is independent of the choice of $\omega \in \mathcal{M}_{\text{inv}}$. Moreover, for any $\omega_1, \omega_2$, there exists a element $\sigma$ in the unipotent radical $R_u$ such that $\sigma^* V_{\omega_1} = V_{\omega_2}$. Let $K$ be a maximal compact subgroup which contains isometric group of $\omega$ generated by $1 \text{m} V$. If $\omega$ is a $K$-invariant metric, then there is a real-valued Hamiltonian function $\theta_{\text{Re} V}$ such that

$$\ell_{\text{Re} V} \omega = \sqrt{-1} \partial \bar{\partial} \theta_{\text{Re} V}.$$

Let $\mathcal{H}_K(M, \omega)$ be the space of $K$-invariant Kähler potentials

$$(3.2) \quad \mathcal{H}_K(M, \omega) = \{ \varphi \in C_0^\infty(M) \mid \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0, L_{\text{Re} V}(\omega_\varphi) = 0 \}.$$

If $\sigma(t)$ is the holomorphic group generated by the real part of $V$, i.e. $\text{Re} V = \sigma^* \left( \frac{\partial}{\partial \varphi} \sigma \right)$, then there is a smooth function $\rho(t) \in \mathcal{H}(M, \omega)$ such that

$$\sigma(t)^* \omega = \omega + \sqrt{-1} \partial \bar{\partial} \rho(t).$$

Let $\varphi(t)$ be the solution of Calabi flow and $\psi(t) = \sigma(-t)^*(\varphi(t) - \rho(t))$. Then $\sigma(t)^* \omega_\psi = \omega_\varphi$. When the initial Kähler potential is $K$-invariant, the solution $\varphi(t)$ remains to be $K$-invariant. Moreover, we have

$$\ell_{\text{Re} V} \omega_t = \sqrt{-1} \partial \bar{\partial} \theta_{\text{Re} V}(t) = \sqrt{-1} \partial \bar{\partial} (\theta_{\text{Re} V} + \text{Re} V(\psi))$$

The function $\psi(t)$ is called a solution of modified Calabi flow, which is given by the equation

$$(3.3) \quad \frac{\partial \psi}{\partial t} = S(\omega_\psi) - S - \theta_{\text{Re} V}(\psi).$$

The proof of (3.3) can be found in Huang-Zheng [41]. We call $S - S - \theta_{\text{Re} V}$ the modified scalar curvature of $\omega$. The modified Calabi-Futaki invariant is defined in $\mathcal{H}(M, \omega)$

$$\mathcal{F}(Y) = - \int_M \theta_X(S - S - \theta_{\text{Re} V}) \omega^n.$$

It equals to zero when $M$ admits extremal metric $\omega$ and we have $S + \theta_{\text{Re} V} = S$. Particularly, if it is restricted in $\mathcal{H}_K(M, \omega)$, the modified Calabi-Futaki invariant becomes

$$(3.4) \quad \mathcal{F}(Y) = - \int_M \theta_X(S - S - \theta_{\text{Re} V}) \omega^n.$$
The modified $K$-energy is well-defined for $K$-invariant Kähler metrics

$$\tilde{\nu}(\varphi) = -\int_0^1 \int_M \dot{\varphi}(S - \theta_{\text{Re} V}(\varphi)) \omega^n \, dt.$$ \hfill (3.5)

We can define the modified Calabi energy by

$$\tilde{C}a(\omega) = \int_M (S - \theta_{\text{Re} V})^2 \omega^n.$$ \hfill (3.6)

We denote the operator $L$ by

$$L(u) := u_{i\bar{j}j\bar{i}} = \Delta^2 u + R_{ij}u_{ji} + S_{,i}u_{,j}.$$\hfill (3.8)

Then the evolution of the modified Calabi energy can be written by

$$\partial_t \int_M \psi^2 \omega^n + 2 \int_M \psi L\psi \omega^n,$$ \hfill (3.7)

and the proof of (3.7) can be found in Section 3 in Huang-Zheng\[41\].

4. ESTIMATES ALONG CALABI FLOW

4.1. Short time estimates. The Calabi flow is a fourth order quasi-linear parabolic equation of Kähler potentials. When the initial data is a smooth Kähler potential, the short time existence of the Calabi flow follows from the theory of the quasi-linear parabolic equation. However, the proof of local existence with the Hölder continuous initial data is quite different from the case of smooth initial data. Da Prato-Grisvard\[28\], Angenent\[2\] and others developed the abstract theory of local existence for the fully nonlinear parabolic equation. They\[28\] constructed the continuous interpolation spaces so that the linearized operator stays in certain classes. This theory has been applied to prove the well-posedness of the Calabi flow with $C^{3,\alpha}$ initial potentials and the pseudo-Calabi flow with $C^{2,\alpha}$ initial potentials (see Theorem 3.2 in Chen-He\[15\] and Theorem 4.1 and Proposition 4.42 in Chen-Zheng\[26\]. We accumulate the results as following.

**Proposition 4.1.** Suppose that the initial Kähler potential $\varphi(0)$ satisfies that for integer $l \geq 3$ and positive constant $\lambda$,

$$\omega_{\varphi(0)} \geq \lambda \omega, \quad |\varphi(0)|_{C^{l,\alpha}} \leq A,$$

then the Cauchy problem for the Calabi flow has a unique solution within the maximal existence time $T$. Moreover, there exists a time $t_0 = t_0(\lambda, A, n, \omega)$ such that for any $t \in [0, t_0]$ we have

$$\omega_t \geq \frac{\lambda}{2} \omega, \quad |\varphi(t)|_{C^{l,\alpha}} \leq 2A.$$\hfill (3.9)

For any $t_0 > \epsilon > 0$ and any $k \geq l + 1$, there also exists a constant $C(\lambda, A, k, \epsilon, n, \omega)$, such that

$$|\varphi(t)|_{C^{k,\alpha}} \leq C, \quad \forall t \in [\epsilon, t_0].$$\hfill (3.10)

4.2. The first eigenvalue estimate along the Calabi flow. In this subsection, we will estimate the first eigenvalue of the operator $L_t$ along the Calabi flow when $M$ has no non-zero holomorphic vector fields. Along the Ricci flow, the study of the eigenvalues of Laplacian and some other operators has been carried out by Cao\[7\], Li\[43\], Perelman\[47\], Wu-Wang-Zheng\[63\] etc.
Proposition 4.2. Let $\mu_1(t)$ be the first eigenvalue of the $L$ operator along the Calabi flow. For any constants $\Lambda, \delta > 0$ and positive function $\epsilon(t)$, if the solution of the Calabi flow satisfies
\begin{equation}
|\text{Ric}|(t) \leq \Lambda, \quad |\nabla \bar{\nabla} S|(t) \leq \epsilon(t), \quad \forall \ t \in [0, T]
\end{equation}
then we have
\begin{equation}
(\mu_1(t) + \Lambda^2) \geq (\mu_1(0) + \Lambda^2) e^{-26 \int_0^t \epsilon(t) \, dt}, \quad \forall \ t \in [0, T].
\end{equation}

Proof. For any $t \in [0, T]$, consider the smooth functions with the normalization condition $\int_M f(x, t)^2 \omega^n_t = 1$. We define the nonnegative auxiliary function
\begin{equation}
\mu(f, t) = \int_M f(x, t) L_t f(x, t) \omega^n_t = \|\nabla f\|^2.
\end{equation}
It is easy to see that $\mu(f, t) \geq \mu_1(t)$ for all $t \in [0, T]$ and the equality holds if and only if $f$ is the eigenfunction of the first eigenvalue. We calculate the derivative at $t$.
\begin{equation}
\frac{d}{dt} \mu(f, t) = 2 \int_M \frac{d}{dt} f L_t f \omega^n_t + \int_M f \left( \frac{\partial}{\partial t} L_t \right) f \omega^n_t + \int_M f L_t \Delta_t S \omega^n_t.
\end{equation}

At $t = t_0$, let $f(x, t_0)$ be the $i$-th eigenfunction of $L_{t_0}$ with respect to the eigenvalue $\mu(f, t_0)$ satisfying
\begin{equation}
L_{t_0} f(x, t_0) = \mu(f, t_0) f(x, t_0).
\end{equation}
Using the normalization condition, we have
\begin{equation}
\frac{d}{dt} \mu(f, t_0) = \int_M f \left( \frac{\partial}{\partial t} L_t \right) f \omega^n_t = \int_M f (-S_m \dddot{f} - (f \dddot{S}, \dddot{f})_{\ddot{\gamma} \ddot{\gamma}} - (f, \dddot{\dddot{S}}, \dddot{\dddot{f}})_{\ddot{\gamma} \ddot{\gamma} \ddot{\gamma}}) \omega^n_t := (I_1 + I_2 + I_3).
\end{equation}

Lemma 4.3. The following inequality holds.
\begin{equation}
\int_M |f_{\dddot{m} \dddot{k} \dddot{k}}|^2 \leq (2 \mu + \Lambda^2) \int_M |\nabla f|^2.
\end{equation}

Proof. We have by the Ricci identity,
\begin{equation}
\int_M |f_{\dddot{m} \dddot{k} \dddot{k}}|^2 = \int_M f_{\dddot{m} \dddot{k} \dddot{k}} (f_{\dddot{l} \dddot{m} \dddot{n}} + R_{\dddot{m} \dddot{n} \dddot{f}_n})
= \int_M -f_{\dddot{m} \dddot{k} \dddot{k}} f_{\dddot{l} \dddot{l}} + f_{\dddot{m} \dddot{k} \dddot{k}} R_{\dddot{m} \dddot{n} \dddot{f}_n}
= \int_M -\mu \cdot f \Delta_t f + f_{\dddot{m} \dddot{k} \dddot{k}} R_{\dddot{m} \dddot{n} \dddot{f}_n}
\leq \int_M \mu |\nabla f|^2 + \frac{1}{2} \int_M |f_{\dddot{m} \dddot{k} \dddot{k}}|^2 + \frac{\Lambda^2}{2} \int_M |\nabla f|^2.
\end{equation}
Thus, we have
\begin{equation}
\int_M |f_{\dddot{m} \dddot{k} \dddot{k}}|^2 \leq (2 \mu + \Lambda^2) \int_M |\nabla f|^2.
\end{equation}
Lemma 4.4. The following inequality holds.

\[ \int_M |\tilde{f} \tilde{m} |^2 \leq (2\mu + \Lambda^2) \int_M |\nabla f|^2. \] (4.5)

Proof. We have by the Ricci identity,

\[ \int_M |\tilde{f} \tilde{m} |^2 = \int_M \tilde{f} \tilde{m} ((\Delta f)_n + R_{\bar{m}p} f_p) \]
\[ = \int_M -\tilde{f} \tilde{m} \Delta f + \tilde{f} \tilde{m} R_{\bar{m}p} f_p \]
\[ \leq -\int_M \mu \cdot f \Delta f + \frac{1}{2} \int_M |\tilde{f} \tilde{m} |^2 + \frac{\Lambda^2}{2} \int_M |\nabla f|^2 \]
\[ = (\mu + \frac{\Lambda^2}{2}) \int_M |\nabla f|^2 + \frac{1}{2} \int_M |\tilde{f} \tilde{m} |^2. \]

Thus, (4.5) is proved.

Lemma 4.5. The following inequalities hold:

\[ \| \nabla f \|^2 < 4(\mu^2 + \Lambda), \]
\[ \max \{ \| f_{\bar{m}k} \|^2, \| f_{\bar{m}m} \|^2 \} < 8(\mu^2 + \Lambda)^3. \]

where \( C \) is a universal constant.

Proof. Applying the Cauchy-Schwarz inequality, we obtain

\[ \mu(f, t) = \int_M (\Delta f)^2 + R_{ij} f_{ij} f + S_{,m} f_{\tilde{m}} f \]
\[ \geq \| \Delta f \|^2 - \Lambda \| \nabla \Delta f \|_2 - \Lambda \| \Delta f \|_2 - \Lambda \| \nabla f \|^2 \]
\[ \geq \frac{1}{2} \| \Delta f \|^2 - 5\Lambda^2, \]

where we used the Cauchy-Schwarz inequality in the last two inequalities. Thus, we have

\[ \| \nabla f \|^2 < \| \Delta f \|_2 < 4(\mu^2 + \Lambda). \]

So we have from (4.4) and (4.5)

\[ \max \{ \int_M |f_{\bar{m}k} |^2, \int_M |f_{\bar{m}m} |^2 \} < 8(\mu^2 + \Lambda)^3. \]

The lemma is proved.

Now we estimate \( I_1 \) at time \( t \):

\[ I_1 = \int_M -f S_{,m} f_{\bar{m}k} n \omega_t = \int_M \left( f_n S_{,m} f_{\bar{m}k} + f(\Delta S)_{,m} f_{\bar{m}k} \right) \omega_t^n \]
\[ = \int_M \left( f_n S_{,m} f_{\bar{m}k} - f_m \Delta S f_{\bar{m}k} - f \Delta S f_{\bar{m}k m} \right) \omega_t^n \]
\[ = \int_M \left( f_n S_{,m} f_{\bar{m}k} - f_m \Delta S f_{\bar{m}k} - \mu f^2 \Delta S \right) \omega_t^n. \] (4.7)
Thus, from (4.4) we have \( I_1 \):
\[
|I_1| \leq 2\epsilon(t) \cdot \|\nabla f\|_2 \|f_{\bar{m}kk}\|_2 + \epsilon(t)\mu.
\]
Similarly, we can estimate \( I_2 \) as follows:
\[
|I_2| = \left| \int_M f_{\bar{m}} S_{mkk} f_{kl}\right| \leq \epsilon(t) \|\nabla f\|_2 \|f_{\bar{m}kk}\|_2.
\]
Now we estimate \( I_3 \).
\[
I_3 = \int_M -f(f_{\bar{m}n} R_{m\bar{p}p}, n) \omega_t^n = \int_M f_n f_{\bar{m}n} R_{m\bar{p}p} \omega_t^n = \int_M f_n f_{\bar{m}n} \Delta \omega_t^n.
\]
\[
= -\int_M f_n f_{\bar{m}n} \Delta \omega_t^n - \int_M f_{nm} f_{\bar{m}n} \Delta \omega_t^n.
\]
So we have
\[
|I_3| \leq \epsilon(t) \|\nabla f\|_2 \|f_{\bar{m}nm}\|_2 + \epsilon(t)\mu.
\]
Combining the above inequalities, we have the differential inequality
\[
\frac{d}{dt} \mu(f, t_0) > -26\epsilon(t)(\mu + \Lambda^2)
\]
holds.

Since \( \mu(f, t) \) is a smooth function of \( t \), there is a small neighborhood \( [t_0 - \delta, t_0 + \delta] \) of \( t_0 \) such that for all \( t \in [t_0 - \delta, t_0 + \delta] \) the inequality (4.3) holds. Therefore, for any \( t_1 \in (t_0 - \delta, t_0) \) we have
\[
\log(\mu(f, t_0) + \Lambda^2) \geq \log(\mu(f, t_1) + \Lambda^2) - 26 \int_{t_1}^{t_0} \epsilon(t) \, dt.
\]
Choose \( \mu(f, t_0) = \mu_1(t_0) \), while \( \mu(f, t_1) \geq \mu_1(t_1) \), we get the inequality
\[
\log(\mu_1(t_0) + \Lambda^2) \geq \log(\mu_1(t_1) + \Lambda^2) - 26 \int_{t_1}^{t_0} \epsilon(t) \, dt.
\]
Since \( t_0 \) is arbitrary, the inequality (4.9) holds for any \( 0 \leq t_1 \leq t_0 \leq T \). So we obtain
\[
(\mu_1(t) + \Lambda^2) \geq (\mu_1(0) + \Lambda^2)e^{-26 \int_0^t \epsilon(t) \, dt}, \quad \forall t \in [0, T].
\]
The proposition is proved.

4.3. The first eigenvalue estimate with bounded Kähler potentials. In this subsection, we will show the exponential decay of the Calabi energy on a given time interval where the Kähler potential is bounded along the Calabi flow. First, we estimate the eigenvalue of the operator \( L \).

Lemma 4.6. Let \( \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi \) be a Kähler metric with the Kähler potential in the space
\[
\varphi \in C_A := \left\{ \varphi \in C^\infty(M, \mathbb{R}) \left| \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \ |\varphi|_{C^{4,\alpha}} \leq A \right. \right\}.
\]
Then there exists a constant \( \mu = \mu(\omega, A) \) such that for any smooth function \( f(x) \in W^{2,2}(M, \omega) \) with
\[
\int_M f \omega_\varphi^n = 0, \quad \int_M X(h) \omega_\varphi^n = 0, \quad \forall X \in \mathfrak{h}_0(M),
\]
(4.10)
where $h$ is a function determined by the equation $\Delta \varphi h = f$ and the normalization condition $\int_M h \omega_{\varphi}^n = 0$, the following inequality holds:

$$\int_M |\nabla \nabla f|^2 \omega_{\varphi}^n \geq \mu \int_M f^2 \omega_{\varphi}^n.$$  

**Proof.** Without lose of generality, we assume $f$ is a smooth function. We prove the lemma by the contradiction argument. Suppose the conclusion fails, we can find a sequence of functions $\varphi_m \in C^A$ and functions $f_m \in C^\infty(M)$ satisfying the properties

\begin{align}
(4.11) \quad \int_M |f_m|^2 \omega_{\varphi_m}^n = 1, \quad \int_M f_m \omega_{\varphi_m}^n = 0, \quad \int_M \Delta \varphi_m \omega_{\varphi_m}^n = 0, \quad \forall \varphi_0(M),
\end{align}

where $\varphi_m$ is a solution of the equation $\Delta_m \varphi_m = f_m$ with $\int_M h_m \omega_{\varphi_m}^n = 0$ and

\begin{align}
(4.12) \quad \int_M |\nabla \nabla f|^2 \omega_{\varphi_m}^n = \mu_m \rightarrow 0.
\end{align}

Here $\Delta_m$ denotes the Laplacian operator of the metric $\omega_{\varphi_m}$. Since $\varphi_m \in C_A$, we can assume that $\varphi_m \rightarrow \varphi_\infty$ in $C^{4, \alpha'}(M)$ for some $\alpha' < \alpha$. Note that

\begin{align}
\int_M (\Delta_m f_m)^2 \omega_{\varphi_m}^n &= \int_M |\nabla \nabla f|^2 \omega_{\varphi_m}^n + \int_M \text{Ric}(\omega_{\varphi_m})(\nabla f_m, \nabla f_m) \omega_{\varphi_m}^n \\
&\leq \mu_m + C_1(A, \omega) \int_M |\nabla f_m|^2 \omega_{\varphi_m}^n \\
&\leq \mu_m + C_2(A, \omega) \int_M f_m^2 \omega_{\varphi_m}^n + \frac{1}{2} \int_M (\Delta_m f_m)^2 \omega_{\varphi_m}^n,
\end{align}

which implies that

\begin{align}
(4.13) \quad \int_M (\Delta_m f_m)^2 \omega_{\varphi_m}^n \leq C_3(A, \omega).
\end{align}

Combining (4.11)–(4.13), there is a subsequence of functions $\{f_m\}$, which we denote by $\{f_{mk}\}$, converges weakly in $W^{2,2}(M, \omega_{\varphi_\infty})$:

$$f_{mk} \rightharpoonup f_\infty, \quad \text{in} \quad W^{2,2}(M, \omega_{\varphi_\infty}), \quad k \rightarrow +\infty.$$

By the Sobolev embedding theorem, we have

\begin{align}
(4.14) \quad \nabla_{m_k} f_{mk} \rightarrow \nabla_\infty f_\infty, \quad \text{in} \quad L^2(M, \omega_{\varphi_\infty}), \quad \text{and} \quad f_{mk} \rightarrow f_\infty \quad \text{in} \quad L^2(M, \omega_{\varphi_\infty}).
\end{align}

In particular, we have

\begin{align}
(4.15) \quad \int_M f_{mk}^2 \omega_{\varphi_\infty} = 1, \quad \int_M f_\infty \omega_{\varphi_\infty} = 0.
\end{align}

By (4.14) the functions $h_{mk}$ subconverges to a function $h_\infty$ in $W^{3,2}(M, \omega_{\varphi_\infty})$, which implies that

\begin{align}
(4.16) \quad \int_M \Delta \varphi_\infty \omega_{\varphi_\infty} = 0, \quad \forall \varphi_0(M).
\end{align}

On the other hand, we have

\begin{align}
\int_M |\nabla \nabla f|^2 \omega_{\varphi_\infty}^n &\leq \liminf_{k \rightarrow +\infty} \int_M |\nabla \nabla f_{mk}|^2 \omega_{\varphi_\infty}^n \\
&\leq \liminf_{k \rightarrow +\infty} \int_M |\nabla_{m_k} \nabla_{m_k} f_{mk}|^2 \omega_{m_k}^n = 0.
\end{align}
Thus, $X_\infty := \nabla_\infty f_\infty$ is a holomorphic vector field in $h_0(M)$, and by (4.16) we have

$$0 = \int_M X_\infty (h_\infty) \omega_\varphi^n = - \int_M f_\infty \Delta_\infty h_\infty \omega_{\varphi_\infty}^n = - \int_M f^2_\infty \omega_{\varphi_\infty}^n,$$

which contradicts (4.15). The lemma is proved.

A direct corollary of Lemma 4.6 is the following result:

**Lemma 4.7.** Let $\omega_t (t \in [0, T])$ be a solution of Calabi flow. Suppose that the Futaki invariant of $[\omega]$ vanishes. If the metric satisfies $|\varphi(t)|_{C^{1, \alpha}} \leq A$, then there is a constant $\mu = \mu(A, \omega)$ such that

$$\frac{\partial}{\partial t} C\alpha(t) \leq -\mu C\alpha(t), \quad \forall t \in [0, T].$$

**Proof.** Direct calculation shows that

$$\frac{d}{dt} C\alpha(t) = -\int_M |\nabla \nabla \dot{\varphi}|^2 \omega_\varphi^n.$$ 

In Lemma 4.6 choose $f = S - \bar{S} = \dot{\varphi}$. Since the Futaki invariant vanishes, the second condition in (4.10) holds. Therefore, there is a constant $\mu = \mu(A, \omega)$ such that

$$\int_M |\nabla \nabla \dot{\varphi}|^2 \omega_\varphi^n \geq \mu \int_M (S - \bar{S})^2 \omega_\varphi^n, \quad t \in [0, T].$$

The lemma follows directly from the above inequalities.

We have a similar result for the modified Calabi flow.

**Lemma 4.8.** Let $\omega_t (t \in [0, T])$ be a solution of modified Calabi flow. Suppose that the modified Futaki invariant of $[\omega_t]$ vanishes. If the metric satisfies $|\psi(t)|_{C^{1, \alpha}} \leq A$, then there is a constant $\mu = \mu(A, \omega)$ such that

$$\frac{\partial}{\partial t} \tilde{C}\alpha(t) \leq -\mu \tilde{C}\alpha(t), \quad \forall t \in [0, T].$$

**Proof.** By (3.7) we obtain the evolution of the modified Calabi energy along the modified Calabi flow,

$$\frac{d}{dt} \tilde{C}\alpha(t) = -2 \int_M |\nabla \nabla \dot{\psi}|^2 \omega_\psi^n.$$ 

In Lemma 4.6 let $f = \psi = S_\psi - \bar{S} + \theta_{R\text{ev}}(\psi)$, the vanishing modified Futaki invariant guarantee the condition (4.10). So there is a constant $\mu = \mu(A, \omega)$ such that

$$\int_M |\nabla \nabla \dot{\psi}|^2 \omega_\psi^n \geq \mu \int_M \dot{\psi}^2 \omega_\psi^n, \quad t \in [0, T].$$

The lemma follows directly from (3.3) and (3.6).

4.4. **The first eigenvalue estimate with bounded curvature.** In this subsection, we will show the exponential decay of the Calabi energy on a given time interval where the Riemann curvature tensor and Sobolev constant are bounded. Here, we assume that $M$ has non-zero holomorphic vector fields. The argument needs the pre-stable condition, which is defined as follows (cf. Chen-Li-Wang [23], Phong-Sturm [48]).

**Definition 4.9.** The complex structure $J$ of $M$ is called pre-stable, if no complex structure of the orbit of diffeomorphism group contains larger (reduced) holomorphic automorphism group.
**Lemma 4.10.** Suppose that \((M, J)\) is pre-stable. For any \(\Lambda, K > 0\), if the metric \(\omega \in \Omega\) satisfies
\[
|R_m(\omega)| \leq \Lambda, \quad C_S(\omega) \leq K, \tag{4.17}
\]
then there exists a constant \(\mu = \mu(\Lambda, K, \omega) > 0\) such that for any smooth function \(f(x) \in C^\infty(M)\) with
\[
\int_M f \omega^n = 0, \quad \int_M X(h) \omega^n = 0, \quad \forall X \in h_0(M), \tag{4.18}
\]
where \(h\) is a function determined by the equation \(\Delta f = f\), the following inequality holds:
\[
\int_M |\nabla \nabla f|^2 \omega^n \geq \mu \int_M f^2 \omega^n. \tag{4.19}
\]

**Proof.** The argument is almost identical to that of Lemma 4.6. See also the proof of Theorem 4.16 in Chen-Li-Wang [23]. □

A direct corollary is the following

**Lemma 4.11.** Let \(\omega_t(t \in [0, T])\) be a solution of Calabi flow. Suppose that \((M, J)\) is pre-stable and the Futaki invariant of \(\omega_t\) vanishes. If the metric satisfies \(|R_m(t)| \leq A\) and \(C_S(t) \leq K\) then there is a constant \(\mu = \mu(A, K, \omega)\) such that
\[
\partial_t Ca(t) \leq -\mu Ca(t), \quad \forall t \in [0, T].
\]

We have an analogous result for the modified Calabi flow.

**Lemma 4.12.** Let \(\omega_t(t \in [0, T])\) be a solution of modified Calabi flow. Suppose that \((M, J)\) is pre-stable and the modified Futaki invariant of \(\omega_t\) vanishes. If the metric satisfies \(|R_m(t)| \leq A\) and \(C_S(t) \leq K\) then there is a constant \(\mu = \mu(A, K, \omega)\) such that
\[
\partial_t \tilde{Ca}(t) \leq -\mu \tilde{Ca}(t), \quad \forall t \in [0, T].
\]

### 4.5. Decay estimates of higher order curvature

In this subsection, we will show the decay of the higher order derivatives of the scalar curvature under the decay assumption of the Calabi energy. First, we recall the Sobolev inequality:

**Lemma 4.13.** For any integer \(p > 2n\) and Kähler metric \(\omega\), there is a constant \(C_{S} = C_{S}(p, \omega)\) such that
\[
\max_M |f| \leq C_{S}\left(\int_M (|f|^p + |\nabla f|^p \omega^n)^{\frac{1}{p}}\right). \tag{4.20}
\]

To estimate the higher derivatives of the curvature, we need the interpolation formula of Hamilton in [39]:

**Lemma 4.14.** ([39]) Let \(n = \dim_C M\). For any tensor \(T\) and \(1 \leq j \leq k - 1\), we have
\[
\int_M |\nabla^j T|^2 \omega^n \leq C \cdot \max_M |T|^{\frac{2k-j}{k}} \int_M |\nabla^k T|^2 \omega^n, \tag{4.21}
\]
\[
\int_M |\nabla^j T|^2 \omega^n \leq C\left(\int_M |\nabla^k T|^2 \omega^n\right)^{\frac{j}{k}} \left(\int_M |T|^2 \omega^n\right)^{1-\frac{j}{k}} \tag{4.22}
\]
where \(C = C(k, n)\) is a constant.

Combining Lemma 4.13 with Lemma 4.14, we have the following result, which gives the higher order estimates in terms of the integral norms of the tensor:
Lemma 4.15. For any integer \( i \geq 1 \) and Kähler metric \( \omega \), there exists a constant \( C = C(\omega_S, i) > 0 \) such that for any tensor \( T \), we have

\[
\max_M |\nabla^i T|^2 \leq C \cdot \max_M |T|^2 \left( \int_M |T|^2 \omega^n \right)^{-\frac{1}{2(n+1)}}.
\]

\[
= \left( \int_M |\nabla^{4(n+1)} T|^2 \omega^n + \int_M |\nabla^{4(n+1)(i+1)} T|^2 \omega^n \right)^{\frac{1}{2(n+1)}}.
\]

Proof. Choosing \( f = |\nabla^i T|^2 \) and \( p = 2(n+1) \) in Lemma 4.13 we have

\[
\max_M |\nabla^i T|^2 \leq C_S \left( \int_M (|\nabla^i T|^4(n+1) + |\nabla f|^{2(n+1)} \omega^n) \right)^{\frac{1}{2(n+1)}}.
\]

On From the Kato’s inequality we have

\[
|\nabla f| = 2|\nabla^i T| \cdot |\nabla |\nabla^i T|| \leq 2|\nabla^i T| \cdot |\nabla^{i+1} T| \leq |\nabla^i T|^2 + |\nabla^{i+1} T|^2,
\]

also,

\[
(4.21) \quad \max_M |\nabla^i T|^2 \leq C \left( \int_M (|\nabla^i T|^4(n+1) + |\nabla^{i+1} T|^4(n+1)) \omega^n \right)^{\frac{1}{2(n+1)}},
\]

where \( C = C(n, C_S) \). Letting \( k = 2(n+1)i, j = i \) in the first inequality in Lemma 4.14 we have

\[
\int_M |\nabla^{i+1} T|^4(n+1) \omega^n \leq C \cdot \max_M |T|^{2k+2} \int_M |\nabla^{2(n+1)} T|^2 \omega^n,
\]

while \( k = 2(n+1)(i+1), j = i+1 \), we have

\[
\int_M |\nabla^{i+1} T|^4(n+1) \omega^n \leq C \cdot \max_M |T|^{2k+2} \int_M |\nabla^{2(n+1)(i+1)} T|^2 \omega^n.
\]

Taking \( k = 4(n+1)i \) and \( j = 2(n+1)i \) in the second inequality in Lemma 4.14 we have

\[
\int_M |\nabla^{2(n+1)i} T|^2 \omega^n \leq C \cdot \left( \int_M |\nabla^{2(n+1)i} T|^2 \right)^{\frac{1}{2}} \left( \int_M |T|^2 \omega^n \right)^{\frac{1}{2}},
\]

Combining this with (4.21), we have

\[
\max_M |\nabla^i T|^2 \leq C \cdot \max_M |T|^{2n+1+1} \left( \int_M (|\nabla^{2(n+1)i} T|^2 + |\nabla^{2(n+1)(i+1)} T|^2 \omega^n \right)^{\frac{1}{2(n+1)}},
\]

\[
\leq C \cdot \max_M |T|^{2n+1+1} \left( \int_M |T|^2 \omega^n \right)^{\frac{1}{2(n+1)}} \left( \int_M |\nabla^{4(n+1)i} T|^2 \omega^n + \int_M |\nabla^{4(n+1)(i+1)} T|^2 \omega^n \right)^{\frac{1}{2(n+1)}},
\]

The lemma is proved.

Now, we can show the decay of higher order derivatives of the scalar curvature.

Lemma 4.16. Given constants \( \Lambda, K, T > 0 \) and a positive function \( \epsilon(t) \). If \( \omega_\epsilon(t \in [0, T]) \) is a solution of the Calabi flow with

\[
(4.22) \quad |Rm| \leq \Lambda, \quad C_S(t) \leq K, \quad C_a(t) \leq \epsilon(t), \quad \forall t \in [0, T],
\]
then for any integer \(i \geq 1\) and any \(t_0 \in (0, T)\) there exists a constant \(C = C(t_0, i, \Lambda, K, n, V) > 0\) such that

\[
|\nabla^i S| \leq C(t_0, i, \Lambda, K, n, V)\epsilon(t(t_0, i, \Lambda, K, n, V)\epsilon(t) \frac{1}{\bar{t}^{i+1}}), \quad \forall t \in [t_0, T].
\]

**Proof.** By Theorem 3.1 in Chen-He [21], for any \(t \in [0, T]\) we have the inequality

\[
\partial_t \int_M |\nabla^k Rm|^2 \omega^n_t \leq -\frac{1}{2} \int_M |\nabla^{k+2} Rm|^2 \omega^n_t + C \int_M |Rm|^2 \omega^n_t,
\]

where \(C = C(k, \Lambda, n)\) is a constant. To estimate the higher order derivatives of the curvature tensor, we follow the argument in Chen-He [21] to define

\[
F_k(t) = \sum_{i=0}^k t^i \int_M |\nabla^i Rm|^2(t) \omega^n_t.
\]

Using Lemma 4.14 we have

\[
\partial_t F_k(t) \leq -\frac{1}{4} \sum_{i=0}^k t^i \int_M |\nabla^{i+2} Rm|^2 \omega^n_t + C \int_M |Rm|^2 \omega^n_t,
\]

where \(C = C(k, \Lambda, n)\). This implies that

\[
F_k(t) \leq \int_M |Rm|^2(0) \omega^n_0 + C \int_0^t dt \int_M |Rm|^2(t) \omega^n_t \leq C(k, \Lambda, n, V)(1 + t).
\]

Thus, for any integer \(k \geq 1\) we have

\[
\int_M |\nabla^k Rm|^2(t) \omega^n_t \leq \frac{C(n, k, \Lambda, V)(1 + t)}{t^k}, \quad \forall t \in (0, T].
\]

Combining this with Lemma 4.15 for the function \(T = S - S\) and integer \(i \geq 1\), we have for any \(t_0 \in (0, T)\),

\[
\max_M |\nabla^i S|^2 \leq C(i, K, \Lambda, n, V)Ca(t) \frac{1}{\bar{t}^{i+1}} \cdot \left( \int_M |\nabla^{4(n+1)i} S|^2 \omega^n t + \int_M |\nabla^{4(n+1)(i+1)} S|^2 \omega^n t \right) \frac{1}{\bar{t}^{i+1}}, \quad \forall t \in [t_0, T].
\]

A direct corollary of Lemma 4.16 is the following:

**Lemma 4.17.** Given any constants \(A > 0, C > 1, T > 0\) and a positive function \(\epsilon(t)\). If \(\omega_i(t \in [0, T])\) is a solution of the Calabi flow with

\[
\frac{1}{c} \omega \leq \omega_i \leq c\omega, \quad |\varphi(t)|_{C^{1,\alpha}} \leq A, \quad Ca(t) \leq \epsilon(t), \quad \forall t \in [0, T],
\]

then for any integer \(i \geq 1\) and any \(t_0 \in (0, T)\) there exists \(C = C(i, c, A, n, V, \omega) > 0\) such that

\[
|\nabla^i S| \leq C(t_0, i, c, A, n, V, \omega)\epsilon(t(t) \frac{1}{\bar{t}^{i+1}}), \quad \forall t \in [t_0, T].
\]
5. The Calabi flow of the Kähler potentials

5.1. Proof of Theorem 1.2

Proof of Theorem 1.2. For any $\lambda, \epsilon > 0$, if the initial metric $\omega_0 := \omega + \sqrt{-1} \partial \bar{\partial} \varphi_0$ satisfies the condition (1.3), then by Proposition 4.1 there exist constants $\tau = \tau(\lambda, \lambda_0, \omega) > 0$ and constants $c = c(\lambda, \lambda_0, \omega), B = B(\lambda, \lambda_0, \omega)$ such that

$$\omega_\tau \in A(c, B, \epsilon) := \left\{ \omega_\varphi \left| \frac{1}{c} \omega \leq \omega_\varphi \leq c \omega, \quad |\varphi|_{C^{4,\alpha}} \leq B, \quad C(\omega_\varphi) \leq \epsilon \right\}. $$

Similarly, we can also choose $t_0 = t_0(\lambda, \lambda_0, \omega) > \tau$ such that $\omega_{t_0} \in A(2c, 2B, \epsilon)$.

Now we start from the time $\tau$. Suppose

$$T := \sup \left\{ t > 0 \left| \omega_s \in A(6c, 6B, \epsilon), \quad \forall s \in [\tau, t] \right\} < +\infty. $$

Lemma 5.1. There exists small $\epsilon = \epsilon(c, B) > 0$ such that there is a constant $t_0 > \tau$ such that the solution satisfies

$$\omega_t \in A(3c, 3B, \epsilon), \quad \forall t \in [t_0, T].$$

Proof. Since the solution $\varphi(t)$ satisfies $|\varphi(t)|_{C^{4,\alpha}} \leq 6B$ for $t \in [\tau, T)$ and the Futaki invariant vanishes, by Lemma 4.1 the Calabi energy decays exponentially

$$\frac{d}{dt} C(\omega_t) \leq -\mu C(\omega_t), \quad t \in [\tau, T]$$

where $\mu = \mu(6B, \omega)$. It follows that $C(\omega_t) \leq ce^{-\mu(t-\tau)}$ for any $t \in [\tau, T]$. Combining this with Lemma 4.17 the higher order derivatives of the scalar curvature have the estimates

$$|\nabla^i S|(t) \leq C_1(t_0, i, 6c, 6B, n, \omega) e^{\frac{1}{4c} \sqrt{\mu} \tau + \frac{1}{4c} \sqrt{\mu} t_0} e^{-\frac{1}{2c \sqrt{\mu}} \mu(t-t_0)}, \quad \forall t \in [t_0, T],$$

where $t_0$ is chosen in step (1). By the equation of Calabi flow, we have

$$\left| \frac{d}{dt} \omega_t \right| \leq |\nabla S| \leq C_1(t_0, i, 6c, 6B, n, \omega) e^{\frac{1}{4c} \sqrt{\mu} \tau + \frac{1}{4c} \sqrt{\mu} t_0} e^{-\frac{1}{2c \sqrt{\mu}} \mu(t-t_0)}, \quad \forall t \in [t_0, T].$$

It follows that

$$e^{-C_2(t_0, c, B, n, \omega)} e^{\frac{1}{4c} \sqrt{\mu} \tau + \frac{1}{4c} \sqrt{\mu} t_0} \omega_{t_0} \leq \omega_t \leq e^{C_2(t_0, c, B, n, \omega)} e^{\frac{1}{4c} \sqrt{\mu} \tau + \frac{1}{4c} \sqrt{\mu} t_0} \omega_{t_0}, \quad \forall t \in [t_0, T].$$

If we choose $\epsilon$ sufficiently small, then

$$\frac{1}{3c} \omega \leq \omega_t \leq 3c \omega, \quad \forall t \in [t_0, T].$$

Now we estimate $|\varphi|_{C^{4,\alpha}}$. In fact, along the Calabi flow we have

$$\frac{d}{dt} (|\nabla^i \varphi|^2) \leq c(n)(|\nabla \nabla S| \cdot |\nabla^i \varphi|^2 + |\nabla^i S| \cdot |\nabla^i \varphi|),$$

where $c(n)$ is a constant depending only on $n$. Note that (5.3) can be written as

$$\frac{d}{dt} \log(1 + |\nabla^i \varphi|) \leq c(n)(|\nabla \nabla S| + |\nabla^i S|).$$

Using the estimate (5.2), we have

$$1 + |\nabla^i \varphi|(t) \leq e^{C(t_0, i, 6c, 6B, n, \omega)} e^{\frac{1}{4c} \sqrt{\mu} \tau + \frac{1}{4c} \sqrt{\mu} t_0} (1 + |\nabla^i \varphi|(t_0)), \quad \forall t \in [t_0, T].$$

Since $\varphi(t_0) \in A(2c, 2B, \epsilon)$, the estimates (5.4) imply $|\varphi|_{C^{4,\alpha}}(t) \leq 3B(t \in [t_0, T])$ for sufficiently small $\epsilon$. The lemma is proved. □
By Lemma 5.1 we can extend the solution $\omega_t$ to $[0, T + \delta]$ for some $\delta = \delta(c, B) > 0$ such that

$$\omega_t \in \mathcal{A}(6c, 6B, \epsilon), \quad \forall \ t \in (t_0, T + \delta],$$

which contradicts (5.1). Therefore, all derivatives of $\varphi$ are uniformly bounded for all time $t > 0$ by Lemma 4.1, and the Calabi energy decays exponentially by Lemma 4.7. Thus, the Calabi flow converges exponentially fast to a constant scalar curvature metric. The theorem is proved.

Similarly, using Lemma 4.8 we have the following result for the modified Calabi flow. The proof is parallel to that of Theorem 1.2 and we omit the details here.

**Theorem 5.2.** Let $(M, \omega)$ be a compact Kähler manifold with vanishing modified Futaki invariant. For any $\lambda, \Lambda > 0$, there is a constant $\epsilon = \epsilon(\lambda, \Lambda, \omega)$ such that for any $K$-invariant metric $\omega_\varphi \in [\omega]$ satisfying

$$\omega_\varphi \geq \lambda \omega, \quad |\varphi|_{C^{1,\alpha}} \leq \Lambda, \quad \tilde{C}_\omega(\omega_\varphi) \leq \epsilon,$$

the modified Calabi flow with the initial metric $\omega_\varphi$ exists for all time and converges exponentially fast to an extremal Kähler metric.

**5.2. Proof of Theorem 1.4 and Corollary 1.3**

**Proof of Theorem 1.4.** By the assumptions of Theorem 1.4, the metric $\omega_0 = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_0$ satisfies the following conditions

$$Ric(\omega_0) \geq -\lambda \omega_0, \quad ||S||_{L^p(\omega_0)} \leq \Lambda, \quad C_S(\omega_0) \leq K, \quad \nu_\omega(\omega_0) \leq \inf_{\omega' \in \Omega} \nu_\omega(\omega') + \epsilon.$$

Therefore, according to Theorem 1.1, the corresponding Kähler potential

$$|\varphi_0|_{C^{3,\alpha}} \leq C(\lambda, \Lambda, K, \omega), \quad \omega_{\varphi_0} \geq c \omega > 0.$$

We use the Calabi flow to smooth this Kähler potential. Let $\varphi(t)$ be the Calabi flow with the initial data $\varphi_0$. According to Proposition 4.1, there exists $t_0 > 0$ such that $|\varphi(t_0)|_{C^{3,\alpha}}$ is bounded by a constant depending on $\omega$ and $|\varphi_0|_{C^{3,\alpha}}$. Moreover, $\varphi(t_0)$ satisfies

$$\omega_{\varphi(t_0)} \geq \tilde{c} \cdot \omega$$

for some constant $\tilde{c} > 0$. Since the $K$-energy is decreasing along the Calabi flow, the $K$-energy of $\varphi(t_0)$ is not larger than $\inf_{\omega' \in \Omega} \nu_\omega(\omega') + \epsilon$.

In order to apply Theorem 1.4 we need to show the Calabi energy of $\varphi(t_0)$ is small. This follows from the next lemma.

**Lemma 5.3.** Assume that $\varphi$ satisfies $|\varphi|_{C^{3,\alpha}} \leq B$ for a fixed constant $B$. Then for any constant $\delta > 0$, there is a positive constant $\epsilon$ depending on $B$ and $\delta$ such that if $\nu_\omega(\omega_0) \leq \inf_{\omega' \in \Omega} \nu_\omega + \epsilon$, then $C\omega(\varphi) \leq \delta$.

**Proof.** We argue this by contradiction. Suppose not, there are $\delta_0 > 0$ and a sequence of positive constants $\epsilon_i \to 0$ and $\varphi_i$ satisfying

$$C\omega(\varphi_i) > \delta_0 > 0, \quad |\varphi_i|_{C^{3,\alpha}} \leq B, \quad \omega_{\varphi_i} \geq c \omega, \quad \nu_\omega(\omega_i) \leq \inf_{\omega' \in \Omega} \nu_\omega + \epsilon_i.$$

Thus, we can take a subsequence of $\varphi_i$ such that $\varphi_i \to \varphi_\infty$ in $C^{4,\alpha'}$ for some $\alpha' < \alpha$ with $\nu_\omega(\omega_\infty) = \inf_{\omega' \in \Omega} \nu_\omega$. Thus, $\omega_\infty$ is a $C^{4,\alpha'}$ cscK metric with $C\omega(\varphi_\infty) \geq \delta_0 > 0$, which is impossible.

□
Therefore, we apply Theorem 1.2 to obtain Theorem 1.4.

Using the "modified" version of Theorem 1.1, the proof of Theorem 1.4 also works for the modified Calabi flow and extK metrics. Thus, we have the result:

**Theorem 5.4.** Let $(M, \omega)$ be a compact Kähler manifold for which the modified K-energy is I-proper. For any constants $\lambda, \Lambda, Q > 0$ and $p > n$, there is a constant $\epsilon = \epsilon(\lambda, \Lambda, Q, p, \omega)$ such that if there exists a $K$-invariant metric $\omega_0 \in [\omega]$ satisfying the following conditions

$$\text{Ric}(\omega_0) \geq -\lambda \omega_0, \quad \|S(\omega_0)\|_{L^p(\omega)} \leq \Lambda, \quad CS(\omega_0) \leq Q, \quad \tilde{\nu}_\omega(\omega_0) \leq \inf_{\omega' \in \Omega} \tilde{\nu}_\omega(\omega') + \epsilon,$$

then the modified Calabi flow with the initial metric $\omega_0$ exists for all time and converges exponentially fast to an extK metric.

Combining Theorem 1.1 with Theorem 1.2, we have

**Proof of Corollary 1.3.** Since the $K$-energy is decreasing along the Calabi flow, all the conditions in Theorem 1.1 are satisfied for the evolving metrics. By Proposition 4.1 the Calabi flow has long time solution and there exist uniform constants $\lambda, C > 0$ such that for any $t \in (t_0, \infty)$ ($t_0 > 0$),

$$\omega_t \geq \lambda \omega, \quad |\varphi(t)|_{C^{k,\alpha}} \leq C.$$

We claim that $\lim_{t \to +\infty} Ca(t) = 0$. In fact, by (5.8) there is a sequence of metrics $\omega_{t_i}(t_i \to +\infty)$ converging to a limit metric $\omega_\infty$ smoothly and $\omega_\infty$ satisfies

$$\int_M |\nabla \nabla S(\omega_\infty)|^2 \omega_n = 0.$$

Therefore, $\omega_\infty$ is an extK metric. Since the $K$-energy is proper, the Futaki invariant vanishes and $\omega_\infty$ is a cscK metric. Since the Calabi energy is decreasing along the flow, we have $\lim_{t \to +\infty} Ca(t) = 0$.

Thus, the Calabi energy is sufficiently small when $t$ is large enough. By Theorem 1.2, the flow converges exponentially fast to a cscK metric. Similar arguments also work for the modified Calabi flow, and we omit the details here.

The proof of Corollary 1.3 also works for the modified Calabi flow and extremal Kähler metrics. Here we state the result and omit its proof.

**Corollary 5.5.** Let $(M, \Omega)$ be an $n$-dimensional compact Kähler manifold for which the modified $K$-energy is I-proper in the Kähler class $\Omega$. If the $L^p$-norm of the scalar curvature for some $p > n$, the Sobolev constant and the lower bound of Ricci curvature are uniformly bounded along the flow, then the modified Calabi flow with a $K$-invariant initial metric converges exponentially fast to an extK metric.

On Fano surfaces the conditions on Sobolev constant in Corollary 1.3 can be removed. It was observed in Tian-Viaclovsky [60] that the Sobolev constant of the cscK metric is essentially bounded by the positive Yamabe invariant. The Yamabe invariant is positive when the Kähler class $\Omega$ stays in the interior of Tian’s cone

$$3C_1^2 > 2 \frac{(C_1 \cdot \Omega)^2}{\Omega^2}.$$
Here $C_1$ denotes the first Chern class $C_1(M)$ for simplicity. This idea was generalized to the extK metrics (cf. Chen-Weber [25] and Chen-Lebrun-Weber [22]) in generalized Tian’s cone
\[
48\pi^2 C_1^2 > A(\Omega) := 32\pi^2 (C_1^2 + \frac{1}{3} (C_1 \cdot \Omega)^2) + \frac{1}{3} \|F\|^2.
\]
The last term is the norm of the Calabi-Futaki invariant [19] [36]. Chen-He [20] proved in Lemma 2.3 that if the minimum of the extremal Hamiltonian potential of the all $K$-invariant Kähler metric is positive and the Calabi energy of the initial $K$-invariant Kähler metric is less than $A(\Omega)$, then the Sobolev constant is bounded. That shows that on Fano surfaces, the third condition in our Theorem 1.1 could be verified when $Ca < A(\Omega)$.

Combining the Sobolev constants with the classification of the blowing up bubbles, Chen-He [20] [21] proved the convergence of the Calabi flow on the toric Fano surfaces under the restriction on the initial Calabi energy. Our next corollary shows that the curvature obstructions of the convergence are the $L^p$-norm of the scalar curvature and the lower bound of the Ricci curvature.

**Corollary 5.6.** On a compact Fano surface for which the modified $K$-energy is $I$-proper. Assume that the initial Kähler metric is $K$-invariant and has Calabi energy less than $A(\Omega)$. If the $L^p$-norm of the scalar curvature for some $p > 2$ and the lower bound of the Ricci curvature are uniformly bounded along the flow, then the Calabi flow converges exponentially fast to an extK metric.

6. Calabi flow of the Kähler metrics

6.1. Proof of Theorem 1.6

**Proof of Theorem 1.6.** Given any constants $\Lambda, K, \delta, \epsilon > 0$, we define
\[
\mathcal{B}(\Lambda, K, \delta, \epsilon) = \left\{ \omega_\phi \mid |Rm| (\omega_\phi) \leq \Lambda, C_S(\omega_\phi) \leq K, \mu_1(\omega_\phi) \geq \delta, Ca(\omega_\phi) \leq \epsilon \right\}.
\]
By the assumption, we assume that the Calabi flow with the initial metric $\omega_0$ is of type II $(\tau, \Lambda)$ and $\omega_0 \in \mathcal{B}(\Lambda, K, \delta, \epsilon)$.

**Lemma 6.1.** There is a $\tau_0 = \tau_0(\tau, \Lambda, K, \delta) > 0$ such that for any $t \in [0, \tau_0]$ we have
\[
\omega_t \in \mathcal{B}(2\Lambda, 2K, \frac{2\delta}{3}, \epsilon).
\]

**Proof.** Since $\omega_t$ is of type $(\tau, \Lambda)$, we have $|Rm(t)| + |\nabla \nabla S| \leq \Lambda$ for any $t \in [0, \tau]$. Thus, by the equation of Calabi flow the evolving metrics satisfy the inequality
\[
\left| \frac{\partial g_{ij}}{\partial t} \right| \leq \Lambda, \quad \forall t \in [0, \tau].
\]
Therefore, we have
\[
e^{-\Lambda t} \omega_0 \leq \omega_t \leq e^{\Lambda t} \omega_0, \quad \forall t \in [0, \tau].
\]
Clearly, we can choose $\tau_0 \in (0, \tau)$ small such that $C_S(\omega_t) \leq 2K$ for any $t \in [0, \tau_0]$. By Proposition 4.2 the first eigenvalue $\mu_1(t) \geq \frac{2}{3} \delta$ for $t \in [0, \tau_0]$ when $\tau_0$ is small. The lemma is proved.

To extend the solution, we define
\[
T := \sup \left\{ t > 0 \mid \omega_t \in \mathcal{B}(6\Lambda, 6K, \frac{1}{4} \delta, \epsilon) \right\}.
\]
Suppose $T < +\infty$. We have the following lemma.
Lemma 6.2. There exists $\epsilon_0 = \epsilon_0(\Lambda, K, \delta, n) > 0$ such that
\[
\omega_t \in B(3\Lambda, 3K, \frac{1}{2}\delta, \epsilon_0), \quad \forall \ t \in [0, T].
\]

Proof. Since the eigenvalue $\mu_1(t) \geq \frac{1}{4}\delta$ for any $t \in [0, T]$, the Calabi energy decays exponentially:
\[
Ca(t) \leq ce^{-\frac{1}{4}\delta t}, \quad \forall \ t \in [0, T].
\]
By Lemma 4.16 for any integer $i \geq 1$ we have
\[
|\nabla^i S(t)| \leq C(\tau_0, i, 6\Lambda, 6K, n) \epsilon e^{-\frac{1}{8}n(3\Lambda+6K+1)} \epsilon, \quad \forall \ t \in [\tau_0, T].
\]
Thus, using the equation of Calabi flow as in Lemma 6.1 we can choose $\epsilon$ small such that
\[
\mu_1(t) \geq \frac{1}{2}\delta, \quad \forall \ t \in [0, T].
\]
By the evolution equation (3.1) of $Rm$ we have
\[
\frac{\partial}{\partial t} |Rm|^2(t) = Rm \ast \nabla^4 S + Rm \ast Rm \ast \nabla^2 S.
\]
Therefore, the following inequality holds:
\[
\frac{\partial}{\partial t} |Rm|(t) \leq |\nabla^4 S| + |Rm| |\nabla^2 S|.
\]
It follows that
\[
|Rm|(t) \leq |Rm|(\tau_0) + C(\tau_0, \Lambda, K, n, \delta) \cdot \epsilon e^{-\frac{1}{8}n(3\Lambda+6K+1)} \epsilon, \quad \forall \ t \in [\tau_0, T],
\]
where we chosen $\epsilon$ small in the last inequality. Thus, we have $\omega_t \in B(3\Lambda, 3K, \frac{1}{2}\delta, \epsilon_0)$ for all $t \in [\tau_0, T]$. The lemma is proved. \qed

By Lemma 6.2 we can extend the solution $\omega_t$ to $[0, T + \delta']$ for some $\delta' > 0$ such that
\[
\omega_t \in B(6\Lambda, 6K, \frac{1}{4}\delta, \epsilon_0), \quad \forall \ t \in [0, T + \delta'],
\]
which contradicts the definition (6.1). Therefore, all derivatives of $\varphi$ are uniformly bounded for all time $t > 0$ by Lemma 4.1 and the Calabi energy decays exponentially by Lemma 4.7. Thus, the Calabi flow converges exponentially fast to a constant scalar curvature metric. The theorem is proved. \qed

6.2. Proof of Theorem 1.7

Proof of Theorem 1.7 The proof is almost the same as that of Theorem 1.6. Here we sketch the details.

Given constants $\Lambda, K, \delta > 0$, we define the set
\[
C(\Lambda, K, \epsilon) = \left\{ \omega_\varphi \mid |Rm|(\omega_\varphi) \leq \Lambda, \ C_S(\omega_\varphi) \leq K, \ Ca(\omega_\varphi) \leq \epsilon \right\}.
\]
By the assumption, we assume that the Calabi flow with the initial metric $\omega_0$ is of type $I(\tau, \Lambda)$ and $\omega_0 \in C(\Lambda, K, \epsilon)$. Following the same argument as in the proof of Lemma 6.1 we have
Lemma 6.3. There is a $\tau_0 = \tau_0(\tau, \Lambda, K) > 0$ such that for any $t \in [0, \tau_0]$ we have $\omega_t \in C(2\Lambda, 2K, \epsilon)$.

To extend the solution, we define

$$T := \sup \left\{ t > 0 \mid \omega_t \in C(6\Lambda, 6K, \epsilon_0) \right\}.$$

Suppose $T < +\infty$. We have the following lemma.

Lemma 6.4. There exists $\epsilon_0 = \epsilon_0(\Lambda, K, \omega) > 0$ such that for any $t \in [0, T]$ we have $\omega_t \in C(3\Lambda, 3K, \epsilon_0)$.

Proof. Since $(M, J)$ is pre-stable, by Lemma 4.11 the Calabi energy decays exponentially:

$$Ca(t) \leq e^{-\delta t}, \quad \forall t \in [0, T],$$

where $\delta = \delta(6\Lambda, 6K, \omega) > 0$. By Lemma 4.16 for any integer $i \geq 1$ we have

$$|\nabla^i S|(t) \leq C(\tau_0, i, 6\Lambda, 6K, n, \omega) e^{\frac{i}{12n+17}} e^{-\frac{\delta}{3(6n+17)}}, \quad \forall t \in [\tau_0, T].$$

Thus, using the equation of Calabi flow as in Lemma 6.1 we can choose $\epsilon$ small such that $\omega_t \in C(3\Lambda, 3K, \epsilon_0)$ for all $t \in [\tau_0, T]$. The lemma is proved.

By Lemma 6.4 we can extend the solution $\omega_t$ to $[0, T + \delta']$ for some $\delta' > 0$ such that

$$\omega_t \in C(6\Lambda, 6K, \epsilon_0), \quad \forall t \in [0, T + \delta'],$$

which contradicts the definition (6.2). Therefore, all derivatives of $\varphi$ are uniformly bounded for all time $t > 0$ by Lemma 4.1 and the Calabi energy decays exponentially by Lemma 4.7. Thus, the Calabi flow converges exponentially fast to a constant scalar curvature metric. The theorem is proved.

We have the following analogous result for the modified Calabi flow.

Theorem 6.5. Let $(M, \omega)$ be an $n$-dimensional compact Kähler manifold. Assume that $M$ is pre-stable. For any $\tau, \Lambda, K > 0$, there is a constant $\epsilon = \epsilon(\tau, \Lambda, K, n, \omega) > 0$ such that if the solution $\omega_t$ of the Calabi flow with any $K$-invariant initial metric $\omega_0 \in [\omega]$ satisfies the following properties:

(a) the modified Calabi flow $\omega_t$ is of type $(\tau, \Lambda)$;
(b) $C_S(\omega_0) \leq K$, $\bar{C}a(\omega_0) \leq \epsilon$,

the Calabi flow $\omega_t$ exists for all the time and converges exponentially fast to an extremal Kahler metric.
6.3. Discussion of \((\tau, \Lambda)\) Calabi flow. In this subsection, we give some conditions on the initial metric such that the Calabi flow is of type \(I(\tau, \Lambda)\) or \(II(\tau, \Lambda)\). We introduce some definitions.

**Definition 6.6.** For any \(\Lambda_1, \Lambda_2, K > 0\), we define \(\mathcal{F}(\Lambda_1, \Lambda_2, K)\) the set of Kähler metrics \(\omega_0 \in \Omega\) satisfying the following conditions

\(\sum_{i=0}^{2} |\nabla^i Rm|(\omega_0) \leq \Lambda_1; \quad (1)\)

\(\sum_{i=1}^{N} \int_{M} |\nabla^i Rm|^2(\omega_0) \leq \Lambda_2\) where \(N = 28(n + 1); \quad (2)\)

\(C_S(\omega) \leq K. \quad (3)\)

**Lemma 6.7.** Given \(\Lambda_1, \Lambda_2, K > 0\). For any \(\omega_0 \in \mathcal{F}(\Lambda_1, \Lambda_2, K)\) there exists \(\tau = \tau(\Lambda_1, \Lambda_2, K, \omega)\) and \(\Lambda(\Lambda_1, \Lambda_2, K, \omega)\) such that the solution \(\omega_t\) of Calabi flow with the initial metric \(\omega_0\) is of type \((\tau, \Lambda)\). In other words, \(\omega_t\) satisfies

\[|Rm|(t) + |\nabla S|(t) \leq \Lambda, \quad t \in [0, \tau].\]

Moreover, we can choose \(\tau\) small such that

\[|Rm|(t) \leq 2|Rm|(0), \quad C_S(t) \leq 2C_S(0), \quad t \in [0, \tau].\]

**Proof.** Let

\[T = \sup\left\{ t > 0 \left| \omega_s \in \mathcal{F}(2\Lambda_1, 2\Lambda_2, 2K) \quad \forall s \in [0, t]\right\}\right.\]

We would like to give a lower bound of \(T\).

1. To estimate the metric \(\omega_t\), we observe that

\[|\frac{\partial}{\partial t} \omega_t| = |\nabla \nabla S| \leq 2\Lambda_1, \quad \forall t \in [0, T]\]

which implies that

\[e^{-2\Lambda_1 t} \omega_0 \leq \omega_t \leq e^{2\Lambda_1 t} \omega_0, \quad \forall t \in [0, T].\]

Thus, if \(t \leq C(\Lambda_1, K)\), we have \(C_S(t) \leq 2K\).

2. Since the curvature tensor is bounded for \(t \in [0, T]\), by (4.23) we have

\[\sum_{k=0}^{N} \int_{M} |\nabla^k Rm|^2(t) \omega_t^n \leq \sum_{k=0}^{N} \int_{M} |\nabla^k Rm|^2(0) \omega_0^n + (N + 1)C(n, k, 2\Lambda_1)V(2\Lambda_1)^2 t \leq \Lambda_2 + (N + 1)C(n, k, 2\Lambda_1)V(2\Lambda_1)^2 t, \quad \forall t \in [0, T].\]

Thus, \(\sum_{i=1}^{N} \int_{M} |\nabla^k Rm|^2(t) \omega_t^n \leq 2\Lambda_2\) if

\[t \leq \frac{\Lambda_2}{(N + 1)C(n, k, 2\Lambda_1)V(2\Lambda_1)^2}.\]

3. By Lemma 4.15 there is a constant \(C_1(c, i, \omega, \Lambda_1, \Lambda_2)\) such that

\[\max_{M} |\nabla^i S|(t) \leq C_1(c, i, \omega, \Lambda_1, \Lambda_2), \quad 1 \leq i \leq 6, \quad t \in [0, T].\]

By the evolution equation (3.1) of \(Rm\), we have the inequality

\[\frac{\partial}{\partial t} |Rm|(t) \leq |\nabla^3 S| + |Rm| |\nabla^2 S|.\]

It follows that

\[|Rm|(t) \leq |Rm|(0) + C_2(c, \omega, \Lambda_1, \Lambda_2)t, \quad \forall t \in [0, T].\]
We have similar estimates for higher order derivatives of $Rm$:

$$\frac{\partial}{\partial t} |\nabla^i Rm| \leq |\nabla^{i+1} S|, \quad i = 1, 2.$$ 

Therefore, we have

$$\sum_{i=0}^{2} |\nabla^i Rm|(t) \leq \sum_{i=0}^{2} |\nabla^i Rm|(0) + C_2(c, \omega, \Lambda_1, \Lambda_2)t \leq 2\Lambda_1, \quad t \in \left[0, \frac{\Lambda_1}{C_3}\right].$$

Combining the above estimates, we have

$$T \geq \min \left\{ C(\Lambda_1, K), \frac{\Lambda_2}{(N+1)C(n, k, 2\Lambda_1)V(2\Lambda_1)^2 \frac{\Lambda_1}{C_3}} \right\}. $$

The lemma is proved.

Therefore, we can replace the condition (a) in Theorem 1.6 and Theorem 1.7 by assuming the initial metric in $F(\Lambda_1, \Lambda_2, K)$. It is interesting to find a simpler condition to replace the type $(\tau, \Lambda)$ condition.

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