Algebraic Nature of Shape-Invariant and Self-Similar Potentials

A. B. Balantekin*, M. A. Cândido Ribeiro†,
Department of Physics, University of Wisconsin
Madison, Wisconsin 53706 USA

A. N. F. Aleixo‡
Department of Physics, University of Wisconsin
Madison, Wisconsin 53706 USA,
Instituto de Física, Universidade Federal do Rio de Janeiro, RJ - Brazil§
(March 19, 2018)

Abstract

Self-similar potentials generalize the concept of shape-invariance which was originally introduced to explore exactly-solvable potentials in quantum mechanics. In this article it is shown that previously introduced algebraic approach to the latter can be generalized to the former. The infinite Lie algebras introduced in this context are shown to be closely related to the q-algebras. The associated coherent states are investigated.

*Electronic address: baha@nucth.physics.wisc.edu
†Electronic address: aribeiro@nucth.physics.wisc.edu
‡Electronic address: aleixo@nucth.physics.wisc.edu
§Permanent address.
I. INTRODUCTION

Supersymmetric quantum mechanics has been shown to be a useful technique to explore exactly solvable problems in quantum mechanics \[1\]. Introducing the function

\[
W(x) \equiv -\frac{\hbar}{\sqrt{2m}} \left[ \frac{\Psi_0'(x)}{\Psi_0(x)} \right],
\]

where \(\Psi_0(x)\) is the ground-state wave-function of the Hamiltonian \(\hat{H}\), and the operators

\[
\hat{A} \equiv W(x) + \frac{i}{\sqrt{2m}} \hat{p},
\]

\[
\hat{A}^\dagger \equiv W(x) - \frac{i}{\sqrt{2m}} \hat{p},
\]

we can show that

\[
\hat{A} | \Psi_0 \rangle = 0
\]

and

\[
\hat{H} - E_0 = \hat{A}^\dagger \hat{A}.
\]

An integrability condition called shape-invariance was introduced by Gendenshtein \[2\] and was cast into an algebraic form by Balantekin \[3\]. The shape-invariance condition can be written as

\[
\hat{A}(a_1) \hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2) \hat{A}(a_2) + R(a_1),
\]

where \(a_{1,2}\) are a set of parameters. The parameter \(a_2\) is a function of \(a_1\) and the remainder \(R(a_1)\) is independent of \(\hat{x}\) and \(\hat{p}\). Not all exactly solvable potentials are shape-invariant \[4\]. In the cases studied so far the parameters \(a_1\) and \(a_2\) are either related by a translation \[4,5\] or a scaling \[6\]. Introducing the similarity transformation that replaces \(a_1\) with \(a_2\) in a given operator

\[
\hat{T}(a_1) \hat{O}(a_1) \hat{T}^\dagger(a_1) = \hat{O}(a_2)
\]

and the operators

\[
\hat{B}_+ = \hat{A}^\dagger(a_1) \hat{T}(a_1)
\]

\[
\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1) \hat{A}(a_1),
\]

the Hamiltonian takes the form

\[
\hat{H} - E_0 = \hat{B}_+ \hat{B}_-,
\]
The Lie algebra associated by the shape-invariance is defined with the commutation relations

\[ [\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1) R(a_1) \hat{T}(a_1) \equiv R(a_0), \tag{1.11} \]

and

\[ [\hat{B}_+, R(a_0)] = [R(a_1) - R(a_0)] \hat{B}_+, \tag{1.12} \]

\[ [\hat{B}_+, \{R(a_1) - R(a_0)\} \hat{B}_+] = \{[R(a_2) - R(a_1)] - [R(a_1) - R(a_0)]\} \hat{B}^2, \tag{1.13} \]

and the Hermitian conjugates of the relations given in Eq. (1.12) and Eq. (1.13). In general there is an infinite number of such commutation relations, hence the appropriate Lie algebra is infinite-dimensional. In some special cases where the parameters are related by translation it is possible to reduce this infinite-dimensional algebra to a finite dimensional one \[\text{[3,7,8]}\]. In this paper we explore the relationship between \(q\)-algebras and the cases where the parameters are related by scaling.

**II. COHERENT STATES**

Since the operator \(\hat{B}_-\) satisfies the relation

\[ \hat{B}_- | \psi_0 \rangle = 0, \tag{2.1} \]

and the excited states can be written in the form

\[ | \psi_n \rangle \propto \hat{B}_n^+ | \psi_0 \rangle, \tag{2.2} \]

the operator \(\hat{B}_-\) does not have a left inverse and the operator \(\hat{B}_+\) does not have a right inverse. However a right inverse for \(\hat{B}_-\)

\[ \hat{B}_- \hat{B}_-^{-1} = 1 \tag{2.3} \]

and a left inverse for \(\hat{B}_+\)

\[ \hat{B}_+^{-1} \hat{B}_+ = 1 \tag{2.4} \]

can be defined. Similarly in the Hilbert space of the eigenstates of the Hamiltonian, the inverse of \(\hat{H}\) does not exist, but

\[ \hat{H}^{-1} \hat{B}_+ = \hat{B}_-^{-1} \tag{2.5} \]

does. Also introducing

\[ \hat{Q}^\dagger = \hat{H}^{-1/2} \hat{B}_+ \tag{2.6} \]

and its Hermitian conjugate

\[ \hat{Q} = (\hat{Q}^\dagger)^\dagger = \hat{B}_- \hat{H}^{-1/2} \tag{2.7} \]
one can show that
\[ \hat{Q} \hat{Q}^\dagger = \hat{1}. \tag{2.8} \]
The normalized excited states can then be written as
\[ | \Psi_n \rangle = (\hat{Q}^+)^n | \Psi_0 \rangle, \tag{2.9} \]
provided that the ground state is normalized, i.e. \( \langle \Psi_0 | \Psi_0 \rangle = 1 \).

We introduce the coherent state for a shape-invariant potential as
\[ | z \rangle = | 0 \rangle + z \hat{B}_-^\dagger | 0 \rangle + z^2 \hat{B}_-^2 | 0 \rangle + \ldots \]
\[ = \frac{1}{1 - z \hat{B}_-^\dagger} | 0 \rangle, \tag{2.10} \]
where we used the short-hand notation \( | 0 \rangle \equiv | \Psi_0 \rangle \). One can easily show that this state in an eigenstate of the operator \( \hat{B}_- \):
\[ \hat{B}_- | z \rangle = z | z \rangle \tag{2.11} \]
and satisfies the condition
\[ (\hat{B}_- - z) \frac{\partial}{\partial z} | z \rangle = | z \rangle. \tag{2.12} \]
The state \( | z \rangle \) coincides with the coherent state defined in Ref. [9] using a generalized exponential function. When the Lie algebra associated with the shape-invariant potential is SU(1,1) [3,7], this is not the standard coherent state introduced in [10], but the state introduced by Barut and Girardello [11].

If a forced harmonic oscillator is in the ground state for \( t = 0 \), it evolves into the harmonic oscillator coherent state. We must emphasize that the coherent states described here, in general, do not have such a simple dynamical interpretation. To illustrate this point we consider the time-dependent Hamiltonian
\[ \hat{h}(t) = \hat{B}_+ \hat{B}_- + f(t) \left[ e^{iR(a_1) t/\hbar} \hat{B}_+ + \hat{B}_- e^{-iR(a_1) t/\hbar} \right], \tag{2.13} \]
where \( f(t) \) is an arbitrary function of time. The solution of the time-evolution equation
\[ i\hbar \frac{\partial \hat{u}(t)}{\partial t} = \hat{h}(t) \hat{u}(t) \tag{2.14} \]
can be written as
\[ \hat{u}(t) = \exp \left\{ -i \frac{\hbar}{\hbar} \hat{B}_+ \hat{B}_- t \right\} \hat{u}_I(t). \tag{2.15} \]
Substituting Eq. (2.14) into Eq. (2.13) one can show that \( \hat{u}_I(t) \) satisfies the equation
\[ i\hbar \frac{\partial \hat{u}_I(t)}{\partial t} = f(t) \left[ \hat{B}_+ + \hat{B}_- \right] \hat{u}_I(t). \tag{2.16} \]
The solution of Eq. (2.14) can be immediately written to be
\[ \hat{u}_I(t) = \exp \left\{ -i \frac{\hbar}{\hbar} \int_0^t f(t') dt' \left[ \hat{B}_+ + \hat{B}_- \right] \right\}. \tag{2.17} \]
Hence under the time-evolution the ground state evolves into the state
\[ | \Psi, t \rangle = \hat{u}_I(t) | 0 \rangle, \tag{2.18} \]
which is not equivalent to the state given in Eq. (2.10).
III. SELF-SIMILAR POTENTIALS AND $Q$-ALGEBRAS

Shabat [12] and Spiridonov [13] discussed reflectionless potentials with an infinite number of bound states. These self-similar potentials are shown to be shape-invariant in Ref. [6]. In this case the parameters are related by a scaling:

$$a_n = q^{n-1} a_1 .$$

Barclay et al. studied such shape-invariant potentials in detail [6]. In the simplest case studied by them the remainder of Eq. (1.6) is given by

$$R(a_1) = c a_1 ,$$

where $c$ is a constant and the operator introduced in Eq. (1.7) by

$$\hat{T}(a_1) = \exp \left\{ (\log q) a_1 \frac{\partial}{\partial a_1} \right\} .$$

Hence the energy eigenvalue of the $n$-th excited state is

$$E_n = R(a_1) + R(a_2) + \ldots + R(a_n)$$
$$= (1 + q + q^2 + \ldots + q^{n-1}) ca_1$$
$$= \frac{1 - q^n}{1 - q} ca_1$$

which is the spectra of quantum oscillator [14]. Introducing the scaled operators

$$\hat{K}_\pm = \sqrt{q} \hat{B}_\pm$$

one can show that the commutation relations of Eqs. (1.11), (1.12) and (1.13) take the form

$$[\hat{K}_-, \hat{K}_+] = R(a_1)$$

and

$$[\hat{K}_+, R(a_1)] = (q - 1) R(a_1) \hat{K}_+ .$$

Note that the algebra associated with the self-similar potentials is not a finite Lie algebra as $\hat{K}_+$ does not commute with $R(a_1) \hat{K}_+$:

$$[\hat{K}_+, (q - 1)^n R(a_1) \hat{K}_+] = (q - 1)^{n+1} R(a_1) \hat{K}_+^{n+1} .$$

Further introducing the operators

$$\hat{S}_+ = \hat{K}_+ R(a_1)^{-1/2}$$

and

$$\hat{S}_- = (\hat{S}_+)^\dagger = R(a_1)^{-1/2} \hat{K}_- ,$$
using Eq. (3.6) one can show that the standard $q$-deformed oscillator relation is satisfied
\begin{equation}
\hat{S}_- \hat{S}_+ - q \hat{S}_+ \hat{S}_- = 1. \tag{3.11}
\end{equation}

In the most general case for a self-similar potential the function $W(x)$ of Eq. (1.11) satisfies the condition [12,13]
\begin{equation}
W(x) \xrightarrow[a_1 \to a_2]{} \sqrt{q}W(\sqrt{q}x), \tag{3.12}
\end{equation}
or equivalently
\begin{equation}
\hat{A}^\dagger(x), \hat{A}(x) \xrightarrow[a_1 \to a_2]{} \sqrt{q}\hat{A}^\dagger(\sqrt{q}x), \sqrt{q}\hat{A}(\sqrt{q}x). \tag{3.13}
\end{equation}

Inserting Eq. (3.13) into Eq. (1.6) one obtains the q-deformed form of Eq. (1.6)
\begin{equation}
\hat{A}(x)\hat{A}^\dagger(x) - q\hat{A}^\dagger(\sqrt{q}x)\hat{A}(\sqrt{q}x) = R(a_1). \tag{3.14}
\end{equation}

Introducing the operators [13]
\begin{equation}
\hat{C} = \hat{A}(x)e^{-\frac{1}{2}px \frac{d}{dx}}, \tag{3.15}
\end{equation}
and
\begin{equation}
\hat{C} = e^{\frac{1}{2}px \frac{d}{dx}}\hat{A}^\dagger(x), \tag{3.16}
\end{equation}
where $q = e^p$, Eq. (3.14) can be rewritten as
\begin{equation}
\hat{C}\hat{C}^\dagger - q\hat{C}^\dagger\hat{C} = R(a_1). \tag{3.17}
\end{equation}

Note that an algebraic approach to the self-similar potentials was already introduced in Refs. [7,8]. Here we would like to establish that our algebra is identical to that in Ref. [8]. To this end we introduce
\begin{equation}
\hat{J}_3 = -\frac{1}{p} \log a_0, \tag{3.18}
\end{equation}
Using Eq. (3.18), Eq. (1.11) can be written as
\begin{equation}
[\hat{B}_-, \hat{B}_+] = c \exp (-p\hat{J}_3). \tag{3.19}
\end{equation}

Using Eq. (1.7), one can show that for an arbitrary function $f(a_n)$ of the parameters $a_n$ we can write
\begin{equation}
f(a_n)\hat{B}_+ = \hat{B}_+ f(a_{n-1}) \tag{3.20}
\end{equation}
and
\begin{equation}
f(a_n)\hat{B}_- = \hat{B}_- f(a_{n+1}). \tag{3.21}
\end{equation}

Using Eqs. (3.20) and (3.21) one can easily prove the commutation relation
\[ [\hat{J}_3, \hat{B}_\pm] = \pm \hat{B}_\pm. \] (3.22)

Eqs. (3.19) and (3.22) represent the algebra introduced in Ref. [8]. This algebra is a deformation of the standard \( SO(2,1) \) algebra.

The coherent state is easy to construct. The term multiplying \( z^n \) in Eq. (2.10) is

\[ z^n \hat{B}^{-n} |0\rangle = z^n (\hat{H}^{-1} \hat{B}_+)^n |0\rangle = [E_n(E_n - E_{n-1})(E_n - E_{n-2}) \ldots (E_n - E_1)]^{-1/2} |n\rangle, \] (3.23)

where \( |n\rangle \) is the short-hand notation for the \( n \)-th excited state \( |\Psi_n\rangle \) the energy of which is \( E_n \). Inserting Eq. (3.4) into Eq. (3.23) one can write down the coherent state as

\[ |z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[R(a_1)]^n}} \frac{(1 - q)^{n/2} q^{-n(n-1)/4}}{\sqrt{(q;q)_n}} |n\rangle, \] (3.24)

where the \( q \)-shifted factorial \( (q;q)_n \) is defined as \( (z;q)_0 = 1 \) and \( (z;q)_n = \prod_{j=0}^{n-1}(1 - zq^j) \), \( n = 1, 2, \ldots \). One observes that the norm of this state belongs to the one-parameter family of \( q \)-exponential functions considered by Floreanini et al. [16]. An alternative approach to the coherent states for the \( q \)-algebras was given in Ref. [14] and was used to construct path integrals in Ref. [17].

ACKNOWLEDGMENTS

This work was supported in part by the U.S. National Science Foundation Grant No. PHY-9605140 at the University of Wisconsin, and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation. M.A.C.R. acknowledges the support of Fundação de Amparo à Pesquisa do Estado de São Paulo (Contract No. 96/3240-5). A.N.F.A. acknowledges the support of Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (Contract No. BEX0610/96-8).
REFERENCES

[1] For a recent review see Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 251 267.
[2] Gendenshtein L 1983 Pis’ma Zh. Eksp. Teor. Fiz. 38 299.
[3] Balantekin A B 1998 Phys. Rev. A 57 4188.
[4] Cooper F, Ginocchio J N and Khare A 1987 Phys. Rev. D 36 2458.
[5] Chuan C 1991 J. Phys. A 24 L 1165.
[6] Khare A and Sukhatme U 1993 J. Phys. A 26 L 901; Barclay D et al. 1993 Phys. Rev. A 48 2786.
[7] Gangopadhyaya A, Mallow J V and Sukhatme U, hep-th/9804191.
[8] Chaturvedi S, Dutt R, Gangopadhyaya A, Panigrahi P, Rasinariu C and Sukhatme U P 1998 Phys. Lett. A 248 109; Gangopadhyaya A, Mallow J V, Rasinariu C and Sukhatme U P, hep-th/9810074.
[9] Fukui T and Aizawa N 1993 Phys. Lett. A 180 308.
[10] Perelomov A M 1972 Commun. Math. Phys. 26 222.
[11] Barut A O and Girardello L 1971 Commun. Math. Phys. 21 41.
[12] Shabat A 1992 Inverse Prob. 8 303.
[13] Spiridonov V P 1992 Phys. Rev. Lett. 69 398.
[14] Biedenharn L C 1989 J. Phys. A 22 L873; Macfarlane A J 1989 J. Phys. A 22 4581.
[15] Ellinas D 1994 Czech. J. Phys. 44 1009.
[16] Floreanini R, Le Tourneux J and Vinet L 1995 J. Phys. A 28 l287; see also Atakishiyev N M 1996 J. Phys. A 29 l223.
[17] Ellinas D 1993 J. Phys. A 26 L543.