THE SIMPLICIAL COALGEBRA OF CHAINS UNDER THREE DIFFERENT NOTIONS OF WEAK EQUIVALENCE

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Abstract. We study the simplicial coalgebra of chains on a simplicial set with respect to three notions of weak equivalence. To this end, we construct three model structures on the category of reduced simplicial sets for any commutative ring $R$. The weak equivalences are given by: (1) an $R$-linearized version of categorical equivalences, (2) maps inducing an isomorphism on fundamental groups and an $R$-homology equivalence between universal covers, and (3) $R$-homology equivalences. Analogously, for any field $F$, we construct three model structures on the category of connected simplicial cocommutative $F$-coalgebras. The weak equivalences in this context are (1') maps inducing a quasi-isomorphism of dg algebras after applying the cobar functor, (2') maps inducing a quasi-isomorphism of dg algebras after applying a localized version of the cobar functor, and (3') quasi-isomorphisms. Building on previous work of Goerss in the context of (3)–(3'), we prove that, when $F$ is algebraically closed, the simplicial $F$-coalgebra of chains defines a homotopically full and faithful left Quillen functor for each pair of model categories. More generally, when $F$ is a perfect field, we compare the three pairs of model categories in terms of suitable notions of homotopy fixed points with respect to the absolute Galois group of $F$.

1. Introduction

1.1. Overview. A central problem in algebraic topology is to understand how much information about a category of topological spaces, considered up to a specified notion of weak equivalence, is preserved by a particular functorial invariant. Quillen proved that a suitable version of the chains functor defines an equivalence of homotopy theories between simply-connected spaces, considered up to rational homotopy equivalence, and simply-connected cocommutative differential graded (dg)
Q-coalgebras, considered up to quasi-isomorphism \cite{Sullivan}. In the context of spaces of finite type, Sullivan developed an effective version of this theory, suitable for geometric applications, in terms of simply-connected commutative dg Q-algebras and their minimal models \cite{Goerss}. For a field \( F \) of arbitrary characteristic, Goerss \cite{Goerss} studied spaces (simplicial sets) up to \( F \)-homology equivalence by means of simplicial cocommutative \( F \)-coalgebras, considered up to quasi-isomorphism. Goerss considered the functor
\[
F[-] : \mathbf{sSet} \to \mathbf{sCoCoalg}_F, \quad X \mapsto F[X],
\]
which associates to any space \( X \) its simplicial cocommutative \( F \)-coalgebra of chains \( F[X] \) with the coproduct induced by the diagonal map \( X \to X \times X \) and showed that \( F[-] \) is homotopically full and faithful when \( F \) is algebraically closed. This implies that any space may be recovered, up to \( F \)-localization in the sense of Bousfield \cite{Bousfield}, from its simplicial cocommutative \( F \)-coalgebra of chains through a derived version of the right adjoint of \( F[-] \), also known as the *functor of \( F \)-points*, which is given by
\[
P : \mathbf{sCoCoalg}_F \to \mathbf{sSet}, \quad C \mapsto \mathbf{sCoCoalg}_F(F, C),
\]
where \( F \) denotes the constant simplicial object at \( F \). This result is of particular importance over \( F_p \), the algebraic closure of the field \( F_p \) of \( p \)-elements. Both Quillen and Goerss used the framework of model categories to describe the necessary homotopical constructions. Furthermore, Mandell \cite{Mandell} proved a similar statement for finite type nilpotent spaces using the \( E_\infty \)-dg-algebra of singular cochains and also obtained an intrinsic description of the essential image of this functor. In another direction and using different techniques, Yuan \cite{Yuan} recently described a full and faithful integral model for finite type nilpotent spaces up to weak homotopy equivalence using the \( E_\infty \)-ring spectrum of spherical cochains.

These models fail to capture the information of the fundamental group in complete generality. In fact, for any commutative ring \( R \), the \( R \)-localization of a space drastically changes the fundamental group. In order to retain the information about the fundamental group, we will consider instead spaces up to \( \pi_1 \)-\( R \)-equivalence. A \( \pi_1 \)-\( R \)-equivalence is a map of (based, connected) spaces inducing an isomorphism on fundamental groups and an \( R \)-homology equivalence between the universal covers \cite{Dwyer}. For instance, a \( \pi_1 \)-\( \mathbb{Z} \)-equivalence is the same as a weak homotopy equivalence. In the present article, we introduce a corresponding notion of weak equivalence for connected simplicial cocommutative \( R \)-coalgebras, which we call \( \hat{\Omega} \)-quasi-isomorphism. To define this notion, we use a suitable localization procedure for simplicial coalgebras at a set of 1-simplices together with the classical cobar construction for dg coalgebras. It turns out there are good homotopy theories for these notions: we construct model category structures for reduced simplicial sets up to \( \pi_1 \)-\( R \)-equivalence and, when \( R = F \) is a field, for connected simplicial cocommutative \( F \)-coalgebras up to \( \hat{\Omega} \)-quasi-isomorphism. Moreover, the corresponding adjunction of categories
\[
F[-] : \mathbf{sSet}_0 \rightleftarrows \mathbf{sCoCoalg}_F^0 : \mathcal{P}
\]
defines a Quillen adjunction between these model categories. Furthermore, we prove that the left Quillen functor \( F[-] \) is homotopically full and faithful when \( F \) is an algebraically closed field. We also study the Quillen adjunction (1.1) in the more general case where \( F \) is a perfect field with absolute Galois group \( G \). In
In this case, we prove that the derived unit transformation of the Quillen adjunction (1.1) may be identified with the canonical map into the homotopy $G$-fixed points, interpreted appropriately in the homotopy theory of simplicial discrete $G$-sets up to $\pi_1$-$F$-equivalence. This is a significant improvement of the main result of [42], where a detection result was shown, namely, that for any field $F$, two (fibrant) spaces $X$ and $Y$ are $\pi_1$-$F$-equivalent if and only if the simplicial cocommutative coalgebras $F[X]$ and $F[Y]$ are $\Omega$-quasi-isomorphic. The present article may be read independently from [13] and [42].

The homotopy theory of connected simplicial cocommutative $F$-coalgebras up to $\hat{\Omega}$-quasi-isomorphism fits strictly between two other homotopy theories: (1) connected simplicial cocommutative coalgebras up to $\Omega$-quasi-isomorphism, and (2) connected simplicial cocommutative coalgebras up to quasi-isomorphism. In the present article, we also construct a model category for (1) and explain its relation to a linearized version of the Joyal model category for reduced simplicial sets [22, 27, 9]. On the other hand, the homotopy theory for (2) is the connected version of the model category studied by Goerss [13] and is related to the Bousfield model category of reduced simplicial sets up to homology equivalence. The main results of the present article are phrased in such a way that the parallelism between three corresponding Quillen adjunctions is emphasized.

The main motivation for this work is to understand the strength of a particular invariant for homotopy types (in this case the simplicial cocommutative coalgebra of chains) with respect to different notions of weak equivalence. We are particularly interested in notions of weak equivalence arising from the homotopy theory of algebraic structures governed by a dg operad, since these usually yield flexible and explicit theories. One of our eventual goals is to develop a similar analysis for the invariant of the simplicial cocommutative coalgebra of integral chains considered up to $\hat{\Omega}$-quasi-isomorphism (cf. [20] for the case of the $E_\infty$ dg algebra of integral cochains up to quasi-isomorphism). It was conjectured in [42] that this invariant faithfully detects whether two homotopy types are equivalent. A refined form of this conjecture will be addressed in subsequent work.

1.2. Summary of results. We now summarize our main results. For any commutative ring $R$, we consider the following three notions of weak equivalence on the category $sSet_0$ of reduced simplicial sets (simplicial sets with a single vertex):

(1) Let $\Lambda : sSet_0 \to \mathsf{dgAlg}_R$ be the restriction to $sSet_0$ of the left adjoint of the dg nerve functor $N_{dg} : \mathsf{dgCat}_R \to sSet$ from dg categories to simplicial sets (see Section 2.6). A map $f : X \to Y$ in $sSet_0$ is called an $R$-categorical equivalence if it becomes a quasi-isomorphism of dg $R$-algebras after applying the functor $\Lambda : sSet_0 \to \mathsf{dgAlg}_R$.

This is a linearized version of the notion of categorical equivalence (or Joyal equivalence) between reduced simplicial sets. We denote by $W_{J,R}$ the class of $R$-categorical equivalences.

(2) A map $f : X \to Y$ in $sSet_0$ is called a $\pi_1$-$R$-equivalence if it induces an isomorphism between the fundamental groups $\pi_1(f) : \pi_1(X) \xrightarrow{\cong} \pi_1(Y)$ and an $R$-homology isomorphism between the universal covers $H_\ast(f; R) : H_\ast(X; R) \xrightarrow{\cong} H_\ast(Y; R)$. We denote by $W_{\pi_1,R}$ the class of $\pi_1$-$R$-categorical equivalences.
(3) A map \( f : X \to Y \) in \( \text{sSet}_0 \) is called an \( R \)-equivalence if it induces an isomorphism in \( R \)-homology \( H_*(f; R) : H_*(X; R) \cong H_*(Y; R) \). We denote by \( W_R \) the class of \( R \)-equivalences.

The category of reduced simplicial sets is a convenient framework to make certain pointed constructions functorial (such as the fundamental group and the universal cover). Moreover, the reduced setting will be convenient in order to establish key connections between (1)–(2) and the cobar construction. Since a \( Z \)-equivalence between simply-connected spaces is a weak homotopy equivalence, a \( \pi_1 \)-\( Z \)-equivalence is the same as a weak homotopy equivalence between reduced simplicial sets. Also note that the class of \( \pi_1 \)-equivalences is strictly contained in the class of \( R \)-equivalences that induce a \( \pi_1 \)-isomorphism (see [16, Example 4.35]).

We construct three model category structures on \( \text{sSet}_0 \), one for each of the above notions of weak equivalence.

**Theorem 1** (see Theorem 7.1.1). Let \( R \) be a commutative ring. The category \( \text{sSet}_0 \) admits three left proper combinatorial model category structures, denoted by \( (\text{sSet}_0, R\text{-cat}-\text{eq.}), (\text{sSet}_0, \pi_1\text{-R-\text{eq.}}), \) and \( (\text{sSet}_0, R\text{-eq.}) \), which have the monomorphisms as cofibrations and \( W_{I,R}, W_{\pi_1,R}, \) and \( W_R \) as weak equivalences, respectively. Furthermore, we have strict inclusions \( W_{I,R} \subseteq W_{\pi_1,R} \subseteq W_R \).

The model category \( (\text{sSet}_0, \pi_1\text{-R-\text{eq.}}) \) is the classical Kan–Quillen model structure on reduced simplicial sets. Moreover, as part of our discussion of the model category \( (\text{sSet}_0, \pi_1\text{-R-\text{eq.}}) \), we prove a result similar to the classical fracture theorem [7, 2.21], but now fully taking into account the fundamental group (Theorem 7.1.1). More specifically, this result describes the weak homotopy type of a reduced simplicial set as a homotopy pullback in terms of fibrant replacements in \( (\text{sSet}_0, \pi_1\text{-Q-\text{eq.}}) \) and \( (\text{sSet}_0, \pi_1\text{-F}_p\text{-\text{eq.}}) \) for all prime numbers \( p \).

One of the goals of the present article is to model each of the three homotopy theories of Theorem 1 using connected simplicial cocommutative coalgebras via the functor of simplicial chains. For each of the three notions of weak equivalence in \( \text{sSet}_0 \), we define an analogous notion in the category \( \text{sCoCoalg}_{\text{R}}^0 \) of connected simplicial cocommutative coalgebras:

1. A map \( f : C \to C' \) in \( \text{sCoCoalg}_{\text{R}}^0 \) is called an \( \Omega \)-quasi-isomorphism if it becomes a quasi-isomorphism of dg algebras after applying the normalized chains functor followed by the cobar functor \( \text{Cobar} : \text{dgCoalg}_{\text{R}}^0 \to \text{dgAlg}_{\text{R}} \) (see Section 2.6). We denote by \( W_{\Omega} \) the class of \( \Omega \)-quasi-isomorphisms.

2. A map \( f : C \to C' \) in \( \text{sCoCoalg}_{\text{R}}^0 \) is called an \( \text{\hat{\Omega}} \)-quasi-isomorphism if it becomes an \( \text{\hat{\Omega}} \)-quasi-isomorphism after first localizing (in an appropriate way) \( C \) and \( C' \) at the set of 1-simplices \( P(C)_1 \subseteq C_1 \) and \( P(C')_1 \subseteq C'_1 \), respectively (see Section 3.4). We denote by \( W_{\text{\hat{\Omega}}} \) the class of \( \text{\hat{\Omega}} \)-quasi-isomorphisms.

3. A map \( f : C \to C' \) in \( \text{sCoCoalg}_{\text{R}}^0 \) is called a quasi-isomorphism if the induced map on normalized chains induces an isomorphism on homology. We denote by \( W_{\text{qi},1} \) the class of quasi-isomorphisms.

When \( R = \mathbb{F} \) is a field, we construct three corresponding model category structures on \( \text{sCoCoalg}_{\text{F}}^0 \).

**Theorem 2** (see Theorem 7.3.1). Let \( \mathbb{F} \) be a field. The category \( \text{sCoCoalg}_{\text{F}}^0 \) admits three left proper combinatorial model category structures, denoted by \( (\text{sCoCoalg}_{\text{F}}^0, \Omega\text{-q.i.}) \),
(sCoCoalg^{0}_{F}, \hat{\Omega}-q.i.), and (sCoCoalg^{0}_{F}, q.i.), which have the injective maps as cofibrations and \(W_{\Omega}, W_{\hat{\Omega}}, \) and \(W_{q.i.}\) as weak equivalences, respectively. Furthermore, we have strict inclusions \(W_{\Omega} \subseteq W_{\hat{\Omega}} \subseteq W_{q.i.}\).

The proofs of Theorems 1 and 2 rely on a useful method for constructing combinatorial model category structures which is based on J. Smith’s recognition theorem and may be of independent interest. This general method is discussed in Section 6 and can be read independently of the rest of the article.

We then compare the model categories of Theorems 1 and 2. For this comparison, a key result is that \(\pi_{1}-R\)-equivalences can be completely described in terms of \(\hat{\Omega}\)-quasi-isomorphisms; see Theorem 5.3.4. We show that we have three Quillen adjunctions (Proposition 7.3.3):

\[
(1.2) \quad F[\cdot] : (sSet^{0}_{F}, F\text{-}cat, eq.) \rightleftarrows (sCoCoalg^{0}_{F}, \Omega\text{-}q.i.) : P
\]

(1.3) \(F[\cdot] : (sSet^{0}_{F}, \pi_{1}\text{-}F\text{-}eq.) \rightleftarrows (sCoCoalg^{0}_{F}, \hat{\Omega}\text{-}q.i.) : P\)

(1.4) \(F[\cdot] : (sSet^{0}_{F}, F\text{-}eq.) \rightleftarrows (sCoCoalg^{0}_{F}, q.i.) : P\).

When \(F\) is an algebraically closed field, or more generally a perfect field, we prove the following statements about these Quillen adjunctions.

**Theorem 3** (see Theorem 8.2.1 and Corollary 8.3.6). If \(F\) is algebraically closed, then each of the three Quillen adjunctions (1.2)–(1.4) is homotopically full and faithful.

More generally, if \(F\) is a perfect field with absolute Galois group \(G\), then the derived unit transformation of each of the three Quillen adjunctions can be identified with the canonical map

\[X \to (\delta(X))^{hG}\]

from \(X\) to the homotopy \(G\)-fixed points of \(\delta(X)\), where \(\delta(X)\) denotes \(X\) equipped with the trivial \(G\)-action and the homotopy \(G\)-fixed points functor \((-)^{hG}\) is interpreted appropriately in each case.

The proof of Theorem 3 makes use of the structure theory of coalgebras over a field and also relies on the construction of three corresponding model structures on the category of simplicial discrete \(G\)-sets for an arbitrary profinite group \(G\). The statement about the Quillen adjunction (1.4) (for sSet) was shown by Goerss [13]; this result was extended to the context of simplicial presheaves of coalgebras (with respect to the local model structures) by the first–named author [38] and it has also been shown for motivic homotopy theory in [15].

1.3. **Organization of the paper.** In Section 2 we review some categorical and algebraic preliminaries and fix the notation and terminology. Section 3 is a recollection several known model category structures used throughout the article.

In Section 4 we develop an explicit theory for the derived (or homotopical) localization of simplicial coalgebras along a set of 1-simplices, which is later used to define the functor \(\hat{\Omega}\) in Section 5. In Section 5 we define and study the three notions of weak equivalence for connected simplicial cocommutative coalgebras (\(\hat{\Omega}\)-quasi-isomorphisms, \(\hat{\Omega}\)-quasi-isomorphisms, and quasi-isomorphisms) and the three
corresponding notions for reduced simplicial sets ($R$-categorical equivalences, $\pi_1$-$R$-equivalences, and $R$-equivalences).

In Section 6 we describe a general method for constructing combinatorial model category structures, based on ideas of J. Smith, which will be used later in Sections 7 and 8. In Section 7 we establish the existence of three model structures on the category of reduced simplicial sets (Theorem 1), three model structures on the category of connected simplicial cocommutative coalgebras (Theorem 2) and obtain the three corresponding Quillen adjuctions $\text{(1.2)-(1.4)}$. In addition, in Section 7 we also prove a fiberwise version of the fracture theorem for arbitrary pointed connected weak homotopy types which fits nicely in the context of the model categories $(\text{sSet}_0, \pi_1$-$R$-eq.) for $R = \mathbb{Q}, \mathbb{F}_p$ ($p$ prime).

Finally, in Section 8 we review some key facts about the structure theory of coalgebras and prove our main comparison results (Theorem 3) about the three Quillen adjunctions (1.2)-(1.4) in the case where $F$ is an algebraically closed field (Theorem 8.2.1) or a perfect field (Theorem 8.3.4 and Corollary 8.3.6).

In Appendix A we give a detailed proof of the fact that for the natural cylinder construction $C \mapsto \text{Cyl}(C)$ in connected simplicial coalgebras, the canonical projection map $\text{Cyl}(C) \rightarrow C$ is an $\Omega$-quasi-isomorphism for any $C \in \text{sCoCoalg}_{\mathbb{F}}^0$. This is a key step in the proof of the existence of the model category structure $(\text{sCoCoalg}_{\mathbb{F}}^0, \Omega$-q.i.) shown in Section 7.

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2. Preliminaries

In this section, we recall some categorical and algebraic preliminaries.

2.1. Simplicial objects. Let $\Delta$ be the category whose objects are non-empty finite ordinals $\{[n] = \{0, \ldots, n\} \mid n \in \mathbb{N}\}$ and morphisms are order-preserving maps. The morphisms in $\Delta$ are generated by coface maps $\partial^i : [n-1] \rightarrow [n]$ and codegeneracy maps $s^i : [n+1] \rightarrow [n]$ for $i = 0, \ldots, n$ and these maps satisfy the usual cosimplicial identities.

A simplicial object in a category $C$ is a functor $F : \Delta^{\text{op}} \rightarrow C$, where $\Delta^{\text{op}}$ denotes the opposite category of $\Delta$. These form a category, denoted by $\text{sC}$, with morphisms being natural transformations. For any $F \in \text{sC}$ we write $F_n = F([n])$. Thus any $F \in \text{sC}$ is determined by the data of objects $F_0, F_1, F_2, \ldots \in C$ together with face maps $F(\partial^i) = \partial_i : F_n \rightarrow F_{n-1}$ and degeneracy maps $F(s^i) = s_i : F_n \rightarrow F_{n+1}$ in $C$ satisfying the usual simplicial identities.

Simplicial objects in $\text{Set}$, the category of sets, are called simplicial sets. We denote by $\text{sSet}_0$ the full subcategory of $\text{sSet}$ whose objects are all simplicial sets $S$ such that $S_0$ is a singleton. The objects of $\text{sSet}_0$ are called reduced simplicial sets.
2.2. Simplicial coalgebras. Fix a commutative ring $R$ and write $\otimes := \otimes_R$. Let $\text{Coalg}_R$ be the category of counital coassociative $R$-coalgebras. More precisely, the objects in $\text{Coalg}_R$ are triples $(C, \Delta, \epsilon)$ where $C$ is an $R$-module, $\Delta: C \to C \otimes C$ is an $R$-linear map, called the coproduct, satisfying
\[(\Delta \otimes \text{id}_C) \circ \Delta = (\text{id}_C \otimes \Delta) \circ \Delta,\]
and $\epsilon: C \to R$ is an $R$-linear map, called the counit, satisfying
\[(\text{id}_C \otimes \epsilon) \circ \Delta = \text{id}_C = (\epsilon \otimes \text{id}_C) \circ \Delta.\]

The morphisms in $\text{Coalg}_R$ are $R$-linear maps of $R$-modules preserving the coproduct and the counit. We denote by $\text{CoCoalg}_R$ the category of cocommutative $R$-coalgebras, namely, the full subcategory of $\text{Coalg}_R$ consisting of those coalgebras $(C, \Delta)$ for which $\Delta$ is cocommutative, i.e., it satisfies $\Delta = \tau \circ \Delta$ where $\tau: C \otimes C \to C \otimes C$ is the switch map $\tau(x \otimes y) = y \otimes x$.

In this article we will consider simplicial (cocommutative) coalgebras, i.e., simplicial objects in $\text{Coalg}_R$ ($\text{CoCoalg}_R$) as models for different homotopy theories. We note that the data of a simplicial coalgebra $C: \Delta^\text{op} \to \text{Coalg}_R$ is equivalent to a simplicial $R$-module $C$ equipped with a map of simplicial $R$-modules $\Delta: C \to C \otimes C$, where $(C \otimes C)_n = C_n \otimes C_n$. We denote by $\text{sCoalg}^0_R$ the full subcategory of $\text{sCoalg}_R$ consisting of those simplicial objects $C: \Delta^\text{op} \to \text{Coalg}_R$ for which $C_0 = (R, \Delta_0)$ where $\Delta_0: R \xrightarrow{\sim} R \otimes R$ is defined by $\Delta_0(1) = 1 \otimes 1$. The objects of $\text{sCoalg}^0_R$ are called connected simplicial coalgebras. The category $\text{sCoCoalg}_R^0$ is defined similarly.

2.3. Simplicial chains. Any simplicial set $X \in \text{sSet}$ gives rise to a simplicial cocommutative coalgebra $R[X]: \Delta^\text{op} \to \text{CoCoalg}_R$. The $R$-module $R[X]_n$ is defined to be $R[X_n]$, the free $R$-module generated by the set $X_n$. The face and degeneracy maps of $R[X]$ are induced by those of $X$, by functoriality. The coproduct maps
\[\Delta_n: R[X_n] \to R[X_n] \otimes R[X_n]\]
are induced by the diagonal maps
\[\Delta: X_n \to X_n \times X_n, \ x \mapsto (x, x),\]
by functoriality. This construction gives rise to a functor
\[R[-]: \text{sSet} \to \text{sCoCoalg}_R\]
called the simplicial $(R)$-chains functor. The simplicial $R$-chains functor is clearly given by the corresponding functor $R[-]: \text{Set} \to \text{CoCoalg}_R$ by passing to simplicial objects. We also obtain a restricted functor
\[R[-]: \text{sSet}_0 \to \text{sCoCoalg}_R^0.\]

The functor of simplicial $R$-chains has a right adjoint
\[P: \text{sCoCoalg}_R \to \text{sSet},\]
called the functor of $(R)$-points (or the set-like elements functor), which is defined pointwise by
\[P(C)_n = \text{Hom}_{\text{CoCoalg}}(R, C_n),\]
where $R$ is considered as a cocommutative $R$-coalgebra (as indicated above). More explicitly,
\[P(C)_n \cong \{ x \in C_n \mid \Delta_n(x) = x \otimes x \text{ and } \epsilon(x) = 1 \}.\]
We also obtain a restricted functor
\[ \mathcal{P} : \text{sCoCoalg}_R^0 \to \text{sSet}_0. \]
When \( R \) has no non-trivial idempotents, the unit of the adjunction \( (R[-], \mathcal{P}) \) is a natural isomorphism
\[ X \cong \mathcal{P}(R[X]) \]
for any simplicial set \( X \). In particular, the simplicial \( R \)-chains functor \( R[-] : \text{Set} \rightleftharpoons \text{CoCoalg}_R : \mathcal{P} \) before passing to simplicial objects.

2.4. Differential graded algebras and coalgebras. A differential graded (dg) \( R \)-module, or chain complex for short, consists of the data \( (M, d) \) where \( M \) is a \( \mathbb{Z} \)-graded \( R \)-module and \( d : M \to M \) a linear map of degree \(-1\) satisfying \( d \circ d = 0 \). If \( M = (M, d_M) \) and \( N = (N, d_N) \) are chain complexes, then \( M \otimes N \) is the chain complex with \( (M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes N_j \) and differential \( d_{M \otimes N} = d_M \otimes \text{id}_N + \text{id}_M \otimes d_N \).

A dg associative unital \( R \)-algebra, or dg algebra for short, consists of the data \( (A, d, \mu) \) where \( (A, d) \) is a dg \( R \)-module and \( \mu : A \otimes A \to A \) is a degree 0 associative unital product for which \( d \) is a derivation, i.e.,
\[ d \circ \mu = \mu \circ (d \otimes \text{id}_A) + \mu \circ (\text{id}_A \otimes d). \]
Throughout the article we use the Koszul sign rule when applying graded maps to elements. For instance, writing \( \mu(a \otimes b) = ab \), the above equation means \( d(ab) = d(a)b + (-1)^{|a|}ad(b) \), where \( |a| \) denotes the degree of \( a \). \( R \) is regarded as a dg algebra concentrated in degree 0. We denote by \( \text{dgAlg}_R \) the category of (\( \mathbb{Z} \)-graded) dg algebras with morphisms the degree 0 maps that preserve the differential and multiplicative structures.

A dg coassociative counital \( R \)-coalgebra, or dg coalgebra for short, consists of the data \( (N, \partial, \Delta) \) where \( (N, \partial) \) is a dg \( R \)-module and \( \Delta : N \to N \otimes N \) is a degree 0 coassociative counital product for which \( \partial \) is a coderivation, i.e.,
\[ \Delta \circ \partial = (\partial \otimes \text{id}_N) \circ \Delta + (\text{id}_N \otimes \partial) \circ \Delta. \]
We denote by \( \text{dgCoalg}_R \) the category of dg coalgebras with morphisms the degree 0 maps that preserve the structure. If \( N \) is a dg \( R \)-coalgebra, a coaugmentation is a map of dg coalgebras \( e : R \to N \), where \( R \) is the dg coalgebra concentrated in degree 0 with coproduct determined by the isomorphism \( R \cong R \otimes R \). A coaugmented dg coalgebra is a dg coalgebra equipped with a coaugmentation. For any coaugmented dg coalgebra \( (N, \partial, \Delta, e) \) write \( \overline{N} := N/e(R) \) and denote by \( \overline{\partial} : \overline{N} \to \overline{N} \) and \( \overline{\Delta} : \overline{N} \to \overline{N}^{\otimes 2} \) the induced differential and coproduct, respectively. A conilpotent dg coalgebra is a coaugmented dg coalgebra \( (N, \partial, \Delta, e) \) such that
\[ \overline{N} = \bigcup_{n=1}^{\infty} \ker(\overline{\Delta}^n), \]
where \( \overline{\Delta}^n : \overline{N} \to \overline{N}^{\otimes n+1} \) denotes the \( n \)-times iterated coproduct. Denote by \( \text{dgCoalg}_R^{\text{c}} \) the category whose objects are conilpotent dg coalgebras and morphisms maps of dg coalgebras that preserve coaugmentations.
2.5. **Normalized chains and coalgebra structures.** Any simplicial coassociative coalgebra \( C \) gives rise to a \( dg \) coassociative coalgebra \( \mathcal{N}_*(C) \) defined as follows. Given a simplicial coassociative coalgebra \( C \) with coproduct \( \Delta: C \to C \otimes C \), let \((\mathcal{N}_*(C), \partial)\) be the chain complex obtained as the quotient of chain complexes \((N_*(C), \partial)/(D_*(C), \partial)\), where \( N_n(C) = C_n \), the differential

\[
\partial = \sum_i (-1)^i \partial_i: N_*(C) \to N_{*+1}(C)
\]

is given by the alternating sum of the face maps of \( C \), and \( D_*(C) \subset N_*(C) \) is the sub-complex generated by the image of the degeneracy maps of \( C \). The chain complex \((\mathcal{N}_*(C), \partial)\) becomes a \( dg \) coalgebra when equipped with the coproduct

\[
\Delta: \mathcal{N}_*(C) \to \mathcal{N}_*(C) \otimes C \xrightarrow{AW} \mathcal{N}_*(C) \otimes \mathcal{N}_*(C).
\]

The map \( AW \) is the *Alexander-Whitney natural transformation*, which is given on any \( x \otimes y \in (C \otimes C)_n = C_n \otimes C_n \) by

\[
AW(x \otimes y) = \sum_{p=1}^{n+1} d_p \ldots d_n(x) \otimes d_0^{p-1}(y),
\]

where each \( d_i \) is a face map of \( C \). The construction

\[
(C, \Delta) \mapsto (\mathcal{N}_*(C), \partial, \Delta)
\]

extends to a functor

\[
\mathcal{N}_*: sCoalg_R \to \text{dgCoalg}_R
\]

called the *normalized chains* functor. We have an induced functor

\[
\mathcal{N}_*: sCoalg^0_R \to \text{dgCoalg}_R^c,
\]

where, for any \( C \in sCoalg^0_R \), we equip \( \mathcal{N}_*(C) \) with the natural coaugmentation map \( e: R \cong N_0(C) \to \mathcal{N}_*(C) \). These functors preserve colimits, since these are computed in the categories of simplicial modules and chain complexes, respectively.

2.6. **The cobar construction.** We now recall the definition of the *cobar construction* (or the *cobar functor*):

\[
\text{Cobar}: \text{dgCoalg}_R^c \to \text{dgAlg}_R.
\]

For any \( N \in \text{dgCoalg}_R^c \), the underlying graded algebra of \( \text{Cobar}(N) \) is the tensor algebra

\[
T(s^{-1}\overline{N}) = R \oplus s^{-1}\overline{N} \oplus (s^{-1}\overline{N} \otimes s^{-1}\overline{N}) \oplus (s^{-1}\overline{N} \otimes s^{-1}\overline{N} \otimes s^{-1}\overline{N}) \oplus \ldots,
\]

where \( s^{-1} \) denotes the functor which shifts the grading by \(-1\), namely, \((s^{-1}\overline{N})_i = \overline{N}_{i+1}\). The differential

\[
D: T(s^{-1}\overline{N}) \to T(s^{-1}\overline{N})
\]

is defined by extending the induced map

\[
-s^{-1} \circ \partial \circ s^{-1} + (s^{-1} \otimes s^{-1}) \circ \Delta \circ s^{-1}: s^{-1}\overline{N} \to T(s^{-1}\overline{N})
\]

as a derivation to all of \( T(s^{-1}\overline{N}) \). The equation \( D \circ D = 0 \) is equivalent to the three properties \( \partial^2 = 0 \), \( \partial \) is a coderivation of \( \Delta \), and \( \Delta \) is coassociative. The functor \( \text{Cobar} \) preserves colimits. Moreover, regarding \( \text{Cobar}(N) \) as an *augmented* \( dg \) algebra, the cobar functor becomes the left adjoint of the Koszul duality Quillen equivalence (when working over a field) with right adjoint being the classical bar construction (see, for example, [23, 34]).
We consider the following composition of functors
\[
\Lambda(-; R) := \text{Cobar} \circ \mathcal{N} \circ R[-]: s\text{Set}_0 \to \text{dgAlg}_R.
\]

The category \(\text{dgCat}_R\) of dg categories, i.e., categories enriched in chain complexes of \(R\)-modules, is a many-object version of the category \(\text{dgAlg}_R\) of dg algebras; specifically, \(\text{dgAlg}_R\) is a full subcategory of \(\text{dgCat}_R\) spanned by the categories with a single object. Analogously, we may view \(s\text{Set}\) as a many-object version of \(s\text{Set}_0\). Lurie \[26, \text{Construction 1.3.1.6}\] constructed a right adjoint functor

\[
\mathcal{N}_{dg}: \text{dgCat}_R \to s\text{Set}
\]
called the \textit{differential graded nerve functor}. The following result relates the cobar functor to the differential graded nerve functor.

**Theorem 2.6.1.** \[11, \text{Theorems 6.1 and 7.1}\] The functor
\[
\Lambda(-; R): s\text{Set}_0 \to \text{dgAlg}_R \subset \text{dgCat}_R
\]
is naturally isomorphic to the restriction of the left adjoint of \(\mathcal{N}_{dg}: \text{dgCat}_R \to s\text{Set}\) to reduced simplicial sets.

In particular, we have an induced adjunction
\[
\Lambda(-; R): s\text{Set}_0 \rightleftarrows \text{dgAlg}_R: \mathcal{N}_{dg},
\]
where we used the same notation for the restriction of the dg nerve functor to dg algebras.

For any \(X \in \text{sSet}_0\), we denote by \(C_*(\Omega|X|; R)\) the dg \(R\)-algebra of singular \(R\)-chains on the based Moore loop space of the geometric realization \(|X|\). The following is one of the main results of \[11\] and extends a classical theorem of Adams to the non-simply-connected case.

**Theorem 2.6.2.** \[11, \text{Proposition 8.2 and Corollary 9.2}\] Let \(X \in \text{sSet}_0\) be a Kan complex. Then the dg \(R\)-algebras \(\Lambda(X; R)\) and \(C_*(\Omega|X|; R)\) are naturally quasi-isomorphic. In particular, there is a natural isomorphism
\[
H_0(\Lambda(X; R)) \cong R[\pi_1(|X|)],
\]
where \(R[\pi_1(|X|)]\) denotes the group ring of the fundamental group \(\pi_1(|X|)\) at the base point given by the single element of \(X_0\).

For simplicity, we denote by
\[
\Omega: \text{sCoCoalg}^0_R \to \text{dgAlg}_R
\]
the functor defined as the composition
\[
\Omega = \text{Cobar} \circ \mathcal{N}_*.
\]
The functor \(\Omega\) preserves colimits.
The following diagram summarizes the categories and functors discussed in this section.

\[
\begin{array}{ccc}
\text{sSet}_0 & \xrightarrow{R[-]} & \text{sCoalg}^0_R \\
\downarrow & & \downarrow \\
\text{sSet}_0 & \xrightarrow{R[-]} & \text{sCoalg}^0_R \\
& \xrightarrow{N} & \text{dgCoalg}^0_R \\
& \xrightarrow{\text{Cobar}} & \text{dgAlg}_R.
\end{array}
\]

\[\Delta(-; R) = 0 \circ R[-] \]

3. Review of some model categories

In this section, we recall several known model categories that will be used throughout the article. We will assume basic knowledge of model category theory; standard references for the subject are [18], [19]. For the theory of combinatorial model categories, see also [4], [10], [27, A.2.6], [39].

3.1. The Joyal model structure on sSet. Following [27, 1.1.5], we recall the definition of the functor \( C: \text{sSet} \rightarrow \text{Cat}_{\text{sSet}} \), where \( \text{Cat}_{\text{sSet}} \) denotes the category of simplicial categories, i.e., categories enriched over the (cartesian monoidal) category of simplicial sets. For the standard \( n \)-simplex \( \Delta^n \in \text{sSet} \), we define \( C(\Delta^n) \in \text{Cat}_{\text{sSet}} \) by letting:

1. \( \text{Obj} \mathcal{C}(\Delta^n) = \{0, 1, \ldots, n\} \).
2. If \( i, j \in \{0, 1, \ldots, n\} \), then \( C(\Delta^n)(i, j) = \begin{cases} \emptyset & \text{if } i > j \\ N(P_{i,j}) \cong (\Delta^1)^{(j-i-1)} & \text{if } i < j \\ \Delta^n & \text{if } i = j \end{cases} \) where \( N \) denotes the nerve functor and \( P_{i,j} \) is the poset (regarded as category) of subsets \( U \subseteq \{0, 1, \ldots, n\} \) with \( i, j \in U \), ordered by inclusion of subsets.
3. For \( 0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n \), the composition:
   \[ \mathcal{C}(\Delta^n)(i_1, i_2) \times \cdots \times \mathcal{C}(\Delta^n)(i_{k-1}, i_k) \rightarrow \mathcal{C}(\Delta^n)(i_1, i_k) \]
   is induced by
   \[ P_{i_1, i_2} \times \cdots \times P_{i_{k-1}, i_k} \rightarrow P_{i_1, i_k} \]
   \[ (U_1, \ldots, U_{k-1}) \mapsto U_1 \cup \cdots \cup U_{k-1} \]
   The assignment \([n] \mapsto \mathcal{C}(\Delta^n)\) defines a cosimplicial object in \( \text{Cat}_{\text{sSet}} \). Then the functor \( \mathcal{C}: \text{sSet} \rightarrow \text{Cat}_{\text{sSet}} \) is the (essentially) unique colimit-preserving extension of \( \mathcal{C}(\Delta^\bullet): \Delta \rightarrow \text{Cat}_{\text{sSet}} \), i.e., \( \mathcal{C} \) is given for every \( S \in \text{sSet} \) by
   \[ \mathcal{C}(S) = \operatorname{colim}_{\sigma: \Delta^n \rightarrow S} \mathcal{C}(\Delta^n). \]

We have a functor
   \[ \pi_0: \text{Cat}_{\text{sSet}} \rightarrow \text{Cat} \]
   given by applying the path components functor on each object of morphisms. For any \( S \in \text{sSet} \), \( \pi_0 \mathcal{C}(S) \) is called the homotopy category of \( S \).

We also recall that \( \vert - \vert: \text{sSet} \rightarrow \text{Top} \) denotes the geometric realization functor from simplicial sets to topological spaces. A map of simplicial sets \( f: S \rightarrow S' \) is
a weak homotopy equivalence if \(|f|: |S| \to |S'|\) is a weak homotopy equivalence of topological spaces in the classical sense.

**Theorem 3.1.1.** [[22] Section 6], [[27] Theorems 2.2.5.1 and 2.4.6.1] There is a left proper combinatorial model category structure on \(sSet\) such that

1. a morphism \(f: S \to S'\) is a weak equivalence if \(\pi_0\mathcal{C}(f): \pi_0\mathcal{C}(S) \to \pi_0\mathcal{C}(S')\) is an essentially surjective functor and for every pair \(x, y \in S_0\), the induced map
   \[\mathcal{C}(f): \mathcal{C}(S)(x, y) \to \mathcal{C}(S)(f(x), f(y))\]
   is a weak homotopy equivalence of simplicial sets;
2. a morphism is a cofibration if it is a monomorphism;
3. the fibrant objects are the quasi-categories, i.e. the simplicial sets \(Q\) with the property that any inner horn \(f: \Lambda^k_n \to Q\), \(0 < k < n\), can be extended to \(\tilde{f}: \Delta^n \to Q\).

We call this model category structure the Joyal model structure on simplicial sets and a weak equivalence in the Joyal model structure a categorical equivalence. There is also an induced model structure on \(sSet_0\), as shown in [[9] Lemma 3.2], where the weak equivalences and the cofibrations are defined analogously.

**3.2. The Kan–Quillen model structure on \(sSet\).** The following is a classical result due to Quillen [[35]].

**Theorem 3.2.1.** There is a proper combinatorial model category structure on \(sSet\) such that

1. the weak equivalences are the weak homotopy equivalences;
2. a morphism is a cofibration if it is a monomorphism;
3. the fibrant objects are the Kan complexes, i.e. the simplicial sets \(K\) with the property that any horn \(f: \Lambda^k_n \to K\) can be extended to \(\tilde{f}: \Delta^n \to K\) for every \(0 \leq k \leq n\).

We call the above model category structure the Kan–Quillen model structure on simplicial sets. The Kan–Quillen model category is a model for the homotopy theory of spaces. There is an induced model structure on \(sSet_0\), which models connected pointed homotopy types, with the same weak equivalences and cofibrations (see, for example, [[14] Section V.6]).

The Kan–Quillen model structure is a left Bousfield localization of the Joyal model structure at the map \(\Delta^1 \to \Delta^0\). Thus every categorical equivalence is a weak homotopy equivalence and every weak homotopy equivalence between Kan complexes is a categorical equivalence. In fact, something slightly stronger also holds:

**Proposition 3.2.2.** Let \(J: sSet \to sSet\) be a fibrant replacement in the Joyal model structure.

1. If the homotopy category of \(S \in sSet\) is a groupoid, then \(J(S)\) is a Kan complex.
2. If \(f: S \to S'\) is a weak homotopy equivalence in \(sSet\) and the homotopy categories of \(S\) and \(S'\) are groupoids, then \(f\) is a categorical equivalence.

**Proof.** (1) follows from [[21] Corollary 1.4] which states that a quasi-category is a Kan complex if and only if its homotopy category is a groupoid. (2) follows from
(1) and from the fact that a weak homotopy equivalence between Kan complexes is a categorical equivalence.

We can also describe the weak homotopy equivalences in terms of the functor \( C \) as follows. Denote by \( \text{Gpd}_{sSet} \) the category of simplicial groupoids, i.e., simplicial objects in the category of groupoids \( \mathbf{G} : \Delta^{op} \to \text{Gpd} \) with a constant simplicial set of objects. Let

\[ L : \text{Cat}_{sSet} \to \text{Gpd}_{sSet} \]

be the functor from simplicial categories to simplicial groupoids that formally inverts every morphism in a simplicial category, i.e., the left adjoint of the full and faithful embedding \( \text{Gpd}_{sSet} \hookrightarrow \text{Cat}_{sSet} \). Then the weak homotopy equivalences may be described by localizing categorical equivalences as follows:

**Proposition 3.2.3.** [11] [31, Corollary 4.8] A map \( f : S \to S' \) in \( sSet \) is a weak homotopy equivalence if and only if \( \pi_0 L \mathcal{C}(f) : \pi_0 L \mathcal{C}(S) \to \pi_0 L \mathcal{C}(S') \) is an essentially surjective functor of groupoids and for every pair \( x, y \in S_0 \), the induced map of simplicial sets

\[ L \mathcal{C}(f) : L \mathcal{C}(S)(x, y) \to L \mathcal{C}(S')(f(x), f(y)) \]

is a weak homotopy equivalence.

### 3.3. The Bousfield model structure on \( sSet \)

The following model category is a special case of well-known result due to Bousfield [5].

**Theorem 3.3.1.** Let \( R \) be a commutative ring. There is a left proper combinatorial model category structure on \( sSet \) such that

1. A morphism \( f : S \to S' \) is a weak equivalence if
   \[ N_* (R[f]) : N_* (R[S]) \to N_* (R[S']) \]
   is a quasi-isomorphism;
2. A morphism is a cofibration if it is a monomorphism;
3. A morphism is a fibration if it has the right lifting property with respect to the trivial cofibrations.

We call the above model structure the **Bousfield model structure** on simplicial sets and we denote it by \((sSet, R\text{-eq})\). We will call a weak equivalence in the Bousfield model structure an **\( R \)**-equivalence. The Bousfield model structure is a left Bousfield localization of the Kan-Quillen model structure. A fibrant replacement in the Bousfield model structure is called **\( R \)**-localization. The Bousfield model structure induces also a model structure on \( sSet_0 \), which is a left Bousfield localization of the corresponding Kan-Quillen model structure on \( sSet_0 \).

### 3.4. Model structures on \( sCoCoalg_{R} \)

Let \( R \) be a commutative ring. Let \( \kappa \) be a regular cardinal such that \( \text{CoCoalg}_{R} \) is locally \( \kappa \)-presentable. For example, this is satisfied if \( \kappa > \max\{|R|, \aleph_0\} \) [31] [38]. Let \( \mathcal{I}_\kappa \) denote the set of maps \( i : A \to B \) in \( \text{CoCoalg}_{R} \) between \( \kappa \)-presentable objects whose underlying map of simplicial \( R \)-modules is a monomorphism. The following result is a special case of the model category shown by the first–named author [38] and generalizes the model category shown by Goerss [13] for the case of fields.

**Theorem 3.4.1.** [38] Theorem A] There is a left proper combinatorial model category structure on \( sCoCoalg_{R} \) such that
(1) a morphism \( f: C \to C' \) is a weak equivalence if
\[
\mathcal{N}_*(f): \mathcal{N}_*(C) \to \mathcal{N}_*(C')
\]
is a quasi-isomorphism;

(2) the class of cofibrations \( \text{Cof}(\mathcal{I}_\kappa) \) is cofibrantly generated by the set \( \mathcal{I}_\kappa \);

(3) a morphism is a fibration if it has the right lifting property with respect to the trivial cofibrations.

This model category will be denoted by \( (\text{sCoCoalg}_R, \text{q.i.}) \). We also refer to [43] for a refinement of this model structure which applies to simplicial cocommutative flat coalgebras over Prüfer domains.

It is easy to see that the adjunction \( R[-]: (\text{sSet}, R\text{-eq}) \rightleftarrows (\text{sCoCoalg}_R, \text{q.i.}) : \mathcal{P} \) is a Quillen adjunction between model categories. As shown in [38], this adjunction can also be used to induce a model structure on \( \text{sCoCoalg}_R \) transferred from the Kan–Quillen model structure on \( \text{sSet} \).

**Theorem 3.4.2.** [38, Theorem 5.4] Suppose that \( R \) has no non-trivial idempotents. There is a right proper combinatorial model category structure on \( \text{sCoCoalg}_R \) such that

1. a morphism \( f: C \to C' \) is a weak equivalence if \( \mathcal{P}(f): \mathcal{P}(C) \to \mathcal{P}(C') \) is a weak homotopy equivalence;
2. a morphism \( f: C \to C' \) is a fibration if \( \mathcal{P}(f): \mathcal{P}(C) \to \mathcal{P}(C') \) is a Kan fibration;
3. a morphism is a cofibration if it has the left lifting property with respect to the trivial fibrations; the class of cofibrations is cofibrantly generated by the set of morphisms \( \{R[i_n] | i_n: \partial \Delta^n \subseteq \Delta^n, n \geq 0\} \).

Moreover, with respect to this model structure on \( \text{sCoCoalg}_R \) and the Kan–Quillen model structure on \( \text{sSet} \), the adjunction
\[
R[-]: \text{sSet} \rightleftarrows \text{sCoCoalg}_R : \mathcal{P}
\]
is a Quillen equivalence.

### 3.5. Model structures on \( \text{dgCat}_R \) and \( \text{dgAlg}_R \)

We recall the model category structure on the category \( \text{dgCat}_R \) of small dg categories over a commutative ring \( R \) which was shown by Tabuada [47, 48]. We denote by \( H_0: \text{dgCat}_R \to \text{Cat}_R \) the functor which is defined by applying \( H_0 \) to the morphism chain complexes, where \( \text{Cat}_R \) is the category of small categories enriched in \( R \)-modules.

**Theorem 3.5.1.** [47] There is a right proper combinatorial model category structure on \( \text{dgCat}_R \) such that

1. a morphism \( F: C \to C' \) is a weak equivalence if the induced functor
\[
H_0(F): H_0(C) \to H_0(C')
\]
is essentially surjective, and for every pair of objects \( x, y \in C \) the induced map
\[
F: C(x,y) \to C'(F(x), F(y))
\]
is a quasi-isomorphism of chain complexes;

(2) a morphism \( F: C \to C' \) is a fibration if for every pair of objects \( x, y \in C \), the chain map \( F: C(x,y) \to C'(F(x), F(y)) \) is degreewise surjective; moreover, for every isomorphism \( F(x) \to z \) in \( H_0(C') \), there is a lift to an isomorphism \( x \to y \) in \( H_0(C) \).
Tabuada’s model category induces a model category structure on the category of dg algebras, where the latter are considered as dg categories with one object. This model category structure was shown by Hinich [18] and Jardin [20] (see also [34, Section 9.1]). (The model category of [20] applies to the case of non-negatively graded dg algebras using the cohomological grading, so the cofibrations in this case must be modified accordingly; see also [48, Section 3] for the analogous model category of dg categories enriched in non-negatively graded chain complexes.)

**Theorem 3.5.2.** [18, 20] There is a right proper combinatorial model structure on $\text{dgAlg}_R$ such that

1. a morphism $f: (A, d) \to (A', d')$ is a weak equivalence if $f$ is a quasi-isomorphism;
2. a morphism $f: (A, d) \to (A', d')$ a fibration if $f$ is degreewise surjective.

We will refer to this model category as the projective model structure on dg algebras. It is easy to see that this model category is finitely combinatorial using the following descriptions of the cofibrations. Denoting by $T(x)$ the free dg algebra generated by an element $x$ in degree $n$, and by $S(x)$ the free graded $R$-algebra $R[x]$ generated by an element $x$ in degree $n$ and equipped with the trivial differential, then the map $R \to T(x)$ generates the trivial cofibrations, and the maps $R \to T(x), R \to S(x), S(dx) \to T(x)$ generate the cofibrations. We deduce the following characterization of a class of cofibrations in this model structure (cf. [34, Section 9.1]): given dg algebras $(A, d)$ and $(A', d')$ which vanish in negative degrees, then a morphism of dg algebras $f: (A, d) \to (A', d')$ is a cofibration in the projective model category if and only if it is a retract of a morphism of the form 

$$(A, d) \to (A\{(x_{n,\alpha} | n \in \mathbb{N}, \alpha \in I_n}\), d'),$$

where $A\{(x_{n,\alpha} | n \in \mathbb{N}, \alpha \in I_n}\}$ denotes the algebra obtained by freely adjoining to $A$ elements in degree $n$, for any $n \geq 0$,

$$\{x_{n,\alpha} | n \in \mathbb{N}, \alpha \in I_n\},$$

for some indexing set $I_n$, in such a way that $d'(x_{n,\alpha}) \in A\{(x_{m,\beta} | m < n, \beta \in I_m\}$. Note that a cofibration in $\text{dgAlg}_R$ is also a cofibration in $\text{dgCat}_R$.

**Remark 3.5.3.** By [26, Proposition 1.3.1.20] (see also [18]), the differential graded nerve functor $\text{N}_{dg}: \text{dgCat}_R \to \text{sSet}$ is a right Quillen functor between the model category of Tabuada and the Joyal model category. The induced adjunction (Theorem 2.6.1)

$$\Lambda(-; R): \text{sSet}_0 \rightleftarrows \text{dgAlg}_R: \text{N}_{dg},$$

is again a Quillen adjunction where the model structure on $\text{sSet}_0$ is induced by the Joyal model structure and the one on $\text{dgAlg}_R$ is given by Theorem 3.5.2. To see this, note that the left adjoint preserves cofibrations (using the description above) and weak equivalences (using Theorem 2.6.1 and [26, 1.3.1.20]). In particular, the morphism

$$\Lambda(f; R): \Lambda(S; R) \to \Lambda(S'; R)$$

is a quasi-isomorphism of dg algebras for every categorical equivalence $f: S \to S'$. 
4. Homotopical localization

In this section we describe explicit and functorial models for certain localizations in the contexts of reduced simplicial sets and connected simplicial cocommutative coalgebras.

4.1. Simplicial localization. Denote by \( S^1 := \Delta^1 / \partial \Delta^1 \in \mathrm{sSet}_0 \) the reduced simplicial set with exactly two non-degenerate simplices, one in each of the dimensions 0 and 1. The homotopy category of \( S^1 \) consists of a single object and a single non-identity endomorphism generates its endomorphisms.

**Definition 4.1.1.** A localization (or weak group completion) of \( S^1 \) is a map of reduced simplicial sets

\[
i : S^1 \hookrightarrow X
\]

which is a trivial cofibration in the Kan–Quillen model structure and the homotopy category of \( X \) is a groupoid (with one object).

**Example 4.1.2.** The following are examples of localizations of \( S^1 \).

1. Any fibrant replacement of \( S^1 \) in the Kan-Quillen model structure in \( \mathrm{sSet}_0 \).
2. Let \( J = (\bullet \leftrightarrow \bullet) \) denote the connected groupoid with two objects and no non-trivial automorphisms and let \((\sigma : \Delta^1 \rightarrow N(J))\) be a non-degenerate 1-simplex. For \((\Delta^1 \rightarrow S^1)\) the non-degenerate 1-simplex of \( S^1 \), we consider the pushout \( X = S^1 \cup_{\Delta^1} N(J) \). Then the canonical map \( i : S^1 \hookrightarrow X \) is a localization of \( S^1 \). Note that \( X \) is not an quasi-category in this case (for example, there is no composition \( ([\sigma] \circ [\sigma]) \) in \( X \)).

Let \( \mathrm{sSet}_0^+ \) denote the category of marked reduced simplicial sets. The objects of \( \mathrm{sSet}_0^+ \) are pairs \((S, W)\) where \( S \in \mathrm{sSet}_0 \) and \( W \) is a subset of 1-simplices in \( S \). A morphism \( f : (S, W) \rightarrow (S', W') \) is a map of simplicial sets \( f : S \rightarrow S' \) such that \( f(W) \subseteq W' \). Given \((S, W) \in \mathrm{sSet}_0^+ \), we will identify the subset \( W \subseteq S_1 \) with the associated map \( \bigvee W \rightarrow S \).

**Definition 4.1.3.** Let \( i : S^1 \hookrightarrow X \) be a localization of \( S^1 \). The simplicial localization functor (or weak group completion functor) with respect to \( i : S^1 \hookrightarrow X \),

\[
\mathcal{K}_i : \mathrm{sSet}_0^+ \rightarrow \mathrm{sSet}_0,
\]

is defined for every \((S, W) \in \mathrm{sSet}_0^+ \) by the following pushout diagram of reduced simplicial sets

\[
\begin{array}{ccc}
\bigvee S^1 & \longrightarrow & S \\
\bigvee W \downarrow & & \downarrow \\
\bigvee X & \longrightarrow & \mathcal{K}_i(S, W).
\end{array}
\]

Note that \( \mathcal{K}_i(S, W) \) is a homotopy pushout in the Joyal model structure. The following proposition shows that the functor \( \mathcal{K}_i \) is essentially independent of the choice of the localization \( i \).

**Proposition 4.1.4.** Let \( i : S^1 \hookrightarrow X \) and \( i' : S^1 \hookrightarrow X' \) be two localizations of \( S^1 \). Then, for any \((S, W) \in \mathrm{sSet}_0^+ \), the reduced simplicial sets \( \mathcal{K}_i(S, W) \) and \( \mathcal{K}_{i'}(S, W) \) are equivalent via a natural zigzag of categorical equivalences.
Proof. We consider the pushout:

\[
\begin{array}{ccc}
S^1 & \to & X' \\
\downarrow & & \downarrow \\
X & \leftarrow & X \cup_{S^1} X'.
\end{array}
\]

Every map in the diagram is a weak homotopy equivalence (and a monomorphism). The homotopy categories of the simplicial sets \(X, X'\) and \(X \cup_{S^1} X'\) are groups. In particular, the composite map \(S^1 \to X \cup_{S^1} X'\) is again a localization of \(S^1\). By Proposition 3.2.2(2), it follows that the natural zigzag of maps

\[X \to X \cup_{S^1} X' \leftarrow X',\]

consists of categorical equivalences. It follows that \(K_\iota(S, W)\) and \(K_{\iota'}(S, W)\) are connected by a natural zigzag of categorical equivalences, since they are defined by homotopy pushouts in the Joyal model structure. \(\square\)

4.2. Derived localization of simplicial coalgebras. We define the category \(\text{sCoCoalg}^{0+}_R\) of marked connected simplicial cocommutative coalgebras as follows.

The objects are pairs \((C, P)\) where \(C \in \text{sCoCoalg}^0_R\) and \(P \subseteq \mathcal{P}(C_1)\) is a subset of set-like elements in \(C_1\). The morphisms \(f: (C, P) \to (C', P')\) are given by morphisms of simplicial coalgebras \(f: C \to C'\) such that \(f(P) \subseteq P'\). We will often identify the marking \(P \subseteq \mathcal{P}(C_1)\) with the canonical map in \(\text{sCoCoalg}^0_R\), which is adjoint to the inclusion of the marked loops \(\bigvee_p S^1 \to \mathcal{P}(C)\).

**Definition 4.2.1.** Let \(\iota: S^1 \hookrightarrow X\) be a localization of \(S^1\). The derived localization functor with respect to \(\iota: S^1 \hookrightarrow X\),

\[\mathcal{L}_\iota: \text{sCoCoalg}^{0+}_R \to \text{sCoCoalg}_R^{0}\]

is defined for every \((C, P) \in \text{sCoCoalg}^{0+}_R\) by the pushout diagram of simplicial coalgebras

\[
\begin{array}{ccc}
R[\bigvee_P S^1] & \longrightarrow & C \\
\downarrow & & \downarrow \\
R[\bigvee_P X] & \longrightarrow & \mathcal{L}_\iota(C, P),
\end{array}
\]

where the top horizontal arrow is induced by the inclusion \(P \hookrightarrow C_1\).

**Remark 4.2.2.** Whenever \(\Omega(C)\) is left proper as a dg algebra, for instance when \(\Omega(C)\) is cofibrant as a dg algebra or flat as an \(R\)-module (see \[\boxcheck\] Definition 2.5)), the dg algebra \(\Omega(\mathcal{L}_\iota(C, P))\) is a model for the derived localization of \(\Omega(C)\) at the set of 0-cycles determined by \(P\).

In fact, under this hypothesis, the square of dg algebras obtained by applying \(\Omega\) to the above diagram is a homotopy pushout in \(\text{dgAlg}_R\) equipped with the projective model structure. The dg algebra map

\[\Omega(R[\bigvee_P S^1]) \to \Omega(R[\bigvee_P X])\]
is a cofibrant replacement of $R(P) \to R(P, P^{-1})$ in the induced model structure on the category $R(P) \downarrow \text{dgAlg}_R$. Thus, assuming that $\Omega(C)$ is a left proper dg algebra, it is enough to make this replacement to compute the desired homotopy pushout, see [7, Section 3.3]. We also refer to [9] for interesting connections between the derived localization of dg algebras and the homotopy theory of monoids.

As in the case of reduced simplicial sets, the functor $L_\iota$ is independent of the choice of localization $\iota$ in the following sense.

**Proposition 4.2.3.** Let $\iota: S^1 \hookrightarrow X$ and $\iota': S^1 \hookrightarrow X'$ be two localizations of $S^1$ and let $(C, P) \in \text{sCoCoalg}^0_{R}$. Suppose $\Omega(C)$ is a left proper dg algebra. Then the simplicial coalgebras $L_\iota(C, P)$ and $L_{\iota'}(C, P)$ may be connected by a zig-zag of maps each of which induces a quasi-isomorphism of dg algebras after applying the functor $\Omega$.

**Proof.** From the proof of Proposition [4.1.4], we know that $X$ and $X'$ are categorically equivalent, thus $\bigvee_p X$ and $\bigvee_p X'$ can also be connected by a zig-zag of categorical equivalences. This zig-zag induces a zig-zag of quasi-isomorphisms of dg algebras upon applying $\Lambda(-; R) = \tilde{\iota} \circ R[-]$ (Remark 3.5.3). The result now follows from the fact that, if $\Omega(C)$ is left proper, the square of Definition 4.2.1 defines a homotopy pushout of dg algebras after applying $\Omega$ (see also Remark 4.2.2). □

**Remark 4.2.4.** Note that $N_\ast(L_\iota(C, P))$ and $N_\ast(L_{\iota'}(C, P))$ are also connected by a zig-zag of quasi-isomorphisms (without any further hypotheses on $C$).

Simplicial localization and derived localization are compatible as follows.

**Proposition 4.2.5.** For any $(S, W) \in \text{sSet}_0^+$, there is a natural isomorphism of simplicial coalgebras $L_\iota(R[S], W) \cong R[K_\iota(S, W)]$.

**Proof.** This follows immediately from the definitions, since the left adjoint functor $R[-]$ preserves pushouts. □

Let $(-)^\#: \text{sSet}_0 \to \text{sSet}_0^+$ be the functor which equips a reduced simplicial set with the maximal marking, i.e. $S^2 = (S, S_1)$. For a fibrant replacement functor $J: \text{sSet}_0 \to \text{sSet}_0$ in the Joyal model structure, Proposition 3.2.2 shows that the natural map $S \to J(K_\iota(S^2))$ is a fibrant replacement in the Kan-Quillen model structure. Consider the functor

$$\mathfrak{X}: \text{sCoCoalg}^0_R \to \text{sCoCoalg}^0_R$$

defined by

$$\mathfrak{X}(C) = (C, \mathcal{P}(C_1))$$

where $\mathcal{P}(C_1)$ is the set of set-like elements in the coalgebra $C_1$.

**Proposition 4.2.6.** The following diagram commutes

\[
\begin{array}{ccc}
\text{sSet}_0 & \xrightarrow{R[-]} & \text{sCoCoalg}_R^0 \\
\downarrow_{K_\iota \circ (-)^\#} & & \downarrow_{L_\iota \circ \mathfrak{X}} \\
\text{sSet}_0 & \xrightarrow{R[-]} & \text{sCoCoalg}_R^0 \\
\end{array}
\]

**Proof.** This follows from Proposition [4.2.5] using that $\mathfrak{X}(R[S]) = (R[S], S_1)$. □
5. Three notions of weak equivalence

In this section, study the properties of the following three different notions of weak equivalence in $s\text{CoCoalg}_R^0$.

1. The standard class $W_{q,i}$ of quasi-isomorphisms in $s\text{CoCoalg}_R^0$ (or $s\text{CoCoalg}_R$). The homotopy theory of simplicial $R$-coalgebras relative to $W_{q,i}$ and its connection with the $R$-local homotopy theory of spaces have been studied in [13] (for fields) and in [38] (for general presheaves of commutative rings).

2. The class $W_\Omega$ of $\Omega$-quasi-isomorphisms (or cobar quasi-isomorphisms) in $s\text{CoCoalg}_R^0$. This is a smaller class of weak equivalences than $W_{q,i}$ and its properties have also been studied in [42].

3. The class $W_{\hat{\Omega}}$ of $\hat{\Omega}$-quasi-isomorphisms in $s\text{CoCoalg}_R^0$ (or localized cobar quasi-isomorphisms). This class of weak equivalences lies strictly between $W_\Omega$ and $W_{q,i}$. This class determines a homotopy theory on $s\text{CoCoalg}_R^0$ which is more suitable for the comparison with the usual homotopy theory of reduced simplicial sets.

We also relate each of the above notions to their corresponding counterparts on $s\text{Set}_0$.

5.1. Quasi-isomorphisms. The following definitions are well known.

**Definition 5.1.1.** A map $f: C \to C'$ in $s\text{CoCoalg}_R$ is a quasi-isomorphism if the induced map of dg coalgebras $N_*(f): N_*(C) \to N_*(C')$ induces an isomorphism on homology. We denote the class of quasi-isomorphisms in $s\text{CoCoalg}_R$ (or $s\text{CoCoalg}_R^0$) by $W_{q,i}$.

**Definition 5.1.2.** A map $f: X \to X'$ in $s\text{Set}$ is an $R$-equivalence if it induces an isomorphism $H_*(f; R): H_*(X; R) \cong H_*(X'; R)$ on homology with $R$-coefficients. We denote the class of $R$-equivalences in $s\text{Set}$ (or $s\text{Set}_0$) by $W_R$.

We record the following obvious statement in order to emphasize the parallelism with analogous statements in the next sections.

**Proposition 5.1.3.** A map $f: X \to X'$ in $s\text{Set}$ is an $R$-equivalence if and only if $R[f]: R[X] \to R[X']$ is a quasi-isomorphism in $s\text{CoCoalg}_R$.

5.2. $\Omega$-quasi-isomorphisms. Recall the notation $\Omega = \text{Cobar} \circ N_*$.

**Definition 5.2.1.** A map $f: C \to C'$ in $s\text{CoCoalg}_R^0$ is an $\Omega$-quasi-isomorphism if the map of dg algebras $\Omega(f): \Omega(C) \to \Omega(C')$ is a quasi-isomorphism. We denote by $W_\Omega$ the class of $\Omega$-quasi-isomorphisms of connected simplicial cocommutative $R$-coalgebras.

This notion of weak equivalence is well studied in the context of dg coalgebras [24] and has been used extensively in Koszul duality [31], [25]. A first observation is that this notion is strictly stronger than quasi-isomorphism.

**Proposition 5.2.2.** Let $C$ and $C'$ be connected simplicial cocommutative flat $R$-coalgebras. If $f: C \to C'$ is an $\Omega$-quasi-isomorphism, then $f: C \to C'$ is a quasi-isomorphism, but not vice versa.

**Proof.** This is shown for fields in [25] Proposition 2.4.2. The same proof applies here too.
The notion of $\Omega$-quasi-isomorphism is related to the following linearized version of categorical equivalences between reduced simplicial sets.

**Definition 5.2.3.** A map $f : S \to S'$ in $s\text{Set}_0$ is an $R$-categorical equivalence if the map between dg algebras

$$\Lambda(f; R) : \Lambda(S; R) \to \Lambda(S'; R)$$

is a quasi-isomorphism of dg algebras. We denote the class of $R$-categorical equivalences in $s\text{Set}_0$ by $W_{f,R}$ (the subscript “J” stands for “Joyal”). Note that, by Theorem 2.6.1 and Remark 3.5.3, every categorical equivalence is also an $R$-categorical equivalence.

**Proposition 5.2.4.** A map $f : S \to S'$ in $s\text{Set}_0$ is an $R$-categorical equivalence if and only if $R[f] : R[S] \to R[S']$ is an $\Omega$-quasi-isomorphism of connected simplicial cocommutative $R$-coalgebras. In particular, if $f : S \to S'$ is a categorical equivalence then $R[f] : R[S] \to R[S']$ is an $\Omega$-quasi-isomorphism.

**Proof.** This follows directly from the definitions of $\Omega$-quasi-isomorphism and $R$-categorical equivalence, since $\Lambda(-; R) = \Omega \circ R[-]$. $\square$

**Remark 5.2.5.** Given $S \in s\text{Set}_0$, we denote by $C_\ast(\Omega|S|; R)$ the normalized singular chains dg $R$-algebra on the topological monoid of based (Moore) loops in $|S|$. For any simplicial localization $S \to K_\ast(S^2)$, we have a natural quasi-isomorphism of dg algebras

$$\Lambda(K_\ast(S^2); R) \simeq C_\ast(\Omega|S|; R).$$

Thus there is a natural isomorphism of algebras $H_0(\Lambda(K_\ast(S^2))) \cong R[\pi_1(|S|)]$, where the latter is the fundamental group algebra. These follow easily by using the fact that $K_\ast(S^2)$ is categorically equivalent to a Kan complex together with Proposition 5.2.4, and then applying Theorem 2.6.2. See also [9, Corollary 4.2].

5.3. $\hat{\Omega}$-quasi-isomorphisms. Fix a localization $\iota : S^1 \to \Sigma^1$ of $S^1$ and define the localized cobar functor as the composition

$$\hat{\Omega} = \Omega \circ \mathcal{L} \circ \mathfrak{X} : s\text{CoCoalg}_R^{0} \to \text{dgAlg}_R.$$

A similar construction was proposed in [24, Section 1.2] under the name of extended cobar construction in the context of dg coalgebras. However, the extended cobar construction is not functorial as defined, since it depends on a basis for the degree 1 summand of the underlying graded $R$-module. By considering simplicial coalgebras, we may obtain the desired set of degree 1 elements functorially through the functor of points.

**Definition 5.3.1.** A map $f : C \to C'$ in $s\text{CoCoalg}_R^{0}$ is an $\hat{\Omega}$-quasi-isomorphism if $\hat{\Omega}(f) : \hat{\Omega}(C) \to \hat{\Omega}(C')$ is a quasi-isomorphism of dg algebras. We denote by $\mathcal{W}\hat{\Omega}$ the class of $\hat{\Omega}$-quasi-isomorphisms in $s\text{CoCoalg}_R^{0}$.

The motivation for the above definition comes from the fact that a map of simplicial sets $f : S \to S'$ is a weak homotopy equivalence if and only if the induced map

$$K_\ast(f^2) : K_\ast(S^2) \to K_\ast(S'^2)$$

obtained by localizing all 1-simplices (see Proposition 4.2.0) is a categorical equivalence. Given the compatibility between simplicial localization and derived localization (Propositions 4.2.5 and 4.2.0), we have the following immediate relationship...
between $\Omega$-quasi-isomorphisms and $\hat{\Omega}$-quasi-isomorphisms for the simplicial chains on reduced simplicial sets.

**Proposition 5.3.2.** Let $f: S \to S'$ be a map of reduced simplicial sets. Then $R[f]: R[S] \to R[S']$ is an $\hat{\Omega}$-quasi-isomorphism if and only if

$$R[K_n(f^*)]: R[K_n(S^2)] \to R[K_n(S'^2)]$$

is an $\Omega$-quasi-isomorphism.

Based on Theorem 5.3.4. we may give a concrete description of the $\hat{\Omega}$-quasi-isomorphisms arising from maps of reduced simplicial sets. This description is one the key results for the comparison between the respective homotopy theories of reduced simplicial sets and connected simplicial cocommutative coalgebras.

**Definition 5.3.3.** A map $f: S \to S'$ in $sSet_0$ is a $\pi_1$-$R$-equivalence if it induces an isomorphism between fundamental groups

$$\pi_1(|f|): \pi_1(|S|) \cong \pi_1(|S'|)$$

and the induced map between the universal covers

$$\bar{f}: |S| \to |S'|$$

is an $R$-equivalence. We denote the class of $\pi_1$-$R$-equivalences in $sSet_0$ by $\mathcal{W}_{\pi_1}$.

**Theorem 5.3.4.** A map of reduced simplicial sets $f: S \to S'$ is a $\pi_1$-$R$-equivalence if and only if $R[f]: R[S] \to R[S']$ is an $\hat{\Omega}$-quasi-isomorphism.

**Proof.** We recall that for any $X \in sSet_0$ there is a natural isomorphism of graded algebras

$$\Theta_X: H_*(\Omega[X]; R) \otimes R[\pi_1(X)] \cong H_*(\Omega[X]; R).$$

Furthermore, $\Theta_X$ restricts to an isomorphism of bialgebras $R[\pi_1(X)] \cong H_0(\Omega[X]; R)$ in degree 0.

We write $\pi_1 = \pi_1(|S|)$ and $\pi'_1 = \pi_1(|S'|)$ in order to simplify the notation. We consider the following commutative diagram

$$
\begin{array}{ccc}
H_*(\Omega[K_n(S^2)]; R) \otimes R[\pi_1] & \overset{\Theta}{\cong} & H_*(\Omega[K_n(S^2)]; R) \\
\downarrow & & \downarrow \\
H_*(\Omega[K_n(S'^2)]; R) \otimes R[\pi'_1] & \overset{\Theta}{\cong} & H_*(\Omega[K_n(S'^2)]; R)
\end{array}
$$

where every horizontal map is an isomorphism and the vertical maps are induced by the map $f$. Note that the vertical maps are isomorphisms in degree 0 if and only if $f$ induces an isomorphism on group rings $R[\pi_1(|f|)]: R[\pi_1] \to R[\pi'_1]$, which holds if and only if $f$ induces an isomorphism $\pi_1(|f|): \pi_1 \to \pi'_1$ on fundamental groups.

Since the canonical maps $S \to K_n(S^2)$ and $S' \to K_n(S'^2)$ are weak homotopy equivalences, the left vertical map is an isomorphism if and only if $f$ is a $\pi_1$-isomorphism and $\Omega[f]$ is an $R$-equivalence. The latter property implies that $|f|$ is also an $R$-equivalence. Moreover, using the Eilenberg–Moore spectral sequence, the condition about $\Omega[f]$ being an $R$-equivalence is equivalent to $|f|$ being an $R$-equivalence, since $|f|$ is a map of simply-connected spaces. (Alternatively, one could
also appeal to Theorem 2.6.2. Thus, the left vertical map is an isomorphism if and only if \( f \) is a \( \pi_1 \)-equivalence.

On the other hand, by Proposition 5.3.2, the right vertical map is an isomorphism if and only if \( R[f] \) is an \( \hat{\Omega} \)-quasi-isomorphism. The result follows. \( \square \)

**Proposition 5.3.5.** The classes of morphisms \( W_{J,R}, W_{\pi_1,R}, W_R \) in \( s\text{Set}_0 \) satisfy the following strict inclusions

\[
W_{J,R} \subset W_{\pi_1,R} \subset W_R.
\]

**Proof.** The first inclusion follows from Theorem 5.3.4, Proposition 5.2.4 and Remark 4.2.2. To see that the inclusion is strict, note that \( \iota : S^1 \to S^1 \) is a \( \pi_1 \)-equivalence, but not an \( R \)-categorical equivalence. The second strict inclusion is well known. (Alternatively, this also follows from Theorem 5.3.4 and Proposition 5.2.2.) \( \square \)

When \( R = \mathbb{F} \) is a field, by Proposition 4.2.3, the quasi-isomorphism type of the dg algebra \( \hat{\Omega}(C) \) is independent of the choice of the localization \( \iota : S^1 \to S^1 \). This is the context in which we will work in Subsection 7.3. Over a field, the notion of \( \hat{\Omega} \)-quasi-isomorphism is strictly stronger than \( \hat{\Omega} \)-quasi-isomorphism. In fact, we have the following inclusions of classes of weak equivalences.

**Proposition 5.3.6.** Let \( \mathbb{F} \) be a field. The classes of morphisms \( W_i, W_{i,i}, \) and \( W_{q.i} \) in \( s\text{CoCoalg}_0^\mathbb{F} \) satisfy the following strict inclusions

\[
W_i \subset W_{i,i} \subset W_{q.i}.
\]

**Proof.** When working over a field, the dg algebra \( \hat{\Omega}(C) \) is a model for the derived localization of \( \hat{\Omega}(C) \) at the set of 0-cycles given by \( \mathcal{P}(C_1) \), as explained in Remark 4.2.2. Since derived localization is invariant under quasi-isomorphisms of dg algebras, it follows that if a map \( f : C \to C' \) in \( s\text{CoCoalg}_0^\mathbb{F} \) induces a quasi-isomorphism \( \hat{\Omega}(f) : \hat{\Omega}(C) \to \hat{\Omega}(C') \), then \( \hat{\Omega}(f) : \hat{\Omega}(C) \to \hat{\Omega}(C') \) is a quasi-isomorphism, so \( W_i \subset W_{i,i} \). The inclusion is clearly strict (e.g., the map \( \mathbb{F} \lbrack i \rbrack : \mathbb{F}[S^1] \to \mathbb{F}[S'] \) is an \( \hat{\Omega} \)-quasi-isomorphism, but not an \( \hat{\Omega} \)-quasi-isomorphism).

Now let \( f : C \to C' \) be a map in \( s\text{CoCoalg}_0^\mathbb{F} \) inducing a quasi-isomorphism \( \hat{\Omega}(f) : \hat{\Omega}(C) \to \hat{\Omega}(C') \), that is, the map

\[
\hat{\Omega}(\mathcal{L}_i(\mathcal{X}(f))) : \hat{\Omega}(\mathcal{L}_i(\mathcal{X}(C))) \to \hat{\Omega}(\mathcal{L}_i(\mathcal{X}(C')))
\]

is a quasi-isomorphism of dg algebras. Then, by Proposition 5.2.2, the map

\[
\mathcal{L}_i(\mathcal{X}(f)) : \mathcal{L}_i(\mathcal{X}(C)) \to \mathcal{L}_i(\mathcal{X}(C'))
\]

is a quasi-isomorphism between (the normalized chain complexes of the) simplicial coalgebras. For any \( C \in s\text{CoCoalg}_0^\mathbb{F} \), the natural map \( C \to \mathcal{L}_i(\mathcal{X}(C)) \) is a quasi-isomorphism of simplicial coalgebras. It follows that \( f : C \to C' \) is a quasi-isomorphism of simplicial coalgebras. Hence, \( W_{i,i} \subset W_{q.i} \). The inclusion is clearly strict, since there are \( \mathbb{F} \)-equivalences \( f : S \to S' \) in \( s\text{Set}_0 \) which are not \( \pi_1 \)-isomorphisms (cf. Proposition 5.3.5). \( \square \)

6. A METHOD FOR CONSTRUCTING MODEL STRUCTURES

In this section we will discuss a useful and general method for constructing model structures on a locally presentable category \( \mathcal{C} \) that are (left-)induced by a combinatorial model category \( \mathcal{M} \) along an accessible functor \( F : \mathcal{C} \to \mathcal{M} \). This
method follows the proof in [38, Section 4] and is based on Smith’s recognition theorem for combinatorial model category structures. We will review this method in a general abstract setting, as this will allow us to give a uniform treatment of the construction of model structures in the next section.

6.1. **Using Smith’s recognition theorem.** Let \( \mathcal{C} \) be a locally presentable category, let \( \mathcal{M} \) be a combinatorial model category, and let \( F: \mathcal{C} \rightarrow \mathcal{M} \) be an accessible functor. Let \( \kappa \) be a fixed regular cardinal such that:

(a) \( \mathcal{C} \) is locally \( \kappa \)-presentable;
(b) \( \mathcal{M} \) is \( \kappa \)-combinatorial, that is, the category \( \mathcal{M} \) is locally \( \kappa \)-presentable and there are generating sets of cofibrations and trivial cofibrations that are given by morphisms between \( \kappa \)-presentable objects;
(c) \( F \) is \( \kappa \)-accessible.

Then we consider the following classes of morphisms in \( \mathcal{C} \).

- **Weak equivalences:** \( W_F = F^{-1}(W_M) \). In other words, the class of weak equivalences \( W_F \) is the inverse image of the weak equivalences \( W_M \) in \( \mathcal{M} \) under the functor \( F \). The class of weak equivalences \( W_M \) is accessible and accessibly embedded in \( \mathcal{M} \) (see [4, 37, 40]). Since \( F \) is accessible, it follows that \( W_F \) is also accessible and accessibly embedded in \( \mathcal{C} \). Moreover, \( W_F \) satisfies the 2-out-of-3 property.

- **Cofibrations:** Consider a set of morphisms in \( \mathcal{C} \), \( I \subseteq \{ i: A \rightarrow B \mid A, B \text{ are } \kappa \text{-presentable}, F(i) \text{ is a cofibration in } \mathcal{M} \} \).

The class of cofibrations \( \text{Cof}(I) \) in \( \mathcal{C} \) is the cofibrant closure of \( I \), that is, the smallest class of morphisms in \( \mathcal{C} \) which contains \( I \) and is closed under pushouts, retracts, and transfinite compositions.

For most applications, it will suffice to let \( I \) be exactly the set of of morphisms between \( \kappa \)-presentable objects (one from each isomorphism class) which become cofibrations in \( \mathcal{M} \) after applying \( F \).

By Smith’s recognition theorem (see [4, Theorem 1.7], [38, Theorem 4.1]), we have the following immediate conclusion.

**Theorem 6.1.1.** Let \( \mathcal{C}, \mathcal{M}, \) and \( F: \mathcal{C} \rightarrow \mathcal{M} \) be as above. Then the classes of morphisms in \( \mathcal{C} \),

- Weak equivalences: \( W_F \)
- Cofibrations: \( \text{Cof}(I) \)

determine a combinatorial model category structure on \( \mathcal{C} \) if and only if the following conditions are satisfied:

1. \( I - \text{inj} \subseteq W_F \).
2. The class of trivial cofibrations \( \text{Cof}(I) \cap W_F \) is closed under pushouts and transfinite compositions.

6.2. **Techniques.** For the applications of Theorem 6.1.1, it is useful to identify specific properties that ensure conditions (1) and (2) and are easy to verify in practice. First, we observe that the following conditions on \( F \) ensure that condition (2) of Theorem 6.1.1 is satisfied.

**Proposition 6.2.1.** Let \( \mathcal{C}, \mathcal{M}, \) and \( F: \mathcal{C} \rightarrow \mathcal{M} \) be as in Theorem 6.1.1. Condition (2) holds if \( F \) preserves colimits. More generally, condition (2) holds if the following weaker assumptions are satisfied:
(i) $F$ sends every pushout square in $\mathcal{C}$

\[
\begin{array}{ccc}
A & \rightarrow & C \\
\downarrow & & \downarrow \\
B & \rightarrow & D
\end{array}
\]

where $i: A \rightarrow B$ is in $\text{Cof}(I) \cap W_F$, to a homotopy pushout square in $\mathcal{M}$.

(ii) $F$ sends every transfinite composition in $\mathcal{C}$,

\[A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_\lambda \rightarrow \cdots \rightarrow \colim_\lambda A_\lambda,\]

where each morphism $A_i \rightarrow A_{i+1}$ is in $\text{Cof}(I) \cap W_F$, to a homotopy colimit diagram in $\mathcal{M}$.

Condition (1) of Theorem 6.1.1 is generally more difficult to verify in practice and some ad hoc argument that uses the particularities of the context might be necessary. The following proposition states some general properties which ensure that condition (1) is satisfied (cf. [38, Lemma 4.4]).

**Proposition 6.2.2.** Let $\mathcal{C}$, $\mathcal{M}$, and $F: \mathcal{C} \rightarrow \mathcal{M}$ be as in Theorem 6.1.1. Suppose that every morphism $f: X \rightarrow Y$ between $\kappa$-presentable objects in $\mathcal{C}$ admits a factorization $f = \pi i$ such that:

(a) $i \in \text{Cof}(I)$;
(b) $F(\pi)$ is a trivial fibration in $\mathcal{M}$.

Then condition (1) of Theorem 6.1.1 is satisfied.

**Proof.** The proof is the same argument as in [38, Proposition 4.5]. □

**Remark 6.2.3.** The technical assumptions of Proposition 6.2.2 are often satisfied simply by using an appropriate mapping cylinder factorization. (This was the case in [38, Lemma 4.4].) Property (b) is often the most tricky property to verify in practice. For that reason, it is desirable to choose $\mathcal{M}$ appropriately. (For example, this was chosen to be the projective model category of simplicial $R$-modules in [38].) Let us mention that in the case where $I$ is the maximal choice, i.e. the set of all morphisms between $\kappa$-presentable objects in $\mathcal{C}$ which become cofibrations in $\mathcal{M}$, then (a) clearly holds as long as:

(a$_1$) $F(i)$ is a cofibration in $\mathcal{M}$;
(a$_2$) the codomain of $i$ is $\kappa$-presentable in $\mathcal{C}$, whenever $X$ and $Y$ are $\kappa$-presentable.

**Proposition 6.2.4.** Let $\mathcal{C}$, $\mathcal{M}$, and $F: \mathcal{C} \rightarrow \mathcal{M}$ be as in Theorem 6.1.1 and suppose that conditions (1) and (2) of Theorem 6.1.1 are satisfied. Then

$F: \mathcal{C} \rightarrow \mathcal{M}$

is a left Quillen functor (with respect to the model category structure of Theorem 6.1.1) if and only if $F$ preserves small colimits.

**Proof.** First note that $F$ preserves weak equivalences by definition. By the (left) special adjoint functor theorem for locally presentable categories (see [11, 27, 33]), $F$ is a left adjoint if and only if $F$ preserves small colimits. In this case, it is immediate that $F$ also preserves cofibrations. □
On the other hand, condition (1) of Theorem 6.1.1 is certainly satisfied when (for some set $I$) the class $I - \text{inj}$ is already known to be contained in a class of weak equivalences $W_C$ which is preserved by $F$. More specifically, Smith’s recognition theorem ([14, Theorem 1.7], [38, Theorem 4.1]) has the following immediate consequence for the existence of left Bousfield localizations of combinatorial model categories.

**Theorem 6.2.5.** Let $(\mathcal{C}, \mathrm{Cof}_C, W_C, \mathrm{Fib}_C)$ and $(\mathcal{M}, \mathrm{Cof}_M, W_M, \mathrm{Fib}_M)$ be combinatorial model categories and let $F: \mathcal{C} \rightarrow \mathcal{M}$ be an accessible functor which preserves the weak equivalences.

Then the left Bousfield localization of $\mathcal{C}$ with weak equivalences $W_F := F^{-1}(W_M)$ exists, and it is again a combinatorial model category, if and only if $\mathrm{Cof}_C \cap W_F$ is closed under pushouts and transfinite compositions.

In particular, this holds if $F$ is additionally a left Quillen functor.

### 6.3. Homotopically full and faithful Quillen functors.

For our main comparison results between spaces and simplicial coalgebras in Section 8, we will be interested in the following property of a Quillen adjunction.

**Proposition 6.3.1.** Let $F: \mathcal{M} \rightleftarrows \mathcal{N}: G$ be a Quillen adjunction of model categories. The following are equivalent:

1. For every pair of cofibrant objects $X, Y \in \mathcal{M}$, the induced map of (derived) mapping spaces
   \[ \text{map}^h_{\mathcal{M}}(X, Y) \rightarrow \text{map}^h_{\mathcal{N}}(F(X), F(Y)) \]
   is a weak homotopy equivalence.
2. The derived unit map
   \[ X \rightarrow G(F(X)^f) \]
   is a weak equivalence for every cofibrant object $X \in \mathcal{M}$. (Here the morphism $F(X) \rightarrow F(X)^f$ is a functorial fibrant replacement in $\mathcal{N}$.)
3. The unit transformation $\text{Id} \Rightarrow RG \circ LF$ of the derived adjunction
   \[ LF: \text{Ho}(\mathcal{M}) \rightleftarrows \text{Ho}(\mathcal{N}): RG \]
   is a natural isomorphism.
4. The left derived functor
   \[ LF: \text{Ho}(\mathcal{M}) \rightarrow \text{Ho}(\mathcal{N}) \]
   is full and faithful.

**Definition 6.3.2.** Let $F: \mathcal{M} \rightleftarrows \mathcal{N}: G$ be a Quillen adjunction. We say that $F$ is homotopically full and faithful if the equivalent conditions of Proposition 6.3.1 are satisfied.

The following simple criterion will suffice for our applications in Section 8.

**Proposition 6.3.3.** Let $F: \mathcal{M} \rightleftarrows \mathcal{N}: G$ be a Quillen adjunction. Suppose that $F$ is full and faithful and $G$ preserves weak equivalences. Then $F$ is homotopically full and faithful.

**Proof.** $F$ is full and faithful if and only if the unit transformation $\text{Id} \Rightarrow G \circ F$ is a natural isomorphism. Since $G$ preserves weak equivalences, it is immediate that $(F, G)$ satisfies property (2) of Proposition 6.3.1. \qed
7. Model structures on $sSet_0$ and $sCoCoalg^0_F$

In this section we will apply the method of Section 6 to establish the existence of three model category structures on reduced simplicial sets and their corresponding counterparts for the category of connected simplicial cocommutative coalgebras over a field. These model structures are associated with the classes of weak equivalences discussed in Section 5.

Fix a localization $ι: S^1 \rightarrow S^1$ of $S^1$ and define the *localized cobar functor* as the composition $\hat{\Omega} = Ω\circ L_i\circ F: sCoCoalg^0_R \rightarrow dgAlg_R$ (as in Subsection 5.3).

7.1. Three model structures on $sSet_0$. The following result proves the existence of three model category structures on reduced simplicial sets: the first one is a linearized version of the Joyal model category (restricted to reduced simplicial sets); the second one is a left Bousfield localization of the first, which involves considering the fundamental group as in the Kan–Quillen model category; finally, the third one corresponds to the Bousfield model category for $R$-homology equivalences to reduced simplicial sets.

**Theorem 7.1.1.** Let $sSet_0$ denote the category of reduced simplicial sets and let $R$ be a commutative ring.

1. There is a left proper combinatorial model category structure on $sSet_0$ with the monomorphisms as cofibrations and the $R$-categorical equivalences $W_{J,R}$ as weak equivalences. This model category is a left Bousfield localization of the Joyal model category structure on $sSet_0$.

   We denote this model category by $(sSet_0, R$-cat.eq$.)$.

2. There is a left proper combinatorial model category structure on $sSet_0$ with the monomorphisms as cofibrations and the $π_1$-$R$-equivalences $W_{π_1,R}$ as weak equivalences. This model category is a left Bousfield localization of the model category in (1) and of the Kan–Quillen model category structure on $sSet_0$.

   We denote this model category by $(sSet_0, π_1$-R$-eq$.)$.

3. There is a left proper combinatorial model category structure on $sSet_0$ with the monomorphisms as cofibrations and the $R$-equivalences $W_R$ as weak equivalences. This model category is a left Bousfield localization of the model category in (2) and is induced by the Bousfield model category structure on $sSet$ restricted to $sSet_0$.

   We denote this model category by $(sSet_0, R$-eq$.)$.

**Proof.** We recall that $Λ(-;R): sSet_0 \rightarrow dgAlg_R$ is a left Quillen functor when $sSet_0$ is equipped with model structure induced by the Joyal model category and $dgAlg_R$ has the model structure of Theorem 3.5.2 (see Remark 3.5.3). Note that $Λ(-;R)$ preserves all weak equivalences, since every object in $sSet_0$ is cofibrant. Then, (1) follows directly by applying Theorem 6.2.5 to the model categories $C = sSet_0$ and $M = dgAlg_R$, and the functor $F = Λ(-;R)$.

We will now prove (2). By Theorem 5.3.2 we have

$$W_{π_1,R} = (\hat{Ω} \circ R[{-}])^{-1}(W_{dgAlg_R})$$.
where \( W_{\text{dgAlg}_R} \) denotes the class of quasi-isomorphisms of dg algebras. To prove (2), we will apply Theorem 6.2.5 in the case of the model categories 
\[ \mathcal{C} = (\text{sSet}_0, \text{R-cat.eq.}) \text{ (as in (1))} \]
and the functor \( F = \hat{\Omega} \circ R[-]: \text{sSet}_0 \to \text{dgAlg}_R \). \( F \) is a composition of functors which preserve filtered colimits, therefore \( F \) also preserves filtered colimits.

Next we verify that \( F \) preserves weak equivalences. If \( f: S \to S' \) is an \( \text{R-cat.} \) categorical equivalence, then \( \hat{\Omega}R[f]: \hat{\Omega}R[S] \to \hat{\Omega}R[S'] \) is a quasi-isomorphism of flat \( \text{dgR} \)-algebras. Since \( \hat{\Omega}R[-] \) is the derived localization of the \( \text{R-flat} \) \( \text{dgR} \)-algebra \( \Omega R[-] \) at the 0-cycles determined by the set of 1-simplices (Remark 4.2.2), and derived localization is invariant under quasi-isomorphisms, it follows that the induced map \( \hat{\Omega}R[f]: \hat{\Omega}R[S] \to \hat{\Omega}R[S'] \) is a quasi-isomorphism, too. (Alternatively, we can use Propositions 4.2.5 and 4.2.6 and argue that the simplicial localization at the set of 1-simplices preserves \( \text{R-cat.} \) categorical equivalences.)

Even though \( F \) does not preserve pushouts in general, the required property that \( \text{Cof}_{\mathcal{C}} \cap W_{\pi_1-\text{R}} \) is closed under pushouts still holds, since given a pushout in \( \text{sSet}_0 \)
\[
\begin{array}{ccc}
S & \longrightarrow & X \\
\downarrow^i & & \downarrow^j \\
S' & \longrightarrow & X',
\end{array}
\]
where \( i \) is a monomorphism and a \( \pi_1\)-\( \text{R-equivalence} \), then we observe that \( j \) is again a monomorphism and a \( \pi_1\)-\( \text{R-equivalence} \). We see similarly (directly) that \( \text{Cof}_{\mathcal{C}} \cap W_{\pi_1-\text{R}} \) is closed under transfinite compositions. This completes the proof of (2).

Finally, (3) follows also by applying Theorem 6.2.5 to the case where \( \mathcal{C} = (\text{sSet}_0, \pi_1-\text{R-equ.}) \) (as in (2)), \( \mathcal{M} = \text{Ch}_R \) is the category of (non-negatively graded) chain complexes of \( \text{R-modules} \) equipped with the projective model structure, and \( F: \text{sSet}_0 \to \text{Ch}_R \) is the normalized chains functor. □

We summarize the relationships between the various model category structures on \( \text{sSet}_0 \) in the following diagram of left Quillen functors. Every arrow indicates the identity functor on \( \text{sSet}_0 \).

\[
\begin{array}{ccc}
(\text{sSet}_0, \text{Joyal}) & \longrightarrow & (\text{sSet}_0, \text{Kan-Quillen}) \\
\downarrow \quad & \quad \downarrow \\
(\text{sSet}_0, \text{R-cat.eq.}) & \longrightarrow & (\text{sSet}_0, \pi_1-\text{R-equ.}) \longrightarrow (\text{sSet}_0, \text{R-equ.}).
\end{array}
\]

### 7.2. A fiberwise fracture theorem.

Homotopy types may be studied in terms of their localizations (or completions) over \( Q \) and over \( \mathbb{F}_p \) at each prime \( p \). The fracture theorem expresses a (nilpotent) homotopy type as a homotopy pullback of its \( Q \)-localization, \( \mathbb{F}_p \)-localizations, and \( Q \)-localization of its \( \mathbb{F}_p \)-localizations [18, 14, 40].

We will show a version of the fracture theorem over \( \pi_1 \) for all homotopy types using the homotopy theories of Theorem 7.1.1(2).

Let \( P \) denote the set of prime numbers. We denote by
\[
X \to X_{R/\pi_1}
\]
a functorial fibrant replacement of \( X \in \text{sSet}_0 \) in the model category \((\text{sSet}_0, \pi_1\text{-R.eq.})\) of Theorem 7.1.1.

**Theorem 7.2.1.** For every \( X \in \text{sSet}_0 \), the canonical square in \( \text{sSet}_0 \)

\[
\begin{array}{ccc}
X & \longrightarrow & \prod_{p \in P} X_{\mathbb{F}_p/\pi_1} \\
\downarrow & & \downarrow \\
X_{\mathbb{Q}/\pi_1} & \longrightarrow & (\prod_{p \in P} X_{\mathbb{F}_p/\pi_1})_{\mathbb{Q}/\pi_1}
\end{array}
\]  

is a homotopy pullback (with respect to the Kan–Quillen model structure). Moreover, suppose we are given a homotopy pullback in \( \text{sSet}_0 \)

\[
\begin{array}{ccc}
X & \longrightarrow & \prod_{p \in P} X_p \\
\downarrow & & \downarrow f \\
X_0 & \longrightarrow & Y
\end{array}
\]

where the following hold:

(i) \( X_p \) is fibrant in \((\text{sSet}_0, \pi_1\text{-F}_p\text{-eq.})\);
(ii) \( X_0 \) and \( Y \) are fibrant in \((\text{sSet}_0, \pi_1\text{-Q}\text{-eq.})\);
(iii) \( f \) is a \( \pi_1\text{-Q}\text{-equivalence} \) in \((\text{sSet}_0, \pi_1\text{-Q}\text{-eq.})\);
(iv) \( g \) induces \( \pi_1\text{-isomorphisms} \) for each \( p \in P \),

\[ \pi_1(X_0) \xrightarrow{\pi_1(g)} \pi_1(Y) \xrightarrow{\pi_1(f)} \pi_1(\prod_{p \in P} X_p) \rightarrow \pi_1(X_p). \]

Then the square can be identified up to weak homotopy equivalence with the square (*) above.

For the proof of Theorem 7.2.1 we will make use of the following lemma about fibrant objects in \((\text{sSet}_0, \pi_1\text{-R.eq.})\). Note that every fibrant object in \((\text{sSet}_0, \text{R.eq.})\) is also fibrant in \((\text{sSet}_0, \pi_1\text{-R.eq.})\).

**Lemma 7.2.2.** Let \( R \) be \( \mathbb{F}_p \) (for some prime \( p \)) or \( \mathbb{Q} \). Let \( Z \) be a reduced Kan complex and let \( p_1 : Z \rightarrow P_1(Z) \) be the map to the first Postnikov truncation of \( Z \). If \( Z \) is fibrant in \((\text{sSet}_0, \pi_1\text{-R.eq.})\), then the homotopy fiber of \( p_1 \) (with respect to the Kan–Quillen model structure) is fibrant in \((\text{sSet}_0, \text{R.eq.})\).

**Proof.** We may assume that \( p_1 : Z \rightarrow P_1(Z) \) is a fibration between fibrant objects in \((\text{sSet}_0, \text{Kan–Quillen})\) (see [13] Section V.6; esp. 6.6–6.8]). It is easy to see directly that \( P_1(Z) \) is fibrant/local in \((\text{sSet}_0, \pi_1\text{-R.eq.})\). By general results on left Bousfield localizations [18] Proposition 3.3.16, it follows that \( Z \) is fibrant in \((\text{sSet}_0, \pi_1\text{-R.eq.})\) if and only if \( p_1 \) is a (local) fibration in \((\text{sSet}_0, \pi_1\text{-R.eq.})\). In this case, the (homotopy) fiber \( Z \) of \( p_1 \), which is simply the universal cover of \( Z \), is also fibrant in \((\text{sSet}_0, \pi_1\text{-R.eq.})\).

Thus it suffices to prove the following special case of the lemma: a simply-connected \( Z \) is fibrant in \((\text{sSet}_0, \pi_1\text{-R.eq.})\) if and only if \( Z \) is fibrant in \((\text{sSet}_0, \text{R.eq.})\). A fibrant replacement \( Z \rightarrow Z_R \) in \((\text{sSet}_0, \text{R.eq.})\) is also a \( \pi_1\text{-R-equivalence} \), because \( Z_R \) is again simply-connected. Since \((\text{sSet}_0, \text{R.eq.})\) is a left Bousfield localization of \((\text{sSet}_0, \pi_1\text{-R.eq.})\), it follows that \( Z_R \) is also fibrant in \((\text{sSet}_0, \pi_1\text{-R.eq.})\); then the map \( Z \rightarrow Z_R \) is a weak homotopy equivalence. Therefore, \( Z \) is fibrant in \((\text{sSet}_0, \text{R.eq.})\), as required. \( \square \)
Proof. (Theorem 7.2.1) We consider the canonical map from the square (†) of Theorem 7.2.1 to the square (††) below obtained after applying the first Postnikov truncation pointwise; explicitly, the Postnikov truncation of (†) can be identified with the following square:

\[
\begin{array}{ccc}
B\pi_1 X & \to & \prod_{p \in P} B\pi_1 X \\
\Delta \downarrow & & \Delta \\
B\pi_1 X & \to & \prod_{p \in P} B\pi_1 X. \\
\end{array}
\]

The square (††) is obviously a homotopy pullback (with respect to the Kan–Quillen model structure). Then consider the induced square which consists of the homotopy fibers of the canonical map of squares from (†) to (††): the resulting square is, up to weak homotopy equivalence, the square of the universal covers of (†). Using Lemma 7.2.2 this square agrees with the classical fracture homotopy pullback of the universal covering $\tilde{X}$ of $X$ (see [30, Theorem 13.1.4], [9, 13]). Then the first claim of Theorem 7.2.1 follows from looking at the long exact sequences of homotopy groups.

For the second claim, we consider again the map of squares from (†) to the (homotopy commutative) square:

\[
\begin{array}{ccc}
B\pi_1 X_0 & \to & \prod_{p \in P} B\pi_1 X_p \\
\pi_1(f) \downarrow & & \pi_1 (g) \\
B\pi_1 X_0 & \to & B\pi_1 Y, \\
\end{array}
\]

which, using the assumptions (iii)–(iv), can be identified up to homotopy equivalence with the homotopy pullback:

\[
\begin{array}{ccc}
B\pi_1 X_0 & \to & \prod_{p \in P} B\pi_1 X_0 \\
\Delta \downarrow & & \Delta \\
B\pi_1 X_0 & \to & \prod_{p \in P} B\pi_1 X_0. \\
\end{array}
\]

Passing to the homotopy fibers of the maps from (†) to (††), we obtain a homotopy pullback:

\[
\begin{array}{ccc}
X' & \to & \prod_{p \in P} \tilde{X}_p \\
\downarrow & & \downarrow \\
X_0 & \to & \tilde{Y}. \\
\end{array}
\]

Then it follows from Lemma 7.2.2 and the corresponding uniqueness property of the classical fracture square (see [30, Theorem 13.1.5]) that (≈) is the classical fracture homotopy pullback of the space $X'$. In particular, $X'$ is simply-connected, so the map $X \to B\pi_1 X_0$ is 2-connected, $X'$ is the universal cover of $X$, and $X \to X_0$ is a $\pi_1$-isomorphism. Therefore, we conclude that the maps $X \to X_0$ and $X \to X_p$ are fibrant replacements in the respective model categories. □
7.3. Three model structures on $sCoCoalg^0_{F}$. The following result proves the existence of three model category structures on $sCoCoalg^0_{F}$ where $F$ is a field. These model structures correspond to the classes of weak equivalences $W_{i}, W_{\hat{i}}, W_{q,i}$, respectively, as discussed in Section 5.

We say that a morphism $f: C \to C'$ in $sCoCoalg^0_{F}$ is injective if the underlying map of simplicial $F$-vector spaces is injective degreewise. We note that a monomorphism in $sCoCoalg^0_{F}$ (or just $CoCoalg^0_{F}$) need not be injective in general (see [2]).

Fix a localization $\iota: S^1 \to S^1$ of $S^1$ and define the localized cobar functor as the composition $\hat{\Omega} = \Omega \circ L \circ X: sCoCoalg^0_{F} \to dgAlg_{F}$ (as in Subsection 5.3).

Theorem 7.3.1. Let $F$ be a field and let $sCoCoalg^0_{F}$ denote the category of connected simplicial cocommutative $F$-coalgebras.

(1) There is a left proper combinatorial model category structure on $sCoCoalg^0_{F}$ with the injective maps as cofibrations and the $\Omega$-quasi-isomorphisms $W_{\Omega}$ as weak equivalences.

We denote this model category by $(sCoCoalg^0_{F}, \Omega{-}q.i.)$.

(2) There is a left proper combinatorial model category structure on $sCoCoalg^0_{F}$ with the injective maps as cofibrations and the $\hat{\Omega}$-quasi-isomorphisms $W_{\hat{\Omega}}$ as weak equivalences. This model category is a left Bousfield localization of the model category in (1).

We denote this model category by $(sCoCoalg^0_{F}, \hat{\Omega}{-}q.i.)$.

(3) There is a left proper combinatorial model category structure on $sCoCoalg^0_{F}$ with the injective maps as cofibrations and the quasi-isomorphisms $W_{q.i}$ as weak equivalences. This model category is a left Bousfield localization of the model category in (2). Moreover, it is induced by the model category of Theorem 3.4.1 restricted to $sCoCoalg^0_{F}$ (for $\kappa = \aleph_0$).

We denote this model category by $(sCoCoalg^0_{F}, q.i.)$.

Proof. We will prove (1) using Theorem 6.1.1. Since every $F$-coalgebra is a filtered colimit of its finite-dimensional subcoalgebras [10], it follows that the categories $CoCoalg^0_{F}$ and $sCoCoalg^0_{F}$ are locally finitely presentable. Let $I$ denote the set of injective maps $i: A \to B$ in $sCoCoalg^0_{F}$ between finitely presentable objects. We claim that $Cof(I)$ consists of the class of injective maps in $sCoCoalg^0_{F}$. Clearly every map in $Cof(I)$ is injective. Conversely, given an injective map $i: C \to D$ in $sCoCoalg^0_{F}$, we apply the small object argument to obtain a factorization:

$$C \overset{j}{\to} E \overset{q}{\to} D,$$

such that $j \in Cof(I)$ and $q \in I{-}\text{inj}$. Then we consider the following lifting problem:

$$\begin{array}{cccc}
C & \overset{j}{\to} & E & \\
\downarrow & & \downarrow & \\
D & \overset{q}{\to} & D
\end{array}$$

and the poset $Z$ whose elements are pairs $(Z, h)$ given by an “intermediate” simplicial coalgebra $(C \subseteq Z \subseteq D)$ and an “intermediate” solution to the lifting problem $h: Z \to E$. We define $(Z_1, h_1) \leq (Z_2, h_2)$ if $Z_1 \subseteq Z_2$ and $h_2$ extends $h_1$. Clearly
\[ Z \neq \emptyset \] and every chain in \( Z \) has an upper bound. By Zorn’s lemma, \( Z \) has a maximal element \((C', \varepsilon)\). If \( C' \neq D \), then there is a simplicial finite-dimensional subcoalgebra \( D' \subseteq D \) such that the injective map \( C' \cap D' \subseteq D' \) is in \( I \) and \( C' \cap D' \neq D' \); the existence of \( D' \) uses the fact that every \( \mathbb{F} \)-coalgebra is the filtered colimit of its finite-dimensional subcoalgebras. Since \( q \in I - \text{inj} \), we can then extend \( h \) to the simplicial “intermediate” subcoalgebra \( C' \subseteq C' \cap C' \cap D' \subseteq D \), contradicting the maximality of \((C', \varepsilon)\). Therefore, \( C' = D \) and the lift \( h \) exhibits \( i \) as a retract of \( j \), so \( i \in \text{Cof}(I) \), as claimed.

Let \( \text{dgAlg}_F \) be the finitely combinatorial model category of Theorem 3.5.2. We will apply the method of Theorem 6.1.1 to the functor \( F \): \( \text{sCoCoalg}^0_F \to \text{dgAlg}_F \). We recall that \( F \) preserves all colimits; therefore, it is finitely accessible. In addition, using the explicit description of the cofibrations in \( \text{dgAlg}_F \), we see directly that for any injective map \( i: A \to B \) in \( \text{sCoCoalg}^0_F \), the map \( \Omega(i) : \Omega(A) \to \Omega(B) \) is a cofibration in the model category \( \text{dgAlg}_F \) (cf. [34, Section 9]).

Condition (2) of Theorem 6.1.1 is satisfied using Proposition 6.2.1 and the fact that \( F \) preserves colimits. Condition (1) of Theorem 6.1.1 will be shown using the criterion in Proposition 6.2.2 for the standard mapping cylinder factorization. Namely, the cylinder object is given by the functor

\[ \text{Cyl}: \text{sCoCoalg}^0_F \to \text{sCoCoalg}^0_F, \ C \mapsto \text{Cyl}(C), \]

defined by the pushout

\[
\begin{array}{ccc}
\mathbb{F} \otimes \mathbb{F}[\Delta^1] & \xrightarrow{e \otimes \text{id}} & \mathbb{F} \otimes \mathbb{F}[\Delta^1] \\
\downarrow & & \downarrow \\
\mathbb{F} \otimes \mathbb{F}[\Delta^0] & \cong & \mathbb{F} \\
\text{Cyl}(C) & \xrightarrow{h} & \text{Cyl}(C)
\end{array}
\]

where \( \mathbb{F} \) is considered as a constant simplicial \( \mathbb{F} \)-coalgebra and \( e: \mathbb{F} \to C \) denotes the (implicit) coaugmentation map. In other words, as simplicial \( \mathbb{F} \)-vector space, we have:

\[ \text{Cyl}(C) = \mathbb{F} \otimes \text{coker}(e \otimes \text{id}_{\mathbb{F}[\Delta^1]} : \mathbb{F} \otimes \mathbb{F}[\Delta^1] \to \mathbb{F} \otimes \mathbb{F}[\Delta^1]). \]

The \( n \)-simplices of \( \Delta^1 \) may be labeled by sequences \([0\ldots011\ldots1]\) consisting of \( r \) consecutive \( 0 \)'s and \( s \) consecutive \( 1 \)'s for some non-negative integers \( r \) and \( s \) satisfying \( r + s = n + 1 \). We denote any such simplex by \([0^n1^s] \in (\Delta^1)_n \). Using this notation, define natural inclusion maps \( i_\varepsilon : C \to \text{Cyl}(C) \), for \( \varepsilon = 0, 1 \), by \( i_\varepsilon(x) = x \otimes [\varepsilon]^{n+1} \) for any \( x \in C_n \). We also have a natural projection map \( q: \text{Cyl}(C) \to C \) determined by \( q(x \otimes [0^n1^s]) = x \).

For every map \( f: C \to C' \) in \( \text{sCoCoalg}^0_F \), we define the mapping cylinder of \( f \) by the pushout

\[
\begin{array}{ccc}
C & \xrightarrow{0 \oplus \text{id}_C} & C \oplus_F C \\
\downarrow^{f} & & \downarrow^{\text{id}_C \oplus f} \\
C' & \xrightarrow{\delta} & \text{M}(f).
\end{array}
\]

Then there is a canonical factorization of \( f \),

\[ C \xrightarrow{\delta = h \circ i} \text{M}(f) \overset{p}{\to} C', \]

where \( p \) is induced by \( f \), the identity map on \( C' \), and \( \text{Cyl}(C) \overset{j}{\to} C \overset{f}{\to} C' \).
We claim that the conditions (a)–(b) of Proposition 6.2.2 are satisfied for this factorizations. First, we see from the construction that the map \( i \) is injective (because \( s \) is). Moreover, assuming that \( C \) and \( C' \) are finitely presentable in \( s\text{CoCoalg}_F^0 \), then so is \( M(f) \); so Proposition 6.2.2(a) is satisfied.

For Proposition 6.2.2(b), we will show that the map \( p: M(f) \to C' \) becomes a trivial fibration in \( \text{dgAlg}_F \) after applying \( \Omega \). First, since \( p \) admits a section \( s' : C' \to C \oplus_F C' \), it follows that \( \Omega(p) \) is surjective (a fibration in \( \text{dgAlg}_F \)). Then it suffices to show that \( s' \) is an \( \Omega \)-quasi-isomorphism. To see this, it suffices to prove that \( \Omega(i_1) : \Omega(C) \to \Omega(\text{Cyl}(C)) \) is a trivial cofibration of dg algebras, since \( \Omega(s') \) is a pushout of this map. Moreover, since \( q \circ i_1 = \text{id}_C \), it suffices to observe that the map \( q : \text{Cyl}(C) \to C \) is an \( \Omega \)-quasi-isomorphism. We prove this in Proposition A.1 in Appendix A.

To prove (2) we will apply Theorem 6.2.5 to the functor \( \hat{\Omega} : s\text{CoCoAlg}_F^0 \to \text{dgAlg}_F \) with respect to the model category structures of (1) and Theorem 3.5.2. First, it is easy to see that \( \hat{\Omega} = \Omega \circ L_\iota \circ \mathcal{X} \) is the composition of finitely accessible functors; therefore, \( \hat{\Omega} \) is finitely accessible. By Proposition 5.3.6, we have \( W_\Omega \subset W_{\hat{\Omega}} \), so \( \hat{\Omega} \) preserves weak equivalences. Thus, the assumptions of Theorem 6.2.5 are satisfied.

It remains to show that the class \( \text{Cof}(I) \cap W_{\hat{\Omega}} \) is closed under pushouts and transfinite compositions. Note that \( \hat{\Omega} \) preserves cofibrations: if \( i : A \to B \) is an injective map in \( s\text{CoCoAlg}_F^0 \), then so is the induced map \( (L_\iota \mathcal{X})(i) \), and hence \( \hat{\Omega}(i) \) is a cofibration. Then, since \( \hat{\Omega} \) commutes with transfinite compositions, it follows that \( \text{Cof}(I) \cap W_{\hat{\Omega}} \) is closed under transfinite compositions. We will have to argue differently for the case of pushouts, because \( \hat{\Omega} \) does not preserve pushouts in general. Let \( i : A \to B \) be in \( \text{Cof}(I) \cap W_{\hat{\Omega}} \) and \( f : A \to C \) an arbitrary map in \( s\text{CoCoAlg}_F^0 \). We must show that the horizontal map in the diagram below, induced by the injective map \( C \to B \oplus_A C \), is an \( \Omega \)-quasi-isomorphism:

\[
\begin{array}{ccc}
L_\iota(\mathcal{X}(C)) & \longrightarrow & L_\iota(\mathcal{X}(B \oplus_A C)) \\
\downarrow & & \downarrow \\
L_\iota(\mathcal{X}(B)) \oplus_{L_\iota(\mathcal{X}(A))} L_\iota(\mathcal{X}(C)) & & \\
\end{array}
\]

Note that \( L_\iota \circ \mathcal{X} \) does not preserve pushouts, since there might be set-like elements in \( B \oplus_A C \) (in degree 1) that do not come from \( B \) or \( C \). However, the vertical map in the above diagram is still an \( \hat{\Omega} \)-quasi-isomorphism, since both simplicial coalgebras differ by pushouts along the map \( F[d] : F[S^1] \to F[S^1] \), one for each missing set-like element. Moreover, the diagonal map in the diagram is an \( \Omega \)-quasi-isomorphism (and, consequently, an \( \hat{\Omega} \)-quasi-isomorphism by Proposition 5.3.6) since the injective map \( L_\iota(\mathcal{X}(A)) \to L_\iota(\mathcal{X}(B)) \) is an \( \Omega \)-quasi-isomorphism by assumption. It follows that the horizontal map is an \( \hat{\Omega} \)-quasi-isomorphism. Since the set-like elements of degree 1 in the domain and target of the map are already inverted, the map is an \( \Omega \)-quasi-isomorphism, as required. This completes the proof of (2).
The proof of (3) is obtained similarly (and more easily) by applying Theorem 6.2.5 to the normalized chains functor,

\[ (\text{CoCoalg}_F^0, \hat{\Omega}-\text{q.i.}) \xrightarrow{\mathcal{N}} \text{dgCoalg}_F^c \xrightarrow{\text{forgetful}} \text{Ch}_F, \]

where \( \text{Ch}_F \) denotes the category of chain complexes of \( F \)-vector spaces, equipped with the usual combinatorial model structure that is determined by the monomorphisms and the quasi-isomorphisms. Note that the normalized chains functor preserves weak equivalences, since \( W_{\hat{\Omega}} \subset W_{\text{q.i.}} \) by Proposition 5.3.6.

\[ \square \]

Remark 7.3.2. The proof of Theorem 7.3.1(3) is based on the existence of the model category \( (\text{CoCoalg}_F^0, \hat{\Omega}-\text{q.i.}) \) and differs from the proof of Theorem 3.4.1 given in [38]. On the other hand, following directly the proof of Theorem 3.4.1 (based on Theorem 6.1.1 and Propositions 6.2.1–6.2.2) and using the mapping cylinder factorization from the proof of Theorem 7.3.1(1), one can show that the model category of Theorem 7.3.1(3) exists for arbitrary commutative rings \( R \).

We note that the proof of Theorem 7.3.1(1) used special properties about simplicial coalgebras and dg algebras over fields; for instance, that an injection \( A \to B \) induces a cofibration of dg algebras \( \hat{\Omega}(A) \to \hat{\Omega}(B) \).

The three pairs of model categories established in Theorems 7.1.1 and 7.3.1 fit into Quillen adjunctions as follows.

Proposition 7.3.3. Let \( F \) be a field. The adjunctions:

\[ (\text{CoCoalg}_F^0, \hat{\Omega}-\text{q.i.}) \rightleftarrows (\text{dgCoalg}_F^c, \hat{\Omega}-\text{q.i.}) \]

are Quillen adjunctions.

Proof. The functor \( F[-] : \text{sSet}_0 \to \text{CoCoalg}_F^0 \) sends monomorphisms to injective maps, that is, \( F[-] \) preserves cofibrations. Then the result follows from Proposition 5.2.4, Theorems 5.3.4, 7.1.1 and 7.3.1.

\[ \square \]

Remark 7.3.4. The left adjoint \( \hat{\Omega} : \text{CoCoalg}_F^0 \to \text{dgAlg}_F^c \) is a left Quillen functor with respect to the model category structures of Theorem 7.3.1(1) and Theorem 7.5.2.

8. Comparison between \text{sSet}_0 and \text{CoCoalg}_F^0

8.1. Coalgebraic preliminaries. We recall some fundamental results about the structure theory of coalgebras over a perfect field. We refer the reader to [46] [124] for more details.

Let \( F \) be a field. A cocommutative \( F \)-coalgebra \( A \) is simple if it has no non-trivial subcoalgebras; simple \( F \)-coalgebras are necessarily finite-dimensional. The étale subcoalgebra \( \text{Ét}(A) \) of a cocommutative \( F \)-coalgebra \( A \) is the (direct) sum of all simple subcoalgebras of \( A \).

If \( F \) is algebraically closed, then \( F \) itself is the only simple \( F \)-coalgebra up to isomorphism. In this case, the étale subcoalgebra of a cocommutative \( F \)-coalgebra can be identified with the canonical counit map

\[ F[\mathcal{P}(A)] \subseteq A \]
which is associated with the adjunction $\mathbb{F}[-] : \text{Set} \rightleftarrows \text{CoCoalg}_{\mathbb{F}} : \mathcal{P}$.

More generally, if $\mathbb{F}$ is a perfect field, the étale subcoalgebra of a cocommutative $\mathbb{F}$-coalgebra $A$ can be identified as follows. Let $\overline{\mathbb{F}} = \mathbb{F}$ be the algebraic closure of $\mathbb{F}$ and let $G$ denote the (profinite) absolute Galois group. Then the étale subcoalgebra of $\overline{A} = A \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is canonically identified with the $\overline{\mathbb{F}}$-subcoalgera $\overline{\mathbb{F}}[\mathcal{P}(\overline{A})]$ generated by the $\overline{\mathbb{F}}$-points of $\overline{A}$. Moreover, this set of $\overline{\mathbb{F}}$-points

$$\mathcal{P}(\overline{A}) = \text{CoCoalg}_{\overline{\mathbb{F}}}(\overline{A}, A \otimes_{\mathbb{F}} \overline{\mathbb{F}})$$

admits a natural $G$-action, which extends to a $G$-action on the $\overline{\mathbb{F}}$-coalgebra $\overline{\mathbb{F}}[\mathcal{P}(\overline{A})]$; this $G$-action is compatible with the $G$-action on $\overline{A}$. The $G$-fixed points of $\overline{\mathbb{F}}[\mathcal{P}(\overline{A})]$ define an $\mathbb{F}$-subcoalgebra of $A$ and there is a natural isomorphism of $\mathbb{F}$-coalgebras:

$$\text{Et}(A) \cong \mathbb{F}[\mathcal{P}(\overline{A})]^G. \tag{8.1}$$

The inclusion of the étale subcoalgebra can be expressed also in terms of the counit of an adjunction. Let $\text{Set}(G)$ denote the category of $G$-sets (= discrete topological spaces with a continuous $G$-action). We recall that a $G$-action on a set $X$ is continuous if and only if the stabilizer of each element $x \in X$ is an open subgroup of $G$. This is equivalent to requiring that

$$X = \colim_{H \in \mathcal{O}_G^{op}} X^H, \tag{8.2}$$

where $X^H$ denotes the $H$-fixed points of $X$ and $\mathcal{O}_G^{opp}$ denotes the opposite of the category of open subgroups $H$ of $G$ with inclusions as morphisms. Note that the poset $\mathcal{O}_G^{opp}$ is filtered. We recall that any open subgroup of a profinite group has finite index.

For any $G$-set $S$, the $G$-fixed points of the $\mathbb{F}$-coalgebra $\mathbb{F}[S]$ form an $\mathbb{F}$-coalgebra (cf. (8.3) below), so we obtain a functor

$$\mathbb{F}[-]^G : \text{Set}(G) \to \text{CoCoalg}_{\mathbb{F}},$$

with admits a right adjoint, given by a functor of $\mathbb{F}$-points, defined by

$$\mathcal{P}_G : \text{CoCoalg}_{\mathbb{F}} \to \text{Set}(G), \quad A \mapsto \text{CoCoalg}_{\mathbb{F}}(\mathbb{F}, A \otimes_{\mathbb{F}} \mathbb{F}).$$

Then the inclusion of the étale subcoalgebra $\text{Et}(A) \subseteq A$ can be identified with the counit of this adjunction. For any $G$-set $S$, there is an isomorphism of $\mathbb{F}$-coalgebras (cf. [13, Lemma 4.3]):

$$\mathbb{F}[S]^G \otimes_{\mathbb{F}} \mathbb{F} \cong \mathbb{F}[S] \tag{8.3}$$

and therefore the unit transformation of the adjunction,

$$S \to \mathcal{P}_G(\mathbb{F}[S]^G),$$

is a natural isomorphism. In particular, the left adjoint $\mathbb{F}[-]^G$ is full and faithful.

We will need the following fundamental property of the étale subcoalgebra only in the case where $\mathbb{F}$ is algebraically closed. For the proof, we refer the reader to [13, Section 2] and [15, Section 2].

**Theorem 8.1.1.** Let $A$ be a cocommutative $\mathbb{F}$-coalgebra over a perfect field $\mathbb{F}$. The inclusion $\text{Et}(A) \subseteq A$ admits a natural retraction of coalgebras.
8.2. Algebraically closed fields. Based on Theorem \S 8.1.1, we can now apply the

criterion of Proposition 6.3.3 to the Quillen adjunctions of Proposition 7.3.3.

Theorem 8.2.1. Let \( \mathbb{F} \) be an algebraically closed field. The left Quillen functors

\begin{align*}
(1) \quad \mathbb{F}[-] : (sSet_{0}, \mathbb{F} \text{-cat.eq.}) & \to (\text{CoCoalg}_{\mathbb{F}}^{0}, \Omega \text{-q.i.}) \\
(2) \quad \mathbb{F}[-] : (sSet_{0}, \pi_{1}^{-}\mathbb{F} \text{-eq.}) & \to (\text{CoCoalg}_{\mathbb{F}}^{0}, \widehat{\Omega} \text{-q.i.}) \\
(3) \quad \mathbb{F}[-] : (sSet_{0}, \mathbb{F} \text{-eq.}) & \to (\text{CoCoalg}_{\mathbb{F}}^{0}, \text{q.i.})
\end{align*}

are homotopically full and faithful.

Proof. This is similar to the proof in [13] which essentially treated case (3). For any field \( \mathbb{F} \), the functor \( \mathbb{F}[-] \) is (1-categorically) full and faithful. Thus, by Proposition 6.3.3, it suffices to show that \( P : \text{CoCoalg}_{\mathbb{F}}^{0} \to sSet_{0} \) preserves weak equivalences in each case. Given a weak equivalence \( f : C \to C' \) in \( \text{CoCoalg}_{\mathbb{F}}^{0} \) (in any of the three cases) and assuming that \( \mathbb{F} \) is algebraically closed, then it follows from Theorem 8.1.1 that the map \( \mathbb{F}[P(f)] \) is a retract of \( f \). Therefore, \( \mathbb{F}[P(f)] \) is again a weak equivalence. Then, \( P(f) \) is a weak equivalence (in any of the three cases) as required, using Proposition 5.2.4 (for (1)), Theorem 5.3.4 (for (2)), and Proposition 5.1.3 (for (3)). \( \square \)

Remark 8.2.2. An analogue of Theorem 8.2.1(3) for simplicial presheaves with

respect to the local model structures was shown in [35] and for the motivic homotopy

theory in [15].

Corollary 8.2.3. Let \( \mathbb{F} \) be an algebraically closed field and let \( X \) and \( Y \) be reduced

simplicial sets.

\begin{enumerate}
\item Suppose \( \mathbb{F}[X] \cong \mathbb{F}[Y] \) in \( \text{Ho}(\text{CoCoalg}_{\mathbb{F}}^{0}, \Omega \text{-q.i.}) \). Then \( X \cong Y \) in \( \text{Ho}(sSet_{0}, \mathbb{F} \text{-cat.eq.}) \).
\item Suppose \( \mathbb{F}[X] \cong \mathbb{F}[Y] \) in \( \text{Ho}(\text{CoCoalg}_{\mathbb{F}}^{0}, \widehat{\Omega} \text{-q.i.}) \). Then \( X \cong Y \) in \( \text{Ho}(sSet_{0}, \pi_{1}^{-}\mathbb{F} \text{-eq.}) \).
\item Suppose \( \mathbb{F}[X] \cong \mathbb{F}[Y] \) in \( \text{Ho}(\text{CoCoalg}_{\mathbb{F}}^{0}, \text{q.i.}) \). Then \( X \cong Y \) in \( \text{Ho}(sSet_{0}, \mathbb{F} \text{-eq.}) \).
\end{enumerate}

Remark 8.2.4. A version of Corollary 8.2.3 was shown in [12] assuming that

\( X \) and \( Y \) are Kan complexes and using the standard (non-localized) cobar functor.

8.3. Perfect fields. Let \( \mathbb{F} \) be a perfect field, let \( \mathbb{F} \subseteq \overline{\mathbb{F}} \) be its algebraic closure and let \( G \) denote the (profinite) absolute Galois group of \( \mathbb{F} \). The analogue of Theorem 8.2.1 for perfect fields applies to the adjunction:

\[ \mathbb{F}[-]^{G} : sSet(G)_{0} \rightleftarrows \text{CoCoalg}_{G}^{0} : P_{G}. \]

Here \( sSet(G)_{0} \) denotes the category of reduced simplicial \( G \)-sets. We recall that the left adjoint \( \mathbb{F}[-]^{G} \) is full and faithful.

In analogy with Theorem 7.1.1 we construct three model structures on \( sSet(G)_{0} \)

for any profinite group \( G \). For a fixed commutative ring \( R \), we say that a map \( f : S \to S' \) in \( sSet(G)_{0} \) is an \( R \)-categorical equivalence (resp. \( \pi_{1}^{-}R \)-equivalence, \( R \)-equivalence) if the underlying map of reduced simplicial sets is so. We denote by

\[ \delta : sSet_{0} \rightleftarrows sSet(G)_{0} : (-)^{G} \]
the adjunction induced by the trivial $G$-action functor $\delta$ and the $G$-fixed points functor $(-)^G$.

**Theorem 8.3.1.** Let $R$ be a commutative ring and $G$ a profinite group.

1. There is a left proper combinatorial model category structure on $sSet(G)_0$ with the monomorphisms as cofibrations and the $R$-categorical equivalences, denoted by $W(G)_{1,R}$, as weak equivalences. We denote this model category by $(sSet(G)_0, R$-cat.eq.).

2. There is a left proper combinatorial model category structure on $sSet(G)_0$ with the monomorphisms as cofibrations and the $\pi_1$-$R$-equivalences, denoted by $W(G)_{\pi_1,R}$, as weak equivalences. This model category is a left Bousfield localization of the model category in (1). We denote this model category by $(sSet(G)_0, \pi_1$-$R$-eq.).

3. There is a left proper combinatorial model category structure on $sSet(G)_0$ with the monomorphisms as cofibrations and the $R$-equivalences, denoted by $W(G)_R$, as weak equivalences. This model category is a left Bousfield localization of the model category in (2). We denote this model category by $(sSet(G)_0, R$-eq.).

Moreover, the adjunctions

1. $\delta: (sSet_0, R$-cat.eq.) $\rightleftarrows (sSet(G)_0, R$-cat.eq.): $(-)^G$

2. $\delta: (sSet_0, \pi_1$-$R$-eq.) $\rightleftarrows (sSet(G)_0, \pi_1$-$R$-eq.): $(-)^G$

3. $\delta: (sSet_0, R$-eq.) $\rightleftarrows (sSet(G)_0, R$-eq.): $(-)^G$

are Quillen adjunctions.

**Proof.** We will apply Theorem 6.1.1 to the forgetful functor $U: sSet(G)_0 \to sSet_0$ and the model categories of Theorem 7.1.1. First note that $U$ preserves colimits; in particular, $U$ is accessible.

We recall that the larger category $sSet(G)$ is a (category of simplicial objects in a) Grothendieck topos; either directly or by using this fact, it is easy to conclude that $sSet(G)_0$ is also locally presentable. Moreover, the class of monomorphisms in $sSet(G)$, as in any topos, is cofibrantly generated by a set of monomorphisms $\overline{I}$. For every $i: A \to B$ in $\overline{I}$, let $i_0: A_0 \to B_0$ be the induced monomorphism in $sSet(G)_0$, where $(-)_0$ denotes the 0-simplices as a constant simplicial object. Let $\overline{I}$ denote the set of monomorphisms obtained from the morphisms in $\overline{I}$ in this way. We observe that a map in $sSet(G)$ between reduced simplicial $G$-sets has the right lifting property with respect to $\overline{I}$ if and only if it is has the right lifting property with respect to $I$. Therefore, by the retract argument, the set of monomorphisms $\overline{I}$ is a generating set for the monomorphisms in $sSet(G)_0$. In more detail, for every monomorphism $f$ in $sSet(G)_0$, there is a factorization $f = pi$ where $i \in Cof(\overline{I})$ and $p \in \overline{I} - inj$ using the small object argument; then, as observed above, $p$ has the right lifting property with respect to $f$, hence $f$ is a retract of $i$.

It suffices to prove that $\overline{I} - inj$ consists of categorical equivalences. Then the result will follow from Theorem 6.1.1 (for all three cases simultaneously).

Let $p: X \to Y$ be a morphism in $sSet(G)_0$ which has the right lifting property with respect to the monomorphisms. Using similar arguments as above, it follows that $p$ has the right lifting property in $sSet(G)$ with respect to the monomorphisms
in $\text{sSet}(G)$. For every open subgroup $H$ of $G$ and any $n \geq 0$, the map $i \times \text{id} : \partial \Delta^n \times G/H \to \Delta^n \times G/H$ is a monomorphism in $\text{sSet}(G)$. Thus the right lifting property of $p$ implies that the map
\[ p^H : X^H \to Y^H \]
is a trivial fibration for any such $H$. Using the continuity of the $G$-action on $X$ and $Y$, it follows that $p$ is the filtered colimit of the maps $p^H$ where $H$ ranges over the open subgroups of $G$ (with finite index). Since filtered colimits of trivial fibrations are again trivial fibrations, we conclude that $p$ is a trivial fibration between the underlying simplicial sets, therefore also a categorical equivalence, as required.

It now follows immediately from the definition of cofibrations and weak equivalences that the adjunction
\[ \delta : \text{sSet}_0 \rightleftarrows \text{sSet}(G)_0 : (-)^G \]
becomes a Quillen adjunction in all three contexts. \hfill \square

**Remark 8.3.2.** The model category structure on $\text{sSet}(G)$ analogous to Theorem 8.3.1 (3), as well as to (2) for the case $R = \mathbb{Z}$, was also constructed by Goerss [12]. Note that the above proof works for any class of weak equivalences $\mathcal{W}$ in $\text{sSet}(G)_0$ (or in $\text{sSet}(G)$) such that: (i) $\mathcal{W}$ is determined by the weak equivalences in a combinatorial model category using a colimit–preserving functor and (ii) $\mathcal{W}$ contains the maps which are categorical equivalences between the underlying simplicial sets.

**Proposition 8.3.3.** Let $F$ be a perfect field with algebraic closure $F \subseteq \overline{F}$ and (profinite) absolute Galois group $G$. The adjunctions
\[ \overline{F}[-]^G : (\text{sSet}(G)_0, \text{F-cat.eq.}) \rightleftarrows (\text{sCoCoalg}_{\overline{F}}^0, \Omega_{\text{q.i.}}) : \mathcal{P}_G \]

(1) \[ \overline{F}[-]^G : (\text{sSet}(G)_0, \pi_1, \text{F-equ.}) \rightleftarrows (\text{sCoCoalg}_{\overline{F}}^0, \hat{\Omega}_{\text{q.i.}}) : \mathcal{P}_G \]

(2) \[ \overline{F}[-]^G : (\text{sSet}(G)_0, \pi_1, \text{F-equ.}) \rightleftarrows (\text{sCoCoalg}_{\overline{F}}^0, \text{q.i.}) : \mathcal{P}_G \]

(3) are Quillen adjunctions.

**Proof.** The left adjoint functor clearly preserves cofibrations. Note that the three notions of weak equivalence in $\text{sSet}(G)_0$ (or just $\text{sSet}_0$) are the same if we replace $F$ by $\overline{F}$. Moreover, the functor
\[ \neg \otimes F : \text{sCoCoalg}_{\overline{F}}^0 \to \text{sCoCoalg}_{\overline{F}}^0 \]

preserves and detects each one of the three notions of weak equivalence. In fact, for (1) and (2) this follows since for any $C \in \text{sCoCoalg}_{\overline{F}}^0$, we have an isomorphism $\Omega(C \otimes F) \cong \Omega(C) \otimes \overline{F}$ and for (3) it is just a consequence of the fact that tensoring with a field preserves and detects quasi-isomorphisms. Then (8.3.3) shows that the left adjoints preserve weak equivalences. \hfill \square

We can now state the analogue of Theorem 8.2.1 for perfect fields.

**Theorem 8.3.4.** Let $F$ be a perfect field and let $\overline{F} \subseteq \overline{F}$ be its algebraic closure with (profinite) absolute Galois group $G$. Then the left Quillen functors
\[ \overline{F}[-]^G : (\text{sSet}(G)_0, \text{F-cat.eq.}) \to (\text{sCoCoalg}_{\overline{F}}^0, \Omega_{\text{q.i.}}) \]

(1) \[ \overline{F}[-]^G : (\text{sSet}(G)_0, \pi_1, \text{F-equ.}) \to (\text{sCoCoalg}_{\overline{F}}^0, \hat{\Omega}_{\text{q.i.}}) \]

(2) \[ \overline{F}[-]^G : (\text{sSet}(G)_0, \pi_1, \text{F-equ.}) \to (\text{sCoCoalg}_{\overline{F}}^0, \text{q.i.}) \]

(3)
are homotopically full and faithful.

**Proof.** The proof is similar to the proof of Theorem 8.2.1, now using the fact that the functor $F[-]^G$ is (1-categorically) full and faithful (see the discussion in 8.1). In fact, by Proposition 6.3.3 it suffices to show that $P_G: \sCoCoalg^0_F \to \sSet(G)_0$ preserves weak equivalences in each of the cases.

Given a weak equivalence $f: C \to C'$ in $\sCoCoalg^0_F$, then so is the induced map $f \otimes \text{id}: C \otimes_F \overline{F} \to C' \otimes_F \overline{F}$ (in any of the three cases). By Theorem 8.1.1 (for $\overline{F}$), it follows that the map $\overline{F}[P(f \otimes \text{id})]$ is a retract of $f \otimes \text{id}$, since this is the map between the étale subcoalgebras degreewise. So $\overline{F}[P(f \otimes \text{id})]$ is a weak equivalence (in any of the three cases). But $\overline{F}[P(f \otimes \text{id})]$ is simply the underlying map of $P_G(f)$ between the simplicial $\overline{F}$-coalgebras (after forgetting the $G$-action). This means that $P_G(f)$ is a weak equivalence (in any of the three cases) as required, using Proposition 5.2.4 (for (1)), Theorem 5.3.4 (for (2)), and Proposition 5.1.3 (for (3); we note that this case is due to Goerss [13].) □

**Remark 8.3.5.** An analogue of Theorem 8.3.4(3) for simplicial presheaves with respect to the local model structures was shown in [38] and for the motivic homotopy theory in [15].

The Quillen adjunction $F[-]: \sSet_0 \rightleftarrows \sCoCoalg^0_F: \mathcal{P}$ factors as the composition of two adjunctions:

$$
\delta: \sSet_0 \rightleftarrows \sSet(G)_0: (-)^G
$$

$$
\overline{F}[\cdot]^G: \sSet(G)_0 \rightleftarrows \sCoCoalg^0_F: \mathcal{P}_G
$$

and these are Quillen adjunctions in any of the three cases considered, as stated in Theorem 8.3.1 and Proposition 8.3.3. Since the derived unit transformation of the last Quillen adjunction is a natural isomorphism by Theorem 8.3.4 in each one of the three cases, we obtain the following identification of the derived unit transformation of the composite Quillen adjunction.

**Corollary 8.3.6.** Let $\mathbb{F}$ be a perfect field with algebraic closure $\overline{\mathbb{F}} \subseteq \overline{\mathbb{F}}$ and (profinite) absolute Galois group $G$.

1. The derived unit transformation of the Quillen adjunction

$$
\overline{F}[\cdot]: (\sSet_0, \text{F-cat.eq.}) \rightleftarrows (\sCoCoalg^0_F, \Omega\text{-q.i.}): \mathcal{P}
$$

is canonically identified with the derived unit transformation of the Quillen adjunction

$$
\delta: (\sSet_0, \text{F-cat.eq.}) \rightleftarrows (\sSet(G)_0^F, \text{F-cat.eq.}): (-)^G.
$$

2. The derived unit transformation of the Quillen adjunction

$$
\overline{F}[\cdot]: (\sSet_0, \pi_1\text{-F.eq.}) \rightleftarrows (\sCoCoalg^0_F, \text{F\text{}-q.i.}): \mathcal{P}
$$

is canonically identified with the derived unit transformation of the Quillen adjunction

$$
\delta: (\sSet_0, \pi_1\text{-F.eq.}) \rightleftarrows (\sSet(G)_0^F, \pi_1\text{-F.eq.}): (-)^G.
$$

3. The derived unit transformation of the Quillen adjunction

$$
\overline{F}[\cdot]: (\sSet_0, \text{F\text{}-eq.}) \rightleftarrows (\sCoCoalg^0_F, \text{q.i.}): \mathcal{P}
$$

is canonically identified with the derived unit transformation of the Quillen adjunction

$$
\delta: (\sSet_0, \text{F\text{}-eq.}) \rightleftarrows (\sSet(G)_0^F, \text{F\text{}-eq.}): (-)^G.
$$
is canonically identified with the derived unit transformation of the Quillen adjunction
\[ \delta : (s\text{Set}_0, \mathbb{F}]-\mathbb{E}q.) \rightleftarrows (s\text{Set}(G)^0, \mathbb{F}]-\mathbb{E}q.) : (-)^G \]

In other words, the derived unit transformation is identified in each case with the canonical map into the homotopy G-fixed points \( X \rightarrow (\delta(X))^hG \) (where \((-)^hG\) is interpreted in the appropriate way in each model category).

Remark 8.3.7. In [13 Proposition 1.5] the space of homotopy fixed points \( \delta(X)^hG \) corresponding to Corollary 8.3.6(3) is described in more detail in certain special cases. For instance, if \( G \) is a finite group and \( X \) is fibrant in \( (s\text{Set}_0, \mathbb{F}]-\mathbb{E}q.) \), then \( \delta(X)^hG \) is the usual homotopy fixed point space of \( X \).

When \( \mathbb{F} = \mathbb{F}_p \) and \( X \) is simply–connected, then \( \delta(X)^hG \) is the free loop space of the \( p \)-completion of \( X \). If \( \mathbb{F} \) has characteristic zero and \( X \) is simply–connected, then \( \delta(X)^hG \) is the rational localization of \( X \). Analogous results hold also in the context of Corollary 8.3.6(2) using Lemma [12].

Appendix A. Cylinder objects for simplicial coalgebras

Let \( R \) be a commutative ring. For any \( C \in s\text{CoCoalg}_R^0 \), define \( \text{Cyl}(C) \) as in the pushout diagram \( (Cyl) \) in the proof of Theorem 7.3.1 considered in the category of simplicial cocommutative \( R \)-coalgebras. In this appendix, we prove the following result that was used in the proof of Theorem 7.3.1(1).

Proposition A.1. For any \( C \in s\text{CoCoalg}_R^0 \), the natural projection map
\[ q : Cyl(C) \rightarrow C \]
is an \( \Omega \)-quasi-isomorphism.

For simplicity, denote by \( q = \mathcal{N}_*(q) : \mathcal{N}_*(Cyl(C)) \rightarrow \mathcal{N}_*(C) \) the map on normalized chains induced by the projection \( q : Cyl(C) \rightarrow C \), and by \( i_1 = \mathcal{N}_*(i_1) : \mathcal{N}_*(C) \rightarrow \mathcal{N}_*(Cyl(C)) \) the map induced by the inclusion \( i_1 : C \rightarrow Cyl(C) \). To prove the above proposition, we will:

- construct a chain homotopy \( H : \mathcal{N}_*(Cyl(C)) \rightarrow \mathcal{N}_{*+1}(Cyl(C)) \) between the composition \( i_1 \circ q \) and the identity map \( \text{id}_{\mathcal{N}_*(Cyl(C))} \), and then
- extend \( H \) to a chain homotopy \( H_0 : \Omega(Cyl(C)) \rightarrow \Omega(Cyl(C)) \) between the identity map \( \text{id}_{\Omega(Cyl(C))} \) and the composition \( \Omega(i_1) \circ \Omega(q) \). This involves verifying that \( H \) is a \((i_1 \circ q, \text{id})\)-coderivation, i.e. that the equation
\[ (H \otimes (i_1 \circ q) + \text{id} \otimes H) \circ \Delta = \Delta \circ H \]
is satisfied.

Using the notation introduced in the proof of Theorem 7.3, we may represent any element in \( \text{Cyl}(C)_n \) by a linear combination of elements of the form \( x \otimes [0^{k+1}1^{n-k}] \). For simplicity we will write \( x \otimes [0^{k+1}1^{n-k}] = (x, [0^{k+1}1^{n-k}]) \). Note that
\[ [0^{k+1}1^{n-k}] = s_{n-1} \ldots s_k \ldots s_0[01], \]
where the \( s_j \)'s denote simplicial degeneracy maps, \([01] \in (\Delta^1)_1 \) is the unique non-degenerate 1-simplex, and \( \hat{s}_k \) means we omit this map from the composition.
Define a degree +1 map

\[ H : \mathcal{N}_*(\text{Cyl}(C)) \to \mathcal{N}_{*+1}(\text{Cyl}(C)) \]

given on any representative \((x, [0^{k+1}1^n]) \in \text{Cyl}(C)_n\) by the formula

\[
(A.1.1) \quad H(x, [0^{k+1}1^n]) := \sum_{j=0}^{k} (-1)^j (s_{k-j}x, [0^{k-j+1}1^{n-k+j+1}]) = \\
\sum_{j=0}^{k} (-1)^j (s_{k-j}x, s_n \ldots s_{k-j} \ldots s_0[01]).
\]

**Remark A.2.** Since we are working with normalized chains we have \(H(x, [1^{n+1}]) = 0\) for any \(x \in C_n\), \(n \geq 0\). Also note \(H(x, [0]) = H(x, [1]) = 0\). Moreover, if \(n > 0\), we have

\[ H(x, [0^{n+1}]) = (\mathcal{N}_*(p) \circ \text{EZ})(x \otimes [01]), \]

where

\[ \text{EZ} : \mathcal{N}_*(C) \otimes \mathcal{N}_*(R[\Delta]) \to \mathcal{N}_*(C \otimes R[\Delta]) \]

is the Eilenberg-Zilber map and \(p : C \otimes R[\Delta] \to \text{Cyl}(C)\) the map in the pushout defining \(\text{Cyl}(C)\).

An easy computation yields that \(H\) satisfies the chain homotopy equation

\[ H \circ \partial + \partial \circ H = i_1 \circ q - \text{id}, \]

where \(\partial : \mathcal{N}_*(C) \to \mathcal{N}_{*-1}(C)\) is the normalized chains differential. Then we define

\[ H_0 : \omega_*(\text{Cyl}(C)) \to \omega_*(\text{Cyl}(C)) \]

by

\[ H_0 := \sum_{i,j \geq 0} (s^{-1} \circ (i_1 \circ q) \circ s^{+1}) \otimes (s^{-1} \circ H \circ s^{+1}) \circ (\text{id}) \otimes. \]

We will show that

\[ D H_0 + H_0 D = \text{id} - \omega(i_1) \circ \omega(q), \]

where \(D : \omega_*(\text{Cyl}(C)) \to \omega_*(\text{Cyl}(C))\) is the differential of the dg algebra \(\omega_*(\text{Cyl}(C))\), but we first make a general observation.

The following notion was introduced in [32, Section 1.11].

**Definition A.3.** Let \(N = (N, \partial_N, \Delta_N)\) and \(N' = (N', \partial_{N'}, \Delta_{N'})\) be two dg coalgebras and \(f, g : N \to N'\) two morphisms of dg coalgebras. A degree +1 map \(F : N \to N'\) between underlying graded modules is said to be an \((f, g)\)-coderivation if the equation

\[ (F \otimes f + g \otimes F) \circ \Delta_N = \Delta_{N'} \circ F \]

is satisfied.

**Lemma A.4.** Let \(N = (N, \partial_N, \Delta_N)\) and \(N' = (N', \partial_{N'}, \Delta_{N'})\) be two conilpotent dg \(R\)-coalgebras. Suppose \(f, g : N \to N'\) are two morphisms of dg coalgebras and let \(F : N \to N'\) be a chain homotopy between \(f\) and \(g\) (of degree +1). If \(F\) is an \((f, g)\)-coderivation, then the map

\[ F_0 : \text{Cobar}(N) \to \text{Cobar}(N') \]
defined by

\[ F_D := \sum_{i,j \geq 0} \mathcal{F}^{\otimes i} \circ \mathcal{F} \circ \mathcal{G}^{\otimes j}, \]

where \( \mathcal{F} = s^{-1} \circ f \circ s + 1, \mathcal{F} = s^{-1} \circ F \circ s + 1, \) and \( \mathcal{G} = s^{-1} \circ g \circ s + 1, \) is a chain homotopy between \( \text{Cobar}(g) = \sum_n \mathcal{G}^{\otimes n} \) and \( \text{Cobar}(f) = \sum_n \mathcal{F}^{\otimes n}. \)

Proof. Let \( D_N \) and \( D_{N'} \) be the differentials of \( \text{Cobar}(N) \) and \( \text{Cobar}(N'), \) respectively. After using that \( f \) and \( g \) are maps of dg coalgebras to cancel terms, we obtain

\[
D_{N'} F_D + F_D D_N = \\
\sum -\mathcal{F}^{\otimes i} \otimes \partial_N, \circ \mathcal{F} \otimes \mathcal{G}^{\otimes j} + \sum \mathcal{F}^{\otimes i} \otimes \Delta_N, \circ \mathcal{F} \otimes \mathcal{G}^{\otimes j} \\
+ \sum -\mathcal{F}^{\otimes i} \circ \partial_N \otimes \mathcal{G}^{\otimes j} + \sum \mathcal{F}^{\otimes i} \otimes (\mathcal{F} \circ \mathcal{H} + \mathcal{G}) \circ \Delta_N \otimes \mathcal{G}^{\otimes j}.
\]

Since \( F \) is a \((f, g)\)-coderivation, the second and fourth terms in the above sum cancel and we obtain

\[
D_{N'} F_D + F_D D_N = \\
\sum -\mathcal{F}^{\otimes i} \otimes \partial_N, \circ \mathcal{F} \otimes \mathcal{G}^{\otimes j} + \sum -\mathcal{F}^{\otimes i} \otimes \partial_N \otimes \mathcal{G}^{\otimes j+1} = \\
\sum \mathcal{F}^{\otimes i} \otimes \mathcal{G}^{\otimes j+1} + \sum \mathcal{G}^{\otimes n} = \text{Cobar}(g) - \text{Cobar}(f).
\]

\[ \square \]

In order to apply Lemma \( \text{A.4} \) with \( f = i_1 \circ q, g = \text{id}_{\mathcal{N}_c(Cyl(C))}, \) and \( F = H \) as defined in \( \text{A.1.1} \) we must verify the following.

Proposition A.5. The map \( H : \mathcal{N}_c(Cyl(C)) \rightarrow \mathcal{N}_{c+1}(Cyl(C)) \) defined in \( \text{A.1.1} \) is a \((i_1 \circ q, \text{id})\)-coderivation.

Proof. We prove that

\begin{equation}
(\mathcal{H} \otimes (i_1 \circ q) + \text{id} \otimes \mathcal{H}) \circ \Delta = \Delta \circ \mathcal{H},
\end{equation}

where \( \Delta : \mathcal{N}_c(Cyl(C)) \rightarrow \mathcal{N}_c(Cyl(C)) \otimes \mathcal{N}_c(Cyl(C)) \) is the Alexander-Whitney coproduct. On any \((x, \sigma) \in \text{Cyl}(C)_n, \) where \( \sigma \in \mathcal{A}_n \), the Alexander-Whitney coproduct is given by

\begin{equation}
\Delta(x, \sigma) = \sum_{p=1}^{n+1} (d_p \ldots d_n x', d_p \ldots d_n \sigma) \otimes (d_0^{p-1} x'', d_0^{p-1} \sigma),
\end{equation}

where we have written \( \Delta_n(x) = \sum (x') \otimes x'' = x' \otimes x'' \) for the coproduct of the cocommutative coalgebra \( C_n. \)

Note that on any \( n \)-simplex \( \sigma = [0^{k+1}1^{n-k}] \in \mathcal{A}_n \), we can split the Alexander-Whitney coproduct into two sums

\begin{equation}
\sum_{p=1}^{n+1} d_p \ldots d_n [0^{k+1}1^{n-k}] \otimes d_0^{p-1} [0^{k+1}1^{n-k}] = \\
\sum_{p=k+2}^{k+1} [0^p] \otimes [0^{k+2-p}1^{n-k}] + \sum_{p=k+2}^{n+1} [0^{k+1}1^{p-k-1}] \otimes [1^{n+2-p}].
\end{equation}
For any \((x, [0^{k+1} 1^{n-k}]) \in \text{Cyl}(C)_n\), the right hand side of \(\text{A.5.1}\) is given by
\[
\Delta H(x, [0^{k+1} 1^{n-k}]) = \\
\sum_{p=1}^{n+2} \sum_{j=0}^{k} (-1)^j (d_p \ldots d_{n+1}s_{k-j}x', d_p \ldots d_{n+1}[0^{k-j+1} 1^{n+1-k+j}]) \\
\otimes (d_0^{p-1}s_{k-j}x'', d_0^{p-1}[0^{k-j+1} 1^{n+1-k+j}]).
\]

Using \(\text{A.5.3}\), we split the above sum into two sums (I) and (II):

sum (I) is given by
\[
\sum_{p=k-j+1}^{k} \sum_{j=0}^{k} (-1)^j (d_p \ldots d_{n+1}s_{k-j}x', [0^p]) \otimes (d_0^{p-1}s_{k-j}x'', [0^{k-j+2-p} 1^{n+1-k+j}]),
\]
and sum (II) by
\[
\sum_{p=1}^{n+2} \sum_{j=0}^{k} (-1)^j (d_p \ldots d_{n+1}s_{k-j}x', [0^{k-j+1} 1^{p-k-1}]) \otimes (d_0^{p-1}s_{k-j}x'', [1^{n+3-p}]).
\]

The simplicial identities, together with the fact that \(H(y, [1^{r+1}]) = 0\) for any \(y \in C_r\), yield that, up to the Koszul sign rule, sum (I) equals:
\[
(id \otimes H)\Delta (x, [0^{k+1} 1^{n-k}]) = \\
(id \otimes H)(\sum_{p=1}^{k+1} (d_p \ldots d_n x', [0^p]) \otimes (d_0^{p-1} x'', [0^{k+2-p} 1^{n-k}])) =
\]
\[
\sum_{j=0}^{k+1-p} \sum_{p=1}^{k+1-j} (-1)^{p-1+j} (d_p \ldots d_n x', [0^p]) \otimes (s_{k+1-p-j} d_0^{p-1} x'', [0^{k+2-p-j} 1^{n-k+j+1}]).
\]

Finally, we use \(\text{A.5.2} \text{A.5.3}\) and the formula
\[
(i_1 \circ q)(y, [0^r 1^s]) = (y, [1^{r+s}])
\]
to compute
\[
(H \otimes (i_1 \circ q))\Delta (x, [0^{k+1} 1^{n-k}]) = \\
\sum_{p=1}^{n+1} H(d_p \ldots d_n x', [0^{k+1} 1^{n-k}]) \otimes (i_1 \circ q)(d_0^{p-1} x'', [0^{k+2-p} 1^{n-k}]) =
\]
\[
\sum_{p=1}^{k+1} H(d_p \ldots d_n x', [0^p]) \otimes (i_1 \circ q)(d_0^{p-1} x'', [0^{k+2-p} 1^{n-k}]) +
\]
\[
\sum_{p=k+2}^{n+1} H(d_p \ldots d_n x', [0^{k+1} 1^{p-k-1}]) \otimes (i_1 \circ q)(d_0^{p-1} x'', [1^{n+2-p}]) =
\]
\[
\sum_{p=1}^{k+1} \sum_{j=0}^{p-1} (-1)^j (s_{p-1-j} d_p \ldots d_n x', [0^{p-j} 1^{j+1}]) \otimes (d_0^{p-1} x'', [1^{n+2-p}]) +
\]
\[
\sum_{p=k+2}^{n+1} \sum_{j=0}^{k} (-1)^j (s_{k-j} d_p \ldots d_n x', [0^{k+1-j} 1^{p-k+j}]) \otimes (d_0^{p-1} x'', [1^{n+2-p}]).
\]
If we reindex the above two sums by setting $i = j - p + 1 + k$ and $q = p + 1$ in the first sum and $q = p + 1$ in the second, we obtain
\[
\sum_{q=2}^{k+2} \sum_{i=k+2-q}^{k} (-1)^{q+i-k}(s_{k-i}d_{q-1} \ldots d_{n}x', [0^{k+1-i}1^{q-k+i-1}]) \otimes (d_{0}^{q-2}x'', [1^{n+3-q}])
\]
\[+ \sum_{q=k+3}^{n+2} \sum_{j=0}^{k} (-1)^{j}(s_{k-j}d_{q-1} \ldots d_{n}x', [0^{k+1-j}1^{q-k+j-1}]) \otimes (d_{0}^{q-2}x'', [1^{n+3-q}]).\]

This is exactly sum (II) after using the simplicial identities.

We may now conclude the main result of this appendix.

Proof of Proposition A.5. Lemma A.4 together with Proposition A.5 imply that $H_{0} : \Omega(Cyl(C)) \to \Omega(Cyl(C))$ is a chain homotopy between the maps id_{\Omega(Cyl(C))} and Cobar(i_{1} \circ q) = \Omega(i_{1} \circ q) = \Omega(i_{1}) \circ \Omega(q)$. Since $\Omega(q) \circ \Omega(i_{1}) = id_{\Omega(Cyl(C))}$, it follows that both maps of dg algebras $\Omega(q)$ and $\Omega(i_{1})$ are chain homotopy inverses to each other and, consequently, quasi-isomorphisms. \hfill \Box

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