A POSSIBLE HIDDEN SYMMETRY AND GEOMETRICAL SOURCE OF THE PHASE IN THE CKM MATRIX

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Abstract

Based on the present data, the three CKM angles may construct a spherical surface triangle whose area automatically provides a "holonomy" phase. By assuming this geometrical phase to be that in the CKM matrix determined by an unknown hidden symmetry, we compare the theoretical prediction on $\epsilon$ with data and find they are consistent within error range. We also suggest restrictions for the Wolfenstein parameters explicitly, the agreement will be tested by more precise measurements in the future. At least we can claim that such geometrical phase should be part of the weak phase appearing in the CKM matrix, even not the whole.

PACS number(s): 11.30.Er, 12.10.Ck, 13.25.+m
A Possible Hidden Symmetry and Geometrical Source of The Phase in The CKM Matrix

Although more than thirty years have elapsed since the discovery of CP violation [1], our understanding about the source of CP violation is still very poor. In the Minimal Standard Model (MSM), CP violation is due to the presence of a weak phase in the Cabibbo-Kabayashi-Maskawa (CKM) matrix [2,3]. Up to now, all the experimental results are in good agreement with MSM. Nevertheless, the correctness of CKM mechanism is far from being proved. Search for the source of CP violation is a profound and hard task in high energy physics [4, 5, 6, 7, 8]. In this work we restrict ourselves in the framework of MSM and see if we can find something which was missing in previous studies. First, let us review what we have learned about the CKM matrix.

(1) Considering all constraints on the matrix elements, for three generations, there are three independent angles and a weak phase which cannot be rotated away or absorbed into the quark wavefunctions. The phase is also independent of the three angles in principle. From another point, much effort has been made to understand the source of the three rotation angles and the phase.

Fritzsch [9, 10] noticed that because the eigenstates of the weak interaction are not the quark mass-eigenstates, there should be a unitary transformation to connect the two bases. It would establish a certain relation between the quark masses and the weak interaction mixing angles, while a weak CP phase is embedded explicitly.

From the general theory of Kabayashi-Maskawa [2], we know that there can exist a phase factor in the three-generation CKM matrix and it cannot be rotated away by re-defining the phases of quarks, but we can ask whether there is an intrinsic relation between the phase and the three rotation angles.

In Fritzsch’s theory, the CKM matrix comes from diagonalizing the U-type and D-type quark mass matrices as it is possible that there are certain horizontal relations between different generations of quarks. The proposed symmtry has undergone some modifications for fitting data, especially the top-quark mass. These horizontal relations which determine the off-diagonal elements of the mass matrices which are proportional to a typical quantity \( \lambda \). It may hint that there is a broken horizontal symmetry and the scale of breaking is related to \( \lambda \).

Concretely, supposing \( V_d \) and \( V_u \) diagonalize the mass matrices for d-type and u-type
quarks respectively. $V_{KM} \equiv V_u^d V_d$ is the CKM matrix and can be written as

$$
V_{KM} = \begin{pmatrix}
c_1 & -s_1 c_3 & -s_1 s_3 \\
s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\
s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta}
\end{pmatrix}
$$

(1)

with the standard notations $s_i = \sin \theta_i$ and $c_i = \cos \theta_i$.

It is noted that here we adopt the original form of the CKM parametrization. There are some other parametrization ways, for example the Wolfenstein’s and that recommended by the data group, but it is believed that physics does not change when adopting various parametrizations.

It is well known that the KM parametrization can be viewed as a product of three Eulerian rotation matrices and a phase matrix

$$
V_{KM} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_2 & -s_2 \\
0 & s_2 & c_2
\end{pmatrix}
\begin{pmatrix}
c_1 & -s_1 & 0 \\
s_1 & c_1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -e^{i\delta}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_3 & s_3 \\
0 & -s_3 & c_3
\end{pmatrix}
$$

(2)

People have noticed that the weak CP phase $\delta$, which cannot be eliminated in the three generation CKM matrix by any means, is introduced artificially and seems to have nothing to do with the three "rotation" angles. Anyway, such a fact does not seem to be natural.

As this concept is widely accepted, we may ask if there may exist a hidden symmetry which can relate the weak phase with the three rotation angles? In general, there is nothing to forbid it and of course nothing to confirm it either, we are only discuss a possible source. Only one thing we are sure is that there indeed exists a geometrical phase and it can serve as one of the sources to the CP phase.

(2) Based on observation, the recently measured $\theta_1, \theta_2, \theta_3$ satisfy

$$
\theta_i + \theta_j \geq \theta_k, \quad i, j, k = 1, 2, 3,
$$

and if we only take the positive values of $\sin \theta_i$ as $0 \leq \theta_i \leq \pi/2$ (i=1,2,3), then

$$
\theta_1 + \theta_2 + \theta_3 \leq \frac{3\pi}{2}.
$$

Therefore the three angles can construct a spherical surface triangle on a unit sphere in the Hilbert space.

The three angles correspond to three arcs on the unit sphere and they enclose an area $\delta$. The $\delta$ and the three angles have a definite relation

$$
\cos \frac{\delta}{2} = \frac{1 + \cos \theta_1 + \cos \theta_2 + \cos \theta_3}{4 \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \cos \frac{\theta_3}{2}}.
$$

(3)
The geometrical meaning of the area is clear. The three vertices A, B and C correspond to $\angle A$, $\angle B$ and $\angle C$. At each vertex, there are two tangents along the two adjacent arcs as defining the positive directions of arcs anti-clockwisely. If one moves one tangent $\vec{t}_1^A$ along the arc $AB$ to vertex B and then it becomes to $\vec{t}_2^B$. Rotate $\vec{t}_2^B$ anti-clockwisely to $\vec{t}_1^B$ by an angle $\pi - \angle B$, then let it move to vertex C along arc $BC$ and rotate $\vec{t}_2^C$ to $\vec{t}_1^C$, finally move it back to vertex A and the resultant $\vec{t}_1^A$ which spans an angle $\pi - \angle A$ with the original vector $\vec{t}_1^A$. Geometrically, the three angles $\alpha_1, \alpha_2, \alpha_3$ which transform $\vec{t}_1^{A,B,C}$ to $\vec{t}_2^{A,B,C}$ respectively have the relation

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi - \angle A + \pi - \angle B + \pi - \angle C = 3\pi - (\angle A + \angle B + \angle C)$$

(4)

$$= 3\pi - (\pi + \delta) = 2\pi - \delta,$$

(5)

where $\delta$ is exactly that area enclosed by the three arcs and is called the "angular excess". It is noted that for a planar triangle, there is no such "angular excess", so the $\delta$-phase is obviously caused by the curved space characteristics, i.e., the affine connection.

Thus $\delta$ represents the tangent transformation along the spherical surface triangle, so it can be the variable of an U(1) holonomy transformation group. So it automatically corresponds to a phase $e^{i\delta}$, which is a geometrical phase.

The three Euler angles $\theta_{12}, \theta_{23}, \theta_{31}$ bridge the three generation quark flavors and the adopted $\theta_1, \theta_2, \theta_3$ are nothing new but an alternative parametrization scheme, so the U(1) phase $e^{i\delta}$ would appear in the CKM matrix afterwards.

As well-known, for a naive $O(3)$ rotation group, a geometric phase can automatically arise while two non-uniaxial successive rotation transformations being performed \cite{18, 19, 20, 21, 22, 23}. For instance, $R_x(\theta_1)$ denotes a clockwise rotation about the x-axis by $\theta_1$, while $R_y(\theta_2)$ is about the y-axis by $\theta_2$. Supposing on a unit-sphere surface, the positive z-axis intersects with the surface at a point A, after performing these two sequential operations $R_y(\theta_2)R_x(\theta_1)$, the point-A would reach point-B via an intermediate point-C, by contrast, one can connect A and B by a single rotation $R_{\hat{\xi}}(\theta_3)$, where $R_{\hat{\xi}}(\theta_3)$ denotes a clockwise rotation about the $\hat{\xi}$-axis by $\theta_3$. The geometric meaning can be depicted in a more obvious way is that if one chooses an arbitrary tangent vector $\hat{\alpha}$ at point-A which would rotate to $\hat{\alpha}'$ and $\hat{\alpha}''$ by $R_y(\theta_2)R_x(\theta_1)$ and $R_{\hat{\xi}}(\theta_3)$ respectively, then one can find that $\hat{\alpha}'$ does not coincide with $\hat{\alpha}''$, but deviates by an extra rotation. Concretely, if one writes down the rotation in the adjoint representation of $O(3)$, he can find

$$R_{\hat{\eta}}(\delta)R_{\hat{\xi}}(\theta_3) = R_y(\theta_2)R_x(\theta_1),$$

(6)

where $R_{\hat{\eta}}(\delta)$ represents a counterclockwise rotation about the $\hat{\eta}$-axis by $\delta$. This resultant
phase is obviously non-removable.

(3) Assuming that this geometrical U(1) phase is the weak phase in the CKM matrix, namely, it means that there is no any underlying physical principle to cause the weak phase in the matrix, but only this holonomy phase plays role of the weak phase. Thus we can find some consequential deductions which would be tested by its phenomenological applications in comparison with corresponding experimental data.

(i) A test from $\epsilon$ in $K^0 - \bar{K}^0$ system.

So far the only reliably measured CP violation quantity is $\epsilon$ in the K-system and the mechanism causing $K^0 - \bar{K}^0$ mixing has already been well studied in the framework of MSM. Except an unknown B-factor, one can evaluate $\epsilon$ in terms of the CP phase $\delta$ as

$$|\epsilon| \approx \cos \theta_2 \sin \theta_2 \sin \theta_3 \sin \delta \left[ \frac{\sin^2 \theta_2 (1 + \eta \log \eta) - \cos^2 \theta_2 \eta (1 + \log \eta)}{\sin^4 \theta_2 + \cos^4 \theta_2 \eta - 2 \sin^2 \theta_2 \cos^2 \theta_2 \eta \log \eta} \right],$$

where $\eta = m_c^2/m_t^2$.

The inputs of $|V_{ij}|$ are taken from the data book [27] and

$$m_c = 1.5 \text{ GeV}, \quad m_t(m_t^2) = 176 \text{ GeV}, \quad |\epsilon| = 2.3 \times 10^{-3},$$

with all the given errors.

By using eq.(3) we obtain

$$\delta = 0.01138 \pm 0.0049,$$

while extracting the $\delta$–value from eq.(7), it is

$$\delta = 0.0044 \pm 0.0030.$$

Therefore, one can notice that considering the experimental error tolerance, the two obtained values are roughly consistent. Since the extraction of $\delta$ from the data $\epsilon$ still depends on the evaluations of concerned hadronic transition matrix elements which are not reliable so far, namely, we cannot handle the non-perturbative QCD effects satisfactorily, the deviation between two $\delta$–values is reasonable and tolerable.

Anyhow, this phenomenological application of eq.(3) does not contradict to the data, in other words, may obtain some sort of support from this comparison.

(ii) The ranges of the Wolfenstein’s parameters.
Now, let us have a look at the possible ranges of the Wolfenstein’s parameters if the assumption that the $\delta$ given in eq.(3) is the weak phase in the CKM matrix, is valid. In the Wolfenstein’s parametrization, the CKM matrix reads as

$$V_W = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta + i\frac{1}{2}\lambda^2) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 - i\eta A^2\lambda^4 & A\lambda^2(1 + i\eta\lambda^2) \\ \lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \quad (8)$$

So, it is necessarily to convert the new constraint Eq.(1) which represented by four angles $\delta, \theta_1, \theta_2, \theta_3$ into the one represented by Wolfenstein’s parameters $A, \lambda, \rho, \eta$. It is easy to do so if we take use of the following translation prescription between KM’s and Wolfenstein’s parameters

$$s_1 \approx \lambda, \quad c_1 \approx 1 - \frac{\lambda^2}{2} \quad (9)$$

$$s_2 \approx \lambda^2 A[(\rho - 1)^2 + \eta^2] \quad (10)$$

$$s_3 \approx (\rho^2 + \eta^2)^{1/2} A\lambda^2 \quad (11)$$

$$\sin\delta \approx \frac{\eta}{(\rho^2 + \eta^2)^{1/2} [(\rho - 1)^2 + \eta^2]^{1/2}}. \quad (12)$$

From Eq.(1), we obtain

$$\sin\delta = \frac{(1 + \cos\theta_1 + \cos\theta_2 + \cos\theta_3)\sqrt{\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 - 2(1 - \cos\theta_1 \cos\theta_2 \cos\theta_3)}}{(1 + \cos\theta_1)(1 + \cos\theta_2)(1 + \cos\theta_3)}. \quad (13)$$

Substituting Eqs.(9-11) to Eq.(13) and expanding the right hand side of Eq.(13) in powers of $\lambda$, with a little more complicated calculation, when approximate to the order of $\lambda^5$, we get

$$\sin\delta = \frac{A\sqrt{(1 - \rho)^2 + \eta^2 + (\rho^2 + \eta^2)}}{2\sqrt{2}} \lambda^3 + \frac{A[(1 - \rho)^2 + \eta^2 + (\rho^2 + \eta^2) - 2A^2(1 - 2\rho)^2]}{2^4\sqrt{2}(1 - \rho)^2 + \eta^2 + (\rho^2 + \eta^2)} \lambda^5 \quad (14)$$

Identify the right hand sides of Eq.(12) and Eq.(14), we have

$$\frac{\eta}{(\rho^2 + \eta^2)^{1/2} [(1 - \rho)^2 + \eta^2]^{1/2}} \approx \frac{A\sqrt{(1 - \rho)^2 + \eta^2 + (\rho^2 + \eta^2)}}{2\sqrt{2}} \lambda^3. \quad (15)$$

Here, in comparison with $\lambda^3$, we have neglected the term of order $\lambda^5$. Eq.(15) is the new constraint on CP-violation and quark-mixing represented by Wolfenstein’s parameters approximate to the order of $\lambda^3$.

In following, we want to give a simple numerical analysis. Let

$$x = (\rho^2 + \eta^2)^{1/2} \quad (16)$$
\( y = [(1 - \rho)^2 + \eta^2]^{1/2} \) \hspace{1cm} (17)

then
\[
\eta = \frac{1}{2} \sqrt{2(x^2 + y^2) - (x^2 - y^2)^2 - 1}
\] \hspace{1cm} (18)

Substituting Eqs.(16-18) to Eq.(15), we arrive
\[
\frac{\sqrt{2(x^2 + y^2) - (x^2 - y^2)^2 - 1}}{xy} = \frac{A\lambda^3}{\sqrt{2}\sqrt{x^2 + y^2}}
\] \hspace{1cm} (19)

Fixing \( \lambda = 0.22 \) and \( A = 0.808 \pm 0.058 \), if we take \( y = 0.54 \sim 1.40 \) as input, then \( 0.22 \sim 0.46 \) for \( x \) is permitted. Hence, we find that the results are well in agreement with the experimental analysis.

\[
x = \sqrt{\rho^2 + \eta^2} = 0.34 \pm 0.12
\] \hspace{1cm} (20)

and
\[
y = \sqrt{(1 - \rho)^2 + \eta^2} = 0.97 \pm 0.43.
\] \hspace{1cm} (21)

Here we draw a graph of \( x^2 \) versus \( y^2 \) using the relation given in eq.(19) and we obtain two branches of solutions where we let \( A \) to be deviated from its central value of 0.808 by a small fractions and the curves in fact are narrow bands.

From the figure, one can see that within the experimental error ranges of \( x \) and \( y \), there exist solutions. It indicates that the results do not contradict to the CKM matrix elements measurement.

(4) Our conclusion and discussion.

In this work, based on observation of the measured values of the CKM matrix elements and consideration of a possible hidden symmetry, we study a possibility that the weak phase in the CKM matrix is due to a geometrical reason which is fully determined by the three rotation angles.

In this assumption, the geometrical phase takes responsibility of all the roles of the weak phase in the CKM matrix. Obviously, there can be some physical mechanisms which can also result in the weak phase and present data cannot eliminate this possibility at all. What we show in this work is that the geometrical phase does exist and can be a part of the phase at the CKM matrix, moreover the bald assumption cannot eliminate existence of other physical sources of the weak phase at all. If this geometrical phase can be the whole of the CKM phase or only a part of it should wait for more precise experimental measurements in the future. If the predicted curves drawn in Fig.1, do not have common
range with the measure data, it would indicate that the geometrical phase is only a part, no matter small or large, of the phase in the CKM matrix.  

In conclusion, we can claim a possible source of the CKM phase due to the geometrical reason and need to wait for future experiments to test if it is the only source or only one of the sources, i.e. there should exist other physical sources to cause this CKM phase.

**Acknowledgment**: This work is partly supported by National Natural Science Foundation of China. We would like to thank Dr. Z.Z. Xing for helpful comments and Dr. X.G. He, Dr. Y.L. Wu for fruitful discussions.

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1 In our previous works, hep-ph/9711330 and hep-ph/9711293 our claim on the geometrical phase went too extreme, that we attributed existence of the phase to geometrical reason and it does not have any proof or enough decisive support from data.
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