Gauss–Bonnet brane-world cosmology without $Z_2$-symmetry

Kenichiro Konya

Institute for Cosmic Ray Research, University of Tokyo, Kashiwa 277-8582, Japan

E-mail: konya@icrr.u-tokyo.ac.jp

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Abstract

We consider a single 3-brane situated between two bulk spacetimes that possess the same cosmological constant, but whose metrics do not possess a $Z_2$-symmetry. On each side of the brane, the bulk is a solution to Gauss–Bonnet gravity. This asymmetry modifies junction conditions, so new terms arise in the Friedmann equation. If $Z_2$-breaking terms become dominant, they behave as a cosmological constant at early times for one case and might remove the initial singularity for the other case. However, we show that these new terms cannot become the dominant ones under the usual conditions when our brane is outside an event horizon. We also show that any brane-world scenarios of this type revert to a $Z_2$-symmetric form at late times, and hence rule out certain proposed scenarios.

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1. Introduction

There has been considerable interest in the brane-world scenario. In this scenario, our universe is modelled by a 3-brane embedded in a higher dimensional bulk spacetime. Of particular interest is the Randall–Sundrum (RS) model, where a single brane is embedded in a five-dimensional anti-de Sitter (AdS$_5$) spacetime [1]. Although the fifth dimension is noncompact, the graviton is localized at low energies on the brane due to the warped geometry of the bulk.

In four dimensions, the Einstein tensor is the only second-rank tensor which satisfies the following conditions:

- it is symmetric,
- it has vanishing divergence,
- it depends only on the metric and its first two derivatives,
- it is linear in the second derivatives of the metric.
However, in $D > 4$ dimensions, the Lovelock tensor satisfies similar conditions: it is symmetric, divergence free, second order in the metric, but quasi-linear in the second derivatives of the metric [2, 3]. Thus, a natural extension to the RS model is to include such higher order curvature invariants in the bulk action. The Lovelock tensor arises from the variation of the Gauss–Bonnet (GB) term:

$$L_{\text{GB}} = R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd}.$$  

String theory also provides us with a more compelling reason to include the GB term. The GB term appears as the next-to-leading order correction in the heterotic string effective action, and is ghost free [4]. Moreover, the graviton is localized in the GB brane world [5] and deviations from Newton’s law at low energies are less pronounced than in the RS model [6]. This term is a topological invariant in four dimensions; however, in AdS$_4$ gravity the addition of this term has nontrivial consequences [26].

Brane cosmologies with and without the GB term have been investigated [7, 10–12]. Most brane-world scenarios assume a $Z_2$-symmetry about our brane. This is motivated by a model derived from $M$-theory proposed by Horava and Witten [8]. However, many recent papers examine models that are not directly derived from $M$-theory: for example, there has been much interest in the one infinite extra-dimension proposal. There are at least two ways in which the asymmetry might arise [9]. It is therefore interesting to analyse a brane-world model without this symmetry [10, 11, 13, 15].

The rest of this paper is organized as follows: in section 2 we present the complete setup we consider; in section 3 we investigate the cosmological consequences without $Z_2$-symmetry; in section 4 some conclusions are drawn.

2. Einstein equations

We consider two 5D bulk spacetimes, $\mathcal{M}_L$ and $\mathcal{M}_R$, separated by a single 3-brane. $\mathcal{M}_{L,R}$ is a solution to Gauss–Bonnet gravity with a cosmological constant, $\Lambda < 0$. This scenario is described by the following action:

$$S = S_{\text{grav}} + S_{\text{brane}},$$  

$$S_{\text{grav}} = \sum_{i=L,R} \frac{1}{2\kappa^2} \int_{\mathcal{M}_i} d^5x \sqrt{-g} \left( R - 2\Lambda + \alpha L_{\text{GB}} \right)$$

$$+ \frac{1}{\kappa^2} \int_{\partial\mathcal{M}_i} d^4x \sqrt{-h} (K + 2\alpha [J - 2\tilde{G}^{ab} K_{ab}]),$$

$$S_{\text{brane}} = \int_{\text{brane}} d^4x \sqrt{-h} L_{\text{brane}}.$$  

Here, $\kappa^2$ is the 5D gravitational constant and $\alpha > 0$ is a Gauss–Bonnet coupling. $h_{ab}$ is the induced 4D metric and is defined by

$$h_{ab} = g_{ab} - n_a n_b,$$

where $n^a$ is the spacetime unit vector, and it points away from the surface and into the adjacent space. The second term in $S_{\text{grav}}$ is a boundary term required for a well-defined action principle [14]. $\tilde{G}_{ab}$ is the 4D Einstein tensor on the brane corresponding to $h_{ab}$. $K$ is the trace of the extrinsic curvature, defined by $K_{ab} = h^{c}{}_{a} h^{d}{}_{b} \nabla_c n_d$ where $\nabla$ is the covariant derivative associated with the bulk metric $g_{ab}$. $J$ is the trace of

$$J_{ab} = \frac{1}{2} (2 K K_{ab} K^c_b + K_{cd} K^{cd} K_{ab} - 2 K_{ac} K^{cd} K_{db} - K^2 K_{ab}).$$
The variation of the action $S$ gives

$$G_{ab} + 2\alpha H_{ab} + \Lambda g_{ab} = \kappa^2 S_{ab} \delta(\partial M),$$  \hspace{1cm} (7)

where

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R,$$  \hspace{1cm} (8)

$$H_{ab} = R R_{ab} - 2 R_{ac} R^c_b - 2 R^{cd} R_{a}^{\ \ c b d} + R^{cde}_a R_{b c d e} - \frac{1}{2} g_{ab} L_{GB},$$  \hspace{1cm} (9)

and

$$S_{ab} = -2 \frac{\delta L_{\text{brane}}}{\delta h_{ab}} + h_{ab} L_{\text{brane}}.$$  \hspace{1cm} (10)

Conservation of energy momentum of the matter on the brane follows from the Gauss–Codazzi equations. Thus, we have [12]

$$\nabla^a S_{ab} = 0,$$  \hspace{1cm} (11)

where $\nabla^a$ is the covariant derivative associated with the brane metric $h_{ab}$.

### 2.1. The bulk

We assume that our brane is homogeneous and isotropic. Then, from the generalized Birkhoff’s theorem, the bulk metric can be written in the following form [15]:

$$ds^2 = -f_{L, R}(a) \, dt^2 + \frac{da^2}{f_{L, R}(a)} + a^2 \Omega_{ij} \, dx^i \, dx^j,$$  \hspace{1cm} (12)

where $\Omega_{ij}$ is the three-dimensional metric of space with constant curvature $k = -1, 0, 1$. For the time being, let us drop the indices $L, R$, as the following analysis will apply to both sides of the brane. The solution of the field equation $G_{ab} + 2\alpha H_{ab} + \Lambda g_{ab} = 0$ is given by

$$f(a) = k + \frac{a^2}{4\alpha} \left( 1 \mp \sqrt{1 + \frac{4}{3} \alpha \Lambda + \frac{8 \alpha \mu}{a^4}} \right),$$  \hspace{1cm} (13)

with $\mu$ being an arbitrary constant. Other solutions do exist for special values of $k, \Lambda$ and $\alpha$ [15], but we will not consider them here. The constant $\mu$ is related to the black hole (BH) mass by the relation

$$M_{BH} = \frac{3V \mu}{k^2},$$  \hspace{1cm} (14)

where $V$ is the volume of the 3D space [16]. In order to avoid a classical instability [17], we must have $\mu \geq 0$.

From equation (13), we find that the metric has a singularity at $a = 0$. For the $(-)$ branch, this singularity is covered by an event horizon if $k \leq 0$ or $k = 1$ and $\mu \geq 2\alpha$. For such a case, the event horizon $(a = a_h)$ is

$$a_h^2 = \frac{3k}{\Lambda} + \sqrt{\frac{9 k^2 + 12 k^2 \alpha \Lambda - 6 \mu \Lambda}{\Lambda^2}}.$$  \hspace{1cm} (15)

This is not the case for the $(+)$ branch. So, to shield this naked singularity we must cut the spacetime off at some small value of $a$ for the $(-)$ branch with $k = 1$ and $\mu < 2\alpha$, and for the $(+)$ branch. This can be done by introducing a second brane at $a \sim M_{col}^{-1}$ [18].
2.2. The brane

We define the position of the brane as \( a = a(\tau) \) and \( t = t(\tau) \) which is parametrized by the proper time on the brane. Then the induced metric of the 3-brane is

\[
ds^2 = -d\tau^2 + a(\tau)^2 \Omega_{ij} \, dx^i \, dx^j.
\]

(16)

The tangent vector of the brane is \( n_{L,R} = (i_{L,R}, 0, 0, 0, \dot{a}) \), where the dot denotes the derivative with respect to a proper time \( \tau \). The normal vector is \( n_{L,a} = (-\dot{a}, 0, 0, i_L, 0) \), \( n_{R,a} = (\dot{a}, 0, 0, -i_R, 0) \). Normalization of \( n^a \) implies

\[-f_{L,R}^2 i_{L,R}^2 + \dot{a}^2 = -f_{L,R}. \]

(17)

The requirement that the metrics must be continuous at the brane implies \[11\]

\[
t_L = t_R \frac{f_R}{f_L} \left( \frac{\dot{a}^2 + f_L}{\dot{a}^2 + f_R} \right). \]

(18)

From equation (7), we obtain the generalized Israel’s junction condition \[12, 19\]

\[\left[ K_{\mu\nu} \right]_+ - h_{\mu\nu} [K]_+ + 2\alpha(3J_{\mu\nu})_+ - h_{\mu\nu} [J]_+ - 2P_{\mu
u\rho\sigma} [K^\rho\sigma]_- = -\kappa^2 S_{\mu\nu}, \]

(19)

where

\[P_{\mu
u\rho\sigma} = R_{\mu
u\rho\sigma} + 2h_{[\nu} R_{\rho]\mu} + 2h_{[\nu} R_{\rho]\mu] + Rh_{[\nu} h_{\rho\mu]} \]

(20)

is the divergence-free part of the Riemann tensor. We have introduced \([X]_- \equiv X_R - X_L \).

We take the brane matter to be a perfect fluid, so \( S_{ab} = (\rho + p)u_a u_b + \rho h_{ab} \). The \( (\tau, \tau) \) component of equation (19) is then

\[
\frac{3f_{L,R}}{a} + \frac{3f_{L,L}}{a} + 2\alpha \left[ \frac{2f_{L,R}^3}{a^2} - \frac{2f_{L,L}^3}{a^2} + 6(\dot{a}^2 + k) \right] \left( \frac{f_{R,R}}{a^3} + \frac{f_{L,L}}{a^3} \right) = \kappa^2 \rho.
\]

(21)

From equations (13), (17) and (21), we have a cubic equation for \( H^2 \):

\[
A^3 - 9 \left( b_L^{2/3} - b_R^{2/3} \right)^2 \frac{1}{256\alpha^2 \kappa^4 \rho^2} A^2 + \left[ -\frac{3}{128\alpha^2} + \frac{3}{512\alpha^3} \right] \frac{b_L^{2/3} - b_R^{2/3}}{4096\alpha^4 \kappa^4 \rho^2} A - \left[ \frac{\kappa^4 \rho^2}{256\alpha^2} + \frac{b_L + b_R}{512\alpha^3} \right] = 0,
\]

(22)

where

\[
A = H^2 + \frac{k}{a^2} + \frac{1}{4\alpha},
\]

(23)

\[
b_{L,R}^{1/3} = \left( 1 + \frac{4}{3} \alpha \Lambda + \frac{8\alpha \mu_{L,R}}{a^2} \right)^{1/2},
\]

(24)

The \( \mp \) signs in equation (22) correspond to those in equation (13).

3. The effect of no \( Z_2 \)-symmetry

In this section we study the cosmological effects of no \( Z_2 \)-symmetry. Here, the GB term may be considered as the lowest-order stringy correction to the 5D Einstein action. In this case, \( \alpha |R|^2 < |R| \), so that

\[\alpha < l^2,
\]

(25)

where \( l \) is the bulk curvature scale, \( |R| \sim l^{-2} = (1 \mp \sqrt{1 + 4\alpha \Lambda/3}) / 4\alpha \). For the \( + \) branch, this reduces to the RS relation \( l^{-2} = -\Lambda / 6 \) when \( \alpha = 0 \). Note that there is an upper limit to
the GB coupling:
\[ \alpha < \frac{l^2}{4} \quad (\text{)} \text{ branch,} \]
\[ \frac{l^2}{4} < \alpha < \frac{l^2}{2} \quad (\text{+}) \text{ branch.} \]

These conditions are consistent with equation (25).

3.1. The (−) branch

We first consider the (−) branch and we assume \( k = 0 \) for simplicity. The Friedmann equation is shown in appendix A.1. We find that there are more \( Z_2 \)-breaking terms in the GB case than in the RS case. This is of course because including the GB term gives rise to extra terms in equation (7). From equations (15) and (A.9), we find that the third term in equation (24) is always smaller than the other terms. Then the \( Z_2 \)-breaking terms outside the square root of equation (A.2) approximately scale as \( 1/(\rho^2 a^8) \), where \( n = 1, 2, 3, 4 \). Therefore, when \( \sigma > \lambda \) (here, we take the usual assumption that \( \rho = \sigma + \lambda \) where \( \sigma \) is the ordinary matter energy density and \( \lambda \) is the brane tension), the \( Z_2 \)-breaking terms behave like a positive cosmological constant term for a radiation-dominant universe (\( \sigma = \gamma/a^4 \)). So we might have an inflation without any other fields in the early universe. At late times (\( \sigma < \lambda \)) the \( Z_2 \)-breaking terms rapidly decrease. Thus, the effects of \( Z_2 \)-breaking terms are no longer significant and we obtain the standard cosmology.

In order to obtain the \( Z_2 \)-breaking term dominant regime, one of the \( Z_2 \)-breaking terms must be larger than all the \( Z_2 \)-symmetry terms during \( \sigma > \lambda \). From this requirement, we have
\[ C^2 = a(\mu_L - \mu_R)^2/k^4 \rho^2 b^{1/3} > 1/18. \]

Here, the third term on the right-hand side of equation (A.6) scales just like radiation. Hence, it is called the dark radiation. Dark radiation affects both big-bang nucleosynthesis and the cosmic microwave background. Accordingly, such observations constraint on \( \mu \) [21]:
\[ \frac{\mu_L + \mu_R}{2b^{1/3} a^4} < \frac{\pi^2 \Delta g_* T_N^4}{30 3M_{pl}^2}, \]

where \( \Delta g_* \) is the deviation from the number of effective relativistic degrees of freedom (\( g_* \)) and \( T_N \) is the temperature of nucleosynthesis. At the time of nucleosynthesis, the energy density is
\[ \sigma(t_N) = \frac{\gamma}{a^4(t_N)} = \frac{\pi^2 g_* T_N^4}{30}. \]

From equations (28) and (29),
\[ \mu_L + \mu_R < \frac{2b^{1/3} \Delta g_*}{3M_{pl} g_*} \sim \frac{1}{10M_{pl}^2}, \]

where we have taken the standard values \( g_* = 10.75 \) and \( \Delta g < 2 \). Using equations (30), (A.7)–(A.9), we find that \( C^2 \) is bounded above:
\[ C^2 \lesssim \frac{\alpha \mu^2}{b^{1/3} k^4 \gamma^2} \sim \frac{1 - b^{1/3}}{400b} \lesssim 8 \times 10^{-4}. \]

Note that \( C^2 > 1/18 \) is needed for the \( Z_2 \)-breaking term dominant regime. Thus, we find that the \( Z_2 \)-symmetric term is always dominant in general, and we have the usual evolution of the
universe (see figure 1). If we permit $\mu_L \approx -\mu_R$, this nucleosynthesis constraint is easy to satisfy, and we can have an inflation regime due to the $Z_2$-breaking term. However, there is a classical instability in this case.

3.2. The (+) branch

In this section we consider the (+) branch. It was claimed that this branch solution is classically unstable to small perturbations and yields a graviton ghost [23, 15]. However, it turns out that this branch solution is classically stable to small perturbations due to the positive Abbott–Deser (AD) energy [24]. Note that the (+) branch is not well defined at $\alpha = 0$. So, there is a significant difference from Einstein gravity. Therefore, it is interesting to study this branch.

In this branch, the usual assumption that the GB energy scale is greater than the RS energy scale is inconsistent with equation (27) (see appendix A.2). So, we investigate the case that the GB energy scale is smaller than the RS energy scale.

Case 1 ($m_{8,RS}^{\mu} \gtrsim 2m_{8,GB}^{\mu}$). The Friedmann equation is the one where we have changed the sign of $b_{L,R,H}^{1/3}$ in the (−) branch, and is shown in appendix A.2. Note that there is no dark radiation term in equation (A.15). Therefore, we will be able to have the $Z_2$-breaking term dominant universe. We do not have an event horizon in this branch. Thus, the third term in equation (24) can be larger than the other terms, not as in the (−) branch. Then the $Z_2$-breaking terms outside the square root of equation (A.11) approximately scale as $(-1)^{n+1}/(\rho^{2n}d^{6n})$, where $n = 1, 2, 3, 4$. Therefore, the $Z_2$-breaking term will behave like a positive cosmological constant term for the radiation-dominant universe. At late times, the $Z_2$-breaking terms rapidly dampen and standard cosmology is recovered.

Here, because of the change of sign of $b_{L,R,H}^{1/3}$, $H^2 + k/a^2$ can be negative if the $Z_2$-breaking or the dark radiation term is large (see figure 2). If $k \geq 0$, this is inconsistent with the positivity of $H^2 + k/a^2$. It follows that $a$ is bounded from below as $a^6 \geq 8\mu \sqrt{2\alpha \mu /k^2 \lambda^2}$. Therefore, we can avoid an initial singularity without introducing a second brane. For the case of $k < 0$, if $\mu \geq 300\alpha$, $a$ is bounded from below as the case of $k \geq 0$. In such a case, we cannot have an inflation due to the $Z_2$-breaking. If $\mu < 300\alpha$, $a$ is not bounded from below and we have an inflation and an unusual evolution of the universe.

Figure 1. Illustrating the relationship between $H^2$ and $T$. The solid line denotes the $Z_2$-breaking case and the dotted line denotes the $Z_2$-symmetric one. The adopted parameters are $\alpha = 10^{20} (\text{GeV})^{-2}$, $\Lambda = -2.4 \times 10^{-23} (\text{GeV})^2$, $\gamma = 2.9 \times 10^{-35} \times (g_*/10.35) (\text{GeV})^2$ and $C^2 = 10^{-2}$. 
Case 2 \(1 \lesssim m_{\text{RS}}^8 / m_{\text{GB}}^8 \lesssim 2\). From the Friedmann equation (A.18), we find that there is a dark radiation term and Newton’s constant evolves. The evolution of the Newton constant has an effect on various theories such as cosmology, astrophysics, geophysics, etc. Thus, \(\mu\) is restricted from such observations (see [25] and references therein) in addition to the restriction for the dark radiation term. From these restrictions, we find that the \(Z_2\)-breaking terms cannot become the dominant ones.

4. Conclusions

In this paper, we have considered the Gauss–Bonnet brane-world cosmology without \(Z_2\)-symmetry. Relaxing the \(Z_2\)-symmetry gives more extra terms than the RS case.

For the \((-\) branch, these \(Z_2\)-breaking terms cannot become dominant outside the event horizon in general. If these terms can be dominant, they behave as a cosmological constant at early times.

For the \((+\) branch, if \(m_{\text{RS}} < m_{\text{GB}}\), the late time cosmology is not compatible with the standard one. So, we have considered the \(m_{\text{RS}} > m_{\text{GB}}\) case. In this case, the late time cosmology is compatible with the standard one. In the case of \(m_{\text{RS}}^8 > 2m_{\text{GB}}^8\), we have shown that the initial singularity problem might be avoided by the \(Z_2\)-breaking terms. In the case of \(1 < m_{\text{RS}}^8 / m_{\text{GB}}^8 < 2\), we have shown that the Newton constant evolves and we cannot have the \(Z_2\)-breaking term dominant universe because of the restriction for the dark radiation and Newton constant.

We also note that the effects of the \(Z_2\)-breaking terms decrease with time like the RS case [13]. Therefore, the scenario without this symmetry at late times is not viable.

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Appendix A. Friedmann equation

A.1. The (−) branch

We first consider the (−) branch. The single real solution of the cubic equation (22) is

\[ H^2 + \frac{k}{\alpha^2} + \frac{1}{4\alpha} - \frac{3(b_{L}^{2/3} - b_{R}^{2/3})^2}{2\alpha^2\kappa^4\rho^2} = c_+ + c_-, \tag{A.1} \]

where

\[ c_+ = \frac{1}{2\kappa} \left[ \frac{25\kappa^4\rho^2}{\alpha^2} + \frac{214(b_{L} + b_{R})}{\alpha^3} + \frac{28}{\alpha^2\kappa^4\rho^2}(b_{L}^{1/3} - b_{R}^{1/3})^2 \times (17b_{L}^{4/3} + 34b_{L}b_{R}^{1/3} + 42b_{L}^{2/3}b_{R}^{2/3} + 34b_{L}b_{R}^{1/3} + 17b_{L}^{4/3}) \right. \]

\[ \left. + \frac{2b^8\kappa^8}{\alpha^2} (b_{L}^{1/3} - b_{R}^{1/3})^4 \left( b_{L}^{1/3} + b_{R}^{1/3}\right)^3 (b_{L}^{2/3} + b_{R}^{1/3}b_{R}^{1/3} + b_{R}^{2/3}) + \frac{3^3(b_{L}^{2/3} - b_{R}^{2/3})^6}{\alpha^8\kappa^{12}\rho^6} \right] \]

\[ \pm \left( 25b_{L}^{4/3} + 50b_{L}b_{R}^{1/3} + 42b_{L}^{2/3}b_{R}^{2/3} + 50b_{L}b_{R}^{1/3} + 25b_{R}^{4/3} \right) \]

\[ + \frac{2^2}{\alpha^2\kappa^8\rho^2} (b_{L}^{1/3} - b_{R}^{1/3})^4 (b_{L}^{1/3} + b_{R}^{1/3}) (19b_{L}^{4/3} + 57b_{L}b_{R}^{1/3} + 64b_{L}^{2/3}b_{R}^{2/3} + 57b_{L}b_{R}^{1/3} + 19b_{R}^{4/3}) \]

\[ + \frac{2}{\alpha^2\kappa^{12}\rho^6} (b_{L}^{1/3} - b_{R}^{1/3})^8 (b_{L}^{1/3} + b_{R}^{1/3})(2b_{L}^{2/3} + 5b_{L}b_{R}^{1/3} + 2b_{R}^{2/3}) \right]^{1/3} \right]. \tag{A.2} \]

Equations (11), (A.1) and (A.2) are sufficient to determine the cosmic dynamics on the brane if an equation of state is specified for the matter source. Such an analysis can be simplified by another form of solution:

\[ H^2 + \frac{k}{\alpha^2} + \frac{1}{4\alpha} - \frac{3(b_{L}^{2/3} - b_{R}^{2/3})^2}{2\alpha^2\kappa^4\rho^2} = 2q \cosh \frac{2}{3}x, \tag{A.3} \]

where

\[ q = \frac{1}{2\kappa} \left[ \frac{2^5\kappa^4\rho^2}{\alpha^2} + \frac{2^7}{\alpha^2\kappa^4\rho^2}(b_{L}^{1/3} - b_{R}^{1/3})^2 \times (b_{L} + 2b_{L}^{2/3}b_{R}^{1/3} + 2b_{L}^{1/3}b_{R}^{2/3} + b_{R}) + \frac{9(b_{L}^{2/3} - b_{R}^{2/3})^4}{\alpha^4\kappa^8\rho^4} \right]^{1/2} \]. \tag{A.4} \]

\[ \cosh 2x = \left[ \frac{2^{15}\alpha^4k_{16}\rho^8 + 2^{14}(b_{L} + b_{R})\alpha^3k_{12}\rho^6 + 2^8(b_{L}^{1/3} - b_{R}^{1/3})^2 \times (17b_{L}^{4/3} + 34b_{L}b_{R}^{1/3} + 42b_{L}^{2/3}b_{R}^{2/3} + 34b_{L}b_{R}^{1/3} + 17b_{L}^{4/3})\alpha^2k^8\rho^4 \right. \]

\[ + 2^8 \cdot 9(b_{L}^{1/3} - b_{R}^{1/3})^4 \left( b_{L}^{1/3} + b_{R}^{1/3}\right)^3 (b_{L}^{2/3} + b_{L}^{1/3}b_{R}^{1/3} + b_{R}^{2/3})\alpha^4\kappa^4\rho^2 \]

\[ + 27(b_{L}^{2/3} - b_{R}^{2/3}) \right]/ \left( 2^{24}\alpha^6k_{12}\rho^6q^4 \right). \tag{A.5} \]
These solutions reduce to the GB case with $Z_2$-symmetry when $b_L = b_R$. So the absence of the $Z_2$-symmetry gives rise to the above extra terms in the Friedmann equation. We find that the effects of the $Z_2$-breaking terms decrease with increasing time. This can be seen in the RS case [13]. Therefore, the scenario without this symmetry at late times [22] is not viable even in the GB case.

The standard form of the Friedmann equation must be recovered at low energy. Setting $k^4\alpha\rho^2$ and $\alpha\mu L^4$ as small variables, the Friedmann equation is approximated as

$$H^2 + \frac{k}{a^2} = \frac{b^{1/3} - 1}{4\alpha} + \frac{k^4(\sigma + \lambda)^2}{36b^{2/3}} + \frac{\mu_L + \mu_R}{2b^{1/3}a^4} + \frac{9(\mu_L - \mu_R)^2}{4k^4(\sigma + \lambda)^2a^8}, \quad (A.6)$$

where $b^{1/3} = \sqrt[3]{1+4}\lambda L/\Lambda$. We assume that the GB energy scale $(m_{GB} = (2b/k^4\alpha)^{1/3})$ is greater than the RS energy scale $(m_{RS} = \lambda^{1/4}) [20]$. This assumption comes from the consideration that the GB term is a correction to the RS gravity.

Equation (A.6) reduces to the RS case without $Z_2$-symmetry when $\alpha = 0$ [13]. In order to obtain standard cosmology at late times, we need to make the identification

$$\kappa^2_4 = \frac{1}{M_{pl}^2} = \frac{\kappa^4_\lambda}{6b^{2/3}}, \quad (A.7)$$

where $M_{pl}$ is the 4D reduced Planck mass. The 4D cosmological constant vanishes when the tension satisfies

$$\lambda = \frac{3}{2}(1 - b^{1/3})\frac{1}{\alpha\kappa^2_4}. \quad (A.8)$$

By equations (A.7) and (A.8), the requirement $(m_{GB} > m_{RS})$ becomes

$$\alpha < \frac{l^2}{22}. \quad (A.9)$$

This condition is consistent with equation (26).

A.2. The (+) branch

Next, we consider the (+) branch. The solution of the cubic equation (22) is the one where we have changed the sign of $b_{L,R}^{1/3}$ in the (−) branch. Thus, the effects of the $Z_2$-breaking terms also decrease with increasing time in this case. The Friedmann equation now becomes

$$H^2 + \frac{k}{a^2} + \frac{1}{4\alpha} - \frac{3(b_L^{1/3} - b_R^{1/3})^2}{2^6\alpha^2\kappa^4\rho^2} = c_+ + c_- \quad (A.10)$$

where

$$c_{\pm} = \frac{k^4}{2\alpha^2} \left\{ \frac{2^{14}k^4\rho^2}{\alpha^4} + \frac{2^6}{\alpha^4\kappa^4\rho^6} \left( b_L^{1/3} - b_R^{1/3} \right)^2 \right\} \times \left( 17b_L^{4/3} + 34b_L b_R^{1/3} + 42b_L^{2/3}b_R^{1/3} + 34b_L^{3/3}b_R + 17b_R^{4/3} \right)$$

$$- \frac{2^6}{\alpha^4\kappa^4\rho^6} \left( b_L^{1/3} - b_R^{1/3} \right)^2 \left( b_L^{1/3} + b_R^{1/3} \right)^3 \left( b_L^{2/3} + b_R^{2/3} \right)^2 + \frac{3^3}{\alpha^4\kappa^4\rho^6} \left( b_L^{2/3} - b_R^{2/3} \right)^6$$

$$ \pm 2^{12}\kappa^4 \rho \left[ \frac{2^{14}k^4\rho^2}{\alpha^4} - \frac{2^{12}(b_L + b_R)}{\alpha^5} + \frac{2^8}{\alpha^4\kappa^4\rho^6} \left( b_L^{1/3} - b_R^{1/3} \right)^2 \right] \times \left( 25b_L^{1/3} + 50b_L b_R^{1/3} + 42b_L^{2/3}b_R^{1/3} + 50b_L^{3/3}b_R + 25b_R^{4/3} \right)$$

$$\times \left( b_L^{1/3} + b_R^{1/3} \right) \left( 19b_L^{4/3} + 57b_L b_R^{1/3} + 64b_L^{2/3}b_R^{1/3} + 57b_L^{3/3}b_R + 19b_R^{4/3} \right)$$
Another form of solution is easy to handle:

\[ H^2 + \frac{k}{a^2} + \frac{1}{4\alpha} - \frac{3(b_{l/3}^2 - b_{R/3}^2)}{2^9\alpha^4 k^4 \rho^2} = 2q \cosh \frac{2}{3} x, \quad (A.12) \]

where

\[ q = \frac{1}{2^8} \left[ \frac{2^9(b_{l/3}^2 + b_{R/3}^2)}{\alpha^2} - \frac{2^7}{\alpha^4 k^4 \rho^2} (b_{l/3} - b_{R/3})^2 \right. \]
\[ \left. \times (b_{L} + 2b_{2/3} b_{R}^3 + 2b_{L} b_{R}^3 b_{R} + b_{R}) + \frac{9(b_{l/3}^2 - b_{R/3}^2)}{\alpha^4 k^4 \rho^2} \right]^{1/2}, \quad (A.13) \]

\[ \cosh 2x = \left[ 2^{15} \alpha^4 k^4 \rho^8 - 2^{14}(b_{L} + b_{R}) \alpha^3 k^2 \rho^6 + 2^8(b_{l/3}^2 - b_{R/3}^2)^2 \right. \]
\[ \left. \times (17b_{l/3}^2 + 34b_{L} b_{R}^3 + 42b_{L} b_{R}^2 b_{R} + 34b_{L} b_{R} + 17b_{R}^2) \alpha^2 k^2 \rho^4 \right. \]
\[ \left. - 2^6 \cdot 9(b_{l/3}^2 - b_{R/3}^2) (b_{l/3} + b_{R}) (b_{l/3} b_{R} + b_{R}^3) (b_{R}^3 + b_{R}^3 b_{R} + b_{R}^2) \alpha k^4 \rho^2 \right. \]
\[ \left. + 27(b_{l/3}^2 - b_{R/3}^2) \right] / (2^{24} \alpha^4 k^4 \rho^4 q^4). \quad (A.14) \]

The requirement that one should recover the conventional cosmology leads to the relation \( k_0^2 = k^4 \lambda / 6b^{2/3} \). The 4D effective cosmological constant vanishes when \( \lambda = \frac{1}{4} (1 - b^{1/3}) \). Using these relations, we find that the assumption \( (m_{\text{GB}} > m_{\text{RS}}) \) is inconsistent with (27).

Here, this assumption comes from the consideration that the GB term is a correction to the RS gravity. However, in this case the RS model is not recovered for \( \alpha = 0 \). So the condition \( (m_{\text{GB}} < m_{\text{RS}}) \) might be allowed. Let us study such a case.

Case 1 \((m_{\text{RS}} > 2m_{\text{GB}})\). Setting \((k^4 \alpha \rho^2)^{-1}\) and \(\alpha \mu / a^4\) as small variables, the Friedmann equation is approximated as

\[ H^2 + \frac{k}{a^2} = -\frac{1}{4\alpha} + \left( \frac{k^2 \rho}{16\alpha} \right)^{2/3} - \frac{1}{12\alpha} \left( \frac{b^{3/2}}{2\alpha k^2 \rho^2} \right)^{2/3} \sim -\frac{1}{4\alpha} + \left( \frac{k^2 \lambda}{16\alpha} \right)^{2/3} \]
\[ - \frac{1}{12\alpha} \left( \frac{b^{3/2}}{2\alpha k^4 \lambda^2} \right)^{2/3} + \left( \frac{2}{3\lambda} \left( \frac{k^2}{16\alpha} \right)^{2/3} - \frac{1}{9\alpha \lambda} \left( \frac{b^{3/2}}{2\alpha k^4 \lambda^2} \right)^{2/3} \right) \sigma. \quad (A.15) \]

In order to obtain the standard form, we get the following relations:

\[ k_0^2 = \frac{2}{\lambda} \left( \frac{k^2 \lambda}{16\alpha} \right)^{2/3} - \frac{1}{3\alpha \lambda} \left( \frac{b^{3/2}}{2\alpha k^4 \lambda^2} \right)^{2/3}, \quad (A.16) \]
\[ k^4 \lambda^2 = \frac{1}{3\alpha B^{2/3}} (16 + 4\beta + 8 B^{1/3} + \beta B^{1/3} + 4 B^{2/3} + B^{3/2} (12 + 3 B^{1/3} + B^{2/3})), \quad (A.17) \]

where \( \beta = \sqrt{b} (16b + 9) \) and \( B = 8 + 9b + 3\beta \). By equation (A.16), we find that the assumption \( (m_{\text{RS}} > 2m_{\text{GB}}) \) is consistent with equation (27) (see figure A1).
Case 2 \((1 \lesssim m_{RS}^8/m_{GB}^8 \lesssim 2)\). At late times, the Friedmann equation is approximated as

\[
H^2 + \frac{k}{a^2} = \frac{7b^{1/3}}{36\alpha} + \frac{\kappa^4\lambda^2}{36b^{2/3}} = \frac{\alpha\kappa^4\lambda^2(\mu_L + \mu_R)}{9b^{4/3}a^4} + \frac{7(\mu_L + \mu_R)}{18b^{1/3}a^4} \left(\frac{\kappa^4\lambda}{18b^{2/3}} - \frac{2\alpha\kappa^4\lambda(\mu_L + \mu_R)}{9b^{4/3}a^4}\right) + \frac{1}{36b^{2/3}} - \frac{\alpha(\mu_L + \mu_R)}{9b^{4/3}a^4}\right) \kappa^4\sigma^2. \tag{A.18}
\]

In order to obtain the standard form of the Friedmann equation, we get the following relations:

\[
\kappa_2^2 = \frac{\kappa^4\lambda}{6b^{2/3}} = \frac{2\alpha\kappa^4\lambda(\mu_L + \mu_R)}{3b^{4/3}a^4}, \tag{A.19}
\]

\[
\kappa^4\lambda^2 = \frac{9b^{2/3} - 7b}{\alpha}. \tag{A.20}
\]

where \(a_*\) is the scale factor. From equation (A.20), we find that the assumption \((1 \lesssim m_{RS}^8/m_{GB}^8 \lesssim 2)\) is satisfied if \(5/11 \lesssim \alpha l^{-2} \lesssim 17/36\). This condition is consistent with equation (27).

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