ON SPECTRAL THEORY FOR SCHRÖDINGER OPERATORS WITH OPERATOR-VALUED POTENTIALS

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Abstract. Given a complex, separable Hilbert space $\mathcal{H}$, we consider differential expressions of the type
\[ \tau = -(d^2/dx^2) + V(x), \]
with $x \in (a, \infty)$ or $x \in \mathbb{R}$. Here $V$ denotes a bounded operator-valued potential $V(\cdot) \in \mathcal{B}(\mathcal{H})$ such that $V(\cdot)$ is weakly measurable and the operator norm $\|V(\cdot)\|_{\mathcal{B}(\mathcal{H})}$ is locally integrable.

We consider self-adjoint half-line $L^2$-realizations $H_\alpha$ in $L^2((a, \infty); dx; \mathcal{H})$ associated with $\tau$, assuming $a$ to be a regular endpoint necessitating a boundary condition of the type $\sin(\alpha)u'(a) + \cos(\alpha)u(a) = 0$, indexed by the self-adjoint operator $\alpha = \alpha^* \in \mathcal{B}(\mathcal{H})$. In addition, we study self-adjoint full-line $L^2$-realizations $H$ of $\tau$ in $L^2(\mathbb{R}; dx; \mathcal{H})$. In either case we treat in detail basic spectral theory associated with $H_\alpha$ and $H$, including Weyl–Titchmarsh theory, Green's function structure, eigenfunction expansions, diagonalization, and a version of the spectral theorem.

1. Introduction

The principal topic of this paper centers around basic spectral theory, including Weyl–Titchmarsh theory, Green’s function structure, eigenfunction expansions, diagonalization, and a version of the spectral theorem for self-adjoint Schrödinger operators with bounded operator-valued potentials on a half-line as well as on the full real line. More precisely, given a complex, separable Hilbert space $\mathcal{H}$, we consider differential expressions $\tau$ of the type
\[ \tau = -(d^2/dx^2) + V(x), \]
with $x \in (a, \infty)$ or $x \in \mathbb{R}$, and $V$ a bounded operator-valued potential $V(\cdot) \in \mathcal{B}(\mathcal{H})$ such that $V(\cdot)$ is weakly measurable and the operator norm $\|V(\cdot)\|_{\mathcal{B}(\mathcal{H})}$ is locally integrable. The self-adjoint operators in question are then half-line $L^2$-realizations $H_\alpha$ in $L^2((a, \infty); dx; \mathcal{H})$, with $a$ assumed to be a regular endpoint for $\tau$, and hence with appropriate boundary conditions at $a$ (cf. (1.2)) on one hand, and full-line $L^2$-realizations of $\tau$ in $L^2(\mathbb{R}; dx; \mathcal{H})$ on the other.

The case of Schrödinger operators with operator-valued potentials under various continuity or smoothness hypotheses on $V(\cdot)$, and under various self-adjoint boundary conditions on bounded and unbounded open intervals, received considerable attention in the past. In the special case where $\dim(\mathcal{H}) < \infty$, that is, in the case of Schrödinger operators with matrix-valued potentials, the literature is so voluminous that we cannot possibly describe individual references and hence we primarily refer to the monographs [3], [108], and the references cited therein. We note that the finite-dimensional case, $\dim(\mathcal{H}) < \infty$, as discussed in [21], is of considerable
interest as it represents an important ingredient in some proofs of Lieb–Thirring inequalities (cf. [73]). In addition, the constant coefficient case, where $\tau$ is of the special form $\tau = -d^2/dx^2 + A$, has received overwhelming attention. But since this is not the focus of this paper we just refer to [62], [63] Chs. 3, 4, [86], and the literature cited therein.

In the particular case of Schrödinger-type operators corresponding to the differential expression $\tau = -(d^2/dx^2) + A + V(x)$ on a bounded interval $(a, b) \subset \mathbb{R}$ with either $A = 0$ or $A$ a self-adjoint operator satisfying $A \geq cI_H$ for some $c > 0$, unique solvability of boundary value problems, the asymptotic behavior of eigenvalues, and trace formulas in connection with various self-adjoint realizations of $\tau = -(d^2/dx^2) + A + V(x)$ on a bounded interval $(a, b)$ are discussed, for instance, in [11], [12], [13], [17], [59], [61], [64], [65], [91], [93] (for the case of spectral parameter dependent separated boundary conditions, see also [5], [7], [18]).

For earlier results on various aspects of boundary value problems, spectral theory, and scattering theory in the half-line case $(a, b)$, see also [5], [7], [18] with either $A$ or $V$ to $B$. For example, the constant coefficient case, where $\tau$ is in the limit point case at the end point $b$, is in [121]). While our treatment of initial value problems associated with $\tau$ in [121] was originally inspired by the one in Saitô [112], we do permit a more general local behavior of $V(\cdot)$. With respect to spectral theory for self-adjoint half-line realizations of $\tau$ in $L^2((a, \infty); dx; \mathcal{H})$, we refer to the fundamental paper by Gorbachuk [59]. Our treatment in this context again permits more general potentials $V(\cdot)$, we also provide all details in connection with the derivation of (1.1) (cf. (1.11)–(1.19), not present in [59]. Our $2 \times 2$ block operator approach in Section 5 in connection with full-line realizations of $\tau$ in $L^2(\mathbb{R}; dx; \mathcal{H})$, with special emphasis on the structure of the Green’s function (5.16) and the Weyl–Titchmarsh matrix (5.18) appears to be new, in particular, Theorems 5.2 and 5.4, represent the principal new results in this paper in this operator-valued setting.

Next we briefly turn to the content of each section: Section 2 recalls our basic results in [59] on the initial value problem associated with Schrödinger operators with bounded operator-valued potentials. We use this section to introduce some of the basic notation employed subsequently and note that our conditions on $V(\cdot)$ (cf. Hypothesis 2.7) are the most general to date with respect to the local behavior of the latter. Also Section 3 is of preparatory nature. Again following our detailed treatment in [59], we introduce maximal and minimal operators associated with $\tau = -(d^2/dx^2) + V(\cdot)$ on the interval $(a, b) \subset \mathbb{R}$ (eventually aiming at the case of a half-line $(a, \infty)$), and assuming that the left end point $a$ is regular for $\tau$ and that $\tau$ is in the limit point case at the end point $b$ we discuss a family of self-adjoint extensions $H_{\alpha}$ in $L^2((a, b); dx; \mathcal{H})$ corresponding to boundary conditions of the type

$$\sin(\alpha)u'(a) + \cos(\alpha)u(a) = 0,$$

(1.2)

indexed by the self-adjoint operator $\alpha = \alpha^* \in \mathcal{B}(\mathcal{H})$ with $u \in \mathcal{H}$ lying in the domain of the maximal operator $H_{\text{max}}$ corresponding to $\tau$. In addition, we recall elements of Weyl–Titchmarsh theory, culminating in the introduction of the operator-valued Weyl–Titchmarsh function $m_{\alpha}(\cdot) \in \mathcal{B}(\mathcal{H})$ and the Green’s function $G_{\alpha}(z, \cdot, \cdot) \in \mathcal{B}(\mathcal{H})$ of $H_{\alpha}$. Section 4 then presents our first set of principal spectral results for the right half-line $(a, \infty)$, denoting the corresponding self-adjoint right half-line operator in $L^2((a, \infty); dx; \mathcal{H})$ by $H_{+, \alpha}$: Theorem 4.2 and especially, Theorem 4.5, then yield a diagonalization of $H_{+, \alpha}$ and contain its underlying generalized
eigenfunction expansion, including a description of support properties of the $B(H)$-valued half-line spectral measure $d\rho_{+,\alpha}$. In particular, they illustrate the spectral theorem for $F(H_{+,\alpha})$, $F \in C(\mathbb{R})$. Our final Section 4 then derives the analogous results for full-line Schrödinger operators $H$ in $L^2(\mathbb{R}; dx; H)$, employing a $2 \times 2$ block operator representation of the associated Weyl–Titchmarsh $M_{\alpha}(\cdot, x_0)$-matrix and its $B(H^2)$-valued spectral measure $d\Omega_{\alpha}(\cdot, x_0)$, decomposing $\mathbb{R}$ into a left and right half-line with reference point $x_0 \in \mathbb{R}$, $(-\infty, x_0] \cup [x_0, \infty)$. The latter decomposition is familiar from the scalar and matrix-valued ($\dim(H) < \infty$) special cases. Our principal new results, Theorems 5.4 and 5.5 again yield a diagonalization of $H$ and the corresponding generalized eigenfunction expansion, illustrating the spectral theorem for $F(H)$ and support properties of $d\Omega_{\alpha}(\cdot, x_0)$. Appendix A collects basic facts on operator-valued Herglotz functions, some of which are of interest in their own right. Appendix B recalls several equivalent definitions of direct integrals of Hilbert spaces and constructions of the model Hilbert space $L^2(\mathbb{R}; d\Sigma; K)$ associated with a $B(K)$-valued measure $d\Sigma$ described in [53] and [57] and also describes a new connection with a construction due to Saitō [112]. The topics in both appendices are frequently used throughout this manuscript and we hope they render this paper sufficiently self-contained.

We should also add that while this paper completes our project on Schrödinger operators with bounded operator-valued potentials, it simultaneously represents the basis for the next step in this program: This step aims at certain classes of unbounded operator-valued potentials $V$, applicable to multi-dimensional Schrödinger operators in $L^2(\mathbb{R}^n; dx^n)$, $n \in \mathbb{N}, n \geq 2$, generated by differential expressions of the type $\Delta + V(\cdot)$. It was precisely the connection between multi-dimensional Schrödinger operators and one-dimensional Schrödinger operators with unbounded operator-valued potentials which originally motivated our interest in this circle of ideas. This connection was already employed by Kato [72] in 1959; for more recent applications of this connection between one-dimensional Schrödinger operators with unbounded operator-valued potentials and multi-dimensional Schrödinger operators we refer, for instance, to [2], [35], [69], [79], [88], [89], [90], [110], [111], [113]–[116], and the references cited therein.

Finally, we comment on the notation used in this paper: Throughout, $\mathcal{H}$ denotes a separable, complex Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ (linear in the second argument) and $\| \cdot \|_{\mathcal{H}}$, respectively. The identity operator in $\mathcal{H}$ is written as $I_{\mathcal{H}}$. We denote by $\mathcal{B}(\mathcal{H})$ (resp., $\mathcal{B}_c(\mathcal{H})$) the Banach space of linear bounded (resp., compact) operators in $\mathcal{H}$. The domain, range, kernel (null space), resolvent set, and spectrum of a linear operator will be denoted by $\text{dom}(\cdot)$, $\text{ran}(\cdot)$, $\ker(\cdot)$, $\rho(\cdot)$, and $\sigma(\cdot)$, respectively. The closure of a closable operator $S$ in $\mathcal{H}$ is denoted by $\overline{S}$.

By $\mathcal{B}(\mathbb{R})$ we denote the collection of Borel subsets of $\mathbb{R}$.

2. THE INITIAL VALUE PROBLEM ASSOCIATED WITH SCHRÖDINGER OPERATORS WITH OPERATOR-VALUED POTENTIALS REVISITED

In this section we recall the basic results about initial value problems for second-order differential equations of the form $-y'' + Qy = f$ on an arbitrary open interval $(a, b) \subseteq \mathbb{R}$ with a bounded operator-valued coefficient $Q$, that is, when $Q(x)$ is a bounded operator on a separable, complex Hilbert space $\mathcal{H}$ for a.e. $x \in (a, b)$. In fact, we are interested in two types of situations: In the first one $f(x)$ is an element
of the Hilbert space $\mathcal{H}$ for a.e. $x \in (a, b)$, and the solution sought is to take values in $\mathcal{H}$. In the second situation, $f(x)$ is a bounded operator on $\mathcal{H}$ for a.e. $x \in (a, b)$, as is the proposed solution $y$.

All results recalled in this section were proved in detail in [56].

We start with some necessary preliminaries: Let $(a, b) \subseteq \mathbb{R}$ be a finite or infinite interval and $X$ a Banach space. Unless explicitly stated otherwise (such as in the context of operator-valued measures in Herglotz representations, cf. Appendix A), integration of $X$-valued functions on $(a, b)$ will always be understood in the sense of Bochner (cf., e.g., [15, p. 6–21], [42, p. 44–50], [67, p. 71–86], [87, Ch. III], [125, p. 71–86]). In particular, if $p \geq 1$, the symbol $L^p((a, b); dx; X)$ denotes the set of equivalence classes of strongly measurable $X$-valued functions which differ at most on sets of Lebesgue measure zero, such that $\|f(\cdot)\|_X^p \in L^1((a, b); dx)$. The corresponding norm in $L^p((a, b); dx; X)$ is given by

$$\|f\|_{L^p((a, b); dx; X)} = \left( \int_{(a, b)} dx \|f(x)\|_X^p \right)^{1/p}$$

(2.1)

and $L^p((a, b); dx; X)$ is a Banach space.

If $\mathcal{H}$ is a separable Hilbert space, then so is $L^2((a, b); dx; \mathcal{H})$ (see, e.g., [19, Subsects. 4.3.1, 4.3.2], [28, Sect. V.5]) for details). In particular, if $p \geq 1$, the symbol $L^p((a, b); dx; X)$ denotes the set of equivalence classes of strongly measurable $X$-valued functions which differ at most on sets of Lebesgue measure zero, such that $\|f(\cdot)\|_X^p \in L^1((a, b); dx)$. The corresponding norm in $L^p((a, b); dx; X)$ is given by

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Following a frequent practice (cf., e.g., the discussion in [14 Sect. III.1.2]), we will call elements of \( W^{1,1}([c, d]; dx; \mathcal{X}) \) strongly absolutely continuous \( \mathcal{X} \)-valued functions on \([c, d]\) (resp., strongly locally absolutely continuous \( \mathcal{X} \)-valued functions on \((a, b)\)), but caution the reader that unless \( \mathcal{X} \) possesses the Radon–Nikodym (RN) property, this notion differs from the classical definition of \( \mathcal{X} \)-valued absolutely continuous functions (we refer the interested reader to [42 Sect. VII.6] for an extensive list of conditions equivalent to \( \mathcal{X} \) having the RN property). Here we just mention that reflexivity of \( \mathcal{X} \) implies the RN property.

In the special case where \( \mathcal{X} = \mathbb{C} \), we omit \( \mathcal{X} \) and just write \( L^p_{(loc)}((a, b); dx) \), as usual.

**A Remark on notational convention:** To avoid possible confusion later on between two standard notions of strongly continuous operator-valued functions \( F(x), x \in (a, b) \), that is, strong continuity of \( F(\cdot)h \) in \( \mathcal{H} \) for all \( h \in \mathcal{H} \) (i.e., pointwise continuity of \( F(\cdot) \)), versus strong continuity of \( F(\cdot) \) in the norm of \( B(\mathcal{H}) \) (i.e., uniform continuity of \( F(\cdot) \)), we will always mean pointwise continuity of \( F(\cdot) \) in \( \mathcal{H} \). The same pointwise conventions will apply to the notions of strongly differentiable and strongly measurable operator-valued functions throughout this manuscript. In particular, and unless explicitly stated otherwise, for operator-valued functions \( Y \), the symbol \( Y' \) will be understood in the strong sense; similarly, \( y' \) will denote the strong derivative for vector-valued functions \( y \).

We start by recalling the following elementary, yet useful lemma:

**Lemma 2.1.** Let \((a, b) \subseteq \mathbb{R}\). Suppose \( Q: (a, b) \to B(\mathcal{H}) \) is a weakly measurable operator-valued function with \( \|Q(\cdot)\|_{B(\mathcal{H})} \in L^1_{loc}((a, b); dx) \) and \( g: (a, b) \to \mathcal{H} \) is (weakly) measurable. Then \( Qg \) is (strongly) measurable. Moreover, if \( g \) is strongly continuous, then there exists a set \( E \subset (a, b) \) with zero Lebesgue measure, depending only on \( Q \), such that for every \( x_0 \in (a, b) \setminus E \),

\[
\lim_{t \downarrow 0} \frac{1}{t} \int_{x_0}^{x_0+t} dx \|Q(x)g(x) - Q(x_0)g(x_0)\| = 0, \tag{2.6}
\]

In particular,

\[
s\lim_{t \downarrow 0} \frac{1}{t} \int_{x_0}^{x_0+t} dx Q(x)g(x) = Q(x_0)g(x_0). \tag{2.7}
\]

In addition, the set of Lebesgue points of \( Q(\cdot)g(\cdot) \) can be chosen independently of \( g \).

In connection with [27], we also refer to [42 Theorem II.2.9], [67 Subsect. III.3.8], [125 Theorem V.5.2].

**Definition 2.2.** Let \((a, b) \subseteq \mathbb{R}\) be a finite or infinite interval and \( Q: (a, b) \to B(\mathcal{H}) \) a weakly measurable operator-valued function with \( \|Q(\cdot)\|_{B(\mathcal{H})} \in L^1_{loc}((a, b); dx) \), and suppose that \( f \in L^1_{loc}((a, b); dx; \mathcal{H}) \). Then the \( \mathcal{H} \)-valued function \( y: (a, b) \to \mathcal{H} \) is called a (strong) solution of

\[
-y'' + Qy = f \tag{2.8}
\]

if \( y \in W^{2,1}_{loc}((a, b); dx; \mathcal{H}) \) and \( (2.8) \) holds a.e. on \((a, b)\).

We recall our notational convention that vector-valued solutions of \( (2.8) \) will always be viewed as strong solutions.
Moreover, the following properties hold:

where the exceptional set \( E \) problem

\[ \text{Corollary 2.5.} \]

\( a, b \) on \((a, b)\) and only if

\[ \text{Theorem 2.3.} \]

the smoothness hypotheses on Chs. III, VII] and [43, Ch. 10], but we emphasize again that our approach minimizes if \( Y \)

\[ \text{Definition 2.4.} \]

One verifies that \( Q : (a, b) \to B(H) \) satisfies the conditions in Definition 2.2 if and only if \( Q^* \) does (a fact that will play a role later on, cf. the paragraph following 2.10).

For fixed \( x, x_0 \) and only if 

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For classical references on initial value problems we refer, for instance, to [36 Chs. III, VII] and [38, Ch. 10], but we emphasize again that our approach minimizes the smoothness hypotheses on \( V \) and \( f \).

\[ \text{Definition 2.4.} \]

\( (a, b) \subseteq \mathbb{R} \) be a finite or infinite interval and assume that \( F, Q : (a, b) \to B(H) \) are two weakly measurable operator-valued functions such that \( ||F(\cdot)||_{B(H)}, ||Q(\cdot)||_{B(H)} \in L^1_{\text{loc}}((a, b); dx) \). Then the \( B(H) \)-valued function \( Y : (a, b) \to B(H) \) is called a solution of

\[ \text{Definition 2.4.} \]

\[ \text{Definition 2.4.} \]

\[ \text{Corollary 2.5.} \]

\( (a, b) \subseteq \mathbb{R} \) be a finite or infinite interval, \( x_0 \in (a, b), z \in \mathbb{C}, Y_0, Y_1 \in B(H) \), and suppose \( F, V : (a, b) \to B(H) \) are two weakly measurable operator-valued functions with \( ||V(\cdot)||_{B(H)}, ||F(\cdot)||_{B(H)} \in L^1_{\text{loc}}((a, b); dx) \). Then there is a unique \( B(H) \)-valued solution \( Y(z, \cdot, x_0) : (a, b) \to B(H) \) of the initial value problem

\[ \text{Corollary 2.5.} \]

where the exceptional set \( E \) is of Lebesgue measure zero and independent of \( z \).

Moreover, the following properties hold:
(i) For fixed \( x_0 \in (a, b) \) and \( z \in \mathbb{C} \), \( Y(z, x, x_0) \) is continuously differentiable with respect to \( x \) on \( (a, b) \) in the \( \mathcal{B}(\mathcal{H}) \)-norm.

(ii) For fixed \( x_0 \in (a, b) \) and \( z \in \mathbb{C} \), \( Y'(z, x, x_0) \) is strongly differentiable with respect to \( x \) on \( (a, b) \setminus E \).

(iii) For fixed \( x_0, x \in (a, b) \), \( Y(z, x, x_0) \) and \( Y'(z, x, x_0) \) are entire in \( z \) in the \( \mathcal{B}(\mathcal{H}) \)-norm.

Various versions of Theorem 2.3 and Corollary 2.5 exist in the literature under varying assumptions on \( V \) and \( Q \). For instance, the case where \( V(\cdot) \) is continuous in the \( \mathcal{B}(\mathcal{H}) \)-norm and \( F = 0 \) is discussed in [66, Theorem 6.1.1]. The case, where \( \|V(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}([a, \infty]; dx) \) for all \( c > a \) and \( F = 0 \) is discussed in detail in [66]. It appears that a measurability assumption of \( V(\cdot) \) in the \( \mathcal{B}(\mathcal{H}) \)-norm is missing in the basic set of hypotheses of [112]. Our extension to \( V(\cdot) \) weakly measurable and \( \|V(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}([a, b]; dx) \) in [66] may well be the most general one published to date.

**Definition 2.6.** Pick \( c \in (a, b) \). The endpoint \( a \) (resp., \( b \)) of the interval \( (a, b) \) is called **regular** for the operator-valued differential expression \(-d^2/dx^2 + Q(\cdot)\) if it is finite and if \( Q \) is weakly measurable and \( \|Q(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}([a, c]; dx) \) (resp., \( \|Q(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}([c, b]; dx) \)) for some \( c \in (a, b) \). Similarly, \(-d^2/dx^2 + Q(\cdot)\) is called **regular at** \( a \) (resp., **regular at** \( b \)) if \( a \) (resp., \( b \)) is a regular endpoint for \(-d^2/dx^2 + Q(\cdot)\).

We note that if \( a \) (resp., \( b \)) is regular for \(-d^2/dx^2 + Q(x)\), one may allow for \( x_0 \) to be equal to \( a \) (resp., \( b \)) in the existence and uniqueness Theorem 2.3.

If \( f_1, f_2 \) are strongly continuously differentiable \( \mathcal{H} \)-valued functions, we define the Wronskian of \( f_1 \) and \( f_2 \) by

\[
W_\ast(f_1, f_2)(x) = (f_1(x), f_2(x))_\mathcal{H} - (f_1'(x), f_2'(x))_\mathcal{H}, \quad x \in (a, b).
\]

If \( f_2 \) is an \( \mathcal{H} \)-valued solution of \(-y'' + Qy = 0\) and \( f_1 \) is an \( \mathcal{H} \)-valued solution of \(-y'' + Q^*y = 0\), their Wronskian \( W_\ast(f_1, f_2)(x) \) is \( x \)-independent, that is,

\[
\frac{d}{dx}W_\ast(f_1, f_2)(x) = 0, \quad \text{for a.e. } x \in (a, b).
\]

Equation (2.13) will show that the right-hand side of (2.14) actually vanishes for all \( x \in (a, b) \).

We decided to use the symbol \( W_\ast(\cdot, \cdot) \) in (2.14) to indicate its conjugate linear behavior with respect to its first entry.

Similarly, if \( F_1, F_2 \) are strongly continuously differentiable \( \mathcal{B}(\mathcal{H}) \)-valued functions, their Wronskian is defined by

\[
W(F_1, F_2)(x) = F_1(x)F_2'(x) - F_1'(x)F_2(x), \quad x \in (a, b).
\]

Again, if \( F_2 \) is a \( \mathcal{B}(\mathcal{H}) \)-valued solution of \(-Y'' + QY = 0\) and \( F_1 \) is a \( \mathcal{B}(\mathcal{H}) \)-valued solution of \(-Y'' + Q^*Y = 0\) (the latter is equivalent to \(-Y'' + Q^*Y = 0\) and hence can be handled in complete analogy via Theorem 2.3 and Corollary 2.5 replacing \( Q \) by \( Q^* \)) their Wronskian will be \( x \)-independent,

\[
\frac{d}{dx}W(F_1, F_2)(x) = 0, \quad \text{for a.e. } x \in (a, b).
\]

Our main interest is in the case where \( V(\cdot) = V(\cdot)^* \in \mathcal{B}(\mathcal{H}) \) is self-adjoint, that is, in the differential equation \( \tau \eta = z \eta \), where \( \eta \) represents an \( \mathcal{H} \)-valued,
Thus the operator $\Theta$ has a left inverse given by

$$\tau = -(d^2/dx^2) + V(\cdot).$$

(2.17)

To this end, we now introduce the following basic assumption:

**Hypothesis 2.7.** Let $(a, b) \subseteq \mathbb{R}$, suppose that $V : (a, b) \rightarrow \mathcal{B}(\mathcal{H})$ is a weakly measurable operator-valued function with $\|V(\cdot)\|_{\mathcal{B}(\mathcal{H})} \in L^1_{\text{loc}}((a, b); dx)$, and assume that $V(x) = V(x)^*$ for a.e. $x \in (a, b)$.

Moreover, for the remainder of this section we assume that $\alpha \in \mathcal{B}(\mathcal{H})$ is a self-adjoint operator,

$$\alpha = \alpha^* \in \mathcal{B}(\mathcal{H}).$$

(2.18)

Assuming Hypothesis [2.7] and [2.13], we introduce the standard fundamental systems of operator-valued solutions of $\tau y = zy$ as follows: Since $\alpha$ is a bounded self-adjoint operator, one may define the self-adjoint operators $A = \sin(\alpha)$ and $B = \cos(\alpha)$ via the spectral theorem. One then concludes that $\sin^2(\alpha) + \cos^2(\alpha) = I_H$ and $[\sin\alpha, \cos\alpha] = 0$ (here $[\cdot, \cdot]$ represents the commutator symbol). The spectral theorem implies also that the spectra of $\sin(\alpha)$ and $\cos(\alpha)$ are contained in $[-1, 1]$ and that the spectra of $\sin^2(\alpha)$ and $\cos^2(\alpha)$ are contained in $[0, 1]$. Given such an operator $\alpha$ and a point $x_0 \in (a, b)$ or a regular endpoint for $\tau$, we now define $\theta_\alpha(z, x_0, x_0), \phi_\alpha(z, x_0)$ as those $\mathcal{B}(\mathcal{H})$-valued solutions of $\tau Y = zY$ (in the sense of Definition 2.4) which satisfy the initial conditions

$$\theta_\alpha(z, x_0, x_0) = \phi_\alpha(z, x_0, x_0) = \cos(\alpha), \quad -\phi_\alpha(z, x_0, x_0) = \theta_\alpha'(z, x_0, x_0) = \sin(\alpha).$$

(2.19)

By Corollary 2.5(iii), for any fixed $x, x_0 \in (a, b)$, the functions $\theta_\alpha(z, x, x_0)$ and $\phi_\alpha(z, x, x_0)$ as well as their strong $x$-derivatives are entire with respect to $z$ in the $\mathcal{B}(\mathcal{H})$-norm. The same is true for the functions $z \mapsto \theta_\alpha(z, x_0, x_0)^*$ and $z \mapsto \phi_\alpha(z, x_0, x_0)^*$. Since $\theta_\alpha(z, x_0, x_0)^*$ and $\phi_\alpha(z, x_0, x_0)^*$ satisfy the adjoint equation $-Y'' + YV = zY$ and the same initial conditions as $\theta_\alpha$ and $\phi_\alpha$, respectively, one obtains the following identities from the constancy of Wronskians:

$$\theta_\alpha'(z, x, x_0)^*\theta_\alpha(z, x, x_0) - \theta_\alpha(z, x, x_0)^*\theta_\alpha'(z, x, x_0) = 0,$$

(2.20)

$$\phi_\alpha'(z, x, x_0)^*\phi_\alpha(z, x, x_0) - \phi_\alpha(z, x, x_0)^*\phi_\alpha'(z, x, x_0) = 0,$$

(2.21)

$$\phi_\alpha'(z, x, x_0)^*\alpha_\alpha(z, x, x_0) - \phi_\alpha(z, x, x_0)^*\theta_\alpha'(z, x, x_0) = \mathbb{I}_H,$$

(2.22)

$$\theta_\alpha'(z, x, x_0)^*\phi_\alpha(z, x, x_0) - \theta_\alpha(z, x, x_0)^*\phi_\alpha'(z, x, x_0) = \mathbb{I}_H.$$

(2.23)

Equations [2.20]–[2.23] are equivalent to the statement that the block operator

$$\Theta_\alpha(z, x_0) = \begin{pmatrix} \theta_\alpha(z, x, x_0) & \phi_\alpha(z, x_0, x_0) \\ \phi_\alpha'(z, x, x_0) & \phi_\alpha'(z, x_0, x_0) \end{pmatrix}$$

(2.24)

has a left inverse given by

$$\begin{pmatrix} \phi_\alpha'(z, x_0, x_0)^* & -\phi_\alpha(z, x_0, x_0)^* \\ -\phi_\alpha'(z, x_0, x_0)^* & \theta_\alpha'(z, x_0, x_0)^* \end{pmatrix}.$$

(2.25)

Thus the operator $\Theta_\alpha(z, x_0)$ is injective. It is also surjective as will be shown next: Let $(f_1, g_1)^T$ be an arbitrary element of $\mathcal{H} \oplus \mathcal{H}$ and let $y$ be an $\mathcal{H}$-valued
solution of the initial value problem
\[
\begin{align*}
\tau y &= zy, \\
y(x_1) &= f_1, \quad y'(x_1) = g_1,
\end{align*}
\] (2.26)
for some given \(x_1 \in (a, b)\). One notes that due to the initial conditions specified in (2.19), \(\Theta_\alpha(z, x_0, x_0)\) is bijective. We now assume that \((f_0, g_0)^\top\) are given by
\[
\Theta_\alpha(z, x_0, x_0) \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} = \begin{pmatrix} y(x_0) \\ y'(x_0) \end{pmatrix}.
\] (2.27)
The existence and uniqueness Theorem 2.3 then yields that
\[
\Theta_\alpha(z, x_1, x_0) \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}.
\] (2.28)
This establishes surjectivity of \(\Theta_\alpha(z, x_1, x_0)\) which therefore has a right inverse too, also given by (2.28). This fact then implies the following identities:
\[
\begin{align*}
\phi_\alpha(z, x, x_0)\theta_\alpha(z, x, x_0)^* - \theta_\alpha(z, x, x_0)\phi_\alpha(z, x, x_0)^* &= 0, \\
\phi'_\alpha(z, x, x_0)\theta'_\alpha(z, x, x_0)^* - \theta'_\alpha(z, x, x_0)\phi'_\alpha(z, x, x_0)^* &= 0, \\
\phi''_\alpha(z, x, x_0)\theta''_\alpha(z, x, x_0)^* - \theta''_\alpha(z, x, x_0)\phi''_\alpha(z, x, x_0)^* &= I_\mathcal{H}, \\
\theta_\alpha(z, x, x_0)\phi'_\alpha(z, x, x_0)^* - \phi_\alpha(z, x, x_0)\theta'_\alpha(z, x, x_0)^* &= I_\mathcal{H}.
\end{align*}
\] (2.29)-(2.32)
Having established the invertibility of \(\Theta_\alpha(z, x_1, x_0)\) we can now show that for any \(x_1 \in (a, b)\), any \(\mathcal{H}\)-valued solution of \(\tau y = zy\) may be expressed in terms of \(\theta_\alpha(z, \cdot, x_1)\) and \(\phi_\alpha(z, \cdot, x_1)\), that is,
\[
y(x) = \theta_\alpha(z, x, x_1)f + \phi_\alpha(z, x, x_1)g
\] (2.33)
for appropriate vectors \(f, g \in \mathcal{H}\) or \(\mathcal{B}(\mathcal{H})\).

We also recall several versions of Green’s formula (also called Lagrange’s identity).

Lemma 2.8. Let \((a, b) \subseteq \mathbb{R}\) be a finite or infinite interval and \([x_1, x_2] \subset (a, b)\).
(i) Assume that \(f, g \in W_{2,1}^1((a, b); dx; \mathcal{H})\). Then
\[
\int_{x_1}^{x_2} dx \left[ ((\tau f)(x), g(x))_{\mathcal{H}} - (f(x), (\tau g)(x))_{\mathcal{H}} \right] = W_s(f, g)(x_2) - W_s(f, g)(x_1).
\] (2.34)
(ii) Assume that \(F : (a, b) \to \mathcal{B}(\mathcal{H})\) is absolutely continuous, that \(F'\) is again differentiable, and that \(F''\) is weakly measurable. Also assume that \(\|F''\|_{\mathcal{H}} \in L_{\text{loc}}^1((a, b); dx)\) and \(g \in W_{2,1}^1((a, b); dx; \mathcal{H})\). Then
\[
\int_{x_1}^{x_2} dx \left[ (\tau F^*)(x)g(x) - F(x)(\tau g)(x) \right] = (F'g' - Fg')(x_2) - (F'g' - Fg')(x_1).
\] (2.35)
(iii) Assume that \(F, G : (a, b) \to \mathcal{B}(\mathcal{H})\) are absolutely continuous operator-valued functions such that \(F', G'\) are again differentiable and that \(F'', G''\) are weakly measurable. In addition, suppose that \(\|F''\|_{\mathcal{H}}, \|G''\|_{\mathcal{H}} \in L_{\text{loc}}^1((a, b); dx)\). Then
\[
\int_{x_1}^{x_2} dx \left[ (\tau F^*)(x)G(x) - F(x)(\tau G)(x) \right] = (F'G' - F'G)(x_2) - (F'G' - F'G)(x_1).
\] (2.36)
3. HALF-LINE WEYL–TITCHMARSH THEORY FOR SCHröDINGER OPERATORS WITH OPERATOR-VALUED POTENTIALS REVISITED

In this section we recall the basics of Weyl–Titchmarsh theory for self-adjoint Schrödinger operators \( H_\alpha \) in \( L^2((a, b); dx; \mathcal{H}) \) associated with the operator-valued differential expression \( \tau = -(d^2/dx^2) + V(\cdot) \), assuming regularity of the left endpoint \( a \) and the limit point case at the right endpoint \( b \) (see Definition 3.4). We discuss the existence of Weyl–Titchmarsh solutions, introduce the corresponding Weyl–Titchmarsh \( m \)-function, and determine the structure of the Green’s function of \( H_\alpha \).

All results recalled in this section were proved in detail in \([56]\). As before, \( \mathcal{H} \) denotes a separable Hilbert space and \((a, b)\) denotes a finite or infinite interval. One recalls that \( L^2((a, b); dx; \mathcal{H}) \) is separable (since \( \mathcal{H} \) is) and that \( (f, g)_{L^2((a, b); dx; \mathcal{H})} = \int_a^b dx \left( f(x), g(x) \right)_{\mathcal{H}}, \quad f, g \in L^2((a, b); dx; \mathcal{H}). \) (3.1)

Assuming Hypothesis 2.7 throughout this section, we are interested in studying certain self-adjoint operators in \( L^2((a, b); dx; \mathcal{H}) \) associated with the operator-valued differential expression \( \tau = -(d^2/dx^2) + V(\cdot) \). These will be suitable restrictions of the maximal operator \( H_{\text{max}} \) in \( L^2((a, b); dx; \mathcal{H}) \) defined by

\[
H_{\text{max}} f = \tau f,
\]

\[
f \in \text{dom}(H_{\text{max}}) = \{ g \in L^2((a, b); dx; \mathcal{H}) \mid g \in W^{2,1}_{\text{loc}}((a, b); dx; \mathcal{H}); \tau g \in L^2((a, b); dx; \mathcal{H}) \}. \] (3.2)

We also introduce the operator \( \dot{H}_{\text{min}} \) in \( L^2((a, b); dx; \mathcal{H}) \) as the restriction of \( H_{\text{max}} \) to the domain

\[
\text{dom}(\dot{H}_{\text{min}}) = \{ g \in \text{dom}(H_{\text{max}}) \mid \text{supp}(u) \text{ is compact in } (a, b) \}. \] (3.3)

Finally, the minimal operator \( H_{\text{min}} \) in \( L^2((a, b); dx; \mathcal{H}) \) associated with \( \tau \) is then defined as the closure of \( \dot{H}_{\text{min}} \),

\[
H_{\text{min}} = \overline{\dot{H}_{\text{min}}}. \] (3.4)

Next, we intend to show that \( H_{\text{max}} \) is the adjoint of \( \dot{H}_{\text{min}} \) (and hence that of \( H_{\text{min}} \)), implying, in particular, that \( H_{\text{max}} \) is closed. To this end, we first establish the following two preparatory lemmas for the case where \( a \) and \( b \) are both regular endpoints for \( \tau \) in the sense of Definition 2.6.

**Lemma 3.1.** In addition to Hypothesis 2.7, suppose that \( a \) and \( b \) are regular endpoints for \( \tau \). Then

\[
\ker(H_{\text{max}} - zI_{L^2((a, b); dx; \mathcal{H})}) = \{ [\theta_0(z, \cdot, a)f + \phi_0(z, \cdot, a)g] \in L^2((a, b); dx; \mathcal{H}) \mid f, g \in \mathcal{H} \},
\] (3.5)

is a closed subspace of \( L^2((a, b); dx; \mathcal{H}) \).

Of course, if \( \mathcal{H} \) is finite-dimensional (e.g., in the scalar case, \( \dim(\mathcal{H}) = 1 \)), then \( \ker(H_{\text{max}} - zI_{L^2((a, b); dx; \mathcal{H})}) \) is finite-dimensional and hence automatically closed.
Lemma 3.2. In addition to Hypothesis 2.7 suppose that a and b are regular endpoints for $\tau$. Denote by $H_0$ the linear operator in $L^2((a,b); dx; H)$ defined by the restriction of $H_{\max}$ to the space
\[
\text{dom}(H_0) = \{ g \in \text{dom}(H_{\max}) \mid g(a) = g(b) = g'(a) = g'(b) = 0 \}. \tag{3.6}
\]
Then
\[
\ker(H_{\max}) = \overline{\text{ran}(H_0)}, \tag{3.7}
\]
that is, the space of solutions $u$ of $\tau u = 0$ coincides with the orthogonal complement of the collection of elements $\tau u_0$ satisfying $u_0 \in \text{dom}(H_0)$.

Theorem 3.3. Assume Hypothesis 2.7. Then the operator $\hat{H}_{\min}$ is densely defined. Moreover, $H_{\max}$ is the adjoint of $H_{\min}$,
\[
H_{\max} = (H_{\min})^*. \tag{3.8}
\]
In particular, $H_{\max}$ is closed. In addition, $\hat{H}_{\min}$ is symmetric and $H_{\max}^*$ is the closure of $\hat{H}_{\min}$, that is,
\[
H_{\max}^* = \overline{\hat{H}_{\min}} = H_{\min}. \tag{3.9}
\]

Lemmas 3.4 3.5 and Theorem 3.3 under additional hypotheses on $V$ (typically involving continuity assumptions) are of course well-known and go back to Rofe-Beketov [105, 106] (see also [63, Sect. 3.4], [108, Ch. 5]).

In the special case where $a$ and $b$ are regular endpoints for $\tau$, the operator $H_0$ introduced in (3.6) coincides with the minimal operator $H_{\min}$.

Using the dominated convergence theorem and Green’s formula 2.34 one can show that $\lim_{x \to a} W_*(u,v)(x)$ and $\lim_{x \to b} W_*(u,v)(x)$ both exist whenever $u, v \in \text{dom}(H_{\max})$. We will denote these limits by $W_*(u,v)(a)$ and $W_*(u,v)(b)$, respectively. Thus Green’s formula also holds for $x_1 = a$ and $x_2 = b$ if $u$ and $v$ are in $\text{dom}(H_{\max})$, that is,
\[
(H_{\max}u,v)_{L^2((a,b); dx; H)} - (u,H_{\max}v)_{L^2((a,b); dx; H)} = W_*(u,v)(b) - W_*(u,v)(a). \tag{3.10}
\]
This relation and the fact that $H_{\min} = H_{\max}^*$ is a restriction of $H_{\max}$ show that
\[
\text{dom}(H_{\min}) = \{ u \in \text{dom}(H_{\max}) \mid W_*(u,v)(b) = W_*(u,v)(a) = 0 \}
\]
for all $v \in \text{dom}(H_{\max})$. \tag{3.11}

Definition 3.4. Assume Hypothesis 2.7. Then the endpoint $a$ (resp., $b$) is said to be of limit-point type for $\tau$ if $W_*(u,v)(a) = 0$ (resp., $W_*(u,v)(b) = 0$) for all $u, v \in \text{dom}(H_{\max})$.

Next, we introduce the subspaces
\[
D_z = \{ u \in \text{dom}(H_{\max}) \mid H_{\max}u = zu \}, \quad z \in \mathbb{C}. \tag{3.12}
\]
For $z \in \mathbb{C} \setminus \mathbb{R}$, $D_z$ represent the deficiency subspaces of $H_{\min}$. Von Neumann’s theory of extensions of symmetric operators implies that
\[
\text{dom}(H_{\max}) = \text{dom}(H_{\min}) + D_i + D_{-i}. \tag{3.13}
\]
where $+$ indicates the direct (but not necessarily orthogonal direct) sum.

We now set out to determine the self-adjoint restrictions of $H_{\max}$ assuming that $a$ is a regular endpoint for $\tau$ and $b$ is of limit-point type for $\tau$.

Hypothesis 3.5. In addition to Hypothesis 2.7 suppose that $a$ is a regular endpoint for $\tau$ and $b$ is of limit-point type for $\tau$. 
Theorem 3.6. Assume Hypothesis 3.5 If \( H \) is a self-adjoint restriction of \( H_{\text{max}} \), then there is a bounded and self-adjoint operator \( \alpha \in \mathcal{B}(\mathcal{H}) \) such that
\[
\text{dom}(H) = \{ u \in \text{dom}(H_{\text{max}}) \mid \sin(\alpha)u'(a) + \cos(\alpha)u(a) = 0 \}. \tag{3.14}
\]
Conversely, for every \( \alpha \in \mathcal{B}(\mathcal{H}) \), \( \text{dom}(H) \) gives rise to a self-adjoint restriction of \( H_{\text{max}} \) in \( L^2((a,b); dx; \mathcal{H}) \).

Henceforth, under the assumptions of Theorem 3.6, we denote the operator \( H \) in \( L^2((a,b); dx; \mathcal{H}) \) associated with the boundary condition induced by \( \alpha = \alpha^* \in \mathcal{B}(\mathcal{H}) \), that is, the restriction of \( H_{\text{max}} \) to the set
\[
\text{dom}(H_\alpha) = \{ u \in \text{dom}(H_{\text{max}}) \mid \sin(\alpha)u'(a) + \cos(\alpha)u(a) = 0 \} \tag{3.15}
\]
by \( H_\alpha \). For a discussion of boundary conditions at infinity, see, for instance, [85], [92], and [107].

Our next goal is to construct the square integrable solutions \( Y(z, \cdot) \in \mathcal{B}(\mathcal{H}) \) of \( \tau Y = zY \), \( z \in \mathbb{C} \setminus \mathbb{R} \), the \( \mathcal{B}(\mathcal{H}) \)-valued Weyl–Titchmarsh solutions, under the assumptions that \( a \) is a regular endpoint for \( \tau \) and \( b \) is of limit-point type for \( \tau \).

Fix \( c \in (a,b) \) and \( z \in \rho(H_\alpha) \). For any \( f_0 \in H \) let \( f = f_0\chi_{[a,c]} \in L^2((a,b); dx; \mathcal{H}) \) and \( u(f_0, z, \cdot) = (H_\alpha - zI_{L^2((a,b); dx; \mathcal{H})})^{-1}f \in \text{dom}(H_\alpha) \). By the variation of constants formula,
\[
u(f_0, z, x) = \theta_\alpha(z, x, a)\left( g(z) + \int_0^x dx' \phi_\alpha(z, x', a)^* f_0 \right) + \phi_\alpha(z, x, a)\left( h(z) - \int_0^x dx' \theta_\alpha(z, x', a)^* f_0 \right) \tag{3.16}
\]
for suitable vectors \( g(z) \in \mathcal{H} \), \( h(z) \in \mathcal{H} \). Since \( u(f_0, z, \cdot) \in \text{dom}(H_\alpha) \), one infers that
\[
g(z) = -\int_a^c dx' \phi_\alpha(z, x', a)^* f_0, \quad z \in \rho(H_\alpha), \tag{3.17}
\]
and that
\[
h(z) = \cos(\alpha)u'(f_0, z, a) - \sin(\alpha)u(f_0, z, a) + \int_a^c dx' \theta_\alpha(z, x', a)^* f_0, \quad z \in \rho(H_\alpha). \tag{3.18}
\]

Lemma 3.7. Assume Hypothesis 3.5 and suppose that \( \alpha \in \mathcal{B}(\mathcal{H}) \) is self-adjoint. In addition, choose \( c \in (a,b) \) and introduce \( g(\cdot) \) and \( h(\cdot) \) as in (3.17) and (3.18). Then the maps
\[
C_{1,\alpha}(c, z) : \mathcal{H} \to \mathcal{H}, \quad f_0 \mapsto g(z), \quad C_{2,\alpha}(c, z) : \mathcal{H} \to \mathcal{H}, \quad f_0 \mapsto h(z), \tag{3.19}
\]
are linear and bounded. Moreover, \( C_{1,\alpha}(c, \cdot) \) is entire and \( C_{2,\alpha}(c, \cdot) \) is analytic on \( \rho(H_\alpha) \). In addition, \( C_{1,\alpha}(c, z) \) is boundedly invertible if \( z \in \mathbb{C} \setminus \mathbb{R} \) and \( c \) is chosen appropriately.

Using the bounded invertibility of \( C_{1,\alpha}(c, z) \) we now define
\[
\psi_\alpha(z, x) = \theta_\alpha(z, x, a) + \phi_\alpha(z, x, a)C_{2,\alpha}(c, z)C_{1,\alpha}(c, z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x \in [a, b), \tag{3.20}
\]
still assuming Hypothesis 3.5 and \( \alpha = \alpha^* \in \mathcal{B}(\mathcal{H}) \). By Lemma 3.7, \( \psi_\alpha(\cdot, x) \) is analytic on \( z \in \mathbb{C} \setminus \mathbb{R} \) for fixed \( x \in [a, b) \).
Since $\psi_\alpha(z, \cdot)f_0$ is the solution of the initial value problem
\[
\tau y = zy, \quad y(c) = u(f_0, z, c), \quad y'(c) = u'(f_0, z, c), \quad z \in \mathbb{C}\setminus\mathbb{R},
\]
the function $\psi_\alpha(z, x)C_{1,\alpha}(z, c)f_0$ equals $u(f_0, z, x)$ for $x \geq c$, and thus is square integrable for every choice of $f_0 \in \mathcal{H}$. In particular, choosing $c \in (a, b)$ such that $C_{1,\alpha}(z, c)^{-1} \in \mathcal{B}(\mathcal{H})$, one infers that
\[
\int_a^b dx \|\psi_\alpha(z, x)f\|^2_{\mathcal{H}} < \infty, \quad f \in \mathcal{H}, \quad z \in \mathbb{C}\setminus\mathbb{R}.
\]
Every $\mathcal{H}$-valued solution of $\tau y = zy$ may be written as
\[
y = \theta_\alpha(z, \cdot, a)f_{\alpha, a} + \phi_\alpha(z, \cdot, a)g_{\alpha, a},
\]
with
\[
f_{\alpha, a} = (\cos \alpha)y(a) + (\sin \alpha)y'(a), \quad g_{\alpha, a} = -(\sin \alpha)y(a) + (\cos \alpha)y'(a).
\]
Hence we can define the maps
\[
C_{1,\alpha, z} : \mathcal{D}_z \to \mathcal{H}, \quad \begin{pmatrix} D_z \to \mathcal{H}, \\ \theta_\alpha(z, \cdot, a)f_{\alpha, a} + \phi_\alpha(z, \cdot, a)g_{\alpha, a} \mapsto f_{\alpha, a} \end{pmatrix}
\]
\[
C_{2,\alpha, z} : \mathcal{D}_z \to \mathcal{H}, \quad \begin{pmatrix} D_z \to \mathcal{H}, \\ \theta_\alpha(z, \cdot, a)f_{\alpha, a} + \phi_\alpha(z, \cdot, a)g_{\alpha, a} \mapsto g_{\alpha, a} \end{pmatrix}
\]

Lemma 3.8. Assume Hypothesis \[3.5\] suppose that $\alpha \in \mathcal{B}(\mathcal{H})$ is self-adjoint, and let $z \in \mathbb{C}\setminus\mathbb{R}$. Then the operators $C_{1,\alpha, z}$ and $C_{2,\alpha, z}$ are linear bijections and hence
\[
C_{1,\alpha, z}, C_{1,\alpha, z}^{-1}, C_{2,\alpha, z}, C_{2,\alpha, z}^{-1} \in \mathcal{B}(\mathcal{H}).
\]

At this point we are finally in the position to define the Weyl–Titchmarsh $m$-function for $z \in \mathbb{C}\setminus\mathbb{R}$ by setting
\[
m_\alpha(z) = C_{2,\alpha, z}C_{1,\alpha, z}^{-1}, \quad z \in \mathbb{C}\setminus\mathbb{R}.
\]

Theorem 3.9. Assume Hypothesis \[3.5\] and that $\alpha \in \mathcal{B}(\mathcal{H})$ is self-adjoint. Then
\[
m_\alpha(z) \in B(\mathcal{H}), \quad z \in \mathbb{C}\setminus\mathbb{R},
\]
and $m_\alpha(\cdot)$ is analytic on $\mathbb{C}\setminus\mathbb{R}$. Moreover,
\[
m_\alpha(z) = m_\alpha(\overline{z})^*, \quad z \in \mathbb{C}\setminus\mathbb{R}.
\]

Thus, the $\mathcal{B}(\mathcal{H})$-valued function $\psi_\alpha(z, \cdot)$ in \[3.20\] can be rewritten in the form
\[
\psi_\alpha(z, x) = \theta_\alpha(z, x, a) + \phi_\alpha(z, x, a)m_\alpha(z), \quad z \in \mathbb{C}\setminus\mathbb{R}, \quad x \in [a, b].
\]
In particular, this implies that $\psi_\alpha(z, \cdot)$ is independent of the choice of the parameter $c \in (a, b)$ in \[3.20\]. Following the tradition in the scalar case ($\dim(\mathcal{H}) = 1$), we will call $\psi_\alpha(z, \cdot)$ the Weyl–Titchmarsh solution associated with $\tau Y = zY$.

We remark that, given a function $u \in \mathcal{D}_z$, the operator $m_0(z)$ assigns the Neumann boundary data $u'(a)$ to the Dirichlet boundary data $u(a)$, that is, $m_0(z)$ is the $(z$-dependent) Dirichlet-to-Neumann map.

With the aid of the Weyl–Titchmarsh solutions we can now give a detailed description of the resolvent $R_{z,\alpha} = (H_\alpha - zI_{L^2((a,b),dx;\mathcal{H})})^{-1}$ of $H_\alpha$. 


Theorem 3.10. Assume Hypothesis \[3.5\] and that \( \alpha \in \mathcal{B}(\mathcal{H}) \) is self-adjoint. Then the resolvent of \( H_{\alpha} \) is an integral operator of the type

\[
(H_{\alpha} - zI_{L^2((a,b);dx;\mathcal{H})})^{-1}u(x) = \int_{a}^{b} dx' G_{\alpha}(z, x, x') u(x'),
\]

(3.32)

with the \( \mathcal{B}(\mathcal{H}) \)-valued Green’s function \( G_{\alpha}(z, \cdot, \cdot) \) given by

\[
G_{\alpha}(z, x, x') = \begin{cases} 
\phi_{\alpha}(z, x, a)\psi_{\alpha}(\tau, x')^*, & a \leq x \leq x' < b, \\
\psi_{\alpha}(z, x)\phi_{\alpha}(\tau, x', a)^*, & a \leq x' \leq x < b, 
\end{cases} \quad z \in \mathbb{C}\setminus\mathbb{R}.
\]

(3.33)

One recalls from Definition A.1 that a nonconstant function \( N : \mathbb{C}_+ \to \mathcal{B}(\mathcal{H}) \) is called a (bounded) operator-valued Herglotz function, if \( z \mapsto (u, N(z)u)_{\mathcal{H}} \) is analytic and has a non-negative imaginary part for all \( u \in \mathcal{H} \).

Theorem 3.11. Assume Hypothesis \[3.5\] and suppose that \( \alpha \in \mathcal{B}(\mathcal{H}) \) and \( \beta \in \mathcal{B}(\mathcal{H}) \) are self-adjoint. Then the \( \mathcal{B}(\mathcal{H}) \)-valued function \( m_{\alpha}(\cdot) \) is an operator-valued Herglotz function and explicitly determined by the Green’s function for \( H_{\alpha} \) as follows,

\[
m_{\alpha}(z) = (-\sin(\alpha), \cos(\alpha))^T \begin{pmatrix} G_{\alpha}(z, a, a) & G_{\alpha, a}(z, a, a) \\ G_{\alpha, x}(z, a, a) & G_{\alpha, x, a}(z, a, a) \end{pmatrix} \begin{pmatrix} \sin(\alpha) \\ -\sin(\alpha) \end{pmatrix},
\]

(3.34)

where we denoted

\[
G_{\alpha, x}(z, a, a) = \lim_{x' \to a} \frac{\partial}{\partial x} G_{\alpha}(z, x, x'),
\]

\[
G_{\alpha, x, a}(z, a, a) = \lim_{x' \to a} \frac{\partial}{\partial x} G_{\alpha}(z, x, x'),
\]

\[
G_{\alpha, x, a}(z, a, a) = \lim_{x' \to a} \frac{\partial}{\partial x} \frac{\partial}{\partial x'} G_{\alpha}(z, x, x'),
\]

(3.35)

(the strong limits referring to the strong operator topology in \( \mathcal{H} \)). In addition, \( m_{\alpha}(\cdot) \) extends analytically to the resolvent set of \( H_{\alpha} \).

Moreover, \( m_{\alpha}(\cdot) \) and \( m_{\beta}(\cdot) \) are related by the following linear fractional transformation,

\[
m_{\beta} = (C + Dm_{\alpha})(A + Bm_{\alpha})^{-1},
\]

(3.36)

where

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.
\]

(3.37)

We also mention that \( G_{\alpha}(\cdot, x, x) \) is a bounded Herglotz operator in \( \mathcal{H} \) for each \( x \in (a, b) \), as is clear from \[2.29\], \[3.31\], \[3.33\], and the Herglotz property of \( m_{\alpha} \).

Remark 3.12. The Weyl–Titchmarsh theory established in this section is modeled after right half-lines \((a, b) = (0, \infty)\). Of course precisely the analogous theory applies to left half-lines \((-\infty, 0)\). Given the two half-line results, one then establishes the full-line result on \( \mathbb{R} \) in the usual fashion with \( x = 0 \) a reference point and a \( 2 \times 2 \) block operator formalism as in the well-known scalar or matrix-valued cases; we omit further details at this point as the basic results will explicitly be derived in Section \[5\].
Spectral Theory and Operator-Valued Potentials on the Half-Line

In this section we develop the basic spectral theory for Schrödinger operators \( H_{+} \) in \( L^{2}((a, \infty); dx; \mathcal{H}) \) on right a half-line \((a, \infty)\) with a bounded operator-valued potential coefficient in some complex, separable Hilbert space \( \mathcal{H} \), and with a regular left endpoint \( a \). We focus on a diagonalization of \( H_{+} \) and the corresponding generalized eigenfunction expansion, including a description of support properties of the underlying \( \mathcal{B}(\mathcal{H}) \)-valued half-line spectral measure. In particular, we illustrate the spectral theorem for \( F(H_{+}) \), \( F \in C(\mathbb{R}) \) (cf. Theorems 4.2 and 4.3).

In the special scalar and matrix-valued cases where \( \dim(\mathcal{H}) < \infty \), the material of this section is standard. In particular, we refer to \([22, 23, 34\ Ch. 9, 45\ Sect. XIII.5], [46, Ch. 2, 47, Sect. III.10], [48, 66, Ch. 10], [49, 68, 74, 80, 81, Ch. 124\ Sects. 7–10]\), in the scalar case (i.e., for \( \dim(\mathcal{H}) = 1 \)) and to \([33, 108, Ch. 1, Appendix A]\), in the scalar-valued case (i.e., for \( \dim(\mathcal{H}) < \infty \)). While there exist a variety of results in the operator-valued case (i.e., for \( \dim(\mathcal{H}) = \infty \)), \([59, 63, Chs. 3, 4, 66\ Sect. 10.7, 88, 89, 90, 108\ Ch. 2, 112, 113, 114, 115, 116, 119, 121]\), typically, under varying regularity hypotheses on \( V(\cdot) \), we emphasize that under our general Hypothesis 2.7 the results obtained in this section are new.

We start with the following useful result, a version of Stone’s formula in the weak sense (cf., e.g., \([15, p. 1203]\)).

**Lemma 4.1.** Let \( T \) be a self-adjoint operator in a complex separable Hilbert space \( \mathcal{K} \) (with inner product denoted by \( \langle \cdot, \cdot \rangle_{\mathcal{K}} \), linear in the second factor) and denote by \( \{E_{T}(\lambda)\}_{\lambda \in \mathbb{R}} \) the family of self-adjoint right-continuous spectral projections associated with \( T \), that is, \( E_{T}(\lambda) = \chi(-\infty,\lambda](T), \lambda \in \mathbb{R} \). Moreover, let \( f, g \in \mathcal{K}, \lambda_{1}, \lambda_{2} \in \mathbb{R}, \lambda_{1} < \lambda_{2} \), and \( F \in C(\mathbb{R}) \). Then,

\[
\langle f, F(T)E_{T}((\lambda_{1}, \lambda_{2}])g \rangle_{\mathcal{K}} = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_{1}+\delta}^{\lambda_{2}+\delta} d\lambda F(\lambda) \left[ \langle f, (T-(\lambda+i\varepsilon)I_{\mathcal{K}})^{-1}g \rangle_{\mathcal{K}} - \langle f, (T-(\lambda-i\varepsilon)I_{\mathcal{K}})^{-1}g \rangle_{\mathcal{K}} \right]. \tag{4.1}
\]

**Proof.** First, assume \( F \geq 0 \). Then

\[
(F(T)^{1/2}E_{T}((\lambda_{1}, \lambda_{2}])f, (T-zI_{\mathcal{K}})^{-1}F(T)^{1/2}E_{T}((\lambda_{1}, \lambda_{2}])f)_{\mathcal{K}} = \int_{\mathbb{R}} d(F(T)^{1/2}f, E_{T}(\lambda)F(T)^{1/2}f)_{\mathcal{K}} \chi_{(\lambda_{1}, \lambda_{2}]}(\lambda)(\lambda-z)^{-1}
\]

\[
= \int_{\mathbb{R}} d(f, E_{T}(\lambda)f)_{\mathcal{K}} F(\lambda)\chi_{(\lambda_{1}, \lambda_{2}]}(\lambda)(\lambda-z)^{-1}
\]

\[
= \int_{\mathbb{R}} \frac{d(F(T)^{1/2}\chi_{(\lambda_{1}, \lambda_{2}]}(T)f, E_{T}(\lambda)F(T)^{1/2}\chi_{(\lambda_{1}, \lambda_{2}]}(T)f)_{\mathcal{K}}}{(\lambda-z)}, \quad z \in \mathbb{C}_{+}, \tag{4.2}
\]

is a Herglotz function and hence \( F = f \) for \( g = f \) follows from the standard Stieltjes inversion formula in the scalar case. If \( F \) is not nonnegative, one decomposes \( F \) as \( F = (F_{1} - F_{2}) + i(F_{3} - F_{4}) \) with \( F_{j} \geq 0 \), \( 1 \leq j \leq 4 \) and applies \( (4.2) \) to each \( j \in \{1, 2, 3, 4\} \). The general case \( g \neq f \) then follows from the case \( g = f \) by polarization. \( \square \)
Next, we replace the interval \((a, b)\) in Sections 2 and 3 by the right half-line \((a, \infty)\) and indicate this change with the additional subscript \(+\) in \(H_{+, \alpha}, m_{+, \alpha} (\cdot), \) \(d\rho_{+, \alpha} (\cdot), \) etc., to distinguish these quantities from the analogous objects on the left half-line \((-\infty, a)\) (later indicated with the subscript \(-\)), which are needed in our subsequent Section 4.

Our aim is to relate the family of spectral projections, \(\{E_{H_{+, \alpha}} (\lambda)\}_{\lambda \in \mathbb{R}},\) of the self-adjoint operator \(H_{+, \alpha}\) and the \(\mathcal{B}(\mathcal{H})\)-valued spectral function \(\rho_{+, \alpha} (\lambda), \lambda \in \mathbb{R},\) which generates the operator-valued measure \(d\rho_{+, \alpha}\) in the Herglotz representation \((4.3)\) of \(m_{+, \alpha} :\)

\[
m_{+, \alpha} (z) = c_{+, \alpha} + \int_{\mathbb{R}} d\rho_{+, \alpha} (\lambda) \left[ \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right], \quad z \in \mathbb{C} \setminus \sigma (H_{+, \alpha}),
\]

where

\[
c_{+, \alpha}^* = c_{+, \alpha} \in \mathcal{B}(\mathcal{H}),
\]

and \(d\rho_{+, \alpha}\) is a \(\mathcal{B}(\mathcal{H})\)-valued measure satisfying

\[
\int_{\mathbb{R}} \frac{d(e, \rho_{+, \alpha} (\lambda) e)_{\mathcal{B}(\mathcal{H})}}{1 + \lambda^2} < \infty,
\]

for all \(e \in \mathcal{H}\) (cf. Appendix A for details on Nevanlinna–Herglotz functions).

We first note that for \(F \in C(\mathbb{R}),\)

\[
(f, F(H_{+, \alpha}) g)_{L^2((a, \infty); dx; \mathcal{H})} = \int_{\mathbb{R}} d(f, E_{H_{+, \alpha}} (\lambda) g)_{L^2((a, \infty); dx; \mathcal{H})} F (\lambda),
\]

\[
f, g \in \text{dom}(F(H_{+, \alpha}))
\]

\[
= \left\{ h \in L^2((a, \infty); dx; \mathcal{H}) \left| \int_{\mathbb{R}} d\|E_{H_{+, \alpha}} (\lambda) h\|^2_{L^2((a, \infty); dx; \mathcal{H})} |F (\lambda)|^2 < \infty \right\}.
\]

Equation \((4.6)\) extends to measurable functions \(F\) and holds also in the strong sense, but the displayed weak version will suffice for our purpose.

In the following, \(C^\infty_0 ((c, d); \mathcal{H}), -\infty < c < d < \infty,\) denotes the usual space of infinitely differentiable \(\mathcal{H}\)-valued functions of compact support contained in \((c, d).\)

**Theorem 4.2.** Assume Hypothesis 2.7 and let \(f, g \in C^\infty_0 ((a, \infty); \mathcal{H}),\) \(F \in C(\mathbb{R}),\) and \(\lambda_1, \lambda_2 \in \mathbb{R},\) \(\lambda_1 < \lambda_2.\) Then,

\[
(f, F(H_{+, \alpha}) E_{H_{+, \alpha}} ((\lambda_1; \lambda_2]) g)_{L^2((a, \infty); dx; \mathcal{H})} = (\hat{f}_{+, \alpha}, M F M_{\lambda_1; \lambda_2} \hat{g}_{+, \alpha})_{L^2(\mathbb{R}; d\rho_{+, \alpha}; \mathcal{H})},
\]

where we introduced the notation

\[
\hat{h}_{+, \alpha} (\lambda) = \int_{\mathbb{R}} dx \phi_0 (\lambda, x, \alpha) h (x), \quad \lambda \in \mathbb{R}, \quad h \in C^\infty_0 ((a, \infty); \mathcal{H}),
\]

and \(M_{G}\) denotes the maximally defined operator of multiplication by the function \(G \in C(\mathbb{R})\) in the Hilbert space \(L^2(\mathbb{R}; d\rho_{+, \alpha}; \mathcal{H})\),

\[
(M_{G} \hat{h}) (\lambda) = G (\lambda) \hat{h} (\lambda) \quad \text{for } \rho_{+, \alpha}-a.e. \; \lambda \in \mathbb{R},
\]

\[
\hat{h} \in \text{dom}(M_{G}) = \{ \hat{k} \in L^2(\mathbb{R}; d\rho_{+, \alpha}; \mathcal{H}) \left| G \hat{k} \in L^2(\mathbb{R}; d\rho_{+, \alpha}; \mathcal{H}) \right. \}
\]

\(^{1}\)We recall that \(L^2(\mathbb{R}; d\rho_{+, \alpha}; \mathcal{H})\) is a convenient abbreviation for the Hilbert space \(L^2(\mathbb{R}; d\rho_{+, \alpha}; \rho_{+, \alpha}),\) discussed in detail in Appendix B with \(d\rho_{+, \alpha}\) a control measure for the \(\mathcal{B}(\mathcal{H})\)-valued measure \(d\rho_{+, \alpha}.\) One recalls that \(M_{\rho_{+, \alpha}} \in \mathcal{S}((\mathcal{H})_{\lambda \in \mathbb{R}})\) is generated by \(\Lambda (\mathcal{H})\) (or by \(\Delta (\{e_n\}_{n \in \mathbb{N}})\) for any complete orthonormal system \(\{e_n\}_{n \in \mathbb{N}, \in \mathbb{H}}\)).
Here $\rho_{+, \alpha}$ generates the operator-valued measure in the Herglotz representation of the operator-valued Weyl–Titchmarsh function $m_{+, \alpha}(\cdot) \in \mathcal{B}(\mathcal{H})$ (cf. (4.3)).

**Proof.** The point of departure for deriving (4.7) is Stone’s formula (4.1) applied to $T = H_{+, \alpha}$,

$$
(f, F(H_{+, \alpha})E_{H_{+, \alpha}}((\lambda_1, \lambda_2])g)_{L^2((a, \infty); dx; \mathcal{H})} = \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda F(\lambda) \left[ (f, (H_{+, \alpha} - (\lambda + i\varepsilon) I)^{-1} g)_{L^2((a, \infty); dx; \mathcal{H})} - (f, (H_{+, \alpha} - (\lambda - i\varepsilon) I)^{-1} g)_{L^2((a, \infty); dx; \mathcal{H})} \right].
$$

(4.10)

Expressing the resolvent in (4.10) in terms of the Green’s function (3.33) then yields the following:

$$
(f, F(H_{+, \alpha})E_{H_{+, \alpha}}((\lambda_1, \lambda_2])g)_{L^2((a, \infty); dx; \mathcal{H})} = \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda F(\lambda) \times \int_a^\infty dx \left\{ \left[ (f(x), \psi_{+, \alpha}(\lambda + i\varepsilon, x) \int_a^x dx' \phi_{\alpha}(\lambda - i\varepsilon, x', a)^* g(x'))_{\mathcal{H}} + (f(x), \phi_{\alpha}(\lambda + i\varepsilon, x, a) \int_x^\infty dx' \psi_{+, \alpha}(\lambda - i\varepsilon, x', a)^* g(x'))_{\mathcal{H}} \right]
$$

$$
- \left[ (f(x), \psi_{+, \alpha}(\lambda - i\varepsilon, x) \int_a^x dx' \phi_{\alpha}(\lambda + i\varepsilon, x', a)^* g(x'))_{\mathcal{H}} + (f(x), \phi_{\alpha}(\lambda - i\varepsilon, x, a) \int_x^\infty dx' \psi_{+, \alpha}(\lambda + i\varepsilon, x', a)^* g(x'))_{\mathcal{H}} \right] \right\}.
$$

(4.11)

Freely interchanging the $dx$ and $dx'$ integrals with the limits and the $d\lambda$ integral (since all integration domains are finite and all integrands are continuous), and inserting expression (3.33) for $\psi_{+, \alpha}(z, x)$ into (4.11), one obtains

$$
(f, F(H_{+, \alpha})E_{H_{+, \alpha}}((\lambda_1, \lambda_2])g)_{L^2((a, \infty); dx; \mathcal{H})} = \int_a^\infty dx \left( \int_a^x dx' \left\{ \int_a^\infty d\lambda F(\lambda) \left[ \left[ \theta_{\alpha}(\lambda, x, a) + \phi_{\alpha}(\lambda, x, a)m_{+, \alpha}(\lambda + i\varepsilon) \phi_{\alpha}(\lambda, x', a)^* - \left[ \theta_{\alpha}(\lambda, x, a) + \phi_{\alpha}(\lambda, x, a) m_{+, \alpha}(\lambda - i\varepsilon) \phi_{\alpha}(\lambda, x', a)^* \right] g(x') \right] \right.ight.
$$

$$
+ \int_x^\infty dx' \left[ \phi_{\alpha}(\lambda, x, a) \left[ \theta_{\alpha}(\lambda, x', a)^* + m_{+, \alpha}(\lambda - i\varepsilon)^* \phi_{\alpha}(\lambda, x', a)^* \right] - \phi_{\alpha}(\lambda, x, a) \left[ \theta_{\alpha}(\lambda, x', a)^* + m_{+, \alpha}(\lambda + i\varepsilon)^* \phi_{\alpha}(\lambda, x', a)^* \right] \right] g(x') \right\} \right). \quad (4.12)
$$

Here we employed the fact that for fixed $x \in [a, \infty)$, $\theta_{\alpha}(z, x, a)$ and $\phi_{\alpha}(z, x, a)$ are entire with respect to $z$, that $\theta_{\alpha}(z, \cdot, a), \phi_{\alpha}(z, \cdot, a) \in W^{1,1}([a, c]; \mathcal{H})$ for all $c > a$, and hence that

$$
\theta_{\alpha}(\lambda \pm i\varepsilon, x, a) = \theta_{\alpha}(\lambda, x, a) \pm i\varepsilon(d/dz)\theta_{\alpha}(z, x, a)|_{z=\lambda} + O(\varepsilon^2),
$$

$$
\phi_{\alpha}(\lambda \pm i\varepsilon, x, a) = \phi_{\alpha}(\lambda, x, a) \pm i\varepsilon(d/dz)\phi_{\alpha}(z, x, a)|_{z=\lambda} + O(\varepsilon^2) \quad (4.13)
$$

\[\]

\[\]
with $O(\varepsilon^2)$ being uniform with respect to $(\lambda, x)$ as long as $\lambda$ and $x$ vary in compact subsets of $\mathbb{R} \times [a, \infty)$. Moreover, we used that for all $f, g \in \mathcal{H}$ (cf. Theorem A.4(v)),

$$
\varepsilon |(f, m_{+, \alpha}(\lambda + i\varepsilon)g)|_\mathcal{H} \leq C(\lambda_1, \lambda_2, \varepsilon_0, f, g) \quad \text{for} \quad \lambda \in [\lambda_1, \lambda_2], \quad 0 < \varepsilon \leq \varepsilon_0.
$$

In particular, utilizing (4.13) and (4.14), $\phi_\alpha(\lambda \pm i\varepsilon, x, a)$ and $\theta_\alpha(\lambda \pm i\varepsilon, x, a)$ have been replaced by $\phi_\alpha(\lambda, x, a)$ and $\theta_\alpha(\lambda, x, a)$ under the $d\lambda$ integrals in (4.12). Canceling appropriate terms in (4.12), simplifying the remaining terms, and using $m_{+, \alpha}(z) = m_{+, \alpha}(z)^*$ then yield

$$
(f, F(H_{+, \alpha})E_{H_{+, \alpha}}((\lambda_1, \lambda_2])g)_{L^2((a, \infty); dx; \mathcal{H})}
$$

$$
= \int_a^\infty dx \int_a^\infty dx' \times \lim \lim_{\delta \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda F(\lambda)
$$

$$
\times (\phi_\alpha(\lambda, x, a)^* f(x), \text{Im}(m_{+, \alpha}(\lambda + i\varepsilon)) \phi_\alpha(\lambda, x', a)^* g(x'))_{\mathcal{H}}.
$$

Using the fact that by (A.12)

$$
\int_{(\lambda_1, \lambda_2]} d\rho_{+, \alpha}(\lambda) h = \rho_{+, \alpha}((\lambda_1, \lambda_2)]h = \lim \lim_{\delta \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda \text{Im}(m_{+, \alpha}(\lambda + i\varepsilon))h,
$$

and hence that

$$
\int_{\mathbb{R}} d\rho_{+, \alpha}(\lambda) h(\lambda) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\mathbb{R}} d\lambda \text{Im}(m_{+, \alpha}(\lambda + i\varepsilon))h(\lambda), \quad h \in C_0(\mathbb{R}; \mathcal{H}),
$$

(14.16)

$$
\int_{(\lambda_1, \lambda_2]} d\rho_{+, \alpha}(\lambda) k(\lambda) = \lim \lim_{\delta \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda \text{Im}(m_{+, \alpha}(\lambda + i\varepsilon))k(\lambda), \quad k \in C(\mathbb{R}; \mathcal{H}),
$$

(14.17)

and with $C_0(\mathbb{R}; \mathcal{H})$ the space of continuous compactly supported $\mathcal{H}$-valued functions on $\mathbb{R}$ one concludes that

$$
(f, F(H_{+, \alpha})E_{H_{+, \alpha}}((\lambda_1, \lambda_2])g)_{L^2((a, \infty); dx; \mathcal{H})}
$$

$$
= \int_a^\infty dx \int_a^\infty dx' \int_{(\lambda_1, \lambda_2]} F(\lambda)(\phi_\alpha(\lambda, x, a)^* f(x), d\rho_{+, \alpha}(\lambda) \phi_\alpha(\lambda, x', a)^* g(x'))_{\mathcal{H}}
$$

$$
= \int_{(\lambda_1, \lambda_2]} F(\lambda) (\hat{f}_{+, \alpha}(\lambda, d\rho_{+, \alpha}(\lambda)) \tilde{g}_{+, \alpha}(\lambda))_{\mathcal{H}}
$$

(14.19)

using (4.18) and interchanging the $dx$, $dx'$ and $d\rho_{+, \alpha}$ integrals once more. We note that $\hat{f}_{+, \alpha}, \tilde{g}_{+, \alpha} \in L^2(\mathbb{R}; d\rho_{+, \alpha}; \mathcal{H})$ by Lemma 3.16 and Theorem 3.17.

Remark 4.3. Theorem 4.2 is of course well-known in the scalar case (i.e., where $\dim(\mathcal{H}) = 1$), see, for instance, the extensive list of references in [55]. In the matrix-valued case (i.e., if $\dim(\mathcal{H}) < \infty$) we refer, for instance, to Gorbachuk [59] and in the operator-valued case (where $\dim(\mathcal{H}) = \infty$) to Gorbachuk [59] under more restrictive regularity assumptions on the potential $V(\cdot)$ and without providing details in the steps leading from (4.12) to (4.19).
Remark 4.4. The effortless derivation of the link between the family of spectral projections $E_{H_{+,\alpha}}(\cdot)$ and the operator-valued spectral function $\rho_{+,\alpha}(\cdot)$ of $H_{+,\alpha}$ in Theorem 4.2 applies equally well to half-line Dirac-type operators, Hamiltonian systems, half-lattice Jacobi operators, and CMV operators (cf. [55, 58] and the literature cited therein). In the context of operator-valued potential coefficients of half-line Schrödinger operators this strategy has already been used by M. L. Gorbachuk [59] in 1966.

Actually, one can improve on Theorem 4.2 and remove the compact support restrictions on $f$ and $g$ in the usual way. To this end one considers the map

$$
\hat{U}_{+,\alpha} : \{ C_0^\infty((a,\infty);\mathcal{H}) \to L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H}) \}
\begin{align*}
\hat{h} &\mapsto \hat{h}_{+,\alpha}(\cdot) = \int_a^\infty dx \phi_\alpha(\cdot, x, a)^* h(x).
\end{align*}
(4.20)
$$

Taking $f = g$, $F = 1$, $\lambda_1 \downarrow -\infty$, and $\lambda_2 \uparrow \infty$ in (4.7) then shows that $\hat{U}_{+,\alpha}$ is a densely defined isometry in $L^2((a,\infty);dx;\mathcal{H})$, which extends by continuity to an isometry on $L^2((a,\infty);dx;\mathcal{H})$. The latter is denoted by $U_{+,\alpha}$ and given by

$$
U_{+,\alpha} : \{ L^2((a,\infty);dx;\mathcal{H}) \to L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H}) \}
\begin{align*}
\hat{h} &\mapsto \hat{h}_{+,\alpha}(\cdot) = \text{l.i.m.}_{\lambda \to \infty} \int_a^b dx \phi_\alpha(\cdot, x, a)^* h(x),
\end{align*}
(4.21)
$$

where l.i.m. refers to the $L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H})$-limit.

The calculation in (4.19) also yields

$$
(E_{H_{+,\alpha}}((\lambda_1,\lambda_2])g)(x) = \int_{(\lambda_1,\lambda_2]} \phi_\alpha(\lambda, x, a) d\rho_{+,\alpha}(\lambda) \hat{g}_{+,\alpha}(\lambda), \quad g \in C_0^\infty((a,\infty);\mathcal{H})
$$
(4.22)

and subsequently, (4.22) extends to all $g \in L^2((a,\infty);dx;\mathcal{H})$ by continuity. Moreover, taking $\lambda_1 \downarrow -\infty$ and $\lambda_2 \uparrow \infty$ in (4.22) using

$$
\text{s-lim}_{\lambda \downarrow -\infty} E_{H_{+,\alpha}}(\lambda) = 0, \quad \text{s-lim}_{\lambda \uparrow \infty} E_{H_{+,\alpha}}(\lambda) = I_{L^2((a,\infty);dx;\mathcal{H})},
$$
(4.23)

where

$$
E_{H_{+,\alpha}}(\lambda) = E_{H_{+,\alpha}}((-\infty,\lambda]), \quad \lambda \in \mathbb{R},
$$
(4.24)

then yields

$$
g(\cdot) = \text{l.i.m.}_{\mu_1 \downarrow -\infty, \mu_2 \uparrow \infty} \int_{\mu_1}^{\mu_2} \phi_\alpha(\lambda, \cdot, a) d\rho_{+,\alpha}(\lambda) \hat{g}_{+,\alpha}(\lambda), \quad g \in L^2((a,\infty);dx;\mathcal{H})
$$
(4.25)

where l.i.m. refers to the $L^2((a,\infty);dx;\mathcal{H})$-limit.

In addition, one can show that the map $U_{+,\alpha}$ in (4.21) is onto and hence that $U_{+,\alpha}$ is unitary (i.e., $U_{+,\alpha}$ and $U^{-1}_{+,\alpha}$ are isometric isomorphisms between $L^2((a,\infty);dx;\mathcal{H})$ and $L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H})$) with

$$
U^{-1}_{+,\alpha} : \{ L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H}) \to L^2((a,\infty);dx;\mathcal{H}) \}
\begin{align*}
\hat{h} &\mapsto \text{l.i.m.}_{\mu_1 \downarrow -\infty, \mu_2 \uparrow \infty} \int_{\mu_1}^{\mu_2} \phi_\alpha(\lambda, \cdot, a) d\rho_{+,\alpha}(\lambda) \hat{h}(\lambda).
\end{align*}
(4.26)
$$

To show this we denote the operator defined in (4.20) temporarily by $V_{+,\alpha}$ and first claim that $V_{+,\alpha}$ is bounded: Indeed, one computes for all $\hat{f} \in C_0^\infty(\mathbb{R};\mathcal{H})$ and
Next, suppose that $f$ is injective.

Taking $s$-lim

\[
\|V_{+,\alpha}\|=\begin{cases} \sup_{g\in L^2((a,\infty);dx;\mathcal{H})} \frac{|(g, V_{+,\alpha}\hat{f})_{L^2((a,\infty);dx;\mathcal{H})}|}{\|g\|_{L^2((a,\infty);dx;\mathcal{H})}} \\ \leq \sup_{g\in L^2((a,\infty);dx;\mathcal{H})} \frac{|U_{+,\alpha}g\|_{L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H})}}{\|g\|_{L^2((a,\infty);dx;\mathcal{H})}} \|\hat{f}\|_{L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H})} \end{cases}
\]

and hence $\|V_{+,\alpha}\|\leq 1$. By (4.29),

\[
V_{+,\alpha}U_{+,\alpha}=I_{L^2((a,\infty);dx;\mathcal{H})}.
\]

To prove that $U_{+,\alpha}$ is onto, and hence unitary, it thus suffices to prove that $V_{+,\alpha}$ is injective.

Let $\hat{f}\in L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H})$, $\lambda_1,\lambda_2\in \mathbb{R}$, $\lambda_1<\lambda_2$, and consider

\[
(H_{+,\alpha} - zI_{L^2((a,\infty);dx;\mathcal{H})}) \int_{\lambda_1}^{\lambda_2} \phi_{\alpha}(\lambda,\cdot,a)(\lambda-z)^{-1}d\rho_{+,\alpha}(\lambda)\hat{f}(\lambda)
\]

\[
= \int_{\lambda_1}^{\lambda_2} \phi_{\alpha}(\lambda,\cdot,a)d\rho_{+,\alpha}(\lambda)\hat{f}(\lambda), \quad z\in \mathbb{C}_+.
\]

Then,

\[
\int_{\lambda_1}^{\lambda_2} \phi_{\alpha}(\lambda,\cdot,a)(\lambda-z)^{-1}d\rho_{+,\alpha}(\lambda)\hat{f}(\lambda)
\]

\[
= (H_{+,\alpha} - zI_{L^2((a,\infty);dx;\mathcal{H})})^{-1}\int_{\lambda_1}^{\lambda_2} \phi_{\alpha}(\lambda,\cdot,a)d\rho_{+,\alpha}(\lambda)\hat{f}(\lambda), \quad z\in \mathbb{C}_+.
\]

Taking $s$-lim$_{\lambda_1\downarrow-\infty,\lambda_2\uparrow\infty}$ in (4.31) implies

\[
V_{+,\alpha}(\cdot - z)^{-1}\hat{f} = (H_{+,\alpha} - zI_{L^2((a,\infty);dx;\mathcal{H})})^{-1}V_{+,\alpha}\hat{f}, \quad z\in \mathbb{C}_+.
\]

Next, suppose that $\hat{f}_0\in \ker(V_{+,\alpha})$, and let $\{\hat{f}_n\}_{n\in \mathbb{N}}\subset L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H})$ such that $\text{supp}(\hat{f}_n)$ is compact for each $n\in \mathbb{N}$ and $\lim_{n\uparrow\infty} \|\hat{f}_0 - \hat{f}_n\|_{L^2(\mathbb{R};d\rho_{+,\alpha};\mathcal{H})} = 0$. Then,

\[
(V_{+,\alpha}(\cdot - z)^{-1}\hat{f}_n)(x) = ((H_{+,\alpha} - zI_{L^2((a,\infty);dx;\mathcal{H})})^{-1}V_{+,\alpha}\hat{f}_n)(x), \quad x > a, \quad z\in \mathbb{C}_+, \quad n\in \mathbb{N},
\]
and thus for all \( y \in [a, \infty) \), and arbitrary \( e \in \mathcal{H} \),

\[
\int_a^y dx \int_{\mathbb{R}} (e, \phi_\alpha(\lambda, x, a) (\lambda - z)^{-1} d\rho_{+, \alpha}(\lambda) \hat{f}_n(\lambda))_{\mathcal{H}} = \int_a^y dx \int_{\mathbb{R}} (\phi_\alpha(\lambda, x, a)^* e, d\rho_{+, \alpha}(\lambda) (\lambda - z)^{-1} \hat{f}_n(\lambda))_{\mathcal{H}} = \int_{\mathbb{R}} \left( \int_a^y dx \phi_\alpha(\lambda, x, a)^* e, d\rho_{+, \alpha}(\lambda) (\lambda - z)^{-1} \hat{f}_n(\lambda) \right)_{\mathcal{H}} = \int_a^y dx (e, (H_{+, \alpha} - zI_{L^2((a, \infty); dx; \mathcal{H})})^{-1} V_{+, \alpha} \hat{f}_n(x))_{\mathcal{H}}. \tag{4.34}
\]

Noticing that

\[
\int_a^\infty dx \phi_\alpha(\cdot, x, a)^* \chi_{[a, y]}(x)e = (U_{+, \alpha} \chi_{[a, y]}e)(\cdot) \in L^2(\mathbb{R}; d\rho_{+, \alpha}; \mathcal{H}), \tag{4.35}
\]

and taking \( n \uparrow \infty \) in (4.34) then results in

\[
\lim_{n \uparrow \infty} \int_a^y dx \int_{\mathbb{R}} (e, \phi_\alpha(\lambda, x, a) d\rho_{+, \alpha}(\lambda) (\lambda - z)^{-1} \hat{f}_n(\lambda))_{\mathcal{H}} = \int_a^y dx \int_{\mathbb{R}} (e, \phi_\alpha(\lambda, x, a) d\rho_{+, \alpha}(\lambda) (\lambda - z)^{-1} \hat{f}_0(\lambda))_{\mathcal{H}} = \int_{\mathbb{R}} (\lambda - z)^{-1} \int_a^y dx (e, \phi_\alpha(\lambda, x, a) d\rho_{+, \alpha}(\lambda) \hat{f}_0(\lambda))_{\mathcal{H}} = \lim_{n \uparrow \infty} \int_a^y dx (e, (H_{+, \alpha} - zI_{L^2((a, \infty); dx; \mathcal{H})})^{-1} V_{+, \alpha} \hat{f}_n)(x))_{\mathcal{H}} = \int_a^y dx (e, (H_{+, \alpha} - zI_{L^2((a, \infty); dx; \mathcal{H})})^{-1} V_{+, \alpha} \hat{f}_0)(x))_{\mathcal{H}} = 0, \]

\( y \in [a, \infty) \), \( z \in \mathbb{C}_+ \), \( e \in \mathcal{H} \).

Applying the Stieltjes inversion formula to the (finite) complex-valued measure in the 3rd line of (4.33), given by,

\[
\int_a^y dx \int_{\mathbb{R}} (e, \phi_\alpha(\lambda, x, a) d\rho_{+, \alpha}(\lambda) \hat{f}_0(\lambda))_{\mathcal{H}}, \tag{4.37}
\]

implies for all \( \lambda_1, \lambda_2 \in \mathbb{R} \), \( \lambda_1 < \lambda_2 \), and \( e \in \mathcal{H} \),

\[
\int_{(\lambda_1, \lambda_2]} \int_a^y dx (e, \phi_\alpha(\lambda, x, a) d\rho_{+, \alpha}(\lambda) \hat{f}_0(\lambda))_{\mathcal{H}} = 0, \quad y \in [a, \infty). \tag{4.38}
\]

Differentiating (4.38) repeatedly with respect to \( y \), noting that \( \phi_\alpha(\lambda, y, a) \) and \( \phi'_\alpha(\lambda, y, a) \) are continuous in \( (\lambda, y) \in \mathbb{R} \times [a, \infty) \), and using the dominated convergence theorem, one concludes that for all \( y \in [a, \infty) \), \( e \in \mathcal{H} \),

\[
\int_{(\lambda_1, \lambda_2]} (e, \phi_\alpha(\lambda, y, a) d\rho_{+, \alpha}(\lambda) \hat{f}_0(\lambda))_{\mathcal{H}} = 0, \quad \int_{(\lambda_1, \lambda_2]} (e, \phi'_\alpha(\lambda, y, a) d\rho_{+, \alpha}(\lambda) \hat{f}_0(\lambda))_{\mathcal{H}} = 0. \tag{4.39}
\]
Using (4.19), the fact that \( \hat{f}_0 \cdot \chi_{(\lambda_1, \lambda_2)} \in L^2(\mathbb{R}; \rho; H) \), and the dominated convergence theorem once again then implies

\[
0 = \int_{(\lambda_1, \lambda_2)} (e, \phi_{\alpha}(\lambda, a, \lambda, \alpha) \rho_{+\alpha}(\lambda) \hat{f}_0(\lambda))_H \\
= - \int_{(\lambda_1, \lambda_2)} (\sin(\alpha) e, \rho_{+\alpha}(\lambda) \hat{f}_0(\lambda))_H \\
= \int_{(\lambda_1, \lambda_2)} (\cos(\alpha) e, \rho_{+\alpha}(\lambda) \hat{f}_0(\lambda))_H \\
= \int_{(\lambda_1, \lambda_2)} (e_1, \rho_{+\alpha}(\lambda) \hat{f}_0(\lambda))_H. \quad (4.40)
\]

Taking \( e = \sin(\alpha) e_1 \) in (4.40) and \( e = \cos(\alpha) e_1 \) in (4.41) with an arbitrary \( e_1 \in H \) and subtracting (4.40) from (4.41) then gives

\[
0 = \int_{(\lambda_1, \lambda_2)} (e_1, \rho_{+\alpha}(\lambda) \hat{f}_0(\lambda))_H. \quad (4.42)
\]

Since the interval \((\lambda_1, \lambda_2)\) was chosen arbitrary, (4.42) implies

\[
\hat{f}_0(\lambda) = 0 \rho_{+\alpha}-a.e., \quad (4.43)
\]

and hence \( \ker(V_{+\alpha}) = \{0\} \). Thus \( U_{+\alpha} \) is onto.

We recall that the essential range of \( F \) with respect to a scalar measure \( \mu \) is defined by

\[
\text{ess.ran}_\mu(F) = \{ z \in \mathbb{C} \mid \text{for all } \varepsilon > 0, \mu(\{ \lambda \in \mathbb{R} \mid |F(\lambda) - z| < \varepsilon \}) > 0 \}, \quad (4.44)
\]

and that \( \text{ess.ran}_{\rho_{+\alpha}}(F) \) for \( F \in C(\mathbb{R}) \) is then defined to be \( \text{ess.ran}_{\mu_{+\alpha}}(F) \) for any control measure \( \rho_{+\alpha} \) of the operator-valued measure \( d\rho_{+\alpha} \). Given a complete orthonormal system \( \{e_n\}_{n \in I} \) in \( H \) (\( I \subseteq \mathbb{N} \) an appropriate index set), a convenient control measure for \( d\rho_{+\alpha} \) is given by

\[
\mu_{+\alpha}(B) = \sum_{n \in I} 2^{-n} (e_n, \rho_{+\alpha}(B)e_n)_H, \quad B \in \mathcal{B}(\mathbb{R}). \quad (4.45)
\]

We sum up these considerations in a variant of the spectral theorem for (functions of) \( H_{+\alpha} \).

**Theorem 4.5.** Assume Hypothesis 2.1 and suppose \( F \in C(\mathbb{R}) \). Then,

\[
U_{+\alpha} F(H_{+\alpha}) U_{+\alpha}^{-1} = M_F I_H \quad (4.46)
\]

in \( L^2(\mathbb{R}; \rho_{+\alpha}; H) \) (cf. (4.20)). Moreover,

\[
\sigma(F(H_{+\alpha})) = \text{ess.ran}_{\rho_{+\alpha}}(F), \quad (4.47)
\]

\[
\sigma(H_{+\alpha}) = \text{supp}(d\rho_{+\alpha}), \quad (4.48)
\]

and the multiplicity of the spectrum of \( H_{+\alpha} \) is at most equal to \( \dim(H) \).

**Proof.** First, we note that (4.46) follows from Theorem 4.2 and the discussion following it. The fact (4.47) is a special case of (4.48) and hence only the latter requires a proof.

Since \( F(H_{+\alpha}) \) is unitarily equivalent to the operator of multiplication by \( F(\cdot) \) in \( L^2(\mathbb{R}; \rho_{+\alpha}; H) \), it suffices to check that \( M_{F(\cdot)} I_H \) is not boundedly invertible whenever \( z \in \text{ess.ran}_{\rho_{+\alpha}}(F) \). Fix an arbitrary \( z \in \text{ess.ran}_{\rho_{+\alpha}}(F) \) and \( \varepsilon > 0 \). Since \( F \in C(\mathbb{R}) \), the set \( \{ \lambda \in \mathbb{R} \mid |F(\lambda) - z| < \varepsilon \} \) is open and hence is a countable
union of disjoint open intervals. By (4.43) there is a bounded interval $B \subset \{ \lambda \in \mathbb{R} \mid |F(\lambda) - z| < \varepsilon \}$ such that $\rho_{\pm \alpha}(B) \neq 0$ and hence there is also a nonzero vector $h \in \mathcal{H}$ such that $(h, \rho_{\pm \alpha}(B)h)_\mathcal{H} \neq 0$. Then $\chi_B h \in L^2(\mathbb{R}; d\rho_{\pm \alpha}; \mathcal{H})$ with
\begin{equation}
\|\chi_B h\|_{L^2(\mathbb{R}; d\rho_{\pm \alpha}; \mathcal{H})}^2 = (h, \rho_{\pm \alpha}(B)h)_\mathcal{H} > 0
\end{equation}
and
\begin{equation}
\|M_{F-z}\chi_B h\|_{L^2(\mathbb{R}; d\rho_{\pm \alpha}; \mathcal{H})} \leq \varepsilon \|\chi_B h\|_{L^2(\mathbb{R}; d\rho_{\pm \alpha}; \mathcal{H})}.
\end{equation}
Since $\varepsilon > 0$ is arbitrary, this implies that $M_{F-z}I_\mathcal{H}$ is not boundedly invertible in $L^2(\mathbb{R}; d\rho_{\pm \alpha}; \mathcal{H})$.

Conversely, assume $z \in \mathbb{R}\setminus \text{ess.ran}_{\rho_{\pm \alpha}}(F)$. Then by (4.44), (4.45), there exists $\varepsilon > 0$ such that for any interval $B \subset \{ \lambda \in \mathbb{R} \mid |F(\lambda) - z| < \varepsilon \}$ one has $\mu_{\pm \alpha}(B) = \rho_{\pm \alpha}(B) = 0$. Then for any $g \in \text{dom}(M_F) \subset L^2(\mathbb{R}; d\rho_{\pm \alpha}; \mathcal{H})$,
\begin{equation}
\|M_{F-z}g\|_{L^2(\mathbb{R}; d\rho_{\pm \alpha}; \mathcal{H})} \geq \varepsilon \|g\|_{L^2(\mathbb{R}; d\rho_{\pm \alpha}; \mathcal{H})},
\end{equation}
that is, $M_{F-z}I_\mathcal{H}$ is boundedly invertible in this case.

Using the identity function $F(z) = z$ it follows from (4.40) that the multiplicity of the spectrum of $H_{\pm \alpha}$ is equal to that of $M_zI_\mathcal{H}$ which is at most $\dim(\mathcal{H})$. \hfill \square

5. Spectral Theory of Schrödinger Operators with Operator-Valued Potentials on the Real Line

In our final section we develop basic spectral theory for full-line Schrödinger operators $H$ in $L^2(\mathbb{R}; dx; \mathcal{H})$, employing a $2 \times 2$ block operator representation of the associated Weyl–Titchmarsh matrix and its $\mathcal{B}(\mathcal{H}^2)$-valued spectral measure, decomposing $\mathbb{R}$ into a left and right half-line with reference point $x_0 \in \mathbb{R}$, $(-\infty, x_0] \cup [x_0, \infty)$. The latter decomposition is familiar from the scalar and matrix-valued $(\dim(\mathcal{H}) < \infty)$ special cases. Our principal new results, Theorems 5.2 and 5.4, again yield a diagonalization of $H$ and the corresponding generalized eigenfunction expansion, illustrating the spectral theorem for $F(H)$ and support properties of the underlying spectral measure.

In the special scalar case where $\dim(\mathcal{H}) < \infty$, the material of this section is standard and various parts of it can be found, for instance, in [23, [34, Ch. 9], [41 Sect. XIII.5], [46 Ch. 2], [48, [66 Ch. 10], [68], [74], [80], [81 Ch. 2], [97 Ch. VI], [101 Ch. 6], [113 Chs. II, III], [123 Sects. 7–10]. However, in the infinite-dimensional case, $\dim(\mathcal{H}) = \infty$, the principal results obtained in this section are new.

We make the following basic assumption throughout this section.

**Hypothesis 5.1.** (i) Assume that
\begin{equation}
V \in L^1_{\text{loc}}(\mathbb{R}; dx; \mathcal{H}), \quad V(x) = V(x)^* \quad \text{for a.e. } x \in \mathbb{R}
\end{equation}
(ii) Introducing the differential expression $\tau$ given by
\begin{equation}
\tau = -\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{R},
\end{equation}
we assume $\tau$ to be in the limit point case at $+\infty$ and at $-\infty$.

Associated with the differential expression $\tau$ one introduces the self-adjoint Schrödinger operator $H$ in $L^2(\mathbb{R}; dx; \mathcal{H})$ by
\begin{equation}
Hf = \tau f,
\end{equation}
\[ f \in \text{dom}(H) = \{ g \in L^2(\mathbb{R}; dx; \mathcal{H}) \mid g, g' \in W^{1,1}_{\text{loc}}(\mathbb{R}; dx; \mathcal{H}); \tau g \in L^2(\mathbb{R}; dx; \mathcal{H}) \}. \]
As in the half-line context we introduce the $\mathcal{B}(\mathcal{H})$-valued fundamental system of solutions $\phi_\alpha(z, \cdot, x_0)$ and $\theta_\alpha(z, \cdot, x_0)$, $z \in \mathbb{C}$, of

$$ (\tau \psi)(z, x) = z\psi(z, x), \quad x \in \mathbb{R} \tag{5.4} $$

with respect to a fixed reference point $x_0 \in \mathbb{R}$, satisfying the initial conditions at the point $x = x_0$,

$$ \phi_\alpha(z, x_0, x_0) = -\theta'_\alpha(z, x_0, x_0) = -\sin(\alpha), $$

$$ \phi'_\alpha(z, x_0, x_0) = \theta_\alpha(z, x_0, x_0) = \cos(\alpha), \quad \alpha = \alpha^* \in \mathcal{B}(\mathcal{H}). \tag{5.5} $$

Again we note that by Corollary 2.5 (iii), for any fixed $x, x_0 \in \mathbb{R}$, the functions $\theta_\alpha(z, x, x_0)$ and $\phi_\alpha(z, x, x_0)$ as well as their strong $x$-derivatives are entire with respect to $z$ in the $\mathcal{B}(\mathcal{H})$-norm. The same is true for the functions $z \mapsto \theta_\alpha(z, x, x_0)$ and $z \mapsto \phi_\alpha(z, x, x_0)^*$.

Moreover, by (2.23),

$$ W(\theta_\alpha(\tau, x_0), \phi_\alpha(z, \cdot, x_0))(x) = I_\mathcal{H}, \quad z \in \mathbb{C}. \tag{5.6} $$

Particularly important solutions of (5.4) are the Weyl–Titchmarsh solutions $\psi_{\pm, \alpha}(z, \cdot, x_0)$, $z \in \mathbb{C} \setminus \mathbb{R}$, uniquely characterized by

$$ \psi_{\pm, \alpha}(z, \cdot, x_0)f \in L^2([x_0, \pm \infty); dx; \mathcal{H}), \quad f \in \mathcal{H}, $$

$$ \sin(\alpha)\psi_{\pm, \alpha}''(z, x_0, x_0) + \cos(\alpha)\psi_{\pm, \alpha}'(z, x_0, x_0) = I_\mathcal{H}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{5.7} $$

The crucial condition in (5.7) is again the $L^2$-property which uniquely determines $\psi_{\pm, \alpha}(z, \cdot, x_0)$ up to constant multiples by the limit point hypothesis of $\tau$ at $\pm \infty$. In particular, for $\alpha = \alpha^*$, $\beta = \beta^* \in \mathcal{B}(\mathcal{H})$,

$$ \psi_{\pm, \alpha}(z, \cdot, x_0) = \psi_{\pm, \beta}(z, \cdot, x_0)C_{\pm}(z, \alpha, \beta, x_0) \tag{5.8} $$

for some coefficients $C_{\pm}(z, \alpha, \beta, x_0) \in \mathcal{B}(\mathcal{H})$. The normalization in (5.7) shows that $\psi_{\pm, \alpha}(z, \cdot, x_0)$ are of the type

$$ \psi_{\pm, \alpha}(z, x, x_0) = \theta_\alpha(z, x, x_0) + \phi_\alpha(z, x, x_0)m_{\pm, \alpha}(z, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x \in \mathbb{R}, \tag{5.9} $$

for some coefficients $m_{\pm, \alpha}(z, x_0) \in \mathcal{B}(\mathcal{H})$, the Weyl–Titchmarsh $m$-functions associated with $\tau, \alpha$, and $x_0$ (cf. Theorem 3.9).

Next, we show that $\pm m_{\pm, \alpha}(\cdot, x_0)$ are operator-valued Herglotz functions. It follows from (5.7) and (5.9) that the Wronskian of $\psi_{\pm, \alpha}(\overline{\tau_1}, x, x_0)$ and $\psi_{\pm, \alpha}(z_2, x, x_0)$ satisfies

$$ W(\psi_{\pm, \alpha}(\overline{\tau_1}, x_0, x_0), \psi_{\pm, \alpha}(z_2, x_0, x_0)) = m_{\pm, \alpha}(z_2, x_0) - m_{\pm, \alpha}(\overline{\tau_1}, x_0)^*, \tag{5.10} $$

$$ \frac{d}{dx}W(\psi_{\pm, \alpha}(\overline{\tau_1}, x_0, x_0), \psi_{\pm, \alpha}(z_2, x_0, x_0)) = (z_1 - z_2)\psi_{\pm, \alpha}(\overline{\tau_1}, x_0)^* \psi_{\pm, \alpha}(z_2, x_0, x_0), \quad z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}. \tag{5.11} $$

Hence, using the limit point hypothesis of $\tau$ at $\pm \infty$ and the $L^2$-property in (5.7) one obtains

$$ (z_2 - z_1) \int_{x_0}^{\pm \infty} dx \left( \psi_{\pm, \alpha}(\overline{\tau_1}, x, x_0)f, \psi_{\pm, \alpha}(z_2, x, x_0)g \right)_\mathcal{H} $$

$$ = (f, [m_{\pm, \alpha}(z_2, x_0) - m_{\pm, \alpha}(\overline{\tau_1}, x_0)^*]g)_\mathcal{H}, \quad f, g \in \mathcal{H}, \quad z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}. \tag{5.12} $$

Setting $z_1 = z_2 = z$ in (5.12), one concludes

$$ m_{\pm, \alpha}(z, x_0) = m_{\pm, \alpha}(\overline{\tau}, x_0)^*, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{5.13} $$
Choosing $f = g$ and $z_2 = z$, $z_1 = \tau$ in (5.12), one also infers
\[
\text{Im}(z) \int_{x_0}^{x_0} dx \|\psi_{\pm,\alpha}(z, x, x_0)f\|_H^2 = \{f, \text{Im}[m_{\pm,\alpha}(z, x_0)]f\}_H, \quad f \in \mathcal{H}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]
(5.14)

Since $m_{\pm,\alpha}(\cdot, x_0)$ are analytic on $\mathbb{C} \setminus \mathbb{R}$, (5.14) yields that $\pm m_{\pm,\alpha}(\cdot, x_0)$ are operator-valued Herglotz functions.

In the following we abbreviate the Wronskian of $\psi_{+\alpha}((\Omega, x, x_0)^* \text{ and } \psi_{-\alpha}(z, x, x_0)$ by $W(z)$. It follows from the identities (2.20), (2.23) and (5.13) that
\[
W(z) = W(\psi_{+\alpha}((\Omega, x, x_0)^*, \psi_{-\alpha}(z, x, x_0))
\]
\[
= m_{-\alpha}(z, x_0) - m_{+\alpha}(z, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]
(5.15)

The Green’s function $G(z, x, x')$ of the Schrödinger operator $H$ then reads
\[
G(z, x, x') = \psi_{\pm,\alpha}(z, x, x_0)W(z)^{-1}\psi_{\pm,\alpha}((\Omega, x, x_0)^*, \quad x \leq x', \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]
(5.16)

Thus,
\[
((H - zI)\mathcal{H}^{-1}f)(x) = \int_{\mathbb{R}} dx' G(z, x, x')f(x'), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x \in \mathbb{R}, \quad f \in L^2(\mathbb{R}; dx; \mathcal{H}).
\]
(5.17)

Next, we introduce the $2 \times 2$ block operator-valued Weyl–Titchmarsh $m$-function, $M_{\alpha}(z, x_0) \in \mathcal{B}(\mathcal{H}^2)$,
\[
M_{\alpha}(z, x_0) = (M_{\alpha, j, j'}(z, x_0))_{j, j' = 0, 1}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
(5.18)
\[
M_{\alpha, 0, 0}(z, x_0) = W(z)^{-1},
\]
(5.19)
\[
M_{\alpha, 0, 1}(z, x_0) = 2^{-1}W(z)^{-1}[m_{-\alpha}(z, x_0) + m_{+\alpha}(z, x_0)],
\]
(5.20)
\[
M_{\alpha, 1, 0}(z, x_0) = 2^{-1}[m_{-\alpha}(z, x_0) + m_{+\alpha}(z, x_0)]W(z)^{-1},
\]
(5.21)
\[
M_{\alpha, 1, 1}(z, x_0) = m_{+\alpha}(z, x_0)W(z)^{-1}m_{-\alpha}(z, x_0)
\]
\[
m_{-\alpha}(z, x_0)W(z)^{-1}m_{+\alpha}(z, x_0).
\]
(5.22)

$M_{\alpha}(z, x_0)$ is a $\mathcal{B}(\mathcal{H}^2)$-valued Herlglotz function with representation
\[
M_{\alpha}(z, x_0) = C_{\alpha}(x_0) + \int_{\mathbb{R}} d\Omega_{\alpha}(\lambda, x_0) \left[ \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right], \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

\[
C_{\alpha}(x_0) = C_{\alpha}(x_0)^*, \quad \int_{\mathbb{R}} \left( e, d\Omega_{\alpha}(\lambda, x_0)e \right)_{\mathcal{B}(\mathcal{H}^2)} < \infty, \quad e \in \mathcal{K}.
\]
(5.23)

In addition, the Stieltjes inversion formula for the nonnegative $\mathcal{B}(\mathcal{H}^2)$-valued measure $d\Omega_{\alpha}(\cdot, x_0)$ reads
\[
\Omega_{\alpha}((\lambda_1, \lambda_2], x_0) = \frac{1}{\pi} \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda \text{Im}(M_{\alpha}(\lambda + i\varepsilon, x_0)), \quad \lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 < \lambda_2.
\]
(5.24)

In particular, $d\Omega_{\alpha}(\cdot, x_0)$ is a $2 \times 2$ block operator-valued measure with $\mathcal{B}(\mathcal{H})$-valued entries $d\Omega_{\alpha, \ell, \ell'}(\cdot, x_0)$. $\ell, \ell' = 0, 1$. Since the diagonal entries of $M_{\alpha}$ are Herglotz functions, the diagonal entries of the measure $d\Omega_{\alpha}(\cdot, x_0)$ are nonnegative $\mathcal{B}(\mathcal{H})$-valued measures. The off-diagonal entries of the measure $d\Omega_{\alpha}(\cdot, x_0)$ naturally admit decompositions into a linear combination of four nonnegative measures.

We note that in formulas (5.23), (5.24) one can replace $z \in \mathbb{C} \setminus \mathbb{R}$ by $z \in \mathbb{C} \setminus \sigma(H)$.
Next, we relate the family of spectral projections, \( \{ E_H(\lambda) \}_{\lambda \in \mathbb{R}} \), of the self-adjoint operator \( H \) and the \( 2 \times 2 \) operator-valued increasing spectral function \( \Omega_\alpha(\lambda, x_0) \), \( \lambda \in \mathbb{R} \), which generates the \( B(\mathcal{H}^2) \)-valued measure \( d\Omega_\alpha(\cdot, x_0) \) in the Herglotz representation \([5.23]\) of \( M_\alpha(z, x_0) \).

We first note that for \( f, g \in L^2(\mathbb{R}; dx; \mathcal{H}) \), applied to \( T = H \),

\[
\begin{align*}
(f, F(H)g)_{L^2(\mathbb{R}; dx; \mathcal{H})} &= \int_{\mathbb{R}} d(f, E_H(\lambda)g)_{L^2(\mathbb{R}; dx; \mathcal{H})} F(\lambda), \\
f, g \in \text{dom}(F(H)) &= \left\{ h \in L^2(\mathbb{R}; dx; \mathcal{H}) \mid \int_{\mathbb{R}} d\|E_H(\lambda)h\|_{L^2(\mathbb{R}; dx; \mathcal{H})}^2 |F(\lambda)|^2 < \infty \right\}.
\end{align*}
\]

**Theorem 5.2.** Let \( \alpha \in [0, \pi) \), \( f, g \in C_0^\infty(\mathbb{R}; \mathcal{H}) \), \( F \in C(\mathbb{R}) \), \( x_0 \in \mathbb{R} \), and \( \lambda_1, \lambda_2 \in \mathbb{R} \), \( \lambda_1 < \lambda_2 \). Then,

\[
(f, F(H)E_H(\lambda, \lambda_2)]g)_{L^2(\mathbb{R}; dx; \mathcal{H})} = (f_\alpha(\cdot, \lambda_0), M_FM_{\lambda(\lambda_1, \lambda_2]}b_\alpha(\cdot, \lambda_0))_{L^2(\mathbb{R}; d\Omega_\alpha(\cdot, x_0); \mathcal{H}^2)}
\]

where we introduced the notation

\[
\begin{align*}
\hat{h}_{\alpha,0}(\lambda, x_0) &= \int_{\mathbb{R}} dx \theta_{\alpha}(\lambda, x, x_0)^* h(x), \\
\hat{h}_{\alpha,1}(\lambda, x_0) &= \int_{\mathbb{R}} dx \phi_{\alpha}(\lambda, x, x_0)^* h(x), \\
\hat{h}_\alpha(\lambda, x_0) &= (\hat{h}_{\alpha,0}(\lambda, x_0), \hat{h}_{\alpha,1}(\lambda, x_0))^T, \\
\hat{h} \in \text{dom}(M_G) &= \left\{ k \in L^2(\mathbb{R}; d\Omega_\alpha(\cdot, x_0); \mathcal{H}^2) \mid Gk \in L^2(\mathbb{R}; d\Omega_\alpha(\cdot, x_0); \mathcal{H}^2) \right\}.
\end{align*}
\]

**Proof.** The point of departure for deriving \(5.26\) is again Stone’s formula \(4.1\).
Insertion of (5.16) and (5.17) into (5.29) then yields the following:

\[
(f, F(H)E_H((\lambda_1, \lambda_2))g)_{L^2(\mathbb{R}; dx; H)} &= \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda F(\lambda) \int dx \\
\times \left\{ (f(x), \psi_{+, \alpha}(\lambda + i\varepsilon, x, x_0)W(\lambda + i\varepsilon)^{-1} \int_{-\infty}^{\infty} dx' \psi_{-, \alpha}(\lambda - i\varepsilon, x', x_0)^* g(x'))_H \\
+ (f(x), \psi_{-, \alpha}(\lambda + i\varepsilon, x, x_0)W(\lambda + i\varepsilon)^{-1} \int_{-\infty}^{\infty} dx' \psi_{+, \alpha}(\lambda - i\varepsilon, x', x_0)^* g(x'))_H \\
- (f(x), \psi_{+, \alpha}(\lambda - i\varepsilon, x, x_0)W(\lambda - i\varepsilon)^{-1} \int_{-\infty}^{\infty} dx' \psi_{-, \alpha}(\lambda + i\varepsilon, x', x_0)^* g(x'))_H \\
- (f(x), \psi_{-, \alpha}(\lambda - i\varepsilon, x, x_0)W(\lambda - i\varepsilon)^{-1} \int_{-\infty}^{\infty} dx' \psi_{+, \alpha}(\lambda + i\varepsilon, x', x_0)^* g(x'))_H \right\}.
\]

Freely interchanging the \(dx\) and \(dx'\) integrals with the limits and the \(d\lambda\) integral (since all integration domains are finite and all integrands are continuous), and inserting the expressions (5.9) for \(\psi_{\pm, \alpha}(z, x, x_0)\) into (5.30), one obtains

\[
(f, F(H)E_H((\lambda_1, \lambda_2))g)_{L^2(\mathbb{R}; dx; H)} = \int_{\mathbb{R}} dx \left( f(x), \left\{ \int_{-\infty}^{\infty} dx' \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda F(\lambda) \\
\times \left[ \theta_{\alpha}(\lambda, x, x_0) + \phi_{\alpha}(\lambda, x, x_0)m_{+, \alpha}(\lambda + i\varepsilon, x_0) \right] \\
\times W(\lambda + i\varepsilon)^{-1} \left[ \theta_{\alpha}(\lambda, x', x_0)^* + m_{-, \alpha}(\lambda + i\varepsilon, x_0)^* \phi_{\alpha}(\lambda, x', x_0)^* \right] g(x') \\
- \left[ \theta_{\alpha}(\lambda, x, x_0) + \phi_{\alpha}(\lambda, x, x_0)m_{+, \alpha}(\lambda - i\varepsilon, x_0) \right] \\
\times W(\lambda - i\varepsilon)^{-1} \left[ \theta_{\alpha}(\lambda, x', x_0)^* + m_{-, \alpha}(\lambda - i\varepsilon, x_0)^* \phi_{\alpha}(\lambda, x', x_0)^* \right] g(x') \right] \\
+ \int_{\mathbb{R}}^{\infty} dx' \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda F(\lambda) \\
\times \left[ \theta_{\alpha}(\lambda, x, x_0) + \phi_{\alpha}(\lambda, x, x_0)m_{-, \alpha}(\lambda + i\varepsilon, x_0) \right] \\
\times W(\lambda + i\varepsilon)^{-1} \left[ \theta_{\alpha}(\lambda, x', x_0)^* + m_{+, \alpha}(\lambda + i\varepsilon, x_0)^* \phi_{\alpha}(\lambda, x', x_0)^* \right] g(x') \\
- \left[ \theta_{\alpha}(\lambda, x, x_0) + \phi_{\alpha}(\lambda, x, x_0)m_{-, \alpha}(\lambda - i\varepsilon, x_0) \right] \\
\times W(\lambda - i\varepsilon)^{-1} \left[ \theta_{\alpha}(\lambda, x', x_0)^* + m_{+, \alpha}(\lambda - i\varepsilon, x_0)^* \phi_{\alpha}(\lambda, x', x_0)^* \right] g(x') \right) \right)_H.
\]

Here we employed (5.13), the fact that for fixed \(x \in \mathbb{R}, \theta_{\alpha}(z, x, x_0)\) and \(\phi_{\alpha}(z, x, x_0)\) are entire with respect to \(z\), that \(\theta_{\alpha}(z, \cdot, x_0), \phi_{\alpha}(z, \cdot, x_0) \in W^{1,1}_{loc}(\mathbb{R}; H)\), and hence that

\[
\theta_{\alpha}(\lambda \pm i\varepsilon, x, x_0) = \theta_{\alpha}(\lambda, x, x_0) \pm i\varepsilon (d/dz)\theta_{\alpha}(z, x, x_0)|_{z=\lambda} + O(\varepsilon^2),
\]

\[
\phi_{\alpha}(\lambda \pm i\varepsilon, x, x_0) = \phi_{\alpha}(\lambda, x, x_0) \pm i\varepsilon (d/dz)\phi_{\alpha}(z, x, x_0)|_{z=\lambda} + O(\varepsilon^2)
\]

with \(O(\varepsilon^2)\) being uniform with respect to \((\lambda, x)\) as long as \(\lambda \) and \(x\) vary in compact subsets of \(\mathbb{R}\). Moreover, we used that

\[
\varepsilon \|M_{\alpha}(\lambda + i\varepsilon, x_0)\|_{B(H^2)} \leq C(\lambda_1, \lambda_2, \varepsilon_0, x_0), \quad \lambda \in [\lambda_1, \lambda_2], \quad 0 < \varepsilon \leq \varepsilon_0,
\]

\[
\varepsilon \|\text{Re}(M_{\alpha}(\lambda + i\varepsilon, x_0))\|_{B(H^2)} \equiv o(1), \quad \lambda \in \mathbb{R},
\]

(5.33)
In particular, utilizing \((5.13), (5.32), (5.33)\), and the elementary facts

\[
\text{Im} \left[ m_{\pm,\alpha}(\lambda + i\varepsilon, x_0)W(\lambda + i\varepsilon)^{-1} \right] = \frac{1}{2} \text{Im} \left[ \left( m_{-\alpha}(\lambda + i\varepsilon, x_0) + m_{+\alpha}(\lambda + i\varepsilon, x_0) \right)W(\lambda + i\varepsilon)^{-1} \right],
\]

\[
\text{Im} \left[ W(\lambda + i\varepsilon)^{-1}m_{\pm,\alpha}(\lambda + i\varepsilon, x_0) \right] = \frac{1}{2} \text{Im} \left[ W(\lambda + i\varepsilon)^{-1}\left( m_{-\alpha}(\lambda + i\varepsilon, x_0) + m_{+\alpha}(\lambda + i\varepsilon, x_0) \right) \right], \quad \lambda \in \mathbb{R}, \; \varepsilon > 0,
\]

\(\phi_\alpha(\lambda \pm i\varepsilon, x, x_0)\) and \(\theta_\alpha(\lambda \pm i\varepsilon, x, x_0)\) under the \(d\lambda\) integrals in \((5.31)\) have immediately been replaced by \(\phi_\alpha(\lambda, x, x_0)\) and \(\theta_\alpha(\lambda, x, x_0)\). Collecting appropriate terms in \((5.31)\) then yields

\[
(f, F(H)E_H((\lambda_1, \lambda_2))g)_{L^2(\mathbb{R}; dx; \mathcal{H})} = \int_{\mathbb{R}} dx \left( \int_{\mathbb{R}} dx' \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda \text{Im}(M_{\alpha,\ell,\ell'}(\lambda + i\varepsilon, x_0)) \right) \cdot \left( \theta_\alpha(\lambda, x, x_0) \text{Im} \left[ W(\lambda + i\varepsilon)^{-1} \right] \phi_\alpha(\lambda, x', x_0)^* \right.
\]

\[
+ \left. 2^{-1} \theta_\alpha(\lambda, x, x_0) \text{Im} \left[ W(\lambda + i\varepsilon)^{-1} \right] \phi_\alpha(\lambda, x', x_0)^* \right)
\]

\[
\times \left. \{ \text{Im} \left[ m_{-\alpha}(\lambda + i\varepsilon, x_0) + m_{+\alpha}(\lambda + i\varepsilon, x_0) \right] W(\lambda + i\varepsilon)^{-1} \} \theta_\alpha(\lambda, x', x_0)^* \right)
\]

\[
+ \phi_\alpha(\lambda, x, x_0) \times \left. \text{Im} \left[ m_{-\alpha}(\lambda + i\varepsilon, x_0) + m_{+\alpha}(\lambda + i\varepsilon, x_0) \right] W(\lambda + i\varepsilon)^{-1} \right] \phi_\alpha(\lambda, x', x_0)^* g(x') \right) \cdot \mathcal{H}.
\]

Since by \((5.22)\) \((\ell, \ell' = 0, 1)\)

\[
\int_{(\lambda_1, \lambda_2)} d\Omega_{\alpha,\ell,\ell'}(\lambda, x_0) = \Omega_{\alpha,\ell,\ell'}((\lambda_1, \lambda_2), x_0)
\]

\[
= \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda \text{Im}(M_{\alpha,\ell,\ell'}(\lambda + i\varepsilon, x_0)),
\]

one also has

\[
\int_{\mathbb{R}} d\Omega_{\alpha,\ell,\ell'}(\lambda, x_0) h(\lambda) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\mathbb{R}} d\lambda \text{Im}(M_{\alpha,\ell,\ell'}(\lambda + i\varepsilon, x_0)) h(\lambda), \quad h \in C_0(\mathbb{R}; \mathcal{H}),
\]

\[
(5.37)
\]

\[
\int_{(\lambda_1, \lambda_2)} d\Omega_{\alpha,\ell,\ell'}(\lambda, x_0) k(\lambda) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda \text{Im}(M_{\alpha,\ell,\ell'}(\lambda + i\varepsilon, x_0)) k(\lambda),
\]

\[
k \in C(\mathbb{R}; \mathcal{H}), \quad (5.38)
\]
Then using (5.18)–(5.22), (5.27), and interchanging the \(dx, \, dx'\) and \(d\Omega_{\alpha,\ell,\ell'}(\cdot,x_0)\), \(\ell, \ell' = 0, 1\), integrals once more, one concludes from (5.35)

\[
(f, F(H)E_H((\lambda_1, \lambda_2)]g)_{L^2(\mathbb{R}; dx; \mathcal{H})} = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dx' \int_{(\lambda_1, \lambda_2]} F(\lambda) \\
\times \left\{ \theta_\alpha(\lambda, x, x_0) d\Omega_{\alpha,0,0}(\lambda, x_0) \theta_\alpha(\lambda, x', x_0)^* \\
+ \phi_\alpha(\lambda, x, x_0) d\Omega_{\alpha,0,1}(\lambda, x_0) \phi_\alpha(\lambda, x', x_0)^* \\
+ \phi_\alpha(\lambda, x, x_0) d\Omega_{\alpha,1,0}(\lambda, x_0) \theta_\alpha(\lambda, x', x_0)^* \\
+ \phi_\alpha(\lambda, x, x_0) d\Omega_{\alpha,1,1}(\lambda, x_0) \phi_\alpha(\lambda, x', x_0)^* \right\} g(x') \bigg\}\mathcal{H} \\
= \int_{(\lambda_1, \lambda_2]} F(\lambda) \left( \tilde{f}_\alpha(\lambda, x_0), d\Omega_{\alpha}(\lambda, x_0) \tilde{g}_\alpha(\lambda, x_0) \right)_{\mathcal{H}^2}. \\
(5.39)
\]

\[\square\]

**Remark 5.3.** Again we emphasize that the idea of a straightforward derivation of the link between the family of spectral projections \(E_H(\cdot)\) and the \(2 \times 2\) block operator-valued spectral function \(\Omega_{\cdot}()\) of \(H\) in Theorem 5.2 can already be found in [68] as pointed out in Remark 4.4. It applies equally well to Dirac-type operators and Hamiltonian systems on \(\mathbb{R}\) (see the extensive literature cited, e.g., in [33]) and to Jacobi and CMV operators on \(\mathbb{Z}\) (cf. [24] and [58]).

As in the half-line case, one can improve on Theorem 5.2 and remove the compact support restrictions on \(f\) and \(g\) in the usual way. To this end one considers the map

\[
\tilde{U}_\alpha(x_0) : \left\{ C^0_0(\mathbb{R}) \rightarrow L^2(\mathbb{R}; d\Omega_{\alpha}(\cdot,x_0); \mathcal{H}^2) \right\} \\
h \mapsto \tilde{h}_\alpha(\cdot,x_0) = \left( \tilde{h}_{\alpha,0}(\cdot, x_0), \tilde{h}_{\alpha,1}(\cdot, x_0) \right)^\top, \\
(5.40)
\]

\[
\tilde{h}_{\alpha,0}(\cdot, x_0) = \int_{\mathbb{R}} dx \theta_\alpha(\lambda, x, x_0)^* h(x), \quad \tilde{h}_{\alpha,1}(\cdot, x_0) = \int_{\mathbb{R}} dx \phi_\alpha(\lambda, x, x_0)^* h(x).
\]

Taking \(f = g\), \(F = 1\), \(\lambda_1 \downarrow -\infty\), and \(\lambda_2 \uparrow \infty\) in (5.26), then shows that \(\tilde{U}_\alpha(x_0)\) is a densely defined isometry in \(L^2(\mathbb{R}; dx; \mathcal{H})\), which extends by continuity to an isometry on \(L^2(\mathbb{R}; dx; \mathcal{H})\). The latter is denoted by \(U_\alpha(x_0)\) and given by

\[
U_\alpha(x_0) : \left\{ L^2(\mathbb{R}; dx; \mathcal{H}) \rightarrow L^2(\mathbb{R}; d\Omega_{\alpha}(\cdot,x_0); \mathcal{H}^2) \right\} \\
h \mapsto \hat{h}_\alpha(\cdot, x_0) = \left( \hat{h}_{\alpha,0}(\cdot, x_0), \hat{h}_{\alpha,1}(\cdot, x_0) \right)^\top, \\
(5.41)
\]

\[
\hat{h}_{\alpha,0}(\cdot, x_0) = \frac{\tilde{h}_{\alpha,0}(\cdot, x_0)}{\tilde{h}_{\alpha,1}(\cdot, x_0)} = \text{l.i.m.}_{a\downarrow -\infty, b\uparrow \infty} \left( \int_a^b dx \theta_\alpha(\cdot, x, x_0)^* h(x) \right)^, \\
\hat{h}_{\alpha,1}(\cdot, x_0) = \int_a^b dx \phi_\alpha(\cdot, x, x_0)^* h(x),
\]

where l.i.m. refers to the \(L^2(\mathbb{R}; d\Omega_{\alpha}(\cdot, x_0); \mathcal{H}^2)\)-limit.
The calculation in (5.39) also yields

\[
(E_H((\lambda_1, \lambda_2))g)(x) = \int_{(\lambda_1, \lambda_2]} (\theta_\alpha(\lambda, x, x_0), \phi_\alpha(\lambda, x, x_0)) d\Omega_\alpha(\lambda, x_0) \widehat{g}_\alpha(\lambda, x_0)
\]

\[
= \int_{(\lambda_1, \lambda_2]} \left[ \theta_\alpha(\lambda, x, x_0) d\Omega_{\alpha,0,0}(\lambda, x_0) \widehat{g}_{\alpha,0}(\lambda, x_0)
+ \theta_\alpha(\lambda, x, x_0) d\Omega_{\alpha,0,1}(\lambda, x_0) \widehat{g}_{\alpha,1}(\lambda, x_0) + \phi_\alpha(\lambda, x, x_0) d\Omega_{\alpha,1,0}(\lambda, x_0) \widehat{g}_{\alpha,0}(\lambda, x_0)
+ \phi_\alpha(\lambda, x, x_0) d\Omega_{\alpha,1,1}(\lambda, x_0) \widehat{g}_{\alpha,1}(\lambda, x_0) \right],
\]

\[ g \in C_0^\infty(\mathbb{R}) \tag{5.42} \]

and subsequently, (5.42) extends to all \( g \in L^2(\mathbb{R}; dx; \mathcal{H}) \) by continuity. Moreover, taking \( \lambda_1 \downarrow -\infty \) and \( \lambda_2 \uparrow \infty \) in (5.42) and using

\[
s-\lim_{\lambda_1 \to -\infty} E_H(\lambda) = 0, \quad s-\lim_{\lambda_1 \to \infty} E_H(\lambda) = I_{L^2(\mathbb{R}; dx; \mathcal{H})},\tag{5.43}
\]

where

\[
E_H(\lambda) = E_H(\langle -\infty, \lambda \rangle), \quad \lambda \in \mathbb{R}, \tag{5.44}
\]

then yield

\[
g(\cdot) = \text{l.i.m.}_{\mu_1 \downarrow -\infty, \mu_2 \uparrow \infty} \int_{\mu_1, \mu_2} (\theta_\alpha(\lambda, \cdot, x_0), \phi_\alpha(\lambda, \cdot, x_0)) d\Omega_\alpha(\lambda, x_0) \widehat{g}_\alpha(\lambda, x_0)
\]

\[
= \text{l.i.m.}_{\mu_1 \downarrow -\infty, \mu_2 \uparrow \infty} \int_{\mu_1}^{\mu_2} \left[ \theta_\alpha(\lambda, \cdot, x_0) d\Omega_{\alpha,0,0}(\lambda, x_0) \widehat{g}_{\alpha,0}(\lambda, x_0)
+ \theta_\alpha(\lambda, \cdot, x_0) d\Omega_{\alpha,0,1}(\lambda, x_0) \widehat{g}_{\alpha,1}(\lambda, x_0) + \phi_\alpha(\lambda, \cdot, x_0) d\Omega_{\alpha,1,0}(\lambda, x_0) \widehat{g}_{\alpha,0}(\lambda, x_0)
+ \phi_\alpha(\lambda, \cdot, x_0) d\Omega_{\alpha,1,1}(\lambda, x_0) \widehat{g}_{\alpha,1}(\lambda, x_0) \right],
\]

\[ g \in L^2(\mathbb{R}; dx; \mathcal{H}), \tag{5.45} \]

where l.i.m. refers to the \( L^2(\mathbb{R}; dx; \mathcal{H}) \)-limit. In addition, one can show that the map \( U_\alpha(x_0) \) in (5.41) is onto and hence unitary with

\[
U_\alpha(x_0)^{-1} : \left\{ L^2(\mathbb{R}; d\Omega_{\cdot, x_0}; \mathcal{H}^2) \to L^2(\mathbb{R}; dx; \mathcal{H}) \right\}
\]

\[
h_\alpha(\cdot) = \text{l.i.m.}_{\mu_1 \downarrow -\infty, \mu_2 \uparrow \infty} \int_{\mu_1}^{\mu_2} (\theta_\alpha(\lambda, \cdot, x_0), \phi_\alpha(\lambda, \cdot, x_0)) d\Omega_\alpha(\lambda, x_0) \widehat{h}_\alpha(\lambda). \tag{5.46} \]

Indeed, denoting the operator defined in (5.46) temporarily by \( V_\alpha(x_0) \), one can closely follow the arguments in the corresponding half-line case in (4.27)–(4.28).

After first proving that \( V_\alpha(x_0) \) is bounded, one then assumes that \( \hat{f} = (f_0, f_1)^T \in \ker(V_\alpha(x_0)) \subset L^2(\mathbb{R}; d\Omega_\alpha(x_0); \mathcal{H}^2) \). As in the half-line case, one can show using the dominated convergence theorem that this implies that for all \( \lambda \in \mathbb{R} \) and \( e \in \mathcal{H} \),

\[
\int_{(\lambda_1, \lambda_2]} (e_1(\theta_\alpha(\lambda, x, x_0), \phi_\alpha(\lambda, x, x_0))) d\Omega_\alpha(\lambda, x_0) \hat{f}(\lambda) = 0,
\]

\[
\int_{(\lambda_1, \lambda_2]} (e_1(\theta_\alpha(\lambda, x, x_0), \phi_\alpha(\lambda, x, x_0))) d\Omega_\alpha(\lambda, x_0) \hat{f}(\lambda) = 0. \tag{5.47}
\]

Using the fact that \( \hat{f}, \chi_\lambda \alpha, \lambda \alpha(e_0, e_1)^T \in L^2(\mathbb{R}; d\Omega_\alpha(x_0); \mathcal{H}^2) \) for all \( \lambda_1 < \lambda_2 \), \( e_0, e_1 \in \mathcal{H} \), that \( \theta_\alpha(\lambda, x, x_0), \phi_\alpha(\lambda, x, x_0), \theta_\alpha(\lambda, x, x_0), \phi_\alpha(\lambda, x, x_0) \) are continuous with respect to \( (\lambda, x) \in \mathbb{R}^2 \), and the dominated convergence theorem once again,
one finally concludes that for all \( e \in \mathcal{H} \),
\[
\int_{(\lambda_1, \lambda_2]} \langle e, (\theta_\alpha(\lambda, x_0, x_0), \phi_\alpha(\lambda, x_0, x_0)) \rangle d\Omega_\alpha(\lambda, x_0) \hat{f}(\lambda) \rangle_{\mathcal{H}} = 0.
\]
(5.48)
\[
\int_{(\lambda_1, \lambda_2]} \langle e, (\theta'_\alpha(\lambda, x_0, x_0), \phi'_\alpha(\lambda, x_0, x_0)) \rangle d\Omega_\alpha(\lambda, x_0) \hat{f}(\lambda) \rangle_{\mathcal{H}} = 0.
\]
(5.49)

The sum of (5.48) with \( e = \cos(\alpha)e_1 - \sin(\alpha)e_2 \) and (5.49) with \( e = \sin(\alpha)e_1 + \cos(\alpha)e_2 \) then yields,
\[
\int_{(\lambda_1, \lambda_2]} \langle (e_1, e_2)^\top, d\Omega_\alpha(\lambda, x_0) \hat{f}(\lambda) \rangle_{\mathcal{H}}^2 = 0, \quad (e_1, e_2)^\top \in \mathcal{H}^2.
\]
(5.50)

Since the interval \((\lambda_1, \lambda_2] \) was chosen arbitrary, (5.50) implies
\[
\hat{f}(\lambda) = 0 \quad \Omega_\alpha(\cdot, x_0) \text{-a.e.,}
\]
(5.51)

and hence that \( V_\alpha(x_0) \) is injective and thus \( U_\alpha(x_0) \) is onto.

We sum up these considerations in a variant of the spectral theorem for (functions of) \( H \).

**Theorem 5.4.** Let \( F \in C(\mathbb{R}) \) and \( x_0 \in \mathbb{R} \). Then,
\[
U_\alpha(x_0) F(H) U_\alpha(x_0)^{-1} = M_F
\]
(5.52)
in \( L^2(\mathbb{R}; d\Omega_\alpha(\cdot, x_0); \mathcal{H}^2) \) (cf. [5.28]). Moreover,
\[
\sigma(F(H)) = \text{ess.ran}_{\Omega_\alpha}(F),
\]
(5.53)
\[
\sigma(H) = \text{supp}(d\Omega_\alpha(\cdot, x_0)),
\]
(5.54)

and the multiplicity of the spectrum of \( H \) is at most equal to \( \dim(\mathcal{H})^2 = 2 \dim(\mathcal{H}) \).

**Proof.** The proof of the theorem is analogous to the one given for Theorem 4.7. \( \square \)

**Appendix A. Basic Facts on Operator-Valued Herglotz Functions**

In this appendix we review some basic facts on (bounded) operator-valued Herglotz functions (also called Nevanlinna, Pick, \( R \)-functions, etc.), applicable to \( m_\alpha \) and \( G_\alpha(\cdot, x, x), x \in (a, b) \), discussed in the bulk of this paper. For additional details concerning the material in this appendix we refer to [56].

In the remainder of this appendix, let \( \mathcal{H} \) be a separable, complex Hilbert space with inner product denoted by \( \langle \cdot, \cdot \rangle_\mathcal{H} \).

**Definition A.1.** The map \( M : \mathbb{C}_+ \to \mathcal{B}(\mathcal{H}) \) is called a bounded operator-valued Herglotz function in \( \mathcal{H} \) (in short, a bounded Herglotz operator in \( \mathcal{H} \)) if \( M \) is analytic on \( \mathbb{C}_+ \) and \( \text{Im}(M(z)) \geq 0 \) for all \( z \in \mathbb{C}_+ \).

Here we follow the standard notation
\[
\text{Im}(M) = (M - M^*)/(2i), \quad \text{Re}(M) = (M + M^*)/2, \quad M \in \mathcal{B}(\mathcal{H}).
\]
(A.1)

Note that \( M \) is a bounded Herglotz operator if and only if the scalar-valued functions \((u, Mu)_{\mathcal{H}} \) are Herglotz for all \( u \in \mathcal{H} \).
As in the scalar case one usually extends $M$ to $\mathbb{C}_-$ by reflection, that is, by defining
\[ M(z) = M(\overline{z})^*, \quad z \in \mathbb{C}_-. \tag{A.2} \]
Hence $M$ is analytic on $\mathbb{C}\setminus\mathbb{R}$, but $M|_{\mathbb{C}_-}$ and $M|_{\mathbb{C}_+}$, in general, are not analytic continuations of each other.

Of course, one can also consider unbounded operator-valued Herglotz functions, but they will not be used in this paper.

In contrast to the scalar case, one cannot generally expect strict inequality in $\text{Im}(M(\cdot)) \geq 0$. However, the kernel of $\text{Im}(M(\cdot))$ has simple properties:

**Lemma A.2.** Let $M(\cdot)$ be a bounded operator-valued Herglotz function in $\mathcal{H}$. Then the kernel $\mathcal{H}_0 = \ker(\text{Im}(M(z)))$ is independent of $z \in \mathbb{C}_+$. Consequently, upon decomposing $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, $\mathcal{H}_1 = \mathcal{H}_0^\perp$, $\text{Im}(M(\cdot))$ takes on the form
\[ \text{Im}(M(z)) = \begin{pmatrix} 0 & 0 \\ 0 & N_1(z) \end{pmatrix}, \quad z \in \mathbb{C}_+, \tag{A.3} \]
where $N_1(\cdot) \in \mathcal{B}(\mathcal{H}_1)$ satisfies
\[ N_1(z) > 0, \quad z \in \mathbb{C}_+. \tag{A.4} \]

For a proof of Lemma A.2 see, for instance, [41, Proposition 1.2 (i)] (alternatively, the proof of [44] Lemma 5.3 in the matrix-valued context extends to the present infinite-dimensional situation).

Next we recall the definition of a bounded operator-valued measure (see, also [24, p. 319], [44], [103]):

**Definition A.3.** Let $\mathcal{H}$ be a separable, complex Hilbert space. A map $\Sigma : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$, with $\mathcal{B}(\mathbb{R})$ the Borel $\sigma$-algebra on $\mathbb{R}$, is called a bounded, nonnegative, operator-valued measure if it satisfies the following conditions (i) and (ii):
\begin{itemize}
  \item[(i)] $\Sigma(\emptyset) = 0$ and $0 \leq \Sigma(B) \in \mathcal{B}(\mathcal{H})$ for all $B \in \mathcal{B}(\mathbb{R})$.
  \item[(ii)] $\Sigma(\cdot)$ is strongly countably additive (i.e., with respect to the strong operator topology in $\mathcal{H}$), that is,
  \[ \Sigma(B) = \limsup_{N \to \infty} \sum_{j=1}^N \Sigma(B_j) \tag{A.5} \]
  whenever $B = \bigcup_{j \in \mathbb{N}} B_j$, with $B_k \cap B_\ell = \emptyset$ for $k \neq \ell$, $B_k \in \mathcal{B}(\mathbb{R})$, $k, \ell \in \mathbb{N}$.
\end{itemize}

$\Sigma(\cdot)$ is called an (operator-valued) spectral measure (or an orthogonal operator-valued measure) if additionally the following condition (iii) holds:
\begin{itemize}
  \item[(iii)] $\Sigma(\cdot)$ is projection-valued (i.e., $\Sigma(B)^2 = \Sigma(B)$, $B \in \mathcal{B}(\mathbb{R})$) and $\Sigma(\mathbb{R}) = I_\mathcal{H}$.
\end{itemize}
\begin{itemize}
  \item[(iii)] Let $f \in \mathcal{H}$ and $B \in \mathcal{B}(\mathbb{R})$. Then the vector-valued measure $\Sigma(\cdot) f$ has finite variation on $B$, denoted by $V(\Sigma f; B)$, if
  \[ V(\Sigma f; B) = \sup \left\{ \sum_{j=1}^N \|\Sigma(B_j)f\|_\mathcal{H} \right\} < \infty, \tag{A.6} \]
  where the supremum is taken over all finite sequences $\{B_j\}_{1 \leq j \leq N}$ of pairwise disjoint subsets on $\mathbb{R}$ with $B_j \subseteq B$, $1 \leq j \leq N$. In particular, $\Sigma(\cdot) f$ has finite total variation if $V(\Sigma f; \mathbb{R}) < \infty$. 

We recall that due to monotonicity considerations, taking the limit in the strong operator topology in (A.5) is equivalent to taking the limit with respect to the weak operator topology in \( \mathcal{H} \).

We also note that integrals of the type (A.7)–(A.10) below are now taken with respect to an operator-valued measure, as opposed to the Bochner integrals we used in the bulk of this paper, Sections 2–5.

For relevant material in connection with the following result we refer the reader, for instance, to [1–4, 10, 20, 24 Sect. VI.5, 30 Sect. I.4, 31, 32, 37, 39–41, 45 Sects. XIII.5–XIII.7, 68, 76, 77, 82, 83, 84, 97 Ch. VI, 98, 99, 100, 117, 120, 123 Sects. 8–10].

**Theorem A.4.** ([10, 30 Sect. I.4, 117]) Let \( M \) be a bounded operator-valued Herglotz function in \( \mathcal{H} \). Then the following assertions hold:

(i) For each \( f \in \mathcal{H} \), \((f, M(\cdot)f)_{\mathcal{H}}\) is a (scalar) Herglotz function.

(ii) Suppose that \( \{e_j\}_{j \in \mathbb{N}} \) is a complete orthonormal system in \( \mathcal{H} \) and that for some subset of \( \mathbb{R} \) having positive Lebesgue measure, and for all \( j \in \mathbb{N} \), \((e_j, M(\cdot)e_j)_{\mathcal{H}}\) has zero normal limits. Then \( M \equiv 0 \).

(iii) There exists a bounded, nonnegative \( \mathcal{B}(\mathcal{H}) \)-valued measure \( \Omega \) on \( \mathbb{R} \) such that the Nevanlinna representation

\[
M(z) = C + Dz + \int_{\mathbb{R}} \frac{d\Omega(\lambda)}{1 + \lambda^2} \frac{1 + \lambda z}{\lambda - z} \quad (A.7)
\]

\[
= C + Dz + \int_{\mathbb{R}} d\Omega(\lambda) \left[ \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right], \quad z \in \mathbb{C}_+ , \quad (A.8)
\]

\[
\tilde{\Omega}((-\infty, \lambda]) = \text{s-lim}_{\varepsilon \downarrow 0} \int_{-\infty}^{\lambda + \varepsilon} d\Omega(t) 1 + t^2 , \quad \lambda \in \mathbb{R}, \quad (A.9)
\]

\[
\tilde{\Omega}(\mathbb{R}) = \text{Im}(M(i)) = \int_{\mathbb{R}} d\Omega(\lambda) \frac{1}{1 + \lambda^2} \in \mathcal{B}(\mathcal{H}), \quad (A.10)
\]

\[
C = \text{Re}(M(i)), \quad D = \text{s-lim}_{\eta \uparrow \infty} \frac{1}{i\eta} M(i\eta) \geq 0, \quad (A.11)
\]

holds in the strong sense in \( \mathcal{H} \). Here \( \tilde{\Omega}(B) = \int_B \left(1 + \lambda^2\right)^{-1} d\Omega(\lambda), B \in \mathcal{B}(\mathbb{R}) \).

(iv) Let \( \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 < \lambda_2 \). Then the Stieltjes inversion formula for \( \Omega \) reads

\[
\Omega([\lambda_1, \lambda_2]) f = \pi^{-1} s\text{-lim}_{\delta \downarrow 0} s\text{-lim}_{\varepsilon \downarrow 0} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda \text{Im}(M(\lambda + i\varepsilon)) f, \quad f \in \mathcal{H}. \quad (A.12)
\]

(v) Any isolated poles of \( M \) are simple and located on the real axis, the residues at poles being nonpositive bounded operators in \( \mathcal{B}(\mathcal{H}) \).

(vi) For all \( \lambda \in \mathbb{R} \),

\[
s\text{-lim}_{\varepsilon \downarrow 0} \lambda \in \text{Re}(M(\lambda + i\varepsilon)) = 0, \quad (A.13)
\]

\[
\Omega(\{\lambda\}) = s\text{-lim}_{\varepsilon \downarrow 0} \varepsilon \text{Im}(M(\lambda + i\varepsilon)) = -i \text{s\text{-lim}}_{\varepsilon \downarrow 0} \varepsilon M(\lambda + i\varepsilon). \quad (A.14)
\]

(vii) If in addition \( M(z) \in \mathcal{B}_\infty(\mathcal{H}), z \in \mathbb{C}_+ \), then the measure \( \Omega \) in (A.7) is countably additive with respect to the \( \mathcal{B}(\mathcal{H}) \)-norm, and the Nevanlinna representation (A.7), (A.8) and the Stieltjes inversion formula (A.12) as well as (A.13), (A.14) hold with the limits taken with respect to the \( \| \cdot \|_{\mathcal{B}(\mathcal{H})} \)-norm.

(viii) Let \( f \in \mathcal{H} \) and assume in addition that \( \Omega(\cdot) f \) is of finite total variation. Then...
for a.e. \( \lambda \in \mathbb{R} \), the normal limits \( M(\lambda + i0)f \) exist in the strong sense and

\[
\lim_{\varepsilon \downarrow 0} M(\lambda + i\varepsilon)f = M(\lambda + i0)f = H(\Omega(\cdot)f)(\lambda) + i\pi \Omega'(\lambda)f,
\]

(A.15)

where \( H(\Omega(\cdot)f) \) denotes the \( \mathcal{H} \)-valued Hilbert transform

\[
H(\Omega(\cdot)f)(\lambda) = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{t - \lambda} d\Omega(t)f = \lim_{\delta \downarrow 0} \int_{|t - \lambda| \geq \delta} \frac{1}{t - \lambda} d\Omega(t)f.
\]

(A.16)

As usual, the normal limits in Theorem A.4 can be replaced by nontangential ones.

The nature of the boundary values of \( M(\lambda + i0) \) when for some \( p > 0 \), \( M(z) \in L^p(\mathcal{H}) \), \( z \in \mathbb{C}_+ \), was clarified in detail in [27], [94], [95], [96].

Using an approach based on operator-valued Stieltjes integrals, a special case of Theorem A.4 was proved by Brodskii [30, Sect. I.4]. In particular, he proved the analog of the Herglotz representation for operator-valued Caratheodory functions. More precisely, if \( F \) is analytic on \( D \) (the open unit disk in \( \mathbb{C} \)) with nonnegative real part \( \text{Re}(F(w)) \geq 0 \), \( w \in D \), then \( F \) is of the form

\[
F(w) = i \text{Im}(F(0)) + \int_{\partial D} d\Upsilon(\zeta) \frac{\zeta + w}{\zeta - w}, \quad w \in D,
\]

(A.17)

\[
\text{Re}(F(0)) = \Upsilon(\partial D),
\]

with \( \Upsilon \) a bounded, nonnegative \( \mathcal{B}(\mathcal{H}) \)-valued measure on \( \partial D \). The result (A.17) can also be derived by an application of Naimark’s dilation theory (cf. [10] and [49, p. 68]), and it can also be used to derive the Nevanlinna representation (A.7), (A.8) (cf. [10], and in a special case also [30, Sect. I.4]). Finally, we also mention that Shmuly’ian [117] discusses the Nevanlinna representation (A.7), (A.8); moreover, certain special classes of Nevanlinna functions, isolated by Kac and Krein [70] in the scalar context, are studied by Brodskii [30, Sect. I.4] and Shmuly’ian [117].

For a variety of applications of operator-valued Herglotz functions, see, for instance, [1], [4], [16], [29], [32], [39]–[41], [53], [83]–[86], [117], and the literature cited therein.

**Appendix B. Direct Integrals and the Construction of the Model Hilbert Space \( L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \)**

In this appendix we recall the construction of a model Hilbert space \( L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \) (and related Banach spaces \( L^p(\mathbb{R}; wd\Sigma; \mathcal{K}) \), \( p \geq 1 \), \( w \) an appropriate scalar nonnegative weight function) as discussed in detail in [53] and slightly extended in [57]. Variants of this construction are of importance in the bulk of this paper.

For proofs of the results in this appendix we refer to [53] and [57]; as general background literature for the topic to follow, we refer to the theory of direct integrals of Hilbert spaces as presented, for instance, in [19] Ch. 4], [28] Ch. 7], [44] Ch. II], [122] Ch. XII].

Throughout this section we make the following assumptions:

**Hypothesis B.1.** Let \( \mu \) denote a \( \sigma \)-finite Borel measure on \( \mathbb{R} \), \( \mathcal{B}(\mathbb{R}) \) the Borel \( \sigma \)-algebra on \( \mathbb{R} \), and suppose that \( \mathcal{K} \) and \( \mathcal{K}_\lambda \), \( \lambda \in \mathbb{R} \), denote separable, complex Hilbert spaces such that the dimension function \( \mathbb{R} \ni \lambda \mapsto \dim(\mathcal{K}_\lambda) \in \mathbb{N} \cup \{ \infty \} \) is \( \mu \)-measurable.
Assuming Hypothesis [B.1] let $\mathcal{S}(\{K_\lambda\}_{\lambda \in \mathbb{R}})$ be the vector space associated with the Cartesian product $\prod_{\lambda \in \mathbb{R}} K_\lambda$ equipped with the obvious linear structure. Elements of $\mathcal{S}(\{K_\lambda\}_{\lambda \in \mathbb{R}})$ are maps
\[ f \in \mathcal{S}(\{K_\lambda\}_{\lambda \in \mathbb{R}}), \quad \mathbb{R} \ni \lambda \mapsto f(\lambda) \in K_\lambda, \quad (B.1) \]
in particular, we identify $f = \{f(\lambda)\}_{\lambda \in \mathbb{R}}$.

**Definition B.2.** Assume Hypothesis [B.1] A measurable family of Hilbert spaces $\mathcal{M}$ modeled on $\mu$ and $\{K_\lambda\}_{\lambda \in \mathbb{R}}$ is a linear subspace $\mathcal{M} \subset \mathcal{S}(\{K_\lambda\}_{\lambda \in \mathbb{R}})$ such that $f \in \mathcal{M}$ if and only if the map $\mathbb{R} \ni \lambda \mapsto (f(\lambda), g(\lambda))_{K_\lambda} \in \mathbb{C}$ is $\mu$-measurable for all $g \in \mathcal{M}$. Moreover, $\mathcal{M}$ is said to be generated by some subset $\mathcal{F}$, $\mathcal{F} \subset \mathcal{M}$, if for every $g \in \mathcal{M}$ we can find a sequence of functions $h_n \in \text{lin.span}\{\chi_B f \in \mathcal{S}(\{K_\lambda\}) \mid B \in \mathcal{B}(\mathbb{R}), f \in \mathcal{F}\}$ with $\lim_{n \to \infty} \|g(\lambda) - h_n(\lambda)\|_{K_\lambda} = 0$ $\mu$-a.e.

We note that we shall identify functions in $\mathcal{M}$ which coincide $\mu$-a.e.; thus $\mathcal{M}$ is more precisely a set of equivalence classes of functions. The definition of $\mathcal{M}$ was chosen with its maximality in mind and we refer to Lemma [B.4] and for more details in this respect. An explicit construction of an example of $\mathcal{M}$ will be given in Theorem [B.8].

**Remark B.3.** The following properties are proved in a standard manner:
(i) If $f \in \mathcal{M}$, $g \in \mathcal{S}(\{K_\lambda\}_{\lambda \in \mathbb{R}})$ and $g = f$ $\mu$-a.e. then $g \in \mathcal{M}$.
(ii) If $\{f_n\}_{n \in \mathbb{N}} \in \mathcal{M}$, $g \in \mathcal{S}(\{K_\lambda\}_{\lambda \in \mathbb{R}})$ and $f_n(\lambda) \to g(\lambda)$ as $n \to \infty$ $\mu$-a.e. (i.e., $\lim_{n \to \infty} \|f_n(\lambda) - g(\lambda)\|_{K_\lambda} = 0$ $\mu$-a.e.) then $g \in \mathcal{M}$.
(iii) If $\phi$ is a scalar-valued $\mu$-measurable function and $f \in \mathcal{M}$ then $\phi f \in \mathcal{M}$.
(iv) If $f \in \mathcal{M}$ then $\mathbb{R} \ni \lambda \mapsto \|f(\lambda)\|_{K_\lambda} \in [0, \infty)$ is $\mu$-measurable.

**Lemma B.4 (B.3).** Assume Hypothesis [B.1] Suppose that $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{S}(\{K_\lambda\}_{\lambda \in \mathbb{R}})$ is such that
(a) $\mathbb{R} \ni \lambda \mapsto (f_m(\lambda), f_n(\lambda))_{K_\lambda} \in \mathbb{C}$ is $\mu$-measurable for all $m, n \in \mathbb{N}$.
(b) $\{f_n(\lambda)\}_{n \in \mathbb{N}} \subset K_\lambda$.
In particular, any orthonormal basis $\{e_n(\lambda)\}_{n \in \mathbb{N}}$ in $K_\lambda$ will satisfy (a) and (b). Setting $\mathcal{M} = \{g \in \mathcal{S}(\{K_\lambda\}_{\lambda \in \mathbb{R}}) \mid (f_n(\lambda), g(\lambda))_{K_\lambda} \text{ is } \mu\text{-measurable for all } n \in \mathbb{N}\}$, (B.2) one has the following facts:
(i) $\mathcal{M}$ is a measurable family of Hilbert spaces.
(ii) $\mathcal{M}$ is generated by $\{f_n\}_{n \in \mathbb{N}}$.
(iii) $\mathcal{M}$ is the unique measurable family of Hilbert spaces containing the sequence $\{f_n\}_{n \in \mathbb{N}}$.
(iv) If $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ is any sequence satisfying (b) then $\mathcal{M}$ is generated by $\{g_n\}_{n \in \mathbb{N}}$.

Next, let $w$ be a $\mu$-measurable function, $w > 0$ $\mu$-a.e., and consider the space
\[ L^2(\mathbb{R}; wd\mu; \mathcal{M}) = \left\{ f \in \mathcal{M} \left| \int_{\mathbb{R}} w(\lambda)d\mu(\lambda) \|f(\lambda)\|^2_{K_\lambda} < \infty \right. \right\} \quad (B.3) \]
with its obvious linear structure. On $L^2(\mathbb{R}; wd\mu; \mathcal{M})$ one defines a semi-inner product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}; wd\mu; \mathcal{M})}$ and hence a semi-norm $\|\cdot\|_{L^2(\mathbb{R}; wd\mu; \mathcal{M})}$ by
\[ (f, g)_{L^2(\mathbb{R}; wd\mu; \mathcal{M})} = \int_{\mathbb{R}} w(\lambda)d\mu(\lambda) \langle f(\lambda), g(\lambda) \rangle_{K_\lambda}, \quad f, g \in L^2(\mathbb{R}; wd\mu; \mathcal{M}). \quad (B.4) \]
That (B.4) defines a semi-inner product immediately follows from the corresponding properties of \((\cdot, \cdot)_{\kappa_\lambda}\) and the linearity of the integral. Next, one defines the equivalence relation \(\sim\), for elements \(f, g \in \hat{L}^2(\mathbb{R}; wd\mu; \mathcal{M})\) by

\[ f \sim g \text{ if and only if } f = g \text{ } \mu\text{-a.e.} \quad (B.5) \]

and hence introduces the set of equivalence classes of \(\hat{L}^2(\mathbb{R}; wd\mu; \mathcal{M})\) denoted by

\[ L^2(\mathbb{R}; wd\mu; \mathcal{M}) = \hat{L}^2(\mathbb{R}; wd\mu; \mathcal{M}) / \sim. \quad (B.6) \]

In particular, introducing the subspace of null functions

\[ \mathcal{N}(\mathbb{R}; wd\mu; \mathcal{M}) = \{ f \in \hat{L}^2(\mathbb{R}; wd\mu; \mathcal{M}) \mid \|f(\lambda)\|_{\kappa_\lambda} = 0 \text{ for } \mu\text{-a.e. } \lambda \in \mathbb{R} \} = \{ f \in \hat{L}^2(\mathbb{R}; wd\mu; \mathcal{M}) \mid \|f\|_{L^2(\mathbb{R}; wd\mu; \mathcal{M})} = 0 \}, \quad (B.7) \]

\(L^2(\mathbb{R}; wd\mu; \mathcal{M})\) is precisely the quotient space \(\hat{L}^2(\mathbb{R}; wd\mu; \mathcal{M}) / \mathcal{N}(\mathbb{R}; wd\mu; \mathcal{M})\). Denoting the equivalence class of \(f \in \hat{L}^2(\mathbb{R}; wd\mu; \mathcal{M})\) temporarily by \([f]\), the semi-inner product on \(L^2(\mathbb{R}; wd\mu; \mathcal{M})\)

\[ ([f], [g])_{L^2(\mathbb{R}; wd\mu; \mathcal{M})} = \int_{\mathbb{R}} w(\lambda)d\mu(\lambda)(f(\lambda), g(\lambda))_{\kappa_\lambda}, \quad (B.8) \]

is well-defined (i.e., independent of the chosen representatives of the equivalence classes) and actually an inner product. Thus, \(L^2(\mathbb{R}; wd\mu; \mathcal{M})\) is a normed space and by the usual abuse of notation we denote its elements in the following again by \(f, g\), etc. Moreover, \(L^2(\mathbb{R}; wd\mu; \mathcal{M})\) is also complete:

**Theorem B.5.** Assume Hypothesis [H.1]. Then the normed space \(L^2(\mathbb{R}; wd\mu; \mathcal{M})\) is complete and hence a Hilbert space. In addition, \(L^2(\mathbb{R}; wd\mu; \mathcal{M})\) is separable.

That \(L^2(\mathbb{R}; wd\mu; \mathcal{M})\) is complete was shown in [19, Subsect. 4.1.2], [28, Sect. 7.1], and more recently, in [53]. Separability of \(L^2(\mathbb{R}; wd\mu; \mathcal{M})\) is proved in [28, Sect. 7.1] (see also [19, Subsect. 4.3.2]).

**Remark B.6.** A similar construction defines the Banach spaces \(L^p(\mathbb{R}; wd\mu; \mathcal{M})\), \(p \geq 1\).

Thus, \(L^2(\mathbb{R}; wd\mu; \mathcal{M})\) corresponds precisely to the direct integral of the Hilbert spaces \(\kappa_\lambda\) with respect to the measure \(wd\mu\) (see, e.g., [19, Ch. 4], [28, Ch. 7], [44, Ch. II], [122, Ch. XII]) and is frequently denoted by \(\int_{\mathbb{R}} w(\lambda)d\mu(\lambda)\kappa_\lambda\).

Having reviewed the construction of \(L^2(\mathbb{R}; wd\mu; \mathcal{M}) = \int_{\mathbb{R}} w(\lambda)d\mu(\lambda)\kappa_\lambda\) in connection with a scalar measure \(wd\mu\), we now turn to the case of operator-valued measures and recall the following definition (we refer, for instance, to [19, Sects. 1.2, 3.1, 5.1], [24, Sect. VII.2.3], [28, Ch. 6], [12, Ch. I], [13, Ch. X], [8] for operator-valued measures):

**Definition B.7.** Let \(\mathcal{H}\) be a separable, complex Hilbert space. A map \(\Sigma : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})\), with \(\mathfrak{B}(\mathbb{R})\) the Borel \(\sigma\)-algebra on \(\mathbb{R}\), is called a bounded, nonnegative, operator-valued measure if the following conditions (i) and (ii) hold:

(i) \(\Sigma(\emptyset) = 0\) and \(0 \leq \Sigma(B) \in \mathcal{B}(\mathcal{H})\) for all \(B \in \mathfrak{B}(\mathbb{R})\).

(ii) \(\Sigma(\cdot)\) is strongly countably additive (i.e., with respect to the strong operator
topology in \( \mathcal{H} \)), that is,
\[
\Sigma(B) = s\text{-}\lim_{N \to \infty} \sum_{j=1}^{N} \Sigma(B_j)
\]  
whenever \( B = \bigcup_{j \in \mathbb{N}} B_j \), with \( B_k \cap B_\ell = \emptyset \) for \( k \neq \ell \), \( B_k \in \mathfrak{B}(\mathbb{R}) \), \( k, \ell \in \mathbb{N} \).

Moreover, \( \Sigma(\cdot) \) is called an (operator-valued) spectral measure (or an orthogonal operator-valued measure) if additionally the following condition (iii) holds:

(iii) \( \Sigma(\cdot) \) is projection-valued (i.e., \( \Sigma(B)^2 = \Sigma(B) \), \( B \in \mathfrak{B}(\mathbb{R}) \)) and \( \Sigma(\mathbb{R}) = I_\mathcal{H} \).

In the following, let \( \Sigma : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{K}) \) be a bounded nonnegative measure, that is, \( \Sigma \) satisfies requirements (i) and (ii) in Definition [2.7]. Denoting \( T = \Sigma(\mathbb{R}) \), one has
\[
0 \leq \Sigma(B) \leq T \in \mathcal{B}(\mathcal{K}), \quad B \in \mathfrak{B}(\mathbb{R}),
\]
and hence
\[
\|\Sigma(B)^{1/2}\xi\|_\mathcal{K} \leq \|T^{1/2}\xi\|_\mathcal{K}, \quad \xi \in \mathcal{K},
\]
shows that
\[
\ker(T) = \ker(T^{1/2}) \subseteq \ker(\Sigma(B)^{1/2}) = \ker(\Sigma(B)), \quad B \in \mathfrak{B}(\mathbb{R}).
\]

We will use the orthogonal decomposition
\[
\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1, \quad \mathcal{K}_0 = \ker(T), \quad \mathcal{K}_1 = \ker(T)^\perp = \text{ran}(\overline{T}),
\]
and identify \( f_0 = (f_0, 0)^T \in \mathcal{K}_0 \) and \( f_1 = (0, f_1)^T \in \mathcal{K}_1 \). In particular, with \( f = (f_0, f_1)^T \), one has \( \|f\|_\mathcal{K}^2 = \|f_0\|_{\mathcal{K}_0}^2 + \|f_1\|_{\mathcal{K}_1}^2 \). Then \( T \) permits the \( 2 \times 2 \) block operator representation
\[
T = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix}, \quad 0 \leq T_1 \in \mathcal{B}(\mathcal{K}_1), \quad \ker(T_1) = \{0\},
\]
with respect to the decomposition [1.13]. By [1.12] one concludes that \( \Sigma(B) \), \( B \in \mathfrak{B}(\mathbb{R}) \), is necessarily of the form
\[
\Sigma(B) = \begin{pmatrix} 0 & D^* \\ D & \Sigma_1(B) \end{pmatrix}, \quad \text{for some } 0 \leq \Sigma_1(B) \in \mathcal{B}(\mathcal{K}_1), \quad D \in \mathcal{B}(\mathcal{K}_0, \mathcal{K}_1),
\]
with respect to the decomposition [1.13]. The computation
\[
0 = \Sigma(B) \begin{pmatrix} f_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & D^* \\ D & \Sigma_1(B) \end{pmatrix} \begin{pmatrix} f_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Df_0 \end{pmatrix}, \quad f_0 \in \mathcal{K}_0,
\]
yields \( D = 0 \) as \( f_0 \in \mathcal{K}_0 \) was arbitrary. Thus, \( \Sigma(B) \), \( B \in \mathfrak{B}(\mathbb{R}) \), is actually also of diagonal form
\[
\Sigma(B) = \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_1(B) \end{pmatrix}, \quad \text{for some } 0 \leq \Sigma_1(B) \in \mathcal{B}(\mathcal{K}_1),
\]
with respect to the decomposition [1.13]. Moreover, let \( \mu \) be a control measure for \( \Sigma \) (equivalently, for \( \Sigma_1 \)), that is,
\[
\mu(B) = 0 \quad \text{if and only if } \Sigma(B) = 0 \quad \text{for all } B \in \mathfrak{B}(\mathbb{R}).
\]
(E.g., \( \mu(B) = \sum_{n \in \mathcal{I}} 2^{-n} \langle e_n, \Sigma(B)e_n \rangle_{\mathcal{K}} \), \( B \in \mathfrak{B}(\mathbb{R}) \), with \( \{e_n\}_{n \in \mathcal{I}} \) a complete orthonormal system in \( \mathcal{K}, \mathcal{I} \subseteq \mathbb{N}, \) an appropriate index set.)
The following theorem was first stated in [53] under the implicit assumption that $\Sigma(\mathbb{R}) = T = I_K$. The general case $T \in \mathcal{B}(K)$, explicitly permitting the existence of a nontrivial kernel of $T$ was recently discussed in [57].

**Theorem B.8.** Let $K$ be a separable, complex Hilbert space, $\Sigma : \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(K)$ a bounded, nonnegative operator-valued measure, and $\mu$ a control measure for $\Sigma$. Then there are separable, complex Hilbert spaces $\mathcal{K}_\lambda$, $\lambda \in \mathbb{R}$, a measurable family of Hilbert spaces $\mathcal{M}_\Sigma$ modelled on $\mu$ and $\{K_\lambda\}_{\lambda \in \mathbb{R}}$, and a bounded linear map $\Lambda \in \mathcal{B}(K, L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma))$, satisfying

$$\|\Lambda\|_{\mathcal{B}(K, L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma))} = \|T^{1/2}\|_{\mathcal{B}(K)},$$

and

$$\ker(\Lambda) = \ker(T),$$

so that the following assertions (i)–(iii) hold:

(i) For all $B \in \mathfrak{B}(\mathbb{R})$, $\xi, \eta \in K$,

$$(\eta, \Sigma(B)\xi)_K = \int_B d\mu(\lambda) ((\Lambda\eta)(\lambda), (\Lambda\xi)(\lambda))_{K_\lambda},$$

in particular,

$$(\eta, T\xi)_K = \int d\mu(\lambda) ((\Lambda\eta)(\lambda), (\Lambda\xi)(\lambda))_{K_\lambda}.$$  

(ii) Let $I = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$, or $I = \mathbb{N}$. $\bar{\Lambda}_i\{e_n\}_{n \in I}$ generates $\mathcal{M}_\Sigma$, where $\{e_n\}_{n \in I}$ denotes any sequence of linearly independent elements in $K$ with the property $\lim \text{span}\{e_n\}_{n \in I} = K$. In particular, $\bar{\Lambda}(K)$ generates $\mathcal{M}_\Sigma$.

(iii) For all $B \in \mathfrak{B}(\mathbb{R})$ and $\xi \in K$,

$$\bar{\Lambda}(S(B)\xi) = \{\chi_B(\lambda)(\bar{\Lambda}\xi)(\lambda)\}_{\lambda \in \mathbb{R}},$$

where (cf. (B.14) and (B.17))

$$S(B) = \begin{pmatrix} I_{K_0} & 0 \\ 0 & T_1^{-1/2} \Sigma_1(B) T_1^{1/2} \end{pmatrix}, \quad S(\mathbb{R}) = I_K,$$

with respect to the decomposition (B.13).

Next, we recall that the construction in Theorem B.8 is essentially unique:

**Theorem B.9 ([53]).** Suppose $\mathcal{K}_\lambda', \lambda \in \mathbb{R}$ is a family of separable complex Hilbert spaces, $\mathcal{M}'$ is a measurable family of Hilbert spaces modelled on $\mu$ and $\{\mathcal{K}_\lambda\}$, and $\Lambda' \in \mathcal{B}(K, L^2(\mathbb{R}; d\mu; \mathcal{M}'))$ is a map satisfying (i), (ii), and (iii) of Theorem B.8. Then for $\mu$-a.e. $\lambda \in \mathbb{R}$ there is a unitary operator $U_\lambda : \mathcal{K}_\lambda \to \mathcal{K}_\lambda'$ such that $f = \{f(\lambda)\}_{\lambda \in \mathbb{R}} \in \mathcal{M}_\Sigma$ if and only if $\{U_\lambda f(\lambda)\}_{\lambda \in \mathbb{R}} \in \mathcal{M}'$ and for all $\xi \in K$,

$$\Lambda'(\xi)(\lambda) = U_\lambda(\Lambda_\xi)(\lambda) \quad \mu\text{-a.e.}$$

**Remark B.10.** (i) Without going into further details, we note that $\mathcal{M}_\Sigma$ depends of course on the control measure $\mu$. However, a change in $\mu$ merely effects a change in density and so $\mathcal{M}_\Sigma$ can essentially be viewed as $\mu$-independent.

(ii) With $0 < \omega$ a $\mu$-measurable weight function, one can also consider the Hilbert space $L^2(\mathbb{R}; d\omega \mu; \mathcal{M}_\Sigma)$. In view of our comment in item (i) concerning the mild dependence on the control measure $\mu$ of $\mathcal{M}_\Sigma$, one typically puts more emphasis on the operator-valued measure $\Sigma$ and hence uses the more suggestive notation $L^2(\mathbb{R}; d\omega \Sigma; K)$ instead of the more precise $L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma)$ in this case.
Next, let
\[ V = \text{lin.span}\{e_n \in \mathcal{K} | n \in \mathcal{I}\}, \quad \mathcal{V} = \mathcal{K}, \quad (B.26) \]
and define
\[ \mathcal{V}_\Sigma = \text{lin.span}\{\chi_B(\xi) \Delta e_n \in L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma) | B \in \mathfrak{B}(\mathbb{R}), n \in \mathcal{I}\}. \quad (B.27) \]
The fact that \( \{\Delta e_n\}_{n \in \mathcal{I}} \) generates \( \mathcal{M}_\Sigma \) then implies that \( \mathcal{V}_\Sigma \) is dense in the Hilbert space \( L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma) \), that is,
\[ \sum_{\Sigma} = L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma). \quad (B.28) \]

Since the operator-valued distribution function \( \Sigma(\cdot) \) has at most countably many discontinuities on \( \mathbb{R} \), denoting by \( \mathfrak{S}_\Sigma \) the corresponding set of discontinuities of \( \Sigma(\cdot) \), introducing the set of intervals
\[ \mathcal{B}_\Sigma = \{(\alpha, \beta] \subset \mathbb{R} | \alpha, \beta \in \mathbb{R}\setminus\mathfrak{S}_\Sigma\}, \quad (B.29) \]
the minimal \( \sigma \)-algebra generated by \( \mathcal{B}_\Sigma \) coincides with the Borel algebra \( \mathfrak{B}(\mathbb{R}) \). Hence one can introduce
\[ \widetilde{\mathcal{V}}_\Sigma = \text{lin.span}\{\chi_{(\alpha, \beta]}(\xi) \Delta e_n \in L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma) | \alpha, \beta \in \mathbb{R}\setminus\mathfrak{S}_\Sigma, n \in \mathcal{I}\}, \quad (B.30) \]
which still retains the density property in \( (B.28) \), that is,
\[ \sum_{\Sigma} = L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma). \quad (B.31) \]

In the following we briefly describe an alternative construction of \( L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \) used by Berezanskii [24] Sect. VII.2.3] in order to identify the two constructions.
Introduce
\[ C_{0,0}(\mathbb{R}; \mathcal{K}) = \left\{ u : \mathbb{R} \to \mathcal{K} \left| u(\cdot) \text{ is strongly continuous in } \mathcal{K}, \text{supp}(u) \text{ is compact}, \quad \dim\left(\bigcup_{\lambda \in \mathbb{R}} \text{ran}(u(\lambda))\right) < \infty\right.\right\}. \quad (B.32) \]
On \( C_{0,0}(\mathbb{R}; \mathcal{K}) \) one can introduce the semi-inner product
\[ (u, v)_{L^2(\mathbb{R}; d\Sigma; \mathcal{K})} = \int_{\mathbb{R}} (u(\lambda), d\Sigma(\lambda)v(\lambda)) \mathcal{K}, \quad u, v \in C_{0,0}(\mathbb{R}; \mathcal{K}), \quad (B.33) \]
where the integral on the right-hand side of \( (B.33) \) is well-defined in the Riemann–Stieltjes sense. Introducing the kernel of this semi-inner product by
\[ \mathcal{N} = \{ u \in C_{0,0}(\mathbb{R}; \mathcal{K}) | (u, u)_{L^2(\mathbb{R}; d\Sigma; \mathcal{K})} = 0\}, \quad (B.34) \]
Berezanskii [24] Sect. VII.2.3] obtains the separable Hilbert space \( L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \) as the completion of \( C_{0,0}(\mathbb{R}; \mathcal{K})/\mathcal{N} \) with respect to the inner product in \( (B.33) \) as
\[ L^2(\mathbb{R}; d\Sigma; \mathcal{K}) = \overline{C_{0,0}(\mathbb{R}; \mathcal{K})}/\mathcal{N}. \quad (B.35) \]
In particular,
\[ ([u], [v])_{L^2(\mathbb{R}; d\Sigma; \mathcal{K})} = \int_{\mathbb{R}} (u(\lambda), d\Sigma(\lambda)v(\lambda)) \mathcal{K}, \quad u, v \in C_{0,0}(\mathbb{R}; \mathcal{K}), \quad (B.36) \]
and (cf. also [34 Corollary 2.6]) \( (B.36) \) extends to piecewise continuous \( \mathcal{K} \)-valued functions with compact support as long as the discontinuities of \( u \) and \( v \) are disjoint from the set \( \mathfrak{S}_\Sigma \) (the set of discontinuities of \( \Sigma(\cdot) \)).

Since Kats’ work in the case of a finite-dimensional Hilbert space \( \mathcal{K} \) (cf. [71], [73] and also Fuhrman [50] Sect. II.6] and Rosenberg [109]), and especially in the work
of Malamud and Malamud [84], who studied the general case $\dim(\mathcal{K}) \leq \infty$, it has become customary to interchange the order of taking the quotient with respect to the semi-inner product and completion in this process of constructing $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$. More precisely, in this context one first completes $C_{0,0}(\mathbb{R}, \mathcal{K})$ with respect to the semi-inner product \[ L^2(\mathbb{R}; d\Sigma; \mathcal{K}) = C_{0,0}(\mathbb{R}; \mathcal{K}) \] to obtain a semi-Hilbert space

and then takes the quotient with respect to the kernel of the underlying semi-inner product, as described in method (I) of [57] Appendix A. Berezanski’s approach in [24] Sect. VII.2.3] corresponds to method (II) discussed in [57] Appendix A. The equivalence of these two methods is not stated in these sources, but was spelled out explicitly in [57].

Next we will recall that Berezanskii’s construction of $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ (and hence the corresponding construction by Kats (if $\dim(\mathcal{K}) < \infty$) and by Malamud and Malamud (if $\dim(\mathcal{K}) \leq \infty$) is equivalent to the one in [84] and hence to that outlined in Theorem B.8. For this purpose we recall that it was shown in the proof of Theorem 2.14 in [84] that

is dense in $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$.

**Theorem B.11.** The Hilbert spaces $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ and $L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma)$ are isometrically isomorphic with isomorphism $\mathcal{U}_\Sigma$ defined as follows:

\[
\mathcal{U}_\Sigma : \mathcal{V}_\Sigma \to \mathcal{V}_\Sigma,
\chi(\alpha,\beta)(\cdot)e_n \mapsto \chi(\alpha,\beta)(\cdot)e_n, \quad \alpha, \beta \in \mathbb{R} \setminus \mathcal{G}_\Sigma, n \in \mathcal{I},
\]

establishes the densely defined isometry between the Hilbert spaces $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ and $L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma)$ which extends by continuity to the unitary map

\[
\mathcal{U}_\Sigma = \mathcal{U}_\Sigma^* : L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \to L^2(\mathbb{R}; d\mu; \mathcal{M}_\Sigma).
\]

As a result, dropping the additional “hat” on the left-hand side of (B.35), and hence just using the notation $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ for both Hilbert space constructions is consistent.

We continue this section by yet another approach originally due to Gel’fand and Kostyuchenko [51] and Berezanski [24] Ch. VII. In this context we also refer to Berezanski [25] Sect. 2.2], Berezansky, Sheftel, and Us [26] Ch. 15], Birman and Entina [27], Gel’fand and Shilov [28] Ch. IV], and M. Malamud and S. Malamud [33], [84]: Introducing an operator $K \in B_2(\mathcal{H})$ with $\ker(K) = \ker(K^*) = \{0\}$, one has the existence of the weakly $\mu$-measurable nonnegative operator-valued function $\Psi_K(\cdot)$ with values in $B_1(\mathcal{H})$, such that

\[
(f, \Sigma(B))_\mathcal{H} = \int_B d\mu(t) \langle \Psi_K(t)^{1/2}K^{-1}f, \Psi_K(t)^{1/2}K^{-1}g \rangle_\mathcal{H},
\]

\[
f, g \in \text{dom}(K^{-1}), \quad B \in \mathfrak{B}(\mathbb{R}), \quad B \text{ bounded},
\]

with

\[
\Psi_K(\cdot) = \frac{dK^*\Sigma K}{d\mu}(\cdot) \quad \mu\text{-a.e.}
\]
In fact, the derivative $\Psi_K$ exists in the $B_1(\mathcal{H})$-norm (cf. [27] and [83], [84]). Introducing the semi-Hilbert space $\tilde{\mathcal{H}}_t$, $t \in \mathbb{R}$, as the completion of $\text{dom}(K^{-1})$ with respect to the semi-inner product

$$(f, g)_{\tilde{\mathcal{H}}_t} = (\Psi_K(t)^{1/2}K^{-1}f, \Psi_K(t)^{1/2}K^{-1}g)_{\tilde{\mathcal{H}}_t}, \quad f, g \in \text{dom}(K^{-1}), \quad t \in \mathbb{R},$$

(B.43)

factoring $\tilde{\mathcal{H}}_t$ by the kernel of the corresponding semi-norm $\text{ker}(\| \cdot \|_{\tilde{\mathcal{H}}_t})$ then yields the Hilbert space $\mathcal{H}_t = \tilde{\mathcal{H}}_t / \text{ker}(\| \cdot \|_{\tilde{\mathcal{H}}_t})$, $t \in \mathbb{R}$. One can show (cf. [82], [84]) that

$$L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \text{ and } \int_\mathbb{R} d\mu(t) \mathcal{H}_t \text{ are isometrically isomorphic,} \quad (B.44)$$

yielding yet another construction of $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$.

Finally, we will discuss one more characterization of $L^2(\mathbb{R}; d\Sigma; \mathcal{K})$ which is used in Sections 4 and 5 and closely patterned after work by Saitô [112].

**Definition B.12.** Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$.

(i) Assume that $Q : [\lambda_1, \lambda_2] \to \mathcal{B}(\mathcal{K})$, $u : [\lambda_1, \lambda_2] \to \mathcal{K}$, and $\rho : [\lambda_1, \lambda_2] \to \mathcal{B}(\mathcal{K})$. Denote by $\Delta$ a finite subdivision of $[\lambda_1, \lambda_2]$ of the form $\lambda_1 = \eta_0 < \eta_1 < \cdots < \eta_n = \lambda_2$. The norm of $\Delta$, denoted by $\| \Delta \|$, is defined by $\| \Delta \| = \max_{0 \leq j \leq n-1} |\eta_{j+1} - \eta_j|$. If the limit

$$\lim_{\| \Delta \| \to 0} \sum_{j=0}^{n-1} Q(\eta_j^i) [\rho(\eta_{j+1}) - \rho(\eta_j)] u(\eta_j^i), \quad (B.45)$$

or

$$\lim_{\| \Delta \| \to 0} \sum_{j=0}^{n-1} Q(\eta_j^i) [u(\eta_{j+1}) - u(\eta_j)], \quad (B.46)$$

with $\eta_j^i \in [\eta_j, \eta_{j+1}]$, $0 \leq j \leq n - 1$, exists in the sense of weak convergence in $\mathcal{K}$ independently of the choice of subdivision $\Delta$ and the choice of $\eta_j^i$, $0 \leq j \leq n - 1$, then the limit is denoted by

$$\int_{[\lambda_1, \lambda_2]} Q(\lambda) \, d\rho(\lambda) \, u(\lambda), \quad (B.47)$$

or

$$\int_{[\lambda_1, \lambda_2]} Q(\lambda) \, du(\lambda), \quad (B.48)$$

respectively.

(ii) Suppose that for any $\lambda_1 \in (\lambda_0, \lambda_2]$ the integral (B.45) or (B.46) exists in the sense described in item (i), and that

$$\lim_{\lambda_1 \downarrow \lambda_0} \int_{[\lambda_1, \lambda_2]} Q(\lambda) \, d\rho(\lambda) \, u(\lambda), \quad (B.49)$$

or

$$\lim_{\lambda_1 \downarrow \lambda_0} \int_{[\lambda_1, \lambda_2]} Q(\lambda) \, du(\lambda), \quad (B.50)$$

exist in the sense of weak convergence in $\mathcal{K}$. Then one defines the integral over the interval $(\lambda_1, \lambda_2]$ by

$$\int_{(\lambda_1, \lambda_2]} Q(\lambda) \, d\rho(\lambda) \, u(\lambda) = \lim_{\lambda_1 \downarrow \lambda_1} \int_{[\lambda_1, \lambda_2]} Q(\lambda') \, d\rho(\lambda') \, u(\lambda'), \quad (B.51)$$
or

\[ \int_{(\lambda_1, \lambda_2]} Q(\lambda) \, d\nu(\lambda) = \lim_{\lambda \downarrow \lambda_1} \int_{[\lambda, \lambda_2]} Q(\lambda) \, d\nu(\lambda). \]  

(B.52)

**Lemma B.13** ([112]). Let \( \lambda_1, \lambda_2 \in \mathbb{R} \), \( \lambda_1 < \lambda_2 \), \( Q \in C^1([\lambda_1, \lambda_2], \mathcal{B}(\mathcal{K})) \), \( u \in C^1([\lambda_1, \lambda_2], \mathcal{K}) \), and \( \rho : [\lambda_1, \lambda_2] \to \mathcal{B}(\mathcal{K}) \), such that for some constant \( C > 0 \), \( \|\rho(\lambda)\|_{\mathcal{B}(\mathcal{K})} \leq C \), \( \lambda \in [\lambda_1, \lambda_2] \). Suppose, in addition, that

for all \( f \in \mathcal{K} \), \( (f, [Q'(\cdot)\rho(\cdot)u(\cdot) + Q(\cdot)\rho(\cdot)u'(\cdot)])_{\mathcal{K}} \) is Riemann integrable on \([\lambda_1, \lambda_2] \).

Then \( \int_{[\lambda_1, \lambda_2]} Q(\lambda) \, d\rho(\lambda) \, u(\lambda) \) exists in the sense of Definition B.12(i), and

\[
\left( f, \int_{[\lambda_1, \lambda_2]} Q(\lambda) \, d\rho(\lambda) \, u(\lambda) \right)_{\mathcal{K}} = (f, Q(\lambda_1)u(\lambda_1))_{\mathcal{K}} - (f, Q(\lambda_2)u(\lambda_2))_{\mathcal{K}} - \int_{\lambda_1}^{\lambda_2} d\lambda \left( f, [Q'(\lambda)\rho(\lambda)u(\lambda) + Q(\lambda)\rho(\lambda)u'(\lambda)] \right)_{\mathcal{K}}, \quad f \in \mathcal{K}. 
\]  

(B.54)

**Lemma B.14** ([112]). Let \( \lambda_1, \lambda_2 \in \mathbb{R} \), \( \lambda_1 < \lambda_2 \), \( Q \in C^1([\lambda_1, \lambda_2], \mathcal{B}(\mathcal{K})) \), and \( v : [\lambda_1, \lambda_2] \to \mathcal{K} \), such that for some constant \( C > 0 \), \( \|v(\lambda)\|_{\mathcal{K}} \leq C \), \( \lambda \in [\lambda_1, \lambda_2] \). Suppose, in addition, that

for all \( f \in \mathcal{K} \), \( (f, Q'(\cdot)v(\cdot))_{\mathcal{K}} \) is Riemann integrable on \([\lambda_1, \lambda_2] \).

Then \( \int_{[\lambda_1, \lambda_2]} Q(\lambda) \, d\nu(\lambda) \) exists in the sense of Definition B.12(i), and

\[
\left( f, \int_{[\lambda_1, \lambda_2]} Q(\lambda) \, d\nu(\lambda) \right)_{\mathcal{K}} = (f, Q(\lambda_1)v(\lambda_1))_{\mathcal{K}} - (f, Q(\lambda_2)v(\lambda_2))_{\mathcal{K}} - \int_{\lambda_1}^{\lambda_2} d\lambda \left( f, Q'(\lambda)v(\lambda) \right)_{\mathcal{K}}, \quad f \in \mathcal{K}. 
\]  

(B.56)

**Definition B.15.** Let \( \lambda_1, \lambda_2 \in \mathbb{R} \), \( \lambda_1 < \lambda_2 \).

(i) Assume that \( v, w : [\lambda_1, \lambda_2] \to \mathcal{K} \), and that \( \Sigma : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{K}) \) is a bounded operator-valued measure as defined in Definition B.17. Denote by \( \Delta \) a finite subdivision of \([\lambda_1, \lambda_2] \) of the form \( \lambda_1 = \eta_0 < \eta_1 < \cdots < \eta_n = \lambda_2 \) as in Definition B.12 with norm \( \|\Delta\| \). If the limit

\[
\lim_{\|\Delta\| \to 0} \sum_{j=0}^{n-1} (v(\eta_j'), [\Sigma(\eta_{j+1}) - \Sigma(\eta_j)]w(\eta_j'))_{\mathcal{K}}, \quad (B.57)
\]

with \( \eta_j' \in [\eta_j, \eta_{j+1}] \), \( 0 \leq j \leq n - 1 \), exists independently of the choice of subdivision \( \Delta \) and the choice of \( \eta_j' \), \( 0 \leq j \leq n - 1 \), then the limit is denoted by

\[
\int_{[\lambda_1, \lambda_2]} (v(\lambda), d\Sigma(\lambda) \, w(\lambda))_{\mathcal{K}}, \quad (B.58)
\]

(ii) Suppose that for any \( \lambda_1 \in (\lambda_0, \lambda_2] \) the integral [B.58] exists, and that

\[
\lim_{\lambda_1 \downarrow \lambda_0} \int_{[\lambda_1, \lambda_2]} (v(\lambda), d\Sigma(\lambda) \, w(\lambda))_{\mathcal{K}} \exists. \tag{B.59}
\]

Then one defines the integral over the interval \((\lambda_1, \lambda_2] \) by

\[
\int_{(\lambda_1, \lambda_2]} (v(\lambda), d\Sigma(\lambda) \, w(\lambda))_{\mathcal{H}} = \lim_{\lambda_1 \downarrow \lambda_1} \int_{[\lambda_1, \lambda_2]} (v(\lambda), d\Sigma(\lambda) \, w(\lambda))_{\mathcal{H}}, \quad (B.60)
\]
Lemma B.16 ([112]). Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, $v, w \in C^1([\lambda_1, \lambda_2], \mathcal{K})$, and $\Sigma : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{K})$ a bounded operator-valued measure as defined in Definition B.15. Then \( \int_{[\lambda_1, \lambda_2]} (v(\lambda), d\Sigma(\lambda) w(\lambda))_{\mathcal{K}} \) exists and

\[
\int_{[\lambda_1, \lambda_2]} (v(\lambda), d\Sigma(\lambda) w(\lambda))_{\mathcal{K}} = (v(\lambda_2), \Sigma(\lambda_2)w(\lambda_2))_{\mathcal{K}} - (v(\lambda_1), \Sigma(\lambda_1)w(\lambda_1))_{\mathcal{K}} - \int_{\lambda_1}^{\lambda_2} d\lambda \left[ (v'(\lambda), \Sigma(\lambda)w(\lambda))_{\mathcal{K}} + (v(\lambda), \Sigma(\lambda)w'(\lambda))_{\mathcal{K}} \right].
\]  

(B.61)

Next, consider the vector space \( \mathcal{D}_0 \) defined by

\[
\mathcal{D}_0 = \left\{ u : \mathbb{R} \to \mathcal{K} \bigg| \text{supp}(u) \text{ compact; } u \text{ is left-continuous and has only discontinuities of the first kind; there exists } \{\lambda_j(u)\}_{1 \leq j \leq N}, \lambda_1 < \lambda_2 < \lambda_N, \right. \\
\left. \text{such that } \begin{cases} u(\lambda), & \lambda \in (\lambda_j, \lambda_{j+1}], \\
\lim_{\lambda \downarrow \lambda_j} u(\lambda'), & \lambda = \lambda_j, \end{cases} \text{ is strongly continuously differentiable on } [\lambda_j, \lambda_{j+1}], 1 \leq j \leq N - 1, \text{ and } u(\lambda) = 0 \text{ for } \lambda \in \mathbb{R}\setminus[\lambda_1, \lambda_N]. \right\}
\]  

(B.62)

Given $\Sigma$ as in Definition B.15, Saitō [112] then introduces the semi-inner product on \( \mathcal{D}_0 \times \mathcal{D}_0 \) by

\[
(v, w)_{\mathcal{D}_0} = \sum_{\gamma_k} \int_{(\gamma_k, \gamma_{k+1}]} (v(\lambda), d\Sigma(\lambda)w(\lambda))_{\mathcal{K}}, \quad v, w \in \mathcal{D}_0,
\]  

(B.63)

where \( \{\gamma_k\}_{1 \leq k \leq K} \) represents the discontinuities of \( v \) and \( w \), appropriately ordered with respect to magnitude. Introducing the subspace of null functions by \( \mathcal{N}_{\Sigma, \mathcal{D}_0} = \{ u \in \mathcal{D}_0 | (u, u)_{\mathcal{D}_0} = 0 \} \), the completion of \( \mathcal{D}_0 / \mathcal{N}_{\Sigma, \mathcal{D}_0} \) becomes a Hilbert space denoted by

\[
L^2(\mathbb{R}; d\Sigma; \mathcal{K})_{\mathcal{D}_0} = \overline{\mathcal{D}_0 / \mathcal{N}_{\Sigma, \mathcal{D}_0}}.
\]  

(B.64)

Next, we recall that \( u : \mathbb{R} \to \mathcal{K} \) is called a step function if

\[
u(\lambda) = \begin{cases} u_j \in \mathcal{K}, & \lambda \in (\alpha_j, \beta_j], 1 \leq j \leq N, \\
0, & \lambda \in \mathbb{R}\setminus\bigcup_{j=1}^N (\alpha_j, \beta_j), \end{cases}
\]  

(B.65)

Denoting the set of step functions by \( \mathcal{D}_{\text{step}} \), then clearly \( \mathcal{D}_{\text{step}} \subset \mathcal{D}_0 \), and one can prove that

\[
\overline{\mathcal{D}_{\text{step}}} = L^2(\mathbb{R}; d\Sigma; \mathcal{K})_{\mathcal{D}_0}.
\]  

(B.66)

It remains to show that Saitō’s space \( L^2(\mathbb{R}; d\Sigma; \mathcal{K})_{\mathcal{D}_0} \) and \( L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \) discussed in Theorem B.5 and Remark B.10 (ii) are isometrically isomorphic. We will show this by proving that Saitō’s construction \( L^2(\mathbb{R}; d\Sigma; \mathcal{K})_{\mathcal{D}_0} \), actually, coincides with Berezanskii’s construction, \( L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \) in [B.39]:

**Theorem B.17.** The Hilbert spaces \( L^2(\mathbb{R}; d\Sigma; \mathcal{K})_{\mathcal{D}_0} \) and \( L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \) coincide.
Proof. By \( \text{(B.38)} \), \( \mathcal{D}_{\text{step}} \) is dense in \( L^2(\mathbb{R}; d\Sigma; \mathcal{K}) \). Since \( \mathcal{D}_{\text{step}} \) is also dense in \( L^2(\mathbb{R}; d\Sigma; \mathcal{K})_S \), the elementary fact
\[
\|\chi(\alpha, \beta) e_n\|^2_{L^2(\mathbb{R}; d\Sigma; \mathcal{K})_S} = \int_{\alpha}^{\beta} d(e_n, \Sigma(\lambda) e_n)_{\mathcal{K}}
\]
completes the proof.

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