WARPED PRODUCT SKEW SEMI-INARIANT SUBMANIFOLDS OF ORDER 1 OF A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

HAKAN METE TAŞTAN

Abstract. We introduce warped product skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold. We give a necessary and sufficient condition for skew semi-invariant submanifold of order 1 to be a locally warped product. We also prove that the invariant distribution which is involved in the definition of the submanifold is integrable under some restrictions. Moreover, we find an inequality between the warping function and the squared norm of the second fundamental form for such submanifolds. Equality case is also discussed.

1. INTRODUCTION

The theory of submanifolds is one of the most popular research area in differential geometry. In an almost Hermitian manifold, its almost complex structure determines several types of submanifolds. For example, holomorphic (invariant) submanifolds and totally real (anti-invariant) submanifolds are determined by the behavior of the almost complex structure. In the first case the tangent space of the submanifolds is invariant under the action of the almost complex structure. In the second case the tangent space of the submanifolds is anti-invariant, that is, it is mapped into the normal space. A. Bejancu [4] introduced the notion of CR-submanifolds of a Kählerian manifold as a natural generalization of invariant and anti-invariant submanifolds. A CR-submanifold is said to be proper if it is neither invariant nor anti-invariant. The theory of CR-submanifolds has been a most interesting topics since then. Slant submanifolds are another generalization of invariant and anti-invariant submanifolds. This type submanifolds is defined by B.Y. Chen [9]. Since then such submanifolds have been studied by many geometers (see [3, 8, 17] and references therein). If a slant submanifold is neither invariant nor anti-invariant then it is said to be proper. We observe that a proper CR-submanifold is never a slant submanifold. In [18], N. Papaghiuc introduced the notion of semi-slant submanifolds obtaining CR-submanifolds and slant submanifolds as special cases. A. Carriazo [8], introduced bi-slant submanifolds which is a generalization of semi-slant submanifolds. One of the classes of such submanifolds is that of anti-slant submanifolds. This type submanifolds are also generalization of slant and CR-submanifolds. However, B. Şahin [23] called these submanifolds as hemi-slant submanifolds because of that the name anti-slant seems to refer that it has no slant factor. He also observed that there is no inclusion between proper

2000 Mathematics Subject Classification. Primary 53B25; Secondary 53C55.

Key words and phrases. locally product manifold, warped product submanifold, skew semi-invariant submanifold, invariant distribution, slant distribution.
hemi-slant submanifolds and proper semi-slant submanifolds. We note that hemi-
slant submanifolds are also studied under the name of pseudo-slant submanifolds
(see [14] [27]).

Skew CR-submanifolds of a Kählerian manifold are first defined by G.S. Ronsse
in [19]. Such submanifolds are a generalizations of bi-slant submanifolds. Conse-
quently, invariant, anti-invariant, CR, slant, semi-slant and hemi-slant submanifolds
are particular cases of skew CR-submanifolds. We notice that CR-submanifolds
in Kählerian manifolds correspond to semi-invariant submanifolds [5] in locally
product Riemannian manifolds. Therefore, skew CR-submanifolds in Kählerian
manifolds correspond to skew semi-invariant submanifolds in locally product Rie-
mannian manifolds. For the fundamental properties and further studies of skew
CR-submanifolds; see [19] and [26]. Skew semi-invariant submanifolds of a locally
product Riemannian manifold were studied first by X. Liu and F.-M. Shao in [16].

The notion of warped product was initiated by R.L. Bishop and B. O'Neill [6].
Let $M_1$ and $M_2$ be two Riemannian manifolds with Riemannian metrics $g_1$ and $g_2$
respectively. Let $f$ be positive differentiable function on $M_1$. The warped product
$M = M_1 \times_f M_2$ of $M_1$ and $M_2$ is the Riemannian manifold $(M_1 \times M_2, g)$, where
$$g = g_1 + f^2 g_2.$$ More explicitly, if $U \in T_p M$, then
$$\|U\|^2 = \|d\pi_1(U)\|^2 + (f^2 \circ \pi_1)\|d\pi_2(U)\|^2,$$
where $\pi_i, i = 1, 2$, are the canonical projections $M_1 \times M_2$ onto $M_1$ and $M_2$ respec-
tively. The function $f$ is called the warping function of the warped product. If the
warping function is constant, then the manifold $M$ is said to be trivial. It is also
known that $M_1$ is totally geodesic and $M_2$ is totally umbilical from [6]. For a
warped product $M_1 \times_f M_2$, we denote by $D_1$ and $D_2$ the distributions given by
the vectors tangent to leaves and fibers respectively. Thus, $D_1$ is obtained from
tangent vectors to $M_1$ via horizontal lift and $D_2$ is obtained by tangent vectors of
$M_2$ via vertical lift. Let $U$ be a vector field on $M_1$ and $V$ be vector field on $M_2$,
then from Lemma 7.3 of [6], we have

$$(1.1) \quad \nabla_U V = \nabla_V U = U(\ln f)V,$$
where $\nabla$ is the Levi-Civita connection on $M_1 \times_f M_2$.

Warped product submanifolds have been studying very actively since B.Y. Chen
[10] introduced the notion of CR-warped product in Kählerian manifolds. In fact,
different type warped product submanifolds of different kinds structures are studied
last thirteen years. For example; see [2] [15] [21] [22] [23] [24] [27]. Most of the stud-
ies related to this topic can be found in the survey book [11]. Recently, B. Şahin
[24] introduced the notion of skew CR-warped product submanifolds of Kählerian
manifolds which is a generalization of different kind warped product submanifolds
studied by many authors. We note that warped product skew CR-submanifolds of
a cosymplectic manifold were studied in [19].

In this paper, we define and study warped product skew semi-invariant subman-
ifolds of order 1 of a locally product Riemannian manifold. We give an illustrate
example and prove a characterization theorem for the mixed totally geodesic proper
skew semi-invariant submanifold using some lemmas. In general, the invariant dis-
tribution of a submanifold is not integrable in a locally product Riemannian man-
ifold. However, we prove that the invariant distribution of a warped product skew
semi-invariant submanifold of order 1 is integrable in a locally product Riemannian
manifold under some restrictions. Finally, we obtain an inequality between the
warping function and the squared norm of the second fundamental form for such
submanifolds. Equality case is also considered.

2. preliminaries

Let \((\bar{M}, g, F)\) be a locally product Riemannian manifold or, (briefly, l.p.R. man-
ifold). It means that \([28]\) \(\bar{M}\) has a tensor field \(F\) of type (1, 1) on \(\bar{M}\) such that,
\[
\forall \bar{U}, \bar{V} \in T\bar{M}, \quad F^2 = I, \quad (F \neq \pm I), \quad g(F\bar{U}, F\bar{V}) = g(\bar{U}, \bar{V}) \quad \text{and} \quad (\nabla_{\bar{U}} F)\bar{V} = 0 ,
\]
where \(g\) is the Riemannian metric, \(\nabla\) is the Levi-Civita connection on \(\bar{M}\) and \(I\) is
the identifying operator on the tangent bundle \(T\bar{M}\) of \(\bar{M}\).

Let \(M\) be a submanifold of a l.p.R. manifold \((\bar{M}, g, F)\) as an isometrically im-
mersed. Let \(\nabla\) and \(\nabla^\perp\) be the induced, and induced normal connection in
\(M\) and the normal bundle \(T^\perp M\) of \(M\), respectively. Then for all \(U, V \in TM\) and \(\xi \in T^\perp M\) the Gauss and Weingarten formulas are given by
\[
\nabla_U V = \nabla_U V + h(U, V)
\]
and
\[
\nabla_U \xi = -A_\xi U + \nabla^\perp_U \xi
\]
where \(h\) is the second fundamental form of \(M\) and \(A_\xi\) is the Weingarten endomor-
phism associated with \(\xi\). The second fundamental form \(h\) and the shape operator
\(A\) related by
\[
\quad g(h(U, V), \xi) = g(A_\xi U, V) .
\]
The mean curvature vector field \(H\) is given by \(H = \frac{1}{m}(\text{trace } h)\), where \(\text{dim}(M) = m\).
The submanifold \(M\) is called totally geodesic in \(\bar{M}\) if \(h = 0\) and minimal if \(H = 0\).
If \(h(U, V) = g(U, V)H\) for all \(U, V \in TM\), then \(M\) is totally umbilical.

3. skew semi-invariant submanifolds of order 1
of a locally product Riemannian manifold

Let \(\bar{M}\) be a l.p.R. manifold with Riemannian metric \(g\) and almost product struc-
ture \(F\). Let \(M\) be Riemannian submanifold isometrically immersed in \(\bar{M}\). For any
\(U \in TM\), we write
\[
FU = TU + NU .
\]
Here \(TU\) is the tangential part of \(FU\), and \(NU\) is the normal part of \(FU\). Similarly, for any \(\xi \in T^\perp M\), we put
\[
F\xi = t\xi + \omega\xi ,
\]
where \( t\xi \) is the tangential part of \( F\xi \), and \( \omega\xi \) is the normal part of \( F\xi \).

Using (2.1) and (3.1), we have \( g(T^2U,V) = g(T^2V,U) \) for all \( U, V \in TM \). It says that \( T^2 \) is a symmetric operator on the tangent space \( T_pM, p \in M \). Therefore its eigenvalues are real and diagonalizable. Moreover, its eigenvalues are bounded by 0 and 1. For each \( p \in M \), we set

\[
D_p^\lambda = \text{Ker}\{T^2 - \lambda^2(p)I\}_p,
\]

where \( I \) is the identity endomorphism and \( \lambda(p) \) belongs to closed interval \([0, 1]\) such that \( \lambda^2(p) \) is an eigenvalue of \( T^2_p \). Since \( T^2_p \) is symmetric and diagonalizable, there is some integer \( k \) such that \( \lambda_1^2(p), ..., \lambda_k^2(p) \) are distinct eigenvalues of \( T^2_p \) and \( T_pM \) can be decomposed as a direct sum of mutually orthogonal eigenspaces, i.e.

\[
T_pM = D_p^{\lambda_1} \oplus \ldots \oplus D_p^{\lambda_k}.
\]

For \( i \in \{1, ..., k\} \), \( D_p^{\lambda_i} \) is a \( T \)-invariant subspace of \( T_pM \). We note that \( D^0_p = \text{Ker}T_p \) and \( D^1_p = \text{Ker}N_p \). \( D^0_p \) is the maximal anti-\( F \)-invariant subspace of \( T_pM \) where as \( D^p \) is the maximal \( F \)-invariant subspace of \( T_pM \). We denote the distributions \( D^0 \) and \( D^1 \) by \( D^\perp \) and \( D^T \), respectively from now on.

**Definition 3.1.** \((16)\) Let \( M \) be a submanifold of a l.p.R. manifold \( \bar{M} \). Then \( M \) is said to be a generic submanifold if there exists an integer \( k \) and functions \( \lambda_i, i \in \{1, ..., k\} \) defined on \( M \) with values in \((0, 1)\) such that

(i) Each \( \lambda_i^2(p), i \in \{1, ..., k\} \) is a distinct eigenvalue of \( T^2_p \) with

\[
T_pM = D_p^{\perp} \oplus D_p^{T} \oplus D_p^{\lambda_1} \oplus \ldots \oplus D_p^{\lambda_k}
\]

for \( p \in M \).

(ii) The dimension of \( D^\perp \), \( D^T \) and \( D^{\lambda_i}, 1 \leq i \leq k \) are independent of \( p \in M \). Moreover, if each \( \lambda_i \) is constant on \( M \), then we say that \( M \) is a skew semi-invariant submanifold of \( \bar{M} \).

In view of Definition 3.1, we observe that the following special cases.

Let \( M \) be a skew semi-invariant submanifold of a l.p.R. manifold \( \bar{M} \) as in Definition 3.1. Then

(a) If \( k = 0 \) and \( D^\perp = \{0\} \), then \( M \) is an invariant submanifold \([1]\).

(b) If \( k = 0 \) and \( D^T = \{0\} \), then \( M \) is an anti-invariant submanifold \([1]\).

(c) If \( k = 0 \), then \( M \) is a semi-invariant submanifold \([5]\).

(d) If \( D^\perp = \{0\} = D^T \) and \( k = 1 \), then \( M \) is a slant submanifold \([20]\).

(e) If \( D^\perp = \{0\}, D^T \neq \{0\} \) and \( k = 1 \), then \( M \) is a semi-slant submanifold \([20]\).

(f) If \( D^T = \{0\}, D^\perp \neq \{0\} \) and \( k = 1 \), then \( M \) is a hemi-slant submanifold \([25]\).

(g) If \( D^\perp = \{0\} = D^T \) and \( k = 2 \), then \( M \) is a bi-slant submanifold \([8]\).
Definition 3.2. A submanifold $M$ of a l.p.R. manifold $\bar{M}$ is called a *skew semi-invariant submanifold of order 1*, if $M$ is a skew semi-invariant submanifold with $k=1$.

In this case, we have
\begin{equation}
TM = D^\perp \oplus DT \oplus D^\theta,
\end{equation}
where $D^\theta = D^{\lambda_1}$ and $\lambda_1$ is constant. We say that a skew semi-invariant submanifold of order 1 is *proper*, if $D^\perp \neq \{0\}$ and $DT \neq \{0\}$.

A slant submanifold $M$ of a l.p.R. manifold $\bar{M}$ is characterized by
\begin{equation}
T^2U = \lambda U
\end{equation}
such that $\lambda \in [0, 1]$, where $U \in TM$, for details; see [20]. Moreover, if $\theta$ is the slant angle of $M$, then we have $\lambda = \cos^2 \theta$. On the other hand, for any slant submanifold $M$ of a l.p.R. manifold $\bar{M}$, we have
\begin{align*}
(a) & \quad T^2 + tN = I, \quad (b) \quad \omega^2 + Nt = I, \\
(c) & \quad NT + \omega N = 0, \quad (d) \quad Tt + t\omega = 0.
\end{align*}

For the proof of (3.5); see [25].

Throughout this paper, the letters $V, W$ will denote the vector fields of the anti-invariant distribution $D^\perp$, $U, Z$ will denote the vector fields of the slant distribution $D^\theta$ and $X, Y$ will denote the vector fields of the invariant distribution $DT$.

For the further study of skew semi-invariant submanifold of order 1 of a l.p.R. manifold, we need to following lemmas.

Lemma 3.3. Let $M$ be a proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $\bar{M}$. Then we have,
\begin{align}
&\quad g(\nabla_V W, X) = -g(A_{FW} V, FX), \\
&\quad g(\nabla_V Z, X) = -\csc^2 \theta \{g(A_{NTZ} V, X) + g(A_{NZ} V, FX)\}, \\
&\quad g(\nabla_Z V, X) = -g(A_{FV} Z, FX), \quad V, W \in D^\perp, Z \in D^\theta \text{ and } X \in DT.
\end{align}

Proof. Using (2.2) and (2.1), we have $g(\nabla_V W, X) = g(\nabla_V FW, FX)$ for $V, W \in D^\perp$ and $X \in DT$. Hence, using (2.3), we get (3.6). In a similar way, we have $g(\nabla_V Z, X) = g(\nabla_V FZ, FX)$, where $Z \in D^\theta$. Then using (3.1) and (2.1), we obtain $g(\nabla_V Z, X) = g(\nabla_V FTZ, X) + g(\nabla_V NZ, FX)$. Hence, using (3.1) and (2.3), we arrive at $g(\nabla_V Z, X) = g(\nabla_V T^2 Z, X) + g(\nabla_V N(TZ), FX) - g(A_{NZ} V, FX)$. With the help of (3.4), (2.2) and (2.3), we get (3.7). Similarly, one can obtain (3.8). □

Lemma 3.4. Let $M$ be a proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $\bar{M}$. Then we have,
\begin{align}
&\quad g(\nabla_U Z, X) = -\csc^2 \theta \{g(A_{NTZ} U, X) + g(A_{NZ} U, FX)\}, \\
&\quad g(\nabla_X Y, Z) = \csc^2 \theta \{g(A_{NTZ} X, Y) + g(A_{NZ} X, FY)\}, \\
&\quad g(\nabla_X Y, V) = g(A_{FV} X, FY), \quad V, W \in D^\perp, Z \in D^\theta \text{ and } X \in DT.
\end{align}
for $X, Y \in D^T$, $U, Z \in D^\theta$ and $V \in D^\perp$.

Proof. Let $U, Z \in D^\theta$ and $X \in D^T$. Then using (2.2), (2.1) and (3.1), we have $g(\nabla_U Z, X) = g(\nabla_U F Z, F X) = g(\nabla_U T Z, F X) + g(\nabla_U N Z, F X)$. Again, using (2.1) and (2.3), we obtain $g(\nabla_U Z, X) = g(\nabla_U F T Z, X) - g(A_{NZ} U, F X)$. Here, if we use (3.5)-(a) and (3.4), then we get $g(\nabla_U Z, X) = \cos^2 \theta g(\nabla_U Z, X) + g(\nabla_U N T Z, X) - g(A_{NZ} U, F X)$. After some calculation, we find (3.9). For the proof of (3.11), using (2.2), (2.1) and (3.1), we have $g(\nabla_X Y, Z) = g(\nabla_X F Y, F Z) = g(\nabla_X F Y, T Z) + g(\nabla_X F Y, N Z)$ for $X, Y \in D^T$ and $Z \in D^\theta$. Again, using (2.1) and (2.3), we obtain $g(\nabla_X Y, Z) = g(\nabla_X Y, F T Z) + g(h(X, F Y), N Z)$. With the help of (3.5)-(a) and (3.4), we get $g(\nabla_X Y, Z) = \cos^2 \theta g(\nabla_X Y, Z) + g(\nabla_X N T Z, N Z) + g(h(X, F Y), N Z)$. Upon direct calculation, we find (3.10). In a similar way, we can obtain (3.11).

Lemma 3.5. Let $M$ be a proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $M$. Then we have,

\begin{align*}
(3.12) \quad & g(\nabla_V X, Z) = \csc^2 \theta \{g(A_{NTZ} V, X) + g(A_{NZ} V, F X)\}, \\
(3.13) \quad & g(\nabla_U Z, V) = \sec^2 \theta \{g(A_{FV} U, T Z) + g(A_{NTZ} U, V)\}, \\
(3.14) \quad & g(\nabla_X V, Z) = \sec^2 \theta \{g(A_{FV} X, T Z) + g(A_{NTZ} X, V)\},
\end{align*}

for $X \in D^T$, $U, Z \in D^\theta$ and $V \in D^\perp$.

Proof. Using (2.2), (2.1) and (3.1), we have $g(\nabla_V X, Z) = g(\nabla_V F X, F Z) = -g(\nabla_V F X, F X) = -g(\nabla_V T Z, F X) - g(\nabla_V N Z, F X)$. Again, using (2.1) and (2.3), we obtain $g(\nabla_V X, Z) = -g(\nabla_V F T Z, X) + g(A_{NZ} V, F X)$. Here, using (3.1) and (3.5)-(a), we get $g(\nabla_V X, Z) = -\cos^2 \theta g(\nabla_V X, Z) - g(\nabla_V N T Z, X) + g(A_{NZ} V, F X)$. According to direct calculation, we arrive at $g(\nabla_V X, Z) = \cos^2 \theta g(\nabla_V X, Z) + g(A_{NTZ} V, X) + g(A_{NZ} V, F X)$ which gives (3.12). On the other hand, for any $U, Z \in D^\theta$ and $V \in D^\perp$, using (2.2), (2.1) and (3.1), we have $g(\nabla_U Z, V) = g(\nabla_U T Z, F V) + g(\nabla_U N Z, F V)$. Hence, using (2.2) and (2.1), we obtain $g(\nabla_U Z, V) = g(h(U, T Z), F V) + g(\nabla_U F N Z, V)$. Here, if we use (3.2) and (2.1), we get $g(\nabla_U Z, V) = g(A_{FV} U, T Z) + g(\nabla_U N Z, V) + g(\nabla_U \omega N Z, V)$. With the help of (3.5)-(a), (3.5)-(c), (3.4) and (2.3), we arrive at $g(\nabla_U Z, V) = g(A_{FV} U, T Z) + g(\nabla_U (1 - \cos^2 \theta) Z, V) + g(A_{NTZ} U, V)$. Upon direct calculation, we find (3.13). Similarly, we can obtain (3.14).

4. Warped Product Skew Semi-Invariant Submanifolds of Order 1 of a Locally Product Riemannian Manifold

In this section, we consider a warped product submanifold of type $M = M_1 \times M_T$ in a l.p.R. manifold $M$, where $M_1$ is a hemi-slan manifold and $M_T$ is an invariant submanifold. Then, it is clear that $M$ is a proper skew semi-invariant submanifold of order 1 of $M$. Thus, from definition of hemi-slan submanifold and skew semi-invariant submanifold of order 1, we have

\begin{equation}
TM = D^\theta \oplus D^\perp \oplus D^T.
\end{equation}

In particular, if $D^\theta = \{0\}$, then $M$ is a warped product semi-invariant submanifold [21]. If $D^\perp = \{0\}$, then $M$ is a warped product semi-slan submanifold [22].
Remark 4.1. From Theorem 3.1 of [21], we know that there are no proper warped product semi-invariant submanifolds of type \( M_T \times_f M_1 \) of a l.p.R. manifold \( \bar{M} \) such that \( M_T \) is invariant submanifold and \( M_1 \) is anti-invariant submanifold of \( \bar{M} \). On the other hand, from Theorem 3.1 of [22], we know that there is no proper warped product submanifold in the form \( M_T \times_f M_0 \) of a l.p.R. manifold \( \bar{M} \) such that \( M_0 \) is a proper slant submanifold and \( M_T \) is an invariant submanifold of \( \bar{M} \). Thus, we conclude that there is no warped product skew semi-invariant submanifold of order 1 of type \( M_T \times_f M_1 \) of a l.p.R. manifold \( \bar{M} \) such that \( M_1 \) is a semi-slant submanifold and \( M_T \) is an invariant submanifold of \( \bar{M} \).

We now present an example of warped product semi-invariant submanifold of order 1 of type \( M_1 \times_f M_T \) in a l.p.R. manifold.

Example 4.2. Consider the locally product Riemannian manifold \( \mathbb{R}^{10} = \mathbb{R}^5 \times \mathbb{R}^5 \) with usual metric \( g \) and almost product structure \( F \) defined by

\[
F(\partial_i) = \partial_i, \quad F(\partial_j) = -\partial_j,
\]

where \( i \in \{1, \ldots, 5\}, j \in \{6, \ldots, 10\} \), \( \partial_k = \frac{\partial}{\partial x_k} \), and \( (x_1, \ldots, x_{10}) \) are natural coordinates of \( \mathbb{R}^{10} \). Let \( M \) be a submanifold of \( \bar{M} = (\mathbb{R}^{10}, g, F) \) given by

\[
\phi(x, y, z, u, v) = (x + y, x - y, x\cos u, x\sin u, z, -z, x, \frac{2}{\sqrt{3}}y, x\cos v, x\sin v),
\]

where \( x > 0 \).

Then, we easily see that the local frame of \( TM \) is spanned by

\[
\phi_x = \partial_1 + \partial_2 + \cos u \partial_3 + \sin u \partial_4 + \partial_5 + \cos \theta \partial_6 + \sin \theta \partial_{10},
\]

\[
\phi_y = \partial_1 - \partial_2 + \frac{2}{\sqrt{3}} \partial_8, \quad \phi_z = \partial_3 - \partial_6,
\]

\[
\phi_u = -x\sin u \partial_3 + x\cos u \partial_4, \quad \phi_v = -x\sin \theta \partial_9 + x\cos \theta \partial_{10}.
\]

Then by direct calculation, we see that \( D^\theta = \text{span}\{\phi_x, \phi_y\} \) is a slant distribution with slant angle \( \theta = \arccos \frac{1}{\sqrt{5}} \) and \( D^\perp = \text{span}\{\phi_z\} \) is an anti-invariant distribution since \( F(\phi_z) \) is orthogonal to \( TM \). Moreover, \( D^T = \text{span}\{\phi_u, \phi_v\} \) is an invariant distribution. Thus, we conclude that \( M \) is a proper skew semi-invariant submanifold of order 1 of \( \bar{M} \). Furthermore, one can easily see that \( D^\theta \oplus D^\perp \) and \( D^T \) are integrable. If we denote the integral submanifolds \( D^\theta, D^\perp \) and \( D^T \) by \( M_\theta, M_\perp \) and \( M_T \), respectively, then the induced metric tensor of \( M \) is

\[
ds^2 = 5dx^2 + \frac{10}{3}dy^2 + 2dz^2 + x^2(du^2 + dv^2)
= g_{M_\theta} + g_{M_\perp} + x^2 g_{M_T}.
\]

Thus, \( M = (M_\theta \times M_\perp) \times_x M_T \) is a warped product skew semi-invariant submanifold of order 1 of \( \bar{M} \) with warping function \( f = x \).

Let \( D^\theta \) and \( D^T \) be slant and invariant distributions on \( M \), respectively. Then we say that \( M \) is \( (D^\theta, D^T) \) mixed totally geodesic if \( h(Z, X) = 0 \), where \( Z \in D^\theta \) and \( X \in D^T \) [19].

Before giving a necessary and sufficient condition for skew semi-invariant submanifold of order 1 to be a locally warped product, we recall that the S. Hiepko’s result [13], (cf. [12], Remark 2.1): Let \( D_1 \) be a vector subbundle in the tangent bundle of a Riemannian manifold \( M \) and let \( D_2 \) be its normal bundle. Suppose that the two distributions are involutive. If we denote by \( M_1 \) and \( M_2 \) the integral
manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, then $M$ is locally isometric to warped product $M_1 \times_f M_2$ if the integral manifold $M_1$ is totally geodesic and the integral manifold $M_2$ is an extrinsic sphere, in other word, $M_2$ is a totally umbilical submanifold with a parallel mean curvature vector.

**Theorem 4.3.** Let $M$ be a $(\mathcal{D}^\theta, \mathcal{D}^T)$ mixed totally geodesic proper skew semi-invariant submanifold of order 1 with integrable distribution $\mathcal{D}^T$ of a l.p.R. manifold $M$. Then $M$ is a locally warped product submanifold if and only if

$$
A_{FV}FX = -V[\sigma]X
$$

and

$$
A_{NZ}FX + A_{NZT}X = -Z[\sigma]\sin^2\theta X
$$

for $X \in \mathcal{D}^T$, $Z \in \mathcal{D}^\theta$, $V \in \mathcal{D}^\perp$ and a function $\sigma$ defined on $M$ such that $Y[\sigma] = 0$ for $Y \in \mathcal{D}^T$.

**Proof.** Let $M = M_1 \times_f M_2$ be a $(\mathcal{D}^\theta, \mathcal{D}^T)$ mixed totally geodesic warped product proper skew semi-invariant submanifold of order 1 with integrable distribution $\mathcal{D}^T$ of a l.p.R. manifold $M$. Then using (3.6) and (3.8), we have $g(A_{FV}FX, FX) = 0$ and $g(A_{FV}Z, FX) = 0$ for any $V, W \in \mathcal{D}^\perp$, $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$. Since $A$ is self-adjoint, we deduce that $A_{FV}FX$ has no components in $TM_1$. So $A_{FV}FX \in \mathcal{D}^T$.

Thus, using (2.2), (2.1) and (1.1), from (3.10), we have $g(A_{FV}FX, Y) = -g(N_YFV, FX) = -g(N_YV, X) = -g(N_YV, X) = -V(\ln f)g(Y, X)$. Which proves (4.2).

Since $M$ is $(\mathcal{D}^\theta, \mathcal{D}^T)$ mixed totally geodesic for any $Z \in \mathcal{D}^\theta$ and $X \in \mathcal{D}^T$, we have $g(A_{NZ}X, Z) = 0$. It means that $A_{NZ}X$ has no components in $\mathcal{D}^\theta$. On the other hand, from Lemma 3.3 of [25], we know that $TZ \in \mathcal{D}^\theta$ for any $Z \in \mathcal{D}^\theta$. Thus, using this fact and (1.1), from (3.11), we get $g(A_{NZ}X, V) = 0$, that is, $A_{NZ}X$ has no components in $\mathcal{D}^\perp$. Thus, from (1.1), we conclude that $A_{NZ}X \in \mathcal{D}^T$.

Also, we have $A_{NZ}X \in \mathcal{D}^T$. Then, for $X, Y \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$, with the help of (1.1), from (3.10), we have $g(A_{NZ}Y, X) + g(A_{NZ}FY, X) = -\sin^2\theta g(N_YX, Y) = -\sin^2\theta(\ln f)g(Y, X)$. This proves (4.3). Moreover, $Y(\ln f) = 0$ for a warped product proper skew semi-invariant submanifold of order 1, we obtain $\sigma = \ln f$.

Conversely, suppose that $M$ is $(\mathcal{D}^\theta, \mathcal{D}^T)$ mixed totally geodesic proper skew semi-invariant submanifold of order 1 with integrable distribution $\mathcal{D}^T$ of a l.p.R. manifold $M$ such that (4.2) and (4.3) hold. We know from Theorem 4.6 of [25], $\mathcal{D}^\perp$ is always integrable. So, we have $g(N_YWX, X) = 0$ for $V, W \in \mathcal{D}^\perp$ and $X \in \mathcal{D}^T$. Using this fact, (1.1), (1.3) and (3.7), (3.9), it is not difficult to see that $M_1$ is totally geodesic in $M$. Let $M_T$ be the integral manifold of $\mathcal{D}^T$ and $h_T$ be the second fundamental form of $M_T$ in $M$. From (2.2), we have $g(h_T(X, Y), V) = g(N_XY, V)$ for $X, Y \in \mathcal{D}^T$ and $V \in \mathcal{D}^\perp$. Then, (3.11) imply that $g(h_T(X, Y), V) = g(A_{FV}FY, X)$. Thus, using (4.2), we obtain

$$
g(h_T(X, Y), V) = -V[\sigma]g(Y, X)
$$

Similarly, from (2.2), we have $g(h_T(X, Y), Z) = g(N_XY, Z)$ for $X, Y \in \mathcal{D}^T$ and $Z \in \mathcal{D}^\theta$. Using (3.11), we obtain $g(h_T(X, Y), Z) = \csc^2\theta(g(A_{NZ}Y, X) + g(A_{NZ}FX, X))$. Thus, from (4.3), we get

$$
g(h_T(X, Y), Z) = -Z[\sigma]g(X, Y)
$$
Thus, for any $E = V + Z \in TM_1$, from (4.3) and (4.5), we arrive at
\begin{equation}
(4.6) \quad g(h_T(X, Y), E) = g(h_T(X, Y), V) + g(h_T(X, Y), Z).
\end{equation}

Last equation (4.6) says that $M_T$ is totally umbilical in $M$. Let denote by $\text{grad}^1 \sigma$ and $\text{grad}^0 \sigma$ the gradient of $\sigma$ on $D^1$ and $D^0$, respectively. From (4.6), we write
\begin{equation}
(4.7) \quad h_T(X, Y) = -\{\text{grad}^1 \sigma + \text{grad}^0 \sigma\}g(X, Y).
\end{equation}

Thus, for any $E = V + Z \in TM_1$, we have
\begin{align*}
g(\nabla_X (\text{grad}^1 \sigma + \text{grad}^0 \sigma), E) &= g(\nabla_X \text{grad}^1 \sigma, E) + g(\nabla_X \text{grad}^0 \sigma, E) \\
&= \{Xg(\text{grad}^1 \sigma, V) - g(\text{grad}^1 \sigma, \nabla_X E)\} \\
+ &\{Xg(\text{grad}^0 \sigma, Z) - g(\text{grad}^0 \sigma, \nabla_X E)\} \\
&= X[V[\sigma]] - g(\text{grad}^1 \sigma, \nabla_X E) + X[Z[\sigma]] - g(\text{grad}^0 \sigma, \nabla_X E).
\end{align*}

Upon direct calculation, we arrive at
\begin{align*}
g(\nabla_X (\text{grad}^1 \sigma + \text{grad}^0 \sigma), E) &= \{X[Z[\sigma]] - [X, Z][\sigma] + g(\text{grad}^1 \sigma, \nabla_Z X)\} \\
&+ \{X[V[\sigma]] - [X, V][\sigma] + g(\text{grad}^0 \sigma, \nabla_V X)\}. \quad \text{After some calculation, we get}
\end{align*}

\begin{align*}
g(\nabla_X (\text{grad}^1 \sigma + \text{grad}^0 \sigma), E) &= \{Z[X[\sigma]] + g(\text{grad}^1 \sigma, \nabla_Z X) + V[X[\sigma]] + g(\text{grad}^0 \sigma, \nabla_V X)\}.
\end{align*}

Since $X[\sigma] = 0$, from the last equation, we derive
\begin{align*}
g(\nabla_X (\text{grad}^1 \sigma + \text{grad}^0 \sigma), E) &= -g(\nabla_Z \text{grad}^1 \sigma, X) - g(\nabla_V \text{grad}^0 \sigma, X).
\end{align*}

Here, we know that $\nabla_Z \text{grad}^1 \sigma, \nabla_V \text{grad}^0 \sigma \in TM_1$, since $M_1$ is totally geodesic. Hence, we obtain $g(\nabla_X (\text{grad}^1 \sigma + \text{grad}^0 \sigma), E) = 0$. It means that $\text{grad}^1 \sigma + \text{grad}^0 \sigma$ is parallel in $M$. This fact and (4.7) imply that $M_T$ is an extrinsic sphere. This completes the proof.

\section{5. A Chen-type Inequality for Warped Product Skew Semi-invariant Submanifolds of Order 1}

In this section, we prove that the invariant distribution which is involved in the definition of the warped product proper skew semi-invariant submanifolds of order 1 of a l.p.R. manifold is integrable under some restrictions. We also give an inequality similar to Chen’s inequality \cite{10} for the squared norm of the second fundamental form in terms of the warping function for such submanifolds. We first give the following two lemmas for later use.
Lemma 5.1. Let $M = M_1 \times_f M_T$ be a warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $M$. Then we have,
\begin{equation}
(5.1) 
g(h(X,V),FW) = 0
\end{equation}
and
\begin{equation}
(5.2) 
g(h(X,V),NZ) = 0,
\end{equation}
for $X \in D^T$, $Z \in D^\theta$ and $V, W \in D^\perp$.

Proof. For any $V, W \in D^T$ and $X \in D^T$, using (2.2), (2.1) and (1.1), we get $g(h(X,V),FW) = g(\nabla_X F, W) = g(\nabla_X F, W) = V(\ln f)g(FX, W) = 0$, since $g(FX, W) = 0$. Hence (5.1) follows. In a similar way, using (2.2), (2.1), (3.1) and (1.1), we have
\[
g(h(X,V),NZ) = g(\nabla_X V, NZ) = g(\nabla_X X, FZ) - g(\nabla_X X, TZ)
\]
\[
= g(\nabla_X FX, Z) - g(\nabla_X FX, TZ)
\]
\[
= V(\ln f)g(FX, Z) - V(\ln f)g(X, TZ) = 0,
\]
since $g(FX, Z) = 0$ and $g(X, TZ) = 0$. \qed

Lemma 5.2. Let $M = M_1 \times_f M_T$ be a warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $M$. Then we have,
\begin{equation}
(5.3) 
g(h(X, FY), FV) = -V(\ln f)g(X, Y)
\end{equation}
and
\begin{equation}
(5.4) 
g(h(X, Y), NZ) = T Z(\ln f)g(X, Y)
\end{equation}
for $X, Y \in D^T$, $Z \in D^\theta$ and $V \in D^\perp$.

Proof. Using (2.2) and (2.1), we have $g(h(X, FY), FV) = g(\nabla_X FY, FV) = g(\nabla_X Y, V) = g(\nabla_X Y, V) = -g(\nabla_X V, Y)$ for any $X, Y \in D^T$ and $V \in D^\perp$. Hence, using (1.1), we get easily (5.3). Last assertion (5.4) follows from Lemma 3.1-(ii) of [2] by using linearity. \qed

Theorem 5.3. Let $M = M_1 \times_f M_T$ be an $(q + m)$-dimensional warped product proper skew semi-invariant submanifold of order 1 of a l.p.R. manifold $M$ of dimension $2q + m$, where $\dim(M_1) = q$ and $\dim(M_T) = m$. Then the invariant distribution $D^T$ of $M_T$ is integrable.

Proof. For any $X, Y \in D^T$, $Z \in D^\theta$ and $V \in D^\perp$, using (5.3) and (1.1), we get $g(h(X, FY), FV) = g(h(FX, Y), FV)$ and $g(h(X, FY), NZ) = g(h(FX, Y), NZ)$, since $g(X, FY) = g(FX, Y)$. Hence, we conclude that $h(X, FY) = h(FX, Y)$, since $T^\perp M = F D^\perp \oplus N D^\theta$, where $T^\perp M$ is the normal bundle of $M$ in $M$. Thus, our assertion immediately comes from Theorem 1 of [5]. \qed

Let $M$ be a $(k + n + m)$-dimensional warped product proper skew semi-invariant submanifold of order 1 of a $(2k + 2n + m)$-dimensional l.p.R. manifold $M$. We choose a canonical orthonormal basis $\{e_1, \ldots, e_m, \bar{e}_1, \ldots, \bar{e}_k, \bar{e}_1, \ldots, \bar{e}_n, \bar{e}_1, \ldots, \bar{e}_n, \bar{e}_1, \ldots, \bar{e}_n\}$ such that $\{e_1, \ldots, e_m\}$ is an orthonormal basis of $D^T$, $\{\bar{e}_1, \ldots, \bar{e}_k\}$ is an orthonormal basis of $D^\theta$, $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is an orthonormal basis of $D^\perp$, $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is an orthonormal basis of $N D^\theta$ and $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is an orthonormal basis of $F D^\perp$. 


Remark 5.4. In view of (2.1), we can observe that \( \{F e_1, ..., F e_m\} \) is also an orthonormal basis of \( D^T \). On the other hand, with the help of the equations (3.5) and (3.6) of [25], we can see that \( \{\sec \theta T e_1, ..., \sec \theta T e_k\} \) is also an orthonormal basis of \( D^\theta \) and \( \{\csc \theta N e_1, ..., \csc \theta N e_k\} \) is also an orthonormal basis of \( N D^\theta \).

We now state the main result of this section.

Theorem 5.5. Let \( M = M_1 \times f M_T \) be a \((k + n + m)\)-dimensional warped product proper skew semi-invariant submanifold of order 1 of a \((2k + 2n + m)\)-dimensional l.p.R. manifold \( \bar{M} \). Then the squared norm of the second fundamental form of \( M \) satisfies

\[
\|h\|^2 \geq m\{\|\nabla^\perp (\ln f)\|^2 + \cot^2 \theta \|\nabla^\theta (\ln f)\|^2\},
\]

where \( m = \dim(M_T) \), \( \nabla^\perp (\ln f) \) and \( \nabla^\theta (\ln f) \) are gradients of \( \ln f \) on \( D^\perp \) and \( D^\theta \), respectively. If the equality case of (5.7) holds, then \( M_1 \) is a totally geodesic submanifold of \( M \) and \( M \) is mixed totally geodesic. Moreover, \( M_T \) can not be minimal.

Proof. In view of decomposition (4.1), the squared norm of the second fundamental form \( h \) can be decomposed as

\[
\|h\|^2 = \|h(D^T, D^T)\|^2 + \|h(D^\theta, D^\theta)\|^2 + \|h(D^\perp, D^\perp)\|^2 + 2\|h(D^T, D^\perp)\|^2 + 2\|h(D^T, D^\theta)\|^2 + 2\|h(D^\perp, D^\theta)\|^2.
\]

Which can be written as follows:

\[
\begin{align*}
\|h\|^2 &= \sum_{i,j=1}^{m} \sum_{a=1}^{n} g(h(e_i, e_j), F \tilde{e}_a)^2 + \sum_{i,j=1}^{m} \sum_{r=1}^{k} g(h(e_i, e_j), e_r^*)^2 \\
&\quad + \sum_{a,b,c=1}^{n} g(h(\tilde{e}_a, \tilde{e}_b), F \tilde{e}_c)^2 + \sum_{a,b=1}^{n} \sum_{r=1}^{k} g(h(\tilde{e}_a, \tilde{e}_b), e_r^*)^2 \\
&\quad + \sum_{r,s=1}^{k} \sum_{a=1}^{n} g(h(\tilde{e}_r, \tilde{e}_s), F \tilde{e}_a)^2 + \sum_{r,s=1}^{k} g(h(\tilde{e}_r, \tilde{e}_s), e_q^*)^2 \\
&\quad + 2 \sum_{i=1}^{m} \sum_{a,b=1}^{n} g(h(e_i, \tilde{e}_a), F \tilde{e}_b)^2 + 2 \sum_{i=1}^{m} \sum_{a=1}^{n} \sum_{r=1}^{k} g(h(e_i, \tilde{e}_a), e_r^*)^2 \\
&\quad + 2 \sum_{i=1}^{m} \sum_{r=1}^{k} \sum_{a=1}^{n} g(h(e_i, \tilde{e}_r), F \tilde{e}_a)^2 + 2 \sum_{i=1}^{m} \sum_{r,s=1}^{k} g(h(e_i, \tilde{e}_r), e_s^*)^2 \\
&\quad + 2 \sum_{r=1}^{k} \sum_{a,b=1}^{n} g(h(\tilde{e}_r, \tilde{e}_a), F \tilde{e}_b)^2 + 2 \sum_{r,s=1}^{k} \sum_{a=1}^{n} g(h(\tilde{e}_r, \tilde{e}_a), e_s^*)^2.
\end{align*}
\]

Here, using (5.1)-(5.3) and Remark 5.4, we have

\[
\sum_{i,j=1}^{m} \sum_{a=1}^{n} g(h(e_i, e_j), F \tilde{e}_a)^2 = \sum_{i,j=1}^{m} \sum_{a=1}^{n} (-\tilde{e}_a (\ln f) g(e_i, e_j))^2
\]

and

\[
\sum_{i,j=1}^{m} \sum_{r=1}^{k} g(h(e_i, e_j), e_r^*)^2 = \sum_{i,j=1}^{m} \sum_{r=1}^{k} g(h(e_i, e_j), N \tilde{e}_r)^2 \csc^2 \theta.
\]
Also, using (5.4) from (5.8), we get

\[(5.9) \quad \sum_{i,j=1}^{m} \sum_{r=1}^{k} g(h(e_i, e_j), e_r^*)^2 = \sum_{i,j=1}^{m} \sum_{r=1}^{k} (T\tilde{e}_r(\ln f)g(e_i, e_j))^2 \csc^2 \theta.\]

Using (5.7) and (5.9) from (5.6), we get

\[(5.10) \quad \|h\|^2 \geq m\|\nabla^\perp (\ln f)\|^2 + \sum_{i,j=1}^{m} \sum_{r=1}^{k} (T\tilde{e}_r(\ln f)g(e_i, e_j))^2 \csc^2 \theta.\]

In view of Remark 5.4, we replace \(\tilde{e}_r\) by \(\sec \theta T\tilde{e}_r\) in the last term of (5.10) and using (3.4), we have

\[(5.11) \quad \sum_{i,j=1}^{m} \sum_{r=1}^{k} (T\tilde{e}_r(\ln f)g(e_i, e_j))^2 \csc^2 \theta = m \cot^2 \theta \|\nabla^\theta (\ln f)\|^2.\]

Thus, using (5.11) in (5.10), we find (5.5).

Next, if the equality case of (5.5) holds, then from (5.6), we have

\[(5.12) \quad h(D^\perp, D^\perp) = 0, \quad h(D^\theta, D^\theta) = 0, \quad h(D^\perp, D^\theta) = 0\]

and

\[(5.13) \quad h(D^T, D^\perp) = 0, \quad h(D^T, D^\theta) = 0.\]

Since \(M_1\) is totally geodesic in \(M\), from (5.12) it follows that \(M_1\) is also totally geodesic in \(M\). On the other hand (5.13) imply that \(M\) is mixed totally geodesic. Finally, if we suppose that \(M\) is minimal, then from (5.3) and (5.4), we conclude that \(\|\nabla (\ln f)\| = 0\), which is a contradiction. \(\square\)

Remark 5.6. Theorem 5.5 coincides with Theorem 4.2 of [21] in case \(D^\theta = \{0\}\). In other word, Theorem 5.5 is a generalization of Theorem 4.2 of [21].

References

1. T. Adati, Submanifolds of an almost product manifold, Kodai Math. J. 4 (1981), no. 2, 327–343.
2. F.R. Al-Solamy and M.A. Khan, Warped product submanifolds of Riemannian product manifolds, Abstract and Applied Analysis (2012), Article ID 724898, 12 pages, doi:10.1155/2012/724898.
3. K. Arslan, A. Carriazo, B. Y. Chen and C. Murathan, On slant submanifolds of neutral Kaehler manifolds, Taiwanese J. Math. 17 (2010), no. 2, 561-584.
4. A. Bejancu, CR-Submanifolds of Kaehler manifold I, Proc. Amer. Math. Soc. 69 (1978), 135–142.
5. A. Bejancu, Semi-invariant submanifolds of locally product Riemannian manifolds, An. Univ. Timisoara Ser. Stint. Mat. Al. 22 (1984), no. 1-2, 3–11.
6. R.L. Bishop and B. O’Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc. 145 (1969), 1–49.
7. J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, Slant submanifolds in Sasakian manifolds, Glasgow Math. J., 42 (2000), 125–138.
8. A. Carriazo, Bi-slant immersions, in: Proc. ICRAMS 2000, Kharagpur, India, 2000, 88–97.
9. B.Y. Chen, Slant immersions, Bull. Austral. Math. Soc., 41 (1990), no. 1 , 135-147.
10. B.Y. Chen, Geometry of warped product in Kaehler manifolds, Monatsh. Math. 133 (2001), 177-195.
11. B.Y. Chen, Geometry of warped product submanifolds: a survey, *J. Adv. Math. Stud.* 6 (2013), 1-43.
12. F. Dillen and S. Nölker, Semi-parallellity multi rotation surfaces and the helix property, *J. Reine Angew Math.* 435 (1993), 33-63.
13. S. Hiepko, Eine innere Kennzeichnung der verzerrten Produkte, *Math. Ann.,* 241 (1979), 209-215.
14. V.A. Khan and M.A. Khan, Pseudo-slant submanifolds of a Sasakian manifold, *Indian J. Pure Appl. Math.,* 38 (2007), 31–42.
15. S.M. Khursheed Haider, M. Thakur, and Advin, Warped product skew CR-submanifolds of a Cosymplectic manifold, *Lobachevskii J. Math.* 33 (2012), no. 3, 262-273.
16. X. Liu and Fang-Ming Shao, Skew semi-invariant submanifolds of a locally product manifold, *Portugaliae Math.* 56 (1999), no. 3, 319–327.
17. A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Sci. Roumanie,* 39 (1996), 183-198.
18. N. Papaghiuc, Semi-slant submanifolds of a Kählerian manifold, *Ann. Şt. Al. I. Cuza Univ. Iaşi,* 40 (1994), 55–61.
19. G.S. Ronsse, Generic and skew CR-submanifolds of a Kähler manifold, *Bull. Inst. Math. Acad. Sinica,* 18 (1990), 127–141.
20. B. Şahin, Slant submanifolds of an almost product Riemannian manifold, *J. Korean Math. Soc.* 43 (2006), no. 4, 717-732.
21. B. Şahin, Warped product Semi-invariant submanifolds of a locally product Riemannian manifold, *Bull. Math. Soc. Sci. Roumanie,* 49(97) (2006), No.4, 383-394.
22. B. Şahin, Warped product Semi-slant submanifolds of a locally product Riemannian manifold, *Studia Sci. Math. Hungarica,* 46 (2009), No.2, 169-184.
23. B. Şahin, Warped product submanifolds of a Kähler manifold with a slant factor, *Ann. Pol. Math.* 95 (2009), no. 3, 207–226.
24. B. Şahin, Skew CR-warped product submanifolds of a Kähler manifolds, *Math. Commun.* 15 (2010), no. 1, 189–204.
25. H.M. Taştan and F. Özdemir, The geometry of hemi-slant submanifolds of a locally product Riemannian manifold, (submitted) (2014), [arXiv:1405.6687v1 [math. DG]].
26. M.M. Tripathi, Generic submanifolds of generalized complex space forms, *Publ. Math. Debrecen,* 50 (1997), no. 3-4, 373–392.
27. S. Uddin, A.Y.M. Chi, Warped product pseudo-slant submanifolds of nearly Kaeahler manifolds, *An. St. Univ. Ovidius Constanta,* 19(3) (2011), 195-204.
28. K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore, 1984.