Comment on “Quantum key agreement protocol”

Nayana Das\textsuperscript{1,*} and Ritajit Majumdar\textsuperscript{2,+}

\textsuperscript{1}Applied Statistics Unit, Indian Statistical Institute, India
\textsuperscript{2}Advanced Computing & Microelectronics Unit, Indian Statistical Institute, India
\textsuperscript{*}dasnayana92@gmail.com, \textsuperscript{+}majumdar.ritajit@gmail.com

Abstract. The first two party Quantum Key Agreement (QKA) protocol, based on quantum teleportation, was proposed by Zhou et al. (Electronics Letters 40.18 (2004): 1149-1150). In this protocol, to obtain the key bit string, one of the parties use a device to obtain inner product of two quantum states, one being unknown, and the other one performs Bell measurement. However, in this article, we show that it is not possible to obtain a device that would output the inner product of two qubits even when only one of the qubit is unknown. This is so because existence of such device would imply perfectly distinguishing among four different states in a two-dimensional vector space. This is not permissible in quantum mechanics. Furthermore, we argue that existence of such a device would also imply violation of the “No Signalling Theorem” as well. Finally, we also comment that this protocol is not a valid key agreement protocol at all.

Keywords: Quantum key agreement, No signalling, Inner product, Non orthogonal states.

1 Introduction

The emergence of quantum computing has left a significant impression on the field of cryptography. Since the time it was realized that quantum computers have the potential to decipher public key cryptosystems in polynomial time \cite{1}, researchers have looked into protocols which make use of quantum mechanical laws in order to provide better security. The first ever quantum cryptographic protocol, to generate a shared secret key between two parties, was proposed by C. H. Bennett and G. Brassard in 1984 \cite{2}. Quantum key distribution \cite{2,3,4}, quantum secret sharing \cite{5,6,7,8}, quantum secure direct communication \cite{9,10} and quantum key agreement (QKA) \cite{11,12,13,14} are some of the domains of quantum cryptography, in which the use of quantum mechanical laws has proven to be more efficient and secure than the existing protocols.

Key agreement is one of the basic requirements of cryptography, which allows two or more parties to agree on a same secret key by exchanging their information over public channels. The difference between key distribution and key agreement protocol is as follows: in key distribution protocol, one party can determine a key
and distribute it to the other legitimate parties. But in a key agreement protocol, all the parties involved in the protocol contribute their information equally in order to generate a shared secret key. In 1976, Diffie and Hellman first proposed a classical key agreement protocol \cite{15}, whose security is based on the assumption that the discrete logarithm problem is computationally hard. But, in 1999, Shor proposed a quantum algorithm, which can solve the discrete logarithm problem in polynomial time \cite{1}. Thus some key agreement protocol was in need, such that, the security of that protocol does not depend on computation complexity of some mathematical problem.

The first two party quantum key agreement protocol was proposed by Zhou et al. \cite{11}, who used quantum teleportation \cite{16} protocol without the classical communication in order to generate the key bit string between the two parties. However, in this article, we show that this protocol violates one fundamental law of quantum mechanics, namely - (i) the impossibility of distinguishing among linearly dependent set of states \cite{17}, and one universal principle namely (ii) No signalling theorem \cite{18}.

Measurement of a quantum state is always associated with an orthonormal basis. Upon measurement, the quantum state collapses to one of the bases vector. Two quantum states $|\psi\rangle$ and $|\phi\rangle$ are said to be orthogonal if their inner product is zero, i.e. $\langle\psi|\phi\rangle = 0$. If $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, and also the states are known to the observer, then he can prepare a measurement $M$ in the basis $\{|\psi\rangle, |\phi\rangle\}$. When one of the two states $|\psi\rangle$ and $|\phi\rangle$ is provided to the observer, he can apply the measurement $M$, and the outcome will determine the state uniquely. However if the states $|\psi\rangle$ and $|\phi\rangle$ are non-orthogonal, they can not be distinguished reliably and if one consider a linearly dependent set of states (three or more than three different states in two dimension), the probability of distinguishing among them is zero.

No signalling theorem states that, it is not possible to transfer a signal instantaneously using quantum correlation. In other words, quantum mechanics does not have any contradiction with the theory of relativity, and the speed of transmission of information over a quantum channel is bounded by the speed of light. It can be noted that in quantum teleportation, one of the parties need to send two classical bits of information to the other party. This sending of classical information provides a bound to the speed of information transmission by this protocol.

In the key agreement protocol proposed by Zhou et al. \cite{11}, both of these fundamental laws are violated, which makes this protocol invalid. Moreover this is not a valid key agreement protocol as one of the parties alone can determine the secret key without being noticed by the other party. Furthermore, if this party is dishonest, he can even make the other party decide on a key according to his own choice.

The rest of the paper is arranged as follows. In the next section we briefly describe the QKA protocol of Zhou et al. \cite{11}. In Section 3 we individually elaborate the fallacies of this QKA protocol, and thus show why this protocol will fail to work. Finally a short conclusion is given in Section 4.
2 Brief Review of Zhou et al.’s Protocol

In this section, we briefly describe the “Quantum key agreement” protocol [11], proposed by Zhou et al. in 2004. This is a two party QKA protocol, in which two parties, namely, Alice and Bob, want to establish a private key over a public channel by using the technique of quantum teleportation. The steps of the protocol are as follows:

1. First Alice prepares a two qubit maximally entangled state, also called the Singlet state, \(|\Psi^-\rangle_{ab} = \frac{1}{\sqrt{2}}(|0\rangle_a |1\rangle_b - |1\rangle_a |0\rangle_b\) in her lab, where \(a\) and \(b\) denote the first and second particles respectively. She keeps the first particle \(a\) with her and sends the second particle \(b\) to Bob. At the end of this step, Alice and Bob share a maximally entangled state between them.

2. After receiving the particle \(b\) from Alice, Bob prepares two identical single qubit states \(|\phi_c\rangle\) and \(|\phi_d\rangle\), where \(|\phi_c\rangle = |\alpha\rangle = \alpha |0\rangle + \beta |1\rangle\), and Bob knows the values of \(\alpha\) and \(\beta\). The composite state of three particles \(a, b\) and \(c\) is

\[|\Psi\rangle_{abc} = |\Psi^-\rangle_{ab}|\phi_c\rangle,\]

which can be written as

\[|\Psi\rangle_{abc} = \frac{1}{2}[-(\alpha |0\rangle + \beta |1\rangle)_a |\Psi^-\rangle_{bc} + (\alpha |0\rangle - \beta |1\rangle)_a |\Psi^+\rangle_{bc} - (\beta |0\rangle + \alpha |1\rangle)_a |\Psi^-\rangle_{bc} + (\beta |0\rangle - \alpha |1\rangle)_a |\Psi^+\rangle_{bc}]\]

where \(|\Phi\rangle_{bc} = \frac{1}{\sqrt{2}}(|0\rangle_b |0\rangle_c \pm |1\rangle_b |1\rangle_c\) and \(|\Psi^\pm\rangle_{bc} = \frac{1}{\sqrt{2}}(|0\rangle_b |1\rangle_c \pm |1\rangle_b |0\rangle_c\).

3. Bob measures the particles \(b\) and \(c\) in Bell basis \(B = \{ |\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle \}\), and depending on the measurement outcome, the state of Alice’s particle (let it be \(|\phi_a\rangle\)) collapses on one of these four states \((\alpha |0\rangle + \beta |1\rangle)_a, (\alpha |0\rangle - \beta |1\rangle)_a, (\beta |0\rangle + \alpha |1\rangle)_a\) and \((\beta |0\rangle - \alpha |1\rangle)_a\) with equal probability. That is, the state of the particle \(c\) is teleported to Alice. However, Bob does not disclose the outcome of his measurement, and hence neither Alice, nor any eavesdropper, knows the state of the qubit \(|\phi_a\rangle\). At the same time Bob sends the qubit \(|\phi_d\rangle\) to Alice through a private optical fibre.

4. The measurement results \(|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle\) and \(|\Psi^-\rangle\) denote the resultant key bits 11, 10, 01 and 00 respectively.

5. After receiving the particle \(d\) from Bob, Alice calculates \(\langle\phi_a|\phi_d\rangle\), i.e., the inner product of the two states of the particles \(a\) and \(d\). If there is no eavesdropper, then the possible results are \(\alpha^2 + \beta^2 = 1, \alpha^2 - \beta^2, 2\alpha\beta, 0\) and these results correspond to the key bits 00, 01, 10, 11 respectively. The relation between the outcomes and the key bits are shown in Table 1. Bob chooses the values of \(\alpha\) and \(\beta\) such that \(\alpha^2 - \beta^2 \neq 2\alpha\beta \neq 0, 1\). Thus Alice and Bob both obtain their key bits.

In 2009, Tsai and Hwang [14] pointed out that the above protocol is not a key agreement protocol, since if Bob is malicious, then he can establish his
Table 1. Relation between the measurement outcomes and the values of the key bits

| measurement outcome of Bob | State of $|\phi\rangle_a$ | $\langle \phi_a | \phi_d \rangle$ | Key |
|----------------------------|-------------------------|-------------------------------|-----|
| $|\Psi^-\rangle_{bc}$      | $(\alpha |0\rangle + \beta |1\rangle)_a$ | 1                             | 00  |
| $|\Psi^+\rangle_{bc}$      | $(\alpha |0\rangle - \beta |1\rangle)_a$ | $\alpha^* - \beta^*$         | 01  |
| $|\Phi^-\rangle_{bc}$      | $(\beta |0\rangle + \alpha |1\rangle)_a$ | $2\alpha\beta$               | 10  |
| $|\Phi^+\rangle_{bc}$      | $(\beta |0\rangle - \alpha |1\rangle)_a$ | 0                             | 11  |

preferable pre-defined key as the secret key, without being detected. To do this, Bob first measures the particles $b$ and $c$ in the Bell basis, and after observing the measurement result, he prepares the particle $d$ such that the result of the inner product implies his preferable key.

3 Fallacies of this QKA Protocol

In this section, we show that the QKA protocol by Zhou et al. violates two fundamental properties of quantum mechanics. We elaborate on each of them individually and then argue why the above protocol is not a valid key agreement protocol as well.

3.1 Implication on non-orthogonal state discrimination

In the final stage of the protocol, Bob sends a qubit $|\phi_d\rangle = \alpha |0\rangle + \beta |1\rangle$ to Alice. He also teleports the state $|\phi_c\rangle$ to Alice, but does not disclose the outcome of his Bell measurement. In this stage, Alice has the qubit $|\phi_a\rangle$, which is in one of the following states

$$
|\phi^1_a\rangle = |\phi_c\rangle = \alpha |0\rangle + \beta |1\rangle \\
|\phi^2_a\rangle = \sigma_x |\phi_c\rangle = \alpha |1\rangle + \beta |0\rangle \\
|\phi^3_a\rangle = \sigma_z |\phi_c\rangle = \alpha |0\rangle - \beta |1\rangle \\
|\phi^4_a\rangle = \sigma_z \sigma_x |\phi_c\rangle = \alpha |1\rangle - \beta |0\rangle
$$

Now, Alice uses her device to calculate the inner product $\langle \phi^i_a | \phi_d \rangle$, where $\alpha$ and $\beta$ are known and the outcome determines the state $|\phi^i_a\rangle$, as shown in Table 1. We argue that, given the two qubits $|\phi^i_a\rangle$ and $|\phi_d\rangle$, it is not possible for Alice to physically calculate this inner product.

**Lemma 1.** There exists no device which can calculate the inner product $\langle \phi^i_a | \phi_d \rangle$, $i = 1, 2, 3, 4$.

**Proof.** Let us assume that there a device which can calculate the inner product mentioned above. The four possible qubit states $|\phi^1_a\rangle$, $|\phi^2_a\rangle$, $|\phi^3_a\rangle$ and $|\phi^4_a\rangle$ reside in a two dimensional Hilbert Space. Since it is not possible for four qubits to be
mutually orthogonal to each other in a two dimensional vector space, therefore, the four qubit states cannot be mutually orthogonal to each other for any value of $\alpha, \beta$. As discussed in Sec. 1, it is not possible to distinguish among linearly dependent set of quantum states with a single copy. However, in this protocol, since the inner product is different for the different possible states of $|\phi_i\rangle$, if such device exists it can perfectly distinguish between the four different states, which violates the postulate of quantum mechanics. Therefore, such a device cannot exist.

This implies that it is not possible to physically calculate the inner product $\langle \phi \mid \phi \rangle$ given the two qubits.

3.2 Violation of No signalling theorem

In order to show how the existence of such above mentioned device violates no-signalling condition, we consider a two qubits singlet state $|\Psi^-\rangle$ shared between Alice and Bob. This state has the property that it retains the same form in every spin direction [17]. Therefore, for any two qubit states $|\psi\rangle, |\bar{\psi}\rangle$, such that $\langle \psi | \bar{\psi} \rangle = 0$, the singlet state can be expressed as

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle |\bar{\psi}\rangle - |\bar{\psi}\rangle |\psi\rangle) \quad (1)$$

We assume $\alpha, \beta \in \mathbb{R}$ and then from the previous subsection one easily note that $\langle \phi_1^i | \phi_4^i \rangle = 0$ and $\langle \phi_2^i | \phi_3^i \rangle = 0$. Therefore,

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\phi_1^a\rangle |\phi_4^a\rangle - |\phi_4^a\rangle |\phi_1^a\rangle) \quad (2)$$

$$= \frac{1}{\sqrt{2}}(|\phi_2^a\rangle |\phi_3^a\rangle - |\phi_3^a\rangle |\phi_2^a\rangle) \quad (3)$$

Now consider that Bob randomly decides to measure his qubit of the shared singlet state $|\Psi^-\rangle$ either in $M_1 = \{|\phi_1^a\rangle, |\phi_4^a\rangle\}$ basis, or in $M_2 = \{|\phi_2^a\rangle, |\phi_3^a\rangle\}$ basis. However, Bob does not disclose his choice of measurement basis, nor the outcome of his measurement.

If Bob measures his qubit in $M_1$ basis, then the density matrix of Alice’s qubit is the following;

$$\rho_{14} = \frac{1}{2}(|\phi_1^a\rangle \langle \phi_1^a| + |\phi_4^a\rangle \langle \phi_4^a|). \quad (4)$$

Whereas, if Bob measures in $M_2$ basis, then the density matrix of Alice’s qubit is the following;

$$\rho_{23} = \frac{1}{2}(|\phi_2^a\rangle \langle \phi_2^a| + |\phi_3^a\rangle \langle \phi_3^a|). \quad (5)$$

Since Alice and Bob can have an an arbitrary spatial distance between them, if Alice can distinguish between $\rho_{14}$ and $\rho_{23}$, then she can determine the measurement basis chosen by Bob without any classical communication from Bob. This would violate the No signalling theorem.
However, in the protocol by Zhou et al. [11], Alice is equipped with a device, which can distinguish among the four different states $|\phi_1^a\rangle$, $|\phi_2^a\rangle$, $|\phi_3^a\rangle$, and $|\phi_4^a\rangle$. Therefore, with this device Alice can also uniquely determine the state of her qubit after the measurement of Bob. If the state of Alice's qubit is (i) $|\phi_1^a\rangle$ or $|\phi_4^a\rangle$, then Bob must have measured in $M_1$ basis, and if the state is (ii) $|\phi_2^a\rangle$ or $|\phi_3^a\rangle$, then Bob must have measured in $M_2$ basis. Hence Alice can determine the measurement chosen by Bob without any classical communication by the later. This obviously violates the No signalling theorem. Even some weaker device that helps Alice distinguish between the two possible preparations ($\rho_{14}$ and $\rho_{23}$) of the same density matrix would also imply violation of the no signaling principle [17].

4 Conclusion

In this comment, we make an analysis of the first two party QKA protocol proposed by Zhou et al. in 2004 [11]. We observed that though the resources used in this protocol, namely one ebit of entanglement and physical transfer of one qubit, may be sufficient for key agreement for some properly chosen protocol, the suggested protocol is basically flawed. The protocol is not consistent with allowed physical operation in quantum mechanics. More importantly, the existence of such protocol would imply violation of “No signaling condition" obeyed by all physical theories including quantum mechanics.

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