Predictive Neutrino Spectrum in Minimal
SO(10) Grand Unification

K.S. Babu*
Bartol Research Institute
University of Delaware
Newark, DE 19716

and

R.N. Mohapatra†
Department of Physics
University of Maryland
College Park, MD 20742

Abstract

We show that minimal SO(10) Grand Unification models where the fermions have Yukawa couplings to only one (complex) $10$ and one $126$ of Higgs scalars lead to a very predictive neutrino spectrum. This comes about since the standard model doublet contained in the $126$ of Higgs (needed for the see–saw mechanism) receives an induced vacuum expectation value at tree–level, which, in addition to correcting the bad asymptotic mass relations $m_d = m_e$ and $m_s = m_\mu$, also relates the Majorana neutrino mass matrix to observables in the charged fermion sectors. We find that (i) the $\nu_e - \nu_\mu$ mixing angle relevant for the solar neutrinos can be considerably smaller than the Cabibbo angle and lies in the range $\sin \theta_{e\mu} = 0 - 0.3$, (ii) $\nu_e - \nu_\tau$ mixing is $\sin \theta_{e\tau} \simeq 3|V_{td}| \simeq 0.05$, (iii) the $\nu_\mu - \nu_\tau$ mixing angle is large, $\sin \theta_{\mu\tau} \simeq 3|V_{cb}| = 0.12-0.16$, and (iv) $m_{\nu_\tau}/m_{\nu_\mu} \geq 10^3$, implying that $\nu_\mu - \nu_\tau$ oscillations should be accessible to forthcoming experiments.

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It is quite possible that the deficit of solar neutrinos reported in the Chlorine, Kamiokande, SAGE, and GALLEX experiments is an indication that the neutrinos have masses and mixings very much like the quarks. The observed deficit can be explained in terms of neutrino oscillations in two different ways: (i) long wave length vacuum oscillation, and (ii) resonant matter oscillation (the Mikheyev-Smirnov-Wolfenstein (MSW) effect). Assuming a two-flavor $\nu_e - \nu_\mu$ oscillation, in the former case, the neutrino masses and mixing angle should satisfy (at 90% CL) $\Delta m^2 \simeq (0.5$ to 1.1) $\times 10^{-10}$ eV$^2$ and $\sin^2 2\theta_{e\mu} \simeq (0.75$ to 1). In case of MSW, on the other hand, there are two allowed windows that fit all of the experimental data: (a) the small mixing angle non-adiabatic solution, which requires $\Delta m^2 \simeq (0.3$ to 1.2) $\times 10^{-5}$ eV$^2$ and $\sin^2 2\theta_{e\mu} \simeq (0.4$ to 1.5) $\times 10^{-2}$, and (b) the large angle solution with $\Delta m^2 \simeq (0.3$ to 5) $\times 10^{-5}$ eV$^2$ and $\sin^2 2\theta_{e\mu} \simeq (0.5$ to 0.9). In all these cases, barring an unlikely scenario of near mass degeneracy among neutrinos, either $\nu_\mu$ or $\nu_\tau$ should have mass in the $(10^{-5}$ to $10^{-3})$ eV range.

A well-known and elegant explanation for the origin of such tiny neutrino masses is the see-saw mechanism, wherein the light neutrino masses scale inversely with the $B-L$ breaking Majorana mass $M$ of the right-handed neutrino: $m_\nu \simeq m^2/M$, $m$ being the neutrino Dirac mass. The smallness of $m_\nu$ is then understood in terms of the heaviness of the $B-L$ breaking scale. The solar neutrino puzzle indicates that the $B-L$ scale is in the $(10^{12} - 10^{16})$ GeV range.

All of the observations above, viz., non-zero neutrino masses, the see-saw mechanism, and a high $B-L$ scale, fit rather naturally in grand unified models based on the gauge group $SO(10)$. In its non-supersymmetric version, experimental constraints from proton life-time and the weak mixing angle $\sin^2 \theta_W$ require that $SO(10)$ breaks not directly into the standard model, but at least in two steps. In a two-step breaking scheme, the left-right symmetric intermediate scale is around $10^{12}$ GeV. In supersymmetric $SO(10)$, on the other hand, there is no need for an intermediate scale, $SO(10)$ can break directly to the standard model at around $10^{16}$ GeV.

To confront $SO(10)$ models with the solar neutrino data, one must make
precise predictions of the neutrino masses and mixing angles. This requires, however, detailed information of the Dirac neutrino mass matrix as well as the Majorana matrix. In grand unified theories (Guts), it is possible to relate the quark masses with the lepton masses. For example, in $SU(5)$ Gut, the charge $-1/3$ quark mass matrix is equal to the charged lepton matrix at the unification scale. Most $SO(10)$ models retain this feature; in addition, they also relate the neutrino Dirac mass matrix to the charge $2/3$ quark matrix. However, there is no simple way, in general, to relate the heavy Majorana matrix $M$ to the charged fermion observables.

The purpose of this Letter is to show that in a class of minimal $SO(10)$ models, in fact, not only the Dirac neutrino matrix, but the Majorana matrix also gets related to observables in the charged fermion sector. This leads to a very predictive neutrino spectrum, which we analyze. We use a simple Higgs system with one (complex) $10$ and one $126$ that have Yukawa couplings to fermions. The $10$ is needed for quark and lepton masses, the $126$ is needed for the see–saw mechanism. Crucial to the predictivity of the neutrino spectrum is the observation that the standard model doublet contained in the $126$ receives an induced vacuum expectation value (vev) at tree–level. In its absence, one would have the asymptotic mass relations $m_b = m_r$, $m_s = m_\mu$, $m_d = m_e$, as in minimal $SU(5)$. While the first relation would lead to a successful prediction of $m_b$ at low energies, the last two are in disagreement with observations. The induced vev of the standard doublet of $126$ corrects these bad relations and at the same time also relates the Majorana neutrino mass matrix to observables in the charged fermion sector, leading to a predictive neutrino spectrum.

We shall consider non–Susy $SO(10)$ breaking to the standard model via the $SU(2)_L \times SU(2)_R \times SU(4)_C \equiv G_{224}$ chain as well as Susy-$SO(10)$ breaking directly to the standard model. Our predictions on the neutrino mass ratios and the mixing angles are essentially independent of the chain of descent, it only affects the overall scale of neutrino masses.

The breaking of $SO(10)$ via $G_{224}$ is achieved by either a $54$ or a $210$ of Higgs. The $210$ also breaks the discrete $D$–parity, the $54$ preserves it.
$D$–parity is a local discrete $Z_2$–subgroup of $SO(10)$, under $D$, a fermion field $f$ transforms into its charge conjugate $f^c$. Breaking of $D$–parity at the Gut scale makes the see–saw mechanism natural.\(^{12}\) The second stage of symmetry breaking goes via the $\mathbf{126}$. Finally, the electro–weak symmetry breaking proceeds via the $\mathbf{10}$. Note that in Susy–$SO(10)$, the first two symmetry breaking scales coalesce into one.

Let us turn attention to the fermion–Higgs Yukawa couplings of the model. Denoting the three families of fermions belonging to $\mathbf{16}$–dimensional spinor representation of $SO(10)$ by $\psi_a$, $a = 1 – 3$, the complex $\mathbf{10}$–plet of Higgs by $H$, and the $\mathbf{126}$–plet of Higgs by $\Delta$, the Yukawa couplings can be written down as

$$L_Y = h_{ab} \psi_a \psi_b H + f_{ab} \psi_a \psi_b \Delta + H.C.$$ (1)

Note that since the $\mathbf{10}$–plet is complex, one other coupling $\psi_a \psi_b \overline{H}$ is allowed in general. In Susy–$SO(10)$, the requirement of supersymmetry prevents such a term. In the non–Susy case, we forbid this term by imposing a $U(1)_{PQ}$ symmetry, which may anyway be needed in order to solve the strong CP problem.

The $\mathbf{10}$ and $\mathbf{126}$ of Higgs have the following decomposition under $G_{224}$:

$$\begin{align*}
\mathbf{126} &\to (1,1,6) + (1,3,10) + (3,1,\overline{10}) + (2,2,15) \\
\mathbf{10} &\to (1,1,6) + (2,2,1)
\end{align*}$$ (2)

Denote the $(1,3,10)$ and $(2,2,15)$ components of $\Delta(\mathbf{126})$ by $\Delta_R$ and $\Sigma$ respectively and the $(2,2,1)$ component of $H(\mathbf{10})$ by $\Phi$. The vev $<\Delta_R^0 > \equiv v_R \sim 10^{12}$ GeV breaks the intermediate symmetry down to the standard model and generates Majorana neutrino masses given by $fv_R$. $\Phi$ contains two standard model doublets which acquire non–zero vev’s denoted by $\kappa_u$ and $\kappa_d$ with $\kappa_{u,d} \sim 10^2$ GeV. $\kappa_u$ generates charge $2/3$ quark as well as Dirac neutrino masses, while $\kappa_d$ gives rise to $-1/3$ quark and charged lepton masses.

Within this minimal picture, if $\kappa_u$, $\kappa_d$ and $v_R$ are the only vev's contributing to fermion masses, in addition to the $SU(5)$ relations $m_b = m_{\tau}$, $m_s =$
\( m_\mu, m_d = m_e \), eq. (1) will also lead to the unacceptable relations \( m_u : m_c : m_t = m_d : m_s : m_b \). Moreover, the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix will be identity. Fortunately, within this minimal scheme, we have found that there are new contributions to the fermion mass matrices which are just of the right order of magnitude to correct these bad relations. To see this, note that the scalar potential contains, among other terms, a crucial term

\[
V_1 = \lambda \Delta \Sigma \Delta H + H.C. \tag{3}
\]

Such a term is invariant under the \( U(1)_{PQ} \) symmetry. It will be present in the Susy \( SO(10) \) as well, arising from the \( 210 \) \( F \)-term. This term induces vev’s for the standard doublets contained in the \( \Sigma \) multiplet of \( 126 \). The vev arises through a term \( \Sigma_R \Sigma \Phi \) contained in \( V_1 \). This induced vev can also be seen by analyzing the one-loop graph involving neutrinos which generates a divergent contribution to such a term.

We can estimate the magnitudes of the induced vev’s of \( \Sigma \) (denoted by \( v_u \) and \( v_d \) along the up and down directions) assuming the survival hypothesis to hold:

\[
v_{u,d} \sim \lambda \left( \frac{v^2_R}{M^2_{\Sigma}} \right) \kappa_{u,d}. \tag{4}\]

Suppose \( M_U \sim 10^{15} \text{ GeV} \), \( M_I \sim 3 \times 10^{12} \text{ GeV} \) and \( M_\Sigma \sim 10^{14} \text{ GeV} \), consistent with survival hypothesis, then \( v_u \) and \( v_d \) are of order 100 MeV, in the right range for correcting the bad mass relations. We emphasize that there is no need for a second fine-tuning to generate such induced vev’s. In the Susy version, since there is no intermediate scale at all, the factor \( (v^2_R/M^2_{\Sigma}) \) is not a suppression, so the induced vev’s can be as large as \( \kappa_{u,d} \).

We are now in a position to write down the quark and lepton mass matrices of the model:

\[
\begin{align*}
M_u &= h\kappa_u + f v_u \\
M_d &= h\kappa_d + f v_d \\
M^D_\nu &= h\kappa_u - 3f v_u \\
M^I_\nu &= h\kappa_d - 3f v_d \quad \text{for} \quad M^M_\nu = f v_R . \tag{5}
\end{align*}
\]

Here \( M^D_\nu \) is the Dirac neutrino matrix and \( M^M_\nu \) is the Majorana mass matrix.
Before proceeding, we should specify the origin of CP violation in the model. We shall assume that it is spontaneous or soft, that will keep the number of parameters at a minimum. The Higgs sector described above already has enough structure to generate realistic CP violation either softly or spontaneously. The Yukawa coupling matrices $h$ and $f$ in this case are real and symmetric. Although there will be three different phases in the vev’s (one common phase for $\kappa_u$ and $\kappa_d$ and one each for $v_u$ and $v_d$), only two combinations enter into the mass matrices, as the overall phase can be removed from each sector. We shall bring these two phases into $v_u$ and $v_d$ and hence forth denote them by $v_u e^{i\alpha}$ and $v_d e^{i\beta}$.

To see the predictive power of the model as regards the neutrino spectrum, note that we can choose a basis where one of the coupling matrices, say $h$, is real and diagonal. Then there are 13 parameters in all, not counting the superheavy scale $v_R$: 3 diagonal elements of the matrix $h\kappa_u$, 6 elements of $fv_u$, 2 ratios of vev’s $r_1 = \kappa_d/\kappa_u$ and $r_2 = v_d/v_u$, and the two phases $\alpha$ and $\beta$. These 13 parameters are related to the 13 observables in the charged fermion sector, viz., 9 fermion masses, 3 quark mixing angles and one CP violating phase. The light neutrino mass matrix will then be completely specified in terms of other physical observables and the overall scale $v_R$. That would lead to 8 predictions in the lepton sector: 3 leptonic mixing angles, 2 neutrino mass ratios and 3 leptonic CP violating phases.

The relations of eq. (5) hold at the intermediate scale $M_I$ where quark–lepton symmetry and left–right symmetry are intact. There are calculable renormalization corrections to these relations below $M_I$. The quark and charged lepton masses as well as the CKM matrix elements run between $M_I$ and low energies. The neutrino masses and mixing angles, however, do not run below $M_I$, since the right-handed neutrinos have masses of order $M_I$ and decouple below that scale. The predictions in the neutrino sector should then be arrived at by first extrapolating the charged fermion observables to $M_I$.

We shall present results for the non–Susy $SO(10)$ model with the $G_{224}$ intermediate symmetry. We fix the intermediate scale at $M_I = 10^{12} \text{GeV}$ and use the one–loop standard model renormalization group equations to track
the running of the gauge couplings between $M_Z$ and $M_I$. For Susy–SO(10), the results are similar, we shall postpone details to a forthcoming longer paper.\cite{14}

To compute the renormalization factors, we choose as low energy inputs the gauge couplings at $M_Z$ to be

$$\alpha_1(M_Z) = 0.01688 \quad ; \quad \alpha_2(M_Z) = 0.03322 \quad ; \quad \alpha_3(M_Z) = 0.12 \quad .$$

For the light quark (running) masses, we choose values listed in Ref. (15):

$$m_u(1\text{ GeV}) = 5.1 \pm 1.5\text{ MeV} \quad \quad m_d(1\text{ GeV}) = 8.9 \pm 2.6\text{MeV}$$
$$m_s(1\text{ GeV}) = 175 \pm 55\text{ MeV} \quad \quad m_c(m_c) = 1.27 \pm 0.05\text{ GeV}$$
$$m_b(m_b) = 4.25 \pm 0.1\text{ GeV} \quad .$$

The top–quark mass will be allowed to vary between 100 and 200 GeV.

Between 1 GeV and $M_Z$, we use two–loop QCD renormalization group equations for the running of the quark masses and the $SU(3)_C$ gauge coupling,\cite{15} treating particle thresholds as step functions. From $M_Z$ to $M_I$, the running factors are computed semi–analytically both for the fermion masses and for the CKM angles by using the one–loop renormalization group equations for the Yukawa couplings and keeping the heavy top–quark contribution.\cite{16} The running factors, defined as $\eta_i = m_i(M_I)/m_i(1\text{ GeV})$ for light quarks ($u,d,s$) are $\eta(u,c,t) = (0.227, 0.241, 0.452)$, $\eta(d,s,b) = (0.232, 0.232, 0.295)$, $\eta(e,\mu,\tau) = 0.912$ for the case of $m_t = 130\text{ GeV}$ and $\eta(u,c,t) = (0.237, 0.252, 0.482)$, $\eta(d,s,b) = (0.242, 0.242, 0.302)$, $\eta(e,\mu,\tau) = 0.952$ for $m_t = 150\text{ GeV}$. The (common) running factors for the CKM angles (we follow the parameterization advocated by the Particle Data Group) $S_{23}$ and $S_{13}$ are 1.054 for $m_t = 130\text{ GeV}$ and 1.077 for $m_t = 150\text{ GeV}$. The Cabibbo angle $S_{12}$ and the KM phase $\delta_{KM}$ are essentially unaltered.

Let us first analyze the mass matrices of eq. (5) in the limit of CP conservation. We shall treat spontaneous CP violation arising through the phases of the vev’s $v_u e^{i\alpha}$ and $v_d e^{i\beta}$ as small perturbations. This procedure will be justified a posteriori. In fact, we find that realistic fermion masses, in particular the first family masses, require these phases to be small.
We can rewrite the mass matrices $M_l, M_D^\nu$ and $M_M^\nu$ of eq. (5) in terms of the quark mass matrices and three ratios of vev’s – $r_1 = \kappa_d/\kappa_u$, $r_2 = v_d/v_u$, $R = v_u/v_R$:

\[
M_l = \frac{4r_1r_2}{r_2 - r_1}M_u - \frac{r_1 + 3r_2}{r_2 - r_1}M_d ,
\]

\[
M_D^\nu = \frac{3r_1 + r_2}{r_2 - r_1}M_u - \frac{4}{r_2 - r_1}M_d ,
\]

\[
M_M^\nu = \frac{1}{Rr_1 - r_2}M_u - \frac{1}{Rr_1 - r_2}M_d .
\]

(8)

It is convenient to go to a basis where $M_u$ is diagonal. In that basis, $M_d$ is given by $M_d = V M_d^{\text{diagonal}} V^T$, where $M_d^{\text{diagonal}} = \text{diagonal}(m_d, m_s, m_b)$ and $V$ is the CKM matrix. One sees that $M_l$ of eq. (8) contains only physical observables from the quark sector and two parameters $r_1$ and $r_2$. In the CP–conserving limit then, the three eigen–values of $M_l$ will lead to one mass prediction for the charged fermions. To see this prediction, $M_l$ needs to be diagonalized. Note first that by taking the Trace of $M_l$ of eq. (8), one obtains a relation for $r_1$ in terms of $r_2$ and the charged fermion masses. This is approximately $r_1 \simeq (m_\tau + 3m_b)/4m_t$ (as long as $r_2$ is larger than $m_b/m_t$).

Since $|m_b| \simeq |m_\tau|$ at the intermediate scale to within 30% or so, depending on the relative sign of $m_b$ and $m_\tau$, $r_1$ will be close to either $m_b/m_t$ or to $(m_b/2m_t)$. Note also that if $r_2 \gg r_1$, $M_l$ becomes independent of $r_2$, while $M_D^\nu$ retains some dependence:

\[
M_l \simeq 4r_1M_u - 3M_d , \quad M_D^\nu \simeq M_u - \frac{4}{r_2}M_d .
\]

(9)

This means that the parameter $r_2$ will only be loosely constrained from the charged fermion sector.

We do the fitting as follows. For a fixed value of $r_2$, we determine $r_1$ from the Tr($M_l$) using the input values of the masses and the renormalization factors discussed above. $M_l$ is then diagonalized numerically. There will be two mass relations among charged fermions. Since the charged lepton masses are precisely known at low energies, we invert these relations to predict the $d$–quark and $s$–quark masses. The $s$–quark mass is sensitive to the muon mass,
the $d$–mass is related to the electron mass. This procedure is repeated for other values of $r_2$. For each choice, the light neutrino masses and the leptonic CKM matrix elements are then computed using the see–saw formula.

We find that there are essentially three different solutions. A two–fold ambiguity arises from the unknown relative sign of $m_b$ and $m_\tau$ at $M_I$. We have found acceptable solutions for both signs. Our numerical fit shows that the loosely constrained parameter $r_2$ cannot be smaller than 0.1 or so, otherwise the $d$–quark mass comes out too small. Now, the light neutrino spectrum is sensitive to $r_2$ only when $r_2 \sim 4m_s/m_c \sim \pm0.4$, since the two terms in $M_D^{\nu}$ of eq. (9) become comparable (for the second family) then. For larger values of $r_2$, the first term in $M_D^{\nu}$ dominates and the light neutrino spectrum becomes independent of $r_2$. Two qualitatively different solutions are obtained depending on whether $r_2$ is near $\pm0.4$ or not.

Numerical results for the three different cases are presented below. The input values of the CKM mixing angles are chosen for all cases to be $S_{12} = -0.22$, $S_{23} = 0.052$, $S_{13} = 6.24 \times 10^{-3}$. Since $\delta_{KM}$ has been set to zero for now, we have allowed for the mixing angles to have either sign. Not all signs result in acceptable quark masses though. Similarly, the fermion masses can have either sign, but these are also restricted. The most stringent constraint comes from the $d$–quark mass, which has a tendency to come out too small. Acceptable solutions are obtained when $\theta_{23}$, $\theta_{13}$ are in the first quadrant and $\theta_{12}$ in the fourth quadrant. We use the precisely known charged lepton masses at low energies and the running factors discussed above to arrive at the values of the masses and mixing angles at $M_I$.

Solution 1:

Input : $m_u(1\ GeV) = 4\ MeV$, $m_c(m_c) = 1.22\ GeV$, $m_t = 150\ GeV$

\[ m_b(m_b) = -4.35\ GeV,\ r_1 = -1/51.7,\ r_2 = 2.0\]

Output : $m_d(1\ GeV) = 6.4\ MeV$, $m_s(1\ GeV) = 164\ MeV$

\[ (m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}) = R \left( 2.0 \times 10^{-2}, 8.7, -2.2 \times 10^4 \right)\ GeV\]
\[ V_{KM}^{\text{lepton}} = \begin{pmatrix} 0.9522 & 0.3051 & 0.0123 \\ -0.2991 & 0.9400 & -0.1637 \\ -0.0615 & 0.1522 & 0.9864 \end{pmatrix}. \] (10)

Solution 2:

Input : \( m_u(1 \text{ GeV}) = 4 \text{ MeV}, \ m_c(m_c) = 1.22 \text{ GeV}, \ m_t = 150 \text{ GeV} \)
\[ m_b(m_b) = -4.35 \text{ GeV}, \ r_1 = -1/51.4, \ r_2 = 0.2 \]
Output : \( m_d(1 \text{ GeV}) = 5.6 \text{ MeV}, \ m_s(1 \text{ GeV}) = 175 \text{ MeV} \)
\[ (m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}) = R \left( 6.3 \times 10^{-4}, 2.1, -2.7 \times 10^3 \right) \text{ GeV} \]
\[ V_{KM}^{\text{lepton}} = \begin{pmatrix} 0.9969 & 0.0506 & -0.0607 \\ -0.0585 & 0.9890 & -0.1359 \\ 0.0532 & 0.1390 & 0.9889 \end{pmatrix}. \] (11)

Solution 3:

Input : \( m_u(1 \text{ GeV}) = 3.5 \text{ MeV}, \ m_c(m_c) = 1.27 \text{ GeV}, \ m_t = 130 \text{ GeV} \)
\[ m_b(m_b) = -4.35 \text{ GeV}, \ r_1 = -1/101.8, \ r_2 = -0.5 \]
Output : \( m_d(1 \text{ GeV}) = -5.24 \text{ MeV}, \ m_s(1 \text{ GeV}) = -173 \text{ MeV} \)
\[ (m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}) = R \left( 4.1 \times 10^{-2}, 1.1, 6.2 \times 10^3 \right) \text{ GeV} \]
\[ V_{KM}^{\text{lepton}} = \begin{pmatrix} 0.9954 & -0.0794 & -0.0521 \\ 0.0704 & 0.9853 & -0.1559 \\ 0.0637 & 0.1515 & 0.9864 \end{pmatrix}. \] (12)

Solution 1 corresponds to choosing \( r_1 \sim m_b/m_t \). All the charged lepton masses are negative in this case. Since \( r_2 \) is large, the Dirac neutrino matrix is essentially \( M_u \), which is diagonal; so is the Majorana matrix. All the leptonic mixing angles arise from the charged lepton sector. Note that the predictions for \( m_d \) and \( m_s \) are within the range quoted in eq. (7). The mixing angle \( \sin \theta_{\nu_e-\nu_\mu} \) relevant for solar neutrinos is 0.30, close to the Cabibbo angle. Such a value may already be excluded by a combination of all solar neutrino data taken at the 90% CL (but not at the 95% CL). Actually, within the model, there is a more stringent constraint. Note that the \( \nu_\mu - \nu_\tau \) mixing angle is large, it is approximately \( 3|V_{cb}| \simeq 0.16 \). For that large a mixing, constraints from \( \nu_\mu - \nu_\tau \) oscillation experiments imply\(^7\) that \( |m_{\nu_\tau}^2 - m_{\nu_\mu}^2| \leq 4 \text{ eV}^2 \).
Solution 1 also has $m_{\nu_e}/m_{\nu_\mu} \simeq 2.5 \times 10^3$, requiring that $m_{\nu_\mu} \leq 0.8 \times 10^{-3} \text{ eV}$. This is a factor of 2 too small for $\nu_e - \nu_\mu$ MSW oscillation for the solar puzzle (at the 90% CL), but perhaps is not excluded completely, once astrophysical uncertainties are folded in. If $\nu_\tau$ mass is around $2 \times 10^{-3} \text{ eV}$, $\nu_e - \nu_\tau$ oscillation may be relevant, that mixing angle is $\simeq 3|V_{td}| \simeq 6\%$. It would require the parameter $R = v_u/v_R \sim 10^{-16}$ or $v_R \sim 10^{16} \text{ GeV}$ for $v_u \sim 1 \text{ GeV}$. Such a scenario fits well within Susy–SO(10), but not in the non–Susy $G_{224}$ chain. Note that $\nu_\tau$ mass is negative, a transformation $\nu_\tau \rightarrow i\nu_\tau$ will make it positive.

Solution 2 differs from 1 in that $r_2$ is smaller, $r_2 = 0.2$. The $1 - 2$ mixing in the neutrino sector is large in this case, so it can cancel the Cabibbo like mixing arising from the charged lepton sector. As we vary $r_2$ from around 0.2 to 0.6, this cancellation becomes stronger, the $\nu_e - \nu_\mu$ mixing angle becoming zero for a critical value of $r_2$. For larger $r_2$, the solution will approach Solution 1. The $\nu_\mu - \nu_\tau$ mixing angle is still near $3|V_{cb}|$, so as before, $m_{\nu_e} \leq 2 \text{ eV}$. From the $\nu_\tau/\nu_\mu$ mass ratio, which is $1.3 \times 10^3$ in this case, we see that $m_{\nu_\mu} \leq 1.6 \times 10^{-3} \text{ eV}$. This is just within the allowed range\(^7\) (at 95% CL) for small angle non–adiabatic $\nu_e - \nu_\mu$ MSW oscillation, with a predicted count rate of about 50 SNU for the Gallium experiment. Note that there is a lower limit of about 1 eV for the $\nu_\tau$ mass in this case. Forthcoming experiments should then be able to observe $\nu_\mu - \nu_\tau$ oscillations. A $\nu_\tau$ mass in the (1 to 2) eV range can also be cosmologically significant, it can be at least part of the hot dark matter. In Susy $SO(10)$, $\nu_e - \nu_\tau$ oscillation (the relevant mixing is about $3|V_{td}| \simeq 5\%$), could account for the solar neutrino puzzle.

Solution 3 corresponds to choosing $r_1 \sim (m_b/2m_t)$. All charged lepton masses are positive in this case. The sign of $r_2$ has been chosen to get small $\sin \theta_{e-\mu}$. (For other values of $r_2$, the results are similar to Solution 1.) However, the mass ratio $\nu_\tau/\nu_\mu$ is $\sim 6 \times 10^3$, and $\sin \theta_{\mu\tau} \simeq 3V_{cb}$ so $\nu_e - \nu_\mu$ oscillation cannot be responsible for solar MSW. As in other cases, $\nu_e - \nu_\tau$ MSW oscillation with a 6% mixing is a viable possibility.

Observe that none of the solution generates $\nu_e - \nu_\mu$ mixing large enough for the vacuum oscillation for solar neutrinos. Similarly, the puzzle with atmospheric neutrinos cannot be explained in this minimal scheme in terms
of $\nu_\mu - \nu_\tau$ oscillation, the relevant mixing is not large enough. (For an $SO(10)$–based explanation of this phenomenon, see Ref. 18).

Let us now re-instate the CP–violating phases $\alpha$ and $\beta$ in the vev’s perturbatively. Small values of the phases are sufficient to account for realistic CP violation in the quark sector. We shall present details for the case of Solution 2 only, others are similar. We also tried to fit all the charged fermion masses and mixing angles for large phases, but found no consistent solution.

First we make a basis transformation to go from the basis where $M_u$ is diagonal to one where the matrix $h_{\kappa_u}$ is diagonal. It is easier to introduce phases in that basis. For $\alpha = 3.5^0$, $\beta = 4.5^0$, the CP–violating parameter $J$ for the quark system$^{19}$ is $J \simeq 1 \times 10^{-5}$, which is sufficient to accommodate $\epsilon$ in the neutral $K$ system. The leptonic CP violating phases are correspondingly small, for eg., the analog of $J$ is $J_l \simeq 7 \times 10^{-5}$. These small phases modify the first family masses slightly, but the effect is less than 10%. Our predictions for the neutrino mixing angles are essentially unaltered.

In summary, we have presented a class of minimal $SO(10)$ grand unified models where the light neutrino masses and mixing angles are predictable in terms of observables in the charged fermion sector. Our approach here has been orthogonal to some other recent attempts$^{20,18}$ based on grand unification, we have kept the Higgs sector as simple as possible and followed its consequences. We have found three different types of solutions for the neutrino spectrum. In Solution 1, the $\nu_e - \nu_\mu$ mixing angle is near the Cabibbo angle, while Solutions 2 and 3 have it much smaller. In all cases, $\nu_e - \nu_\tau$ mixing angle is predicted to be near $3|V_{td}| \simeq 0.05$ and $\nu_\mu - \nu_\tau$ mixing angle is $\simeq 3|V_{cb}| \simeq 0.15$ with the mass ratio $m_{\nu_e}/m_{\nu_\mu} \geq 10^3$. If the solar neutrino puzzle is due to small angle non–adiabatic MSW, as in Solution 2, $\nu_\mu - \nu_\tau$ oscillation should be observable in the forthcoming experiments.

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