THE THIN OBSTACLE PROBLEM FOR SOME VARIABLE COEFFICIENT DEGENERATE ELLIPTIC OPERATORS

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Abstract. In this paper, we establish the optimal interior regularity and the $C^{1,\gamma}$ smoothness of the regular part of the free boundary in the thin obstacle problem for a class of degenerate elliptic equations with variable coefficients.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this paper we prove the optimal interior regularity and the $C^{1,\gamma}$ local regularity of the regular part of the free boundary in the following degenerate thin obstacle problem with variable coefficients:

$$
\begin{cases}
\text{div}(\vert y \vert^a A(x) \nabla X U) = 0, & \text{in } B^+_1, \\
\min\{U(x,0) - \psi(x), -\partial_y^a U(x,0)\} = 0 & \text{on } B_1,
\end{cases}
$$

where for $x \in \mathbb{R}^n$, $y > 0$, we have indicated $X = (x, y) \in \mathbb{R}^{n+1}_+$, and defined $\partial_y^a U(x,0) \overset{\text{def}}{=} \lim_{y \to 0^+} y^a \partial_y U(x,y)$. The function $\psi$ is called the thin obstacle since it is defined in the thin set $\mathbb{R}^n \times \{0\}$. What makes the problem (1.1) degenerate is the presence of the weight $|y|^a = \text{dist}(X, \{y = 0\})^a$, where the parameter $a$ is allowed to range in the interval $(-1, 1)$.

We recall that the coincidence set is $\Lambda_\psi(U) = \{x \in B_1 \mid U(x,0) = \psi(x)\}$, and that the free boundary $\Gamma_\psi(U)$ is the topological boundary (in the relative topology of $B_1$) of the set $\Lambda_\psi(U)$. While we refer the reader to Section 2 for a detailed account of notations and hypothesis, here we confine ourselves to mention that throughout the present work we assume that the matrix-valued function $A$ is uniformly elliptic with Lipschitz continuous coefficients satisfying (2.1) below. We emphasise that the interest in studying a problem such as (1.1) with variable

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coefficients is not merely academic: in concrete situations the separating thin manifold is not necessarily flat, and if one flattens it one is led to a problem of the form (1.1).

With this being said, our first main result concerning the optimal interior regularity is the following.

**Theorem 1.1.** Assume $0 \leq a < 1$. Let $U$ be a solution to (1.1) with $\psi \in C^{1,1}$. Then $U \in C^{\frac{3-\alpha}{2}}(\overline{B^+_{r/2}})$, $\nabla_x U \in C^{\frac{1+a}{2}}(\overline{B^+_{r/2}})$ and $y^a U_y \in C^{\frac{1+a}{2}}(\overline{B^+_{r/2}})$.

To state our second main result we need to introduce the notion of regular free boundary points. An equivalent definition of such points based on Almgren type frequency function is given in Section 6 below, see Definition 6.1. We say that a free boundary point $(x_0, 0) \in \Gamma_\psi(U)$ is regular if there exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \leq \limsup_{r \to 0} \frac{\|U - \psi\|_{L^\infty(B_r(x_0))}}{r^{\frac{3-\alpha}{2}}} \leq \beta.$$  

We denote by $\Gamma^{\frac{3-a}{2}}_\psi(U)$ the set of all regular free boundary points, and call it the regular set. The following is our second main result.

**Theorem 1.2.** Suppose that $0 \leq a < 1$ and let $U$ be as in Theorem 1.1 above. Then, $\Gamma^{\frac{3-a}{2}}_\psi(U)$ is a relatively open subset of $\Gamma_\psi(U)$. After possibly a translation and rotation of the coordinate axes in the thin space $\mathbb{R}^n \times \{0\}$, the set $\Gamma^{\frac{3-a}{2}}_\psi(U)$ is locally given as a graph

$$x_n = g(x_1, \ldots, x_{n-1}),$$

with $g \in C^{1+\gamma}$.

We emphasise that the interior regularity claimed in Theorem 1.1 is best possible, even when $A = I$. In this context the optimal interior regularity as well as the $C^1$ smoothness of the regular set were established in the pioneering work [10] in the full range $-1 < a < 1$. We also mention that for the case $a = 0$ Theorems 1.1 and 1.2 were first respectively established in [24] and [21]. More recently, and still for the case $a = 0$, the authors of [27] have succeeded in treating the more general case of almost minimisers and Hölder variable coefficients.

For variable coefficients thin obstacle problems such as (1.1) prior to the present paper there have been no contributions to the optimal interior regularity or the $C^{1,\gamma}$ regularity of the regular free boundary when $a \neq 0$. One of the things that makes the analysis particularly challenging is the lack of those fundamental initial results such as the Hölder continuity up to the thin set of the solution $U$, its weighted Neumann derivative $y^a \partial_y U$ and that of $\nabla_x U$.

As it is well-known, such results represent the main building blocks in the study of lower-dimensional obstacle problems. Once they are available, the next challenge is to develop suitable monotonicity formulas that play a critical role in the blowup analysis.

The aim of the present paper is to fill this gap, at least when $0 \leq a < 1$. The range $-1 < a < 0$ remains presently open, but we hope to return to this question in a future study. We mention that the limitation $a \geq 0$ in Theorems 1.1 and 1.2 stems from Theorem 3.4 below and that, with the exception of such technical result, the remainder of the work in the present paper covers the full range $-1 < a < 1$ without any changes. It is also worth recalling here that at a local level the thin obstacle problem (1.1) is known to be equivalent to the following nonlocal obstacle problem

$$\min \{(-\text{div}(B(x)\nabla)^s u, u - \psi) = 0, \quad 0 < s < 1. \}$$
The connection between the parameters $a$ in problem (1.1) and $s$ in (1.2) is given by $s = \frac{1-a}{2}$ and the matrix-valued function $B(x)$ is connected to $A(x)$ by formula (2.1) below. With this in mind, it is evident that Theorems 1.1 and 1.2 only presently cover the range $s \in (0, \frac{1}{2}]$ in (1.2), leaving open the remaining interval $\frac{1}{2} < s < 1$.

The paper is organised as follows. In Section 2, we introduce some basic notations and gather some preliminary results which will be subsequently needed in our work. Theorem 2.3 is the main regularity result about odd solutions. We stress that it cannot be extracted from the existing works. Section 3 constitutes one of the essential new contributions of the present paper. Its central results are Theorems 3.7 and 3.11 which provide the above mentioned regularity theorems which are necessary to develop the analysis in the reminder of our work. Our approach relies on a delicate adaptation of the method of Campanato coupled with a new quantitative regularity estimate for the constant coefficient problem studied in [10]. Section 4 is devoted to proving Theorem 4.20. The latter is a new Almgren type monotonicity formula for (1.1) that generalises the one in [24] for the case $a = 0$ and which plays a fundamental role in the rest of the paper. In Section 5 by combining such a monotonicity formula with the a priori estimates established in Section 3 we use a blowup analysis to establish the optimal regularity in Theorem 1.1. Finally, in Section 6 we prove a Weiss type monotonicity formula which, together with the epiperimetric inequality obtained in [21], allows us to obtain the $C^{1,\gamma}$ regularity of the regular part of the free boundary in Theorem 1.2.

In closing, we mention that the theory of thin obstacle problems is by now quite well developed and has several important ramifications. We refer the interested reader to [8, 1, 2, 4, 10, 20, 32, 24, 21, 17, 29, 15, 22, 9, 14, 30, 18, 3, 23, 26, 5, 27, 6] and the references therein.

2. Preliminaries

In this section we introduce the notations and gather some preliminary results which will be needed in our work. We consider the thick space $\mathbb{R}^{n+1}$ with generic variable $X = (x, y)$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, and let $|X| = (|x|^2 + y^2)^{\frac{1}{2}}$. The thin space $\mathbb{R}^n \times \{0\}$ will be routinely identified with $\mathbb{R}^n$. We denote by $\mathbb{B}_r = \{X \in \mathbb{R}^{n+1} | |X| < r\}$ the ball of radius $r$ centred at the origin in the thick space, and we indicate with $\mathbb{B}^+_r = \{X \in \mathbb{R}^{n+1} | |X| < r, y > 0\}$ its upper part. The symbol $\partial^- \mathbb{B}_r$ will indicate the corresponding lower part of $\mathbb{B}_r$. We denote by $\mathcal{S}_r = \partial \mathbb{B}_r = \{X \in \mathbb{R}^{n+1} | |X| = r\}$ the sphere of radius $r$ in the thick space, and we indicate with $\mathcal{S}^+_r = \mathcal{S}_r \cap \{y > 0\}$ its upper part. We use the notation $\mathcal{B}_r = \{(x, 0) \in \mathbb{B}_r | |x| < r\}$ for the unit ball in the thin space $\mathbb{R}^n$. We assume that $X \to A(x) = [a_{ij}(x)]$ is a given symmetric, uniformly elliptic matrix-valued function of the form

\begin{equation}
(2.1) 
a_{ij}(x) = \sum_{i,j=1}^n b_{ij}(x)e_i \otimes e_j + e_{n+1} \otimes e_{n+1},
\end{equation}

where $b_{ij}$’s are Lipschitz continuous and independent of $y$. This assumption will remain in force throughout the rest of the paper and will not be repeated further.

Given a number $a \in (-1, 1)$, and a function $\psi$ in $\mathcal{B}_1$, known as the thin obstacle, we consider the problem of finding a function $U$ in $\mathbb{B}^+_1 \cup B_1$ such that:

\begin{equation}
(2.2) 
\begin{cases}
\text{div}_X(y^a A(x) \nabla_X U) = 0, & \text{in } \mathbb{B}^+_1, \\
\min\{U(x, 0) - \psi(x), -\partial_y U(x, 0)\} = 0 & \text{on } B_1,
\end{cases}
\end{equation}

where we have defined

\begin{equation}
(2.3) 
\partial_y^a U(x, 0) \overset{\text{def}}{=} \lim_{y \to 0^+} y^a \partial_y U(x, y).
\end{equation}
For notational convenience, we will hereafter write \( \text{div} \) and \( \nabla \) for respectively \( \text{div}_X \) and \( \nabla_X \). Also, it will be important for the reader to keep in mind that in the applications of the divergence theorem to the domain \( \mathbb{B}_1^+ \) the orientation of the outer unit normal is opposite to that used in (2.3). In this respect, We explicitly note for subsequent use that if we denote by \( \nu \) the outer unit normal to the boundary \( \partial \mathbb{B}_1^+ = S_1^+ \cup B_1 \) of the upper half-ball, then from (2.1) we have \( A(x)\nu = -\epsilon_n \) in \( B_1 \), and consequently for a function \( U \) we have in \( B_1 \)

\[
(2.4) \quad \langle \nabla U, A(x)\nu \rangle = -\partial_0^n U(x,0).
\]

For later purposes we now consider in the ball \( \mathbb{B}_1 \) the degenerate pde in (2.2), but with a non-zero right-hand side of the form

\[
(2.5) \quad \text{div}(|y|^\alpha A(x)\nabla V) = |y|^\alpha f,
\]

where \( f \in L^\infty(\mathbb{B}_1) \). By a solution to (2.5) we mean a weak solution. For the next result see [33, Theorem 1.2].

**Theorem 2.1.** Let \( V \) be an even in \( y \) solution to (2.5). Then, \( V \in C^{1,\alpha}_{loc}(\mathbb{B}_1) \) for any \( \alpha \in (0,1) \) and the following estimate holds

\[
||V||_{C^{1,\alpha}(\mathbb{B}_1^+)} \leq C \left(||V||_{L^2(\mathbb{B}_1,|y|^\alpha dX)} + ||f||_{L^\infty(\mathbb{B}_1)}\right),
\]

where \( C > 0 \) depends also on \( \alpha \).

Our next result, Theorem 2.3, concerns regularity of odd solutions. In preparation for it we establish the following crucial intermediate result.

**Lemma 2.2.** Let \( V \) be a solution to (2.5) such that \( V \equiv 0 \) on \( \mathbb{B}_1 \cap \{y = 0\} \). Then, given \( \beta < \min\{1 - a, 1\} \), there exists a \( C^{\beta+a} \) function \( b : \mathbb{B}_1^+ \to \mathbb{R} \), and a constant \( C \) in the form

\[
C = \tilde{C}(n, \alpha, \beta, ||A||_{C^{0,1}})(||V||_{L^2(\mathbb{B}_1,|y|^\alpha dX)} + ||f||_{L^\infty(\mathbb{B}_1)}),
\]

such that the following estimate holds for every \( (x_0,0) \in \mathbb{B}_1^+ \cap \{y = 0\} \) and \( r < \frac{1}{4} \),

\[
(2.6) \quad \int_{\mathbb{B}_1^+(x_0,0)} (V(X) - b(x_0)|y|^{1-a})^2 y^\alpha dX \leq Cr^{n+1+a+2(1+\beta)}.
\]

**Proof.** The proof is divided into several steps. We first establish (2.6) when \( (x_0,0) = (0,0) \). Furthermore, by a change of coordinates we can also assume that \( A(0,0) = \mathbb{I} \).

**Step 1:** We begin by making the observation that when \( f \equiv 0 \) and \( A \equiv \mathbb{I} \) the function \( g = y^\alpha V_y \) can be evenly extended across \( \{y = 0\} \) so that it is a solution of

\[
(2.7) \quad \text{div}(|y|^{-\alpha}\nabla g) = 0.
\]

From Theorem 2.1 it follows in particular that up to the thin set \( \{y = 0\} \) we have \( y^\alpha V_y \in C^\gamma \) for all \( 0 < \gamma \leq 1 \), with bounds depending only on \( \int_{\mathbb{B}_1^+} V^2 y^\alpha dX \). From the proof of [12, Theorem 4.1 (2)] (more precisely, by applying [12, Lemma 4.6] in the limit \( k \to \infty \) with \( \beta_0 = \alpha - a \) where \( \alpha \in (0,1) \)), and using \( V(x,\cdot) \equiv 0 \), it follows that given \( \beta_0 < \min\{1 - a, 1\} \), there exists \( C \) depending also on \( \beta_0 \) such that the following decay estimate holds at any arbitrary point \( (x_0,0) \in B_1^+ \times \{y = 0\} \) for all \( r < \frac{1}{4} \),

\[
(2.8) \quad \int_{\mathbb{B}_1^+(x_0,0)} \left(V(X) - \frac{1}{1-a} \partial_y^n V_x(x_0,0)|y|^{1-a}\right)^2 y^\alpha dX \leq Cr^{n+1+a+2(1+\beta_0)}.
\]
**Step 2:** We now make the following claim: given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any \( V \) which solves (2.5), with

\[
||(V)||_{L^2(B^+_1, |y|^a dX)} \leq 1, \quad ||A(x) - I||_{C^{0,1}} \leq \delta, \quad ||f||_{L^\infty} \leq \delta,
\]

there exists \( V_0 \) which solves

\[
\begin{cases}
\text{div}(v^a \nabla V_0) = 0 \text{ in } B^+_1 \\
V_0 = 0 \text{ on } \{y = 0\},
\end{cases}
\]

with \( ||V_0||_{L^2(B^+_1, |y|^a dX)} \leq 1 \), such that

\[
\int_{B^+_1} (V - V_0)^2 y^a dX \leq \varepsilon.
\]

The proof of this claim follows by a standard argument by contradiction as that of Corollary 3.3 in [12].

**Step 3:** Next, we claim that there exist universal \( \delta, \lambda \in (0, 1) \) such that if (2.9) holds, then for some constant \( b_0 \) with universal bounds, for all \( \beta < \min\{1 - a, 1\} \) we have

\[
\int_{B^+_\lambda} (V - b_0 y^{1-a})^2 y^a dX \leq \lambda^{n+1+a+2(1+\beta)}.
\]

To establish (2.12) we first choose some \( \beta_0 \) such that \( \beta < \beta_0 < \min\{1 - a, 1\} \). Then, we note that for a given \( \varepsilon > 0 \), the estimate (2.11) holds for some \( V_0 \) which solves (2.10), provided the conditions in (2.9) are satisfied for an appropriate \( \delta \) depending on \( \varepsilon \). Subsequently, given such a \( \beta_0 \), we have that the estimate (2.8) holds for \( V_0 \). Thus it follows from (2.8) and (2.11) that, for \( b_0 = \frac{\partial_2^X V_0(x_0, 0)}{1-a} \), we have that for any \( \lambda < \frac{1}{2} \) the following estimate holds

\[
\int_{B^+_\lambda((x_0, 0))} (V(X) - b_0 y^{1-a})^2 y^a dX \leq C \lambda^{n+1+a+2(1+\beta_0)} + C\varepsilon.
\]

Since \( \beta_0 > \beta \) we can now choose \( \lambda > 0 \) such that

\[
C \lambda^{n+1+a+2(1+\beta_0)} = \frac{\lambda^{n+1+a+2(1+\beta)}}{2}.
\]

Subsequently, we choose \( \varepsilon > 0 \) such that \( C\varepsilon = \frac{\lambda^{n+1+a+2(1+\beta)}}{2} \) which decides the choice of \( \delta \) and thus (2.12) follows.

**Step 4:** We now show that, under the assumptions (2.9), for \( \delta, \lambda \) as in **Step 3** we have that for every \( k = N \) there exists \( b_k \) such that the following holds

\[
\begin{cases}
\int_{B^+_\lambda} (V - b_k y^{1-a})^2 y^a dX \leq \lambda^{k(n+1+a+2(1+\beta))}, \\
|b_k - b_{k+1}| \leq C \lambda^{k(\beta+a)}.
\end{cases}
\]

We prove (2.14) by induction. We note that the case \( k = 1 \) is proven in **Step 3**. Assume then that (2.14) hold up to some \( k \geq 2 \). We let

\[
\tilde{V}(X) = \frac{V(\lambda^k X) - b_k \lambda^{k(1-a)} y^{1-a}}{\lambda^{k(1+\beta)}}.
\]

Since (2.14) holds for \( k \) it follows by a change of variable that

\[
||(\tilde{V})||_{L^2(B^+_1, |y|^a dX)} \leq 1.
\]
Moreover, $\tilde{V}$ solves
\begin{equation}
\begin{cases}
\text{div}(y^a A_k(x) \nabla \tilde{V}) = y^a f_k \text{ in } \mathbb{B}_1^+,

\tilde{V} = 0 \text{ on } \{y = 0\},
\end{cases}
\tag{2.15}
\end{equation}
where $A_k(x) = A(\lambda^k x)$ and $f_k(X) = \lambda^{k(1-\beta)} f(\lambda^k X)$. By $\lambda^k, \beta < 1$, we see that the conditions in (2.9) are satisfied and thus applying the conclusion of Step 3 to $\tilde{V}$ we infer that there exists some $b_0$ with universal bounds such that
\begin{equation}
\int_{\mathbb{B}_1^+((x_0,0))} (\tilde{V}(X) - b_0 y^{1-a})^2 y^a dX \leq \lambda^{n+1+a+2(1+\beta)}.
\tag{2.16}
\end{equation}
By scaling back to $V$, and letting $b_{k+1} = b_k + \lambda^{k(\beta+a)} b_0$, we see that (2.14) is satisfied for $k + 1$. By induction we infer that (2.14) holds for all $k$.

Step 5: Given $V$ as in the hypothesis of the lemma, and defining $V_{r_0}(X) = V(r_0 X)$, we note that $V_{r_0}$ solves
\[
\text{div}(y^a A(r_0 x) \nabla V_{r_0}) = y^a r_0^2 f(r_0 x).
\]
Therefore, by choosing $r_0$ small enough we can ensure that $||A(r_0 \cdot) - I||_{C^{0,1}} \leq \delta$, where $\delta$ is as in (2.9). Subsequently, by letting
\[
W = \frac{V_{r_0}}{||V_{r_0}||_{L^2(\mathbb{B}_1^+, y^a dX)} + \frac{r_0^2 ||f||_{L^\infty}}{\delta}},
\]
we see that $W$ solves (2.5) and that all the assumptions in (2.9) are satisfied. We can thus let $W$ be our new $V$ and then by applying the conclusion of Step 4, we have that the estimate (2.14) holds for $W$. The estimate (2.6) for $W$ follows from (2.14) by a standard real analysis argument with $b(0) = \lim_{k \to \infty} b_k$. In conclusion, the estimate (2.6) holds for $V$ at $(0,0)$. By translation we finally infer that (2.6) holds for every $(x_0,0) \in \mathbb{B}_1^+ \cap \{y = 0\}$. The $C^{\beta+a}$ Hölder continuity of the function $b$ also follows in a standard way.

We can now prove the relevant regularity result for odd solutions that is needed in this work. We emphasise that in the proof of the next theorem we cannot appeal to [34, Theorem 1.6] because that result requires $f$ to have certain decay near $y = 0$ which does not generically hold in our situation.

**Theorem 2.3.** Let $V$ be an odd in $y$ solution to (2.5). Then, $\nabla_x V(\cdot, y) \in C^\beta(\mathbb{B}_h^+) \setminus \{0\}$ for all $0 < \beta < \min\{1-a,1\}$. Moreover, the following quantitative estimate holds
\begin{equation}
||\nabla_x V||_{C^\beta(\mathbb{B}_h^+)} \leq C(n, a, \beta, ||A||_{C^{0,1}}) \left(||V||_{L^2(\mathbb{B}_1, |y|^a dX)} + ||f||_{L^\infty(\mathbb{B}_1)}\right).
\tag{2.17}
\end{equation}

Furthermore, when $f(x,y) \equiv f(x)$ for $y > 0$ (i.e., when $f$ is independent of $y$ when $y > 0$), we have that $y^a V_y \in C^\alpha(\mathbb{B}_h^+)$ for all $0 < \alpha < \min\{1+a,1\}$ and the following estimate holds
\begin{equation}
||y^a V_y||_{C^\alpha(\mathbb{B}_h^+)} \leq C(n, a, \alpha, ||A||_{C^1}) \left(||V||_{L^2(\mathbb{B}_1, |y|^a dX)} + ||f||_{L^\infty(\mathbb{B}_1)}\right).
\tag{2.18}
\end{equation}

**Proof.** We first note that, given $\beta < \min\{1-a,1\}$, in view of Lemma 2.2 the estimate (2.6) holds for some $b \in C^{\beta+a}(B_{1/2})$. Before proceeding further we also remark that, as
Let now
\begin{equation}
\text{div}(|y|^a A(x) \nabla w) = |y|^a f.
\end{equation}

Let now \(X_1 = (x_1, y_1)\) and \(X_2 = (x_2, y_2)\) be two points in \(B^+_{\frac{r}{2}}\). Without loss of generality we assume that \(y_1 \leq y_2\). There are two cases:

1. \(|X_1 - X_2| \leq \frac{1}{2} y_1\);
2. \(|X_1 - X_2| \geq \frac{1}{2} y_1\).

If (1) occurs, then applying (2.6) with \(r = \frac{y_1}{2}\), it ensues that the following \(L^2\) bound is satisfied by \(w_1(X) \overset{\text{def}}{=} V(X) - b(x_1)y^{1-a}\)

\begin{equation}
\int_{B_{\frac{r}{2}}(X_1)} |w_1|^2 |y|^a \leq C y_1^{n+1+a+2(1+\beta)}.
\end{equation}

We then note that the rescaled function

\begin{equation}
W_0(X) = w_1(x_1 + y_1 x, y_1 y)
\end{equation}
solves in \(B^+_{\frac{r}{2}}((0,1))\) a uniformly elliptic PDE with Lipschitz principal part, bounded drift and scalar term bounded by \(||f||_{L^\infty} y_1^2\). From the classical theory we thus have that the following Hölder estimate holds:

\begin{equation}
|\nabla_x W_0(X) - \nabla_x W_0((0,1))| \leq C \left[ \left( \int_{B^+_{\frac{r}{2}}((0,1))} W_0^2 \, dX \right)^{\frac{1}{2}} + ||f||_{L^\infty} y_1^2 \right] |X - (0,1)|^\beta.
\end{equation}

Keeping in mind that
\[
\nabla_x W_0(X) = y_1 \nabla_x w_1(x_1 + y_1 x, y_1 y) = y_1 \nabla_x V(X),
\]
we obtain from (2.22)

\begin{equation}
|\nabla_x V(X_1) - \nabla_x V(X_2)| = |\nabla_x w_1(X_1) - \nabla_x w_1(X_2)|
\leq C \left[ \left( \frac{1}{y_1^{n+1}} \int_{B^+_{\frac{r}{2}}(X_1)} |w_1|^2 \right)^{\frac{1}{2}} + ||f||_{L^\infty} y_1^2 \right] \frac{|X_1 - X_2|^\beta}{y_1^{1+\beta}}
\leq C \left[ \left( \frac{1}{y_1^{n+a+1}} \int_{B^+_{\frac{r}{2}}(X_1)} |y_1|^a |w_1|^2 \right)^{\frac{1}{2}} + ||f||_{L^\infty} y_1^2 \right] \frac{|X_1 - X_2|^\beta}{y_1^{1+\beta}} \leq C |X_1 - X_2|^\beta.
\end{equation}

Note that in the second inequality in (2.23) we have used that \(y \sim y_1\) in \(B_{\frac{r}{2}}(X_1)\). Also, in the last inequality we have used the decay estimate (2.20).

Suppose now that (2) occurs. We note that, for \(i = 1, 2\), the function \(w_i(X) \overset{\text{def}}{=} V(X) - b(x_i)y^{1-a}\) solves the pde (2.19) in \(B_{\frac{r}{2}}(X_i)\). After rescaling as in (2.21), and using (2.20) (which also holds for \(w_2\) with \(y_1\) replaced by \(y_2\)), from the classical gradient estimates we obtain that the following gradient bound is satisfied

\begin{equation}
|\nabla_x V(X_i)| = |\nabla_x w_i(X_i)| \leq C y_i^\beta.
\end{equation}
The triangle inequality now gives
\[
|y_2| = |X_2 - x_2| \leq |X_2 - X_1| + |X_1 - x_1| + |x_1 - x_2| \\
= |X_2 - X_1| + |y_1| + |x_1 - x_2| \leq 6|X_2 - X_1|.
\]
Using (2.24) and (2.25) we thus find
\[
|\nabla_x V(X_1) - \nabla_x V(X_2)| \leq |\nabla_x V(X_1)| + |\nabla_x V(X_2)| \leq C|X_1 - X_2|^{\beta},
\]
which shows that \( \nabla_x V \in C^\beta(\mathbb{R}^n) \) for all \( \beta < \min\{1 - a, 1\} \). Moreover, the estimate in (2.17) is seen to hold as well.

We now establish (2.18) when \( f(x, y) \equiv f(x) \). Given \( \alpha < \min\{1 + a, 1\} \), we let \( \beta = \alpha - a \). Then, we have that \( \beta < \min\{1 - a, 1\} \). We observe that, since \( f \) is independent of \( y \), for each point \( (x_0, 0) \in B_\frac{1}{2} \times \{y = 0\} \), we have that for \( y > 0 \) the function \( h = y^a V_y - (1 - a)b(x_0) \) solves
\[
\text{div}(y^{-a} A(x) \nabla h) = 0,
\]
where \( b \) is as in Lemma 2.2. Given \( X_0 = (x_0, y_0) \in \mathbb{B}_1^+ \), since \( v = V - y^{1-a}b(x_0) \) solves
\[
\text{div}(y^a A(x) \nabla v) = y^a f,
\]
from the energy estimate applied to \( v \) in \( \mathbb{B}_{\frac{n+1-a+2\beta}{2}}(X_0) \) it follows that the following inequality holds
\[
\int_{\mathbb{B}_{\frac{n+1-a+2\beta}{2}}(X_0)} h^2 y^{-a} \leq \int_{\mathbb{B}_{\frac{n+1-a+2\beta}{2}}(X_0)} |\nabla v|^2 y^a \leq \frac{C}{y_0^n} \int_{\mathbb{B}_{\frac{n+1-a+2\beta}{2}}(X_0)} (v^2 + y^a f^2) y^a.
\]
Using the decay estimate (2.6) we thus obtain the following bound from (2.28)
\[
\int_{\mathbb{B}_{\frac{n+1-a+2\beta}{2}}(X_0)} h^2 y^{-a} \leq C y_0^{\frac{n+1+a+2\beta}{2}},
\]
where \( C \) also depends on \( ||f||_{L^\infty} \). Observing now that \( h \) solves (2.26), by rescaling as in (2.21) we note that the rescaled function solves a uniformly elliptic PDE in \( \mathbb{B}_1((0, 1)) \). We can thus apply the Moser type subsolution estimate to the rescaled function and then by scaling back we obtain the following bound
\[
|h(X_0)| = |y^a V_y(X_0) - (1 - a)b(x_0)| \leq C \left( \frac{1}{y_0^{n+1-a}} \int_{\mathbb{B}_{\frac{n+1-a+2\beta}{2}}(X_0)} h^2 y^{-a} \right)^{\frac{1}{2}} \\
\leq C y_0^{\beta+a} = C y_0^\alpha,
\]
where in the last inequality in (2.30) we have used the estimate (2.29), and also the fact that in \( \mathbb{B}_{\frac{n+1-a+2\beta}{2}}(X_0) \) we have that \( y \sim y_0 \), whereas in the last equality we have used that \( \beta + a = \alpha \).

With the estimate (2.30) in hand we now show that \( y^a V_y \) is in \( C^{0, \alpha} \). Again, let \( X_1 = (x_1, y_1) \) and \( X_2 = (x_2, y_2) \) be two points in \( \mathbb{B}_{\frac{1}{2}}^+ \). Without loss of generality we assume that \( y_1 \leq y_2 \). There are two cases:

(a) \( |X_1 - X_2| \leq \frac{1}{2} \); \\
(b) \( |X_1 - X_2| \geq \frac{1}{2} \).

If (a) occurs, then \( X_2 \in \mathbb{B}_{\frac{1}{2}}^+(X_1) \) and the function \( h_1 = y^a V_y - (1 - a)b(x_1) \) solves an equation of the type (2.26) in \( \mathbb{B}_{\frac{1}{2}}^+(X_1) \). Again by rescaling as in (2.21) we note that the rescaled
function satisfies a uniformly elliptic PDE with Lipschitz coefficients in \( B_{1/2}^+ \). Arguing as in (2.22)-(2.23) we thus obtain
\[
|y^a V_y(X_1) - y^a V_y(X_2)| = |h_1(X_1) - h_1(X_2)| 
\leq \frac{C}{y_1^a} \left( \frac{1}{y_1^{a+1}} \int_{B_{1/2}^+} h^2 y^{-a} \right)^{\frac{1}{2}} |X_1 - X_2|^\alpha \leq C |X_1 - X_2|^\alpha.
\]

We note that in the first inequality in (2.31) we have used that \( y \sim y_1 \) in \( B_{1/2}^+ \). Moreover, in the last inequality in (2.31) we have used the decay estimate in (2.29) with \( y_0 \) replaced by \( y_1 \) and also the fact that \( \alpha = \beta + a \).

Suppose now (b) occurs. In this case we first observe that the estimate (2.30) holds when \( X_0 \) is replaced by either \( X_1 \) or \( X_2 \). More precisely, we have the following inequality
\[
|y^a V_y(X_i) - (1 - a)b(x_i)| \leq C y_1^a, \quad i = 1, 2.
\]

Moreover, from (2.25) we also have that \( |y_2| \leq 6|X_1 - X_2| \). Consequently, we obtain
\[
|y^a V_y(X_1) - y^a V_y(X_2)| \leq |y^a V_y(X_1) - (1 - a)b(x_1)| + |y^a V_y(X_2) - (1 - a)b(x_2)|
+ (1 - a)|b(x_1) - b(x_2)| \leq C y_1^a + C y_2^a + C|x_1 - x_2|^\alpha \leq \tilde{C}|X_1 - X_2|^\alpha.
\]

We mention that in (2.33) we have used (2.32) and the \( C^{0, \alpha} \) estimate for the function \( b \). The estimate (2.18) thus follows.

We also need the following H"older estimate for odd solutions which follows from [34, Theorem 1.6, part 1].

**Theorem 2.4.** Let \( V \) be an odd solution to (2.5) in \( B_{1/2}^+ \), with \( f = 0 \). Then for any \( \alpha < \min\{1 - a, 1\} \) we have that \( V \in C^{0, \alpha}(\overline{B_{1/2}^+}) \) and the following estimate holds
\[
||V||_{C^{0, \alpha}(\overline{B_{1/2}^+})} \leq C||V||_{L^2(B_{1/2}; |y|^a dX)}.
\]

3. \( W^{2,2} \) type estimates and H"older regularity of \( \nabla_x U, y^a U_y \).

As it is by now well-known, see the works [4], [10], [20], [21], two crucial ingredients in the study of the thin obstacle problem (2.2) are: a) the monotonicity of Almgren and Weiss type functionals; and b) the subsequent blow-up analysis. Both a) and b) critically rely on a priori H"older estimates for \( y^a U_y, \nabla_x U \) similar to those for the case \( A = I \). In this section we establish the \( W^{2,2} \) and \( C^{1, \alpha} \) estimates that will be essential to our study of (2.2). The following is our first result. For brevity, we will use the notation \( U_i = U_{x_i}, i = 1, ..., n \), to indicate the tangential partial derivatives.

**Theorem 3.1.** Let \( U \) be a solution to (2.2), with \( \psi \in C^2 \). Then, the following estimate holds
\[
\sum_{i=1}^n \int_{B_{1/2}^+} |\nabla U_i|^2 y^a dX + \int_{B_{1/2}^+} (y^a U_y) y^2 y^{-a} dX \leq C \int_{B_{1/2}^+} (U^2 + |\nabla U|^2 + 1) y^a dX,
\]
where \( C = C(||\psi||_{C^2}, ||A||_{C^{0,1}}, n) > 0 \).

**Proof.** To establish (3.1) we first note that (2.2) is equivalent to the minimisation problem
\[
\min_{V \in K_{\psi, U}} \int_{B_1} \langle A(x) \nabla V, \nabla V \rangle y^a dX,
\]
where
\( (3.3) \quad K_{\psi, U} = \{ V \in W^{1,2}(\mathbb{B}^+, y^a dX) \mid V(x, 0) \geq \psi(x), \ V = U \text{ on } S_1^+ \} .\)

By subtracting off the obstacle \( \psi \) from the solution, we observe that \( (2.2) \) can be reduced to the following non-homogeneous thin obstacle problem with zero obstacle
\[
(3.4) \quad \begin{cases}
\text{div}(y^a A(x) \nabla U) = y^a f, & \text{in } B_1^+,
\min\{U(x, 0), -\partial_y^a U(x, 0)\} = 0 & \text{on } B_1,
\end{cases}
\]
where \( f \in L^\infty(\mathbb{B}^+) \) and is independent of \( y \). To study \( (3.4) \) we now introduce a one-parameter family of functions \( \beta_\varepsilon : \mathbb{R} \to (-\infty, 0] \), such that \( \beta_\varepsilon(s) \equiv 0 \) for \( s \geq 0 \), \( \beta_\varepsilon' \geq 0 \), and \( \beta_\varepsilon(s) = \varepsilon + \frac{s}{\varepsilon} \), for \( s \leq -2\varepsilon^2 \). In a standard way, \( (3.4) \) can now be approximated by solutions to the following penalised problems
\[
(3.5) \quad \begin{cases}
\text{div}(y^a A(x) \nabla U_\varepsilon) = y^a f^\varepsilon, & \text{in } B_1^+,
U_\varepsilon = U & \text{on } S_1^+,
\partial_y^a U_\varepsilon = \beta_\varepsilon(U_\varepsilon),
\end{cases}
\]
where \( f^\varepsilon \) is a smooth mollification of \( f \) (see for instance [32, Chap. 9] for the case \( a = 0 \), or also [5, Sec. 2]). Using \( (2.4) \) we see that the weak form of \( (3.5) \) translates into the equation
\[
(3.6) \quad \int_{\mathbb{B}_1^+} (A(x) \nabla U_\varepsilon, \nabla \zeta) y^a dX + \int_{\mathbb{B}_1^+} \zeta f^\varepsilon y^a = - \int_{B_1} \beta_\varepsilon(U_\varepsilon) \zeta dx,
\]
which is requested to hold for any test function \( \zeta \in W^{1,2}(\mathbb{B}_1^+, y^a dX) \) that is smooth and is independent of \( y \). To study \( (3.5) \) it follows by a standard difference quotient type argument that \( y^a |\nabla U_{x_k}^\varepsilon|^2 \in L^2_{\text{loc}}(\mathbb{B}_1^+) \) for \( k = 1, 2, ..., n \). Henceforth, we let \( U_k^\varepsilon = U_{x_k}^\varepsilon \). If we fix \( k \in \{1, ..., n\} \) and use \( \zeta = \eta_k = \eta_{x_k} \) as a test function in \( (3.6) \), we obtain
\[
(3.7) \quad \int_{\mathbb{B}_1^+} (A(x) \nabla U_\varepsilon, \nabla \eta_k) y^a dX + \int_{\mathbb{B}_1^+} \eta_k f^\varepsilon y^a = - \int_{B_1} \beta_\varepsilon(U_\varepsilon) \eta_k dx.
\]
If we integrate by parts with respect to \( x_k \) in the integrals \( \int_{\mathbb{B}_1^+} (A(x) \nabla U_\varepsilon, \nabla \eta_k) y^a dX \) and \( \int_{B_1} \beta_\varepsilon(U_\varepsilon) \eta_k dx \) in \( (3.7) \), we find
\[
(3.8) \quad \int_{\mathbb{B}_1^+} (A \nabla U_k^\varepsilon, \nabla \eta) y^a + \int_{\mathbb{B}_1^+} (B_{ij})_k^\varepsilon \eta_{ij} y^a - \int f^\varepsilon \eta_k y^a = - \int_{B_1} \beta_\varepsilon(U_\varepsilon) \eta_k y^a.
\]
If we choose
\[
(3.9) \quad \eta = U_k^\varepsilon \tau^2,
\]
where \( \tau \in C_0^\infty(\mathbb{B}_1^+ \cup B_1) \), keeping in mind that \( \beta_\varepsilon' \geq 0 \) we see that the term in the right-hand side of \( (3.8) \) is non-positive. Consequently, using the uniform ellipticity and the bounds on the derivatives of \( A \), Young’s inequality, and by summing over \( k = 1, ..., n \), we obtain the following estimate
\[
(3.10) \quad \sum_{k=1}^n \int_{\mathbb{B}_1^+} |\nabla U_k^\varepsilon|^2 \tau^2 y^a \leq C \int_{\mathbb{B}_1^+} (|\nabla U_k^\varepsilon|^2 + (f^\varepsilon)^2)(\tau^2 + |\nabla \tau|^2)y^a.
\]
It is worth noting here that, although the derivative $\eta_k$ of the function in (3.9) is not a legitimate test function, nevertheless the argument leading to the estimate (3.10) can be justified by first taking incremental quotients of the type

$$\eta_{k,h} = \frac{\eta(X + h\epsilon_k) - \eta(X)}{h},$$

and then letting $h \to 0$. Finally, using the equation (3.5) satisfied by $U^\epsilon$ we have

$$\tag{3.11} (y^a U^\epsilon_y)^2 \leq 2y^{2a} \sum_{i=1}^{n} (U^\epsilon_{x_i})^2 \leq 2y^{2a} \sum_{k=1}^{n} |\nabla_x U_k^\epsilon|^2.$$ Combining the estimates (3.10) and (3.11) we obtain a bound for $\int ((y^a U^\epsilon_y)^2 \tau^2 y^{-a} dX$ which is uniform with respect to $\epsilon$. Finally, letting $\epsilon \to 0$ in such bound and in (3.10), we obtain the desired estimate (3.1).

\[\square\]

**Remark 3.2.** We note that the estimate (3.1) can be localised. Also, taking $U^\epsilon \tau^2$ as a test function in the weak formulation of (3.5), and using $s\beta(s) \geq 0$, we find

$$\tag{3.12} \int |\nabla U^\epsilon|^2 \tau^2 y^a \leq C \int ((U^\epsilon)^2 + (f^\epsilon)^2)(\tau^2 + |\nabla \tau|^2) y^a.$$ Passing to the limit as $\epsilon \to 0$ we deduce that one can get rid of the term involving $\int |\nabla U|^2 y^a dX$ from the right-hand side of the inequality in (3.1). The estimate (3.1) also implies that $\partial_y U_y$ exists as a $L^2$ function on the thin set $\{y = 0\}$, and moreover we have a.e. on $\{y = 0\}$,

$$\tag{3.13} U \partial_y^2 U_y = 0.$$

Our next result is Theorem 3.4 below that provides a quantitative gradient estimate for solutions to the homogeneous thin obstacle problem which plays a critical role in the proof of the subsequent Theorem 3.7, see (3.51) below. Theorem 3.7, in turn, plays a key role in the proof of Theorem 3.11. We begin with the following version of the Poincaré inequality.

**Lemma 3.3.** Let $v \in W^{1,2}(B^+_\rho \setminus B^+_\rho/2; y^a dX)$. Assume that for some $\gamma > 0$ one has

$$\mathcal{H}^n \left( \{ x \in B_\rho \setminus B_{\rho/2} \mid v(x,0) = 0 \} \right) \geq \gamma \rho^n.$$ Then there exists $C = C(n,a,\gamma) > 0$ such that

$$\int_{B^+_\rho \setminus B^+_\rho/2} v^2 y^a \leq C \rho^2 \int_{B^+_\rho \setminus B^+_\rho/2} |\nabla v|^2 y^a.$$ \[Proof.\] By rescaling, it suffices to assume $\rho = 1$. We argue by contradiction and assume that the conclusion of the lemma does not hold. Then there exists a sequence $\{ v_k \} \in W^{1,2}(B^+_1 \setminus B^+_1/2; y^a dX)$ such that $\mathcal{H}^n \left( \{ x \in B_1 \setminus B_1/2 \mid v_k(x,0) = 0 \} \right) \geq \gamma$, $\int_{B^+_1 \setminus B^+_1} v_k^2 y^a = 1$ and $\int_{B^+_1 \setminus B^+_1} |\nabla v_k|^2 y^a \to 0$. Since $w(X) = y^2$ is an $A_2$-weight, using the extension and compactness theorems in [13] it follows that, up to a subsequence, we have $v_k \to v_0$ in $L^2(B^+_1 \setminus B^+_1/2; y^a dX)$, with $\int_{B^+_1 \setminus B^+_1} v_0^2 y^a = 1$. Since $\int_{B^+_1 \setminus B^+_1} |\nabla v_k|^2 y^a \to 0$, we must have $v_0 \equiv c$ with $c \neq 0$. By the compactness of the trace operator established in [31] it follows that, possibly up to a further subsequence, we have

$$\int_{B_1 \setminus B_1/2} |v_k - v_0|^2 \to 0.$$
However, since
\[ \int_{B_1 \setminus B_{\frac{1}{2}}} |v_k - v_0|^2 \geq c^2 \mathcal{H}^n \{v_k(x, 0) = 0\} \geq \gamma c^2, \]
this leads to a contradiction, thus establishing the lemma.

\[ \square \]

**Theorem 3.4.** Assume that \( a \geq 0 \) and let \( V \) be the solution to the Signorini problem (2.2) in \( \mathbb{B}_R^+ \) with \( \psi \equiv 0 \) and \( A = I \). Then there exists \( \alpha > 0 \) such that the following estimate holds for any \( 0 < \rho < R \)

\[ (3.14) \quad \int_{\mathbb{B}_R^+} (y^a V_y - \langle y^a V_y \rangle_\rho)^2 y^{-a} \leq C \left( \frac{\rho}{R} \right)^{n+1-a+2\alpha} \int_{\mathbb{B}_R^+} |\nabla V|^2 y^a. \]

**Proof.** The following is a delicate adaption of an argument of Kinderlehrer in [28], where a similar estimate is proven for the case when \( a = 0 \). By considering \( V_R(X) = V(RX) \), we may assume that \( R = 1 \). Furthermore, with this assumption in place, it suffices to establish (3.14) for \( \rho \leq \frac{1}{2} \). In fact, since we always have

\[ \int_{\mathbb{B}_R^+} (y^a V_y - \langle y^a V_y \rangle_\rho)^2 y^{-a} \leq 2 \int_{\mathbb{B}_R^+} |\nabla V|^2 y^a, \]

if \( \rho > \frac{1}{2} \) the estimate (3.14) is valid in a trivial fashion. We first claim that \( V \) satisfies the following estimates for any \( \rho < \frac{1}{2} \)

\[ (3.15) \quad \int_{\mathbb{B}_R^+} \frac{|\nabla V_x|}{|X|} y^{-a} \leq \frac{C}{\rho^{n+1-a}} \int_{\mathbb{B}_R^+ \setminus \mathbb{B}_R} (V_x)^2 y^a, \quad i = 1, \ldots, n, \]

and

\[ (3.16) \quad \int_{\mathbb{B}_R^+} \frac{|\nabla (y^a V_y)|}{|X|} y^{-a} \leq \frac{C}{\rho^{n+1-a}} \int_{\mathbb{B}_R^+ \setminus \mathbb{B}_R} (V_y)^2 y^a. \]

We start with proving (3.15). As before, we approximate \( V \) with the solutions \( V^\varepsilon \) to the following penalised problems with Neumann condition

\[ (3.17) \quad \begin{cases} \text{div}(y^a \nabla V^\varepsilon) = 0 & \text{in } \mathbb{B}_1^+, \\ \partial^a_y V^\varepsilon = \beta \varepsilon (V^\varepsilon) & \text{on } B_1, \end{cases} \]

whose weak formulation is

\[ (3.18) \quad \int_{\mathbb{B}_1^+} \langle \nabla V^\varepsilon, \nabla \zeta \rangle y^a dX = - \int_{B_1} \beta \varepsilon (V^\varepsilon) \zeta dx, \]

for every \( \zeta \in W^{1,2}(\mathbb{B}_1^+, y^a dX) \) such that \( \zeta \equiv 0 \) on \( \mathbb{S}_1^+ \). Let \( G = \frac{1}{|X|^{n+1-a}} \) and for \( 0 < c < \rho \) consider the truncated functions \( G_c = \min \{ G, \frac{1}{|X|^{n+1-a}} \} \) (at the end we will let \( c \to 0 \)). We notice for subsequent purposes that \( \nabla G_c \equiv 0 \) in \( \mathbb{B}_c \), and that

\[ (3.19) \quad \text{div}(y^a \nabla G_c) = 0 \quad \text{in } \mathbb{B}_1^+ \setminus \mathbb{B}_c, \quad \partial^a_y G_c = 0 \quad \text{in } (\mathbb{B}_1 \setminus \mathbb{B}_c) \cap \{ y = 0 \}. \]

Given \( i \in \{1, \ldots, n\} \), we choose as test function \( \zeta = \eta x_i \) in (3.18), with

\[ \eta = (V^\varepsilon)_{x_i} G_c. \]
where \( \tau \in C_0^\infty(\mathbb{B}_1^+ \cup B_1) \) is such that \( \tau \equiv 1 \) in \( \mathbb{B}_1^+ \), and \( \tau \equiv 0 \) outside \( \mathbb{B}_p^+ \). As in (3.6)-(3.8) above, after substituting such a test function in the weak formulation (3.18), we integrate by parts with respect to \( x_i \) obtaining

\[
(3.20) \quad \int_{\mathbb{B}_1^+} (|\nabla (V^\varepsilon)|_{x_i}^2 G_c \tau^2 + 2\tau (\nabla \tau, \nabla (V^\varepsilon))_{x_i} (V^\varepsilon)_{x_i} G_c + V^\varepsilon_{x_i} \langle \nabla V^\varepsilon, \nabla G_c \rangle \tau^2) y^a = - \int_{B_1} \beta^i (V^\varepsilon) (V^\varepsilon)_{x_i} G_c \tau^2 \leq 0.
\]

Writing the third integral in the left-hand side in (3.20) as \( \frac{1}{2} \int_{\mathbb{B}_1^+} \langle \nabla (V^\varepsilon)_{x_i}^2, \nabla G_c \rangle \tau^2 y^a \), and integrating by parts on the set \( \mathbb{B}_1^+ \setminus \mathbb{B}_c^+ \) (where as we have noted the integrand is supported), we obtain

\[
(3.21) \quad \frac{1}{2} \int_{\mathbb{B}_1^+} \langle \nabla (V^\varepsilon)_{x_i}^2, \nabla G_c \rangle \tau^2 y^a = \frac{n - 1 + a}{2e^{\alpha}} - \alpha^2 \int_{\mathbb{B}_1^+} (V^\varepsilon_{x_i}^2) y^a - \int_{\mathbb{B}_1^+ \setminus \mathbb{B}_c^+} (V^\varepsilon_{x_i}^2) \langle \nabla G_c, \nabla \tau \rangle y^a \geq - \int_{\mathbb{B}_1^+ \setminus \mathbb{B}_c^+} (V^\varepsilon_{x_i}^2) |\nabla G_c| |\nabla \tau| y^a.
\]

Note that in (3.21) we have used both equations in (3.19). Using (3.21) in (3.20), and also using the numerical inequality \( 2\alpha \beta \leq \frac{1}{2} \alpha^2 + 2 \beta^2 \) to estimate

\[
2 \int_{\mathbb{B}_1^+} \tau (\nabla \tau, \nabla (V^\varepsilon))_{x_i} (V^\varepsilon)_{x_i} G_c y^a \leq \frac{1}{2} \int_{\mathbb{B}_1^+} |\nabla (V^\varepsilon)|_{x_i}^2 G_c \tau^2 y^a + 2 \int_{\mathbb{B}_1^+} |(V^\varepsilon)|_{x_i}^2 |\nabla \tau|^2 G_c y^a,
\]

we finally obtain from (3.20) that the following inequality holds,

\[
(3.22) \quad \int_{\mathbb{B}_1^+} |\nabla (V^\varepsilon)|_{x_i}^2 G_c \tau^2 y^a \leq C \int_{\mathbb{B}_1^+} \left( (V^\varepsilon_{x_i})^2 |\nabla G_c| |\nabla \tau| + (V^\varepsilon_{x_i})^2 |\nabla \tau|^2 G_c \right) y^a,
\]

for some universal \( C > 0 \). Using now that \( |\nabla \tau| \leq \frac{C}{|\tau|} \), and also that \( \nabla \tau \) is supported in \( \mathbb{B}_2 \setminus \mathbb{B}_p \), by first letting \( \varepsilon \to 0 \) and then \( \tau \to 0 \), we conclude from (3.22) that (3.15) is valid.

We next prove (3.16). For that, we crucially use that \( w^\varepsilon = y^a (V^\varepsilon) y \) solves the following problem for the conjugate equation with Dirichlet condition

\[
(3.23) \quad \begin{cases}
\text{div}(y^{-a} \nabla w^\varepsilon) = 0, & \text{in } \mathbb{B}_1^+, \\
w^\varepsilon(\cdot, 0) = \beta^a (V^\varepsilon), & \text{on } B_1.
\end{cases}
\]

In this respect we observe that, arguing similarly to the proof of Theorem 3.1, one can show that \( w^\varepsilon \in W^{1,2}(\mathbb{B}_1^+, y^{-a} dX) \). Once this is done, a computation shows that \( w^\varepsilon \) satisfies (3.23).

Let now \( \tilde{G} = \frac{1}{|\tau|^{n-1-\tau}} \), and for \( c > 0 \) also consider \( \tilde{G}_c = \min \left( \tilde{G}, \frac{1}{c^{n-1-\tau}} \right) \) (as before, we will eventually let \( c \to 0 \)). Using the equation in (3.23), we now observe that for any \( \delta > 0 \) the following holds

\[
(3.24) \quad \int_{\mathbb{B}_1^+ \cap \{y > \delta\}} \text{div}(y^{-a} \nabla w^\varepsilon) \eta = 0,
\]
where \( \eta = w^\varepsilon \hat{G}_c \tau^2 \) and, as in the proof of (3.15), \( \tau \in C_0^\infty (\mathbb{B}_1^+ \cup B_1) \) is such that \( \tau \equiv 1 \) in \( \mathbb{B}_1^+ \), and \( \tau \equiv 0 \) outside \( \mathbb{B}_2^+ \). Integrating by parts in (3.24) we obtain

\[
(3.25) \quad \int_{\mathbb{B}_1^+ \cap \{y > \delta\}} \left( |\nabla w|^2 \hat{G}_c \tau^2 + 2w^\varepsilon \langle \nabla w^\varepsilon, \nabla \tau \rangle \hat{G}_c + w^\varepsilon \langle \nabla w^\varepsilon, \nabla \hat{G}_c \rangle \tau^2 \right) y^{-a} = - \int_{\mathbb{B}_1^+ \cap \{y = \delta\}} w^\varepsilon w^\varepsilon \hat{G}_c \tau^2 y^{-a} = \int_{\mathbb{B}_1^+ \cap \{y = \delta\}} \Delta_x V^\varepsilon w^\varepsilon \hat{G}_c \tau^2 .
\]

Note that in the last equality in (3.25) we have used the definition \( w^\varepsilon = y^\alpha (V^\varepsilon)_y \) and the equation (3.17) satisfied by \( V^\varepsilon \). By the continuity of \( \Delta_x V^\varepsilon, w^\varepsilon \) and \( \tau^2 \) up to \( \{y = 0\} \) and Lebesgue dominated convergence theorem, letting \( \delta \to 0 \) we deduce from (3.25)

\[
(3.26) \quad \int_{\mathbb{B}_1^+} \left( |\nabla w|^2 \hat{G}_c \tau^2 + 2w^\varepsilon \langle \nabla w^\varepsilon, \nabla \tau \rangle \hat{G}_c + w^\varepsilon \langle \nabla w^\varepsilon, \nabla \hat{G}_c \rangle \tau^2 \right) y^{-a} = \int_{B_1} \Delta_x V^\varepsilon w^\varepsilon \hat{G}_c \tau^2 = \int_{B_1} \Delta_x V^\varepsilon \beta^a (V^\varepsilon) \hat{G}_c \tau^2 .
\]

Integrating by parts in the integral in the right-hand side of the latter equality we obtain

\[
(3.27) \quad \int_{\mathbb{B}_1^+} \left( |\nabla w|^2 \hat{G}_c \tau^2 + 2w^\varepsilon \langle \nabla w^\varepsilon, \nabla \tau \rangle \hat{G}_c + w^\varepsilon \langle \nabla w^\varepsilon, \nabla \hat{G}_c \rangle \tau^2 \right) y^{-a} \leq 0,
\]

where we have let \( w = y^\alpha V_y \). The third integral in the left-hand side of (3.27) can be handled similarly to (3.21) using the fact that \( \text{div}(y^{-a} \nabla \hat{G}_c) = 0 \) in \( \mathbb{B}_1^+ \setminus \mathbb{B}_c^+ \). Arguing as in (3.21), (3.22) we thus obtain for a universal \( C > 0 \)

\[
\int_{\mathbb{B}_1^+} \left( |\nabla w|^2 \hat{G}_c \tau^2 y^{-a} \right) \leq C \int \left( w^2 |\nabla \hat{G}_c| |\nabla \tau| + w^2 |\nabla \tau|^2 \hat{G}_c \right) y^{-a},
\]

from which (3.16) follows by letting \( c \to 0 \). We now introduce a notation for the quantities in the left-hand sides of (3.15) and (3.16),

\[
I_i(\rho) = \int_{\mathbb{B}_c^+} \frac{|\nabla V_{x_i}|^2}{|x|^{n-1-a}} y^a, \quad i = 1, \ldots, n, \quad \quad I_y(\rho) = \int_{\mathbb{B}_c^+} \frac{|\nabla (y^2 V_y)|^2}{|x|^{n-1-a}} y^{-a}.
\]

For later use we observe that there exists a universal constant \( C > 0 \) such that

\[
(3.28) \quad I_y(\rho) \leq C \sum I_i(\rho)
\]
For this it suffices to observe that the equation $\text{div}_X (y^a \nabla_X V) = 0$ satisfied by $V$ in $\mathbb{B}^+_1$ implies

$$y^{-a} |\nabla (y^a V_y)|^2 \leq C \sum_{i=1}^{n} y^a |\nabla V_{x_i}|^2,$$

and also that $a \geq 0$ gives $\frac{1}{|X|^{\frac{n}{n-1}-\frac{a}{2}}} \leq \frac{1}{|X|_n^{\frac{n}{n-1}-\frac{a}{2}}}$ in $\mathbb{B}^+_1$. It is clear that (3.28) immediately follows from these observations and the definitions of $I_i(\rho)$ and $I_y(\rho)$.

Now, since $\partial_y^\alpha V \nabla_x V \equiv 0$ in $B_1$, given any $\rho \in (0, 1/4)$ we have

$$\mathcal{H}^n(B_{2\rho} \setminus B_{\rho}) = \mathcal{H}^n(\{x \in B_{2\rho} \setminus B_{\rho} | \partial_y^\alpha V(x, 0) \nabla_x V(x, 0) = 0\}$$

$$\leq \mathcal{H}^n(\{x \in B_{2\rho} \setminus B_{\rho} | \nabla_x V(x, 0) = 0\}) + \mathcal{H}^n(\{x \in B_{2\rho} \setminus B_{\rho} | \frac{\partial_y^\alpha V(x, 0)}{2}\}).$$

Therefore, either

(a) $\mathcal{H}^n(\{x \in B_{2\rho} \setminus B_{\rho} | \nabla_x V(x, 0) = 0\}) \geq \frac{1}{2} \mathcal{H}^n(B_{2\rho} \setminus B_{\rho}),$

must hold, or

(b) $\mathcal{H}^n(\{x \in B_{2\rho} \setminus B_{\rho} | \nabla_x V(x, 0) = 0\}) \geq \frac{1}{2} \mathcal{H}^n(B_{2\rho} \setminus B_{\rho}).$

If (b) occurs then by Lemma 3.3, applied to $y^a V_y$ in $\mathbb{B}^+_1 \setminus \mathbb{B}_\rho^+$, we can bound from above the integral in the right-hand side in (3.16) by $C(I_y(2\rho) - I_y(\rho))$. Here, we have used the fact that on the set $\mathbb{B}^+_2 \setminus \mathbb{B}_\rho$ we have $\frac{1}{|X|_n^{\frac{n}{n-1}-\frac{1}{2}}} \sim \frac{1}{\rho^{n-1}}$. On the other hand, if (a) occurs then applying Lemma 3.3 to $\nabla_x U$ we obtain that for all $i=1,\ldots,n$ the integral in the right-hand side of (3.15) can be bounded from above by $C(I_i(2\rho) - I_i(\rho))$. In conclusion, we have shown that for $\rho \in (0, 1/4)$ either $I_y(\rho) \leq C(I_y(2\rho) - I_y(\rho))$, or $I_i(\rho) \leq C(I_i(2\rho) - I_i(\rho))$ for $i = 1,\ldots,n$. Equivalently, either

$$I_i(\rho) \leq \frac{C}{C+1}I_i(2\rho) \text{ for } i = 1,\ldots,n, \quad \text{or} \quad I_y(\rho) \leq \frac{C}{C+1}I_y(2\rho).$$

Iterating these inequalities on a dyadic sequence of radii $\rho_k = 2^{-k}$ we deduce that with $\gamma = \frac{1}{2} \log_2 (\frac{C+1}{C})$ and for any $\rho \in (0, 1/4)$, either

(3.29) $$I_y(\rho) \leq C\rho^\gamma I_y(1/4)$$

is true, or

(3.30) $$I_i(\rho) \leq C\rho^\gamma I_i(1/4), \text{ for } i = 1,\ldots,n.$$  

Suppose that (3.30) hold. In such case we obtain from (3.28)

(3.31) $$I_y(\rho) \leq C \sum_{i=1}^{n} I_i(\rho) \leq C\rho^\gamma \sum_{i=1}^{n} I_i(1/4) \leq C\rho^\gamma \int_{\mathbb{B}^+_1} |\nabla V|^2 y^a,$$

where in the last inequality we have used the energy estimate in (3.15) with the choice $\rho = 1/4$. If instead (3.29) holds, then we have

$$I_y(\rho) \leq C\rho^\gamma I_y(1/4) \leq C\rho^\gamma \int_{\mathbb{B}^+_1} |\nabla V|^2 y^a,$$

where in the second inequality we have applied (3.16) with $\rho = 1/4$. In both cases (3.31) holds and, since $n-1-a \geq 1-a \geq 0$, this implies in particular that

(3.32) $$\frac{1}{\rho^{n-1-a}} \int_{\mathbb{B}^+_1} |\nabla (y^a V_y)|^2 y^{-a} \leq C\rho^\gamma \int_{\mathbb{B}^+_1} |\nabla V|^2 y^a.$$  

Combining (3.32) with the weighted Poincaré inequality in [16] the desired estimate (3.14) now follows with $a = \frac{n}{2}$ and $R = 1$.

□
Remark 3.5. We stress that by translation the estimate (3.14) continues to hold for balls centred at any point of the thin set \( B_1 \). We also note that, although from [10] one knows that \( y^aV_y \in C^{1-\nu} \) up to the thin set \( B_1 \), yet the quantitative estimate (3.14) does not seem to follow from the results in that paper.

We next recall the following real analysis lemma due to Campanato and Morrey, see [25, Lemma 2.1 on p. 86]. It will be needed in the proof of Theorem 3.7.

Lemma 3.6. Let \( \varphi : [0, \infty) \to [0, \infty) \) be such that \( s \leq t \implies \varphi(s) \leq \varphi(t) \). Suppose that for every \( 0 < \rho \leq R < R_0 \) one has

\[
\varphi(\rho) \leq A \left( \left( \frac{\rho}{R} \right)^\gamma + \varepsilon \right) \varphi(R) + BR^\beta,
\]

where \( A, \alpha, \beta, \varepsilon \geq 0 \), with \( \beta < \gamma \). There exists \( \varepsilon_0 = \varepsilon_0(A, \gamma, \beta) \) such that if \( \varepsilon < \varepsilon_0 \) one has for \( 0 < \rho \leq R < R_0 \)

\[
\varphi(\rho) \leq C \left( \left( \frac{\rho}{R} \right)^\beta \varphi(R) + B\rho^\beta \right),
\]

where \( C = C(A, \gamma, \beta) \geq 0 \).

The next result asserts the Hölder regularity of \( U \) and \( y^aU_y \) up to the thin set \( \{ y = 0 \} \).

Theorem 3.7. Let \( U \) be a solution to (3.4) with \( a \geq 0 \). Then there exists \( \beta > 0 \) such that \( U, y^a U_y \in C^{\beta}(\mathbb{B}_R^+) \).

Proof. Without loss of generality we assume again that \( A(0, 0) = I \). The proof is divided into three steps.

Step 1: We first show that for any \( 0 < \sigma < 1 \) there exists \( R_\sigma > 0 \) such the following estimate holds for all \( 0 < \rho < R_\sigma \)

\[
\int_{\mathbb{B}_\rho^+} |\nabla U|^2 y^a \leq C \rho^{\alpha-1-\alpha+2\sigma} \int_{\mathbb{B}_R^+} (|\nabla U|^2 + 1) y^a,
\]

where \( C = C(n, a, ||f||_{L^\infty}) > 0 \). Let \( 0 < R < 1 \) to be fixed sufficiently small subsequently, and denote by \( V \) the minimiser of the energy

\[
\int_{\mathbb{B}_R^+} |\nabla W|^2 y^a
\]

over all \( W \geq 0 \) at \( \{ y = 0 \} \) such that \( W = U \) on \( \mathbb{S}_R^+ \). From the fact that \( V \) minimises (3.36), we obtain

\[
\int_{\mathbb{B}_R^+} (\langle \nabla V, \nabla (V - U) \rangle) y^a \leq 0.
\]

Since \( U \) minimises the energy corresponding to the Euler-Lagrange equation (3.4), we find

\[
\int_{\mathbb{B}_R^+} (\langle A(x) \nabla U, \nabla (V - U) \rangle + f(V - U)) y^a \geq 0.
\]

This inequality can be rewritten as follows

\[
\int_{\mathbb{B}_R^+} (\langle (A(x) - I) \nabla U, \nabla (V - U) \rangle) y^a + \int_{\mathbb{B}_R^+} (\langle \nabla U, \nabla (V - U) \rangle + f(V - U)) y^a \geq 0.
\]
From (3.38) we trivially obtain

\[ \int_{\mathbb{B}_R^+} (\langle A(x) - I \rangle \nabla U, \nabla (V - U) \rangle) y^a + \int_{\mathbb{B}_R^+} (\langle \nabla V, \nabla (V - U) \rangle + f(V - U) \rangle) y^a \]

\[ \geq \int_{\mathbb{B}_R^+} |\nabla (V - U)|^2 y^a. \]

Using (3.37) in (3.39), and also the fact that the Lipschitz continuity of the matrix \( A \) implies

\[ \| A - I \|_{L^\infty(\mathbb{B}_R^+)} \leq CR, \]

we find

\[ \int_{\mathbb{B}_R^+} |\nabla (V - U)|^2 y^a \leq CR \int_{\mathbb{B}_R^+} (\langle \nabla V, \nabla (V - U) \rangle) y^a + \int_{\mathbb{B}_R^+} f(V - U) y^a. \]

Using Young’s inequality, for every \( \delta > 0 \) the right-hand side in (3.40) can be bounded from the above in the following way

\[ CR \int_{\mathbb{B}_R^+} (\langle \nabla V, \nabla (V - U) \rangle) y^a + \int_{\mathbb{B}_R^+} f(V - U) y^a \]

\[ \leq C \delta \int_{\mathbb{B}_R^+} |\nabla (V - U)|^2 y^a + \frac{CR^2}{\delta} \int_{\mathbb{B}_R^+} |\nabla V|^2 y^a + \frac{C \delta}{R^2} \int_{\mathbb{B}_R^+} (V - U)^2 y^a + \frac{CR^2}{\delta} \int_{\mathbb{B}_R^+} f^2 y^a, \]

where \( C > 0 \) is universal. In the last inequality in (3.41) we have used \( \int_{\mathbb{B}_R^+} |\nabla V|^2 y^a \leq \int_{\mathbb{B}_R^+} |\nabla U|^2 y^a \), which follows from the fact that \( V \) minimises the Dirichlet energy in the class of competitors containing \( U \). Since \( V = U \) on \( \{|x| = R\} \), applying to \( V - U \) the Poincaré inequality in [16], we can estimate

\[ \frac{C \delta}{R^2} \int_{\mathbb{B}_R^+} (V - U)^2 y^a \leq C' \delta \int_{\mathbb{B}_R^+} |\nabla (V - U)|^2 y^a, \]

where \( C' > 0 \) is another universal constant. Using (3.42) in (3.41), and finally choosing \( \delta \) small enough so that the integral \( (C' \delta + C \delta) \int |\nabla (V - U)|^2 y^a \) can be absorbed in the left-hand side of (3.40), we can finally assert that the following inequality holds for a new \( C > 0 \)

\[ \int_{\mathbb{B}_R^+} |\nabla (V - U)|^2 y^a \leq CR^2 \int_{\mathbb{B}_R^+} (|\nabla U|^2 + f^2) y^a. \]

Observe now that for any \( 0 < \rho < R \) we have trivially

\[ \int_{\mathbb{B}_\rho^+} |\nabla U|^2 y^a \leq C \int_{\mathbb{B}_\rho^+} (|\nabla (V - U)|^2 + |\nabla V|^2) y^a. \]

It is at this point that we make critical use of the assumption \( a \geq 0 \) as this limitation is present in [26, Lemma 3.3], which we now use, obtaining

\[ \int_{\mathbb{B}_R^+} |\nabla V|^2 y^a \leq \left( \frac{\rho}{R} \right)^{n+1-a} \int_{\mathbb{B}_R^+} |\nabla V|^2 y^a \leq \left( \frac{\rho}{R} \right)^{n+1-a} \int_{\mathbb{B}_R^+} |\nabla U|^2 y^a. \]
Inserting (3.43), (3.45) in (3.44), and using the fact that \( f \in L^\infty \), we find

\[
\int_{B^+_\rho} |\nabla U|^2 y^a \leq C \left( \frac{\rho}{R} \right)^{n+1-a} \int_{B^+_R} |\nabla U|^2 y^a + C R^2 \int_{B^+_R} (|\nabla U|^2 + f^2) y^a
\]

\[
\leq A \left( \frac{\rho}{R} \right)^{n+1-a} \int_{B^+_R} |\nabla U|^2 y^a + A R^2 \int_{B^+_R} |\nabla U|^2 y^a + BR^{n+3+a},
\]

where \( A > 0 \) is universal and \( B > 0 \) is a universal constant that also depends of the \( L^\infty \) norm of \( f \). Fix now \( \sigma \in (0,1) \). Since we can assume without restriction that \( R < 1 \), and since \( n + 3 + a > n - 1 - a + 2\sigma \), it is clear that (3.46) trivially implies the following inequality

\[
\int_{B^+_\rho} |\nabla U|^2 y^a \leq A \left( \frac{\rho}{R} \right)^{n+1-a} \int_{B^+_R} |\nabla U|^2 y^a + AR^2 \int_{B^+_R} |\nabla U|^2 y^a + BR^{n+1-a+2\sigma},
\]

We now define

\[
\varphi(\rho) = \int_{B^+_\rho} |\nabla U|^2 y^a.
\]

Keeping in mind that \( R \leq \delta \), we can express (3.47) in the following form:

\[
\varphi(\rho) \leq A \left( \left( \frac{\rho}{R} \right)^\gamma + \varepsilon \right) \varphi(R) + BR^\beta
\]

where \( \varepsilon = R^2 \),

\[
\gamma = n + 1 - a, \quad \text{and} \quad \beta = n - 1 - a + 2\sigma.
\]

Noting that \( 0 < \beta < \gamma \), by Lemma 3.6 we infer that, given \( \sigma \in (0,1) \), there exists \( R_\sigma = R_\sigma(A,n,a) > 0 \) such that for every \( 0 < \rho \leq R \leq R_\sigma \) one has

\[
\int_{B^+_\rho} |\nabla U|^2 y^a \leq C \left[ \left( \frac{\rho}{R} \right)^{n+1-a+2\sigma} \int_{B^+_R} |\nabla U|^2 y^a + BR^{n+1-a+2\sigma} \right].
\]

Now by letting \( R \to R_\sigma \) we conclude from (3.48) that (3.35) holds.

**Step 2:** We next prove that there exists \( \beta > 0 \) such that for all \( \rho \) small enough one has

\[
\int_{B^+_\rho} (y^a U_y - \langle y^a U_y \rangle_\rho)^2 y^{-a} \leq C \rho^{n+1-a+2\beta},
\]

where for a function \( f \) we have indicated with \( \langle f \rangle_\rho = \frac{1}{\int_{B^+_\rho} y^{-a} dX} \int_{B^+_\rho} f(X) y^{-a} dX \) the integral average of \( f \) in \( B^+_\rho \) with respect to the measure \( y^{-a} dX \). To establish (3.49) we apply (3.35) with \( \rho = R \) sufficiently small. Again, let \( V \) be the minimiser to (3.36) corresponding to this choice of \( R \). We note that in view of the \( W^{2,2} \) type estimates in Theorem 3.1, \( \partial_y^2 U \) exists as a \( L^2 \) function at \( y = 0 \). The triangle inequality now gives for any \( 0 < \rho < R \),

\[
\int_{B^+_\rho} (y^a U_y - \langle y^a U_y \rangle_\rho)^2 y^{-a} \leq C \left( \int_{B^+_\rho} (y^a U_y - y^a V_y)^2 y^{-a} + \int_{B^+_\rho} (y^a V_y - \langle y^a V_y \rangle_\rho)^2 y^{-a} + \int_{B^+_\rho} \langle \langle y^a V_y \rangle_\rho \rangle - \langle y^a U_y \rangle_\rho \rangle_\rho \rangle_\rho \right)
\]

\[
= C((I) + (II) + (III)),
\]

where we have slightly abused the notation in writing for instance \( \int_{B^+_\rho} \langle y^a V_y \rangle_\rho y^{-a} \), instead of the more rigorous \( \int_{B^+_\rho} \langle (\cdot)^a V_y \rangle_\rho y^{-a} \). We trivially estimate

\[
(I) \leq \int_{B^+_\rho} |\nabla (U - V)|^2 y^a.
\]
Jensen inequality and the fact that \( \int_{B^+} y^{-a} \, dX = C(n, a) \rho^{n+1-a} \) give

\[
(III) \leq C \int_{B^+} (U_y - V_y)^2 y^a \leq C \int_{B^+} |\nabla(U - V)|^2 y^a.
\]

Combining estimates, we find

\[
(3.50) \quad \int_{B^+} (y^a U_y - \langle y^a U_y \rangle)^2 y^{-a} \leq C \left( \int_{B^+} (y^a V_y - \langle y^a V_y \rangle) \right)^2 y^{-a} + \int_{B^+} |\nabla U - \nabla V|^2 y^a.
\]

To control the first integral in the right-hand side of (3.50) we invoke Theorem 3.4 that gives for some \( \alpha > 0 \)

\[
(3.51) \quad \int_{B^+} (y^a V_y - \langle y^a V_y \rangle)^2 y^{-a} \leq C \left( \frac{\rho}{R} \right)^{n+1-a+2\alpha} \int_{B^+} |\nabla V|^2 y^a.
\]

Substituting (3.51) in (3.50) we obtain

\[
(3.52) \quad \int_{B^+} (y^a U_y - \langle y^a U_y \rangle)^2 y^{-a} \leq C \left( \frac{\rho}{R} \right)^{n+1-a+2\alpha} \int_{B^+} |\nabla V|^2 y^a + C \int_{B^+} |\nabla U - \nabla V|^2 y^a
\]

\[
\leq C \left( \frac{\rho}{R} \right)^{n+1-a+2\alpha} \int_{B^+} |\nabla U|^2 y^a + C \int_{B^+} (|\nabla U|^2 + f^2) y^a,
\]

where in the second inequality we have used (3.43) and \( \int_{B^+} |\nabla V|^2 y^a \leq \int_{B^+} |\nabla U|^2 y^a \), which follows from the fact that \( V \) minimises the Dirichlet energy in the class of competitors containing \( U \). Finally, using (3.35) in (3.52) (with \( \rho \) replaced by \( R \)) we deduce that for any \( 0 < \sigma < 1 \) there exists \( C_\sigma > 0 \) (depending also on the \( W^{1,2}(B^+_1, y^a dX) \) norm of \( U \) and the \( L^\infty(B^+_1) \) norm of \( f \)) such that

\[
(3.53) \quad \int_{B^+} (y^a U_y - \langle y^a U_y \rangle)^2 y^{-a} \leq C_\sigma \left( \left( \frac{\rho}{R} \right)^{n+1-a+2\alpha} R^{n-1-a+2\sigma} + R^{n+1-a+2\sigma} \right).
\]

At this point, we fix \( 0 < \varepsilon < \alpha^{-1} \). Having done this, we now let \( \sigma = 1 - \varepsilon \alpha \), so that \( 0 < \sigma < 1 \), and we finally fix a number \( \tau \) such that

\[
(3.54) \quad \frac{n + 1 - a + 1 - \sigma}{n + 1 - a + 2\sigma} < \tau < \frac{2\alpha - (1 - \sigma)}{2(1 - \sigma + \alpha)}.
\]

It is clear that \( 0 < \tau < 1 \). Therefore, if we let \( R = \rho^\tau \) then \( 0 < \rho < R \) and we obtain from (3.53)

\[
(3.55) \quad \int_{B^+} (y^a U_y - \langle y^a U_y \rangle)^2 y^{-a} \leq C(\rho^{n+1-a+2\alpha} - 2\tau(1-\sigma)) + \rho^{(n+1-a+2\sigma)}).
\]

The choice (3.54) allows to conclude that

\[
\rho^{2\alpha - 2\tau(1-\sigma)} \leq \rho^{1-\sigma}, \quad \text{and} \quad \rho^{(n+1-a+2\sigma)} \leq \rho^{n+1-a+1-\sigma}.
\]

If therefore we set \( \beta = 1 - \sigma \), then we can reformulate (3.55) as follows

\[
\int_{B^+} (y^a U_y - \langle y^a U_y \rangle)^2 y^{-a} \leq C \rho^{n+1-a+2\beta},
\]

which establishes the decay estimate (3.49).

Step 3: Using (3.49) from Step 2 we finally show that \( y^a U_y \) is \( C^\beta \) Hölder continuous up to the thin set \( B_1 \). We first note that, by translation, the estimate (3.49) continues to hold for
balls centred at any point in $B_1$. Similarly to the argument in the proof of (3.16), we now observe that $w = y^a U_y$ solves in $\mathbb{B}_1^+$ the following conjugate equation
\[
\text{div}(y^{-a}A(x)\nabla w) = 0.
\]
By applying the Campanato type result in [26, Theorem A.1] from (3.49) we infer the existence of $h(x) \in C^\beta(B_1)$ such that at every $(x,0) \in B_1$ and $0 \leq r \leq 1/2$ one has
\[
\int_{\mathbb{B}_1^+(x,0)} (w - h(x))^2 y^{-a} \leq Cr^{n+1-a+2\beta}.
\]
We now observe that for any point $X = (x_1, y_1) \in \mathbb{B}_{1/4}^+$ we have $\mathbb{B}_{y_1/2}(X_1) \subset \mathbb{B}_1^+$. The inclusion $\mathbb{B}_{y_1/2}(X_1) \subset \mathbb{B}_1$ is a trivial consequence of the triangle inequality, whereas the inclusion $\mathbb{B}_{y_1/2}(X_1) \subset \{y > 0\}$ follows from the fact that if $X = (x, y) \in \mathbb{B}_{y_1/2}(X_1)$, then we have $|y - y_1| \leq \frac{y_1}{2}$, and therefore in particular $y \geq \frac{y_1}{2} > 0$. Let now $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ be two arbitrary points in $\mathbb{B}_{1/4}^+$. Without loss of generality we may assume that $y_1 \leq y_2$. There are two cases:
1. $|X_1 - X_2| = \frac{y_1}{4}$;
2. $|X_1 - X_2| \geq \frac{y_1}{4}$.
Suppose (1) occurs. In this case, we make use of the fact that $\tilde{w}_1 = w - h(x_1)$ solves in $\mathbb{B}_1^+$ the equation
\[
\text{div}(y^{-a}A(x)\nabla \tilde{w}_1) = 0.
\]
Since the triangle inequality gives $\mathbb{B}_{y_1/2}(X_1) \subset \mathbb{B}_{y_1/4}^+(x_1,0)$, and since $\frac{3}{2}y_1 \leq \frac{3}{2} \leq \frac{4}{3}$, we can apply (3.56) to infer
\[
\int_{\mathbb{B}_{y_1/2}(X_1)} \tilde{w}_1^2 y^{-a} \leq y_1^{n+1-a+2\beta}.
\]
Next, we note that the rescaled function
\[
W_1(x,y) = \tilde{w}_1(x_1 + y_1 x, y_1 y)
\]
solves in $\mathbb{B}_{\frac{3}{4}}^+(0,1)$ the differential equation
\[
\text{tr}(B(x)\nabla_2^2 W_1) + \partial_{yy} W_1 - \frac{a}{y} \partial_y W_1 = 0,
\]
where $\nabla_2^2 W_1$ denotes the Hessian of $W_1$ and the Lipschitz matrix-valued function $B(x) = [b_{ij}(x)]_{i,j=1}^n$ is as in (2.1). Since $\mathbb{B}_{\frac{3}{4}}^+(0,1) \subset \{X = (x,y) \in \mathbb{R}^{n+1} \mid \frac{1}{4} < y < \frac{3}{2}\}$, it is clear (3.60) is a uniformly elliptic pde with Lipschitz principal part and bounded drift. From the classical theory we infer that for $X = (x,y) \in \mathbb{B}_{\frac{3}{4}}^+(0,1)$ the following Hölder estimate holds for $W_1$
\[
|W_1(X) - W_1(0,1)| \leq C|X - (0,1)|^\beta \left( \int_{\mathbb{B}_{\frac{3}{4}}^+(0,1)} W_1^2 \right)^{\frac{1}{\gamma}}.
\]
We note that $W_1(0,1) = \tilde{w}_1(X_1)$, and elementary considerations show that if $X = (x,y) \in \mathbb{B}_{\frac{3}{4}}^+(0,1)$, then $X_2 = (x_2, y_2) = (x_1 + y_1 x, y_1 y) \in \mathbb{B}_{\frac{3}{4}}(X_1)$. Rewriting the above inequality
for $W_1$ in terms of $\tilde{w}_1$ we obtain
\[
|w(X_1) - w(X_2)| = |\tilde{w}_1(X_1) - \tilde{w}_1(X_2)| \leq \left( \frac{C}{y_1^{n+1-a}} \int_{B_{y_1}(x_1)} \tilde{w}_1^2 \, y^{-a} \right)^{\frac{1}{2}} \frac{|X_1 - X_2|^\beta}{y_1^\beta} \leq C|X_1 - X_2|^\beta.
\]

In the second inequality above we have used the fact that in the ball $B_{y_1}(X_1)$ one has $y \sim y_1$, whereas in the last inequality we used the estimate (3.58).

If instead (2) occurs, then letting $\tilde{w}_i = w - h(x_i)$ for $i = 1, 2$, again we note that $W_i = \tilde{w}_i(x_i + y_i x, y_i y)$ solves a uniformly elliptic pde of the type (3.60) in $B_{\frac{3}{2}}((0, 1))$. By the classical elliptic estimates applied to $W_i$, and rewritten in terms of $\tilde{w}_i$, we obtain
\[
|\tilde{w}_i(X_i)| \leq \left( \frac{C}{y_i^{n+1-a}} \int_{B_{y_i}(x_i)} \tilde{w}_i^2 \, y^{-a} \right)^{\frac{1}{2}} \leq C y_i^\beta.
\]

In (3.61) we have used the fact that the decay estimate (3.58) also holds for $\tilde{w}_2$ (when $y_1$ is replaced by $y_2$). Also, by an application of triangle inequality we obtain from (2)
\[
y_2 = |X_2 - (x_2, 0)| \leq |X_2 - X_1| + |X_1 - (x_1, 0)| + |x_1 - x_2| \leq 6|X_2 - X_1|.
\]

Using (3.61), (3.62) and the $C^\beta$ Hölder continuity of $h$, we conclude that the following inequality holds,
\[
|w(X_1) - w(X_2)| \leq |w(X_1) - h(x_1)| + |h(x_1) - h(x_2)| + |w(X_2) - h(x_2)| 
\leq C y_1^\beta + C y_2^\beta + C|x_1 - x_2|^\beta \leq C|X_1 - X_2|^\beta.
\]

This shows that $w = y^\beta U_y \in C^\beta(B_{\frac{3}{2}})$, which, in particular, implies that $\partial_y^\alpha U \in L^\infty(B_{\frac{1}{2}})$. The fact that $U \in C^\beta(B_{\frac{1}{2}})$ for some $\beta' > 0$ now follows by a Moser type iteration argument as in [35], see also the proof of Theorem 5.1 in [7].

Before proceeding we introduce the extended free boundary of $U$, 
\[
\Gamma^*(U) = \{(x, 0) \in B_1 \mid U(x, 0) = \partial_y^\alpha U(x, 0) = 0\},
\]
and note that $\Gamma(U) \subset \Gamma^*(U)$. We next show that at every point of $\Gamma^*(U)$ the solution $U$ separates from the obstacle at a rate $1 + \sigma$ for every $\sigma < \frac{1-a}{2}$. This is accomplished by a compactness argument.

**Lemma 3.8.** For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $U$ solves (3.4) in $\mathbb{B}_1^+$, $0 \in \Gamma^*(U)$, $||U||_{L^\infty(\mathbb{B}_1^+)} \leq 1$ and $||A - I||_{C^{0,1}}$, $||f||_{L^\infty} \leq \delta$, then one can find $V$ that solves (3.4) with $A = I$ and $f = 0$, with $||V||_{L^\infty(\mathbb{B}_{1/4}^+)} \leq 1$, $0 \in \Gamma^*(V)$, and such that
\[
||U - V||_{L^\infty(\mathbb{B}_{3/4}^+)} \leq \varepsilon.
\]
Proof. We argue by contradiction and assume the existence of $\varepsilon_0 > 0$ and of a sequence of triplets $(U^k, A^k, f^k)_{k=1}^\infty$ such that for every $k \in \mathbb{N}$ the function $U^k$ solves

$$\begin{cases}
\text{div}(g^o A^k(x) \nabla U_k) = g^o f_k, & \text{in } \mathbb{B}_1^+,
\min\{U_k(x,0), -\partial_0^o U_k(x,0)\} = 0 & \text{on } B_1,
\end{cases} \tag{3.65}$$

with

$$||U^k||_{L^\infty(\mathbb{B}_1^+)} \leq 1, ||A^k - I||_{C^{0,1}} \leq \frac{1}{k}, ||f^k||_{L^\infty} \leq \frac{1}{k}, 0 \in \Gamma^*(U^k),$$

and such that

$$||U^k - V||_{L^\infty(\mathbb{B}_1^+)} > \varepsilon_0 \tag{3.66}$$

for every $V$ that solves (3.4) with $A = I, f = 0$, and such that $||V||_{L^\infty(\mathbb{B}_1^+)} \leq 1$. By the H"older estimates up to the thin set of $V$ and $\partial_0^o V$ in Theorem 3.7 we also have $0 \in \Gamma^*(V)$. From the uniform $W^{2,2}$ type estimates for the sequence $U^k$ in $\mathbb{B}_1^+$, which follow from (3.1) and (3.12), as well as the uniform Hölder estimates for $U^k, \partial_0^o U^k$ which are a consequence of Theorem 3.7, there exists a subsequence which we continue to denote by $U_k$, such that $U_k \rightarrow U_0$ uniformly in $\mathbb{B}_1^+$, $||U_0||_{L^\infty(\mathbb{B}_1^+)} \leq 1$, where $0 \in \Gamma^*(U_0), U_0$ solves (3.4) with $A = I$, and $f = 0$. This contradicts (3.66) for large enough $k$'s. The conclusion of the lemma thus follows.

From the previous result we obtain the following corollary. Before stating it we make an observation. Suppose that $V$ that solves (3.4) with $A = I$ and $f = 0$, with $||V||_{L^\infty(\mathbb{B}_1^+)} \leq 1, 0 \in \Gamma^*(V)$. Since $V(0) = 0$ and $V \geq 0$ on the thin set, we infer that it must be $\nabla_x V(0) = 0$. We can thus apply the optimal $\frac{3-\alpha}{2}$ decay estimate in [10, Theorem 6.7] to infer the existence of a universal constant $C > 0$ such that for every $0 < r < 1/4$

$$||V||_{L^\infty(B_1^+)} \leq Cr^{\frac{3-\alpha}{2}}. \tag{3.67}$$

We will use (3.67) momentarily.

**Corollary 3.9.** Suppose that $U$ solves (3.4) in $\mathbb{B}_1^+$, that $||U||_{L^\infty(\mathbb{B}_1^+)} \leq 1$ and $0 \in \Gamma^*(U)$. For every $\sigma < \frac{1-\alpha}{2}$ there exist universal $\delta, \lambda \in (0,1/4)$, depending on $\sigma$, such that if $||A - I||_{C^{0,1}}, ||f||_{L^\infty} \leq \delta$, then

$$||U||_{L^\infty(\mathbb{B}_1^+)} \leq \lambda^{1+\sigma}. \tag{3.68}$$

**Proof.** Given $\sigma < \frac{1-\alpha}{2}$ we choose $\lambda < 1/4$ (depending only on the universal constant $C > 0$ in (3.67) and on $\sigma$) such that $C\lambda^{\frac{3-\alpha}{2}} \leq \lambda^{1+\sigma}$. If we let $\varepsilon = \frac{\lambda^{1+\sigma}}{2}$, then from Lemma 3.8 we infer the existence of $\sigma = \delta(\varepsilon) = \delta(C,\sigma) > 0$ such that if the hypothesis in the lemma are verified, then in correspondence of such $\sigma$ there exists $V$ that solves (3.4) with $A = I$ and $f = 0$, with $||V||_{L^\infty(\mathbb{B}_1^+)} \leq 1, 0 \in \Gamma^*(V)$, and such that (3.64) be true. Since from what has been observed above for such $V$ the estimate (3.67) is in force, the triangle inequality combined with (3.64) and (3.67) gives

$$||U||_{L^\infty(\mathbb{B}_1^+)} \leq ||U - V||_{L^\infty(\mathbb{B}_1^+)} + ||V||_{L^\infty(\mathbb{B}_1^+)} \leq \varepsilon + C\lambda^{\frac{3-\alpha}{2}} = \frac{\lambda^{1+\sigma}}{2} + C\lambda^{\frac{3-\alpha}{2}} \leq \lambda^{1+\sigma},$$

which provides the desired conclusion (3.68) for $U$.

□
From Corollary 3.9 we obtain the following almost optimal decay result at any point of the extended free boundary.

**Lemma 3.10.** Let $U$ be a solution to the Signorini problem (3.4) such that $||U||_{L^\infty(\mathbb{B}_t^+)} \leq 1$ and assume that $0 \in \Gamma^*(U)$. Given any $0 < \sigma < \frac{1-a}{2}$ there exists a constant $C = C(n,a,\sigma) > 0$ such that for every $r \in (0,1/4)$ one has

$$||U||_{L^\infty(\mathbb{B}_t^+)} \leq Cr^{1+\sigma}. \tag{3.69}$$

**Proof.** Without loss of generality we may assume that $A(0) = I$. Given $0 < \sigma < \frac{1-a}{2}$, let $\lambda, \delta$ be the universal constants (depending on $\sigma$) whose existence is claimed in Corollary 3.9. If we let $U_r(X) = U(rx)$, then $U_r$ solves (3.4) corresponding to $A_r(X) = A(rx)$ and $f_r(X) = r^2 f(rx)$, and moreover $0 \in \Gamma^*(U_r)$. Since by the Lipschitz continuity of the matrix-valued function $X \rightarrow A(X)$ we have for each $X = (x,y) \in \mathbb{B}_t^+$:

$$|A_r(X) - I| = |A(rx) - A(0)| \leq Lr|x| \leq Lr,$$

it is clear that there exists $r_0 > 0$ (depending on $\delta$ above, and therefore on $\sigma$) such that for $r \in (0,r_0]$ the functions $A_r$ and $f_r$ fulfil the constraint

$$||A_r - I||_{C^0,1}, ||f_r||_{L^\infty} \leq \delta. \tag{3.70}$$

In view of Corollary 3.9, applied to $U_{r_0}$, this allows to conclude that for every $\lambda \in (0,1/4)$

$$||U_{r_0}||_{L^\infty(\mathbb{B}_t^+)} \leq \lambda^{1+\sigma}. \tag{3.71}$$

By rescaling it is clear that it suffices to prove (3.69) for $U_{r_0}$. Therefore henceforth, to simplify the notation, we drop the subscript $r_0$ and indicate $U_{r_0}, A_{r_0}, f_{r_0}$ with $U, A, f$. With this being said, we now claim that for every $k \in \mathbb{N}$ the following estimate holds

$$||U||_{L^\infty(\mathbb{B}_t^+)} \leq \lambda^{k(1+\sigma)}. \tag{3.71}$$

Once (3.71) is established, the conclusion of the lemma follows by a standard real analysis argument observing that for any $r < 1/4$ we can find $k \in \mathbb{N} \cup \{0\}$, such that $\lambda^{k+1} < r \leq \lambda^k$. This gives

$$||U||_{L^\infty(\mathbb{B}_t^+)} \leq ||U||_{L^\infty(\mathbb{B}_{\lambda^k t}^+)} \leq \lambda^{k(1+\sigma)} = \frac{1}{\lambda^{1+\sigma}} \lambda^{(k+1)(1+\sigma)} \leq \frac{1}{\lambda^{1+\sigma}} r^{1+\sigma}.$$

To achieve (3.69) it thus suffices to take $C = \frac{1}{\lambda^{1+\sigma}}$. We are left with proving (3.71). We proceed by induction. First, note that from Corollary 3.9 the estimate holds for $k = 1$ (for $k = 0$ (3.71) follows trivially from the assumption $||U||_{L^\infty(\mathbb{B}_t^+)} \leq 1$). Assume now that (3.71) hold up to some $k \geq 2$. Letting

$$\tilde{U} = \frac{U(\lambda^k X)}{\lambda^{k(1+\sigma)}}, \tag{3.72}$$

we note that $\tilde{U}$ solves a Signorini problem of the type (3.4) with $\tilde{A} = A(\lambda^k x)$ and $\tilde{f} = \lambda^{k(1-\sigma)} f(\lambda^k x)$, and also $0 \in \Gamma^*(\tilde{U})$. Since $\sigma, \lambda < 1$, we observe that thanks to (3.70) and (3.71) (which does hold for such $k$ thanks to the inductive assumption), the hypothesis of Corollary 3.9 is satisfied. Consequently, (3.68) holds for $\tilde{U}$. After scaling back to $U$, which in turn implies the following

$$||U||_{L^\infty(\mathbb{B}_{\lambda^{k+1} t}^+)} \leq \lambda^{(k+1)(1+\sigma)}. \tag{3.73}$$

By induction we conclude that (3.71) does hold for all $k$, thus completing the proof of the lemma. \[\square\]
We finally establish the second main regularity result of this section, the (sub-optimal) a priori $C^\alpha$ regularity of $\nabla_x U$ up to the thin set.

**Theorem 3.11.** Let $a \in [0, 1)$ and $U$ be a solution of (2.2) with an obstacle $\psi \in C^2$. Then $\nabla_x U \in C^\alpha(\mathbb{B}_2^+)$ for some $\alpha > 0$.

**Proof.** By subtracting the obstacle from $U$ we can assume without loss of generality that $U$ solves (3.4). Since for such case we note that since the $C^\beta$ continuity of $U, y^a U_y$ up to $\{y = 0\}$ has already been established in Theorem 3.7, we are only left with proving the Hölder continuity of $\nabla_x U$. Given $X \in \mathbb{B}_2^+$, we let $d(X) = d(X, \Gamma^*(U))$. We note that, if such set is non-empty, since there is no point of $\Gamma^*(U)$ inside $\mathbb{B}_{d(X)}(X) \cap \{y = 0\}$, either $\partial_y^a U$ or $U$ must vanish identically in this set. Otherwise, the subsets where $U > 0$ and $\partial_y^a U < 0$ would separate the connected set $\mathbb{B}_{d(X)}(X) \cap \{y = 0\}$. By even or odd reflection across $\{y = 0\}$ (depending on whether $\partial_y^a U \equiv 0$, or $U \equiv 0$) we infer that $U$ solves in $\mathbb{B}_{d(X)}(X)$

$$\text{(3.74)} \quad \text{div}(|y|^a A(x) \nabla U) = |y|^a f.$$ 

Moreover, if we fix $\sigma < \frac{1-a}{2}$ then from Lemma 3.10 we have

$$\text{(3.75)} \quad ||U||_{L^\infty(\mathbb{B}_{d(X)}(X))} \leq C d(X)^{1+\sigma}.$$ 

Using the scaled version of the gradient estimates in Theorem 2.1 or Theorem 2.3, depending on whether $U$ has been reflected in an even or odd way across $\{y = 0\}$ in $\mathbb{B}_{d(X)}(X)$, we deduce from (3.75) that the following holds,

$$\text{(3.76)} \quad |\nabla_x U(X)| \leq C d(X)^\sigma.$$ 

We now take points $X^1, X^2 \in \mathbb{B}_2^+$, set $\delta = |X^1 - X^2|$, and let $d_i = d(X^i, \Gamma^*(U))$ for $i = 1, 2$. Without loss of generality we assume that $d_1 \geq d_2$. There exist two possibilities: (a) $\delta \geq \frac{1}{8} d_1$; or, (b) $\delta < \frac{1}{8} d_1$. If (a) occurs, it trivially follows from (3.76) that

$$|\nabla_x U(X^1) - \nabla_x U(X^2)| \leq |\nabla_x U(X^1)| + |\nabla_x U(X^2)|$$

$$\leq C d_1^\sigma + C d_2^\sigma \leq C \delta^\sigma.$$ 

If instead (b) occurs, then we have $X^2 \in \mathbb{B}_{d_1}(X^1)$. As before, we again note that either $U$ or $\partial_y^a U$ vanishes identically in $\mathbb{B}_{d_1}(X) \cap \{y = 0\}$. Therefore after an odd or even reflection of $U$ across $\{y = 0\}$ in $\mathbb{B}_{d_1}(X)$ (depending on whether $U$ or $\partial_y^a U$ vanishes), we obtain that $U$ solves an equation of the type (3.74). From the $C^\alpha$ estimate of $\nabla_x U$ in Theorem 2.1 or Theorem 2.3 it follows that for some $0 < \alpha < \sigma$, the following holds

$$\text{(3.77)} \quad |\nabla_x U(X^1) - \nabla_x U(X^2)| \leq \frac{C}{d_1^{1+\alpha}}(||U||_{L^\infty(\mathbb{B}_{d_1}(X))} + d_1^2 ||f||_{L^\infty}) \delta^\alpha \leq C \delta^\alpha.$$ 

Note that in the second inequality in (3.77) we have used the decay estimate in (3.75) for $X = X_1$ and the fact that $\alpha < \sigma$. In both cases a) or b) we obtain for some $\alpha > 0$

$$|\nabla_x U(X^1) - \nabla_x U(X^2)| \leq C|X^1 - X^2|^\alpha,$$

thus reaching the sought for conclusion. □
4. Monotonicity Formulas

In this section we establish a variant of Almgren’s monotonicity which is the crucial tool in the blowup analysis required to establish the optimal regularity of solutions. We continue to indicate a generic point in the thick space by $X = (x,y) \in \mathbb{R}^{n+1}$, and we set $r = r(X) = |X|$. For notational convenience we will sometimes denote the operator $\text{div}(|y|^a A(x) \nabla)$ by $L_a$. Throughout this section and in the remainder of the paper we will assume, without restriction, that $A(0) = I$. This can always be accomplished by a suitable linear transformation of the coordinates. We now state our first lemma which can be verified by a standard computation.

**Lemma 4.1.** For $r \neq 0$ one has

\[
L_ar = \text{div}(|y|^a A(x) \nabla r) = \frac{n + a}{r}|y|^a + O(|y|^a).
\]

In particular, $L_ar \in L^1(B_1)$.

In the following we will need the function

\[
\mu(X) = \tilde{\mu}(X)|y|^a \overset{\text{def}}{=} \frac{\langle A(x)X, X \rangle}{|X|^2}|y|^a = \langle A(x)\nabla r, \nabla r \rangle |y|^a.
\]

The properties of the function $\tilde{\mu}(X) = \langle A(x)\nabla r, \nabla r \rangle$ are summarised in [24, Lemma 4.2] and will be used in the sequel without further specific reference. Let $U$ be the solution to the thin obstacle problem (3.4) in $\mathbb{B}^+_1$. After an even reflection in $y$ across $\{y = 0\}$ we have that $U$ solves in the distributional sense

\[
\begin{cases}
\text{div}(|y|^a A(x) \nabla U) = |y|^a f + 2\partial_y^a U \mathcal{H}^n(\{y = 0\}) \\
U \partial_y^a U \equiv 0.
\end{cases}
\]

For any $r \in (0,1)$ we now define the height function of $U$ in $\mathbb{S}_r$ as

\[
H(r) = \int_{\mathbb{S}_r} U^2 \mu d\sigma,
\]

where $\mu$ is as in (4.2). We also set

\[
B(r) = \int_{\mathbb{B}_r} U^2 |y|^a dX.
\]

The Dirichlet integral of $U$ in $\mathbb{B}_r$ is defined as

\[
D(r) = \int_{\mathbb{B}_r} \langle A(x)\nabla U, \nabla U \rangle |y|^a dX.
\]

Finally, we denote by

\[
I(r) = \int_{\mathbb{S}_r} U \langle A\nabla U, \nu \rangle |y|^a d\sigma
\]

the total energy of $U$ in $\mathbb{B}_r$. We next recall a well-known trace inequality. For its proof we refer the interested reader to e.g. [19, Lemma 14.4].

**Lemma 4.2.** There exists a universal constant $C = C(n, a, \lambda, \Lambda) > 0$, such that for $r > 0$ and $U \in W^{1,2}(\mathbb{B}, |y|^a dX)$. Then, one has

\[
H(r) \leq C \left[ \frac{1}{r} B(r) + rD(r) \right],
\]
The following lemma concerns the first variation of the height function $H$.

**Lemma 4.3.** The function $H$ is absolutely continuous and for a.e. $r \in (0, 1)$ one has

$$H'(r) = 2I(r) + \int_{S_r} U^2 L_a |X|.$$ 

**Proof.** We follow a by now standard approximation argument that crucially uses the continuity up to the thin set $\{y = 0\}$ of the functions $U, \nabla_x U, y^a U_y$, see Theorems 3.7 and 3.11. By first integrating in the region $B_r \cap \{|y| > \varepsilon\}$, and then letting $\varepsilon \to 0$, by an application of the divergence theorem using the Signorini condition $U \partial_y U = 0$, we can express the height function as the following solid integral

$$H(r) = \int_{B_r} \text{div} \left( |y|^a U^2 A \frac{X}{|X|} \right).$$

From (4.10) we obtain

$$H(r) = \int_{B_r} U^2 \text{div} \left( |y|^a A \frac{X}{|X|} \right) + 2|y|^a U \langle A \nabla U, \frac{X}{|X|} \rangle.$$ 

The desired conclusion (4.9) now follows from (4.11) by an application of the coarea formula. \qed

Using (4.3) and Theorem 3.11 again, it is easy to recognise that $I(r)$ and $D(r)$ are related as follows.

**Lemma 4.4.** For every $r \in (0, 1)$ we have

$$I(r) = D(r) + \int_{B_r} U f |y|^a.$$ 

Following the analysis of the case $a = 0$ in [24], in order to control the second integral in the right-hand side of (4.9) we now introduce some quantities which play a critical auxiliary role.

**Definition 4.5.** Let $U$ be a solution of (3.4). Consider the function $G : (0, 1] \to (0, \infty)$ defined for any $r \in (0, 1]$ by

$$G(r) = \begin{cases} \frac{\int_{S_r} U^2 L_a |X|}{\int_{B_r} U^2 \mu(X)} & \text{if } H(r) \neq 0, \\ \frac{n + a}{r} & \text{if } H(r) = 0. \end{cases}$$

**Lemma 4.6.** There exists a universal constant $\beta \geq 0$ such that for any $r \in (0, 1)$:

$$\frac{n + a}{r} - \beta \leq G(r) \leq \frac{n + a}{r} + \beta.$$ 

**Proof.** When $r \in (0, 1]$ is such that $H(r) = 0$ the desired conclusion follows trivially from the definition of $G(r)$. Since $\bar{\mu}(X) = O(1)$, and also

$$\frac{L_r}{\bar{\mu}} = \left( \frac{n + a}{r} + O(1) \right),$$
we infer that there exists a universal constant $\beta \geq 0$ such that
\[
\frac{n + a}{r} - \beta \leq \frac{La}{\mu} \leq \frac{n + a}{r} + \beta.
\]
This implies
\[
\left(\frac{n + a}{r} - \beta\right) \int_{S_r} U^2 \mu \leq \int_{S_r} U^2 La \leq \left(\frac{n + a}{r} + \beta\right) \int_{S_r} U^2 \mu,
\]
which concludes the proof.

Next, with $U$ being the solution of (3.4), and $G$ as in Definition 4.5, following [24] we introduce the functions $\psi : (0, 1] \to (0, \infty)$ and $\sigma : (0, 1] \to (0, \infty)$ respectively defined by the Cauchy problems:

(4.13) \[
\begin{cases}
\frac{d}{dr} \log(\psi(r)) = \frac{\psi'(r)}{\psi(r)} = G(r) \\
\psi(1) = 1
\end{cases}
\]

and

(4.14) \[
\begin{cases}
\sigma'(r) - \frac{\psi'(r)}{\psi(r)} + \frac{n - 1 + a}{r} = 0 \\
\sigma(1) = 1
\end{cases}
\]

Lemma 4.7. There exists a universal constant $\beta \geq 0$ such that if $r \in (0, 1)$ one has
\[
\frac{n + a}{r} - \beta \leq \frac{d}{dr} \log(\psi(r)) \leq \frac{n + a}{r} + \beta,
\]
and therefore
\[
e^{-\beta(1-r)r^{n+a}} \leq \psi(r) \leq e^{\beta(1-r)r^{n+a}}.
\]
This implies, in particular, that $\psi(0^+) = 0$. For the function $\sigma(r)$ we have $\sigma(r) = \frac{\psi(r)}{r^{1-a}}$, and so
\[
e^{-\beta(1-r)r^{n+a}} \leq \sigma(r) \leq e^{\beta(1-r)r}
\]
for $0 < r < 1$. In particular, $\sigma(0^+) = 0$.

Proof. The first inequality is a consequence of Lemma 4.6. For the first half of the second inequality, we note that integrating the first one over $(r, 1)$, we have
\[
\log(\psi(1)) - \log(\psi(r)) \leq (n + a)\left(\log(1) - \log(r) + \beta(1 - r)\right) \implies \psi(r) \geq e^{-\beta(1-r)r^{n+a}}.
\]
Same steps to obtain the second-half of the second inequality. For the third one, we observe that
\[
\log(\sigma(1)) - \log(\sigma(r)) = \log(\psi(1)) - \log(\psi(r)) - (n - 1 + a)\left(\log(1) - \log(r)\right),
\]
which implies $\log(\sigma(r)) = \log(\psi(r)r^{-(n-1+a)})$ and thus $\sigma(r) = \psi(r)r^{-(n-1+a)}$.

Lemma 4.8. There exists a universal constant $r_0$ such that the function $r \mapsto \sigma(r)$ is increasing on $(0, r_0)$.
Proof. By Lemma 4.7 we know that
\[
\frac{\sigma'(r)}{\sigma(r)} = \frac{\psi'(r)}{\psi(r)} - \frac{n - 1 + a}{r} = G(r) - \frac{n - 1 + a}{r} \geq \frac{n + a}{r} - \beta - \frac{n - 1 + a}{r} = \frac{1}{r} - \beta.
\]
If we take \( r_0 < \beta^{-1} \), we obtain \( \frac{\sigma'(r)}{\sigma(r)} \geq 0 \).

We now note that, if we consider the numbers: \( \alpha^- = \lim \inf_{r \to 0^+} \frac{\sigma(r)}{r} \) and \( \alpha^+ = \lim \sup_{r \to 0^+} \frac{\sigma(r)}{r} \), then we obviously have
\[
0 < e^{-\beta} \leq \alpha^- \leq \alpha^+ \leq e^\beta.
\]
The following lemma will be needed in the proof of optimal regularity of solutions to \((3.4)\).

**Lemma 4.9.** One has for \( r \in (0,1) \)
\[
\left| \frac{\sigma(r)}{r} - \alpha^\pm \right| \leq \beta e^\beta r.
\]
In particular, we have \( \alpha^+ = \alpha^- \) and thus, in particular, it exists
\[
\alpha \overset{\text{def}}{=} \lim_{r \to 0^+} \frac{\sigma(r)}{r} > 0.
\]

**Proof.** We start with the preliminary observation
\[
\left| \frac{d}{dr} \log \frac{\sigma(r)}{r} \right| = \left| \frac{d}{dr} \left( \log(\sigma(r)) - \log r \right) \right| = \left| \frac{\sigma'(r)}{\sigma(r)} - \frac{1}{r} \right| = \left| \frac{\psi'(r)}{\psi(r)} - \frac{n - 1 + a}{r} - \frac{1}{r} \right| = \left| \frac{\psi'(r)}{\psi(r)} - \frac{n + a}{r} \right| \leq \beta,
\]
where in the latter inequality we have used Lemma 4.7. If we define \( h(r) = \log \frac{\sigma(r)}{r} \), then by Lemma 4.7 and the fact \( r \in (0,1) \), we have
\[
\left| \frac{d}{dr} \frac{\sigma(r)}{r} \right| = |h'(r)| e^{h(r)} = \left| \frac{d}{dr} \log \frac{\sigma(r)}{r} \right| \frac{\sigma(r)}{r} \leq \beta \frac{\sigma(r)}{r} \leq \beta e^{\beta(1-r)} = \beta e^\beta.
\]
If we set \( g(r) = \frac{\sigma(r)}{r} \), and fix \( 0 < \varepsilon < r < 1 \), we have
\[
|g(r) - g(\varepsilon)| = \left| \int_\varepsilon^r g'(\tau)d\tau \right| \leq \int_\varepsilon^r \beta e^\beta d\tau = \beta e^\beta (r - \varepsilon) \leq \beta e^\beta r,
\]
which implies \( g(\varepsilon) - \beta e^\beta r \leq g(r) \leq g(\varepsilon) + \beta e^\beta r \). Taking \( \lim \inf_{\varepsilon \to 0^+} \) and \( \lim sup_{\varepsilon \to 0^+} \) in the above inequalities, we obtain
\[
\alpha^\pm - \beta e^\beta r \leq g(r) \leq \alpha^\pm + \beta e^\beta r \implies \left| \frac{\sigma(r)}{r} - \alpha^\pm \right| \leq \beta e^\beta r,
\]
for \( r \in (0,1) \). We conclude observing that
\[
0 \leq \alpha^+ - \alpha^- \leq (\alpha^+ - g(r)) - (\alpha^- - g(r)) \leq |\alpha^+ - g(r)| + |\alpha^- - g(r)| \leq 2\beta e^\beta r \to 0
\]
as \( r \to 0^+ \).

In the subsequent steps we will need the following two lemmas.
Lemma 4.10. Let $U$ be the solution of (3.4) with $U(0) = 0$. Then,

$$
\int_{S_r} U^2 \mu \leq Cr \int_{B_r} \langle A \nabla U, \nabla U \rangle |y|^a + Cr^{n+4+a},
$$

where $C > 0$ is a universal constant depending on $||f||_{\infty}$.

Proof. Note that from $f \in L^\infty(B_1)$ we deduce that $U$ is a supersolution to

$$
div(|y|^a A(x) \nabla U) \leq C|y|^a.
$$

Keeping (4.2) in mind, we set

$$
L(r) = \int_{S_r} U \mu.
$$

We have

$$
L'(r) = \frac{n+a}{r} L(r) + O(1) L(r) + \int_{S_r} U \nu \tilde{\mu}.
$$

This can be further rewritten as

$$
L'(r) = \left( \frac{n+a}{r} + O(1) \right) L(r) + \int_{S_r} |y|^a < A \nabla U, \nu > + \int_{S_r} |y|^a < \nabla U, \tilde{\mu} \nu - A \nu >
$$

$$
= \left( \frac{n+a}{r} + O(1) \right) L(r) + \int_{S_r} \text{div}(|y|^a A \nabla U) + \int_{S_r} |y|^a < \nabla U, \tilde{\mu} \nu - A \nu >
$$

$$
\leq \left( \frac{n+a}{r} + O(1) \right) L(r) + Cr^{n+1+a} + \int_{S_r} |y|^a < \nabla U, \tilde{\mu} \nu - A \nu >,
$$

where in the last inequality we have used (4.16). Now is easily checked that the vector

$k = \tilde{\mu} \nu - A \nu$ is tangential to the sphere $S_r$ and thus by applying divergence theorem on the sphere, we deduce from (4.17) that the following holds,

$$
L'(r) \leq \frac{n+a}{r} L(r) + O(1) L(r) + Cr^{n+1+a} + \int_{S_r} \text{div}_{S_r}(k)|y|^a + \int_{S_r} U < \nabla |y|^a, k >
$$

$$
\leq \frac{n+a}{r} L(r) + O(1) L(r) + Cr^{n+1+a}.
$$

Over here we used the fact that $\text{div}_{S_r} k = O(1)$ since $A$ is Lipschitz and $A(0) = I$ and also that

$$
< \nabla y^a, k > = ay^{a-1}(\tilde{\mu} - 1)\frac{y}{r} \leq Cy
$$

since $(\tilde{\mu} - 1) = O(r)$. From (4.18) we obtain that with

$$
L_0(r) = \frac{L(r)}{r^{n+a}}
$$

we have that for some universal $C > 0$

$$
r \to e^{-Cr} L_0(r) - Cr^2
$$

is non-increasing from which it follows that

$$
\frac{1}{r^{n+a}} \int_{S_r} U \mu \leq CU(0) + Cr^2, \; r \in (0, 1).
$$

Using the super mean value inequality in (4.19), one can argue as in the proof of [10, Lemma 2.13] to deduce the validity of (4.15).
Corollary 4.11. Let \( U \) be the solution of (3.4) such that \( U(0) = 0 \). Then,

\[
\int_{B_r} U^2 |y|^a \leq Cr^2 \int_{B_r} \langle A\nabla U, \nabla U \rangle |y|^a + Cr^{n+a+5}.
\]

Proof. Keeping in mind that \( \mu(X) \leq \lambda^{-1} |y|^a \), integrating (4.15) between (0, r) we obtain

\[
\lambda \int_{B_r} U^2 |y|^a \leq \int_0^r \int_{S_{r'}} U^2 \mu du \leq C \int_0^r \int_{S_{r'}} \langle A\nabla U, \nabla U \rangle |y|^a du + C \int_0^r u^{n+a+4} du.
\]

By integrating by parts, we then observe that

\[
\int_{B_r} U^2 \mu \leq C' r^2 \int_{B_r} \langle A\nabla U, \nabla U \rangle |y|^a - C' \int_0^r \rho^2 \int_{S_{r'}} \langle A\nabla U, \nabla U \rangle |y|^a du + C'' r^{5+a+n} = C' r^2 \int_{B_r} \langle A\nabla U, \nabla U \rangle |y|^a + C'' r^{5+a+n} \leq C' r^2 \int_{B_r} \langle A\nabla U, \nabla U \rangle |y|^a + C'' r^{5+a+n}.
\]

The conclusion thus follows.

Given \( \delta \in (0, 1) \) and a universal constant \( r_0 > 0 \) (which will also depend on \( \delta \)), we now introduce the sets:

\[
\Lambda_{r_0} = \{ r \in (0, r_0) \mid H(r) > \psi(r)r^{3+\delta} \},
\]

\[
\Gamma_{r_0} = \{ r \in (0, r_0) \mid H(r) > e^{-\beta} r^{3+\delta+n+a} \},
\]

where \( \beta \geq 0 \) is the constant in Lemma 4.7.

Lemma 4.12. One has the inclusion \( \Lambda_{r_0} \subseteq \Gamma_{r_0} \). In particular, \( H(r) \neq 0 \) for every \( r \in \Lambda_{r_0} \).

Proof. By Lemma 4.7, let \( r \in \Lambda_{r_0} \), we have

\[
H(r) > \psi(r)r^{3+\delta} \geq e^{-\beta(1-r)} r^{3+\delta+n+a} \geq r^{3+\delta+n+a},
\]

which implies \( r \in \Gamma_{r_0} \). The second part of the statement is an obvious consequence of the first one.

Lemma 4.13. Assume that \( U(0) = 0 \). There exists a universal \( r_0 > 0 \), depending also on \( \delta \in (0, 1) \), such that:

\[
H(r) \leq 2CrD(r) \quad r \in \Gamma_{r_0}
\]

where \( C \) is the same as in (4.15).

Proof. By (4.15) we get \( H(r) \leq CrD(r) + Cr^{4+a+n} \). Then, if \( r \in \Gamma_{r_0} \), we get:

\[
r^{4+a+n} = r^{3+\delta+a+n} r^{1-\delta} \leq e^{\beta} H(r) r^{1-\delta} \implies H(r) \leq CrD(r) + C' r^{1-\delta} H(r).
\]

By taking \( r < r_0 < (\frac{1}{2C})^{\frac{1}{1-\delta}} \), we note that \( C' r^{1-\delta} \leq \frac{1}{2} \) which in turn implies that

\[
H(r) \leq CrD(r) + \frac{H(r)}{2} \implies \frac{H(r)}{2} \leq CrD(r) \implies H(r) \leq 2CrD(r).
\]

\[\square]
**Corollary 4.14.** Suppose that \( U(0) = 0 \). There exists a universal constant \( r_0 > 0 \), depending also on \( \delta \in (0,1) \), such that:

\[
[r^{n+3+\alpha} \leq 2Ce^\beta r^{1-\delta} D(r) \quad r \in \Gamma_{r_0}.
\]

**Proof.** By Lemma 4.13, we have for \( r \in \Gamma_{r_0} \) such that:

\[
r^{3+n+\alpha} = r^{-\delta} r^{3+n+\alpha+\delta} \leq e^\beta r^{-\delta} H(r) \leq 2Ce^{-\beta} r^{1-\delta} D(r).
\]

\( \square \)

**Lemma 4.15.** Let \( U(0) = 0 \). There exists a universal \( r_0 > 0 \), depending on \( \delta \in (0,1) \) and \( ||f||_\infty \), such that if \( r \in \Gamma_{r_0} \) then:

\[
I(r) \geq \frac{D(r)}{2}
\]

**Proof.** By (4.12) we need to prove

\[
\left| \int_{B_r} Uf|y|^a \right| \leq \frac{D(r)}{2}.
\]

Note that,

\[
D(r) = I(r) - \int_{B_r} Uf|y|^a \leq I(r) + \int_{B_r} Uf|y|^a \leq I(r) + \frac{D(r)}{2}.
\]

By Cauchy-Schwartz, since \( f \in L^\infty \):

\[
\left| \int_{B_r} Uf|y|^a \right| \leq C \int_{B_r} U|y|^{a/2}|y|^{a/2} \leq C \left( \int_{B_r} |y|^a \right)^{1/2} \left( \int_{B_r} U^2|y|^a \right)^{1/2} \leq Cr^{\frac{a+1}{2}} \left( \int_{B_r} U^2|y|^a \right)^{1/2}.
\]

Now, by Corollary 4.11 we have:

\[
\int_{B_r} U^2|y|^a \leq Cr^2 \langle A\nabla U, \nabla U \rangle |y|^a + Cr^{5+a+n}.
\]

Thus,

\[
\left| \int_{B_r} Uf|y|^a \right| \leq Cr^{\frac{a}{2}} r^{\frac{n+1}{2}} \left( r^2 \langle A\nabla U, \nabla U \rangle |y|^a + r^{5+a+n} \right)^{\frac{1}{2}} \leq Cr^{\frac{n+1+a}{2}} \left( r[D(r)]^{1/2} + r^{\frac{5+a+n}{2}} \right) = Cr^{\frac{n+1+a}{2}} \left( r[D(r)]^{1/2} + r^{\frac{5+a+n}{2}} \right) = C\left(r^{\frac{n+3+a}{2}}[D(r)]^{1/2} + r^{n+a+3} \right).
\]

(4.25)

\[
\implies \left| \int_{B_r} Uf|y|^a \right| \leq C\left(r^{\frac{n+3+a}{2}}[D(r)]^{1/2} + r^{n+a+3} \right).
\]

Notice, now, that \( \forall c_1, c_2 > 0 \) and \( \forall \varepsilon > 0 \), we have:

(4.26)

\[
\left( \sqrt{\varepsilon c_1} - \sqrt{\frac{c_2}{\varepsilon}} \right)^2 = \varepsilon c_1 + \frac{c_2}{\varepsilon} - 2\sqrt{\varepsilon c_1 c_2} \geq 0,
\]

\[
\implies \varepsilon c_1 + \frac{c_2}{\varepsilon} \geq 2\sqrt{\varepsilon c_1 c_2} \geq \sqrt{c_1 c_2},
\]

\[
\implies \sqrt{c_1 c_2} + c_2 \leq c_1 + \left( \frac{1}{\varepsilon} + 1 \right)c_2.
\]

This means that \( \forall \varepsilon > 0 \) we have:

\[
\left| \int_{B_r} Uf|y|^a \right| \leq C\varepsilon D(r) + C\left(\frac{1}{\varepsilon} + 1\right)r^{n+a+3}.
\]
Theorem 4.17. Using the equation (4.29) we now use the following Rellich type identity
\[ D(r) = \frac{D(r)}{4} + 2e^\beta C_1 (4C + 1)r^{1-\delta} D(r). \]

We now let \( r_0 = \left( \frac{1}{3} \right) \left( \frac{1}{2e^\beta C_1 (4C + 1)} \right)^{\frac{1}{1-\delta}}. \) Thus,
\[ r < r_0 \implies 2e^\beta C_1 (4C + 1)r^{1-\delta} \leq \frac{1}{4}. \]

Therefore for every \( r \in \Gamma_{r_0}, \) it follows that
\[ \left| \int_{B_r} Uf |y|^a \right| \leq \frac{D(r)}{2}. \]

Before proceeding further, we need to compute the derivative of the total energy of \( U \) introduced in (4.6). We need the following lemma which can be verified by a standard computation keeping in mind the definition of the function \( \tilde{\mu} \) in (4.2).

Lemma 4.16. Consider the vector field \( Z = \frac{A(x)X}{\tilde{\mu}(x)}. \) We have
\[ \partial_i Z_j = \delta_{ij} + O(r), \quad \text{div} \, Z = (n+1) + O(r). \]

Our next result concerns the first variation of \( D(r). \)

Theorem 4.17.
\[ D'(r) = 2 \int_{S_r} (\frac{(A(x)\nabla U, \nu)}{\tilde{\mu}})^2 |y|^a + \left( \frac{n-1+a}{r} + O(1) \right) D(r) - \frac{2}{r} \int_{B_r} \langle Z, \nabla U \rangle f |y|^a. \]

Proof. First, by the coarea formula we see that
\[ D'(r) = \int_{S_r} \langle A \nabla U, \nabla U \rangle |y|^a. \]

Since \( <Z, \nu> = r \) on \( S_r, \) by an application of the divergence theorem we deduce
\[ D'(r) = \int_{B_r} \text{div}((|y|^a \langle A \nabla U, \nabla U \rangle Z). \]

We now use the following Rellich type identity
\[ \text{div}(|y|^a \langle A \nabla U, \nabla U \rangle Z) = 2 \text{div}(|y|^a \langle Z, \nabla U \rangle A \nabla U) \]
\[ = \text{div}(Z)|y|^a \langle A \nabla U, \nabla U \rangle + |y|^a Z_t \partial_a z_j \partial_j U \partial_k U \]
\[ + Z_{n+1}a|y|^{a-2} \langle A \nabla U, \nabla U \rangle - 2\langle Z, \nabla U \rangle \text{div}(|y|^a A \nabla U) - 2\partial_t Z_k a_{ij} \partial_j U \partial_k U. \]

Using the equation (4.3) satisfied by \( U \) the identity in (4.30) and Lemma 4.16, we obtain from (4.29) that the following holds,
\[ D'(r) = 2 \int_{S_r} (\frac{(A(x)\nabla U, \nu)}{\tilde{\mu}})^2 |y|^a + \left( \frac{n-1+a}{r} + O(1) \right) D(r) - \frac{2}{r} \int_{B_r} \langle Z, \nabla U \rangle f |y|^a \]
\[ - \frac{2}{r} \int_{B_r \cap \{y=0\}} \langle Z, \nabla U \rangle \partial_y U. \]
We note that the formal computation leading to (4.31) can again be justified by a limiting type argument as before using the continuity of $U, \nabla_x U, y^a U_y$ up to $\{y = 0\}$ as well as the $W^{2,2}$ type estimates for $U$. Finally by noting that at $\{y = 0\}$,

(4.32) \[
\langle Z, \nabla U \rangle \partial_y^a U = \langle x, \nabla_x U \rangle \partial_y^a U \equiv 0,
\]

thanks to the complementarity condition in (4.3), we thus conclude by using (4.32) in (4.31) that (4.27) holds.

\[\square\]

**Theorem 4.18.** Let $U$ be the solution of (3.4). Then, for a.e. $r \in (0, 1)$ we have

(4.33) \[
I'(r) = 2 \int_{S_r} \frac{(A(x, \nabla U, \nu))^2}{m} |y|^a + \left( \frac{n - 1 + a}{r} + O(1) \right) I(r) + \int_{S_r} U f |y|^a - \frac{2}{r} \int_{S_r} \langle Z, \nabla U \rangle f |y|^a.
\]

**Proof.** By Lemma 4.4 we have that $I(r) = D(r) + \int_{S_r} U f |y|^a$ and thus by (4.27):

\[
I'(r) = D'(r) + \int_{S_r} U f |y|^a = 2 \int_{S_r} \frac{(A(x, \nabla U, \nu))^2}{m} |y|^a + \left( \frac{n - 1 + a}{r} + O(1) \right) D(r) - \frac{2}{r} \int_{S_r} \langle Z, \nabla U \rangle f |y|^a + \int_{S_r} U f |y|^a.
\]

Observing now that by (4.12), $D(r) = I(r) - \int_{S_r} |y|^a U f$, we obtained the desired conclusion. \[\square\]

Following [24], we next introduce certain quantities that play a key role in the analysis of the monotonicity properties of the frequency. We consider

\[
M(r) = \frac{H(r)}{\psi(r)}, \quad J(r) = \frac{I(r)}{\psi(r)},
\]

and define the **generalised frequency** as

\[
\Phi(r) = \frac{\sigma(r) J(r)}{M(r)},
\]

where $\sigma$ is defined by (4.14).

**Theorem 4.19.** Assume that $U(0) = 0$. Given $\delta \in (0, 1)$, there exist universal constants $r_0, K' > 0$ such that the function $r \mapsto e^{K'r^{1-\delta}} \Phi(r)$ is non-decreasing on $\Gamma_{r_0}$. Precisely, for every $r \in \Gamma_{r_0}$ we have

\[
\frac{d}{dr} \log \Phi(r) = \frac{\Phi'(r)}{\Phi(r)} \geq -\frac{K'}{r^{2+\delta}}.
\]

**Proof.** We begin by computing the derivatives of $M(r)$ and $J(r)$. One has

\[
M'(r) = -\frac{\psi'(r) H(r)}{[\psi(r)]^2} + \frac{H'(r)}{\psi(r)} = -\frac{\psi'(r) H(r)}{[\psi(r)]^2} + \frac{1}{\psi(r)} (2I(r) + \int_{S_r} U^2 L_a |X|) = \frac{1}{\psi(r)} \left[ \int_{S_r} U^2 L_a |X| - \frac{\psi'(r) J(r)}{\psi(r)} \right] + \frac{2I(r)}{\psi(r)},
\]
where we have used Lemma 4.3. Keeping in mind that if \( r \in \Gamma_{r_0} \) we have \( H(r) \neq 0 \), by (4.13) and Definition 4.5 we have at every \( r \in \Gamma_{r_0} \): 
\[
\frac{\psi'(r)}{\psi(r)} = G(r) = \frac{I_{\mathcal{B}_r} U^2 L_a |X|}{H(r)},
\]
or equivalently, 
\[
\frac{\psi'(r)}{\psi(r)} H(r) - \int_{\mathcal{B}_r} U^2 L_a |X| = 0.
\]
This implies at every \( r \in \Gamma_{r_0} \),
\[
M'(r) = \frac{2I(r)}{\psi(r)} = 2J(r), \quad M'(r) = \frac{2J(r)}{M(r)}.
\]
Moreover, by (4.33) and the fact that \( I(r) = J(r)\psi(r) \), we have:
\[
J'(r) = \frac{1}{\psi(r)} I'(r) - \frac{\psi'(r)}{[\psi'(r)]^2} I(r) = \frac{1}{\psi(r)} \left[ 2 \int_{\mathcal{B}_r} \frac{(A(x) \nabla U, \nu)^2}{\mu} |y|^a + \left( \frac{n - 1 + a}{r} + O(1) \right) I(r) \right] \psi(r) - \int_{\mathcal{B}_r} U f |y|^a - \frac{2}{\psi(r)} \int_{\mathcal{B}_r} \langle Z, \nabla U \rangle f |y|^a - \frac{\psi'(r)}{[\psi'(r)]^2} I(r)
\]
\[
= \left( \frac{n - 1 + a}{r} + O(1) \right) J(r) + \frac{1}{\psi(r)} \left[ 2 \int_{\mathcal{B}_r} \frac{(A(x) \nabla U, \nu)^2}{\mu} |y|^a \right] + \int_{\mathcal{B}_r} U f |y|^a - \frac{2}{\psi(r)} \int_{\mathcal{B}_r} \langle Z, \nabla U \rangle f |y|^a
\]
\[
= \left( \frac{n - 1 + a}{r} + O(1) \right) J(r) + \frac{1}{\psi(r)} \left[ 2 \int_{\mathcal{B}_r} \frac{(A(x) \nabla U, \nu)^2}{\mu} |y|^a \right] + \int_{\mathcal{B}_r} U f |y|^a - \frac{2}{\psi(r)} \int_{\mathcal{B}_r} \langle Z, \nabla U \rangle f |y|^a
\]
We next compute \( \Phi'(r) / \Phi(r) \). By the definition (4.6), we have:
\[
\Phi'(r) = \frac{\sigma'(r)}{\sigma(r)} + \frac{J'(r)}{J(r)} - \frac{M'(r)}{M(r)} = \frac{\sigma'(r)}{\sigma(r)} + \frac{J'(r)}{J(r)} - 2 \frac{J(r)}{M(r)}
\]
\[
= \frac{\sigma'(r)}{\sigma(r)} - \frac{\psi'(r)}{\psi(r)} + \frac{n - 1 + a}{r} + O(1) + \frac{1}{\psi(r)} \left[ 2 \int_{\mathcal{B}_r} \frac{(A(x) \nabla U, \nu)^2}{\mu} |y|^a \right] \psi(r) + \int_{\mathcal{B}_r} U f |y|^a - \frac{2}{\psi(r)} \int_{\mathcal{B}_r} \langle Z, \nabla U \rangle f |y|^a - 2 \frac{J(r)}{M(r)}
\]
\[
= O(1) + \frac{1}{\psi(r)} \left( \frac{n - 1 + a}{r} \right) \int_{\mathcal{B}_r} U f |y|^a - \frac{2}{\psi(r)} \int_{\mathcal{B}_r} \langle Z, \nabla U \rangle f |y|^a - 2 \frac{J(r)}{M(r)}
\]
Notice that we have:
\[
\frac{2}{\psi(r)} \int_{\mathcal{B}_r} \frac{(A(x) \nabla U, \nu)^2}{\mu} |y|^a - \frac{2 J(r)}{M(r)} \geq 0 \iff \frac{2}{\psi(r)} \int_{\mathcal{B}_r} \frac{(A(x) \nabla U, \nu)^2}{\mu} |y|^a - \frac{J(r)}{H(r)} \geq 0 \iff (I(r))^2 \leq \left( \int_{\mathcal{B}_r} \frac{(A(x) \nabla U, \nu)^2}{\mu} |y|^a \right) H(r)
\]
which in turn is true by Cauchy-Schwartz inequality. Indeed:
\[
\left( \int_{\mathcal{B}_r} U \langle A(x) \nabla U, \nu \rangle |y|^a \right)^2 = \left( \int_{\mathcal{B}_r} U \langle A(x) \nabla U, \nu \rangle |y|^a \right)^2 \left( \int_{\mathcal{B}_r} U ^2 |y|^a \right)^2 \leq \int_{\mathcal{B}_r} \frac{(A(x) \nabla U, \nu)^2}{\mu} |y|^a \int_{\mathcal{B}_r} U^2 \mu |y|^a
\]
\[
= H(r) \int_{\mathcal{B}_r} \frac{(A(x) \nabla U, \nu)^2}{\mu} |y|^a.
\]
where we have used \( \bar{\mu} |y|^a = \mu \). Thus, it follows

\[
\frac{\Phi'(r)}{\Phi(r)} \geq \frac{1}{I(r)} \left( -\frac{2}{r} \int_{B_r} (Z, \nabla U)f |y|^a - \left( \frac{n-1+a}{r} + O(1) \right) \int_{S_r} Uf |y|^a + \int_{S_r} Uf |y|^a \right) + O(1).
\]

Now, we want to prove:

\[
\frac{1}{I(r)} \left( -\frac{2}{r} \int_{B_r} (Z, \nabla U)f |y|^a - \left( \frac{n-1+a}{r} + O(1) \right) \int_{B_r} Uf |y|^a + \int_{S_r} Uf |y|^a \right) \geq -Kr^{-\frac{1+\delta}{2}}.
\]

By (4.23) and (4.25) and since for every \( r \in \Gamma_{\mu_0} \) we have \( r^{1-\delta} = r^{\frac{1+\delta}{2}} \leq Cr^{\frac{1+\delta}{2}} \), therefore:

\[
\left| \int_{B_r} Uf |y|^a \right| \leq C(r^{\frac{n+1+a}{2}}D(r)^{\frac{1}{2}} + r^{3+n+a}) \leq C'r^{-\frac{1+\delta}{2}}D(r) + Cr^{-\frac{1+\delta}{2}}D(r) \leq C'r^{-\frac{1+\delta}{2}}D(r).
\]

Then using (4.24) we obtain \( |\int_{B_r} Uf |y|^a| \leq Cr^{-\frac{1+\delta}{2}}I(r) \). Therefore,

\[
\left| \frac{1}{rI(r)} \int_{B_r} Uf |y|^a \right| \leq Cr^{-\frac{1+\delta}{2}} \Rightarrow -\frac{1}{I(r)} \left( \frac{n-1+a}{r} + O(1) \right) \int_{B_r} Uf |y|^a \geq -C'r^{-\frac{1+\delta}{2}}.
\]

Recalling that \( f \in L^\infty \), we have:

\[
\left| \frac{2}{r} \int_{B_r} (Z, \nabla U)f |y|^a \right| = \frac{2}{r} \int_{B_r} \langle \frac{A(x)}{\mu}X, \nabla U \rangle f |y|^a \leq \frac{C}{r} \int_{B_r} |X|\langle A(x) \nabla r, \nabla U \rangle |y|^a |
\]

\[
\leq \frac{C}{r} \int_{B_r} |X|\langle (A(x) \nabla r, \nabla U) \rangle \frac{1}{2} \langle (A(x) \nabla U, \nabla U) \rangle \frac{1}{2} |y|^a |
\]

\[
\leq \frac{C}{r} \int_{B_r} (\int_{\rho}^{2r} \int_{0}^{\rho} \rho^{2+n+a} d\rho) \frac{1}{2} [D(r)]^{\frac{1}{2}} \leq C'r^{-\frac{3+a+n}{2}}[D(r)]^{\frac{1}{2}}.
\]

Now by (4.23) we have \( r^{\frac{3+a+n}{2}} \leq r^{\frac{1+\delta}{2}}[D(r)]^{\frac{1}{2}} \). Thus by using (4.24), we deduce:

\[
\left| \frac{2}{r} \int_{B_r} (Z, \nabla U)f |y|^a \right| \leq C'r^{-\frac{1+\delta}{2}}[D(r)] \leq C'r^{-\frac{1+\delta}{2}}[I(r)]
\]

\[
\Rightarrow \frac{2}{rI(r)} \int_{B_r} (Z, \nabla U)f |y|^a \leq C'r^{-\frac{1+\delta}{2}}
\]

\[
\Rightarrow -\frac{2}{rI(r)} \int_{B_r} (Z, \nabla U)f |y|^a \geq -C'r^{-\frac{1+\delta}{2}}.
\]

Moreover:

\[
|\int_{S_r} Uf |y|^a| \leq C|\int_{S_r} U |y|^a| = C|\int_{S_r} U \frac{\sqrt{\mu}}{\sqrt{\rho}} |y|^{a/2}|y|^{a/2}| \leq C \left( \int_{S_r} U^2 \frac{|y|^a}{\mu} \right)^{\frac{1}{2}} \left( \int_{S_r} \frac{|y|^a}{\mu} \right)^{\frac{1}{2}}
\]

\[
= C \left( \int_{S_r} U^2 \mu \right)^{\frac{1}{2}} \left( \int_{S_r} \frac{|y|^a}{\mu} \right)^{\frac{1}{2}} \leq C \left( \int_{S_r} U^2 \mu \right)^{\frac{1}{2}} \int_{r}^{r+\frac{1}{2}} r^{\frac{n+a}{2}} = C \left( H(r) \right)^{\frac{1}{2}} r^{\frac{n+a}{2}}
\]

\[
\leq C[D(r)]^{\frac{1}{2}} r^{\frac{n+a+1}{2}} = C[D(r)]^{\frac{1}{2}} r^{\frac{n+a+1}{2}} r^{-1} \leq [D(r)]^{\frac{1}{2}} r^{-\frac{1+\delta}{2}} [D(r)]^{\frac{1}{2}} \leq Cr^{-\frac{1+\delta}{2}}I(r),
\]

\[
\Rightarrow \frac{1}{I(r)} \int_{S_r} Uf |y|^a \geq -Cr^{-\frac{1+\delta}{2}}.
\]
Note that in the above inequality, we also used (4.22) and (4.24).
Thus we have proved that for \( r \in \Gamma_{r_0} \),
\[
T(r) := \frac{1}{I(r)} \left( -\frac{2}{r} \int_{B_r} |y|^a \langle Z, \nabla U \rangle f - \left( \frac{n-1+a}{r} + O(1) \right) \int_{B_r} |y|^a U f + \int_{B_r} |y|^a U f \right) \geq -Kr^{-\frac{1+a}{2}},
\]
which implies that \( \frac{\Phi'(r)}{\Phi(r)} \geq -K'r^{-\frac{1+a}{2}} \) since \( |\Phi'(r)/\Phi(r) - T(r)| \leq C \) for all \( r \) small enough. This concludes the proof. \( \square \)

With Theorem 4.19 in hands, by an analogous argument as in the proof of Theorem 5.12 in [24], we obtain the following monotonicity result.

**Theorem 4.20.** Assume \( U(0) = 0 \). With \( r_0, K' \) as in Theorem 4.19 corresponding to some choice of \( \delta \in (0, 1) \), we have that
\[
r \mapsto N(r) \equiv \frac{\sigma(r)}{2} e^{K' r^{-\frac{1+a}{2}}} \frac{d}{dr} \log \max(M(r), r^{3+\delta})
\]
is non-decreasing in \((0, r_0)\). In particular \( N(0+) \) exists.

We also need to work with the following quantity
\[
(4.34) \quad \tilde{N}(r) \equiv \frac{r}{\sigma(r)} N(r).
\]
Now it follows from Lemma 4.9 and Theorem 4.20 that the following holds.

**Corollary 4.21.** Let \( \tilde{N}(r) \) be defined as in (4.34). Then \( \tilde{N}(0+) \) exists.

## 5. Optimal regularity

We now choose \( \delta \) in Theorem 4.20 such that \( 3 + \delta > 3 - a \). Our next result concerns the optimal decay of \( U \) near a free boundary point.

**Theorem 5.1.** Let \( U \) be a solution to (3.4) and let \( X_0 = (x_0, 0) \in \Gamma(U) \). Then we have that
\[
(5.1) \quad |U(X)| \leq C|X - X_0|^\frac{3-a}{2}
\]
for some universal constant \( C \).

**Proof.** Without loss of generality we may assume that \( X_0 = (0, 0) \) and it suffices to show that
\[
(5.2) \quad ||U||_{L^\infty(B_+^1)} \leq Cr^{\frac{3-a}{2}}.
\]
By rotation of coordinates, we may assume that \( A(0) = I \). Let \( d_r = M(r)^\frac{1}{2} \) and consider the following Almgren type rescalings \( U_r(X) = \frac{U(rX)}{d_r} \). Note that \( U_r \) solves the Signorini problem in (3.4) corresponding to \( A_r(X) = A(rX) \) and \( f_r = r^{\frac{a}{2}} f(rX) \) and \( 0 \in \Gamma(U_r) \). We note that the Lipschitz norm of \( A_r \) is bounded from above by the Lipschitz norm of \( A \). Now given the validity of Lemma 4.9 as well as the monotonicity result in Theorem 4.20, by an analogous blowup argument (which uses Theorem 3.1, Theorem 3.7 and Theorem 3.11) as in the proof of Lemma 6.3 in [24] for \( a = 0 \) (see also the proof of Lemma 6.2 in [10] for \( a \in (-1, 1) \) and \( A = I \)) we obtain that \( \tilde{N}(0+) \geq \frac{3-a}{2} \) with \( \tilde{N} \) as in (4.34). We note that this crucially utilizes the fact that up to a subsequence, \( U_r \rightarrow U_0 \) which is a homogeneous solution to the Signorini problem in (3.4) with \( A = I \) and \( f = 0 \) and that the homogeneity of \( U_0 \geq \frac{3-a}{2} \), thanks to
Theorem 5.7 in [10]. Then by using the monotonicity result in Theorem 4.20 and by arguing as in the proof of Lemma 6.4 in [24] we obtain that
\[(5.3) \quad H(r) \leq Cr^{n+3}.\]
for \(r \in (0, r_0)\) with \(r_0\) as in Theorem 4.20 above. From (5.3) it follows that
\[(5.4) \quad \int_{B_r} |y|^a U^2 \leq Cr^{n+4}.\]
Now we note that \(U^+\) and \(U^-\) are subsolutions to
\[(5.5) \quad L_a v \geq -|y|^a C\]
where \(C = ||f||_{L^{\infty}}\). This is seen by arguing as in Lemma 2.5 in [24]. Then we note that from (5.5), it follows that \(w = U^+ + \frac{C}{2(1+a)} y^2\) solves
\[(5.6) \quad L_a w \geq 0.\]
Moreover using (5.4) it is seen that
\[(5.7) \quad \int_{B_r} |y|^a w^2 \leq Cr^{n+4}.\]
Thus from the subsolution estimates as in [16], we have that
\[\sup_{B_{r/2}} w \leq Cr^{\frac{3-a}{2}}\]
from which we obtain that
\[\sup_{B_{r/2}} U^+ \leq Cr^{\frac{3-a}{2}}.\]
And analogous argument holds for \(U^-\) and we thus conclude that (5.2) holds.

We also note that the following gap of frequency follows from Theorem 5.7 in [10] which concerns the degree of homogeneous global solutions to the constant coefficient Signorini problem (i.e. \(A = I\)) with zero obstacle.

**Lemma 5.2.** Let \(0 \in \Gamma(U)\) and assume that \(A(0) = I\). Then either \(\tilde{N}(0^+) = \frac{3-a}{2}\) or \(\tilde{N}(0^+) \geq \frac{3+a}{2}\).

We now proceed with the proof of optimal regularity as stated in Theorem 1.1.

**Proof of Theorem 1.1.** The proof is similar to that of Theorem 3.11 in view of the improved decay estimate in (5.1). We nevertheless provide the complete details. By subtracting off the obstacle, we may assume that \(U\) solves (3.4) with \(f\) independent of \(y\). Given \(X \in \mathbb{H}_1^+\), let \(d(X) = d(X, \Gamma(U))\). We note that in \(\mathbb{B}_{d(X)}(X) \cap \{y = 0\}\), either \(\partial_y^a U\) or \(U\) identically vanishes. Therefore by even or odd reflection, we have that \(U\) solves in \(\mathbb{B}_{d(X)}(X)\),
\[\text{div}(|y|^a A(x) \nabla U) = |y|^a f\]
and moreover from (5.1) we have,
\[(5.8) \quad ||U||_{L^{\infty}(\mathbb{B}_{d(X)})} \leq Cd(X)^{\frac{3-a}{2}}.\]
Then from the estimate (3.75) it follows by using scaled versions of the estimates in Theorem 2.1 or 2.3 that the following gradient bounds holds,
\[(5.9) \quad |\nabla_x U(X)| \leq Cd(X)^{\frac{1-a}{2}}.\]
We now take points $X^1, X^2$ and let $d_i = d(X^i, \Gamma(U))$ for $i = 1, 2$. Without loss of generality assume that $d_1 \geq d_2$. We also set $\delta = |(X^1 - X^2)|$. There exist two possibilities: (a) $\delta \geq \frac{1}{\delta} d_1$; or, (b) $\delta < \frac{1}{\delta} d_1$. If (a) occurs, it follows from (5.9) that
\[
|\nabla_x U(X^1) - \nabla_x U(X^2)| \leq \frac{C}{d_1^{\frac{1-a}{2}}} + \frac{C}{d_2^{\frac{1-a}{2}}} \leq C|\delta|^{\frac{1-a}{2}}.
\]
If (b) occurs, then we have that $X^2 \in B_{d_1}(X^1)$. It follows from the rescaled estimates in Theorem 2.1 or 2.3 (corresponding to $\beta = \frac{1-a}{2}$) that the following holds,
\[
|\nabla_x U(X^1) - \nabla_x U(X^2)| \leq \frac{C}{d_1^{\frac{1-a}{2}}} (|U|_{L^\infty(B_{d_1}(X^1))} + d_2^2 ||f||_{L^\infty}) |\delta|^{\frac{1-a}{2}}
\]
\[
\leq C|\delta|^{\frac{1-a}{2}}.
\]
where we also used the decay estimate in (5.8) above. Thus in both cases, we obtain,
\[
|\nabla_x U(X^1) - \nabla_x U(X^2)| \leq C|X^1 - X^2|^{\frac{1-a}{2}}.
\]
We now establish the optimal Hölder regularity for $y^a U_y$. Again given $X \in \mathbb{B}_1^\perp$, we note that either $U \equiv 0$ or $\partial_y U \equiv 0$ on $\mathbb{B}_{d(X)}(X) \cap \{y = 0\}$. If $y^a U_y \equiv 0$, then $w = y^a U_y$ can be oddly reflected across $\{y = 0\}$ so that it solves
\[
\text{div}(|y|^{-a} A \nabla w) = 0
\]
in $\mathbb{B}_{d(X)}(X)$. We then claim that the following decay estimate holds for $y^a U_y$ near a free boundary point,
\[
|y^a U_y(X)| \leq C d(X)^{\frac{1+a}{2}}.
\]
The estimate (5.11) is a consequence of the following Moser type estimate as in [16],
\[
|y^a U_y(X)| \leq \frac{C}{d(X)^{\frac{a+1}{2}}} \left( \int_{\mathbb{B}_{d(X)/4}(X)} |y|^{-a} (|y|^a U_y)^2 \right)^{\frac{1}{2}},
\]
combined with the following energy estimate,
\[
\int_{\mathbb{B}_{d(X)/4}(X)} |y|^{-a} (|y|^a U_y)^2 \leq \frac{C}{d(X)^2} \int_{\mathbb{B}_{d(X)/2}(X)} |y|^a U^2
\]
and the decay estimate for $U$ as in (5.8). If instead $U \equiv 0$ in $\mathbb{B}_{d(X)}(X) \cap \{y = 0\}$, then we can use the estimate (2.18) and the decay for $U$ in (5.8) to again deduce that (5.11) holds.

We now establish the $C^{1+a}$ regularity of $y^a U_y$ in $\mathbb{B}_1^\perp$. Again we take points $X^1, X^2$ and let $d_i = d(X^i, \Gamma(U))$ for $i = 1, 2$. Without loss of generality assume that $d_1 \geq d_2$. We also set $\delta = |(X^1 - X^2)|$. There exist two possibilities: (a) $\delta \geq \frac{1}{\delta} d_1$; or, (b) $\delta < \frac{1}{\delta} d_1$. If (a) occurs, the using the decay estimate in (5.11) we obtain,
\[
|y^a U_y(X^1) - y^a U_y(X^2)| \leq |y^a U_y(X^1)| + |y^a U_y(X^2)| \leq C \left( d_1^{\frac{1+a}{2}} + d_2^{\frac{1+a}{2}} \right) \leq C|\delta|^{\frac{1+a}{2}}.
\]
If instead (b) occurs, then we have that $X^2 \in B_{d_1}(X^1)$. Then as before, we note that either $U$ or $\partial_y U$ vanishes identically on $\mathbb{B}_{d_1}(X^1) \cap \{y = 0\}$. If $U \equiv 0$ on $B_{d_1}(X^1) \cap \{y = 0\}$, then
from the scaled version of the estimate (2.18) in Theorem 2.3 (corresponding to \( \alpha = \frac{1+a}{2} \)) we have that,

\[
|y^a U_y(X^1) - y^a U_y(X^2)| \leq \frac{C}{d_1^{\alpha+a}} \left( ||U||_{L^\infty(\mathbb{B}_{d_1}(X^1))} + d_1^2 ||f||_{L^\infty} \right) \delta^{\frac{1+a}{2}} \leq C \delta^{\frac{1+a}{2}},
\]

where in the last inequality in (5.12) above, we used the decay estimate (5.8) for \( X = X^1 \). On the other hand, if \( \partial_y^2 U \equiv 0 \) on \( \mathbb{B}_{d_1}(X^1) \setminus \{y = 0\} \), then we can extend \( y^a U_y \) in an odd way across \( \{y = 0\} \) so that it is an odd in \( y \) solution to (5.10) in \( \mathbb{B}_{d_1}(X^1) \). Now from the rescaled estimate in Theorem 2.4 (corresponding to \( \alpha = \frac{1+a}{2} \)) we get,

\[
|y^a U_y(X^1) - y^a U_y(X^2)| \leq \frac{C}{d_1^{\alpha+a}} ||y^a U_y||_{L^\infty(\mathbb{B}_{d_1}(X^1))} \delta^{\frac{1+a}{2}} \leq C \delta^{\frac{1+a}{2}},
\]

where in the last inequality in (5.13) above, we used the estimate (5.11) for \( X = X^1 \). The conclusion thus follows.

\[\square\]

**Remark 5.3.** We note that it is not true that the solution is \( C^{1,s} \) in the \( y \) variable. See for instance Remark 4.5 in [10] for further discussion on this aspect.

6. Smoothness of the regular set of the free boundary

We now define the notion of regular points to (2.2). Let \( (x_0, 0) \in \Gamma(U) \). Let \( U_{x_0}(x,y) = U(x_0 + A^\frac{1}{a}(x_0)x, y) \), \( A_{x_0}(x,y) = A_{x_0}^{-\frac{1}{a}}(x_0)A(x_0 + A^\frac{1}{a}(x_0)x)A_{x_0}^{-\frac{1}{a}}(x_0) \). Under this normalization, we have that \( U_{x_0} \) solves (2.2) corresponding to the new matrix \( A_{x_0} \) and moreover we have that \( 0 \in \Gamma(U_{x_0}) \) and \( A_{x_0}(0) = I \). Again by subtracting off the obstacle, we have that \( U_{x_0} \) solves a problem of the type (3.4). We thus have \( N, \hat{N} \) have limits at 0 defined with respect to the new operator \( A_{x_0} \) and for notational convenience, we denote such quantities by \( N_{x_0}, \hat{N}_{x_0} \) etc.

**Definition 6.1.** Let \( U \) be a solution of (3.4). We say that \( 0 \in \Gamma(U) \) is a regular free boundary point if \( \hat{N}(0^+) = \frac{3-a}{2} \). Likewise, we say that \( X_0 = (x_0, 0) \) is regular if \( \hat{N}_{x_0}(0^+) = \frac{3-a}{2} \). We denote by \( \Gamma_{\frac{3-a}{2}}(U) \) the set of all regular free boundary points and we call it the regular set of \( U \).

For the analysis of the regular set we will need the following result which generalises [21, Theorem 4.3].

**Theorem 6.2** (Weiss type monotonicity formula). Given a solution \( U \) to (3.4), such that \( 0 \in \Gamma_{\frac{3-a}{2}}(U) \), define

\[
W(U, r) = W(r) = \frac{\sigma(r)}{r^{3-a}} \{ J(r) - \frac{3-a}{2r} M(r) \}.
\]

There exist universal constants \( C, r_0 > 0 \), depending on \( ||f||_{L^\infty(B_1)} \), such that for any \( 0 < r < r_0 \) one has:

\[
\frac{d}{dr} (W(U, r) + Cr^{\frac{1+a}{2}}) \geq \frac{2}{r^{n+2}} \int_{S_r} \left( \frac{\langle A \nabla U, \nu \rangle}{\sqrt{\mu}} - \frac{3-a}{2r} \sqrt{\mu} U \right)^2 |y|^a \geq \frac{2}{r^{n+2}} \int_{S_r} \left( \frac{\langle A \nabla U, \nu \rangle}{\sqrt{\mu}} - \frac{3-a}{2r} \sqrt{\mu} U \right)^2 |y|^a.
\]

In particular, there exists \( C > 0 \) such that the function \( r \mapsto W(U, r) + C r^{\frac{1+a}{2}} \) is monotone increasing and therefore the limit \( W(U, 0^+) := \lim_{r \to 0^+} W(U, r) \) exists.
Proof. Differentiating (6.1) we find
\[
\frac{d}{dr} W(r) = \frac{\sigma(r)}{r^{3-a}} \left[ \left( J'(r) - J(r) \right) + \frac{3-a}{2r^2} M(r) - \frac{3-a}{2r} M'(r) \right] + \left( \frac{\sigma'(r)}{r^{3-a}} - \frac{3-a}{r^4-a} \right) \left( J(r) - \frac{3-a}{2r} M(r) \right) = \sigma(r) \left[ \frac{\sigma'(r)}{\sigma(r)} - \frac{3-a}{r} \right] \left( J(r) - \frac{3-a}{2r} M(r) \right) + \left( J'(r) + \frac{3-a}{2r^2} M(r) - \frac{3-a}{2r} M'(r) \right).
\]

After some easy computations, by recalling the expression we found for \( J'(r) \) in the proof of Theorem 4.19, we get
\[
\frac{d}{dr} W(r) = \sigma(r) \left[ \left( \frac{\psi'(r)}{\psi(r)} - \frac{n-1+a}{r} - \frac{3-a}{r} \right) J(r) - \frac{3-a}{2r} M(r) + \left( \frac{n-1+a}{r} - \psi'(r) \right) J(r) + O(1) \right] + \frac{1}{\psi(r)} \left( 2 \int_{S_r} \frac{(\langle A\nabla U, \nu \rangle)^2}{\mu} |y|^a + \int_{S_r} U f |y|^a - \frac{n-1+a}{r} + O(1) \right) \int_{S_r} U f |y|^a - \frac{3-a}{2r} M'(r) + \frac{3-a}{2r^2} M(r). \]

Now from the proof of Theorem 4.19 we have that \( M'(r) = 2J(r) \) and hence using this we obtain,
\[
\frac{d}{dr} W(r) = \sigma(r) \left[ \left( \frac{\psi'(r)}{\psi(r)} - \frac{n-1+a}{r} - \frac{3-a}{r} + \frac{n-1+a}{r} - \psi'(r) + O(1) \right) J(r) + \frac{2}{\psi(r)} \int_{S_r} \langle A\nabla U, \nu \rangle^2 \frac{|y|^a}{\mu} + \left( \frac{3-a}{2r^2} - \frac{3-a}{2r} \psi'(r) - \frac{n-1+a}{r} - \frac{3-a}{r} \psi'(r) \right) M(r) - \frac{1}{\psi(r)} \left( \frac{n-1+a}{r} + O(1) \right) \int_{S_r} U f |y|^a + \frac{2}{r \psi(r)} \int_{S_r} \langle Z, \nabla U \rangle f |y|^a \right] + \frac{1}{\psi(r)} \int_{S_r} U f |y|^a.
\]

Proceeding further, we get,
\[
\frac{d}{dr} W(r) = \sigma(r) \left[ 2 \left( -\frac{3-a}{r} + O(1) \right) J(r) + \frac{2}{\psi(r)} \int_{S_r} \frac{(\langle A\nabla U, \nu \rangle)^2}{\mu} |y|^a \right] + \frac{3-a}{2r^2} \left( 1 - \frac{r \psi'(r)}{\psi(r)} - (n-1+a) - 3+a \right) M(r) - \frac{1}{\psi(r)} \left( \frac{n-1+a}{r} + O(1) \right) \int_{S_r} U f |y|^a + \frac{2}{r} \int_{S_r} \langle Z, \nabla U \rangle f |y|^a - \int_{S_r} U f |y|^a \right]\]
\[
= \sigma(r) \left[ 2 \left( -\frac{3-a}{r} + O(1) \right) J(r) + \frac{2}{\psi(r)} \int_{S_r} \frac{(\langle A\nabla U, \nu \rangle)^2}{\mu} |y|^a + \frac{3-a}{2r^2} \left( -\frac{r \psi'(r)}{\psi(r)} + 3+n \right) M(r) - \frac{1}{\psi(r)} \left( \frac{n-1+a}{r} + O(1) \right) \int_{S_r} U f |y|^a + \frac{2}{r} \int_{S_r} \langle Z, \nabla U \rangle f |y|^a - \int_{S_r} U f |y|^a \right]\]
\[
= \frac{\sigma(r)}{r^{3-a}} \left[ \left( \frac{\psi'(r)}{\psi(r)} - \frac{n-1+a}{r} - \frac{3-a}{r} \right) J(r) - \frac{3-a}{2r} M(r) + \left( \frac{n-1+a}{r} - \psi'(r) \right) J(r) + O(1) \right] + \frac{1}{\psi(r)} \left( 2 \int_{S_r} \frac{(\langle A\nabla U, \nu \rangle)^2}{\mu} |y|^a + \int_{S_r} U f |y|^a - \frac{n-1+a}{r} + O(1) \right) \int_{S_r} U f |y|^a - \frac{3-a}{2r} M'(r) + \frac{3-a}{2r^2} M(r). \]
Now from (4.13) and Lemma 4.6, we observe that \( \frac{\psi'(r)}{\psi(r)} = \frac{n+a}{r} + O(1) \implies \frac{\psi'(r)}{\psi(r)} = n + a + O(r) \). Subsequently by recalling the definitions of \( J(r) \) and \( M(r) \) we obtain,

\[
\begin{align*}
\frac{d}{dr} W(r) &= \frac{\sigma(r)}{r^{3-a}} \left[ 2 \left( -\frac{3-a}{r} + O(1) \right) J(r) + \frac{2}{\psi(r)} \int_{S_r} \frac{(A\nabla U, \nu)^2}{\mu} |y|^a + \frac{3-a}{2r^2} (-n-a+3+n+O(r)) M(r) \right] \\
& \quad - \frac{1}{\psi(r)} \left[ \left( \frac{n-1+a}{r} + O(1) \right) \int_{B_r} U f |y|^a + \frac{2}{r} \int_{B_r} \langle Z, \nabla U \rangle f |y|^a - \int_{S_r} U f |y|^a \right] \\
& = \frac{2\sigma(r)}{r^{3-a} \psi(r)} \left[ \left( -\frac{3+a}{r} + O(1) \right) I(r) + \int_{S_r} \frac{(A\nabla U, \nu)^2}{\mu} |y|^a + \frac{3-a}{2r^2} (1 + O(r)) H(r) \right] \\
& \quad + \frac{\sigma(r)}{r^{3-a} \psi(r)} \left[ - \left( \frac{n-1+a}{r} + O(1) \right) \int_{B_r} U f |y|^a - \frac{2}{r} \int_{B_r} \langle Z, \nabla U \rangle f |y|^a + \int_{S_r} U f |y|^a \right].
\end{align*}
\]

We also have,

\[
\int_{S_r} \left( \frac{(A\nabla U, \nu)}{\sqrt{\mu}} - \frac{3-a}{2r} U \sqrt{\mu} \right)^2 |y|^a = \int_{S_r} \frac{(A\nabla U, \nu)^2}{\mu} |y|^a + \frac{3-a}{2r^2} \int_{S_r} U^2 \frac{\mu}{\mu} |y|^a \\
\quad - \frac{3-a}{r} \int_{S_r} U (A\nabla U, \nu) |y|^a \\
\quad = - \frac{3-a}{r} I(r) + \int_{S_r} \frac{(A\nabla U, \nu)^2}{\mu} |y|^a + \frac{3-a}{2r^2} H(r).
\]

Thus

\[
\begin{align*}
\frac{d}{dr} W(r) &= \frac{2\sigma(r)}{r^{3-a} \psi(r)} \left[ \int_{S_r} \left( \frac{(A\nabla U, \nu)}{\sqrt{\mu}} - \frac{3-a}{2r} \sqrt{\mu} U \right)^2 |y|^a + O(1) I(r) + \frac{O(1)}{r} H(r) \right] \\
& \quad + \frac{\sigma(r)}{r^{3-a} \psi(r)} \left[ - \left( \frac{n-1+a}{r} + O(1) \right) \int_{B_r} U f |y|^a - \frac{2}{r} \int_{B_r} \langle Z, \nabla U \rangle f |y|^a + \int_{S_r} U f |y|^a \right].
\end{align*}
\]

Now since \( \frac{\psi(r)}{r^{3-a} \psi(r)} = \frac{1}{r^{1+n}} \), therefore we have that \( \frac{\sigma(r)}{r^{3-a} \psi(r)} = \frac{1}{r^{1+n}} \). Using this we obtain,

\[
\begin{align*}
\frac{d}{dr} W(r) &= \frac{2}{r^{2+n}} \left[ \int_{S_r} \left( \frac{(A\nabla U, \nu)}{\sqrt{\mu}} - \frac{3-a}{2r} \sqrt{\mu} U \right)^2 |y|^a + O(1) I(r) + \frac{O(1)}{r} H(r) \right] \\
& \quad + \frac{1}{r^{2+n}} \left[ - \left( \frac{n-1+a}{r} + O(1) \right) \int_{B_r} U f |y|^a - \frac{2}{r} \int_{B_r} \langle Z, \nabla U \rangle f |y|^a + \int_{S_r} U f |y|^a \right].
\end{align*}
\]

Now by applying the Cauchy-Schwarz inequality and also by using Theorem 5.1 we have,

\[
|I(r)| \leq \int_{S_r} |U| |\langle A\nabla U, \nu \rangle| |y|^a \leq \left( \int_{S_r} U^2 |y|^a \right)^{\frac{1}{2}} \left( \int_{S_r} (A\nabla U, \nu)^2 |y|^a \right)^{\frac{1}{2}} \leq C r^{(n+a)/2} \int_{S_r} \langle A\nabla U, \nu \rangle^2 |y|^a \frac{1}{2}.
\]

Now again since \( 0 \in \Gamma(U) \) which in particular implies that \( y^a U_y(0) = \nabla_x U(0) = 0 \), therefore using Theorem 1.1 we infer that the following estimate holds,

\[
|y|^a (A\nabla U, \nu) \leq C |y|^a |\nabla U| \leq C (|y|^a |\nabla_x U| + |y|^a |\partial_y U|) \leq C r^{\frac{1+n}{2}}.
\]
Thus $O(1)I(r) = O(r^{n+2})$. Also by (5.3) we have $\frac{O(1)H(r)}{r} = O(r^{n+2})$ and hence we obtain,
\[
\frac{d}{dr} W(r) = \frac{2}{r^{2+n}} \int_{S_r} \left( \frac{\langle A\nabla U, \nu \rangle}{\sqrt{\mu}} - 3 - a \frac{\sqrt{\mu U}}{2r} \right)^2 |y|^a + O(1)
\]
\[
+ \frac{1}{r^{2+n}} \left[ - \left( \frac{n-1+a}{r} + O(1) \right) \int_{B_r} U f^a - \frac{2}{r} \int_{B_r} \langle Z, \nabla U \rangle f |y|^a + \int_{S_r} U f |y|^a \right].
\]
Again by Theorem 1.1, we have,
\[
\left| - \frac{2}{r} \int_{B_r} \langle Z, \nabla U \rangle f |y|^a \right| \leq \frac{2}{r} \int_{B_r} |Z| |\nabla U| |f| |y|^a \leq \frac{C}{r} r^{\frac{1+a}{2}} r^{n+1} = C r^{n+1+\frac{a}{2}}
\]
and by Theorem 5.1 we also have,
\[
\left| - \left( \frac{n-1+a}{r} + O(1) \right) \int_{B_r} |y|^a U f + \int_{S_r} |y|^a U f \right| \leq C r^{n+1+\frac{a}{2}}.
\]
Now since $\frac{a-1}{2} < 0$, thus we have that $O(1) \geq -C' \geq -C'' \frac{a-1}{2}$ and so we finally obtain,
\[
\frac{d}{dr} W(r) \geq \frac{2}{r^{2+n}} \int_{S_r} \left( \frac{\langle A\nabla U, \nu \rangle}{\sqrt{\mu}} - 3 - a \frac{\sqrt{\mu U}}{2r} \right)^2 |y|^a - C r^{\frac{a-1}{2}}
\]
which concludes the proof. 

Proof of Theorem 1.2. Now given the Weiss type monotonicity as in Theorem 6.2, together with the epiperimetric inequality established in [22] (see Theorem 4.2 in [22]), we can argue as in [21] and [22] to conclude that locally $\Gamma_{\frac{a}{n+a}}$ is a $C^{1,\gamma}$ graph for some $\gamma > 0$. 

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