Generalization in Ball Banach Function Spaces of Brezis–Van Schaftingen–Yung Formulae with Applications to Fractional Sobolev and Gagliardo–Nirenberg Inequalities

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Abstract
Let $X$ be a ball Banach function space on $\mathbb{R}^n$. In this article, under the mild assumption that the Hardy–Littlewood maximal operator is bounded on the associated space $X'$ of $X$, the authors prove that, for any $f \in C^2_c(\mathbb{R}^n)$,

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \frac{f(\cdot) - f(y)}{|\cdot - y|^{\frac{n}{q} + 1}} \right\|_{X} \sim \|\nabla f\|_X
\]

with the positive equivalence constants independent of $f$, where $q \in (0, \infty)$ is an index depending on the space $X$, and $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. Particularly, when $X := L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$, the above estimate holds true for any given $q \in [1, p]$, which when $q = p$ is exactly the recent surprising formula of H. Brezis, J. Van Schaftingen, and P.-L. Yung, and which even when $q < p$ is new. This generalization has a wide range of applications and, particularly, enables the authors to establish new fractional Sobolev and Gagliardo–Nirenberg inequalities in various function spaces, including Morrey spaces, mixed-norm Lebesgue spaces, variable Lebesgue spaces, weighted Lebesgue spaces, Orlicz spaces, and Orlicz-slice (generalized amalgam) spaces, and, even in all these special cases, the obtained results are new. The proofs of these results strongly depend on the Poincaré inequality, the extrapolation, the exact operator norm on $X'$ of the Hardy–Littlewood maximal operator, and the geometry of $\mathbb{R}^n$.

1 Introduction

It is well known that, for any given $s \in (0, 1)$ and $p \in [1, \infty)$, the *homogeneous fractional Sobolev space* $W^{s,p}(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^n$ having the following finite Gagliardo semi-norm

\[
\|f\|_{W^{s,p}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}} := \left\| \frac{f(x) - f(y)}{|x - y|^{\frac{n}{p} + s}} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^n)}.
\]

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These spaces play a key role in harmonic analysis and partial differential equations (see, for instance, [11, 53, 60, 15, 16, 54]).

A well-known drawback of the Gagliardo semi-norm in (1.1) is that one can not recover the homogeneous Sobolev semi-norm \( \| \nabla f \|_{L^p(\mathbb{R}^n)} \) when \( s = 1 \), in which case the integral in (1.1) is infinite unless \( f \) is a constant (see [9, 12]), here and thereafter, for any differentiable function \( f \) on \( \mathbb{R}^n \), \( \nabla f \) denotes the gradient of \( f \), namely,

\[
\nabla f := \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).
\]

An important approach to recover \( \| \nabla f \|_{L^p(\mathbb{R}^n)} \) out of the Gagliardo semi-norms is due to Bourgain et al. [10] who in particular proved that, for any given \( p \in [1, \infty) \) and for any \( f \in W^{1,p}(\mathbb{R}^n) \),

\[
\lim_{s \rightarrow 1} \left(1 - s\right) \| f \|_{W^{s,p}(\mathbb{R}^n)}^p = C_{(p,n)} \| \nabla f \|_{L^p(\mathbb{R}^n)}^p,
\]

where \( C_{(p,n)} \) is a positive constant depending only on \( p \) and \( n \). Very recently, Brezis et al. [13] discovered an alternative way to repair this defect by replacing the \( L^p \) norm in (1.1) with the weak \( L^p \) quasi-norm, namely, \( \| \cdot \|_{L^{p,\infty}(\mathbb{R}^n \times \mathbb{R}^n)} \). For any given \( p \in [1, \infty) \), Brezis et al. in [13] proved that there exist positive constants \( C_1 \) and \( C_2 \) such that, for any \( f \in C_c^{\infty}(\mathbb{R}^n) \),

\[
C_1 \| \nabla f \|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{f(x) - f(y)}{|x - y|^{\frac{p}{p} + 1}} \right\|_{L^{p,\infty}(\mathbb{R}^n \times \mathbb{R}^n)} \leq C_2 \| \nabla f \|_{L^p(\mathbb{R}^n)},
\]

where

\[
\left\| \frac{f(x) - f(y)}{|x - y|^{\frac{p}{p} + 1}} \right\|_{L^{p,\infty}(\mathbb{R}^n \times \mathbb{R}^n)} := \sup_{\lambda \in (0, \infty)} \lambda \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|f(x) - f(y)|}{|x - y|^{\frac{p}{p} + 1}} > \lambda \right\}^{\frac{1}{p}},
\]

here and thereafter, the symbol \( |E| \) denotes the Lebesgue measure of a measurable set \( E \subset \mathbb{R}^m \) for any given \( m \in \mathbb{N} \), and the symbol \( C_c^{\infty}(\mathbb{R}^n) \) the set of all infinitely differentiable functions on \( \mathbb{R}^n \) with compact support. The equivalence (1.2) in particular allows Brezis et al. in [13] to derive some surprising alternative estimates of fractional Sobolev and Gagliardo–Nirenberg inequalities in some exceptional cases involving \( W^{1,1}(\mathbb{R}^n) \), where the anticipated fractional Sobolev and Gagliardo–Nirenberg inequalities fail. For later discussions, we use the Fubini theorem to write the weak \( L^p \)-norm in (1.3) and the corresponding Gagliardo semi-norm in (1.1), respectively, as

\[
\left\| \frac{f(x) - f(y)}{|x - y|^{\frac{p}{p} + 1}} \right\|_{L^{p,\infty}(\mathbb{R}^n \times \mathbb{R}^n)} = \sup_{\lambda \in (0, \infty)} \lambda \left\{ \int_{\mathbb{R}^n} \left| y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^{\frac{p}{p} + 1} \right| dx \right\}^{1/p}
\]
and
\[
\|f\|_{W^{n,p}(\mathbb{R}^n)} = \left\| \left[ \int_{\mathbb{R}^n} \left| \frac{f(x) - f(y)}{y^n + \lambda^p} \right| dy \right]^{\frac{1}{p}} \right\|_{L^p(\mathbb{R}^n)}.
\]

Consequently, the estimate (1.2) takes the following form: for any \( f \in C^\infty_c(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \left| y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^{\frac{n+1}{p}} \right| dx \right]^{1/p} \sim \|\nabla f\|_{L^p(\mathbb{R}^n)}
\]
with the positive equivalence constants independent of \( f \). More related works can be found in [30, 14].

Let us also give a few comments on the proof of (1.2) in [13]. The proof of the lower bound is relatively simpler. Indeed, a substantially sharper lower bound was obtained in [13], using a method of rotation and the Taylor remainder theorem. On the other hand, as was pointed out in [13], the stated upper bound for any given \( p \in (1, \infty) \) can be easily deduced from the following Lusin–Lipschitz inequality in [8]: for any differentiable function \( f \) and any \( x, y \in \mathbb{R}^n \),
\[
|f(x) - f(y)| \leq |x - y| |\nabla f(x)| + |\nabla f(y)|,
\]
where the implicit positive constant is independent of \( x, y, \) and \( f \). Here and thereafter, the Hardy–Littlewood maximal operator \( \mathcal{M} \) is defined by setting, for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) (the set of all locally integrable functions on \( \mathbb{R}^n \)) and \( x \in \mathbb{R}^n \),
\[
\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,
\]
where the supremum is taken over all balls \( B \subset \mathbb{R}^n \) containing \( x \). Thus, the hard core of the proof of the upper bound in (1.2) concerns the case \( p = 1 \). The proof in [13], which actually works for the full range \( p \in [1, \infty) \), uses the Vitali covering lemma in one variable, and a method of rotation. Thus, the rotation invariance of the space \( L^p(\mathbb{R}^n) \) seems to play a vital role in the proof of (1.2) in [13].

The main purpose in this article is to give an essential extension of the main results [particularly, the equivalence (1.4)] of [13]. Such extensions are fairly nontrivial because our setting typically involves function spaces that are neither rotation invariance nor translation invariance. We use the symbol \( C^2_s(\mathbb{R}^n) \) to denote the set of all twice continuously differentiable functions on \( \mathbb{R}^n \) with compact support. Somewhat surprisingly, even returning to the standard Lebesgue spaces \( L^p(\mathbb{R}^n) \), we have the following new estimate (see Theorem 4.15 below): for any given \( 1 \leq q \leq p < \infty \) and for any \( f \in C^2_s(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \left| y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^{\frac{n+1}{q}} \right| d^n x \right]^{\frac{1}{q}} \sim \|\nabla f\|_{L^p(\mathbb{R}^n)},
\]
where the positive equivalence constants are independent of \( f \). In the case of \( p = q \), (1.7) is exactly the surprising estimate (1.4) of Brezis et al. [13].

Our main result extends the results of Brezis et al. [13] to a wide class of function spaces on \( \mathbb{R}^n \), including Morrey spaces, mixed-norm Lebesgue spaces, variable Lebesgue spaces, weighted Lebesgue spaces, Orlicz spaces, and Orlicz-slice spaces (see, respectively, Subsections 4.1 through 4.6 below for their histories and definitions). We treat these spaces in a uniform manner in the setting of ball quasi-Banach function spaces recently introduced by Sawano et al. [68]. Ball quasi-Banach function spaces are quasi-Banach spaces of measurable functions on \( \mathbb{R}^n \) in which the quasi-norm is related to the Lebesgue measure on \( \mathbb{R}^n \) in an appropriate way (see Definition 2.7 below). These function spaces play an important role in many branches of analysis. They are less restrictive than the classical Banach function spaces introduced in the book [7, Chapter 1]. For more studies on ball quasi-Banach function spaces, we refer the reader to [65, 64, 68, 77, 78, 17] for the Hardy space associated with ball quasi-Banach function spaces, to [79, 33, 75] for the boundedness of operators on ball quasi-Banach function spaces, and to [41, 42, 76, 37, 72] for the applications of ball quasi-Banach function spaces.

Our aim in this part is to establish the following analogue of (1.4) for the quasi-norm \( \| \cdot \|_X \) of a ball quasi-Banach function space \( X \) (see Theorem 3.4 below): for any \( f \in C_c^2(\mathbb{R}^n) \),

\[
(1.8) \quad \sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |x - y|^q \right\} \right\|_X \sim \| \nabla f \|_X,
\]

where \( q \in (0, \infty) \) is an index depending on \( X \) and the positive equivalence constants are independent of \( f \). In particular, returning to the special case of \( X = L^p(\mathbb{R}^n) \), we obtain the estimate (1.7), which when \( p = q \) is just (1.2) obtained in [13], and which when \( q < p \) seems new. Similarly to the case of \( X = L^p(\mathbb{R}^n) \) as in [13], (1.8) also allows us to extend the fractional Sobolev and Gagliardo–Nirenberg inequalities to the setting of ball quasi-Banach function spaces (see Corollaries 3.9 and 3.11 below).

The formula (1.8) gives an equivalence between the Sobolev semi-norm and a quantity involving the difference of the function \( f \). It is quite remarkable that such an equivalence holds true for a ball Banach function space \( X \). Indeed, finding an appropriate way to characterize smoothness of functions via their finite differences is a notoriously difficult problem in approximation theory, even for some simple weighted Lebesgue space in one dimension (see [48, 52] and the references therein). A major difficulty comes from the fact that difference operators \( \Delta_h f := f(\cdot + h) - f(\cdot) \) for any \( h \in \mathbb{R}^n \) are no longer bounded on general weighted \( L^p \) spaces. It turns out that, via using the extrapolation in [22] and the exact operator norm on the associate space of \( X \) of the Hardy–Littlewood maximal operator, the estimate (1.8) in \( X \) follows from the following estimates in weighted Lebesgue spaces with Muckenhoupt weights.

**Theorem 1.1.** Let \( p \in [1, \infty) \) and \( \omega \in A_1(\mathbb{R}^n) \). Then, for any \( f \in C_c^2(\mathbb{R}^n) \),

\[
(1.9) \quad \sup_{\lambda \in (0, \infty)} \lambda \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{E_f(\lambda, p)}(x, y) \omega(x) \, dx \sim \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
\]

where, for any \( \lambda \in (0, \infty) \),

\[ E_f(\lambda, p) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^q \} \]

and the positive equivalence constants are independent of \( f \).
Indeed, we prove a promoted version of Theorem 1.1 (see Theorem 2.3 below). The $A_p(\mathbb{R}^n)$-condition on the weights $\omega$ in Theorem 1.1 is necessary in some sense in the case of $n = 1$ (see Theorem 2.5 below). Note that, unlike the integral $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \cdot \cdot \cdot \cdot \cdot dx \, dy$ in the estimate (1.2), the integral $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \cdots \cdot dy \omega(x) \, dx$ in (1.9) is not symmetric with respect to $x$ and $y$, which causes additional technical difficulties in the proof of the upper bound in (1.9). Indeed, the proof of the upper estimate in (1.9) is fairly nontrivial. On one hand, the proof of (1.2) in the unweighted case in the article [13] is based on a method of rotation, and seems to be inapplicable in the weighted case here. On the other hand, using the Lusin-Lipschitz inequality (1.5) would give the stated upper estimate in (1.9) is fairly nontrivial. On one hand, the proof of (1.2) in the unweighted case as in [13], we use several adjacent systems of dyadic cubes in $\mathbb{R}^n$ and hence the geometry of $\mathbb{R}^n$ (see, for instance, [50, Section 2.2]) to overcome these obstacles.

The remainder of this article is organized as follows.

Section 2 is devoted to the proof of Theorem 1.1, which characterizes the Sobolev semi-norm in weighted Lebesgue spaces. To prove Theorem 1.1, we prove a more general result (see Theorem 2.3 below), which plays an essential role in the proof of Theorem 3.4. First, we establish the lower estimate of Theorem 2.3 in ball quasi-Banach function spaces (see Theorem 2.14 below), which is a part of Theorem 3.4. In the proof of the upper estimate in Theorem 2.3, as was aforementioned, since the integral in the estimate (2.1) in Theorem 2.3 is not symmetric, the method of rotation seems to be inapplicable in the weighted case here. Instead of applying the Vitali covering lemma, we use several adjacent systems of dyadic cubes in $\mathbb{R}^n$ and hence the geometry of $\mathbb{R}^n$ (see, Lemma 2.18 below) to overcome these obstacles. Finally, we prove Theorem 2.5 which shows that the $A_p(\mathbb{R}^n)$-condition on the weight $\omega$ in Theorem 1.1 is necessary in some sense in the case of $n = 1$.

In Section 3, we generalize (1.2) to ball Banach function spaces. However, the calculations in [13] need to use the following three crucial properties of $L^p(\mathbb{R}^n)$, which are not available for ball Banach function spaces: the rotation invariance, the translation invariance, and the explicit expression of the norm. Borrowing some ideas from the extrapolation theorem in [22], using Theorem 2.3 and the exact operator norm on the associate space of $X$ of the Hardy–Littlewood maximal operator, we establish the characterization of the Sobolev semi-norm in ball Banach function spaces (see Theorems 3.4 and 3.7 below). As applications, we also establish alternative fractional Sobolev and Gagliardo–Nirenberg inequalities in ball Banach function spaces (see Corollaries 3.9 and 3.11 below).

In Section 4, we apply all these results obtained in Section 3, respectively, to $X := M^p_\rho(\mathbb{R}^n)$ (the Morrey space), $X := L^{p/(n)}(\mathbb{R}^n)$ (the variable Lebesgue space), $X := L^p_j(\mathbb{R}^n)$ (the mixed-norm Lebesgue space), $X := L^p(\mathbb{R}^n)$ (the weighted Lebesgue space), $X := L^p(\mathbb{R}^n)$ (the Orlicz space), or $X := (E^p_\rho)_j(\mathbb{R}^n)$ (the Orlicz-slice space or the generalized amalgam space), all these results are totally new.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We always denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\alpha, \beta, \ldots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \beta, \ldots$. The symbol $f \lesssim g$ means that $f \leq C g$. If $f \leq g$ and $g \leq f$, we then write $f \sim g$. If $f \leq C g$ and $g = h$ or $g \leq h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \sim g = h$ or $f \sim g \sim h$. We use $0$ to denote the origin of $\mathbb{R}^n$. If $E$ is a subset of $\mathbb{R}^n$, we denote by $1_E$ its
characteristic function and, for any measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$, and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, let
\[
\int_E f(x) \, dx := \frac{1}{|E|} \int_E f(x) \, dx =: f_E.
\]
For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and
\[
B := B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty).
\]
(1.10)

For any $\alpha \in (0, \infty)$ and any ball $B := B(x_B, r_B) \in \mathbb{R}^n$, with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, let $\alpha B := B(x_B, \alpha r_B)$. Finally, for any $q \in [1, \infty]$, we denote by $q'$ its conjugate exponent, namely, $1/q + 1/q' = 1$.

2 Estimates in weighted Lebesgue spaces

In this section, we establish the characterization of the Sobolev semi-norm in the weighted Lebesgue space (see Theorem 2.3 below), which is just Theorem 1.1 when $p = q$. We should point out that Theorem 2.3 plays a vital role in the proof of Theorem 3.4 below. Moreover, we show that the $A_p(\mathbb{R}^n)$ condition in Theorem 1.1 is sharp in some sense (see Theorem 2.5 below).

Let us first recall the notion of Muckenhoupt weights $A_p(\mathbb{R}^n)$ (see, for instance, [29]).

**Definition 2.1.** An $A_p(\mathbb{R}^n)$-weight $\omega$, with $p \in [1, \infty)$, is a nonnegative locally integrable function on $\mathbb{R}^n$ satisfying that, when $p \in (1, \infty)$,
\[
[\omega]_{A_p(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \frac{1}{\omega(x)} \, dx \right)^{1/q} < \infty
\]
and, when $p = 1$,
\[
[\omega]_{A_1(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q \omega(x) \, dx \left[ \|\omega^{-1}\|_{L^\infty(Q)} \right] < \infty,
\]
where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

Moreover, let $A_{\infty}(\mathbb{R}^n) := \bigcup_{p \in [1, \infty)} A_p(\mathbb{R}^n)$.

**Definition 2.2.** Let $p \in [0, \infty)$ and $\omega \in A_{\infty}(\mathbb{R}^n)$. The weighted Lebesgue space $L^p_\omega(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^n$ such that
\[
\|f\|_{L^p_\omega(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right]^{1/p} < \infty.
\]

**Theorem 2.3.** Let $p \in [1, \infty)$ and $q \in (0, \infty)$ satisfy $n(\frac{1}{p} - \frac{1}{q}) < 1$. Assume that $\omega \in A_1(\mathbb{R}^n)$. Then there exist positive constants $C_1$, $C_2$, and $C_{[\omega]_{A_1(\mathbb{R}^n)}}$ such that, for any $f \in C^2_0(\mathbb{R}^n)$,
\[
C_1 \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \leq \sup_{\lambda \in (0, \infty)} \lambda^p \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} 1_{E_f(\lambda q)}(x, y) \, dy \right)^{\frac{p}{q'}} \omega(x) \, dx
\]
(2.1)
\[
\leq C_2 C_{[\omega]_{A_1(\mathbb{R}^n)}} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
\]
Remark 2.4. (i) As a consequence of Theorem 2.3, we obtain Theorem 1.1.

(ii) Let $p \in [1, \infty)$, $q \in (0, \infty)$, and $n \max\{0, \frac{1}{q} - 1, \frac{1}{p} - \frac{1}{q}\} < s < 1$. By [36, Theorem 3], we know that, if $f \in L^\text{min}[p,q]_\text{loc}(\mathbb{R}^n)$, then $f \in F^s_{p,q}(\mathbb{R}^n)$ if and only if

\[
I := \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left( \int_{\mathbb{R}^n} \frac{|f() - f(y)|^q}{|x - y|^{n+sq}} \, dy \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty
\]

and, moreover, in this case,

\[(2.3) \quad I \sim \|f\|_{F^s_{p,q}(\mathbb{R}^n)}
\]

with the positive equivalence constants independent of $f$, where $F^s_{p,q}(\mathbb{R}^n)$ denotes the classical Triebel–Lizorkin space (see [74, Section 2.3] for the precise definition). Moreover, using (i) and (ii) of Theorem 3.2 below, we conclude that, when $s = 1$, $p \in [1, \infty)$, and $q \in [1, p]$,

\[
\left\| \left( \int_{\mathbb{R}^n} \frac{|f() - f(y)|^q}{|x - y|^{n+sq}} \, dy \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} = \infty
\]

unless $f$ is a constant. Thus, the Gagliardo quasi-semi-norm in (2.3) can not recover the Triebel–Lizorkin quasi-semi-norm $\| \cdot \|_{F^1_{p,q}(\mathbb{R}^n)}$ when $s = 1$, and, in this sense, the assumption $n(\frac{1}{p} - \frac{1}{q}) < 1$ in Theorem 2.3 seems to be sharp.

Replacing the strong type quasi-norm in (2.3) by the weak type quasi-norm, and using Theorem 2.3 with $\omega = 1$, we find that, for any $f \in C^2_\omega(\mathbb{R}^n)$,

\[
\|f\|_{L^p(\mathbb{R}^n)} + \sup_{\lambda \in (0,\infty)} \lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_\lambda}(x,y) \, dy \right]^\frac{q}{p} \, dx \right\}^{\frac{1}{p}} \sim \|f\|_{F^1_{p,q}(\mathbb{R}^n)}
\]

with the positive equivalence constants independent of $f$. This indicates that Theorem 2.3 is a perfect replacement of (2.3) in the critical case $s = 1$.

(iii) Let $p \in [1, \infty)$ and $q \in (0, \infty)$ satisfy $n(\frac{1}{p} - \frac{1}{q}) < 1$. Let $\tilde{W}^1_{L^p(\mathbb{R}^n),q_1}(\mathbb{R}^n)$ be the weak-type space $\tilde{W}^1_{X,q_1}(\mathbb{R}^n)$ in Definition 3.1 below with $X := L^p_n(\mathbb{R}^n)$. Assume that $q_1, q_2 \in (0, \infty)$ satisfy $n(\frac{1}{p} - \frac{1}{q_1}) < 1$ and $n(\frac{1}{p} - \frac{1}{q_2}) < 1$. From Theorem 2.3, it follows that

\[
\tilde{W}^1_{L^p_n(\mathbb{R}^n),q_1}(\mathbb{R}^n) \cap C^2_\omega(\mathbb{R}^n) = \tilde{W}^1_{L^p_n(\mathbb{R}^n),q_2}(\mathbb{R}^n) \cap C^2_\omega(\mathbb{R}^n)
\]
with equivalent quasi-norms. Thus, when $q \in (0, \infty)$ satisfies $n\left(\frac{1}{p} - \frac{1}{q}\right) < 1$, the space $W^{1}_{\ell^q,L^\infty,\omega}(\mathbb{R}^n) \cap C^2_\infty(\mathbb{R}^n)$ is independent of $q$.

(iv) For the purpose of our applications later, we only consider $A_1(\mathbb{R}^n)$-weights here. However, our proof actually works equally well for more general $A_p(\mathbb{R}^n)$-weights. For instance, a slight modification of the proofs in this section shows that (2.1) holds true for any given $1 \leq p = q < \infty$ and $\omega \in A_{\text{min}\{p,1+\frac{2}{n}\}}(\mathbb{R}^n)$.

**Theorem 2.5.** Let $p \in [1, \infty)$ and $\omega$ be a non-negative function on $\mathbb{R}$. Assume that there exists a positive constant $C_1$ such that, for any $f \in C^1(\mathbb{R})$ satisfying that $f'$ has compact support,

$$\sup_{\lambda \in (0, \infty)} \lambda^p \int_{\mathbb{R}^2} 1_{E_f(\lambda,p)}(x,y) \omega(x) \, dx \, dy \leq C_1 \int_{\mathbb{R}} |f'(x)|^p \omega(x) \, dx,$$

where $E_f(\lambda, p)$ is as in (2.2) for any $\lambda \in (0, \infty)$. Then $\omega \in A_p(\mathbb{R})$.

**Remark 2.6.** In one dimension, Theorem 2.5 implies that the $A_1(\mathbb{R})$ condition in Theorem 1.1 with $p = 1$ is sharp.

The proof of Theorem 2.3 is given in Subsections 2.1 and 2.2. In Subsection 2.3, we prove Theorem 2.5.

### 2.1 Proof of Theorem 2.3: lower estimate

In this subsection, we prove a generalization of the lower estimate of Theorem 2.3 on ball quasi-Banach function spaces, which plays an essential role in the proof of Theorem 3.4 below.

First, we recall some preliminaries on ball quasi-Banach function spaces introduced in [65]. Denote by the symbol $\mathcal{M}(\mathbb{R}^n)$ the set of all measurable functions on $\mathbb{R}^n$.

**Definition 2.7.** A quasi-Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$ is called a *ball quasi-Banach function space* if it satisfies

(i) $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;

(ii) $|g| \leq |f|$ almost everywhere implies that $\|g\|_X \leq \|f\|_X$;

(iii) $0 \leq f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$;

(iv) $B \in \mathcal{B}$ implies that $1_B \in X$, where $\mathcal{B}$ is as in (1.10).

Moreover, a ball quasi-Banach function space $X$ is called a *ball Banach function space* if the norm of $X$ satisfies the triangle inequality: for any $f, g \in X$,

$$\|f + g\|_X \leq \|f\|_X + \|g\|_X,$$

and that, for any $B \in \mathcal{B}$, there exists a positive constant $C(B)$, depending on $B$, such that, for any $f \in X$,

$$\int_B |f(x)| \, dx \leq C(B) \|f\|_X.$$
Remark 2.8. Observe that, in Definition 2.7, if we replace any ball $B$ by any bounded measurable set $E$, we obtain its another equivalent formulation.

The following notion of the associate space of a ball Banach function space can be found, for instance, in [7, Chapter 1, Definitions 2.1 and 2.3].

Definition 2.9. For any ball Banach function space $X$, the associate space (also called the Köthe dual) $X'$ is defined by setting

$$X' := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{X'} := \sup_{g \in X, \|g\|_X = 1} \|fg\|_{L^1(\mathbb{R}^n)} < \infty \right\},$$

where $\| \cdot \|_{X'}$ is called the associate norm of $\| \cdot \|_X$.

Remark 2.10. By [65, Proposition 2.3], we know that, if $X$ is a ball Banach function space, then its associate space $X'$ is also a ball Banach function space.

The following lemma is just [79, Lemma 2.6].

Lemma 2.11. Let $X$ be a ball Banach function space. Then $X$ coincides with its second associate space $X''$. In other words, a function $f$ belongs to $X$ if and only if it belongs to $X''$ and, in that case,

$$\|f\|_X = \|f\|_{X''}.$$

The following Hölder inequality is a direct corollary of both Definition 2.7(i) and (2.4) (see [7, Theorem 2.4]).

Lemma 2.12. Let $X$ be a ball Banach function space and $X'$ its associate space. If $f \in X$ and $g \in X'$, then $fg$ is integrable and

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_X \|g\|_{X'}.$$

Definition 2.13. Assume that $X$ is a ball quasi-Banach function space and $p \in (0, \infty)$. The $p$-convexification $X^p$ of $X$ is defined by setting $X^p := \{ f \in \mathcal{M}(\mathbb{R}^n) : |f|^p \in X \}$ equipped with the quasi-norm $\|f\|_{X^p} := \|f^p\|^1/p$.

We have the following generalization of the lower estimate of Theorem 2.3 on any ball quasi-Banach function space $X$.

Theorem 2.14. Let $X$ be a ball quasi-Banach function space and $q \in (0, \infty)$. Then, for any $f \in C^2(\mathbb{R}^n)$,

$$\liminf_{\lambda \to \infty} \lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda,q)\}}(y) \, dy \right\|_{X^q} \geq \frac{K(q,n)}{n} \|\nabla f\|_X^q,$$

where $E_f(\lambda,q)$ for any $\lambda \in (0, \infty)$ is as in (2.2), and

$$K(q,n) := \int_{S^{n-1}} |\xi| \cdot e|\xi| \, d\sigma(\xi).$$
with $e$ being some unit vector in $\mathbb{R}^n$. Moreover, if $X^\frac{1}{2}$ is a ball Banach function space, then, for any $f \in C_c^2(\mathbb{R}^n)$,

\begin{equation}
(2.6) \quad \lim_{\lambda \to 0} \lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n : (x,y) \in E_f(x,\lambda,q)\}}(y) \, dy \right\|_{X^\frac{1}{2}} = \frac{K(q,n)}{n} \|\nabla f\|_q^q.
\end{equation}

Proof. Let $q \in (0,\infty)$, $f \in C_c^2(\mathbb{R}^n)$, and $M \in (0,\infty)$ be such that $\text{supp} f \subseteq B(0,M)$. Let $K := B(0,M + 1)$. In order to prove (2.6), we first prove (2.5). For any $\lambda \in (0,\infty)$, $x \in \mathbb{R}^n$, and $\xi \in S^{n-1}$, let

$$F_f(x,\xi,\lambda,q) := \left\{ t \in (0,\infty) : \frac{|f(x + t\xi) - f(x)|^q}{t^q} > \lambda^q t^p \right\}.$$

Then, by the Fubini Theorem, we have, for any $\lambda \in (0,\infty)$ and $x \in \mathbb{R}^n$,

\begin{equation}
(2.7) \quad \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n : (x,y) \in E_f(x,\lambda,q)\}}(y) \, dy = \int_{\mathbb{S}^{n-1}} \int_0^\infty 1_{F_f(x,\xi,\lambda,q)}(t) t^{n-1} \, dt \, d\sigma(\xi).
\end{equation}

Since $f \in C_c^2(\mathbb{R}^n)$, it follows that there exist constants $L_1 \in (\|\nabla f\|_{L^\infty(\mathbb{R}^n)},\infty)$ and $L_2 \in (0,\infty)$ such that, for any $t \in (0,\infty)$, $x \in \mathbb{R}^n$, and $\xi \in S^{n-1}$,

\begin{equation}
(2.8) \quad |f(x + t\xi) - f(x)| \leq L_1 t
\end{equation}

and

\begin{equation}
(2.9) \quad |f(x + t\xi) - f(x) - t\nabla f(x) \cdot \xi| \leq L_2 t^2.
\end{equation}

Let $\lambda \in (L_1,\infty)$. By (2.8), we conclude that, for any $t \in (0,\infty)$, $x \in \mathbb{R}^n$, and $\xi \in S^{n-1}$,

\begin{equation}
(2.10) \quad \frac{|f(x + t\xi) - f(x)|^q}{t^q} \leq L_1^q t^p
\end{equation}

and hence

\begin{equation}
(2.11) \quad F_f(x,\xi,\lambda,q) = \left\{ t \in (0,(L_1/\lambda)^\frac{p}{q}) : \frac{|f(x + t\xi) - f(x)|^q}{t^q} > \lambda^q t^p \right\}.
\end{equation}

From (2.9), we deduce that, for any $t \in (0,\infty)$, $x \in \mathbb{R}^n$, and $\xi \in S^{n-1}$,

$$|\xi \cdot \nabla f(x)| - tL_2 \leq \frac{|f(x + t\xi) - f(x)|}{t} \leq |\xi \cdot \nabla f(x)| + tL_2,$$

which, combined with (2.10) and (2.11), implies that, for any $t \in (0, (L_1/\lambda)^\frac{p}{q})$,

$$A_f(x,\xi,\lambda,q) \leq \frac{|f(x + t\xi) - f(x)|^q}{t^q} \leq A_f^*(x,\xi,\lambda,q),$$

where

$$A_f^*(x,\xi,\lambda,q) := \max \left\{ |\xi \cdot \nabla f(x)| - (L_1/\lambda)^\frac{q}{p} L_2, 0 \right\}^q$$

and

$$A_f^+(x,\xi,\lambda,q) := \min \left\{ |\xi \cdot \nabla f(x)| + (L_1/\lambda)^\frac{q}{p} L_2, L_1^q \right\}.$$
Using this, we conclude that, for any \( x \in \mathbb{R}^n \) and \( \xi \in \mathbb{S}^{n-1} \),

\[
F_{\tilde{x}}(x, \xi, \lambda, q) \subset F_f(x, \xi, \lambda, q) \subset F_{\tilde{f}}(x, \xi, \lambda, q),
\]

where

\[
F_{\tilde{x}}(x, \xi, \lambda, q) := \{ t \in (0, \infty) : A_{\tilde{x}}^+(x, \xi, \lambda, q) > \lambda^q t^q \}.
\]

We now show that, for any \( \lambda \in (L_1, \infty) \), \( \xi \in \mathbb{S}^{n-1} \), and \( x \in K^C \),

\[
F_f(x, \xi, \lambda, q) = \emptyset
\]

and hence, for any \( x \in \mathbb{R}^n \),

\[
\int_{\mathbb{S}^n} \int_0^\infty 1_{f_f(x, \xi, \lambda, q)(t)} t^{p_n-1} dt d\sigma(\xi) = \int_{\mathbb{S}^n} \int_0^\infty 1_{f_f(x, \xi, \lambda, q)(t)} t^{p_n-1} dt d\sigma(\xi) 1_{K}(x).
\]

Indeed, by (2.11), for any \( \lambda \in (L_1, \infty) \), we have \( F_f(x, \xi, \lambda, q) \subset (0, 1) \). Thus, for any \( \lambda \in (L_1, \infty) \), \( x \in K^C \), \( \xi \in \mathbb{S}^{n-1} \), and \( t \in F_f(x, \xi, \lambda, q) \subset (0, 1) \), we obtain \( x + t\xi \in B(0, M)^C \) and hence \( f(x + t\xi) = 0 \). This implies that, for any \( \lambda \in (L_1, \infty) \), \( \xi \in \mathbb{S}^{n-1} \), and \( x \in K^C \),

\[
F_f(x, \xi, \lambda, q) = \emptyset,
\]

which completes the proof of (2.13). By (2.7), (2.12), and Definition 2.7(ii) together with \( X \) being a ball quasi-Banach function space, we have

\[
\lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda q)\}(y)} dy \right\|_{X^q}^{1 \over 1 - q} \geq \lambda^q \left\| \int_{\mathbb{S}^n} \int_0^\infty 1_{f_f(x, \xi, \lambda, q)(t)} t^{p_n-1} dt d\sigma(\xi) \right\|_{X^q}^{1 \over 1 - q} \geq \frac{1}{n} \left\| \int_{\mathbb{S}^n} A_{\tilde{x}}^-(x, \xi, \lambda, q) d\sigma(\xi) \right\|_{X^q}^{1 \over 1 - q}.
\]

Notice that the function \( \lambda \to A_{\tilde{x}}^-(x, \xi, \lambda) \) is increasing on \( (0, \infty) \) and

\[
\lim_{\lambda \to \infty} A_{\tilde{x}}^-(x, \xi, \lambda, q) = \|\xi \cdot \nabla f(x)\|^q,
\]

and that \( K(q, n) \) is independent of \( e \). From this, letting \( \lambda \to \infty \) in (2.14), and using Definition 2.7(iii), we know that

\[
\lim_{\lambda \to \infty} A_{\tilde{x}}^-(x, \xi, \lambda, q) = \|\xi \cdot \nabla f(x)\|^q,
\]

This finishes the proof of (2.5).
Next, we prove that, if $X^\frac{1}{q}$ is a ball Banach function space, then, for any $f \in C^2_\varepsilon(\mathbb{R}^n)$,

\begin{equation}
\limsup_{\lambda \to \infty} \lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n : \langle \cdot, y \rangle \in E_f(|\lambda|d)}(y) \, dy \right\|_{X^\frac{1}{q}} \leq \frac{K(q,n)}{n} \|\nabla f\|_{X}^q.
\end{equation}

(2.16)

Recall that, for any $\theta \in (0, 1)$, there exists a positive constant $C_\theta$ such that, for any $a, b \in (0, \infty)$,

\begin{equation}
(a + b)^q \leq (1 + \theta)a^q + C_\theta b^q
\end{equation}

(see, for instance, [12, p. 699]). Obviously, we have, for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{S}^{n-1}$,

\[|\xi \cdot \nabla f(x)| \leq \|\nabla f\|_{L^\infty(\mathbb{R}^n)} < L_1.\]

From this and (2.17), we deduce that, for any $\lambda \in (L_1, \infty)$ sufficiently large,

\[A^+_\lambda(x, \xi, \lambda, q) = [||\xi \cdot \nabla f(x)| + (L_1/\lambda)^{\frac{q}{2}}L_2]^q \leq (1 + \theta)||\xi \cdot \nabla f(x)||^q + C_\theta (L_1/\lambda)^{\frac{q^2}{2}}L_2^q.\]

By this, (2.7), (2.12), (2.13), and the assumption that $X^\frac{1}{q}$ is a ball Banach function space, we conclude that, for any $\lambda \in (L_1, \infty)$,

\[\lambda^q \left\| \int_{\mathbb{R}^n} 1_{\{y \in \mathbb{R}^n : \langle \cdot, y \rangle \in E_f(|\lambda|d)}(y) \, dy \right\|_{X^\frac{1}{q}} \leq \frac{K(q,n)}{n} \|\nabla f\|_{X}^q.
\]

(2.16)

Letting $\lambda \to \infty$ and $\theta \to 0$, we then finish the proof of (2.16), which, combined with (2.15), completes the proof of Theorem 2.14. □

As an immediate consequence of Theorem 2.14 with $X$ replaced by $L^p_\omega(\mathbb{R}^n)$, we have the following conclusion.

**Corollary 2.15.** Let $p \in [1, \infty)$, $q \in (0, \infty)$, and $\omega \in A_\infty(\mathbb{R}^n)$. Then, for any $f \in C^2_\varepsilon(\mathbb{R}^n)$,

\[\liminf_{\lambda \to \infty} \lambda^p \left\| \int_{\mathbb{R}^n} 1_{E_f(^{\lambda})}(x, y) \, dy \right\|^{\frac{q}{p}} \omega(x) \, dx \geq \left[ \frac{K(q,n)}{n} \right]^{\frac{q}{p}} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx.
\]

As a consequence of Corollary 2.15, we obtain the lower estimate of Theorem 2.3. Observe that, in Corollary 2.15, we do not need the assumption $n(\frac{1}{p} - \frac{1}{q}) < 1$. 


2.2 Proof of Theorem 2.3: upper estimate

We begin with recalling some conclusions about Muckenhoupt weights $A_p(\mathbb{R}^n)$. The following lemma is a part of [29, Proposition 7.1.5].

**Lemma 2.16.** Let $p \in [1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$. Then the following statements hold true.

(i) For any $\lambda \in (1, \infty)$ and any cube $Q \subset \mathbb{R}^n$, one has $\omega(\lambda Q) \leq [\omega]_{A_p(\mathbb{R}^n)} \lambda^p \omega(Q)$;

(ii) $$\left[ \frac{\int_Q |f(t)| \, dt}{\omega(Q)} \right]^{\frac{1}{p}} \leq \sup_{Q \in L_\omega(\mathbb{R}^n)} \sup_{f \in A_\omega L_p(\mathbb{R}^n)} \frac{\int_Q |f(t)|^p \omega(t) \, dt}{\omega(Q)}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

The following lemma is a part of [29, Theorem 7.1.9].

**Lemma 2.17.** Let $p \in (1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$. Then there exists a positive constant $C$, independent of $\omega$, such that, for any $f \in L_\omega^p(\mathbb{R}^n)$,

$$\|M(f)\|_{L_p^\omega(\mathbb{R}^n)} \leq C [\omega]_{A_p(\mathbb{R}^n)}^{\frac{1}{p}} \|f\|_{L_p^\omega(\mathbb{R}^n)},$$

where $M$ is as in (1.6).

For the proof of the upper bound in (2.1), we need to use several adjacent systems of dyadic cubes, which can be found, for instance, in [50, Section 2.2].

**Lemma 2.18.** For any $\alpha \in \{0, \frac{1}{2}, \frac{3}{2}\}^n$, let

$$\mathcal{D}^\alpha := \{2^j(k + [1, 0]^n + (-1)^j \alpha) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

Then

(i) for any $Q, Q' \in \mathcal{D}^\alpha$ with $\alpha \in \{0, \frac{1}{2}, \frac{3}{2}\}^n$, $Q \cap Q' \in \{0, Q, Q'\}$;

(ii) for any ball $B \subset \mathbb{R}^n$, there exist an $\alpha \in \{0, \frac{1}{2}, \frac{3}{2}\}^n$ and a $Q \in \mathcal{D}^\alpha$ such that $B \subset Q \subset CB$, where the positive constant $C$ depends only on $n$.

The following Poincaré inequality is just [27, Lemma 4.1].

**Lemma 2.19.** Let $p \in [1, \infty)$. Then there exists a positive constant $C_{(n)}$, depending only on $n$, such that, for any $B(x, r) \subset \mathbb{R}^n$ with $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, $g \in C^1(B(x, r))$, and $z \in B(x, r)$,

$$\int_{B(x, r)} |g(y) - g(z)|^p \, dy \leq C_{(n)} r^{n+1-p} \int_{B(x, r)} |\nabla g(y)|^p |y - z|^{1-n} \, dy.$$

The following lemma is a direct consequence of Lemma 2.19.
Lemma 2.20. Let \( B = B(x, r) \subset \mathbb{R}^n \) be a ball with \( x \in \mathbb{R}^n \) and \( r \in (0, \infty) \), and \( B_1 \in \mathbb{B} \) such that \( x \in B_1 \subset B \). Then there exists a positive constant \( C_{(n)} \), depending only on \( n \), such that, for any \( f \in C^1(B) \),
\[
|f(x) - f_{B_1}| \leq C_{(n)} r \sum_{j=0}^{\infty} 2^{-j} \int_{2^{-j}B} |\nabla f(z)| \, dz.
\]

Proof. Let \( f \), \( x \), and \( B \) be as in this lemma. By Lemma 2.19, we have
\[
|f(x) - f_{B_1}| \leq \frac{1}{|B_1|} \int_{B_1} |f(x) - f(y)| \, dy
\]
\[
\leq \int_{B_1} |\nabla f(z)||z - x|^{1-n} \, dz \leq \int_B |\nabla f(z)||z - x|^{1-n} \, dz
\]
\[
\leq \sum_{j=0}^{\infty} (2^{-j} r)^{1-n} \int_{2^{-j-1}r \leq |z| < 2^{-j} r} |\nabla f(z)| \, dz
\]
\[
\leq \sum_{j=0}^{\infty} (2^{-j} r) \int_{2^{-j} B} |\nabla f(z)| \, dz.
\]
This finishes the proof of Lemma 2.20. \( \square \)

We state the upper estimate of Theorem 2.3 in the following separate theorem. Observe that, differently from the lower estimate of Theorem 2.3 (see Corollary 2.15), here we need the additional assumption \( n(\frac{1}{p} - \frac{1}{q}) < 1 \).

Theorem 2.21. Let \( p \in [1, \infty) \) and \( q \in (0, \infty) \) satisfy \( n(\frac{1}{p} - \frac{1}{q}) < 1 \). Assume that \( \omega \in A_1(\mathbb{R}^n) \). Then there exist positive constants \( C_2 \) and \( C_{(\omega)A_1(\mathbb{R}^n)} \) such that, for any \( f \in C^2_c(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(x, y) \, dy \left| \omega(x) \right| \, dx \leq C_2 C_{(\omega)A_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
\]
where, for any \( \lambda \in (0, \infty) \), \( E_f(\lambda, q) \) is as in (2.2), the positive constants \( C_1 \) and \( C_2 \) are independent of \( \omega \), and the positive constant \( C_{(\omega)A_1(\mathbb{R}^n)} \) increases as \([\omega]_{A_1(\mathbb{R}^n)} \) increases.

To prove Theorem 2.21, for any \( x, y \in \mathbb{R}^n \), let \( B_{xy} := B(\frac{x+y}{2}, |x-y|) \). For any \( q \in (0, \infty) \) and \( f \in C^2_c(\mathbb{R}^n) \), let
\[
E^{(1)}_f(1, q) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f_{B_{xy}}| \geq 2^{-1} |x-y|^{\frac{q}{q-1}} \right\}
\]
and
\[
E^{(2)}_f(1, q) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(y) - f_{B_{xy}}| \geq 2^{-1} |x-y|^{\frac{q}{q-1}} \right\}.
\]
We need the following several lemmas.
Lemma 2.22. Let $p \in [1, \infty)$, $q \in (0, \infty)$, and $\omega \in A_1(\mathbb{R}^n)$. Then there exists a positive constant $C$ such that, for any $f \in C_c^1(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} 1_{E_f(1,q)}(x, y) \, dy \right)^{\frac{p}{q}} \omega(x) \, dx \leq C[\omega]_{A_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
$$

where $E_f(1,q)$ is as in (2.18) and the positive constant $C$ is independent of $\omega$.

Proof. Let $\varepsilon \in (0,1)$ and $f \in C_c^1(\mathbb{R}^n)$. Indeed, to prove this lemma, we only need to consider the case $\varepsilon = \frac{1}{2}$. However, some of the estimates below with an arbitrarily given $\varepsilon \in (0,1)$ are needed in the proof of Lemma 2.23 below and hence we give the details here.

By Lemma 2.20 with $B := B(x, 2|x-y|)$ and $B_1 := B(x+\varepsilon, |x-y|) =: B_{x,y}$, we know that there exists a positive constant $c_1$, depending only on $n$, such that, for any $x, y \in \mathbb{R}^n$,

$$
|f(x) - f_{B_{x,y}}| \leq c_1 |x-y| \sum_{j=0}^{\infty} 2^{-j} \int_{B(x,2^{-j}|x-y|)} |\nabla f(z)| \, dz.
$$

This implies that, for any $(x, y) \in E_f(1,q)$, there exists a $j \in \mathbb{Z}_+$ such that

$$
\int_{B(x,2^{-j}|x-y|)} |\nabla f(z)| \, dz > c_2 2^j (1-\varepsilon)|x-y|^{n/q},
$$

where the positive constant $c_2$ depends only on $n$ and $\varepsilon$. For any $j \in \mathbb{Z}_+$, let

$$
B_j := B(x, 2^{-j+1}|x-y|).
$$

Applying Lemma 2.18 to the ball $B_j := B(x, 2^{-j+1}|x-y|)$, we know that there exists some $\alpha \in \{0, \frac{1}{2}, \frac{3}{4}\}^n$ and $Q \in \mathcal{D}^\alpha$ such that $B_j \subset Q_j \subset CB_j$, where the positive constant $C$ depends only on $n$. This implies that, if (2.21) is satisfied, then there exists a positive constant $c_3$, depending only on $n, q, \varepsilon$, such that

$$
\int_{Q_j} |\nabla f(z)| \, dz > c_3 2^j (1-\varepsilon)|2^j Q_j|^{\frac{1}{n}}.
$$

For any $\alpha \in \{0, \frac{1}{2}, \frac{3}{4}\}^n$ and $j \in \mathbb{Z}_+$, we denote by the symbol $\mathcal{A}_\alpha^j$ the collection of all dyadic cubes $Q \in \mathcal{D}^\alpha$ which satisfies (2.22) with $Q_j$ replaced by $Q$, where $\mathcal{D}^\alpha$ is as in Lemma 2.18. Clearly, from (2.22), it is easy to deduce that, for any $\alpha \in \{0, \frac{1}{2}, \frac{3}{4}\}^n$ and $j \in \mathbb{Z}_+$, $\sup_{Q \in \mathcal{A}_\alpha^j} l(Q) < \infty$ with $l(Q)$ for any $Q \in \mathcal{A}_\alpha^j$ being the side length of $Q$. Thus, every cube $Q \in \mathcal{A}_\alpha^j$ is contained in a dyadic cube in $\mathcal{A}_\alpha^j$ that is maximal with respect to set inclusion. For any $\alpha \in \{0, \frac{1}{2}, \frac{3}{4}\}^n$, we denote by the symbol $\mathcal{A}_\alpha^{j, \text{max}}$ the collection of all dyadic cubes in $\mathcal{A}_\alpha^j$ that are maximal with respect to set inclusion. Clearly, the maximal dyadic cubes in $\mathcal{A}_\alpha^{j, \text{max}}$ are pairwise disjoint. For any $(x, y) \in E_f(1,q)$, since $(x, y) \in B_j \times 2^j B_j$, it follows that $(x, y) \in Q \times 2^j Q$ for some $Q \in \mathcal{A}_\alpha^{j, \text{max}}$. Thus, we have

$$
E_f(1,q) \subset \bigcup_{j=0}^{\infty} \bigcup_{\alpha \in \{0, \frac{1}{2}, \frac{3}{4}\}^n} \bigcup_{Q \in \mathcal{A}_\alpha^{j, \text{max}}} (Q \times 2^j Q),
$$
which implies that, for any $x \in \mathbb{R}^n$,

$$
\int_{\mathbb{R}^n} 1_{E_j^{(1)}(1,q)}(x,y) \, dy \leq \sum_{j=0}^{\infty} \sum_{\alpha \in \{0,1,2\}^n} \sum_{Q \in \mathcal{A}_{n,\max}} 1_Q(x)[2^j Q].
$$

Now, we prove (2.20) by considering two cases on $q$.

Case 1) $q \in (0, p]$. In this case, by (2.23), the Minkowski inequality on $L^{\frac{2}{q}}(\mathbb{R}^n)$, and the fact that the dyadic cubes in $\mathcal{A}_{n,\max}^j$ are pairwise disjoint, we obtain

$$
1 = \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_j^{(1)}(1,q)}(x,y) \, dy \right]^{\frac{q}{2}} \omega(x) \, dx \right\}^{\frac{2}{q}} \leq \sum_{j=0}^{\infty} \sum_{\alpha \in \{0,1,2\}^n} \left\{ \int_{\mathbb{R}^n} \left[ \sum_{Q \in \mathcal{A}_{n,\max}} 2^j Q 1_Q(x) \right]^{\frac{q}{2}} \omega(x) \, dx \right\}^{\frac{2}{q}} \leq \sum_{j=1}^{\infty} \sum_{\alpha \in \{0,1,2\}^n} \left\{ \sum_{Q \in \mathcal{A}_{n,\max}} 2^j Q^{\frac{q}{2}} \omega(Q) \right\}^{\frac{2}{q}}.
$$

From (2.22), Lemma 2.16(ii) with $\omega$ regarded as an $A_p(\mathbb{R}^n)$ weight, and the fact that $[\omega]_{A_p(\mathbb{R}^n)} \leq [\omega]_{A_1(\mathbb{R}^n)}$, we deduce that, for any $Q \in \mathcal{A}_{n,\max}^j$,

$$
|2^j Q|^{\frac{q}{2}} \leq c_3^{-p} [\omega]_{A_1(\mathbb{R}^n)} 2^{-j(1-\varepsilon)p} \frac{1}{\omega(Q)} \int_Q |\nabla f(z)|^p \omega(z) \, dz.
$$

This, combined with (2.24), implies that

$$
I \leq [\omega]_{A_1(\mathbb{R}^n)}^{q/p} \sum_{j=0}^{\infty} \sum_{\alpha \in \{0,1,2\}^n} \left[ \sum_{Q \in \mathcal{A}_{n,\max}} 2^{-j(1-\varepsilon)p} \int_Q |\nabla f(z)|^p \omega(z) \, dz \right]^{\frac{q}{p}} \leq [\omega]_{A_1(\mathbb{R}^n)}^{q/p} \sum_{j=0}^{\infty} 2^{-j(1-\varepsilon)q} \left[ \int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z) \, dz \right]^{\frac{q}{p}} \leq [\omega]_{A_1(\mathbb{R}^n)}^{q/p} \left[ \int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z) \, dz \right]^{\frac{q}{p}},
$$

where, in the second step, we used the fact that the dyadic cubes in $\mathcal{A}_{n,\max}^j$ are pairwise disjoint. This gives the desired estimate (2.20) for any given $q \in (0, p]$.

Case 2) $q \in (p, \infty)$. In this case, recall that, for any $r \in (0, 1]$ and $\{a_j\}_{j \in \mathbb{Z}_+} \subset (0, \infty)$,

$$
\left( \sum_{j \in \mathbb{Z}_+} a_j \right)^r \leq \sum_{j \in \mathbb{Z}_+} a_j^r.
$$
By this, (2.23), (2.25), and the fact that the dyadic cubes in $\mathcal{A}_{n,\text{max}}^j$ are pairwise disjoint, we conclude that

$$
\int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E^j_f(1,q)}(x,y) \, dy \right]^{\frac{p}{q}} \omega(x) \, dx \\
\leq \sum_{j=0}^{\infty} \sum_{\alpha \in \{0,1,2\}^n} \sum_{Q \in \mathcal{A}_{n,\text{max}}^j} \omega(Q) |2^j Q|^{\frac{p}{q}} \\
\leq [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^{\infty} \sum_{\alpha \in \{0,1,2\}^n} \sum_{Q \in \mathcal{A}_{n,\text{max}}^j} 2^{-j(1-\varepsilon)p} \int_Q |\nabla f(z)|^p \omega(z) \, dz \\
\leq [\omega]_{A_1(\mathbb{R}^n)} 3^n \sum_{j=0}^{\infty} 2^{-j(1-\varepsilon)p} \int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z) \, dz.
$$

This gives the desired estimate (2.20) for any given $q \in (p, \infty)$, which completes the proof of Lemma 2.22. \qed

**Lemma 2.23.** Let $p \in [1, \infty)$ and $q \in (0, \infty)$ satisfy $n\left(\frac{1}{p} - \frac{1}{q}\right) < 1$. Let $\omega \in A_1(\mathbb{R}^n)$. Then there exist positive constants $C$ and $C_{(\omega)_{A_1(\mathbb{R}^n)}}$ such that, for any $f \in C_c^1(\mathbb{R}^n)$,

$$
(2.27) \quad \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E^j_f(1,q)}(x,y) \, dy \right]^{\frac{p}{q}} \omega(x) \, dx \leq CC_{(\omega)_{A_1(\mathbb{R}^n)}} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx,
$$

where $E^j_f(1,q)$ is as in (2.19), the positive constant $C$ is independent of $\omega$, and the positive constant $C_{(\omega)_{A_1(\mathbb{R}^n)}}$ increases as $[\omega]_{A_1(\mathbb{R}^n)}$ increases.

Before we prove Lemma 2.23, we need the following lemma, which can be found in [22, p. 18].

**Lemma 2.24.** Let $p \in [1, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$. For any $g \in L^p_\omega(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$
R_g(x) := \sum_{k=0}^{\infty} 2^{k||M||_{L^p_\omega(\mathbb{R}^n) \to L^p_\omega(\mathbb{R}^n)}^2} M_k^k g(x),
$$

where, for any $k \in \mathbb{N}$, $M_k := M \cdots \circ M$ is $k$ iterations of the Hardy–Littlewood maximal operator, and $M_k^k g(x) := |g(x)|$. Then, for any $g \in L^p_\omega(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

(i) $|g(x)| \leq R_g(x)$;

(ii) $R_g \in A_1(\mathbb{R}^n)$ and $[R_g]_{A_1(\mathbb{R}^n)} \leq 2||M||_{L^p_\omega(\mathbb{R}^n) \to L^p_\omega(\mathbb{R}^n)}$, where $||M||_{L^p_\omega(\mathbb{R}^n) \to L^p_\omega(\mathbb{R}^n)}$ denotes the operator norm of $M$ mapping $L^p_\omega(\mathbb{R}^n)$ to $L^p_\omega(\mathbb{R}^n)$;

(iii) $||R_g||_{L^p_\omega(\mathbb{R}^n)} \leq 2||g||_{L^p_\omega(\mathbb{R}^n)}$.

**Proof of Lemma 2.23.** Let $\varepsilon \in (0,1)$ be a sufficiently small absolute constant. By an argument similar to that used in the proof of Lemma 2.22, we know that there exist positive constants $C_1$ and $C_2$, depending only on $n$, $q$, and $\varepsilon$, such that, for any $j \in \mathbb{Z}_+$, $\alpha \in \{0,\frac{1}{2},\frac{3}{2}\}^n$, and $Q \in \mathcal{A}_{n,\text{max}}^j$,

$$
C_1 2^{(1-\varepsilon)p} |2^j Q|^{\frac{p}{q}} < \int_Q |\nabla f(z)| \, dz.
$$
\(2.28\) \[ |2^j Q|^{\frac{p}{q}} \leq C_2[\omega]_{A_1(\mathbb{R}^n)} 2^{-j(1-\varepsilon)p} \frac{1}{\omega(Q)} \int_Q |\nabla f(z)|^p \omega(z) \, dz, \]

and

\(2.29\) \[ E_f^{(2)}(1, q) \subset \bigcup_{j=0}^{\infty} \bigcup_{\alpha \in \{0, \frac{1}{2}, 1\}^n} \bigcup_{Q \in \mathcal{A}_{\text{max}}} (2^j Q \times Q), \]

where \(\mathcal{A}_{\text{max}}^j\) is the same as in the proof of Lemma 2.22. Using \(2.29\), we conclude that, for any \(x \in \mathbb{R}^n\),

\(2.30\) \[ \int_{\mathbb{R}^n} 1_{E_f^{(2)}(1, q)}(x, y) \, dy \leq \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, 1\}^n} \sum_{Q \in \mathcal{A}_{\text{max}}} |Q|^{\frac{p}{q}} \omega(2^j Q). \]

Now, we prove \(2.27\) by considering two cases on \(q\).

Case 1) \(q \in [p, \infty)\) and \(n(\frac{1}{p} - \frac{1}{q}) < 1\). In this case, by \(2.26\), \(2.30\), \(2.28\), and Lemma 2.16(i), we find that

\(2.31\) \[ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} 1_{E_f^{(2)}(1, q)}(x, y) \, dy \right|^{\frac{p}{q}} \omega(x) \, dx \]

\[ \leq \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, 1\}^n} \sum_{Q \in \mathcal{A}_{\text{max}}} |Q|^{\frac{p}{q}} \omega(2^j Q) \]

\[ \leq [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, 1\}^n} \sum_{Q \in \mathcal{A}_{\text{max}}} 2^{j(1-\varepsilon)p} \omega(Q) \]

\[ \leq [\omega]_{A_1(\mathbb{R}^n)} \sum_{j=0}^{\infty} \sum_{\alpha \in \{0, \frac{1}{2}, 1\}^n} \sum_{Q \in \mathcal{A}_{\text{max}}} 2^{j[(\frac{1}{p} - \frac{1}{q}) - (1-\varepsilon)]} \int_Q |\nabla f(z)|^p \omega(z) \, dz \]

\[ \leq [\omega]_{A_1(\mathbb{R}^n)} 3^n \sum_{j=0}^{\infty} 2^{j[(\frac{1}{p} - \frac{1}{q}) - (1-\varepsilon)]} \int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z) \, dz \]

\[ \leq [\omega]_{A_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla f(z)|^p \omega(z) \, dz, \]

where, in the fourth step, we took \(\varepsilon \in (0, 1)\) sufficiently small so that \(n(\frac{1}{p} - \frac{1}{q}) < 1 - \varepsilon\). This can be done because \(n(\frac{1}{p} - \frac{1}{q}) < 1\). This finishes the proof of \(2.27\) in this case.

Case 2) \(q \in (0, p)\). In this case, let \(r := \frac{p}{q}, r' := \frac{1}{r-1}\), and \(\mu(x) := [\omega(x)]^{1-r'}\) for any \(x \in \mathbb{R}^n\). Since \(\omega \in A_1(\mathbb{R}^n) \subset A_r(\mathbb{R}^n)\), it follows that \(\mu \in A_{r'}(\mathbb{R}^n)\) and

\(2.32\) \[ [\omega^{1-r'}]_{A_{r'}(\mathbb{R}^n)} = [\omega]_{A_1(\mathbb{R}^n)} \leq [\omega]_{A_r(\mathbb{R}^n)} \]

(see, for instance, [29, (4) and (6) of Proposition 7.1.5]). It is known that \([L_\omega(\mathbb{R}^n)]' = L'_{\mu}(\mathbb{R}^n)\), where \([L_\omega(\mathbb{R}^n)]'\) denotes the associated space of \(L_\omega(\mathbb{R}^n)\) as in Definition 2.9 (see [25, Theorem
2.7.4). From this, Lemma 2.11 with $X := L^r_ω(\mathbb{R}^n)$, Definition 2.9 with $X := L^r_μ(\mathbb{R}^n)$, and Lemma 2.24(i), we deduce that

\[
(2.33) \quad \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathbf{1}_{E^{(1, q)}_f}(x, y) \, dy \right]^r \omega(x) \, dx \right\}^{\frac{1}{r}} \leq \sup_{\|g\|_{L^r_μ(\mathbb{R}^n)} = 1} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathbf{1}_{E^{(1, q)}_f}(x, y) \, dy \right] \| \nabla f(x) \|^p \, dx \leq \sup_{\|g\|_{L^r_μ(\mathbb{R}^n)} = 1} [Rg]^2_{A_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla f(x)|^q Rg(x) \, dx,
\]

where, in the last step, we used (2.31) with $p := q$ and $ω := Rg$. On the other hand, by the Hölder inequality, (ii) and (iii) of Lemma 2.24, Lemma 2.17, and (2.32), we know that, for any $g \in L^r_μ(\mathbb{R}^n)$ with $\|g\|_{L^r_μ(\mathbb{R}^n)} = 1$,

\[
[Rg]^2_{A_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla f(x)|^q Rg(x) \, dx \leq [Rg]^2_{A_1(\mathbb{R}^n)} \|g\|_{L^r_μ(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{q}} \leq \|M\|^2_{L^r_μ(\mathbb{R}^n) \to L^r_μ(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{q}} \leq |μ|^2_{A^p_1(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{q}} \leq [ω]^2_{A_1(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{q}}.
\]

This, combined with (2.33), implies that

\[
(2.34) \quad \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathbf{1}_{E^{(1, q)}_f}(x, y) \, dy \right]^r \omega(x) \, dx \right\}^{\frac{1}{r}} \leq [ω]^2_{A_1(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx \right\}^{\frac{1}{q}}.
\]

This finishes the proof of (2.27) in this case.

Let

\[
C_{([ω]_{A_1(\mathbb{R}^n)})} := \begin{cases} [ω]^2_{A_1(\mathbb{R}^n)}, & q \in [p, \infty) \text{ and } \eta(\frac{1}{p} - \frac{1}{q}) < 1, \\ [ω]^{(2p)/q}_{A_1(\mathbb{R}^n)}, & q \in (0, p). \end{cases}
\]

It is easy to see that $C_{([ω]_{A_1(\mathbb{R}^n)})}$ increases as $[ω]_{A_1(\mathbb{R}^n)}$ increases. By (2.31) and (2.34), we have

\[
\int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \mathbf{1}_{E^{(1, q)}_f}(x, y) \, dy \right]^q \omega(x) \, dx \leq C_{([ω]_{A_1(\mathbb{R}^n)})} \int_{\mathbb{R}^n} |\nabla f(x)|^p \omega(x) \, dx.
\]
This finishes the proof of Lemma 2.23.

Finally, we prove Theorem 2.21.

**Proof of Theorem 2.21.** Without loss of generality, we may assume that \( \lambda = 1 \) because, otherwise, we can replace \( f \) by \( f/\lambda \) for any \( \lambda \in (0, \infty) \). Then \( E_f(1, q) \subset E_f^{(1)}(1, q) \cup E_f^{(2)}(1, q) \), where \( E_f^{(1)}(1, q) \) and \( E_f^{(2)}(1, q) \) are, respectively, as in (2.18) and (2.19). Thus, it suffices to prove the corresponding upper estimates, respectively, for the sets \( E_f^{(1)}(1, q) \) and \( E_f^{(2)}(1, q) \), which are done in Lemmas 2.22 and 2.23. This finishes the proof of Theorem 2.21. \( \square \)

Now, we can complete the proof of Theorem 2.3.

**Proof of Theorem 2.3.** As a consequence of Theorem 2.21 and Corollary 2.15, we immediately obtain the desired conclusions of this theorem, which completes the proof of Theorem 2.3. \( \square \)

### 2.3 Proof of Theorem 2.5

**Proof of Theorem 2.5.** Assume that there exists a positive constant \( C_1 \) such that, for any \( f \in C^1(\mathbb{R}) \) satisfying that \( f' \) has compact support,

\[
\sup_{\lambda \in (0, \infty)} A^p \int_{\mathbb{R}^2} 1_{E_f(\lambda, p)}(x, y) \omega(x) \, dx \, dy \leq C_1 \int_{\mathbb{R}} |f'(x)|^p \omega(x) \, dx,
\]

where, for any \( \lambda \in (0, \infty) \), \( p \in [1, \infty) \) and any measurable function \( f, E_f(\lambda, p) \) is as in (2.2) and the constant \( C_1 \) is independent of \( f \). We now show that \( \omega \in A_p(\mathbb{R}) \) with \( p \in [1, \infty) \). Observe that the inequality (2.35) is both dilation and translation invariance; that is, for any \( \delta \in (0, \infty) \) and \( x_0 \in \mathbb{R} \), both the weights \( \omega(\delta x) \) and \( \omega(x-x_0) \) satisfy (2.35) with the same constant \( C_1 \). This, combined with Lemma 2.22(ii), implies that, to prove \( \omega \in A_p(\mathbb{R}) \), it suffices to show that there exists a positive constant \( C \), depending only on \( C_1 \) such that, for any nonnegative function \( g \in L^1_{\text{loc}}(\mathbb{R}) \),

\[
|\int_{-1}^{1} g(x) \, dx| \leq \frac{C}{\omega([-1, 1])} \int_{-1}^{1} |g(x)|^p \omega(x) \, dx.
\]

To show (2.36), we first prove that, for any \( 0 \leq g \in C^{\infty}(\mathbb{R}) \),

\[
|\int_{-1}^{1} g(x) \, dx| \leq \frac{C_1 6^{p+1}}{4\omega(I_0)} \int_{-1}^{4} |g(x)|^p \omega(x) \, dx,
\]

where \( I_0 := [-3, -1] \cup [1, 3] \). Let \( \eta \in C^{\infty}(\mathbb{R}) \) be such that \( \eta(x) \in [0, 1] \) for any \( x \in \mathbb{R}^n \), \( \eta(x) = 1 \) for any \( x \in [-3, 3] \), and \( \eta(x) = 0 \) for any \( x \in \mathbb{R} \) with \( |x| \in [4, \infty) \). For any \( x \in \mathbb{R} \), let

\[
f(x) := \int_{-\infty}^{x} g(t) \eta(t) \, dt.
\]

Clearly, \( f \in C^{\infty}(\mathbb{R}) \) and \( \text{supp } f' \subset [-4, 4] \). Let \( \lambda := 6^{1-\frac{1}{p}} \int_{-1}^{1} g(t) \, dt \). Then, for any \( x \in [-3, -1] \) and \( y \in [1, 3] \), we have

\[
|f(y) - f(x)| = \int_{x}^{y} g(t) \, dt \geq \int_{-1}^{1} g(t) \, dt = 6^{1-\frac{1}{p}} \lambda \geq \lambda |x-y|^\frac{1}{p+1}.
\]
This, together with the symmetry implies that

\[(1, 3] \times [-3, -1] \cup [-3, -1] \times [1, 3]) \subset E_f(\lambda, p),\]

Thus, using this and (2.35), we have

\[
4\lambda^p \int_{I_0} \omega(x) \, dx \leq \lambda^p \int_{\mathbb{R}^2} \omega(x) I_{E_f(\lambda, p)}(x, y) \, dx \, dy \\
\leq C_1 \int_{\mathbb{R}} |f'(x)|^p \omega(x) \, dx \leq C_1 \int_{-4}^4 |g(x)|^p \omega(x) \, dx. 
\]

This proves (2.37).

Second, we show that, for any nonnegative locally integrable function \(g\),

\[(2.38) \quad \left[ \int_{-1}^1 g(x) \, dx \right]^p \leq \frac{C_16^{p+1}}{4\omega(I_0)} \int_{-1}^1 |g(x)|^p \omega(x) \, dx. \]

Without loss of generality, we may assume that \(g\) is bounded because, otherwise, one may replace \(g\) by \(\min\{g, n\}\) for any \(n \in \mathbb{N}\), and then apply the monotone convergence theorem. Let \(\varphi \in \mathcal{C}_c(\mathbb{R})\) be such that \(\varphi(t) \geq 0\) for any \(t \in \mathbb{R}\), \(\varphi(t) = 0\) for any \(t \in \mathbb{R}\) with \(|t| \geq 1\), and \(\int_{\mathbb{R}} \varphi(t) \, dt = 1\). For any \(\varepsilon \in (0, \infty)\) and \(t \in \mathbb{R}\), let \(\varphi_\varepsilon(t) := \varepsilon^{-1} \varphi(t/\varepsilon)\) and

\[g_\varepsilon(t) := (g I_{[-1, 1]}) * \varphi_\varepsilon(t) = \int_{-1}^1 g(u) \varphi_\varepsilon(t-u) \, du. \]

Then \(0 \leq g_\varepsilon \in \mathcal{C}_c(\mathbb{R})\) and, using (2.37), we obtain

\[(2.39) \quad \left[ \int_{-1}^1 g_\varepsilon(x) \, dx \right]^p \leq \frac{1}{\omega(I_0)} \int_{-4}^4 |g_\varepsilon(x)|^p \omega(x) \, dx. \]

Since, for almost every \(t \in \mathbb{R}\),

\[
\lim_{\varepsilon \to 0} g_\varepsilon(t) = g(t) I_{[-1, 1]}(t)
\]

and

\[
\sup_{\varepsilon \in (0, \infty)} \|g_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leq \sup_{\varepsilon \in (0, \infty)} \|\varphi_\varepsilon\|_{L^1(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} < \infty,
\]

from (2.39) and Lebesgue dominated convergence theorem, we deduce (2.38).

Finally, we prove (2.36). Let \(g := 1\) and \(I := [-1, 1]\). By (2.38), we know that \(\frac{\omega(I)}{\omega(I_0)} \geq c_1\), where \(c_1 := \frac{2}{C_13^{p+1}}\). Thus,

\[
\omega(2I) \leq \omega(I_0) + \omega(I) \leq (1 + 1/c_1) \omega(I). 
\]

Since (2.35) is both dilation and translation invariance for the weight \(\omega\), it follows that the inequality \(\omega(2I) \leq (1 + 1/c_1) \omega(I)\) holds true for any compact interval \(I \subset \mathbb{R}\). By this, we know that

\[
\omega([-1, 1]) \leq \omega([-4, 0]) + \omega([0, 4]) \\
\leq (1 + 1/c_1)\{\omega([-1, -3]) + \omega([1, 3])\} \\
= (1 + 1/c_1) \omega(I_0).
\]

This, combined with (2.38), implies that (2.36) holds true. This finishes the proof of Theorem 2.5. \[\square\]
3 Estimates in ball Banach function spaces

In this section, we establish the Brezis–Van Schaftingen–Yung formulae in ball Banach function space (see Theorem 3.4 below). As applications, we also obtain some fractional Sobolev and Gagliardo–Nirenberg type inequalities in ball Banach function spaces.

We begin with introducing the following notions of homogeneous (weak) Triebel–Lizorkin-type spaces.

**Definition 3.1.** Let \( q \in (0, \infty), s \in [0, \infty), \) and \( X \) be a ball Banach function space.

(i) The *homogeneous Triebel–Lizorkin-type space* \( \dot{F}^s_{X,q}(\mathbb{R}^n) \) is defined to be the set of all measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\|f\|_{\dot{F}^s_{X,q}(\mathbb{R}^n)} := \left\| \left[ \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x-y|^{n+sq}} \, dy \right]^\frac{1}{q} \right\|_X < \infty.
\]

(ii) The *homogeneous weak Triebel–Lizorkin-type space* \( \dot{W}^s_{X,q}(\mathbb{R}^n) \) is defined to be the set of all measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\|f\|_{\dot{W}^s_{X,q}(\mathbb{R}^n)} := \sup_{\lambda \in (0, \infty)} \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, s)}(x, y) \, dy \right]^\frac{1}{q} \right\|_X < \infty,
\]

where, for any \( \lambda \in (0, \infty), \)

\[
E_f(\lambda, q, s) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x) - f(y)| > \lambda |x-y|^{\frac{s}{q}+1}\}.
\]

Similarly to Brezis et al. [13, (1.2)] (see also [9, 12]), we have the following conclusions on the “drawback” of \( \dot{F}^s_{X,q}(\mathbb{R}^n) \).

**Theorem 3.2.** Let \( X \) be a ball Banach function space and \( s, q \in (0, \infty) \). Assume that \( X^\frac{1}{q} \) is a ball Banach function space.

(i) If \( q \in [1, \infty), s \in [1, \infty), \) and \( f \in \dot{F}^s_{X,q}(\mathbb{R}^n) \), then \( f \) is a constant function.

(ii) If \( q \in (0, 1), sq \in [1, \infty), \) and \( f \in \dot{F}^s_{X,q}(\mathbb{R}^n) \), then \( f \) is a constant function.

**Proof.** Let \( X \) be a ball Banach function space, \( s, q \in (0, \infty), \) and \( f \in \dot{F}^s_{X,q}(\mathbb{R}^n) \). By Lemma 2.11 and Definition 2.9, we have

\[
(3.2) \quad \|f\|_{\dot{F}^s_{X,q}(\mathbb{R}^n)} = \sup_{g \in \dot{F}^{s/q}_{X^{1/q}}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x-y|^{n+sq}} \, dy = \sup_{g \in \dot{F}^{s/q}_{X^{1/q}}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^q}{|x-y|^{n+sq}} \, dy + \int_{\mathbb{R}^n} \frac{|f(x) - f(x-h)|^q}{|h|^{n+sq}} \, dhg(x) \, dx.
\]
For any $N \in (0, \infty)$, let $g := 1_{B(0, N)}/\|1_{B(0, N)}\|_{L^{1/q'}}$. Using (3.2), we conclude that, for any $N \in (0, \infty)$,
\[
\int_{|x| < N} \int_{|y| < r} \frac{|f(x) - f(x - h)|^q}{|h|^{q+q'}} \, dh \, dx < \infty.
\]
From this, we deduce that, for any $N \in (0, \infty)$ and $r \in (0, N)$,
\[
(3.3) \quad \infty > \int_{|x| < N} \int_{|h| < r} \frac{|f(x) - f(x - h)|^q}{|h|^{q+q'}} \, dh \, dx \\
\geq \sum_{j=0}^{\infty} 2^{j(n+q)} r^{-(n+q)} \int_{2^{j+1}r \leq |h| < 2^jr} \int_{|x| < N} |f(x) - f(x - h)|^q \, dx \, dh.
\]
We first prove (i). Let $q \in [1, \infty)$, $s \in [1, \infty)$, and $f \in \mathcal{P}^{s}_{X,q}(\mathbb{R}^n)$. Recall the discrete Hölder inequality that, for any $m \in \mathbb{Z}_+$ and $(a_j)_{j=1}^m \subset (0, \infty)$,
\[
(3.4) \quad \left\{ \sum_{j=1}^m a_j \right\}^q \leq m^{q-1} \left( \sum_{j=1}^m a_j^q \right).
\]
By this, we obtain, for any $j \in \mathbb{Z}_+$,
\[
(3.5) \quad \int_{2^{-1}r \leq |h| < r} \int_{|x| < N-r} |f(x) - f(x - h)|^q \, dx \, dh \\
= 2^{jn} \int_{2^{-(j+1)}r \leq |h| < 2^{-j}r} \int_{|x| < N-r} |f(x) - f(x - 2^j h)|^q \, dx \, dh \\
= 2^{jn} \int_{2^{-(j+1)}r \leq |h| < 2^{-j}r} \int_{|x| < N-r} \left| \sum_{i=0}^{2^{j-1}} [f(x - ih) - f(x - (i+1)h)] \right|^q \, dx \, dh \\
\leq 2^{jn+jq-j} \sum_{i=0}^{2^{j-1}} \int_{2^{-(j+1)}r \leq |h| < 2^{-j}r} \int_{|x| < N-r} |f(x - ih) - f(x - (i+1)h)|^q \, dx \, dh \\
\leq 2^{jn+jq-j} \sum_{i=0}^{2^{j-1}} \int_{2^{-(j+1)}r \leq |h| < 2^{-j}r} \int_{|x| < N-r+2^{-j}r} |f(x) - f(x - h)|^q \, dx \, dh \\
\leq 2^{jn+jq} \int_{2^{-(j+1)}r \leq |h| < 2^{-j}r} \int_{|x| < N} |f(x) - f(x - h)|^q \, dx \, dh,
\]
which, combined with (3.3), implies that, for any $N \in (0, \infty)$ and $r \in (0, N)$,
\[
\sum_{j=0}^{\infty} 2^{j(n-1)q} \int_{2^{-1}r \leq |h| < r} \int_{|x| < N-r} |f(x) - f(x - h)|^q \, dx \, dh < \infty.
\]
From this and $(s-1)q \in [0, \infty)$, we deduce that, for any $N \in (0, \infty)$ and $r \in (0, N)$,
\[
\int_{2^{-1}r \leq |h| < r} \int_{|x| < N-r} |f(x) - f(x - h)|^q \, dx \, dh = 0.
\]
Using this and letting \( N \to \infty \), we then obtain, for any \( r \in (0, \infty) \),
\[
\int_{2^{-1}r \leq |h| < r} \int_{\mathbb{R}^n} |f(x) - f(x-h)|^q \, dx \, dh = 0.
\]
By this, we further conclude that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(x-h)|^q \, dx \, dh = 0,
\]
which implies that \( f \) is a constant function on \( \mathbb{R}^n \). This finishes the proof of (i).

As for (ii), let \( q \in (0, 1) \), \( sq \in [1, \infty) \), and \( f \in \dot{W}^{s, q}_{X, q}(\mathbb{R}^n) \). By an argument similar to that used in the proof of (3.5) with (3.4) replaced by (2.26), we have, for any \( j \in \mathbb{Z}_+ \),
\[
\int_{2^{-1}r \leq |h| < r} \int_{[x < N-r]} |f(x) - f(x-h)|^q \, dx \, dh \\
\leq 2^{jn} \int_{2^{(j+1)n} \leq |h| < 2^j \varepsilon} \int_{[x < N]} |f(x) - f(x-h)|^q \, dx \, dh,
\]
which, combined with (3.3), implies that, for any \( N \in (0, \infty) \) and \( r \in (0, N) \),
\[
\sum_{j=0}^{\infty} 2^{j(sq-1)} \int_{2^{-1}r \leq |h| < r} \int_{[x < N-r]} |f(x) - f(x-h)|^q \, dx \, dh < \infty.
\]
From this and \( sq - 1 \in [0, \infty) \), we deduce that, for any \( r \in (0, \infty) \),
\[
\int_{2^{-1}r \leq |h| < r} \int_{\mathbb{R}^n} |f(x) - f(x-h)|^q \, dx \, dh = 0.
\]
This implies that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(x-h)|^q \, dx \, dh = 0,
\]
which further implies that \( f \) is a constant function on \( \mathbb{R}^n \). This finishes the proof of (ii) and hence of Theorem 3.2.

\( \square \)

**Remark 3.3.** Let \( q \in [1, \infty) \), \( s := 1 \), and \( X := L^q(\mathbb{R}^n) \). Then, in this case, Theorem 3.2 coincides with [13, (1.2)] (see also [9, 12]).

One of the main targets in this section is to prove the equivalence (1.8) in a ball Banach function space \( X \) under some mild assumptions on \( X \) and \( p \). Theorem 3.2 justifies the use of the semi-norm \( \|f\|_{\dot{W}^{s, q}_{X, q}(\mathbb{R}^n)} \) instead of \( \|f\|_{X^s(\mathbb{R}^n)} \) in the equivalence (1.8) as follows.

**Theorem 3.4.** Let \( p \in [1, \infty) \) and \( q \in (0, \infty) \) satisfy \( n\left(\frac{1}{p} - \frac{1}{q}\right) < 1 \). Assume that \( X \) is a ball quasi-Banach function space, \( X^{1/p} \) a ball Banach function space, and \( M \) as in (1.6) bounded on its associate space \( (X^{1/p})' \). Then there exist positive constants \( C_1, C_2 \), and \( C_3 \) such that, for any \( f \in C^2_{\dot{W}^{1, p}_{X, p}(\mathbb{R}^n)} \),
\[
C_1 \|\nabla f\|_X \leq \|f\|_{\dot{W}^{s, q}_{X, q}(\mathbb{R}^n)} \leq C_2 \|M\|_{\dot{X}^{1/p}, \dot{X}^{1/q}} \|\nabla f\|_X,
\]
\[
\text{for } n \geq 2.
\]
where the positive constants \( C_1 \) and \( C_2 \) depend only on \( p, q, \) and \( n \), and the positive constant \( C(\|M\|_{L^p(\mathbb{R}^d)}) \) depending only on \( \|M\|_{(X^1/p)'} \to (X^1/p)', p \), and \( q \), increases as \( \|M\|_{(X^1/p)'} \to (X^1/p)' \) increases, and \( C(\cdot) \) is continuous on \((0, \infty)\).

To prove Theorem 3.4, we need the following conclusion whose proof is a slight modification of [22, p. 18] via replacing \( L^p_0(\mathbb{R}^d) \) in [22, p. 18] by \( X \); we omit the details.

**Lemma 3.5.** Let \( X \) be a ball Banach function space. Assume that the Hardy–Littlewood maximal operator \( M \) is bounded in \( X \). For any \( g \in X \) and \( x \in \mathbb{R}^n \), let
\[
R_xg(x) := \sum_{k=0}^{\infty} \frac{M^k(x)}{2k\|M\|_{X \to X}^k},
\]
where, for any \( k \in \mathbb{N} \), \( M^k := M \circ \cdots \circ M \) is \( k \)-iterations of the Hardy–Littlewood maximal operator, and \( M^0(g(x)) := |g(x)| \). Then, for any \( g \in X \) and \( x \in \mathbb{R}^n \),

(i) \( |g(x)| \leq R_xg(x) \);

(ii) \( R_xg \in A_1(\mathbb{R}^n) \) and \( [R_xg]_{A_1(\mathbb{R}^n)} \leq 2\|M\|_{X \to X} \), where \( \|M\|_{X \to X} \) denotes the operator norm of \( M \) mapping \( X \) to \( X \);

(iii) \( \|R_xg\|_X \leq 2\|g\|_X \).

Now, we are in a position to prove Theorem 3.4.

**Proof of Theorem 3.4.** Let \( f \in C^2(\mathbb{R}^n) \). The stated lower estimate in (3.6) is proved by Theorem 2.14. Next, we prove the upper estimate in (3.6). Let \( Y := X^1/p \). Then both \( Y \) and \( Y' \) are ball Banach function spaces. By Lemmas 2.12 and 3.5(iii), we have
\[
\sup_{\|g\|_{Y'} \leq 1} \left[ \int_{\mathbb{R}^n} |\nabla f(x)|^p R_yg(x) \, dx \right]^{1/p} \leq \|\nabla f\|_{Y'}^{1/p} \sup_{\|g\|_{Y'} \leq 1} \|R_yg\|_{Y'}^{1/p} \leq 2\|\nabla f\|_X.
\]

On the other hand, using Lemma 3.5(ii) and Theorem 2.3 with \( \omega \) replaced by \( R_yg \), we know that there exist positive constants \( C \) and \( C_1(\|R_yg\|_{Y', Y'}) \) such that, for any \( g \in Y' \) with \( \|g\|_{Y'} \leq 1 \),
\[
\left[ \int_{\mathbb{R}^n} |\nabla f(x)|^p R_yg(x) \, dx \right]^{1/p} \geq C^{-1}C_1(\|R_yg\|_{Y', Y'}) \sup_{\lambda \in (0, \infty)} \lambda \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_{f(x,y)}(\lambda, q)}(x,y) \, dy \right]^{\frac{p}{q}} R_yg(x) \, dx \right\}^{1/p} \geq C^{-1}C_1(\|M\|_{Y'^{-1}}) \sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_{f(x,y)}(\lambda, q)}(x,y) \, dy \right]^{\frac{p}{q}} R_yg(x) \, dx \right]^{1/p},
\]
where \( E_{f(x,y)}(\lambda, q) \) for any \( \lambda \in (0, \infty) \) is as in (2.2), the positive constant \( C(\|M\|_{Y'^{-1}}) \) increases as \( \|M\|_{Y'^{-1}} \) increases, \( C(\cdot) \) is continuous on \((0, \infty)\), and the positive constant \( C \) is independent of \( \|M\|_{Y'^{-1}} \). Thus, to prove (3.6), it suffices to show that, for any \( \lambda \in (0, \infty) \),
\[
(3.7) \quad \sup_{\|g\|_{Y'} \leq 1} \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} 1_{E_{f(x,y)}(\lambda, q)}(x,y) \, dy \right]^{\frac{p}{q}} R_yg(x) \, dx \right\}^{1/p}
\]
we deduce that 

where the positive equivalence constants depend only on \( p \). Indeed, from Lemmas 2.12 and 3.5(iii), we deduce that

\[
(3.8) \quad \sup_{\|y\| \leq 1} \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{p}{q}} R_Y g(x) \right\|_p \\
\leq \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{p}{q}} \right\|_p \sup_{\|y\| \leq 1} \|R_Y g\|_{Y'} \\
\leq 2 \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{p}{q}} \right\|_p.
\]

On the other hand, by (i) and (iii) of Lemma 3.5, and Lemma 2.11, we obtain

\[
\sup_{\|y\| \leq 1} \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{p}{q}} R_Y g(x) \right\|_p \\
\geq \sup_{\|y\| \leq 1} \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{p}{q}} g(x) \right\|_p \\
= \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{p}{q}} \right\|_{Y'} = \left\| \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda, q)}(\cdot, y) \, dy \right]^{\frac{p}{q}} \right\|_{X}. 
\]

This, combined with (3.8), implies that (3.7) holds true, which completes the proof of Theorem 3.4. \( \square \)

**Remark 3.6.**

(i) Let \( p, q \in [1, \infty) \) with \( p = q \), and \( X := L^p(\mathbb{R}^n) \). Then, in this case, Theorem 3.4 coincides with [13, Theorem 1.1].

(ii) Let \( p \in [1, \infty) \). Assume that \( q_1, q_2 \in (0, \infty) \) satisfy \( n\left(\frac{1}{p} - \frac{1}{q_1}\right) < 1 \) and \( n\left(\frac{1}{p} - \frac{1}{q_2}\right) < 1 \). From Theorem 3.4, it follows that

\[
WF_{X,q_1}^1(\mathbb{R}^n) \cap C^2_c(\mathbb{R}^n) = WF_{X,q_1}^1(\mathbb{R}^n) \cap C^2_c(\mathbb{R}^n)
\]

with equivalent quasi-norms. Thus, when \( q \in (0, \infty) \) satisfies \( n\left(\frac{1}{p} - \frac{1}{q}\right) < 1 \), the space \( WF_{X,q}^1(\mathbb{R}^n) \cap C^2_c(\mathbb{R}^n) \) is independent of \( q \).

When \( p = 1 \) and the Hardy–Littlewood maximal operator \( M \) in (1.6) is not known to be bounded on its associate space \( X' \), Theorem 3.4 does not work anymore in this case; instead of this, we have the following conclusion.
Theorem 3.7. Let $X$ be a ball quasi-Banach function space. Assume that, for any $\theta \in (0, 1)$, $X^{1/\theta}$ is a ball Banach function space, $M$ as in (1.6) bounded on its associate space $(X^{1/\theta})'$, and

$$
\lim_{\theta \to 1} \sup_{\theta \in (0, 1)} \|M\|_{(X^{1/\theta})' \to (X^{1/\theta})'} < \infty.
$$

Assume that $q \in (0, \infty)$ and $n(1 - \frac{1}{q}) < 1$. Then, for any $f \in C_c^2(\mathbb{R}^n)$,

$$
||f||_{WF_{X,q}^1(\mathbb{R}^n)} \sim ||\nabla f||_X,
$$

where the positive equivalence constants are independent of $f$.

Proof. Let $f \in C_c^2(\mathbb{R}^n)$. From Theorem 2.14, it follows that

$$
||\nabla f||_X \leq ||f||_{WF_{X,q}^1(\mathbb{R}^n)},
$$

Thus, to complete the proof of this theorem, it remains to prove that

$$
||f||_{WF_{X,q}^1(\mathbb{R}^n)} \leq ||\nabla f||_X.
$$

To this end, it suffices to show that, for any $\lambda \in (0, \infty),

$$
\lambda \left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_X^\frac{1}{\theta} \leq ||\nabla f||_X,
$$

where $E_f(\lambda, q)$ for any $\lambda \in (0, \infty)$ is as in (2.2). By Theorem 3.4 and the fact that, for any $\theta \in (0, 1)$, $X^{1/\theta}$ is a ball Banach function space and that $M$ as in (1.6) is bounded on its associate space $(X^{1/\theta})'$, we conclude that

$$
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_X^\frac{1}{\theta} \leq C\|M\|_{(X^{1/\theta})' \to (X^{1/\theta})'}\|\nabla f\|_{X^{1/\theta}},
$$

which further implies that, for any $\lambda \in (0, \infty),

$$
\lambda^{1/\theta} \left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_X^\frac{1}{\theta} \leq C\lambda^{1/\theta} \|M\|_{(X^{1/\theta})' \to (X^{1/\theta})'}\|\nabla f\|_{X^{1/\theta}}.
$$

Let $\{\theta_m\}_{m \in \mathbb{N}} \subset (0, 1)$ satisfy $\lim_{m \to \infty} \theta_m = 1$. From this, (3.13), Definition 2.7(iii), (3.9), and the fact that $C\lambda^{1/\theta}$ is continuous on $(0, \infty)$, together with $X$ being a quasi-Banach space, we deduce that, for any $\lambda \in (0, \infty),

$$
\left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_X^\frac{1}{\theta} = \lim_{m \to \infty} \inf_{j \geq m} \left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_X^\frac{1}{\theta} = \lim_{m \to \infty} \sup_{j \geq m} \left\| \left\{ y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\} \right\|_X^\frac{1}{\theta}.
$$
from Definition 2.7(ii) and Theorem 3.7, we deduce that

\[ \liminf_{m \to \infty} \left\| y \in \mathbb{R}^n : (\cdot, y) \in E_f(\lambda, q) \right\|_{X} \leq \lambda^{-1} \limsup_{m \to \infty} \left\| \nabla f \right\|_{X} \]

\[ \leq \lambda^{-1} C(\limsup_{m \to \infty} \|M\|_{L^q(\mathbb{R}^n)}) \limsup_{m \to \infty} \left\| \nabla f \right\|_{X} \]

which implies (3.12) and hence (3.11) hold true. This finishes the proof of Theorem 3.7. \( \square \)

**Remark 3.8.**

(i) In the case when \( q := 1 \) and \( X := L^1(\mathbb{R}^n) \), Theorem 3.7 is just [13, Theorem 1.1]. Moreover, in Section 4, Theorem 3.7 is used to solve the endpoint case of concrete examples of ball Banach function spaces.

(ii) We should point out that we do not need the assumption (3.9) in the proof of the lower estimate of (3.10).

As a consequence of Theorem 3.4, we obtain the following fractional Sobolev type inequality on ball Banach function spaces.

**Corollary 3.9.** Let \( q_1 \in [1, \infty] \), \( \theta \in [0, 1] \), and \( q \in [1, q_1] \) satisfy \( \frac{1}{q} = \frac{1 - \theta}{q_1} + \theta \). Let \( X \) be as in Theorem 3.7.

(i) If \( q_1 \in [1, \infty) \), then there exists a positive constant \( C \) such that, for any \( f \in C^2(\mathbb{R}^n) \),

\[ \|f\|_{W^q_{\theta, q}(\mathbb{R}^n)} \leq C\|f\|_{\mathbb{R}^n}^{1 - \theta}\|\nabla f\|_{X}^{\theta}. \]

(ii) If \( q_1 = \infty \), then there exists a positive constant \( C \) such that, for any \( f \in C^2(\mathbb{R}^n) \),

\[ \|f\|_{W^q_{\theta, q}(\mathbb{R}^n)} \leq C\|f\|_{\mathbb{R}^n}^{1 - \theta}\|\nabla f\|_{X}^{\theta}. \]

**Proof.** Let \( \theta, q_1, q, f, \) and \( X \) be as in this corollary. We first prove (ii). Let \( q_1 := \infty \). Then \( q\theta = 1 \). For any \( \lambda, r, s \in (0, \infty) \), let \( E_f(\lambda, r, s) \) be as in (3.1). Since, for any \( \lambda \in (0, \infty) \),

\[ E_f\left(\lambda, \frac{1}{\theta}, \theta, \right) \subseteq E_f\left(\lambda^{1/\theta}, \left(\frac{1 - \theta}{\left(2\|f\|_{\mathbb{R}^n}\right)^{(1 - \theta)/\theta}}, 1, 1\right)\right), \]

from Definition 2.7(ii) and Theorem 3.7, we deduce that

\[ \|f\|_{W^q_{\theta, q}(\mathbb{R}^n)}^{1/\theta} \leq \sup_{\lambda \in (0, \infty)} \lambda^{1/\theta} \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda^{1/\theta}, \theta, \cdot)}(\cdot, y) \, dy \right\|_{X} \]

\[ \leq \sup_{\lambda \in (0, \infty)} \lambda^{1/\theta} \left\| \int_{\mathbb{R}^n} \mathbf{1}_{E_f(\lambda^{1/\theta}, \theta, \cdot)}(\cdot, y) \, dy \right\|_{X} \]
This, together with the fact that $\|\cdot\|_A$ Choose and (3.16)
we deduce that, for any (3.17)
where $\lambda$
This finishes the proof of (ii) of this corollary.

Then we prove (i). Assume $q_1 \in [1, \infty)$. Let $A \in (0, \infty)$ be a constant which is specified later. Since, for any $x, y \in \mathbb{R}^n$,

$$\frac{|f(x) - f(y)|}{|x - y|^{\frac{n+\theta}{\theta}}} = \left[ \frac{|f(x) - f(y)|}{|x - y|^{\frac{n}{\theta}}} \right]^{1-\theta} \left[ \frac{|f(x) - f(y)|}{|x - y|^{n+1}} \right]^\theta,$$

we deduce that, for any $\lambda \in (0, \infty)$,

$$E_f(\lambda, q, \theta) \subset \left[ E_f \left( A^{-\theta} \lambda, q_1, 0 \right) \cup E_f \left( A^{1-\theta} \lambda, 1, 1 \right) \right].$$

This, together with the fact that $\|\cdot\|_X$ is a quasi-norm, implies that, for any $\lambda \in (0, \infty)$,

$$\begin{align*}
(3.16) \quad \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, \theta)}(\cdot, y) \, dy \right\|_X^{\frac{q}{q_1}} & \leq \left\| \int_{\mathbb{R}^n} 1_{E_f(A^{-\theta} \lambda, q_1, 0)}(\cdot, y) \, dy \right\|_X^{\frac{q}{q_1}} + \left\| \int_{\mathbb{R}^n} 1_{E_f(A^{1-\theta} \lambda, 1, 1)}(\cdot, y) \, dy \right\|_X^{\frac{q}{q_1}} \\
& \leq \left( \frac{A^{\theta} G^\theta}{\lambda} \right)^{\frac{q}{q_1}} + \left( \frac{H}{A^{1-\theta} \lambda} \right)^{\frac{1}{q}}.
\end{align*}$$

where

$$G := \sup_{\lambda \in (0, \infty)} \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(\cdot, y) \, dy \right\|_X^{\frac{q_1}{q_1}}$$

and

$$H := \sup_{\lambda \in (0, \infty)} \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, 1, 1)}(\cdot, y) \, dy \right\|_X.$$ 

Choose $A \in (0, \infty)$ such that

$$\left( \frac{A^{\theta} G^\theta}{\lambda} \right)^{\frac{q}{q_1}} = \left( \frac{H}{A^{1-\theta} \lambda} \right)^{\frac{1}{q}}.$$

This, combined with (3.16), implies that

$$\begin{align*}
(3.17) \quad \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, \theta)}(\cdot, y) \, dy \right\|_X^{\frac{q}{q_1}} & \leq \left( \frac{A^{\theta} G^\theta}{\lambda} \right)^{\frac{q}{q_1}} + \left( \frac{H}{A^{1-\theta} \lambda} \right)^{\frac{1}{q}} \\
& \sim \left( \frac{A^{\theta} G^\theta}{\lambda} \right)^{\frac{q}{q_1}} \sim A^{-G} \lambda^{1-\theta} H^\theta.
\end{align*}$$
From this and Theorem 3.4, we deduce that

\begin{equation}
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > \lambda |x - y|^{\frac{a}{|\alpha|}} \right\} \right\|_{X^q} \lesssim G^{1-\theta} \| \nabla f \|_{X^q}^{\theta}.
\end{equation}

Next, we prove that, for any \( f \in C_c^2(\mathbb{R}^n) \),

\begin{equation}
G \lesssim \| f \|_{X^1},
\end{equation}

which, combined with (3.18), then completes the proof of (i) of this corollary.

To this end, by Lemma 2.11 and Lemma 3.5(i), we have, for any \( \lambda \in (0, \infty) \) and \( \theta \in (0, 1) \),

\begin{equation}
\left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 1)}(\cdot, y) \, dy \right\|_{X^{1/q_1}} = \sup_{\|g\|_{X^{1/q_1}} = 1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 1)}(x, y) \, dy \, dx \\
\leq \sup_{\|g\|_{X^{1/q_1}} = 1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 1)}(x, y) \, dy \, R_{X^{1/q_1}}(x) \, dx.
\end{equation}

For any \( x, y \in \mathbb{R}^n \), let

\[ D_f(\lambda, q_1)_1 := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(x)| > \lambda |x - y|^{|\alpha|}/2\} \]

and

\[ D_f(\lambda, q_1)_2 := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |f(y)| > \lambda |x - y|^{|\alpha|}/2\}. \]

Observe that, for any \( \lambda \in (0, \infty) \), \( E_f(\lambda, q_1, 0) \subset D_f(\lambda, q_1)_1 \cup D_f(\lambda, q_1)_2 \). From this, (ii) and (iii) of Lemma 3.5, and the definition of \( A_1(\mathbb{R}^n) \), we deduce that, for any \( \lambda \in (0, \infty) \), \( \alpha \in (0, 1) \), and \( g \in (X^{1/q_1})' \) with \( \|g\|_{(X^{1/q_1})'} = 1 \),

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(x, y) \, dy \, R_{X^{1/q_1}}(x) \, dx \\
\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\{(x, y) \in D_f(\lambda, q_1)_1\}} \, dy \, R_{X^{1/q_1}}(x) \, dx \\
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\{(x, y) \in D_f(\lambda, q_1)_2\}} \, R_{X^{1/q_1}}(x) \, dx \, dy \\
\leq \lambda^{-q_1} \int_{\mathbb{R}^n} |f(x)|^{q_1} R_{X^{1/q_1}} g(x) \, dx + [R_{X^{1/q_1}} g]_{A_1(\mathbb{R}^n)} \lambda^{-q_1} \int_{\mathbb{R}^n} |f(y)|^{q_1} R_{X^{1/q_1}} g(y) \, dy \\
\lesssim \left[ 1 + \|M\|_{(X^{1/q_1})' \rightarrow (X^{1/q_1})} \right] \lambda^{-q_1} \|f\|_{X^{1/q_1}} \|R_{X^{1/q_1}} g\|_{X^{1/q_1}} \\
\lesssim \left[ 1 + \|M\|_{(X^{1/q_1})' \rightarrow (X^{1/q_1})} \right] \lambda^{-q_1} \|f\|_{X^{1/q_1}}.
\]

By this and (3.20), we know that there exists a positive constant \( C \) such that, for any \( \alpha \in (0, 1) \), \( \lambda \in (0, 1) \), and \( f \in C_c^2(\mathbb{R}^n) \),

\begin{equation}
\lambda \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q_1, 0)}(\cdot, y) \, dy \right\|_{X^{1/q_1}} \leq C \left[ 1 + \|M\|_{(X^{1/q_1})' \rightarrow (X^{1/q_1})} \right] \|f\|_{X^{1/q_1}}.
\end{equation}
Let \( \{\alpha_m\}_{m \in \mathbb{N}} \subset (0, 1) \) be such that \( \lim_{m \to \infty} \alpha_m = 1 \). Using this, Definition 2.7(iii), (3.21), and (3.9), together with \( X \) being a quasi-Banach space, we conclude that, for any \( \lambda \in (0, \infty) \),

\[
\left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda q, 0)}(\cdot, y) \, dy \right\|_X
= \left\| \lim_{m \to m} \inf_{j \geq m} \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda q, 0)}(\cdot, y) \, dy \right]^{1/\alpha_j} \right\|_X
\leq \lim_{m \to m} \inf \left[ \int_{\mathbb{R}^n} 1_{E_f(\lambda q, 0)}(\cdot, y) \, dy \right]^{1/\alpha_m} \|f\|_{L^{q_1}(\mathbb{R}^n)} \leq \lambda^{-q_1} \limsup_{m \to \infty} \|f\|_{L^{q_1}(\mathbb{R}^n)} \leq \lambda^{-q_1} \|f\|_{L^{q_1}(\mathbb{R}^n)}.
\]

This implies that (3.19) holds true. By (3.19) and (3.18), it follows that (3.14) holds true, which completes the proof of (i) and hence of Corollary 3.9.

**Remark 3.10.**

(i) In the case when \( q_1 := \infty \), \( \theta := 1/q \), and \( X := L^1(\mathbb{R}^n) \), Corollary 3.9 is just [13, Corollary 5.1].

(ii) Let \( q_1 := \infty \). In this case, when we replace the weak type norm \( \| \cdot \|_{\mathcal{W}^{\theta}_{X_0}(\mathbb{R}^n)} \) in (3.15) by the strong type norm \( \| \cdot \|_{\mathcal{F}^{\theta}_{X_0}(\mathbb{R}^n)} \), (3.15) may not hold true (see, for instance, [13, (5.3)]). In this sense, (3.15) with \( q_1 = \infty \) seems to be sharp.

(iii) Let \( q_1 \in [1, \infty) \). We should point out that, if the weak type norm \( \| \cdot \|_{\mathcal{W}^{\theta}_{X}(\mathbb{R}^n)} \) in (3.14) is replaced by the strong type norm \( \| \cdot \|_{\mathcal{F}^{\theta}_{X}(\mathbb{R}^n)} \), it is still unclear whether or not Corollary 3.9(i) still holds true.

(iv) In Corollary 3.9, instead of \( X \) being as in Theorem 3.7, if \( X \) is a ball Banach function and \( \mathcal{M} \) is bounded on \( X' \), then the same conclusions of Corollary 3.9 still hold true, which can be proved by a slight modification of the proof of Corollary 3.9 with Theorem 3.7 replaced by Theorem 3.4.

Similarly, using Theorem 3.4, we obtain the following fractional Gagliardo–Nirenberg type inequality on ball Banach function spaces.
Corollary 3.11. Let $s_1 \in [0, 1)$, $q_1 \in (1, \infty)$, and $\theta \in (0, 1)$. Let $s \in (s_1, 1)$ and $q \in (1, q_1)$ satisfy $s = (1 - \theta)s_1 + \theta$ and $\frac{1}{q} = \frac{1 - \theta}{q_1} + \theta$. Let $X$ be as in Theorem 3.7. Then there exists a positive constant $C$ such that, for any $f \in C_2^\infty(\mathbb{R}^n)$,

$$
\|f\|_{WF_{q,q}^{s,q}(\mathbb{R}^n)} \leq C\|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)}^{1-q_1} \|\nabla f\|_X^q \leq C\|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)}^{1-q_1} \|\nabla f\|_X^q.
$$

Proof. Let $s_1, q_1, \theta, s, q, f$, and $X$ be as in this corollary. For any $\lambda, r, s \in (0, \infty)$ and $x \in \mathbb{R}^n$, let $E_f(\lambda, r, s)$ be as in (3.1). Let

$$
G_1 := \sup_{x \in (0, \infty)} \lambda \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, s, x)}(\cdot, y) \, dy \right\|_{X^{s_1}}^{1/q_1} = \|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)}
$$

and

$$
H_1 := \sup_{x \in (0, \infty)} \lambda \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, 1, 1)}(\cdot, y) \, dy \right\|_X = \|f\|_{WF_{q,q}^{1,1}(\mathbb{R}^n)}.
$$

By an argument similar to that used in the proof of (3.17) with $E_f(\lambda, q, \theta)$ and $E_f(\lambda, q_1, 0)$ replaced, respectively, by $E_f(\lambda, q, s)$ and $E_f(\lambda, q_1, s_1)$, we conclude that

$$
\left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, s)}(\cdot, y) \, dy \right\|_X \lesssim \frac{1}{\lambda^q} G_1^{q(1-\theta)} H_1^{\theta}.
$$

From this, Theorem 3.7, and the fact that $\|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)} \leq \|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)}$, we deduce that

$$
\|f\|_{WF_{q,q}^{s,q}(\mathbb{R}^n)}^{q} = \sup_{x \in (0, \infty)} \lambda^q \left\| \int_{\mathbb{R}^n} 1_{E_f(\lambda, q, s)}(\cdot, y) \, dy \right\|_X \lesssim G_1^{q(1-\theta)} H_1^{\theta} \approx \|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)}^{q(1-\theta)} \|\nabla f\|_{X}^{\theta} \lesssim \|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)}^{q(1-\theta)} \|\nabla f\|_{X}^{\theta}.
$$

This finishes the proof of Corollary 3.11. \qed

Remark 3.12. 
(i) In the case when $X := L^1(\mathbb{R}^n)$, Corollary 3.11 is just [13, Corollary 5.2].

(ii) In Corollary 3.11, if we take $s_1 := 0$, we then have, for any $f \in C_2^\infty(\mathbb{R}^n)$,

$$
(3.22) \|f\|_{WF_{q,q}^{s,q}(\mathbb{R}^n)} \lesssim \|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)} \|\nabla f\|_X
$$

with the implicit positive constant independent of $f$. From (3.19), it follows that, for any $f \in C_2^\infty(\mathbb{R}^n)$, $\|f\|_{WF_{q,q}^{s,q}(\mathbb{R}^n)} \lesssim \|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)}$ with the implicit positive constant independent of $f$. This, together with (3.22), further implies that, for any $f \in C_2^\infty(\mathbb{R}^n)$, $\|f\|_{WF_{q,q}^{s,q}(\mathbb{R}^n)} \lesssim \|f\|_{WF_{q,q}^{s_1,q_1}(\mathbb{R}^n)} \|\nabla f\|_X$ with the implicit positive constant independent of $f$, which is just Corollary 3.9. Thus, Corollary 3.9 can regarded as a corollary of Corollary 3.11 in the critical case when $s_1 = 0$.

(iii) Similarly to Remark 3.10(iv), in Corollary 3.11, instead of $X$ being as in Theorem 3.7, if $X$ is a ball Banach function and $M$ is bounded on $X$, then the same conclusions of Corollary 3.11 still hold true, which can be proved by a slight modification of the proof of Corollary 3.11 with Theorem 3.7 replaced by Theorem 3.4.
4 Applications

In this section, we apply Theorems 3.4 and 3.7, Corollaries 3.9 and 3.11, respectively, to six concrete examples of ball Banach function spaces, namely, Morrey spaces (see Subsection 4.1 below), mixed-norm Lebesgue spaces (see Subsection 4.2 below), variable Lebesgue spaces (see Subsection 4.3 below), weighted Lebesgue spaces (see Subsection 4.4 below), Orlicz spaces (see Subsection 4.5 below), and Orlicz-slice spaces (see Subsection 4.6 below).

4.1 Morrey spaces

For $0 < r \leq \alpha < \infty$, the Morrey space $M^\alpha_r(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^n$ with the finite semi-norm

$$\|f\|_{M^\alpha_r(\mathbb{R}^n)} := \sup_{B \in \mathfrak{B}} |B|^{1/n-1/r} \|f\|_{L^r(B)}.$$ 

These spaces were introduced in 1938 by Morrey [57] in order to study the regularity of solutions to partial differential equations. They have important applications in the theory of elliptic partial differential equations, potential theory, and harmonic analysis (see, for instance, [2, 19, 43, 66, 67, 73]). As was indicated in [68, p. 86], the Morrey space $M^\alpha_r(\mathbb{R}^n)$ is a ball Banach function space, but is not a Banach function space in the terminology of Bennett and Sharpley [7].

It is known that, for $1 < r \leq \alpha < \infty$, the associate space $X'$ of the Morrey space $X := M^\alpha_r(\mathbb{R}^n)$ is a block space, on which the Hardy–Littlewood maximal function $M$ is bounded (see, for instance, [69, Theorem 4.1], [18, Theorem 3.1], and [31, Lemma 5.7]). By this and [68, p. 86], we know that, for any given index $p \in [1, r)$, $X^{1/p} = M^{\alpha/p}_r(\mathbb{R}^n)$ is a ball Banach function space and $M$ is bounded on its associate space $(X^{1/p})'$. Thus, all the assumptions of Theorem 3.4 are satisfied for $X := M^\alpha_r(\mathbb{R}^n)$ with $1 < r \leq \alpha < \infty$, and any $p \in [1, r)$. Moreover, by the proof of [18, Theorem 3.1], we know that, when $1 \leq r \leq \alpha < \infty$ and $\theta \in (0, 1)$, for any $f \in (M^{\alpha/p}_r(\mathbb{R}^n))'$,

$$\|Mf\|_{(M^{\alpha/p}_r(\mathbb{R}^n))'} \leq \frac{r}{\theta} \|f\|_{(M^{\alpha/p}_r(\mathbb{R}^n))'},$$

where the implicit positive constant depends only on $n$, and $M$ is as in (1.6). Thus, all the assumptions of Theorem 3.7, and Corollaries 3.9 and 3.11 are satisfied for $X := M^\alpha_r(\mathbb{R}^n)$ with $1 \leq r \leq \alpha < \infty$. Using Theorems 3.4 and 3.7, we obtain the following conclusions.

Theorem 4.1. Let $1 \leq r \leq \alpha < \infty$ and $q \in (0, \infty)$ satisfy $n(\frac{1}{r} - \frac{1}{q}) < 1$. Then, for any $f \in C^1_c(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda |B|^{\frac{n}{r} - \frac{1}{q}} \left( \int_B \|f(x) - f(y)\| dA \right)^{\frac{1}{r}} = \|\nabla f\|_{M^\alpha_r(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of $f$.

Proof. Let $r$, $\alpha$, $q$, and $f$ be as in this theorem. We prove (4.1) by considering two cases on $r$.

Case 1) $r \in (1, \infty)$. In this case, since $n(\frac{1}{r} - \frac{1}{q}) < 1$, it follows that there exists a $p \in [1, r)$ satisfying $n(\frac{1}{p} - \frac{1}{q}) < 1$. From this and the above discussion, it follows that all the assumptions of
Theorem 3.4 are satisfied for \( X := \mathcal{M}^q_r(\mathbb{R}^n) \). By Theorem 3.4, we know that (4.1) holds true. This finishes the proof of this theorem in this case.

Case 2) \( r = 1 \). In this case, we then have \( n(1 - \frac{1}{q}) < 1 \). From this and the above discussion, it follows that all the assumptions of Theorem 3.7 are satisfied for \( X := \mathcal{M}^q_r(\mathbb{R}^n) \). In this case, by Theorem 3.7, we know that (4.1) holds true. This finishes the proof of Theorem 4.1.

\[ \square \]

**Remark 4.2.** Let \( r \in [1, \infty) \) and \( q = \alpha = r \). In this case, Theorem 4.1 is just [13, Theorem 1.1].

Using Corollaries 3.9 and 3.11, we obtain the following conclusions.

**Corollary 4.3.** Let \( 1 \leq r \leq \alpha < \infty, q_1 \in [1, \infty] \), and \( \theta \in [0, 1] \). Let \( q \in [1, q_1] \) satisfy \( \frac{1}{q} = \frac{1-\theta}{q_1} + \theta \).

(i) If \( q_1 \in [1, \infty) \), then there exists a positive constant \( C \) such that, for any \( f \in C^2(\mathbb{R}^n) \),
\[
\sup_{B \in \mathbb{B}} A(B)^{-\frac{1}{q} + \frac{1}{q_1}} \left[ \int_B \left| \{ y \in \mathbb{R}^n : |f(x) - f(y)| > A|x-y|^{\frac{n+\theta}{q}} \} \right|^r dx \right]^{1/rq} 
\leq C \| f \|_{\mathcal{M}^{q_1}_{\mathcal{M}^q_r}(\mathbb{R}^n)}^{\theta} \| \nabla f \|_{\mathcal{M}^q_r(\mathbb{R}^n)}.
\]

(ii) If \( q_1 = \infty \), then there exists a positive constant \( C \) such that, for any \( f \in C^2(\mathbb{R}^n) \),
\[
\sup_{B \in \mathbb{B}} A(B)^{-\frac{1}{q} + \frac{1}{q_1}} \left[ \int_B \left| \{ y \in \mathbb{R}^n : |f(x) - f(y)| > A|x-y|^{\frac{n+\theta}{q}} \} \right|^r dx \right]^{1/rq} 
\leq C \| f \|_{L^\infty(\mathbb{R}^n)}^{\theta} \| \nabla f \|_{\mathcal{M}^q_r(\mathbb{R}^n)}.
\]

**Corollary 4.4.** Let \( 1 \leq r \leq \alpha < \infty, s_1 \in (0, 1), q_1 \in (1, \infty) \), and \( \theta \in (0, 1) \). Let \( s \in (s_1, 1) \) and \( q \in (1, r) \) satisfy \( s = (1-\theta)s_1 + \theta \) and \( \frac{1}{q} = \frac{1-\theta}{q_1} + \theta \). Then Corollary 3.11 holds true with \( X \) replaced by \( \mathcal{M}^q_r(\mathbb{R}^n) \).

**Remark 4.5.** We point out that the Gagliardo–Nirenberg type inequality in the Sobolev–Morrey space related to the Riesz potential was established by Sawano et al. in [70]. The Gagliardo–Nirenberg type inequalities in the Sobolev-Morrey spaces, as given in Corollaries 4.3 and 4.4, appear new.

### 4.2 Mixed-norm Lebesgue spaces

For a given vector \( \vec{r} := (r_1, \ldots, r_n) \in (0, \infty)^n \), the **mixed-norm Lebesgue space** \( L^\vec{r}(\mathbb{R}^n) \) is defined to be the set of all measurable functions \( f \) on \( \mathbb{R}^n \) with the finite quasi-norm
\[
\|f\|_{L^\vec{r}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |f(x_1, \ldots, x_n)|^{r_1} dx_1 \right\}^\frac{1}{r_1},
\]
where the usual modifications are made when \( r_i = \infty \) for some \( i \in \{1, \ldots, n\} \). Here and throughout this subsection, let \( r_- := \min(r_1, \ldots, r_n) \). The study of mixed-norm Lebesgue spaces can be traced
Theorem 4.6. Let \( \vec{r} := (r_1, \ldots, r_n) \in (1, \infty)^n \) and \( q \in (0, \infty) \) satisfy \( n(\frac{1}{r_i} - \frac{1}{q}) < 1 \). Then, for any \( f \in C^2_c(\mathbb{R}^n) \),

\[
(4.2) \quad \sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^{\frac{1}{q} + 1} \right\} \right\|_{L^q(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^q(\mathbb{R}^n)},
\]

where the positive equivalence constants are independent of \( f \).

Proof. Let \( \vec{r}, q, \) and \( f \) be as in this theorem. Since \( n(\frac{1}{r_i} - \frac{1}{q}) < 1 \), there exists a \( p \in [1, r_-) \) satisfying \( n(\frac{1}{r_i} - \frac{1}{q}) < 1 \). From this and the above discussion, it follows that all the assumptions of Theorem 3.4 are satisfied for \( X := L^p(\mathbb{R}^n) \). By Theorem 3.4, we conclude that (4.2) holds true. This finishes the proof of Theorem 4.6. \( \square \)

Remark 4.7. In the case of mixed-norm Lebesgue spaces in Theorem 4.6, the reason why the case of the boundary \( r_- = 1 \) was excluded is that the Hardy–Littlewood maximal operator \( M \) may not be bounded on \( L^p(\mathbb{R}^n) \) if \( 1 < r_- < r_+ := \max \{ r_1, \ldots, r_n \} = \infty \), as was pointed out in [40, Remark 3.3].

Using Corollaries 3.9 and 3.11, we obtain the following conclusions.

Corollary 4.8. Let \( \vec{r} := (r_1, \ldots, r_n) \in (1, \infty)^n \), \( q_1 \in (1, \infty] \), and \( \theta \in [0, 1] \). Let \( q \in [1, q_1] \) satisfy \( \frac{1}{q} = \frac{1 - \theta}{q_1} + \theta \).

(i) If \( q_1 \in (1, \infty) \), then there exists a positive constant \( C \) such that, for any \( f \in C^2_c(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^{\frac{1}{q_1} + 1} \right\} \right\|_{L^{q_1}(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^{q_1}(\mathbb{R}^n)},
\]

(ii) If \( q_1 = \infty \), then there exists a positive constant \( C \) such that, for any \( f \in C^2_c(\mathbb{R}^n) \),

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^{\frac{1}{q_1} + 1} \right\} \right\|_{L^{q_1}(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^{q_1}(\mathbb{R}^n)}.
\]
Corollary 4.9. Let \( r := (r_1, \ldots, r_n) \in (1, \infty)^n, s_1 \in (0, 1), q_1 \in (1, \infty), \) and \( \theta \in (0, 1). \) Let \( s \in (s_1, 1) \) and \( q \in (1, r_-) \) satisfy \( s = (1 - \theta)s_1 + \theta \) and \( \frac{1}{\theta} = \frac{1 - \theta}{q_1} + \theta. \) Then Corollary 3.11 holds true with \( X \) replaced by \( L^r(\mathbb{R}^n). \)

Remark 4.10. To the best of our knowledge, the Gagliardo–Nirenberg type inequalities of Corollaries 4.8 and 4.9 on the mixed-norm Sobolev space are totally new.

4.3 Variable Lebesgue spaces

Let \( r : \mathbb{R}^n \to (0, \infty) \) be a nonnegative measurable function. Let

\[ r_- := \text{ess inf}_{x \in \mathbb{R}^n} r(x) \quad \text{and} \quad r_+ := \text{ess sup}_{x \in \mathbb{R}^n} r(x). \]

A function \( r : \mathbb{R}^n \to (0, \infty) \) is said to be globally log-Hölder continuous if there exist an \( r_\infty \in \mathbb{R} \) and a positive constant \( C \) such that, for any \( x, y \in \mathbb{R}^n, \)

\[ |r(x) - r(y)| \leq \frac{C}{\log(e + 1/|x - y|)} \quad \text{and} \quad |r(x) - r_\infty| \leq \frac{C}{\log(e + |x|)}. \]

The variable Lebesgue space \( L^{(r)}(\mathbb{R}^n) \) associated with the function \( r : \mathbb{R}^n \to (0, \infty) \) is defined to be the set of all measurable functions \( f \) on \( \mathbb{R}^n \) with the finite quasi-norm

\[ \|f\|_{L^{(r)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left[ \frac{|f(x)|}{\lambda} \right]^{r(x)} \, dx \leq 1 \right\}. \]

If \( 1 < r_- \leq r_+ < \infty, \) then \( (L^{(r)}(\mathbb{R}^n), \| \cdot \|_{L^{(r)}(\mathbb{R}^n)}) \) is a Banach function space in the terminology of Bennett and Sharpley [7] and hence also a ball Banach function space. If, in addition, \( r : \mathbb{R}^n \to (0, \infty) \) is globally log-Hölder continuous, then the Hardy–Littlewood maximal operator is bounded on the space \( L^{(r)}(\mathbb{R}^n), \) as was shown in [1, Theorem 1.7]. For related results on variable Lebesgue spaces, we refer the reader to [58, 59, 49, 21, 23, 24]. Furthermore, by [21, Theorem 2.80], we know that, for \( X := L^{(r)}(\mathbb{R}^n) \) and any given \( p \in [1, r_-), \) \( X^{1/p} \) is a ball Banach function space and \( M \) bounded on the associate space \( (X^{1/p})'. \)

Thus, all the assumptions of Theorem 3.4 are satisfied for \( X := L^{(r)}(\mathbb{R}^n) \) and any given \( p \in [1, r_-). \) Moreover, when function \( r(\cdot) \) is globally log-Hölder continuous, \( 1 \leq r_- \leq r_+ < \infty, \) and \( \theta \in (1/2, 1), \) by the proof of [25, Theorem 4.3.8], we know that, for any \( f \in [L^{(r)}(\mathbb{R}^n)]', \)

\[ \|Mf\|_{[L^{(r)}(\mathbb{R}^n)]'} \leq \frac{r^*_\theta}{\theta} \|f\|_{[L^{(r)}(\mathbb{R}^n)]'}, \]

where the implicit positive constant depends only on \( n \) and \( r(\cdot), \) and \( M \) is as in (1.6). Thus, all the assumptions of Theorem 3.7 and Corollaries 3.9 and 3.11 are satisfied for \( X := L^{(r)}(\mathbb{R}^n) \) when \( r(\cdot) \) is globally log-Hölder continuous and \( 1 \leq r_- \leq r_+ < \infty. \) Using Theorems 3.4 and 3.7, similarly to the proof of Theorem 4.1, we obtain the following conclusions.

Theorem 4.11. Let \( r : \mathbb{R}^n \to (0, \infty) \) be globally log-Hölder continuous. Assume that \( 1 \leq r_- \leq r_+ < \infty \) and \( q \in (0, \infty) \) satisfy \( n \left( \frac{1}{r_-} - \frac{1}{q} \right) < 1. \) Then, for any \( f \in C^2_c(\mathbb{R}^n), \)

\[ \sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda \cdot |y|^{r+1} \right\} \right\|_{L^{(r)}(\mathbb{R}^n)} \sim \| \nabla f \|_{L^{(r)}(\mathbb{R}^n)}, \]

where the positive equivalence constants are independent of \( f. \)
Using Corollaries 3.9 and 3.11, we obtain the following conclusions.

**Corollary 4.12.** Let \( r : \mathbb{R}^n \to (0, \infty) \) be globally log-Hölder continuous. Let \( 1 \leq \bar{r}_- \leq \bar{r}_+ < \infty \), \( q_1 \in [1, \infty) \), and \( \theta \in [0, 1] \). Let \( q \in [1, q_1] \) satisfy \( \frac{1}{q} = \frac{1}{q_1} + \theta \).

(i) If \( q_1 \in [1, \infty) \), then there exists a positive constant \( C \) such that, for any \( f \in \mathcal{C}^2_c(\mathbb{R}^n) \),

\[
\sup_{A \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(y) - f(y)| > A \lambda \theta + |y| \right\} \right\|_{L^{q'(\mathbb{R}^n)}}^\frac{1}{q'} \leq C \|f\|_{L^{q'(\mathbb{R}^n)}}^{1-q} \|\nabla f\|^q_{L^{q}(\mathbb{R}^n)}.
\]

(ii) If \( q_1 = \infty \), then there exists a positive constant \( C \) such that, for any \( f \in \mathcal{C}^2_c(\mathbb{R}^n) \),

\[
\sup_{A \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(y) - f(y)| > A \lambda \theta + |y| \right\} \right\|_{L^{q'(\mathbb{R}^n)}}^\frac{1}{q'} \leq C \|f\|_{L^{q'(\mathbb{R}^n)}}^{1-q} \|\nabla f\|^q_{L^{q}(\mathbb{R}^n)}.
\]

**Corollary 4.13.** Let \( r : \mathbb{R}^n \to (0, \infty) \) be globally log-Hölder continuous. Let \( 1 \leq \bar{r}_- \leq \bar{r}_+ < \infty \), \( s_1 \in (0, 1) \), \( q_1 \in (1, \infty) \), and \( \theta \in (0, 1) \). Let \( s \in (s_1, 1) \) and \( q \in (1, \bar{r}_-) \) satisfy \( s = (1 - \theta)s_1 + \theta \) and \( \frac{1}{q} = \frac{1}{q_1} + \theta \). Then Corollary 3.11 holds true with \( X \) replaced by \( L^{q}(\mathbb{R}^n) \).

**Remark 4.14.** We point out that a different Gagliardo–Nirenberg type inequality was established in the variable Sobolev spaces related to the Riesz potential in [47, 56]. However, Corollaries 4.12 and 4.13 appear new.

### 4.4 Weighted Lebesgue spaces

Recall that, for any given \( 1 \leq r \leq \infty \) and any given weight \( \omega \) on \( \mathbb{R}^n \), \( L^r_\omega(\mathbb{R}^n) \) denotes the weighted Lebesgue space with respect to the measure \( \omega(x) \, dx \) on \( \mathbb{R}^n \) (see Definition 2.2). It is worth pointing out that a weighted Lebesgue space with an \( A_\infty(\mathbb{R}^n) \)-weight may not be a Banach function space; see [68, Section 7.1]. As is well known, for any given \( r \in (1, \infty) \), the Hardy–Littlewood maximal operator

\[
(4.3) \quad \mathcal{M} \text{ is bounded on } L^r_\omega(\mathbb{R}^n) \text{ if and only if } \omega \in A_r(\mathbb{R}^n).
\]

(see, for instance, [3, Theorem 3.1(b)]). Moreover, if \( r \in (1, \infty) \) and \( \omega \in A_\infty(\mathbb{R}^n) \), then \( [L^r_\omega(\mathbb{R}^n)]' = L^{r'}_{\omega_{1/r'}(\mathbb{R}^n)} \), where \( [L^r_\omega(\mathbb{R}^n)]' \) denotes the associated space of \( L^r_\omega(\mathbb{R}^n) \) as in Definition 2.9 (see [25, Theorem 2.7.4]). From this, [68, Section 7.1], (4.3), and the observation that \( \omega \in A_r(\mathbb{R}^n) \) if and only if \( \omega^{1/r'} \in A_{r'}(\mathbb{R}^n) \), it follows that for the space \( X := L^r_\omega(\mathbb{R}^n) \) with \( r \in [1, \infty) \), \( X^{1/p} \) is a ball Banach space and \( \mathcal{M} \) bounded on \( (X^{1/p})' \) for any given \( p \in [1, r) \) and \( \omega \in A_{r/p}(\mathbb{R}^n) \). Thus, the assumptions of Theorem 3.4 are satisfied. Moreover, when \( p \in [1, \infty) \) and \( \omega \in A_{r/p}(\mathbb{R}^n) \), by Lemma 2.17, we know that, for any \( \theta \in (0, 1) \) and \( f \in [L^{p/\theta}(\mathbb{R}^n)]' \),

\[
\|\mathcal{M}\|_{[L^{p/\theta}(\mathbb{R}^n)]'} \leq [\omega]_{A_{r/p}(\mathbb{R}^n)} \|f\|_{[L^{p/\theta}(\mathbb{R}^n)]'},
\]

where the implicit positive constant depends only on \( n \) and \( \mathcal{M} \) is as in (1.6). Thus, all the assumptions of Theorem 3.7 and Corollaries 3.9 and 3.11 are satisfied. Using Theorems 3.4 and 3.7, similarly to the proof of Theorem 4.1, we obtain the following conclusions.
Theorem 4.15. Let $1 \leq p \leq r < \infty$, $\omega \in A_{r/p}(\mathbb{R}^n)$, and $q \in (0, \infty)$ satisfy $n(\frac{1}{p} - \frac{1}{q}) < 1$. Then, for any $f \in C^2_\omega(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \left\{ \lambda |x - y|^{\frac{n+1}{q}} \right\}^q \omega(x) \, dx \right]^{1/q} \sim \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

where the positive equivalence constants are independent of $f$.

Using Corollaries 3.9 and 3.11, we obtain the following conclusions.

Corollary 4.16. Assume that $p \in [1, \infty)$, $\omega \in A_p(\mathbb{R}^n)$, $q_1 \in [1, \infty)$, and $\theta \in [0, 1]$. Let $q \in [1, q_1]$ satisfy $\frac{1}{q} = \frac{1}{q_1} + \theta$.

(i) If $q_1 \in [1, \infty)$, then there exists a positive constant $C$ such that, for any $f \in C^2_\omega(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \left\{ |f(x) - f(y)| > \lambda |x - y|^{\theta} \right\}^\theta \omega(x) \, dx \right]^{\frac{1}{\theta}} \leq C \|f\|_{L^{\theta q_1}(\mathbb{R}^n)} \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

(ii) If $q_1 = \infty$, then there exists a positive constant $C$ such that, for any $f \in C^2_\omega(\mathbb{R}^n)$,

$$\sup_{\lambda \in (0, \infty)} \lambda \left[ \int_{\mathbb{R}^n} \left\{ |f(x) - f(y)| > \lambda |x - y|^{\theta} \right\}^\theta \omega(x) \, dx \right]^{\frac{1}{\theta}} \leq C \|f\|_{L^{\theta}(\mathbb{R}^n)} \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

Corollary 4.17. Let $s_1 \in (0, 1)$, $q_1 \in (1, \infty)$, and $\theta \in (0, 1)$. Let $s \in (s_1, 1)$ and $q \in (1, q_1)$ satisfy $s = (1 - \theta)s_1 + \theta$ and $\frac{1}{q} = \frac{1}{q_1} + \theta$. Assume that $\omega \in A_p(\mathbb{R}^n)$. Then Corollary 3.11 holds true with $X$ replaced by $L^p_\omega(\mathbb{R}^n)$.

Remark 4.18. We pointed out that the Gagliardo–Nirenberg type inequality in the weighted Sobolev space related to the Riesz potential was obtained in [26, 61]. However, to the best of our knowledge, the Gagliardo–Nirenberg type inequalities of Corollaries 4.16 and 4.17 on the weighted Sobolev space are totally new.

4.5 Orlicz spaces

First, we describe briefly some necessary notions and facts on the Orlicz spaces. A non-decreasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is called an Orlicz function if $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0, \infty)$, and $\lim_{t \to \infty} \Phi(t) = \infty$. An Orlicz function $\Phi$ is said to be of lower (resp., upper) type $r$ for some $r \in \mathbb{R}$ if there exists a positive constant $C_{(r)}$ such that, for any $t \in [0, \infty)$ and $s \in (0, 1)$ (resp., $s \in [1, \infty)$),

$$\Phi(st) \leq C_{(r)} s^r \Phi(t).$$
In the remainder of this subsection, we always assume that \( \Phi : [0, \infty) \to [0, \infty) \) is an Orlicz function with positive lower type \( r^-_\Phi \) and positive upper type \( r^+_\Phi \). The Orlicz norm \( \|f\|_{L^\Phi(\mathbb{R}^n)} \) of a measurable function \( f \) on \( \mathbb{R}^n \) is then defined by setting
\[
\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.
\]
Accordingly, the Orlicz space \( L^\Phi(\mathbb{R}^n) \) is defined to be the set of all measurable functions \( f \) on \( \mathbb{R}^n \) with finite norm \( \|f\|_{L^\Phi(\mathbb{R}^n)} \). It is easy to prove that the Orlicz space \( L^\Phi(\mathbb{R}^n) \) is a quasi-Banach function space (see [68, Section 7.6]). As is well known that the Hardy–Littlewood maximal operator \( M \) is bounded on the Orlicz space \( L^\Phi(\mathbb{R}^n) \) if \( 1 < r^-_\Phi \leq r^+_\Phi < \infty \) (see, for instance, [71, Theorem 1.2.1]). Thus, by the dual theorem of \( L^\Phi(\mathbb{R}^n) \) (see, for instance, [68, Subsection 7.8]), we know that, if \( 1 < r^-_\Phi \leq r^+_\Phi < \infty \) and \( p \in [1, r^-_\Phi) \), then, for \( X := L^\Phi(\mathbb{R}^n) \), \( X^{1/p} \) is a ball Banach function space and \( M \) bounded on \( (X^{1/p})' \). Thus, all the assumptions of Theorem 3.4 are satisfied for \( X := L^\Phi(\mathbb{R}^n) \) with \( 1 < r^-_\Phi \leq r^+_\Phi < \infty \) and any given \( p \in [1, r^-_\Phi) \). Moreover, by the proof of [71, Theorem 1.2.1], we know that, when \( 1 \leq r^-_\Phi \leq r^+_\Phi < \infty \) and \( \theta \in (0, 1) \), for any \( f \in ([L^\Phi(\mathbb{R}^n)]^{1/\theta})' \),
\[
\|Mf\|_{([L^\Phi(\mathbb{R}^n)]^{1/\theta})'} \leq (3C_{r^-_\Phi})^{\frac{\theta}{2}} \|f\|_{([L^\Phi(\mathbb{R}^n)]^{1/\theta})'},
\]
where the implicit positive constant depends only on \( n \), and \( M \) is as in (1.6). Thus, the assumptions of Theorem 3.7 and Corollaries 3.9 and 3.11 are satisfied for \( X := L^\Phi(\mathbb{R}^n) \) with \( 1 \leq r^-_\Phi \leq r^+_\Phi < \infty \). Using Theorems 3.4 and 3.7, similarly to the proof of Theorem 4.1, we obtain the following conclusions.

**Theorem 4.19.** Let \( \Phi \) be an Orlicz function with positive lower type \( r^-_\Phi \) and positive upper type \( r^+_\Phi \). Let \( 1 \leq r^-_\Phi \leq r^+_\Phi < \infty \) and \( q \in (0, \infty) \) satisfy \( n(\frac{1}{r^-_\Phi} - \frac{1}{q}) < 1 \). Then, for any \( f \in C^2_c(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^{\frac{n+1}{q}} \right\} \right\|_{L^\Phi(\mathbb{R}^n)} \sim \|\nabla f\|_{L^\Phi(\mathbb{R}^n)},
\]
where the positive equivalence constants are independent of \( f \).

Using Corollaries 3.9 and 3.11, we obtain the following conclusions.

**Corollary 4.20.** Assume that \( \Phi \) is an Orlicz function with positive lower type \( r^-_\Phi \) and positive upper type \( r^+_\Phi \). Let \( 1 \leq r^-_\Phi \leq r^+_\Phi < \infty \), \( q_1 \in [1, \infty] \), and \( \theta \in [0, 1] \). Let \( q \in [1, q_1] \) satisfy \( \frac{1}{q} = \frac{1-\theta}{q_1} + \theta \).

(i) If \( q_1 \in [1, \infty) \), then there exists a positive constant \( C \) such that, for any \( f \in C^2_c(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^{\frac{n+1}{q}} \right\} \right\|_{L^\Phi(\mathbb{R}^n)} \leq C \|f\|_{L^{q_1,\Phi}(\mathbb{R}^n)} \|\nabla f\|_{L^\Phi(\mathbb{R}^n)}^{q}. 
\]

(ii) If \( q_1 = \infty \), then there exists a positive constant \( C \) such that, for any \( f \in C^2_c(\mathbb{R}^n) \),
\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| \left\{ y \in \mathbb{R}^n : |f(\cdot) - f(y)| > \lambda |\cdot - y|^{\frac{n+1}{q}} \right\} \right\|_{L^\Phi(\mathbb{R}^n)} \leq C \|f\|_{L^{\infty,\Phi}(\mathbb{R}^n)} \|\nabla f\|_{L^\Phi(\mathbb{R}^n)}^{q}. 
\]
Corollary 4.21. Let $\Phi$ be an Orlicz function with positive lower type $r^-_\Phi$ and positive upper type $r^+_\Phi$. Let $1 \leq r^-_\Phi \leq r^+_\Phi < \infty$, $s_1 \in (0, 1)$, $q_1 \in (1, \infty)$, and $\theta \in (0, 1)$. Let $s \in (s_1, 1)$ and $q \in (1, r^-_\Phi)$ satisfy $s = (1 - \theta)s_1 + \theta$ and \( \frac{1}{q} = \frac{1}{q_1} + \theta \). Then Corollary 3.11 holds true with $X$ replaced by $L^\Phi (\mathbb{R}^n)$.

Remark 4.22. We pointed out that the Gagliardo–Nirenberg type inequality in the Sobolev–Orlicz space related to the Riesz potential was obtained in [44, 45, 55]. However, to the best of our knowledge, the Gagliardo–Nirenberg type inequalities of Corollaries 4.20 and 4.21 on the Sobolev–Orlicz space are totally new.

4.6 Orlicz-slice spaces

First, we give the definition of the Orlicz-slice spaces and describe briefly some related facts. Throughout this subsection, we assume that $\Phi : [0, \infty) \to [0, \infty)$ is an Orlicz function with positive lower type $r^-_\Phi$ and positive upper type $r^+_\Phi$. For any given $t$, $r \in (0, \infty)$, the Orlicz-slice space $(E'^r_\Phi)(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^n$ with the finite quasi-norm

\[
\|f\|_{(E'^r_\Phi)(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \left[ \frac{\|f 1_{B(x,t)}\|_{L^q(\mathbb{R}^n)}}{\|1_{B(x,t)}\|_{L^q(\mathbb{R}^n)}} \right]^r dx \right\}^{\frac{1}{r}}.
\]

The Orlicz-slice spaces were introduced in [80] as a generalization of the slice spaces of Auscher and Mourougoulo [4, 5] and the Wiener amalgam spaces in [34, 46, 32]. According to [80, Lemma 2.28] and [79, Remark 7.41(i)], the Orlicz-slice space $(E'^r_\Phi)(\mathbb{R}^n)$ is a ball Banach function space, but in general is not a Banach function space. Furthermore, according to [80, Lemma 4.4], all the assumptions of Theorem 3.4 are satisfied for $X := (E'^r_\Phi)(\mathbb{R}^n)$ with $1 < r^-_\Phi \leq r^+_\Phi < \infty$ and $p \in [1, \min(r^-_\Phi, r))$. Moreover, by the proof of [80, Theorem 2.20], we know that, when $1 \leq r^-_\Phi < r^+_\Phi < \infty$, $r \in [1, \infty)$, and $\theta \in (0, 1)$, for any $f \in [(E'^r_\Phi)(\mathbb{R}^n)]'$,

\[\|Mf\|_{[(E'^r_\Phi)(\mathbb{R}^n)]'} \lesssim \left( 3C_{r^-_\Phi} \right)^{-\frac{r^-_\Phi}{p}} + \frac{r}{\theta} \|f\|_{[(E'^r_\Phi)(\mathbb{R}^n)]'},\]

where the implicit positive constant depends only on $n$, and $M$ is as in (1.6). Thus, all the assumptions of Theorem 3.7 and Corollaries 3.9 and 3.11 are satisfied for $X := (E'^r_\Phi)(\mathbb{R}^n)$ with $1 \leq r^-_\Phi \leq r^+_\Phi < \infty$ and $r \in [1, \infty)$. Using Theorems 3.4 and 3.7, similarly to the proof of Theorem 4.1, we obtain the following conclusions.

Theorem 4.23. Let $t \in (0, \infty)$, $r \in [1, \infty)$, and $\Phi$ be an Orlicz function with positive lower type $r^-_\Phi$ and positive upper type $r^+_\Phi$. Let $1 \leq r^-_\Phi \leq r^+_\Phi < \infty$ and $q \in (0, \infty)$ satisfy $n(\frac{1}{\min(r^-_\Phi, r)} - \frac{1}{q}) < 1$. Then, for any $f \in C^2_c (\mathbb{R}^n)$,

\[
\sup_{\lambda \in (0, \infty)} \lambda \left\| 1_{\{y \in \mathbb{R}^n : \|f(\cdot) - f(y)\| > \lambda |y|^{\frac{q-1}{q}}\}} \right\|^\frac{1}{r} \lesssim \|\nabla f\|_{(E'^r_\Phi)(\mathbb{R}^n)},
\]

where the positive equivalence constants are independent of $f$.

Using Corollaries 3.9 and 3.11, we obtain the following conclusions.
Corollary 4.24. Assume that $t \in (0, \infty)$, $r \in (1, \infty)$, and $\Phi$ be an Orlicz function with positive lower type $r_\Phi^-$ and positive upper type $r_\Phi^+$. Let $1 \leq r_\Phi^- \leq r_\Phi^+ < \infty$, $q_1 \in [1, \infty]$, and $\theta \in [0, 1]$. Let $q \in [1, q_1]$ satisfy $\frac{1}{q} = \frac{1}{q_1} + \theta$.

(i) If $q_1 \in [1, \infty)$, then there exists a positive constant $C$ such that, for any $f \in C^2_c(\mathbb{R}^n)$,

$$
\sup_{A \in (0, \infty)} A \left\| \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > A|x - y|^\frac{\theta}{q} \right\} \right\|_{L^{\Phi}(E_R)} \leq C C^{1-\theta} \| f \|_{L^{1, \Phi}(E_R)} \| \nabla f \|_{L^{\Phi}(E_R)}.
$$

(ii) If $q_1 = \infty$, then there exists a positive constant $C$ such that, for any $f \in C^2_c(\mathbb{R}^n)$,

$$
\sup_{A \in (0, \infty)} A \left\| \left\{ y \in \mathbb{R}^n : |f(x) - f(y)| > A|x - y|^\frac{\theta}{q} \right\} \right\|_{L^{\infty}(E_R)} \leq C C^{1-\theta} \| f \|_{L^{1, \infty}(E_R)} \| \nabla f \|_{L^{\infty}(E_R)}.
$$

Corollary 4.25. Let $t \in (0, \infty)$, $r \in (1, \infty)$, and $\Phi$ be an Orlicz function with positive lower type $r_\Phi^-$ and positive upper type $r_\Phi^+$. Let $1 \leq r_\Phi^- \leq r_\Phi^+ < \infty$, $q_1 \in (1, \infty)$, and $\theta \in (0, 1)$. Let $s \in (s_1, 1)$ and $q \in (1, \min\{r_\Phi^-, r_\Phi^+\})$ satisfy $s = (1 - \theta)s_1 + \theta$ and $\frac{1}{q} = \frac{1}{q_1} + \theta$. Then Corollary 3.11 holds true with $X$ replaced by $(E_R^\Phi)_s$.

Remark 4.26. To the best of our knowledge, the Gagliardo–Nirenberg type inequalities of Corollaries 4.24 and 4.25 on the Sobolev–Orlicz–slice space are totally new.

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