Comparative Study on the Fractal Dimensions of Soil Particle Size

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Abstract. The application of fractal theory provides a new approach for the study of soil science. The calculation of fractal dimension is the basis. The purpose of the study is to find a suitable method for calculating the fractal dimension of the soil particle size and to analyze how the fractal dimension reflects soil properties. We mostly use comparative analysis to study. Comparing the mass fractal dimension method with volume fractal dimension method, we analyze their advantages and disadvantages, and the fractal dimension performance of the soil in different land types.

1. Introduction
Fractal theory is a branch of mathematics that has emerged in recent years. It can effectively describe complex and irregular things in nature, and is a new method of dealing with nonlinear problems. With the development of fractal theory, its application in various domains has also increased. Since the 1990s, in China, fractal theory has been widely applied to geography [1]. So far, fractal methods have made great progress in geography, cartography, soil science, remote sensing and other geographical branches. Among them, there are a lot of researches on the application of fractal theory in soil science. These studies have different perspectives, but most of them involve the soil particle size fractal dimension.

2. Calculation of Fractal Dimension of Soil Particle Size
Fractal dimension of soil, as an important parameter of soil, is the basis for studying soil properties, and its calculation method has long been a focus of scholars’ concerns. Application of Fractal Mathematics to Soil Water Retention Estimation, a study of soil fractals by Tyler S W and Wheatcraft S W, shows that the distribution of soil particle size follows the rules [2]:

\[ NR_i^D = c \]  

In the formula, \( R_i \) is the particle radius of level \( i \), \( N \) is the number of particles having a particle size larger than \( R_i \), \( D \) is the fractal dimension of the soil particle, and \( c \) is a constant. Since the number of soil particles \( N \) is involved in this formula and it is difficult to measure directly through experiments, the calculation of fractal dimension \( D \) also encounters a bottleneck. In subsequent studies, scholars have overcome this difficulty by replacing the number of soil particles with other experimentally derived parameters such as soil volume and soil quality. And then, they got two kinds
of calculation formulas that can be practically applied in experiments, which are volume fractal dimension formula and mass fractal dimension formula. Nowadays, they are used widely. These two fractal dimensions are different in the calculation method, idea and properties distinction.

2.1. Volume Fractal Dimension

The volume fractal dimension comes from Mandelbrot, the founder of the fractal theory. He established the particle size fractal feature model in 1983, but the particles are limited to two-dimensional space. Afterwards, Tyler promoted on the basis of the two-dimensional model and established a volume fractal dimension model in three-dimensional space. However, due to the limitations of technical conditions at the time, he did not apply this formula to actual operations. The domestic scholars Wang Guoliang, Huang Guanhua et al. organized and formed a complete calculation model of the fractal dimension of the soil volume based on this, and used to solve practical problems.

It is assumed that the porous medium soil with a self-similar structure consists of particles of different sizes. In a two-dimensional plane, the area $A$ occupied by particles larger than a certain characteristic dimension $d$ is [3]:

$$A(\delta > d) = A_0 \left[ 1 - \left( \frac{d}{d_{\text{max}}} \right)^{2-D} \right]$$  \hspace{1cm} (2)

In the formula, $\delta$ is the measurement scale, $A_0$ is the area occupied by all the soil particles, $d_{\text{max}}$ is the maximum particle size. If the above formula is extended to three-dimensional space, the volume of soil particles larger than a certain particle size $d_i$ ($d_i > d_{i+1}, i = 1,2,3,\cdots$), the larger the index, the smaller the particle size) is:

$$V(\delta > d_i) = V_0 \left[ 1 - \left( \frac{d_i}{d_{\text{max}}} \right)^{3-D} \right]$$  \hspace{1cm} (3)

Similar to the above formula, $d_i$ is a specific particle size, in which $V_0$ is the total volume of all soil, $d_{\text{max}}$ is the maximum particle size.

In practical experiments, the particles in the soil analysis between the two sieve fractions $d_i$ and $d_{i+1}$ are usually expressed by their arithmetic mean $\overline{d_i}$, then the volume greater than a given average particle diameter $\overline{d_i}$ can be expressed as:

$$V(\delta > \overline{d_i}) = V_0 \left[ 1 - \left( \frac{\overline{d_i}}{d_{\text{max}}} \right)^{3-D} \right]$$  \hspace{1cm} (4)

It is easy to know,

$$V(\delta < \overline{d_i}) = V_0 - V(\delta > \overline{d_i}) = V_0 - V_0 \left[ 1 - \left( \frac{\overline{d_i}}{d_{\text{max}}} \right)^{3-D} \right] = V_0 \left( \frac{\overline{d_i}}{d_{\text{max}}} \right)^{3-D}$$  \hspace{1cm} (5)

The general representation of the volume fractal dimension formula is:

$$\frac{V(\delta < \overline{d_i})}{V_0} = \left( \frac{\overline{d_i}}{d_{\text{max}}} \right)^{3-D}$$  \hspace{1cm} (6)

This formula was originally proposed by Tyler and Wheatcraft when studying the soil particle size distribution (PSD), which is called the volume fractal dimension ($D_v$). However, due to the fact that the relevant technology is not mature enough when this formula have appeared, the PSD information of the soil particle volume can not be obtained accurately. Therefore, it is not used in actual measurement. Later, with the development of related technologies, the problem of extracting the PSD
information of soil particle volume was resolved. Without converting volume into mass for calculation, of course, it is unscientific, the formula can be truly used. Currently, Laser Diffraction (LD), a method for obtaining PSD information of soil particle volume, is widely used [4].

2.2. Mass Fractal Dimension

The mass fractal dimension is attributed to the foreign scholar Tyler and the domestic scholar Yang Peiling. Tyler proposed the formula of the volume fractal dimension. However, after encountering obstacles in the specific operation, the volume can only be converted into mass calculation. Therefore, the mass fractal dimension appears. Yang Peiling has expanded and improved it on this basis.

They assumed that the uniform density of all particle sizes of the soil is $\rho = \rho_i$. The quality PSD information is obtained by pipette method and hydrometer method. Thus, we convert the required volume information into quality information and calculate it. This assumption is obviously not practical, therefore, the mass fractal dimension obtained is very different from the true value.

Tyler’s initial volume formula was [5]:

$$ V(\delta > \bar{d}_i) = V_0 \left[ 1 - \left( \frac{\bar{d}_i}{d_{\max}} \right)^{3-D} \right] $$

Assuming that all soil densities are uniform $\rho = \rho_i$, masses greater than a given average particle size are:

$$ M(\delta > \bar{d}_i) = \rho V(\delta > \bar{d}_i) = \rho V_0 \left[ 1 - \left( \frac{\bar{d}_i}{d_{\max}} \right)^{3-D} \right] $$

When the average particle size $\bar{d}_i = 0$, you can calculate the total mass $M_0$ of all soil particles:

$$ M_0 = M(\delta > 0) = \rho V(\delta > 0) = \rho V_0 \left[ 1 - \left( \frac{0}{d_{\max}} \right)^{3-D} \right] = \rho V_0 $$

The above two formulas can be obtained:

$$ M(\delta > \bar{d}_i) = M_0 \left[ 1 - \left( \frac{\bar{d}_i}{d_{\max}} \right)^{3-D} \right] $$

Then,

$$ \frac{M(\delta > \bar{d}_i)}{M_0} = 1 - \left( \frac{\bar{d}_i}{d_{\max}} \right)^{3-D} $$

and because of,

$$ \frac{M(\delta < \bar{d}_i)}{M_0} = 1 - \frac{M(\delta > \bar{d}_i)}{M_0} $$

general expression for the fractal dimension of the mass is:

$$ \frac{M(\delta < \bar{d}_i)}{M_0} = \left( \frac{\bar{d}_i}{d_{\max}} \right)^{3-D_m} $$

2.3. Comparison Between Volume Fractal Dimension And Mass Fractal Dimension

In the actual measurement, the calculation steps using the mass fractal dimension $D_m$ and the volume fractal dimension $D_v$ are approximately the same [5]:

(1) Firstly, the soil used for the experiment was pretreated and classified according to the soil particle size classification standard of the US Soil Texture Classification System: clay (<0.002mm), silt (0.002mm-0.05mm), very fine sand (0.05mm-0.1mm), fine sand (0.1mm-0.25mm), medium sand
(0.25mm-0.5mm), coarse sand (0.5mm-1mm), extremely coarse sand (1mm-2mm) total 7 levels.

(2) Secondly, the arithmetic mean value is used to represent the average particle size of each stage, which is calculated from the upper and lower limits of the above grade intervals.

\[
\bar{d}_i = \frac{d_i + d_{i+1}}{2}
\]  

(14)

(3) Thirdly, we measured the cumulative mass \( M(\delta < \bar{d}_i) \) or volume \( V(\delta < \bar{d}_i) \) less than each particle size, the total mass \( M_0 \) or volume \( V_0 \) of the soil, and the ratio between the average particle size of the soil at each stage and the maximum particle size \( \bar{d}_i \). And then the double logarithm of the equal sign on both sides should be found according to the above formula, that is,

\[
\lg \left[ \frac{V(\delta < \bar{d}_i)}{V_0} \right] = (3 - D_v) \log(\frac{\bar{d}_i}{d_{\max}}) \quad \text{or} \quad \lg \left[ \frac{M(\delta < \bar{d}_i)}{M_0} \right] = (3 - D_m) \log(\frac{\bar{d}_i}{d_{\max}})
\]

(15)

(4) Finally, according to the double logarithmic linear relation, the least square method is used to fit the straight line. After the slope \( \alpha_m \) or \( \alpha_v \) of the fitted line is obtained, the fractal dimensions \( D_m \) and \( D_v \) can be expressed as

\[
D_m = 3 - \alpha_m \quad \text{or} \quad D_v = 3 - \alpha_v
\]  

(16)

The measurement methods used for calculating the mass fractal dimension formula and the volume fractal dimension formula are different. In the calculation using the volume fractal dimension formula, the laser diffraction technique (LD method) is usually used. When the mass fractal dimension formula is used for the calculation, the straw method is frequently used. Through a large number of comparative studies conducted by previous researchers, it found that the soil clay content obtained by measuring the soil volume PSD information by the LD method was lower than the straw method. Exactly, the clay is one of the main components that reflect the nature of the soil, and it also has a great influence on the fractal dimension. From the past studies, it is known that the soil clay content is positively correlated with the fractal dimension of the soil, which leads to the result that, comparing with the one using the fractal dimension of the mass the value using the fractal dimension of the volume is smaller. In addition to the errors in the measurement of the clay content, the premise of the fractal dimension, if the mass itself, the same density of all soils, is also an important cause of the inaccuracy using the mass fractal dimension. It is precisely because this hypothesis is too idealistic that many scholars question this formula.

In general, the volume fractal dimension formula is more objective and reasonable than the mass fractal dimension formula. The key lies in the measurement method of soil volume. At present, the most used and relatively most reasonable measurement method is the LD method, but this method still cannot accurately reflect soil volume PSD information. Because the error caused by shortcomings in the measurement leads to a certain difference between the final fractal dimension and the actual value.

3. Insufficiency and Prospect of the Application of Fractal Theory in Soil Science

The application of fractal theory in many fields has brought new ideas, new methods and new progress to lot of research work. However, there are still some deficiencies, as is the use of fractal theory in soil science.

First, the fractal theory itself is still not mature enough in the development process. The identification and judgment methods of fractals, related parameters of fractal theory, and calculation methods need to be further improved, the understanding of fractals is not systematic and comprehensive.

Second, there are deficiencies in the measurement of the fractal dimension of soil particle size distribution. The two most commonly used fractal dimension measurement formulas have their
advantages and disadvantages. The fact that the mass fractal dimension assumes the same density for all soils is its biggest drawback, although the volume fractal dimension avoids this defect. However, because the method of extracting soil volume PSD information is not advanced enough, it causes human error in the initial information extraction, which makes the calculated volume fractal dimension different from the actual value.

Third, the fact that soil properties are reflected by fractal theory has the disadvantages of a single method and limited content. In a large number of articles on the application of fractal theory to soil science, almost all of them are based on the calculation of the fractal dimension of soil particles, and then a linear regression on the fractal dimension and soil particles to reflect the texture of soil. The study of other properties has not been thorough enough.

The further research and method development of fractal theory, as well as the improvement of soil volume information extraction technology, are the basis for the future development of fractal theory in soil science. This is a key issue in the application of fractal theory applied to soil science. The response of fractal dimension to soil properties has great room for development in the future. Most of the studies in the present study are conducted on the soil of a certain area with different depths or different types of vegetation coverage, but there are few comparative analyses of soils in different regions. In the future, we can start from the macro level, compare and analyze the soil in different regions, and combine with geography knowledge to deeply analyze the causes of the formation of different types of soil. In addition, the relationship between the soil fractal dimension and the properties of the soil itself could be extended to the connection with the plant growth status, reflecting the growth status of the plants in different periods, and researching the different dimensions of the fractal dimension for the most suitable soil for planting crops, etc.

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