Optimal strategies of radial velocity observations in planet search surveys

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ABSTRACT

Applications of the theory of optimal design of experiments to radial velocity planet search surveys are considered. Different optimality criteria are discussed, basing on the Fisher, Shannon, and Kullback-Leibler informations. Algorithms of optimal scheduling of RV observations for two important practical problems are considered. The first problem is finding the time for future observations to yield the maximum improvement of the precision of exoplanetary orbital parameters and masses. The second problem is finding the most favourable time for distinguishing alternative orbital fits (the scheduling of discriminating observations).

These methods of optimal planning are demonstrated to be potentially efficient for multi-planet extrasolar systems, in particular for resonant ones. In these cases, the optimal dates of observations are often concentrated in quite narrow time segments.

Key words: methods: statistical - stars: planetary systems - surveys

1 INTRODUCTION

Since the discovery of the first extrasolar planet orbiting the main-sequence star 51 Pegasii (Mayor & Queloz 1995), more than 300 planets orbiting other stars were found, and about 30 extrasolar systems containing at least two planets are known (see The Extrasolar Planets Encyclopaedia by J. Schneider, www.exoplanet.eu). The majority of these planets was discovered using the radial velocity (hereafter RV) method. Many multi-planet systems show interesting dynamical behaviour like mean-motion resonances and apsidal corotation locks. The sharp understanding of dynamical regimes in these systems is of a great importance. For example, this information may provide significant constraints on the planetary migration processes (Beaugé et al. 2006).

Unfortunately, orbital parameters and masses of many of the extrasolar planets are still poorly known due to a lack and/or a non-uniform sampling of RV observations. Estimations of parameters may be too uncertain or alternative models of the systems may be allowed. Unfortunately, such uncertainties are especially inherent in multi-planetary configurations. It seems that strictly justified strategies of allocating observations are still not widely used in the current practice of RV planet searches. The aim of the present work is to describe several mathematically strict ways of optimal planning of RV observations, based on the general theory of optimal design of experiments (e.g., Ermakov et al. 1983; Ermakov & Zhiglyavskii 1987; Bard 1974, chapter 10), and with particular attention to refining orbits in multi-planet configurations. In Section 2 several criteria of observing schedule optimality are discussed. In Section 3 algorithms of optimal planning of RV observations are described in details. In Section 4 the region of their applicability is discussed. In Section 5 the efficiency of these algorithms is demonstrated using data for real planetary systems.

2 OVERVIEW OF OPTIMALITY CRITERIA

Let us denote $N$ RV measurements of a given star by $v_{\text{meas},n}$, the variances of the corresponding RV errors by $\sigma^2_{\text{meas},n}$. We assume that the RV errors are statistically independent and follow Gaussian distributions with zero means and with specified variances. The timings of the observations are denoted by $t_n$. We also assume that the RV model (corresponding to a certain orbital configuration of the planetary system orbiting the star) is given by $\mu(t, \theta)$, where $\theta$ is the vector of $d$ parameters $\theta$ of the RV curve for a given star (the majority of these parameters characterises the planetary system orbiting this star). Based on the RV time series $(t_n, v_{\text{meas},n}, \sigma_{\text{meas},n})$ and on the model $\mu(t, \theta)$, we can construct an estimation $\theta^*$ of the vector $\theta$. Since the observations suffer from random errors, the estimations $\theta^*$ contains random errors as well. New observations would decrease them. Our task is to find some ‘optimal’ schedule for these new observations, which would lead to a maximum improvement in the precision of the estimations. But firstly we should clarify the sense in which we consider a given
strategy of scheduling the observations as ‘optimal’. The treatment of the optimality is not unique.

Let us write down the elements of the $d \times d$ Fisher information matrix associated with the parameters $\theta$:

$$Q_{ij}(\theta) = \frac{1}{\sigma_{\hat{\mu} \hat{\mu}_n}^2} \left| \frac{\partial \hat{\mu}_n}{\partial \theta_i} \frac{\partial \hat{\mu}_n}{\partial \theta_j} \right|_{\theta_n}.$$  

(1)

The Fisher information matrix characterises the degree of statistical determinability of the parameters $\theta$. The larger is $Q$, the larger are the derivatives $\partial \hat{\mu}_n/\partial \theta_i$, reflecting the sensitivity of the RV model to small shifts of the parameters, and the smaller is the domain of uncertainty associated with $\theta$. For example, it is well-known [Lehman 1983 § 6.4] that if the estimation $\theta^*$ is a (non-linear) least-squares estimation and the number of observations $N$ grows infinitely, then $\theta^*$ is asymptotically unbiased (i.e., its mathematical expectation $E\theta^*$ tends to the true values of $\theta$), and its probability distribution tends to a multivariate Gaussian one with the variance-covariance matrix $\text{Var}\theta^*$ tending to the matrix $C = Q^{-1}$ (that is, in the coordinate notation, $(\text{Var}\theta^*)_{ij} = E(\theta^*\theta^*) - E\theta^*E\theta^* \simeq C_{ij}$). When $N$ is smaller, the distribution of the least-squares estimations may be significantly non-Gaussian. For example, it may be multimodal so that the data may allow several alternative orbital fits. In this case, the variance-covariance matrix of $\theta^*$ does not characterise the confidence domain of $\theta$ in advance, but the Fisher information matrix $Q(\theta)$ still can be used to characterise the degree of ‘peakiness’ of the distribution of $\theta^*$ in the given point $\theta$.

Therefore, the information matrix $Q$ constitutes the basis for a series of optimality criteria. For arbitrary timings $t_1, t_2, \ldots, t_m$ for $m$ new observations, we can predict the new information matrix $Q$ using the formula similar to (1), but containing $m$ extra terms. To obtain an optimal schedule, we should maximize the elements of this matrix over $t_i$. Different criteria from this group pay different attention to different elements of the information matrix (see, e.g., §2.1 by Ermakov et al. 1983 and §3.1 by Ermakov & Zhiglyavkii 1985). Let us select a few popular criteria of optimality that may suit our needs:

(i) **D-optimality.** This criterion considers the determinant $\text{det}Q$ (the D-information) as an objective function to be maximized. In the large-sample asymptotics ($N \to \infty$), the quantity $\sqrt{\text{det}Q} = 1/\sqrt{\text{det}C}$ is inversely proportional to the volume of the uncertainty ellipsoid associated with the estimations $\theta^*$. Therefore, the D-optimal schedule lead to a maximum decrease of this volume (provided the scheduled observations are performed).

(ii) **Generalised D-optimality (D$_s$-optimality).** This criterion is used to refine the precision of certain function of the parameters $\theta$. Introducing the vector of $s$ quantities $\eta = \eta(\theta)$ to be refined, we can calculate the matrix

$$K = \left( \frac{\partial \eta}{\partial \theta} \right)^T C \left( \frac{\partial \eta}{\partial \theta} \right),$$  

(2)

which approximates asymptotically (for $N \to \infty$) the variance-covariance matrix of $\eta(\theta^*)$. We can also predict the new matrix $K$ for the time after making the new observations, by means of substituting in the eq. (2), instead of the matrix $C$, the matrix $C = Q^{-1}$. The generalised D-optimality criterion seeks to minimize $\text{det}K$.

(iii) **L-optimality (linear optimality).** This criterion seeks to minimize certain linear combination of the elements of $C$. Mathematically, we need to minimize the quantity

$$\text{Tr}(LC) = \sum_{i,j=1}^N L_{ij}C_{ij},$$

where the positive definite or semi-definite matrix $L$ is fixed a priori. This criterion may be used, for instance, to minimize a weighted average of the variances of the estimations. The criterion of L-optimality minimizes the mathematical expectation of the quadratic loss function $\sum_{i,j=1}^N L_{ij}\Delta \theta_i \Delta \theta_j$, associated with the random errors $\Delta \theta$ of the estimations $\theta^*$.

There is some obstacle in the use of the criteria of optimality discussed above. It comes from the fact that all of these criteria depend on the values of the parameters $\theta$, which are unknown. To overcome this obstacle, one of the following approaches can be used (Ermakov & Zhiglyavki [1985] §§5.2.5.3):

(i) **Sequential approach.** In this approach, we simply substitute the current estimations $\theta^*$. Therefore, after obtaining more and more measurements, we refine the estimations together with the optimality criterion.

(ii) **Bayesian approach.** In this approach, we average the adopted objective function using some weight function of $\theta$. This weight function may represent the current prior (posterior with respect to observations which are already made) distribution of $\theta$. The averaged objective function can be further maximized (minimized) to obtain the corresponding optimal schedule.

(iii) **Minimax approach.** In this approach, we maximize (minimize) the minimum (maximum) value of the objective function within the domain to which $\theta$ is supposed to belong.

From the computational view point, the sequential approach is more efficient, since it does not require extra integrations or maximizations. However, it requires a bigger amount of ‘priming’ observations, which are needed to obtain a definite and, desirably, nearly Gaussian, starting estimation $\theta^*$. The Bayesian approach is formally free from this limitation, but it has two well-known disadvantages: the complexity of the appearing integrals over $\theta$ and the ambiguity concerning the choice of prior distributions. The main disadvantage of the minimax approach is the presence of extra multi-dimensional optimisation, which is easy to perform in simplest cases only.

There is another group of optimality criteria, which deal with the Shannon information (negative Shannon entropy)

$$I = - \int p(\theta) \ln p(\theta) d\theta,$$

associated with certain probability density $p(\theta)$. The Bayesian approach of maximizing the Shannon information associated with the posterior distribution of $\theta$ was considered by Ford (2008). This method also suffers from the two mentioned disadvantages of the Bayesian algorithms. Although for some simplified RV models Ford (2008) proposed a way of decreasing the amount of calculations required in extra integrations, the abilities of the Bayesian criteria still remain limited, especially for multi-planet systems with large numbers of free parameters.

Since the uncertainty of the estimations $\theta^*$ decreases for larger time series, the sequential and Bayesian approaches become asymptotically equivalent when $N \to \infty$. The same property of the asymptotic equivalence holds true for the D-optimality criterion and the Shannon information criterion. The reason for this equivalence comes from the fact that when $N \to \infty$, the distribution of $\theta$ tends to the mul-
tivaraiate Gaussian one with \( \text{Var} \theta^* \simeq \mathbf{C} \). The resulting Shannon information can be uniquely expressed via the \( D \)-information, according to the relation \( F = \ln \sqrt{\det \mathbf{Q}} + \text{const.} \). Therefore, for relatively large \( N \), the best choice is to use the sequential optimality criterion for refining the orbital configurations of planetary systems.

In RV planet searches, we often deal with two or more roughly equally likely orbital solutions for the planetary system. The optimality criteria discussed above are insensitive to this multiplicity. They favour to futher increasing of the ‘peakiness’ of the modes of the \( \chi^2 \) function or of the likelihood function (or of the posterior distribution of \( \theta \)), but they may be less useful for discriminating one of the peaks. In the case of multiple alternative orbital fits, we should use another optimality criterion, which is based on the Kullback-Leibler discriminating information (see §5.5 by Barnakov & Zhiglyavskii [1985] and §10.5.1 by Bard [1974]). Before we describe this criterion, let us introduce some extra definitions. Given the current estimations \( \theta = \theta^* \), we can make a prediction of the RV at any time \( t \) as \( v = \mu(t, \theta^*) \). Since \( \theta^* \) incorporate random errors, the RV prediction should also contain a random component leading to an uncertainty of the predicted value of \( v \). For each orbital fit we may construct its own RV prediction. After that, we can define the Kullback-Leibler informations

\[
I_{(2)}(v) = \int p_1(v) \ln \frac{p_1(v)}{p_2(v)} dv, \quad I_{(1)}(v) = \int p_2(v) \ln \frac{p_2(v)}{p_1(v)} dv. \tag{3}
\]

Here, the probability densities \( p_{1,2}(v) \) describe the distribution of the RV prediction for the one of two orbital fits. It is not hard to see that the quantities \( I \) represent the mathematical expectations of the likelihood ratio statistic considering the first or the second model as true. To obtain optimal dates for discriminating observations, we need to maximize \( I_{(1)}, I_{(2)}, \) or some their combination.

In this paper, we aim to describe rules of optimal planning of RV measurements for multi-planet extrasolar systems. For such systems, the initial amounts of observations, needed in the sequential approach, are usually available. Thus we adopt below the sequential approach to construct detailed algorithms for \( D \)- and \( L \)-optimal scheduling and for optimal scheduling of discriminating observations.

### 3 ALGORITHMS OF OPTIMAL SCHEDULING

#### 3.1 D-optimal scheduling

If we obtain an extra, \( (N + 1) \text{th} \), RV observation at time \( \tau \), the precision of the estimations \( \theta^* \) increases. The new information matrix \( \mathbf{Q} \) is defined by the formula similar to (1), but containing an extra, \( (N + 1) \text{th} \), term in the summation. The \( D \)-optimality criterion seeks to find \( \tau \) that provides the maximum value of \( \det \mathbf{Q} \). Let us write down the asymptotic \( (N \to \infty) \) variance of the RV prediction \( v \), using the following analogue of the formula (2):

\[
\sigma_{\text{pred}}^2(t, \theta) = \frac{\partial \mu}{\partial \theta} \mathbf{C} \left( \frac{\partial \mu}{\partial \theta} \right)^\top = \sum_{i,j=1}^d C_{ij} \frac{\partial \mu}{\partial \theta_i} \frac{\partial \mu}{\partial \theta_j}. \tag{4}
\]

According to [Bard 1974], § 10.3), the maximization of \( \det \mathbf{Q} \) is equivalent to the maximization of the RV prediction variance \( \sigma_{\text{pred}}^2(\tau) \), which should be calculated in the approximation (4). That is, the optimal time for the future RV measurement corresponds to the most uncertain RV prediction. This rule is quite clear: to improve our knowledge, we need to make observations when our predicting abilities are mostly limited. On contrary, we profit little from observations made when the observed quantity is well predictable.

When merging data from several observatories, it may be more useful to maximize the full variance of the deviation of the future measurement from the RV prediction:

\[
\sigma^2(\tau) = \sigma_{\text{pred}}^2(\tau) + \sigma_{\text{meas}}^2, \tag{5}
\]

where \( \sigma_{\text{meas}}^2 \) is the variance of the future RV measurement (expected for a given observatory). At last, we define the non-dimensional function \( J^2(\tau) = \det \mathbf{Q}/\det \mathbf{Q}^* \), which can be transformed to a more simple form

\[
J^2(\tau) \equiv \frac{\det \mathbf{Q}}{\det \mathbf{Q}^*} = \frac{\sigma^2(\tau)}{\sigma_{\text{meas}}^2} = 1 + \frac{\sigma_{\text{pred}}^2(\tau)}{\sigma_{\text{meas}}^2} \geq 1. \tag{6}
\]

This relation is derived in [Bard 1974, §10.3) and in the Appendix [A] of the present paper. The identity (6) means that the value of \( J(\tau) \) tells us how much the volume of the uncertainty ellipsoid associated with \( \theta^* \) would decrease after making the extra observation at time \( \tau \).

Often, it is not necessary to refine the whole set of parameters \( \theta \). Instead, we may be interested in improving the precision of only some of these parameters or in refining a certain function of \( \theta \). For instance, we have no direct need in refining the estimation of the velocity of the barycentre of the planetary system (the constant velocity term in the RV model). We may want to refine orbital elements of only certain planets in the system. We may want to refine only some combinations of the parameters of the system.

Let us assume that we need to improve the precision of the vector \( \eta = \eta(\theta) \). Linear (asymptotic \( N \to \infty \)) approximation to the variance-covariance matrix of \( \eta \) is given by (2) The matrix \( \mathbf{K} \) can be defined using the same formula but with \( \mathbf{C} \) changed by \( \mathbf{C} \). Now we need to minimize \( \det \mathbf{K} \) instead of \( \det \mathbf{C} \). In the case when the matrices \( \mathbf{K} \) and \( \mathbf{K} \) are not degenerated, we can extend the definition of \( J \) according to \( J^2(\tau) = \det \mathbf{K}/\det \mathbf{K} \). As it is shown in Appendix [A] the function \( J(\tau) \) can be again rewritten in a more simple form:

\[
J^2(\tau) = \frac{\sigma^2(\tau)}{\sigma_{\eta}^2(\tau)} \geq 1, \tag{7}
\]

where \( \sigma_{\eta}^2(\tau) \) is the conditional variance of the difference (RV prediction − actual future RV measurement), calculated under condition of fixed \( \eta \). This conditional variance can be calculated using the formulas (1) and (4), but substituting, instead of the matrix \( \mathbf{C} \), the corresponding conditional variance-covariance matrix \( \mathbf{C}_{\eta} \) of \( \theta^* \):

\[
\mathbf{C}_{\eta} = \mathbf{C} - \mathbf{A}^\top \mathbf{K}^{-1} \mathbf{A} \quad \text{with} \quad \mathbf{A} = \frac{\partial \eta}{\partial \theta} \mathbf{C}. \tag{8}
\]

Here, the matrix \( \mathbf{A} \) represents the asymptotic approximation to the cross-covariance matrix of \( \eta \) and \( \theta \). Note that when \( \eta \equiv \theta \), we have \( \mathbf{A} = \mathbf{K} = \mathbf{C} \), \( \mathbf{C}_{\eta} = 0 \), and \( \sigma_{\eta} = \sigma_{\text{meas}}, \) as we could expect. If the parameters in \( \eta \) represent a subset of the parameters in \( \theta \) then the matrices \( \mathbf{K} \) and \( \mathbf{A} \) represent certain submatrices of the matrix \( \mathbf{C} \) and the calculations are much simplified. In this case, all of the elements in the matrix \( \mathbf{C}_{\eta} \) are zero, except for those corresponding to the variances and mutual correlations of the parameters \( \eta \).
Some difficulties arise when the matrix $K$ is degenerated. This may take place when some of the variables in $\eta$ are dependent and, hence, the matrix $\partial \eta / \partial \theta$ is not of full rank. Such a case quite can be met in practice and we need to process it correctly. In this case $\text{det} K = \text{det} \bar{K} = 0$ and $r < \dim \eta$ axes of the uncertainty ellipsoid of $\eta$ vanish. However, the volume of this ellipsoid in the subspace of the rest $(\dim \eta - r)$ axes does not vanish yet. This volume is proportional to the product of all non-zero eigenvalues of $K$ (recall that the product of all eigenvalues of a matrix is equal to its determinant). Denoting this product as $\text{det} K$, we can define $J^2(\tau) = D K / D K$. Note that this revised definition incorporates the previous one as a special case. Again, the function $J(\tau)$ can be transformed to the more simple form $\eta$. However, now we cannot calculate the inverse matrix $K^{-1}$ in the eq. $\theta$. Instead, we should substitute the pseudoinverse matrix $K^+$. This pseudoinverse matrix can be calculated via the eigendecomposition of $K$ (see, e.g., Appendix A in [Bard 1974] for a brief summary of this procedure and further references).

In the general case, we need to find the time range when the values of the function $\eta$ are large. The physical sense of this rule is intuitively clear again: to improve the precision of a given set of parameters, we ought to make observations when the uncertainty of our prediction is large, in comparison with the same uncertainty calculated under assumption that the parameters to be refined are known exactly.

It is worth noting that the variances $\sigma^2_{\text{pred}}(\tau)$, $\sigma^2_{\eta \text{ pred}}(\tau)$, and, hence, the values of $J(\tau)$ are invariable with respect to arbitrary smooth non-degenerated re-parametrisation. That is, if we define the transformations parameters $\theta_1 = \theta_1(\eta)$ and $\eta_1 = \eta_1(\eta)$ having non-zero Jacobians (i.e., $\text{det}(\partial \eta_1 / \partial \theta_1) \neq 0$ and $\text{det}(\partial \eta / \partial \eta_1) \neq 0$) in the point $\theta^*$, the values of $J(\tau)$ calculated by $\theta_1, \eta_1$ would exactly coincide with those calculated by $\theta, \eta$.

### 3.2 L-optimal scheduling

We may also be interested in constructing an $L$-optimal schedule. Now the objective function to be minimized by $\tau$ is $\text{Tr}(L \mathbf{C})$. We can define the non-dimensional gain function $l(\tau) = \text{Tr}(L \mathbf{C}) / \text{Tr}(L \mathbf{C})$ to be maximized. Using identity $\mathbf{A}^2$ from Appendix A we can write down the expression

$$\frac{1}{l(\tau)} = 1 - \frac{c(\tau)^T L c(\tau)}{\sigma^2(\tau) \text{Tr}(L \mathbf{C})}, \tag{9}$$

where the vector $c(\tau)$ has elements $c_i = \sum_{j=1}^d G_{ij} \frac{\partial \theta_j}{\partial \eta_1}$ and represents the asymptotic covariation of $\theta^*$ and the RV prediction at $t = \tau$.

When we want to refine the vector of parameters $\eta(\theta)$, we need to use some generalisation of the function $\mathbf{A}$. In this case, we need to minimize the function $\text{Tr}(L \mathbf{K})$ by $\tau$. We redefine $l(\tau) = \text{Tr}(L \mathbf{K}) / \text{Tr}(L \mathbf{K})$ and use the first of the formulae $\mathbf{A}^3$ to write down

$$\frac{1}{l(\tau)} = 1 - \frac{a(\tau)^T L a(\tau)}{\sigma^2(\tau) \text{Tr}(L \mathbf{K})} = 1 - \frac{c(\tau)^T L c(\tau)}{\sigma^2(\tau) \text{Tr}(L \mathbf{C})}, \tag{10}$$

where the vector $a(\tau)$ having elements $a_i = \sum_{j=1}^d A_{ij} \frac{\partial \theta_j}{\partial \eta_1}$ represents the asymptotic covariation of $\eta(\theta^*)$ and the RV prediction at $t = \tau$. The matrix $L$ is equal to $(\frac{\partial \eta}{\partial \theta^*})^T L (\frac{\partial \eta}{\partial \theta^*})$. This generalised $L$-optimality rule represents the usual one with the matrix $L$ changed by $\mathbf{M}$. Note that now possible degeneracy of the matrix $K$ does not produce any obstacles.

The $L$-optimal criterion is invariable with respect to non-degenerated changes of variables only if the matrix $L$ is transformed in accordance: $L_1 = (\frac{\partial \eta}{\partial \theta^*})^T L (\frac{\partial \eta}{\partial \theta^*})$.}

### 3.3 Scheduling discriminating observations

Let us now assume that we have two alternative RV models $\mu_1(t, \theta_1)$ and $\mu_2(t, \theta_2)$ describing two different orbital configurations of the planetary system. The sets of parameters, $\theta_1$ and $\theta_2$, do not necessarily coincide and the dimensions of the models, dim $\theta_1$ and dim $\theta_2$, may be different as well. Using the approach from the previous subsection, we can calculate the predictions $v_1(\tau)$ and $v_2(\tau)$ along with the full variances $\sigma^2_1(\tau)$ and $\sigma^2_2(\tau)$, according to $\mathbf{A}$. Then we write down the expected information for discriminating between the models (Bard 1974, §10.5), $J_{12}(\tau)$, as

$$J_{12} = -1 + \frac{1}{2} \frac{\sigma^2_1}{\sigma^2_2} + \frac{\sigma^2_2}{\sigma^2_1} \left(1 + \frac{1}{2} \frac{\sigma^2_1}{\sigma^2_2}ight) \left(\frac{v_1 - v_2}{2}\right)^2. \tag{11}$$

The function $\mathbf{A}$ represents the sum of the Kullback-Leibler informations $I_{12}$ and $I_{21}$, calculated from $\mathbf{A}$ under assumption that the distributions of $v_i$ are close to Gaussian. The largest values of $J_{12}(\tau)$ correspond to the most promising time for future observations to rule out one of the alternative models. This rule means that we need to make RV observations when the two models imply largely different predictions of the radial velocity. Simultaneously, the uncertainties of these predictions should not be too large, in order to avoid statistically insignificant differences. The combination $\mathbf{B}$ takes into account both these requirements.

The property of the invariance of $J(\tau)$ with respect to a re-parametrisation is valid for the function $J_{12}(\tau)$ as well.

### 3.4 Scheduling multiple observations

We may need to plan several observations simultaneously. This problem may arise, for instance, when we wish to plan (at least preliminarily) a whole set of observation allocated for a coming observing season for a given star.

Let us denote the timings of $m$ future observations by $\tau_1, \tau_2, \ldots, \tau_m$. Using the same approach as in the previous subsections, we can make RV predictions forming the vector $v$ (dim $v = m$) and calculate the full variance-covariance matrix $V_{\text{pred}}$ of this vector. Thus the $m \times m$ matrix $V_{\text{pred}}$ should contain the variances and cross covariances of the RV predictions. Also, we can calculate the conditional $m \times m$ variance-covariance matrix $V_{\eta \text{ pred}}$ of the vector $v$, taken under condition of fixed $\eta$. Corresponding variance-covariance matrices of full deviations of the future RV measurements from their predictions can be calculated as

$$V = \sigma^2_m + V_{\text{pred}}, \quad V_\eta = \sigma^2_m I + V_{\eta \text{ pred}} \tag{12}$$

with $I$ being the identity matrix. Now we can write down the extension of the function $J(\tau)$ to multiple observations:

$$J^2(\tau_1, \tau_2, \ldots, \tau_m) \equiv \frac{D K}{D \tau} = \frac{\text{det} V}{\text{det} V_\eta} \geq 1. \tag{13}$$

The generalisation of the function $l(\tau)$ can be calculated according to the equality $\mathbf{C}$.
where the matrix $W$ is calculated in the same way as the variance-covariance matrix $V_{\text{pred}}$ of RV predictions but substituting, instead of the matrix $C$, the matrix $CMC$ (with $M$ given in Section 3.2). The generalised discriminating information looks like

$$J_{12}(\tau_1, \tau_2, \ldots, \tau_m) = -m + \frac{1}{2} \text{Tr}(V_1^{-1}V_2 + V_2^{-1}V_1) + \frac{1}{2}(v_2 - v_1)^T (V_1^{-1} + V_2^{-1})(v_2 - v_1),$$

where subscripts of $V$ and $v$ refer to the alternative models.

The main difficulty in using time allocation rules based on the functions may be connected with too large number of timings $\tau_i$ to be found. Perhaps, there is no big obstacles to scan a two-dimensional grid of $(\tau_1, \tau_2)$ directly when $m = 2$. For $m \geq 3$, such direct scanning requires too intensive computations and is not practical. We may use here the following algorithm. At first, the one-dimensional objective function $\Phi_1(\tau_1)$ is constructed (it may be $J(\tau_1)$, $l(\tau_1)$, or $J_{12}(\tau_1)$, depending on our aims). Using the direct one-dimensional search of the maximum, the date $\tau_1$ for one of the future RV measurements is obtained. Then the two-dimensional planning function, $\Phi_2(\tau_1, \tau_2)$, is plotted, but the date $\tau_1$ is fixed at the value obtained in previous step. Therefore, again we can use a one-dimensional search of the maximum. As a result, we obtain the second date $\tau_2$. The values of $\tau_1$ and $\tau_2$ may be further adjusted using some non-linear maximization algorithm. Then we construct the function $\Phi_3(\tau_1, \tau_2, \tau_3)$ with $\tau_1$ and $\tau_2$ fixed, obtain the optimal value of $\tau_3$, adjust the whole array $\tau_1, \tau_2, \tau_3$, and so on until the full set of $m$ optimal dates is found. We may break this sequence if an extra observation does not provide enough gain. The released time can be used to observe other stars.

### 4 APPLICABILITY OF THE ALGORITHMS

Of course, there is no statistical method that can be applied in every practical situation. The algorithms described above require the following conditions to be satisfied:

(i) The estimations $\theta^*$ are obtained using the least-squares approach or the approach used to obtain them is verified to be equivalent in the sense of planning the observations.

(ii) The number of existing observations should be sufficient, so that the estimations $\theta^*$ are (approximately) unbiased and their joint distribution can be approximated by a multivariate Gaussian one.

(iii) The equations of the RV model, $\mu(t, \theta)$, and of the parameters to be refined, $\eta(\theta)$, can be linearised in the uncertainty ellipsoid surrounding the vector of the estimations.

The first condition is usually satisfied in practice. Here, it is worth mentioning the paper where an approach, other than the least-squares one, was proposed for determination of orbital parameters and masses of exoplanets. This maximum-likelihood approach incorporates a built-in estimation of the so-called RV jitter to be taken into account in the estimations of planetary parameters. A careful analysis shows that the modification of the rules of optimal planning for this case is easy and straightforward.

This is provided by the fact that the cross elements in the Fisher information matrix, corresponding to the estimations of $\theta$ and of the RV jitter, vanish. In fact, the rules of planning remain the same, but with the clause that the RV jitters should not enter in the vector $\theta$. Instead, they should only be added to the values of $\sigma_{\text{mean}}^2$.

The second condition is satisfied for well-conditioned situations, when there is a single clear best-fitting orbital model of the system or the alternative models are well separated in the parametric space. The cases when RV data allow multiple locally best-fitting orbital solutions located suspiciously close to each other, correspond to the ill-conditioned situation. In such cases, no one of the alternative orbital solutions may represent a good approximation to the real configuration of the system. The local maxima of the likelihood function (or local minima of the weighted r.m.s.) represent only ‘ripples’ produced by the lack of the data. The formal uncertainties of the estimations may be unrealistic and underestimated. New RV measurements may change the orbital solutions dramatically. In this situation, the estimations of parameters of a planetary system are strongly biased and their distribution is far from the Gaussian one. Then the optimal planning algorithms should be used with care. We should track the sensitivity of their results to what orbital solution we adopt, to the functional model of the RV curve, to the set of free parameters. Seemingly, the extrasolar systems HD82943 (Mayor et al. 2003) and HD37124 (Vogt et al. 2005) may represent such cases.

Different methods can be used to assess the reliability of orbital fits, needed to justify the use of the scheduling rules described above. Beaugé et al. (2008) performed a series of orbital fits of the system of HD82943 with truncated RV datasets, in order to pronounce the sensitivity of current orbital configuration to future RV measurements. Another, probably more rapid, approach was used by Balev (2008a) for the system of HD37124. It uses the condition number of the Fisher information matrix $Q$ (or, speaking more precisely, of the scaled information matrix having elements $Q_{ij}/\sqrt{Q_{ii}Q_{jj}}$) to assess the degree of ‘ravineness’ of the graph of the likelihood function.

The third condition is tightly connected with the second one. For robust cases, both these conditions should hold true asymptotically, when $N$ grows (because then all uncertainties tend to zero). For a finite $N$, the temporal region of their validity is limited, mainly due to the non-linear dependence on orbital periods of planets. It is admissible to use the linear methods of optimal scheduling during the time much less than the span of the RV time series (say, less than one third of the total time span). Attempts of prediction of optimal dates in a more distant future may be unsafe.

Not every kind of non-linearity and non-gaussianity of the parameters can make the linear theory of planning the observations unreliable. For instance, when the orbital eccentricity of a planet is small, its distribution may be non-Gaussian, though the orbital configuration is well-determined. This is a typical situation for the systems containing a hot Jupiter planet: the true orbital eccentricity $e$ of the hot Jupiter may be so small that even a very precise determination is unable to detect its deviation from zero. The formal (linear) estimation may look like $e = 0.01 \pm 0.01$, implying the argument of the periastron $\omega$ is ill-determined.
In this case, we can make the following change of variables:

\[ x = e \cos \omega, \quad y = e \sin \omega. \]

The dependence of the radial velocity on the new pair of parameters \( x, y \) is almost linear and all necessary equations can be perfectly linearised with respect to these new variables. The joint distribution of estimations of \( x, y \) is much closer to a bivariate Gaussian (peaked near zero). In practical calculations, it is not necessary to perform such change explicitly, thanks to the invariance property of the functions \( J(\tau) \) and \( J_{12}(\tau) \). We can calculate \( J(\tau) \) and \( J_{12}(\tau) \) in the common way, using linear approximations with the initial (non-linear and non-Gaussian) set of parameters. As it was discussed above, the result is exactly the same as for the new (almost linear and Gaussian) set.

We also note that rules described in Section 3 do not account for possible sunlight or moonlight contamination: the formal maximum of \( J(\tau) \) or \( J_{12}(\tau) \) may lie beyond the observing window of a given star. Therefore, we should search for maximum of \( J \) and \( J_{12} \) within the admissible dates only.

5 APPLICATIONS

5.1 Gliese 876: an orbital resonance

The first (Jovian) planet in this system was discovered by Delfosse et al. (1998) using the spectrograph ELODIE and shortly after this confirmed by Marcy et al. (1998), basing on RV observations at Keck observatory. Some further, Marcy et al. (2001) announced the second Jovian planet, which was trapped in the 2/1 mean-motion resonance. Now this system is believed to host three planets: the very low-mass (\( \sim 7.5 \) Earth mass) planet d on a short-period (\( P_d \approx 2 \) days) orbit and two Jovian planets b and c, trapped in the 2/1 mean-motion resonance with \( P_b \approx 30 \) days and \( P_c \approx 60 \) days (Rivera et al. 2005). It is important that the gravitational interactions between planets b and c were directly observed in the RV curve: after 8 years of observations, the orbital periastra of these planets have completed a full revolution. It makes possible to determine the inclination of the system to the sky plane (it was estimated by about 50°), but simultaneously it introduces extra statistical uncertainties and correlations between different parameters. In addition, the orbital period of the planet c is close to lunar cycles. All these facts make precise determination of orbital parameters and masses in the system more difficult. Although currently the orbits in this system are constrained well, we may be interested in further refining the estimations and suppressing the correlations between different parameters of this system.

We adopt the three-planet RV model taking into account planetary perturbations. The orbits are assumed to lie in a common plane, and its inclination to the sky plane is treated as an extra free parameter. Therefore, the RV model have total of 17 free parameters \( \Theta \): four osculating orbital elements and mass for the planets b, five similar parameters for the planet c, five ones for the planet d, the orbital inclination and the constant velocity term. The RV dataset was published in Rivera et al. (2005) and consists of \( N = 155 \) Keck RV measurements with typical internal RV uncertainties \( \sim 4 \) m/s. After obtaining the best-fitting estimations \( \Theta^* \), we can apply the algorithm of the D-optimal scheduling discussed above. Here and in further examples, to obtain a more accurate estimation of the RV error variance \( \sigma_{\text{meas}}^2 \) expected from the observation being scheduled, we use the maximum-likelihood approach of estimating the RV jitter, described in Baluev (2008).

In Fig. 1 the function \( J(\tau) \) for this system is plotted, for the case when \( \eta \) incorporates the 10 parameters of planets b and c and the common orbital inclination (case I), and for the case when \( \eta \) incorporates only the common orbital inclination (case II). We can see that these graphs are peaked. The sequence of peaks shows approximately monthly periodicity, probably due to an interaction between the RV periodicities inspired by the resonant planet ‘c’ and ‘b’ and the aliasing inspired by lunar cycles. During the observing season following the last published observations, the heights of the largest peaks corresponded to about 30% decreasing of the total volume of the uncertainty ellipsoid of \( \eta \) (in the case I) and about 10% increasing of the accuracy of the orbital inclination (in the case II). Such increase of precision would be provided by a single RV measurement only. It is important that these peaks are not necessarily centered in the regions where the observations cannot be done due to the moonlight contamination. Typically, the peaks have semi-widths of about one week only. If made three days before or after a maximum of \( J(\tau) \), the observations would not yield much gain. We can see that, during the last season when the star was observed, some peaks of \( J(\tau) \) were not covered by observations. Possibly, the observations might be distributed more efficiently, if this graph was constructed at that time. Also, a series of tempting peaks of \( J(\tau) \) can be seen in further seasons. However, the probability to observe in these narrow segments is quite small until the optimal planning algorithms are not used.

We can see that, according to the graphs in Fig. 1, the most tempting time ranges for making the RV observations in the nearest future are located in the end of August, 2008 and in the end of October, 2008. Unfortunately, any more precise statement of the optimal dates would be unreliable, because our analysis incorporates only published RV measurements of 3 – 4 years ago, whereas their time span is only 8 years. It may be dangerous to predict the optimal dates which are spaced from the last real measurement by about half of the actual time span. Moreover, it is possible that the actual RV measurements spanning the last three observing seasons of GJ876, significantly shift the positions of the peaks of \( J(\tau) \). Nevertheless, the existence of favourable observation dates for GJ876 in the nearest future is clear. The precise dates can be determined on the basis of the up-to-date RV dataset, including the unpublished measurements taken in the observing seasons of 2005-2007.

5.2 HD208487: an alias ambiguity

The first, roughly half Jupiter mass, planet in this system was discovered by Tinney et al. (2005) on the basis of RV observations made at the Anglo-Australian Observatory. Its

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1 As it can be seen in Figs. 1 and 2 the actual RV observations always fall near minima on the graphs. This fact does not mean that all these observations were sampled so inefficiently. This means that extra observations would yield little gain in these positions, because they would only duplicate the existing observations.
Figure 1. The graphs of the function $J(\tau)$ for the planetary system of GJ876. The broken segments of the graph correspond to the time when observations are impossible (when the star is absent on the night sky or is contaminated by the moonlight). The crosses on the time axis mark positions of several last RV measurements. Top panel: the vector $\eta$ incorporates 10 parameters of the planets b and c and the common orbital inclination to the sky plane. Bottom panel: $\eta$ incorporates only the orbital inclination. See text for more details.

Figure 2. The graph of the information for discrimination between two orbital models for the star HD208487 described in the text. The thin broken segments of the graph correspond to the time when observations are impossible. The crosses on the time axis mark positions of three last RV measurements.

orbital period is close to 130 days. Gregory (2005, 2007) found an extra periodicity in the RV data for this star. The orbital period of the putative second planet was estimated by about 1000 days, and its mass was estimated to be close to the mass of the first planet. Wright et al. (2007) found two alternative solutions, one is consistent with that from Gregory (2007), and another one corresponds to the orbital period of the second planet of about 28.6 days.

The periodogram of the latest RV data for HD208487 ($N = 35$ measurements published by Butler et al. 2006 and having the internal precision $\sim 5$ m/s) indeed shows two high peaks, near 28.6 days and near 900 days. The difference between the respective frequencies corresponds to a period of about 29.5 days, clearly indicating the aliasing connected with the full moon / new moon cycle. The false alarm probability associated with the higher peak (which is near 28.6 days) can be estimated using analytic bounds from Balucinska-Church (2008a) by less than 1%. Thus, some extra periodicity is probably present, but it is not fully clear which periodogram peak is real and which one is its alias. Naturally, we might ask, when the new RV observations should be made for them to rule out one of these alternatives.

In Fig. 2 the corresponding graph of the function $J_{12}(\tau)$ is plotted. It was constructed using two best-fitting double-Keplerian models of the RV curve, with the orbital period of the putative second planet near 900 days or near 28.6 days. In both models, the number of free parameters of the fit was equal to 11 (5 and 5 usual parameters of the Keplerian RV variation for the two planets plus constant velocity term). We can see that, in the last observing season in 2005, there were many good time segments when extra observations would be highly desirable. In these peaks of $J_{12}(\tau)$, the RV predictions for different orbital models diverged by up to $\sim 20$ m/s. A single observation placed in one of these peaks could rule out one of the alternative models at the significance level of $\sim 2$ sigma or even more. However, three RV measurements actually made during this season did not cover the peaks of $J_{12}(\tau)$. It is remarkable that the subsequent observing seasons offer less opportunities of discrimination between the models.

6 CONCLUSIONS

This paper demonstrates how the general tools of the theory of optimal design of experiments can be used for planning RV observations in planet search surveys. Two important practical problems were considered. In the first one, the observations are required to produce the largest improvement in the precision of estimations of exoplanetary parameters.
In the second one, the observations are planned to yield maximum information for distinguishing between two alternative orbital models of an exoplanetary system. These optimising tools are demonstrated using RV data for several real planetary systems. It is shown that these tools may significantly increase the efficiency of observations in RV planet search surveys. They would be especially useful for multi-planet extrasolar systems, in particular for systems containing planet pairs in a mean-motion resonance, for many of which the orbits are still poorly determined. They include, among others, the well-known systems of GJ876, HD82943, and possibly HD37124. The algorithms of optimal scheduling may also be useful in resolving ambiguities concerning planetary orbital configurations, e.g. the alias ambiguity for the HD208487 system.

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APPENDIX A: SOME MATRIX RELATIONS

Let us consider the matrix \( Q \) given by (1) and the matrix \( \tilde{Q} = Q + gg^T/\sigma_{\text{meas}}^2 \), where the elements of the vector \( g \) are given by \( g_i = \partial\mu_i/\partial\theta_i \). Using the identity \( \det(1 + AB) = \det(1 + BA) \) from [Bard 1974, Appendix A], which is valid for arbitrary matrices \( A \) and \( B \) with matching dimensions, we can obtain the relation

\[
\det \tilde{Q} = \det \left( 1 + Q^{-1}(gg^T) \right) = 1 + \frac{g^TQ^{-1}g}{\sigma_{\text{meas}}^2} \tag{A1}
\]

which directly implies (9). Also, we can use the identity \( (A + xy^T)^{-1} = A^{-1} - A^{-1}xyA^{-1}/(1 + y^TA^{-1}x) \) from [Ermakov & Zhiglyavskii 1987, Appendix 1), which is valid for arbitrary matrix \( A \) and vectors \( x, y \) with matching dimensions, to obtain

\[
\tilde{C} = \left( Q + \frac{gg^T}{\sigma_{\text{meas}}^2} \right)^{-1} = Q^{-1} - \frac{ce^T}{\sigma^2}, \tag{A2}
\]

where \( e = Q^{-1}g \). The relation (9) is a direct consequence of the latter equality. Using (A2), we can also calculate

\[
\tilde{K} = K - \frac{aa^T}{\sigma^2}, \quad \frac{\det \tilde{K}}{\det K} = 1 - \frac{a^TK^{-1}a}{\sigma^2} \tag{A3}
\]

where \( a = Ag \) with matrix \( A \) given in (8). The first of equations (A3) implies the relation (10), and the second one implies the relation (11).

When \( K \) and \( \tilde{K} \) are degenerated due to dependency of the parameters \( \eta \), we can easily check the equality

\[
\frac{\partial K}{\partial K} = 1 - \frac{a^TK^{-1}a}{\sigma^2} \tag{A4}
\]

using (A3) and the eigenvalue decompositions of the matrices \( K \) and \( \tilde{K} \), also bearing in mind that the orthogonal matrix, needed to transform \( K \) to a diagonal form, coincides with that for \( \tilde{K} \).

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