Lectures on Cyclotomic Hecke Algebras

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1 Introduction

The purpose of these lectures is to introduce the audience to the theory of cyclotomic Hecke algebras of type $G(m, 1, n)$. These algebras were introduced by the author and Koike, Broué and Malle independently. As is well known, group rings of Weyl groups allow certain deformation. It is true for Coxeter groups, which are generalization of Weyl groups. These algebras are now known as (Iwahori) Hecke algebras.

Less studied is its generalization to complex reflection groups. As I will explain later, this generalization is not artificial. The deformation of the group ring of the complex reflection group of type $G(m, 1, n)$ is particularly successful.

The theory uses many aspects of very modern development of mathematics: Lusztig and Ginzburg’s geometric treatment of affine Hecke algebras, Lusztig’s theory of canonical bases, Kashiwara’s theory of global and crystal bases, and the theory of Fock spaces which arises from the study of solvable lattice models in Kyoto school.

This language of Fock spaces is crucial in the theory of cyclotomic Hecke algebras. I would like to mention a little bit of history about Fock spaces in the context of representation theoretic study of solvable lattice models. For level one Fock spaces, it has origin in Hayashi’s work. The Fock space we use is due to Misra and Miwa. For higher level Fock spaces, they appeared in work of Jimbo, Misra, Miwa and Okado, and Takeamura and Uglov. We also note that Varagnolo and Vasserot’s version of level one Fock spaces have straight generalization to higher levels and coincide with the Takeamura and Uglov’s one. The Fock spaces we use are different from them. But they are essential in the proofs.

Since the cyclotomic Hecke algebras contain the Hecke algebras of type A and type B as special cases, the theory of cyclotomic Hecke algebras is also useful to study the modular representation theory of finite classical groups of Lie type.

I shall explain theory of Dipper and James, and its relation to our theory. The relevant Hecke algebras are Hecke algebras of type A. In this case, we have an alternative approach depending on the Lusztig’s conjecture on quantum groups, by virtue of Du’s refinement of Jimbo’s Schur-Weyl reciprocity. Even for this rather well studied case, our viewpoint gives a new insight. This

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viewpoint first appeared in work of Lascoux, Leclerc and Thibon. This Fock space description looks quite different from the Kazhdan-Lusztig combinatorics, since it hides affine Kazhdan-Lusztig polynomials behind the scene. Inspired by this description, Goodman and Wenzl have found a faster algorithm to compute these polynomials. Leclerc and Thibon are key players in the study of this type A case. I also would like to mention Schiffman and Vasserot’s work here, since it makes the relation of canonical bases between modified quantum algebras and quantized Schur algebras very clear.

I will refer to work of Geck, Hiss, and Malle a little if time allows, since we can expect future development in this direction. It is relevant to Hecke algebras of type B. Finally, I will end the lectures with Broué’s famous dream.

Detailed references can be found at the end of these lectures. The first three are for overview, and the rest are selected references for the lectures. [i-] implies a reference for the i th lecture.

2 Lecture One

2.1 Definitions

Let \( k \) be a field (or an integral domain in general). We define cyclotomic Hecke algebras of type \( G(m, 1, n) \) as follows.

**Definition 2.1** Let \( v_1, \ldots, v_m, q \) be elements in \( k \), and assume that \( q \) is invertible. The Hecke algebra \( H_n(v_1, \ldots, v_m; q) \) of type \( G(m, 1, n) \) is the \( k \)-algebra defined by the following relations for generators \( a_i \) (\( 1 \leq i \leq n \)). We often write \( H_n \) instead of \( H_n(v_1, \ldots, v_m; q) \). If we want to make the base ring explicit, we write \( H_n/k \).

\[
\begin{align*}
(a_1 - v_1) \cdots (a_1 - v_m) &= 0, \\
(a_i - q)(a_i + 1) &= 0 \quad (i \geq 2) \\
a_1a_2a_1a_2 = a_2a_1a_2a_1, \\
a_i a_j = a_j a_i \quad (j \geq i+2) \\
a_ia_{i-1}a_i &= a_i^{-1}a_ia_{i-1} \quad (3 \leq i \leq n)
\end{align*}
\]

The elements \( L_i = q^{1-i}a_ia_{i-1} \cdots a_2a_1a_2 \cdots a_i \) (\( 1 \leq i \leq n \)) are called (Jucy-) Murphy elements or Hoefsmit elements.

**Remark 2.2** Let \( \hat{H}_n \) be the (extended) affine Hecke algebra associated with the general linear group over a non-archimedian field. For each choice of positive

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I would like to thank all the researchers involved in the development. Good interaction with German modular representation group (Geck, Hiss, Malle; Dipper), British combinatorial modular representation group (James, Mathas, Murphy), French combinatorics group (Lascoux, Leclerc, Thibon), modular representation group (Broué, Rouquier; Vigneras), geometric representation group (Varagnolo, Vasserot, Schiffman) and Kyoto solvable lattice model group (Okado, Takeamura, Uglov) has nourished the rapid development. We still have some problems to solve, and welcome young people who look for problems.

I also thank Kashiwara, Lusztig, Ginzburg for their theories which we use.
root system, we have Bernstein presentation of this algebra. Let \( P = \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_n \) be the weight lattice as usual. We adopt "geometric choice" for the positive root system. Namely \( \{\alpha_i := \epsilon_{i+1} - \epsilon_i\} \) are simple roots. Let \( S \) be the associated set of Coxeter generators (simple reflections). Then \( H_n \) has description via generators \( X_\epsilon (\epsilon \in P) \) and \( T_s (s \in S) \). We omit the description since it is well known. The following mapping gives rise to a surjective algebra homomorphism from \( \hat{H}_n \) to \( H_n \).

\[
X_\epsilon \mapsto L_i, \quad T_s \mapsto a_{i+1}
\]

This fact is the reason why we can apply Lusztig’s theory to the study of cyclotomic Hecke algebras. Since the module theory for \( H_n \) has been developed by different methods, it has also enriched the theory of affine Hecke algebras.

Remark 2.3 Let \( \zeta_m \) be a primitive \( m \)th root of unity. If we specialize \( q = 1, v_i = \zeta_m^{-1} \), we have the group ring of \( G(m, 1, n) \). \( G(m, 1, n) \) is the group of \( n \times n \) permutation matrices whose non zero entries are allowed to be \( m \) th roots of unity. Under this specialization, \( L_i \) corresponds to the diagonal matrix whose \( i \) th diagonal entry is \( \zeta_m \) and whose remaining diagonal entries are 1. We would like to stress two major differences between the group algebra and the deformed algebra \( H_n \).

1. \((L_i - v_1) \cdots (L_i - v_m)\) is not necessarily zero for \( i > 1 \).

2. If we consider the subalgebra generated by Murphy elements, its dimension is not \( m^n \) in general. Further, the dimension depends on parameters \( v_1, \ldots, v_m, q \).

Nevertheless, we have the following Lemma. \( a_w \) is defined by \( a_{i_1} \cdots a_{i_l} \) for a reduced word \( s_{i_1} \cdots s_{i_l} \) of \( w \). It is known that \( a_w \) does not depend on the choice of the reduced word.

Lemma 2.4 \( \{L_{1}^{e_1} \cdots L_{n}^{e_n}a_w | 0 \leq e_i < m, w \in \mathfrak{S}_n\} \) form a basis of \( H_n \).

(How to prove) We consider \( H_n \) over an integral domain \( R \), and show that \( \sum RL_1^{e_1} \cdots L_n^{e_n}a_w \) is a two sided ideal. Then we have that these elements generate \( H_n \) as an \( R \)-module. To show that they are linearly independent, it is enough to take \( R = \mathbb{Z}[q, q^{-1}, v_1, \ldots, v_m] \). In this generic parameter case, we embed the algebra into \( H_n/\mathbb{Q}(q, v_1, \ldots, v_m) \). Then we can construct enough simple modules to evaluate the dimension. ■

An important property of \( H_n \) is the following.

Theorem 2.5 (Malle-Mathas) Assume that \( v_i \) are all invertible. Then \( H_n \) is a symmetric algebra.
(How to prove) Since $\mathcal{H}_n$ is deformation of the group algebra of $G(m,1,n)$, we can define a length function $l(w)$ and $a_w$ for a reduced word of $w$. Unlike the Coxeter group case, $a_w$ does depend on the choice of the reduced word. Nevertheless, the trace function

$$\text{tr}(a_w) = \begin{cases} 
0 & (w \neq \mathbf{1}) \\
1 & (w = \mathbf{1}) 
\end{cases}$$

is well defined. $(u,v) := \text{tr}(uv)$ ($u,v \in \mathcal{H}_n$) gives the bilinear form with the desired properties. ■

**Remark 2.6** We have defined deformation algebras for (not all but most of) other types of irreducible complex reflection groups by generators and relations. ($G(m,p,n)$: the author, other exceptional groups: Broué and Malle.)

The most natural definition of cyclotomic Hecke algebras is given by Broué, Malle and Rouquier. It coincides with the previous definition in most cases.

Let $\mathcal{A}$ be the hyperplane arrangement defined by complex reflections of $W$. For each $C \in \mathcal{A}/W$, we can associate the order $e_C$ of the cyclic group which fix a hyperplane in $C$. Primitive idempotents of this cyclic group are denoted by $\epsilon_j(H)$ ($0 \leq j < e_C$). We set $\mathcal{M} = \mathbb{C}^n \setminus \cup_{H \in \mathcal{H}} H$.

**Definition 2.7** For each hyperplane $H$, let $\alpha_H$ be the linear form whose kernel is $H$. It is defined up to scalar multiple. We fix a set of complex numbers $t_{C,j}$. Then the following partial differential equation for $\mathbb{C}W$-valued functions $F$ on $\mathcal{M}$ is called the (generalized) KZ equation.

$$\frac{\partial F}{\partial x_i} = \frac{1}{2\pi \sqrt{-1}} \sum_{C \in \mathcal{A}/W} \sum_{j=0}^{e_C-1} \sum_{H \in C} \frac{\partial (\log \alpha_H)}{\partial x_i} t_{C,j} \epsilon_j(H) F$$

**Theorem 2.8** (Broué-Malle-Rouquier) Assume that parameters are sufficiently generic. Let $B$ be the braid group attached to $\mathcal{A}$. Then the monodromy representation of $B$ with respect to the above KZ equation factors through a deformation ring of $\mathbb{C}W$. If $W = G(m,1,n)$ for example, it coincides with the cyclotomic Hecke algebra with specialized parameters.

### 2.2 Representations

If all modules are projective modules, we say that $\mathcal{H}_n$ is a semi-simple algebra, and call these representations ordinary representations. We have

**Proposition 2.9** (Ariki(-Koike)) $\mathcal{H}_n$ is semi-simple if and only if $q^i v_j - v_k$ ($|i| < n, j \neq k$) and $1 + q + \cdots + q^i$ ($1 \leq i < n$) are all non zero. In this case, simple modules are parametrized by $m$-tuples of Young diagrams of total size $n$. For each $\lambda = (\lambda^{(m)}, \ldots, \lambda^{(1)})$, the corresponding simple module can be realized on the space whose basis elements are indexed by standard tableaux of shape $\lambda$. The basis elements are simultaneous eigenvectors of Murphy elements, and we have explicit matrix representation for generators $a_i$ ($1 \leq i \leq n$).
These representations are called semi-normal form representations. Hence we have complete understanding of ordinary representations. If $\mathcal{H}_n$ is not semi-simple, representations are called modular representations. A basic tool to get information for modular representations from ordinary ones is "reduction" procedure.

**Definition 2.10** Let $(K, R, k)$ be a modular system. Namely, $R$ is a discrete valuation ring, $K$ is the field of fractions, and $k$ is the residue field. For an $\mathcal{H}_n/K$-module $V$, we take an $\mathcal{H}_n/R$-lattice $V_R$ and set $\overline{V} = V_R \otimes k$. It is known that $\overline{V}$ does depend on the choice of $V_R$, but the composition factors do not depend on the choice of $V_R$. The map between Grothendieck groups of finite dimensional modules given by

$$\text{dec}_{K,k} : K_0(\text{mod-}\mathcal{H}_n/K) \longrightarrow K_0(\text{mod-}\mathcal{H}_n/k)$$

which sends $[V]$ to $[\overline{V}]$ is called a decomposition map. Since Grothendieck groups have natural basis given by simple modules, we have the matrix representation of the decomposition map with respect to these bases. It is called the decomposition matrix. The entries are called decomposition numbers.

In the second lecture, we also consider the decomposition map between Grothendieck groups of $KGL(n,q)$-mod and $kGL(n,q)$-mod.

**Remark 2.11** Decomposition maps are not necessarily surjective even after coefficients are extended to complex numbers. If we take $m = 1, 2$ and $q \in k$ to be zero, we have counter examples. These are called zero Hecke algebras, and studied by Carter. **Note that we exclude the case $q = 0$ in the definition.** In the case of group algebras, the theory of Brauer characters ensures that decomposition maps are surjective.

In the case of cyclotomic Hecke algebras, we have the following result.

**Theorem 2.12 (Graham-Lehrer)** $\mathcal{H}_n$ is a cellular algebra. In particular, the decomposition maps are surjective.

The notion of cellularity is introduced by Graham and Lehrer. It has some resemblance to the definition of quasi hereditary algebras. This is further pursued by König and Changchang Xi.

In this lecture, we follow Dipper, James and Mathas’ construction of Specht modules. We first fix notation.

Let $\lambda = (\lambda^{(m)}, \ldots, \lambda^{(1)})$, $\mu = (\mu^{(m)}, \ldots, \mu^{(1)})$ be two $m$-tuples of Young diagrams. We say $\lambda$ dominates $\mu$ and write $\lambda \succeq \mu$ if

$$\sum_{j > k} |\lambda^{(j)}| + \sum_{j=1}^l \lambda^{(k)}_j \geq \sum_{j > k} |\mu^{(j)}| + \sum_{j=1}^l \mu^{(k)}_j$$

for all $k, l$. This partial order is called dominance order.
For each $\lambda = (\lambda^{(m)}, \ldots, \lambda^{(1)})$, we set $a_k = n - |\lambda^{(1)}| - \cdots - |\lambda^{(k)}|$. We have $n \geq a_1 \geq \cdots \geq a_l > 0$ and $a_k = 0$ for $k > l$ for some $l$. We denote $l$ by $l(a)$. For $a = (a_k)$, we denote by $\mathfrak{S}_a$ the set of permutations which preserve $\{1, \ldots, a_l\}, \ldots, \{a_k + 1, \ldots, a_k-1\}, \ldots, \{a_1 + 1, \ldots, n\}$. We also set

$$u_a = (L_1 - v_1) \cdots (L_{a_1} - v_1) \times (L_1 - v_2) \cdots (L_{a_2} - v_2) \times \cdots \\ \cdots \times (L_{l(a)} - v_{l(a)})$$

Let $t^\lambda$ be the canonical tableau. It is the standard tableau on which $1, \ldots, n$ are filled in by the following rule:

1. $\lambda^{(m)}_1$ are written in the first row of $\lambda^{(m)}$; $\lambda^{(m)}_1 + 1, \ldots, \lambda^{(m)}_1 + \lambda^{(m)}_2$ are written in the second row of $\lambda^{(m)}$; \ldots; $|\lambda^{(m)}| + 1, \ldots, |\lambda^{(m)}| + \lambda^{(m-1)}_1$ are written in the first row of $\lambda^{(m-1)}$; and so on.

The row stabilizer of $t^\lambda$ is denoted by $\mathfrak{S}_\lambda$. We set

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} a_w, \quad m_\lambda = x_\lambda u_a = u_a x_\lambda.$$

Let $t$ be a standard tableau of shape $\lambda$. If the location of $i_k \in \{1, \ldots, n\}$ in $t$ is the same as the location of $k$ in $t^\lambda$, we define $d(t) \in \mathfrak{S}_n$ by $k \mapsto i_k$ ($1 \leq k \leq n$).

**Definition 2.13** Let $*: \mathcal{H}_n \rightarrow \mathcal{H}_n$ be the anti-involution induced by $a_i^* = a_i$. For each pair $(s, t)$ of standard tableaux of shape $\lambda$, we set $m_{st} = a^*_{d(s)} m_\lambda a_{d(t)}$.

**Remark 2.14** $\{m_{st}\}$ form a cellular basis of $\mathcal{H}_n$.

**Proposition 2.15** (Dipper-James-Mathas) Let $(K, R, k)$ be a modular system. We set $\mathcal{I}_\lambda = \sum R m_{st}$ where sum is over pairs of standard tableaux of shape strictly greater than $\lambda$ (with respect to the dominance order). Then $\mathcal{I}_\lambda$ is a two sided ideal of $\mathcal{H}_n/R$.

(How to prove) It is enough to consider straightening laws for elements $a_i m_{st}$ and $m_{st} a_i$. We can then show that $m_{uv}$ appearing in the expression have greater shapes with respect to the dominance order. ■

**Definition 2.16** Set $z_\lambda = m_\lambda \bmod \mathcal{I}_\lambda$. Then the submodule $S^\lambda = z_\lambda \mathcal{H}_n$ of $\mathcal{H}_n/\mathcal{I}_\lambda$ is called a Specht module.

**Theorem 2.17** (Dipper-James-Mathas) $\{z_\lambda a_{d(t)}| t : \text{standard of shape } \lambda\}$ form a basis of $S^\lambda$.

(How to prove) We can show by induction on the dominance order that these generate $S^\lambda$. Hence the collection of all these generate $\mathcal{H}_n$. Thus counting argument completes the proof. ■
Definition 2.18 \( S^\lambda \) is equipped with a bilinear form defined by

\[
\langle z \lambda a_{d(t)}, z \lambda a_{d(s)} \rangle m_\lambda = m_\lambda a_{d(s)} a^*_{d(t)} m_\lambda \mod I_\lambda
\]

Theorem 2.19 (General theory of Specht modules)

(1) \( D^\lambda = S^\lambda / \text{rad}(\langle , \rangle) \) is absolutely irreducible or zero module. \( \{D^\lambda \neq 0\} \) form a complete set of simple \( \mathcal{H}_n \)-modules.

(2) Assume \( D^\mu \neq 0 \) and \([S^\lambda : D^\mu] \neq 0\). Then we have \( \mu \sqsubseteq \lambda \).

Remark 2.20 In the third lecture, we give a criterion for non vanishing of \( D^\lambda \).

Theorem 2.21 (Dipper-Mathas) Let \( \{v_1, \ldots, v_m\} = \bigcup_{i=1}^n S_i \) be the decomposition such that \( v_j, v_k \) are in a same \( S_i \) if and only if \( v_j = v_k q^b \) for some \( b \in \mathbb{Z} \). Then we have

\[
\text{mod-} \mathcal{H}_n \simeq \bigoplus_{n_1, \ldots, n_a} \text{mod-} \mathcal{H}_{n_1} \boxtimes \cdots \boxtimes \text{mod-} \mathcal{H}_{n_a}
\]

where \( \mathcal{H}_n = \mathcal{H}_n(v_1, \ldots, v_m; q) \), \( \mathcal{H}_{n_i} = \mathcal{H}_{n_i}(S_i; q) \), and the sum runs through \( n_1 + \cdots + n_a = n \).

Hence, it is enough to consider the case that \( v_i \) are powers of \( q \).

Remark 2.22 For the classification of simple modules, we can use arguments of Rogawski and Vigneras for the reduction to the case that \( v_i \) are powers of \( q \). Hence we do not need the above theorem for this purpose.

2.3 First application

Let \( k_q^\times = k^\times / (q) \). We assume that \( q \neq 1 \), and denote the multiplicative order of \( q \) by \( r \). A segment is a finite sequence of consecutive residue numbers which take values in \( \mathbb{Z}/r\mathbb{Z} \). A multisegment is a collection segments. Assume that a multisegment is given. Take a segment in the multisegment. By adding \( i (i \in \mathbb{Z}/r\mathbb{Z}) \) to the entries of the segment simultaneously, we have a segment of shifted entries. If all of these \( r \) segments appear in the given multisegment, we say that the given multisegment is periodic. If it never happens for all segments in the multisegment, we say that the given multisegment is aperiodic. We denote by \( \mathcal{M}_r^{ap} \) the set of aperiodic multisegments.

Theorem 2.23 (Ariki-Mathas) Simple modules over \( \mathcal{H}_n / k \) are parametrized by

\[
\mathcal{M}_r^{ap}(k) = \{ \lambda : k_q^\times \to \mathcal{M}_r^{ap} \mid \sum_{x \in k_q^\times} |\lambda(x)| = n \}.
\]
We consider a setting for reduction procedure, and show that a lower bound and an upper bound for the number of simple modules coincide. To achieve the lower bound, we use the integral module structure of the direct sum of Grothendieck groups of $\text{proj}\mathcal{H}_n$ with respect to a Kac-Moody algebra action, which will be explained in the second lecture. The upper bound is achieved by cellularity.

**Remark 2.24** The lower bound can be achieved by a different method. This is due to Vigneras.

Let $F$ be a nonarchimedian local field and assume that the residue field has characteristic different from the characteristic of $k$. We assume that $k$ is algebraically closed. We consider admissible $k$-representations of $GL(n,F)$. We take modular system $(K,R,k)$ and consider reduction procedure.

**Theorem 2.25 (Vigneras)** All cuspidal representations are obtained by reduction procedure. The admissible dual of $k$-representations is obtained from the classification of simple $\hat{H}_n/k$-modules.

Hence we have contribution to the last step of the classification.

**Remark 2.26** Her method is induction from open compact groups and theory of minimal $K$-types. In the characteristic zero case, it is done by Bushnell and Kutzko. Considering $M := \text{ind}_{G,K}(\sigma)$ where $(K,\sigma)$ is irreducible cuspidal distinguished $K$-type, she shows that $\text{End}_{kG}(M)$ is isomorphic to product of affine Hecke algebras, and $M$ satisfies the following hypothesis.

"There exists a finitely generated projective module $P$ and a surjective homomorphism $\beta : P \to M$ such that $\text{Ker}(\beta)$ is $\text{End}_{kG}(P)$-stable."

Then the classification of simple $kG$-modules reduces to that of simple $\text{End}_{kG}(M)$-modules. This simple fact is known as Dipper’s lemma.

### 3 Lecture Two

#### 3.1 Geometric theory

Let $\mathcal{N}$ be the set of $n \times n$ nilpotent matrices, $\mathcal{F}$ be the set of $n$-step complete flags in $\mathbb{C}^n$. We define the **Steinberg variety** as follows.

$$Z = \{(N,F_1,F_2) \in \mathcal{N} \times \mathcal{F} \times \mathcal{F}|F_1,F_2 \text{ are } N\text{-stable}\}$$

$$G := GL(n,\mathbb{C}) \times \mathbb{C}^\times \text{ naturally acts on } Z \text{ via }$$

$$(g,q)(N,F_1,F_2) = (q^{-1}Ad(g)N,gF_1,gF_2).$$

Let $K^G(Z)$ be the Grothendieck group of $G$-equivariant coherent sheaves on $Z$. It is an $\mathbb{Z}[q,q^{-1}]$- algebra via convolution product.
Theorem 3.1 (Ginzburg)  
(1) We have an algebra isomorphism $K^G(Z) \simeq \hat{H}_n$.  
(2) Let us consider a central character of the center $Z[X_{\pm 1}, \ldots, X_{\pm n}]^\mathbb{G}_m[q^\pm]$ induced by $\hat{s} : X_i \mapsto \lambda_i$. By specializing the center via this linear character, we obtain a specialized affine Hecke algebra. Let $s$ be diag$(\lambda_1, \ldots, \lambda_n)$. Then $H_s(Z^{(s,q)}, \mathbb{C})$ equipped with convolution product is isomorphic to the specialized affine Hecke algebra. Here the homology groups are Borel-Moore homology groups, and $Z^{(s,q)}$ are fixed points of $(s,q) \in G$.

Remark 3.2 All simple modules are obtained as simple modules of various specialized affine Hecke algebras.

Theorem 3.3 (Sheaf theoretic interpretation)  
Let $\mathcal{N}$ be $\{ (N, F) \in \mathcal{N} \times \mathcal{F} | F$ is $N$-stable $\}$, $\mu : \mathcal{N} \to \mathcal{N}$ be the first projection. Then  
(1) $H_*(Z^{(s,q)}, \mathbb{C}) \simeq \text{Ext}^*(\mu_*C_{\mathcal{N}(s,q)}, \mu_*C_{\mathcal{N}(s,q)})$.  
(2) Let $\mu_*C_{\mathcal{N}(s,q)} = \bigoplus_{k \in \mathbb{Z}} L_0(k) \otimes IC(O, \mathbb{C})[k]$. Then $L_0 := \bigoplus_{k \in \mathbb{Z}} L_0(k)$ is a simple $H_*(Z^{(s,q)}, \mathbb{C})$-module or zero module. Further, non-zero ones form a complete set of simple $H_*(Z^{(s,q)}, \mathbb{C})$-modules. If $q$ is not a root of unity, all $L_0$ are non-zero. If $q$ is a primitive $r$th root of unity, $L_0 \neq 0$ if and only if $O$ corresponds to a (tuple of) aperiodic multisegments taking residues in $\mathbb{Z}/r\mathbb{Z}$.

In the above theorem, the orbits run through orbits consisting of isomorphic representations of a quiver, which is disjoint union of infinite line quivers or cyclic quivers of length $r$. The reason is that $\mathcal{N}^{(s,q)}$ is the set of nilpotent matrices $N$ satisfying $sNs^{-1} = qN$, which can be identified with representations of a quiver via considering eigenspaces of $s$ as vector spaces on nodes and $N$ as linear maps on arrows. This is the key fact which relates the affine quantum algebra of type $A_\infty$, $A^{(1)}_{r-1}$ and representations of cyclotomic Hecke algebras.

Definition 3.4 Let $C_n$ be the full subcategory of $\text{mod} \hat{H}_n$ whose objects are modules which have central character $\hat{s}$ with all eigenvalues of $s$ being powers of $q$. Set $c_n = X_{e_1} + \cdots + X_{e_n}$. We denote by $P_{c_n, \lambda}(-)$ the exact functor taking generalized eigenspaces of eigenvalue $\lambda$ with respect to $c_n$. We then set  
\[ i - \text{Res}(M) = \bigoplus_{\lambda \in \mathbb{C}} P_{c_{n-1}, \lambda - q^i} \left( \text{Res}_{H_{n-1}}^{H_n} (P_{c_n, \lambda}(M)) \right) \]

This is an exact functor from $C_n$ to $C_{n-1}$. We set $U_n = \text{Hom}_\mathbb{C}(K_0(C_n), \mathbb{C})$, $f_i = (i - \text{Res})^T : U_{n-1} \to U_n$.

I shall give some historical comments here. The motivation to introduce these definitions was Lascoux-Leclerc-Thibon’s observation that Kashiwara’s global basis on level one modules computes the decomposition numbers of Hecke algebras of type $A$ over the field of complex numbers. The above notions for
affine Hecke algebras and cyclotomic Hecke algebras were first introduced by the
author in his interpretation of Fock spaces and action of Chevalley generators
in LLT observation into (graded dual of) Grothendieck groups of these Hecke
algebras and \( i \)-restriction and \( i \)-induction operations. This is the starting point
of a new point of view on the representation theory of affine Hecke algebras
and cyclotomic Hecke algebras. As I will explain below, it allows us to give
a new application of Lusztig’s canonical basis. It triggered intensive studies of
canonical bases on Fock spaces. These are carried out mostly in Paris and Kyoto.
On the other hand, the research on cyclotomic Hecke algebras are mostly lead
by Dipper, James, Mathas, Malle and the author. In the third lecture, these
two will be combined to prove theorems on Specht module theory of cyclotomic
Hecke algebras.

We now state a key proposition necessary for the proof of the next theorem.
In the top row of the diagram, we allow certain infinite sum in \( U^-(g(A_{\infty})) \) in
accordance with infinite sum in \( U_n \). Note that we do not have infinite sum in the
bottom row.

**Proposition 3.5 (Ariki)** There exists a commutative diagram
\[
\begin{array}{c}
U^-(g(A_{\infty})) \\
\uparrow \\
U^-(g(A_{r-1}))
\end{array} \cong \bigoplus_{n \geq 0} U_n / q
\]
\[
\begin{array}{c}
\bigoplus_{n \geq 0} U_n / q \uparrow \\
\cong \bigoplus_{n \geq 0} U_n / \sqrt{q}
\end{array}
\]
such that the left vertical arrow is inclusion, the right vertical arrow is induced
by specialization \( q \rightarrow q \), and the bottom horizontal arrow is an \( U^-(g(A_{r-1})) \)-module isomorphism. Under this isomorphism, canonical basis elements of
\( U^-(g(A_{r-1})) \) map to dual basis elements of \{simple module\}.

(How to prove) We firstly construct the upper horizontal arrow by using PBW-
type basis and dual basis of \{standard module\} of affine Hecke algebras. Here
we use Kazhdan-Lusztig induction theorem. We also use restriction rule for
Specht modules. We then appeal to folding argument. On the left hand side,
we consider this folding in geometric terms. Since only short explanation was
supplied in my original paper, I refer to Varagnolo-Vasserot’s argument instead
for this part. We then use

\[\text{[standard module:simple module]} = \text{[canonical basis:PBW-type basis]}\]

which is a consequence of the Ginzburg’s theorem stated above. ■

We now turn to the cyclotomic case. In this case, we can consider not only
negative part of Kac-Moody algebra, but the action of the whole Kac-Moody
algebra.

**Definition 3.6** Assume that \( v_i = q^\gamma_i \ (1 \leq i \leq m) \) and \( q = \sqrt{T} \). We set
\[
V_n = \text{Hom}_C(K_0(\text{mod-}\mathcal{H}_n), \mathbb{C}), \quad V = \oplus_{n \geq 0} V_n.
\]
We define $c_n = L_1 + \cdots + L_n$. Then we can define
\[
i - \text{Res}(M) = \bigoplus_{\lambda \in \mathbb{K}} P_{c_n, \lambda - q^i} \left( \text{Res}_{\mathcal{H}_{n-1}}^\mathcal{H}_n (P_{c_n, \lambda}(M)) \right),
\]
\[
i - \text{Ind}(M) = \bigoplus_{\lambda \in \mathbb{K}} P_{c_n+1, \lambda + q^i} \left( \text{Ind}_{\mathcal{H}_n}^\mathcal{H}_{n+1} (P_{c_n, \lambda}(M)) \right).
\]
These are exact functors and we can define
\[
e_i = (i - \text{Ind})^T : V_{n+1} \to V_n
\]
\[
f_i = (i - \text{Res})^T : V_{n-1} \to V_n
\]

**Definition 3.7** Let $\mathcal{F} = \oplus \mathbb{C}\lambda$ be a based vector space whose basis elements are $m$-tuples of Young diagrams $\lambda = (\lambda^{(m)}, \ldots, \lambda^{(1)})$.

Assume that $\gamma_i \in \mathbb{Z}/r\mathbb{Z}$ ($1 \leq i \leq m$) are given. We introduce the notion of residues of cells as follows: Take a cell in $\lambda$. If the cell is located on the $(i, j)$th entry of $\lambda^{(k)}$, we say that the cell has residue $-i + j + \gamma_k \in \mathbb{Z}/r\mathbb{Z}$. Once residues are defined, we can speak of removable $i$-nodes and addable $i$-nodes on $\lambda$: Convex corners of $\lambda$ with residue $i$ are called **removable** $i$-nodes. Concave corners of $\lambda$ with residue $i$ are called **addable** $i$-nodes.

We define operators $e_i$ and $f_i$ by $e_i \lambda$ (resp. $f_i \lambda$) being the sum of all $\mu$’s obtained from $\lambda$ by removing (resp. adding) a removable (resp. addable) $i$-node. We can extend this action to make $\mathcal{F}$ an integrable $\mathfrak{g}(\mathcal{A}_{r-1}^{(1)})$-module. (If $r = \infty$, we consider $\mathcal{A}_{r-1}^{(1)}$ as $A_{\infty}$.)

We call $\mathcal{F}$ the **combinatorial Fock space**. Note that the action of the Kac-Moody algebra depends on $(\gamma_1, \ldots, \gamma_m; r)$.

**Theorem 3.8 (Ariki)** We assume $v_i = q^{\gamma_i}$ ($1 \leq i \leq m$), $q = \sqrt[r]{1} \in \mathbb{C}$. We set $\Lambda = \sum_{i=1}^m \Lambda_{\gamma_i}$. Then we have the following.

1. $L(\Lambda) \simeq V = U(\mathfrak{g}(\mathcal{A}_{r-1}^{(1)})) \emptyset \subset \mathcal{F}$.
2. Through this isomorphism, canonical basis elements of $L(\Lambda)$ are identified with dual basis elements of simple modules, and the embedding to $\mathcal{F}$ is identified with the transpose of the decomposition map.

(How to prove) We first consider reduction procedure from semi-simple $\mathcal{H}_n/K$ to $\mathcal{H}_n/k$. Note that this is not achieved by $v_i = q^{\gamma_i}$ and $q$ to $q$. Then $V/K$ can be identified with $\mathcal{F}$. We then consider
\[
U^- (\mathfrak{g}(\mathcal{A}_{r-1}^{(1)})) \emptyset \simeq V \subset \mathcal{F}
\]
\[
\uparrow \\
U^- (\mathfrak{g}(\mathcal{A}_{r-1}^{(1)})) \emptyset \oplus U_n/q = \sqrt[r]{1}
\]
Then the previous proposition and integrality of $\mathcal{F}$ prove the theorem. ■
Remark 3.9 The theorem says that we have a new application of Lusztig’s canonical bases, which is similar to the application of Kazhdan-Lusztig bases of Hecke algebras to Lie algebras (Kazhdan-Lusztig conjecture) and quantum algebras (Lusztig conjecture). It is interesting to observe that the roles of quantum algebras and Hecke algebras are interchanged: in Lusztig’s conjecture, Kazhdan-Lusztig bases of Hecke algebras describe decomposition numbers of quantum algebras at roots of unity; in our case, canonical bases of quantum affine algebras on integrable modules describe decomposition numbers of cyclotomic Hecke algebras at roots of unity. Previously, a positivity result was the only application of canonical bases.

The fact that affine Kazhdan-Lusztig polynomials appear in geometric construction of quantum algebras and affine Hecke algebras was known to specialists. What was new for affine Hecke algebras is the above proposition, particularly its formulation in terms of Grothendieck groups of affine Hecke algebras.

For canonical bases on integrable modules, the theorem was entirely new, since no one knew the “correct” way of taking quotients of affine Hecke algebras to get the similar Grothendieck group description of canonical bases on integrable modules. It was just after cyclotomic Hecke algebras were introduced.

Remark 3.10 Let \((K, R, k)\) be a modular system. If we take semi-perfect \(R\), we can identify \(V\) with \(\oplus_{n \geq 0} K_0(\text{proj-} \mathcal{H}_n)\), the transpose of the decomposition map with the map induced by lifting idempotents, the dual basis elements of simple modules with principal indecomposable modules, respectively. Here \(\text{proj-} \mathcal{H}_n\) denotes the category of finite dimensional projective \(\mathcal{H}_n\)-modules. We often use this description since it is more appealing.

Remark 3.11 If \(m = 1\), namely the Hecke algebra has type A, we have another way to compute decomposition numbers. Let us consider Jimbo’s Schur-Weyl reciprocity. It has refinement by Du, and can be considered with specialized parameters. Let us denote the dimension of the natural representation by \(d\), the endomorphism ring \(\text{End}_{\mathcal{H}_n}(V^\otimes n)\) by \(S_d,n\). This endomorphism ring is called the \(q\)-Schur algebra. Note that Schur functors embed the decomposition numbers of Hecke algebras into those of \(q\)-Schur algebras. Then Du’s result implies that the decomposition numbers of Hecke algebras are derived from those of quantum algebras \(U_q(\mathfrak{g}_d)\) with \(q = \sqrt{\lambda}\). There is a closed formula for decomposition numbers [Weyl module:simple module] of quantum algebras at a root of unity: these are values at 1 of parabolic Kazhdan-Lusztig polynomials for (extended) affine Weyl groups of type A. This formula is known as the Lusztig conjecture for quantum algebras. (This is a theorem of Kazhdan-Lusztig+Kashiwara-Tanisaki. There is another approach for this \(m = 1\) case. This is due to Varagnolo-Vasserot and Schiffman.)

Remark 3.12 The introduction of combinatorial Fock spaces is due to Misra, Miwa and Hayashi, as I stated in the introduction. We will return to their work on \(v\)-deformed Fock spaces in the third lecture.
3.2 Algorithms

For the case $m = 1$, we have four algorithms to compute decomposition numbers. These are LLT algorithm, LT algorithm, Soergel algorithm, and modified LLT algorithm. For general $m$, we have Uglov algorithm.

(1) LLT algorithm

This is due to Lascoux, Leclerc and Thibon. It is based on theorem 3.8. Basic idea is to construct ”ladder decomposition” of restricted Young diagrams. Then it produces basis $\{A(\lambda)\}$ of the level one module $L(\Lambda_0)$. (I will show an example in the lecture. This is a very simple procedure.)

Once $\{A(\lambda)\}$ is given, we can determine canonical basis elements $G(\lambda)$ recursively. We set

$$G(\lambda) = A(\lambda) - \sum_{\mu \subsetneq \lambda} c_{\lambda \mu}(v)G(\mu),$$

and find $c_{\lambda \mu}(v)$ by the following condition.

$$c_{\lambda \mu}(v^{-1}) = c_{\lambda \mu}(v), \quad G(\lambda) \in \lambda + \sum_{\mu \subsetneq \lambda} vZ[v]|\mu$$

Note that we follow the convention that restricted partitions form a basis of $L(\Lambda_0)$.

Remark 3.13 By a theorem of Leclerc, we can also compute decomposition numbers of $q$-Schur algebras by using those of Hecke algebras.

(2) LT algorithm

This is based on Leclerc-Thibon’s involution and Varagnolo- Vasserot’s reformulation of Lusztig conjecture. It has an advantage that we directly compute all decomposition numbers of $q$-Schur algebras.

We use fermionic description of the Fock space. Then a simple procedure on basis elements and straightening laws define bar operation on the Fock space. We then compute canonical basis elements by the characterization

$$\overline{G(\lambda)} = G(\lambda), \quad G(\lambda) \in \lambda + \sum_{\mu \subsetneq \lambda} vZ[v]|\mu$$

(3) Soergel algorithm

It is reformulation of Kazhdan-Lusztig algorithm for parabolic Kazhdan-Lusztig polynomials. Let $\mathcal{A}^+$ be the set of alcoves in the positive Weyl chamber. We consider vector space with basis $\{(A)\}_{A \in \mathcal{A}^+}$. For each simple reflection $s$, we denote by $As$ the adjacent alcove obtained by the reflection. The Bruhat order determines partial order on $\mathcal{A}^+$. Let $C_s$ be the Kazhdan-Lusztig element
corresponding to \( s \) (we use \((T_s - v)(T_s + v^{-1}) = 0\) as a defining relation here). Then the action of \( C_s \) on this space is given by

\[
(A)C_s = \begin{cases} 
(As + v(A)) & (As \in \mathcal{A}^+, As > A) \\
(As + v^{-1}(A)) & (As \in \mathcal{A}^+, As < A) \\
0 & (\text{else})
\end{cases}
\]

We determine Kazhdan-Lusztig basis elements \( G(A) \) recursively. For \( A \in \mathcal{A}^+ \), we take \( s \) such that \( As < A \). Then we find

\[
G(A) = G(As)C_s - \sum_{B < A} c_{A,B}(v)G(B)
\]

by the condition

\[
c_{A,B}(v^{-1}) = c_{A,B}(v), \quad G(A) \in (A) + \sum_{B < A} v\mathbb{Z}[v](B)
\]

(4) modified LLT algorithm
This is an algorithm which improves LLT algorithm. The idea is not to start from the empty Young diagram. This is due to Goodman and Wenzl. Their experiment shows that Soergel’s is better than LLT, and modified LLT is much faster than both.

(5) Uglov algorithm
This is generalization of LT algorithm, and it uses the higher level Fock space introduced by Takemura and Uglov.

### 3.3 Second application

Let us return to the \( q \)-Schur algebra. We summarize the previous explanation as follows.

**Theorem 3.14** If \( q \neq 1 \) is a root of unity in a field of characteristic zero, the decomposition numbers of the \( q \)-Schur algebra are computable.

**Corollary 3.15** (Geck) Let \( k \) be a field. We consider the \( q \)-Schur algebra over \( k \). If the characteristic of \( k \) is sufficiently large, the decomposition numbers of the \( q \)-Schur algebra over \( k \) are computable. Note that we do not exclude \( q = 1 \) here.

It has application to the modular representation theory of \( GL(n,q) \). Let \( q \) be a power of a prime \( p \), the characteristic of \( k \) be \( l \neq p \). We assume that \( k \) is algebraically closed. This case is called non-describing characteristic case. We want to study \( K_0(kGL(n,q)_{\text{mod}}) \).

**Theorem 3.16** (Dipper-James) Assume that the decomposition numbers of \( q^a \)-Schur algebras over \( k \) for various \( a \in \mathbb{Z} \) are known. Then the decomposition numbers of \( GL(n,q) \) in non-describing characteristic case are computable.
We explain how to compute the decomposition numbers of \( G := GL(n,q) \). Let \((K,R,k)\) be an \( l\)-modular system. James has constructed Specht modules for \( RG \). We denote them by \( \{ S_R(s, \lambda) \} \). \( s \) is a semi-simple element of \( G \). If the degree of \( s \) over \( \mathbb{F}_q \) is \( d \), \( \lambda \) run through partitions of size \( n/d \).

1. A complete set of simple \( KG \)-modules is given by

\[
\{ R^G \left( \bigotimes_{1 \leq i \leq N} S_k(s_i, \lambda^{(i)}) \right) \mid \sum d_i|\lambda^{(i)}| = n \}
\]

where \( R^G(-) \) stands for Harish-Chandra induction, \( d_i \) is the degree of \( s_i \), and \( \{s_1, \ldots, s_N\} \) run through sets of distinct semi-simple elements. We use Dipper-James’ formula

\[
[S_k(s, \lambda) : D_k(s, \mu)] = d_{\lambda, \mu}
\]

where \( d_{\lambda, \mu} \) is a decomposition number of the \( q^d \)-Schur algebra. Then we rewrite \( R^G \left( \bigotimes_{1 \leq i \leq N} S_k(s_i, \lambda^{(i)}) \right) \) into sum of \( R^G \left( \bigotimes_{1 \leq i \leq N} D_k(s_i, \mu^{(i)}) \right) \).

2. Let \( t_i \) be the \( l\)-regular part of \( s_i \), \( a_i \) be the degree of \( t_i \), \( \nu^{(i)} \) be the Young diagram obtained from \( \mu^{(i)} \) by multiplying all columns by \( d_i/a_i \). Then we have \( D_k(s_i, \mu^{(i)}) \simeq D_k(t_i, \nu^{(i)}) \). This is also due to Dipper and James. Thus we can rewrite \( R^G \left( \bigotimes_{1 \leq i \leq N} D_k(s_i, \mu^{(i)}) \right) \) into \( R^G \left( \bigotimes_{1 \leq i \leq N} D_k(t_i, \nu^{(i)}) \right) \). Assume that \( t_i = t_j \). Then we use the inverse of the decomposition matrices of \( q^a \)-Schur algebras of rank \( d_i k_i + d_j k_j \) to describe \( D_k(t_i, \nu^{(i)}) \bigotimes D_k(t_j, \nu^{(j)}) \) as an alternating sum of \( S_k(t_i, \eta^{(i)}) \bigotimes S_k(t_j, \eta^{(j)}) \). Then the Harish-Chandra induction of this module is explicitly computable by using Littlewood-Richardson rule. We use the decomposition matrix of the \( q^a \)-Schur algebra of rank \( d_i k_i + d_j k_j \) to rewrite it again into the sum of \( R^G \left( \bigotimes_{1 \leq i \leq N'} D_k(t_i, \kappa^{(i)}) \right) \). Continuing this procedure, we reach the case that all \( t_i \) are mutually distinct.

3. The final result of the previous step already gives the answer since the following set is a complete set of simple \( kG \)-modules.

\[
\{ R^G \left( \bigotimes_{1 \leq i \leq N'} D_k(t_i, \kappa^{(i)}) \right) \mid \sum a_i|\kappa^{(i)}| = n \}
\]

where \( \{t_1, \ldots, t_{N'}\} \) run through sets of distinct \( l\)-regular semi-simple elements.

4 Lecture Three

4.1 Specht modules and \( v\)-deformed Fock spaces

We now \( v\)-deform the setting we have explained in the second lecture. The view point which has emerged is that behind the representation theory of cyclotomic
Hecke algebras, there is the same crystal structure as integrable modules over quantum algebras of type $A_{r-1}^{(1)}$, and this crystal structure is induced by canonical bases of integrable modules. As a corollary to this viewpoint, Mathas and the author have parametrized simple $H_n$-modules over an arbitrary field using crystal graphs. Since the canonical basis is defined in the v-deformed setting, it further lead to intensive study of canonical bases on various v-deformed Fock spaces.

The purpose of the third lecture is to show the compatibility of this crystal structure with Specht module theory. The above mentioned studies on canonical bases on v-deformed Fock spaces are essential in the proof.

Before going to this main topic, I shall mention related work recently done in Vazirani’s thesis. This can be understood in the above context. As I have explained in the second lecture, this viewpoint has origin in Lascoux, Leclerc and Thibon’s work, which I would like to stress here again.

**Theorem 4.1 (Vazirani-Grojnowski)** Let $\tilde{e}_i(M) = \text{soc}(i - \text{Res}(M))$. If $M$ is irreducible, then $\tilde{e}_i(M)$ is irreducible or zero module.

The case $m = 1$ is included in Kleshchev and Brundan’s modular branching rule. It is natural to think that the socle series would explain the canonical basis in the crystal structure. This observation was first noticed by Rouquier as was explained in [2b], and adopted in this Vazirani’s thesis.

We now start to explain how Specht module theory fits in the description of higher level Fock spaces.

Let $F_v = \bigoplus \mathbb{C}(v)\lambda$ be the v-deformed Fock space. It has $U_v(\mathfrak{g}(A_{r-1}^{(1)}))$-module structure which is deformation of $U(\mathfrak{g}(A_{r-1}^{(1)}))$-module structure on $F$. To explain it, we introduce notation.

Let $x$ be a cell on $\lambda = (\lambda^{(m)}, \ldots, \lambda^{(1)})$. Assume that it is the $(a, b)$th cell of $\lambda^{(c)}$. We say that a cell is above $x$ if it is on $\lambda^{(k)}$ for some $k > c$, or if it is on $\lambda^{(c)}$ and the row number is strictly smaller than $a$. We denote the set of addable (resp. removable) $i$-nodes of $\lambda$ which are above $x$ by $A_i^a(x)$ (resp. $R_i^a(x)$). In a similar way, we say that a cell is below $x$ if it is on $\lambda^{(k)}$ for some $k < c$, or if it is on $\lambda^{(c)}$ and the row number is strictly greater than $a$. We denote the set of all addable (resp. removable) $i$-nodes of $\lambda$ which are below $x$ by $A_i^b(x)$ (resp. $R_i^b(x)$). The set of all addable (resp. removable) $i$-nodes of $\lambda$ is denoted by $A_i(\lambda)$ (resp. $R_i(\lambda)$). We then set

$$N_i^a(x) = |A_i^a(x)| - |R_i^a(x)|, \quad N_i^b(x) = |A_i^b(x)| - |R_i^b(x)|$$

$$N_i(\lambda) = |A_i(\lambda)| - |R_i(\lambda)|$$

We denote the number of all 0-nodes in $\lambda$ by $N_0(\lambda)$. Then the $U_v(\mathfrak{g}(A_{r-1}^{(1)}))$-
module structure given to $F_v$ is as follows.

$$
e_i \lambda = \sum_{\lambda/\mu = [\lambda]} v^{-N^v(\lambda/\mu)} \mu, \quad f_i \lambda = \sum_{\mu/\lambda = [\lambda]} v^{N^v(\mu/\lambda)} \mu$$

$$v^h \lambda = v^{N_v(\lambda)} \lambda, \quad v^d \lambda = v^{-N_d(\lambda)} \lambda$$

This action is essentially due to Hayashi.

Set $V_v = U_v(A_{v-1}) \otimes 0$. It is considered as the $v$-deformed space of $V = \oplus_{n \geq 0} K_0(\text{proj-}H_n)$.

**Remark 4.2** If we apply a linear map $(\lambda^{(m)}), \ldots, (\lambda^{(1)}) \mapsto (\lambda^{(1)}), \ldots, (\lambda^{(m)})$, we have Kashiwara’s lower crystal base which is compatible with his coproduct $\Delta_–$.

On the other hand, if an anti-linear map $(\lambda^{(m)}), \ldots, (\lambda^{(1)}) \mapsto (\lambda^{(m)}), \ldots, (\lambda^{(1)})$ is applied, we have Lusztig’s basis at $\infty$ which is compatible with his coproduct. We denote it by $F_–^\gamma$.

Set $L = \oplus \mathbb{Q}[v] \lambda$ and $B = \{ \lambda \text{ mod } v \}$. Then it is known that $(L, B)$ is a crystal base of $F_v$. We nextly set $L_0 = V_v \cap L$, and $B_0 = (L_0/\langle vL_0 \rangle) \cap B$. Then general theory concludes that $(L_0, B_0)$ is a crystal base of $V_v$.

**Definition 4.3** We say that $\lambda$ is $(\gamma_1, \ldots, \gamma_m; r)$-Kleshchev if $\lambda \text{ mod } v \in B_0$. We often drop parameters and simply says $\lambda$ is Kleshchev.

It has the following combinatorial definition. We say that a node on $\lambda$ is **good** if there is $i \in \mathbb{Z}/r\mathbb{Z}$ such that if we read addable $i$-nodes (write A in short) and removable $i$-nodes (write R in short) from the top row of $\lambda^{(m)}$ to the bottom row of $\lambda^{(1)}$ and do RA deletion as many as possible, then the node sits in the left end of the remaining R’s. (I will give an example in the lecture.)

**Definition 4.4** $\lambda$ is called $(\gamma_1, \ldots, \gamma_m; r)$-**Kleshchev** if there is a standard tableau $T$ of shape $\lambda$ such that for all $k$, $[k]$ is a good node of the subtableau $T_{\leq k}$ which consists of nodes $[1] \ldots [k]$ by definition.

**Theorem 4.5 (Ariki)** We assume that $v_i = q^{v_i}, q = \sqrt{T}$. Then $D^\lambda \neq 0$ if and only if $\lambda$ is $(\gamma_1, \ldots, \gamma_m; r)$-Kleshchev.

(How to prove) We show that canonical basis elements $G(\lambda)$ ($\lambda$=Kleshchev) have the form

$$G(\lambda) = \lambda + \sum_{\mu \not\subset \lambda} c_{\lambda \mu}(v) \mu$$

On the other hand, the Specht module theory constructed by Dipper-James-Mathas shows that the principal indecomposable module $P^\lambda$ for $D^\lambda \neq 0$ has the form

$$P^\lambda = S^\lambda + \sum_{\mu \not\subset \lambda} m_{\lambda \mu} S^\mu$$
Comparing these, and recalling that $\lambda \in \mathcal{F}$ is identified with $S^\lambda$, we have the result. ■

To know the form of $G(\lambda)$, we have to understand higher level $v$-deformed Fock spaces.

**Definition 4.6** Take $\gamma = (\gamma_1, \ldots, \gamma_m) \in (\mathbb{Z}/r\mathbb{Z})^m$. If $\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m) \in \mathbb{Z}^m$ satisfies $\tilde{\gamma}_k \mod r = \gamma_k$ for all $k$, we say that $\tilde{\gamma}$ is a lift of $\gamma$.

**Theorem 4.7** (Takemura-Uglov) For each $\tilde{\gamma} \in \mathbb{Z}^m$, we can construct higher level $v$-deformed Fock space, whose underlying space is the same as $\mathcal{F}$. It has geometric realization due to Varagnolo and Vasserot. For reader’s convenience, I also add it here. Let $V$ be a $\mathbb{Z}$-graded $\mathbb{C}$-vector space whose dimension type is $(d_i)_{i \in \mathbb{Z}}$. We denote by $V_{\tilde{i}} = \oplus_{j \in \tilde{i}} V_j$. We set $V_{\tilde{i}} \geq i = \oplus_{j \geq i} V_j$. Let

$$E_V = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_\mathbb{C}(V_i, V_{i+1}), \quad E_{V_{\tilde{i}}} = \bigoplus_{\tilde{i} \in \mathbb{Z}/r\mathbb{Z}} \text{Hom}_\mathbb{C}(V_{\tilde{i}}, V_{\tilde{i}+1}).$$

and define $E_{V_{\tilde{i}},V} = \{x \in V | x(V_{\tilde{j} \geq i}) \subseteq V_{\tilde{j} \geq i}\}$. Then we have a natural diagram

$$E_V \xleftarrow{\kappa} E_{V_{\tilde{i}},V} \xrightarrow{\iota} E_V$$

We consider $\gamma_d := \kappa \mu^*[\text{shift}]$. Then it defines a map from the derived category which is used to construct $\mathcal{U}_{\tilde{\gamma}}(g(A^{(1)}_1))$ to the derived category which is used to construct $\mathcal{U}_{\tilde{\gamma}}(g(A_\infty))$. Let $\eta$ be anti-involutions on both quantum algebras which sends $f_i$ to $f_j$ respectively.

Recall that $\mathcal{F}_{\tilde{\gamma},1}$ is a $U_v(g(A_\infty))$-module. We then have the following.

**Theorem 4.8** (Varagnolo-Vasserot) For each $x \in U_{\tilde{\gamma}}(g(A^{(1)}_1-1))$, we set

$$x\lambda = \sum_{d} \eta(\gamma_d(\eta(x))) v^{-\sum_{i<j,i=j} d_i h_j} \lambda$$

Then $\mathcal{F}_{\tilde{\gamma},1}$ becomes an $U_v(g(A^{(1)}_1-1))$-module.

**Remark 4.9** If we take $\tilde{\gamma}_i \gg \tilde{\gamma}_{i+1}$, the canonical basis elements on these three Fock spaces coincide as long as the size of the Young diagrams indexing these canonical basis elements is not too large.

**Remark 4.10** If we take $0 \leq -\tilde{\gamma}_1 \leq \cdots \leq -\tilde{\gamma}_m < r$ in the above Fock space, we have Jimbo-Miwa-Nou-Okado higher level Fock space. This Fock space is the first example of higher level Fock spaces.
By the above remark, we can use these Fock spaces to compute canonical basis elements on $\mathcal{F}_\gamma$ if we suitably care about the choice of $\gamma$.

**Theorem 4.11 (Uglov)** The Takemura-Uglov Fock space has a bar operation such that $\overline{\emptyset} = \emptyset$, $\overline{f_\lambda} = f_\overline{\lambda}$ and $\overline{\lambda}$ has the form $\lambda + (\text{higher terms})$ with respect to a dominance order.

The relation between the dominance order in the above theorem and the dominance order we use is well understood by using "abacus". As a conclusion, we can prove that $G(\lambda) = \lambda + \sum_{\mu \triangleleft \lambda} c_{\lambda \mu}(\nu)\mu$ as desired.

We have explained that how crystal base theory on higher level Fock spaces fits in the modular representation theory of cyclotomic Hecke algebras. In particular, Kac $q$-dimension formula gives the generating function of the number of simple $H_n$-modules. Even for type B Hecke algebras, it was new.

### 4.2 Future direction and Broué’s dream

The original motivation of Broué and Malle to introduce cyclotomic Hecke algebras is the study of modular representation theory of finite classical groups of Lie type over fields of non-describing characteristics. For example, Geck, Hiss and Malle’s result towards classification of simple modules inspires many future problems. I may mention more in the lecture on demand.

I would like to end these lectures with Broué’s famous dream. Let $B$ be a block of a group ring of a finite group $G$, and assume that it has an abelian defect group $D$. Let $b$ be the Brauer correspondent in the group ring of $DC_G(D) = C_G(D) \subset N_G(D)$. ($(D, b)$ is called a maximal subpair or Brauer pair.) Then he conjectures that $D^b(B-\text{mod}) \simeq D^b(N_G(D, b)b-\text{mod})$, i.e. $B$ and $N_G(D, b)b$ are derived equivalent (Rickard equivalent). To be more precise on its base ring, let $(K, R, k)$ be a modular system. He conjectures the derived equivalence over $R$.

Let $q$ be a power of a prime $p$, $G = G(q)$ be the general linear group $GL(n, q)$, and $k$ be an algebraically closed field of characteristic $l \neq p$, $(K, R, k)$ be a $l$-modular system. Assume that $l > n$, and take $d$ such that $d|\Phi_d(q)$, $\Phi_d(q) | q^n(q^n-1) \cdots (q-1)$, where $\Phi_d(q)$ is a cyclotomic polynomial. We take $B$ to be a unipotent block. In this case, unipotent blocks are parameterized by $d$-cuspidal pairs $(L(q), \lambda)$ up to conjugacy. Here $L(q)$ is a Levi subgroup, $\lambda$ is an irreducible cuspidal $KL(q)$-module. Further, $D$ is the $l$-part of the center of $L(q)$. $(L(q), \lambda)$ is the centralizer of a "$\Phi_d$-torus" $S(q)$.

If we set $W(D, \lambda) := N_G(D, \lambda)/C_G(D)$, it is isomorphic to $G(d, 1, a)$ for some $a$. $W(D, \lambda)$ is called cyclotomic Weyl group. These are due to Broué, Malle and Michel.

In this setting, Broué, Malle and Michel give an explicit conjecture on the bimodule complex which induces the Rickard equivalence between $B$ and $N_G(D, b)b$. It is given in terms of a variety which appeared in Deligne-Lusztig theory to trivialize a $L(q)$-bundle on a Deligne-Lusztig variety. Going down to the Deligne-Lusztig variety itself, it naturally conjectures the existence of a bimodule complex which induces derived equivalence between $B$ and a deformation ring of the group ring of the semi-direct of $S(q)_l$ with $W(D, \lambda) \simeq G(d, 1, a)$.
This conjecture is supported by the fact that they are isotypic in the sense of Broué.

It is expected that the deformation of $W(D, \lambda)$ is the cyclotomic Hecke algebra we have studied in these lectures. Hence, we expect that cyclotomic Hecke algebras with $m$ not restricted to 1 or 2 will have applications in this field. We remark that the Broué conjecture is not restricted to $GL(n, q)$ only.

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