Explicit minimisation of a convex quadratic under a general quadratic constraint: a global, analytic approach

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Abstract

A novel approach is introduced to a very widely occurring problem, providing a complete, explicit resolution of it: minimisation of a convex quadratic under a general quadratic, equality or inequality, constraint. Completeness comes via identification of a set of mutually exclusive and exhaustive special cases. Explicitness, via algebraic expressions for each solution set. Throughout, underlying geometry illuminates and informs algebraic development. In particular, centrally to this new approach, affine equivalence is exploited to re-express the same problem in simpler coordinate systems. Overall, the analysis presented provides insight into the diverse forms taken both by the problem itself and its solution set, showing how each may be intrinsically unstable. Comparisons of this global, analytic approach with the, intrinsically complementary, local, computational approach of (generalised) trust region methods point to potential synergies between them. Points of contact with simultaneous diagonalisation results are noted.

Keywords  Constrained optimisation, quadratic programming, simultaneous diagonalisation

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1 Introduction

1.1 Background

This paper introduces a novel approach to a very widely occurring problem: minimisation of a convex quadratic under a general quadratic, equality or inequality, constraint. This may arise by itself, or as a component of a larger problem – notably, as a single iteration in minimising a smooth convex objective function under a smooth constraint. In either case, we treat solving the above problem as of interest in itself.

Statistical instances of this problem – our primary motivation – occur, for example, in minimum distance estimation, Bayesian decision theory, generalised linear models with smooth constraints, canonical variate analysis with fewer samples than variables, various forms of oblique Procrustes analysis, Fisher/Guttmann estimation of optimal scores, spline fitting, estimation of Hardy-Weinberg equilibrium, and size and location constraints in iterative missing value procedures in Procrustes analysis. Associated literature is reviewed, united and extended in Albers et al. [4]. Some of these instances are elaborated in Albers et al. [3, 7].

Within the optimisation and numerical analysis literatures, the above problem has particularly close connections with (generalised) trust region ((G)TR) methods, being a special case of the GTRS problem defined in [23]. At the same time, the global, analytic approach presented here is, intrinsically, complementary to the local, computational one in the GTR literature. Valuable in itself, this new approach points to potential synergies with existing methodology, as we briefly indicate in closing.

In the main, GTR methods adapt a nonlinear optimisation protocol to address the problem on hand. Regularity conditions (constraint qualifications) are introduced, as required, under which the Karush-Kuhn-Tucker conditions are necessary for a local optimum, assuming such exists, additional conditions typically being required to ensure their sufficiency. Algorithms are then designed to seek a point satisfying all of these conditions. The literature on TR and GTR methods can be accessed via the excellent accounts presented in [9] and [23, 21, 25] respectively. See also [24, 15, 11, 28, 26, 27, 19, 12, 13], algorithms to handle the extreme forms of ill-conditioning that can occur being presented, for example, in [12, 13]. Recent work in [1] on non-iterative algorithms extends [10], and notes the possibility of non-unique solutions [?, cf.][Martinez. Other related recent work includes [18, 5, 14, 20, 30].

In contrast to the GTR approach, the one introduced here exploits a series of nonsingular affine transformations to re-express the same problem in successively simpler forms, each transformation corresponding to a convenient change of coordinate system. Solving the last of these affinely equivalent forms solves at once the initial problem, via back transformation. Throughout, underlying geometry illuminates and informs algebraic development. This approach offers a complete, explicit resolution of the problem presented at the outset. Completeness comes from analysing (literally, splitting) all of its possible instances into precisely identified, mutually exclusive and exhaustive, special cases. Explicitness comes from providing, in each case, closed-form expressions for the solution set and optimised value attained thereon.

Overall, this new approach highlights the diverse nature of different instances of both problem and solution set. And, moreover, the possibility of intrinsic instability – whereby arbitrarily (hence, undetectably) small changes in problem specification lead to radical changes in the form of problem and/or solution set – pinpointing where this occurs. It may then, in practice, be impossible to be sure which of several forms applies.

Concerning this approach, Critchley [10], cited in [21], provided an early account of the strictly convex form, the general case being first addressed in the technical report [2]. Again, Gower and Dijksterhuis [17] addressed the problem in the context of Procrustes analysis and gave a preliminary algorithm, further worked out in Albers and Gower [6].

Finally, in both approaches, there are direct points of contact with simultaneous diagonalisation results [22] as detailed below and, for example, in [19] in the GTR case.

1.2 Notation and conventions

The following notation and conventions are used. Terms involving arrays of vanishing order are absent. $R^n$ is endowed with the standard Euclidean inner product, inherited by each of its subspaces. Its zero member is denoted by $0_n$, and the span of its first $r$ unit coordinate vectors $S_r$. $M_n$ denotes the set of all $n \times n$ real symmetric matrices, with zero member $O_n$. Subscripts denoting the order of arrays may be omitted when no confusion is possible. Positive (semi-)definiteness of a matrix $A$ is denoted by $A \succ O$ (respectively, $A \succeq O$), the latter terminology here implying that $A$ is singular. The Moore-Penrose inverse of $A$ is denoted by $A^+$.
Finally, diag(., . , .) denotes a (block)diagonal matrix with the diagonal entries listed, while \( \subset \) denotes strict inclusion.

For brevity, straightforward proofs are omitted.

1.3 The general problem \( \mathbb{P}_\omega \)

The general equality-constrained problem is as follows.

Definition 1.1. For \( A, B \) in \( M_{n \times n}, t, b \) in \( \mathbb{R}^n \) and \( k \) in \( \mathbb{R} \), \( A \neq O_n \), writing \( \omega = (A, B, t, b, k) \), the \( n \)-variable problem \( \mathbb{P}_\omega \) is:

\[
\text{find } \min_{x \in \mathbb{R}^n} L_\omega(x), \text{ with } L_\omega(x) := (x - t)'A(x - t),
\]

and \( \hat{X}_\omega := \{ \hat{x} \in X_\omega : L_\omega(\hat{x}) = L_\omega \} \)

subject to \( Q_\omega(x) := x'Bx + 2b'x - k = 0 \),

where the objective (loss) function \( L_\omega(\cdot) \) is convex and the feasible set \( X_\omega := \{ x \in \mathbb{R}^n : Q_\omega(x) = 0 \} \) nonempty; when the solution set \( \hat{X}_\omega \) is nonempty, \( \tilde{L}_\omega = \min \{ L_\omega(x) : x \in X_\omega \} \) may be written as \( \hat{L}_\omega \). The set of all such \( \omega \) is denoted by \( \Omega_n \). Where no confusion is possible, we may omit the subscript \( \omega \).

We say that \( \mathbb{P} \) is (a) centred if the target \( t = 0_n \), (b) a (partitioned) least-squares problem if \( A \) has the form \( \operatorname{diag}(1, ... 1, 0, ..., 0) \), and (c) a full least-squares problem if \( A = I \). For any least-squares problem, we partition

\[
x = \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{10} \\ B_{01} & B_{00} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_0 \end{pmatrix}
\]

(1)

conformably with \( A \), subscripts 1 and 0 connoting its range and null spaces, so that \( L_\omega(x) = \| x_1 - t_1 \|^2 \). In this way, terms with 0 subscripts are absent when \( A = I_n \), so that \( x = x_1, B = B_{11}, t = t_1 \) and \( b = b_1 \).

Remark 1.1. Denoting the rank of \( A \) by \( r \), \( B \succ O \) and \( L(\cdot) \) is strictly convex when \( r = n \), while otherwise \( A \succeq O \), with \( L(x) = 0 \) if and only if \( (x - t) \) lies in the \( (n - r) \)-dimensional null space of \( A \). Symmetry of \( A \) and \( B \) is assumed without loss, while taking \( A \) nonzero avoids one obvious triviality. \( X \neq \emptyset \) avoids another, entailing restrictions on \( (B, b, k) \) characterised (by negation) in Lemma 1.1 below. The overall sign of the constraint coefficients can be reversed at will, \( \mathbb{P}_\omega \) being unchanged under \( (B, b, k) \rightarrow (-B, -b, -k) \).

Additional consistent affine constraints do not require separate treatment: they can be substituted out to arrive at an equivalent instance of \( \mathbb{P}_\omega \) in fewer variables.

Lemma 1.1. Let \( Q(x) := x'Bx + 2b'x - k \), where \( B \in M_{n \times n}, b \in \mathbb{R}^n \) and \( k \in \mathbb{R} \). Let \( b \) have decomposition \( b = Bx_b + b_\perp, x_b := B^{-1}b, \) along the range and null spaces of \( B \), so that \( b_\perp x_b = 0 \), and let \( k_+ := k + b' B^{-1}b \). Then:

\[
Q(x) = (x + x_b)' B(x + x_b) + 2b_\perp'(x + x_b) - k_+.
\]

Accordingly, \( Q(x) = 0 \) does not have a solution if and only if

\[
\begin{align*}
(i): & \quad |B = O|, \\
\text{or (ii):} & \quad |B \text{ is nonsingular}, \quad b_\perp = 0, \quad k_+ \neq 0; \\
\text{or (iii):} & \quad |B \neq O \text{ is singular}, \quad b_\perp = 0, \quad k_+ \neq 0,
\end{align*}
\]

1.4 Overview of paper

The solution to \( \mathbb{P}_\omega \) when \( B = O \) is well-known. For all other \( \omega \in \Omega_n \), the infimal value \( \tilde{L}_\omega \) and solution set \( \hat{X}_\omega \) are given explicitly. The variant of \( \mathbb{P}_\omega \) in which the constraint is relaxed to a weak inequality is also completely resolved. The organisation and principal subsidiary results of the paper are as follows.

Sections 2 and 3 Theorem 2.1 describes affine equivalence on the set of problems \( \{ \mathbb{P}_\omega : \omega \in \Omega_n \} \), Theorem 3.1 showing that every \( \mathbb{P}_\omega \) is affinely equivalent to a centred least-squares problem.

Section 4 deals with the inequality constrained case. We say that two minimisation problems with the same loss function are effectively equivalent if they have the same infimal value attained on the same solution set. A simple continuity argument shows that, for every \( \mathbb{P}_\omega \), its weak inequality constraint variant either has a trivial solution or is effectively equivalent to \( \mathbb{P}_\omega \) itself, Theorem 4.1 specifying exactly when each occurs.
Section 5 discusses the $A \succeq O$ case, Lemma 5.2 showing that every centred least-squares problem is now affinely equivalent to a simplified form. Theorem 6.1 establishes that there are at most three possibilities for such a form, specifying exactly when each occurs. In two of them, distinguished by whether or not $X$ is empty, $L = 0$. In the third, (a) $L > 0$ while (b) a reduced form of $P_\omega$ is effectively equivalent to an induced (centred) full least-squares problem in $r < n$ variables. Remark 5.2 establishes direct points of contact with simultaneous diagonalisation.

Section 6 given the above results, there is no loss in assuming now that $A$ is positive definite, in which case, $P_\omega$ always has a solution (Theorem 6.1). Theorem 6.2 establishing that it is affinely – indeed, linearly – equivalent to a canonical form $P_\omega^{*}$, governed by the spectral decomposition of $B$.

Section 7 exploiting orthogonal indeterminacy within eigenspaces, Theorem 7.1 establishes that solutions to $P_\omega^{*}$ can be characterised in terms of those of a dimension-reduced canonical form $P_\omega^{*\ast}$. There is no loss in restricting attention to regular such forms, denoted by $P_{\omega^{*\ast}}$ (Remark 7.2).

Section 8 presents a series of auxiliary results culminating in Theorem 8.1 specifying the minimised objective function and solution set for any $P_{\omega^{*\ast}}$. This completes our primary objective.

Section 9 discusses intrinsic instability of solution sets and problem forms, while Section 10 concludes the paper with a short discussion.

2 Affine equivalence

Recalling that the nonsingular affine transformations $G_n$ on $R^n$ form a group under composition, whose general member we denote by $g : x \to x_g$, the same instance of $P_\omega$ can be re-expressed in different, affinely equivalent, coordinate systems.

Definition 2.1. For any vector $a$ and for any nonsingular $T$, $g = g(T,a)$ denotes the map $x \to x_g := T^{-1}(x - a)$, inducing $\omega \to \omega_g$ via:

\begin{align*}
t &\to t_g := T^{-1}(t - a) \\
A &\to A_g := T^T A T \\
B &\to B_g := T^T B T \\
b &\to b_g := T'(b + Ba) \\
k &\to k_g := k - a'(2b + Ba).
\end{align*}

For linear maps ($a = 0_n$), we abbreviate $g$ as $g_T$.

Remark 2.1. Note that: (a) the rank and signature of $A$ and $B$ are maintained in those of $A_g$ and $B_g$ respectively, (b) $A$ and $B$ are unchanged under translation ($T = I$), and (c) $k$ is unchanged under linear maps ($a = 0_n$).

Theorem 2.1. For all $\omega \in \Omega_n$, $x \in R^n$ and $g \in G_n$: $L_\omega(x) = L_{\omega_g}(x_g)$ and $Q_\omega(x) = Q_{\omega_g}(x_g)$, so that $x \in X_\omega \iff x_g \in X_{\omega_g}$, $L_\omega = L_{\omega_g}$, $\tilde{x} \in \tilde{X}_\omega \iff \tilde{x}_g \in \tilde{X}_{\omega_g}$ and $\tilde{X}_\omega \neq \emptyset \iff \tilde{X}_{\omega_g} \neq \emptyset$, in which case $L_\omega = \tilde{L}_{\omega_g}$.

In view of Theorem 2.1 we say that $P_\omega$ and $P_{\omega_g}$ are affinely equivalent, writing $\omega \sim \omega_g$. Again, if $g = g(T,a)$ with $T$ orthogonal, we call $P_\omega$ and $P_{\omega_g}$ Euclideanally equivalent.

Recall that $\omega \to \omega_+ := (A, -B, t, -b, -k)$ also leaves $P_\omega$ unchanged. For later use (Section 6), we note here

Lemma 2.1. $P_\omega \sim P_{\omega_g}$ commutes with $P_\omega \sim P_{\omega_g}$, $g \in G$.

3 Centred least-squares form

We characterise here the set of centred least-squares problems to which a given problem $P_\omega$ is affinely equivalent. Partitioning $T$ conformably with $B$, as in [1], let $T$ denote $\{ T : T_{11}$ is orthogonal, $T_{10} = O$ and $T_{00}$ is nonsingular $\}$, noting that $T$ forms a group under multiplication.

Theorem 3.1. Let $\omega \in \Omega_n$ and $T_A := U_A \text{diag}(D_A^{-1}, I_{n-r})$ where $A$ has spectral decomposition $A = U_A \text{diag}(D_A^2, O_{n-r})U_A^T$ with $U_A$ orthogonal and $D_A$ diagonal, positive definite. Then:

(i) $P_{\omega_0}$, $g_0 = g(T_A, t)$, is a centred least-squares problem;
We denote by $P_4$ Inequality contrained variant $X := \{ x \in \mathbb{R}^n : Q(x) = 0 \} \neq \emptyset$ replaced by $X_{\leq} := \{ x \in \mathbb{R}^n : Q(x) \leq 0 \} \neq \emptyset$, its infimal value and solution set being denoted by $L_{\leq}$ and $\hat{X}_{\leq}$ respectively. The reverse inequality is accommodated by changing the overall sign of $(B, h, k)$.

Affine equivalence generalises at once to the inequality constrained case, as does Theorem 3.1. Accordingly, in discussing $P_{\leq}$, there is again no loss in restricting attention to the centred least-squares case, when $L(x) = \|x_1\|^2$.

**Theorem 4.1.** Let $P_{\leq}$ be an inequality constrained, centred least-squares problem.

(i) When $A \succ 0$:
- If $Q(0_n) \leq 0$, $L_{\leq} = 0$ and $\hat{X}_{\leq} = \{0_n\}$;
- otherwise, $X \neq \emptyset$ and $P_{\leq}$ is effectively equivalent to $P$.

(ii) When $A \preceq 0$, putting $X_{0,\leq} := \{ x_0 \in \mathbb{R}^{n-r} : Q(0', x_0') \leq 0 \}$:
- if $X_{0,\leq} \neq \emptyset$, $L_{\leq} = 0$ and $\hat{X}_{\leq} = \{ (0', x_0') : x_0 \in X_{0,\leq} \}$;
- otherwise, $X \neq \emptyset$ and $P_{\leq}$ is effectively equivalent to $P$.

**Proof.** The proof is similar in both cases.

(i) If $Q(0_n) \leq 0$, the result is immediate. Else, $Q(0_n) > 0$, continuity of $Q(\cdot)$ ensuring that, $\forall x \in X_{\leq}$, $Q(x) < 0 \Rightarrow \exists 0 < \kappa < 1$ with $Q(\kappa x) = 0$.

(ii) If $X_{0,\leq} \neq \emptyset$, the result is immediate. Else, $Q(0', x_0') > 0 \forall x_0 \in \mathbb{R}^{n-r}$ while, $\forall x \in X_{\leq}$, $Q(x) < 0 \Rightarrow \exists 0 < \kappa < 1$ with $Q(\kappa x_1', x_0') = 0$.

\[ \square \]

**5 Solving $P_{\omega}$ when $A$ is positive semi-definite**

In this section, we take $A$ positive semi-definite ($r < n$), so that $x_0$ occurs in the constraint but not in the objective function. Accordingly, any centred least-squares problem takes an associated reduced form, an immediate lemma providing geometric insight.

Denoting orthogonal projection of $\mathbb{R}^n$ onto $S_r$ by $P : x \to x_1$, we have:

**Definition 5.1.** For any centred least-squares problem $P_{\omega}$ with $r < n$, its reduced form is:

\[
\text{find } L_1 := \inf \{ L_1(x_1) : x_1 \in X_1 \} , L_1(x_1) := \|x_1\|^2 , X_1 := P(X) .
\]

**Remark 5.1.** Note that $X_1$ is (a) nonempty, since $X$ is nonempty, and (b) given by

\[ X_1 = \{ x_1 \in \mathbb{R}^r : X_0(x_1) \neq \emptyset \} \]

where $X_0(x_1) := \{ x_0 \in \mathbb{R}^{n-r} : Q(x_1', x_0') = 0 \}$.

**Lemma 5.1.** Let $P_{\omega}$ be a centred least-squares problem with $r < n$. Then $\forall x \in \mathbb{R}^n$, $L(x) = L_1(x_1)$ while $x \in X \Leftrightarrow \{ x_1 \in X_1 , x_0 \in X_0(x_1) \}$, so that $L_1 = L$, while $(x_1', x_0')$ solves $P_{\omega} \Leftrightarrow [x_1 \text{ solves } [3]$ and $x_0 \in X_0(x_1) \}$.

Geometrically, the reduced form seeks the infimal (squared) distance from the origin in $S_r$ to the orthogonal projection of the conic $X$ onto that subspace, its solution set $\hat{X}_1$ being the orthogonal projection of $\hat{X}$ onto $S_r$.

To help solve the reduced form $[3]$, we introduce a simplifying linear transformation, via a decomposition of the null space of $A$ according to its intersections with the range and null spaces of $B_{00}$.
Definition 5.2. Let $\mathbb{P}_\omega$ be a centred least-squares problem with $r < n$. Then, $\mathbb{P}_\omega$ is said to be in simplified form if, for some $0 \leq s_0 \leq n - r$ and for some nonsingular diagonal $\Gamma_0$ of order $s_0$, $B$ has the partitioned form:

\[
B = \begin{pmatrix}
B_{11} & C_{10} & O \\
C_{10}^\prime & O_{n-r-s_0} & O \\
O & O & \Gamma_0
\end{pmatrix}.
\] (4)

Accordingly, any term involving $\Gamma_0$ is absent if and only if $B_{00} = O$, and any involving $C_{10}$ if and only if $B_{00}$ is nonsingular – in particular, if $B$ is (positive or negative) definite.

Lemma 5.2. Let $\mathbb{P}_\omega$ be a centred least-squares problem with $r < n$. Then, $\exists T \in \mathcal{T}$ with $\mathbb{P}_{\omega \gamma T}$ in simplified form.

Proof. Let $B_{00}$ have rank $0 \leq s_0 \leq n - r$ and spectral decomposition $U_0 \text{diag}(O_{n-r-s_0}, \Gamma_0)U_0^\prime$, with $U_0$ orthogonal and $\Gamma_0$ nonsingular diagonal. Define $C_{10}$ and $D_{10}$ implicitly via

\[
\text{diag}(I_r, U_0)B \text{diag}(I_r, U_0) = \begin{pmatrix}
B_{11} & C_{10} & D_{10} \\
C_{10}^\prime & O_{n-r-s_0} & O \\
D_{10} & O & \Gamma_0
\end{pmatrix}
\]

and put

\[
T := \text{diag}(I_r, U_0) \begin{pmatrix}
I_r & O & O \\
O & I_{n-r-s_0} & O \\
-\Gamma_0^{-1}D_{10} & O & I_{s_0}
\end{pmatrix}.
\] (5)

Then $T \in \mathcal{T}$ and so, by Theorem 3.1, $\mathbb{P}_{\omega \gamma T}$ is a centred least-squares problem in simplified form.

We note in passing that, while preserving simplified form, a further linear transformation establishes direct points of contact with simultaneous diagonalisation.

Remark 5.2. If $\mathbb{P}_\omega$ is in simplified form, while $B_{11}$ in [4] has spectral decomposition:

\[
B_{11} = U_1 \text{diag}(O_{r-s_1}, \Gamma_1)U_1^\prime, \quad \Gamma_1 \text{ nonsingular},
\]

the further transformation $x \to T^{-1}x$ with $T := \text{diag}(U_1, I_{n-r}) \in \mathcal{T}$ induces:

\[
B \to \begin{pmatrix}
O_{r-s_1} & O & E_{10} & O \\
O & \Gamma_1 & F_{10} & O \\
E_{10} & F_{10} & O_{n-r-s_0} & O \\
O & O & O & \Gamma_0
\end{pmatrix}
\]

in which $(E_{10}, F_{10}) = U_1^\prime C_{10}$ (6)

so that, using again Theorem 3.1, $\mathbb{P}_{\omega \gamma T}$, $q = qr$, remains in its simplified form. This can be seen as extending Newcomb [22] who showed that any two symmetric matrices, neither of which is indefinite, can be simultaneously diagonalised. For, there is no loss in restricting attention to matrices that, like $A$, are either positive definite or positive semi-definite, which, if true of $B$, entails that $E_{10}$ and $F_{10}$ in [6] are absent or zero respectively.

Returning to the mainstream, the following additional terms are used.

Definition 5.3. For any centred least-squares problem $\mathbb{P}_\omega$ in simplified form, we sub-partition $x_0$ and $b_0$ so that

\[
x_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} \quad \text{and} \quad B_{00} = \begin{pmatrix} O_{n-r-s_0} & O \\
O & \Gamma_0
\end{pmatrix}
\]

conform. Accordingly, any term involving $y_0$ or $c_0$ is absent if and only if $B_{00}$ is nonsingular; and any involving $z_0$ or $d_0$ if and only if $B_{00} = O$.

For given $x_1$, $Q(\cdot)$ depends quadratically on $z_0$, but only linearly on $y_0$, since

\[
Q(x) = (z_0 + \Gamma_0^{-1}d_0)\Gamma_0(z_0 + \Gamma_0^{-1}d_0) + 2(C'_{10}x_1 + c_0)'y_0 + Q_1(x_1)
\]

in which $Q_1(x_1) := x_1'B_{11}x_1 + 2b_1'x_1 - k_1$, with $k_1 = k_1(\omega) := k + d_0'\Gamma_0^{-1}d_0$. If $B_{00} \neq O$, $Z_0(\alpha) := \{z_0 \in \mathbb{R}^{s_0} : (z_0 + \Gamma_0^{-1}d_0)\Gamma_0(z_0 + \Gamma_0^{-1}d_0) = \alpha\}$, $\alpha \in \mathbb{R}$, so that $\alpha(\Gamma_0) := \{\alpha : Z_0(\alpha) \neq \emptyset\}$ is $R$, $[0, \infty)$ or $(-\infty, 0)$ according as $\Gamma_0$ is indefinite, positive definite or negative definite, while $Z_0(0) = \{-\Gamma_0^{-1}d_0\}$ if $\Gamma_0$ is definite. If $B_{00}$ is singular and $c_0 \neq 0$, $\alpha(y_0) := k_1 - 2c_0'y_0$, while $y_0(c_0) := k_1c_0/(2||c_0||^2)$, so that $\alpha(y_0(c_0)) = 0$. 

5
In view of previous results, when \( r < n \), it suffices to solve \( P_\omega \) or, giving also each \( X_0(x_1) (x_1 \in X_1) \), its reduced form \( \{0\} \) - for centred least-squares problems in simplified form.

**Definition 5.4.** Let the infimum of \( x \) the corresponding sets \( X \) over all \( (0) \) be a centred least-squares problem in simplified form. Then, \( \omega_1 := (I_r, B_{11}, 0_r, b_1, k_1) \) is called the **projected form** of \( \omega = (\text{diag}(I_r, O), B, 0_n, b, k) \), and we say that \( P_\omega \) admits:

(a) a **perfect solution** if \( L(x) = 0 \) for some \( x \in X \);

(b) an **essentially perfect solution** if \( L(x) > 0 \) (\( x \in X \)), but \( L = 0 \);

(c) a **projected, yet imperfect, reduced form** if (i) \( X_{\omega_1} \neq \emptyset \), so that \( \omega_1 \in \Omega_r \), and (ii) its reduced form is effectively equivalent to \( P_{\omega_1} \):

\[
\text{find } \inf \{L_1(x_1) : x_1 \in X_{\omega_1} \}, \quad X_{\omega_1} = \{x_1 \in R^r : Q_1(x_1) = 0\},
\]

in which \( k_1 \neq 0 \).

**Example 5.1.** Examples of these three possibilities are the problems, with \( r = 1 \) and \( n = 2 \), of finding the infimum of \( x_1^2 \) over all \( (x_1, x_0)' \) satisfying, respectively, (a) \( x_1^2 - x_0^2 = -1 \), (b) \( x_1 x_0 = 0 \) and (c) \( x_1^2 - x_0^2 = +1 \), the corresponding sets \( X_1 \) being \( R, R \setminus \{0\} \) and \( \{x_1 : x_1^2 \geq 1\} \).

Indeed, there are no other possibilities. Denoting the closure and boundary of \( X_1 \) by \( cl(X_1) \) and \( \partial X_1 \), we have:

**Theorem 5.1.** Let \( P_\omega \) with \( r < n \) be a centred least-squares problem in simplified form. There are at most three, mutually exclusive, possibilities:

(a) \( P_\omega \) admits a perfect solution,

(b) \( P_\omega \) admits an essentially perfect solution,

(c) \( P_\omega \) admits a projected, yet imperfect, reduced form,

whose occurrences are characterised thus:

(a) occurs \( \iff 0_r \in X_1 \iff [L = 0, \hat{X} \neq \emptyset] \),

(b) occurs \( \iff 0_r \in \partial X_1 \iff [L = 0, \hat{X} = \emptyset] \),

(c) occurs \( \iff 0_r \notin cl(X_1) \iff L > 0 \),

these arising as follows:

(i) If \( s_0 = 0 \):

\( (a) \) occurs \( \iff [k_1 = 0, c_0 = 0] \) or \( c_0 \neq 0 \), \( X_0(0) \) being \( R^{n-r} \) or \( \{y_0(c_0)\} \) respectively. Otherwise, \( [k_1 \neq 0, c_0 = 0] \), with (c) or (b) occurring according as \( C_{10} \) does, or does not, vanish. When (c) occurs, \( X_1 = X_{\omega_1} \) with each \( X_0(x_1) = R^{n-r} \).

(ii) If \( s_0 = n - r \):

\( (a) \) occurs \( \iff k_1 = 0 \) or \( k_1 \Gamma_0 \) has a positive eigenvalue,

\( X_0(0) \) being \( Z_0(0) \) or \( Z_0(k_1) \) respectively, while (b) does not occur. Thus, (c) occurs \( \iff (-k_1 \Gamma_0) > O \), when \( X_1 = \{x_1 \in R^r : k_1 Q_1(x_1) \geq 0\} \) with each \( X_0(x_1) = \{x_0 : z_0 \in Z_0(-Q_1(x_1))\} \).

(iii) If \( 0 < s_0 < n - r \):

\( (a) \) occurs \( \iff [k_1 = 0, c_0 = 0] \) or \( [k_1 \Gamma_0 \text{ has a positive eigenvalue}, c_0 = 0] \) or \( c_0 \neq 0 \),

\( X_0(0) \) comprising all \( x_0 \) with \( z_0 \) in \( Z_0(0), Z_0(k_1) \) or \( \cup_{\alpha(y_0) \in \alpha(\Gamma_0)} Z_0(\alpha(y_0)) \), respectively. Otherwise, \( [(-k_1 \Gamma_0) > O, c_0 = 0] \), with (c) or (b) occurring according as \( C_{10} \) does, or does not, vanish. When (c) occurs, \( X_1 \) and each \( X_0(x_1) \) are as in (ii).
Proof. Using Lemma 5.1, the characterisations of the occurrence of (a) and of (b) are immediate. Since
\[ 0, \notin cl(X_1) \Leftrightarrow \exists \eta > 0 \text{ such that } \|x_1\| < \eta \Rightarrow x_1 \notin X_1 \] (8)
so, too, is the fact that (c) occurs \( \Rightarrow 0, \notin cl(X_1) \Leftrightarrow L > 0 \). Recall that
\[ x_1 \notin X_1 \Leftrightarrow Q(x', x'_0)' \neq 0 \text{ for all } x_0 \in R^{n-r}. \] (9)
Using the \( x_1 = 0 \) instances of (7) and (9), and identifying the former as an instance of the identity (2),
Lemma 1.1 gives:
\[ 0, \notin X_1 \Leftrightarrow \begin{cases} (\text{i}): & [s_0 = 0, \quad c_0 = 0, \quad k_1 \neq 0], \\ or (\text{ii}): & [s_0 = n - r, \quad (k_1 \Gamma_0) \succ O], \\ or (\text{iii}): & [0 < s_0 < n - r, \quad (k_1 \Gamma_0) \succ O, \quad c_0 = 0] \end{cases} \] Similarly, using also (8), we have:
\[ 0, \notin X_1 \Leftrightarrow [0, \notin X_1 \text{ and } C_{10} \text{ is either absent } (s_0 = n - r) \text{ or zero}]. \]
Suppose \( 0, \notin cl(X_1) \). If also \( s_0 = 0, \quad Q(x) = Q_1(x_1) \), so that \( X_{\omega_1} = X_1 \neq 0 \) \text{ and the reduced form (3) of } P_{\omega}
is \( P_{\omega_1} \). Thus, (c) occurs, while each \( X_0(x_1) = R^{n-r} \). If, instead, we also have \( s_0 > 0, \quad (k_1 \Gamma_0) \succ O \) and
\[ k_1 Q(x) = k_1 Q_1(x_1) - (z_0 + \Gamma_0^{-1} d_0)' (k_1 \Gamma_0) (z_0 + \Gamma_0^{-1} d_0), \]
so that \( X_1 = \{ x_1 : k_1 Q_1(x_1) \geq 0 \} \). Since \( X_1 \neq 0 \) while \( k_1 \neq 0 \), Theorem 4.1 establishes that (c) again occurs, while each \( X_0(x_1) \) has the form stated. The forms taken by \( X_0(0) \) when (a) occur are immediate. \( \square \)

Corollary 5.1. Under the hypotheses of Theorem 5.1, and adopting its terminology, it is necessary for (b) to occur that \( B \) be indefinite.

Proof. If \( B \) is not indefinite, \( C_{10} \) is either absent or zero. Geometrically, a hyperbolic quadratic constraint is necessary for \( P \) to admit an essentially perfect solution.

As Theorem 5.1 shows – and as the example of (b) given immediately before it illustrates – it is possible that \( P_{\omega} \) has no solution when \( r < n \), reflecting the fact that, although \( X \) is closed, its projection \( X_1 = P(X) \) may not be. In contrast, a simple compactness argument, used in proving Theorem 6.1 below, shows that \( P_{\omega} \) always has a solution when \( r = n \).

6 Canonical form
Given the above results, it suffices to solve the centred, full least-squares form \( A = I, \quad t = 0 \) of problem \( P_{\omega} \).

The affinely constrained case being well-known, there is no essential loss in also requiring \( B \neq O \). Reversing if necessary the overall sign of \( (B, b, k) \), and recalling Lemma 2.1, there is no loss in further assuming that \( B \) has a positive eigenvalue.

Such a problem always has a solution.

Theorem 6.1. \( \hat{X}_{\omega} \neq 0 \) for any full least-squares problem \( P_{\omega} \).

Proof. For any \( \bar{x} \in X_{\omega}, \quad P_{\omega} \) is essentially equivalent to minimising \( \|x - t\|^2 \) over the compact set \( X_{\omega} \cap \{ x \in R^n : \|x - t\| \leq \|\bar{x} - t\| \} \).

Geometrically, there is always a shortest normal from \( t \) to the conic \( X \).

We introduce next the notion of a canonical form, a full least-squares form of \( P_{\omega} \) in which the constraint has been simplified according to the spectral decomposition of \( B \). In it, \( v_m := (1, 0'_{m-1})' \) denotes the first unit coordinate vector in \( R^m \).

Definition 6.1. Let \( P_{\omega} \) be a centred, full least-squares problem in which \( B \) has a positive eigenvalue. Let \( B \) have rank \( s \) and distinct nonzero eigenvalues \( \gamma_1 > \ldots > \gamma_q (1 \leq q \leq s) \), so that \( \gamma_1 > 0 \). For each \( i = 1, \ldots, q \), let \( \gamma_i \) have multiplicity \( m_i \geq 1 \) and let the orthogonal projection of \( b \) onto the corresponding eigenspace have length \( l_i \geq 0 \). Let \( m_0 := n - s \geq 0 \) denote the dimension of the null space of \( B \), and \( l_0 \geq 0 \) the length of the orthogonal projection of \( b \) onto this subspace.
With $\varepsilon := l_0$ and $\delta = (\delta_i) \in R^q$, $\delta_i := |\gamma_i|^{-1}l_i$, let $\omega^* := (A^*, B^*, t^*, b^*, k^*)$ in which $A^* = I$, $B^* = \text{diag}(O_{m_0}, \Gamma)$ where $\Gamma := \text{diag}(\gamma_1 I_{m_1}, \ldots, \gamma_q I_{m_q})$.

$$t^* = \begin{pmatrix} 0_{m_0} \\ \delta_1 u_{m_1} \\ \vdots \\ \delta_q u_{m_q} \end{pmatrix}, b^* = \begin{pmatrix} \varepsilon u_{m_0} \\ 0_{m_1} \\ \vdots \\ 0_{m_q} \end{pmatrix}$$

and $k^* = k + \sum_{i=1}^q |\gamma_i|^{-1}l_i^2$. Then, we call $\mathbb{P}_{\omega^*}$ the canonical form of $\mathbb{P}_{\omega}$.

The reason for this terminology is made clear by the following result.

**Theorem 6.2.** $\mathbb{P}_{\omega}$ is Euclideanly equivalent to $\mathbb{P}_{\omega^*}$.

**Proof.** By the spectral decomposition of $B$, there is an orthogonal matrix $T_B$ with $T_B^* B T_B = \text{diag}(O_{m_0}, \Gamma)$, where $T_B$ is unique up to postmultiplication by $U \in U := \{ \text{diag}(U_0, U_1, \ldots, U_q) : U^T U_i = I_{m_i} (0 \leq i \leq q) \}$.

Let $T_B^* b = (c', d'_1, \ldots, d'_q)'$ and $T_B^* B T_B$ conform, so that $||c|| = l_0$ while $||d_i|| = l_i$ $(1 \leq i \leq q)$, and let now $U = UB \in U$ be such that the first column of $U_0$ is $c/||c||$ if $c \neq 0$ and of $U_i$ $(1 \leq i \leq q)$ is $d_i/||d_i||$ if $d_i \neq 0$.

Finally, let $g : x \mapsto x_g = U_B^T \{ T_B^* x + (0', \gamma_1^{-1} d_1', \ldots, \gamma_q^{-1} d_q') \}$. Then, using Lemma 1.1, $\mathbb{P}_{\omega}$ is Euclideanly equivalent to $\mathbb{P}_{\omega^*} = \mathbb{P}_{\omega^*}$. \hfill $\square$

**Remark 6.1.** Viewed geometrically, the isometry inducing $\omega \rightarrow \omega^*$ so translates, rotates and reflects coordinate axes that:

(a) $B^*$, determining the quadratic part of the constraint, becomes diagonal,

(b) $b^*$, determining its linear part, vanishes when $B^*$ is nonsingular and, otherwise, has at most one nonzero element – its coordinate $\varepsilon \geq 0$ on the first dimension of the null space of $B^*$ – and:

(c) the target $t^*$ is orthogonal to this subspace, and has at most one nonzero element associated with each nonzero eigenvalue of $B^*$ – its coordinate $\delta_i \geq 0$ on the first dimension of the corresponding eigenspace.

As is explicit in the proof of Theorem 6.2, the simple structure enjoyed by a canonical form within each eigenspace of $B^*$ is made possible by a well-known orthogonal indeterminacy there. This simple structure invites a natural dimension reduction, further exploiting this same indeterminacy.

### 7 Dimension-reduced canonical form

Solutions to $\mathbb{P}_{\omega^*}$ can be characterised in terms of those of a dimension-reduced canonical form $\mathbb{P}_{\omega^**}$, having just one variable for each distinct eigenvalue of $B^*$. Variables $z = (z_i) \in R^q$ enter the constraint quadratically.

When $B^*$ is singular, an additional scalar variable $y$ associated with its null space enters the constraint linearly.

A unified account is provided in which we denote these variables by $w \in R^2$, whether $B^*$ is nonsingular $(m_0 = 0)$ or not $(m_0 > 0)$. In the former case, $\nu = q$ and $w = z$. In the latter, $\nu = q + 1$ and $w = (y, z')'$.

**Definition 7.1.** Adopting the assumptions and notation of Definition 6.1 let $\Gamma^* := \text{diag}(\gamma_1, \ldots, \gamma_q)$, so that $\Gamma^*$ is either positive definite ($\gamma_q > 0$) or nonsingular indefinite ($\gamma_1 > 0 > \gamma_q$, requiring $q > 1$).

If $B^*$ is singular, let $w_0 := (0, 0')'$ and $\Delta := \text{diag}(0, \Gamma^*)$ conform with $w := (y, z')'$, while $d := \varepsilon u_{q+1}$. Otherwise, put $w := z$, $w_0 := \delta$, $\Delta := \Gamma^*$ and $d := 0$. Then, the dimension-reduced canonical form of $\mathbb{P}_{\omega}$ is $\mathbb{P}_{\omega^**}$, $w^** := (A^*, B^*, t^*, b^*, k^*)$, with $A^* = I_w$, $B^* = \Delta$, $t^* = w_0$, $b^* = d$ and $k^* = k^*$. That is, the problem $\mathbb{P}_{\omega^**}$ is to:

$$\text{find } L^* := \inf_{w \in R^2} \{ L^*(w), L^*(w) := ||w - w_0||^2 \},$$

and $\hat{W} := \{ \hat{w} \in W : L^*(\hat{w}) = L^* \}$

subject to $Q^*(w) := w^T \Delta w + 2d^T w - k^* = 0$, \hfill (10)

where the feasible and solution sets $W := \{ w \in R^2 : Q^*(w) = 0 \}$ and $\hat{W}$ are nonempty, as $X_w$ and $\hat{X}_w$ are so. We may write $\hat{w} \in \hat{W}$ as $\hat{z} = (\hat{z}_i)$ $(m_0 = 0)$, or $(\hat{y}, \hat{z}')'$ $(m_0 > 0)$. Reflecting the form of the conic, when $d = 0$, we call the constraint in $\mathbb{P}_{\omega^**}$ elliptic or hyperbolic according as $\gamma_q > 0$ or $\gamma_q < 0$. When $d \neq 0$, we call it parabolic-elliptic or parabolic-hyperbolic in the same two cases.
Definition 7.2. A dimension-reduced canonical form is called defined next.

**Remark** Let \( \hat{x} := (\tilde{x}_0, \tilde{x}'_1, \ldots, \tilde{x}'_q) \), where \( \tilde{x}_i \in \mathbb{R}^{m_i} \). Then, \( \hat{x} \) solving \( \mathbb{P}_{\omega^{**}} \) if and only if

\[
\tilde{x}_0 = \hat{y} u_{m_0} \text{ and, for } i = 1, \ldots, q, \quad \begin{cases}
\tilde{x}_i = \hat{z}_i u_{m_i}, & \text{if } \delta_i > 0 \\
|\tilde{x}_i|^2 = \hat{z}_i^2, & \text{if } \delta_i = 0
\end{cases},
\]

where \( \hat{w} = (\hat{y}, \hat{z})' \) solves \( \mathbb{P}_{\omega^{**}} \). In particular, \( L^* = L \).

Clearly, \( \delta_i > 0 \Rightarrow \hat{z}_i \geq 0 \). Moreover, Theorem 7.1 has the immediate

**Corollary 7.1.** \( \hat{w} \) solving \( \mathbb{P}_{\omega^{**}} \) determines more than one solution to \( \mathbb{P}_{\omega^{**}} \) \( \iff \delta_i = 0 \) and \( \hat{z}_i \neq 0 \) for some \( i \), in which case the sign of \( \hat{z}_i \) is indeterminate, while \( \hat{x}_i \) is orthogonally indeterminate.

**Example 7.1.** A clear example of Corollary 7.1 is when the problem \( \mathbb{P}_{\omega^{**}} \) is to minimise the (squared) distance to the unit sphere in \( \mathbb{R}^n \) from its centre, the origin \( 0_n \). The solution set \( \hat{X}_{\omega^{**}} \) is, of course, the unit sphere, while \( \mathbb{P}_{\omega^{**}} \) is one-dimensional with \( \hat{X}_{\omega^{**}} = \{ \pm 1 \} \).

**Remark 7.2.** If \( B^* \) is singular, but \( b^* \) has zero component in its null space – that is, if \( m_0 > 0 \), but \( \varepsilon = 0 \) – the linear term in the constraint vanishes so that the optimal \( \hat{y} = 0 \), reducing \( \mathbb{P}_{\omega^{**}} \) to the corresponding \( m_0 = 0 \) case. It suffices, then, to solve dimension-reduced canonical forms which are regular in the sense defined next.

**Definition 7.2.** A dimension-reduced canonical form is called regular if either \( m_0 = 0 \) or \( m_0 > 0 \) and \( \varepsilon > 0 \). We denote such forms by \( \mathbb{P}_{\omega^{**}} \).

Since, trivially, \( m_0 = 0 \Rightarrow \varepsilon = 0 \), we have at once

**Remark 7.3.** For any regular dimension-reduced canonical form \( \mathbb{P}_{\omega^{**}} \):

\[
\varepsilon > 0 \iff m_0 > 0 \iff d \neq 0 \iff \text{the linear term in } Q^*(\cdot) \text{ does NOT vanish},
\]

in which case, substituting out the constraint, \( \mathbb{P}_{\omega^{**}} \) can be rephrased as the unrestricted minimisation of a quartic in \( z \).

### 8 Solving \( \mathbb{P}_{\omega^{**}} \)

We solve any regular dimension-reduced canonical form via a series of simple, insightful, auxiliary results. The first shows that a Lagrangian approach establishes sufficient conditions for this.

#### 8.1 Sufficient conditions

The Lagrangian \( \mathcal{L} := L^*(w) - \lambda Q^*(w) \) for \( \mathbb{P}_{\omega^{**}} \) has normal equations

\[
[I - \lambda \Delta]w = (w_0 + \lambda d)
\]

(11)

and Hessian \( H := 2[I - \lambda \Delta] \).

**Definition 8.1.** We refer to \( \Lambda := \{ \lambda : H \text{ has no negative eigenvalues} \} \) as the admissible region – that is:

\[
\Lambda := (-\infty, \gamma_1^{-1}] \text{ if } \gamma_q > 0, \text{ while } \Lambda := [\gamma_q^{-1}, \gamma_1^{-1}] \text{ if } \gamma_q < 0
\]

(12)

– its interior, denoted by \( \Lambda^\circ \), being where \( H \succ O \). For each \( \lambda \in \Lambda \),

\[
W_N(\lambda) := \{ w \in \mathbb{R}^n : (11) \text{ holds} \},
\]

so that \( W(\lambda) := W \cap W_N(\lambda) \) denotes the set of feasible vectors, if any, obeying the normal equations.
Lemma 8.1. For any $\lambda \in \Lambda$, $W(\lambda) \neq \emptyset \Rightarrow W(\lambda) = \hat{W}$. Indeed, if $w_N \in W(\lambda)$,

$$L^* = \|w_N - w_0\|^2 \text{ and } \hat{W} = \{w \in W : H(w - w_N) = 0\} = W(\lambda).$$

In particular, if $\lambda \in \Lambda^o$, $w_N$ uniquely solves $P_{F^*}$.

Proof. It suffices to note that, by the Cosine Law:

$$\forall w \in W, L^*(w) - L^*(w_N) = (w - w_N)'[I - \lambda \Delta](w - w_N) \geq 0.$$

In view of Lemma 8.1, $P_{F^*}$ is completely resolved if we can find, in explicit form, a nonempty set $W(\lambda)$ for any $\lambda \in \Lambda$.

8.2 Feasible solutions to the normal equations

Characterising when feasible solutions to the normal equations exist requires extensions of terminology established in Section 7. Continuing its unified presentation of $n = q$ and $n = q + 1$, terms involving any of $y$, $\hat{y}$ or, introduced below, $\tilde{y}_1$ or $\tilde{y}_q$ are absent by convention if $m_0 = 0$.

We distinguish between interior and boundary points of $\Lambda$. Accordingly, recalling [12], there are either two or three cases to consider according as $\gamma_q > 0$ or $\gamma_q < 0$. We call these Cases A ($\lambda \in \Lambda^o$), $B_1$ ($\lambda = \gamma_1^{-1}$) and $B_q$ ($\lambda = \gamma_q^{-1}$), this last arising when and only when $\gamma_q < 0$. By convention this last condition – that the constraint (10) be, at least in part, hyperbolic – is implicit whenever any of the terms defined only in Case $B_q$ is mentioned. In particular, this applies to $\tilde{y}_q$, $\tilde{z}(q)$, $\Gamma^*(q)$ and $f(q)$, defined next.

Definition 8.2. Let $P_{F^*}$ be a regular dimension-reduced canonical form and let $\lambda \in \Lambda$.

Case A ($\lambda \in \Lambda^o$) : $w(\lambda) := [I - \lambda \Delta]^{-1}(w_0 + \lambda \delta)$ denotes the unique solution to the normal equations, while the function $f : \Lambda^o \to R$ is defined by $f(\lambda) := Q^*(w(\lambda))$. That is, for all $\lambda \in \Lambda^o$:

$$f(\lambda) := \sum_i (1 - \lambda \gamma_i)^{-2} \gamma_i \delta_i^2 + 2 \varepsilon \lambda - k^*.$$ (13)

$f$ and $\overline{f}$ respectively denote the infimum and supremum of $\{f(\lambda) : \lambda \in \Lambda^o\}$.

Case $B_1$ ($\lambda = \gamma_1^{-1}$) : $\tilde{y}_1 := \delta_{\gamma_1}^{-1} (m_0 > 0)$, $\tilde{z}(1) \in R^{q-1}$ has general element $[1 - (\gamma_i / \gamma_1)^{1}]^{-1} \delta_i$ ($i = 2, ..., q$), while $f_1 := \tilde{z}(1)^{\top} \Gamma^*(1) \tilde{z}(1) + 2 \varepsilon \tilde{y}_1 - k^*$ where $\Gamma^* \equiv \text{diag}(\gamma_1, \Gamma^*_1)$.

Case $B_q$ ($\lambda = \gamma_q^{-1}$) : when and only when $\gamma_q < 0$, $\tilde{y}_q := \delta_{\gamma_q}^{-1} (m_0 > 0)$, $\tilde{z}(q) \in R^{q-1}$ has general element $[1 - (\gamma_i / \gamma_q)]^{-1} \delta_i$ ($i = 1, ..., q - 1$), while $f_q := \tilde{z}(q)^{\top} \Gamma^*_q \tilde{z}(q) + 2 \varepsilon \tilde{y}_q - k^*$ where $\Gamma^* \equiv \text{diag}(\Gamma^*_q, \gamma_q)$.

Remark 8.1. The constraint $f(\lambda) = 0$ is polynomial in $\lambda$.

Key properties of $f : \Lambda^o \to R$ are summarised in

Proposition 8.1. For any $P_{F^*}$ :

if $\delta = 0_q$ and $\varepsilon = 0$, $f(\lambda) = -k^*$ for all $\lambda \in \Lambda^o$, so that $\overline{f} = -k^* = \overline{f}$;

else, if either $\delta$ or $\varepsilon$ does not vanish, $f$ is continuous and strictly increasing, $\overline{f} < f$ being given by:

$$\overline{f} = f_1 \quad \text{or} \quad + \infty \quad \text{according as } \delta_1 = 0 \text{ or } \delta_1 > 0, \text{ while}$$

$$f = f_q \quad \text{or} \quad - \infty \quad \text{according as } \delta_q = 0 \text{ or } \delta_q > 0, \quad (\gamma_q < 0),$$

$$\overline{f} = -k^* \quad \text{or} \quad - \infty \quad \text{according as } \varepsilon = 0 \text{ or } \varepsilon > 0, \quad (\gamma_q > 0).$$

Consider now Case A. Regularity of $P_{F^*}$ and Proposition 8.1 give at once

Lemma 8.2. $w(\lambda)$ does not depend upon $\lambda \Leftrightarrow [\delta = 0_q \text{ and } \varepsilon = 0] \Leftrightarrow w(\lambda) = 0_q$ for every $\lambda \in \Lambda^o \Leftrightarrow f = -k^* = \overline{f}$.

Again, using Lemma 8.1, we have at once
Lemma 8.3. Let \( \lambda \in \Lambda^\circ \). Then,

\[
W(\lambda) = \begin{cases} 
\{w(\lambda)\} & \text{if } f(\lambda) = 0 \\
\emptyset & \text{if } f(\lambda) \neq 0
\end{cases}
\]

so that: \( f(\lambda_1) = f(\lambda_2) = 0 \Rightarrow W(\lambda_1) = W(\lambda_2) = \hat{W} \Rightarrow w(\lambda_1) = w(\lambda_2) \).

In view of Lemma 8.3, we may define \( \hat{w}_o \) as the common value of \( w(\lambda) \) among all solutions to \( f(\lambda) = 0 \), when at least one such exists. Putting \( W_o := \cup_{\lambda \in \Lambda^\circ} W(\lambda) \), Lemmas 8.1 and 8.3 now give

Lemma 8.4. \( W_o = \begin{cases} 
\hat{W} = \{\hat{w}_o\} & \text{if } \exists \lambda \text{ with } f(\lambda) = 0 \\
\emptyset & \text{else.}
\end{cases}
\]

Combining the above auxiliary results makes \( \hat{w}_o \) explicit and, with it, the following summary of Case A.

Proposition 8.2. For any \( \mathbb{P}_{\pi^*} \), there are three possibilities:
(a) If \( f < 0 < \tilde{f} \), \( f(\lambda) = 0 \) has a unique solution \( \hat{\lambda} \) and \( \hat{W} = W_o = \{w(\hat{\lambda})\} \).
(b) If \( f = 0 = \tilde{f} \), \( f(\lambda) = 0 \) for every \( \lambda \in \Lambda^\circ \) and \( \hat{W} = W_o = \{0_q\} \).
(c) In all other cases, \( f(\lambda) = 0 \) has no solutions and \( W_o = \emptyset \).

Remark 8.2. When it exists, computing \( \hat{\lambda} \) is straightforward (see Albers et al. [4, 3]).

In view of Proposition 8.2, we refer to \( W_o \) as a potential Lagrangian solution set, this potential being realised if and only if it is non-empty. Turning now to the boundary cases, there are two other potential Lagrangian solution sets \( W_1 := W(\gamma_1^{-1}) \) and \( W_q := W(\gamma_q^{-1}) \), this second definition being made when and only when \( \gamma_q < 0 \). The hessian \( H \) being singular here, the normal equations (11) have either no solution, or a unique solution for all but one member of \( w \), which they leave unconstrained. We have

Proposition 8.3. For any \( \mathbb{P}_{\pi^*} \) :

(1) Case \( B_1 \) \( (\lambda = \gamma_1^{-1}) : W_1 \neq \emptyset \Leftrightarrow [\delta_1 = 0 \text{ and } f_1 \leq 0] \), in which case:

\[
\hat{W} = W_1 = \{(y_1, z_1, z_1') : z_1^2 = (-f_1)/\gamma_1\}.
\]

(2) Case \( B_q \) \( (\lambda = \gamma_q^{-1} < 0) : W_q \neq \emptyset \Leftrightarrow [\delta_q = 0 \text{ and } f_q \geq 0] \), in which case:

\[
\hat{W} = W_q = \{(y_q, z_q', z_q') : z_q^2 = f_q/(-\gamma_q)\}.
\]

Proof. \( W_N(\gamma_1^{-1}) \neq \emptyset \Leftrightarrow \delta_1 = 0 \) in which case:

\[
w \in W_N(\gamma_1^{-1}) \Leftrightarrow [y = \hat{y}_1 \text{ and } z(1) = \hat{z}_{(1)}], \text{ where } z \equiv (z_1, z_1')',
\]

\( z_1 \) being unconstrained. Thus,

\[
W_1 \neq \emptyset \Leftrightarrow [\delta_1 = 0 \text{ and } \exists z_1 \text{ with } \gamma_1 z_1^2 + f_1 = 0] \Leftrightarrow [\delta_1 = 0 \text{ and } f_1 \leq 0].
\]

(1) now follows from Lemma 8.1. The proof of (2) is entirely similar.

In summary, \( \mathbb{P}_{\pi^*} \) has either two \( (\gamma_q > 0) \) or three \( (\gamma_q < 0) \) types of potential Lagrangian solution set – \( W_o, W_1 \) and \( W_q \) – Propositions 8.1 to 8.3 together establishing precisely when these potentials are realised and, in each case, what the corresponding solution set \( \hat{W} \) then is.

8.3 The minimised objective function and the solution set

We are now ready to solve any regular dimension-reduced canonical form \( \mathbb{P}_{\pi^*} \) in which, by definition, either \( m_0 = 0 \) or \( [m_0 > 0 \text{ and } \varepsilon > 0] \).

To aid geometric interpretation, recall that, under the assumptions and notation of Definition 6.1
(a) \( \varepsilon := l_0 \geq 0 \) denotes the length of the orthogonal projection of \( b \) onto the null space of \( B \), whose dimension is \( m_0 \geq 0 \);
(b) for each distinct nonzero eigenvalue \( \gamma_1 > \ldots > \gamma_q \) of \( B \), \( \delta \in \mathbb{R}^q \) has general element \( \delta_i := l_i/|\gamma_i| \) in which
\( l_i \geq 0 \) denotes the length of the orthogonal projection of \( b \) onto the corresponding eigenspace of \( B \), whose dimension is \( m_i \geq 1 \). Accordingly:

- the linear part of the constraint vanishes \( \iff m_0 = 0 \iff \varepsilon = 0 \),
- the origin is feasible \( \iff k^* = 0 \),
- the origin is the target \( \iff \delta = 0 \), and:
- the constraint is, at least in part, elliptic \( \iff \gamma_q > 0 \).

We distinguish three mutually exclusive and exhaustive types of regular dimension-reduced canonical form \( \mathbb{P}_{\text{r}} \). The first two are trivial.

We call \( \mathbb{P}_{\text{r}} \) non-Lagrangian if \([m_0 = 0, k^* = 0, \delta \neq 0, \text{ and } \gamma_q > 0]\), (so that, in particular, \( W_q \) is undefined). Geometrically, if the feasible set is the origin, the target being a positive distance away. As Theorem 8.1 establishes, in this case, both \( W_o \) and \( W_1 \) are empty. That is, the constraint and normal equations are inconsistent, so that \( \mathbb{P}_{\text{r}} \) is not amenable to Lagrangian solution.

We call \( \mathbb{P}_{\text{r}} \) multiply-Lagrangian if \( f = 0 = f^* \). That is, if \([m_0 = 0, k^* = 0 \text{ and } \delta = 0_q]\). Geometrically, if the target is the origin, through which the conic defining the constraint passes. As Theorem 8.1 establishes, in this case, \( \hat{W} \) coincides with each of the nonempty sets \( W_o, W_1 \) and, when defined, \( W_q \).

Finally, we call \( \mathbb{P}_{\text{r}} \) singly-Lagrangian if it is neither non-Lagrangian nor multiply-Lagrangian. Algebraically, if either \([m_0 = 0, k^* = 0, \text{ and } \gamma_q < 0]\), or \([m_0 = 0 \text{ and } k^* \neq 0]\), or \([m_0 > 0 \text{ and } \varepsilon > 0]\). As Theorem 8.1 establishes, in this case, exactly one of \( W_o, W_1 \) and \( W_q \) is nonempty, and so provides the solution set required.

**Theorem 8.1.** The minimised objective function \( L^* \) and solution set \( \hat{W} \) for a regular dimension-reduced canonical form \( \mathbb{P}_{\text{r}} \) are as follows.

(a) If \( \mathbb{P}_{\text{r}} \) is non-Lagrangian, \( W_o = W_1 = \emptyset \), while:

\[
L^* = \sum_{i=1}^{q} \gamma_i^{-2} l_i^2 > 0 \quad \text{attained on } \hat{W} = W = \{q_o\}.
\]

(b) If \( \mathbb{P}_{\text{r}} \) is multiply-Lagrangian:

\[
L^* = 0 \quad \text{attained on } \hat{W} = W_o = W_1 = W_q = \{q_o\}.
\]

(c) Otherwise, if \( \mathbb{P}_{\text{r}} \) is singly-Lagrangian, \( \hat{W} \) is the unique nonempty member of \( \{W_o, W_1, W_q\} \). Specifically, \( f \) and \( \bar{f} \) being as in Proposition 8.1

- if \( \gamma_q > 0 \), \( f < 0 \), while \( \hat{W} = W_o \) or \( W_1 \) according as \( \bar{f} > 0 \) or \( \bar{f} \leq 0 \);
- if \( \gamma_q < 0 \), \( \hat{W} = W_o \), \( W_1 \) or \( W_q \) according as \( \bar{f} < 0 \), \( \bar{f} \geq 0 \), or \( \bar{f} < 0 \).

When \( \hat{W} = W_o \),

\[
L^* = \hat{\lambda}^2 \{l_0^2 + \sum_{i=1}^{q} (1 - \hat{\lambda} \gamma_i)^{-2} l_i^2 \},
\]

attained at \( w = w(\hat{\lambda}) \), where \( \hat{\lambda} \) uniquely solves \( f(\lambda) = 0 \).

When \( \hat{W} = W_1 \),

\[
L^* = (\gamma_1^{-1})^2 \{l_0^2 + \sum_{i=2}^{q} (1 - \gamma_1^{-1} \gamma_i)^{-2} l_i^2 \} + \zeta_1^2,
\]

attained at \( w = (\hat{y}_1, \pm \zeta_1, \hat{z}_{(1)}') \), where \( \zeta_1 := \sqrt{(-f_1)/\gamma_1} \geq 0 \).

When \( \hat{W} = W_q \),

\[
L^* = (\gamma_q^{-1})^2 \{l_0^2 + \sum_{i=1}^{q-1} (1 - \gamma_q^{-1} \gamma_i)^{-2} l_i^2 \} + \zeta_q^2,
\]

attained at \( w = (\tilde{y}_q, \hat{z}_{(q)}', \pm \zeta_q) \), where \( \zeta_q := \sqrt{f_q/(\gamma_q)} > 0 \).

**Proof.** The result follows from detailed, straightforward application of the auxiliary results of Section 8.2 noting the following:

(a) Here, \( W_o = W_1 = \emptyset \) as \( f = 0 \) and \( f_1 > 0 \) while, by definition, \( W = \{0_q\} \).

(b) Here, \( w_0 = \hat{w}_o = 0_q \) and \( \hat{z}_{(1)} = \hat{z}_{(q)} = 0_{q-1} \), while \( f_1 = f_q = 0 \).
The characterisation of non-uniqueness of solution to
Proof.

Consider, for example, minimising the distance from the origin to the ellipse

\[ z_1^2 + z_2^2 = 1 \]

in which cases solutions to \( P_{\omega} \) are unique up to orthogonal indeterminacy of (a) \( \hat{x}_1 \) or (b) \( \hat{x}_q \) respectively.

\[ \text{Corollary 8.1.} \quad \text{Let } P_{\omega} \text{ be a canonical form whose dimension-reduced form } P_{\omega^*} \text{ is regular. Then:} \]

(1) \( P_{\omega^*} \) has a unique solution whenever \( P_{\omega^*} \) does.

(2) \( P_{\omega^*} \) does not have a unique solution if, and only if, it is singly-Lagrangian and either:

(a) \( [f_1 < 0, \delta_1 = 0] \), when its solutions are unique up to the sign of \( \hat{z}_1 \neq 0 \); or:

(b) \( [f_q > 0, \delta_q = 0] \), when its solutions are unique up to the sign of \( \hat{z}_q \neq 0 \),

\[ \text{Theorem 8.1 and its Corollary 8.1 complete our primary objective, providing the minimised objective function and solution set of any regular dimension-reduced } P_{\omega^*} \text{ and so, via Theorem 7.1 of any initial canonical form } P_{\omega^*}. \]

Two worked examples of this overall approach are given in [5].

9 Intrinsic instability

The above analysis of any, equality or inequality constrained, problem \( P_{\omega} \) rests on several partitions of possibilities. Passage between different members of a partition can involve movement between equality and inequality of the same two reals. Or, again, between weak (\( \leq \)) and strict (\( < \)) versions of the same inequality.

As a result, both the form of the solution set and, indeed, of the problem itself can be intrinsically unstable under arbitrarily small perturbations of problem parameters. Whether or not the solution set \( \hat{X}_\omega \) varies continuously with \( \omega \) across such partition boundaries is implicit in the above analysis. In particular, in its summative Theorems 4.1, 5.1, 7.1 and 8.1 and their key Corollaries 7.1 and 8.1. The discontinuities involved can be dramatic, as the following instances illustrate.

Potential synergetic uses of these analytical insights in connection with efficient numerical optimisation methods are noted in the closing discussion (Section 10).

9.1 Instability of the form of the solution set

We begin with a marked instance of the passage from non-unique to unique solutions. As in Example 7.1, consider minimisation of the distance to a sphere from its centre, whose solution set is, of course, the sphere itself. Changing this to minimising the same distance from \( \omega \) other point, the solution suddenly becomes unique. This instability corresponds to the passage from \( \delta_1 = 0 \) to \( \delta_1 > 0 \) in Corollary 8.1.

In the hyperbolic case, there can be extreme directional instability, due to the orthogonality of the eigenspaces of \( \gamma_1 \) and \( \gamma_q \). For instance, consider minimisation of the distance from the origin to the hyperbola \( z_1^2 - z_2^2 = k^* \), both of whose eigenvalues, \( \pm 1 \), are simple. When \( k^* = 0 \), this comprises the lines \( z_2 = \pm z_1 \) which meet at the (repeated) solution (0, 0)', while \( P_{\omega^*} \) is multiply-Lagrangian. Otherwise, it comprises two branches with these lines as asymptotes, \( P_{\omega^*} \) being singly-Lagrangian. For \( k^* > 0 \), these branches meet the \( z_1 \)-axis at the twin solutions \( \pm (\sqrt{k^*}, 0)' \) while, for \( k^* < 0 \), they meet the \( z_2 \)-axis at the twin solutions \( \pm (0, \sqrt{-k^*})' \). Thus, no matter how small we take \( k^*_+ > 0 > k^*_− \), the twin solutions for \( k^*_+ \) are always orthogonal to those for \( k^*_− \). In terms of Theorem 8.1, the solution set \( \hat{W} \) changes from \( W_1 \) to \( W_0 \) via \( W_\delta \) as \( k^* \) goes from positive to negative via zero.

Geometrically, it is clear that this same extreme directional instability can arise in the elliptic case. Consider, for example, minimising the distance from the origin to the ellipse \( z_1^2 + z_2^2/k^2 = 1 \) \( (k > 0) \) so that, when \( k = 1 \), the solution set is the ellipse itself. Then, with \( k_+ > 1 > k_− \), no matter how close we take...
\( \kappa_+ \) and \( \kappa_- \) to 1 – and so, to each other – the solution set for \( \kappa_+ \) is \( \pm(1,0)' \) while, for \( \kappa_- \), it remains in the directions \( \pm(0,1)' \) orthogonal to these.

9.2 Instability of the form of the problem

Intrinsic instability of the form of the problem comes, itself, in a variety of forms, as we illustrate.

As a first instance, consider the variant of Example 5.1(b) with constraint \((x_1 + x_2)x_0 = 1\) for given \( \xi_1 \neq 0 \). Whereas the perfect solution here has \( x_1 = 0 \) for every \( \xi_1 \), it has \( \hat{x}_0 = 1/\xi_1 \), which tends to \( +\infty \) as \( \xi_1 \to 0_+ \) and to \(-\infty \) as \( \xi_1 \to 0_- \). Such instability holds generally. Indeed, Theorem 5.1 shows that, if \( \mathbb{P}_\omega \) admits an essentially perfect solution, it is intrinsically unstable under small constraint perturbations. And, relatedly, that the occurrence of essentially perfect solutions to \( \mathbb{P}_\omega \) is itself intrinsically unstable, depending as it does on the vanishing of the constraint parameter \( c_0 \). Replacing \( c_0 \) by \( C_{10} \), the same is true of reducibility of \( \mathbb{P}_\omega \) when \( B_{00} \) is singular. That is, when \( A \) is positive semi-definite, there is intrinsic instability at the boundaries between the three possibilities – \( \mathbb{P}_\omega \) admits a perfect solution, \( \mathbb{P}_\omega \) admits an essentially perfect solution, or \( \mathbb{P}_\omega \) admits a projected, yet imperfect, reduced form – their occurrences being characterised in Theorem 5.1.

As a second instance, any positive semi-definite \( A \) is arbitrarily close to a positive definite matrix \( A(\kappa) \), \( \kappa > 0 \) – for example, \( A(\kappa) = A + \kappa I \) – analysis of the problem changing from that of Section 5 to that discussed in the sequel.

Similar intrinsic instabilities in problem form occur at the boundaries between members of the following partitions made for positive definite \( A \):

(i) the four possible forms of constraint – elliptic, hyperbolic, or the partly parabolic variant of either – as detailed in Definition 7.1

(ii) in the dimension-reduced canonical form \( \mathbb{P}_\omega^{\ast\ast} \), whether it is regular or not – as detailed in Definition 7.2

(iii) the three possibilities – non-Lagrangian, multiply Lagrangian and singly Lagrangian – as detailed in Section 8.3

10 Discussion

The current approach to the problem addressed in this paper, and the more generally applicable approach of generalised trust region (GTR) methods, broadly reviewed at the outset, may be compared as follows. Their intrinsic complementarity is evident and raises two substantial questions for future research.

Being derivative-based, the GTR approach is, intrinsically, local. A local optimum is identified implicitly, and then sought numerically, via simultaneous satisfaction of an appropriate set of conditions. These may include conditions additional to those of the problem itself (notably, those introduced to ensure numerical stability). Instances of the problem not satisfying such conditions require separate resolution. Under certain such additional constraint qualifications, it may be possible to characterise global optimality: see, for example, 23, 21, 25. Unless a local optimum can be shown to be unique, identifying a global optimum requires repeated use of one or more algorithms. Of course, at most a finite number of local optima may be obtained in this way. The desired trade-off to be struck between the speed, accuracy and numerical stability of the overall computational procedure depends on context and purpose.

In contrast, the approach presented here is, intrinsically, global in three distinct senses: (a) it partitions all possible instances of the problem, and of its solution set, without exception; (b) its simplifying transformations are applied to the whole space; and: (c) its solution sets comprise global optima only. These sets are specified explicitly in closed algebraic form, affording both insight and interpretation. In particular, it is shown that, depending on precisely defined circumstances, there may be 0, 1, 2, or more – indeed, uncountably many – global optima, geometry illuminating when each possibility occurs. Overall, the current global, analytic approach throws a distinctive algebraic and geometric spotlight on the diverse nature of different instances of both problem and solution set; and, on the possibility of intrinsic instability of both, pinpointing precisely when this occurs: specifically, when passage between its different partition members involve distinctions that cannot be drawn in the floating-point world, yet lead to radical changes in the form of either the solution set, or of problem itself.

Two substantial questions arise at once:
1. can the current global, analytic approach be extended to accommodate possible non-convexity of the objective function?

2. can the intrinsic complementarity it shares with the local, computational GTR approach be exploited, releasing potential synergies between them?

Motivations for examining these questions in future work, and first pointers towards possible answers, are briefly offered in closing.

Concerning (1), broadening the applicability of the current approach in this way would bring it closer to that of GTR methods. One natural strategy here would work with affine transformations such that the partitioned least-squares form \( \text{diag}(I_r, O_{n-r}) \) of \( A \) is replaced by \( \text{diag}(I_{r+}, O_{n-r}, -I_{r-}) \), for some affine invariants \( r_+ \geq 0, \ r_- \geq 0 \) with \( r_+ + r_- = r \).

Concerning (2), generic motivation here comes from seeking to capitalise on the distinctive advantages of both approaches: the former providing, in principle, global solution sets that are complete, exact, insightful and interpretable; the latter providing, in practice, algorithms whose specification and deployment can be varied to deliver a balance of speed, accuracy and numerical stability well-suited to a given context and purpose. Especially strong when the present problem arises as one iteration of a general purpose optimisation procedure, specific motivation here includes the need to handle the possibility of various forms of intrinsic – not just numerical – instability highlighted by the present global analysis and illustrated in Section 9 among these, the possibility of undetectably small changes leading to solutions in orthogonal directions is particularly striking (see Section 9.1). One natural strategy here would be to modify existing computational procedures so that, when numerical checks informed by global analysis indicate its appropriateness, multiple forms of problem or solution set are entertained. This strategy has the potential advantage of identifying several, quite different, locally-promising search directions at each iteration.

Concerning both, it will be of interest to see how far, and by what means, progress can best be made in addressing these challenging, open questions.

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A Two worked examples

We provide with two worked examples and a short conclusion. The examples differ only by a single element of $A$, yet involve completely different approaches to their solution. As such, they illustrate one of the intrinsic instabilities noted in Section 9. Namely, that at the boundary between $A \succeq O$ where Section 5 applies, and $A \succ O$ where Sections 6 to 8 pertain.

Whereas this generic type of instability can of course arise whatever the scale of the problem, to make things explicit, we work here with $n = 3$ and with values of $\omega$ for which exact calculations are fairly straightforward. Specifically, we take $k = 1$, $b = 0_3$, $t = (1, 1, 1)'$, $B = I_3$ and

$$A = A(\kappa) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 + \kappa \end{pmatrix}, \ \kappa \geq 0,$$

so that $A(0) \succeq O$, while $A(\kappa) \succ O$ for all $\kappa > 0$. While reporting exact results, we also indicate where there is – or is not – numerical uncertainty, especially concerning which member of a partition of possibilities the problem on hand belongs to.

Examples A.1 and A.2 concern $\kappa = 0$ and $\kappa = 3/2$ respectively. In the first case, one of the computed eigenvalues of $A$ will be within machine accuracy of zero; in the second, all will be clearly positive. As $\kappa \to 0+$, both problem forms will be flagged up. Numerically, $B$ is clearly positive definite.

Example A.1. Here, $A = A(0)$ clearly has rank $r = 2$, and we follow the approach of Section 3.

Analysis begins with a three-stage transformation: first, to centred – here, full – least-squares form, using (5) of Lemma 5.2; and third, to simultaneous diagonal form, via the further linear transformation of Remark 5.2. Overall, this transformation is $x \to x_\omega = T^{-1}(x - t)$ where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

inducing $\omega \to \omega_\omega$ with $t_\omega = 0_3$ and $A_\omega = T'AT = \text{diag}(1, 1, 0)$. These two components of $\omega_\omega$ are known (and so do not require calculation), as is the diagonal nature of $B$. To within machine accuracy, $B_\omega = T'BT = \text{diag}(1, \frac{1}{2}, 1)$, $b_\omega = (1, 0, -\sqrt{2})'$ and $k_\omega = -2$.

Next, we apply Theorem 5.1 to $\mathbb{P}_{\omega_\omega}$, dropping the subscript $g$ where notationally convenient. Identifying terms in Definitions 5.2 and 5.3 for this transformed member of $\Omega_3$, and noting that the third diagonal element $B_\omega$ is clearly positive, we have $s_0 = 1$, so that $B_0 = \Gamma_0 = (1)$, while $b_0 = d_0 = (-\sqrt{2})$. In particular, $s_0 = m - r$, so that part (ii) of Theorem 5.1 applies. Moreover, to within machine accuracy, $k_1 = 0$ and $Z_0(0) = \{\sqrt{2}\}$, so that $\mathbb{P}_{\omega_\omega}$ admits perfect solution $\hat{x}_\omega = (0_3, \sqrt{2})'$, $\hat{L}_{\omega_\omega} = 0$.

Finally, invoking Theorem 2.1, we transform back to conclude that $\mathbb{P}_\omega$ has perfect solution $\hat{x} = t + T\hat{x}_\omega = (1, 0, 0)'$, $\hat{L}_\omega = 0$.

Example A.2. Here, $A = A(3/2)$ clearly has full rank, and we follow Sections 6 to 8.

Analysis begins with a two-stage transformation: first, to centred – here, full – least-squares form, using again Theorem 3.1(i); second, to canonical form, via the Euclidean transformation of Theorem 6.2. Overall, apart from a constant, this transformation is $x \to x_\omega = T^{-1}x$ where

$$T = \begin{pmatrix} 0 & -1 & 0 \\ -\sqrt{\frac{2}{5}} & 0 & \frac{1}{\sqrt{15}} \\ -\sqrt{\frac{2}{5}} & 0 & -\frac{2}{\sqrt{15}} \end{pmatrix}$$

inducing $\omega \to \omega_\omega$ in which $A_\omega = T'AT$ is $I_3$ and $B_\omega = T'BT$ is diagonal. To within machine accuracy, $B_\omega = \text{diag}(2, 1, \frac{1}{2})$, which clearly has distinct, positive, eigenvalues. Accordingly, $\mathbb{P}_{\omega_\omega}$ is, itself, dimension-reduced and regular. Finally, in the notation of Definition 6.1, we compute that $\delta = (3/\sqrt{10}, 1, \sqrt{3/5})'$ and $k^* = 1$.

We solve the regular dimension-reduced canonical form $\mathbb{P}_{\omega_\omega}$ via the steps described in Section 8. Since, clearly, $m_0 = 0$ and $k^* \neq 0$, $\mathbb{P}_{\omega_\omega}$ is singly-Lagrangian. Here, $A^* = (-\infty, \gamma_1^{-1}) = (-\infty, \frac{1}{4})$, the function
Figure 1: Display of the functional form of $f(\lambda)$. The vertical lines are at $\lambda = l^o$ and $\lambda = u^o$, defined in the text.

$f : \Lambda^o \rightarrow R$ as given in Definition 8.2 taking the form:

$$f(\lambda) = \frac{9}{5(1-2\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{9}{5(3-\lambda)^2} - 1.$$ (14)

Again, it is numerically clear that $\gamma_3 > 0$, $\varepsilon = 0$ and $\delta_1 > 0$ so that, by Proposition 8.1, $f = -k^* = -1 < 0$ and $\overline{f} = +\infty$. Accordingly (Proposition 8.2), $\widehat{W} = W_\circ = \{w(\hat{\lambda})\}$ where $\hat{\lambda}$ is the unique root of $f(\lambda) = 0$. Albers et al [4] show how to reduce $\Lambda^o$ to a finite interval $(l^o, u^o)$, $u^o := \gamma_1^{-1}$, containing $\hat{\lambda}$. Here, $l^o = \frac{1}{2} - 3\sqrt{3}/5$. This is visualised in Figure 1 where the functional form (14) is plotted over an interval containing $(l^o, u^o)$, its two vertical lines being at $\lambda = l^o$ and $\lambda = u^o$, while there are vertical asymptotes at $\lambda = \gamma_i^{-1}$, $(i = 1, 2, 3)$.

Numerical solution over $(l^o, u^o)$ – for example, by the bisection method – is straightforward, yielding here the approximate value $\hat{\lambda} \approx -0.527$.

Finally, substituting in the relevant part of Theorem 8.1(c), and back transforming $\hat{x}_g \rightarrow \hat{x}$, yields the final solution $\hat{x} \approx (0.655, 0.414, 0.632)'$, $\hat{L}_\omega \approx 0.370$. 