Robust Output Feedback Consensus for Networked Heterogeneous Nonlinear Negative-Imaginary Systems with Free Body Motion

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Abstract—This paper presents a framework to address the robust output feedback consensus problem for networked heterogeneous nonlinear Negative-Imaginary (NI) systems with free body dynamics. The aim of this paper is to complete and extend the results in previous papers on robust output feedback consensus for multiple heterogeneous nonlinear NI systems so that the systems in the network are allowed to have free body motion. A subclass of NI systems called Output Strictly Negative-Imaginary (OSNI) systems are applied as controllers to ensure that the outputs of the nonlinear NI plants converge to the same limit trajectory. The definitions of nonlinear NI systems and nonlinear OSNI systems are extended and a new stability result is developed for the interconnection of a single nonlinear NI system and a single nonlinear OSNI system. Robust output feedback consensus is addressed by establishing a similar stability result for the interconnection of networked NI systems and networked OSNI systems.

Index Terms—nonlinear Negative-Imaginary systems, free body motion, heterogeneous systems, consensus, robust control.

I. INTRODUCTION

Negative-Imaginary (NI) systems theory was introduced by Lanzon and Petersen in [1] and [2] to address a robust control problem for flexible structures and has attracted much attention among control theory researchers (see [3]–[9]). Typical NI systems are systems with colocated force actuators and position sensors. Positive-Real (PR) systems theory [10] cannot be applied to the control of such systems in general. An NI system was initially defined in [1] to be a stable system with a frequency response \( F(j\omega) \) satisfying

\[
|jF(j\omega) - F(j\omega^*)| \geq 0
\]

for all \( \omega > 0 \). Examples of such systems arise in lightly damped structures [11]–[13] and nano-positioning systems [14]–[17]. The stability of an NI system with transfer function matrix \( F(s) \) can be guaranteed by applying a strictly Negative-Imaginary (SNI) controller with transfer function matrix \( G(s) \) such that the DC gain condition

\[
\lambda_{\max}(F(0)G(0)) < 1
\]

is satisfied [1].

The definition of NI systems in [1], [2] was extended in [9] to include systems with poles in the closed left half of the complex plane except at the origin. The definition has been extended again in [18] to include systems with poles at the origin. Systems with free body motion such as single integrators and double integrators were included in this new definition and the new robust stability result for NI systems in [18]. The original definition of NI systems has also been recently extended to include nonlinear systems [19] and some stability results were established for nonlinear NI systems in [19] and [20]. However, systems with free body motion are excluded in the nonlinear NI definition in [19].

The output feedback consensus problem has been widely studied over the past decade (see [13], [21]–[27]). However, there are many restrictions in the existing literature, such as requiring plants to be single-input single-output (SISO) systems, or minimum phase linear time-invariant (LTI) systems, or full state feedback second-order systems. In [8], the robust cooperative control problem for multiple heterogeneous linear NI systems is addressed. In particular, [8] models the communication between multiple NI systems using an undirected connected graph, where the plants correspond to the nodes of the graph and the SNI controllers correspond to the edges of the graph. Each controller takes the difference between the outputs of the plants connected to it as input and feeds back its output to the plants so that the outputs of all of the plants will converge to the same limit trajectory. This is called robust output feedback consensus. The theoretical result presented in [8] was applied in [28]–[32] to address some real-world cooperative control problems. However, the result in [8] is restricted to linear systems only and cannot solve cooperative control problems for nonlinear systems. Motivated by nonlinear NI systems theory, the result in [8] has been recently extended in [33] to provide a method for robust output feedback consensus for networked heterogeneous nonlinear NI systems without free body motion.

The contribution of this paper involves establishing a stability result and providing a control framework to achieve robust output feedback consensus for nonlinear NI systems with free body motion, which can be regarded as a complement to the nonlinear NI system results in [19], [20] and [33]. This paper can also be considered as an extension of the previous studies in [18] and [8] from linear systems to nonlinear systems. In particular, this paper provides a new definition for nonlinear NI systems, which includes systems with free body motion. Nonlinear OSNI systems are also defined to allow a direct feedthrough from their inputs to their outputs, which is in contrast to the previous definition in [20]. The input feedthrough term in an OSNI controller makes it possible for a Lyapunov function for the closed-loop interconnection of a nonlinear NI system with free body motion and a nonlinear OSNI system to be positive definite. Robust output feedback consensus is achieved for multiple heterogeneous nonlinear NI systems with possible free body motion using a control framework similar to that in [33]. However, the control framework is refined and a new proof is given due to the new definition for nonlinear NI systems. The proposed control framework is robust with respect to uncertainty in the system models for both the nonlinear NI plants and the nonlinear OSNI controllers.
Notation: The notation in this paper is standard. \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers, respectively. \( \mathbb{R}^{m \times n} \) and \( \mathbb{C}^{m \times n} \) denote the spaces of real and complex matrices of dimension \( m \times n \), respectively. \( A^T \) and \( A^* \) denote the transpose and complex conjugate transpose of a matrix \( A \), respectively. \( \lambda_{\max}(\cdot) \) denotes the maximum eigenvalue of a matrix with only real eigenvalues. \( \overline{a} \) denotes a constant vector or scalar. \( I_n \) is the \( n \times n \) identity matrix. For a nonlinear dynamical system \( H \) with input \( u \) and output \( y \), \( y = H(u) \) describes its input-output relationship. \( \text{diag}(\{a_1, a_2, \cdots, a_l\}) \) represents a diagonal matrix with the values \( a_1, a_2, \cdots, a_l \) on its diagonal.

Graph Theory Preliminaries: \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{v_1, v_2, \cdots, v_n\} \) and \( \mathcal{E} = \{e_1, e_2, \cdots, e_l\} \subseteq \mathcal{V} \times \mathcal{V} \), describes an undirected graph with \( N \) nodes and \( l \) edges. The corresponding symmetric adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{N \times N} \) is defined so that \( a_{ij} = 0 \) and \( \forall i \neq j, a_{ij} = 1 \) if \( (v_i, v_j) \in \mathcal{E} \) and \( a_{ij} = 0 \) otherwise. A sequence of unrepeated edges in \( \mathcal{E} \) that joins a sequence of nodes in \( \mathcal{V} \) defines a path. An undirected graph is connected if there is a path between every pair of nodes. Given an undirected graph \( \mathcal{G} \), a corresponding directed graph can be obtained by defining a direction for each edge of \( \mathcal{G} \). The incidence matrix \( Q = [q_{ev}] \in \mathbb{R}^{l \times N} \) of a directed graph is defined so that the elements in \( Q \) are given by

\[
q_{ev} := \begin{cases} 
1 & \text{if } v \text{ is the initial vertex of edge } e, \\
-1 & \text{if } v \text{ is the terminal vertex of edge } e, \\
0 & \text{if } v \text{ does not belong to edge } e.
\end{cases}
\]

In this paper, the initial and terminal vertices of an edge in a directed graph can both send information to each other via the corresponding controller. For an undirected graph \( \mathcal{G} \), the choice of a corresponding directed graph is not unique. However, the Laplacian matrix \( L_N \) of \( \mathcal{G} \) has the following relationship with the incidence matrix \( Q \) of any directed graph corresponding to \( \mathcal{G}: L_N = Q^T Q \).

II. AN INITIAL STABILITY RESULT

In this section, new definitions of nonlinear NI and nonlinear OSNI systems are provided and a new stability result is established for the interconnection of a single nonlinear NI system and a single nonlinear OSNI system. The new stability result is applicable to nonlinear NI systems with free body motion, which are excluded in the previous stability result given in [20].

Consider the following general nonlinear system:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)); \\
y(t) &= h(x(t)) + Du(t),
\end{align*}
\]

where \( f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a Lipschitz continuous function, \( h: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a class \( C_1 \) function and \( D \in \mathbb{R}^{m \times m} \) is a symmetric matrix; i.e., \( D = D^T \).

Definition 1: The system (1), (2) is said to be a nonlinear negative-imaginary (NI) system if there exists a positive semidefinite storage function \( V: \mathbb{R}^n \rightarrow \mathbb{R} \) of class \( C^1 \) such that

\[
V(x(t)) \leq u(t)^T \dot{y}(t)
\]

for all \( t \geq 0 \), where \( \dot{y}(t) = h(x(t)) \).

In contrast to Definition 3 in [19], which excludes linear NI systems with poles at the origin, Definition 1 now includes all linear NI systems satisfying the definition given in [18] by allowing the storage function of the system to be positive semidefinite instead of positive definite. For example, a single integrator with the state-space model \( \dot{x}_{SI} = u_{SI}, \ y_{SI} = x_{SI} \) and a double integrator with the state-space model \( \dot{x}_{DI1} = x_{DI2}, \ \dot{x}_{DI2} = u_{DI1}, \ y_{DI1} = x_{DI1} \) satisfy the linear NI definition in [18]. However, they do not satisfy the nonlinear NI Definition 3 in [19] because there does not exist a positive definite storage function that satisfies the NI property (3). However, according to Definition 1 in this paper, they are both nonlinear NI systems, with \( V_{SI} = 0 \) and \( V_{DI} = \frac{1}{2}x_{DI2}^2 \) being their storage functions, respectively.

Definition 2: The system (1), (2) is said to be a nonlinear output strictly negative-imaginary (OSNI) system if there exists a positive semidefinite storage function \( V: \mathbb{R}^n \rightarrow \mathbb{R} \) of class \( C^1 \) and a constant \( \epsilon > 0 \) such that

\[
\dot{V}(x(t)) \leq u(t)^T \dot{y}(t) - \epsilon \| \dot{y}(t) \|^2
\]

for all \( t \geq 0 \), where \( \dot{y}(t) = h(x(t)) \). Here, the quantity \( \epsilon \) quantifies the level of output strictness of the system.

In this paper, nonlinear OSNI systems defined as in Definition 2 are applied as controllers to achieve the robust output feedback consensus for networked heterogeneous nonlinear NI systems defined as in Definition 1. First, we provide a stability result for a single feedback interconnection of nonlinear NI systems.

Consider a multiple-input multiple-output (MIMO) nonlinear NI system \( H_1 \) with the following state-space model:

\[
\begin{align*}
\dot{x}_1(t) &= f_1(x_1(t), u_1(t)); \\
y_1(t) &= h_1(x_1(t)),
\end{align*}
\]

where \( f_1: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a Lipschitz continuous function and \( h_1: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a class \( C^1 \) function.

For the system \( H_1 \) with the state-space model (5), (6), we suppose the following assumption is satisfied.

Assumption I: When the system \( H_1 \) is in steady state; i.e., \( u_1(t) \equiv \bar{u}_1, \ x_1(t) \equiv \bar{x}_1 \) and \( y_1(t) \equiv \bar{y}_1 \), we have \( \bar{u}_1^T \bar{y}_1 \geq 0 \).

For nonlinear NI systems, Assumption I corresponds to the property of linear NI systems stated in Lemma 2 in [1].

Consider a MIMO nonlinear OSNI system \( H_2 \) with the following state-space model:

\[
\begin{align*}
\dot{x}_2(t) &= f_2(x_2(t), u_2(t)); \\
y_2(t) &= h_2(x_2(t)) + D_2 u_2(t),
\end{align*}
\]

where \( f_2: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a Lipschitz continuous function, \( h_2: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a class \( C^1 \) function and \( D_2 \in \mathbb{R}^{m \times m} \) is a symmetric matrix; i.e., \( D_2 = D_2^T \).

For the system \( H_2 \) with the state-space model (7), (8), we suppose that the following assumption is satisfied.

Assumption II: When the system \( H_2 \) is in steady state; i.e., \( u_2(t) \equiv \bar{u}_2, \ x_2(t) \equiv \bar{x}_2 \) and \( y_2(t) \equiv \bar{y}_2 \), we have \( \bar{u}_2^T \bar{y}_2 \leq -\gamma |\bar{u}_2| |\bar{y}_2| \) with \( \gamma > 0 \).

It might be observed that as nonlinear OSNI systems belong to a subclass of nonlinear NI systems, Assumption II seems
to have a conflicting relationship with Assumption I. In fact, Assumption II can be satisfied because of the term $D_2 u_2(t)$ in the output equation (6), and it corresponds to the inequality (61) for linear NI systems in [13].

In addition, both of the systems $H_1$ and $H_2$ are assumed to satisfy the following assumptions. For the system $H_1$ with input $u_1(t)$, state $x_1(t)$ and output $y_1(t) = h_1(x_1(t))$ described by the state-space model (3), (4) and the system $H_2$ with input $u_2(t)$, state $x_2(t)$ and the auxiliary output $\hat{y}_2(t) = h_2(x_2(t))$ described by the state-space model (7), (8), we suppose for $i = 1$ and 2, the following conditions are satisfied.

Assumption III: Over any time interval $[t_a, t_b]$ where $t_b > t_a$, $h_i(x_i(t))$ remains constant if and only if $x_i(t)$ remains constant; i.e., $\dot{h}_i(x_i(t)) \equiv 0 \iff \dot{x}_i(t) \equiv 0$. Moreover, $h_i(x_i(t)) \equiv 0 \iff x_i(t) \equiv 0$.

Assumption IV: Over any time interval $[t_a, t_b]$ where $t_b > t_a$, $x_i(t)$ remains constant only if $u_i(t)$ remains constant; i.e., $x_i(t) \equiv \bar{x}_i \implies u_i(t) \equiv \bar{u}_i$. Moreover, $x_i(t) \equiv 0 \implies u_i(t) \equiv 0$.

In the case of linear systems, Assumption III corresponds to observability and Assumption IV corresponds to the $B$ matrix in the realisation $(A, B, C, D)$ of the linear system having full column rank.

**Theorem 1:** Consider the closed-loop positive feedback interconnection of the system $H_1$ with state-space model (3), (4) and $H_2$ with state-space model (7), (8), as shown in Fig. 1. Suppose that Assumptions I-IV are satisfied, and the storage function, defined as

$$W(x_1, x_2) := V_1(x_1) + V_2(x_2) - h_1(x_1)^T h_2(x_2)$$

$$- \frac{1}{2} h_1(x_1)^T D_2 h_1(x_1),$$

(9)

is positive definite, where $V_1(x_1)$ and $V_2(x_2)$ are positive semidefinite storage functions that satisfy (3) for the system $H_1$ and (4) for the system $H_2$, respectively. Then, the closed-loop interconnection of the systems $H_1$ and $H_2$ is asymptotically stable.

**Proof:** According to the nonlinear NI property (3) for the system $H_1$, the nonlinear OSNI property (4) for the system $H_2$ and the system setting $u_1(t) \equiv y_2(t)$ and $u_2(t) \equiv y_1(t)$ in Fig. 1, we have

$$\dot{V}_1(x_1) \leq u_1^T \dot{y}_1$$

$$= y_1^T \dot{y}_1$$

$$= [h_2(x_2) + D_2 u_2]^T \dot{h}_1(x_1)$$

$$= [h_2(x_2) + D_2 y_1]^T \dot{h}_1(x_1)$$

$$= [h_2(x_2) + D_2 h_1(x_1)]^T \dot{h}_1(x_1),$$

(10)

and

$$\dot{V}_2(x_2) \leq u_2^T \dot{y}_2 - \epsilon |\dot{y}_2|^2$$

$$= y_2^T \dot{y}_2 - \epsilon |\dot{y}_2|^2$$

$$= h_1(x_1)^T \dot{h}_2(x_2) - \epsilon |\dot{h}_2(x_2)|^2.$$  

(11)

We obtain the time derivative of the storage function $W(x_1, x_2)$ in (9) using (10) and (11):

$$\dot{W}(x_1, x_2) = \dot{V}_1(x_1) + \dot{V}_2(x_2) - h_1(x_1)^T h_2(x_2)$$

$$- h_1(x_1)^T h_2(x_2) - h_1(x_1)^T D_2 h_1(x_1)$$

$$\leq [h_2(x_2) + D_2 h_1(x_1)]^T \dot{h}_1(x_1)$$

$$+ h_1(x_1)^T h_2(x_2) - \epsilon |\dot{h}_2(x_2)|^2$$

$$- \dot{h}_1(x_1)^T h_2(x_2) - h_1(x_1)^T D_2 h_1(x_1)$$

$$= - \epsilon |\dot{h}_2(x_2)|^2$$

$$\leq 0.$$  

(12)

From this it follows that $\dot{W}(x_1, x_2) = 0$ is only possible when $\dot{h}_2(x_2) = 0$. Hence, $\dot{W}(x_1, x_2)$ can remain zero only if $h_2(x_2)$ remains zero; i.e., $\dot{W}(x_1, x_2) = 0 \implies \dot{h}_2(x_2) = 0$. According to Assumptions III and IV, $\dot{h}_2(x_2(t)) = 0 \implies \dot{x}_2(t) = 0 \implies u_2(t) = \bar{u}_2$. Hence, the system $H_2$ is in steady-state. According to the system setting in Fig. 1, $y_1(t) = u_2(t)$. Hence, using Assumptions III and IV, $\dot{y}_1(t) = 0 \implies \dot{x}_1(t) = 0 \implies \dot{u}_1(t) = \bar{u}_1$. Thus, the system $H_1$ is also in steady-state. Then, according to Assumption II, we have

$$u_2^T \dot{y}_2 \geq -\gamma \bar{u}_2^2.$$  

(13)

If $\bar{u}_2 = 0$, then $u_2^T \dot{y}_2 = 0$. According to Assumptions III and IV, and the system setting in Fig. 1, $\bar{u}_2 = 0 \implies \dot{y}_2 = 0 \implies \dot{x}_2 = 0$. Hence, in this case, the system is in equilibrium. Otherwise, if $\bar{u}_2 \neq 0$, we have

$$u_2^T \dot{y}_2 < 0.$$  

(14)

Also, according to Assumption I, we have

$$\bar{u}_1 \dot{y}_1 \geq 0.$$  

(15)

According to the system setting in Fig. 1, we have $\bar{u}_1 = \bar{y}_2$ and $\dot{y}_1 = \dot{\bar{y}}_2$. Hence, (14) can be rewritten as

$$\bar{u}_2^T \dot{y}_2 \geq 0,$$

which contradicts (13). Thus, we can conclude that $\dot{W}(x_1, x_2)$ cannot remain zero unless $x_1 = x_2 = 0$. Thus, according to LaSalle’s invariance principle, $W(x_1, x_2)$ will keep decreasing until $W(x_1, x_2) = 0$. Hence, the equilibrium at $(x_1, x_2) = (0, 0)$ of the closed-loop interconnection is asymptotically stable.
III. ROBUST OUTPUT FEEDBACK CONSENSUS

Consider $N$ heterogeneous nonlinear systems $H_{p_i}$ ($i = 1, 2, \cdots, N$) described as

\begin{align*}
\dot{x}_{p_i}(t) &= f_{p_i}(x_{p_i}(t), u_{p_i}(t)); \quad (15) \\
y_{p_i}(t) &= h_{p_i}(x_{p_i}(t)), \quad (16)
\end{align*}

where $f_{p_i} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are Lipschitz continuous functions and $h_{p_i} : \mathbb{R}^n \to \mathbb{R}^m$ are class $C^1$ functions. These systems operate independently in parallel and each of them has its own input $u_{p_i} \in \mathbb{R}^m$ and output $y_{p_i} \in \mathbb{R}^m$, $(i = 1, 2, \cdots, N)$, which is shown in Fig. 2. The subscript “$p$” indicates that this system will play the role of a plant in what follows. We combine the inputs and outputs respectively as the vectors $U_p = [u^T_{p1}, u^T_{p2}, \cdots, u^T_{pN}] \in \mathbb{R}^{Nm \times 1}$ and $Y_p = [y^T_{p1}, y^T_{p2}, \cdots, y^T_{pN}]^T = [h_{p1}(x_{p1})^T, h_{p2}(x_{p2})^T, \cdots, h_{pN}(x_{pN})^T]^T \in \mathbb{R}^{Nm \times 1}$, respectively.

![Fig. 2. System $H_p$: a nonlinear system consisting of $N$ independent and heterogeneous nonlinear NI systems $H_{p_i}$ ($i = 1, 2, \cdots, N$), with independent inputs and outputs combined as the input and output of the networked system $H_p$.](image)

Let us consider the networked plants connected according to the graph network topology $H_p$ as shown in Fig. 3, where $Q$ is the incidence matrix of a directed graph that represents the communication links between the heterogeneous nonlinear NI plants.

![Fig. 3. Heterogeneous nonlinear NI plants connected according to the directed graph network topology.](image)

For the system $\hat{H}_p$ shown in Fig. 3, we have the following lemma:

**Lemma 1:** If the systems $H_{p_i}$ are nonlinear NI systems for all $i = 1, 2, \cdots, N$, then the system $\hat{H}_p$ is also a nonlinear NI system.

**Proof:** According to Definition 1, each nonlinear NI system $H_{p_i}$ ($i = 1, 2, \cdots, N$) must have a corresponding positive semidefinite storage function $V_{p_i}(x_{p_i})$ such that $V_{p_i}(x_{p_i}) \leq u_{p_i}^T y_{p_i}$, where $x_{p_i}$ is the state of the system $H_{p_i}$. We define the storage function for the system $\hat{H}_p$ as $\dot{V}_p = \sum_{i=1}^{N} V_{p_i}(x_{p_i})$, which is positive semidefinite. Then

\[
\dot{V}_p = \sum_{i=1}^{N} V_{p_i}(x_{p_i}) \leq \sum_{i=1}^{N} u^T_{p_i} y_{p_i} = U_p^T Y_p.
\]

Let $\hat{U}_p$ and $\hat{Y}_p$ denote the input and output of the system $\hat{H}_p$, respectively. According to the system setting in Fig. 3, we have

\[
U_p = (Q^T \otimes I_m) \hat{U}_p, \quad \text{and} \quad \hat{Y}_p = (Q \otimes I_m) Y_p.
\]

Therefore, we have

\[
U_p^T Y_p = [(Q^T \otimes I_m) \hat{U}_p]^T Y_p = \hat{U}_p^T (Q \otimes I_m) Y_p = \hat{U}_p^T \hat{Y}_p.
\]

According to (17) and (18), we obtain the nonlinear NI inequality for the system $H_p$:

\[
\dot{\hat{V}}_p \leq \hat{U}_p^T \hat{Y}_p.
\]

Therefore, $\hat{H}_p$ is a nonlinear NI system.

Now we give a definition of output feedback consensus for a network of systems as shown in Fig. 2.

**Definition 3:** A distributed output feedback control law achieves output feedback consensus for a network of systems if $|y_{p_i}(t) - y_{p_j}(t)| \to 0$ as $t \to +\infty$, $\forall i, j \in \{1, 2, \cdots, N\}$.

Consider a series of heterogeneous nonlinear OSNI systems $H_{ck}$ ($k = 1, 2, \cdots, l$) applied as controllers corresponding to the edges in the network. The OSNI controllers have the following state-space models:

\begin{align*}
\dot{x}_{ck}(t) &= f_{ck}(x_{ck}(t), u_{ck}(t)); \quad (20) \\
y_{ck}(t) &= h_{ck}(x_{ck}(t)) + D_{ck} u_{ck}(t), \quad (21)
\end{align*}

where $f_{ck} : \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}^q$ are Lipschitz continuous functions, $h_{ck} : \mathbb{R}^q \to \mathbb{R}$ are class $C^1$ functions and $D_{ck} \in \mathbb{R}^{m \times q}$ are symmetric matrices. These systems operate independently in parallel and each of them has its own input $u_{ck} \in \mathbb{R}$ and output $y_{ck} \in \mathbb{R}$, $k = 1, 2, \cdots, l$, which is shown in Fig. 4. We combine the inputs and outputs respectively as the vectors $U_c = [u^T_{c1}, u^T_{c2}, \cdots, u^T_{cl}] \in \mathbb{R}^{lm \times 1}$ and $Y_c = [y^T_{c1}, y^T_{c2}, \cdots, y^T_{cl}]^T = \Pi_c + D_c U_c \in \mathbb{R}^{lm \times 1}$, where

\[
\Pi_c = \begin{bmatrix} h_{c1}(x_{c1}) \\ h_{c2}(x_{c2}) \\ \vdots \\ h_{cl}(x_{cl}) \end{bmatrix} \in \mathbb{R}^{lm \times 1}, \quad (22)
\]

and

\[
D_c = \text{diag}\{D_{c1}, D_{c2}, \cdots, D_{cl}\} \in \mathbb{R}^{lm \times lm}. \quad (23)
\]

For the system $H_c$, we have the following lemma.

**Lemma 2:** If the systems $H_{ck}$ are nonlinear OSNI systems for all $k = 1, 2, \cdots, l$, then the system $H_c$ is a nonlinear OSNI system.

**Proof:** For every nonlinear OSNI system $H_{ck}$, we have a positive semidefinite storage function $V_{ck}(x_{ck})$ and a constant $\epsilon_k > 0$ such that

\[
V_{ck}(x_{ck}) \leq u^T_{ck} y_{ck} - \epsilon_k |y_{ck}|^2, \quad (24)
\]

where $x_{ck}$ and $y_{ck}$ are the state and output of the system $H_{ck}$, respectively.
where \( \tilde{y}_{ck} = h_{ck}(x_{ck}) \) and \( \epsilon_k \) is the level of output strictness of the system \( H_{ck} \). For the system \( \mathcal{H}_c \), we define its storage function \( V_c \) as the sum of the storage functions of all the networked controllers; i.e.,

\[
V_c := \sum_{k=1}^{l} V_{ck}(x_{ck}),
\]

which is positive semidefinite. The time derivative of \( V_c \) is:

\[
\dot{V}_c = \sum_{k=1}^{l} \dot{V}_{ck}(x_{ck}) \\
\leq \sum_{k=1}^{l} u_{T_k} \tilde{y}_{ck} - \sum_{k=1}^{l} \epsilon_k |\tilde{y}_{ck}|^2 \\
\leq \sum_{k=1}^{l} u_{T_k} \tilde{y}_{ck} - \epsilon_{min} \sum_{k=1}^{l} |\tilde{y}_{ck}|^2 \\
= U_c^T \Pi_c - \epsilon_{min} ||\Pi_c||^2, \tag{25}
\]

where \( \epsilon_{min} = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_l\} \). Hence, the system \( \mathcal{H}_c \) satisfies the definition of a nonlinear OSNI system and \( \epsilon_{min} \) quantifies a level of output strictness of the system. This completes the proof.

Now consider the closed-loop positive feedback interconnection of the networked plants shown in Fig. 3 and the networked controllers shown in Fig. 4 which is depicted in Fig. 5. In this paper, robust output consensus of heterogeneous nonlinear NI plants is achieved by constructing a control system with the block diagram shown in Fig. 5 and choosing suitable controllers that satisfy certain conditions. The control framework in Fig. 5 is very similar to the framework used in [33]. The connections between the plants and controllers can be best visualised using the undirected graph, as shown in the example in Fig. 6.

The nodes \( p_i \) (\( i = 1, \ldots, 5 \) in this example) represent the heterogeneous nonlinear NI plants, while the heterogeneous nonlinear OSNI controllers \( c_k \) (\( k = 1, \ldots, 5 \) in this example) correspond to the edges. Given any directed graph corresponding to the graph in Fig. 6 with the incidence matrix \( Q \), each edge will have a direction. Also, each edge corresponds to a controller and two plants. The corresponding connection between the plants and the controller is as shown in Fig. 7. The controller takes the difference between the outputs of the plants as its input and feeds back its output to the plants with a positive or negative sign corresponding to the edge direction. Each plant takes the sum of all the outputs of the controllers connected to it as its input with corresponding signs.

For the system \( \hat{\mathcal{H}}_p \) with input \( \hat{U}_p(t) \) and output \( \hat{Y}_p(t) \), we make the following assumption.

Assumption V: Given a constant input \( \hat{U}_p(t) \equiv \hat{U}_p \) to the system \( \hat{\mathcal{H}}_p \), if its output is also constant; i.e., \( \hat{Y}_p(t) \equiv \hat{Y}_p \), then \( \hat{U}_p \) and \( \hat{Y}_p \) satisfy \( \hat{U}_p^T \hat{Y}_p \geq 0 \).

In fact, Assumption I implies Assumption V in the case that all the plants \( H_{pi} \) are in steady state. This is because when all the plants satisfy Assumption I and are in steady state, we have \( \hat{U}_p^T \hat{Y}_p \geq 0 \) and according to the system setting in Fig. 5 we have \( \hat{U}_p^T \hat{Y}_p = (Q^T \otimes I_m)^T \hat{U}_p^T \hat{Y}_p = \hat{U}_p^T (Q \otimes I_m) \hat{Y}_p = \hat{U}_p^T \hat{Y}_p \) similarly to (18). Hence \( \hat{U}_p^T \hat{Y}_p \geq 0 \). However, Assumption V is assumed for the networked systems \( \mathcal{H}_p \) in the following theorem instead of assuming Assumption I for each individual plant because Assumption V also allows for the situation in which the input and output of the system \( \hat{\mathcal{H}}_p \) are constant, but the individual plants \( H_{pi} \) are not all in steady state. This situation is possible because the matrix \( Q \otimes I_m \) makes the difference between the outputs of the plants.
Under constant inputs, if the plants oscillate with a constant difference between their outputs, then this situation is allowed under Assumption V.

**Theorem 2:** Consider an undirected connected graph $G$ that models the communication links for a network of heterogeneous nonlinear NI systems $H_{pi}$ ($i = 1, 2, \cdots, N$) as shown in Fig. 2 and any directed graph corresponding to $G$ with the incidence matrix $Q$. Also, consider the heterogeneous nonlinear OSNI control laws $H_{ck}$ ($k = 1, 2, \cdots, l$) for all of the edges. Suppose Assumptions III and IV are satisfied for the plants $H_{pi}$, Assumptions II, III and IV are satisfied for the controllers $H_{ck}$ (with $\gamma = \gamma_k$ for the controller $H_{ck}$ in Assumption II) and Assumption V is satisfied by the system $H_{p}$. Also, suppose the storage function, defined as

\[
\dot{W} := \dot{V}_p + V_c - \dot{Y}_p^T \Pi_c - \frac{1}{2} \dot{Y}_p^T D_c \dot{Y}_p,
\]

is positive definite, where $\dot{V}_p$ and $V_c$ are positive semidefinite storage functions that satisfy (19) for the system $H_p$ and (25) for the system $H_c$, respectively. Here, $\dot{Y}_p$ is the output of the system $H_p$. $\Pi_c$ and $D_c$ are terms in the output $Y_c$ of the system $H_c$ and are defined in (22) and (23). Then robust output feedback consensus can be achieved via the protocol

\[
U_p = (Q^T \otimes I_m)H_c((Q \otimes I_m)Y_p),
\]

or equivalently,

\[
u_{pi} = \sum_{k=1}^{l} q_{ki} H_{ck} \left( \sum_{j=1}^{N} q_{kj} y_{pj} \right)
\]

for each plant $p_i$, as shown in Fig. 3 where $\sum_{j=1}^{N} q_{kj} y_{pj}$ represents the difference between the outputs of the plants connected by the edge $e_k$.

**Proof:** According to (19), (25) and the system setting $\dot{U}_p \equiv Y_c$ and $U_c \equiv \dot{Y}_p$ shown in Fig. 6 we have

\[
\dot{V}_p \leq \dot{Y}_p^T \dot{Y}_p = Y_p^T [\Pi_c + D_c U_c] = \dot{Y}_p^T [\Pi_c + D_c \dot{Y}_p],
\]

and

\[
\dot{V}_c \leq \dot{U}_p^T \Pi_c - \epsilon_{min} |\dot{\Pi}_c|^2 = \dot{Y}_p^T \dot{\Pi}_c - \epsilon_{min} |\dot{\Pi}_c|^2.
\]

According to (26), (27) and the symmetry of $D_c$ in (23), the time derivative of the storage function $W$ satisfies the following inequality:

\[
\dot{W} = \dot{V}_p + V_c - \dot{Y}_p^T \Pi_c - \dot{Y}_p^T \dot{\Pi}_c - \frac{1}{2} \dot{Y}_p^T (D_c + D_c^T) \dot{Y}_p \\
\leq \dot{Y}_p^T [\Pi_c + D_c \dot{Y}_p] + \dot{Y}_p^T \dot{\Pi}_c - \epsilon_{min} |\dot{\Pi}_c|^2 - \dot{Y}_p^T \Pi_c \\
- \dot{Y}_p^T \dot{\Pi}_c - \dot{Y}_p^T D_c \dot{Y}_p \\
\leq \epsilon_{min} |\dot{\Pi}_c|^2 \\
\leq 0.
\]

Hence, $H_{ck}$ is in steady-state for all $k = 1, 2, \cdots, l$. We have $U_c(t) = \dot{U}_c$ and $Y_c(t) \equiv \dot{Y}_c$. According to the system setting in Fig. 5 that $\dot{U}_p(t) \equiv Y_c(t)$ and $U_c(t) \equiv \dot{Y}_p(t)$, we also have $\dot{U}_p(t) \equiv \dot{U}_p$ and $\dot{Y}_p(t) \equiv \dot{Y}_p$. According to Assumption V, we have

\[
\dot{U}_p^T \dot{Y}_p \geq 0.
\]

According to Assumption II, we have

\[
\dot{U}_c^T Y_c = \sum_{k=1}^{l} \bar{u}_{ck}^T y_{ck} \leq \sum_{k=1}^{l} |\gamma_k| |\bar{u}_{ck}|^2 \leq -\gamma_{min} |\dot{U}_c|^2;
\]

where $\gamma_{min} = \min\{\gamma_1, \gamma_2, \cdots, \gamma_l\}$. In the case that $\dot{U}_c \neq 0$, we have

\[
\dot{U}_c^T Y_c < 0
\]

which contradicts (29) because $\dot{U}_c^T Y_c = \dot{U}_c^T \dot{Y}_p$. In the case that $\dot{U}_c = 0$, all connected plants have the difference between their system outputs being zero. Hence, output consensus has already been achieved. Otherwise, $\dot{W}$ cannot remain zero. According to LaSalle’s invariance principle, $W$ will keep decreasing until either $\dot{U}_c = 0$ or $W = 0$. Thus, output consensus is achieved in both cases. This completes the proof.

**IV. ILLUSTRATIVE EXAMPLE**

**Fig. 8.** An undirected and connected graph consisting of three nodes.

Consider three nonlinear NI plants connected by the graph shown in Fig. 8. We choose the directions of the edges as $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_3)$. Then the incidence matrix of the directed graph corresponding to $G$ is $Q = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

The plants are nonlinear single integrators which have the following state-space models:

Plant $i$ ($i = 1, 2, 3$): \begin{align*}
\dot{x}_{pi}(t) &= \mu_1 u_{pi}^3(t); \\
y_{pi}(t) &= x_{pi}(t).
\end{align*}

where $\mu_4$ is a constant coefficient and $\mu_1 = 1, \mu_2 = 3$ and $\mu_3 = 2$. The storage functions for these three plants are all $V_{pi}(x_{pi}) = 0$. The states $x_{p1}, x_{p2}$ and $x_{p3}$ of the three plants have initial values $30, 2$ and $-8$, respectively. Two different nonlinear OSNI controllers are applied to achieve output feedback consensus. The state-space models of the controllers are:

Controller 1:

\[
\begin{align*}
\dot{x}_{c1}(t) &= -5x_{c1}(t) - 3x_{c1}^3(t) + u_{c1}(t); \\
y_{c1}(t) &= x_{c1}(t) - u_{c1}(t); \\
\end{align*}
\]

Controller 2:

\[
\begin{align*}
\dot{x}_{c2}(t) &= -8x_{c2}(t) - 2x_{c2}^3(t) + u_{c2}(t); \\
y_{c2}(t) &= x_{c2}(t) - u_{c2}(t). \\
\end{align*}
\]
The storage functions of Controller 1 and Controller 2 are $V_{c1} = \frac{5}{2}x_{1}^2 + \frac{3}{4}x_{1}^4$ and $V_{c2} = 4x_{2}^2 + \frac{1}{2}x_{2}^4$, respectively.

The system $\mathcal{H}_{c1}$ in this example is:

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{u}_1^3 - 3(-\hat{u}_1 + \hat{u}_2)^3; \\
\dot{\hat{x}}_2 &= (\hat{u}_1 + \hat{u}_2)^3 + 2\hat{u}_2^4; \\
\hat{y}_1 &= \hat{x}_1; \\
\hat{y}_2 &= \hat{x}_2.
\end{align*}
\]

The storage function of the entire system is

\[
\dot{\mathcal{W}} = \frac{5}{2}x_{c1}^2 + \frac{3}{4}x_{c1}^4 + 4x_{c2}^2 + \frac{1}{2}x_{c2}^4 - \hat{x}_1x_{c1}
- \hat{x}_2x_{c2} + \frac{1}{2}\hat{x}_1^2 + \frac{1}{2}\hat{x}_2^2
= \begin{bmatrix} \hat{x}_1 \\ x_{c1} \end{bmatrix}
\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} \\
\end{bmatrix}
\begin{bmatrix} \hat{x}_1 \\ x_{c1} \end{bmatrix}
+ \begin{bmatrix} \hat{x}_2 \\ x_{c2} \end{bmatrix}
\begin{bmatrix}
\frac{1}{2} & -\frac{3}{4} \\
-\frac{3}{4} & \frac{1}{2} \\
\end{bmatrix}
\begin{bmatrix} \hat{x}_2 \\ x_{c2} \end{bmatrix}
+ \frac{3}{4}x_{c1}^4 + \frac{1}{2}x_{c2}^4,
\]

which is positive definite. Assumptions II-V are satisfied. Output feedback consensus is achieved, as shown in Fig. 9. Because of the cubic nonlinearity in the plants, their outputs converge quickly at first and then slowly when they are close to the limit value. Therefore, a log scale is used for the time axis in the plot shown in Fig. 9.

![Output Feedback Consensus](image)

**Output Feedback Consensus**

- **Plant 1**
- **Plant 2**
- **Plant 3**

Fig. 9. Output feedback consensus for three nonlinear single integrator plants.

V. CONCLUSION

This paper provides a control framework to achieve robust output feedback consensus for networked heterogeneous nonlinear NI systems including systems with free body motion, using nonlinear OSNI controllers. New definitions for nonlinear NI systems and nonlinear OSNI systems are given and a stability result is established for the simple feedback interconnection of a nonlinear NI plant and a nonlinear OSNI controller. A networked control framework is then considered by modelling the communication topology between systems as a connected graph, where the plants are nodes and the controllers are edges. The network of nonlinear NI plants and the network of OSNI controllers are proved to be a nonlinear NI system and a nonlinear OSNI system, respectively. Under reasonable assumptions, output feedback consensus is established for the networked heterogeneous nonlinear NI systems, and the result is robust against variations in the system models of both the plants and controllers provided that the relevant nonlinear NI and nonlinear OSNI properties are preserved. Finally, an example is given to demonstrate the proposed result on a consensus problem to which earlier results are not applicable.

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