UNIQUE CONTINUATION ON CONVEX DOMAINS

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Abstract: In this paper, we obtain estimates on the quantitative strata of the critical set of non-trivial harmonic functions $u$ which vanish continuously on $V \subset \partial \Omega$, a relatively open subset of the boundary of a convex domain $\Omega \subset \mathbb{R}^n$. In particular, these estimates improve dimensional estimates on $\{|\nabla u| = 0\}$ both in $V \subset \partial \Omega$ and as it approaches $V \cap \Omega$. These estimates are not obtainable by naively combining interior and boundary estimates and represent a significant improvement upon existing results for boundary analytic continuation in the convex case.

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1. Introduction

Unique continuation is a fundamental property for functions which solve the Laplace and related linear equations. A closely related problem is that of boundary unique continuation: given a domain $\Omega \subset \mathbb{R}^n$ and a function $u$ which is harmonic in $\Omega$ and vanishes continuously on $V \subset \partial \Omega$, how large can the set $\{Q \in V : |\nabla u| = 0\}$ be if $u \neq 0$? Boundary unique continuation is closely tied to the Cauchy problem and questions of well-posedness and stability of solutions to boundary value problems (see, for instance, [Tat03] and [ARRV09]).

In this paper, we address two questions. First, we address the question of boundary unique continuation for harmonic functions on convex domains. Second, we also address the question of how the critical set $\{|\nabla u| = 0\} \cap \Omega$ approaches $V \subset \partial \Omega$. We follow the approach of Garofalo and Lin [GL87] insofar as we make essential use of the Almgren frequency function. And, because we want to obtain results on the full critical set $\{|\nabla u| = 0\}$, we use packing estimates inspired by Cheeger, Naber, and Valtorta [CNV15]. These tools allow us to obtain results the strata of the critical set $\{p \in \Omega : |\nabla u(p)| = 0\}$ as it approaches $V \subset \partial \Omega$. 
1.1. Background on Boundary Unique Continuation for Harmonic Functions. For dimensions $n \geq 3$, Bourgain and Wolff [BW90] have constructed an example of a function, $u : \mathbb{R}^n_+ \to \mathbb{R}$, which is harmonic in $\mathbb{R}^n_+ \setminus C^1$ up to the boundary $\mathbb{R}^{n-1} \subset \mathbb{R}^n$, and for which both $u$ and $\nabla u$ vanish on a set of positive surface measure. This result has been generalized by Wang [Van95] to $C^{1,\alpha}$ domains, $\Omega \subset \mathbb{R}^n$, for $n \geq 3$. However, the sets of positive measure for which these functions vanish are not open.

In general, the following question posed by Lin in [Lin91] is still open.

**Question 1.1.** Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an open, connected Lipschitz domain. If $u$ is a harmonic function which vanishes continuously on a relatively open set $V \subset \partial \Omega$, does

$$\mathcal{H}^{n-1}(\{x \in V : |\nabla u| = 0\}) > 0$$

imply that $u$ is the zero function?

If $u$ is non-negative, the techniques of PDEs on non-tangentially accessible (NTA) domains give a comparison principle [Dah77], which allows us to say that the norm of the normal derivative is point-wise comparable to the density of the harmonic measure with respect to the surface measure $d\sigma$. Additionally, for Lipschitz domains it is well-known that the harmonic measure is mutually absolutely continuous with respect to $d\sigma$. These two facts then imply that if the normal derivative vanishes on a set of positive (surface) measure, then $u$ must be identically 0.

The challenge is for harmonic functions $u$ which change sign. For such functions, the aforementioned techniques fail completely because we cannot apply the Harnack principle. Authors have therefore approached this problem by asking for additional regularity. In [Lin91], Lin proves that for $C^{1,1}$ domains, $\Omega \subset \mathbb{R}^n$, for $n \geq 2$, if $u$ is a non-constant harmonic function which vanishes on an open set $V \cap \partial \Omega$, then $\dim_{\mathcal{H}}(\{x \in V \cap \partial \Omega : |\nabla u| = 0\}) \leq n - 2$. Similar results were later shown by Adolfsson and Escauriaza for domains with locally $C^{1,\alpha}$ boundary [AE97]. Relatedly, Kukavica and Nystöm showed that $\mathcal{H}^{n-1}(\{x \in V : |\nabla u| = 0\}) > 0$ implies that $u \equiv 0$ if $\partial \Omega$ is $C^1$ Dini [KN98]. Recently, this result has been greatly improved. Kenig and Zhou [KZ21] employed powerful techniques from [NV17] and [dLMSV16], have shown that for $C^1$ Dini domains, the $(n-2)$-generalized singular set $\{u = 0 = |\nabla u|\}$ has finite $(n-2)$-dimensional upper Minkowski content.

For merely convex domains $\Omega \subset \mathbb{R}^n$, Adolfsson, Escauriaza, and Kenig showed that if $u$ is a harmonic function in $\Omega$ which vanishes continuously on a relatively open set $V \subset \partial \Omega$, then if $\{x \in V \cap \partial \Omega : |\nabla u| = 0\}$ has positive surface measure, $u$ must be a constant function [AEK95]. The method of attack pursued in [AEK95] (and [Lin91], [AE97], [KN98]) was centered on showing that the harmonic function is “doubling” on the boundary in the following sense. If $\Omega \subset \mathbb{R}^n$, then there exists an absolute constant $M < \infty$ such that for all $B_{2r}(Q) \cap \partial \Omega \subset V$

$$\int_{B_{2r}(Q) \cap \Omega} u^2 \, dx \leq M \int_{B_{r}(Q) \cap \Omega} u^2 \, dx.$$

This doubling property allows the authors to show that the normal derivative is an $A_2$-Muckenhoupt weight with respect to surface measure, a kind of quantified version of mutual absolute continuity. It is well-known that if $u$ vanishes in a surface ball $\Delta_r(Q)$ and the normal derivative of $u$ is a 2-weight with respect to surface measure, then either $\{Q' \in \Delta_r(Q) : |\nabla u| = 0\}$ has measure zero or $\{Q' \in \Delta_r(Q) : |\nabla u| > 0\}$ has measure zero. The
improvement from measure to dimension bounds in [AE97] and [Lin91] comes from applying an additional Federer dimension-reduction type argument.

Recently, Tolsa has answered Question 1.1 in the affirmative for $C^1$ domains and Lipschitz domains with small Lipschitz constant [Tol20].

In this paper, we restrict our investigation to convex domains $\Omega$ and obtain bounds upon the full generalized critical set $\{|\nabla u| = 0\}$. We note that this includes $\{x \in \partial \Omega : |\nabla u| = 0\}$ and $\{x \in \Omega : |\nabla u| = 0\}$.

Theorem 1.2. Let $\Omega$ be a convex domain and $u \in C^0(\overline{\Omega})$ be a non-constant function such that $\Delta u = 0$ in $\Omega$. Let $V \subset \partial \Omega$ be a relatively open set. If $u = 0$ on $V$, then for any compact subset $K \subset V$, there exists a radius $0 < r(K)$ such that

$$\dim \mathcal{H}(B_r(K) \cap \{|\nabla u| = 0\} \setminus \text{sing}(\partial \Omega)) \leq \dim \mathcal{M}(B_r(K) \cap \{|\nabla u| = 0\} \setminus \text{sing}(\partial \Omega)) \leq n - 2.$$ 

Furthermore

$$\overline{\dim \mathcal{M}}(B_r(K) \cap \{|\nabla u| = 0\} \cap \overline{\Omega}) \leq n - 2.$$

The content of Theorem 1.2 is two-fold. First, consider the results restricted to the boundary $\{|\nabla u| = 0\} \cap V \subset \partial \Omega$. Alberti [Alb94] proved (among other things) that the singular set of a convex function is a $C^2$ $(n - 2)$-rectifiable set, which implies that the geometric singular set of a convex body satisfies $\dim \mathcal{H}(\text{sing}(\partial \Omega)) \leq n - 2$. Thus, Theorem 1.2 combined with [Alb94] implies

$$\dim \mathcal{H}(B_r(K) \cap \{|\nabla u| = 0\}) \leq n - 2,$$

which gives a strong improvement on the results of [AEK95], which proved that in this situation $\mathcal{H}^{n-1}(V \cap \{|\nabla u| = 0\}) = 0$.

Second, Theorem 1.2 provides new insight into how the critical set interacts with $\partial \Omega$. Returning to [Alb94], Alberti also proved that the singular set of a convex function may be prescribed to be any $C^2$ $(n - 2)$-rectifiable set. Thus, it can happen that $\overline{\dim \mathcal{M}}(\text{sing}(\partial \Omega)) = n - 1$. On the other hand, from the interior perspective, [NV14] proved finite $(n - 2)$-dimensional upper Minkowski content bounds on $\{|\nabla u| = 0\}$ in the interior. But, naive application of these estimates degenerate as one approaches the boundary because upper Minkowski dimension is not stable under countable unions. Considering $\{|\nabla u| = 0\} \cap \Omega$ as it approaches $\partial \Omega$, it was unknown whether or not $\{|\nabla u| = 0\} \cap \Omega$ could oscillate wildly and have positive $(n - 1)$-upper Minkowski dimension like $\text{sing}(\partial \Omega)$, would remain $(n - 2)$-upper Minkowski dimensional like the interior, or if something in between these two held. Theorem 1.2 proves that the set $\{|\nabla v| = 0\} \cap \Omega$ cannot oscillate too wildly as it approaches $\partial \Omega$ and $\{|\nabla u| = 0\} \cap \Omega \cap K$ inherits its upper Minkowski dimension bounds from the interior rather than the boundary.

It is still an open question whether or not the $(n - 2)$-upper Minkowski content of $\{|\nabla u| = 0\} \cap \partial \Omega \setminus \text{sing}(\partial \Omega)$ is finite.

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2. Definitions and Main Results

Theorem 1.2 is a corollary to the Technical Theorem (Theorem 2.10) and a containment result (Lemma 2.11). In order to state these results, we need the following definitions.

We define the following class of domains.

**Definition 2.1. (A normalized class of convex domains)** Let $D(n)$ be the collection of connected, open domains $\Omega \subset \mathbb{R}^n$ which satisfy the following conditions:

1. $0 \in \partial \Omega$.
2. $\Omega \cap B_2(0)$ is convex.
3. $\Omega \cap (B_2(0))^c \neq \emptyset$.

One of the key tools of this paper will be an Almgren frequency function, introduced by Almgren in [Alm79].

**Definition 2.2. (Almgren frequency function)** Let $r > 0$, $\Omega \subset \mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}$, such that $u \in C(B_{2r}(0)) \cap W^{1,2}(B_{2r}(p))$ and $p \in \Omega$. We define the following quantities:

$$H_{\Omega}(p, r, u) := \int_{\partial B_r(p)} |u - u(p)|^2 d\sigma.$$  
$$D_{\Omega}(p, r, u) := \int_{B_r(p)} |\nabla u|^2 dx.$$  
$$N_{\Omega}(p, r, u) := \frac{r D_{\Omega}(p, r, u)}{H_{\Omega}(p, r, u)}.$$

**Remark 2.3. (Invariances of the Almgren frequency function)** This normalized version of the Almgren frequency function is invariant in the following senses. Let $a, b, c \in \mathbb{R}$ with $a, r \neq 0$. If $w(x) = au(bx + p) + c$ and $T_{p,b,\Omega} = \frac{1}{b}(\Omega - p)$ then

$$N_{\Omega}(p, r, u) = N_{T_{p,b,\Omega}}(0, b^{-1}r, w).$$

We now define the class of functions in which we will work in this paper.

**Definition 2.4. (A class of functions)** Let $\mathcal{A}(n, \Lambda)$ be the set of functions, $u : \mathbb{R}^n \to \mathbb{R}$, which have the following properties:

1. $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic in a convex domain, $\Omega \in D(n)$.
2. $u \in C(B_2(0))$, $u = 0$ on $\Omega^c \cap B_2(0)$, and $u$ is non-constant.
3. $N_{\Omega}(0, 2, u) \leq \Lambda$.

We shall use rescalings which are adapted to the quantitative stratification methods introduced by Cheeger and Naber in [CN13] for studying the regularity of stationary harmonic maps and minimal currents.

**Definition 2.5. (Rescalings)** Let $u \in \mathcal{A}(n, \Lambda)$, and let $\Omega$ be its associated domain. We define the rescaled function, $T_{x,r}u$ of $u$ at a point $x \in B_1(0)$ at scale $0 < r < 1$ by

$$T_{x,r}u(y) := \frac{u(x + ry) - u(x)}{\left(\int_{\partial B_1(0) \cap \Omega} (u(x + ry) - u(x))^2 d\sigma(y)\right)^{1/2}}.$$

In the case that the denominator is zero, we define $T_{x,r}u = \infty$.

We shall break with established convention and denote the rescalings of $\Omega$ in analogy with the rescaling of functions. Let $T_{p,r,\Omega} := \frac{\Omega - p}{r}$ and $T_{p,r,\partial \Omega} := \frac{\partial \Omega - p}{r}$.
The geometry we wish to capture with the rescalings $T_{x,r}f$ are encoded in their translational symmetries.

**Definition 2.6.** (The class of blow-up profiles) Let $u \in C(\mathbb{R}^n)$. We say $u$ is 0-symmetric if $u$ satisfies one of the following conditions.

1. $u(x) = \phi(x - p) + c$ for some function $\phi$ which is homogeneous and harmonic in a convex cone, $\Omega' \in \mathcal{D}(n)$, some point $p \in \mathbb{R}^n \cap \Omega'$, and some $c \in \mathbb{R}$.

2. $u(x) = (x \cdot \delta - \delta^2) + c$ for some function $\delta$ which is homogeneous and harmonic in a convex cone, $\Omega' \in \mathcal{D}(n)$, some point $p \in \mathbb{R}^n \cap \Omega'$, and some $c \in \mathbb{R}$.

We will say that $u$ is $k$-symmetric if $u$ is 0-symmetric and there exists a $k$-dimensional subspace $V$ such that $u(x + y) = u(x)$ for all $x \in \mathbb{R}^n$ and all $y \in V$.

We now define the quantitative version of symmetry which describes how close to being $k$-symmetric a function is in a ball, $B_r(x) \subset \mathbb{R}^n$.

**Definition 2.7.** (Quantitative symmetry) For any $u \in \mathcal{A}(n, \Lambda)$ with associated domain $\Omega$, $u$ will be called $(k, \epsilon, r, p)$-symmetric if there exists a $k$-symmetric function $P$ such that

1. $\int_{\partial B_1(0)} |P|^2 d\sigma = 1$.
2. $\int_{B_1(0) \cap \Omega} |T_{r, p}u - P|^2 dV < \epsilon$.

We shall say that $u$ is $(k, \delta_0, r, p)$-symmetric with respect to a $k$-dimensional subspace $V$ if there is a $k$-symmetric function $P$ which verifies that $u$ is $(k, \delta_0, r, p)$-symmetric such that $P(x + y) = P(x)$ for all $x \in V$.

**Definition 2.8.** (Quantitative Generalized Critical Strata) Let $u \in \mathcal{A}(n, \Lambda)$ with $\Omega \in \mathcal{D}(n)$ its associated domain. For $0 < \epsilon$, $0 < r \leq 1$, and integer $0 \leq k \leq n - 1$ we denote the $(k, \epsilon, r)$-generalized critical strata of $u$ by $\mathcal{C}^k_{\epsilon, r}(u)$, and we define it by

$$\mathcal{C}^k_{\epsilon, r}(u) := \{x \in \overline{\Omega} : u \text{ is not } (k + 1, \epsilon, s, x)\text{-symmetric for all } s \leq 1\}.$$ We shall also use the notation $\mathcal{C}^k_{\epsilon}(u)$ for $\mathcal{C}^k_{\epsilon, 1}(u)$.

The quantitative strata of the generalized critical set behave very well under $L^2$-convergence of functions. That is, if $u_i \in \mathcal{A}(n, \Lambda)$ and $x_i \in B_1(0) \cap \Omega$ such that $x_i \to x \in B_1(0)$, then if $u_i$ is $(k, \epsilon, r_i, x_i)$-symmetric and $T_{x_{i,1}}u_i \to u_\infty$ in $L^2(B_2(0))$ then $u_\infty$ is $(k, \epsilon, 1, 0)$-symmetric.

As a special case, this implies that the $\mathcal{C}^k_{\epsilon, r}(u)$ are closed.

It is immediate from the definitions that $\mathcal{C}^k_{\epsilon, r}(u) \subset \mathcal{C}^{k'}_{\epsilon', r'}(u)$ if $k \leq k'$, $\epsilon' \leq \epsilon$, $r \leq r'$.

**Definition 2.9.** (Qualitative Generalized Critical Set) Let $u \in \mathcal{A}(n, \Lambda)$ with $\Omega \in \mathcal{D}(n)$ its associated domain. Using the quantitative generalized critical strata, we define the generalized critical set of $u$ $\mathcal{C}^{n-2}(u) := \cup_{n} \mathcal{C}^{n-2}_{\eta, r}(u)$. In turn, we define the strata of the generalized critical set as follows $\mathcal{C}^k(u) := \cup_{\eta} \mathcal{C}^k_{\eta, r}(u)$.

We shall use the convention that for any $A \subset \mathbb{R}^n$, $B_r(A) = \{x \in \mathbb{R}^n : d(A, x) < r\}$. Recall that we can define upper Minkowski s-content by

$$(1) \quad \mathcal{M}^{s, \ast}(A) = \limsup_{r \to 0} \frac{\text{Vol}(B_r(A))}{\omega_{n-s} r^{n-s}}$$

and upper Minkowski dimension as

$$\overline{\dim}_M(A) = \inf\{s : \mathcal{M}^{s, \ast}(A) = 0\} = \sup\{s : \mathcal{M}^{s, \ast}(A) > 0\}.$$
Theorem 2.10. (Technical Theorem) Let \( u \in A(n, \Lambda) \), then for any \( r_0 > 0 \) and all \( 0 < r_0 < r \)
\[
(2) \quad \Vol \left( B_r \left( \mathcal{C}^k_{\mathcal{C}^k_{r_0}(u)} \cap B_{\frac{1}{4}}(0) \right) \right) \leq C(n, \Lambda, k, \epsilon) r^{n-k-\epsilon}.
\]
In particular, letting \( r_0 \to 0 \)
\[
\mathcal{H}^{k+\epsilon} \left( \mathcal{C}^k_{\epsilon}(u) \cap B_{\frac{1}{4}}(0) \right) \leq \mathcal{M}^{*,k+\epsilon} \left( \mathcal{C}^k_{\epsilon}(u) \cap B_{\frac{1}{4}}(0) \right) \leq C(n, \Lambda, k, \epsilon)
\]

Lemma 2.11. (Containment) There exists an \( 0 < \epsilon = \epsilon(n, \Lambda) \) such that
\[
(3) \quad \left( \mathcal{C}^{n-2}(u) \cap \Omega \right) \cap B_{\frac{1}{8}}(0) \subset \mathcal{C}^{n-2}_\epsilon(u)
\]
and
\[
(4) \quad \left( \mathcal{C}^{n-2}(u) \cap \partial \Omega \setminus \operatorname{sing}(\partial \Omega) \right) \cap B_{\frac{1}{8}}(0) \subset \mathcal{C}^{n-2}_\epsilon(u)
\]

Proof of Theorem 1.2 assuming Lemma 2.11 For each point \( x \in K \subset V \), there is a radius \( 0 < r \) such that \( B_{4r}(x) \cap \partial \Omega \subset V \). Since \( K \) is compact, we may find a finite subcover \( \{B_r(x_i)\} \). Thus, Lemma 2.11 implies that in each \( B_r(x_i) \)
\[
\inf \{ s : \mathcal{M}^{s}(B_{r_i}(x_i)) \cap \{ \nabla u = 0 \} \setminus \operatorname{sing}(\partial \Omega) < \infty \} \leq n - 2.
\]
Since upper Minkowski dimension is stable under finite unions, the first claim of Theorem 1.2 holds. The second follows from an identical argument using Lemma 2.11. \( \square \)

2.1. Outline of the Paper. The structure of this paper is roughly in four parts. Section 3 and Section 4 use the geometric techniques of [GLS74], [HL] (and many, many others) to establish that the Almgren frequency is monotonically non-decreasing and bounded on \( \{ u = 0 \} \). Section 5 uses these results to establish compactness of \( \{T_{p,r}u\} \) for \( u(p) = 0 \). Section 3 extends these results to \( p \in \Omega \) such that \( u(p) \neq 0 \).

The second part of this paper is devoted to obtaining geometric control upon \( \mathcal{C}^k_{\mathcal{C}^k_{r}(u)} \). The general idea is to employ the usual “frequency pinching” (Lemma 7.2) and cone-splitting results (Lemma 7.3). However, because we are considering \( \mathcal{N}_\Omega(p, r, u) \) at points \( p \) such that \( u(p) \neq 0 \), the Almgren frequency is not monotonitic. This is overcome by proving that if dist(\( p, \{ u = 0 \} \)) << \( r \), then
\[
\mathcal{N}_\Omega(p, 1, u) - \mathcal{N}_\Omega(p, 1/10, u) \leq \gamma,
\]
for \( 0 < \gamma \) sufficiently small implies that \( u \) is \((0, \epsilon, 1, p)\)-symmetric. In Corollary 7.6 we prove that if \( x \in \mathcal{C}^k(u) \) then \( \{ p \in \Omega : u(0, \delta, r, p) \text{-symmetric} \cap B_{r}(x) \} \) is contained in a tubular neighborhood of a \( k \)-plane \( L^k \).

The third part of this paper is devoted to obtaining packing estimates to prove Theorem 2.10. To do so, we use the tools of [CNV15], which do not require restricting to a level set or the delicate machinery which powers the finer estimates of [dLMSV16]. The fact that we do not control the tilt of approximating \( L^k \) at different scales accounts for the \((k+\epsilon)\)-dimensional results.

The fourth part of this paper is devoted to proving the containment results which prove Lemma 2.11.

Throughout this paper, the constant \( C \) will by ubiquitous and represent different constants even within the same string of inequalities. A constant written \( C(n, \Lambda) \) will only depend upon \( n \) and \( \Lambda \), but each instantiation may represent a distinct constant.
3. The Almgren frequency function

In this section, we develop crucial properties of the Almgren frequency function. The main results of this Section are the monotonicity of the Almgren frequency on \( p \in \{u = 0\} \cap \overline{\Omega} \).

We now note some of the elementary properties of \( H_{\Omega}(p, r, u) \), \( D_{\Omega}(p, r, u) \), \( N_{\Omega}(p, r, u) \), and their derivatives.

**Lemma 3.1.** Let \( u \in A(n, \Lambda) \) and \( p \in \overline{\Omega} \cap B_1(0) \) and all \( 0 < r < 1 \)

\[
\frac{d}{dr} H_{\Omega}(p, r, u) = \frac{n - 1}{r} H_{\Omega}(p, r, u) + 2 D_{\Omega}(p, r, u) + 2u(p) \int_{\partial \Omega^c \cap B_r(p)} \nabla u \cdot \bar{n} d\sigma
\]

\[
\frac{d}{dr} D_{\Omega}(p, r, u) = \frac{n - 2}{r} D_{\Omega}(p, r, u) + 2 \int_{\partial B_r(p)} (\nabla u \cdot \bar{n})^2 d\sigma
\]

\[
+ \int_{\partial \Omega^c \cap B_r(p)} (Q - p) \cdot \bar{n} (\nabla u \cdot \bar{n})^2 d\sigma(Q)
\]

\[
\frac{d}{dr} \ln(H_{\Omega}(p, r, u)) = \frac{n - 1}{r} + \frac{2}{r} N_{\Omega}(p, r, u) + 2 \frac{u(p) \int_{\partial \Omega^c \cap B_r(p)} \nabla u \cdot \bar{n} d\sigma}{H_{\Omega}(p, r, u)},
\]

where \( \bar{n} \) is the unit outer normal of the relevant domains.

**Proof.** In the interior setting, for \( B_r(p) \subset \Omega \), these identities follow from straightforward computation. (5) follows from the change of variables, \( y \to rx + p \), and the divergence theorem. (6) relies upon the Rellich-Necas Identity,

\[
\text{div}(X|\nabla u|^2) = 2\text{div}((X \cdot \nabla u) \nabla u) + (n - 2)|\nabla u|^2,
\]

the divergence theorem, and the fact that \( u \) vanishes on the boundary. The last two equations follow immediately from (5). Without exception, the standard interior computations go through identically for radii for which \( B_r(p) \cap \partial \Omega \neq \emptyset \), where we also use the identity

\[
\int_{B_r(p)} (u - u(p)) \Delta u = u(p) \int_{\partial \Omega^c \cap B_r(p)} \nabla u \cdot \bar{n} d\sigma,
\]

where \( \bar{n} \) is the outer unit normal vector to \( \Omega \). See [HL] Theorem 2.2.3, Corollary 2.2.5 and [AEK95] (Proof of the Doubling Property) for details. \( \square \)

The following lemma records a useful identity which follows from the previous lemma by straightforward computation.

**Lemma 3.2.** For \( u \in A(n, \Lambda) \) and \( p \in \overline{\Omega} \cap B_1(0) \) and all \( 0 < r < 1 \), \( \frac{d}{dr} N_{\Omega}(p, r, u) \) may be decomposed into four terms

\[
\frac{d}{dr} N_{\Omega}(p, r, u) = N_1'(r) + N_2'(r) + N_3'(r) + N_4'(r),
\]
where
\begin{align*}
N_1'(r) &= \frac{1}{H_\Omega(p, r, u)^2} 2r \left[ H_\Omega(p, r, u) \int_{\partial B_r(p) \cap \Omega} (\nabla u \cdot \tilde{\eta})^2 d\sigma - \left( \int_{\partial B_r(p) \cap \Omega} (u - u(p))(\nabla u \cdot \tilde{\eta}) d\sigma \right)^2 \right] \\
N_2'(r) &= \frac{1}{H_\Omega(p, r, u)} r \int_{\partial \Omega \cap B_r(p)} (Q - p) \cdot \tilde{\eta}(\nabla u \cdot \tilde{\eta})^2 d\sigma(Q) \\
N_3'(r) &= 2N_\Omega(p, r, u) \frac{u(p)}{H_\Omega(p, r, u)} \int_{\partial \Omega \cap B_r(p)} \nabla u \cdot \tilde{\eta} d\sigma \\
N_4'(r) &= \frac{2r}{H_\Omega(p, r, u)^2} \left( u(p) \int_{\partial \Omega \cap B_r(p)} \nabla u \cdot \tilde{\eta} d\sigma \right)^2,
\end{align*}
and \( \tilde{\eta} \) is the unit outer normal.

**Lemma 3.3.** Let \( u \in A(n, \Lambda) \) and \( p \in \overline{\Omega} \cap B_1(0) \cap \{ u = 0 \} \) and all \( 0 < r < 1 \),
\[
N_1'(r) = \frac{2r}{H_\Omega(p, r, u)^2} \left[ \frac{H_\Omega(p, r, u) \int_{\partial B_r(p) \cap \Omega} (\nabla u \cdot \tilde{\eta})^2 d\sigma - \left( \int_{\partial B_r(p) \cap \Omega} (u - u(p))(\nabla u \cdot \tilde{\eta}) d\sigma \right)^2}{H_\Omega(p, r, u)^2} \right] \\
= \frac{2}{rH_\Omega(p, r, u)} \left( \int_{\partial B_r(p) \cap \Omega} |\nabla u \cdot (y - p) - N_\Omega(p, r, u)(u(y) - u(p))|^2 d\sigma(y) \right).
\]

**Proof.** Recall that by the Cauchy-Schwarz inequality, we have that for \( \lambda = \langle w, v \rangle \ |
\|
|v|^2 - \lambda |v|^2 = |w|^2 - |\langle w, v \rangle|^2. \]

Choosing \( w = \nabla u \cdot (y - p) \) and \( v = u - u(p) \), we have
\[
N_1'(r) = H_\Omega(p, r, u)^{-1} 2r \left( \int_{\partial B_r(p) \cap \Omega} |(u)_\nu - \frac{1}{r} \lambda(p, r, u)(u - u(p))|^2 d\sigma \right) \\
= \frac{2}{rH_\Omega(p, r, u)} \left( \int_{\partial B_r(p) \cap \Omega} |\nabla u \cdot (y - p) - \lambda(p, r, u)(u(y) - u(p))|^2 d\sigma(y) \right),
\]
where
\[
(11) \quad \lambda(p, r, u) := \frac{\int_{\partial B_r(p) \cap \Omega} (u(y) - u(p))\nabla u \cdot (y - p) d\sigma(y)}{H_\Omega(p, r, u)}.
\]
The Divergence theorem then implies that for \( u(p) = 0 \), \( \lambda(p, r, u) = N_\Omega(p, r, u) \). \( \Box \)

**Lemma 3.4.** (*Monotonicity*) Let \( u \in A(n, \Lambda) \) and \( p \in \{ u = 0 \} \cap \overline{\Omega} \cap B_1(0) \), then \( N_\Omega(p, r, u) \) is monotonically non-decreasing in \( 0 < r < 1 \).

**Proof.** Recall Lemma 3.2. Note that \( N_1'(r) \) is non-negative by the Cauchy-Schwarz inequality. Furthermore, because \( N_2'(r) \) is non-negative because \( \partial \Omega \) is a convex surface and \( \tilde{\eta} \) is the outer normal, \( (Q - p) \cdot \tilde{\eta} \geq 0 \) for all \( p \in \Omega \cap B_2(0) \) and all \( Q \in \partial \Omega \cap B_2(0) \). Observe that \( N_3'(r) = N_4'(r) = 0 \) because \( u(p) = 0 \). Therefore, \( \frac{d}{dr} N_\Omega(p, r, u) \) is non-negative. \( \Box \)
4. The Zero set: Uniform Frequency Bounds

The main result in this section is Lemma 4.1 which gives a uniform bound on the Almgren frequency function for all \( p \in \Omega \cap B_{\frac{1}{2}}(0) \) for which \( u(p) = 0 \) and all \( 0 < r \leq \frac{1}{2} \). We begin with a few basic results.

**Lemma 4.1.** (Doubling) Let \( u \in A(n, \Lambda) \) with \( p \in B_1(0) \cap \{u = 0\} = \overline{\Omega} \). For any \( 0 < s < S \leq 1 \)

\[
H_\Omega(p, S, u) \leq \left( \frac{S}{s} \right)^{(n-1) + 2N_\Omega(p, S, u)} H_\Omega(p, s, u).
\]

**Proof.** Recalling Equations (5) and (7)

\[
\ln \left( \frac{H_\Omega(p, S, u)}{H_\Omega(p, s, u)} \right) = \ln \left( \frac{H_\Omega(p, S, u)}{H_\Omega(p, s, u)} \right) - \ln \left( H_\Omega(p, s, u) \right)
\]

\[
= \int_s^S \frac{H_\Omega'(p, r, u)}{H_\Omega(p, r, u)} \, dr
\]

\[
= \int_s^S \frac{n-1}{r} + \frac{2}{r} N_\Omega(p, r, u).
\]

We bound \( N_\Omega(p, r, u) \) by \( N_\Omega(p, S, u) \) using Lemma 3.4. Plugging in these bounds, we have that for \( r \in [s, S] \)

\[
\ln \left( \frac{H_\Omega(p, S, u)}{H_\Omega(p, s, u)} \right) \leq [(n - 1) + 2N_\Omega(p, S, u)] \ln(r)|_s^S.
\]

Evaluating and exponentiating gives the desired result. \qed

**Remark 4.2.** Because \( N_\Omega(p, r, u) \) is monotonic for \( p \in B_1(0) \cap \{u = 0\} \cap \overline{\Omega} \), we can also extract the inequality

\[
\ln \left( \frac{H_\Omega(p, S, u)}{H_\Omega(p, s, u)} \right) \geq [(n - 1) + 2N_\Omega(p, s, u)] \ln(r)|_s^S,
\]

which leads to

\[
H_\Omega(p, s, u) \leq \left( \frac{S}{S} \right)^{(n-1) + 2N_\Omega(p, s, u)} H_\Omega(p, S, u).
\]

If \( S = 1 \) and \( u = T_{p,r}u \), then we have that for all \( 1 > s > 0 \)

\[
H_\Omega(0, s, T_{p,r}u) \leq s^{(n-1) + 2N_\Omega(0, T_{p,r}u)}.
\]

We are now ready for the main result of this section.

**Lemma 4.3.** (Uniform bound on frequency) Let \( u \in A(n, \Lambda) \), as above. There is a constant, \( C_1(n, \Lambda) \) such that for all \( p \in \{u = 0\} \cap \overline{\Omega} \cap B_{\frac{1}{2}}(0) \) and all \( r \in (0, \frac{1}{2}) \)

\[
N_\Omega(p, r, u) \leq C_1(n, \Lambda).
\]

**Proof.** Recall that \( 0 \in \partial \Omega \) and that the Almgren frequency function is invariant under rescalings. Therefore, we normalize our function \( u \) by the rescaling \( v = T_{0,1}u \).
Therefore, applying Lemma 4.1 to $Q = 0$, letting $r = cR$, and integrating both sides with respect to $R$ from 0 to $S$, we have that for any $c \in (0, 1)$

\[
\int_{B_{S}(0)} |v|^2 dV \leq \int_{0}^{S} \left( \frac{1}{c} \right)^{(n-1)+2N_{\Omega}(0,R,v)} \int_{\partial B_{cR}(0)} |v|^2 d\sigma dR
\]

\[
\leq \left( \frac{1}{c} \right)^{(n-1)+2N_{\Omega}(0,S,v)} \int_{0}^{S} \int_{\partial B_{cR}(0)} |v|^2 d\sigma dR.
\]

Thus, letting $S = 1$ and $c = \frac{1}{16}$ and dividing by $\omega_n$ we obtain

\[
(16) \quad \int_{B_{\frac{1}{4}}(0)} |v|^2 dV \leq 16^{2N_{\Omega}(0,1,v)} \int_{\frac{B_{\frac{1}{4}}}{\frac{1}{16}}(0)} |v|^2 dV.
\]

Thus, for any $p \in \{v = 0\} \cap \Omega \cap B_{\frac{1}{4}}(0)$ by inclusion

\[
\int_{B_{1}(0)} |v|^2 dV \geq \int_{B_{\frac{3}{4}}(p)} |v|^2 dV, \quad \int_{B_{\frac{1}{4}}(0)} |v|^2 dV \leq \int_{B_{\frac{1}{4}}(p)} |v|^2 dV.
\]

Therefore, substituting these bounds into (16)

\[
(17) \quad \int_{B_{\frac{3}{4}}(p)} |v|^2 dV \leq 16^{2N_{\Omega}(0,1,v)} \left( \int_{B_{\frac{1}{4}}(p)} |v|^2 dV \right).
\]

Now, we wish to bound $\int_{B_{\frac{3}{4}}(p)} |v|^2 dV$ from below and $\int_{B_{\frac{1}{4}}(p)} |v|^2 dV$ from above. By (5), if $v(p) = 0$, $\frac{d}{dr} \int_{\partial B_{r}(p)} |v|^2 d\sigma \geq 0$. Thus, for all $p \in \{v = 0\} \cap \Omega \cap B_{\frac{1}{4}}(0)$, we bound

\[
\int_{B_{\frac{3}{4}}(p)} |v|^2 dV \geq \int_{\frac{3}{4}}^{\frac{1}{2}} \int_{\partial B_{r}(p)} |v|^2 d\sigma dr \geq c \int_{\frac{1}{2}}^{\partial B_{\frac{1}{4}}(p)} |v|^2 d\sigma.
\]

\[
\int_{B_{\frac{1}{4}}(p)} |v|^2 dV \leq \int_{0}^{\frac{1}{4}} \int_{\partial B_{\frac{1}{2}}(p)} |v|^2 d\sigma dr \leq c \int_{\frac{1}{4}}^{\partial B_{\frac{1}{4}}(p)} |v|^2 d\sigma.
\]

Plugging the above bounds into (17) and dividing, we obtain

\[
(18) \quad \frac{\int_{\partial B_{\frac{3}{4}}(p)} |v|^2 d\sigma}{\int_{\partial B_{\frac{1}{4}}(p)} |v|^2 d\sigma} \leq C(n)16^{2N_{\Omega}(0,1,v)}
\]

for all $p \in \{v = 0\} \cap \Omega \cap B_{\frac{1}{4}}(0)$.
Recalling Equation (7) and Lemma 3.4, we see
\[
\ln \left( \frac{1}{v^{2}d\sigma} \right) - \ln \left( \frac{1}{v^{2}d\sigma} \right) = \int_{\frac{1}{v}}^{\frac{1}{r}} \frac{d}{dr} \ln \left( \frac{1}{r^{n-1}H_{\Omega}(p, r, v)} \right) dr
\]
\[
= \int_{\frac{1}{v}}^{\frac{1}{r}} N_{\Omega}(p, r, v) dr
\]
\[
\geq 2N_{\Omega}(p, \frac{1}{2}, v) \left( \ln \left( \frac{5}{8} \right) - \ln \left( \frac{9}{16} \right) \right)
\]
\[
\geq 2cN_{\Omega}(p, \frac{1}{2}, v).
\]
Thus, (18) gives us that
\[
2c \left[ N_{\Omega}(p, \frac{1}{2}, v) \right] \leq \ln \left( \frac{1}{v^{2}d\sigma} \right) \leq \ln \left( \frac{1}{v^{2}d\sigma} \right) \leq \ln (C(n)(16)^{2N_{\Omega}(0,1,v)})
\]
\[
= 2N_{\Omega}(0,1,v)\ln(16) + C(n)
\]
\[
\leq 2\Lambda \ln(16) + C(n).
\]

Now, Lemma 3.4 gives that for \( \frac{1}{2} > s > 0 \) \( N_{\Omega}(p, 1/2, v) \geq N_{\Omega}(p, s, v) \). Since \( N_{\Omega}(p, r, v) = N_{\Omega}(p, r, u) \), we have the desired claim.

5. The Zero Set: Compactness

The uniform bounds on the Almgren frequency function allow us to prove compactness results on the collection of rescaling \( \{T_{p,r}u\} \). The main results of this Section are weak compactness (Lemma 5.4), the geometric non-degeneracy of the domains \( \Omega \) (Corollary 5.2).

We now state a sequence of preliminary corollaries to Lemma 4.3. We shall denote the \( C^{0,\gamma}(B_{1}(0)) \)-norm by
\[
||u||_{C^{0,\gamma}(B_{1}(0))} := ||u||_{C^{0}(B_{1}(0))} + \sup_{x \neq y \in B_{1}(0)} \frac{|u(x) - u(y)|}{|x - y|^{\gamma}}.
\]

Lemma 5.1. (Uniform Holder continuity) Let \( u \in A(n, \Lambda), Q \in B_{1}(0) \cap \partial \Omega, \) and \( r \in (0, 1/2] \). Then
\[
||T_{Q,r}u||_{C^{0,\gamma}(B_{1}(0))} \leq C(n, \Lambda).
\]

We defer the proof of this statement to the Appendix A. The techniques are standard.

Corollary 5.2. (Non-degeneracy of domains) Let \( u \in A(n, \Lambda) \) and \( \Omega \in D(n) \) it’s associated convex domain. There exists a constant, \( 0 < c = c(\Lambda, n) \) such that for all \( Q \in \partial \Omega \cap B_{\frac{1}{2}}(0) \) and \( 0 < r \leq \frac{1}{2} \), \( \partial B_{1}(0) \cap T_{Q,r}\Omega \) is a relatively open convex surface with
\[
\mathcal{H}^{n-1}(\partial B_{1}(0) \cap T_{Q,r}\Omega) > c.
\]
Proof. That \( \partial B_r(Q) \cap \Omega \) is relatively open and relatively convex is immediate from the definition of \( \Omega \). By Lemma 5.4, we see that \( \max_{B_i(0)} |T_{Q,r}u(x)| \leq C(n, \Lambda) \) and \( H_{T_{Q,r},\Omega}(0,1, T_{Q,r}u) = 1 \). Furthermore, we have that

\[
H_{T_{Q,r},\Omega}(0,1, T_{Q,r}u) \leq H^{n-1}(\partial B_1(0) \cap T_{Q,r}\Omega)C^2.
\]

Therefore, \( H^{n-1}(\partial B_1(0) \cap T_{Q,r}\Omega) \geq C^{-2} = c \). \( \square \)

**Corollary 5.3.** For all \( u \in A(n, \Lambda) \), \( Q_0 \in \partial\Omega \cap B_{1/4}(0) \), and \( 0 < r \leq \frac{1}{2} \), the following estimate holds. Let \( Q \in T_{Q_0,r}\partial\Omega \cap B_1(0) \) and let \( L_Q \) be a supporting hyperplane to \( Q \in T_{Q_0,r}\partial\Omega \). Then for all \( p \in T_{Q_0,r}\Omega \cap B_{\frac{1}{2}}(Q) \)

\[
|T_{Q_0,r}u(p)| \leq C(n, \Lambda)\text{dist}(p, L_Q).
\]

In particular, \( |\nabla T_{Q_0,r}u(Q)| \leq C(n, \Lambda) \) for all \( Q \in \partial T_{Q_0,r}\Omega \cap B_{\frac{1}{2}}(0) \).

**Proof.** Let \( \mathbb{H}_Q \) be the half-space with boundary \( L_Q \) which contains \( T_{Q_0,r}\Omega \). Consider the Dirichlet problem

\[
\Delta \phi = 0 \quad \text{in} \quad \mathbb{H}_Q \cap B_1(Q),
\]

\[
\phi = \begin{cases} C(n, \Lambda) & \text{on} \ \partial B_{\frac{1}{2}}(Q) \cap T_{Q_0,r}\Omega \\ 0 & \text{on} \ \partial(B_1(Q) \cap \mathbb{H}_Q) \setminus (\partial B_{\frac{1}{2}}(Q) \cap T_{Q_0,r}\Omega), \end{cases}
\]

where we choose \( C(n, \Lambda) \) to be the same constant in Lemma 5.1 for which we have \( \sup_{\partial B_1(0)} |T_{Q_0,r}u| \leq C(n, \Lambda) \). Note that for any \( Q \in \partial\Omega \mathbb{H}_Q \cap B_{1/2}(Q) \) is a Weiner regular domain and the boundary data is piecewise continuous, so a unique solution \( \phi \) must exist.

By the maximum principle \( T_{Q_0,r}u \leq \phi \) in \( T_{Q_0,r}\Omega \cap B_{\frac{1}{2}}(Q) \). We now argue that \( \phi \) is comparable to a linear function in \( B_{\frac{1}{2}}(Q) \cap \mathbb{H}_Q \).

Let \( L \) be the affine linear function with \( \{L = 0\} = L_Q \) such that

\[
\max_{\partial B_{1/2}(Q)} L = \max_{\partial B_{1/2}(Q)} \phi = C(n, \Lambda).
\]

By [JKS2] Theorem 5.1., there is a constant \( C \) such that we have that for all \( x \in B_{\frac{1}{2}}(Q) \cap \mathbb{H}_Q \)

\[
\phi(x) \leq CL(x),
\]

where \( C \) depends only upon the geometry of \( B_{\frac{1}{2}}(Q) \cap \mathbb{H}_Q \). Since this geometry is always a half-ball, this constant is uniform. Therefore, we have that for all \( x \in B_{\frac{1}{2}}(Q) \cap \mathbb{H}_Q \)

\[
\phi(x) \leq C\phi(x) \leq C2C(n, \Lambda)\text{dist}(x, L_Q).
\]

Thus, for \( p \in T_{Q_0,r}\Omega \cap B_{\frac{1}{2}}(Q) \), we have

\[
T_{Q_0,r}u(p) \leq \phi(p) \leq C(n, \Lambda)\text{dist}(x, L_Q).
\]

Applying this argument to \( \pm T_{Q_0,r}u \), we obtain the desired estimate. \( \square \)

**Lemma 5.4.** (Preliminary Compactness) Let \( u_i \in A(n, \Lambda) \), \( Q_i \in \partial\Omega_i \cap B_{1/4}(0) \), and \( 0 < r_i \leq \frac{1}{4} \). Then there exists a subsequence (also indexed by \( i \)) such that

1. \( T_{Q_i,r_i}u_i \to u_\infty \) in \( C_0^{n,\gamma}(B_1(0)) \).
2. If we define \( \Omega_\infty = \text{Interior}(|u_\infty| > 0) \), then \( T_{Q_\infty,r_i}\Omega_\infty \cap B_1(0) \to \Omega_\infty \cap B_1(0) \) in the Hausdorff metric on compact subsets and \( \Omega_\infty \) is a non-degenerate convex domain with \( 0 \in \partial\Omega_\infty \) which satisfies the same non-degeneracy as Corollary 5.2.
(3) $u_\infty$ is harmonic in $\Omega_\infty$.

Proof. By definition, $T_{Q_i, r_i} u_i(0) = 0$. Therefore, Lemma 5.1 implies the first convergence result by Arzela-Ascoli. Note that since $H_{T_{Q_i, r_i}, \Omega_i}(0, 1, T_{Q_i, r_i} u_i) = 1$ for all $i$, $H_{\Omega_\infty}(0, 1, u_\infty) = 1$.

By taking a further subsequence, we may assume that $\lim \infty \{ |u_\infty| > \epsilon \} = 0$. Therefore, $\Omega_\infty$ is non-degenerate. For all $y \in \Omega_\infty$ and all $0 < \delta$ there exists a point $x \in \Omega_\infty$ such that $|x - y| \leq \delta$ and $|u_\infty(x)| > \epsilon$ for some $\epsilon > 0$. Since $x \in \text{lim} T_{Q_i, r_i} \Omega_i$ and $0 < \delta$ was arbitrary, $\Omega_\infty \subset \text{lim} B_\delta(T_{Q_i, r_i})$ for all $\delta > 0$.

To see the converse containment, we observe that for any $x_0 \in \text{Int}(\Omega')$, if $B_\delta(x_0) \subset \Omega'$ then for all sufficiently large $i B_{\delta/2}(x) \subset T_{Q_i, r_i} \Omega_i$ and $T_{Q_i, r_i} u_i$ is uniformly bounded in $W^{1,2}(B_\delta(x_0))$. Therefore, $T_{Q_i, r_i} u_i$ converge to a harmonic function in $\text{Int}(\Omega')$. By the convexity of $T_{Q_i, r_i} \Omega_i$ for all $i$, $\text{Int}(\Omega')$ is connected. Now, let $0 < \delta$ and suppose that for all sufficiently large $i$ there exists $x_i \in T_{Q_i, r_i} \Omega_i \setminus B_\delta(\Omega_\infty) \cap B_1(0)$. By passing to a subsequence, we may assume $x_i \to x_\infty \in B_1(0) \cap \Omega' \setminus B_{\delta/2}(\Omega_\infty)$ Since $u_\infty = 0$ on $B_{\delta/2}(x_\infty)$ it must be that $\lim_{i \to \infty} \sup_{B_{\delta/2}(x_\infty)} |T_{Q_i, r_i} u_i| = 0$. Since the convexity of $\Omega_\infty \subset B_1(0)$ in the Hausdorff metric on compact subsets. Convexity and the conclusion of Corollary 5.2 are preserved under this mode of convergence, and so (2) is proved.

Now that we know that $\Omega' = \Omega_\infty$, the previous argument proves (3), as well. \qed

6. Estimates Off the Zero Set

In this section we prove that analogs of the results of Sections 4 and 5 hold for all $p \in B_2(0) \cap \overline{\Omega}$ and $0 < r < \frac{1}{4}$. The key to obtaining estimates off $\{ u = 0 \}$ is the following technical lemma.

Lemma 6.1. (Technical Lemma) For any $u \in \mathcal{A}(n, \Lambda)$, $Q \in B_2(0)$, and $0 < r < \frac{1}{4}$, there is a constant $0 < c(n, \Lambda)$ such that for all $y \in B_{1/2}(0)$,

$$c(n, \Lambda) < H_{T_{Q, 2r}, \Omega}(y, \frac{1}{4}, T_{Q, 2r} u) < C(n, \Lambda).$$

Proof. Note that the upper bound follows directly from Lemma 5.1. To show the lower bound, we argue by compactness. Suppose that there is a sequence of functions, $u_i \in \mathcal{A}(n, \Lambda)$, points $Q_i \in \partial \Omega_i \cap B_1(0)$ and radii $0 < r_i < \frac{1}{4}$ such that there exist points $y_i \in B_{1/2}(0) \cap \overline{\Omega}$ for which

$$H_{T_{Q_i, 2r_i}, \Omega_i}(y_i, \frac{1}{4}, T_{Q_i, 2r_i} u_i) \leq 2^{-i}.$$

Letting $i \to \infty$, by Lemma 5.4 there exists a subsequence $T_{Q_i, 2r_i} u_j$ which converges to a Hölder continuous function, $u_\infty$, which is harmonic in a non-degenerate convex domain, $\Omega_\infty$. Note that $u_\infty$ vanishes on the boundary of $\partial \Omega_\infty \cap B_s(0)$. Similarly, we may take subsequences such that $y_j \to y_\infty$. Note that Hölder convergence implies $H_{\Omega_\infty}(0, 1, u_\infty) = 1$. Since we have that $H_{\Omega_\infty}(y_j, \frac{1}{4}, u_\infty) = 0$, it must be that $u_\infty = u_\infty(y_\infty)$ on $\partial B_{1/4}(y) \cap \Omega_\infty$. If $\partial B_{1/4}(y) \subset \Omega_\infty$, then $u_\infty \equiv u_\infty(y_\infty)$ in $\Omega_\infty$. This contradicts $u_\infty(0) = 0$ and $H_{\Omega_\infty}(0, 1, u_\infty) = 1$. If $\partial B_{1/4}(y)$
intersects $\partial \Omega_\infty$, then $u_\infty(y_\infty) = 0$, since $u_\infty$ must vanish continuously on $\partial \Omega_\infty$. However, this forces $u_\infty \equiv 0$, which contradicts $H_{\Omega_\infty}(0, 1, u_\infty) = 1$. \hfill \Box

**Lemma 6.2. (Bounding the Almgren Frequency)** Let $u \in A(n, \Lambda)$, $p \in B_{\frac{1}{8}}(0) \cap \overline{\Omega}$ and $0 < r \leq \frac{1}{8}$. Then, there is a constant, $C_2 = C_2(n, \Lambda) < \infty$ such that

$$N_\Omega(p, r, u) \leq C_2.$$

**Proof.** For $p \in \{u = 0\} \cap \overline{\Omega}$ Lemma 4.3 proves the desired inequality. Let $p \in \Omega$ be such that $u(p) \neq 0$. Let $0 < \delta = \text{dist}(p, \partial \Omega)$ and $Q \in \partial \Omega$ such that $|p − Q| = \delta$. If $B_r(p) \subset \Omega$ we use the monotonicity of the Almgren frequency function to reduce to considering $B_\delta(p)$. We let $\tau = \max\{r, \delta\} \leq \frac{1}{8}$.

Consider $T_{Q,4r}u$. We note that

$$C_1(n, \Lambda) \geq N_{T_{Q,4r}\Omega}(0, 1, T_{Q,4r}u) = \int_{B_{\frac{1}{8}}(0)} |\nabla T_{Q,4r}u|^2 d\nu \geq \frac{\int_{\partial B_{\frac{1}{8}}(0)} |\nabla T_{Q,4r}u|^2 d\tau}{\int_{B_{\frac{1}{8}}(T_{Q,4r},\tau)} |\nabla T_{Q,4r}u|^2 d\nu}.$$

On the other hand, Lemma 6.1 implies that

$$H_{T_{Q,4r}\Omega}(T_{Q,4r}p, \frac{1}{4}, T_{Q,4r}u) \geq c(n, \Lambda).$$

Therefore, we have $N_\Omega(p, r, u) = \frac{D_{T_{Q,4r}\Omega}(T_{Q,4r}p, \frac{1}{4}, T_{Q,4r}u)}{4H_{T_{Q,4r}\Omega}(T_{Q,4r}p, \frac{1}{4}, T_{Q,4r}u)} \leq C(n, \Lambda)$.

**Lemma 6.3. (Lipschitz Bounds)** For $u \in A(n, \Lambda)$, for all $Q \in \partial \Omega \cap B_{\frac{1}{8}}(0)$ and all radii, $0 < r \leq \frac{1}{8}$, $T_{Q,r}u \in \text{Lip}(B_1(0))$ with uniform Lipschitz constant $\text{Lip}(T_{Q,r}u) \leq C(n, \Lambda)$.

**Proof.** Since $T_{Q,r}u$ is continuous and constant outside of $T_{Q,r}\Omega$, we reduce to bounding $\nabla T_{Q,r}u$ at interior points $y \in T_{Q,r}\Omega \cap B_1(0)$. Note that by our definition of the rescalings (Definition 2.5) and Lemma 4.1

$$|T_{Q,r}u(y)| = \left(\frac{1}{\frac{1}{4r} - H_{\Omega}(Q, 4r, u)}\right)^{1/2} |T_{Q,4r}u(y')| \leq 4^{2C_1(n, \Lambda)} |T_{Q,4r}u(y')| \leq C(n, \Lambda)|T_{Q,4r}u(y')|,$$

where $y' = \frac{1}{4}y$.

Note that $y' \in B_{\frac{1}{8}}(0) \cap T_{Q,4r}\Omega$. Let $\delta = \text{dist}(y', T_{Q,4r}\partial \Omega)$. Therefore, $\nabla T_{Q,4r}u(y') = \int_{B_{\delta}(y')} \nabla T_{Q,4r}u d\nu$. Recall that $|\nabla u|$ is subharmonic, and therefore by Lemma 6.2

$$|\nabla T_{Q,4r}u(y')| \leq \int_{B_{\delta}(y')} |\nabla T_{Q,4r}u| d\nu \leq \left(\int_{B_{\delta}(y')} |\nabla T_{Q,4r}u|^2 d\nu\right)^{1/2} \leq \left(C_2(n, \Lambda)\delta^{-2}\int_{\partial B_{\delta}(y')} (T_{Q,4r}u - T_{Q,4r}u(y'))^2 d\tau\right)^{1/2}.$$
Now, let $Q' \in T_{Q,4r} \partial \Omega$ be a point such that $\delta = |y' - Q'|$ and let $y'' = y'' + Q'$. Now, we translate the domain by $Q$.

\[
\int_{\partial B_\delta(y')} (T_{Q,4r} u - T_{Q,4r} u(y'))^2 d\sigma = \int_{\partial B_\delta(y'')} (T_{Q,4r} u(x + Q) - T_{Q,4r} u(y'' + Q))^2 d\sigma.
\]

Note that $T_{Q,4r} u(x + Q') \in \mathcal{A}(n, C_1(n, \Lambda))$. Now, by Corollary 5.3 applied to $T_{Q,4r} u(x + Q) \in \mathcal{A}(n, C_1(n, \Lambda))$ with $Q_0 = 0$ we bound

\[
\int_{\partial B_\delta(y'')} (T_{Q,4r} u(x + Q) - T_{Q,4r} u(y'' + Q))^2 d\sigma(x)
\]

\[
\leq \int_{\partial B_\delta(y'')} (4C(n, C_1(n, \Lambda))\delta)^2 d\sigma = (4C(n, C_1(n, \Lambda))\delta)^2.
\]

Thus, we have that,

\[
|\nabla T_{Q,r} u(y)| \leq C(n, \Lambda)|T_{Q,4r} u(y')|
\]

\[
\leq C(n, \Lambda) \left( C_1(n, \Lambda)\delta^{-2} \left( \int_{\partial B_\delta(y)} (T_{Q,r} u - T_{Q,r} u(y))^2 d\sigma \right) \right)^{\frac{1}{2}}
\]

\[
\leq C(n, \Lambda)C_1(n, \Lambda)^{\frac{1}{2}} \frac{1}{\delta}(4C(n, C_1(n, \Lambda))\delta) \leq C(n, \Lambda).
\]

\[
\square
\]

We now prove that the Almgren frequency is a function of uniformly bounded variation.

**Lemma 6.4.** *(Bounded Variation)* Let $u \in \mathcal{A}(n, \Lambda)$ and $p \in B_{\bar{r}}(0) \cap \overline{\Omega}$. Then, there is a constant $C_3 = C_3(n, \Lambda) < \infty$ such that for all $0 < r \leq \frac{1}{8}$

\[
\text{var}(N_\Omega(p, r, u), [0, \frac{1}{8}]) \leq C_3.
\]

**Proof.** We estimate the variation by a “rays of the sun” argument. Since $N_\Omega(p, r, u)$ is monotone increasing and bounded for $p \in \{u = 0\} \cap \overline{\Omega}$, we argue for $u(p) \neq 0$. Again, we let $\delta = \text{dist}(p, \partial \Omega)$.

\[
\text{var}(N_\Omega(p, r, u), [0, \frac{1}{8}]) \leq 2 \int_0^{\frac{1}{8}} N_3'(r) dr + |N_\Omega(p, 0^+, u) - N_\Omega(p, \frac{1}{8}, u)|
\]

\[
\leq 2 \int_0^{\frac{1}{8}} 2N_\Omega(p, r, u) \frac{1}{H_\Omega(p, r, u)} \int_{B_r(p) \cap \partial \Omega} u(p) \nabla u \cdot \bar{\eta} d\sigma dr + 2C_2.
\]

Now, if we let $Q_0 \in \partial \Omega$ be a point such that $|p - Q_0| = \delta$ we may calculate by Lemma 6.1 Lemma 4.1 Lemma 5.3 and Lemma 6.3

\[
\frac{1}{H_\Omega(p, r, u)} \int_{B_r(p) \cap \partial \Omega} u(p) \nabla u \cdot \bar{\eta} d\sigma \leq C(n, \Lambda) \int_{B_1(T_{Q_0}\cdot r, p) \cap \overline{T_{Q_0}\cdot r, \partial \Omega}} T_{Q_0,r} u(y) \frac{1}{r} \nabla T_{Q_0,r} u \cdot \bar{\eta}_{T_{Q_0},r} d\sigma
\]

\[
\leq C(n, \Lambda) \int_{B_1(T_{Q_0}\cdot p) \cap \overline{T_{Q_0}\cdot r, \partial \Omega}} \frac{\delta}{r^2} d\sigma \leq C(n, \Lambda) \frac{\delta}{r^2}.
\]
Therefore, by Arzela-Ascoli, there exists a subsequence which converges in $C(n, \Lambda)$. By continuity of measures and Lemma 6.3, for all $\epsilon > 0$, we can find a $\tau(\Lambda, n, \epsilon)$ independent of $T_{p_i, r_i}u_j$, such that
\[
\int_{B_1(0) \cap B_r(\partial \Omega_\infty)} |\nabla T_{p_i, r_i}u_j|^2 dV \leq \epsilon.
\]
Therefore, using the notation $\partial \Omega_{j, \tau} = B_r(T_{p_i, r_i} \partial \Omega_j)$,
\[
\lim_{j \to \infty} D_{\Omega}(1, 0, T_{p_i, r_i}u_j) = \lim_{j \to \infty} \int_{B_1(0)} |\nabla T_{p_i, r_i}u_j|^2 dV
\leq \lim_{j \to \infty} \int_{B_1(0) \cap \partial \Omega_{j, \tau}} |\nabla T_{p_i, r_i}u_j|^2 dV + \lim_{j \to \infty} \int_{B_1(0) \cap \partial \Omega_{j, \tau}} |\nabla T_{p_i, r_i}u_j|^2 dV
\leq \lim_{j \to \infty} \int_{B_1(0) \setminus B_{\epsilon/2}(\partial \Omega_\infty)} |\nabla T_{p_i, r_i}u_j|^2 dV + \epsilon
\leq D_{\Omega_{\infty}}(1, 0, u_\infty) + \epsilon.
\]
where the last equality follows from $W^{1,2}$-convergence of harmonic functions in the region $B_1(0) \setminus B_r(\partial \Omega_\infty)$. Since $\epsilon > 0$ was arbitrary, we have that $\lim_{j \to \infty} D_{\Omega_j}(1,0,T_{p_j,r_j} u_j) \leq D_{\Omega_\infty}(1,0,u_\infty)$. The other inequality follows from the same trick or from lower semi-continuity. Thus, $\lim_{j \to \infty} D(1,0,T_{Q_j,r_j} u_j) = D_{\Omega_\infty}(1,0,u_\infty)$. This implies strong convergence.

Corollary 6.6. (Convergence of the Almgren frequency) For $u_j \in \mathcal{A}(n, \Lambda)$, $p_j \in B_\frac{1}{8}(0) \cap \overline{\Omega}_i$, and $r_j \in (0, \frac{1}{8}]$, there exists a subsequence and a limit function such that

\begin{equation}
N_{T_{p_j,r_j}} \Omega_i(0,1,T_{p_j,r_j} u_j) \to N_{\Omega_\infty}(0,1,u_\infty).
\end{equation}

Proof. The continuous convergence of $T_{p_j,r_j} u_j$ in $B_1(0)$ and the strong convergence $\nabla T_{p_j,r_j} u_j$ in $B_1(0)$ give the desired convergence. By Corollary 6.6. (Convergence of the Almgren frequency) Theorem 5.9.6, we have that for all $\Omega_2 \in \mathcal{A}(n, \Lambda)$, $p \in B_\frac{1}{8}(0) \cap \overline{\Omega}_i$, and $r \in (0, \frac{1}{8}]$, the sequence of functions $T_{p,r} u_j$ will be harmonic in the region $B_1(0) \setminus B_r(\partial \Omega_\infty)$. By $C^{0,\gamma}(B_1(0))$ convergence of harmonic functions, $u_\infty$ is therefore harmonic in $B_1(0) \setminus B_r(\partial \Omega_\infty)$. Letting $\epsilon \to 0$ gives the desired statement.

7. Geometric Control

The main results of this section are two “quantitative rigidity” results about homogeneous harmonic functions. Both are essentially consequences of the compactness obtained in Lemma 6.5.

Lemma 7.1. Let $u \in \mathcal{A}(n, \Lambda)$. Let $p \in \overline{\Omega} \cap B_\frac{1}{8}(0)$ and $0 < r \leq \frac{1}{8}$. If

\begin{equation}
N_\Omega(p,r,u) = N_\Omega(p,r/10,u),
\end{equation}

and either $u(p) = 0$ or $B_r(p) \subset \Omega$ then $u$ is $(0,0,s,p)$-symmetric for all $0 < s \leq 1$.

Proof. The hypotheses imply that $N_\Omega(p,s,u)$ is a constant for all $r/10 \leq s \leq r$. Furthermore, using the notation in Lemma 3.3.

\begin{equation}
N'_1(s) = N'_2(s) = N'_3(s) = N'_4(s) = 0 \text{ for all } r/10 \leq s \leq r.
\end{equation}

Thus, by Lemma 3.3. we have that for all $y \in \partial B_s(p)$

\begin{equation}
\nabla u \cdot (y - p) = \partial_r u
+ N_\Omega(p,s,u)(u - u(p)).
\end{equation}

Since $N_\Omega(p,s,u)$ is a constant for all $r/10 \leq s \leq r$, this becomes a separable ODE in polar coordinates, and $u - u(p) = v(s,\theta) = s^{N_\Omega(p,r,u)} u(\theta)$. Since $\Omega \cap B_r(p)$ is open, unique continuation implies that $u - u(p)$ is a homogeneous function of degree $N_\Omega(p,r,u)$ in $\Omega \cap B_1(p)$. In particular, $u$ is $(0,0,s,p)$-symmetric for all $0 < s \leq 1$. 

\hfill $\square$
Standard quantitative rigidity results usually prove that if $N_{\Omega}(p, r, u)$ is almost constant ($|N_{\Omega}(p, 1, u) - N_{\Omega}(p, r, u)| \leq \delta(\epsilon)$), then $u$ is almost a homogeneous harmonic polynomial ($|\|T_{p,1}u - P\|_{L^2(B_1(0))}| \leq \sqrt{\epsilon}$). However, these results rely essentially upon the monotonicity of $N_{\Omega}(p, r, u)$. If $N_{\Omega}(p, r, u)$ is not monotonic, then $N_{\Omega}(p, 1, u) = N_{\Omega}(p, r, u)$ does not imply that the Almgren frequency is constant. In fact, even if $N_{\Omega}(p, r, u)$ is constant for $1/10 < r < 1$, if $u(p) \neq 0$, it is not clear that $u$ would be homogeneous. To overcome this technical issue we consider $p$ which are merely very close to \{u = 0\}.

**Lemma 7.2.** (Quantitative Rigidity) Let $u \in A(n, \Lambda)$, as above. Let $p \in B_{\frac{1}{8}}(0) \cap \Omega$ and $0 < r \leq \frac{1}{8}$. For every $\delta > 0$, there is an $0 < \gamma_0 = \gamma_0(n, \Lambda, \delta)$ such that for any $0 < \gamma \leq \gamma_0$ if

$$|N_{\Omega}(0, 1, T_{p,r}u) - N_{\Omega}(0, 10, T_{p,r}u)| \leq \gamma$$

and either $\text{dist}(p, \{u = 0\}) \leq \gamma r$ or $B_1(0) \subset T_{p,r} \Omega$ then $T_{p,r}u$ is $(0, \delta, 1, 0)$-symmetric.

**Proof.** We argue by contradiction. Assume that there exists a $\delta > 0$ such that there is a sequence of functions, $u_i \in A(n, \Lambda)$, and points, $p_i \in B_{\frac{1}{8}}(0) \cap \Omega$, radii $0 < r_i < \frac{1}{8}$, such that $\text{dist}(p_i, \{u_i = 0\}) \leq r_i 2^{-i}$ and

$$|N_{\Omega_i}(0, 1, T_{p_i,r_i}u_i) - N_{\Omega_i}(0, 10, T_{p_i,r_i}u_i)| \leq 2^{-i}$$

but that no $T_{p_i,r_i}u_i$ is $(0, \delta, 1, 0)$-symmetric.

By Lemma 6.5 we have that there exists a subsequence such that $T_{p_j,r_j}u_j$ converges strongly in $W_{1,2}^1(B_1(0))$ to a function $u_\infty$. By Corollary 6.7 we know that $u_\infty$ is harmonic in a convex domain $\Omega_\infty$. Furthermore, by Lemma 6.5(1) $u_\infty(0) = 0$ and $u_\infty = 0$ on $\partial \Omega_\infty$. By Lemma 6.2 and the proof of Corollary 6.6 applied to $T_{p_j,r_j}u_j$, we have that $\lim_{j \to \infty} N_{\Omega_j}(0, r, T_{p_j,r_j}u_j) = N_{\Omega_\infty}(0, r, u_\infty) \in [0, C_2(n, \Lambda)]$, and $N_{\Omega_\infty}(0, r, u_\infty)$ is constant for $1/10 \leq r \leq 1$. Therefore, by Lemma 7.4 $u_\infty$ is $(0, 0, 1, 0)$-symmetric. This contradicts our assumption that no $T_{p_i,r_i}u_i$ is $(0, \delta, 1, 0)$-symmetric.

The case that $B_1(0) \subset T_{p_i,r_i} \Omega_i$, repeat the argument to obtain the same contradiction. □

**Remark 7.3.** By the scale invariance of the Almgren frequency, Lemma 7.2 implies that for all $0 < \delta$, if $0 < \gamma \leq \gamma_0(n, \Lambda, \delta)$, then

$$|N_{\Omega}(p, r, u) - N_{\Omega}(p, r/10, u)| \leq \gamma$$

and either $\text{dist}(p, \{u = 0\}) \leq \gamma r$ or $B_r(p) \subset \Omega$ implies $u$ is $(0, \delta, r, p)$-symmetric.

Next, we obtain a “cone-splitting” result. The prototypical example of a result like this is the following proposition. See [HL] Theorem 4.1.3 for the proof of similar results.

**Proposition 7.4.** Let $P : \mathbb{R}^n \to \mathbb{R}$ be a 0-symmetric function. Let $k \leq n - 2$. If $P$ is symmetric with respect to some $k$-dimensional subspace $V$ and $P$ is homogeneous with respect to some point $x \notin V$, then $P$ is $(k+1)$-symmetric with respect to $\text{span}\{x, V\}$.

We now prove a similar result for our almost-symmetric functions $u \in A(n, \Lambda)$.

**Lemma 7.5.** (Cone-splitting) Let $0 \leq k \leq n - 2$ be an integer and $u \in A(n, \Lambda)$. For any fixed $\epsilon, \tau > 0$ and any $0 < r \leq 1/8$, there is a $0 < \delta_0 = \delta_0(n, \tau, \epsilon, \Lambda)$ such that for $0 < \delta < \delta_0$, the following holds. If $p \in \Omega \cap B_{\frac{1}{8}}(0)$ and $u$ is $(k, \delta, r, p)$-symmetric with respect to a $k$-dimensional subspace $V$ and $(0, \delta, r, x)$-symmetric for some $x \in B_r(p) \setminus B_{rr}(V + p)$, then $v$ is $(k + 1, \epsilon, r, p)$-symmetric.
Proof. Assume that there exists a $\delta, \tau > 0$ for which there exist a sequence of $0 < r_i \leq 1$, function, $u_i \in \mathcal{A}(n, \Lambda)$ and points $\{p_i\}$ for which $u_i$ is $(k, i^{-1}, r_i, p_i)$-symmetric with respect to some $V_i$ and $(0, i^{-1}, r_i, x_i)$-symmetric for some $x_i \in B_{r_i}(p_i) \setminus B_{r}(V_i + p_i)$, but that all $u_i$ are not $(k + 1, \delta, r_i, p_i)$-symmetric.

By considering $T_{p_i, r_i}u_i$ and applying Lemma 6.5 and Lemma 6.7 there exists a function $u_\infty \in L^2(B_1(0))$ such that a subsequence $T_{p_j, r_j}u_j \to u_\infty$ in the senses of the lemma. Note that $u_\infty$ is non-degenerate. Taking further subsequences, we may reduce to a sequence for which

$$T_{p_j, r_j}V_j \to V, \quad x_j \to x \in \overline{B_1(0)} \setminus B_r(V).$$

Note that $u_\infty$ is $(k, 0, 1, 0)$-symmetric with respect to $V$, and harmonic. Therefore, $u_\infty$ is $(k + 1, 0, 1, 0)$-symmetric. Since $u_i \to u$ in $L^2(B_2(0))$, we have our contradiction. By taking the smallest $0 < \delta_0$ for $0 \leq k \leq n - 2$, we eliminate the dependence upon $k$.

Corollary 7.6. Let $k \leq n - 2$ and $u \in \mathcal{A}(n, \Lambda)$. For any fixed $\epsilon, \tau > 0$ and any $0 < r \leq 1$, there is a $0 < \delta_0 = \delta_0(n, \tau, \epsilon, \Lambda)$ such that for $0 < \delta < \delta_0$, the following holds. If $p \in \overline{\Omega} \cap B_\frac{1}{4}(0)$ and $u$ is $(0, \delta, r, p)$-symmetric but not $(k + 1, \epsilon, r, p)$-symmetric then there exists an affine $k$-plane $V$ such that

$$\{x \in B_r(p) : u \text{ is } (0, \delta, r, x)\text{-symmetric}\} \subset B_{\tau r}(V).$$

Proof. Let $0 < \epsilon_1, \ldots, \epsilon_{n-1}$ be small parameters to be chosen later. To prove the lemma, we inductively apply Lemma 7.5. That is, suppose that there was another point $x_1 \in B_r(p) \setminus B_{r_1}(p)$ such that $u$ was $(0, \delta, r, x)$-symmetric, then Lemma 7.5 implies that for any $0 < \epsilon_1$ there is a $0 < \delta_0(n, \tau, \epsilon_1, \Lambda)$ such that if $\delta \leq \delta_0(n, \tau, \epsilon_1, \Lambda)$ then $u$ is $(1, \epsilon_1, r, p)$-symmetric with respect to some affine 1-plane $V_1$. Inducting up, assume that $u$ is $(i, \epsilon_i, r, p)$-symmetric with respect to $V_i$. Then, for any $0 < \epsilon_{i+1}$ there exists a $\delta_0(n, \tau, \epsilon_{i+1}, \Lambda)$ such that if we can find an $x_{i+1} \in B_{r}(p) \setminus B_{\tau r}(V_i)$ such that $u$ is $(0, \delta, r, x)$-symmetric and $0 < \delta, \epsilon_i \leq \delta_0(n, \tau, \epsilon_{i+1}, \Lambda)$, then $u$ is $(i + 1, \epsilon_{i+1}, r, p)$-symmetric with respect to $V_i$.

Therefore, choosing $0 < \epsilon_{n-1} \leq \delta_0(n, \tau, \epsilon, \Lambda)$ and $0 < \epsilon_{n-1} \leq \delta_0(n, \tau, \epsilon, \Lambda)$ for all $i = 1, \ldots, n - 1$, we obtain $0 < \delta_0(n, \tau, \epsilon, \Lambda) = \delta_0(n, \tau, \epsilon, \Lambda)$ as desired. Since it is assumed that such that $u$ is not $(k + 1, \epsilon, r, p)$-symmetric, this procedure must terminate before step $k + 1$ and therefore there must exist an affine $k$-plane such that the claim of the corollary holds.

8. The Covering and its Properties

The lemmata in the previous section allow us to inductively define a covering with the right packing conditions. Quantitative rigidity allows us to prove a “Quantitative Differentiation” lemma that bounds the number of scales across which the frequency can change by more than some threshold $\gamma > 0$. Cone splitting on the other hand, will give us good geometric control of the singular set at scales for which $v$ is close to a homogeneous harmonic polynomial. Together, these things will give us the necessary packing conditions. First, we describe the covering.

8.1. The General Construction. Let $u \in \mathcal{A}(n, \Lambda)$, and let $\epsilon, r > 0$, $k \leq n - 2$, and $N \in \mathbb{N}$ be given. We use the notation $\rho_i = 10^{-i}$. In this section we describe a general procedure which will produce a cover of $C_{\epsilon, r}^k(u) \cap B_{\frac{1}{10}}(0)$ by balls of radius $\rho_N$. 


We shall argue that there is a non-empty.

Proof. Let \( 0 < \delta_0 \). We shall refer to \( \delta_0 \) as the sorting threshold. For any \( i \in \mathbb{N} \) we can assign to each \( x \in C_{\epsilon, r}^k(u) \cap B_{\frac{1}{\rho}}(0) \) an \( i \)-tuple \( T^i(x) \) according to the rule

\[
(T^i(x))_j = \begin{cases} 
1 & \text{if } D(u, x, \rho_j) \geq \delta_0 \\
0 & \text{if } D(u, x, \rho_j) < \delta_0.
\end{cases}
\]

For any \( T_i \) we shall use \( |T^i| \) to denote the sum of the entries. Note that there is a partial ordering on the set of these \( i \)-tuples. That is, if \( k < i \), we can say that \( T^k < T^i \) if \( (T^k)_j = (T^i)_j \) for all \( j \in \{1, 2, ..., k\} \).

Now, we partition our set according to these \( i \)-tuples. For any given \( i \)-tuple, \( T^i \in \{0, 1\}^i \), we define

\[
E(T^i) = \{ x \in C_{\epsilon, r}^k(u) \cap B_{\frac{1}{\rho}}(0) : T^i(x) = T^i \}.
\]

It follows immediately from the definitions that \( E(T^i) \subset E(T^k) \) if and only if \( T^k < T^i \).

We now define our covering inductively. For \( i = 1 \), we let \( C_{\epsilon, r}^k(T^i) = B_{\frac{1}{\rho}}(0) \) for both 1-tuples \( T^i \in \{0, 1\}^1 \). Now, assume that \( i \in \mathbb{N}, i < N \), and \( C_{\epsilon, r}^k(T^i) \) has been defined and consists of balls of radius \( \rho_i \). Within each ball \( B_{\rho_i}(y) \subset C_{\epsilon, r}^k(T^i) \) partition the set \( B_{\rho_i}(y) \cap E(T^i) \) into the sets \( E(T^{i+1}) \) for \( T^i < T^{i+1} \) such that \( T^i < T^{i+1} \). For either such \( T^{i+1} \), take a minimal covering of \( B_{\rho_i}(y) \cap E(T^{i+1}) \) by balls of radius \( \rho^{i+1} \) centered at points in \( B_{\rho_i}(y) \cap E(T^{i+1}) \). The union of these balls is \( C_{\epsilon, r}^k(T^{i+1}) \).

For some \( i \)-tuples, the set \( E(T^i) \) may be empty. In this case, we simply allow the corresponding collection of balls, \( C_{\epsilon, r}^k(T^i) \), be empty.

If \( i = N \), we terminate the procedure. Note that for any sorting threshold \( 0 < \delta_0 \) and \( N \in \mathbb{N} \) this procedure defines a sequence of collections such that

\[
C_{\epsilon, r}^k(u) \cap B_{\frac{1}{\rho}}(0) \subseteq \bigcup_{T^N B_{\rho_N}(y) \in C_{\epsilon, r}^k(T^N)} B_{\rho_N}(y).
\]

8.2. Properties of the Construction. Now, we argue that there is a choice of sorting threshold \( 0 < \delta_0 \) with the desired properties.

Lemma 8.1. Let \( u \in A(n, \Lambda) \). Let \( 0 < \delta' \) be the sorting threshold in the construction above. For any \( i \in \mathbb{N} \) there are at most \( N^{D(n, \Lambda, \delta')} \) nonempty sets \( E(T^i) \) such that \( E(T^i) \) is non-empty.

Proof. Let \( 0 < \delta' \) be given. Let \( 0 < \gamma_0(n, \Lambda, \delta') \ll 1 \) as in Lemma 7.2 and Remark 7.3.

Now, decompose \( \Omega = \bigcup_{j=0}^{\infty} A_j(\Omega) \) where

\[
A_j(\Omega) = \Omega \cap B_{j+1}(\partial \Omega) \setminus B_j(\partial \Omega).
\]

We shall argue that there is a \( D = D(n, \Lambda, \delta') \) such that \( |T^N(p)| \leq D \) for all \( p \in \overline{\Omega} \cap B_{10}(0) \). If the claim is true, then if \( N \leq D \) there are at most \( 2^N \leq N^D \) \( N \)-tuples with \( |T^N| \leq D \). And, if \( N \geq D \) there are at most \( \binom{N}{D} \) many \( N \)-tuples with \( |T^N| \leq D \). Since \( \binom{N}{D} \leq N^D \) we have the desired claim: \( C_{\epsilon, r}^k(u) \cap B_{1/10}(0) \) is contained in the union of at most \( N^D \) nonempty sets \( E(T^N) \) and covered by at most \( N^D \) collections \( C_{\epsilon, r}^k(T^N) \).
To prove the claim, we argue by cases. Let \( p \in E(T^N) \). If \( p \in \partial \Omega \), then \( u(p) = 0 \) and 
\[
|T^N(p)| \leq |\{ i \in \mathbb{N} : |N_\Omega(p, \rho_i, u) - N_\Omega(p, \rho_{i+1}, u)| \geq \gamma_0(n, \Lambda, \delta') \}| \leq C_3(n, \Lambda)\gamma_0^{-1}.
\]
for \( \gamma_0(n, \Lambda, \delta') \) as in Remark 7.3.

Now, assume that \( p \in A_j(\Omega) \). Note that by Remark 7.3,
\[
|N_\Omega(p, \rho_i, u) - N_\Omega(p, \rho_{i+1}, u)| \leq \gamma_0(n, \Lambda, \delta')
\]
still implies \( (T^N(p))_i = 0 \) if \( B_{\rho_i}(p) \subset \Omega \) or if \( \text{dist}(p, \partial \Omega) \leq \gamma_0(n, \Lambda, \delta')\rho_i \). But, the conditions \( B_{\rho_i}(p) \subset \Omega \) or \( \text{dist}(p, \partial \Omega) \leq \gamma_0(n, \Lambda, \delta')\rho_i \) only fail for \( i \in [(j - 1)\frac{\ln(\gamma_0)}{\ln(\rho_0)}, (j + 1)\frac{\ln(\gamma_0)}{\ln(\rho_0)}] \). Therefore
\[
|T^N(p)| \leq \frac{C_3(n, \Lambda)}{\gamma_0(n, \Lambda, \delta')} + 3\frac{\ln(\gamma_0(n, \Lambda, \delta'))}{\ln(\rho_0)} = D(n, \Lambda, \delta')
\]
independent of \( j \). This proves the claim.

Remark 8.2. If, later, we choose \( 0 < \gamma \leq \gamma_0(n, \Lambda, \delta') \) so that \( \gamma = \gamma(n, \Lambda, \varepsilon) \), then the statement of Lemma 8.1 holds with a new \( D = D(n, \Lambda, \varepsilon) \).

We now prove that this construction satisfies the claimed packing condition.

Lemma 8.3. Let \( u \in A(n, \Lambda), 0 < \varepsilon, r < 1, \) and \( k \leq n - 2. \) Let \( \delta' = \delta_0(n, 1/10, \varepsilon, \Lambda) \) as in Corollary 7.6. Let \( \gamma_0(n, \Lambda, \delta') \) be as in Lemma 7.2.

Then for all \( N \in \mathbb{N} \) with \( \rho_N \geq r \) there exist constants \( 0 < c_1, c_2 \) depending only on the ambient dimension \( n \) and a constant \( D(n, \Lambda, \varepsilon) \) such that each collection \( C_{\varepsilon,r}(T^N) \) consists of at most \( (c_1\rho_1^{-n})^D(c_2\rho_1^{-k})^{N-D} \) balls of radius \( \rho_N \).

Proof. For any given \( N \)-tuple \( T^N \) for which \( E(T^N) \) is non-empty, let \( T^i < T^N \). Now, let \( B_{\rho_i}(x) \in C_{\varepsilon,r}(T^i) \). Consider the set
\[
A := B_{\rho_i}(x) \cap E(T^N).
\]
We argue by cases.

Case 1. \( (T^N)_i = 0 \). In this case, \( u \) is \((0, \delta_0, \rho_i, x)\)-symmetric. By applying Corollary 7.6 we see that \( A \subset B_{\rho_i}(x) \cap B_{\rho_{i+1}}(V^k) \) for some \( k \)-dimensional plane \( V^k \). Thus, the minimal covering from the construction can cover \( A \subset B_{\rho_i}(x) \cap B_{\rho_{i+1}}(V^k) \) by at most \( c_1(n)\rho_1^{-k} \) balls of radius \( \rho_{i+1} \).

Case 2. \( (T^N)_i = 1 \). In this case, we have no control. Therefore, the minimal covering of \( A \) described in the construction could consist of at most \( c_2(n)\rho_1^{-\gamma} \) balls of radius \( \rho_{i+1} \).

Carrying this process through, by the proof of Lemma 8.1 Case 2 can only happen at most \( D(n, \Lambda, \varepsilon) \) times. Thus, \( C_{\varepsilon,r}(T^N) \) is a collection of at most \( (c_1\rho_1^{-n})^D(c_2\rho_1^{-k})^{N-D} \) balls of radius \( \rho_N \), as claimed.

8.3. Proof of Theorem 2.10

Proof. Let \( 0 < \varepsilon, r < 1, \) and \( k \leq n - 2. \) Let \( \delta_0 = \delta_0(n, 1/10, \varepsilon, \Lambda) \) as in Corollary 7.6. Let \( 0 < \gamma_0(n, \Lambda, \delta_0) \) be as in Lemma 7.2. Recalling \( c_2(n) \) from Lemma 8.3 let
\[
0 < \gamma \leq \min\{\gamma_0(n, \Lambda, \delta_0), c_2^{-1}\} < 1.
\]

The construction given in Section 8.1 gives a covering of \( B_{\frac{1}{10}}(0) \cap C_{\varepsilon,r}(u) \) by balls of radius \( \rho_N \). Doubling the radius of these balls is sufficient, then, to cover \( B_{\frac{1}{10}}(0) \cap B_{\rho_N}(C_{\varepsilon,r}(u)) \).
Thus, by Remark \[8.2\] for \( D = D(n, \Lambda, \epsilon) \)

\[
\text{Vol}(B_{1/10}^k(0) \cap B_{\rho_N}(C_{e,r}(u))) \leq N^D (c_1 \rho_1^{-n} c_2 \rho_1^{-k})^{N-D} (\omega_n 2^{p^N})^n
\leq c(n) N^D c_1^D c_2^{-D} (\omega_n 2^N)^n \rho_N^{n-k} \rho_1^{-D(n-k)}.
\]

Estimating \( \rho_1^{-D(n-k)} \leq C(n, \Lambda, \epsilon) \) and \( N^D \leq C(n, \Lambda, \epsilon) c_2(n)^N \) for all \( N \in \mathbb{N} \) and recalling from our choice of \( 0 < \gamma \) that \( c_2(n) \leq \gamma^{-\frac{1}{2}} \), we obtain

\[
\text{Vol}(B_{1/10}^k(0) \cap B_{\rho_N}(C_{e,r}(u))) \leq C(n, \Lambda, \epsilon) c_2(n)^{2N-D} c_1(n)^D (\omega_n 2^N)^n \rho_N^{n-k}
\leq C(n, \Lambda, \epsilon) \gamma^{-N} \rho_N^{n-k}
\leq C(n, \Lambda, \epsilon) \rho_N^{n-k-\epsilon}.
\]

Thus, for any \( 0 < r \) we may find an \( N \in \mathbb{N} \) such that \( \rho_{N+1} \leq r < \rho_N \). Thus, we may estimate

\[
\text{Vol}(B_{1/10}^k(0) \cap B_r(C_{e,r}(u))) \leq C(n, \Lambda, \epsilon) \rho_N^{n-k-\epsilon}
\leq C(n, \Lambda, \epsilon) (10r)^{n-k-\epsilon} \leq C'(n, \Lambda, \epsilon) (r)^{n-k-\epsilon}.
\]

Covering \( B_{1/10}^k(0) \) \( C(n) \) many such balls of radius \( 1/10 \) and repeating the argument produces \( [2] \).

\[\square\]

9. Containment

We now turn to proving Lemma \[2.11\] which follows from the rigidity and continuity of the Almgren frequency function.

**Lemma 9.1.** Let \( u \in \mathcal{A}(n, \Lambda) \). Suppose that \( Q \in \partial \Omega \cap B_{1/4}(0) \) and there exists a \( p \in C^{n-2}(u) \) such that \( B_r(p) \subset \Omega \cap B_{1/4}^k(0) \) and \( p \subset B_{2r}(Q) \). Then there is an \( 0 < \delta(n, \Lambda) \) such that \( N_{\Omega}(Q, 4r, u) \geq 1 + \delta \).

**Proof.** Suppose that the statement is false. Then, there exists a sequence of \( u_i, Q_i, p_i \) and \( 0 < r_i \) such that \( B_{r_i}(p_i) \subset \Omega \cap B_{1/4}^k(0) \) for which

\[
N_{T_{Q_i,4r_i}^i, \alpha_i}(0, 1, T_{Q_i,4r_i}^i, u_i) \leq 1 + 2^{-i}.
\]

By Lemma \[6.5\] we may extract a subsequence \( T_{Q_j,4r_j}^j, u_j \) which converges strongly in \( W^{1,2}(B_1(0)) \) to a function \( u_\infty \). Similarly, we may assume that \( T_{Q_j,4r_j}^j, p_j \to p \) and that \( B_{1/4}(p) \subset \Omega_\infty \cap B_{1/4}(0) \).

Observe that by Lemma \[6.6\] \( N_{\Omega_\infty}(0, 1, u_\infty) = 1 \). Therefore, \( N_{\Omega_\infty}(0, r, u_\infty) \) is constant for \( 0 < r \leq 1 \). By Lemma \[7.1\] \( u_\infty \) must be a piecewise linear function. Therefore, \( N_{\Omega_\infty}(p, 1/4, u_\infty) = 1 \). However, \( N_{\Omega_\infty}(p_j, r_j, u_j) \geq 2 \) for all \( j \) which implies by Lemma \[6.6\] that \( N_{\Omega_\infty}(p, 1/4, u_\infty) \geq 2 \). This is a contradiction. \[\square\]

**Lemma 9.2.** Let \( u \in \mathcal{A}(n, \Lambda) \). Suppose that \( p \in \overline{\Omega} \cap B_{1/4}(0) \) and an \( 0 < r \leq 1/4 \) such that \( N_{\Omega}(p, r, u) \geq 2 \). If either \( p \in \partial \Omega \) or \( B_r(p) \subset \Omega \) then, there is an \( 0 < \epsilon = \epsilon(n, \Lambda) \) such that \( u \) is not \( (n-1, \epsilon, r, p) \)-symmetric.

**Proof.** Suppose that the statement is false. Then, there exists a sequence of \( u_i, p_i \) and \( 0 < r_i \) such that \( N_{T_{p_i,r_i}^i, \alpha_i}(0, 1, T_{p_i,r_i}^i, u_i) \geq 2 \) and \( u_i \) is \( (n-1, 2^{-i}, r, p) \)-symmetric. By Lemma \[6.5\] we may extract a subsequence \( T_{p_j,r_j}^j, u_j \) which converges strongly in \( W^{1,2}(B_1(0)) \) to a function \( u_\infty \). By Lemma \[6.6\] \( N_{\Omega_\infty}(0, 1, u_\infty) \geq 2 \). But, \( u_\infty \) is \( (n-1, 0, 1, 0) \)-symmetric and hence piece-wise linear.
If either $B_1(0) \subset \Omega_\infty$ or $0 \in \partial \Omega$ then $u_\infty$ being piecewise linear implies $N_{\Omega_\infty}(0, 1, u_\infty) = 1$. This is a contradiction. \hfill \square

9.1. Proof of Lemma 2.11.

\begin{proof}
First, we prove (3). Because for all integers $k$ and all $0 < \epsilon$ the set $C_\epsilon^k(u)$ is closed, we reduce to proving that there is an $0 < \epsilon(n, \Lambda)$ such that $C_{\epsilon(n, \Lambda)}^{n-2}(u) \cap B_{1/8}(0) \subset C_{\epsilon(n, \Lambda)}^{n-2}(u)$. Suppose that this containment is false. Then, there would exist a sequence of functions $u_i \in A(n, \Lambda)$, points $p_i \in C_{n-2}(u_i) \cap B_{\delta/4}(0)$ and scales $0 < r_i \leq 1$ such that $u_i$ is $(n - 1, 2^{-i}, p_i, r_i)$-symmetric. We rescale to the functions $T_{p_i, r_i} u_i$. By Lemma 6.5 there exists a subsequence (also indexed by $i$) and a $(n - 1)$-symmetric function $u_\infty$ such that $T_{p_i, r_i} u_i \to u_\infty$ strongly in $W^{1,2}(B_1(0))$ and $C^0(B_1(0))$. Note that for all $Q \in \partial \Omega_\infty$ and all $B_r(p) \subset \Omega_\infty$ $N_{\Omega_\infty}(Q, r, u_\infty) = N_{\Omega_\infty}(p, r, u_\infty) = 1$.

If $B_2(0) \in \Omega_\infty$ then for all sufficiently large $i \in \mathbb{N}$, $N_{\Omega_\infty}(p_i, \delta/2r_i, u_i) \geq 2$. Letting $i \to \infty$, by Lemma 6.6 $N_{\Omega_\infty}(0, \delta/2, u_\infty) \geq 2$ which contradicts $u_\infty$ being piecewise linear.

If $0 \in \partial \Omega_\infty$, then for each $p_i$ we let $Q_i \in \partial \Omega_\infty$ be such that $|p_i - Q_i| = \text{dist}(p_i, \partial \Omega)$. By Lemma 9.1 there exists a $0 < \delta(n, \Lambda)$ such that $N_{\Omega_\infty}(Q_i, 4|Q_i - p_i|, u_i) \geq 1 + \delta$ for all $i \in \mathbb{N}$. For sufficiently large $i$ $4|Q_i - p_i| \leq r_i$ and so by Lemma 3.4 for sufficiently large $i$ $N_{\Omega_\infty}(Q_i, r_i, u_i) \geq 1 + \delta$. Letting $i \to \infty$ we obtain that by Lemma 6.6 that $N_{\Omega_\infty}(0, 1, u_\infty) > 1 + \delta$. This is a contradiction. Thus, there exists an $0 < \epsilon(n, \Lambda)$ such that $C_{\epsilon(n, \Lambda)}^{n-2}(u) \cap \Omega \cap B_{1/8}(0) \subset C_{\epsilon(n, \Lambda)}^{n-2}(u)$.

For prove (4, we note that if $Q \in C_{\epsilon(n, \Lambda)}^{n-2}(u) \cap \partial \Omega \setminus \text{sing}(\partial \Omega)$, then letting $r \to 0$, we may extract a subsequence such that $T_{Q, r} u \to u_\infty$ in the sense of Lemma 5.4 and Lemma 6.3. By the monotonicity of the Almgren frequency, Lemma 6.6 and by considering $T_{Q, r} u$ for any $0 < c < 1$ we see that $N_{\Omega_\infty}(0, r, u_\infty) \equiv \lim_{r \to 0^+} N_{\Omega}(Q, r, u)$. Thus, Lemma 7.1 implies that $u_\infty$ is a homogeneous function which is harmonic in $\Omega_\infty$. Since $Q \notin \text{sing}(\partial \Omega)$, $\Omega_\infty$ is a half-space and we may extend $u_\infty$ to an entire, homogeneous harmonic function by reflection. Since $Q \in C_{\epsilon(n, \Lambda)}^{n-2}(u)$ this polynomial must be a non-linear homogeneous harmonic polynomial and $N_{\Omega_\infty}(Q, r, u) \geq 2$. By Lemma 3.4 then $N_{\Omega}(Q, r, u) \geq 2$ for all $0 < r \leq 1$ and Lemma 9.2 gives the claim. \hfill \square

10. Appendix A: Hölder Continuity

In this section, we provide a proof of Lemma 5.1. First, some standard results.

Definition 10.1. A bounded domain, $\Omega \subset \mathbb{R}^n$, is said to be of class $S$ if there exist numbers $0 < c_0 \leq 1$ and $0 < r_0$ such that for all $Q \in \partial \Omega$ and all $0 < r \leq r_0$

$$\mathcal{H}^n(B_r(Q) \cap \Omega^c) \geq c_0 \mathcal{H}^n(B_r(Q)).$$

Lemma 10.2. (Bounding the Supremum, \cite[Lemma 1.1.22]{Ken94}) Let $\Omega$ be a domain of class $S$. Let $Q \in \partial \Omega \cap B_1(0)$ and $0 < r \leq \frac{1}{2}$. Let $u$ be a function which is harmonic in $\Omega$ such that $u \in C(B_{2r}(Q) \cap \Omega)$ and $u \equiv 0$ on $B_{2r}(Q) \cap \partial \Omega$. There exists a $c(n)$ such that for any $p \in B_r(Q) \cap \Omega$

$$\max_{B_r(Q) \cap \Omega} |u| \leq c(n) \left( \int_{B_{2r}(Q) \cap \Omega} u^2 \, dx \right)^{\frac{1}{2}}.$$

Lemma 10.3. (Hölder continuity up to the Boundary, \cite[Corollary 1.1.24]{Ken94}) Let $\Omega$ be a domain of class $S$. Let $Q \in \partial \Omega \cap B_1(0)$ and $0 < r \leq \frac{1}{2}$. Let $u$ be a function which is
harmonic in $\Omega$, $u \in C(B_{2r}(Q) \cap \Omega)$, $u \equiv 0$ on $B_{2r}(Q) \cap \partial \Omega$, and $u \geq 0$. There exists a $c(n)$ and an exponent $0 < \alpha(n) \leq 1$ such that for any $p \in B_{r}(Q) \cap \Omega$

\[
u(p) \leq c(n) \left( \frac{|p - Q|}{r} \right)^{\alpha} \sup \{u(y) : y \in B_{2r}(Q)\}.
\]

**Theorem 10.4.** (Oscillation in the Interior, [HKM93 Theorem 6.6]) Suppose that $u$ is harmonic in $\Omega$. If $0 < r < R < \infty$ are such that $B_{r}(x) \subset B_{R}(x_{0}) \subset \Omega$, then

\[\text{osc}(u, B_{r}(x_{0})) \leq 2^{\alpha} \left( \frac{r}{R} \right)^{\alpha} \text{osc}(u, B_{R}(x_{0})),\]

where $\alpha = \alpha(n) \in (0, 1]$ only depends on $n$.

**Theorem 10.5.** (Hölder Continuity in $B_{2}(0)$, [HKM93 Theorem 6.44]) Suppose that $\Omega_{1}$ is of class $S$ with constant $c_{0} > 0$ and $0 < r_{0} \leq 1$. Let $h \in C^{0}(\Omega_{1})$ be a harmonic function in $\Omega_{1}$. If there are constants $M \geq 0$ and $0 < \alpha \leq 1$ such that

\[|h(x) - h(y)| \leq M|x - y|^{\alpha}\]

for all $x, y \in \partial \Omega_{1}$, then

\[|h(x) - h(y)| \leq M_{1}|x - y|^\gamma\]

for all $x, y \in \overline{\Omega_{1}}$. Moreover, $\gamma = \gamma(n, \alpha, c_{0}) > 0$ and one can choose $M_{1} = 80Mr_{0}^{-2}\max\{1, \text{diam}(\Omega_{1})\}$.

10.1. **Proof of Lemma 5.1.** Let $u \in \mathcal{A}(n, \Lambda)$ with associated domain $\Omega \in \mathcal{D}(n)$, and let $Q_{0} \subset \partial \Omega \cap B_{1}(0)$ and $0 < r_{0} \leq \frac{1}{2}$. First, we claim that $(\int_{B_{2}(0) \cap \Omega}(T_{Q_{0},r_{0}}u)^{2}dx)^{1/2} \leq C(n, \Lambda)$. By the Poincare inequality

\[
\int_{B_{2}(0)} |T_{Q,2r}u - \text{Avg}_{B_{2r}(0)}(T_{Q,r}u)|^{2}dx \leq C(n)|B_{2r}(0)|^{2} \left( \int_{B_{2}(0)} |\nabla T_{Q,r}u|^{2}dx \right)^{1/2} \leq C(n)C(\Lambda, n).
\]

Furthermore, since $\mathcal{H}^{n}(\partial_{c} \cap B_{2}(0)) \geq \frac{1}{10}n\mathcal{H}^{n}(B_{2}(0))$, we estimate

\[|\text{Avg}_{B_{2r}(0)}(T_{Q,r}u)|^{2} \frac{1}{10}n\mathcal{H}^{n}(B_{2}(0)) \leq C(n)C(\Lambda, n).
\]

Thus, $|\int_{B_{2r}(0)}(T_{Q,r}u)| \leq C(n)\sqrt{C(n, \Lambda)}$.

Next, we claim that there exists a $c(n)$ and an exponent $0 < \alpha(n) \leq 1$ such that, for $Q \in T_{Q_{0},r_{0}}\partial \Omega \cap B_{1}(0)$ and $p \in T_{Q_{0},r_{0}}\partial \Omega \cap B_{\frac{1}{2}}(Q)$

\[|T_{Q_{0},r_{0}}u(p)| \leq C(n, \Lambda) \left( \frac{|p - Q|}{r} \right)^{\alpha}.
\]

For any $T_{Q_{0},r_{0}}u$ which changes sign in $T_{Q_{0},r_{0}}\partial \Omega \cap B_{1}(Q)$, we decompose $T_{Q_{0},r_{0}}u = T_{Q_{0},r_{0}}u^{+} - T_{Q_{0},r_{0}}u^{-}$. Note that both $T_{Q_{0},r_{0}}u^{\pm}$ are subharmonic. Let $h_{\pm}$ be the harmonic extension of $T_{Q_{0},r_{0}}u^{\pm}$ to $B_{1}(Q) \cap T_{Q_{0},r_{0}}\Omega$. Note that $B_{1}(Q) \cap T_{Q_{0},r_{0}}\Omega$ is convex, and so is of class $S$. Then, by Lemma [10.3] and the maximum principle

\[h_{\pm}(p) \leq c(n) \left( \frac{|p - Q|}{r} \right)^{\alpha} \sup \{h_{\pm}(y) : y \in \partial(B_{1}(Q) \cap T_{Q_{0},r_{0}}\Omega)\}.\]
By subharmonicity, $T_{Q_0, r_0} u^\pm \leq h_\pm$, respectively. By construction, $h_\pm = T_{Q_0, r_0} u^\pm$ on $\partial (B_1(Q) \cap T_{Q_0, r_0} \Omega)$ and by our first claim and Lemma [10.2]

$$
\sup \{h_\pm(y) : y \in \partial (B_1(Q) \cap T_{Q_0, r_0} \Omega) \} \leq C(n, \Lambda).
$$

Note that this gives uniform control on the oscillation in $T_{Q_0, r_0} \Omega \cap B_2(0)$. This uniform control together with Theorem [10.4] implies that $T_{Q_0, r_0} u$ is locally Hölder on $\partial B_1(0) \cap T_{Q_0, r_0} \Omega$.

Now, we claim that for all $x, y \in \partial (T_{Q_0, r_0} \Omega \cap B_1(0))$

$$
|T_{Q_0, r_0} u(x) - T_{Q_0, r_0} u(y)| \leq C(n, \Lambda)|x - y|^\alpha.
$$

We argue by cases. Suppose that $|x - y| < \max \{ \text{dist}(x, T_{Q_0, r_0} \partial \Omega), \text{dist}(y, T_{Q_0, r_0} \partial \Omega) \}$. Then, there is a ball, $B_r(z) \subset T_{Q_0, r_0} \Omega$ with $|x - y| < r \leq 2|x - y|$ which contains both $x$ and $y$. By Theorem [10.4] and the preceding paragraph, then we have the desired statement.

Suppose that $|x - y| \geq \max \{ \text{dist}(x, T_{Q_0, r_0} \partial \Omega), \text{dist}(y, T_{Q_0, r_0} \partial \Omega) \}$. Let $x_0, y_0 \in T_{Q_0, r_0} \partial \Omega$ be points such that $|x - x_0| = \text{dist}(x, \partial \Omega)$ and $|y - y_0| = \text{dist}(y, \partial \Omega)$. Then

$$
|T_{0,1} u(x) - T_{0,1} u(y)| \leq |T_{0,1} u(x) - T_{0,1} u(x_0)| + |T_{0,1} u(y) - T_{0,1} u(y_0)| \\
\leq C(n, \Lambda)2^\alpha |x - x_0|^{\alpha} + C(n, \Lambda)2^\alpha |y - y_0|^{\alpha} \\
\leq C(n, \Lambda)2^{\alpha+1} \max \{ \text{dist}(x, T_{Q_0, r_0} \partial \Omega), \text{dist}(y, T_{Q_0, r_0} \partial \Omega) \}^{\alpha} \\
\leq C(n, \Lambda)|x - y|^{\alpha}.
$$

This proves the claim. To obtain uniform interior Hölder continuity on the interior of $T_{Q_0, r_0} \Omega \cap B_1(0)$, we invoke Theorem [10.5] with $\Omega_1 = T_{Q_0, r_0} \Omega \cap B_1(0)$.

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