Staircase tableaux, the asymmetric exclusion process, and Askey-Wilson polynomials

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We introduce some combinatorial objects called staircase tableaux, which have cardinality $4^n n!$, and connect them to both the asymmetric exclusion process (ASEP) and Askey-Wilson polynomials. The ASEP is a model from statistical mechanics introduced in the late 1960s, which describes a system of interacting particles hopping left and right on a one-dimensional lattice of $n$ sites with open boundaries. It has been cited as a model for traffic flow and translation in protein synthesis. In its most general form, particles may enter and exit at the left with probabilities $\alpha$ and $\gamma$, and they may exit and enter at the right with probabilities $\beta$ and $\delta$. In the bulk, the probability of hopping left is $q$ times the probability of hopping right. Our first result is a formula for the stationary distribution of the ASEP with all parameters general, in terms of staircase tableaux. Our second result is a formula for the moments of (the weight function of) Askey-Wilson polynomials, also in terms of staircase tableaux. The Askey-Wilson polynomials are a family of $q$-orthogonal polynomials with parameters $a, b, c, d, q$ (14). They reside at the top of the hierarchy of the one-variable $q$-orthogonal polynomial family in the Askey scheme. In the early 1980s, following groundbreaking work of Flajolet (15), Viennot (16) initiated a combinatorial approach to orthogonal polynomials. Since then, combinatorial formulas have been given for the moments of (the weight functions of) many of the polynomials in the Askey scheme, including $q$-Hermite, Tchebycheff, $q$-Laguerre, Charlier, and Al-Salam-Chihara polynomials, see e.g. refs. 17–20. However, until now, no such formula was known for the moments of the Askey-Wilson polynomials.

Staircase tableaux seem to have a great deal of combinatorial interest, as well as a potential connection to geometry. For example, staircase tableaux of size $n$ have cardinality $4^n n!$, and hence are in bijection with doubly signed permutations. In a subsequent paper we will present an explicit bijection, as well as connections to other combinatorial objects. Furthermore, because of the connection to the ASEP, we know that our staircase tableaux have some hidden symmetries which are not at all apparent from their definition. For instance, it is clear from the definition of the ASEP that the model remains unchanged if we reflect it over the $y$-axis, and exchange parameters $\alpha$ and $\beta$, $\gamma$ and $\delta$, and $q$ and $u$. However, the corresponding bijection on the level of tableaux has so far eluded us. Finally, staircase tableaux generalize permutation tableaux, which index certain cells in the nonnegative part of the Grassmannian (13); it would be interesting to better understand the relationship between the tableaux, the ASEP, and the geometry, and potentially generalize it to staircase tableaux.

Here we provide an outline of the proofs. However, details will appear elsewhere.

Discussion

The Asymmetric Exclusion Process (ASEP). The ASEP is often defined using a continuous time parameter (22). However, one can equivalently define it as a discrete-time Markov chain (7), as follows.

**Definition 1:** Let $\alpha$, $\beta$, $\gamma$, $\delta$, $q$, and $u$ be constants such that $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $0 \leq q \leq 1$, and $0 \leq u \leq 1$. The ASEP is the Markov chain on the $2^n$ words in the language $\{\ast, \bullet\}$, with transition probabilities:

- If $X = A \ast \bullet$ and $Y = A \ast \bullet$ then $P_{X} \ast \gamma = \frac{\gamma}{\alpha + \gamma}$ (particle hops right)
- If $Y = \ast B$ and $Y = \ast B$ then $P_{Y, \ast} = \frac{\gamma}{\alpha + \gamma}$ (particle hops left).
- If $X = B \ast$ and $X = B \ast$ then $P_{X, \ast} = \frac{\gamma}{\alpha + \gamma}$.

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• If $X = xB$ and $Y = eB$ then $P_{X,Y} = \frac{1}{n}$.
• If $X = xB$ and $Y = B$ then $P_{X,Y} = \frac{1}{n^2}$.
• Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{x \neq y} P_{X,Y}$.

Note that we will sometimes denote a state of the ASEP as a word in $\{\alpha, \beta, \gamma, \delta\}$ and sometimes as a word in $\{x, \bullet\}^n$. In the former notation, the symbol $x$ denotes the absence of a particle, which one can also think of as a white particle.

See Fig. 1 for an illustration of the four states, with transition probabilities, for the case $n = 2$. The probabilities on the loops are determined by the fact that the sum of the probabilities on all outgoing arrows from a given state is 1.

In the long time limit, the system reaches a steady state where all the probabilities $P_n(\tau_1, \tau_2, \ldots, \tau_n)$ of finding the system in configurations $(\tau_1, \tau_2, \ldots, \tau_n)$ are stationary, i.e. satisfy

$$\frac{d}{dt} P_n(\tau_1, \ldots, \tau_n) = 0.$$ 

Moreover, the stationary distribution is unique, as shown by Derrida, Evans, Hakim, and Pasquier (21).

When the parameters are all positive, it is clear that the ASEP has multiple symmetries, including the following.

• The “left-right” symmetry: if we reflect the ASEP over the $y$ axis, we get back the same model, except that the parameters $\alpha$ and $\delta$, $\beta$ and $\gamma$, and $u$ and $q$ are switched.

• The “arrow-reversal” symmetry: if we exchange black and white particles, we get back the same model, except that the parameters $\alpha$ and $\gamma$, $\beta$ and $\delta$, and $u$ and $q$ are switched.

• The “particle-hole” symmetry: if we compose the above two symmetries, i.e. reflect the ASEP over the $y$ axis and exchange black and white particles, we get back the same model, except that $\alpha$ and $\beta$, and $\gamma$ and $\delta$ are switched.

These symmetries imply results about the stationary distribution.

**Observation 1.** The steady state probabilities satisfy the following identities:

$$P_n(\tau_1, \ldots, \tau_n) = P_n(\tau_1, \ldots, \tau_1\text{ with }\alpha\text{ or }\delta\text{ exchanged}),$$

$$P_n(\tau_1, \ldots, \tau_n) = P_n(1 - \tau_1, \ldots, 1 - \tau_n\text{ with }\alpha\text{ or }\delta\text{ exchanged}),$$

$$P_n(\tau_1, \ldots, \tau_n) = P_n(1 - \tau_1, \ldots, 1 - \tau_1\text{ with }\alpha\text{ or }\delta\text{ exchanged}).$$

Above, the notation $|_{\alpha=\delta}$ indicates that the parameters $\alpha$ and $\delta$ are exchanged.

**Staircase Tableaux and the ASEP** The main combinatorial objects of this paper are some new tableaux which we call **staircase tableaux**. These tableaux generalize permutation tableaux (12, 13) and also Viennot’s alternative tableaux.

Definition 2: A staircase tableau of size $n$ is a Young diagram of “staircase” shape $(n, n - 1, \ldots, 2, 1)$ such that boxes are either empty or labeled with $\alpha$, $\beta$, $\gamma$, or $\delta$, subject to the following conditions:

- no box along the diagonal is empty;
- all boxes in the same row and to the left of a $\beta$ or a $\delta$ are empty;
- all boxes in the same column and above an $\alpha$ or a $\gamma$ are empty.

The type of a staircase tableau is a word in $\{\alpha, \beta, \gamma, \delta\}$ obtained by reading the diagonal boxes from northeast to southwest and writing $\bullet$ for each $\alpha$ or $\delta$, and $\circ$ for each $\beta$ or $\gamma$.

See Fig. 2 for an example.

**Definition 3:** The weight $wt(\mathcal{F})$ of a staircase tableau $\mathcal{F}$ is a monomial in $\alpha, \beta, \gamma, \delta, q,$ and $u$, which we obtain as follows. Every blank box of $\mathcal{F}$ is assigned a $q$ or $u$, based on the label of the closest labeled box to its right in the same row and the label of the closest labeled box below it in the same column, such that

- every blank box which sees a $\beta$ to its right gets a $u$;
- every blank box which sees a $\delta$ to its right gets a $q$;
- every blank box which sees an $\alpha$ or $\gamma$ to its right, and an $\alpha$ or $\delta$ below it, gets a $u$;
- every blank box which sees an $\alpha$ or $\gamma$ to its right, and a $\beta$ or $\gamma$ below it, gets a $q$.

After filling all blank boxes, we define the weight $wt(\mathcal{F})$ to be the product of all labels in all boxes.

Fig. 3 shows that the staircase tableau of Fig. 2 has weight $\alpha^3\beta^2\gamma^3\delta q^4u^3$.

**Remark 1:** The weight of a staircase tableau always has degree $n(n + 1)/2$. For convenience, we will sometimes set $u = 1$, since this results in no loss of information.

Our first main result is the following.

**Theorem 2.** Consider any state $\tau$ of the ASEP with $n$ sites, where the parameters $\alpha, \beta, \gamma, \delta, q,$ and $u$ are general. Set $Z_n = \sum_{\mathcal{F}} wt(\mathcal{F})$, where the sum is over all staircase tableaux of size $n$. Then $Z_n$ is the partition function for the ASEP, and the steady state probability that the ASEP is at state $\tau$ is precisely

$$\frac{\sum_{\mathcal{F}} wt(\mathcal{F})}{Z_n},$$

where the sum is over all staircase tableaux $\mathcal{F}$ of type $\tau$.

Fig. 4 illustrates Theorem 2 for the state $\bullet \bullet$ of the ASEP. All staircase tableaux $\mathcal{F}$ of type $\bullet \bullet$ are shown. It follows that the steady state probability of $\bullet \bullet$ is

$$\frac{\alpha^3u + \delta^3q + q\delta u + \alpha\delta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \alpha\beta^2}{Z_2}.$$

**A Moment Formula for Askey-Wilson Polynomials.** The Askey-Wilson polynomial is a $q$-orthogonal polynomial with four free parameters besides $q$. It resides at the top of the hierarchy of the one-variable $q$-orthogonal polynomial family in the Askey scheme (14, 22).

Fig. 5 illustrates Theorem 2 for the state $\bullet \bullet$ of the ASEP.
The $q$-shifted factorial is defined by

$$(a_1,a_2,\ldots,a_n;q)_n = \prod_{r=0}^{n-1}(1-a_rq^r),$$

and the basic hypergeometric function $\phi^{[a_1,\ldots,a_n]}_{[b_1,\ldots,b_n]}(q,z)$ is defined by

$$\sum_{k=0}^{\infty} (a_1,\ldots,a_n)_k (b_1,\ldots,b_n)_k (-1)^k q^{\binom{k}{2}} z^k.$$

The Askey-Wilson polynomial $P_n(x) = P_n(x;a,b,c,d;q)$ is explicitly defined to be

$$a^n(ab,ac,ad;q)_n\phi\left[q^{-n},q^{n-1}abcd,ae^{\theta},ae^{-\theta};q,q\right],$$

with $x = \cos \theta$ for $n \in \mathbb{Z}$. For $|a|,|b|,|c|,|d| < 1$, using $z = e^{i\theta}$, the orthogonality is expressed by

$$\int_C \frac{dz}{4\pi i} w\left(z+z^{-1}\right)\frac{P_n(z+z^{-1})}{2} = \delta_{mn},$$

where the integral contour $C$ is a closed path which encloses the poles at $z = aq^k, bq^k, cq^k, dq^k (k \in \mathbb{Z})$ and excludes the poles at $z = (aq^k)^{-1}, (bq^k)^{-1}, (cq^k)^{-1}, (dq^k)^{-1} (k \in \mathbb{Z})$, and where

$$w(\cos \theta) = \frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{\theta},ae^{-\theta},be^{\theta},be^{-\theta};q)_\infty}\frac{(1-q^{-n}abcd,q,ab,ac,ad,bc,bd,cd;q)_n}{(1-q^{-n}abcd,abcd;q)_n}\frac{(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}.$$ 

(In the other parameter region, the orthogonality is continued analytically.)

The moments of the weight function of the Askey-Wilson polynomials (which are commonly referred to as simply the moments of the Askey-Wilson polynomials) are defined by

$$\mu_k = \int_C \frac{dz}{4\pi i} w\left(z+z^{-1}\right)\left(z+z^{-1}\right)^k.$$ 

The second main result of this paper is a combinatorial formula for the moments of the Askey-Wilson polynomials. In Theorem 3 below, we use the substitution

$$\begin{align*}
\alpha &= \frac{1-q}{1+ac+a+c}, & \beta &= \frac{1-q}{1+bd+b+d}, \\
\gamma &= \frac{-(1-q)ac}{1+ac+a+c}, & \delta &= \frac{-(1-q)bd}{1+bd+b+d}.
\end{align*}$$

Recall that $Z_\ell = \sum \phi(w(\gamma)), \sum$ over all staircase tableaux of size $\ell$. 

**Theorem 3.** The $k$th moment of the Askey-Wilson polynomials is given by

$$\mu_k = h_0 \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(1-q)^j}{2} \prod_{i=0}^{j} (ai - \gamma b q^i).$$

We will prove Theorem 3 in a separate publication, by combining Theorem 2 with the results of ref. 6.

**A More Flexible Matrix Ansatz.** One of the most powerful techniques for studying the ASEP is the so-called matrix ansatz, an ansatz given by Derrida, Evans, Hakim, and Pasquier (21) as a tool for solving the steady state probabilities $P_n(\tau_1,\ldots,\tau_n)$ of the ASEP. In this section we recall their matrix ansatz, then give a slight generalization which is a primary tool for proving Theorem 2.

For convenience, we set $u = 1$. Also, we define unnormalized weights $f_n(\tau_1,\ldots,\tau_n)$, which are equal to the $P_n(\tau_1,\ldots,\tau_n)$ up to a constant:

$$P_n(\tau_1,\ldots,\tau_n) = f_n(\tau_1,\ldots,\tau_n)/Z_n,$$

where $Z_n$ is the partition function $\sum f_n(\tau_1,\ldots,\tau_n)$. The sum denoting $Z_n$ is over all possible configurations $\tau \in \{0,1\}^n$. Derrida et al showed the following (21).

**Theorem 4.** Suppose that $D$ and $E$ are matrices, $V$ is a column vector, and $W$ is a row vector, such that the following conditions hold:

$$DE - qED = D + E, \quad \beta DV - \delta EV = V,$$

$$aWE - \gamma WD = W.$$ 

Then for any state $\tau = (\tau_1,\ldots,\tau_n)$ of the ASEP,

$$f_n(\tau) = W \left( \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) \right) V.$$ 

Note that $\prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E)$ is simply a product of $n$ matrices $D$ or $E$ with matrix $D$ at position $i$ if site $i$ is occupied ($\tau_i = 1$). Also note that Theorem 4 implies that $Z_n = W(D + E) V.$

We now give a more flexible version of Theorem 4.

**Theorem 5.** Let $\{\alpha_n\}_{n \geq 0}$ be a family of constants. Suppose that $W$ and $V$ are row and column vectors, and $D$ and $E$ are matrices such that for any words $X$ and $Y$ in $D$ and $E$, we have

1. $WX(\alpha X - qED) YV = \lambda_{|X|+|Y|+1} WX(D+E) YV$;
2. $\beta WXV - \delta XYE = \lambda_{|Y|+1} WXV$;
3. $\alpha WXYV - \gamma WDYV = \lambda_{|X|+1} WYV$. 

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(Here \( |X| \) is the length of \( X \)). Then for any state \( \tau = (\tau_1, \ldots, \tau_n) \) of the ASEP,
\[
f_n(\tau) = W \left( \prod_{i=1}^{n} (\tau_i D + (1 - \tau_i) E) \right) V.
\]

In the next section we will explain how to use Theorem 5 to prove Theorem 2.

A Sketch of the Proof of Theorem 2. Our method for proving Theorem 2 is to define vectors \( W, V \) and matrices \( D, E \); prove that they have the requisite combinatorial interpretation in terms of staircase tableaux; and check that they satisfy the relations of Theorem 5, with \( \lambda_0 = 1 \) and \( \lambda_n = \alpha \beta - \gamma \delta \phi^{n-1} \) for \( n \geq 1 \).

**Definition 4:** We define “row” and “column” vectors \( W = (W_{i,k})_{i,k} \) and \( V = (V_{j,\ell})_{j,\ell} \) and matrices \( D = (D_{i,j,k,\ell})_{i,j,k,\ell} \) and \( E = (E_{i,j,k,\ell})_{i,j,k,\ell} \) (where \( i, j, k, \ell \) range over the nonnegative integers) by the following:
\[
W_{i,k} = \begin{cases} 
1 & \text{if } i = k = 0, \\
0 & \text{otherwise}, 
\end{cases} \
V_{j,\ell} = 1 \text{ always.}
\]
\[
D_{i,j,k,\ell} = \begin{cases} 
0 & \text{if } i < j \text{ or } \ell < k + 1, \\
\alpha \beta & \text{if } i = j - 1 \text{ and } k = \ell = 0, \\
\delta \gamma & \text{if } i = j, k = 0 \text{ and } \ell = 1, \\
\delta (D_{i,j-1,k-1,\ell} + E_{i,j-1,k-1,\ell}) + D_{i,j-k,1,\ell} & \text{else.}
\end{cases}
\]
\[
E_{i,j,k,\ell} = \begin{cases} 
0 & \text{if } i < j \text{ or } \ell < k + 1, \\
\beta \alpha & \text{if } i = j \text{ and } k = 0, \\
\gamma \delta & \text{if } i = j, k = 0 \text{ and } \ell = 1, \\
\beta (D_{i,j-1,k-1,\ell} + E_{i,j-1,k-1,\ell}) + q E_{i,j-1,1,\ell} & \text{else.}
\end{cases}
\]

We say that a row of a staircase tableau \( \tau \) is **indexed by \( \beta \)** if the leftmost box in that row which is not occupied by a \( q \) or \( u \) is a \( \beta \). Note that every box to the left of that \( \beta \) must be a \( u \). Similarly, we will talk about rows which are **indexed by \( \delta \)**; in this case, every box to the left of that \( \delta \) must be a \( q \). We will also talk about row \( \alpha / \gamma \), which is shorthand for rows which are indexed by \( \alpha \) or \( \gamma \).

It is not too hard to give a combinatorial interpretation to products of our matrices and vectors. In what follows, if \( X \) is a word in \( D \)'s and \( E \)'s, we say that its **type** is the corresponding word in \( s \)'s and \( s \)’s.

**Theorem 6.** If \( X \) is a word in \( D \)'s and \( E \)'s, then
- \( (WX)_\tau = \sum_{\tau'} \text{wt}(\tau') \), where the sum is over all staircase tableaux of type \( X \) which have \( j \) rows indexed by \( \delta \) and \( \ell \) rows indexed by \( \alpha / \gamma \) (and hence \( |X| - j - \ell \) rows indexed by \( \beta \));
- \( WXV = \sum_{\tau'} \text{wt}(\tau') \), where the sum is over all staircase tableaux of type \( X \).

We want to show that our matrices and vectors \( D, E, W, V \) satisfy the relations of Theorem 5. It is easy to give a combinatorial proof that they satisfy relation (3) of Theorem 5; however, we have not found a combinatorial proof that they satisfy relations (1) or (2). Instead, our strategy is to give an algebraic proof that they satisfy the following two identities, which suffices to show that they satisfy relations (1) and (2) of Theorem 5.

**Proposition 7.** For all nonnegative integers \( j \) and \( \ell \) we have
1. \( (WXDE)_\tau = q (WXED)_\tau + \alpha (WX (D + E))_\tau - \gamma \delta \phi^{1 - |j|} (WX (D + E))_{j-1,\ell} \).
2. \( \beta (WXD)_\tau = \delta (WXE)_\tau + \alpha (WXX)_\tau - \gamma \delta \phi^{1 - |j|} (WX)_\tau - \gamma \delta \phi^{1 - |\ell|} (WXE)_{j-1,\ell-1} \).

**Open Problems.** This work suggests a variety of open problems, which we briefly mention below.

**Specializing the Askey-Wilson moment formula.**

**Problem 8.** Show directly that the moment formula of Theorem 3 recovers already-known moment formulas for specializations or limiting cases of Askey-Wilson polynomials.

**A combinatorial proof of the relations of the matrix ansatz.**

**Problem 9.** Give a combinatorial proof that the matrices \( D, E \) and vectors \( V, W \) defined in the previous section satisfy relations (1) and (2) of Theorem 5.2.

When \( q = t \), or one of \( \alpha, \beta, \gamma, \delta \) is 0, this problem is easy.

**Symmetries in the ASEP.** As mentioned in the Introduction, the ASEP has a “left-right symmetry.” (It also has an “arrow-reversal symmetry” and a “particle-hole symmetry,” which is the composition of the other two.)

**Problem 10.** Find an involution on staircase tableaux which corresponds to the “left-right symmetry” in the ASEP.

**Lifting the ASEP to a Markov chain on staircase tableaux.**

**Problem 11.** Find a Markov chain on the set of all staircase tableaux of size \( n \) which projects to the ASEP in the sense of ref. 11, such that the steady state probability of state \( \tau \) is proportional to \( wt(T) \). Such an approach would give a completely combinatorial proof of Theorem 2. (This was done in ref. 11 for \( \gamma = \delta = 0 \).

**Keeping track of where the particles come from.** In the ASEP, a black particle enters from either the left (at rate \( \alpha \)) or from the right (at rate \( \delta \)). Similarly, a “hole” (or a white particle) enters from either the left (at rate \( \gamma \)) or from the right (at rate \( \beta \)). We might therefore hope to define a more refined Markov chain on \( 4^n \) states, which “projects” to the ASEP and keeps track of where the black and white particles came from. Indeed, the staircase tableaux themselves seem to be keeping track of more information than just the color of the particles.

**Problem 12.** Fix a lattice of \( n \) sites, and define a Markov chain on \( 4^n \) states (words of length \( n \) on the alphabet \( \{\alpha, \beta, \gamma, \delta\} \) with the following properties:
- particles labeled \( \alpha \) and \( \gamma \) always enter from the left, and particles labeled \( \beta \) and \( \delta \) always enter from the right;
- the Markov chain projects to the ASEP in the sense of ref. 11;
- the steady state probability of state \( (\tau_1, \ldots, \tau_n) \) is proportional to the generating function for all staircase tableaux whose border is \( (\tau_1, \ldots, \tau_n) \).

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