Approximating Bounded Job Start Scheduling with Application in Royal Mail Deliveries under Uncertainty

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Abstract Motivated by mail delivery scheduling problems arising in Royal Mail, we study a generalization of the fundamental makespan scheduling \( P||C_{\text{max}} \) problem which we call the bounded job start scheduling problem. Given a set of jobs, each specified by an integer processing time \( p_j \), that have to be executed non-preemptively by a set of \( m \) parallel identical machines, the objective is to compute a minimum makespan schedule subject to an upper bound \( g \leq m \) on the number of jobs that may simultaneously begin per unit of time. We show that Longest Processing Time First (LPT) algorithm is tightly 2-approximate. After proving that the problem is strongly \( \mathcal{NP} \)-hard even when \( g = 1 \), we elaborate on improving the 2-approximation ratio for this case. We distinguish the classes of long and short instances satisfying \( p_j \geq m \) and \( p_j < m \), respectively, for each job \( j \). We show that LPT is 5/3-approximate for the former and optimal for the latter. Then, we explore the idea of scheduling long jobs in parallel with short jobs to obtain tightly satisfied packing and bounded job start constraints. For a broad family of instances excluding degenerate instances with many very long jobs, we derive a 1.985-approximation ratio. For general instances, we require machine augmentation to obtain better than 2-approximate schedules. Finally, we exploit machine augmentation and lexicographic optimization, which is useful for \( P||C_{\text{max}} \) under uncertainty, to propose a two-stage robust optimization approach for bounded job start scheduling under uncertainty.

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tainty aiming in good trade-offs in terms of makespan and number of used machines. We substantiate this approach numerically using Royal Mail data.

**Keywords** Bounded job start scheduling · Approximation algorithms · Robust scheduling · Mail deliveries

## 1 Introduction

Royal Mail provides mail collection and delivery services for all United Kingdom (UK) addresses. With a small van fleet (as of January 2019) of 37,000 vehicles and 90,000 drivers delivering to 27 million locations in UK, efficient resource allocation is essential to guarantee the business viability. The backbone of the Royal Mail distribution network is a three-layer hierarchical network with 6 regional distribution centers serving 38 mail centers. Each mail center receives, processes, and distributes mail for a large geographically-defined area via 1,250 delivery offices, each serving disjoint sets of neighboring post codes. Mail is collected in mail centers, sorted by region, and forwarded to an appropriate onward mail center, making use of the regional distribution centers for cross-docking purposes. From the onward mail center it is transferred to the final delivery office destination. This process has to be completed within 12 to 16 hours for 1st class post and 24 to 36 hours for 2nd class post depending on when the initial collection takes place.

In a delivery office, post is sorted, divided into routes, and delivered to addresses using a combination of small fleet vans and walked trolleys. Allocating delivery itineraries to vans is critical. Each delivery office has a van exit gate which gives an upper bound the number of vehicles that can depart per unit of time. Thus, we deal with scheduling a set $J$ of jobs (delivery itineraries), each associated with an integer processing time $p_j$, on $m$ parallel identical machines (vehicles), s.t. the makespan, i.e. the last job completion time, is minimized. Parameter $g$ imposes an upper bound on the number of jobs that may simultaneously begin per unit of time. Each job has to be executed non-preemptively, i.e. by a single machine in a continuous time interval without interruptions. We refer to this problem as the *Bounded Job Start Scheduling Problem (BJSP)*.

 BJSP is strongly related to the fundamental makespan scheduling problem, a.k.a. $P||C_{\text{max}}$ [10]. BJSP generalizes $P||C_{\text{max}}$ as the problems are equivalent when $g = m$. Furthermore, $P||C_{\text{max}}$ is the BJSP relaxation obtained by dropping the BJSP constraint. Note that the $P||C_{\text{max}}$ optimal solution is a factor $\Omega(m)$ from the BJSP optimum. To see this, take an arbitrary $P||C_{\text{max}}$ instance and construct a BJSP one with $g = 1$, by adding a large number of unit jobs. The BJSP optimal schedule requires time intervals during which $m - 1$ machines are idle at each time while the $P||C_{\text{max}}$ optimal schedule is perfectly balanced and all machines are busy until the last job completes. On the positive side, we may easily convert any $\rho$-approximation algorithm for $P||C_{\text{max}}$ into $2\rho$-approximation algorithm for BJSP using naive bounds. Given that $P||C_{\text{max}}$
admits a PTAS, we obtain an $O(n^{1/\epsilon} \cdot \text{poly}(n))$-time $(2+\epsilon)$-approximation algorithm for BJSP. Here, a main goal is to obtain tighter performance guarantees. Similarly to $P||C_{\text{max}}$, provably good BJSP solutions must attain low imbalance $\max_i \{T - T_i\}$, where $T$ and $T_i$ are the makespan and completion time of machine $i$, respectively. Because of the BJSP constraint, feasible schedules may require idle machine time before all jobs have begun. Therefore, BJSP exhibits the additional difficulty of effectively bounding the total idle period $\sum_{t \leq r}(m - |A_t|)$, where $r$ and $A_t$ are the last job start time and set of jobs executed during $[t, t + 1)$, respectively.

BJSP relaxes the scheduling problem with forbidden sets, i.e. non-overlapping constraints, where subsets of jobs cannot run in parallel [21]. For the latter problem, better than 2-approximation algorithms are ruled out, unless $P = \mathcal{NP}$ [21]. Even when there is a strict order between jobs in the same forbidden set, the scheduling with forbidden sets problem is equivalent to the precedence-constrained scheduling problem $P|\text{prec}|C_{\text{max}}$ and cannot be approximated by a factor lower than $(2 - \epsilon)$, assuming a variant of the unique games conjecture [23]. Also, BJSP relaxes the scheduling with forbidden job start times problem, where no job may begin at certain time points, which does not admit constant-factor approximation algorithms [5, 8, 18, 20]. Despite the commonalities with the aforementioned literature, to the authors’ knowledge, there is a lack of approximation algorithms for scheduling problems with bounded job starts.

**Contributions and Paper Organization** Section 2 formally defines BJSP, proves the problem’s $\mathcal{NP}$-hardness, and derives an $O(\log n)$ integrality gap for a natural integer programming formulation. Section 3 investigates Longest Processing Time First (LPT) algorithm and derives a tight 2-approximation ratio. We thereafter explore improving this ratio for the special case $g = 1$. Section 2 shows that BJSP is strongly $\mathcal{NP}$-hard even when $g = 1$. Several of our arguments can be extended to arbitrary $g$, but focusing on $g = 1$ avoids many floors, ceilings, and simplifies our presentation. Furthermore, any Royal Mail instance can be converted to $g = 1$ using small discretization.

Section 4 distinguishes long versus short instances. An instance $\langle m, J \rangle$ is long if $p_j \geq m$ for each $j \in J$ and short if $p_j < m$ for all $j \in J$. This distinction comes from the observation that idle time occurs mainly because of (i) simultaneous job completions for long jobs and (ii) limited allowable parallel job executions for short jobs. Section 4 proves that LPT is $5/3$-approximate for long instances and optimal for short instances. A key ingredient for establishing the ratio in the case of long instances is a concave relaxation for bounding idle machine time. Section 4 also obtains an improved approximation ratio for long instances, when the maximum job processing time is relatively small, using the Shortest Processing Time First (SPT) algorithm.

Greedy scheduling, e.g. LPT and SPT, which sequences long jobs first and short jobs next, or vice versa, cannot achieve an approximation ratio better than 2. Section 5 proposes Long-Short Mixing (LSM), which devotes a certain number of machines to long jobs and uses all remaining machines for
short jobs. By executing the two job types in parallel, LSM achieves a 1.985-approximation ratio for a broad family of instances. For degenerate instances with many very long jobs, we require constant-factor machine augmentation, i.e. $f_m$ machines where $f > 1$ is constant, to achieve a strictly lower than 2-approximation ratio.

Because Royal Mail delivery scheduling is subject to uncertainty, Section 6 exploits machine augmentation and lexicographic optimization for $P||C_{\text{max}}$ under uncertainty [15, 22] to construct a two-stage robust optimization approach for the BJSP under uncertainty. Section 7 substantiates the approach empirically using Royal Mail data. Section 8 concludes with a collection of intriguing future directions.

2 Problem Definition and Preliminary Results

An instance $I = \langle m, J \rangle$ of the Bounded Job Start Scheduling Problem (BJSP) is specified by a set $M = \{1, \ldots, m\}$ of parallel identical machines and a set $J = \{1, \ldots, n\}$ of jobs. A machine may execute at most one job per unit of time. Job $j \in J$ is associated with an integer processing time $p_j$. Each job should be executed non-preemptively, i.e. in a continuous time interval without interruptions, by a single machine. BJSP parameter $g$ imposes an upper bound on the number of jobs that may begin per unit of time. The goal is to assign each job $j \in J$ to a machine and decide its starting time so that this BJSP constraint is not violated and the makespan, i.e. the time at which the last job completes, is minimized. Consider a feasible schedule $S$ with makespan $T$. We denote the start time of job $j$ by $s_j$. Each job $j$ must be entirely executed during the interval $[s_j, C_j)$, where $C_j = s_j + p_j$ is the completion time of $j$. So, $T = \max_{j \in J} \{C_j\}$. Job $j$ is alive at time $t$ if $t \in [s_j, C_j)$. Let $A_t = \{j : \{s_j, C_j\} \cap [t - 1, t) \neq \emptyset, j \in J\}$ and $B_t = \{j : s_j \in [t - 1, t), j \in J\}$ be the set of alive and beginning jobs during time unit $t$, respectively. Schedule $S$ is feasible only if $|A_t| \leq m$ and $|B_t| \leq g$, for all $t$.

BJSP is strongly $NP$-hard because it becomes equivalent with $P||C_{\text{max}}$ in the special case where $g = \min \{m, n\}$. Theorem 1 shows that BJSP is strongly $NP$-hard also when $g = 1$.

**Theorem 1** BJSP is strongly $NP$-hard in the special case $g = 1$.

**Proof** We present an $NP$-hardness proof from 3-Partition. Given a set $A = \{a_1, \ldots, a_{3m}\}$ and a parameter $B \in \mathbb{Z}^+$ s.t. $a_j \in \mathbb{Z}^+, B/4 \leq a_j \leq B/2$, for $j \in \{1, \ldots, 3m\}$, and $\sum_{j=1}^{3m} a_j = mB$, the 3-Partition problem asks whether there exists a partition of $A$ into $m$ subsets $S_1, \ldots, S_m$ s.t. $\sum_{j \in S_i} a_j = B$ for each $i \in \{1, \ldots, m\}$. Given an instance of 3-Partition, construct a BJSP instance $I = \langle m, J \rangle$ with $n = 3m$ jobs of processing time $p_j = n^2 a_j$, for $j \in \{1, \ldots, 3m\}$, and BJSP parameter $g = 1$. W.l.o.g., $n^2 > 3n$. We show that $A$ admits a 3-Partition iff there exists a feasible schedule $S$ of makespan $T < n^2 B + n^2$ for $I$. 
Suppose that $A$ admits a 3-Partition $S_1, \ldots, S_m$. Because $B/4 < a_j < B/2$, for $j \in \{1, \ldots, 3m\}$, $S_i$ contains exactly three elements, i.e. $|S_i| = 3$, for each $i \in \{1, \ldots, m\}$. We fix some arbitrary order $1, \ldots, 3m$ of all jobs and construct a schedule $S$ for $I$ where all jobs in $S_i$ are executed by machine $i \in \mathcal{M}$. The job starting times are decided greedily. In particular, let $T_i$ be the last job completion time in machine $i$ just before assigning job $j$. If no job has been assigned to $i$, then $T_i = 0$. We set $s_j$ equal to the earliest time slot after $T_i$ at which no job begins in any machine, i.e. min$\{t : |B_t| < 1, t > T_i\}$. Now, let $T_i$ be the last completion time in machine $i$, once the greedy procedure has been completed. Consider any job $j \in S_i$ and let $j' \in S_i$ be the last job executed before $j$ in machine $i$. If no job is executed before $j$, then $s_j \leq n$. Otherwise, by construction, we have $|B_{s_j}| = 1$, for every $t \in [C_{s_j} + 1, s_j - 1]$. Hence, $s_j - C_{s_j} \leq n - 1$. Since $|S_i| = 3$ and $|$[{$t : |B_t| = 1, t \in D$}|] $\leq n$, we conclude $T_i \leq \sum_{j \in S_i} p_j + 3n = n^2 \left( \sum_{j \in S_i} a_j \right) + 3n = n^2B + 3n < n^2B + n^2$.

So schedule $S$ attains makespan $T < n^2B + n^2$.

$\Leftarrow$ : Suppose that there exists a feasible schedule $S$ of makespan $T < n^2B + n^2$ for $I$. We argue that each machine executes exactly three jobs. Suppose for contradiction that machine $i \in \mathcal{M}$ executes a subset $S_i$ of jobs with $|S_i| \geq 4$. Denote by $T_i = \max\{C_j : j \in S_i\}$ the last job completion time in machine $i$. Then, $T_i \geq \sum_{j \in S_i} p_j = n^2 \left( \sum_{j \in S_i} a_j \right)$. Because $a_j \in \mathbb{Z}_+$ and $a_i > B/4$, it must be the case that $\sum_{j \in S_i} a_j \geq B + 1$. Hence, $T_i \geq n^2B + n^2$, which is a contradiction on the fact that $T_i \leq T$. Thus, schedule $S$ defines a partitioning of the jobs into $m$ subsets $S_1, \ldots, S_m$ s.t. $|S_i| = 3$, for each $i \in \mathcal{M}$. We claim that $\sum_{j \in S_i} a_i = B$. Otherwise, there would be a machine $i \in \mathcal{M}$ with $\sum_{j \in S_i} a_i \geq B + 1$ and we would obtain a contradiction using similar reasoning to before. We conclude that $A$ admits a 3-Partition.

Next, we investigate the integrality gap of a natural integer programming formulation. To obtain this integer program, we partition the time horizon into a set $D = \{1, \ldots, \tau\}$ of unit-length discrete time slots. Time slot $t \in D$ corresponds to time interval $[t-1, t)$. We may naively choose $\tau = \sum_{j \in \mathcal{J}} p_j$, but smaller $\tau$ values are possible using tighter makespan upper bounds. For simplicity, this manuscript assumes discrete time intervals $[s, t] = \{s, s+1, \ldots, t-1, t\}$, i.e. of integer length. Interval $[1, \tau]$ is the time horizon. In integer programming Formulation [4], binary variables decide a starting time for each job. Binary variable $x_{j,s}$ is 1 if job $j \in \mathcal{J}$ begins at time slot $s \in D$, and 0 otherwise. Continuous variable $T$ corresponds to the makespan. If job $j$ starts at $s$, then it is performed exactly during the time slots $s, s+1, \ldots, s + p_j - 1$. Hence, job $j$ is alive at time slot $t$ iff it has begun at one among the time slots in the set $A_j,t = \{t-p_j+1, t-p_j+2, \ldots, t\}$. To complete before the time horizon ends, job $j$ must begin at a time slot in the set $E_j = \{1, 2, \ldots, \tau - p_j + 1\}$. Finally, denote by $\mathcal{J}_s = \{j : s \in F_j, j \in \mathcal{J}\}$ the eligible subset of jobs at $s$, i.e. the ones that may be feasibly begin at time slot $s$ without exceeding the time
Expression (1a) minimizes makespan. Constraints (1b) enforce that the makespan is equal to the last job completion time. Constraints (1c) ensure that at most \( m \) machines are used at each time slot \( t \). Constraints (1d) require that each job \( j \) is scheduled. Constraints (1e) express the BJSP constraint.

**Theorem 2** The fractional relaxation of integer programming formulation \((1)\) has integrality gap \( O(\log n) \).

**Proof** Consider an instance with \( m \) machines, \( n = m \) jobs of processing time \( p_j = 1 \) for each \( j \in J \), and BJSP parameter \( g = m \). For this instance, the LP solution sets \( x_{j,s} = \frac{1}{s \cdot \sum_{t=1}^{\frac{1}{t}}} \) for each \( j,s \). The LP fractional solution is feasible as at each time, no more than \( m \) job pieces are feasibly executed (and begin), while the cost is \( \max \{sx_{j,s}\} = \frac{1}{\sum_{t=1}^{\frac{1}{t}}} \). On the contrary, the optimal integral solution has makespan 1.

**3 LPT Algorithm**

Longest Processing Time first algorithm (LPT) schedules the jobs on a fixed number \( m \) of machines w.r.t. the order \( p_1 \geq \ldots \geq p_n \). Recall that \( |A_t| \) and \( |B_t| \) is the number of alive and beginning jobs, respectively, at time slot \( t \in D \).

We say that time slot \( t \in D \) is available if \( |A_t| < m \) and \( |B_t| < g \). LPT schedules the jobs greedily w.r.t. their sorted order. Each job \( j \) is scheduled in the earliest available time slot, i.e. at \( s_j = \min \{t : |A_t| < m, |B_t| < g, t \in D\} \). **Theorem 3** proves a tight approximation ratio of 2 for LPT.

**Theorem 3** LPT is 2-approximate for minimizing makespan and this ratio is tight.
Proof Denote by $S$ and $S^*$ the LPT and a minimum makespan schedule, respectively. Let $\ell$ be the job completing last in $S$, i.e. $T = s_\ell + p_\ell$. For each time slot $t \leq s_\ell$, either $|A_t| = m$, or $|A_t| < m$. Since $\ell$ is scheduled at the earliest available time slot, for each $t \leq s_\ell$ s.t. $|A_t| < m$, we have $|B_t| = g$. Let $\lambda$ be the total length of time s.t. $|A_t| < m$ in $S$. Because of the BJSP constraint, exactly $g$ jobs begin per unit of time, which implies that $\lambda \leq \lceil \frac{\ell}{g} \rceil$. Therefore, schedule $S$ has makespan:

$$T = s_\ell + p_\ell \leq \frac{1}{m} \sum_{j \neq \ell} p_j + \lambda + p_\ell \leq \frac{1}{m} \sum_{j=1}^{n} p_j + \left( \left\lceil \frac{\ell}{g} \right\rceil + p_\ell \right).$$

Denote by $s^*_j$ the starting time of job $j$ in $S^*$ and let $\pi_1, \ldots, \pi_n$ the job indices ordered in non-decreasing schedule $S^*$ starting times, i.e. $s^*_1 \leq \ldots \leq s^*_n$. Because of the BJSP constraint, $s^*_j \geq \lceil [j/g] \rceil$. In addition, there exists $j' \in [j,n]$ s.t. $p_{\pi_{j'}} > p_j$. Thus, $\max_{j'=j} \{s^*_j + p_{\pi_{j'}}\} \geq \lceil [j/g] \rceil + p_j$, for $j = 1, \ldots, n$. Hence, $S^*$ has makespan:

$$T^* \geq \max \left\{ \frac{1}{m} \sum_{j=1}^{n} p_j, \max_{j=1}^{n} \left\{ \left\lceil \frac{j}{g} \right\rceil + p_j \right\} \right\}.$$

We conclude that $T \leq 2T^*$.

For the tightness of the analysis, consider instance $I = \langle m, J \rangle$ with $m(m-1)$ long jobs of processing time $p$, where $p = \omega(m)$ and $m = \omega(1)$, $m(p-m)$ unit jobs, and BJSP parameter $g = 1$. LPT schedules the long jobs into $m-1$ batches, each one with exactly $m$ jobs. All jobs of a batch are executed in parallel for their greatest part. In particular, the $i$-th job of the $k$-th batch is executed by machine $i$ starting at time slot $(k-1)p + i$. All unit jobs are executed sequentially by machine 1 starting at $(m-1)p + 1$. Observe that $S$ is feasible and has makespan $T = (m-1)p + m(p-m) = (2m-1)p - m^2$. The optimal solution $S^*$ schedules all jobs in $m$ batches. The $k$-th batch contains $(m-1)$ long jobs and $(p-m+1)$ unit jobs. Specifically, the $i$-th long job is executed by machine $i$ beginning at $(k-1)p + i$, while all short jobs are executed consecutively by machine $m$ starting at $(k-1)p + m$ and completing at $kp$. Schedule $S^*$ is feasible and has makespan $T^* = mp$. Because $\frac{m}{p} \to 0$ and $\frac{1}{m} \to 0$, i.e. both approach zero, $T \to 2T^*$.

4 Long and Short Instances

This section assumes that $g = 1$, but several of the arguments can be extended to arbitrary $g$. From an application viewpoint, any Royal Mail instance can be converted to $g = 1$ using small discretization.
4.1 Longest Processing Time First

We consider two natural classes of BJSP instances for which LPT achieves an approximation ratio better than 2. Instance \( \langle m, J \rangle \) is (i) long if \( p_j \geq m \) for each \( j \in J \) and (ii) short if \( p_j < m \) for every \( j \in J \). This section proves that LPT is 5/3-approximate for long instances and optimal for short instances.

Consider a feasible schedule \( S \) and let \( r = \max_{j \in J} \{s_j\} \) be the last job start time. We say that \( S \) is a compact schedule if it holds that either (i) \( |A_t| = m \), or (ii) \( |B_t| = 1 \), for each \( t \in [1, r] \). Lemma 1 shows the existence of an optimal compact schedule and derives a lower bound on the optimal makespan.

**Lemma 1** For each instance \( I = \langle m, J \rangle \), there exists a feasible compact schedule \( S^* \) which is optimal. Let \( J^k = \{j : p_j \geq m, j \in J\} \). If \( |J^k| \geq m \), then \( S^* \) has makespan \( T^* \geq \frac{1}{m} \left( \frac{m(m-1)}{2} + \sum_{j=1}^n p_j \right) \).

**Proof** For the first part, among the set of all optimal schedules, pick the schedule \( S^* \) lexicographically minimizing the vector of job start times sorted in non-decreasing order. We claim that \( S^* \) is compact. Assume that this is not the case. Then, there exists a time \( t \in [1, r] \) such that \( |A_t| < m \) and \( |B_t| < 1 \). Consider the earliest such time \( t \). Moreover, let \( t' \) be the earliest time \( t' > t \) satisfying either \( |A_t| = m \), or \( |B_t| = 1 \). Clearly, there exists a job \( j \in J \) such that \( s_j = t' \). If we decrease the job \( j \) start time by one unit of time, we obtain a feasible compact schedule \( S' \) with makespan \( T' \leq T^* \), where \( T^* \) is the schedule \( S^* \) makespan, and \( S' \) has a lexicographically smaller vector of job start times than \( S^* \) which is a contradiction.

The second part requires lower bounding the total idle machine time \( \Lambda = \sum_{t=1}^r (m - |A_t|) \) for each feasible schedule \( S \). By the BJSP constraint, \( |B_t| = 1 \) and, hence, \( m - |A_t| \geq m - t \), for each \( t \in \{1, \ldots, m\} \). This is because all machines are idle before the first time slot. Hence, we obtain the idle machine time lower bound \( \Lambda \geq \sum_{t=1}^m (m - t) = \frac{m(m-1)}{2} \), by considering time interval \([1, m]\). A simple packing argument implies that schedule \( S \) has makespan \( T \geq \frac{1}{m} \left( \frac{m(m-1)}{2} + \sum_{j=1}^n p_j \right) \).

Next, we analyze LPT in the case of long instances. Similar to the Lemma 1 proof, we may show that LPT produces a compact schedule \( S \). We partition the interval \([1, r]\) into a sequence \( P_1, \ldots, P_k \) of maximal periods satisfying the following invariant: for each \( q \in \{1, \ldots, k\} \), either (i) \( |A_t| < m \) for each \( t \in P_q \) or (ii) \( |A_t| = m \) for each \( t \in P_q \). That is, there is no pair of time slots \( s, t \in P_q \) such that \( |A_s| < m \) and \( |A_t| = m \). We call \( P_q \) a slack period if \( P_q \) satisfies (i), otherwise, \( P_q \) is a full period. For a given period \( P_q \) of length \( \lambda_q \), denote by \( A_q = \sum_{t \in P_q} (m - |A_t|) \) the idle machine time. Note that \( A_q = 0 \), for each full period \( P_q \). Lemma 2 upper bounds the total idle machine time of slack periods in the LPT schedule \( S \), except the very last period \( P_k \). When \( P_k \) is slack, the length \( \lambda_k \) of \( P_k \) is upper bounded by Lemma 3.
Lemma 2  Let \( k' = k - 1 \). Consider a long instance \( I = \langle m, J \rangle \), with \( |J| \geq m \), and the LPT schedule \( S \). It holds that (i) \( \lambda_q \leq m - 1 \) and (ii) \( A_q \leq \frac{\lambda_q(\lambda_q - 1)}{2} \) for each slack period \( P_q \), where \( q \in \{1, \ldots, k'\} \). Furthermore, (iii) \( \sum_{q=1}^{k'} A_q \leq \frac{nm}{2} \).

Proof For (i), let \( P_q = [s, t] \) be a slack time period in \( S \) and assume for contradiction that \( \lambda_q \geq m \), i.e. \( t \geq s + m - 1 \). Given that \( p_j \geq m \) for each \( j \in J \), we have \( \{j : s_j \in [s, s + m - 1], j \in J\} \subseteq \mathcal{A}_{s+m-1} \). That is, all jobs starting during \([s, s + m - 1]\) are alive at time \( s + m - 1 \). Since \( P_q \) is a slack period, \( |A_u| < m \) holds for each \( u \in [s, s + m - 1] \). Because \( S \) is compact, \( |B_u| = 1 \), i.e. exactly \( g = 1 \) jobs begin, at each \( u \in [s, s + m - 1] \). This implies \( |A_{s+m-1}| \geq m \), contradicting the fact that \( P_q \) is a maximal slack period.

For (ii), consider the partitioning \( A_u = A_{u}^l + A_{u}^r \) for each time slot \( u \in P_q = [s, t] \), where \( A_{u}^l \) and \( A_{u}^r \) is the set of all jobs at time \( u \) completing inside \( P_q \) and after \( P_q \), i.e. \( C_j \in [s, t] \) and \( C_j > t \), respectively. Since \( \lambda_q \leq m - 1 \), every job \( j \) beginning during \( P_q \), i.e. \( s_j \in [s, t] \), must complete after \( P_q \), i.e. \( C_j > t \). We modify schedule \( S \) by removing every job \( j \) completing inside \( P_q \), i.e. \( C_j \leq t \). Clearly, the modified schedule \( S' \) has increased idle time \( A_q' \) during \( P_q \).

For (iii), consider slack period \( P_q \), for \( q \in \{1, \ldots, k'\} \). By (i), \( \lambda_q \leq m - 1 \). Since at most \( g = 1 \) jobs begin at each \( t \in P_q \), \( \sum_{q=1}^{k'} \lambda_q \leq n - 1 \). By (ii), Concave Program (2) upper bounds \( \sum_{q=1}^{k'} A_q \).

\[
\max_{\lambda \in \{0,1\}^{k'}} \sum_{q=1}^{k'} \frac{\lambda_q(\lambda_q - 1)}{2} \quad \text{(2a)}
\]

\[
1 \leq \lambda_q \leq m \quad q \in \{1, \ldots, k'\} \quad \text{(2b)}
\]

\[
\sum_{q=1}^{k'} \lambda_q \leq n \quad \text{(2c)}
\]

Assume w.l.o.g. that \( n/m \) is integer. If \( k' \leq n/m \), by setting \( \lambda_q = m \), for \( q \in \{1, \ldots, k'\} \), we obtain \( \sum_{q=1}^{k'} \lambda_q(\lambda_q - 1)/2 \leq k'm(m-1)/2 \leq nm/2 \). If \( k' > n/m \), we argue that the solution \( \lambda_q = m \), for \( q \in \{1, \ldots, n/m\} \), and \( \lambda_q = 0 \), otherwise, is optimal for Concave Program (2). In particular, for any solution \( \lambda < \lambda_q, \lambda_{q'} < m \) such that \( q \neq q' \), we may construct a modified solution with higher objective value by applying Jensen’s inequality \( f(\lambda) + f(\lambda') \leq f(\lambda + \lambda') \) for any \( \lambda, \lambda' \in [0, \infty) \), with respect to the single variable, convex function \( f(\lambda) = \lambda(\lambda-1)/2 \). If \( \lambda_q + \lambda_{q'} \leq m \), we may set \( \tilde{\lambda}_q = \lambda_q + \lambda_{q'} \) and \( \tilde{\lambda}_{q'} = 0 \). Otherwise, \( m < \lambda_q + \lambda_{q'} \leq 2m \) and we set \( \tilde{\lambda}_q = m \) and
\( \hat{\lambda}' = \lambda_q + \lambda_{q'} - m \). In both cases, \( \lambda_{q''} = \hat{\lambda}_{q''} \), for each \( q'' \in \{1, \ldots, k'\} \setminus \{q, q'\} \).

Clearly, \( \hat{\lambda} \) attains higher objective value than \( \lambda \), for Concave Program \((2)\).

**Lemma 3** Suppose that \( P_k \) is a slack period and let \( J_k \) be the set of jobs beginning during \( P_k \). Then, it holds that \( \lambda_k \leq \frac{1}{m} \sum_{j \in J_k} p_j \).

**Proof** Because \( P_k \) is a slack period, it must be the case that \( |B_u| = 1 \), for each \( u \in P_k \). Since we consider long instances, \( p_j \geq m \), for each \( j \in J \). Therefore, \( \lambda_k \leq \frac{1}{m} \sum_{j \in J_k} p_j \).

**Theorem 4** LPT is 5/3-approximate for long instances.

**Proof** Denote the LPT and optimal schedules by \( S \) and \( S^* \), respectively. Let \( \ell \in J \) be a job completing last in \( S \), i.e. \( T = s_\ell + p_\ell \). Recall that LPT sorts jobs s.t. \( p_1 \geq \ldots \geq p_n \). W.l.o.g. we assume that \( \ell = \arg \min_{j \in J} \{ p_j \} \). Indeed, we may discard every job \( j < \ell \) and bound the algorithm’s performance w.r.t. instance \( I = (m, J \setminus \{ \ell : j \leq \ell, j \in J \}) \). Let \( \hat{S} \) and \( \hat{S}^* \) be the LPT and an optimal schedule attaining makespan \( T \) and \( T^* \), respectively, for instance \( \hat{I} \). Showing that \( T \leq (5/3)T^* \) is sufficient for our purposes because \( T = T \) and \( \hat{T} \leq T^* \). We distinguish two cases based on whether \( p_n > T^*/3 \), or \( p_n \leq T^*/3 \).

Case 1 \( (p_n > T^*/3) \): We claim \( T \leq (3/2)T^* \). Initially, observe that \( n \leq 2m \). Otherwise, there would be a machine \( i \in \mathcal{M} \) executing at least \( |S_i^*| \geq 3 \) jobs, say \( j, j', j'' \in J \), in \( S^* \). This machine would have last job completion time \( T_i^* \geq p_j + p_{j'} + p_{j''} > T^* \), a contradiction. If \( n \leq m \), LPT clearly has makespan \( T = T^* \). So, consider \( n > m \). Then, some machine executes at least two jobs in \( S^* \), i.e. \( p_n \leq T^*/2 \). To prove our claim, it suffices to show \( s_n \leq T^* \), i.e. job \( n \) starts before or at \( T^* \). Let \( c = \max_{1 \leq j \leq m} \{ C_j \} \) be the time when the last among the \( m \) biggest jobs completes. If \( s_n \leq c \), then \( s_n \leq \max_{1 \leq j \leq m} \{ j + p_j \} \leq T^* \).

Otherwise, let \( \lambda = s_n - c \). Because \( n \leq 2m \), it must be the case that \( \lambda \leq m \). Furthermore, \( |A_\lambda| < m \) and, thus, \( |B_t| = 1 \), for each \( t \in [c + 1, s_n - 1] \). That is, exactly one job begins per unit of time during \( [c + 1, s_n] \). Due to the LPT ordering, these are exactly the jobs \{\( n - \lambda, \ldots, n \}\}. Since \( \lambda \leq m \) and \( p_j \geq m \), at least \( m - h \) units of time of job \( n - h \) are executed from time \( s_n \) onwards, for \( h \in [1, \ldots, \lambda] \). Thus, the total processing load which executed not earlier than \( s_n \) is \( \mu \geq \sum_{h=1}^{\lambda} (m - h) \). On the other hand, at most \( m - h \) machines are idle at time slot \( c + h \), for \( h \in [1, \ldots, \lambda] \). So, the total idle machine time during \( [m + 1, s_n - 1] \) is \( \lambda \leq \sum_{h=1}^{\lambda} (m - h) \). We conclude that \( \mu \geq \lambda \) which implies that \( s_n \leq \frac{m(m-1)}{2} + \frac{1}{m} \sum_{j \in J} p_j \). By Lemma 1, our claim follows.

Case 2 \( (p_n \leq T^*/3) \): In the following, Equalities \((3b)-(3d)\) hold because job \( n \) completes last and by the definition of alive jobs. Inequalities \((3a)-(3d)\) use a simple packing argument with job processing times and machine idle time taking into account: \((3a)\) \((3b)\) \((3c)\) \((3d)\) property \( \sum_{q=1}^{k'} \lambda_q \leq \frac{nn}{2} \), and \((3d)\) the bound \( T^* \geq \max \{ \frac{1}{m} \sum_{j \in J} p_j, n + p_n, 3p_n \} \).

\[
T = s_n + p_n = \frac{1}{m} \left( \sum_{t=1}^{s_n} |A_t| + \sum_{t=m-|A_t|}^{s_n} (m - |A_t|) \right) + p_n \tag{3a}
\]
\[
\leq \frac{1}{m} \left( \sum_{i=1}^{n} p_i + \sum_{q=1}^{k'} \Lambda_q \right) + p_n \quad (3b)
\]
\[
\leq \frac{1}{m} \sum_{i=1}^{n} p_i + \frac{n}{2} + p_n \quad (3c)
\]
\[
\leq \frac{5}{3} T^*. \quad (3d)
\]

We complement Theorem 4 with a long instance \( I = (m, J) \) where LPT is \( 3/2 \)-approximate and leave closing the gap between the two as an open question. Instance \( I \) contains \( m+1 \) jobs, where \( p_j = 2m - j \), for \( j \in \{1, \ldots, m\} \), and \( p_{m+1} = m \). In the LPT schedule \( S \), job \( j \) is executed at time \( s_j = j + 1 \), for \( j \in \{1, \ldots, m\} \), and \( s_{m+1} = 2m - 1 \). Hence, \( T = 3m - 1 \). But an optimal schedule \( S^* \) assigns job \( j \) to machine \( j+1 \) at time \( s_j = j + 1 \), for \( j \in \{1, \ldots, m-1\} \). Moreover, jobs \( m \) and \( m+1 \) are assigned to machine 1 beginning at times \( s_m = 1 \) and \( s_{m+1} = m \), respectively. Clearly, \( S^* \) has makespan \( T^* = 2m \).

Theorem 5 completes this section with a simple argument on the LPT performance for short instances.

\textbf{Theorem 5} LPT is optimal for short instances.

\textit{Proof} Let \( p_1 \geq \ldots \geq p_n \) be the LPT job ordering and suppose that job \( \ell \) completes last in LPT schedule \( S \). We claim that job \( \ell \) begins at time slot \( s_\ell = \lceil \ell/g \rceil \) in \( S \). Due to the BJSP constraint, we have \( s_\ell \geq \lceil \ell/g \rceil \). Assume that the last inequality is strict. Then, \( |A_t| = m \) for some time slot \( t < s_\ell \). So, there exists job \( j \in A_t \) with \( s_j = t - m + 1 \). Since \( p_j \leq m - 1 \), we get \( C_j < t \) which contradicts \( j \in A_t \). Because \( T^* \geq \lceil j/g \rceil + j \) for each \( j \in J \), the theorem follows.

### 4.2 Shortest Processing Time First

This section investigates the performance of Long Job Shortest Processing Time First Algorithm (LSPT). LSPT orders the jobs as follows: (i) each long job precedes every short job, (ii) long jobs are sorted according to Shortest Processing Time First (SPT), and (iii) short jobs are sorted as in LPT. LSPT schedules jobs greedily, in the same vein as LPT, with this new job ordering.

For long instances, when the largest processing time \( p_{\max} \) is relatively small compared to the average machine load, Theorem 6 shows that LSPT achieves an approximation ratio better than the 5/3, i.e. the approximation ratio Theorem 4 obtains for LPT. From a worst-case analysis viewpoint, the main difference between LSPT and LPT is that the former requires significantly lower idle machine time until the last job start, but at the price of much higher difference between the last job completion times in different machines.

\textbf{Theorem 6} LSPT is 2-approximate for minimizing makespan. For long instances, LSPT is \( (1 + \min\{1, 1/\alpha\}) \)-approximate, where \( \alpha = (\frac{1}{m} \sum_{j \in J} p_j)/p_{\max}. \)
Proof Let \( S \) be the LSPT schedule and suppose that it attains makespan \( T \). Moreover, denote by \( \ell \in S \) the job completing last, i.e. \( C_\ell = T \). Similarly to the Lemma 4 proof, \( S \) is compact. We distinguish two cases based on whether \(|J^L| < m\), or \(|J^L| \geq m\). In the former case, every job \( j \in J^L \) satisfies \( s_j \leq p_{\text{max}} \). Using similar reasoning to the Theorem 3 proof, we get \( T \leq p_{\text{max}} + n + p_{\text{min}} \).

In latter case, let \( t' = \min\{t : |A_t| < m, t > m\} \). That is, \( t' \) is the earliest time slot strictly after time \( t = m \) when at least one machine is idle. We claim that \( s_j < t' \) for each \( j \in J^L \), i.e. every long job begins before \( t' \) and there is no idle machine time during \([m, t']\). Assume for contradiction that there is some job \( j \in J^L \) such that \( s_j > t' \). Since, there is an idle machine at \( t' \) and long jobs remain to begin after \( t' \), at least two jobs \( j', j'' \in J^L \) complete simultaneously at \( t' - 1 \), i.e. \( C_{j'} = C_{j''} = t' - 1 \). Because of the BJSP constraint \(|B_i| \leq 1\), it must by the case that \( s_{j'} \neq s_{j''} \). W.l.o.g. \( s_{j'} < s_{j''} \). By the SPT ordering, we also have that \( p_{j'} \leq p_{j''} \). Thus, we get the contradiction \( C_{j'} < C_{j''} \). Our claim implies that \( t' \leq \frac{m(m-1)}{2} + \frac{1}{n} \sum_{j \in J} p_j \).

Next, consider two subcases based on whether \( \ell \in J^L \), or \( \ell \in J^S \). If \( \ell \in J^L \), then \( T \leq t' + p_{\text{max}} \). Otherwise, if \( \ell \in J^S \), then \( T \leq t' + n + p_{\text{min}} \). In both subcases,

\[
T \leq \frac{m(m-1)}{2} + \frac{1}{n} \sum_{j=1}^{n} p_j + \max\{p_{\text{max}}, n + p_{\text{min}}\}.
\]

Obviously, the optimal solution satisfies:

\[
T^* \geq \max\left\{ \frac{1}{m} \sum_{j=1}^{n} p_j, p_{\text{max}}, n + p_{\text{min}} \right\}.
\]

In all cases, \( T \leq 2T^* \). For long instances, i.e. the case \( \ell \in J^L \), LSPT is \((1 + \min\{1, 1/\alpha\})\)-approximate.

5 Parallelizing Long and Short Jobs

This section proposes the Long Short Job Mixing Algorithm (LSM) to compute 1.985-approximate schedules for a broad family of instances, e.g. with at most \([5m/6]\) jobs of processing time (i) \( p_j > (1 - \epsilon)(\sum_j p_j) \), or (ii) \( p_j > (1 - \epsilon)(\max_j (j' + p_{j'}) \) assuming non-increasing \( p_j \)’s, for sufficiently small constant \( \epsilon > 0 \). For degenerate instances with more than \([5m/6]\) jobs of processing time \( p_j > T^*/2 \), where \( T^* \) is the optimal objective value, LSM requires constant machine augmentation to achieve an approximation ratio lower than 2. There can be at most \( m \) such jobs. In the Royal Mail application, machine augmentation [6] [12] [19] adds more delivery vans. For simplicity, we also assume that \( m = \omega(1) \), but the approximation ratio can be adapted for smaller values of \( m \). However, we require that \( m \geq 7 \).
LSM attempts to construct a feasible schedule where long jobs are executed in parallel with short jobs. LSM uses $m^L < m$ machines for executing long jobs. The remaining $m^S = m - m^L$ machines are reserved for short jobs. Carefully selecting $m^L$ allows to obtain a good trade-off in terms of (i) delaying long jobs and (ii) achieving many short job starts at time slots when many long jobs are executed in parallel. Here, we set $m^L = \lceil 5m/6 \rceil$. Before formally presenting LSM, we modify the notions of long and short jobs by setting $J^L = \{ j : p_j \geq m^L, j \in J \}$ and $J^S = \{ j : p_j < m^L, j \in J \}$. Both $J^L$ and $J^S$ are sorted in non-increasing processing time order. LSM schedules jobs greedily by traversing time slots in increasing order. Let $A^t_L$ be the set of alive long jobs at time slot $t \in D$. For $t = 1, \ldots, \tau$, LSM proceeds as follows: (i) if $|A^t_L| < m$, then the next long job begins at $t$, (ii) if $|J^L| = 0$ or $m^L \leq |A^t| < m$, LSM schedules the next short job to start at $t$, else (iii) LSM considers the next time slot. From a complementary viewpoint, LSM partitions the machines $M$ into $M^L = \{ i : i \leq m^L, i \in M \}$ and $M^S = \{ i > m^L, i \in M \}$. LSM prioritizes long jobs on machines $M^L$ and assigns only short jobs to machines $M^S$. A job may undergo processing on machine $i \in M^S$ only if all machines in $M^L$ are busy. The Algorithm 1 pseudocode describes LSM.

**Algorithm 1** Long-Short Mixing (LSM)

Sort $J^L = \{ j \in J : p_j \geq m^L \}$ in non-increasing order.  
Sort $J^S = \{ j \in J : p_j < m^L \}$ in non-increasing order.  
for $t = 1, \ldots, \tau$ do  
if $|A_t| < m$ then  
if $|A^t_L| < m$ and $|J^t| > 0$ then  
$j' = \text{arg max}_{j \in J^t} \{ p_j \}$  
$J^L = J^L \setminus \{ j' \}$  
else if $|J^S| > 0$ then  
$j' = \text{arg max}_{j \in J^S} \{ p_j \}$  
$J^S = J^S \setminus \{ j' \}$  
In either of the above cases, set $s_{j'} = t$.

Theorem 7 shows that LSM achieves a better approximation ratio than LPT, i.e. strictly lower than 2, for a broad family of instances.

**Theorem 7** LSM is 1.985-approximate (i) for instances with no more than $\lceil 5m/6 \rceil$ jobs s.t. $p_j > (1 - \epsilon) \max \{ \frac{1}{m} \sum_j p_j, \max_j \{ j' + p_j \} \}$ for sufficiently small $\epsilon > 0$, and (ii) for general instances using 1.2-machine augmentation.

**Proof** Let $S$ be the LSM schedule and $\ell = \text{arg max} \{ C_j : j \in J \}$ the job completing last. That is, $S$ has makespan $T = C_\ell$. For notational convenience, given a subset $P \subseteq D$ of time slots, denote by $\lambda(P) = |P|$ and $\mu(P) = \sum_{t \in P} |A_t|$ the number of time slots and executed processing load, respectively, during $P$. Furthermore, let $n^L = |J^L|$ and $n^S = |J^S|$ be the number of long and short jobs, respectively. We distinguish two cases based on whether (i) $\ell \in J^S$ or (ii) $\ell \in J^L$. 

---

The text continues discussing the properties and implications of the LSM algorithm, including its approximation ratio and extensions to general instances.
Next, we upper bound a linear combination of \( \lambda \)
(taking into account the fact that certain short jobs begin during a subset \( B \) of time slots. By definition, \( \lambda(B) \leq n^S \)). We claim that \( \lambda(B) \geq (m^S/m^L)(\lambda(F^L) + \lambda(F^S)) \). For this, consider the time slots \( F^L \) as a continuous time period by disregarding intermediate \( B \) time slots. Partition this \( F^L \) time period into subperiods of equal length \( m^L \). Note that no long job begins during \( F^L \) and the machines in \( M^S \) may only execute small jobs in \( S \). Because of the greedy nature of LSM and the fact that \( p_j < m^L \) for \( j \in J^S \), there are at least \( m^S \) short jobs in each subperiod. Hence, our claim is true and we obtain that
\[
\lambda(B) \leq n^S - (m^S/m^L)(\lambda(F^L) + \lambda(F^S)),
\]
or equivalently:
\[
m^S \lambda(F^L) + m^S \lambda(F^S) + m^L \lambda(B) \leq m^L n^S.
\]

Subsequently, we upper bound a linear combination of \( \lambda(F^L) \), \( \lambda(B^L) \), and \( \lambda(F^S) \) using a simple packing argument. The part of the LSM schedule for long jobs is exactly the LPT schedule for a long instance with \( m^L \) jobs and \( m^L \) machines. If \( \mathcal{A}_L^L \) is the set of jobs in \( J^L \), we make the convention that \( \mu(B) \) does not contain any load of jobs beginning in the maximal slack period completed at time \( r^L \). Observe that \( \mu(F^L) = m^L \lambda(F^L) \) and \( \mu(F^S) = m \lambda(F^S) \). Additionally, by Lemma 2 we get that \( \mu(B) \geq m^L \mu(B)/2 \), except possibly the very last slack period. Then, \( \mu(F^L) + \mu(B^L) + \mu(F^S) \leq \sum_{j \in J} p_j \). Hence, we obtain:
\[
m^L \lambda(F^L) + \frac{1}{2} m^L \lambda(B^L) + m \lambda(F^S) \leq \sum_{j \in J} p_j.
\]

Summing Expressions 3 and 6,
\[
(m^L + m^S) \lambda(F^L) + \frac{1}{2} m^L \lambda(B^L) + (m + m^S) \lambda(F^S) + m^L \lambda(B) \leq \sum_{j \in J} p_j + m^L n^S.
\]
We distinguish two subcases based on whether $\lambda(F^L) + \frac{1}{2} \left( \frac{m^L}{m} \right) \lambda(B^L) + \lambda(F^S) + \frac{m^L}{m} \lambda(B^S) \leq \frac{1}{m} \sum_{j \in J} p_j + \frac{m^L}{m} n^S$.

(7)

We distinguish two subcases based on whether $\lambda(F^L) + \lambda(F^S) \geq 5n^S/6$ or not. Obviously, $\lambda(B^L) \leq n^L$. In the former subcase, Inequality (5) gives $\lambda(B^S) \leq (1 - \frac{5n^S}{6m}) n^S$. Using Inequality (7), Expression (4) becomes:

$$T \leq \frac{1}{m} \sum_{j \in J} p_j + \left( 1 - \frac{m^L}{2m} \right) n^L + \left[ \left( \frac{m^L}{m} \right) + \left( 1 - \frac{m^L}{m} \right) \left( 1 - \frac{5m^S}{6m^L} \right) \right] n^S + p_t.$$  

For $m^L = \lceil 5m/6 \rceil$, we have (i) $5/6 \leq m^L/m \leq 5/6 + 1/m$ and (ii) $m^S/m^L \geq \frac{1/6 - 1/m}{1/6 + 1/m}$. Given $m = \omega(1)$,

$$T \leq \frac{1}{m} \sum_{j \in J} p_j + \left( 1 - \frac{5}{12} \right) n^L + \left[ \frac{5}{6} + \left( 1 - \frac{5}{6} \right) \left( 1 - \frac{1}{5} \right) \right] n^S + p_t.$$  

Note that an optimal solution $S^*$ has makespan:

$$T^* \geq \max \left\{ \frac{1}{m} \sum_{j \in J} p_j, n^L + n^S + p_t \right\}.$$  

Because the instance is mixed with long and short jobs and we consider the case $\ell \in J^S$, we have $p_t \geq T^*/2$. Therefore, $T \leq \left( 1 + \frac{29}{60} + \left( \frac{29}{60} \right)^{1/2} \right) T^* \leq 1.985T^*$. Now, consider the opposite subcase where $\lambda(F^L) + \lambda(F^S) \leq 5n^S/6$. Given $\lambda(B^L) \leq n^L$ and $\lambda(B^S) \leq n^S$, expression (4) becomes $T \leq \frac{1}{6} (n^S + n^L + p_t) \leq 1.835 \cdot T^*$.

Case $\ell \in J^L$ Recall that $A^L_t$ and $B^L_t$ are the sets of long jobs which are alive and begin, respectively, at time slot $t$. Furthermore, $r^L = \max \{ s_j : j \in J^L \}$ is the last long job starting time. Because LSM greedily uses $m^L$ machines for long jobs, either $|A^L_t| = m^L$, or $|B^L_t| = 1$, for each $t \in [1,r^L]$. So, we may partition time slots $\{1, \ldots , r^L\}$ into $F^L = \{ t : |A^L_t| = m \}$ and $B^L = \{ t : |A^L_t| < m, |B^L_t| = 1 \}$ and obtain:

$$T \leq \lambda(F^L) + \lambda(B^L) + p_t.$$  

Because $m^L$ long jobs are executed at each time slot $t \in F^L$,

$$\lambda(F^L) \leq \frac{1}{m^L} \left[ \sum_{j \in J^L} p_j - \mu(B^L) \right].$$
Then, Lemma 2 implies that $\mu(B^L) \geq n^L m^L / 2$. Furthermore, $\lambda(B^L) \leq n^L$. Therefore, by considering Lemma 3, we obtain:

$$T \leq \frac{m}{m^L} \left( \frac{1}{m} \sum_{j \in J} p_j \right) + \frac{1}{2} (n^L + p^L) + \frac{1}{2} p^L.$$

In the case $p^L \leq T^*/2$, since $T^* \geq n^L + p^L$, we obtain an approximation ratio of $(\frac{m}{m^L} + \frac{3}{4}) \leq 1.95$, when $m^L = \lceil 5m/6 \rceil$, given that $m = \omega(1)$.

Next, consider the case $p^L > T^*/2$. Let $J^V = \{ j : p_j > T^*/2 \}$ be the set of very long jobs and $n^V = |J^V|$. Clearly, $n^V \leq m$. By using resource augmentation, i.e. allowing LSM to use $m' = \lceil 6m/5 \rceil$ machines, we guarantee that LSM assigns at most one job $j \in J^V$ in each machine. The theorem follows.

Remark If $\lceil 5m/6 \rceil < |J^V| \leq m$, LSM does not achieve an approximation ratio better than 2, e.g. as illustrated by an instance consisting of only $J^V$ jobs. Assigning two such jobs on the same machine is pathological. Thus, better than 2-approximate schedules require assigning all jobs in $J^V$ to different machines.

6 Dealing with Uncertainty

This section exploits machine augmentation, which computes provably near-optimal solutions in Section 5, and Lexicographic Optimization (LexOpt) [15], for robust solutions to $P || C_{\text{max}}$, to design a robust optimization approach for BJSP under uncertainty. Section 6.1 summarizes a collection of guidelines from the Royal Mail experience shaping our investigations to account for uncertainty. Section 6.2 formally describes our uncertainty setting. Section 6.3 presents the proposed approach for generating robust BJSP schedules.

6.1 Guidelines from the Royal Mail Experience

Royal Mail deliveries are subject to uncertainty. Once a delivery has begun, it might finish earlier or later than its anticipated completion time. Optimizing under uncertainty with hard bounds on the makespan or the number of machines produces conservative solutions compared to nominal ones obtained with full input knowledge. So, backup vehicles and driver overtimes are regular practices. The Royal Mail aims to use a small number of vehicles while respecting employees’ working hours. We employ two-stage decision-making: the first stage computes a feasible, efficient schedule for an initial nominal problem instance before the day begins. The second stage repairs the original solution accounting for uncertainty. In particular, delivery start times and assignments of duties to vehicles can be adapted in an online manner to account for this uncertainty. Significantly modifying the initial solution may diminish
6.2 Uncertainty Setting

Figure 1 illustrates the two-stage setting for solving BJSP under uncertainty. The Figure 1 setting is most similar to the [17] recoverable robustness setting, but also has connections to other two-stage optimization problems under uncertainty [1, 3, 11]. Stage 1 computes a feasible, efficient schedule $S$ for an initial nominal instance $I$ of the problem with a set $J$ of jobs and vector of processing times $p$. After stage 1, there is uncertainty realization and a different, perturbed vector $\tilde{p}$ of processing times becomes known. Stage 2 transforms $S$ into a feasible, efficient solution $\tilde{S}$ for the perturbed instance $\tilde{I}$ with vector $\tilde{p}$ of processing times. Designing and analyzing a two-stage robust optimization method necessitates (i) defining the uncertainty set of the problem and (ii) quantifying the solution robustness.

Scheduling under uncertainty may involve different perturbation types. We study processing time variations, i.e. $p_j$ may be perturbed by $f_j > 0$ to become $\tilde{p}_j = f_j p_j$. If $f_j > 1$, then job $j$ is delayed. If $f_j < 1$, job $j$ completes early. Instance $I$ is modified by a perturbation factor $F > 1$ when $1/F \leq f_j \leq F$. Uncertainty set $U_F(I)$ contains every instance $\tilde{I}$ that can be obtained by disturbing $I$ w.r.t. perturbation factor $F$.

Our two-stage robust optimization approach aims to attain good trade-offs between the number of used machines and makespan. To this end, stage 1 imposes a global deadline $D$ for upper bounding makespan and computes a robust solution with a small number of vehicles. Stage 2 imposes the stage 1 job starts to avoid many deadline violations and dynamically allocates jobs to machines. Stage 2 may require more machines than the initial schedule. To quantify the initial solution robustness, we aim for a low price of robustness [4, 2, 9, 24]. Let $\tilde{S}$ and $\tilde{S}^*$ be the final obtained solution and the nominal optimal solution obtained with full input knowledge, respectively. The price
of robustness is \( V(\tilde{S}) / V(\tilde{S}^*) \), where \( V(\cdot) \) denotes the number of machines in a given schedule. The stage 1 global deadline and stage 2 fixed decisions both avoid large makespan constraint deviations.

6.3 Two-Stage Robust Scheduling Approach

This section proposes a two-stage robust optimization for solving BJSP under uncertainty with: (i) an exact method producing initial solution \( S \) and (ii) a recovery strategy restoring \( S \) after instance \( \tilde{I} \) is revealed.

6.3.1 Exact LexOpt Scheduling with Machine Augmentation (Stage 1)

To develop an integer programming formulation for producing the initial solution \( S \), we define a schedule’s characteristic value. Denote by \( V(S) \) be the number of machines in \( S \). In addition, associate the weight \( w_j(S) = 2C_j(S) \) to each job \( j \in \mathcal{J} \), where \( C_j(S) \) is the job \( j \) completion time, and let \( W(S) = \sum_{j \in \mathcal{J}} w_j(S) \) be the sum of job weights in \( S \). The characteristic value \( F(S) \) of schedule \( S \) is the weighted sum:

\[
F(S) \triangleq V(S) + \theta \cdot W(S),
\]

where \( \theta > 0 \) is a parameter specifying the relevant importance between \( V(S) \) and \( W(S) \). Section 7 selects the \( \theta \) value empirically.

Similarly to [15], computing a schedule of minimal characteristic value in stage 1 enables more efficient two-stage BJSP solving under uncertainty. Because the job start times are not modified in stage 2, if a minimal of machines is used in the initial schedule \( S \), many new job starts may occur after uncertainty realization due to job delays. So, the number of machines might be high in the final solution \( \tilde{S} \). On the other hand, if a large number of machines is used in the stage 1 schedule \( S \), a small number of new job overlaps may occur in the stage 2 schedule \( \tilde{S} \), but the final solution number of machines is already high. Minimizing \( W(S) = \sum_{j \in \mathcal{J}} w_j(S) \) lexicographically minimizes the sum of job completion times and produces robust schedules in terms of minimizing makespan [15]. Minimizing the characteristic value generates a robust schedule with small machine augmentation if the parameter \( \theta \) is carefully selected.

A schedule of minimal characteristic value can be computed with integer programming formulation (8). Variable \( v \) corresponds to the number of machines and parameter \( w_{j,t} = 2^{t} \) is the weight of job \( j \) if it completes at time slot \( t \).

\[
\begin{align*}
\min_{x_{j,s}, v, w} & \quad v + \theta \left( \sum_{j \in \mathcal{J}} \sum_{s \in D} x_{j,s} w_{j,s+p_j} \right) \\
\text{subject to} & \quad v \geq x_{j,s} \quad j \in \mathcal{J}, s \in D \\
& \quad x_{j,s}(s + p_j) \leq D \quad j \in \mathcal{J}, s \in D
\end{align*}
\]

(8a)
Expression (8a) minimizes the characteristic value. Constraints (8b) limit the active machines to the total number of machines. Constraints (8c) force all jobs to complete before the deadline. Constraints (8d) ensure that at most \(v\) machines are used at each time slot \(t\). Constraints (8e) require that each job \(j\) is scheduled. Constraints (8f) express the BJSP constraint. Numerical instabilities create tractability issues for Integer Program (8). We circumvent this issue by reducing the number of lexicographic objectives. To avoid many lexicographic objectives, round the completion times. In particular, we divide the time horizon into a set of \(\ell\) time periods. We set itinerary weights \(w_j = 2^\ell\) if job \(j\) completes at time period \(\ell\). If solution \(S\) minimizes \(\sum_{j \in J} w_j\), then it (approximately) lexicographically minimizes the job completion times.

6.3.2 Rescheduling Strategy (Stage 2)

A rescheduling strategy transforms an initial schedule \(S\) for the nominal problem instance \(I\) into a final schedule \(\tilde{S}\) for the perturbed instance \(\tilde{I}\). To satisfy the requirement that schedule \(\tilde{S}\) should stay close to \(S\), we distinguish between binding and free optimization decisions similarly to [15]. Let \(x_{j,s}\) and \(y_{i,j}\) be binary variables specifying whether job \(j \in J\) begins at time slot \(t \in D\) and is executed by machine \(i \in M\), respectively. We consider rescheduling with restricted job start times and flexible job-to-machine assignments. Definition 1 formalizes this fact with binding and free optimization decisions.

**Definition 1** Let \(S\) be the initial schedule in BJSP under uncertainty.

- **Binding decisions** \(\{x_{j,t} : j \in J, t \in F_j\}\) are variable evaluations determined from \(S\) in the rescheduling process.
- **Free decisions** \(\{y_{i,j} : i \in M, j \in J\}\) are variable evaluations not determined from \(S\) but needed to recover feasibility.

Enforcing binding decisions ensures a limited number of initially planned solution modifications. Note that first-stage decisions remain critical in this context. The proposed recovery strategy sets \(x_{j,s}(S) = x_{j,s}(\tilde{S})\). On the other hand, job-to-machine assignments are decided in an online manner. For \(t = 1, \ldots, \tau\), the jobs \(\{j : x_{j,t}(S) = 1\}\) are assigned to the lowest-indexed available machines. A machine is available at \(t\) if it executes no jobs. This assignment derives the \(y_{i,j}(\tilde{S})\) values.
7 Numerical Results

This section describes numerical findings for BJSP under uncertainty using Royal Mail data. Section 7.1 discusses the derivation of Royal Mail BJSP instances and historical schedules. Section 7.2 evaluates the historical schedules number of machines, i.e. Royal Mail vans, and sensitivity. Finally, Section 7.3 presents a numerical assessment of the two-stage robust optimization approach. We run all computations on a 2.5GHz Intel Core i7 processor with an 8 GB RAM memory running macOS Mojave 10.14.6. Our implementations use Python 3.6.8, Pyomo 5.6.1, and solve the integer programming instances with CPLEX 12.8. A recent MEng thesis considers several of these contexts in greater detail [7].

7.1 Generation of Benchmark Instances and Historical Schedules

We use historical data from three Royal Mail delivery offices which we refer to as (i) DO-A, (ii) DO-B, and (iii) DO-C. Part of the data is encrypted for confidentiality protection. We consider a continuous time period of 78, 111, and 111 working days for DO-A, DO-B, and DO-C, respectively. A BJSP instance is the set of all jobs performed by a single delivery office during one date. So, we examine 300 instances in total. A job corresponds to a delivery in a set of neighboring postal codes. The data is a list of jobs, each specified by a (i) unique identifier, (i) delivery office, (iii) date, (iv) vehicle plate number, (v) begin time, and (v) completion time. Below, we give more details for generating the benchmark instances and the actual schedules realized by Royal Mail.

A BJSP instance is defined by a (i) time horizon, (ii) time discretization, (iii) number of available vehicles, (iv) set of jobs, and (v) BJSP parameter. A simple data inspection shows that, among all jobs, 92% run during [06:00,19:00] in DO-A, 91% are executed during [05:00,19:30] in DO-B, and 93% are implemented during [05:30,19:30] in DO-C. A scatter plot illustration clearly demonstrates that these boundaries specify the time horizon for each delivery office after dropping outliers [16]. The time horizon boundaries might be violated by both the historical schedules and our two-stage robust optimization method. We use a time discretization of $\delta = 15$ minutes. The number of available vehicles is the number of distinct plate numbers used by each delivery office. We set the processing time of job $j \in J$ equal to $p_j = \lceil (C_j - s_j)/\delta \rceil$, where $s_j$ and $C_j$ is start and completion time of $j$ in the corresponding historical schedule. The minimal processing time is 30 minutes, but a job may last for a number of hours. A scatter plot illustration shows that the distribution of processing times follows a similar pattern on a weekly basis [16]. This observation supports using robust optimization to deal with the Royal Mail BJSP instances under uncertainty. BJSP parameter $g$ is set equal to the maximum number of jobs beginning in a time interval $\delta$ units of time after ignoring few outliers.

The Royal Mail data include a historical schedule for each BJSP instance. Such a schedule is associated with (i) job start times, (ii) makespan, and (iii)
number of used machines. Begin times are rounded down to the closest multiple of $\delta$ for time discretization. After rounding, the makespan is the time at which the last job completes. The number of vehicles is the maximal number of jobs running simultaneously. We note that solutions in this form do not explicitly specify job-to-machine assignments. However, once the job start times are known, feasible assignments can be computed with simple interval scheduling algorithms [13].

7.2 Evaluation of Historical Schedules

This section evaluates the Royal Mail historical schedules (i) efficiency in number of used machines and (ii) sensitivity with respect to processing time and (iii) BJSP parameter variations.

For part (i), we solve each BJSP instance by feeding the corresponding MILP model to CPLEX. In these MILP models, we set $\theta = 0$ to minimize the number of used machines. Figure 2 compares the number of machines in the Royal Mail historical schedules and the CPLEX solutions. We observe that nominal optimal solutions save at least 10, 25, and 10 vehicles per day compared to historical schedules for DO-A, DO-B, and DO-C, respectively. This finding is a strong indication that more efficient fleet management might be possible in Royal Mail delivery offices.

For part (ii), we create a set of perturbed instances. In particular, for each original instance $I$, we create one perturbed instance $\tilde{I}$ where the processing time of each job $j \in J$ is decreased by a factor $f_j = 0.5$. We reduce the processing times to guarantee feasibility. For both instances $I$ and $\tilde{I}$, we employ CPLEX to solve the corresponding MILP formulations with $\theta = 0$. Figure 3 compares the number of used machines obtained for the original and perturbed instances. Not surprisingly, doubling itinerary durations results in a proportional increase on the number of used vehicles in the nominal optimal solution. But, Figure 3 exhibits an important consequence of uncertainty in Royal Mail fleet management. Disturbances amplify the difference in number of used machines between different days for one delivery office. This situation leads to inefficient machine utilization.

For part (iii), we investigate the effect of modifying the BJSP parameter for each delivery office. Figure 4 depicts the obtained results. Adding BJSP constraints, especially in the DO-B case, may significantly increase the number of used machines. This outcome motivates further investigations on scheduling with BJSP constraints.

7.3 Robustness Assessment

The current section evaluates the two-stage robust optimization method presented in Section 6.3. Specifically, we argue that low characteristic value results more robust BJSP schedules. To this end, we adopt the experimental setup in [15]. For each BJSP instance $I$, we generate a collection $C(I)$
of feasible, diverse schedules using the CPLEX solution pool feature. Let \( F^*(I) = \min\{F(S) : S \in C(I)\} \) and \( V^*(I) = \min\{V(S) : S \in C(I)\} \) be the minimum characteristic value and number of machines achieved for instance \( I \) among all schedules in \( C(I) \). For each \( S \in C(I) \), we set a normalized characteristic value \( F^N(S) = F(S) / F^*(I) \) and normalized number of used machines \( V^N(S) = V(S) / V^*(I) \). Figure 2 is a scatter plot with a pair \((V^N(S), F^N(S))\) for each schedule and for all instances. We observe that the smaller the initial schedule characteristic value is, the best the final solution we get in terms of number of vehicles.

8 Conclusion

This manuscript initiates study of the bounded job start scheduling problem (BJSP), e.g. as arising in Royal Mail deliveries. This project is part of our
Fig. 3: Line chart comparing the number of used machines between the original instances and instances where the job processing times have been halved.

larger aims towards approximation algorithms for process systems engineering [14]. The main contributions are (i) better than 2-approximation algorithms for various cases of the problem and (ii) a two-stage robust optimization approach for BJSP under uncertainty based on machine augmentation and lexicographic optimization, whose performance is substantiated empirically. We conclude with a collection of future directions. Because BJSP relaxes scheduling problems with non-overlapping constraints for which better than 2-approximation algorithms are impossible under widely adopted conjectures, the existence of an algorithm with an approximation ratio strictly better than 2 which does not use resource augmentation is an intriguing open question. A positive answer combining LSM algorithm with a new algorithm specialized to instances with many very long jobs is possible. Moreover, analyzing the price of robustness of the proposed two-stage robust optimization approach may provide new insights for effectively solving BJSP under uncertainty. Finally, our findings
Fig. 4: Line chart comparing the number of vehicles between instances with different BJSP parameters.

demonstrate a strong potential for more efficient Royal Mail resource allocation by using vehicle sharing between different delivery offices. The BJSP scheduling problem where multiple delivery offices are integrated in a unified setting and vehicle exchanges are performed on a daily basis consists a promising direction for fruitful investigations. In this context, recent work on car pooling might be useful.

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Fig. 5: Scatter plots comparing the initial solution weighted value with the final solution number of vehicles.

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