The Picard Group of Various Families of $(\mathbb{Z}/2\mathbb{Z})^4$-invariant Quartic K3 Surfaces

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Abstract

The subject of this paper is the study of various families of quartic K3 surfaces which are invariant under a certain $(\mathbb{Z}/2\mathbb{Z})^4$ action. In particular, we describe families whose general member contains 8, 16, 24 or 32 lines as well as the 320 conics found by Eklund [Ekl10] (some of which degenerate into the mentioned lines). The second half of this paper is dedicated to finding the Picard group of a general member of each of these families, and describing it as a lattice. It turns out that for each family the Picard group of a very general surface is generated by the lines and conics lying on said surface.

1 Introduction

Consider the $(\mathbb{Z}/2\mathbb{Z})^4$ subgroup of $\text{Aut}(\mathbb{P}^3_{\mathbb{Q}})$ generated by the four transformations

$$[x : y : z : w] \mapsto [y : x : w : z], [z : w : x : y], [x : y : -z : -w], [x : -y : z : -w].$$

In this paper we look at various families of quartic surfaces in $\mathbb{P}^3_{\mathbb{Q}}$ which are invariant under these transformations. The family of all such quartics is known to be parameterised by $\mathbb{P}^4$ and has been studied extensively by [BN94, Ekl10]. Barth and Nieto, [BN94], studied the moduli space of invariant quartics which contain a line, while Eklund [Ekl10] looked at those that contain a conic. It turns out that a very general invariant quartic contains at least 320 conics, and Eklund uses this, among other tools, to calculate the Picard group of a very general invariant quartic. Barth and Nieto the locus of invariant quartics containing lines to be a quintic threefold $N_5 \subset \mathbb{P}^4$, plus the tangent cones of the 10 singular points of $N_5$. Eklund studied the quartic surfaces parameterised by $N_5$, so in this paper we look at the surfaces parameterised by the cones.

In particular we consider:

- a four dimensional family $\mathcal{X}$,
- a three dimensional family $\mathcal{X}_{C,D,E}$,
- a two dimensional family $\mathcal{X}_{C,D}$,
- a one dimensional family $\mathcal{X}_{B}$,
- a one dimensional family $\mathcal{X}_{C}$,
- a specific quartic K3 surface $Y$,
- and the Fermat quartic, $F_4$.

For each of these families, we look at the lines a very general member contains. We use these and the 320 conics that Eklund found to calculate the Picard group of a very general member. Our main result (Theorem 4.9) can be summarised as follows:

Theorem (Summarised Theorem 4.9).

- A very general member of $\mathcal{X}$ contains no lines, and has Picard rank 16,
• A very general member of $X_{C, D, E}$ contains exactly 8 lines, and has Picard rank 17,
• A very general member of $X_{C, D}$ contains exactly 16 lines, and has Picard rank 18,
• A very general member of $X_{B}$ contains exactly 24 lines, and has Picard rank 19,
• A very general member of $X_{C}$ contains exactly 32 lines, and has Picard rank 19,
• The surface $Y$ contains exactly 32 lines, and has Picard rank 20,
• The Fermat quartic, $F_4$, contains exactly 48 lines, and has Picard rank 20.

Possibly except for the surface $Y$, the Picard group is generated by the lines and conics lying on the surface. In all cases, we decompose the Picard group into known lattices.

Remark. The result about a very general member of $X$ having Picard rank 16, with the Picard group generated by the conics, was already proven by Eklund [Ekl10, Thm 3.5, Cor 7.4] but in this paper we prove this using a different method.

The fact that the Fermat quartic has 48 lines, which generate the Picard group of rank 20 is a classical result. We will use that result in our proof of Theorem 4.9.

We note that Theorem 4.9 fits nicely with the fact that certain moduli spaces of K3 surfaces whose Picard group contains a fixed lattice $M$ have dimension $20 - \text{rank}(M)$. I.e., in each of the above, a Picard group of rank $r$, fits nicely with a $20 - r$ dimensional family.

In Section 2 we review the notations and results we need for lattices. In Section 3 we start by introducing the notations we will use and review the known results. We finish the section by introducing the families containing lines, that we will study in the rest of the paper.

In Section 4 we look at the Picard group of the families. First by calculating the Picard number, then by proving that in each case the Picard group is generated by the conics and lines. We finish by putting everything together and using the results from Section 2 to prove Theorem 4.9.

Note. Throughout this paper, most calculations, point counting and lattice manipulation were done using the computer algebra package Magma [BCP97, V2.21-4].

2 Lattices

In this paper a lattice, $L$, is a free abelian group of finite rank equipped with a symmetric, non-degenerate, bilinear form $\langle , \rangle : L \times L \to \mathbb{Z}$. We say it has signature $(b_+, b_-)$ if for some basis $\{ e_i \}$ of $L \otimes \mathbb{R}$ we have

$$\langle e_i, e_j \rangle = \begin{cases} 1 & i = j, i \in \{ 1, \ldots, b_+ \} \\ -1 & i = j, i \in \{ b_+ + 1, \ldots, b_+ + b_- \} \\ 0 & i \neq j \end{cases}.$$  

A lattice is positive definite if it has signature $(b_+, 0)$, negative definite if it has signature $(0, b_-)$, and indefinite otherwise. A lattice, $L$, is even if $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$. Let $\{ e_i \}$ be a basis for $L$, then a Gram matrix of $L$ (with respect to $\{ e_i \}$) is the matrix $(\langle e_i, e_j \rangle)_{i,j}$. The discriminant of $L$, denoted Disc($L$), is the determinant of a Gram matrix, which is invariant under change of basis. A lattice is unimodular if it has discriminant $\pm 1$. 


Example. Consider the following Dynkin diagrams:

\[ A_n := \quad \begin{array}{cccc} e_1 & e_2 & e_{n-1} & e_n \\ \end{array} \quad , \]

\[ D_n := \quad \begin{array}{cccc} e_1 & e_2 & e_{n-3} & e_{n-2} \\ & e_{n-1} & \end{array} \quad , \]

\[ E_8 := \quad \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ & & e_5 & e_6 \\ & e_7 & \end{array} \quad . \]

Each diagram defines a (root) lattice, with basis \( \{ e_i \} \) and bilinear form

\[ \langle e_i, e_j \rangle = \begin{cases} 2 & i = j \\ -1 & e_i \neq e_j \\ 0 & \text{otherwise} \end{cases} \ . \]

Another example of a lattice is the hyperbolic plane lattice, denoted \( U \), which is the unique (up to isomorphism) rank 2 even indefinite unimodular lattice. For some basis, its Gram matrix is

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \]

Given a lattice \( L \) with basis \( \{ e_i \} \) and \( m \in \mathbb{Z} \), we denote by \( L \langle m \rangle \) the lattice with basis \( \{ e_i \} \) and bilinear form \( \langle e_i, e_j \rangle_L \langle m \rangle = m \langle e_i, e_j \rangle_L \). By abuse of notation, we denote the rank one lattice with bilinear form \( \langle e_1, e_1 \rangle = m \) by \( \langle m \rangle \). If \( L_1 \) and \( L_2 \) are two lattices with basis \( \{ e_i \} \), \( \{ f_i \} \) respectively, we denote by \( L_1 \oplus L_2 \) the lattice with basis \( \{ e_i \} \cup \{ f_i \} \) and bilinear form given by \( \langle e_i, f_j \rangle = 0 \). We will say that a lattice \( L \) decomposes into \( L_1, \ldots, L_n \) if \( L \cong L_1 \oplus \cdots \oplus L_n \).

We say a lattice \( L_1 \) is a sublattice of a lattice \( L_2 \) if it is a subset of \( L_2 \) and if the bilinear form of \( L_2 \) restricted on \( L_1 \) agrees with the bilinear form of \( L_1 \). A sublattice is said to be primitive if \( L_2/L_1 \) is torsion free. If \( L_1 \) is a full-rank sublattice of \( L_2 \), i.e. \( \text{rank}(L_1) = \text{rank}(L_2) \), then we call \( L_2 \) an overlattice of \( L_1 \). Note that in such case \( \text{Disc}(L_1)/\text{Disc}(L_2) = [L_2 : L_1]^2 \).

In Section 4 we try to find a decomposition of lattices into \( A_n \langle m \rangle, D_n \langle m \rangle, E_8 \langle m \rangle \) and \( U \langle m \rangle \), to do so we will need some extra invariants. We may extend the bilinear form \( \langle \, , \rangle \) on \( L \otimes \mathbb{Q} \) linearly to \( L \otimes \mathbb{Q} \) and define the dual lattice (which is often not a lattice with respect to our definition):

\[ L^* := \text{Hom}(L, \mathbb{Z}) \cong \{ x \in L \otimes \mathbb{Q} : \langle x, y \rangle \in \mathbb{Z} \forall y \in L \} . \]

Definition 2.1. The discriminant group of a lattice \( L \) is the finite abelian group \( A_L := L^*/L \). We denote by \( \ell(A_L) \) the minimal number of generators of \( A_L \).

The discriminant group comes with a bilinear form, \( b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z} \) defined by \( b_L(x + L, y + L) \mapsto \langle x, y \rangle_L \mod \mathbb{Z} \).

For even lattices, we define the discriminant form, \( q_L : A_L \to \mathbb{Q}/2\mathbb{Z} \) by \( x + L \mapsto \langle x, x \rangle_L \mod 2\mathbb{Z} \).

The following theorem of Nikulin will help identify the lattices we will find:

Theorem 2.2 (Nikulin [Nik80] Cor. 1.13.3). If a lattice \( L \) is even, indefinite with \( \text{rank}(L) > \ell(A_L) + 2 \), then \( L \) is determined up to isometry by its rank, signature and discriminant form.
With that theorem in mind, we write down in Table 1 a summary of the rank, signature and discriminant form for the lattices $U$, $E_8$, $A_n$ ($m$), $D_n$ ($m$) and $\langle 2m \rangle$.

|   | $U$ | $E_8$ | $A_n$ ($m$) | $D_{2n}$ ($m$) | $D_{2n+1}$ ($m$) | $\langle 2m \rangle$ |
|---|-----|-------|-------------|---------------|----------------|-------------------|
| Rank | 2   | 8     | $n$         | $2n$          | $2n+1$        | $1$               |
| Sgn  | (1,1) | (8,0) | $\{(n,0) : m > 0\}$ | $\{(2n,0) : m > 0\}$ | $\{(2n+1,0) : m > 0\}$ | $\{(1,0) : m > 0\}$ |
| Disc | 1   | 1     | $(n+1) \cdot m^2$ | $4 \cdot m^{2n}$ | $4 \cdot m^{2n+1}$ | $2m$               |
| $A_L$ | $-$ | $-$   | $C_{(n+1)m} \times C_{m-1}^2$ | $C_{2m}^2 \times C_{m}^{2n-2}$ | $C_{4m} \times C_{m}^{2n}$ | $C_{2m}$          |
| $q_L$ | $-$ | $-$   | $\left\{ \frac{n}{m} \text{ or } \frac{n(n-1)}{m} \right\}$ | $\left\{ \frac{2n+1}{4m}, \frac{2n}{2m}, \frac{2n}{m} \right\}$ | $\left\{ \frac{1}{2m} \right\}$ |                   |

Table 1: Invariants of Lattices

The row $q_L$ lists the values of $q_L(x_i)$ where $x_i$ are chosen generators of $A_L$, i.e., $A_L = \langle x_1 \rangle \times \cdots \times \langle x_{\ell(A)} \rangle$. Therefore it only encodes partial information of the discriminant form and not the whole of it, but it encodes enough to rule out (in most cases) whether a summand occurs. As $U$ and $E_8$ have trivial discriminant group, we use following theorem of Nikulin to identify copies of $U$ and $E_8$ sitting inside a given lattice.

**Theorem 2.3** ([Nik80, Cor 1.13.15]). Let $L$ be an even lattice of signature $(b_+, b_-)$.

- If $b_+ \geq 1, b_- \geq 1$ and $b_+ + b_- \geq 3 + \ell(A_L)$ then $L \cong U \oplus T$ for some $T$.
- If $b_+ \geq 1, b_- \geq 8$ and $b_+ + b_- \geq 9 + \ell(A_L)$ then $L \cong E_8 \langle -1 \rangle \oplus T$ for some $T$.

We note that we can not always have a decomposition of lattices into $A_n$ ($m$), $D_n$ ($m$), $E_8$ ($m$) and $U$ ($m$). When this happens, we express our lattices as full rank sublattices of a lattice composed of $A_n$ ($m$), $D_n$ ($m$), $E_8$ ($m$) and $U$ ($m$). For this we will use:

**Theorem 2.4.** ([Nik80] Prop 1.4.1) Let $L$ be an even lattice. Then there is a natural bijection between isotropic subgroups $G$ of $A_L$ (subgroups on which the discriminant form $q_L$ satisfies $q_L(g) = 0$ for all $g \in G$) and overlattices $L_G$ of $L$.

Furthermore, the discriminant form $q_{L_G}$ is given by the discriminant form $q_L$ restricted to $G^\perp / G$, where orthogonality is with respect to $b_L$.

### 3 The Families and Lines

We will be studying the variety $X \subset \mathbb{P}^3_{[x,y,z,w]} \times \mathbb{P}^4_{[A,B,C,D,E]}$ defined by the following equation over $\overline{\mathbb{Q}}$:

$$A(x^4 + y^4 + z^4 + w^4) + Bxyzw + C(x^2y^2 + z^2w^2) + D(x^2z^2 + y^2w^2) + E(x^2w^2 + y^2z^2) = 0.$$ 

We view $X$ as a family of quartic surfaces over $\mathbb{P}^3$ parametrised by points $[A, B, C, D, E]$ in $\mathbb{P}^4$.

**Notation.** We will use $X_p$ and $[A, B, C, D, E]$ to denote the quartic surface parametrised by the point $p = [A, B, C, D, E] \in \mathbb{P}^4$.

**Note.** If $X_p$ is a smooth quartic surface, then it is a K3 surface.

Consider the group $\Omega$ acting on $\mathbb{P}^3 \times \mathbb{P}^4$ generated by the following five elements: the point $[x, y, z, w, A, B, C, D, E]$ is sent to

- $[x, y, z, -w, A, -B, C, D, E]$,
Denote these five elements by $\phi_1, \phi_2, \phi_3, \phi_4$ and $\phi_5$ respectively. The group $\Omega$ fixes $X$. While it is a rather large group with order $2^4 \cdot 6!$, we can pick out a normal subgroup $\Gamma$, which is generated by the following four elements

- $\gamma_1 := \phi_3^2 \phi_3 \phi_5^2$,
- $\gamma_2 := \phi_2^2 \phi_3 \phi_5$,
- $\gamma_3 := \phi_4$,
- $\gamma_4 := \phi_3^2 \phi_3$.

The group $\Gamma$ consists of all elements of $\Omega$ which fix $\mathbb{P}^4_{[A,B,C,D,E]}$ in $\mathbb{P}^3 \times \mathbb{P}^4$. In particular upon picking a point $p \in \mathbb{P}^4$ we have that $\Gamma$ is a subgroup of $\text{Aut}(X_p)$ (when projecting the elements of $\Gamma$ onto the $\mathbb{P}^3_{[x,y,z,w]}$ component). Explicitly, when regarding $\Gamma$ as acting on $\mathbb{P}^3$, we have that its generators are

$$[x, y, z, w] \mapsto \begin{cases} [y, x, w, z] & \gamma_1 \\ [z, w, x, y] & \gamma_2 \\ [x, y, -z, -w] & \gamma_3 \\ [x, -y, z, -w] & \gamma_4 \end{cases}.$$  

From this we know that $\Gamma \cong C_2^4$. We calculate that $\Omega/\Gamma \cong S_6$, but $\Omega \not\cong C_2^4 \times S_6$ because in particular $\Omega$ has trivial centre. We now consider the cases when $X_p$ is not a smooth surface using the following proposition taken from [Ek10] Prop 2.1.

**Proposition 3.1.** Let $p = [A, B, C, D, E] \in \mathbb{P}^4$. The surface $X_p$ is singular if and only if

$$\Delta \cdot A \cdot q+ C \cdot q- C \cdot q+ D \cdot q- D \cdot q+ E \cdot q- E \cdot p+0 \cdot p+1 \cdot p+2 \cdot p+3 \cdot p-0 \cdot p-1 \cdot p-2 \cdot p-3 = 0,$$

where:

$$\Delta = 16 A^3 + AB^2 - 4A(C^2 + D^2 + E^2) + 4CDE$$  \hspace{1cm} (1)

- $q_+ C = 2A + C$
- $q_+ D = 2A + D$
- $q_+ E = 2A + E$
- $q_- C = 2A - C$
- $q_- D = 2A - D$
- $q_- E = 2A - E$
- $p_{+0} = 4A + B + 2C + 2D + 2E$
- $p_{+1} = 4A + B + 2C - 2D - 2E$
- $p_{+2} = 4A + B - 2C + 2D - 2E$
- $p_{+3} = 4A + B - 2C - 2D + 2E$
- $p_{-0} = 4A - B + 2C + 2D + 2E$
- $p_{-1} = 4A - B + 2C - 2D - 2E$
- $p_{-2} = 4A - B - 2C + 2D - 2E$
- $p_{-3} = 4A - B - 2C - 2D + 2E$.

**Definition 3.2.** The surface $S_3 = \{\Delta = 0\} \subset \mathbb{P}^4$ is the Segre cubic. We shall refer to the 15 hyperplanes in $\mathbb{P}^4$ defined by the 15 equations

$$\{A, p_{\pm j}, q_{\pm \alpha} : \alpha \in \{C, D, E\}, j \in \{0, 1, 2, 3\}\}$$

above as the singular hyperplanes.
The Segre cubic has 10 singular points, namely:

\[ [1, 0, -2, -2, 2], [1, 0, -2, 2, -2], [1, 0, 2, -2, -2], [1, 0, 2, 2, 2], [0, -2, 1, 0, 0], [0, 2, 1, 0, 0], [0, -2, 0, 1, 0], [0, 2, 0, 1, 0], [0, -2, 0, 0, 1], [0, 2, 0, 0, 1]. \]

We shall denote these 10 points by \( q_i, i \in \{1, \ldots, 10\} \), as ordered above. These 10 points have associated quartics in \( \mathbb{P}^3 \), which turns out to be quadrics of multiplicity two. We denote the quadric associated to the point \( q_i \) by \( Q_i \). Explicitly they are:

\[
\begin{align*}
&x^2 - y^2 - z^2 + w^2 = 0, \quad x^2 - y^2 + z^2 - w^2 = 0, \quad x^2 + y^2 - z^2 - w^2 = 0, \quad x^2 + y^2 + z^2 + w^2 = 0, \\
&xy - zw = 0, \quad xy + zw = 0, \quad xz - yw = 0, \quad xz + yw = 0, \quad xw - yz = 0, \quad xw + yz = 0.
\end{align*}
\]

**Definition 3.3.** We define the **Nieto quintic**, \( N_5 \subseteq \mathbb{P}^4_{[A,B,C,D,E]} \), by the equation

\[ 4A^3(48A^2 - B^2) - A(32A^2 - B^2)(C^2 + D^2 + E^2) - 4A(C + D + E)(C + D - E)(C - D + E)(-C + D + E) + B^2CDE = 0. \]

The Nieto quintic was studied by Barth and Nieto when they were looking at K3 surfaces in \( \mathcal{X} \) containing lines. In particular, they proved in [BN94, Section 7 and 8] the following proposition.

**Proposition 3.4.** Let \( p \in \mathbb{P}^4 \), then the surface \( X_p \) contains a line, \( L \), if and only if \( p \) is in \( N_5 \) or in one of the 10 tangent cones to the isolated singular points of \( N_5 \) (i.e., the 10 nodes of \( S_5 \)).

*In the case where \( p \) lies on the tangent cone of \( q_i \), then \( L \) lies on \( Q_i \).*

As Eklund studies in detail the K3 surfaces defined by a point lying on the Nieto quintic [Ekl10], we study here those surfaces defined by a point lying on the 10 tangent cones. We first make the following remark:

**Remark.** The four roots of the equation \( f = x^4 + cx^2 + 1 \) are of the form

\[ \alpha = \frac{1}{2} \left( \sqrt{-c+2} + \sqrt{-c-2} \right). \]

To see this, note that \( \alpha^2 = \frac{1}{4}(-c + \sqrt{c^2 - 4}) \) which solves \( y^2 + cy + 1 \).

**Proposition 3.5.** Let \( p \in \mathbb{P}^4 \) lie on one of the 10 tangent cones to the isolated singular points of \( N_5 \), away from \( N_5 \) and the 15 singular planes. Then the surface \( X_p \) contains eight lines. *In the case where \( p \) lies on a unique tangent cone, \( X_p \) contains exactly eight lines.*

**Proof.** If \( p \in \mathbb{P}^4 \setminus N_5 \) lies on a unique tangent cone, say \( q_i \), then by Proposition 3.4 all the lines lying on \( X_p \) must be lines lying on \( Q_i \).

We first prove that when \( p = [A,B,C,D,E] \in \mathbb{P}^4 \) lies on the tangent cone of the point \( q_6 \), there are exactly eight lines lying on \( Q_6 \cap X_p \subseteq \mathbb{P}^3 \). By [BN94, 3.2], we have that \( p \) satisfies the equation \( AB - 2AC + DE = 0 \). The quadric \( Q_6 : xy + wz = 0 \) has the following lines (for any \( \alpha \in \mathbb{K}^* \))

- \( x + \alpha z = y - \alpha^{-1}w = 0 \),
- \( x + \alpha w = y - \alpha^{-1}z = 0 \),
- \( x = z = 0 \),
- \( x = w = 0 \),
- \( y = z = 0 \),
- \( y = w = 0 \).
Note that the last four lines can not lie on $X_p$, as $p$ does not lie on the 15 singular planes (hence $A \neq 0$). Now $X_p \cap \{x + \alpha z = y - \alpha^{-1} w = 0\}$ is defined by the equations:

\[
\begin{align*}
x + \alpha z &= 0, \\
y - \alpha^{-1} w &= 0, \\
(A \alpha^4 + D \alpha^2 + A) \left( z^4 + \frac{w^4}{\alpha^4} \right) + (E \alpha^4 + (2C - B) \alpha^2 + E) \frac{z^2 w^2}{\alpha^2} &= 0.
\end{align*}
\]

As $AB - 2AC + DE = 0$ implies

\[
E \alpha^4 + (2C - B) \alpha^2 + E = E \alpha^4 + \frac{DE}{A} \alpha^2 + E = \frac{E}{A} (A \alpha^4 + D \alpha^2 + A),
\]

we have that the last equation becomes

\[
(A \alpha^4 + D \alpha^2 + A) \left( z^4 + \frac{w^4}{\alpha^4} \right) + (A \alpha^4 + D \alpha^2 + A) \frac{E z^2 w^2}{A \alpha^2} = 0.
\]

This is identically zero if and only $A \alpha^4 + D \alpha^2 + A = 0$. Hence there are exactly four lines of the form $x + \alpha z = y - \alpha^{-1} w = 0$ on $X_p$, corresponding to the four roots of $A \alpha^4 + D \alpha^2 + A = 0$. We can run through exactly the same process for lines of the form $x + \alpha w = y - \alpha^{-1} z = 0$ and find that this time $\alpha$ needs to solve $A \alpha^4 + E \alpha^2 + A = 0$. Hence by letting

\[
\alpha = \frac{\sqrt{A}}{2A} (\sqrt{q-D} + \sqrt{-q+D})
\]

\[
\beta = \frac{\sqrt{A}}{2A} (\sqrt{q-E} + \sqrt{-q+E})
\]

we have the eight lines

- $x + \alpha z = y - \alpha^{-1} w = 0$,
- $x - \alpha z = y + \alpha^{-1} w = 0$,
- $x + \beta w = y - \beta^{-1} z = 0$,
- $x - \beta w = y + \beta^{-1} z = 0$,
- $x + \alpha^{-1} z = y - \alpha w = 0$,
- $x - \alpha^{-1} z = y + \alpha w = 0$,
- $x + \beta^{-1} w = y - \beta z = 0$,
- $x - \beta^{-1} w = y + \beta z = 0$,

which lie on our surface $X_p$, and these are the only lines on $X_p \cap Q_6$. To finish the proof, we use the group $\Omega$ acting on $\mathbb{P}^3 \times \mathbb{P}^4$. This group permutes the 15 singular planes and the 10 points $q_i$ as follows:

- $\phi_1$ acts as $(p_{+0}, p_{-0})(p_{+1}, p_{-1})(p_{+2}, p_{-2})(p_{+3}, p_{-3})$ and as $(q_5, q_6)(q_7, q_8)(q_9, q_{10})$,
- $\phi_2$ acts as $(q_{+D}, q_{+E})(q_{-D}, q_{-E})(p_{+2}, p_{-2})(p_{+3}, p_{-3})$ and as $(q_1, q_2)(q_7, q_9)(q_8, q_{10})$,
- $\phi_3$ acts as $(q_{+C}, q_{+D})(q_{-C}, q_{-D})(p_{+1}, p_{+2})(p_{-1}, p_{-2})$ and as $(q_2, q_3)(q_5, q_7)(q_6, q_8)$,
- $\phi_4$ acts as $(q_{+D}, q_{-D})(q_{+E}, q_{-E})(p_{+0}, p_{-1})(p_{+0}, p_{+1})(p_{+2}, p_{-3})(p_{-2}, p_{+3})$ and as $(q_1, q_2)(q_3, q_4)(q_5, q_6)$,
- $\phi_5$ acts as $(A, q_{+C})(q_{+D}, p_{+0})(q_{-D}, p_{-1})(q_{+E}, p_{-0})(q_{-E}, p_{+1})(p_{+2}, p_{-3})$ and as $(q_1, q_5)(q_2, q_6)(q_7, q_{10})$.

Hence by applying the appropriate element $\phi \in \Omega$ on the above eight lines, we get the equations of the eight lines lying on the surface $X_{\phi(p)}$. We have listed the equations of the lines in Table 2 of Appendix A.

\[\square\]
Using the fact that the eight lines comes from the two different rulings of the quadric (one set using $\alpha$, the other $\beta$), it is not hard to see that the lines come in two sets of four skew lines. Furthermore each line from one set intersects each of the four lines in the other set.

Finally, using the explicit equations, we note that given two (not necessarily distinct) lines in one set, $L_1$ and $L_2$, and two in the other set $M_1$ and $M_2$, there exists a unique $\gamma \in \Gamma$ interchanging $L_1$ with $L_2$ and $M_1$ with $M_2$.

We can use Proposition 3.5 to find various families containing $8, 16, 24, 32$ and $48$ lines.

Lemma 3.6.

- A very general surface in the family $[A, (DE - 2AC)/A, C, D, E]$ contains exactly 8 lines. We denote this family by $X_{C,D,E}$.

- A very general surface in the family $[A, 0, C, D, 2AC/D]$ contains exactly 16 lines. We denote this family by $X_{C,D}$.

- A very general surface in the family $[A, B(2A - B)/A, B, B, B]$ contains exactly 24 lines. We denote this family by $X_B$.

- A very general surface in the family $[A, 0, C, 0, 0]$ contains exactly 32 lines. We denote this family by $X_C$.

- The surface $[\sqrt{-3}, 12(\sqrt{-3} - 1), 6, 6, -6]$ contains exactly 32 lines. We denote this surface by $Y$.

- The Fermat quartic $[1, 0, 0, 0, 0]$ contains exactly 48 lines. We denote this surface by $F_4$.

Up to an action of $\Omega$, there are no other families whose very general member is smooth and lies on the tangent cones to one of the points $q_i$.

Proof. Note that for each family, a very general point will not be on $N_5$, hence if for each family a very general member lie on $m$ distinct tangent cones, then it will contain $8m$ lines as claimed.

Recall that $\Omega$ acts on the 10 points $q_i$, and hence on the 10 tangent cones. For each $m \in \{1, \ldots, 10\}$, we find representatives of the action of $\Omega$ on sets of size $m$. For example, when $m = 2$, as $\Omega$ is two-transitive, we have the representative $\{q_1, q_2\}$, for $m = 3$, we have two representative $\{q_1, q_2, q_3\}$ and $\{q_2, q_4, q_5\}$. Starting from $m = 10$ to 1, for each representative we intersect the corresponding tangent cones. We look at its irreducible components and discard any that is a subset of $L$ (the union of the 15 singular hyperplanes), any component remaining give us a family that we list. This also verifies that our list is complete. This calculation is available online [Bou].

We illustrate how the families fit together with the following diagram. The lines show which family is a subfamily of another family.
4 The Picard Group

We now turn to proving that the Picard rank of the families given above are those claimed by (the summarised) Theorem 4.9. Note that we already know this to be true for the Fermat quartic, \( x^4 + y^4 + z^4 + w^4 \) (see for example [AS83]) and the family \( X \), parameterised by \( \mathbb{P}^4 \) (Ekl10). To achieve this, for each family we will bound the rank from below and above. To bound the Picard rank from below, we use the following theorem proven in [Ekl10, Thm 4.3].

**Theorem 4.1.** A very general K3 surface \( X \) from the family \( X \) contains at least 320 smooth conics.

The equations of the conics can be listed explicitly in terms of the point \([A, B, C, D, E] \in \mathbb{P}^4\) giving the surface \( X \), more details can be found in [Bou15]. As the lines and conics are elements of \( \text{Pic}(X) \), they form a sublattice of it. Hence by using the explicit equations of the 320 conics and \( 8m \) lines, we can calculate their intersection matrix. The rank of said matrix, which is the rank of the sublattice generated by lines and conics, is a lower bound to the rank of the Picard group.

To calculate an upper bound, we use three main ideas:

### 4.1 Reduction at a good prime

The first idea is due to Van Luijk [vL07] which we briefly recap here.

**Theorem 4.2.** Let \( X \) be a K3 surface defined over a number field \( K \). Choose a finite prime \( p \subseteq \mathcal{O}_K \) of good reduction for \( X \). Let \( R = (\mathcal{O}_K)_p \) and \( k \) its residue field. Fix an algebraic closure \( \overline{K} \) of \( K \), \( \overline{R} \) the integral closure of \( R \) in \( \overline{K} \), and let \( \overline{k} = \overline{R}/p \) be the algebraic closure of \( k \). There are natural injective homomorphisms

\[
\text{NS}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \hookrightarrow \text{NS}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \hookrightarrow H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(1))
\]

of finite dimensional vector space over \( \mathbb{Q}_\ell \). The second injection respects the Galois action \( \text{Gal}(\overline{k}/k) \).

**Proposition 4.3.** Let \( X \) be a K3 surface defined over a finite field \( \mathbb{F}_q \) with \( q = p^r \). Let \( F_q : X \to X \) be the absolute Frobenius map of \( X \), which acts on the identity on points, and by \( x \mapsto x^p \) on the structure sheaf. Set \( \Phi_q = F_q^r \) and let \( \Phi_q^* \) denote the automorphism on \( H^2_{\text{ét}}(X, \mathbb{Q}_\ell) \) induced by \( \Phi_q \times 1 \) acting on \( X_{\overline{\mathbb{Q}_q}} \). Then the rank of \( \text{NS}(X_{\overline{\mathbb{Q}_q}}) \) is bounded above by the number of eigenvalues \( \lambda \) of \( \Phi_q^* \) for which \( \lambda/q \) is a root of unity (counted with multiplicity).
Hence, given a K3 surface over a number field \( K \), its Picard rank, \( \rho(X) \), is bounded above by eigenvalues in a certain form of \( \Phi_q^* \). Such eigenvalues can be read off from the characteristic polynomial, \( f_q(x) \), of \( \Phi_q^* \). To calculate the characteristic polynomial we use the Lefschetz formula:

\[
\text{Tr} \left( (\Phi_q^*)^i \right) = \# X_k(\mathbb{F}_q) - 1 - q^{2i},
\]

and the following lemma:

**Lemma 4.4** (Newton’s Identity). Let \( V \) be a vector space of dimension \( n \) and \( T \) a linear operator on \( V \). Let \( t_i \) denote the trace of \( T^i \). Then the characteristic polynomial of \( T \) is equal to

\[
f_T(x) = \det(x \cdot \text{id} - T) = x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_n
\]

where the \( c_i \) are given recursively by \( c_1 = -t_1 \) and

\[
-kc_k = t_k + \sum_{i=1}^{k-1} c_it_{k-i}.
\]

So in theory, since \( n = 22 \) as \( X \) is a K3 surface, we can calculate the characteristic polynomial by counting points over \( \mathbb{F}_q \) for \( i = 1, \ldots, 22 \). But this is computationally infeasible. To make the computation more feasible we use the fact that from the Weil conjectures we have the functional equation

\[
p^{22}f_q(x) = \pm x^{22}f_q(p^2/x).
\]

Second of all, in our cases we have an explicit submodule \( M \subseteq \text{NS}(\mathcal{X}) \) of rank \( r \), namely the one generated by the lines and conics lying on \( X \). Hence we can calculate the characteristic polynomial \( f_M(x) \) of Frobenius acting on \( M \). Since \( f_M(x) | f_q(x) \), we can compute two possible polynomials \( f_{q,+}(x) \) and \( f_{q,-}(x) \) (one for each possible sign in the functional equation) by counting points on \( X_k(\mathbb{F}_q^t) \) for \( i = 1, \ldots, (22 - r)/2 \).

Explicitly, suppose \( f_M(x) = \prod_j g_j(x)^{c_j} \) with \( \deg(g_j) = d_j \), hence \( \sum d_jc_j = r \). Note that \( f'_q(x) = f_M(x)h'(x) + f'_M(x)h(x) \), hence if \( c_j > 1 \) then \( g_j(x)|f'_q(x) \), and in general \( g_j(x) \) divides the \((e_j-1)\)th derivative of \( f_q(x) \). Therefore, we can use the roots of \( M \) to construct \( r/2 \) linear equations in the 11 coefficients of \( f_q(x) \) (by assuming \( f_q(x) \) satisfies one of the functional equation). Hence we just need to count points on \( X_k(\mathbb{F}_q^t) \) for \( i = 1, \ldots, (22 - r)/2 \) to be able to use linear algebra and find the 11 coefficients of \( f_q(x) \). Note that when we assume the negative functional equations, we have in fact only 10 coefficients of \( f_q(x) \), as \( c_{11} = 0 \). Hence, we end up not using all the information from \( f_M(x) \), therefore it is possible to construct \( f_q(x) \) such that \( f_M(x) \nmid f_q(x) \). This is a contradiction, meaning that \( f_q(x) \) satisfies the positive functional equation and not the negative.

Finally, note that by rescaling \( f_q(x) \) by \( f_q(x/p) \), we just need to count the roots which are also roots of unity.

### 4.2 Artin-Tate conjecture

Unfortunately, as the roots come in conjugate pairs, the above method can only ever give an even upper bound. The following proposition can potentially reduce the upper bound by one more than the above bound.

**Proposition 4.5.** Let \( X \) be a K3 surface defined over a number field \( K \) and let \( p \) and \( p' \) be two primes of good reductions. Suppose that \( \rho(X) = \rho(\overline{X}) = n \) but the discriminants \( \text{Disc(\text{NS}(X_p))} \) and \( \text{Disc(\text{NS}(\overline{X}_{p'}))} \) are different in \( \mathbb{Q}^*/(\mathbb{Q}^*)^2 \). Then \( \rho(X) < n \).

**Proof.** By the above, we know that \( \rho(X) \leq n \). If \( \rho(X) = n \), then \( \text{NS}(\overline{X}) \) is a full rank sublattice of \( \text{NS}(\overline{X}_p) \) and \( \text{NS}(\overline{X}_{p'}) \). But in that case, as elements of \( \mathbb{Q}^*/(\mathbb{Q}^*)^2 \), all three discriminants should be equal, which is a contradiction to the hypothesis. \( \square \)

As the proposition requires us to calculate the discriminants of \( \text{NS}(X_p) \) and \( \text{NS}(\overline{X}_{p'}) \) we use the following conjecture:
**Conjecture 4.6** (Artin - Tate). Let $X$ be a K3 surface over a finite field $\mathbb{F}_q$. Let $\rho$ and $\text{Disc}$ denote respectively the rank and discriminant of the Picard group defined over $\mathbb{F}_q$. Then

$$|\text{Disc}| = \lim_{T \to q} \frac{\Phi(T)}{(T-q)^{21-\rho \# \text{Br}(X)}}$$

Here $\Phi$ is the characteristic polynomial of Frob on $H^2_{et}(X_{\mathbb{Q}_q}, \mathbb{Q}_l)$. Finally, $\text{Br}(X)$ is the Brauer group of $X$.

In the case when $q$ is odd, then the above conjecture has been proven to be true (using the fact that it follows from the Tate Conjecture \cite{Mil75}, which has been proven for K3 surfaces \cite{Nyg83, NO85, Cha13, Mau14, MP15}). Furthermore in the case the conjecture is true we have that $\# \text{Br}(X)$ is a square. Hence, by picking $q$ large enough so that $\rho(X_q) = \rho(X_{\mathbb{Q}})$, we can find $|\text{Disc}|$ as an element of $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.

### 4.3 Only finitely many singular K3 surfaces

Suppose that a general member of the family $\mathcal{Y}$ has Picard rank at least 19 and the family $\mathcal{Y}$ is parameterised by a one dimensional curve. The third idea uses the fact that, up to $\mathbb{Q}$-isomorphism, there only finitely many K3 surfaces over $\mathbb{Q}$ which are singular, i.e., with Picard rank 20. Hence if a very general member of $\mathcal{Y}$ has Picard rank 20, then every member of $\mathcal{Y}$ is singular. Therefore $\mathcal{Y}$ parametrises a set of isomorphic surfaces. If we can show that there are two $\mathbb{Q}$-surfaces in $\mathcal{Y}$ which are not isomorphic, then a very general member of the family $\mathcal{Y}$ has Picard rank at most 19 (as it can not be 20).

We implement this by noting that in each of the cases we are interested in, the Fermat quartic, $F_4$, belongs to our family $\mathcal{Y}$. Furthermore the Fermat quartic is supersingular over algebraically closed fields of characteristic 3 mod 4, i.e., $\rho(F_{4,p}) = 22$ for $p \equiv 3 \pmod{4}$ \cite{Tat65}. Hence if there is another surface in $\mathcal{Y}$ with $\rho(X_p) = 20$ over a prime $p \equiv 3 \pmod{4}$, then $F_4$ and $X$ are not isomorphic (since their specialisations to the field $\mathbb{F}_p$ are not isomorphic, as they have different Picard rank).

With all these tools we tackle the following proposition:

**Proposition 4.7.**

- A very general surface in the family $\mathcal{X}$ has Picard rank 16,
- A very general surface in the family $\mathcal{X}_{C,D,E}$ has Picard rank 17,
- A very general surface in the family $\mathcal{X}_{C,D}$ has Picard rank 18,
- A very general surface in the family $\mathcal{X}_{B}$ has Picard rank 19,
- A very general surface in the family $\mathcal{X}_{C}$ has Picard rank 19,
- The surface $Y$ is singular.

**Proof.** To get the lower bound we want to calculate the intersection matrix of the conics and lines lying on a very general member of each family. The lines and conics are defined over a degree $2^{10}$ field extension, hence calculating the intersection matrix is computationally infeasible. Instead we do the calculations over finite fields. Pick $X$ in one of the families (call it $\mathcal{X}$) and let $p$ be a prime of good reduction. Then we know that the conics and lines of $X_{\mathbb{F}_p}$ are defined over $\mathbb{F}_{p^2}$ (due to having explicit equations and there are only two square classes in $\mathbb{F}_p$) and so we calculate with ease the intersection matrix. By Theorem 4.2 $\text{NS}(X_{\mathbb{F}_p}) \otimes \mathbb{Q}_\ell \to \text{NS}(X_{\mathbb{Q}_p}) \otimes \mathbb{Q}_\ell$ is injective, so the intersection matrix of the lines and conics over $\mathbb{F}_{p^2}$ is the same as the intersection matrix of the lines and conics over $\mathbb{Q}$. Furthermore, as the set of surfaces in $\mathcal{X}$, which reduce to $X_{\mathbb{F}_p}$ is Zariski open, the intersection matrix calculated is the same as the intersection matrix of a very general member of $\mathcal{X}$.

As the intersection matrix is a large matrix, we have included in Appendix B a full rank minor of the matrix for each family (in particular, the lower bound is the dimension of said minor). We work (see \cite{Bou}) through the families in reverse order from the list above.
• As $M_Y$ has rank 20, we know that $\rho(Y) = 20$ and hence the surface $Y$ is singular.

• As $M_C$ has rank 19, we know that a very general surface $X_C$ of $\mathcal{X}_C$ has $\rho(X_C) \geq 19$. Using the idea in Subsection 4.3 we see that the surface $X_B$, associated to the point $[1,0,0,0,0]$, is the Fermat quartic so it is supersingular over $\mathbb{F}_{19}$. On the other hand consider the surface $X_5$, associated to the point $[1,0,5,0,0]$, over $\mathbb{F}_{19}$. The characteristic polynomial of Frobenius acting on conics and lines on $X_5$ is $f(x) = (x-1)^{10}(x+1)^9$. Hence we just need to count points over $\mathbb{F}_{19}$ and $\mathbb{F}_{19^2}$ to find the two possible characteristic polynomials for $\Phi_{19}$. We find, after rescaling, $f_{19,+}(x) = \frac{1}{13}(x-1)^{10}(x+1)^9(19x^2 - 22x + 19)$ and a contradiction for $f_{19,-}(x) = \frac{1}{19}(x-1)^9(x+1)^9(19x^3 - 22x^2 - 22x + 19)$ as $f(x) \nmid f_{19,-}(x)$. As $X_2$ is not supersingular, $X_0$ and $X_2$ are not isomorphic over $\mathbb{F}_{19}$. Therefore a very general surface in $\mathcal{X}_C$ has Picard number 19.

• As $M_B$ has rank 19, we know that a very general surface $X_B$ of $\mathcal{X}_B$ has $\rho(X_B) \geq 19$. Using the idea in Subsection 4.3 we see that the surface $X_2$, associated to the point $[1,1,1,1,1]$, over $\mathbb{F}_{19}$. The characteristic polynomial of Frobenius acting on conics and lines on $X_2$ is $f(x) = (x-1)^{16}(x+1)^3$. After point counting over $\mathbb{F}_{19}$ and $\mathbb{F}_{19^2}$ we find the possible two characteristic polynomials for $\Phi_{19}$, namely $f_{19,+}(x) = \frac{1}{19}(x-1)^{16}(x+1)^4(19x^2 - 18x + 19)$ and a contradiction for $f_{19,-}(x) = \frac{1}{19}(x-1)^{15}(x+1)^3(19x^4 - 18x^3 - 18x + 19)$. As $X_1$ is not supersingular, $X_2$ and $X_1$ are not isomorphic over $\mathbb{F}_{19}$. Therefore a very general surface in $\mathcal{X}_B$ has Picard number 19.

• As $M_{C,D}$ has rank 18, we know that a very general surface $X_{C,D}$ of $\mathcal{X}_{C,D}$ has $\rho(X_{C,D}) \geq 18$. We use the idea in Subsection 4.3 and find a surface whose reduction at a prime $p$ gives an upper bound of 18. To make point counting easier, we will work over $\mathbb{F}_{13}$ and the surface $X_{4,1}$, associated to the point $[1,0,4,1,8]$. Our first step is to calculate the characteristic polynomial of Frobenius acting on conics and lines, which is $f(x) = (x-1)^{10}(x+1)^8$. After point counting over $\mathbb{F}_{13}$ and $\mathbb{F}_{13^2}$ we find the two possible characteristic polynomials for $\Phi_{13}$, namely $f_{13,+}(x) = \frac{1}{13}(x-1)^{10}(x+1)^8(13t^4 + 12t^3 + 14t^2 + 12t + 13)$ and a contradiction for $f_{13,-}(x) = \frac{1}{13}(x-1)^9(x+1)^9(13t^4 - 14t^3 + 16t^2 + 14t + 13)$ (since $f(x) \nmid f_{13,-}(x)$). Hence $\rho(X_{4,1}) \leq 18$, so a very general surface in $\mathcal{X}_{C,D}$ has Picard number 18.

• As $M_{C,D,E}$ has rank 17, we know that a very general surface $X_{C,D,E}$ of $\mathcal{X}_{C,D,E}$ has $\rho(X_{C,D,E}) \geq 17$. We use the idea in Subsection 4.3 and find a surface whose reduction at two primes $p$ and $p'$ gives an upper bound of 17. We work with the surface $X_{3,5,7}$, associated to the point $[1,29,3,5,7]$, over the fields $\mathbb{F}_{13}$ and $\mathbb{F}_{19}$. The characteristic polynomial of Frobenius acting on conics and lines on $X_{13}$ is $f_{13}(x) = (x-1)^9(x+1)^9$ and over $\mathbb{F}_{19}$ is $f_{19}(x) = (x-1)^9(x+1)^8$. We find the following possible characteristic polynomials (after rescaling):

| $\mathbb{F}_{13}$ | $\mathbb{F}_{19}$ |
|------------------|------------------|
| $f_{13,+}$       | $\frac{1}{13}(x-1)^{10}(13x^4 + 22x^2 + 13)$ |
| $f_{13,-}$       | $\frac{1}{13}(x-1)^9(x+1)^9(13x^4 + 26x^3 + 48x^2 + 26x + 13)$ |
| $f_{19,+}$       | $\frac{1}{19}(x-1)^{10}(x+1)^8(19x^3 + 32x^2 + 42x^2 + 32x + 19)$ |
| $f_{19,-}$       | $\frac{1}{19}(x-1)^9(x+1)^9(19x^4 - 6x^3 + 16x^2 - 6x + 19)$ |

We then apply the idea in Subsection 4.2 by working over $\mathbb{F}_{13^2}$ and $\mathbb{F}_{19^2}$. We find that, up to squares, the discriminants are as follow:

| $\text{Disc.}_{\mathbb{F}_{13^2}}$ | $\text{Disc.}_{\mathbb{F}_{19^2}}$ |
|-------------------------------|-------------------------------|
| 13                            | $13 \cdot 17 \cdot 61$       |
| 18691                         | 75011                         |

As these four discriminants are all different elements in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ we have $\text{Disc}(\NS(X_{3,5,7,\mathbb{F}_{13^2}})) \neq \text{Disc}(\NS(X_{3,5,7,\mathbb{F}_{19^2}}))$ and so a very general surface in $\mathcal{X}_{C,D,E}$ has Picard number 17.

• As $M$ has rank 16, we know a very general surface $X$ of $\mathcal{X}$ has $\rho(X) \geq 16$. We use the idea in Subsection 4.3 and find a surface whose reduction at a prime $p$ gives an upper bound of 16. We work over $\mathbb{F}_{19}$ and let $X$ be
the surface defined by the point $[1, 2, 7, 11, 13]$. We calculate that the characteristic polynomial of Frobenius acting on conics and lines is $f(x) = (x - 1)^8(x + 1)^8$, hence we need to count points over $F_{19}$, $F_{19^2}$ and $F_{19^3}$ to find the two possible characteristic polynomials for $\Phi_{19}$. We find that, after rescaling

$$f_{19, +}(x) = \frac{1}{19} (x - 1)^8 (x + 1)^8 (19x^6 + 10x^5 + 29x^4 + 12x^3 + 29x^2 + 10x + 19),$$

and a contradiction for

$$f_{19, -}(x) = \frac{1}{19} (x - 1)^7 (x + 1)^9 (19x^6 - 28x^5 + 47x^4 - 64x^3 + 47x^2 - 28x + 19),$$

as $f(x) \nmid f_{19, -}(x)$. Hence a very general surface in $\mathcal{X}$ has Picard number 16.

Now that we know the rank of the Picard group of a very general member of each family, we can prove the following proposition:

**Proposition 4.8.** For a very general member of the families $\mathcal{X}, \mathcal{X}_{C,D,E}, \mathcal{X}_{C,D}, \mathcal{X}_B$ and $\mathcal{X}_C$, as well as the Fermat quartic, $F_4$, the Picard groups are generated by lines and conics.

In particular the matrices $M, M_{C,D,E}, M_{C,D}, M_B$ and $M_C$ as defined in Appendix B define the Picard group of a very general member of the families $\mathcal{X}, \mathcal{X}_{C,D,E}, \mathcal{X}_{C,D}, \mathcal{X}_B$ and $\mathcal{X}_C$ respectively.

**Proof.** First note that if $L_1 \hookrightarrow L_2$ is primitive, then no overlattice $L'$ of $L_1$ can be a sublattice of $L_2$. Let $\mathcal{X}$ and $\mathcal{Y}$ are two families of K3 surfaces, with $\mathcal{Y}$ a subfamily of $\mathcal{X}$. If $\mathcal{X}$ and $\mathcal{Y}$ denote a very general member of $\mathcal{X}$ and $\mathcal{Y}$, then $\text{Pic}(\mathcal{X}) \hookrightarrow \text{Pic}(\mathcal{Y})$ as the elements of $\text{Pic}(\mathcal{X})$ must specialise to elements of $\text{Pic}(\mathcal{Y})$.

With this in mind we start with the proven fact (see for example [SSvL10]) that the Picard group of the Fermat quartic, denoted by Pic($F_4$), is generated by lines. Upon calculating the Gram matrix of the 48 lines on $F_4$, we find that the Picard group has discriminant $-64$. On the other hand we calculate the following Gram matrix, which we denote $M_{F_4}$, generated by 16 conics and four lines (each line coming from a different set of eight lines associated to a point $q_i$):

$$
\begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 2 & -2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\
2 & 0 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 & 1 & 0 & 1 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & -2 & 2 & -2 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -2 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -2 & 1 & 1 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -2 
\end{pmatrix}
$$

$M_{F_4}$ has determinant $-64$ and hence does represent Pic($F_4$).
Let $X_C$ be a very general surface in $X_C$. We extracted the matrix $M_C$, from the intersection matrix of the lines and conics on $X_C$, by looking at the lines and conics which specialise to a subset of the 16 conics and four lines that lie on the Fermat quartic (which makes sense since $X_9 \subset X_C$ is the Fermat quartic). We ended up with 16 conics and three lines (which must come from three different sets of eight lines) and hence we have a rank 19, i.e. full rank, sublattice of $\text{Pic}(X_C)$. Notice that $M_C$ is a minor of $M_{F_4}$ (just remove the last row and column), and the lines and conics generating $M_C$ specialise to those generating the corresponding minor of $M_{F_4}$. Hence the lattice defined by $M_C$ is a primitive sublattice of $\text{Pic}(F_4)$. If $M_C$ did not define $\text{Pic}(X_C)$ then $\text{Pic}(X_C)$ would be an overlattice of $M_C$. Furthermore by the remark at the beginning of the proof $\text{Pic}(X_C)$ would be a sublattice of $\text{Pic}(F_4)$. This is a contradiction to the fact that $M_C$ is already a primitive sublattice of $\text{Pic}(F_4)$. Hence the lattice defined by $M_C$, which is generated by lines and conics, is $\text{Pic}(X_C)$.

Similarly we extracted $M_{C,D}$ from the intersection matrix using $M_C$ (and note it is a minor of $M_C$ by removing the last row and column), $M_{C,D,E}$ using $M_{C,D}$ (a minor of $M_{C,D}$ by removing the last row and column) and $M$ using $M_{C,D,E}$. Hence by the same argument, they represent respectively $\text{Pic}(X_{C,D})$, $\text{Pic}(X_{C,D,E})$ and $\text{Pic}(X)$. Finally, we extracted $M_B$ from $F_4$ using the same process (and notice it is a minor of $F_4$ by removing column and row 18), finishing the proof.

**Notation.** Let $M, N$ be matrices, then we use $MN$ to mean $N^T \cdot M \cdot N$.

We now have all the tools to prove our main result

**Theorem 4.9.** Let $p = [A,B,C,D,E] \in \mathbb{P}^4$ define the quartic $X_p : A(x^4 + y^4 + z^4 + w^4) + Bxyzw + C(x^2y^2 + z^2w^2) + D(x^2z^2 + y^2w^2) + E(x^2w^2 + y^2z^2) \subset \mathbb{P}^3$. Then

- A very general member of family parameterised by $\mathbb{P}^4$ contains no lines, has Picard rank 16 and Picard group isomorphic to $E_8 \langle -1 \rangle \oplus U \oplus D_5 \langle -2 \rangle \oplus \langle -4 \rangle$,

- A very general member of the family parameterised by $[A, (DE-2AC)/A, C, D, E]$ contains exactly eight lines, has rank 17 and Picard group isomorphic to $E_8 \langle -1 \rangle \oplus U \oplus A_2 \langle -2 \rangle \oplus (D_4 \langle -1 \rangle \oplus \langle -2 \rangle)^N$, with

$$N = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2
\end{pmatrix},$$

- A very general member of the family parameterised by $[A, 0, C, D, 2AC/D]$ contains exactly 16 lines, has rank 18 and Picard group isomorphic to $E_8 \langle -1 \rangle \oplus U \oplus A_7 \langle -1 \rangle^{14,2} \oplus \langle -8 \rangle$, with $I_{4,2} = \text{Diag}([1, 1, 1, 1, 1, 1, 1]),$

- A very general member of the family parameterised by $[A, B(2A-B)/A, B, B, B]$ contains exactly 24 lines, has rank 19 and Picard group isomorphic to $E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle^{19,2} \oplus \langle -40 \rangle,$

- A very general member of the family parameterised by $[A, 0, C, 0, 0]$ contains exactly 32 lines, has rank 19 and Picard group isomorphic to $E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle^{18,2} \oplus \langle -8 \rangle$ with $I_{8,2} = \text{Diag}([1, 1, 1, 1, 1, 1, 1, 2]),$

- The surface defined by the point $[\sqrt{3}, 12(\sqrt{3} - 1), 6, 6, -6]$ contains exactly 32 lines, has rank 20 and Picard group isomorphic either to $E_8 \langle -1 \rangle^{18} \oplus U \oplus \langle -4 \rangle \oplus \langle -24 \rangle$ or to $E_8 \langle -1 \rangle^{18} \oplus U \oplus \langle -4 \rangle \oplus \langle -6 \rangle$ (but not both),

- The Fermat quartic defined by the point $[1, 0, 0, 0, 0]$ contains exactly 48 lines, has rank 20 and Picard group isomorphic to $E_8 \langle -1 \rangle^{18} \oplus U \oplus \langle -8 \rangle \oplus \langle -8 \rangle$.

Possibly except for the point $[\sqrt{3}, 12(\sqrt{3} - 1), 6, 6, -6]$, the Picard group is generated by the lines and conics lying on the surface.
**Proof.** The claim about the number of lines each very general member contains is proven in Lemma 3.6 while the rank is proven in Proposition 4.7. Apart from the surface defined by $[\sqrt{3}, 12(\sqrt{3} - 1), 6, 6, -6]$, the claim that

As all the Picard groups are even and indefinite, and in each case the rank is large enough, we can apply Theorem 2.2 to each of our Picard groups. Specifically one can check (see [Bou]) that the discriminant form, rank and signature of the lattices defined by $M, M_{C,D,E}, M_{C,D}, M_B, M_C$ and $M_E$ are the same as the discriminant form, rank and signature of the lattice

- \( E_8 \langle -1 \rangle \oplus U \oplus (A_2 \langle -2 \rangle \oplus (D_4 \langle -1 \rangle \oplus (2)^3) \),
- \( E_8 \langle -1 \rangle \oplus U \oplus A_2 \langle -2 \rangle \oplus (D_4 \langle -1 \rangle \oplus (2)^3 \rangle \),
- \( E_8 \langle -1 \rangle \oplus U \oplus A_7 \langle -1 \rangle \langle 2 \rangle \oplus (2)^3 \),
- \( E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle \oplus (40) \),
- \( E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle \langle 2 \rangle \oplus (2)^3 \),
- and \( E_8 \langle -1 \rangle \langle 2 \rangle \oplus U \oplus (2)^3 \rangle \) respectively.

For the surface \( Y \), defined by $[\sqrt{3}, 12(\sqrt{3} - 1), 6, 6, -6]$, the lattice defined by \( M_Y \) is isomorphic to \( E_8 \langle -1 \rangle \langle 2 \rangle \oplus U \oplus (2)^3 \rangle \). While we don’t know that the lattice defined by \( M_Y \) is the Picard group of \( Y \), we know that it is a full rank sublattice of it. One can then use Theorem 2.2 to find all overlattices of it, of which there is only one, and use Theorem 2.2 to identify said lattice using its discriminant form, rank and signature.

Recall that at the end of Section 3, we had a diagram illustrating the various subfamilies of \( X \) containing lines and how they fitted together. Here we reproduce the same diagram where instead of the families, we put together the Picard group of the generic member of each family (except for the surface \( Y \), where we put the two possible Picard groups), and instead of the dimension of each family we put the rank of the Picard group.
4.4 Method

We include here two examples of how the isomorphic lattices were found for Theorem 4.9 which the reader might find useful. Those two examples illustrate the two different approaches we took in identifying the lattices.

We start with the lattice defined by $M$, i.e. the Picard group of a very general member $X$ of $\mathcal{X}$. We know that $M$ has signature $(1,15)$ and rank 16. We calculate its discriminant group to be $C_2^7 \times C_4 \times C_8$, and $\text{Pic}(X)$ has discriminant $-512$ (this concurs with the proof of [Ekl10, Thm 7.3, Cor 7.4]). By Theorem 2.3, we see that we can fit in one copy of $E_8(-1)$ and one copy of $U$ in $\text{Pic}(X)$, i.e., $\text{Pic}(X) = U \oplus E_8(-1) \oplus T$ where $T$ is a lattice with the same discriminant group and discriminant form as $\text{Pic}(X)$, but with signature $(0,6)$.

Recall that $A_L$ denotes the discriminant group of a lattice $L$, and $q_L$ its discriminant form. We calculate the discriminant form and find that:

- If $x \in A_{\text{Pic}(X)}$ has order 2 then $q_{\text{Pic}(X)}(x) \in \{0,1\}$,
- If $x \in A_{\text{Pic}(X)}$ has order 4 then $q_{\text{Pic}(X)}(x) \in \{-\frac{7}{8}, -\frac{5}{8}, \frac{1}{8}, \frac{3}{8}\}$,
- If $x \in A_{\text{Pic}(X)}$ has order 8 then $q_{\text{Pic}(X)}(x) \in \{-\frac{7}{8}, -\frac{5}{8}, \frac{1}{8}, \frac{3}{8}\}$.

As the lattice $(-4)$ has discriminant form $-\frac{1}{4}$ and discriminant group $C_4$, we guess that it appears as one of the summands of $\text{Pic}(X)$. Using Table H and the fact we need negative definite lattices, we see that the $C_8$ factor could arise from $A_7$ ($-1$), $A_3$ ($-2$), $(-8)$ or $D_{2n+1}$ ($-2$). As $A_7$ ($-1$) has too large of a rank (greater than six), and both $A_3$ ($-2$) and $(-8)$ have an element of order 8 with discriminant form $-\frac{7}{8}$ and $-\frac{1}{8}$ respectively, they can not be a factor of $\text{Pic}(X)$. On the other hand, $D_5$ ($-2$) does not give any obvious contradiction while having discriminant group $C_2^7 \times C_8$. We guess that it is a factor of $\text{Pic}(X)$. Hence putting everything together we check that $\text{Pic}(X) \cong U \oplus E_8(-1) \oplus D_5(-2) \oplus (-4)$. It is easy to see they have the same rank and signature; and a calculation checks they have the same discriminant form, namely both discriminant group have a basis $\{g_1, \ldots, g_6\}$ such that the discriminant form is given by

$$M_{q_L}(a_{ij}) = \begin{cases} q_L(g_i + g_j) & i \neq j \\ q_L(g_i) & i = j \end{cases}$$

$$= \begin{pmatrix} 0 & 1 & 1 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 1 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}.$$  

Our second example is with the lattice defined by $M_C$, i.e. the Picard group of a very general member $X_C$ of $\mathcal{X}_C$. We know that $M_C$ has signature $(1,18)$ and rank 19. We calculate that $\text{Pic}(X_C)$ has discriminant 128 and discriminant group $C_2^7 \times C_8$. By Theorem 2.3 we know that $\text{Pic}(X_C) \cong E_8(-1) \oplus U \oplus T$, where $T$ is a lattice of signature $(0,9)$ with discriminant group $C_2^7 \times C_8$ and discriminant form as:

- If $x \in A_{\text{Pic}(X_C)}$ has order 2 then $q_{\text{Pic}(X_C)}(x) \in \{0\}$,
- If $x \in A_{\text{Pic}(X_C)}$ has order 4 then $q_{\text{Pic}(X_C)}(x) \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1\}$,
- If $x \in A_{\text{Pic}(X_C)}$ has order 8 then $q_{\text{Pic}(X_C)}(x) \in \{-\frac{5}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{7}{8}\}$.

As there is no negative definite lattice in Table H which gives a copy of $C_4$ without giving an element of discriminant form $\frac{2n+1}{2}$ for some $n$, we deduce that $T$ can not be written simply in terms of scaled root lattices. Instead we use Theorem 2.3 to find an overlattice of $\text{Pic}(X_C)$ that we can identify. In particular, if we let $\{e_i\}$ be the basis given by $M_C$, then $\frac{1}{2}(e_4 + e_5 + e_{10} + e_{11} + e_{13} + e_{14}) \in A_{\text{Pic}(X_C)}$ has order two and discriminant form zero. This generates an isotropic subgroup of $A_{\text{Pic}(X_C)}$ and gives a corresponding index two overlattice. This overlattice, $L$, has discriminant group $C_2^7 \times C_8$ and discriminant form given as:
• If \( x \in A_L \) has order 2 then \( q_L(x) \in \{-\frac{1}{2}, 0, \frac{1}{2}\} \).
• If \( x \in A_L \) has order 4 then \( q_L(x) \in \{-\frac{1}{4}, 0, \frac{1}{4}\} \).
• If \( x \in A_L \) has order 8 then \( q_L(x) \in \{-\frac{5}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{7}{8}\} \).

Following our first example this allows us to guess that \( L \cong E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle \oplus \langle -8 \rangle \). We check that is the case, as they both have rank 19, signature (1,18) and discriminant form given by

\[
M_{q_L}(a_{ij}) = \begin{cases}
q_L(g_i + g_j) & i \neq j \\
q_L(g_i) & i = j
\end{cases}
= \begin{pmatrix}
0 & 1 & -\frac{1}{8} \\
1 & 0 & -\frac{7}{8} \\
-\frac{1}{8} & -\frac{7}{8} & \frac{5}{8}
\end{pmatrix}.
\]

Knowing that \( \text{Pic}(X) \) is an index two full rank sublattice of \( E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle \oplus \langle -8 \rangle \), we enumerate the index two full rank sublattices of \( E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle \oplus \langle -8 \rangle \) until we find one that has the same discriminant form as \( \text{Pic}(X) \).
## A The equations of the lines

The following table gives the equations of the 8 lines lying on the point \( p = [A, B, C, D, E] \in \mathbb{P}^5 \) depending on which tangent cone its lies on.

| Tangent cone to the point | Conics associated | Lines |
|---------------------------|-------------------|-------|
| \( q_1 \) \( x^2 - y^2 - z^2 + w^2 \) | \( 2\sqrt{q+c}x + \sqrt{p-1}z + \sqrt{-p+6}w = 2\sqrt{q+c}y + \sqrt{-p+6}z + \sqrt{p-1}w = 0 \) | \( 2\sqrt{q+c}x + \sqrt{p+1}z + \sqrt{-p-6}w = 2\sqrt{q+c}y + \sqrt{-p-6}z + \sqrt{p+1}w = 0 \) |
| \( q_2 \) \( x^2 - y^2 + z^2 - w^2 \) | \( 2\sqrt{q+c}x + \sqrt{-p+6}z + \sqrt{p-1}w = 2\sqrt{q+c}y + \sqrt{-p-6}z + \sqrt{p+1}w = 0 \) | \( 2\sqrt{q+c}x + \sqrt{-p-6}z + \sqrt{p-1}w = 2\sqrt{q+c}y - \sqrt{-p-6}z - \sqrt{p+1}w = 0 \) |
| \( q_3 \) \( x^2 + y^2 - z^2 - w^2 \) | \( 2\sqrt{q-c}x + \sqrt{p+3}z + \sqrt{-p-2}w = 2\sqrt{q-c}y - \sqrt{p+3}z + \sqrt{-p-2}w = 0 \) | \( 2\sqrt{q-c}x + \sqrt{-p-2}z + \sqrt{-p+3}w = 2\sqrt{q-c}y - \sqrt{-p-2}z - \sqrt{-p+3}w = 0 \) |
| \( q_4 \) \( x^2 + y^2 + z^2 + w^2 \) | \( 2\sqrt{q-c}x + \sqrt{-p-2}z + \sqrt{-p+3}w = 2\sqrt{q-c}y - \sqrt{-p-2}z - \sqrt{-p+3}w = 0 \) | \( 2\sqrt{q-c}x + \sqrt{-p+2}z + \sqrt{-p-3}w = 2\sqrt{q-c}y + \sqrt{-p+2}z - \sqrt{-p-3}w = 0 \) |
| \( q_5 \) \( xy - zw \) | \( 2\sqrt{Ax + \sqrt{q+D} - \sqrt{-q+D}} \) | \( z = 2\sqrt{Ay + \sqrt{q-D} - \sqrt{q+D}} \) |
| \( q_6 \) \( xy + zw \) | \( 2\sqrt{Ax} + \sqrt{q-D} + \sqrt{-q+D} \) | \( z = 2\sqrt{Ay} + \sqrt{q-D} - \sqrt{-q+D} \) |
| \( q_7 \) \( xz - yw \) | \( 2\sqrt{Ax} + \sqrt{q-c} - \sqrt{-q-c} \) | \( y = 2\sqrt{Az} + \sqrt{q-c} + \sqrt{-q-c} \) |
| \( q_8 \) \( xz + yw \) | \( 2\sqrt{Ax} + \sqrt{q-c} + \sqrt{-q-c} \) | \( y = 2\sqrt{Az} - \sqrt{q-c} + \sqrt{-q-c} \) |
| \( q_9 \) \( xw - yz \) | \( 2\sqrt{Ax} + \sqrt{-q+D} + \sqrt{q-D} \) | \( z = 2\sqrt{Aw} - \sqrt{-q-c} - \sqrt{q+c} \) |
| \( q_{10} \) \( xw + yz \) | \( 2\sqrt{Ax} + \sqrt{q-c} - \sqrt{-q+c} \) | \( z = 2\sqrt{Aw} + \sqrt{q-c} + \sqrt{-q+c} \) |

Table 2: Equations of lines
### B List Of Gram Matrices

For a very general member of the family $\mathcal{X}$, a full rank minor of minimal discriminant is

\[
M = \begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & -2 & 0 & 2 & 0 & 1 & 1 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 2 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & -2 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2
\end{pmatrix}
\]

For a very general member of the family $\mathcal{X}_{C,D,E}$, a full rank minor of minimal discriminant is

\[
M_{C,D,E} = \begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 1 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 \\
2 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 & 1 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & -1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2
\end{pmatrix}
\]
For a very general member of the family $X_{C,D}$, a full rank minor of minimal discriminant is

$$M_{C,D} = \begin{pmatrix}
-2 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 0 \\
0 & -2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 0 \\
2 & 1 & 1 & 0 & 1 & 2 & 1 & -2 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & -2 & 2 & 2 & 1 & 1 & 1 & 0 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 & 1 \\
2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -2 \\
\end{pmatrix}$$

For a very general member of the family $X_B$, a full rank minor of minimal discriminant is

$$M_B = \begin{pmatrix}
-2 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 0 \\
0 & -2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 0 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 & 1 & 1 & 1 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & -2 \\
\end{pmatrix}$$

20
For a very general member of the family $X_C$, a full rank minor of minimal discriminant is

$$M_C = \begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 1 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 2 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 0 & 2 & 2 & -2 & 2 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -2 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\
\end{pmatrix}$$

For the surface $Y$, a full rank minor of minimal discriminant is

$$M_Y = \begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 0 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 0 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & -2 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -2 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

21
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