$p$-Schatten commutators of projections

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Abstract
Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fixed orthogonal decomposition of the complex separable Hilbert space $\mathcal{H}$ in two infinite-dimensional subspaces. We study the geometry of the set $\mathcal{P}_p$ of selfadjoint projections in the Banach algebra

$$\mathcal{A}_p = \{ A \in B(\mathcal{H}) : [A, E_+] \in B_p(\mathcal{H}) \},$$

where $E_+$ is the projection onto $\mathcal{H}_+$ and $B_p(\mathcal{H})$ is the Schatten ideal of $p$-summa-
ble operators ($1 \leq p < \infty$). The norm in $\mathcal{A}_p$ is defined in terms of the norms of the matrix entries of the operators given by the above decomposition. The space $\mathcal{P}_p$ is shown to be a differentiable $C^\infty$ submanifold of $\mathcal{A}_p$, and a homogeneous space of the group of unitary operators in $\mathcal{A}_p$. The connected components of $\mathcal{P}_p$ are characterized, by means of a partition of $\mathcal{P}_p$ in nine classes, four discrete classes, and five essential classes: (1) the first two corresponding to finite rank or co-rank, with the connected components parametrized by these ranks; (2) the next two discrete classes carrying a Fredholm index, which parametrizes their components; (3) the remaining essential classes, which are connected.

Keywords  Projections · Schatten $p$-ideals

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1 Introduction

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be an orthogonal decomposition of the complex separable Hilbert space $\mathcal{H}$ in two infinite-dimensional closed subspaces, with corresponding projections $E_+$ and $E_-$. Consider the algebra:

$$\mathcal{A}^p := \{ A \in \mathcal{B}(\mathcal{H}) : [A, E_+] = AE_+ - E_+A \in \mathcal{B}_p(\mathcal{H}) \},$$

where $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators in $\mathcal{H}$, and $\mathcal{B}_p(\mathcal{H})$ is the Schatten ideal of $p$-summable operators ($1 \leq p < \infty$). $\mathcal{A}^p$ is a $*$-Banach algebra with a suitable norm ($*$ is the usual adjoint). The purpose of this paper is the study of the set $\mathcal{P}^p$ of selfadjoint projections in $\mathcal{A}^p$:

$$\mathcal{P}^p = \{ P \in \mathcal{A}^p : P^2 = P^* = P \}.$$

It is known [5, 12] that the set of idempotents ($Q$ such that $Q^2 = Q$) of a Banach algebra is a complemented submanifold of the algebra. Here, we show that also the set of selfadjoint idempotents is a submanifold of the algebra, in the case of the algebra $\mathcal{A}^p$ (it can be proved to hold for arbitrary $*$-Banach algebras). The case $p = 2$ was extensively treated in [4].

We characterize the connected components of $\mathcal{P}^p$. First, we see that $\mathcal{P}^p$ is partitioned in nine classes, four discrete classes $\mathcal{D}_j$, $1 \leq j \leq 4$, and five essential classes $\mathcal{E}_j$, $1 \leq j \leq 5$. The first two discrete classes correspond to the projections of finite rank or finite co-rank, and its connected components are characterized by these numbers. The next two discrete classes are more interesting, and correspond to the so-called $p$-restricted Grassmannian, associated with $E_+$ and $E_-$, respectively. Projections in a restricted Grassmannian carry an integer Fredholm index, which in turn parametrizes the connected components of $\mathcal{D}_3$ and $\mathcal{D}_4$ (as with the former two, one passes from one class to the other with the symmetry $P \mapsto P^\perp = 1 - P$, and thus, the geometric and topological properties of both pairs are similar). The remaining essential classes are shown to be connected.

Examples of discrete projections (in the $p = 1$ restricted Grassmannian) of the decomposition $L^2(\mathbb{D}) = H^2(\mathbb{D}) \oplus H^2(\mathbb{D})$, where $H^2(\mathbb{D})$ is the Hardy space of the disk, are the projections onto the subspaces $fH^2(\mathbb{D})$, for $f$ a smooth function of modulus one. The index given by (minus) the winding number of $f$.

Examples of essential projections, again for $p = 1$, are given for the decomposition $L^2(\mathbb{R}^n) = L^2(\Omega) \oplus L^2(\Omega^c)$, where $\Omega \subset \mathbb{R}^n$ is a measurable set with positive finite measure. In this setting, the projection $FE_+F^{-1}$ ($F =$ Fourier–Plancherel transform), onto the space of functions in $L^2(\mathbb{R}^n)$ with Fourier transform supported in $\Omega$, is an essential projection.

This study is a continuation of [2], where the ideal of compact operators was considered. Some of the techniques and results are similar in both contexts, the main difficulty in the case at hand ($p$-Schatten ideals) is that the structure algebra $\mathcal{A}^p$ is a Banach algebra, whereas in the compact case, it is a C*-algebra. For instance, we need to prove the smooth local structure of the group $\mathcal{U}_\mathcal{A}^p$ of unitary operators in $\mathcal{A}^p$, which acts in $\mathcal{P}^p$. 

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We do not know if the geodesics of the Grassmann manifold of $\mathcal{H}$, lying in $\mathcal{P}^p$ (that is, with initial velocity in $\mathcal{A}^p$), are short for the Finsler metric given by the norm of the Banach algebra $\mathcal{A}^p$. However, we show that for the case of the discrete classes $\mathbb{D}_3$ and $\mathbb{D}_4$ (corresponding to the $p$-restricted Grassmannian), the connected component containing a given projection $P$ is a submanifold of the affine space $P + \mathcal{B}_p^p(\mathcal{H})$, which carries naturally the Schatten $p$-norm. With the Finsler metric given by this norm, the geodesics of the full Grassmannian of $\mathcal{H}$, lying inside this component, are short.

2 Preliminaries

For $1 \leq p < \infty$, we denote by $\mathcal{B}_p^p(\mathcal{H})$ the ideal of $p$-Schatten operators in $\mathcal{H}$, i.e., $\mathcal{B}_p^p(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : \text{Tr}(|T|^p) < \infty \}$, with its norm $\| T \|_p = \text{Tr}^{1/p}(|T|^p)$. We fix an orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

with corresponding projections $E_+$ and $E_-$. We make the assumption that both $\mathcal{H}_+$ and $\mathcal{H}_-$ are infinite-dimensional. Denote by $\mathcal{P}(\mathcal{H}) = \mathcal{P}$ the set of all orthogonal projections in $\mathcal{H}$. We are interested in the set:

$$\mathcal{P}^p_{\mathcal{H}_+} = \mathcal{P}^p := \{ P \in \mathcal{P} : [P, E_+] \in \mathcal{B}_p^p(\mathcal{H}) \}.$$ 

Accordingly, we denote by:

$$\mathcal{A}^p_{\mathcal{H}_+} = \mathcal{A}^p := \{ A \in \mathcal{B}(\mathcal{H}) : [A, E_+] \in \mathcal{B}_p^p(\mathcal{H}) \}.$$ 

With $\mathcal{A}^p_{\mathcal{H}_+}$ and $\mathcal{A}^p_{\mathcal{H}_-}$, we denote, respectively, the sets of selfadjoint and anti-Hermitian elements of $\mathcal{A}^p$.

In terms of the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, operators in $\mathcal{H}$ can be written as $2 \times 2$ matrices. It is clear that elements of $\mathcal{A}^p$ are characterized as those matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

such that $A_{12} \in \mathcal{B}_p^p(\mathcal{H}_-, \mathcal{H}_+)$ and $A_{21} \in \mathcal{B}_p^p(\mathcal{H}_+, \mathcal{H}_-)$.

We endow $\mathcal{A}^p$ with the following norm:

$$\| A \|_{\infty,p} := \| A_{11} \| + \| A_{22} \| + \| A_{12} \|_p + \| A_{21} \|_p. \tag{1}$$

It is easy to see that $\mathcal{A}^p$ is a Banach space with this norm. Also, it is clear that it is an algebra. This can be seen by elementary matrix computations, or noting that if $A, B \in \mathcal{A}^p$, then:

$$[AB, E_+] = ABE_+ - AE_+B + AE_+B - E_+AB$$

$$= A[B, E_+] - [A, E_+]B \in \mathcal{B}_p^p(\mathcal{H}).$$

With the norm $\| . \|_{\infty,p}$ just defined, it is elementary that:
\[\|AB\|_{\infty,p} \leq \|A\|_{\infty,p}\|B\|_{\infty,p}\]

for \(A, B \in \mathcal{A}_p\), i.e., \(\mathcal{A}_p\) is a Banach algebra. Note also that if \(A \in \mathcal{A}_p\), then \(A^* \in \mathcal{A}_p\) and \(\|A^*\|_{\infty,p} = \|A\|_{\infty,p}\). It is also clear that \(\|\|_{\infty,p}\) is not a C*-norm. However, the inclusion

\[(\mathcal{A}_p, \|\|_{\infty,p}) \hookrightarrow (\mathcal{B}(\mathcal{H}), \|\|)\]

is continuous, so that \(\mathcal{A}_p\) is a Banach subalgebra of \(\mathcal{B}(\mathcal{H})\).

Let us denote by \(\mathcal{G}_p\) the group of invertible elements in \(\mathcal{A}_p\). It is usually known in the literature [11] as one of the reduced groups (acting in the restricted Grassmannian; see, for instance, [4]). For instance, it is known that if \(G \in \mathcal{G}_p\), then its diagonal entries (in the \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\) matrix) are \(p\)-Fredholm operators (i.e., operators which are invertible modulo the ideal \(\mathcal{B}_p(\mathcal{H})\)). As such, to \(G \in \mathcal{G}_p\), an index can be attached, namely, the Fredholm index of the 1, 1-entry. As a consequence, \(\mathcal{G}_p\) is disconnected, and its connected components are parametrized by this index.

Let us denote by \(\mathcal{U}_p := \{U \in \mathcal{G}_p : U \text{ is unitary in } \mathcal{H}\}\).

The following elementary property of \(\mathcal{G}_p\) shall be very useful, it states that the usual polar decomposition of (invertible) elements of \(\mathcal{A}_p\) stays in \(\mathcal{A}_p\).

**Proposition 2.1** If \(G \in \mathcal{G}_p\) and \(G = U|G|\) is the polar decomposition, then \(|G|, U \in \mathcal{G}_p\) (i.e., \(U \in \mathcal{U}_p\)). Moreover, \(G\) and \(U\) share the same index.

**Proof** It suffices to show that \(|G| \in \mathcal{G}_p\). Clearly, \(G^*G \in \mathcal{G}_p\). Denote by \(\sigma_{\mathcal{B}(\mathcal{H})}(G^*G)\) and \(\sigma_{\mathcal{A}_p}(G^*G)\) the spectra of \(G^*G\) in \(\mathcal{B}(\mathcal{H})\) and \(\mathcal{A}_p\), respectively. Since \(\mathcal{A}_p \subset \mathcal{B}(\mathcal{H})\) is a Banach subalgebra, its follows that (see, for instance, [10]):

\[\sigma_{\mathcal{B}(\mathcal{H})}(G^*G) \subset \sigma_{\mathcal{A}_p}(G^*G) \quad \text{and} \quad \partial \sigma_{\mathcal{A}_p}(G^*G) \subset \partial \sigma_{\mathcal{B}(\mathcal{H})}(G^*G)\]

Moreover, since \(G^*G\) is positive and invertible, \(\sigma_{\mathcal{B}(\mathcal{H})}(G^*G) \subset (0, +\infty)\). Then,

\[\sigma_{\mathcal{B}(\mathcal{H})}(G^*G) = \partial \sigma_{\mathcal{B}(\mathcal{H})}(G^*G)\]

Thus:

\[\sigma_{\mathcal{B}(\mathcal{H})}(G^*G) = \sigma_{\mathcal{A}_p}(G^*G) = \sigma \subset [\delta, +\infty)\]

for some \(\delta > 0\). Denote by \(\log(z)\) the usual complex log function (discontinuous in the negative real axis). Note that \(\log(z)\) is analytic on an open neighbourhood of \(\sigma\), and thus, \(\log(G^*G)\) is defined in \(\mathcal{A}_p\) by means of the usual holomorphic functional calculus in Banach algebras. Let \(C = \exp(\frac{1}{2} \log(G^*G)) \in \mathcal{A}_p\). Note that if one regards \(G^*G\) as an element in \(\mathcal{B}(\mathcal{H})\), \(C \in \mathcal{B}(\mathcal{H})\) is the usual (positive) square root of \(G^*G\), i.e., \(C = (G^*G)^{1/2} = |G|\).

The set of positive elements in \(\mathcal{G}_p\) is convex, and, therefore, connected. Thus, \(G\) and \(U\) lie in the same connected component of \(\mathcal{G}_p\). \(\square\)
3 Regular structure of $\mathcal{P}^p$

In this section, we show that the set $\mathcal{P}^p$ of orthogonal projections in $\mathcal{A}^r$ is a complemented $C^\infty$ submanifold of $\mathcal{A}_{ab}^r$. It is known that the set of idempotents of a Banach algebra is a complemented submanifold of the algebra (see [5] or [12]). Here, we are dealing with selfadjoint idempotents.

First, note that the group $\mathcal{U}_\mathcal{A}^r$ is a Banach–Lie group, whose Banach–Lie algebra is $\mathcal{A}_{ab}^r$.

**Theorem 3.1** The group $\mathcal{U}_\mathcal{A}^r$ is a Banach–Lie group and a complemented submanifold of $\mathcal{A}^r$. Its Banach–Lie algebra is $\mathcal{A}_{ab}^r$.

**Proof** The exponential map $\exp : \mathcal{A}^p \to \mathcal{G}_\mathcal{A}^r$, $\exp(X) = e^X$, is a local diffeomorphism, and there exists a radius $0 < r < 1$ such and an open subset $0 \in \mathcal{W} \subset \mathcal{A}^p$, such that:

$$\exp : \mathcal{W} \to \{ G \in \mathcal{A}^p : \|G - 1\|_{\infty,p} < r \}$$

is a diffeomorphism. Its inverse is the usual log series. When restricted to $\mathcal{A}_{ab}^p$, it takes values in $\mathcal{U}_\mathcal{A}^r$, which is a complemented (real) subspace of $\mathcal{A}^r$. Thus, to obtain a local chart for $\mathcal{U}_\mathcal{A}^r$ around 1, it suffices to show that elements in $\mathcal{U}_\mathcal{A}^r$ close enough to 1 are of the form $e^X$ for some $X \in \mathcal{A}_{ab}^r$ close to 0. In fact, if $U \in \mathcal{U}_\mathcal{A}^r$ satisfies $\|U - 1\|_{\infty,p} < r(< 2)$, since:

$$\|U - 1\| \leq \|U - 1\|_{\infty,p} < 2,$$

the spectrum $\sigma_B(U)$ is contained in an arc $\{e^{i\theta} : |\theta| \leq \theta_0 < \pi\}$. Thus, by a similar argument as in Proposition 2.1:

$$\sigma_{\mathcal{A}^r}(U) = \sigma_{\mathcal{B}(\mathcal{H})}(U).$$

It follows that $\log(U) \in \mathcal{A}_{ab}^p \cap \mathcal{W}$. One obtains local charts around other elements of $\mathcal{U}_\mathcal{A}^r$ translating this chart around 1, by means of the left action of this group on itself.

It is clear that the group operations (multiplication and taking adjoint) are smooth: these operations are smooth in the whole Banach algebra $\mathcal{A}^r$. \hfill $\square$

Note that if $P \in \mathcal{P}^r$, then $P^\perp := 1 - P$ also belongs to $\mathcal{P}^r$. Let $P \in \mathcal{P}^r$ and $A \in \mathcal{A}^p$, denote by:

$$S_{P,A} = AP + (1 - A)P^\perp \in \mathcal{A}^p.$$

**Lemma 3.2** There exists an open neighbourhood:

$$\mathcal{W}_P = \{ A \in \mathcal{A}^p : \|A - P\|_{\infty,p} < r_P \}$$

(for a given $r_P > 0$) of $P$ in $\mathcal{A}^r$, such that if $A \in \mathcal{W}_P$, then $S_{P,A} \in \mathcal{G}_\mathcal{A}^r$. 

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Proof If $A = P$, then $S_{P,P} = 1$, so for $A \in \mathcal{A}_h'$ close enough to $P$, $S_{P,A}$ remains in $\mathcal{G}_{\mathcal{A}_p}$, which is open in $\mathcal{A}_p$.

If $A = Q \in \mathcal{P}_p$, then $S_{P,Q}$ is a standard element used to intertwine $P$ and $Q$: clearly:

$$S_{P,Q}P = PQ = QS_{P,Q}. $$

If, additionally, $Q$ belongs to $\mathcal{W}_P$, then $S_{P,Q}$ is invertible and:

$$Q = S_{P,Q}PS_{P,Q}^{-1}. $$

It is also a standard procedure to use $U_{P,Q}$, the unitary part in the polar decomposition $S_{P,Q} = U_{P,Q}|S_{P,Q}|$, to obtain a unitary that intertwines:

$$Q = U_{P,Q}PU_{P,Q}^*. $$

Thus, a continuous map is defined:

$$\mu_P : \mathcal{P}_p \cap \mathcal{W}_P \to \mathcal{U}_{\mathcal{A}_p}, \quad \mu(Q) = U_{P,Q}. \quad (2) $$

Remark 3.3 Let us denote by:

$$\pi_P : \mathcal{U}_{\mathcal{A}_p} \to \mathcal{P}_p, \quad \pi_P(U) = UPU^*. $$

Clearly, $\pi_P$ is a continuous map, whose image is the unitary orbit:

$$\mathcal{O}_P = \{ UPU^* : U \in \mathcal{U}_{\mathcal{A}_p} \} $$

of $P$. The map $\mu_P$ of (2) is a continuous local cross section for $\pi_P$:

$$\pi_P(\mu_P(Q)) = Q, \quad \text{for } Q \in \mathcal{P}_p \cap \mathcal{W}_P. $$

By translating this section using the left action of $\mathcal{U}_{\mathcal{A}_p}$ on itself, one obtains local cross sections on neighbourhoods of any point $P'$ in $\mathcal{P}_p$.

By general topological considerations, this fact implies that the orbit $\mathcal{O}_P$ is open and closed in $\mathcal{P}_p$, and, therefore, a union of connected components. This union is necessarily discrete, because as we have seen, close projection belongs to the same orbit.

Let us show that $\mathcal{P}_p$ is a complemented submanifold of $\mathcal{A}_h'$ (and since $\mathcal{A}_p$ is a (real) complemented subspace of $\mathcal{A}_p$, $\mathcal{P}_p$ is also a complemented submanifold of $\mathcal{A}_p$). Also, the same argument shows that the map $\pi_P : \mathcal{U}_{\mathcal{A}_p} \to \mathcal{O}_P$ is a submer- sion. To prove this fact, we shall use the next general result, which can be found in [9], and is a consequence of the implicit function theorem in Banach spaces.
Lemma 3.4 Let $G$ be a Banach–Lie group acting smoothly on a Banach space $X$. For a fixed $x_0 \in X$, denote by $\pi_{x_0} : G \to X$ the smooth map $\pi_{x_0}(g) = g \cdot x_0$. Suppose that:

1. $\pi_{x_0}$ is an open mapping, regarded as a map from $G$ onto the orbit:
   \[ \mathcal{O}_{x_0} := \{ g \cdot x_0 : g \in G \} \]
   of $x_0$ (with the relative topology of $X$).
2. The differential $\left( d\pi_{x_0} \right)_1 : (TG)_1 \to X$ splits: its null space and range are closed complemented subspaces.

Then, the orbit $\mathcal{O}_{x_0}$ is a smooth submanifold of $X$, and the map:
\[ \pi_{x_0} : G \to \mathcal{O}_{x_0} \]

is a smooth submersion.

Here, smooth means $C^\infty$.

Theorem 3.5 $P^p$ is a $C^\infty$ complemented submanifold of $\mathcal{A}_h^p$, and for any $P \in P^p$ the map:
\[ \pi_p : \mathcal{U}_{\mathcal{A}^p} \to \mathcal{O}_p \]
is a $C^\infty$ submersion.

Proof Let us use Lemma 3.4 in our context, namely: $X = \mathcal{A}_h^p$, $G = \mathcal{U}_{\mathcal{A}^p}$, and $x_0 = P$. Clearly, $\pi_p$ is an open mapping, because it has local continuous cross sections. Let us denote by $\Pi = (d\pi_p)_1 : \mathcal{A}_{ah}^p \to \mathcal{A}_h^p$ and prove that it splits. The cross section $\mu_p$ can be extended to a map defined in $\mathcal{W}_p$ (which is open in $\mathcal{A}_h^p$); let:
\[ \tilde{\mu}_p : \mathcal{W}_p \to \mathcal{U}_{\mathcal{A}^p}, \tilde{\mu}_p(A) = S_{P,A} |S_{P,A}|^{-1}, \]
i.e., $\tilde{\mu}_p(A)$ is the unitary part in the polar decomposition of the invertible element $S_{P,A}$. Clearly, $\tilde{\mu}_p$ is a $C^\infty$ extension of $\mu_p$, defined on an open set in $\mathcal{A}_h^p$. Let us denote by $\Sigma = (d\tilde{\mu}_p)_p$. Clearly $\Sigma : \mathcal{A}_h^p \to \mathcal{A}_{ah}^p$. Note that, since $\pi_p$ takes values in $P^p$, on a neighbourhood of $1 \in \mathcal{U}_{\mathcal{A}^p}$, one has:
\[ \pi_p \tilde{\mu}_p \pi_p = \pi_p \mu_p \pi_p = \pi_p. \]
Differentiating this identity at 1, one gets:
\[ \Pi \Sigma \Pi = \Pi \quad \text{in } \mathcal{A}_{ah}^p. \]
This implies that both $\Pi \Sigma$ and $\Sigma \Pi$ are idempotent operators, acting $\mathcal{A}_h^p$ and $\mathcal{A}_{ah}^p$, respectively. Thus, $R(\Pi \Sigma) \subset \mathcal{A}_h^p$ is complemented, and note that:
\[ R(\Pi \Sigma) \subset R(\Pi) = R(\Pi \Sigma \Pi) \subset R(\Pi \Sigma), \]
i.e., \( R(\Pi \Sigma) = R(\Pi) \). Similarly, \( N(\Sigma \Pi) = N(\Pi) \) is complemented in \( \mathcal{A}_p \), i.e., \( \Pi = (d\pi_p)_1 \) splits.

\[ \square \]

Remark 3.6 In particular, the tangent space \( (T^pP)_p \) at \( P \in \mathcal{P} \) is given by:

\[ (T^pP)_p = \{ [X, P] : X \in \mathcal{A}_p \}. \]

4 Spectral picture of projections in \( \mathcal{P}_p \)

If \( P \in \mathcal{P}_p \) has matrix (in terms of \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \)):

\[ P = \begin{pmatrix} x & a \\ a^* & y \end{pmatrix}, \]

then the fact that \( P^2 = P \geq 0 \) implies that \( x, y \geq 0, x - x^2 = aa^*, y - y^2 = a^*a \), and \( xa + ay = a \). Moreover, \( a \in B_p(\mathcal{H}_-, \mathcal{H}_+) \). Let us state the following elementary consequences of these relations:

Lemma 4.1 With the above notations, one has that \( \|a\| \leq 1/2 \), and the eigenvalues of \( x \) and \( y \) are of the form:

\[ t^+ = \frac{1}{2} + \sqrt{\frac{1}{4} - s^2} \quad \text{or} \quad t^- = \frac{1}{2} - \sqrt{\frac{1}{4} - s^2}, \]

where \( s \leq \frac{1}{2} \) is a singular value of \( a \). One or both \( t_+, t_- \) may occur.

Proof Clearly, \( \|x\| \leq 1 \) and \( \|y\| \leq 1 \). Then:

\[ \|a\|^2 = \|aa^*\| = \|x - x^*\| = \sup(t - t^2 : t \in \sigma(x)) = \sup(t - t^2 : t \in [0, 1]) = \frac{1}{4}. \]

Again, \( x - x^2 = aa^* \), and the fact that \( aa^* \) is compact, imply that the elements \( t \) in the spectrum of \( x \) which are neither 0 nor 1 (which correspond to the spectral value 0 for \( aa^* \)) are (finite or countable many) eigenvalues. Moreover, if \( t \neq 0, 1 \) is an eigenvalue of \( x \), then:

\[ t - t^2 = s^2, \]

for \( s \) a singular value of \( a \). Then, either \( t = t^+ = \frac{1}{2} + \sqrt{\frac{1}{4} - s^2} \) or \( t = t^- = \frac{1}{2} - \sqrt{\frac{1}{4} - s^2} \). The same facts hold for \( y \). \( \square \)

Note that the biggest singular value \( s = \frac{1}{2} \) of \( a \) corresponds to \( t^+ = t^- = \frac{1}{2} \).
The next result, which was proven for compact commutators in [2], holds also in this context, and clarifies the relation between (the multiplicities of) the eigenvalues of $x$ and $y$. We include the proof, because it is elementary and straightforward.

**Lemma 4.2** If $\lambda \neq 0, 1$ is an eigenvalue of $y$, then $1 - \lambda$ is an eigenvalue of $x$, and the operator $a|_{N(y - \lambda 1_{\mathcal{H}_-})}$ maps $N(y - \lambda 1_{\mathcal{H}_-})$ isomorphically onto $N(x - (1 - \lambda)1_{\mathcal{H}_+})$. Thus, in particular, these eigenvalues have the same multiplicity. Moreover:

$$aP_{N(y - \lambda 1_{\mathcal{H}_-})} = P_{N(x - (1 - \lambda)1_{\mathcal{H}_+})}a.$$

**Proof** Let $\xi \in \mathcal{H}$, $\xi \neq 0$, such that $y\xi = \lambda \xi$ (with $\lambda \neq 0, 1$). Then, using the relation $a = xa + ay$, one has:

$$a\xi = xa\xi + ay\xi = xa\xi + \lambda a\xi,$$

i.e. $xa\xi = (1 - \lambda)a\xi$.

Also note that:

$$N(a) = N(a^*a) = N(y - y^2) = N(y) \oplus N(y - 1_{\mathcal{H}_-}),$$

and thus, $a\xi \neq 0$ is an eigenvector for $x$, with eigenvalue $1 - \lambda$, and the map $a|_{N(y - \lambda 1_{\mathcal{H}_-})}$ is injective from $N(y - \lambda 1_{\mathcal{H}_-})$ to $N(x - (1 - \lambda)1_{\mathcal{H}_+})$. Therefore:

$$\dim(N(y - \lambda 1_{\mathcal{H}_-})) \leq \dim(N(x - (1 - \lambda)1_{\mathcal{H}_+}).$$

By a symmetric argument, using $a^*$ (and the relation $ya^* + a^*x = a^*$), one obtains equality.

Pick now an arbitrary $\xi \in \mathcal{H}_-$, $\xi = \xi_1 + \xi_2$, with $\xi_1 \in N(y - \lambda 1_{\mathcal{H}_-})$ and $\xi_2 \perp N(y - \lambda 1_{\mathcal{H}_-})$. Then:

$$aP_{N(y - \lambda 1_{\mathcal{H}_-})}\xi = a\xi_1.$$

On the other hand:

$$P_{N(x - (1 - \lambda)1_{\mathcal{H}_+})}a\xi_1 = a\xi_1,$$

by the fact proven above. Let us see that $P_{N(x - (1 - \lambda)1_{\mathcal{H}_+})}a\xi_2 = 0$, which would prove our claim. Since $\xi_2 \perp N(y - \lambda 1_{\mathcal{H}_-})$, $\xi_2 = \sum_{l \geq 2} \eta_l + \eta_0 + \eta_1$, where $\eta_l$, $l \geq 2$, are eigenvectors of $y$ corresponding to eigenvalues $\lambda_l$ different from 0, 1 and $\lambda$, $\eta_0 \in N(y)$, $\eta_1 \in N(y - 1_{\mathcal{H}_-})$ (where these two latter may be trivial). Note then that $\eta_0, \eta_1 \in N(a)$, and thus:

$$a\xi_2 = \sum_{l \geq 2} \lambda a\eta_l,$$

where the (non nil) vectors $\lambda a\eta_l$ are eigenvectors of $x$ corresponding to eigenvalues $1 - \lambda_l$, different from 0, 1 and $1 - \lambda$. Thus, $P_{N(x - (1 - \lambda)1_{\mathcal{H}_+})}a\xi_2 = 0$. $\square$
Remark 4.3

1. In the notation above, this Lemma says that to \( t^+ \) of \( x \) corresponds \( t^- = 1 - t^+ \) of \( y \), and vice versa, with the same multiplicity.

2. If \( a \) has infinite rank, \( s = s_n \) form a sequence in \( \ell^p \). If there are infinitely many \( t^- \), then they form a sequence in \( \ell^q \). If there are infinitely many \( t^+ \), they form a sequence \( t^+_n \), such that \( 1 - t^+_n \) belongs to \( \ell^q \). Indeed, note that near the origin, \( f(s) = \frac{1}{2} - \sqrt{\frac{1}{4} - s^2} = s^2 + o(s^4) \).

We will characterize the connected components of \( \mathcal{P}' \). To this effect, it shall be useful and clarifying to consider the \( * \)-homomorphism:

\[
\pi : \mathcal{B}(H) \to \mathcal{B}(H)/K(H)
\]

onto the Calkin algebra \( \mathcal{B}(H)/K(H) \). Note that \( \pi(E_+) \), \( \pi(E_-) \) are non trivial projections with \( \pi(E_+) + \pi(E_-) = 1 \). Thus, elements in \( \mathcal{B}(H)/K(H) \) can be written as \( 2 \times 2 \) matrices in terms of this sum. Let us write:

\[
\pi(E_+) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \pi(E_-) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Then, if \( P \in \mathcal{P}' \), it follows that:

\[
\pi(P) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix},
\]

where \( e, f \) are projections in \( \mathcal{B}(H)/K(H) \) with \( e \leq \pi(E_+) \) and \( f \leq \pi(E_-) \).

Lemma 4.4 Let \( P, Q \in \mathcal{P}' \) in the same connected component, say:

\[
\pi(P) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \quad \text{and} \quad \pi(Q) = \begin{pmatrix} e' & 0 \\ 0 & f' \end{pmatrix}.
\]

Then, there exists a curve \( \gamma(t) = \begin{pmatrix} e(t) & 0 \\ 0 & f(t) \end{pmatrix} \) of projections in \( \mathcal{B}(H)/K(H) \), such that \( \gamma(0) = \pi(P) \) and \( \gamma(1) = \pi(Q) \).

Proof Let \( P(t) \) be a continuous path in \( \mathcal{P}' \) with \( P(0) = P \) and \( P(1) = Q \). Note that, in particular, \( P(t) \) is continuous in the norm topology of \( \mathcal{B}(H) \), and clearly, \( \pi(P(t)) \) has diagonal matrix with respect to \( \pi(E_+) + \pi(E_-) = 1 \). \( \square \)

Recall that there are three classes of projections in \( \mathcal{B}(H)/K(H) \) modulo unitary equivalence: 0, 1, and \( p \not= 0, 1 \). Also, two projections are connected by a continuous path of projections if and only if they are unitarily equivalent.

Remark 4.5 The projections in \( \mathcal{P}' \) can be classified in the following nine types:
1. P belongs to $\mathcal{D}_1$ if $\pi(P) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$;
2. P belongs to $\mathcal{D}_2$ if $\pi(P) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;
3. P belongs to $\mathcal{D}_3$ if $\pi(P) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$;
4. P belongs to $\mathcal{D}_4$ if $\pi(P) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$;
5. P belongs to $\mathcal{E}_1$ if $\pi(P) = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$, where $e \neq 0, 1$;
6. P belongs to $\mathcal{E}_2$ if $\pi(P) = \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$, where $f \neq 0, 1$;
7. P belongs to $\mathcal{E}_3$ if $\pi(P) = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$, where $e \neq 0, 1$;
8. P belongs to $\mathcal{E}_4$ if $\pi(P) = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}$, where $f \neq 0, 1$;
9. P belongs to $\mathcal{E}_5$ if $\pi(P) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$, where $e, f \neq 0, 1$.

We call the classes $\mathcal{D}_i$ discrete and $\mathcal{E}_j$ essential.

Summarizing, one has the following expression for an arbitrary projection in $\mathcal{P}^\alpha$.

We make here a slight change of notation. Without loss of generality, we assume that $\mathcal{H} = \mathcal{L} \times \mathcal{L}$, $\mathcal{H}_+ = \mathcal{L} \times 0$ and $\mathcal{H}_- = 0 \times \mathcal{L}$.

**Theorem 4.6** If $P = \begin{pmatrix} x & a \\ a^* & y \end{pmatrix} \in \mathcal{P}^\alpha \subset \mathcal{B}(\mathcal{L} \times \mathcal{L})$, then:

$$P = \left( \sum_n \alpha_n P_n + \sum_m \beta_m Q_m + E_1 \right) \left( \sum_k \lambda_k \xi_k^* \otimes \xi_k + \sum_l \mu_l \eta_l \otimes \eta_l' \right) \sum_n (1 - \alpha_n)P'_n + \sum_m (1 - \beta_m)Q'_m + E'_1,$$

where

- The spectrum of $x$ (in $\mathcal{B}(\mathcal{L})$) consists of two strictly monotone (eventually finite) sequences $\alpha_n, \beta_m$, such that $\frac{1}{2} > \alpha_n \to 0, \frac{1}{2} \leq \beta_m \to 1$, plus, eventually, 0 and 1, which may or may not be eigenvalues. The spectrum of $y$ consists of $1 - \alpha_n, 1 - \beta_m$ and eventually 0 and 1 (with similar considerations).
- $r(P_n) = r(P'_n), r(Q_m) = r(Q'_m)$. These multiplicities are finite.
- $r(P_m)\alpha_m$ and $1 - r(Q_m)\beta_m$ belong to $\ell^2$; $\lambda_k = \sqrt{\alpha_k - \alpha_k^2}$ and $\mu_l = \sqrt{\beta_l - \beta_l^2}$.
- $E_1$ and $E'_1$ denote the spectral projections of $x$ and $y$, respectively, corresponding to the spectral value 1. They can be nil, finite or infinite, and are unrelated.
- $\{\xi_k : k \geq 1\}, \{\xi'_k : k \geq 1\}, \{\eta_l : l \geq 1\}, \{\eta'_l : l \geq 1\}$ are orthonormal systems which span, respectively.
and consists of eigenvectors of \( x \) and \( y \) in the following manner:

\[
x_{\xi_k} = \alpha_{n(k)}\xi_k \quad \text{and} \quad x_{\eta_l} = \beta_{m(l)}\eta_l,
\]

\[
y_{\xi'_k} = (1 - \alpha_{n(k)})\xi'_k \quad \text{and} \quad y_{\eta'_l} = (1 - \beta_{m(l)})\eta'_l.
\]

\[\bigoplus_{n \geq 1} R(P_n), \quad \bigoplus_{m \geq 1} R(Q_m) \quad \text{and} \quad \bigoplus_{m \geq 1} R(Q'_m),\]

5 Halmos decomposition

Given two projections, in this case \( P \) and \( E_+ \), the space \( \mathcal{H} \) can be decomposed in 5 orthogonal subspaces which reduce \( P \) and \( E_+ \), namely:

\[
\mathcal{H} = (R(P) \cap \mathcal{H}_+) \oplus (N(P) \cap \mathcal{H}_-) \oplus (R(P) \cap \mathcal{H}_-) \oplus (N(P) \cap \mathcal{H}_+) \oplus \mathcal{H}_0,
\]

where \( \mathcal{H}_0 \), the orthogonal complement of the sum of the first 4, is usually called the \textit{generic part} of \( P \) and \( E_+ \). In [7], Halmos proved that there is a unitary isomorphism between \( \mathcal{H}_0 \) and a product space \( L \times L \), and a positive operator \( \Gamma \) with trivial null space and \( \|\Gamma\| \leq \pi/2 \), acting in \( L \), such that the reductions \( E_0^+ \) and \( P_0 \) of \( E_+ \) and \( P \) to \( \mathcal{H}_0 \) are unitarily equivalent to (respectively):

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},
\]

where \( C = \cos(\Gamma) \) and \( S = \sin(\Gamma) \). It can be shown that \( P_0 \) and \( E_+^0 \) are unitarily equivalent:

\[
e^{iX} E_+^0 e^{-iX} = P_0,
\]

where \( X = X_{E_+^0, P_0} = \begin{pmatrix} 0 & -i\Gamma \\ i\Gamma & 0 \end{pmatrix} \).

In this decomposition of \( \mathcal{H} \), the commutator has the form:

\[
[E_+, P] = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & CS \\ -CS & 0 \end{pmatrix}.
\]

Therefore:

\[\text{Proposition 5.1} \ P \in \mathcal{P}' \text{ if and only if } CS \in \mathcal{B}_p(L). \text{ Moreover, this means the spectrum of } \Gamma \text{ has the form } \{\gamma^+_n : n \geq 1\} \cup \{\gamma^-_k : k \geq 1\}, \text{ where } \pi/4 \leq \gamma^+_n < \pi/2 \text{ and } 0 < \gamma^-_k < \pi/4 \text{ are strictly monotone (eventually finite) sequences:}
\]

\[
\Gamma = \sum_{k \geq 1} \gamma^-_k G^-_k + \sum_{n \geq 1} \gamma^+_n G^+_n.
\]
for $G^+_n G^-_k$ mutually orthogonal projections in $\mathcal{L}$, of ranks $r(G^+_n) = r^+_n < \infty$ and $r(G^-_k) = r^-_k < \infty$, and:

$$\{ r^-_k \gamma^-_k \}, \{ \pi/2 - r^+_n \gamma^+_n \} \in \ell^p.$$

**Proof** The first assertion is clear. Note that $CS = \cos(\Gamma) \sin(\Gamma) = \frac{1}{2} \sin(2\Gamma)$. Thus, the facts that $\sin(2\Gamma) \in B_\infty(\mathcal{L})$ and $0 \leq 2\Gamma \leq \pi$ mean that the spectrum of $\Gamma$ consists of eigenvalues which accumulate only (eventually) at 0 and $\pi$. Since $C$ and $S$ have trivial null spaces, neither 0 nor $\pi/2$ are eigenvalues of $\Gamma$. \hfill $\Box$

### 5.1 Examples

1. Let $\mathcal{H} = L^2(\mathbb{T})$, $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ with normalized Lebesgue measure, and $\mathcal{H}_+ = H^2(\mathbb{D})$ the Hardy space of the disk. Let $\varphi : \mathbb{T} \to \mathbb{C}$ be non-vanishing and $C^1$. Then, the projection $P_{\varphi H^2(\mathbb{D})}$ onto $\varphi H^2(\mathbb{D})$ belongs to the restricted Grassmannian given by the subspace $\mathcal{H}^2(\mathbb{D})$ (see [11]), with $[P_+, P_{\varphi H^2(\mathbb{D})}] \in \mathcal{B}_1(L^2(\mathbb{T}))$. Thus, $P_{\varphi H^2(\mathbb{D})} \in \mathbb{D}_3$.

2. Let $\mathcal{H} = L^2(\mathbb{R}^n)$ (with Lebesgue measure), $\Omega \subset \mathbb{R}^n$ a measurable set with $|\Omega| < \infty$ and $\mathcal{H}_+ = L^2(\Omega)$, regarded as the subspace of $L^2(\mathbb{R}^n)$ of classes of functions with essential support contained in $\Omega$. Denote by $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the Fourier–Plancherel transform. Put $P = FE_+ F^{-1}$, which projects onto functions whose Fourier transform is supported in $\Omega$. It is known (see for instance [6]) that $E_+ P \in \mathcal{B}_1(L^2(\mathbb{R}^n))$. Thus, in particular:

$$[E_+, P] = E_+ P - P E_+ = E_+ P - (E_+ P)^* \in \mathcal{B}_1(L^2(\mathbb{R}^n)).$$

Moreover, Lenard proved in [8] that:

$$R(E_+) \cap R(P) = R(E_+) \cap N(P) = N(E_+) \cap R(P) = \{0\},$$

and that $N(E_+) \cap N(P)$ is infinite-dimensional. Therefore, since $E_+ P E_+$, $E_+ P E_-$ and $E_- P E_+$ are compact, it follows that $\pi(P)$ is of the form:

$$\pi(P) = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix},$$

with $p \neq 0, 1$, because $P$ is an infinite rank projection, with $N(P)$ infinite dimensional. That is, $P \in \mathbb{E}_2$.

3. Let $B \in \mathcal{B}_p(\mathcal{L})$, and put $\mathcal{H} = \mathcal{L} \times \mathcal{L}$ and $\mathcal{H}_+ = \mathcal{L} \times 0$. Consider the idempotent (non-orthogonal projection):

$$E_B = E = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix}$$
with \( R(E) = \mathcal{H}_+ \). Consider \( P_{R(E^*)} \) the orthogonal projection onto \( R(E^*) \). It is known (see, for instance, [1]), that if \( Q \) is an idempotent operator, then \( P_{R(Q)} = Q(Q + Q^* - 1)^{-1} \). In our case, note that

\[
(E^* + E - 1)^2 = \begin{pmatrix} 1 + BB^* & 0 \\ 0 & 1 + B^*B \end{pmatrix},
\]
so that:

\[
P = P_{R(E^*)} = (E^* + E - 1)^{-1} = E^*(E^* + E - 1)(E^* + E - 1)^{-2} = \begin{pmatrix} (1 + BB^*)^{-1} & B(1 + B^*B)^{-1} \\ B^*(1 + BB^*)^{-1} & B^*B(1 + B^*B)^{-1} \end{pmatrix}.
\]

Note that the 1, 2 entry \( B(1 + B^*B)^{-1} \) belongs to \( \mathcal{B}_p(\mathcal{H}) \), with singular values which have the same asymptotic behaviour as those of \( B \). Clearly, \( \pi(P) \) is of the form \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), i.e., \( P \in \mathbb{D}_3 \). The index (of the 1, 1 entry) of \( P \) is 0.

6 The classes \( \mathbb{D}_3 \) and \( \mathbb{D}_4 \)

Let us show that \( \mathbb{D}_3 \) coincides with the \( p \)-restricted Grassmannian induced by \( E_+ \). Recall that (see, for instance, [11]) the \textit{\( p \)-restricted Grassmannian} \( \text{Gr}_p^{\text{res}}(E_+) \) relative to \( E_+ \), is the space projections \( P \) in \( \mathcal{H} \), such that:

- \( E_+\big|_{R(P)} : R(P) \to \mathcal{H} \in \mathcal{B}(R(P), \mathcal{H}) \) is a \( p \)-Fredholm operator (i.e., there exists \( S \in \mathcal{B}(\mathcal{H}, R(P)) \), such that \( SE_+\big|_{R(P)} = 1 + M \) and \( E_+\big|_{R(P)}S = 1 + N, \) for \( M \in \mathcal{B}_p(R(P)), N \in \mathcal{B}_p(\mathcal{H}) \), and
- \( E_-\big|_{R(P)} : R(P) \to \mathcal{H} \in \mathcal{B}_p(R(P), \mathcal{H}) \).

The components of the restricted Grassmannian are parametrized by \( k \in \mathbb{Z} \), where \( k \) is the index of the operator \( E_+\big|_{R(P)} : R(P) \to \mathcal{H} \in \mathcal{B}(R(P), \mathcal{H}) \):

\[
\text{Gr}_p^{\text{res},k}(E_+) = \{ P \in \text{Gr}_p^{\text{res}} : \text{ind}(E_+\big|_{R(P)} : R(P) \to \mathcal{H}) = k \}.
\]

In particular, note that \( E_+ \in \text{Gr}_p^{\text{res},0}(E_+) \).

The coincidence of the \( p \)-restricted Grassmannian of \( \mathcal{H}_+ \) and \( \mathbb{D}_3 \) follows from this result:

**Theorem 6.1** Denote by \( \mathcal{O}(E_+) \) the unitary orbit of \( E_+ \) under the action of \( \mathcal{U}_{\mathcal{A}^p} \), \( \mathcal{O}(E_+) = \{ UE_+U^* : U \in \mathcal{U}_{\mathcal{A}^p} \} \). Then:

\[ \mathcal{O}(E_+) = \{ P \in \mathcal{P}^p : P - E_+ \in \mathcal{B}_p(\mathcal{H}) \} = \{ P \in \mathcal{P} : P - E_+ \in \mathcal{B}_p(\mathcal{H}) \}. \]

**Proof** If \( P = UE_+U^* \) for some \( U \in \mathcal{U}_{\mathcal{A}^p} \), then:

\[
P - E_+ = UE_+U^* - E_+ = (UE_+ - E_+U)U^* = [U, E_+]U^* \in \mathcal{B}_p(\mathcal{H}).
\]
Clearly, \{P ∈ \mathcal{P} : P − E_+ ∈ B_\rho(\mathcal{H})\} ⊂ \{P ∈ \mathcal{P} : P − E_+ ∈ B_\rho(\mathcal{H})\}. Suppose that
\(P ∈ \mathcal{P}\), such that \(P − E_+ ∈ B_\rho(\mathcal{H})\). It is easy to see that:
\[N(P − E_+) = R(P) \cap \mathcal{H}_+ ⊕ N(P) \cap \mathcal{H}_-\.

Also, it is clear that both summands reduce \(P\) and \(E_+\). Then, \(\mathcal{H}' = N(P − E_+)\) reduces \(P\) and \(E_+\). Denote by \(P'\) and \(E'_+\) the reductions. It is straightforward that \([P', E'_+] ∈ B_\rho(\mathcal{H}')\). Since \(P' − E'_+\) is selfadjoint and has trivial null space, if one performs the polar decomposition:
\[P' − E'_+ = V'|P' − E'_+|,
\]
the isometric part \(V'\) is a symmetry (a selfadjoint unitary) in \(\mathcal{H}'\). Also, the fact that \(S' = P' − E'_+\) satisfies \(S'E'_+ = E'_+S'\) implies that \(V'\) intertwines \(E'_+\) and \(P'\):
\[V'E'_+V' = P'.\]
Then, it also follows that \(V'\) belongs to:
\[\{X' ∈ \mathcal{B}(\mathcal{H}') : [X', E'_+] ∈ B_\rho(\mathcal{H}')\},\]
the algebra \(\mathcal{A}'\) in \(\mathcal{H}'\) corresponding to the reduced projection \(E'_+\). Indeed:
\[V'E'_+ − E'_+V' = (V'E'_+V' − E'_+V')V' ∈ B_\rho(\mathcal{H}').\]

Consider now the unitary operator (in fact symmetry) \(V\) of \(\mathcal{H}\), which is given in terms of the decomposition \(\mathcal{H} = \mathcal{H}' ⊕ (R(P) \cap \mathcal{H}_+) ⊕ (N(P) \cap \mathcal{H}_-)\) is given by:
\[V' ⊕ 1 ⊕ 1\]

Note that in this same decomposition, \(P\) and \(E_+\) are given by:
\[P = P' ⊕ 1 ⊕ 0\quad \text{and} \quad E_+ = E'_+ ⊕ 1 ⊕ 0.\]

Then:
\[\left[ V, E_+ \right] = (V'E'_+ − E'_+V') ⊕ 0 ⊕ 0 ∈ B_\rho(\mathcal{H}),\]
and
\[VE_+V = (V'E'_+V') ⊕ 1 ⊕ 0 = P.\]

\[\square\]

**Corollary 6.2** \[\mathfrak{D}_3 = \mathcal{O}(E_+).\]

**Proof** \(P ∈ \mathcal{O}(E_+),\) if and only if \(P − E_+ ∈ B_\rho(\mathcal{H}),\) and then, \(\pi(P − E_+) = 0,\) i.e., \(\pi(P) = \pi(E_+).\) Conversely, in \(\pi(P) = \pi(E_+),\) then \(P − E_+\) is compact. Using a suitable unitary isomorphism, we may suppose (as in Theorem 4.6) \(\mathcal{H} = \mathcal{L} × \mathcal{L}\) and \(\mathcal{H}_+ = \mathcal{L} × 0,\) and use the spectral picture of \(P ∈ \mathcal{P}\). Note that the assumption that \(P − E_+\) is compact implies that \(x − 1\) and \(y\) are compact. Thus, following the notation of Theorem 4.6, one has that there are finitely many \(\alpha_n\) and that \(1 − \beta_n\) is a sequence in \(\ell^2\). It follows that \(P − E_+ ∈ B_\rho(\mathcal{H}).\) \[\square\]
Remark 6.3 Note that, in particular, these facts imply that $D_3 \in E_+ + B_p(H)$, i.e., $D_3$ is contained in the affine space obtained as a translation of $B_p(H)$. Thus, the tangent spaces belong naturally inside $B_p(H)$. We shall profit from this condition, to endow the manifold $D_3$ with the natural Finsler metric, which consists in considering the $p$-norm at every tangent space.

Remark 6.4 In a similar fashion (or using the symmetry $P \mapsto P^\perp$), one proves that:

$$D_4 = \mathcal{O}(E_-) = \{ P \in \mathcal{P} : P - E_- \in B_p(H) \},$$

which coincides with the $p$-restricted Grassmannian $G^p_{\text{res}}(E_-)$ induced by $E_-$. Similarly, one can consider the Finsler $p$-norm structure in $D_4$.

In general, if $P$ and $Q$ are projections, $\|P - Q\| \leq 1$.

Proposition 6.5 If $P \in \mathcal{T}'$ satisfies that $\|P - E_+\| < 1$, then $P \in D_3$. Similarly, if $\|P - E_-\| < 1$, then $P \in D_4$.

Proof Recall Halmos decomposition. Clearly, $\|P - E_+\| < 1$ implies that $R(P) \cap H_- = N(P) \cap H_+ = \{0\}$. Indeed, a unit vector $\xi \in R(P) \cap H_-$ satisfies $\|(P - E_+)\xi\| = \|\xi\| = 1$, and thus, $R(P) \cap H_- = \{0\}$; and similarly for the other intersection. Note that:

$$P - E_+ = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \left( -S^2 CS \atop CS S^2 \right).$$

and therefore, $(P - E_+)^2 = \left( S^2 0 \atop 0 S^2 \right)$. Then, $\|P - E_+\| < 1$ implies that the spectrum of $\Gamma$ cannot accumulate at $\pi/2$, and, therefore, only accumulates only at the origin.

Thus, analysing the spectral picture of $P = \left( C^2 CS \atop CS S^2 \right)$ according to Theorem 4.6, it is clear that the eigenvalues of $x = C^2$ accumulate only at 1 and the eigenvalues of $y = S^2$ accumulate only at the origin.

The proof for the case $\|P - E_-\| < 1$ is analogous. $\square$

Corollary 6.6 If $P \in \mathcal{E}_j$, $1 \leq j \leq 5$, then $\|P - E_+\| = \|P - E_-\| = 1$.

6.1 Finsler metric in the discrete classes

In this subsection, we examine the relationship between the $p$-Finsler metric of $D_3$ and $D_4$, i.e., the metric which arises when endowing each tangent space of these manifolds with the $p$-norm, with the ambient metric given by the norm $\|\|_{\infty,p}$ of $A^p$.

Let us reason with $D_3$, the same facts hold analogously for $D_4$. 

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Let us first compare the Finsler metric with the metric induced by the $p$ norm of the affine space $E_+ + B_p(H)$ (if $X = E_+ + A, Y = E_+ + B \in E_+ + B_p(H)$, the $p$ distance $\|X - Y\|_p = \|A - B\|_p < \infty$ is defined).

In [3], it was proven that if $P, Q$ lie in the same component of the $p$-restricted Grassmannian $G_{res,k}^p(E_+) = \mathbb{D}_3$, then there exists a minimal geodesic of the form $\delta(t) = e^{itX}Pe^{-itX}$, with $X^* = X \in B_p(H)$ $P$-codiagonal and $\|X\| \leq \pi/2$, such that $\delta(1) = Q$, so that the geodesic distance is $d_p(P, Q) = \|X\|_p$. We recall that $d_p$ is formally defined as:

$$d_p(P, Q) = \inf \left\{ \int_I \|\dot{\gamma}(t)\|_p \; dt : \gamma : I \to \mathcal{T}^p \text{ is smooth with endpoints } P, Q \right\}.$$  

**Proposition 6.7** With the current notations, if $P, Q$ lie in the same component of $\mathbb{D}_3$ (resp. $\mathbb{D}_4$), then:

$$\frac{2}{\pi} d_q(P, Q) \leq \|P - Q\|_p \leq d_p(P, Q).$$

**Proof** The inequality $\|P - Q\| \leq d_p(P, Q)$ is clear: if one takes the infimum among all smooth curves with values in $E_+ + B_p(H)$, which is an affine space, one obtains the norm distance $\|P - Q\|_p$.

Let $X = X^* \in B_p(H)$ be the exponent of the geodesic joining $P$ and $Q$: $\|X\| \leq \pi/2$, $X$ is $P$-codiagonal, and $Q = e^{iX}Pe^{-iX}$. Note that:

$$\|P - Q\|_p = \|P - e^{iX}Pe^{-iX}\|_p = \|(Pe^{iX} - e^{iX}P)e^{-iX}\|_p = \|(P, e^{iX})\|_p.$$  

Since $X$ is $P$-codiagonal, $P$ commutes the even powers of $X$, and thus:

$$[P, X^{2n+1}] = PX^{2n}X - X^{2n}XP = X^{2n}(PX - XP) = X^{2n}[P, X].$$

It follows that:

$$[P, e^{iX}] = \left[ P, 1 + iX - \frac{1}{2}X^2 - \frac{i}{3!}X^3 + \frac{1}{4!}X^4 + \cdots \right]$$

$$= i \left\{ [P, X] - \frac{1}{3!} [P, X^3] + \frac{1}{5!} [P, X^5] - \cdots \right\}$$

$$= i \left\{ 1 - \frac{1}{3!}X^2 + \frac{1}{5!}X^4 - \cdots \right\} [P, X]$$

$$= i \; \text{sinc} \; (X)[P, X],$$

where sinc denotes the cardinal sine function, which is the entire function given by $\text{sinc}(t) = \frac{\sin(t)}{t}$ (sinc $(0) = 1$). It $|t| \leq \pi/2$, and this function verifies that:

$$\frac{2}{\pi} \leq \text{sinc} \; (t) \leq 1.$$  

In particular, since $X$ is selfadjoint with spectrum in $[-\pi/2, \pi/2]$, $S = \text{sinc} \; (X)$ is an invertible operator, with $\|S^{-1}\| \leq \frac{\pi}{2}$. Therefore:
that is:

\[ \frac{2}{\pi} \|X\|_p \leq \|S^{-1}SX\|_p \leq \|S^{-1}\| \|SX\|_p \leq \frac{\pi}{2} \|SX\|_p, \]

Remark 6.8 By the above remarks, if \( P, Q \in \mathbb{D}_3 \), then \( P - Q \in B_p(\mathcal{H}) \). Denote \( P - Q = A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \). Since \( A_{ij} = E_{ij}A \) for appropriate elementary (partial isometric) operators, it is clear that \( A_{ij} \in B_p(\mathcal{H}) \).

Moreover, \( \|A_{ij}\|_p = \|E_{ij}A\|_p \leq \|E_{ij}\| \|A\|_p = \|A\|_p \). Then:

\[ \|P - Q\|_{\infty,p} = \|A_{11}\| + \|A_{22}\| + \|A_{12}\|_p + \|A_{12}^*\|_p \leq \|A_{11}\|_p + \|A_{22}\|_p + \|A_{12}\|_p + \|A_{12}^*\|_p \leq 4\|P - Q\|_p. \]

However, these two metrics are not equivalent in \( \mathbb{D}_3 \). Indeed, fix an orthonormal basis \( \{f_n\} \) for \( \mathcal{H}_- \), and consider \( P = E_+ + D \) and \( Q = E_+ + F \), where \( D, F \leq E_- \), project onto mutually orthogonal subspaces generated by finite (disjoints) subsets of the basis \( \{f_n\} \). Then, \( \|P - Q\|_{\infty,p} = \|D - F\| = 1 \), whereas \( \|P - Q\|_p = (\text{rank}(E) + \text{rank}(F))^{1/p} \). Since these ranks are arbitrary, the metrics are non-equivalent.

7 Connectedness of the essential classes

In this section, we prove our main result in the classes \( \mathbb{E}_i, 1 \leq i \leq 5 \), namely, that each of these spaces is connected. The proof of this result is similar to the proof of the analogous result in [2]. We shall sketch the argument, emphasizing only the necessary modifications.

Recall from Theorem 4.6, the form of a projection \( P = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in \mathcal{P}^p \):

\[ P = \begin{pmatrix} \sum_{n} \alpha_n P_n + \sum_{m} \beta_m Q_m + E_1 \\ \sum_{k} \lambda_k \xi_k \otimes \xi_k + \sum_{l} \mu_l \eta_l \otimes \eta_l \end{pmatrix} \begin{pmatrix} \sum_{k} \lambda_k \xi_k \otimes \xi_k + \sum_{l} \mu_l \eta_l \otimes \eta_l \\ \sum_{n} (1 - \alpha_n)P_n' + \sum_{m} (1 - \beta_m)Q_m' + E_1' \end{pmatrix}, \]

where the relevant facts we need now are that \( r(P_n) = r(P_n') < \infty \), \( r(Q_m) = r(Q_m') < \infty \), and \( \alpha_n, 1 - \beta_m \) belong to \( \ell^2 \). Consider the projection:
where \( N \) and \( N' \) are the projections onto the nullspaces of \( x \) and \( y \), respectively.

The first step of the argument is the following.

**Lemma 7.1** The operator \( B = P + P_0 - 1 \) is invertible in \( \mathcal{A}^p \), and belongs to the connected component of the identity.

**Proof** In [2], Lemma 5.1, it was shown that \( B \) is invertible, that its commutator with \( E \) is compact, and that its 1, 1 entry is invertible. Here, \( [B, P_+] \) is the \( P_+ \)-codiagonal matrix whose non nil entries are those of \( P \), and therefore, \( B \in \mathcal{A}^p \). Clearly, it belongs to the component of zero index. \( \square \)

The operator \( B \) is selfadjoint, and satisfies \( BP = P_0B \), which is equivalent to \( BPB^{-1} = P_0 \). Then, the unitary part \( V \) in the polar decomposition \( B = V|B| \) satisfies \( VPV^* = P_0 \). Clearly, \( V \) also belongs to the connected component of the identity in \( \mathcal{U}_{\mathcal{A}^p} \). It follows that \( P \) and \( P_0 \) belong to the same connected component of \( \mathcal{P}^p \).

The next step is to show that any pair of diagonal essential projections in the same class \( \mathbb{E}_i \) are conjugate by an element in the connected component of the identity of the same essential class.

Let \( F, G \) be two projections, which are diagonal with respect to \( E_+ \), both in the same essential class.

- If \( F, G \in \mathbb{E}_1 \) are of the form:
  \[
  F = \begin{pmatrix} P_+ & 0 \\ 0 & F_- \end{pmatrix}, \quad G = \begin{pmatrix} P'_+ & 0 \\ 0 & G_- \end{pmatrix},
  \]
  where \( P_+, P'_+ \) are projections of infinite rank and co-rank, and \( F_-, G_- \) are of finite rank. One can show that \( F \) and \( G \) are unitarily equivalent to the projection:
  \[
  \begin{pmatrix} P_+ & 0 \\ 0 & 0 \end{pmatrix},
  \]
  with a unitary in operator in \( \mathcal{A}^p \), which belongs to the connected component of the identity. First note that with an unitary of the form \( \begin{pmatrix} U_+ & 0 \\ 0 & 0 \end{pmatrix} \), one can connect \( G \) with: \( \begin{pmatrix} P_+ & 0 \\ 0 & G_- \end{pmatrix} \). That is, we may suppose \( P'_+ = P_+ \). Next, we construct the unitary operator \( U \) given in the proof of Lemma 5.2 of [2], which is a finite rank perturbation of the identity, and therefore also in the connected component of the identity of the invertible group of \( \mathcal{A}^p \). This unitary connects \( F \) to: \( \begin{pmatrix} P_+ & 0 \\ 0 & 0 \end{pmatrix} \), and then, \( \mathbb{E}_1 \) is connected.
• The connectedness of $\mathbb{E}_3$ can be obtained by noting that the map
\[ P \mapsto P^\perp = 1 - P \]
transforms $\mathbb{E}_1$ into $\mathbb{E}_3$.
• The case of $\mathbb{E}_2$ is analogous to the case of $\mathbb{E}_1$, and therefore, the case of $\mathbb{E}_4$ also follows.
• If $F, G \in \mathbb{E}_5$, they are of the form:
\[
F = \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix}, \quad G = \begin{pmatrix} G_+ & 0 \\ 0 & G_- \end{pmatrix},
\]
with $F_\pm, G_\pm$ of infinite rank and co-rank. Clearly, these projections are unitarily equivalent with a diagonal unitary matrix, which, therefore, belong to the connected component of the identity in the invertible group of $\mathcal{A}^\sigma$.

Thus, we have the following:

**Theorem 7.2.** The classes $\mathbb{E}_i, 1 \leq i \leq 5$ are connected. Each of these spaces is the orbit of a fixed (diagonal) projection in the corresponding class, under the action of the unitary operators in the connected component of the identity in $\mathcal{A}^\sigma$.

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