Theoretical analysis of the optimization of power transfer from half loop source in a simple cavity

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Abstract. A model that illustrate the optimal placement of coupler inside an RF cavity is proposed. The chosen geometry of the cavity are rectangular and cylindrical cavity. A small half loop of localized current source which acts as a coupler is placed on the side of the cavity. The half loop is then rotated along an axis perpendicular to the side of the cavity. An angle which gives maximum transferred power for both cavities is then sought. The result is then compared between cavities, which is then used to estimate the optimized configuration of a half loop coupler inside an RF cavity.

1. Introduction
The Center for Accelerator Science and Technology (PSTA) is currently developing a cyclotron intended for medical use. The cyclotron, called DECY-13, is designed to accelerate negative hydrogen ion beam up to 13 Mega electronvolts. The ion is then bombarded into enriched water, which contain heavier oxygen isotope. The bombarded oxygen will become an unstable fluorine isotope, which will emit positron that can be used to image cancer tissues inside a person [1,2].

A cyclotron works by accelerating a beam of particles using time varying electric field. A magnet is placed each above and below the trajectory plane of the beam. The magnet will bend the trajectory of the accelerated beam, so that it is possible for the beam to be accelerated many times by a single accelerating zone, making it economical. Compared to other accelerating device such as a synchrotron, a cyclotron has a lower maximum energy but a higher beam current [3]. Thus, cyclotron is widely used to produce radioisotopes [4].

DECY-13 is based on earlier cyclotron first developed by Korean accelerator researcher [5,6]. The development of DECY-13 is currently on the stage of optimizing transferred power from the electromagnetic field generator into the radio frequency (RF) cavity. An unoptimized power transfer might cause the transferred power to be reflected back into the generator, which will cause the generator to break. Earlier effort of DECY-13 team to optimize power transfer was done by [7] using impedance matching analysis, by considering a lumped circuit representing the accelerating cavity. The main problem of this analysis is that the spatial variation of electromagnetic field cannot be obtained. Also, the value of one of the parameters of interest in that analysis, the mutual inductance between coupler and cavity, is unknown. This paper intends to solve the problem from that paper using proper analysis via Maxwell equations.

Unfortunately, a realistic analysis of RF cavity is very difficult. An easier way is to use simulation program, but the optimization will be hard to get, since the realistic RF simulation is computationally resource demanding. In this paper, to get an idea about how coupler positioning inside a cavity is
optimized, only a simple cavity and coupler are considered. Two simplest cavities that come to mind would be either cylindrical or rectangular prism perfectly conducting cavities. The simplest coupler would be a thin line made of conducting material. But to mimic the coupler design of DECY-13, a thin half loop conductor is considered instead. Then, the position of the coupler is varied inside the cavity, and the maximum electromagnetic excitation is sought. It is argued that the maximum electromagnetic excitation would correspond to maximal mutual induction in previous analysis.

2. Calculating the Optimal Excited Electromagnetic Amplitude
In this section, the amplitude of excited electromagnetic field inside a cavity is considered. To calculate the amplitude, one need to understand how an electromagnetic source is coupled to excited electromagnetic fields. The formalism is given on the section B of the appendix. The premise is as follows, given the form of excited field $E_\lambda$, a form of $d\ell$ along with some constraint, which maximizes the integral is sought. For this paper, the source is a thin half loop wire, assumed to have spatially homogenous electrical current (valid if the wavelength is much longer than the radius of the wire). The relation between half loop parameter and the coordinate of the cavity is then sought. Most of the time the result is very difficult to be solved analytically. If that happens, numerical method is used to see the optimized form.

The half loop source is assumed to have a radius $r_0$ and is placed on the side of the rectangular cavity and the base of the cylinder cavity for rectangular and cylindrical cavities, respectively. It is easier to transform the coordinate of the electromagnetic fields on each cavity in terms of coordinate of the loop.

![Figure 1. Half loop source diagram.](image)

As mentioned earlier, the form of $d\ell$ which gives maximum value of the integral is sought. In practice, since the shape of the source is already decided (a thin half loop), then the variation would be the orientation of the source, indicated by the value of the azimuthal angle $\alpha$ and the position of the source. If the integral is simple enough, then it is possible to use analytical method. But if the integral is complicated, then it would be best to use numerical integral, then plot the excited amplitude for each value of azimuthal angle. The first one to be considered is rectangular cavity.

2.1. The optimization of the excited electromagnetic amplitude by a half loop source in a rectangular cavity
The calculation proceeds as follows, first, the integral $d\ell$ need to be defined. Assuming that one end of the half loop is placed on $yz$-plane fixed on $x, y, z = (0, h_1, h_2)$ inside the cavity, then $d\ell$ can be expressed as

$$d\ell = (\cos \theta \hat{x} - \sin \theta \sin \alpha \hat{y} - \sin \theta \cos \alpha \hat{z}) r_0 d\theta$$

where the coordinate transformation related to the half loop is

$$x = r_0 \sin \theta \quad y = h_1 + r_0 (1 + \cos \theta) \sin \alpha \quad z = h_2 + r_0 (1 + \cos \theta) \cos \alpha$$

The illustration is shown in figure 2.
Figure 2. Half loop position and orientation inside a rectangular cavity.

Then, the integral as expressed by equation (A33) is done along the half loop, where integral is done with respect to $\theta$ and the value of $\alpha$ is fixed. The electromagnetic field of the chosen mode (preferably the lowest one for a given cavity) is then expressed in terms of the coordinate of the loop.

Generally, the integral would be very difficult to calculate analytically since there will be sine function inside another sine function on the integrand. To do numerical calculation, the value of the used parameters need to be specified. For the case at hand, the length, width, and height are given by $a = 1$ m, $b = 0.8$ m, and $c = 3$ m, respectively. For a cavity not filled with dielectric or paramagnetic material, the value of permittivity and permeability of free space are given by $\varepsilon_0 \approx 8.854 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ H/m. The lowest possible frequency in this case is attributed to TE$_{001}$ mode, but no electric field will be excited in this mode. The next lowest frequency are shared by TE$_{101}$ and TM$_{101}$ mode, but only TE$_{101}$ that is going to be excited (as evident from the electromagnetic components in the appendix). The frequency for this mode in this cavity is $f \approx 158$ MHz. Setting a reasonable value of $h_1$, $h_2$, and $r_0$ such as $h_1 = 0.4$ m, and $r_0 = 0.05$ m, and assuming the current to be simply $1$ A, the plot of azimuthal angle vs relative excitation amplitude for $h_2 = 0.1$ m, $h_2 = 0.5$ m, and $h_2 = 1.5$ m is then sought. To make calculation even simpler, the term outside the integral is assumed to be equal to unity, thus the term “relative amplitude” which is denoted by $R$, since it is a comparison between amplitudes of the same parameters but different rotating azimuthal angle $\alpha$. The results are as follow

![Image](image_url)

Figure 3. Relative Amplitude $R$ vs Azimuthal Angle $\alpha$ for rectangular cavity with: (a) $h_2 = 0.1$ m, (b) $h_2 = 1.5$ m, (c) $h_2 = 2.9$ m.

2.2. The optimization of the excited electromagnetic amplitude by a half loop source in a cylindrical cavity

The steps are the same with previous calculation. Assuming that the loops hovers inside the cavity (realistically, the half loop wire will need a support, but it is neglected here) with one of the end at $\rho, \phi, z = (h_1, \pi, h_2)$ is being held fix, where the half loop faces away from the nearest side of the cylinder. The half loop is then rotated with an angle $\alpha$ along an axis which is perpendicular to the side of the cylinder and passes through the fixed point. $dl$ for cylindrical coordinate can then be expressed as

$$dl = d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{z}$$  (3)
The cylindrical coordinate is related to the half loop parameter by

\[ \rho = \sqrt{(r_0 \sin \theta - h_1)^2 + (r_0(1 + \cos \theta) \sin \alpha)^2} \]  \hspace{1cm} (4)

\[ \phi = \tan^{-1} \left( \frac{r_0(1+\cos \theta) \sin \alpha}{r_0 \sin \theta - h_1} \right) \]  \hspace{1cm} (5)

\[ z = h_2 + r_0(1+\cos \theta) \cos \alpha \]  \hspace{1cm} (6)

The illustration is given by Figure 4.

**Figure 4.** Half loop position and orientation inside cylindrical cavity.

Same as before, the integral would be really challenging to be solved analytically. Thus, it would be more sensible to use numerical method. Most of the time, the interesting frequency is the lowest one. For this cavity, the lowest mode is given by TM\textsubscript{010} with frequency \( f \approx 229.5 \) MHz. Assuming the length and the radius of the cylindrical cavity to be \( d = 4 \) m and \( a = 0.5 \) m respectively, the radius of the half loop to be \( r_0 = 0.05 \) m, and \( h_1 = 0.4 \), the plot of relative excitation amplitude vs rotated angle \( \alpha \) for \( h_2 = 0.1 \) m, \( h_2 = 1 \) m, and \( h_2 = 2 \) m is given by

**Figure 5a.** Relative Amplitude R vs Azimuthal Angle \( \alpha \) for cylindrical cavity with: (a) \( h_1 = 0.4 \) m and \( h_2 = 0.1 \) m, (b) \( h_1 = 0.4 \) m and \( h_2 = 1 \) m, (c) \( h_1 = 0.4 \) m and \( h_2 = 2 \) m, (d) \( h_1 = 0.1 \) m and \( h_2 = 0.1 \) m
3. Discussion
There are several things that need to be pointed out regarding the results obtained above. The first obvious one is that indeed for half loop source placed that way in a cavity, the orientation of the half loop will affect the amplitude of excited electromagnetic fields. As can be seen from Figure 4a to Figure 4d, it is that at some special angle (in this case $\alpha = 0.5\pi$) no electromagnetic field will be excited, at all, even if the frequency of the source matches the intended frequency of the cavity. The opposite happens for rectangular cavity, where no excitation is experienced under $\alpha = 0$ and $\alpha = \pi$. This is greatly influenced by the mode that was going to be excited, since the direction and the shape of the electromagnetic field will generally be different.

Second, the amplitude of excited of electromagnetic field generally depends on the position of the source, as can be seen from Figure 3a – Figure 3c. The profile of the amplitude as shown by the graphs, changes. When the source is placed at the center of the sides of the cavity, which is the case of Figure 3b, the amplitude is much greater compared when the source is placed at the ends of the side of the cavity. Although that is not always the case (depends on the mode). It can be seen from Figure 5a – Figure 5c, that although the half loop is translated along z-axis, there is no change of profile of the amplitude. But when the source is moved closer to the axis of the cylinder, the amplitude becomes greater (this particular result is provided to illustrate the case), although the profile stays the same.

From the results above, the maximum amplitude indeed is only obtained for some specific angle, and a specific position for a given mode and shape of the source. Thus, when designing an accelerating cavity, one need to consider the best placement possible for the source, by keeping in mind what mode of the cavity that is going to be utilized. The rotational (in this case azimuthal) adjustment is generally easier to be done compared to placement adjustment.

The analysis of power transfer as indicated by maximum excited amplitude in this paper can still be further refined. This paper only assumes perfectly conducting cavity, while most of cavity used for cyclotron are ordinary conductor, which dissipates some heat. This dissipative part will give an additional term on the left-hand side of equation (A45). This makes the calculation of excitation of amplitude more direct (not relative amplitude anymore) since the amplitude of $C_\lambda$ in this case will not be infinite. For a more realistic analysis, a more geometrically complex cavity can be considered, presumably a case that include drift tube inside the cavity. Unfortunately, some geometrically more complex cavity will require full numerical analysis which makes optimizing generally more difficult (one calculation may take relatively long time).

The result above was obtained via equation (A45), limiting the consideration only at the coupling strength given by the integral of dot product of excited field and the electrical current. Some authors have different method of calculating coupling strength. One notable author is Condon in his paper [10,11], where the coupling strength is calculated via vector potential. Same method is also given by Wangler in [9]. The problem is, it is generally more difficult to calculate the excited vector potential inside a cavity, while the calculation of electromagnetic fields inside a cavity is straightforward (albeit incredibly difficult for complex cavity). Another author which calculated the coupling strength using electromagnetic field is Jackson [8]. But the method is used on a waveguide, which is somewhat different from an RF cavity.

4. Conclusion
From the result obtained in the previous chapter, there are several important points that can be drawn. The amplitude of excited electromagnetic fields greatly depends on the orientation and the placement of the electromagnetic source. The orientation and the placement of the source need to be adjusted in accordance to the shape and the intended mode of the cavity. It is possible that there will be no electromagnetic excitation at all for some specific source orientation. For rectangular cavity, it was found that the optimum configuration is when $\alpha = 0.5\pi$ and $h_2 = 1.5$ m, while for cylindrical cavity, it is obtained when $\alpha = 0$, $\pi$ and $h_1 = 0.1$ m.
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APPENDIX

In this part, some of derivation needed to understand the elaboration given on the paper are shown. The first part explains how to calculate the harmonically oscillating excited electromagnetic field inside z-symmetrical cavity.

A. The harmonically oscillating electromagnetic field in a z-symmetrical cavity.

The formalism given on this section follows from Jackson’s classical electrodynamics book[8], which is elaborated further in the appendix. The electromagnetic fields inside a perfectly conducting cavity obey Maxwell equations with some boundary conditions. Assuming that the field is harmonically oscillating and that there is no electrical charge nor current around the cavity, the Maxwell equations can be written as

\begin{align*}
\nabla \cdot \vec{E} &= 0 \quad (A1) \\
\nabla \times \vec{E} &= i\omega \mu \vec{H} \quad (A2) \\
\nabla \cdot \vec{H} &= 0 \quad (A3) \\
\nabla \times \vec{H} &= -i\omega \epsilon \vec{E} \quad (A4)
\end{align*}

while the boundary condition is given by noting that for a given conducting surface, these equations apply

\begin{align*}
\hat{n} \times \vec{E} &= 0 \quad (A5) \\
\hat{n} \cdot \vec{H} &= 0 \quad (A6)
\end{align*}

\(\hat{n}\) is the normal vector perpendicular to the surface.

From manipulating equation (A1) – (A4), it can be seen that the electric and magnetic field satisfy wave equation, specifically

\begin{align*}
(\nabla^2 + \mu \epsilon \omega^2)\vec{E} &= 0 \quad (A7) \\
(\nabla^2 + \mu \epsilon \omega^2)\vec{H} &= 0 \quad (A8)
\end{align*}

The cavity is specifically also assumed to be symmetrical in z-axis. This means that the z dependence of the electromagnetic field should satisfy standing wave equation. Thus

\[
\vec{E}(x, y, z) = \vec{E}(x, y)(\cos(k_z z) + \sin(k_z z))
\]  

(A9)

where \(k_z\) is the wave number in z direction. The magnetic field also obey similar equation. The wave equation can thus be rewritten into

\begin{align*}
(\nabla^2 - k_z^2 + \mu \epsilon \omega^2)\vec{E}(x, y) &= 0 \quad (A10) \\
(\nabla^2 - k_z^2 + \mu \epsilon \omega^2)\vec{H}(x, y) &= 0 \quad (A11)
\end{align*}

From equation (A1) – (A4) it is also evident that for z-symmetrical cavity, it is possible to state non-z components of electromagnetic field (which is collectively called the transversal components) in terms of z components of electromagnetic field (which is called the longitudinal component). Separating the transversal and longitudinal component, such as
\[ \nabla = \nabla_t + \hat{\mathbf{z}} \frac{\partial}{\partial z} \quad \vec{E} = \vec{E}_t + \hat{\mathbf{z}}E_z \quad \vec{H} = \vec{H}_t + \hat{\mathbf{z}}H_z \]  

(A12)

Thus the four Maxwell equation can be rewritten into

\[ \nabla_t \cdot \vec{E}_t = -\frac{\partial E_z}{\partial z} \quad \nabla_t \cdot \vec{H}_t = -\frac{\partial H_z}{\partial z} \]  

(A13)

\[ \hat{\mathbf{z}} \cdot (\nabla_t \times \vec{E}_t) = i\mu \omega H_z \quad \hat{\mathbf{z}} \cdot (\nabla_t \times \vec{H}_t) = -i\varepsilon \omega E_z \]  

(A14)

\[ \left( \hat{\mathbf{z}} \times \frac{\partial \vec{E}_t}{\partial z} \right) - \hat{\mathbf{z}} \times (\nabla_t E_z) = i\mu \omega \vec{H}_t \quad \left( \hat{\mathbf{z}} \times \frac{\partial \vec{H}_t}{\partial z} \right) - \hat{\mathbf{z}} \times (\nabla_t H_z) = -i\varepsilon \omega \vec{E}_t \]  

(A15)

Using equation (A9), (A10), and (A11) along with six equations above, the transverse component of electric and magnetic field can be stated in terms of the longitudinal component.

\[ \vec{E}_t = \frac{1}{k_z^2 - \mu \varepsilon^2} \left( i\mu \omega \hat{\mathbf{z}} \times (\nabla_t H_z) - \nabla_t \frac{\partial E_z}{\partial z} \right) \]  

(A16)

\[ \vec{H}_t = \frac{1}{k_z^2 - \mu \varepsilon^2} \left( i\varepsilon \omega \hat{\mathbf{z}} \times (\nabla_t E_z) + \nabla_t \frac{\partial H_z}{\partial z} \right) \]  

(A17)

The specific function of \( E_z \) and \( H_z \) can actually be freely chosen, as long as the boundary condition from equation (A5) and (A6) are satisfied. One of the simplest choices for longitudinal electromagnetic fields are \( E_z \) and \( H_z \) equal to zero, which is called the Transverse Electromagnetic (TEM) mode. For a cavity not composed of layered smaller cavities, this means that the electromagnetic field is equal to zero everywhere, as can be seen from equation (A16) and (A17). It is more convenient to set either of \( E_z \) or \( H_z \) equal to zero, then use the boundary condition (A5) or (A6). For example, by setting \( H_z = 0 \) and using boundary condition from equation (A5) on the sides of the cavity (\( \vec{n} = \hat{x} \) and \( \vec{n} = \hat{y} \)) the Transverse Magnetic (TM) mode is obtained, which is

\[ H_z = 0 \quad E_z \big|_S = 0 \]  

(A18)

If instead \( E_z \) is being set equal to zero but \( H_z \) cannot be zero everywhere, then by using the boundary condition along with

\[ E_z = 0 \quad \frac{\partial H_z}{\partial n} \big|_S = 0 \]  

(A19)

which is often called the Transverse Electric (TE) mode.

\section*{A.1. The excited electromagnetic fields in a rectangular cavity}

In this subsection, one of the simple cavity shapes, the rectangular shape, is considered. The electromagnetic field that satisfy the boundary condition, specifically chosen to be TM or TE modes are sought. The frame of the rectangle is assumed to coincide with the coordinate axis (in this case, cartesian coordinate is used) where the frame that coincides with x-axis goes from \( x = 0 \) to \( x = a \), the y-axis from \( y = 0 \) to \( y = b \) and z-axis from \( z = 0 \) to \( z = d \). Only \( E_z \) for TM modes, and \( H_z \) for TE modes are shown here (the rest of the components are shown in the appendix). For TM modes, from equation (A10) and equation (A5), the components of electromagnetic field need to satisfy

\[ E_z(x,y,z) = E_{z0} \sin \left( \frac{mn \pi x}{a} \right) \sin \left( \frac{np \pi y}{b} \right) \cos \left( \frac{p \pi z}{c} \right) \]  

(A20)

\[ E_x(x,y,z) = \frac{E_{z0} \pi^2 \rho z c}{a \left( \rho^2 \pi^2 - \mu \varepsilon^2 c^2 \right)} \cos \left( \frac{mn \pi x}{a} \right) \sin \left( \frac{np \pi y}{b} \right) \sin \left( \frac{p \pi z}{c} \right) \]  

(A21)
The excited electromagnetic fields in a cylindrical cavity

The next simple cavity that will be considered is the cylindrical cavity. It is assumed that the length of the cylinder is \( l \) and its radius is \( a \). It is also assumed that the center of the base of the cylinder coincide with the center of the coordinate, while the cylinder is directed along the \( z \) axis. For TM case, the components of electromagnetic fields is given by

\[
E_y(x, y, z) = \frac{E_{20} np^2 c}{b(p^2 \pi^2 - \mu \omega^2 c^2)} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{p \pi z}{c} \right) \tag{A22}
\]

\[
H_x(x, y, z) = -\frac{i E_{20} \omega \mu \pi^2}{b(p^2 \pi^2 - \mu \omega^2 c^2)} \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{p \pi z}{c} \right) \tag{A23}
\]

\[
H_y(x, y, z) = \frac{i E_{20} \omega \mu \pi^2}{a(p^2 \pi^2 - \mu \omega^2 c^2)} \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{p \pi z}{c} \right) \tag{A24}
\]

while for TE mode, the components of electromagnetic fields are given by

\[
H_z(x, y, z) = H_{20} \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{p \pi z}{c} \right) \tag{A25}
\]

\[
H_x(x, y, z) = -\frac{H_{20} np c \pi^2}{b(p^2 \pi^2 - \mu \omega^2 c^2)} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \cos \left( \frac{p \pi z}{c} \right) \tag{A26}
\]

\[
H_y(x, y, z) = -\frac{H_{20} np c \pi^2}{b(p^2 \pi^2 - \mu \omega^2 c^2)} \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \cos \left( \frac{p \pi z}{c} \right) \tag{A27}
\]

\[
E_x(x, y, z) = \frac{i H_{20} \mu \omega \pi^2}{b(p^2 \pi^2 - \mu \omega^2 c^2)} \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \sin \left( \frac{p \pi z}{c} \right) \tag{A28}
\]

\[
E_y(x, y, z) = -\frac{i H_{20} \mu \omega \pi^2}{a(p^2 \pi^2 - \mu \omega^2 c^2)} \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \sin \left( \frac{p \pi z}{c} \right) \tag{A29}
\]

The imaginary number on some of the electromagnetic field components written above indicates that the phase of those particular components differ from the real one by ninety degrees (since the time dependency is assumed to be \( \exp(i \omega t) \), which means that all of the components have real and imaginary part if time dependency is included).

All of the equations above are obtained by using Maxwell equations harmonically varying with respect to time which satisfy the boundary condition. Thus, in such cavity (expressed by its boundary condition) only combination of such modes are allowed. It will be shown below that for arbitrary greater than zero value of \( a, b, \) and \( c \), only a set of frequencies are allowed. By plugging the components of electromagnetic field for TM and TE modes into equation (A7) and (A8), it is evident that for both modes the frequency must satisfy

\[
\omega = \omega_{mnp} = \frac{1}{\sqrt{\varepsilon \mu}} \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 + \left( \frac{p \pi}{c} \right)^2 \right] \tag{A30}
\]

A.2. The excited electromagnetic fields in a cylindrical cavity

The next simple cavity that will be considered is the cylindrical cavity. It is assumed that the length of the cylinder is \( l \) and its radius is \( a \). It is also assumed that the center of the base of the cylinder coincide with the center of the coordinate, while the cylinder is directed along the \( z \) axis. For TM case, the components of electromagnetic fields is given by

\[
E_z(\rho, \phi, z) = E_{20} I_m \left( x_{mn} \frac{\rho}{a} \right) \cos \left( \frac{p \pi z}{l} \right) (\cos m \phi + i \sin m \phi) \tag{A31}
\]

\[
E_{\rho}(\rho, \phi, z) = \frac{E_{20} \rho \pi l}{p^2 \pi^2 - \mu \omega^2 \pi^2} \frac{\partial J_m \left( x_{mn} \frac{\rho}{a} \right)}{\partial \rho} \sin \left( \frac{p \pi z}{l} \right) (\cos m \phi + i \sin m \phi) \tag{A32}
\]

\[
E_{\phi}(\rho, \phi, z) = \frac{E_{20} \rho \pi l m}{p^2 \pi^2 - \mu \omega^2 \pi^2} J_m \left( x_{mn} \frac{\rho}{a} \right) \sin \left( \frac{p \pi z}{l} \right) (-\sin m \phi + i \cos m \phi) \tag{A33}
\]
\[ H_\rho(\rho, \phi, z) = \frac{-i \varepsilon E_{\rho 0} l^2 \omega}{\rho (p^2 \pi^2 - \mu \omega^2 z^2)} J_m \left( x_{mn} \frac{\rho}{a} \right) \cos \left( \frac{p \pi z}{l} \right) (-\sin m \phi + i \cos m \phi) \]  
(A34)

\[ H_\phi(\rho, \phi, z) = \frac{i \varepsilon E_{\phi 0} l^2 \omega}{p^2 \pi^2 - \mu \omega^2 z^2} \frac{\partial J_m \left( x_{mn} \frac{\rho}{a} \right)}{\partial \rho} \cos \left( \frac{p \pi z}{l} \right) (\cos m \phi + i \sin m \phi) \]  
(A35)

where \( J_m \) is the Bessel function of order \( m \), \( x_{mn} \) is the \( n \)th root of \( J_m \), and \( m, n, p = 0, 1, 2, ... \). The allowed frequency of this mode is given by

\[ \omega_{mn p} = \sqrt{\frac{1}{\mu \varepsilon} \left( \frac{x_{mn}^2}{a^2} + \left( \frac{p \pi}{l} \right)^2 \right)} \]  
(A36)

The components of electromagnetic field for TE case is given by

\[ H_x(\rho, \phi, z) = H_{x0} J_m \left( x_{mn} \frac{\rho}{a} \right) \sin \left( \frac{p \pi z}{l} \right) (\cos m \phi + i \sin m \phi) \]  
(A37)

\[ H_\rho(\rho, \phi, z) = \frac{H_{x0} \pi m l}{\rho (p^2 \pi^2 - \mu \omega^2 z^2)} \frac{\partial J_m \left( x_{mn} \frac{\rho}{a} \right)}{\partial \rho} \cos \left( \frac{p \pi z}{l} \right) (\cos m \phi + i \sin m \phi) \]  
(A38)

\[ H_\phi(\rho, \phi, z) = \frac{H_{x0} \pi m l}{\rho (p^2 \pi^2 - \mu \omega^2 z^2)} J_m \left( x_{mn} \frac{\rho}{a} \right) \cos \left( \frac{p \pi z}{l} \right) (-\sin m \phi + i \cos m \phi) \]  
(A39)

\[ E_\rho(\rho, \phi, z) = \frac{-i \mu \omega H_{x0} l^2 m}{\rho (p^2 \pi^2 - \mu \omega^2 z^2)} J_m \left( x_{mn} \frac{\rho}{a} \right) \sin \left( \frac{p \pi z}{l} \right) (-\sin m \phi + i \cos m \phi) \]  
(A40)

\[ E_\phi(\rho, \phi, z) = \frac{i \mu \omega H_{x0} l^2 m}{p^2 \pi^2 - \mu \omega^2 z^2} \frac{\partial J_m \left( x_{mn} \frac{\rho}{a} \right)}{\partial \rho} \sin \left( \frac{p \pi z}{l} \right) (\cos m \phi + i \sin m \phi) \]  
(A41)

with \( x_{mn}' \) is the \( n \)th roots of \( \partial J_m / \partial \rho \) which for the case at hand, means that \( \partial J_m / \partial \rho = 0 \) at \( \rho = a \) in accordance with the boundary condition of TE case. The frequency of this mode is given by

\[ \omega_{mn p} = \sqrt{\frac{1}{\mu \varepsilon} \left( \frac{x_{mn}^2}{a^2} + \left( \frac{p \pi}{l} \right)^2 \right)} \]  
(A42)

B. The coupling between electromagnetic source and excited electromagnetic fields

Having derived the excited electromagnetic fields inside rectangular and cylindrical cavity, an electromagnetic source placed inside the cavity will be considered next. But to do that, it would be more convenient to talk about the orthogonality condition of the electromagnetic field inside the cavity. Let \( \lambda \) and \( \sigma \) be the representation of mode number \( mn p \) such that \( \vec{E}_{mn p} \rightarrow \vec{E}_\lambda \) for either TM or TE mode. The orthogonality condition for

\[ \int_V \vec{E}_\lambda \cdot \vec{E}_\sigma \, d\tau = A_\lambda B_{\lambda \sigma} \]  
(B1)

where \( d\tau \) is the element of spatial volume, and \( A_\lambda \) and \( B_{\lambda \sigma} \) are some constant which depend on the value of \( \lambda \) (the normalization here is not that important, because the sought quantity are relative amplitude between excited fields for a given mode, thus both \( E_{\rho 0} \) and \( H_{\rho 0} \) for TM and TE case respectively can still be used). Obviously, \( \vec{E}_\lambda \) is orthogonal to \( \vec{H}_\lambda \) for any \( \lambda \) and \( \sigma \).

Since the components of fields are all orthogonal and form a complete basis [9] (although it’s not normalized, which will be explained why it won’t be necessary), any shape of electromagnetic field can be expanded in terms of \( \vec{E}_\lambda \) and \( \vec{H}_\lambda \), specifically
\[ \vec{E}(\vec{x}, t) = \sum_{\lambda} C_{\lambda}(t) \vec{E}_{\lambda}(\vec{x}) \]
\[ \vec{H}(\vec{x}, t) = \sum_{\lambda} D_{\lambda}(t) \vec{H}_{\lambda}(\vec{x}) \]

(B2)

with \( C_{\lambda} \) and \( D_{\lambda} \) are some time-dependent functions. Now, for those relation to work, both \( \vec{E}_{\lambda} \) and \( \vec{H}_{\lambda} \) actually need to be normalized. But since every mode are in different frequency (neglecting the possibility of degeneracy at higher frequency, since most of the time the most important mode are the lowest ones), and for most cases the source only radiate electromagnetic field harmonically in a single frequency, then only a mode with correct frequency is needed. This also means that for a source that emit a single frequency, then normalization of excited electromagnetic field is not necessary.

For a source specified by a current density \( J \), the amplitude of excitation \( C_{\lambda}(t) \) should satisfy

\[ \ddot{C}_{\lambda} + \omega_{\lambda}^2 C_{\lambda} = \frac{\int \frac{\partial J}{\partial t} \cdot \vec{E}_{\lambda} d\tau}{\int \vec{E}_{\lambda} \cdot \vec{E}_{\lambda} d\tau} \]

(B3)

For a more realistic model, the left-hand side should also include damping term, which is neglected here. The right-hand side acts like a driving force, with time dependence entirely encoded in \( J \). If the current density varies harmonically with respect to time with a frequency of \( \omega_d \), for a lossless cavity, will have and infinite amplitude for \( \omega_d = \omega_{\lambda} \), assuming that the right hand side of equation (A32) is nonzero. This is an implication that resonance between source and cavity. Thus, assuming that resonance is achieved by frequency matching, the maximum amplitude of the right-hand side term is sought. It can be easily seen that even if the driving frequency matched the frequency of the cavity, when \( \int \frac{\partial J}{\partial t} \cdot \vec{E}_{\lambda} d\tau = 0 \) there won’t be any excitation inside the cavity. For some cases, the source can be approximated as a thin wire, which is easier to deal with. For thin wire, the current is then written as

\[ \int \frac{\partial J}{\partial t} \cdot \vec{E}_{\lambda} d\tau \rightarrow \int \frac{\partial I}{\partial t} \vec{E}_{\lambda} \cdot d\vec{l} \]

(B4)