Anisotropic background for two fluids: Matter and holographic dark energy

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Abstract. We discuss a spatially homogeneous and anisotropic Bianchi type-I space-time with two fluids as the content of the Universe: Matter and holographic dark energy in the framework of general relativity. To get the exact solutions of Einstein’s field equations, we choose the scale factor as a hyperbolic function, specifically, \(a(t) = \sinh^{1/n}(\gamma t)\), where \(\gamma\) and \(n > 0\) are arbitrary constants, which gives us a time-dependent deceleration parameter. Then, we study our cosmological model under the conditions of parameters as: \(\gamma\) fixed and \(n > 1\).

Our cosmological solutions led to an early deceleration phase followed by the current observed acceleration phase. Further, the anisotropic parameter and some other physical parameters are discussed. We conclude that our cosmological model is consistent with the results of recent astronomical observations.

Keywords. Bianchi type-I space-time—general relativity—varying deceleration parameter—holographic dark energy.

1. Introduction

Recent observations (Filippenko & Riess 1998; Perlmutter et al. 1998, 1999; de Bernardis et al. 2000; Hanany et al. 2000; Tonry et al. 2003; Clocchiatti et al. 2006; Blake et al. 2011; Anderson et al. 2012; Padmanabhan et al. 2012; Hinshaw et al. 2013) indicate that our Universe has entered an accelerated expansion phase. According to Einstein’s general relativity (GR), the cause of such acceleration is the presence of a component of unknown nature called dark energy (DE), which has negative pressure and represents 68% of the total density of the Universe, it behaves like a repulsive gravity. Its nature remains unknown today. It may simply be the cosmological constant (\(\Lambda\)) induced by GR which would have a non-zero value. This cosmological constant has an equation of state (EoS) parameter \(\omega = -1\) and is considered to be very consistent with the observation data. In front of the difficulties linked to its theoretically predicted order of magnitude with respect to that of the observed vacuum energy (Zlatev et al. 1999), other dynamical models of DE have been proposed, such as quintessence (Carroll 1998; Turner 2002), phantom (Caldwell 2002), \(k\)-essence (Chiba et al. 2000), tachyons (Padmanabhan 2002), Chaplygin gas (Kamenshchik et al. 2001), etc. There occurs another type of DE models, in which, we do not need to introduce any other form of energy, and this approach is called modified gravity theories (MGT), i.e., the accelerating expansion of the Universe can be caused by a modification in gravity. Moreover, GR is not valid on cosmological scales of matter in the Universe. The most famous of these theories are: \(f(T)\) gravity, \(f(G)\) gravity, \(f(R, G)\) gravity, \(f(R, T)\) gravity, \(f(R, T, R_{\mu\nu}T^{\mu\nu})\) gravity and \(f(T, T)\) gravity, where \(T\) is the trace of energy-momentum tensor (or it could be torsion), \(G\) is the Gauss–Bonnet (GB) invariant and \(R_{\mu\nu}\) is Ricci tensor (Linder 2010; Harko et al. 2011; Myrzakulov 2011; de Laurentis et al. 2015).

To know the nature of DE, the holographic dark energy (HDE) provides a more reliable framework for its simplicity and reasonableness. The coincidence problem can be easily solved for some interactive models of HDE. This model is considered as an application of the holographic principle (HP) to the problem of DE. The HP was first suggested by ‘t Hooft G’t (1993) in the background of black hole physics, then in a cosmological context, another version of HP was proposed by
Fischler & Susskind (1998). In the background of the DE problem, the HP tells us that all physical quantities in the Universe, including the density of DE ($\rho_\Lambda$), can be described by a few quantities on boundary of the Universe. It is clear that it is given in terms of two physical quantities, namely the reduced Planck mass ($M_p$) and the cosmological length scale ($L$) as $\rho_\Lambda \approx c^2 M_p^2 L^{-2}$ (Fischler & Susskind 1998). Next, a relationship was proposed, which combines the HDE density ($\rho/L\Lambda_1$) and the Hubble parameter ($H$) as $\rho_\Lambda \propto H^2$, it does not contribute to the current accelerated expansion of the Universe (Li 2004). For purely dimensional reasons, Granda & Oliveros (2008) proposed a new infrared cut-off for the HDE density of the form $\rho_\Lambda \approx \alpha H^2 + \beta H$, where $\alpha$ and $\beta$ are constants. They show that this new model of DE represents the accelerated expansion of the Universe and is consistent with current observational data. In several works, Sarkar & Mahanta (2013) and Sarkar (2014, 2016) have investigated the HDE in various contexts. In addition, Samanta (2013) in his work, studied the homogeneous and anisotropic Bianchi type-V Universe filled with matter and HDE components, and a correspondence between the HDE and quintessence DE are also established. Recently, Dubey et al. (2019) has evaluated Tsallis holographic dark energy (THDE) and infrared cut-off for the Hubble horizon in the anisotropic Universe using hybrid expansion law (HEL).

The anisotropic Universe has attracted the attention of many researchers because anisotropy played an important role in the early moments of cosmic evolution. In addition, the possibility of an anisotropy phase at the beginning of the Universe followed by an isotropy phase was supported by the observations. Several researchers have studied homogeneous and anisotropic Bianchi models, such as the spatially homogeneous and anisotropic Bianchi type-I model, which is a direct generalization of the FLRW Universe with a scale factor in each spatial direction (Koussour & Bennai 2021, 2022a, b). In this study, we analyze a spatially homogeneous and anisotropic Bianchi type-I space-time with two fluids as the content of the Universe: matter and holographic dark energy in the framework of GR. Moreover, to find the exact solutions of the field equations and some physical parameters, we assume the scale factor as a hyperbolic function, specifically, $a(t) = \sinh^{1/n} (\gamma t)$, where $\gamma$ and $n$ are free model parameters, which give us a time-dependent deceleration parameter (DP).

The present paper is organized as follows: In Section 2, we present the field equations for the Bianchi type-I Universe and defined some physical and geometrical parameters to solve the field equations in the same section. In Section 3, we solve the field equations by assuming a hyperbolic function of the scale factor. Finally, in Sections 3 and 4, we discuss the jerk parameter and conclude our results, respectively.

2. Metric and basic field equations

In our analysis, we consider a spatially homogeneous and anisotropic Bianchi type-I metric (Sarkar & Mahanta 2013)

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2,$$  \hspace{1cm} (1)

where $A(t), B(t)$ and $C(t)$ are the directional scale factors, functions of cosmic time $t$ only. We will follow the same steps in the literature, and first, we write expressions for the physical and geometrical parameters that we will use here to solve Einstein’s field equations for the metric of Equation (1).

The average scale factor $a$ of the Bianchi type-I space-time is given by:

$$a = (ABC)^{1/3}. \hspace{1cm} (2)$$

The spatial volume $V$ of the Universe is defined as:

$$V = a^3 = ABC. \hspace{1cm} (3)$$

Further, the directional Hubble parameters $H_1$, $H_2$ and $H_3$ are respectively,

$$H_1 = \frac{\dot{A}}{A}, \hspace{0.5cm} H_2 = \frac{\dot{B}}{B}, \hspace{0.5cm} H_3 = \frac{\dot{C}}{C}. \hspace{1cm} (4)$$

The average Hubble parameter is defined as:

$$H = \frac{1}{3} (H_1 + H_2 + H_3). \hspace{1cm} (5)$$

From Equations (2)–(5), we find:

$$H = \frac{1}{3} \frac{\dot{V}}{V} = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right). \hspace{1cm} (6)$$

Other physical parameters, the expansion scalar ($\theta$), average anisotropic parameter ($A_m$) and shear scalar ($\sigma^2$), are defined for the Bianchi type-I metric (1), as:

$$\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}, \hspace{1cm} (7)$$

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2, \hspace{1cm} (8)$$

$$\sigma^2 = \frac{1}{2} \left[ \left( \frac{\dot{A}}{A} \right)^2 + \left( \frac{\dot{B}}{B} \right)^2 + \left( \frac{\dot{C}}{C} \right)^2 \right] - \frac{\theta^2}{6}. \hspace{1cm} (9)$$
where $\Delta H_i = H_i - H$ and $H_i (i = 1, 2, 3)$ represent the directional Hubble parameters.

The Einstein’s field equation (with $8\pi G = 1$ and $c = 1$) is given by:

$$ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -(T_{\mu\nu} + \overline{T}_{\mu\nu}), $$

(10)

where $R_{\mu\nu}$ is the Ricci tensor, $R$ is the Ricci scalar and $T_{\mu\nu}, \overline{T}_{\mu\nu}$ are the energy-momentum tensors of matter and HDE, respectively. These energy-momentum tensors are defined as:

$$ T_{\mu\nu} = \rho_m u_{\mu} u_{\nu} $$

(11)

and

$$ \overline{T}_{\mu\nu} = (\rho_\Lambda + p_\Lambda) u_{\mu} u_{\nu} + g_{\mu\nu} p_\Lambda, $$

(12)

where $\rho_m, \rho_\Lambda$ are the energy densities of the matter, respectively, while $p_\Lambda$ is the pressure of the HDE.

The Einstein’s field Equation (10) with Equations (11) and (12) for the metric (1) leads to the following system of field equations:

$$ \frac{\dot{A}}{A} \frac{\dot{B}}{B} + \frac{\dot{B} \dot{C}}{BC} + \frac{\dot{C} \dot{A}}{CA} = \rho_m + \rho_\Lambda, $$

(13)

$$ \frac{\ddot{A}}{A} + \frac{\dot{B}^2}{AB} + \frac{\dot{A} \dot{B}}{AB} = -p_\Lambda, $$

(14)

$$ \frac{\ddot{B}}{B} + \frac{\dot{C} \dot{A}}{CA} = -p_\Lambda, $$

(15)

$$ \frac{\ddot{C}}{C} + \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{C} \dot{A}}{CA} = -p_\Lambda, $$

(16)

where \( (\cdot) \) represents a derivative with respect to cosmic time.

Now, subtracting Equation (16) from Equation (15), we get:

$$ \frac{d}{dt} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) + \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 0. $$

(17)

Using Equation (3), we can write Equation (17) in the form:

$$ \frac{d}{dt} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) + \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) \frac{\dot{V}}{V} = 0. $$

(18)

By integrating the above equation, we get

$$ \frac{\dot{A}}{A} = d_1 \exp \left( k_1 \int \frac{dt}{V} \right), $$

(19)

where $d_1$ and $k_1$ are constants of integration.

Similarly, subtracting Equation (14) from Equation (13), and Equation (13) from Equation (15), we find:

$$ \frac{\dot{B}}{B} = d_2 \exp \left( k_2 \int \frac{dt}{V} \right), $$

(20)

$$ \frac{\dot{C}}{C} = d_3 \exp \left( k_3 \int \frac{dt}{V} \right), $$

(21)

where $d_2, d_3, k_2$ and $k_3$ are constants of integration.

From Equation (3), one can obtain the relation between the constants $d_1, d_2, d_3, k_1, k_2$ and $k_3$ as $d_2 = d_1 d_3$, $k_2 = k_1 + k_3$.

From Equations (19)–(21), the directional scale factors $A(t), B(t)$ and $C(t)$ can be explicitly written in terms of the the average scale factor $a(t)$ as:

$$ A(t) = l_1 a \exp \left( m_1 \int a^{-3} dt \right), $$

(22)

$$ B(t) = l_2 a \exp \left( m_2 \int a^{-3} dt \right), $$

(23)

$$ C(t) = l_3 a \exp \left( m_3 \int a^{-3} dt \right), $$

(24)

where $l_1, l_2, l_3$ and $m_1, m_2, m_3$ are the constants that satisfy the following two relations:

$$ l_1 l_2 l_3 = 1, \quad m_1 + m_2 + m_3 = 0. $$

(25)

Now by using Equations (13)–(16) and the barotropic EoS $p_\Lambda = \omega_\Lambda \rho_\Lambda$, we obtained the continuity equation as:

$$ \dot{\rho}_m + \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \rho_m + \dot{\rho}_\Lambda \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \left( 1 + \omega_\Lambda \right) \rho_\Lambda = 0. $$

(26)

For two fluids anisotropic: matters and HDE, the continuity Equation (26) can be written as:

$$ \dot{\rho}_m + \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \rho_m = 0 $$

(27)

and

$$ \dot{\rho}_\Lambda + \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \left( 1 + \omega_\Lambda \right) \rho_\Lambda = 0, $$

(28)

respectively.

### 3. Cosmological solutions of the model

Taking into account the new form proposed in Granda & Oliveros (2008), we assume the following HDE density for our analysis:

$$ \rho_\Lambda = 3(\alpha H^2 + \beta \dot{H}), $$

(29)

where $\alpha, \beta$ are constants that must satisfy the constraints imposed by the present observational data, $H$ is the average Hubble parameter and $M_{p}^{-2} = 8\pi G = 1$. This new HDE model for energy density introduced by Granda
& Oliveros (2008) may be important in comprehending the evolution of the Universe especially, an anisotropic Universe. As mentioned in the ‘introduction’ section, the advantage of this model is that it predicts the accelerated expansion of the Universe and is coherent with the present observational data. In addition, for the characteristic length scale, \( L \) (infrared cut-off) found in the expression of HDE, there are several possible options in the literature, such as the Hubble horizon, future event horizon or particle horizon (Xu 2009). Recently, Chen & Jing (2009) have modified this new HDE model in which energy density of HDE contains the second order derivative of Hubble’s parameter with regard to cosmic time and named it as modified holographic Ricci dark energy.

Using Equation (29) in Equation (28), the EoS parameter for HDE is obtained as:

\[
\omega_A = -1 - \frac{2\alpha H \dot{H} + \beta \ddot{H}}{3H(\alpha H^2 + \beta \dot{H})}.
\]  

(30)

To solve these field equations, we assume the cosmological scale factor as a hyperbolic function,

\[
a(t) = \sinh^{1/n}(\gamma t),
\]

(31)

where \( \gamma \) and \( n > 0 \) are arbitrary constants. The motivation to choose the scale factor obtained in Equation (31) is that it produces a time-dependent deceleration parameter. It belongs to a class of models that describe the transition of the Universe from early decelerated phase to the recent accelerating phase as indicated by the recent observations in cosmology. The derivation and the motivation to choose such scale factor has already been described in details by Chawla et al. (2012). The role of two fluids minimally coupled in the evolution of the dark energy parameter has been investigated by Pradhan (2014) with the help of the hyperbolic solution of the scale factor. Esmaeili & Mishra (2018) constructed the cosmological model in \( f(R, T) \) theory of gravity in a Bianchi type-VIh Universe by using the hyperbolic scale factor. Recently, Pradhan et al. (2019) proposed the hyperbolic form to examine the physical behavior of the transition of the anisotropic Bianchi type-I perfect fluid cosmological models from early decelerating to the current accelerating phase in the framework of \( f(R, T) \) gravity. In addition, Singh & Lallek (2022) studied the background of flat FLRW metric in the framework of \( f(Q, T) \) gravity theory and considered two cosmological models by taking the parameterization of the scale factor as a hyperbolic function.

Using Equation (31) for the average scale factor in Equations (22)–(24), we obtain the directional scale factors of the following form (Ahmed & Pradhan 2014):

\[
A(t) = l_1 \sinh^{1/n}(\gamma t) \times \exp \left[ \frac{m_1(-1)^{\frac{m+3}{2\gamma}}}{2\gamma} \cos(\gamma t)F(t) \right],
\]

(32)

\[
B(t) = l_2 \sinh^{1/n}(\gamma t) \times \exp \left[ \frac{m_2(-1)^{\frac{m+3}{2\gamma}}}{2\gamma} \cos(\gamma t)F(t) \right],
\]

(33)

\[
C(t) = l_3 \sinh^{1/n}(\gamma t) \times \exp \left[ \frac{m_3(-1)^{\frac{m+3}{2\gamma}}}{2\gamma} \cos(\gamma t)F(t) \right],
\]

(34)

where

\[
F(t) = 1 + \frac{1}{6} \left( 1 + \frac{3}{n} \right) \cosh^2(\gamma t) + \frac{3}{40} \left( 1 + \frac{3}{n} \right) \times \left( 1 + \frac{1}{n} \right) \cosh^4(\gamma t) + o[\cosh(\gamma t)]^6.
\]

(35)

The directional Hubble parameters \( H_i \) and the average Hubble parameter \( H \) become:

\[
H_1 = \frac{\gamma}{n} \coth(\gamma t) + \frac{m_1}{\sinh^{3/n}(\gamma t)},
\]

(36)

\[
H_2 = \frac{\gamma}{n} \coth(\gamma t) + \frac{m_2}{\sinh^{3/n}(\gamma t)},
\]

(37)

\[
H_3 = \frac{\gamma}{n} \coth(\gamma t) + \frac{m_3}{\sinh^{3/n}(\gamma t)},
\]

(38)

\[
H = \frac{\gamma}{n} \coth(\gamma t).
\]

(39)

The expansion scalar \( \theta \) and shear scalar \( \sigma^2 \) are obtained as:

\[
\theta = 3H = \frac{3\gamma}{n} \coth(\gamma t),
\]

(40)

\[
\sigma^2 = \left( m_1^2 + m_2^2 + m_3^2 \right) \left( \frac{1}{\sinh^{3/n}(\gamma t)} \right)^2.
\]

(41)

From Equations (39)–(41), we can see that the Hubble parameter, scalar expansion and scalar shear were diverged at \( t = 0 \) and approach to zero at \( t \to \infty \). Now, using Equation (31) into Equation (3), we get the spatial volume of the Universe as:

\[
V = \sinh^{3/n}(\gamma t).
\]

(42)

From the above equation, it is clear that the spatial volume of our model increases exponentially with cosmic time and zero at initial time \( t = 0 \). In addition, it shows that the evolution of our Universe commences from a big bang scenario. The average scale factor in
Equation (31) is also zero at the early epoch of the Universe. Therefore, our model has a singularity of type point (MacCallum 1971).

The average anisotropy parameter $A_m$ is given as:

$$A_m = \left( \frac{m_1^2 + m_2^2 + m_3^2}{3} \right) \left( \frac{n}{\gamma \coth(\gamma t) \sinh^{3/2}(\gamma t)} \right)^2. \tag{43}$$

From Figure 1, it is clear that the average anisotropic parameter $A_m$ is a decreasing function of cosmic time, which tends towards zero at $t \to \infty$. This indicates that our cosmological model contains a transition from the early anisotropic Universe to the current isotropic Universe as DE starts to dominate the energy density of the Universe, this characteristic is consistent with recent observations. In addition, all model parameters are chosen based on the constraints imposed by the current observational data. In the literature, the model parameters are constrained by using one of the available datasets, such as 31 points of the Hubble datasets, 6 points of the baryon acoustic oscillations (BAO) datasets and 580 points from type Ia supernovae (SNe Ia). According to the analysis in Nagpal et al. (2019), the free parameter $n$ is fit with the observational data. The constrained value of $n$ are obtained as 1.5176, 1.5907, 1.5009, 1.5396 and 1.5060 corresponding to the Hubble $H(z)$, SNe Ia, BAO, $H(z) + $SNe Ia and $H(z) + $SNe Ia + BAO datasets.

Several recent observational data have shown that a positive value of the DP ($q > 0$) describes a decelerating Universe, and a negative value ($q < 0$) describes the acceleration of the cosmic expansion, other observational data from SNe Ia has shown that the current Universe in the acceleration phase and the value of the DP is confined to a range: $-1 \leq q < 0$. The DP is defined as:

$$q = -1 + \frac{d}{dt} \left( \frac{1}{H} \right). \tag{44}$$

Using Equation (39), the DP for our cosmological model is:

$$q = n[1 - \tanh^2(\gamma t)] - 1. \tag{45}$$

From Equation (45), we can find the relation between the parameters of the model $n$ and $\gamma$ as follows:

$$\gamma t_0 = \tanh^{-1} \left( \frac{n - q_0 - 1}{n} \right)^{1/2}, \tag{46}$$

where $t_0$ is the present time and $q_0$ is the present value of DP, to analyse the behavior of certain parameters, we consider $t_0 = 13.8$ GYs and $q_0 = -0.54$ (Mamon & Das 2017). In addition, using the relation which connects the average scale factor and the redshift $a = a_0(1 + z)^{-1}$, where $a_0$ is the present value of the scale factor, i.e., at $z = 0$, we obtained the following expression:

$$H(z) = \frac{\gamma \coth \left( \sinh^{-1} \left( \frac{n - (q_0 + 1)}{(z + 1)^{2n}(q_0 + 1)} \right) \right)}{n}, \tag{48}$$

$$q(z) = n - 1 - n \times \left[ \tanh \left( \sinh^{-1} \left( \frac{n - (q_0 + 1)}{(z + 1)^{2n}(q_0 + 1)} \right) \right) \right]^2. \tag{49}$$

From Equation (45), it is clear that $q > 0$ for $t < (1/\gamma) \tanh^{-1}(1 - 1/n)^{1/2}$ and $q < 0$ for $t > (1/\gamma) \tanh^{-1}(1 - (1/n))^{1/2}$, and it predicts the transition phase, i.e., $q = 0$ at $t = (1/\gamma) \tanh^{-1}(1 - 1/n)^{1/2}$.

In Nagpal et al. (2019), it is shown that for $0 < n \leq 1$, the model is in the deceleration phase, while for $n > 1$, the model of Universe exhibits a phase transition from early decelerating phase to present accelerating phase, which is in good agreement with the results of recent observations. Thus, we can choose a value of $n$ which gives us the physical behavior of the DP consistent with the observation data. Figure 2 shows the behavior of the DP in terms of redshift, in which the parameter $\gamma$ is fixed and three values of the parameter $n$, especially, 1.4, 1.5 and 1.556 corresponding to the values of the transition redshift $z_{tr} = 0.57, 0.63$ and 0.75, respectively. The redshift transition values $z_{tr}$ for our cosmological model are consistent with the observational data (Capozziello et al. 2014, 2015; Farooq et al. 2017) (Figure 3).
Using Equation (39) in Equation (29), we get the HDE energy density
\[
\rho_\Lambda = \frac{3\gamma^2}{n^2} [n\beta + \alpha \coth^2(\gamma t) - n\beta \coth^2(\gamma t)].
\] (50)

Again, using Equation (39) in Equation (27), we get the matter energy density:
\[
\rho_m = c \exp \left( -\frac{3}{n} (\ln(e^{2\gamma t} - 1) - \gamma t) \right),
\] (51)

where \( c \) is a constant of integration.

Using Equation (39) in Equation (30), we get the EoS parameter of the HDE:
\[
\omega_\Lambda = -1 + \frac{2n(\coth^2(\gamma t) - 1)(\alpha - n\beta)}{3n\beta + 3\alpha \coth^2(\gamma t) - 3n\beta \coth^2(\gamma t)}.
\] (52)

Figure 4 shows that the energy densities of matter and HDE are positive, decreasing the functions of cosmic time. These densities start with an infinite value at the beginning of cosmic time \( t \to 0 \) and approach zero at the end time \( t \to \infty \). Figure 5(a) indicates that the EoS parameter is a decreasing function with cosmic time for the values of the constant \( n > 1 \), which starts from the quintessence region \(-1 < \omega_\Lambda < -\frac{1}{3}\), it remains constant in this region for the initial time and approaches the value \( \omega_\Lambda = -1 \) (LCDM model) in the future. From Figure 5(b), the present values of the EoS parameter corresponding to \( n = 1.3, 1.4 \) and 1.5 are \( \omega_0 = -0.92, -0.94 \) and \(-0.98\), respectively. These values are in excellent agreement with the observations (Aghanim et al. 2020).
Let \( r \) be the coincidence parameter and defined as
\[
 r = \frac{\rho_\Lambda}{\rho_m}. 
\]
Hence, by using Equations (50) and (51), the coincidence parameter becomes:
\[
 r = \frac{\rho_\Lambda}{\rho_m} = \frac{3\gamma^2}{n^2} \left[ n\beta + \alpha \coth^2(\gamma t) - n\beta \coth^2(\gamma t) \right] c \exp\left[ -\frac{3}{n} (\ln(e^{2\gamma t} - 1) - \gamma t) \right] - n\beta \coth^2(\gamma t) + \alpha. 
\] (53)

Figure 6(a) indicates the behavior of the coincidence parameter \( r \) as a function of cosmic time \( t \). From the figure, the current value of the coincidence parameter, i.e., \( t_0 = 13.798 \) GYs is consistent with the current value extracted from the observation data (Aghanim et al. 2020). Further, it is useful to use yet another notation, the abundances, also called the density parameters, it represents the proportion of each component in the Universe. The total energy density parameter, i.e., \( \Omega = \Omega_m + \Omega_\Lambda \) takes three values: \( \Omega > 1 \), \( \Omega = 1 \), \( \Omega < 1 \) correspond to the open, flat and closed Universe, respectively. The matter density parameter \( \Omega_m \) and HDE density parameter \( \Omega_\Lambda \) are defined by:
\[
 \Omega_m = \frac{\rho_m}{3H^2} \quad \text{and} \quad \Omega_\Lambda = \frac{\rho_\Lambda}{3H^2}. 
\] (54)

Using Equations (39), (50), (51) and (54), we get the total density parameter as:
\[
 \Omega = \Omega_m + \Omega_\Lambda \\
= \frac{cn^2}{3\gamma^2} \exp\left[ -(3/n)(\ln(e^{2\gamma t} - 1) - \gamma t) \right] \coth^2(\gamma t) + n\beta \left( \frac{1}{\coth^2(\gamma t)} - 1 \right) + \alpha. 
\] (55)

Figure 6(b) represents the evolution of the total energy density parameter as a function of cosmic time \( t \), and it appears that its value is large in the first era of the Universe, while begin to approach \( \Omega \sim 1 \) in the last era of Universe, which causes our cosmological model to predict a flat Universe at a later time, as indicated by the recent astronomical observations.
4. Jerk parameter

As it is known in the literature, the jerk parameter is one of the fundamental physical quantities to describe the dynamics of the Universe. The jerk parameter is a dimensionless third derivative of the scale factor \( a \) with respect to cosmic time \( t \) and is defined as (Visser 2005; Rapetti et al. 2007):

\[
 j = \frac{\dddot{a}}{aH^3}. \tag{56}
\]

Equation (56) can be written in terms of the deceleration parameter \( q \) as:

\[
 j = q + 2q^2 - \frac{\dot{q}}{H}. \tag{57}
\]

Using Equations (39) and (45), the jerk parameter for our cosmological model is:

\[
 j = 1 + n(2n - 3)(\gamma t)^2. \tag{58}
\]

To study the behavior of the jerk parameter \( j \), it is better to express in terms of redshift \( z \):

\[
 j(z) = 1 + \frac{n(2n - 3)}{1 + 2.17391(n - 0.46)(1 + z)^{-2n}}. \tag{59}
\]

For the \( \Lambda \)CDM model, the value of the jerk parameter is \( j = 1 \). According to the \( \Lambda \)CDM model, the Universe shifts from the early deceleration phase to the current acceleration phase with a positive jerk parameter \( j_0 > 0 \) and a negative DP \( q_0 < 0 \). From Figure 7, it is clear that the jerk parameter remains positive in various cases for \( n > 1 \) and approaches 1 later. The current jerk parameter value \( j_0 \) is positive. Thus, for \( n = 1.3 \) and 1.4, our cosmological model can be expected to adopt the behavior of another DE model instead of the \( \Lambda \)CDM model, while for \( n = 1.5 \), our cosmological model is similar to the \( \Lambda \)CDM model.

5. Conclusions

In this work, we investigated a spatially homogeneous and anisotropic Bianchi type-I Universe with two fluids as the content of the Universe: matter and HDE in the framework of GR. We considered a scale factor as a hyperbolic function, specifically, \( a(t) = \sinh^{1/n}(\gamma t) \), where \( \gamma \) and \( n > 0 \) are arbitrary constants, which gives us a time-dependent deceleration parameter. Then, we derived the Einstein’s field equations for Bianchi type-I Universe. We found the exact solutions for our cosmological model. Further, to obtain the Universe moving from early decelerating phase to present accelerating phase, we choose the value of \( n > 1 \) (Nagpal et al. 2019). In addition, we have investigated the behavior of anisotropic parameter and deceleration parameter for the three values of model parameters of \( n \), i.e., \( n = 1.3 \), 1.4 and 1.5. The evolution of the deceleration parameter in Figure 1 indicates that our cosmological model contains a transition from the early anisotropic Universe to the current isotropic Universe as DE starts to dominate the energy density of the Universe and the deceleration parameter in Figure 2 shows a phase transition from early decelerating phase to current accelerating phase, which is in good concurrence with the results of recent observations. The values of the transition redshift corresponding to the three values of model parameter \( n \) are \( z_{tr} = 0.57, 0.63 \) and 0.75, respectively.

In addition, we have investigated the behavior of the energy densities of matter and HDE for the three values of model parameter \( n \). From Figure 4, we observed that the energy densities of matter and HDE are positive decreasing functions of cosmic time. They start with an infinite value at the beginning of cosmic time \( t \to 0 \) and approach zero at the end time \( t \to \infty \). These results are consistent with the expansion of the Universe. Additional results of our cosmological model show that under certain conditions, the ratio of the HDE density to the energy density of matter \( r \) increases with the expansion of the Universe, i.e., that the Universe moves from the era of the domination of matter to the era of the domination of DE. This is good for the problem of cosmic coincidence. Further, the EoS parameter presented in Figure 5 indicates that the two fluids of the model behaves like quintessence dark energy at present. The value of the EoS parameter at present epoch \( z = 0 \) for dark energy obtained by Planck collaboration is \( \omega_0 = -1.026 \pm 0.032 \) (Aghanim et al. 2020). In
our analysis, we found $\omega_0 = -0.92, -0.94$ and $-0.98$ corresponding to the three values of model parameter $n$, respectively, which is in good agreement with Planck measurements. Finally, the evolution of the total density parameter presented in Figure 6 indicates that our cosmological observations is close to 1 in the current time, which leads to a flat Universe. We also found that the jerk parameter is similar to $\Lambda$CDM model in the future.

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References

Aghanim N., Akrami Y., Ashdown M., et al. 2020, Astronomy & Astrophysics, 641, A6
Ahmed N., Pradhan A. 2014, International Journal of Theoretical Physics, 53, 289
Anderson L., Aubourg E., Bailey S., et al. 2012, Monthly Notices of the Royal Astronomical Society, 427, 3435
Blake C., Kazin E. A., Beutler F., et al. 2011, Monthly Notices of the Royal Astronomical Society, 418, 1707
Caldwell R. R. 2002, Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics, 545, 23
Capozziello S., Farooq O., Luongo O., Ratra B., et al. 2014, Physical Review D, 90, 044016
Capozziello S., Luongo O., Saridakis E. N., et al. 2015, Physical Review D, 91, 124037
Carroll S. M. 1998, Physical Review Letters 82, 896
Chawla C., Mishra R. K., Pradhan A., et al. 2012, The European Physical Journal Plus 127, 1
Chen S., Jing J. 2009, Physics Letters B, 679, 2
Chiba T., Okabe T., Yamaguchi M., et al. 2000, Physical Review D, 62, 8
Clocchiatti A., Schmidt B. P., Filippenko A. V., et al. 2006, The Astrophysical Journal, 642, 1
de Bernardis P., Ade P. A. R., Bock J. J., et al. 2000, Nature, 404, 955
de Laurentis M., Paolella M., Capozziello S., et al. 2015, Physical Review D, 91
Dubey V. C., Srivastava S., Sharma U. K., Pradhan A., et al. 2019, Pramana — Journal of Physics, 93, 1
Esmaeili F. M., Mishra B. 2018, Journal of Astrophysics & Astronomy, 39, 5
Farooq O., Madiyar F. R., Crandall S., Ratra B., et al. 2017, The Astrophysical Journal, 835, 26
Filippenko A. V., Riess A. G. 1998, Physics Reports, 307, 31
Fischler W., Susskind L. 1998, Holography and Cosmology, arXiv:hep-th/9806039
Granda L. N., Oliveros A. 2008, Physics Letters B, 669, 275
Hanany S., Ade P., Balbi A., et al. 2000, The Astrophysical Journal, 545, L5
Harko T., Lobo F. S. N., Nojiri S., Odintsov S. D., et al. 2011, Physical Review D, 84, 1
Hinshaw G., Larson D., Komatsu E., et al. 2013, Astrophysical Journal, Supplement Series, 208, 19
Hooft G’t. 1993, gr-qc/9310026
Kamenshchik A., Moschella U., Pasquier V., et al. 2001, Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics, 511, 265
Koussour M., Bennai M. 2021, International Journal of Geometric Methods in Modern Physics, 19, 03
Koussour M., Bennai M. 2022a, International Journal of Modern Physics A, 37
Koussour M., Bennai M. 2022b, Afrka Matematika, 33, 1
Li M. 2004, Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics, 603, 1
Linder E. V. 2010, Physical Review D, 81, 127301
MacCallum M. A. H. 1971, Communications in Mathematical Physics, 20, 57
Mamon A. A., Das S. 2017, The European Physical Journal C, 77, 1
Myrzakulov R. 2011, The European Physical Journal C, 71, 1
Nagpal R., Singh J. K., Beesham A., Shabani H., et al. 2019, Annals of Physics 405, 234
Padmanabhan N., Xu X., Eisenstein D. J., et al. 2012, Monthly Notices of the Royal Astronomical Society, 427, 2132
Padmanabhan T. 2002, Physical Review D, 66, 021301
Perlmutter S., Aldering G., della Valle M., et al. 1998, Nature, 391, 51
Perlmutter S., Aldering G., Goldhaber G., et al. 1999, The Astrophysical Journal, 517, 565
Pradhan A. 2014, Indian Journal of Physics, 88, 215
Pradhan A., Tiwari R. K., Beesham A., Zia R., et al. 2019, The European Physical Journal Plus, 134, 1
Rapetti D., Allen S. W., Amin M. A., Blandford R. D., et al. 2007, Monthly Notices of the Royal Astronomical Society, 375, 1510
Samanta G. C. 2013, International Journal of Theoretical Physics, 52, 4389
Sarkar S., Mahanta C. R. 2013, International Journal of Theoretical Physics, 52, 1482
Sarkar S. 2014, Astrophysics and Space Science, 349, 985
Sarkar S. 2016, International Journal of Theoretical Physics, 55, 481
Singh G. P., Lalke A. R. 2022, Indian Journal of Physics, 1
Tonry J. L., Schmidt B. P., Barris B., et al. 2003, The Astrophysical Journal, 594, 1
Turner M. S. 2002, A Spacetime Odyssey, 180
Visser M. 2005, General Relativity and Gravitation, 37, 1541
Xu L. 2009, Journal of Cosmology and Astroparticle Physics, 09, 016
Zlatev I., Wang L., Steinhardt P. J., et al. 1999, Physical Review Letters, 82, 896