THE DIMENSION OF SOME AFFINE DELIGNE-LUSZTIG VARIETIES

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ABSTRACT. We prove Rapoport’s dimension conjecture for affine Deligne-Lusztig varieties for $GL_n$ and superbasic $b$. From this case the general dimension formula for affine Deligne-Lusztig varieties for special maximal compact subgroups of split groups follows, as was shown in a recent paper by Görtz, Haines, Kottwitz, and Reuman.

1. INTRODUCTION

Let $k$ be a finite field with $q = p^r$ elements and let $\overline{k}$ be the algebraic closure. Let $F = k((t))$ and let $L = \overline{k}((t))$. Let $O_F$ and $O_L$ be the valuation rings. We denote by $\sigma : x \mapsto x^q$ the Frobenius of $\overline{k}$ over $k$ and also of $L$ over $F$.

Let $G$ be a split connected reductive group over $k$. Let $A$ be a split maximal torus of $G$ and $W$ the Weyl group of $A$ in $G$. For $\mu \in X_*(A)$ let $t^\mu$ be the image of $t \in G_m(F)$ under the homomorphism $\mu : G_m \to A$. Let $B$ be a Borel subgroup of $G$ containing $A$. We write $\mu_{\text{dom}}$ for the dominant element in the orbit of $\mu \in X_*(A)$ under the Weyl group of $A$ in $G$.

We recall the definitions of affine Deligne-Lusztig varieties from [Ra1], [GHKR]. Let $G$ be a split connected reductive group over $k$, compare [K1]. For nonempty affine Deligne-Lusztig varieties the dimension is given by the following formula. Note that there is a simple criterion by Kottwitz and Rapoport (see [KR]) to decide whether an affine Deligne-Lusztig variety is nonempty.

There is a canonical $J(F)$-action on $X_\mu(b)$.

Let $\rho$ be the half-sum of the positive roots of $G$. By $\text{rk}_F$ we denote the dimension of a maximal $F$-split subtorus. Let $\text{def}_G(b) = \text{rk}_F G - \text{rk}_F J$. Let $\nu \in X_*(A)_Q$ be the Newton point of $b$, compare [K1]. For nonempty affine Deligne-Lusztig varieties the dimension is given by the following formula. Note that there is a simple criterion by Kottwitz and Rapoport (see [KR]) to decide whether an affine Deligne-Lusztig variety is nonempty.

**Theorem 1.1.** Assume that $X_\mu(b)$ is nonempty. Then

$$\dim(X_\mu(b)) = \langle \rho, \mu - \nu \rangle - \frac{1}{2} \text{def}_G(b).$$

Rapoport conjectured this in [Ra2]. Conjecture 5.10 in a different form. For the reformulation compare [K2]. In [Re2], Reuman verifies the formula for some small groups and $b = 1$. For $G = GL_n$, minuscule $\mu$ and over $\mathbb{Q}_p$ rather than over a function field, the Deligne-Lusztig

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Then $b \sigma$ by $d$. We may therefore assume that all $\mu$.

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2. Notation and conventions

From now on we use the following notation: Let $G = GL_h$ and let $A$ be the diagonal torus. Let $B$ be the Borel subgroup of lower triangular matrices. For $\mu, \mu' \in X_*(A)_Q$ we say that $\mu \preceq \mu'$ if $\mu' - \mu$ is a non-negative linear combination of positive coroots. As we may identify $X_*(A)_Q$ with $Q^h$, this induces a partial ordering on the latter set. An element $\mu = (\mu_1, \ldots, \mu_h) \in X_*(A) \cong \mathbb{Z}^h$ is dominant if $\mu_1 \leq \cdots \leq \mu_h$.

Let $N = L^h$ and let $M_0 \subset N$ be the lattice generated by the standard basis $e_0, \ldots, e_{h-1}$. Then $K = GL_h(O_L) = \text{Stab}(M_0)$ and $g \mapsto gM_0$ defines a bijection

$$X_\mu(b)(K) \cong \{ M \subset N \text{ lattice } | \text{inv}(M, b \sigma(M)) = t^\mu \}.$$  

We define the volume of $M = gM_0 \subset X_\mu(b)$ to be $v_t(\det(g))$.

We assume $b$ to be superbasic. The Newton point $\nu \in X_*(A)_Q \cong Q^h$ of $b$ is then of the form $\nu = (\nu_1, \ldots, \nu_h) \in Q^h$ with $(m, h) = 1$. For $i \in \mathbb{Z}$ define $e_i$ by $e_{i+m} = te_i$. We choose $b$ to be the representative of its $\sigma$-conjugacy class that maps $e_i$ to $e_{i+m}$ for all $i$. For superbasic $b$, the condition that the affine Deligne-Lusztig variety is nonempty, namely $\nu \preceq \mu$, is equivalent to $\sum \mu_i = m$. From now on we assume this.

For each central $\alpha \in X_*(A)$ there is the trivial isomorphism

$$X_\mu(b) \to X_{\mu+\alpha}(t^\alpha b).$$

We may therefore assume that all $\mu_i$ are nonnegative. For the lattices in (2.1), this implies that $b \sigma(M) \subset M$.

In the following we will abbreviate the right hand side of the dimension formula for $X_\mu(b)$ by $d(b, \mu)$.

The set of connected components of $X$ is isomorphic to $\mathbb{Z}$, an isomorphism is given by mapping $g \in GL_h(L)$ to $v_t(\det(g))$. Let $X_\mu(b)^i$ be the intersection of the affine Deligne-Lusztig variety with the $i$-th connected component of $X$. Let $\pi \in GL_h(L)$ with $\pi(e_i) = e_{i+1}$ for all
$i \in \mathbb{Z}$. Then $\pi$ commutes with $b\sigma$, and defines isomorphisms $X_\mu(b)^i \rightarrow X_\mu(b)^{i+1}$ for all $i$. Thus it is enough to determine the dimension of $X_\mu(b)^0$.

For superbasic $b$, an element of $J(F)$ is determined by its value at $e_0$. More precisely, $J(F)$ is the multiplicative subgroup of a central simple algebra over $F$. Hence $\det g(b) = h - 1$. If $v_t(\det(g)) = j$ for some $g \in J(F)$, then $g$ induces isomorphisms between $X_\mu(b)^j$ and $X_\mu(b)^{j+1}$ for all $j$. On $X_\mu(b)^0$, we have an action of \{$g \in J(F) \mid v_t(\det(g)) = 0\} = J(F) \cap \text{Stab}(M_0)$.

**Remark 2.1.** To a vector $\psi = (\psi_i) \in \mathbb{Q}^h$ we associate the polygon in $\mathbb{R}^2$ that is the graph of the piecewise linear continuous function $f : [0, h] \rightarrow \mathbb{R}$ with $f(0) = 0$ and slope $\psi_i$ on $[i-1, i]$. One can easily see that $d(b, \mu)$ is equal to the number of lattice points below the polygon corresponding to $\nu$ and (strictly) above the polygon corresponding to $\mu$.

### 3. Extended semi-modules

In this section we describe the combinatorial invariants which are used to decompose $X_\mu(b)^0$.

**Definition 3.1.** Let $m$ and $h$ be coprime positive integers. A semi-module for $m, h$ is a subset $A \subset \mathbb{Z}$ that is bounded below and satisfies $m + A \subset A$ and $h + A \subset A$. Let $B = A \setminus (h + A)$. The semi-module is called normalized if $\sum_{a \in B} a = \frac{h(h-1)}{2}$.

(2) Let $\nu = (\frac{m_0}{h}, \ldots, \frac{m_n}{h}) \in \mathbb{Q}^h$. Let $\mu = (\mu'_1, \ldots, \mu'_n) \in \mathbb{N}^h$ not necessarily dominant with $\nu \leq \mu$. A semi-module $A$ for $m, h$ is of type $\mu'$ if the following condition holds: Let $b_0 = \min(b \in B)$ and let inductively $b_i = b_{i-1} + m - \mu'_i h \in \mathbb{Z}$ for $i = 1, \ldots, h$. Then $b_0 = b_h$ and $\{b_i \mid i = 0, \ldots, h - 1\} = B$.

**Remark 3.2.** Semi-modules are also used by de Jong and Oort in [JO] to define a stratification of the moduli space of $p$-divisible groups whose rational Dieudonné modules are simple of slope $m$. In this case $\mu$ is minuscule, and they use semi-modules for $m, h - m$ to decompose the moduli space.

**Lemma 3.3.** If $A$ is a semi-module, then its translate $-\sum_{a \in B} a \frac{h}{h} + h - \frac{1}{2} + A$ is the unique normalized translate of $A$. It is called the normalization of $A$. There is a bijection between the set of normalized semi-modules for $m, h$ and the set of possible types $\mu' \in \mathbb{N}^h$ with $\nu \leq \mu'$.

**Proof.** For the first assertion one only has to notice that the fact that the $h$ elements of $B$ are incongruent modulo $h$ implies that $\sum_{a \in B} a - \frac{h(h-1)}{2}$ is divisible by $h$. For the second assertion let $A$ be a normalized semi-module, let $b_0 = \min\{a \in B\}$ and let inductively $b_i = b_{i-1} + m - \mu'_i h$ where $\mu'_i$ is maximal with $b_i \in A$. Then $b_h = b_0$ and $\{b_i \mid i = 0, \ldots, h - 1\} = B$. From $b_0 < b_h$ for $i_0 = 1, \ldots, h - 1$ we obtain $\sum_{i=1}^{i_0} (m - \mu'_i h) > 0$ for all $i_0 < h$. Similarly, $b_0 = b_h$ implies $\sum_{i=1}^{h} \mu'_i = m$. This shows $\nu \leq \mu'$. As $m + A \subset A$, the $\mu'_i$ are nonnegative. Given $\mu'$ as above, the corresponding normalized semi-module $A$ can be constructed as follows: Let $b_0 = 0$, and inductively $b_i = b_{i-1} + m - \mu'_i h$. Then $A$ is the normalization of $\{b_i + ah \mid a \in \mathbb{N}, 0 \leq i < h\}$. \(\Box\)

**Definition 3.4.** Let $m$ and $h$ be as before and let $\mu = (\mu_i) \in \mathbb{N}^h$ be dominant with $\sum \mu_i = m$. An extended semi-module $(A, \varphi)$ for $\mu$ is a normalized semi-module $A$ for $m, h$ together with a function $\varphi : \mathbb{Z} \rightarrow \mathbb{N} \cup \{-\infty\}$ with the following properties:

1. $\varphi(a) = -\infty$ if and only if $a \notin A$.
2. $\varphi(a + h) \geq \varphi(a) + 1$ for all $a$.
3. $\varphi(a) \leq \max\{n \mid a + m - nh \in A\}$ for all $a \in A$. If $b \in A$ for all $b \geq a$, then the two sides are equal.
4. There is a decomposition of $A$ into a disjoint union of sequences $a_j^1, \ldots, a_j^h$ with $j \in \mathbb{N}$ and the following properties:
   a. $\varphi(a_{j+1}^1) = \varphi(a_{j}^1) + 1$
   b. If $\varphi(a_j^1 + h) = \varphi(a_j^1) + 1$, then $a_{j+1}^1 = a_j^1 + h$. Otherwise $a_{j+1}^1 > a_j^1 + h$. 


(c) The $h$-tuple $(\varphi(a_0^i))$ is a permutation of $\mu$.

An extended semi-module such that equality holds in (3) for all $a \in A$ is called cyclic.

Let $A$ be a normalized semi-module for $m$, $h$ and let $\mu'$ be its type. Let $\mu = \mu_{\text{dom}}$. Let $\varphi$ be such that (1) holds and that we have equality in (3) for all $a \in A$. Then in (2) the two sides are also equal for all $a \in A$. A decomposition of $A$ as in (4) is given by putting all elements into one sequence that are congruent modulo $h$. Hence $(A, \varphi)$ is a cyclic extended semi-module for $\mu$, called the cyclic extended semi-module associated to $A$.

**Example 3.5.** We give an explicit example of a non-cyclic extended semimodule for $m = 4$, $h = 5$, and $\mu = (0, 0, 1, 2, 1)$. Let $A$ be the normalized semi-module of type $(0, 0, 1, 2, 1)$. Then $B = A \setminus (5 + A)$ consists of $-2, -1, 2, 5, 6$. Let $\varphi(-1) = 0$ and $\varphi(a) = \max\{n \mid a + m - nh \in A\}$ if $a \in A \setminus \{-1\}$. See also Figure 1 that shows elements of $A$ marked by crosses and the corresponding values of $\varphi$. A decomposition of $A$ is given as follows: There are $h$ sequences as given by the elements of $A$ congruent to $-2, 2,$ and $5$ modulo $5$, respectively. The forth sequence is given by all elements congruent to $4$ modulo $5$ and greater than $-1$. The last sequence consists of the remaining elements $-1$ and $6, 11, 16, \ldots$.

\[
\begin{array}{cccccccc}
a & -3 & 2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
\varphi(a) & \cdot & \cdot & \cdots & -\infty & 0 & 0 & -\infty & -\infty & 0 & 1 & 2 & 1 & 1 & 2 & 2 & \cdots \\
\end{array}
\]

**Figure 1.** A non-cyclic extended semi-module

**Lemma 3.6.** If $(A, \varphi)$ is an extended semi-module for $\mu$, and if $\mu^0$ is the type of $A$, then $\mu^0_{\text{dom}} \preceq \mu$. If $\mu^0_{\text{dom}} = \mu$, then $(A, \varphi)$ is a cyclic extended semi-module.

**Proof.** Let $(A, \varphi_0)$ be the cyclic extended semi-module associated to $A$. Let 

\[
\{x_1, \ldots, x_n\} = \{a \in A \mid \varphi(a + h) > \varphi(a) + 1\}
\]

with $x_i > x_{i+1}$ for all $i$. For $i \in \{1, \ldots, n\}$ let

\[
\varphi_i(a) = \begin{cases} 
-\infty & \text{if } a \notin A \\
\varphi(a) & \text{if } a \geq x_i \\
\varphi_i(a + h) - 1 & \text{else.}
\end{cases}
\]

We show that $(A, \varphi_i)$ is an extended semi-module for some $\mu^i$ with $\mu^i_{\text{dom}} \preceq \mu^0_{\text{dom}}$ and $\mu^i_{\text{dom}} \neq \mu^0_{\text{dom}}$ for all $i \geq 1$. As $\varphi_0 = \varphi$, it then follows that $\mu^0_{\text{dom}} \preceq \mu_{\text{dom}} = \mu$ with equality if and only if $n = 0$, that is if $\varphi$ is cyclic.

The decomposition of $(A, \varphi_i)$ is defined as follows: For $a < x_i$, the successor of $a$ is $a + h$. Otherwise it is the successor from the decomposition of $(A, \varphi)$. From the properties of the decompositions for $\varphi_0$ and $\varphi$ one deduces that the decomposition satisfies the required properties. Let $n_i \geq 0$ be maximal with $x_i - n_i h \in A$ and let $\alpha_i = \varphi(x_i + h) - 1 - \varphi(x_i) > 0$. Thus $\varphi_i$ is obtained from $\varphi_{i-1}$ by subtracting $\alpha_i$ from the values at $x_i, x_i - h, \ldots, x_i - n_h h$. From $\mu^{i-1}$ we obtain $\mu^i$ by replacing the two entries $\varphi_{i-1}(x_i - n_i h) = \varphi_{i-1}(x_i) - n_i$ and $\varphi_{i-1}(x_i) - \alpha_i + 1$ (which is the value of $\varphi$ of the successor of $x_i$ in the sequence corresponding to $\varphi_i$) by $\varphi_{i-1}(x_i) - \alpha_i - n_i$ and $\varphi_{i-1}(x_i) + 1$. As $\varphi_{i-1}(x_i) - n_i, \varphi_{i-1}(x_i) - \alpha_i + 1 \in (\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i) + 1)$, we have $\mu^i_{\text{dom}} \preceq \mu_{\text{dom}}^i$ and $\mu^i_{\text{dom}} \neq \mu^0_{\text{dom}}$. \hfill \Box

**Corollary 3.7.** If $\mu$ is minuscule, then all extended semi-modules for $\mu$ are cyclic.
Proof. Let \((A, \varphi)\) be such an extended semi-module. Let \(\mu'\) be the type of \(A\). Then \(\mu'_{\text{dom}} \leq \mu\), thus \(\mu'_{\text{dom}} = \mu\). Hence the assertion follows from the preceding lemma. \(\square\)

Lemma 3.8. There are only finitely many extended semi-modules \((A, \varphi)\) for each \(\mu\).

Proof. Let \(\mu'\) be the type of the semi-module \(A\). As \(\mu'_{\text{dom}} \leq \mu\), there are only finitely many possible types and corresponding normalized semi-modules. For fixed \(A\), the third condition for extended semi-modules determines all but finitely many values of \(\varphi\). For the remaining values we have \(0 \leq \varphi(a) \leq \max\{n \mid a + m - nh \in A\}\). Thus for each \(A\) there are only finitely many possible functions \(\varphi\) such that \((A, \varphi)\) is an extended semi-module for \(\mu\). \(\square\)

4. The decomposition of the affine Deligne-Lusztig variety

Let \(M \in X_\mu(b)^0\) be a lattice in \(N\). In this section we associate to \(M\) an extended semi-module for \(\mu\). This leads to a paving of \(X_\mu(b)^0\) by finitely many locally closed subschemes. For minuscule \(\mu\), this decomposition of the set of lattices is the same as the one constructed by de Jong and Oort in [JO], compare also [N], Section 5.1.

Let \(m\) and \(h\) be as in Section 2. Let \(v \in N\) and recall that \(te_i = e_i + h\). Then we can write \(v = \sum_{i \in \mathbb{Z}} \alpha_i e_i\) with \(\alpha_i \in \mathbb{F}\) and \(\alpha_i = 0\) for small \(i\). Let
\[
I : N \setminus \{0\} \to \mathbb{Z}
\]
\[
v \mapsto \min\{i \mid \alpha_i \neq 0\}.
\]
For a lattice \(M \in X_\mu(b)^0\) we consider the set
\[
A = A(M) = \{I(v) \mid v \in M \setminus \{0\}\}.
\]
Then \(A(M)\) is bounded below and \(h + A(M) \subset A(M)\). As \(b\sigma(M) \subset M\), we have \(m + A(M) \subset A(M)\), thus \(A(M)\) is a semi-module for \(m, h\). We have
\[
\text{vol}(M) = |N \setminus (A \cap N)| - |A \setminus (N \cap A)| = 0.
\]
This implies that \(\sum_{a \in B} a = \sum_{i=0}^{h-1} i\), thus \(A\) is normalized.

Let further
\[
\varphi = \varphi(M) : \mathbb{Z} \to \mathbb{N} \cup \{-\infty\}
\]
\[
a \mapsto \begin{cases} 
\max\{n \mid \exists v \in M \text{ with } I(v) = a, t^{-n}b\sigma(v) \in M\} & \text{if } a \in A(M) \\
-\infty & \text{else}.
\end{cases}
\]
Note that by the definition of \(A(M)\), the set on the right hand side is nonempty. As \(b\sigma(M) \subset M\), the values of \(\varphi\) are indeed in \(\mathbb{N} \cup \{-\infty\}\).

Lemma 4.1. Let \(M \in X_\mu(b)^0\). Then \((A(M), \varphi(M))\) is an extended semi-module for \(\mu\).

Proof. We already saw that \(A(M)\) is a normalized semi-module. We have to check the conditions on \(\varphi\). The first condition holds by definition. Let \(v \in M\) with \(I(v) = a\) be realizing the maximum for \(\varphi(a)\). Then \(tv \in M\) with \(I(tv) = a + h\) implies that \(\varphi(a + h) \geq \varphi(a) + 1\), which shows (2). Let \(v \in M\) with \(I(v) = a\) and \(t^{-\varepsilon(a)}b\sigma(v) \in M\). Then \(I(t^{-\varepsilon(a)}b\sigma(v)) = a + m - \varphi(a)h \in A(M)\), whence the first part of (3). Let \(b \in A\) for all \(b \geq a\). Let \(n_0 = \max\{n \mid a + m - nh \in A\}\). Let \(v' \in M\) with \(I(v') = a + m - n_0 h\) and let \(v = (b\sigma)^{-1}(t^{n_0}v') \in N\). Then \(I(v) = a\), thus \(v = \sum_{b \geq a} \alpha_b e_b\) for some \(\alpha_b \in \mathbb{F}\). As \(b \in A\) for all \(b \geq a\), we also have \(e_b \in M\) for all such \(b\). Thus \(v \in M\) with \(t^{-n_0}b\sigma(v) = v' \in M\). Hence \(\varphi(a) = n_0\). It remains to show (4). For \(a \in \mathbb{Z}\) and \(\varphi_0 \in \mathbb{N}\) let
\[
\tilde{V}_{a, \varphi_0} = \{v \in M \mid v = 0 \text{ or } I(v) \geq a, t^{-\varphi_0}b\sigma(v) \in M\}
\]
and \(V_{a, \varphi_0} = \tilde{V}_{a, \varphi_0} / \tilde{V}_{a, \varphi_0 + 1}\). Then \(V_{a, \varphi_0}\) is a \(\mathbb{F}\)-vector space of dimension \(\{|a' \geq a \mid \varphi(a') = \varphi_0\}|\).

We construct the sequences by inductively sorting all elements \(a \in A\) with \(\varphi(a) \leq \varphi_0\) for some \(\varphi_0\): For \(\varphi_0 = \min\{\varphi(a) \mid a \in A\}\) we take each element \(a\) with this value of \(\varphi\) as the first element.
of a sequence. (At the end we will see that we did not construct more than \( h \) sequences.) We now describe the induction step from \( \varphi_0 \) to \( \varphi_0 + 1 \): If \( v_1, \ldots, v_j \) is a basis of \( V_{a,\varphi_0} \) for some \( a \), then the \( tv_j \) are linearly independent in \( V_{a+h,\varphi_0+1} \). Thus \( \dim V_{a,\varphi_0} \leq \dim V_{a+h,\varphi_0+1} \) for every \( a \). Hence there are enough elements \( a \in A \) with \( \varphi(a) = \varphi_0 + 1 \) to prolong all existing sequences such that conditions (a) and (b) are satisfied. We take the \( a \in A \) with \( \varphi(a) = \varphi_0 + 1 \) that are not already in some sequence as first elements of new sequences. Inductively, this constructs sequences with properties (a) and (b). To show (c), let \( a < b_0 \). Then

\[
| \{ i \mid \mu_i = n \} | = \dim \mathbb{F} V_{a,n} - \dim \mathbb{F} V_{a-h,n-1} = | \{ a_0' \mid \varphi(a_0') = n \} | .
\]

This also shows that we constructed exactly \( h \) sequences.

For each extended semi-module \((A,\varphi)\) for \( \mu \) let

\[
S_{A,\varphi} = \{ M \subset N \text{ lattice} \mid A(M) = A, \varphi(M) = \varphi \} \subset X.
\]

**Lemma 4.2.** The sets \( S_{A,\varphi} \) are contained in \( X_\mu(b)^0 \). They define a decomposition of \( X_\mu(b)^0 \)

\[
\text{into finitely many disjoint locally closed subschemes. Especially,} \quad \dim X_\mu(b)^0 = \max \{ \dim S_{A,\varphi} \}.
\]

**Proof.** The last property in the definition of an extended semi-module shows that \((A,\varphi)\) determines \( \mu \). Thus \( S_{A,\varphi} \subseteq X_\mu(b)^0 \). Using Lemma 4.3 and Lemma 4.1 it only remains to show that the subschemes are locally closed. The condition that \( a \in A(M) \) is equivalent to \( \dim(M \cap \langle e_a, e_{a+1}, \ldots \rangle)/(M \cap \langle e_{a+1}, e_{a+2}, \ldots \rangle) = 1 \). This is clearly locally closed. If \( a \) is sufficiently large, it is contained in all extended semi-modules for \( \mu \) and if \( a \) is sufficiently small, it is not contained in any extended semi-module for \( \mu \). Thus fixing \( A \) is an intersection of finitely many locally closed conditions on \( X_\mu(b)^0 \), hence locally closed. Similarly, it is enough to show that \( \varphi(a) < n \) for some \( a \in A \) and \( n \in \mathbb{N} \) is an open condition on \( \{ M \in X \mid b \sigma(M) \subset M, A(M) = A \} \subset X \). But this condition is equivalent to

\[
(\langle e_i \mid i \geq a \rangle \cap M \cap t^n(b \sigma)^{-1}(M))/\langle e_i \mid i \geq a + 1 \rangle = (0),
\]

which is an open condition. \( \square \)

Let \((A,\varphi)\) be an extended semi-module for \( \mu \). Let

(1.4) \[
\mathcal{V}(A,\varphi) = \{ (a, b) \in A \times A \mid b > a, \varphi(a) > \varphi(b) > \varphi(a-h) \}.
\]

**Theorem 4.3.** \((1)\) Let \( A \) and \( \varphi \) be as above. There exists a nonempty open subscheme \( U(A,\varphi) \subseteq \mathbb{A}^{\mathcal{V}(A,\varphi)} \) and a morphism \( U(A,\varphi) \rightarrow S_{A,\varphi} \) that induces a bijection between the set of \( \mathbb{F} \)-valued points of \( U(A,\varphi) \) and \( S_{A,\varphi} \). Especially, \( \dim(S_{A,\varphi}) = | \mathcal{V}(A,\varphi) | \).

(2) If \((A,\varphi)\) is a cyclic extended semi-module, then \( U(A,\varphi) = \mathbb{A}^{\mathcal{V}(A,\varphi)} \).

**Proof.** We denote the coordinates of a point \( x \) of \( \mathbb{A}^{\mathcal{V}(A,\varphi)} \) by \( x_{a,b} \) with \((a, b) \in \mathcal{V}(A,\varphi)\). To define a morphism \( \mathbb{A}^{\mathcal{V}(A,\varphi)} \rightarrow X \), we describe the image \( M(x) \) of a point \( x \in \mathbb{A}^{\mathcal{V}(A,\varphi)}(R) \) where \( R \) is a \( \mathbb{F} \)-algebra. For each \( a \in A \) we define an element \( v(a) \in N_R = N \otimes_\mathbb{F} R \) of the form \( v(a) = \sum_{b \geq a} \alpha_b e_b \) with \( \alpha_a = 1 \). The \( R[[t]] \)-module \( M(x) \subset N_R \) will then be generated by the \( v(a) \). We want the \( v(a) \) to satisfy the following relations: For \( a \in h + A \) we want

(4.2) \[
v(a) = tv(a-h) + \sum_{(a,b) \in \mathcal{V}(A,\varphi)} x_{a,b} v(b).
\]

Let \( y = \max \{ b \in B \} \). If \( a = y \) we want

(4.3) \[
v(a) = e_a + \sum_{(a,b) \in \mathcal{V}(A,\varphi)} x_{a,b} v(b).
\]
For all other elements $a \in B$, we want the following equation to hold: Let $a' \in A$ be minimal with $a' + m - \varphi(a')h = a$. Then $v' = t^{-\varphi(a')b}\sigma(v(a')) \in N_R$ with $I(v') = a$. Let

$$v(a) = v' + \sum_{(a,b) \in V(A,\varphi)} x_{a,b}v(b).$$

Claim 1. For every $x \in A^V(A,\varphi)(R)$ there are uniquely determined $v(a) \in N_R$ for all $a \in A$ satisfying (1.3) to (1.4).

We set

$$v(a) = \sum_{j \in \mathbb{N}} \alpha_{a,j}c_{a,j}$$

with $\alpha_{a,j} \in R$ and $\alpha_{a,0} = 1$ for all $a$. We solve the equations by induction on $j$. Assume that the $\alpha_{a,j}$ are determined for $j \leq j_0$ and such that the equations for $v(a)$ hold up to summands of the form $\beta_{a,j}$ with $j > a + j_0$. To determine the $\alpha_{a,j_0+1}$, we write $a = y + im \ (\text{mod } h)$ and proceed by induction on $i \in \{0, \ldots, h-1\}$. For $i = 0$ and $a = y$, the coefficient $\alpha_{a,j_0+1}$ is the uniquely determined element such that (1.3) holds up to summands of the form $\beta_{a,j}$ with $j > j_0 + 1$. Note that by induction on $j$ and as $b > a$, the coefficient of $e_{y+j_0+1}$ on the right hand side of the equation is determined. For $a = y + nh$ with $n > 0$, the coefficients are similarly defined by (1.3). For $i > 0$ and $a \in A$ minimal in this congruence class, the coefficient is determined by (1.4). Here, the coefficient of $e_{a+j_0+1}$ on the right hand side of each equation is determined by induction on $i$ and $j$. For larger $a$ in this congruence class we use again (1.2).

By passing to the limit on $j$, we obtain the uniquely defined $v(a) \in N_R$ solving the equations.

Claim 2. Let $M(x) = \{v(a) \mid a \in A\}R[[y]]$. Then at each specialization of $x$ to a $K$-valued point $y$ we have $A = A(M(y))$ and $\varphi(M(y))(a) \geq \varphi(a)$ for all $a$.

From the definition of $M$ we immediately obtain $A \subseteq A(M(y))$. To show equality consider an element $v = \sum_a \alpha_a v(a) \in M(y) = M$. Write $v = \sum_{i\in\mathbb{Z}} b_i e_i$ with $b_i \in \bar{K}$. Let $i_0 = \min\{I(\alpha_a v(a))\}$. If $b_{i_0} \neq 0$, then $I(v) = i_0 \in A$. Otherwise we consider $\sum_{\{a \in \mathbb{Z} I(\alpha_a v(a)) = i_0\}} \alpha_a v(a)$. Note that $I(v(a)) = i_0 \ (\text{mod } h)$ for all $a$ occurring in the sum. Then (1.2) shows that this sum can be written as a sum of $v(b)$ with $b > i_0$. Thus we may replace $i_0$ by a larger number. As $i \in A$ for all sufficiently large $a$, this shows that $I(v) \in A$, so $A(M(y)) = A$.

Let $x \in A^V(A,\varphi)(R)$ and let $M = M(x)$. We show that $t^{-\varphi(a)}b\sigma(v(a)) \in M$ for all $a$. This means that $\varphi(M)(a) \geq \varphi(a)$ for all $a$. Consider the elements $a' \in A$ that are minimal with $a' + m - \varphi(a')h = a$ for some $a \in B \setminus \{y\}$. For these elements, the assertion follows from (1.4).

If $a$ is minimal with $a + m - \varphi(a)h = y$, then $I(t^{-\varphi(a)}b\sigma(v(a))) = y$. As all $e_i$ with $i \geq y$ are in $M$, this element is also contained in $M$. If $\varphi(a) = \varphi(a-h) + 1$ then $v(a) = tv(a-h)$ and the assertion holds for $a - h$ if and only if it holds for $h$. From this, we obtain the claim for all $a \in A$ with $\varphi(a) = \max\{n\mid a + m - nh \in A\}$. Especially, it follows for all sufficiently large elements of $A$. It remains to prove the claim for the finitely many elements $a \in A$ with $\max\{n\mid a + m - nh \in A\} > \varphi(a)$. We use decreasing induction on $a$: Let $a$ be in this set, and assume that we know the assertion for all $a' > a$. From (1.2) we obtain that

$$t^{-\varphi(a)}b\sigma(v(a)) = t^{-\varphi(a)-1}b\sigma(tv(a)) = t^{-\varphi(a)-1}b\sigma(v(a+h)) - \sum_{b > a+h, \varphi(b) \geq \varphi(a)+1} x_{a+h,b}v(b)).$$

By induction, the right hand side is in $M$ and Claim 2 is shown.

As all $\mu_i$ are nonnegative, we constructed a morphism from $A^V(A,\varphi)$ to the subscheme $X_A$ of $X$ defined by $X_A(K) = \{M \mid A(M) = A, b\sigma(M) \subseteq M\}$.

Claim 3. There is a nonempty open subscheme $U(A, \varphi)$ of $A^V(A,\varphi)$ that is mapped to $S_{A,\varphi}$, if $(A, \varphi)$ is cyclic, then $U(A, \varphi) = A^V(A,\varphi)$.

In general we do not have $\varphi(M)(a) = \varphi(a)$ for all $a$. The proof of Lemma (1.2) shows that $\varphi(M)(a) \leq \varphi(a)$ is an open condition on $X_A$, and thus on $A^V(A,\varphi)$. Let $U(A, \varphi)$ be
the corresponding open subscheme, which is then mapped to $S_{A,\varphi}$. We have to show that it is nonempty, thus to construct a point in $\mathbb{A}^V(A,\varphi)$ where the corresponding function $\varphi(M)$ is equal to $\varphi$. If $\varphi(a) = \max\{n \mid a + m - nh \in A\}$, then $\varphi(M)(a) = \varphi(a)$. Especially, the two functions are equal for all $a$ if $(A, \varphi)$ is cyclic. In this case $U(A, \varphi) = \mathbb{A}^V(A,\varphi)$. If $\varphi(a) + 1 = \varphi(a + h)$ and if $\varphi(M)(a + h) = \varphi(a + h)$, then $\varphi(M)(a + h) - 1 \geq \varphi(M)(a) \geq \varphi(a)$ implies that $\varphi(M)(a) = \varphi(a)$. Thus it is enough to find a point where $\varphi(M)(a) = \varphi(a)$ for all $a \in A$ with $\varphi(a + h) > \varphi(a) + 1$. For each such $a$ let $b_a$ be the successor in a decomposition of $(A, \varphi)$ into sequences. Then $(a + h, b_a) \in \mathcal{V}(A, \varphi)$. Let $x_{a+h,b_a} = 1$ for these pairs and choose all other coefficients to be 0. Then for this point and $a$ as before we have that $\varphi(M)(a) = \varphi(b_a) - 1 = \varphi(a)$. Thus $U(A, \varphi)$ is nonempty.

Claim 4. The map $U(A, \varphi) \to S_{A,\varphi}$ defines a bijection on $\mathbb{k}$-valued points.

More precisely, we have to show that for each $M \in S_{A,\varphi}$ there is exactly one $x \in U(A, \varphi)(\mathbb{k})$ such that $M$ contains a set of elements $v(a)$ for $a \in A$ with $I(v(a)) = a$ and satisfying \eqref{equation:12} to \eqref{equation:4} for this $x$. The argument is similar as the construction of $v(a)$ for given $x$. By induction on $j$ we will show the following assertion: There exist $x^j = (x^j_{a,b}) \in U(A, \varphi)(\mathbb{k})$ and $v_j(a) \in M$ for all $a$ with $t^{-\varphi(a)} b \sigma(v_j(a)) \in M$ and which satisfy equations \eqref{equation:12} to \eqref{equation:4} for $x^j$ up to summands of the form $\beta_n e_n$ with $n > a + j$. Furthermore the $x^j_{a,b}$ with $b - a \leq j$ and the coefficients of $e_n$ in $v_j(a)$ for $n \leq a + j$ will be chosen independently of $j$ and only depending on $M$.

For $j = 0$ choose any $x^0 = U(A, \varphi)(\mathbb{k})$ and $v_0(a) \in M$ with $I(v_0(a)) = a$, first coefficient 1 and $t^{-\varphi(a)} b \sigma(v_0(a)) \in M$. The existence of these $v_0(a)$ follows from $M \in X_\mu(b)$. Assume that the assertion is true for some $j_0$. For $n \leq j_0$ let $x^{j_0+1}_{a,a+n} = x^{j_0}_{a,a+n}$. We proceed again by induction on $i$ to define the coefficients for $a \equiv y + im \pmod{h}$. Let $a = y$. Choose the coefficients $x^{j_0+1}_{a,y+n}$ with $n > j_0$ such that

$$v_{j_0+1}(y) = e_y + \sum_{(y,y+n) \in \mathcal{V}(A,\varphi)} x^{j_0+1}_{y,y+n} v_{j_0}(y+n)$$

satisfies $t^{-\varphi(y)} b \sigma(v_{j_0+1}(y)) \in M$. The definition of $\varphi = \varphi(M)$ shows that such coefficients exist and from $\varphi(y+n) < \varphi(y)$ it follows that they are unique. For the other elements $v(a)$ we proceed similarly: For those with $a - h \notin A$ we use equation \eqref{equation:4}, on the right hand side with the values from the induction hypothesis, to define the new $v_{j_0+1}(a)$. For $a \in h + A$ we use \eqref{equation:13}. As we know that $t^{-\varphi(a-h)} b \sigma(tv_{j_0}(a - h)) \in M$, it is sufficient to consider the $b > a$ with $\varphi(a - h) < \varphi(b) < \varphi(a)$. At each step the coefficient of $e_{a+j_0+1}$ of the right hand side is already defined by the induction hypothesis. It only depends on the $x^{j_0}_{a,a+n}$ and the coefficients of $e_{b+n}$ of $v_{j_0}(b)$ with $n \leq j_0$, hence only on $M$. The coefficients of $x^{j_0+1}$ are given by requiring that $t^{-\varphi(a)} b \sigma(v_{j_0+1}(a)) \in M$.

5. Combinatorics

In this section we estimate $|\mathcal{V}(A,\varphi)|$ to determine the dimension of the affine Deligne-Lusztig variety $X_\mu(b)$.

Remark 5.1. For cyclic extended semi-modules we have $\varphi(a+h) = \varphi(a) + 1$ for all $a \in A$. Thus

$$\mathcal{V}(A,\varphi) = \{(b_i, b) \mid b_i \in B, b \in A, b > b_i, \varphi(b) < \varphi(b_i)\}$$

where $B = A \setminus (h + A)$.

Proposition 5.2. Let $(A, \varphi)$ be the cyclic extended semi-module associated to the normalized semi-module $A$ of type $\mu$. Then $|\mathcal{V}(A,\varphi)| = d(b, \mu)$. 

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Proof. Recall that by \( b_0 \) we denote the minimal element of \( A \) or \( B \). Let \( b_i \) be as in the definition of the type of \( A \) and let \( b_h = b_0 \). First we show that

\[
\mathcal{V}(A, \varphi) \to \mathbb{Z}
\]

\[
(b_i, b_h) \mapsto b - b_i + b_h
\]

induces a bijection between \( \mathcal{V}(A, \varphi) \) and \( \{ a \in A \mid a > b_h \} \). Let \( b \in A \) for some \( b > b_i \). Then \( b - b_i + b_{i+1} \notin A \) if and only if \( (b_i, b) \in \mathcal{V}(A, \varphi) \). Let \( b_{i_0} = \max\{b_i \in B \} \). We have \( b \in A \) for all \( b > b_{i_0} \). Thus for every \( b > b_h \) with \( b \notin A \), there is an element \( (b_i, b - b_h + b_i) \in \mathcal{V}(A, \varphi) \) for some \( h > i \geq i_0 \). Hence \( \{ a \notin A \mid a > b_h \} \) is in the image of the map. To show that it is injective and that its image is contained in \( \{ a \notin A \mid a > b_h \} \), it is enough to show that \( (b_i, b) \in \mathcal{V}(A, \varphi) \) implies that \( b - b_i + b_j \notin A \) for all \( j \in \{i+1, \ldots, h\} \). Indeed, this ensures that \( (b_j, b - b_i + b_j) \notin \mathcal{V}(A, \varphi) \) for all such \( j \) and that \( b - b_i + b_h \notin A \). We write \( b = b_i + ah \) for some \( l \) and \( \alpha \). Recall that \( \varphi(b_i) = \mu_{i+1} \). As \( b_i, b_i + ah \in \mathcal{V}(A, \varphi) \), we have \( \mu_{i+1} + \alpha < \mu_{i+1} \).

Especially, \( l < i \). This implies \( \mu_{i+1} + \cdots + \mu_{i+\beta} + \alpha < \mu_{i+1} + \cdots + \mu_{i+\beta} \) for all \( \beta \leq h-i \). Using the recurrence for the \( b_j \), one sees that this implies \( b - b_i + b_{i+\beta} \notin A \) for all \( \beta < h-i \).

It remains to count the elements of \( \{ a \notin A \mid a > b_0 \} \). As \( h + A \subseteq A \), we have

\[
| \{a \notin A \mid a > b_0 \} | = \left( \sum_{i=0}^{h-1} b_i - b_0 - i \right) \cdot \frac{1}{h}
\]

From the construction of \( A \) from its type we obtain

\[
\left( \sum_{i=0}^{h-1} \sum_{j=1}^{i} (m - \mu_j h) - i \right) \cdot \frac{1}{h}
\]

\[
= \left( \sum_{i=0}^{h-1} \sum_{j=1}^{i} \frac{m}{h} - \mu_j \right) - \frac{h-1}{2}
\]

\[
= d(b, \mu).
\]

\[ \square \]

Theorem 5.3. Let \( (A, \varphi) \) be an extended semi-module for \( \mu \). Then \( |\mathcal{V}(A, \varphi)| \leq d(b, \mu) \).

Proof of Theorem 5.3 for cyclic extended semi-modules. We write \( B = \{b_0, \ldots, b_{h-1}\} \) as in the definition of the type \( \mu' \) of \( A \). As the extended semi-module is assumed to be cyclic, \( \mu' \) is a permutation of \( \mu \). Using Remark 5.1 we see

\[
|\mathcal{V}(A, \varphi)| = |\{(b_i, a) \in B \times A \mid a > b_i, \varphi(a) < \varphi(b_i)\}|
\]

\[
= \sum_{(b_i, b_j) \in B \times B \mid b_j > b_i, \mu'_j + 1 < \mu'_{i+1}} \mu'_{i+1} - \mu'_{j+1} + \sum_{(b_i, b_j + ah) \mid b_j < b_i < b_j + ah, \mu'_j + 1 > \mu'_{j+1} + \alpha}
\]

We refer to these two summands as \( S_1 \) and \( S_2 \).

Let \( (\tilde{b}_0, \tilde{\mu}_1), \ldots, (\tilde{b}_{h-1}, \tilde{\mu}_h) \) be the set of pairs \( (b_0, \mu'_1), \ldots, (b_{h-1}, \mu'_h) \), but ordered by the size of \( b_i \). That is, \( b_i < b_{i+1} \) for all \( i \). Let

\[
f : B \to B
\]

\[
b_i \mapsto b_{i+1} = b_i + m - \mu'_{i+1} h
\]

where we identify \( b_h \) with \( b_0 \). This defines a permutation of \( B \). From the ordering of the \( \tilde{b}_i \) we obtain \( \sum_{i=0}^{h_0} f(\tilde{b}_i) \geq \sum_{i=0}^{i_0} \tilde{b}_i \) for all \( i_0 \). As \( f(\tilde{b}_i) = \tilde{b}_i + m - \tilde{\mu}_{i+1} h \), this is equivalent to \( \sum_{i=0}^{h+1} \tilde{\mu}_i \leq (i_0 + 1)\frac{m}{h} \) for all \( i_0 \). We thus have \( \nu \leq \tilde{\mu} \leq \mu \).
Recall the interpretation of $d(b, \mu)$ from Remark 2.4. We show that $S_1$ is equal to the number of lattice points above $\mu$ and on or below $\tilde{\mu}$. The second summand $S_2$ will be less or equal to the number of lattice points above $\tilde{\mu}$ and below $\nu$. Then the theorem follows for cyclic extended semi-modules.

We have $S_1 = \sum_{i<j} \max\{\tilde{\mu}_{i+1} - \tilde{\mu}_{j+1}, 0\}$. Consider this sum for any permutation $\tilde{\mu}$ of $\mu$. If we interchange two entries $\tilde{\mu}_i$ and $\tilde{\mu}_{i+1}$ with $\tilde{\mu}_i > \tilde{\mu}_{i+1}$, the sum is lessened by the difference of these two values. There are also exactly $\tilde{\mu}_i - \tilde{\mu}_{i+1}$ lattice points on or below $\tilde{\mu}_i$ and above the polygon corresponding to the permuted vector. If $\tilde{\mu} = \mu$, both $S_1$ and the number of lattice points above $\mu$ and on or below $\tilde{\mu}$ are 0. Thus by induction $S_1$ is equal to the claimed number of lattice points.

The last step is to estimate $S_2$. It is enough to construct a decreasing sequence (with respect to $\leq$) of $\psi^i \in \mathbb{Q}^h$ for $i = 0, \ldots, h - 1$ with $\psi^0 = \tilde{\mu}$ and $\psi^{h-1} = \nu$ such that the number of lattice points above $\psi^i$ and on or below $\psi^{i+1}$ is greater or equal to the number of pairs $(\tilde{b}_{i+1}, \tilde{b}_j + ah)$ contributing to $S_2$. Note that $\psi^i$ will no longer be lattice polygons. Let $f_i : B \to B$ be defined as follows: For $j > i$ let $f_i(b_j) = f(b_j)$. Let $\{f_i(b_j) \mid 0 \leq j \leq i\}$ be the set of $f(b_j)$, but sorted increasingly. Let $\psi^i = (\psi^i_j)$ be such that $f_i(b_j) = b_j + m - \psi^i_{j+1}h$, i.e.

$$
\psi^i_{j+1} = \frac{b_j + m - f_i(b_j)}{h} = \frac{m}{h} - \frac{f_i(b_j) - b_j}{h}.
$$

Similarly as for $\nu \leq \tilde{\mu}$ one can show that

$$
\nu \leq \psi^{i+1} \leq \psi^i \leq \tilde{\mu}
$$

for all $i$. As $f_0 = f$ and $f_{h-1} = id$, we have $\psi^0 = \tilde{\mu}$ and $\psi^{h-1} = \nu$. It remains to count the lattice points between $\psi^i$ and $\psi^{i+1}$. To pass from $f_i$ to $f_{i+1}$ we have to interchange the value $f(\tilde{b}_{i+1})$ with all larger $f_i(b_j)$ with $j \leq i$. Thus to pass from the polygon associated to $\psi^i$ to the polygon of $\psi^{i+1}$ we have to change the value at $j$ by $(f_i(b_j) - f(\tilde{b}_{i+1}))/h$, and that for all $j \leq i$ with $f_i(b_j) > f(\tilde{b}_{i+1})$. Thus there are at least

$$
\sum_{j \leq i, f_i(b_j) > f(\tilde{b}_{i+1})} \left| \frac{f_i(b_j) - f(\tilde{b}_{i+1})}{h} \right| = \sum_{j \leq i, f(b_j) > f(\tilde{b}_{i+1})} \left| \frac{f(b_j) - f(\tilde{b}_{i+1})}{h} \right|
$$

lattice points above $\psi^i$ and on or below $\psi^{i+1}$. For fixed $i$ and $j < i + 1$, the set of pairs $(\tilde{b}_{i+1}, \tilde{b}_j + ah)$ contributing to $S_2$ is in bijection with $\{a \geq 1 \mid f(\tilde{b}_j) - ah > f(\tilde{b}_{i+1})\}$. The cardinality of this set is at most $\frac{f(\tilde{b}_{i+1}) - f(\tilde{b}_j + ah)}{h}$ which proves that $S_2$ is not greater than the number of lattice points between $\tilde{\mu}$ and $\nu$.

Example 5.4. We give an example of a cyclic semi-module $(A, \varphi)$ where the type of $A$ is not dominant but where $|V(A, \varphi)| = d(b, \mu)$. Let $m = 4$, $h = 5$, and $\mu = (0, 0, 1, 1, 2)$. Let $(A, \varphi)$ be the cyclic extended semi-module associated to the normalized semi-module of type $(0, 0, 1, 2, 1)$. Note that $A$ is the same semi-module as in Example 3.5. Then the dimension of the corresponding subscheme is

$$
|V(A, \varphi)| = |\{(-1, 2), (5, 6), (5, 7)\}| = d(b, \mu).
$$

**Proof of Theorem 5.3.** Let $(A, \varphi)$ be an extended semi-module for $\mu$. Let $\varphi_1$ and $\mu^i$ be the sequences constructed in the proof of Lemma 3.5. By induction on $i$ we show that $|V(A, \varphi_1)| \leq d(b, \mu^i)$. For $i = 0$, the extended semi-module $(A, \varphi_0)$ is cyclic, hence the assertion is already shown.
We use the notation of the proof of Lemma 3.6. The description of the difference between $\mu^i$ and $\mu^{i-1}$ given there shows that

$$d(b, \mu^i) - d(b, \mu^{i-1}) = \sum_{l=1}^{h} \sum_{j=1}^{l} (\mu^{l-1}_{\text{dom}, j} - \mu^i_{\text{dom}, j})$$

$$= \left( \{ \{ \mu^{l-1}_{\text{dom}, j} \in (\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i) + 1) \} \mid -1 \right) \cdot \min \{\alpha_i, n_i + 1\}.$$

We denote this difference by $\Delta$. To show that $|\mathcal{V}(A, \varphi_i)| - |\mathcal{V}(A, \varphi_{i-1})| \leq \Delta$ we use the decomposition into sequences $a^j_i$ of the extended semi-module $(A, \varphi_{i-1})$. Using the definition of $\mathcal{V}(A, \varphi)$ and the description of the difference between $\varphi_i$ and $\varphi_{i-1}$ from the proof of Lemma 3.6, one obtains

$$|\mathcal{V}(a, \varphi_i)| - |\mathcal{V}(a, \varphi_{i-1})| = S_1 + S_2 + S_3$$

where

$$S_1 = \{ (x_i + h, b) \mid b \in A, b > x_i + h, \varphi_{i-1}(x_i) + 1 > \varphi_{i-1}(b) > \varphi_{i-1}(x_i) - \alpha_i \}$$

$$S_2 = \{ (b, x_i - \delta h) \mid b > x_i - \delta h, \delta \in \{0, \ldots, n_i\}, \varphi_{i-1}(x_i) - \delta - \alpha_i < \varphi_{i-1}(b) \leq \varphi_{i-1}(x_i) - \delta \}$$

$$S_3 = \{ (x_i - n_i h, b) \mid b > x_i - n_i h, \varphi_{i-1}(x_i) - n_i > \varphi_{i-1}(b) \geq \varphi_{i-1}(x_i) - n_i - \alpha_i \}.$$

Here we used that $a \leq x_i$ implies that $\varphi_{i-1}(a + h) = \varphi_{i-1}(a) + 1$. For each sequence $a^j_i$ of the extended semi-module $(A, \varphi_{i-1})$ we use $S_{1,j}$, $S_{2,j}$, and $S_{3,j}$ for the contributions of pairs with $b \in \{a^j_i\}$ to the three summands. Furthermore we write $S^j = S_{1,j} + S_{2,j} + S_{3,j}$. We show the following assertions: If $\varphi_{i-1}(a^0_i) \in (\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i) + 1)$ or if $a^0_i = x_i - n_i h$, then $S^j = 0$. Otherwise, $S^j \leq \min \{\alpha_i, n_i + 1\}$. Then the theorem follows from property (4c) of extended semi-modules.

To determine the $S^j$, we consider the following cases:

**Case 1:** $\varphi_{i-1}(a^0_i) \geq \varphi_{i-1}(x_i) + 1$. In this case it is easy to see that $S_{1,j} = S_{2,j} = S_{3,j} = 0$.

**Case 2:** $a^0_i > x_i$. This implies that $S_{2,j} = 0$. If $\varphi_{i-1}(a^0_i) \leq \varphi_{i-1}(x_i) - n_i - \alpha_i$, then $S_{1,j} + S_{3,j} = \alpha_i - \alpha_i = 0$. Let now $\varphi_{i-1}(a^0_i) \in (\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i) + 1)$. Then

$$S_{1,j} + S_{3,j} \leq \{ a^j_i \mid \varphi_{i-1}(x_i) + 1 > \varphi_{i-1}(a^j_i) \geq \max \{\varphi_{i-1}(x_i) - \alpha_i + 1, \varphi_{i-1}(x_i) - n_i \} \}.$$

As $\varphi_{i-1}(a^j_{i+1}) = \varphi_{i-1}(a^j_i) + 1$ for all $j$, the right hand side is less or equal to $\min \{\alpha_i, n_i + 1\}$.

**Case 3:** $a^0_i = x_i - n_i h$. This sequence starts with $x_i - n_i h, \ldots, x_i, x_i + h$. (Recall that the sequences $\{a^j_i\}$ for $\varphi_{i-1}$ are of this easy form with stepwidth $h$ as long as $a^j_i \leq x_i < x_i - 1$.) Note that within such a sequence $a^j_i > a^j_j$ implies $\varphi_{i-1}(a^j_i) > \varphi_{i-1}(a^j_j)$. Hence this special sequence does not make any contribution, as in $S^j$ we only consider pairs where both elements are in the sequence starting with $x_i - n_i h$.

**Case 4:** $a^0_i < x_i$, but not congruent to $x_i$ modulo $h$. Again $a^j_{i+1} = a^j_i + h$ if $a^j_i \leq x_i$. We first assume that $\varphi_{i-1}(a^0_i) \leq \varphi_{i-1}(x_i) - n_i - \alpha_i$. Then $S_{2,j} = 0$. Assume that $b = a^j_i$ contributes to $S_{1,j}$. Then $j \geq n_i + 1$ and $a^j_i > x_i + h$. If $a^0_i < x_i - n_i h$, then $|x_i - n_i h, x_i + h|$ contains $n_i + 1$ elements of the sequence. Thus in all cases $a^j_{i-n_i-1} > x_i - n_i h$. This element then leads to a contribution to $S_{3,i}$, as $\varphi_{i-1}(a^0_i) = \varphi_{i-1}(a^j_i) - n_i - 1$. In the other direction, if $a^j_i$ contributes to $S_{3,i}$, then $a^j_{i+n_i+1}$ contributes to $S_{1,j}$. Thus $S^j = 0$. We now assume that $\varphi_{i-1}(a^0_i) \in (\varphi_{i-1}(x_i) - \alpha_i - n_i, \varphi_{i-1}(x_i) + 1)$. Let $n$ be maximal with $a^0_n = a^0_n + nh < x_i$. Then
we have
\[ S_{1,l} = \{ a_j^l \mid j \geq n + 1, \varphi_{i-1}(x_i) \geq \varphi_{i-1}(a_0^l) + j > \varphi_{i-1}(x_i) - \alpha_i \} \]
\[ S_{2,l} = \{ a_j^l \mid 0 \leq j \leq \min(n, n_i), \varphi_{i-1}(x_i) \geq \varphi_{i-1}(a_0^l) + j > \varphi_{i-1}(x_i) - \alpha_i \} \]
\[ S_{3,l} = \{ a_j^l \mid j \geq \max(n - n_i + 1, 0), \varphi_{i-1}(x_i) - n_i > \varphi_{i-1}(a_0^l) + j \} \]
\[ = \{ a_j^l \mid j > \max(n + 1, n_i), \varphi_{i-1}(x_i) \geq \varphi_{i-1}(a_0^l) + j \} . \]
Thus
\[ S_l \leq S_{1,l} + S_{2,l} \leq \{ j \mid \varphi_{i-1}(x_i) \geq \varphi_{i-1}(a_0^l) + j > \varphi_{i-1}(x_i) - \alpha_i \} = \alpha_i. \]

If \( n + 1 \geq n_i \), then \( S_{1,l} + S_{3,l} \leq 0 \). Thus \( S_l \leq S_{2,l} \leq n_i + 1 \). If \( n_i > n + 1 \) then \( S_{1,l} + S_{3,l} \leq n_i - n - 1 \) and \( S_{2,l} \leq n + 1 \). Hence in both cases \( S_l \leq \min(\alpha_i, n_i + 1) \).

Example 5.5. Example 3.3 describes a non-cyclic extended semi-module \( (A, \varphi) \) for \( \mu = (0, 0, 0, 2, 2) \) such that
\[ |\mathcal{V}(A, \varphi)| = |\{(5, 6), (5, 7), (4, 6), (4, 7)\}| = d(b, \mu). \]

Proof of Theorem 1.1. Lemma 4.2 and Theorem 4.3 imply that \( \dim X_{\mu}(b) = \max |\mathcal{V}(A, \varphi)| \).
In Proposition 6.2 we give a pair with \( |\mathcal{V}(A, \varphi)| = d(b, \mu) \). Theorem 53 show that the maximum is at most \( d(b, \mu) \). Together we obtain \( \dim X_{\mu}(b) = d(b, \mu) \).

6. Irreducible components

Corollary 6.1. Let \( G = GL_h \), let \( b \) be superbasic and \( \nu \leq \mu \). Then the action of \( J(F) \) on the set of irreducible components of \( X_{\mu}(b) \) has only finitely many orbits.

Proof. It is enough to consider the intersection of the orbits with the set of irreducible components of \( X_{\mu}(b) \). Theorem 4.3 implies that each \( S_{A,\varphi} \) is irreducible. Thus the Corollary follows from Lemma 3.8.

Example 6.2. We give two examples to show that even for superbasic \( b \), the irreducible components of \( X_{\mu}(b) \) are in general not permuted transitively by \( J(F) \). The description of \( J(F) \) in Section 2 implies that \( A(gM) = A(M) \) and \( \varphi(gM) = \varphi(M) \) for each \( g \in J(F) \) with \( v_1(\det(g)) = 0 \). First we consider the example \( m = 4, h = 5, \) and \( \mu = (0, 0, 1, 1, 2) \). It is enough to find two extended semi-modules for \( \mu \) leading to subspaces of dimension \( d(b, \mu) = 3 \). Indeed, the subschemes corresponding to different extended semi-modules are disjoint and lead to irreducible components in different \( J(F) \)-orbits. One such extended semi-module is the cyclic extended semi-module considered in Proposition 3.2. A second extended semi-module \( (A, \varphi) \) is given in Example 3.4. Here, \( A \) is of type \((0, 0, 1, 2, 1)\), hence different from the semi-module considered before.

For the second example let \( m = 4, h = 5, \) and \( \mu = (0, 0, 2, 2) \). Here the two extended semi-modules for \( \mu \) leading to subspaces of dimension \( d(b, \mu) = 4 \) are the ones considered in Proposition 5.2 and Examples 3.4 and 5.5. The corresponding semi-modules are different as they are of type \( (0, 0, 2, 2) \) and \((0, 0, 1, 2, 1)\).

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