Axiomatizing some small classes of set functions

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Abstract
In this note we axiomatize the class of rudimentary functions, one of
primitive recursive functions, one of safe recursive set functions, one of
predicatively computable functions augmented with an ι-operator, and
relativized classes of these.

1 Introduction
In this note we axiomatize some small classes of set-theoretic functions. Roughly
speaking, by an axiomatization of a class of functions, we mean to give a formal
system in which (demonstrably) Σ₁-definable functions are exactly functions in
the class. We consider the classes of rudimentary functions in [7],
primitive recursive functions in [8], safe recursive set functions in [3], predicatively com-
putable set functions in [1] augmented with an ι-operator, and relativized classes of these.

Let $T_0$ denote the fragment of set theory obtained from the Kripke-Platek
set theory by deleting Foundation schema. First it is shown that a set-theoretic
function is Σ₁-definable in $T_0$ iff it is rudimentary. Also this is extended to
relativized class. Namely let $G$ be a collection of (hereditarily) Π₁-definable
functions, cf. Definition 2.1 and $T_0(G)$ denote the theory $T_0$ in the language
$L(G) = \{\in, =\} \cup \{g : g \in G\}$ with the defining axiom for the function symbol $g$
for functions $g \in G$. In $T_0(G)$, $\Delta_0$-Separation and $\Delta_0$-Collection are extended
to bounded formulas $\Delta_0(G)$ in the expanded language $L(G)$. Then we see that
a set-theoretic function is $\Sigma_1(G)$-definable in $T_0(G)$ iff it is rudimentary in $G$,
cf. Theorem 2.6.1a. This yields readily that a predicate is $\Delta_1(G)$-definable in
$T_0(G)$ iff it is rudimentary in $G$, cf. Corollary 3.2.1.

Second let $T_1(G)$ denote the fragment obtained from $T_0(G)$ by adding $\Sigma_1(G)$-Foundation schema. M. Rathjen [9] showed that for collections $G$ of $\Delta_0$-definable
functions, a set-theoretic function is $\Sigma_1$-definable in $T_1(G)$ iff it is primitive
recursive in $G$. This is extended to collections $G$ of (hereditarily) Π₁-definable
functions in this note. Again this yields that a predicate is $\Delta_1$-definable in $T_1(G)$
iff it is primitive recursive in $G$, cf. Theorem 2.6.1b and Corollary 3.2.2.
Third we axiomatize the class of safe recursive set functions in $\mathbb{R}$, and the relativized class. We expand the language to $\mathcal{L}(\mathcal{D}, \mathcal{G}) = \mathcal{L}(\mathcal{G}) \cup \{\mathcal{D}\}$ by augmenting a predicate $\mathcal{D}$, denoting a transitive class for normal arguments. Then a fragment $T_2^2(\mathcal{G})$ is obtained from $T_1(\mathcal{G})$ in the language $\mathcal{L}(\mathcal{D}, \mathcal{G})$ by restricting $\Sigma_1(\mathcal{G})$-Foundation schema to $\Sigma_1(\mathcal{G})$-formulas $\exists \varphi(x, a)$ for $x \in \mathcal{D}$. Namely an instance of the schema $\Sigma_1^1(\mathcal{G})$-Foundation schema runs as follows

$$\forall y \in \mathcal{D}[\forall x \in y \exists a \varphi(x, a) \rightarrow \exists a \varphi(y, a)] \rightarrow \forall y \in \mathcal{D} \exists a \varphi(y, a)$$

for $\Delta_0(\mathcal{G})$-formula $\varphi$. This means that the whole universe for safe arguments need not to be well-founded. The predicate $\mathcal{D}$ does not occur in $\Delta_0(\mathcal{G})$-formulas. Moreover an inference rule 'infer $\exists a \in \mathcal{D} \varphi(x, a)$ from $\exists a \varphi(x, a)$' ($\Sigma_1(\mathcal{G})$-Submodel Rule) is added for $x \in \mathcal{D}$ and $\Delta_0(\mathcal{G})$-formulas $\varphi$. Then it is shown that a set-theoretic function $\bar{f}(\bar{x}/\bar{a})$ is $\Sigma_1^1(\mathcal{G})$-definable in $T_2^2(\mathcal{G})$ iff it is in the class SRSF($\mathcal{G}$) of safe recursive set functions in $\mathcal{G}$, and a predicate is $\Delta_0^2(\mathcal{G})$-definable in $T_2^2(\mathcal{G})$ iff it is in the class SRSF($\mathcal{G}$), cf. Theorem 2.2 and Corollary 3.3.

The 'only-if' part of each of these three characterizations is proved by a witnessing argument due to S. Buss [4]. The idea is that given implication $\exists a \varphi(x, a) \rightarrow \exists b \psi(x, b)$ of $\Sigma_1$-formulas, find a function $f$ such that for any $x, a$, $\varphi(x, a) \rightarrow \psi(x, f(x, a))$. However a naive approach does not work. Consider the case when $\exists a \in c \varphi(x, a) \rightarrow \exists b \psi(x, b)$ is derived from $a \in c \land \varphi(x, a) \rightarrow \exists b \psi(x, b)$, where $c = c(x)$ is a set depending on $x$. Supposing that we have a function $f$ in hand such that $a \in c \land \varphi(x, a) \rightarrow \psi(x, f(x, a))$, we need a function $g(x)$ such that $\exists a \in c \varphi(x, a) \rightarrow \psi(x, g(x))$. If $\exists a \in c \varphi(x, a)$, then $g(x)$ can be arbitrary. But if $\exists a \in c \varphi(x, a)$, we need in general to pick an $a \in c$ such that $\varphi(x, a)$, and put $g(x) = f(x, a)$. This involves a choice function $a = a(x) \in \{a \in c(x) : \varphi(x, a)\}$ as pointed out by A. Beckmann. Putting it in another way, we can prove even logically $\exists x \in a(x = x) \rightarrow \exists x(x \in a)$, and its witnessing function would be nothing but a choice function $c$, $\exists x \in a(x = x) \rightarrow c(x) \in a$. Let us try alternatively to find a function $g$ denoting a non-empty set of witnesses. Namely for any $x, a$, $a \in c \land \varphi(x, a) \rightarrow \emptyset \neq g(x, a) \subset \{b : \psi(x, b)\}$. Then $\exists a \in c \varphi(x, a) \rightarrow \emptyset \neq f(x) \subset \{b : \psi(x, b)\}$ for $f(x) = \bigcup (g(x, a) : a \in c, \varphi(x, a))$. The function $f$ is defined from $g$ (and a bounded formula $\varphi$) by Bounded Union. Each of three classes is closed under it. Assume that $\exists a \varphi(x, a)$ is derivable in one of these fragments. We can find a function $g(x)$ in the relevant class such that $\emptyset \neq g(x) \subset \{a : \varphi(x, a)\}$. Since the set $\{a : \varphi(x, a)\}$ is a singleton for each $x$, $f(x) = \cup g(x)$ enjoys $\forall x \varphi(x, f(x))$ as desired.

### 1.1 Predicatively computable set functions with $i$-operator

Next let us turn to the class PCSF of predicatively computable set functions in $\mathbb{R}$. It is easy to see that we can weaken $\Delta_0$-Collection to $\Delta^0_0$-Collection, $\forall y \in \mathcal{D}[\forall x \in y \exists a \varphi(x, a) \rightarrow \exists c \forall x \in \exists a \in c \varphi(x, a)]$, to $\Sigma_1$-define functions in the class PCSF. This means that $y \in \mathcal{D}$ is a ‘domain’ of functions $f(x_1, \ldots, x_n/\bar{a})$ with respect to normal arguments $x_i \in y$. However the class PCSF is not closed under Bounded Union evidently unless PCSF=SRSF. Let us stick to the
uniqueness $\exists a$ in the whole derivation, not only in the end-formula. The class PCSF is enlarged with an $\iota$-operator as follows.

It is shown that each polynomial time computable function on finite binary strings is in the class PCSF, and conversely each PCSF function on hereditarily finite sets is polynomial time computable.

**Theorem 1.1** ([1])

For each (definition of) function $f(x_1, \ldots, x_n/a_1, \ldots, a_m) \in \text{PCSF}$ and for any hereditarily finite sets $\vec{X} = X_1, \ldots, X_n$ and $\vec{A} = A_1, \ldots, A_m$, the size of the transitive closure of $f(\vec{X}/\vec{A})$ is bounded by the sum of a polynomial of the sizes of the transitive closures of $X_i$ and the sizes of the transitive closures of $A_i$:

$$\text{card}(\text{TC}(f(\vec{X}/\vec{A}))) \leq p_f(\text{card}(\text{TC}(X_1)), \ldots, \text{card}(\text{TC}(X_n)))+\sum_i \text{card}(\text{TC}(A_i))$$

for a polynomial $p_f(\vec{x})$ with positive integer coefficients.

**Theorem 1.2** ([1])

Each polynomial time computable function on finite binary strings is in the class PCSF. Conversely each function in the class PCSF is polynomial time computable when the function is restricted to hereditarily finite sets.

It seems to us that there remains some room for the class PCSF to extend holding Theorems 1.1 and 1.2 also for the extension, and keeping the extensionality of functions under coding as contrasted with a choice function $a \neq \emptyset \Rightarrow c(-/a) \in a$, cf. Remark after Corollary 2 in [1].

Actually A. Beckmann, et. al. [4] introduced such an extension PCSF' by the following schema, cf. Remark before Proposition 1 of [1]:

(Normal Separation)

$$f(\vec{x}/\vec{a}, c) = \{b \in c : g(\vec{x}/\vec{a}, b) \neq \emptyset\}$$

This means that for any $g(\vec{x}/\vec{a}, b) \in \text{PCSF}'$, the function $f$ is in $\text{PCSF}'$. The extended class $\text{PCSF}'$ is seen to be closed under Russell’s $\iota$-operator in a restricted case as follows. The $\iota$-operator describes an object $\iota x.A(x)$ for a predicate $A(x)$: $\iota x.A(x)$ denotes the unique element $x$ enjoying $A(x)$ if there exists a unique such $x$. Otherwise put $\iota x.A(x) = \emptyset$\footnote{We don’t assume here the derivability of the fact $\exists x \ A(x)$ in introducing the $\iota$-expression $\iota x.A(x)$.} In other words, $b = \iota x.A(x)$ iff either $A(b) \land \exists! x.A(x)$ or $b = \emptyset \land \neg \exists! x.A(x)$. As shown in Theorem 30 of [4], if $g$ is in $\text{PCSF}'$, then so is $f(\vec{x}/\vec{a}, c) = \{b \in c : g(\vec{x}/\vec{a}, b) \neq \emptyset\}$, since $f(\vec{x}/\vec{a}, c) = \bigcup\{b \in c : g(\vec{x}/\vec{a}, b) \neq \emptyset\}$ if $\exists! b \in c(g(\vec{x}/\vec{a}, b) \neq \emptyset)$, $f(\vec{x}/\vec{a}, c) = \emptyset$ otherwise.

Let us extend the class PCSF by definite descriptions alternatively. We obtain a class $\text{PCSF}''$ closed under $(i):$ if $g \in \text{PCSF}'$, then so is the following $f$.

$$(i) \quad f(\vec{x}/\vec{a}, c) = \iota d(\exists b \in c(g(\vec{x}/\vec{a}, b) = d)).$$
This means that when the range $g'' c = \{g(\vec{x}/\vec{a}, b) : b \in c\}$ is a singleton, $f(\vec{x}/\vec{a}, c)$ denotes the unique element, and $f(\vec{x}/\vec{a}, c) = \emptyset$ otherwise.

Next let us relativize the class PCSF by a collection $G$ of functions. We assume that arguments of each function $g(\vec{x}/\vec{a})$ in $G$ are divided in normal arguments $\vec{x}$ and safe arguments $\vec{a}$. In other words even if $g(\vec{x}/-) = h(-/\vec{x})$ for any $\vec{x}$, the classes PCSF($\{g\}$) and PCSF($\{h\}$) may differ. The class PCSF($G$) is obtained from functions in $G$, some rudimentary set functions on safe arguments by safe composition scheme and predicative set (primitive) recursion scheme a là Bellantoni-Cook [3].

Initial functions in $\text{PCSF}^-(G)$ are functions $g(\vec{x}/\vec{a})$ in the collection $(G)$, (Projection) for each argument besides the initial functions in $\text{PCSF}^-$. The class $\text{PCSF}^-(G)$ is closed under (Safe Composition) and the following ($\Delta_0$-Separation): Stratified formulas with respect to variables $\vec{x}$ and $\vec{a}$ with $\vec{x} \cap \vec{a} = \emptyset$ are generated as follows.

1. Stratified terms with respect to $\vec{x}$ and $\vec{a}$ are defined as follows.
   (a) Each variable in $\vec{x} \cup \vec{a}$ is stratified with respect to $\vec{x}$ and $\vec{a}$.
   (b) For a function $f(\vec{y}/\vec{b}) \in \text{PCSF}^-(G)$ and variables $\vec{y} \subset \vec{x}$ and $\vec{b} \subset \vec{a}$, the term $f(\vec{y}/\vec{b})$ is stratified with respect to $\vec{x}$ and $\vec{a}$.

2. Any literals $t \in s, t \notin s, t = s, t \neq s$ are stratified with respect to $\vec{x}$ and $\vec{a}$ if $t$ and $s$ are stratified terms with respect to $\vec{x}$ and $\vec{a}$.

3. If $\varphi_0, \varphi_1$ are stratified with respect to $\vec{x}$ and $\vec{a}$, then so are $\varphi_0 \lor \varphi_1$ and $\varphi_0 \land \varphi_1$.

4. If $\varphi$ is stratified with respect to $\vec{x} \cup \{y\}$ and $\vec{a}$ and $t$ is a stratified term with respect to $\vec{x}$ and $\vec{a}$, then $\exists y \in t \varphi$ and $\forall y \in t \varphi$ are stratified with respect to $\vec{x}$ and $\vec{a}$.

5. If $\varphi$ is stratified with respect to $\vec{x}$ and $\vec{a} \cup \{b\}$ and $t$ is a stratified term with respect to $\vec{x}$ and $\vec{a}$, then $\exists b \in t \varphi$ and $\forall b \in t \varphi$ are stratified with respect to $\vec{x}$ and $\vec{a}$.

($\Delta_0$-Separation) $f(\vec{x}/\vec{a}, c) = \{b \in c : \theta(\vec{x}, \vec{a}, b)\}$

where $\theta(\vec{x}, \vec{a}, b)$ is a bounded formula which is stratified with respect to $\vec{x}$ and $\vec{a} \cup \{b\}$.

The class $\text{PCSF}'(G)$ is then obtained from $\text{PCSF}^-(G)$ by operating (Safe Composition), (Predicative Set Recursion) and (i).

Remark. Note that in the absolute case $G = \emptyset, \text{PCSF}^-(\emptyset) = \text{PCSF}^-$, since $h(-/\vec{a}) \neq 0$ is $\Delta_0$, and conversely for any $\Delta_0$-formula $\theta(\vec{a})$ there exists an $h \in \text{PCSF}^-$ such that $\theta(\vec{a}) \Leftrightarrow h(-/\vec{a}) \neq 0$, cf. Propositions 1 and 2.5 in [1]. However in the relativised case it is unclear whether or not $\text{PCSF}^-(G)$ with ($\Delta_0$-Separation) is closed under (Safe Separation) (or one should call it
A formal system $T_3^D(\mathcal{G})$ is a union of increasing formal systems $T_3^{(n)}(\mathcal{G})$ in a language $\mathcal{L}^{(n)}$. First let $\mathcal{L}^{(0)} = \{\in,=,\mathcal{D},\text{TC}\} \cup \{g : g \in \mathcal{G}\}$ with a function symbol $\text{TC}(x)$ for the transitive closure of $x \in \mathcal{D}$. The axiom for it states $\forall x \in \mathcal{D}\text{trcl}(\text{TC}(x), x, a)$, where

\[
\begin{align*}
\text{trcv}(u, x) &\iff \forall y \in x(y \in u) \land \forall y \in u \forall z \in y(z \in u) \\
\text{trcl}(y, x, a) &\iff \text{trcv}(u, x) \land [\text{trcv}(u, x) \rightarrow \forall z \in y(z \in a)]
\end{align*}
\]

Assume that an expanded language $\mathcal{L}^{(n)}$ of $\mathcal{L}^{(0)}$ has been defined. Let $\Delta_0(\mathcal{L}^{(n)}$ denote the set of bounded formulas in $\mathcal{L}^{(n)}$. A formula $\exists a \varphi$ with $\varphi \in \Delta_0(\mathcal{L}^{(n)})$ is said to be a $\Sigma_1(\mathcal{L}^{(n)})$-formula. A formal system $T_3^{(n)}(\mathcal{G})$ in $\mathcal{L}^{(n)}$ is obtained from $T_3^D(\mathcal{G})$ as follows.

1. Add the axiom $\forall x \in \mathcal{D}\text{trcl}(\text{TC}(x), x, a)$ for transitive closure.

2. Add an axiom $\forall \vec{x} \in \mathcal{D}\forall \vec{a} \theta_1(\vec{x}, \vec{a}, f(\vec{x}, \vec{a}))$ for each function symbol $f \in \mathcal{L}^{(n)} - \mathcal{L}^{(n-1)}$ for $n > 0$, where $\theta_1$ is a $\Sigma_1(\mathcal{L}^{(n-1)})$-formula associated with the function symbol $f$ so that $T_3^{(n-1)}(\mathcal{G}) \vdash \forall \vec{x} \in \mathcal{D}\forall \vec{a} \theta_1(\vec{x}, \vec{a}, b)$.  

3. Separation axiom schema is available for any $\Delta_0(\mathcal{L}^{(n)})$-formula.

4. $\Delta_0(\mathcal{G})$-Collection, $\Sigma_1^D(\mathcal{G})$-Foundation schema, and $\Sigma_1(\mathcal{G})$-Submodel Rule are replaced by their ‘unique’ versions in the language $\mathcal{L}^{(n)}$.

\[
(\Delta_0(\mathcal{L}^{(n)})) \text{-Replacement} \quad \forall y \in \mathcal{D}[\forall x \in y \exists a \varphi(x, x) \rightarrow \exists \vec{x} \vec{x} \in y \varphi(\vec{x}, x')].
\]

\[
(\Sigma_1^D(\mathcal{L}^{(n)})) \text{-Foundation} \quad \forall y \in \mathcal{D}[\forall x \in y \exists a \varphi(x, x) \rightarrow \exists a \varphi(y, a)] \rightarrow \\
\forall y \in \mathcal{D}\exists a \varphi(y, a).
\]

\[
(\Sigma_1(\mathcal{L}^{(n)})) \text{-Submodel Rule} \quad \forall \vec{x} \in \mathcal{D}\exists y \in \mathcal{D} \varphi(\vec{x}, y)
\]

In each of three axiom schemata, $\varphi$ is a $\Delta_0(\mathcal{L}^{(n)})$-formula.

Enlarge the language $\mathcal{L}^{(n)}$ to get $\mathcal{L}^{(n+1)}$ by adding function symbols $f(\vec{x}/\vec{a})$ when $T_3^{(n)}(\mathcal{G}) \vdash \forall \vec{x} \in \mathcal{D}\forall \vec{a} \exists b \theta_1(\vec{x}, \vec{a}, b)$ for $\theta_1 \in \Sigma_1(\mathcal{L}^{(n)})$.

This means that a function symbol for each $\Sigma_1(\mathcal{L}^{(n)})$-definable function in $T_3^{(n)}(\mathcal{G})$ is introduced in the next stage $T_3^{(n+1)}(\mathcal{G})$ in which the introduced function symbol may occur in bounded formulas of Replacement, Foundation and Submodel rule.

Finally $T_3^D(\mathcal{G}) = \bigcup_n T_3^{(n)}(\mathcal{G})$ in the language $\mathcal{L}^{(\omega)} = \bigcup_n \mathcal{L}^{(n)}$.

It seems to us that $a = \text{TC}(x)$, or specifically $a \subset \text{TC}(x)$ is a $\Sigma$-formula, but not a $\Sigma_1$-formula, to which Foundation is not available. This is the reason why we assume the existence of the transitive closure $\text{TC}(x)$ for $x \in \mathcal{D}$.

Obviously $\Sigma_1^D(\mathcal{L}^{(n)}))$-Foundation yields $\Delta_0^D(\mathcal{L}^{(n)})$-Foundation axiom, $\forall y[\forall x \in y \theta(x) \rightarrow \theta(y)] \rightarrow \forall y \theta(y)$ for each $\Delta_0(\mathcal{L}_n)$-formula $\theta$ by letting $\varphi(x, a) \equiv$
\((\theta(x) \land a = b)\) with a parameter \(b\).

Although it is easy to see that each function in \(\text{PCSF}^i(G)\) is \(\Sigma_1!(\mathcal{L}^{(\omega)})\)-definable in \(T_3^P(G)\), it is not a routine work to prove the converse because of the uniqueness conditions, which involve unbounded universal quantifiers, i.e., \(\forall a, b[\varphi(a) \land \varphi(b) \rightarrow a = b]\). To control the unbounded universal quantifiers, we introduce classes \(X\), i.e., \(\forall a, b\) is restricted to \(\forall a, b \in X\). A function \(f_X(x/\bar{a})\) may depend on classes \(X\), but \(f_X\) becomes a \(\text{PCSF}^i(G)\)-function for each classes \(X\) defined in a restricted way. It turns out that we can associate a condition on classes \(X\) and a witnessing function \(f_X\) depending uniformly on \(X\) for each derivable implication of \(\Sigma_1!(G)\)-formulas. We conclude that a set function is in \(\text{PCSF}^i(G)\) iff it is \(\Sigma_1!(\mathcal{L}^{(\omega)})\)-definable in \(T_3^P(G)\), cf. Theorem 4.2.

## 2 Fragments of set theory

First let us introduce collections of set-theoretic functions, to which classes of functions are relativized.

### Definition 2.1

1. Let \(\mathcal{L}\) be a language obtained from the language \(\mathcal{L}_{-1} = \{\in, =\}\) by adding some function symbols. \(\Delta_0(\mathcal{L})\) denotes the set of bounded formulas in \(\mathcal{L}\), and \(\Sigma_1(\mathcal{L})\) the set of \(\Sigma_1\)-formulas \(\exists a \theta\) in \(\mathcal{L}\) with \(\theta \in \Delta_0(\mathcal{L})\). The set \(\Sigma_k(\mathcal{L})\) of formulas is defined similarly for \(k > 0\).

2. A collection \(G\) of hereditarily \(\Pi_1\)-definable set-theoretic functions is generated recursively as follows. For \(n \geq 0\), let \(G_n\) be a collection of \(\Pi_1(\mathcal{L}_{n-1})\)-definable functions with \(\mathcal{L}_{-1} = \{\in, =\}\). This means that for each \(g \in G_n\), a \(\Pi_1(\mathcal{L}_{n-1})\)-formula \(\theta_g\) is assigned so that for any \(\bar{a} = (a_0, \ldots, a_{m-1})\) and any \(b\),

\[
  g(\bar{a}) = b \iff V \models \theta_g(\bar{a}, b) \tag{1}
\]

The language \(\mathcal{L}_n\) is then obtained from the language \(\mathcal{L}_{n-1}\) by adding a function symbol \(g\) for each \(g \in G_n\), \(\mathcal{L}_n = \mathcal{L}_{n-1} \cup \{g : g \in G_n\}\). Then \(G = \bigcup_n G_n\) is a collection of hereditarily \(\Pi_1\)-definable functions over the language \(\mathcal{L}(G) := \bigcup_n \mathcal{L}_n\).

We write \(\Delta_0(G)\) for \(\Delta_0(\mathcal{L}(G))\), and \(\Sigma_1(G)\) for \(\Sigma_1(\mathcal{L}(G))\), etc.

Note that any \(\Sigma_1\)-definable function is \(\Pi_1\)-definable.

Examples of hereditarily \(\Pi_1\)-definable set-theoretic functions are the power set \(\mathcal{P}(a)\) of \(a\), '\(a\) is a cardinal', '\(a\) is the next cardinal above \(b\)' and \(\omega_n\), etc.

Let \(G\) be a collection of hereditarily \(\Pi_1\)-definable set-theoretic functions over a language \(\mathcal{L}(G)\). In this section let us introduce two fragments \(T_i(G)\) \((i = 0, 1)\) of set theory over \(\mathcal{L}(G)\), and a fragment \(T_2^P(G)\) over a language \(\mathcal{L}(\mathcal{D}, G) = \mathcal{L}(G) \cup \{\mathcal{D}\}\) with a unary predicate symbol \(\mathcal{D}\).

### Definition 2.2

1. \(T_0(G)\) denotes the Kripke-Platek set theory minus Foundation schema in the language \(\mathcal{L}(G)\). Namely its axioms are extensionality,
null set, pair, union, $\Delta_0(\mathcal{G})$-Separation schema, and equality axioms for function symbols $g \in \mathcal{L}(\mathcal{G})$, $\forall \bar{x}\forall \bar{y}[\bar{x} = \bar{y} \rightarrow g(\bar{x}) = g(\bar{y})]$ and $\forall \bar{x}\theta_g(\bar{x}, g(\bar{x}))$
where $\theta_g$ is a $\Pi_1(\mathcal{G})$-formula assigned to the function $g$ with 1.

2. $T_1(\mathcal{G})$ is obtained from $T_0(\mathcal{G})$ by adding Foundation schema restricted to $\Sigma_1(\mathcal{G})$-formulas $\varphi$, $\forall b[\forall a \in b \varphi(a) \rightarrow \varphi(b)] \rightarrow \forall b \varphi(b)$.

For the next definition, let us suppose that arguments of each function $g(\bar{x}/\bar{a})$ in $\mathcal{G}$ are divided to normal arguments $\bar{x}$ and safe arguments $\bar{a}$.

**Definition 2.3** Let us introduce some axiom schemata and related inference rules. Below, e.g., in $\Delta_0(\mathcal{G})$-Separation schema $\Delta_0(\mathcal{G})$ denotes the class of bounded formulas in the language $\mathcal{L}(\mathcal{G})$, and similarly for the classes $\Sigma_1(\mathcal{G})$, $\Pi_1(\mathcal{G})$.

This means that the predicate $D$ does not occur in any $\Delta_0(\mathcal{G})$-formulas.

$x, y, z, \ldots$ are variables ranging over elements in the class $D$, while $a, b, c, \ldots$ are variables ranging over the universe.

($\mathcal{G}$) For each $g(\bar{x}/\bar{a}) \in \mathcal{L}(\mathcal{G})$,
\[
\forall \bar{x} \subset D \forall \bar{a} \theta_g(\bar{x}, \bar{a}, g(\bar{x}, \bar{a}))
\]
where $\theta_g$ is a $\Pi_1(\mathcal{G})$-formula such that 1.

(transitivity)
\[
\forall a, b[b \in a \rightarrow D(a) \rightarrow D(b)].
\]

($\Delta_0(\mathcal{G})$)-Collection For each $\Delta_0(\mathcal{G})$-formula $\varphi$
\[
\forall b[\forall e \in b \exists a \varphi(e, a) \rightarrow \exists c \forall e \in c \varphi(e, a)]
\]
where in $\varphi$ parameters $\bar{d}$ may occur.

($\Sigma_1^1(\mathcal{G})$)-Foundation) For each $\Delta_0(\mathcal{G})$-formula $\varphi$
\[
\forall y[\forall x \in y \exists a \varphi(x, a) \rightarrow \exists a \varphi(y, a)] \rightarrow \forall y \exists a \varphi(y, a)
\]
where $\varphi$ may have parameters $\bar{d}$.

($\Sigma_1(\mathcal{G})$)-Submodel Rule) For each $\Delta_0(\mathcal{G})$-formula $\varphi(\bar{x}, a)$ whose free variables are among the list $\bar{x} \cup \{a\}$
\[
\forall \bar{x} \exists a \varphi(\bar{x}, a)
\]
\[
\forall \bar{x} \exists y \varphi(\bar{x}, y).
\]
This is a shorthand for
\[
\forall \bar{x}[D(\bar{x}) \rightarrow \exists a \varphi(\bar{x}, a)]
\]
\[
\forall \bar{x}[D(\bar{x}) \rightarrow \exists y(D(y) \land \varphi(\bar{x}, y))]
\]
This rule says that ‘infer $\exists y \varphi(\bar{x}, y)$ from $\exists a \varphi(\bar{x}, a)$’ if $\exists a \varphi(\bar{x}, a)$ is derivable without assumptions.

A related inference rule in the context of arithmetic was investigated by Spoors and Wainer [10].
Definition 2.4 \( T_2^0(G) \) is obtained from the theory \( T_0(G) \) in the expanded language \( \mathcal{L}(D, G) \) with the axioms (G), the axiom (transitivity) and the equality axiom, \( \forall a, b(a = b \rightarrow D(a) \rightarrow D(b)) \), and by adding (\( \Delta_0(G)\)-Collection), (\( \Sigma^0_1(G)\)-Foundation) and (\( \Sigma_1(G)\)-Submodel Rule). Note that in \( \Delta_0(G)\)-Separation schema, the predicate \( D \) does not occur.

Definition 2.5 1. Let \( T \) be one of the fragments \( T_i(G) \) \( (i = 0, 1) \). We say that a set-theoretic function \( f(\bar{a}) \) is \( \Sigma_i(G)\)-definable in \( T \) if there exists a \( \Sigma_1(G)\) -formula \( \varphi(\bar{a}, b) \) in \( \mathcal{L}(G) \) such that \( T \vdash \forall \bar{a} \exists !b \varphi(\bar{a}, b) \) and \( f(\bar{a}) = b \iff \models \varphi(\bar{a}, b) \) for any \( \bar{a}, b \).

2. We say that a set-theoretic function \( f(\bar{x}/\bar{a}) \) is \( \Sigma^0_1(G)\)-definable in \( T_2^0(G) \) if there exists a \( \Sigma_1(G)\)-formula \( \varphi(\bar{x}/\bar{a}, b) \) in \( \mathcal{L}(G) \) such that \( T_2^0(G) \vdash \forall \bar{x} \in G \exists !b \varphi(\bar{x}/\bar{a}, b) \) and \( f(\bar{x}/\bar{a}) = b \iff \models \varphi(\bar{x}/\bar{a}, b) \) for any \( \bar{x}, \bar{a}, b \).

Now our first theorem runs as follows.

Theorem 2.6 1. Let \( G \) be a collection of hereditarily \( \Pi_1\)-definable set-theoretic functions \( g(\bar{a}) \), and \( f \) a set-theoretic function.

(a) \( f(\bar{x}) \) is rudimentary in \( G \) in the sense of Jensen \[7\] iff \( f(\bar{x}) \) is \( \Sigma_1(G)\)-definable in \( T_0(G) \).

(b) \( f(\bar{x}) \) is primitive recursive in \( G \) in the sense of Jensen-Karp \[8\] iff \( f(\bar{x}) \) is \( \Sigma_1(G)\)-definable in \( T_1(G) \).

2. Let \( G \) be a collection of hereditarily \( \Pi_1\)-definable set-theoretic functions \( g(\bar{y}/\bar{b}) \), and \( f(\bar{x}/\bar{a}) \) a set-theoretic function.

\( f(\bar{x}/\bar{a}) \in SRSF(G) \) iff \( f(\bar{x}/\bar{a}) \) is \( \Sigma^0_1(G)\)-definable in \( T_2^0(G) \).

2.1 \( \Sigma_1(G)\)-definability in fragments

First let us verify the easy halves in Theorem 2.6.

The set of rudimentary-in-\( G \) functions are generated from functions in \( G \), projections, pair, difference \( a - b \) by operating composition and (Bounded Union):

\( f(\bar{x}, z) = \bigcup \{ g(\bar{x}, y) : y \in z \} \).

\( \Sigma_1(G)\)-definability of rudimentary-in-\( G \) functions in \( T_0(G) \), the easy half of Theorem 2.6a is ready to see. For the bounded union, assume that \( g(\bar{x}, y) = a \) is defined by a \( \Sigma_1(G)\)-formula \( \varphi_g(\bar{x}, y, a) \) (in \( T_0(G) \)). We have \( \forall y \in z \exists a \varphi_g(\bar{x}, y, a) \).

By \( \Delta_0(G)\)-Collection there exists a \( b \) such that \( \forall y \in z \exists a \in b \varphi_g(\bar{x}, y, a) \). Then \( f(\bar{x}, z) = c \) is defined by the \( \Sigma_1(G)\)-formula \( \exists b(\forall y \in z \exists a \in b(\varphi_g(\bar{x}, y, a) \land a \in c) \land \forall a \in c \exists y \in z \varphi_g(\bar{x}, y, a)) \), i.e., \( c = \cup \{ a \in b : \exists y \in z \varphi_g(\bar{x}, y, a) \} = \cup \{ a \in b : \exists y \in z \forall d \in b(\varphi_g(\bar{x}, y, d) \rightarrow a = d) \} \).
The set of primitive-recursive-in-\( \mathcal{G} \) functions is generated from functions in \( \mathcal{G} \), projections, null, conditional, and \( M(a, b) = a \cup \{b\} \), and operating composition and set recursion:

\[
f(x, y) = h(x, \bar{y}, \{f(z, y) : z \in x\}).
\]

Again it is easy to see the \( \Sigma_1(\mathcal{G}) \)-definability of primitive-recursive-in-\( \mathcal{G} \) functions in \( T_1(\mathcal{G}) \), the easy half of Theorem 2.6 [11]. An inspection in pp. 24-28 of [2] shows that \( \Sigma_1(\mathcal{G}) \)-Foundation together with \( \Delta_0(\mathcal{G}) \)-Collection suffices for the existence of the transitive closure \( TC(x) \) of \( x \), and \( \Sigma \)-recursion of functions.

The class \( \text{SRSF} \) is obtained from Gandy-Jensen rudimentary set functions on safe arguments by safe composition schema and predicative set (primitive) recursion schema. The remaining easy half of Theorem 2.6 [2] for the class \( \text{SRSF} \) of safe recursive set functions in [3] is seen as follows. We see that the class of \( \Sigma^P_1(\mathcal{G}) \)-definable functions in \( T^P_1(\mathcal{G}) \) is closed under (Predicative Set Recursion), and the class is closed under (Bounded Union), cf. the proof of Theorem 2.6 [11] using \( \Delta_0(\mathcal{G}) \)-Collection. (\( \Sigma_1(\mathcal{G}) \)-Submodel Rule) suffices to show the closure under (Safe Composition) \( f(\bar{x}/a) = h(\bar{r}(\bar{x}/-)/\bar{l}(\bar{x}/a)) \).

**Lemma 2.7** Each \( f(\bar{x}/a) \in \text{SRSF}(\mathcal{G}) \) is \( \Sigma^P_1(\mathcal{G}) \)-definable in \( T^P_1(\mathcal{G}) \).

**Proof.** By induction on the construction of \( f \).

It is clear that each initial rudimentary function (projections, difference \( a - b \), and pair \( \{a, b\} \)) is \( \Sigma^P_1(\mathcal{G}) \)-definable in \( T^P_1(\mathcal{G}) \). \( \Delta_0(\mathcal{G}) \)-Collection suffices to show the closure of \( \Sigma^P_1(\mathcal{G}) \)-definability under (Bounded Union).

Let \( f(\bar{x}a) = h(\bar{r}(\bar{x}/-)/\bar{l}(\bar{x}/a)) \) be defined by (Safe Composition) from \( h, \bar{r} \) and \( \bar{l} \), and \( \varphi_h, \varphi_{\bar{r}} \) and \( \varphi_{\bar{l}} \) be \( \Sigma(\mathcal{G}) \)-formulas for \( h, \bar{r} \) and \( \bar{l} \), resp. Let

\[
\varphi_f(\bar{x}, \bar{a}, b) := \exists F \exists H[\varphi_r(\bar{x}, \bar{r}) \wedge \varphi_f(\bar{x}, \bar{a}, \bar{l}) \wedge \varphi_h(\bar{r}, \bar{l}, b)]
\]

where \( \varphi_r(\bar{x}, \bar{r}, \bar{c}) := \bigwedge_i \varphi_r(\bar{c}i)(\bar{x}, \bar{r}; ci) \) for \( r(\bar{x}/-)) = (r_i)i \) and \( \bar{r} = (r_i)i \).

By IH \( T^P_1(\mathcal{G}) \) proves \( \forall \bar{x} \in D \exists \bar{r} \varphi_r(\bar{x}, \bar{r}) \), \( \forall \bar{r} \in D \exists \bar{l} \varphi_{\bar{l}}(\bar{r}, \bar{l}) \) and similarly for \( \bar{l}(\bar{x}/a) \). Then by the inference rule (\( \Sigma_1(\mathcal{G}) \)-Submodel Rule) we have in \( T^P_1(\mathcal{G}) \), \( \forall \bar{x} \in D \exists \bar{r} \varphi_r(\bar{x}, \bar{r}) \). Hence \( \forall \bar{x} \in D \exists \bar{r} \varphi_f(\bar{x}, \bar{a}, b) \).

As for \( T_1(\mathcal{G}) \), \( \Sigma^P_1(\mathcal{G}) \)-Foundation together with \( \Delta_0(\mathcal{G}) \)-Collection suffices to show the existence of the transitive closure \( TC(x) \) of \( x \in D \). By (\( \Sigma_1(\mathcal{G}) \)-Submodel Rule) we have \( TC(x \cup \{x\}) \in D \).

Let \( f(x, \bar{y}/a) = h(x, \bar{y}/a, \{f(z, \bar{y}/a) : z \in x\}) \) be defined by (Predicative Set Recursion) from \( h \), and \( \varphi_h \) be a \( \Sigma_1(\mathcal{G}) \)-formula for \( h \). We have \( \forall z, \bar{y} \in D \exists \bar{c} \varphi_h(z, \bar{y}, \bar{a}, e, b) \) and \( h(z, \bar{y}/a, e) = b \iff \varphi_h(z, \bar{y}, \bar{a}, e, b) \).

Let

\[
\varphi(x, \bar{y}, \bar{a}, b) := (c \text{ is a function on } TC(x \cup \{x\}) ) \land (2)
\]

\[
\forall z \in TC(x \cup \{x\}) \varphi_h(z, \bar{y}, \bar{a}, c'z, c'z) \land (c'x = b)
\]

where for \( c'x = \{c'z : z \in x\} \). Then \( f(x, \bar{y}/a) = b \iff \exists c \varphi(x, \bar{y}, \bar{a}, b) \).

Modulo \( \Delta_0(\mathcal{G}) \)-Collection \( \varphi \) is equivalent to a \( \Sigma_1(\mathcal{G}) \)-formula.

We show \( \forall \bar{y} \in D \exists \bar{d}r \forall x \in D \exists \bar{c}(b, c) \varphi(x, \bar{y}, \bar{a}, b, c) \). There is nothing to prove for the uniqueness of \( b \).
Let \( \theta(x, c) \coloneqq \varphi(x, \bar{y}, \bar{a}, c'x; c) \). Suppose \( x_0 \in \mathcal{D} \), and let \( d_0 \in \mathcal{D} \) be the transitive closure of \( x_0 \cup \{ x_0 \} \). We show \( \exists c \theta(x_0, c) \). We have \( d_0 \subset \mathcal{D} \) by (transitivity).

The uniqueness of \( c \), i.e., \( \theta(x, c) \land \theta(x, d) \to c = d \) follows from \( (\Delta^P_0(\mathcal{G})\text{-Foundation}) \).

Suppose \( \forall z \in x \cap d_0 \exists c \theta(z, c) \). We show \( \exists c \theta(x, c) \) assuming \( x \in d_0 \). Then by \( (\Sigma^P_1(\mathcal{G})\text{-Foundation}) \) we have \( \forall x \in \mathcal{D} \cap d_0 \exists c \theta(x, c) \). Hence \( \exists c \theta(x_0, c) \).

By \( (\Delta^P_0(\mathcal{G})\text{-Collection}) \) pick \( d \) so that \( \forall z \in x \cap d_0 \exists c \in d(\theta(z, c))^{(d)} \). Let 
\[ e = \{ c \in d : \exists z \in x(\theta(z, c))^{(d)} \} \]
by \( \Delta^P_0(\mathcal{G})\text{-Separation} \), and \( c_0 = \cup e \). Then \( c_0 \) is seen to be a function on \( \text{TC}(x) \). For \( z_1, z_2 \in x \) and \( z \in d_1 \cap d_2 \) with \( d_i = \text{dom}(c_i), c_i = c'_i z_i (i = 1, 2) \), if \( c_1 \upharpoonright z = c_2 \upharpoonright z \), then \( \varphi_h(z, \bar{y}, \bar{a}, c'_1 z, c'_2 z) \) for \( i = 1, 2 \). Hence \( c'_2 z = c'_2 z \). \( (\Delta^P_0(\mathcal{G})\text{-Foundation}) \) with \( d_1 \cup d_2 \subset d_0 \subset \mathcal{D} \) yields \( c_1 \upharpoonright (d_1 \cap d_2) = c_2 \upharpoonright (d_1 \cap d_2) \). Hence \( c_0 \) is a function.

Let \( b \) be such that \( \varphi_h(x, \bar{y}, \bar{a}, c'_0 x, b) \), and let \( c_x = c_0 \cup \{ (x, b) \} \). Then \( \theta(x, c_x) \) as desired.

We see that \( (\Sigma_1(\mathcal{G})\text{-Submodel Rule}) \) in \( T_2^P(\mathcal{G}) \) of Theorem \( \ref{2.6} \) cannot be replaced by the axiom. Let \( T_1^P(\mathcal{G}) \) denote the fragment in the language \( \mathcal{L}(\mathcal{D}, \mathcal{G}) \) obtained from \( T_2^P(\mathcal{G}) \) by weakening \( (\Delta^P_0(\mathcal{G})\text{-Collection}) \) to the following \( (\Delta^P_0(\mathcal{G})\text{-Collection}) \) and strengthening \( (\Sigma_1(\mathcal{G})\text{-Submodel Rule}) \) to the following axiom \( (\Sigma_1(\mathcal{G})\text{-Submodel}) \):

\[ \text{(\Delta^P_0(\mathcal{G})\text{-Collection}) For each } \Delta^P_0(\mathcal{G})\text{-formula } \varphi \]
\[ \forall y [ \forall x \in y \exists a \varphi(x, a) \to \exists c \forall x \in y \exists a \in c \varphi(x, a) ] \]
\[ \text{where in } \varphi \text{ parameters } \bar{d} \text{ may occur. Formerly the axiom should be } \]
\[ \forall d \forall y \in \mathcal{D} [ \forall x \in y \exists a \varphi(x, a, \bar{d}) \to \exists c \forall x \in y \exists a \in c \varphi(x, a, \bar{d}) ] . \]

\[ \text{(\Sigma_1(\mathcal{G})\text{-Submodel}) For each } \Delta^P_0(\mathcal{G})\text{-formula } \varphi(\bar{x}, a) \text{ whose free variables are } \]
\[ \text{among the list } \bar{x} \cup \{ a \} \]
\[ \forall \bar{x} \exists a \varphi(\bar{x}, a) \to \exists y \varphi(\bar{x}, y) . \]

This is a shorthand for
\[ \forall \bar{x} \subset \mathcal{D} [ \exists a \varphi(\bar{x}, a) \to \exists y \in \mathcal{D} \varphi(\bar{x}, y) ] . \]

Corollary \( \ref{2.8} \) shows that any primitive-recursive-in-\( \mathcal{G} \) function is \( \Sigma^P_1(\mathcal{G}) \)-definable in \( T^P_1(\mathcal{G}) \). For example, the set \( ^n x \) of all functions from any natural number \( n \) to sets \( x \in \mathcal{D} \) is seen to exist provably in \( T^P_1(\mathcal{G}) \). By Theorem \( \ref{1.1} \) the primitive recursive set function \( (n, x) \mapsto ^n x \) is not in \( \text{PCS} \) nor even in \( \text{PCS} \).

\[ \text{Corollary 2.8 Let } \mathcal{G} \text{ be a collection of } \Pi_1-\text{definable set-theoretic functions } g(\bar{x}). \]
\[ \text{A set function } f(\bar{x}) \text{ is primitive recursive in } \mathcal{G} \text{ iff } f(\bar{x}/-) \text{ is } \Sigma^P_1(\mathcal{G}')\text{-definable in } T^P_1(\mathcal{G}') \text{ for } \mathcal{G}' := \{ g(\bar{x}/-) : g(\bar{x}) \in \mathcal{G} \} . \]
The converses of Theorem 2.6 are proved by a witnessing argument. Values of a witnessing function \( f \) introduce an individual constant 0 or denoted ∅. Now observe that the class \( D \) is a model of \( T_1(G) \) provably in \( T^P(G) \) in the sense that \( T^P(G) \) proves \( \varphi^D \) for each axiom \( \varphi \) in \( T_1(G) \) using the axiom (\( \Sigma_1(G) \)-Submodel), where \( \varphi^D \) denotes the sentence obtained from the sentence \( \varphi \) by restricting any quantifiers to \( D \).

Note here that the relativized axiom \((\forall \theta g(x, g(x/\neg)))^D\) for the function \( g \in G \) follows from the axiom \( \forall \theta g(x, g(x/\neg)) \) since \( \theta g \) is a \( \Pi_1 \)-formula and \( D \) is transitive. Hence \( T^P(G) \vdash \forall \theta g(x, g(x/a)) \), and this shows that \( f(x/\neg) \) is \( \Sigma^P_1(G) \)-definable in \( T^P(G) \).

Conversely if \( T^P(G) \vdash \forall \theta g(x, g(x/a)) \), then \( T^P(G) + (V = D) \vdash \forall \theta g(x, g(x/a)) \) a fortiori. Therefore \( T_1(G) \vdash \forall \theta g(x, g(x/a)). \)

### 3 \( \Sigma_1(G) \)-definable functions

The converses of Theorem 2.6 are proved by a witnessing argument. Values of a witnessing function \( f(x) \) for a \( \Sigma_1 \)-formula \( \exists a \varphi(x, a) \) is a non-empty set of witnesses for \( \varphi(x, a) \), i.e., \( \emptyset \neq f(x) \subset \{ a : \varphi(x, a) \} \).

#### 3.1 \( \Sigma_1(G) \)-definable functions in \( T_0(G) \) and in \( T_1(G) \)

Let us formulate \( T_i(G) \) for \( i = 0, 1 \) in a one-sided sequent calculus. Let us introduce an individual constant 0 or denoted ∅ for the empty set.

Terms\(^2\) are denoted \( t, s, \ldots \). Literals are \( t \in s, t \not\in s, t = s, t \neq s \). Formulas are built from literals by propositional connectives \( \lor, \land, \lor \), bounded quantifiers \( \exists x, \forall x \) and unbounded quantifiers \( \exists x, \forall x \). Thus each formula is in negation normal form, and the negation \( \neg \varphi \) is defined recursively by de Morgan’s law and elimination of double negations.

Sequents are finite sets of formulas, and denoted by \( \Gamma, \Delta, \ldots \). \( \Gamma, \Delta \) denotes the union \( \Gamma \cup \Delta \), and \( \Gamma, A \) the union \( \Gamma \cup \{ A \} \). A finite set \( \Gamma \) of formulas is intended to denote the disjunction \( \biglor \Gamma := \biglor \{ A : A \in \Gamma \} \).

Axioms or initial sequents of \( T_i(G) \) are logical ones \( \Gamma, \neg L, L \) for literals \( L \).

Inference rules of \( T_i(G) \) are divided to logical ones and non-logical ones. Logical ones are \( (\lor), (\land), (b\exists), (b\forall) \) for introducing bounded quantifiers, \( (\exists) \), \( (\forall) \) for introducing unbounded quantifiers and (cut).

\[
\begin{align*}
&\frac{\Gamma, A_0, A_1}{\Gamma, A_0 \lor A_1} \quad (\lor) \\
&\frac{\Gamma, t \in a \quad \Gamma, A(t)}{\Gamma, \exists b \in a A(b)} \quad (b\exists) \\
&\frac{\Gamma, b \notin a, A(b)}{\Gamma, \forall b \in a A(b)} \quad (b\forall) \\
&\frac{\Gamma, A(t)}{\Gamma, \exists b \in a A(b)} \quad (\exists) \\
&\frac{\Gamma, A(b)}{\Gamma, \forall b \in a A(b)} \quad (\forall) \\
&\frac{\Gamma, \neg C \quad \Gamma, \Delta}{\Gamma, \Delta} \quad (cut)
\end{align*}
\]

\(^2\)Terms are variables and the constant 0.

\(^3\)\( b \in 0 : \varphi <=> b \in 0 \) and \( \forall a \in 0 \varphi <=> a \not\in 0 \). The abbreviations are applied for \( \lor 0 := 0 \) and \( \{ b \in 0 : \varphi(b, a) \} := 0 \).
In \((b\forall)\) and \((\forall)\), \(b\) is the \textit{eigenvariable} and does not occur freely in \(\Gamma \cup \{\forall b \in a \ A(b)\}\).

Moreover inference rules \((b\exists\forall)\) and \((b\forall\exists)\) for introducing bounded quantifiers with \(\Pi_1(G)\) or \(\Sigma_1(G)\) matrices are added for conveniences. For \(\Delta_0(G)\)-formula \(\varphi\),

\[
\frac{\Gamma, s \in t \quad \Gamma, \varphi(s, a)}{\Gamma, \exists x \in t \forall y \varphi(x, y)} \quad (b\exists\forall)
\]

where \(a\) is the \textit{eigenvariable} and does not occur freely in \(\Gamma \cup \{\exists x \in t \forall y \varphi(x, y)\}\).

\[
\frac{\Gamma, x \notin t \quad \exists x \in t \exists y \neg \varphi(x, y)}{\Gamma, \forall x \in t \exists y \neg \varphi(x, y)} \quad (b\forall\exists)
\]

where \(x\) is the \textit{eigenvariable} and does not occur freely in \(\Gamma \cup \{\forall x \in t \exists y \neg \varphi(x, y)\}\).

Non-logical ones are as follows.

1. (a)

\[
\frac{\neg(t \in a \land s \in a \land \forall x \in a(x = t \lor x = s))}{\Gamma} \quad (\text{pair})
\]

where \(a\) is the \textit{eigenvariable} and does not occur freely in \(\Gamma \cup \{t, s\}\).

(b)

\[
\frac{\neg(\forall a \in t \forall b \in a(b \in c) \land \forall b \in c \exists a \in t(b \in a))}{\Gamma} \quad (\text{union})
\]

where \(c\) is the \textit{eigenvariable} and does not occur freely in \(\Gamma \cup \{t\}\).

(c) For each \(\Delta_0(G)\)-formula \(\varphi\),

\[
\frac{\neg(\forall x \in a(x \in t \land \varphi(x)) \land \forall x \in t(\varphi(x) \rightarrow x \in a))}{\Gamma} \quad (\Delta_0(G)\)-Sep)
\]

where \(a\) is the \textit{eigenvariable} and does not occur freely in \(\Gamma \cup \{\forall x \varphi(x), t\}\).

2. For \(g(\bar{x}) \in G\), let \(\theta_g(\bar{x}, a) \equiv (\forall c \psi_g(\bar{x}, a, c))\) be a \(\Pi_1(G)\)-formula assigned to \(g\) as in \(\Pi\). Then

\[
\frac{\exists c \neg \psi_g(\bar{t}, g(\bar{t}), c)}{\Gamma} \quad (g)
\]

3. \(\Delta_0(G)\)-Coll

\[
\frac{\Gamma, x \notin t \quad \exists x \in t \forall a \in c \varphi(x, a)}{\Gamma} \quad (\Delta_0(G)\)-Coll)
\]

where \(\varphi\) is a \(\Delta_0(G)\)-formula, and \(x\) and \(c\) are the \textit{eigenvariables} and does not occur freely in \(\Gamma \cup \{\forall x \in t \exists a \varphi(x, a)\}\).
4. The following inference rule \((\Sigma_1(G)-\text{Fund})\) is only for \(T_1(G)\).

\[
\frac{\neg \forall x \in y \exists a \varphi(x,a), \exists a \varphi(y,a), \Gamma \vdash \neg \varphi(t,a), \Gamma}{\Gamma} (\Sigma_1(G)-\text{Fund})
\]

where \(\varphi\) is a \(\Delta_0(G)\)-formula, and \(y\) and \(a\) are the eigenvariables and do not occur freely in \(\Gamma \cup \{ t = t, \forall x \exists a \varphi(x,a) \}\).

A \(\Sigma(G)\)-formula is either a \(\Sigma_1(G)\)-formula or a formula \(\forall x \in t \sigma\) for a \(\Sigma_1(G)\)-formula \(\sigma\).

Suppose that \(T_i(G) \vdash \forall x \exists y f(x,b)\). Let \(\text{Eq}\) denote the set of equality axioms \(\forall a[a = a], \forall a, b, c[a = b \rightarrow a = c \rightarrow b = c], \forall a, b, c[a = b \rightarrow b = c \rightarrow a \in c]\), and equality axioms for function symbols in \(L\). Also let \(\text{Ext} = \{ \forall a, b [\forall c \in a(c \in b) \wedge \forall c \in b(c \in a) \rightarrow a = b] \}\) for the extensionality. There exists a derivation of \(\neg Eq, \neg \text{Ext}, \forall x \exists y f(x,b)\) in the sequent calculus for \(T_1(G)\). Eliminate \((\text{cut})'s\) to get a cut-free derivation of the sequent \(\neg Eq, \neg \text{Ext}, \exists b \varphi_f(x,b)\). Then any formula occurring in it is one of the followings:

1. a \(\Sigma_1(G)\)-formula, which is in the end-sequent \(\neg Eq, \neg \text{Ext}, \exists b \varphi_f(x,b)\), or arises from \(\exists c \varphi_f(b, f(b), c)\) in the upper sequents of \((G), \exists a \varphi(x,a)\) in the upper sequents of \((\Delta_0(G)-\text{Coll})\) and from \(\exists a \varphi(y,a)\) in the upper sequents of \((\Sigma_1(G)-\text{Fund})\).

2. a \(\Pi(G)\)-formula, which arises from \(\neg \forall x \in y \exists a \varphi(x,a)\) in the upper sequents of \((\Sigma_1(G)-\text{Fund})\).

3. a \(\Delta_0(G)\)-formula.

In particular there occurs no \(\forall\) in the derivation though some hidden \(\forall\) may occur in \((b \exists \forall)\). Moreover we can assume that any free variable occurring in the derivation is either a variable \(x_i \in \bar{x}\) in the end-formula \(\exists b \varphi_f(x,b)\) or an eigenvariable. Otherwise substitute \(\emptyset\) for redundant free variables.

Let \(\varphi(x)\) be either a \(\Sigma_1(G)\)-formula or a \(\Sigma(G)\)-formula. A \(\Delta_0(G)\)-formula \(w_{\varphi}(x,b)\) is defined as follows. Let \(b\) be a variable not occurring in \(\varphi(x)\).

1. \(w_{\varphi}(x,b) : \varphi(x)\) if \(\varphi(x)\) is a \(\Delta_0(G)\)-formula.

2. If \(\varphi(x)\) is a \(\Sigma_1(G)\)-formula \(\exists c \psi(x,c)\) for a \(\Delta_0(G)\)-formula \(\psi\), then \(w_{\varphi}(x,b) : [\emptyset \neq b \subset \{ c : \psi(x,c) \}]\).

3. If \(\varphi(x)\) is a \(\Sigma(G)\)-formula \(\forall x \in y \exists c \psi(x,c)\) for a \(\Delta_0(G)\)-formula \(\psi\), then \(w_{\varphi}(x,b) : [\emptyset \neq b \subset \{ c : \psi(x,c) \}]\).

Let \(\Gamma = \{ \varphi_i : i < n \}\) be a set of \(\Sigma(G)\)-formulas, and \(b = \{ b_i : i < n \}\) be fresh variables. Then \(w_{\Gamma}(b) := \{ w_{\varphi_i}(b_i) : i < n \}\), and \(\neg \Gamma := \{ \neg \varphi_i : i < n \}\).

The following Lemma 3.1 yields the converses of Theorem 2.6.1a and of Theorem 2.6.1b.
Lemma 3.1 Let $\Gamma$ be a finite set of $\Sigma(G)$-formulas, $\Delta$ a finite set of $\Sigma_1(G)$-formulas, and $\bar{a}$ be a list of free variables occurring in $\Gamma \cup \Delta$.

Let $\bar{b}$ and $\bar{c}$ be fresh variables. Assume that $\neg\Gamma, \Delta$ is derivable in the sequent calculus for $T_0(G)$ [derivable in the sequent calculus for $T_1(G)$].

Then there exists a list of functions $\tilde{f}(\bar{a}, \bar{b})$ which are rudimentary-in-$G$ [primitive-recursive-in-$G$], resp. such that for any $\bar{b}$ and $\bar{a}$,

$$\bigwedge w_\Gamma(\bar{b}) \rightarrow \bigvee w_\Delta(\tilde{f}(\bar{a}, \bar{b}))$$

holds (in $V$), where $w_\Delta(\tilde{f}(\bar{a}, \bar{b}))$ is obtained from $w_\Delta(\bar{c})$ by replacing $\bar{c}$ by $\tilde{f}(\bar{a}, \bar{b})$.

**Proof.** Given a cut-free derivation of the $\neg\Gamma, \Delta$, we show the lemma by induction on the length of the derivation. In the proof we need some facts on rudimentary functions/relations in $\mathbb{I}$.

**Case 0.** Consider the case when two occurrences of a formula is contracted. When the formula is in $\neg\Gamma$, use a projection to get $f(\bar{x}, \bar{b}, c, c)$ for $w_\bar{c} \varphi$ and $w_\bar{d} \varphi$. Otherwise $w_\bar{e} \varphi \rightarrow w_\bar{f} \varphi$, where $e$ is defined by cases. Note that any $\Delta_0(G)$-relation is rudimentary-in-$G$.

**Case 1.** Consider the case when the last rule is one of $(pair)$, $(union)$ and $(\Delta_0(G)-Sep)$. For example consider the case $\neg(\forall c \in a(c \in t \land \varphi(c)) \land \forall c \in t(\varphi(c) \rightarrow c \in a))$, $\neg\Gamma, \Delta$ $(\Delta_0(G)-Sep)$

where $\varphi$ is a $\Delta_0(G)$-formula, and $a$ is the eigenvariable and does not occur freely in $\neg\Gamma \cup \{\forall c \varphi(c), t\}$.

Let $h(\bar{a}, \bar{b}, a)$ be a witnessing function of the upper sequent. For the rudimentary-in-$G$ function $g(\bar{a}) = \{c \in t : \varphi(c)\}$, $f(\bar{a}, \bar{b}) = h(\bar{a}, \bar{b}, g(\bar{a}))$ is a witnessing function for $\Gamma$.

**Case 2.** Consider the case when the last rule is one of $(g)$ for a $g(\bar{x}) \in G$.

$$\exists c \psi_g(\bar{t}, g(\bar{t}), c), \neg\Gamma, \Delta$$

where $\theta_g(\bar{x}, a) \equiv (\forall c \psi_g(\bar{x}, c))$ is a $\Pi_1(G)$-formula assigned to $g$ as in $\mathbb{I}$.

By IH we have some witnessing functions $f(\bar{a}, \bar{b})$ for the upper sequent $\{\exists c \psi_g(\bar{t}, g(\bar{t}), c)\} \cup \neg\Gamma$. Since $\forall c \psi_g(\bar{t}, g(\bar{t}), c)$ holds, $f(\bar{a}, \bar{b})$ witnesses also the lower sequent.

**Case 3.** Consider the case when the last rule is an $(\exists)$.

$$\neg\Gamma, \Delta, \varphi(s)$$

$$\neg\Gamma, \Delta, \exists a \varphi(a)$$

$(\exists)$
\[ f(\bar{a}, \bar{b}) = \{ s \} \] is a witness for \( \varphi \).

**Case 4.** Consider the case when the last rule is a \((b\exists\forall)\) introducing a \(\Pi(\mathcal{G})\)-formula.

\[
\frac{s \in t, \neg \varphi(s, d), \neg \varphi, \Delta}{\neg \forall x \in t \exists a \varphi(x, a), \neg \varphi(s, d), \neg \varphi, \Delta, \Delta (b\exists\forall)}
\]

where \( d \) is an eigenvariable, and \( s \) is a term such that any variable occurring in it occurs in \( \{ \neg \forall x \in t \exists a \varphi(x, a) \} \cup \neg \varphi \). Let us assume that \( \Gamma = \emptyset \) and \( \Delta = \{ \exists c \theta(\bar{a}, c) \} \) for a \( \Delta_0(\mathcal{G})\)-formula \( \theta \).

\[
\frac{s \in t, \exists c \theta(\bar{a}, c), \neg \varphi(s, d), \exists c \theta(\bar{a}, c)}{\neg \forall x \in t \exists a \varphi(x, a), \exists c \theta(\bar{a}, c), \Delta (b\exists\forall)}
\]

By IH we have an \( h \) such that 
\[
\forall x \in t \varphi(s, d) \rightarrow \emptyset \neq h(\bar{a}, d) \subset \{ c : \theta(\bar{a}, c) \}
\]
for any \( d \). Then for \( f(\bar{a}, b) = \bigcup \{ h(\bar{a}, d) : d \in b's \} \) with rudimentary \( b's \), we have 
\[
\forall x \in t \emptyset \neq b'x \subset \{ d : \varphi(x, d) \} \rightarrow \emptyset \neq f(\bar{a}, b) \subset \{ c : \theta(\bar{a}, c) \}.
\]

**Case 5.** Consider the case when the last rule is a \((b\forall)\).

\[
\frac{a \notin t, \varphi(a), \neg \varphi, \Delta}{\forall c \in t \varphi(c), \neg \varphi, \Delta (b\forall)}
\]

where \( a \) is an eigenvariable. Let \( \Gamma = \{ \sigma \} \) and \( \Delta = \{ \exists d \theta(\bar{a}, d) \} \) for a \( \Sigma(\mathcal{G})\)-formula \( \sigma \) and a \( \Delta_0(\mathcal{G})\)-formula \( \theta \). By IH we have an \( h \) such that \( w_{\sigma}(b) \land a \in t \rightarrow \varphi(a) \lor \emptyset \neq h(\bar{a}, b, a) \subset \{ d : \theta(\bar{a}, d) \} \). Suppose \( w_{\sigma}(b) \). Then for \( f(\bar{a}, b) = \bigcup \{ h(\bar{a}, b, a) \cap \{ d : \theta(\bar{a}, d) \} : a \in t \} \) with \( t = t(\bar{a}) \), we have either \( \forall c \in t \varphi(c) \lor \emptyset \neq f(\bar{a}, b) \subset \{ d : \theta(\bar{a}, d) \} \). If \( h \) is rudimentary-in-\( \mathcal{G} \), then so is \( f \) by bounded union.

**Case 6.** Consider the case when the last rule is a \((\Delta_0(\mathcal{G})\text{-Coll})\).

\[
\frac{\neg \varphi, \Delta}{\exists x \in t \forall a \in c \neg \varphi(x, a), \neg \varphi, \Delta, \Delta (\Delta_0(\mathcal{G})\text{-Coll})}
\]

where \( x \) and \( c \) are eigenvariables.

Let \( \Gamma = \{ \sigma \} \) and \( \Delta = \{ \exists d \theta(\bar{a}, d) \} \) for a \( \Sigma(\mathcal{G})\)-formula \( \sigma \) and a \( \Delta_0(\mathcal{G})\)-formula \( \theta \).

\[
\frac{\neg \varphi, \Delta}{\exists x \in t \forall a \in c \neg \varphi(x, a), \neg \varphi, \Delta, \Delta (\Delta_0(\mathcal{G})\text{-Coll})}
\]

By IH we have some \( h, k \) such that \( w_{\sigma}(b) \land x \in t \rightarrow \emptyset \neq h(\bar{a}, x, b) \subset \{ a : \varphi(x, a) \} \lor \emptyset \neq k(\bar{a}, x, b) \subset \{ d : \theta(\bar{a}, d) \} \). Suppose \( w_{\sigma}(b) \).

If \( \exists x \in t(\emptyset \neq k(\bar{a}, x, b) \subset \{ d : \theta(\bar{a}, d) \}) \), then \( \bigcup \{ k(\bar{a}, x, b) \cap \{ d : \theta(\bar{a}, d) \} : x \in t \} \) for the term \( t = t(\bar{a}) \) is a desired one, i.e., \( \emptyset \neq \bigcup \{ k(\bar{a}, x, b) \cap \{ d : \theta(\bar{a}, d) \} : x \in t \} \subset \{ d : \theta(\bar{a}, d) \} \).
Suppose \( \neg \exists x \in t(\emptyset \neq k(\bar{a}, x, b) \subset \{d : \theta(\bar{a}, d)\}) \). Then \( \forall x \in t(\emptyset \neq h(\bar{a}, x, b) \subset \{a : \varphi(x, a)\}) \), and hence for \( c_1 = \bigcup \{h(\bar{a}, x, b) : x \in t\} \) we obtain \( \forall x \in t \exists a \in c_1 \varphi(x, a) \).

On the other hand we have a \( j \) such that if \( \forall x \in t \exists a \in c \varphi(x, a) \), then \( \emptyset \neq j(\bar{a}, b, c) \subset \{d : \theta(\bar{a}, d)\} \) for any \( c \). We have \( \forall x \in t \exists a \in c \varphi(x, a) \). Hence \( \emptyset \neq j(\bar{a}, b, c_1) \subset \{d : \theta(\bar{a}, d)\} \), and \( j(\bar{a}, b, c_1) \) is a desired one. To sum up, for the function

\[
f(\bar{a}, b) = \begin{cases} \\
\bigcup \{k(\bar{a}, x, b) \cap \{d : \theta(\bar{a}, d)\} : x \in t\} & \text{if } \exists x \in t(\emptyset \neq k(\bar{a}, x, b) \subset \{d : \theta(\bar{a}, d)\}) \\
j(\bar{a}, b) \bigcup \{h(\bar{a}, x, b) : x \in t\} & \text{otherwise}
\end{cases}
\]

we obtain \( w_\sigma(b) \rightarrow \emptyset \neq f(\bar{a}, b) \subset \{d : \theta(\bar{a}, d)\} \). When all of \( h, k, j \) are rudimentary-in-\( \mathcal{G} \), then so is \( f \) by bounded union.

**Case 7.** Finally consider the case when the last rule is a \( \Sigma_1(\mathcal{G}) \)-Foundation. For an eigenvariable \( y \)

\[
\neg \forall x \in y \exists a \varphi(x, a), \exists a \varphi(y, a), \neg \Gamma, \Delta \quad \neg \varphi(t, a), \neg \Gamma, \Delta \quad (\Sigma_1(\mathcal{G})\text{-Fund})
\]

For simplicity let \( \Gamma = \{\sigma\} \) and \( \Delta = \{\exists d \theta(\bar{a}, d)\} \) for a \( \Sigma(\mathcal{G}) \)-formula \( \sigma \) and a \( \Delta_0(\mathcal{G}) \)-formula \( \theta \).

\[
\neg \forall x \in y \exists a \varphi(x, a), \exists a \varphi(y, a), \neg \sigma, \exists \exists d \theta(\bar{a}, d) \quad \neg \varphi(t, a), \neg \sigma, \exists \exists d \theta(\bar{a}, d) \quad (\Sigma_1(\mathcal{G})\text{-Fund})
\]

By IH we have some \( h, k \) such that for any \( b : y \rightarrow V \) and any \( c \) if \( \forall x \in y[\emptyset \neq b'x \subset \{a : \varphi(x, a)\}] \) and \( w_\sigma(c) \), then either \( \emptyset \neq h(\bar{a}, y, c, b) \subset \{a : \varphi(y, a)\} \) or \( \emptyset \neq k(\bar{a}, y, c, b) \subset \{d : \theta(\bar{a}, d)\} \). Suppose \( w_\sigma(c) \).

Let \( g(\bar{a}, y, c) = h(\bar{a}, y, c, g \downharpoonright y) \) for \( g \downharpoonright y = \{x, g(\bar{a}, x, c) : x \in y\} \), and \( k_1(\bar{a}, y, c) = k(\bar{a}, y, c, g \downharpoonright y) \).

If \( \exists x \in \text{TC}(t \cup \{t\})[\emptyset \neq k_1(\bar{a}, x, c) \subset \{d : \theta(\bar{a}, d)\}] \), then \( \emptyset \neq \bigcup \{k_1(\bar{a}, x, c) \cap \{d : \theta(\bar{a}, d)\} : x \in \text{TC}(t \cup \{t\})\} \subset \{d : \theta(\bar{a}, d)\} \) for \( t \equiv t(\bar{a}) \).

Otherwise we see that \( \forall x \in \text{TC}(t \cup \{t\})[\emptyset \neq g(\bar{a}, x, c) \subset \{a : \varphi(x, a)\}] \) by induction on \( x \). In particular \( \emptyset \neq g(\bar{a}, t, c) \subset \{a : \varphi(t, a)\} \). On the other hand we have a \( p \) such that for any \( c \) if \( w_\sigma(c) \) and \( \varphi(t, a) \), then \( \emptyset \neq p(\bar{a}, c, a) \subset \{d : \theta(\bar{a}, d)\} \) for any \( a \). Thus for \( q(\bar{a}, c) = \bigcup \{p(\bar{a}, c, a) : a \in g(\bar{a}, t, c)\} \), we obtain \( \emptyset \neq q(\bar{a}, c) \subset \{d : \theta(\bar{a}, d)\} \).

To sum up, for the function

\[
f(\bar{a}, c) = \begin{cases} \\
\{k(\bar{a}, x, c, g \downharpoonright x) \cap \{d : \theta(\bar{a}, d)\} : x \in \text{TC}(t \cup \{t\})\} & \text{if } \exists x \in \text{TC}(t \cup \{t\})(\emptyset \neq k_1(\bar{a}, x, c) \subset \{d : \theta(\bar{a}, d)\}) \\
\bigcup \{p(\bar{a}, c, a) : a \in g(\bar{a}, t, c)\} & \text{otherwise}
\end{cases}
\]

we obtain \( w_\sigma(c) \rightarrow \emptyset \neq f(\bar{a}, c) \subset \{d : \theta(\bar{a}, d)\} \). If all of \( h, k, p \) are primitive-recursive-in-\( \mathcal{G} \), then so are \( g, k_1 \) and \( f \).
Let us finish the proof of the converses of Theorem 2.6.11 and of Theorem 2.6.12. Assume that $f(x)$ is $\Sigma_1(G)$-definable in $T_1(G)$ ($i = 0, 1$), and let $\psi_f(x, b, c)$ be a $\Delta_0(G)$-formula such that $T_1(G) \models \forall x \exists b \exists c \psi_f(x, b, c)$, and $f(x) = b$ iff $\exists c \psi_f(x, b, c)$. Let $1st(d) = \{b \in \cup d : \exists c \in \cup d(b, c) = d\}$, i.e., $1st(-/(b, c)) = b$, and $2nd(d) = \{c \in \cup d : \exists b \in \cup d(b, c) = d\}$, i.e., $2nd(-/(b, c)) = c$. Each of $1st$ and $2nd$ is a rudimentary function. Then there exists a derivation of $\neg Eq, \neg Ext, \forall x \exists d \psi_f(x/1st(d), 2nd(d))$ in the sequent calculus for $T_1(G)$.

By Lemma 3.1 pick a function $g(x)$ such that either $\neg Eq \lor \neg Ext$ or $g(x) \neq \emptyset$ consists of pairs $(b, c)$ such that $\psi_f(x, b, c)$. Since the axioms of equality and of extensionality hold, we obtain $\emptyset \neq h(x) = \{1st(d) : d \in g(x)\} \subset \{b : \exists c \psi_f(x, b, c)\}$. Finally by the uniqueness of $b$, we conclude $\exists c \psi_f(x, b, c)$ for $f(x) = b = \cup h(x)$. If $g$ is rudimentary-in-$G$ [primitive-recursive-in-$G$], then so is $f$.

**Corollary 3.2** Let $G$ be a collection of hereditarily $\Pi_1$-definable set-theoretic functions $g(a)$, and $Q(x)$ a predicate on sets.

1. $Q(x)$ is rudimentary in $G$ iff $Q(x)$ is $\Delta_0(G)$ iff $Q(x)$ is $\Delta_1(G)$-definable in $T_0(G)$.

2. $Q(x)$ is primitive recursive in $G$ iff $Q(x)$ is $\Delta_1(G)$-definable in $T_1(G)$.

**Proof.** Consider Corollary 3.2.1. Let $Q(x)$ be a $\Delta_1(G)$-definable predicate in $T_0(G)$. Pick $\Sigma_1(G)$-formulas $\varphi_0, \varphi_1$ so that $T_0(G) \models \forall x \exists a \varphi_0(x, a) \iff \exists a \varphi_1(x, a)$, and $\exists a \varphi_1(x, a) \iff Q(x)$. Then $T_0(G) \models \exists a \varphi_0(x, a) \lor \exists a \varphi_1(x, a)$.

By Lemma 3.1 pick rudimentary-in-$G$ functions $f_0, f_1$ so that for any $x$, either $(\emptyset \neq f_0(x) \subset \{a : \varphi_0(x, a)\})$ or $(\emptyset \neq f_1(x) \subset \{a : \varphi_1(x, a)\})$. Then $(\emptyset \neq f_1(x) \subset \{a : \varphi_1(x, a)\}) \rightarrow \exists a \in f_1(x) \varphi_1(x, a) \rightarrow \exists a \varphi_1(x, a) \rightarrow \neg \exists a \varphi_0(x, a) \rightarrow \neg (\emptyset \neq f_1(x) \subset \{a : \varphi_1(x, a)\})$. Hence $\exists a \in f_1(x) \varphi_1(x, a) \iff \exists a \varphi_1(x, a)$.

Finally we see that each rudimentary-in-$G$ function $f$ is simple-in-$G$ in the sense that if $\varphi$ is a $\Delta_0(G)$-relation, then so is $\varphi(f(x))$. □

### 3.2 $\Sigma_1^D(G)$-definable functions in $T_2^D(G)$

As in subsection 3.1, $T_2^D(G)$ is formulated in a one-sided sequent calculus.

Inference rules $(b \exists^D \forall)$ and $(b \forall^D \exists)$ for introducing bounded quantifiers on $D$ with $\Pi_1(G)$ or $\Sigma_1(G)$ matrices are added for conveniences. For $\Delta_0(G)$-formula $\varphi$,

\[
\frac{\Gamma, \neg D(t), s \in t \quad \Gamma, \neg D(t), \varphi(s, a)}{\Gamma, \neg D(t), \exists x \in t \forall a \varphi(x, a)} \quad (b \exists^D \forall)
\]

where $a$ is the eigenvariable and does not occur freely in $\Gamma \cup \{\neg D(t), \exists x \in t \forall a \varphi(x, a)\}$.

\[
\frac{\Gamma, \neg D(t), x \notin t \quad \exists a \neg \varphi(x, a)}{\Gamma, \neg D(t), \forall x \in t \exists a \neg \varphi(x, a)} \quad (b \forall^D \exists)
\]
where \( x \) is the **eigenvariable** and does not occur freely in \( \Gamma \cup \{ \neg D(t), \forall x \in t \exists a \neg \varphi(x, a) \} \).

Non-logical ones are (pair), (union), \( (\Delta_0(G))\)-Sep, and \( (\Delta_0(G))\)-Coll, as for \( T_i(G) \). As for Foundation, \( (\Sigma_1^P(G))\)-Fund) is added.

\[
\frac{\neg D(y), \forall x \in y \exists a \varphi(x, a), \exists a \varphi(y, a), \Gamma \neg \varphi(t, a), \Gamma}{\neg D(t), \Gamma} \quad (\Sigma_1^P(G)\text{-Fund})
\]

where \( y \) and \( a \) are eigenvariables.

Inference rules \( \mathfrak{g} \) for \( g(x/a) \in G \) are modified as follows:

\[
\frac{\exists c \neg \psi_g(t, s, g(t/s), c), \Gamma}{\neg D(t), \Gamma} \quad (\mathfrak{g})
\]

where \( \psi_g(x, a, b) \equiv (\forall c \psi_g(x, a, b, c)) \) be a \( \Pi_1(G)\)-formula, cf. \([1]\).

Moreover \( (\Sigma_1(G)\text{-Submodel Rule}) \) is added. Note that \( \neg Eq \cup \neg Ext \) are \( \Sigma_1\)-sentences, and we can add these to \( (\Sigma_1(G)\text{-Submodel Rule}) \).

Inference rules for equality and transitivity of \( D \) are added.

\[
\frac{s = t, \Gamma \neg D(s), \Gamma}{\neg D(t), \Gamma} \quad (EqD)
\]

\[
\frac{s \in t, \Gamma \neg D(s), \Gamma}{\neg D(t), \Gamma} \quad (TrD)
\]

Let \( T_{2,n}^P(G) \) denote a subsystem of the sequent calculus for \( T_2^P(G) \) such that \( T_{2,n}^P(G) \vdash \theta \) iff there exists a sequent calculus \( T_2^P(G) \)-proof of \( \theta \) in which the number of nesting of the inference rules \( (\Sigma_1(G)\text{-Submodel Rule}) \) are at most \( n \)-times.

The converse of Theorem 2.6.2 is proved by induction on \( n \) using the following Lemma 3.3.

Let \( \Phi = \{ \varphi_i(x, a) : i = 1, \ldots, n \} (n \geq 0) \) be a list of \( \Delta_0(G)\)-formulas such that variables occurring in \( \varphi_i(x, a) \) are among the list \( x_1 \cup \{ a \} \).

Consider the following inference rule for each \( \varphi_i \in \Phi \).

\[
\frac{\neg D(y), \neg \varphi_i(t, y), \Gamma}{\neg D(t), \Gamma} \quad (\varphi_i)
\]

where \( y \) is the **eigenvariable** and does not occur freely in \( \Gamma \cup \{ \neg D(t), \exists y \varphi_i(t, y) \} \).

This inference rule says that \( \forall x \in D \exists y \in D \varphi_i(x, y) \).

Then a sequent calculus \( T_{2,0}^P(G) + \Phi \) is obtained from the sequent calculus for \( T_2^P(G) \) by dropping the inference rule \( (\Sigma_1(G)\text{-Submodel Rule}) \) and adding the rule \( (\varphi_i) \) for each \( \varphi_i \in \Phi \).

Given a derivation of \( \neg Eq, \neg Ext, \forall x \subset D \forall \bar{a} \exists b \varphi_f(x, \bar{a}, b) \) in the sequent calculus for \( T_{2,0}^P(G) + \Phi \), eliminate (cut)'s to get a cut-free derivation of the sequent \( \neg Eq, \neg Ext, \neg D(x), \exists b \varphi_f(x, \bar{a}, b) \).
As for $T_{i}(\mathcal{G})$, it suffices to show the following Lemma 3.3 to prove the converse of Theorem 2.6.2.

A $\Sigma^{P}(\mathcal{G})$-formula is either a $\Sigma_{1}(\mathcal{G})$-formula or a formula $\forall x \in t \sigma$ for a $\Sigma_{1}(\mathcal{G})$-formula $\sigma$ in an environment $t \in \mathcal{D}$.

**Lemma 3.3** Let $\Gamma$ be a finite set of $\Sigma^{P}(\mathcal{G})$-formulas, $\Delta$ a finite set of $\Sigma_{1}(\mathcal{G})$-formulas, and $\vec{x}, \vec{a}$ be a list of free variables occurring in $\Gamma \cup \Delta$. Also let $\vec{t} = \vec{f}(\vec{x})$ be a list of terms whose variables are among the list $\vec{x}$.

Let $\vec{b}$ and $\vec{c}$ be fresh variables. Assume that $\neg \mathcal{D}(\vec{t}), \neg \Gamma, \Delta$ is derivable in the sequent calculus for $T^{P}_{2,0}(\mathcal{G}) + \Phi$. Moreover assume that for each $\varphi \in \Phi$ there exists a function $f_{i}(\vec{x}, \vec{t})$ in $\text{SRSC}(\mathcal{G})$ such that

$$\forall \vec{x}, \theta \neq f_{i}(\vec{x}, \vec{t}) \subseteq \{y : \varphi_{i}(\vec{x}, y)\}$$

is true.

Then there exists a list of functions $\vec{f}(\vec{x}/\vec{a}, \vec{b}) \subseteq \text{SRSF}(\mathcal{G})$ such that for any $\vec{b}, \vec{a}$ and $\vec{x}$,

$$\bigwedge w_{T}(\vec{b}) \rightarrow \bigvee w_{\Delta}(\vec{f}(\vec{x}/\vec{a}, \vec{b}))$$

holds (in $V$).

**Proof.** This is seen as in Lemma 3.1. Some comments are in order. In Case 1 for $(\Delta_{0}(\mathcal{G})-\text{Sep})$, $g(\vec{x}/\vec{a}) = \{c \in t : \varphi(\vec{x}, \vec{a}, c)\}$ is in $\text{SRSC}(\mathcal{G})$. Hence so is $f(\vec{x}/\vec{a}, \vec{b}) = h(\vec{x}/\vec{a}, \vec{b}, g(\vec{x}/\vec{a}))$ by (Safe Composition). In Case 4-Case 6 we need (Bounded Union), e.g., (3) becomes here

$$f(\vec{x}/\vec{a}, \vec{b}) = \begin{cases} \bigcup \{k(\vec{x}/\vec{a}, x, b) \cap \{d : \theta(\vec{x}, \vec{a}, d)\} : x \in t \} \\
\text{if } \exists x \in t(\emptyset \neq k(\vec{x}/\vec{a}, x, b) \subseteq \{d : \theta(\vec{x}, \vec{a}, d)\}) \\
\text{otherwise}
\end{cases}$$

In Case 7 for $(\Sigma^{P}_{1}(\mathcal{G})-\text{Fund})$

$$\neg \mathcal{D}(y), \forall x \in y \exists a \varphi(x, a), \exists a \varphi(y, a), \neg \Gamma, \Delta \rightarrow \neg \varphi(t, a), \neg \Gamma, \Delta$$

Let $\Gamma = \{\sigma\}$, $\Delta = \{\exists d \theta(\vec{x}, \vec{a}, d)\}$ for a $\Sigma^{P}(\mathcal{G})$-formula $\sigma$ and a $\Delta_{0}(\mathcal{G})$-formula $\theta$. Assume for any $b : y \rightarrow V$ and any $c$ if $\forall x \in y[\emptyset \neq b' \subseteq \{a : \varphi(x, a)\}]$ and $w_{\sigma}^{\mathcal{G}}(\vec{x}, \vec{a})$, then either $\emptyset \neq h(\vec{x}, y/\vec{a}, c, b) \subseteq \{a : \varphi(y, a)\}$ or $\emptyset \neq k(\vec{x}, y/\vec{a}, c, b) \subseteq \{d : \theta(\vec{x}, \vec{a}, d)\}$. Then by (Predicative Set Recursion) let $g(\vec{x}, y/\vec{a}, c) = h(\vec{x}, y/\vec{a}, c, g \upharpoonright y)$ for $g | y = \{(x, g(\vec{x}, x/\vec{a}, c)) : x \in y\}$, and $k_{1}(\vec{x}, y/\vec{a}, c) = k(\vec{x}, y/\vec{a}, c, g \upharpoonright y)$ by (Safe Composition). Also note that $T_{i}(t \cup \{t\}/-) + \text{Fund}$ is allowed for $t = t(\vec{x})$ with $\mathcal{D}(\vec{x})$.

There is a new case.

**Case 8.** Consider the case when the last rule is a $(\varphi_{i})$ with the eigenvariable $y$.

$$\neg \mathcal{D}(y), \neg \varphi_{i}(\vec{t}, y), \neg \Gamma, \Delta$$

$$\neg \mathcal{D}(\vec{t}), \neg \Gamma, \Delta$$

(\varphi_{i})
For simplicity let us assume that $\vec{t}_i$ is a list of variables $\vec{x}_i \subseteq \vec{x}$, and $\Gamma = \{ \sigma \}$, $\Delta = \{ \exists \vec{c} \theta(\vec{x}, \vec{a}, \vec{c}) \}$ for a $\Sigma^P(\mathcal{G})$-formula $\sigma$ and a $\Delta^0(\mathcal{G})$-formula $\theta$. By IH we have for an $h \in \text{SRSF}(\mathcal{G})$ such that $\varphi_i(\vec{x}_i, y) \land u^h_i(\vec{x}, \vec{a}) \rightarrow \emptyset \neq h(\vec{x}, y/\vec{a}, b) \subset \{ c : \theta(\vec{x}, \vec{a}, c) \}$ for any $y$. On the other hand we have $\emptyset \neq f_i(\vec{x}, i/-) \subset \{ y : \varphi_i(\vec{x}_i, y) \}$ by the assumption $\mathbf{1}$. Hence for $f(\vec{x}/\vec{a}, b) = \bigcup \{ h(\vec{x}, y/\vec{a}, b) : y \in f_i(\vec{x}_i/-) \}$ by (Safe Composition), we obtain $u^b_i(\vec{x}, \vec{a}) \rightarrow \emptyset \neq f(\vec{x}/\vec{a}, b) \subset \{ c : \theta(\vec{x}, \vec{a}, c) \}$. □

Let us finish the proof of the converse of Theorem 2.14. Assume that $f(\vec{x}/\vec{a})$ is $\Sigma_1(\mathcal{G})$-definable in $T^P_2(\mathcal{G})$, and let $\psi_f(\vec{x}, \vec{a}, b, c)$ be a $\Delta_0(\mathcal{G})$-formula such that $T^2(\mathcal{G}) \vdash \forall \vec{x} \in \mathcal{D}\forall \vec{a}!\exists b \exists c \psi_f(\vec{x}, \vec{a}, b, c)$, and $\forall \vec{x} \in \mathcal{D}\forall \vec{a}!\exists d \psi_f(\vec{x}, \vec{a}, 1st(d), 2nd(d))$. By IH on $n$ we have a function $f_i(\vec{x}/-)$ in $\text{SRSF}(\mathcal{G})$ enjoying the assumption $\mathbf{1}$. Therefore by Lemma 3.3 pick a function $g(\vec{x}/\vec{a})$ such that either $\neg Eq \lor \neg Ext$ or $g(\vec{x}/\vec{a}) \neq \emptyset$ consists of pairs $(b, c)$ such that $\psi_f(x, a, b, c)$. We obtain $\emptyset \neq h(\vec{x}/\vec{a}) = \bigcup \{ 1st(d) : d \in g(\vec{x}/\vec{a}) \} \subset \{ b : \exists c \psi_f(\vec{x}, \vec{a}, b, c) \}$. The uniqueness of $b$ yields $\exists c \psi_f(\vec{x}, \vec{a}, b, c)$ for $f(\vec{x}/\vec{a}) = b = \bigcup h(x/a)$.

A predicate $Q(\vec{x}/\vec{a})$ on sets is said to be $\Delta^P_2(\mathcal{G})$-definable in $T^P_2(\mathcal{G})$ if there are $\Sigma_1(\mathcal{G})$-formulas $\varphi_i (i = 0, 1)$ such that $T^2(\mathcal{G}) \vdash \forall \vec{x} \in \mathcal{D}\forall \vec{a}! (\neg \varphi_0(\vec{x}, \vec{a}) \leftrightarrow \varphi_1(\vec{x}, \vec{a}))$, and $Q(\vec{x}/\vec{a})$ iff $\varphi_1(\vec{x}, \vec{a})$.

**Corollary 3.4** Let $\mathcal{G}$ be a collection of hereditarily $\Pi_1$-definable set-theoretic functions $g(\vec{x}/\vec{a})$, and $Q(\vec{x}/\vec{a})$ a predicate on sets. $Q(\vec{x}/\vec{a})$ is in $\text{SRSF}(\mathcal{G})$ iff $Q(\vec{x}/\vec{a})$ is $\Delta^P_2(\mathcal{G})$-definable in $T^P_2(\mathcal{G})$.

**Proof.** This is seen as in Corollary 3.2. □

## 4 $\Sigma_1!(L^{(\omega)})$-definable functions in $T^D_3(\mathcal{G})$

In this section some elementary fact in $\mathbf{1}$ are assumed.

**Definition 4.1** We say that a set-theoretic function $f(\vec{x}/\vec{a})$ is $\Sigma^P_1!(L^{(\omega)})$-definable in $T^P_3(\mathcal{G})$ if there exists a $\Sigma_1!(L^{(\omega)})$-formula $\varphi(\vec{x}/\vec{a}, b)$ in $L^{(\omega)}$ such that $T \vdash \forall \vec{x} \in \mathcal{D}\forall \vec{a}! b \varphi(\vec{x}/\vec{a}, b) \land f(\vec{x}/\vec{a}) = b \iff V \models \varphi(\vec{x}/\vec{a}, b)$ for any $\vec{x}, \vec{a}, b$.

**Theorem 4.2** Let $\mathcal{G}$ be a collection of hereditarily $\Pi_1$-definable set-theoretic functions $g(\vec{x}/\vec{a})$, and $f$ a set-theoretic function. $f(\vec{x}/\vec{a}) \in \text{PCSF}^e(\mathcal{G})$ iff $f(\vec{x}/\vec{a})$ is $\Sigma^P_1!(L^{(\omega)})$-definable in $T^P_3(\mathcal{G})$. □
Next let us show the easy half of Theorem 4.2.

Lemma 4.3 Each $f(\bar{x}/\bar{a}) \in \text{PCS}^\dagger(\mathcal{G})$ is $\Sigma^P_1(\mathcal{L}(\omega))$-definable in $T^P_3(\mathcal{G})$.

Proof. This is seen as in Lemma 2.7.

For a $\text{PCS}^\dagger(\mathcal{G})$-function $f$, show that in $T^P_3(\mathcal{G})$, $f$ is $\Delta_0(\mathcal{G})$-definable together with its simplicity-in-$\mathcal{G}$.

Let $f(\bar{x}/\bar{a}) = h(\bar{r}(\bar{x}/-) / \bar{t}(\bar{x}, \bar{a}))$ be defined by (Safe Composition) from $h$, $\bar{r}$ and $\bar{t}$, and $h$, $\bar{r}$ and $\bar{f}$ function symbols for $h$, $\bar{r}$ and $\bar{f}$, resp. Let $\varphi_f(\bar{x}, \bar{a}, b) : \iff h(\bar{r}(\bar{x}/-) / \bar{t}(\bar{x}, \bar{a})) = b$. Then by the inference rule ($\Sigma(\mathcal{G})$-Submodel Rule) we have $h(\bar{r}(\bar{x}/-) / \bar{t}(\bar{x}, \bar{a})) = b$, so the transitive closure of this relation is $\Delta_0(\mathcal{G})$-definable.

Let $f(x, y/\bar{a}) = h(x, y/\bar{a}, \{f(z, y/\bar{a}) : z \in x\})$ be defined by (Predicative Set Recursion) from $h$. Let

$$
\varphi(x, y, \bar{a}, b, c) : \iff (c \text{ is a function on } \text{TC}(x \cup \{x\})) \land \\
\forall z \in \text{TC}(x \cup \{x\}) h(z, y, \bar{a}, c''z) = c'z \land (c'x = b)
$$

where for $c''x = \{c'z : z \in x\} = \{b \in \cup \cup c : \exists z \in x, z, b \in c\}$. Then $f(x, y/\bar{a}) = b$ iff $\exists c\varphi(x, y, \bar{a}, b, c)$. Note that $\varphi$ is a $\Delta_0(\mathcal{L}(\omega))$-formula.

We show $\forall \varphi \in \mathcal{D} \forall \varphi \in \mathcal{D} \exists \varphi(x, y, \bar{a}, b, c)$. There is nothing to prove for the uniqueness of $b$.

Let $\theta(x, c) : \equiv \varphi(x, y, \bar{a}, c', x; c)$. Suppose $x_0 \in D$, and let $d_0 \in D$ be the transitive closure of $x_0 \cup \{x_0\}$. We show $\exists c \theta(x_0, c)$. We have $d_0 \in D$ by (transitivity).

The uniqueness of $c$, i.e., $\theta(x, c) \land \theta(x, d) \rightarrow c = d$ follows from ($\Delta^P_0(\mathcal{G})$-Foundation).

Suppose $\forall z \in x \cap d_0 \exists c \theta(z, c, c)$. We show $\exists c \theta(x, c)$ assuming $x \in d_0$. Then by ($\Sigma^P_1(\mathcal{G})$-Foundation) we have $\forall x \in D \cap d_0 \exists c \theta(x, c)$. Hence $\exists c \theta(x_0, c)$.

By ($\Sigma^P_1(\mathcal{G})$-Replacement), pick an $e$ so that $\forall z \in x \cap d_0 \theta(z, e'z)$. Let $c_0 = \cup \{e'z : z \in x\}$. Then $c_0$ is seen to be a function on $\text{TC}(x)$. For $z_1, z_2 \in x$ and $z \in d_1 \cap d_2$ with $d_i = \text{dom}(c_i)$, $c_i = e'z_i$ ($i = 1, 2$), if $c_1 \cup z = c_2 \cup z$, then $\varphi_h(z, y, \bar{a}, c''\bar{z}, c'\bar{z})$ for $i = 1, 2$. Hence $c''\bar{z} = c'_\bar{z}$. ($\Delta^P_0(\mathcal{G})$-Foundation) with $d_1 \cup d_2 \subset d_0 \subset D$ yields $c_1 \cup (d_1 \cap d_2) = c_2 \cup (d_1 \cap d_2)$. Hence $c_0$ is a function.

Let $b$ be such that $\varphi_h(x, y, \bar{a}, c''\bar{z}, c''\bar{z})$, and let $c_x = c_0 \cup \{x\}$. Then $\theta(x, c_x)$ as desired.

Finally let $f(\bar{x}/\bar{a}, c) = id(\exists b \in c g(\bar{x}/\bar{a}, b) = d))$ be defined by (4) from $g$. Let $\varphi_f(\bar{x}, \bar{a}, c, d) : \iff \exists e(\exists b \in c g(\bar{x}, \bar{a}, b) = d = e) \lor (e = d = \emptyset \land (c \neq \emptyset \rightarrow \exists b, c = c_0 \land (b \neq b_1 \land (\forall \bar{a} g(\bar{x}, \bar{a}, b, d)))$. $\varphi_f$ is a $\Sigma_1(\mathcal{L}(\omega))$-formula.

Remark. Consider (Normal Separation), $f(\bar{x}/\bar{a}, c) = \{b \in c : h(\bar{x}/\bar{a}, b) \neq 0\}$ in $\text{PCS}^\dagger$ for $\Sigma^P_1(\mathcal{G})$-definable function $h$, then $T^P_3(\mathcal{G})$ proves the existence of $f(\bar{x}/\bar{a}, c)$ from $\Delta^P_0(\mathcal{G})$-Separation. However $\{b \in c : h(\bar{x}/\bar{a}, b) \neq 0\} = d \not\implies \forall b \in d[b \in c \land h(\bar{x}/\bar{a}, b) \neq 0] \land \forall b \in c [h(\bar{x}/\bar{a}, b) \neq 0 \rightarrow b \in d]$ seems not to be a $\Sigma_1(\mathcal{G})$-relation due to the bounded universal quantifiers $\forall b \in d, \forall b \in c$ whose scope contains an unbounded existential quantifier.
4.1 A witnessing argument for \( T^D_3(\mathcal{G}) \)

Up to here our proofs work since \textit{bounded union} is available for rudimentary functions and safe recursive set functions. Therefore we need to modify the proofs for \( \text{PCSF}^n \). In what follows \( n \) denotes a fixed natural number.

Let us formulate \( T^{(n)}_3(\mathcal{G}) \) in a one-sided sequent calculus as for \( T^D_3(\mathcal{G}) \).

Inference rules (\( \exists ! \)) and (\( \forall ! \)) for introducing quantifiers \( \exists !, \forall ! \) are added for conveniences.

\[
\begin{align*}
\Gamma, \varphi(t) & \quad \Gamma, \text{Unique}_{a,b}^n(\varphi) \quad (\exists !) \\
\Gamma & \quad \exists ! a \varphi(a) \\
\end{align*}
\]
\[
\begin{align*}
\Gamma, \neg \varphi(b) & \quad \neg \text{Unique}_a(\varphi) \\
\Gamma & \quad \forall ! a \neg \varphi(a) \\
\end{align*}
\]

where \( a, b \) are the eigenvariables, \( \text{Unique}_{a,b}^n(\varphi) := \{ \neg \varphi(a), \neg \varphi(b), a = b \} \) and \( \text{Unique}_a(\varphi) := (\forall a, b(\varphi(a) \land \varphi(b) \rightarrow a = b)) \). Thus \( \forall ! a \neg \varphi(a) \Leftrightarrow \neg \exists ! a \varphi(a) \).

Inference rules (\( b \in D \forall \)) and (\( b \in D \exists \)) are replaced by their unique versions (\( b \in D \forall ! \)) and (\( b \in D \exists ! \)), resp. For \( \Delta_0(\mathcal{L}(n)) \)-formula \( \varphi \),

\[
\begin{align*}
\Gamma & \quad \neg \forall ! (t), s \in t \quad \neg \exists ! (t), \varphi(s, a), \neg \text{Unique}_a(\neg \varphi(s, a)) \\
\Gamma & \quad \neg \exists ! (t), \exists x \in t \exists ! a \varphi(x, a) \\
\end{align*}
\]

where \( a \) is the eigenvariable and does not occur freely in \( \Gamma \cup \{ \neg \exists ! (t), \forall x \in t \exists ! a \varphi(x, a) \} \).

Non-logical inference rules are (\( \text{Eq}D \)), (\( \text{Trcl}D \)), (\( g \)) for \( g(\vec{x}/\vec{a}) \in \mathcal{G} \).

Also (\( \text{trcl} \)) for transitive closure, (\( f \)) for \( f \in \mathcal{L}(n) \), (\( \Sigma^P_1(\mathcal{L}(n)) \)-Fund), (\( \Delta^P_0(\mathcal{L}(n)) \)-Repl) and (\( \Sigma_1(\mathcal{L}(n)) \)-Submodel Rule) are added.

\[
\begin{align*}
\neg \text{trcl}(\text{TC}(t), t, s), \Gamma & \quad \text{trcl} \quad \neg \theta_t(\vec{i}, \vec{s}, f(\vec{i}/\vec{s})), \Gamma \\
\neg \exists ! (t), \Gamma & \quad \text{f} \\
\end{align*}
\]

where \( \theta_t \in \Delta_0(\mathcal{L}(n-1)) \), \( t \in \mathcal{L}(n) \) and \( T^{(n-1)}_3(\mathcal{G}) \vdash \forall \vec{x} \in D \forall \vec{a} \exists ! b \theta_t(\vec{x}, \vec{a}, b) \).

For an eigenvariables \( y \) and a \( \Delta_0(\mathcal{L}(n)) \)-formula \( \varphi \)

\[
\begin{align*}
y \notin \text{TC}(t \cup \{ t \}), \neg \forall x \in y \exists ! a \varphi(x, a), \exists ! a \varphi(y, a), \Gamma & \quad \neg \exists ! a \varphi(t, a), \Gamma \\
\neg \exists ! (t), \Gamma & \quad \text{Fund} \\
\end{align*}
\]

For eigenvariables \( x, c \) and a \( \Delta_0(\mathcal{L}(n)) \)-formula \( \varphi \)

\[
\begin{align*}
\Gamma, x \notin t, \exists ! a \varphi(x, a) & \quad \neg \forall x \in t \varphi(x, c x), \Gamma \\
\neg \exists ! (t), \Gamma & \quad \text{Repl} \\
\end{align*}
\]

For each \( \Delta_0(\mathcal{L}(n)) \)-formula \( \varphi(\vec{x}, a) \) whose free variables are among the list \( \vec{x} \cup \{ a \} \)

\[
\begin{align*}
\forall \vec{x} \subset D \exists ! a \varphi(\vec{x}, a) & \quad \forall \vec{x} \subset D \exists y \in D \varphi(\vec{x}, y) \\
\end{align*}
\]

\( \text{Fund} \), (\( \Sigma_1(\mathcal{L}(n)) \)-Submodel Rule).
The converse of Theorem 4.2 is proved by main induction on \( n \) with subsidiary induction on the number of nested applications of \((\Sigma_1 ! (\mathcal{L}(n))) - \text{Submodel Rule}\) as in the proof of Theorem 2.6.2.

We see from MIH that each function symbol \( f \) in the language \( \mathcal{L}(n) \) denotes a function in \( \text{PCSF}^f (\mathcal{G}) \).

Let \( \Phi = \{ \varphi_i (\bar{x}_i, a) : i = 1, \ldots, m \} \ (m \geq 0) \) be a list of \( \Sigma_1 ! (\mathcal{L}(n)) \)-formulas such that variables occurring in \( \varphi_i (\bar{x}_i, a) \) are among the list \( \bar{x}_i \cup \{ a \} \). Then for each \( \varphi_i \in \Phi \).

\[
\frac{\neg D(y), \neg \varphi_i (t_i, y), \Gamma}{\neg D(t_i), \Gamma} (\varphi_i)
\]

where \( y \) is the eigenvariable and does not occur freely in \( \Gamma \cup \{ \neg D(t_i), \exists y \varphi_i (t_i, y) \} \).

\( T_{3,0} (\mathcal{G}) + \Phi \) is obtained from \( T_{3} (\mathcal{G}) \) by dropping the inference rule \((\Sigma_1 ! (\mathcal{L}(n)) - \text{Submodel Rule})\) and adding the rule \((\varphi_i)\) for each \( \varphi_i \in \Phi \).

Given a derivation of \( \neg Eq, \neg Ext, \forall \bar{x} \in D \forall a \exists b \varphi_f (\bar{x}, \bar{a}, b) \) in the sequent calculus for \( T_{3} (\mathcal{G}) + \Phi \), eliminate (cut)'s to get a cut-free derivation of the sequent \( \neg Eq, \neg Ext, \neg D(\bar{x}), \exists! b \varphi_f (\bar{x}, \bar{a}, b) \). Then any formula occurring in it is one of the followings:

1. a \( \Sigma_1 ! (\mathcal{G}) \)-formula, which is in the end-sequent \( \neg D(\bar{x}), \exists! b \varphi_f (\bar{x}, \bar{a}, b) \), or arises from \( \exists! a \varphi (x, a) \) in the upper sequents of \((\Delta_0^P (\mathcal{L}(n)) - \text{Repl})\) and from \( \exists! a \varphi(y, a) \) in the upper sequents of \((\Sigma_1^P ! (\mathcal{L}(n)) - \text{Fund})\).

2. Negated formulas \( \neg \forall x \in y \exists! a \varphi (x, a) \) and \( \neg \exists! a \varphi(t, a) \) in the upper sequents of \((\Sigma_1^P ! (\mathcal{L}(n)) - \text{Fund})\) and \( \neg \forall x \in t \varphi(x, c'x) \) of \((\Delta_0^P (\mathcal{L}(n)) - \text{Repl})\), \( \neg \varphi_i (t_i, y) \) in the upper sequents of \((\varphi_i)\) and \( \text{Unique}_{a, b} (\varphi) \) in \( (\exists!) \) for \( \Delta_0 (\mathcal{L}(n)) \)-formulas \( \varphi \).

3. Negated formulas \( \neg \text{Unique}_{a} (\varphi) \) in the upper sequents of \((\forall!)\) for \( \Delta_0 (\mathcal{L}(n)) \)-formulas \( \varphi \).

4. a negative literal \( \neg D(t) \).

5. a \( \Delta_0 (\mathcal{L}(n)) \)-formula.

Unbounded universal quantifiers in \( \exists! a \varphi(x, a) \) and unbounded existential quantifiers in \( \neg \text{Unique}_{a} (\varphi) \) are restricted to classes, which are generated as follows.

1. Each singleton \( \{ f(\bar{x}/\bar{a}) \} \) for \( f \in \text{PCSF}^f (\mathcal{G}) \) is a class.

2. For classes \( X, Y \), \( X \cup Y \) is a class.

3. If \( X(a) \) is a class and \( f \in \text{PCSF}^f (\mathcal{G}) \), then \( \bigcup \{ X(a) : a \in f(\bar{x}/\bar{a}) \} \) is a class.

In models of \( T_3^P (\mathcal{G}) \) a class may be a proper class. Each class is defined by a formula in a special form, \( \text{condition} \).
\textbf{Definition 4.4} Let $\mathcal{L}(\text{PCSF}'(G))$ be the language \{\varepsilon, =\} \cup \{f : f \in \text{PCSF}'(G)\} \cup \{x_i, a_i : i \in \omega\}$ with function symbol $f$ for each function $f$, and two sorted variables $x_i, a_i$. Variables $x_i$ are intended to vary through elements in the predicates $D$, and $a_i$ through the universe. (Stratified) Terms in $\mathcal{L}(\text{PCSF}'(G))$ are generated as follows. Each variable is a term. If $t_1, \ldots, t_k$ and $s_1, \ldots, s_m$ are terms, and variables occurring in each $t_i$ are $x$-variables, then $f(t_1, \ldots, t_k/s_1, \ldots, s_m)$ is a term for $f(x_1, \ldots, x_k/a_1, \ldots, a_m) \in \text{PCSF}'(G)$.

Let $\ast$ be a symbol not in the language. Then the set of condition is generated recursively as follows.

1. For each term $t$ (in $\mathcal{L}(\text{PCSF}'(G))$), $t = \ast$ is a condition.
2. If $\lambda_i (i = 0, 1)$ are conditions, then so is $\lambda_0 \lor \lambda_1$.
3. If $\lambda(a)$ is a condition and $t$ is a term, then $\exists a \in t \lambda(a)$ is a condition.

For condition $\lambda(s)$, let $X_{\lambda} = \{d : \lambda(s) = d\}$ denote the class defined by $\lambda$.

Each condition is a disjunction of formulas in the following form:

$$\lambda(\vec{x}, \vec{a}, \ast) \equiv \exists c_1 \in f_1(\vec{x}/\vec{a}) \exists c_2 \in f_2(\vec{x}/\vec{a}, c_1) \cdots \exists c_n \in f_k(\vec{x}/\vec{a}, \vec{c}_{k-1}) [h(\vec{x}/\vec{a}, \vec{c}_k) = \ast]$$

where $\vec{c}_i = (c_1, \ldots, c_{i-1})$ and $f_1, f_2, \ldots, f_k \in \text{PCSF}'(G)$ with $k \geq 0$.

\textbf{Definition 4.5} For formulas $\varphi(d)$ and classes $X_{\lambda}$, $\forall d \in X_{\lambda} \varphi(d)$ denotes the formula defined as follows. If $\lambda \equiv (t = \ast)$, then, $\forall d \in X_{t=\ast} \varphi(d) \iff \varphi(t)$. For disjunction $\lambda_0 \lor \lambda_1$ of conditions $\lambda_i$, $\forall d \in X_{\lambda_0 \lor \lambda_1} \varphi(d) \iff \bigwedge_{i=0,1} \forall d \in X_{\lambda_i} \varphi(d)$. Finally let $X_{\lambda}$ be a class defined from a condition $\lambda$ in \cite{4}. Then $\forall d \in X_{\lambda} \varphi(d)$ denotes the formula

$$\forall d \in X_{\lambda} \varphi(d) \iff \forall c_1 \in f_1(\vec{x}/\vec{a}) \forall c_2 \in f_2(\vec{x}/\vec{a}, c_1) \cdots \forall c_k \in f_k(\vec{x}/\vec{a}, \vec{c}_{k-1}) \varphi(h(\vec{x}/\vec{a}, \vec{c}_k))$$

Note that if $\varphi$ is a bounded formula in the language $\mathcal{L}(\text{PCSF}'(G))$, then so is the formula $\forall d \in X_{G} \varphi(d)$. Also note that the characteristic function $\chi_{\varphi}$ of bounded formulas in the expanded language is a PC SF'-function.

In what follows $X$ varies through classes defined by conditions.

A witness $b$ of a $\Sigma_1 !(\mathcal{L}')$-formula $\exists a \varphi$ with respect to classes $X$ is a unique witness in $X$, i.e., $\varphi(b) \land \forall a \in X (\varphi(a) \rightarrow a = b)$.

A $\Sigma_1 !(\mathcal{L}'(n))$-formula is either a $\Sigma_1 !(\mathcal{L}'(n))$-formula or a formula $\forall x \in t \sigma$ for a $\Sigma_1 !(\mathcal{L}'(n))$-formula $\sigma$ in an environment $t \in D$. $w^X_{\varphi}(b)$ for a $\Sigma_1 !(\mathcal{L}'(n))$-formula $\varphi$ is a bounded formula in the language $\mathcal{L}(\text{PCSF}'(G))$ for each class $X$.

1. $w^X_{\varphi}(b) : \varphi$ if $\varphi$ is a $\Delta_0(\mathcal{L}'(n))$-formula.
2. If $\varphi$ is a $\Sigma_1 !(\mathcal{L}'(n))$-formula $\exists c \psi(c)$ for a $\Delta_0(\mathcal{L}'(n))$-formula $\psi$, then

$$w^X_{\varphi}(b) : \psi(b) \land \text{Unique}^X_{c}(\psi(b))$$

where

$$\text{Unique}^X_{c}(\psi(b)) : b \in X \land \forall c \in X (\psi(c) \rightarrow b = c)$$
3. If \( \varphi \) is a formula \( \forall x \in y \exists c \, \psi(x, c) \) for a \( \Delta_0(\mathcal{L}^{(n)}) \)-formula \( \psi \), then \( w^X_\varphi(b) \) iff \( b \) is a function on \( y \) such that \( \forall x \in y[w^X_\varphi(y, y, x, c) (b'x)] \), i.e., \( \forall x \in y[\psi(x, b'x) \land \text{Unique}_c^X(\psi(x, b'x))] \).

**Definition 4.6** Let \( X \) be a variable ranging over classes. \( \text{PCSF}^X_X(G) \) denotes a set of set functions depending on classes \( X \) recursively defined as follows.

1. \( \text{PCSF}^X(G) \subset \text{PCSF}^X_X(G) \).
2. \( \text{PCSF}^X_X(G) \) is closed under (Safe Composition) and (Predicative Set Recursion).
3. When \( f \) is defined from \( j, k, g, h \in \text{PCSF}^X_X(G) \) and a \( \varphi(\bar{x}, \bar{a}) \in \Sigma_1(G) \) by definition by cases
   \[
   f(\bar{x}/\bar{a}) = \begin{cases} 
   j(\bar{x}/\bar{a}) & \text{if } \forall x \in g(\bar{x}/\bar{a})[w^X_\varphi(h(x, \bar{a}))] \\
   k(\bar{x}/\bar{a}) & \text{otherwise}
   \end{cases}
   \]
   then \( f \in \text{PCSF}^X_X(G) \).

Each \( f \in \text{PCSF}^X_X(G) \) denotes a function in \( \text{PCSF}^X(G) \) depending uniformly on classes \( X \).

**Proposition 4.7** Let \( f_X(\bar{x}/\bar{a}) \) be a function in \( \text{PCSF}^X_X(G) \) and \( \lambda(\bar{x}, \bar{a}) \) a condition. In the definition of \( f_X \) replace the ‘variable’ \( X \) by the class \( X_\lambda \). It results in a function \( F(\bar{x}/\bar{a}) = f_{X_\lambda(x,a)}(\bar{x}/\bar{a}) \) in \( \text{PCSF}^X(G) \).

**Proof.** This is seen from Definition 4.6 and the fact that \( w^X_\varphi(b) \) is a bounded formula with \( \text{PCSF}^X(G) \)-functions. \( \square \)

The following Lemma 4.8 yields the converse of Theorem 4.2. For a condition \( \lambda \), \( w^X_\varphi(b) \Leftrightarrow w^X_{\varphi_1}(b) \). For a finite set \( \Delta = \{ \varphi_i : i < m \} \) of \( \Sigma_1(\mathcal{L}^{(n)}) \)-formulas, a list of functions \( \bar{f}(\bar{x}/\bar{a}, \bar{b}) = (f_i(\bar{x}/\bar{a}, \bar{b}) : i < m) \), and a condition \( \lambda \), \( w^X_\Delta(\bar{f}(\bar{x}/\bar{a}, \bar{b})) = \{ w^X_{\varphi_i}(f_i(\bar{x}/\bar{a}, b)) : i < m \} \).

For a finite set of formulas \( \Delta_u = \{ \text{Unique}_{\lambda_1}(\theta_i) : i < m \} \) with \( \Delta_0(\mathcal{L}^{(n)}) \)-formulas \( \theta_i \) and a condition \( \lambda \), \( \Delta_u^\lambda := \{ \text{Unique}_{\lambda_1}^\lambda(\theta_i) : i < m \} \), where

\[-\text{Unique}_{\lambda_1}^\lambda(\theta) \iff \exists a, b \in X_\lambda(\theta(a) \land \theta(b) \land a \neq b) \]

**Definition 4.8** For conditions \( \lambda \) and a sequent \( S \), \( E(\lambda; S) \) denotes a set of conditions (envelope of \( \lambda \) with respect to \( S \)) obtained from \( \lambda \) by applying the following three operations:

1. \( \mu(\bar{y}, \bar{b}, *) \rightarrow \mu \lor (t(\bar{y}, \bar{b}) = *) \) for terms \( t \) over \( \mathcal{L}^{(n)} \) in variables \( \bar{y}, \bar{b} \).
2. \( (\mu_0(\bar{y}, \bar{b}, *), \mu_1(\bar{y}, \bar{b}, *)) \rightarrow \mu_0 \lor \mu_1 \).
3. \( \mu(\bar{y}, \bar{b}, d, *) \rightarrow \exists d \in t \mu \) for terms \( t \) over \( \mathcal{L}^{(n)} \) in variables \( \bar{y}, \bar{b} \) for which the negative literal \( d \notin t \) is in \( S \).
Lemma 4.9 Let $\Delta$ be a finite set of $\Sigma_1(\mathcal{L}^{(n)})$-formulas, and $\Gamma$ a finite set of $\Sigma^0_1(\mathcal{L}^{(n)})$-formulas. Let $\Delta_u$ be a finite set of negated formulas $\neg\text{Unique}_u(\theta_i)$ for $\Delta_0(\mathcal{L}^{(n)})$-formulas $\theta_i$. Let $\Psi$ be a finite set of $\Sigma_1(\mathcal{L}^{(n)})$-formulas. Also let $\vec{x}, \vec{a}$ be a list of free variables occurring in $\Gamma \cup \Delta_u \cup \Delta \cup \Psi$, $\vec{f} = \vec{f}(\vec{x})$ a list of terms whose variables are among the list $\vec{x}$, and $\vec{b}$ and $\vec{c}$ fresh variables.

Moreover assume that for each $\Phi \in PCSF$ is true. Let $\vec{F}$ be a list of functions.

Assume that a sequent $S = \neg D(\vec{f}) \cup \neg \Gamma \cup \Delta_u \cup \Delta \cup \Psi$ is derivable in $T^{(n)}_A(G) + \Phi$. Moreover assume that for each $\varphi_i \in \Phi$ there exists a function $f_i(\vec{x}_i/-) \in PCSF(G)$ such that

$$\forall \vec{x}_i \varphi_i(\vec{x}_i, f_i(\vec{x}_i/-))$$

is true.

Then there exist a condition $\lambda(\vec{x}/\vec{a}, \vec{b})$ which depends only on $\vec{x}, \vec{a}, \vec{b}$, and a list of functions $f_X(\vec{x}/\vec{a}, \vec{b}) \in PCSF_X(G)$ such that for any condition $\mu \in E(\lambda; \mathcal{S})$

$$\bigwedge w^\mu_1(\vec{b}) \rightarrow \bigvee \Delta^\mu_u \cup \bigvee w^\mu_1(\vec{f}_{X_u}(\vec{x}/\vec{a}, \vec{b})) \wedge \bigvee \Psi$$

holds (in $V$) for any $\vec{x}, \vec{a}, \vec{b}$.

Proof. We see from MIH on $n$ that each function symbol $f$ in the language $\mathcal{L}^{(n)}$ denotes a function in $PCSF(G)$.

We show how to modify the condition $\lambda_0$ for upper sequents of inference rules to one $\lambda$ for the lower sequent. When an inference rule has two upper sequents with no eigenvariables, the conditions $\lambda_0, \lambda_1$ for upper sequents can be merged to their disjunction $\lambda_0 \vee \lambda_1$. As our proof goes, it is clear that (7) holds for any $\mu \in E(\lambda; \mathcal{S})$ if once it holds for $\lambda$. Since $f_X$ does not depend on conditions $\lambda, \mu$, it gives a uniform 'solution'.

Note that $\vec{f}$ may depend on $\lambda$ in Case 0, Case 9 and Case 10 below. For brevity’s sake, let us write $f_X = f_{X, \lambda}$.

Case 0. The case when two occurrences of a formula $\varphi$ is contracted in $\Delta$. Let $e$ be defined by cases from $c, d$ and a bounded formula $w^\lambda_{\varphi}(c)$, whose characteristic function is in $PCSF(G)$. Then $w^\lambda_{\varphi}(c) \vee w^\lambda_{\varphi}(d) \rightarrow w^\lambda_{\varphi}(e)$. Specifically

$$e = \begin{cases} c & \text{if } w^\lambda_{\varphi}(c) \\ d & \text{otherwise} \end{cases}$$

Case 1. Consider the case when the last rule is one of (pair), (union) and ($\Delta_0(\mathcal{L}^{(n)})$-Sep). For example consider the case

$$\neg(\forall c \in a(c \in t \wedge \varphi(c)) \wedge \forall c \in t (\varphi(c) \rightarrow c \in a)), \neg \Gamma, \Delta_u, \Delta, \Psi$$

$$(\Delta_0(\mathcal{L}^{(n)})$$-Sep)

where $\varphi$ is a $\Delta_0(\mathcal{L}^{(n)})$-formula, and $a$ is the eigenvariable.

For a condition $\lambda_0(a) = \lambda_0(\vec{x}/\vec{a}, a)$ for the upper sequent, let $\lambda = \lambda(\vec{x}/\vec{a}) = \lambda_0(\vec{x}/\vec{a}, g(\vec{x}/\vec{a}))$ for $g(\vec{x}/\vec{a}) = \{ c \in t : \varphi(c) \}$. By IH there exist witnessing functions $F(\vec{x}/\vec{a}, \vec{a}, \vec{b}) = \vec{f}_{\lambda_0(a)}(\vec{x}/\vec{a}, \vec{a}, \vec{b})$ of the upper sequent. Substitute $g(\vec{x}/\vec{a})$ for
the variable $a$, we obtain witnessing functions of the lower $F(a/\bar{a}, g(a/\bar{a}), \hat{b}) = f_{\lambda_0(a)}(a/\bar{a}, g(a/\bar{a}), \hat{b})$ since $\forall a-(\forall c \in a(c \in t \land \varphi(c)) \land \forall c \in t(c \rightarrow c \in a))$ is false. $\lambda$ is a desired condition for the lower.

**Case 2.** Consider the case when the last rule is one of (g) for a $g(x/\bar{a}) \in \mathcal{G}$ or (trcl) or (f) for $f \in \mathcal{L}(n)$. For $\Phi = -\Gamma \cup \Delta_u \cup \Delta \cup \Psi$,

$$
\frac{\exists c - \psi_g(\bar{i}, g(\bar{i}), c), \Phi}{\Phi} - \text{trcl}(\text{TC}(t), t, s), \Phi \quad \frac{\text{trcl}(\text{TC}(t), t, s), \Phi}{\Phi} - \theta_t(\bar{i}, \bar{s}, f(\bar{i}/\bar{s})), \Phi \quad \frac{\Phi}{\Phi} - \text{trcl}(t), \Phi}
$$

where $\theta_g(\bar{x}, a) \equiv (\forall c \psi_g(\bar{x}, a, c))$ is a $\Pi_1(\mathcal{G})$-formula assigned to $g$ as in (g), and $\theta_t a \Delta_0(\mathcal{L}(n))$-formula assigned to $f$ so that $\theta_t(\bar{x}, a, b) \iff f(\bar{x}/\bar{a}) = b$. Let the false $\Sigma_1(\mathcal{L}(n))$-formulas $\exists c - \psi_g(\bar{i}, g(\bar{i}), c), - \text{trcl}(\text{TC}(t), t, s), - \theta_t(\bar{i}, \bar{s}, f(\bar{i}/\bar{s}))$ alone, i.e., put these in the $\Psi$-part in the upper sequent. Then witnessing functions and condition of the upper sequent are also ones of the lower sequent.

**Case 3.** Consider the case when the last rule is a $(\varphi_i)$ with the eigenvariable $y$.

$$
\frac{- \Delta(y), - \varphi_i(\bar{i}, y), - \Gamma, \Delta_u, \Delta, \Psi}{\Delta(y), - \Gamma, \Delta_u, \Delta, \Psi} (\varphi_i)
$$

As in the **Case 1** and the **Case 8** of Lemma 3.3 (**Safe Composition**) with the assumption $\exists \varphi_i(\bar{x}_i, f_i(\bar{x}_i/-))$, yields witnessing functions and condition of the lower sequent from ones of the upper sequent by substituting $f_i(\bar{x}_i/-)$ for $y$.

**Case 4.** Consider the case when the last rule is an $(\exists \mu)$. **Case 4.1.** The introduced formula is a negated formula $- \text{Unique}_u(\varphi)$ in $\Delta_u$.

$$
\frac{- \Gamma, \Delta_u, \Delta, \Psi, - \psi(s_0) \land \varphi(s) \land s_0 \neq s_1}{- \Gamma, \Delta_u, \Delta, \Psi, - \text{Unique}_u(\varphi)} (\exists \mu)
$$

For $- \text{Unique}_u^2(\varphi)$ it suffices to have $\{s_0, s_1\} \subset X_\lambda$. For a condition $\lambda_0$ of the upper sequent, $\lambda \leftrightarrow \lambda_0 \lor (s_0 = s) \lor (s_1 = s)$ for the lower sequent. Then $\{s_0, s_1\} \subset X_\lambda$, and this is the only requirement which is needed here. Therefore if $\mu$ is obtained by weakening the condition $\lambda$, i.e., $X_\lambda \subset X_\mu$, then the same holds for $\mu$. For example $\lambda \lor (t = s)$ with any $t, s$. Let $\exists a \in t \lambda(a)$ in an environment $a \in t$. Specifically let $s(a) \in X_{\lambda(a)}$ and $a \in t$. Then $s(a) \in X_{\exists a \in t \lambda(a)} = \bigcup_{a \in t} X_{\lambda(a)}$.

**Case 4.2.** The introduced formula $\exists \psi(c)$ is in $\Psi$ for a $\Delta_0(\mathcal{L}(n))$-formula $\psi$.

$$
\frac{- \Gamma, \Delta_u, \Delta, \Psi, \psi(t)}{- \Gamma, \Delta_u, \Delta, \Psi, \exists \psi(c)} (\exists \mu)
$$

Witnessing functions and condition of the upper remain ones of the lower.

**Case 5.** Consider the case when the last rule is an $(\exists \mu !)$. The introduced formula $\exists! \varphi(a)$ is in $\Delta$.

$$
\frac{- \Gamma_0, \Delta_{0u}, \Delta_0, \Psi_0, \varphi(s)}{- \Gamma_1, \Delta_{1u}, \Delta_1, \Psi_1, \text{Unique}_u^2(\varphi)} (\exists \mu !)
$$
where $\Gamma = \Gamma_0 \cup \Gamma_1$, etc., and Unique$_a^{a,b}$($\varphi$) = \{\neg \varphi(a), \neg \varphi(b), a = b\} with eigenvariables $a, b$.

For simplicity let us assume $\Gamma_0 = \{\sigma\}$ and $\Delta_0 = \Delta_1 = \Psi_0 = \Gamma_1 = \Delta_1 = \Psi_1 = \emptyset$ with a $\Sigma_1!(\mathcal{L}^{(n)})$-formula $\sigma$ and a $\Delta_0(\mathcal{L}^{(n)})$-formula $\varphi$.

$$
-\sigma, \varphi(s) \quad \text{Unique}_a^{a,b}(\varphi) \\
-\sigma, \exists! a \varphi(a) (\exists!)
$$

Let $\lambda \leftrightarrow \lambda_0 \vee (s = *)$ by augmenting $s$. Assume $w!^\lambda_\sigma(b)$. Then by IH we have $\varphi(s)$. On the other hand we have $\varphi(a_0) \wedge \varphi(a_1) \rightarrow a_0 = a_1$ for any $a_0, a_1$. In particular $\varphi(a_0) \rightarrow s = a_0$. Then $f(\bar{x}/\bar{a}, b) = s$ is a witness for the outermost $\exists a$ in $\varphi$, i.e., if $w!^\lambda_\sigma(b)$, then $w!^\lambda_\sigma a \varphi(a)(f(\bar{x}/\bar{a}, b))$ since $s \in X_\lambda$.

**Case 6.** Consider the case when the last rule is $(\forall!)$ with an eigenvariable $b$. The introduced formula $\neg \exists a \varphi(a)$ is in $\Gamma$.

$$
-\Gamma, \Delta_0, \Delta, \Psi, \neg \varphi(b), \neg \text{Unique}_a(\varphi) \\
-\Gamma, \Delta_0, \Delta, \Psi, \neg \exists a \varphi(a) (\forall!)
$$

For simplicity let us assume that $\Delta_0 = \Psi = \emptyset$, $\Gamma = \{\sigma\}$ and $\Delta = \{\theta\}$ with $\Sigma_1!(\mathcal{L}^{(n)})$-formulas $\sigma, \theta$ and a $\Delta_0(\mathcal{L}^{(n)})$-formula $\varphi$.

$$
-\varphi(b), \neg \text{Unique}_a(\varphi), -\sigma, \theta \\
-\exists a \varphi(a), -\sigma, \theta (\forall!)
$$

Let $\lambda = \lambda_0$. By IH we have an $h$ such that for any $b, c$, if $\varphi(b)$ and $w!^\lambda_\sigma(c)$, then either $w!^\lambda_\sigma(h(\bar{x}/\bar{a}, b, c))$ or $\exists a_0, a_1 \in X_\lambda(\bigwedge_{i=0}^1 \varphi(a_i) \wedge a_0 \neq a_1)$.

Suppose $w!^\lambda_\sigma(c)$ and $w!^\lambda_\sigma a \varphi(a)$. In particular $b$ is the unique witness for the fact $\varphi(b)$ in $X_\lambda$. Hence $\forall d \in X_\lambda(\varphi(d) \rightarrow d = b)$, and $a_i = b$ if $\bigwedge_{i=0}^1 \varphi(a_i)$ with $a_0, a_1 \in X_\lambda$. Thus we obtain $w!^\lambda_\sigma h(\bar{x}/\bar{a}, b, c))$. Note that the variable $b$ in $\lambda_0$ is a free variable occurring in the upper sequent, while it denotes an arbitrary witness in $\lambda$.

**Case 7.** Consider the case when the last rule is a $(b \exists \exists! \forall!)$. The introduced formula $\neg \forall x \in t \exists a \psi(x, a)$ is in $\neg \Gamma$ with a $\Delta_0(\mathcal{L}^{(n)})$-formula $\psi$.

$$
-\Gamma, \Delta_0, \Delta, \Psi, s \in t \\
-\Gamma, \Delta_0, \Delta, \Psi, -\psi(s, a), -\text{Unique}_a(\psi(s, a)) \\
-\Gamma, \Delta_0, \Delta, \Psi, \neg D(t), \neg \forall x \in t \exists a \psi(x, a) (b \exists \exists! \forall!)
$$

Let us assume $\Delta_0 = \Psi = \emptyset$, $\Gamma = \{\forall x \in t \exists a \psi(x, a)\}$ and $\Delta = \{\theta\}$ with a $\Sigma_1!(\mathcal{L}^{(n)})$-formula $\theta$, and the eigenvariable $a$.

$$
\begin{align*}
  s & \in t, \theta, -\psi(s, a), -\text{Unique}_a(\psi(s, a)), \theta \\
  & \quad \neg D(t), \neg \forall x \in t \exists a \psi(x, a), \theta (b \exists \forall!)
\end{align*}
$$

Let $\lambda = \lambda_0(b's)$, which is obtained from $\lambda_0(a)$ by substituting $b's$ for $a$. By IH we have some $h_0, h_{1,x}$ such that either $s \in t$ or $w!^\lambda_\theta a \varphi(h_0(\bar{x}/\bar{a}))$, and $\psi(s, a) \rightarrow
[-Unique_λ^0(ψ(s, a₀)) \lor w^1_λ(ψ(λ_0(a₀)(x/ā, a)))] for any a, where h_1λ₀(a₀) = h_1X_λψ(a). Suppose w^1_λ(∀a \in s \exists a \psi(b))

If \( s \not\in t \), then \( f(\bar{x}/\bar{a}, b) = h_0(\bar{x}/\bar{a}) \) does the job. In what follows assume \( s \in t \). We have \( \psi(s, b's) \), and hence either \(-Unique^λ_0(ψ(s, a₀)) \) or \( w^1_θ(h_1λ(\bar{x}/\bar{a}, b's)) \). Moreover \( b's \in X_λ \) is the unique witness for the fact \( \psi(s, b's) \). Therefore \( \forall c \in X_λ(ψ(s, c) \to c = b's) \). Thus \( Unique^λ_0(ψ(s, a₀)) \), and \( w^1_θ(h_1λ(\bar{x}/\bar{a}, b's)) \) is obtained.

To sum up, for the function

\[
f(\bar{x}/\bar{a}, b) = \begin{cases} 
h_0(\bar{x}/\bar{a}) & \text{if } s \not\in t \\
h_1λ(\bar{x}/\bar{a}, b's) & \text{otherwise}
\end{cases}
\]

we obtain \( w^1_θ(∀a \in s \exists a \psi(\bar{x}/\bar{a}, b)) \rightarrow w^1_θ(f(\bar{x}/\bar{a}, b)) \).

**Case 8.** Consider the case when the last rule is a \((b∀)\). Then the introduced formula \(-∃c \in t \varphi(c)\) is a \( Δ_0(\mathcal{L}^{(n)})\)-formula.

\[
\frac{-Γ, Δ_u, Δ, \Psi, d \not\in t, \neg\varphi(d) \quad (b∀)}{-Γ, Δ_u, Δ, \Psi, \neg∃c \in t \varphi(c) \quad (b∀)}
\]

Let \( Γ = \{σ\}, Γ_u = Ψ = \emptyset \) and \( Δ = \{θ\} \) for \( Σ_1!(\mathcal{L}^{(n)})\)-formulas \( σ, θ \).

\[
\frac{d \not\in t, \neg\varphi(d), \negσ, θ}{\neg∃c \in t \varphi(c), \negσ, θ \quad (b∀)}
\]

where \( d \) is an eigenvariable.

Let \( λ \Leftrightarrow ∃d \in t λ_0(d) ∈ E(λ_0) \) for a condition of the lower. Without loss of generality we can assume that the literal \( d \not\in t \) is contained in any sequents occurring in the subderivation of \( d \not\in t, \neg\varphi(d), \negσ, θ \). Hence the condition \( λ_0 \) can be revised to the \( λ \) for the upper sequent.

By IH we have an \( h_λ \) such that if \( w^1_θ(b), \varphi(d) \) and \( d \in t \), then \( w^1_θ(h_λ(\bar{x}/\bar{a}, d, b)) \).

Let \( f(\bar{x}/\bar{a}, b) = ϵ[∃d \in t(ϕ(d) ∧ h_λ(\bar{x}/\bar{a}, d, b) = e)] \). Suppose \( w^1_θ(b) \) and \( ∃c \in t \varphi(c) \). Pick a \( d \in t \) such that \( ϕ(d) \). Since each \( e = h_λ(\bar{x}/\bar{a}, d, b) \in X_λ \) for such a \( d \) is the witness for the fact \( w^1_θ(h_λ(\bar{x}/\bar{a}, d, b)) \) uniquely in \( X_λ \), we obtain \( ∃d \in t(ϕ(d) \to f(\bar{x}/\bar{a}, b) = h_λ(\bar{x}/\bar{a}, d, b)) \). Hence \( w^1_θ(f(\bar{x}/\bar{a}, b)) \) as desired.

**Case 9.** Consider the case when the last rule is a \( (Δ_0^P(\mathcal{L}^{(n)})\text{-Repl})\).

\[
\frac{-Γ_0, Δ_u, Δ, Ψ, x \not\in t, \exists a \varphi(x, a) \quad -Γ_1, Δ_u, Δ, Ψ, \forall x \in t \varphi(x, c', x)}{-Γ, Δ_u, Δ, Ψ, \neg D(t) \quad (Δ_0^P(\mathcal{L}^{(n)})\text{-Repl})}
\]

Let \( Γ_1 = Δ_u = Ψ = \emptyset, Γ_0 = \{σ\} \) and \( Δ = \{θ\} \) for \( Σ_1!(\mathcal{L}^{(n)})\)-formulas \( σ, θ \). \( \varphi \) is a \( Δ_0(\mathcal{L}^{(n)})\)-formula.

\[
\frac{-σ, x \not\in t, \exists a \varphi(x, a), θ \quad -σ, x \not\in t \varphi(x, c', x), θ \quad (Σ^P!(G)\text{-Repl})}{-D(t), \negσ, θ}
\]

where \( x \) and \( c \) are eigenvariables.
Let $\lambda \iff \exists x \in t \lambda_0(x)$ be a condition for the lower. By IH we have some $h_\lambda, k_\lambda$ such that if $w^\lambda_\phi(b)$, then either $w^\lambda_{\exists a \varphi(x)}(h_\lambda(x, x/\vec{a}, b))$ or $w^\lambda_{\exists b}(k_\lambda(x, x/\vec{a}, b))$ for any $x \in t$. Suppose $w^\lambda_{\phi}(b)$.

If $-\forall x \in t[w^\lambda_{\exists a \varphi(x)}(h_\lambda(x, x/\vec{a}, b))]$, then $\exists x \in t[w^\lambda_{\phi}(k_\lambda(x, x/\vec{a}, b))]$. Let $C = \{k_\lambda(x, x/\vec{a}, b) : x \in t\}$ for the term $t = t(\vec{x})$. $C$ is a set such that $\{d \in C : w^\lambda_{\phi}(d)\} \subseteq X_\lambda$ is a singleton.

Otherwise for $c'_1 x = h_\lambda(x, x/\vec{a}, b)$ we obtain $\forall x \in t[w^\lambda_{\exists a \varphi(x)}(c'_1 x)]$. On the other hand we have a $j_\lambda$ such that if $\forall x \in t \varphi(x, c', x)$, then $w^\lambda_{\phi}(j_\lambda(\vec{x}/\vec{a}, c))$ for any $c$. $\forall x \in t[w^\lambda_{\exists a \varphi(x)}(c'_1 x)]$ yields $\forall x \in t \varphi(x, c'_1 x)$. Hence $w^\lambda_{\phi}(j_\lambda(\vec{x}/\vec{a}, c))$.

To sum up, for the function

$$f(\vec{x}/\vec{a}, b) = \begin{cases} j_\lambda(\vec{x}/\vec{a}, c_1) \\
\forall x \in t[w^\lambda_{\exists a \varphi(x)}(h_\lambda(x, x/\vec{a}, b))] \\
\bigcup\{k_\lambda(x, x/\vec{a}, b) : w^\lambda_{\phi}(k_\lambda(x, x/\vec{a}, b)), x \in t\}
\end{cases}$$

we obtain $w^\lambda_{\phi}(b) \rightarrow w^\lambda_{\phi}(f(\vec{x}/\vec{a}, b))$ with $c_1 = \{(x, h_\lambda(x, x/\vec{a}, b)) : x \in t\}$.

**Case 10.** Consider the case when the last rule is a $\Sigma^1_\phi(\mathcal{L}^{(n)})$-Foundation.

$$\frac{\gamma \not\in t', \not\exists \gamma \not\in y \exists a \varphi(x, a), \exists a \varphi(y, a), -\not\exists a \varphi(t, a)}{\not\exists \gamma, \not\exists a \varphi(t, a), -\not\exists a \varphi(t, a), -\not\exists a \varphi(t, a), -\not\exists a \varphi(t, a)} (\Sigma^1_\phi(\mathcal{L}^{(n)})\text{-Fund})$$

where $t' = TC(t \cup \{t\})$ for the term $t = t(\vec{x})$. For simplicity let $\Delta_\gamma = \Psi = \emptyset$, $\Gamma = \{\gamma\}$ and $\Delta = \{\theta\}$ with $\Sigma^1_\phi(\mathcal{L}^{(n)})$-formulas $\sigma, \theta$ and a $\Delta_\gamma(\mathcal{L}^{(n)})$-formula $\varphi$. For an eigenvariable $y$

$$\not\exists \gamma \not\in y \exists a \varphi(x, a), \exists a \varphi(y, a), -\not\exists a \varphi(t, a), -\not\exists a \varphi(t, a), -\not\exists a \varphi(t, a) (\Sigma^1_\phi(\mathcal{L}^{(n)})\text{-Fund})$$

By IH we have some $h_X, k_X$ and a condition $\lambda_0 = \lambda_0(y, b)$ such that for any $b : y \rightarrow V$, $y$ and $c$ if $\forall x \in y w^{\lambda_0(y, b)}(h_{\lambda_0(y, b)}(y, x/\vec{a}, c, b))$ or $w^{\lambda_0(y, b)}(k_{\lambda_0(y, b)}(y, x/\vec{a}, c, b))$. Let $g(y, x/\vec{a}, c) = h_{\lambda_0(y, b)}(y, x/\vec{a}, c, b_0)$ for $b_0 = g[y] = \{x, g(x, x/\vec{a}, c), \varphi) : x \in y\}$.

Then for any $y$, if $\forall x \in y w^{\lambda_0(y, b)}(h_{\lambda_0(y, b)}(y, x/\vec{a}, c, b_0))$ or $w^{\lambda_0(y, b)}(k_{\lambda_0(y, b)}(y, x/\vec{a}, c, b_0))$.

Let $\lambda \iff \exists y \in t' \lambda_0(y, b_0)$ for the lower. Then for any $y \in t'$, if $\forall x \in y w^{\lambda_0(y, b_0)}(h_{\lambda_0(y, b_0)}(y, x/\vec{a}, c, b_0))$ or $w^{\lambda_0(y, b_0)}(k_{\lambda_0(y, b_0)}(y, x/\vec{a}, c, b_0))$, then either $w^{\lambda_0(y, b_0)}(h_{\lambda_0(y, b_0)}(y, x/\vec{a}, c, b_0))$ or $w^{\lambda_0(y, b_0)}(k_{\lambda_0(y, b_0)}(y, x/\vec{a}, c, b_0))$.

Now let $K(y, x/\vec{a}, c) = k_X(y, x/\vec{a}, c, b_0)$. We obtain for any $y \in t'$, if $\forall x \in y w^{\lambda_0(y, b_0)}(g'_x)$ and $w^{\lambda_0}(c)$, then either $w^{\lambda_0(y, b_0)}(g(y, x/\vec{a}, c))$ or $w^{\lambda_0}(K(y, x/\vec{a}, c))$. Suppose $w^{\lambda_0}(c)$. If $-\not\exists a \not\in t' \not\exists a \not\in y w^{\lambda_0(y, b_0)}(g(x, x/\vec{a}, c))$, then $\exists x \in t' w^{\lambda_0}(K(x, x/\vec{a}, c))$.

For the set $C = \{K(x, x/\vec{a}, c) : x \in t'\}$, $\{d \in C : w^\lambda_{\phi}(d)\} \subseteq X_\lambda$ is a singleton.

Otherwise we obtain $w^{\lambda_0}_{\exists a \varphi(x)}(c_1)$ for $c_1 = g(t, x/\vec{a}, c)$. On the other hand we have $p = p_\lambda$ such that for any $e$ if $w^{\lambda_0}(c)$ and $w^{\lambda_0}_{\exists a \varphi(t)}(e)$, then $w^\lambda_{\phi}(p(x, x/\vec{a}, c))$. Thus for $g(x, x/\vec{a}, c) = p(x, x/\vec{a}, c, e_1)$, we obtain $w^\lambda_{\phi}(g(x, x/\vec{a}, c))$.
To sum up, for the function

\[
 f(\vec{x}/\vec{a}, c) = \begin{cases} 
 p(\vec{x}/\vec{a}, g(x, \vec{x}/\vec{a}, c)) & \text{if } \forall x \in t'[u^A_{\vec{a}, \varphi(x)}(g(x, \vec{x}/\vec{a}, c)) (t' = TC(t \cup \{t\})) \\
 \bigcup \{K(x, \vec{x}, x/\vec{a}, c) : u^A_{\vec{a}}(K(x, \vec{x}, x/\vec{a}, c)), x \in t'\} & \text{otherwise}
\end{cases}
\]

we obtain \( u^A_{\vec{a}}(c) \rightarrow u^A_{\vec{a}}(f(\vec{x}/\vec{a}, c)) \).

This completes a proof of Lemma 4.9.

Let us finish the proof of the converse of Theorem 4.2. Assume that \( f(\vec{x}/\vec{a}) \) is \( \Sigma_1^D(\mathcal{G}) \)-definable in \( T_3^D(\mathcal{G}) \), and let \( \psi_f(\vec{x}/\vec{a}, b, c) \) be a \( \Delta_0(\mathcal{L}^{(\omega)}) \)-formula such that \( T_3^D(\mathcal{G}) \vdash \forall \vec{x} \in D \forall a \exists b \exists c \psi_f(\vec{x}, \vec{a}, b, c) \), and \( f(\vec{x}/\vec{a}) = b \) if \( \exists c \psi_f(\vec{x}, \vec{a}, b, c) \). Then there exists a derivation of \( \forall \vec{x} \in D \forall a \exists b \exists c \psi_f(\vec{x}, \vec{a}, b, c) \) in the sequent calculus for \( T_3^n(\mathcal{G}) \) for an \( n \). The converse of Theorem 4.2 is proved by main induction on \( n \) with subsidiary induction on the number of nested applications of \( \Sigma_1^1(\mathcal{L}^{(n)}) \)-Submodel Rule as in the proof of Theorem 2.32.

SIH yields the assumption (6) in Lemma 4.9. Also we see from MIH that each function symbol \( f \) in the language \( \mathcal{L}^{(n)} \) denotes a function in PCSF\(^n\)(\( \mathcal{G} \)).

By Lemma 4.9 pick a condition \( \lambda \) and a PCSF\(^n\)(\( \mathcal{G} \))-function \( g_{\chi_\lambda}(\vec{x}/\vec{a}) \) such that \( g_{\chi_\lambda}(\vec{x}/\vec{a}) \) is a pair \( \langle b, c \rangle \) with \( \psi_f(\vec{x}/\vec{a}, b, c) \). Then \( f(\vec{x}/\vec{a}) = 1st(-/g_{\chi_\lambda}(\vec{x}/\vec{a})) \in \text{PCSF}\(^n\)(\( \mathcal{G} \)) as desired.

**Problem.** It is open for us how to axiomatize PCSF\(^n\)-predicates.

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