Some characterizations of the preimage of $A_{\infty}$ for the Hardy-Littlewood maximal operator and consequences

Álvaro Corvalán
acorvala@ungs.edu.ar

November 6, 2017

Abstract

The purpose of this paper is to give some characterizations of the weight functions $w$ such that $Mw \in A_{\infty}$. We show that for those weights to be in $A_{\infty}$ ensures to be in $A_{1}$. We give a criterion in terms of the local maximal functions $m_{\lambda}$ and we present a pair of applications, one of them similar to the Coifman-Rochberg characterization of $A_{1}$ but using functions of the form $(f^\#)^{\delta}$ and $(m_{\lambda} u)^{\delta}$ instead of $(Mf)^{\delta}$.

INTRODUCTION

In this work we look at some characterizations of the weights $u$ such that $Mu \in A_{\infty}$. This question is mentioned as open in [CU-P] and that paper refers the reader to [CU] for partial results for monotonic functions in $\mathbb{R}$, and at our knowledge no previous work brings explicitly a complete result. We will show that if for a weight $u$ we have that $Mu \in A_{\infty}$, actually we must have that $Mu \in A_{1}$. From a result due to Neugebauer it is known that those weights can be characterized for a pointwise condition for the maximal operator: $(M(u^{r})(x))^{\frac{1}{r}} \leq CMu(x)$ for some $C > 0, r > 1$ and $\forall x \in \mathbb{R}^{n}$, so it is immediately satisfied for a weight belonging to any reverse Hölder class -this means that $(u^{r}_{Q})^{\frac{1}{r}} \leq C(u_{Q})$ for some $C > 0, r > 1$ and any cube $Q$ with sides parallel to the coordinate axes. Notwithstanding, some weaker conditions, for instance: $u \in weak - A_{\infty}$, allows to satisfy the condition of Neugebauer. We will also present another condition in terms of the size of sub-level sets, by means the use of some useful pointwise inequalities found by A. Lerner, involving the sharp maximal operator $u^{\#}$, the local maximal function $m_{\lambda}(u)$ and the Hardy-Littlewood maximal operator $Mu$. The resulting condition is weaker but quite similar to certain characterization for $A_{\infty}$ weights -in [DMO] it is proven that this characterization, equivalent to $A_{\infty}$ for standard cubes, is weaker, for general bases, than most of the usual definitions for $A_{\infty}$ classes-. An interesting consequence that we can derive from this result is a characterization of the $A_{1}$ weights similar to the construction of Coifman and Rochberg in terms
of \( k(x)(Mf(x))^\delta \) with \( k \) and \( k^{-1} \) belonging to \( L^\infty \), but involving \( u^\# \) and \( m_\lambda(u) \) instead of \( Mf(x) \). As another consequence for those weights \( u \) such that \( Mu \in A_\infty \) and hence \( Mu \in A_1 \) we can improve some known inequalities for singular integral operators.

The weights belonging to \( A_\infty \) can be described by several conditions. In the reference [DMO] many of them are enumerated; all of them are equivalent for the usual Muckenhoupt weights for the maximal operator associated with the bases of cubes whose sides are parallel to the coordinate axes (or associated with balls), but they can provide different types of weights for other bases. Here we deal with the usual bases of cubes (with sides parallel to the coordinate axes) and the corresponding Muckenhoupt weights. But we might translate some of the results for other bases for which the following condition describe \( A_\infty \) as the union of \( A_p \) classes and for which it holds those properties that we use relating the corresponding weights and the \( A_p \) constants.

Summarizing the main results are:

**Proposition 1** If \( u \) is any weight, \( Mu \in A_\infty \iff Mu \in A_1 \)

**Criterion 2** Let \( u \) a weight function in \( \mathbb{R}^n \), \( Mu \in A_\infty \) if and only if there exists \( s > 1 \) and \( C_0 > 0 \) such that \( (Mu^s)^\frac{1}{s}(x) \leq C_0Mu(x) \).

**Criterion 3** Let’s \( u \) a weight function, \( Mu \in A_\infty \) if and only if for any \( \lambda \in (0,1) \) it holds that \( m_\lambda(Mu) \approx M(Mu) \).

**Theorem 4** Let \( u \) a weight function. Then \( Mu \in A_\infty \) if and only if (2) holds, that is:

\[
Mu \in A_\infty \iff \exists \alpha > 0, \beta \in (0,1) : \left| \{ y \in Q_x : Mu(y) \leq \alpha (Mu)_Q \} \right| \leq \beta |Q_x|
\]

for almost every \( x \in \mathbb{R}^n \) for some cube \( Q_x \ni x \), and for every cube \( Q \) to which \( x \) belongs.

**Theorem 5** (1) If \( 0 < \delta < 1 \), \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( u \in A_1 \) and \( c, d \) non-negative constants then \( (c \cdot f^\#(x) + d \cdot m_\lambda u(x))^\delta \in A_1 \).

(2) Conversely, if \( w \in A_1 \) then there are \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), \( u \in A_1 \), non-negative constants \( c \) and \( d \), and \( k(x) \) with \( k, k^{-1} \in L^\infty \) such that \( w(x) = k(x) \left( c \cdot f^\#(x)^\delta + d \cdot m_\lambda u(x)^\delta \right) \).

**PRELIMINARIES**

Here \( M \) is the (non-centered) Hardy-Littlewood maximal operator for the bases of cubes with sides parallel to the co-ordinate axes; so if \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) we have:

\[
Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(z) \, dz
\]
A weight \( w \) is a non-negative locally integrable function in \( \mathbb{R}^n \). A weight \( w \in A_p \) class for \( 1 < p < \infty \) if and only if

\[
[A_p] := \sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \left[ \frac{1}{|Q|} \int_Q w^{-\frac{p}{p-1}} \right]^{p-1} < +\infty
\]

A weight \( w \in A_1 \) if and only if

\[
Mw(x) \leq Cw(x) \text{ a.e.} x \in \mathbb{R}^n
\]

and \([A_1]\) is the minimal constant \( C \) such that this inequality occurs.

We will note \( f(Q) = \int_Q f(x) \, dx \) and \( f_Q = \frac{f(Q)}{|Q|} \).

We also recall the statement of an useful result due to Coifman, R. and Rochberg, R. in characterizing \( A_1 \) weights:

**Theorem 6** (1) Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) be such that \( Mf(x) < \infty \text{ a.e.} \) and \( 0 \leq \delta < 1 \), then \( w(x) = Mf(x)^\delta \) is in \( A_1 \). Also the \( A_1 \) constant depends only on \( \delta \).

(2) Conversely, if \( w \in A_1 \) then there are \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( k(x) \) with \( k \) and \( k^{-1} \) both belonging to \( L^\infty \) such that \( w(x) = k(x)Mf(x)^\delta \).

The proof can be found in [D] (or see [C-R] for the original work), using a suitable decomposition of \( f \) and Kolmogorov’s inequality for proving (1). The point (2) is quite elementary.

We collect some known properties that we will use. The first of which can be easily obtained using the definition of \( A_p \) classes and the definition of \([A_p]\) constants, and Hölder’s inequality (see [D], for instance):

A) \( A_p \subset A_q \) if \( p < q \) and \([w]_{A_q} \leq [w]_{A_p} \).

B) \( w \in A_p \) if and only if \( w^{\frac{1}{p'}} \in A_{\frac{p}{p'}} \).

C) If \( w_0, w_1 \in A_1 \) then \( w_0 w_1^{1-p} \in A_p \).

Another property that we will need is the reciprocal of property C). That property (P. Jones’ Factorization Theorem) it’s very much deeper than the previous (see for instance [S]).

D) If \( w \in A_p \) there exists \( w_0, w_1 \in A_1 \) such that \( w = w_0 w_1^{1-p} \).

Finally, one last property that we will need is:

E) If \( w \in A_p \) there is \( \alpha > 1 \) such that \( w^{\alpha} \in A_p \).

This latter property is usually proved by means the use of reverse Hölder inequalities that \( A_p \) weights satisfy (see [D],[G] or [G-R]), but it can be obtained easily from the Coifman-Rochberg construction: if \( w \in A_1 \) by (2) is \( w(x)^\alpha = k(x)^\alpha Mf(x)^\delta \) and taking \( 1 < \alpha < \frac{p}{p-1} \) we have from (1) that \( Mf(x)^\delta \in A_1 \)

and then

\[
Mw(x)^\alpha \leq M \left( \|k\|_{A_1}^\alpha Mf(x)^\delta \right) \leq [(Mf)^\delta]_{A_1} \|k\|^\alpha_{\infty} \left( Mf(x)^\delta \right) \leq
\]

So \( w(x)^\alpha \in A_p \) with \([w]_{A_p} \leq [(Mf)^\delta]_{A_1} \|k\|^\alpha_{\infty} \|k^{-1}\|^\alpha_{\infty} \). On the other hand for \( p > 1 \) and \( w \in A_p \) by property D) we have \( w = w_0 w_1^{1-p} \) with \( w_0, w_1 \in A_1 \) and
for \( j = 0, 1 \) we write \( w_j (x) = k_j (x) M f_j (x)^\delta \) and for \( 1 < \alpha < \min \left\{ \frac{1}{\delta} \right\} \) we have that \( w_0^\alpha, w_1^\alpha \in A_1 \) and using C) we have that \( w^\alpha = w_0^\alpha (w_1^\alpha)^{1-p} \in A_p \).

By property A, the \( A_p \) classes are nested, so it is well defined the class \( A_{\infty} = \bigcup_{p<\infty} A_p \).

A characterization of a weight \( w \) for belonging to \( A_{\infty} \) is the following:

\[
w \in A_{\infty} \iff \exists \alpha, \beta \in (0, 1) : |\{ y \in Q : w(y) \leq \alpha \cdot w_Q \}| \leq \beta \cdot |Q| \quad (1)
\]

for every cube \( Q \) (see for instance [DMO] for this and other characterizations for general bases).

We will prove that for a weight \( u \) there is a necessary and sufficient condition for \( Mu \) to belong to \( A_{\infty} \) with a statement quite similar to (1).

\[
Mu \in A_{\infty} \iff \exists \alpha > 0, \beta \in (0, 1) : \left| \{ y \in Q_x : Mu(y) \leq \alpha \cdot (Mu)_Q \} \right| \leq \beta \cdot |Q_x| \quad (2)
\]

for \( x \in \mathbb{R}^n \text{ a.e. and for some cube } Q_x \supset x, \text{ and for every cube } Q \text{ to which } x \text{ belongs.}

**SOME RESULTS**

The first step is the following proposition that shows that if \( Mu \in A_{\infty} \) indeed \( Mu \in A_1 \), and then because \( A_1 \subset A_{\infty} \) we have that \( Mu \in A_{\infty} \iff Mu \in A_1 \).

So, what we have to do is to characterize the weights \( u \) such that \( Mu \in A_1 \).

**Remark 7** Of course \( A_1 \subsetneq A_{\infty} \), so there are weights \( w \) such that \( w \in A_{\infty} \) and \( w \notin A_1 \). The lemma tells us that being in \( A_{\infty} \) is the same as being in \( A_1 \) for those weights \( w \) such that \( w = Mu \) for some weight \( u \).

**Proposition 8** If \( u \) is any weight, \( Mu \in A_{\infty} \iff Mu \in A_1 \)

**Proof.** The implication \( Mu \in A_1 \implies Mu \in A_{\infty} \) is trivial because \( A_1 \subset A_{\infty} \).

It remains to show that if \( Mu \in A_{\infty} \implies Mu \in A_1 \).

If \( Mu \in A_{\infty} = \bigcup_{p<\infty} A_p \), we have that \( Mu \in A_p \) for some \( p \geq 1 \). If \( p = 1 \) there is nothing to prove. Let \( p > 1 \). Because the result of Coifman and Rochberg we have that \( (Mu)^\delta \in A_1 \) for any \( \delta \) with \( 0 \leq \delta < 1 \) and any \( u \) locally integrable but generally does not occur that \( Mu \in A_1 \), actually we are in the process of proving that if we additionally have that \( Mu \in A_p \), in fact \( Mu \in A_1 \).

We need the following result (see, for instance, [Rudin, ej 5 d] Chap 3): For a measure space \( (\Omega, \mu) \) with measure \( \mu (\Omega) = 1 \) and \( \left( \int_\Omega |f|^r d\mu \right)^{\frac{1}{r}} < \infty \) for some \( r > 0 \), we have that

\[
\lim_{r \to 0^+} \left( \int_\Omega |f|^r d\mu \right)^{\frac{1}{r}} = \exp \left( \int_\Omega \log (|f|) d\mu \right).
\]
Let’s observe that using that $\mu(\Omega) = 1$ and Hölder Inequality we obtain 
\[
\left( \int_{\Omega} |f|^r \, d\mu \right)^{\frac{1}{r}} \geq \left( \int_{\Omega} |f|^{r_2} \, d\mu \right)^{\frac{1}{r_2}} \text{ if } r_1 \geq r_2.
\]
So for $r > 0$ we have that 
\[
\left( \int_{\Omega} |f|^r \, d\mu \right)^{\frac{1}{r}} \geq \exp \left( \int_{\Omega} \log (|f|) \, d\mu \right) = \lim_{r \to 0^+} \left( \int_{\Omega} |f|^r \, d\mu \right)^{\frac{1}{r}}.
\]

Now for every $q > p$, using that 
\[
\sup_Q \frac{Mu(Q)}{|Q|} \left( \frac{1}{|Q|} \int_Q Mu(x)^{-\frac{q}{q-1}} \, dx \right)^{q-1} = [Mu]_{A_q} \leq [Mu]_{A_p}
\]
(property A), we obtain that for any cube $Q$:
\[
\frac{Mu(Q)}{|Q|} \left( \frac{1}{|Q|} \int_Q Mu(x)^{-\frac{q}{q-1}} \, dx \right)^{q-1} \leq [Mu]_{A_p} < \infty
\]

If $q$ tends to infinity then $\frac{1}{q-1}$ tends to $0^+$, so taking $r = \frac{1}{q-1}$ and applying the result from above for $f = w^{-1}$, $\Omega = Q$ and $d\mu = \frac{dx}{|Q|}$, we have 
\[
\lim_{q \to +\infty} \left( \frac{1}{|Q|} \int_Q Mu(x)^{-\frac{q}{q-1}} \, dx \right)^{q-1} = \exp \left( \int_Q \log \left( Mu(x)^{-1} \right) \, dx \right)
\]
\[
= \exp \left( \int_Q - \log (Mu(x)) \, dx \right) = \frac{1}{\exp \left( \int_Q \log (Mu(x)) \, dx \right)}
\]

Taking limit in 
\[
\frac{Mu(Q)}{|Q|} \left( \frac{1}{|Q|} \int_Q Mu(x)^{-\frac{q}{q-1}} \, dx \right)^{q-1} \leq [Mu]_{A_p},
\]
we have that 
\[
\frac{Mu(Q)}{|Q|} \cdot \frac{1}{\exp \left( \int_Q \log (Mu(x)) \, dx \right)} \leq [Mu]_{A_p},
\]
so 
\[
\frac{Mu(Q)}{|Q|} \leq [Mu]_{A_p} \cdot \exp \left( \int_Q \log (Mu(x)) \, dx \right)
\]

Additionally, the observation from above applied for $f = Mu$ gives us that for any $r > 0$ it holds that 
\[
\left( \frac{1}{|Q|} \int_Q (Mu)^{r} \, dx \right)^{\frac{1}{r}} \geq \exp \left( \int_Q \log (Mu(x)) \, dx \right)
\]

Thus 
\[
\frac{Mu(Q)}{|Q|} \leq [Mu]_{A_p} \cdot \exp \left( \int_Q \log (Mu(x)) \, dx \right) \leq [Mu]_{A_p} \left( \frac{1}{|Q|} \int_Q |Mu|^r \, dx \right)^{\frac{1}{r}}
\]
\[ \frac{M u(Q)}{|Q|} \leq [Mu]_{A_p} \left( \frac{1}{|Q|} \int_Q |Mu|^r \, dx \right)^{\frac{1}{r}}. \]

Taking \( r = \delta \) with \( 0 \leq \delta < 1 \) and using that for such \( \delta \) it holds that \((Mu)^\delta = (Mu)^{A_1} \in A_1 \) and then
\[ \frac{1}{|Q|} \int_Q |Mu|^r \, dx \leq [(Mu)^r]_{A_1} \cdot (Mu(x))^r \]
a.e for every \( x \in Q \).

So we have a.e for \( x \in Q \)
\[ \frac{M u(Q)}{|Q|} \leq [Mu]_{A_p} \left( \frac{1}{|Q|} \int_Q |Mu|^r \, dx \right)^{\frac{1}{r}} \leq [(Mu)^r]_{A_1} \cdot [(Mu)^r]_{A_1} \cdot (Mu(x))^r \]
\[ = [(Mu)^r]_{A_1} \cdot [(Mu)^r]_{A_1} \cdot (Mu(x)) \]

Taking \( C = [(Mu)^r]_{A_1} \cdot [(Mu)^r]_{A_1} \cdot (Mu(x))^r \) independent of \( Q \), for every \( Q \) we obtain that
\[ \frac{M u(Q)}{|Q|} \leq C \cdot Mu(x) \]
a.e for \( x \in Q \).

Then almost everywhere for \( x \in \mathbb{R}^n \) we have that
\[ M (Mu)(x) = \sup_{Q \ni x} \frac{M u(Q)}{|Q|} \leq C \cdot Mu(x) \]
, that is
\[ M (Mu)(x) \leq C \cdot Mu(x) \]
and then we obtain that \( Mu \in A_1 \).

The previous proposition together with a lemma due to Neugebauer (published in [CU]) enables us to give a characterization of all the weights \( u \) such that \( Mu \in A_\infty \). Until a few years ago this was an open problem with interesting consequences for improving some two-weight inequalities for several operators, including maximal, vector-valued an Calderon-Zygmund ones (see [CU-P]).

For completitude we transcribe below the lemma of Neugebauer and its easy proof, in [CU] the lemma is considered in \( \mathbb{R} \) but it works mutatis mutandi for \( \mathbb{R}^n \).

**Lemma 9 (Neugebauer)** For a weight \( u \) it holds that \( Mu \in A_1 \) if and only if there exists \( s > 1 \) and \( C_0 > 0 \) such that \((Mu)^s(x) \leq C_0 \cdot Mu(x)\)
Proof. If such \( s > 1 \) exists then \( \frac{1}{s} < 1 \) and the Coifman-Rochberg characterization of \( A_1 \) weights tells us that \( (Mu^s)^{\frac{1}{s}} \) is in \( A_1 \), so \( M \left( (Mu^s)^{\frac{1}{s}} \right) \leq C_1 (Mu^s)^{\frac{1}{s}} \), and using the hypothesis and the fact that by Hölder: \( Mu \leq (Mu^s)^{\frac{1}{s}} \), we obtain \( M(Mu) \leq M \left( (Mu^s)^{\frac{1}{s}} \right) \leq C_1 (Mu^s)^{\frac{1}{s}} \leq C_1.C.Mu \), and then \( M(Mu) \leq C.Mu \), that is \( Mu \in A_1 \).

Reciprocally if \( Mu \in A_1 \) then \( Mu \) satisfies a reverse Hölder inequality (RHI), that means that for some \( s > 1 \) and \( C > 0 \) it holds for any cube \( Q \)

\[
\left( \frac{1}{|Q|} \int_Q Mu^s \right)^{\frac{1}{s}} \leq C, \frac{1}{|Q|} \int_Q Mu
\]

and taking suprema over the cubes we have:

\[ (Mu^s)^{\frac{1}{s}} \leq C.Mu \]

As we have already mention the lemma and the proposition above, which says that \( Mu \in A_\infty \) if and only if it actually belongs to \( A_1 \), provide us with the following characterization of the weights \( u \) such that \( Mu \in A_\infty \):

**Criterion 10** Let \( u \) a weight function in \( \mathbb{R}^n \), \( Mu \in A_\infty \) if and only if there exists \( s > 1 \) and \( C_0 > 0 \) such that \( (Mu^s)^{\frac{1}{s}} (x) \leq C_0.Mu(x) \).

Let’s observe that with have got a bound for the constant \([Mu]_{A_1} \), that is

\[
[Mu]_{A_1} \leq [(Mu)^\Gamma]_{A_1} \cdot ([((Mu)^\Gamma)]_{A_1})^{\frac{1}{s}} \]

Because the previous proposition the weights \( u \) with \( Mu \) in \( A_\infty \) are those for which there are some \( C > 0 \) such that

\[
M(Mu) (x) \leq C \cdot Mu(x) \text{ a.e.}
\]

**SOME FURTHER DEFINITIONS AND PROPERTIES**

Now we will use some pointwise inequalities for certain maximal operators to weaken the above condition. We need a couple of definitions:

**Definition 11** If \( f \in L^1_{loc} (\mathbb{R}^n) \) the sharp maximal function of Fefferman-Stein \( f^\# \) is defined by

\[
f^\# (x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f (x) - f_Q| dx
\]

**Definition 12** \( BMO (\mathbb{R}^n) = \{ f \in L^1_{loc} (\mathbb{R}^n) : f^\# \in L^\infty (\mathbb{R}^n) \} \) is the space of functions with bounded mean oscillation, and \( \| f \|_{BMO} = \| f^\# \|_\infty \).
Remark 13 \[ \|\|_{BMO} \] is a seminorm for \( BMO(\mathbb{R}^n) \) since \( \|f\|_\infty = 0 \) if and only if \( f \) is constant (a.e.). It is usual to identify \( BMO \) with its quotient with the class of almost everywhere constant functions and then \( \|\|_{BMO} \) becomes a norm.

Notation 14 For a measurable function \( f : \mathbb{R}^n \to \mathbb{R} \), the non-increasing rearrangement of \( f \) is \( f^* \). That is, for \( t \geq 0 \)
\[
 f^*(t) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| \leq t\}
\]
We use the convention that \( \inf\emptyset = \infty \).

Remark 15 An equivalent way to define \( f^*(t) \) is
\[
 f^*(t) = \sup_{|E| = t} \inf_{x \in E} |f(x)|
\]
where \( E \) are measurable sets.

Remark 16 Non-increasing rearrangements of functions from measure spaces \( (X, \mu) \) can be defined in the same way replacing \( \mathbb{R}^n \) by \( X \) and the Lebesgue measure \( \|\| \) by \( \mu \). Much more details and results can be found in [BS].

Definition 17 If \( f \) is a measurable function and \( \lambda \in (0, 1) \) the local maximal functions \( m_\lambda(f) \) are defined by
\[
 m_\lambda f(x) = \sup_{Q \ni x} (f\chi_Q)^*(\lambda |Q|)
\]
Let’s point out some basic properties of \( f^* \), \( m_\lambda f(x) \), and \( f^\# \), immediate from their definitions:
(i) \( f^\#(x) \leq 2Mf(x) \)
(ii) If \( c > 0 \) then \( (cf)^*(t) = c(f)^*(t) \)
(iii) If \( f(x) \geq g(x) \) a.e. then \( f^*(t) \geq g^*(t) \) for every \( t \).
(iv) Using iii) if \( f(x) \geq g(x) \) a.e. then \( m_\lambda(f)(x) \geq m_\lambda(g)(x) \) everywhere.
(v) If \( c > 0 \) using ii) we have \( m_\lambda(cf)(x) = cm_\lambda(f)(x) \).
We will also need the somewhat less trivial inequalities:

Lemma 18 (vi) \( m_\lambda(f)(x) \geq |f(x)| \) that holds at every Lebesgue point of \( f \), so a.e. if \( f \in L^1_{loc}(\mathbb{R}^n) \).

Proof. We will need to remember a definition and a known result of Real Analysis. The definition is the following: a sequence \( \{E_i\}_{i \in \mathbb{N}} \) of Borel sets of \( \mathbb{R}^n \) is said to shrink to \( x \) nicely if there is a number \( \alpha > 0 \) such that there is a sequence of cubes of \( \mathbb{R}^n \) centered at \( x \) of radii \( r_i \to 0 \), \( \{Q(x,r_i)\}_{i \in \mathbb{N}}, \) such that
\[
 E_i \subset Q(x,r_i) \text{ and } |E_i| \geq \alpha |Q(x,r_i)|.
\]
The result is: if \( x \in \mathbb{R}^n \) is a Lebesgue point of \( f \in L^1_{loc}(\mathbb{R}^n) \) and \( \{E_i\}_{i \in \mathbb{N}} \) is a sequence of sets that shrinks to \( x \) nicely then
\[
 f(x) = \lim_{i \to \infty} \frac{1}{|E_i|} \int_{E_i} f(z) \, dz
\]
(see [Rudin], theorem 7.10 - changing cubes for balls and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ instead of $f \in L^1(\mathbb{R}^n)$ the proof still works).

Now for any positive $\tau$ with $\tau < 1$, using the definitions of non-increasing rearrangements and $m_\lambda$ we have that

$$\forall Q \ni x : |\{ y \in Q : |f(y)| > \tau m_\lambda f(x) \}| \geq \lambda |Q|$$

so if we take $r_i = \frac{1}{i} \to 0$ and we name $\{ E_i \}_{i \in \mathbb{N}} = \{ y \in Q(x,r_i) : |f(y)| \leq \tau m_\lambda f(x) \}$

then $E_i = Q(x,r_i) \setminus \{ y \in Q(x,r_i) : |f(y)| > \tau m_\lambda f(x) \}$

and we obtain that

$$|E_i| = |Q(x,r_i) \setminus \{ y \in Q(x,r_i) : |f(y)| > \tau m_\lambda f(x) \}| \geq Q(x,r_i) - \lambda |Q(x,r_i)|$$

that is

$$|E_i| \geq (1 - \lambda) . |Q(x,r_i)|$$

and then $\{ E_i \}_{i \in \mathbb{N}}$ is a sequence of sets that shrinks to $x$ nicely. But now, with these sets $E_i$ we can apply the mentioned result for any Lebesgue point to obtain:

$$f(x) = \lim_{i \to \infty} \frac{1}{|E_i|} \int_{E_i} f(z) \, dz \leq \lim_{i \to \infty} \frac{1}{|E_i|} \int_{E_i} \tau m_\lambda f(x) \, dz$$

and using $|f(x)|$ instead of $f(x)$:

$$|f(x)| \leq \lim_{i \to \infty} \frac{\tau m_\lambda f(x)}{|E_i|} \int_{E_i} \, dz = \lim_{i \to \infty} \frac{\tau m_\lambda f(x)}{|E_i|} |E_i| = \tau m_\lambda f(x)$$

Then

$$|f(x)| \leq \tau m_\lambda f(x)$$

for every Lebesgue point of $f$ and then almost everywhere.

(vii) For any $\lambda \in (0,1)$ there is a constant $c_{\lambda,n}$ (depending only of $\lambda$ and $n$) such that for all $u \in L^1_{\text{loc}}$ and $x \in \mathbb{R}^n$ we have ([L]):

$$m_\lambda (Mu)(x) \leq c_{\lambda,n} u^#(x) + Mu(x)$$

(viii) Observe that using vii) and applying ii) to $f = Mu$ we obtain $m_\lambda (Mu)(x) \leq c.Mu(x)$ a.e. for some $c > 0$.

(ix) $m_\lambda (Mu)$ and $Mu$ are pointwise equivalent a.e. (we will write $m_\lambda (Mu) \approx Mu$ for that situation) that is that there are positive constants $A$ and $B$ such
that $m_{\lambda}(Mu)(x) \leq A.Mu(x)$ and $Mu(x) \leq B.m_{\lambda}(Mu(x))$ a.e., we obtain this taking $A = c$ in viii), and $B = 1$ in vi).

(x) It’s immediate from the definition of $M$ that $Mf(x) \geq f(x)$ a.e.

(xi) We will also use a pointwise inequality (see [L2]) that goes in the opposite direction of vii): for any $u \in L^1_{loc}$ and $x \in \mathbb{R}^n$ we have:

SOME MORE RESULTS

Another criterion for characterization of the weights $u$ such that $Mu \in A_\infty$ follows from our Proposition 1 and from inequality vii) :

Criterion 19 Let’s $u$ a weight function, $Mu \in A_\infty$ if and only if for any $\lambda \in (0, 1)$ it holds that $m_{\lambda}(Mu) \approx M(Mu)$.

Proof. By Proposition 1 we have $Mu \in A_\infty \iff Mu \in A_1$, so $Mu \in A_\infty$ if and only if there is some $C > 0 : M(Mu)(x) \leq C.Mu(x)$ a.e. and using that $M(f)(x) \geq f(x)$ a.e. for $f \in L^1_{loc}$ we have that $M(Mu)(x) \geq Mu(x)$ a.e., and then ix) gives us that $Mu \in A_\infty \iff Mu \in A_1 \iff Mu \approx M(Mu) \iff m_{\lambda}(Mu) \approx M(Mu)$. ■

Remark 20 We can observe that it is enough that $m_{\lambda}(Mu) \approx M(Mu)$ for some $\lambda \in (0, 1)$ to obtain that $Mu \in A_\infty$ and then $m_{\lambda}(Mu) \approx M(Mu)$ for every $\lambda \in (0, 1)$.

Remark 21 Because of viii) for any $u$ we always can ensure for a suitable $c > 0 : m_{\lambda}(Mu)(x) \leq c.Mu(x) \leq c.M(Mu)(x)$, that is $m_{\lambda}(Mu)(x) \leq c.M(Mu)(x)$ a.e.; thus, by the criterion above, a condition necessary and sufficient, on $u$, for $Mu$ to belong to $A_\infty$ is the existence of a constant $C > 0$ such that $M(Mu)(x) \leq C.m_{\lambda}(Mu)(x)$ a.e.

As we mentioned in the introduction now we want to prove that (2) is a necessary and sufficient condition on a weight $u$ for $Mu$ to be in $A_\infty$.

A condition like (2) but applied for an arbitrary weight $w$ instead of $Mu$ is weaker than (1), that is, if $w \in A_\infty$ then $w$ satisfies the following:

Condition 22 (LocalAINF) \( \exists \alpha_1 > 0, \beta_1 \in (0, 1) \) such that for almost every $x \in \mathbb{R}^n$ exists a cube $Q_x \ni x$ that $\forall Q \ni x$ verifies that: $|\{y \in Q_x : w(y) \leq \alpha_1 w_Q\}| \leq \beta_1 |Q_x|$

To see this implication let’s remember that $w \in A_\infty$ if and only if $w$ satisfies:

Condition 23 (CAINF) \( \exists \alpha, \beta \in (0, 1) : \forall Q \text{ cube we have } |\{y \in Q : w(y) \leq \alpha w_Q\}| \leq \beta |Q|$

Now, if $w \in A_\infty$ we fix some $k \in (0, 1)$, for instance $k = \frac{1}{2}$, and for any $x$ we take a cube $Q_x \ni x$ such that $w_{Q_x} = w_{Q_x(w)} \geq k.Mw(x)$. So let $\alpha_1 = \alpha.k$ and for any $Q \ni x$ we have that $\{y \in Q_x : w(y) \leq \alpha_1 w_Q\} \subset \{y \in Q_x : w(y) \leq \alpha_1 Mw(x)\}$

10
then applying the previous condition to $Q_x$

$$\left| \{ y \in Q_x : w(y) \leq \alpha_1 w_{Q_x} \} \right| \leq \left| \{ y \in Q_x : w(y) \leq \frac{\alpha_1}{k} w_{Q_x} \} \right|$$

so the condition (LocalAINF) is fulfilled with $\alpha_1 = \alpha.k$, $\beta_1 = \beta$ and the $Q_x$ selected for which $\frac{w(Q_x)}{|x|} \geq k.Mw(x)$.

Then we have that it also holds:

Although the condition (LocalAINF) is weaker than $A_\infty$ for a general weight when it is applied to a weight that is the maximal function of another weight, that is if $w = Mu$ then the condition (LocalAINF) implies $A_\infty$, so they are equivalent conditions for $Mu$ weights.

**Theorem 24** Let $u$ a weight function. Then $Mu \in A_\infty$ if and only if (3) holds, that is:

$$Mu \in A_\infty \iff \exists \alpha > 0, \beta \in (0, 1) : \left| \{ y \in Q_x : Mu(y) \leq \alpha. (Mu)_Q \} \right| \leq \beta. |Q_x|$$

for almost every $x \in \mathbb{R}^n$ for some cube $Q_x \ni x$, and for every cube $Q$ to which $x$ belongs.

**Proof.** Because the previous remark $Mu \in A_\infty$ if and only if there exists a positive constant $B$ and $\lambda \in (0, 1)$ :

$$M(Mu)(x) \leq B.m_\lambda(Mu(x))$$

(3)  
a.e. So to guarantee $Mu \in A_\infty$ is equivalent to have.

$$\alpha.M(Mu)(x) \leq m_\lambda(Mu(x))$$

(4)  
for some $\alpha > 0$ and almost every $x \in \mathbb{R}^n$. Now using the definition of $m_\lambda$ we have that (3) is equivalent to say that for almost every $x \in \mathbb{R}^n$

$$\exists Q_x \ni x : (Mu.\chi_{Q_x})^*(\lambda. |Q_x|) \geq \alpha. (Mu)_Q$$

for every cube $Q \ni x$. Now by the definition of non-increasing rearrangements this means that for a.e. $x \in \mathbb{R}^n$

$$\exists Q_x \ni x : \left| \{ y \in Q_x : Mu(y) > \alpha. (Mu)_Q \} \right| > \lambda. |Q_x|$$

for every cube $Q \ni x$, or, taking complements respect $Q_x$ and naming $\beta = (1 - \lambda) \in (0, 1)$, we have that (3) and therefore $Mu \in A_\infty$ is equivalent to the existence of $\alpha > 0, \beta \in (0, 1)$ such that for almost every $x \in \mathbb{R}^n$ there is some $Q_x \ni x$:

$$\exists Q_x \ni x : \left| \{ y \in Q_x : Mu(y) \leq \alpha. (Mu)_Q \} \right| \leq \beta. |Q_x|$$

for every cube $Q \ni x$. ■
Example 25 It's easy to see that a class of weights functions $u$ such that $Mu \in A_\infty$ is the class $A_\infty$ itself, that is $M (A_\infty) \subset A_\infty$, and by our first proposition in fact $M (A_\infty) \subset A_1$. Indeed we can provide an elementary proof of this using the previous theorem and the characterization $\mathfrak{U}$ of $A_\infty$ weights: We fix some $k \in (0, 1)$, and for any $x$ we take a cube $Q_x$ such that $\frac{Mu(Q_x)}{|Q_x|} \geq k.M (Mu) (x)$; because $\mathfrak{U}$ and the fact that $u \in A_\infty$ we have $\alpha_1, \beta_1$ such that for any cube $Q$ it holds: $\{y \in Q : u (y) \leq \alpha_u \mu_Q\} \leq \beta_1, |Q|$. Then for $Q = Q_x$, $\alpha = \frac{\mu(Q_x)}{k}$, $\beta = \beta_1$ and for any $Q \ni x$, and using the trivial inclusions due to the inequalities $\frac{Mu(Q)}{|Q|} \geq k.M (Mu) (x); \quad MMu (z) \geq Mu (z)$ a.e. and $Mu (z) \geq u (z)$ a.e. we get:

$$\left| \{y \in Q_x : Mu (y) \leq \alpha \frac{Mu (Q)}{|Q|} \right| \leq \left| \{y \in Q_x : Mu (y) \leq \alpha \frac{Mu (Q)}{|Q|} \} \right| \leq \left| \{y \in Q_x : u (y) \leq \alpha \frac{Mu (Q_x)}{|Q_x|} \} \right| \leq \beta |Q_x|$$

that is we have

$$\left| \{y \in Q_x : Mu (y) \leq \alpha \frac{Mu (Q)}{|Q|} \right| \leq \beta |Q_x|$$

Example 26 Actually for those functions there are shorter way to prove that $Mu \in A_1$: Because the Hölder's inequality we have that for all $r > 1$:

$$\frac{1}{|Q|} \int_Q u (x) \leq \left( \frac{1}{|Q|} \int_Q u^r (x) \right)^\frac{1}{r}$$

, and taking suprema

$$Mu (x) \leq (M (u^r) (x))^{\frac{1}{r}}$$

. Now for the Coifman-Rochberg characterization of $A_1$ weights for any locally integrable function $g$ and $\delta \in [0, 1)$ we have that $Mg(x)\delta \in A_1$ and then $(M (u^\delta) (x))^{\frac{1}{\delta}} \in A_1$, therefore for some constant $C > 1$:

$$MMu (x) \leq M \left( (M (u^\delta) (x))^{\frac{1}{\delta}} \right) \leq C \left( M (u^\delta) (x) \right)^{\frac{1}{\delta}}$$

a.e. But if $u \in A_\infty$ then $u \in A_p$ for some $p \geq 1$, and then it satisfy a reverse Hölder inequality (see $[D]$) for some $r > 1$, that is

$$\left( \frac{1}{|Q|} \int_Q u^r (x) \right)^\frac{1}{r} \leq C \frac{1}{|Q|} \int_Q u (x)$$

for certain $C > 0$, thus

$$\left( M (u^r) (x) \right)^{\frac{1}{r}} \leq C Mu (x)$$
and then
\[ MMu(x) \leq C.Mu(x) \]
a.e. That is \( M \mu \in A_1 \). We remark that this way requires two strong results: characterization of \( A_1 \) and the reverse Hölder inequality for \( A_p \) weights, while proposition 1 is elementary.

**Example 27** A larger class of weights that \( M \) sends to \( A_1 \) are the weak \(-\) \( A_\infty \) weights.

We recall that \( u \in A_\infty \) if and only if there exists positive constants \( C \) and \( \delta \) such that for any cube \( Q \) and any measurable \( E \subset Q \):
\[ u(E) \leq C \left( \frac{|E|}{|Q|} \right)^\delta u(Q) \]

Let’s give the definition of weak \(-\) \( A_\infty \) weights: \( u \in \text{weak} - A_\infty \) if and only if there exists positive constants \( C \) and \( \delta \) such that for any cube \( Q \) and any measurable \( E \subset Q \):
\[ u(E) \leq C \left( \frac{|E|}{|Q|} \right)^\delta u(2Q) \quad (5) \]

**Remark 28** It’s easy to prove that we can replace the factor 2 with any constant \( k > 1 \) obtaining an equivalent definition of weak \(-\) \( A_\infty \).

**Remark 29** It’s clear that if \( u \in A_\infty \) then \( u \in \text{weak} - A_\infty \) if and only if there exists positive constants \( C \) and \( \delta \) such that for any cube \( Q \):
\[ u(E) \leq C \left( \frac{|E|}{|Q|} \right)^\delta u(Q) \]

It’s a known result that an equivalent condition for \( u \) to be in \( A_\infty \) is to belong to a RHI class, that means that for some \( r > 1 \) and \( C > 0 \) it holds for any cube \( Q \)
\[ \left( \frac{1}{|Q|} \int_Q u^r \right)^\frac{1}{r} \leq C \frac{1}{|Q|} \int_Q u \]

**Remark 30** Let’s remark that those weights that belongs to weak \(-\) \( A_\infty \) but that don’t belong to \( A_\infty \) are always non-doubling weights.

**Remark 31** A corollary that we can obtain immediately taking suprema on the RHI condition for \( A_\infty \) weights is that for any \( x \in \mathbb{R}^n \)
\[ (M (u^r)(x))^\frac{1}{r} \leq C.Mu(x) \]

It can be obtained for weak \(-\) \( A_\infty \) weights a condition analogous to RHI as we can see in the next:

**Lemma 32** If \( u \in \text{weak} - A_\infty \) there are some \( r > 1 \) and \( C > 0 \) such that for any cube \( Q \)
\[ \left( \frac{1}{|Q|} \int_Q u^r \right)^\frac{1}{r} \leq C \frac{1}{|2Q|} \int_{2Q} u \]
**Proof.** Let $Q$ any cube and $E_t = \{ x \in Q : u(x) > t \}$. Now, applying the definition of $E_t$ and we have $t. |E_t| \leq u(E_t) \leq C \frac{|E_t|}{|Q|} \cdot u(2Q)$. Hence, using $|2Q| = 2^n |Q|$ and incorporating the factor $2^n$ to the constant $C$:

$$t. |E_t|^{1-\delta} \leq C. |Q|^{1-\delta} \cdot \frac{u(2Q)}{|2Q|}$$

so

$$|E_t| \leq C. t^{\frac{1}{r-1}}. |Q|. \left( \frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}}$$

Now we use this inequality in the layer-cake formula. Let’s be $k \in (0, \infty)$ that we will chose later:

$$\int_Q u^r = \int_0^\infty r t^{-1} |E_t| \, dt = \int_0^\infty r t^{-1} |E_t| \, dt = \int_0^k r t^{-1} |E_t| \, dt + \int_k^\infty r t^{-1} |E_t| \, dt$$

then

$$\int_Q u^r \leq \int_0^k r t^{-1} |Q| \, dt + C \int_k^\infty r t^{-1} |Q| \cdot \left( \frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}} \, dt$$

that is:

$$\int_Q u^r \leq |Q| \cdot t^r_0 + C \cdot |Q| \cdot \left( \frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}} \cdot r \cdot \frac{r}{1-\delta} \cdot t^{r-\frac{1}{1-\delta}}$$

then, for $r : 1 < r < \frac{1}{1-\delta}$ we get:

$$\frac{1}{|Q|} \int_Q u^r \leq k^r + C \cdot \frac{r}{1-\delta} \cdot \left( \frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}} \cdot k^{r-\frac{1}{1-\delta}}$$

Now choosing $k = \frac{u(2Q)}{|2Q|}$ it results:

$$\frac{1}{|Q|} \int_Q u^r \leq \left( \frac{u(2Q)}{|2Q|} \right)^r + C \cdot \frac{r}{1-\delta} \cdot \left( \frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}} \cdot \left( \frac{u(2Q)}{|2Q|} \right)^{r-\frac{1}{1-\delta}}$$

, hence

$$\frac{1}{|Q|} \int_Q u^r \leq \left( C \cdot \frac{r}{1-\delta} \right) \left( \frac{u(2Q)}{|2Q|} \right)^r$$

and renaming the constant we have:

$$\left( \frac{1}{|Q|} \int_Q u^r \right)^{\frac{1}{r}} \leq C \cdot \frac{u(2Q)}{|2Q|}$$
Corollary 33  From the previous lemma it’s obvious that the pointwise inequality
\[(M(u^r)(x))^\frac{1}{r} \leq C.Mu(x) \]  \hspace{1cm} (6)
still remains true for weak – \(A_\infty\) weights and using Neugebauer’s Lemma the weights \(u \in \text{weak} – A_\infty\) satisfy that \(Mu \in A_1\).

Remark 34  Actually the condition:
\[ \left( \frac{1}{|Q|} \int_Q u^r \right)^{\frac{1}{r}} \leq C. \frac{1}{|2Q|} \int_{2Q} u \]  \hspace{1cm} (7)
characterizes the weak – \(A_\infty\) weights; it can be proved that the converse of the previous lemma is also true, nevertheless we will not need here that result. As we mentioned in a previous remark we can replace the constant 2 for any \(k > 1\), so \(u \in \text{weak} – A_\infty\) iff there exists some positive constant \(C\) such that for any \(k > 1\) and every cube \(Q\)
\[ \left( \frac{1}{|Q|} \int_Q u^r \right)^{\frac{1}{r}} \leq C. \frac{1}{|kQ|} \int_{kQ} u \]  \hspace{1cm} (7)

Remark 35  We have already seen that \(A_\infty \subset \text{weak} – A_\infty \subset M^{-1}(A_\infty)\) where we denote \(M^{-1}(A_\infty)\) the class of weights \(u\) such that \(Mu \in A_\infty\).

It’s interesting to observe that this question has a close relationship with another one involving the weighted Fefferman-Stein inequality in \(L^p(w)\):
\[ \|f\|_{L^p(w)} \leq c \|f\|_{L^p(w)}^\# \quad (1 < p < \infty) \]  \hspace{1cm} (8)
for some \(c > 0\), and for every \(f \in L^p\) such that \(f \in \mathcal{S}_0(\mathbb{R}^n)\), where \(\mathcal{S}_0(\mathbb{R}^n)\) is the space of measurable functions \(f\) on \(\mathbb{R}^n\) such that for any \(t > 0\)
\[ \mu_f(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}| < \infty \]

The inequality (8) is equivalent to many interesting others, for instance, with the same hypothesis of (8)
\[ \|Mf\|_{L^p(w)} \leq c \|f\|_{L^p(w)}^\# \quad (1 < p < \infty) \]
or for some \(c > 0\), \(r > 1\) and for any \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\)
\[ \int_{\mathbb{R}^n} M_{p,r}(f, w) |f| \, dx \leq c \int_{\mathbb{R}^n} (Mf)^p \, wx \quad (1 < p < \infty) \]  \hspace{1cm} (9)
where \(M_{p,r}(f, w) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f| \right)^{p-1} \left( \frac{1}{|Q|} \int_Q w^r \right)^{\frac{1}{r}}\). The equivalence of those inequalities is proven in [L3].

Related to the -at our knowledge- open question about for which weights the former inequalities hold are the following inclusions of nested classes: \(A_\infty \subset \text{weak} – A_\infty \subset C_{p+\varepsilon} \subset C_p\) where \(\varepsilon > 0\) and \(C_p\) condition means that there exists \(c, \delta > 0\) such that for any cube \(Q\) and any measurable \(E \subset Q\)
\[ u(E) \leq c \left( \frac{|E|}{|Q|} \right)^{\delta} \int_{\mathbb{R}^n} (M_{XQ})^p \, u \]
Remember that for \( u \in A_1 \) for any cube \( Q \) and any measurable \( E \subset Q \)
\[
u(E) \leq c \left( \frac{|E|}{|Q|} \right) \delta u(Q) = c \left( \frac{|E|}{|Q|} \right) \delta \int_{\mathbb{R}^n} (\chi_Q)^p u
\]

and for weak–\( A_\infty \) weights: \( u \in \text{weak–} A_\infty \) if and only if there exists positive constants \( C \) and \( \delta \) such that for any cube \( Q \) and any measurable \( E \subset Q \) :
\[
u(E) \leq C \left( \frac{|E|}{|Q|} \right) \delta \int_{\mathbb{R}^n} (\chi_Q)^p u
\]

and the mentioned inclusion are obvious. It can be found in [L3] (see also [Y]) that \( C_p \) is necessary and \( C_{p+\varepsilon} \) is sufficient for \( \mathbb{R} \) or \( \mathbb{R}^n \), and in [L3] is introduced a new sufficient condition \( C_p \) instead of \( C_{p+\varepsilon} \) but it is not known if \( C_p \) or \( C_{p+\varepsilon} \) are necessary conditions.

The inclusion relations from \( A_\infty \subset \text{weak–} A_\infty \subset M^{-1}(A_\infty) \) and \( A_\infty \subset \text{weak–} A_\infty \subset C_{p+\varepsilon} \subset C_p \) and the former inequalities seems to be close linked: For instance \( u \in C_p \) is necessary for \( \mathbb{R} \) and \( \mathbb{R}^n \) implies that for any \( Q \) we have that
\[
\left( \frac{1}{|Q|} \int_Q u^+ \right)^\frac{1}{p} \leq C \left( \frac{1}{|Q|} \int_Q (M\chi_Q)^p u, \text{ which is a bit weaker than } \left( \frac{1}{|Q|} \int_Q u^+ \right)^\frac{1}{p} \leq C \left( \frac{1}{|Q|} \int_Q (\chi_{2Q})^p u \right) \text{ that it is equivalent to weak–} A_\infty.
\]

Additionally in [L3] is proven that \( C_p \) is necessary for \( \int_{\mathbb{R}^n} M_{p,r}(f,w) |f| dx \leq c \int_{\mathbb{R}^n} (Mf)^p wdx \), that is \( \boxed{\text{[2]}} \) implies \( C_p \).

On the other hand, using the lemma of Neugebauer telling us \( (Mu^+)^\frac{1}{p} (x) \leq C.Mu(x) \) for \( u \in M^{-1}(A_\infty) \) for some \( C > 0, r > 1 \) and the definition of \( M_{p,r}(f,w) \) we obtain that if \( u \in M^{-1}(A_\infty) \) then
\[
M_{p,r}(f,w)(u) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f|^p \right)^{\frac{1}{p}} \left( \frac{1}{|Q|} \int_Q u^+ \right)^{\frac{1}{p}} \leq \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f|^p \right)^{\frac{1}{p}} M_r u(x) \leq \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f|^p \right)^{\frac{1}{p}} C.Mu(x) \leq (Mf)^{p-1}(x).C.Mu(x)
\]
and then integrating we have:
\[
\int_{\mathbb{R}^n} M_{p,r}(f,w) |f| dx \leq c \int_{\mathbb{R}^n} (Mf)^p Mu dx \tag{10}
\]
(compare with \( \boxed{\text{[7]} \). So we have that \( M^{-1}(A_\infty) \) implies \( \boxed{\text{[10]}} \) and \( \boxed{\text{[7]} \) implies \( C_p \).

**A PAIR OF APPLICATIONS**

**Application 36** Using the criterion: \( Mu \in A_\infty \) if and only if for any \( \lambda \in (0,1) \) it holds that \( m_\lambda(Mu) \approx M(Mu) \) we can derive from this result a characterization of the \( A_1 \) weights similar to the construction of Coifman and Rochberg.
First of all we introduce the definition of the local sharp maximal operator; for \(0 < \lambda < 1\) we define:
\[
M_{\lambda}^\# f (x) = \sup_{Q \ni x} \inf_{c} \left( (f - c) \chi_{Q} \right)^{+} \left( \lambda |Q| \right)
\]

The sharp maximal function have a role quite similar to the Hardy-Littlewood maximal operator for the local sharp maximal functions because there are positive constants \(c_1\) and \(c_2\) such that for \(f \in L_{1,loc}^1\):
\[
c_1 MM_{\lambda}^\# f(x) \leq f^\#(x) \leq c_2 MM_{\lambda}^\# f(x)
\]
(see [J-T]).

Let Lemma 37 an statement similar to the first one of the Coifman-Rochberg theorem:

\[ \text{Lemma 37} \]

Proof. For \(c_1 \left( MM_{\lambda}^\# f(x) \right)^{\delta} \leq f^\#(x)^{\delta} \leq c_2 \left( MM_{\lambda}^\# f(x) \right)^{\delta} \) and \(MM_{\lambda}^\# f(x)\) in \(A_1\) because the mentioned result of Coifman and Rochberg. Now \(Mf^\#(x)^{\delta} \leq M \left( c_2 \left( MM_{\lambda}^\# f(x) \right)^{\delta} \right) \leq c_2 \left( MM_{\lambda}^\# f(x) \right) \leq (MM_{\lambda}^\# f(x))^{\delta} \) with constant \(C = c_2 \left( MM_{\lambda}^\# f(x) \right) \), so \(f^\#(x)^{\delta} \in A_1\). \(\blacksquare\)

We don't know if any \(w \in A_1\) always could be written as \(k(x) f^\#(x)^{\delta}\) for suitable \(f \in L_{1,loc}^1\); \(0 < \delta < 1\) and \(k, k^{-1} \in L^{\infty}\), but we can obtain a result similar to the second part of Coifman-Rochberg theorem if we added a multiple of the local maximal function \(m_{\lambda}\):

\[ \text{Proposition 38} \]

Proof. If \(w \in A_1\) we can use the property E) to take \(\alpha > 1\) such that \(w^\alpha \in A_1\). Thus \(M(w^\alpha) \in A_1\). Now using for \(w^\alpha\) the above criterion that establishes that \(Mu \in A_1\) if and only if \(m_{\lambda}(Mu) \approx M(Mu)\) and then in such situation: \(m_{\lambda}(Mu) \approx M(M(w^\alpha)) \approx M(w^\alpha) \approx w^\alpha\), also we have that \(Mw \approx w\) because \(w \in A_1\) and also using the pointwise inequalities mentioned in xi) and vii): \(m_{\lambda}(Mu)(x) \leq c_{\lambda,n}Mu(x)\) and \(m_{\lambda}(Mu)(x) \leq c_{\lambda,n}Mu(x)\), for \(u = w^\alpha\) we have:
\[
w(x)^{\alpha} \leq M(w^\alpha)(x) \leq c_{\lambda,n} \cdot (w^\alpha)^{\#}(x) + m_{\lambda}(w^\alpha)(x)
\]

Then with \(\delta = \frac{1}{\alpha}\) it is \(0 < \delta < 1\) and \(\alpha \delta = 1\). Also we will use property (i): \(u^\# \leq 2Mu\) pointwise, properties (vi) \((|f(x)| \leq m_{\lambda}f(x))\) and (x) \((f(x) \leq Mf(x))\) and that if \(f(x) \leq g(x)\) a.e. for positive functions then \(Mf(x) \leq Mg(x)\) and \(m_{\lambda}(f)(x) \leq m_{\lambda}(g)(x)\) a.e.

Further we use the sublinearity of \(M\) and the facts that \(w^\alpha\) and \(w\) are in \(A_1\) and then because the criterion, we can use that for \(w \in A_1\) then \(MW \in A_1\) too and it occurs that \(m_{\lambda}(MW) \approx M(Mw) \approx Mw \approx w\). We will number or rename
the constants that appear. Also we will use that $M \left( (Mu^\alpha)^\delta \right) \leq C (Mu^\alpha)^\delta$ (because $(Mf)^\delta \in A_1$ by Coifman-Rochberg). So we get:

$$w(x) \leq \left( c_1 \cdot (w^\alpha)^\#(x) + m_\lambda (w^\alpha)(x) \right)^\delta$$

$$\leq c_2 \cdot \left( (w^\alpha)^\#(x) \right)^\delta + (m_\lambda (w^\alpha)(x))^\delta$$

$$\leq M \left( c_2 \cdot \left( (w^\alpha)^\#(x) \right)^\delta + (m_\lambda (w^\alpha)(x))^\delta \right)$$

$$\leq c_2 \cdot M \left( \left( (w^\alpha)^\#(x) \right)^\delta \right) + M \left( (m_\lambda (w^\alpha)(x))^\delta \right)$$

$$\leq c_3 M \left( M (w^\alpha)(x)^\delta \right) + M \left( (m_\lambda (Mu^\alpha)(x))^\delta \right)$$

$$\leq c_3 M \left( c_4 (w^\alpha)(x)^\delta \right) + M \left( c_5 (w^\alpha)(x)^\delta \right)$$

Thus we obtain:

$$w(x) \leq c_6 M w(x) + c_7 M w(x) = c_6 M w(x) \leq Cw(x)$$

and then $k(x) = \frac{\frac{w(x)}{c_2.((w^\alpha)^\#)^\delta}}{c_1.((w^\alpha)^\#)^\delta + (m_\lambda w^\alpha(x))^\delta}$ satisfy that $k \in L^\infty$ and $k^{-1} \in L^\infty$

So $w(x) = k(x) \left( c. \left( (w^\alpha(x))^\# \right)^\delta + d. (m_\lambda w^\alpha(x))^\delta \right)$ with $k, k^{-1} \in L^\infty$ and $\delta \in (0, 1)$ for $c = c_2$ and $d = 1$.

On the other hand we have:

**Lemma 39** If $0 < \delta < 1$ and $u \in A_1$ then $(m_\lambda u(x))^\delta \in A_1$

**Proof.** Using that $u \in A_1$, then $Mu \in A_1$ and $m_\lambda (Mu) \approx M (Mu) \approxMu \approx u$ and that $(MMu)^\delta \in A_1$ (by Coifman-Rochberg theorem) we have the following inequalities with multiplicative constants that we will be renumbering -:

$M \left( (m_\lambda u)^\delta \right) \leq M \left( (m_\lambda Mu)^\delta \right) \leq M \left( (C_1 MMu)^\delta \right) = C_2 M \left( (MMu)^\delta \right) \leq C_3 (MMu)^\delta \leq C_4 (m_\lambda (Mu))^\delta \leq C_5 (m_\lambda (C_4 u))^\delta = C_5 (m_\lambda u)^\delta$ and then we get that $(m_\lambda u)^\delta \in A_1$.

**Remark 40** It’s elementary that if $v_1, v_2$ are non-negative functions with $v_1, v_2 \in A_1$ and if $c$ and $d$ are non-negative constants then $cv_1 + dv_2 \in A_1$ and $|cv_1 + dv_2|_{A_1} \leq \max\{|v_1|_{A_1}, |v_2|_{A_1}\}$.

Compiling the last two lemmas, the proposition and the previous remark we have a theorem similar to the Coifman-Rochberg result:
Theorem 41 (1) If $0 < \delta < 1$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $u \in A_1$ and $c, d$ non-negative constants then $(c \cdot f^\#(x) + d \cdot m_\lambda u(x))^{\delta} \in A_1$.

(2) Conversely, if $w \in A_1$ then there are $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $u \in A_1$, non-negative constants $c$ and $d$, and $k(x)$ with $k, k^{-1} \in L^\infty$ such that $w(x) = k(x) \left(c \cdot f^\#(x)^{\delta} + d \cdot m_\lambda u(x)^{\delta}\right)$.

Proof. The first statement is consequence of the latter remark and from the lemmas telling us that $f^\#(x)^{\delta}$ and $(m_\lambda u(x))^{\delta}$ are in $A_1$ for $f \in L^1_{\text{loc}}$ and $u \in A_1$.

The second was obtained in the latter proposition for $f = u = w^\alpha$ taking a suitable $\alpha > 1$ such that $w^\alpha \in A_1$. The existence of that $\alpha$ is guaranteed by property E.

Remark 42 The previous result, like the Coifman-Rochberg Theorem, presents a class of functions, included in $A_1$, such that any $A_1$ weight differs from some element of that class only by a factor function $k(x)$ that it is bounded and bounded away from zero, that is $k, k^{-1} \in L^\infty$. Another remarkable example is given by the functions in the image of an operator obtained by means of a variant of the Rubio de Francia algorithm.

The usual construction (see for instance [G2]) involves some sublinear operator bounded in $L^p(\mu)$ with $p \geq 1$ for certain measure $\mu$ and it is defined for $f \in L^p(\mu)$ by:

$$Rf(x) = \sum_{k=0}^{\infty} \frac{T^k f(x)}{\left(2 \|T\|_{p, \mu}\right)^k}$$

where $T^0$ is the identity and $T^k = T \circ T \circ ... \circ T$, $k$ times. Some basic properties of $R$ are:

i) $f(x) \leq Rf(x)$ a.e.

ii) $\|Rf\|_{p, \mu} \leq 2 \|f\|_{p, \mu}$

iii) $T(Rf)(x) \leq 2 \|T\|_{p, \mu} Rf(x)$ a.e.

For $T = M$, the Hardy-Littlewood maximal operator and the usual Lebesgue measure in $\mathbb{R}^n$ the third property means $M \left(Rf\right)(x) \leq 2 \|M\|_p Rf(x)$ thus $Rf \in A_1$ for any $f \in L^p$ with $\|Rf\|_{A_1} \leq 2 \|M\|_p$. For to characterize the whole $A_1$ might be necessary to change this procedure for to avoid the issue about the belonging to $L^p$ (for instance if $f \in L^1$ then $Mf$ is never in $L^1$ except when $f$ is identically 0). Notwithstanding we can give the following:

Proposition 43 $u \in A_1$ if and only if there are $C > 0$, $f \in L^1_{\text{loc}}$ and $k(x)$ with $k, k^{-1} \in L^\infty$ such that $w(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{C_k}$ is well defined, $w \in A_1$ and $u(x) = k(x) \cdot w(x)$.

Proof. The proof is almost trivial. The "if" part is immediate because if $u(x) = k(x) \cdot w(x)$ with $w \in A_1$ and $k, k^{-1} \in L^\infty$ then

$$Mu(x) \leq \|k\|_\infty M(w)(x) \leq \|k\|_\infty \|w\|_{A_1} w(x)$$
\[
\leq \|k\|_\infty [u]_{A_1} \|k^{-1}\|_\infty k(x) \cdot w(x) \leq [u]_{A_1} \|k\|_\infty \|k^{-1}\|_\infty u(x)
\]
that is \( u \in A_1 \) and \([u]_{A_1} \leq [u]_{A_1} \|k\|_\infty \|k^{-1}\|_\infty\).

For the "only if" part let's take \( f = u \in L^1_{loc} \) (because \( u \) is a weight), \( C = 2[u]_{A_1}, w(x) = Ru(x) := \sum_{k=0}^{\infty} \frac{M^k u(x)}{2^k [u]_{A_1}} \) and \( k(x) = \frac{u(x)}{R u(x)} \).

Iterating we have that \( M^k u(x) \leq [u]_{A_1}^k u(x) \) a.e. and then

\[
0 \leq \frac{M^k u(x)}{2^k [u]_{A_1}} \leq \frac{u(x) [u]_{A_1}^k}{2^k} = \frac{u(x)}{2^k} \text{ a.e.}
\]
Thus \( \sum_{k=0}^{\infty} \frac{M^k u(x)}{2^k [u]_{A_1}} \) is convergent a.e., \( w(x) = Ru(x) = \sum_{k=0}^{\infty} \frac{M^k u(x)}{2^k [u]_{A_1}} \) is well defined and \( Mw(x) \leq \sum_{k=0}^{\infty} \frac{M^{k+1} u(x)}{2^k [u]_{A_1}} \leq \sum_{k=0}^{\infty} \frac{[u]_{A_1} M^k u(x)}{2^k [u]_{A_1}} = [u]_{A_1} w(x), \) that is \( w \in A_1 \) and \([u]_{A_1} \leq [u]_{A_1}\).

Finally

\[
u(x) \leq w(x) = \sum_{k=0}^{\infty} \frac{M^k u(x)}{2^k [u]_{A_1}} \leq \sum_{k=0}^{\infty} \frac{[u]_{A_1}^k u(x)}{2^k} = u(x) \sum_{k=0}^{\infty} \frac{1}{2^k} = 2u(x)
\]
so \( 1 \leq \frac{w(x)}{u(x)} \leq 2 \) and then \( k(x) = \frac{u(x)}{w(x)} \) satisfies that \( k, k^{-1} \in L^\infty \) with \( \|k\|_\infty \leq 1 \)
and \( \|k^{-1}\|_\infty \leq 2 \), thus \( u(x) = k(x) \cdot w(x) \) with \( w = \sum_{k=0}^{\infty} \frac{M^k u(x)}{2^k [u]_{A_1}} \in A_1 \) and \( k, k^{-1} \in L^\infty \). □

**Application 44** For those weights \( u \) such that \( Mu \in A_\infty \) and hence \( Mu \in A_1 \) we can improve some known inequalities for singular integral operators. For instance if \( T \) is a Calderón-Zygmund singular integral operator (see [G] for a definition) the following weighted inequalities were proved for \( 1 < p < \infty \) by C. Pérez ([P]) -previously J.M. Wilson obtained the first inequality for \( 1 < p < 2 \):

\[
\int_{\mathbb{R}^n} |Tf|^p u \leq C_p \int_{\mathbb{R}^n} |f|^p M^{[p]+1} u
\]
and then

\[
u \left( \{ x \in \mathbb{R}^n : |Tf(x)| > \lambda \} \right) \leq \frac{C_p}{\lambda^p} \int_{\mathbb{R}^n} |f|^p M^{[p]+1} u
\]
the last one for the case \( p = 1 \) looks:

\[
u \left( \{ x \in \mathbb{R}^n : |Tf(x)| > \lambda \} \right) \leq \frac{C_2}{\lambda} \int_{\mathbb{R}^n} |f|^2 M^2 u
\]
where \([p]\) is the integer part of \( p \) and \( M^k \) is the \( k \)-th iterate composition of \( M \).
The strong inequality is sharp in the sense that \([p]+1\) cannot be replaced by \([p]\), and the weak case is sharp when \( p \) is not an integer and it is an open question.
at our knowledge- if it is possible to replace $M^{[p]+1}$ with $M^{[p]}$ if $p \in \mathbb{N}$ -and $M^2$ with $M$ in the last inequality.-

Now for a weight $u$ such that $Mu \in A_\infty$ we have that actually $Mu \in A_1$ and then there are a constant $C > 0$ such that for almost every $x \in \mathbb{R}^n$:

$M^2 u(x) \leq C.Mu(x)$, and using that if in almost everywhere $f(x) \leq g(x)$ then $Mf(x) \leq Mg(x)$, we can iterate in $M^2 u(x) \leq C.Mu(x)$ to obtain $M^k u(x) \leq C^k Mu(x)$, then with $C = C^p_k$ we have for the Calderón-Zygmund singular integral operators and the weights $u$ with $Mu \in A_\infty$:

$$\int_{\mathbb{R}^n} |Tf|^p u \leq C \int_{\mathbb{R}^n} |f|^p Mu$$

$$u \left( \{x \in \mathbb{R}^n : |Tf (x)| > \lambda \} \right) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f|^p Mu$$

for any $1 < p < \infty$.

REFERENCES

[BS] Bennett C and Robert Sharpley R, Interpolation of Operators. Pure and Applied Mathematics Series, Vol 129, 1988

[C-R] Coifman, R., Rochberg, R. Another characterization of B.M.O., Proc. Amer. Math. Soc. 1980

[CU] Cruz-Uribe, D. Cruz-Uribe, SFO, Piecewise monotonic doubling measures, Rocky Mtn. J. Math. 26 (1996),1-39.

[CU-P] D. Cruz-Uribe, SFO, Pérez C. Two weight extrapolation via the maximal operator, Journal of Functional Analysis, 2000

[D] Duoandikoetxea, J. Fourier Analysis, Graduate studies in Mathematics, AMS 2001

[DMO] Duoandikoetxea J, Martín-Reyes F, Ombrosi S., On the $A_\infty$ conditions for general bases, Mathematische Zeitschrift, 2016

[G] Grafakos, Classical Fourier Analysis, Graduate studies in Mathematics, Springer, 2000

[G2] Grafakos, Modern Fourier Analysis, Graduate studies in Mathematics, Springer, 2009

[G-R] Garcia-Cuerva, J., and Rubio de Francia, J. L. Weighted Norm Inequalities and Related Topics, North Holland, New York, 1985.

[J-T] B. Jawerth, A. Torchinsky, Local sharp maximal functions, J. Approx. Theory 43 (1985) 231–270

[L] Lerner, A, On some pointwise estimates for maximal and singular integral operators, Studia Math 138 (2000)

[L2] Lerner, A. On some pointwise inequalities, J. Math. Anal. Appl. 289 (2004)

[L3] Lerner, A. Some remarks on the Fefferman-Stein inequality, Journal d’Analyse Mathématique October 2010, Volume 112, Issue 1, pp 329–349

[P] Pérez C. Weighted norm inequalities for singular integral operators. C. Pérez. Journal of the London mathematical society 49, 1994
[S] E. Stein, Harmonic analysis real-variable methods, orthogonality, and oscillatory integrals, Princeton Univ. Press, Princeton, NJ, 1993

[Y] Yabuta K., Sharp maximal function and Cp condition, Arch. Math. 55 (1990), 151–155.