Lagrangian formulation for the infinite spin 
$N = 1$ supermultiplets in $d = 4$

I.L. Buchbinder$^{ab*}$, M.V. Khabarov$^{cd†}$, T.V. Snegirev$^{ac‡}$, Yu.M. Zinoviev$^{cd§}$

$^a$Department of Theoretical Physics, Tomsk State Pedagogical University, 
Tomsk, 634061, Russia

$^b$National Research Tomsk State University, Tomsk 634050, Russia

$^c$Institute for High Energy Physics of National Research Center "Kurchatov Institute" 
Protvino, Moscow Region, 142281, Russia

$^d$Moscow Institute of Physics and Technology (State University), 
Dolgoprudny, Moscow Region, 141701, Russia

$^e$National Research Tomsk Polytechnic University, Tomsk 634050, Russia

Abstract

We provide an explicit Lagrangian construction for the massless infinite spin $N = 1$ supermultiplet in four dimensional Minkowski space. Such a supermultiplet contains a pair of massless bosonic and a pair of massless fermionic infinite spin fields with properly adjusted dimensionful parameters. We begin with the gauge invariant Lagrangians for such massless infinite spin bosonic and fermionic fields and derive the supertransformations which leave the sum of their Lagrangians invariant. It is shown that the algebra of these supertransformations is closed on-shell.

$^{*}$joseph@tspu.edu.ru
$^{†}$maksim.khabarov@ihep.ru
$^{‡}$snegirev@tspu.edu.ru
$^{§}$Yurii.Zinoviev@ihep.ru
1 Introduction

Infinite spin fields (see \cite{1,2} and references therein) have attracted a lot of interest recently. A number of different approaches for their description were proposed \cite{3–22}. Also the investigations of possible interactions for such fields started \cite{23–25}. At the same time there exists just a few results on the supermultiplets containing such particles \cite{11,21,22}, while their classification is already known for a long time \cite{1}.

One of the interesting features of the infinite spin fields is that being massless they however depend on some dimensionful parameter, related with the value of the second Casimir operator of Poincare group. In many respects such fields can be considered as a limit of massive higher spin fields, where mass \( m \to 0 \) and spin \( s \to \infty \) so that \( \mu = ms \to \text{const} \).

In particular, it appears that the gauge invariant formalism for the description of massive higher spin fields \cite{26–29} can be successfully applied for the description of infinite spin fields as well \cite{9–11,13}. Taking into account that such gauge invariant formalism remains to be the only effective way to construct massive higher spin supermultiplets \cite{30–33}, it seems natural to apply this approach to the construction of the infinite spin supermultiplets.

In this note we provide an explicit on-shell Lagrangian realization for the massless \( N = 1 \) infinite spin supermultiplet in flat four dimensional Minkowski space. Our paper is organized as follows. In Section 2 we give the gauge invariant Lagrangians for the massless bosonic and fermionic infinite spin fields. Then in Section 3 we find supertransformations which leave the sum of their free Lagrangians invariant and such that the algebra of the supertransformations is closed on-shell.

Notations and conventions We use the same frame-like multispinor formalism as in \cite{32}, where all objects are one or zero forms having some number of dotted \( \dot{\alpha} = 1, 2 \) and undotted \( \alpha = 1, 2 \) completely symmetric indices. Coordinate-free description of flat Minkowski space is given by the external derivative \( d \) and a background one-form frame \( e^{\alpha \dot{\alpha}} \), as well as by basic two, three and four forms:

\[
E_2^{\alpha \beta}, \quad E_2^{\dot{\alpha} \dot{\beta}}, \quad E_3^{\alpha \dot{\alpha}}, \quad E_4,
\]

which are defined as follows:

\[
\begin{align*}
e^{\alpha \dot{\alpha}} \wedge e^{\beta \dot{\beta}} &= \varepsilon^{\alpha \beta} E^{\dot{\alpha} \dot{\beta}} + \varepsilon^{\dot{\alpha} \dot{\beta}} E^{\alpha \beta}, \\
E^{\alpha \beta} \wedge e^{\gamma \dot{\alpha}} &= \varepsilon^{\alpha \gamma} E^{\beta \dot{\alpha}} + \varepsilon^{\beta \gamma} E^{\alpha \dot{\alpha}}, \\
E^{\dot{\alpha} \dot{\beta}} \wedge e^{\alpha \dot{\gamma}} &= -\varepsilon^{\dot{\alpha} \dot{\gamma}} E^{\alpha \dot{\beta}} - \varepsilon^{\dot{\beta} \dot{\gamma}} E^{\alpha \dot{\alpha}}, \\
E^{\alpha \dot{\alpha}} \wedge e^{\beta \dot{\beta}} &= \varepsilon^{\alpha \beta} \varepsilon^{\dot{\alpha} \dot{\beta}} E.
\end{align*}
\]

2 Kinematics of infinite spin fields

In this work we use a description for the infinite spin bosonic and fermionic fields based on the gauge invariant formalism for the massive higher spin fields \cite{26–29}, which has already been successfully applied for the infinite spin fields as well \cite{9–11,13}. 
2.1 Infinite spin boson

An infinite spin bosonic field in $d = 4$ contains an infinite number of helicities $0 \leq l < \infty$, so for the gauge invariant formulation we introduce a number of physical and auxiliary one-forms $f^{\alpha(k)\hat{\alpha}(k)}$, $\Omega^{(k+1)\hat{\alpha}(k-1)} + h.c.$, $1 \leq k < \infty$, as well as zero-forms $B^{\alpha(2)} + h.c.$ and one-form $A$ for the helicities $\pm 1$, while helicity 0 is described by two zero-forms $\pi^{\alpha\hat{\alpha}}$ and $\varphi$.

The general ansatz for the corresponding bosonic Lagrangian $\mathcal{L}_B$ is written as follows:

$$\mathcal{L}_B = \sum_{k=1}^{\infty} (-1)^{k+1} \left[ (k + 1) \Omega^{(k)\hat{\alpha}(k-1)} E_\beta \gamma \Omega^{(k)\hat{\alpha}(k-1)} \right]$$

$$+ \left[ (k - 1) \Omega^{(k+1)\hat{\alpha}(k-2)} E_\beta \gamma \Omega^{(k+1)\hat{\alpha}(k-2)} \right] + 2\Omega^{(k)\hat{\alpha}(k-1)} e^{\beta\hat{\beta}} d_f \Omega^{(k)\hat{\alpha}(k-1)} + h.c.]$$

$$+ 4[E B^{\alpha(2)} B_{\alpha(2)} + E B^{\hat{\alpha}(2)} B_{\hat{\alpha}(2)}] + 2[E^{\alpha(2)} B_{\alpha(2)} dA - E^{\hat{\alpha}(2)} B_{\hat{\alpha}(2)} dA]$$

$$- 6 E^{\alpha,\hat{\alpha}} \pi_{\alpha,\hat{\alpha}} - 12 E^{\alpha,\hat{\alpha}} \pi_{\alpha,\hat{\alpha}} d\varphi$$

$$+ \sum_{k=1}^{\infty} (-1)^{k+1} a_k \left[ \Omega^{(k)(2)\hat{\alpha}(2)} E^{(2)} f_{\alpha(k)\hat{\alpha}(k)} \right]$$

$$+ \frac{k}{(k+2)} \Omega^{(k+1)\hat{\alpha}(k-1)} E^{(2)} f_{\alpha(k+1)\hat{\alpha}(k-1)\hat{\beta}(2)} + h.c.]$$

$$- a_0 [E^{\alpha(2)} E_{\alpha(2)} A - \Omega^{(2)} E_{\alpha(2)} A] - 2a_0 [B^{\alpha\hat{\beta}} E^{\hat{\alpha}} \hat{\alpha} f_{\beta,\hat{\alpha}} + B^{\hat{\alpha}\hat{\beta}} E^{\alpha} \hat{\alpha} f_{\alpha,\hat{\alpha}}] + a_0 E^{\alpha\hat{\alpha}} \pi_{\alpha,\hat{\alpha}} A$$

$$+ \sum_{k=1}^{\infty} (-1)^{k+1} \left[ b_k f^{(k-1)\hat{\beta}(k)} E^{(2)} f_{\alpha(k-1)\hat{\alpha}(k)} + h.c. \right]$$

$$+ \frac{a_0 a_0}{2} E^{\alpha\hat{\alpha}} f_{\alpha,\hat{\alpha}} \varphi + 3a_0^2 E \varphi^2. (1)$$

Here $a_k$, $b_k$ are arbitrary dimensional coefficients providing the mixture of the different helicities into one multiplet. The Lagrangian $\mathcal{L}_B$ has a common structure for the massive higher spin gauge invariant description, namely it contains the usual kinetic and mass-like terms for all the helicity components as well as the cross-terms connecting the nearest neighbours. Such a structure follows from the requirement that we still must have all (appropriately modified) gauge symmetries, which our helicity components initially possessed. The ansatz for these modified gauge transformations (consistent with the structure of the ansatz for the Lagrangian $\mathcal{L}_B$) has the form:

$$\delta \Omega^{(k+1)\hat{\alpha}(k-1)} = d\eta^{(k+1)\hat{\alpha}(k-1)} + e^{\beta\hat{\alpha}} \eta^{(k+1)\beta\hat{\alpha}(k-2)} + \frac{a_{k-1}}{k+1}(k+2) e^{\alpha\hat{\alpha}} \eta^{(k)\hat{\alpha}(k-2)} + \frac{b_k}{2(k+1)} e^{\alpha\hat{\beta}} \xi^{(k)\hat{\beta}(k-1)}$$

$$\delta f^{\alpha(k)\hat{\alpha}(k)} = d\xi^{\alpha(k)\hat{\alpha}(k)} + e^{\beta\hat{\alpha}} \eta^{(k+1)\beta\hat{\alpha}(k-1)} + e^{\alpha\hat{\beta}} \eta^{(k+1)\hat{\alpha}(k-1)\hat{\beta}}$$

$$+ \frac{k a_k}{2(k+2)} e^{\beta\hat{\alpha}} \xi^{(k)\beta\hat{\alpha}(k)} + \frac{a_{k-1}}{2(k+1)} e^{\alpha\hat{\beta}} \xi^{(k)\hat{\alpha}(k-1)\hat{\beta}}$$

$$\delta \Omega^{(2)} = d\eta^{(2)} + \frac{a_1}{2} \left[ e^{\beta\hat{\beta}} \eta^{(2)\beta\hat{\beta}} + \frac{b_1}{4} e^{\alpha\hat{\alpha}} \xi^{\alpha\hat{\alpha}} \right]$$

$$\delta f^{\alpha\hat{\alpha}} = d\xi^{\alpha\hat{\alpha}} + e^{\alpha\hat{\beta}} \eta^{\alpha\hat{\beta}} + e^{\alpha\hat{\alpha}} \eta^{\alpha\hat{\beta} \hat{\beta}} + \frac{a_1}{6} e^{\beta\hat{\beta}} \xi^{\alpha\beta\hat{\beta}} - \frac{a_0}{4} e^{\alpha\hat{\alpha}} \xi^{\alpha\hat{\alpha}}$$

$$\delta B^{(2)} = \frac{a_0}{2} \eta^{(2)} \ , \ \delta A = d\xi - \frac{a_0}{2} e^{\alpha\hat{\alpha}} \xi^{\alpha\hat{\alpha}},$$

$$\delta \pi^{\alpha\hat{\alpha}} = - \frac{a_0 a_0}{24} \xi^{\alpha\hat{\alpha}} \ , \ \delta \varphi = \frac{a_0}{12} \xi.$$
The invariance of the Lagrangian under these gauge transformations leads to the following relations on the parameters:

\[(k + 2)b_k = (k - 1)b_{k-1},\]

\[\frac{k}{4(k + 2)}a_k^2 - \frac{(k - 1)}{4(k + 1)}a_{k-1}^2 + b_k = 0,\]

\[\frac{1}{12}a_1^2 - \frac{1}{4}a_0^2 + b_1 - 4\lambda^2 = 0, \quad \tilde{a}_0^2 = 72b_1.\]

A general solution for these relations depends on two parameters. We choose \(a_0\) and \(b_1\) and obtain:

\[a_k^2 = \frac{(k + 2)}{k}[a_0^2 - \frac{6k(k + 3)}{(k + 1)(k + 2)}b_1], \quad b_k = \frac{6b_1}{k(k + 1)(k + 2)}.\]  

(3)

For the resulting Lagrangian be hermitian (and the theory to be unitary) we must have \(a_k^2 \geq 0\) for all \(k\). This leads to the two types of solutions [13,27]:

- The solutions with the whole spectrum of helicities \(0 \leq k < \infty\), which requires \(a_0^2 \geq 6b_1\). For \(a_0^2 > 6b_1\) they correspond to the tachyonic fields, while for \(a_0^2 = 6b_1\) we obtain a massless infinite spin boson (the case we are mostly interested in):

\[a_k^2 = \frac{2a_0^2}{k(k + 1)}, \quad b_k = \frac{a_0^2}{k(k + 1)(k + 2)}.\]  

(4)

- The second type of solutions have the spectrum \(s \leq l < \infty\)

\[a_s = 0 \quad \Rightarrow \quad a_k^2 = -\frac{12(k + 2)(k + s + 3)(k - s)}{k(k + 1)(s + 1)(s + 2)}b_1.\]

The positivity of \(a_k^2\) requires \(b_1\) to be negative, so all these solutions are tachyonic.

To simplify our construction of the supermultiplet we do not introduce any supertransformations for the auxiliary fields \(\Omega, b\) and \(\pi\). Instead, all calculations are done up to the terms proportional to the auxiliary field equations of motion that is equivalent to the following "zero torsion conditions":

\[\mathcal{T}^{\alpha(k)\dot{\alpha}(k)} = df^{\alpha(k)\dot{\alpha}(k)} + e_\beta \Omega^{\alpha(k)\dot{\alpha}(k-1)} + e_\dot{\beta} \Omega^{\alpha(k-1)\dot{\alpha}(k)} + \frac{ka_k}{2(k + 2)}e_\beta f^{\alpha(k)\dot{\alpha}(k)} + \frac{a_{k-1}}{2(k + 1)}e^{\alpha\dot{\alpha}}f^{\alpha(k-1)\dot{\alpha}(k-1)} \approx 0,\]

\[\mathcal{T}^{\alpha\dot{\alpha}} = df^{\alpha\dot{\alpha}} + e_\beta \Omega^{\alpha\beta} + e_{\dot{\beta}} \Omega^{\alpha\dot{\beta}} + \frac{a_1}{6}e^{\alpha\dot{\beta}}f^{\alpha\dot{\beta} \dot{\alpha}} - \frac{a_0}{4}e^{\alpha\dot{\alpha}}A \approx 0,\]  

(5)

\[\mathcal{T} = dA + 2(E_{\alpha(2)}B^{\alpha(2)} + E_{\dot{\alpha}(2)}B^{\alpha(2)}) - \frac{a_0}{2}e^{\alpha\dot{\alpha}}f^{\alpha\dot{\alpha}} \approx 0,\]

\[\mathcal{C} = d\varphi + e_{\alpha\dot{\alpha}}\pi^{\alpha\dot{\alpha}} - \frac{a_0}{12}A \approx 0.\]

As for the supertransformations for the physical fields \(f, A\) and \(\varphi\), the corresponding variation of the Lagrangian can be compactly written as follows

\[\delta \mathcal{L}_B = 2i \sum_{k=1}^{\infty} (-1)^k \mathcal{R}^{\alpha(k)\dot{\alpha}(k-1)}e_\beta \delta f^{\alpha(k)\dot{\alpha}(k-1)} + \delta E_{\alpha(2)}C^{\alpha(2)} \delta A + 12iE_{\alpha\dot{\alpha}}C^{\alpha\dot{\alpha}} \delta \varphi + h.c.\]  

(6)

3
where we introduced "curvatures":
\[
\mathcal{R}^{\alpha(k+1)\dot{\alpha}(k-1)} = d\Omega^{\alpha(k+1)\dot{\alpha}(k-1)} + e_\beta^{\dot{\alpha}}\Omega^{\alpha(k+1)\beta\dot{\alpha}(k-2)} + \frac{a_k}{2} e_\beta^{\dot{\alpha}}\Omega^{\alpha(k+1)\beta \dot{\alpha}(k-1)\dot{\beta}} + \frac{a_k}{k+1} e^{\dot{\alpha}} \Omega^{\alpha(k)\dot{\alpha}(k-2)} + \frac{b_k}{2(k+1)} e^{\dot{\alpha}} f^{\alpha(k)\dot{\alpha}(k-1)\dot{\beta}},
\]
\[
\mathcal{R}^{\alpha(2)} = d\Omega^{\alpha(2)} + \frac{a_1}{2} e_\beta^{\dot{\alpha}}\Omega^{\alpha(2)\beta\dot{\beta}} + \frac{b_1}{4} e^{\dot{\alpha}} f^{\alpha\dot{\beta}} - \frac{a_0}{4} E^\alpha_\beta B^{\alpha\beta} - \frac{a_0}{24} E^{\alpha(2)} \varphi,
\]
\[
C^{\alpha(2)} = dB^{\alpha(2)} - \frac{a_0}{2} \Omega^{\alpha(2)} - \frac{\tilde{a}_0}{24} e^{\dot{\alpha}} \pi^{\alpha\dot{\beta}},
\]
\[
C^{\alpha\dot{\alpha}} = d\pi^{\alpha\dot{\alpha}} + \frac{a_0}{24} f^{\alpha\dot{\alpha}} - \frac{\tilde{a}_0}{12} (e^{\dot{\alpha}} B^{\alpha\beta} + e^{\alpha} B^{\dot{\beta}\dot{\alpha}}) + \frac{a_0^2}{8} e^{\alpha} \varphi.
\]

2.2 Infinite spin fermion

For the gauge invariant description of the infinite spin fermionic field we need one-forms \( \Phi^{\alpha(k+1)\dot{\alpha}(k)} \), \( \Phi^{\alpha(k)\dot{\alpha}(k+1)} \), \( 0 \leq k < \infty \), as well as zero-forms \( \phi^{\alpha} \), \( \phi^{\dot{\alpha}} \). The general ansatz for the corresponding Lagrangian \( \mathcal{L}_F \) is written as follows:

\[
\mathcal{L}_F = \sum_{k=0}^{\infty} (-1)^{k+1} \Phi^{\alpha(k)\dot{\alpha}(k)} e_\beta^{\dot{\alpha}} d\Phi^{\alpha(k)\dot{\alpha}(k)\dot{\beta}} - \phi^\alpha E^\alpha_\dot{\alpha} d\phi^{\dot{\alpha}}
\]
\[
+ \sum_{k=1}^{\infty} (-1)^{k+1} c_k \Phi^{\alpha(k+1)\dot{\alpha}(k)} \Omega^{\beta(2)\dot{\alpha}(k)} E^{\beta(2)\dot{\alpha}(k)} + c_0 \Phi^{\alpha\dot{\alpha}} E^\alpha_\dot{\alpha} \phi^{\dot{\alpha}} + h.c.
\]
\[
+ \sum_{k=0}^{\infty} (-1)^{k+1} d_k [(k+2) \Phi^{\alpha(k)\dot{\alpha}(k)} E^{\beta\gamma} \Phi^{\alpha(k)\gamma\dot{\alpha}(k)} - k \Phi^{\alpha(k+1)\dot{\alpha}(k-1)\dot{\beta}} E^{\beta\gamma} \Phi^{\alpha(k+1)\dot{\alpha}(k-1)\dot{\beta}} + h.c.]
\]
\[
+ 2d_0 E\phi^\alpha \phi^{\dot{\alpha}} + h.c.
\]

Here \( c_k, d_k \) are the dimensionful coefficients providing the mixture of the different helicities into one multiplet. The Lagrangian \( \mathcal{L}_F \) has the same common structure as in the bosonic case. The ansatz for the supertransformations (consistent with that for the Lagrangian) \( \mathcal{L}_F \) has the form:

\[
\delta \Phi^{\alpha(k+1)\dot{\alpha}(k)} = d\eta^{\alpha(k+1)\dot{\alpha}(k)} + c_{k+1} e_\beta^{\dot{\alpha}} \eta^{\alpha(k+1)\beta\dot{\alpha}(k)\dot{\beta}} + 2d_k e^{\alpha} \dot{\beta} \eta^{\alpha(k)\beta\dot{\alpha}(k)\dot{\beta}} + \frac{c_k}{k+2} e^{\alpha\dot{\alpha}} \eta^{\alpha(k)\dot{\alpha}(k)\dot{\beta}},
\]
\[
\delta \Phi^{\alpha(k)\dot{\alpha}(k+1)} = d\eta^{\alpha(k)\dot{\alpha}(k+1)} + c_{k+1} e_\beta^{\dot{\alpha}} \eta^{\alpha(k)\beta\dot{\alpha}(k+1)\dot{\beta}} + 2d_k e^{\alpha} \dot{\beta} \eta^{\alpha(k)\beta\dot{\alpha}(k+1)\dot{\beta}} + \frac{c_k}{k+2} e^{\alpha\dot{\alpha}} \eta^{\alpha(k)\dot{\alpha}(k+1)\dot{\beta}},
\]
\[
\delta \Phi^\alpha = d\eta^\alpha + c_1 e_\beta^{\dot{\alpha}} \eta^{\alpha\beta\dot{\beta}} + 2d_0 e^{\alpha} \dot{\beta} \eta^{\dot{\beta}}, \quad \delta \phi^\alpha = c_0 \eta^\alpha,
\]
\[
\delta \Phi^{\dot{\alpha}} = d\eta^{\dot{\alpha}} + c_1 e_\beta^{\dot{\alpha}} \eta^{\beta\dot{\alpha}\dot{\beta}} + 2d_0 e^{\dot{\alpha}} \dot{\beta} \eta^{\dot{\beta}}, \quad \delta \phi^{\dot{\alpha}} = c_0 \eta^{\dot{\alpha}}.
\]

The invariance of the Lagrangian \( \mathcal{L}_F \) under these gauge transformations leads to the following relations on the parameters:

\[
(k+2)d_k = kd_{k-1}, \quad k \geq 1,
\]
\[
c_{k+1}^2 - c_k^2 + 4(2k + 3)d_k^2 = 0,
\]
\[
2c_1^2 - c_0^2 + 24d_0^2 = 0.
\]
General solution again depends on the two parameters. We choose $c_0$ and $d_0$ and obtain:

$$c_k^2 = \frac{c_0^2}{2} - \frac{16k(k+2)}{(k+1)^2}d_0^2, \quad d_k = \frac{2d_0}{(k+1)(k+2)}.$$  \hfill (10)

As in the bosonic case we have two types of solution \[10\] [13].

- Solutions with the whole spectrum of helicities $1/2 \leq l < \infty$, which requires $c_0^2 \geq 32d_0^2$. Most of them are tachyonic, while for $c_0^2 = 32d_0^2$ we obtain a massless infinite spin fermion

$$c_k^2 = \frac{c_0^2}{2(k+1)^2}, \quad d_k = \pm \frac{c_0}{2\sqrt{2}(k+1)(k+2)}.$$  \hfill (11)

- Solutions with the spectrum $s + 1/2 \leq l < \infty$

$$c_s = 0 \quad \Rightarrow \quad c_k^2 = -\frac{16(k+s+2)(k-s)}{(k+1)^2(s+1)^2}d_0^2,$$

where the positivity of $c_k^2$ requires $d_0^2$ to be negative and hence $d_0$ to be imaginary.

Note, that in the fermionic case all tachyonic solutions require imaginary masses so that the Lagrangian is not hermitian. Thus in what follows we restrict ourselves with the massless infinite spin bosons and fermions.

As in the bosonic case, the variation of the Lagrangian $L_F$ under the arbitrary transformations for the physical fields can be compactly written as follows:

$$\delta L_F = \sum_{k=0}^{\infty} (-1)^k \mathcal{F}_{\alpha(k)\beta\dot{\alpha}(k)} e_{\beta}^\dot{\alpha} \delta \Phi^{\alpha(k)\dot{\alpha}(k)} - C_{\alpha} E_{\beta}^\alpha \delta \phi_{\beta} + h.c. \quad \text{(12)}$$

where we introduced gauge invariant "curvatures":

$$\mathcal{F}_{\alpha(k+1)\dot{\alpha}(k)} = d \Phi^{\alpha(k+1)\dot{\alpha}(k)} + c_{k+1} e_{\beta\dot{\beta}}^{\alpha(k+1)\beta\dot{\beta}(k)} \Phi^{\alpha(k)\dot{\beta}(k)} \dot{\beta} + 2d_k e_{\beta\dot{B}}^{\alpha(k)\beta\dot{\alpha}(k)} \Phi^{\alpha(k)\dot{\alpha}(k-1)} \dot{\beta},$$

$$\mathcal{F}_{\alpha} = d \Phi^{\alpha} + c_{1} e_{\beta\dot{\beta}}^{\alpha\beta\dot{\beta}} \Phi^{\alpha\beta\dot{\beta}} + 2d_0 e_{\beta\dot{\beta}}^{\alpha} \Phi^{\beta\dot{\beta}} - \frac{c_0}{3} E_{\beta}^\alpha \phi_{\beta},$$

$$C_{\alpha} = d \phi^{\alpha} - c_0 \Phi^{\alpha} + 2d_0 e_{\beta\dot{\beta}}^{\alpha} \phi^{\beta\dot{\beta}}.$$  \hfill (13)

### 3 Infinite spin supermultiplet

In this section we construct a supermultiplet containing infinite spin bosonic and fermionic fields. Let us consider one massless infinite spin boson with the Lagrangian (11) and the parameters (4) and one massless infinite spin fermion with the Lagrangian (18) and the parameters (11). Taking into account close similarity between the gauge invariant description
for massive finite spin fields and massless infinite spin ones, we take the same general ansatz for the supertransformations as in [32]. Namely, for the bosonic components we take

$$\delta \Phi^\alpha \delta \Omega^\gamma = \alpha_k \Phi^\alpha \delta \Omega^\gamma - \bar{\alpha}_k \Phi^\alpha \delta \Omega^\gamma$$

\[+\alpha' k \Phi^\alpha \delta (k-1) \delta \Omega^\gamma - \bar{\alpha}' k \Phi^\alpha \delta (k-1) \delta \Omega^\gamma,\]

$$\delta A = \alpha_0 \Phi^\alpha \zeta_\alpha - \bar{\alpha}_0 \Phi^\alpha \zeta_\alpha + \alpha'_0 \zeta_\alpha \Phi^\alpha - \bar{\alpha}'_0 \zeta_\alpha \Phi^\alpha,$$

$$\delta \Phi = \bar{\alpha}_0 \Phi^\alpha \zeta_\alpha - \bar{\alpha}_0 \Phi^\alpha \zeta_\alpha,$$

while for the fermions respectively

$$\delta \Phi^\alpha = \beta_k \Omega^\alpha \zeta_\beta + \gamma_k \Phi^\alpha \zeta_\beta$$

\[+\beta'_k+1 \Omega^\alpha \zeta_\beta + \gamma'_k+1 \Phi^\alpha \zeta_\beta,\]

$$\delta \Phi = \beta_0 \Phi^\alpha \zeta_\beta + \gamma_0 \Phi^\alpha \zeta_\beta + \gamma_0 \Phi^\alpha \zeta_\beta,$$

$$\delta \Phi^\alpha = \beta_0 \Phi^\alpha \zeta_\beta + \gamma_0 \Phi^\alpha \zeta_\beta + \gamma_0 \Phi^\alpha \zeta_\beta,$$

where $$\zeta_\alpha, \zeta_\beta$$ are the anticommuting supersymmetry transformation parameters. These transformations contain the undefined yet complex coefficients.

Using the general expressions for the variation of the bosonic Lagrangian (6) we obtain:

$$\delta \mathcal{L}_B = \sum_{k=1} (-1)^k [4i \alpha_k \Phi^\alpha (k-1) \zeta_\beta + \gamma_k \Phi^\alpha (k-1) \zeta_\beta]$$

\[+4i \alpha'_k \Phi^\alpha (k-1) \zeta_\beta + \gamma'_k \Phi^\alpha (k-1) \zeta_\beta + ... + h.c.\]

where dots stand for the contributions of the lower spin components. Similarly, using the analogous expression for the fermionic Lagrangian (12) we get:

$$\delta \mathcal{L}_F = \sum_{k=1} (-1)^k [-k \beta_k \Phi^\alpha (k-1) \zeta_\beta + \gamma_k \Phi^\alpha (k-1) \zeta_\beta]$$

\[-k \beta'_k \Phi^\alpha (k-1) \zeta_\beta + \gamma'_k \Phi^\alpha (k-1) \zeta_\beta + ... + h.c.\]

Now we have to find the explicit expressions for all the coefficients $$\alpha, \beta, \gamma$$ such that $$\delta (\mathcal{L}_B + \mathcal{L}_F) = 0$$. The general technique here is essentially the same as the one described in [32], the main difference being in the explicit form of Lagrangian parameters (14) and (11). This produces the following results.

- All parameters $$\alpha$$ and $$\gamma$$ can be expressed in terms of $$\beta$$:

$$\alpha_k = \frac{i k}{4} \bar{\beta}_k, \quad \alpha'_k = \frac{i}{4 k} \bar{\beta}'_k,$$

$$\alpha_0 = -\frac{i}{4 k} \bar{\beta}_0, \quad \alpha_0' = -\frac{i}{8} \bar{\beta}'_0,$$

$$\gamma_k = 2d_{k+1} \bar{\beta}_k, \quad \gamma'_k = 2d_k \bar{\beta}'_k,$$

$$\gamma_0 = -d_0 \bar{\beta}_0, \quad \gamma_0' = -\frac{12}{a_0} \bar{\beta}_0, \quad \gamma_0'' = -\frac{a_0}{8} \bar{\beta}_0.$$
Note that these relations are purely kinematical and are the same as in the massive case.

- We obtain an important relation on the two dimensionful parameters, one from bosonic sector and another one from fermionic sector.

\[ a_0 = c_0 \] (18)

As it is well known, supersymmetry requires that all the members of massive supermultiplets have the same mass. Our fields are massless but they are characterized by the dimensionful parameters (related with the value of the second Casimir operator of Poincare group). So it seems natural that supersymmetry requires that these parameters have to be related.

- At last, we obtain a very simple (in comparison with the massive case) solution for the remaining parameters:

\[
\begin{align*}
\beta_k &= \frac{1}{\sqrt{k}} \rho, \quad \beta'_k = \sqrt{k} \rho', \\
\beta_0 &= \sqrt{2} \rho, \quad \beta'_0 = 2 \rho', \quad \tilde{\beta}_0 = -\sqrt{6} \rho,
\end{align*}
\] (19)

where

\[ \rho' = \pm \tilde{\rho}. \]

Here ± sign corresponds to that of the fermionic mass terms. Note also, that in our multispinor formalism real (imaginary) values of \( \beta \) correspond to parity-even (parity-odd) bosonic fields. Thus we have four independent solutions.

So we managed to find the supertransformations which leave the sum of the bosonic and fermionic Lagrangians invariant, \( \delta(\mathcal{L}_B + \mathcal{L}_F) = 0 \). However, the explicit calculations of the commutator for supertransformations (14), (15) show that their superalgebra is not closed even on-shell. To see the reason we briefly discuss a relation between massive supermultiplets and massless infinite spine ones. We remember that in four dimensional Minkowski space there are two massive higher spin \( N = 1 \) supermultiplets

\[
\begin{pmatrix}
s + \frac{1}{2} & s' \\
s - \frac{1}{2} & s'
\end{pmatrix}, \quad \begin{pmatrix}
s + \frac{1}{2} & s + \frac{1}{2}
\end{pmatrix}.
\]

Each of them contains four fields. The highest spin in the first case is fermionic and the highest spin in the second case is bosonic. In both cases the \( s \) and \( s' \) are integer and equal. Label \( ' \) means that the corresponding bosonic field has the opposite parity in comparison with another bosonic field from the same supermultiplet [30]. Each multiplet is characterized on-shell by equal number of bosonic and fermionic degrees of freedom. As we know, the massless infinite spin fields can be obtained from the massive one in the limit where \( m \rightarrow 0, \ s \rightarrow \infty, \ ms \rightarrow const \), therefore it is natural to consider that the massless infinite spin supermultiplet must appear as the analogous limit from the massive one. Moreover, the limit for the two types of the massive supermultiplets seems to be the same since in both cases we get infinite
number of all possible helicities. Thus to construct an infinite spin supermultiplet we need four fields

\[
\begin{pmatrix}
\Phi_+ \\
\Phi_-
\end{pmatrix},
\]

where \(f_+ (f_-)\) denotes parity-even (parity-odd) boson, while the signs of the \(\Phi_{\pm}\) correspond to that of mass terms \(d_k\) in \(L_F\) \((\ref{8})\). In particular, it means that we need all four solutions given above \((\ref{19})\) and then the complete Lagrangian for the supermultiplet under consideration have to take the form

\[
L = L_B^{(+)} + L_B^{(-)} + L_F^{(+)} + L_F^{(-)},
\]

where \(L_B^{(\pm)}\) is the Lagrangian \(L_B\) \((\ref{11})\) expressed in terms of \(f_{\pm}\) and corresponding auxiliary fields \(\Omega_{\pm}\), and \(L_F^{(\pm)}\) corresponds to the Lagrangian \(L_F\) \((\ref{8})\) expressed in terms \(\Phi_{\pm}\) respectively.

To simplify the presentation of the results, let us introduce the notations for bosonic field variables:

\[
f = f_+ + if_-,
\]

\[
\tilde{f} = f_+ - if_-,
\]

\[
\Omega = \Omega_+ + i\Omega_,
\]

\[
\bar{\Omega} = \Omega_+ - i\Omega_-
\]

Also we introduce new fermionic variables:

\[
\Phi = \Phi_+ + \Phi_-,
\]

\[
\tilde{\Phi} = \Phi_+ - \Phi_-,
\]

so that the fermionic mass terms in Lagrangian for the infinite spin supermultiplet now have the Dirac form:

\[
\Phi_+ \Phi_+ - \Phi_- \Phi_- \Rightarrow \tilde{\Phi} \Phi.
\]

In notations \((\ref{21}), (\ref{22})\) the supertransformations can be written in a very compact form:

\[
\begin{align*}
\delta f^{\alpha(k)\dot{\alpha}(k)} &= \frac{i}{2\sqrt{k}} \rho \Omega^{\alpha(k)\dot{\alpha}(k)} \zeta_{\beta} + \frac{i}{2\sqrt{k}} \rho \bar{\Phi}^{\alpha(k-1)\dot{\alpha}(k)} \zeta_{\beta}, \\
\delta \tilde{f}^{\alpha(k)\dot{\alpha}(k)} &= \frac{i}{2\sqrt{k}} \rho \Phi^{\alpha(k)\dot{\alpha}(k)} \zeta_{\beta} + \frac{i}{2\sqrt{k}} \rho \bar{\Phi}^{\alpha(k)\dot{\alpha}(k-1)} \zeta_{\beta},
\end{align*}
\]

\[
\begin{align*}
\delta \Phi^{\alpha(k+1)\dot{\alpha}(k)} &= \frac{2}{\sqrt{k}} \rho \Omega^{\alpha(k+1)\dot{\alpha}(k-1)} \zeta_{\dot{\beta}} + 4\sqrt{k+1} d_k \rho f^{\alpha(k+1)\dot{\alpha}(k)} \zeta_{\dot{\beta}}, \\
\delta \tilde{\Phi}^{\alpha(k+1)\dot{\alpha}(k)} &= 2\sqrt{k+1} \rho \Omega^{\alpha(k+1)\dot{\alpha}(k)} \zeta_{\beta} + \frac{4d_k+1}{\sqrt{k}} \rho \tilde{f}^{\alpha(k)\dot{\alpha}(k)} \zeta_{\beta},
\end{align*}
\]

and similarly for the lower spin components. Here \(\rho\) is the only free (real) parameter which determines the normalization of the superalgebra.

Direct calculations show that the algebra of these supertransformations is indeed closed on-shell (it is instructive to compare the structure of the results with the zero torsion conditions \((\ref{11})\)):

\[
\frac{1}{i\rho^2} [\delta_1, \delta_2] f^{\alpha(k)\dot{\alpha}(k)} = \Omega^{\alpha(k)\dot{\beta}(k-1)} \zeta_{\beta} + \Omega^{\alpha(k)\dot{\alpha}(k)\dot{\beta}} \zeta_{\dot{\beta}},
\]

8
\[ + \frac{k a_k}{2(k+2)} f^{\alpha(k)\beta\dot{\alpha}(k)\dot{\beta}} \dot{\xi}_\beta + \frac{a_{k-1}}{2k(k+1)} f^{\alpha(k-1)\dot{\alpha}(k-1)} \xi^{\dot{\alpha}}, \]

\[ \frac{1}{i \rho^2} [\delta_1, \delta_2] f^{\alpha\dot{\alpha}} = \Omega^{\alpha\beta} \dot{\xi}_\beta + \Omega^{\dot{\alpha}\dot{\beta}} \dot{\xi}_\beta + \frac{a_1}{6} f^{\alpha\beta\dot{\alpha}\dot{\beta}} \dot{\xi}_\beta - \frac{a_0}{4} A \xi^{\alpha\dot{\alpha}}, \]  

\[ \frac{1}{i \rho^2} [\delta_1, \delta_2] A = -2 \epsilon_{\dot{\beta} \dot{\beta} 1} [B^{\alpha\beta} \dot{\xi}_\alpha + B^{\dot{\alpha}\dot{\beta}} \dot{\xi}_\alpha] - \frac{a_0}{2} f^{\alpha\dot{\alpha}} \dot{\xi}_{\alpha\dot{\alpha}}, \]

\[ \frac{1}{i \rho^2} [\delta_1, \delta_2] \varphi = \pi^{\alpha\dot{\alpha}} \dot{\xi}_{\alpha\dot{\alpha}}, \]

where the translation parameter \( \xi^{\alpha\dot{\alpha}} \) is defined by

\[ \xi^{\alpha\dot{\alpha}} = \xi^{\alpha_1 \dot{\alpha}_2} - \xi^{\alpha_2 \dot{\alpha}_1}. \]

Supertransformations (23) and (24) are the final results connecting two bosonic and two fermionic infinite spins in one infinite spin supermultiplet. The corresponding invariant Lagrangian has form (20) expressed in terms of new field variables (21), (22).

### 4 Conclusion

We have constructed the Lagrangian formulation for the massless infinite spin \( N = 1 \) supermultiplet in four dimensional Minkowski space. Such supermultiplet consists of the two bosonic (with opposite parities) and two fermionic infinite spin fields with the properly adjusted dimensionful parameters. We provide the gauge invariant Lagrangian formulation for the massless infinite spin boson and fermions which depends on one dimensionful parameter. Then we construct the supertransformations which leave the sum of the four Lagrangians invariant and such that the algebra of these transformations is closed on-shell. We note that although our construction was based on assumption that correct massless infinite spin supermultiplet is obtained as the special limit of higher spin massive supermultiplet, we have derived both supertransformations (23), (24) and the invariant Lagrangian (20). The algebra of the supertransformations is closed on-shell. The results in a whole is completely consistent with the properties of \( N = 1 \) supersymmetric theories formulated in component approach.

We want to emphasize a power and universality of the gauge invariant approach for derivation of the Lagrangian formulation for higher spin fields possessing the massive or dimensionful parameters. The approach under consideration works perfectly both for massive bosonic and fermionic field theories and for infinite spin field theories. Also it allows to develop successfully the corresponding supergeneralizations as it was demonstrated in the works [32], [33] and in this work.

In this paper we constructed the supertransformation whose algebra is closed on-shell. Such a situation is typical for component formulation of the supersymmetric field theory where supersymmetry is not manifest. The manifest supersymmetry is achieved in the framework of superfield approach (see e.g. [34]). It would be extremely interesting to develop a superfield approach to Lagrangian formulation of the supersymmetric infinite spin field theory and obtain off-shell supersymmetry. First step in this direction has been made in the work [22] although the problem is open on a whole.
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