Semi-closed form prices of barrier options in the Hull-White model

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In this paper we derive semi-closed form prices of barrier (perhaps, time-dependent) options for the Hull-White model, i.e., where the underlying follows a time-dependent OU process with a mean-reverting drift. Our approach is similar to that in (Carr and Itkin, 2020) where the method of generalized integral transform is applied to pricing barrier options in the time-dependent OU model, but extends it to an infinite domain (which is an unsolved problem yet). Alternatively, we use the method of heat potentials for solving the same problems. By semi-closed solution we mean that first, we need to solve numerically a linear Volterra equation of the first kind, and then the option price is represented as a one-dimensional integral. Our analysis shows that computationally our method is more efficient than the backward and even forward finite difference methods (if one uses them to solve those problems), while providing better accuracy and stability.

Introduction

The Hull-White model since it was invented in (Hull and White, 1990a) became to be very popular among practitioners for modeling interest rates and credit. That is because it is relatively simple and allows for negativity prices (while for a long time this behaviour was treated as a deficiency of the model, nowadays this became its advantage). The model could be calibrated to the given term-structure of interest rates and to the prices or implied volatilities of caps, floors or European swaptions since the mean-reversion level and volatility are functions of time. Under the Hull-White model the prices of Zero-coupon bonds and European Vanilla options are known in closed form, (Andersen and Piterbarg, 2010). However, for exotic options, e.g., highly liquid barrier options, these prices are not known yet in closed form. Therefore, various numerical methods are used to obtain such prices.

In this paper we present an analytical solution of this problem, and demonstrate that it significantly accelerates computation of the prices and, accordingly, calibration of the model. Our contribution is twofold. First, we solve the problem of pricing Barrier options in semi-closed form and provide the resulting expressions not known yet in the literature. Second, we solve this problem by two methods.
The first one is the method of heat potentials, which came to mathematical finance due to A. Lipton who borrowed it from mathematical physics (see references in the next section). The other method is a method of generalized integral transform, also known in physics, and introduced to mathematical finance in (Carr and Itkin, 2020). However, this method solves the problem where the underlying is defined at the domain $S \in [0, y(t)]$ with $S$ being the stock price, and $y(t)$ being the time-dependent barrier. Contrary, in this paper we consider a complimentary domain $r \in (y(t), \infty)$ where the solution is not known yet. Therefore, our paper fills this gap, and the constructed solution can be applied to a wide class of problems, both in physics and finance.

1 The model

In this section we consider a one-factor short interest rate model, first introduced in (Hull and White, 1990a), and named after the authors as the Hull-White model. The model assumes dynamics of the short interest rate $r_t$ to follow the Ornstein-Uhlenbeck (OU) process with time-dependent coefficients

$$dr_t = \kappa(t)[\theta(t) - r_t]dt + \sigma(t)dW_t, \quad r_{t=0} = r. \quad (1)$$

Here $t \geq 0$ is the time, $\kappa(t) > 0$ is the constant speed of mean-reversion, $\theta(t)$ is the mean-reversion level, $\sigma(t)$ is the volatility of the process, $W_t$ is the standard Brownian motion under the risk-neutral measure. This model is also popular for modeling prices of the plain vanilla and exotic options. In particular, in this paper we consider a Down-and-Out barrier option with the time-dependent lower barrier $L(t)$ where the underlying is a zero-coupon bond with maturity $S$ and price $F(r, t, S)$.

1.1 The underlying - Zero Coupon Bond

We assume that the short interest rate evolves in time as in Eq. (1). It is known that $F(r, t, S)$ under a risk-neutral measure solves a linear parabolic partial differential equation (PDE), (Privault, 2012)

$$\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2(t) \frac{\partial^2 F}{\partial r^2} + \kappa(t)[\theta(t) - r] \frac{\partial F}{\partial r} = rF. \quad (2)$$

This equation should be solved subject to the terminal condition

$$F(r, S, S) = 1, \quad (3)$$

and the boundary conditions

$$F(0, t, S) = g(t), \quad F(r, t, S) \bigg|_{r=\infty} = 0, \quad (4)$$

where $g(t)$ is some function of the time $t$. See, for instance, (Zhang and Yang, 2017) and references therein.

To find $g(t)$, recall that according to (Ekström and Tysk, 2011) for single-factor models that predict nonnegative short rates, no second boundary condition is required for Eq. (2) if $r_t > 0$, $\forall t \geq 0$ in Eq. (1), i.e., the boundary $r_t = 0$ is not attainable. Otherwise, the PDE itself at this point serves as the boundary condition. In particular, as applied to our OU process, it reads

$$\left. \left( \frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2(t) \frac{\partial^2 F}{\partial r^2} + \kappa(t)\theta(t) \frac{\partial F}{\partial r} \right) \right|_{r=0} = 0. \quad (5)$$

However, since Eq. (1) is the OU process, it allows zero or even negative interest rates, i.e. in this model $r \in \mathbb{R}$. Therefore, the PDE Eq. (2) must be solved subject to the second boundary condition either at
$r \rightarrow -\infty$, or at some artificial left boundary $r = r_{\min} < 0$. It is not obvious, however, how to setup this boundary condition at $r_{\min}$.

On the other hand, since the Hull-White model belongs to the class of affine models, (Andersen and Piterbarg, 2010), the solution of Eq. (2) can be represented in the form

$$F(r,t,S) = A(t,S)e^{B(t,S)r}. \quad (6)$$

Substituting this expression into Eq. (2) and separating the terms proportional to $r$, we obtain two equations to determine $A(t,S), B(t,S)$

$$\frac{\partial B(t,S)}{\partial t} = B(t,S)\kappa(t) + 1, \quad (7)$$

$$2\frac{\partial A(t,S)}{\partial t} = -A(t,S)B(t,S) \left[ 2\theta(t)\kappa(t) + B(t,S)\sigma(t)^2 \right].$$

To obey the terminal condition Eq. (3), the first PDE in Eq. (7) should be solved subject to the terminal condition $B(S,S) = 0$, and the second one - to $A(S,S) = 1$. The solution reads

$$B(t,S) = e^{\int_0^t \kappa(x) \, dx} \int_S^t e^{-\int_0^x \kappa(q) \, dq} \, dx, \quad (8)$$

$$A(t,S) = \exp \left[ -\frac{1}{2} \int_S^t B(x,S) \left( 2\theta(x)\kappa(x) + B(x,S)\sigma^2(x) \right) \, dx \right].$$

It can be seen that $B(t,S) < 0$ if $t < S$. Therefore, $F(r,t,S) \rightarrow 0$ when $r \rightarrow -\infty$.

## 2 Down-and-Out barrier option

Let us consider a Down-and-Out barrier Call option. It is known that the price of the option $C(r,t)$ written on this bond under a risk-neutral measure solves the same PDE as in Eq. (2)

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2(t) \frac{\partial^2 C}{\partial r^2} + \kappa(t)\theta(t) - r \frac{\partial C}{\partial r} = rC. \quad (9)$$

This PDE should be solved subject to the terminal condition at the option maturity $T \leq S$, and some boundary conditions provided to guarantee a uniqueness of the solution. The terminal condition reads

$$C(r,T) = (F(r,T,S - K))^+, \quad (10)$$

where $K$ is the option strike.

By the contract definition, the lower barrier $L_F$ is set on the Zero coupon bond (ZCB) price. In other words, at the barrier we have the following condition

$$C(r,t) = 0 \quad \text{if } F(r,t,S) = L_F. \quad (11)$$

Since the ZCB price $F(r,t,S)$ is known in closed form in Eq. (6), the above condition can be reformulated in the $r$ domain by solving the equation

$$F(r,t,S) = A(t,S)e^{B(t,S)r} = L_F,$$

with respect to $r$. This yields the following equivalent barrier $L(t)$ in the $r$ domain

$$L(t) = \frac{1}{B(t,S)} \log \left( \frac{L_F}{A(t,S)} \right) > 0, \quad (12)$$
where it is assumed that \( L_F > A(t, S) \). Thus, the boundary condition to Eq. (9) now reads
\[
C(L(t), t) = 0.
\] (13)

Hence, in this case the barrier becomes time-dependent. Accordingly, this allows a natural generalization of Eq. (11) to make the barrier \( L_F \) being also dependent of time, i.e. \( L_F(t) \).

At the second boundary as \( r \to \infty \) the ZCB price tends to zero, see Eq. (6), and, therefore, the Call option price tends to zero. Thus, we set
\[
C(r, t) \bigg|_{r \to \infty} = 0.
\] (14)

Our goal now is to build a series of transformations to transform Eq. (9) to the heat equation.

### 2.1 Transformation to the heat equation

To transform the PDE Eq. (2) to the heat equation we first rewrite Eq. (9) in the form
\[
\frac{\partial C}{\partial t} = -\frac{1}{2} \sigma^2(t) \frac{\partial^2 C}{\partial r^2} - \left[ -\kappa(t)r + \kappa(t)\theta(t) \right] \frac{\partial C}{\partial r} + rC.
\] (15)

This equation belongs to the type of equations considered in (Polyanin, 2002), Section 3.8.7. It is shown there that by transformation
\[
C(r, t) = \exp[\alpha(t)r + \beta(t)]u(x, \tau), \quad \tau = \phi(t), \quad x = r\psi(t) + \xi(t),
\] (16)

Eq. (15) can be reduced to the heat equation
\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.
\] (17)

Here,
\[
\psi(t) = C_1 \exp \left( \int_0^t \kappa(q)dq \right),
\] (18)
\[
\phi(t) = \frac{1}{2} \int_t^T \sigma^2(q)\psi^2(q)dq + C_2,
\]
\[
\alpha(t) = \psi(t) \int_0^t \frac{1}{\psi(q)}dq + C_3\psi(t),
\]
\[
\beta(t) = -\frac{1}{2} \int_0^t \alpha(q) \left[ 2\kappa(q)\theta(q) + \sigma^2(q)\alpha(q) \right] dq + C_4,
\]
\[
\xi(t) = -\int_0^t \left[ \kappa(q)\theta(q) + \sigma^2(q)\alpha(q) \right] \psi(q)dq + C_5,
\]

where \( C_1, \ldots, C_5 \) are some constants. In our case we can choose \( C_1 = 1, C_2 = C_3 = C_4 = C_5 = 0 \).

The Eq. (17) should be solved subject to the initial condition
\[
\begin{align*}
\bar{F}(x, T, S) & = A(T, S) \exp \left[ \frac{B(T, S)}{\psi(T)}(x - \xi(T)) \right], \\
\end{align*}
\] as it follows from Eq. (10), Eq. (6) and Eq. (16).
The boundary conditions should be set at the new domain \( x \in \Omega : [y(\tau), \infty) \), where \( y(\tau) = L(t(\tau))\psi(t(\tau)) + \xi(t(\tau)) \), and \( t(\tau) \) is the inverse map \( t \rightarrow \tau \). The latter can be found explicitly by solving the second equation in Eq. (18)

\[
\tau = \frac{1}{2} \int_t^T \sigma^2(q)\psi^2(q) dq,
\]

with respect to \( t \).

Accordingly, the conditions at the boundaries of \( \Omega \) can be obtained from Eq. (14), Eq. (13) and read

\[
u(x, \tau) \bigg|_{x \rightarrow \infty} = 0, \quad u(y(\tau), \tau) = 0.
\]

### 2.2 Solution of the barrier pricing problem

As applied to equities, the problem of pricing barrier options, where the underlying follows a time-dependent OU process, was considered in (Carr and Itkin, 2020). There the authors utilized and extended a method of generalized integral transform actively elaborated on by the Russian mathematical school to solve parabolic equations at the domain with moving boundaries, see eg., (Kartashov, 1999) and references therein. However, in (Carr and Itkin, 2020; Kartashov, 1999) those problems were solved at the domain \( x \in [0, y(\tau)] \) while in this paper we have to deal with the infinite domain \( \Omega \). For that kind of domains the above method is not fully elaborated yet since there is a problem with constructing the inverse transform. Some examples where the inverse transform is not necessary can be found in (Kartashov, 1999, 2001).

Therefore, in this Section we will use the method of heat potentials (while the method of generalized integral transform is considered in Section 2.3). This method is known in the theory of heat equation for a long time, see, eg., (Tikhonov and Samarskii, 1963; Friedman, 1964.; Kartashov, 2001) and references therein. The first use of this method in mathematical finance is due to (Lipton, 2002) for pricing path-dependent options with curvilinear barriers, and more recently in (Lipton and Kaushansky, 2018; Lipton and de Prado, 2020) (also see references therein).

Recall, that we have to solve the heat equation in Eq. (17) with the initial condition in Eq. (19) and the boundary conditions in Eq. (21). Since this problem has an inhomogeneous initial condition, the method of heat potentials cannot be directly applied. Therefore, let us represent \( u(x, \tau) \) in the form

\[
u(x, \tau) = q(x, \tau) + \frac{1}{2\sqrt{\pi \tau}} \int_y(0) u(x', 0)e^{-\frac{(x-x')^2}{4\tau}} dx'.
\]

Here the second term in the RHS is the solution of the heat equation in Eq. (17) at the infinite domain with the initial condition in Eq. (10). But due to domain of definition \( x \in \Omega \), we moved the left boundary from \(-\infty\) to \( y(0)\).

The function \( q(x, \tau) \) solves the problem

\[
\frac{\partial q(x, \tau)}{\partial \tau} = \frac{\partial^2 q(x, \tau)}{\partial x^2},
\]

\[
q(x, 0) = 0, \quad y(0) < x < \infty,
\]

\[
q(x, \tau) \bigg|_{x \rightarrow \infty} = 0, \quad q(y(\tau), \tau) = \phi(\tau),
\]

\[
\phi(\tau) = \frac{-1}{2\sqrt{\pi \tau}} \int_y(0) u(x', 0)e^{-\frac{(y(\tau)-x')^2}{4\tau}} dx'.
\]

This problem is similar to that in Eq. (17), Eq. (10), Eq. (21), but now with a homogeneous initial condition. Therefore, the solution can be found by the method of heat potentials.
Following the general idea of this method, we are looking for the solution of Eq. (23) in the form of a generalized heat potential

\[
q(x, \tau) = \frac{1}{4\sqrt{\pi}} \int_0^\tau \psi(k) \frac{x - y(k)}{\sqrt{(\tau - k)^3}} e^{-\frac{(x - y(k))^2}{4(\tau - k)}} dk,
\]

where \( \psi(k) \) is the heat potential density. It is easy to check that thus defined function \( q(x, \tau) \) solves the first line of Eq. (23), and satisfies the initial condition and the vanishing condition at \( x \to \infty \). Also, from Eq. (24) at the barrier \( x = y(\tau) \) we must have, (Tikhonov and Samarskii, 1963)

\[
2\phi(\tau) = \psi(\tau) + \frac{1}{2\sqrt{\pi}} \int_0^\tau \psi(k) \frac{y(\tau) - y(k)}{\sqrt{(\tau - k)^3}} e^{-\frac{(y(\tau) - y(k))^2}{4(\tau - k)}} dk,
\]

since for \( x = y(\tau) \) function \( q(x, \tau) \) is discontinuous, and its limiting value at \( x = y(\tau) + 0 \) is equal to \( \phi(\tau) \).

The Eq. (25) is a linear Volterra equations of the second kind, (Polyanin and Manzhirov, 2008). Since \( \phi(\tau) \) is a continuously-differentiable function, Eq. (25) has a unique continuous solution for \( \psi(\tau) \).

The Eq. (25) can be attacked twofold. First, it can be efficiently solved numerically. By a standard approach, the integral in the RHS is approximated using some quadrature rule with \( N \) nodes in \( k \), and the solution is obtained at \( M \) nodes in \( \tau \). Since the kernel is proportional to Gaussian, the discrete sum approximating the integral can be computed with linear complexity \( O(N + M) \) using the Fast Gauss Transform, see eg., (Spivak et al., 2010). The final solution can be obtained by using Pickard’s iterations. Another approach is discussed in Section 4.

Second, if in Eq. (20) \( \tau(0) \) is small, we can approximate a curvilinear boundary \( y(\tau) \) by a linear function

\[
y(\tau) = a + b\tau, \quad a = y(0), \quad b = \frac{y(\tau(0)) - y(0)}{y(\tau(0))}.
\]

Then the integral kernel becomes a function of the difference \( \tau - k \), and so the integrand is a convolutional function. Thus, Eq. (25) can be solved by using the Laplace or \( G \)-transforms. For instance, with allowance for Eq. (26) let us re-write Eq. (25) in the form, (Kartashov, 2001)

\[
\phi_1(\tau) = \psi_1(\tau) + \frac{b}{2\sqrt{\pi}} \int_0^\tau \frac{\psi_1(k)}{\sqrt{\tau - k}} dk;
\]

\[
\phi_1(\tau) = 2\phi(\tau)e^{b^2\tau/4}, \quad \psi_1(k) = \psi(k)e^{b^2k/4}.
\]

This is the Abel integral equation of the second kind, (Polyanin and Manzhirov, 2008), which can be solved in closed form by using the Laplace transform. Since \( \phi(0) = 0 \), the solution reads

\[
\psi_1(\tau) = \mathcal{F}(\tau) + \frac{b^2}{4} \int_0^\tau e^{\frac{k}{\sqrt{\tau - k}}} \mathcal{F}(k) dk, \quad \mathcal{F} = \phi_1(\tau) = \frac{b}{2\sqrt{\pi}} \int_0^\tau \frac{\phi(k)}{\sqrt{\tau - k}} dk.
\]

In case where the linear approximation is too crude, this expression can be used as a smart initial guess for the function \( \psi(\tau) \) which is needed by the iterative numerical method described in above.

Once Eq. (25) is solved and the function \( \psi(\tau) \) is found, the final solution reads

\[
u(x, \tau) = \frac{1}{4\sqrt{\pi}} \int_0^\tau \psi(k) \frac{x - y(k)}{\sqrt{(\tau - k)^3}} e^{-\frac{(x - y(k))^2}{4(\tau - k)}} dk + \frac{1}{2\sqrt{\pi}} \int_{y(0)}^\infty u(x', 0) e^{-\frac{(x - x')^2}{4\tau}} dx'.
\]
The second integral can be further simplified with allowance for the definition of $u(x, 0)$ in Eq. (10). This yields

\[
\frac{1}{2\sqrt{\pi}} \int_{y(0)}^{\infty} u(x', 0) e^{-\frac{(x-x')^2}{4\tau}} \, dx' = \frac{1}{2} e^{-\beta(T)} \left[ e^{A_2 x + A_1 A(T, S)} \left( \text{Erf} \left( \frac{x - y(0) + 2\tau A_2}{2\sqrt{\tau}} \right) - \text{Erf} \left( \frac{x - K_1 + 2\tau A_2}{2\sqrt{\tau}} \right) \right) ight] - K e^{B_2 x + B_1} \left( \text{Erf} \left( \frac{x - y(0) + 2\tau B_2}{2\sqrt{\tau}} \right) - \text{Erf} \left( \frac{x - K_1 + 2\tau B_2}{2\sqrt{\tau}} \right) \right),
\]

\[A_1 = \frac{A_2}{\psi(T)} \left[ \tau(B(T, S) - \alpha(T)) - \xi(T)\psi(T) \right], \quad A_2 = \frac{B(T, S) - \alpha(T)}{\psi(T)},
\]

\[B_1 = \frac{B_2}{\psi(T)} \left[ -\tau \alpha(T) - \xi(T)\psi(T) \right], \quad B_2 = -\frac{\alpha(T)}{\psi(T)},
\]

\[K_1 = \max \left\{ \xi(T) + \frac{\psi(T)}{B(T, S)} \log \left( \frac{K}{A(T, S)} \right), y(0) \right\}.
\]

Also, by definition in Eq. (23), the function $\phi(\tau)$ can be represented in closed form just by substituting $x = y(\tau)$ into Eq. (29) and multiplying by -1.

### 2.3 Second solution of the barrier pricing problem

In this Section we solve the same problem but using the method of generalized integral transform. This method was invented by the Russian mathematical school in the 20th century starting from A.V. Luikov, and then by B.Ya. Lyubov, E.M. Kartashov, and some others, see a detailed survey in (Kartashov, 1999). However, as mentioned in (Kartashov, 2001), the solution for a semi-infinite domain is not known yet, while some recommendations were given on how one can try to proceed. Therefore, to the best of authors’ knowledge, the solution presented in this Section is new and compliments the method of heat potentials presented in Section 2.2.

We attack this problem by introducing the following integral transform

\[
\tilde{u}(p, \tau) = \int_{y(\tau)}^{\infty} u(x, \tau) e^{-\tau^{\frac{1}{2}} x} \, dx,
\]

where $p = a + i\omega$ is a complex number with $\text{Re}(p) = \beta > 0$, and $-\pi/4 < \text{arg}(\sqrt{p}) < \pi/4$. Let us multiply both parts of Eq. (17) by $e^{-\tau^{\frac{1}{2}} x}$ and then integrate on $x$ from $y(\tau)$ to $\infty$:

\[
\frac{\partial}{\partial \tau} \int_{y(\tau)}^{\infty} u(x, \tau) e^{-\tau^{\frac{1}{2}} x} \, dx + u(y(\tau), \tau) e^{-\tau^{\frac{1}{2}} y} \right|_{x=y(\tau)}^{x=\infty} = \frac{\partial u}{\partial x} e^{-\tau^{\frac{1}{2}} x} \bigg|_{x=y(\tau)}^{x=\infty} + p \int_{y(\tau)}^{\infty} u(x, \tau) e^{-\tau^{\frac{1}{2}} x} \, dx.
\]

By taking into account the boundary conditions in Eq. (21), Eq. (31) can be represented as the following Cauchy problem

\[
\frac{d\tilde{u}}{d\tau} - p\tilde{u} = \Psi(\tau) e^{-\tau^{\frac{1}{2}} y},
\]

\[
\tilde{u}(p, 0) = \int_{y(0)}^{\infty} u(x, 0) e^{-\tau^{\frac{1}{2}} x} \, dx, \quad \Psi(\tau) = -\frac{\partial u(x, \tau)}{\partial x} \bigg|_{x=y(\tau)}.
\]
where \( u(x,0) \) is given in Eq. (19). The solution of this problem reads

\[
\bar{u}(p,\tau) = e^{p\tau} \int_0^\tau \Psi(s)e^{-ps}e^{-\sqrt{\tau}s}\,ds + e^{p\tau} \int_{y(0)}^{\infty} u(z,0)e^{-\sqrt{\tau}z}\,dz. \tag{33}
\]

By analogy with (Carr and Itkin, 2020), the function \( \Psi(\tau) \) solves the Fredholm equation of the first type

\[
\int_0^\infty \Psi(\tau)e^{-p\tau-\sqrt{\tau}\tau}\,d\tau = F(p), \tag{34}
\]

with

\[
F(p) = -e^{\alpha(T)\psi(T) - \beta(T)} \left[ A(T,S)e^{-\beta(T)}e^{-f_1(p,T)y_0 - f_1(p,T)K_1} - K e^{-f_2(p,T)y_0 - f_2(p,T)K_1} \right] f_1(p,T) - f_2(p,T) = \sqrt{p} + \frac{\alpha(T)}{\psi(T)} \tag{35}
\]

where \( K_1 \) is defined in Eq. (29).

To construct the inverse transform, recall that the solution of the heat equation \( \mathcal{L}u(x,\tau) = 0, \mathcal{L} = \partial/\partial\tau - \partial^2/\partial x^2 \) in the half-plane domain \( x \in (0,\infty) \) can be expressed via the Fourier sine integral, (Cannon and Browder, 1984)

\[
\int_0^\infty \alpha(\xi)e^{-\xi^2\tau}\sin(\xi x)\,d\xi.
\]

Therefore, by analogy we look for the inverse transform of \( \bar{u} \) (or for the solution of Eq. (32) in terms of \( u \)) to be an oscillatory integral of the form

\[
u(x,\tau) = \int_0^{\infty} \alpha(\xi,\tau) \sin[\xi(x-y(\tau))]\,d\xi, \tag{36}\]

where \( \alpha(\xi,\tau) \) is some function to be determined. Note, that this definition automatically respects the vanishing boundary conditions for \( u(x,\tau) \). We assume that this integral converges absolutely and uniformly \( \forall x \in [y(\tau),\infty) \) for any \( \tau > 0 \).

Applying Eq. (30) to both parts of Eq. (36) and integrating yields

\[
\bar{u}(p,\tau) = \int_{y(\tau)}^{\infty} e^{-\sqrt{\tau}s} \int_0^{\infty} \alpha(\xi,\tau) \sin(\xi(x-y(\tau)))\,d\xi\,dx = e^{-\sqrt{\tau}s} \int_0^{\infty} \alpha(\xi,\tau) \frac{\xi\,d\xi}{\xi^2 + p}. \tag{37}\]

Replacing \( \bar{u}(p,\tau) \) with the solution found in Eq. (33), we obtain

\[
\int_0^{\infty} \alpha(\xi,\tau) \frac{\xi\,d\xi}{\xi^2 + p} = \int_0^\tau \Psi(s)e^{p(\tau-s)}e^{\sqrt{\tau}(y(\tau)-y(s))}\,ds + e^{p\tau} \int_{y(0)}^{\infty} u(z,0)e^{\sqrt{\tau}(y(\tau)-z)}\,dz. \tag{38}\]

Now, similar to inverse operator methods, like the inverse Laplace transform, we need an analytic continuation of the transform parameter \( p \) into the complex plane. Let us integrate both sides of Eq. (38) on \( p \) along the so-called keyhole contour presented in Fig. 1. In more detail, this contour can be described as follows. It starts with a big symmetric arc \( \Gamma \) around the origin with the radius \( R \); extending to two horizontal line segments \( l_1, l_4 \) (a cut around the line \( \text{Im} p = 0, \text{Re} p > 0 \)); connecting to two small semicircles \( \gamma_\varepsilon \) around the origin with the radius \( \varepsilon \ll 1 \); then extending to two vertical line segments up to points \( \text{Im} p = \pm \xi \); then again two horizontal parallel line segments \( l_1, l_2 \) at \( \text{Im} p = \pm \xi \), which end points are connected to the arc \( \Gamma \) with a cut at \( \text{Im} p = -\xi^2 \) (it consists of two vertical line segments and two semi-circles \( \gamma_\tau \) with the radius \( \varepsilon \)), such that the whole contour is continuous.

Using a standard technique, we take a limit \( \varepsilon \to 0, R \to \infty \), so in this limit the contour takes the form as depicted in Fig. 2. It has a horizontal cut along the positive real line with point \( p = 0 \) excluded.
from the area inside the contour; another vertical cut at \( \text{Re} \, p = -\xi^2 \) with the point \( p = -\xi^2 \) lying inside the contour; and a branch cut \( l_1, l_2 \) of the multivalued function \( \sqrt{p} \) at \( p = -\xi^2 \). Also, in this limit \( l_7 \to 0, l_8 \to 0 \), but in Fig. 2 we left them as it is for a better readability.

Now we are ready to compute the integrals in Eq. (38). That one in the LHS is regular everywhere inside this contour except a single pole \( p = -\xi^2 \). By the residue theorem, we obtain

\[
\oint_{\gamma} \left( \int_{0}^{\infty} \alpha(\xi, \tau) \frac{\xi \, d\xi}{\xi^2 + p} \right) \, dp = -2\pi i \int_{0}^{\infty} \xi \alpha(\xi, \tau) \, d\xi. \tag{39}
\]

The integral on the RHS doesn’t have any singularity inside the contour \( \gamma \), however, it has several cuts. As can be easily checked, the integrals along the segments \( l_3 \) and \( l_4 \) cancel out, as well as those along \( l_7 \) and \( l_8 \), and those along \( l_5 \) and \( l_6 \). The integral along the contour \( \Gamma \) tends to zero if \( R \to \infty \) due to Jordan’s lemma. Hence, the only remaining integrals are those along the horizontal semi-infinite lines \( l_1 \) and \( l_2 \). They read

\[
\int_{l_1} \left( \int_{0}^{\tau} \Psi(s) e^{\xi(\tau-s)} e^{\sqrt{p}(y(\tau)-y(s))} \, ds + e^{\xi \tau} \int_{y(0)}^{\infty} u(z, 0) e^{\sqrt{p}(y(\tau)-z)} \, dz \right) \, dp \\
= -2 \int_{0}^{\infty} \xi \left( \int_{0}^{\tau} \Psi(s) e^{-\xi^2(\tau-s)} e^{i\xi(y(\tau)-y(s))} \, ds + e^{-\xi^2 \tau} \int_{y(0)}^{\infty} u(z, 0) e^{i\xi(y(\tau)-z)} \, dz \right) \, d\xi, \tag{40}
\]

\[
\int_{l_2} \left( \int_{0}^{\tau} \Psi(s) e^{\xi(\tau-s)} e^{\sqrt{p}(y(\tau)-y(s))} \, ds + e^{\xi \tau} \int_{y(0)}^{\infty} u(z, 0) e^{\sqrt{p}(y(\tau)-z)} \, dz \right) \, dp \\
= 2 \int_{0}^{\infty} \xi \left( \int_{0}^{\tau} \Psi(s) e^{-\xi^2(\tau-s)} e^{-i\xi(y(\tau)-y(s))} \, ds + e^{-\xi^2 \tau} \int_{y(0)}^{\infty} u(z, 0) e^{-i\xi(y(\tau)-z)} \, dz \right) \, d\xi. \tag{41}
\]
Semi-closed form solutions for barrier options...

![Contour of integration \( \gamma \) of Eq. (38) in a complex plane of \( p \) at \( \varepsilon \to 0, R \to \infty \).](image)

The RHS of Eq. (38) is equal to a sum of these integrals

\[
4i \int_0^\infty \xi \left[ \int_0^\tau \Psi(s)e^{-\xi^2(\tau-s)} \sinh (i\xi(y(\tau) - y(s))) \, ds + e^{-\xi^2\tau} \int_{y(0)}^\infty u(z,0) \sin (\xi(y(\tau) - z)) \, dz \right] \, d\xi.
\]

Equating the LHS and RHS provides an explicit representation of \( \alpha(\xi,\tau) \)

\[
\alpha(\xi,\tau) = -\frac{2}{\pi} \left[ \int_0^\tau \Psi(s)e^{-\xi^2(\tau-s)} \sin (\xi(y(\tau) - y(s))) \, ds + e^{-\xi^2\tau} \int_{y(0)}^\infty u(z,0) \sin (\xi(y(\tau) - z)) \, dz \right]
\]  

Finally, we substitute Eq. (42) into Eq. (38) and take into account the identity, (Gradshtein and Ryzhik, 2007)

\[
\int_0^\infty e^{-\beta x^2} \sin (ax) \sin (bx) \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left( e^{-\frac{(a-b)^2}{4\beta}} - e^{-\frac{(a+b)^2}{4\beta}} \right), \quad \beta > 0,
\]

which yields

\[
u(x,\tau) = \frac{1}{2\sqrt{\pi}} \int_0^\tau \frac{\Psi(s)}{\sqrt{\tau-s}} \left( e^{-\frac{(x-y(s))^2}{4(\tau-s)}} - e^{-\frac{(x-2y(\tau)+y(s))^2}{4(\tau-s)}} \right) \, ds
\]

\[
+ \frac{1}{2\sqrt{\pi}} \int_{y(0)}^\infty u(z,0) \left( e^{-\frac{(x-z)^2}{4\tau}} - e^{-\frac{(x-2y(\tau)+z)^2}{4\tau}} \right) \, dz.
\]

Thus, we obtained another representation of the solution which reads a bit different from that in Eq. (28), despite the general ansatz looks similar (the solution is a sum of two integrals, one in time \( \tau \) and the other one in space \( x \) of the initial condition with a Gaussian weight). The difference can be attributed to
the different definitions of function $\Psi(k)$, as in Eq. (24) it is the heat potential density, while in Eq. (32) this is the gradient of the solution at $x = y(\tau)$. The first one is determined by the solution of the Volterra equation of the second kind Eq. (25), and the second one - by the solution of the Fredholm equation of the first kind in Eq. (34). However, the latter can either be transformed to the Volterra equation of the second kind. For doing that, one needs to differentiate Eq. (43) on $x$, and then let $x = y(\tau)$. This yields

$$
\Psi(\tau) = \frac{1}{2\sqrt{\pi}} \int_0^\tau \Psi(s) \frac{y(\tau) - y(s)}{(\tau - s)^{3/2}} e^{-\frac{(y(\tau) - y(s))^2}{4(\tau - s)}} ds + \frac{1}{2\sqrt{\pi} \tau^3} \int_{\nu(0)}^{\infty} u(z, 0)(y(\tau) - z)e^{-\frac{(z - y(\tau))^2}{4\tau}} dz. \quad (44)
$$

3 Double barrier options

Similar to Section 2.2, the method of heat potentials can be used to price double barrier options with the lower barrier $L_F$ and the upper barrier $H_F$. In this case, we have the following problem to solve

$$
\frac{\partial u(x, \tau)}{\partial \tau} = \frac{\partial^2 u(x, \tau)}{\partial x^2}, \quad (45)
$$

$$
u(x, \tau) = u(x, \tau), \quad y(0) < x < z(0),
$$

$$
u(y(\tau), \tau) = u(z(\tau), \tau) = 0,
$$

where $z(\tau) = H(t(\tau))\psi(t) + \xi(t)$ is the moving upper boundary, $H(t(\tau))$ is defined in Eq. (12) by replacing $L_F$ with $H_F$, and $u(x, 0)$ is defined in Eq. (19).

Similar to Eq. (22) we represent the solution in the form

$$
u(x, \tau) = q(x, \tau) + \frac{1}{2\sqrt{\pi} \tau} \int_{\nu(0)}^{z(0)} u(x', 0)e^{-\frac{(x-x')^2}{4\tau}} dx', \quad (46)
$$

so function $q(x, \tau)$ solves a problem with homogeneous initial condition

$$
\frac{\partial q(x, \tau)}{\partial \tau} = \frac{\partial^2 q(x, \tau)}{\partial x^2}, \quad (47)
$$

$$
q(x, 0) = 0, \quad y(0) < x < z(0),
$$

$$
q(y(\tau), \tau) = -\phi_2(\tau), \quad q(z(\tau), \tau) = -\psi_2(\tau),
$$

$$
\phi_2(\tau) = -\frac{1}{2\sqrt{\pi}} \int_{\nu(0)}^{z(0)} u(x', 0)e^{-\frac{(y(\tau) - x')^2}{4\tau}} dx', \quad \psi_2(\tau) = -\frac{1}{2\sqrt{\pi}} \int_{\nu(0)}^{z(0)} u(x', 0)e^{-\frac{(z(\tau) - x')^2}{4\tau}} dx'.
$$

Again, we are looking for the solution of Eq. (47) in the form of a generalized heat potential

$$
q(x, \tau) = \frac{1}{4\sqrt{\pi}} \int_0^\tau \frac{1}{\sqrt{(\tau - k)^3}} \left( (x - y(k))\Psi(k)e^{-\frac{(x - y(k))^2}{4(\tau - k)}} + (x - z(k))\Phi(k)e^{-\frac{(x - z(k))^2}{4(\tau - k)}} \right) dk, \quad (48)
$$

where $\Psi(k), \Phi(k)$ are the heat potential densities. They solve a system of two Volterra equations of the second kind

$$
2\phi_2(\tau) = \Psi(\tau) + \frac{1}{2\sqrt{\pi}} \int_0^\tau \left( \Psi(k)\frac{y(\tau) - y(k)}{(\tau - k)^{3/2}} e^{-\frac{(y(\tau) - y(k))^2}{4(\tau - k)}} + \Phi(k)\frac{y(\tau) - z(k)}{(\tau - k)^{3/2}} e^{-\frac{(y(\tau) - z(k))^2}{4(\tau - k)}} \right) dk, \quad (49)
$$

$$
2\psi_2(\tau) = \Phi(\tau) + \frac{1}{2\sqrt{\pi}} \int_0^\tau \left( \Psi(k)\frac{z(\tau) - y(k)}{(\tau - k)^{3/2}} e^{-\frac{(z(\tau) - y(k))^2}{4(\tau - k)}} + \Phi(k)\frac{z(\tau) - z(k)}{(\tau - k)^{3/2}} e^{-\frac{(z(\tau) - z(k))^2}{4(\tau - k)}} \right) dk,
$$

where functions $\psi_2(\tau), \phi_2(\tau)$ can be expressed in closed form, similar to Eq. (29). This system, again can be solved by the Variational Iteration Method (VIM), see (Wazwaz, 2011) with a linear complexity by using the FGT. Once this is done, the solution of our double barrier problem is found.
4 Numerical example

To test performance and accuracy of our method we run a test where the explicit form of parameters $\kappa(t), \theta(t), \sigma(t)$ is chosen as

$$\begin{align*}
\kappa(t) &= \kappa_0, \\
\theta(t) &= \theta_0 e^{-\theta_k t}, \\
\sigma(t) &= \sigma_0 e^{-\sigma_k t},
\end{align*}$$

(50)

with $\kappa_0, \theta_0, \sigma_0, \theta_k, \sigma_k$ being constants. With these definitions all functions in Eq. (18) can be found in closed form.

We approach pricing of Down-and-Out barrier Call option written on a ZCB in two ways. First, we solve the PDE in Eq. (9) by using a finite-difference scheme of the second order in space and time. We use the Crank-Nicolson scheme with few first Rannacher steps on a non-uniform grid compressed close to the barrier level at $t = 0$, in more detail, see (Itkin, 2017). Second, we use the method of heat potentials (HP) and solve the same problem as this is described in Section 2.2. To solve the Volterra equation in Eq. (25) we approximate the kernel on a rectangular grid $M \times M$, and the integral using the trapezoidal rule, which gives rise to the following system of linear equations

$$\|2\phi\| = (I + P)\|\Psi\|.$$

(51)

Here $\|\Psi\|$ is the vector of discrete values of $\Psi(\tau), \tau \in [0, \tau(0)]$ on a grid with $M$ nodes, $\|\phi\|$ is a similar vector of $\phi(\tau)$, $I$ is the unit $M \times M$ matrix, and $P$ is the $M \times M$ matrix of the kernel values on the same grid. Due to the specific structure of the Gaussian kernel, matrix $P$ is lower triangular. Therefore, solution of Eq. (51) can be done with complexity $O(M^2)$. An important point here is that the kernel (and so the matrix $P$) doesn’t depend of strikes $K$, but only the function $\phi(\tau)$. Therefore, Eq. (51) can be solved simultaneously for all strikes by inverting the matrix $I + P$ with the complexity $O(M^2)$, and then multiplying it by vectors $\|\phi\|_k, k = 1, \ldots, \bar{k}, \bar{k}$ is the total number of strikes. Hence, the total complexity is $O(kM^2)$.

We emphasize, that this algorithm of solving the Volterra equation Eq. (25) doesn’t require iterations, as that which makes use of the FGT. To compare both algorithms, note that complexity of the matrix algorithm is $O(M^2)$ versus $O(2nM)$ - the complexity of the iterative algorithm which requires $n$ iterations to converge. Therefore, if $M$ is relatively small, e.g., $M = 20$, and the number of iterations is near 10, both algorithms are of the same complexity.

For this test parameters of the model are presented in Table 1.

| $r_0$ | $\kappa_0$ | $\theta_0$ | $\sigma_0$ | $\theta_k$ | $\sigma_k$ | $H_F$ | $S$ |
|-------|-------------|-------------|-------------|-------------|-------------|-------|-----|
| 0.07  | 1.0         | 0.08        | 0.2         | 0.3         | 0.2         | 0.87  | 7   |

Table 1: Parameters of the test.

We run the test for a set of maturities $T \in [1/12, 0.3, 0.5, 1]$ and strikes $K \in [0.06, 0.08, 0.1, 0.15, 0.2, 0.3]$. The Down-and-Out barrier Call option prices computed in such an experiment are presented in Fig. 3. In Fig. 4 the relative errors (in percents) between these prices obtained by using the HP method and the FD solver are presented as a function of the option strike $K$ and maturity $T$. Here to provide a comparable accuracy we run the FD solver with 200 nodes in space $r$ and 201 steps in time $t$. Otherwise the quality of the FD solution is poor.

A relatively large error about 8% at simultaneously high maturities and strikes is due to the very low computed price of the option, which is 1.99 cents for the FD method, and 2.17 cents for the HP method, respectively. Otherwise, the error is about 1-3%. Also, since in this test we have ceson a fixed barrier in the ZCB price space, the corresponding barrier in the $r$ space is moving down. Therefore, the
computed option price could vanish for strikes close to the barrier at short maturities, but contrary have some positive value for the same strikes at longer maturities.

Obviously, since the FD solver needs a high number of nodes in both space and time to achieve a reasonable accuracy, the cost for this is speed. Suppose that this solver uses $N_r$ nodes in space $r$, 

Figure 3: Down-and-Out barrier Call option price computed by using the HP method.

Figure 4: The relative difference in % of the Down-and-Out barrier Call option prices computed by using the HP method and the FD solver with 201 nodes in space $r$ and time $t$. 
and $M_t$ nodes in time $t$. Then the total complexity of solving the forward PDE to simultaneously get option prices for all given strikes $K_i$, $i = 1, \ldots, \bar{k}$ and maturities $T_j$, $j = 1, \ldots, \bar{m}$ is $O(N_r \times M_t)$. This should be compared with the complexity of the HP method which is $O(\bar{k} \bar{m} M^2)$. Since, as we saw, $N_r = M_t = 200$, $\bar{k} = 6$, $\bar{m} = 4$, $M = 20$, the HP method should be four times faster than the FD method. In reality, our test with 24 points in the $K \times T$ space shows that the elapsed time of the HP method is 20 mls, while for the FD method this is 146 mls which is 7 times slower. Decreasing the size of the FD grid to $100 \times 100$ nodes also decreases the elapsed time to 21 mls, but by the cost of increasing a relative error up to $\pm 15\%$ for a wide range of maturities and strikes. Thus, overall, the method of HP demonstrates, at least same performance as the forward FD solver.

5 Discussion

In Sections 2.2, 2.3 we constructed semi-closed form solutions for the prices of Down-and-Out barrier Call option $C_{dao}$ where the underlying is a Zero-Coupon Bond, and the interest rate dynamics follows the Hull-White model. Obviously, using the parity for barrier options, (Hull, 1997), the price of the Up-and-Out barrier Call option $C_{uao}$ can be found as $C_{dao} = C_{van} - C_{uao}$, where $C_{van}$ is the price of the European vanilla Call option in the Hull-White model. Since this model allows closed-form solutions for European options on Zero-coupon bonds, (Andersen and Piterbarg, 2010), our solution also provides a closed form solution for $C_{uao}$. The double barrier case was also considered in Section 3.

As shown in Section 2.2, this solution also covers the case when the barrier is some arbitrary function of time, as this just slightly modifies the definition of function $y(\tau)$.

From the computational point of view the proposed solution is very efficient as this is shown in Section 4. Using theoretical analysis justified by a test example we conclude that our method is, at least, of the same complexity, or even faster than the forward FD method. On the other hand, our approach provides high accuracy in computing the options prices, as this is regulated by the order of a quadrature rule used to discretize the kernel. Therefore, the accuracy of the method in $x$ space can be easily increased by using high-order quadratures. For instance, using the Simpson instead of the trapezoid rule doesn’t affect the complexity of our method, while increasing the accuracy for the FD method is not easy (i.e., it significantly increases the complexity of the method, e.g., see (Itkin, 2017)).

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