The Robust Merton Problem of an Ambiguity Averse Investor

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Abstract

We derive a closed form portfolio optimization rule for an investor who is diffident about mean return and volatility estimates, and has a CRRA utility. The novelty is that confidence is here represented using ellipsoidal uncertainty sets for the drift, given a volatility realization. This specification affords a simple and concise analysis, as the optimal portfolio allocation policy is shaped by a rescaled market Sharpe ratio, computed under the worst case volatility. The result is based on a max-min Hamilton-Jacobi-Bellman-Isaacs PDE, which extends the classical Merton problem and reverts to it for an ambiguity-neutral investor.

Keywords: Robust optimization, Merton problem, volatility uncertainty, ellipsoidal uncertainty on mean returns, Hamilton-Jacobi-Bellman-Isaacs equation.

AMS subject classifications: 91G10, 91B25, 90C25, 90C46, 90C47

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1 Introduction

Traditionally, financial modelling heavily relies on the choice of an underlying probability measure $P$, which is chosen to incorporate the statistical and stochastic nature of market price movements. As early back as the works of Bachelier, Samuelson and Black, Scholes and Merton, the underlying risk factors—such as stock prices or interest rates—have been modeled as Markovian diffusions (with possible jumps) under $P$. However, as has become quite agreed upon, the complexity of the global economic and financial dynamics render impossible the precise identification of the probability law of the evolution of the risk factors. Unavoidably, financial modelling is inherently subject to model uncertainty, which also appears under the appellation of Knightian uncertainty.

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In the presence of model uncertainty, one may admit various degrees of severity. One may deal with model misspecification only at the level of the equivalence class of \( P \), or go beyond and take into account a family of non dominated models. The core issue of portfolio optimization has been widely investigated over the last twenty years in the multiple priors context. The investor has a pessimistic view of the odds, and takes a max-min (also known as robust) approach to the problem, first minimizing a utility functional over the priors and afterwards maximizing over the investment strategies. We are aware of only a few results in the non dominated case, notably Hernández-Hernández and Schied \[9\] and the recent preprints by Nutz \[15\] in full generality but discrete time, and Lin and Riedel \[19\] in a diffusion context. On the contrary, in the dominated priors case there is a rich literature. We content ourselves with citing Chen and Epstein \[3\], Garlappi et alii \[11\], Maenhout \[13\], Föllmer et alii \[10\] for a comprehensive review and references, and, more recently, the work by Owari \[16\].

In such an active environment, the present note offers a resolution of the robust non-dominated Merton problem, which is both simple and mathematically rigorous. The main novelties of the present contribution lie in the form of the uncertainty set and in the accommodation for market incompleteness. We assume that the asset prices process is an \( N \)-dimensional diffusion, and the driving Wiener process is \( d \)-dimensional with \( d \geq N \). The investor is diffident about the constant drift and volatility estimates \( \hat{\mu} \) and \( \hat{\sigma} \). Thus, she considers as plausible all the variance-covariance matrices lying in a given compact set and, for a given realization of \( \sigma \), she considers all the drifts which take values in a ellipsoid centered at \( \hat{\mu} \):

\[
U_\epsilon(\sigma) = \{ u \in \mathbb{R}^N \mid (u - \hat{\mu})'(\Sigma)^{-1}(u - \hat{\mu}) \leq \epsilon^2 \},
\]

in which \( \epsilon > 0 \) is the radius of ambiguity and \( \Sigma = \sigma \sigma' \) is the variance-covariance matrix.

The merits of an ellipsoidal representation for the ambiguous drifts has been amply demonstrated and discussed in \[11\], \[12\] for the robust mean-variance optimization. The problem of worst-case (max-min) robust portfolio choice is a well-studied problem (see e.g., \[4, 5, 6, 7, 8, 11, 12, 17, 18\] for robust portfolio optimization in single period problems) under different representations of ambiguity. Intuitively, the non-linear but simple geometry of ellipsoids offers robustness that avoids a worst case which is a corner solution. This is the case in a polyhedral hyper-rectangle or box representation, as in Lin and Riedel where the drift (as well as volatility) is allowed to vary in a box \([\mu, \mu]\). In the dominated setup, the assumption of \( k \)-ignorance in \[3\] also amounts to a box representation for the drift.

At the same time, the choice of \( U_\epsilon(\sigma) \) preserves tractability. Citing Fabozzi et al. \[6\]: “The coefficient realizations are assumed to be close to the forecasts, but they may deviate. They are more likely to deviate from their (instantaneous) means if their variability (measured by their standard deviation) is higher, so deviations from the mean are scaled by the inverse of the covariance matrix of the uncertain coefficients. The parameter \( \epsilon \) corresponds to the overall amount of scaled deviations of the realized returns from the forecasts against which the investor would like to be protected.”
Another appealing feature of taking model uncertainty into account is that it offers a theoretical solution to the equity premium puzzle. As noted by Mehra and Prescott [14], the high levels of historical equity premium and the simultaneous moderate equity demand seem to be implied by unreasonable levels of risk aversion. Their conclusion was skeptical on the ability of a frictionless Arrow-Debreu economy to account for such empirical evidence. However, the works by Abel [1] and Cecchetti, Lam and Mark [2] addressed the equity premium puzzle by relaxing the hypothesis that the investor perfectly knows the probability law. The key point is that, in the multiple priors setup, the optimal equity demand depends on two aversion components: risk and ambiguity aversion. In accordance to these results, and the subsequent [3], [13] and [19], we find that robustness of decisions lowers the optimal demand of equity since the ambiguity and risk averse investor effectively behaves like a risk averse investor with an increased risk aversion coefficient. Precisely, in a CRRA utility case with relative risk aversion parameter $R$, the optimal relative portfolio is given by

$$\pi = \frac{(\overline{H} - \epsilon)^+}{R\overline{H}} \Sigma^{-1} (\hat{\mu} - r1)$$

in which $\Sigma$ is the worst case variance-covariance matrix, and $\overline{H}$ is the Sharpe ratio computed under $\Sigma$ (see Proposition 1, 2 and Section 4). So, when the ambiguity radius is too high, namely it exceeds the worst case Sharpe ratio, the investor refrains from investing in the risky assets and puts all the money in the safe asset. The opposite case is when there is no uncertainty in the drift, that is $\epsilon = 0$, and no uncertainty on volatility as well: then the optimal solution reverts to the Merton relative portfolio, $\pi_M := \frac{1}{R} \Sigma^{-1} (\hat{\mu} - r1)$.

The paper is organized as follows. Section 2 contains the (diffusion) model specifications in the non dominated case, together with the general version of the Martingale Principle needed here. However, to derive an abstract max-min PDE from the Martingale Principle, some conditions on the volatility structure must be imposed in the fully incomplete market model, as in [9] for the case of a traded asset with coefficients depending on an underlying, non traded, asset. The focus here is on the complete market case, to wit the volatility is a square matrix. This allows for a simpler, yet effective, analysis. In fact, the HJB-Isaacs PDE formulation shows that the investor is observationally equivalent to one who has distorted, worst case, beliefs on the parameters. In Section 3, a representative investor with CCRA utility is considered, and we assume that ambiguity is present only in the drift, i.e. the priors are all equivalent. This is an interesting case per se, since the drift is subject to imprecision in estimations to a much greater extent than volatility. There, we solve and provide the explicit solutions to the robust problems for the infinite and finite horizon planning. Finally, we apply these findings in Section 4 to some examples in the genuinely non dominated setup.
2 The Merton problem under ambiguity aversion

Consider the problem of an agent investing in \( n \) risky assets and a riskless asset. Specifically, we work under the Black-Scholes-Merton market model assumptions. Namely, the riskless rate \( r \) is constant and the risky assets dynamics are, for each \( i = 1, \ldots, d \):

\[
    dS^i_t = S^i_t \left( \mu^i dt + \sum_{j=1}^{d} \sigma^{ij} dW^j_t \right)
\]

where \( \sigma^{ij} \) and \( \mu^i \) are constants and \( W \) is a standard, \( d \)-dimensional Brownian motion on a filtered space \( (\Omega, (F_t)_{t \geq 0}, P) \). Assume that \( d \geq n \), so the market is allowed to be incomplete.

In matrix-vector form, the above equation becomes:

\[
    dS_t = \text{Diag}(S_t)(\mu dt + \sigma dW_t)
\]

where by \( \text{Diag}(S_t) \) we denote the diagonal \( n \times n \) matrix with \( i \)-th diagonal element equal to \( S^i_t \). In addition, \( \sigma \) is required to have full rank, so that the variance-covariance matrix \( \Sigma = \sigma \sigma' \) is invertible. Here and in what follows, \( ' \) denotes the transpose operation.

Given the initial endowment \( w_0 \), the investor is allowed to trade and consume in a self-financing way. To be explicit, let \( h = (h_t)_t \) denote the \( n \)-dimensional progressively measurable process, representing the number of shares of each asset held in portfolio, and let the progressively measurable, nonnegative, scalar process \( c \) indicate the consumption stream. Assume also that \( \int_0^\cdot c_s ds \) is finite \( P \)-a.s. Then, the wealth process is governed by the following stochastic differential equation:

\[
    dw_t = (rw_t + h^t \text{Diag}(S_t)(\mu - r \textbf{1} - c_t) dt + h^t \text{Diag}(S_t) \sigma dW_t
\]

in which \( \textbf{1} \) is the \( d \)-vector with all components equal to one. It is convenient to recast the wealth equation by the vector process \( \theta \) of cash value allocated in each risky asset, i.e. \( \theta_t := \text{Diag}(S_t)h_t \). Thus,

\[
    dw_t = (rw_t + \theta^t_t(\mu - r \textbf{1} - c_t) dt + \theta^t \sigma dW_t.
\]

The pair \( (\theta_t, c_t) \) is admissible for the initial wealth \( w_0 \) if the wealth process \( w_t \) given by (2) remains \( P \)-a.s. non-negative at all times. Let \( \mathcal{A}^P(w_0) \) be the set of all admissible \( (\theta, c) \) pairs for initial wealth \( w_0 \). Note that the admissible set depends only on the equivalence class of \( P \).

The agent is then trying to choose \( (\theta, c) \in \mathcal{A}^P(w_0) \), so as to maximize the expected utility from running consumption and terminal wealth:

\[
    \sup_{(\theta, c) \in \mathcal{A}^P(w_0)} \mathbb{E}[\int_0^T u(t, c_t) dt + u(T, w_T)]
\]

The utility function \( u \) is assumed to be concave, increasing in the second argument and measurable in the first. This class of stochastic control problems is known under the name of Merton.
problem. It includes a number of specific cases, among which the infinite horizon planning. In fact, if $T = \infty$, $w_{\infty} := \limsup_{t \to \infty} w_t$ and $u(\infty, \cdot) = 0$ the above optimization problem becomes:

$$\sup_{(\theta, c) \in \mathcal{A}^F(w_0)} \mathbb{E}[\int_0^\infty u(t, c_t) dt].$$

So far, the exposition is classical, and can be found in many textbooks. The reader is referred to the new [20, Chapter 1], for a remarkably didactic approach.

However, things change quite a bit if the agent is diffident about the (constant) estimates $\hat{\mu}$ and full rank matrix $\hat{\sigma}$, for the drift and volatility matrix of the risky assets respectively. Assume from now on that $\Omega$ is the Wiener space of continuous functions, with the natural filtration $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$. Our investor assumes that the ‘true’ volatility $\sigma$ is a progressively measurable matrix, and such that the variance-covariance $\Sigma = \sigma \sigma'$ takes values in some fixed compact set $K$ of $n \times n$ invertible matrices, containing $\hat{\Sigma}$:

$$S := \{ \sigma \in \mathbb{R}^{n \times d} | \Sigma \in K \}.$$

Let us denote by $S = \{ \sigma \text{ progr mis} | \sigma_t(\omega) \in S \text{ for all } \omega, t \}$. This choice is in line with empirical practice, as $\Sigma$ is the estimated object, not the volatility $\sigma$. The uncertain drift is also assumed to be progressively measurable, and for a given realization of $\sigma$ it is allowed to vary in

$$U_\epsilon(\sigma) = \{ u \in \mathbb{R}^n | (u - \hat{\mu})' \Sigma^{-1} (u - \hat{\mu}) \leq \epsilon^2 \text{ for all } \omega, t \},$$

that is, in an ellipsoid centered at $\hat{\mu}$ with radius $\epsilon$. Let us denote the set of plausible drifts and volatilities by

$$\Upsilon := \{ (\mu, \sigma) \text{ progr. meas.} | \sigma \in S, \mu_t(\omega) \in U_\epsilon(\sigma_t(\omega)) \}$$

and let $\Upsilon_\sigma$ denote its $\sigma$-section. Different choices of $(\mu, \sigma) \in \Upsilon$ correspond to considering different probabilities on the Wiener space, namely those under which the risky assets evolve with the prescribed coefficients. These probabilities are orthogonal to each other across the sections $\Upsilon_\sigma$.

For a fixed choice of the process $\sigma$ however, the Girsanov theorem ensures that all the vector processes $\mu \in \Upsilon_\sigma$ correspond to probabilities that are equivalent to each other on $\mathcal{F}_t$ for all $t > 0$. We would like to describe these equivalent changes of measure as a function of $\mu$. To this end, let us select a reference probability corresponding to $(\hat{\mu}, \sigma)$ and call it $P^{\hat{\mu}, \sigma}$. Market incompleteness implies that the probability in $\Upsilon_\sigma$ under which the risky assets evolve with drift $\mu$ is not unique. However, such measures can be fully parametrized as probability changes with respect to $P^{\hat{\mu}, \sigma}$. And a minimal choice (see also Remark [1] below) is selecting for each $\mu$ the probability $P^{\mu, \sigma}$ corresponding to the measure change given by the Doléans exponential $\mathcal{E}(\int_0^\cdot \varphi^\mu dW)$, where

$$\varphi^\mu_t := \sigma'_{\Sigma_t^{-1}} (\mu_t - \hat{\mu}).$$
Such selection does not reduce generality, since what matters in our context of expected utility maximization are only the distributional properties of the risky assets. Therefore, we have now a one-to-one correspondence between elements of $\Upsilon$ and probabilities on $(\Omega, \mathcal{F})$, namely the possible prior models are given by

$$
P = \{P^{\mu, \sigma} \mid (\mu, \sigma) \in \Upsilon\}.
$$

Now, the wealth process evolves under each $P^{\mu, \sigma}$ according to

$$
dw_t = (rw_t + \theta_t'(\mu_t - r1) - ct)dt + \theta_t'\sigma_t dW,
$$

where $W$ is a $P^{\mu, \sigma}$-standard Brownian motion. Finally, let us call the investment/consumption pair $(\theta, c)$ (robust) admissible for the initial positive wealth $w_0$ if in addition to the measurability and integrability assumptions already made at the beginning of this section, the wealth process remains non-negative for all $P \in \mathcal{P}$:

$$
A_{rob}(w_0) := \cap_{P \in \mathcal{P}} A^P(w_0) = \cap_{\sigma \in \mathcal{S}} A^{P^{\hat{\mu}, \sigma}}(w_0).
$$

The equality on the rhs holds because, for a given $\sigma$, the admissible class is invariant for different choices of $\mu \in \Upsilon_\sigma$. The ambiguity averse investor takes a prudential worst case approach, and faces the following robust Merton problem:

$$
u_{opt}(w_0) := \sup_{(\theta, c) \in A_{rob}(w_0)} \inf_{(\mu, \sigma) \in \Upsilon} \mathbb{E}^{(\mu, \sigma)} \left[ \int_0^T u(t, c_t)dt + u(T, w_T) \right]. \tag{3}
$$

It is clear that more conservative portfolio choices are made when the uncertainty set $\Upsilon$ is larger, while an ambiguity-neutral investor sets $\mathcal{S} = \{\hat{\sigma}\}$ and $\epsilon$ equal to zero, thus facing a classical Merton problem for the model $P^{\hat{\mu}, \hat{\sigma}}$.

**Remark 1** Fix $\sigma \in \mathcal{S}$. Any change of drift, say from $\hat{\mu}$ to $\mu$, corresponds to a change of measure from $P^{\hat{\mu}, \hat{\sigma}}$ to a probability $\tilde{P}$ with density process $\frac{d\tilde{P}}{dP}$ given by a Doléans exponential $\mathcal{E}(\int_0^\cdot \varphi dW)$. The suitable $\varphi$s can be characterized as those in the form:

$$
\varphi_t = \varphi^\mu_t + \psi_t
$$

in which $\psi$ is a (sufficiently integrable) progressively measurable process, belonging $dt \otimes dP^{\hat{\mu}, \hat{\sigma}}$-a.e. to $\ker(\sigma_t(\omega))$. Chen and Epstein call the process $\varphi$ the market price of ambiguity. Denote this class of probabilities by $\mathcal{P}^\sigma$. Elementary optimization shows $\varphi^\mu$ is minimal, in the sense that it has the smallest pointwise $\mathbb{R}^n$-norm among $\mathcal{P}^\sigma$:

$$
\|\varphi^\mu_t(\omega)\|^2 = \min_{P \in \mathcal{P}^\sigma} \|\varphi_t(\omega)\|^2 dt \otimes dP^{\hat{\mu}, \sigma} - a.e.
$$

Thus, the ellipsoidal ambiguity on the drift in $\Upsilon_\sigma$ can be recast into a (non convex!) condition on the market price of uncertainty $\varphi$, namely

$$
\mu \in \Upsilon_\sigma \iff \min_{P \in \mathcal{P}^\sigma} \|\varphi_t(\omega)\|^2 \leq \epsilon^2 dt \otimes dP^{\hat{\mu}, \sigma} - a.e.
$$
This should be contrasted with the ubiquitous conditions in the literature (see e.g., [9], [3]) which require the market price of uncertainty to be valued in a convex set.

The resolution of [3] is based on the next robust version of the verification theorem. Although formulated for the sets $P, \Upsilon$, its validity is general and does not rely on any specific parametrization of the set of prior models.

**Theorem 1**

For a shorthand, call $\nu = (\mu, \sigma)$ a generic element of $\Upsilon$. Suppose that:

1. there exists a function $V : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$, which is continuous on $[0, T] \times \mathbb{R}^+$ and $C^{1,2}$ on $[0, T) \times \mathbb{R}^+$, verifying $V(T, \cdot) = u(T, \cdot)$;

2. for any $(\theta, c)$ there exists an optimal solution $\nu(\theta, c) \in \Upsilon$ of the inner minimization in [3], such that

   $$Y_t = Y_{t}^{\nu(\theta, c)} \equiv V(t, w_t) + \int_{0}^{t} u(s, c_s)ds$$

   is a $P^{\nu(\theta, c)}$-supermartingale;

3. there exist some $(\bar{\theta}, \bar{c}) \in A_{rob}(w_0)$ such that the corresponding $\bar{Y}$ is a $P^{\nu(\bar{\theta}, \bar{c})}$-martingale.

Then $(\bar{\theta}, \bar{c})$ is optimal for the problem [3] and $V(0, w_0)$ is the optimal value function, namely $u_{opt}(w_0) = V(0, w_0)$.

**Proof:** The proof is a simple modification of the Davis-Varaiya Martingale Principle of Optimal Control [20, Theorem 1.1]. In fact, by the supermartingale property of $\bar{Y}$ under $P^{\nu(\bar{\theta}, \bar{c})}$, and by $V(T, \cdot) = u(T, \cdot)$, we have:

$$E^{\nu(\theta, c)}[Y_T] = E^{\nu(\theta, c)}[\int_{0}^{T} u(s, c_s)ds + u(T, w_T)] \leq Y_0 = V(0, w_0).$$

Taking the supremum over $A_{rob}(w_0)$ gives $u_{opt}(w_0) = \sup_{A_{rob}(w_0)} E^{\nu(\theta, c)}[\int_{0}^{T} u(s, c_s)ds + u(T, w_T)] \leq V(0, w_0)$. Since by assumption for some $\nu = (\bar{\theta}, \bar{c})$ the process $\bar{Y}$ is a martingale under $P^{\nu(\bar{\theta}, \bar{c})}$, then $E^{\nu}[Y_T] = Y_0 = V(0, w_0)$ and the conclusions immediately follow.

Now, the usage of the verification theorem to solve the ambiguity-averse investor’s problem is quite intuitive. Given a specific utility function, one looks for a function $V$ satisfying the premises of the theorem. Using Itô’s formula, any process $Y$ as in [4] verifies under $P^{\nu}$ the following SDE:

$$dY_t = \{u(t, c_t) + V_t + V_w(rw_t + \theta'_t(\mu t - r) - c_t) + \frac{1}{2}\theta'^{2}_{t} \Sigma t \theta_t V_{ww}\} dt + V_{w} \theta'_t \sigma_t dW.$$  

To make $Y$ a supermartingale under every $P^{\nu(\theta, c)}$, and a martingale for some $P^{\nu(\theta^*, c^*)}$, the maximum over $(\theta, c) \in \mathbb{R}^n \times \mathbb{R}^+$ of the minimum of $\nu \in \Upsilon$ must be equated to zero. At this point, some other specific structure on $\sigma$ must be assumed in the fully incomplete market case,
like e.g., dependence on a correlated nontraded asset as in [3].

In the rest of the paper however, we focus on the complete market case, i.e. we assume $\sigma$ is a square matrix. Then, a max-min nonlinear PDE arises from (5):

$$\max_{(\theta, c) \in \mathbb{R}^n \times \mathbb{R}^+} \min_{\sigma \in S, \mu \in U_{\epsilon}(\sigma)} \left[ u(t, c) + V_t + V_w(rw + \theta'(\mu - r \mathbf{1}) - c) + \frac{1}{2} \theta' \Sigma \theta V_{ww} \right] = 0,$$

which is of the Hamilton-Jacobi-Bellman-Isaacs type. In the following, we simply refer to it as the robust HJB equation. The minimization of the drifts for $\sigma$ fixed gives:

$$\min_{\mu \in \mathbb{R}^n} \{ \theta' \mu : (\mu - \hat{\mu})' \Sigma^{-1} (\mu - \hat{\mu}) \leq \epsilon^2 \},$$

which is a simple exercise in Karush-Kuhn-Tucker necessary and sufficient conditions. When $\theta \neq 0$, the optimal solution is

$$\mu(\theta) = \hat{\mu} - \epsilon \frac{\Sigma \theta}{\theta' \Sigma \theta}.$$ 

Substituting it back in the robust HJB, we get

$$\max_{(\theta, c) \in \mathbb{R}^n \times \mathbb{R}^+} \min_{\sigma \in S} \left[ u(t, c) + V_t + V_w(rw + \theta' (\hat{\mu} - r \mathbf{1}) - \epsilon \sqrt{\theta' \Sigma \theta} - c) + \frac{1}{2} \theta' \Sigma \theta V_{ww} \right] = 0, \quad (6)$$

which covers also the case $\theta = 0$. Since the value function $V$ will be increasing (and concave) in the wealth $w$, we are minimizing a concave function over $S$:

$$\min_{\sigma \in S} \left( -V_w \epsilon \sqrt{\theta' \Sigma \theta} + \frac{V_{ww}}{2} \theta' \Sigma \theta \right). \quad (7)$$

Notice that the function to be optimized depends on $\sigma$ only via the quadratic form $\theta' \Sigma \theta$, and that its derivative wrt $y := \theta' \Sigma \theta$ is positive as $V_w > 0, V_{ww} < 0$. Therefore, the optimizers are not unique in general and compactness of $S$ is of essence. The actual computations depend on the specification of $S$, and may be quite easy when the set $S$ is defined precisely in terms of constraints on the quadratic form, as we show in Section 4.

Let us denote by $\bar{\sigma}(\theta)$ an optimizer, where $\bar{\sigma}(\theta)$ is the Cholesky factorization of an optimal varcov matrix $\bar{\Sigma}$. Since the optimal value of quadratic form $\theta' \bar{\Sigma}(\theta) \theta$ does not depend on the specific choice of $\bar{\sigma}(\theta)$, one obtains the PDE

$$\max_{(\theta, c) \in \mathbb{R}^n \times \mathbb{R}^+} \left[ u(t, c) + V_t + V_w(rw + \theta' (\bar{\mu} - r \mathbf{1}) - \epsilon \sqrt{\theta' \bar{\Sigma}(\theta) \theta} - c) + \frac{1}{2} \theta' \bar{\Sigma}(\theta) \theta V_{ww} \right] = 0, \quad (8)$$

In the main applications we present in Section 3 and 4, $\bar{\Sigma}(\theta) = \Sigma$, namely a constant. When this is the case, the above equation is equivalently viewed as stemming from the worst-case $(\bar{\mu}, \bar{\sigma})$, in which $\bar{\sigma}$ is the Cholesky factorization of $\bar{\Sigma}$, $\bar{\mu} = \hat{\mu} - \epsilon \frac{\Sigma \theta}{\sqrt{\theta' \Sigma \theta}}$, and using the worst couple in the wealth equation:

$$dw_t = (rw_t + \theta'_t (\hat{\mu}_t - r \mathbf{1}) - \epsilon \sqrt{\theta'_t \Sigma \theta_t - c_t})dt + \theta'_t \bar{\sigma}dW. \quad (9)$$
Therefore, the general problem (3) becomes equivalent to a robust utility maximization when there is (ellipsoidal) uncertainty in the drift only.

The techniques employed to solve (5) are then standard, and rely on educated guesses at the form of the solution. If the solution $V$ of the robust HJB can be found explicitly, then it is the candidate to be the value function we are looking for. Finally, to conclude that $V$ is indeed the value function, one must check that it verifies items 1, 2 and 3 in Theorem 1.

3 The robust power utility problem with non ambiguous $\sigma$

We assume in this Section that the investor has a power utility function, and that there is no uncertainty on the (constant) square volatility matrix, namely $\mathcal{S} = \{\hat{\sigma}\}$. Lack of uncertainty in the volatility may be empirically justified by the consideration that mean returns are subject to imprecision to a much higher extent than volatilities. We provide explicit solutions both when the planning horizon is finite and infinite. To avoid notation overload, and for the next usage in Section 4, we drop the hat over $\sigma$. Also, we denote by

$$\mathcal{U}_\epsilon := \{\mu \text{ progr meas} | \mu_t(\omega) \in U_\epsilon(\sigma) \text{ for all } \omega\},$$

and by $\mathbb{E}^\mu$ the expectation under $P^{\mu,\sigma}$.

3.1 The infinite horizon planning

3.1.1 Resolution of the robust HJB equation

Let us assume the investor has CRRA power utility from intertemporal consumption:

$$u(t, x) = e^{-\rho t} x^{1-R} \frac{1}{1-R},$$

where $\rho$ and $R \neq 1$ are positive constants, modeling the time impatience rate and relative risk aversion respectively. In the infinite horizon case, we wish to find the solution of:

$$u_{opt}(w_0, \epsilon) = \sup_{(\theta, c) \in A_{rob}(w_0)} \inf_{\mu \in \mathcal{U}_\epsilon} \mathbb{E}^\mu \left[ \int_0^\infty e^{-\rho s} \frac{C_s^{1-R}}{1-R} ds \right], \quad (10)$$

when the problem is well-posed, i.e. when it has a finite value.\footnote{Assume for the moment that this is the case and also that both the inner infimum (for a fixed $(\theta, c) \in A_{rob}(w_0)$) and the outer supremum are attained. The properties of the problem imply, exactly as in the classic case, that a guess at the value function takes the form

$$V(t, w) = \gamma_c^{-R} u(t, w).$$

\footnote{When $\mathcal{S} = \{\sigma\}$, we remark that $A_{rob}(w_0)$ coincides with the classic set of admissible plans $A^{\mu,\sigma}(w_0)$.}
The positive constant $\gamma_\epsilon$ has to be determined, and we use $\epsilon$ as subscript to highlight the dependence on the radius of ambiguity $\epsilon$. With this guess, let us solve (8). The optimization over $c$ trivially results in

$$\bar{c} = \gamma_\epsilon w,$$

with

$$\max_c \{ u(t, c) - cV_w \} = e^{-\rho t} R \frac{1}{1 - R} (\gamma_\epsilon w)^{1 - R}.$$

The residual optimization is

$$\max_\theta \left[ e^{-\rho t} R \frac{1}{1 - R} (\gamma_\epsilon w)^{1 - R} + V_t + V_w (rw + \theta'(\hat{\mu} - r1) - \epsilon \sqrt{\theta' \Sigma \theta}) + \frac{1}{2} \theta' \Sigma \theta V_{ww} \right].$$

The function to be maximized is concave in $\theta$, and smooth in $\mathbb{R}^n \setminus \{0\}$. The first order conditions are thus necessary and sufficient for optimality in $\theta \neq 0$. So, by equating the gradient to zero we obtain:

$$\theta(s) = \frac{-sv_w}{sv_{ww} - V_w} \Sigma^{-1}(\hat{\mu} - r1),$$

where $s := \sqrt{\theta' \Sigma \theta}$. We are left with

$$s^2 = \theta(s)' \Sigma \theta(s)$$

Set

$$H := \sqrt{(\hat{\mu} - r1)' \Sigma^{-1}(\hat{\mu} - r1)}, \quad H_\epsilon := H - \epsilon.$$

The above equation has a positive root, given by:

$$\bar{s} = -\frac{V_w H_\epsilon}{V_{ww}}$$

if and only if $H_\epsilon > 0$. If $H_\epsilon \leq 0$, the optimal solution is necessarily $\bar{\theta} = 0$. Finally, if $H_\epsilon^+$ denotes the positive part of $H_\epsilon$, the following is a compact way of writing the optimal solution in both cases:

$$\bar{\theta} = \frac{H_\epsilon^+}{R \Sigma^{-1}} (\hat{\mu} - r1)$$

Now, $\gamma_\epsilon$ is found by substituting these $\bar{c}$ and $\bar{\theta}$ back into (8) and solving for the constant. Straightforward calculations result in:

$$\gamma_\epsilon = \frac{\rho + (R - 1)(r + \frac{1}{2} (H_\epsilon^+)^2)}{R} \quad (11)$$

which for $\epsilon = 0$ falls back to the constant $\gamma_0 = \frac{\rho + (R - 1)(r + \frac{1}{2} H_0^2)}{R}$ of the classic case. Therefore, the value function $V$ of the problem is found as

$$V(t, w) = \gamma_\epsilon^{-R} u(t, w).$$

This of course holds as long as $\gamma_\epsilon > 0$, which is shown below to be a necessary and sufficient condition for the well posedness of the robust Merton problem.
3.1.2 The verification and comparison with the classic Merton problem

Proposition 1 The infinite-horizon robust Merton problem under ellipsoidal ambiguity of mean returns:

\[ u_{\text{opt}}(w_0, \epsilon) = \sup_{(\theta, c) \in \mathcal{A}_{\text{rob}}(w_0)} \inf_{\mu \in \mathcal{U}} \mathbb{E}^\mu \left[ \int_0^\infty e^{-\rho s} \frac{c_s^{1-R}}{1-R} ds \right] \]

is well posed if and only if \( \gamma_\epsilon \) in (11) is strictly positive. In this case, the optimal value is

\[ u_{\text{opt}}(w_0, \epsilon) = V(0, w_0) = \frac{\gamma_\epsilon}{1-R} \frac{(w_0)^{1-R}}{1-R}, \]

and the optimal controls are:

\[ \tilde{\theta}_t = \tilde{w}_t \pi_\epsilon, \quad \tilde{c}_t = \gamma_\epsilon \tilde{w}_t, \]

with optimal portfolio proportions vector given by

\[ \pi_\epsilon := \frac{H^+}{R} \Sigma^{-1} (\hat{\mu} - r 1) \]

The worst case drift is constant:

\[ \hat{\mu} := \mu(\tilde{\theta}) = \mu - \epsilon \frac{\Sigma}{\pi_\epsilon' \Sigma \pi_\epsilon}, \]

and the optimal wealth process has \( P_{\hat{\mu}, \sigma} \) dynamics given by

\[ \tilde{w}_t = w_0 \exp \left( \pi_\epsilon \sigma W_t + (r + \frac{(H^+_\epsilon)^2 (2R - 1)}{2R^2} - \gamma_\epsilon) t \right). \quad (12) \]

Proof: The proof is split into two steps.

1. If \( \gamma_\epsilon \leq 0 \) then \( u_{\text{opt}}(w_0, \epsilon) = \infty \). Note first that this case can only happen when \( 0 < R < 1 \).

   The proof here closely follows the lines of [20, Section 1.6].

1-a) Assume \( \gamma_\epsilon < 0 \). Then, consider a couple of controls that are both proportional to the wealth:

\[ \theta_t = \tilde{w}_t \pi, \quad c_t = \lambda \tilde{w}_t \text{ with } \lambda > 0. \]

If we substitute them into (7), then the solution is the positive wealth

\[ \tilde{w}_t = w_0 \exp \left( (r + \pi' (\hat{\mu} - r 1) - \epsilon \sqrt{\pi' \Sigma \pi} - \lambda - \frac{1}{2} \pi' \Sigma \pi) t + \pi' \sigma W_t \right) \]

so that:

\[ u_{\text{opt}}(w_0, \epsilon) \geq \mathbb{E}^\mu(\tilde{\theta}) \left[ \int_0^\infty e^{-\rho t} \frac{\lambda^{1-R}}{1-R} (\tilde{w}_t)^{1-R} dt \right]. \]

An application of the stochastic Fubini’s theorem shows that the latter is proportional to

\[ \int_0^\infty \exp \left( t \left( -\rho + (1 - R)(r + \pi' (\hat{\mu} - r 1) - \epsilon \sqrt{\pi' \Sigma \pi} - \lambda - \frac{R}{2} \pi' \Sigma \pi) \right) \right) dt. \]
If there exist some \((\pi, \lambda)\) for which the exponent is positive, then the integral diverges and the value function is infinite. For fixed \(\lambda\), the maximum over \(\pi\) in the exponent is attained for \(\pi = \pi_\epsilon\), and the value is

\[-\rho + (1 - R)(r + \frac{(H^+)^2}{2R} - \lambda) = -R\gamma \epsilon - \lambda(1 - R),\]

which is positive for \(\lambda\) small enough.

1-b) If \(\gamma \epsilon = 0\), take \(\theta_t = w_t \pi_\epsilon\), and \(c_t = \frac{k}{1+t} w_t\) for some constant \(k > 0\). This choice leads to

\[u_{opt}(w_0, \epsilon) \geq \int_0^\infty e^{-\rho t} \frac{1}{1 - R} \frac{k^{1-R}}{(1+t)^{1-R}} E^{\mu(\theta)}[(\bar{\pi}_t)^{1-R}] dt.\]

Now, \(E^{\mu(\theta)}[(\bar{w}_t)^{1-R} e^{-\rho t}] = e^{-(1-R) \int_0^t \frac{k}{1+s} ds} = e^{-(1-R)k \ln(1+t)} = \frac{1}{(1+t)^{(k(1-R))}}\) when \(t \to \infty\). Therefore, the integrand is asymptotic to \((1 + t)^{-(k+1)(1-R)}\) and hence the integral diverges if e.g., \(k = \frac{1}{1-R} - 1\).

2. If \(\gamma _\epsilon > 0\), then the optimal processes/value are as given in the statement of the proposition. Substituting the candidate optimal controls \((\bar{\theta}_t, \bar{c}_t)\) into (9), and solving for the candidate optimal wealth one gets (12), which can be further simplified to:

\[\bar{w}_t = w_0 \exp \left( \pi_\epsilon W_t + \frac{1}{1 - R} \frac{1}{(1+t)^{1-R}} E^{\mu(\theta)}[(\bar{\pi}_t)^{1-R}] \right).\]

The process \(\bar{w}\) is a deterministic scaling of Geometric Brownian motion, as well as the process \(\bar{v}_t := (\bar{w}_t)^{1-R}\). Now,

\[(1 - R)\bar{Y}_t := (1 - R) [V(t, \bar{w}_t) + \int_0^t u(s, \bar{c}_s) ds] = \gamma _\epsilon e^{-\rho t} \bar{v}_t + \int_0^t e^{-\rho s} (\gamma _\epsilon)^{1-R} \bar{v}_s ds.\]

has integrable maximal functional in every compact \([0, T]\) and by construction has zero drift term. Henceforth, \(\bar{Y}\) is a \(P^{\mu, \sigma}\) martingale.

For what concerns other admissible controls \((\theta_t, c_t)\), the process

\[Y_t = V(t, w_t) + \int_0^t u(s, c_s) ds\]

with \(w\) as in (9), is by construction a diffusion, which has the same sign as \((1 - R)\), and which has non positive drift under \(P^{\mu(\theta), \sigma}\).

• If \(0 < R < 1\), then any such \(Y\) is positive. By writing (5) under \(P^{\mu(\theta), \sigma}\), it is immediate to realize that \(Y\) is a positive, decreasing scaling of a positive local martingale, namely the Doléans exponential of the process \(X\) defined by:

\[X := \int_0^\cdot \gamma _\epsilon e^{-\rho s} w_s^{-R} \bar{Y}_s \sigma dW.\]
Henceforth, $Y$ is a $P^{\mu(\theta),\sigma}$-supermartingale. In addition, $V(\infty, \cdot) = u(\infty, \cdot) = 0$ so that the conditions of the Verification Theorem [1] are satisfied, and the proof in this case is complete.

- If $R > 1$, any $Y$ is negative. A simple modification of the argument just used in the $0 < R < 1$ case only shows that $Y$ is a local supermartingale. Therefore in this case we show the optimality of $((\bar{\theta}, \bar{c}), \bar{\mu})$ in another way. To this end, note that the martingale property of $\bar{Y}$ as above, together with standard minimax inequalities, gives

$$
\mathbb{E}^{\bar{\mu}} \left[ \int_0^\infty e^{-\rho t} \frac{(\bar{c}_t)^{1-R}}{1-R} dt \right] = \gamma(\bar{\mu}) \frac{w_0^{1-R}}{1-R} \leq u_{opt}(w_0, \epsilon) \leq \inf_{\mu \in \mathcal{U}_\epsilon} \sup_{(\theta, c) \in \mathcal{A}_{rob}(w_0)} \mathbb{E}^\mu \left[ \int_0^\infty e^{-\rho t} \frac{(c_t)^{1-R}}{1-R} dt \right],
$$

so if we prove that the first value on the left is equal to the value of the last problem on the RHS, we are done. This is quite an easy task. In fact, for a fixed constant $\mu \in \mathcal{U}_\epsilon$ the inner supremum is a standard Merton problem. Hence,

$$
\sup_{(\theta, c) \in \mathcal{A}_{rob}(w_0)} \mathbb{E}^\mu \left[ \int_0^\infty e^{-\rho t} \frac{(c_t)^{1-R}}{1-R} dt \right] = (\gamma(\mu)) - R \frac{w_0^{1-R}}{1-R}
$$

in which we pose $\gamma(\mu) := \frac{\rho+(R-1)(\nu+\frac{1}{2}(H(\mu))^2)}{R^2}$. with $H(\mu) := \sqrt{(\mu - r1)^{-1} \Sigma^{-1}(\mu - r1)}$. The residual minimization:

$$
\inf_{\mu \in \mathcal{U}_\epsilon} (\gamma(\mu)) - R \frac{w_0^{1-R}}{1-R}
$$

is then a simple exercise, the minimizer being $\bar{\mu}$, so that $\gamma(\mu) = \gamma(\bar{\mu}) = \gamma_\epsilon$, which ends the proof.

Let us remark that the optimal portfolio $\bar{\theta}$ preserves the form of the Merton’s Mutual Fund theorem. In fact, the optimal portfolio consists of an allocation between two fixed mutual funds, namely the riskless asset and the fund of risky assets given by $\Sigma^{-1}(\bar{\mu} - r1)$. At each time point the optimal relative allocation of wealth is now dependent on the ambiguity aversion of the investor in addition to his/her risk aversion through the coefficient:

$$
\frac{H_+}{RH^2}.
$$

The above allocation naturally collapses to the Merton allocation $\frac{1}{R}$ for $\epsilon = 0$. In case the radius of ambiguity $\epsilon$ is greater than or equal to the market Sharpe ratio $H$, the optimal control policy
is not to invest at all into the risky assets. Since $H^+ / RH \leq \frac{1}{R}$, the robust Merton portfolio $\pi_\epsilon$ has smaller positions in absolute value with respect to the classical Merton portfolio. To wit, both long and short positions are shrunk with respect to the ambiguity-neutral portfolio. As expected, and already anticipated in the Introduction, robustness in the decisions lowers the optimal demand on equity, and thus offer a theoretical basis for a possible explanation of the equity premium puzzle.

The consumption in the ambiguity averse case may be increased or curtailed, depending on the sign of $R - 1$. In fact, when the classical problem and its ambiguity averse counterpart with $\epsilon > 0$ are both well posed, if $0 < R < 1$ then $\gamma_\epsilon > \gamma_0 > 0$, while if $R > 1$ the opposite inequality chain holds.

3.2 The finite horizon planning for non ambiguous $\sigma$

Now the investor has a CRRA power utility both from intertemporal and terminal consumption at time $T < \infty$:

$$u(t, w) = e^{-\rho t} \frac{w^{1-R}}{1-R} \quad \text{for} \quad 0 \leq t < T \quad \text{and} \quad u(T, w) = A \frac{w^{1-R}}{1-R}$$

in which $A$ is a fixed positive constant. Here, we set the deterministic scaling of the CRRA power utility identical to that of the infinite horizon case to better highlight the similarities, but everything stated below holds also if $e^{-\rho t}$ is replaced by an integrable, positive and deterministic function $h(t)$. We then wish to find the solution of:

$$u_{\text{opt}}(w_0, \epsilon) = \sup_{(\theta, c) \in A_{\text{rob}}(w_0)} \inf_{\mu \in \mathcal{U}_\epsilon} \mathbb{E}^\mu \left[ \int_0^T e^{-\rho s} \frac{c_s^{1-R}}{1-R} ds + A \frac{w_T^{1-R}}{1-R} \right]$$

Using the scaling properties of the CRRA utility, the guess to the value function is of the form $V(t, w) = f(t) \frac{w^{1-R}}{1-R}$ for some positive, differentiable function satisfying $f(T) = A$. The HJB equation now looks like

$$\max_{(\theta, c) \in \mathbb{R}^n \times \mathbb{R}^+} \left[ e^{-\rho t} \frac{c^{1-R}}{1-R} + f(t) \frac{w^{1-R}}{1-R} + f(t) w^{-R} (\mu w' - \theta' \Sigma \theta - c) - \frac{R}{2} f(t) w^{-R-1} \theta' \Sigma \theta \right] = 0.$$
With the substitution $f(t) = g(t)^R$, the ODE can be linearized and easily solved:

\[
g(t) = A_\pi \exp \left( \frac{k_\epsilon}{R} (T - t) \right) + e^{-\frac{k_\epsilon}{R} t} \int_t^T \exp \left( \frac{k_\epsilon - \rho}{R} s \right) ds.
\]

Comparing this to the solution of [20, Section 2.1] the only changes are: 1) the constant $k_\epsilon$, in which $H^2$ is replaced by $(H^+)^2$ and 2) the optimal portfolio allocation, which is identical to the robust allocation case of the previous section. Obviously for an ambiguity neutral investor with $\epsilon = 0$ we fall back to the finite horizon solution of the Merton problem.

Let us conclude by summing up the results just found, leaving the verification to the reader.

**Proposition 2** The finite horizon robust Merton problem under ellipsoidal ambiguity of mean returns

\[
u_{\text{opt}}(w_0, \epsilon) = \sup_{(\theta, c) \in A_{\text{rob}}(w_0)} \inf_{\mu \in \mathcal{U}_{\epsilon}} \mathbb{E}^{\mu} \left[ \int_0^T e^{-\rho s} \frac{\epsilon^{1-R}}{1-R} ds + A \frac{w_T^{1-R}}{1-R} \right],
\]

is always well-posed, and admits the optimal controls:

\[
\bar{\theta}_t = \frac{\bar{w}_t H^+ \Sigma^{-1} (\hat{\mu} - r \mathbf{1})}{\bar{w}_t \pi^{\epsilon}}, \quad \bar{c}_t = \frac{\bar{w}_t e^{-\frac{k_\epsilon}{R} t}}{g(t)},
\]

where

\[
g(t) = A_\pi \exp \left( \frac{k_\epsilon}{R} (T - t) \right) + e^{-\frac{k_\epsilon}{R} t} \int_t^T \exp \left( \frac{k_\epsilon - \rho}{R} s \right) ds,
\]

and $k_\epsilon = (1 - R) (r + \frac{(H^+)^2}{2R})$. The optimal $\bar{\mu} = \mu(\bar{\theta}) = \mu - \epsilon \frac{\Sigma}{\sqrt{\pi^{\epsilon} \Sigma \pi^{\epsilon}}} \pi^{\epsilon}$, and the optimal wealth process $\bar{w}$ has dynamics under $P^{\bar{\mu}, \sigma}$ given by:

\[
\bar{w}_t = w_0 \exp \left[ \left( \frac{r + (H^+)^2}{2R^2} (R - 1) \right) t + \int_0^t \frac{e^{-\frac{k_\epsilon}{R} s}}{g(s)} ds + \pi^{\epsilon} \sigma W_t \right].
\]

### 4 Examples with ambiguous $\sigma$

In all the following examples, the volatilities are square, full rank, matrices.

**Example 1 (The uncorrelated case)** Suppose that estimated volatility matrix $\hat{\sigma}$ is diagonal. To wit, the risky assets returns are (instantaneously) uncorrelated. Further, we suppose that the ambiguity does not affect correlations, namely the ambiguity set $S$ is that of diagonal matrices, whose diagonal $\Sigma$ lies in some product $[\sigma_1^2, \sigma_1] \times \ldots \times [\sigma_n^2, \sigma_n]$, with $\inf_i \sigma_i > 0$ and $\sigma_i \leq \hat{\sigma}_i \leq \sigma_i$. This is exactly the case examined by Lin and Riedel [19], where the problem is treated via a G-Brownian motion technique.
For a fixed ambiguity radius $\epsilon > 0$ on the drift, the solution of the residual inner minimization over $\sigma$ in the max-min HJB becomes a triviality with this diagonal uncertainty specification. The unique worst case volatility is constant and it is the ‘highest’ one, $\bar{\sigma} = \text{Diag}(\sigma_1, \ldots, \sigma_n)$ and does not depend on $\theta$. Therefore, the general problem becomes equivalent to a robust utility maximization with volatility $\bar{\sigma}$ and ellipsoidal uncertainty on the drift only, with radius $\epsilon$. To give an explicit example, in the power utility case one ends up in solving (10) or (13) with $\sigma = \bar{\sigma}$. It is clear then that the verifications are identical to the ones just seen in the previous Section. The resulting optimal relative portfolio is also constant:

$$\pi_\epsilon(\bar{\sigma}) = \frac{\overline{H}_\epsilon^+}{R\overline{H}}(\Sigma)^{-1}(\hat{\mu} - r1)$$

in which $\overline{H} = \sqrt{(\hat{\mu} - r1)\Sigma^{-1}(\hat{\mu} - r1)}$, $\overline{H}_\epsilon^+ = (\overline{H} - \epsilon)^+$. 

**Example 2 (Upper bound on the quadratic form $\Sigma$)** This example can be seen as relaxation of the previous one, in the sense that we do not impose constraints separately on each of the eigenvalues of $\Sigma$, nor we assume that $\Sigma$ is diagonal. We simply restrict the quadratic form induced by $\Sigma$ not to exceed a given threshold $\overline{\lambda}^2 > 0$ on the unit sphere, with $\overline{\lambda} \geq \hat{\lambda}_M$, the latter being the maximum eigenvalue of $\hat{\sigma}$. This amounts to imposing the same bound on the maximum eigenvalue of $\Sigma$. Precisely, the volatility is assumed to be valued in $S := \{\sigma \in \mathbb{R}^{n \times n} \mid 0 < x'\Sigma x \leq \overline{\lambda}^2\|x\|^2 \text{ for all } x \in \mathbb{R}^n, x \neq 0\}$

The minimizers $\bar{\sigma}$ of the inner minimization in (6) are volatilities $\sigma$ with maximum eigenvalue equal to $\overline{\lambda}$, and such that $\theta$ is an eigenvector relative to $\overline{\lambda}$. Finally, the robust HJB boils down to the concave maximization:

$$\max_{(\theta,c) \in \mathbb{R}^n \times \mathbb{R}^+} \left[ u(t,c) + V_t + V_w(rw + \theta'(\hat{\mu} - r1) - \epsilon\overline{\lambda}\|\theta\| - c) + \frac{1}{2}\overline{\lambda}^2\|\theta\|^2V_{ww} \right] = 0$$

Therefore, the ambiguous volatility problem is observationally equivalent to a Merton problem with volatility matrix equal to $\overline{\lambda}I$, and drift uncertainty radius $\epsilon$. So, from here one proceeds exactly as in Section 3. The optimal relative portfolio is thus

$$\pi_\epsilon(\bar{\sigma}) = \frac{\left(\frac{1}{\overline{\lambda}}\|\hat{\mu} - r1\| - \epsilon\right)^+}{\frac{R}{\overline{\lambda}}\|\hat{\mu} - r1\|} \frac{1}{\overline{\lambda}^2}(\hat{\mu} - r1) = \frac{\|\hat{\mu} - r1\| - \overline{\lambda}\epsilon)^+}{R\|\hat{\mu} - r1\|} \frac{1}{\overline{\lambda}^2}(\hat{\mu} - r1)$$

**Remark 2** Another interesting case of ambiguity specification on volatility is based on defining a ball around an estimate of the variance/covariance matrix. However, it does not lead to a closed form portfolio rule.
The ambiguity on volatility is represented here as membership to the set \( S = \{ \Sigma \geq 0 : \| \Sigma - \hat{\Sigma} \|_F \leq \delta \} \) for some estimate \( \hat{\Sigma} \) of the variance/covariance matrix and a positive parameter \( \delta \). The inner optimization problem \( \max_{\Sigma \in S} V_w \epsilon \sqrt{\theta' \Sigma \theta} - \frac{V_{ww}}{2} \theta' \Sigma \theta \) is equivalently posed as the matrix optimization problem in the space of symmetric \( n \times n \) matrices:

\[
\max_{\Sigma \in S} V_w \epsilon \sqrt{\langle \theta' \Sigma \rangle} = \frac{V_{ww}}{2} \langle \theta' \Sigma \rangle
\]

where \( \langle X, Y \rangle = \text{Tr}(X^T Y) \) is the trace product (inner product in the space of symmetric \( n \times n \) matrices). Denoting rank-one matrix \( \theta \theta' \) (obtained from the dyadic product) as \( \Theta \), and passing to variables \( X = \Sigma - \hat{\Sigma} \) we have the equivalent quadratically constrained optimization (convex) problem in matrix variable \( X \):

\[
\max_{X} \beta(\epsilon) \sqrt{\langle \Theta, X \rangle} + \langle \Theta, \hat{\Sigma} \rangle + \alpha \langle \Theta, X \rangle
\]

subject to

\[
\langle X, X \rangle \leq \delta^2
\]

where we defined \( \alpha = -\frac{V_{ww}}{2} \) and \( \beta(\epsilon) = V_w \epsilon \) for convenience and we omitted momentarily the term \( \alpha \langle \Theta, \hat{\Sigma} \rangle \).

From the first-order optimality conditions, after some straightforward algebra we obtain the following optimal (worst-case) matrix

\[
\Sigma = \hat{\Sigma} + \frac{1}{2\lambda}(\alpha + \frac{\beta}{2\xi}) \Theta
\]

where \( A = \langle \Theta, \hat{\Sigma} \rangle \) and \( B = \| \Theta \|_F \), \( \xi = \sqrt{A + \delta B} \), \( \lambda = \frac{1}{4}(\frac{2\alpha \sqrt{A + \delta B} + \beta B}{\delta \sqrt{A + \delta B}}) \). Note that \( \Sigma \) is positive definite. Unfortunately, substitution of the worst-case matrix into the robust HJB equation yields a fourth-order polynomial which does not lead to a closed-form portfolio rule. Nonetheless, a numerical procedure can be used to find the optimal portfolio rule.

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\(^2\)It is more convenient to work on \( \Sigma \) rather than on \( \sigma \); \( \| Y \|_F \) denotes the Frobenius norm of a \( m \times n \) matrix \( Y \), and is equal to \( \text{Tr}(Y^T Y) \).
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