A refinement of a Hardy type inequality for negative exponents, and sharp applications to Muckenhoupt weights on \( \mathbb{R} \)

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Abstract

We prove a sharp integral inequality that generalizes the well known Hardy type integral inequality for negative exponents. We also give sharp applications in two directions for Muckenhoupt weights on \( \mathbb{R} \). This work refines the results that appear in [9].

1 Introduction

In 1920, Hardy has proved (as one can see in [2] or [3]) the following inequality which is known as Hardy’s inequality

**Theorem A.** If \( p > 1 \), \( a_n \geq 0 \) and \( A_n = a_1 + a_2 + \ldots + a_n, \ n \in \mathbb{N}^* \), then

\[
\sum_{n=1}^{\infty} \left( \frac{A_n}{n} \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p.
\]  

(1.1)

Moreover, inequality (1.1) is best possible, that is the constant on the right side cannot be decreased.

In 1926, Copson generalized in [1] Theorem A by replacing the arithmetic mean of a sequence by a weighted arithmetic mean. More precisely, he proved the following

**Theorem B.** Let \( p > 1 \), \( a_n, \lambda_n > 0 \) for \( n = 1, 2, \ldots \). Further suppose that \( \Lambda_n = \sum_{i=1}^{n} \lambda_i \) and \( A_n = \sum_{i=1}^{n} \lambda_i a_i \). Then

\[
\sum_{n=1}^{\infty} \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p,
\]

(1.2)

where the constant involved in (1.2) is best possible.
Certain generalizations of (1.1) have been given in [6], [7] and elsewhere. For example, one can see in [8] further generalizations of Hardy’s and Copson’s inequalities be replacing means by more general linear transforms. Theorem A has a continued analogue which is the following

**Theorem C.** If $p > 1$ and $f : [0, +\infty) \to \mathbb{R}^+$ is $L^p$-integrable, then

$$
\int_0^\infty \left( \frac{1}{t} \int_0^t f(u) \, du \right)^p \, dt \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f^p(t) \, dt. \tag{1.3}
$$

The constant in the right side of (1.3) is best possible.

It is easy to see that Theorems A and C are equivalent, by standard approximation arguments which involve step functions. Now as one can see in [4], there is a continued analogue of (1.3) for negative exponents, which is presented there without a proof. This is described in the following

**Theorem D.** Let $f : [a, b] \to \mathbb{R}^+$. Then for every $p > 0$ the following is true

$$
\int_a^b \left( \frac{1}{t-a} \int_a^t f(u) \, du \right)^{-p} \, dt \leq \left( \frac{p+1}{p} \right)^p \int_a^b f^{-p}(t) \, dt. \tag{1.4}
$$

Moreover (1.4) is best possible.

In [9], a generalization of (1.4) has been given, which can be seen in the following

**Theorem E.** Let $p \geq q > 0$ and $f : [a, b] \to \mathbb{R}^+$. The following inequality is true and sharp

$$
\int_a^b \left( \frac{1}{t-a} \int_a^t f(u) \, du \right)^{-p} \, dt \leq \left( \frac{p+1}{p} \right)^q \int_a^b \left( \frac{1}{t-a} \int_a^t f(u) \, du \right)^{-p+q} f^{-q}(t) \, dt. \tag{1.5}
$$

What is proved in fact in [9] is a more general weighted discrete analogue of (1.5) which is given in the following

**Theorem F.** Let $p \geq q > 0$ and $a_n, \lambda_n > 0$ for $n = 1, 2, \ldots$. Define $A_n$ and $\Lambda_n$ as in Theorem A. Then

$$
\sum_{n=1}^\infty \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^{-p} \leq \left( \frac{p+1}{p} \right)^q \sum_{n=1}^\infty \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^{-p+q} a_n^{-q}. \tag{1.6}
$$

Certain applications exist for the above two theorems. One of them can be seen in [9], concerning Muckenhoupt weights. In this paper we generalize and refine inequality (1.5) by specifying the integral of $f$ over $[a, b]$. We also assume, for simplicity reasons, that $f$ is Riemann integrable on $[a, b]$. More precisely we will prove the following
Theorem 1. Let $p \geq q > 0$ and $f : [a, b] \to \mathbb{R}^+$ with $\frac{1}{b-a} \int_a^b f = \ell$. Then the following inequality is true
\[
\int_a^b \left( \frac{1}{t-a} \int_a^t f(u) \, du \right)^{-p} \, dt \leq \left( \frac{p+1}{p} \right)^q \int_a^b \left( \frac{1}{t-a} \int_a^t f(u) \, du \right)^{-p+q} f^{-q}(t) \, dt - \frac{q}{p+1} (b-a) \cdot \ell^{-p}.
\] (1.7)

Moreover, inequality (1.7) is sharp if one considers all weights $f$ that have mean integral average over $[a, b]$ equal to $\ell$.

What we mean by noting that (1.7) is sharp is the following: The constant in front of the integral on the right side cannot be decreased, while the one in front of $\ell^{-p}$ cannot be increased. These facts will be proved below. In fact more is true as can be seen in the following

Theorem 2. Let $p \geq q > 0$ and $a_n, \lambda_n > 0$, for every $n = 1, 2, \ldots$. Define $A_n$ and $\Lambda_n$ as above. Then the following inequality holds for every $N \in \mathbb{N}$.
\[
\sum_{n=1}^N \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^{-p} \leq \left( \frac{p+1}{p} \right)^q \sum_{n=1}^N \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^{-p+q} a_n^{q-1} - \frac{q}{p+1} \Lambda_N \left( \frac{A_N}{\Lambda_N} \right)^{-p}.
\] (1.8)

In Section 2 we describe the proof of Theorem 2 and we also prove the validity and the sharpness of (1.7). Moreover if one wants to study the whole topic concerning generalization of inequalities (1.1) or (1.2), can see [5] and [10].

In the last section we prove an application of Theorem 1. More precisely we prove the following

Theorem 3. Let $\varphi : [0, 1) \to \mathbb{R}^+$ be non-decreasing satisfying the following Muckenhoupt type inequality
\[
\left( \frac{1}{t} \int_0^t \varphi(y) \, dy \right)^{1/p} \left( \frac{1}{t} \int_0^t \varphi^{-1/(q-1)}(y) \, dy \right)^{q-1} \leq M_t,
\] (1.9)
for every $t \in (0, 1]$, where $q > 1$ is fixed and $M \geq 1$ is given. Let now $p_0 \in (1, q)$ be defined as the solution of the following equality:
\[
\frac{q-p_0}{q-1} (M p_0)^{1/(q-1)} = 1.
\] (1.10)

Then for every $p \in (p_0, q]$ the following inequality
\[
\frac{1}{t} \int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(p-1)} \, ds \leq \frac{1}{t} \int_0^t \varphi \left( \frac{p}{q} \right)^{-1/(p-1)} \frac{1}{K'} \left( \frac{q}{q-1} \right)^2
\] (1.11)

is true, for every $t \in (0, 1]$, where $c = M^{1/(q-1)}$ and $K' = K'(p, q, c) = \frac{1}{p^{1/(q-1)}} - c \frac{q-p}{q-1}$. It is also sharp for $t = 1$. 

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The above theorem implies immediately the following

**Corollary.** Let \( \varphi \) be as in Theorem 3. Then the following inequality is true for every \( t \in (0, 1] \) and every \( p \in (p_0, q] \).

\[
\left( \frac{1}{t} \int_0^t \varphi^{-1/(p-1)} \right)^{p-1} \left( \frac{1}{t} \int_0^t \varphi \right) \leq \left[ \frac{1}{K'} c \frac{q}{p} \left( \frac{p - 1}{q - 1} \right)^2 \right]^{p-1}
\]

This gives us the best possible range of \( p \)'s for which the Muckenhoupt condition (1.9) still holds, under the hypothesis of (1.9).

The above corollary is the content of [9] but with another constant. Thus by proving Theorem 3 we refine the results in [9] by improving the constants that appear there and by giving certain sharp inequalities that involve Muckenhoupt weights on \( \mathbb{R} \).

### 2 The Hardy inequality

**Proof of Theorem 2.**

Let \( p \geq q > 0 \) and \( a_n, \lambda_n > 0 \), for every \( n \in \mathbb{N}^* \). We define \( \Lambda_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n \), \( A_n = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n \), for \( n = 1, 2, \ldots \). We shall prove inequality (1.8). In order to do this we will give two Lemmas that are stated below. We follow [9].

**Lemma 1.** Under the above notation the following inequality holds for every \( n \in \mathbb{N}^* \).

\[
\left( \frac{p + 1}{p} \right)^q a_n^{-q} \left( \frac{A_n}{\Lambda_n} \right)^{-(p+q)} + p \left( \frac{p}{p + 1} \right)^{q/p} a_n^{q/p} \left( \frac{A_n}{\Lambda_n} \right)^{-(p-q)/p} \geq (p + 1) \left( \frac{A_n}{\Lambda_n} \right)^{-p}.
\]

**Proof.** It is well known that the following inequality holds

\[
y_1^{-p} + p y_1 y_2^{-p-1} - (p + 1) y_2^{-p} \geq 0,
\]

for every \( y_1, y_2 > 0 \).

This is in fact an immediate consequence of the inequality

\[
y^{-p} + p y \geq (p + 1), \text{ for every } y, p \geq 0.
\]

Inequality (2.3) is true in view of Young’s inequality which asserts that for every \( t, s \) nonnegative the following inequality is true

\[
\frac{1}{q} t^q + \frac{1}{q'} s^{q'} \geq ts
\]

whenever \( q \) is greater than 1, and \( q' \) is such that \( \frac{1}{q} + \frac{1}{q'} = 1 \). Then by choosing \( q = p + 1 \) in Young’s inequality, and setting \( t = \frac{1}{y} \) we obtain (2.3).
If we apply (2.3) when \( y = y_1/y_2 \) we obtain (2.2). Now we apply (2.2) when

\[
y_1 = \left( \frac{p}{p+1} \right)^{1+q/p} a_n^{q/p} \left( \frac{A_n}{\Lambda_n} \right)^{1-q/p} \quad \text{and} \quad y_2 = \left( \frac{p}{p+1} \right) \frac{A_n}{\Lambda_n}.
\]

Then as it is easily seen (2.1) is immediately proved. Our proof of Lemma 1 is now complete.

As a consequence of Lemma 1 we have (by summing the respective inequalities) that:

\[
\left( \frac{p+1}{p} \right)^q \sum_{n=1}^{N} \lambda_n a_n^{-q} \left( \frac{A_n}{\Lambda_n} \right)^{-p+q} + p \left( \frac{p}{p+1} \right)^q \sum_{n=1}^{N} \lambda_n a_n^{q/p} \left( \frac{A_n}{\Lambda_n} \right)^{-p-q/p} \geq \left( p+1 \right) \sum_{n=1}^{N} \left( \frac{A_n}{\Lambda_n} \right)^{-p} \lambda_n,
\]

for every \( N \in \mathbb{N}^* \).

We proceed to the proof of

**Lemma 2.** Under the above notation the following inequality is true for every \( N \in \mathbb{N}^* \)

\[
\sum_{n=1}^{N} \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^{-p} - \left( \frac{p}{p+1} \right)^N \sum_{n=1}^{N} \lambda_n a_n \left( \frac{A_n}{\Lambda_n} \right)^{-p-1} \geq \frac{A_n}{p+1} \left( \frac{A_n}{\Lambda_n} \right)^{-p}.
\]

**Proof.** We follow [9].

For \( N = 1 \), inequality (2.6) is in fact equality. We suppose now that it is true with \( N-1 \) in place of \( N \). We will prove that it is also true for the choice of \( N \).

Define \( S_N = \sum_{n=1}^{N} \left[ \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^{-p} - \left( \frac{p}{p+1} \right) \lambda_n a_n \left( \frac{A_n}{\Lambda_n} \right)^{-p-1} \right] = \sum_{n=1}^{N-1} \left[ \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^{-p} - \left( \frac{p}{p+1} \right) \lambda_n a_n \left( \frac{A_n}{\Lambda_n} \right)^{-p-1} \right] + \lambda_N \left( \frac{A_n}{\Lambda_n} \right)^{-p} - \left( \frac{p}{p+1} \right) (A_N - A_{N-1}) \left( \frac{A_n}{\Lambda_n} \right)^{-p-1}.
\]

By our induction step we obviously see that

\[
S_N \geq \frac{A_{N-1}}{p+1} \left( \frac{A_{N-1}}{\Lambda_{N-1}} \right)^{-p} + \lambda_N \left( \frac{A_N}{\Lambda_N} \right)^{-p} - \left( \frac{p}{p+1} \right) (A_N - A_{N-1}) \left( \frac{A_N}{\Lambda_N} \right)^{-p-1} = \frac{A_{N-1}}{p+1} \left( \frac{A_{N-1}}{\Lambda_{N-1}} \right)^{-p} + \lambda_N \left( \frac{A_N}{\Lambda_N} \right)^{-p} - \left( \frac{p}{p+1} \right) \frac{A_N}{\Lambda_N} \left( \frac{A_N}{\Lambda_N} \right)^{-p-1} + \frac{A_{N-1}}{p+1} \left( \frac{A_{N-1}}{\Lambda_{N-1}} \right)^{-p-1}.
\]

(2.8)
We use now inequality (2.2) in order to find a lower bound for the expression in brackets in (2.8). We thus have
\[ p \left( \frac{A_{N-1}}{A_N} \right)^{p-1} \geq - \left( \frac{A_{N-1}}{A_N} \right)^p + (p + 1) \left( \frac{A_N}{A_N} \right)^p. \] (2.9)

We use (2.9) in (2.8) and obtain that
\[ S_N \geq \frac{A_{N-1}}{p + 1} \left( \frac{A_{N-1}}{A_N} \right)^p + \lambda_N \left( \frac{A_N}{A_N} \right)^p - (p + 1) \left( \frac{A_N}{A_N} \right)^p + \left( \frac{A_{N-1}}{p + 1} \right)^p \left( \frac{A_N}{A_N} \right)^p = \left( \frac{A_N}{A_N} \right)^p \left( \lambda_N - \frac{p}{p + 1} \Lambda_N + \Lambda_N \right) = \frac{\Lambda_N}{p + 1} \left( \frac{A_N}{A_N} \right)^p \]
that is (2.6) holds. In this way we derived inductively the proof of our Lemma. \[ \square \]

We consider now the quantity
\[ y = \sum_{n=1}^{N} \lambda_n a_n^{q/p} \left( \frac{A_n}{A_n} \right)^{p-q/p}. \] (2.10)

Then \[ y = \sum_{n=1}^{N} \lambda_n \left[ a_n^{q/p} \left( \frac{A_n}{A_n} \right)^{q-q/p} \right] \left[ \frac{A_n}{A_n} \right]^{-p+q}. \] Suppose that \( p > q \). The case \( p = q \) will be discussed in the end of the proof. Applying Hölder’s inequality now in the above sum with exponents \( r = \frac{p}{q} \) and \( r' = \frac{p}{p-q} \), we have as a consequence that
\[
y \leq \left\{ \sum_{n=1}^{N} \lambda_n a_n \left( \frac{A_n}{A_n} \right)^{-p+1} \right\}^{1-\frac{p}{q}} \left\{ \sum_{n=1}^{N} \lambda_n \left( \frac{A_n}{A_n} \right)^{-p} \right\}^{-\frac{p}{q}} \leq \left\{ \frac{p+1}{p} \sum_{n=1}^{N} \lambda_n \left( \frac{A_n}{A_n} \right)^{-p} - \frac{1}{p} \Lambda_N \left( \frac{A_N}{A_N} \right)^{-p} \right\}^{1-\frac{p}{q}} \left\{ \sum_{n=1}^{N} \lambda_n \left( \frac{A_n}{A_n} \right)^{-p} \right\}^{1-\frac{p}{q}}, \]
(2.11)
in view of Lemma 2.

We set now \( z = \sum_{n=1}^{N} \lambda_n a_n^{-q} \left( \frac{A_n}{A_n} \right)^{-p+q} \) and \( x = \sum_{n=1}^{N} \lambda_n \left( \frac{A_n}{A_n} \right)^{-p} \). Because of (2.11) we have that
\[ y \leq \left\{ \frac{p+1}{p} x - \frac{1}{p} \Lambda_N \left( \frac{A_N}{A_N} \right)^{-p} \right\}^{1-\frac{p}{q}} \cdot x^{1-\frac{p}{q}}. \] (2.12)
By setting now $c = \Lambda_N \left( \frac{4N}{AN} \right)^{-p}$, we have because of (2.13),

$$y \leq \left( \frac{p + 1}{p} x - \frac{c}{p} \right)^{\frac{q}{p}} x^{1-\frac{q}{p}} = \left( \frac{p + 1}{p} \right)^{\frac{q}{p}} \left[ \frac{x - c}{p + 1} \right]^{\frac{q}{p}} x^{1-\frac{q}{p}}. \quad (2.13)$$

Note that by (2.13) the quantity $x - \frac{c}{p+1}$ is positive, that is $x > \frac{c}{p+1}$. Now because of Lemma 1, it is immediate that

$$(p + 1)^{\frac{q}{p}} z + p \left( \frac{p}{p + 1} \right)^{\frac{q}{p}} x^{1-\frac{q}{p}} \geq (p + 1) x \quad \Rightarrow$$

$$(p + 1)^{\frac{q}{p}} z + p \left( \frac{p}{p + 1} \right)^{\frac{q}{p}} \left[ x - \frac{c}{p + 1} \right]^{\frac{q}{p}} x^{1-\frac{q}{p}} =$$

$$= x + \left\{ p x - p \left[ x - \frac{c}{p + 1} \right]^{\frac{q}{p}} x^{1-\frac{q}{p}} \right\} = x + p G(x), \quad (2.14)$$

where $G(x)$ is defined for $x > \frac{c}{p+1}$, by $G(x) = x - \left[ x - \frac{c}{p+1} \right]^{\frac{q}{p}} x^{1-\frac{q}{p}}$.

By (2.14) now we obtain

$$(p + 1)^{\frac{q}{p}} z - x \geq p G(x) \geq p \inf \left\{ G(x) : x > \frac{c}{p + 1} \right\}, \quad (2.15)$$

We will now find the infimum in the above relation. Note that

$$G'(x) = 1 - \left( 1 - \frac{q}{p} \right) x^{-\frac{q}{p}} \left( x - \frac{c}{p + 1} \right)^{\frac{q}{p}} - x^{1-\frac{q}{p}} \left( \frac{q}{p} \right) \left( x - \frac{c}{p + 1} \right)^{\frac{q}{p}-1} =$$

$$= 1 - \left( 1 - \frac{q}{p} \right) \left( 1 - \frac{c}{(p + 1) x} \right)^{\frac{q}{p}} - \frac{q}{p} \left( 1 - \frac{c}{(p + 1) x} \right)^{\frac{q}{p}-1}. \quad (2.16)$$

We consider now the function

$$H(t) = 1 - \left( 1 - \frac{q}{p} \right) t^\frac{q}{p} - \frac{q}{p} t^{\frac{q}{p}-1}, \quad t \in (0, 1).$$

Then $H'(t) = -q/p^2 \left( 1 - \frac{q}{p} \right) \frac{q}{p} (t - 1) > 0$, for every $t \in (0, 1)$. Thus $H(t)$ is strictly increasing $\Rightarrow H(0) \leq H(1) = 0, \forall t \in (0, 1)$. By setting now $t = 1 - \frac{c}{(p + 1)x}$, we conclude that the expression in the right of (2.16) is negative, that is $G'(x) \leq 0, \forall x > \frac{c}{p+1} \Rightarrow G$ is decreasing in $\left( \frac{c}{p+1}, +\infty \right)$. Thus $G(x) \geq \lim_{x \to +\infty} G(x) = \ell$.

Then $\ell = \lim_{x \to +\infty} \left[ x - x^{1-\frac{q}{p}} \left( x - \frac{c}{p + 1} \right)^{\frac{q}{p}} \right] = \lim_{x \to +\infty} \frac{1 - \left( 1 - \frac{c}{(p + 1)x} \right)^{\frac{q}{p}}}{\frac{q}{p}} =$

$$= \left. \lim_{y \to 0^+} \frac{1 - (1 - \frac{c}{y(p+1)})^{\frac{q}{p}}}{\frac{q}{p}} \right|_{y = x} = -\frac{q}{p} \left( -\frac{c}{p+1} \right) = \frac{4c}{p(p+1)}, \text{ by applying the De L'Hospital}$$
rule. Thus we have by (2.15) that 
\[ \left( \frac{p+1}{p} \right)^q z - x \geq p \frac{q c}{p(p+1)} = \frac{q c}{p+1}, \]
which gives inequality (1.8), by the definitions of \( x, z \) and \( c \).

The proof of Theorem 2 in the case \( p > q \) is complete. The case \( p = q \) is also true by continuity reasons, that is by letting \( p \to q^+ \) in (1.8). \( \square \)

**Proof of Theorem 1.**

We first prove the validity of (1.7). We simplify the proof by considering the case where \( a = 0 \) and \( b = 1 \). We consider also the case where \( f : [0, 1] \to \mathbb{R}^+ \) is continuous. The general case for Riemann integrable functions can be handled by using approximation arguments which involve sequences of continuous functions. We suppose that \( \int_0^1 f = \ell \). We define \( F : (0, 1] \to \mathbb{R}^+ \) by
\[ F(t) = \frac{1}{t} \int_0^t f(u) \, du. \]
Then
\[ \int_0^1 \left( \frac{1}{t} \int_0^t f(u) \, du \right)^{-p} \, dt = \int_0^1 (F(t))^{-p} \, dt. \]
The integral above can be approximated by Riemann sums of the following type:
\[ \sum_{n=1}^{2^k} \frac{1}{2^k} (F(n/2^k))^{-p} = \frac{1}{2^k} \sum_{n=1}^{2^k} \left( \frac{\sum_{i=1}^n a_i^{(k)}}{n} \right)^{-p}, \]
where the quantities \( a_i^{(k)} \) are defined as follows:
\[ a_i^{(k)} = 2^k \int_{i/2^k}^{(i+1)/2^k} f. \]
for \( i = 1, \ldots, 2^k \). We use now inequality (1.8). Thus the sum that appears above is less or equal than
\[ \left( \frac{p+1}{p} \right)^q \frac{1}{2^k} \sum_{n=1}^{2^k} \left( \frac{\sum_{i=1}^n a_i^{(k)}}{n} \right)^{-p} - p \frac{q}{p+1} (\sum_{n=1}^{2^k} a_n^{(k)})^{-q} - \frac{q}{p+1} \ell^{-p}. \]

Now we obviously have that \( \sum_{i=1}^{2^k} a_i^{(k)} = \ell \), while since \( f \) is continuous, for every \( n = 1, \ldots, 2^k \) there exists \( b_n^{(k)} \in [\frac{n-1}{2^k}, \frac{n}{2^k}] \), such that \( a_n^{(k)} = f(b_n^{(k)}) \). Thus the quantity that appears above equals
\[ \left( \frac{p+1}{p} \right)^q \frac{1}{2^k} \sum_{n=1}^{2^k} (F(n/2^k))^{-p} q (f(b_n^{(k)}))^{-q} - \frac{q}{p+1} \ell^{-p}. \]

Now, by continuity reasons, and by the choice of \( b_n^{(k)} \), the quantity above approximates
\[ \left( \frac{p+1}{p} \right)^q \frac{1}{2^k} \sum_{n=1}^{2^k} (F(b_n^{(k)}))^{-p} q (f(b_n^{(k)}))^{-q} - \frac{q}{p+1} \ell^{-p}. \]
as \( k \to \infty \). It is clear now that this last quantity approximates the right side of (1.7), as \( k \to \infty \).

We now prove the sharpness of (1.7). Let \( \ell > 0 \) be fixed and \( p \geq q > 0 \).

We consider for any \( a \in \left( -\frac{1}{p}, 0 \right) \) the following function \( g_a(t) = \ell (1 - a) t^{-a} \), \( t \in [0, 1] \). It is easy to see that \( \int_0^1 g_a = \ell, \int_0^t g_a = \frac{1}{1-a} g_a(t) \) for every \( t \in (0, 1] \) and that \( \int_0^1 g_a^{-p} = \ell^{-p(1-a)} \). We consider now the difference

\[
L_a = \int_0^1 \left( \frac{1}{t} \int_0^t g_a \right)^{-p} \, dt - \left( \frac{p+1}{p} \right)^q \int_0^1 \left( \frac{1}{t} \int_0^t g_a \right)^{-p+q} g_a^{-q}(t) \, dt.
\]

It equals to (because of the above properties that \( g_a \) satisfy)

\[
L_a = \ell^{-p} \left[ 1 - (1 - a)^{-q} \left( \frac{p+1}{p} \right)^q \right].
\]

We let \( a \to -\frac{1}{p}^+ \) and we conclude that

\[
\lim_{a \to -\frac{1}{p}^+} L_a = \ell^{-p} q (1 - a)^{-q-1} \left[ (-1) \left( \frac{p+1}{p} \right) \right] = -\frac{q}{p+1} \ell^{-p}.
\]

In this way we derived the sharpness of (1.7).

The proof of Theorem 1 is complete. \( \Box \)

3 Proof of Theorem 3

Let \( \varphi : [0, 1) \to \mathbb{R}^+ \) be non decreasing satisfying the inequality

\[
\left( \frac{1}{t} \int_0^t \varphi \right) \left( \frac{1}{t} \int_0^t \varphi^{-1/(q-1)} \right)^{q-1} \leq M, \quad (3.1)
\]

for every \( t \in (0, 1] \), where \( q \) is fixed such that \( q > 1 \) and \( M > 0 \). We assume also that there exists an \( \varepsilon > 0 \) such that \( \varphi(t) \geq \varepsilon > 0, \forall t \in [0, 1) \). The general case can be handled using this one, by adding a small constant \( \varepsilon > 0 \) to \( \varphi \).

We need the following from [9].

**Lemma A.** Let \( \psi : (0, 1) \to [0, +\infty) \), such that \( \lim_{t \to 0^+} t^{[\psi(t)]^a} = 0 \), where \( a \in \mathbb{R}, a > 1 \) and \( \psi(t) \) is continuous and monotone on \( (0, 1) \). Then the following is true for any \( a \in (0, 1) \).

\[
a \int_0^u \psi^{a-1}(t) [t \psi(t)]^a \, dt = u \psi^a(u) + (a - 1) \int_0^u \psi^{a}(t) \, dt. \quad (3.2)
\]

We refer to [9] for the proof.
We continue the proof of Theorem 3. We set \( h : [0, 1) \to \mathbb{R}^+ \) by \( h(t) = \varphi^{-1/(q-1)}(t) \). Then obviously \( h \) satisfies \( h(t) \leq \varepsilon^{-1/(q-1)} \), \( \forall t \in [0, 1) \). Let also \( p_0 \in [1, q] \) be defined such that

\[
\frac{q - p_0}{q - 1} \left( M p_0 \right)^{1/(q-1)} = 1.
\]

Let also \( p \in (p_0, q] \). Define \( \psi \) by \( \psi(t) = \frac{1}{t} \int_0^t \varphi^{-1/(q-1)} \). Then by Lemma A we get for \( a = \frac{q-1}{p-1} > 1 \), the following:

\[
\frac{q - 1}{p - 1} \int_0^t \varphi^{-1/(q-1)}(s) \left( \frac{1}{s} \int_0^s \varphi^{-1/(q-1)} \right)^{\frac{q-1}{p-1}} ds - \left( \frac{q - p}{p - 1} \right) \int_0^t \left( \frac{1}{s} \int_0^s \varphi^{-1/(q-1)} \right)^{\frac{q - 1}{p - 1}} ds = t \left( \frac{1}{t} \int_0^t \varphi^{-1/(q-1)} \right)^{\frac{q - 1}{p - 1}}. \tag{3.3}
\]

Define for every \( y > 0 \) the following function of the variable of \( x \in [y, +\infty) \)

\[
g_y(x) = \frac{q - 1}{q - p} y x^{(q-p)/(p-1)} - x^{(q-1)/(p-1)}. \tag{3.4}
\]

Then \( g'_y(x) = \frac{q - 1}{p - 1} x^{(q-1)/(p-1)} - 2(y - x) \leq 0 \), \( \forall x \geq y \). Then \( g_y \) is strictly decreasing on \([y, +\infty)\).

So if \( y \leq x \leq w \implies g_y(x) \geq g_y(w) \). For every \( s \in (0, t] \) set now

\[
x = \frac{1}{s} \int_0^s \varphi^{-1/(q-1)}, \quad y = \varphi^{-1/(q-1)}(s), \quad c = M^{1/(q-1)}, \quad \text{and} \quad z = \left( \frac{1}{s} \int_0^s \varphi \right)^{-\frac{q-1}{q-p}}.
\]

Note that by (3.1) the following is true \( y \leq x \leq cz =: w \). Thus

\[
g_y(x) \geq g_y(w) \implies
\]

\[
\frac{q - 1}{q - p} \varphi^{-1/(q-1)}(s) \left( \frac{1}{s} \int_0^s \varphi^{-1/(q-1)} \right)^{\frac{q-1}{p-1}} - \left( \frac{1}{s} \int_0^s \varphi^{-1/(q-1)} \right)^{\frac{q-1}{p-1}} \geq
\]

\[
\geq \frac{q - 1}{q - p} \varphi^{-1/(q-1)}(s) \left( \frac{1}{s} \int_0^s \varphi \right)^{\frac{q-1}{p-1}} - c^{\frac{q-1}{p-1}} \left( \frac{1}{s} \int_0^s \varphi \right)^{\frac{q-1}{p-1}}. \tag{3.5}
\]

Integrating (3.5) on \( s \in (0, t] \) we get

\[
\frac{q - 1}{q - p} \int_0^t \varphi^{-1/(q-1)}(s) \left( \frac{1}{s} \int_0^s \varphi \right)^{-\frac{q-1}{p-1}} ds \cdot c^{\frac{q-1}{p-1}} \leq
\]

\[
\leq \frac{q - 1}{q - p} \int_0^t \varphi^{-1/(q-1)}(s) \left( \frac{1}{s} \int_0^s \varphi^{-1/(q-1)} \right)^{\frac{q-1}{p-1}} ds - \int_0^t \left( \frac{1}{s} \int_0^s \varphi^{-1/(q-1)} \right)^{\frac{q-1}{p-1}} ds + c^{\frac{q-1}{p-1}} \int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-(q-1)/(p-1)} ds \tag{3.6}
\]

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Thus (3.6) gives

\[
\frac{q - 1}{q - p} \int_0^t \varphi^{-1/(q-1)}(s) \left( \frac{1}{s} \int_0^s \varphi^{-1/(q-1)} \right)^{\frac{q-1}{q-1}} ds - \int_0^t \left( \frac{1}{s} \int_0^s \varphi^{-1/(q-1)} \right)^{\frac{q-1}{q-1}} ds
\]

\[
= \frac{p - 1}{q - p} \frac{1}{t^{(q-p)/(p-1)}} \left( \int_0^t \varphi^{-1/(q-1)} \right)^{\frac{q-1}{q-1}}
\]

(3.7)

Thus (3.6) gives

\[
c^{q-1} \frac{q - 1}{q - p} \int_0^t \varphi^{-1/(q-1)}(s) \left( \frac{1}{s} \int_0^s \varphi \right)^{-\frac{1}{q-1}} ds \leq
\]

\[
\leq c^{q-1} \int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(q-1)} ds + \frac{p - 1}{q - p} t \left( \frac{1}{t} \int_0^t \varphi^{-1/(q-1)} \right)^{(q-1)/(p-1)}. \tag{3.8}
\]

But

\[
\left[ \frac{1}{t} \int_0^t \varphi^{-1/(q-1)} \right]^{(q-1)/(p-1)} \leq M^{1/(p-1)} \left( \frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)}
\]

\[
c^{q-1} \frac{q - 1}{q - p} \int_0^t \varphi^{-1/(q-1)}(s) \left( \frac{1}{s} \int_0^s \varphi \right)^{-\frac{1}{q-1}} ds \leq
\]

\[
\leq c^{q-1} \int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(q-1)} ds + \frac{p - 1}{q - p} t M^{1/(p-1)} \left( \frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)} \implies
\]

\[
A_1 := \frac{q - 1}{q - p} \int_0^t \varphi^{-1/(q-1)}(s) \left( \frac{1}{s} \int_0^s \varphi \right)^{-\frac{1}{q-1}} ds \leq
\]

\[
\leq c \int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(q-1)} ds + \frac{p - 1}{q - p} M^{1/(p-1)} t \left( \frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)}. \tag{3.9}
\]

Now by using Theorem II we get

\[
\int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-\frac{1}{q-1}} ds \leq
\]

\[
\left( 1 + \frac{p - 1}{p-1} \right)^{\frac{q-1}{p-1}} \int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-\frac{1}{q-1}} \varphi^{-\frac{1}{q-1}}(s) ds - \frac{1}{1 + \frac{1}{p-1}} t \left( \frac{1}{t} \int_0^t \varphi \right)^{-\frac{1}{q-1}} =
\]

\[
= p \int_0^t A_1 \frac{q - p}{q - 1} \frac{1}{(q - 1) p} t \left( \frac{1}{t} \int_0^t \varphi \right)^{-\frac{1}{q-1}}. \tag{3.10}
\]
Thus in view of (3.10), (3.9) becomes

\[ A_1 \leq cp^{1/(q-1)} A_1 \frac{q-p}{q-1} - c \frac{p-1}{(q-1)p} t \left( \frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)} + \\
+ \frac{p-1}{q-p} \frac{M^{1/(p-1)}}{c(q-p)/(p-1)} \left( \frac{1}{t} \int_0^1 \varphi \right)^{-1/(p-1)} \]

\[ \left[ 1 - cp^{1/(q-1)} \frac{q-p}{q-1} \right] A_1 \leq \left[ \frac{M^{1/(p-1)}}{c(q-p)/(p-1)} \frac{p-1}{q-p} - c \frac{p-1}{(q-1)p} \right] \cdot t \left( \frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)} \]

\[ \implies K(p, q, c) \left[ \frac{1}{t} \int_0^t \varphi^{-1/(p-1)}(s) \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(p-1) + 1/(q-1)} ds \right] \leq \\
\leq \left( \frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)} \left( \frac{p-1}{q-1} \right)^2 c \frac{q}{p} \tag{3.11} \]

where \( K = K(p, q, c) = 1 - cp^{1/(q-1)} \frac{q-p}{q-1} > 0, \forall p \in (p_0, q] \).

As a consequence (3.11) gives

\[ K \left[ \frac{1}{t} \int_0^t \varphi^{-1/(p-1)}(s) \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(p-1) + 1/(q-1)} ds \right] \leq \\
\leq \left( \frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)} \left( \frac{p-1}{q-1} \right)^2 c \frac{q}{p}. \tag{3.12} \]

Now we use the inequality

\[ \frac{1}{t} \int_0^t \varphi^{-1/(p-1)}(s) \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(p-1) + 1/(q-1)} ds \geq \\
\geq \left[ \frac{1/(p-1)}{1 + (1/(p-1))} \right]^{(q-1)/(p-1)} \cdot \frac{1}{t} \int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(p-1)} ds \]

which is true because of Theorem 3. Thus (3.12) gives

\[ \frac{K'}{t} \int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(p-1)} ds \leq \left( \frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)} \left( \frac{p-1}{q-1} \right)^2 c \frac{q}{p} \tag{3.13} \]

where \( K' = \frac{K}{p^{1/(q-1)}}, K = 1 - cp^{1/(q-1)} \frac{q-p}{q-1} \).

Thus the inequality stated in Theorem 3 is proved.

We need to prove the sharpness of (3.13). We consider \( a \) such that \( 0 < a < q-1 \) and the function \( \varphi_a : (0, 1] \to \mathbb{R}^+ \) defined by \( \varphi_a(t) = t^a, t \in (0, 1] \). The function
\( \varphi_a \) is strictly increasing and 
\[
\frac{1}{t} \int_0^t \varphi_a = \frac{1}{t^{a+1}} = \frac{1}{a+1} \varphi_a(t), \ \forall t \in (0, 1],
\]
while 
\[
\int_0^t \varphi_a^{-1/(q-1)} = \frac{1}{1-a/(q-1) t^{1-a/(q-1)}}.
\]
Thus 
\[
\left( \frac{1}{t} \int_0^t \varphi_a \right) \left[ \frac{1}{t} \int_0^t \varphi_a^{-1/(q-1)} \right]^{q-1} = \left[ \frac{q-1}{q-1-a} \right]^{q-1} \int_0^t \varphi_a^{-a/(q-1)}
\]
and 
\[
\left( \frac{1}{t} \int_0^t \varphi_a \right) = \frac{1}{a+1} \left( \frac{q-1}{q-1-a} \right)^{q-1} =: M(q, a)
\]
and 
\[
c_a = c(q, a) = [M(q, a)]^{1/(q-1)} = \left[ \frac{q-1}{(q-1)-a} \right] \frac{1}{(a+1)^{1/(q-1)}}.
\]
Let now \( p \in (p_0, q] \) and suppose additionally that \( a < p-1 \) so that \( \int_0^1 \varphi_a^{-1/(p-1)} = (p-1)/(p-1-a) \). We prove the sharpness of (1.11) for \( t = 1 \). That is we prove that the inequality 
\[
\frac{K'}{t} \int_0^t \left( \frac{1}{s} \int_0^s \varphi \right)^{-1/(p-1)} \mathrm{d}s \leq \left( \frac{1}{t} \int_0^t \varphi \right)^{-1/(p-1)} = \frac{c}{p} \left( \frac{p-1}{q-1} \right)^2
\]
becomes sharp for \( t = 1 \). Obviously if 
\[
I_a = \frac{1}{1-a/(p-1)} \int_0^1 \varphi_a^{-1/(p-1)} = (1+a)^{1/(p-1)} \frac{1}{1-a/(p-1)} \quad \text{while} \quad \left( \int_0^1 \varphi_a \right)^{-1/(p-1)} = \left( \frac{1}{a+1} \right)^{-1/(p-1)}. \]
Thus in order to prove the sharpness of the above inequality we just need to prove that the following is true
\[
\left[ \frac{1}{p^{1/(q-1)}} - \frac{q-p}{q-1} c_a \right] \left( \frac{p-1}{(p-1)-a} \right)^2 \approx c_a \left( \frac{p-1}{q-1} \right)^2 \quad \text{as} \quad a \to (p-1)^- \quad \iff \quad
\left[ \frac{1}{p^{1/(q-1)}} - \frac{q-p}{q-1} \frac{1}{(1+a)^{1/(p-1)}} \left( \frac{q-1}{(q-1)-a} \right) \right] \frac{1}{p} \left( \frac{p-1}{q-1} \right)^{-1/(p-1)} \approx \frac{q}{p} \left( \frac{p-1}{q-1} \right)^2 \left( 1+a \right)^{-1/(p-1)} \left( q-1-a \right)^{-1/(p-1)} \quad \text{as} \quad a \to (p-1)^-.
\]
(3.14)
Let then \( a \to (p-1)^- \) or equivalently \( x := (a+1) \to p^- \). Then for the proof of (3.14) we just need to note that 
\[
\left[ p^{1/(q-1)} - \frac{q-p}{q-x} \frac{1}{x^{1/(q-1)}} \right] \approx \frac{q}{p} \left( \frac{p-1}{q-1} \right)^{-1} \frac{1}{p^{1/(q-1)} q^{-p}} \quad \text{as} \quad x \to p^-,
\]
which is a simple application of De L’Hospital’s rule.
The proof of Theorem 3 is now complete.
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