Abstract

In this paper we obtain a rule for the derivative of the product and quotient of $\psi$-exponential generating functions for a Ward calculus. The main idea is to define a $\mathbb{C}$-algebra of Fontané products defined on a $\psi$-ring of Ward which is compatible with the ordinary calculus and with the $q$-calculus and then obtain the above rules. In addition, we obtain a general Leibniz rule by means of defining a binomial operator which is the analogue of the binomials coefficients.

Keywords: ring of Ward-Fontané, Fontané product, Ward’s calculus, Leibniz’s rule

Mathematics Subject Classification: 11B65, 11B39, 13F25

1 Introduction

Fontané in [1] published a paper in which he generalized the binomial coefficients by replacing $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$, consisting of natural numbers, with $\binom{n}{k}_\psi = \frac{\psi_n \psi_{n-1} \cdots \psi_{n-k+1}}{\psi_1 \psi_2 \cdots \psi_k}$, formed by an arbitrary sequence $\psi = \{\psi_n\}$ of real or complex numbers. He gave a fundamental recurrence relation for these coefficients such that when we make $\psi_n = n$ we recover the ordinary binomial coefficients and when we make $\psi_n = [n] = \frac{q^n - 1}{q - 1}$ we recover the $q$-binomial coefficients studied by Gauss, Euler, Jackson and others.

Subsequently, Ward in [10] developed a calculus on sequences $\psi = \{\psi_n\}$ with $\psi_0 = 0$, $\psi_1 = 1$ and $\psi_n \neq 0$ for all $n \geq 1$, and thus generalized the ordinary calculus and the $q$-calculus of Jackson [4], [5]. From his work came a third much-studied calculus known as the Fibonomial calculus, where $\psi_n = F_n$ is the Fibonacci sequence defined recursively by $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$. For more details on some works on this subject see [2], [3], [7], [8], [9]. However, the latter calculus is not complete, and any other with $\psi_n \neq n$ and $\psi_n \neq [n]$, because they still have neither the derivative of the product of functions nor the chain rule defined. In this paper we will address the problem of constructing a product and quotient rule for a calculus on sequences.
Let $\psi = \{\psi_n\}$ be a sequence of complex numbers with $\psi_0 = 0$, $\psi_1 = 1$ and $\psi_n \neq 0$ for $n \geq 1$. We define the $\psi$-factorial numbers as $\psi_0! = \psi_1 \psi_2 \cdots \psi_n$, where $\psi_0! = 1$. If $\psi_n = [n]$, we obtain the $q$-factorials and if $\psi_n = F_n$, we obtain the $F$-factorials $F_n! = F_1 F_2 \cdots F_n$ with $F_0! = 1$. The $\psi$-binomial coefficients are given by

$$\left( \frac{n}{k} \right)_\psi = \frac{\psi_n}{\psi_k \psi_{n-k}!},$$

which satisfy the symmetry relation

$$\left( \frac{n}{k} \right)_\psi = \left( \frac{n}{n-k} \right)_{\psi^{-1}}, \quad 0 \leq k \leq n. \quad (1)$$

When $\psi_n$ is the Fibonacci sequence, the $\psi_n$-binomial coefficients are known as Fibonomial coefficients.

Now we define the $\psi$-difference

$$\psi_n - \psi_k = F(n, k) \psi_{n-k}$$

where $F(n, k)$ are appropriate complex numbers. When $\psi_n = n$, then $F(n, k) = 1$ and when $\psi_n = [n]$, then $F(n, k) = q^k$. The recurrence relation provided by Fontané [1] is given by

$$\left( \frac{n+1}{k} \right)_\psi = \left( \frac{n}{k-1} \right)_\psi + F(n+1, k) \left( \frac{n}{k} \right)_\psi. \quad (2)$$

If we change $n$ to $n-k$ and apply (1), we will find that

$$\left( \frac{n+1}{k} \right)_\psi = \left( \frac{n}{k} \right)_\psi + F(n+1, n-k+1) \left( \frac{n}{k-1} \right)_\psi \quad (3)$$

is the symmetric version of (2).

On the other hand, if we make $k = n$ in (2), we will obtain the recurrence relation that the sequence $\psi_n$ must fulfill

$$\psi_{n+1} = \psi_n + F(n+1, n).$$

Thus, we have

$$F(n+1, n) = \begin{cases} 1, & \text{si } \psi_n = n; \\ q^n, & \text{si } \psi_n = [n]; \\ F_{n-1}, & \text{si } \psi_n = F_n. \end{cases}$$

Now we define the $\psi$-derivative as $D_\psi x^n = \psi_n x^{n-1}$. When $\psi_n = n$, we get the ordinary derivative $D$ and when $\psi_n = [n]$, we get the $q$-Jackson derivative

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$
and the respective product rules are

\[ D(f(x)g(x)) = Df(x) \cdot g(x) + f(x)Dg(x) \]

and

\[ D_q(f(x)g(x)) = D_qf(x) \cdot g(x) + f(qx)Dg(x) \]
\[ = D_qf(x) \cdot g(qx) + f(x)Dg(x). \]

In [5] was defined the linear operator \( M_q \) given by \( M_q[f(x)] = f(qx) \). Then the \( q \)-derivative of the above product becomes

\[ D_q(f(x)g(x)) = D_qf(x) \cdot g(x) + M_q[f(x)]Dg(x) \]
\[ = D_qf(x)M_q[g(x)] + f(x)Dg(x) \]

and we can notice that the \( q \)-derivative of the product of two functions involves the ordinary product of functions together with the action of \( M_q \). With this in mind we want to construct a family of products of \( \psi \)-exponential generating functions such that when \( \psi_n = [n] \), then such products are defined in terms of the operator \( M_q \).

This article is divided as follows. In the second section we will define the base ring for our constructions, i.e., the \( \psi \)-ring \( W_{\psi,\mathbb{C}}[[x]] \) of \( \psi \)-exponential generating functions. On this ring we will define a family of Fontané products of \( \psi \)-exponential generating functions by using the numbers \( F(p, n, k) \) as the kernel of each product. Subsequently we show that this family of products define an algebra of products, defined on the ring \( W_{\psi,\mathbb{C}}[[x]] \), with coefficients in \( \mathbb{C} \). In the third section we will use the numbers \( F(p, n, k) \) to construct the \( \psi \)-ring opposites of the above. In the fourth section we will give Leibniz’s \( \psi \)-rule for Fontané products of \( \psi \)-exponential generating functions and then generalize it in the next section. Finally, in the last section we will give the \( \psi \)-rule for the quotient of \( \psi \)-exponential generating functions. Throughout this article we will suppose that \( \mathbb{N} \) contains zero.

2 Algebra of products defined over the \( \psi \)-ring of Ward

Now denote \( W_{\psi,\mathbb{C}}[[x]] \) the set of formal power series of the form \( \sum_{n=0}^{\infty} a_n \frac{x^n}{\psi_n!} \) with coefficients in \( \mathbb{C} \). It is clear that \( (W_{\psi,\mathbb{C}}[[x]], +, \cdot) \) is a ring with sum and product of ordinary series, that is,

\[ f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) \frac{x^n}{\psi_n!}, \]

and

\[ f(x) \cdot g(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{a_kb_{n-k}}{\psi_n} \frac{x^n}{\psi_n!}, \]
where \( f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{\psi^n}, \) \( g(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{\psi^n} \in W_{\psi, C}[[x]]. \) The ring \( W_{\psi, C}[[x]] \) will be called \( \psi \)-ring of Ward of \( \psi \)-exponential generating functions. For the case \( \psi_n = n \) the Ward ring reduces to a Hurwitz ring (see [6]). When \( \psi_n = [n], \) then we will call \( W_{q, C}[[x]] \) the \( q \)-ring of Ward and when \( \psi_n = F_n, \) we will call \( W_{F, C}[[x]] \) the \( F \)-ring of Ward or Ward’s Fibonomial ring.

Next we define the Fontané product of \( \psi \)-exponential generating functions.

**Definition 1.** For all \( i, j \in \mathbb{N} \) with \( j < i \) and for all \( f, g \in W_{\psi, C}[[x]] \) we define the product \( *_{i,j} : W_{\psi, C}[[x]] \times W_{\psi, C}[[x]] \to W_{\psi, C}[[x]] \) by

\[
f *_{i,j} g = \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n + i, k + j) \binom{n}{k} a_k b_{n-k} \frac{x^n}{\psi^n}.
\]

(4)

We also define the product \( *_{\infty, \infty} : W_{\psi, C}[[x]] \times W_{\psi, C}[[x]] \to W_{\psi, C}[[x]] \) by

\[
f *_{\infty, \infty} g = fg,
\]

(5)

i.e., \( *_{\infty, \infty} \) is the ordinary product in \( W_{\psi, C}[[x]]. \) We will call the products \( *_{i,j} \) the Fontané products.

When \( \psi_n = n, \) then the Fontané products reduces to the ordinary product of series in \( W_{C}[[x]]. \) Now let \( W_{C} \) denote the ring of sequences with elements in the field \( C \) and let \( \rho_x \) the map \( \rho_x : W_{\psi} \to W_{\psi, C}[[x]] \) defined by

\[\rho_x(\{a_n\}_{n \in \mathbb{N}}) = \sum_{n=0}^{\infty} a_n \frac{x^n}{\psi^n} .\]

In other words, \( \rho_x \) maps the ring of sequences \( W_{C} \) to the ring of \( \psi \)-exponential generating functions \( W_{\psi, C}[[x]]. \) Let \( F_{i,j} \) and \( G_{i,j} \) the sequences

\[F_{i,j} = \{F(n + i, n + j)\}_{n=0}^{\infty}
\]

and

\[G_{i,j} = \{F(n + i, j)\}_{n=0}^{\infty}
\]

and define the maps \( M_{i,j} : W_{\psi, C}[[x]] \to W_{\psi, C}[[x]] \) by \( M_{i,j}(\rho_x(a)) = \rho_x(F_{i,j} \cdot a) \) and \( L_{i,j} : W_{\psi, C}[[x]] \to W_{\psi, C}[[x]] \) by \( L_{i,j}(\rho_x(a)) = \rho_x(G_{i,j} \cdot a) \) for all \( a \in W_{C}, \) where \( \cdot \) is the component-wise product of sequences in \( W_{C}. \) When \( j = 0, G_{i,0} = \{1,1,1,\ldots\}. \) That is, the map \( L_{i,0} \) is the identity map. With this in mind, we show that the set \( W_{\psi, C}[[x]] \) with the Fontané product is a ring.

**Theorem 1.** If \( \psi_n = n, \) then \( (W_{\psi, C}[[x]], +, *_{i,j}) \) is a commutative ring with unit. If \( \psi_n \neq n, \) then the set \( (W_{\psi, C}[[x]], +, *_{i,j}) \) is a non-associative and non-commutative ring. For all \( f(x) \) en \( W_{\psi, C}[[x]] \) it holds that \( f(x) *_{i,j} e = e M_{i,j}(f(x)) \) and \( e *_{i,j} f(x) = e L_{i,j}(f(x)) \) with \( e = \frac{1}{F(0)} \). When \( \psi_n = [n], \) \( e \) is an unit on left-hand side in \( W_{q, C}[[x]]. \) In general, if \( j = 0, \) then \( 1 \) is an unit on the left-hand side in \( W_{\psi, C}[[x]]. \) We will call \( (W_{\psi, C}[[x]], +, *_{i,j}) \) the \( \psi \)-ring of Ward-Fontané with product \( *_{i,j}. \)
Proof. Clearly \((W_{\psi,C}[[x]],+,\cdot,\cdot)\) is an abelian group. If \(\psi_n = n\), then \((W_{\psi,C}[[x]],+,\cdot,\cdot)\) is a commutative ring with unit. Now suppose that \(\psi_n = [n]\). Then \(f(x) \cdot_i \cdot_j g(x) = q^j f(qx)g(x)\). On the one hand

\[
(f(x) \cdot_i \cdot_j g(x)) \cdot_i \cdot_j h(x) = (q^j f(qx)g(x)) \cdot_i \cdot_j h(x) = q^{2j} f(q^2 x)g(qx)h(x)
\]

and on the other hand

\[
f(x) \cdot_i \cdot_j (g(x) \cdot_i \cdot_j h(x)) = f(x) \cdot_i \cdot_j (q^j g(qx)h(x)) = q^{2j} f(qx)g(qx)h(x).
\]

Therefore the product \(\cdot_i \cdot_j\) is not associative. It is also easy to show that \((W_{q,C}[[x]],+,\cdot,\cdot)\) is not commutative. Now, for all \(f(x) \in W_{q,C}[[x]]\) it holds that

\[
f(x) \cdot_i \cdot_j e = e \sum_{n=0}^{\infty} F(n+i,n+j) a_n \frac{x^n}{\psi_n!} = eM_{i,j}(f(x)) = f(qx)
\]

and

\[
e \cdot_i \cdot_j f(x) = e \sum_{n=0}^{\infty} F(n+i,j) a_n \frac{x^n}{\psi_n!} = eL_{i,j}(f(x)) = f(x)
\]

Then \(e\) is an unit on the left. Moreover, it is satisfied that \(f \cdot_i \cdot_j (g + h) = f \cdot_i \cdot_j g + f \cdot_i \cdot_j h\) and \((f + g) \cdot_i \cdot_j h = f \cdot_i \cdot_j h + g \cdot_i \cdot_j h\). Thus \((W_{q,C}[[x]],+,\cdot,\cdot)\) is a ring non-commutative, non-associative with unit on left. We will now do the proof for any sequence other than \([n]\). We will show that the product \(\cdot_i \cdot_j\) is non-associative. On the one hand,

\[
(f \cdot_i \cdot_j g) \cdot_i \cdot_j h = \left( \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n+i,k+j) \binom{n}{k} a_k b_{n-k} \frac{x^n}{\psi_n!} \right) \cdot_i \cdot_j \sum_{n=0}^{\infty} c_n \frac{x^n}{\psi_n!} = \sum_{n=0}^{\infty} \sum_{i,j} F(n+i,l+j) F(n+i,k+j) \binom{n}{l} \binom{l}{k} a_k b_{l-k} c_{n-l} \frac{x^n}{\psi_n!}
\]

and on the other hand

\[
f \cdot_i \cdot_j (g \cdot_i \cdot_j h) = \sum_{n=0}^{\infty} a_n \frac{x^n}{\psi_n!} \cdot_i \cdot_j \left( \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n+i,k+j) \binom{n}{k} b_k c_{n-k} \frac{x^n}{\psi_n!} \right) = \sum_{n=0}^{\infty} \sum_{i,j} F(n+i,l+j) F(n-l+i,k+j) \binom{n}{k} \binom{n-l}{k} a_i b_{l-i} c_{n-l} \frac{x^n}{\psi_n!}
\]

Now it suffices to note that for \(n \geq 1\) the coefficients of \((f \cdot_i \cdot_j g) \cdot_i \cdot_j h\) and \(f \cdot_i \cdot_j (g \cdot_i \cdot_j g)\) do not match. In the same way it is proved that \((W_{\psi,C}[[x]],+,\cdot,\cdot)\) is not commutative. Now, making \(j = 0\) in Eq.(6) and Eq.(7) we obtain

\[
1 \cdot_i \cdot_j f(x) = \sum_{n=0}^{\infty} F(n+i,0) a_n \frac{x^n}{\psi_n!} = f(x)
\]
and
\[ f(x) \ast_{i,0} 1 = \sum_{n=0}^{\infty} F(n + i, n)n! x^n \psi_n! = M_{i,0}(f(x)) \]
and thus 1 is a unit on the left. If \( j \neq 0 \), then there is no unit with respect to the product \( \ast_{i,j} \). Finally, a proof for the distributive property is similar to the one done for the \( \psi = [n] \) case. \( \square \)

**Proposition 1.** For all \( f, g \) in \( W_{\psi, C}[[x]] \) and for all \( \alpha \in C \) we have

1. \( (\alpha f) \ast_{i,j} g = f \ast_{i,j} (\alpha g) = \alpha f \ast_{i,j} g \).
2. If \( \psi_n = [n] \), then \( \alpha \ast_{i,j} f \ast_{i,j} g = \alpha \ast_{i,j} f \ast_{i,j} g \).

**Proof.** The proof of 1 is trivial and this result tells us that the \( \psi \)-ring \( W_{\psi, C}[[x]] \) is a \( C \)-algebra. To prove 2 we will keep in mind that \( (f \ast_{i,j} g) \ast_{i,j} h = q^{2j} f(q^2 x)g(qx)h(x) \) and \( f \ast_{i,j} (g \ast_{i,j} h) = q^{2j} f(qx)g(qx)h(x) \). Then by making \( f(x) = \alpha \) we obtain the desired result. \( \square \)

**Remark 1.** Result 2 in the above proposition does not hold in general for all \( \psi \)-ring \( W_{F, C}[[x]] \). The latter can be checked for the \( F \)-ring \( W_{F, C}[[x]] \). On the other hand, result 1 allows us to obtain a field \( C_\psi \) isomorphic to \( C \), where \( \alpha \ast_{i,j} \beta = F(i, j)\alpha\beta \) with unit \( e = \frac{1}{F(i, j)} \).

If we concatenate Fontané products, we will obtain new products as shown below.

**Definition 2.** For all \( i_1, i_2, j_1, j_2 \) with \( j_1 < i_1, j_2 < i_2 \) and for all \( f, g \) in \( W_{\psi, C}[[x]] \) we define the product \( \ast_{i_1, j_1} \ast_{i_2, j_2} : W_{\psi, C}[[x]] \times W_{\psi, C}[[x]] \rightarrow W_{\psi, C}[[x]] \) by

\[ f \ast_{i_1, j_1} \ast_{i_2, j_2} g = \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n + i_1, k + j_1)F(n + i_2, k + j_2) \binom{n}{k} a_k b_{n-k} \frac{x^n}{\psi_n!}. \]  

(8)

In general

\[ f \left( \prod_{h=1}^{l} \ast_{i_h, j_h} \right) g = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \prod_{h=1}^{l} F(n + i_h, k + j_h) \binom{n}{k} a_k b_{n-k} \frac{x^n}{\psi_n!}. \]  

(9)

with \( i_h, j_h \in N, j_h < i_h \) and \( \prod_{h=1}^{l} \ast_{i_h, j_h} = \ast_{i_{i_1}, j_{i_1}} \cdots \ast_{i_{i_l}, j_{i_l}} \).

Set \( i = (i_1, \ldots, i_l) \) and \( j = (j_1, \ldots, j_l) \). Now define the maps \( M_{i,j} : W_{\psi, C}[[x]] \rightarrow W_{\psi, C}[[x]] \) by \( M_{i,j} = M_{i_1, j_1} \cdots M_{i_l, j_l} \) and \( L_{i,j} : W_{\psi, C}[[x]] \rightarrow W_{\psi, C}[[x]] \) by \( L_{i,j} = L_{i_1, j_1} \cdots L_{i_l, j_l} \), where the product is by composition of maps. The proof of the following theorem is similar to the proof of the theorem \( \square \) and will therefore be omitted.
Theorem 2. If \( \psi_n = n \), then \((W_{\psi,\mathbb{C}}[[x]], +, \prod_{h=1}^{l} *_{i_h,j_h})\) is a commutative ring with unit. If \( \psi_n \neq n \), then the set \((W_{\psi,\mathbb{C}}[[x]], +, \prod_{h=1}^{l} *_{i_h,j_h})\) is a non-associative and non-commutative ring. For all \( f \) from (12) it follows that

\[
f(x) \left( \prod_{h=1}^{l} *_{i_h,j_h} \right) e = e M_{i,j}(f(x))
\]

and

\[
e \left( \prod_{h=1}^{l} *_{i_h,j_h} \right) f(x) = e L_{i,j}(f(x))
\]

with \( e = \frac{1}{\prod_{h=1}^{l} F(i_h,j_h)} \). When \( \psi_n = [n] \), \( e \) is an unit on left-hand side in \( W_{\psi,\mathbb{C}}[[x]] \). In general, if \( j_h = 0 \) for \( 1 \leq h \leq l \), then \( 1 \) is an unit on the left-hand side in \( W_{\psi,\mathbb{C}}[[x]] \). We will call \((W_{\psi,\mathbb{C}}[[x]], +, \prod_{h=1}^{l} *_{i_h,j_h})\) the \( \psi \)-ring of Ward-Fontané with product \( \prod_{h=1}^{l} *_{i_h,j_h} \).

We conclude this section by constructing a \( \mathbb{C} \)-algebra of products defined on the set \( W_{\psi,\mathbb{C}}[[x]] \). We will note that the structure of this \( \mathbb{C} \)-algebra depends on the nature of the sequence \( \psi \).

Theorem 3. \((\mathbb{C}_{\psi}\{*_{i,j}\}, \boxplus, \cdot)\), \( j < i \), is an \( \mathbb{C} \)-algebra of products \( *_{i,j} \) defined over the \( \psi \)-ring of Ward \( W_{\psi}[[x]] \) with sum \( \boxplus \) formally defined as

\[
f(*_{i_1,j_1} \boxplus *_{i_2,j_2}) g = f *_{i_1,j_1} g + f *_{i_2,j_2} g, \quad (10)
\]

product \( \cdot \) given by concatenation, i.e.

\[
*_{i_1,j_1} \cdot *_{i_2,j_2} = *_{i_1,j_1} *_{i_2,j_2}, \quad (11)
\]

and scalar product

\[
f(\alpha *_{i,j}) g = (\alpha f) *_{i,j} g = f *_{i,j} (\alpha g), \quad (12)
\]

for all \( f, g \in W_{\psi,\mathbb{C}}[[x]] \) and all \( \alpha \in \mathbb{C} \).

Proof. From (12) it follows that \( f(0 *_{i,j}) g = 0 \). Now denote \( *_0 = 0 *_{i,j} \) to then show that \( *_0 \) is the neutral element in \( \mathbb{C}_{\psi}\{*_{i,j}\} \). This follows because

\[
f(*_{i,j} \boxplus *_0) g = f *_{i,j} g + f *_0 g = f *_{i,j} g + f(0 *_{k,l}) g = f *_{i,j} g + 0 = f *_{i,j} g
\]

and thus \( *_{i,j} \boxplus *_0 = *_0 \boxplus *_{i,j} = *_{i,j} \). With the above given, we can show the existence of neutral elements in \( \mathbb{C}_{\psi}\{*_{i,j}\} \). We define \( \boxplus \) as \( *_{i_1,j_1} \boxplus *_{i_2,j_2} = *_{i_1,j_1} \boxplus (-*_{i_2,j_2}) \). Then

\[
f(*_{i,j} \boxplus *_{i,j}) g = f *_{i,j} g - f *_{i,j} g = 0 = f *_0 g
\]
for all \( f, g \in W_{\psi,C}[[x]] \). Then \( *_{i,j} \square *_{i,j} = *_{0} \) so \( *_{i,j} \) and \(-*_i{}^j_{\cdot}\) are additive inverses in \( C_{i,j} \). It is easily shown that the sum \( \square \) is commutative. Then the set \((C_{\psi}\{*_{i,j}\}, \square)\) is an abelian group. On the other hand, it follows easily from (11) that

\[
(*_{i,j} *_{i_2,j_2}) *_{i_3,j_3} = *_{i_1,j_1} (*_{i_2,j_2} *_{i_3,j_3})
\]

and the concatenation is associative. In addition, it is easy to notice that the product \( *_{\infty, \infty} \) is the unit in \( C_{\psi}\{*_{i,j}\} \), because

\[
f *_{\infty, \infty} *_{i,j} g = f *_{i,j} *_{\infty, \infty} g = f *_{i,j} g,
\]

and from here \( *_{\infty, \infty} *_{i,j} = *_{i,j} *_{\infty, \infty} = *_{i,j} \). The commutativity is evident. Finally we have that

\[
*_{i,j} (*_{k,l} \square *_{r,s}) = *_{i,j} *_{k,l} \square *_{r,s}
\]

and therefore \( C_{\psi}\{*_{i,j}\} \) is a commutative \( C \)-algebra.

When \( \psi_n = n \), then \( C_{\psi}\{*_{i,j}\} = \{*_{\infty, \infty}\} \), i.e., the algebra \( C_{\psi}\{*_{i,j}\} \) reduces to the ordinary product in \( W_C[[x]] \). When \( \psi_n = [n] \) we have that \( f(x) *_{i,j} g(x) = q^i f(qx)g(x) \) and it is noted that this product does not depend on \( i \). Then

\[
\begin{align*}
f *_{1,0} g &= f *_{2,0} g = f *_{3,0} = \cdots = f(qx)g(x) \\
f *_{2,1} g &= f *_{3,1} g = f *_{4,1} = \cdots = qf(qx)g(x) \\
f *_{3,2} g &= f *_{4,2} g = f *_{5,2} = \cdots = q^2 f(qx)g(x) \\
&\vdots
\end{align*}
\]

for all \( f, g \in W_{q,C}[[x]] \) and a fixed \( j \). Where \( (W_{q,C}[[x]], +, *_{r,j}) \) is isomorphic to \((W_{q,C}[[x]], +, *_{s,j})\). Likewise

\[
\left(W_{q,C}[[x]], +, \prod_{i=1}^{l} *_{a_i,b_i}\right) \cong \left(W_{q,C}[[x]], +, \prod_{i=1}^{l} *_{c_i,d_i}\right)
\]

provided that \( b_1 + \cdots + b_l = d_1 + \cdots + d_l \). On the other hand, when \( \psi_n \neq n \) and \( \psi_n \neq [n] \) we have that \((W_{\psi,C}[[x]], +, *_{a,b}) \neq (W_{\psi,C}[[x]], +, *_{c,d})\) for all \( a, b, c, d \in \mathbb{N} \) with \( a > b \) and \( c > d \).

### 3 The opposite of the \( \psi \)-rings of Ward-Fontané

In this section we will construct the opposite \( \psi \)-rings of Ward-Fontané, where this time we will use the equation (13) for the definition of the Fontané products.

**Definition 3.** For all \( i, j \) with \( j < i \) and for all \( f, g \in W_{\psi,C}[[x]] \) we define the product \( *_{i,j} : W_{\psi,C}[[x]] \times W_{\psi,C}[[x]] \to W_{\psi,C}[[x]] \) as

\[
f *_{i,j} g = \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n + i, n - k + j) \binom{n}{k} a_k b_{n-k} x^n \psi_n!
\]

We will call \( *_{i,j} \) the opposite Fontané product.

[1]: #8
Theorem 4. For all \(i_1, i_2, j_1, j_2 \in \mathbb{N}\) with \(j_1 < i_1, j_2 < i_2\) and for all \(f, g \in W_{\psi, \mathbb{C}}[[x]]\) we define the product \(*_{i_1,j_1 \cdot i_2,j_2} : W_{\psi, \mathbb{C}}[[x]] \times W_{\psi, \mathbb{C}}[[x]] \to W_{\psi, \mathbb{C}}[[x]]\) as

\[
f *_{i_1,j_1 \cdot i_2,j_2} g = \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n + i_1, n - k + j_1) F(n + i_2, n - k + j_2) \binom{n}{k} \psi a_k b_{n-k} x^n \psi_n!
\]

In general, we define by concatenation the product \(\prod_{i=1}^{l} *_{a_i, b_i}\) as

\[
f \left( \prod_{i=1}^{l} *_{a_i, b_i} \right) g = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \prod_{i=1}^{l} F(n + a_i, n - k + b_i) \binom{n}{k} \psi a_k b_{n-k} x^n \psi_n!
\]

with \(a_i, b_i \in \mathbb{N}\), \(b_i < a_i\) and \(\prod_{i=1}^{l} *_{a_i, b_i} = *_{a_1, b_1} \cdots *_{a_l, b_l}\).

The following are the symmetric versions of the theorems 1, 2 and 3.

Theorem 4. If \(\psi_n = n\), then \((W_{\psi, \mathbb{C}}[[x]], +, *_{i,j})\) is a commutative ring with unit. If \(\psi_n \neq n\), then the set \((W_{\psi, \mathbb{C}}[[x]], +, *_{i,j})\) is a non-associative and non-commutative ring. For all \(f(x) \in W_{\psi, \mathbb{C}}[[x]]\) it holds that \(f(x) *_{i,j} e = e \lambda_{i,j}(f(x))\) and \(e *_{i,j} f(x) = e \mu_{i,j}(f(x))\) with \(e = \frac{1}{F(i,j)}\). When \(\psi_n = [n]\), \(e\) is an unit on right-hand side in \(W_{q, \mathbb{C}}[[x]]\). In general, if \(j = 0\) then \(1\) is an unit on the right-hand side in \(W_{\psi, \mathbb{C}}[[x]]\). We will call \((W_{\psi, \mathbb{C}}[[x]], +, *_{i,j})\) the opposite \(\psi\)-ring of Ward-Fontané with product \(*_{i,j}\).

Theorem 5. If \(\psi_n = n\), then \((W_{\psi, \mathbb{C}}[[x]], +, \prod_{h=1}^{l} *_{i_h,j_h})\) is a commutative ring with unit. If \(\psi_n \neq n\), then the set \((W_{\psi, \mathbb{C}}[[x]], +, \prod_{h=1}^{l} *_{i_h,j_h})\) is a non-associative and non-commutative ring. For all \(f(x) \in W_{\psi, \mathbb{C}}[[x]]\) it holds that

\[
f(x) \left( \prod_{h=1}^{l} *_{i_h,j_h} \right) e = e \lambda_{i,j}(f(x))
\]

and

\[
e \left( \prod_{h=1}^{l} *_{i_h,j_h} \right) f(x) = e \mu_{i,j}(f(x))
\]

with \(e = \frac{1}{\prod_{h=1}^{l} F(i_h,j_h)}\). When \(\psi_n = [n]\), \(e\) is an unit on right-hand side in \(W_{q, \mathbb{C}}[[x]]\). In general, if \(j_h = 0\) for \(1 \leq h \leq l\), then \(1\) is an unit on the right-hand side in \(W_{\psi, \mathbb{C}}[[x]]\). We will call \((W_{\psi, \mathbb{C}}[[x]], +, \prod_{h=1}^{l} *_{i_h,j_h})\) the opposite \(\psi\)-ring of Ward-Fontané with product \(\prod_{i=1}^{l} *_{a_i, b_i}\).

Theorem 6. \((\mathbb{C}_\psi, \{ *_{i,j} \}, \oplus, \cdot\) is an \(\mathbb{C}\)-algebra of products \(*_{i,j}\) defined over \(W_{\psi, \mathbb{C}}[[x]]\) with sum \(\oplus\) formally defined by

\[
f(\otimes_{i_1,j_1} \oplus \otimes_{i_2,j_2}) g = f \otimes_{i_1,j_1} g + f \otimes_{i_2,j_2} g.
\]

product \(\cdot\) given by concatenation, i.e.

\[
\otimes_{i_1,j_1} \cdot \otimes_{i_2,j_2} = \otimes_{i_1,j_1 \cdot i_2,j_2}
\]
and scalar product
\[ f(\alpha \ast_{i,j} g) = (\alpha f) \ast_{i,j} g = f \ast_{i,j} (\alpha g) \]
for all \( f, g \in W_{\psi, C}[[x]] \) and all \( \alpha \in \mathbb{C} \).

It is clear that \( f \prod_{i=1}^{l} a_{i,b_{i}} g = g \prod_{i=1}^{l} a_{i,b_{i}} f \) for all \( f, g \in W_{\psi, C}[[x]] \). Then the rings \((W_{\psi,C}[[x]],+,\prod_{i=1}^{l} a_{i,b_{i}})\) and \((W_{\psi,C}[[x]],+,\prod_{i=1}^{l} a_{i,b_{i}})\) are opposite rings and therefore isomorphic.

4 A rule for the \( \psi \)-derivative of the product of \( \psi \)-exponential generating functions

In this section we will find the \( \psi \)-derivative of the \( C \psi \)-products of \( \psi \)-exponential generating functions in \( W_{\psi, C}[[x]] \). In particular we will compute the \( \psi \)-derivative of the ordinary product of functions from which the ordinary case and the \( q \)-analogue are deduced.

**Theorem 7.** Take \( f, g \in W_{\psi, C}[[x]] \). Then the \( \psi \)-derivative of the product \( f \ast_{i,j} g \) on the set \( W_{\psi, C}[[x]] \) is given by
\[
D_{\psi}(f \ast_{i,j} g) = D_{\psi}f \ast_{i+1,j+1} g + f \ast_{i+1,j} *_{1,0} D_{\psi}g. \tag{16}
\]

**Proof.** Set \( f(x) = \sum_{n=0}^{\infty} a_{n}(t^{n}/\psi_{n}!) \) and \( g(x) = \sum_{n=0}^{\infty} b_{n}(t^{n}/\psi_{n}!) \). Then
\[
D_{\psi}(f(x) \ast_{i,j} g(x)) = D_{\psi} \left( \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n+i,k+j) \binom{n}{k} a_{k} b_{n-k} \frac{x^{n}}{\psi_{n}!} \right)
= \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} F(n+i+1,k+j) \binom{n+1}{k} a_{k} b_{n+1-k} \frac{x^{n}}{\psi_{n}!}.
\]

We take series for \( k = 0 \) and for \( k = n+1 \) from above sum
\[
D_{\psi}(f(x) \ast_{i,j} g(x)) = \sum_{n=0}^{\infty} F(n+i+1,j) a_{0} b_{n+1} \frac{x^{n}}{\psi_{n}!}
+ \sum_{n=0}^{\infty} \sum_{k=1}^{n} F(n+i+1,k+j) \binom{n+1}{k} a_{k} b_{n+1-k} \frac{x^{n}}{\psi_{n}!}
+ \sum_{n=0}^{\infty} F(n+i+1,n+1+j) a_{n+1} b_{0} \frac{x^{n}}{\psi_{n}!}.
\]

Now we will use the equation (2) to separate the sum of the middle into two
summands

\[ D_\psi(f(x) *_{i,j} g(x)) = \sum_{n=0}^{\infty} F(n + i + 1, j) a_0 b_{n+1} \frac{x^n}{\psi_n!} + \]
\[ \sum_{n=0}^{\infty} \sum_{k=1}^{n} F(n + i + 1, k + j) \left[ \binom{n}{k-1}_\psi + F(n + 1, k) \binom{n}{k}_\psi \right] a_k b_{n+1-k} \frac{t^n}{\psi_n!} \]
\[ + \sum_{n=0}^{\infty} F(n + i + 1, n + 1 + j) a_{n+1} b_0 \frac{x^n}{\psi_n!}. \]

Then

\[ D_\psi(f(x) *_{i,j} g(x)) = \sum_{n=0}^{\infty} F(n + i + 1, j) a_0 b_{n+1} \frac{x^n}{\psi_n!} \]
\[ + \sum_{n=0}^{\infty} \sum_{k=1}^{n} F(n + i + 1, k + j) \left( \binom{n}{k-1}_\psi \right) a_k b_{n+1-k} \frac{t^n}{\psi_n!} \]
\[ + \sum_{n=0}^{\infty} \sum_{k=1}^{n} F(n + i + 1, k + j) F(n + 1, k) \binom{n}{k}_\psi a_k b_{n+1-k} \frac{t^n}{\psi_n!} \]
\[ + \sum_{n=0}^{\infty} F(n + i + 1, n + 1 + j) a_{n+1} b_0 \frac{x^n}{\psi_n!}. \]

Now we rearrange the second sum for \( k \)

\[ D_\psi(f(x) *_{i,j} g(x)) = \sum_{n=0}^{\infty} F(n + i + 1, j) a_0 b_{n+1} \frac{x^n}{\psi_n!} \]
\[ + \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} F(n + i + 1, k + j + 1) \binom{n}{k}_\psi a_{k+1} b_{n-k} \frac{t^n}{\psi_n!} \]
\[ + \sum_{n=0}^{\infty} \sum_{k=1}^{n} F(n + i + 1, k + j) F(n + 1, k) \binom{n}{k}_\psi a_k b_{n+1-k} \frac{t^n}{\psi_n!} \]
\[ + \sum_{n=0}^{\infty} F(n + i + 1, n + 1 + j) a_{n+1} b_0 \frac{x^n}{\psi_n!}. \]

Finally, we join the first sum with the third sum and the second sum with the fourth sum

\[ D_\psi(f(x) *_{i,j} g(x)) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n + i + 1, k + j + 1) \binom{n}{k}_\psi a_{k+1} b_{n-k} \frac{t^n}{\psi_n!} \]
\[ + \sum_{n=0}^{\infty} \sum_{k=0}^{n} F(n + i + 1, k + j) F(n + 1, k) \binom{n}{k}_\psi a_k b_{n+1-k} \frac{t^n}{\psi_n!} \]
\[ = D_\psi f(x) *_{i+1,j+1} g(x) + f(x) *_{i+1,j} *_{1,0} D_\psi g(x). \]

In this way we reach the desired result. \( \square \)
For the opposite product \(*_{i,j}\) we obtain the following result

**Theorem 8.** Take \(f, g \in W_{\psi,C}[[x]]\). Then the \(\psi\)-derivative of the product \(f \ast_{i,j} g\) in the set \(W_{\psi,C}[[x]]\) is given by

\[
D_\psi(f \ast_{i,j} g) = D_\psi f \ast_{i+1,j} \ast_{1,0} g + f \ast_{i+1,j+1} D_\psi g.
\] (17)

Now we will obtain the \(\psi\)-derivative of the ordinary product of functions

**Theorem 9.** Take \(f, g \in W_{\psi,C}[[x]]\). Then the \(\psi\)-derivative \(D_\psi\) of the product \(fg\) is

\[
D_\psi(fg) = f(1) \ast_{1,0} D_\psi g(x) + D_\psi f(x) \cdot g(x)
\] (18)

and

\[
D_\psi(fg) = D_\psi f(x) \ast_{1,0} g(x) + f(x)D_\psi g(x).
\] (19)

**Proof.** We use a proof similar to the proof of the theorem.

The \(\psi\)-derivative of the product of functions is well defined as we can notice by making \(g(x) = 1\) and then interchanging \(f\) with \(g\).

\[
D_\psi(f(x)) = D_\psi(f(x) \cdot 1) = f(x) \ast_{1,0} 0 + D_\psi f(x) = D_\psi f(x)
\]

and

\[
D_\psi(f(x)) = D_\psi(1 \cdot f(x)) = 1 \ast_{1,0} D_\psi f(x) = D_\psi f(x).
\]

On the other hand, when \(\psi_n = n\), then we obtain Leibniz’s ordinary rule for the product of functions

\[
D(f(x)g(x)) = Df(x) \cdot g(x) + f(x)Dg(x).
\]

and when \(\psi_n = [n]\), then we obtain the \(q\)-rule of Leibniz

\[
D_q(f(x)g(x)) = D_qf(x) \cdot g(x) + f(x) \ast_{1,0} D_qg(x)
\]

and by symmetry

\[
D_q(f(x)g(x)) = f(x)D_qg(x) + D_qf(x) \ast_{1,0} g(x)
\]

and

\[
D_q(f(x)g(x)) = f(x)D_qg(x) + D_qf(x)g(qx)
\]

thus reaching the two known results for the \(q\)-derivative of the product of functions. When \(\psi_n = F_n\), we obtain the \(F\)-analogue of Leibniz’s rule

\[
D_F(f(x)g(x)) = D_F f(x) \cdot g(x) + f(x) \ast_{1,0} D_F g(x)
\]

and

\[
D_F(f(x)g(x)) = f(x)D_F g(x) + D_F f(x) \ast_{1,0} g(x).
\]
Example 1. We will use the $F$-rule of product to show that $D_F(x^{n+m}) = F_{n+m}x^{n+m-1}$. On the one hand, using the product $*:1,0$ we have

$$D_F(x^{n+m}) = D_F(x^n x^m) = x^n *_{1,0} F_m x^{m-1} + F_n x^{n+m-1} = (F_m F(n + m, n) + F_n)x^{n+m-1} = (F_m \frac{F_{n+m} - F_n}{F_m} + F_n)x^{n+m-1} = F_{n+m} x^{n+m-1}.$$ 

On the other hand, with the product $*:1,0$ we obtain

$$D_F(x^{n+m}) = D_F(x^n x^m) = F_n x^{n-1} *_{1,0} x^m + F_m x^{n+m-1} = (F_n F(n + m, m) + F_m)x^{n+m-1} = (F_n \frac{F_{n+m} - F_m}{F_n} + F_m)x^{n+m-1} = F_{n+m} x^{n+m-1}.$$ 

Now we will give Leibniz’s rule for the products $\prod_{i=1}^l *_{a_i, b_i}$ and $\prod_{i\leq j} *_{a_i, b_j}$ of functions

**Theorem 10.** Take $f, g$ in $W_\psi, \mathbb{C}[[x]]$. Then

$$D_\psi \left( f \prod_{i=1}^l *_{a_i, b_i} g \right) = D_\psi f \left( \prod_{i=1}^l *_{a_i, b_i + 1} \right) g + f \left( \prod_{i=1}^l *_{a_i, b_i, *_{1,0}} \right) D_\psi g,$$

$$D_\psi \left( f \prod_{i=1}^l *_{a_i, b_i} g \right) = f \left( \prod_{i=1}^l *_{a_i, b_i + 1} \right) D_\psi g + D_\psi f \left( \prod_{i=1}^l *_{a_i, b_i, *_{1,0}} \right) g.$$ 

**Proof.** The proof is analogous to that given for the theorem \[Q.E.D.\]

When $\psi_n = \lfloor n \rfloor$ we get the following result

$$D_q \left( f(x) \prod_{i=1}^l *_{a_i, b_i} g(x) \right) = D_q \left( q^{b_1 + \cdots + b_l} f(q^l x) g(x) \right) = q^{b_1 + \cdots + b_l + 1} D_q f(q^l x) \cdot g(x) + q^{b_1 + \cdots + b_l} D_q g(x).$$ 

Then

$$D_q(f(q^l x) g(x)) = q^l D_q f(q^l x) \cdot g(x) + f(q^l x) D_q g(x).$$ 

If we make $g(x) = 1$, we get

$$D_q f(q^l x) = q^l (D_q f)(q^l x)$$

which is the chain rule for the function $f(q^l x)$. We ended with the following result.
Theorem 11. Take \( f, g \in W_{\psi,C}[[x]] \). Then
\[
D_{\psi}(f \ast_{i_1,j_1} \ast_{i_2,j_2} g) = D_{\psi}f \ast_{i_1+1,j_1+1} \ast_{i_2+1,j_2+1} g + f \ast_{i_1+1,j_1} \ast_{i_2+1,j_2} \ast_{1,0} D_{\psi}g;
\]
\[
D_{\psi}(f \ast_{i_1,j_1} \ast_{i_2,j_2} g) = f \ast_{i_1+1,j_1+1} \ast_{i_2+1,j_2+1} D_{\psi}g + D_{\psi}f \ast_{i_1+1,j_1} \ast_{i_2+1,j_2} \ast_{1,0} g.
\]

5 Leibniz’s general \( \psi \)-rule

In this section we will find the \( n \)-th \( \psi \)-derivative of the ordinary product of \( \psi \)-exponential generating functions in \( W_{\psi,C}[[x]] \). For the ordinary derivative it is known that Leibniz’s general rule is
\[
D(f(x)g(x)) = \sum_{k=0}^{n} \binom{n}{k} D^k f(x) D^{n-k} g(x)
\]
and the \( q \)-analogue of this rule is
\[
D_{q}(f(x)g(x)) = \sum_{k=0}^{n} \binom{n}{k}_q D^k_{q} f(q^k x) D^{n-k}_q g(x).
\]

The idea of our construction of a general product \( \psi \)-rule is to use an analogous of binomial coefficients defined on \( C \{ *_{i,j} \} \) such that we can recover the above identities.

Definition 5. We define the binomial operator \( \binom{n}{k}_q \) as the operator \( \binom{n}{k}_q : W_{\psi,C}[[x]] \times W_{\psi,C}[[x]] \to W_{\psi,C}[[x]] \) satisfying
\[
\begin{align*}
\binom{n}{0}_q &= 0, \\
\binom{n}{k}_q &= \rho \binom{n-1}{k-1}_q \oplus \sigma \binom{n-1}{k-1}_q, \quad 1 \leq k \leq n-1, \ n \geq 2, \\
\binom{n}{n}_q &= \prod_{i=1}^{n} *_{i,0}, \ n \geq 1,
\end{align*}
\]
where \( \rho, \sigma : C \{ *_{i,j} \} \to C \{ *_{i,j} \} \) are linear maps given by
\[
\rho(*_{a_1,b_1} *_{a_2,b_2} \cdots *_{a_l,b_l}) = *_{a_1+1,b_1+1} *_{a_2+1,b_2+1} \cdots *_{a_l+1,b_l+1}, \\
\sigma(*_{a_1,b_1} *_{a_2,b_2} \cdots *_{a_l,b_l}) = *_{a_1+1,b_1} *_{a_2+1,b_2} \cdots *_{a_l+1,b_l} *_{1,0}.
\]

In Table 1 we can see some values of binomial operators.
Table 1: Analog of Pascal’s Triangle

| $n$ | 0   | 1     | 2       | 3       |
|-----|-----|-------|---------|---------|
| 0   | *$x$,$x$ |   |   |  |
| 1   | *$x$,$x$ | *$1,0$ |   |   |
| 2   | *$x$,$x$ | *$1,0$⊕*$2,1$ | *$1,0$*$2,0$ |   |
| 3   | *$x$,$x$ | *$1,0$*$2,1$*$3,2$ | *$1,0$*$2,0$⊕*$1,0$*$3,1$⊕*$2,1*$3,1$ | *$1,0$*$2,0$*$3,0$ |

If we compare the above definition with the theorem [10], we notice that the maps $\rho$ and $\sigma$ are related to the $\psi$-derivative of the product of $\psi$-exponential power series. In the following theorem we will give the $\psi$-rule general of Leibniz.

**Theorem 12.** Take $f, g \in W_{\psi}[x]$. Then the $\psi$-rule general of Leibniz is

$$D^\psi_n(f(x)g(x)) = \sum_{k=0}^{n} \left< \frac{n}{k} \right> (D^\psi_{n-k}f(x), D^\psi_kg(x)) \quad (23)$$

**Proof.** The proof will be by induction. Assume true for $n$ and let us compute the $\psi$-derivative of $D^\psi_n(f(x)g(x))$. Applying the Theorem [10] to the product $\left< \frac{n}{k} \right> (D^\psi_{n-k}f(x), D^\psi_kg(x))$ we obtain

$$D^\psi_{n+1}(f(x)g(x)) = D^\psi \left( \sum_{k=0}^{n} \left< \frac{n}{k} \right> (D^\psi_{n-k}f(x), D^\psi_kg(x)) \right)$$

$$= D^\psi \left( \sum_{k=0}^{n} D^\psi_{n-k}f(x) \left< \frac{n}{k} \right> D^\psi_kg(x) \right)$$

$$= \sum_{k=0}^{n} \left( D^\psi_{n+1-k}f(x) \rho \left< \frac{n}{k} \right> D^\psi_kg(x) \right.$$

$$\left. + D^\psi_{n-k}f(x) \sigma \left< \frac{n}{k} \right> D^\psi_{k+1}g(x) \right)$$

$$= \sum_{k=0}^{n} D^\psi_{n+1-k}f(x) \rho \left< \frac{n}{k} \right> D^\psi_kg(x)$$

$$+ \sum_{k=0}^{n} D^\psi_{n-k}f(x) \sigma \left< \frac{n}{k} \right> D^\psi_{k+1}g(x).$$

Now we rewrite the second series for $k$ and draw summands from the two series,
i.e,

\[ D^{n+1}_\psi(f(x)g(x)) = \sum_{k=0}^{n} D^{n+1-k}_\psi f(x) \rho \left\langle \frac{n}{k} \right\rangle_\ast D^k_\psi g(x) \]

\[ + \sum_{k=1}^{n+1} D^{n+1-k}_\psi f(x) \sigma \left\langle \frac{n}{k-1} \right\rangle_\ast D^k_\psi g(x) \]

\[ = D^{n+1}_\psi f(x) \rho \left\langle \frac{n}{0} \right\rangle_\ast g(x) \]

\[ + \sum_{k=1}^{n} D^{n+1-k}_\psi f(x) \rho \left\langle \frac{n}{k} \right\rangle_\ast D^k_\psi g(x) \]

\[ + \sum_{k=1}^{n} D^{n+1-k}_\psi f(x) \sigma \left\langle \frac{n}{k-1} \right\rangle_\ast D^k_\psi g(x) \]

\[ + f(x) \sigma \left\langle \frac{n}{n} \right\rangle_\ast D^{n+1}_\psi g(x). \]

As

\[ \rho \left\langle \frac{n}{0} \right\rangle_\ast = \rho(\ast x, x) = \left\langle \frac{n+1}{0} \right\rangle_\ast \]

and

\[ \sigma \left\langle \frac{n}{n} \right\rangle_\ast = \sigma \left( \prod_{i=1}^{n} \ast_{i,0} \right) = \prod_{i=1}^{n} \ast_{i+1,0} \ast_{1,0} = \left\langle \frac{n+1}{n+1} \right\rangle_\ast, \]

then

\[ D^{n+1}_\psi(f(x)g(x)) = D^{n+1}_\psi f(x) \left\langle \frac{n+1}{0} \right\rangle_\ast g(x) \]

\[ + \sum_{k=1}^{n} D^{n+1-k}_\psi f(x) \rho \left\langle \frac{n}{k} \right\rangle_\ast D^k_\psi g(x) \]

\[ + \sum_{k=1}^{n} D^{n+1-k}_\psi f(x) \sigma \left\langle \frac{n}{k-1} \right\rangle_\ast D^k_\psi g(x) \]

\[ + f(x) \left\langle \frac{n+1}{n+1} \right\rangle_\ast D^{n+1}_\psi g(x) \]

\[ = \left\langle \frac{n+1}{0} \right\rangle_\ast (D^{n+1}_\psi f(x), g(x)) \]

\[ + \sum_{k=1}^{n} \left( \rho \left\langle \frac{n}{k} \right\rangle_\ast \sigma \left\langle \frac{n}{k-1} \right\rangle_\ast \right) (D^{n+1-k}_\psi f(x), D^k_\psi g(x)) \]

\[ + \left\langle \frac{n+1}{n+1} \right\rangle_\ast (f(x), D^{n+1}_\psi g(x)). \]
Then by definition of \( \langle n \rangle_k \) we get to
\[
D^{n+1}_\psi(f(x)g(x)) = \langle n + 1 \rangle_* \left( D^{n+1}_\psi f(x), g(x) \right) \\
+ \sum_{k=1}^{n} \langle n + 1 \rangle_k \left( D^{n+1-k}_\psi f(x), D^k_\psi g(x) \right) \\
+ \langle n + 1 \rangle \left( f(x), D^{n+1}_\psi g(x) \right)
\]
\[
= \sum_{k=0}^{n+1} \langle n + 1 \rangle_k \left( D^{n+1-k}_\psi f(x), D^k_\psi g(x) \right)
\]
as it was intended to be shown. \( \Box \)

In the following propositions we will find special values of \( \langle n \rangle_k \). The proofs will be done by induction on \( n \).

**Proposition 2.** For all \( n \geq 1 \)
\[
\langle n \rangle_1 = \bigoplus_{i=0}^{n-1} *_{i+1,i}.
\] (24)

**Proof.** Suppose that \( \langle n \rangle_1 = \bigoplus_{i=0}^{n-1} *_{i+1,i} \). Using the Definition 5 we obtain.
\[
\langle n + 1 \rangle_1 (f, g) = \left( \sigma \langle n \rangle_0 \bigoplus \rho \langle n \rangle_1 \right) (f, g)
\]
\[
= \left( \sigma *_{\infty, \infty} \bigoplus \rho \left( \bigoplus_{i=0}^{n-1} *_{i+1,i} \right) \right) (f, g)
\]
\[
= \left( *_{\infty, \infty} *_{1,0} \bigoplus \bigoplus_{i=0}^{n-1} *_{i+2,i+1} \right) (f, g)
\]
\[
= \left( *_{1,0} \bigoplus \bigoplus_{i=0}^{n} *_{i+1,i} \right) (f, g)
\]
\[
= \left( \bigoplus_{i=0}^{n} *_{i+1,i} \right) (f, g).
\]
Then \( \langle n+1 \rangle_1 = \bigoplus_{i=0}^{n} *_{i+1,i} \). \( \Box \)

**Proposition 3.** For all \( n \geq 2 \)
\[
\langle n \rangle_2 = \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{n-2-i} *_{i+j+2,i+j} *_{i+1,i}.
\] (25)
Proof. Suppose true that

$$\langle n \rangle \begin{array}{ll} _2 \end{array} \begin{array}{ll} _n \end{array} \begin{array}{ll} i=0 \end{array} \begin{array}{ll} j=0 \end{array} *_{i+j+2,i+j} *_{i+1,i}.$$ 

Then

$$\langle n + 1 \rangle \begin{array}{ll} _2 \end{array} \begin{array}{ll} _n \end{array} \begin{array}{ll} _2 \end{array} \begin{array}{ll} _n \end{array} \begin{array}{ll} i=0 \end{array} \begin{array}{ll} j=0 \end{array} \begin{array}{ll} i+j+2,i+j \end{array} *_{i+1,i}.$$ 

Thus

$$\langle n + 1 \rangle \begin{array}{ll} _2 \end{array} \begin{array}{ll} _n \end{array} \begin{array}{ll} _2 \end{array} \begin{array}{ll} _n \end{array} \begin{array}{ll} i=0 \end{array} \begin{array}{ll} j=0 \end{array} \begin{array}{ll} i+j+2,i+j \end{array} *_{i+1,i}.$$ 

and the result is true for all $n$. 

When $\langle n \rangle \begin{array}{ll} _k \end{array}$ is a product in $W_C[[x]]$, then

$$\langle n \rangle \begin{array}{ll} _k \end{array} (f, g) = \begin{array}{ll} n \end{array} \begin{array}{ll} _k \end{array} f \cdot g.$$ 

and thus we recover the general ordinary Leibniz formula. In the following result we show that it is also possible to recover the $q$-analogue of that rule.

**Theorem 13.** If $\psi_n = [n]$, then

$$\langle n \rangle \begin{array}{ll} _k \end{array} (f(x), g(x)) = \begin{array}{ll} n \end{array} \begin{array}{ll} _k \end{array} q^f(q^k x) g(x).$$ (26)

**Proof.** Using recursively the equation (21) we have

$$\langle n \rangle \begin{array}{ll} _k \end{array} = \bigoplus_{i=1}^{k} \sigma^{i-1} \rho \begin{array}{ll} n-i \end{array} \begin{array}{ll} _k-i+1 \end{array} \begin{array}{ll} _k \end{array} \begin{array}{ll} n-1 \end{array} \begin{array}{ll} _k \end{array} \begin{array}{ll} 0 \end{array}.$$

First we calculate $\sigma^k \begin{array}{ll} n-1 \end{array} \begin{array}{ll} 0 \end{array} \begin{array}{ll} _0 \end{array}$. We have

$$\sigma^k \begin{array}{ll} n-1 \end{array} \begin{array}{ll} 0 \end{array} (f(x), g(x)) = \begin{array}{ll} n \end{array} \begin{array}{ll} _k \end{array} q^f(q^k x) g(x).$$
By a similar calculation using Propositions 2 and 3 it is proved that
\[ \sigma^{k-1} \rho \left\langle \frac{n-k}{1} \right\rangle (f(x), g(x)) = q[n] f(q^k x) g(x), \]
\[ \sigma^{k-2} \rho \left\langle \frac{n-k+1}{2} \right\rangle (f(x), g(x)) = q^2 \left( \frac{n-k+1}{2} \right)_q f(q^k x) g(x), \]
and in general
\[ \rho \left\langle \frac{n-1}{k} \right\rangle (f(x), g(x)) = q^k \left( \frac{n-1}{k} \right)_q f(q^k x) g(x). \]

Now for (5)
\[ \left\langle \frac{n}{k} \right\rangle (f(x), g(x)) = \sum_{i=1}^{k} q^{k-i+1} \left( \frac{n-i}{k-i+1} \right)_q f(q^k x) g(x) \]
\[ = \left( \frac{n}{k} \right)_q f(q^k x) g(x) \]
where we have used recursively (2) with \( \psi_n = [n]. \)

From the above theorem and Theorem 12 it follows that
\[ D^n_q (f(x) g(x)) = \sum_{k=0}^{n} \left\langle \frac{n}{k} \right\rangle (D^{n-k}_q f(x), D^k_q g(x)) \]
\[ = \sum_{k=0}^{n} \left( \frac{n}{k} \right)_q D^{n-k}_q f(q^k x) D^k_q g(x). \]

We conclude by showing the following example.

**Example 2.** We want to calculate the \( n \)-th derivative of the function \( x e^x \psi \), where
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{\psi_n!}. \]

Using the Theorem 12 we have
\[ D^n\psi e^x = \sum_{k=0}^{n} \left\langle \frac{n}{k} \right\rangle (D^{n-k}_\psi x, D^k\psi e^x) \]
\[ = \left\langle \frac{n}{n} \right\rangle (x, D^n\psi e^x) + \left\langle \frac{n}{n-1} \right\rangle (D_\psi x, D^{n-1}\psi e^x) \]
\[ = x \left( \prod_{i=1}^{n} \psi_i \right) e^x + 1 \left\langle \frac{n}{n-1} \right\rangle e^x. \]

When \( \psi_m = m \), the \( n \)-th derivative of \( x e^x \) is \( D^n(x e^x) = (n + x)e^x \). When \( \psi_m = [m] \), then \( D^n q(x e^x) = (q^n x + [n])e^x \).
6 Quotient rule for the $\psi$-derivative

We conclude by showing the $\psi$-derivative of the quotient $f/g$.

Theorem 14. Take $f, g \in W_{\psi, C}[[x]]$. Then

$$D_\psi \left( \frac{f(x)}{g(x)} \right) = \frac{D_\psi f(x) - \frac{f(x)}{g(x)} \ast_{1,0} D_\psi (g(x))}{g(x)}$$

Proof. If we apply the Theorem 9 to

$$\frac{f(x)}{g(x)} \cdot g(x) = f(x),$$

we obtain

$$\frac{f(x)}{g(x)} \ast_{1,0} D_\psi (g(x)) + g(x)D_\psi \left( \frac{f(x)}{g(x)} \right) = D_\psi f(x)$$

and thus

$$D_\psi \left( \frac{f(x)}{g(x)} \right) = \frac{D_\psi f(x) - \frac{f(x)}{g(x)} \ast_{1,0} D_\psi (g(x))}{g(x)}.$$

When $\psi_n = n$,

$$D \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)Df(x) - f(x)Dg(x)}{g^2(x)}.$$

And if $\psi_n = [n]$, then

$$D_q \left( \frac{f(x)}{g(x)} \right) = \frac{g(qx)D_qf(x) - f(qx)D_qg(x)}{g(x)g(qx)}.$$

On the other hand, by Theorem 9

$$D_\psi \left( \frac{f(x)}{g(x)} \right) \ast_{1,0} g(x) = \frac{g(x)D_\psi f(x) - f(x)D_\psi g(x)}{g(x)}.$$

Thus, we obtain

$$D_q \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(x)g(qx)}.$$

Finally, we obtain the $\psi$-derivative of the reciprocal $1/g$ of the function $g \in W_{\psi, C}[[x]]$. 

20
Corollary 1.

\[ D_\psi \left( \frac{1}{g(x)} \right) = -\frac{1}{g(x)} \left( \frac{1}{g(x)} \ast_{1,0} D_\psi (g(x)) \right) \]

Example 3.

\[ D_\psi \left( \frac{1}{e^x} \right) = -\frac{1}{e^x} \left( \frac{1}{e^x} \ast_{1,0} e^x \right) \]

When \( \psi_n = [n] \),

\[ D_q \left( \frac{1}{e^x} \right) = -\frac{1}{e^x} \left( \frac{1}{e^x} \ast_{1,0} e^x \right) \]

\[ = -\frac{1}{e^x} \left( e^x e^x \right) \]

\[ = -\frac{1}{e^x}. \]

With \( \psi_n = n \), we obtain \( D(e^{-x}) = -e^x \).

Example 4. Define \( \sin_\psi x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/\psi_{2n+1}! \) and \( \cos_\psi x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/\psi_{2n}! \).

Then

\[ D_\psi \left( \frac{\sin_\psi x}{\cos_\psi x} \right) = \frac{\cos_\psi x + \frac{\sin_\psi x}{\cos_\psi x} \ast_{1,0} \sin_\psi x}{\cos_\psi x} \]

and

\[ D_\psi \left( \frac{1}{\cos_\psi x} \right) = \frac{1}{\cos_\psi x} \left( \frac{\sin_\psi x}{\cos_\psi x} \ast_{1,0} \sin_\psi x \right) \]

References

[1] G. Fontané, Généralisation d’une formule connue, Nouv. Ann. Math., 15(1915) 112.

[2] A.T. Benjamin and S.S. Plott, A combinatorial approach to Fibonomial coefficients, Fibonacci Quart., 46/47(2008/2009), 7-9

[3] H.W. Gould, The bracket function and Fontané-Ward generalized binomial coefficients with application to Fibonomial coefficients, Fibonacci Quart., 7(1969) 23-40

[4] F.H. Jackson, On q-functions and a certain difference operator, Tans. Roy. Soc. Edin., 46(1908), 253-281

[5] Kac V. and Pokman C., Quantum Calculus, Universitext, Spinger-Verlag, 2002
[6] Keigher W.F. On the ring of Hurwitz series, Communications in Algebra, 25:6(1997) 1845-1859.

[7] Kus S., Tuglu N. and Kim T., Bernoulli F-polynomials and Fibo-Bernoulli matrices, Adv Differ Equ, 145(2019)

[8] Trojovský P., On some identities for the Fibonomial coefficients via generating function, Discrete Applied Mathematics, 155(15)(2007), 2017-2024

[9] Tuglu N. and Yesil F. and E. Kocer and M. Dziemiańczuk, The F-Analogue of Riordan Representation of Pascal Matrices via Fibonomial Coefficients, J. Appl. Math., 2014(2014) 1-6

[10] M. Ward., A calculus of sequences, Am. J. Math. 58(1936) 255-266.

E-mail address, R. Orozco: rj.orozco@uniandes.edu.co