Testing the equal-time angular-averaged consistency relation of the gravitational dynamics in $N$-body simulations

Takahiro Nishimichi$^1$, Patrick Valageas$^2$

$^1$Institut d’Astrophysique de Paris & UPMC (UMR 7095), 98, bis boulevard Arago, 75014, Paris, France
$^2$Institut de Physique Théorique, CEA Saclay, 91191 Gif-sur-Yvette, cedex, France

(Dated: February 17, 2014)

We explicitly test the equal-time consistency relation between the angular-averaged bispectrum and the power spectrum of the matter density field, employing a large suite of cosmological $N$-body simulations. This is the lowest-order version of the relations between $(l+n)$-point and $n$-point polyspectra, where one averages over the angles of $l$ soft modes. This relation depends on two wave numbers, $k’$ in the soft domain and $k$ in the hard domain. We show that it holds up to a good accuracy, when $k’/k \ll 1$ and $k’$ is in the linear regime, while the hard mode $k$ goes from linear ($0.1\, h\, \text{Mpc}^{-1}$) to nonlinear ($1.0\, h\, \text{Mpc}^{-1}$) scales. On scales $k \lesssim 0.4\, h\, \text{Mpc}^{-1}$, we confirm the relation within a $\sim 5\%$ accuracy, even though the bispectrum can already deviate from leading-order perturbation theory by more than $30\%$. We further show that the relation extends up to nonlinear scales, $k \sim 1.0\, h\, \text{Mpc}^{-1}$, within an accuracy of $\sim 10\%$.

I. INTRODUCTION

The large-scale structure of the Universe provides us with a wealth of information on the initial conditions of the Universe as well as the underlying gravity theory that governs the time evolution on sufficiently large scales [1, 2]. A classic tool for discussing its statistical properties are the polyspectra of the matter density field at a given time (the Fourier transforms of the $n$-point correlation functions) [3, 4]. The power spectrum, the lowest-order polyspectrum, has played a central role to test cosmological models and determine their parameters precisely. Standard models of the early universe predict almost Gaussian initial conditions, in agreement with a number of observational probes (e.g., measures of cosmic microwave background anisotropies [5]). However, even if the initial conditions are perfectly Gaussian, the cosmic density field at late times exhibits non-Gaussian features acquired through the nonlinear gravitational dynamics.

The polyspectra induced by gravity can be analytically derived order by order using standard perturbation-theory techniques (see [4] for a review). In these calculations, an approximate treatment is usually adopted that greatly simplifies the structure of the basic equations. That is, the combination $\Omega_\text{m}/f^2$ is replaced with unity, where $\Omega_\text{m}$ is the time-dependent matter density parameter and $f \equiv \Delta \ln D_+/\Delta \ln a$ is the linear growth-rate, with $D_+$ being the linear growing mode. This approximation is exact in the Einstein-de Sitter universe and sufficiently accurate in most other cosmological models based on General Relativity, because i) one usually recovers Einstein-de Sitter at early times and ii) over the realistic range of cosmological parameters one has $f \simeq \Omega_\text{m}^\gamma$ with $\gamma \simeq 0.5$ [3]. When this approximation is applied, all the dependence on the cosmological parameters is absorbed by the linear growth rate $D_+$, and the time dependence of the solution is also fully encapsulated in $D_+$. This simplifies perturbative computations because one can factor the time and scale dependence of high-order diagrams (e.g., the contribution of order $n$ to the power spectrum scales as $D_+^{2n}$).

Beyond perturbation theory, several articles have recently been devoted to the study of exact “consistency relations” that remain valid in the nonperturbative regime, independently of the small-scale physics (including baryon or star-formation processes) [6, 13]. They relate the $(l+n)$-point correlation, with $l$ modes in the linear regime (soft domain) and $n$ modes at much higher wave numbers (hard domain) that can be in the nonlinear regime, to the $n$-point correlation (with $l$ linear power spectrum prefactors). These results can be interpreted as the response of small structures (i.e., each element in the cosmic web such as walls, filaments or halos) to an initial density perturbation on much larger scales. More precisely, they derive from the equivalence principle, which ensures that all particles and structures fall in the same fashion in a gravitational potential force with a constant gradient. Then, at leading order a large scale perturbation of the initial conditions merely transports smaller scale structures without distortions. Thus, a detection of a violation of these consistency relations would signal a deviation from Gaussian initial conditions, significant decaying modes, or a departure from General Relativity.

In the standard scenario, the kinematic consistency relations discussed above vanish at equal times (because equal-time statistics cannot distinguish a uniform translation of the system). By going to the next order, and taking an angular average over the soft modes, Refs. [14, 15] derived angular-averaged consistency relations that remain non-trivial even at equal times. Because this involves the dynamics of small-scale structures in a gravitational potential with a uniform curvature (the order beyond a constant gradient), this probes the physics beyond the equivalence principle and it is sensitive to the details of the dynamics. In particular, the explicit relations one obtains only hold for dark matter...
(i.e., they would be violated by non-gravitational processes) and within the approximation $\Omega_m/f^2 \simeq 1$, which enables to relate the dynamics associated with different backgrounds (which correspond to different large-scale curvatures). However, within these approximations they remain valid in the nonperturbative regime.

In this study, we examine the validity of these angular-averaged relations by employing a large set of cosmological $N$-body simulations. We focus on the lowest-order consistency relation for the angular-averaged matter bispectrum, which is the most interesting one in practice. Ref. [14] has already checked this relation for the bispectrum explicitly at the leading order of perturbation theory. The aim of this study is to see how higher-order corrections enter both sides of the equation and how accurately the relation is recovered on smaller scales (i.e., whether and by how much nonlinearities amplify the inaccuracy due to the approximation $\Omega_m/f^2 \simeq 1$).

This paper is organized as follows. We briefly review the angular-averaged consistency relations and its validation at tree level in Sec. II. We then present the simulation analysis in Sec. III starting from the detail of the simulations in Sec. IIIA and next showing our results for the consistency relation in Sec. IIIb. We finally give a summary of the paper in Sec. IV.

II. THE ANGULAR-AVERAGED CONSISTENCY RELATIONS

We briefly summarize the angular-averaged consistency relations in this section. We also review the perturbative expressions for the relevant spectra here.

A. General cases

Because of statistical homogeneity, polyspectra contain a Dirac factor $\delta_D$ that we can factor out by defining

$$\langle \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_n} \rangle = \delta_D (k_1 + \cdots + k_n) \langle \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_n} \rangle', \quad (1)$$

where $\langle \ldots \rangle$ is the statistical average over the Gaussian initial conditions and the prime in $(\ldots)'$ denotes the average in Fourier space without the Dirac factor. We denote the nonlinear density contrast in Fourier space by $\tilde{\delta}$, with a wave vector shown by the subscript. In a similar fashion, we also consider mixed spectra, $\langle \tilde{\delta}_{L,k_1} \ldots \tilde{\delta}_{L,k_1} \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_n} \rangle'$, which cross-correlate the nonlinear density contrast $\tilde{\delta}_L$ with the linear density contrast $\tilde{\delta}_L$. Here $\tilde{\delta}_L$ is the linear growing mode that also defines the Gaussian initial conditions (we assume that usual that decaying modes have had time to become negligible).

Integrating over the direction of the linear wave numbers $k_j'$, we introduce the angular-averaged mixed polyspectra by

$$\int \prod_{j=1}^\ell \frac{d\Omega k_j'}{4\pi} \langle \tilde{\delta}_{L,k_1} \ldots \tilde{\delta}_{L,k_1} \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_n} \rangle'_{k_j' \to 0} = \langle \tilde{\delta}_{L,k_1} \ldots \tilde{\delta}_{L,k_1} \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_n} \rangle'_{k_j' \to 0}, \quad (2)$$

where the limit $k_j' \to 0$ is taken for all the $\ell$ wave numbers with a prime, while obeying the constraint $\sum_j k_j' + \sum_i k_i = 0$ (associated with statistical homogeneity).

When the soft wave numbers satisfy the hierarchy $k_j' \ll k_{j+1}'$ and within the approximation $\Omega/f^2 \simeq 1$, the angular-averaged consistency relation at equal times states that Eq. (2) can be expressed in terms of the $n$th order polyspectrum as [14, 15]

$$\langle \tilde{\delta}_{L,k_1} \ldots \tilde{\delta}_{L,k_1} \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_n} \rangle'_{k_j' \to 0} = \mathcal{L}_1' \ldots \mathcal{L}_\ell' \cdot \langle \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_n} \rangle'. \quad (3)$$

In the right hand side, the operators $\mathcal{L}_j'$ are given by

$$\mathcal{L}_j' = P_L(k_j') \left[ 1 + \frac{13}{21} \frac{\partial}{\partial \ln D_+} - \frac{1}{3} \sum_{m=j+1}^\ell \frac{\partial}{\partial \ln k_m} - \frac{1}{3} \sum_{i=1}^n \frac{\partial}{\partial \ln k_i} \right], \quad (4)$$

where $P_L$ is the initial matter power spectrum linearly extrapolated to the time of interest. (Because these operators do not commute the ordering in the above relation only holds for the hierarchy of soft wave numbers $k_1' \ll k_2' \ll \cdots \ll k_\ell'$.) Because we take the limit $k_j' \to 0$ in Eq. (3) and we recover linear theory on large scales, we can replace the linear density fields by the nonlinear ones and write

$$\langle \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_1} \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_n} \rangle'_{k_j' \to 0} = \mathcal{L}_1' \ldots \mathcal{L}_\ell' \cdot \langle \tilde{\delta}_{k_1} \ldots \tilde{\delta}_{k_n} \rangle'. \quad (5)$$

B. Bispectrum

The simplest example of the relation [15] relates the angular-averaged bispectrum to the power spectrum. This corresponds to $\ell = 1$ and $n = 2$, namely,

$$\tilde{B}(k';k)_{k_j' \to 0} = P_L(k') \left[ 1 + \frac{13}{21} \frac{\partial}{\partial \ln D_+} - \frac{1}{3} \frac{\partial}{\partial \ln k} \right] P(k), \quad (6)$$

where we denote

$$\tilde{B}(k';k) \equiv \langle \tilde{\delta}_{k'} \tilde{\delta}_{k-k'/2} \tilde{\delta}_{k-k'/2} \rangle', \quad P(k) \equiv \langle \tilde{\delta}_{k} \tilde{\delta}_{-k} \rangle', \quad (7)$$

for the angular-averaged bispectrum and the power spectrum (taking care of the constraint $\sum_j k_j' + \sum_i k_i = 0$).
associated with the Dirac factor in Eq. (1) due to statistical homogeneity. Because of statistical isotropy the spectra in Eq. (2) no longer have any dependence on the direction of \( \mathbf{k} \). Since higher-order polyspectra are increasingly noisy in general, in practice the main application of these consistency relations is the lowest-order one, for the angular-averaged bispectrum. We thus focus on the consistency relation (6) in this study and we test the low-\( k' \) asymptotic behavior with a large set of cosmological \( N \)-body simulations.

C. Tree-level perturbation theory

The relation (4) has been checked by Ref. [14] at leading order of perturbation theory. At this order, all we need is the second-order kernel of the matter density field [4]:

\[
F_2^2 (k_1, k_2) = \frac{5}{7} \frac{1}{k_1 k_2} \left( \frac{k_1 + k_2}{k_2} \right) + \frac{2}{7} \frac{(k_1 - k_2)^2}{k_1 k_2^2},
\]

where we applied the approximation \( \Omega_m/f^2 = 1 \). The time dependence of the kernel function (8) is actually very small [4], and for instance approximately given by \( \Omega_m^{-2/63} - 1 \) in case of \( \Omega_m \gtrsim 0.1 \) for open universes without cosmological constant. At tree order, the bispectrum, \( B = \langle \delta_{k_1} \delta_{k_2} \delta_{k_3} \rangle' \), can be written as

\[
B(k_1, k_2, k_3) = 2 F_2^2 (k_1, k_2) P_L(k_1) P_L(k_2) + \text{cyclic},
\]

where (cyclic) stands for two more terms given as the cyclic permutation over the three wavevectors. Then, taking the angular average of the tree-level bispectrum (9) as in Eq. (7) gives

\[
\bar{B}(k'; k) = P_L(k') \left[ \frac{47}{21} - \frac{1}{3} \frac{\partial}{\partial \ln k} \right] P_L(k) + \mathcal{O} ((k'/k)^2).
\]

Using \( P_L(k, t) \propto D_+ (t)^2 \), this confirms the consistency relation (6) within the validity of perturbation theory at the leading order.

III. SIMULATION ANALYSIS

We now describe the simulations that we analyze in this study. We also present the method to measure the relevant statistical quantities and discuss the reliability of the measurements. We finally show how accurately the consistency relation (6) is recovered in the simulations.

A. Setup of the simulations

We use two sets of cosmological simulations in this paper. The first set of simulations has already been used in [16]. Employing \( 1024^3 \) particles, each of the 60 independent random realizations covers a comoving volume of \((2048 h^{-1} \text{Mpc})^3\). The total simulation volume of \( 515 h^{-3} \text{Gpc}^3 \) enables precise measurements of statistical quantities. These simulations are designed to calibrate analytical models of the matter power spectrum based on renormalized perturbation theory approaches on large scales (i.e., \( k \gtrsim 0.3 h \text{Mpc}^{-1} \)) and the systematic error as well as the statistical error are controlled very well on these scales to meet the requirements (see also [17, 18] for more on the convergence).

However, because of their rather poor spatial resolution, it is known that the power spectrum on smaller scales is systematically smaller than it should be. Although this systematic error is at most \( \sim 2\% \) at \( k = 0.4 h \text{Mpc}^{-1} \), almost independently of redshift, it increases toward smaller scales. The error reaches \( 4\% \) at \( k \sim 0.7 h \text{Mpc}^{-1} \). Since our target accuracy in this study is about \( 5\% \) and, what is more, the consistency relation is less trivial on smaller scales (where we go beyond lowest-order perturbation theory), we decided to run new simulations with a better spatial resolution. We run 512 independent realizations of \( 512^3 \)-particles simulations, each of which has a cubic volume of \((512 h^{-1} \text{Mpc})^3\). This allows us to double the dynamic range in wave number toward smaller scales, though the total simulation volume of these new simulations is only about \( 13\% \) of the low resolution simulations. For simplicity, we adopt the simulations of [16] for the discussion on scales \( k \leq 0.4 h \text{Mpc}^{-1} \), while the new high-resolution simulations are used on smaller scales.

The cosmological model in both set of simulations is a flat-\( \Lambda \text{CDM} \) model with the parameters \( \Omega_m = 0.279 \), \( \Omega_b/\Omega_m = 0.165 \), \( h = 0.701 \), \( n_s = 0.96 \) and \( \sigma_8 = 0.816 \), which is consistent with the five-year observation by the WMAP satellite [19]. The combination \( \Omega_m/f^2 \) in this cosmology is shown in Fig. 1. The ratio is very close to unity at high redshifts, \( z \gtrsim 1 \), and reaches about 1.15 at \( z = 0 \). In this paper, we test the consistency relation in our simulations at the redshifts \( z = 1 \) and \( z = 0.35 \), at which the ratio \( \Omega_m/f^2 \) departs from unity by \( 2.7\% \) and \( 7.5\% \), respectively. However, the polyspectra at a given time are affected not only by the value of \( \Omega_m/f^2 \) at that time but by its evolution history up to that epoch. This further decreases the inaccuracy due to the approximation \( \Omega_m/f^2 \simeq 1 \) on the power spectrum and bispectrum, as found in previous perturbative studies [4].

B. Left-hand side: measurement of the bispectrum

We first describe our method to measure the angular-averaged bispectrum in this subsection. We assign particles onto \( 1024^3 \) grid points using the Cloud-in-Cells interpolation scheme (e.g., [20]) and apply the fast Fourier transformation to obtain the density field in Fourier space. We then correct the smoothing effect arising from the grid assignment by dividing by the CIC kernel func-
We next take an average of cubic products of the density fields to have an estimate of $B$ defined in Eq. (7):

$$B(k'; k) = \frac{V^2}{N_{\text{tri}}^{k', k}} \sum_{k' - \Delta k' / 2 < |k'| < k' + \Delta k' / 2} \sum_k \text{Re} \left[ \tilde{\delta}_{k'} \tilde{\delta}_{k - k'} / 2 \tilde{\delta}_{-k - k'} / 2 \right],$$

(11)

where $V$ stands for the simulation volume, $N_{\text{tri}}^{k', k}$ is the number of triangles for the wave number bin specified by $k'$ and $k$, and the summation is taken over modes $k'$ and $k$ that satisfy $k' - \Delta k' / 2 \leq |k'| < k' + \Delta k' / 2$ and $k - \Delta k / 2 \leq |k| < k + \Delta k / 2$, respectively. We choose $\Delta k' = 0.004 \, h^{-1} \text{Mpc}$ and $\Delta k = 0.02 \, h^{-1} \text{Mpc}$ for the low-resolution simulations and $\Delta k' = 0.005 \, h^{-1} \text{Mpc}$ and $\Delta k = 0.02 \, h^{-1} \text{Mpc}$ for the high-resolution ones. Because we are now working on a periodic system with finite volume, the density field $\delta_k$ is dimensionless, unlike the one in the previous section for continuous Fourier transforms. In Eq. (11), note also that we take the angular average not only over $k'$ but also over $k$, in order to increase the statistics and suppress the statistical error level [24].

We finally take the average over different realizations to obtain our final estimate of the angular-averaged bispectrum and we record the variance among realizations, divided by the square root of the number of realizations minus unity (i.e., the standard error on the average values), to estimate the statistical error.

The resultant bispectrum is plotted in Figs. 2 and 3 at $z = 1$ and 0.35, respectively. We plot in the top panel the angular-averaged bispectrum, $B(k'; k)$, as a function of wave number $k'$ for several fixed values of $k$ as written in the legend. The filled symbols show the measurements from the low-resolution simulations while the open ones depict those from the high-resolution simulations.

We also show in solid line the perturbation-theory prediction at the tree level [i.e., Eq. (9)] for $k = 0.1$ and $0.2 \, h^{-1} \text{Mpc}$, while the result at $k = 0.2 \, h^{-1} \text{Mpc}$ reveals a lack of amplitude in the analytical curve. This discrepancy is more important at $z = 0.35$ (10 to 20% depending on $k'$, and more evident at larger $k'$). We omit analytical curves at $k \geq 0.3 \, h^{-1} \text{Mpc}$ to avoid making the plot busy, but the discrepancy between the model and the simulations is even greater on these scales (a factor of two or more). Thus, we conclude that the applicable wave number range of the tree-level perturbation theory is limited to $k \lesssim 0.1 \, h^{-1} \text{Mpc}$, both at $z = 0.35$ and 1.

In the top panels, we plot both filled and open circles at $k = 0.4 \, h^{-1} \text{Mpc}$ to check the consistency between the two sets of simulations. The ratio of the bispectrum measured from the two sets of simulations is shown in the bottom panel in which we interpolate the results of low-resolution simulations with a cubic spline function to evaluate the values at $k'$ at which the measurements of the high-resolution simulations are available. We plot the statistical error only on the numerator since the denominator is better converged statistically thanks to the larger simulation volume. Though some data points depart from unity, we do not see a clear systematic trend. From this comparison, we conclude that the bispectrum measured from the low-resolution simulations has converged within $\sim 5\%$ accuracy at this wave number.
number. The systematic error due to the low spatial resolution should be smaller at smaller $k$. We also expect that a similar accuracy is achieved for high-resolution simulations at $k \sim 0.8 \, h \, \text{Mpc}^{-1}$ though the measurement at $k = 1.0 \, h \, \text{Mpc}^{-1}$ may have a larger statistical error.

Finally, the middle panels of Figs. 2 and 3 plot the fractional error on $\hat{B}(k'k)$ measured from the simulations (we adopt the same symbols as in the top panels). Since we fix the bin width, $\Delta k'$ and $\Delta k$, the number of available Fourier-space triangles increases with $k'$ and $k$, resulting in a smaller error at smaller scale for filled symbols (i.e., low-resolution simulations). Also, the error level is higher for high-resolution simulations, which cover a smaller volume than the low-resolution ones. The decrement of the error as a function of $k$ for the same set of simulations is only marginal, especially at $z = 0.35$, due to significant covariance among different modes on small scales. Eventually, at $k > \sim 0.4 \, h \, \text{Mpc}^{-1}$, we do not observe clear dependence of the statistical error on $k$ for high-resolution simulations (i.e., open symbols which are mostly overlapping with each other). On these scale, the statistical error is mostly determined by that in the soft mode $\tilde{\delta}_k$, and one does not gain much when one adds more hard modes $\delta_k$.

The typical statistical error level on the angular-averaged bispectrum is roughly one per cent, which allows us a meaningful test of the consistency relation. We are especially interested in $\hat{B}$ at the limit of small $k'$ and the low-resolution simulations, which cover a total volume of $515 \, h^{-3} \, \text{Gpc}^3$, provide us with measurements of the angular-averaged bispectrum down to $k' \sim 0.01 \, h \, \text{Mpc}^{-1}$ with an error level of several per cent. On the other hand, although the available data points are limited, the high-resolution simulations enable us to test the consistency relation with a statistical error of $\sim 3\%$ down to smaller scales where non-perturbative corrections to the density field are important.

C. Right-hand side: measurement of the power spectrum and its derivatives

We next describe our method to measure the right-hand side of Eq. (6). The three terms are explained one by one in the following, and we then summarize the accuracy of the measurements of the sum of them.

1. Nonlinear power spectrum

The measurement of the nonlinear power spectrum is rather straightforward after we have given the explanation for the bispectrum. The procedure is exactly the same as in Sec. III B up to the density field in Fourier space with the correction of the smoothing effect. This time, we take

$$P(k) = \frac{V}{N_k^{\text{mode}}} \sum_k |\delta_k|^2,$$

where $N_k^{\text{mode}}$ stands for the number of Fourier modes in the $k$ bin. In the summation, we consider modes $k - \Delta k/2 \leq |k| < k + \Delta k/2$, and we adopt $\Delta k = \ldots$

Finally, the middle panels of Figs. 2 and 3 plot the fractional error on $\hat{B}(k'k)$ measured from the simulations (we adopt the same symbols as in the top panels). Since we fix the bin width, $\Delta k'$ and $\Delta k$, the number of available Fourier-space triangles increases with $k'$ and $k$, resulting in a smaller error at smaller scale for filled symbols (i.e., low-resolution simulations). Also, the error level is higher for high-resolution simulations, which cover a smaller volume than the low-resolution ones. The decrement of the error as a function of $k$ for the same set of simulations is only marginal, especially at $z = 0.35$, due to significant covariance among different modes on small scales. Eventually, at $k > \sim 0.4 \, h \, \text{Mpc}^{-1}$, we do not observe clear dependence of the statistical error on $k$ for high-resolution simulations (i.e., open symbols which are mostly overlapping with each other). On these scale, the statistical error is mostly determined by that in the soft mode $\tilde{\delta}_k$, and one does not gain much when one adds more hard modes $\delta_k$.

The typical statistical error level on the angular-averaged bispectrum is roughly one per cent, which allows us a meaningful test of the consistency relation. We are especially interested in $\hat{B}$ at the limit of small $k'$ and the low-resolution simulations, which cover a total volume of $515 \, h^{-3} \, \text{Gpc}^3$, provide us with measurements of the angular-averaged bispectrum down to $k' \sim 0.01 \, h \, \text{Mpc}^{-1}$ with an error level of several per cent. On the other hand, although the available data points are limited, the high-resolution simulations enable us to test the consistency relation with a statistical error of $\sim 3\%$ down to smaller scales where non-perturbative corrections to the density field are important.

C. Right-hand side: measurement of the power spectrum and its derivatives

We next describe our method to measure the right-hand side of Eq. (6). The three terms are explained one by one in the following, and we then summarize the accuracy of the measurements of the sum of them.

1. Nonlinear power spectrum

The measurement of the nonlinear power spectrum is rather straightforward after we have given the explanation for the bispectrum. The procedure is exactly the same as in Sec. III B up to the density field in Fourier space with the correction of the smoothing effect. This time, we take

$$P(k) = \frac{V}{N_k^{\text{mode}}} \sum_k |\delta_k|^2,$$

where $N_k^{\text{mode}}$ stands for the number of Fourier modes in the $k$ bin. In the summation, we consider modes $k - \Delta k/2 \leq |k| < k + \Delta k/2$, and we adopt $\Delta k = \ldots$
0.005 \, h \, \text{Mpc}^{-1} for both sets of simulations. The results are shown by thin solid lines in the top panels of Figs. 4 and 5 at \( z = 1 \) and 0.35, respectively. We here plot the results of the low-resolution simulations, but the high-resolution simulations almost coincide with the low-resolution simulations (see later discussion for the convergence of the power spectrum).

The statistical error on the measured power spectrum is plotted in the second and the third panel for the low- and high-resolution simulations, respectively. Similarly to the bispectrum, the error level decreases with wave number since we fix the bin width \( \Delta k \) and thus we can access more Fourier modes at larger \( k \). Since the covariance between different modes grows with \( k \) and time, the \( k \)-dependence of the fractional error is shallower than \( k^{-1} \) expected for uncorrelated measures.

2. Time derivative

Estimating the time derivative of the power spectrum from the simulation data is less trivial. We adopt the following procedure in this study. Instead of preparing multiple snapshots at slightly different redshifts, we work on a single snapshot of the positions and velocities of simulation particles. We slightly displace the positions of particles according to their velocities:

\[
x(t + \Delta t) = x(a + \Delta a) = x(t) + \mathcal{H}^{-1}(t)v(t)\Delta a, \quad (13)
\]

where \( x \) and \( v \) are the position and velocity of a particle in comoving coordinate and \( \mathcal{H} = da/dt \). We repeat the same procedure as before and measure the power spectrum after applying the above displacement. We finally take the combination to estimate the derivative term:

\[
\frac{dP(k)}{d\ln D_+} = \frac{P(k; a + \Delta a/2) - P(k; a - \Delta a/2)}{\ln D_+(a + \Delta a/2) - \ln D_+(a - \Delta a/2)}, \quad (14)
\]

This procedure can be justified as long as \( \Delta a \) is small, and we adopt \( \Delta a = 0.02 \), which gives a converged result.

The measurement and its error is plotted in Figs. 4 and 5 in dashed line. This term dominates the other terms over the whole range of wave number plotted in the figures. The fractional error plotted in the middle two panels behave similarly to that on the nonlinear power spectrum at small \( k \), and is slightly larger on small scales reflecting the stronger nonlinearity in the momentum field than in the density field [26].

3. Wave number derivative

We compute the last term in the right-hand side of Eq. (6) using the cubic spline fitting to the power spectrum measured above. Our choice of \( \Delta k = 0.005 \, h \, \text{Mpc}^{-1} \) is fine enough to evaluate the derivative without introducing a severe interpolation error. The measured derivative term shown in dashed line in Figs. 4 and 5 exhibits a clear feature of baryon acoustic oscillations. Note that we show the absolute value of this term as it is negative over most of the plotted wave number range. The fractional error on this term estimated from the scatter among realizations is the largest among the three terms probably because this term involves an interpolation and the derivative operation is not local in \( k \), but the error level is still several per cent over the most part of the plotted wave number range thanks to the large statistics.

4. Sum of the three terms

Adding up the three terms already discussed and multiplying by the linear power spectrum, we finally obtain an estimate of the right-hand side of Eq. (6). We plot the sum of the three terms as the bold solid lines in Figs. 4 and 5. The statistical error estimated from the scatter among realizations shown in the middle two panels is controlled below 1% level both in the low- and high-resolution simulations on \( k \gtrsim 0.05 \, h \, \text{Mpc}^{-1} \). This error level is in between that on the wave number-derivative term (dotted) and the time-derivative term (dashed). Since the former is smallest among the three terms, its large error does not ruin the quality of the sum of the three terms. Thus the statistical error on the left-hand side (i.e., the angular-averaged bispectrum) of Eq. (6), dominates over that in the right-hand side in checking the consistency relation in what follows.

In order to see a rough estimate of the systematic error on the measurement of the right-hand side of Eq. (6), we also plot in the bottom panels of Figs. 4 and 5 the ratio of the sum of the three terms measured from the

FIG. 5: Same as Fig. 4 but at \( z = 0.35 \).
high- and the low-resolution simulations. As we have discussed in a series of our previous studies, the low-resolution simulations are affected by the numerical error arising from the finiteness of the spatial resolution, and the power spectrum on small scales is underestimated by a few per cent at $k \gtrsim 0.3 \, h \text{Mpc}^{-1}$. The ratio plotted in the bottom panels is actually larger than unity on these scales. The systematic error is more important at $z = 0.35$ and it amounts to $\sim 5\%$ on the smallest scales of the plot. In reality, however, we do not use the low-resolution simulations in the test at $k > 0.4 \, h \text{Mpc}^{-1}$, and the systematic error is at most $3\%$ over the wave number range where we use these simulations. We also expect a similar quality in the measurement from the high-resolution simulations at $k \sim 1 \, h \text{Mpc}^{-1}$. This is again because the two sets of simulations have a factor of two difference in the mean inter-particle distance, and this distance serves as a rough indicator of the scale where the simulation starts to be affected by finiteness effects.

D. Results

Now we are in a position to discuss the validity of the consistency relation (9) between the angular-averaged bispectrum and the power spectrum of the matter density field. We consider the ratio of the two sides of Eq. (9), measure this combination from each realization, and then take the average over realizations, which is plotted in Figs. 6 and 7 respectively at $z = 1$ and $z = 0.35$.

The left four panels in each of the two figures show the measurement from the low-resolution simulations covering a larger volume ($0.1 \, h \text{Mpc}^{-1} \leq k \leq 0.4 \, h \text{Mpc}^{-1}$), while the right panels show that from high-resolution simulations ($0.4 \, h \text{Mpc}^{-1} \leq k \leq 1.0 \, h \text{Mpc}^{-1}$), as a function of the soft wave number $k'$. We also plot the ratio expected from the tree-level perturbation theory (solid lines) and the ratio of the measured bispectrum to the tree-order prediction (9) in the left upper two panels (dashed lines). The filled circles correspond to the bispectrum obtained from the nonlinear density fields measured at the redshift of interest, $\langle \delta_{l,k} \delta_{k-k'/2} \delta_{-k-k'/2} \rangle'$ as in Eq. (7), whereas the empty triangles correspond to the mixed bispectrum $\langle \delta_{l,k} \delta_{k-k'/2} \delta_{-k-k'/2} \rangle'$, where we cross-correlate two nonlinear fields with one linear field, as in Eq. (3).

In agreement with Figs. 2 and 3, the dashed lines show that tree-level perturbation theory only gives an accurate prediction for the bispectrum for $k'$ and $k$ below $\sim 0.1 \, h \text{Mpc}^{-1}$. When $k = 0.2 \, h \text{Mpc}^{-1}$, it underestimates the bispectrum by about $10\%$, and for higher $k$ the discrepancy becomes greater and can reach a factor two or more (it no longer appears in these panels because it is out of range). This shows that the panels with $k \geq 0.3 \, h \text{Mpc}^{-1}$ are beyond the lowest-order perturbative regime and that we test the consistency relation (9) in a nontrivial regime, beyond the perturbative check of...
Even though lowest-order perturbation theory cannot predict the bispectrum itself for $k \gtrsim 0.2 \ h \ Mpc^{-1}$, higher-order corrections partly cancel in the ratio between both sides of Eq. (6) and this ratio remains well described by lowest-order perturbation theory up to $k' \sim 0.07 \ h \ Mpc^{-1}$ in all panels, where $k \leq 1 \ h \ Mpc^{-1}$. This also agrees with previous studies that found that the reduced bispectrum, defined as $B(k_1, k_2, k_3) / [P(k_1)P(k_2) + \text{cyc.}]$, is more robust and shows smaller deviations from the perturbative prediction than the bispectrum itself [3]. In particular, for $k \lesssim 0.3 \ h \ Mpc^{-1}$, lowest-order perturbation theory is able to reproduce the first deviations from unity of the consistency-relation ratio, at $k' \sim 0.06 \ h \ Mpc^{-1}$, which may be either positive or negative, depending on scales. In terms of the consistency relation (6), these departures signal that the ratio $k''/k$ is not small enough to reach the low-$k'$ asymptotic behavior. At higher $k'$, the behavior is the same in all panels and the ratio grows with $k'$. On the other hand, on large scales, $k' \lesssim 0.04 \ h \ Mpc^{-1}$, the ratio is consistent with unity. Given the large statistics of the low-resolution simulations, we basically confirm the validity of the consistency relation (6) within 5% on these scales.

Then, the results of the high-resolution simulations shown in the right panels, though they have a larger scatter, show a similar trend as that at $k = 0.3$ or $0.4 \ h \ Mpc^{-1}$ found in the low-resolution simulations. At the joint wave number, $k = 0.4 \ h \ Mpc^{-1}$, the overall $k'$ dependence is consistent with the low-resolution ones. Although some data points are away from unity in the top right panel, the pattern looks not systematic except for large $k'$ ($\gtrsim 0.07 \ h \ Mpc^{-1}$) where the ratio is an increasing function of $k'$. The $k'$ dependence in the other three panels is quite similar to that in the top right panel. The coherence of the zigzag pattern among the four panels might be explained by the fact that we always use the same set of soft modes $\delta_{k'}$ for different hard wave numbers.

Note that on these scales, non-perturbative corrections such as shell crossing or the one-halo term in the halo model start to kick in (see e.g., [13, 21, 23]). However, in agreement with theoretical expectations, they do not lead to an increasingly large deviation from unity of the low-$k'$ limit. Indeed, the consistency relation (6) only relies on the approximate symmetry associated with the approximation $\Omega_m/f^2 \simeq 1$, and within this approximation it remains valid beyond shell crossing on highly nonlinear scales for $k$. Nonlinearities might amplify the sensitivity to this approximation, but this seems not to be the case in the range of scales shown in Figs. 2 and 3 except maybe in the panels at $z = 0.35$.

As expected, the filled circles and the empty triangles match on large scales as we recover linear theory. The differences that appear for $k' \gtrsim 0.06 \ h \ Mpc^{-1}$ show that the soft mode density contrast $\delta_{k'}$ begins to receive non-negligible nonlinear corrections. These contributions violate the consistency relation because the latter is actually derived for the mixed polyspectra, as in Eq. (3), and the form (5) makes use of the additional approximation $\delta_{k'} \approx \delta_{L,k'}$. Therefore, we would expect that the consistency relation is better satisfied when we do not introduce this additional approximation and consider the mixed bispectrum, shown by the empty triangles. The left panels do not show that the range of validity of the consistency relation is extended when we use the mixed bispectrum, but this could be due to the fact that the condition $k' \ll k$ is violated. On the other hand, the right panels at $z = 1$, with a lower scale ratio $k''/k$, show a broader range of validity of the consistency relation when we use the mixed bispectrum, in agreement with these theoretical expectations. The right panels at $z = 0.35$ also show a broader plateau, as expected, but with a small negative offset. The comparison between the two panels with $k = 0.4 \ h \ Mpc^{-1}$ suggests that some of this offset may be due to some numerical error or lack of power in the simulations, but this is not very conclusive given the large error bars. To put it the other way around, the consistency relation (6) might be useful to check the quality of the simulations. In this sense, our simulations meet the statistical error level in most of the cases, while there might exist a systematic error on small scales and low redshifts. However, even on these smaller scales at $z = 0.35$ we recover the consistency relation within a $\sim 10\%$ accuracy.

**IV. SUMMARY**

We have conducted a first numerical test of the angular-averaged consistency relation (5) by exploiting a large suite of cosmological $N$-body simulations. We focus on the lowest-order example of the relation ($\ell = 1$ and $n = 2$), which expresses the angular-averaged bispectrum in terms of the soft-mode and hard-mode power spectra. The large total volume of the simulations allows us to conduct a qualitative discussion on the validity of this relation.

We confirm that the relation is recovered within 5%, beyond the validity range of the tree-level perturbation theory (4), for $k \lesssim 0.4 \ h \ Mpc^{-1}$. On the other hand, these scales remain within the range of higher-order perturbation theories so that the validity of the consistency relation is not surprising, because it is well known that the approximation $\Omega_m/f^2$ used in most perturbative schemes is sufficiently accurate on these scales [4].

Beyond this regime, we find that the validity range of the consistency relation extends to smaller scales, $k \lesssim 1 \ h \ Mpc^{-1}$, where non-perturbative effects are not negligible [22, 23]. We confirm this at the 10% level. We check that the condition $k''/k \ll 1$ is not sufficient for the consistency relation, and the soft mode $\delta_{k'}$ must be in the linear regime. Using the mixed bispectrum pro-
vides a more direct connection with the theory and our results suggest that this also extends the validity range of the consistency relation. However, such a quantity can only be measured in numerical simulations and not from observations of the real universe.

We leave further discussions on, for instance, the effect of nonlinear bias or the usefulness of the relation to detect primordial non-Gaussianity to future studies. Also, it might be interesting to see how baryonic effects alter the relation between different spectra in hydro-dynamical simulations.

Acknowledgments

T. N. is supported by Japan Society for the Promotion of Science (JSPS) Postdoctoral Fellowships for Research Abroad. This work is supported in part by the French Agence Nationale de la Recherche under Grant ANR-12-BS05-0002. The numerical calculations in this work were carried out on Cray XC30 at Center for Computational Astrophysics, CfCA, of National Astronomical Observatory of Japan.

[1] P. J. E. Peebles, *Principles of Physical Cosmology* (1993).
[2] R. Laureijs, J. Amiaux, S. Arduini, J. Augerès, J. Brinchmann, R. Cole, M. Cropper, C. Dabin, L. Ducat, A. Ealet, et al., ArXiv e-prints (2011), 1110.3193.
[3] P. J. E. Peebles, *The large-scale structure of the universe* (1980).
[4] F. Bernardeau, S. Colombi, E. Gaztaña, and R. Scoccimarro, Phys. Rep. 367, 1 (2002), arXiv:astro-ph/0112551.
[5] Planck Collaboration, P. A. R. Ade, N. Aghanim, C. Armitage-Caplan, M. Arnaud, M. Ashdown, F. Atrio-Barandela, J. Aumont, C. Baccigalupi, A. J. Banday, et al., ArXiv e-prints (2013), 1303.5084.
[6] A. Kehagias and A. Riotto, Nuclear Physics B 873, 514 (2013), 1302.0130.
[7] M. Peloso and M. Pietroni, JCAP 5, 031 (2013), 1302.0223.
[8] P. Creminelli, J. Noreña, M. Simonović, and F. Vernizzi, JCAP 12, 025 (2013), 1309.3557.
[9] A. Kehagias, J. Noreña, H. Perrier, and A. Riotto, ArXiv e-prints (2013), 1311.0786.
[10] M. Peloso and M. Pietroni, ArXiv e-prints (2013), 1310.7915.
[11] P. Creminelli, J. Gleyzes, M. Simonović, and F. Vernizzi, ArXiv e-prints (2013), 1311.0290.
[12] P. Valageas, ArXiv e-prints (2013), 1311.1236.
[13] P. Creminelli, J. Gleyzes, L. Hui, M. Simonović, and F. Vernizzi, ArXiv e-prints (2013), 1312.6074.
[14] P. Valageas, ArXiv e-prints (2013), 1311.4286.
[15] A. Kehagias, H. Perrier, and A. Riotto, ArXiv e-prints (2013), 1311.5524.
[16] A. Taruya, F. Bernardeau, T. Nishimichi, and S. Codis, Physical Review D 86, 103528 (2012), 1208.1191.
[17] T. Nishimichi, A. Shirata, A. Taruya, K. Yahata, S. Saito, Y. Suto, R. Takahashi, N. Yoshida, T. Matsubara, N. Sugiyama, et al., Publ. Astron. Soc. Japan 61, 321 (2009), 0810.0813.
[18] P. Valageas and T. Nishimichi, Astronomy & Astrophysics 527, A87 (2011), 1009.0597.
[19] E. Komatsu, J. Dunkley, M. R. Nolta, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. Limon, L. Page, et al., Astrophys. J. Suppl. 180, 330 (2009), 0803.0547.
[20] R. W. Hockney and J. W. Eastwood, *Computer Simulation Using Particles* (1981).
[21] P. Valageas and T. Nishimichi, Astronomy & Astrophysics 532, A4 (2011), 1102.0641.
[22] P. Valageas, T. Nishimichi, and A. Taruya, Phys. Rev. D 87, 083522 (2013), 1302.4533.
[23] P. Valageas, Phys. Rev. D 88, 083524 (2013), 1308.6755.
[24] Once we take the angular average for $k'$, $\bar{B}(k'; k)$ no longer has any angular dependence on $k$ and thus the additional angular average over $k$ does not change the expectation value.
[25] Remember that the high-resolution simulations have twice better spatial resolution than the low-resolution ones.
[26] Note that the time derivative of the density field is equivalent to the momentum field from the continuity equation.