Behavioral Intervention and Non-Uniform Bootstrap Percolation

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Abstract

Bootstrap percolation is an often used model to study the spread of diseases, rumors, and information on sparse random graphs. The percolation process demonstrates a critical value such that the graph is either almost completely affected or almost completely unaffected based on the initial seed being larger or smaller than the critical value. In this paper, we consider behavioral interventions, that is, once the percolation has affected a substantial fraction of the nodes, an external advisory suggests simple policies to modify behavior (for example, asking vertices to reduce contact by randomly deleting edges) in order to stop the spread of false information or disease. We analyze some natural interventions and show that the interventions themselves satisfy a similar critical transition.

To analyze intervention strategies we provide the first analytic determination of the critical value for basic bootstrap percolation in random graphs when the vertex thresholds are nonuniform and provide an efficient algorithm. This result also helps solve the problem of “Percolation with Coinflips” when the infection process is not deterministic – which has been a criticism about the model. We also extend the results to “clustered” random graphs thereby extending the classes of graphs considered. In these graphs the vertices are grouped in a small number of clusters, the clusters model a fixed communication network and the edge probability is dependent if the vertices are in “close” or “far” clusters. We present simulations for both basic percolation and interventions that support our theoretical results.

1 Introduction

Bootstrap percolation is a model of choice in many contexts modeling spread of information, diseases, etc.

Definition 1 (Bootstrap percolation). Let $G$ be a graph with $n$ vertices and $r : V(G) \rightarrow \mathbb{N}$ drawn from some family. Given $G$, select $\varphi$ vertices uniformly at random without replacement and mark them as infected. Vertex $u$ becomes infected when it has $r(u)$ or more infected neighbors. $G$ is declared to be infected if $n - o(n)$ vertices are infected. The central question is to determine the existence and quantify the parameter $\Phi$ such that the graph exhibits a sharp dichotomy. That is, for any fixed $\epsilon > 0$, if $\varphi > (1 + \epsilon)\Phi$ then $G$ becomes infected with probability vanishingly close to 1, and if $\varphi < (1 - \epsilon)\Phi$, $G$ does not become infected with probability vanishingly close to 1.

An extensive and rich literature exists on the topic of bootstrap percolation which we discuss shortly. In this work we focus on decentralized behavioral intervention strategies. Suppose that the infection is propagating sufficiently slowly. After the percolation has spread to $\lambda n$ nodes the nodes are instructed to behave differently (e.g., communicate less, become less susceptible to...
new information) which leads to increases of \( r(u) \). Alternatively, the nodes reduce contact \( r(u) \) which corresponds to dropping edges at random. If the infection has spread reasonably then the graph already has a “residual state” and not all interventions are useful \( r(u) \) thus we need to quantify how even simple transformations affect the percolation, and such analysis does not exist in the literature. The natural questions we ask here are: Does the required intervention also exhibit a sharp phase transition between failure and success? Can that region of transition be explicitly quantified?

We envision the primary application of large scale intervention to be useful in the domain of swarm of particles or agents which choose a random network to interact with each other \([22]\). The intervention in this context arise from the following: if a large fraction of the swarm is showing undesirable behavior which is spreading \( r(u) \) what is the effort required to stabilize such a system? The same question can be asked for communication networks of machines which often adopt a random topology for communication and efficiency purposes. In particular we focus on a hierarchically clustered graph where \( n/k \) vertices are in each cluster and the communication between two nodes in different clusters is an independent random variable which only depends on the finite cluster topology defined on the \( k \) supernodes. We intentionally do not discuss networks with power law distribution in this paper because in such graphs, the spread shows a phase transition and the graph is infected almost immediately or not at all \([2]\). In particular the infection spreads to all high degree nodes in generation 1, then to a large percentage of the graph in generation 2. Intervention is difficult to imagine in such a context given such a dramatic change in the number of infected nodes a single step. Moreover in the context of swarms or social networks for machines, power law behavior is unlikely to be desirable from the perspective of communication bottlenecks.

**Challenges and Context.** Perhaps unsurprisingly, to prove sharp dichotomy results for intervention, we need to strengthen and extend existing results for bootstrap percolation for random graphs to many natural generations of independent interest. Consider:

**(a1) Non-Uniform Thresholds.** The overwhelming majority of the literature focuses on uniform constant thresholds, that is, \( r(u) = r \) is the same constant for all vertices. Even for the simplest possible random graph model, the Erdős-Rényi model, classic results such as that of Janson et al \([16]\) (see also \([21]\)) only provide bounds for this uniform case. This has to be remedied to provide twosided analysis of interventions \( r(u) \) – because at the time the intervention happens, there is already residual state \( r(u) \) (the set of infected vertices). For a healthy vertex \( u \) with 3 infected neighbors, the threshold is now \( r(u) - 3 \). This corresponds to a distributional specification of \( r(u) \) which is a natural problem.

**(a2) Small number of early adopters or easily influenced/susceptible nodes.** Moreover the existing literature on bootstrap percolation focuses on the case where the vertex thresholds are greater than 2. This is understandable, because threshold 1 correspond to a connectivity. In particular for a Erdős-Rényi graph \( G(n,p) \) if \( np > 1 \) then there exists a giant connected component. Therefore if the fraction of threshold vertices is \( \zeta_1 \) and we have \( np\zeta_1 > 1 \) then we will have a giant connected component in the subgraph induced by the threshold 1 vertices and percolation will be instantaneous. However the existing literature does not handle the complementary and natural regime where \( np\zeta_1 \leq 1 - \beta \) for some \( \beta > 0 \) and \( \zeta_1 \ll 1 \) – that is, we have a few (non-negligible) “early adopters” who are influenced as soon as they are in contact with a new idea or “easily” susceptible individuals who fall sick at first contact, but the remainder of the vertices exhibit the key bootstrap percolation property of waiting to see more evidence of sufficient contact.

**(a3) Non-Deterministic Transitions.** The behavior of bootstrap percolation that a node deterministically becomes infected when \( r(u) \) of its neighbors are infected have often been criticized.
A slightly modified but very natural model is Percolation with Coin Flips: an individual node $u$ becomes susceptible (but not infected) after contact with $s(u)$ infected nodes. Each subsequent contact with an infected node infects $u$ with probability $z(u)$, say determined by an independent coin flip. Intuitively node $u$ behaves like having a threshold of $r(u) \approx s(u) + 1/z(u)$ but the transitions are not deterministic. However as discussed in the example above an expected threshold of $5r(u)/4$ can be worse than a deterministic threshold of $r(u)$, and therefore the intuition $r(u) \approx s(u) + 1/z(u)$ is not usable for analysis. No analysis of this natural problem of percolation with coinflips exists in the literature to date.

(a4) Hierarchical Networks with Few Levels. No quantitative analysis of sharp dichotomy for bootstrap percolation exists for random graphs which are hierarchical in nature – even for hierarchies which are just two levels! While Erdős-Rényi Graphs certainly are not often a sufficient model of behavior, hierarchical models can model complex phenomenon [9, 10]. Note however that multilevel iterative products lead to power law behavior [20].

We note that there has been studies on stopping the spread of infection in social networks – however those strategies have typically been (i) centralized or before the fact, i.e., before the disease starts spreading, see [17] and references therein; or (ii) 0/1 vaccination, i.e., a node is removed from the graph or unmodified, see [19] and references therein. None of those approaches solve (a1)–(a3). We discuss the result of Janson et al [16] (see also [21]) before proceeding further, other related work which are somewhat orthogonal to the line of inquiry in this paper is discussed at the end of the section. In the $G(n,p)$ notation for Erdős-Rényi graphs, an edge between a pair of vertices is present with probability $p$ (independent of other edges). Under a set of standard assumptions, such as $pn = \omega(1)$ (a slowly growing function of $n$) the dichotomy occurs when

$$\left(1 - \frac{1}{r}\right)\left(\frac{(r-1)!}{np^r}\right)^{1/(r-1)}$$

vertices are seeded initially. Here $r(u) = r$ for all $n$ vertices. The results on non-uniform thresholds are minimal. Watts [23] studied the case on Erdős-Rényi graphs where $r(u) = c \deg(u)$ for some $c \in [0,1]$. Amini [1] studied the case on random graphs of a given degree sequence where $r(u) = g(\deg(u))$ and $g$ is a fixed deterministic function – however the results in that paper demonstrate the existence of a sharp dichotomy and explicitly leave open the computational question. Note however that such arguments cannot work if $g()$ is changed midway through the percolation as is the case in interventions. In the context of fixed graphs with $r(u) = r$ for all vertices, Holroyd [15] proved a bound on $a$ that leads to infection on the 2-dimensional grid, which was later improved by Gravner et al [14]. Balogh et al proved a corresponding bound for the 3-dimensional grid [5], and later proved a general bound for the $d$-dimensional grid [4]. Other results have been found for hypercubes [3], tori [13], expander graphs [11], homogeneous trees [12], regular trees [6] and $d$-regular graphs [7]. For an arbitrary $G$, approximating the minimum $\Phi$ that leads to infection within small factors is hard under reasonable complexity assumptions [8].

Results and Techniques. We discuss basic percolation problems (a1)–(a4) in 1.1. We discuss interventions next in Section 1.2 and provide proofs of sharp dichotomy and consider several simulations in Section 1.3.

1.1 Results for Basic Percolation Problems

We resolve the three scenarios (a1)–(a4) posited above. In particular, we analyze a Templated Multisection graph where the vertex specific thresholds $r(u)$ satisfy $1 \leq r(u) \leq r_m$ for some constant $r_m$. The Templated Multisection graph is defined as:
Definition 2 (Templated Multisection Graph). Let \([k] = \{0, 1, \ldots, k−1\}\). Suppose that we are provided a finite template graph \(F\) which is a undirected regular graph on \([k]\). The neighborhood of vertex \(i \in [k]\) is given by the function \(N_F : [k] \rightarrow 2^{[k]}\); where \(i \in N_F(j)\) iff \(j \in N_F(i)\). Suppose \(|N_F(i)| = k_p \leq k\). Define \(G\) to be a graph with \(n\) vertices evenly partitioned into \(k\) clusters of \(n/k\) vertices each. Let \(\chi(u)\) denote the index of the partition \(u\) belongs to. If \(\chi(u) \in N_F(\chi(v))\), include edge \((u, v)\) with probability \(p\). Otherwise, include edge \((u, v)\) with probability \(q\). We denote the family of graphs defined in this process as \(TM(F, n, k_p, k_q, p, q)\) where \(k_q + k_p = k\).

While some of the results in this paper will extend to clusters of non-uniform sizes (provided each cluster is large), we omit their discussion in the interest of brevity. The \(TM(F, n, k_p, k_q, p, q)\) family is illustrated by the following:

- Erdős-Rényi graphs. This corresponds to a single cluster, \(k = 1\) and \(N_F(0) = \{0\}\). In this case \(k_q = q = 0\).
- The Planted Multisection graph is a generalization to \(k\) clusters with \(N_F(i) = \{i\}\). In this case \(k_q = k − 1\).
- Any succinctly described constant degree graph can be used as the template graph – since the intuitive purpose of \(F\) is to determine the communication behavior of nodes in the clusters. Of particular interest is the “ring” type communication where \(N_F(i) = \{i ± a \mod k\}\) for \(|a| \leq \ell\) which defines a ring of \(k\) vertices each node connected to \(2\ell\) closest neighbor. We can also explicitly use any fixed size small world graph.

Notation: We use \(\eta = n/k\) to denote the number of vertices in a cluster and use \(\phi = k_p p + k_q q\). The parameters \(\eta, \phi\) correspond to \(n, p\) in the Erdős-Rényi model. We say \(u\) is ‘near’ \(v\) if \(\chi(u) \in N_F(\chi(v))\) and \(u\) is ‘far’ from \(v\) if \(\chi(u) \notin N_F(\chi(v))\). \(Bin(x, \lambda)\) denotes a binomial distribution with \(x\) elements and per trial probability of success \(\lambda\).

Theorem 1 (Proved in Section 2\(^\text{[2]}\)). Let \(r_m = O(1)\) and fix \(\delta, \beta, \epsilon > 0\). Let \(\{\zeta_r\}_{r=1}^{m}, \sum_{r=1}^{m} \zeta_r = 1\) define a distribution such that \(\zeta_1 < 2\zeta_3/3\). Fix \(q < p < 1/2\). Given a graph \(G\) from the family \(TM(F, n, k_p, k_q, p, q)\), with sufficiently many nodes \(n \geq n_0(\delta, \beta, \epsilon, k)\) for each \(u\), assign \(u\) threshold \(r\) with probability \(\zeta_r\). If \(\phi = pk_p + qk_q\), note \(\eta \phi\) is the expected degree. Define

\[
\pi_r(t) = \Pr[Bin(k_p t, p) + Bin(k_q t, q) \geq r]
\]

\[
A(t) = \sum_{r=1}^{r_m} \zeta_r \pi_r(t)
\]

\[
f(\varphi, t) = (n - \varphi) A(t) - kt + \varphi
\]

\[
t^*(\varphi) = \arg\min_{t \leq 1/(3\phi)} f(\varphi, t)
\]

\[
\Phi = \min_{\varphi} \{\varphi | \forall t \leq 1/(3\phi), f(\varphi, t) \geq 0\}
\]

\[
t^* = t^*(\Phi)
\]

Assume \((i)\) \(\eta \phi \zeta_1 \leq 1 - \beta\), i.e., the expected number of threshold 1 vertices adjacent to a node is small\(^\text{[2]}\)\((ii)\) \(\eta \phi = o(\sqrt{n})\), i.e., the graph is not dense otherwise percolation is immediate. Then

- If \(\delta \phi t \leq 1/3\) then \(A(t)\) is convex. Moreover \(t^* \geq \frac{3n}{2k(\zeta_1 \phi)} \rightarrow \infty\) as \(n \rightarrow \infty\).

- Suppose we choose \(\varphi\) vertices uniformly at random and set them as infected. If \(\varphi < (1 - \epsilon) \Phi\) then \(G\) does not become becomes infected with probability at least \(1 - O(\epsilon^{-2}/(t^* \beta^2(r_m+1)))\).
  If \(\varphi > (1 + \epsilon) \Phi\) then an absolute constant fraction of the nodes in \(G\) become infected with

\(^2\text{If} \eta \phi \zeta_1 > 1\) then discussion in (a2) applies.
probability at least \(1 - O(e^{-2/(t^r \beta^2(r_m + 1))})\) (slightly larger constant). Moreover if the expected degree \(\phi\) is a slowly growing function then with same expression of probability close to 1, the percolation does not stop till \(\eta - o(\eta)\) nodes are infected.

The probabilities of convergence with only absolute constants in the \(O()\) all evaluate to \(1 - O\left(\frac{t^{O(r_m)k(\eta \phi)^2}}{e^{t^r \beta^2 r_m + 3}}\right)\) and is (inverse) polynomially close to 1 when the expected degree \(\eta \phi\) is \(o(\sqrt{n})\).

Theorem 1 follows the argument template of Janson et al. [16], but differs significantly in the internal analysis. In case of uniform thresholds and a single cluster, it was sufficient to approximate the value of \(r\) as the value of \(O\).

Theorem 1.1 relies on the construction of a Martingale and a reverse Martingale for a fixed uniform threshold \(r(u) = r\) for an Erdős-Rényi graph. We show that we can construct similar martingales as the value of \(r\) is varied, even as the graph has multiple clusters. Martingales are invariant under addition and thus the key is to bound their step sizes. However the addition of martingales for different thresholds is manageable only with more precise approximations of \(A(t)\). The changes in the internal analysis reflects this key difference. However the surprising insight of the overall proof is that even though the percolation is nonlinear in the connectivity parameter, \(\Phi\) is linear in the distribution parameters, and the effects are separable!

The next result is a consequence of the separability and distributional result proven in Theorem 1. The basic intuition is that the coins can be “preflipped” ahead of time to reduce percolation with coinflips to a distribution over percolation with non-uniform thresholds.

**Corollary 2 (Percolation with Coinflips).** For \(G = TM(F, n, k_p, k_q, p, q)\), suppose the distribution of thresholds of is chosen as follows: a vertex becomes susceptible after \(s(u)\) neighbors of \(u\) have been infected. Subsequent to becoming susceptible, a vertex \(u\) becomes infected with probability \(z(u) > \delta > 0\) as soon as a new neighbor becomes infected where \(\delta \in [0, 1]\) is a constant or when \(r_m\) neighbors are infected. Given this setting we can determine the percolation threshold \(\Phi\) explicitly, even for non-uniform \(z(u)\). Observe that if \(s(u) \geq 1\) for a large fraction of the nodes then the condition \(\zeta_1 \leq 2\zeta_2/3\) holds, e.g., if \(s(u) \geq 1\) for \(2/3\) of the nodes then \(\zeta_1 \leq 2\zeta_2/3\) if all \(z(u) \leq 1/2\).

**Summary:** We ran several simulations to verify the application of Theorem 1 in the context of multiple graphs which are “small network of clusters”.

We focused on graphs with \(n = 10000\) nodes and rings with 10, 20 clusters, the 3-D cube with 8 clusters alongside the standard \(G(n, p)\) graph. We varied the distribution of the threshold in simple ways so that the results can be verified conceptually. The theorem and the simulations agreed and these are presented in Section 1.3. The network did not affect the thresholds significantly, but the expected degree and the distribution of the vertex thresholds had a strong impact as predicted. We now focus on intervention strategies.

### 1.2 Results for Interventions

We now discuss percolation problems when we have a chance to modify the behavior of edges or vertices. The classic example is that after the infection has spread to \(\lambda n\) individuals, better health practices are announced, i.e., which reduce contact (edges) or increase the thresholds of susceptibility and infection for a vertex. We consider the following interventions:

- **Bolster.** This corresponds to assigned every vertex with threshold \(r\) a new threshold \(r'\) from a distribution \(\zeta'(r)\). Note that \(\zeta'(r_1)\) may be different from \(\zeta'(r_2)\).
• **Delay.** This corresponds to modifying \( z(u) \) to \( z'(u) \) in the coinflips model for the remainder of the percolation. This is a special case of **Bolster**.

• **Sequester.** This corresponds to dropping the edges between healthy and the infected vertices independently with probability \( 1 - \alpha_p \) for the edges corresponding to neighborhoods in \( F \) and \( 1 - \alpha_q \) for the other edges. The edges remain dropped throughout the percolation. Edges connecting two healthy vertices are unaffected.

• **Diminish.** This corresponds to permanently dropping the edges between all vertices in the graph with probability \( 1 - \alpha_p \) for the edges corresponding to neighborhoods in \( F \) and \( 1 - \alpha_q \) for the other edges.

Note that the standard strategy of **vaccination** corresponds to setting \( r(u) = \infty \) for a healthy node \( u \), or equivalently, removing that node \( u \). There is a large literature on this removal behavior, see \cite{19} and references therein. Edge removal strategies have been considered in the literature, see \cite{17} and references therein. However none of the existing strategies can express the adaptive nature of the four interventions we consider.

We assume that the original graph \( G \) is chosen so that the assumptions in Theorem 1 are satisfied (\( G \) is sufficiently sparse) and that \( G \) has no threshold-1 vertices. We define \( \tau \) to be the generation at which the intervention is applied. Let \( I(\tau) \) be the set of infected vertices at generation \( \tau \) and define \( I(\tau - 1) \) similarly. Let \( H(r) \) be the set of threshold-\( r \) healthy vertices. This is all the data we need to determine whether the intervention is successful.

**Theorem 3** (Proved in Section 3). Assume \( n, p, q, r, \lambda, I(\tau), I(\tau - 1), H(r) \) are known and that \( I(\tau) < k/(3\phi) \). Given either \( \zeta(\tau) \) (for **Bolster**), \( z' \) (for **Delay**), or \( \alpha_p \) and \( \alpha_q \) (for **Diminish** and **Sequester**), it is possible to determine whether the intervention is successful and \( G \) becomes infected with probability \( 1 - 1/poly(n) \).

When \( I(\tau) > k/(3\phi) \), \( I(\tau + 1) \) consists of a positive fraction of the nodes. In other words, the intervention has occurred too late and there are too many infected nodes to do a meaningful analysis. The key idea of the proof is that knowing \( I(\tau), I(\tau - 1), H(\tau) \), we define \( H_a \) to be the set of infected vertices with an infected neighbor. \( \Pr[u \in H_a] \) can be explicitly calculated. Using this information, we create a new TM graph \( J \) with \( |H| \) vertices and the same cluster structure as \( G \). For every vertex \( u \in H_a \), we add a vertex \( v \) with threshold \( r'(u) = a \) to \( J \). In this way, the probability \( J \) becomes infected is equal to the probability \( G \) becomes infected, and we can use Theorem 1.

Less formally, when the intervention is applied to \( G \), every vertex has a “residual” state which we can estimate. Interestingly, we can estimate this information knowing only \( I(\tau) \) and \( I(\tau - 1) \); we do not need to know any information about the early generations, this corresponds to the memoryless martingale behavior of the basic percolation processes.

This construction also showcases the need to resolve (a1) and (a2). Even if \( G \) has uniform thresholds, \( J \) will have non-uniform thresholds and \( J \) will have a small number of easily influenced threshold-1 vertices.

### 1.3 Simulations

To test Theorem 1 and Corollary 2, we performed several different simulations. The parameter \( \epsilon = 0.1 \) and for each setting we repeated the experiments for 50 graphs and for each graph the experiment was repeated 50 times. We use the color **green** to denote the cases where the percolation stopped (graph was mostly healthy) and color **red** when the infection spread exceeded 90% of the nodes. The number of vertices was always 10000. We also varied \( k = \{1, 10, 20\} \). For \( k > 1 \) we
chose the ring topology with $\ell = 1$ (which corresponds to $k_p = 3$). Figure 1 consider nonuniform thresholds and for $k = 1$ we chose $p = 10/n$. For $k = 10$ we set $p = 50/(3n)$ and $q = 50/(7n)$. For $k = 20$, we chose $p = 100/(3n)$ and $q = 100/(17n)$. Note that for all cases $pk_p \eta = 5$ and $qk_q \eta = 5$ where $\eta = n/k$, that is the average degree is 5 within the cluster and 5 outside the cluster. We repeat the same settings as in Figure 1 for percolation with coinflips and the results are in Figure 2. The case for $k = 10$ was close to the two extreme cases shown. Again the results are parallel those in Figure 1 even though $r_m = 20$.

(a) Erdős-Rényi $k = 1$ case  
(b) Ring of $k = 10$ clusters  
(c) Ring of $k = 20$ clusters

Figure 1: Non Uniform Thresholds: the $x$ axis indicates the fraction of vertices which have threshold 3 the remainder have threshold 2. The two lines correspond to $(1 - \epsilon)\Phi, (1 + \epsilon)\Phi$ when the threshold was $\Phi$. Each of the points correspond to the average of 50 experiments and the color depends on the fraction of times the percolation succeeded (red) or stopped (green). For the chosen settings Theorem 1 predicts that the thresholds would be the same for $k = 10$ and $k = 20$. The $k = 1$ case is different but the underlying random variables are close in distribution.

(a) $k = 1$ case  
(b) Ring of $k = 20$ clusters

Figure 2: Percolation with coinflips: For interpretability we considered $s(u) = 1$ and $z(u) = z$ for all vertices. We do not show the $k = 10$ case which is similar to the above two. $r_m$ was set to 20. The $x$ axis indicates the coin probability.

We consider unbalanced setups where the expected degree of a node within a cluster is different from the expected degree of the node outside, i.e., $pk_p \neq qk_q$, in Figure 3a. In Figure 3b we consider the clusters arranged as the vertices of a 3-D cube where we have $k = 8$ clusters and $k_p = 4$. In all cases the result is consistent with the prediction of Theorem 1. Although the network structure did not affect the critical values and the percolation, as long as the expected degree was the same, changing the expected degree had a much greater impact. In Figure 4 we change the average degree parameter (and the threshold to be between 3 and 4) — as predicted, the effect is clearly seen on the critical value of percolation.
Figure 3: Different topologies of graphs, Average degree is 10 in both cases. In (3a) \( p = \frac{70}{3n}, q = \frac{30}{7n} \), whereas in (3b) \( p = \frac{15}{n} \) and \( q = \frac{5}{n} \).

Figure 4: Ring of \( k = 10 \) clusters, thresholds between 3 and 4: we vary the total (expected) degree of the nodes. We kept the y-axis scale the same for comparison.

Interventions. Figure 5 and Figure 6 show the results of BOLSTER. In both simulations, \( n = 10000, \lambda = .1, k = 1, r = 2 \) uniformly. Figure 5 depicts an Erdős-Rényi graph with \( k = 1, p = \frac{7}{n} \). Figure 6 depicts a ring graph with \( k = 10, p = \frac{4}{3000}, q = \frac{3}{7000} \). In both figures, green dots imply percolation stopped and red dots imply the spread was complete. The blue dots and line correspond to \( 1 - \epsilon \) times the expected cutoff point, where \( \epsilon = .1 \). The black x’s and line correspond to \( 1 + \epsilon \) times the estimated cutoff.

Figure 5: Graph is Erdős-Rényi. The x-axis corresponds to \( I(\tau) \) plus the number of just infected vertices. Each vertical line corresponds to a graph (50 such graphs) and for each value of \( \alpha \) we show the average of 50 trials. The blue dots correspond to the estimated lower bound with \( \epsilon = .1 \) and the x corresponds to the estimated upper bound. Strategy A seems to be better amenable to estimation. Green dots imply that the percolation stopped and red implies that the spread was complete.
We consider two possible BOLSTER interventions. For BOLSTER-A (Figure 5a and 6a), with probability $\alpha$ we increase the threshold by 1 (to 3) and with probability $1 - \alpha$ we increase the threshold by 2 (to 4). For BOLSTER-B (Figure 5b and 6b), with probability $1/2 + \alpha/2$ we increase the threshold by 1 (to 3) and with probability $1/2 - \alpha/2$ we increase the threshold by 2 (to 5). Note that for both interventions, the expected post-intervention threshold is $4 - \alpha$ and that $\alpha = 0$ corresponds to the strongest possible intervention.

The first interesting result is that BOLSTER-A is substantially more powerful than BOLSTER-B (the intervention is successful with a higher value of $\alpha$). For example, having 50% threshold-3 vertices and 50% threshold-5 vertices is worse than having 100% threshold-4 vertices; the vulnerable threshold-3 vertices become infected and then they infect the threshold-5 vertices.

The second interesting result is that for two graphs $G_1$ and $G_2$, there are times when $G_1$ has more infected nodes than $G_2$ but it is easier to stop the infection on $G_1$ than in $G_2$. This is because there are two factors that determine the effectiveness of the intervention: $I(\tau)$ and $I(\tau) - I(\tau - 1)$. Thus, instead of having a two-dimensional decision boundary, we have a three-dimensional boundary. We illustrate this boundary in Figure 7, which is the three-dimensional version of Figure 5a (BOLSTER-A and $k = 1$).

The final interesting result is that BOLSTER-B is substantially nosier than BOLSTER-A. Pulling apart a single simulation reveals why. In Figure 8, we zoom in on a single graph and compare various hypothetical interventions. The x-axis corresponds to the value of $\alpha$, and black line corresponds
to the hypothetical value of $I(\tau)$ that would lead to the spread of infection given this value of $\alpha$. The actual value of $I(\tau)$ along with simulated results is depicted by the red-green line. When the black theoretical value is greater than the actual $I(\tau)$, we expect the percolation to stop (and the result-line should be green). When the black theoretical value is less than $I(\tau)$, we expect the percolation to spread (and the result-line should be red).

Figure 8: Black line corresponds to hypothetical value of $I(\tau)$ that would lead to spread of percolation. Red-green line corresponds to the actual value of $I(\tau)$. Green dots imply percolation stopped and red dots imply the spread was complete. Notice that when the black value is greater than $I(\tau)$, the percolation stops but when the black value is less than $I(\tau)$, the percolation spreads.

Notice that the BOLSTER-A theoretical line is substantially steeper than the BOLSTER-B line. This is the reason the BOLSTER-B intervention is so noisy, and also the reason why BOLSTER-A is a stronger intervention than BOLSTER-B; small changes in $\alpha$ dramatically improve the strength of the intervention.

We can perform the same analysis on DIMINISH and SEQUESTER. Recall that DIMINISH deletes every edge with probability $1 - \alpha$, whereas SEQUESTER only deletes edges connected to an infected vertex. Figure 9 shows these results, using $p = 15/n$ and $r = 3$ uniformly.

Figure 9: The $x$-axis and vertical lines have the same meaning as in Figure 5. Here the edges are dropped with probability $1 - \alpha$.

Notice that DIMINISH is a substantially stronger intervention than SEQUESTER which is expected as DIMINISH deletes all edges whereas SEQUESTER only deletes a subset of the edges. The figure for $k = 10$ is similar and we omit it for space. Instead, for $k = 10$, $p = 9/3000$ and $q = 6/3000$, we will perform a similar analysis as Figure 8 and zoom in on a single graph. When $\alpha_p = \alpha_q = \alpha$, we obtain Figure 10.
Figure 10: Edges are dropped with probability \(1 - \alpha_p\) or \(1 - \alpha_q\) where \(\alpha_p = \alpha_q = \alpha\). The lines have the same meaning as Figure 8.

Our results also hold for the case where \(\alpha_p \neq \alpha_q\), which is depicted in Figure 11 (using the same graph as Figure 10 for ease of comparison).

Figure 11: Edges are dropped with probability \(1 - \alpha_p\) or \(1 - \alpha_q\) where \(\alpha_p = \alpha\) and \(\alpha_q = (2/3)\alpha\). The lines have the same meaning as Figure 8.

Summary. We can construct various intervention strategies and accurately predict whether the percolation will halt or spread. We can also compare various intervention strategies to determine which strategy is more effective.

2 Proof of Theorem 1

Recall the definition of Templated Multisection graphs in Definition 2 in Page 4.

Definition 3. Let \(\phi = pk_p + qk_q\) and \(\pi_r(t) = \Pr[Bin(k_p t, p) + Bin(k_q t, q) \geq r]\)

\[
A(t) = \sum_{r=1}^{\min} \zeta_r \pi_r(t)
\]

\[
f(\varphi, t) = (n - \varphi)A(t) - kt + \varphi
\]

\[
t^*(\varphi) = \arg\min_{t \leq 1/(3\phi)} f(\varphi, t)
\]

\[
\Phi = \min_{\varphi} \{\varphi | \forall t \leq 1/(3\phi), f(\varphi, t) \geq 0\}
\]

Note \(t^* = t^*(\Phi)\). Observe that \(\eta \phi\) is the expected degree.

Theorem 4 follows from Theorem 4 and Theorem 5. Theorem 4 proves the existence of \(\Phi\) and Theorem 5 shows that this seed value shows the desired sharp dichotomy.

Theorem 4 (Proved in Section 2.1.1). If \(\zeta_1 < 2\zeta_2/3, p, q \leq 1/2\) and \(\phi t \leq 1/3\) then \(A(t)\) is convex. If \(\varphi_1 < \varphi_2\) then \(t^*(\varphi_1) \leq t^*(\varphi_2)\). Moreover if for some constant \(\beta > 0\) we have \(\zeta_1 \eta \phi \leq 1 - \beta\) and \(\eta \phi = o(\sqrt{3n/k})\) then \(t^* \geq \frac{\beta n}{2k(\phi \eta)^2} \to \infty\) as \(n \to \infty\).
Theorem 5. Let $\delta, \epsilon > 0$. Let $G$ be a TM graph with sufficiently large number of nodes $n \geq n_0(\delta, \beta, \epsilon, k)$. Suppose we choose $\varphi$ vertices uniformly at random and set them as infected. If $\varphi < (1-\epsilon)\Phi$ then $G$ does not become becomes infected with probability at least $1 - O(\epsilon^{-2}/(t^*\beta^2(\rho_{m+1})))$. If $\varphi > (1+\epsilon)\Phi$ then an absolute constant fraction of the nodes in $G$ become infected with probability at least $1 - O(\epsilon^{-2}/(t^*\beta^2(\rho_{m+1}))) - O(1/t^*)$. Moreover if the expected degree $\eta$ is a slowly growing function then with probability $1 - O(\epsilon^{-2}/(t^*\beta^2(\rho_{m+1}))) - O(1/t^*) - 1/\eta^{O(1)} = 1 - O(\epsilon^{-2}/(t^*\beta^2(\rho_{m+1})))$, the percolation does not stop till $\eta - o(\eta)$ nodes are infected.

Forced Linearizations. We begin our proof of Theorem 5 by defining two notions: Halting and Cheating three-stage percolations. Halting percolation is pessimistic: it stops the moment it encounters a problem. If $G$ is infected by halting percolation, it will be infected.

Definition 4 (HALTING THREE-STAGE PERCOLATION). Let $G$ be a Templated Multisection graph where every vertex $u$ has threshold $r(u)$. Vertices can have three states: healthy, latent, and contagious. At timestep 0, select $\varphi$ uniformly across the graph and mark them as latent. Mark all other vertices as healthy. At every timestep, choose one latent vertex in every cluster and mark it as contagious. Then, every healthy vertex $u$ with $r(u)$ or more contagious neighbors become latent. The process terminates the first time any cluster has zero latent vertices.

Our second definition is Cheating Three-Stage Percolation. Cheating percolation is optimistic: it cheats by making vertices contagious even if they have fewer neighbors (than the corresponding thresholds) infected. If $G$ is not infected by cheating percolation, it will not be infected.

Definition 5 (CHEATING THREE-STAGE PERCOLATION). Use the same initialization as Halting Three-Stage Percolation. At every timestep, choose one latent vertex in every cluster and mark it as contagious. If there are no latent vertices in a cluster, instead choose one healthy vertex in that cluster and mark it as contagious. The process terminates the first time every cluster has zero latent vertices.

For $k = 1$ the two definitions coincide and are the same. Theorem 5 follows from Lemma 12 and Lemma 9. The next lemma addresses the growth in $\pi_r(t)$.

Lemma 6 (Proved in Section 2.1.2). For any $p \geq q, x \geq 1$ and any $t \geq 4r$ with $\phi xt \leq 1/3$, we have $\pi_r(zt) \leq 3 \left(\frac{4x}{3(1-p)}\right)^r \pi_r(t)$ and if $3x(1-p) > 4$ then $\pi_r(t) \leq 4 \left(\frac{4}{3x(1-p)}\right)^r \pi_r(zt)$.

Definition 6. Let $\varphi_i^r$ be the number of seeded vertices in cluster $i$ with threshold $r$. Let $S_i^r(t)$ to be the number of non-seeded vertices in cluster $i$ that have threshold $r$ and have $r$ or more infected neighbors then $S_i^r(t)$ is a random variable which is $\text{Bin}(\eta_r - \varphi_i^r, \pi_r(t))$ where $\eta_r$ is the number of vertices with threshold $r$. Note $\mathbb{E}[\eta_r] = \zeta_r n/k$. Let $S_i^\gamma(t) = \sum_{t=2}^{\rho_{m+1}} S_i^r(t)$ and $S(t) = \sum_i S_i^\gamma(t)$.

The arguments in [16] for a fixed threshold can be modified to prove the next lemma, it pretends that the percolation for different thresholds are proceeding simultaneously. For a fixed threshold the derivation uses a martingale argument and Doob’s $L_2$ inequality which bounds the deviation of the entire trajectory from the expectation. However martingales are preserved under addition – we bound the per step maximum value for which we use Lemma 6 (first part).

Lemma 7 (Proved in Section 2.1.3). Let $t_0 = \frac{96}{(1-p)^2\beta^2} t^*$. For $q \leq p \leq 1/2$ and all fixed $\gamma, \beta > 0$ if $m \geq m_0(\beta, \gamma, k)$ which is sufficiently large, with probability at least $1 - c_0/(\gamma^2 \beta^2(\rho_{m+1} t^*))$, for any absolute constant $c_0$, simultaneously for all $i$,

$$\sup_{1 \leq i \leq t_0} \left| S_i(t) - \left(\eta - \frac{\varphi}{k}\right) A(t) \right| \leq \gamma t^*$$
The next lemma follows from using the second part of Lemma \[6\]

**Lemma 8.** \( \Phi \) is not too small, i.e., \( \frac{9(1-p)^2 \beta^2}{128} kt^* \leq \Phi \). Note \( p \leq 1/2 \) and \( \beta > 0 \).

**Proof.** Recall \( f(\varphi, t) = (n - \varphi) A(t) - kt + \varphi \) and \( t^*(\varphi) \) is the value of \( t \) that minimizes \( f(\varphi, t) \).

Consider decreasing \( \Phi \) to be a fractional value \( \Phi' \) such that

\[
f(\Phi', t^*) = (n - \Phi') A(t^*) - kt^* + \Phi' = 0 \quad \forall t \leq 1/(3\Phi) f(\Phi', t) \geq 0
\]

Now \( (n - \Phi') A(t^*) = kt^* - \Phi' \). Set \( z = \frac{32}{9(1-p)^2} \) \( \) and \( f(\Phi', t^*/z) \geq 0 \) rewrites as

\[
0 \leq f(\Phi', t^*/z) = (n - \Phi') A(t^*/z) - kt^*/z + \Phi' = (n - \Phi') \zeta_1 \pi_1(t^*/z) + (n - \Phi') \sum_{r=2}^{r_m} \zeta_r \pi_r(t^*/z) - kt^*/z + \Phi' \quad \text{(Expanding A)}
\]

Now for \( t \geq 1 \), we have \( \pi_1(t) = 1 - (1 - p)^{k_\Phi t} (1 - q)^{k_\Phi t} \leq 1 - (1 - pk_t)(1 - qk_t) \leq pk_t + qk_t = t \phi \) (note \( (1-p)^z \geq 1 - pz \) for all \( z \geq 1 \)). Therefore

\[
0 \leq n\zeta_1 \phi t^*/z + (n - \Phi') \sum_{r=2}^{r_m} \zeta_r \pi_r(t^*/z) - kt^*/z + \Phi'
\]

\[
\leq (n - \Phi') \sum_{r=2}^{r_m} \zeta_r \pi_r(t^*/z) - \beta kt^*/z + \Phi' \quad \text{(Since } \zeta_1 \phi n/k \leq (1 - \beta)\text{)}
\]

\[
\leq \frac{64}{9z^2(1-p)^2} (n - \Phi') \sum_{r=2}^{r_m} \zeta_r \pi_r(t^*) - \beta kt^*/z + \Phi' \quad \text{(Using Lemma } [6] \text{ with } x = 1/z.\text{)}
\]

\[
= \frac{64}{9z^2(1-p)^2} (n - \Phi') A(t^*) - \beta kt^*/z + \Phi'
\]

\[
\leq \beta kt^*/(2z) - \beta kt^*/z + \Phi' = -\beta kt^*/(2z) + \Phi' \quad \text{(Expanding A)}
\]

\[
\phi \leq (1 - \epsilon) \Phi \text{ then the cheating percolation stops with probability } 1 - O(\epsilon^2/(t^* \beta^{2(r_m+1)})) \text{ for all sufficiently large } n.
\]

**Proof.** Consider decreasing \( \Phi \) to be a fractional value \( \Phi' \) such that \( \forall t \leq 1/(2p) f(\Phi', t^*) \geq 0 \) and

\[
f(\Phi', t^*) = (n - \Phi') A(t^*) - kt^* + \Phi' = 0
\]

Note \( \Phi' \geq \Phi - 1 \). (This step is also helpful in proving Lemma \[8\].) Observe that \( \Phi' \leq kt^* \) since \( A(t) \) is non-negative. Therefore \( nA(t^*) \leq kt^* + \Phi' A(t^*) \leq 2kt^* \). But notice that we assumed \( \eta \phi \geq 4 \) and \( t\phi \leq 1/3 \) and therefore \( kt^* \leq n/12 \). Therefore \( A(t^*) \leq 1/4 \).

Let \( \varphi = \Phi' - c\Phi \) (differs from \( (1 - \epsilon) \Phi \) by at most \( 1 \)) and \( t = t^*(\varphi) \). From Theorem \[4\] \( t \leq t^* \).

\[
(n - \Phi') A(t^*) - kt^* + \Phi' = 0 \implies (n - \varphi) A(t^*) - kt^* + \varphi = -(\Phi' - \varphi)(1 - A(t^*))
\]

\[
(n - \varphi) A(t^*) - kt^* + \varphi \leq -(\Phi' - \varphi)(1 - A(t^*))
\]

The last line follows from the fact that in the range \([t, t^*]\) the function \( f(\varphi, t) \) is increasing. But since \( A(t^*) \leq 1/4 \) we now have that

\[
(n - \varphi) A(t^*) - kt^* + \varphi \leq -3\epsilon \Phi/4
\]
Using $\gamma = \frac{e^9(1-p)^2\beta^2}{128}$ in Lemma 7 with probability $1 - \mathcal{O}(\epsilon^2/t^* \beta^{2(r_m+1)})$ for every cluster $i$, we have

$$\sup_{1 \leq i \leq t_0} |S^i(t) - \mathbb{E}[S^i(t)]| \leq \frac{e^9(1-p)^2\beta^2}{128} t^* \leq \frac{\epsilon}{2k} \Phi \quad \text{(Using Lemma 8)}$$

Therefore $S^i(t) \leq \mathbb{E}[S^i(t)] + \frac{\epsilon}{2k} \Phi$. Let $\varphi^i$ be the number of seed vertices in cluster $i$. Using Chernoff bounds we can assert that with probability $(1 - \delta/(2k))$, $\varphi^i \leq (1 + \epsilon/8)\varphi/k$ - observe that this result will hold when $\varphi \geq \frac{3k}{\epsilon} \ln \frac{2k}{\delta}$ but $\varphi = \Omega(t^*) \rightarrow \infty$. Therefore using union bound with probability at least $1 - \delta$, for every cluster $i$,

$$S^i(t) + \varphi^i \leq \mathbb{E}[S^i(t)] + (1 + \epsilon) \frac{1}{k} \Phi + \frac{\epsilon}{2k} \Phi = \frac{1}{k} \left( (n - \varphi) A(t) + \varphi - k t^* + \frac{5\epsilon}{8} \Phi \right) + t \leq \frac{1}{k} \left( - \frac{3\epsilon}{4} \Phi + \frac{5\epsilon}{8} \Phi \right) + t < t$$

Therefore with probability at least $1 - \delta$ the percolation stops in every cluster before $t$.

The large seed case: In the other case we show that if the percolation survives sufficiently past the bottleneck region then it leads to complete percolation. Note that for the following lemma we can assume that we started with a seed $\overline{\varphi} = (1 + \epsilon)\Phi$ and $\epsilon$ is small. If the seed size is larger we can simply ignore the remaining nodes. The proof is broken into three lemmas, culminating in Lemma 12.

Lemma 10 (Proved in Section 2.1.4). When the seed size is $\overline{\varphi} = (1 + \epsilon)\Phi$, $\epsilon \leq 1/9$ and we have reached $t = \min\{\frac{96}{(1-p)^2\beta^2} t^*, 1/(3\Phi)\}$ then with probability $1 - \mathcal{O}(\frac{1}{\overline{\varphi}})$, $S^i(t) > t - a$ for all $t \in [\frac{96}{(1-p)^2\beta^2} t^*, 1/(3\Phi)]$, i.e., as $n$ increases and $t^* \rightarrow \infty$ the percolation continues till $t = 1/(3\Phi)$.

Lemma 11 (Proved in Section 2.1.5). If the percolation has continued till $t = (3\Phi)^{-1}$ then with probability $1 - 1/n^{\Omega(1)}$, the percolation does not stop till a constant fraction of the graph is infected. Moreover if the expected degree $\phi \eta$ is a slowly growing function then with probability $1 - 1/n^{\Omega(1)}$, the percolation does not stop till $\eta - o(\eta)$ nodes are infected.

Lemma 12. If the expected degree is at least 2 and $\varphi > (1 + \epsilon)\Phi$ then for sufficiently large $n$ with probability $1 - \mathcal{O}(\epsilon^{-2}/\beta^{2(r_m+1)} t^*)$ the halting percolation continues till an absolute constant fraction of the nodes are infected. Moreover if the expected degree $\phi \eta$ is a slowly growing function then with probability $1 - \mathcal{O}(\epsilon^{-2}/\beta^{2(r_m+1)} t^*)$, the percolation does not stop till $\eta - o(\eta)$ nodes are infected.

Proof. As in the proof of Lemma 9 let $\overline{\varphi} = \Phi' + ct^*$ and $\overline{t} = t^*(\overline{\varphi})$. By Theorem 4, $t^* \leq \overline{t}$. Suppose that $(n - \overline{\varphi})A(\overline{t}) - k\overline{t} + \overline{\varphi} \leq 3\epsilon/4\Phi$ then since $\overline{t}, \Phi/k$ are at most $1/(3\Phi) \leq nk/6$ when $\epsilon \leq 1$ we again have $A(\overline{t}) \leq 1/4$. Suppose not. Then assume for contradiction,

$$(n - \overline{\varphi})A(\overline{t}) - \overline{t} + \overline{\varphi} < \frac{3\epsilon}{4} \Phi \leq \frac{3\epsilon}{4} t^* \leq \frac{3\epsilon}{4} \frac{1}{3\Phi} \leq \frac{3\epsilon}{4} \frac{\eta}{3\Phi} \leq \frac{3\epsilon}{4} \frac{n}{6k}$$

(assuming that the expected degree is at least 2) which implies (since $A(t) \leq 1$ for all $t$ and $\overline{\varphi}(1 - A(t)) \geq 0$)

$$nA(\overline{t}) \leq \frac{3\epsilon}{4} \frac{n}{6k} + \overline{t} \leq \frac{3\epsilon}{4} \frac{n}{6k} + \frac{n}{6k} \leq \frac{3\epsilon}{4} \frac{n}{6k} + \frac{n}{6k} \leq \frac{n}{4}$$

which implies that $A(\overline{t}) \leq 1/4$ when $\epsilon \leq 1$. Now using definition of $\Phi'$, at $t = \overline{t}$,
\[(n - \Phi')A(\bar{t}) - \bar{t} + \Phi' \geq 0 \implies (n - \varphi)A(\bar{t}) - \bar{t} + \varphi \geq (\varphi - \Phi')(1 - A(\bar{t}))\]
\[\implies (n - \varphi)A(\bar{t}) - \bar{t} + \varphi \geq \frac{3\epsilon}{4} \Phi\]

which is a contradiction. Since \(\bar{t}\) was the minimum,
\[(n - \varphi)A(t) - t + \varphi \geq \frac{3\epsilon}{4} \Phi \quad \forall t \leq 1/(3\varphi) \quad (1)\]

Again as in the proof of Lemma 9, with probability \(1 - O(\epsilon^2/t^*\beta^2(r+1))\) for every cluster \(i\),
\[
\sup_{1 \leq i \leq t_0} \left| S^i(t) - E[S^i(t)] \right| \leq \frac{e9(1-p)^2\beta^2}{128} t^* \leq \frac{\epsilon}{2k} \Phi
\]

Therefore with probability \(1 - O(\epsilon^2/t^*\beta^2(r+1))\) for all \(i\) and \(t \leq t_0\)
\[
S^i(t) + \varphi^i \geq \frac{1}{k} \left( (n - \varphi)A(t) + \varphi \right) - \frac{\epsilon}{2k} \Phi
\]
\[\geq \frac{1}{k} \left( \frac{3\epsilon}{4} \Phi - \frac{\epsilon}{2} \Phi \right) + t \geq t \quad \text{(Using Equation (1))}
\]

which implies that the percolation does not stop till \(t_0\). To complete the proof we now use Lemmas 10 and 11.

2.1 Omitted Proofs

2.1.1 Proof of Theorem 4

Definition 3. Let \(\phi = pk_p + qk_q\) and \(\pi_r(t) = \Pr[Bin(k_p, t, p) + Bin(k_q, t, q) \geq r]\)
\[
A(t) = \sum_{r=1}^{\infty} \zeta_r \pi_r(t) \quad f(\varphi, t) = (n - \varphi)A(t) - kt + \varphi
\]
\[
t^*(\varphi) = \arg\min_{t \leq 1/(3\varphi)} f(\varphi, t) \quad \Phi = \min_{\varphi} \{\varphi | \forall t \leq 1/(3\varphi), f(\varphi, t) \geq 0\}
\]

Note \(t^* = t^*(\Phi)\). Observe that \(\eta \Phi\) is the expected degree.

Theorem 4. If \(\zeta_1 < 2\zeta_2/3, p, q \leq 1/2\) and \(\phi t \leq 1/3\) then \(A(t)\) is convex. If \(\varphi_1 < \varphi_2\) then \(t^*(\varphi_1) \leq t^*(\varphi_2)\). Moreover if for some constant \(\beta > 0\) we have \(\zeta_1 \eta \Phi \leq 1 - \beta\) and \(\eta \Phi = o(\sqrt{\beta n/k})\) then \(t^* \geq \frac{3n}{2k(\eta \Phi)^2} \to \infty\) as \(n \to \infty\).

Theorem 4 follows from the next two theorems. We state both theorems and prove the latter theorem (Theorem 14) first since the former (Theorem 13) is a detailed verification of properties of binomial coefficients and Theorem 14 relies on Theorem 13.

Theorem 13. When \(\zeta_1 \leq 2\zeta_2/3 p, q \leq 1/2\) and \(\phi t \leq 1/3\) then \(A(t)\) is convex.

Theorem 14. If \(\varphi_1 < \varphi_2\) then \(t^*(\varphi_1) \leq t^*(\varphi_2)\). Further when \(\zeta_1 \leq 2\zeta_2/3, \) and for some constant \(\beta > 0\) (i) the expected degree \(\eta \Phi\) satisfies \(\eta \Phi \leq \sqrt{\beta n/k}\) and (ii) the fraction \(\zeta_1\) of vertices with threshold 1 satisfies \(\zeta_1 \eta \Phi \leq 1 - \beta\) then as \(n \to \infty\) implies \(t^* \geq \frac{3n}{2k(\eta \Phi)^2} \to \infty\).
Proof. (Of Theorem 14.) We use Theorem 13 to prove $A(t)$ to be a convex function. Note $A(0) = 0$. Suppose that we could find another convex function $\tilde{A}(t)$ such that $\tilde{A}(0) = 0$ and for all $t \leq 1/(3\phi)$, $\tilde{A}(t) \geq A(t)$.

Define $\tilde{f}(\varphi, t) = (n - \varphi)\tilde{A}(t) - kt + \varphi$ and let $\Phi' = \min_{\varphi}{\varphi | \forall t \leq 1/(3\phi), \tilde{f}(\varphi, t) \geq 0}$. Note that $\Phi' \leq \Phi$ since for all $\varphi$ we have $\tilde{f}(\varphi, t) \geq f(\varphi, t)$.

Now for a fixed seed $\varphi$, the value $t^*(\varphi)$ (extended to the reals from integers) corresponds to the point where the slope of $(n - \varphi)A(t)$ is $k$. Consider simultaneously the functions $(n - \Phi')\tilde{A}(t), (n - \Phi')A(t)$ and $(n - \Phi)A(t)$ and the corresponding points where they are tangent to a line with slope $k$. This is shown if Figure 12.

![Figure 12: Bounding $t^*$](image)

Then $t^*(\Phi')$ is bounded below by $t'(\Phi')$ and $t^*(\Phi')$ is bounded above by $t^*(\Phi)$. Therefore if we prove that $t'(\Phi') \to \infty$ then $t^* = t^*(\Phi) \to \infty$ as well. Observe that this observation also proves if $\varphi_1 < \varphi_2$ then $t^*(\varphi_1) \leq t^*(\varphi_2)$.

To choose $\tilde{A}$ we first observe that if we increase $\zeta_2$ by $\Delta$ and decrease any $\zeta_r$ by $\Delta$ for any $r \geq 3$ then $A(t)$ does not decrease and continues to remain convex (note that we continue to satisfy the constraint involving $\zeta_1$). This implies that we can assume $\zeta_2 = 1 - \zeta_1$ and let this new function be $A_1(t)$ and by construction $A_1(t) \geq A(t)$. Now

$$A(t) \leq A_1(t) = \zeta_1 \pi_1(t) + (1 - \zeta_1) \pi_2(t)$$
$$= 1 - \Pr[\text{Bin}(k_p t, p) + \text{Bin}(k_q t, q) = 0] - (1 - \zeta_1) \Pr[\text{Bin}(k_p t, p) + \text{Bin}(k_q t, q) = 1]$$
$$\leq 1 - e^{-\phi t} - (1 - \zeta_1) \phi t e^{-\phi t} + 2p^2 k_p t + 2q^2 k_q t \quad \text{(Le Cam’s Theorem [18])}$$

where the last step follows from Le Cam’s Theorem of approximating the sum of bernoulli distributions by a Poisson process. In this case we were summing $kt$ Bernoulli processes of which $k_p t$ had probability $p$ and $k_q t$ had probability $q$. Now $2p^2 k_p + 2q^2 k_q \leq 2\phi^2$ and therefore we set:

$$\tilde{A}(t) = 1 - e^{-\phi t} - (1 - \zeta_1) \phi t e^{-\phi t} + 2\phi^2 t$$

It is immediate that $A(t) \leq \tilde{A}(t)$ and

$$\tilde{A}'(t) = \zeta_1 \phi e^{-\phi t} + (1 - \zeta_1) \phi^2 t e^{-\phi t} + 2\phi^2$$
$$\tilde{A}''(t) = \phi^2 e^{-\phi t} [ - \zeta_1 + (1 - \zeta_1) - \phi t (1 - \zeta_1) ]$$

Based on $\phi t \leq 1/3$ and $\zeta_1 \leq 2(1 - \zeta_1)/3$ we have $\tilde{A}''(t) \geq 0$ and $\tilde{A}$ is convex. At $t = t'(\Phi')$

$$k = (n - \Phi')\tilde{A}'(t'(\Phi')) \leq (n - \Phi') \left[ \zeta_1 \phi + \phi^2 (t'(\Phi') + 2) \right]$$

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Using $\zeta_1 \eta \phi \leq (1 - \beta)$ and $\eta = n/k$ we get

$$t'(\Phi') + 2 \geq \frac{1}{\varphi^2} \left[ k \frac{1}{n - \Phi'} - \frac{(1 - \beta)k}{n} \right] = \frac{1}{\varphi^2} \frac{\beta nk + (1 - \beta)\Phi' \kappa}{(n - \Phi')n} \geq \frac{1}{\varphi^2} \frac{\beta nk}{k(\eta \phi)^2}$$

The theorem follows. \hfill \Box

**Proof. (Of Theorem 13)** We begin with some notation.

**Definition 7.** Define $B_i(t) = \Pr[\text{Bin}(k_i p_t, p) = i]$, $C_j(t) = \Pr[\text{Bin}(k_j q_t, q) = j]$ and $D_r(t) = \Pr[\text{Bin}(k_r p_t + \text{Bin}(k_q t, q) = r]$. Observe that

$$D_r(t) = \sum_{i=0}^r B_i(t) C_{r-i}(t)$$

Note $\pi_r(t) = \sum_{r' \geq r} D_{r'}(t)$.

Let $c_1 = \ln(1 - p)$. Note $c_1 < 0$ and when $p \leq 1/2$ $c_1 = -p - \frac{p^2}{2} - \frac{p^3}{3} \cdots \geq -3p/2$. Now $B_i(t) = (k_i^p)^p (1 - p)^{k_i - 1}$ and

$$B'_i(t) = \frac{d B_i(t)}{dt} = \binom{k_i^p}{i} p^i (1 - p)^{k_i - 1} \left[ c_1 + \frac{i-1}{j_1^k p_t - j} \right] k_p$$

$$B''_i(t) = \binom{k_i^p}{i} p^i (1 - p)^{k_i - 1} \left[ c_1^2 + 2c_1 \left( \sum_{j=0}^{i-1} \frac{1}{k_p t - j} \right) + \left( \sum_{j=0}^{i-1} \frac{1}{k_p t - j} \right)^2 \right] k_p^2$$

The above implies that $B'_i(t) \geq 0$ for $i \geq 1$ and $k_p p_t \leq \frac{1}{3}$. Further for $i \geq 2$

$$B''_i(t) \geq \binom{k_i^p}{i} p^i (1 - p)^{k_i - 1} \left[ c_1^2 + \frac{2c_1 i}{k_p t - i + 1} + \frac{2}{\sum_{j=1}^{i-1} \sum_{j'=0}^{i-1} (k_p t - j)(k_p t - j')} \right] k_p^2$$

which is positive for $i \geq 2$ and $k_p c_1 \geq -1/2$. By the exact same argument $C_j(t) \geq 0$ for $j \geq 1$ and $C''_j(t) \geq 0$ for all $j \geq 2$. Now consider $B_i(t) C_0(t)$ for $i \geq 2$, let $c_2 = \ln(1 - q)$.

$$\frac{d B_i(t) C_0(t)}{dt} = \binom{k_i^p}{i} p^i (1 - p)^{k_i - 1} \left( 1 - q \right)^k q_t \left[ k_p c_1 + \sum_{j=0}^{i-1} \frac{k_p}{k_p t - j} + k_q c_2 \right]$$

$$\frac{d^2 B_i(t) C_0(t)}{dt^2} = \binom{k_i^p}{i} p^i (1 - p)^{k_i - 1} \left( 1 - q \right)^k q_t \left[ \left( k_p c_1 + \sum_{j=0}^{i-1} \frac{k_p}{k_p t - j} + k_q c_2 \right)^2 - \sum_{j=0}^{i-1} \frac{k_p^2}{(k_p t - j)^2} \right]$$
To prove that \( \frac{d^2 B_i(t) C_0(t)}{dt^2} \geq 0 \) for \( i \geq 2 \) it suffices to show that

\[
\left[ k_p c_1 + \sum_{j=0}^{i-1} \frac{k_p}{k_p t - j} + k_q c_2 \right]^2 - \sum_{j=0}^{i-1} \frac{k_p^2}{(k_p t - j)^2} \geq 0
\]

The left hand side expands to

\[
(k_p c_1 + k_q c_2)^2 + 2(k_p c_1 + k_q c_2) \sum_{j=0}^{i-1} \frac{kp}{k_p t - j} + \sum_{j=0}^{i-1} \frac{kp}{k_p t - j} \left( \sum_{j'=0, j' \neq j}^{i-1} \frac{k_p}{k_p t - j'} \right)
\]

(2)

The first term is positive and for \( i \geq 2 \) for any \( j \) there exists a \( j \neq j' \) and

\[
2(k_p c_1 + k_q c_2) + \frac{kp}{k_p t - j} \geq 2 \left( -\frac{3p}{2} k_p - \frac{3q}{2} k_q \right) + \frac{1}{t} = \frac{1 - 3\phi t}{t} \geq 0
\]

(3)

and therefore for \( i \geq 2 \) we have \( \frac{d^2 B_i(t) C_0(t)}{dt^2} \geq 0 \). Note that the same argument holds for \( B_0(t) C_j(t) \) for \( j \geq 2 \) and \( \frac{d^2 B_0(t) C_j(t)}{dt^2} \geq 0 \) in that case as well. Finally consider \( B_i(t) C_1(t) \) for \( i \geq 1 \).

\[
B_i(t) C_1(t) = qk_t \left( k_p t \right)^i (1 - p)^{k_p t - i} (1 - q)^{k_q t - 1}
\]

\[
\frac{dB_i(t) C_1(t)}{dt} = qk_t \left( k_p t \right)^i (1 - p)^{k_p t - i} (1 - q)^{k_q t - 1} \left[ k_p c_1 + \sum_{j=0}^{i-1} \frac{k_p}{k_p t - j} + k_q c_2 + \frac{1}{t} \right]
\]

\[
\frac{d^2 B_i(t) C_1(t)}{dt^2} = B_i(t) C_1(t) \left[ k_p c_1 + \sum_{j=0}^{i-1} \frac{k_p}{k_p t - j} + k_q c_2 + \frac{1}{t} \right]^2 - \sum_{j=0}^{i-1} \frac{k_p^2}{(k_p t - j)^2} - \frac{1}{t^2}
\]

Using an expansion similar to Equation 2 we wish to prove

\[
(k_p c_1 + k_q c_2 + \frac{1}{t})^2 + 2 \left( k_p c_1 + k_q c_2 + \frac{1}{t} \right) \sum_{j=0}^{i-1} \frac{k_p}{k_p t - j} + \sum_{j=0}^{i-1} \frac{k_p}{k_p t - j} \left( \sum_{j'=0, j' \neq j}^{i-1} \frac{k_p}{k_p t - j'} \right) = \frac{1}{t^2} \geq 0
\]

But the left hand side is greater than

\[
(k_p c_1 + k_q c_2 + \frac{1}{t})^2 + 2 \left( k_p c_1 + k_q c_2 + \frac{1}{t} \right) \frac{1}{t} - \frac{1}{t^2}
\]

which rewrites to

\[
(k_p c_1 + k_q c_2)^2 + \frac{2}{t} \left( \frac{1}{t} + 2(k_p c_1 + k_q c_2) \right) \geq (k_p c_1 + k_q c_2)^2 + \frac{2}{t} \left( \frac{1}{t} - 3\phi t \right) \geq 0
\]

But the term \( k_q q \geq 0 \) whereas \( c_1, c_2 < 0 \) and therefore by the exact same logic as in Equation 3

\[
2(k_p c_1 + k_q c_2 + k_q q) + \frac{k_p}{k_p t - j'} \geq 2 \left( -\frac{3p}{2} k_p - \frac{3q}{2} k_q \right) + \frac{1}{t} = \frac{1 - 3\phi t}{t} \geq 0
\]

Therefore \( \frac{d^2 B_i(t) C_1(t)}{dt^2} \geq 0 \) for \( i \geq 1 \). Likewise \( \frac{d^2 B_1(t) C_j(t)}{dt^2} \geq 0 \) for \( j \geq 1 \).
Now for any \( D_{r'}(t) \) with \( r' \geq 2 \) observe that
\[
D_{r'}(t) = B_{r'}(t)C_0(t) + B_{r'-1}(t)C_1(t) + \left( \sum_{j=2}^{r'-2} B_{r'-j}(t)C_j(t) \right) + B_1(t)C_{r'-1}(t) + B_0(t)C_{r'}(t)
\]
The middle sum is a sum of product of convex functions and every other term is proven to be convex as above. Therefore \( \pi_r(t) = \sum_{r'' \geq r} D_{r''}(t) \) is convex in \( t \) for \( r \geq 2 \).

Therefore to show that \( A(t) = \sum_{r=1}^{\infty} \zeta_1 \pi_r(t) \) is convex, we can ignore the terms corresponding to \( r \geq 3 \). The term corresponding to \( r = 1 \) is not convex however – but we will show that
\[
A(t) = \zeta_1 \pi_1(t) + \zeta_2 \pi_2(t)
\]
is convex, which will complete the proof that \( A(t) \) is convex. Set \( a = \zeta_1 / (\zeta_1 + \zeta_2) \) and note
\[
\frac{A(t)}{\zeta_1 + \zeta_2} = 1 - (1 - p)^{k_p t}(1 - q)^{k_q t} - (1-a) \left[ pk_p t(1-p)^{k_p t-1}(1-q)^{k_q t} + (1-p)^k p q_k t(1-q)^{k_q t-1} \right]
\]
\[
= 1 - (1 - p)^{k_p t}(1 - q)^{k_q t} - b(k t)(1-a)(1-p)^{k_p t}(1-q)^{k_q t}
\]
where
\[
b = \frac{pk_p + qk_q}{k} \geq \frac{1}{k} \left( pk_p + qk_q \right) \quad \text{and} \quad b \leq \frac{1}{k} \left( pk_p + qk_q \right) + 2 \max\{p^2, q^2\} = \phi/k + 2 \max\{p^2, q^2\}
\]
Let \( (1-z) = \left( (1-p)^{k_p t}(1-q)^{k_q t} \right)^{1/k} \) which implies \( (1-z)^{kt} = (1-p)^{k_p t}(1-q)^{k_q t} \). Let \( c = \ln(1-z) \).
\[
c = \frac{1}{k} \left( k_p \ln(1-p) + k_q \ln(1-q) \right) \geq -\frac{1}{k} \left( pk_p + qk_q + p^2 k_p + q^2 k_q \right) \geq -b - \max\{p^2, q^2\}
\]
now
\[
\frac{A(t)}{\zeta_1 + \zeta_2} = 1 - (1-z)^{kt} - b(k t)(1-a)(1-z)^{kt}
\]
\[
\Rightarrow \quad \frac{A'(t)}{(\zeta_1 + \zeta_2) k} = -c(1-z)^{kt} - b(1-a)(1-z)^{kt} - b(k t)c(1-a)(1-z)^{kt}
\]
\[
\Rightarrow \quad \frac{A''(t)}{(\zeta_1 + \zeta_2) k^2} = -c^2(1-z)^{kt} - 2bc(1-a)(1-z)^{kt} - c^2 b(k t)(1-a)(1-z)^{kt}
\]
Therefore
\[
\frac{A''(t)}{-c(1-z)^{kt}(\zeta_1 + \zeta_2) k^2} = c + 2(1-a)b + cb(k t)(1-a)
\]
\[
\geq c + 2(1-a)b + \phi tc(1-a) + 2ckt(1-a) \max\{p^2, q^2\} \quad \text{since } c < 0
\]
\[
\geq 2(1-a)b + \left( \frac{4}{3} - \frac{1}{3}a \right) c + ck(1-a) \max\{p^2, q^2\} \quad \text{since } \phi t \leq \frac{1}{3}, c < 0
\]
\[
\geq \left( \frac{2}{3} - \frac{5a}{3} \right) b - \left( \frac{4}{3} - \frac{1}{3}a \right) \max\{p^2, q^2\} + ck(1-a) \max\{p^2, q^2\} \quad (4)
\]
Observe that all the terms involving \( p^2, q^2 \) are \( o(b) \) and \( A'' \geq 0 \) when \( a \leq 2/5 \) which is true when \( \zeta_1 \leq 2\zeta_2/3 \). Therefore \( A(t) \) is convex. \( \square \)
2.1.2 Proof of Lemma 6

**Lemma 6.** For any \( p \geq q, x \geq 1 \) and any \( t \geq 4r \) with \( \phi xt \leq 1/3 \), we have \( \pi_r(xt) \leq 3 \left( \frac{4x}{3(1-p)} \right)^r \pi_r(t) \) and if \( 3x(1-p) > 4 \) then \( \pi_r(t) \leq 4 \left( \frac{4x}{3(1-p)} \right)^r \pi_r(xt) \).

**Proof.** Consider an \( r' \geq r, x \geq 1 \), and \( xt \) is an integer \( \leq 1/(3\phi) \)

\[
\Pr[Bin(k_p,tx,p) + Bin(k_q,tx,q) = r'] = \sum_{i=0}^{r'} \binom{k_p}{i} p^i (1-p)^{xtk_p-i} \binom{k_q}{r'-i} q^{r'-i} (1-q)^{xtk_q-r'+i}
\]

\[
\leq \sum_{i=0}^{r'} \binom{\binom{k_p}{i}}{i} p^i (1-p)^{xtk_p-i} \binom{\binom{k_q}{r'-i}}{r'-i} q^{r'-i} (1-q)^{xtk_q-r'+i}
\]

\[
= \frac{\binom{\binom{k_p}{i}}{i}}{i!} (1-p)^{xtk_p} (1-q)^{xtk_q}
\] (Equation 5)

In the case \( y \geq 4r \) note that \( \binom{y}{r} \geq (y-r)^r/r! \geq (3y/4)^r/r! \). Thus in the case \( xt \geq 4r \) we get

\[
\Pr[Bin(k_p,tx,p) + Bin(k_q,tx,q) = r] = \sum_{i=0}^{r} \binom{k_p}{i} p^i (1-p)^{xtk_p-i} \binom{k_q}{r-i} q^{r-i} (1-q)^{xtk_q-r+i}
\]

\[
\geq \sum_{i=0}^{r} \binom{\binom{k_p}{i}}{i} p^i (1-p)^{xtk_p} \binom{\binom{k_q}{r-i}}{r-i} q^{r-i} (1-q)^{xtk_q}
\]

\[
= \frac{\binom{\binom{k_p}{i}}{i}}{i!} (1-p)^{xtk_p} (1-q)^{xtk_q}
\] (Equation 6)

Therefore when \( 3xt\phi \leq 1 \), we have:

\[
\frac{\phi xt}{(\frac{4}{3})^r r!} (1-p)^{xtk_p} (1-q)^{xtk_q} \leq \Pr[Bin(k_p,tx,p) + Bin(k_q,tx,q) = r](Equation \[7]\]
\leq \pi_r(xt)
\leq \sum_{r' \geq r} \Pr[Bin(k_p,tx,p) + Bin(k_q,tx,q) = r']
\leq \sum_{r' \geq r} \frac{\binom{\phi xt}{r'}}{r!} (1-p)^{xtk_p} (1-q)^{xtk_q}(Equation \[6]\)
\leq \frac{\binom{\phi xt}{r'}}{r!} (1-p)^{xtk_p} (1-q)^{xtk_q} \sum_{r' \geq r} \left( \frac{\phi xt}{1-p} \right)^{r'-r}
\]

\[
\leq \left( \frac{3}{2-3p} \right) \frac{\binom{\phi xt}{r'}}{r!} (1-p)^{xtk_p} (1-q)^{xtk_q}
\]

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Summarizing the above we get for $x \geq 1, xt \geq 4r$:

\[
\left(\frac{\phi_xt}{r}\right)^r (1 - p)^{xt_k}(1 - q)^{xt_k} \leq \pi_r(xt) \leq \left(\frac{3}{2 - 3p}\right) \left(\frac{(\phi_xt)^r}{r!}\right)(1 - p)^{xt_k}(1 - q)^{xt_k} \tag{9}
\]

But this immediately implies that (using Equation 8 in the second part):

\[
\pi_r(xt) \leq \left(\frac{3}{2 - 3p}\right) \left(\frac{4x}{3(1 - p)}\right)^r (\phi_xt)^r \frac{r!}{r!} (1 - p)^{xt_k}(1 - q)^{xt_k} \leq 2 \left(\frac{4x}{3(1 - p)}\right)^r \pi_r(t)
\]

and if $3x(1 - p) > 4$,

\[
\pi_r(t) \leq 2 \left(\frac{4}{3x(1 - p)}\right)^r \pi_r(xt)(1 - p)^{(x-1)xt_k}(1 - q)^{(x-1)xt_k} \\
\leq 2 \left(\frac{4}{3x(1 - p)}\right)^r \pi_r(xt)e^{2(x-1)xt_k}e^{2(x-1)xt_k} (\text{Since } e^{2y} \geq 1 + y \text{ for } y \in [0, \frac{1}{2}].) \\
\leq 2 \left(\frac{4}{3x(1 - p)}\right)^r \pi_r(xt)e^{2(x-1)\phi t} \leq 3 \left(\frac{4}{3x(1 - p)}\right)^r e^{2/3}\pi_r(xt) \\
\leq 4 \left(\frac{4}{3x(1 - p)}\right)^r \pi_r(xt)
\]

The lemma follows. \qed

2.1.3 Proof of Lemma 7

Lemma 15. For any fixed $r > 0$, define the stochastic processes

\[
\xi(t) = \frac{S_i^t(t) - \mathbb{E}[S_i^t(t)]}{1 - \pi_r(t)} \quad \xi_{rev}(t) = \frac{S_i^t(t) - \mathbb{E}[S_i^t(t)]}{\pi_r(t)}
\]

$\xi(t)$ is a martingale (i.e., $\mathbb{E}[\xi(t + 1)] = \xi(t)$ for all $t$) and $\xi_{rev}$ is a reverse martingale (i.e., $\mathbb{E}[\xi_{rev}(t - 1)] = \xi_{rev}(t)$ for all $t$).

Proof. Let $V_r^i$ be the set of vertices in cluster $i$ with threshold $r$ that were not seeded. For $u \in V_r^i$, let $Y_u$ be the time at which $u$ becomes infected (set $Y_u = \infty$ if $u$ never becomes infected). Then $\pi_r(t) = \Pr[Y_u \leq t]$ and $S_i^t(t) = \sum_u \mathbb{1}[Y_u \leq t]$ where $\mathbb{1}[]$ is the indicator function. The terms $\mathbb{1}[Y_u \leq t]$ are independent and identically distributed, so it suffices to show $\xi_u$ is a martingale, where

\[
\xi_u(t) = \frac{\mathbb{1}[Y_u \leq t] - \Pr[Y_u \leq t]}{1 - \Pr[Y_u \leq t]} = 1 - \frac{\mathbb{1}[Y_u > t]}{1 - \Pr[Y_u \leq t]} = 1 - \frac{\mathbb{1}[Y_u > t]}{1 - \pi_r(t)} \tag{10}
\]

If $Y_u \leq t$, then $\xi_u(t + 1) = 1$ and $\xi_u$ is a martingale. If $Y_u > t$,

\[
\xi_u(t + 1) = \begin{cases} 
\frac{-\pi_r(t+1)}{1-\pi_r(t+1)} & \text{if } Y_u > t + 1 \\
1 & \text{if } Y_u = t + 1 
\end{cases} \quad \Pr[Y_u > t + 1 | Y_u > t] = \frac{1 - \pi_r(t+1)}{1 - \pi_r(t)} \\
\Pr[Y_u = t + 1 | Y_u > t] = \frac{\pi_r(t+1)}{1 - \pi_r(t)}
\]
and \( \mathbb{E}[\xi_u(t + 1)] = \xi_u(t) \), so \( \xi_u \) is a martingale. Summing over all \( u \), \( \xi(t) = \sum_u \xi_u(t) \) is also a martingale. To show \( \xi_{rev} \) is a reverse martingale, it suffices to show \( \xi_v \) is reverse martingale, where

\[
\xi_v(t) = \frac{1[Y_u \leq t] - \mathbb{Pr}[Y_u \leq t]}{\mathbb{Pr}[Y_u \leq t]} = \frac{1[Y_u \leq t]}{\mathbb{Pr}[Y_u \leq t]} - 1 = \frac{1[Y_u \leq t] - \pi_r(t)}{\pi_r(t)} \tag{11}
\]

If \( Y_u > t \), then \( \xi_v(t) = \xi_v(t - 1) = -1 \) and \( \xi_v \) is a reverse martingale. If \( Y_u \leq t \),

\[
\xi_v(t - 1) = \begin{cases} 
\frac{1 - \pi_r(t - 1)}{\pi_r(t - 1)} & \text{if } Y_u \leq t - 1 \quad \mathbb{Pr}[Y_u \leq t - 1 | Y_u \leq t] = \frac{\pi_r(t - 1)}{\pi_r(t)} \\
-1 & \text{if } Y_u = t \quad \mathbb{Pr}[Y_u = t | Y_u \leq t] = \frac{\pi_r(t) - \pi_r(t - 1)}{\pi_r(t)}
\end{cases}
\]

and \( \mathbb{E}[\xi_v(t - 1)] = \xi_v(t) \), so \( \xi_v \) is a reverse martingale and so is \( \sum_v \xi_v \).

**Lemma 16.** For any fixed \( r > 0 \) and any \( t_0 \),

\[
\mathbb{E} \left( \sup_{0 < t \leq t_0} \left| S^i_r(t) - \mathbb{E}[S^i_r(t)] \right|^2 \right) \leq 16\eta^i \pi_r(t_0)
\]

**Proof.** Let \( \eta^i_r, \varphi^i_r \) be the number of vertices in cluster \( i \) with threshold \( r \) and the vertices (with threshold \( r \)) which were seeded. Note that for a fixed \( \eta^i_r, \varphi^i_r \), \( \mathbb{Var}[S^i_r(t)] = (\eta^i_r - \varphi^i_r)\pi_r(t)(1 - \pi_r(t)) < \eta^i_r\pi_r(t) \). Therefore taking the expectation over \( \eta^i_r \), we get \( \mathbb{Var}[S^i_r(t)] \leq \eta^i_r\pi_r(t) \).

Define \( \xi \) and \( \xi_{rev} \) to be the martingales from Lemma 15. We will be using Doob’s \( L^p \) maximal inequality which states that for a Martingale \( M(t), p > 1 \) and any \( \tau \geq 1 \),

\[
\mathbb{E} \left( \sup_{t \leq \tau} |M(t)|^p \right) \leq \left( \frac{p}{p - 1} \right)^p \mathbb{E}[|M(\tau)|^p]
\]

We will use the inequality for \( p = 2 \). We break the proof into two cases.

**Case I** \( \pi_r(t_0) \leq 1/2, t \leq t_0 \). We apply Doob’s Inequality on \( \xi \) with \( \tau = t_0 \) to get

\[
\mathbb{E} \left( \sup_{t \leq t_0} |S^i_r(t) - \mathbb{E}[S^i_r(t)]|^2 \right) \leq \mathbb{E}[\sup_{t \leq t_0} |\xi(t)|^2] \leq 4\mathbb{E}[|\xi(t_0)|^2] = 4\mathbb{Var}[S^i_r(t_0)] \leq 8\eta^i_r\pi_r(t_0) \tag{12}
\]

**Case II** \( \pi_r(t_0) > 1/2, t \leq t_0 \). Observe that \( \pi_r(0) = 0 \) and \( \pi_r(t) \) is monotonic nondecreasing. Let \( t_1 \) be the largest integer such that \( \pi_r(t_1) \leq 1/2 \). Then using exactly the same argument as in Equation 12 we have

\[
\mathbb{E} \left( \sup_{t \leq t_1} |S^i_r(t) - \mathbb{E}[S^i_r(t)]|^2 \right) \leq 8\eta^i_r\pi_r(t_1) \leq 8\eta^i_r\pi_r(t_0) \tag{13}
\]

We use \( \xi_{rev} \) to get

\[
\mathbb{E} \left( \sup_{t \geq t_1 + 1} |S^i_r(t) - \mathbb{E}[S^i_r(t)]|^2 \right) \leq 4 \frac{\mathbb{Var}[S^i_r(t_1 + 1)]}{\pi_r(t_1 + 1)^2} \leq 8\eta^i_r(1 - \pi_r(t_1 + 1))\pi_r(t_1 + 1) \leq 8\eta^i_r\pi_r(t_0) \tag{14}
\]

Since

\[
\mathbb{E} \left( \sup_{t \geq 0} |S^i_r(t) - \mathbb{E}[S^i_r(t)]|^2 \right) \leq \mathbb{E} \left( \sup_{t \leq t_1} |S^i_r(t) - \mathbb{E}[S^i_r(t)]|^2 \right) + \mathbb{E} \left( \sup_{t \geq t_1 + 1} |S^i_r(t) - \mathbb{E}[S^i_r(t)]|^2 \right)
\]

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And we apply Equations 13 and 14 to get that
\[
\mathbb{E} \left[ \left( \sup_{t \geq 0} |S^i(t) - \mathbb{E}[S^i(t)]| \right)^2 \right] \leq 8\eta\zeta_r \pi_r(t_0) + 8\eta\zeta_r (1 - \pi_r(t_0)) < 8\eta\zeta_r < 16\pi_r(t_0)\eta\zeta_r
\]
(15)

The lemma follows.

We can now prove the main result of this subsubsection.

**Lemma 7.** Let \( t_0 = \frac{96}{(1-p)^3\beta^2} t^* \) and suppose \( t^* \to \infty \) as \( n \to \infty \). For \( q \leq p \leq 1/2 \) and all fixed \( \gamma, \beta > 0 \) if \( n \geq n(\beta, \gamma, k) \) which is sufficiently large, with probability at least \( 1 - c_0/(\gamma^2 \beta^{2m} t^*) \) for some absolute constant \( c_0 \), simultaneously for all \( i \),
\[
\sup_{1 \leq t \leq t_0} \left| S^i(t) - \left( \eta - \frac{\varphi}{k} \right) A(t) \right| \leq \gamma t^*
\]

**Proof.** Applying Lemma 16 and Lemma 6 we get:
\[
\mathbb{E} \left[ \sup_{1 \leq t \leq t_0} |S^i(t) - \mathbb{E}[S^i(t)]|^2 \right] = 16\eta\zeta_r \pi_r(t_0) \leq 48 \left( \frac{128}{(1-p)^3\beta^2} \right)^r \eta\zeta_r \pi_r(t^*)
\]
(16)
We then use the triangle inequality over \( r = 1 \ldots r_m \),
\[
\mathbb{E} \left[ \sup_{1 \leq t \leq t_0} |S^i(t) - \mathbb{E}[S^i(t)]|^2 \right] \leq 48 \left( \frac{128}{(1-p)^3\beta^2} \right)^{r_m} \sum_{r=1}^{r_m} \eta\zeta_r \pi_r(t^*) \leq 48 \left( \frac{128}{(1-p)^3\beta^2} \right)^{r_m} \frac{n}{k} A(t^*)
\]
But \( n A(t^*) \leq t^* \) (Definition 3) and \( t^* \to \infty \), thus for sufficiently large \( n \), therefore for \( \delta = c_0/(\gamma^2 \beta^{2m} t^*) \):
\[
\mathbb{E} \left[ \sup_{1 \leq t \leq t_0} |S^i(t) - \mathbb{E}[S^i(t)]|^2 \right] \leq 48 \left( \frac{128}{(1-p)^3\beta^2} \right)^{r_m} \frac{t^*}{k} \leq \frac{\delta \gamma^2 (t^*)^2}{2k}
\]
Therefore using Markov inequality, for every \( i \) with probability \( 1 - \delta/(2k) \) we have
\[
\sup_{0 \leq t \leq t_0} \left| S^i(t) - \mathbb{E}[S^i(t)] \right|^2 \leq \gamma^2 (t^*)^2
\]
The lemma follow from the union bound over all \( i \) and taking the square root.

**2.1.4 Proof of Lemma 10.**

**Lemma 10.** When the seed size is \( \varphi = (1+\epsilon)\Phi \), \( \epsilon \leq 1/9 \) and we have reached \( t = \min\{ \frac{96}{(1-p)^3\beta^2} t^*, 1/(3\phi) \} \) then with probability \( 1 - O(1/\Phi^2) \), \( S^i(t) > t - a \) for all \( t \in \left[ \frac{96}{(1-p)^3\beta^2} t^*, 1/(3\phi) \right] \), i.e., as \( n \) increases and \( t^* \to \infty \) the percolation continues till \( t = 1/(3\phi) \).

**Proof.** Let \( F^i(t) = \sum_{r=1}^{r_m} (Bin(\eta_r - \varphi_r^i, \pi_r(t)) + \varphi_r^i) \). Recall that \( \eta_r \) is the number of vertices with threshold \( r \) and \( \varphi_r^i \) are the number of seeded vertices with threshold \( r \) in cluster \( i \). Note that \( \varphi = \varphi \). We will show \( F^i(t) > (1+\epsilon)t \) for all \( t \) with probability \( 1 - O(1/\Phi^2) \). Since the percolation has proceeded to at least \( 2t^*/\beta \) observe that
\[
(n - \varphi)A(2t^*/\beta) + \varphi \geq 2kt^*/\beta
\]
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Expanding $A$
\[(n - \varphi)\zeta_1 \pi_1(2t^*/\beta) + (n - \varphi) \sum_{r=2}^{r_m} \zeta_r \pi_r(2t^*/\beta) \geq 2kt^*/\beta - \varphi\]

Now for $t \geq 1$, we have $\pi_1(t) = 1 - (1 - p)kp(1 - q)kq^t \leq 1 - (1 - pk_p t)(1 - qk_q t) \leq pk_p t + qk_q t = t \phi$ (note $(1 - p)^z \geq 1 - pz$ for all $z \geq 1$). Moreover $\zeta_1 \phi = \zeta_1 \phi n/k \leq (1 - \beta),$
\[2kt^*/\beta - \varphi \leq n \zeta_1 \phi 2t^*/\beta + (n - \varphi) \sum_{r=2}^{r_m} \zeta_r \pi_r(2t^*/\beta) \leq (1 - \beta)2kt^*/\beta + (n - \varphi) \sum_{r=2}^{r_m} \zeta_r \pi_r(2t^*/\beta)\]

But $\Phi \leq k t^*$ (Definition 3) and $\varphi \leq (1 + \epsilon)kt^* \leq 10kt^*/9$ therefore
\[(n - \varphi) \sum_{r=2}^{r_m} \zeta_r \pi_r(2t^*/\beta) \geq 8kt^*/9 \tag{17}\]

Let $t_j = 2zt^*/\beta$. Note $z \geq \frac{48}{(1-p)\beta}$. We use Lemma 6
\[
\mathbb{E}[F^i(t_j)] \geq \sum_{r=1}^{r_m} (\eta - \varphi/k) \zeta_r \pi_r(t_j) \geq \frac{1}{k} (n - \varphi) \sum_{r=2}^{r_m} \zeta_r \pi_r(t_j) \\
\geq 9(1 - p)^2 z^2 \frac{1}{64} \frac{1}{k} (n - \varphi) \sum_{r=2}^{r_m} \zeta_r \pi_r(2t^*/\beta) \quad \text{(Using Lemma 6)} \\
\geq 9(1 - p)^2 z^2 \frac{8t^*}{64} \frac{8t^*}{9} \quad \text{(Using Equation 17)} \geq 3t_j
\]

We apply Chebyshev’s inequality, note $\text{Var}[F^i(t)] \leq \mathbb{E}[F^i(t)]$.
\[
\Pr[F(t_j) \leq 2t_j] \leq \Pr \left[ F(t_j) \leq \frac{2}{3} \mathbb{E}[F(t_j)] \right] \leq \frac{\text{Var}[F(t_j)]}{((1/3)\mathbb{E}[F(t_j)])^2} \leq \frac{9}{\mathbb{E}[F(t_j)]]} \leq 3/t_j
\]

We now use union bounds over the intervals – note that the sum of the probabilities of stopping in each interval telescopes to a total of $O(1/t^*)$. The lemma follows.

\[\square\]

2.1.5 Proof of Lemma 11

Lemma 11. If the percolation has continued till $t = (3\phi)^{-1}$ then with probability $1 - 1/n^{O(1)}$, the percolation does not stop till a constant fraction of the graph is infected. Moreover if the expected degree $\phi \eta$ is a slowly growing function then with probability $1 - 1/n^{O(1)}$, the percolation does not stop till $\eta - o(\eta)$ nodes are infected.

Proof. The analysis corresponds to several different intervals.

Statement 1. For all $t \in [(3\phi)^{-1}, c_2 \eta]$, $S^i(t) > c_2 \eta > t$, where $c_2$ is some small constant.

We can pessimistically pretend that every vertex has been assigned threshold $r_m$, and the probability that a vertex is healthy is at most $\pi_{r_m}(t) \geq \pi_{r_m}(t)$ when $t = 1/(3\phi)$. Consider $t \geq 4r_m$
and since \( p, q \leq 1/2 \),

\[
\pi_{r_m}(t) \geq \sum_{i=0}^{r_m} \binom{k_p t}{i} \left( \binom{k_q t}{r_m-i} \right) p^i q^{r_m-i} \frac{1}{r_m!} \left( \frac{3k_p t}{4} \right)^i \left( \frac{3k_q t}{4} \right)^{r_m-i} \left( 1 \right)^{3k_p t - r_m + i} \left( 1 - q \right)^{k_q t - r_m + i} \left( 1 \right)^{3k_p t - r_m + i}
\]

\[
\geq \sum_{i=0}^{r_m} \frac{1}{i!} \left( \frac{3k_p t}{4} \right)^i \left( \frac{3k_q t}{4} \right)^{r_m-i} \left( 1 \right)^{3k_p t - r_m + i} \left( 1 - q \right)^{k_q t - r_m + i} \left( 1 \right)^{3k_p t - r_m + i} \left( 1 \right)^{3k_p t - r_m + i} \left( 1 \right)^{3k_p t - r_m + i}
\]

for some constant \( c_1 > 0 \). This implies that \( \mathbb{E}[Bin(\eta - \varphi, \pi_{r_m}(t))] + \varphi > c_1 \eta \). An application of Chernoff bounds provides a \( c_2 \) such that \( Bin(\eta, c_2) > c_2 \eta \) with probability \( 1 - 1/n^\Omega(1) \). This implies that once \((3\phi)^{-1}\) nodes are infected, in the very next generation a constant fraction of the cluster is infected. In the remainder we will try to bound the number of nodes who do not get infected and show that the number is small. This part is identical to the proof in [16]; because the analysis now switches to the vertices who continue to survive – and that analysis does not depend on the threshold of infection. The first part of that analysis is:

**Statement 2.** For all \( t \in [c_2 \eta, \eta - c_4 / \phi] \), for some constant \( c_3 \) the percolation does not stop with probability \( 1 - 1/\eta^\Omega(1) \).

The probability of being healthy is bounded above \( 1 - \pi_{r_m} \), irrespective of the threshold.

\[
1 - \pi_{r_m}(t) = \sum_{r=0}^{r_m-1} \sum_{i=0}^{r} \binom{k_p t}{i} \binom{k_q t}{r-i} \left( p^i q^{r_m-i} \right) \frac{1}{r_m!} \left( \frac{3k_p t}{4} \right)^i \left( \frac{3k_q t}{4} \right)^{r_m-i} \left( 1 \right)^{3k_p t - r_m + i} \left( 1 - q \right)^{k_q t - r_m + i} \left( 1 \right)^{3k_p t - r_m + i} \left( 1 \right)^{3k_p t - r_m + i} \left( 1 \right)^{3k_p t - r_m + i} \left( 1 \right)^{3k_p t - r_m + i}
\]

for some absolute constant \( c_3 \). Now the expected number of remaining healthy vertices is \( (\eta - t)(1 - \pi_{r_m}(t)) \) which is \( c_3 / \phi \) in expectation. Therefore we can again apply Chernoff bound to prove that the percolation proceeds to \( n - c_4 / \phi \) nodes with probability \( 1 - 1/\eta^\Omega(1) \) (Note \( c_4 > c_3 \).)

**Statement 3.** If \( \phi \eta \) is a slowly growing function \( \Rightarrow r_m \) then the percolation does not stop in the range for \( t \in [\eta - c_4 / \phi, \eta - o(\eta)] \) with probability \( 1 - 1/\eta^\Omega(1) \).

We reuse Equation [18] once more and get

\[
1 - \pi_{r_m}(t) \leq e^{r_m(p+q)} \sum_{r=0}^{r_m-1} \frac{(\phi t)^r}{r!} e^{-\phi t} \leq e^{r_m(p+q)} \sum_{r=0}^{r_m-1} \frac{(\eta \phi - c_4)^r}{r!} e^{-\eta \phi + c_4} = o(1)
\]

The lemma follows. \( \square \)

### 3 Intervention Strategies

Recall from Definition 2 that for a vertex \( u \) in a Templated Multisection graph, \( v \) is near \( u \) if \( u \) and \( v \) are connected with probability \( p \) and \( v \) is far from \( u \) if \( u \) and \( v \) are connected with probability \( q \).
**Definition 8** (Set of Healthy Vertices). For a fixed generation \( \tau \), we define \( \mathcal{H} \) to be the set of healthy vertices and define \( \mathcal{H}(r) \) to be the set of healthy vertices with threshold \( r \). We define \( \mathcal{H}_a \) to be the set of healthy vertices with exactly \( a \) infected neighbors. When \( G \) is a Templated Multisection graph, we define \( \mathcal{H}_{bc} \) to be the set of healthy vertices with exactly \( b \) infected neighbors and exactly \( c \) far infected neighbors.

**Definition 9** (Set of Infected Vertices). We define \( \mathcal{I}(\tau) \) to be the set of infected vertices at generation \( \tau \) and define \( \mathcal{I}(\tau - 1) \) similarly.

**Lemma 17** (Proved in Section 3.3). We can calculate \( \Pr[u \in \mathcal{H}_a \mid u \in \mathcal{H}, r(u) = r] \) and \( \Pr[u \in \mathcal{H}_{bc} \mid u \in \mathcal{H}, r(u) = r] \) using \( \mathcal{I}(\tau) \) and \( \mathcal{I}(\tau - 1) \). Additionally, if \( \mathcal{I}(\tau) < k/(3\phi) \), then \( \Pr[u \in \mathcal{H}_{a+1}] < (2/3) \Pr[u \in \mathcal{H}_a] \).

**Theorem 3.** Assume \( n, p, q, r, \lambda, \mathcal{I}(\tau), \mathcal{I}(\tau - 1), \mathcal{H}(r) \) are known and that \( \mathcal{I}(\tau) < k/(3\phi) \). Given either \( \zeta(r) \) (for BOLSTER), \( z' \) (for DELAY), or \( \alpha_p \) and \( \alpha_q \) (for DIMINISH and SEQUESTER), it is possible to determine whether the intervention is successful and \( G \) becomes infected with probability \( 1 - 1/poly(n) \).

Note that it is virtually impossible to apply an intervention when exactly \( \lambda n \) vertices are infected, for example the case where \( \mathcal{I}(10) = 0.95\lambda n \) and \( \mathcal{I}(11) = 1.05\lambda n \). As a result, small changes in \( \lambda \) may have no impact on whether the intervention is successful or not; the true determining factor is the size of \( \mathcal{I}(\tau) \) and \( \mathcal{I}(\tau + 1) \).

Also consider two different graphs \( G_1 \) and \( G_2 \) and let \( \mathcal{I}_i(t) \) denote the number of infected vertices in \( G_i \) at time \( t \). If \( |\mathcal{I}_1(\tau)| < |\mathcal{I}_2(\tau)| \) and \( |\mathcal{I}_1(\tau + 1) - \mathcal{I}_1(\tau)| < |\mathcal{I}_2(\tau + 1) - \mathcal{I}_2(\tau)| \) then \( |\mathcal{I}_1(\tau + 1)| < |\mathcal{I}_2(\tau + 1)| \) and every intervention that successfully stops the percolation on \( G_2 \) will also stop the percolation on \( G_1 \). However if \( |\mathcal{I}_1(\tau + 1)| < |\mathcal{I}_2(\tau + 1)| \) but \( |\mathcal{I}_1(\tau + 1) - \mathcal{I}_1(\tau)| > |\mathcal{I}_2(\tau + 1) - \mathcal{I}_2(\tau)| \), there is no guarantee that an intervention that stops the percolation on \( G_1 \) stops the intervention on \( G_2 \). See also the discussing about Figure 7.

### 3.1 BOLSTER and DELAY Intervention

We begin by formally defining BOLSTER intervention.

**Definition 10** (BOLSTER intervention). Define the intervention generation \( \tau \) to be \( \tau = \min_t |\mathcal{I}(t + 1)| > \lambda n \) and for every \( r \), let \( \zeta(r) \) be a distribution on \( [r, \ldots, r_m] \). Every non-infected vertex with threshold \( r \) will be assigned a new threshold from distribution \( \zeta(r) \).

For the first \( \tau - 1 \) generations, we run the standard bootstrap percolation process. Then the sequence of events are:

1. Generation \( \tau \) begins. Every vertex counts its infected neighbors. Every vertex with \( r(u) \) or more infected neighbors becomes infected. Note that \( |\mathcal{I}(\tau)| < \lambda n \).

2. Generation \( \tau + 1 \) begins. Every vertex counts its infected neighbors. Every vertex \( u \) with \( r(u) \) or more infected neighbors becomes infected.

3. Every vertex \( u \) with less than \( r(u) \) infected neighbors is assigned a new threshold from distribution \( \zeta'(r(u)) \). Note that \( |\mathcal{I}(\tau + 1)| > \lambda n \).

**Definition 11** (DELAY intervention). DELAY intervention is a special case of BOLSTER intervention where \( \zeta'_j(r) = (1 - z'(u))^{3-r} z'(u) \).
Note that with this definition Bolster intervention cannot save a vertex that is about to become infected; if \( u \) has \( r(u) \) infected neighbors when the intervention is applied it still becomes infected. Our results focus on the definition above, at the end of the section we describe how to modify the results using either of the following alternate definitions.

**Modification 1.** Bolster intervention is allowed to save vertices. In the definition above, when generation \( \tau + 1 \) begins, every vertex is assigned a new threshold from \( \zeta' \) before checking whether vertices become infected; essentially we swap steps 2 and 3 in the definition.

Modification 1 is substantially stronger than our definition of Bolster it is in fact so strong that it is difficult to generate interesting simulation data (even the weakest interventions are always successful).

**Modification 2.** Bolster intervention is allowed to weaken vertices, and \( \zeta'(r) \) is allowed to be a distribution on \([2, r'_m] \) instead of \([r, r'_m] \).

If \( \zeta'(r) \) is any distribution on \([r, r'_m] \), our analysis holds. If \( \zeta'(r) \) is allowed to be a distribution on \([2, r'_m] \), then badly chosen \( \zeta' \) may lead to problems (as discussed in the end of the section). One example of a badly chosen intervention is \( \zeta'(r) = r - 1 \), the ‘intervention’ that reduces every vertex’s threshold by 1.

We now begin the proof of Theorem 3 for Bolster, i.e. determining whether Bolster stops the spread of percolation. We construct a new graph \( J \) of the same ‘type’ as \( G \) but with \(|H| \) vertices. We will choose thresholds for the vertices of \( J \) so that the probability \( J \) becomes infected is equal to the probability \( G \) becomes infected. For every \( u \in H_a \), let \( r'(u) \) denote the new threshold of \( u \).

\( r(u) \) becomes infected when it has \( r'(u) - a \) infected neighbors in \( H \) to go along with its \( a \) infected neighbors in \( I(\tau) \). Thus, we will add a vertex \( v \) to \( J \) with threshold \( r'(u) - a \). \( u \) becomes infected when it has \( r'(u) - a \) neighbors in \( J \). In this way, we encode the information about \( I(\tau) \) into the thresholds of \( J \), and the probability \( u \in H \) becomes infected is equal to the probability that \( v \in J \) becomes infected.

For example, \( v \) has threshold 2 if \( r'(u) = 2 \) and \( u \) had zero infected neighbors or \( r'(u) = 3 \) and \( u \) had one infected neighbors -or- \( r'(u) = 4 \) and \( u \) had two infected neighbors and so on.

Formally, let \( G = TM(F, n, k_p, k_q, p, q) \) and \( J = (F, |H|, k_p, k_q, p, q) \). Recall we can calculate \( \Pr[u \in H_a \mid r(u) = r] \) using Lemma 17. We now define \( j_s \) and \( \varphi \) as follows.

\[
 j_s = \sum_{r=2}^{r_m} \sum_{a=0}^{r} \frac{|H'(r)|}{|H|} \Pr[u \in H_a \mid r(u) = r] \ast 1[a < r] \ast \zeta'_{s+a}(r) \quad \varphi = |H| - |H| \sum_{s=1}^{r'_m} j_s \quad (19)
\]

We will let \( j_1, j_2, \ldots, j_{r'_m} \) be the distribution used to assign thresholds and \( \varphi \) will be the number of seed vertices. We then use Theorem 1 to determine whether \( J \) becomes infected. If \( J \) becomes infected with polynomially high probability, then \( G \) also becomes infected and with that same probability and the intervention is not successful. If \( J \) does not become infected, then the intervention is successful.

In order to apply Theorem 1 we need to confirm that \( j_1 < (2/3) j_2 \). Note that

\[
 j_1 = \sum_{r=2}^{r_m} \frac{|H'(r)|}{|H|} \Pr[u \in H_{r-1} \mid r(u) = r] \zeta'_r(r) \\
 j_2 = \sum_{r=2}^{r_m} \frac{|H(r)|}{|H|} \left( \Pr[u \in H_{r-2} \mid r(u) = r] \zeta'_r(r) + \Pr[u \in H_{r-1} \mid r(u) = r] \zeta'_{r+1}(r) \right)
\]
By Lemma 17, $\Pr[u \in H_{r-1}] < (2/3) \Pr[u \in H_{r-2}]$. This immediately implies that $j_1 < (2/3)j_2$ and that Theorem 1 applies. To use Modification 1, remove the $1[a < r]$ part of Equation 19. To use Modification 2, no changes to Equation 19 are necessary. However, note that with Modification 2, $j_1$ is not guaranteed to be less than $(2/3)j_2$, depending on the distribution $\zeta'$, it might be the case that $j_1 > (2/3)j_2$. At this point, Theorem 1 no longer applies.

3.2 Diminish and Sequester Intervention

We begin by formally defining Diminish intervention, which corresponds to the idea of deleting edges randomly.

**Definition 12 (Diminish intervention).** Define the intervention generation $\tau$ to be $\tau = \min_r |E(t)| > \lambda n$. Let $\zeta'$ be a distribution of new vertex thresholds. For the first $\tau - 1$ generations, we run the standard bootstrap percolation process. Then the sequence of events are:

1. Generation $\tau - 1$ begins. Every vertex counts its infected neighbors. Every vertex with $r(u)$ or more infected neighbors becomes infected.

2. Generation $\tau$ begins. Delete that edge connecting two near vertices with probability $1 - \alpha_p$. Delete every edge connecting two far vertices with probability $1 - \alpha_q$. After edges are deleted, every vertex counts its infected neighbors. Every vertex with $r(u)$ infected neighbors in $G'$ becomes infected.

**Definition 13 (Sequester intervention).** Sequester intervention is defined similarly to Diminish but edges connecting two healthy vertices are never deleted. Edges connected to an infected vertices are deleted with probability $1 - \alpha_p$ or $1 - \alpha_q$.

Note that unlike Bolster intervention, we do allow Diminish to ‘save’ vertices about to be infected. Let $G'$ be the post-intervention graph after edges are deleted and define $H'_a$ to be the set of healthy vertices that have exactly $a$ infected vertices in $G'$. Define $H'_{b,c}$ similarly. When $G$ is an Erdos-Reyni graph, we get

$$\Pr[u \in H'_{a} | r(u) = r, u \in H] = \sum_{d=a}^{\infty} \Pr[Bin(d, \alpha) = a] \Pr[u \in H_{a} | r(u) = r, u \in H]$$

When $G$ is a TM graph, we get

$$\Pr[u \in H'_{b,c} | r(u) = r, u \in H] = \sum_{d=b}^{\infty} \sum_{e=c}^{\infty} \Pr[Bin(d, \alpha_p) = b] \Pr[Bin(e, \alpha_q) = c] \Pr[u \in H_{b,c} | r(u) = r, u \in H]$$

and $\Pr[u \in H'_{a}] = \sum_{b+c=a} \Pr[u \in H'_{b,c}]$. The remainder of the analysis is very similar to the Bolster case, but using $H'_{a}$ instead of $H_{a}$. We construct a new graph $J$, and for every $u \in H'_{a}$, we add a vertex $v$ to $J$ with threshold $r(u) - a$. $J$ will be a TM graph with $\alpha_{pq}p$ and $\alpha_{q}q$ edge probability instead of $p$ and $q$. Formally, let $G = TM(F, n, k_p, k_q, p, q)$ and $J = (F, |H|, k_p, k_q, \alpha_{pq}, \alpha_{q}q)$.

$$j_s = \sum_{r=2}^{r_m} \sum_{a=0}^{r} \frac{|H'(r)|}{|H|} \Pr[u \in H'_a | r(u) = r] * 1[a < r] \quad \varphi = |H| - |H| \sum_{s=1}^{r_m} j_s$$

We then use Theorem 1 on $J$ to determine whether the intervention is successful or not. For Sequester intervention, we define $J = (F, |H|, k_p, k_q, p, q)$ but use the same distribution of $j_s$.
3.3 Proof of Lemma 17

As a warp up, we first consider the easier case where $G$ is an Erdős-Rényi graph and $r(u)$ is known.

**Statement 1.** When $G$ is an Erdős-Rényi graph, $\Pr[u \in \mathcal{H}_a \mid u \in \mathcal{H}, r(u) = r]$ can be estimated using $\mathcal{I}(\tau)$ and $\mathcal{I}(\tau - 1)$.

**Proof.** We define $B_i(x) = \Pr[Bin(x, p) = i]$. For any sets $S_1$ and $S_2$, define $\sigma(S_1, S_2)$ to be the number of edges connecting $S_1$ and $S_2$. We begin by breaking $\Pr[u \in HH_a]$ into its component parts. For the remainder of the proof, we will suppress the conditional $u \in \mathcal{H}$, $r(u) = r$ in the interest of space; all probabilities are conditioned on knowing $u \in \mathcal{H}$, and $r(u) = r$.

\[
\Pr[u \in \mathcal{H}_a \mid r(u) = r] = \Pr[\sigma(u, \mathcal{I}(\tau)) = a] = \sum_{d=0}^{r-1} \Pr[\sigma(u, \mathcal{I}(\tau - 1)) = d] \Pr[\sigma(u, \mathcal{I}(\tau) - \mathcal{I}(\tau - 1)) = a - d]
\]

If $u \in \mathcal{H}$, $u$ cannot be connected to $r$ or more vertices in $\mathcal{I}(\tau - 1)$; if $u$ was connected to $r$ or more vertices, $u$ would belong to $\mathcal{I}(\tau)$ and it would not belong to $\mathcal{H}$. We can capture this constraint with a conditional binomial.

\[
\Pr[\sigma(u, \mathcal{I}(\tau - 1)) = d] = \frac{B_d(|\mathcal{I}(\tau - 1)|)}{\sum_{i=1}^{r-1} B_i(|\mathcal{I}(\tau - 1)|)}
\]

In contrast, if $v \in \mathcal{I}(\tau) - \mathcal{I}(\tau - 1)$, then $u$ and $v$ are connected with probability $p$ independent of the other edges, so

\[
\Pr[\sigma(u, \mathcal{I}(\tau) - \mathcal{I}(\tau - 1)) = d - a] = B_{d-a}(|\mathcal{I}(\tau) - \mathcal{I}(\tau - 1)|)
\]

The statement follows from the last three equations. \hfill \Box

We now consider the case where $G$ is a Templated Multisection graph. Conceptually, the ideas underlying the proof are identical to the Erdős-Rényi graph.

**Statement 2.** When $G$ is a Templated Multisection graph, $\Pr[u \in \mathcal{H}_a \mid u \in \mathcal{H}, r(u) = r]$ and $\Pr[u \in \mathcal{H}_{b,c} \mid u \in \mathcal{H}, r(u) = r]$ can be estimated using $\mathcal{I}(\tau)$ and $\mathcal{I}(\tau - 1)$.

**Proof.** We define $B_i(x) = \Pr[Bin(x, p) = i]$ and $C_i(x) = \Pr[Bin(x, q) = i]$ . Define $\sigma(S_1, S_2)$ to be the number of edges connecting $S_1$ and $S_2$. For a fixed vertex $u$, we break $\mathcal{I}(t)$ into its two component parts.

\[
\mathcal{I}^{near}(\tau) = \mathcal{I}(\tau) \cap \{v : v \text{ is near } u\} \quad \mathcal{I}^{far}(\tau) = \mathcal{I}(\tau) \cap \{v : v \text{ is far from } u\}
\]

and define similar expressions for $\tau - 1$. We begin by breaking $\Pr[u \in \mathcal{H}_{b,c}]$ into its component parts. For the remainder of the proof, we will suppress the conditional $r(u) = r$ in the interest of space; all probabilities are conditioned on knowing $r(u) = r$.

\[
\Pr[u \in \mathcal{H}_{b,c} \mid r(u) = r] = \Pr[\sigma(u, \mathcal{I}^{close}(\tau)) = b \wedge \Pr[\sigma(u, \mathcal{I}^{far}(\tau)) = c]
\]
Note that these terms are not independent.

\[
\begin{align*}
\Pr[\sigma(u,T^{\text{near}}(\tau)) = b] &= \sum_{d=0}^{b} \Pr[\sigma(u,T^{\text{near}}(\tau - 1)) = d] \Pr[\sigma(u,T^{\text{near}}(\tau) - T^{\text{near}}(\tau - 1)) = b - d] \\
\Pr[\sigma(u,T^{\text{far}}(\tau)) = c] &= \sum_{e=0}^{c} \Pr[\sigma(u,T^{\text{far}}(\tau - 1)) = e] \Pr[\sigma(u,T^{\text{far}}(\tau) - T^{\text{far}}(\tau - 1)) = c - e]
\end{align*}
\]

\[
\sigma(u,T^{\text{near}}(\tau) - T^{\text{near}}(\tau - 1)) \text{ is independent from the other terms, as is } \sigma(u,T^{\text{far}}(\tau) - T^{\text{far}}(\tau - 1)).
\]

Thus, we get

\[
\Pr[u \in H_{b,c}] = \sum_{d=0}^{b} \sum_{e=0}^{c} \left( \Pr[\sigma(u,T^{\text{near}}(\tau - 1)) = d \wedge \sigma(u,T^{\text{far}}(\tau - 1)) = e] \times \Pr[\sigma(u,T^{\text{near}}(\tau) - T^{\text{near}}(\tau - 1)) = b - d] \times \Pr[\sigma(u,T^{\text{far}}(\tau) - T^{\text{far}}(\tau - 1)) = c - e] \right)
\]

Observe \(\sigma(u,T^{\text{near}}(\tau - 1)) + \sigma(u,T^{\text{far}}(\tau - 1)) \leq r - 1\). We can use a modified equation similar to the conditional binomial to get

\[
\Pr[\sigma(u,T^{\text{near}}(\tau - 1)) = d] \wedge \sigma(u,T^{\text{far}}(\tau - 1)) = e] = \frac{B_d(|T^{\text{near}}(\tau - 1)|) \times C_e(|T^{\text{far}}(\tau - 1)|)}{\sum_{f+g \leq r-1} B_f(|T^{\text{near}}(\tau - 1)|) \times C_g(|T^{\text{far}}(\tau - 1)|)}
\]

Note that if \(v \in T^{\text{near}}(\tau) - T^{\text{near}}(\tau - 1)\), then \(u\) and \(v\) are connected with probability \(p\) independent of all other edges (and similarly for \(T^{\text{far}}\) and \(q\)). Thus,

\[
\begin{align*}
\Pr[\sigma(u,T^{\text{near}}(\tau) - T^{\text{near}}(\tau - 1)) = b - d] &= B_{b-d}(|T^{\text{near}}(\tau) - T^{\text{near}}(\tau - 1)|) \\
\Pr[\sigma(u,T^{\text{far}}(\tau) - T^{\text{far}}(\tau - 1)) = c - e] &= C_{c-e}(|T^{\text{far}}(\tau) - T^{\text{far}}(\tau - 1)|)
\end{align*}
\]

Combining the previous equations gives the formula for \(\Pr[u \in H_{b,c}]\). The first part of the statement follows from \(\Pr[u \in H_{a}] = \sum_{b+c=a} \Pr[u \in H_{b,c}]\).

We now begin working for the second part of Lemma 17. The following lemma follows from facts about binomials.

**Lemma 18.** Define \(B_r(t) = \Pr[\text{Bin}(k_pt,p) = r]\), \(C_r(t) = \Pr[\text{Bin}(k_qt,q) = r]\) and \(D_r(t) = \Pr[\text{Bin}(k_pt,p) + \text{Bin}(k_qt,q) = r]\). Then for \(t > r\),

\[
\begin{align*}
B_{r+1}(t) &= \frac{k_pt - r}{r+1} \frac{p}{1-p} B_r(t) \leq tk_p p(1-p)^{-1} B_r(t) \\
C_{r+1}(t) &< \frac{k_qt - r}{r+1} \frac{q}{1-q} C_r(t) \leq tk_q q(1-q)^{-1} C_r(t) \\
D_{r+1}(t) &< \delta t (1 - \max \{p, q\})^{-1} D_r(t)
\end{align*}
\]
Proof. We begin with the proof for $B_{r+1}(t)$.

$$B_{r+1}(t) = \left( \frac{k pt}{r + 1} \right) p^{r+1}(1-p)^{k pt-r-1} = \frac{k pt - r}{r + 1} \left( \frac{p}{1 - p} \right)^r (1-p)^{k pt-r}$$

$$= \frac{k pt - r}{r + 1} \frac{p}{1 - p} B_r(t) < t k p B_r(t)$$

The proof of $C_{r+1}(t)$ follows similar logic. For $D_{r+1}(t)$, we get

$$D_{r+1}(t) = \sum_{i=0}^{r+1} B_i(t) C_{r+1-i}(t) = \sum_{i=0}^{r} B_i(t) C_{r+1-i}(t) + B_{r+1}(t) C_0(t)$$

$$\leq \frac{t k_p q}{1 - q} \sum_{i=0}^{r} B_i(t) C_{r-i}(t) + \frac{t k_p p}{1 - p} B_r(t) C_0(t) \leq \frac{t k_p q}{1 - q} \sum_{i=0}^{r} B_i(t) C_{r-i}(t) + \frac{t k_p p}{1 - p} \sum_{i=0}^{r} B_i(t) C_{r-i}(t)$$

$$\leq \frac{\phi t}{1 - \max\{p, q\}} \sum_{i=0}^{r} B_i(t) C_{r-i}(t) = \frac{\phi t}{1 - \max\{p, q\}} D_r(t)$$

We are now ready to prove the second half of Lemma 17.

Statement 3. If $\mathcal{I}(\tau) < k/(3\phi)$, then $|\mathcal{H}_{a+1}| < (2/3)|\mathcal{H}_a|$.

Proof. Let $t = \mathcal{I}(\tau)/k$. If $\mathcal{I}(\tau) < k/(3\phi)$, then $t < 1/(3\phi)$ and $\phi t < 1/3$. This implies $\phi t (1 - \max\{p, q\}) < 2/3$. For every $u \in \mathcal{H}$, $\Pr[u \in \mathcal{H}_a] = D_a(t)$.

$$\Pr[u \in \mathcal{H}_{a+1}] = D_{a+1}(t) < \frac{\phi t}{1 - \max\{p, q\}} D_a(t) < \frac{2}{3} D_a(t) < \frac{2}{3} \Pr[u \in \mathcal{H}_a]$$

If $\mathcal{I}(\tau) > k/(3\phi)$ and $t = \mathcal{I}(\tau)/k$, then $t > 1/(3\phi)$. At this point, Lemma 11 Statement 1 applies and $\mathcal{I}(\tau + 1)$ consists of a positive fraction of the nodes.

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