SYNTHESIS OF LOSSLESS ELECTRIC CIRCUITS BASED ON PRESCRIBED JORDAN FORMS

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ABSTRACT. We advance here an algorithm of the synthesis of lossless electric circuits such that their evolution matrices have the prescribed Jordan canonical forms subject to natural constraints. Every synthesized circuit consists of a chain-like sequence of LC-loops coupled by gyrators. All involved capacitances, inductances and gyrator resistances are either positive or negative with values determined by explicit formulas. A circuit must have at least one negative capacitance or inductance for having a nontrivial Jordan block for the relevant matrix.

1. INTRODUCTION

This work is motivated by an interest to electromagnetic and optical systems exhibiting Jordan eigenvector degeneracy, which is a degeneracy of the system evolution matrix when not only some eigenvalues coincide but also the corresponding eigenvectors coincide as well. Another way to describe the eigenvector degeneracy of a matrix is by acknowledging that there is no a basis in the relevant vector space made of eigenvectors of the matrix. Such degenerate system states are quite often are referred to as exceptional points of degeneracy (EPDs), [Kato, II.1]. A particularly important class of applications of EPDs is sensing, [CheN], [PeLiXu], [Wie], [Wie1]. Other potential applications include (i) enhancement of the gain in active systems, [VPFC], [MLSPL], [OVFC], [OVFC1], [OTC], and (ii) directivity of antennas, [OthCap]. A variety of systems have been suggested that exhibit EPDs in space for waveguide structures and time for circuits. These systems are based on: (i) non-Hermitian parity-time (PT) symmetric coupled systems, which are systems with balanced loss and gain, [BenBoe], [RKEC], [OGC]; (ii) coupled resonators [SLZEK], [HMHCK], [HHWGECK], (iii) electronic circuits involving dissipation, [SteHeSch].

Systems with EPDs in cited above literature commonly involve loss and gain elements suggesting that they might be essential to the existence of EPDs, see for instance [Berr]. It turns out though that the presence of loss and gain elements in a system is not necessary for having EPD regimes. An interesting system without loss and gain elements has been proposed in [KNAC] where the authors demonstrate that EPDs can exist for a single LC resonator with time-periodic modulation. Our own studies in [FigTWTbk] show that an analytical model of traveling wave tube (TWT) has the Jordan eigenvector degeneracy at some points of the system dispersion relation. This TWT system is governed by a Lagrangian and consequently it is a perfectly conservative system. Inspired by those studies we raised a question if simple lossless (perfectly conservative) circuits exist such that their evolution matrices exhibit the Jordan eigenvector degeneracy. We answered to the question positively by constructing circuits with prescribed degeneracies.

Our primary goal here is to synthesize a lossless electric circuit so that its evolution matrix $\mathcal{H}$ has a prescribed Jordan canonical form $\mathcal{J}$ subject to natural constraints considered later on. Hence by the definition of the Jordan canonical form, $\mathcal{H} = S \mathcal{J} S^{-1}$ where $S$ is an invertible matrix and $\mathcal{J}$ is a block diagonal matrix of the form.

Key words and phrases. Electric circuit, electric network, synthesis, negative capacitance, negative inductance, gyrator, Lagrangian, Hamiltonian, exceptional points of degeneracy (EPD), Jordan block, lossless.
\( \mathcal{J} = \begin{bmatrix} J_{n_1}(\zeta_1) & \cdots & 0 & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \ddots & J_{n_{q-1}}(\zeta_{q-1}) & 0 \\ 0 & \cdots & 0 & J_{n_q}(\zeta_q) \end{bmatrix}, \quad J_n(\zeta) = \begin{bmatrix} \zeta & 1 & \cdots & 0 & 0 \\ 0 & \zeta & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \zeta & 1 \\ 0 & 0 & \cdots & 0 & \zeta \end{bmatrix} \)

\( \zeta \) are real or complex numbers, and \( J_n(\zeta) \) is the so-called Jordan block which is \( n \times n \) matrix. For \( n = 1 \) the matrix \( J_n(\zeta) = \zeta \) turns just into number \( \zeta \).

As to the evolution matrix \( \mathcal{H} \) we assume that the circuit evolution is governed by the following linear equation

\[
\partial_t X = \mathcal{H} X
\]

where \( X \) is \( 2n \) dimensional vector-column describing the circuit state and \( \mathcal{H} \) is \( 2n \times 2n \) matrix where \( n > 1 \) is an integer. The particular choice of the dimensions is explained by our desire to have an underlying Lagrangian and Hamiltonian structure so that equation (1.2) will be the Hamilton evolution equation. Consequently, \( 2n \times 2n \) matrix \( \mathcal{H} \) is going to be a Hamiltonian matrix and we will refer to it as the circuit evolution matrix or just circuit matrix, see Section 7. To meet the dimension requirements of the evolution equation (1.2) the circuit topological structure is expected to have \( n \) fundamental loops or f-loops for short, see Section 8. The circuit state then is described by the corresponding \( n \) time dependent charges \( q_k(t) \) which are the time integrals of the relevant loop currents \( \partial_t q_k(t) \). Hamiltonian formulations of the dynamics of LC circuits has been studied, see for instance [Masc] and references therein.

The eigenvalue problem associated with the evolution evolution (1.2) is

\[
\mathcal{H} X = sX, \quad s = i\omega,
\]

where \( \omega \) is the frequency. Notice that the eigenvalue (spectral parameter) \( s \) is pure imaginary for real frequencies.

As to the prescribed Jordan canonical form \( \mathcal{J} \) we are rather interested in the simplest possible systems exhibiting nontrivial Jordan blocks than systems that can have arbitrary Jordan canonical form allowed for Hamiltonian matrices. It turns out that if the Jordan canonical form \( \mathcal{J} \) of the circuit matrix \( \mathcal{H} \) has a nontrivial Jordan block then the circuit must have at least one negative capacitance or inductance, see Section 7.5. The Jordan forms associated with Hamiltonian matrices must satisfy certain constraints considered in Section 4. The origin of the constraints is the fundamental property of a Hamiltonian matrix \( \mathcal{H} \) to be similar to \( -\mathcal{H}^T \) which is the transposed to \( \mathcal{H} \) matrix. This special property of a Hamiltonian matrix combined with the general statement that every square matrix \( M \) is similar to the transposed to it matrix \( M^T \) impose the following constraints on the spectral structure of matrix \( \mathcal{H} \): (i) if \( s \) in an eigenvalue of \( \mathcal{H} \) then \( -s \) is its eigenvalue as well; (ii) the Jordan blocks corresponding to the eigenvalues \( s \) and \( -s \) have the same structure. If in addition to that the entries of the Hamiltonian matrix \( \mathcal{H} \) are real-valued then the following properties hold: (i) if \( s \) in an eigenvalue of \( \mathcal{H} \) then \( -s \), \( \bar{s} \) and \( -\bar{s} \), where \( \bar{s} \) is complex-conjugate to \( s \), are its eigenvalues as well; (ii) the Jordan blocks corresponding to \( s \), \( -s \), \( \bar{s} \) and \( -\bar{s} \) have the same structure. We refer to the listed properties as Hamiltonian spectral symmetry, see Sections 4, 7.6. Apart from the Hamiltonian spectral symmetry the Jordan structure of Hamiltonian matrices can be arbitrary, [ArnGiv 2.2]. Our approach to the generation of Hamiltonian and the corresponding Hamiltonian matrices is intimately related to the Hamiltonian canonical forms, see Section 15 and references therein.
Another significant mathematical input to the synthesis of the simplest possible systems exhibiting nontrivial Jordan blocks comes from the property of a square matrix \( M \) to be cyclic (also called non-derogatory), see Section 11 and references therein. We remind that a square matrix \( M \) is called cyclic (or non-derogatory) if the geometric multiplicity of each of its eigenvalues is exactly 1, or in other words, if every eigenvalue of \( M \) has exactly one eigenvector. Consequently, if a square matrix \( M \) is cyclic its Jordan form \( J_M \) is completely determined by its characteristic polynomial \( \chi(s) = \det(sI - M) \) where \( I \) is the identity matrix of the relevant dimension. Namely, every eigenvalue \( s_0 \) of \( M \) of multiplicity \( m \) is associated with the single Jordan block \( J_m(s_0) \) in the Jordan form \( J_M \) of \( M \). Consequently, for a cyclic matrix \( M \) its characteristic polynomial \( \chi(s) = \det(sI - M) \) encodes all the information about its Jordan form \( J_M \). Another property of any cyclic matrix \( M \) associated with the a monic polynomial \( \chi \) is that it is similar to the so-called companion matrix \( C_\chi \) defined by simple explicit expression involving the coefficients of the polynomial \( \chi \), see Section 11 and references therein. Companion matrix \( C_\chi \) is naturally related to the high-order differential equation \( \chi(\partial_t)x(t) = 0 \) where \( x(t) \) is a complex-valued function of \( t \), see Sections 11 and 13. This fact underlines the relevance of the cyclicity property to the evolution of simpler systems described by higher order differential equations for a scalar function. In light of the above discussion, we focus on cyclic Hamiltonian matrices \( \mathcal{H} \) for they lead to the simplest circuits with the evolution matrices \( \mathcal{H} \) having nontrivial Jordan forms \( J \).

Suppose the prescribed Jordan form \( J \) is an \( 2n \times 2n \) matrix subject to the Hamiltonian spectral symmetry and the cyclicity conditions. The synthesis of a circuit associated with \( J \) involves the following steps. We introduce first the characteristic polynomial \( \chi(s) = \det(sI - J) \) which is an even monic polynomial \( \chi(s) \) of the degree \( 2n \). We consider then the companion to \( \chi(s) \) matrix \( C \), see Section 11 which by the design has \( J \) as its Jordan form, that is

\[
C = \mathcal{Y} J \mathcal{Y}^{-1},
\]

where the columns of matrix \( \mathcal{Y} \) form the so-called Jordan basis of the companion matrix \( C \) associated with the characteristic polynomial \( \chi(s) = \det(sI - J) \), see Section 11. We proceed with an introduction of our principal Hamiltonian \( \mathcal{H} \), see Section 4 and recover from it \( 2n \times 2n \) Hamiltonian matrix \( \mathcal{H} \) that governs the system evolution according to equation (1.2). As the result of our particular choice of the Hamiltonian \( \mathcal{H} \) the corresponding to it Hamiltonian matrix \( \mathcal{H} \) is similar to the companion matrix \( C \) and consequently it has exactly the same Jordan form \( J \) as \( C \). In particular, we construct an \( 2n \times 2n \) matrix \( T \) such that

\[
C = T^{-1} \mathcal{H} T, \quad \mathcal{H} = \mathcal{Z} J \mathcal{Z}^{-1}, \quad \mathcal{Z} = T \mathcal{Y},
\]

where the columns of matrix \( \mathcal{Z} \) form a Jordan basis of the evolution matrix \( \mathcal{H} \). The relations (1.4) and (1.5) between involved matrices are considered in Section 5.

To relate the constructed Hamiltonian \( \mathcal{H} \) to a circuit we introduce the corresponding to it Lagrangian \( \mathcal{L} \). Finally, based on the Lagrangian \( \mathcal{L} \) we design the relevant to it circuit, see Section 2. Consequently, this circuit evolution is governed by equation (1.2) with cyclic Hamiltonian matrix \( \mathcal{H} \) that has the prescribed \( J \) as its Jordan form. Each of the described steps of the circuit synthesis and the quantities constructed in the process provide insights into the circuit features.

In the light of our studies we can revisit now the question whether the presence of the balanced loss and gain is essential for achieving an electric circuit governed by the evolution matrix with nontrivial Jordan form. We have succeeded in constructing lossless circuits associated with nontrivial Jordan forms. Each of these circuits though must involve at least one negative capacitance or inductance. If we take a look at the physical implementations of negative capacitance and inductance provided in Section 8.2 we find that they involve matched positive and negative resistances. Based on this we may conclude that (i) the presence of the balanced loss and gain is
essential for achieving negative values for the capacitance and the inductance; (ii) the presence of at least one capacitor or inductor of negative value of the capacitance or inductance respectively is necessary for achieving a lossless electric circuit associated with nontrivial Jordan form.

The structure of the paper is as follows. In Section 2 we show our principal circuit tailored to the desired Jordan form $\mathcal{J}$ subject to natural constraints. In Section 3 we introduce special circuits tailored to specially chosen characteristic polynomials $\chi(s)$ and the corresponding Jordan forms made of exactly two Jordan blocks of the size 2, 3 and 4. In Section 4 we provide our strategy for the synthesis of circuits associated with the desired Jordan forms. Section 5 is devoted to the analysis of our principal circuit Hamiltonian which is the basis to the circuit synthesis. In Section 6 we consider examples of the principal circuit Hamiltonian and significant matrices. Section 7 provides aspects of the Lagrangian and Hamiltonian formalisms as well important properties of Hamiltonian matrices. In Section 8 we review basic elements of the electric networks and their elements including gyrators and negative capacitances and inductances. Sections 10-15 are devoted to a number of mathematical subjects needed for our analysis. In Section 16 we provide the list of notations used throughout the paper.

2. Principal circuit

Leaving the technical details of the circuit synthesis to the following sections we present here our principal circuit design that implements the desired Jordan form $\mathcal{J}$ of the circuit evolution matrix $\mathcal{H}$. Quite remarkably the topology of circuits associated with different Jordan forms is essentially the same. The difference between the circuits is in: (i) the number of involved $LC$-loops; (ii) particular values of the involved capacitances, inductances and gyration resistances. Fig. 2.1 shows our principal circuit made of $n$ $LC$-loops coupled by gyrators. Quantities $L_j$, $C_j$ and $G_j$ are respectively inductances, capacitances and gyrator resistances.

![Image of the principal circuit made of n LC-loops.](image)

**Figure 2.1.** The principal circuit made of $n$ $LC$-loops. Notice the difference between the left and the right connections for the gyrators and $LC$-loops. It is explained by the non-reciprocity of the gyrators and is designed to be consistent with (i) the standard port assignment and selection of positive directions for the loop currents and the gyrator; (ii) the sign of gyration resistance as shown in Fig. 8.2 and equations (8.2). The values of quantities $L_j$, $C_j$ and $G_j$ are determined by equations (2.7)-(2.8) relating them to the coefficients of the relevant polynomial.

To simplify equations throughout the paper we introduce the following dimensionless version of some of the involved quantities

\begin{equation}
\tilde{t} = \omega_0 t, \quad \tilde{C}_j = \frac{C_j}{|C_n|}, \quad \tilde{L}_j = \omega_0^2 |C_n| L_j, \quad \tilde{G}_j = G_j |C_n| \omega_0, \quad \tilde{\mathcal{L}} = |C_n| \mathcal{L},
\end{equation}

where $\omega_0 > 0$ is a unit of frequency and $1 \leq j \leq n$ and $\tilde{\mathcal{L}}$ is the scaled Lagrangian. To have less cluttered formulas we actually omit “hat” from $\tilde{\mathcal{L}}$, $\tilde{t}$, $\tilde{C}_j$, $\tilde{L}_j$ and $\tilde{G}_j$ and simply remember from now on that we use the relevant letters for the dimensionless quantities and the scaled Lagrangian.

The principal circuit Lagrangian associated with the principal circuit depicted in Fig. 2.1 is
\[ (2.2) \quad \mathcal{L} = \sum_{k=1}^{n} \frac{L_k (\partial_k q_k)^2}{2} - \sum_{k=1}^{n} \frac{(q_k)^2}{2C_k} + \sum_{k=1}^{n-1} G_k (q_k \partial_k q_{k+1} - q_{k+1} \partial_k q_k), \]

where \( q_k \) and \( i_k \) are respectively the charges and the currents associated with LC-loops of the principal circuit depicted in Fig. 2.1. The corresponding Euler-Lagrange (EL) equations are

\[ (2.3) \quad L_1 \partial_t^2 q_1 - G_1 \partial_t q_2 + \frac{q_1}{C_1} = 0, \quad L_n \partial_t^2 q_n + G_{n-1} \partial_t q_{n-1} + \frac{q_n}{C_n} = 0, \]

\[ (2.4) \quad L_k \partial_t^2 q_k + G_{k-1} \partial_t q_{k-1} - G_k \partial_t q_{k+1} + \frac{q_k}{C_k} = 0, \quad 1 < k < n. \]

It is well known that the EL equations (2.3) and (2.4) represent the Kirchhoff voltage law for each of the \( n \) f-loops, see Section 8. Indeed, each term in these equations is associated with the voltage drop for the relevant electric element as it can be verified by comparison with the voltage-current relations reviewed in Section 8.1. As to the Kirchhoff current law, one finds that it is already satisfied for the relevant electric element as it can be verified by comparison with the voltage-current expressions for the circuit electric inductances, capacitances and gyrations resistances in

\[ (2.5) \quad \chi (s) = \text{det} \{ s2n - \mathcal{J} \} = s^{2n} + (-1)^n \sum_{k=1}^{n} a_{n-k} s^{2(n-k)} = s^{2n} + (-1)^n \left( a_{n-1} s^{2(n-1)} + a_{n-2} s^{2(n-2)} + \cdots + a_0 \right), \]

where parameters \( a_k \) are real-valued and satisfy

\[ (2.6) \quad -\infty < a_k < \infty, \quad 0 \leq k \leq n-1; \quad a_0 \neq 0. \]

The Jordan form \( \mathcal{J} \) has to satisfy some a priori symmetry conditions to be associated with a Hamiltonian matrix \( \mathcal{H} \). In particular, its characteristic polynomial \( \chi (s) \) has to be even polynomial as indicated by equations (2.5) and its parameters \( a_k \) must be as described in relations (2.6). The details on indicated properties of matrices \( \mathcal{J} \) and \( \mathcal{H} \) are provided in Section 5.

We relate then the principal circuit to the characteristic polynomial \( \chi (s) \) by the setting up the following expressions for the circuit electric inductances, capacitances and gyrations resistances in
terms of the coefficients $a_k$ of the polynomial $\chi(s)$:

\begin{equation}
L_j = \frac{1}{(-1)^{j-1}a_{j-1}}, \quad 1 \leq j \leq n; \quad L_1 = \frac{1}{a_0},
\end{equation}

\begin{equation}
C_j = (-1)^{j-1}a_j, \quad G_j = \frac{1}{(-1)^{j-1}a_j} \quad 1 \leq j \leq n-1, \quad C_n = -1.
\end{equation}

Notice, that the equations (2.7)-(2.8) imply the following identities

\begin{equation}
C_jL_{j+1} = -1, \quad C_jG_j = 1, \quad 1 \leq j \leq n-1,
\end{equation}

as well as following expressions for coefficients $a_j$ in terms of the circuit parameters

\begin{equation}
a_j = \frac{1}{(-1)^j L_{j+1}}, \quad 0 \leq j \leq n-1, \quad a_j = (-1)^{j-1} \frac{1}{G_j} = (-1)^{j-1} C_j, \quad 1 \leq j \leq n-1.
\end{equation}

Under assumptions that the circuit elements values satisfy equations (2.7)-(2.8) the Lagrangian $L$ defined by equations (2.2) is related to our principal Hamiltonian $H$ defined by equations (4.3). The relationship between $L$ and $H$ is as follows. The Lagrangian $L'$ obtained from $H$ by the Legendre transformation has exactly the same EL equations as the Lagrangian $L$, see Sections 5, 7.1.

We show in Section 13 that any solution to the EL equations (2.3) and (2.4) satisfies also the scalar differential equation

\begin{equation}
\chi(\partial_t)q_k(t) = 0, \quad 1 \leq k \leq n,
\end{equation}

indicating that the circuit Hamiltonian matrix $\mathcal{H}$ is cyclic and is determined by the characteristic polynomial $\chi(s)$ defined by equations (2.5).

2.1. Principle circuit for two loops. The principal circuit for two-loops is shown in Fig. 2.2. It is the simplest case of our principal circuit that carries most of the significant properties of the general case. The the general form (2.5) of the characteristic polynomial for $n = 2$ turns into

\begin{equation}
\chi(s) = s^4 + a_1s^2 + a_0.
\end{equation}

The general form (2.2) of principal circuit Lagrangian yields for $n = 2$

\begin{equation}
L = \frac{L_1(\partial_tq_1)^2}{2} + \frac{L_2(\partial_tq_2)^2}{2} - \frac{(q_1)^2}{2C_1} - \frac{(q_2)^2}{2C_2} + \frac{G_1(q_1\partial_tq_2 - q_2\partial_tq_1)}{2},
\end{equation}

and the corresponding EL equations are

\begin{equation}
L_1\partial_t^2q_1 - G_1\partial_tq_2 + \frac{q_1}{C_1} = 0, \quad L_2\partial_t^2q_2 + G_1\partial_tq_1 + \frac{q_2}{C_2} = 0.
\end{equation
In particular, as the consequence of equations (2.9)-(2.14) as well as the data in Table 1 the following identities hold

\[
C_1 L_2 = -1, \quad C_1 G_1 = 1; \quad a_0 = \frac{1}{L_1}, \quad a_1 = -\frac{1}{L_2} = \frac{1}{G_1} = C_1.
\]

The set of values of the principal circuit elements described by equations (2.7)-(2.8) for \(n = 2\) are listed in Table 1. The significant circuit matrices in this case are as follows:

**Table 1. Circuit elements values, \(n = 2\)**

| \(k\) | \(1\) | \(2\) |
|-------|------|------|
| \(L_k\) | \(\frac{1}{a_0}\) | \(-\frac{1}{a_1}\) |
| \(C_k\) | \(a_1\) | \(-1\) |
| \(G_k\) | \(\frac{1}{a_1}\) | \(\frac{1}{a_1}\) |

\[
(2.15) \quad H = \begin{bmatrix}
0 & 0 & a_0 & 0 \\
1 & 0 & 0 & -a_1 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-a_0 & 0 & -a_1 & 0
\end{bmatrix}, \quad T = \begin{bmatrix}
0 & -a_0 & 0 & 0 \\
-a_0 & 0 & -a_1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

The determinants of the above matrices are as follows

\[
(2.17) \quad \det \{H\} = \det \{C\} = a_0, \quad \det \{T\} = a_0^2.
\]

The similarity between matrices \(H\) and \(C\) takes here the form

\[
(2.18) \quad T^{-1} H T = C,
\]

and can be verified by showing that \(HT = CT^{-1}\) based on expressions (2.16) for the involved matrices.

Solutions to the EL equations (2.14) according to equations (2.6) satisfy the following scalar differential equation

\[
(2.19) \quad \left( \partial_t^4 + a_1 \partial_t^2 + a_0 \right) q_k(t) = 0, \quad k = 1, 2.
\]

### 3. Special circuits

We define special circuits as implementations of our principal circuit tailored to specially chosen characteristic polynomials \(\chi(s)\) to achieve the desired Jordan forms. Namely, we are interested in

\[
(3.1) \quad \chi(s) = (s^2 - a^2)^n, \quad (s^2 + b^2)^n, \quad n = 2, 3, 4,
\]

for real \(a\) and \(b\) corresponding to the Jordan form \(J\) made of two Jordan blocks \(J_n(\pm a)\) and \(J_n(\pm bi)\) respectively, where \(J_n(s)\) is \(n \times n\) matrix defined by equation (1.1). We are also interested in

\[
(3.2) \quad \chi(s) = [(s - \zeta) (s - \bar{\zeta}) (s + \zeta) (s + \bar{\zeta})]^2, \quad \zeta = a + bi,
\]

where \(a \neq 0\) and \(b \neq 0\) corresponding to the Jordan form \(J\) made of 4 Jordan blocks \(J_2(\pm a \pm bi)\) and \(J_2(\pm a \mp bi)\).

When considering special circuits we evaluate the significant matrices \(H, J, C, Y, Z\) and \(T\) related by equations (1.4) and (1.5), see Sections 1.5.
3.1. Special circuits for two real or pure imaginary eigenvalues and \( n = 2 \). This is the simplest case demonstrating nontrivial Jordan forms and for that reason we study it greater detail. The special circuit shown in Fig. 2.2 has 2 f-loops. To get the desired Jordan form we use one of the polynomials

\[
\chi(s) = (s^2 - a^2)^2, \quad (s^2 + b^2)^2,
\]

and assign to the circuit elements the values provided in Table 2.

**Table 2.** Circuit elements values, \( n = 2 \)

| \( k \) | \( 1 \) | \( 2 \) | \( k \) | \( 1 \) | \( 2 \) |
|-----|-----|-----|-----|-----|-----|
| \( L_k \) | \( \frac{1}{a^2} \) | \( \frac{1}{2a^2} \) | \( L_k \) | \( \frac{1}{2b^2} \) | \( \frac{1}{2b^2} \) |
| \( C_k \) | \( -2a^2 \) | \( -1 \) | \( C_k \) | \( 2b^2 \) | \( -1 \) |
| \( G_k \) | \( -\frac{1}{2a^2} \) | \( \frac{1}{2b^2} \) | \( G_k \) | \( \frac{1}{2b^2} \) | \( \frac{1}{2b^2} \) |
| \( r; n \) | \( \pm a; 2 \) | \( r; n \) | \( \pm bi; 2 \) |

In the case of the first polynomial \( \chi(s) \) that has real roots \( \pm a \) the circuit significant matrices are as follows:

\[
\mathcal{H} = \begin{bmatrix}
0 & 0 & a^4 & 0 \\
1 & 0 & 0 & 2a^2 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix}
a & 1 & 0 & 0 \\
0 & a & 0 & 0 \\
0 & 0 & -a & 1 \\
0 & 0 & 0 & -a
\end{bmatrix}, \quad T = \begin{bmatrix}
0 & -a^4 & 0 & 0 \\
-a^4 & 0 & 2a^2 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix}
-4 & -4 & a^5 & -a^4 \\
4a^3 & -a^4 & 4a^3 & -4a^3 \\
-2a & -a^2 & -2a & 2a \\
a^3 & 3a^2 & -3a^3 & 3a^2
\end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix}
0 & -b^4 & 0 & 0 \\
b^4 & 0 & -2b^2 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

In the case of the second polynomial \( \chi(s) \) that has pure imaginary roots \( \pm bi \) the significant circuit matrices are

\[
\mathcal{H} = \begin{bmatrix}
0 & 0 & bi^4 & 0 \\
1 & 0 & 0 & -2bi^2 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix}
bi & 1 & 0 & 0 \\
0 & bi & 0 & 0 \\
0 & 0 & -bi & 1 \\
0 & 0 & 0 & -bi
\end{bmatrix}, \quad T = \begin{bmatrix}
0 & -b^4 & 0 & 0 \\
b^4 & 0 & -2b^2 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix}
1 & 0 & 1 & 0 \\
bi & 1 & -bi & 1 \\
-b^2 & 2bi & -b^2 & -2bi \\
-b^3i & -3b^2 & b^3i & -3b^2
\end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix}
-4 & -4 & b^5i & -b^4 \\
b^4 & 4b^3i & b^4 & 4b^3i \\
-2b & -2bi & b^2 & 2bi \\
-b^3i & -3b^2 & b^3i & -3b^2
\end{bmatrix}.
\]
3.2. **Special circuit for two real or pure imaginary eigenvalues and** \( n = 3 \). This special circuit shown in Fig. 3.1 has 3 f-loops. It provides for two Jordan blocks of order 3 for circuit elements values as in Table 3. The corresponding polynomial is

\[
\chi(s) = (s^2 - a^2)^3, \quad (s^2 + b^2)^3,
\]

In the case when the roots of \( \chi(s) \) are real numbers \( \pm a \) the circuit matrices are

\[
H = \begin{bmatrix}
0 & 0 & 0 & a^6 & 0 & 0 \\
1 & 0 & 0 & 0 & 3a^4 & 0 \\
0 & 1 & 0 & 0 & 0 & 3a^2 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}, \quad J = \begin{bmatrix}
a & 1 & 0 & 0 & 0 & 0 \\
0 & a & 1 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & -a & 1 & 0 \\
0 & 0 & 0 & 0 & -a & 1 \\
0 & 0 & 0 & 0 & 0 & -a
\end{bmatrix},
\]

\[
F = \begin{bmatrix}
a^8 & 2a^7 & a^6 & a^8 & -2a^7 & a^6 \\
-2a^7 & -8a^6 & -9a^5 & 2a^7 & -8a^6 & 9a^5 \\
a^6 & 6a^5 & 15a^4 & a^6 & -6a^5 & 15a^4 \\
a^3 & 3a^2 & 3a & -a^3 & 3a^2 & -3a \\
-a^4 & -4a^3 & -6a^2 & -a^4 & 4a^3 & -6a^2 \\
a^5 & 5a^4 & 10a^3 & -a^5 & 5a^4 & -10a^3
\end{bmatrix}.
\]

In the case when the roots of \( \chi(s) \) are pure imaginary numbers \( \pm bi \) the circuit matrices are

\[
H = \begin{bmatrix}
0 & 0 & 0 & -b^6 & 0 & 0 \\
1 & 0 & 0 & 0 & 3b^4 & 0 \\
0 & 1 & 0 & 0 & 0 & -3b^2 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}, \quad J = \begin{bmatrix}
bi & 1 & 0 & 0 & 0 & 0 \\
b & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & bi & 0 & 0 & 0 \\
0 & 0 & 0 & -bi & 1 & 0 \\
0 & 0 & 0 & 0 & -bi & 1 \\
0 & 0 & 0 & 0 & 0 & -bi
\end{bmatrix}.
\]
3.3. Special circuit for two real or pure imaginary eigenvalues and \( n = 4 \). This special circuit shown in Fig. has 4 f-loops. It provides for two Jordan blocks of order 4 for circuit elements values as in Tables 4 and 5.

![Figure 3.2. Special circuit for 3 LC-loops.](image)

The values of circuit elements are provided in Tables 4 and 5.

**Table 4. Circuit elements values, \( n = 4 \)**

| \( k \) | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| \( L_k \) | \( \frac{1}{a^8} \) | \( \frac{1}{4a^6} \) | \( \frac{1}{6a^4} \) | \( \frac{1}{4a^2} \) |
| \( C_k \) | \( -4a^6 \) | \( -6a^4 \) | \( -4a^2 \) | \( -1 \) |
| \( G_k \) | \( -\frac{1}{4a^6} \) | \( -\frac{1}{6a^4} \) | \( -\frac{1}{4a^2} \) |  |
| \( r; \, n \) | \( \pm a; \, 4 \) |  |

The corresponding polynomial is

\[
\chi(s) = (s^2 - a^2)^4, \quad (s^2 + b^2)^4.
\]

In the case when the roots of \( \chi(s) \) are real numbers \( \pm a \) the circuit matrices are

\[
(3.15) \quad \mathcal{L} = \begin{bmatrix}
    0 & 0 & 0 & a^8 & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & 4a^6 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 6a^4 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 4a^2 \\
    0 & 0 & 0 & 0 & -1 & 0 & 0 \\
    0 & 0 & 0 & 0 & -1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & -1 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix}
a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -a & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -a & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a & 1
\end{bmatrix}.
\]
In the case when the roots of \( \chi(s) \) are pure imaginary numbers \( \pm bi \), the circuit matrices are

\[
\mathcal{X} = \begin{bmatrix}
-a^{11} & -3a^{10} & -3a^9 & -a^8 & a^{11} & -3a^{10} & 3a^9 & -a^8 \\
3a^{10} & 14a^9 & 23a^8 & 16a^7 & 3a^{10} & -14a^9 & 23a^8 & -16a^7 \\
-3a^9 & -19a^8 & -48a^7 & -56a^6 & 3a^9 & -19a^8 & 48a^7 & -56a^6 \\
a^8 & 8a^7 & 28a^6 & 56a^5 & a^8 & -8a^7 & 28a^6 & -56a^5 \\
a^7 & -4a^6 & -6a^5 & -4a^4 & 4a^3 & -6a^2 & 4a \\
a^6 & 5a^5 & 10a^4 & 10a^3 & -a^5 & 5a^4 & -10a^3 & 10a^2 \\
a^5 & -6a^4 & -15a^3 & -20a^2 & -6a^5 & -15a^4 & 20a^3 \\
a^4 & 7a^3 & 21a^2 & 35a^1 & 7a^6 & -21a^5 & 35a^4
\end{bmatrix}
\]

\[
\chi(s) = [(s - \zeta)(s - \bar{\zeta})(s + \zeta)(s + \bar{\zeta})]^2 = (a^2 + b^2 + 2bs + s^2)(a^2 + b^2 - 2bs + s^2)^2, \quad \zeta = a + bi,
\]

where \( a \neq 0 \) and \( b \neq 0 \). The values of the circuit elements are provided in Tables 6. We have computed all significant matrices but the number of their entries combined with the length of their expressions are too large to be displayed here.
Table 6. Circuit elements values, $n = 2$

| $k$ | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|
| $L_k$ | $\frac{1}{(a^2+b^2)^2}$ | $\frac{1}{4(a^2-b^2)(a^2+b^2)^2}$ | $\frac{1}{2(3a^4+3b^4-a^2b^2)}$ | $-\frac{1}{4(a^2-b^2)}$ |
| $C_k$ | $4 \left( a^2 - b^2 \right) \left( a^2 + b^2 \right)^2$ | $-2 \left( 3a^4 + 3b^4 - a^2b^2 \right)$ | $4 \left( a^2 - b^2 \right)$ | $-1$ |
| $G_k$ | $\frac{1}{4(a^2-b^2)(a^2+b^2)^2}$ | $-\frac{1}{2(2a^2b^2-3a^4-3b^4)}$ | $\frac{1}{2(2a^2b^2-3a^4-3b^4)}$ | $\frac{1}{4(a^2-b^2)}$ |
| $r; n$ | $a \pm bi$, $-a \pm bi$ | 2 |

4. Circuit synthesis strategy and elements

The first goal of our synthesis process is to construct a Hamiltonian system governed by the evolution equation (1.2) with the circuit matrix $H$ having the prescribed Jordan canonical form subject to natural constraints. Consequently, the circuit matrix $H$ has to be a Hamiltonian matrix, that is a matrix obtained from a quadratic Hamiltonian $H_{\text{principal}}$ is fundamental to the synthesis of all special circuits we construct and we refer to it as principal Hamiltonian.

To achieve the desired Jordan form for the circuit matrix $H$ we introduce the characteristic polynomial $\chi_H(s) = \chi_J(s)$ and find its coefficients. Having coefficients $a_k$ of the polynomial $\chi_H(s)$ as in equations (2.5) and (2.6) we define the Hamiltonian $H_a$ by the following explicit expression

\[(4.3) \quad H_a = \sum_{k=1}^{n} p_{k+1} q_k + \frac{1}{2} \sum_{k=1}^{n} (-1)^{k-1} a_{k-1} p_k^2 + \frac{1}{2} q_n^2,\]

\[(4.4) \quad -\infty < a_k < \infty, \quad 0 \leq k \leq n-1; \quad a_0 \neq 0,\]

Notice that the system parameters $a_k$ can be negative and positive. The particular choice of signs in expression (4.3) is a matter of convenience. Hamiltonian $H_a$ defined by equations (4.3) is fundamental to the synthesis of all special circuits we construct and we refer to it as principal Hamiltonian.

The principal Hamiltonian matrix $H_a$ that corresponds to the principal Hamiltonian $H_a$ has the following properties (see Section 3 for details):

- the corresponding to $H_a$ Hamiltonian matrix $H_a$ has the polynomial $\chi(s)$ defined by equations (2.5) as its characteristic polynomial $\chi_H(s)$, and, consequently, the set of the distinct roots $s_j$ of the polynomial is exactly the set of all distinct eigenvalues of the circuit matrix $H_a$, that is $\text{spec}(H_a) = \{s_j\}$;
- since $a_0 \neq 0$ we have $s_j \neq 0$ for every $j$;
- the spectrum $\text{spec}(H_a)$ satisfies Hamiltonian spectral symmetry condition (4.1).
the circuit matrix $\mathcal{H}_a$ is cyclic (nonderogatory), that is it the geometric multiplicity of every eigenvalue $s_j$ is exactly one, and every $s_j$ is associated with the single Jordan block $J_{n_j}(s_j)$ of the size $n_j$ which is the algebraic multiplicity of eigenvalue $s_j$; in other words there is always a single Jordan block for each distinct eigenvalue; the cyclicity property is an integral part of the construction yielding simpler Jordan forms;

- if a non-zero $\zeta \in \text{spec}(\mathcal{H}_a)$ is real or pure imaginary then the Jordan form $\mathcal{J}_a$ of the Hamiltonian matrix $\mathcal{H}_a$ has two Jordan blocks $J_n(\zeta)$ and $J_n(-\zeta)$ of the matching size $n$ where $n$ is the multiplicity of $\zeta$ as the a root of the polynomial $\chi(s)$.

- if $\zeta \in \text{spec}(\mathcal{H}_a)$ and $\zeta = a + bi$ with $a \neq 0$ and $b \neq 0$ then the Jordan form $\mathcal{J}_a$ of the Hamiltonian matrix $\mathcal{H}_a$ has four Jordan blocks $J_n(\pm a \pm bi)$ and $J_n(\pm a \mp bi)$ of the matching size $n$ where $n$ is the multiplicity of $\zeta$ as the a root of the polynomial $\chi(s)$.

Making particular choices of $a_k$ for the principal Hamiltonian $\mathcal{H}_a$ allows to achieve the desired Jordan forms. With that in mind we introduce the following specific polynomials for real numbers non-zero numbers $a$ and $b$:

$$\chi(s) = (s^2 - a^2)^n, \quad (s^2 + b^2)^n, \quad (4.5)$$

$$\chi(s) = \left[s^4 + 2(b^2 - a^2)s^2 + (a^2 + b^2)^2\right]^n. \quad (4.6)$$

Notice that polynomials in equations (4.5) have respectively two real roots $\pm a$ and two pure imaginary roots $\pm bi$ of multiplicities $n$, whereas the polynomial in equation (4.6) has four roots $\pm a \pm bi$ and $\pm a \mp bi$ of multiplicities $n$. The Jordan forms $\mathcal{J}_a$ of system matrices $\mathcal{H}_a$ associated with the polynomials in equations (4.5) and (4.6) are respectively

$$\mathcal{J}_a = \left[\begin{array}{cc} J_n(\zeta) & 0 \\ 0 & J_n(-\zeta) \end{array}\right], \quad \mathcal{J}_a = \left[\begin{array}{cccc} J_n(\zeta) & 0 & 0 & 0 \\ 0 & J_n(-\zeta) & 0 & 0 \\ 0 & 0 & J_n(\zeta) & 0 \\ 0 & 0 & 0 & J_n(-\zeta) \end{array}\right], \quad \zeta = a, bi, \quad (4.7)$$

where Jordan block $J_n(\zeta)$ is defined by equation (1.1).

We summarize now the important points of the analysis in Sections 2 and 3 in the following statement.

**Theorem 1** (principal circuit). Suppose that the principal circuit depicted in Fig. 2.1 has its element values defined by equations (2.7)-(2.8). Then the dynamics of the principal circuit is governed by the principal Lagrangian $L$ defined by equation (2.2) and the principal Hamiltonian $H$ defined by defined by equations (4.3). The corresponding EL equations (2.9) and (2.10) represent the Kirchhoff voltage law, whereas the Kirchhoff current is enforced by the selection of $n$ involved $f$-loops and currents $\partial q_k$.

The relevant Hamiltonian matrix $\mathcal{H}$ is cyclic (non-derogatory), and its characteristic polynomial $\chi(s)$ is defined by equations (2.12). The Jordan form $\mathcal{J}$ of matrix $\mathcal{H}$ is completely determined by $\chi(s)$. In particular, each distinct root $s_j$ of $\chi(s)$ of the multiplicity $n_j$ is represented in $\mathcal{J}$ by the single Jordan block $J_{n_j}(s_j)$ of the matching size $n_j$.

For particular choices of the monic polynomial $\chi(s)$ as described in equations (4.5) and (4.6) one obtains circuits associated with the Jordan forms represented respectively in equations (4.7).

## 5. The principal Hamiltonian and Lagrangian

Suppose that the system configuration is described by time-dependent $n$-dimensional vector-column $q$ and its dynamics is governed by a Hamiltonian $\mathcal{H} = \mathcal{H}(p, q)$ where $p$ is the system momentum which is an $n$-dimensional vector-column just as the configuration vector $q$. Suppose
now that the Hamiltonian \( \mathcal{H} \) is defined by equations (4.3). To present the system information in a compact matrix form we recast the representation of the Hamiltonian (4.3) as

\[
\mathcal{H}_a = \frac{1}{2} X^T M_H X, \quad X = \begin{bmatrix} q \\ p \end{bmatrix}, \quad M_H = \begin{bmatrix} -\pi_n & K_n \\ K_n^T & D_a \end{bmatrix},
\]

where \( D_a \) and \( \pi_n \) are diagonal \( n \times n \) matrices defined by

\[
\begin{pmatrix} a_0 & 0 & \cdots & 0 & 0 \\ 0 & -a_1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & (-1)^{n-3} a_{n-2} \end{pmatrix}, \quad \pi_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix},
\]

and \( K_n \) is \( n \times n \) nilpotent matrix defined by

\[
K_n = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.
\]

We also make use of the Jordan block \( J_n (\zeta) \) of the size \( n \) defined by

\[
J_n (\zeta) = \zeta \mathbb{I}_n + K_n = \begin{bmatrix} \zeta & 1 & \cdots & 0 & 0 \\ 0 & \zeta & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \zeta & 1 \end{bmatrix},
\]

where \( \mathbb{I}_n \) is \( n \times n \) identity matrix.

The evolution equations for the principal Hamiltonian \( \mathcal{H}_a \), defined be equations (5.1), are

\[
\partial_t X = \mathcal{H}_a X, \quad \mathcal{H}_a = \mathbb{J} M_H, \quad \mathbb{J} = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix},
\]

where the system state vector \( X \) and matrix \( M_H \) are defined by (4.3), and consequently

\[
\mathcal{H}_a = \mathbb{J} M_H = \begin{bmatrix} K_n^T & -D_a \\ \pi_n & -K_n \end{bmatrix}.
\]

With an eigenvalue problem in mind we introduce matrix

\[
s \mathbb{I}_{2n} - \mathcal{H}_a = \begin{bmatrix} -J_n^T (-s) & -D_a \\ -\pi_n & J_n (s) \end{bmatrix},
\]

and find then the corresponding characteristic function is equal to

\[
\chi_a (s) = \det \{ s \mathbb{I}_{2n} - \mathcal{H}_a \} = s^{2n} + (-1)^n \sum_{k=1}^{n} a_{n-k} s^{2(n-k)} = \\
= s^{2n} + (-1)^n (a_{n-1} s^{2(n-1)} + a_{n-2} s^{2(n-2)} + \cdots + a_0).
\]
To see that representation (5.8) for $\chi_a(s)$ holds we apply formula (14.3) to the right-hand side of equation (5.7) and obtain

$$
(5.9) \quad \chi_a(s) = \det\{sI_{2n} - \mathcal{H}_a\} = \det\{J_n^T(-s)\} \det\{-J_n(s) + \pi_n[J_n^{-1}(-s)]^T D_a\}.
$$

We use then equations (5.2) and (5.18) to evaluate the right-hand side of equation (5.9) and arrive at the formula (5.8).

We introduce now the so-called companion to the polynomial $\chi_a(s)$, see Section III which is $2n \times 2n$ matrix defined by

$$
(5.10) \quad \mathcal{C}_a = \begin{bmatrix}
0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
c_0 & c_1 & 0 & \cdots & c_{n-1} & 0
\end{bmatrix}, \quad c_k = (-1)^{n-k} a_k, \quad 0 \leq k \leq n.
$$

Notice that the eigenvalue problem for the companion matrix $\mathcal{C}_a$ has the following explicit form solution, see Section III

$$
(5.11) \quad \mathcal{C}_a Y(s) = sY(s), \quad Y(s) = \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{2n-2} \\ s^{2n-1} \end{bmatrix}, \quad Y[k] = s^{k-1}, \quad 1 \leq k \leq 2n, \quad \chi_a(s) = 0,
$$

where evidently vector polynomial $Y(s)$ is uniquely determined by the corresponding eigenvalue $s$. If all eigenvalues $s_j$, $1 \leq j \leq 2n$ of the companion matrix $\mathcal{C}_a$ are different the set of the corresponding eigenvectors $Y(s_j)$ form a basis that diagonalize matrix $\mathcal{C}_a$. In the general case we introduce an $2n \times 2n$ matrix $\mathcal{Y}_a$ as the generalized Vandermonde matrix defined by equations (11.10), (11.11). Then according to Proposition 14 we have

$$
(5.12) \quad \mathcal{C}_a = \mathcal{Y}_a \mathcal{J}_a \mathcal{Y}_a^{-1},
$$

where $\mathcal{J}_a$ is the Jordan form of the companion matrix $\mathcal{C}_a$. We refer to $\mathcal{Y}_a$ as Jordan basis matrix for matrix $\mathcal{C}_a$. In the special case of distinct eigenvalues matrix $\mathcal{Y}_a$ turns into the standard Vandermonde matrix defined by equation (11.13), that is a matrix formed by column-vectors $Y(s_j)$ as in equation (5.11).

Notice also that it follows from equations (5.8) and (5.10) that

$$
(5.13) \quad \det \{\mathcal{H}_a\} = \det \{\mathcal{C}_a\} = (-1)^n a_0.
$$

Let us turn now to the eigenvalue problem for the system matrix $\mathcal{H}_a$. In view of equation (5.7) an eigenvector $Z$ of $\mathcal{H}_a$ satisfies

$$
(5.14) \quad \begin{bmatrix} -J_n^T(-s) & -D_a \\ -\pi_n & J_n(s) \end{bmatrix} Z(s) = 0, \quad Z(s) = \begin{bmatrix} q(s) \\ p(s) \end{bmatrix},
$$
or equivalently
\begin{align}
(5.15) & \quad J_n^T (-s) q (s) + D_a p (s) = 0, \\
(5.16) & \quad -\pi_n q (s) + J_n (s) p (s) = 0.
\end{align}

Notice first that \( \pi_n q (s) \neq 0 \) otherwise we consequently obtain \( p (s) = 0 \) from equation (5.16) and then \( q (s) = 0 \) from equation (5.15) implying \( Z (s) = 0 \) contradicting that \( Z (s) \) is an eigenvector. Using that we normalize \( q (s) \) by the following assumption
\begin{equation}
(5.17) \quad \pi_n q (s) = e_n \left[ e_n^T q (s) \right] = s^{2n} e_n, \quad \text{or equivalently} \quad q_n (s) = e_n^T q (s) = s^{2n}.
\end{equation}

This particular choice of normalization makes the components of eigenvectors to be polynomials of \( s \) rather then rational functions. Combining the explicit formula
\begin{equation}
(5.18) \quad [J_n (s)]^{-1} = \begin{bmatrix}
\frac{1}{s} & -\frac{1}{s^2} & \frac{1}{s^3} & \cdots & \frac{(-1)^{n-1}}{s^n} \\
0 & \frac{1}{s} & -\frac{1}{s^2} & \cdots & \frac{(-1)^{n-2}}{s^{n-1}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \frac{1}{s} \\
0 & 0 & \cdots & 0 & -\frac{1}{s^2}
\end{bmatrix},
\end{equation}

with equations (5.16) and (5.18) we readily obtain
\begin{equation}
(5.19) \quad p (s) = s^{2n} (J_n (s))^{-1} e_n = \begin{bmatrix}
(-1)^{n-1} s^n \\
(-1)^n s^{n+1} \\
\vdots \\
-s^{2n-2} \\
s^{2n-1}
\end{bmatrix}.
\end{equation}

Then plugging expression (5.19) into equation (5.15) yields
\begin{equation}
(5.20) \quad q (s) = \left[ -J_n^T (-s) \right]^{-1} D_a p = -s^{2n} \left[ -J_n^T (-s) \right]^{-1} D_a (J_n (s))^{-1} e_n.
\end{equation}

Using equations (5.18), (5.19) and (5.20) we obtain the following expressions for the components of \( q (s) \) and \( p (s) \)
\begin{equation}
(5.21) \quad q_j (s) = (-1)^{n-1} \sum_{k=1}^{j} a_{k-1} s^{n-j+2(k-1)}, \quad p_j (s) = (-1)^{n+j} s^{n+j-1}, \quad 1 \leq j \leq n.
\end{equation}

Consequently we get the following representation for eigenvector \( Z (s) \)
\begin{equation}
(5.22) \quad \mathcal{H}_a Z (s) = s Z (s), \quad Z (s) = \begin{bmatrix}
q (s) \\
p (s)
\end{bmatrix} = \begin{bmatrix}
q_1 (s) \\
\vdots \\
q_n (s) \\
p_1 (s) \\
\vdots \\
p_n (s)
\end{bmatrix} = \sum_{k=0}^{2n-1} Z_k s^k,
\end{equation}

where \( q (s) \) and \( p (s) \) are defined by equations (5.21).

Notice that according to equations (5.21) and (5.22) the eigenvector \( Z (s) \) of the system matrix \( \mathcal{H}_a \) is uniquely determined by the corresponding eigenvalue \( s \). Evidently, \( Z (s) \) is a vector polynomial of \( s \) with vector coefficients \( Z_k \) which are determined by expressions (5.21) for vectors \( q (s) \) and \( p (s) \).
Comparing equations (5.22) and (5.11) we arrive with the following relationship between eigenvectors $Z(s)$ and $Y(s)$

\begin{equation}
Z(s) = T_a Y(s), \quad T_a = [Z_0 | Z_1 | \ldots | Z_{2n-1}], \quad \text{col}(T_a, k) = Z_{k-1}, \quad 1 \leq k \leq 2n - 1.
\end{equation}

Notice that $2n \times 2n$ matrix $T_a$ in equations (5.11) is defined by its columns which are the vector coefficients $Z_k$ of the vector polynomial $Z(s)$. Just as the system matrix $\mathcal{H}_a$ and the companion matrix $\mathcal{C}_a$ matrix $T_a$ is completely defined by the system parameters $a_k$ and hence by the polynomial $\chi_a(s)$. An analysis showed that $T_a$ is $2 \times 2$ upper triangular block matrix, with blocks of the dimension $n \times n$, and based on that one can establish that

\begin{equation}
\det\{T_a\} = a_0^n.
\end{equation}

The significance of matrix $T_a$ is that it provides for the similarity relation between the system matrix $\mathcal{H}_a$ and its companion matrix $\mathcal{C}_a$, that is

\begin{equation}
\mathcal{C}_a = T_a^{-1} \mathcal{H}_a T_a.
\end{equation}

Equations (6.2)-(6.3) and (6.6)-(6.7) show examples of matrices $\mathcal{H}_a$, $\mathcal{C}_a$ and $T_a$ for the cases $n = 3, 4$.

Notice then if we introduce $2n \times 2n$ matrix

\begin{equation}
\mathcal{Z}_a = T_a \mathcal{Y}_a
\end{equation}

use it in combination with equations (5.12) we obtain

\begin{equation}
\mathcal{H}_a = \mathcal{Z}_a \mathcal{J}_a \mathcal{Z}_a^{-1},
\end{equation}

where $\mathcal{J}_a$ is the Jordan form of the companion matrix $\mathcal{C}_a$ and hence of the system matrix $\mathcal{H}_a$ as well. We refer to $\mathcal{Z}_a$ as Jordan basis matrix for matrix $\mathcal{H}_a$.

The principal Lagrangian $\mathcal{L}_a$ obtained from the principal Hamiltonian $\mathcal{H}_a$ by the Legendre transformation is

\begin{equation}
\mathcal{L}_a = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{a_{k-1}} v_{k+1} q_k + \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{a_{k-1}} v_k^2 + \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^k}{a_k} q_k^2 + \frac{1}{2} q_n^2,
\end{equation}

\[v_k = \partial_t q_k, \quad 1 \leq k \leq n.\]

An equivalent to $\mathcal{L}_a$ version of it with the skew-symmetric gyroscopic part is the following Lagrangian

\begin{equation}
\mathcal{L}'_a = \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{a_{k-1}} (v_{k+1} q_k - v_k q_{k+1}) + \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{a_{k-1}} v_k^2 + \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^k}{a_k} q_k^2 + \frac{1}{2} q_n^2,
\end{equation}

\[v_k = \partial_t q_k, \quad 1 \leq k \leq n.\]

The equivalency between two Lagrangians defined by equations (5.28) and (5.29) is understood as that the corresponding EL equations are same, see Section 7.1.

6. Examples of the significant matrices for the principal Hamiltonian

We show in this section explicit form of matrices $\mathcal{H}_a$, $\mathcal{C}_a$ and $T_a$ related to the principal Hamiltonian defined by equations (4.3), (4.4). The expressions of these matrices are somewhat different for even and odd $n$, and with that in mind we consider two case of $n = 3$ and $n = 4$. 
6.1. The principal Hamiltonian and significant matrices for \( n = 3 \). The principal Hamiltonian and the corresponding characteristic polynomials for \( n = 4 \) are respectively

\[
\mathcal{H} = \sum_{k=1}^{3} p_{k+1} q_k + \frac{1}{2} \sum_{k=1}^{3} (-1)^{k-1} a_{k-1} p_k^2 + \frac{1}{2} q_3^2, \quad \chi(s) = s^6 - a_2 s^4 - a_1 s^2 - a_0.
\]

The significant matrices in this case are as follows:

\[
\mathcal{H} = \begin{bmatrix} 0 & 0 & 0 & a_0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -a_1 & 0 \\ 0 & 1 & 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ a_0 & 0 & a_1 & 0 & a_2 & 0 \end{bmatrix},
\]

\[
T = \begin{bmatrix} 0 & 0 & a_0 & 0 & 0 & 0 \\ 0 & a_0 & 0 & a_1 & 0 & 0 \\ a_0 & 0 & a_1 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]

6.2. The principal Hamiltonian and significant matrices for \( n = 4 \). The principal Hamiltonian and the corresponding characteristic polynomials for \( n = 4 \) are respectively

\[
\mathcal{H} = \sum_{k=1}^{4} p_{k+1} q_k + \frac{1}{2} \sum_{k=1}^{4} (-1)^{k-1} a_{k-1} p_k^2 + \frac{1}{2} q_4^2, \quad \chi(s) = s^8 + a_3 s^6 + a_2 s^4 + a_1 s^2 + a_0.
\]

The significant matrices in this case are as follows:

\[
\mathcal{H} = \begin{bmatrix} 0 & 0 & 0 & a_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -a_0 & 0 & -a_1 & 0 & -a_2 & 0 & -a_3 \end{bmatrix},
\]

\[
T = \begin{bmatrix} 0 & 0 & 0 & -a_0 & 0 & 0 & 0 \\ 0 & 0 & -a_0 & 0 & -a_1 & 0 & 0 \\ 0 & -a_0 & 0 & -a_1 & 0 & -a_2 & 0 \\ -a_0 & 0 & -a_1 & 0 & -a_2 & 0 & -a_3 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]
7. Lagrangian and Hamiltonian Structures for Linear Systems

We provide here basic facts on the Lagrangian and Hamiltonian structures for linear systems.

7.1. Lagrangian. Lagrangian $L$ for a linear system is a quadratic function (bilinear form) of the system state $Q = [q_r]_{r=1}^n$ (column vector) and its time derivatives $\partial_t Q$, that is

$$L = \mathcal{L}(Q, \partial_t Q) = \frac{1}{2} \left[ Q \partial_t Q \right]^T M_L \left[ Q \partial_t Q \right], \quad M_L = \begin{bmatrix} -\eta & \theta^T \\ \theta & \alpha \end{bmatrix},$$

(7.1)

where $T$ denotes the matrix transposition operation, and $\alpha, \eta$ and $\theta$ are $n \times n$-matrices with real-valued entries. In addition to that, we assume matrices $\alpha, \eta$ to be symmetric, that is

$$\alpha = \alpha^T, \quad \eta = \eta^T.$$

(7.2)

Consequently,

$$L = \frac{1}{2} \partial_t Q^T \alpha \partial_t Q + \partial_t Q^T \theta Q - \frac{1}{2} Q^T \eta Q.$$

Then by Hamilton’s principle, the system evolution is governed by the EL equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \partial_t Q} \right) - \frac{\partial L}{\partial Q} = 0,$$

(7.3)

which, in view of equation (7.3) for the Lagrangian $L$, turns into the following second-order vector ordinary differential equation (ODE):

$$\alpha \partial_t^2 Q + (\theta - \theta^T) \partial_t Q + \eta Q = 0.$$

(7.4)

Notice that matrix $\theta$ enters equation (7.5) through its skew-symmetric component $\frac{1}{2} (\theta - \theta^T)$ justifying as a possibility to impose the skew-symmetry assumption on $\theta$, that is

$$\theta^T = -\theta.$$

(7.6)

Indeed, the symmetric part $\theta_s = \frac{1}{2} (\theta + \theta^T)$ of the matrix $\theta$ is associated with a term to the Lagrangian which can be recast as is the complete (total) derivative, namely $\frac{1}{2} \partial_t (Q^T \theta_s Q)$. It is a well known fact that adding to a Lagrangian the complete (total) derivative of a function of $Q$ does not alter the the EL equations. Namely, the EL equations are invariant under the Lagrangian gauge transform $L \rightarrow L + \partial_t F(q, t)$, \cite{Scheck, 2.9, 2.10}, \cite{LanLifM, I.2}.

Under the assumption (7.6) equation (7.5) turns into its version with the skew-symmetric $\theta$

$$\alpha \partial_t^2 Q + 2\theta \partial_t Q + \eta Q = 0, \quad \text{if } \theta^T = -\theta.$$

(7.7)

It turns out though that our our principal Lagrangian that corresponds to the principal Hamiltonian by the Legendre transformation does not have skew-symmetric $\theta$ satisfying (7.6). For this reason we don’t impose the condition of skew-symmetry on $\theta$.

The EL equations are the second order ODE. The standard way to reduce them to the equivalent first order ODE yields

$$\partial_t Y = \mathcal{L} Y, \quad Y = \begin{bmatrix} Q \\ \partial_t Q \end{bmatrix},$$

(7.8)

where

$$\mathcal{L} = \begin{bmatrix} 0 & -\alpha^{-1} (\theta - \theta^T) \\ -\alpha^{-1} \eta & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -\alpha^{-1} \end{bmatrix} \begin{bmatrix} 0 & \eta \\ \theta & -\theta^T \end{bmatrix}.$$

(7.9)

With the spectral analysis of equation (7.5) in mind we can recast it as

$$A(\partial_t) Q = 0, \quad A(s) = \alpha s^2 + 2\theta s + \eta.$$

(7.10)
where evidently $A(s)$ is the $n \times n$-matrix polynomial.

7.2. Hamiltonian. An alternative to equations (7.8) and (7.9) way to replace the second-order vector ODE (7.5) with the first-order one with the Hamilton equations associated with the Hamiltonian $\mathcal{H}$ defined by

\begin{equation}
\mathcal{H} = \mathcal{H}(P, Q) = P^T \partial_t Q - \mathcal{L}(Q, \partial_t Q), \quad P = \frac{\partial \mathcal{L}}{\partial \partial_t Q} = \alpha \partial_t Q + \theta Q.
\end{equation}

Notice that the second equation in (7.11) implies the following relations between the velocity and momentum vectors:

\begin{equation}
\partial_t Q = \alpha^{-1} (P - \theta Q), \quad P = \alpha \partial_t Q + \theta Q.
\end{equation}

Consequently

\begin{equation}
\mathcal{H}(P, Q) = \frac{1}{2} \left[(P - \theta Q)^T \alpha^{-1} (P - \theta Q) + Q^T \eta Q\right] = \frac{1}{2} \partial_t Q^T \alpha \partial_t Q + \frac{1}{2} Q^T \eta Q.
\end{equation}

Notice also that equations (7.12) imply

\begin{equation}
\begin{bmatrix}
Q \\
P
\end{bmatrix} = \begin{bmatrix}
\mathbb{I} & 0 \\
\theta & \alpha
\end{bmatrix} \begin{bmatrix}
Q \\
\partial_t Q
\end{bmatrix}, \quad \begin{bmatrix}
Q \\
\partial_t Q
\end{bmatrix} = \begin{bmatrix}
-\alpha^{-1} & \eta \\
0 & \alpha^{-1}
\end{bmatrix} \begin{bmatrix}
P \\
Q
\end{bmatrix}.
\end{equation}

$\mathcal{H}$ can be interpreted as the system energy which is a conserved quantity, that is

\begin{equation}
\partial_t \mathcal{H}(P, Q) = 0.
\end{equation}

The function $\mathcal{H}(P, Q)$ defined by (7.13) can be recast into the following form

\begin{equation}
\mathcal{H}(P, Q) = \frac{1}{2} \begin{bmatrix}
Q \\
P
\end{bmatrix}^T M_H \begin{bmatrix}
Q \\
P
\end{bmatrix},
\end{equation}

where $M_H$ is the $2n \times 2n$ matrix having the block form

\begin{equation}
M_H = \begin{bmatrix}
\theta^T \alpha^{-1} \theta + \eta & -\theta^T \alpha^{-1} \\
-\alpha^{-1} \theta & \alpha^{-1}
\end{bmatrix} = \begin{bmatrix}
\mathbb{I} & -\theta^T \\
0 & \mathbb{I}
\end{bmatrix} \begin{bmatrix}
\eta & 0 \\
0 & \alpha^{-1}
\end{bmatrix} \begin{bmatrix}
\mathbb{I} & 0 \\
-\theta & \mathbb{I}
\end{bmatrix},
\end{equation}

where $\mathbb{I}$ is the identity $n \times n$-matrix. The Hamiltonian form of the Euler-Lagrange equation (7.4) reads

\begin{equation}
\partial_t u = J M_H u, \quad u = \begin{bmatrix}
Q \\
P
\end{bmatrix}, \quad J = \begin{bmatrix}
0 & \mathbb{I} \\
-\mathbb{I} & 0
\end{bmatrix}.
\end{equation}

Matrix $J$ defined in equations (7.18) is called unit imaginary matrix and it satisfies \textbf{[BernM]} 3.1

\begin{equation}
J = -J^T = -J^{-1}.
\end{equation}

Notice that in view of equations (7.17), (7.18) we have

\begin{equation}
J M_H == \begin{bmatrix}
-\alpha^{-1} \theta & \alpha^{-1} \\
-\theta^T \alpha^{-1} \theta - \eta & \theta^T \alpha^{-1}
\end{bmatrix}.
\end{equation}

Then the corresponding to Hamilton vector equation (7.18) matrix similar to the companion polynomial matrix $C_A(s) = sB - A$ in (12.5) is

\begin{equation}
C(J M_H; s) = s \begin{bmatrix}
\mathbb{I} & 0 \\
0 & \mathbb{I}
\end{bmatrix} - J M_H == \begin{bmatrix}
s + \alpha^{-1} \theta & -\alpha^{-1} \\
\theta^T \alpha^{-1} \theta + \eta & s - \theta^T \alpha^{-1}
\end{bmatrix}.
\end{equation}

Let us introduce matrix

\begin{equation}
\mathcal{H} = J M_H.
\end{equation}
Notice that in view of equations (7.17), (7.19) we have $M_H^T = M_H$ and

$$[\mathcal{H}]^T = -M_H J = -J [-J M_H] J = -J^{-1} [\mathcal{H}] J,$$

implying that the transposed to $\mathcal{H}$ matrix $[\mathcal{H}]^T$ is similar to $-\mathcal{H}$.

7.3. **Relationship between the Lagrangian and Hamiltonian.** Notice that under assumption that $\alpha^{-1}$ exists according to equations (7.1) and (7.17) we have

$$M_L = \begin{bmatrix} -\eta & \theta^T \\ \theta & \alpha \end{bmatrix}, \quad M_H = \begin{bmatrix} \eta_H & \theta_H^T \\ \theta_H & \alpha_H \end{bmatrix} = \begin{bmatrix} \theta^T \alpha^{-1} \theta + \eta & -\theta^T \alpha^{-1} \\ -\alpha^{-1} \theta & \alpha^{-1} \end{bmatrix},$$

implying

$$\alpha_H = \alpha^{-1}, \quad \theta_H = -\alpha^{-1} \theta, \quad \eta_H = \theta^T \alpha^{-1} \theta + \eta,$$

and

$$\alpha = \alpha_H^{-1}, \quad \theta = -\alpha_H^{-1} \theta, \quad \eta = \eta_H - \theta_H^T \alpha_H^{-1} \theta_H.$$

7.4. **Lagrangian and Hamiltonian for higher order ODEs.** If the Lagrangian $L$ depends on higher order derivatives as in

$$L = \frac{1}{2} \left[ x^2 + \sum_{m=0}^{n-1} a_m x_m^2 \right], \quad x_m = \partial^m_t x,$$

then the corresponding equations for its extremals are [ArnGiv 1.2.3, 3.1.4]

$$\partial_t^{2n} x + \sum_{m=0}^{n-1} (-1)^{n-m} a_m \partial_t^{2m} x = 0.$$

7.5. **Positive energy case.** The main point of this section is that in the case when the energy is non-negative, that is $\mathcal{H}(P, Q) \geq 0$, then the system spectral properties are ultimately determined by a self-adjoint, and hence diagonalizable, operator $\Omega$ defined by equations (7.33). The argument is as follows, [FigWel14]. Suppose that

$$\alpha = \alpha^T \geq 0, \quad \eta = \eta^T \geq 0.$$

Then representations (7.16), (7.17) combined with the inequalities (7.2) and (7.27) imply

$$\mathcal{H}(P, Q) \geq 0 \quad \text{and} \quad M_H = M_H^T \geq 0.$$

Notice that matrix $M_H$ can be recast as

$$M_H = K^T K,$$

where the matrix $K$ is the block matrix

$$K = \begin{bmatrix} K_q & 0 \\ 0 & K_p \end{bmatrix} \begin{bmatrix} I & 0 \\ -\theta & I \end{bmatrix} = \begin{bmatrix} K_q & 0 \\ -K_p \theta & K_p \end{bmatrix}, \quad K_q = \sqrt{\eta}, \quad K_p = \sqrt{\alpha^{-1}},$$

which manifestly takes into account the gyroscopic term $\theta$. Here $\sqrt{\alpha}$ and $\sqrt{\eta}$ denote the unique positive semidefinite square roots of the matrices $\alpha$ and $\eta$, respectively. In particular, it follows from the properties (7.2) and the proof of [ReSi1, S VI.4, Theorem VI.9] that $K_p, K_q$ are $n \times n$ matrices with real-valued entries with the properties

$$K_p = K_p^T > 0, \quad K_q = K_q^T \geq 0.$$

If we introduce now the force variable

$$v = K u,$$
then the evolution equation (7.18) can be recast into the following form

\[
\frac{\partial}{\partial t} v = -i\Omega v, \quad \Omega = \Omega^* = iKJK^T = \begin{bmatrix} 0 & i\Phi \\ -i\Phi^T & \Omega_p \end{bmatrix}, \quad \Omega_p = -i2K_p\theta K_p^T, \quad \Phi = KqK^T.
\]

where \(\Omega\) is evidently a self-adjoint operator.

7.6. **Symplectic and Hamiltonian matrices basics.** Hamiltonian matrices arise naturally as the matrices governing the evolution of Hamiltonian systems, see Section 7.2.

Let \(J \in \mathbb{R}^{2n \times 2n}\) be unit imaginary matrix defined by equations (7.18). It satisfies the identities (7.19).

**Definition 2** (Symplectic matrix). A matrix \(T \in \mathbb{R}^{2n \times 2n}\) is called *symplectic* if it satisfies the following identity [Mey, 3.1]:

\[
T^TJ = J, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.
\]

It readily follows from equations (7.34) and (7.19) that symplectic matrix \(T\) is nonsingular and

\[
T^{-1} = -JT^TJ.
\]

It is also evident that \(T\) is symplectic if and only if matrices \(T^{-1}\) and \(T^T\) are symplectic.

Evidently symplectic matrices in \(\mathbb{R}^{2n \times 2n}\) form a group.

**Definition 3** (Hamiltonian matrix). A matrix \(M \in \mathbb{R}^{2n \times 2n}\) is called *Hamiltonian* (or infinitesimally symplectic) if it satisfies the following identity [Mey, 3.1]:

\[
J^{-1}M^TJ = -M, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},
\]

The Hamiltonian matrix property (7.35) is evidently equivalent to the symmetry of the matrix \(JM\), that is,

\[
(JM)^T = JM.
\]

In other words, a Hamiltonian matrix \(A\) is a matrix of the form

\[
A = JA, \quad A^T = A.
\]

Since the definition of Hamiltonian matrix involves a transposed matrix the following general statement it is of importance to know that a matrix over the field of complex numbers is always similar to its transposed [HorJohn, 3.2.3].

**Proposition 4** (Similarity of a matrix and its transposed). Let \(A \in \mathbb{C}^{n \times n}\). There exists a nonsingular complex symmetric matrix \(S\) such that \(A^T = SAS^{-1}\).

The following statement provides different equivalent descriptions of a Hamiltonian matrix [Mey 3.1]:

**Proposition 5** (Hamiltonian matrix). The following are equivalent: (i) \(M\) is Hamiltonian, (ii) \(M = JA\) where \(A\) is symmetric, and (iii) \(JA\) is symmetric. Moreover, if \(M\) and \(K\) are Hamiltonian, then so are \(M^T\), \(\alpha M, \alpha \in \mathbb{R}\), \(M \pm K\), and \([M, K] \equiv MK - KM\).

The following representation holds for a Hamiltonian matrix \(A\) [BernM, 3.1]:

\[
A = JA, \quad A^T = A.
\]
Proposition 6 (Hamiltonian matrix). A matrix \( \mathcal{A} \in \mathbb{C}^{2n \times 2n} \) is a Hamiltonian matrix if and only if there exist matrices \( A, B, C \in \mathbb{F}^{n \times n} \) such that \( B \) and \( C \) are symmetric and

\[
\mathcal{A} = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}, \quad B = B^T, \quad C = C^T.
\]

The set of all Hamiltonian matrices forms a Lie algebra.

In fact, a matrix over the field of complex numbers is always similar to its transposed [HorJohn 3.2.3].

Proposition 7 (Similarity of a matrix and its transposed). Let \( A \in \mathbb{C}^{n \times n} \). There exists a nonsingular complex symmetric matrix \( S \) such that \( A^T = SAS^{-1} \).

The proof of Proposition 7 can be obtained from the matrix similarity to its Jordan canonical form.

Important spectral properties of Hamiltonian matrices and their canonical forms are studied in [ArnGiv, 2.2], [LauMey], [Mey, 3.3, 4.6, 4.7]. As to the more detailed spectral properties of Hamiltonian matrices the following statements holds.

Proposition 8 (Jordan structure of a real Hamiltonian matrix). The characteristic polynomial of a real Hamiltonian matrix is an even polynomial. Thus if \( \zeta \) is an eigenvalue of a Hamiltonian matrix, then \( -\zeta, \bar{\zeta} \) and \( -\bar{\zeta} \) are also its eigenvalues with the same multiplicity. The entire Jordan block structure is same for \( \zeta, -\zeta, \bar{\zeta} \) and \( -\bar{\zeta} \).

8. A Sketch of the Basics of Electric Networks

For the sake of self-consistency, we provide in this section basic information on the basics of the electric network theory and relevant notations.

Electrical networks is a well established subject represented in many monographs. We present here basic elements of the electrical network theory following mostly to [BalBic, 2], [Cau], [SesRee]. The electrical network theory constructions are based on the graph theory concepts of branches (edges), nodes (vertices) and their incidences. This approach is efficient in loop (fundamental circuit) analysis and the determination of independent variables for the Kirchhoff current and voltage laws - the subjects relevant to our studies here.

We are particularly interested in conservative electrical network which is a particular case of an electrical network composed of electric elements of three types: capacitors, inductors and gyrators. We remind that a capacitor or an inductor are the so-called two-terminal electric elements whereas a gyrator is four-terminal electric element as discussed below. We assume that capacitors and inductors can have positive or negative respective capacitances and inductances.

8.1. Circuit elements and their voltage-current relationships. The elementary electric network (circuit) elements of interest here are a capacitor, an inductor, a resistor and a gyrator, [BalBic 1.5, 2.6], [Cau, App.5.4], [Iza, 10]. These elements are characterized by the relevant voltage-current relationships. These relationships for the capacitor, inductor and resistor are respectively as follows [BalBic 1.5], [Rich, 3-Circuit theory], [SesBab, 1.3]:

\[
I = C \partial_t V, \quad V = L \partial_t I, \quad V = RI,
\]

where \( I \) and \( V \) are respectively the current and the voltage, and real \( C, L \) and \( R \) are called respectively the capacitance, the inductance and the resistance. The voltage-current relationship for the gyrator depicted in Fig. 8.2 are

\[
(a) : \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -GI_2 \\ GI_1 \end{bmatrix}, \quad (b) : \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} GI_2 \\ -GI_1 \end{bmatrix},
\]
where $I_1$, $I_2$ and $V_1$, $V_2$ are respectively the currents and the voltages, and quantity $G$ is called the gyration resistance.

The common graphic representations of the network elements are depicted in Figures 8.1 and 8.2. The arrow next to the symbol $G$ in Fig. 8.2 shows the direction of gyration.

The gyrator has the so-called inverting property as shown in Fig. 8.3 [BalBic 1.5], [Iza 10], [Dorf 29.1]. Namely, when a capacitor or an inductor connected to the output port of the gyrator it behaves as an inductor or capacitor respectively with the following effective values

\begin{equation}
L_{ef} = G^2 C, \quad C_{ef} = \frac{L}{G^2}.
\end{equation}

Notice that the voltage-current relationships in the second equation in (8.2) can be obtained from the first equation in (8.2) by substituting $-G$ for $G$. The gyrator is a device that accounts for physical situations in which the reciprocity condition does not hold. The voltage-current relationships in equations (8.2) show that the gyrator is a non-reciprocal circuit element. In fact, it is antireciprocal. Notice, that the gyrator, like the ideal transformer, is characterized by a single parameter $G$, which is the gyration resistance. The arrows next to the symbol $G$ in Fig. 8.2(a) and (b) show the direction of gyration.

**Figure 8.1.** Capacitance, inductance and resistance.

**Figure 8.2.** Gyrator.

**Figure 8.3.** Gyrator.
Along with the voltage $V$ and the current $I$ variables we introduce the charge variable $Q$ and the momentum (per unit of charge) variable $P$ by the following formulas

\begin{align}
Q(t) &= \int I(t) \, dt, \quad I(t) = \partial_t Q, \\
P(t) &= \int V(t) \, dt, \quad V(t) = \partial_t P.
\end{align}

We introduce also the energy stored variable $W$, [Rich, Circuit Theory]. Then the voltage-current relations (8.1) and the stored energy $W$ can be represented as follows:

\begin{align}
\text{capacitor:} & \quad V = \frac{Q}{C}, \quad I = \partial_t Q = CV = C\partial_t P; \\
W &= \frac{1}{2}VQ = \frac{Q^2}{2C} = \frac{CV^2}{2} = \frac{C(\partial_t P)^2}{2};
\end{align}

\begin{align}
\text{inductor:} & \quad V = L\partial_t I, \quad P = LI = L\partial_t Q, \quad \partial_t Q = \frac{P}{L}; \\
W &= \frac{PI}{2} = \frac{LI^2}{2} = \frac{L(\partial_t Q)^2}{2} = \frac{P^2}{2L};
\end{align}

\begin{align}
\text{resistor:} & \quad V = RI, \quad P = RQ.
\end{align}

The Lagrangian associated with the network elements are as follows [GantM, 9], [Rich, 3]:

\begin{align}
\text{capacitor:} & \quad L = \frac{Q^2}{2C}, \quad \text{inductor:} \quad L = \frac{L(\partial_t Q)^2}{2}, \\
\text{gyrator:} & \quad L = GQ_1\partial_t Q_2, \quad L = \frac{G(Q_1\partial_t Q_2 - Q_2\partial_t Q_1)}{2}.
\end{align}

Notice that the difference between two alternatives for the Lagrangian in equations (8.12) is $\frac{1}{2}G\partial_t(Q_1Q_2)$ which is evidently the complete time derivative. Consequently, the EL equation are the same for both Lagrangians, see Section 7.1.

8.2. Circuits of negative impedance, capacitance and inductance. There are a number of physical devices that can provided for negative capacitances and inductances needed for our circuits [Dorf, 29]. Following to [Iza, 10] we show below circuits in Fig. 8.4 that utilize operational amplifiers to achieve negative impedance, capacitance and inductance respectively. The currents and voltages for circuits depicted in Fig. 8.4 are respectively as follows: (i) for negative impedance as in Fig. 8.4(a)

\begin{align}
V_{\text{in}} &= -ZI, \quad V_o = 2V_{\text{in}}, \quad I_1 = I_2 = \frac{V_{\text{in}}}{R};
\end{align}

(ii) for negative capacitance as in Fig. 8.4(b)

\begin{align}
V_{\text{in}} &= Z_{\text{in}}I, \quad Z_{\text{in}} = -\frac{i}{\omega C}, \quad V_o = 2V_{\text{in}}, \quad I_1 = I_2 = \frac{V_{\text{in}}}{R};
\end{align}

(iii) for negative inductance as in Fig. 8.4(c)

\begin{align}
V_{\text{in}} &= Z_{\text{in}}I, \quad Z_{\text{in}} = -\frac{i}{\omega C}, \quad V_o = 2V_{\text{in}}, \quad I_1 = I_2 = \frac{V_{\text{in}}}{R};
\end{align}
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\[ V_\text{in} = Z_{\text{in}} I, \quad Z_{\text{in}} = -i \omega R^2 C, \quad I_1 = I_2 = \frac{V_\text{in}}{R}, \quad V_\text{o} = V_\text{in} \left(1 + \frac{1}{i \omega R C}\right). \]

8.3. Topological aspects of the electric networks. We follow here mostly to \cite{BalBic2}. The purpose of this section is to concisely describe and illustrate relevant concepts with understanding that the precise description of all aspects of the concepts is available in \cite{BalBic2}.

To describe topological (geometric) features of the electric network we use the concept of linear graph defined as a collection of points, called nodes, and line segments called branches, the nodes being joined together by the branches as indicated in Fig. 8.2 (b). Branches whose ends fall on a node are said to be incident at the node. For instance, Fig. 8.2 (b) branches 1, 2, 3, 4 are incident at node 2. Each branch in Fig. 8.2 (b) carries an arrow indicating its orientation. A graph with oriented branches is called an oriented graph. The elements of a network associated with its graph have both a voltage and a current variable, each with its own reference. In order to relate the orientation of the branches of the graph to these references the convention is made that the voltage and current of an element have the standard reference - voltage-reference “plus” at the tail of the current-reference arrow. The branch orientation of a graph is assumed to coincide with the associated current reference as shown in Figures 8.1 and 8.2.

We denote the number of branches of the network by \( N_b \geq 2 \), and the number of nodes by \( N_n \geq 2 \).

A subgraph is a subset of the branches and nodes of a graph. The subgraph is said to be proper if it consists of strictly less than all the branches and nodes of the graph. A path is a particular subgraph consisting of an ordered sequence of branches having the following properties:

1. At all but two of its nodes, called internal nodes, there are incident exactly two branches of the subgraph.
2. At each of the remaining two nodes, called the terminal nodes, there is incident exactly one branch of the subgraph.
3. No proper subgraph of this subgraph, having the same two terminal nodes, has properties 1 and 2.

A graph is called connected if there exists at least one path between any two nodes. We consider here only connected graphs such as shown in Fig. 8.5 (b).

A loop (cycle) is a particular connected subgraph of a graph such that at each of its nodes there are exactly two incident branches of the subgraph. Consequently, if the two terminal nodes of a
path coincide we get a “closed path”, that is a loop. In Fig. 8.5 (b) branches 7, 1, 3, 5 together with nodes 1, 2, 3, and 4 form a loop. We can specify a loop by an either the ordered list of the relevant branched or the ordered list of the relevant nodes.

We remind that each branch of the network graph is associated with two functions of time $t$: its current $I(t)$ and its voltage $V(t)$. The set of these functions satisfy two Kirchhoff’s laws, [BalBic 2.2], [Cau 2], [Rich Circuit Theory], [SesRee 1]. The Kirchhoff current law (KCL) states that in any electric network the sum of all currents leaving any node equals zero at any instant of time. The Kirchhoff voltage law (KVL) states that in any electric network, the sum of voltages of all branches forming any loop equals zero at any instant of time. It turns out that the number of independent KCL equations is $N_n - 1$ and the number KVL equations is $N_{fl} = N_b - N_n + 1$ (the first Betti number [Cau 2], [SesRee 2.3]).

![Figure 8.5. The network (a) and its graph (b). There are 4 nodes marked by small disks (black). In (b) there are 3 twigs identified by bolder (black) lines and labeled by numbers 1, 3, 5. There are 4 links identified by dashed (red) lines and labeled by numbers 2, 4, 6, 7. There also 4 oriented f-loops formed by the branches as shown.

There is an important concept of a tree in the network graph theory [BalBic 2.2], [Cau 2.1] and [SesRee 2.3]. A tree, known also as complete tree, is defined as a connected subgraph of a connected graph containing all the nodes of the graph but containing no loops as illustrated in Fig. 8.5 (b). The branches of the tree are called twigs and those branches that are not on a tree are called links [BalBic 2.2]. The links constitute the complement of the tree, or the cotree. The decomposition of the graph into a tree and cotree is not a unique.

The system of fundamental loops or system of f-loops for short, [BalBic 2.2], [Cau 2.1] and [SesRee 2.3], is of particular importance to our studies. The system of time-dependent charges (defined as the time integrals of the currents) associated with the system of f-loops provides a complete set of independent variables. When the network tree is selected then every link defines the containing it f-loop. The orientation of an f-loop is defined by the orientation of the link it contains. Consequently, there are as many of f-loops in as there are links, and

(8.16) \[ \text{number of f-loops} : N_{fl} = N_b - N_n + 1. \]

The number $N_{fl}$ of f-loops defined by equation (8.16) quantifies the connectivity of the network graph, and it is known in the algebraic topology as the first Betti number [Cau 2], [SesRee 2.3], [Witt].

The discussed concepts of the graph of an electric network such as the tree, twigs, links and f-loops are illustrated in Fig. 8.5. In particular, there are 4 nodes marked by small disks (black).
In Fig. 8.5 (b) there are 3 twigs identified by bolder (black) lines and labeled by numbers 1, 3, 5. There are 4 links identified by dashed (red) lines and labeled by numbers 2, 4, 6, 7. There also 4 oriented f-loops formed by the branches as follows: (1) 7, 1, 3, 5; (2) 2, 1; (3) 4, 3; (4) 6, 5. These representations of the f-loops as ordered lists of branches identify the corresponding links as number in the first position in every list.

One also distinguishes simpler planar networks with graphs that can be drawn so that lines representing branches do not intersect. The graph of a general electric network does not have to be planar though. Networks with non-planar graphs can still be represented graphically with more complex display arrangements or algebraically by the incidence matrices, [BalBic 2.2].

9. Conclusions

We developed here complete mathematical theory allowing to synthesize circuits with evolution matrices exhibiting prescribed Jordan canonical forms subject to natural constraints. In particular, we synthesized simple lossless circuits associated with pairs of Jordan blocks of size 2, 3 and 4, analyzed all their significant properties and derived closed form algebraic expressions for all significant matrices. Importantly, the elements of the constructed circuits involve negative capacitances and/or inductances. Naturally, those negative values are needed for chosen fixed frequencies only and that is beneficiary for efficiently achieving them based on operational amplifier converters.

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10. Appendix A: Jordan canonical form

We provide here very concise review of Jordan canonical forms following mostly to [Hale, III.4], [HorJohn, 3.1,3.2]. As to a demonstration of how Jordan block arises in the case of a single n-th order differential equation we refer to [ArnODE, 25.4].

Let $A$ be an $n \times n$ matrix and $\lambda$ be its eigenvalue, and let $r(\lambda)$ be the least integer $k$ such that $N[(A - \lambda I)^k] = N[(A - \lambda I)^{k+1}]$, where $N[C]$ is a null space of a matrix $C$. Then we refer to $M_{\lambda} = N[(A - \lambda I)^{r(\lambda)}]$ is the generalized eigenspace of matrix $A$ corresponding to eigenvalue $\lambda$. Then the following statements hold, [Hale, III.4].

**Proposition 9** (generalized eigenspaces). Let $A$ be an $n \times n$ matrix and $\lambda_1, \ldots, \lambda_p$ be its distinct eigenvalues. Then generalized eigenspaces $M_{\lambda_1}, \ldots, M_{\lambda_p}$ are linearly independent, invariant under the matrix $A$ and

$$
\mathbb{C}^n = M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_p}.
$$

Consequently, any vector $x_0$ in $\mathbb{C}^n$ can be represented uniquely as

$$
x_0 = \sum_{j=1}^{p} x_{0,j}, \quad x_{0,j} \in M_{\lambda_j},
$$

and

$$
\exp \{At\} x_0 = \sum_{j=1}^{p} e^{\lambda_j t} p_j(t),
$$
where column-vector polynomials $p_j(t)$ satisfy

$$p_j(t) = \sum_{k=0}^{r(\lambda_j) - 1} (A - \lambda_j I)^k \frac{t^k}{k!} x_{0,j}, \quad x_{0,j} \in M_{\lambda_j}, \quad 1 \leq j \leq p. \quad (10.4)$$

For a complex number $\lambda$ a Jordan block $J_r(\lambda)$ of size $r \geq 1$ is a $r \times r$ upper triangular matrix of the form

$$J_r(\lambda) = \lambda I_r + K_r = \begin{bmatrix}
\lambda & 1 & \cdots & 0 & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda & 1 \\
0 & 0 & \cdots & 0 & \lambda
\end{bmatrix}, \quad J_1(\lambda) = [\lambda], \quad J_2(\lambda) = \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}. \quad (10.5)$$

The special Jordan block $K_r = J_r(0)$ defined by equation $(10.6)$ is a nilpotent matrix that satisfies the following identities

$$K_r = J_r(0) = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}. \quad (10.6)$$

A general Jordan $n \times n$ matrix $J$ is defined as a direct sum of Jordan blocks, that is

$$J = \begin{bmatrix}
J_{n_1}(\lambda_1) & 0 & \cdots & 0 & 0 \\
0 & J_{n_2}(\lambda_2) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & J_{n_{q-1}}(\lambda_{n_{q-1}}) & 0 \\
0 & 0 & \cdots & 0 & J_{n_q}(\lambda_{n_q})
\end{bmatrix}, \quad n_1 + n_2 + \cdots + n_q = n, \quad (10.7)$$

where $\lambda_j$ need not be distinct. Any square matrix $A$ is similar to a Jordan matrix as in equation $(10.8)$ which is called Jordan canonical form of $A$. Namely, the following statement holds, [HorJohn 3.1].

**Proposition 10** (Jordan canonical form). Let $A$ be an $n \times n$ matrix. Then there exists a nonsingular $n \times n$ matrix $Q$ such that the following block-diagonal representation holds

$$Q^{-1}AQ = J \quad (10.9)$$

where $J$ is the Jordan matrix defined by equation $(10.8)$ and $\lambda_j, \ 1 \leq j \leq q$ are not necessarily different eigenvalues of matrix $A$. Representation $(10.9)$ is known as the Jordan canonical form of
matrix $A$, and matrices $J_j$ are called Jordan blocks. The columns of the $n \times n$ matrix $Q$ constitute the Jordan basis providing for the Jordan canonical form (10.9) of matrix $A$.

A function $f(J_r(s))$ of a Jordan block $J_r(s)$ is represented by the following equation [MeyCD 7.9], [BernM 10.5]

$$f(J_r(s)) = \begin{bmatrix} f(s) & \partial f(s) & \frac{\partial^2 f(s)}{2} & \cdots & \frac{\partial^{r-1} f(s)}{(r-1)!} \\ 0 & f(s) & \partial f(s) & \cdots & \frac{\partial^{r-2} f(s)}{(r-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & f(s) & \partial f(s) \\ 0 & 0 & \cdots & 0 & f(s) \end{bmatrix}.$$ \hspace{1cm} (10.10)

Notice that any function $f(J_r(s))$ of the Jordan block $J_r(s)$ is evidently an upper triangular Toeplitz matrix.

There are two particular cases of formula (10.10) which can be also derived straightforwardly using equations (10.7):

$$\exp\{K_r t\} = \sum_{k=0}^{r-1} \frac{t^k}{k!} K_r^k = \begin{bmatrix} 1 & \frac{t^2}{2!} & \cdots & \frac{t^{r-1}}{(r-1)!} \\ 0 & 1 & \cdots & \frac{t^{r-2}}{(r-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & t \end{bmatrix},$$ \hspace{1cm} (10.11)

$$[J_r(s)]^{-1} = \sum_{k=0}^{r-1} s^{-k-1} (-K_r)^k = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \cdots & \frac{(-1)^{r-1}}{s^{r-1}} \\ 0 & \frac{1}{s} & \cdots & \frac{(-1)^{r-2}}{s^{r-2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s} & \frac{-1}{s^2} \end{bmatrix},$$ \hspace{1cm} (10.12)

11. Appendix C: Companion matrix and cyclicity condition

The companion matrix $C(a)$ for monic polynomial

$$a(s) = s^\nu + \sum_{1 \leq k \leq \nu} a_{\nu-k} s^{\nu-k}$$ \hspace{1cm} (11.1)

where coefficients $a_k$ are complex numbers is defined by [BernM 5.2]

$$C(a) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_0 & -a_1 & \cdots & -a_{\nu-2} & -a_{\nu-1} \end{bmatrix}. \hspace{1cm} (11.2)

Notice that

$$\det \{C(a)\} = (-1)^\nu a_0.$$ \hspace{1cm} (11.3)
An eigenvalue is called cyclic (nonderogatory) if its geometric multiplicity is 1. A square matrix is called cyclic (nonderogatory) if all its eigenvalues are cyclic [BernM, 5.5].

**Definition 11** (cyclic eigenvalue and matrix). An eigenvalue is called cyclic (nonderogatory) if its geometric multiplicity is 1. A square matrix is called cyclic (nonderogatory) if all its eigenvalues are cyclic.

The following statement provides different equivalent descriptions of a cyclic matrix [BernM, 5.5].

**Proposition 12** (criteria for a matrix to be cyclic). Let $A \in \mathbb{C}^{n \times n}$ be an $n \times n$ matrix with complex-valued entries. Let $\text{spec}(A) = \{\zeta_1, \zeta_2, \ldots, \zeta_r\}$ be the set of all distinct eigenvalues and $k_j = \text{ind}_A(\zeta_j)$ is the largest size of Jordan block associated with $\zeta_j$. Then the minimal polynomial $\mu_A(s)$ of the matrix $A$, that is a monic polynomial of the smallest degree such that $\mu_A(A) = 0$, satisfies

$$(11.4) \quad \mu_A(s) = \prod_{j=1}^{r} (s - \zeta_j)^{k_j}.$$ 

Furthermore, the following statements are equivalent:

1. $\mu_A(s) = \chi_A(s) = \det\{sl - A\}$.
2. $A$ is cyclic.
3. For every $\zeta_j$ the Jordan form of $A$ contains exactly one block associated with $\zeta_j$.
4. $A$ is similar to the companion matrix $C(\chi_A)$.

**Proposition 13** (companion matrix factorization). Let $a(s)$ be a monic polynomial having degree $\nu$ and $C(a)$ is its $\nu \times \nu$ companion matrix. Then, there exist unimodular $\nu \times \nu$ matrices $S_1(s)$ and $S_2(s)$, such that

$$(11.5) \quad sI_\nu - C(a) = S_1(s) \begin{bmatrix} I_{\nu-1} & 0_{(\nu-1)\times 1} \\ 0_{1\times(\nu-1)} & a(s) \end{bmatrix} S_2(s).$$

Consequently, $C(a)$ is cyclic and

$$(11.6) \quad \chi_{C(a)}(s) = \mu_{C(a)}(s) = a(s).$$

The following statement summarizes important information on the Jordan form of the companion matrix and the generalized Vandermonde matrix, [BernM, 5.16], [LanTsi 2.11], [MeyCD, 7.9].

**Proposition 14** (Jordan form of the companion matrix). Let $C(a)$ be an $n \times n$ companion matrix of the monic polynomial $a(s)$ defined by equation (11.1). Suppose that the set of distinct roots of polynomial $a(s)$ is $\{\zeta_1, \zeta_2, \ldots, \zeta_r\}$ and $\{n_1, n_2, \ldots, n_r\}$ is the corresponding set of the root multiplicities such that

$$(11.7) \quad n_1 + n_2 + \cdots + n_r = n.$$ 

Then

$$(11.8) \quad C(a) = RJR^{-1},$$

where

$$(11.9) \quad J = \text{diag}(J_{n_1}(\zeta_1), J_{n_2}(\zeta_2), \ldots, J_{n_r}(\zeta_r))$$

is the the Jordan form of companion matrix $C(a)$ and $n \times n$ matrix $R$ is the so-called generalized Vandermonde matrix defined by

$$(11.10) \quad R = [R_1 | R_2 | \cdots | R_r],$$
where $R_j$ is $n \times n_j$ matrix of the form

\begin{equation}
R_j = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\zeta_j & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_j^{n-2} & (n-2) & \zeta_j^{n-3} & \cdots & (n-2) \\
\zeta_j^{n-1} & (n-1) & \zeta_j^{n-2} & \cdots & (n-1)
\end{bmatrix}.
\end{equation}

(11.11)

As a consequence of representation (11.9) $C(a)$ is a cyclic matrix.

As to the structure of matrix $R_j$ in equation (11.11), if we denote by $Y(\zeta_j)$ its first column then it can be expressed as follows [LanTsi, 2.11]:

\begin{equation}
R_j = \left[ Y^{(0)} | Y^{(1)} | \cdots | Y^{(n_j-1)} \right], \quad Y^{(m)} = \frac{1}{m!} \partial_s^m Y(\zeta_j), \quad 0 \leq m \leq n_j - 1.
\end{equation}

(11.12)

In the case when all eigenvalues of a cyclic matrix are distinct then the generalized Vandermonde matrix turns into the standard Vandermonde matrix

\begin{equation}
V = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\zeta_1 & \zeta_2 & \cdots & \zeta_n \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_1^{n-2} & \zeta_2^{n-2} & \cdots & \zeta_n^{n-2} \\
\zeta_1^{n-1} & \zeta_2^{n-1} & \cdots & \zeta_n^{n-1}
\end{bmatrix}.
\end{equation}

(11.13)

### 12. Appendix D: Matrix Polynomials

An important incentive for considering matrix polynomials is that they are relevant to the spectral theory of the differential equations of the order higher than 1, particularly the Euler-Lagrange equations which are the second-order differential equations in time. We provide here selected elements of the theory of matrix polynomials following mostly to [GoLaRo, II.7, II.8], [Baum, 9]. General matrix polynomial eigenvalue problem reads

\begin{equation}
A(s) x = 0, \quad A(s) = \sum_{j=0}^{\nu} A_j s^j, \quad x \neq 0,
\end{equation}

(12.1)

where $s$ is complex number, $A_k$ are constant $m \times m$ matrices and $x \in \mathbb{C}^m$ is $m$-dimensional column-vector. We refer to problem (12.1) of funding complex-valued $s$ and non-zero vector $x \in \mathbb{C}^m$ as polynomial eigenvalue problem.

If a pair of a complex $s$ and non-zero vector $x$ solves problem (12.1) we refer to $s$ as an eigenvalue or as a characteristic value and to $x$ as the corresponding to $s$ eigenvector. Evidently the characteristic values of problem (12.1) can be found from polynomial characteristic equation

\begin{equation}
\det \{ A(s) \} = 0.
\end{equation}

(12.2)

We refer to matrix polynomial $A(s)$ as regular if det $\{ A(s) \}$ is not identically zero. We denote by $m_s$ the multiplicity (called also algebraic multiplicity) of eigenvalue $s_0$ as a root of polynomial det $\{ A(s) \}$. In contrast, the geometric multiplicity of eigenvalue $s_0$ is defined as dim $\{ \ker \{ A(s_0) \} \}$, where ker $\{ A \}$ defined for any square matrix $A$ stands for the subspace of solutions $x$ to equation $Ax = 0$. Evidently, the geometric multiplicity of eigenvalue does not exceed its algebraic one, see Corollary 17.

It turns out that the matrix polynomial eigenvalue problem (12.1) can be always recast as the standard “linear” eigenvalue problem, namely

\begin{equation}
(sB - A)x = 0,
\end{equation}

(12.3)
where $mν \times mν$ matrices $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{B} = \begin{bmatrix} \mathbb{I} & 0 & \cdots & 0 & 0 \\ 0 & \mathbb{I} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbb{I} & 0 \\ 0 & 0 & \cdots & 0 & A_ν \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \mathbb{I} & \cdots & 0 & 0 \\ 0 & 0 & \mathbb{I} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -A_0 & -A_1 & \cdots & -A_{ν-2} & -A_{ν-1} \end{bmatrix}$$

with $\mathbb{I}$ being $m \times m$ identity matrix. Matrix $\mathbf{A}$, particularly in monic case, is often referred to as companion matrix. In the case of monic polynomial $A(λ)$, when $A_ν = \mathbb{I}$ is $m \times m$ identity matrix, matrix $\mathbf{B} = \mathbb{I}$ is $mν \times mν$ identity matrix. The reduction of original polynomial problem (12.1) to an equivalent linear problem (12.3) is called linearization.

The linearization is not unique, and one way to accomplish is by introducing the so-called known “companion polynomial” which is $mν \times mν$ matrix

$$C_A(σ) = s\mathbf{B} - \mathbf{A} = \begin{bmatrix} s\mathbb{I} & -\mathbb{I} & \cdots & 0 & 0 \\ 0 & s\mathbb{I} & -\mathbb{I} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -A_0 & A_1 & \cdots & -A_{ν-2} & -A_{ν-1} \\ A_0 & A_1 & \cdots & A_{ν-2} & sA_ν + A_{ν-1} \end{bmatrix}$$

Notice that in the case of the EL equations the linearization can be accomplished by the relevant Hamilton equations.

To demonstrate the equivalency between the eigenvalue problems for $mν \times mν$ companion polynomial $C_A(σ)$ and the original $m \times m$ matrix polynomial $A(σ)$ we introduce two $mν \times mν$ matrix polynomials $\mathbf{E}(σ)$ and $\mathbf{F}(σ)$. Namely,

$$\mathbf{E}(σ) = \begin{bmatrix} E_1(σ) & E_2(σ) & \cdots & E_{ν-1}(σ) & 1 \\ -\mathbb{I} & 0 & 0 & \cdots & 0 \\ 0 & -\mathbb{I} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & -\mathbb{I} & 0 \end{bmatrix}, \quad \det \{ \mathbf{E}(σ) \} = 1,$$

where $m \times m$ matrix polynomials $E_j(σ)$ are defined by the following recursive formulas

$$E_ν(σ) = A_ν, \quad E_{j-1}(σ) = A_{j-1} + sE_j(σ), \quad j = ν, \ldots, 2.$$  

Matrix polynomial $\mathbf{F}(σ)$ is defined by

$$\mathbf{F}(σ) = \begin{bmatrix} \mathbb{I} & 0 & \cdots & 0 & 0 \\ -s\mathbb{I} & \mathbb{I} & 0 & \cdots & 0 \\ 0 & -s\mathbb{I} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathbb{I} & 0 \\ 0 & 0 & \cdots & -s\mathbb{I} & \mathbb{I} \end{bmatrix}, \quad \det \{ \mathbf{F}(σ) \} = 1.$$

Notice, that both matrix polynomials $\mathbf{E}(σ)$ and $\mathbf{F}(σ)$ have constant determinants readily implying that their inverses $\mathbf{E}^{-1}(σ)$ and $\mathbf{F}^{-1}(σ)$ are also matrix polynomials. Then it is straightforward to
verify that

\[ E(s) C_A(s) F^{-1}(s) = E(s) (sB - A) F^{-1}(s) = \begin{bmatrix}
A(s) & 0 & \cdots & 0 & 0 \\
0 & I & 0 & \cdots & 0 \\
0 & 0 & \ddots & \cdots & \vdots \\
\vdots & \vdots & \ddots & I & 0 \\
0 & 0 & \cdots & 0 & I
\end{bmatrix} \]

(12.9)

The identity (12.9) where matrix polynomials \( E(s) \) and \( F(s) \) have constant determinants can be viewed as the definition of equivalency between matrix polynomial \( A(s) \) and its companion polynomial \( C_A(s) \).

Let us take a look at the eigenvalue problem for eigenvalue \( s \) and eigenvector \( x \in \mathbb{C}^{m\nu} \) associated with companion polynomial \( C_A(s) \), that is

(12.10) \[ (sB - A)x = 0, \quad x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{\nu-1} \end{bmatrix} \in \mathbb{C}^{m\nu}, \quad x_j \in \mathbb{C}^m, \quad 0 \leq j \leq \nu - 1, \]

where

(12.11) \[ (sB - A)x = \begin{bmatrix}
sx_0 - x_1 \\
sx_1 - x_2 \\
\vdots \\
\sum_{j=0}^{\nu-2} A_j x_j + (sA_\nu + A_{\nu-1}) x_{\nu-1}
\end{bmatrix}. \]

With equations (12.10) and (12.11) in mind we introduce the following vector polynomial

(12.12) \[ x_s = \begin{bmatrix} x_0 \\ sx_0 \\ \vdots \\ s^{\nu-2}x_0 \\ s^{\nu-1}x_0 \end{bmatrix}, \quad x_0 \in \mathbb{C}^m. \]

Not accidentally, the components of the vector \( x_s \) in its representation (12.12) are in evident relation with the derivatives \( \partial_t^j (x_0 e^{st}) = s^j x_0 e^{st} \). That is just another sign of the intimate relations between the matrix polynomial theory and the theory of systems of ordinary differential equations, see Section 13.

**Theorem 15** (eigenvectors). Let \( A(s) \) as in equations (12.1) be regular, that \( \det \{ A(s) \} \) is not identically zero, and let \( m\nu \times m\nu \) matrices \( A \) and \( B \) be defined by equations (12.2). Then the following identities hold

(12.13) \[ (sB - A)x_s = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ A(s) x_0 \end{bmatrix}, \quad x_s = \begin{bmatrix} x_0 \\ sx_0 \\ \vdots \\ s^{\nu-2}x_0 \\ s^{\nu-1}x_0 \end{bmatrix}, \]
linear case

(12.14) \[ \det \{ A(s) \} = \det \{ sB - A \}, \quad \det \{ B \} = \det \{ A_\nu \}, \]

where \( \det \{ A(s) \} = \det \{ sB - A \} \) is a polynomial of the degree \( mn \) if \( \det \{ B \} = \det \{ A_\nu \} \neq 0. \)

There is one-to-one correspondence between solutions of equations \( A(s)x = 0 \) and \( (sB - A)x = 0. \)

Namely, a pair \( s, x \) solves eigenvalue problem \( (sB - A)x = 0 \) if and only if the following equalities hold

(12.15) \[ x = x_s = \begin{bmatrix} x_0 \\ sx_0 \\ \vdots \\ s^{\nu-2}x_0 \\ s^{\nu-1}x_0 \end{bmatrix}, \quad A(s)x_0 = 0, \quad x_0 \neq 0; \quad \det \{ A(s) \} = 0. \]

Proof. Polynomial vector identity (12.13) readily follows from equations (12.11) and (12.12). Identities (12.14) for the determinants follow straightforwardly from equations (12.12), (12.15) and (12.19). If \( \det \{ B \} = \det \{ A_\nu \} \neq 0 \) then the degree of the polynomial \( \det \{ sB - A \} \) has to be \( mn \) since \( A \) and \( B \) are \( mn \times mn \) matrices.

Suppose that equations (12.15) hold. Then combining them with proven identity (12.13) we get \( (sB - A)x_s = 0 \) proving that expressions (12.13) define an eigenvalue \( s \) and an eigenvector \( x = x_s. \)

Suppose now that \( (sB - A)x = 0 \) where \( x \neq 0. \) Combing that with equations (12.11) we obtain

(12.16) \[ x_1 = sx_0, \quad x_2 = sx_1 = s^2x_0, \ldots, \quad x_{\nu-1} = s^{\nu-1}x_0, \]

implying that

(12.17) \[ x = x_s = \begin{bmatrix} x_0 \\ sx_0 \\ \vdots \\ s^{\nu-2}x_0 \\ s^{\nu-1}x_0 \end{bmatrix}, \quad x_0 \neq 0, \]

and

(12.18) \[ \sum_{j=0}^{\nu-2} A_jx_j + (sA_\nu + A_{\nu-1})x_{\nu-1} = A(s)x_0. \]

Using equations (12.17) and identity (12.13) we obtain

(12.19) \[ 0 = (sB - A)x = (sB - A)x_s = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ A(s)x_0 \end{bmatrix}. \]

Equations (12.19) readily imply \( A(s)x_0 = 0 \) and \( \det \{ A(s) \} = 0 \) since \( x_0 \neq 0. \) That completes the proof. \( \square \)

Remark 16 (characteristic polynomial degree). Notice that according to Theorem 15 the characteristic polynomial \( \det \{ A(s) \} \) for \( m \times m \) matrix polynomial \( A(s) \) has the degree \( mn \), whereas in linear case \( sI - A_0 \) for \( m \times m \) identity matrix \( I \) and \( m \times m \) matrix \( A_0 \) the characteristic polynomial \( \det \{ sI - A_0 \} \) is of the degree \( m \). This can be explained by observing that in the non-linear case of \( m \times m \) matrix polynomial \( A(s) \) we are dealing effectively with many more \( m \times m \) matrices \( A \) than just a single matrix \( A_0. \)
Another problem of our particular interest related to the theory of matrix polynomials is eigenvalues and eigenvectors degeneracy and consequently the existence of non-trivial Jordan blocks, that is Jordan blocks of dimensions higher or equal to 2. The general theory addresses this problem by introducing so-called “Jordan chains” which are intimately related to the theory of system of differential equations expressed as \( A (\partial_t) x (t) = 0 \) and their solutions of the form \( x (t) = p (t) e^{st} \) where \( p (t) \) is a vector polynomial, see Section \([13]\) and \([\text{GoLaRo}, 1, II], [\text{Baum}, 9]\). Avoiding the details of Jordan chains developments we simply notice that an important to us point of Theorem \([15]\) is that there is one-to-one correspondence between solutions of equations \( A (s) x = 0 \) and \((sB - A) x = 0\), and it has the following immediate implication.

**Corollary 17** (equality of the dimensions of eigenspaces). Under the conditions of Theorem \([15]\) for any eigenvalue \( s_0 \), that is \( \{A (s_0)\} = 0 \), we have

\[
\text{dim} \{\ker \{s_0 B - A\}\} = \text{dim} \{\ker \{A (s_0)\}\}.
\]

In other words, the geometric multiplicities of the eigenvalue \( s_0 \) associated with matrices \( A (s_0) \) and \( s_0 B - A \) are equal. In view of identity \([12.20]\) the following inequality holds for the (algebraic) multiplicity \( m (s_0) \)

\[
m (s_0) \geq \text{dim} \{\ker \{A (s_0)\}\}.
\]

The next statement shows that if the geometric multiplicity of an eigenvalue is strictly less than its algebraic one than there exist non-trivial Jordan blocks, that is Jordan blocks of dimensions higher or equal to 2.

**Theorem 18** (non-trivial Jordan block). Assuming notations introduced in Theorem \([15]\) let us suppose that the multiplicity \( m (s_0) \) of eigenvalue \( s_0 \) satisfies

\[
m (s_0) > \text{dim} \{\ker \{A (s_0)\}\}.
\]

Then the Jordan canonical form of companion polynomial \( C_A (s) = sB - A \) has at least one nontrivial Jordan block of the dimension exceeding 2.

In particular, if

\[
\text{dim} \{\ker \{s_0 B - A\}\} = \text{dim} \{\ker \{A (s_0)\}\} = 1,
\]

and \( m (s_0) \geq 2 \) then the Jordan canonical form of companion polynomial \( C_A (s) = sB - A \) has exactly one Jordan block associated with eigenvalue \( s_0 \) and its dimension is \( m (s_0) \).

The proof of Theorem \([18]\) follows straightforwardly from the definition of the Jordan canonical form and its basic properties. Notice that if equations \([12.23]\) hold that implies that the eigenvalue 0 is cyclic (nonderogatory) for matrix \( A (s_0) \) and eigenvalue \( s_0 \) is cyclic (nonderogatory) for matrix \( B^{-1}A \) provided \( B^{-1} \) exists, see Section \([11]\).

13. **Appendix B: Vector differential equations and the Jordan canonical form**

In this section we relate the vector ordinary equations to the matrix polynomials reviewed in Section \([12]\) following to \([\text{GoLaRo}, 5.1, 5.7], [\text{Hale}, III.4], [\text{MeyCD}, 7.9]\).

Equation \( A (s) x = 0 \) with polynomial matrix \( A (s) \) defined by equations \([12.1]\) corresponds to the following \( m \)-vector \( \nu \)-th order ordinary differential

\[
A (\partial_t) x (t) = 0, \quad \text{where} \quad A (\partial_t) = \sum_{j=0}^{\nu} A_j \partial_t^j,
\]

where \( A_j \) are \( m \times m \) matrices. Then differential equation \([13.1]\) can be recast in standard fashion as \( m \nu \)-vector first order differential equation

\[
B \partial_t Y (t) = A Y (t),
\]
where \( A \) and \( B \) are \( m\nu \times m\nu \) companion matrices defined by equations (12.4) and (13.3)

\[
Y(t) = \begin{bmatrix}
    x(t)
    \frac{\partial x(t)}{\partial t}
    \vdots
    \frac{\partial^{\nu-2} x(t)}{\partial t^{\nu-2}}
    \frac{\partial^{\nu-1} x(t)}{\partial t^{\nu-1}}
\end{bmatrix}
\]

is \( m\nu \)-column-vector function.

In the case when \( A_{\nu} \) is an invertible \( m \times m \) matrix equation (13.2) can be recast further as

\[
\partial_t Y(t) = \dot{A} Y(t),
\]

where

\[
\dot{A} = \begin{bmatrix}
0 & I & \cdots & 0 & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-A_0 & -A_1 & \cdots & -A_{\nu-2} & -A_{\nu-1}
\end{bmatrix}, \quad \dot{A}_j = A_{\nu}^{-1} A_j, \quad 0 \leq \nu - 1.
\]

Notice one can interpret equation (13.4) as particular case of equation (13.2) where matrices \( A_{\nu} \) and \( B \) are identity matrices of the respective dimensions \( m \times m \) and \( m\nu \times m\nu \), and that polynomial matrix \( A(s) \) defined by equations (12.1) becomes monic matrix polynomial \( \dot{A}(s) \), that is

\[
\dot{A}(s) = Is^\nu + \sum_{j=0}^{\nu - 1} \dot{A}_j s^j, \quad \dot{A}_j = A_{\nu}^{-1} A_j, \quad 0 \leq \nu - 1.
\]

Notice that in view of equation (13.3) one recovers \( x(t) \) from \( Y(t) \) by the following formula

\[
x(t) = P_1 Y(t), \quad P_1 = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \end{bmatrix},
\]

where \( P_1 \) evidently is \( m \times m\nu \) matrix.

Observe also that, [GoLaRo2, Prop. 5.1.2], [LanTsi, 14]

\[
\begin{bmatrix} A(s) \end{bmatrix}^{-1} = P_1 \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}^{-1} R_1, \quad P_1 = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 \\
0 \\
\vdots \\
0 & I
\end{bmatrix},
\]

where \( P_1 \) and \( R_1 \) evidently respectively \( m \times m\nu \) and \( m\nu \times m \) matrices.

The general form for the solution to vector differential equation (13.4) is

\[
Y(t) = \exp \left\{ \dot{A} t \right\} Y_0, \quad Y_0 \in \mathbb{C}^{m\nu}.
\]

Then using the formulas (13.7), (13.9) and Proposition 9 we arrive the following statement.

**Proposition 19** (solution to the vector differential equation ). Let \( \dot{A} \) be \( m\nu \times m\nu \) companion matrix defined by equations (13.5), \( \zeta_1, \ldots, \zeta_p \) be its distinct eigenvalues, and \( M_{\zeta_1}, \ldots, M_{\zeta_p} \) be the
corresponding generalized eigenspaces of the corresponding dimensions \( r(\zeta_j), 1 \leq j \leq p \). Then the \( m\nu \) column-vector solution \( Y(t) \) to differential equation \((13.4)\) is of the form

\[
Y(t) = \exp \left\{ \hat{A}t \right\} Y_0 = \sum_{j=1}^{p} e^{\zeta_j t} p_j(t), \quad Y_0 = \sum_{j=1}^{p} Y_{0,j}, \quad Y_{0,j} \in M_{\zeta_j},
\]

where \( m\nu \)-column-vector polynomials \( p_j(t) \) satisfy

\[
p_j(t) = \sum_{k=0}^{r(\zeta_j)-1} \frac{t^k}{k!} \left( \hat{A} - \zeta_j I \right)^k Y_{0,j}, \quad 1 \leq j \leq p.
\]

Consequently, the general \( m \)-column-vector solution \( x(t) \) to differential equation \((13.1)\) is of the form

\[
x(t) = \sum_{j=1}^{p} e^{\zeta_j t} P_1 p_j(t), \quad P_1 = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}.
\]

Notice that \( \chi_{\hat{A}}(s) = \det \left\{ sI - \hat{A} \right\} \) is the characteristic function of the matrix \( \hat{A} \). Then using notations of Proposition 19 we obtain

\[
\chi_{\hat{A}}(s) = \prod_{j=1}^{p} (s - \zeta_j)^{r(\zeta_j)}.
\]

Notice also that for any values of complex-valued coefficients \( b_k \) we have

\[
(\partial_t - \zeta_j)^{r(\zeta_j)} \left[ e^{\zeta_j t} p_j(t) \right] = 0, \quad p_j(t) = \sum_{k=0}^{r(\zeta_j)-1} b_k t^k,
\]

implying together with representation \((13.13)\)

\[
\chi_{\hat{A}}(\partial_t) \left[ e^{\zeta_j t} p_j(t) \right] = 0, \quad p_j(t) = \sum_{k=0}^{r(\zeta_j)-1} b_k t^k.
\]

Combing now Proposition 19 with equation \((13.15)\) we obtain the following statement.

**Corollary 20** (property of a solution to the vector differential equation). Let \( x(t) \) be the general \( m \)-column-vector solution \( x(t) \) to differential equation \((13.1)\). Then \( x(t) \) satisfies

\[
\chi_{\hat{A}}(\partial_t) x(t) = 0.
\]

14. **Appendix F: Block matrices and their inverses**

The statements on block matrices below are useful for our studies [BernM, 2.8].

**Proposition 21** (factorization of a block matrix). Let \( A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m}, C \in \mathbb{C}^{p \times n}, D \in \mathbb{C}^{p \times m}, \) and assume \( A \) is nonsingular. Then

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix},
\]

and

\[
\text{Rank} \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\} = n + \text{Rank} \left\{ D - CA^{-1}B \right\}.
\]
If furthermore, \( m = p \), that is \( A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m}, C \in \mathbb{C}^{m \times n}, D \in \mathbb{C}^{m \times m} \), then

\[
\det \begin{bmatrix} A & B \\
C & D \end{bmatrix} = \det A \det \{ D - CA^{-1}B \}.
\]

15. Appendix G: Canonical forms for quadratic Hamiltonian

Canonical forms of Hamiltonian matrices and Hamiltonians is a well-studied subject \[\text{ArnMec, App. 6}\], \[\text{ArnGiv, 2.4}\], \[\text{Miln, 3.3, 4.6}\], \[\text{Mey, 3.3, 4.6, 4.7}\]. In particular, an approach to the canonical forms due to D. Galin is as follows \[\text{ArnMec, App. 6}\].

Hamiltonian associated with a pair of Jordan blocks of order \( n \) with real eigenvalues \( \pm a \) is as follows

\[
H = -a \sum_{j=1}^{n} p_j q_j + \sum_{j=1}^{n-1} p_j q_{j+1}.
\]

Hamiltonian associated with a pair of Jordan blocks of odd order \( 2n + 1 \) with purely imaginary eigenvalues \( \pm bi \), that is \( b \) is real, is one of the following two nonequivalent types:

\[
H = \pm \frac{1}{2} [H_1 - H_2] - \sum_{j=1}^{2n} p_j q_{j+1}, \quad H_1 = \sum_{j=1}^{n} \left( b^2 p_{2j} q_{2n-2j+2} + p_{2j} q_{2n-2j+2} \right),
\]

\[
H_2 = \sum_{j=1}^{n+1} \left( b^2 p_{2j-1} q_{2n-2j+3} + p_{2j-1} q_{2n-2j+3} \right).
\]

Hamiltonian associated with a pair of Jordan blocks of even order \( 2n \) with purely imaginary eigenvalues \( \pm bi \), that is \( b \) is real, is one of the following two nonequivalent types:

\[
H = \pm \frac{1}{2} [H_1 - H_2] - b^2 \sum_{j=1}^{n} p_{2j} q_{2j-1} + \sum_{j=1}^{n} p_{2j} q_{2j-1}.
\]

\[
H_1 = \sum_{j=1}^{n} \left( \frac{1}{b^2} q_{2j-1} q_{2n-2j+1} + q_{2j} q_{2n-2j+2} \right),
\]

\[
H_2 = \sum_{j=1}^{n-1} \left( b^2 p_{2j+1} q_{2n-2j+2} + p_{2j+1} q_{2n-2j+2} \right).
\]

In particular for \( n = 1 \) the above formula turns into

\[
H = \pm \frac{1}{2} \left( \frac{1}{b^2} q_1^2 + q_2^2 \right) - b^2 p_1 q_2 + p_2 q_1.
\]

Hamiltonian associated with a quadruple of Jordan blocks of order \( n \) with eigenvalues \( \pm a \pm bi \), is as follows

\[
H = -a \sum_{j=1}^{2n} p_j q_j + b \sum_{j=1}^{n} (p_{2j-1} q_{2j} - p_{2j} q_{2j-1}) + \sum_{j=1}^{2n-2} p_j q_{j+2}.
\]

Canonical forms of Hamiltonians according to \[\text{ArnGiv, 2.4}\] are as follows.

Hamiltonian associated with a pair of Jordan blocks of order \( n \) with real or pure imaginary eigenvalues \( \pm \chi \) is as follows
\[(15.6) \quad H = \pm \left[ \sum_{j=1}^{n-1} p_j q_{j+1} + \frac{p_n^2}{2} - \frac{1}{2} \sum_{j=1}^{n} \left( \begin{array}{c} n \\ j-1 \end{array} \right) \chi^{2(n-j+1)} q_j^2 \right], \text{ where} \]

\[(15.7) \quad \left( \begin{array}{c} n \\ j \end{array} \right) = C_j^n = \frac{n!}{j!(n-j)!} \text{ where is the binomial coefficient.} \]

Hamiltonian associated with a quadruple of Jordan blocks of order \( m = \frac{n}{2} \) where \( n \) is an even positive integer with eigenvalues \( \pm a \pm bi \), is as follows

\[(15.8) \quad H = \sum_{j=1}^{n-1} p_j q_{j+1} + \frac{p_n^2}{2} - \frac{1}{2} \sum_{j=1}^{n} a_{j-1} q_j^2, \quad n = 2m, \]

where

\[(15.9) \quad \sum_{j=0}^{2m} a_j \zeta^j = \left[ \zeta^2 + 2(a^2 - b^2) \zeta + (a^2 + b^2)^2 \right]^m. \]

16. Appendix H: Notations

- \( \mathbb{C}^{n \times m} \) is a set of \( n \times m \) matrices with complex-valued entries.
- \( \mathbb{R}^{n \times m} \) is a set of \( n \times m \) matrices with real-valued entries.
- \( \text{spec} (A) \) is the set of all distinct eigenvalues of a \( n \times n \) matrix \( A \).
- \( \text{ind}_A (\lambda) \) is defined for an eigenvalue \( \lambda \) of a \( n \times n \) matrix \( A \) to be the largest size of Jordan block associated with \( \lambda \).
- \( \chi_A (s) = \det \{ sI - A \} \) is the characteristic polynomial of a \( \nu \times \nu \) matrix \( A \).
- \( \mu_A (s) \) is the minimal polynomial of a \( n \times n \) matrix \( A \), that is the smallest degree polynomial such that \( \mu_A (A) = 0 \).
- \( \text{col} (A, k) \) is \( k \)-th column of matrix \( A \).
- \( \text{row} (A, k) \) is \( k \)-th row of matrix \( A \).
- \( I_\nu \) is \( \nu \times \nu \) identity matrix.
- \( J, J_\nu \) is \( 2\nu \times 2\nu \) unit imaginary matrix.
- \( M^T \) is a matrix transposed to matrix \( M \).
- \( \text{diag} (b_1, b_2, \ldots, b_n) \) is \( n \times n \) diagonal matrix with indicated entries.
- \( \text{diag} (B_1, B_2, \ldots, B_n) \) is diagonal block-matrix where \( B_j \) are square matrices.
- \( [R_1 | R_2 | \cdots | R_r] \) is an \( n \times m \) matrix formed by putting next to each other in a row \( n \times m_j \) matrices \( R_j, 1 \leq j \leq r \) where \( m = m_1 + \cdots + m_r \).
- \( L_j, C_j \) and \( G_j \) are respectively inductances, capacitances and gyrator resistances.
- EL stands for the Euler-Lagrange (equations).

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