The purpose of the present paper is to generalize the method of master equation\(^1\) to more complex circuits that include also capacitors and/or inductors. Our main goal is to understand the behavior of complex stochastic circuits on average, as each particular realization of the circuit dynamics is not representative of the circuit behavior overall (as it depends on the particular realization of switching probabilities). The difficulty stems from the fact that in any particular realization of circuit dynamics, the switching of memristive components depends on the values of reactive variables, and vice-versa. Therefore, the statistical description is also necessary for the description of reactive variables. For this purpose, we utilize a set of probability distribution functions.

This paper is organized in the following sections. In Sec.\(\text{II}\) the model of binary and multi-state stochastic memristors is presented. Section\(\text{III}\) is the main part of this work, where the evolution equations for a binary memristor-capacitor circuit are derived from the Chapman-Kolmogorov equation (Subsec.\(\text{III A}\)), and their analytical solutions are found (Subsec.\(\text{III B}\)). The application of Sec.\(\text{III}\) approach to more complex circuits is discussed in Sec.\(\text{IV}\) that concludes.

\section{Stochastic Memristor Model}

In the present paper we consider discrete stochastic memristors, whose resistance is characterized by \(G\) values \(R_i, i = 0, \ldots, G - 1\). The transitions between two states, \(i\) and \(j\), are described in terms of the voltage-dependent transition rates, \(\gamma_{i ightarrow j}(V_M) \equiv \gamma_{ij}(V_M)\), where \(V_M\) is the voltage across the memristor. Only transitions between the adjacent states are allowed (e. g., \(0 \rightarrow 1, 1 \rightarrow 2, \ldots\), at \(V_M > 0\), etc.). The transition from one boundary state to another occurs sequentially through all intermediate states.

The binary stochastic memristors\(^1\hit{1}\) are the simplest case of discrete stochastic memristors. Experiments have shown\(^6\hit{6}\) that in some electrochemical metallization cells the switching can be described by a probabilistic model with voltage-dependent switching rates

\[ \gamma_{0 \rightarrow 1}(V_M) = \begin{cases} (\tau_0 e^{-V_M/V_0})^{-1}, & V_M > 0 \\ 0, & \text{otherwise} \end{cases} , \quad (1) \]

\[ \gamma_{1 \rightarrow 0}(V_M) = \begin{cases} (\tau_1 e^{-V_M/V_1})^{-1}, & V_M < 0 \\ 0, & \text{otherwise} \end{cases} . \quad (2) \]
Here, $\tau_{0(1)}$ and $V_{0(1)}$ are constants. The interpretation of the above equations is following: The probability to switch from $R_{off} \equiv R_0$ (state 0) to $R_{on} \equiv R_1$ (state 1) within the infinitesimal time interval $\Delta t$ is $\gamma_{0\rightarrow 1}(V_M)\Delta t$. The probability to switch in the opposite direction is defined similarly. Equations similar to Eqs. (1)-(2) can be used for multi-state ($G > 2$) stochastic memristors.

### III. BINARY MEMRISTOR-CAPACITOR CIRCUIT

In this Section we consider the circuit shown in Fig. 1(a), where $M$ is a stochastic binary memristor (described by Eqs. (1) and (2)), and $C$ is a linear capacitor. It is assumed that the circuit is driven by a time-dependent voltage $V(t)$.

#### A. Dynamical equations

The circuit description is based on the probability distribution functions, $p_i(q,t)$, $i = 0, 1$. The meaning of these functions is that $p_i(q,t)\Delta q$ is the probability to find the memristor in state $R_{M} = R_i$, and capacitor charge $q$ in the interval from $q$ to $q + \Delta q$ at time $t$. For the full probabilistic description of memristive switching it is necessary to introduce the transition probabilities

$$P(R_k, q_2, t_2 | R_j, q_1, t_1) \equiv P_{kj}(q_2, t_2 | q_1, t_1)$$

from the memristor state $R_j$ and capacitor charge $q_1$ at time $t_1$ to the memristor state $R_k$ and capacitor charge $q_2$ at time $t_2$. The Chapman-Kolmogorov equations describing the Markov process of stochastic switching can be written as

$$p_k(q, t + \tau) = \sum_{j=0}^{M-1} \int_{-\infty}^{+\infty} P_{kj}(q, t + \tau | r, t) p_j(r, t) dr. \quad (3)$$

In order to satisfy the normalization condition at any time $t$

$$\sum_{k=0}^{M-1} \int_{-\infty}^{+\infty} p_k(q, t) dq = 1, \quad (4)$$

the transition probabilities must satisfy for any $r, j, t$, and $\tau > 0$ the corresponding conditions

$$\sum_{k=0}^{M-1} \int_{-\infty}^{+\infty} P_{kj}(q, t + \tau | r, t) dq = 1. \quad (5)$$

It is easy to find the transition probabilities for infinitesimally small $\tau > 0$. There are four transition probabilities as the memristive system is binary. For example, the transitional probability for switching from $R_0$ to $R_1$ is equal to

$$P_{10}(q, t + \tau | r, t) = \gamma_{0\rightarrow 1} \left( V(t) - \frac{r}{C} \right) \tau \times \delta \left( q - r - \frac{\tau}{R_0} \left( V(t) - \frac{r}{C} \right) \right), \tau \rightarrow +0, \quad (6)$$

where $V(t) - r/C$ is the voltage drop across the memristor, and the Dirac delta function represents the change in the capacitor charge that is defined by Kirchhoff’s law

$$I \approx \frac{q - r}{\tau} = \frac{1}{R_0} \left( V(t) - \frac{r}{C} \right), \tau \rightarrow +0. \quad (7)$$

Then, for the transitional probability from $R_0$ to $R_0$ (no switching) we can write (similarly to Eq. (6))

$$P_{00}(q, t + \tau | r, t) = \left[ \frac{1}{R_0} \left( V(t) - \frac{r}{C} \right) \tau \right] \times \delta \left( q - r - \frac{\tau}{R_0} \left( V(t) - \frac{r}{C} \right) \right), \tau \rightarrow +0. \quad (8)$$

The transitional probabilities $P_{12}$ and $P_{22}$ are obtained by replacing $0 \rightarrow 1$ and $1 \rightarrow 0$ in Eqs. (6) and (8). Note that such expressions for the transitional probabilities satisfy normalization conditions (5). Also we should note that Eqs. (6) and (8) are valid only up to the first order of magnitude of time $\tau$, i.e. we should omit $\tau^2$-terms or higher while using these equations.

By substituting Eqs. (6) and (8) into the Chapman-Kolmogorov equation (3) with $k = 0$, and expanding it with respect to $\tau$ up to the first order in magnitude, we get the following partial differential equation for the probability distribution function $p_0(q, t)$

$$\frac{\partial p_0(q, t)}{\partial t} + \frac{\partial}{\partial q} \left( \frac{V_M}{R_0} p_0(q, t) \right)
$$

$$= \gamma_{1\rightarrow 0} \left( V(t) - \frac{r}{C} \right) \tau \times \delta \left( q - r - \frac{\tau}{R_0} \left( V(t) - \frac{r}{C} \right) \right), \tau \rightarrow +0. \quad (9)$$

where $V_M = V(t) - q/C$ is the voltage across the memristor.

The other equation is obtained by replacing $0 \rightarrow 1$ and $1 \rightarrow 0$ in Eq. (9):

$$\frac{\partial p_1(q, t)}{\partial t} + \frac{\partial}{\partial q} \left( \frac{V_M}{R_1} p_1(q, t) \right)
$$

$$= \gamma_{0\rightarrow 1} \left( V(t) - \frac{r}{C} \right) \tau \times \delta \left( q - r - \frac{\tau}{R_0} \left( V(t) - \frac{r}{C} \right) \right), \tau \rightarrow +0. \quad (10)$$

The system of Eqs. (9) and (10) must be supplemented with the initial distributions: $p_0(q, t = 0) = f(q)$ and $p_1(q, t = 0) = g(q)$.

![FIG. 1. (a) Schematics of the binary memristor-capacitor circuit. (b) Evolution scheme for $p_0(q, t) \Delta q$. The interval $\Delta q$ is defined by two vertical lines. The arrows represent the flow of probability density.](image-url)
We note that Eqs. (9) and (10) can be considered as generalized continuity equations. Schematically, the interpretation of various terms in Eq. (9) is presented in Fig. 1(b). The second term in the left-hand side of Eq. (9) describes the flow of probability density through the boundaries of a small charge interval $\Delta q$ (the horizontal arrows in Fig. 1(b)). The right-hand side of Eq. (9) represents the flow of probability density between $p_0(q, t)$ and $p_1(q, t)$ (the vertical arrows in Fig. 1(b)).

**B. Analytical solutions**

While the fundamental solution of Eqs. (9) and (10) delivers the complete description of circuit behavior, it cannot be found analytically in a closed form for an arbitrary $V(t)$. Thus we confine ourselves to some important particular cases, when such solution can be found. They are i) the limit of small voltages when no transition occurs, and ii) unidirectional switching case, when transitions go only from one state to another.

1. **No switching case**

For those moments of time $t$, capacitor charge $q$, and applied voltage $V(t)$ when no switching practically occurs, Eqs. (9) and (10) can be simplified to the following independent equations:

$$
\frac{\partial p_0(q, t)}{\partial t} + \frac{\partial}{\partial q} \left[ \frac{V_M}{R_0} p_0(q, t) \right] = 0, \quad (11)
$$

$$
\frac{\partial p_1(q, t)}{\partial t} + \frac{\partial}{\partial q} \left[ \frac{V_M}{R_1} p_1(q, t) \right] = 0. \quad (12)
$$

The general solution of Eqs. (11) and (12) can be found by the method of characteristics and presented as

$$
p_0(q, t) = e^{\frac{V_M}{R_0} t} f \left( q e^{\frac{V_M}{R_0} t} - \int_0^t e^{\frac{V_M}{R_0} \tau} \frac{V(\tau)}{R_0} d\tau \right), \quad (13)
$$

$$
p_1(q, t) = e^{\frac{V_M}{R_1} t} g \left( q e^{\frac{V_M}{R_1} t} - \int_0^t e^{\frac{V_M}{R_1} \tau} \frac{V(\tau)}{R_1} d\tau \right), \quad (14)
$$

where $f(q)$ and $g(q)$ are two arbitrary functions (initial conditions). We reiterate that Eqs. (13) and (14) with $p_0(q, 0) = f(q)$ and $p_1(q, 0) = g(q)$ provide the full solution in the case when the transitions between the states can be neglected.

To illustrate the above solution, we consider the step-like initial probability distribution

$$
p_0(q, 0) = \begin{cases} \frac{1}{q_0 - q_0}, & \text{for } q \in (q_\alpha, q_\beta), \\ 0, & \text{otherwise}, \end{cases} \quad (15)
$$

$$
p_1(q, 0) = 0. \quad (16)
$$

Then in accordance with Eq. (13) we get the following expression for the charge probability distribution at any moment of time:

$$
p_0(q, t) = \begin{cases} e^{\frac{V_M}{R_0} t}, & \text{for } q \in (q_\alpha(t), q_\beta(t)), \\ 0, & \text{otherwise}, \end{cases} \quad (17)
$$

$$
p_1(q, t) = \begin{cases} 0, & \text{for } q \in (q_\alpha(t), q_\beta(t)), \\ e^{\frac{V_M}{R_1} t}, & \text{otherwise}, \end{cases} \quad (18)
$$

where

$$
q_i(t) = q_i e^{-\frac{V_M}{R_0} t} \int_0^t e^{\frac{V_M}{R_0} \tau} V(\tau) d\tau, \quad (19)
$$

with $i = \alpha, \beta$. Eq. (19) is similar to the time-dependence of capacitor charge in the classical RC-circuit subjected to the voltage $V(t)$.

From Eq. (18) we see that the dissipative nature of memristor-capacitor circuit leads to the exponential narrowing of the probability distributions with time. At long times, $t \gg CR_0$, the distribution approaches Dirac delta function

$$
p_0(q, t) = \delta \left( q - \int_0^t e^{\frac{V_M}{R_0} \tau} V(\tau) d\tau \right). \quad (20)
$$

It is clear Eq. (20) is valid for any initial distribution $f(q)$ in the absence of resistance switching events.

Moreover, the initially deterministic state will remain deterministic in the absence of switchings. For $q(t = 0) = q_0$ and $p_1(q, t = 0) = 0$, Eqs. (13) and (14) lead to

$$
p_0(q, t) = \delta \left( q - q_0 e^{-\frac{V_M}{R_0} t} - \int_0^t e^{\frac{V_M}{R_0} \tau} V(\tau) d\tau \right), \quad (21)
$$

$$
p_1(q, t) = 0. \quad (22)
$$

According to Eq. (21), the evolution of the capacitor charge is

$$
q(t) = q_0 e^{-\frac{V_M}{R_0} t} + \int_0^t e^{\frac{V_M}{R_0} \tau} V(\tau) d\tau. \quad (23)
$$

2. **Unidirectional switching**

Next we consider the region in $q-t$ phase plane, where the transitions from $R_1$ to $R_0$ are forbidden, but the opposite transitions may occur. In this situation Eqs. (9) and (10) can be rewritten as

$$
\frac{\partial p_0(q, t)}{\partial t} + \frac{\partial}{\partial q} \left[ \frac{V_M}{R_0} p_0(q, t) \right] = -\gamma_{0 \to 1} (V_M) p_0(q, t), \quad (24)
$$

$$
\frac{\partial p_1(q, t)}{\partial t} + \frac{\partial}{\partial q} \left[ \frac{V_M}{R_1} p_1(q, t) \right] = \gamma_{0 \to 1} (V_M) p_0(q, t). \quad (25)
$$

Eqs. (24) and (25) are valid when $\gamma_{0 \to 0} (V_M) = 0$. The general solution of Eqs. (24) and (25) can be derived by using the method of characteristics. As a result we get
\[ p_0(q, t) = e^{\frac{t}{\tau_{00}}} \delta\left(q - q(t)\right) e^{-\frac{C R_0}{V_a} \left[ E_i \left( \frac{V_a - q(t)}{V_0} \right) - E_i \left( \frac{V_a - q(t)}{V_0} \right) \right]}, \]

\( \text{where } E_i(x) \text{ is the exponential integral function and} \)

\[ q(t) = q_0 e^{-\frac{t}{\tau_{00}}} + V_a C \left( 1 - e^{-\frac{t}{\tau_{00}}} \right). \]

The integral of Eq. (28) over \( q \) from minus to plus infinity gives the probability to find the memristor in the state \( 0 \) at time \( t \):

\[ p_0(t) = e^{-\frac{C R_0}{V_a} \left[ E_i \left( \frac{V_a - q_0}{V_0} \right) - E_i \left( \frac{V_a - q_0}{V_0} \right) \right]} \]

which are valid in the region \( q < CV(t) \), where \( \gamma_{1 \to 0} (V_M) = 0 \), and \( f(q) \) and \( g(q) \) are two arbitrary functions.

By the interchanging indexes \( 0 \to 1 \) and \( 1 \to 0 \) in Eqs. (26) and (27) we can also find the solutions of Eqs. (9) and (10) for the region \( q > CV(t) \), where \( \gamma_{0 \to 1} (V_M) = 0 \). Note that if there are no transitions at all \( (\gamma_{0 \to 1} = 0) \), then the results (26) and (27) coincide with Eqs. (13) and (14).

As an example of the theory above let us consider the case of deterministic initial conditions and constant applied voltage: \( q(t) = 0 \), \( p_0(q, t) = 0 \), and \( V(t) = V_a \). Then, Eq. (26) simplifies to

\[ p_0(q, t) = e^{\frac{t}{\tau_{00}}} \delta\left(q - q(t)\right) e^{-\frac{C R_0}{V_a} \left[ E_i \left( \frac{V_a - q(t)}{V_0} \right) - E_i \left( \frac{V_a - q(t)}{V_0} \right) \right]}, \]

where \( E_i(x) \) is the exponential integral function and

\[ q(t) = q_0 e^{-\frac{t}{\tau_{00}}} + V_a C \left( 1 - e^{-\frac{t}{\tau_{00}}} \right). \]

\[ \text{(29)} \]

The mean switching time for the dynamics in Fig. 2 can be calculated using

\[ \langle T_1 \rangle = \frac{1}{p_1(t^*)} \int_0^{t^*} t \, dp_1(t) \, dt, \]

\[ \text{where } t^* \text{ is the characteristic saturation time for the fast initial dynamics of } p_1(t). \]

\[ \text{Taking } t^* = 1 \text{ s, one can find that } \langle T_1 \rangle = 5.3 \text{ ms, which is in perfect agreement with the behavior in Fig. 2.} \]

The exponential relaxation with the mean switching time \( \langle T_1 \rangle \) provides an excellent approximation for the initial interval of fast evolution of \( p_0(t) \), see Fig. 3. On long times, the relaxation can be approximated by \( p_0(t) \approx p_0(t^*) \exp(-t/\tau_0). \)

Moreover, by using Eq. (30) one can find \( p_0(t^*) \) in the most interesting parameter region \( (V_a - q_0/C) \gg V_0 \) corresponding to the initial interval of fast evolution. This is accomplished employing the asymptotic expansion of the exponential integral function \( E_i(x) = e^x/(x-1) (1 + O(1/x^2)) \), where \( x \to +\infty \). For the time interval \( \langle T_1 \rangle \ll t \ll \tau_0 \), we can omit the second term in the square brackets in Eq. (30) and use the asymptotic expansion for the first one. This leads us to the following expression for the probability of no switching event

\[ p_0(t^*) = \exp \left[ -\frac{C R_0}{\tau_0} \frac{e^{\frac{V_a - q_0/C}{V_0}}}{V_a - q_0/C - 1} \right]. \]

\[ \text{(32)} \]

For the same parameter values as in Fig. 2, Eq. (32) gives \( p_0(t^*) \approx 0.447 \), which is in an excellent agreement with the result \( p_0(t^*) = 0.446 \) obtained numerically from Eq. (30).
where the memristive switching rates corresponds to the flip of a single memristor and depend on the voltage across the same memristor in a particular circuit configuration. In order to close Eq. (33) we need to express the full derivatives \( \dot{Q}_j \) and \( \dot{I}_j \) as functions of \( R, Q, I \) by using the Kirchhoff’s circuit laws. If there are no transitions, then the RHS of Eq. (33) turns to zero and this equation coincides with the continuity equation for the probability density \( p_i(Q, I, t) \) as it should be.

In conclusion, we have introduced a powerful analytical approach to model heterogeneous stochastic circuits. A simple example was considered in detail and the recipe to apply the approach to other circuits has been formulated. Compared to the traditional Monte Carlo simulations, the proposed approach can be used to derive analytical expressions describing the circuit dynamics on average.

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**FIG. 3.** The probability to find the memristive device in state \( 0, p_0(t) \) is well approximated by an exponential decay curve based on \( \langle T_1 \rangle \) (Eq. (31)). The exponential decay curve was obtained using \( \langle T_1 \rangle = 5.3 \) ms and \( t^* = 1 \) s. The probability \( p_0(t) \) curve is the same as in Fig. 2.

\[
\frac{\partial p_i(Q, I, t)}{\partial t} + \sum_j \frac{\partial}{\partial Q_j} \left[ Q_j p_i(Q, I, t) \right] + \sum_j \frac{\partial}{\partial I_j} \left[ I_j p_i(Q, I, t) \right] = \sum_{j \neq i} \left[ \gamma_{ij} p_j(Q, I, t) - \gamma_{ij} p_i(Q, I, t) \right],
\]

IV. DISCUSSION

The method used to model the memristor-capacitor circuit above can be straightforwardly extended to other circuits. Consider, for instance, a circuit composed of \( N \) \( G \)-state stochastic memristors, \( K \) capacitors and \( J \) inductors. In the general case, the simulation of such a circuit requires \( G^N \) probability distribution functions of \( K + J \) variables and time. Circuits with symmetries may require less functions to represent their states (see Ref. [11] for examples). The number of independent reactive variables can be less than \( K + J \). For example, if the external voltage \( E(t) \) is applied directly across some capacitor, then its charge is not an independent variable.

It is anticipated that the general evolution equation can be written similarly to Eqs. (9) and (10). Introducing the sets of capacitive and inductive variables, \( Q \) and \( I \), the evolution equation for a particular state \( i \) is formulated as

\[ p_i(t_{\text{cap}}) = \frac{\partial p_i}{\partial t} + \sum_j \frac{\partial}{\partial Q_j} \left[ Q_j p_i \right] + \sum_j \frac{\partial}{\partial I_j} \left[ I_j p_i \right] = \sum_{j \neq i} \left[ \gamma_{ij} p_j - \gamma_{ij} p_i \right], \]

where \( \gamma_{ij} \) is well approximated by an exponential decay curve \( \langle T_1 \rangle \).

**REFERENCES**

1. O. Chua and S. M. Kang, Proc. IEEE 64, 209 (1976).
2. S. Kvatinsky, M. Ramadan, E. G. Friedman, and A. Kolodny, IEEE Transactions on Circuits and Systems II: Express Briefs 62, 786 (2015).
3. Y. V. Pershin, S. La Fontaine, and M. Di Ventra, Phys. Rev. E 80, 021926 (2009).
4. P. Strachan, A. C. Torrezan, F. Miao, M. D. Pickett, J. J. Yang, W. Yi, G. Medeiros-Ribeiro, and R. S. Williams, IEEE Transactions on Electron Devices 60, 2194 (2013).
5. I. Valov, R. Waser, J. R. Jameson, and M. N. Kozicki, Nanotechnology 22, 254003 (2011).
6. S. H. Jo, K. H. Kim, and W. Lu, Nano letters 9, 496 (2009).
7. S. Gaba, P. Sheridan, J. Zhou, S. Choi, and W. Lu, Nanoscale 5, 5872 (2013).
8. S. Gaba, P. Knag, Z. Zhang, and W. Lu, in 2014 IEEE International Symposium on Circuits and Systems (ISCAS) (IEEE, 2014) pp. 2592-2595.
9. S. Gaba, P. Knag, Z. Zhang, and W. Lu, in 2014 IEEE International Symposium on Circuits and Systems (ISCAS) (IEEE, 2014) pp. 2592-2595.
10. Y. V. Pershin, S. La Fontaine, and M. Di Ventra, Phys. Rev. E 80, 021926 (2009).
11. P. Strachan, A. C. Torrezan, F. Miao, M. D. Pickett, J. J. Yang, W. Yi, G. Medeiros-Ribeiro, and R. S. Williams, IEEE Transactions on Electron Devices 60, 2194 (2013).
12. I. Valov, R. Waser, J. R. Jameson, and M. N. Kozicki, Nanotechnology 22, 254003 (2011).
13. S. H. Jo, K.-H. Kim, and W. Lu, Nano letters 9, 496 (2009).
14. S. Gaba, P. Sheridan, J. Zhou, S. Choi, and W. Lu, Nanoscale 5, 5872 (2013).
15. S. Gaba, P. Knag, Z. Zhang, and W. Lu, in 2014 IEEE International Symposium on Circuits and Systems (ISCAS) (IEEE, 2014) pp. 2592-2595.
16. S. Gaba, P. Knag, Z. Zhang, and W. Lu, Nano letters 9, 496 (2009).
17. S. Gaba, P. Sheridan, J. Zhou, S. Choi, and W. Lu, Nanoscale 5, 5872 (2013).
18. S. Gaba, P. Knag, Z. Zhang, and W. Lu, in 2014 IEEE International Symposium on Circuits and Systems (ISCAS) (IEEE, 2014) pp. 2592-2595.