Quantum mechanics offers a new, promising perspective for computer science. Quantum computers are believed to hold a computational advantage over classical ones. One of the most spectacular examples of this quantum speed-up is Shor’s famous algorithm for factoring large numbers. The executing time of Shor’s algorithm scales polynomially with the size of the problem, whereas the best known classical algorithm—General Number Field Sieve scales subexponentially. Of course the notion of quantum speed-up is not absolute unless it is judged by comparing optimal quantum and classical algorithms. However, finding lower bounds for NP problems is not easy in general. Thus, in order to prove the advantage of quantum computers in a rigorous way a special model of computation, namely the oracle model of computation (OMC) was introduced. In the OMC, algorithmic complexity is identified with query complexity, *i.e.* the number of oracle calls required for solving a problem. Within this model, quantum and classical oracles are essentially different. Thus, strictly speaking, reliable comparison of quantum and classical algorithms is not possible. To overcome this problem the notion of correspondence between quantum and classical oracles is commonly used.

In this Letter, we show that this correspondence can not in general be unique. As a consequence, we propose a modified procedure for reliable comparison of quantum and classical algorithms within the OMC. Within this framework it turns out that quantum speed-up offered by some algorithms is just an artefact of the ambiguity of the previously used correspondence. Our arguments also shed some light on the role of entanglement in quantum speed-up.

Let us consider the question of “quantizing” a given classical operation. As an example to clear notions, suppose the classical operation is the (one bit) NOT gate which converts a bit \(a\) into its compliment \((1 – a)\), \((a = 0,1)\). It seems natural to choose as a quantum counterpart of this gate the \(\sigma_x\) Pauli operator

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Indeed, the transformation invoked by this matrix on choosing the quantum bit in the form \(|a\rangle\langle a|\), where \(|a\rangle = \begin{pmatrix} 1 - a \\ a \end{pmatrix}\) is exactly

\[
|a\rangle \xrightarrow{\sigma_x} \chi_{1-a}.
\]

On the other hand, the \(\sigma_z\) Pauli operation

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

also implements the NOT gate

\[
\eta_a \xrightarrow{\sigma_z} \eta_{1-a}
\]

provided that the computational basis states are chosen in a different way, namely \(|a\prime\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -(-1)^a \end{pmatrix}\). This simple example illustrates the dependence of the quantum-classical correspondence on the choice of computational basis states. Furthermore, even after choosing a given computational basis there remains a different ambiguity described below. Suppose we choose the bit \(\chi_a\), then the operation \(\sigma_x\)

\[
\sigma_x = \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}
\]

implies the bit compliment

\[
|a\rangle \xrightarrow{\sigma_x} \chi_{1-a}
\]

for any \(\theta, \phi\). Let us emphasize that the common convention of choosing \(\theta = \phi = 0\) is arbitrary and can not be justified simply by correspondence. This is because physical states are represented by rays not vectors in Hilbert space.
The argument given above is bidirectional, i.e. to a given quantum operation there may correspond different classical operations. For example, the $\sigma_z$ operation corresponds to the classical identity
\[
\chi_a \xrightarrow{\sigma_z} \chi_a
\]
as well as NOT operation
\[
\eta_a \xrightarrow{\sigma_z} \eta_{1-a}.
\]
In general, we say that a classical reversible operation $O$ transforming $m$-bits into $m$-bits according to the rule
\[
\vec{a} \xrightarrow{O} \vec{b},
\]
where $z = (z_1, z_2, \ldots, z_m), \vec{a} = (a_1, a_2, \ldots, a_m)$ corresponds to a quantum operation $U$ transforming $m$-qubits into $m$-qubits iff
\[
\rho_{\vec{a}} \xrightarrow{U} \rho_{\vec{b}}
\]
for $\rho_{\vec{a}} = \rho_{\vec{a}_1} \otimes \rho_{\vec{a}_2} \otimes \cdots \otimes \rho_{\vec{a}_m}$. This correspondence is based on the formal identification of the action of the classical and quantum operations on their respective computational states. Note that each of the single qubit quantum computational states may be chosen arbitrarily and may be different for each qubit as long as they are pure and satisfy the orthogonality condition $\rho_{\vec{a}_j} \otimes \rho_{\vec{1}_1} = 0$.

Let us now consider the so-called standard oracle which is a $n+1$-qubit unitary operation defined below
\[
|\vec{x}\rangle|y\rangle \xrightarrow{U_{S}^f} |\vec{x}\rangle|y \oplus f(\vec{x})\rangle,
\]
where $|\vec{x}\rangle = \otimes_j |x_j\rangle$ denotes an $n$-bit quantum register $(x_j, y = 0, 1)$ and $f: \{0, 1\}^n \mapsto \{0, 1\}$. On choosing the computational states of the form $\rho_{\vec{x}_j} = \chi_{x_j}, \rho_y = \chi_y$, the action of the oracle is
\[
\rho_{\vec{x}} \otimes \rho_y \xrightarrow{U_{S}^f} \rho_{\vec{x}} \otimes \rho_y \oplus f(\vec{x}),
\]
where $\rho_{\vec{x}} = \otimes_j \rho_{\vec{x}_j}$. Thus, a natural corresponding classical counterpart $O_{S}^f$ of this oracle transforms $n+1$ bits as follows:
\[
(\vec{x}), (y) \xrightarrow{O_{S}^f} (\vec{x}), (y \oplus f(\vec{x})),
\]
where $\vec{x} = (x_1, x_2, \ldots, x_j)$. As explained earlier, this classical oracle need not be a unique counterpart of $U_{S}^f$. Indeed, by choosing a different set of computational states of the form $\rho_{\vec{x}_1} = \eta_{x_1}, \rho_{\vec{x}_j} = \chi_{x_j}$, $(j \neq 1), \rho_y = \eta_y$ the quantum oracle implements the transformation
\[
\rho_{\vec{x}} \otimes \rho_y \xrightarrow{U_{S}^f} \rho_{\vec{x} \oplus \vec{c}} \otimes \rho_y,
\]
where $\vec{c} = (c, 0, \ldots, 0)$ and $c = f(0, x_2, \ldots, x_n) \oplus f(1, x_2, \ldots, x_n)$. The classical counterpart corresponding to this transformation of computational states is now
\[
O_{A}^f
\]
\[
(\vec{x}), (y) \xrightarrow{O_{A}^f} (\vec{x} \oplus \vec{c}), (y).
\]

So, there are at least two manifestly different classical oracles corresponding to the standard quantum oracle. It is interesting to see what effect this ambiguity in the quantum classical correspondence has on the quantum speed-up of oracle problems. Let us focus on the well known PARITY problem which generalizes the original Deutsch problem. In the oracle setting, this problem requires deciding whether $\sum_{\vec{x}} f(\vec{x})$ is even or odd. The optimal classical algorithm to the standard classical oracle $O_{A}^f$ requires $N = 2^n$ queries to solve this problem, whereas it suffices to query the quantum oracle $U_{S}^f$ just $N/2$ times. Hence, the quantum speed-up exhibited by $U_{S}^f$ as compared to $O_{A}^f$ is simply by a constant factor. On the other hand, the classical oracle $O_{A}^f$ requires exactly the same number of queries $N/2$ as the quantum oracle. Thus, there is no quantum speed-up at all when the oracles $U_{S}^f$ and $O_{A}^f$ are compared.

As another example, consider the slightly modified Bernstein-Vazirani problem. The oracle function $f : \{0, 1\}_n \mapsto \{0, 1\}$ is promised to be of the form $f(\vec{x}) = k_0 + \vec{k} \cdot \vec{x}$, where $\vec{k} = (k_1, k_2, \ldots, k_n)$ is the $n$-bit string to be identified. Notice that both the Deutsch problem and the original Bernstein-Vazirani (BV) problem are special cases of the stated problem when $n = 1$ and $k_0 = 0$ respectively. Comparison of the standard classical oracle $O_{S}^{f_b}$ which optimally requires $n+1$ calls with the quantum oracle $U_{S}^f$ which requires only a single call yields linear quantum speed-up. Now, let us choose a different set of computational states for the quantum oracle of the form $\rho_{\vec{x}_j} = \eta_{x_j}$ (for all $j$) and $\rho_y = \eta_y$. The action of the quantum oracle on these states is
\[
\rho_{\vec{x}} \otimes \rho_y \xrightarrow{U_{S}^f} \rho_{\vec{x} + \vec{k}} \otimes \rho_y.
\]

Hence, the classical oracle corresponding to this transformation is $O_{B}^f$
\[
(\vec{x}), (y) \xrightarrow{O_{B}^f} (\vec{x} \oplus \vec{k}), (y).
\]

Obviously a single call to the classical oracle $O_{B}^f$ suffices to solve the promise problem. Again, we conclude that there exists a classical oracle corresponding to the quantum oracle which is just as efficient.

Let us now comment on the interpretation of the above simple examples. In the usual scenario, one starts with the standard classical oracle which is then replaced by its quantum counterpart. Our results do not question the fact that the quantum oracle may provide a more
efficient solution to a formulated oracular problem than the standard classical oracle. However, it is the algorithms not that the oracles that should be compared. As mentioned in the introduction, since quantum and classical oracles are completely different, strict comparison of the algorithms that call these oracles is meaningless. If indeed this comparison is made, then the source of the advantage of the better “quantum” solution could be hidden within the quantum oracle itself, although it may seem to be manifested in the quantum algorithm. Indeed providing the quantum oracle may be equivalent to providing different (non-standard) classical oracles.

To clarify further the notion of quantum speed-up in the OMC, suppose Alice and Bob (who is constrained to use only classical operations on logical bits) compete with each other to get a quicker solution to a given oracular problem. Suppose both Alice and Bob are given the same classical device (oracle). In this case, Alice cannot use quantum mechanical operations to her advantage since one cannot construct a quantum oracle given a closed classical black-box. Now suppose both Alice and Bob are given the same quantum oracle. Quantum speed-up occurs when Alice can provide a more efficient solution than Bob (this is indeed the case, e.g. in Grover’s search algorithm). In the BV problem however, Alice will manage a quicker solution only if Bob is additionally forced to use a particular (inefficient) encoding of logical states. Suppose the quantum oracle is implemented by an optical system closed in a black box whose input and output ports consist of optical fibres. Assume the logical bits to be encoded in the polarization of light. The classical nature of Bob’s state implies that he can use only two orthogonal polarization states, e.g. vertical and horizontal. Notice that the number of steps Bob needs to solve the problem (n or 1) depends just on the orientation of the device (0° or 45° respectively).

Thus, we pose the question whether the speed-up in oracle problems is genuine quantum speed-up or just the result of the interplay between two classical oracles. Answering this question requires a refined procedure of comparing quantum and classical oracles. Here, we postulate the detection of genuine quantum speed-up by comparing the quantum oracle to its best possible corresponding classical counterpart.

Our considerations also resolve the apparent puzzle of “infinite” quantum speed-up in BV algorithms. From Eqs. (10) and (17), notice that the query bit is not transformed at all. Therefore, especially in experimental realizations, the query bit is completely excluded and the BV circuit is implemented as a controlled-$f$ phase shift oracle,

$$|\bar{x}\rangle \overset{U_f}{\longrightarrow} (-1)^{\bar{x} \cdot \bar{k}} |\bar{x}\rangle$$  \hspace{1cm} (18)

The standard classical counterpart of this oracle does not allow the extraction of any information about $\bar{k}$, since it is simply the Identity oracle:

$$|\bar{x}\rangle \overset{O^f_{\bar{k}}}{\longrightarrow} (-1)^{\bar{k} \cdot \bar{x}} |\bar{x}\rangle.$$ \hspace{1cm} (19)

Since the quantum oracle recovers the value of $\bar{k}$ in a single query, it would seem that there is “infinite” quantum speed-up for this oracle setting. However, notice that there exists a different classical counterpart

$$|\bar{x}\rangle \overset{O^f_{\bar{k}}}{\longrightarrow} (-1)^{\bar{k} \cdot \bar{x}} |\bar{x}\rangle \oplus |\bar{k}\rangle$$ \hspace{1cm} (20)

which also recovers $\bar{k}$ in a single call and thus resolves the puzzle.

Below, we sketch the general problem of finding all possible classical counterparts of an arbitrary unitary and solve it for the simplest case of 2-qubit unitaries. A general reversible classical oracle acting on $m$-bits is a permutation $O$ of all $2^m$ possible input strings. On the other hand, a quantum oracle is of course a general $m$-qubit unitary operation $U$. When does a classical oracle $O$ correspond to a quantum oracle $U$? As discussed in the introductory part of this Letter (see Eq. (14)), for characterizing classical counterparts, one must consider generalized permutation unitaries $P$ whose non-zero entries are unit modulus complex numbers. We say that a unitary matrix $U$ has a classical counterpart $O$ (in accordance with Eqs. (10) and (11)) iff $U$ is locally equivalent to $P = DO$, i.e.

$$\left(\otimes L_i^{(1)} \right) U \left(\otimes L_i^{(2)} \right) = P,$$ \hspace{1cm} (21)

where all $L_i^{(1)}$, $L_i^{(2)}$ are single qubit operations and $D$ is a diagonal unitary matrix. The problem of finding all possible classical counterparts is a particular subset of the general problem of local equivalence of unitary operations. Unfortunately, no general solution to this problem has been obtained so far.

In the simplest case of 2-qubit unitaries $U$, three real parameters completely characterize local equivalence. A computationally appealing choice of these parameters are given by Makhlin [R]

$$\alpha = \text{Re} \frac{\text{Tr}^2 V}{16 \det U},$$ \hspace{1cm} (22)

$$\beta = \text{Im} \frac{\text{Tr}^2 V}{16 \det U},$$ \hspace{1cm} (23)

$$\gamma = \frac{\text{Tr}^2 V - (\text{Tr} V)^2}{4 \det U},$$ \hspace{1cm} (24)

where $V = W^T W$, $W = Q^T U Q$ and

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}.$$ \hspace{1cm} (25)
In particular, these parameters uniquely classify equivalence classes of all 2-qubit generalized permutations and thus the set $CC(U)$ of all classical counterparts of a given unitary $U$. For compactness, it is convenient to first divide the group of permutations $S_4$ into 6 cosets with respect to the subgroup of local permutations $S_2 \otimes S_2 = \{\sigma_2^j \otimes \sigma_2^k | j, k = 0, 1\}$. These cosets may then be identified with their respective representatives chosen as follows: $I$, SWAP, CNOT$_{12}$, CNOT$_{21}$, SWAT$_{12}$, SWAT$_{21}$, where SWAT $\equiv$ SWAP $\cdot$ CNOT. The classes of $CC(U)$ are identified in Table I. There are four non-trivial classes and one empty class.

**TABLE I: Complete classification of classical counterparts of 2-qubit unitaries**

| $\alpha$ | $\beta$ | $\gamma$ | CC(U) |
|----------|---------|---------|-------|
| $\alpha \neq 0$ | 0 | 1 | $I, CNOT_{12}, CNOT_{21}$ |
| 0 | 0 | $-1$ | $\{SWAP, SWAT_{12}, SWAT_{21}\}$ |
| $\alpha > 0$ | 0 | $1 + 2\alpha$ | $\{I\}$ |
| $\alpha < 0$ | 0 | $-1 + 2\alpha$ | $\{SWAP\}$ |

Finally, let us turn to the important question of the source of quantum speed-up. Although quantum entanglement is believed to be the key to quantum speed-up, there is no proof that this is indeed the case. For example, BV problem is a commonly mentioned case where quantum speed-up seems to be obtained without entanglement. The PARITY problem solution also does not use any entanglement. We believe, that our notion of genuine quantum speed-up may help clarify the role of entanglement as a necessary constituent of quantum over classical algorithmic superiority. In the examples mentioned above we have been able to show that there is actually no genuine quantum speed-up where entanglement is absent. Moreover, in examples such as Grover’s problem and the Deutsch-Jozsa problem where entanglement is crucial, we have not been able to report finding corresponding classical oracles that diminish the quantum speed-up. Of course, in order to prove the link between genuine quantum speed-up and entanglement the non-trivial task of finding all the classical counterparts of an arbitrary multi-qubit quantum unitary operation must be solved.

Summarizing, we have shown that the common procedure for comparing quantum and classical oracles is ambiguous. This has led us to introduce the notion of genuine quantum speed-up which allows reliable comparison of quantum and classical oracles. As an example, we have shown that the Bernstein-Vazirani and PARITY problems do not exhibit genuine quantum speed-up.

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