Abstract. In this work, we mainly focus on the energy-supercritical nonlinear Schrödinger equation,

\[ i\partial_t u + \Delta u = \mu |u|^p u, \quad (t, x) \in \mathbb{R}^{d+1}, \]

with \( \mu = \pm 1 \) and \( p > \frac{4}{d-2} \).

We prove that for radial initial data with high frequency, if it is outgoing (or incoming) and in rough space \( H^{s_1}(\mathbb{R}^d) \) \((s_1 < s_c)\) or its Fourier transform belongs to \( W^{s_2,1}(\mathbb{R}^d) \) \((s_2 < s_c)\), the corresponding solution is global and scatters forward (or backward) in time. We also construct a class of large global and scattering solutions starting with many bubbles, which are mingled with in the physical space and separate in the frequency space. The analogous results are also valid for the energy-subcritical cases.
1. Introduction

We study the Cauchy problem for the following nonlinear Schrödinger equation (NLS) on \( \mathbb{R} \times \mathbb{R}^d \):

\[
\begin{cases}
    i\partial_t u + \Delta u = \mu |u|^p u, \\
    u(0, x) = u_0(x),
\end{cases}
\]

(1.1)

with \( \mu = \pm 1 \) and \( p > 0 \). Here \( u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) is a complex-valued function. \( \mu = 1, -1 \) denotes the nonlinearity is defocusing and focusing, respectively. The class of solutions to equation (1.1) is invariant under the scaling

\[
u(t, x) \to \nu(\lambda t, \lambda x)
\]

(1.2)

which maps the initial data as

\[
u(0) \to \nu_\lambda(0) := \lambda^{d/2}u_0(\lambda x) \quad \text{for} \quad \lambda > 0.
\]

Denote

\[
s_c = \frac{d}{2} - \frac{2}{p}.
\]

Then the scaling leaves \( \dot{H}^{s_c} \) norm invariant, that is,

\[
\|u\|_{\dot{H}^{s_c}} = \|u_\lambda\|_{\dot{H}^{s_c}},
\]

which is called critical regularity \( s_c \). It is also considered as the lowest regularity that problem (1.1) is well-posed for general \( H^s(\mathbb{R}^d) \)-data, since one can always find some special initial datum belonging to \( H^s(\mathbb{R}^d) \), \( s < s_c \) such that the problem (1.1) is ill-posed. Note that \( \dot{H}^{s_c}(\mathbb{R}^d) \hookrightarrow L^{p_c}(\mathbb{R}^d) \) with \( p_c = \frac{dp}{2} \), and

\[
\|u\|_{L^{p_c}} = \|u_\lambda\|_{L^{p_c}},
\]

then one naturally takes \( L^{p_c}(\mathbb{R}^d) \) as the critical Lebesgue space.

The \( H^1 \)-solution of equation (1.1) also enjoys the mass, momentum and energy conservation laws, which read

\[
\begin{align*}
M(u(t)) &:= \int |u(x, t)|^2 \, dx = M(u_0), \\
P(u(t)) &:= \text{Im} \int u(x, t) \nabla u(x, t) \, dx = P(u_0), \\
E(u(t)) &:= \int |\nabla u(x, t)|^2 \, dx + \frac{2\mu}{p+2} \int |u(x, t)|^{p+2} \, dx = E(u_0).
\end{align*}
\]

(1.3)

The well-posedness and scattering theory for Cauchy problem (1.1) with initial data in \( H^s(\mathbb{R}^d) \) are extensively studied. The local well-posedness theory follows from a standard fixed point argument, implying that for all \( u_0 \in H^s(\mathbb{R}^d) \) with \( s \geq s_c \), there exists \( T_0 > 0 \) such that its corresponding solution \( u \in C([0, T_0], H^s(\mathbb{R}^d)) \). In fact, the above \( T_0 \) depends on \( \|u_0\|_{H^s(\mathbb{R}^d)} \) when \( s > s_c \) and also the profile of \( u_0 \) when \( s = s_c \). Some of the results can be found in Cazenave and Weissler [10]. Such argument can be applied directly to prove the global well-posedness for solutions to equation (1.1) with small initial data in \( H^s(\mathbb{R}^d) \) with \( s \geq s_c \). It is of great interest to consider the large initial data problem of NLS for supercritical case \( s_c > 1 \), since all of known conservations are below the critical scaling regularity, few of the results on the long time behavior of the large data solutions were established, even the initial datum are smooth enough.
Recently, conditional global and scattering results with assumption that
\[ u \in L^\infty_t(I, \dot{H}^{s_c}(\mathbb{R}^d)) \]
were considered by many authors, which was started from [27, 28] and then developed by [7, 15, 19, 20, 29, 32, 33, 34, 35, 36, 42, 43, 45, 46, 59] and references therein. An important result indicated from these works is that if the initial data \( u_0 \in \dot{H}^{s_c}(\mathbb{R}^d) \) and the solution has priori estimate
\[
\sup_{0 < t < T_{\text{out}}(u_0)} \| u \|_{\dot{H}^{s_c}(\mathbb{R}^d)} < +\infty, \tag{1.4}
\]
then \( T_{\text{out}}(u_0) = +\infty \) and the solution scatters in \( \dot{H}^{s_c}(\mathbb{R}^d) \); here \( [0, T_{\text{out}}(u_0)) \) is the forward maximal interval for existence of the solution. Consequently, these results give the blowup criterion that the lifetime only depends on the critical norm \( \| u \|_{L^\infty_t \dot{H}^{s_c}(\mathbb{R}^d)} \).

In this paper, we consider the large global solution for the energy-supercritical nonlinear Schrödinger equation. The results obtained here are unconditional ones and are also valid for the energy-subcritical cases. It is the third part of our series of works on large global and scattering solution of nonlinear Schrödinger equation. In fact, in the first part of our series of works [1], we considered the global solution for the mass-subcritical nonlinear Schödinger equation in the critical space \( \dot{H}^{s_c}(\mathbb{R}^d) \) and proved that for radial initial data with compact support in space, the corresponding solution is global in time in \( \dot{H}^{s_c}(\mathbb{R}^d) \). In the second part of our series of works [2], we considered the global solution for the the defocusing mass-supercritical, energy-subcritical nonlinear Schödinger equation in the supercritical space \( \dot{H}^{s}(\mathbb{R}^d), s < s_c \) and proved that under some restrictions on \( d \) and \( p \), there exists \( s_0 < s_c \), such that any function in \( H^{s_0}(\mathbb{R}^d) \) with the support away from the origin, it has an incoming/outgoing decomposition. Moreover, the outgoing part of the initial data leads to the global well-posedness and scattering forward in time; while the incoming part of initial data leads to the global well-posedness and scattering backward in time.

The literature for the energy-supercritical is very limited compared to the energy-critical and subcritical cases. As mentioned previously, the main reason is the lack of the conservation laws beyond \( H^1(\mathbb{R}^d) \) Sobolev space. Besides the conditional global results described above, the unconditional results on the energy-supercritical equations are mostly from the wave equations, see [3, 4, 21, 22, 37, 25, 39, 44, 53, 17, 18, 51, 57, 60] and cited references. The first results from Tao [53], who proved global well-posedness and scattering for radial initial data for the logarithmically supercritical defocusing wave equation. The proof employs the Morawetz estimate and time slicing argument which is critical in \( H^1(\mathbb{R}^d) \), following [23]. Then Roy [17, 48] further proved the scattering of solutions to the log-log-supercritical defocusing wave equations. Recently, Bulut and Dodson [8] extended the work of Tao in the radially symmetric setting to a partially symmetric setting. Struwe [51] proved the global well-posedness for the exponential nonlinear wave equation in two dimension. Furthermore, it was first proved by Li [39] that the \((3 + 1)\)-Skyrme problem, which is energy-supercritical, is globally well-posed with hedgehog solutions for arbitrary large initial data. The work was followed by Geba and Grillakis [21, 22] to study the classical equivariant Skyrme model and the 2 + 1-dimensional equivariant Faddeev model, both of which are the energy-supercritical models. Very recently, large outgoing solution for the nonlinear wave equation was constructed in the papers of the first and the third authors [3, 4], by using the explicit formula of the outgoing and incoming components of the radial linear wave flow in three dimension. But such formulism does not exist for the Schrödinger equation. See also [16, 17, 18, 14] for
the blowing up results for the energy-supercritical wave equations. When it comes to the Schrödinger setting, the numerical investigation is from [13], who considered the equation,

\[ i\partial_t u + \Delta u = |u|^4 u, \quad x \in \mathbb{R}^5, \]

and found a class of global solutions which are large and uniformly bounded in \( H^2(\mathbb{R}^5) \). Yet, Tao [55] showed that there exist a class of defocusing nonlinear Schrödinger systems which the solutions can blow up in finite time by suitably constructing the initial datum (see also [54] for the analogous results for the wave equations). In the focusing case, Merle, Raphaël, and Rodnianski [41] shows the existence of type II blowing-up solutions for the energy-supercritical nonlinear Schrödinger equation. See also Wang [58], who constructed a class of quasi-periodic solutions to the energy-supercritical nonlinear Schrödinger equation. 

In this paper, we construct classes of initial data, which could be arbitrarily large in critical Sobolev space \( \dot{H}^{s_c}(\mathbb{R}^d) \) such that its corresponding solutions of (1.1) exist globally in time and scatter. In particular, we construct the global large solutions verifying the result in [13] theoretically. To state the first result, we introduce two indices as follows which are denoted by

\[
s_1 = \max\{ \frac{s_c}{d} - \varepsilon_0, s_c - \frac{4d-1}{4d-2} + \frac{2}{d} + \varepsilon_0 \},
\]

and

\[
s_2 = \max\{ -\varepsilon_0, s_c - \frac{d-2}{2(d-1)} - \frac{d-1}{2d-1} + \varepsilon_0 \},
\]

where \( \varepsilon_0 > 0 \) is a small fixed constant. Denote by \( \dot{W}^{s_1,1}(\mathbb{R}^d) \) the space of functions such that

\[
\| h \|_{\dot{W}^{s_1,1}(\mathbb{R}^d)} : = \| \langle \xi \rangle^s h \|_{L^1(\mathbb{R}^d)}
\]

and let it finite. Then we have the following result.

**Theorem 1.1.** Let \( p \geq \frac{4}{d}, d = 3, 4, 5 \) and \( \mu = \pm 1 \). Assume that there exists a small constant \( \delta_0 \) such that the radial function \( f \) satisfies

\[
\| \chi_{\leq 1} f \|_{\dot{H}^{s_c}(\mathbb{R}^d)} + \| \chi_{\geq 1} f \|_{\dot{H}^{s_1}(\mathbb{R}^d)} \leq \delta_0;
\]

and the radial function \( g \) satisfies

\[
supp g \in \{ x : |x| \leq 1 \}, \quad \text{and} \quad \| \langle \xi \rangle^{s_2} \hat{g} \|_{L^1(\mathbb{R}^d)} \leq \delta_0.
\]

Then if the initial date of equation (1.1) is of form

\[
u_0 = f_+ + g \quad \text{(or} \quad u_0 = f_- + g),
\]

the corresponding solution \( u \) exists globally forward (or backward) in time and

\[
u \in C(\mathbb{R}^+; \dot{H}^{s_1}(\mathbb{R}^d) + \dot{W}^{s_2,1}(\mathbb{R}^d)) \quad \text{(or} \quad u \in C(\mathbb{R}^-; \dot{H}^{s_1}(\mathbb{R}^d) + \dot{W}^{s_2,1}(\mathbb{R}^d))).
\]

Here \( f_+ \) and \( f_- \) are the modified outgoing and incoming components of \( f \) respectively, which are defined in Definition 6.1 with

\[
f = f_+ + f_-.
\]

Moreover, there exists \( u_{0+} \in \dot{H}^{s_1}(\mathbb{R}^d) + \dot{W}^{s_2,1}(\mathbb{R}^d) \) (or \( u_{0-} \in \dot{H}^{s_1}(\mathbb{R}^d) + \dot{W}^{s_2,1}(\mathbb{R}^d) \)), such that when \( t \to +\infty \) (or \( t \to -\infty \)),

\[
\lim_{t \to +\infty} \| u(t) - e^{it\Delta} u_{0+} \|_{\dot{H}^{s_c}(\mathbb{R}^d)} = 0 \quad \text{(or} \quad \lim_{t \to -\infty} \| u(t) - e^{it\Delta} u_{0-} \|_{\dot{H}^{s_c}(\mathbb{R}^d)} = 0).
\]
Remark 1.2. It is worth noting that $s_1 < s_c$, which means that $f_+$ (or $f_-$) in initial data can be localized in the supercritical Sobolev space $\dot{H}^s(\mathbb{R}^d)$ for some $s < s_c$. Indeed, if $f \in \dot{H}^s(\mathbb{R}^d)$, we have the fact that at least one of $f_+$ and $f_-$ belongs to $\dot{H}^s(\mathbb{R}^d)$ since

$$f = f_+ + f_-.$$ 

In particular, if we only consider the high frequency of incoming and outgoing waves and their supercritical space-time estimates. In particular, the localization of the initial data, as done by Bourgain [5], may not work.

Let $a_0 = \max\{\frac{s(d-1)}{d}, \frac{4d-1}{4d-2} - \frac{2}{d} - 2\varepsilon_0\}$ be a positive constant. We have the following corollary.

**Corollary 1.3.** Let $p \geq \frac{4}{d}$, $d = 3, 4, 5$ and $\mu = \pm 1$. Then there exist $N_0 > 0$ and a small constant $\delta_0$, such that for given $N > N_0$ and any radial function $f$ satisfying

$$\|\chi_{\leq 1} f\|_{\dot{H}^s(\mathbb{R}^d)} + \|P_{\leq N}(\chi_{> 1} f)\|_{\dot{H}^s(\mathbb{R}^d)} \leq \delta_0, \quad \|P_{\geq N}(\chi_{> 1} f)\|_{\dot{H}^s(\mathbb{R}^d)} \leq N^{a_0},$$

the solution $u$ to the equation (1.1) with the initial data

$$u_0 = f_+ \quad \text{(or} \quad u_0 = f_- \text{)}$$

exists globally forward (or backward) in time and

$$u \in C(\mathbb{R}^+; \dot{H}^s(\mathbb{R}^d)) \quad \text{(or} \quad u \in C(\mathbb{R}^-; \dot{H}^s(\mathbb{R}^d))).$$

Moreover, there exists $u_{0+} \in \dot{H}^s(\mathbb{R}^d)$ (or $u_{0-} \in \dot{H}^s(\mathbb{R}^d)$), such that when $t \to +\infty$ (or $t \to -\infty$),

$$\lim_{t \to +\infty} \|u(t) - e^{it\Delta} u_{0+}\|_{\dot{H}^s(\mathbb{R}^d)} = 0 \quad \text{(or} \quad \lim_{t \to -\infty} \|u(t) - e^{it\Delta} u_{0-}\|_{\dot{H}^s(\mathbb{R}^d)} = 0).$$

On the other hand, we note that $s_2 < s_c$ in Theorem 1.1. The function $g$ in initial data is localized in $\dot{W}^{s_2,1}(\mathbb{R}^d)$, which needs much less derivatives than $\dot{H}^s(\mathbb{R}^d)$. Moreover, one notices that the initial data can be large in $L^\infty(\mathbb{R}^d)$ if $p$ is close to $\frac{4}{d}$ and $s_2 < 0$. Indeed, we have the following corollary.

**Corollary 1.4.** Let $d = 3, 4, 5$, $0 \leq s_c < \frac{d-2}{2(d-1)} + \frac{d-1}{2d-1}$ and $\mu = \pm 1$. Then there exist $N_0$ and $b_0 > 0$, such that for given $N > N_0$ any radial function $g$ satisfying

$$\text{supp } g \in \{x : |x| \leq 1\}, \quad g = P_{\geq N} g, \quad \text{and} \quad \|\hat{g}\|_{L^1(\mathbb{R}^d)} \leq N^{b_0}, \quad (1.8)$$

the solution $u$ to the equation (1.1) with the initial data $u_0 = g$ exists globally in time and $u \in C(\mathbb{R}; \dot{W}^{s_2,1}(\mathbb{R}^d))$. Moreover, there exists $u_{0\pm} \in \dot{W}^{s_2,1}(\mathbb{R}^d)$, such that when $t \to \pm\infty$,

$$\lim_{t \to \pm\infty} \|u(t) - e^{it\Delta} u_{0\pm}\|_{\dot{H}^s(\mathbb{R}^d)} = 0.$$

Notice that in Corollary 1.4, we require $s_c < \frac{d-2}{2(d-1)} + \frac{d-1}{2d-1}$. In fact, the result is also valid for $s_c \geq \frac{d-2}{2(d-1)} + \frac{d-1}{2d-1}$ if one impose extra regularity on $g$. This corollary applies also in the focusing case, and therefore, the argument based on pseudo-conformal identity (that uses the localization of the initial data, as done by Bourgain [5]) may not work.

The key ingredients we rely on for the proof of Theorem 1.1 are as follows. The first one is the estimates obtained in Section 3 below, which regards as the decomposition of the incoming and outgoing waves and their supercritical space-time estimates. In particular, the
estimates obtained imply that the incoming/outgoing solution has the “smoothing effect” as follows: any \( \epsilon > 0 \),
\[
\| e^{it\Delta} (\chi_{\geq 1} f) \|_{L^2_t L^\infty_x (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \| \chi_{\geq 1} f \|_{H^\epsilon (\mathbb{R}^d)}.
\]
The second one is that if the function \( g \) is compactly supported, then we have the estimates with the “smoothing effect” in the following sense: any \( q \geq 2 \),
\[
\| e^{it\Delta} g \|_{L^q_t L^\infty_x (\mathbb{R} \times \mathbb{R}^d)} \lesssim \| \langle \xi \rangle^{-\frac{d-2}{q(d-1)}} \hat{g} \|_{L^1_\xi (\mathbb{R}^d)}.
\]
Then we consider the equation for \( w = u - v_L \) where \( u \) is the solution to equation (1.1) and \( v_L \) is the linear solution with the initial data \( f_+ + g \). It is easy to see that \( w \) obeys the equation of
\[
i\partial_t w + \Delta w = \mu |u|^p u.
\]
By using bootstrap argument and the space-time estimates for \( v_L \), we could prove Theorem 1.1. We also note that the choice of working spaces to this problem would be another obstacle, since one may find that the space-time estimates are not standard.

Now we introduce our second main result, which is about the global solution for initial data which consists of many bubbles, which are mingled within the physical space but separate in the frequency space.

Before stating our result, we introduce the hypothesis on the initial data.

**Assumption 1.5.** Given a constant \( \epsilon \in (0, 1] \). We assume that \( h \) is of form
\[
h = \sum_{k=0}^{+\infty} h_k,
\]
with
\[
supp \hat{h}_k = \{ \xi : 2^k \leq |\xi| \leq (1 + \epsilon)2^k \}.
\]
where \( h_k \in \dot{H}^{s_c} (\mathbb{R}^d) \) and there exists an absolute constant \( \alpha_0 > 0 \), such that
\[
\| h \|_{\dot{H}^{s_c} (\mathbb{R}^d)} \leq \epsilon^{-\alpha_0}.
\]

Now our second main theorem is stated as follows.

**Theorem 1.6.** Let \( d \geq 1, p > \frac{4}{d} \) and \( \mu = \pm 1 \). Then there exists some constant \( \epsilon_0 \in (0, 1] \) such that \( h \) satisfies Assumption 1.5 with respect to \( \epsilon \) and \( \psi_0 \in H^{s_c} (\mathbb{R}^d) \) satisfies
\[
\| \psi_0 \|_{H^{s_c} (\mathbb{R}^d)} \leq \epsilon
\]
for \( \epsilon \in (0, \epsilon_0] \), the solution \( u \) to the equation (1.1) with the initial data
\[
u_0 = \psi_0 + h
\]
exists globally in time and \( u \in C_t H^{s_c}_x (\mathbb{R} \times \mathbb{R}^d) \). Furthermore, there exists \( u_{0\pm} \in \dot{H}^{s_c} (\mathbb{R}^d) \) such that
\[
\lim_{t \to \pm \infty} \| u(t) - e^{it\Delta} u_{0\pm} \|_{H^{s_c} (\mathbb{R}^d)} = 0.
\]

**Remark 1.7.** Under Assumption 1.5, the function \( h \) in initial data can be regarded as a combination of the bubbles \( h_k \) which are separated in frequency space. Hence, \( h \) can be
arbitrary large in $\hat{H}^s(\mathbb{R}^d)$ by choosing $\epsilon$ small enough. We take one bubble case as an example, for any arbitrary large $L$, let

$$\hat{h}(\xi) = e^{-\frac{1}{2} + \frac{\alpha_0}{2}} \chi_{\leq 1} \left( \frac{|\xi| - 1}{\epsilon} \right),$$

then $\|h\|_{\hat{H}^s(\mathbb{R}^d)} = L$, for $\epsilon \sim L^{-\frac{s_c}{d}}$.

The key observation in the proof of Theorem 1.6 is based on the following estimate

$$\|\nabla^\gamma e^{it\Delta} h_k\|_{L^p_x \cap L^q_t \cap (\mathbb{R} \times \mathbb{R}^d)} \lesssim \epsilon^{\frac{s_c}{2} - \frac{d}{q}} \|h_k\|_{\hat{H}^s(\mathbb{R}^d)}$$

with $\frac{d+2}{q} - \gamma = \frac{d}{2} - s_c$. Since the norms are scaling invariant, we believe that the above estimate is nontrivial. The key observation is due to the weak topology of the space-time norm compared to the Sobolev norm, and the narrow belt restriction on the frequency.

Organization of the paper. In Section 2, we give some preliminaries. This includes some basic lemmas, some estimates on the linear Schrödinger operator. Moreover, we recall the definition of the incoming/outgoing waves and their basic properties which were obtained in our previous paper [2]. In Sections 3 and 4, we establish the estimates on linear flow. In Section 5, we give some spacetime estimates on many bubbles case. In Section 6, we give the proof of the main theorems.

2. Preliminary

2.1. Notations. We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \leq CY$ for some constant $C > 0$. If $C$ depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_a Y$ denotes the assertion that $X \leq C(a)Y$ for some $C(a)$ depending on $a$. We use $O(Y)$ to denote any quantity $X$ such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$.

For small constant $\epsilon > 0$, the notations $a+$ and $a-$ denote $a+\epsilon$ and $a-\epsilon$, respectively. The fractional derivative is given by $|\nabla|^a = (-\partial_x^2)^{a/2}$. Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz function space on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ its topological dual space. Let $h \in \mathcal{S}'(\mathbb{R}^{d+1})$, we use $\|h\|_{L^q_x L^r_t}$ to denote the mixed norm $\left( \int \|h(\cdot, t)\|_{L^r_t}^q \, dt \right)^{\frac{1}{q}}$, and $\|h\|_{L^q_x L^r_t} := \|h\|_{L^q_x L^r_t}$. Sometimes, we use the notation $q' := \frac{a}{q-1}$.

Throughout this paper, we use $\chi_{\leq a}$ for $a \in \mathbb{R}^+$ to be the smooth function

$$\chi_{\leq a}(x) = \begin{cases} 1, & |x| \leq a, \\ 0, & |x| \geq \frac{11}{10} a. \end{cases}$$

Moreover, we denote $\chi_{\geq a} = 1 - \chi_{\leq a}$ and $\chi_{a \leq \leq b} = \chi_{\leq b} - \chi_{\leq a}$. We denote $\chi_a = \chi_{\leq 2a} - \chi_{\leq a}$ and $\chi_{\sim a} = \chi_{\frac{a}{2} \leq \leq 4a}$ for short. For any interval $\Omega \subset \mathbb{R}$, we denote $I_\Omega$ as its characteristic function

$$I_\Omega(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \not\in \Omega. \end{cases}$$
Also, we need some Fourier operators. For each number \( N > 0 \), we define the Fourier multipliers \( P_{\leq N}, P_{> N}, P_N \) as
\[
\hat{P}_{\leq N} f(\xi) := \chi_{\leq N}(\xi) \hat{f}(\xi), \\
\hat{P}_{> N} f(\xi) := \chi_{> N}(\xi) \hat{f}(\xi), \\
\hat{P}_N f(\xi) := \chi_N(\xi) \hat{f}(\xi),
\]
and similarly \( P_{< N} \) and \( P_{\geq N} \). We also define
\[
\tilde{P}_N := P_{N/2} + P_N + P_{2N}.
\]
We will usually use these multipliers when \( N \) are dyadic numbers (that is, of the form \( 2^k \) for some integer \( k \)).

2.2. Basic lemmas. First of all, we introduce the following Sobolev embedding theorem for radial function, see \cite{56} for example.

Lemma 2.1. Let \( \alpha, q, p, s \) be the parameters which satisfy
\[
\alpha > -\frac{d}{q}; \quad \frac{1}{q} \leq \frac{1}{p} \leq \frac{1}{q} + s; \quad 1 \leq p, q \leq \infty; \quad 0 < s < d
\]
with
\[
\alpha + s = d\left(\frac{1}{p} - \frac{1}{q}\right).
\]
Moreover, at most one of the equalities hold:
\[
p = 1, \quad p = \infty, \quad q = 1, \quad q = \infty, \quad \frac{1}{p} = \frac{1}{q} + s.
\]
Then for any radial function \( u \) such that \( |\nabla|^s u \in L^p(\mathbb{R}^d) \),
\[
\| |\cdot|^\alpha u \|_{L^q(\mathbb{R}^d)} \lesssim \| |\nabla|^s u \|_{L^p(\mathbb{R}^d)}.
\]

The second lemma is the following fractional Leibniz rule, see \cite{6, 30, 38} and references therein.

Lemma 2.2. Let \( 0 < s < 1, 1 < p \leq \infty, \) and \( 1 \leq p_1, p_2, p_3, p_4 \leq \infty \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), \( \frac{1}{p} = \frac{1}{p_3} + \frac{1}{p_4} \), and let \( f, g \in \mathcal{S}(\mathbb{R}^d) \), then
\[
\| |\nabla|^s (fg) \|_{L^p} \lesssim \| |\nabla|^s f \|_{L^{p_1}} \| g \|_{L^{p_2}} + \| |\nabla|^s g \|_{L^{p_3}} \| f \|_{L^{p_4}}.
\]

Consequently, we have the following elementary inequality, one can see \cite{1} for the proof.

Lemma 2.3. For any \( a > 0, 1 < p \leq \infty, 0 \leq \gamma < \frac{d}{p}, \) and \( |\nabla|\gamma g \in L^p(\mathbb{R}^d) \),
\[
\| |\nabla|^\gamma (\chi_{\leq a} g) \|_{L^p(\mathbb{R}^d)} \lesssim \| |\nabla|^\gamma g \|_{L^p(\mathbb{R}^d)}.
\]
Here the implicit constant is independent on \( a \). The same estimate holds for \( \chi_{> a} g \).

We need the following mismatch result, which is helpful in commuting the spatial and the frequency cutoffs.
Lemma 2.4 (Mismatch estimates, [10]). Let $\phi_1$ and $\phi_2$ be smooth functions obeying
\[ |\phi_j| \leq 1 \quad \text{and} \quad \text{dist}(\text{supp}\phi_1, \text{supp}\phi_2) \geq A, \]
for some large constant $A$. Then for $\sigma > 0$, $M \leq 1$ and $1 \leq r \leq q \leq \infty$,
\[ \|\phi_1|\nabla|^{\sigma}P_{\leq M}(\phi_2f)\|_{L^r_x} + \|\phi_1|\nabla|^{\sigma-1}P_{\leq M}(\phi_2f)\|_{L^r_x} \lesssim A^{-\frac{\sigma}{q}} + \|\phi_2f\|_{L^q_x}; \quad (2.2) \]
\[ \|\phi_1P_{\leq M}(\phi_2f)\|_{L^r_x} \lesssim M^{-m}A^{-m}\|f\|_{L^q_x}, \quad \text{for any } m \geq 0. \quad (2.3) \]

Furthermore, we also need the following Leibnitz-type formula, which is proved in [1].

Lemma 2.5. Let $f \in \mathcal{S}(\mathbb{R}^d)$ be a vector-valued function and $g \in \mathcal{S}(\mathbb{R}^d)$ be a scalar function. Then for any integer $N$,
\[ \nabla_{\xi} \cdot (f \nabla_{\xi})^{N-1} \cdot (fg) = \sum_{I_1,\ldots,I_N \in \mathbb{R}^d, j \in \mathbb{R}^d; \ |I_1| + \cdots + |I_N| + |j| = N} C_{I_1,\ldots,I_N,j} \partial_{\xi}^{I_1}f \cdots \partial_{\xi}^{I_N}f \partial_{\xi}^j g, \]
where we have used the notations
\[ \nabla_{\xi} = \{\partial_{\xi_1}, \ldots, \partial_{\xi_d}\}; \quad \partial_{\xi}^j = \partial_{\xi_1}^{j_1} \cdots \partial_{\xi_d}^{j_d}, \quad \text{for any } l = \{l_1, \ldots, l_d\} \in \mathbb{R}^d. \]

2.3. Linear Schrödinger operator. Let the operator $S(t) = e^{it\Delta}$ be the linear Schrödinger flow, that is,
\[ (i\partial_t + \Delta)S(t) \equiv 0. \]
The following are some fundamental properties of the operator $e^{it\Delta}$. The first is the explicit formula, see for example [7].

Lemma 2.6. For all $\phi \in \mathcal{S}(\mathbb{R}^d)$, $t \neq 0$,
\[ S(t)\phi(x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4it} \phi(y) \, dy. \]
Moreover, for any $r \geq 2$,
\[ \|S(t)\phi\|_{L^r_x(\mathbb{R}^d)} \lesssim |t|^{-d(\frac{1}{2} - \frac{1}{r})} \|\phi\|_{L^r_x(\mathbb{R}^d)}. \]

The following is the standard Strichartz estimate, see for example [24].

Lemma 2.7. Let $I$ be a compact time interval and let $u : I \times \mathbb{R}^d \to \mathbb{R}$ be a solution to the inhomogeneous Schrödinger equation
\[ iu_t - \Delta u + F = 0. \]
Then for any $t_0 \in I$, any pairs $(q_j, r_j), j = 1, 2$ satisfying
\[ q_j \geq 2, \ r_j \geq 2, \ \text{and} \ \frac{2}{q_j} + \frac{d}{r_j} = \frac{d}{2}, \]
the following estimates hold,
\[ \|u\|_{C(I;L^2_x(\mathbb{R}^d))} + \|u\|_{L^{q_j}_tL^{r_j}_x(I \times \mathbb{R}^d)} \lesssim \|u(t_0)\|_{L^2_x(\mathbb{R}^d)} + \|F\|_{L^{q_j}_tL^{r_j}_x(I \times \mathbb{R}^d)}. \]

Further, we need the following inhomogeneous Strichartz estimate which is not sharp but sufficient for this paper, see for examples [9] and [26].
Lemma 2.8. Let $I$ be a compact time interval and $t_0 \in I$. Assume that $2 < r < \frac{2d}{d-2}$ $(2 < r \leq \infty$ if $d = 1)$, and $1 < q, \tilde{q} < \infty$ satisfy
\[
\frac{1}{q} + \frac{1}{\tilde{q}} = d(\frac{1}{2} - \frac{1}{r}).
\]
Then the following estimates hold,
\[
\left\| \int_0^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{L^q_x L^\tilde{q}_r(I \times \mathbb{R}^d)} \lesssim \left\| F \right\|_{L^q_x L^\tilde{q}_r(I \times \mathbb{R}^d)}.
\]

We also need the special Strichartz estimate for radial data, which was first proved in [49] and then developed in [11, 24].

Lemma 2.9 (Radial Strichartz estimates). Let $g \in L^2(\mathbb{R}^d)$ be a radial function, and let the triple $(q, r, \gamma)$ satisfy
\[
\gamma \in \mathbb{R}, \; q \geq 2, \; r > 2, \; \frac{2}{q} + \frac{2d-1}{r} < \frac{d-1}{2}, \; \text{and} \; \frac{2}{q} + \frac{d}{r} = \frac{d}{2} + \gamma,
\]
moreover, when $\frac{d+2}{d-1} \leq q \leq \infty$, the penultimate inequality allows equality. Then
\[
\left\| \left\| \nabla e^{it\Delta} g \right\|_{L^q_t L^r_x(\mathbb{R}^d)} \right\| \lesssim \| g \|_{L^2(\mathbb{R}^d)}.
\]
Furthermore, let $F \in L^q_x L^r_x(\mathbb{R}^{d+1})$ be a radial function in $x$, then
\[
\left\| \int_0^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{L^q_t L^\tilde{q}_r(\mathbb{R}^{d+1})} + \left\| \left\| \nabla^{-\gamma} \int_0^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{L^q_t L^\tilde{q}_r(\mathbb{R}^{d+1})} \right\| \lesssim \left\| F \right\|_{L^q_x L^r_x(\mathbb{R}^{d+1})},
\]
where the triples $(q, r, \gamma), (\tilde{q}, \tilde{r}, -\gamma)$ satisfy (2.1).
Let
\[ K(r) = \chi_{\geq 1}(r) \left[ -\frac{1}{2\pi ir} - \frac{d-3}{(2\pi ir)^3} \right], \quad \text{for } d = 3, 4, 5. \]

Then the form \( J(r) - K(r) \) has some good properties as follows.

**Lemma 2.10.** Let \( d = 3, 4, 5 \), then there exist functions \( a(r) \) and \( \eta(\theta) \) satisfying
\[
a(r) = O(\langle r \rangle^{-5}); \quad \eta(\theta) \in C^3 \left( [0, \frac{\pi}{2}] \right),
\]
such that
\[
J(r) - K(r) = \int_0^{\frac{\pi}{2}} e^{2\pi ir \sin \theta} \chi_{\geq \pi / 6}(\theta) \cos d^{-2} \theta \, d\theta + a(r) \int_0^{\frac{\pi}{2}} e^{2\pi ir \sin \theta} \eta(\theta) \, d\theta.
\]

Next, we define the incoming and outgoing decomposition in terms of the deformed Fourier transform as follows.

**Definition 2.11.** Let \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) be a radial function. We define the incoming component of \( f \) as
\[
f_{\text{in}}(r) = r^{-d/2} \int_0^{+\infty} \left( J(-\rho r) + K(\rho r) \right) \rho^{d-1} \mathcal{F} f(\rho) \, d\rho;
\]
the outgoing component of \( f \) as
\[
f_{\text{out}}(r) = r^{-d/2} \int_0^{+\infty} \left( J(\rho r) - K(\rho r) \right) \rho^{d-1} \mathcal{F} f(\rho) \, d\rho.
\]

A very important property of the outgoing/incoming component of function \( f \) is that the following identity
\[ f = f_{\text{in}} + f_{\text{out}} \]
holds, which is obtained directly from the definition above. Some further properties will be stated in the rest of this section.

The following lemma shows that if \( f \) is supported outside of a ball, then \( f_{\text{out/in}} \) is also almost supported outside of the ball.

**Lemma 2.12.** Let \( \mu(d) = 2 \) if \( d = 3, 4 \), and \( \mu(5) = 3 \). Suppose that \( \text{supp} f \subset \{ x : |x| \geq 1 \} \), then
\[
\| \chi_{\leq 4}(P_{\geq 1} f)_{\text{out/in}} \|_{H^{\mu(d)}(\mathbb{R}^d)} \lesssim \| P_{\geq 1} f \|_{H^{-1}(\mathbb{R}^d)}.
\]

We also need the boundedness of incoming/outgoing projection on \( \dot{H}^s(\mathbb{R}^d) \).

**Lemma 2.13.** Suppose that \( f \in L^2(\mathbb{R}^d) \), then for any \( k \in \mathbb{Z}^+ \) and \( s \in [0, 1] \),
\[
\| \chi_{\geq 4}(P_{2^k} f)_{\text{out/in}} \|_{\dot{H}^s(\mathbb{R}^d)} \lesssim 2^{k s} \| f \|_{L^2(\mathbb{R}^d)}.
\]

Here the implicit constant is independent on \( k \).

The last one is the following simplified form of \( f_{\text{in/out}} \), which follows from Lemma 2.10.
Lemma 2.14. Let $k$ be an integer. Suppose that $f \in L^2(\mathbb{R}^d)$ with $\text{supp} f \subset \{ x : |x| \geq 1 \}$, then

$$(P_{2^k} f)_{\text{out/in}}(r) = r^{-\frac{d-1}{2}+2} \int_{0}^{+\infty} e^{2\pi ir \sin \theta} \chi_{\geq \frac{r}{2}}(\theta) \cos^{d-2} \theta \, d\theta \cdot \chi_{2^{k-1} \leq \cdot \leq 2^{k+1}}(\rho) \mathcal{F}(P_{2^k} f)(\rho) \rho^{d-1} \, d\rho + R(P_{2^k} f),$$

with

$$\| \chi_{\geq \frac{r}{2}} R(P_{2^k} f) \|_{H^{\mu(d)}(\mathbb{R}^d)} \lesssim 2^{-k} \| f \|_{H^{1}(\mathbb{R}^d)}.$$ 

Proof. We only sketch the proof here, one can see details in [2]. It suffices to give the formula for $(P_{2^k} f)_{\text{in}}$ since the proof for $(P_{2^k} f)_{\text{out}}$ shares essentially the same procedures.

According to the support of $f$, we may write $\mathcal{F}(P_{2^k} f)$ as the standard Fourier transform of $|x|^{-\frac{d-1}{2}} P_{2^k} \chi_{\frac{r}{2}} f$. Hence, using the mismatch estimates in Lemma 2.14 we have

$$(P_{2^k} f)_{\text{in}}(r) = r^{-\frac{d-1}{2}+2} \int_{0}^{+\infty} \left( J(-\rho r) + K(\rho r) \right) \rho^{d-1} \chi_{2^{k-1} \leq \cdot \leq 2^{k+1}}(\rho) \mathcal{F}(P_{2^k} f)(\rho) \, d\rho + h_{1,k}^1,$$

with

$$\| h_{1,k}^1 \|_{H^{\mu(d)}(\mathbb{R}^d)} \lesssim 2^{-k} \| f \|_{H^{1}(\mathbb{R}^d)}.$$ 

Moreover, by using Lemma 2.10 we further write

$$r^{-\frac{d-1}{2}+2} \int_{0}^{+\infty} \left( J(-\rho r) + K(\rho r) \right) \rho^{d-1} \chi_{2^{k-1} \leq \cdot \leq 2^{k+1}}(\rho) \mathcal{F}(P_{2^k} f)(\rho) \, d\rho$$

$$= r^{-\frac{d-1}{2}+2} \int_{0}^{+\infty} e^{2\pi ir \sin \theta} \chi_{\geq \frac{r}{2}}(\theta) \cos^{d-2} \theta \, d\theta \chi_{2^{k-1} \leq \cdot \leq 2^{k+1}}(\rho) \mathcal{F}(P_{2^k} f)(\rho) \rho^{d-1} \, d\rho + h_{1,k}^2,$$

with

$$\| \chi_{\geq \frac{r}{2}} h_{1,k}^2 \|_{H^{\mu(d)}(\mathbb{R}^d)} \lesssim 2^{-k} \| f \|_{H^{1}(\mathbb{R}^d)}.$$ 

Let $R(P_{2^k} f) = h_{1,k}^1 + h_{1,k}^2$ and then it satisfies the desired estimate. Hence we finish the proof. \hfill \Box

3. Estimates on the incoming/outgoing linear flow

In this section, we give the estimates on the linear flow when the initial data is the outgoing or incoming. Let $f$ be a radial function satisfying $\text{supp} f \subset \{ x : |x| \geq 1 \}$. We abuse the notation and write $f = \chi_{\geq 1} f$ for simplicity. In the following, we focus on the estimates on the outgoing part, the estimates on the incoming part are similar. Fixing an integer $k_0 \geq 0$, we consider the estimates on $e^{it\Delta} (P_{\geq 2^{k_0}} f)_{\text{out}}$. For function $v$, we will use the notation $v_L = e^{it\Delta} v$ for short in the following.

Due to Lemmas 2.12 and 2.14 we may write

$$(P_{\geq 2^{k_0}} f)_{\text{out},L} = (P_{\geq 2^{k_0}} f)_{\text{out},L}^I + (P_{\geq 2^{k_0}} f)_{\text{out},L}^{II},$$
where
\[
\left( P_{\geq 2^{k_0}} f \right)^I_{\text{out}, L} = \left( \chi_{\leq 4} (P_{\geq 2^{k_0}} f)_{\text{out}} \right)_L + \sum_{k=k_0}^{\infty} \chi_{\leq c(1+2^k t)} \cdot \left( \chi_{\geq 4} \mathcal{R} \left( P_{2^k} f \right) \right)_L \\
+ \sum_{k=k_0}^{\infty} \chi_{\leq c(1+2^k t)} \cdot e^{it\Delta} \left( \chi_{\geq 4} (r^{-\frac{d}{2}+2} e^{2\pi i \rho r \sin \theta}) \cdot \chi_{\geq \frac{1}{2}} (P_{2^k} f)_{\text{out}} \right)_L.
\]

and
\[
\left( P_{\geq 2^{k_0}} f \right)^{II}_{\text{out}, L} = \sum_{k=k_0}^{\infty} \chi_{\geq c(1+2^k t)} \cdot \left( \chi_{\geq \frac{1}{4}} (P_{2^k} f)_{\text{out}} \right)_L.
\]

Here \( c \) is a small positive constant. Then the estimates on \( \left( P_{\geq 2^{k_0}} f \right)^I_{\text{out}, L} \) and \( \left( P_{\geq 2^{k_0}} f \right)^{II}_{\text{out}, L} \) are included in the following two lemmas.

**Lemma 3.1.** Let \( d = 3, 4, 5, \ s \in [0, \mu(d)], q \geq 2, r \geq 2 \) and \( \frac{2}{q} + \frac{d}{r} = \frac{d}{2} \). Then the following estimates hold,
\[
\left\| \| \nabla \|^s \left( P_{\geq 2^{k_0}} f \right)^I_{\text{out}, L} \right\|_{L^q_t L^r_x (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \left\| f \right\|_{H^{-1}(\mathbb{R}^d)}.
\]

**Proof.** We shall consider the estimates on the three pieces in (3.1). For the first two pieces, by using Lemmas 2.12, 2.14 and 2.7 we have
\[
\left\| \| \nabla \|^s \left( \chi_{\leq 4} (P_{\geq 2^{k_0}} f) \right)_{\text{out}, L} \right\|_{L^q_t L^r_x (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \left\| f \right\|_{H^{-1}(\mathbb{R}^d)}.
\]

and
\[
\left\| \| \nabla \|^s \sum_{k=k_0}^{\infty} \chi_{\leq c(1+2^k t)} \cdot \left( \chi_{\geq 4} \mathcal{R} \left( P_{2^k} f \right) \right)_L \right\|_{L^q_t L^r_x (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \left\| f \right\|_{H^{-1}(\mathbb{R}^d)}.
\]

It remains to consider the estimates for the third piece in (3.1). To do this, we first claim that for \( j = 0, 1, 2, 3, \)
\[
\left\| \| \nabla \|^j \left( \chi_{\leq c(1+2^k t)} \cdot e^{it\Delta} \left( \chi_{\geq \frac{1}{4}} (r^{-\frac{d}{2}+2} e^{2\pi i \rho r \sin \theta}) \right) \right) \right\| \lesssim \left\langle t \right\rangle^{-\frac{d}{2}} \rho^{-j}.
\]

To prove the claim (3.5), we first use the formula in Lemma 2.6 and write
\[
e^{it\Delta} \left( \chi_{\geq \frac{1}{4}} (r^{-\frac{d}{2}+2} e^{2\pi i \rho r \sin \theta}) \right) = \frac{C}{t^\frac{d}{2}} \int_{\mathbb{R}^d} e^{i \frac{|x|^2}{4t} + i 2\pi i \rho |y| \sin \theta} \chi_{\geq \frac{1}{4}} (y) y^{-\frac{d-1}{2}+2} dy
\]
\[
= \frac{C}{t^\frac{d}{2}} e^{\frac{|x|^2}{4t}} \int_{\mathbb{R}^d} e^{-i \frac{|x|^2}{4t} + i 2\pi i \rho |y| \sin \theta} \chi_{\geq \frac{1}{4}} (y) y^{-\frac{d-1}{2}+2} dy
\]
\[
= \frac{C}{t^\frac{d}{2}} e^{\frac{|x|^2}{4t}} \left[ \int_{|\omega|=1} + \int_0^{\infty} \right] e^{i \phi(r)} \chi_{\geq \frac{1}{4}} (r) r^{-\frac{d-1}{2}+2} dr d\omega,
\]

where \( C \in \mathbb{C} \) may vary line to line, and \( \phi(r) = -\frac{\omega \cdot x}{2t} + \frac{r^2}{4t} + 2\pi \rho \sin \theta \). Then it is easy to see that
\[
\phi'(r) = -\frac{x \cdot \omega}{2t} + \frac{r}{2t} + 2\pi \rho \sin \theta, \quad \phi''(r) = \frac{1}{2t}, \quad \text{and} \quad \phi^{(j)}(r) = 0, \quad j \geq 3.
\]
Note that when $|x| \leq c(1+2^k t)$, $\rho \sim 2^k$, $r \geq \frac{1}{2}$ and $\sin \theta \geq \frac{1}{4}$, by choosing $c$ small enough, we have
\[
\phi' (r) \geq \frac{1}{4} \left( \frac{r}{t} + \pi \rho \right).
\]
(3.7)
Then using the formula,
\[
e^{i \phi (r)} = \frac{1}{i \phi' (r)} \partial_r \left( e^{i \phi (r)} \right),
\]
and integrating by parts $K$ times, we have that for some $c_K \in \mathbb{C},$
\[
\chi_{\leq c(1+2^k t)} \cdot e^{it \Delta} \left( \chi_{\geq \frac{1}{4}} (r) \cdot \frac{d-1}{2} + e^{2\pi i \rho r \sin \theta} \right) (x) = \chi_{\leq c(1+2^k t)} \cdot \frac{c_K}{t^{\frac{d}{2}}} e^{\frac{|x|^2}{4t}} \int_0^{+\infty} e^{i \phi (r)} \partial_r \left( \frac{1}{\phi' (r)} \partial_r \right)^{K-1} \left[ \frac{1}{\phi' (r)} r^{d-1} \chi_{\geq \frac{1}{4}} (r) \right] dr d\omega.
\]
(3.8)
Notice that it follows from Lemma 2.39 that
\[
\partial_r \left( \frac{1}{\phi' (r)} \partial_r \right)^{K-1} \left[ \frac{1}{\phi' (r)} r^{d-1} \chi_{\geq \frac{1}{4}} (r) \right] = \sum_{l_1, \ldots, l_K \in \mathbb{R}^d; l_1 + \cdots + l_K + 1^{15} = K} C_{l_1, \ldots, l_K} \partial_r^{l_1} \frac{1}{\phi' (r)} \partial_r^{l_2} \frac{1}{\phi' (r)} \cdots \partial_r^{l_K} \frac{1}{\phi' (r)} \partial_r^{l_{15}} \left[ r^{d-1} \chi_{\geq \frac{1}{4}} (r) \right],
\]
which combined with (3.6) and (3.7) implies
\[
\left| \partial_r \left( \frac{1}{\phi' (r)} \partial_r \right)^{K-1} \left[ \frac{1}{\phi' (r)} r^{d-1} \chi_{\geq \frac{1}{4}} (r) \right] \right| \leq t^{\frac{d}{2}} (t)^{\frac{d}{2} - 10} \rho^{-10} \chi_{\geq \frac{1}{4}} (r).
\]
Then inserting the above estimate into (3.8), we obtain (3.5) for $j = 0,$
\[
\left| \chi_{\leq c(1+2^k t)} \cdot e^{it \Delta} \left( \chi_{\geq \frac{1}{4}} (r) \cdot \frac{d-1}{2} + e^{2\pi i \rho r \sin \theta} \right) \right| \lesssim (t)^{\frac{d}{2}} \rho^{-15}.
\]
The estimates on the $j$-th derivative in (3.5) share similar arguments, since when the derivatives hit $\chi_{\leq c(1+2^k t)}$ and $\chi_{\geq \frac{1}{4}} (r) r^{\frac{d-1}{2} +}$, the estimates would become better, and when the derivatives hit $e^{2\pi i \rho r \sin \theta}$, it only increases the power of $\rho$. Hence we finish the proof of (3.5).

It follows from Hölder’s inequality and (3.5) that
\[
\left| \nabla^2 \sum_{k=k_0}^{\infty} \chi_{\leq c(1+2^k t)} \cdot e^{it \Delta} \left( \chi_{\geq \frac{1}{4}} (r) \cdot \frac{d-1}{2} + e^{2\pi i \rho r \sin \theta} \right) \right| \lesssim (t)^{\frac{d}{2}} 2^{-5k} \| \chi_{\geq 2^k (\rho)} F (P_{2^k} (\chi_{\geq 1} f)) \|_{L^2}.
\]
We next prove that
\[
\| \chi_{\geq 2^k (\rho)} F (P_{2^k} (\chi_{\geq 1} f)) \|_{L^2} \lesssim 2^{-\frac{d+1}{2} k} \| P_{2^k} (\chi_{\geq 1} f) \|_{L^2 (\mathbb{R}^d)}.
\]
(3.9)
To this end, notice that
\[
F (P_{2^k} (\chi_{\geq 1} f)) = \mathcal{F} \left( |x|^{\frac{d+1}{2}} P_{2^k} (\chi_{\geq 1} f) \right),
\]
where \( \mathcal{F} \) is the standard Fourier transformation, then by Plancherel identity, Hölder’s and Bernstein’s inequalities and Lemma 2.4, we have

\[
\| \chi_{2^{k}}(\rho) \mathcal{F}(P_{2^{k}}(\chi_{\geq 1}f)) \|_{L^{2}_{\rho}} \lesssim 2^{-\frac{d-1}{2}k} \| \mathcal{F}(P_{2^{k}}(\chi_{\geq 1}f)) \|_{L^{2}_{\rho}(\mathbb{R}^{d})}
\]

\[
\lesssim 2^{-\frac{d-1}{2}k} \| |x|^{\frac{d-1}{2}k} P_{2^{k}}(\chi_{\geq 1}f) \|_{L^{2}_{\rho}(\mathbb{R}^{d})}
\]

\[
\lesssim 2^{-\frac{d-1}{2}k} \left( \| |x|^{\frac{d-1}{2}} \chi_{\leq 2^{k}} P_{2^{k}}(\chi_{\geq 1}f) \|_{L^{2}(\mathbb{R}^{d})} + \| |x|^{\frac{d-1}{2}} \chi_{\leq 2^{k}} P_{2^{k}}(\chi_{\geq 1}f) \|_{L^{2}(\mathbb{R}^{d})} \right)
\]

\[
\lesssim 2^{-\frac{d-1}{2}k} \| P_{2^{k}}(\chi_{\geq 1}f) \|_{L^{2}(\mathbb{R}^{d})},
\]

which implies (3.3). Hence, we obtain

\[
\left\| \nabla \left[ \sum_{k=k_{0}}^{\infty} \chi_{\leq c(1+2^{k}t) \cdot e^{it\Delta}} \left( \chi_{\geq \frac{1}{2^{k}} t} \right) e^{\frac{i}{2}k+2} \int_{0}^{\infty} \int_{0}^{\pi} e^{2\pi i \rho \sin \theta} \chi_{\geq \frac{1}{2}^{k}}(\theta) \cos^{d-2} \theta \ d\theta \right. \right.
\]

\[
\left. \cdot \chi_{2^{k-1} \leq \leq 2^{k+1}}(\rho) \mathcal{F}(P_{2^{k}}f)(\rho) d\rho \right] \left( t \right) \lesssim 2^{-\frac{d-1}{2}k} \| P_{2^{k}}(\chi_{\geq 1}f) \|_{L^{2}(\mathbb{R}^{d})},
\]

By using Hölder’s inequality again, we obtain

\[
\left\| \nabla \left[ \sum_{k=k_{0}}^{\infty} \chi_{\leq c(1+2^{k}t) \cdot e^{it\Delta}} \left( \chi_{\geq \frac{1}{2^{k}} t} \right) e^{\frac{i}{2}k+2} \int_{0}^{\infty} \int_{0}^{\pi} e^{2\pi i \rho \sin \theta} \chi_{\geq \frac{1}{2}^{k}}(\theta) \cos^{d-2} \theta \ d\theta \right. \right.
\]

\[
\left. \cdot \chi_{2^{k-1} \leq \leq 2^{k+1}}(\rho) \mathcal{F}(P_{2^{k}}f)(\rho) d\rho \right] \left. \right\|_{L^{2}_{t}L^{q}_{x}(\mathbb{R}^{+} \times \mathbb{R}^{d})} \lesssim \| f \|_{H^{-1}(\mathbb{R}^{d})}.
\]

Hence we finish the proof. □

We next consider the Strichartz estimates for \( \left( P_{2^{k_{0}}f} \right)^{II}_{out,L} \) (see (3.2) its definition). Let \( \sigma_{0} \) be a positive constant satisfying

\[
\frac{1}{\sigma_{0}} = \frac{2d+1}{4d-2} - \frac{2}{d}.
\]

Then we have the following lemma.

**Lemma 3.2.** Let \( d = 3, 4, 5 \) and \( (q, r) \) be one of the following pairs

\[
(\infty, 2), \ (\infty, \frac{dp}{2}), \ (\infty, -, \frac{dp}{2}), \ (2p, dp), \ (2, \sigma_{0}). \tag{3.10}
\]

Then we have

\[
\left\| \left( P_{2^{k_{0}}f} \right)^{II}_{out,L} \right\|_{L^{2}_{t}L^{q}_{x}(\mathbb{R}^{+} \times \mathbb{R}^{d})} \lesssim \| f \|_{H^{\frac{dp}{2}}(\mathbb{R}^{d})}. \tag{3.11}
\]

Furthermore, for any \( \beta \in [0, 1] \), it holds that

\[
\left\| \nabla^{\beta} \left( P_{2^{k_{0}}f} \right)^{II}_{out,L} \right\|_{L^{2}_{t}L^{q}_{x}(\mathbb{R}^{+} \times \mathbb{R}^{d})} \lesssim \| f \|_{H^{\frac{dp}{2}+\beta}(\mathbb{R}^{d})}. \tag{3.12}
\]
Proof. By Lemma 2.1, we have
\[
\left\| \left( P_{\geq 2^k_0} f \right)_{out} \right\|_{L_t^q L_x^r([\mathbb{R}^+ \times \mathbb{R}^d])} \lesssim \sum_{k=k_0}^\infty \left\| \chi_{\geq c(1+2^k t)} \cdot \left( \chi_{\geq \frac{1}{k}} (P_{\geq 2^k} f)_{out} \right)_L \right\|_{L_t^q L_x^r([\mathbb{R}^+ \times \mathbb{R}^d])}
\lesssim \sum_{k=k_0}^\infty \left\| \left( 1 + 2^k t \right)^{-\alpha} \right\| \left( \chi_{\geq \frac{1}{k}} (P_{\geq 2^k} f)_{out} \right)_L \right\|_{L_t^q L_x^r([\mathbb{R}^+])}
\lesssim \sum_{k=k_0}^\infty \left\| \left( 1 + 2^k t \right)^{-\alpha} \right\|_{L_t^q([\mathbb{R}^+])} \cdot \left\| \chi_{\geq \frac{1}{k}} (P_{\geq 2^k} f)_{out} \right\|_{\dot{H}_t^s([\mathbb{R}^d])}.
\]
Here we set \( \alpha = (d-1) \left( \frac{1}{2} - \frac{1}{r} \right) \) and \( s = \frac{1}{2} - \frac{1}{q} \). Note that for any \((q, r)\) satisfies (3.10) and \( q\alpha > 1 \),
\[
\left\| \left( 1 + 2^k t \right)^{-\alpha} \right\|_{L_t^q([\mathbb{R}^+])} \lesssim 2^{-\frac{k}{q} t}. \]
Moreover, it follows from Lemma 2.13 that
\[
\left\| \chi_{\geq \frac{1}{k}} (P_{\geq 2^k} f)_{out} \right\|_{\dot{H}_t^s([\mathbb{R}^d])} \lesssim 2^{ks} \left\| P_{\geq 2^k} f \right\|_{L_t^2([\mathbb{R}^d])}.
\]
Thus we have
\[
\left\| \left( P_{\geq 2^k_0} f \right)_{out} \right\|_{L_t^q L_x^r([\mathbb{R}^+ \times \mathbb{R}^d])} \lesssim \left\| f \right\|_{H_t^\frac{1}{2} \cdot \dot{H}_x^\frac{s}{2} ([\mathbb{R}^d])}.
\]
In particular, when \((q, r) = (2, \sigma_0)\), it follows
\[
\left\| \left( P_{\geq 2^k_0} f \right)_{out} \right\|_{L_t^q L_x^r([\mathbb{R}^+ \times \mathbb{R}^d])} \lesssim \left\| f \right\|_{H_t^\frac{1}{2} \cdot \dot{H}_x^\sigma_0 ([\mathbb{R}^d])}. \tag{3.13}
\]
Notice that for the pairs \((q, r)\) in (3.10), the maximal value of \( \frac{1}{2} - \frac{1}{r} - \frac{1}{q} \) is \( \frac{1}{2} - \frac{2}{dp} = \frac{s}{d} \), which finishes the proof for (3.11).

The proof for (3.12) follows from similar method as above, we only sketch the proof. Write
\[
|\nabla|^\beta \left[ \chi_{\geq c(1+2^k t)} e^{it\Delta} \left( \chi_{\geq \frac{1}{k}} (P_{\geq 2^k} f)_{out} \right) \right]
\]
\[= P_{\leq 1} |\nabla|^\beta \left[ \chi_{\geq c(1+2^k t)} e^{it\Delta} \left( \chi_{\geq \frac{1}{k}} (P_{\geq 2^k} f)_{out} \right) \right] \tag{3.14}
\]
\[+ \chi_{\leq c^2(1+2^k t)} P_{\geq 1} |\nabla|^\beta \left[ \chi_{\geq c(1+2^k t)} e^{it\Delta} \left( \chi_{\geq \frac{1}{k}} (P_{\geq 2^k} f)_{out} \right) \right] \tag{3.15}
\]
\[+ \chi_{\geq c^2(1+2^k t)} P_{\geq 1} |\nabla|^\beta \left[ \chi_{\geq c(1+2^k t)} e^{it\Delta} \left( \chi_{\geq \frac{1}{k}} (P_{\geq 2^k} f)_{out} \right) \right]. \tag{3.16}
\]
By using the Bernstein inequality and (3.13), we have
\[
\left\| (3.14) \right\|_{L_t^q L_x^r([\mathbb{R}^+ \times \mathbb{R}^d])} \lesssim \left\| f \right\|_{H_t^{-\frac{1}{2}} \cdot \dot{H}_x^\sigma_0 ([\mathbb{R}^d])}.
\]
Furthermore, it follows from mismatch estimate (see Lemma 2.4) that
\[
\left\| (3.15) \right\|_{L_t^q L_x^r([\mathbb{R}^+ \times \mathbb{R}^d])} \lesssim \left\| f \right\|_{H_t^{-10}([\mathbb{R}^d])}.
\]
As for the term (3.16), using the similar method as the one used in the proof of (3.11) and Lemma 2.3, we obtain that it can be controlled by \( \left\| f \right\|_{H_t^{\frac{1}{2}} \cdot \dot{H}_x^\sigma_0 ([\mathbb{R}^d])} \). Hence we finish the proof. \qed
4. Linear flow estimates for compactly supported functions

In this section, we shall prove the Strichartz estimates for $e^{it\Delta}g$ with $g$ satisfying the assumptions in Theorem 1.1. The main result is stated as follows.

**Proposition 4.1.** Suppose that the suitable smooth function $g$ satisfies that

$$\text{supp } g \subset \{x : |x| \leq 1\},$$

then for any $\varrho \geq 2$, $\sigma \geq 2$ with $\frac{\varrho}{\sigma} + \frac{d}{\sigma} \leq \frac{d}{2}$, $\frac{1}{\varrho} + \frac{d-1}{\sigma} < \frac{d-1}{2}$,

$$\|e^{it\Delta}g\|_{L_t^\varrho L_x^\sigma(\mathbb{R}^d)} \lesssim \|\langle \xi \rangle^{-\frac{d-2}{2(\varrho-1)}} g\|_{L_t^1(\mathbb{R}^d)}.$$  \hspace{1cm} (4.1)

The proof of the proposition shall be divided into several parts. First of all, we show that the estimate holds for low frequency.

**Lemma 4.2.** Let $g \geq 2$, $\sigma \geq 2$ with $\frac{\varrho}{\sigma} + \frac{d}{\sigma} \leq \frac{d}{2}$. Then

$$\|e^{it\Delta}(\chi_{\leq 1}P_{\leq 1}g)\|_{L_t^\varrho L_x^\sigma(\mathbb{R}^d)} \lesssim \|\chi_{\leq 1}g\|_{L^1}.$$  \hspace{1cm} (4.2)

**Proof.** Let $s = \frac{d}{2} - \frac{2}{\varrho} - \frac{d}{\sigma} > 0$. It follows from Lemma 2.7 that

$$\|e^{it\Delta}(\chi_{\leq 1}P_{\leq 1}g)\|_{L_t^\varrho L_x^\sigma(\mathbb{R}^d)} \lesssim \|\chi_{\leq 1}P_{\leq 1}g\|_{H^s(\mathbb{R}^d)}. \hspace{1cm} (4.3)$$

We only need to show that

$$\|\chi_{\leq 1}P_{\geq 1}g\|_{H^s(\mathbb{R}^d)} \lesssim \|\chi_{\leq 1}g\|_{L^1(\mathbb{R}^d)}. \hspace{1cm} (4.4)$$

Indeed, it follows from interpolation, Hölder’s and Berstein’s inequalities that for $0 \leq s \leq 2$,

$$\|\chi_{\leq 1}P_{\geq 1}g\|_{H^s(\mathbb{R}^d)} \lesssim \|\chi_{\leq 1}P_{\geq 1}g\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}} \|\Delta(\chi_{\leq 1}P_{\leq 1}g)\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}} \lesssim \|P_{\geq 1}g\|_{L^2(|x| \leq 1)} + \|P_{\leq 1}g\|_{L^2(|x| \leq 1)} \|\nabla P_{\leq 1}g\|_{L^2(|x| \leq 1)}^{\frac{1}{2}} \|\Delta P_{\leq 1}g\|_{L^2(|x| \leq 1)}^{\frac{1}{2}} \lesssim \|P_{\geq 1}g\|_{L^\infty}. \hspace{1cm} (4.5)$$

We can then finish the proof by applying the Young inequality.

It remains the proof of the estimates for high frequency $P_{\geq 1}g$. By dyadic decomposition, we have

$$\chi_{\leq 1}P_{\geq 1}g = \sum_{N \geq 1} \chi_{\leq 1}P_N g.$$  \hspace{1cm} (4.6)

Hence in the following, we only consider the function with localized frequency. The following lemma shows that for high frequency and short time, we can gain regularity for the free flow in suitable Strichartz norms.

**Lemma 4.3.** Assume that $\varrho \geq 1$, $\sigma \geq 1$, $N \geq 1$ and $\gamma \in (0, 1]$. Then

$$\|e^{it\Delta}(\chi_{\leq 1}P_N g)\|_{L_t^\varrho L_x^\sigma([-8N^{-\gamma}, 8N^{-\gamma}] \times \mathbb{R}^d)} \lesssim N^{-\frac{\varrho}{2}} \|\chi_{\sim N}g\|_{L^1}. \hspace{1cm} (4.7)$$
Proof. By Hölder’s inequality, we only need to prove
\[ \left\| e^{it\Delta} (\chi \leq 1) P_N g \right\|_{L^\infty_t L^2_x([-8N^{-\gamma}, 8N^{-\gamma}] \times \mathbb{R}^d)} \lesssim \left\| \chi_N \hat{g} \right\|_{L^1_t}. \]
Moreover, it follows from interpolation that
\[ \left\| e^{it\Delta} (\chi \leq 1) P_N g \right\|_{L^\infty_t L^2_x([-8N^{-\gamma}, 8N^{-\gamma}] \times \mathbb{R}^d)} \lesssim \left\| e^{it\Delta} (\chi \leq 1) P_N g \right\|_{L^2_t L^4_x([-8N^{-\gamma}, 8N^{-\gamma}] \times \mathbb{R}^d)}^{\frac{1}{2}} \left\| e^{it\Delta} (\chi \leq 1) P_N g \right\|_{L^\infty_t L^4_x([-8N^{-\gamma}, 8N^{-\gamma}] \times \mathbb{R}^d)}^{\frac{1}{2}}. \]
Hence, it reduces to give control the $L^1_t$-norm and $L^\infty_t$-norm of $e^{it\Delta} (\chi \leq 1) P_N g$, which should be uniform in time. To this end, we write
\[ 1 = \chi_0(x, y) + \sum_{j=1}^\infty \chi_j(x, y), \]
for which
\[ \chi_0(x, y) = \chi \leq 1\left( \frac{y - x + 2t\xi}{2|t|^{\frac{1}{2}}} \right); \]
and for $j \geq 1$,
\[ \chi_j(x, y) = \chi \leq 1\left( \frac{y - x + 2t\xi}{2^{j+1}|t|^{\frac{1}{2}}} \right) - \chi \leq 1\left( \frac{y - x + 2t\xi}{2^j|t|^{\frac{1}{2}}} \right). \]
By using the same formula for $e^{it\Delta}$ as in Lemma 2.6 we write
\[ e^{it\Delta} (\chi \leq 1) P_N g = \frac{1}{(4\pi it)^\frac{d}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t} + iy \xi} \chi \leq 1(y) \chi_N(\xi) \hat{g}(\xi) dyd\xi. \]
We further split $e^{it\Delta} (\chi \leq 1) P_N g$ into the following parts,
\[ \frac{1}{(4\pi it)^\frac{d}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t} + iy \xi} \chi_0(x, y) \chi \leq 1(y) \chi_N(\xi) \hat{g}(\xi) dyd\xi \]
\[ + \sum_{j=1}^\infty \frac{1}{(4\pi it)^\frac{d}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t} + iy \xi} \chi_j(x, y) \chi \leq 1(y) \chi_N(\xi) \hat{g}(\xi) dyd\xi. \]
For (4.5), by the definition of $\chi_0$, we have
\[ |y - x + 2t\xi| \lesssim |t|^{\frac{1}{2}}. \]
Then it follows from the Hölder inequality that
\[ \left\| (4.5) \right\|_{L_1} \lesssim \frac{1}{|t|^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left| \chi_0(x, y) \right|_{L^\infty_x} \chi \leq 1(y) \chi_N(\xi) \hat{g}(\xi) dyd\xi \]
\[ \lesssim \left\| \chi_N \hat{g} \right\|_{L^1_t}. \tag{4.7} \]
On the other hand, notice that
\[ \left\| (4.5) \right\|_{L^1_t} \lesssim \frac{1}{|t|^{\frac{d}{2}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\| \chi_0(\cdot, y) \right\|_{L^\infty_x} \chi \leq 1(y) \chi_N(\xi) \hat{g}(\xi) dyd\xi, \]
and
\[ \left\| \chi_0(\cdot, y) \right\|_{L^1_t} \lesssim |t|^{\frac{d}{2}}, \]
then we obtain
\[ \left\| (4.5) \right\|_{L^1_t} \lesssim \left\| \chi_N \hat{g} \right\|_{L^1_t}, \]
which implies that
\[ \| (4.5) \|_{L_\infty^2 L_t^1} \lesssim \| \chi_N \hat{g} \|_{L^1}. \]  
(4.8)

Interpolation between (4.7) and (4.8), we obtain that for any \( \sigma \geq 1 \),
\[ \| (4.5) \|_{L_\infty^2 L_t^1} \lesssim \| \chi_N \hat{g} \|_{L^1}. \]  
(4.9)

We next consider the estimate for (4.6). For simplicity, denote by
\[ (4.6) \]  

For each \( j \), denote by
\[ \phi(y) = -\frac{x \cdot y}{2t} + \frac{|y|^2}{4t} + y \cdot \xi, \]  
(4.10)

and
\[ \partial_{jk} \phi(y) = \frac{\delta_{jk}}{2t}, \quad \partial_{jkh} \phi(y) = 0, \quad \text{for any } j, k, h \in \{1, 2, 3\}. \]  
(4.11)

Then
\[ (4.6) \]  

By using the identity
\[ e^{i\phi} = \nabla_y e^{i\phi} \cdot \frac{\nabla_y \phi}{i|\nabla_y \phi|^2}, \]  
(4.12)

and integrating by parts \( K \) times, we obtain that for some \( C_K \in \mathbb{C} \),
\[ (4.6) \]  

We shall first prove that
\[ \left| \nabla_y \cdot \left( \frac{\nabla_y \phi}{i|\nabla_y \phi|^2} \nabla_y \right)^K \chi_j (x, y) \chi_{\leq 1}(y) \right| \lesssim 2^{-2jK} \chi_{\sim 1} \left( \frac{y - x + 2t \xi}{2|t|^\sigma} \right) \chi_{\leq 1}(y). \]  
(4.14)

To this end, by using Lemma 2.5, we may expand the left-hand side of (4.14) as
\[ \sum_{l_1, \ldots, l_K \in \mathbb{R}^d, l' \in \mathbb{R}^d, \sum_{l_j \leq |l_1| + \ldots + |l_K| + |l'| = K} C_{l_1, \ldots, l_K, l'} \partial_y^K \left( \frac{\nabla_y \phi}{i|\nabla_y \phi|^2} \right) \partial_y^l \left( \frac{\nabla_y \phi}{i|\nabla_y \phi|^2} \right) \phi_j (x, y) \chi_{\leq 1}(y). \]  

It follows from (4.11) that
\[ \left| \partial_y^l \left( \frac{\nabla_y \phi}{i|\nabla_y \phi|^2} \right) \right| \lesssim \frac{\| \nabla_y \phi \|_{|l| + 1}}{|\nabla_y \phi|_{|l| + 1}} \lesssim \frac{1}{|l||l'||\nabla_y \phi|_{|l| + 1}}. \]
Hence we obtain
\[
|\partial_y^l \left( \frac{\nabla_y \phi}{i \nabla_y \phi^2} \right) \cdots \partial_y^K \left( \frac{\nabla_y \phi}{i \nabla_y \phi^2} \right)| \lesssim \frac{1}{|t|^{l_1} \cdots |t_K| |\nabla_y \phi|^{l_1} + \cdots + |t_K| + K}.
\] (4.15)

By the definition of \( \chi_j \), we have
\[
|y - x + 2t\xi| \sim 2^l |t|^{\frac{1}{2}},
\]
which combined with (4.10) implies
\[
|\nabla_y \phi| \gtrsim 2^l |t|^{-\frac{1}{2}}.
\]
Moreover, one has
\[
|\partial_y^l (\chi_j(x, y) \chi_{\leq 1}(y))| \lesssim 2^{-j|l'|} |t|^{-\frac{1}{2}|l'|} (1 + 2^{|l'|} |t|^{\frac{1}{2}|l'|}) \chi_{\sim 1} \left( \frac{y - x + 2t\xi}{2^l |t|^{\frac{1}{2}}} \right) \chi_{\leq 1}(y).
\]
Then combining the estimates above, we have that
\[
|\partial_y^l \left( \frac{\nabla_y \phi}{i \nabla_y \phi^2} \right) \cdots \partial_y^K \left( \frac{\nabla_y \phi}{i \nabla_y \phi^2} \right) \partial_y^l' (\chi_j(x, y) \chi_{\leq 1}(y))| \
\lesssim \frac{1 + 2^{|l'|} |t|^{\frac{1}{2}|l'|}}{2^{|l'|} |t|^{l_1 + \cdots + |t_K|} + |t_K| |\nabla_y \phi|^{l_1} + \cdots + |t_K| + K} \cdot \chi_{\sim 1} \left( \frac{y - x + 2t\xi}{2^l |t|^{\frac{1}{2}}} \right) \chi_{\leq 1}(y),
\]
which gives (4.14).

Now by using (4.14) and the fact
\[
|y - x + 2t\xi| \sim 2^l |t|^{\frac{1}{2}},
\]
we obtain that
\[
\left\| \nabla_y^l \left( \frac{\nabla_y \phi}{i \nabla_y \phi^2} \nabla_y \right)^{K-1} \cdot \left( \frac{\nabla_y \phi}{i \nabla_y \phi^2} \chi_j(x, y) \chi_{\leq 1}(y) \right) \right\|_{L^2} + \left\| \nabla_y^l \left( \frac{\nabla_y \phi}{i \nabla_y \phi^2} \nabla_y \right)^{K-1} \cdot \left( \frac{\nabla_y \phi}{i \nabla_y \phi^2} \chi_j(x, y) \chi_{\leq 1}(y) \right) \right\|_{L^\infty} \lesssim 2^{-Kl + 3j} |t|^{\frac{1}{2}}.
\]
Then it follows that
\[
\left\| (4.16)_j \right\|_{L^2} + \left\| (4.16)_j \right\|_{L^\infty} \lesssim K 2^{3j - Kj} \| \chi_N \hat{g} \|_{L^1}.
\]
Choosing \( K = 4 \) and using interpolation, we get that for any \( \sigma \geq 1 \),
\[
\left\| (4.16)_j \right\|_{L^\infty} \lesssim 2^{-j} \| \chi_N \hat{g} \|_{L^1},
\]
and then
\[
\left\| (4.16) \right\|_{L^\infty} \lesssim \| \chi_N \hat{g} \|_{L^1}. \tag{4.16}
\]
Combining (4.9) and (4.16), we obtain
\[
\| e^{it\Delta} (\chi_{\leq 1} P_N g) \|_{L^\infty_t L^2_x((-\frac{8}{N^2}, \frac{8}{N^2}) \times \mathbb{R}^d)} \lesssim \| \chi_N \hat{g} \|_{L^1}.
\] (4.16)
Hence, we finish the proof.

The second lemma shows that the linear flow $e^{it\Delta}$ become outgoing after a short time.

**Lemma 4.4.** Let $N \geq 1$. Then for any $\gamma \in (0, 1]$ and any $t$ with $|t| \geq 8N^{-\gamma}$,
\[
|\chi_{\leq \frac{1}{4}N|t|}(x)\left(e^{it\Delta} (\chi_{\leq 1} P_N g)\right)(x)| \lesssim < t >^{-\frac{3}{2}} N^{-100}\|\chi_N \tilde{g}\|_{L^1}.
\]

**Proof.** As in the proof of the previous lemma, denote by $\phi$ the phase function. Then it follows from (4.12) that
\[
e^{it\Delta} (\chi_{\leq 1} P_N g) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi(y)} \chi_{\leq 1}(y) \chi_N(\xi) \hat{g}(\xi) \, dyd\xi.
\]
Notice that when $|y| \leq \frac{3}{4}, |\xi| \geq \frac{3}{4} N, t \geq \frac{8}{N}$ and $|x| \leq \frac{3}{16} N|t|$, we have
\[
|\nabla_y \phi(y)| \gtrsim N.
\]
By applying the identity (4.12) and integrating by parts in $y$-variable $K$ times, we obtain
\[
e^{it\Delta} (\chi_{\leq 1} P_N g) = C_K \frac{|\gamma|}{|t|^\frac{d}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi(y)} \nabla_y \cdot \left( \frac{\nabla_y \phi}{|\nabla_y \phi|^2} \chi_{\leq 1}(y) \right) K^{-1}
\]
for some constant $C_K \in \mathbb{C}$. Note that by Lemma 2.5 one has
\[
\nabla_y \cdot \left( \frac{\nabla_y \phi}{|\nabla_y \phi|^2} \chi_{\leq 1}(y) \right) = \sum_{l_1, \ldots, l_K, l' \in \mathbb{R}^d; \, |l_j| \leq \frac{1}{2}; \, |l_1| + \cdots + |l_K| + |l'| = K} C_{l_1, \ldots, l_K, l'} \partial_y^{l_1} \left( \frac{\nabla_y \phi}{|\nabla_y \phi|^2} \right) \cdots \partial_y^{l_K} \left( \frac{\nabla_y \phi}{|\nabla_y \phi|^2} \right) \frac{1}{t} \nabla_y \cdot \chi_{\leq 1}(y).
\]
Then by using the estimates in (4.15), (4.18) and the definition for $\chi_{\leq 1}$, we obtain that
\[
\left| \partial_y^{l_1} \left( \frac{\nabla_y \phi}{|\nabla_y \phi|^2} \right) \cdots \partial_y^{l_K} \left( \frac{\nabla_y \phi}{|\nabla_y \phi|^2} \right) \right| \lesssim \frac{1}{|t||l_1| + \cdots + |l_K||l_1 + \cdots + |l_K| + K} \lesssim \frac{1}{(|t|N)|l_1| + \cdots + |l_K|} \lesssim N^{-K},
\]
and
\[
\partial_y^{l'} \chi_{\leq 1}(y) \lesssim \chi_{\leq 1}(y),
\]
which imply that
\[
\left| \nabla_y \cdot \left( \frac{\nabla_y \phi}{|\nabla_y \phi|^2} \right) K^{-1} \cdot \left( \frac{\nabla_y \phi}{|\nabla_y \phi|^2} \chi_{\leq 1}(y) \right) \right| \lesssim N^{-K} \chi_{\leq 1}(y).
\]
Hence it follows (4.19) and the estimate above that
\[
\left| e^{it\Delta} (\chi_{\leq 1} P_N g) \right| \lesssim_K \frac{1}{|t|^{\frac{d}{2}}} N^{-K} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{\leq 1}(y) d\eta(y) dy N(\xi) |\hat{g}(\xi)| d\xi \\
\lesssim_K \frac{1}{|t|^{\frac{d}{2}}} N^{-K} \| \chi_N \hat{g} \|_{L^1}.
\]
Moreover, since $|t| \gtrsim \frac{1}{N}$, we have
\[
\left| e^{it\Delta} (\chi_{\leq 1} P_N g) \right| \lesssim_K \frac{1}{|t|^{\frac{d}{2}}} N^{-K+\frac{d}{2}} \| \chi_N \hat{g} \|_{L^1}.
\]
Now by choosing $K$ large enough, we could obtain the desired estimate. \hfill \box

We next prove the estimate for linear flow out the ball $B(0, \frac{1}{8} N|t|)$, which is a consequence of the radial Sobolev embedding theorem.

**Lemma 4.5.** For any $\rho, \sigma \geq 1$ satisfying $\frac{1}{\rho} + \frac{d-1}{\sigma} \leq \frac{d-1}{2}$, and any $\gamma \geq 0$, the following estimate holds,
\[
\left\| \chi_{\geq \frac{1}{2} N|t|} (x) e^{it\Delta} (\chi_{\leq 1} P_N g) \right\|_{L^1_t L^2_x ((|t|\geq 8 N^{-\gamma}) \times \mathbb{R}^d)} \lesssim N - (d-1)(\frac{1}{2} - \frac{1}{\rho}) \left\| \nabla \chi_{\leq 1} P_N g \right\|_{L^2(\mathbb{R}^d)}
\]

**Proof.** It follows from the radial Sobolev embedding (see Lemma 2.1) that
\[
\left\| \chi_{\geq \frac{1}{2} N|t|} (x) e^{it\Delta} (\chi_{\leq 1} P_N g) \right\|_{L^2_t L^2_x (\mathbb{R}^d)} \lesssim (N|t|)^{-\frac{2}{d-1}} \left\| \nabla \chi_{\leq 1} P_N g \right\|_{L^2(\mathbb{R}^d)}
\]
Similarly to the proof of (4.3), we have
\[
\left\| \nabla \chi_{\leq 1} P_N g \right\|_{L^2(\mathbb{R}^d)} \lesssim N \frac{1}{N^{\frac{d}{2}} - \frac{1}{2}} \| \chi_{\leq 1} P_N g \|_{L^1(\mathbb{R}^d)}.
\]
Then it follows from Hölder’s and Young’s inequalities that
\[
\| \chi_{\leq 1} P_N g \|_{L^1(\mathbb{R}^d)} \lesssim \| \chi_{\leq 1} P_N g \|_{L^\infty(\mathbb{R}^d)} \lesssim \| \chi \hat{g} \|_{L^1(\mathbb{R}^d)},
\]
which leads to
\[
\left\| \nabla \chi_{\leq 1} P_N g \right\|_{L^2(\mathbb{R}^d)} \lesssim N \frac{1}{N^{\frac{d}{2}} - \frac{1}{2}} \| \chi \hat{g} \|_{L^1(\mathbb{R}^d)}.
\]
Combining the above three inequalities with (4.20), we obtain that
\[
\left\| \chi_{\geq \frac{1}{2} N|t|} (x) e^{it\Delta} (\chi_{\leq 1} P_N g) \right\|_{L^1_t L^2_x (\mathbb{R}^d)} \lesssim N^{-\frac{(d-2)}{2}} - (d-1)\frac{1}{\rho - \frac{1}{\sigma}} \left\| \chi \hat{g} \right\|_{L^1(\mathbb{R}^d)}.
\]
Note that
\[
\frac{1}{\rho} < (d-1)\left(\frac{1}{2} - \frac{1}{\sigma}\right),
\]
then it follows
\[
\left\| \chi_{\geq \frac{1}{2} N|t|} (x) e^{it\Delta} (\chi_{\leq 1} P_N g) \right\|_{L^1_t L^2_x (\mathbb{R}^d)} \lesssim N^{-\frac{(d-2)}{2}} - (d-1)\frac{1}{\rho - \frac{1}{\sigma}} \left\| \chi \hat{g} \right\|_{L^1(\mathbb{R}^d)},
\]
which finishes the proof. \hfill \box

By Lemmas 4.3, 4.5, we could prove the Strichartz estimates for each frequency of function $g$, which is stated as follows.
Corollary 4.6. For any $\varrho \geq 1, \sigma \geq 2$ satisfying $\frac{1}{\varrho} + \frac{d-1}{\sigma} < \frac{d-1}{2}$, we have

$$\left\| e^{it\Delta} \left( \chi_{\leq 1} P_N g \right) \right\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^d)} \lesssim N^{-\frac{d-2}{(d-1)p}} \left\| \chi N \hat{g} \right\|_{L^1}.$$  \hspace{1cm} (4.21)

Proof. First, we claim that for $\gamma \in (0, 1]$,

$$\left\| e^{it\Delta} \left( \chi_{\leq 1} P_N g \right) \right\|_{L_t^p L_x^q(\{|t| \geq 8N^{-\gamma} \} \times \mathbb{R}^d)} \lesssim N^{[(d-1)\gamma -(d-2)] \left( \frac{1}{2} - \frac{1}{\sigma} \right) - \frac{\gamma}{\varrho}} \left\| \chi N \hat{g} \right\|_{L^1}. \hspace{1cm} (4.22)$$

Indeed, it follows from Lemma 4.12 that

$$\left\| \chi_{\leq 1} F_N \right\|_{L_t^p L_x^q(\{|t| \geq 8N^{-\gamma} \} \times \mathbb{R}^d)} \lesssim \left\| (N|t|)^{-\frac{d}{2}} \chi_{\leq 1} \hat{F}_N(x) \right\|_{L_t^p L_x^q(\{|t| \geq 8N^{-\gamma} \} \times \mathbb{R}^d)} \lesssim \left\| (t)^{-\frac{d}{2} + \frac{\gamma}{2} - N^{100} + \frac{\gamma}{2}} \right\|_{L_t^p \mathbb{R}^d} \chi_N \hat{g} \lesssim N^{-50} \left\| \chi N \hat{g} \right\|_{L^1},$$

which together with Lemma 4.3 gives (4.21).

Now we recall from Lemma 4.3 that for $\gamma \in (0, 1]$,

$$\left\| e^{it\Delta} \left( \chi_{\leq 1} P_N g \right) \right\|_{L_t^p L_x^q([-8N^{-\gamma}, 8N^{-\gamma}] \times \mathbb{R}^d)} \lesssim N^{-\frac{\gamma}{2}} \left\| \chi \sim \hat{g} \right\|_{L^1}. \hspace{1cm} (4.22)$$

Choosing $\gamma = \frac{d-2}{d-1}$ and then

$$[(d-1)\gamma -(d-2)] \left( \frac{1}{2} - \frac{1}{\sigma} \right) - \frac{\gamma}{\varrho} = -\frac{\gamma}{\varrho},$$

Hence by (4.21) and (4.22), we obtain that

$$\left\| e^{it\Delta} \left( \chi_{\leq 1} P_N g \right) \right\|_{L_t^p L_x^q(\{|t| \geq 8N^{-\gamma} \} \times \mathbb{R}^d)} + \left\| e^{it\Delta} \left( \chi_{\leq 1} P_N g \right) \right\|_{L_t^p L_x^q([-8N^{-\gamma}, 8N^{-\gamma}] \times \mathbb{R}^d)} \lesssim N^{-\frac{d-2}{(d-1)p}} \left\| \chi \sim \hat{g} \right\|_{L^1},$$

which gives the desired estimates and completes the proof of the corollary. \quad \square

Proof of Proposition 4.1. Since $g$ is supported in the unit ball, we abuse the notation and write $g = \chi_{\leq 1} g$. Then by the Littlewood-Paley decomposition,

$$\left\| e^{it\Delta} g \right\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^d)} \leq \left\| e^{it\Delta} \left( \chi_{\leq 1} P_N g \right) \right\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^d)} + \sum_{N \geq 1} \left\| e^{it\Delta} \left( \chi_{\leq 1} P_N g \right) \right\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^d)}.$$

By Lemma 4.2 and Lemma 4.6, it is further controlled by

$$\left\| \chi_{\leq 1} \hat{g} \right\|_{L^1(\mathbb{R}^d)} + \sum_{N \geq 1} N^{-\frac{d-2}{(d-1)p}} \left\| \chi N \hat{g} \right\|_{L^1(\mathbb{R}^d)} \lesssim \left\| \left( \frac{d-2}{(d-1)p} + \frac{\gamma}{2} \right) \right\|_{L^1(\mathbb{R}^d)}.$$ 

This finishes the proof of the proposition. \quad \square
In this section, we introduce the Strichartz estimates for the free flow $e^{it\Delta}h$ with $h$ satisfying Assumption 1.5. The following is the main result of this section.

**Proposition 5.1.** Let $s_c > 0$, $\gamma \in [0, s_c)$ and $q_\gamma = \frac{2(d+2)}{d-2(s_c-\gamma)}$. Assume that $h$ is the function satisfying Assumption 1.5. Then

$$
\|\nabla |e^{it\Delta}h\|_{L^q_t(L^r_x)} \lesssim \epsilon^{\frac{s_c-\gamma}{d}} \|h\|_{\dot{H}^s(\mathbb{R}^d)}.
$$

The proof of this proposition is followed by the lemma stated as follows, in which we obtain the smallness of the space-time estimates.

**Lemma 5.2.** Let $h_k$ be a component of $h$ in Assumption 1.5. Under the same assumptions as in Proposition 5.1, Then

$$
\|\nabla |e^{it\Delta}h_k\|_{L^q_t(L^r_x)} \lesssim \epsilon^{\frac{s_c-\gamma}{d}} \|h_k\|_{\dot{H}^s(\mathbb{R}^d)}.
$$

**Remark 5.3.** The weak result when $\epsilon = 1$ is known from the Strichartz estimate. When $\epsilon$ is small, then the smallness for the space-time estimates is due to the weak topology of the Sobolev norm compared to the Sobolev norm, and the restriction on frequency.

**Proof of Lemma 5.2.** By the assumption of $h_k$ in Assumption 1.5, we have

$$
h_k = P_{\Omega_k} h_k,
$$

where $P_{\Omega_k}$ is the projection satisfying

$$
\tilde{P}_{\Omega_k} g(\xi) = \chi_{\Omega_k}(\xi) \hat{g}(\xi)
$$

with

$$\chi_{\Omega_k}(\xi) = \chi_{\leq \xi}(\frac{\xi}{(1+\epsilon)2^k}) - \chi_{\leq \xi}(\frac{\xi}{2^k}).$$

Note that

$$
\|\nabla |e^{it\Delta}h_k\|_{L^q_t(L^r_x)} = \|P_{\Omega_k} e^{it\Delta} P_{\Omega_k} h_k\|_{L^q_t(L^r_x)}
$$

and

$$
P_{\Omega_k} g = \mathcal{F}^{-1}(\chi_{\Omega_k}) * g.$$

Hence, it follows from Young’s inequality that

$$
\|\nabla |e^{it\Delta} P_{\Omega_k} h_k\|_{L^q_t(L^r_x)} = \|\mathcal{F}^{-1}(\chi_{\Omega_k}) * e^{it\Delta} |\nabla |h_k\|_{L^q_t(L^r_x)}
\lesssim \|\mathcal{F}^{-1}(\chi_{\Omega_k})\|_{L^r_x} \|e^{it\Delta} P_{\Omega_k} |\nabla |h_k\|_{L^q_t(L^r_x)},
$$

where $r, \tilde{r}$ are the parameters satisfying

$$\frac{1}{r} = \frac{1}{2} - \frac{2}{d q_\gamma}, \quad \frac{1}{\tilde{r}} = 1 - \frac{s_c - \gamma}{d}.$$

Note that $1 < \tilde{r} < 2$ and

$$\mathcal{F}^{-1}(\chi_{\Omega_k}) = ((1+\epsilon)2^k)^d \mathcal{F}^{-1}(\chi_{\leq 1})(1+\epsilon)2^k x - (2^k)^d \mathcal{F}^{-1}(\chi_{\leq 1})(2^k x).$$
Then we have
\[ \| \mathcal{F}^{-1}(\chi_{\Omega_k}) \|_{L^1(\mathbb{R}^d)} \leq \left\| \left( 1 + \epsilon \right) 2^k \mathcal{F}^{-1}(\chi_{\leq 1}) \left( 1 + \epsilon \right) 2^k x \right\|_{L^1(\mathbb{R}^d)} + \left\| (2^k)^d \mathcal{F}^{-1}(\chi_{\leq 1}) (2^k_x) \right\|_{L^1(\mathbb{R}^d)} \leq \| \mathcal{F}^{-1}(\chi_{\leq 1}) \|_{L^1(\mathbb{R}^d)} \lesssim 1; \]
and
\[ \| \mathcal{F}^{-1}(\chi_{\Omega_k}) \|_{L^2(\mathbb{R}^d)} = \| \chi_{\Omega_k} \|_{L^2(\mathbb{R}^d)} \lesssim \| \chi_{2^k \leq \cdot \leq (1 + \epsilon) 2^k} \|_{L^2(\mathbb{R}^d)} \lesssim \epsilon^{\frac{1}{2}} 2^{\frac{dk}{2}}. \]
which imply that
\[ \| \mathcal{F}^{-1}(\chi_{\Omega_k}) \|_{L^p(\mathbb{R}^d)} \lesssim \| \mathcal{F}^{-1}(\chi_{\Omega_k}) \|^{\frac{2}{2} - 1}_{L^1(\mathbb{R}^d)} \| \mathcal{F}^{-1}(\chi_{\Omega_k}) \|^{\frac{2}{2}}_{L^2(\mathbb{R}^d)} \lesssim \epsilon \frac{2^{(s_c - \gamma)k}}{\epsilon}. \] (5.3)
On the other hand, by Lemma 2.9 we have
\[ \| \| \nabla \|^{\gamma} e^{it\Delta} P_{\Omega_k} h_k \|_{L^\gamma_t L^q_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \| \| \nabla \|^{\gamma} h_k \|_{L^2(\mathbb{R}^d)} \lesssim 2^{-(s_c - \gamma)} \| h_k \|_{H^{s_c}(\mathbb{R}^d)}. \] (5.4)
Then it follows from (5.2)-(5.4) that
\[ \| \| \nabla \|^{\gamma} e^{it\Delta} P_{\Omega_k} h_k \|_{L^\gamma_t L^q_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \epsilon^{\frac{2^{(s_c - \gamma)k}}{\epsilon}} \| h_k \|_{H^{s_c}(\mathbb{R}^d)}. \]
Hence we finish the proof. \( \square \)

Now we turn to prove Proposition 5.1.

Proof of Proposition 5.1. Using the Littlewood-Paley characterization of Lebesgue space, we have
\[ \| \| \nabla \|^{\gamma} e^{it\Delta} h \|_{L^\gamma_t L^q_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \left\| \left( \sum_{k=0}^{\infty} \| \nabla \|^{\gamma} e^{it\Delta} h_k \|^{2}_{L^q_x(\mathbb{R} \times \mathbb{R}^d)} \right)^{\frac{1}{2}} \right. \]
\[ \lesssim \left. \left( \sum_{k=0}^{\infty} \| \nabla \|^{\gamma} e^{it\Delta} h_k \|^{2}_{L^q_t L^q_x(\mathbb{R} \times \mathbb{R}^d)} \right)^{\frac{1}{2}}. \]
Applying Lemma 5.2 we further bound it by
\[ \epsilon^{\frac{2^{(s_c - \gamma)k}}{\epsilon}} \left( \sum_{k=0}^{\infty} \| h_k \|^{2}_{H^{s_c}(\mathbb{R}^d)} \right)^{\frac{1}{2}} = \epsilon^{\frac{2^{(s_c - \gamma)k}}{\epsilon}} \| h \|_{H^{s_c}(\mathbb{R}^d)}, \]
which finishes the proof. \( \square \)

6. Proof of the main theorems

In this section, we prove Theorems 1.1 and 1.6.

6.1. The proof of Theorem 1.1. In this subsection, we mainly focus on the proof of Theorem 1.1 which is based on the estimates obtained in Section 3 and 4 and continuity argument. Before proving the theorem, we first introduce the modified incoming/outgoing components of radial function, the appropriate working spaces, the homogeneous and inhomogeneous Strichartz estimates.
6.1.1. Definitions of the modified incoming and outgoing components. First of all, we define the modified incoming and outgoing components, say $f_+$ and $f_-$, of the function $f$. For this purpose, we split the function $f$ as follows,

$$f = (1 - \chi_{\geq 1})f + (1 - P_{\geq 1})\chi_{\geq 1}f + P_{\geq 1}\chi_{\geq 1}f.$$  

**Definition 6.1.** Let the radial function $f \in S(\mathbb{R}^d)$. We define the modified outgoing component of $f$ as

$$f_+ = \frac{1}{2}(1 - \chi_{\geq 1})f + \frac{1}{2}(1 - P_{\geq 1})\chi_{\geq 1}f + (P_{\geq 1}\chi_{\geq 1}f)_{\text{out}},$$

the modified incoming component of $f$ as

$$f_- = \frac{1}{2}(1 - \chi_{\geq 1})f + \frac{1}{2}(1 - P_{\geq 1})\chi_{\geq 1}f + (P_{\geq 1}\chi_{\geq 1}f)_{\text{in}}.$$  

We note that such definitions can be extended to more general functions. It follows from the definitions that

$$f = f_+ + f_-.$$  

Hence, if $f \in H^s(\mathbb{R}^d)$, then at least one of $f_+$ and $f_-$ belongs to $H^s(\mathbb{R}^d)$.

6.1.2. Definitions of working space. For simplicity, we introduce the following working spaces. We denote $X(I)$ for $I \subset \mathbb{R}^+$ to be the space under the norms

$$\|h\|_{X(I)} := \|h\|_{L_t^\infty H_x^s(I \times \mathbb{R}^d)} + \|h\|_{L_t^2 L_x^p(I \times \mathbb{R}^d)} + \|h\|_{L_t^{2(d+2)p/(d+2)} L_x^{2d(d+2)p/d-8}(I \times \mathbb{R}^d)} + \|\nabla^s h\|_{L_t^{2(d+2)p/(d+2)} L_x^{2d(d+2)p/d-8}(I \times \mathbb{R}^d)}.$$  

Moreover, we denote $X_0(I)$ for $I \subset \mathbb{R}^+$ to be the space under the norms

$$\|h\|_{X_0(I)} := \|h\|_{L_t^{\infty-} L_x^{2+}(I \times \mathbb{R}^d)} + \|h\|_{L_t^{2(d+2)p/(d+2)} L_x^{2d(d+2)p/d-8}(I \times \mathbb{R}^d)} + \|h\|_{L_t^{2(d+2)p/(d+2)} L_x^{2d(d+2)p/d-8}(I \times \mathbb{R}^d)}.$$  

Then we denote $Y(I)$ to be the space under the norms

$$\|h\|_{Y(I)} := \|h\|_{X(I)} + \|h\|_{X_0(I)}.$$  

Moreover, we need the spaces, $Z_f(I)$ and $Z_g(I)$ which are tailor-made for $f_{\text{out}}$ and $g$ respectively. Before defining the spaces, let $\epsilon$ be a fixed small positive constant and $\sigma_0, \gamma$ be the constants satisfying

$$\frac{2d+1}{4d-2} - \frac{2}{d} \leq \sigma_0 \leq \gamma = \frac{d-1}{2d-1} - \epsilon.$$  

Also, we denote $\infty^- = \frac{1}{\epsilon}$. Now we define $Z_f(I)$ to be the space under the norm

$$\|h\|_{Z_f(I)} := \|h\|_{L_t^{2p} L_x^{2p}(I \times \mathbb{R}^d)} + \sup_{q \in [\frac{d}{2} - \epsilon] \cap \mathbb{N}} \|h\|_{L_t^{2q} L_x^{\frac{2q}{d}}(I \times \mathbb{R}^d)} + \|h\|_{L_t^{\infty-} L_x^{s+}(I \times \mathbb{R}^d)} + \|h\|_{L_t^{2d} L_x^{2d}(I \times \mathbb{R}^d)};$$  

and we define $Z_g(I)$ to be the space under the norm

$$\|h\|_{Z_g(I)} := \|h\|_{L_t^{2d} L_x^{2d}(I \times \mathbb{R}^d)} + \|h\|_{L_t^{2d} L_x^{\infty}(I \times \mathbb{R}^d)} + \|h\|_{L_t^{2d} L_x^{\infty}(I \times \mathbb{R}^d)} + \sup_{q \in [\frac{d}{2} - \epsilon] \cap \mathbb{N}} \|\nabla^{s-\gamma} h\|_{L_t^{q} L_x^{\frac{q}{d}}(I \times \mathbb{R}^d)}.$$.  

6.1.3. The linear estimates. For convenience, we rewrite the initial data $u_0$ as

$$u_0 = \psi_0 + (P_{\geq 1} \chi_{\geq 1} f)_{\text{out}} + g,$$

where

$$\psi_0 = \frac{1}{2} \chi_{\leq 1} f + \frac{1}{2} P_{\leq 1} \chi_{\geq 1} f.$$  

By (1.5), we claim that $\psi_0 \in H^{s_c}(\mathbb{R}^d)$ with

$$\|\psi_0\|_{H^{s_c}(\mathbb{R}^d)} \lesssim \delta_0. \quad (6.1)$$  

Indeed,

$$\|\chi_{\leq 1} f\|_{L^2(\mathbb{R}^d)} \lesssim \|\chi_{\leq 1} f\|_{L^{p_c}(\mathbb{R}^d)} \lesssim \|\chi_{\leq 1} f\|_{H^{s_c}(\mathbb{R}^d)}.$$

Hence, $\chi_{\leq 1} f \in H^{s_c}(\mathbb{R}^d)$ and

$$\|\chi_{\leq 1} f\|_{H^{s_c}(\mathbb{R}^d)} \lesssim \delta_0.$$

Moreover, since the index $s_1$ in (1.5) is smaller than $s_c$, one has

$$\|P_{\leq 1} \chi_{\geq 1} f\|_{H^{s_c}(\mathbb{R}^d)} \lesssim \|\chi_{\geq 1} f\|_{H^{s_1}(\mathbb{R}^d)} \lesssim \delta_0.$$

Hence we obtain the claim (6.1).

We still need some linear estimates related to $(P_{\geq 1} \chi_{\geq 1} f)_{\text{out}}$ and $g$. Denote by $f_{\text{out}, L}$ the linear flow of

$$\left\{ \begin{array}{l}
i \partial_t \phi + \Delta \phi = 0, \\
\phi(0, x) = (P_{\geq 1} \chi_{\geq 1} f)_{\text{out}}(x). \end{array} \right.$$

that is, $f_{\text{out}, L}(t) = e^{it\Delta} (P_{\geq 1} \chi_{\geq 1} f)_{\text{out}}$. Similarly, we denote $g_L(t) = e^{it\Delta} g$. Next we shall introduce the estimates of $f_{\text{out}, L}(t)$ and $g_L$ in spaces $Z_f(I)$ and $Z_g(I)$, respectively. Such estimates can be obtained by using the results in Subsections 3 and 4.

**Lemma 6.2.** Suppose that $f$ is a radial function verifying (1.5), then

$$\|f_{\text{out}, L}\|_{Z_f(\mathbb{R}^+)} \lesssim \delta_0.$$

**Proof.** Due to the decomposition in the beginning of Section 3 we write

$$f_{\text{out}, L} = (P_{\geq 1} \chi_{\geq 1} f)_{\text{out}, L}^I + (P_{\geq 1} \chi_{\geq 1} f)_{\text{out}, L}^{II}.$$

Let us first deal with the estimates for $(P_{\geq 1} \chi_{\geq 1} f)_{\text{out}, L}^I$. Notice that it follows from Lemma 3.1 that

$$\|
abla^s (P_{\geq 1} \chi_{\geq 1} f)_{\text{out}, L}^I \|_{L^2_t L^{p_c} \times \mathbb{R}^d}^2 + \left( (P_{\geq 1} \chi_{\geq 1} f)_{\text{out}, L}^I \right)_{L^{p_c} H^{s_c}} \lesssim \|P_{\geq 1} \chi_{\geq 1} f\|_{H^{-1}(\mathbb{R}^d)},$$

which combined with interpolation implies

$$\|
abla (P_{\geq 1} \chi_{\geq 1} f)_{\text{out}, L}^I \|_{L^2_t L^{p_c}} + \sup_{q \in [2, \frac{d+2}{d-2}]} \left( P_{\geq 1} \chi_{\geq 1} f \right)_{\text{out}, L}^I \|_{L^q H^{s_c}} \lesssim \|P_{\geq 1} \chi_{\geq 1} f\|_{H^{-1}(\mathbb{R}^d)}.$$

(6.2)
Now we turn to the estimates for \( (P_{\geq 1}^{\gamma_{\geq 1}} f)^{II}_{out,L} \). We need to prove that
\[
\left\| (P_{\geq 1}^{\gamma_{\geq 1}} f)^{II}_{out,L} \right\|_{L^p_t L^q_x (\mathbb{R}^+ \times \mathbb{R}^d)} + \sup_{q \in [2, \frac{2d}{d-1}]} \left\| (P_{\geq 1}^{\gamma_{\geq 1}} f)^{II}_{out,L} \right\|_{L^\infty_t L^q_x (\mathbb{R}^+ \times \mathbb{R}^d)} + \left\| (P_{\geq 1}^{\gamma_{\geq 1}} f)^{II}_{out,L} \right\|_{L^\infty_t L^q_x (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \left\| P_{\geq 1}^{\gamma_{\geq 1}} f \right\|_{H^{s_1} (\mathbb{R}^d)},
\]
which give (6.3). Hence, we finish the proof. \( \square \)

**Lemma 6.3.** Suppose that \( g \) is a radial function verifying (1.6), then
\[
\left\| gL \right\|_{L^q (\mathbb{R})} \lesssim \delta_0.
\]

**Proof.** Let \((\varrho, \sigma)\) be one of the following pairs
\[
\left( \frac{2(d+2)}{d}, \frac{2(d+2)}{d} \right), \quad \left( \frac{(d+2)p}{2}, \frac{(d+2)p}{2} \right), \quad (2p, dp), \quad (\infty-, q)
\]
with \(q \in (2, \infty)\). It is easy to see that every pair above satisfies \( \frac{2}{\varrho} + \frac{d}{\sigma} \leq \frac{d}{2} \) and \( \frac{1}{\varrho} + \frac{d-1}{\sigma} < \frac{d-1}{2} \). Then by Proposition 4.1 for the constant \( \varepsilon_0 \) in \( s_2 \) (see definition above Theorem 1.1), we have that
\[
\left\| gL \right\|_{L^\frac{2(d+2)}{d} (\mathbb{R} \times \mathbb{R}^d)} + \left\| gL \right\|_{L^\frac{2(d+2)p}{2} (\mathbb{R} \times \mathbb{R}^d)} + \left\| gL \right\|_{L^{2p} L^{dp} (\mathbb{R} \times \mathbb{R}^d)} + \sup_{q \in [2, \frac{2d}{d-1}]} \left\| gL \right\|_{L^\infty_t L^q_x (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \left\| \xi^{-\varepsilon_0} \right\|_{L^1 (\mathbb{R}^d)};
\]
and for any \( q \geq \frac{2d}{d-2} \),
\[
\left\| \nabla x^{-2} gL \right\|_{L^\infty_t L^q_x (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \left\| \xi^{x^{-2} - \frac{d-2}{2(d-1)} +} \right\|_{L^1 (\mathbb{R}^d)}.
\]
Let the constant \( \varepsilon_0 \) can also be chosen such that
\[
s_c - \gamma - \frac{d-2}{2(d-1)} = s_c - \frac{d-2}{2(d-1)} - \frac{d-1}{2d-1} + \varepsilon_0.
\]
Then by assumptions on \( g \) (see (1.6)), we finish the proof. \( \square \)
6.1.4. The proof of Theorem 1.1. Denote by \( \psi = u - f_{\text{out}, L} - g_L \), then \( \psi \) obeys the following equation,

\[
\begin{aligned}
   i \partial_t \psi + \Delta \psi &= |u|^p u, \\
   \psi(0, x) &= \psi_0(x).
\end{aligned}
\]

It follows from the Duhamel formula that

\[
\psi(t) = e^{-it\Delta} \psi_0 + \int_0^t e^{-i(t-s)\Delta} (|u|^p u) \, ds. \tag{6.4}
\]

By using the standard Strichartz estimate given in Lemma 2.7 and (6.1), we have

\[
\|e^{-it\Delta} \psi_0\|_{Y(\mathbb{R})} \lesssim \|\psi_0\|_{H^{s_c}(\mathbb{R}^d)} \lesssim \delta_0. \tag{6.5}
\]

It remains to consider the estimates for nonlinear part. The following lemma deals with the estimate of the nonlinear term in \( X_0(I) \).

**Lemma 6.4.** Assume that \( d = 3, 4, 5, p \geq \frac{4}{3}, 0 \in I \) and \( \psi \in X(I) \), then

\[
\left\| \int_0^t e^{-i(t-s)\Delta} (|u|^p u) \, ds \right\|_{X_0(I)} \lesssim \|\psi\|_{X_0(I)} \left\| |\psi| \right\|_{X(I)} + \delta_0 \|\psi\|_{X(I)}^p + \delta_0^p \left\| \psi \right\|_{X_0(I)} + \delta_0^{p+1}.
\]

**Proof.** Notice that by using Lemmas 6.2 and 6.3, we have

\[
\| f_{\text{out}, L} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} + \| f_{\text{out}, L} \|_{L_t^\infty L_x^2 L_y^p(I \times \mathbb{R}^d)} \lesssim \delta_0 \tag{6.6}
\]

and

\[
\| g_L \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} + \| g_L \|_{L_t^\infty L_x^2 L_y^p(I \times \mathbb{R}^d)} \lesssim \delta_0. \tag{6.7}
\]

Since \( u = f_{\text{out}, L} + g_L + \psi \), it follows from (6.6) and (6.7) that

\[
\| u \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim \| \psi \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} + \| f_{\text{out}, L} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} + \| g_L \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim \| \psi \|_{X_0(I)} + \delta_0
\]

and

\[
\| u \|_{L_t^\infty L_x^2 L_y^p(I \times \mathbb{R}^d)} \lesssim \| \psi \|_{L_t^\infty L_x^2 L_y^p(I \times \mathbb{R}^d)} + \| f_{\text{out}, L} \|_{L_t^\infty L_x^2 L_y^p(I \times \mathbb{R}^d)} + \| g_L \|_{L_t^\infty L_x^2 L_y^p(I \times \mathbb{R}^d)} \lesssim \| \psi \|_{X(I)} + \delta_0.
\]

Then by applying Lemma 2.7, we have

\[
\left\| \int_0^t e^{-i(t-s)\Delta} (|u|^p u) \, ds \right\|_{X_0(I)} \lesssim \left\| |u|^p u \right\|_{L_t^\infty L_x^2 L_y^p(I \times \mathbb{R}^d)} \lesssim \left\| u \right\|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \left\| |u|^p \right\|_{L_t^\infty L_x^2 L_y^p(I \times \mathbb{R}^d)} \lesssim \| \psi \|_{X_0(I)} \| \psi \|_{X(I)}^p + \delta_0 \| \psi \|_{X(I)}^p + \delta_0^p \| \psi \|_{X_0(I)} + \delta_0^{p+1}.
\]

which implies the desired estimate. Hence we finish the proof. \( \square \)

We still need to estimate the nonlinear term in \( X(I) \), which require to deal with the high-order derivatives. By Lemma 2.2, we expand the nonlinearity and write

\[
|\nabla|^s (|u|^p u) = O ( (|\psi|^p + |f_{\text{out}, L}|^p + |g_L|^p) (|\nabla|^s \psi + |\nabla|^s f_{\text{out}, L} + |\nabla|^s g_L) ) , \tag{6.8}
\]
for any $s \geq 0$. We shall consider each term separately. Here we abuse the notations and denote $O(f_1f_2)$ to be the product of the functions that it holds in the sense of Hölder’s inequality, that is,
\[
\|O(f_1f_2)\|_{L^q} \lesssim \|f_1\|_{L^{q_1}} \|f_2\|_{L^{q_2}}, \quad \text{with} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad q, q_1, q_2 \in (1, +\infty).
\]

To consider the estimates of nonlinear term in $X(I)$, it follows from the Strichartz estimates in radial case (see Lemma 2.9) that
\[
\left\| \int_0^t e^{i(t-s)\Delta} (|u|^p u) \, ds \right\|_{X(I)} \lesssim \left\| |\nabla|^s \gamma |u|^p u \right\|_{L_t^\infty L_x^{\frac{4d+2}{d-1}}(\mathbb{R}^d)}(I\times\mathbb{R}^d), \tag{6.9}
\]
where $\gamma = \frac{d-1}{2d-1}$. The estimates of the right hand side of (6.9) would be done in the following three lemmas.

**Lemma 6.5.** Assume that $d = 3, 4, 5$, $p \geq \frac{d}{2}$, $0 \in I$ and $\psi \in X(I)$. Then
\[
\left\|(|\psi|^p + |f_{out,L}|^p + |gL|^p)|\nabla|^s \gamma \psi| \right\|_{L_t^{\frac{4d+2}{d-1}}L_x^{\frac{4d+2}{d-1}}(\mathbb{R}^d)} \lesssim \delta_0^{p+1} + \|\psi\|_{X(I)}^{p+1}.
\]

**Proof.** Notice that it follows from Hölder’s inequality that
\[
\left\|(|\psi|^p + |f_{out,L}|^p + |gL|^p)|\nabla|^s \gamma \psi| \right\|_{L_t^{\frac{4d+2}{d-1}}L_x^{\frac{4d+2}{d-1}}(\mathbb{R}^d)} \lesssim \left\| |\nabla|^s \gamma \psi| \right\|_{L_t^{\infty}L_x^{\frac{4d+2}{d-1}}(\mathbb{R}^d)}(\|\psi\|_{L_t^{2p}L_x^{dp}(\mathbb{R}\times\mathbb{R}^d)} + \|f_{out,L}\|_{L_t^{2p}L_x^{dp}(\mathbb{R}\times\mathbb{R}^d)} + \|gL\|_{L_t^{2p}L_x^{dp}(\mathbb{R}\times\mathbb{R}^d)}),
\]
where $q_1$ is the parameter satisfying
\[
\frac{1}{q_1} = \frac{1}{d} - \frac{d-1}{d(2d-1)} +
\]
and
\[
\left\| |\nabla|^s \gamma \psi| \right\|_{L_t^{\infty}L_x^{\frac{4d+2}{d-1}}(\mathbb{R}^d)} \lesssim \left\| |\nabla|^s \psi| \right\|_{L_t^{2p}L_x^{dp}(\mathbb{R}^d)}.
\]

Thus we have
\[
\left\|(|\psi|^p + |f_{out,L}|^p + |gL|^p)|\nabla|^s \gamma \psi| \right\|_{L_t^{\frac{4d+2}{d-1}}L_x^{\frac{4d+2}{d-1}}(\mathbb{R}^d)} \lesssim \left\| |\nabla|^s \psi| \right\|_{L_t^{\infty}L_x^{\frac{4d+2}{d-1}}(\mathbb{R}^d)}(\|\psi\|_{L_t^{2p}L_x^{dp}(\mathbb{R}\times\mathbb{R}^d)} + \|f_{out,L}\|_{L_t^{2p}L_x^{dp}(\mathbb{R}\times\mathbb{R}^d)} + \|gL\|_{L_t^{2p}L_x^{dp}(\mathbb{R}\times\mathbb{R}^d)}).
\]

On the other hand, it follows from Lemma 6.2 and Lemma 6.3 that
\[
\|f_{out,L}\|_{L_t^{2p}L_x^{dp}(\mathbb{R}\times\mathbb{R}^d)} + \|gL\|_{L_t^{2p}L_x^{dp}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \delta_0,
\]

Then we could obtain the desired estimate by the Cauchy-Schwartz inequality. This gives the proof of the lemma.

**Lemma 6.6.** Assume that $d = 3, 4, 5$, $p \geq \frac{d}{2}$, $0 \in I$ and $\psi \in Y(I)$. Then
\[
\left\|(|\psi|^p + |f_{out,L}|^p + |gL|^p)|\nabla|^s \gamma f_{out,L}| \right\|_{L_t^{\frac{4d+2}{d-1}}L_x^{\frac{4d+2}{d-1}}(\mathbb{R}^d)} \lesssim \delta_0^{p+1} + \delta_0 \|\psi\|_{X(I)}^{p+1}.
\]
Proof. According to Lemma 6.2, we split \( f_{out,L} \) into two parts,
\[
f_{out,L} = f_{out,L}^I + f_{out,L}^{II},
\]
with
\[
\| f_{out,L}^I \|_{L_t^\infty H_x^{s-c} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \delta_0,
\]
and
\[
\| f_{out,L}^{II} \|_{L_t^2 H_x^{s-c-\gamma} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \delta_0.
\]

For \( f_{out,L}^I \), similarly to the proof of Lemma 6.3, we obtain that
\[
\left\| (|\psi|^p + |f_{out,L}|^p + |g_L|^p) |\nabla|^{s-c} f_{out,L}^I \right\|_{L_t^2 L_x^\frac{4d-p}{p+2} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \delta_0 |\psi|_{X(I)}^p.
\]

As for \( f_{out,L}^{II} \), it follows from Hölder’s inequality that
\[
\left\| (|\psi|^p + |f_{out,L}|^p + |g_L|^p) |\nabla|^{s-c} f_{out,L}^{II} \right\|_{L_t^2 L_x^\frac{4d-p}{p+2} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \delta_0 |\psi|_{X(I)}^p.
\]

Note that \( \| \psi \|_{L_t^\infty L_x^\frac{dp}{d} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \| \psi \|_{L_t^\infty H_x^{s-c}(\mathbb{R}^d)} \). Therefore, by using Lemma 6.2 and Lemma 6.3, we get
\[
\left\| (|\psi|^p + |f_{out,L}|^p + |g_L|^p) |\nabla|^{s-c} g_L \right\|_{L_t^2 L_x^\frac{4d-p}{p+2} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \delta_0 |\psi|_{X(I)}^p,
\]
which combined with the estimate for \( f_{out,L}^I \) implies the desired estimate. \( \square \)

Lemma 6.7. Assume that \( d = 3, 4, 5, p \geq \frac{4}{d}, 0 \in I \) and \( \psi \in Y(I) \). Then
\[
\left\| (|\psi|^p + |f_{out,L}|^p + |g_L|^p) |\nabla|^{s-c} g_L \right\|_{L_t^2 L_x^\frac{4d-p}{p+2} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \delta_0 |\psi|_{X(I)}^p.
\]

Proof. By using Hölder’s inequality, we have
\[
\left\| (|\psi|^p + |f_{out,L}|^p + |g_L|^p) |\nabla|^{s-c} g_L \right\|_{L_t^2 L_x^\frac{4d-p}{p+2} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \left( \left\| |\nabla|^{s-c} g_L \right\|_{L_t^2 L_x^\frac{q_3}{2} (\mathbb{R}^d)} \right) \left( \left\| |\psi|^p \right\|_{L_t^\frac{4d-p}{d} L_x^1 (\mathbb{R}^+ \times \mathbb{R}^d)} + \left\| f_{out,L} \right\|_{L_t^\frac{4d-p}{d} L_x^1 (\mathbb{R}^+ \times \mathbb{R}^d)} + \left\| g_L \right\|_{L_t^\frac{4d-p}{d} L_x^1 (\mathbb{R}^+ \times \mathbb{R}^d)} \right)
\]
with
\[
q_3 = \frac{d(Ad - 2)}{2d^2 - 7d + 4} - .
\]

It is easy to see that \( q_3 > \frac{2d}{d^2} \) when \( d \geq 3 \). Then by Lemma 6.3 and interpolation, we obtain
\[
\left\| |\nabla|^{s-c} g_L \right\|_{L_t^2 L_x^\frac{q_3}{2} (\mathbb{R}^d)} + \left\| g_L \right\|_{L_t^\frac{4d-p}{d} L_x^1 (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \delta_0.
\]
Notice that it follows from Lemma 6.2 that
\[ \| f_{\text{out}, L} \|_{L_t^\infty L_x^{4d} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \delta_0. \]
Hence we have
\[ \left\| (|\psi|^p + f_{\text{out}, L}^p + |gL|^p)|\nabla|^{s_{\gamma}} gL \right\|_{L_t^{\frac{4d}{d-2}} L_x^{\frac{4d}{d+2}} (\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \delta_0^{p+1} + \delta_0 \| \psi \|_{X(I)}^p, \]
which finishes the proof. \(\square\)

Now we turn to close the estimate in \( Y(I) \). It follows from the Duhamel formula \(6.4\) that for any \( I \subset \mathbb{R}^+ \),
\[ \| \psi \|_{Y(I)} \lesssim \| e^{it\Delta} \psi_0 \|_{Y(\mathbb{R}^+)} + \left\| \int_0^t e^{i(t-s)\Delta} (|u|^p u) \, ds \right\|_{X_0(I)} + \left\| \int_0^t e^{i(t-s)\Delta} (|u|^p u) \, ds \right\|_{X(I)}. \]
By using \(6.1\) and Lemma 6.3, Lemma 6.7 we obtain that
\[ \| \psi \|_{Y(I)} \lesssim \delta_0 + \| \psi \|_{Y(I)}^{p+1} + \delta_0 \| \psi \|_{Y(I)}^p + \delta_0^5 \| \psi \|_{Y(I)} + \delta_0^{p+1}, \]
where the implicit constant is independent on \( I \). Then by continuity argument, we prove the global existence of the solution \( \psi \) and
\[ \| \psi \|_{Y(\mathbb{R}^+)} \lesssim \delta_0, \]
We consider the scattering. Denote by
\[ \psi_0^+ = \psi_0 + \int_0^{+\infty} e^{-is\Delta} (|u|^p u) \, ds. \]
It follows
\[ \psi(t) - e^{it\Delta} \psi_0^+ = \int_t^{+\infty} e^{i(s-t)\Delta} (|u|^2 u) \, ds. \]
Then using Lemma 2.7 and the argument as above, we obtain that
\[ \| \psi(t) - e^{it\Delta} \psi_0^+ \|_{H^{s_c}} \lesssim \| \psi \|_{Y([t, +\infty))}^{p+1} \left( \| f_{\text{out}, L} \|_{Z_f([t, +\infty))} + \| gL \|_{Z_g([t, +\infty))} \right) \| \psi \|_{Y([t, +\infty))}^p \]
\[ + \left( \| f_{\text{out}, L} \|_{Z_f([t, +\infty))} + \| gL \|_{Z_g([t, +\infty))} \right) \| \psi \|_{Y([t, +\infty))} \]
\[ + \| f_{\text{out}, L} \|_{Z_f([t, +\infty))} \| gL \|_{Z_g([t, +\infty))}^{p+1}. \]
Since
\[ \| \psi \|_{Y(\mathbb{R}^+)} + \| f_{\text{out}, L} \|_{Z_f(\mathbb{R}^+)} + \| gL \|_{Z_g(\mathbb{R}^+)} < +\infty, \]
we have
\[ \| \psi \|_{Y([t, +\infty))} + \| f_{\text{out}, L} \|_{Z([t, +\infty))} + \| gL \|_{Z_g([t, +\infty))} \rightarrow 0, \quad \text{as} \quad t \rightarrow +\infty. \]
Therefore
\[ \| \psi(t) - e^{it\Delta} \psi_0^+ \|_{H^{s_c}} \rightarrow 0, \quad \text{as} \quad t \rightarrow +\infty. \]
Setting
\[ u_{0+} = (P_{\geq 1} \chi_{\geq 1} f)_{\text{out}} + g + \psi_0^+, \]
we obtain \(1.7\) and thus finish the proof of Theorem \(1.1\).
6.2. **Proof of Theorem 1.6** We denote $h_{k,L}$ to be the linear flow of
\[
\begin{cases}
i\partial_t \phi + \Delta \phi = 0, \\
\phi(0, x) = h_k(x).
\end{cases}
\]
that is, $h_{k,L}(t) = e^{it\Delta}h_k$. Moreover,
\[
h_L = \sum_{k=k_0}^{+\infty} h_{k,L}.
\]
Then it follows from Proposition 5.1 and (1.9) that for $\gamma \in [0, s_c)$,
\[
\|\nabla^\gamma h_L\|_{L^{q\gamma}_t(I \times \mathbb{R}^d)} \lesssim \varepsilon \frac{\|h\|_{H^{s_c}}}{\varepsilon \frac{\|h\|_{H^{s_c}}}}.
\]
Now consider $\psi = u - h_L$, which obeys the following equation,
\[
\begin{cases}
i\partial_t \psi + \Delta \psi = |u|^p u, \\
\psi(0, x) = \psi_0.
\end{cases}
\]
To solve the above equation, we first introduce the working space as follows. We denote $X(I)$ for $I \subset \mathbb{R}$ to be the space under the norm
\[
\|\psi\|_{X(I)} := \varepsilon^{\frac{2\alpha}{d}} \sup_{\gamma \in [0, s_c-\gamma_0]} \|\nabla^\gamma \psi\|_{L^{q\gamma}_t(I \times \mathbb{R}^d)} + \|\nabla^\gamma \psi\|_{L^{q}_{t}L^{q}_{x}(I \times \mathbb{R}^d)} + \|\nabla^\gamma \psi\|_{L^{q}_{t}L^{\infty}_{x}(I \times \mathbb{R}^d)},
\]
where $\gamma_0$ is a parameter decided later and
\[
q_\gamma = \frac{2(d + 2)}{d - 2(s_c - \gamma)}.
\]
It follows from the Duhamel formula that
\[
\psi(t) = e^{it\Delta} \psi_0 + \int_0^t e^{i(t-s)\Delta} (|u|^p u) \, ds.
\]
Then by using Lemma 2.7, Lemma 2.8 and (6.11), we have that for any $I \subset \mathbb{R}$ and $\gamma \in [0, s_c - \gamma_0]$,
\[
\|\nabla^\gamma \psi\|_{L^{q\gamma}_t(I \times \mathbb{R}^d)} \lesssim \|\psi_0\|_{H^{s_c}(\mathbb{R}^d)} + \|\nabla^\gamma(|u|^p u)\|_{L^{q_1}_tL^{q_2}_{x}(I \times \mathbb{R}^d)},
\]
where $q$ satisfy
\[
\frac{1}{q_1} = \frac{d}{2} - \frac{d + 1}{q_\gamma} = \frac{d}{2(d + 2)} + (d + 1)\varepsilon.
\]
Here and in the following, we denote $\varepsilon = s_c - \gamma$ and $\varepsilon_0 = s_c - \gamma_0$ for short. Moreover, we choose $\varepsilon_0$ small enough such that for any $\varepsilon \in [0, \varepsilon_0]$, $q_1 > 1$. By using the H"older inequality and the fractional Leibniz rule in Lemma 2.7, we obtain
\[
\|\nabla^\gamma(|u|^p u)\|_{L^{q_1}_tL^{q_2}_{x}(I \times \mathbb{R}^d)} \lesssim \|\nabla^\gamma u\|_{L^{q_3}_t(I \times \mathbb{R}^d)}\|u\|_{L^{l}_tL^{p}_{x}(I \times \mathbb{R}^d)}^p,
\]
where $q_2, r_2$ satisfy
\[
\frac{1}{q_2} = \frac{2}{(d + 2)p} - \frac{d}{p(d + 2)}\varepsilon; \quad \frac{1}{r_2} = \frac{2}{(d + 2)p} + \frac{2}{p(d + 2)}\varepsilon.
\]
It follows from interpolation and the Sobolev embedding that
\[
\left\| u \right\|_{L^{q_2}_{t} L^{\infty}_{x}(\mathbb{R}^d)} \lesssim \left\| u \right\|_{L^{p_1}_{t} L^{\infty}_{x}(\mathbb{R}^d)}^{\frac{1}{2}} \left\| u \right\|_{L^{p_2}_{t} L^{\infty}_{x}(\mathbb{R}^d)}^{\frac{1}{2}} \left\| u \right\|_{L^{d+2}_{t} L^{2}_{x}(\mathbb{R}^d)} \lesssim \left\| u \right\|_{L^{p_1}_{t} L^{\infty}_{x}(\mathbb{R}^d)}^{\frac{1}{2}} \left\| u \right\|_{L^{p_2}_{t} L^{\infty}_{x}(\mathbb{R}^d)}^{\frac{1}{2}} \left\| u \right\|_{L^{d+2}_{t} L^{2}_{x}(\mathbb{R}^d)},
\]
where \( \theta = \frac{d(d+2)}{2} \varepsilon < 1 \) (after choosing \( \varepsilon_0 \) small enough). Notice that by (1.9) and (6.10), one has
\[
\left\| u \right\|_{L^{\infty}_{t} H^{s}_{x}(\mathbb{R}^d)} \lesssim \left\| h_L \right\|_{L^{p_1}_{t} H^{s}_{x}(\mathbb{R}^d)} + \left\| \psi \right\|_{L^{p_1}_{t} H^{s}_{x}(\mathbb{R}^d)} \lesssim \varepsilon^{-\alpha_0} + \left\| \psi \right\|_{X(I)}
\]
and
\[
\left\| u \right\|_{L^{(d+2)p}_{t} L^{2}_{x}(\mathbb{R}^d)} \lesssim \left\| h_L \right\|_{L^{(d+2)p}_{t} L^{2}_{x}(\mathbb{R}^d)} + \left\| \psi \right\|_{L^{(d+2)p}_{t} L^{2}_{x}(\mathbb{R}^d)} \lesssim \varepsilon^{\frac{2}{p_1}} - \alpha_0 + \varepsilon^{\frac{2}{p_2}} \left\| \psi \right\|_{X(I)}.
\]
Then,
\[
\left\| u \right\|_{L^{q_2}_{t} L^{\infty}_{x}(\mathbb{R}^d)} \lesssim \varepsilon^{\frac{2}{p_1}(1-\theta) - \alpha_0} + \varepsilon^{-\alpha_0 + \frac{2}{p_1}(1-\theta)} \left\| \psi \right\|_{X(I)} \left( \varepsilon^{\frac{2}{p_1}(1-\theta) - \alpha_0} + \varepsilon^{-\alpha_0 + \frac{2}{p_1}(1-\theta)} \left\| \psi \right\|_{X(I)} \right)^{\frac{1}{1-p}}.
\]
which combined with (6.10) and (6.13) gives
\[
\left\| \nabla \right\|_{L^{p_1}_{t} L^{\infty}_{x}(\mathbb{R}^d)} \lesssim \varepsilon^{\frac{2}{p_1}(1-\theta) - \alpha_0} + \varepsilon^{-\alpha_0 + \frac{2}{p_1}(1-\theta)} \left\| \psi \right\|_{X(I)} \left( \varepsilon^{\frac{2}{p_1}(1-\theta) - \alpha_0} + \varepsilon^{-\alpha_0 + \frac{2}{p_1}(1-\theta)} \left\| \psi \right\|_{X(I)} \right)^{\frac{1}{1-p}}.
\]
Thus by (6.12), (6.14) and (1.10), we have
\[
\varepsilon^{-\frac{2}{p_1}} \left\| \nabla \right\|_{L^{p_1}_{t} L^{\infty}_{x}(\mathbb{R}^d)} \lesssim \varepsilon^{\frac{1}{1-p}} + \left\| \psi \right\|_{X(I)} \left( \varepsilon^{\frac{2}{p_1}(1-\theta) - \alpha_0} + \varepsilon^{-\alpha_0 + \frac{2}{p_1}(1-\theta)} \left\| \psi \right\|_{X(I)} \right)^{\frac{1}{1-p}}.
\]
Choosing \( \alpha_0 \) and \( \varepsilon_0 \) small enough, and using the Cauchy-Schwartz inequality, we obtain that
\[
\varepsilon^{-\frac{2}{p_1}} \left\| \nabla \right\|_{L^{p_1}_{t} L^{\infty}_{x}(\mathbb{R}^d)} \lesssim \varepsilon^{1 - \frac{2}{p_1}} + \left\| \psi \right\|_{X(I)} + \left\| \psi \right\|_{X(I)}^{1+p}.
\]
For the other two norms in \( X(I) \), one has
\[
\left\| \nabla \right\|_{L^{p_1}_{t} L^{\infty}_{x}(\mathbb{R}^d)} \lesssim \left\| \psi \right\|_{X(I)} \lesssim \left\| \psi \right\|_{X(I)} + \left\| \psi \right\|_{X(I)}^{1+p}.
\]
Notice that by (6.10), one has
\[
\left\| \nabla \right\|_{L^{p_1}_{t} L^{\infty}_{x}(\mathbb{R}^d)} \lesssim \left\| \nabla \right\|_{L^{p_1}_{t} L^{\infty}_{x}(\mathbb{R}^d)} + \left\| \nabla \right\|_{L^{p_1}_{t} L^{\infty}_{x}(\mathbb{R}^d)} \lesssim \varepsilon^{-\alpha_0} + \left\| \psi \right\|_{X(I)}.
\]
and
\[
\left\| u \right\|_{L^{(d+2)p}_{t} L^{2}_{x}(\mathbb{R}^d)} \lesssim \varepsilon^{\frac{2}{p_1}} - \alpha_0 + \varepsilon^{\frac{2}{p_2}} \left\| \psi \right\|_{X(I)}.
\]
Then by using similar argument as above, we have
\[
\| \nabla^k \psi \|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)} + \| \psi \|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)}^{(d+2)p} \lesssim \epsilon + \epsilon \frac{\| \psi \|_{X(I)}}{\epsilon} + \| \psi \|_{X(I)}^{1+p}.
\]

(6.16)

It follows from (6.15) and (6.16) that
\[
\| \psi \|_{X(I)} \lesssim \epsilon^{1-\frac{2}{d}} + \epsilon + \epsilon \frac{\| \psi \|_{X(I)}}{\epsilon} + \| \psi \|_{X(I)}^{1+p}.
\]

Hence, for some \(a_0 > 0\),
\[
\| \psi \|_{X(I)} \lesssim \epsilon^{a_0} + \| \psi \|_{X(I)}^{1+p}.
\]

Therefore, by the bootstrap argument, we obtain that
\[
\| \psi \|_{X(I)} \lesssim \epsilon^{a_0}.
\]

One can see that the estimate above is uniformly in interval \(I \subset \mathbb{R}\), which gives the global existence of the solution. Further, the scattering statement can be proved in the same way as in the proof of Theorem 1.1. This finishes the proof of Theorem 1.6.

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References

[1] M. Beceanu, Q. Deng, A. Soffer and Y. Wu, Large global solutions for nonlinear Schrödinger equations I, mass-subcritical cases, preprint, \texttt{arXiv:1601.06335v1}.
[2] M. Beceanu, Q. Deng, A. Soffer and Y. Wu, Large global solutions for nonlinear Schrödinger equations II, mass-supercritical, energy-supercritical cases, preprint.
[3] M. Beceanu, and A. Soffer, Large outgoing solutions to supercritical wave equations, preprint, \texttt{arXiv:1601.06335v1}.
[4] M. Beceanu, and A. Soffer, Large initial data global well-posedness for a supercritical wave equation, \texttt{arXiv:1602.08163}.
[5] J. Bourgain, Global solutions of nonlinear Schrödinger equations, V. 46, American Mathematical Society, Colloquium Publications.
[6] J. Bourgain and D. Li. On an endpoint Kato-Ponce inequality. Differential Integral Equations 27 (2014), no. 11-12, 1037–1072.
[7] A. Bulut, The defocusing energy-supercritical cubic nonlinear wave equation in dimension five, preprint, arXiv: 1112.0629.
[8] A. Bulut, and B. Dodson, Global well-posedness for the logarithmically energy-supercritical nonlinear wave equation with partial symmetry, \texttt{arXiv:1807.06582}.
[9] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes in Mathematics, 10, American Mathematical Society, 2003.
[10] T. Cazenave and F. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in \(H^s\), Nonlinear Anal., Theory, Methods and Applications, 14 (1990), 807–836.
[11] Y. Cho, S. Lee, \textit{Strichartz estimates in spherical coordinates}. Indiana Univ. Math. J., 62 (2013), no.3, 991–1020.

[12] M. Christ, J. Colliander, and T. Tao, \textit{Ill-posedness for nonlinear Schrödinger and wave equations}. Preprint \url{arXiv:math/0311048}

[13] J. Colliander, G. Simpson and C. Sulem, \textit{Numerical simulation of the energy-supercritical nonlinear Schrödinger equation}. J. Hyperbolic Differ. Equ. 7 (2010), no. 2, 279–296.

[14] C. Colot, \textit{Type II blow up manifolds for the energy supercritical semilinear wave equation}. Mem. Amer. Math. Soc. 252 (2018), no. 1205, v+163 pp.

[15] B. Dodson, C. Miao, J. Murphy, and J. Zheng, \textit{The defocusing quintic NLS in four space dimensions}, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017), no. 3, 759–787.

[16] R. Donninger, \textit{On stable self-similar blowup for equivariant wave maps}, Comm. Pure Appl. Math. 64 (2011), no. 8, 1095-1147.

[17] R. Donninger, \textit{Stable self-similar blowup in energy supercritical Yang-Mills theory}, Math. Z. 278 (2014), no. 3-4, 1005-1032.

[18] R. Donninger, B. Schörkluber, \textit{Stable blow up dynamics for energy supercritical wave equations}, Tran. Amer. Math. Soc., 366 (2014), 2167–2189.

[19] T. Duyckaerts, C. Kenig and F. Merle, \textit{Scattering for radial, bounded solutions of focusing supercritical wave equations}, International Mathematics Research Notices 2014 (2014), 224–258.

[20] T. Duyckaerts, and T. Roy. \textit{Blow-up of the critical Sobolev norm for nonscattering radial solutions of supercritical wave equations on $\mathbb{R}^3$}, arXiv preprint \url{arXiv:1506.00788} (2015).

[21] D.A. Geba, M.G. Grillakis, \textit{Large data global regularity for the classical equivariant Skyrme model}. arXiv:1707.02917 (2017).

[22] D.A. Geba, M.G. Grillakis, \textit{Large data global regularity for the 2 + 1-dimensional equivariant Faddeev model}. arXiv:1709.00331 (2017).

[23] J. Ginibre, A. Soffer, G. Velo, \textit{The global Cauchy problem for the critical nonlinear wave equation}, J. Funct. Anal., (1992), 110 (1), 96–130.

[24] Z. Guo and Y. Wang, \textit{Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations}, J. Anal. Math. 124 (2014), 1–38.

[25] S. Ibrahim, M. Mohamed, and N. Masmoudi, \textit{Well-and ill-posedness issues for energy supercritical waves}, Anal. PDE, 4 (2011), 341–367.

[26] M. Keel and T. Tao, \textit{Endpoint Strichartz Estimates}, Amer. J. Math. 120 (1998), 955–980.

[27] C. Kenig, F. Merle, \textit{Scattering for $\dot{H}^{-1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions}. Trans. Amer. Math. Soc. 362 (2010), 1937–1962.

[28] C. Kenig, F. Merle, \textit{Nondispersive radial solutions to energy supercritical non-linear wave equations, with applications}, American journal of mathematics, 133 (2011), 1029–1065.

[29] C. Kenig, F. Merle, \textit{Radial solutions to energy supercritical wave equations in odd dimensions}, Discrete Continuous Dynamical Systems-A 31.4 (2011), 362 (2010), no. 2, 361–381.

[30] C. Kenig, G. Ponce and L. Vega, \textit{Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle}, Comm. Pure Appl. Math. 46 (1993), no. 4, 527–620.

[31] C. E. Kenig, G. Ponce, and L. Vega, \textit{On the ill-posedness of some canonical dispersive equations}. Duke Math. J. 106 (2001), no. 3, 617–633.

[32] R. Killip, S. Masaki, J. Murphy, and M. Visan. \textit{Large data mass-subcritical NLS: critical weighted bounds imply scattering}. NoDEA Nonlinear Differential Equations Appl. 24 (2017), no. 4, Art. 38, 33pp.

[33] R. Killip, S. Masaki, J. Murphy, and M. Visan., \textit{The radial mass-subcritical NLS in negative order Sobolev spaces}. arXiv:1709.00331 (2017).

[34] R. Killip and M. Visan, \textit{Energy-supercritical NLS: critical $\dot{H}^{s}$-bounds imply scattering}. Comm. Partial Differential Equations 35 (2010), no. 6, 945–987.

[35] R. Killip and M. Visan, \textit{The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions}, preprint. arXiv:1002.1756.

[36] R. Killip and M. Visan, \textit{The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher}. Amer. J. Math. 132 (2010), no. 2, 361–424.

[37] J. Krieger and W. Schlag, \textit{Large global solutions for energy supercritical nonlinear wave equations on $\mathbb{R}^{3+1}$}, Journal d’Analyse Mathematique, 133 (2017), 91–131.

[38] D. Li, \textit{On Kato-Ponce and fractional Leibniz}, to appear in Rev. Mat. Iberoam., 2017.

[39] D. Li, \textit{Global wellposedness of hedgehog solutions for the (3+1) Skyrme model}, preprint. arXiv:1208.4977.
[40] D. Li, X. Zhang, Regularity of almost periodic modulo scaling solutions for mass-critical NLS and applications. Anal. PDE, 3 (2010), no 2, 175–195.
[41] F. Merle, P. Raphaël and I. Rodnianski, Type II blow up for the energy supercritical NLS. Cambridge Journal of Mathematics, 3 (2015), 439–617.
[42] C. Miao, J. Murphy and J. Zheng, The defocusing energy-supercritical NLS in four space dimensions, J. Funct. Anal. 267 (2014), no. 6, 1662–1724.
[43] C. Miao, Y. Wu and X. Zhang, The defocusing energy-supercritical nonlinear wave equation when $d = 4$, preprint.
[44] S. Miao, L. Pei, P. Yu, On classical global solutions of nonlinear wave equations with large data, preprint, arXiv:1407.4192.
[45] J. Murphy, Intercritical NLS: critical $\dot{H}^s$-bounds imply scattering, SIAM J. Math. Anal. 46 (2014), no. 1, 939–997.
[46] J. Murphy, The radial defocusing nonlinear Schrödinger equation in three space dimensions. Comm. Partial Differential Equations, 40 (2015), no. 2, 265–308.
[47] T. Roy, Scattering above energy norm of solutions of a loglog energy-supercritical Schrödinger equation with radial data, J. Differential Equations 250 (2011), no. 1, 292-319.
[48] T. Roy, Global existence of smooth solutions of a 3D loglog energy-supercritical wave equation, Anal. PDE, 2–3 (2009), 261–280.
[49] S. Shao, Sharp linear and bilinear restriction estimates for paraboloids in the cylindrically symmetric case, Rev. Mat. Iberoam., 25 (2009), 1127–1168.
[50] E. Stein, Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals. Princeton University Press. (1993).
[51] M. Struwe, Global well-posedness of the Cauchy problem for a super-critical nonlinear wave equation in two space dimensions, Mathematische Annalen 350 (2011), 707–719.
[52] T. Tao, On the asymptotic behavior of large radial data for a focusing non-linear Schrödinger equation, Dynamics of PDE, V.1 (2004), 1-47.
[53] T. Tao, Global regularity for a logarithmically supercritical defocusing nonlinear wave equation for spherically symmetric data, J. Hyperbolic Differ. Equ. 4 (2007), 259–266.
[54] T. Tao, Finite-time blowup for a supercritical defocusing nonlinear wave system, Anal. PDE, 9 (2016), 1999–2030.
[55] T. Tao, Finite time blowup for a supercritical defocusing nonlinear Schrödinger system, Anal. PDE, 11 (2017), 383–438.
[56] T. Tao, M. Visan, X. Zhang, Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions. Duke Math. J., 140 (2007), no. 1, 165–202.
[57] J. Wang, P. Yu, A large data regime for nonlinear wave equations, J. Eur. Math. Soc. (JEMS), Vol. 18, Issue 3, 2016, pp. 575–622.
[58] W. Wang, Energy supercritical nonlinear Schrödinger equation: quasiperiodic solutions, Duke Math. J. 165 (2016), no. 6, 1129–1192.
[59] J. Xie and D. Fang, Global well-posedness and scattering for the defocusing $\dot{H}^s$-critical NLS, Chin. Ann. Math. Ser. B 34 (2013), no. 6, 801–842.
[60] S. Yang, Global solutions of nonlinear wave equations with large energy, arXiv:1312.7265.
Department of Mathematics and Statistics, University at Albany SUNY, Earth Science
110, Albany, NY, 12222, USA,
E-mail address: mbeceanu@albany.edu

Department of Mathematics, Hubei Key Laboratory of Mathematical Science, Central
China Normal University, Wuhan 430079, P. R. China,
E-mail address: dengq@mail.ccnu.edu.cn

Department of Mathematics, Hubei Key Laboratory of Mathematical Science, Central
China Normal University, Wuhan 430079, P. R. China, and Department of Mathematics,
Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ, 08854, USA,
E-mail address: soffer@math.rutgers.edu

Center for Applied Mathematics, Tianjin University, Tianjin 300072, P. R. China
E-mail address: yerfmath@gmail.com