NEW APPROACH TO CERTAIN REAL HYPER-ELLIPTIC INTEGRALS

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Abstract. In this paper we treat certain elliptic and hyper-elliptic integrals in a unified way. We introduce a new basis of these integrals coming from certain basis $\phi_n(x)$ of polynomials and show that the transition matrix between this basis and the traditional monomial basis is certain upper triangular band matrix. This allows us to obtain explicit formulas for the considered integrals. Our approach, specified to elliptic case, is more effective than known recursive procedures for elliptic integrals. We also show that basic integrals enjoy symmetry coming from the action of the dihedral group $D_M$ on a real projective line. This action is closely connected with the properties of homographic transformation of a real projective line. This explains similarities occurring in some formulas in popular tables of elliptic integrals. As a consequence one can reduce the number of necessary formulas in a significant way. We believe that our results will simplify programming and computing the hyper-elliptic integrals in various problems of mathematical physics and engineering.

1. INTRODUCTION

The aim of this paper is to give explicit formulas for the integrals of the following form

\[ I_{n,p} = \int \frac{(x-p)^n dx}{\sqrt{Q(x)}}, \quad n \in \mathbb{Z}, \]

where $Q(x)$ is a polynomial of degree $M$ with $M$ real zeroes of multiplicity 1. It is very well known that when $M > 2$ the integrals are in general not elementary. The cases $M = 3$ and $M = 4$ lead to elliptic integrals, whereas for $M \geq 5$ we obtain hyper-elliptic integrals. The computation of the elliptic integrals in general, and of type \((1.1)\) in particular, is based on a recursive formula for these integrals. In this way all elliptic integrals of the form \((1.1)\) can be expressed as linear combinations of fundamental integrals \((2.7)\) and elementary functions. The coefficients at the fundamental integrals are rational functions (cf. \([KK00], [PS97]\)). Nevertheless, the usage of this recursive formula is cumbersome. In this paper we offer a new method for computation of the integrals of type \((1.1)\) for arbitrary $M$ and $n$. We give explicit formulas for this computation. We choose a special basis in the vector space generated by the family of monomials $(x-p)^n$, $n \in \mathbb{Z}$. As the fundamental integrals we choose \(\{I_{-1,p}, \, p \in \mathbb{R}, \, Q(p) \neq 0\}, I_0, \ldots I_{M-2}\). For shorthand, we denote the integrals $I_{i,0}$ as $I_i, \quad i = 0, \ldots, M - 2$. The choice of the basis allows one to compute all the integrals in the
family \( \mathbf{1} \) as sums

\[
\int \frac{x^n dx}{\sqrt{Q(x)}} = \sum_{l=0}^{M-2} B_{l,n} I_l + 2 \sum_{l=M-1}^{n} B_{l,n} x^{l+1-M} \sqrt{Q(x)} + C
\]

for \( n > M - 2 \) and

\[
\int (x - p)^n \frac{dx}{\sqrt{Q(x)}} = \sum_{l=-1}^{M-2} U_{l,n} I_l,p + 2 \sum_{l=n}^{-2} U_{l,n} (x - p)^{l+1} \sqrt{Q(x)} + C
\]

for \( n < -1 \).

Notice that the coefficients \( B_{l,n} \) (resp. \( U_{l,n} \)) constitute the \( n \)-th column of the upper triangular matrix \( B \) (resp. \( U \)) and can be found by an easy recurrence. This recursive procedure in the elliptic case is much simpler than the original one (cf. [PS97]). We illustrate this in the examples 7.1 and 7.2. For a general hyper-elliptic integral our method is very clear and easy to apply. It is also efficient for big values of \( n \).

Symmetries for elliptic integrals and elliptic functions are interesting from both theoretical and computational points of view (cf. [C64], [C04], [C06], [C10]). Analyzing various tables with the elliptic integrals (cf. [AS72], [BF71], [GR00]) we see that there are many similar formulas depending on between which roots there is a variable of integration \( x \). Using appropriate permutations of the roots which constitute the dihedral group \( D_4 \) we can obtain many formulas from the suitable one. To do this we use one point compactification of a real line. Observe that this approach can also be applied in a hyper-elliptic case where we obtain an action of the dihedral group \( D_M \). We show that this action comes from the action of the homographic transformations on \( S^1 \cong \mathbb{P}^1_{\mathbb{R}} \).

2. Recurrence

In this section we prove that any hyper-elliptic integral of type \( \mathbf{1} \) can be expressed in terms of fundamental integrals. This is well known for the elliptic case (cf. [PS97]), the hyper-elliptic case is a straightforward generalisation. We include it for completeness.

It is clear that for our purposes it is enough to consider the following integrals

\[
I_n = \int \frac{x^n dx}{\sqrt{Q(x)}}, \quad \text{for } n \geq 0
\]

and

\[
I_n = \int (x - p)^n dx \frac{dx}{\sqrt{Q(x)}}, \quad \text{for } n < 0.
\]

The key fact we use is that certain combinations of hyper-elliptic integrals are elementary functions. Indeed, consider the derivative of the product \( 2u^{n+1} \sqrt{Q(x)} \), where \( u = x - p \),

\[
Q(x) = \sum_{j=0}^{M} a_j x^j = \sum_{j=0}^{M} b_j u^j, \quad b_j = b_j(p).
\]
We have
\[(2.4) \quad (2u^{n+1}\sqrt{Q(x)})' = \sum_{j=0}^{M} [2(n + 1) + j]b_ju^{j+n}/\sqrt{Q(x)}.
\]

Integrating the formula \[(2.4)\] one obtains
\[(2.5) \quad \sum_{j=0}^{M} [2(n + 1) + j]b_jI_{n+j,p}(x) = 2(x - p)^{n+1}\sqrt{Q(x)} + C.
\]

The equality \[(2.5)\] shows that the integrals \(I_{n,p}\) for \(n \geq 0\) can be recursively written using \(I_{0}, I_{1,p}, \ldots, I_{M-2,p}\) and \(I_{M-1,p}\). Substituting \(n = -1\) into \[(2.4)\] we obtain
\[\sum_{j=0}^{M-1} (j + 1)b_{j+1}I_{j,p}(x) = 2\sqrt{Q(x)} + C.
\]

Thus we can compute \(I_{M-1,p}\) in terms of the integrals \(I_{0}, I_{1,p}, \ldots, I_{M-2,p}\)
\[I_{M-1,p} = \frac{1}{Mb_M} \left( 2\sqrt{Q(x)} - \sum_{j=0}^{M-2} (j + 1)b_{j+1}I_{j,p}(x) \right).
\]

Hence all integrals of the form \(I_{n,p}\) for \(n \geq 0\) can be expressed by means of \(I_{0}, I_{1,p} \ldots I_{M-2,p}\). Naturally,
\[(2.6) \quad I_{n,p} = \sum_{k} (-1)^k \binom{n}{k} p^k I_{n-k}.
\]

So, for \(n \geq 0\), these integrals can be expressed by \(I_{0}, I_{1}, \ldots I_{M-2}\). In order to obtain integrals \(I_{n,p}\) for all integers \(n\), we see that by \[(2.5)\], it is enough to add to the last system, the integrals of the form \(I_{-1,p}\). Hence the integrals \(I_{n,p}\) can be expressed by means of the integrals:
\[(2.7) \quad I_{-1,p}, I_{0}, I_{1}, \ldots I_{M-2}.
\]

In the case when \(p\) is a root of \(Q\) the integral \(I_{M-2}\) can be expressed by means of the remaining integrals \(I_{-1,p}, I_{1} \ldots I_{M-3}\). This is because \(b_0 = 0\) and the formula \[(2.4)\] has one less nonzero terms cf. Example \[(2.3)\]. Equivalently, the integral \(I_{-1,p}\) can be expressed by \(I_{1}, \ldots I_{M-2}\). This shows that one can take, as basic integrals, the following set \(\{I_{-1,p} : p \in \mathbb{R}, Q(p) \neq 0\}, I_{0}, I_{1}, \ldots I_{M-2}\).

Thus we have proved the following:

**Proposition 2.1.** The following integrals
\[\{I_{-1,p} : p \in \mathbb{R}, Q(p) \neq 0\}, I_{0}, I_{1}, \ldots I_{M-2}\]
form a basis for hyper-elliptic integrals \[(2.7)\] i.e. any hyper-elliptic integral can be expressed by a linear combination, with coefficients being rational functions, of the basic integrals and elementary functions.

\[\square\]

The following example illustrates this.
Example 2.1. Let \( Q = b_3 x^3 + b_2 x^2 + b_1 x \). Hence 0 is a root of \( Q \) and
\[
\left( \frac{2}{x} \sqrt{Q(x)} \right)' = \frac{(b_3 x - b_1)}{\sqrt{Q(x)}}.
\]
So, \( b_3 I_1 - b_1 I_{-1} = 2 \sqrt{Q(x)} \). This shows that the integral \( I_{-1} \) can be expressed by \( I_1 \) and vice versa. In this case one can take as the basic integrals either \( \{ I_{-1, p} : p \in \mathbb{R}, Q(p) \neq 0 \} \), \( I_0 \), or \( \{ I_{-1, p} : p \in \mathbb{R}, Q(p) \neq 0 \} \), \( I_0 \), \( I_1 \).

3. Hyper-elliptic integrals for a polynomial factor.

We put \( p = 0 \) and consider \( n \geq -1 \) in (2.4)-(2.5). For this case \( u = x, b_j = a_j \) and shifting properly we have
\[
(2x^{n+1-M} \sqrt{Q(x)})' = \sum_{l=n-M}^{n} (l + n - M + 2)a_{l+M-n}x^l/\sqrt{Q(x)}, \quad n \geq M - 1.
\]
Observe that for \( n = M - 1 \) the summand for \( l = -1 \) in (3.1) equals 0.

Definition 3.1. Define a basis in the polynomial space:
\[
\phi_n(x) = \begin{cases} 
  x^n & \text{for } n < M - 1 \\
  \sum_{l=n-M}^{n} (l + n - M + 2)a_{l+M-n}x^l & \text{for } n \geq M - 1.
\end{cases}
\]

Remark 3.1. We chose the basis (3.2) so that the computation of the integral \( \int \frac{\phi_n(x)dx}{\sqrt{Q(x)}} \) is obvious.

From (3.1) one gets
\[
\int \frac{\phi_n(x)dx}{\sqrt{Q(x)}} = I_n(x) + C \quad \text{for } n < M - 1
\]
and
\[
\int \frac{\phi_n(x)dx}{\sqrt{Q(x)}} = 2x^{n+1-M} \sqrt{Q(x)} + C \quad \text{for } n \geq M - 1.
\]
The transition matrix from the standard basis \( e_n(x) = x^n \) to the new basis \( \phi_n(x) \) is an invertible upper triangular \( \infty \times \infty \) matrix. Call it \( A \). Moreover, the left top block \( (M - 1) \times (M - 1) \) of \( A \) is the unit matrix, and the left lower block of type \( \infty \times (M - 1) \) is the zero matrix. More precisely, we have the following:

Lemma 3.1. The entries of \( A \) are given by the following formula:
\[
A_{l,n} = \begin{cases} 
  \delta_{l,n} & \text{for } n = 0, \ldots, M - 2, \\
  0 & \text{for } l > n \text{ or } l < n - M, \\
  (l + n - M + 2)a_{l+M-n} & \text{for } n - M \leq l \leq n \text{ when } n \geq M \\
  \text{and for } 0 \leq l \leq M - 1 \text{ when } n = M - 1.
\end{cases}
\]
Remark 3.2. In fact, the transition matrix is an upper triangular matrix with the following block structure:

\[ A = \begin{bmatrix} I & C \\ 0 & D \end{bmatrix}, \]

where

\[
C = \begin{bmatrix}
  a_1 & 2a_0 & \cdots & 4a_0 & \cdots & (2M-2)a_0 \\
  2a_2 & 3a_1 & \cdots & 5a_1 & \cdots & (2M+1)a_1 \\
  3a_3 & 4a_2 & \cdots & 6a_2 & \cdots & (2M+2)a_2 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
(2M-1)a_{M-1} & (M+1)a_{M-1} & \cdots & (2M-2)a_{M-2} & \cdots & (2M-2)a_0 \\
(2M)a_M & (M+2)a_M & \cdots & (2M-1)a_M & \cdots & (2M+2)a_0 \\
\end{bmatrix},
\]

and

\[
D = \begin{bmatrix}
  Ma_M & (M+1)a_{M-1} & \cdots & (2M-1)a_1 & 2Ma_0 \\
  (M+2)a_M & (M+3)a_{M-1} & \cdots & (2M+1)a_1 & \cdots & (2M+2)a_0 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
(2M-2)a_M & (2M-3)a_{M-2} & \cdots & (2M-1)a_{M-1} & \cdots & (2M+2)a_0 \\
\end{bmatrix}.
\]

Hence \( \phi_n(x) = \sum_l A_{l,n} x^l \). Let \( B \) be the inverse of \( A \), i.e. \( x^n = \sum_l B_{l,n} \phi_l(x) \). Thus

\[
\int x^n \frac{dx}{\sqrt{Q(x)}} = \sum_{l=0}^{M-2} B_{l,n} I_l + 2 \sum_{l=M-1}^n B_{l,n} x^{l+1-M} \sqrt{Q(x)} + C.
\]

The coefficients \( B_{l,j} \) as the entries of the inverse \( B \) of the upper triangular matrix \( A \) can easily be found taking into account that \( B \) has the following block form:

\[
B = \begin{bmatrix} I & -CD^{-1} \\ 0 & D^{-1} \end{bmatrix}.
\]

Observe that matrices \( C, D \) have band structures. Entries in any non-zero diagonal are multiples of suitable coefficients of the polynomial \( Q \) and they form an arithmetic progression with the difference being a double of a coefficient of the polynomial.

4. Hyper-elliptic integrals for a rational factor.

In a similar way, we determine the integrals of the family:

\[
\int (x-p)^n \frac{dx}{\sqrt{Q(x)}}, \quad n < -1, \ Q(p) \neq 0.
\]

For a chosen \( p \), consider the linear space \( V \) generated by the family of monomials:

\[
f_n(x) = (x-p)^n \quad \text{for} \quad -\infty < n < M - 1.
\]

Taking into account (2.3)-(2.5) we define the following basis of \( V \):
Definition 4.1.

\[ \psi_n(x) = \begin{cases} (x-p)^n & \text{for } -1 \leq n < M-1, \\ \sum_{l=n}^{n+M} (l+n+2) b_{l-n}(x-p)^l & \text{for } n < -1. \end{cases} \]

Hence, from (2.4) one gets

\[ \int \psi_n(x) \sqrt{Q(x)} \, dx = I_{n,p}(x) + C \text{ for } -1 \leq n < M - 1 \]

and

\[ \int \psi_n(x) \sqrt{Q(x)} \, dx = 2(x-p)^{n+1} \sqrt{Q(x)} + C \text{ for } n < -1. \]

We have the following:

Lemma 4.1. Enumerate rows and columns of the transition matrix \( T \) from \( \{f_n(x)\} \) to \( \{\psi_n(x)\} \) by integral indices \( n \leq M - 2 \) in the decreasing order. The matrix \( T \) is an invertible upper triangular \( \infty \times \infty \) matrix. The left top block \( M \times M \) of \( T \) is the unit matrix. More precisely, the matrix elements are as follows:

\[ T_{l,n} = \begin{cases} \delta_{l,n} & \text{for } n = M - 2, \ldots, 0, -1, \\ 0 & \text{for } l < n \text{ and } l > n + M, \\ (l+n+2) b_{l-n} & \text{for } n \leq l \leq n + M, \quad n < -1. \end{cases} \]

\[ \square \]

Remark 4.1. The transition matrix is again upper triangular and has a block structure.

\[ T = \begin{bmatrix} I & Y \\ 0 & W \end{bmatrix}, \]

where

\[ Y = \begin{bmatrix} (M-2)b_M \\ (M-3)b_{M-1} & (M-4)b_M \\ (M-4)b_{M-2} & (M-5)b_{M-1} & (M-6)b_M \\ \vdots & \ddots & \ddots & \ddots \\ 0 b_2 & -b_3 & -2b_4 & \cdots & -(M-2)b_M \\ -b_1 & -2b_2 & -3b_3 & \cdots & -M b_M \end{bmatrix}, \]

and

\[ W = \begin{bmatrix} -2b_0 & -3b_1 & \cdots & -(M+3)b_M \\ -4b_0 & -5b_1 & \cdots & -(M+4)b_M \\ \vdots & \ddots & \ddots & \ddots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \]

So, \( \psi_n(x) = \sum_l T_{l,n} (x-p)^l \). Let \( U \) be the inverse of \( T \), i.e. \( (x-p)^n = \sum_l U_{l,n} \psi_l(x) \). Hence,

(4.1) \[ \int (x-p)^n \sqrt{Q(x)} \, dx = \sum_{l=-1}^{M-2} U_{l,n} I_{l,p} + 2 \sum_{l=n}^{M-2} U_{l,n} (x-p)^{l+1} \sqrt{Q(x)} + C, \]
where $U$ has the following form:

\[(4.2) \quad U = \begin{bmatrix} I & -YW^{-1} \\ 0 & W^{-1} \end{bmatrix}.\]

Similarly as the matrices $C, D$, the matrices $Y$ and $W$ have band structures and entries in any diagonal are multiples of suitable coefficients of the polynomial $Q$ (see (2.3)). We also observe, the analogous to those appearing in the matrices $C$ and $D$, arithmetic progressions on the non-zero diagonals. For expressing (4.1) as a combination of basic integrals one replaces, for a positive $l$, the integral $I_{l,p}$ by (2.6).

5. $D_N$ - action

**Definition 5.1.** We call a sequence of real numbers

\[(5.1) \quad (y_1, \ldots, y_N)\]

cyclically monotonous if there exists a cyclic permutation $\sigma \in S_N$ of length $N$ such that

\[(5.2) \quad (y_{\sigma(1)}, \ldots, y_{\sigma_N})\]

is strictly monotonous in the usual sense. In case where (5.2) is strictly increasing we say that (5.1) is cyclically increasing and analogously we define a cyclically decreasing sequence.

**Remark 5.1.** On a real projective line there exists a canonical positive orientation on $S^1 \cong P^1_R$ coming from the positive direction of the real line $\mathbb{R}$. Notice that a cyclically monotonous sequence of real numbers yields an orientation on $S^1 \cong P^1_R$. In case of a cyclically increasing sequence this orientation is positive (cf. Fig. 1). Cyclically decreasing sequence leads to negative orientation.

Consider the set of permutations $\{\tau_k, \eta_k, k = 1, \ldots, N\} \subset S_N$ defined by the action on the $N$-tuples of real numbers, viewed as the elements of the real projective space $\mathbb{P}^1_\mathbb{R} \cong S^1$:

\( \tau_k(y_1, \ldots, y_N) = (y_{k+1}, y_{k+2}, \ldots, y_N, y_1, \ldots, y_k), \)
\( \eta_k(y_1, \ldots, y_N) = (y_k, y_{k-1}, \ldots, y_1, y_N, \ldots, y_{k+1}). \)

The permutations $\tau_k, \eta_k$ transform cyclically monotonous sequences into cyclically monotonous sequences. In fact $\tau_k$ transform cyclically decreasing (resp. increasing) sequences into cyclically decreasing (resp. increasing) sequences. Permutations $\eta_k$ reverse cyclic monotonicity. This set forms a subgroup of $S_N$ isomorphic to the dihedral group $D_N$.

**Remark 5.2.** Notice that that $\tau_k$ preserves and $\eta_k$ reverses an orientation of $S^1$ derived from a cyclically monotonous sequence.

Let $(a_1, \ldots, a_N)$ be an increasing sequence of roots of a polynomial $Q$. We consider integrals of the form (2.1) and (2.2). In the next section we show how to transform, for a cyclically monotonous sequence of roots of $Q(x)$, hyper-elliptic integrals (2.1) and (2.2) into the Riemann canonical form. The choice of a transformation into the canonical form depends on the interval in which a variable of integration $x$ is supposed to be. Assume $x \in (a_N, \infty)$ or $x \in (-\infty, a_1)$ then to assure that

\[(5.3) \quad x \in (a_k, a_{k+1})\]
we choose the transformation described in Corollary 6.4 for either
\[(5.4) \quad (x_1, \ldots, x_N) = \tau_k(a_1, \ldots, a_N)\]
or
\[(5.5) \quad (x_1, \ldots, x_N) = \eta_k(a_1, \ldots, a_N).\]

With the change of interval to which \(x\) belongs the index \(k\) changes in (5.3) (cf. Fig.1).

The roots \((x_1, x_2, \ldots, x_N)\) divide the projective line \(\mathbb{P}^1_{\mathbb{R}}\) into \(n\) arcs \(L(x_1, x_2), L(x_2, x_3), \ldots, L(x_{N-1}, N), L(x_N, x_1)\), positively or negatively oriented depending on the type of cyclic monotonicity of the sequence \((x_1, x_2, \ldots, x_N)\). Let \(x\) be a number different from the roots. Among all possible cyclically monotonous sequences of the roots we choose as \(x\)-canonical those which satisfy \(x \in L(x_N, x_1)\). There are only two such sequences: one is cyclically increasing, the other is cyclically decreasing. Moreover, each of the \(x\)-canonical sequences can be obtained from any cyclically monotonous sequence of roots by applying a suitable transformation: either \(\tau_k\) or \(\eta_k\).

Accordingly, there is a change in enumeration of the roots due to the condition (5.3). We also see that in the orbit of the action of the group \(D_N\), all possibilities of (5.3) are obtained twice, cf. (5.4) and (5.5).

Recall that the cross-ratio of the numbers \(d_1, d_2, d_3, d_4\) is given by the following formula:

\[
(d_1, d_2; d_3, d_4) = \frac{(d_3 - d_1)(d_4 - d_2)}{(d_3 - d_2)(d_4 - d_1)}.
\]

**Lemma 5.1.** Let \((a, b, c)\) be a cyclically increasing (resp. decreasing). Then the following homographic transformation:

\[f : \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}, \quad f(x) = (b, c; a, x)\]

preserves (resp. reverses) an orientation of the real projective line.

**Proof.** It is enough to notice that for \((a, b, c)\) a cyclically increasing (resp. decreasing) sequence the derivative \(f'(x)\) is positive (resp. negative) for \(x \neq b\). Hence the homography \(f\) as a map \(R \longrightarrow R\) is locally strictly increasing (res. decreasing) depending on the type of cyclic monotonicity of the sequence \((a, b, c)\). Gluing the points \(\pm \infty\), i.e. \([\infty] = \{\pm \infty\}\), (cf. Fig.1) in the domain and range we obtain the assertion. \(\square\)
Lemma 5.2. Let \( x = (x_1, \ldots, x_N) \), \( N \geq 4 \) be a cyclically monotonous sequence. Then the following sequence:

\[
t_k := (x_{N-1}, x_N; x_k) \quad k = 2, \ldots, N - 2
\]

is strictly increasing with terms greater than 1.

Proof. The lemma follows from Lemma 5.1, Remark 5.2 and equalities \((x_{N-1}, x_N; x_1, x_1) = 1, (x_{N-1}, x_N; x_1, x_{N-1}) = [\infty]\).

\[\square\]

6. Riemann canonical form for hyper-elliptic integrals.

Let

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \psi_A(t) = \frac{at + b}{ct + d}.
\]

Denote \( N_A(t) = at + b \) and \( D_A(t) = ct + d \).

Definition 6.1. Define an operator \( r_k(A) \) sending a real function \( f \) of one real variable \( x \) to a real function of one real variable \( t \) by the following formula

\[
(6.1) \quad (r_k(A)f)(t) = D_A^k(t)f(\psi_A(t)).
\]

The following proposition describes basic properties of this operator

Proposition 6.1. Operator \( r_k(A) \) enjoys the following properties

1) For any numbers \( k, l \) and functions \( f, g \) the following equality holds:

\[
(r_k(A)f)(r_l(A)g) = r_{k+l}(fg),
\]

1a) in particular taking \( f \equiv 1 \) in 1) we get:

\[
(r_{k+l}(A)g)(t) = D_A^k(t)(r_l(A)g)(t)
\]

2) Operator \( r_1(A) \) sends a linear polynomial \( P_1(x) = x - x_0 \) to the following linear in \( t \) polynomial:

\[
(r_1(A)P_1)(t) = N(t) - D(t)x_0,
\]

3) \( r_m(A)\prod_{i=0}^{m-1} (x - x_i) = \prod_{i=0}^{m-1} r_1(A)(x - x_i) = \prod_{i=0}^{m-1} (N(t) - D(t)x_i), \)

4) If \( f(x) = \sum_j c_j x^j \) is a polynomial of degree not bigger than \( k \), the function \( r_k(A)f(t) \) is also a polynomial and

\[
(r_k(A)f)(t) = \sum_j c_j N_A^j(t)D_A^{k-j}(t).
\]

Proof. 1) follows from definition 6.1, 2) is a straightforward calculation, 3) follows from 2) and 1). Since \( r_k(A) \) is by definition additive 4) follows from 1’) and 3) applied to \( x^j, j = 0, \ldots, k \).
Consider the following differential form:

\[ \omega_x = \frac{R(x)dx}{|\sqrt{P(x)}|}, \]

where \( P(x) = a_N \prod_{i=1}^{N}(x - x_i), \ N = 2m. \)

**Lemma 6.2.** The pullback of the form \( \omega_x \) by the homographic map \( \psi_A \) has the following form:

\[ (\psi_A^* \omega)_t = \det A (\psi_A^* R)(t) \frac{|D_A^{m-2}(t)|dt}{\sqrt{|(r_{2m}(A)P(t))|}}. \]

**Proof.** Notice that by (6.1) we have \( P(A(t)) = D_A^{-2m}(r_{2m}(A)P(t)) \). The formula follows now by substitution. \( \square \)

**Remark 6.1.** Notice that for the case \( P(A) = 0, \ r_{2m}(A)P(t) \) is a polynomial of a degree \( N - 1 \) where \( N = 2m \) is a degree of \( P \).

Now let \( A \) be such that \( \psi_A(\infty) = x_{N-1}, \psi_A(0) = x_N, \psi_A(1) = x_1. \) Of course \( A \) is defined up to a scalar factor. We can take as \( A \) the matrix given by the following equality:

\[ A = A_{x_{N-1},x_N,x_1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \]

where

\[ a = (x_1 - x_N)x_{N-1}, b = -(x_1 - x_{N-1})x_N, c = x_1 - x_N, d = -(x_1 - x_{N-1}). \]

We have the following:

**Theorem 6.3.** Let \( P(x) = a_N \prod_{i=1}^{2m}(x - x_i) \), where \( (x_1, \ldots, x_{2m}) \) is a cyclically monotonous sequence, be a polynomial of degree \( N = 2m \) and let \( A \) be as in (6.4). Then

\[ (r_N(A)P)(t) = Ct(1-t)(1-k_2t) \cdots (1-k_{N-2}t), \quad 1 > k_2 > \ldots > k_{N-2}, \]

where the variable \( t \) is expressed by \( x \) as the cross-ratio

\[ t = (x_{N-1},x_N;x_1,x) \]

and

\[ k_j = t_j^{-1}, \]

where

\[ t_j = (x_{N-1},x_N;x_1,x_j), \quad j = 2, \ldots, N-2 \]

are roots of \( r_N(A)P \) different from 0 and 1. \( C \) is here a suitable constant cf. (6.6).

**Proof.** Indeed,

\[ P_A(t) = a_N [(at + b) - (ct + d)x_1] \cdots [(at + b) - (ct + d)x_N], \]

where \( a, b, c, d \) are given by (6.4). Hence

\[ (at + b) - (ct + d)x_i = (x_1 - x_N)(x_{N-1} - x_i)t - (x_1 - x_{N-1})(x_N - x_i) \]

and \( k_j = t_j^{-1} \) for \( j = 2, \ldots, 2N - 2. \) By Lemma 5.2 we obtain \( 1 > k_2 > \ldots > k_{N-2}. \)
NEW APPROACH TO CERTAIN REAL HYPER-ELLIPTIC INTEGRALS

Also

\[(6.6) \quad C = a_N(x_N - x_1)(x_N - x_{N-1}) \prod_{j=1}^{N-1} (x_{N-1} - x_1)(x_N - x_j).\]

Lemma 6.4. Let \( \omega_x \) be as in (6.2) and \( A \) - as in (6.4). Then

\[(6.7) \quad \psi_A^* \omega_t = \frac{(\psi_A^* R)(t)|D^{m-2}(t)|\epsilon}{\sqrt{|a_N(x_{N-1} - x_1)^{N-3} \prod_{j=2}^{N-2} (x_N - x_j)|}} \times \]

\[\int \frac{dt}{\sqrt{t(1-t)(1-k_2 t) \cdots (1-k_{N-2} t)}},\]

where \( \epsilon = \text{sgn} \det A. \)

Proof. We have the following equality:

\[\det A = (x_N - x_1)(x_N - x_{N-1})(x_{N-1} - x_1).\]

Now use Theorem 6.3 and Lemma 6.2.

The last lemma one can read as such a formula:

\[(6.8) \quad \int \frac{R(x)dx}{\sqrt{P(x)}} = \frac{\epsilon}{\sqrt{|a_N(x_{N-1} - x_1)^{N-3} \prod_{j=2}^{N-2} (x_N - x_j)|}} \times \]

\[\int \frac{(\psi_A^* R)(t)|D^{m-2}(t)|dt}{\sqrt{t(1-t)(1-k_2 t) \cdots (1-k_{N-2} t)}},\]

Remark. Observe that \( \epsilon = 1 \) (resp. \( \epsilon = -1 \)) for \((x_1, \ldots, x_{N-1}, x_N)\) cyclically increasing (resp. cyclically decreasing).

We finish this section with the following, easy to prove lemma:

Lemma 6.5. Let \( P(t) = t(1-t)(1-k_2 t) \cdots (1-k_{n-2} t) = \sum_{i=1}^{n} a_i t^i. \) Then

\[a_i = (-1)^{i-1} \sigma_{i-1}(1, k_2, \ldots, k_{n-2}),\]

where

\[\sigma_i(u_1, \ldots, u_n) = \sum_{1 \leq s_1 < s_2 < \cdots < s_i \leq n} u_{s_1} \cdots u_{s_i}\]

is an elementary symmetric polynomial of degree \( i \), for \( i > 0 \) and \( \sigma_0(u_1, \ldots, u_n) := 1. \)
7. Deriving some formulas

7.1. Elliptic case. In the case when the degree of the polynomial \( Q_4 := P \) is 4:

\[
Q_4 = a_4 \prod_{i=1}^{4} (x - x_i),
\]

the formula (6.8) can be simplified. We assume that the sequence of roots \((x_1, x_2, x_4, x_5)\) is cyclically monotonous.

\[
\int \frac{R(x)dx}{\sqrt{|Q_4(x)|}} = \frac{1}{\sqrt{|a_4(x_3 - x_1)(x_4 - x_2)|}} \int \frac{R \left( \frac{at+b}{cx+d} \right) dt}{\sqrt{|t(1-t)(1-kt)|}}
\]

Here the substitution \( x = \phi(t) = \frac{at+b}{cx+d} \) is defined by \( \phi(1) = x_1, \phi([\infty]) = x_3, \phi(0) = x_4 \), hence coefficients \( a, b, c, d \) can be given by (6.4) for \( N = 4 \). It is easy to see that the quantity under the square root in the denominator in the expression in front of the integral \( |a_4(x_3 - x_1)(x_4 - x_2)| = |a_4(a_3 - a_1)(a_4 - a_2)| \) is invariant under both \( \tau_k \) and \( \eta_k \).

Let us consider the following three particular cases:

\[
R(x) \equiv 1, \quad R(x) = x, \quad R(x) = \frac{1}{x - p}.
\]

In the sequel, we use properties concerning homographic transformations, described in the Appendix.

For the exponent \( -1 \) we will use the following integral:

\[
P(t, h, k) = \int \frac{dt}{(1 - ht)\sqrt{t(1-t)(1-kt)}} \quad P(t, h, k) = -\frac{1}{h} I_{-1, \frac{1}{h}}(t, k)
\]

where

\[
I_{-1,s}(t, k) = \int \frac{dt}{(t-s)\sqrt{t(1-t)(1-kt)}}
\]

Using formulas from the Appendix we obtain

\[
(7.1) \quad \int \frac{dx}{\sqrt{|Q_4(x)|}} = \frac{\epsilon}{\sqrt{|a_4(x_3 - x_1)(x_4 - x_2)|}} I_0(t, k),
\]

Further, in terms of \( P \) we obtain the following formulas:

\[
(7.2) \quad \int \frac{xdx}{\sqrt{|Q_4(x)|}} = \frac{\epsilon}{\sqrt{|a_4(x_3 - x_1)(x_4 - x_2)|}} (x_3 I_0(t, k) + (x_4 - x_3) P(t, h, k)),
\]

\[
(7.3) \quad \int \frac{dx}{(x - p)\sqrt{|Q_4(x)|}} = \frac{\epsilon}{\sqrt{|a_4(x_3 - x_1)(x_4 - x_2)|}(x_3 - p)(x_4 - p)} ((x_4 - p) I_0(t, k) - (x_4 - x_3) P(t, h_p, k)).
\]

In the formulas (7.1)-(7.3) we put

\[
(7.4) \quad t = (x_3, x_4; x_1, x), \quad k = (x_3, x_4; x_1, x_2)^{-1},
\]
(7.5) \[ h = \frac{x_4 - x_1}{x_3 - x_1}, \quad h_p = (x_3, x_4; x_1, p)^{-1}. \]

The formulas (7.1)-(7.3) can be transformed by the operations \( \tau_i, \eta_i, i = 1, 2, 3, 4 \) which preserve the relative position of roots on \( S^1 \). Thus we see that the dihedral group \( D_4 \) acts freely on these integral formulas. More precisely, every formula of (7.1)-(7.3) determines a regular orbit of this group action.

Now we illustrate the action of the group \( D_4 \) on the basis of elliptic integrals we have chosen.

7.2. **Definite elliptic integrals.**

Definite integrals of the forms \( \frac{dx}{\sqrt{G_4(x)}} \), \( \frac{x dx}{\sqrt{G_4(x)}} \) and \( \frac{dx}{(x-p)\sqrt{G_4(x)}} \) can be written by means of indefinite ones, which were calculated in the previous subsection.

Recall that elliptic integrals of first and third kinds are defined by the following formulas:

(7.6) \[ F(\phi, l) = \int_0^\phi \frac{d\alpha}{\sqrt{1 - l^2\sin^2\alpha}} = \frac{1}{2} \int_0^{\sin^2\phi} \frac{dt}{\sqrt{t(1-t)(1-l^2t)}}. \]

(7.7) \[ \Pi(\phi, h, l) = \int_0^\phi \frac{d\alpha}{(1 - h \sin^2\alpha)\sqrt{1 - l^2\sin^2\alpha}} = \frac{1}{2} \int_0^{\sin^2\phi} \frac{dt}{(1 - ht)\sqrt{(1-t)(1-l^2t)}}. \]

The roots \( (x_1, x_2, x_3, x_4) \) divide the projective line \( \mathbb{P}^1 \mathbb{R} \) into four arcs \( L(x_1, x_2), L(x_2, x_3), L(x_3, x_4) \) and \( L(x_4, x_1) \) (Fig.1. for \( N = 4 \), positively or negatively oriented depending on the type of cyclic monotonicity of the sequence \( (x_1, x_2, x_3, x_4) \). For any \( u \in L(x_4, x_1) \) we also distinguish an oriented sub-arc \( L(x_4, u) \). Naturally, \( \int_{L(x_4, u)} = \int_x^u \) if \( [\infty] \notin L(x_4, u) \). In the case where \( [\infty] \in L(x_4, u) \) we have \( \int_{L(x_4, u)} = \int_4^x + \int_x^{-\infty} \) if the arc has positive orientation, and \( \int_{L(x_4, u)} = \int_x^{-\infty} + \int_x^u \) if it is negatively oriented (cf. Fig.1).

In our notation the equations (7.1)-(7.3) lead to:

(7.8) \[ \int_{L(x_4, u)} \frac{dx}{\sqrt{Q_4(x)}} = \frac{2\epsilon}{\sqrt{|a_4(x_4 - x_2)(x_3 - x_1)|}} F(\nu, q), \]

(7.9) \[ \int_{L(x_4, u)} \frac{x dx}{\sqrt{|Q_4(x)|}} = \frac{2\epsilon}{\sqrt{|a_4(x_4 - x_1)(x_4 - x_2)|}} (x_3 F(\nu, q) + (x_4 - x_3)\Pi(\nu, h, q)), \]

(7.10) \[ \int_{L(x_4, u)} \frac{dx}{(x - p)\sqrt{|Q_4(x)|}} = \frac{2\epsilon}{\sqrt{|a_4(x_3 - x_1)(x_4 - x_2)(x_4 - p)(x_4 - p)|}} ((x_4 - p)F(\nu, q) - (x_4 - x_3)\Pi(\nu, h, q)), \]

where

(7.11) \[ \nu = \arcsin \sqrt{(x_3, x_4; x_1, u)}, \quad q = \sqrt{(x_3, x_4; x_1, x_2)^{-1}} \] (cf. 7.4). Both \( h \) and \( h_p \) are given by (7.5).
Consider the formulas (1)-(8) from the sections 3.147, 3.148, 3.151 of [GR00]. We obtain the cases (8) in the formulas 3.147, 3.148, 3.151 by taking $x_1 = d$, $x_2 = c$, $x_3 = b$, $x_4 = a$ in (7.8)-(7.10). One readily verifies that applying $\tau_i$ and $\eta_i$ for $i = 1, \ldots, 4$ to (7.8)-(7.10) one obtains all of the formulas of 3.147, 3.148 and 3.151. (the orbit of any integral under the of $D_4$-action yield all the formulas in the corresponding section).

In the formulas discussed above we have not used recursive formulas. The expression of an elliptic integral as a combination of basic integrals was connected with both the properties of homographic transformations, described in the Appendix, and the change of variable.

Notice that analogously, the action of $D_4$ can be used for general, much more complicated, elliptic integrals i.e. those which require recurrence. More generally, in order to obtain fewer formulas, one can use the described above $D_N$-action for a hyper-elliptic case.

We do not focus on a recurrence for the elliptic case. Our approach in this case is more efficient than the usual recursive procedures that can be found in the literature cf. [BF71], [PS97]. In the next subsection we give two examples concerning hyper-elliptic integrals which show advantages of our approach.

### 7.3. Examples of computation of hyper-elliptic integrals

We end this section with two examples. In Section 6 we have showed how in general hyper-elliptic case one transforms the integral involving the polynomial $Q$ to the Riemann canonical form. Therefore we will not do it here, but we show how to express an integral as a linear combination of basic integrals. In the first example, we compute the hyper-elliptic integral with a polynomial factor of higher degree. In the second example we consider a hyper-elliptic integral with a rational factor in the Riemann form. In the first example, we compute the hyper-elliptic integral with a polynomial factor of higher degree, In the second example we consider a hyper-elliptic integral with a rational factor in the Riemann form. We included both examples to illustrate how efficient and easy to apply is our approach cf. Remark 7.3.

**Example 7.1.** Let us compute the integral

\[
\int \frac{x^9 \, dx}{\sqrt{Q(x)}}, \quad \text{where} \quad Q(x) = \sum_{j=0}^{7} a_j x^j.
\]

Notice that under the assumption that all roots of a polynomial $Q(x)$ are real and distinct one can use a linear transformation for obtaining the Riemann canonical form of (7.12). We leave the justification of this to the reader. Thus we have $n = 9$ and $M = 7$. However, in the formulas below i.e., (7.14)-(7.15) we keep writing $M$ to make clear how they were derived. According to formula (3.3) we obtain

\[
\int \frac{x^9 \, dx}{\sqrt{Q(x)}} = \sum_{l=0}^{5} B_{l,9} I_l + 2 \sum_{l=6}^{9} B_{l,9} x^{l-6} \sqrt{Q(x)} + C.
\]

We have to find the elements of the 9-th column of the matrix $B$. We start with the diagonal term and move inductively up. According to (3.4) the elements $B_{9,9}, \ldots, B_{6,9}$ are the terms of the matrix $D^{-1}$, whereas the remaining terms $B_{l,9}$, $l = 0, \ldots, 5$ are the elements
of the matrix \(-CD^{-1}\). We have
\[
B_{9,9} = \frac{1}{A_{9,9}} = \frac{1}{D_{9,9}} = \frac{1}{(M+6)a_M},
\]
and further recursively
\[
B_{k,9} = \frac{-1}{D_{k,k}} \sum_{l=k+1}^{9} D_{k,l} B_{l,9}, \quad \text{for } 5 < k < 9.
\]
Thus we obtain the following formulas:
\[
B_{8,9} = -\frac{(M+5)a_{M-1}}{(M+4)(M+6)a_M^2},
\]
\[
B_{7,9} = \frac{1}{(M+2)(M+4)(M+6)a_M^3} \left[(M+3)(M+5)a_{M-1}^2 - (M+4)^2a_{M-2}a_M\right]
\]
\[
B_{6,9} = \frac{-1}{M(M+2)(M+4)(M+6)a_M^4} \left\{(M+1)(M+3)(M+5)a_{M-1}^3 - (M+1)(M+4)^2(M+4)M_{a_{M-2}} - (M+2)(M+3)(M+4)M_{a_{M-3}}a_M^2\right\}.
\]
and
\[
\begin{bmatrix}
B_{0,9} \\
B_{1,9} \\
B_{2,9} \\
B_{3,9} \\
B_{4,9} \\
B_{5,9}
\end{bmatrix} = -
\begin{bmatrix}
a_1 & 2a_0 & 0 & 0 \\
2a_2 & 3a_1 & 4a_0 & 0 \\
3a_3 & 4a_2 & 5a_1 & 6a_0 \\
4a_4 & 5a_3 & 6a_2 & 7a_1 \\
5a_5 & 6a_4 & 7a_3 & 8a_2 \\
6a_6 & 7a_5 & 8a_4 & 9a_3
\end{bmatrix}
\begin{bmatrix}
B_{6,9} \\
B_{7,9} \\
B_{8,9} \\
B_{9,9}
\end{bmatrix}.
\]
Inserting (7.14), (7.15) and (7.16) to (7.13) we obtain an explicit form of the desired integral.

**Example 7.2.** Consider the integral
\[
\int \frac{(t - \frac{3}{2})^{-3}}{\sqrt{t(1-t)(1-\frac{1}{4}t)(1-\frac{1}{3}t)(1-\frac{1}{2}t)}} dt
\]
Then \(P(t) = \sum_{i=0}^{5} a_i t^i = \sum_{i=1}^{5} b_i (t - \frac{3}{2})^i\), where \(a_0 = 0, a_1 = 1, a_2 = -\frac{25}{12}, a_3 = \frac{35}{24}, a_4 = -\frac{5}{12}, a_5 = \frac{5}{24}, b_0 = -\frac{25}{256}, b_1 = \frac{3}{128}, b_2 = -\frac{25}{96}, b_3 = -\frac{5}{48}, b_4 = -\frac{5}{48}, b_5 = \frac{5}{24}.

The subspace \(V_{-2} \subset V\) generated by \((x - \frac{3}{2})^n\), \(n = 3, 2, \ldots, -1, -2, -3\) yields the corresponding invariant subspace of integrals \(I_3, I_2, \ldots, I_0, I^{-\frac{1}{2}}, I^{-\frac{3}{2}}, I^{-\frac{3}{2}}, I^{-\frac{3}{2}}\). Therefore instead of infinite matrices \(Y\) and \(W\) we can take their cuts (denoted by the same letters for simplicity):
\[
W = \begin{bmatrix}
-2b_0 & -3b_1 & 0 & 0 & 0 \\
0 & -4b_0 & 0 & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
\frac{15}{128} & \frac{9}{64} & 0 \\
0 & \frac{15}{64} & 0 \\
\end{bmatrix},
\]
\[
W^{-1} = \begin{bmatrix}
\frac{128}{15} & \frac{64}{64} & 0 \\
0 & \frac{64}{64} & 0 \\
\end{bmatrix},
\]
\[
Y = \begin{bmatrix}
3b_5 & 0 & 0 & 0 & 0 \\
2b_4 & b_5 & \frac{10}{24} & \frac{1}{24} & 0 \\
b_3 & 0b_4 & -\frac{5}{48} & 0 & 0 \\
ob_2 & -b_3 & 0 & \frac{5}{48} & 0 \\
-b_1 & -2b_2 & -\frac{3}{128} & -\frac{25}{48} & 0 \\
\end{bmatrix},
\]
\[
-YW^{-1} = \begin{bmatrix}
\frac{16}{15} & \frac{8}{27} & 0 \\
\frac{16}{15} & \frac{16}{15} & 0 \\
\frac{8}{9} & \frac{4}{9} & 0 \\
0 & 0 & \frac{4}{9} \\
\frac{1}{9} & \frac{1027}{4500} & 0 \\
\end{bmatrix}.
\]
Thus in the following, derived from (4.1), formula:

\[\int \frac{(t - \frac{3}{2})^{-3}}{\sqrt{t(1-t)(1-\frac{1}{4}t)(1-\frac{1}{3}t)(1-\frac{1}{2}t)}} \ dt = \sum_{l=-1}^{3} U_{l,-3} \int \frac{(t - \frac{3}{2})^{l}}{\sqrt{t(1-t)(1-\frac{1}{4}t)(1-\frac{1}{3}t)(1-\frac{1}{2}t)}} \ dt + 2 \sum_{l=-3}^{-2} U_{l,-3} (t - \frac{3}{2})^{l+1} \sqrt{t(1-t)(1-\frac{1}{4}t)(1-\frac{1}{3}t)(1-\frac{1}{2}t)} + C.\]

we should put \(U_{3,-3} = -\frac{8}{25}, U_{2,-3} = \frac{16}{15}, U_{1,-3} = \frac{4}{9}, U_{0,-3} = \frac{4}{9}, U_{1,-3} = \frac{1027}{438}, U_{2,-3} = \frac{64}{25}, U_{3,-3} = \frac{64}{15}\). Further, the formula (2.6) enables one to replace the integrals \(I_{l,p}\) for positive indices \(l\) by appropriate combinations of the basic integrals \(I_l\).

**Remark 7.3.** Notice that one does not need to invert whole matrix \(W\). It is enough to compute an appropriate column of \(W^{-1}\). This is important in numerical calculations involving computations of many hyper-elliptic integrals \(I_{-n,p}\) for big \(n\).

**Remark 7.4.** Let

\[(7.18) \quad F_{D}^{(n)}(a, b_1, \ldots, b_n | x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n=1}^{\infty} \frac{(a)_{i_1+\ldots+i_n}}{(c)_{i_1+\ldots+i_n} i_1! \ldots i_n!} x_1^{i_1} \ldots x_n^{i_n},\]

where \((s)_i\) is a Pochhammer symbol, denote the Lauricella function of type \(D\) in \(n\) variables. Then for and \(c > a > 0\)

\[(7.19) \quad F_{D}^{(n)}(a, b_1, \ldots, b_n | x_1, \ldots, x_n) = K \int_{0}^{1} t^{a-1}(1-t)^{c-a-1}(1-x_1 t_1)^{-b_1} \ldots (1-x_n t_n)^{-b_n} dt\]

where \(K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\). Taking the definite integral in (7.17) over the interval \([0, 1]\) we obtain the following equality for the special values of Lauricella functions:

\[F_{D}^{(4)}\left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} | \frac{2}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}\right) = \sum_{l=-1}^{3} \frac{(-3)^{l+3}}{2} U_{l,-3} F_{D}^{(4)}\left(\frac{1}{2}, -l, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} | \frac{2}{3}, \frac{1}{3}, \frac{1}{2}\right).\]

Notice also that formulas (3.3) and (4.1) lead to many, analogous to the above, formulas for special values of Lauricella functions.

**APPENDIX A.**

In this appendix, for convenience of the reader, we collect some elementary facts concerning real homographic transformations. Let

\[(A.1) \quad \phi(t) = \frac{at + b}{ct + d}, a, b, c, d \in \mathbb{R}\]

be a homographic transformation of \(\mathbb{P}_{\mathbb{R}}^1\). 
Lemma A.1. i) Any homographic transformation \((A.1)\) of \( \mathbb{P}_1 \) can be written in the following form:
\[
\phi(t) = \phi(\infty) + \frac{\phi(0) - \phi(\infty)}{1 - \frac{t}{\phi^{-1}(\infty)}}.
\]
ii) the inverse homographic transformation is given by the following cross-ratio:
\[(A.2)\]
\[
t = \phi^{-1}(x) = (\phi(\infty), \phi(0); \phi(1), x).
\]
Proof. For i) write \( \phi(t) \) in the following form
\[(A.3)\]
\[
\frac{at + b}{ct + d} = \frac{a}{c} + \left( \frac{b}{d} - \frac{a}{c} \right) \frac{1}{1 + \frac{c}{d}t}
\]
and notice that \( \phi(\infty) = \frac{a}{c}, \phi(0) = \frac{b}{d} \) and \( \phi^{-1}(\infty) = -\frac{d}{c}. \)

For ii) put \( x = \phi(0) \) (resp. \( x = \phi(1) \) and \( x = \phi(\infty) \) ) in the cross-ratio \((A.2)\) and check that the value is 0 (resp. 1 and \( \infty \)). □

As a consequence one obtains the following:

Lemma A.2. i) For any \( p \neq \phi(0), \phi(\infty), \)
\[
\frac{1}{\phi(t) - p} = \frac{1}{[\phi(0) - p][\phi(\infty) - p]} \left\{ [\phi(0) - p] + \frac{\phi(\infty) - \phi(0)}{1 - \frac{t}{\phi^{-1}(p)}} \right\},
\]
ii) \[
\frac{1}{\phi(t) - \phi(0)} = \frac{1 - \frac{t}{\phi^{-1}(\infty)}}{\phi(\infty) - \phi(0)}.
\]
iii) \[
\frac{1}{\phi(t) - \phi(\infty)} = \frac{1}{\phi(\infty) - \phi(0)} - \frac{1}{\phi^{-1}(\infty)} - 1.
\]
Proof. Applying A.1. i) for \( \psi(t) = \frac{1}{\phi(t) - p} \) we get A.2. i). The remaining equalities can be verified by a straightforward calculation. □

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