Continuous self-similarity
in parametric piecewise isometries

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Abstract

We exhibit two distinct renormalization scenarios in many-parameter families of piecewise isometries (PWI) of a rhombus. The rotational component, defined over the quadratic field \( \mathbb{K} = \mathbb{Q}(\sqrt{5}) \), is fixed. The translations are specified by affine functions of the parameters, with coefficients in \( \mathbb{K} \). In each case the parameters range over a convex domain.

In one scenario the PWI is self-similar if and only if one parameter belongs to \( \mathbb{K} \), while the other is free. Such a continuous self-similarity is due to the possibility of merging adjacent atoms of an induced PWI, a common phenomenon in the Rauzy-Veech induction for interval exchange transformations.

In the second scenario, the phase space splits into several disjoint (non-convex) invariant components. We show that each component has continuous self-similarity, but due to the transversality of the corresponding foliations, full self-similarity in phase space is achieved if and only if both parameters belong to \( \mathbb{K} \).

All our computations are exact, using algebraic numbers.

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1 Introduction

This work represents a first study of renormalization in many-parameter families of planar piecewise isometries (PWI). These are maps of polygonal domains partitioned into convex sub-domains —called atoms— in such a way that the restriction of the map to each atom is an isometry. The first-return map to any convex sub-domain \(D\) is a new PWI, called the induced PWI on \(D\). If by repeating the induction we obtain a sub-system conjugate to the original one via a suitable group of isometries and homotheties, then we consider the original PWI to be renormalizable.

Recent work on renormalization in two-dimensional parametric families concerned one-parameter deformations of the translational part of a PWI. \[10,16,23\]. Induction is accompanied by a transformation \(s \mapsto r(s)\), where \(s\) is the parameter and \(r\) is the renormalization function. Self-similarity then corresponds to the periodic points of \(r\).

This setting is analogous to Rauzy-Veech induction for interval exchange transformations (IET’s, see \[21, 24, 26\]), which are one-dimensional PWI’s. For an IET, the parameters are a vector of sub-interval lengths together with a permutation, and the renormalization acts on the lengths via an integral matrix. The fixed-point condition for self-similarity is an eigenvalue condition for a product of matrices.

An arithmetical characterisation of self-similarity is provided by the Boshernitzan-Carroll theorem \[5\], which states that if an IET is defined over a quadratic number field (meaning that all intervals’ lengths belong to that field), then inducing on atoms results in only finitely many distinct IETs, up to scaling. In the case of two intervals (a rotation), this theorem reduces to Lagrange’s theorem on the eventual periodicity of the continued fractions coefficients of quadratic irrationals. However, unlike for continued fractions, there are self-similar IETs over fields of larger degree \[3\]. It is also known that in a uniquely ergodic self-similar IET, the scaling constant is a unit in a distinguished ring of algebraic integers \[20\].

In two dimensions general results are scarce \[18,19\]. All early results on renormalization concerned specific models of PWI’s defined over quadratic fields (the field of a PWI is determined by the entries of the rotation matrices and the translation vectors defining the isometries) \[1,2,12,14,22\]. A more intricate form of renormalization has been found in a handful of cubic cases \[9,15\].

The first results on parametric families concerned polygon-exchange transformations, due to Hooper \[10\] (on the measure of the periodic and aperiodic sets in a two-parameter family of rectangle-exchange transformations) and
Schwartz [23] (on the renormalization group of a one-parameter family of polygon-exchange transformations). Subsequently, the present authors [16] studied two one-parameter families of piecewise isometries. Each family has a fixed rotational component defined over a quadratic field ($\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{8})$, respectively), and parameter-dependent translations. It was shown that self-similarity occurs if and only if the parameter belongs to the respective field.

The mapping for the pentagonal model is shown in figure 1. For the parameter $s$ restricted to a suitable interval, there is an induced PWI on a triangular sub-domain (the so-called base triangle), which reproduces itself after scaling and the reparametrisation $s \mapsto r(s)$. After an affine change of parameter, the function $r$ was found to be of L"uroth type —a piecewise affine version of Gauss’s map [4,8,17]. In the pentagonal model, the discontinuities of $r$ accumulate at the origin; in the octagonal case one has $r = f \circ f$, where $f$ has two accumulation points of discontinuities. In both cases $r$ is expanding and preserves the Lebesgue measure.

![Figure 1: One-parameter rhombus map of the pentagonal model of [16]. The rhombus and all directions are fixed. The parameter shifts the inner boundaries of the atoms in such a way that all three atoms retain a reflection symmetry.](image)

The present work deals with two-parameter extensions of the one-parameter pentagonal model of [16] (figure 2). The roles of the parameters is made explicit in figure 3, where we employ a new co-ordinate system in order to restrict the arithmetic to the quadratic field $K = \mathbb{Q}(\sqrt{5})$, rather than a bi-quadratic field —see section 2.6. In the new co-ordinates, the atom boundary lines are of the form

$$u_i \cdot x = b_i, \quad b_i = b_{i,0} + b_{i,1}s_1 + b_{i,2}s_2, \quad b_{i,j} \in \mathbb{Q}(\sqrt{5}) \quad 0 \leq i \leq 4$$

where $s_1, s_2$ are the parameters. We use these co-ordinates for calculations, reserving the original co-ordinates for graphics.
In section 2 below, we develop the necessary geometrical constructs: tiles, dressed domains, isometries, etc. In section 3 we review the renormalization theory of the base triangle developed in [16], with two additions. We extend the isometries to the atoms’ boundaries (which will be needed to glue together atoms in section 4), and we introduce an improved renormalization scheme whose renormalization function has finitely many singularities.

The first renormalization scenario is established in section 4. There we make three successive inductions on triangular sub-domains, to obtain a PWI whose eight distinct return orbits tile the rhombus, apart from the orbits of finitely many periodic domains. We identify a convex polygon Π in the $t,s$ parameter space within which the induced map has only three distinct isometries. By merging neighbouring atoms which share the same isometry, we recover the one-parameter PWI discussed in the previous section. As a result, self-similar dynamics occurs if and only if $s \in \mathbb{K}$, and since $t$ is arbitrary, the self-similarity constraint corresponds to a foliation of Π.

The appearance of a hidden reduction of the number of atoms, which reveals itself only after induction, is present in a simpler form in the Rauzy-Veech induction of interval exchange transformations. In section 5 we show that the occurrence of free parameters in renormalizable IETs can be understood in terms of the properties of the associated translation surfaces [11]. In particular, there are self-similar IETs without free parameters, as long as the number of intervals is greater than three.
In view of this analogy, we conducted an extensive search for a non-degenerate two-parameter family, using two-dimensional sections of a three-parameter system of the $2\pi/5$ rhombus (section 6). We found a weak form of non-degeneracy, resulting from the co-existence of two continuously renormalizable families with transversally intersecting foliations. Each family corresponds to a (non-convex) invariant set in phase space, and hence to an ergodic component of the exceptional set.

In many-parameter families of piecewise isometries over quadratic fields, the existence of an irreducible component in phase space which is self-similar without free parameters has yet to be established.

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2 Preliminaries

Throughout this paper, we let

$$\alpha = \sqrt{5}, \quad \omega = (\alpha + 1)/2, \quad \beta = \omega^{-1} = (\alpha - 1)/2.$$  \hspace{1cm} (1)

The arithmetical environment is the quadratic field $\mathbb{Q}(\omega)$ with its ring of integers $\mathbb{Z}[\omega]$, given by

$$\mathbb{Q}(\omega) = \{x + y\alpha : x, y \in \mathbb{Q}\}, \quad \mathbb{Z}[\omega] = \{m + n\omega : m, n \in \mathbb{Z}\}. \hspace{1cm} (2)$$
The number $\omega$, which is the fundamental unit in $\mathbb{Z}[\omega]$ (see [6, chapter 6]), will determine the scaling under renormalization. The number $\beta = \omega - 1$ is also a unit.

2.1 Planar objects

A tile $X$ with $n$ edges is a convex polygon defined by the half-plane conditions

$$
\mathbf{u}_{m_i} \cdot \mathbf{x} < b_i \quad \text{(excluded edge)}
$$

or

$$
\mathbf{u}_{m_i} \cdot \mathbf{x} \geq b_i \quad \text{(included edge)}
$$

$i = 1, \ldots, n,$

(3)

where $\mathbf{x} = (x,y)$, $b_i \in \mathbb{R}$, and the $\mathbf{u}_m$ are the vectors

$$
\mathbf{u}_m = \left( \cos \frac{2\pi m}{5}, \sin \frac{2\pi m}{5} \right) \quad m \in \{0, \ldots, 4\}.
$$

(4)

For the $i$th edge, defined by $\mathbf{u}_{m_i} \cdot \mathbf{x} = b_i$, we introduce an index $\epsilon_i$, where $\epsilon_i = -1$ if the edge is included in $X$, and $\epsilon_i = 1$ if it is excluded. We then represent $X$ as a triple of $n$-vectors

$$
X = [(m_1, \ldots, m_n), (\epsilon_1, \ldots, \epsilon_n), (b_1, \ldots, b_n)].
$$

(5)

We shall assume that $n$ is minimal, namely that $X$ is not definable by fewer conditions.

A tiling $X$ is a set of disjoint tiles,

$$
X = \{X_1, \ldots, X_N\}
$$

and is associated with a domain $X$ (union of tiles)

$$
X = \bigcup_{k=1}^{N} X_k.
$$

Note that a domain need not be convex, or even connected. Note further that thanks to (3), if a pair of tiles have disjoint interiors but share a common boundary segment, that segment belongs to one and only one tile of the pair. This allows the possibility of gluing together adjacent tiles without disturbing the inclusion relation of the respective edges.
2.2 Similarity group

The transformation properties of planar objects are provided by a group $\mathfrak{G}$ which comprises the rotations and reflections of the symmetry group of the regular pentagon (the dihedral group $D_5$) together with translations in $\mathbb{K}^2$ and real scale transformations.

We adopt the following notation:

- $U_m$: reflection about the line generated by $u_m$.
- $R_m$: rotation by the angle $2m\pi/5$.
- $T_d$: translation by $d \in \mathbb{K}^2$.
- $S_\eta$: scaling by $\eta \in \mathbb{R}^+$.

We write $X \sim Y$ to indicate that $X$ is similar to $Y$, i.e., that $X = G(Y)$ for some $G \in \mathfrak{G}$. As $\mathfrak{G}$ is a group, this is an equivalence relation. Within $\mathfrak{G}$ we distinguish two important subgroups: the isometry group $\mathcal{I}$ generated by rotations, reflections, and translations, and the dynamical group $\mathcal{I}_+$, generated by rotations and translations.

2.3 Dressed domains and sub-domains

A dressed domain is a triple

$$\mathcal{X} = (X, X, \rho),$$

where $X = \{X_1, \ldots, X_N\}$ is a tiling of the domain $X$, and $\rho = \{\rho_1, \ldots, \rho_N\}$, where $\rho_i \in \mathcal{I}_+$ is an orientation-preserving isometry acting on the tile $X_k$.

Under the action of $G \in \mathfrak{G}$, a dressed domain $\mathcal{X}$ transforms as

$$G(\mathcal{X}) = G(X, X, \rho) = (G(X), \{G(X_1), \ldots, G(X_N)\}, G \circ \rho \circ G^{-1})$$

where the conjugacy acts componentwise. To emphasize the association of a mapping $\rho$ with a particular dressed domain $\mathcal{X}$, we use the notation $\rho_{\mathcal{X}}$.

Let $\mathcal{X} = (X, X, \rho_X)$ be a dressed domain, and let $Y$ be a sub-domain of $X$. We denote by $\rho_Y$ the first-return map on $Y$ induced by $\rho_X$. We call the resulting dressed domain $\mathcal{Y} = (Y, Y, \rho_Y)$ a dressed sub-domain of $\mathcal{X}$, and write

$$\mathcal{Y} \triangleleft \mathcal{X}. \quad (7)$$

The dressed sub-domain relation (7) is scale invariant, namely invariant under an homothety. Indeed, if $S_\eta$ denotes scaling by a factor $\eta$, then in the
data specifying a tile, the orientations $m_k$ remain unchanged, while the pentagonal coordinates $b_k$ scale by $\eta$. Moreover, the identity

$$S_\eta T_\eta R_n = T_\eta R_n S_\eta.$$ 

shows that the piecewise isometries $\rho$ scale in the same way. We conclude that the subdomain relation (7) is preserved if the dressed domain parameters are scaled by the same factor for both members.

### 2.4 Parametric dressed domains

In this article we consider continuously deformable dressed domains $X = \mathcal{X}(s)$ called *parametric dressed domains*, depending on a real parameter vector $s = (s_1, \ldots, s_p)$.

These are domains whose tiles $X_k$ and image tiles $\rho_k(X_k)$ depend on $s$ only via the coefficients $b_i$, while the parameters $n$, $m_i$ and $\epsilon_i$ remain fixed [see (5)]. We shall require that the $b_i$’s be affine functions of $s_1, \ldots, s_p$, with coefficients in $\mathbb{Q}(\omega)$. Algebraically, this is expressed as

$$b_i \in \mathbb{S} \quad \mathbb{S} = \mathbb{Q}(\omega) + \mathbb{Q}(\omega)s_1 + \cdots + \mathbb{Q}(\omega)s_p,$$  

where $s_1, \ldots, s_p$ are regarded as indeterminates. The set $\mathbb{S}$ is is a $(p + 1)$-dimensional vector space over $\mathbb{Q}(\omega)$ (a $\mathbb{Q}(\omega)$-module).

The condition (8) gives us affine functions $b_i : \mathbb{R}^p \to \mathbb{R}$

$$b_i(s_1, \ldots, s_p) = b_{i,0} + b_{i,1}s_1 + \cdots + b_{i,p}s_p \quad b_{i,j} \in \mathbb{Q}(\omega).$$  

We define the *bifurcation-free set* $\mathbb{P}(\mathcal{X})$ to be the maximal open set such that all of the edges of all $X_k(s)$ have non-zero lengths. Note that other types of bifurcations may occur if $\mathcal{X}$ is embedded within a larger domain (see section 2.6).

### 2.5 Renormalizable dressed domains

A parametric dressed domain $\mathcal{X}(s)$ is said to be *renormalizable* over an open domain $\Pi \subset \mathbb{R}^p$ if there exists a piecewise smooth map $r : \Pi \to \mathbb{P}$ such that for every choice of $s \in r^{-1}(\Pi)$ the dressed domain $\mathcal{X}(s)$ has a dressed subdomain $\mathcal{Y}$ similar to $\mathcal{X}(r(s))$ which satisfies the recursive tiling property. The function $r$ depends only on $s$, a requirement of scale invariance. In general, we have $\mathcal{Y} = \mathcal{Y}_{i(s)}(s)$, where $i$ is a discrete index. The set $r^{-1}(\Pi)$ need not be connected (even if $i$ is constant), each connected component being a
bifurcation-free domain of $\mathcal{Y}$. (To extend the renormalization function $r$ to the closure of $\Pi$, one must include bifurcation parameter values, as in [16].)

If $s = s_0$ is eventually periodic under $r$, then we say that $\mathcal{X}(s_0)$ is self-similar. A self-similar system has an induced sub-system which reproduces itself on a smaller scale under induction.

Let a parametric dressed domain $\mathcal{X}(s)$ have induced $\mathcal{X}_j(s)$, such that, for $j = 1, \ldots, N$ we have: (i) $\mathcal{X}_j$ is renormalizable over a domain $\Pi_j$; (ii) the $\mathcal{X}_j$ recursively tile $\mathcal{X}$; (iii) the $\Pi_j$ have non-empty intersection $\Pi$. Then we still consider $\mathcal{X}$ renormalizable over $\Pi$.

The definition of renormalizability given above is tailored to our model; it is not the most general possible, and it is local in parameter space. We allow $\mathcal{Y}$ to depend on a discrete index (as in Rauzy induction for interval-exchange transformations —see section 5) to obtain a simpler renormalization function $r$ (section 3). We only require $\mathcal{X}$ to be eventually renormalizable, and we allow $\mathcal{X}$ to have sub-domains with independent renormalization schemes (which is a common phenomenon, see section 6).

2.6 Computations

For computations, we use the Mathematica® procedures listed in the Electronic Supplement [7]. All computations reported in this work are exact, employing integer and polynomial arithmetic, and the symbolic representation of algebraic numbers.

The geometrical objects defined in section 2.1 require arithmetic in a biquadratic field, since only the first component of the vectors $\mathbf{u}_m$ is in $\mathbb{Q}(\omega)$. To circumvent this difficulty, we conjugate our PWI to a map of a square where the clockwise rotation $2\pi/5$ is represented by the following matrix over $\mathbb{Z}[\omega]$

$$
\begin{pmatrix}
0 & 1 \\
-1 & \beta
\end{pmatrix}
$$

where $\beta$ was defined in (1). (This is still a PWI with respect to a non-Euclidean metric.) In the new co-ordinates, the vectors $\mathbf{u}_m$ become

$$
\{(1,0),(0,1),(\beta,0),(-\beta,0),(0,\beta,0),(-\beta,0,\beta,0,0,\beta,0)\}
$$

which belong to $\mathbb{Z}[\omega]^2$. With this representation, all of our calculations can be performed within the module $\mathbb{S}$ defined in [5]. We shall still display our figures in the original coordinates, where geometric relations (especially reflection symmetries) are more apparent to the eye.

In constructing a return map orbit of a domain $\mathcal{X}(s)$ by direct iteration, one determines inclusion and disjointness relations among domains, which
require evaluations of inequalities \((3)\). Since the latter are expressed by affine functions of the parameter \(s\) in some polytope \(\Pi\), it suffices to check the inequalities on the boundary of \(\Pi\). All these boundary values belong to the field \(\mathbb{Q}(\omega)\), and the inequalities can be reduced to integer inequalities.

Typically, \(\mathcal{X}\) will be immersed in a larger domain \(\mathcal{Y}\) (an atom, say). Therefore, in addition to the intrinsic bifurcation-free polytope \(\Pi(\mathcal{X})\) defined in section \([2.4]\), one must also consider the polytope \(\Pi(\mathcal{X}, \mathcal{Y})\) defined by the inclusion \(\mathcal{X}(s) \subset \mathcal{Y}(s)\), as well as intersection of these polytopes.

The recursive tiling property defined in section \([2.5]\) is established by adding up the areas of the tiles of all the orbits, and comparing it with the total area of the parent domain.

With these techniques, we are able to establish rigorously a variety of statements valid over convex sets in parameter space.

3 Base triangle

The base triangle is the simplest one-parameter renormalizable piecewise isometry associated with rotations by \(2\pi/5\); it is self-similar precisely for parameter in the quadratic field \(\mathbb{Q}(\sqrt{5})\). It was instrumental to the proof of renormalizability of a one-parameter rhombus map in \([16]\), and it will appear again in the many-parameter versions presented here.

We develop a variant of the model presented in \([16]\), which includes boundary segments in the tiles and an improved renormalization scheme. The base triangle \(B\) prototype is the following dressed domain (see figure \([4]\):

![Base triangle prototype](image)

Figure 4: Base triangle prototype.
\[ B = (B, (B_1, B_2, B_3), (\rho_1, \rho_2, \rho_3)) \]

where

\[
\begin{align*}
B &= [(1, 0, 2), (-1, 1, 1), (\tau - \omega^2, 0, 0)], \\
B_1 &= [(0, 2, 3), (1, 1, 1), (0, 0, \omega - \omega\tau)], \\
B_2 &= [(1, 4, 0, 3, 2), (-1, 1, 1, -1, 1), (\tau - \omega^2, \omega^2 - \omega\tau, 0, \omega - \omega\tau, 0)], \\
B_3 &= [(1, 0, 4), (-1, 1, -1), (\tau - \omega^2, 0, \omega^2 - \omega\tau)].
\end{align*}
\]

The dynamics is given by a local reflection of each atom about its own symmetry axis, followed by a global reflection about the symmetry axis of \( B \), which can be written as:

\[ \rho_1 = T_{(\omega\tau - \omega, -\omega^2 + \omega^2\tau)} R_2 \]
\[ \rho_2 = T_{(0, \omega^2\tau - \omega^3)} R_3 \]
\[ \rho_3 = T_{(\omega\tau - 2\omega, \omega^2\tau - 2\omega^2)} R_2. \]

Here we have chosen a coordinate system such that the peak of the isosceles triangle is at the origin and the altitude of the atom \( B_3 \) is the parameter \( \tau \), which varies over the interval \((0, 1)\) without the occurrence of a bifurcation. This parameter (together with time-reversal invariance) determines the scale-invariant properties of the dressed domain, since it is related to the ratio \( \eta \) of altitudes of \( B_3 \) and \( B \) by the formula

\[ \eta = \frac{\tau}{\omega^2 - \tau}. \]

As \( \tau \) varies from 0 to 1, \( \eta \) increases from 0 to \( \beta \).

The edges of the domain \( B \) are included or excluded as stipulated in section 2; a vertex joining two included edges is included, but is excluded otherwise. The renormalizability analysis will also require a second base triangle \( \tilde{B} \), differing from \( B \) by a change of sign of all edge coordinates and translation vectors, as well as of the respective \( \epsilon_i \). The dressed domains \( B \) and \( \tilde{B} \) are \( \mathcal{G} \)-inequivalent: not only do they have different boundary conditions, but their interiors differ by a rotation by \( \pi \), not an element of the similarity group.

The renormalizability analysis for the base triangle is summarized in the following lemma:

**Lemma 1** Let \( B \) be as above. The following holds:

(i) For \( 0 < \tau < \beta^2 \), \( B \) has a dressed subdomain \( B_1 \sim B \) which is scaled by a factor \( (1 - \tau)/(\omega^2 - \tau) \) and has shape parameter \( r(\tau) = \omega^2\tau \).
(ii) For $\beta^2 < \tau < \beta$, $\mathcal{B}$ has a dressed subdomain $\mathcal{B}_2 \sim \tilde{\mathcal{B}}$ which is scaled by a factor $\tau/(\omega^2 - \tau)$ and has shape parameter $r(\tau) = \omega^3(\beta - \tau)$.

(iii) For $\beta < \tau < 1$, $\mathcal{B}$ has a dressed subdomain $\mathcal{B}_3 \sim \tilde{\mathcal{B}}$ which is scaled by a factor $\tau/(\omega^2 - \tau)$ and has shape parameter $r(\tau) = \omega^2(1 - \tau)$.

![Figure 5: Renormalization function $r(\tau)$ for base triangles.](image)

The renormalization function $r$ has three branches (see figure 5). In cases (ii) and (iii) one induces on the atom $B_3$, over two disjoint bifurcation-free parameter ranges. Since the size of $B_3$ vanishes as $\tau$ approaches 0, in the range (i) we induce on the triangle $[(1, 0, 2), (-1, 1, 1), (\beta^2 \tau - 1, 0, 0)]$, which is not an atom. This device prevents the occurrence of infinitely many singularities in the renormalization function found in [16].

In each case, the return orbits of the atoms of $\mathcal{B}_i$, together with a finite number of periodic tiles, completely tile the triangle $\mathcal{B}$. For $\tilde{\mathcal{B}}$, the prescriptions (i)-(iii) hold with the roles of $\mathcal{B}$ and $\tilde{\mathcal{B}}$ exchanged. The induction relations are represented as the graph in figure 6.

As in [16], the proof of Lemma 1 is by direct iteration, as discussed in section 2.6. The main difference in the two computational algorithms lies in the procedures used to verify inclusion and disjointness of tiles (see [7]). Specifically, checking the sub-polygon relation $X \subset Y$ requires verifying that no included vertex of $X$ has landed on an excluded edge of $Y$. Similarly, to
Figure 6: Renormalization graph for base triangles. A directed link from $X$ to $Y$ indicates that $X$ has an induced dressed subdomain equivalent to $Y$, subject to the indicated parameter constraint.

delete that $X$ and $Y$ are disjoint, one must check that no included vertex of either tile lies on an included edge of the other.

With reference to lemma 1, we remark that the base triangle $B$, with its definition extended to the two-atom limiting cases $\tau = 0, 1$, is in fact renormalizable also at the parameter values $0, \beta^2, \beta, 1$, with $r(\tau) = 0$ in all these cases (see [7] for the calculations). These additional parameter values are needed to make $B$ renormalizable over the whole interval $[0, 1] \cap \mathbb{Q}(\omega)$.

We shall use this property in sections 4 and 6.

4 Continuous self-similarity

We now turn to the two-parameter rhombus map introduced in section 1—see figures 2 and 3. In suitable coordinates, the dressed domain is given by

$$\mathcal{R} = (R, (R_1, \ldots, R_5), (\rho_{R_1}, \ldots, \rho_{R_5})), $$

with (see figure 2)

$$R = [(0, 1, 0, 1), (-1, -1, 1, 1), (-t, -s, 1 - t, 1 - s)],$$
$$R_1 = [(0, 2, 1), (-1, -1, 1), (-t, -s, 1 - s)],$$
$$R_2 = [(0, 1, 2, 0, 1, 2), (-1, -1, -1, 1, 1), (-t, -t, -1 - s, 1 - t, 1 - s)],$$
$$R_3 = [(0, 2, 1), (1, 1, -1), (1 - t, -1 - s, -t)],$$
$$R_4 = [(0, 1, 2, 1), (-1, -1, -1, 1), (-t, -s, -1 - s, -t)],$$
$$R_5 = [(1, 0, 1, 2), (-1, 1, 1, 1), (-s, 1 - t, -t, -1 - s)],$$

$$\rho_{R_1} = T_{(0,0)} R_4, \quad \rho_{R_2} = T_{(0,1)} R_4, \quad \rho_{R_3} = T_{(0,2)} R_4,$$

(13)

$$\rho_{R_4} = T_{(1,0)} R_4, \quad \rho_{R_5} = T_{(1,2)} R_4.$$
and \((\beta/\alpha, -\beta/\alpha)\). On the boundary of \(\Pi(\mathcal{R})\) given by with \(s = t\), the dressed domain \(\mathcal{R}\) collapses into the one-parameter pentagonal model of \([16]\), and hence is self-similar for all \(s \in \mathbb{Q}(\omega)\) within a suitable interval.

![Figure 7: The first two steps of the triple induction \(\mathcal{R} \triangleright \mathcal{R}_3 \triangleright \mathcal{R}_{31} \triangleright \mathcal{A}\). The third step produces the dressed domain \(\mathcal{A}\) shown in figure 8.](image)

Our goal is to determine a parametric dressed domain \(\mathcal{A}\) with bifurcation-free subdomain \(\Pi(\mathcal{A}) \subset \Pi(\mathcal{R})\) over which \(\mathcal{R}\) is renormalizable. To this end, we choose a specific parameter pair close to the \(s = t\) boundary: \((s_0, t_0) = (-19/200, -1/10)\), and we establish that at this value the renomalization is amenable to a complete analysis (with computer assistance). Specifically, we consider a three-step induction, first on the triangular atom \(\mathcal{R}_3\), followed by two inductions on sub-triangles, as shown in figure 7. The last induction produces the dressed domain \(\mathcal{A}\), shown in figure 8, which is given by:

\[
\mathcal{A} = (A, (A_1, \ldots, A_8), (\rho A_1, \ldots, \rho A_8)),
\]

with

\[
\begin{align*}
A &= [(2, 1, 0), (1, -1, 1), (-1 - s, \beta^4 - s, 1 - s)] \\
A_1 &= [(1, 0, 4), (-1, 1, -1), (\beta^4 - t, 1 - s, \alpha \beta^4 - t)], \\
A_2 &= [(1, 4, 0, 4), (-1, 1, 1, -1), (\beta^4 - t, \alpha \beta^4 - t, 1 - s, \alpha \beta^4 - s)], \\
A_3 &= [(1, 1, 4, 1), (1, 1, -1, -1), (1 - s, \beta^4 - t, \alpha \beta^4 - t, \beta^4 - s)], \\
A_4 &= [(1, 1, 4, 1), (1, 1, -1, -1), (\alpha \beta^4 - t, \beta^4 - t, \alpha \beta^4 - s, \beta^4 - s)], \\
A_5 &= [(2, 1, 4, 0, 3), (1, -1, 1, 1, -1), (-1 - s, \beta^4 - t, \alpha \beta^4 - s, 1 - s, -4 \beta^3 - t)]
\end{align*}
\]
\[ A_6 = [(2, 3, 0, 3), (1, 1, 1, -1), (-1 - s, -4\beta^3 - t, 1 - s, -4\beta^3 - s)], \]
\[ A_7 = [(2, 1, 4, 1), (1, -1, 1, 1), (-1 - s, \beta^4 - s, \alpha\beta^4 - s, \beta^4 - t)], \]
\[ A_8 = [(2, 3, 0), (1, 1, 1), (-1 - s, -4\beta^3 - s, 1 - s)], \]
\[ \rho_{A_1} = \rho_{A_2} = \rho_{A_3} = \rho_{A_4} = T_{(8\beta^3, -2\beta^5)} R_2, \]
\[ \rho_{A_5} = \rho_{A_6} = \rho_{A_7} = T_{(2,1+\beta^5)} R_3, \]
\[ \rho_{A_8} = T_{(2-\beta^6, -\beta^5)} R_2. \]

Figure 8: The dressed domain \( \mathcal{A} \), with its 8 atoms numbered as in (15). The boundaries of the composite atoms \( C_1, C_2, C_3 \) are coloured red, green, and blue, respectively.

We find that \( \Pi(\mathcal{A}) \) is the triangle with vertices
\[ (-1 + \frac{2}{\alpha}, -1 + \frac{2}{\alpha}), \quad (\frac{1}{2}(11 - 5\alpha), \frac{1}{4}(-25 + 11\alpha)), \quad (\frac{1}{2}(11 - 5\alpha), \frac{1}{2}(11 - 5\alpha)), \]
shown in figure 9. One verifies that \( \Pi(\mathcal{A}) \) is adjacent to the line \( s = t \) and that \( (s_0, t_0) \) lies in its interior.

Using direct iteration, we verify that for all \( (s, t) \in \Pi(\mathcal{A}) \) the return orbits of the eight atoms of \( \mathcal{A} \), together with those of 13 periodic tiles, completely tile the rhombus \( R \) (see figure 10).

A decisive simplification of the analysis results from the observation that the four atoms \( A_1, \ldots, A_4 \) of \( \mathcal{A} \) are mapped by the same isometry, and hence, with regard to the first-return map to \( \mathcal{A} \), can be merged into a single triangular tile, \( A_{1234} \). Similarly, atoms \( A_5, A_6, A_7 \) can be merged into a single
reflection-symmetric pentagon, $A_{567}$. The mergers have been suggested in the shading of the tiles in figure 9. The dressed domain thus simplifies into

$$C = (C_1, (C_2, C_3), (\rho_{C_1}, \rho_{C_2}, \rho_{C_3}))$$

$$= (A_1, (A_{567}, A_{1234}), (\rho_{A_1}, \rho_{A_5}, \rho_{A_3})) \quad (17)$$

Moreover, one verifies that over $\Pi(C) = \Pi(A)$, we have $C \sim B$. The intrinsic shape parameter of $C$ can be calculated from the ratio $\eta_C$ of the altitude of $C_3$ to that of $C$:

$$\tau_C = \frac{\omega^2 \eta_C}{1 + \eta_C} = \omega^7 (\alpha s + \beta^3). \quad (18)$$

As we traverse $\Pi(C)$ from left to right, $s$ increases from $2/\alpha - 1$ to $(11 - 5\alpha)/2$, with $\tau_C$ increasing from 0 to 1.

The issue of recursive tiling is now rather subtle. The rhombus is certainly tiled by the return orbits of $C_1, C_2, C_3$, and the 13 periodic tiles which arose in the induction on $A$. (see figure 10). However, the return paths are not the same for all tiles. In (say) $C_3$, the tiles $A_1, A_2, A_3, A_4$ have four distinct 78-step return paths, which go their separate ways, but recombine eventually to form an atom of the dressed domain with a unique isometry. The coincidence of the return times is not necessary to the recombination, as these times could differ by any integer multiples of 5. As a result, the partition of $C_3$ into $A_1, A_2, A_3, A_4$ is relevant to the recursive tiling of the original rhombus, but not to dynamical self-similarity. (We shall encounter again the same phenomenon —recombination with different return times—in section 6.)
Figure 10: Tiling of $\mathcal{R}_3$ by return orbits of the 8 atoms of $A$ (coloured) and 7 periodic tiles (grey). Note that $A_1, A_2, A_3, A_4$, which comprise $C_1$, have distinct return orbits.

The parameter pairs $(s, t) \in \Pi(C)$ corresponding to self-similarity for the rhombus map $\mathcal{R}$ are now determined by the self-similarity of the induced dressed subdomain $C$. In turn, the latter are the values of $s$ for which the base triangle is self-similar, namely $\tau_C(s) \in (0, 1) \cap \mathbb{Q}(\omega)$, while $t$ is unconstrained. Thus, by (18), $\mathcal{R}$ is renormalizable in $\Pi(C)$ if and only if

$$(s, t) \in \Pi(C) \cap (\mathbb{Q}(\omega) \times \mathbb{R}) .$$

This is the main result of this section.
5 Continuous self-similarity in Rauzy induction

Recombination of atoms, and the resulting appearance of a free parameter in self-similarity may seem a coincidental feature of planar PWI’s. This phenomenon is in fact common in the Rauzy-Veech analysis of renormalizable interval exchange transformations (IET’s) [21,24,25].

We fix a half-open interval $\Omega = [0, l)$ and a partition of $\Omega$ into $n$ half-open sub intervals $\Omega_i$. An IET is a piecewise isometry of $\Omega$ which is a translation on each $\Omega_i$. We represent it as a pair $(\pi, \Lambda)$, where $\Lambda = (\lambda_1, \ldots, \lambda_n)$ is the vector of the lengths of the sub-intervals and $\pi$ is the permutation of $\{1, \ldots, n\}$ such that the intervals in the image appear in the order $\pi(1), \ldots, \pi(n)$.

We assume that $\pi$ is irreducible in the sense that $\{1, \ldots, k\}$ is mapped into itself only if $k = n$. If we fix $n$, then $(\pi, \Lambda)$ is a parametric PWI, with discrete and continuous parameters $\pi$ and $\Lambda$, respectively. IET’s which differ only by an overall translation or scale transformation are considered equivalent.

The Rauzy-Veech induction on $(\pi, \Lambda)$ consists of inducing on the larger of the two intervals $\Omega^{(0)} = [0, l-\lambda_n)$ and $\Omega^{(1)} = [0, l-\lambda_{\pi^{-1}(n)}$), denoted by type 0 and type 1 induction, respectively (the case $|\Omega^{(0)}| = |\Omega^{(1)}|$ is excluded from consideration, as in this case the map is not minimal). Induction corresponds to a map $(\pi, \Lambda) \mapsto (\pi', \Lambda')$. Letting

$$(a_i(\pi), A_i(\pi)^{-1} \Lambda) = (\pi', \Lambda') \quad i = 0, 1$$

one finds that $A_i(\pi)^{-1}$ is an $n \times n$ integral matrix (see [20] for explicit expressions for $a_i$ and $A_i(\pi)$).

The permutations $\pi$ of $n$ symbols are then represented as the vertices of the Rauzy graph. Each vertex has two outgoing and two incoming edges, associated with $a_i$ and $a_i^{-1}$, respectively, for $i = 0, 1$. The Rauzy classes

![Rauzy Graph](image)

Figure 11: Rauzy graph for $n = 3$. All permutations are degenerate, and their translation surface is a torus. Any renormalizable IET with three intervals will have a free parameter.
are the connected components of the graph. These IET’s are linked by a sequence of Rauzy inductions, and a self-similar IET corresponds to a path on a Rauzy graph which terminates in a closed circuit, \( \pi_1, \pi_2, \ldots, \pi_p \). Transversing such a circuit produces an induced IET which is a rescaled version of the original one. Its length vector is an eigenvector of a product of matrices \( A_{\pi_k}(\pi_k)^{-1}, k = 1, \ldots, p \), with a scale factor given by the corresponding eigenvalue. In figures 11 and 12 we display the Rauzy graphs for \( n = 3 \) and \( n = 4 \) [25, section 6].

![Rauzy Graph for n=3](image)

![Rauzy Graph for n=4](image)

Figure 12: Rauzy graph for \( n = 4 \). All permutations of the upper component are degenerate. No permutation of the lower component has this property.

Two of the three irreducible permutations of the Rauzy graph for \( n = 3 \) (figure 11) and all of the permutations for the first class for \( n = 4 \) (figure 12) contain a consecutive pair: \((\ldots, j, j + 1, \ldots)\). For those IET’s, the consecutive intervals \( \Omega_j \) and \( \Omega_{j+1} \) have the same translation vector, and hence the interval \( \Omega_j \cup \Omega_{j+1} \) may be merged into a single interval of length \( \lambda'_j = \lambda_j + \lambda_{j+1} \). The \( n \)-interval IET is thus equivalent to an \( n - 1 \)-interval IET. If the latter is self-similar, then the original PWI is also self-similar for any choice of the parameter \( \lambda_j \in [0, \lambda'_j] \).

Accordingly, we say that a permutation is degenerate if it has consecutive pairs or if it acquires this property after a single induction. In the latter case, consecutive atoms of the child IET have distinct return paths in the tiling of the parent.
Such a degeneracy is best understood by representing an IET as Poincaré section of a flow on a translation surface \cite{11,24,25}. The latter is a polygon with $2n$ sides ($n$ is the number of intervals), labelled according to the ordering of the intervals before and after the permutation. The sides that correspond to the same interval have equal length and are parallel, and they are to be identified. A rectilinear flow on the plane will develop conical singularities on the surface, in correspondence to the vertices of the $2n$-gon. While the translation surface is not unique, its genus and singularities (given by the total angle $2\pi(m + 1)$ at the identified vertices) depend only on the Rauzy class. The removable singularities ($m = 0$) correspond to degenerate permutations, and they signal the appearance of free parameters in renormalizability. Since the translation surface does not change under induction, these structures depend only on the Rauzy class to which the permutation belongs. Furthermore, for any $n \geq 4$ there are both degenerate and non-degenerate permutations, e.g., $(n, \ldots, n-1, 1)$ and $(n, n-1, \ldots, 1)$, respectively.

The permutations which acquire consecutive pairs after induction are uniquely of the form $(\ldots, n, k+1, \ldots, k)$ for some $k < n-1$, and all four of its neighbours in a Rauzy graph have consecutive pairs, thanks to the relations

$$a_0 \left((\ldots, n, k+1, \ldots, k)\right) = (\ldots, k+1, k+2, \ldots, k),$$
$$a_0^{-1} \left((\ldots, n, k+1, \ldots, k)\right) = (\ldots, n-1, n, \ldots, k),$$
$$a_1 \left((\ldots, n, k+1, \ldots, k)\right) = (\ldots, n, k, k+1, \ldots),$$
$$a_1^{-1} \left((\ldots, n, k+1, \ldots, k)\right) = (\ldots, n, \ldots, k, k+1).$$

The $n = 3$ class provides the simplest illustration of this phenomenon.
6 Weakly-discrete self-similarity

The analogy with Rauzy induction suggests that there might exist two-parameter planar PWI’s which do not admit self-similarity with free parameters, meaning that both parameters would be algebraically constrained. We shall exhibit a weak form of this property, resulting from the coexistence of two systems with continuous self-similarity.

Our starting point is the four-atom, three-parameter PWI of the $2\pi/5$ rhombus shown in figure 13. The interpretation of the parameters $s, t, u$ is made clear in the conjugate system of figure 14, where the rhombus appears as a unit square. (A similar strategy can be pursued for the five-atom family of figures 2 and 3, but we found that the four-atom family is somewhat easier to work with.)

In convenient coordinates, the dressed domain is

$$\mathcal{R} = (R, (R_1, \ldots, R_4), (\rho_1, \ldots, \rho_4)),$$

with

$$R = \{(0, 1, 0, 1), (-1, -1, 1, 1), (-t, -s, 1 - t, 1 - s)\},$$

$$R_1 = \{(0, 1, 2, 1), (-1, -1, -1, 1), (-t, -t, -s, 1 - s)\},$$

Figure 13: Four-atom rhombus PWI.
Figure 14: Dressed domain \( \mathcal{R} \). Dependence of atoms on parameters \( s, t, u \) is shown.

\[
R_2 = [(0, 1, 2, 1), (1, 1, 1, -1), (1 - t, 1 - s, -s, -t)], \\
R_3 = [(0, 1, 2, 1), (-1, -1, -1, 1), (-t, -s, -1 - s + u, -t)], \\
R_4 = [(0, 1, 2, 1), (1, 1, 1, -1), (1 - t, -t, -1 - s + u, -s)]
\]

(19)

\[
\rho_1 = T_{(0,0)} R_4, \\
\rho_2 = T_{(0,1)} R_4, \\
\rho_3 = T_{(1,1-u)} R_4, \\
\rho_4 = T_{(1,2-u)} R_4.
\]

(20)

From figure 14, we see that the bifurcation-free domain \( \Pi(\mathcal{R}) \subset \mathbb{R}^3 \) is the polytope bounded by the planes \( s - t = 0, s - t = 1, u = 1, s + \beta^2 t = 0, u - s - \beta^2 t = 0, \beta^2 - \beta^2 s - t = 0, \) and \( 1 + \beta^2 s + t - u = 0 \).

This system has a simple one-parameter subsystem on the line \( L \) defined by \( s - t = \beta^2, u = \beta \). We shall consider a two-parameter perturbation of this subsystem in the plane \( u = \beta \) which intersects \( L \). (We have also considered other planes, obtaining other manageable examples: see remarks at the end of this section.)

Setting \( u = \beta \), the parameter polytope reduces to the hexagonal domain shown in figure 17 (left). As done in section 4 we choose a parameter pair close to \( L \) lying within such a domain: \( (s_0, t_0) = (2/5, 1/25) \). By inducing on the trapezoidal atom \( R_1 \), we obtain the parametric dressed domain \( \mathcal{F} \) shown in figure 15. One readily verifies that the return orbits of the eight atoms of \( \mathcal{F} \) completely tile \( \mathcal{R} \), so that the renormalizability of \( \mathcal{R} \) will follow from
that of $\mathcal{F}$. We find that the complete tiling of $F$ by renormalizable dressed sub-domains, given in figure 16 requires the return orbits of three dressed triangles $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, plus seven periodic tiles $\mathcal{P}_i$ (five regular pentagons, one trapezoid, and one rhombus).

Letting

$$\Pi^*(\mathcal{F}) = \bigcap_{i=1}^{3} \Pi(\mathcal{F}_i) \bigcap_{i=1}^{7} \Pi(\mathcal{P}_i)$$

(21)

we find that $\Pi^*(\mathcal{F})$ is the quadrilateral with vertices

$$(\beta^2, 0), \ (\beta^2 + \beta^6/\alpha, \beta^6/\alpha), \ (\beta^2 + \beta^6/\alpha, \beta^4/\alpha), \ (\beta^2, \beta^4/\alpha),$$

shown in figure 17 (right). Note that one of the bounding edges of the parameter domain coincides with the line $L$ which was the starting point of our perturbative exploration.

The dressed domains $\mathcal{F}_1$ and $\mathcal{F}_2$ are base triangles equivalent to the prototype $\mathcal{B}$, with respective shape parameters

$$\tau_1 = \omega^6 \alpha(s - \beta^2), \quad \tau_2 = \omega^4 \alpha t.$$ 

Examination of $\mathcal{F}_3$ shows that its atoms with four and five sides share the same isometry (in spite of having different return paths, and even different return times on the rhombus), and hence can be merged for the purpose of testing renormalizability. After the merger, $\mathcal{F}_3$ is also equivalent to $\mathcal{B}$, with shape parameter

$$\tau_3 = \omega^6 \alpha(s - \beta^2).$$
Figure 16: Tiling of $F$ by return orbits of $F_1$ (red), $F_2$ (green), $F_3$ (blue), and seven periodic tiles (grey).

Since each $F_i$ is renormalizable for $\tau_i \in \mathbb{Q}(\sqrt{5})$, we conclude that $F$ (hence $R$) is renormalizable when all three shape parameters are in $\mathbb{Q}(\sqrt{5})$, i.e., when $(s,t)$ is constrained to belong to $\Pi^*(F) \cap \mathbb{Q}(\sqrt{5})^2$.

Each of the three renormalizable dressed domains $F_i$ provides a sequence of nested coverings of a distinct invariant component of the exceptional set complementary to all periodic points of the rhombus. The number of distinct ergodic components of the exceptional set is thus at least three. For the model of section 4, on the other hand, we believe that there is a single ergodic component.

In closing, we summarize briefly the results of our explorations of the three-parameter space of the four- or five-atom rhombus maps.

Manageable renormalizations are likely to be found in systems specified by parameters of small height. [The height $H(\zeta)$ of the algebraic number $\zeta = (m/n) + (m'/n')\omega$ is defined as $H(\zeta) = \max(|m|, |n|, |m'|, |n'|)$]. Such was the case for the domain $R$, and the two-parameter restriction $u = \beta$ described above. We have considered other planes of small height: $s - t - \omega u + \beta = 0$, $s - t + u - 1 = 0$, etc. Within such planes, and for carefully chosen parameter patches, we have encountered a number of two-parameter renormalizable models. These are characterized by a decomposition of the rhombus into $N$ disjoint dressed domains, each tiled by the return orbits of a single base triangle (provided we ignore the “decorations” produced by the common
Figure 17: Left: The bifurcation-free domains $\Pi(\mathcal{R})$ and $\Pi(\mathcal{F})$ for the parameters $s$ and $t$, with $u = \beta$. Right: Detailed view of $\Pi(\mathcal{F})$, showing the trapezoidal domain $\Pi^*(\mathcal{F})$ of equation (21). The latter is constructed as the intersection of $\Pi(\mathcal{F}_1)$ (semi-transparent green trapezoid), $\Pi(\mathcal{F}_2)$ (semi-transparent blue triangle), and $\Pi(\mathcal{F}_3)$ (semi-transparent red square). The seven periodic tiles $\mathcal{P}_i$ do not contribute any additional constraints. The unperturbed one-parameter model corresponds to the (yellow) line $s - t = \beta^2$, which lies along the south-east boundary of $\Pi(\mathcal{F})$.

For the plane $u = 0$ of the five-atom map, the case $N = 1$ appears to be the norm, so that in those models the exceptional set is likely to be uniquely ergodic. Elsewhere, a proliferation of ergodic components is typical (albeit not universal). Notably, we have found no example of a rigidly self-similar, single component piecewise isometry with two or more parameters. Whether such a dynamical system exists at all remains an important open question.

References

[1] R. Adler, B. Kitchens and C. Tresser, Dynamics of non-ergodic piecewise affine maps of the torus, Ergod. Th. and Dynam. Sys. 21 (2001)
959–999.

[2] S. Akiyama and H. Brunotte and A. Pethő and W. Steiner, Periodicity of certain piecewise affine integer sequences, *Tsukuba J. Math.* **32** (2008) 197–251.

[3] P. Arnoux P. and J. Yoccoz, Construction de difféomorphismes pseudo-Anosov, *C. R. Acad. Sci. Paris* **292** (1981) 75–78.

[4] J. Barrionuevo, R. M. Burton, K. Dajani and C. Kraaikamp, Ergodic properties of generalised Lüroth series, *Acta Arithm.* **LXXIV** (4) (1996) 311–327.

[5] M. D. Boshernitzan and C. R. Carroll, An extension of Lagrange’s theorem to interval exchange transformations over quadratic fields, *Journal d’Analyse Mathématique* **72** (1997) 21–44.

[6] H. Cohn, *Advanced number theory*, Dover, New York (1980). (First published as *A second course in number theory* John Wiley and Sons, New York (1962).)

[7] J. H. Lowenstein and F. Vivaldi, Electronic supplement to this article, https://nyu.box.com/s/idygjzsyt2f5f24tzezvfcn7fi00ri9 (2015).

[8] J. Galambos, *Representations of Real Numbers by Infinite Series*, Lecture Notes in Math. 502, Springer, Berlin (1982).

[9] A. Goetz and G. Poggiaspalla, Rotation by $\pi/7$, *Nonlinearity* **17** (2004) 1787–1802.

[10] W. P. Hooper, Renormalization of polygon exchange maps arising from corner percolation, *Invent. Math.*, **191** (2013) 255-320.

[11] M. Kontsevich and A. Zorich, Connected components of the moduli spaces of Abelian differentials with prescribed singularities, *Invent. Math.* **153** (2003) 631–678.

[12] K. L. Kouptsov, J. H. Lowenstein and F. Vivaldi, Quadratic rational rotations of the torus and dual lattice maps, *Nonlinearity* **15** (2002) 1795–1842.

[13] J. H. Lowenstein, *Pseudochaotic kicked oscillators*, Higher Education Press, Beijing and Springer-Verlag, Berlin (2012).
[14] J. H. Lowenstein, S. Hatjispyros and F. Vivaldi, Quasi-periodicity, global stability and scaling in a model of Hamiltonian round-off, *Chaos* 7 (1997) 49–66.

[15] J. H. Lowenstein, K. L. Kouptsov and F. Vivaldi, Recursive tiling and geometry of piecewise rotations by $\pi/7$, *Nonlinearity* 17 (2004) 371–395.

[16] J. H. Lowenstein and F. Vivaldi, Renormalization of one-parameter families of piecewise isometries, subm. to *Dynamical Systems* arXiv:1405.7918 (2014).

[17] J. Lüroth, Ueber eine eindeutige Entwicklung von Zahlen in eine unendliche Reihe, *Math. Ann.* 21 (1883) 411–423.

[18] G. Poggiaspalla, Auto-similarités dans les systèmes isométriques par morceaux, PhD thesis, Université de la Méditerranée, Aix-Marseille II, (2003).

[19] G. Poggiaspalla, Self-similarity in piecewise isometric systems, *Dynamical Systems*, 21 (2006) 147–189.

[20] G. Poggiaspalla, J. H. Lowenstein and F. Vivaldi, Geometric representation of interval exchange maps over algebraic number fields, *Nonlinearity* 21 (2008) 149–177.

[21] G. Rauzy, Échange d’intervalles et transformations induites, *Acta Arith.*, 34 (1979) 315–328.

[22] R. E. Schwartz, *Outer billiards on kites*, vol. 171 of Annals of Math. studies, Princeton University Press, Princeton (2009).

[23] R. E. Schwartz, *The Octagonal Pet*, Mathematical Surveys and Monographs, Volume 97, American Mathematical Society, (2014).

[24] W. Veech, Gauss measures for transformations on the space of interval exchange maps, *Acta Math.*, 115 (1982) 201-242.

[25] M. Viana, The ergodic theory of interval exchange maps, *Rev. Mat. Complut.* 19 (2006) 7–100.

[26] J.-C. Yoccoz, *Continued fractions algorithms for interval exchange maps: an introduction*, Frontiers in Number Theory, Physics and Geometry, Vol 1, P. Cartier, B. Jula, P. Moussa, P. Vanhove (editors), Springer-Verlag, Berlin 4030437 (2006).