The gravitational self-energy of a spherical shell

The absence of Newtonian black holes and some remarks on the classical electron radius

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Abstract. According to Einstein’s mass-energy equivalence, a body with a given mass extending in a large region of space, will get a smaller mass when confined into a smaller region, because of its own gravitational energy. The classical self-energy problem has been studied in the past in connection with the renormalization of a charged point particle. Still exact consistent solutions have not been thoroughly discussed in the simpler framework of Newtonian gravity. Here we exploit a spherically symmetrical shell model and find two possible solutions, depending on some additional assumption. The first solution goes back to Arnowitt, Deser and Misner (1960). The second is new. When applied to a spherical shell of a given “bare” mass \( M_0 \), both solutions lead to a vanishing “renormalized” mass for a vanishing radius \( R \) of the shell. As a consequence the condition for the existence of a Newtonian black hole will never be met for finite \( R \). When applied to the e.m. mass of a pure static electric charge the second solution yields a new vanishingly small value (10\(^{-55}\) cm) for the classical electron radius.

1 Introduction

In classical Newtonian physics, the gravitational energy of a simple spherically symmetric shell of mass \( M_0 \) and radius \( R \) is

\[
U_0(R) = -G \frac{M_0^2}{2R},
\]

(1)

where \( G \) is the gravitational constant.

Taking into account the mass-energy equivalence from special relativity, this binding energy (negative) is equivalent to a mass defect. Furthermore the equality of inertial and gravitational masses has been tested experimentally even in the presence of sizable mass defects due to large binding energies \([1,2]\). Hence the gravitational mass of the shell will be different from \( M_0 \) and, in turn, (1) should be expressed in terms of the new “renormalized” mass. In the following we shall refer to \( M_0 \) as the “bare” mass and to \( M(R) \) as the renormalized mass, i.e. the resulting mass when \( M_0 \) is uniformly distributed on a spherical shell of radius \( R \). \( M(R) \) takes into account the gravitational self-energy, while \( M_0 \) corresponds to the sum of all the masses that one would obtain tearing the sphere in many small pieces and moving them away from each other\(^1\).

The renormalized mass at the first order in \( c^{-2} \) is immediately obtained from (1)

\[
M(R) \approx M_0 + U_0(R)/c^2 = M_0 \left( 1 - G \frac{M_0}{2Rc^2} \right).
\]

(2)

However the exact calculation of \( M(R) \) from a given bare mass \( M_0 \) is not trivial and rests upon some additional assumptions.

The classical self-energy problem has been studied in the past by R. Arnowitt, S. Deser and C.W. Misner (ADM) \([3–5]\) in the framework of a canonical formulation of General Relativity. Here we wish to consider the problem in the simpler context of Newtonian physics with minimal assumptions. In this sense the present paper has some pedagogical character. Furthermore we limit our discussion to the simple model of a spherical shell. Indeed this model has a decisive feature to unravel the problem: In considering the gravitational interaction of the shell with a test particle and thinking

\(^1\) Here we are disregarding any form of kinetic energy or internal pressure or stress that can also contribute to the mass.
of it as being spherically distributed around the mass shell \(M(R)\) one is in fact led to a double-shell model that simplifies the interaction at the level of two (radial) pointlike particles. This allows us to write a differential equation for \(M(R)\) as a function of the bare mass \(M_0\) (see sects. 3 and 4) whose solution leads to the desired expression for the renormalized mass. While this may be considered as a first step towards the general problem of exactly calculating the gravitational self-energy of a given distribution of mass, still remarkable consequences may be drawn.

We start revisiting in sect. 2 the solution for \(M(R)\) given by ADM. We find however that this solution is inconsistent with the following expression:

\[
\delta U(R) = -G \frac{M(R)\delta m_0}{R},
\]

(3)

for the gravitational interaction energy between the shell and a test particle \(\delta m_0\) settled on its surface. In sect. 3 we relate the reason for the inconsistency to the lack of an appropriate mass renormalization of the test particle due to \(\delta U(R)\) itself. In sect. 4 we propose an alternative solution, consistent with (3). In sect. 5 we examine the behaviour of both solutions in the context of Newtonian black holes (NBH) [6]. We will show that the conditions for the existence of a NBH will never be met for finite \(R\). Finally, in sect. 6, we revisit the old problem of the e.m. mass of the classical electron. We find that the alternative solution for \(M(R)\) allows also for a new definition of the classical electron radius, which adds to the well-known one.

### 2 A consistent solution for the spherical shell and an inconsistency

Given the expression (1) for the gravitational energy of a spherical shell, it seems quite natural to write down the following consistent equation for the renormalized mass \(M(R)\):

\[
M(R) = M_0 - \frac{G M(R)^2}{2 R c^2},
\]

(4)

or, equivalently, for the gravitational self-energy of the shell,

\[
U(R) = -\frac{G(M_0 + U(R)/c^2)^2}{2 R}.
\]

(5)

Defining

\[
R_0 = \frac{G M_0}{c^2},
\]

(6)

one gets the (positive) solution for the mass

\[
M(R) = M_0 \left(1 + \sqrt{1 + \frac{2R_0}{R}}\right) \frac{R}{R_0},
\]

(7)

with the corresponding gravitational self-energy

\[
U(R) = -G \frac{M(R)^2}{2 R} = -M_0 c^2 \left(1 + \frac{R_0}{R} - \frac{1}{\sqrt{1 + \frac{2R_0}{R}}}\right) \frac{R}{R_0}.
\]

(8)

Equation (4) has been considered [5,7] in the framework of the classical theory of the electron. In fact, adding to \(M_0\) the contribution to the mass of the electromagnetic energy \(e^2/2R\) (this time positive), the ensuing solution tends to a finite value when \(R \to 0\),

\[
M(R \to 0) = |e|/\sqrt{G} \equiv m_G,
\]

(9)

independent of \(M_0\). This elegant result exhibits a nice feature of the gravitational self-energy as a natural cutoff for the Coulomb self-energy of a point charge. The result is however numerically too big \((m_G \approx 10^{24} m_e)\) compared to the electron mass. We shall further comment on this.

Now one could naively think that the (now renormalized) shell yields a gravitational field identical to that of a spherical shell with the trivial substitution \(M_0 \to M(R)\), i.e. a gravitational potential function (for \(r \geq R\)),

\[
\Phi(r) = -G \frac{M(R)}{r}.
\]

(10)

Hence the interaction energy with a test particle settled on its surface should be given by (3). If so, the total mass of the system (spherical shell of renormalized mass \(M(R)\) plus test particle \(\delta m_0\) on its surface) is

\[
M_t(R) = M(R) + \delta m_0 + \delta U(R)/c^2 = M(R) + \delta m_0 \left(1 - G \frac{M(R)}{c^2 R}\right).
\]

(11)
Substituting $M(R)$ from (7) one has

$$M_t(R) = M(R) + \delta m_0 \left( 2 - \sqrt{1 + 2R_0/R} \right). \quad (12)$$

Here we come to a contradiction. Indeed, let us suppose that we want to deposit a test particle $\delta m_0$ on the surface of $M(R)$ and let us think about this test mass as being uniformly distributed on a thin spherical shell of radius $r$ centered on the origin, just as $M(R)$. (Note that, neglecting higher orders in $\delta m_0$, we do not worry about self-energy of $\delta m_0$ on its own. In other words, $\delta m(r) \approx \delta m_0$.) Now let us imagine we bring $r$ to $R$ and to stick $\delta m_0$ as a thin film on $M(R)$. Then, viewing the system as a new shell of bare mass $M_0 + \delta m_0$, one has again, from (7),

$$M_t(R) = M(R) + \frac{\partial M(R)}{\partial M_0} \delta m_0 = M(R) + \frac{\delta m_0}{\sqrt{1 + 2R_0/R}}, \quad (13)$$

that does not agree with (12), but at the first order in $R_0/R$ (i.e. in $c^{-2}$). So there is a mistake somewhere.

### 3 Renormalizing the test mass

In sect. 2 we thought about the test particle as being spherically distributed on a thin shell concentric to the shell $M(R)$. We argued that we should not worry about its own self-energy (since we work at first order in $\delta m_0$); however we left aside the possibility of a further renormalization due to the interaction energy $\delta U(R)$ itself. Let us assume, by now, that the whole $\delta U(R)$ be attributed to the test mass, i.e.

$$\delta m_0 \rightarrow \delta m = \delta m_0 + \delta U(R)/c^2. \quad (14)$$

This assumption amounts rewriting (3) as a self-consistent equation for $\delta U(R)$,

$$\delta U(R) = -\frac{G M(R) \left( \delta m_0 + \delta U(R)/c^2 \right)}{R}. \quad (15)$$

From (15),

$$\delta U(R) = -\frac{G M(R) \delta m_0}{R \left( 1 + G M(R)/Rc^2 \right)}, \quad (16)$$

and

$$M_t(R) = M(R) + \frac{\delta m_0}{1 + G M(R)/Rc^2}, \quad (17)$$

at variance with (11). Substituting (7) we get

$$1 + G \frac{M(R)}{Rc^2} = \sqrt{1 + 2R_0/R}, \quad (18)$$

so (17) now agrees with (13) and there is no inconsistency.

In fact the starting equation (4) may be obtained from (17) as follows: Viewing the total system as a new shell with increased bare mass $(M_0 + \delta m_0)$, one can write

$$M_t(R) = M(R) \equiv dM(R), \quad \delta m_0 \equiv dM_0.$$

Hence (17) is a differential equation that yields the mass $M(R)$ of a spherical shell of radius $R$ as a function of its bare mass $M_0$,

$$\frac{dM(R)}{dM_0} = \frac{1}{1 + G \frac{M(R)}{Rc^2}}, \quad (19)$$

whose solution is just (4).

Note that the renormalization of $\delta m_0$ may be equivalently described in terms of a suitable modification of the gravitational potential, that, instead of (10), should be written as (for $r > R$)

$$\Phi(r) \rightarrow \tilde{\Phi}(r) = -\frac{G M(R)}{\left( 1 + G \frac{M(R)}{Rc^2} \right)} \frac{M(R)}{r}, \quad (20)$$

with $M(R)$ given by (7).
4 An alternative solution

In sect. 3 it was proved that renormalizing the mass of the test particle by the interaction energy with the massive shell leads to (7) and (8). However this assumption may appear somewhat arbitrary. Moreover it seems not reconcilable with additivity. In fact let us think about the shell \( M(R) \) as made up by the sum of a large number \( N \) of light overlapping shells each of mass \( \delta m' \)

\[
\sum_i \delta m'_i = N \delta m' \equiv M(R)
\]

and rewrite (3) as the sum of the interaction energies with each sub-shells

\[
\delta U(R) = -G \sum_i \frac{\delta m'_i \delta m_0}{R} \equiv \sum_i \delta u_i(R) = N \delta u(R).
\] (21)

Now for each individual term in the sum in (21) let us perform the renormalization of the test mass as before \((\delta m_0 \to \delta m_0 + \delta u(R)/c^2)\). Then

\[
\delta U(R) = -G \sum_i \frac{\delta m'_i (\delta m_0 + \delta u(R)/c^2)}{R} \approx -G \frac{M(R) \delta m_0}{R},
\] (22)

for large \( N \). Moreover the final result (22) will not change even if the little mass equivalent to the interaction energy were attributed (fully or in part) to the shell \( M(R) \) rather than to the test particle. This is because the ensuing correction to \( \delta U(R) \) comes to be at a higher order in \( \delta m_0 \) in this case.

Then, assuming additivity, one should not change (3) and look for a solution other than (7) and (8). In fact we may exploit the method outlined in sect. 3. Starting from (11) we get the differential equation

\[
\frac{dM(R)}{dM_0} = 1 - G \frac{M(R)}{Rc^2},
\] (23)

whose solution is

\[
\ln \left(1 - G \frac{M(R)}{Rc^2}\right) = -G \frac{M_0}{Rc^2},
\] (24)

\[
M(R) = \frac{Rc^2}{G} \left(1 - \exp \left[-G \frac{M_0}{Rc^2}\right]\right) \equiv M_0 \left(1 - \exp \left[-\frac{R_0}{R}\right]\right) \frac{R}{R_0}.
\] (25)

Instead of (12) we have now

\[
M_t(R) = M(R) + \delta m_0 \exp \left[-\frac{R_0}{R}\right]
\] (26)

and there is, of course, no inconsistency.

5 Newtonian black holes

The possible existence of celestial objects so massive to hold back even light with their gravity goes back to the end of the 18th century [8,9]. Indeed if rays of light were constituted by a flux of tiny particles with a given kinetic energy (as was believed at the time) one would easily get, from the conservation of the mechanical energy (kinetic+potential), the condition for the mass of a spherical body to be heavy enough to prevent the light from escaping its surface. Plugging Einstein’s special relativity into Newtonian gravitation one writes for the total mass of a spherical shell (of renormalized mass \( M(R) \)) plus a test particle \( \delta m_0 \),

\[
M_t(R) = M(R) + \delta m_0 + \delta U(R)/c^2,
\] (27)

where \( \delta U(R) \) is the gravitational interaction when the test particle is settled on the surface of \( M(R) \). For a relativistic particle \( \delta m_0 c^2 \) is to be understood as the full energy of the particle. So, for a photon, it is \( \delta m_0 = \hbar \omega / c^2 \). Then, if

\[
\delta U(R) = -\delta m_0 c^2,
\] (28)

one has from (27) \( M_t(R) = M(R) \), i.e. the total energy of the system with or without \( \delta m_0 \) is the same. This means that a photon, leaving the surface of that sphere, must spend its whole energy \( \hbar \omega \) to get out from the gravitational field and will end its journey with a vanishingly small frequency irrespective of the initial one. Therefore (28) is the up-to-date condition for the existence of a NBH [10].
From (3), neglecting the gravitational self-energy \( (M(R) \equiv M_0) \), one gets (28) at \( R = R_0 = GM_0/c^2 \), but taking it into account changes things drastically. This has been pointed out recently by Christillin [11]. His analysis however rests on the first-order approximation (corresponding to eq. (2) in the case of the shell) and cannot be used when \( U(R)/c^2 \) is comparable to the bare mass \( M_0 \). Here we may easily check that both solutions for \( M(R) \) imply (see (17), (18) and (26))

\[
M_t(R) > M(R),
\]

for any \( R > 0 \). Note that \( M_t(R) = M(R) \) (i.e. eq. (28)) is attained just at the limit \( R \to 0 \).

6 A new classical electron radius

The alternative solution (25) deserves a couple of comments in connection with the classical theory of the electron [12]. To obtain the mass of a spherical shell of radius \( R \), charge \( e \) and bare mass \( M_0 = 0 \), it is enough to make the substitution \( M_0 \to e^2/2Rc^2 \) in (25). One gets

\[
M_e(R) = \frac{e^2R}{G} \left(1 - \exp \left[-G\frac{e^2}{2R^2c^4}\right]\right).
\]

Again we may appreciate the regularizing role of the gravitational self-energy that heals the divergency of the Coulomb self-energy when \( R \to 0 \). Instead of the previously quoted result \( M_e(R \to 0) = m_G = |e|/\sqrt{G} \) [5,7], this time we obtain \( M_e(R \to 0) = 0 \).

Let us now define

\[
R_e \equiv Gm_e/c^2,
\]

and rewrite (30) in terms of \( R_e \) and \( r_e = e^2/m_Gc^2 \) (the usual expression for the classical electron radius [12]). We get

\[
M_e(R) = m_e \frac{R}{R_e} \left(1 - \exp \left[-\frac{r_e R_e}{2R^2}\right]\right).
\]

In order that \( M_e(R) \equiv m_e \), one has to find solutions of

\[
\frac{R}{R_e} \left(1 - \exp \left[-\frac{r_e R_e}{2R^2}\right]\right) = 1.
\]

Numerically, for the electron, it is \( r_e = 2.82 \cdot 10^{-13} \) cm and \( R_e = 0.7 \cdot 10^{-55} \) cm. Then it can be immediately seen that (33) has two distinct solutions \((R_1, R_2)\), almost exactly coincident with \( r_e/2 \) and \( R_e \). Note that, while for the first solution \((R_1 \approx r_e/2)\) gravitational self-energy effects are totally negligible, the opposite happens for \( R_2 \approx R_e \) (see (31)).

7 Concluding remarks

In this paper, taking due account of the mass-energy equivalence (and of the equality of the inertial and gravitational masses), we faced the seemingly trivial problem of calculating the gravitational self-energy of a given distribution of mass. We exploited the simple model of a spherically symmetrical distribution over a shell of radius \( R \). The problem may equivalently be stated as how to calculate the renormalized mass of the given distribution \( M(R) \) from its bare mass \( M_0 \). We displayed two possible solutions. The first one was known since 1960 [5,7], the second is new. The difference between the two is related to specific assumptions regarding the mass renormalization method. The first one requires a modification (see (20)) of the gravitational potential generated by the shell with respect to the well-known expression in Newtonian theory (10). The second solution does not need any modification of (10) and follows from a simple assumption of additivity (see sect. 4).

For a given bare mass \( M_0 \) the renormalized masses \( M(R) \) go to zero when \( R \to 0 \) and the condition (28) for the existence of a NBH is never satisfied for \( R \neq 0 \). Therefore gravitational self-energy effects prevent the formation of a NBH.

When the bare mass \( M_0 \) is substituted with \( e^2/2Rc^2 \) one gets the renormalized gravitational mass of a pure static electric charge uniformly distributed on a spherical shell of radius \( R \). Defining

\[
R_G = |e|\sqrt{G}/c^2 \equiv Gm_G/c^2,
\]

it is, for the solution (7),

\[
M_e(R) = m_G \frac{R}{R_G} \left(-1 + \sqrt{1 + (R_G/R)^2}\right).
\]
which reaches its maximum value $m_G = |e|/\sqrt{G}$ at $R = 0$. On the other hand, from the alternative solution (30), one gets

$$M_e(R) = m_G \frac{R}{R_G} \left( 1 - \exp \left[ -\frac{R_G^2}{2R^2} \right] \right),$$

(36)

that has a maximum of the order of $m_G$ at $R \approx R_G$. Since (36) goes to zero when $R \to 0$, one finds a further solution for the electron mass at a vanishingly small radius ($10^{-55}$ cm).

Indeed, in [7], it was stated that a system cannot have pointlike dimensions at the mass $m_G$ and the minimum possible dimensions for that value are just given by (34). It is remarkable that $m_G$ may also be written as $m_G = \sqrt{\alpha} M_{\text{Planck}}$; likewise $R_G = \sqrt{\alpha} l_{\text{Planck}}$. As emphasized in [13], these quantities are purely classical (the square root of the fine-structure constant $\alpha$ cancels $\bar{\hbar}$) and define the scale at which gravitational and electromagnetic self-energies become comparable.

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References

1. L. Southerns, Proc. R. Soc. A 84, 325 (1910).
2. A.L. Zeeman, Proc. Sect. Sci. K. ned. Akad. Wet. 20, 542 (1917).
3. R. Arnowitt, S. Deser, C.W. Misner, Phys. Rev. Lett. 4, 375 (1960).
4. R. Arnowitt, S. Deser, C.W. Misner, Phys. Rev. 118, 1100 (1960).
5. R. Arnowitt, S. Deser, C.W. Misner, Phys. Rev. 120, 313 (1960).
6. A.K. Raychaudhuri, Gen. Relativ. Gravit. 24, 281 (1992).
7. M.A. Markov, Sov. Phys. JETP 37, 561 (1972).
8. J. Michell, Philos. Trans. 74, 35 (1784).
9. P.S. de Laplace, Exposition du Système du Monde (Paris, 1796).
10. G. Dillon, arXiv:1303.2577 (2013).
11. P. Christillin, Eur. Phys. J. Plus 126, 48 (2011).
12. J.D. Jackson, Classical electrodynamics (Wiley, New York, 1975) Chapt. 17.
13. M. Visser, Phys. Lett. A 139, 99 (1989).