Algebra in superextensions of groups, II: cancelativity and centers

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ABSTRACT. Given a countable group $X$ we study the algebraic structure of its superextension $\lambda(X)$. This is a right-topological semigroup consisting of all maximal linked systems on $X$ endowed with the operation

$$A \circ B = \{C \subset X : \{x \in X : x^{-1}C \in B\} \in A\}$$

that extends the group operation of $X$. We show that the subsemigroup $\lambda^0(X)$ of free maximal linked systems contains an open dense subset of right cancelable elements. Also we prove that the topological center of $\lambda(X)$ coincides with the subsemigroup $\lambda^*(X)$ of all maximal linked systems with finite support. This result is applied to show that the algebraic center of $\lambda(X)$ coincides with the algebraic center of $X$ provided $X$ is countably infinite. On the other hand, for finite groups $X$ of order $3 \leq |X| \leq 5$ the algebraic center of $\lambda(X)$ is strictly larger than the algebraic center of $X$.

Introduction

After the topological proof (see [HS, p.102], [H2]) of Hindman theorem [H1], topological methods become a standard tool in the modern combinatorics of numbers, see [HS], [P]. The crucial point is that any semigroup operation $*$ defined on any discrete space $X$ can be extended to

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a right-topological semigroup operation on $\beta(X)$, the Stone-Čech compactification of $X$. The extension of the operation from $X$ to $\beta(X)$ can be defined by the simple formula:

$$U \ast V = \left\{ \bigcup_{x \in U} x V_x : U \in \mathcal{U}, \{V_x\}_{x \in U} \subset \mathcal{V} \right\}, \quad (1)$$

where $U, V$ are ultrafilters on $X$. Endowed with the so-extended operation, the Stone-Čech compactification $\beta(X)$ becomes a compact right-topological semigroup. The algebraic properties of this semigroup (for example, the existence of idempotents or minimal left ideals) have important consequences in combinatorics of numbers, see [HS], [P].

The Stone-Čech compactification $\beta(X)$ of $X$ is the subspace of the double power-set $\mathcal{P}(\mathcal{P}(X))$, which is a complete lattice with respect to the operations of union and intersection. In $[G_2]$ it was observed that the semigroup operation extends not only to $\beta(X)$ but also to the complete sublattice $G(X)$ of $\mathcal{P}(\mathcal{P}(X))$ generated by $\beta(X)$. This complete sublattice consists of all inclusion hyperspaces over $X$.

By definition, a family $\mathcal{F}$ of non-empty subsets of a discrete space $X$ is called an inclusion hyperspace if $\mathcal{F}$ is monotone in the sense that a subset $A \subset X$ belongs to $\mathcal{F}$ provided $A$ contains some set $B \in \mathcal{F}$. On the set $G(X)$ there is an important transversality operation assigning to each inclusion hyperspace $\mathcal{F} \in G(X)$ the inclusion hyperspace

$$\mathcal{F}^\perp = \{A \subset X : \forall F \in \mathcal{F} (A \cap F \neq \emptyset)\}.$$

This operation is involutive in the sense that $(\mathcal{F}^\perp)^\perp = \mathcal{F}$.

It is known that the family $G(X)$ of inclusion hyperspaces on $X$ is closed in the double power-set $\mathcal{P}(\mathcal{P}(X)) = \{0, 1\}^{\mathcal{P}(X)}$ endowed with the natural product topology.

The extension of a binary operation $*$ from $X$ to $G(X)$ can be defined in the same way as for ultrafilters, i.e., by the formula (1) applied to any two inclusion hyperspaces $\mathcal{U}, \mathcal{V} \in G(X)$. In $[G_2]$ it was shown that for an associative binary operation $*$ on $X$ the space $G(X)$ endowed with the extended operation becomes a compact right-topological semigroup. Besides the Stone-Čech extension, the semigroup $G(X)$ contains many important spaces as closed subsemigroups. In particular, the space

$$\lambda(X) = \{\mathcal{F} \in G(X) : \mathcal{F} = \mathcal{F}^\perp\}$$

of maximal linked systems on $X$ is a closed subsemigroup of $G(X)$. The space $\lambda(X)$ is well-known in General and Categorial Topology as the superextension of $X$, see [vM], [TZ]. Endowed with the extended binary
operation, the superextension $\lambda(X)$ of a semigroup $X$ is a supercompact right-topological semigroup containing $\beta(X)$ as a subsemigroup.

The thorough study of algebraic properties of the superextensions of groups was started in \cite{BGN} where we described right and left zeros in $\lambda(X)$ and detected all groups $X$ with commutative superextension $\lambda(X)$ (those are groups of cardinality $|X| \leq 4$). In \cite{BGN} we also described the structure of the semigroups $\lambda(X)$ for all finite groups $X$ of cardinality $|X| \leq 5$. In \cite{BG3} we shall describe the structure of minimal left ideals of the superextensions of groups. In this paper we concentrate at cancellativity and centers (topological and algebraic) in the superextensions $\lambda(X)$ of groups $X$. Since $\lambda(X)$ is an intermediate subsemigroup between $\beta(X)$ and $G(X)$ the obtained results for $\lambda(X)$ in a sense are intermediate between those for $\beta(X)$ and $G(X)$.

In section 2 we describe cancelable elements of $\lambda(X)$. In particular, we show that for a finite group $X$ all left or right cancelable elements of $\lambda(X)$ are principal ultrafilters. On the other hand, if a group $X$ is countable, then the set of right cancelable elements has open dense intersection with the subsemigroup $\lambda^0(X) \subset \lambda(X)$ of free maximal linked systems, see Theorem 2.4. This resembles the situation with the semigroup $\beta(X) \setminus X$ which contains a dense open subset of right cancelable elements (see \cite[8.10]{HS}), and also with the semigroup $G(X)$ whose right cancelable elements form a subset having open dense intersection with the set $G^\circ(X)$ of free inclusion hyperspaces, see \cite{G2}.

The section 3 is devoted to describing the topological center of $\lambda(X)$. By definition, the topological center of a right-topological semigroup $S$ is the set $\Lambda(S)$ of all elements $a \in S$ such that the left shift $l_a : S \to S$, $l_a : x \mapsto a \ast x$, is continuous. By \cite{HS}, for every group $X$ the topological center of the semigroup $\beta(X)$ coincides with $X$. On the other hand, the topological center of the semigroup $G(X)$ coincides with the subspace $G^\bullet(X)$ of $G(X)$ consisting of inclusion hyperspaces with finite support, see \cite[7.1]{G2}. A similar results holds also for the semigroup $\lambda(X)$: for any at most countable group $X$ the topological center of $\lambda(X)$ coincides with $\lambda^\bullet(X)$, see Theorem 3.4.

The final section 4 is devoted to describing the algebraic center of $\lambda(X)$. We recall that the algebraic center of a semigroup $S$ consists of all elements $s \in S$ that commute with all other elements of $S$. In Theorem 4.2 we shall prove that for any countable infinite group $X$ the algebraic center of $\lambda(X)$ coincides with the algebraic center of $X$. It is interesting to note that for any group $X$ the algebraic centers of the semigroups $\beta(X)$ and $G(X)$ also coincide with the center of the group $X$, see \cite[6.54]{HS} and \cite[6.2]{G2}. In contrast, for finite groups $X$ of cardinality $3 \leq |X| \leq 5$ the algebraic center of $\lambda(X)$ is strictly larger than the
algebraic center of $X$, see Remark 4.4.

1. Inclusion hyperspaces and superextensions

In this section we recall the necessary definitions and facts.

A family $\mathcal{L}$ of subsets of a set $X$ is called a linked system if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{L}$. Such a linked system $\mathcal{L}$ is maximal linked if $\mathcal{L}$ coincides with any linked system $\mathcal{L}'$ on $X$ that contains $\mathcal{L}$. Each (ultra)filter on $X$ is a (maximal) linked system. By $\lambda(X)$ we denote the family of all maximal linked systems on $X$. Since each ultrafilter on $X$ is a maximal linked system, $\lambda(X)$ contains the Stone-Čech extension $\beta(X)$ of $X$. It is easy to see that each maximal linked system on $X$ is an inclusion hyperspace on $X$ and hence $\lambda(X) \subset G(X)$. Moreover, it can be shown that $\lambda(X) = \{ \mathcal{A} \in G(X) : \mathcal{A} = \mathcal{A}^\perp \}$, see $[G_1]$.

By $[G_1]$ the subspace $\lambda(X)$ is closed in the space $G(X)$ endowed with the topology generated by the sub-base consisting of the sets

$$U^+ = \{ \mathcal{A} \in G(X) : U \in \mathcal{A} \} \text{ and } U^- = \{ \mathcal{A} \in G(X) : U \in \mathcal{A}^\perp \}$$

where $U$ runs over subsets of $X$. By $[G_1]$ and $[vM]$ the spaces $G(X)$ and $\lambda(X)$ are supercompact in the sense that any their cover by the sub-basic sets contains a two-element subcover. Observe that $U^+ \cap \lambda(X) = U^- \cap \lambda(X)$ and hence the topology on $\lambda(X)$ is generated by the sub-basis consisting of the sets

$$U^\pm = \{ \mathcal{A} \in \lambda(X) : U \in \mathcal{A} \}, \ U \subset X.$$  

We say that an inclusion hyperspace $\mathcal{A} \in G(X)$

- has finite support if there is a finite family $\mathcal{F} \subset \mathcal{A}$ of finite subsets of $X$ such that each set $A \in \mathcal{A}$ contains a set $F \in \mathcal{F}$;

- is free if for each $A \in \mathcal{A}$ and each finite subset $F \subset X$ the complement $A \setminus F$ belongs to $\mathcal{A}$.

By $G^\bullet(X)$ we denote the subspace of $G(X)$ consisting of inclusion hyperspaces with finite support and $G^\circ(X)$ stands for the subset of free inclusion hyperspaces on $X$. Those two sets induce the subsets

$$\lambda^\bullet(X) = G^\bullet(X) \cap \lambda(X) \text{ and } \lambda^\circ(X) = G^\circ(X) \cap \lambda(X)$$

in the superextension $\lambda(X)$ of $X$. By $[G_1]$, $\lambda^\bullet(X)$ is an open dense subset of $\lambda(X)$ while $\lambda^\circ(X)$ is closed and nowhere dense in $\lambda(X)$. 
Given any semigroup operation \( * : X \times X \to X \) on a set \( X \) we can extend this operation to \( G(X) \) letting 

\[
U \ast V = \left\{ \bigcup_{x \in U} x \ast V_x : U, \{V_x\}_{x \in U} \subset V \right\}
\]

for inclusion hyperspaces \( U, V \in G(X) \). Equivalently, the product \( U \ast V \) can be defined as 

\[
U \ast V = \{ A \subset X : \{ x \in X : x^{-1}A \in V \} \in U \} \tag{2}
\]

where \( x^{-1}A = \{ z \in X : x \ast z \in A \} \). By \([G_2]\) the so-extended operation turns \( G(X) \) into a right-topological semigroup. The structure of this semigroup was studied in details in \([G_2]\). In this paper we shall concentrate at the study of the algebraic structure of the semigroup \( \lambda(X) \) for a group \( X \).

The formula (2) implies that the product \( U \ast V \) of two maximal linked systems \( U \) and \( V \) is a principal ultrafilter if and only if both \( U \) and \( V \) are principal ultrafilters. So we get the following

**Proposition 1.1.** For any group \( X \) the set \( \lambda(X) \setminus X \) is a two-sided ideal in \( \lambda(X) \).

2. Cancelable elements of \( \lambda(X) \)

In this section, given a group \( X \) we shall detect cancelable elements of \( \lambda(X) \).

We recall that an element \( x \) of a semigroup \( S \) is right (resp. left) cancelable if for every \( a, b \in X \) the equation \( x \ast a = b \) (resp. \( a \ast x = b \)) has at most one solution \( x \in S \). This is equivalent to saying that the right (resp. left) shift \( r_a : S \to S, r_a : x \mapsto x \ast a, \) (resp. \( l_a : S \to S, l_a : x \mapsto a \ast x \)) is injective.

**Proposition 2.1.** Let \( G \) be a finite group. If \( C \in \lambda(G) \) is left or right cancelable, then \( C \) is a principal ultrafilter.

**Proof.** Assume that some maximal linked system \( a \in \lambda(G) \setminus G \) is left cancelable. This means that the left shift \( l_a : \lambda(G) \to \lambda(G), l_a : x \mapsto a \circ x, \) is injective. By Proposition 1.1, the set \( \lambda(G) \setminus G \) is an ideal in \( \lambda(G) \). Consequently, \( l_a(\lambda(G)) = a \ast \lambda(G) \subset \lambda(G) \setminus G \). Since \( \lambda(G) \) is finite, \( l_a \) cannot be injective. \( \square \)

Thus the semigroups \( \lambda(X) \) can have non-trivial cancelable elements only for infinite groups \( X \). According to [HS, 8.11] an ultrafilter \( U \in \beta(X) \)
is right cancelable if and only if the orbit \( \{xU : x \in X\} \) is discrete in \( \beta(X) \) if and only if for every \( x \in X \) there is a set \( U_x \in \mathcal{U} \) such that the indexed family \( \{x \ast U_x : x \in X\} \) is disjoint.

This characterization admits a partial generalization to the semigroup \( G(X) \). According to [G2] if an inclusion hyperspace \( A \in G(X) \) is right cancelable in \( G(X) \), then its orbit \( \{x \ast A : x \in X\} \) is discrete in \( G(X) \).

On the other hand, \( \mathcal{A} \) is cancelable provided for every \( x \in X \) there is a set \( A_x \in \mathcal{A} \cap \mathcal{A}^\perp \) such that the indexed family \( \{x \ast A_x : x \in X\} \) is disjoint. The latter means that \( x \ast A_x \cap y \ast A_y = \emptyset \) for any distinct points \( x, y \in X \). This result on right cancelable elements in \( G(X) \) will help us to prove a similar result on the right cancelable elements in the semigroup \( \lambda(X) \).

**Theorem 2.2.** Let \( X \) be a group and \( \mathcal{L} \in \lambda(X) \) be a maximal linked system on \( X \).

1. If \( \mathcal{L} \) is right cancelable in \( \lambda(X) \), then the orbit \( \{x \mathcal{L} : x \in X\} \) is discrete in \( \lambda(X) \) and \( x \mathcal{L} \neq y \mathcal{L} \) for all \( x, y \in X \).

2. \( \mathcal{L} \) is right cancelable in \( \lambda(X) \) provided for every \( x \in X \) there is a set \( S_x \in \mathcal{L} \) such that the family \( \{x \ast S_x : x \in X\} \) is disjoint.

**Proof.** 1. First note that the right cancelativity of a maximal linked system \( \mathcal{L} \in \lambda(X) \) is equivalent to the injectivity of the map \( \mu_X \circ \lambda \bar{R}_\mathcal{L} : \lambda(X) \to \lambda(X) \), see [G2]. We recall that \( \mu_X : \lambda^2(X) \to \lambda(X) \) is the multiplication of the monad \( \lambda = (\lambda, \mu, \eta) \) while \( \bar{R}_\mathcal{L} : \beta(X) \to \lambda(X) \) is the Stone-Čech extension of the right shift \( R_\mathcal{L} : X \to \lambda(X) \), \( R_\mathcal{L} : x \mapsto x \ast \mathcal{L} \). The map \( \bar{R}_\mathcal{L} \) certainly is not injective if \( R_\mathcal{L} \) is not an embedding, which is equivalent to the discreteness of the indexed set \( \{x \ast \mathcal{L} : x \in X\} \) in \( \lambda(X) \).

2. Assume that \( \{S_x\}_{x \in X} \subset \mathcal{L} \) is a family such that \( \{x \ast S_x : x \in X\} \) is disjoint. To prove that \( \mathcal{L} \) is right cancelable, take two maximal linked systems \( \mathcal{A}, \mathcal{B} \in \lambda(X) \) with \( \mathcal{A} \circ \mathcal{L} = \mathcal{B} \circ \mathcal{L} \). It is sufficient to show that \( \mathcal{A} \subset \mathcal{B} \). Take any set \( A \in \mathcal{A} \) and observe that the set \( \bigcup_{a \in A} aS_a \) belongs to \( \mathcal{A} \circ \mathcal{L} = \mathcal{B} \circ \mathcal{L} \). Consequently, there is a set \( B \in \mathcal{B} \) and a family of sets \( \{L_b\}_{b \in B} \subset \mathcal{L} \) such that

\[
\bigcup_{b \in B} bL_b \subset \bigcup_{a \in A} aS_a.
\]

It follows from \( S_b \in \mathcal{L} \) that \( L_b \cap S_b \) is not empty for every \( b \in B \).

Since the sets \( aS_a \) and \( bS_b \) are disjoint for different \( a, b \in X \), the inclusion

\[
\bigcup_{b \in B} b(L_b \cap S_b) \subset \bigcup_{b \in B} bL_b \subset \bigcup_{a \in A} aS_a
\]

implies \( B \subset A \) and hence \( A \in \mathcal{B} \).

\( \square \)
It is interesting to remark that the first item gives a necessary but not sufficient condition of the right cancelability in $\lambda(X)$ (in contrast to the situation in $\beta(X)$).

**Example 2.3.** By [BGN, 6.3], the superextension $\lambda(C_4)$ of the 4-element cyclic group $C_4$ is isomorphic to the direct product $C_4 \times C_1^1$, where $C_1^1 = C_2 \cup \{e\}$ is the 2-element cyclic group with attached external unit $e$ (the latter means that $ex = xe = x$ for all $x \in C_2^1$). Consequently, each element of the ideal $\lambda(C_4) \setminus C_4$ is not cancelative but has the discrete 4-element orbit $\{xL : x \in C_4\}$. In fact all the (left or right) cancelable elements of $\lambda(C_4)$ are principal ultrafilters, see Proposition 2.1.

According to [HS, 8.10], for each infinite group the semigroup $\beta(X)$ contains many right cancelable elements. In fact, the set of right cancelable elements contains an open dense subset of $\beta(X) \setminus X$. A similar result holds also for the semigroup $G(X)$ over a countable group $X$: the set of right cancelable elements of $G(X)$ contains an open dense subset of the subsemigroup $G^0(X)$. Theorem 2.2 will help us to prove a similar result for the semigroup $\lambda(X)$.

**Theorem 2.4.** For each countable group $X$ the subsemigroup $\lambda^0(X)$ of free maximal linked systems contains an open dense subset consisting of right cancelable elements in the semigroup $\lambda(X)$.

**Proof.** Let $X = \{x_n : n \in \omega\}$ be an injective enumeration of the countable group $X$. Given a free maximal linked system $\mathcal{L} \in \lambda^0(X)$ and a neighborhood $O(\mathcal{L})$ of $\mathcal{L}$ in $\lambda^0(X)$, we should find a non-empty open subset of right cancelable elements in $O(\mathcal{L})$. Without loss of generality, the neighborhood $O(\mathcal{L})$ is of basic form:

$$O(\mathcal{L}) = \lambda^0(X) \cap U_0^+ \cap \cdots \cap U_{n-1}^+$$

for some sets $U_1, \ldots, U_{n-1}$ of $X$. Those sets are infinite because $\mathcal{L}$ is free. We are going to construct an infinite set $C = \{c_n : n \in \omega\} \subset X$ that has infinite intersection with the sets $U_i$, $i < n$, and such that for any distinct $x, y \in X$ the intersection $xC \cap yC$ is finite. The points $c_k$, $k \in \omega$, composing the set $C$ will be chosen by induction to satisfy the following conditions:

- $c_k \in U_j$ where $j = k \mod n$;
- $c_k$ does not belong to the finite set

$$F_k = \{z \in X : \exists i, j \leq k \exists l < k \ (x_iz = x_je_l)\}.$$
It is clear that the so-constructed set $C = \{c_k : k \in \omega\}$ has infinite intersection with each set $U_i, i < n$. The choice of the points $c_k$ for $k > j$ implies that $x_i C \cap x_j C \subset \{x_i c_m : m \leq j\}$ is finite.

Now let $C$ be a free maximal linked system on $X$ enlarging the linked system generated by the sets $C$ and $U_0, \ldots, U_{n-1}$. It is clear that $C \in O(\mathcal{L})$. Consider the open neighborhood $$O(C) = O(\mathcal{L}) \cap C^\pm$$ of $C$ in $\lambda^o(X)$.

We claim that each maximal linked system $A \in O(C)$ is right cancelable in $\lambda(X)$. This will follow from Proposition 2.2 as soon as we construct a family of sets $\{A_i\}_{i \in \omega} \in A$ such that $x_i A_i \cap x_j A_j = \emptyset$ for any numbers $i < j$. Observe that the sets $$A_i = C \setminus \{x_i^{-1} x_k c_m : k, m \leq i\}, \quad i \in \omega,$$ have the required property. \hfill \Box

By [HS, 8.34], the semigroup $\beta(\mathbb{Z}) \setminus \mathbb{Z}$ contains an open dense subset consisting of free ultrafilters that are left cancelable in $\beta(\mathbb{Z})$. On the other hand, by [G2, 8.1], the only left cancelable elements of the semigroup $G(\mathbb{Z})$ are principal ultrafilters.

**Problem 2.5.** Is some free maximal linked system left cancelable in the semigroup $\lambda(\mathbb{Z})$?

### 3. The topological center of $\lambda(X)$

In this section we describe the topological center of the superextension $\lambda(X)$ of a group $X$. By definition, the **topological center** of a right-topological semigroup $S$ is the set $\Lambda(S)$ of all elements $a \in S$ such that the left shift $l_a : S \to S$, $l_a : x \mapsto a \ast x$, is continuous.

By [HS, 4.24, 6.54], for every group $X$ the topological center of the semigroup $\beta(X)$ coincides with $X$. On the other hand, the topological center of the semigroup $G(X)$ coincides with $G^\bullet(X)$, see [G2, 7.1]. A similar result holds also for the semigroup $\lambda(X)$: the topological center of $\lambda(X)$ coincides with $\lambda^\bullet(X)$ (at least for countable groups $X$).

To prove this result we shall use so-called detecting ultrafilters.

**Definition 3.1.** A free ultrafilter $D$ on a group $X$ is called **detecting** if there is an indexed family of sets $\{D_x : x \in X\} \subset D$ such that for any $A \subset X$
1. the set \( U_A = \bigcup_{x \in A} xD_x \) has the property: \( U_A \cup yU_A \neq X \) for all \( y \in X \);

2. for every \( D \in \mathcal{D} \) the set \( \{x \in X : xD \subset U_A\} \) is finite and lies in \( A \).

**Lemma 3.2.** On each countable group \( X \) there is a detecting ultrafilter.

**Proof.** Let \( X = \{x_n : n \in \omega\} \) be an injective enumeration of the group \( X \) such that \( x_0 \) is the neutral element of \( X \). For every \( n \in \omega \) let \( F_n = \{x_i, x_i^{-1} : i \leq n\} \). Let \( a_0 = x_0 \) and inductively, for every \( n \in \omega \) choose an element \( a_n \in X \) so that

\[
a_n \notin F_n^{-1}F_nA_{\leq n} \quad \text{where} \quad A_{\leq n} = \{a_i : i \leq n\}.
\]

For every \( n \in \omega \) let \( A_{\geq n} = \{a_i : i \geq n\} \). Let also \( D_0 = \{a_2i : i \in \omega\} \).

Let us show that for any distinct numbers \( n, m \) the intersection \( x_nA_{\geq n} \cap x_mA_{\geq m} \) is empty. Otherwise there would exist two numbers \( i \geq n \) and \( j \geq m \) such that \( x_na_i = x_m a_j \). It follows from \( x_n \neq x_m \) that \( i \neq j \). We lose no generality assuming that \( j > i \). Then \( x_n a_i = x_m a_j \) implies that

\[
a_j = x_m^{-1}x_n a_i \in F_j^{-1}F_j A_{< j},
\]

which contradicts the choice of \( a_j \).

Let \( \mathcal{D} \in \beta(X) \) be any free ultrafilter such that \( D_0 \in \mathcal{D} \) and \( \mathcal{D} \) is not a P-point. To get such an ultrafilter, take \( \mathcal{D} \) to be a cluster point of any countable subset of \( D_0^+ \cap \beta(X) \setminus X \). Using the fact that \( \mathcal{D} \) fails to be a P-point, we can take a decreasing sequence of sets \( \{V_n : n \in \omega\} \subset \mathcal{D} \) having no pseudointersection in \( \mathcal{D} \). The latter means that for every \( D \in \mathcal{D} \) the almost inclusion \( D \subset V_n \) (which means that \( D \setminus V_n \) is finite) holds only for finitely many numbers \( n \).

For every \( n \in \omega \) let \( D_n = V_n \cap A_{\geq n} \cap D_0 \). We claim that the ultrafilter \( \mathcal{D} \) and the family \( (D_n)_{n \in \omega} \) satisfy the requirements of Definition 3.1.

Take any subset \( A \subset \omega \) and consider the set \( U_A = \bigcup_{n \in A} x_n D_n \).

First we verify that \( U_A \cup yU_A \neq X \) for each \( y \in X \). Find \( m \in \omega \) with \( y^{-1} = x_m \) and take any odd number \( k > m \). We claim that \( a_k \notin U_A \cup yU_A \). Otherwise, \( a_k \in x_mD_n \cup x_m^{-1}x_n D_n \) for some \( n \in A \). It follows that \( a_k = x_n a_i \) or \( a_k = x_m^{-1}x_n a_i \) for some even \( i \geq n \). If \( k > i \), then both the equalities are forbidden by the choice of \( a_k \notin F_k^{-1}F_k A_{\leq k} \supset \{x_n a_i, x_m^{-1}x_n a_i\} \). If \( k < i \), then those equalities are forbbiden by the choice of

\[
a_i \notin F_i^{-1}F_i A_{< i} \supset \{x_n^{-1}a_k, x_n^{-1}x_m^{-1}a_k\}.
\]

Therefore, \( U_A \cup yU_A \neq X \).

Next, given arbitrary \( D \in \mathcal{D} \) we show that the set \( S = \{n \in \omega : x_n D \subset U_A\} \) is finite and lies in \( A \). First we show that \( S \subset A \). Assuming
the converse, we could find \( n \in S \setminus A \). Then \( x_n(D \cap D_n) \subset x_nD \subset U_A = \bigcup_{m \in A} x_mD_m \), which is not possible because the set \( x_nD_n \) misses the union \( U_A \). Thus \( S \subset A \). Next, we show that \( S \) is finite. By the choice of the sequence \((V_n)\), the set \( F = \{ n \in \omega : D \cap D_0 \not\subseteq V_n \} \) is finite. We claim that \( S \subset F \). Indeed, take any \( m \in S \). It follows from \( x_mD \subset U_A = \bigcup_{n \in A} x_mD_n \) and \( x_mA_{\geq m} \cap \bigcap_{n \neq m} x_mD_n = \emptyset \) that

\[
x_m(D \cap D_0) \subset^* x_m(D \cap A_{\geq m}) \subset x_mD_m \subset x_mV_m
\]

and hence \( m \in F \). \( \square \)

**Theorem 3.3.** Let \( X \) be a group admitting a detecting ultrafilter \( D \). For a maximal linked system \( A \in \lambda(X) \) the following conditions are equivalent:

1. the left shift \( L_A : G(X) \to G(X) \), \( L_A : \mathcal{F} \mapsto A \circ \mathcal{F} \), is continuous;
2. the left shift \( l_A : \lambda(X) \to \lambda(X) \), \( l_A : \mathcal{L} \mapsto A \circ \mathcal{L} \), is continuous;
3. the left shift \( l_A : \lambda(X) \to \lambda(X) \) is continuous at the detecting ultrafilter \( D \);
4. \( A \in \lambda^*(X) \).

**Proof.** The implications \((1) \Rightarrow (2) \Rightarrow (3)\) are trivial while \((4) \Rightarrow (1)\) follows from Theorem 7.1 [G2] asserting that the topological center of the semigroup \( G(X) \) coincides with \( G^*(X) \). To prove that \((3) \Rightarrow (4)\), assume that the left shift \( l_A : \lambda(X) \to \lambda(X) \) is continuous at the detecting ultrafilter \( D \).

We need to show that \( A \in \lambda^*(G) \). By Theorem 8.1 of [G1], it suffices to check that each set \( A \in \mathcal{A} \) contains a finite set \( F \in \mathcal{A} \).

Since \( D \) is a detecting ultrafilter, there is a family of sets \( \{ D_x : x \in X \} \subset D \) such that for every \( D \in D \) the set \( \{ x \in X : xD \subset \bigcup_{x \in A} xD_x \} \) is finite and lies in \( A \).

Consider the set \( U_A = \bigcup_{x \in X} xD_x \) belonging to the product \( A \circ D \). The continuity of the left shift \( l_A : \lambda(X) \to \lambda(X) \) at \( D \) yields us a set \( D \in D \), such that \( l_A(D^\pm) \subset U_A^\pm \). This means that \( U_A \in A \circ \mathcal{L} \) for any maximal linked system \( \mathcal{L} \in \lambda(X) \) that contains \( D \).

The choice of \( D \) and \( \{ D_x \}_{x \in X} \) guarantees that

\[
S = \{ x \in X : xD \subset U_A \}
\]

is a finite subset lying in \( A \). We claim that there is a maximal linked system \( \tilde{\mathcal{L}} \in \lambda(X) \) such that \( D \in \tilde{\mathcal{L}} \) and \( x^{-1}U_A \notin \tilde{\mathcal{L}} \) for all \( x \notin S \). Such a system \( \tilde{\mathcal{L}} \) can be constructed as an enlargement of the linked system

\[
\mathcal{L} = \{ D, X \setminus x^{-1}U_A : x \in X \setminus S \}.
\]
The latter system is linked because of the definition of $S = \{ x \in X : D \subset x^{-1}U_A \}$ and the property (1) of the family $(D_x)_{x \in X}$ from Definition 3.1.

Take any maximal linked system $\tilde{L}$ containing $L$ and observe that $D \in L$ and

$$\{ x \in X : x^{-1}U_A \in \tilde{L} \} = \{ x \in X : x^{-1}U_A \in L \} = S \subset A.$$ 

Taking into account that $D \in L$, we conclude that $A \circ \mathcal{L} = l_A(\mathcal{L}) \in U_A^{\pm}$ and hence the set $S = \{ x \in X : x^{-1}U_A \in L \} \in \mathcal{A}$. This set $S$ is the required finite subset of $A$ belonging to $\mathcal{A}$. \qed

Combining Theorem 3.3 with Lemma 3.2 we obtain the main result of this section.

**Corollary 3.4.** For any countable group $X$ the topological center of the semigroup $\lambda(X)$ coincides with $\lambda^*(X)$.

**Question 3.5.** Is Theorem 3.4 true for a group $X$ of arbitrary cardinality?

### 4. The algebraic center of $\lambda(X)$

This section is devoted to studying the algebraic center of $\lambda(X)$. We recall that the *algebraic center* of a semigroup $S$ consists of all elements $s \in S$ that commute with all other elements of $S$. Such elements $s$ are called *central* in $S$.

**Lemma 4.1.** Let $X$ be a group with the neutral element $e$. A maximal linked system $\mathcal{A} \in \lambda(X)$ is not central in $\lambda(X)$ provided there are sets $S, T \subset X$ such that

1. $|T| = 3$;
2. for each $A \in \mathcal{A}$ we get $A \cap S \in \mathcal{A}$ and $|A \cap S| \geq 2$;
3. there is a finite set $B \in \mathcal{A}$ such that $BS^{-1} \cap T^{-1}T \subset \{ e \}$.

**Proof.** We claim that $\mathcal{A}$ does not commute with the maximal linked system $T = \{ A \subset X : |A \cap T| \geq 2 \}$. By (3), the maximal linked system $\mathcal{A}$ contains a finite set $B \in \mathcal{A}$ such that $BS^{-1} \cap TT^{-1} \subset \{ e \}$. By (2), we can assume that $B \subset S$ and $B$ is minimal in the sense that each $B' \subset B$ with $B' \in \mathcal{A}$ is equal to $B$. By (2), $|B| \geq 2$. Choose a family $\{ T_b \}_{b \in B}$ of 2-element subsets of $T$ such that $\bigcup_{b \in B} T_b = T$. Such a choice is possible because $|B| \geq 2$. The union $\bigcup_{b \in B} bT_b$ belongs to $A \circ T = T \circ A$ and hence we can find a subset $D \in T$ and a family $\{ A_d \}_{d \in D} \subset A$ with...
\[ \bigcup_{d \in D} dA_d \subset \bigcup_{b \in B} bT_b. \]
By (2), we can assume that each \( A_d \subset S \). Replacing \( D \) by a smaller set, if necessary, we can assume that \( D \subset T \) and \( |D| = 2 \). We claim that \( A_d = B \) for all \( d \in D \) and \( T_b = D \) for all \( b \in B \).

Indeed, take any \( d \in D \) and any \( a \in A_d \). Since \( da \in \bigcup_{x \in D} xA_x \subset \bigcup_{b \in B} bT_b \), there are \( b \in B \) and \( t \in T_b \) with \( da = bt \). Then \( T^{-1}T \ni t^{-1}d = ba^{-1} \in BA \cup \subset BS^{-1} \). Taking into account that \( T^{-1}T \cap BS^{-1} \subset \{e\} \), we conclude that \( t^{-1}d = ba^{-1} \) is the neutral element of \( X \). Consequently, \( a = b \in B \) and \( d = t \in T_b \). Since \( a \in A_d \) was arbitrary, we get \( A \ni A_d \subset B \). The minimality of \( B \in A \) implies that \( A_d = B \). It follows from \( d = t \in T_b \) for \( d \in D \) that \( D \subset T_b \). Since \( |D| = |T_b| = 2 \), we get \( D = T_b \) for every \( b \in B = A_d \). Consequently, \( D = \bigcup_{b \in B} T_b = T \) which contradictions (1). □

By [HS, 6.54], for every group \( X \) the algebraic center of the semigroups \( \beta(X) \) coincides with the center of the group \( X \). Consequently, the semigroup \( \beta(X) \setminus X \) contains no central elements. A similar result holds also for the semigroup \( \lambda(X) \).

**Theorem 4.2.** For any countable infinite group \( X \) the algebraic center of \( \lambda(X) \) coincides with the algebraic center of \( X \).

**Proof.** It is clear that all central elements of \( X \) are central in \( \lambda(X) \). Now assume that a maximal linked system \( C \in \lambda(X) \) is a central element of the semigroup \( \lambda(X) \). Observe that the left shift \( l_C : \lambda(X) \to \lambda(X) \), \( l_C : \mathcal{X} \mapsto C \circ \mathcal{X} \), is continuous because it coincides with the right shift \( r_C : \lambda(X) \to \lambda(X) \), \( r_C : \mathcal{X} \mapsto \mathcal{X} \circ C \). Consequently, \( C \) belongs to the topological center of \( \lambda(X) \). Applying Theorem 3.4, we conclude that \( C \in \lambda^\bullet(X) \). We claim that \( C \) is a principal ultrafilter.

Assuming the converse, consider the family \( C_0 \) of minimal finite subsets in \( C \). Since \( C \in \lambda^\bullet(X) \), the family \( C_0 \) is finite and hence has finite union \( S = \bigcup C_0 \). Take any set \( B \in C_0 \) and observe that \( |B| \geq 2 \) (because \( C \) is not a principal ultrafilter).

Since the group \( X \) is infinite, we can choose a 3-element subset \( T \subset X \) such that \( T^{-1}T \cap BS^{-1} \subset \{e\} \). Now we see that the maximal linked system \( C \) satisfies the conditions of Lemma 4.1 and hence is not central in \( \lambda(X) \), which is a contradiction. □

We do not know if Theorem 4.2 is true for any infinite group \( X \).

**Question 4.3.** Let \( X \) be an infinite group. Does the algebraic center of \( \lambda(X) \) coincides with the algebraic center of \( X \)?

**Remark 4.4.** Theorem 4.2 certainly is not true for finite groups. According to [BGN, §6], for any group \( X \) of cardinality \( 3 \leq |X| \leq 5 \) the
semigroup $\lambda(X)$ contains a central element, which is not a principal ultrafilter.

**Problem 4.5.** Characterize (finite) abelian groups $X$ whose superextensions $\lambda(X)$ have central elements distinct from principal ultrafilters. Have all such groups $X$ cardinality $|X| \leq 5$?

It is interesting to remark that the semigroup $\lambda(X)$ contains many non-principal maximal linked systems that commute with all ultrafilters.

**Proposition 4.6.** Let $X$ be a group and $Y, Z \subset X$ be non-empty subsets such that $yz = zy$ for all $y \in Y$, $z \in Z$. Then for any $\mathcal{L} \in \lambda^*(Y) \subset \lambda^*(X)$ and $\mathcal{U} \in \beta(Z) \subset \beta(X)$ we get $\mathcal{L} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{L}$.

**Proof.** It is sufficient to prove that $\mathcal{L} \circ \mathcal{U} \subset \mathcal{U} \circ \mathcal{L}$. Let $\bigcup_{x \in L} x^* U_x \in \mathcal{L} \circ \mathcal{U}$. Without loss of generality we may assume that $L = \{x_1, \ldots, x_n\}$ is finite, $L \subset Y$ and $U_{x_i} \subset Z$. Denote $V = U_{x_1} \cap \ldots \cap U_{x_n} \in \mathcal{U}$. Then

$$\bigcup_{x \in L} x^* U_x = \bigcup_{x \in L} U_x^* x \supset V^* L \in \mathcal{U} \circ \mathcal{L}.$$ 

It follows that $\bigcup_{x \in L} x^* U_x \in \mathcal{U} \circ \mathcal{L}$ and the proof is complete. ~

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