1 Introduction

In this paper we construct a representation of the blob algebra \[ b_n \] over a ring allowing base change to every interesting (i.e. non–semisimple) specialisation which, in quasihereditary specialisations, passes to a full tilting module.

The Temperley–Lieb algebras are a tower \( T_0(q) \subset T_1(q) \subset \ldots \) of one–parameter finite dimensional algebras \[ 28 \], each with a basis independent of \( q \). These algebras are quasihereditary \[ 3, 8 \] except in case \( q+q^{-1} = 0 \). Accordingly one may in principle construct tilting modules, full tilting modules, and corresponding Ringel duals. In fact, if \( V \) is a free module of rank 2 over the ground ring then \( T_n(q) \) has an action on \( V^{\otimes n} \), and it is straightforward to show (see later) that \( V^{\otimes n} \) is a full tilting module in the quasihereditary cases. Since \( V^{\otimes n} \) exists over the ground ring, the Ringel dual can be constructed without having to pick a specialisation. The cases of \( n \) finite of this dual are a nested sequence of quotients of the quantum group \( U_q \mathfrak{sl}_2 \) \[ 17, 11 \].

This \( q \)–deformable duality and glorious limit structure \[ 16 \] (more usually observed with \( U_q \mathfrak{sl}_2 \) as the starting point) provides the mechanism for massive exchange of representation theoretic information between the two sides \[ 26, 10, 13, 4, 18 \]. In particular the weight theory of \( U_q \mathfrak{sl}_2 \) controls the representation theory of \( T_n(q) \) for all \( n \) simultaneously (as localisations of a global limit).

The blob algebras are a tower \( b_0 \subset b_1 \subset \ldots \) of two–parameter finite dimensional algebras (and \( b_n \supset T_n(q) \)). They are quasihereditary except at a finite set of parameter values. Accordingly one may in principle construct tilting modules and so on. \textit{Ab initio} one would have to expect such a construction to depend on the specialisation, as \textit{indecomposable} tilting modules do \[ 24 \]. On the other hand, it turns out \[ 23 \] that \( b_n \) has an action on \( V^{\otimes 2n} \), and in this paper we show that \( V^{\otimes 2n} \) is a full tilting module in the quasihereditary cases.

Historically, \( T_n(q) \) and \( U_q \mathfrak{sl}_2 \) were studied extensively separately, before the full tilting module/Ringel duality connection was known, but if one side, and the appropriate full tilting module, had been discovered first, the passage to the Ringel dual would rightly have been regarded as quite a significant spin–off! The \( b_n \) tilting property of \( V^{\otimes 2n} \) is a striking result, in as much as it places us in a position analogous to this (as it were, before the discovery of quantum groups).
Suitably prepared, the blob algebra may be regarded as a quotient of the Ariki–Koike algebra, which itself is a quotient of the affine Hecke algebra \([1, 15]\). Thus the representation theory of the blob algebra is part of the representation theory of the Ariki–Koike and of the affine Hecke algebra, this last point being the basic idea of \([21, 22]\) (see also Graham and Lehrer’s analysis \([14]\)). Although \(V^{\otimes 2n}\) can be regarded as a module of the Ariki–Koike algebra, it is not in any obvious way a sum of permutation modules in the usual Ariki–Koike sense \([4, 5, 6]\). In the absence of a natural tensor space (cf. \([2, 27]\)), these permutation modules form the starting point for most tilting related approaches to Ariki–Koike representation theory. Our approach is of an essentially different nature. In particular it gives a weight theory (in the sense mentioned above) for \(b_n\) which, for Ariki–Koike, would imply a structure which it seems very unlikely to possess (see \([23]\)).

The blob algebra, and certain generalisations, have been observed to manifest several indicators of an underlying structure evocative of algebraic Lie theory (such as the role played in their representation theory by alcove geometry — see \([23]\)). We wish to understand the underlying reasons for the extra structure. The Temperley–Lieb paradigm suggests that an appropriately prepared Ringel dual is a good place to look (hence ‘virtual algebraic Lie theory’). For example, the results of \([24]\) suggest that this “dual blob algebra” should be reminiscent of the Kac–Moody quantum algebra \(U_q\hat{\mathfrak{sl}}_2\).

Given the full background, a natural approach to proving that \(V^{\otimes n}\) is a tilting module for \(T_n(q)\) is to use the duality itself (that this module is a tilting module on the dual side is a direct consequence of the general machinery of Donkin \([3, 4, 10]\) et al \([12]\)). The challenge here is that for the blob no such general machinery yet exists. Accordingly we include here a proof in the Temperley–Lieb case which does not appeal to the algebraic Lie theory machinery, but only to quasiheredity. This exercise is motivated only by the need to understand how such a proof might work, for use in the blob case. The blob case is then the main object of this paper (see section 4).

1.1 Preliminaries

The blob algebra \(b_n\) is usually defined in terms of a certain basis of diagrams and their compositions \([22]\), from which it derives its name. This blob algebra is isomorphic to an algebra defined by a presentation \([4]\). We will only need the presentation.

For \(R\) a commutative ring, \(x\) an invertible element in \(R\), \(q = x^2\), and \(\gamma, \delta, e \in R\), define \(b_n^R\) to be the \(R\)–algebra with generators \(\{1, e, U_1, \ldots, U_{n-1}\}\) and relations

\[
U_i U_i = (q + q^{-1}) U_i
\]

\[
U_i U_{i+1} U_i = U_i
\]

\[
U_i U_j = U_j U_i \quad (|i - j| \neq 1)
\]

\[
U_1 e U_1 = \gamma U_1
\]

\[
e e = \delta e
\]

\[
U_i e = e U_i \quad (i \neq 1).
\]

By the isomorphism with the diagram algebra this algebra is a free \(R\)–module (with basis most conveniently described in terms of diagrams, however we do not otherwise need this basis, so we will not recall it here — see \([22]\)).
It will be evident that $e$ can be rescaled to change $\gamma$ and $\delta_e$ by the same factor. Thus, if we require that $\delta_e$ is invertible, then we might as well replace it by 1. (This brings us to the original two-parameter definition of the algebra.)

For $k$ a field which is a $R$-algebra define $b_n = k \otimes_R b_n^R$. It is known [22] that the representation theory of $b_n$ falls into one of three distinct categories, depending on the number of integer values of $a$ for which

$$\gamma[a]_q = \delta_e[a - 1]_q$$

(where $[a]_q$ is the usual $q$-number). If there is no solution then $b_n$ is semisimple and has trivial tilting theory. Accordingly it is convenient to reparameterize into the following form:

$$\gamma = q^{m-1} - q^{-m+1}, \quad \delta_e = q^m - q^{-m}. \quad (4)$$

Note that provided $m$ is integer (which includes all the interesting cases) this parameterization has a lattice in $b_n^{\mathbb{Z}[q,q^{-1}]}$ (i.e. $\gamma, \delta_e$ lie in $\mathbb{Z}[q,q^{-1}]$).

Previously used parameterizations include

$$\gamma = \frac{[m-1]}{[m]}, \quad \delta_e = 1;$$

and

$$\gamma = \pm [m - 1], \quad \delta_e = \pm [m]$$

(see [22], [24], [4] respectively). Our form has the mild disadvantage that it is not a simple rescaling in case $q - q^{-1} = 0$.

## 2 The ‘crypto–tensor’ representations

We now recall the representations $\rho_\chi$ of $b_n$ defined in [23, §6.1].

Set

$$U^\chi = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & \chi \end{array} \right)$$

and $U^\chi = U^\chi(0)$.

Let $V = \text{span}\{v_1, v_2\}$. Let seq $\{1, 2\}$ denote the set of words of finite length in $\{1, 2\}$, seq$_n \{1, 2\}$ the subset of words of length $n$, and seq$_n^r \{1, 2\}$ the subset of this in which the number of 1s is $r$. For $w \in \text{seq}_n \{1, 2\}$ let $\#^1(w)$ denote the number of 1s in $w$ (so $w \in \text{seq}_n^r \{1, 2\}$ implies $\#^1(w) = r$). Then $V^\otimes n$ has basis \{\(v_1 \otimes v_2 \otimes \ldots \otimes v_n \mid i_1 i_2 \ldots i_n \in \text{seq}_n \{1, 2\}\}\}. We will adopt the shorthand of writing the sequence for the basis element. We ascribe the usual lexicographic order to this basis (11,12,21,22 and so on). Let $U^\chi$ act on $V \otimes V$ with respect to this ordering of the basis.

Let $\mu^\chi(U_i) \in \text{End}(V^\otimes n)$ be a matrix acting trivially on every tensor factor except the $i^{th}$ and $(i + 1)^{th}$, where it acts as $+U^\chi$. Write $\mu^\chi(U_i)$ for $\mu^\chi(U_i)$. The Temperley–Lieb algebra $T_n(q)$ is the subalgebra of $b_n$ with generators $\{1, U_1, \ldots, U_{n-1}\}$. The tensor space representation of $T_n(q)$ is given by $\mu^\chi$.

Note that

$$(U^s \otimes U^t)(1 \otimes U^r(\chi) \otimes 1)(U^s \otimes U^t) = \left( \frac{r}{st} + \frac{st}{r} + \frac{t}{s} \chi \right) (U^s \otimes U^t) \quad (5)$$

for any $r, s, t, \chi$ (an explicit calculation).
Suppose henceforth that there is an element $a \in K$ such that $a^4 = -1$. Then $a^2 + a^{-2} = 0$. Set
\[
    r = a^2 q^m \\
    s = a^5 x \\
    t = a^3 x
\]
We have
\[
    r + r^{-1} = a^2 (q^m - q^{-m}) \\
    s + s^{-1} = a^5 x + a^3 x^{-1} \\
    t + t^{-1} = a^3 x + a^5 x^{-1} \\
    st = q \\
    [2]_s[2]_t = [2]_q \\
    \frac{st}{r} + \frac{r}{st} = a^2 (q^{m-1} - q^{1-m})
\]
Then by equation (5) there is an algebra homomorphism
\[
    \rho : b_n[Z,q,q^{-1}](q,m) \longrightarrow \text{End}_{Z[a,x,x^{-1}]}(V \otimes 2^n)
\]
given by
\[
    \rho : e \mapsto a^{-2} \mu^r(U_n) \\
    \rho : U_i \mapsto \mu^s(U_{n-i}) \mu^t(U_{n+i})
\]
for $b_n$ in the form described in equation (4).

There is another algebra homomorphism $\rho'$ defined in exactly the same way except that
\[
    \rho' : e \mapsto a^{-2} \mu^r[2]_r(U_n)
\]
Note that the $T_n(q)$–module $\mu^q$ has manifest direct summands with basis $\text{seq}^r_n\{1,2\}$ $(r = 0,1,...,n)$, called permutation modules. Similarly $\rho$ and $\rho'$ have manifest direct summands with basis $\text{seq}_2^r_n\{1,2\}$ which we will again call permutation modules.

Similarly evidently we have

**Proposition 1** The following are manifest direct sums (i.e. respecting the basis):
\[
    \text{Res}_{T_n}^{T_{n-1}} \mu^q = \mu^q \oplus \mu^q \\
    \text{Res}_{b_n}^{b_{n-1}} \rho = \rho \oplus \rho \oplus \rho \oplus \rho
\]
\[\text{\footnotesize It should be emphasised that although } b_n \text{ is a quotient of the Hecke algebra of type–B (which itself is an Ariki–Koike algebra), the above permutation modules do not coincide with the type–B or Ariki–Koike permutation modules described in \cite{[2], [25]}; in the notation of \cite{[23]} the quotient map sends } g_i + q \text{ to } U_i, \text{ and so } U_i \text{ will map a typical Ariki–Koike permutation module basis vector to a linear combination of precisely two basis vectors (see \cite{[7]}), which is clearly not the case in our situation. As a matter of fact, an Ariki–Koike permutation module is typically not a module for the blob algebra, even if its leading Specht factor is one.}\]
3 A Temperley–Lieb tilting module

The goal of this section is to prove that the tensor space module $V^\otimes n$ is a tilting module for the Temperley–Lieb algebra. This follows from general results (for instance in [10]), but we give here a self-contained argument which later generalizes to the blob algebra representation $\rho$.

We will from now on assume that $[2]_q \neq 0$ over $k$. Then the Temperley-Lieb algebras $T_n = T_n(q)$ are quasihereditary. In fact [19, 24], setting $\epsilon = \frac{1}{[2]_q}U_{n-1}$, we have that $\epsilon$ is part of a heredity chain for $T_n$, and since $\epsilon$ and $T_{n-2} \subset T_n$ commute, $T_n\epsilon$ is a right $T_{n-2}$–module, and indeed

$$\epsilon T_n \epsilon = \epsilon T_{n-2} \cong T_{n-2} \quad (11)$$

Let $F$ be the localization functor

$$F : T_n - \text{mod} \rightarrow T_{n-2} - \text{mod} : M \mapsto \epsilon M$$

and let $G$ be the globalization functor

$$G : T_{n-2} - \text{mod} \rightarrow T_n - \text{mod} : N \mapsto T_n \epsilon \otimes_{T_{n-2}} N$$

Note that $F$ is exact and $G$ is right exact, being the left adjoint of $F$.

3.1 Homological considerations

Since the categories $T_n - \text{mod}$ are quasihereditary they come with standard modules $\Delta_n(\lambda)$, costandards $\nabla_n(\lambda)$, simples $L_n(\lambda)$, their projective covers $P_n(\lambda)$, injective envelopes $I_n(\lambda)$ and tiltings $T_n(\lambda)$ for $\lambda \in \Lambda_n = \{n, n - 2, \ldots, 0/1\}$. (The heredity order $\triangleright$ is the reverse of the natural order on $\Lambda_n$ as a subset of $\mathbb{Z}$.) The following statements can be found in appendix A1 of Donkin’s book [10] (in a much more general setting than ours):

Proposition 2 Assume that $\lambda \in \Lambda_{n-2}$. Then

i) $F L_n(\lambda) = L_{n-2}(\lambda)$.

ii) $F \Delta_n(\lambda) = \Delta_{n-2}(\lambda)$ and $F \nabla_n(\lambda) = \nabla_{n-2}(\lambda)$.

iii) $FP_n(\lambda) = P_{n-2}(\lambda)$ and $FI_n(\lambda) = I_{n-2}(\lambda)$.

Otherwise (i.e. for $\lambda = n$) we have that $F L_n(\lambda) = F \Delta_n(\lambda) = F \nabla_n(\lambda) = 0$.

Our next step is to investigate the application of $G$ to these modules. Write $M \in \mathcal{F}_n(\Delta)$ if $M \in T_n - \text{mod}$ has a standard filtration. Write $(M : \Delta_n(\mu))$ for the multiplicity of $\Delta_n(\mu)$ as a filtration factor of $M$. We need the following Proposition:

Proposition 3 If $M \in \mathcal{F}_{n-2}(\Delta)$, then $GM$ also has a standard filtration. Furthermore the standard multiplicity is

$$(GM : \Delta_n(\mu)) = \begin{cases} (M : \Delta_{n-2}(\mu)) & \text{if } \mu \in \Lambda_{n-2} \\ 0 & \text{otherwise} \end{cases}$$
Proof: By Donkin’s homological criterion, see e.g. [10, A2.2 (iii)], we know that $GM \in \mathcal{F}_n(\Delta)$ if and only if

$$\text{Ext}_{T_n}^1(GM, \nabla_n(\mu)) = 0 \quad \forall \mu \in \Lambda_n$$

Applying $\text{Hom}_{T_n}(GM, -)$ to the short exact sequence $\nabla_n(\mu) \hookrightarrow I_n(\mu) \rightarrow Q_n(\mu)$ (defining $Q_n(\mu)$) yields a long exact sequence whose first terms are

$$0 \rightarrow \text{Hom}_{T_n}(GM, \nabla_n(\mu)) \rightarrow \text{Hom}_{T_n}(GM, I_n(\mu)) \rightarrow \text{Hom}_{T_n}(GM, Q_n(\mu))$$

Assume first that $\mu \in \Lambda_{n-2}$. Then by Proposition 3 and adjointness, the first three terms of this sequence become

$$0 \rightarrow \text{Hom}_{T_{n-2}}(M, \nabla_{n-2}(\mu)) \rightarrow \text{Hom}_{T_{n-2}}(M, I_{n-2}(\mu)) \rightarrow \text{Hom}_{T_{n-2}}(M, FQ_{n}(\mu))$$

which coincides with the beginning of the long exact sequence that arises from applying $\text{Hom}_{T_{n-2}}(M, -)$ to $F\nabla_n(\mu) \hookrightarrow FI_n(\mu) \rightarrow FQ_n(\mu)$. But then by (the easy direction of) the criterion, the last map of (12) is surjective and thus $\text{Ext}_{T_n}^1(GM, \nabla_n(\mu)) = 0$ as claimed.

Assume then that $\mu = n$. When we apply $F$ to the short exact sequence $\nabla_n(I) \hookrightarrow I_n(n) \rightarrow Q_n(n)$ we get an isomorphism $FI_n(n) \cong FQ_n(n)$. So applying $\text{Hom}_{T_n}(GM, -)$ to it, we get

$$0 \rightarrow \text{Hom}_{T_n}(GM, \nabla_n(n)) \rightarrow \text{Hom}_{T_n}(GM, I_n(n)) \rightarrow \text{Hom}_{T_n}(GM, Q_n(n))$$

where by adjointness the last map is an isomorphism, so also in this case $\text{Ext}_{T_n}^1(GM, \nabla_n(\mu)) = 0$, and the criterion applies.

To get the multiplicity statement, recall that since $M \in \mathcal{F}_{n-2}(\Delta)$, $GM \in \mathcal{F}_n(\Delta)$, we have

$$(GM : \Delta_n(\mu)) = \dim \text{Hom}_{T_n}(GM, \nabla_n(\mu)) = \dim \text{Hom}_{T_{n-2}}(M, F\nabla_n(\mu))$$

which is zero for $\mu = n$, while for $\mu \in \Lambda_{n-2}$

$$\dim \text{Hom}_{T_{n-2}}(M, \nabla_{n-2}(\mu)) = (M : \Delta_{n-2}(\mu))$$

and we are done. $\Box$

Remark 1 $G$ does not map $\mathcal{F}_{n-2}(\nabla)$ to $\mathcal{F}_n(\nabla)$.

We may now, incidentally, prove:

Corollary 3.1

$$L^iG\Delta_{n-2}(\lambda) = \begin{cases} \Delta_n(\lambda) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

(13)

Proof: The $i = 0$ case follows from Proposition 3. By applying $G$ to the short exact sequence $K_{n-2}(\lambda) \hookrightarrow P_{n-2}(\lambda) \rightarrow \Delta_{n-2}(\lambda)$, the last terms of the resulting long exact cohomology sequence are as follows:

$$L^iG\Delta_{n-2}(\lambda) \hookrightarrow GK_{n-2}(\lambda) \rightarrow GP_{n-2}(\lambda) \rightarrow G\Delta_{n-2}(\lambda)$$
Now $K_{n-2}(\lambda)$ has a standard filtration, so once again using Proposition \[3\] we get $[GP_{n-2}(\lambda)] = [G\Delta_{n-2}(\lambda)] + [GK_{n-2}(\lambda)]$, where as usual $[M]$ denotes the image of $M \in T_n$-mod in the Grothendieck group. But then $[L^1G\Delta_{n-2}(\lambda)] = 0$ and thus $L^1G\Delta_{n-2}(\lambda) = 0$.

To get the vanishing of the higher $L^iG\Delta_{n-2}(\lambda)$, we use induction from above (with respect to the heredity order) on $\lambda$. If $\lambda$ is maximal $(\lambda = 0 \text{ or } 1)$ then $P_n(\lambda) = \Delta_n(\lambda)$ and there is nothing to prove. Otherwise note that by the long exact sequence, $L^iGK_{n-2}(\lambda) = L^{i+1}G\Delta_{n-2}(\lambda)$ for $i > 0$ so it is enough to show that $L^iGK_{n-2}(\lambda) = 0$ for these $i$. But only $\Delta(\mu)$ with $\mu \triangleright \lambda$ occur in the standard filtration of $K_{n-2}$ so the induction hypothesis applies to them. Let $\nu$ be such that $K_{n-2} \triangleright \Delta(\nu)$. Considering the short exact sequence $K_{n-2}^\nu \hookrightarrow K_{n-2} \twoheadrightarrow \Delta(\nu)$ (defining $K_{n-2}^\nu$), we get that $L^iGK_{n-2}^\nu = L^iGK_{n-2}$ and so on. $\square$

3.2 The main induction

Recall that the set $\text{seq}_n^r \{1, 2\}$ is a basis of a permutation submodule of $V^{\otimes n}$, which we now denote $M_n(2r - n)$ (the argument $2r - n$ counts the excess of 1’s over 2’s). Now let

$$v(r, n) := 111 \ldots 11222 \ldots 22 \in \text{seq}_n^r \{1, 2\}$$

Then $v(r, n)$ generates $M_n(2r - n)$ as a $T_n$-module.

Our argument for showing that $V^{\otimes n}$ is tilting will be an induction on $n$. The inductive step is based on the following Lemma:

Lemma 1 Assume that $n \geq 2$. Then there are isomorphisms in $T_{n-2}$-mod:

$$F(V^{\otimes n}) \cong V^{\otimes n-2}$$

$$FM_n(s) \cong \begin{cases} M_{n-2}(s) & |s| < n \\ 0 & |s| = n \end{cases}$$  \tag{14}  \tag{15}

Proof: $F$ is multiplication by $\epsilon = \frac{1}{[2]q}U_{n-1}$, which acts in the last two factors of $V^{\otimes n}$ through the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has eigenvalues 1 with multiplicity 1 and 0 with multiplicity 3 and therefore is a projection onto a one dimensional space. Let $w_2 \in V^{\otimes 2}$ be an eigenvector to eigenvalue 1 (say $w_2 = q v_1 \otimes v_2 + v_2 \otimes v_1$). Then, cf. proposition \[4\], the (inverse of the) first isomorphism is given by $v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_{n-2}} \mapsto v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_{n-2}} \otimes w_2$.

To get the second isomorphism, note first that the above map clearly induces

$$FM_n(s) \subseteq M_{n-2}(s)$$

But then equality follows from the first isomorphism combined with $V^{\otimes n} = \bigoplus_s M_n(s)$ and the analogous formula for $V^{\otimes n-2}$. $\square$

Note that this Lemma relates level $n$ to $n - 2$. Accordingly our inductive argument for proving that $V^{\otimes n}$ is tilting will require two base cases: $n = 1$ and
Both are straightforward ($n = 1$ it trivial, and for $n = 2$ we have $V^\otimes 2 = 3\Delta_2(2) \oplus \Delta_2(0) = 3\nabla_2(2) \oplus \nabla_2(0)$ by the proof of the Lemma).

We now consider the exact sequence

$$0 \to K_n \to G \circ F(V^\otimes n) \overset{\varphi_n}{\to} V^\otimes n \to C_n \to 0 \quad (16)$$

where $\varphi_n$ is the adjointness map.

**Proposition 4** Assume that $\varphi_n$ is injective for all $n$. Then $V^\otimes n$ and $M_n(r)$ are tilting modules for $T_n$ for all $n, r$.

**Proof.** Since $F \circ G = \text{Id}$, we have that $F \circ G \circ F = F$ and hence $F(C_n) = 0$. Thus, cf. Proposition 3, $C_n$ has only (copies of) the ‘trivial’ one dimensional module (= $\Delta_n(n)$) as composition factors. But then $C_n$ is semisimple by quasiheredity (or otherwise).

Now work by induction on $n$. By the Lemma and the inductive hypothesis $F(V^\otimes n)$ is tilting, so in particular $F(V^\otimes n)$ has a standard filtration, and then so does $G \circ F(V^\otimes n)$ by Proposition 3. But $\varphi_n$ is assumed to be injective, so $K_n = 0$ and (16) becomes a short exact sequence with $V^\otimes n$ in the middle and with extremal terms in $F_n(\Delta)$. But then $V^\otimes n$ too lies in $F_n(\Delta)$. Since the matrices representing the action of $T_n$ on $V^\otimes n$ are selfadjoint with respect to the canonical, non–degenerate form (note from the presentation that the algebra is isomorphic to its opposite), $V^\otimes n$ is contravariant selfdual and so tilting. But then also $M_n(r)$ is tilting as a direct summand of $V^\otimes n$.

We have that $\varphi_n$ injective implies $V^\otimes n$ tilting. As an aside we note that the reverse implication also holds, and rather more generally. Indeed for $M \in F(\Delta)$, the adjointness map $G \circ F(M) \to M$ is injective. This is clear if $M \cong \Delta$ and otherwise it follows by induction on the number of $\Delta$-factors in $M$ using the following commutative diagram

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & G \circ F(C) \\
\downarrow & & \downarrow \\
0 & \to & G \circ F(M) \\
\downarrow & & \downarrow \\
0 & \to & \Delta \\
\end{array}
$$

where standard $\Delta$ is such that $M \to \Delta$, noting that the second row of the diagram is exact because of equation (13).

Now spelling out the definitions, $\varphi_n$ is the multiplication map

$$\varphi_n : T_n U_{n-1} \otimes U_{n-1} T_n U_{n-1} U_{n-1} V^\otimes n \to V^\otimes n$$

The rest of the construction of our inductive step amounts to a careful combinatorial analysis of this map. First of all consider $T_n U_{n-1}$ as a right module over $U_{n-1} T_n U_{n-1}$. As such it is easy to see that it is generated by the elements

$$U_{n-1}, \quad U_{n-2} U_{n-1}, \quad \ldots, \quad U_1 \cdots U_{n-2} U_{n-1} \quad (17)$$

But then any element of $T_n U_{n-1} \otimes U_{n-1} T_n U_{n-1} U_{n-1} V^\otimes n$ can be represented in the form

$$\sum_k U_k \cdots U_{n-2} U_{n-1} \otimes U_{n-1} T_n U_{n-1} v_k \quad (18)$$
for some $v_k \in U_{n-1}V^\otimes n$. We must show that this expression is zero if its image under the multiplication map is zero, i.e. if

$$\sum_k U_k \cdots U_{n-2} U_{n-1} v_k = 0$$

To do this, the following notation will be useful:

Recall that $i_1 i_2 \ldots i_n \in \text{seq}_n \{1, 2\} \text{ is a basis element of } V^\otimes n$. Denote by $i_1 i_2 \ldots i_{k-1}12 i_{k+2} \ldots i_n$ the vector $U_k (i_1 i_2 \ldots i_{k-1}12 i_{k+2} \ldots i_n)$. In other words:

$$i_1 i_2 \ldots i_{k-1}12 i_{k+2} \ldots i_n = q v_1 \otimes v_2 \otimes \cdots \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_i_n + v_1 \otimes v_2 \otimes \cdots \otimes v_2 \otimes v_1 \otimes \cdots \otimes v_i_n$$

To establish usage of this notation we first work out a couple of low rank examples.

**Example 1.** Consider $n = 3$. Then $\{U_1, U_2\}$ generate $T_n$ and $V^\otimes 3$ has dimension 8 and is the direct sum of 4 permutation modules. The injectivity of our map can be checked on each of them. On the one dimensional permutation modules the statement is trivial since $G \circ F$ kills them. Let us then consider the permutation module generated by 112 (that generated by 122 is isomorphic to it). Applying $G \circ F$, we get by (18) the two vectors:

$$U_2 \otimes U_2 T_3 U_2 112 = 1 \otimes U_2 T_3 U_2 112, \quad U_1 U_2 \otimes U_2 T_3 U_2 112 = U_1 \otimes U_2 T_3 U_2 112$$

(NB, $U_2 \otimes 211 = 0$ and $U_2 \otimes 121 \propto U_2 \otimes 112$). The images under the multiplication are

$$112, \quad 121,$$

and these are linearly independent, so also in this case the statement is clear.

**Example 2:** Consider $n = 4$. Then $T_n$ is generated by $\{U_1, U_2, U_3\}$ and $V^\otimes 4$ is the sum of five permutation modules.

Let us consider the permutation module $M_4(0)$ generated by 1122 $\in V^\otimes 4$. Using (18), $G \circ FM_4(0)$ is spanned by the vectors

$$1 \otimes 1212, \quad 1 \otimes 2112, \quad U_2 \otimes 1212, \quad U_2 \otimes 2112, \quad U_1 U_2 \otimes 1212, \quad U_1 U_2 \otimes 2112$$

i.e. by the set $\{U_1 U_2 \otimes w12, \quad U_2 \otimes w12, \quad 1 \otimes w12 \mid w \in \text{seq}_2 \{1, 2\}\}$.  

The images under the multiplication map are the vectors

$$1212, \quad 2112, \quad 1122, \quad 2121, \quad 1212, \quad 1221.$$

Note that these vectors are not independent, since

$$q1212 + 2112 = 1212 = q1212 + 1221$$

On the other hand this ‘trivial’ dependency is the only one, as can seen by a dimension counting: our permutation module has dimension $\binom{4}{2} = 6$ and is generated
by the above vectors together with \(1122\). So there is exactly one relation between them, the one we have pointed out.

Since there is a corresponding dependency amongst the first set of vectors:

\[
q1 \otimes 1212 + 1 \otimes 2112 = 1 \otimes 1212 = U_1 U_2 \otimes 1212 = q U_1 U_2 \otimes 1212 + U_1 U_2 \otimes 2112
\]

our claim is proved in this case as well.

We now turn to the general case.

**Theorem 1**  \(V \otimes n\) and \(M_n(r)\) are tilting modules for all \(n\).

**Proof.** By proposition \([4]\) it is enough to show that \(\varphi_n\) is injective for all \(n\). \(F\) and \(G\) are additive functors, so the claim can be verified on the permutation submodules. By \([18]\), \(G \circ F\) on \(M_n(2r - n)\) is spanned by

\[
\{ X \otimes w12 \mid X \in \{U_1..U_{n-1}, \ U_2..U_{n-1}, \ ... \ U_{n-1}, \ 1\}; \ w \in \text{seq}^{r-1}_{n-2}(1, 2)\}
\]

Let us denote by \(S_n^r\) the set of vectors of the form

\[
i_1 i_2 i_3 \ldots 12 \ldots i_{n-2} i_{n-1} i_n \quad i_k \in \{1, 2\}
\]

inside \(M_n(2r-n)\). Note that these are the images of the above vectors under the multiplication map. A simple counting argument shows that \(|S_n^r| = \binom{n-1}{1} \cdot \binom{n-2}{r-1}\).

Note that \(M_n(2r-n)\) is spanned by \(S_n^r \cup \{v(r,n)\}\). But \(S_n^r\) is not linearly independent. To each vector in \(M_n(2r-n)\) of the form

\[
i_1 i_2 i_3 \ldots 12 \ldots i_{n-1} 1
\]

we may associate a dependency:

\[
q \ldots 12 \ldots 12 \ldots 21 \ldots 12 \ldots = \ldots 12 \ldots 12 \ldots = q \ldots 12 \ldots 12 \ldots + \ldots 12 \ldots 21 \ldots
\]

(19)

Note that each of these dependencies has a preimage in \(G \circ FM_n(2r-n)\):

\[
q U_j..U_{n-2} \otimes \ldots 12 \ldots w_j w_{j+1} \ldots 12 + U_j..U_{n-2} \otimes \ldots 21 \ldots w_j w_{j+1} \ldots 12
\]

\[
= U_j..U_{n-2} \otimes \ldots 12 \ldots w_j w_{j+1} \ldots 12
\]

\[
= U_i..U_{n-2} \otimes \ldots w_i w_{i+1} \ldots 12 \ldots 12 + U_i..U_{n-2} \otimes \ldots w_i w_{i+1} \ldots 21 \ldots 12
\]

(recall that we are tensoring over \(\epsilon_{T_n} = U_{n-1} T_n U_{n-1}\)).

Let \(S_n'\) denote the subset of \(S_n^r\) in which no subsequence 12 appears before the \(12\). We will now show, using the set of linear dependencies above, that all vectors not in the subset \(S_n'\) may be discarded without affecting the spanning property, i.e. \(M_n(2r-n)\) is also spanned by \(S_n' \cup \{v(r,n)\}\).

Let the ‘underlying’ sequence \(u(s) \in \text{seq}_n\{1, 2\}\) of \(s \in S_n^r\) be the sequence obtained by removing the underline from \(s\). Note that any \(s \in S_n^r\) of the form \(.12..12..\) can be written as a linear combination of \(.12..12..\) and \(.21..12..\) and \(.12..21..\) using \([19]\), and that the last two have underlying sequences later in the lexicographic order than \(.12..12..\). Let \(s = .12..12.. \in S_n^r \setminus S_n'\). There may be other pairs 12 before 12 in \(s\), but we may take it that the pair 12 written explicitly is the first such. Using
the above remark we then replace $s$ by a linear combination of an element of $S'_n$ and elements of $S'_n$ not necessarily in $S'_n$, but whose underlying sequence is later in the lexicographic order than $u(s)$. Iterating, we arrive at $22.211211..1$ (or similar) which is in $S'_n$.

Note that $S'_n$ has a natural bijection with $\text{seq}^n \{1, 2\} \setminus \{22..2111..1\}$ (every element of this set has at least one subsequence 12 — just underline the first of these). Thus $S'_n \cup \{v(r, n)\}$ is a basis of $M_n(2r - n)$, and in particular it is linearly independent. This means that all linear dependencies in $S_n$ can be constructed from those of form (19). But each of these dependencies has a preimage in $G \circ FM_n(2r - n)$, so $\varphi_n$ has a trivial kernel. 

\begin{corollary}
$V^\otimes n$ is a full tilting module for $T_n$.
\end{corollary}

\begin{proof}
We proved in the Theorem that $M_n(s)$ is a tilting module for all $n, s$. Now we have the restriction rule

$$\text{Res}^u_{T_n} M_n(s) = M_{n-1}(s-1) \oplus M_{n-1}(s+1) \text{ for } s \in \{-n, -n + 2 \ldots, n - 2, n\}$$

Combining this with the restriction rule for the standard modules [19]

$$[\text{Res}^u_{T_n} \Delta_n(s)] = [\Delta_{n-1}(s+1)] + [\Delta_{n-1}(s-1)] \text{ for } s \in \{0/1\ldots, n - 2, n\}$$

it is easily proved by induction that

$$[M_n(s)] = [\Delta_n(s)] + [\Delta_n(s+2)] + \ldots (s \geq 0)$$

In other words $(M_n(s) : \Delta_n(u)) = 1$ for $u$ less than or equal $s$ in the heredity order; 0 otherwise. But then the tilting module $T_n(s)$ must occur as a summand of $M_n(s)$ (with multiplicity one).
\end{proof}

\section{The blob crypto–tensor module case}

Let us now consider the blob algebra situation. Our overall strategy will be similar to the one used in the previous section. In particular the quasiheredity arguments carry over almost unchanged. On the other hand, the actual calculation requires some new combinatorial ideas.

We keep the condition that $[2]_q \neq 0$, but assume also that $[m]_q \neq 0$ (where $m$ is as in section [11]). In that case the blob algebra $b_n^k = b_n$ is quasihereditary, see [24]. In fact, setting $\epsilon = \frac{1}{[2]_q} U_n$, we have as for $T_n(q)$ that $\epsilon$ is part of a heredity chain for $b_n$ and $\epsilon b_n \epsilon \cong b_{n-2}$. Accordingly the results from the previous section involving quasiheredity hold in this case as well. We state them here indicating the necessary modifications of the previous proofs.

As before we have a localization functor $F$

$$F : b_n \text{-mod} \rightarrow b_{n-2} \text{-mod} : M \mapsto \epsilon M$$

and a globalization functor $G$

$$G : b_{n-2} \text{-mod} \rightarrow b_n \text{-mod} : N \mapsto b_n \epsilon \otimes_{b_{n-2}} N$$

We denote as before the standard (costandard etc.) modules in $b_n$-mod by $\Delta_n(\lambda)$ ($\nabla_n(\lambda)$ etc.), but in this case the parametrizing set is $\Gamma_n = \{-n, -n + 2 \ldots, n - 2, n\}$ [22, 24]. We then have the following version of Proposition 2.

\section*{4 The blob crypto–tensor module case}
Proposition 5 Assume that $\lambda \in \Gamma_{n-2}$. Then

i) $FL_n(\lambda) = L_{n-2}(\lambda)$.

ii) $F\Delta_n(\lambda) = \Delta_{n-2}(\lambda)$ and $F\nabla_n(\lambda) = \nabla_{n-2}(\lambda)$.

iii) $FP_n(\lambda) = P_{n-2}(\lambda)$ and $FI_n(\lambda) = I_{n-2}(\lambda)$.

Otherwise (i.e. for $\lambda = n$ or $\lambda = -n$) we have $FL_n(\lambda) = F\Delta_n(\lambda) = F\nabla(\lambda) = 0$.

We then get as before

Proposition 6 Supposing a module $M \in \mathcal{F}_{n-2}(\Delta)$ (i.e. $M \in b_{n-2} – \text{mod}$ has a standard filtration), then $G(M)$ also has a standard filtration. Furthermore

$$(G(M) : \Delta_n(\mu)) = \begin{cases} (M : \Delta_{n-2}(\mu)) & \text{if } \mu \in \Gamma_{n-2} \\ 0 & \text{otherwise} \end{cases}$$

Proof. The proof is once again an application of the cohomological criterion for standard filtrations. One shows that $\text{Ext}^1_{I_n}(G(M), \nabla_n(\mu)) = 0$ for all $\mu$. The special case $\mu = n$ from the Temperley-Lieb situation now becomes two special cases $\mu = n$ and $\mu = -n$, each of which can be treated as before. □

The cohomological statement \[13\] also carries over:

Corollary 6.1

$$L^iG\Delta_{n-2}(\lambda) = \begin{cases} \Delta_n(\lambda) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$ \hspace{1cm} (20)

The set $\text{seq}^r_{2n}(1, 2)$ is a basis of a permutation module of $\rho$ which we denote $M_n(2r-2n)$. (For example a basis of $M_2(0)$ is $\{1122, 1212, 1221, 2112, 2121, 2211\}$.) Evidently

$$\text{Res}_{b_n+1} M_{n+1}(\lambda) = M_n(\lambda + 2) \oplus 2M_n(\lambda) \oplus M_n(\lambda - 2)$$ \hspace{1cm} (21)

We also have a blob algebra version of \[14\]:

Lemma 2 Assume that $n \geq 2$. Then there are isomorphisms in $b_{n-2} – \text{mod}$:

$$F(V^\otimes 2n) \cong V^\otimes 2(n-2)$$ \hspace{1cm} (22)

$$FM_n(s) \cong \begin{cases} M_{n-2}(s) & |s| < 2n - 2 \\ 0 & |s| = 2n, 2n - 2 \end{cases}$$ \hspace{1cm} (23)

Proof. The Temperley-Lieb argument goes through almost unchanged: $F$ is multiplication by the idempotent $\epsilon = \frac{1}{|2|} U_{n-1}$ which acts only in the first two and last two factors of $V^\otimes 2n$. The isomorphism $V^\otimes 2(n-2) \to F(V^\otimes 2n)$ is given by $w \in V^\otimes 2(n-2) \mapsto ev_1 \otimes w \otimes ev_2$ where $ev_1 \otimes ev_2 \in V^\otimes 4$ is an eigenvector to eigenvalue 1 of our idempotent $\epsilon$. □

We consider also in the blob algebra setting the adjointness map $\varphi_n : G \circ F(V^\otimes 2n) \to V^\otimes 2n$ and get a four term exact sequence:

$$0 \to K_n \to G \circ F(V^\otimes 2n) \xrightarrow{\varphi_n} V^\otimes 2n \to C_n \to 0$$ \hspace{1cm} (24)

Proposition 7 Assume that $\varphi_n$ is injective for all $n$. Then $V^\otimes 2n$ and $M_n(r)$ are tilting modules for $b_n$ for all $n, r$. 

12
Proof. The Temperley–Lieb proof carries over almost verbatim. □

We are then once again left with the task of showing injectivity of the adjointness map \( \varphi_n : G \circ F(V^\otimes 2n) \to V^\otimes 2n \). Let us as before first work out a low rank example (which will this time also be needed in the general argument).

**Example 1.** Consider \( n = 2 \). Then the blob algebra \( b_2 \) acts on \( V^\otimes 4 \) as described in section 2. For example

\[
U_1 1212 = st1212 + s1221 + t2112 + 2121 =: 1212
\]

(NB, the underline notation is refined here to accommodate the definition of \( \rho \)—the position of the underline to left or right of centre determines precisely which linear combination it corresponds to). A generating set of \( b_2 \) is \( U_1 \) and \( U_0 = a^2 e \). Hence the vectors

\[
1 \otimes 1212, \ U_0 \otimes 1212
\]

generate \( G \circ F(V^\otimes 4) \). The images under the multiplication map \( \varphi_2 \) are

\[
1212, \ 1\overline{2}2
\]

(where \( 1\overline{2}2 \) denotes a certain linear combination of \( 1122 \) and \( 2121 : U_01212 = r2121 + 2211 + rst1122 + st1212 \)) which are independent.

More generally, a spanning set for \( G \circ F(V^\otimes 2n) \) is given by

\[
B_n := \{ X \otimes 12w12 \mid X \in \begin{cases}
1, \\
U_{n-2}, \\
U_{n-3}U_{n-2}, \\
\vdots \\
U_0U_1..U_{n-2}
\end{cases} ; w \in \text{seq}_{2n-4}\{1,2\}\}
\]

The images under the multiplication map are vectors of the form \( ..12.. \), where the concatenation of the subsequences indicated by ellipsis is the sequence \( w \) from \( \text{seq}_{2n-4}\{1,2\} \), and the first and second \( 12 \) are equidistant from the left and right hand end respectively; and \( ..\overline{12}.. \), where the concatenation of the subsequences indicated by ellipsis is the sequence from \( \text{seq}_{2n-4}\{1,2\} \). This is a consequence of the following straightforward exercise in the blob algebra relations:

**Lemma 3** Let \( \epsilon = \frac{1}{2} n_{n-1} \). The set \( \{ 1, \ U_{n-2}, \ U_{n-2}U_{n-3}, \ldots, U_{n-2}U_{n-3} \ldots U_0 \} \) generates \( b_n \epsilon \) as a right \( b_n \epsilon \)-module.

It will be evident that linear dependencies arise in general between these vectors, in a way analogous to the \( T_n \) case. For example it is easy to write down a linear dependence involving \( 12121212 \) and \( 12121212 \) (and others).

It is straightforward, using this machinery, to verify that \( \varphi_n \) is injective and hence our module is tilting, up to \( b_4 \) and a little beyond. We will prove injectivity for general \( n \) by a slightly different route.

Define first numbers \( v(n) \) by the recursion \( v(0) = 1, v(1) = 1, v(-1) = 3 \) and

\[
v(n) = \begin{cases}
4v(n-1) - v(n-2) & \text{if } n \geq 2, \\
v(n) = 4v(n+1) - v(n+2) & \text{if } n \leq -2
\end{cases}
\]
**Proposition 8** Let $\rho(n)$ be the representation $\rho$ of $b_n$ on $V^{\otimes 2n}$. Then
1) $\varphi_n : G \circ F(\rho(n)) \to \rho(n)$ is injective, so $\rho(n)$ is a tilting module.
2) $(\rho(n) : \Delta_n(\lambda)) = v(\lambda)$.
3) Set $r_n = \dim(G \circ F(\rho(n)))$. Then $r_1 = 0$, $r_2 = 2$, and

$$r_n = 4r_{n-1} + 4^{n-2} - r_{n-2}.$$ 

**Proof:** By induction on $n$. The case $n = 1$ follows easily from the fact that

$$\rho(1) = V \otimes V \cong \Delta_1(1) \oplus 3\Delta_1(-1)$$

and 1) and 3) of the $n = 2$ case is the calculation done above. The calculation also shows that $\dim F(\rho(2)) = 1$, so we get that $\Delta_2(0)$ occurs in $\rho(2)$ with multiplicity 1. We can then read off the other two multiplicities using $\operatorname{Res}_{b_1}^{b_2} \rho(2) = 4\rho(1)$ and the restriction rules for the standard modules, thus verifying 2). For the reader’s convenience we express this last point in formulas: write $\rho(2)$ in the Grothendieck group as follows

$$\rho(2) = a_2 \Delta_2(2) + a_0 \Delta_2(0) + a_{-2} \Delta_2(-2)$$

Applying $F$ to this expression we get $a_0 = 1$ and applying the restriction functor to it we get

$$4 \rho(1) = \operatorname{Res} \rho(2) = (a_2 + a_0) \Delta_1(1) + (a_{-2} + a_0) \Delta_1(-1)$$

and 2) now follows from $\rho(1) = \Delta_1(1) + 3 \Delta_1(-1)$.

Now assume the Proposition for $n'$ with $n' < n$. Then $F(\rho(n')) = \rho(n-2)$ is tilting and

$$(F(\rho(n)) : \Delta_{n-2}(\lambda)) = (\rho(n-2) : \Delta_{n-2}(\lambda)) = v(\lambda) \text{ if } |\lambda| \leq n - 2$$

But then also

$$(G \circ F(\rho(n)) : \Delta_n(\lambda)) = v(\lambda) \text{ if } |\lambda| \leq n - 2$$

since $G$ is exact on $\mathcal{F}(\Delta)$ and takes standard modules to standard modules. Note [24] the short exact sequence

$$0 \to \Delta_{n-1}(\lambda \pm 1) \to \operatorname{Res}_{b_{n-1}}^{b_n} \Delta_n(\lambda) \to \Delta_{n-1}(\lambda \mp 1) \to 0 \quad (\lambda \mp 1 \triangleleft \lambda) \quad (25)$$

$(\Delta_{n-1}(\nu)$ to be interpreted as 0 if $|\nu| > n - 1)$. We can then calculate $r_n$ as follows

$$r_n = \sum_{\lambda : |\lambda| \leq n-2} v(\lambda) |\Delta_n(\lambda)| = \sum_{\lambda : |\lambda| \leq n-2} v(\lambda) (|\Delta_{n-1}(\lambda + 1)| + |\Delta_{n-1}(\lambda - 1)|) =$$

$$(v(n-2) + v(n-4)) |\Delta_{n-1}(n-3)| + \ldots + (v(-n+2) + v(-n+4)) |\Delta_{n-1}(-n+3)| + v(n-2) |\Delta_{n-1}(n-1)| + v(-n+2) |\Delta_{n-1}(-n+1)|$$

Using the recursion formula for $v(n)$, this becomes

$$4 \left\{ v(n-3) |\Delta_{n-1}(n-3)| + \ldots + v(-n+3) |\Delta_{n-1}(-n+3)| \right\} + v(n-2) + v(-n+2)$$

We now apply the induction hypothesis (part 2) and 3) and get that the first term is equal to $4r_{n-1}$, while the sum $v(n-2) + v(-n+2)$ is equal to

$$\dim \rho(n-2)/G \circ F(\rho(n-2)) = 4^{n-2} - r_{n-2}$$
Combining this we have shown 3) at level $n$.

To prove 2) at level $n$ note first that $F$ annihilates $\rho(n)/G \circ F(\rho(n))$, so it can be written in the Grothendieck group as a sum

$$a_n \Delta_n(n) + a_{-n} \Delta_n(-n)$$

We restrict and apply the formula $\text{Res} \rho(n) = 4 \rho(n - 1)$, and find by comparing coefficients that

$$a_n + v(n - 2) = 4v(n - 1) \quad \text{and} \quad a_{-n} + v(-n + 2) = 4v(-n + 1)$$

Since the $\Delta$-multiplicities of $G \circ F(\rho(n))$ are already known by induction, this shows 2) at level $n$.

Let us now finally prove 1) at level $n$. This is the most tricky part of our proof and involves some interesting combinatorics on sequences. Let $v \in \varphi_n(B_n)$, then the four sequences which occur as summands of $v$ are either of the form

$$\{x12y12z, x12y21z, x21y12z, x21y21z\}$$

where $x, y, z \in \text{seq}\{1, 2\}$, or

$$\{x1122z, x1212z, x2121z, x2211z\}$$

where $x, z \in \text{seq}\{1, 2\}$. Define $u(v) \in \text{seq}_{2n}\{1, 2\}$ to be the lexicographically earliest sequence that occurs as a summand of $v$.

One easily sees from the description of the elements of $\varphi_n(B_n)$ that $u(v)$ satisfies the rule

$$au(v)b = u(ab) \quad (26)$$

for $a, b \in \{1, 2\}$ such that $abv \in \varphi_n(B_{n+1})$.

Now define for all $n \geq 1$ a subset $E_n$ of our representation space $V^\otimes 2n$ as follows:

$$E_1 := \emptyset, \quad E_2 := \{1212, 1122\}$$

then for $n \geq 2$:

$$E_1^n := \{1x1, 1x2, 2x1, 2x2 \mid x \in E_{n-1}\}$$

$$E_2^n := \{12w12 \mid w \in \text{seq}_{2n-4}\{1, 2\} \setminus u(E_{n-2})\}$$

$$E_n := E_1^n \cup E_2^n$$

Consider now the following properties of $E_n$:

**Claim:** i) $|E_n| = r_n$

ii) $E_n$ is a basis of $\varphi_n(G \circ F(\rho(n)))$

iii) $|E_n| = |u(E_n)|$

iv) $u(E_n^1) \cap u(E_n^2) = \emptyset$

Part 1) of the Proposition is a consequence of i) and ii) since we already know that $\dim G \circ F(\rho(n)) = r_n$. In order to prove the claim we again proceed by induction. Since $E_n^1$ are only defined for $n \geq 3$ we take $n = 3$ as base of the induction, but actually i), ii) and iii) also make sense for $n = 1, 2$ and basically follow from the calculations prior to the Proposition: note that

$$u(1212) = 1212, \quad u(1122) = 1122$$
to obtain iii). Now for $n = 3$ we have

$$E_3^1 = \begin{cases} 
12121, & 111221, \\
12122, & 111222, \\
212121, & 211221, \\
212122, & 211222 
\end{cases}$$

while

$$E_3^2 = \begin{cases} 
121112, \\
121212, \\
122112, \\
122212, 
\end{cases}$$

Each element of $E_3^1$ has summands all of which have the same first and last factor, and therefore cannot appear in $E_3^2$. Since there are clearly no duplicates inside the two sets, we get then i). Applying $u$ to the two sets produces the same two sets, with the underlines removed, so also iii) and iv) follow. But then also ii) follows: $u$ picks out the highest summand of the elements, so the matrix relating the vectors of $E_3$ and those of $u$ follow. But then also ii) follows: the first implication since

$$\|E_n^1\| = |u(E_n^1)| \quad \text{and} \quad |E_n^2| = |u(E_n^2)|$$

But then we get iv) $\Rightarrow$ iii) $\Rightarrow$ ii) at level $n$: the first implication since

$$|E_n| = |E_n^1 \cup E_n^2| = |E_n^1| + |E_n^2| = |u(E_n^1)| + |u(E_n^2)| =$$

$$|u(E_n^1) \cup u(E_n^2)| = |u(E_n^1) \cup u(E_n^2)| = |u(E_n)|$$

The second implication, since once again $u$ defines a lower triangular matrix with respect to the order and since $E_n \subset \varphi_n(B_n)$ by the description of $\varphi_n(B_n)$. So let us prove iv). Now

$$u(E_n^1) \cap u(E_n^2) = u(E_n^1) \cap \{12w12 \mid w \in \text{seq}_{2n-4}\{1, 2\} \setminus u(E_{n-2})\}$$

Consider first

$$u(E_n^1) \cap \{12w12 \mid w \in \text{seq}_{2n-4}\{1, 2\}\}$$

Any element of this intersection is on the form $1t2 = u(e)$ where $e \in E_n^1$. But applying $u$ and the rule (26) to the different elements of $E_n^1$ we see that $e = 1x2$ with $x \in E_{n-1}$. Now $u(e)$ is also on the form $12w12$, so $x = 2y1$. But such an $x$ must come from $E_{n-1}^1$ and thus $y \in E_{n-2}$. All in all: $u(e) = 12y12$ with $y \in E_{n-2}$. But then our first intersection is empty proving the last part of the claim.

### 4.1 Fullness and more multiplicities

The standard content of the individual $\rho$-permutation modules $M_n(\lambda)$ may be determined similarly to that of $\rho$, except that we need to recurse all the $\lambda$s together.
By Lemma 2 there is a function $v^\lambda(\mu)$ such that

$$v^\lambda(\mu) = (M_n(\lambda) : \Delta_n(\mu))$$

for any $n$.

By virtue of (21) and (25) we have

$$v^{\lambda - 2}(\mu) + 2v^\lambda(\mu) + v^{\lambda + 2}(\mu) = v^\lambda(\mu + 1) + v^\lambda(\mu - 1) \quad (27)$$

As before, explicit inspection of the smallest cases is sufficient to prime a recursion using this formula to determine all multiplicities. We have for example

| $\lambda$ | 8 6 4 2 0 −2 .. |
|-----------|----------------|
| −4        | 1 7 19 31 37 31 .. |
| −3        | 1 5 9 11 9 5 1 |
| −2        | 1 3 3 3 1 |
| −1        | 1 1 1 |
| $\mu = 0$ | 1 |
| +1        | 1 |
| +2        | 1 1 1 |
| +3        | 1 3 3 3 1 |
| +4        | 1 5 9 11 9 5 1 |
| +5        | 1 7 19 31 37 31 .. |

In the format of this table, for every subpart of form

$$x$$

$$a \quad b \quad c$$

$$y$$

we have $a + 2b + c = x + y$.

Note that $M(\lambda) \cong M(-\lambda)$ and that such a $\lambda$ is necessarily even. Accordingly, call $\{2n, 2n - 2, 2n - 4, .., 0\}$ the set of $M$-weights of $b_n$ — a sufficient set of labels for inequivalent permutation modules.

We do not need a closed formula for all the multiplicities, but rather

**Proposition 9** Restrict attention to $\lambda$ an $M$-weight. Then

$$v^\lambda(\mu) = \begin{cases} 1, & 2\mu = -\lambda \\ 1, & 2\mu = (\lambda + 2) \\ 0, & 0 > 2\mu > -\lambda \\ 0, & 0 < 2\mu < (\lambda + 2) \end{cases}$$

**Proof:** This is the neighbourhood of the domain of zeros (unwritten) in our table above. The template (28) populates this region as claimed with the rows at $\mu = 0, \pm 1$ as base. \(\Box\)

(Another proof follows from noting, for example, that $M_n(2n) = \Delta_n(-n)$ so $(M_n(\lambda) : \Delta_n(\mu)) = 0$ if $0 > 2\mu > -\lambda$ and $(M_n(\lambda) : \Delta_n(\mu)) = 1$ if $2\mu = -\lambda$.)

**Corollary 9.1** The module $\rho$ is full tilting.
Proof: The singleton multiplicities in the expression above give a bijection between the $M$–weights and ordinary weights. Recall [10] that each $T(\mu)$ contains:

- one copy of $\Delta_\mu$, and
- no copy of any other standard module except having weight higher in the heredity order.

The proposition thus implies that $M(\lambda)$ contains no $T(\mu)$ unless $\mu$ lower than (or equal to) the ordinary weight corresponding to $\lambda$; and hence exactly one copy of the indecomposable tilting module associated to the corresponding ordinary weight. 

\[\blacksquare\]

5 On the generic standard content of $\rho'$

The question of tilting for $\rho'$ remains open (our specific combinatorial construction in the proof of injectivity of $\varphi_n$ is particular to $\rho$). For the reasons outlined in [23] it might be useful to know the standard content of $\rho'$ when it is tilting. Just as for $\rho$ we have

\[F(\rho'(n + 2)) \cong \rho'(n)\] \hspace{1cm} (29)

\[\text{Res}^{n+2}_n \rho'(n + 2) = 4 \rho'(n)\]

The argument for (29) in the $\rho'$ case is exactly the same as before.

Under the assumption that $\rho'$ has a standard filtration (as in any semisimple specialisation for example), it follows from Proposition 5 and (29) that there is a function $v' : \mathbb{Z} \to \mathbb{N}$ such that

\[(\rho'(n) : \Delta(\lambda)) = v'(\lambda)\]

(any $n$, $|\lambda| \leq n$, $\lambda - n \equiv 0 \text{ mod.} 2$). Let $\mathcal{M}(i, j) : \mathbb{Z} \to \mathbb{N}$ be $\mathcal{M}(i, j) = \delta_{i,j-1} + \delta_{i,j+1}$, so (from [24])

\[(\text{Res}^{n+1}_n \Delta_{n+1}(\mu) : \Delta_n(\lambda)) = \mathcal{M}(\lambda, \mu)\] \hspace{1cm} (30)

Regarding $\mathcal{M}, v'$ as infinite matrices it follows that

\[\mathcal{M}v' = 4v'\]

which is to say that

\[v'(\lambda + 1) + v'(\lambda - 1) = 4v'(\lambda)\]

Thus $v'$ is determined by recursion from the initial conditions

\[v'(0) = 1, \quad v'(1) = 2, \quad v'(-1) = 2\]

(which may be determined by inspection of the representations themselves — note that it is only these initial conditions which distinguish this analysis from a corresponding one for $\rho$). In case $l > 0$ we may now obtain $v'(l + 1)$ by $v'(l + 1) = 4v'(l) - v'(-l + 1) = 4v'(l) - v'(l - 1)$ ($l < 0$ case similar, or note that $v'(-l) = v'(l)$).

We have

\[
\begin{array}{c|cccccccc}
 l & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 \\
v'(l) & 97 & 26 & 7 & 2 & 1 & 2 & 7 & 26
\end{array}
\]
6 Discussion

In [23] the representations \( \rho \) and \( \rho' \) were introduced, and Martin and Woodcock posed the question of whether these representations are full tilting. We have now answered this question in the affirmative for \( \rho \). (They also asked if the representations are faithful for arbitrary \( k \) — a question we answer in the affirmative in [20], using entirely different techniques.) The primary focus of the original paper, however, was generalisations of the blob algebra. In particular it points out the potential usefulness of corresponding generalisations of \( \rho \). It does not succeed in constructing any. The discovery in the present paper that \( \rho \) is tilting makes it even more desirable to find such generalisations.

Since we have constructed a full tilting module for \( b_n \) we have, formally at least, constructed a Ringel dual, \( B_n = \text{End}_{b_n}(V^{\otimes 2n}) \). Armed with this mechanism (and the associated combinatorics, summarized generically by the truncation

\[
\begin{pmatrix}
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
41 \\
11 \\
3 \\
1 \\
1 \\
1 \\
3 \\
1 \\
41 \\
153
\end{pmatrix} =
\begin{pmatrix}
1 \\
4 \\
16 \\
64 \\
256
\end{pmatrix}
\]

where the \( n^{th} \) matrix row gives the dimensions of standards of \( b_{n-1} \); and the column vector gives their multiplicities, and hence the dimensions of (co)standards of the dual) we can search for Lie theoretic settings (i.e., a familiar presentation) for this dual.

This search will be the subject of a separate paper, but it behoves us to assemble the clues which are now ready to hand. In particular, let us look briefly at the most interesting case in characteristic 0. This means, essentially, \( q \) an \( l^{th} \) root of unity and \( m \) an integer (\( |m| < l \)). (Although the connection with Lie theory is still, for the present, ‘virtual’ we know from [24] that Lie theoretic terminology provides the correct setting for a description of blob representation theory.) Then the alcove structure is as follows. The weight space is \( \mathbb{R} \) and integral weights \( \mathbb{Z} \). The affine Weyl group is generated by a reflection at \( m \) and another at \( m - l \). No ‘wall’ (reflection point) lies at 0, so call the alcove containing 0 the 0–alcove. Label the first alcove on the \( \pm \)-ve side of the 0–alcove the \( \pm 1 \)-alcove. Label all other alcoves by the obvious counting scheme. The blocks are the affine Weyl orbits, and the regular blocks are (up to localisation) Morita equivalent, so we will pick one arbitrarily and relabel weights in it simply by their alcove labels. Then the simple submodule structure of
standard $\Delta(\nu) \ (\nu \geq 0)$ is

(\text{the ladder continues down until truncated by localisation). So far all is taken from [24]. Now consider what we may deduce about the indecomposable tilting module labelled by $\nu$. We have that every simple in the defining standard must be the socle of a costandard. We need then to take a standard filtered closure, and assemble the resultant melange into a contravariant selfdual module. For example:

Here, since layers may contain modules with multiplicity, some of the edges in the graph indicate no more than layer constraints (although they provide a useful guide to the eye). These modules are, of course, far from projective.

Note that although $T_n(q)$ fails to be quasihereditary when $[2] = 0$ this failure is degenerate rather than exceptional, in the sense that if one allows the notion of a single `formal' standard module of dimension 0 then the whole formalism is resurrected (the fact that $V^\otimes n$ itself is not compromised by passing to $[2] = 0$ is a signal of this). Similar statements apply in the blob case and, as mentioned above, in the paper [20] we show that $\rho$ is faithful for arbitrary (not just quasihereditary) specialisations. The questions of tilting and faithfulness for $\rho'$ remain open.

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