Exponentially Harmonic Maps into Spheres

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Abstract: We study smooth exponentially harmonic maps from a compact, connected, orientable Riemannian manifold $M$ into a sphere $S^m \subset \mathbb{R}^{m+1}$. Given a codimension two totally geodesic submanifold $\Sigma \subset S^m$, we show that every nonconstant exponentially harmonic map $\phi : M \rightarrow S^m$ either meets or links $\Sigma$. If $H^1(M, \mathbb{Z}) = 0$ then $\phi(M) \cap \Sigma \neq \emptyset$.

Keywords: exponentially harmonic map; totally geodesic submanifold; Euler-Lagrange equations

1. Introduction

Let $M$ be a compact, connected, orientable $n$-dimensional Riemannian manifold, with the Riemannian metric $g$. Let $\phi : M \rightarrow N$ be a $C^\infty$ map into another Riemannian manifold $(N, h)$. The Hilbert-Schmidt norm of $d\phi$ is $\|d\phi\| = \left[\text{trace}_g (\phi^*h)\right]^{1/2} : M \rightarrow \mathbb{R}$. Let us consider the functional

$$E : C^\infty(M, N) \rightarrow \mathbb{R}, \quad E(\phi) = \int_M \exp \left(\frac{1}{2} \|d\phi\|^2\right) \, dV_g.$$ 

A $C^\infty$ map $\phi : M \rightarrow N$ is exponentially harmonic if it is a critical point of $E$ i.e., $\{dE(\phi_s)/ds\}_{s=0} = 0$ for any smooth 1-parameter variation $\{\phi_s\}_{s<e} \subset C^\infty(M, N)$ of $\phi_0 = \phi$. Exponentially harmonic maps were first studied by J. Eells & L. Lemaire [1], who derived the first variation formula

$$\frac{d}{ds} \{E(\phi_s)\}_{s=0} = -\int_M \exp[e(\phi)] h^\phi(V, \tau(\phi) + \phi_\ast \nabla e(\phi)) \, dV_g$$

where $e(\phi) = \frac{1}{2} \|d\phi\|^2$ and $\tau(\phi) \in C^\infty(\phi^{-1}TN)$ is the tension field of $\phi$ (cf. e.g., [2]). Also $V = (\partial\phi_s/\partial s)_{s=0}$ is the infinitesimal variation induced by the given 1-parameter variation. In particular, the Euler-Lagrange equations of the variational principle $\delta E(\phi) = 0$ are

$$-\Delta \phi^i + \left(\Gamma^i_{jk} \circ \phi\right) \frac{\partial\phi^i}{\partial x^a} \frac{\partial\phi^j}{\partial x^b} G^{\alpha\beta} + \frac{\partial\phi^i}{\partial x^a} \frac{\partial e(\phi)}{\partial x^b} G^{\alpha\beta} = 0$$

where

$$\Delta u = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a} \left(\sqrt{G} G^{\alpha\beta} \frac{\partial u}{\partial x^\alpha}\right), \quad G = \det[G_{\alpha\beta}],$$

is the Laplace-Beltrami operator and $\Gamma^i_{jk}$ are the Christoffel symbols of $h_{ij}$. The (partial) regularity of weak solutions to (1) was investigated by D.M. Duc & J. Eells (cf. [3]) when $N = \mathbb{R}$ and by Y-J. Chiang et al. (cf. [4]) when $N = S^m$. Differential geometric properties of exponentially harmonic maps, including the second variation formula for $E$, were found by M-C. Hong (cf. [5]), J-Q. Hong & Y. Yang (cf. [6]), L-F. Cheung & P-F. Leung (cf. [7]), and Y-J. Chiang (cf. [8]).
The purpose of the present paper is to further study exponentially harmonic maps $\phi$ winding in $N = S^m$, a situation previously attacked in [4], though confined to the case where $M$ is a Fefferman space-time (cf. [9]) over the Heisenberg group $\mathbb{H}_n$ and $\phi : M \to S^m$ is $S^1$ invariant. Fefferman spaces are Lorentzian manifolds and exponentially harmonic maps of this sort are usually referred to as exponential wave maps (cf. e.g., Y-J. Chiang & Y-H. Yang, [10]). Base maps $f : \mathbb{H}_n \to S^m$ associated (by the $S^1$ invariance) to $\phi : M \to S^m$ are $S^1$ invariant. Fefferman spaces turn out to be solutions to degenerate elliptic equations [resembling (cf. [11]) the exponentially harmonic map system (1)] and the main result in [4] is got by applying regularity theory within subelliptic theory (cf. e.g., [12]).

Through this paper, $M$ will be a compact Riemannian manifold and $\phi : M \to S^m$, an exponentially harmonic map. Although the properties of an exponentially harmonic map may differ consistently from those of ordinary harmonic maps (see the emphasis by Y-J. Chiang, [13]), we succeed in recovering, to the setting of exponentially harmonic maps, the result by B. Solomon (cf. [14]) that for any nonconstant harmonic map $\phi : M \to S^m$ from a compact Riemannian manifold either $\phi(M) \cap \Sigma \neq \emptyset$ or $\phi : M \to S^m \setminus \Sigma$ isn’t homotopically null. Here $\Sigma \subset S^m$ is an arbitrary codimension 2 totally geodesic submanifold.

The ingredients in the proof of the exponentially harmonic analog to Solomon’s theorem (see [14]) are (i) setting the Equation (1) in divergence form

$$-
abla^* \left( \exp [e(\phi)] \nabla\phi \right) + 2 e(\phi) \exp [e(\phi)] \phi' = 0$$

(got by a verbatim repetition of arguments in [4]), (ii) observing that $S^m \setminus \Sigma$ is isometric to the warped product manifold $S^{m-1}_+ \times_w S^1$, and (iii) applying the Hopf maximum principle (to conclude that there are no nonconstant exponentially harmonic maps into hemispheres).

2. Exponentially Harmonic Maps into Warped Products

Let $S = L \times \mathbb{R}$, where $L$ is a Riemannian manifold with the Riemannian metric $g_L$. Let $w \in C^\infty(S)$ such that $w(y) > 0$ for any $y \in S$ and let us endow $S$ with the warped product metric

$$h = \Pi_1^* g_L + w^2 \, dt \otimes dt,$$

where $t = \tilde{t} \circ \Pi_2$, $\tilde{t}$ is the Cartesian coordinate on $\mathbb{R}$, and

$$\Pi_1 : S \to L, \quad \Pi_2 : S \to \mathbb{R},$$

are projections. The Riemannian manifold $(S, h)$ is customarily denoted by $L \times_w \mathbb{R}$. Let $\phi : M \to S$ be an exponentially harmonic map and let us set

$$F = \Pi_1 \circ \phi, \quad u = \Pi_2 \circ \phi.$$

We need to establish the following

Lemma 1. Let $M$ be a compact, connected, orientable Riemannian manifold and $\phi = (F, u) : M \to S = L \times_w \mathbb{R}$ a nonconstant exponentially harmonic map. Then $u$ is a solution to

$$\left( w \circ \phi \right) \Delta u + \left( \frac{\partial w}{\partial t} \circ \phi \right) \|\nabla u\|^2$$

$$= \left( w \circ \phi \right) (\nabla u) e(\phi) + 2 (\nabla u)(w \circ \phi).$$

If additionally $\partial w / \partial t = 0$ then $\phi(M) \subset L \times \{t_\phi\}$ for some $t_\phi \in \mathbb{R}.$
Also for an arbitrary test function $\phi \in C^\infty(M)$ we set

$$\phi_s(x) = (F(x) + s \phi(x)), \quad x \in M, \quad |s| < \epsilon,$$

so that $\{ \phi_s \}_{|s| < \epsilon}$ is a 1-parameter variation of $\phi$. For each $x_0 \in M$ let $\{E_a : 1 \leq a \leq n\} \subset C^\infty(U, T(M))$ be a local $g$-orthonormal (i.e., $g(E_a, E_\beta) = \delta_{a\beta}$) frame, defined on an open neighborhood $U \subset M$ of $x_0$. Then

$$\|d\phi_s\|^2 = \text{trace}_g (\phi_s^* h) = \sum_{a=1}^n (\phi_s^* h)(E_a, E_a)$$
on $U$. On the other hand

$$(\phi_s^* h)(X, X) = (F^* g_L)(X, X) + (w \circ \phi_s)^2 \left[ X(u) + sX(\phi) \right]^2$$

for every tangent vector field $X \in \mathfrak{X}(M)$. Formula (3) for $X = E_a$ yields

$$\|d\phi_s\|^2 = \|dF\|^2 + (w \circ \phi_s)^2 \left[ \|\nabla u\|^2 + 2s \, g(\nabla u, \nabla \phi) \right] + s^2 \|\nabla \phi\|^2.$$

Hence (differentiating with respect to $s$)

$$\frac{d}{ds} \{E(\phi_s)\}_{s=0} = \int_M \exp [e(\phi)] \left\{ (w \circ \phi)^2 g(\nabla u, \nabla \phi) + (w \circ \phi) (w \circ \phi) \|\nabla u\|^2 \right\} \, d\nu_g$$

where $w_1 = \partial w / \partial t$. Moreover

$$\exp [e(\phi)] (w \circ \phi)^2 g(\nabla u, \nabla \phi) = \text{div} (\phi \exp [e(\phi)] (w \circ \phi)^2 \nabla u)$$

$$+ \phi \left\{ \exp [e(\phi)] (w \circ \phi)^2 \Delta u - (\nabla u) \left( \exp [e(\phi)] (w \circ \phi)^2 \right) \right\}$$

where div: $\mathfrak{X}(M) \rightarrow C^\infty(M)$ is the divergence operator with respect to the Riemannian volume form

$$d\nu_g = \sqrt{G} \, dx^1 \wedge \cdots \wedge dx^n$$

i.e., $L_X d\nu_g = \text{div}(X) \, d\nu_g$ and $\Delta$ is the Laplace-Beltrami operator (on functions) i.e., $\Delta u = -\text{div}(\nabla u)$. Substitution from (5) into (4) together with Green's lemma yields [by $\{dE(\phi_s) / ds\}_{s=0} = 0$ and the density of $C^\infty(M)$ in $L^2(M)$]

$$\left( w \circ \phi \right) \Delta u + \left( w_1 \circ \phi \right) \|\nabla u\|^2$$

$$= (w \circ \phi) \left( \nabla u \right) e(\phi) + 2 \left( \nabla u \right) (w \circ \phi)$$

which is (2) in Lemma 1. When $w_1 = 0$ Equation (6) is

$$\text{div} \left\{ \exp [e(\phi)] (w \circ \phi)^2 \nabla u \right\} = 0.$$

Equation (7) is part of the Euler-Lagrange system associated to the variational principle $\delta E(\phi) = 0$. Next (by (7))

$$\text{div} \left\{ (w \circ \phi)^2 u \exp [e(\phi)] \nabla u \right\} = \exp [e(\phi)] (w \circ \phi)^2 \|\nabla u\|^2.$$

Let us integrate over $M$ in (8) and use Green's lemma. We obtain

$$\int_M \exp [e(\phi)] (w \circ \phi)^2 \|\nabla u\|^2 \, d\nu_g = 0.$$
yielding (as \( \phi \) is assumed to be nonconstant) \( u(x) = t_\phi \) for some \( t_\phi \in \mathbb{R} \) and any \( x \in M \). Q.e.d.

3. Exponentially Harmonic Maps Omitting a Codimension 2 Sphere Aren’t Null Homotopic

Let \( \Sigma \subset S^n \) be a codimension 2 totally geodesic submanifold. A continuous map \( \phi : M \to S^n \) meets \( \Sigma \) if \( \phi(M) \cap \Sigma \neq \emptyset \) and links \( \Sigma \) if \( \phi(M) \cap \Sigma = \emptyset \) and \( \phi : M \to S^n \setminus \Sigma \) is not null homotopic. The purpose of the section is to establish Theorem 1. Let \( \phi : M \to S^n \) be a nonconstant exponentially harmonic map from a compact, connected, orientable Riemannian manifold \( M \) into the sphere \( S^n \subset \mathbb{R}^{n+1} \). If \( \Sigma \subset S^n \) is a codimension 2 totally geodesic submanifold, then \( \phi \) either meets or links \( \Sigma \).

**Proof.** The proof is by contradiction, i.e., we assume that \( \phi \) doesn’t meet \( \Sigma \) and the map \( \phi : M \to S^n \setminus \Sigma \) is null homotopic. Let \((\xi_i)\) be a system of coordinates on \( \mathbb{R}^{n+1} \) such that \( \Sigma \) is given by the equations \( \xi_1 = \xi_2 = 0 \). Let \( S_+^{n-1} \subset \mathbb{R}^n \) be the hemisphere

\[
S_+^{n-1} = \left\{ y = (y', y_m) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \in S^{n-1}, \ y_m > 0 \right\}.
\]

Let us consider the map

\[
I : S_+^{n-1} \times S^1 \to S^n \setminus \Sigma, \quad I(y, \xi) = (y_m u, y_m v, y'),
\]

\[
y = (y', y_m) \in S_+^{n-1}, \quad \xi = u + iv \in S^1 \subset \mathbb{C}.
\]

Let \( g_N \) denote the canonical Riemannian metric on \( S^N \subset \mathbb{R}^{N+1} \). The map \( I \) is an isometry of \( S_+^{n-1} \times_f S^1 \) onto \( (S^n \setminus \Sigma, g_m) \) with the warping function

\[
f = C^0(S_+^{n-1} \times S^1), \quad f(y, \xi) = y_m.
\]

Let us consider the map \( \tilde{\phi} = I^{-1} \circ \phi \). We need the following. Q.e.d.

**Lemma 2.** Let \( S \) and \( \overline{S} \) be Riemannian manifolds, \( \pi : S \to \overline{S} \) a local isometry, and \( \overline{f} : M \to \overline{S} \) an exponentially harmonic map. Then every map \( f : M \to S \) such that \( \pi \circ f = \overline{f} \) is exponentially harmonic.

**Proof.** Let \( h \) and \( \overline{h} \) be the Riemannian metrics on \( S \) and \( \overline{S} \). For every 1-parameter variation \( \{ f_s \}_{|s| < \epsilon} \) of \( f_0 = f \) we set \( \overline{f}_s = \pi \circ f_s \) so that \( \{ \overline{f}_s \}_{|s| < \epsilon} \) is a 1-parameter variation of \( \overline{f}_0 = \overline{f} \). A calculation relying on \( \pi^* \overline{h} = h \) yields \( E(f_s) = E(\overline{f}_s) \) for every \( |s| < \epsilon \). Q.e.d.

By Lemma 2 the map \( \tilde{\phi} = I^{-1} \circ \phi \) is exponentially harmonic. Let us set

\[
F = \pi_1 \circ \tilde{\phi}, \quad \tilde{u} = \pi_2 \circ \tilde{\phi},
\]

where \( \pi_1 : S_+^{n-1} \times S^1 \to S_+^{n-1} \) and \( \pi_2 : S_+^{n-1} \times S^1 \to S^1 \) are projections. Next let us consider a point \( x_0 \in M \) and set \( \tilde{x}_0 = \tilde{u}(x_0) \in S^1 \). Also, considered the covering map \( p : \mathbb{R} \to S^1, \ p(t) = \exp(2\pi tit) \), pick \( t_0 \in \mathbb{R} \) such that \( p(t_0) = \tilde{x}_0 \). As \( \phi \) is null homotopic, the map \( \tilde{\phi} \) is null homotopic as well. Then

\[
\tilde{u}_* \pi_1(M, x_0) = 0
\]

where \( \pi_1(M, x_0) \) is the first homotopy group of \( M \). Consequently there is a unique smooth function \( u : M \to \mathbb{R} \) such that \( p \circ u = \tilde{u} \) and \( u(x_0) = t_0 \). The map

\[
\psi = (F, u) : M \to S_+^{n-1} \times_{\tilde{u}} \mathbb{R}
\]
is exponentially harmonic [because \( \psi = \pi \circ \tilde{\psi} \) and
\[
\pi = \left( 1_{S^{m-1}_+}, p \right) : S^{m-1}_+ \times_{w} \mathbb{R} \to S^{m-1}_+ \times I \]
is a local isometry, where \( w \in C^\infty(S^{m-1}_+) \) is given by \( w(y) = y_m \). We may then apply Lemma 1 to the map \( \psi \) with \( L = S^{m-1}_+ \) to conclude that
\[
\psi(M) \subset S^{m-1}_+ \times \{ t \}
\]
for some \( t \in \mathbb{R} \). It follows that \( F = \pi_1 \circ \psi : M \to S^{m-1}_+ \) is exponentially harmonic. We shall close the proof of Theorem 1 by showing that exponentially harmonic mappings into \( S^{m-1}_+ \) are constant. \( \square \)

4. Exponentially Harmonic Map System in Divergence Form

Let us consider the \( L^2 \) inner products
\[
(u, v)_{L^2} = \int_M u v \, d v_g, \quad (X, Y)_{L^2} = \int_M g(X, Y) \, d v_g.
\]

Let us think of the gradient \( \nabla \) as a first order differential operator \( \nabla : C^1(M) \to C(T(M)) \) and let \( \nabla^* \) be its formal adjoint, i.e.,
\[
(\nabla^* X, u)_{L^2} = (X, \nabla u)_{L^2}
\]
for any \( X \in C^1(T(M)) \) and \( u \in C^1(M) \). Ordinary integration by parts shows that \( \nabla^* X = -\text{div}(X) \).
Let \( \rho = \exp \left[ e(F) \right] \in C^\infty(M) \). Starting from \( \Delta u = -\text{div}(\nabla u) \) one has
\[
(\rho \Delta u, \varphi)_{L^2} = (\nabla^* \nabla u, \rho \varphi)_{L^2} = (\nabla u, \nabla (\rho \varphi))_{L^2}
\]
\[
= (\nabla^* (\rho \nabla u), \varphi)_{L^2} + \int_M \rho \, g(\nabla u, \nabla \rho) \, d v_g
\]
for any \( \varphi \in C^\infty(M) \), that is
\[
\exp \left[ e(F) \right] \Delta u = \nabla^* \left( \exp \left[ e(F) \right] \nabla u \right) + \exp \left[ e(F) \right] \, g(\nabla u, \nabla e(F)).
\]

Lemma 3. Let \( F : M \to S^{m-1}_+ \) be an exponentially harmonic map and \( F = j \circ F \) where \( j : S^{m-1} \to \mathbb{R}^m \) is the inclusion. If \( F = (F^1, \ldots, F^m) \) then
\[
- \nabla^* \left( \exp \left[ e(F) \right] \nabla F^i \right) + 2 e(F) \, \exp \left[ e(F) \right] F^i = 0
\]
for any \( 1 \leq i \leq m \).

Proof. Let \( y = (y^1, \ldots, y^{m-1}) : S^{m-1}_+ \to \mathbb{B}^{m-1} \) be the projection, where \( \mathbb{B}^{m-1} \subset \mathbb{R}^{m-1} \) is the open unit ball. With respect to this choice of local coordinates, the standard metric \( g_{m-1} \) and its Christoffel symbols are
\[
h_{ij} = \delta_{ij} + \frac{y^i y^j}{1 - |y|^2}, \quad |y|^2 = \sum_{i=1}^{m-1} (y^i)^2,
\]
\[
h^{ij} = \delta^{ij} - y^i y^j, \quad \Gamma^i_{jk} = y^j h_{ik}.
\]
Let us substitute from (13) into (1) [with $\phi^i = F^i$] and take into account

$$e(F) = \frac{1}{2} S^{ab} \frac{\partial F^i}{\partial x^a} \frac{\partial F^k}{\partial x^b} (h_{jk} \circ F).$$

(14)

The exponentially harmonic map system (1) becomes

$$-\Delta F^i + 2 e(F) F^i + g(\nabla e(F), \nabla F^i) = 0, \quad 1 \leq i \leq m - 1.
$$

(15)

Multiplication of (15) by $\exp[e(F)]$ and subtraction from (9) [with $u = F^i$] yields (10) for any $1 \leq i \leq m - 1$.

To see that (15) (and therefore (10)) holds for $i = m$ as well, one first exploits the constraint $(F^m)^2 = 1 - \sum_{i=1}^{m-1} (F^i)^2$ together with (11) and (14) to show that

$$e(F) = \frac{1}{2} \sum_{j=1}^{m} \|\nabla F^j\|^2.$$

Finally, one contracts (15) by $F^i$ and uses once again the constraint together with $\Delta (u^2) = 2\{u \Delta u - \|\nabla u\|^2\}$. Q.e.d.

We may now end the proof of Theorem 1 as follows. Let $F : M \to S^{m-1}$ be an exponentially harmonic map. Let us integrate over $M$ in (10) for $j = m$. Then (by Green’s lemma)

$$\int_M e(F) \exp[e(F)] F^m \, dv = 0$$

and $F^m > 0$ so that

$$0 = e(F) = \frac{1}{2} \sum_{j=1}^{m} \|\nabla F^j\|^2$$

yielding $F^i =$ constant. So $\phi$ is constant as well, a contradiction. \(\square\)

As well known $S^{m-1} \times S^1$ and $S^1$ are homotopically equivalent. Therefore a continuous map $\phi : M \to S^{m-1} \times S^1$ is null homotopic if and only if $\pi_2 \circ \phi : M \to S^1$ is null homotopic. The homotopy classes of continuous maps $M \to S^1$ form an abelian group $\pi_1(M)$ (the Bruschlinski group of $M$) naturally isomorphic to $H^1(M, \mathbb{Z})$. We may conclude that

**Corollary 1.** Let $M$ be a compact, orientable, connected Riemannian manifold with $H^1(M, \mathbb{Z}) = 0$. Then every nonconstant exponentially harmonic map $\phi : M \to S^m$ meets $\Sigma$.

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