1. Introduction

- Consider the following driftless control system

\[ \dot{x}(t) = \sum_{i=1}^{n} a_i(t) u_i(t), \quad x(0) = x_0 \]

subject to the constraints:

1. System (1) satisfy the Lie Algebraic Rank Condition (LARC) everywhere.
2. A global upper bound on the non-holonomic degree is known.
3. The vector fields \( g_i \) for \( i = 1, \ldots, m \) are smooth, i.e., \( C^0(\mathbb{R}^2, \mathbb{R}^2) \).

- Consider the order map of time \( n \)

\[ g_i \circ \tau_{(n-1)} = g_i \circ \tau_{(n-2)} \circ \tau_{(n-3)} \circ \cdots \circ \tau_{a}, \quad g_{\alpha} \in \mathcal{C}_L \text{ a finite alphabet, for all } \alpha \in \mathbb{N}. \]

- Consider the encoder map of time \( n \)

\[ h_i \circ \tau_{(0)} = (y_{i\alpha}, y_{i\alpha}) \]

\[ h_i \circ \tau_{(n+1)} = (y_{i\alpha}, y_{i\alpha}), \quad \alpha \in \mathbb{N}, \]

where \( \alpha \in \mathcal{C}_L \) is the set of functions from \( \{0, \ldots, n\} \) to \( \mathcal{Y} \).

- Let the average data-rate be given by

\[ h = \limsup \frac{1}{n} \log \left( |C_L| \right). \]

In this presentation, \( C_L = \{1, -1\} \), \( \alpha \in \mathcal{C}_L \), and \( \mathcal{Y} = [-1, 1] \).

- Our problem is: Can we find a constructive algorithm that gives us encoder and controller maps such that system (1) is globally asymptotically stabilized with minimum average data-rate?

- The answer is yes and we can do it with average data-rate zero.

2. The Driver

- System (1) can be rewritten in its integral form as

\[ x(t) = x(t_0) + \sum_{t_0 < \tau < t} a_i(t) u_i(t), \]

\[ x(0) = x_0. \]

Define the parameter \( \alpha_0 = \langle \pi_{\alpha_0} \rangle \rangle \rangle y_{i\alpha}, \alpha \in \mathcal{C}_L \). Also, let \( x_0 = \pi_{x_0} \alpha \circ \tau_{(0)}, \alpha \in \mathcal{C}_L \). Furthermore, let \( \gamma_{\alpha} = \langle \pi_{\alpha} \rangle \rangle \rangle y_{i\alpha}, \alpha \in \mathcal{C}_L \). Note that the function \( \langle \pi_{\alpha_0} \rangle \rangle \rangle y_{i\alpha} \) has its image on \( [-1, 1] \) by definition of \( \alpha_0 \), but it is an arbitrary piecewise constant function.

- Therefore the following equation holds

\[ x = x_0 + \alpha_0 \alpha_0. \]

- Consider \( Y : \mathbb{R}^2 \rightarrow \mathbb{R} \) a convex, radially unbounded, \( C^0(\mathbb{R}^2, \mathbb{R}) \), function with Lipschitz derivative around the origin.

The idea is to choose a control law \( 2_{y_{i\alpha_0}} \) such that \( \alpha_0 \) can be made into a decreasing direction for the cost function \( V \) departing from \( x_0 \). This is the idea behind the compass search method [A. R. Conn, K. Scheinberg, and L. N. Vicente, 2000].

If we choose a non-decreasing direction, we can go back, due to the strong connectivity property of driftless systems.

We need to be careful about two things:

- The step size \( \alpha_0 \) needs to be small.
- If the decrease in the function value between two consecutive iterations is not large enough we might get stuck.

To solve the second problem: Introduce a function \( \mu : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with the properties: (1) non-decreasing, (2) continuous, (3) \( \lim_{|\mu| \rightarrow 0} \frac{\mu}{|\mu|} = 0 \).

- Declare the iteration successful if \( (x(x_{i+2}), y_{i\alpha_0}) - (y_{i\alpha_0}) + \mu(x_{i+1}) < 0 \).

- To generate the directions \( y_{i\alpha_0} \in \mathcal{C}_L \), we pick control laws that generate approximations to all Lie brackets up to order \( \mathcal{O} \), i.e., \( \alpha_0 = \alpha_0 \circ \tau_{(1)}, ... \).

3. Parking the Car

- Consider the equations of the double car

\[ \begin{align*}
J_1 &= J_1 \cos(\theta) \\
J_2 &= J_1 \sin(\theta) \\
\dot{\theta} &= \tau_1
\end{align*} \]

where states \( J_1 \) and \( \theta \) represent the \( x \)-\( y \) coordinates of a car, while \( \theta \) is the angle.

- The cost function is \( f(x_1, x_2, \theta) = x_1^2 + x_2^2 + \theta^2 \).

- The controls are the functions \( y_{i\alpha_0} = (0, 0) \) for \( i \in [0, 2], \alpha_{11} = (0, 0), \alpha_{21} = (0, 0) \) for \( i \in [0, 2], \alpha_{12} = (0, 0), \alpha_{22} = (0, 0) \)

- The cost value reaches a plateau due to (i) local convergence properties of direct search methods [A. R. Conn, K. Scheinberg, and L. N. Vicente, 2009, (4)] and (ii) the fact that \( \alpha_{22} \) is zero.

4. Analysis

- It can be shown that the following assumptions are satisfied if the first set of assumptions hold. From this we conclude GAS:

(i)\( \) The level set \( L(x_0) = \{ x \in \mathbb{R}^2 \mid f(x) \leq f(x_0) \} \) is compact.

(ii)\( \) If for some constant \( c > 0 \), the step size at iteration \( k, \alpha_0 \) is such that \( \alpha_0 > c, \) for all \( k \in \mathbb{N} \). Then the algorithm visits only a finite number of points.

(iii)\( \) Let \( \alpha_0 > 0 \) be some fixed constant. The positive spanning sets \( V_{x_0} \) used in the algorithm are chosen from the set

\[ \{ \text{positive spanning } V_{x_0} \mid c(x_{i+1}) > c, \quad \text{for } i \in [0, 2] \} \]

Recall that the coarse measure of a finite positively spanning set \( V \subset \mathbb{R}^2 \) is \( c(V) = \min_{x \in V} \max_{y \in V} \langle \tau_{(0)} \rangle \rangle \rangle y_{i\alpha_0} \).

The gradient \( \nabla f(x) \) is Lipschitz continuous in an open set containing \( L(x_0) \) with Lipschitz constant \( \nu \).

5. Conclusion

- We presented a limited information control algorithm that globally asymptotically stabilizes a subclass of non-holonomic driftless affine systems.

- We proved that the minimum average data-rate for achieving GAS for this class is 0.

- Future works include extending the constructive method presented here to other classes of control systems and providing data-rate theorems for more general classes of systems.