ON THE WARING-GOLDBACH PROBLEM ON AVERAGE

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ABSTRACT. Let $s$, $\ell$ be two integers such that $2 \leq s \leq \ell - 1$, $\ell \geq 3$. We prove that a suitable asymptotic formula for the average number of representations of integers $n = \sum_{i=1}^{s} p_i^\ell$, where $p_i$, $i = 1, \ldots, s$, are prime numbers, holds in short intervals.

1. Introduction

Let $s \geq 2$, $\ell \geq 1$ be two integers, $N$ be a sufficiently large integer and $1 \leq H \leq N$ an integer. Let

$$ r_{s,\ell}(n) = \sum_{\substack{i=1,\ldots,s \leq \ell \leq N \atop n=p_1^\ell+\cdots+p_s^\ell}} \log p_1 \cdots \log p_s, $$

be the number of representation of an integer as a sum of exactly $s$ summands each one is a $\ell$-th prime power. The problem of obtaining an asymptotic formula for $r_{s,\ell}(n)$ is usually called the Waring-Goldbach problem. The history about the results on such a problem is a very long one; we refer to the surveys of Vaughan-Wooley [11] and Kumchev-Tolev [3] for an overview.

A simpler problem is to study the order of magnitude for an average of $r_{s,\ell}(n)$ because the averaging procedure let us to gain non-trivial information in cases in which the classical approaches fail, for example when $s \leq 4\ell((\log \ell + (1/2) \log \log \ell + O(1))$ for $\ell$ large or $s \leq H(\ell)$ for $4 \leq s \leq 10$, where $H(4) \leq 14$, $H(5) \leq 21$, $H(6) \leq 33$, $H(7) \leq 47$, $H(8) \leq 63$, $H(9) \leq 83$, $H(10) \leq 107$, according to [11], page 20. Recently Cantarini, Gambini and Zaccagnini [1] proved that a suitable asymptotic formula in short intervals holds for $\sum_{n=N+1}^{N+H} r_{s,\ell}(n)$ when $s = \ell + 1$, $\ell \geq 2$ and $s = 1, \ell \geq 2$, thus generalizing previous results by Languasco-Zaccagnini [7] ($s = 4, \ell = 3$) and [4] ($s = 2, \ell = 2$).

Here we restrict our attention to the more difficult case in which we have less summands, i.e., when $2 \leq s \leq \ell - 1$, $\ell \geq 3$. We also remark that a more general binary problem is treated by Languasco-Zaccagnini in [6] and [8]. Our first result is

**Theorem 1.** Let $s$, $\ell$ be two integers such that $2 \leq s \leq \ell - 1$, $\ell \geq 3$, $N \geq 2$, $1 \leq H \leq N$ be integers. Then, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that

$$ \sum_{n=N+1}^{N+H} r_{s,\ell}(n) = \frac{\Gamma(1 + 1/\ell)^s}{\Gamma(s/\ell)} H^{\ell s/\ell - 1} + O_{s,\ell}\left(H^{\ell s/\ell - 1} \exp\left(-C\left(\frac{\log N}{\log \log N}\right)^{1/3}\right)\right) \quad \text{as } N \to \infty, $$

uniformly for $N^{1-5/(6\ell)+\varepsilon} \leq H \leq N^{1-\varepsilon}$, where $\Gamma$ is Euler’s function.

As an immediate consequence of Theorem 1 we can say that, for $N$ sufficiently large, every interval of size larger than $N^{1-5/(6\ell)+\varepsilon}$ contains the expected amount of integers which are a sum of exactly $s$ summands, $2 \leq s \leq \ell - 1$, each one is a $\ell$-th prime power, $\ell \geq 3$.

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We remark that the uniformity level for $H$ in Theorem 1 is the expected optimal one given the known density estimates for the non-trivial zeros of the Riemann zeta-function.

Assuming the Riemann Hypothesis holds, we can improve the uniformity range of $H$ since in this case Lemma 4 below holds in the whole unit interval for $\xi$.

**Theorem 2.** Let $s$, $\ell$ be two integers such that $2 \leq s \leq \ell - 1$, $\ell \geq 3$, $N \geq 2$, $1 \leq H \leq N$ be integers and assume the Riemann Hypothesis (RH) holds. Then

$$
\sum_{n=N+1}^{N+H} \frac{r_{s,\ell}(n)}{\Gamma(s/\ell)} HN^{s/\ell-1} + \mathcal{O}(H^{2}N^{s/\ell-2} + H^{1/2}N^{3/(\ell-1)(2\ell)-1/2}(\log N)^{3})
$$

as $N \to \infty$, uniformly for $\infty(N^{1-1/\ell}(\log N)^{6}) \leq H \leq o(N)$, where $f = \infty(g)$ means $g = o(f)$ and $\Gamma$ is Euler’s function.

As an immediate consequence of Theorem 2 we can say that, for $N$ sufficiently large, every interval of size larger than $N^{1-1/\ell+\epsilon}$ contains the expected amount of integers which are a sum of exactly $s$ summands, $2 \leq s \leq \ell - 1$, each one is a $\ell$-th prime power, $\ell \geq 3$. We remark that in this case the $H$-level is essentially optimal given the spacing of the sequence.

In both the proofs of Theorems 1,2 we will use the circle method with the original Hardy-Littlewood generating functions to exploit the easier main term treatment they allow (comparing with the one which would follow using Lemmas 2.3 and 2.9 of Vaughan [10]). Key tools in the proofs of Theorems 1,2 are Lemma 5 and 4 which, respectively, give suitable estimates on the truncated mean-square average for the exponential sum over prime powers $\tilde{S}_{\ell}(\alpha)$ and for the error term in its first order approximation which, by Lemma 2, is directly connected with the non-trivial zeros of the Riemann zeta-function.

2. **Settings**

Let $s \geq 2$, $\ell \geq 1$ be two integers, $N$ be a sufficiently large integer, $1 \leq H \leq N$ an integer, $e(\alpha) = e^{2\pi i \alpha}$, $\alpha \in [-1/2, 1/2]$, $L = \log N$, $z = 1/N - 2\pi i \alpha$,

$$
\tilde{S}_{\ell}(\alpha) := \sum_{n=1}^{\infty} \Lambda(n) e^{-n^2/N} e(n^2 \alpha) \quad \text{and} \quad \tilde{V}_{\ell}(\alpha) := \sum_{p=2}^{\infty} \log p e^{-p^2/N} e(p^2 \alpha).
$$

We remark that

$$
|z|^{-1} \ll \min(N, |\alpha|^{-1}) \quad \text{and} \quad \tilde{S}_{\ell}(\alpha) \ll \tilde{V}_{\ell}(\alpha) N^{1/\ell}, \quad (2)
$$

where the second inequality is a direct consequence of the Prime Number Theorem. We further set

$$
U(\alpha, H) := \sum_{m=1}^{H} e(m\alpha)
$$

and, moreover, we also have the well known numerically explicit inequality

$$
|U(\alpha, H)| \leq \min(H, |\alpha|^{-1}). \quad (3)
$$

We list now the needed preliminary results.

**Lemma 1 (Lemma 3 of [7]).** Let $\ell \geq 1$ be an integer. Then $|\tilde{S}_{\ell}(\alpha) - \tilde{V}_{\ell}(\alpha)| \ll \tilde{V}_{\ell}(\alpha) N^{1/2\ell}$.

**Lemma 2 (Lemma 2 of [7]).** Let $\ell \geq 1$ be an integer, $N \geq 2$ and $\alpha \in [-1/2, 1/2]$. Then

$$
\tilde{S}_{\ell}(\alpha) = \frac{\Gamma(1/\ell)}{\ell z^{-1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma(\rho/\ell) + \mathcal{O}(1),
$$

where $\rho$ runs over the nontrivial zeros of the Riemann zeta-function with $|\Re(\rho)| \leq 1/\ell$. We remark that for $N \to \infty$, uniformly for $\infty(0) \leq H \leq o(N)$, $|\tilde{S}_{\ell}(\alpha)| \ll \tilde{V}_{\ell}(\alpha) N^{1/2\ell}$.
where \( \rho = \beta + i\gamma \) runs over the non-trivial zeros of \( \zeta(s) \) and \( \Gamma \) is Euler’s function.

**Lemma 3.** Let \( N \) be a positive integer and \( \mu > 0 \). Then, uniformly for \( n \geq 1 \) and \( X > 0 \), we have
\[
\int_{-X}^{X} z^{-\mu} e(-n\alpha) \, d\alpha = e^{-n/N} \frac{N^{\mu-1}}{\Gamma(\mu)} + \Theta_{\mu}(\frac{1}{nX^\mu}),
\]
where \( \Gamma \) is Euler’s function.

**Proof.** We remark that the proof is identical to the one of Lemma 4 of [5] but in that case we just stated the lemma in the particular case \( X = 1/2 \). Now we need its full strength and hence, for completeness, we rewrite its proof. We start with the identity
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{iDu}}{(a+iu)^\mu} \, du = \frac{D^{s-1} e^{-aD}}{\Gamma(s)},
\]
which is valid for \( \sigma = \Re(s) > 0 \) and \( a \in \mathbb{C} \) with \( \Re(a) > 0 \) and \( D > 0 \). Letting \( u = -2\pi\alpha \) and taking \( s = \mu \), \( D = n \) and \( a = N^{-1} \) we find
\[
\int_{\mathbb{R}} \frac{e(-n\alpha)}{(N^{-1} - 2\pi i\alpha)^\mu} \, d\alpha = \int_{\mathbb{R}} z^{-\mu} e(-n\alpha) \, d\alpha = \frac{n^{\mu-1} e^{-n/N}}{\Gamma(\mu)}.
\]

For \( 0 < X < Y \) let
\[
I(X, Y) = \int_{X}^{Y} \frac{e^{iDu}}{(a+iu)^\mu} \, du.
\]

An integration by parts yields
\[
I(X, Y) = \left[ \frac{1}{iD} \frac{e^{iDu}}{(a+iu)^\mu} \right]_{X}^{Y} + \frac{\mu}{D} \int_{X}^{Y} \frac{e^{iDu}}{(a+iu)^{\mu+1}} \, du.
\]

Since \( a > 0 \), the first summand is \( \ll_{\mu} D^{-1} X^{-\mu} \), uniformly. The second summand is
\[
\ll_{\mu} \frac{\mu}{D} \int_{X}^{Y} \frac{du}{u^{\mu+1}} = \ll_{\mu} D^{-1} X^{-\mu}.
\]

The result follows. \( \square \)

**Lemma 4 (Lemma 4 of [7]).** Let \( \varepsilon \) be an arbitrarily small positive constant, \( \ell \geq 1 \) be an integer, \( N \) be a sufficiently large integer and \( L = \log N \). Then there exists a positive constant \( c_1 = c_1(\varepsilon) \), which does not depend on \( \ell \), such that
\[
\int_{-\xi}^{\xi} \left| \frac{S_\ell(\alpha)}{\ell \zeta^{1/\ell}} \right|^2 \, d\alpha \ll_{\ell} N^{2/\ell-1} \exp \left( - c_1 \left( \frac{L}{\log L} \right)^{1/3} \right)
\]
uniformly for \( 0 \leq \xi < N^{-1+5/(6\ell)-\varepsilon} \). Assuming RH we get
\[
\int_{-\xi}^{\xi} \left| \frac{S_\ell(\alpha)}{\ell \zeta^{1/\ell}} \right|^2 \, d\alpha \ll_{\ell} N^{1/\ell} \xi L^2
\]
uniformly for \( 0 \leq \xi \leq 1/2 \).

The following two lemmas hold for a real index \( k \) instead of an integral one \( \ell \). The new ingredient we are using here is based on a Tolev’s lemma [9].

**Lemma 5.** Let \( k > 1 \), \( n \in \mathbb{N} \) and \( \tau > 0 \). We have
\[
\int_{-\tau}^{\tau} |S_k(\alpha)|^2 \, d\alpha \ll_k (\tau N^{1/k} + N^{2/(k-1)}) L^3 \quad \text{and} \quad \int_{-\tau}^{\tau} |\overline{S_k}(\alpha)|^2 \, d\alpha \ll_k (\tau N^{1/k} + N^{2/(k-1)}) L^3.
\]
Theorem implies estimate on $\tilde{\cdot}$ of Lemma 5 but follows analogously. □

Lemma 7 of Tolev [9] in the form given in Lemma 5 of [2] on Lemma 6.

Proof. We just prove the first part since the second one follows immediately by remarking that the primes are supported on a thinner set than the prime powers. Let $P = (2NL/k)^{1/k}$. A direct estimate gives $\tilde{S}_k(\alpha) = \sum_{n \leq P} \Lambda(n)e^{-nk/N}e(n^k\alpha) + O_k(L^{1/k})$. Recalling that the Prime Number Theorem implies $S_k(\alpha; t) = \sum_{n \leq t} \Lambda(n)e(n^k\alpha) \ll t$, a partial integration argument gives

$$\sum_{n \leq P} \Lambda(n)e^{-nk/N}e(n^k\alpha) = -\frac{k}{N} \int_1^P t^{k-1}e^{-t^{k/N}}S_k(\alpha; t)\ dt + O_k(L^{1/k}).$$

Using the inequality $(|a| + |b|)^2 \ll |a|^2 + |b|^2$, Cauchy-Schwarz inequality and interchanging the integrals, we get that

$$\int_{-\tau}^{\tau} \left| \tilde{S}_k(\alpha) \right|^2 \ d\alpha \ll \int_{-\tau}^{\tau} \left| \frac{1}{N} \int_1^P t^{k-1}e^{-t^{k/N}}S_k(\alpha; t)\ dt \right|^2 \ d\alpha + L^{2/k}$$

$$\ll \frac{1}{N^2} \left( \int_1^P t^{k-1}e^{-t^{k/N}} \ dt \right)^2 \left( \int_1^P t^{k-1}e^{-t^{k/N}} \int_{-\tau}^{\tau} \left| S_k(\alpha; t) \right|^2 \ d\alpha \ dt \right) + L^{2/k}.$$ 

Lemma 7 of Tolev [9] in the form given in Lemma 5 of [2] on $S_k(\alpha; t) = \sum_{n \leq t} \Lambda(n)e(n^k\alpha)$ implies that $\int_{-\tau}^{\tau} \left| S_k(\alpha; t) \right|^2 \ d\alpha \ll (\tau t + t^{2-k})(\log t)^3$. Using such an estimate and remarking that $\int_1^P t^{k-1}e^{-t^{k/N}} \ dt \ll_k N$, we obtain that

$$\int_{-\tau}^{\tau} \left| \tilde{S}_k(\alpha) \right|^2 \ d\alpha \ll \frac{1}{N} \int_1^P (\tau t + t^{2-k})t^{k-1}e^{-t^{k/N}(\log t)^3} \ dt + L^{2/k}$$

$$\ll_k (\tau N^{1/k} + N^{2/k-1})L^3$$

by a direct computation. This proves the first part of the lemma. □

The last lemma is a consequence of Lemma 5.

Lemma 6. Let $N \in \mathbb{N}$, $k > 1$, $u \geq 1$ and $N^{-u} \leq \omega \leq N^{1/k-1}/L$. Let further $I(\omega) := [-1/2, -\omega] \cup [\omega, 1/2]$. We have

$$\int_{I(\omega)} \left| \tilde{S}_k(\alpha) \right|^2 \frac{d\alpha}{|\alpha|} \ll_k \frac{N^{2/k-1}}{\omega} L^3 \quad \text{and} \quad \int_{I(\omega)} \left| \tilde{V}_k(\alpha) \right|^2 \frac{d\alpha}{|\alpha|} \ll_k \frac{N^{2/k-1}}{\omega} L^3.$$

Let further assume the Riemann Hypothesis, $\ell \geq 1$ be an integer and $N^{-u} \leq \eta \leq 1/2$. Then

$$\int_{I(\eta)} \left| \tilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell \zeta(1/\ell)} \right|^2 \frac{d\alpha}{|\alpha|} \ll_k N^{1/\ell} L^3.$$

Proof. By partial integration and Lemma 5 we get that

$$\int_{-\omega}^{1/2} \left| \tilde{S}_k(\alpha) \right|^2 \frac{d\alpha}{\alpha} \ll \frac{1}{\omega} \int_{-\omega}^{\omega} \left| \tilde{S}_k(\alpha) \right|^2 \ d\alpha + \int_{-1/2}^{1/2} \left| \tilde{S}_k(\alpha) \right|^2 \ d\alpha + \int_{-\xi}^{1/\ell} \left( \int_{-\xi}^{\xi} \left| \tilde{S}_k(\alpha) \right|^2 \ d\alpha \right) \frac{d\xi}{\xi^2}$$

$$\ll_k \frac{L^3}{\omega} \left( \omega N^{1/k} + N^{2/k-1} \right) + \frac{N^{1/k} L^3 + L^3}{\omega} \int_{-\xi}^{1/\ell} \xi N^{1/k} + N^{2/k-1} \frac{d\xi}{\xi^2}$$

$$\ll_k N^{1/k} L^3 |\log(2\omega)| + \frac{N^{2/k-1}}{\omega} L^3 \ll_k N^{2/k-1} L^3$$

since $N^{-u} \leq \omega \leq N^{1/k-1}/L$. A similar computation proves the result in $[-1/2, -\omega]$ too. The estimate on $\tilde{V}_k(\alpha)$ can be obtained analogously. The third estimate requires Lemma 4 instead of Lemma 5 but follows analogously. □
3. Proof of Theorem \[1\]

Let \( s, \ell \) be two integers such that \( 2 \leq s \leq \ell - 1 \) and \( \ell \geq 3 \). Let further \( H > 2B \), where 
\[
B = N^\varepsilon. 
\]
\[\text{(4)}\]

Letting \( I(v) := [-1/2, -v] \cup [v, 1/2] \), where \( 0 < v < 1/2 \), and recalling \[1\], we have
\[
\sum_{n=N+1}^{N+H} e^{-n/N} r_{s, \ell}(n) = \int_{-1/2}^{1/2} \overline{V}_\ell(\alpha)^s U(-\alpha, H)e(-Na) \, d\alpha \\
= \int_{-B/H}^{B/H} \overline{S}_\ell(\alpha)^s U(-\alpha, H)e(-Na) \, d\alpha + \int_{I(B/H)} \overline{S}_\ell(\alpha)^s U(-\alpha, H)e(-Na) \, d\alpha \\
+ \int_{-1/2}^{1/2} \left( \overline{V}_\ell(\alpha)^s - \overline{S}_\ell(\alpha)^s \right) U(-\alpha, H)e(-Na) \, d\alpha = I_1 + I_2 + I_3, \tag{5}\]
say. We need to split the unit interval in this way because in the unconditional case Lemma \[4\] works only on a subset of \([-1/2, 1/2]\); in the remaining part of the interval we need a different estimate based on Lemma \[6\]. By \[3\], the ideal splitting level should be with \( B = 1 \) but that would not be good enough for controlling the estimate in eq. \[6\] below since the expected order of magnitude of the main term is essentially \( c_{s, \ell}HN^{s/\ell-1} \), where \( c_{s, \ell} = \Gamma(1+1/\ell)^s/\Gamma(s/\ell) \). Hence we have to choose a larger \( B \) and, in fact, the choice made in eq. \[4\] is sufficiently good for our purposes. We also remark that \( I_3 \) collects the difference between the contributions from the exponential sums over prime powers \( \overline{S} \) and the one over primes \( \overline{V} \); we’ll see that such contributions are negligible. Finally, we will see that \( I_1 \) contains the “main term” of the problem together with several error terms connected with the distribution of zeros of the Riemann zeta-function; the main term will be evaluated using Lemma \[3\] while the long part needed to estimate the error terms makes use of Lemmas \[4\] and \[5\]. At the end of the proof we will see how to get rid of the \( e^{-n/N} \) weight we have on the left-hand side of \[5\]. Now we evaluate these terms.

3.1. Estimation of \( I_2 \). Since \( s \geq 2 \), using \[2\]-\[3\] and Lemma \[6\] with \( \omega = B/H \), we have that
\[
I_2 \ll (\max |\overline{S}_\ell(\alpha)|^{s-2}) \int_{I(B/H)} |\overline{S}_\ell(\alpha)|^2 \frac{d\alpha}{|\alpha|} \ll_{s, \ell} \frac{H}{B} N^{s/\ell-1} L^3, \tag{6}\]
provided that \( H \gg N^{1-1/\ell} BL \).

3.2. Estimation of \( I_3 \). Clearly
\[
|\overline{V}_\ell(\alpha)^s - \overline{S}_\ell(\alpha)^s| \ll_{s, \ell} |\overline{V}_\ell(\alpha) - \overline{S}_\ell(\alpha)| (|V_\ell(\alpha)| + |\overline{S}_\ell(\alpha)|)^{s-1} \\
\ll_{s, \ell} |\overline{V}_\ell(\alpha) - \overline{S}_\ell(\alpha)| \max(|V_\ell(\alpha)|^{s-1}; |\overline{S}_\ell(\alpha)|^{s-1}).
\]

Hence by Lemma \[1\] we have
\[
I_3 \ll_{s, \ell} \int_{-1/2}^{1/2} \left( |\overline{V}_\ell(\alpha)|^{s-1} + |\overline{S}_\ell(\alpha)|^{s-1} \right) |U(-\alpha, H)| \, d\alpha = N^{1/(2\ell)}(K_1 + K_2), \tag{7}\]
say. If \( s \geq 3 \), by \[2\]-\[3\], Lemmas \[5\]\[6\] with \( \tau = 1/H \), we get
\[
K_2 \ll (\max |\overline{S}_\ell(\alpha)|^{s-3}) \int_{-1/2}^{1/2} |\overline{S}_\ell(\alpha)|^2 |U(-\alpha, H)| \, d\alpha \\
\ll_{\ell} N^{(s-3)/\ell} \left( H \int_{-1/2}^{1/2} |\overline{S}_\ell(\alpha)|^2 \, d\alpha + \int_{I(1/H)} |\overline{S}_\ell(\alpha)|^2 \frac{d\alpha}{|\alpha|} \right) \ll_{\ell} HN^{s/\ell-1/\ell-1} L^3, \tag{8}\]
provided that \( H \gg N^{1-1/\ell} L \). If \( s = 2 \), by (3), the Cauchy-Schwarz estimate, Lemmas 5, 6 with \( \tau = 1/H \), we get
\[
K_2 \ll \left( \int_{-1/2}^{1/2} |\tilde{S}_e(\alpha)|^2 |U(-\alpha, H)| \, d\alpha \right)^{1/2} \left( \int_{-1/2}^{1/2} |U(-\alpha, H)| \, d\alpha \right)^{1/2}
\]
\[
\ll \ell \left( H \int_{-1/2}^{1/2} |\tilde{S}_e(\alpha)|^2 \, d\alpha + \int_{1/H \ell}^{1/H} |\tilde{S}_e(\alpha)|^2 \, d\alpha \right)^{1/2} \left( H \int_{-1/2}^{1/2} \, d\alpha + \int_{1/H \ell} \, d\alpha \right)^{1/2}
\]
\[
\ll \ell H^{1/2} N^{\ell-1/2} L^2,
\]
provided that \( H \gg N^{1-1/\ell} L \). Analogous computations give
\[
K_1 \ll \ell \begin{cases} 
HN^{s/\ell-1/2} L^3 & \text{if } s \geq 3 \\
H^{1/2} N^{1-1/2} L^2 & \text{if } s = 2,
\end{cases}
\]
provided that \( H \gg N^{1-1/\ell} L \). By (10), we can finally write
\[
I_3 \ll_{s, \ell} \begin{cases} 
HN^{s/\ell-1/2} L^3 & \text{if } s \geq 3 \\
H^{1/2} N^{3/2} L^2 & \text{if } s = 2,
\end{cases}
\]
provided that \( H \gg N^{1-1/\ell} L \).

3.3. Evaluation of \( I_1 \). Since \( \Gamma(1 + 1/\ell) = \Gamma(1/\ell) / \ell \), we let \( d_{s, \ell} := \Gamma(1 + 1/\ell)^s / \Gamma(s/\ell) \), say, and we have that
\[
I_1 = \int_{-B/H}^{B/H} d_{s, \ell} u(-\alpha, H) e(-N\alpha) \, d\alpha + \int_{-B/H}^{B/H} \left( \tilde{S}_e(\alpha) - d_{s, \ell} / z^{s/\ell} \right) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]
\[
= J_1 + J_2,
\]
say. By Lemma 3 and a direct calculation we obtain
\[
J_1 = c_{s, \ell} \sum_{n=N+1}^{N+H} n^{s/\ell-1} e^{-n/N} + \Theta_{s, \ell} \left( \frac{H}{N} \right)^{s/\ell} = \frac{c_{s, \ell}}{e} \sum_{n=N+1}^{N+H} n^{s/\ell-1} + \Theta_{s, \ell} \left( \frac{H^2 N^{s/\ell-2} + \frac{H}{N} \left( \frac{H}{B} \right)^{s/\ell}}{} \right)
\]
\[
= \frac{c_{s, \ell}}{e} H^{N^{s/\ell-1}} + \Theta_{s, \ell} \left( \frac{H^2 N^{s/\ell-2} + \frac{H}{N} \left( \frac{H}{B} \right)^{s/\ell}}{} \right).
\]

3.4. Evaluation of \( J_2 \). From now on, we denote \( m_\ell(z) := \Gamma(1 + 1/\ell) z^{-1/\ell} \), so that \( d_{s, \ell} / z^{s/\ell} = m_\ell(z)^s \) and \( \tilde{E}_\ell(\alpha) := \tilde{S}_e(\alpha) - m_\ell(z) \). Using \( f^2 - g^2 = 2g(f - g) + (f - g)^2 \) we obtain
\[
\tilde{S}_e(\alpha)^s - m_\ell(z)^s = \begin{cases} 
\tilde{E}_\ell(\alpha) \sum_{j=0}^{s-1} \tilde{S}_e(\alpha)^{s-1-j} m_\ell(z)^j & \text{if } s \text{ is odd} \\
(2m_\ell(z) \tilde{E}_\ell(\alpha) + \tilde{E}_\ell(\alpha)^2) \sum_{j=0}^{s/2-1} \tilde{S}_e(\alpha)^{s-2-j} m_\ell(z)^{2j} & \text{if } s \text{ is even}
\end{cases}
\]
\[
(14)
\]
We define
\[
\mathcal{A}(d) := \exp \left( d \left( \frac{L}{\log L} \right)^{1/3} \right),
\]
\[
\mathcal{E} := \int_{-B/H}^{B/H} |\tilde{E}_\ell(\alpha)|^2 \, d\alpha \ll \ell N^{2/\ell-1} \mathcal{A}(-c_1),
\]
\[
\mathcal{F} := \int_{-B/H}^{B/H} |\tilde{S}_e(\alpha)|^2 \, d\alpha \ll \ell N^{2/\ell-1} L^3,
\]
\[
(15, 16, 17)
\]
where \( d \) is a real constant and in which the estimates follow respectively using Lemma \( \text{[2]} \) with \( \xi = B/H \) and \( H \geq BN^{1-5/(6\ell)+\varepsilon} \) for (16) and Lemma \( \text{[5]} \) with \( \tau = B/H \) and \( H \geq BN^{1-1/\ell} \) for (17). Moreover using (2), for every integral \( n \geq 2 \) we get

\[
|m_t(z)^u \ll_{u, \ell} N^{(u-2)/\ell} [m_t(z)]^2 \ll_{u, \ell} N^{(u-2)/\ell} (\tilde{S}_t(\alpha))^2 + |\tilde{E}_t(\alpha)|^2). \tag{18}
\]

Hence we have, using also (16)-(17), that

\[
\mathcal{M}(u) := \int_{-B/H}^{B/H} |m_t(z)|^u \, d\alpha \ll_{u, \ell} N^{(u-2)/\ell} (\mathcal{E} + \mathcal{E}') \ll_{u, \ell} N^{u/\ell-1} L^3. \tag{19}
\]

Assume \( s \) is odd. Inserting (14) into (12) and using (3) we get that

\[
J_2 \ll H \sum_{j=0}^{s-1} \int_{-B/H}^{B/H} |\tilde{E}_t(\alpha)| |\tilde{S}_t(\alpha)^{s-1-j}| |n_t(z)|^{j+1} | \, d\alpha = H \sum_{j=0}^{s-1} K_j, \tag{20}
\]

say. If \( 0 \leq j \leq s-2 \), using the Cauchy-Schwarz inequality, (2) and (16)-(17) we have

\[
K_j \ll (\max |m_t(z)|^j)(\max |\tilde{S}_t(\alpha)^{s-2-j}|) \int_{-B/H}^{B/H} |\tilde{E}_t(\alpha)| |\tilde{S}_t(\alpha)| \, d\alpha \ll_{s, \ell} N^{s/\ell-1} \mathcal{A}(-c_1/4), \tag{21}
\]

provided that \( H \geq BN^{1-5/(6\ell)+\varepsilon} \). If \( j = s-1 \), using the Cauchy-Schwarz inequality, (16) and (19) we have

\[
K_{s-1} \ll \int_{-B/H}^{B/H} |\tilde{E}_t(\alpha)| |m_t(z)|^{s-1} \, d\alpha \ll \mathcal{E}^{1/2} \cdot \mathcal{M}(2s-2)^{1/2} \ll_{s, \ell} N^{s/\ell-1} \mathcal{A}(-c_1/4), \tag{22}
\]

provided that \( H \geq BN^{1-5/(6\ell)+\varepsilon} \). Summarizing, using (20)-(22), we obtain, for \( s \) odd, that

\[
J_2 \ll_{s, \ell} H N^{s/\ell-1} \mathcal{A}(-c_1/4), \tag{23}
\]

provided that \( H \geq BN^{1-5/(6\ell)+\varepsilon} \).

Assume \( s \) is even. Inserting (14) into (12) and using (3) we get that

\[
J_2 \ll H \sum_{j=0}^{s/2-1} \int_{-B/H}^{B/H} |\tilde{E}_t(\alpha)| |\tilde{S}_t(\alpha)^{s-2-2j}| |m_t(z)|^{j+1} | \, d\alpha + H \sum_{j=0}^{s/2-1} \int_{-B/H}^{B/H} |\tilde{E}_t(\alpha)|^2 |\tilde{S}_t(\alpha)^{s-2-2j}| |m_t(z)|^{2j} | \, d\alpha = H \sum_{j=0}^{s/2-1} (U_j + V_j), \tag{24}
\]

say. Now we estimate the \( V_j \)'s. Recalling (2) and (16), we have

\[
V_j \ll (\max |m_t(z)|^j)(\max |\tilde{S}_t(\alpha)^{s-2-2j}|) \ll_{s, \ell} N^{s/\ell-1} \mathcal{A}(-c_1), \tag{25}
\]

provided that \( H \geq BN^{1-5/(6\ell)+\varepsilon} \). Now we estimate the \( U_j \)'s. If \( s \geq 4 \) and \( 0 \leq j \leq s/2-2 \), recalling (2), using the Cauchy-Schwarz inequality and (16)-(17), we obtain as in (21) that

\[
U_j \ll (\max |m_t(z)|^{j+1})(\max |\tilde{S}_t(\alpha)^{s-2-2j}|) \int_{-B/H}^{B/H} |\tilde{E}_t(\alpha)| |\tilde{S}_t(\alpha)| \, d\alpha \ll_{s, \ell} N^{s/\ell-1} \mathcal{A}(-c_1/4), \tag{26}
\]
provided that \( H \geq BN^{1-5/(6\ell)+\varepsilon} \). If \( j = s/2 - 1, s \geq 2 \), recalling (2), using the Cauchy-Schwarz inequality, (16) and (19), we have as in (22) that
\[
U_{s/2-1} \ll \int_{-B/H}^{B/H} \left| \mathcal{E}(\alpha) |m_{\ell}(z)| \right|^{s-1} \, d\alpha \ll \varepsilon^{1/2} \mathcal{M}(2s-2)^{1/2} \ll_{s,\ell} N^{s/(\ell-1)} \mathcal{A}(-c_1/4),
\] (27)
provided that \( H \geq BN^{1-5/(6\ell)+\varepsilon} \). Summarizing, using (24)-(27), we obtain, for \( s \) even, that
\[
J_2 \ll_{s,\ell} HN^{s/(\ell-1)} \mathcal{A}(-c_1/4),
\] (28)
provided that \( H \geq BN^{1-5/(6\ell)+\varepsilon} \).

By (12)-(13), (23) and (28) we have that, for every \( \varepsilon > 0 \), there exists \( c_1 = c_1(\varepsilon) > 0 \) such that
\[
I_1 = \frac{c_{s,\ell}}{e} HN^{s/(\ell-1)} + O_{s,\ell}(HN^{s/(\ell-1)} \mathcal{A}(-c_1/4)),
\] (29)
provided that \( H \geq BN^{1-5/(6\ell)+\varepsilon} \). The error terms are dominated by the first one assuming \( H \leq N^{1-\varepsilon} \) and remarking \( B = N^\varepsilon \) by (4). Hence we can write that, for every \( \varepsilon > 0 \), there exists \( C = C(\varepsilon) > 0 \) such that
\[
\sum_{n=N+1}^{N+H} e^{-n/N} r_{s,\ell}(n) = \frac{c_{s,\ell}}{e} HN^{s/(\ell-1)} + O_{s,\ell}(HN^{s/(\ell-1)} \mathcal{A}(-C)),
\] (30)
provided that \( N^{1-5/(6\ell)+2\varepsilon} \leq H \leq N^{1-\varepsilon} \). From \( e^{-n/N} = e^{-1} + O(H/N) \) for \( n \in [N+1, N+H] \), 1 \( \leq H \leq N \), we get that, for every \( \varepsilon > 0 \), there exists \( C = C(\varepsilon) > 0 \) such that
\[
\sum_{n=N+1}^{N+H} r_{s,\ell}(n) = c_{s,\ell} HN^{s/(\ell-1)} + O_{s,\ell}(HN^{s/(\ell-1)} \mathcal{A}(-C)) + O_{s,\ell}\left(\frac{H}{N} \sum_{n=N+1}^{N+H} r_{s,\ell}(n)\right),
\]
provided that \( N^{1-5/(6\ell)+2\varepsilon} \leq H \leq N^{1-\varepsilon} \). Using \( e^{n/N} \leq e^2 \) and (30), the last error term is \( \ll_{s,\ell} H^{2} N^{s/(\ell-2)} \). Hence we get that, for every \( \varepsilon > 0 \), there exists \( C = C(\varepsilon) > 0 \) such that
\[
\sum_{n=N+1}^{N+H} r_{s,\ell}(n) = c_{s,\ell} HN^{s/(\ell-1)} + O_{s,\ell}(HN^{s/(\ell-1)} \mathcal{A}(-C)),
\]
provided that \( N^{1-5/(6\ell)+2\varepsilon} \leq H \leq N^{1-\varepsilon} \). Theorem 1 follows by rescaling \( \varepsilon \) and recalling (15) and \( c_{s,\ell} = \Gamma(1 + 1/\ell)/\Gamma(s/\ell) \).
4. Proof of Theorem

From now on we assume the Riemann Hypothesis holds. The proof runs analogously to the unconditional case we described in section 4 but we can slightly simplify the approach since in this case Lemma 4 holds in the whole unit interval for $\xi$.

Let $s, \ell$ be two integers such that $2 \leq s \leq \ell - 1$ and $\ell \geq 3$. Recalling (1) and $d_{s, \ell} := \Gamma(1 + 1/\ell)^s$, we have

$$
\sum_{n=N+1}^{N+H} e^{-n/N} r_{s, \ell}(n) = \int_{-1/2}^{1/2} \tilde{V}_\ell(\alpha)^s U(-\alpha, H) e(-N\alpha) \, d\alpha
$$

$$
= d_{s, \ell} \int_{-1/2}^{1/2} \frac{U(-\alpha, H)}{z^{s/\ell}} e(-N\alpha) \, d\alpha + \int_{-1/2}^{1/2} \left( \tilde{S}_\ell(\alpha)^s - \frac{d_{s, \ell}}{z^{s/\ell}} \right) U(-\alpha, H) e(-N\alpha) \, d\alpha
$$

$$
+ \int_{-1/2}^{1/2} \left( \tilde{V}_\ell(\alpha)^s - \tilde{S}_\ell(\alpha)^s \right) U(-\alpha, H) e(-N\alpha) \, d\alpha
$$

$$
= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,
$$

(31)

say. This time we collect in $\mathcal{J}_3$ the difference between the contributions from the exponential sums over prime powers $\tilde{S}$ and the one over primes $\tilde{V}$; in fact $\mathcal{J}_3$ is equal to $I_3$ of section 3 and we’ll see that it is a negligible term in this case too. $\mathcal{J}_1$ is directly linked with the “main term” of the problem while $\mathcal{J}_2$ is connected with the distribution of zeros of the Riemann zeta-function. At the end of the proof we will see how to get rid of the $e^{-n/N}$ weight we have on the left-hand side of (31). Now we evaluate these terms.

4.1. Evaluation of $\mathcal{J}_1$. Recalling $c_{s, \ell} := \Gamma(1 + 1/\ell)^s / \Gamma(s/\ell)$, by Lemma 4 and a direct calculation we have

$$
\mathcal{J}_1 = c_{s, \ell} \sum_{n=N+1}^{N+H} n^{s/\ell - 1} e^{-n/N} + O_{s, \ell}(H/\sqrt{N}) = c_{s, \ell} \sum_{n=N+1}^{N+H} n^{s/\ell - 1} + O_{s, \ell}(H^2 N^{s/\ell - 2})
$$

$$
= \frac{c_{s, \ell}}{e} H N^{s/\ell - 1} + O_{s, \ell}(H^2 N^{s/\ell - 2}).
$$

(32)

4.2. Estimate of $\mathcal{J}_2$. Recalling $m_\ell(z) := \Gamma(1 + 1/\ell) z^{-1/\ell}$, so that $d_{s, \ell}/z^{s/\ell} = m_\ell(z)^s$ and $\tilde{E}_\ell(\alpha) := \tilde{S}_\ell(\alpha) - m_\ell(z)$, we define

$$
\mathcal{E} := \int_{-1/2}^{1/2} |\tilde{E}_\ell(\alpha)|^2 |U(-\alpha, H)| \, d\alpha \ll_\ell N^{1/\ell} L^3,
$$

(33)

$$
\mathcal{S} := \int_{-1/2}^{1/2} |\tilde{S}_\ell(\alpha)|^2 |U(-\alpha, H)| \, d\alpha \ll_\ell H N^{2/\ell - 1} L^3
$$

(34)

in which the estimates follow respectively using (3), Lemmas 4 and 6 with $\xi = \eta = 1/H$ for (33) and Lemmas 5 and 6 with $\tau = \omega = 1/H$ and $H \geq N^{1-1/\ell} L$ for (34).

Moreover, using (18), for every integral $u \geq 2$, we have, using also (33)-(34), that

$$
\mathcal{M}(u) := \int_{-1/2}^{1/2} |m_\ell(z)|^u |U(-\alpha, H)| \, d\alpha \ll_{u, \ell} N^{(u-2)/\ell} (\mathcal{S} + \mathcal{E}) \ll_{u, \ell} H N^{u/\ell - 1} L^3,
$$

(35)

since $H \geq N^{1-1/\ell} L$. 

Using (14) for $s$ odd we can write that
\begin{equation}
\mathcal{J}_2 \ll \sum_{j=0}^{s-1} \int_{-1/2}^{1/2} |\tilde{E}_\ell(\alpha)| |\tilde{S}_\ell(\alpha)^s| |m_\ell(z)^j| |U(-\alpha, H)| \, d\alpha = \sum_{j=0}^{s-1} B_j, \tag{36}
\end{equation}
say. If $0 \leq j \leq s-2$, using the Cauchy-Schwarz inequality, (2) and (33)-(34) we have
\begin{equation}
B_j \ll (\max |m_\ell(z)^j|)(\max |\tilde{S}_\ell(\alpha)^{s-2-j}|) \int_{-1/2}^{1/2} |\tilde{E}_\ell(\alpha)| |\tilde{S}_\ell(\alpha)| |U(-\alpha, H)| \, d\alpha \ll_{s, \ell} N^{(s-2)/\ell} \mathcal{E}^{1/2} S^{1/2} \ll_{s, \ell} H^{1/2} N^{s/\ell-1/(2\ell)-1/2} L^3, \tag{37}
\end{equation}
provided that $H \geq N^{1-1/\ell}$. If $j = s-1$ using the Cauchy-Schwarz inequality, (33) and (35) we have
\begin{equation}
B_{s-1} \ll \int_{-1/2}^{1/2} |\tilde{E}_\ell(\alpha)| |m_\ell(z)^j| |U(-\alpha, H)| \, d\alpha \ll_{s, \ell} \mathcal{E}^{1/2} \mathcal{M}_1 (2s-2)^{1/2}
\ll_{s, \ell} H^{1/2} N^{s/\ell-1/(2\ell)-1/2} L^3. \tag{38}
\end{equation}
Hence, using (36)-(38), we obtain, for $s$ odd, that
\begin{equation}
\mathcal{J}_2 \ll_{s, \ell} H^{1/2} N^{s/\ell-1/(2\ell)-1/2} L^3, \tag{39}
\end{equation}
provided that $N^{1-1/\ell} L \leq H \leq N$.

Assume $s$ is even. Using (14) we get that
\begin{equation}
\mathcal{J}_2 \ll \sum_{j=0}^{s/2-1} \int_{-1/2}^{1/2} |\tilde{E}_\ell(\alpha)| |\tilde{S}_\ell(\alpha)^{s-2-2j}| |m_\ell(z)^{2j+1}| |U(-\alpha, H)| \, d\alpha
\ll \sum_{j=0}^{s/2-1} \int_{-1/2}^{1/2} \sum_{j=0}^{s/2-1} |\tilde{E}_\ell(\alpha)^2| |\tilde{S}_\ell(\alpha)^{s-2-2j}| |m_\ell(z)^{2j}| |U(-\alpha, H)| \, d\alpha = \sum_{j=0}^{s/2-1} (C_j + D_j), \tag{40}
\end{equation}
say. Now we estimate the $D_j$’s. Using (2) and (33) we have
\begin{equation}
D_j \ll (\max |m_\ell(z)^{2j}|)(\max |\tilde{S}_\ell(\alpha)^{s-2-2j}|) \mathcal{E} \ll_{s, \ell} N^{(s-1)/\ell} L^3. \tag{41}
\end{equation}
Now we estimate the $C_j$’s. If $s \geq 4$ and $0 \leq j \leq s/2-2$, using the Cauchy-Schwarz inequality, (2) and (33)-(34) we obtain as in (37) that
\begin{equation}
C_j \ll (\max |m_\ell(z)^{2j+1}|)(\max |\tilde{S}_\ell(\alpha)^{s-3-2j}|) \int_{-1/2}^{1/2} |\tilde{E}_\ell(\alpha)| |\tilde{S}_\ell(\alpha)| |U(-\alpha, H)| \, d\alpha
\ll_{s, \ell} N^{(s-2)/\ell} \mathcal{E}^{1/2} S^{1/2} \ll_{s, \ell} H^{1/2} N^{s/\ell-1/(2\ell)-1/2} L^3, \tag{42}
\end{equation}
provided that $H \geq N^{1-1/\ell}$. If $j = s/2-1$, $s \geq 2$, using the Cauchy-Schwarz inequality, (33) and (35) we have as in (38) that
\begin{equation}
C_{s/2-1} \ll \int_{-1/2}^{1/2} |\tilde{E}_\ell(\alpha)| |m_\ell(z)^{s-1}| |U(-\alpha, H)| \, d\alpha \ll_{s, \ell} \mathcal{E}^{1/2} \mathcal{M}_1 (2s-2)^{1/2}
\ll_{s, \ell} H^{1/2} N^{s/\ell-1/(2\ell)-1/2} L^3. \tag{43}
\end{equation}
Summarizing, using (40)-(43), we obtain, for $s$ even, that
\begin{equation}
\mathcal{J}_2 \ll_{s, \ell} N^{(s-1)/\ell} L^3 + H^{1/2} N^{s/\ell-1/(2\ell)-1/2} L^3, \tag{44}
\end{equation}
provided that $N^{1-1/\ell} L \leq H \leq N$. In conclusion, by (39) and (44), we can write
\[ \mathcal{F}_2 \ll_{s, \ell} N^{(s-1)/\ell} L^3 + H^{1/2} N^{s/(\ell-1)/(2\ell)-1/2} L^3, \]
for every $s$, $2 \leq s \leq \ell - 1$, provided that $N^{1-1/\ell} L \leq H \leq N$.

4.3. **Estimate of $\mathcal{F}_3$.** It is clear that $\mathcal{F}_3 = I_3$ of section 3.2. Hence by (11) we obtain
\[ \mathcal{F}_3 \ll_{s, \ell} \begin{cases} H N^{s/(\ell-1)/(2\ell)-1} L^3 & \text{if } s \geq 3 \\ H^{1/2} N^{3/(2\ell)-1/2} L^2 & \text{if } s = 2, \end{cases} \]
provided that $H \gg N^{1-1/\ell} L$.

4.4. **Final words.** By (31)-(32) and (45)-(46), we have
\[ \sum_{n=1}^{N+H} e^{-n/N} r_{s, \ell}(n) = \frac{c_{s, \ell}}{e} H N^{s/\ell-1} + \mathcal{O}_{s, \ell}(H^2 N^{s/\ell-2} + H^{1/2} N^{s/(\ell-1)/(2\ell)-1/2} L^3), \]
which is an asymptotic formula $\infty(N^{1-1/\ell} L^6) \leq H \leq o(N)$. From $e^{-n/N} = e^{-1} + \mathcal{O}(H/N)$ for $n \in [N+1, N+H]$, $1 \leq H \leq N$, we get
\[ \sum_{n=N+1}^{N+H} r_{s, \ell}(n) = c_{s, \ell} H N^{s/\ell-1} + \mathcal{O}_{s, \ell}(H^2 N^{s/\ell-2} + H^{1/2} N^{s/(\ell-1)/(2\ell)-1/2} L^3) + \mathcal{O}_{s, \ell}
\[ \frac{H}{N} \sum_{n=N+1}^{N+H} r_{s, \ell}(n). \]
Using $e^{n/N} \leq e^2$ and (47), the last error term is $\ll_{s, \ell} H^2 N^{s/\ell-2} + H^{3/2} N^{s/(\ell-1)/(2\ell)-3/2} L^3$. Hence we get
\[ \sum_{n=N+1}^{N+H} r_{s, \ell}(n) = c_{s, \ell} H N^{s/\ell-1} + \mathcal{O}_{s, \ell}(H^2 N^{s/\ell-2} + H^{1/2} N^{s/(\ell-1)/(2\ell)-1/2} L^3) \]
uniformly for $\infty(N^{1-1/\ell} L^6) \leq H \leq o(N)$. **Theorem 2** follows by recalling $c_{s, \ell} = \Gamma(1 + 1/\ell)^{\ell}/\Gamma(s/\ell)$.

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