SUPPORTS OF SIMPLE MODULES IN CYCLOTONIC CHEREDNIK CATEGORIES $\mathcal{O}$

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Abstract. The goal of this paper is to compute the supports of simple modules in the categories $\mathcal{O}$ for the rational Cherednik algebras associated to groups $G(\ell,1,n)$. For this we compute some combinatorial maps on the set of simples: wall-crossing bijections and a certain $sl_\infty$-crystal associated to a Heisenberg algebra action on a Fock space.

1. Introduction

We fix positive integers $\ell,n$ and form the wreath-product group $W = G(\ell,1,n) := \mathfrak{S}_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$. This is a complex reflection group acting on $\mathfrak{h} := \mathbb{C}^n$. To the pair $(W,\mathfrak{h})$ we can assign the so called rational Cherednik algebra $H_c$ depending on a parameter $c$ that is a collection of complex numbers, one per each conjugacy class of complex reflections in the group $W$. These algebras were introduced by Etingof and Ginzburg in [EG, Section 4].

As a vector space, $H_c = S(\mathfrak{h}^*) \otimes \mathbb{C} W \otimes S(\mathfrak{h})$, where $S(\mathfrak{h}^*)$, $\mathbb{C} W$, $S(\mathfrak{h})$ are subalgebras in $H_c$. The adjoint actions of $W$ on $S(\mathfrak{h})$, $S(\mathfrak{h}^*)$ are the usual ones, and there is an interesting commutation relation between $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$ depending on the parameter $c$. We will recall a presentation of $H_c$ by generators and relations below, Section 2.1.

One has a distinguished category of $H_c$-modules, the category $\mathcal{O}_c$ introduced in [GGOR, Section 3] to be recalled in Section 2.3. This category consists of all modules that are finitely generated over the subalgebra $S(\mathfrak{h}^*)$ and where $\mathfrak{h}$ acts locally nilpotently. Its simple objects are parameterized by $\text{Irr}(W)$: to $\tau \in \text{Irr}(W)$ we assign the unique simple quotient $L_c(\tau)$ of the Verma module $\Delta_c(\tau) = H_c \otimes_{S(\mathfrak{h})} W \tau$, where $\mathfrak{h}$ acts on $\tau$ by 0.

To each module $M \in \mathcal{O}_c$ we can assign its support, the closed subvariety of $\mathfrak{h}$ defined by the annihilator of $M$ in $S(\mathfrak{h}^*) = \mathbb{C}[\mathfrak{h}]$. The main purpose of this paper is to compute $\text{Supp}(L_c(\tau))$ combinatorially starting from $\tau$ and $c$. In particular, this will yield a classification of the finite dimensional irreducible $H_c$-modules, as those are precisely the modules whose supports are equal to $\{0\}$.

1.1. Known results. First of all, all possible supports of simples are known, this is implicit in [BE, Section 3.8] and explicit in [SV, Section 3.10]. Namely, let $\kappa$ be the component of the parameter $c$ corresponding to the conjugacy class of complex reflections in $W$ intersecting $\mathfrak{S}_n$. The case $\kappa = 0$ is easy and, in what follows, we mostly consider $\kappa \neq 0$. Let $e$ denote the denominator of $\kappa$ presented as an irreducible fraction if $\kappa$ is rational, we take $e = +\infty$ if $\kappa$ is irrational. By [BE], the support of any simple equals $W \Gamma_{p,q}$, where $p,q$ are non-negative integers satisfying $p + eq \leq n$ (in particular, if $e > n$, then $q = 0$) and $\Gamma_{p,q}$ is the subspace of $\mathfrak{h}$ given by

$$\Gamma_{p,q} = \{(x_1,\ldots,x_p,y_1,\ldots,y_1,\ldots,y_q,\ldots,y_q,0,\ldots,0)\},$$

where we have $q$ groups of $e$ equal elements. The numbers $p,q$ are determined from $\Gamma_{p,q}$ uniquely whenever $e > 1$. When $e = 1$, we always take $p = 0$. We write $p(\lambda), q(\lambda)$ (or

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and \( p_c(\lambda), q_c(\lambda) \) if we want to indicate the dependence on the parameter \( c \) for the numbers \( p, q \) such that \( \text{Supp}(L_c(\lambda)) = \text{WT}_{p,q} \).

A combinatorial recipe to compute \( p(\lambda) \) was given in [\text{L3}]. Recall that the irreducible representations of \( G(\ell,1,n) \) are parameterized by the \( \ell \)-multipartitions \( \lambda \) of \( n \), denote this set by \( \mathcal{P}_\ell(n) \). We can encode the remaining \( \ell - 1 \) components of \( c \) as an \( \ell \)-tuple of complex numbers \( s_1, \ldots, s_\ell \) defined up to a common summand, see Section 4.1. There is a \( \hat{\mathfrak{sl}}_c^k \)-crystal structure on \( \mathcal{P}_\ell := \bigsqcup_n \mathcal{P}_\ell(n) \), where \( k \) and the crystal structure itself depend on \( s_1, \ldots, s_\ell \), this will be recalled below in 4.2.2 (when \( e = +\infty \), by \( \hat{\mathfrak{sl}}_{\infty} \) we mean \( \mathfrak{sl}_{\infty} \)). A combinatorial construction of this crystal was given by Uglov in [U, Section 2.2]. According to [\text{L3}, Section 5.5], \( p(\lambda) \) is the depth of \( \lambda \) in the crystal, i.e., the minimal number \( d \) such that any composition of \( d + 1 \) annihilation crystal operators kills \( \lambda \).

Let us explain now what is known about \( q(\lambda) \). First, one can reduce the computation of \( q(\lambda) \) to the case when one has \( k = 1 \), see Section 4.3 below.

The case \( \ell = 1 \) was done in [\text{W4}]. Here we always have \( p + eq = n \). We can divide \( \lambda \) by \( e \) with residue: \( \lambda = e\lambda' + \lambda'' \), where \( \lambda', \lambda'' \) are partitions such that \( |\lambda'| \) is maximal possible, the operations are done part-wise. Then we have \( q(\lambda) = |\lambda'| \), see [\text{W4}, Theorem 1.6].

The number of simples \( L(\lambda) \) with given support was computed in [\text{SY}] (under several restrictions on the parameters that can be removed, as will be explained in 4.2.3). Based on a construction from there, we will see that, when \( k = 1 \), there is a level 1 crystal structure for the algebra \( \mathfrak{sl}_c \) on \( \mathcal{P}_1 \) such that each creation operator adds \( e \) boxes to \( \lambda \) and \( q(\lambda) \) is the depth of \( \lambda \) in this crystal, Lemma 5.2. So, to determine \( q(\lambda) \) combinatorially, it is enough to determine this crystal structure explicitly. We will do this in the present paper. One easy thing to observe is that the \( \hat{\mathfrak{sl}}_c \) and \( \hat{\mathfrak{sl}}_{\infty} \)-crystal commute, see Section 5.1. So it is enough to compute the latter on the singular (=depth 0) elements for the former.

1.2. Main results of this paper. The computation of the level 1 \( \hat{\mathfrak{sl}}_{\infty} \)-crystal consists of two parts. We first compute it explicitly in an asymptotic chamber and then use explicit combinatorial wall-crossing bijections to transfer to the other chambers.

Let us explain what we mean by chambers. Pick two parameters \( c = (\kappa, s_1, \ldots, s_\ell) \) and \( c' = (\kappa', s'_1, \ldots, s'_\ell) \). We say that \( c \) and \( c' \) have integral difference if \( \kappa' - \kappa, \kappa's'_i - \kappa s_i \) are integers for all \( i \). The set of parameters that have integral difference with a given one forms a lattice \( \mathfrak{c}_\mathbb{Z} := \mathbb{Z}^\ell \) in the space of all parameters (this makes sense as stated when \( e > 1 \), and there is a way to extend the definition of the lattice to \( e = 1 \)). There are finitely many hyperplanes (depending on \( n \)) in \( \mathbb{Q} \otimes_\mathbb{Z} \mathfrak{c}_\mathbb{Z} \) that split \( \mathfrak{c}_\mathbb{Z} \) into the union of cones to be called chambers. The walls depend on \( n \) in such a way that, as \( n \) increases, we need to add more walls. The categories \( \mathcal{O}_c \) are the same for any \( c \) in a given chamber, see [\text{L5}, Section 4.2], (it is actually enough to assume that \( c \) lies in an interior of a chamber).

The most interesting case is when \( \kappa \) is a rational number with denominator \( e \) and \( \kappa es_1, \ldots, \kappa es_\ell \) are all integers. We are going to assume this until the end of the section. The general case can be reduced to this one.

There are distinguished chambers to be called asymptotic. These are \( 2\ell! \) chambers containing parameters, where \( s_1, \ldots, s_\ell \) satisfy \( |s_i - s_j| > n \) (the factor of 2 comes from the choice of a sign of \( \kappa \) and \( \ell! \) comes from ordering \( s_1, \ldots, s_\ell \)). Let us suppose that

\[
\kappa < 0, s_1 \gg s_2 \gg \ldots \gg s_\ell.
\]

Note that the condition that \( c \) lies in an asymptotic chamber depends on \( n \). When \( n \) becomes large enough, any given parameter stops lying in the asymptotic chamber.
Then we have the following result (note that we can impose weaker assumptions on \( c \), see Proposition 5.3).

**Proposition 1.1.** Suppose \( c \) is as in (1.1) and let \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \) be a singular multipartition for the \( \hat{\mathfrak{sl}}_c \)-crystal. Then the following holds.

1. \( \lambda^{(\ell)} \) is divisible by \( e \), i.e., there is a partition \( \lambda' \) such that \( \lambda^{(\ell)} = e\lambda' \). We have \( q(\lambda) = |\lambda'| \).

2. The annihilation operator \( \tilde{e}^\infty_i, i \in \mathbb{Z} \), for the \( \mathfrak{sl}_\infty \)-crystal takes \( \lambda \) to the multipartition \( \Lambda \) specified by \( \Lambda^{(j)} = \lambda^{(j)} \) for \( j < \ell \) and \( \Lambda^{(\ell)} = e\lambda' \), where \( \lambda' \) is obtained from \( \lambda' \) by removing an \( i \)-box (if there is no removable \( i \)-box in \( \lambda' \), then we set \( \tilde{e}^\infty_i \lambda = 0 \)).

Now let us explain what happens when we cross a wall. Let \( c, c' \) be two parameters with integral difference lying in two chambers separated by a wall.

**Proposition 1.2.** There is a bijection

\[
\text{wc}_{c' \leftarrow c} : \bigsqcup_{m \leq n} \mathcal{P}_\ell(m) \to \bigsqcup_{k \leq n} \mathcal{P}_\ell(m)
\]

that preserves \( k \) and intertwines the annihilation operators for the \( \hat{\mathfrak{sl}}_c \)- and \( \mathfrak{sl}_\infty \)-crystals. This bijection is given by a combinatorial recipe to be explained below in Section 5.4.

The bijection \( \text{wc}_{c' \leftarrow c} \) has already implicitly appeared in [L5]. There we have established an equivalence \( \text{wc}_{c' \leftarrow c} : D^b(\mathcal{O}_c) \sim \to D^b(\mathcal{O}_{c'}) \) and have shown that it is perverse (in the sense of Chuang and Rouquier). This gives rise to a bijection \( \text{Irr}(\mathcal{O}_c) \to \text{Irr}(\mathcal{O}_{c'}) \) and this is the bijection \( \text{wc}_{c' \leftarrow c} \) that we need.

We would like to emphasize that we only have a relatively explicit combinatorial recipe for a bijection for two neighboring chambers. Of course, for arbitrary chambers we can take the composition of such bijections. But the resulting bijection is going to be complicated. It is unclear whether there is an explicit combinatorial formula to compute \( q(\lambda) \) in an arbitrary chamber.

### 1.3. Content.

Sections 2 and 4 basically do not contain any new results. Sections 3 and 5 are new.

In Section 2 we recall several known results and constructions for general rational Cherednik algebras. In Section 2.1 we recall the definition of these algebras and basic structural results following [EG]. In Section 2.2 we recall basics about highest weight categories. In Section 2.3 we define categories \( \mathcal{O} \). In Section 2.4 we recall Harish-Chandra (shortly, HC) bimodules for rational Cherednik algebras. In Section 2.5 we recall isomorphisms of some completions of rational Cherednik algebras. Section 2.6 deals with induction and restriction functors for categories \( \mathcal{O} \), [BE], and for HC bimodules, [L1]. In the next part, Section 2.7 we elaborate on the chamber decomposition for the space of parameters. Finally, in Section 2.8 we recall wall-crossing functors between categories \( \mathcal{O} \) with different parameters introduced in [L5]. We also recall wall-crossing bijections between the sets of simples in those categories.

Section 3 establishes some further properties of the wall-crossing functors and the wall-crossing bijections. First, in Section 3.1 we show that a wall-crossing bijection \( \text{wc}_{c' \leftarrow c} \) depends not on \( c \) but on the wall being crossed. Second, we show that wall-crossing functors commute with restriction and induction functors, Section 3.2. These are crucial tools to compute these bijections in the cases we need.

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1 An explicit formula was recently obtained by Jacon and Lecouvey in [JL]
In Section 4, we recall a few additional facts about categories $O_c(W)$ for $W = G(\ell, 1, n)$. In Section 4.2 we recall categorical Kac-Moody ([Sh] and [GM]) and Heisenberg, [SV], actions on the cyclotomic categories $O$, as well as the crystal for the Kac-Moody action, [L3]. Finally, Section 4.3 recalls a decomposition of cyclotomic categories $O$ that is used to reduce the computation of supports to some special parameters $c$.

In Section 5 we prove results explained in Section 4.2. In Section 5.1, we introduce an $\mathfrak{sl}_\infty$-crystal on $P_\ell$ and establish some of its basic properties. We compute the corresponding crystal operators in many chambers including asymptotic ones in Section 5.2. Then we show that the wall-crossing bijections commute with the Kac-Moody and Heisenberg crystal operators, Section 5.3. Next, we explain how to compute the wall-crossing bijections through hyperplanes, Section 5.4. We summarize the computations of supports and of Heisenberg operators, Section 5.5 and give an example of computation in Section 5.7.

Section 6 is an appendix describing various developments related to the main body of the paper. In Section 6.1, we consider the computation of supports in the case when $\kappa = 0$. In Section 6.2, we explain how to reduce a computation of supports for the complex reflection groups $G(\ell, r, n)$ to that for $G(\ell, 1, n)$. Finally, in Section 6.3, we explain a conjectural crystal version of the level-rank duality for affine type A Kac-Moody algebras (involving three commuting crystals) and its categorical meaning. It is based on a variant of techniques used in [RSVV] and [L4] combined with an approach of Bezrukavnikov and Yun, [BY] to Koszul duality for Kac-Moody groups.

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2. Cherednik algebras and their categories $O$

2.1. Rational Cherednik algebras. Let $W$ be a complex reflection group and $\mathfrak{h}$ be its reflection representation. For a reflection hyperplane $H$, the pointwise stabilizer $W_H$ is cyclic, let $\ell_H$ be the order of this group. The set of the reflection hyperplanes will be denoted by $\mathfrak{S}$. Let $\alpha_H, \alpha_H^\vee$ denote the eigenvectors for $W_H$ in $\mathfrak{h}^*, \mathfrak{h}$ with non-unit eigencharacters, partially normalized by $\langle \alpha_H, \alpha_H^\vee \rangle = 2$. For a complex reflection $s$ we write $\alpha_s, \alpha_s^\vee$ for $\alpha_H, \alpha_H^\vee$ where $H = \mathfrak{h}^*$. Let $c : S \to \mathbb{C}$ be a function constant on the conjugacy classes. The space of such functions is denoted by $c$, it is a vector space of dimension $|S/W|$.

By definition, [EG, Section 1.4], [GGOR, Section 3.1], the rational Cherednik algebra $H_c(=H_c(W) = H_c(W, \mathfrak{h}))$ is the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*) \# W$ by the following relations:

$$[x, x'] = [y, y'] = 0, \ [y, x] = \langle y, x \rangle - \sum_{s \in S} c(s) \langle x, \alpha_s^\vee \rangle \langle y, \alpha_s \rangle s, \ x, x', y, y' \in \mathfrak{h}. \quad \square$$

2.1.1. Deformation. We would like to point out that $H_c$ is the specialization to $c$ of a $\mathbb{C}[c]$-algebra $H_c$ defined as follows. The space $c^*$ has basis $c_s$ naturally numbered by the conjugacy classes of reflections. Then $H_c$ is the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*) \# W \otimes \mathbb{C}[c]$ by the relations similar to the above but with $c(s) \in \mathbb{C}$ replaced with $c_s \in c^*$. For a commutative algebra $R$ with a $W$-invariant map $c : S \to R$ we can consider the algebra $H_{R,c} = R \otimes_{\mathbb{C}[c]} H_c$. If $R = \mathbb{C}[c^1]$ for an affine subspace $c^1 \subset c$, then we write $H_{c^1}$ instead of $H_{R,c}$.

2After the present paper appeared, the problem of showing that the three crystals commute was solved by Gerber in [Ga] using combinatorial methods.
2.1.2. PBW property and triangular decomposition. Let us recall some structural results about $H_c$. The algebra $H_c$ is filtered with $\deg \mathfrak{h}^* = 0, \deg W = 0, \deg \mathfrak{h} = 1$. The associated graded algebra is $S(\mathfrak{h} \oplus \mathfrak{h}^*)# W, [EG$, Section 1.2]. This yields the triangular decomposition $H_c = S(\mathfrak{h}^*) \otimes CW \otimes S(\mathfrak{h}), [GGOR$, Section 3.2]. The algebra $H_c$ is also filtered with $\deg \mathfrak{c}^* = 1$. We get $H_c = S(\mathfrak{h}^*) \otimes \mathbb{C}[\mathfrak{c}] W \otimes S(\mathfrak{h})$ as a $\mathbb{C}[\mathfrak{c}]$-module.

2.1.3. Euler element. There is an Euler element $h \in H_c$ satisfying $[h, x] = x, [h, y] = -y, [h, w] = 0$. It is constructed as follows. Pick a basis $y_1, \ldots, y_n \in \mathfrak{h}$ and let $x_1, \ldots, x_n \in \mathfrak{h}^*$ be the dual basis. For $s \in S$, let $\lambda_s$ denote the eigenvalue of $s$ in $\mathfrak{h}^*$ different from 1. Then

$$h := \sum_{i=1}^n x_i y_i + \frac{n}{2} - \sum_{s \in S} \frac{2c(s)}{1 - \lambda_s}.$$  

2.1.4. Spherical subalgebras. Consider the averaging idempotent $e := |W|^{-1} \sum_{w \in W} w \in CW \subset H_c$. The spherical subalgebra by definition is $eH_ce$. More generally, let $\chi$ be a one-dimensional character of $W$. Let $e_\chi$ be the corresponding idempotent in $CW$. Form the algebra $e_\chi H_ce_\chi$.

We can also consider spherical subalgebras over $\mathfrak{c}$, we get the algebras $e_\chi H_ce_\chi$. These algebras inherit the filtration from $H_c$, the associated graded coincides with $e_\chi \text{gr } H_ce_\chi$. Let $Z(c)$ denote the center of $\text{gr } H_c$. It was shown by Etingof and Ginzburg in $[EG$, Theorem 3.1] that the map $z \mapsto ze_\chi$ defines an isomorphism $Z(c) \rightarrow \text{gr } e_\chi H_ce_\chi$. In particular, the associated graded algebras of $e_\chi H_ce_\chi$ are all identified.

It turns out that $e_\chi H_ce_\chi \cong eH_ce$, where the isomorphism induces a shift by an element $\bar{\chi} \in \mathfrak{c}$ on $\mathfrak{c}$. The element $\bar{\chi}$ is constructed as follows. We can find elements $h_{H,j} \in \mathbb{C}$ with $j = 0, \ldots, \ell_H - 1$ and $h_{H,j} = h_{H',j}$ for $H' \in WH$ such that

$$c(s) = \sum_{j=1}^{\ell_H-1} \frac{1 - \lambda_s^j}{2}(h_{h^*,j} - h_{h^*,j-1}).$$

Clearly, for fixed $H$, the numbers $h_{H,0}, \ldots, h_{H,\ell_H-1}$ are defined up to a common summand. We can recover the elements $h_{H,i}$ by the formula

$$h_{H,i} = \frac{1}{\ell_H} \sum_{s \in WH \setminus \{1\}} \frac{2c(s)}{\lambda_s - 1} \lambda_s^{-i}.$$

Note that $\sum_{i=0}^{\ell_H-1} h_{H,i} = 0$. We will view $h_{H,i}$ as an element of $\mathfrak{c}^*$ whose value on $c : S \rightarrow \mathbb{C}$ is given by (2.3).

There is a homomorphism $\text{Hom}(W, \mathbb{C}) \rightarrow \prod_{H \in \mathfrak{h}/W} \text{Irr}(W_H)$ given by the restriction. It turns out that this map is an isomorphism, see $[K$, 3.3.1]. So to arbitrary $W$-invariant collection of elements $(a_H)$ with $0 \leq a_H \leq \ell_H - 1$ we can assign the character of $W$ that sends $s$ to $\lambda_s^{-a_H}$. To a character $\chi$ given in this form we assign the element $\bar{\chi} \in \mathfrak{c}$ by $h_{H,i}(\bar{\chi}) = 1 - \frac{a_H}{\ell_H}$ if $i \geq \ell - a_H$ and $-\frac{a_H}{\ell_H}$ if $i < \ell - a_H$.

**Lemma 2.1.** There is an isomorphism $\iota : eH_ce \cong e_\chi H_ce_\chi$ of filtered $\mathbb{C}$-algebras that is the identity on the associated graded algebras and maps $p \in \mathfrak{c}^*$ to $p + \langle \bar{\chi}, p \rangle$.

**Proof.** The isomorphism is constructed in $[BC$, Proposition 5.6] (for a specialized parameter, but our case is similar).
2.2. Highest weight categories. In this section we recall highest weight categories.

Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category equivalent to the category of modules over some finite dimensional associative $\mathbb{C}$-algebra. Let $\Lambda$ be an indexing set for the simples in $\mathcal{C}$, for $\tau \in \Lambda$, we write $L(\tau)$ for the simple object indexed by $\tau$ and $P(\tau)$ for its projective cover. By a highest weight category we mean a triple $(\mathcal{C}, \leq, \{\Delta(\tau)\}_{\tau \in \Lambda})$, where $\leq$ is a partial order on $\Lambda$ and $\Delta(\tau), \tau \in \Lambda$, is a collection of standard objects in $\mathcal{C}$ satisfying the following conditions:

(i) $\text{Hom}_\mathcal{C}(\Delta(\tau), \Delta(\tau')) \neq 0$ implies $\tau \leq \tau'$.
(ii) $\text{End}_\mathcal{C}(\Delta(\tau)) = \mathbb{C}$.
(iii) There is an epimorphism $P(\tau) \to \Delta(\tau)$ whose kernel admits a filtration with successive quotients of the form $\Delta(\tau')$ with $\tau' > \tau$.

2.2.1. Costandard and tilting objects. Recall that in any highest weight category $\mathcal{C}$ one has costandard objects $\nabla(\tau), \tau \in \Lambda$, with $\text{dim} \text{Ext}^i(\Delta(\tau), \nabla(\xi)) = \delta_{i,0}\delta_{\tau,\xi}$.

By a tilting object in $\mathcal{C}$ we mean an object that is both standardly filtered (=admits a filtration with standard quotients) and costandardly filtered. The indecomposable tiltings are in bijection with $\Lambda$. By a tilting generator we mean a tilting that contains every indecomposable tilting as a summand.

2.2.2. Highest weight subcategories. Let $\Lambda_0$ be a poset ideal in $\Lambda$ (i.e., a subset such that, for each $\lambda \in \Lambda_0, \lambda' \leq \lambda$, we have $\lambda' \in \Lambda_0$). Consider the Serre subcategory $\mathcal{C}(\Lambda_0) \subset \mathcal{C}$ spanned by the simples $L(\tau), \tau \in \Lambda_0$. This is a highest weight category with respect to the order restricted from $\Lambda$ and with standard objects $\Delta(\tau), \tau \in \Lambda_0$ (and costandard objects $\nabla(\tau), \tau \in \Lambda_0$). Moreover, a natural functor $D^b(\mathcal{C}(\Lambda_0)) \to D^b(\mathcal{C})$ is a full embedding so that $D^b(\mathcal{C}(\Lambda_0))$ gets identified with the full subcategory $D^b_{\mathcal{C}(\Lambda_0)}(\mathcal{C})$ of all objects with homology in $\mathcal{C}(\Lambda_0)$. As usual, $D^b(\mathcal{C}/\mathcal{C}(\Lambda_0))$ gets identified with $D^b(\mathcal{C})/D^b(\mathcal{C}(\Lambda_0))$.

2.2.3. Ringel duality. Now recall the Ringel duality. Let $\mathcal{C}$ be a highest weight category and let $T$ be a tilting generator. Set $\vee \mathcal{C} := \text{End}_\mathcal{C}(T)^{\text{opp}} \cdot \text{mod}$. Then $\vee \mathcal{C}$ is a highest weight category with standard objects $\vee \Delta(\tau) := \text{Hom}(T, \nabla(\tau))$. The sets $\text{Irr}(\vee \mathcal{C})$ and $\text{Irr}(\mathcal{C})$ are identified and the orders on them are opposite. The functor $\mathcal{R} := R\text{Hom}_\mathcal{C}(T, \bullet)$ is a derived equivalence $D^b(\mathcal{C}) \sim \sim D^b(\vee \mathcal{C})$ called the Ringel duality functor. Note that $\vee(\vee \mathcal{C})$ is naturally identified with $\mathcal{C}^{\text{opp}}$. We write $C'$ for $(\vee \mathcal{C})^{\text{opp}}$ so that $\vee(\vee C') = \mathcal{C}$. So we get a derived equivalence $\mathcal{R}^{-1} : D^b(\mathcal{C}) \sim \sim D^b(\vee C')$ that maps $\Delta(\tau)$ to $\vee(\Delta(\tau))$.

2.3. Categories $\mathcal{O}$. Following [GGOR, Section 3.2], we consider the full subcategory $\mathcal{O}_c(W)$ of $H_c$-mod consisting of all modules $M$ that are finitely generated over $S(\mathfrak{h}^*)$ and such that $\mathfrak{h}$ acts on $M$ locally nilpotently. For example, pick an irreducible representation $\tau$ of $W$. Then the Verma module $\Delta_c(\tau) := H_c \otimes_{S(\mathfrak{h})}\delta W \tau$ (here $\mathfrak{h}$ acts by $0$ on $\tau$) is in $\mathcal{O}_c(W)$.

2.3.1. Supports. To a module $M \in \mathcal{O}_c(W)$ we can assign its support $\text{Supp}(M)$ that, by definition, is the support of $M$ (as a coherent sheaf) in $\mathfrak{h}$. Clearly, $\text{Supp}(M)$ is a closed $W$-stable subvariety. For a parabolic subgroup $W_1 \subset W$, set $X(W) = W\delta W_1$. The support of $M$ is the union of some subvarieties $X(W)$. Moreover, if $M$ is simple, then $\text{Supp}(M) = X(W)$ for some $W$. See [BE, Section 3.8] for the proofs.

2.3.2. Highest weight structure. Let us describe a highest weight structure on $\mathcal{O}_c(W)$, [GGOR, Theorem 2.19]. For the standard objects we take the Verma modules. A partial order on $\Lambda = \text{Irr}(W)$ is introduced as follows. The element $\sum_{s \in S} \frac{2c(s)}{\lambda_s - 1}s \in \mathbb{C} W$ is central so acts by
a scalar, denoted by $c_\tau$ (and called the $c$-function), on $\tau$. We set $\tau < \xi$ if $c_\tau - c_\xi \in \mathbb{Q}_{>0}$ (we could take the coarser order by requiring the difference to lie in $\mathbb{Z}_{>0}$ but we do not need this). We write $\tau \prec c$ if we want to indicate the dependence on the parameter $c$.

The following is established in [GGOR Section 4.3].

**Lemma 2.2.** Let $L \in \operatorname{Irr}(O_c(W))$ and let $T$ be a tilting generator in $O_c(W)$. Then $\dim \mathfrak{h} - \dim \operatorname{Supp}(L)$ coincides with the minimal number $i$ such that $\operatorname{Ext}^i_{O_c(W)}(T, L) \neq 0$.

2.3.3. $K_0$ and characters. We identify $K_0(O_c(W))$ with $K_0(W\text{-mod})$ by sending the class $[\Delta_c(\tau)]$ to $[\tau]$. Further, to a module $M \in O_c(W)$ we can assign its character $\operatorname{ch}(M) := \sum_{a \in C}[M_a]_Wq^a$, where $M_a$ is the generalized eigenspace for the action of $h$ with eigenvalue $a$, and $[M_a]_W$ is the class of $M_a$ in $K_0(W\text{-mod})$.

We have the following easy lemma.

**Lemma 2.3.** Let $M, M'$ be two objects in $O_c(W)$ such that the coefficients of $q^a$ in $\operatorname{ch}M$ and $\operatorname{ch}M'$ coincide for any $a$ of the form $c_\tau, \tau \in \operatorname{Irr}(W)$. Then $[M] = [M']$.

Here and below we write $[M]$ for the class of $M$ in $K_0(O_c(W))$.

2.3.4. Twist by a character of $W$. Now let $\chi$ be a one-dimensional character of $W$. Given $c \in \mathfrak{c}$, define $c^\chi \in \mathfrak{c}$ by $c^\chi(s) = \chi(s)^{-1}c(s)$. We have an isomorphism $\psi_\chi : H_c \simeq H_c^\chi$ given on the generators by $x \mapsto x, y \mapsto y, w \mapsto \chi(w)w$. This gives rise to an equivalence $\psi_\chi\ast : O_c(W) \to O_c^\chi(W)$ that maps $\Delta_c(\tau)$ to $\Delta_c^\chi(\chi \otimes \tau)$.

2.3.5. Deformation. To finish this section, let us note that one can also define the category $O_{R,c}(W)$ for a commutative algebra $R$: it consists of all $H_{R,c}$-modules that are finitely generated over $R \otimes C S(\mathfrak{h}^*)$ and have a locally nilpotent action of $\mathfrak{h}$. If $R = C[\mathfrak{c}']$ for an affine subspace $\mathfrak{c}' \subset \mathfrak{c}$, then we write $O_{c'}(W)$ instead of $O_{R,c}(W)$.

2.4. Harish-Chandra bimodules. Let us introduce Harish-Chandra (shortly, HC) bimodules following [BEG]. We say that an $H_c-H_c$-bimodule $M$ is HC if it is finitely generated and the adjoint actions of $S(\mathfrak{h})^W$ and $S(\mathfrak{h}^*)^W$ (that are subalgebras in both $H_c, H_c^\chi$) are locally nilpotent. For example, the algebra $H_c$ is a HC $H_c$-bimodule.

An equivalent definition is as follows: a bimodule $\mathcal{B}$ is HC if and only if it has a bimodule filtration such that $\operatorname{gr} \mathcal{B}$ is a finitely generated bimodule over $S(\mathfrak{h} \oplus \mathfrak{h}^*)^W$ and the left and the right actions of $S(\mathfrak{h} \oplus \mathfrak{h}^*)^W$ on $\operatorname{gr} \mathcal{B}$ coincide (such a filtration is called good). In other words, $\operatorname{gr} \mathcal{B}$ is a finitely generated $S(\mathfrak{h} \oplus \mathfrak{h}^*)^W$-module. The equivalence of these two definitions was established in [LJ Section 5.4].

2.4.1. Shift bimodules. A further example is provided by translation bimodules introduced in the whole generality in [BC Section 5]. Let $\chi$ be a character of $W$. Recall $e_\chi \in CW$ and $\bar{\chi} \in \mathfrak{c}$ that have appeared in 2.3.4. We get an $H_{c+\bar{\chi}}-H_c$ bimodule

$$\mathcal{B}_{c,\bar{\chi}} := H_{c+\bar{\chi}}e \otimes eH_{c+\bar{\chi}}eH_{c+\bar{\chi}}e \otimes eH_{c+\bar{\chi}}eH_c.$$ 

Similarly, we get the $H_c-H_{c+\bar{\chi}}$ bimodule $\mathcal{B}_{c+\bar{\chi},-\bar{\chi}}$. These bimodules are HC by [LJ Section 3.1].
2.4.2. **Tensor products.** The following result is obtained in [L5] Section 3.4.

**Proposition 2.4.** Let $\mathcal{B}_1$ be a HC $H_{c'}$-$H_c$-bimodule, $\mathcal{B}_2$ be a HC $H_c$-$H_c$-module, and $M \in \mathcal{O}_c(W)$. Then the following is true.

1. $\text{Tor}^{H_{c'}}_{i}(\mathcal{B}_1, \mathcal{B}_2)$ is a HC $H_{c'}$-$H_c$-bimodule for any $i$.
2. $\text{Tor}^{H_c}_{i}(\mathcal{B}_2, M) \in \mathcal{O}_c$ for any $i$.
3. If $M$ is projective, then $\text{Tor}^{H_c}_{i}(\mathcal{B}_2, M) = 0$ for any $i > 0$.

So, for a HC $H_{c'}$-$H_c$-bimodule $\mathcal{B}$, we get a functor $\mathcal{B} \otimes^L_{H_c} \bullet : D^b(\mathcal{O}_c(W)) \to D^b(\mathcal{O}_c(W))$.

2.4.3. **Deformations and supports in $c$.** Let $\psi \in c$. By a HC $(H_c, \psi)$-bimodule we mean a finitely generated $H_c$-bimodule $\mathcal{B}$ with locally nilpotent adjoint actions of $S(\mathfrak{h})^W, S(\mathfrak{h}^*)^W$ and such that $\langle z, b \rangle = \langle z, \psi \rangle b$ for any $z \in c^*, b \in \mathcal{B}$. Let $HC(H_c, \psi)$ denote the category of HC $(H_c, \psi)$-bimodules.

Note that $\mathcal{B} \otimes_{\mathbb{C}_c} \mathbb{C}_c$ is a HC $H_{c^\psi}$-$H_c$-bimodule. Conversely, any HC $H_{c'}$-$H_c$-bimodule belongs to $HC(H_c, c' - c)$. Similarly to (2.4.1) we have bimodules $\mathcal{B}_{\chi, c} \in HC(H_c, \chi), \mathcal{B}_{\chi + \chi, -\chi} \in HC(H_c, -\chi)$.

For $\mathcal{B} \in HC(H_c, \psi)$ we define its right $c$-support $\text{Supp}_c^r(\mathcal{B})$ as the set of all $c \in c$ such that $\mathcal{B} \otimes_{\mathbb{C}_c} \mathbb{C}_c \neq \{0\}$. Completely analogously to [L6] Proposition 2.6], we see that $\text{Supp}_c^r(\mathcal{B})$ is closed.

2.4.4. **HC bimodules over spherical subalgebras.** We can define the notion of a HC $A$-$A'$-bimodule, where $A, A'$ are one of the algebras $H_c, e_\chi H_c e_\chi$ similarly to the above. Note that if $\mathcal{B}$ is a HC $(H_c, \psi)$-bimodule, then $\mathcal{B} e_\chi$ is a HC $H_{c^\psi}$-$e_\chi H_c e_\chi$-bimodule. As before, the tensor product of two HC bimodules is again a HC bimodule.

2.5. **Isomorphisms of completions.** Here we are going to study completions of $H_c$ and their connections to the algebras $H_c(W, h)$, where $W \subset W$ is a parabolic subgroup (the stabilizer of a point in $\mathfrak{h}$).

We pick $b \in \mathfrak{h}$. Set $H^b_c := \mathbb{C}[\mathfrak{h}/W]^h \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c$. This is a filtered algebra (with filtration inherited from $H_c$) containing $\mathbb{C}[\mathfrak{h}/W]^h, H_c$ as subalgebras. We can form the RCA $H_c(W, h) := \mathbb{C}[\mathfrak{c}] \otimes_{\mathbb{C}[\mathfrak{c}]} H_c(W, h)$ for $W$ acting on $\mathfrak{h}$ (here $\mathfrak{c}$ is the parameter space for $W$). Form the completion $H^\wedge_c(W, h) := \mathbb{C}[\mathfrak{h}/W]^h \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c(W, h)$.

It turns out that there is an isomorphism of $H_{c^\psi}$ and the matrix algebra of size $|W/W|$ over $H^\wedge_c(W, h)$, [BE]. We will need an invariant realization of the matrix algebra in terms of centralizer algebras from [BE] Section 3.2].

Let $A$ be an algebra equipped with a homomorphism from $\mathbb{C}[W]$. Consider the space $\text{Fun}_{\mathbb{C}[W]}(W, A)$ of all functions satisfying $f(uw) = uf(w)$ for all $u \in W, w \in W$. This is a free right $A$-module of rank $|W/W|$. Its endomorphism algebra, the centralizer algebra from [BE], will be denoted by $Z(W, W, A)$. Note that $e Z(W, W, A) e = e A e$, where we write $e$ for the trivial idempotent in $\mathbb{C}[W]$. More generally, $e_\chi Z(W, W, A) e_\chi = e_\chi A e_\chi$ for any one-dimensional character $\chi$ of $W$.

The algebras $\text{gr} H_{c^\psi} := \mathbb{C}[\mathfrak{h}/W]^h \otimes_{\mathbb{C}[\mathfrak{h}/W]} \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# W$ and $Z(W, W, \text{gr} H_{c^\psi})$ are naturally isomorphic. The isomorphism $\theta^h$ is given by the following formulas:

$$
\begin{align*}
[\theta^h(F)](w) &= Ff(w), \\
[\theta^h(u)](w) &= f(uw), \\
[\theta^h(v)](w) &= (vw)f(w), \\
F &\in \mathbb{C}[\mathfrak{h}/W]^h, u \in W, v \in \mathfrak{h} \oplus \mathfrak{h}^*.
\end{align*}
$$
In particular, $\theta^0$ restricts to an isomorphism $\mathbb{C}[\mathfrak{h}/W]/^\otimes \sim \mathbb{C}[\mathfrak{h}/W]/^\otimes$.

**Proposition 2.5.** The following is true.

1. There is a $\mathbb{C}[c]$-linear isomorphism $\theta_b : H_2^{\mathfrak{h}} \sim \mathbb{Z}(W/W, \mathbb{H}_2^{\mathfrak{h}}(W, \mathfrak{h}))$ of filtered algebras that gives $\theta^0$ after passing to the associated graded and specializing to $0 \in c$.

2. If $\theta_b$ is another isomorphism with these properties, then there is an invertible element $F \in \mathbb{C}[\mathfrak{h}/W]/^\otimes \otimes (\mathbb{C} \oplus c^*)$ such that $\theta'_b := \theta_b \circ \exp(\text{ad} F)$.

**Proof.** The isomorphism $\theta_b$ is constructed explicitly in [BE] Section 3.3. The existence of $F$ is proved in the same way as in [LL] Lemma 5.2.1. \hfill $\square$

The isomorphism $\theta_b$ restricts to $e_\chi H_2^{\mathfrak{h}} e_\chi \sim e_\chi H_2^{\mathfrak{h}}(W, \mathfrak{h}) e_\chi$ and a straightforward analog of (2) holds. We note that $e_\chi H_2^{\mathfrak{h}} e_\chi = \mathbb{C}[\mathfrak{h}/W]/^\otimes \otimes_{\mathbb{C}[\mathfrak{h}/W]} e_\chi H_2 e_\chi$.

### 2.6. Induction and restriction functors.

An isomorphism of completions in Section 2.5 allows one to define restriction functors $\text{Res}_W^:\mathcal{O}_c(W) \to \mathcal{O}_c(W)$. We write $\mathcal{O}_c(W)$ for the category of all parabolic subgroups of $W$ that gives $\mathcal{O}_c(W)$ and a straightforward analog of (2) holds. We note that $\text{Ind}_W^:\mathcal{O}_c(W) \to \mathcal{O}_c(W)$.

**Functors for categories $\mathcal{O}$:** construction. We start by explaining how to construct the functors $\text{Res}_W^\mathcal{O}_c$, $\text{Ind}_W^\mathcal{O}_c$.

Let $\mathcal{O}_c^{\mathfrak{h}}$ denote the category of all $H_2^{\mathfrak{h}}$-modules that are finitely generated over $\mathbb{C}[\mathfrak{h}/W]/^\otimes$. This category is equivalent to $\mathcal{O}_c(W)$. The equivalence is established as a composition of several intermediate equivalences. First, note that $\mathcal{O}_c(W)$ is equivalent to $\mathcal{O}_c^{\mathfrak{h}}(W, \mathfrak{h})$ via $N \mapsto \mathbb{C}[\mathfrak{h}/W]/^\otimes \otimes_{\mathbb{C}[\mathfrak{h}/W]} N, N \in \mathcal{O}_c(W)$. Next, $\mathcal{O}_c^{\mathfrak{h}}$ is equivalent to $\mathcal{O}_c^{\mathfrak{h}}(W, \mathfrak{h})$ via $M' \mapsto e(W)\theta_b(e(M'))$, where $e(W)$ is the idempotent in $Z(W/W, \mathbb{H}_2^{\mathfrak{h}}(W, \mathfrak{h}))$ given by $[e(W)f](w) = f(w)$ if $w \in W$ and $[e(W)f](w) = 0$, else. Let $\mathcal{F}$ denote the resulting equivalence $\mathcal{O}_c^{\mathfrak{h}} \sim \mathcal{O}_c(W)$.

We have an exact functor $\mathcal{O}_c \to \mathcal{O}_c^{\mathfrak{h}}$ given by $M \mapsto M^{\mathfrak{h}} := \mathbb{C}[\mathfrak{h}/W]/^\otimes \otimes_{\mathbb{C}[\mathfrak{h}/W]} M$. We set $\text{Res}_W^\mathcal{O}_c := \mathcal{F}(\cdot^{\mathfrak{h}})$, this functor is independent of $b$ up to an isomorphism, see [BE] Section 3.7. It was shown in [BE] Section 3.5 that it admits an exact right adjoint functor, the induction functor $\text{Ind}_W^\mathcal{O}_c : \mathcal{O}_c(W) \to \mathcal{O}_c(W)$.

Note that the functors $\text{Res}_W^\mathcal{O}_c$, $\text{Ind}_W^\mathcal{O}_c$ do not depend (up to an isomorphism) on the choice of $\theta_b$, this follows from (2) of Proposition 2.5.

2.6.2. **Functors for categories $\mathcal{O}$:** properties. The functor $\text{Ind}_W^\mathcal{O}_c$ is also left adjoint to $\text{Res}_W^\mathcal{O}_c$, see [SH] Section 2.4 for the proof of this under some restrictions on $W$ and $\ell$ in general. On the level of $K_0$’s, these functors behave like the restriction and the induction for groups, [BE] Section 3.6.

The functors $\text{Ind}$ and $\text{Res}$ are compatible with the supports as follows. Let $L \in \mathcal{O}_c(W)$ be a simple module and $\text{Supp } L = X(W_1)$. Then $\text{Supp } \text{Res}_W^L = \bigcup_{W_1'} X(W_1')$. Here $W_1'$ runs over all parabolic subgroups of $W$ that are conjugate to $W_1$ in $W$. We write $X(W_1')$ for $W_1^b/W_1'$, where $b := \mathfrak{h}/\mathfrak{h}/W$.

**Remark 2.6.** If one knows the classes of $L_\ell(\tau)$ in $K_0(\mathcal{O}_c(W)) = K_0(W \text{-mod})$, then, in principle, one can use the properties above to compute the support of $L_\ell(\tau)$. For example, if $W = G(\ell, 1, n)$, then the classes were computed in [RSV, LL, We]. However, the formulas...
are very involved and so computing the support in this way is much more complicated than what is proposed in the present paper.

**Lemma 2.7.** Let \( L \) be a simple module in the category \( \mathcal{O}(W) \) with \( \text{Supp} \, L = X(W) \). Then \( \text{Ind}^W_L(L) \neq 0 \) and the supports of any submodule and any quotient module of \( \text{Ind}^W_L(L) \) are equal to \( X(W) \).

**Proof.** By [SV, Proposition 2.7], \( \text{Ind}^W_L(L) \neq 0 \) and the support of \( \text{Ind}^W_L(L) \) equals \( X(W) \). Now let \( M \) be a quotient of \( \text{Ind}^W_L(L) \) whose support is a proper subvariety of \( X(W) \). By adjointness of Res and Ind, we get a nonzero homomorphism \( L \to \text{Res}^W_M(\text{M}) \) and so \( X(W) \subset \text{Supp}(\text{Res}^W_M(\text{M})) \). This contradicts the description of the support of restriction given above. \( \square \)

### 2.6.3. Functors for HC bimodules: construction

Let us recall restriction functors for HC bimodules. According to [LL, Section 3.6], there is an exact \( \mathbb{C}[c] \)-linear functor \( \bullet \cdot W_{\vdash} : \text{HC}(H_c(W), \psi) \to \text{HC}(H_c(W), \psi) \).

The functor \( \bullet \cdot W_{\vdash} \) is constructed as follows. By a HC \( \mathcal{H}_{\chi}^b - \mathcal{H}_{\chi}^a \)-bimodule we mean a bimodule \( \mathcal{B}' \) that is equipped with a filtration such that \( \text{gr} \mathcal{B}' \) is a finitely generated \( \mathbb{C}[h/W]h_{W} \otimes \mathbb{C}[h/W] \) \( S(h \oplus h^*)_{W} \)-module. More generally, a HC \( (\mathcal{H}_{\chi}^b, \psi) \)-bimodule is a bimodule \( \mathcal{B} \) (with the same compatibility of left and right \( c^* \)-actions as before) that can be equipped with a bimodule filtration such that \( \text{gr} \mathcal{B} \) is a finitely generated \( \mathbb{C}[h/W]h_{W} \otimes \mathbb{C}[h/W] Z_c \)-module, where we write \( Z_c \) for the center of \( \text{gr} \mathcal{H}_{\chi}^c \). Recall that it follows from [EG, Theorem 3.1], that \( Z_c \) is a graded free deformation of \( S(h \oplus h^*)_{W} \). So any HC \( \mathcal{H}_{\chi}^b - \mathcal{H}_{\chi}^a \)-bimodule is also an HC \( (\mathcal{H}_{\chi}^b, \psi) \)-bimodule.

We have an exact completion functor \( \mathcal{B} \mapsto \mathcal{B}^{\infty} := \mathbb{C}[h/W]h_{W} \otimes \mathbb{C}[h/W] \mathcal{B} \) from \( \text{HC}(H_c, \psi) \) to \( \text{HC}(H_c, \psi) \). The categories \( \text{HC}(H_c, \psi) \) and \( \text{HC}(H_c(W), \psi) \) are equivalent, this is analogous to an equivalence between the categories \( \mathcal{O} \) recalled in 2.6.1.

The categories \( \text{HC}(H_c(W), \psi) \) and \( \text{HC}(H_c(W, \psi)) \) are equivalent as well. An equivalence is constructed as follows: to \( \mathcal{B} \in \text{HC}(H_c(W, \psi)) \) we assign

\[
H_c(W, \psi) \otimes H_c(W, \psi) \mathcal{B}.
\]

A quasi-inverse equivalence sends \( \mathcal{B}' \) to the \( h \)-finite part of the centralizer of \( D(h\mathcal{W}) \subset H_c(W) \) in \( \mathcal{B}' \). Here \( h \) is the Euler element in \( H_c(W) \). The claim that these two functors are quasi-inverse equivalences can be easily deduced from [LL, Section 5.5].

Let \( \mathcal{F} \) denote the resulting equivalence \( \text{HC}(H_c, \psi) \cong \text{HC}(H_c(W, \psi)) \). We set \( \mathcal{B}^{\infty} : = \mathcal{F}(\mathcal{B}^{\infty}) \).

The similar construction works for HC \( H_c e_{\chi} H_c e_{\chi} \)-bimodules, etc. Note that \( (\mathcal{B} e_{\chi})^{\infty} \) is naturally isomorphic to \( (\mathcal{B} e_{\chi})^{\infty} \) (and this is the same for the multiplications by \( e_{\chi} \) from the left).

### 2.6.4. Functors for HC bimodules: properties

The functor \( \bullet \cdot W_{\vdash} \) has the following properties established in [LL, L5].

1) The functor \( \bullet \cdot W_{\vdash} \) intertwines \( \text{Tor}_i \), see [LB, Lemma 3.11].

2) We have a bifunctorial isomorphism \( \text{Res}^W_L(\mathcal{B} \otimes L_c M) \cong \mathcal{B} \cdot W_{\vdash} \otimes L_c \text{Res}^W_L M \), see [LL, Section 5.5].

Let us recall the notion of the associated variety of a HC bimodule. Equip \( \mathcal{B} \in \text{HC}(H_c, \psi) \) with a good filtration. Then for the associated variety, \( \text{V}(\mathcal{B}) \), we take the support of the \( S(h \oplus h^*)_{W} \)-module \( \text{gr} \mathcal{B}/c^* \text{gr} \mathcal{B} \).

Now let us explain how the restriction functors behaves on the associated varieties.
3) The associated variety of $B_{\mathfrak{t},W}$ can be recovered from that of $B$ as follows. Pick a point $v \in \mathfrak{h} \oplus \mathfrak{h}^*$ with stabilizer $W$. The formal neighborhood of $Wv$ in $(\mathfrak{h} \oplus \mathfrak{h}^*)/W$ is naturally identified with the formal neighborhood of 0 in $(\mathfrak{h} \oplus \mathfrak{h}^*)/W$. Then $V(B_{\mathfrak{t},W})$ is uniquely recovered from

$$V(B)^{\mathfrak{h}^*} = ((\mathfrak{h} \oplus \mathfrak{h}^*)/W \times V(B_{\mathfrak{t},W}))^{\alpha}. $$

This is easily seen from the construction.

2.7. Chambers and walls. We consider the $\mathbb{Z}$-lattice and the $\mathbb{Q}$-lattice $c_Z^* \subset c_Q^* \subset c^*$ spanned by the elements $h_{H,i} - h_{H,j}$ and the dual lattices $c_Z \subset c_Q \subset c$. The lattice $c_Z$ is spanned by the elements $\chi$. We will need a certain sublattice in $c_Z$. In [BC, Section 7.2], Berest and Chalykh established a group homomorphism $\mathfrak{t}w : c_Z \to \text{Bij}(\text{Irr}W)$ called the $KZ$ twist. Set $c_Z := \ker \mathfrak{t}w$.

As we have seen in [L5, Lemma 2.6], the function $c \mapsto c_\tau$ is in $c_Q^*$. Define an equivalence relation $\sim$ on $\text{Irr}(W)$ by setting $\tau \sim \tau'$ if $c_\tau = c_{\tau'}$ for every parameter $c$. Now if $\tau \not\sim \xi$, then we have the hyperplane $\Pi_{\tau,\xi}$ in $c$ given by $c_\tau = c_\xi$. All the hyperplanes $\Pi_{\tau,\xi}$ are rational.

Fix a coset $c + c_Z$ and consider $c'$ in this coset. We write $c \prec c'$ if $\tau \leq c, \xi$ implies $\tau \leq c'$, $\xi$. We write $c \sim c'$ if $c \prec c'$ and $c' \prec c$. The equivalence classes are relative interiors in the cones defined by the hyperplane arrangement $\{\Pi_{\tau,\xi}, \tau \not\sim \xi, c_\tau = c_\xi \in \mathbb{Q}\}$ on $c + c_Z$. We note that the faces of the cones do not necessarily intersect $c + c_Z$.

We are mostly interested in the open cones. In what follows, the open cones in this stratification will be called open chambers. For each open chamber we have its opposite chamber, where the order is opposite. Note that if $c$ is Weil generic in $c$ (where, recall, Weil generic means “outside of countable many algebraic subvarieties”), we have just one open chamber, while for a Weil generic $c$ on a rational hyperplane parallel to $\Pi_{\tau,\xi}$ we have exactly two open cones that are opposite to each other.

Here is an important result, [L5] Proposition 4.2, on a category equivalence.

**Proposition 2.8.** Let $c, c'$ be such that $c' - c \in c_Z$. Suppose $c \prec c'$. Then there is a category equivalence $\mathcal{O}_c \cong \mathcal{O}_{c'}$ mapping $\Delta_c(\tau)$ to $\Delta_{c'}(\tau)$ and preserving the supports of the simple modules.

So, basically, we do not need to distinguish between parameters lying in the same open chamber of $c + c_Z$.

2.8. Wall-crossing functors. In this section we recall wall-crossing functors introduced in [L5].

2.8.1. Construction. Let us construct a wall-crossing bimodule $B_{c}(\psi)$ that is a HC $H_{c-\psi}$-$H_c$-bimodule. We assume that $c$ lies in the interior of an open chamber $C$. Pick a face $F$ in that chamber and choose $\psi \in c_Z$ such that $c - \psi$ lies in the chamber $C^-$ that is opposite to $C$ with respect to $F$. This means that $F$ is a face for both $C, C^-$ and there is an interval with end points in $C, C^-$ and a mid-point in $F$. Note that, thanks to Proposition 2.8, we may replace $c$ with a Zariski generic $c'$ with $c' - c \in F \cap c_Z$ without changing the category $\mathcal{O}$.

Let $c'$ denote the affine subspace of $c$ containing $c$, whose associated vector space is spanned by $F$.

Consider a sequence of characters $\chi_1, \ldots, \chi_k$ with $-\psi = \sum_{i=1}^{k} e_i \bar{\chi}_i$, where $e_i \in \{\pm 1\}$. Then set $c'_j := c' + e_1 \bar{\chi}_1 + \ldots + e_j \bar{\chi}_j$ and

$$B_{c',\psi} := B_{c'_{k-1},e_k\bar{\chi}_k \otimes H_{c'_{k-1}}} \ldots \otimes H_{c_2} B_{c'_1,e_2\bar{\chi}_2} \otimes H_{c'_1} B_{c'_0,e_1\bar{\chi}_1}. $$
There is a labelling preserving highest weight equivalence inverse Ringel duality. More precisely, the following holds, see [L5, Theorem 4.1].

**Lemma 2.9.** The specialization $B_c(\psi)$ is well-defined for a Zariski generic $c \in \mathcal{C}'$ meaning that for any other choice of sub-bimodules $B_1 \subset B_2$ (satisfying the assumptions before the lemma) we have $B_2/c \cong B_2/B_1/c$ for a Zariski generic $c \in \mathcal{C}'$.

**Proof.** Set $\tilde{B}_1 = B_1 := B_1 + B_0, \tilde{B}_2 := B_2 + B_1, \tilde{B}_3 := B_2 + B_1$. The bimodules $\tilde{B}_1/c, \tilde{B}_3/c, B_{c,\psi}/\tilde{B}_2/c, B_{c,\psi}/\tilde{B}_2/c$ still have proper associated varieties. Moreover, $B_2/B_1 \to \tilde{B}_2/B_1$ and $B_2/B_1 \to \tilde{B}_2/B_1$. Since the Weil generic fibers of $B_2/B_1, B_2/B_1$ are simple, it follows that these epimorphisms are iso when specialized to $c$. Therefore they are iso when specialized to Zariski generic parameters (this is a consequence of the claim that the $c$-supports of HC bimodules are closed, see [L5, Proposition 5.3]. So we can replace $B_1, B_1, B_2, B_2$ with $\tilde{B}_1, \tilde{B}_1, \tilde{B}_2, \tilde{B}_2$).

Thus we can assume that $B_1 = B_1$. Now we can replace both $B_2, B_2$ with $B_1, B_1, B_2, B_2$ without changing $B_2/B_1/c$ and $B_2/B_1/c$ (for a Zariski generic $c$).

For a Zariski generic $c \in \mathcal{C}'$, define the wall-crossing functor $\mathcal{W}C_{c,\psi,c} := B_c(\psi) \otimes^L_{H_c} \bullet : D^b(O_c) \to D^b(O_{c,\psi})$.

**2.8.2. Properties.** Now let us explain some important properties of wall-crossing functors.

The following is [L5, Proposition 5.3].

**Proposition 2.10.** For a Zariski generic $c \in \mathcal{C}'$, the functor $\mathcal{W}C_{c,\psi,c}$ is a derived equivalence.

Further, it turns out that, for a Weil generic $\hat{c} \in \mathcal{C}'$, the functor $\mathcal{W}C_{\hat{c},\psi,c}$ realizes the inverse Ringel duality. More precisely, the following holds, see [L5, Theorem 4.1].

**Lemma 2.11.** There is a labelling preserving highest weight equivalence $O_{c,\psi} \cong O_{\hat{c}}'$ that intertwines $\mathcal{W}C_{c,\psi,c}$ with the inverse Ringel duality functor $D^b(O_c) \cong D^b(O_{\hat{c}}')$. In particular, $\mathcal{W}C_{c,\psi,c}(\Delta_c(\tau)) = \Delta_{\hat{c},\psi}(\tau)$.

The most important property of $\mathcal{W}C_{c,\psi,c}$ is that it is a perverse equivalence. To define a perverse equivalence, one needs filtrations by Serre subcategories, in our case those will come from certain chains of two sided ideals. We have sequences of two-sided ideals $\{0\} = I_n^c \subset I_n^c \subset \ldots \subset I_0^c \subset H_c$ and $\{0\} = I_n^{c,\psi} \subset I_n^{c,\psi} \subset \ldots \subset I_0^{c,\psi} \subset H_{c,\psi}$ (here $n = \dim h$) that have the following property: for a Weil generic $\hat{c} \in \mathcal{C}'$ the specialization $I_i^c$ is the minimal ideal in $H_c$ with the property $V(H_c/I_i^c) \leq 2i$ and the similar property holds for $I_i^{c,\psi}$. Similarly to Lemma 2.9 this shows that the specializations $I_i^c, I_i^{c,\psi}$ are well-defined for a Zariski generic parameter $c \in \mathcal{C}'$. Moreover, for such a $c$, we have $(I_i^c)^2 = I_i^c$ and $(I_i^{c,\psi})^2 = I_i^{c,\psi}$. So we can consider the Serre subcategories $C_i^c \subset O_c(W), C_i^{c,\psi} \subset O_{c,\psi}(W)$ consisting of all modules annihilated by $I_{n-i}^c, I_{n-i}^{c,\psi}$.

The following claim is [L5, Theorem 6.1].

**Proposition 2.12.** The equivalence $\mathcal{W}C_{c,\psi,c}$ is perverse with respect to the filtrations $C_i^c \subset O_c(W), C_i^{c,\psi} \subset O_{c,\psi}(W)$ meaning that the following holds.

1. $\mathcal{W}C_{c,\psi,c}$ restricts to an equivalence between $D^b_c(O_c(W))$ and $D^b_{c,\psi}(O_{c,\psi}(W))$.

2. Here we write $D^b_c(O_c(W))$ for the full subcategory of $D^b(O_c(W))$ of all objects with homology in $C_i^c$. 
Proof. H implies that $M$ the modules $w_c$ we get a self-bijection $\text{Corollary } 2.13$. From the compatibility of the restriction functors with supports, see 2.6.4. It follows that Wall-crossing bijections are independent of the choice of $3.1$. We will see that this bijection is actually independent of $c$ as long as $c$ is Zariski generic.

Corollary 2.13. The bijection $w_c \psi \rightarrow c$ preserves supports.

Proof. Let $B$ be a HC $H_{c \psi} - L$ bimodule and $L \in \text{Irr}(O_c)$. We claim that $\text{Supp}(\text{Tor}^H_c(B, L)) \subset \text{Supp}(L)$. This follows from the compatibility of Tor’s with the restriction functors and from the compatibility of the restriction functors with supports, see 2.6.4. It follows that $\text{Supp}(w_c \psi L) \subset \text{Supp } L$. The similar property holds for $w_c \psi \psi$ and $w_c \psi \psi$ are bijections, we see that $\text{Supp}(w_c \psi L) = \text{Supp } L$. □

3. Further properties of wall-crossing

3.1. Wall-crossing bijections are independent of the choice of $c$. The goal of this section is to prove the following claim.

Proposition 3.1. The bijection $w_c \psi \rightarrow c$ : $\text{Irr}(W) \rightarrow \text{Irr}(W)$ is independent of the choice of a Zariski generic $c$.

This will allow us to compute some of the wall-crossing bijections for the groups $G(\ell, 1, n)$ in Section 5.4 below. We prove Proposition 3.1 in 3.1.3 after some preliminary considerations.

3.1.1. Wall-crossing on $K_0$. We have the following.

Proposition 3.2. Let $c \in c'$ be Zariski generic. Then the following is true.

1. The complex $w_c \psi \rightarrow c \Delta c(\tau)$ has no higher homology and its class in $K_0$ equals $[\nabla c(\tau)]$.
2. In particular, $w_c \psi \rightarrow c$ gives the identity map $K_0(O_c(W)) \rightarrow K_0(O_c(\psi(W))$ (recall that both are identified with $K_0(\psi\text{-mod})$).

Proof. Let us prove (1). By Lemma 2.11 when $c$ is Weil generic, we have

$$w_c \psi \rightarrow c \Delta c(\tau) = \nabla c(\tau).$$

The objects $H_i(B_c(\psi) \otimes H_c \Delta c(\tau))$ are in $O_c(\psi(W))$ for any $i$. In particular, they are finitely generated over $\mathbb{C}[h][c']$ and hence are generically free over $\mathbb{C}[c']$. Together with (2.11), this implies that $H_i(w_c \psi \rightarrow c \Delta c(\tau)) = 0$ for $i > 0$ and Zariski generic $c$. It is easy to see that the modules $M = H_0(w_c \psi \rightarrow c \Delta c(\tau))$ and $M' = \nabla c(W)$ satisfy the condition of Lemma 2.3 (this condition is preserved by degeneration from $c$ to $c$). This implies the claim about the classes in $K_0$.

(2) follows from (1) and the equality $[\nabla c(\tau)] = [\Delta c(\tau)]$ proved in [GGOR, Proposition 3.3]. □
3.1.2. Pre-orders and classes of degenerations. We will define a pre-order on \( \text{Irr}(W) \) refined by \( \leq c \). Namely, recall that we have fixed a face \( F \) of the chamber of \( c \). Pick \( c^0 \) sufficiently deep inside \( F \cap \xi^c \) and replace \( c \) with \( c + c^0 \). This leads to the same pre-order \( \leq c \) and hence to the equivalent category \( O_c(W) \), see Proposition 2.8. Let \( \hat{c} \) denote a Weil generic element in \( c' \).

We define a pre-order \( \leq_F \) on \( \text{Irr}(W) \) by setting \( \tau \leq_F \tau' \) if \( c^0(\tau) - c^0(\tau') \in \mathbb{Q}_{\geq 0} \). The choice of \( c \) shows that \( \tau \leq c \tau' \) implies \( \tau \leq_F \tau' \). Moreover, if \( \tau \sim_F \tau' \), then \( \tau \leq c \tau' \) is equivalent to \( \tau \leq \hat{c} \tau' \).

We will need the following technical lemma.

**Lemma 3.3.** Let \( \tau \in \text{Irr}(W) \). Then we have the following equality in \( K_0(O_c(W)) = K_0(O_{\hat{c}}(W)) \):

\[
[L_\ell(\tau)] = [L_c(\tau)] + \sum_{\xi \in \tau} n_\xi[L_c(\xi)].
\]

**Proof.** Note that the equivalence classes for \( \sim_F \) are unions of blocks for the category \( O_{\hat{c}}(W) \). Indeed, if \( \tau \not\sim_F \xi \), then \( \hat{c}_\tau - \hat{c}_\xi \not\in \mathbb{Q} \).

We will produce a module \( M'_c \) in \( O_{\hat{c}}(W) \) with the following properties: the specialization \( M'_c \) coincides with \( L_\ell(\tau) \), while the specialization at \( c \) has the class in \( K_0 \) of the form predicted by the statement of the lemma. This \( M'_c \) is obtained as a quotient of \( \Delta_c(\tau) \) in several steps to be described in the next paragraph.

Let \( M_\ell \) be a module in \( O_\ell(W) \) and pick \( \Xi \subset \text{Irr}(W) \). Suppose that \( M_\ell \) is equipped with a \( \mathbb{Z}_{\geq 0} \)-grading \( M_\ell = \bigoplus M_\ell(i) \) that is compatible with the grading on \( H_c \). Note that \( M_\ell(i) \) is a finitely generated \( \mathbb{C}[\ell] \)-module for each \( i \). For each \( \xi \in \Xi \), pick \( d_\xi \in \mathbb{Z}_{\geq 0} \). The \( \mathbb{C}[\ell] \)-module \( \text{Hom}_W(\xi, M_\ell(d_\ell)^b) \) (the superscript indicates all elements annihilated by \( b \)) is finitely generated and hence is generically free. It follows that \( \text{dim} \text{Hom}_W(\xi, M_\ell(d_\ell)^b) \) is generically constant for \( \hat{c} \in \ell' \). Also we can consider the natural homomorphism

\[
\bigoplus_{\xi \in \Xi} \text{Hom}_W(\xi, M_\ell(d_\ell)^b) \otimes_{\mathbb{C}[\ell]} \Delta_\ell(\xi) \to M_\ell.
\]

This homomorphism preserves gradings if we put \( \text{Hom}_W(\xi, M_\ell(d_\ell)^b) \) in degree \( d_\ell \) and grade \( \Delta_\ell(\xi) \) by assigning degree 0 to \( \mathbb{C}[\ell] \otimes \xi \). Its cokernel, denote it by \( M_\ell' \), still lies in \( O_c(W) \) and inherits a grading from \( M_\ell \). So we can replace \( M_\ell \) with \( M_\ell' \).

Now let \( \tau \in \text{Irr}(W) \). We consider \( \Xi = \{ \xi \in \text{Irr}(W) | \xi \sim_F \tau, \xi \leq c \tau \} \). Note that \( d_\ell \) is independent of the choice of \( \hat{c} \in \ell' \). Set \( M_\ell := \Delta_\ell(\tau) \). For any quotient \( M_\ell \) of \( M_\ell' \), the natural inclusion \( \text{Hom}_W(\xi, M_\ell(d_\ell)^b) \to \text{Hom}_W(\xi, M_\ell') \) is an isomorphism. We apply the construction in the previous paragraph to \( \Xi \) and \( M_\ell' \). We get the quotient \( M_\ell' \) of \( M_c \). We then apply the construction again but now to \( M_\ell', \) getting its quotient \( M_\ell^2 \). The quotients will be proper as long as \( M_\ell^2 \) is not simple. We stop when \( M_\ell^k \) is simple, equivalently, \( (M_\ell^k)^b = \tau \). The object \( M_\ell^k \) satisfies \( \text{Hom}_H(\Delta_c(\xi), M_\ell^k) = 0 \) for any \( \xi \in \Xi \). Moreover \( [M_\ell^k] = [M_\ell^k] \) by the construction and \( M_\ell^k \to L_\ell(\tau) \). Let us show that the kernel of \( M_\ell^k \to L_\ell(\tau) \) is filtered with subquotients \( L_\ell(\eta) \), where \( \eta \leq_F \tau \). Consider the Serre subcategory \( O_{c,<\tau}(W) \) spanned by the simples \( L_\ell(\tau'), \tau' \leq c \tau \). The projective objects \( P_{c,<\tau}(\xi), \xi \in \Xi \), in the category \( O_{c,<\tau}(W) \) are filtered with \( \Delta_c(\xi'), \xi' \in \Xi \). We deduce that

\[
\text{Hom}_{O_{c,<\tau}(W)}(P_{c,<\tau}(\xi), M_\ell^k) = 0, \quad \forall \xi \in \Xi.
\]

It follows that, for any simple composition factor \( L_\ell(\tau') \) of \( M_\ell^k \), we have \( \tau' \leq_F \tau \). This completes the proof of the lemma. \( \square \)
3.1.3. Completion of the proof. Now we are ready to prove Proposition 3.3.

Proof of Proposition 3.3. Let us write \( \mathcal{O}_{c, \leq p} (W) \) for the Serre subcategory of \( \mathcal{O}_c (W) \) spanned by the simples \( L_c (\xi) \) with \( \xi \leq p \). We set \( \mathcal{O}_{c, \asymp p} (W) := \mathcal{O}_{c, \leq p} (W) / \mathcal{O}_{c, < p} (W) \).

Recall that \( D^b (\mathcal{O}_{c, \leq p} (W)) \) is a full subcategory of \( D^b (\mathcal{O}_c (W)) \). Since \( \mathfrak{g}_c^\psi \) maps \( \Delta_c (\tau') \) to an object with class \( [\nabla_{c, \psi} (\tau')] \), it restricts to an equivalence \( D^b (\mathcal{O}_{c, \psi, \leq p} (W)) \isom D^b (\mathcal{O}_{c, \psi, \asymp p} (W)) \). Hence it induces a (still perverse) equivalence

\[
D^b (\mathcal{O}_{c, \asymp p} (W)) \isom D^b (\mathcal{O}_{c, \psi, \asymp p} (W))
\]

that is the identity on the level of \( K_0 \) (here \( \mathcal{O}_{c, \asymp p} (W) \) stands for the quotient of \( \mathcal{O}_{c, \leq p} (W) \)) modulo \( \mathcal{O}_{c, < p} (W) \). By Lemma 3.3 the classes of \( [L_c (\tau')] \), \( [L_{c, \psi} (\tau')] \) in these \( K_0 \)'s are independent of \( c \) (of course, as long as \( c \) is Zariski generic).

On the other hand, an easy induction together with \( [\mathfrak{g}_c^\psi] = \text{id} \) shows that \( K_0 (C_i^\gamma) = K_0 (C_i^\gamma) \) (an equality of subgroups in \( K_0 (W \text{-mod}) \)). It follows that, for the filtration subquotients \( C_{c, \asymp p \tau, i}^\psi, C_{c, \asymp p \tau, i}^\gamma \) of the categories \( \mathcal{O}_{c, \asymp p} (W), \mathcal{O}_{c, \psi, \asymp p} (W) \), we also have \( K_0 (C_{c, \asymp p \tau, i}^\gamma) = K_0 (C_{c, \asymp p \tau, i}^\gamma) \). Since the classes \( [L_c (\tau')] \), \( [L_{c, \psi} (\tau')] \) are independent of \( c \), the equality \( K_0 (C_{c, \asymp p \tau, i}^\psi) = K_0 (C_{c, \asymp p \tau, i}^\gamma) \) implies that the labels of simples in \( C_{c, \asymp p \tau, i}^\psi, C_{c, \asymp p \tau, i}^\gamma \) do not depend on \( c \) as long as \( c \) is Zariski generic. Since \( [\mathfrak{g}_c^\psi] \) is independent of \( c \) as well (it is always the identity), we deduce that \( \mathfrak{g}_c^\psi \) is independent of \( c \) too.

3.2. Wall-crossing vs induction and restriction. Here we are going to prove that the restriction functors intertwine the wall-crossing functors.

3.2.1. Shift bimodules and restriction. Recall the shift bimodule \( \mathcal{B}_{c, \psi} \) from Section 2.1. Our goal is to understand the bimodule \( \mathcal{B}_{c, \psi} \). Consider the analog of \( \mathcal{B}_{c, \psi} \) for \( W \), the \( H_{c, \psi} (W) - H_{c'} (W) \)-bimodule \( \mathcal{B}_{c', \psi} \).

Lemma 3.4. We have an isomorphism \( \mathcal{B}_{c, \psi} \cong \mathcal{B}_{c', \psi} \) of \( H_{c, \psi} (W) - H_{c'} (W) \)-bimodules.

Proof. The proof is in several steps.

Step 1. Recall that we have isomorphisms \( \iota : e_\chi H_{c, \psi} e_\chi \isom e_{H_{c, \psi} e_\chi} \) and \( \iota : e_\xi H_{c} (W) e_\chi \isom e_{H_{c} (W) e_\chi} \), see Lemma 3.1. Our goal is to relate these isomorphisms. For this, we will first produce an isomorphism \( \iota' : e_\chi H_{c, \psi} e_\chi \isom e_{H_{c, \psi} e_\chi} \) from \( \iota \).

The isomorphism \( \iota \) induces an isomorphism of completions \( \iota^{ab} : e_{H_{c, \psi} e_\chi} \cong e_{H_{c, \psi} e_\chi} \). Using the isomorphism \( \theta_b \) from Proposition 2.5 we transfer \( \iota^{ab} \) to an isomorphism

\[
\iota^{ab} : e_{H_{c} (W) e_\chi} \cong e_{H_{c} (W) e_\chi}
\]

to be denoted again by \( \iota^{ab} \). This isomorphism preserves the filtrations and is the identity on the associated graded modulo \( c^* \).

We have inclusions \( D(\mathfrak{g}_c^{ab}) \hookrightarrow e_{H_{c} (W) e_\chi} \hookrightarrow e_{H_{c} (W) e_\chi} \). These embeddings are strictly compatible with filtrations and, after passing to the associated graded algebras, coincide with \( \theta^{ab} : C[\mathfrak{g}_c^{ab}] \cong (C[\mathfrak{g}_c^{ab}] W)^{ab} \). One can show that any two such embeddings differ by the conjugation with \( \exp (\text{ad} f) \), where \( f \in C[\mathfrak{g}] \otimes (C \otimes c^*) \), compare to Section 2.5. We can change \( \theta_b \) and assume that \( \iota^{ab} \) intertwines the embeddings of \( D(\mathfrak{g}_c^{ab}) \). So \( \iota^{ab} \) restricts to an isomorphism of the centralizers of \( D(\mathfrak{g}_c^{ab}) \), i.e., to \( \iota' : e_{H_{c, \psi} e_\chi} \cong e_{H_{c, \psi} e_\chi} \).

Let \( h_\chi \in e_{H_{c, \psi} e_\chi} \) denote the Euler elements. Since the \( \iota^{ab} \) preserves the filtrations and is the identity modulo \( c^* \), on the associated graded algebras, we see that \( \iota' (h_\chi) = h + F \), where \( F \in C[\mathfrak{g}_c^{ab}] \otimes (C \otimes c^*) \). From here and the fact that the eigenvalues of...
on the augmentation ideal in \( \mathbb{C}[\hbar/\mathbb{W}] \) are positive we deduce that there is \( F' \in \mathbb{C}[\hbar/\mathbb{W}]^{\times_0} \otimes (\mathbb{C} \oplus \mathfrak{c}^*) \) such that \( {\prime}(\hbar, \nabla) - \exp(\text{ad} F') \hbar \in \mathbb{C} \oplus \mathfrak{c}^* \). Twisting \( \theta_b \) by \( \exp(\text{ad} F') \) (where we view \( \mathbb{C}[\hbar/\mathbb{W}]^{\times_0} \) as a subalgebra of \( \mathbb{C}[\hbar/\mathbb{W}]^\times \) using \( (\theta^0)^{-1} \)) we achieve that \( \prime \) intertwines all tensor products involved in the construction \( \mathfrak{c} \).

**Example.** From the claims that every prime ideal in \( \mathcal{O}_c \) is too fine. We can choose a rougher decomposition, for example, as follows. Recall that we deduce that there is an element \( f \in \mathbb{C}[\hbar/\mathbb{W}] \otimes (\mathbb{C} \oplus \mathfrak{c}^*) \) such that \( \mathfrak{c} = \prime \circ \text{ad}(f) \). Further modifying \( \theta_b \), we may assume that \( \mathfrak{c} = \prime \).

**Step 3.** It follows from Step 2 and the construction of the restriction functor that \( (e_\chi H_c(e))_{\mathbb{W}} \cong e_\chi H_c(e) \), an isomorphism of \( e_\chi H_c(e) \) bi-bimodules. From Step 1 we deduce that \( (e_\chi H_c(e))_{\mathbb{W}} \cong e_\chi H_c(e) \), an isomorphism of \( e_\chi H_c(e) \) bi-bimodules.

**Step 4.** Since the functor \( \bullet_\mathbb{W} \) intertwines all tensor products involved in the construction of \( \mathcal{B}_c, \psi \), we deduce that \( (\mathcal{B}_c, \psi)_{\mathbb{W}} \cong \mathcal{B}_c, \psi \).

### 3.2.2. Wall-crossing functors and restriction

**Proposition 3.5.** Suppose that \( \hat{c}, \hat{c} - \psi \) lie in opposite chambers for both \( \mathbb{W} \) and \( \mathbb{W} \) provided \( \hat{c} \) is Weil generic in \( \hat{c} \). Then we have a natural isomorphism of functors

\[
\mathfrak{M}_{c-\psi\leftarrow c} \circ \text{Res}^\mathbb{W}_{\mathbb{W}} \cong \text{Res}^\mathbb{W}_{\mathbb{W}} \circ \mathfrak{M}_{c-\psi\leftarrow c}.
\]

Here we write \( \mathfrak{M}_{c-\psi\leftarrow c} \) for the wall-crossing functor \( D^b(\mathcal{O}_c(\mathbb{W})) \to D^b(\mathcal{O}_{c-\psi}(\mathbb{W})) \).

**Proof.** It follows from Lemma 3.4 that \( (\mathcal{B}_c, \psi)_{\mathbb{W}} = \mathcal{B}_c, \psi \). By [15] Proposition 4.10, we get \( \mathcal{B}_c(\psi) \cong (\mathcal{B}_c(\psi))_{\mathbb{W}} \). So we can set \( \mathcal{B}_c(\psi) = (\mathcal{B}_c(\psi))_{\mathbb{W}} = (\mathcal{B}_c(\psi))_{\mathbb{W}} \), where \( \mathcal{B}_c, \mathcal{B}_c \) are as in [2.8.1]. This shows \( \mathcal{B}_c(\psi) = (\mathcal{B}_c(\psi))_{\mathbb{W}} \). Using 2) from Section 2.4 we complete the proof.

The next corollary follows from Proposition 3.5 and the adjointness properties of \( \text{Res}^\mathbb{W}_{\mathbb{W}} \) and \( \text{Ind}^\mathbb{W}_{\mathbb{W}} \).

**Corollary 3.6.** Under the assumptions of Proposition 3.5, we have

\[
\mathfrak{M}_{c-\psi\leftarrow c} \circ \text{Ind}^\mathbb{W}_{\mathbb{W}} \cong \text{Ind}^\mathbb{W}_{\mathbb{W}} \circ \mathfrak{M}_{c-\psi\leftarrow c}.
\]

**3.2.3. Assumptions on \( \hat{c} - \psi, \hat{c} \).** We would like to make some comments on the assumption in Proposition 3.5 that \( \hat{c} - \psi, \hat{c} \) are opposite for \( \mathbb{W} \). We basically need to impose this assumption because the decomposition into chambers that we have chosen (according to the \( \mathfrak{c} \)-function) is too fine. We can choose a rougher decomposition, for example, as follows. Recall that we have fixed a coset \( c + \mathfrak{c} \). Let \( \Pi_{\tau, \xi} \) be a hyperplane as in Section 2.7. We say that \( \Pi_{\tau, \xi} \) is essential (for \( c + \mathfrak{c} \)) if, for a Weil generic \( \hat{c} \in c + \Pi_{\tau, \xi} \), the category \( \mathcal{O}_c \) is not semisimple.

**Lemma 3.7.** Let \( c, c - \psi \) lie in chambers that share a wall spanning a non-essential hyperplane. Then \( \mathfrak{M}_{c-\psi\leftarrow c} \) is a highest weight equivalence.

**Proof.** If the category \( \mathcal{O}_c(\mathbb{W}) \) is semisimple, then the algebra \( H_c \) is simple, this follows, for example, from the claims that every prime ideal in \( H_c \) is primitive, while every primitive ideal is the annihilator of a simple module from \( \mathcal{O}_c(\mathbb{W}) \), see [G]. It follows that \( I_{\mathbb{W}}^c = H_c \) and \( I_{\mathbb{W}}^{c-\psi} = H_c \). So \( I_{\mathbb{W}}^c \cong H_c \) and \( I_{\mathbb{W}}^{c-\psi} = H_c \). Therefore \( \mathfrak{M}_{c-\psi\leftarrow c} \) is an equivalence of abelian categories.
It remains to check that \( \mathfrak{M}\!\!c_{c-\psi-c} \Delta_c(\tau) = \Delta_{c-\psi}(\tau) \) for all \( \tau \). The functor \( \mathfrak{M}\!\!c_{c-\psi-c} \) is the identity on \( K_0 \) by Proposition 3.2. So it maps the projective Verma in \( \mathcal{O}_c(W) \) to the projective Verma \( \mathcal{O}_{c-\psi}(W) \) and those have the same labels in the two categories. Then we restrict \( \mathfrak{M}\!\!c_{c-\psi-c} \) to the Serre spans of the remaining Verma and repeat the argument. \( \square \)

So we can partition \( c+x \) into chambers using essential hyperplanes only. It is still unclear whether the condition that \( \hat{c}, \hat{c}-\psi \) lie in opposite chambers for \( W \) implies that for \( W' \), in general. However, this is true when we consider chambers separated by a single essential hyperplane.

**Proposition 3.8.** Let \( c \in \mathfrak{c} \), \( \Pi \) be an essential wall for \( c+x \) and let \( \psi \in \mathfrak{c}_x \) be such that \( c, c-\psi \) are separated by \( \Pi \). Then, after replacing \( c \) with a Zariski generic element of \( c+(\Pi \cap \mathfrak{c}_x) \), we have that either \( \Pi \) is an essential wall for \( W' \) or \( c, c-\psi \) are not separated by any essential wall for \( W \). Moreover, we have a natural isomorphism of functors

\[
\mathfrak{M}\!\!c_{c-\psi-c} \circ \text{Res}^W_{W'} \cong \text{Res}^W_{W'} \circ \mathfrak{M}\!\!c_{c-\psi-c}.
\]

**Proof.** If \( \Pi \) is an essential wall for \( W' \), then our claim follows from Proposition 3.5. Assume that \( \Pi \) is not an essential wall for \( W \). From the proof of Lemma 3.7 it follows that \( \mathcal{B}_c(\psi) = \mathcal{B}_{c-\psi} \). Also it follows that the bimodules \( \mathcal{B}_{1,W}^{1,W} (\mathcal{B}_{c+\Pi,\psi}/\mathcal{B}_{2}^{2,W} )_{1,W} \) are torsion over \( \mathbb{C}[\Pi] \). Therefore \( \mathcal{B}_c(\psi)_{1,W} = \mathcal{B}_{c-\psi} \). The required isomorphism of functors follows from 2) of 2.6.4 \( \square \)

In any case, suppose that \( \hat{c}, \hat{c}-\psi \) lie in opposite chambers for \( W \). Then one can show that \( \mathcal{O}_{c-\psi}(W) \cong \mathcal{O}_c(W)^\vee \) and the functor

\[
\mathcal{B}_{\hat{c}-\psi-\hat{c}} \otimes^{\mathbb{L}}_{\mathcal{H}_c(W)} \mathbb{C}
\]

becomes the inverse Ringel duality under this identification. In particular, this functor becomes \( \mathfrak{M}\!\!c_{c-\psi-c} \) up to pre-composing with a highest weight equivalence, where \( \psi \) is such that \( \hat{c}, \hat{c}-\psi \) lie in opposite chambers for \( W \). We do not prove this claim as we do not need it.

4. Cyclotomic Cherednik categories \( \mathcal{O} \)

4.1. Cyclotomic Cherednik algebras and their categories \( \mathcal{O} \). We now concentrate on the case when \( W = G(\ell,1,n) \). Here we have either one (if \( \ell = 1 \) or \( n = 1 \)) or two conjugacy classes of reflection hyperplanes. Let \( H^1 \) denote the hyperplane \( x_1 = 0 \) and \( H^2 \) denote the hyperplane \( x_1 = x_2 \). We set \( \kappa = -c(s) \), where \( s \) is a transposition in \( \mathfrak{S}_n \subset W \). If \( \kappa = 0 \), then \( \mathcal{H}_c = \mathcal{H}_c(1)^{\otimes n} \# \mathfrak{S}_n \), where we write \( \mathcal{H}_c(1) \) for the rational Cherednik algebra corresponding to \( (\mathbb{Z}/\ell\mathbb{Z}, \mathbb{C}) \). This is a very easy case and we are not going to consider it here (we explain how to compute the supports in Section 6.1). So we assume that \( \kappa \neq 0 \). Define complex numbers \( s_i, i = 1, \ldots, \ell \), by \( h_{H^1,i} = \kappa s_i - \frac{1}{\ell} \) (below we will write \( h_i \) for \( h_{H^1,i} \)). We view \( (s_1, \ldots, s_\ell) \) as an \( \ell \)-tuple of complex numbers defined up to a common summand.

Below we will write \( \mathcal{O}_c(n) \) for the category \( \mathcal{O}_c(G(\ell,1,n)) \). We will write \( \mathcal{O}_c \) for the direct sum \( \bigoplus_{n \geq 0} \mathcal{O}_c(n) \).

4.1.1. \( \text{Irr}(W) \) vs multipartitions. Let us proceed to the combinatorial description of the set \( \text{Irr}(W) \). We can identify this set with the set \( P_\ell(n) \) of \( \ell \)-multipartitions of \( n \) as follows. To a multipartition \( \lambda \), we assign the irreducible \( \prod_{i=1}^\ell G(|\lambda^{(i)}|,1,\ell) \)-module \( \otimes_{i=1}^\ell S_{\lambda^{(i)}} \), where an element \( \eta \) in any copy of \( \mathbb{Z}/\ell\mathbb{Z} \) inside \( \Gamma(\{|\lambda^{(i)}|,1,\ell|) \) acts on the irreducible \( \mathfrak{S}_\lambda^{(i)} \)-module \( S_{\lambda^{(i)}} \) (corresponding to the diagram \( \lambda^{(i)} \)) by \( \exp(2\pi \sqrt{-1} i \eta/\ell) \). The irreducible \( G(\ell,1,n) \)-module corresponding to \( \lambda \) is obtained from \( \otimes_{i=1}^\ell S_{\lambda^{(i)}} \) by induction.
4.1.2. Blocks and order on boxes. Now we are going to describe the blocks of $O_c$. To a box $b$ with column number $x$, row number $y$ and diagram number $i$ we assign its shifted content $\text{cont}(b) := x - y + s_i$. The following is a reformulation of [SV] Lemma 5.16.

**Lemma 4.1.** Two multipartitions $\lambda$ and $\lambda'$ lie in the same block if the multisets \{cont($b$) mod $\kappa^{-1}\mathbb{Z}$, $b \in \lambda$\} and \{cont($b$) mod $\kappa^{-1}\mathbb{Z}$, $b \in \lambda'$\} are the same.

Now let $b = (x, y, i), b' = (x', y', i')$ be two boxes such that their contents modulo $\kappa^{-1}\mathbb{Z}$ are the same (we say that such boxes are equivalent). When $b \sim b'$, we say that $b \leq b'$ if $\kappa \ell \text{cont}(b) - i \geq \kappa \ell \text{cont}(b') - i'$.

4.1.3. Lattices $c_{\mathbb{Z}}$ and $c_{\mathbb{Z}}$. Note that for $c = (\kappa, s_1, \ldots, s_\ell), c' = (\kappa', s'_1, \ldots, s'_\ell)$, the inclusion $c - c' \in c_{\mathbb{Z}}$ means that $\kappa - \kappa' \in \mathbb{Z}$ and $\kappa(s_i - s_j) - \kappa'(s'_i - s'_j) \in \mathbb{Z}$. The sublattice $c_{\mathbb{Z}}$ coincides with $c_{\mathbb{Z}}$ in this case because, for the groups $G(\ell, 1, n)$, the KZ twist is trivial, see [GL 6.4.7] for explanation and references.

4.1.4. Switching $\kappa$ to $-\kappa$. Let $\chi$ be the character of $G(\ell, 1, n)$ that is equal to the sign on $\mathfrak{S}_n$ and is the identity on $\mathbb{Z}/\ell\mathbb{Z}$. Then, if $c \in c, c = (\kappa, s_1, \ldots, s_\ell)$, we have $c^\chi = (-\kappa, -s_1, \ldots, -s_\ell)$. The corresponding equivalence $O_c \overset{\sim}{\to} O_{\chi}$ maps $\Delta_c(\lambda')$ to $\Delta_{\chi}(\lambda')$, where $\lambda'$ denotes the componentwise transpose of the $\ell$-multipartition $\lambda$.

4.2. Categorical actions. In this section we consider categorical actions of Lie algebras on $O_c$ and some related structures.

4.2.1. Categorical Kac-Moody action. There is a categorical type A Kac-Moody action (in the sense of [R2 5.3.7,5.3.8]) on $O_c$ defined in [SH Section 5]. Such an action consists of biadjoint functors $E, F$ (with fixed one sided adjunction morphisms) and functor endomorphisms $X \in \text{End}(E)$ and $T \in \text{End}(E^2)$. We are going to recall the functors $E$ and $F$ and also how they split into the eigen-functors for $X$, but we are not going to construct the other parts of the data.

We set $F := \bigoplus_{n \geq 0} \text{Ind}^{n}_{n+1}$ and $E := \bigoplus_{n \geq 0} \text{Res}^{n}_{n-1}$, where we write $\text{Ind}^{n}_{n+1}$ for the induction functor from $G(\ell, 1, n)$ to $G(\ell, 1, n+1)$ and $\text{Res}^{n}_{n-1}$ has the similar meaning.

Let $z \in C/\kappa^{-1}\mathbb{Z}$. We say that a box $b = (x, y, i)$ is a $z$-box if $x - y + s_i$ is congruent to $z$ modulo $\kappa^{-1}\mathbb{Z}$. For a module $M$ in a block corresponding to a multiset $A$, we define $F_zM$ as the projection of $FM$ to the block corresponding to $A \cup \{z\}$. We define $E_zM$ as the projection of $EM$ to the block corresponding to $A \setminus \{z\}$. Then $(F_z, E_z)$ define a highest weight categorical $\mathfrak{sl}_2$-action on $O_c$, see [L3 Section 4.2].

The functors $E_z, F_z$ give rise to a categorical action of a Kac-Moody algebra $\mathfrak{g}_c$ that is determined as follows. For $i, j \in \{1, \ldots, \ell\}$, we write $i \sim_c j$ if the $i$th and the $j$th diagrams can have equivalent boxes. Then the algebra acting is the product of several copies of $\mathfrak{sl}_e$ (where $e$ is the denominator of $\kappa$), one copy per each equivalence class for $\sim_c$. The complexified $K_0$ is the product $\bigotimes_\alpha \mathcal{F}_\alpha$, where $\alpha$ runs over the equivalence classes for $\sim_c$ and we write $\mathcal{F}_\alpha$ for the level $|\alpha|$ Fock space whose basis is indexed by multipartitions of the form $(\lambda^{(i)})$, where $\lambda^{(i)} = \emptyset$ if $i \notin \alpha$.

4.2.2. Kac-Moody crystal. For a category $C$ equipped with a categorical action of a Kac-Moody algebra $\mathfrak{g}$, the set $\text{Irr}(C)$ comes equipped with the structure of a $\mathfrak{g}$-crystal. Let us recall the construction. Pick a simple $L$. It was checked in [CR Proposition 5.20] (for any categorical $\mathfrak{sl}_2$-action with functors $E_z, F_z$) that the object $E_zL$ has isomorphic head and socle that are simple provided $E_zL \neq 0$. The same is true for $F_zL$. The operator $\partial_z$ sends $L$
to the socle of $E_z L$ provided it is nonzero and to zero else. The operator $\tilde{f}_z$ is defined in a similar fashion.

Now let us recall how to compute the crystal for the $g_c$-action on $O_c$, see \cite{L3} Theorem 5.1. To compute $\tilde{f}_z L_c(\lambda)$ and $\tilde{e}_z L_c(\lambda)$ consider all addable and removable $z$-boxes in $\lambda$. We place them in the decreasing order and write $+$ for an addable box and $-$ for a removable one (we call this collection the $z$-signature of $\lambda$). Then we consequently remove all instances of $++$ getting what we call the reduced signature. The operator $\tilde{f}_z$ adds the box that corresponds to the rightmost remaining $+$, the operator $\tilde{e}_z$ removes the box corresponding to the leftmost remaining $-$. If the reduced signature consists of all $+$’s (resp, all $-$’s), then $\tilde{e}_z \lambda = 0$ (resp., $\tilde{f}_z \lambda = 0$).

4.2.3. Heisenberg action. Now suppose that $\kappa < 0$ is a rational number, while $e \kappa s_1, \ldots, e \kappa s_\ell \in \mathbb{Z}$. Then we get functors $E_i, F_i$, one per residue mod $e$. The based space $(K^c_0(O_c), [\Delta_c(\tau)], \tau \in \mathcal{P}_\ell)$ becomes the level $\ell$ Fock space with multicharge $(s_1, \ldots, s_\ell)$ (note that the Fock space makes sense as an $\hat{sl}_c$-module as long as $s_1, \ldots, s_\ell$ are rational numbers whose denominators are coprime to $e$).

We can realize $W := G(\ell, 1, n - m) \times \mathfrak{S}_m$ as a parabolic subgroup of $W := G(\ell, 1, n)$. We will need some functors obtained from the induction functors from $G(\ell, 1, n - em) \times \mathfrak{S}_m$ to $G(\ell, 1, n)$. Let $\mu$ be a partition of $m$. Consider the functor $A_\mu := \text{Ind}_{W}^{G} \bullet \otimes L^\mu(e \mu)$, where $L^\mu(e \mu)$ stands for the simple in $O_\kappa(\mathfrak{S}_{em})$ indexed by the partition $e \mu := (e \mu_1, e \mu_2, \ldots)$ of $em$. As Shan and Vasserot checked in \cite{SV} Section 5.3, the functors $A_\mu$ commute with $E_i, F_i$ for all $i$ and $\mu$. On the level of $K_0$, the functors $A_\mu$ and their derived right adjoint functors $RA_\mu^*$ give rise to a Heisenberg action.

We note that in \cite{SV} there was an assumption that $e > 2$. It was needed to make sure that the category $O_\kappa(\mathfrak{S}_m)$ is equivalent to the category of modules over the $q$-Schur algebra $S_q(n)$, where $q = \exp(\pi \sqrt{-1} \kappa)$. This is trivial when $e = 1$ (both categories are isomorphic to $\mathfrak{S}_n$-mod) and was established in \cite{L4} Appendix when $e = 2$.

If $\kappa > 0$, we still have a categorical Heisenberg action: we need to set $A_\mu := \text{Ind}_{W}^{G} (\bullet \otimes L^\mu((e \mu)^*))$.

4.3. Decomposition. When $\kappa$ is irrational, the computation of supports was done in \cite{L3}. So we assume that the denominator $e$ of $\kappa$ is finite. Below we will write $O_{\kappa,s}$ for $O_c$ (where $s = (s_1, \ldots, s_\ell)$).

Suppose that there is more than one equivalence class (with respect to $\sim_c$) of the indexes $1, \ldots, \ell$. For an equivalence class $\alpha$, set $s^\alpha := (s_i)_{i \in \alpha}$. Let us write $O_{\kappa,s^\alpha}$ for the category $O$ for $G(|\alpha|, 1, n)$, this category comes equipped with a categorical action of the factor of $g_c$ corresponding to $\alpha$. We take the set of all multipartitions that have zero entries outside of $\alpha$ for the labelling set of $O_{\kappa,s^\alpha}$. So the simples in $\mathfrak{Z}_\alpha O_{\kappa,s^\alpha}$ are labelled by $\mathcal{P}_\ell$. Also note that $\mathfrak{Z}_\alpha O_{\kappa,s^\alpha}$ comes with the tensor product action of $g_c$.

**Lemma 4.2.** There is an equivalence $O_{\kappa,s} \sim \mathfrak{Z}_\alpha O_{\kappa,s^\alpha}$ mapping $\Delta(\lambda)$ to $\Delta(\lambda)$ and strongly equivariant for the $g_c$-action.

The proof is given in \cite{R1} Section 6.4 in the case when $e \neq 2$ and $s_i - s_j$ is not divisible by $e$. This restriction can be removed using techniques from \cite{L3} Section 4.2. The strong equivariance follows from the construction of the equivalence in loc. cit. of the categorical action in \cite{Sh}.

For $\lambda \in \mathcal{P}_\ell$, let $\lambda^\alpha$ denote the collection of components of $\lambda$ in the diagrams of class $\alpha$.

**Corollary 4.3.** We have $p_{\kappa,s}(\lambda) = \sum_\alpha p_{\kappa,s^\alpha}(\lambda^\alpha)$ and $q_{\kappa,s}(\lambda) = \sum_\alpha q_{\kappa,s^\alpha}(\lambda^\alpha)$. 
Proof. The first equality follows from the strong equivariance of the equivalence in Lemma 4.2 combined with the computation of \( p(\lambda) \) from \([L3, \text{Section 5.5}]\). Also note that the codimensions of support of \( L(\lambda) \) in \( O_{\kappa,s} \) and \( \otimes_q O_{\kappa,s^\alpha} \) coincide. This is because these codimensions are recovered from the highest weight structure, Lemma 2.2. From here and the equality \( p_{\kappa,s}(\lambda) = \sum_{\alpha} p_{\kappa,s^\alpha}(\lambda^\alpha) \) we deduce \( q_{\kappa,s}(\lambda) = \sum_{\alpha} q_{\kappa,s^\alpha}(\lambda^\alpha) \).

\[ \square \]

5. Proofs of main results

5.1. Heisenberg crystal. Let \( e \) denote the denominator of \( \kappa \). Suppose \( \kappa e_{s_1}, \ldots, \kappa e_{s_{\ell}} \) are all integers. Here we will define a level 1 \( \mathfrak{sl}_\infty \)-crystal on \( \text{Irr}(O_c) = \mathcal{P}_\ell \) such that \( q_c(\lambda) \) is the depth in this crystal. It appeared implicitly in \([SV, \text{Section 5.6}]\). Shan and Vasserot did not describe this structure as a crystal but they gave a basically equivalent description.

5.1.1. The case \( p_c(\lambda) = 0 \). Let us start by establishing an \( \mathfrak{sl}_\infty \)-crystal structure on the set \( \{ \lambda \in \mathcal{P}_\ell | p_c(\lambda) = 0 \} \).

Let \( \lambda \) be such that \( L_c(\lambda) \) is finite dimensional, equivalently, \( p_c(\lambda) = q_c(\lambda) = 0 \). Then the structure of \( A_\mu L_c(\lambda) \) is as follows, see \([SV, \text{Sections 5.4-5.6}]\): there is a uniquely determined multipartition \( \tilde{a}_\mu \lambda \in \mathcal{P}_\ell(|\lambda| + |\mu|) \) with \( L_c(\tilde{a}_\mu \lambda) \) being a subquotient of \( A_\mu L_c(\lambda) \) satisfying \( p_c(\tilde{a}_\mu \lambda) = 0 \) and \( q_c(\tilde{a}_\mu \lambda) = |\mu| \). Any other subquotient \( L_c(\lambda') \) of \( A_\mu L_c(\lambda) \) satisfies \( p_c(\lambda') = 0, q_c(\lambda') < |\mu| \). The module \( L_c(\lambda') \) cannot occur in the socle or in the head of \( A_\mu L(\lambda) \) by Lemma 2.7.

Further, it is shown in \([SV, \text{Section 5.6}]\), that, for any \( q \in \mathbb{Z}_{>0} \), the map

\[ \mathcal{P}_1(q) \times \{ \lambda \in \mathcal{P}_\ell | p_c(\lambda) = q_c(\lambda) \} \rightarrow \{ \lambda \in \mathcal{P}_\ell | p_c(\lambda) = 0, q_c(\lambda) = q \}, (\mu, \lambda) \mapsto \tilde{a}_\mu \lambda \]

is a bijection. The resulting bijection

\[ \mathcal{P}_1 \times \{ \lambda \in \mathcal{P}_\ell | p_c(\lambda) = q_c(\lambda) = 0 \} \rightarrow \{ \lambda \in \mathcal{P}_\ell | p_c(\lambda) = 0 \} \]

produces a (level 1) \( \mathfrak{sl}_\infty \)-crystal on the target space, carried over from the standard \( \mathfrak{sl}_\infty \)-crystal on \( \mathcal{P}_1 \).

By the construction, \( q_c(\lambda) \) is the depth of \( \lambda \) in the \( \mathfrak{sl}_\infty \)-crystal.

5.1.2. The general case. Now let us extend the \( \mathfrak{sl}_\infty \)-crystal to the whole set \( \text{Irr}(O_c) = \mathcal{P}_\ell \). A crucial step is the following claim.

Proposition 5.1. Let \( \lambda^0 \) be a multipartition with \( p(\lambda^0) = q(\lambda^0) = 0 \), let \( C_e \) be a composition of \( \tilde{f}_i \)'s such that \( \lambda := C_e \lambda^0 \neq 0 \), and let \( \mu \) be a partition. Set \( \lambda^0 := \tilde{a}_\mu \lambda^0 \). Then the head of \( A_\mu L(\lambda) \) is a multiple of \( L(\tilde{\lambda}) \), where \( \tilde{\lambda} := C_e \lambda^0 \).

Proof. First of all, since \( p_c(\lambda^0) = 0 \), the multipartition \( \tilde{\lambda}^0 \) is a singular vertex in the \( \mathfrak{sl}_\infty \)-crystal. Since the weights of \( \lambda^0, \tilde{\lambda}^0 \) coincide, the connected components of the crystal through \( \lambda^0, \tilde{\lambda}^0 \) are isomorphic (and the isomorphism is unique and maps \( \lambda^0 \) to \( \tilde{\lambda}^0 \)). In particular, \( \tilde{\lambda} \neq 0 \).

Now we can prove our claim by the induction on the length of \( C_e \). The case when the length is 0 is trivial. Now suppose that the claim is proved for all lengths less than some \( N \). We are going to prove it for \( C_e \) of length \( N \). First of all, we will modify \( C_e \) without changing \( \lambda, \tilde{\lambda} \) (and therefore preserving the length). Namely, we can find indexes \( i_1, \ldots, i_k \) and positive integers \( n_1, \ldots, n_k \) summing to \( N \) such that

- \( \tilde{e}_{i_1}^{n_1} \tilde{e}_{i_2}^{n_2} \ldots \tilde{e}_{i_k}^{n_k} \lambda = \lambda^0 \).
- \( \tilde{e}_{i_j}^{n_{j+1}} \tilde{e}_{i_{j+1}}^{n_{j+1}} \ldots \tilde{e}_{i_k}^{n_k} \lambda = 0 \) for all \( j \).
We set $C_k := \tilde{e}_{i_k}^{n_k} \ldots \tilde{e}_{i_1}^{n_1}$, it maps $\lambda^0$ to $\lambda$ and $\lambda^0$ to $\tilde{\lambda}$.

Set $\lambda' := \tilde{e}_{i_k}^{n_k} \lambda, \tilde{\lambda}' := \tilde{e}_{i_k}^{n_k} \tilde{\lambda}$, we will write $i$ instead of $i_k$ and $n$ instead of $n_k$ to simplify the notation. By the inductive assumption, the head of $A_\mu L(\lambda')$ is the direct sum of several copies of $L(\tilde{\lambda}')$. By the definition of $f_i$, we have an epimorphism $F_{i}^{(n)} L(\lambda') \twoheadrightarrow L(\lambda)$. Since the functors $A_\mu$ and $F_{i}^{(n)}$ commute, \cite[Proposition 5.15]{SV}, we get an epimorphism $F_{i}^{(n)} A_\mu L(\lambda') \twoheadrightarrow A_\mu L(\lambda)$. By \cite[Lemma 5.11]{CR}, any simple $L(\tilde{\lambda})$ appearing in the head of $F_{i}^{(n)} A_\mu L(\lambda')$ is not killed by $E_i^{(n)}$. By the construction, $E_i L(\lambda') = 0$. Since $E_i$ also commutes with $A_\mu$, we see that $E_i A_\mu L(\lambda') = 0$. It follows that $E_i^{(n)} F_{i}^{(n)} A_\mu L(\lambda')$ is a multiple of $A_\mu L(\lambda')$. But $E_i^{(n)} L(\tilde{\lambda})$ appears in the head of $A_\mu L(\lambda')$. It follows that $\tilde{e}_i \lambda = \tilde{\lambda}'$ and hence $\lambda = \tilde{\lambda}$. This completes the proof.

Using Proposition 5.1 we can now define an $\mathfrak{sl}_{\infty}$-crystal structure commuting with the $\mathfrak{sl}_c$-crystal on the whole set $P_\ell$. Namely, we declare that for each $\lambda$ with $p_c(\lambda) = 0$ a crystal operator $C_\ell$ for $\mathfrak{sl}_c$ with $C_\ell \lambda \neq 0$ is an $\mathfrak{sl}_{\infty}$-crystal embedding of the component of $\lambda$ into $P_\ell$. Thanks to Proposition 5.1, this indeed gives rise to an $\mathfrak{sl}_{\infty}$-crystal on $P_\ell$. Since this crystal arises from the Heisenberg categorical action, we call it the Heisenberg crystal.

**Lemma 5.2.** The number $q_c(\lambda)$ coincides with the depth of $\lambda$ in the $\mathfrak{sl}_{\infty}$-crystal.

**Proof.** Note that $q_c(f_i \lambda') = q_c(\lambda')$ for any $\lambda'$. This follows from Lemma 2.7. Using this and the construction of the $\mathfrak{sl}_{\infty}$-crystal, we reduce the proof to the case when $p_c(\lambda) = 0$. Here our claim follows from 5.1. □

5.2. Computation of the Heisenberg crystal in the asymptotic chambers. Here we are going to compute the $\mathfrak{sl}_{\infty}$-crystal operators under the condition that one of $s_j$ is much less than the others.

**Proposition 5.3.** Suppose that $\kappa < 0$ and $s_j < s_i - N$ for all $i \neq j$ and some $N > 0$. If $|\lambda| < N$ is a multipartition with $p_c(\lambda) = 0$, then $\lambda^{(j)}$ is divisible by $e$ and $q_c(\lambda) = |\lambda^{(j)}|/e$. Further, if $|\lambda| + e|\mu| < N$, then $(a_\mu \lambda)^{(i)} = \lambda^{(i)}$ for $i \neq j$ and $(a_\mu \lambda)^{(j)} = \lambda^{(j)} + e \mu$.

Note that this proposition implies Proposition 1.1.

**Proof.** It follows from \cite[Theorem 5.1]{L3} that, under the assumptions of the proposition, $\lambda^{(j)}$ is divisible by $e$. Indeed, under our assumption on $s_1, \ldots, s_\ell$, the $z$-signature of $\lambda^{(j)}$ will appear in the end of the $z$-signature of $\lambda$, for all $z$. It follows that the reduced signatures of $\lambda^{(j)}$ consist only of $+$’s. It is easy to see that this condition is equivalent to $\lambda^{(j)}$ being divisible by $e$.

Now let us prove that $q_c(\lambda) \geq |\mu|$, where $e \mu = \lambda^{(j)}$. Set $\underaccent{\bar}{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(j-1)}, \emptyset, \lambda^{(j+1)}, \ldots, \lambda^{(\ell)})$. First, let us notice that $\Delta(\lambda)$ is the smallest (in the highest weight order) standard appearing in the filtration of $\text{Ind} \Delta(\lambda) \boxtimes \Delta^A(e \mu)$ (here we have the induction from $G(|\lambda| - e|\mu|, 1, \ell) \times G_{e|\mu|}$ to $G(|\lambda|, 1, n)$). It follows that $L(\lambda)$ is in the head of $\text{Ind} \Delta(\lambda) \boxtimes \Delta^A(e \mu)$ and hence in the head of some object induced from a simple in the category $\mathcal{O}_c(G(|\lambda| - e|\mu|, 1, \ell) \times G_{e|\mu|})$. By Lemma 2.7 $q_c(\lambda) \geq |\mu|$.

On the other hand the number of $\lambda$ with $p_c(\lambda) = 0, q_c(\lambda) = |\mu|$ coincides with the number of $\lambda$ with $p_c(\lambda) = 0, |\lambda^{(j)}| = e|\mu|$. This follows from the fact that (5.1) is a bijection. Therefore $q_c(\lambda) = |\mu|$.

It remains to show that $a_\mu \underaccent{\bar}{\lambda} = \lambda$. Let $\underaccent{\bar}{\lambda}$ be minimal (with $p_c(\lambda) = q_c(\lambda) = 0$ and hence $\lambda^{(j)} = \emptyset$) such that this fails. Again, $L(\lambda)$ appears in the head of $\text{Ind} \Delta(\lambda) \boxtimes \Delta^A(\mu)$. Let $K$
denote the kernel of $\Delta^A(\mu) \to L^A(\mu)$. Then $K$ does not contain any minimally supported object from $\mathcal{O}_c^A(\epsilon|\mu|)$ in the head because the category of such objects is semisimple (see [Wi] Theorem 1.8]) and the head of $\Delta^A(\epsilon\mu)$ is $L^A(\epsilon\mu)$. Lemma 2.7 implies that $L(\lambda)$ does not appear in the head of $\text{Ind} \Delta(\lambda) \boxtimes K$. So $L(\lambda)$ lies in the head of $\text{Ind} \Delta(\lambda) \boxtimes L^A(\epsilon\mu)$. Let us show that the only subquotient $L(\lambda')$ of $\Delta(\lambda)$ such that $L(\lambda)$ lies in the head of $\text{Ind} L(\lambda') \boxtimes L^A(\epsilon\mu)$ is the top quotient $L(\lambda)$. Indeed, if $L(\lambda)$ lies in the head of $\text{Ind} L(\lambda') \boxtimes L^A(\epsilon\mu)$, then $\alpha_{\mu,\lambda'} = \lambda$. It follows from Lemma 2.7 that $p_c(\lambda) = q_c(\lambda') = 0$. As $\lambda' < \lambda$, we get a contradiction with the inductive assumption in the beginning of this paragraph. □

Remark 5.4. The result of the previous proposition can be generalized to $p_c(\lambda) \neq 0$. Here we divide $\lambda^{(j)}$ by $e$ with remainder: $\lambda^{(j)} = e\lambda' + \lambda''$. Then $q_c(\lambda) = |\lambda'|$. This is easily from Proposition 5.3 combined with the fact that the $\mathfrak{sl}_e$ and the $\mathfrak{sl}_\infty$-crystals commute.

5.3. Wall-crossing bijections and crystal operators. Now let us explain an interplay between wall-crossing bijections and crystal operators. We will show that wall-crossing bijections through essential walls (defined in 3.2.3) intertwine the crystal operators for both $\mathfrak{g}_e$- and $\mathfrak{sl}_\infty$-crystals (the latter is considered when all numbers $eKs_1, \ldots, eKs_\ell$ are integral).

First, let us list the essential walls.

Lemma 5.5. The following list gives a complete collection of essential walls for the group $G(\ell, 1, n)$.

(1) $\kappa = 0$ for the parameters $c$, where the $\kappa$-component is a rational number with denominator between 2 and $n$.

(2) $h_i - h_j = km$ with $i \neq j$ and $|m| < n$ – for the parameters $c$ satisfying $s_i - s_j - m \in \kappa^{-1}\mathbb{Z}$.

Proof. The category $\mathcal{O}_c(n)$ is semisimple if and only if all blocks in $\mathcal{P}_t(n)$ consist of one element. The description of blocks is provided in Lemma 4.1. So if $c$ is not of the form described in (1) or (2), then $\mathcal{O}_c$ is semisimple. Conversely, for $c$ as described in (1) and (2), we can find two multipartitions $\lambda, \lambda'$ of the form $\lambda = \lambda\sqcup b, \lambda' = \lambda\sqcup b'$, where $b, b'$ are unequal equivalent boxes. The corresponding wall $\Pi$ is of the form $\Pi_{\lambda, \lambda'}$. It is essential. □

Below we write $\mathfrak{WC}_{c, \psi^{+\psi}}(n)$ for the sum of the wall-crossing functors over all $n$. The summand corresponding to $n$ will be denoted by $\mathfrak{WC}_{c, \psi^{+\psi}}(n)$. The main result here is as follows.

Proposition 5.6. The following statement is true.

(1) Wall-crossing bijections $\mathfrak{wc}_{c, \psi^{+\psi}}$ through essential walls commute with the crystal operators $\tilde{e}_z, \tilde{f}_z$ for $\mathfrak{g}_c$.

(2) Consider the wall in (2) of Lemma 5.5. Then the corresponding wall-crossing bijections $\mathfrak{wc}_{c, \psi^{+\psi}}$ commutes with the crystal operators for $\mathfrak{sl}_\infty$.

(3) Consider the wall in (1). Then $\mathfrak{wc}_{c, \psi^{+\psi}} \circ \tilde{a}_\mu = \tilde{a}_\mu c \circ \mathfrak{wc}_{c, \psi^{+\psi}}$.

Proof. From Proposition 5.8 we deduce that

$$\mathfrak{WC}_{c, \psi^{+\psi}} \circ \text{Res}_{W}^W \cong \text{Res}_{W}^W \circ \mathfrak{WC}_{c, \psi^{+\psi}}. \tag{5.2}$$

Proof of (1). Here we take $W := G(\ell, 1, n - 1)$. Let us show, first, that

$$\mathfrak{WC}_{c, \psi^{+\psi}} \circ E_z \cong E_z \circ \mathfrak{WC}_{c, \psi^{+\psi}}. \tag{5.3}$$

The functor $\mathfrak{WC}_{c, \psi^{+\psi}}$ maps indecomposable projectives to indecomposable objects and $\mathfrak{WC}_{c, \psi^{+\psi}}^{-1}$ maps indecomposable injectives to indecomposable objects. It follows that $\mathfrak{WC}_{c, \psi^{+\psi}}$ maps blocks to blocks.
The block decompositions of $\mathcal{O}_c, \mathcal{O}_{c-\psi}$ on the level of $K_0$ are the same because $\psi$ is integral.

So the functor $\mathcal{M}_c_{\psi}$ preserves the labels of blocks because it acts as the identity on the $K_0$-groups, see Proposition 4.2.1. (5.3) follows now from the construction of the functors $E_z$ in [12.1].

Now let us show that $\mathcal{M}_c_{\psi} \circ \widetilde{c}_i \cong \widetilde{c}_i \circ \mathcal{M}_c_{\psi}$. Since $\mathcal{M}_c_{\psi}$ is a bijection, the claim that it intertwines the crystal operators $\tilde{f}_i$ will follow.

Recall from [23.8.2] that $\mathcal{M}_c_{\psi}$ is a perverse equivalence, let $C^i \subset \mathcal{O}_c, C^i_{\psi} \subset \mathcal{O}_{c-\psi}$ denote the corresponding filtration subcategories. (5.3) implies that the functors $\mathcal{M}_c_{\psi}$ induces the abelian (up to a homological shift) equivalences of the quotients we deduce that $\mathcal{M}_c_{\psi} \circ \widetilde{c}_i \cong \widetilde{c}_i \circ \mathcal{M}_c_{\psi}$.

Proof of (2). Here $\mathcal{W} = G(\ell, 1, n-ek) \times \mathcal{S}_{ek}$. The functor $\mathcal{M}_c_{\psi} : D^b(\mathcal{O}_c \boxtimes \mathcal{O}_{A^\kappa}(em)) \rightarrow D^b(\mathcal{O}_{c-\psi} \boxtimes \mathcal{O}_{A^\kappa}^c(em))$ decomposes as $\mathcal{M}_c_{\psi} \boxtimes \text{id}$. So it follows from (5.2) that

(5.4) $\mathcal{M}_c_{\psi} \circ A_\mu \cong A_\mu \circ \mathcal{M}_c_{\psi}$.

Using part (1) we reduce the proof of (2) to $\mathcal{M}_c_{\psi} \circ \widetilde{a}_\mu(\lambda) = \widetilde{a}_\mu \circ \mathcal{M}_c_{\psi}(\lambda)$ for $\lambda$ satisfying $p_c(\lambda) = q_c(\lambda) = 0$. Set $\lambda' := \mathcal{M}_c_{\psi}(\lambda), \lambda' := \mathcal{M}_c_{\psi}(\lambda), \lambda' := \widetilde{a}_\mu(\lambda')$. We need to show that $\lambda' = \lambda'$. It follows from (5.4) that $A_\mu(C^i_1) \subset C^i_1$. On the other hand, by Corollary 2.13 we have $p_c(\lambda') = q_c(\lambda') = 0$ and $p_c(\lambda') = q_c(\lambda') = 0$. Let $i$ be such that $\lambda \in \text{Irr}(C^i_1/C^i_{i+1})$. Consider the object $M := A_\mu \circ \mathcal{M}_c_{\psi}(L_c(\lambda)) \in D^b(\mathcal{O}_{c-\psi})$. By (P2) from [23.8.2] we have $H_j(M) = 0$ for $j < i$, $H_j(M) \in C^i_{i+1}$ for $j > i$. Moreover, $L_c(\lambda')$ is a unique simple subquotient of $H_i(M)$, and the subquotient exists at all. So we see that $\lambda' = \lambda'$.

Proof of (3). Let $\mathcal{M}_c_{\psi}$ denote the wall-crossing functor for type A categories $\mathcal{O}$ (going from $\kappa$ negative to $\kappa$ positive). By (5.2) we have $\mathcal{M}_c_{\psi} = \mathcal{M}_c_{\psi} \boxtimes \mathcal{M}_c_{\psi}$. Note that $\mathcal{M}_c_{\psi} : D^b(\mathcal{O}_c^{A}(ek)) \rightarrow D^b(\mathcal{O}_c^{A}(ek))$ induces an abelian equivalence with a shift of categories of modules with minimal support (that are subs in the corresponding perverse filtration). We have a self-bijection $\mu \mapsto \mu'$ of $\mathcal{P}_1(k)$ such that $\mathcal{M}_c_{\psi}(L_-(\mu)) = L_+((\mu')^t)[-k(e-1)]$. Similarly to the proof of part (2), part (3) will follow if we check that $\mu' = \mu'$.

First, consider the case when $k = |\mu| = 2$ so that we have two options for $\mu$: either (2) or (1^2). By [23.11] Theorem 1.8, we have

(5.5) $[L_+^{A}(2e)] - [L_+^{A}(e^2)] = \sum_{i=0}^{2e-1} (-1)^i[\Delta_+^{A}(2e - i, 1^i)]$.

Similarly, we have

(5.6) $[L_+^{A}(1^{2e})] - [L_+^{A}(2e^2)] = -\sum_{i=0}^{2e-1} (-1)^i[\Delta_+^{A}(2e - i, 1^i)]$.

By Proposition 5.2, $\mathcal{M}_c_{\psi}$ induces the identity map $K_0(\mathcal{O}_c^A) \rightarrow K_0(\mathcal{O}_c^A)$. So it maps $[L_+^{A}(2e)] - [L_+^{A}(e^2)]$ to $[L_+^{A}(1^{2e})] - [L_+^{A}(1^{2e})]$. It follows that $\mathcal{M}_c_{\psi}(L_+^{A}(2e)) = L_+^{A}(2e)[-2(e-1)], \mathcal{M}_c_{\psi}(L_+^{A}(e^2)) = L_+^{A}(1^{2e})[-2(e-1)]$. So, indeed, in this case, $\mu' = \mu'$. 
Now we are going to prove that $\mu' = \mu^t$ by induction on $|\mu|$ with the base $|\mu| = 2$. Note that
\begin{equation}
\text{Ind}_{\mathfrak{g}_{e(k-1)} \times \mathfrak{g}_e}^{\mathfrak{g}_{ek}} L^A_{\neq}(e^\mu) \boxtimes L^A_{\neq}(e) = \bigoplus_{\mu} L^A_{\neq}(e^\mu),
\end{equation}
where the sum is taken over all $\mu$ obtained from $\mu$ by adding a box. To see this one can use the equivalence of $O_\mu^A(ek)$ with $S_q(ek)$-mod, where $q = \exp(\pi \sqrt{-1}/e)$ and $S_q(ek)$ is the $q$-Schur algebra of degree $ek$. By [SV, Proposition 3.1], the equivalences $O_\mu^A(ek) \sim S_q(ek)$-mod intertwines the induction functors and the tensor product functors. The subcategory of minimally supported modules in $O_\mu^A(ek)$ corresponds to the essential image of the Frobenius $S_1(ek)$-mod $\rightarrow S_\mu$-mod. The Frobenius pull-back intertwines the tensor product functors as well. So the equivalence of the subcategory of minimally supported modules in $O_\mu^A(ek)$ and $\mathfrak{g}_k$-mod given by $L^A_{\neq}(e^\mu) \mapsto S_\mu$ intertwines the induction functors. (5.7) follows. We also have a direct analog of (5.7) for the category $O_\mu^A(ek)$.

It follows from (5.2), (5.7) and its +-analog that $\mu \mapsto \mu'$ is an automorphism of the branching graph for $S_1 \subset S_2 \subset \ldots \subset S_n \subset \ldots$. We can uniquely recover $\mu$ from the collection of all diagrams obtained from $\mu$ by removing a box when $|\mu| > 2$. Namely, if there is more than one diagram in this collection, then for $\mu$ we take the union of these diagrams. Otherwise, $\mu$ is a rectangle and also can be recovered uniquely. This serves as the induction step in our proof of $\mu' = \mu^t$. The proof of (3) is now complete.

In particular, this proposition allows to compute the wall-crossing bijection for the type A categories $O$. Take a partition $\lambda$ and divide it by $e$ with remainder $\lambda = e\lambda' + \lambda''$. The partition $\lambda' = e$-coRestricted meaning that a column of each height appears less than $e$ times. Let $P^e$ denote the set of all $e$-co-restricted partitions. This is the connected component of $\mathfrak{g}$ in the $\mathfrak{sl}_e$-crystal. This set has a remarkable involution called the Mullineux involution, $M$: it is defined as the only map that send $\mathfrak{g}$ to itself and is twisted equivariant with respect to the crystal operators: $M(f_i \lambda) = f_{-i}^t M(\lambda)$.

The next corollary follows from (1) and (3) of Proposition 5.6

**Corollary 5.7.** The bijection $\text{wc}_{+-}^A$ sends $\lambda = e\lambda' + \lambda''$ to $(e(\lambda'') + M(\lambda'))^t$.

**5.4. Computation of wall-crossing bijections.** Here we will use Propositions 3.1 and 3.6 to compute the wall-crossing bijections through the walls described in (2) of Lemma 5.5. More precisely, using Proposition 3.1 we reduce the computation to the case when $c$ is Weil generic in $c'$. Then we use (1) of Proposition 5.6 to do the computation.

For $m \in \mathbb{Z}$, we define the self-bijection $\text{wc}_m$ of $P_n$. For a box $b = (x, y)$ in the first diagram we set $\text{cont}(b) := x - y$, for $b' = (x, y)$ in the second diagram we set $\text{cont}(b') = x - y + m$. We can produce two crystal structures on $P_n$ using the cancellation recipe for addable/removable boxes similar to the one used in [1.2.2] (both our crystals will be special cases of the crystals considered there). The crystal operators $e_i^{[j]}$ (resp., $f_i^{[j]}$) with $i \in \mathbb{Z}, j = 1, 2$, for both crystals will remove (resp., add) boxes with shifted content $i$. What is different for the two structures is the order in which the boxes are listed. For the crystal with $j = 1$, we first list the addable/removable $i$-box from the first diagram and then the box from the second diagram. For the crystal with $j = 2$, we do vice versa. Cancellation of addable/removable boxes (we cancel $-+$) and the definition of the crystal operators is the same as in [1.2.2]
Lemma 5.8. There is a unique bijection $\mathbf{wc}_m : \mathcal{P}_2 \rightarrow \mathcal{P}_2$ that preserves the total number of boxes and intertwines the two sets of crystal operators: $\mathbf{wc}_m(\tilde{c}_i^{[1]} \lambda) = \tilde{c}_i^{[2]} \mathbf{wc}_m(\lambda)$ and $\mathbf{wc}_m(\tilde{f}_i^{[1]} \lambda) = \tilde{f}_i^{[2]} \mathbf{wc}_m(\lambda)$ for all $i \in \mathbb{Z}$.

Proof. Let us describe the bipartitions $\lambda$ annihilated by all $\tilde{c}_i^{[1]}$, $i \in \mathbb{Z}$. For all $i$, the signatures must look like $\varnothing$, $+$, $++$ or $--$. So $\lambda^{(2)} = \varnothing$ and $\lambda^{(1)}$ can have only one removable box, that box must have content $m$. Hence $\lambda^{(1)}$ is a rectangle with opposite vertices being the box $(1,1)$ and a box with content $m$. The similar description works for the operators $\tilde{c}_i^{[2]}$. In that case, $\lambda^{(1)} = \varnothing$ and $\lambda^{(2)}$ is a rectangle with vertex on the diagonal with non-shifted content $-m$. In particular, we see that for each $n$, there is not more than one singular 2-partition of either of the crystals. Since $\mathbf{wc}_m$ has to map singular bi-partitions to singular ones, we see that the requirement that $\mathbf{wc}_m$ preserves the number of boxes determines $\mathbf{wc}_m$ on the singular bi-partitions uniquely. Since $\mathbf{wc}_m$ intertwines the crystal operators, its extension to all 2-partitions is unique as well. \hfill \Box

Now we are ready to describe the wall-crossing bijection through the wall $h_i - h_j = km$.

Proposition 5.9. Let $\Pi$ be the wall defined by $h_i - h_j - km = 0$, let $c$ be a parameter in a chamber adjacent to $\Pi$ and let $c - \psi$ lie in the chamber opposite to that of $c$ with respect to $\Pi$. Suppose that the linear function $h_i - h_j - km$ is positive on $\psi$. Then $\lambda' := \mathbf{wc}_{c - \psi - c}(\lambda)$ is computed as follows: $\lambda^{(k)} = \lambda^{(k)}$ if $k \neq i, j$ and $(\lambda^{(j)}, \lambda^{(i)}) = \mathbf{wc}_m(\lambda^{(j)}, \lambda^{(i)})$.

Proof. Thanks to Proposition 3.1, we reduce the proof to the case when $c$ is Weil generic on $c + \Pi$. In this case, if $b, b'$ are equivalent boxes, we have $b$ in the $i$th diagram and $b'$ in the $j$th diagram, or vice versa or $b, b'$ lie in the same diagram and have the same content. We can add the same number to the $s_1, \ldots, s_\ell$ and assume that $s_j = 0, s_i = m$, while the other $\ell - 2$ numbers are generic. We will have $\ell - 2$ collections of $\mathfrak{sl}_\infty$-crystal operators, each collection acts on its own partition. We will have another $\mathfrak{sl}_\infty$-crystal acting on partitions $i, j$ in one of two ways described above. The claim of this proposition follows now from Proposition 5.6 combined with the uniqueness part of Lemma 5.8. \hfill \Box

5.5. Summary. Let us summarize the computation of $p_c(\lambda)$ and $q_c(\lambda)$.

First, the number $p_c(\lambda)$ is the depth of $\lambda$ in the $g_c$-crystal, see 4.2.2. If $c$ is irrational, then $q_c(\lambda) = 0$, and we are done.

So suppose from now on that $\kappa$ is rational, let $e$ be the denominator. We may assume that $\kappa < 0$, otherwise we switch $(\kappa, s_1, \ldots, s_\ell)$ to $(-\kappa, -s_1, \ldots, -s_\ell)$ and $\lambda$ to $\lambda'$.

We may assume all numbers $\kappa s_1, \ldots, \kappa s_\ell$ are integral, we can reduce to this case using Corollary 1.8. Also we can assume that $p_c(\lambda) = 0$, we can reduce to this case by replacing $\lambda$ with the singular element in the connected component containing $\lambda$ in the $\mathfrak{sl}_c$-crystal.

Now let $|\lambda| = n$. If there is $j$ such that $s_j < s_i - n$ for all $i \neq j$, then $q_c(\lambda) = |\lambda^{(j)}|/e$. In general, we can reduce to the case when there is such $j$ by crossing walls of the form $h_a - h_b = km$. Each time we cross the wall we pick a parameter $(\kappa, s_1, \ldots, s_\ell)$ in the neighboring chamber and modify $\lambda$ by applying the wall-crossing bijection from Proposition 5.9.

5.6. Chambers, combinatorially. Let us fix a parameter $c = (\kappa, s_1, \ldots, s_\ell)$ with $\kappa = -\frac{r}{e}$, where $r > 0, e > 1$ and $r, e$ are coprime. We are going to describe the walls of a “combinatorial” chamber containing $c$. 
Let us start by defining combinatorial chambers. Those will be specified by linear orders of boxes with same residue. Recall that we record a multipartition as a collections of shapes consisting of unit square boxes in \( \ell \) coordinate planes.

A combinatorial chamber will be parameterized by an \( \ell \)-tuple of pairs of integers. Pick a residue \( \alpha \mod e \). The choice of our parameter \( c \) gives a linear ordering on \( \alpha \)-boxes. Pick an interval of length \( \ell \) (it contains exactly one box from each of the \( \ell \) coordinate planes) and record (in a decreasing way) the pairs \( (m_j, i_j) \), \( j = 1, \ldots, k \), where \( m_j \) is the (unshifted) content of the \( j \)th box and \( i_j \) is the number of plane, where this box is located.

Depending on our choice of \( \alpha \) and of the first box in the interval, we will get different sequences that will be called equivalent. The equivalence relation is generated by

\[
((m_1, i_1), \ldots, (m_\ell, i_\ell)) \sim ((m_1 + m, i_1), \ldots, (m_\ell + m, i_\ell)),
\]

\[
((m_1, i_1), \ldots, (m_\ell, i_\ell)) \sim ((m_2, i_2), \ldots, (m_\ell, i_\ell), (m_1 - e, i_1)).
\]

We declare parameters \( (\kappa, s_1, \ldots, s_\ell), (\kappa', s'_1, \ldots, s'_\ell) \) with integral difference (recall that this means \( \kappa' - \kappa \in \mathbb{Z}, rs_i' - rs_i \in e\mathbb{Z} \)) to lie in the same combinatorial chamber if the \( \ell \)-sequences of pairs produced from these parameters are equivalent. Note that for parameters lying in the same combinatorial chamber, the orders on the categories \( \mathcal{O} \) are the same for all \( n \).

Note that the affine Weyl group \( S_\ell \) acts on the set of chambers via

\[
\sigma_j((m_1, i_1), \ldots, (m_\ell, i_\ell)) = ((m_1, i_1), \ldots, (m_{j+1}, i_{j+1}), (m_j, i_j), \ldots, (m_\ell, i_\ell)), 1 < j < \ell,
\]

\[
\sigma_\ell((m_1, i_1), \ldots, (m_\ell, i_\ell)) = ((m_\ell + e, i_\ell), (m_2, i_2), \ldots, (m_1 - e, i_1)).
\]

Here \( \sigma_1, \ldots, \sigma_\ell \) denote the simple reflections in \( S_\ell \).

We say that two combinatorial chambers are neighbors if the corresponding \( \ell \)-tuples of pairs are obtained from one another either by applying some \( \sigma_j \). In other words, neighboring chambers correspond to a minimal perturbation of the order on boxes. Therefore, for \( n \) sufficiently large, each combinatorial chamber is an essential chamber.

Let us explain how to determine walls between neighbor combinatorial chambers, this will allow us to determine which wall-crossing bijection we need to apply to get between these two chambers. Suppose that we have permuted pairs \( (m_j, i_j), (m_{j+1}, i_{j+1}) \). Then the corresponding wall is \( h_{i_j} - h_{i_{j+1}} = \kappa(m_j - m_{j+1}) \). So to get from the initial chamber to its neighbor we will need to apply the bijection \( \nu_{m_\ell} \) with \( m = m_{j+1} - m_j \) to the partitions \( i_j, i_{j+1} \).

Let us finish this section with an example. Let \( \kappa = -\frac{1}{3}, s_1 = 0, s_2 = 1, s_3 = 2 \). Then we get the following triple of pairs: \( (0, 3), (1, 2), (2, 1) \). The three neighbor combinatorial chambers correspond to the following triples:

(i) \( (1, 2), (0, 3), (2, 1) \).
(ii) \( (0, 3), (2, 1), (1, 2) \).
(iii) \( (5, 1), (1, 2), (−3, 3) \).

For a parameter in chamber (i), we can take, for example, \( \kappa' = -\frac{4}{3}, s'_1 = 0, s_2 = 1, s_3 = \frac{1}{2} \). To get to this chamber, we need to apply \( \nu_{c_1} \) to the diagrams number 3 and 2.

5.7. Example of computation. Here we will compute the numbers \( p_\ell(\lambda), q_\ell(\lambda) \) for \( \ell = 2, |\lambda| \leq 3, \kappa < 0, e = 2 \), even \( s_1 \) and \( s_2 \).

5.7.1. Chambers. The essential walls are \( h_1 - h_2 = m \kappa \) for \( m = −2, 0, 2 \) and \( \kappa = 0 \). So we have four chambers with \( \kappa < 0 \). Here are the pairs of integers corresponding to these combinatorial chambers (we only take chambers adjacent to these walls).
The simple modules in 

\[ \text{Supports of simple modules in cyclotomic Cherednik categories } \mathcal{O} \]

5.7.2. Wall-crossing bijections. Let us start with the bijection between (1) and (2) that is \( \mathfrak{w}_{c-2} \) applied to partitions 1 and 2. It is the identity on \( P_{\lambda}(1) \) and \( P_{\lambda}(2) \).

The bijection sends the singular bipartition \( (1^3, \emptyset) \) in chamber (1) to the singular bipartition \( (\emptyset, 3) \) in chamber (2). Further, it sends \( (1^2, 1) \) to \( (1^3, \emptyset), (\emptyset, 3) \) to \( (1, 2) \), and \( (1, 2) \) to \( (1^2, 1) \). Finally, the bijection \( \mathfrak{w}_{c-2} \) fixes all other partitions. The bijection \( \mathfrak{w}_{c_0} \) from chamber (2) to chamber (3) just swaps the components of the partition.

The bijection \( \mathfrak{w}_{c_2} \) from chamber (3) to chamber (4) sends \( (3, \emptyset) \) to \( (\emptyset, 1^3), (\emptyset, 1^3) \) to \( (1, 1^2) \), \( (2, 1) \) to \( (3, \emptyset) \) and \( (1, 1^2) \) to \( (2, 1) \).

5.7.3. Supports. Chamber (1). The \( \mathfrak{sl}_2 \)-crystal looks as follows:

\[
\begin{align*}
\mathfrak{f}_0(\emptyset, \emptyset) &= (\emptyset, 1), \mathfrak{f}_1(\emptyset, \emptyset) = 0. \\
\mathfrak{f}_0(\emptyset, 1) &= (1, 1), \mathfrak{f}_1(\emptyset, 1) = (\emptyset, 1^2). \\
\mathfrak{f}_0(1, \emptyset) &= 0, \mathfrak{f}_1(1, \emptyset) = (1^2, \emptyset). \\
\mathfrak{f}_0(\emptyset, 1^2) &= (\emptyset, 1^2), \mathfrak{f}_1(\emptyset, 1^2) = (\emptyset, (21)). \\
\mathfrak{f}_0(\emptyset, 2) &= (\emptyset, 3), \mathfrak{f}_1(\emptyset, 2) = (\emptyset, (21)). \\
\mathfrak{f}_0(1, 1) &= 0, \mathfrak{f}_1(1, 1) = (1, 1^2). \\
\mathfrak{f}_0(1^2, \emptyset) &= (1^2, 1), \mathfrak{f}_1(1^2, \emptyset) = (21, \emptyset). \\
\mathfrak{f}_0(2, \emptyset) &= (2, 1), \mathfrak{f}_1(2, \emptyset) = 0.
\end{align*}
\]

The following bipartitions have \( p(\lambda) = |\lambda|: (\emptyset, \emptyset), (\emptyset, 1), (\emptyset, 1^2), (1, 1)(\emptyset, 1^3), (\emptyset, (21)), \) and \( (1, 1^2) \).

The following bipartitions have \( p(\lambda) = |\lambda| - 1: (1, \emptyset), (1^2, \emptyset), (1^2, 1), (21, \emptyset). \)

The following bipartitions have \( p(\lambda) = |\lambda| - 2: (\emptyset, 2), (2, \emptyset), (\emptyset, 3), (2, 1). \)

The following bipartitions of 3 have \( p(\lambda) = 0: (3, \emptyset), (1^3, \emptyset), (1, 2). \)

The following bipartitions have \( q(\lambda) = 1: (\emptyset, 2), (1, 2), (\emptyset, 3). \) All other bipartitions have \( q(\lambda) = 0. \)

Chamber (2). The pairs \( (p(\lambda), q(\lambda)) \) are the same as in chamber (1) except the following four cases.

- \( p(1^3, \emptyset) = 2, q(1^3, \emptyset) = 0, \)
- \( p(\emptyset, 3) = q(\emptyset, 3) = 0, \)
- \( p(1, 2) = q(1, 2) = 1, \)
- \( p(1^2, 1) = 0, q(1^2, 1) = 1. \)

Chamber (3). Obtained from chamber (2) by swapping the components of a bipartition.

Chamber (4). Obtained from chamber (1) by swapping the components of a bipartition.

6. Appendix

6.1. Case \( \kappa = 0 \). Here we will explain how to compute the supports of the irreducible modules in \( \mathcal{O}_c(n) \) in the case when \( \kappa = 0 \). In this case \( H_c(n) = H_c(1)^{\otimes n} \# \mathfrak{S}_n \) and so an object in the category \( \mathcal{O}_c(n) \) is the same things as an \( \mathfrak{S}_n \)-equivariant object in \( \mathcal{O}_c(1)^{\otimes n} \).

Recall that in this case \( p_c(\lambda) = 0, \) by convention. The number \( q_c(\lambda) \) is computed as follows. The simple modules in \( \mathcal{O}_c(1) \) are labelled by the numbers from 1 to \( \ell \). Let \( I \) denote the subset
of \(\{1, \ldots, \ell\}\) consisting of the indexes \(i\) such that the corresponding module has dimension of support equal to 1. Then \(q_c(\lambda) = \sum_{i \in \ell} |\lambda^{(i)}|\).

6.2. Groups \(G(\ell, r, n)\). Let \(\ell, n\) be the same as before and let \(r\) divide \(\ell\). Then we can consider the normal subgroup \(G(\ell, r, n)\) consisting of all elements of the form \(\sigma \eta\), where \(\sigma \in \mathfrak{S}_n\) is an arbitrary element and \(\eta = (\eta(1), \ldots, \eta(n)) \in (\mathbb{Z}/\ell \mathbb{Z})^n\) satisfies \(\prod_{i=1}^{n} \eta(i) = 1\). This is a complex reflection group (in its action on \(\mathfrak{h} = \mathbb{C}^n\)). In particular, for \(r = \ell = 2\) we get the Weyl group of type \(D_n\).

The following lemma is elementary.

**Lemma 6.1.** Suppose that \(n > 2\). Then every conjugacy class in \(G(\ell, 1, n)\) contained in \(G(\ell, r, n)\) is a single conjugacy class in the latter. So we have \(\ell/r\) conjugacy classes of reflections in \(G(\ell, r, n)\).

Note that the claim of the lemma is false when \(n = 2\): the class of a transposition from \(\mathfrak{S}_n\) in \(G(\ell, 1, 2)\) is contained in \(G(\ell, r, 2)\) and splits into the union of several conjugacy classes there.

Let \(c \in \mathfrak{c}\) be such that the values of \(c\) on the conjugacy classes not intersecting \(G(\ell, r, n)\) are zero. Define a parameter \(c\) for \(G(\ell, r, n)\) as the restriction of \(c\) to \(G(\ell, r, n) \cap \mathcal{S}\). Let \(H_c\) and \(H_\mathcal{S}\) be the Cherednik algebras for \(G(\ell, 1, n)\) and \(G(\ell, r, n)\), respectively. Note that the group \(G(\ell, 1, n)\) acts on \(H_\mathcal{S}\) by automorphisms. Then we have \(H_c = H_\mathcal{S} \# G(\ell, r, n) G(\ell, 1, n)\). It follows that \(\mathcal{O}_c(n)\) is the category of \(G(\ell, 1, n)\)-equivariant objects in \(\mathcal{O}_\mathcal{S}(G(\ell, r, n))\) (i.e., objects \(M \in \mathcal{O}_\mathcal{S}(G(\ell, r, n))\) that are also \(G(\ell, 1, n)\)-modules such that the actions of \(G(\ell, 1, n) \subset G(\ell, r, n)\), \(H_c\) agree, and \(M\) is a \(G(\ell, 1, n)\)-equivariant module). This reduces questions about characters/supports from \(\mathcal{O}_\mathcal{S}(G(\ell, r, n))\) to \(\mathcal{O}_c(n)\).

6.3. Three commuting crystals. Assume that \(\kappa = -\frac{1}{c}, s_1, \ldots, s_\ell \in \mathbb{Z}\). We write \(s\) for the \(\ell\)-tuple \((s_1, \ldots, s_\ell)\) and \(|s|\) for \(s_1 + \ldots + s_\ell\). We assume, for simplicity, that \(|s| = 0\). We write \(\mathcal{O}_s\) for \(\mathcal{O}_c\).

We have established two commuting crystals on \(\mathcal{P}_c\). They are crystal analogs of the two commuting actions on the Fock space \(\mathcal{F}_s\) of \(\hat{\mathfrak{sl}}_c\) and of the Heisenberg algebra. Recall that one way to realize the Fock space is via the level-rank duality. Namely, \(\sum_{s, |s| = 0} \mathcal{F}_s\) is a module over \(\mathfrak{sl}_c \times \mathfrak{heis} \times \hat{\mathfrak{sl}}_c\), and \(\mathcal{F}_s\) is a weight space for \(\hat{\mathfrak{sl}}_c\), see [U] Section 2.1] for the quantum version of this construction. The name “level-rank duality” is explained by the fact that the representation in \(\sum_{s, |s| = 0} \mathcal{F}_s\) has level \(\ell\) for the algebra \(\mathfrak{sl}_c\) and level \(c\) for the algebra \(\hat{\mathfrak{sl}}_c\). Also we have \(\sum_{s, |s| = 0} \mathcal{F}_s = \sum_{s', |s'| = 0} \mathcal{F}_{s'}\), where \(s' = (s'_1, \ldots, s'_c)\) and \(\mathcal{F}_{s'}\) denotes the level \(c\) Fock space with multi-charge \(s'\) for \(\hat{\mathfrak{sl}}_c\).

We get two commuting crystals for \(\mathfrak{sl}_c\) and \(\hat{\mathfrak{sl}}_c\) (usually realized via abaci). It should be possible to check that the Heisenberg crystal commutes with the \(\hat{\mathfrak{sl}}_c\)-crystal combinatorially (this has been done in [Ge]). What we would like to do, however, is to explain the categorical meaning of these three crystals that should easily imply the commutativity.

It is known that the level-rank duality is categorified by the Koszul duality. [[RSVV] [We]]. Namely, the category \(\bigoplus_{s, |s| = 0} \mathcal{O}_s\) is standard Koszul, and its Koszul dual category is \(\bigoplus_{s', |s'| = 0} \mathcal{O}_{s'}\), where \(\mathcal{O}_{s'}\) stands for the category \(\mathcal{O}\) for the groups \(G(e, 1, ?)\) and the parameters \(\kappa' := -\frac{1}{c}, s'\). Below we are going to sketch a new approach to the Koszul duality that nicely incorporates all three categorical actions (two Kac-Moody actions and one Heisenberg action).
Our approach is based on the work of Bezrukavnikov and Yun, [BY]. They consider geometric versions of (singular, parabolic) affine categories \( \mathcal{O} \) for Kac-Moody Lie algebras, in particular, for \( \mathfrak{sl}_m \). For two compositions \( s, s' \) of \( m \) with \( \ell \) and \( e \) parts, respectively, we have the parabolic-singular category \( \mathcal{O}_{s,s'}^{aff} \), where \( s \) encodes the “parabolicity” and \( s' \) encodes the singularity. We have commuting functors \( E_i, i = 1, \ldots, e \), and \( E'_j, j = 1, \ldots, \ell \), between the categories \( \mathcal{O}_{s,s'}^{aff} \), Gaitsgory’s central functors, that commute with the \( E_i \)'s, \( F_i \)'s and \( E'_j \)'s, \( F'_j \)'s.

The Koszul (or more precisely, Ringel-Koszul) duality is between the categories \( \mathcal{O}_{s,s'}^{aff} \) and \( \mathcal{O}_{s',s}^{aff} \). It switches the functors \( E_i \)'s and \( E'_j \)'s and preserves Gaitsgory's central functors.

Now we need to relate the categories \( \mathcal{O}_{s,s'}^{aff} \) with the categories \( \mathcal{O}_s(n) \). This should be done as in [L4, RSVV]. Namely, one can pick \( m \) large enough and consider a “polynomial truncation” of \( \mathcal{O}_{s,s'}^{aff} \). The Ringel-Koszul duality restricts to the polynomial truncations. We plan to establish an equivalence of \( \mathcal{O}_s(n) \) with the polynomial truncation in a subsequent paper. The functors \( E_i, F_i \) on \( \mathcal{O}_{s,s'}^{aff} \) will become the Kac-Moody categorification functors, while Gaitsgory’s central functors will give rise to the categorical Heisenberg action.

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