Numerical Scheme for Backward Doublay Stochastic Differential Equations with Time Delayed Coefficients

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ABSTRACT

In this paper, we present some assumptions to get the numerical scheme for backward doubly stochastic differential delay equations (shortly-BDSDDEs), and we propose a scheme of BDSDDEs and discuss the numerical convergence and rate of convergence of our scheme.

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1 Introduction

Backward stochastic differential equations (shortly-BSDEs) have been first presented in Pardoux and Peng [16, 17] in order to prove the existence and uniqueness of the adapted solutions and presented a new class of backward doubly stochastic differential equations, further investigations being (see [3, 4, 11, 13]). A lot of mathematicians interested in a numerical methods for approximating solution of BSDEs (see [1, 10, 14, 15, 18, 22]). Xuerong Mao et al. [21] discussed the effects of environmental noise on the delay Lotka-Volterra model. Brahim Boufoussi et al. [2] presented a new class of backward doubly stochastic differential equations, this a new class depend on an integral with respect to an adapted continuous increasing process. Lukasz Delong [5, 6] studied applications of a new class of time-delayed BSDEs and he gives examples of pricing, hedging and portfolio management problems which could be established in the framework of backward stochastic differential delay equation. Wen Lu et al. [19] investigated a class of multivalued backward doubly stochastic differential delay equation, and they proved the existence and uniqueness of the solutions for these equations under Lipschitz condition. Using the Euler-Maruyama method, Xiaotai Wu and Litan Yan [20] defined the numerical solutions of doubly perturbed stochastic delay differential equations driven by Levy process, and they proved the numerical solutions converge to the exact solutions with the local Lipschitz condition. Delong and Imkeller [7] presented a class of BSDEs with time delayed, and they established the existence and uniqueness of a solution for BSDEs with time delayed. Also, they [8] proved the existence and uniqueness as well as the Malliavin’s differentiability of the solution for BSDEs with delayed time. Moreover, Diomande and Matriciuc [9] proved the existence and uniqueness of a solution for multivalued BSDEs with time delayed generators. Besides, Lu and Ren [12] established the existence and uniqueness of the solutions for a class of backward doubly stochastic differential equations with time delayed coefficients under Lipschitz condition.

The purpose of this work is to study the numerical convergent of backward doubly stochastic delay differential equations (shortly-BDSDDEs) that has the following

\[ Y(t) = \xi + \int_t^T f(s, Y(s), Z(s), Y_s, Z_s)ds + \int_t^T g(s, Y(s), Z(s), Y_s, Z_s)dB(s) - \int_t^T Z(s)dW(s) \]  \hspace{1cm} (1)

where \( \{W_t, 0 \leq t \leq T\} \text{ and } \{B_t, 0 \leq t \leq T\} \) are a Brownian motion defined on the probability space \( (\Omega_1, F_1, P_1) \) and \( (\Omega_2, F_2, P_2) \), respectively, and \( T < \infty \) is a finite time horizon. The coefficients \( f \) and \( g \) at time \( s \) and the terminal condition \( \xi \) depend on the past values of a solution \( (Y_s, Z_s) = (Y(s + \theta), Z(s + \theta))_{T \leq \theta < 0} \).

In our work, we extend the approach of BDSDEs in the general case, and introduce some general assumptions on the numerical convergence of backward doubly stochastic differential equations with time delayed coefficients. Furthermore, we present a numerical scheme based on iterative regression functions which are approximated by projection on vector space of functions. Also, we discuss the numerical convergence and rate of convergence of BDSDDEs Lipschitz condition.

The present paper is organized as follows: In section 2, we present some preliminaries that explain the approximation scheme for BDSDDEs. In section 3, we consider the approximation solution of BDSDDEs and prove some problems that useful for our work. In section 4, we have discussed the numerical convergence and rate of convergence of our scheme.

2 Notations, preliminaries and basic assumptions

In this section, we provide some assumptions and space used in the sequel. Therefore, we consider two independent standard \( d \)-dimensional Brownian motions \( \{W_t, 0 \leq t \leq T\} \) and \( \{B_t, 0 \leq t \leq T\} \), defined on the complete probability spaces \( (\Omega_1, F_1, P_1) \) and \( (\Omega_2, F_2, P_2) \), respectively, and a finite time horizon \( T < \infty \). We denote

\[ F_{t,s}^B = \sigma[B_r - B_s, s \leq r \leq t], \quad F_t^W = \sigma[W_r, 0 \leq r \leq t]. \]

Moreover, we consider \( \Omega = \Omega_1 \times \Omega_2 \), \( F = F_1 \otimes F_2 \) and \( P = P_1 \otimes P_2 \). In addition, we put...
where $N$ is the collection of $P$-null sets of $F$. That is to say, the $\sigma$-fields $F_t, 0 \leq t \leq T$, are $P$-complete, and the family of $\sigma$-algebras $F = \{F_t\}_{t \in [0,T]}$ is neither increasing nor decreasing, it is not constitute a filtration.

We consider the Euclidian norm $\| \cdot \|$ in $R^k$ and $R^{k \times d}$, we use the following spaces

1. Let $L^2_{t=0} (R^{k \times d})$ is the space of measurable function $Z : [-T, 0] \rightarrow R^{k \times d}$ such that $\int_{-T}^{0} |Z(t)|^2 \, dt < \infty$.
2. Let $L^2_{t=0} (R^k)$ is the space of measurable function $Y : [-T, 0] \rightarrow R^k$ such that $\sup_{t \leq T} |Y(t)|^2 < \infty$.
3. Let $H^2_T (R^m)$ is the space of $F$-predictable processes $Y : \Omega \times [0, T] \rightarrow R^m$ such that $E \int_{0}^{T} |Y(t)|^2 \, dt < \infty$.
4. Let $S^2_{T} (R^k)$ is the space of $F$-adapted, product measurable processes $Y : \Omega \times [0, T] \rightarrow R^k$ such that $E[\sup_{0 \leq s \leq T} |Y(t)|^2 ] < \infty$.

The spaces $H^2_T (R^{k \times d})$ and $S^2_{T} (R^k)$ are done with the norm $\| Z \|^2_{H^2_T} = E \int_{0}^{T} |Z(t)|^2 \, dt$ and $\| Y \|^2_{S^2_T} = E[\sup_{0 \leq s \leq T} |Y(t)|^2 ]$, respectively. In this paper, we consider the following BDSDE with time delayed coefficients

$$
\left\{ d(Y(t)) = f(t, Y(t), Z(t), Y_t, Z_t) \, dt + g(t, Y(t), Z(t), Y_t, Z_t) \, dB(t) - Z(t) \, dW(t), 0 \leq t \leq T,
Y_T = \xi(Y_T, Z_T), -T \leq t \leq 0,
\right\}
$$

where $f$ and $g$ are Borel-measurable functions at time set depend on the past values of the solution $Y_s = (Y(s + \theta))_{\theta \geq 0}$ and $Z_s = (Z(s + \theta))_{\theta \leq 0}$. We always set $Z(t) = 0$ and $Y(t) = Y(0)$ for $t < 0$. Now, we make the following assumptions

Assumption (H1): There exist a positive constant $K_1$ and for all $-\tau \leq s < t \leq 0$ such that

$$E[|\xi(t) - \xi(s)|^2] \leq K_1(t-s).$$

Assumption (H2): Suppose that $f : \Omega \times [0, T] \times R^k \times R^{k \times d} \times L^\infty_{t=0}(R^k) \times L^2_{t=0}(R^{k \times d}) \rightarrow R^k$ and $g : \Omega \times [0, T] \times L^\infty_{t=0}(R^k) \times L^2_{t=0}(R^{k \times d}) \rightarrow R^{k \times d}$ are product measurable, there exist a positive constants $K_2$, $K_3$ and $K_4$, and a finite measure $\alpha$ on $[-T, 0]$ such that

$$\left| f(t, Y^1, Z^1, Y^1_t, Z^1_t) - f(t, Y^2, Z^2, Y^2_t, Z^2_t) \right|^2 \leq K_2 (|Y^1 - Y^2|^2 + |Z^1 - Z^2|^2) + K_4 \left( \int_{-T}^{0} |Y^1(t + \theta) - Y^2(t + \theta)|^2 \alpha(d\theta) \right) + \int_{-T}^{0} |Z^1(t + \theta) - Z^2(t + \theta)|^2 \alpha(d\theta),$$

and

$$\left| g(t, Y^1, Z^1, Y^1_t, Z^1_t) - g(t, Y^2, Z^2, Y^2_t, Z^2_t) \right|^2 \leq K_3 (|Y^1 - Y^2|^2 + |Z^1 - Z^2|^2) + K_4 \left( \int_{-T}^{0} |Y^1(t + \theta) - Y^2(t + \theta)|^2 \alpha(d\theta) \right) + \int_{-T}^{0} |Z^1(t + \theta) - Z^2(t + \theta)|^2 \alpha(d\theta),$$

for all $t \in [0, T], (Y^1_t, Z^1_t), (Y^2_t, Z^2_t) \in R^k \times R^{k \times d}, (Y^1_t, Z^1_t), (Y^2_t, Z^2_t) \in L^\infty_{t=0}(R^k) \times L^2_{t=0}(R^{k \times d})$.

Assumption (H3)

$$E \int_{0}^{T} |f(t,0,0,0,0)|^2 \, dt < \infty, E \int_{0}^{T} |g(t,0,0,0,0)|^2 \, dt < \infty.$$

Assumption (H4)
\[ f(t, \ldots, \cdot) = 0, \quad g(t, \cdot) = 0, \]

for \( t < 0 \).

Assumption (H5): There exists a positive constant \( K_4 \) such that

\[ |f(Y, Z)|^2 + |g(Y, Z)|^2 \leq K_4 (1 + |Y|^2 + |Z|^2), \]

where \( a \vee b = \max \{a, b\} \).

3 A numerical scheme for BDSDDEs

In this section, we propose a numerical scheme is based upon a descretization of (1). Moreover, for all integers \( n, l \geq 1 \) and \( t \in [0, T] \), let

\[ -\tau = t_{-n} < t_{-n+1} < \cdots < 0 = t_{0} < t_{1} < \cdots < t_{n} = T \]

be a partition of \([ -\tau, T] \), and denote

\[ \delta = \Delta t_{i+1} = t_{i+1} - t_{i} = \frac{T}{n}, 1 \leq i \leq n, \Delta B_{i+1} = B_{i+1} - B_{i}, \Delta W_{i+1} = W_{i+1} - W_{i}, \]

where \( i = 0, 1, \ldots, n - 1 \), and \( \Delta t = \max \{ -\tau : t_{i+1} \} \). Now, on the small interval \([t_i, t_{i+1}] \) the equation

\[ Y_{i} = Y_{i+1} + \int_{t_{i}}^{t_{i+1}} f(s, Y(s), Z(s), Y_{s}, Z_{s}) ds + \int_{t_{i}}^{t_{i+1}} g(s, Y(s), Z(s), Y_{s}, Z_{s}) dB(s) - \int_{t_{i}}^{t_{i+1}} Z(s) dW(s). \]

(2)

We can be approximated by the discrete equation

\[ Y_{i}^{n}(t) = Y_{i+1}^{n} + f(t, Y_{i}^{n}(t), Z_{i}^{n}(t), Y_{i}^{n}(t + \theta), Z_{i}^{n}(t + \theta)) \delta + g(t, Y_{i}^{n}(t), Z_{i}^{n}(t), Y_{i}^{n}(t + \theta), Z_{i}^{n}(t + \theta)) \Delta B_{i+1} - Z_{i}^{n}(t) \Delta W_{i+1}, \]

with \( Y(T) = \xi(T) \) on \(-T \leq t \leq 0\). Therefore, we consider a class of BDSDDEs as the form

\[ Y_{i}^{n}(t) = \xi(T) + \int_{0}^{T} f(s, Y_{i}^{n}(s), Z_{i}^{n}(s), Y_{i}^{n}(s + \theta), Z_{i}^{n}(s + \theta)) ds + \int_{0}^{T} g(s, Y_{i}^{n}(s), Z_{i}^{n}(s), Y_{i}^{n}(s + \theta), Z_{i}^{n}(s + \theta)) dB(s) - \int_{0}^{T} Z_{i}^{n}(s) dW(s). \]

Now, let us define the Euler-Maruyama approximate solution by

\[ \tilde{Y}(t) = \xi(T) + \int_{0}^{T} f(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}(s + \theta), \tilde{Z}(s + \theta)) ds + \int_{0}^{T} g(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}(s + \theta), \tilde{Z}(s + \theta)) dB(s) - \int_{0}^{T} \tilde{Z}(s) dW(s). \]

(3)

Lemma 3.1 Assume the assumptions (H1)-(H4) hold, for all \( n = 0, 1, \ldots, N - 1 \), then it holds that

\[ \tilde{Y}_{n}^{N}(t) = Y_{n}^{N}(t) \] and \( \tilde{Z}_{n}^{N}(t) = \frac{1}{n} E_{n} \left[ \int_{n+1}^{N} \tilde{Z}_{n}^{N}(s) ds \right] = Z_{n}^{N}(t) \).

Proof. From equation

\[ \tilde{Y}_{n}^{N}(t) = \tilde{Y}_{n+1}^{N}(t) + \int_{n}^{n+1} f(s, \tilde{X}_{n}^{N}(s), \tilde{Z}_{n}^{N}(s), \tilde{X}_{n}^{N}(s + \theta), \tilde{Z}_{n}^{N}(s + \theta)) ds + \int_{n}^{n+1} g(s, \tilde{X}_{n}^{N}(s), \tilde{Z}_{n}^{N}(s), \tilde{X}_{n}^{N}(s + \theta), \tilde{Z}_{n}^{N}(s + \theta)) dB(s) - \int_{n}^{n+1} \tilde{Z}_{n}^{N}(s) dW(s) \]

where \( n \leq t \leq n+1 \), we have that

\[ \tilde{Y}_{n}^{N}(t) = \tilde{Y}_{n+1}^{N}(t) + \int_{n}^{n+1} f(s, \tilde{X}_{n}^{N}(s), \tilde{Z}_{n}^{N}(s), \tilde{X}_{n}^{N}(s + \theta), \tilde{Z}_{n}^{N}(s + \theta)) ds \]
\[
+ \int_{t_n}^{t_{n+1}} g(s, \tilde{X}_n(s), \tilde{Y}_n(s), \tilde{Z}_n(s), \tilde{X}_n(s + \theta), \tilde{Y}_n(s + \theta), \tilde{Z}_n(s + \theta))dB(s)
\]

\[- \int_{t_n}^{t_{n+1}} \tilde{Z}_n(s)dW(s),
\]

and then we get that

\[
\tilde{Y}_n(t) = E_n \left[ \int_{t_n}^{t_{n+1}} (g(t, \tilde{X}_n(t), \tilde{Y}_n(t), \tilde{Z}_n(t), \tilde{X}_n(t + \theta), \tilde{Y}_n(t + \theta), \tilde{Z}_n(t + \theta))dW(s)) \right] + hf(t, \tilde{X}_n(t), \tilde{Y}_n(t), \tilde{Z}_n(t), \tilde{X}_n(t + \theta), \tilde{Y}_n(t + \theta), \tilde{Z}_n(t + \theta)).
\]

From equation above, we deduce that

\[
\int_{t_n}^{t_{n+1}} \tilde{Z}_n(s)dW(s)\Delta W_n = \tilde{Y}_n(t)\Delta W_n
\]

\[
+ \int_{t_n}^{t_{n+1}} f(s, \tilde{X}_n(s), \tilde{Y}_n(s), \tilde{Z}_n(s), \tilde{X}_n(s + \theta), \tilde{Y}_n(s + \theta), \tilde{Z}_n(s + \theta))ds\Delta W_n
\]

\[
+ \int_{t_n}^{t_{n+1}} g(s, \tilde{X}_n(s), \tilde{Y}_n(s), \tilde{Z}_n(s), \tilde{X}_n(s + \theta), \tilde{Y}_n(s + \theta), \tilde{Z}_n(s + \theta))dB(s)\Delta W_n
\]

\[- \tilde{Y}_n(t)\Delta W_n.
\]

By taking the expectation, we get that

\[
E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}_n(s)dW(s)\Delta W_n \right]
\]

\[
= E_n \left[ \int_{t_n}^{t_{n+1}} f(s, \tilde{X}_n(s), \tilde{Y}_n(s), \tilde{Z}_n(s), \tilde{X}_n(s + \theta), \tilde{Y}_n(s + \theta), \tilde{Z}_n(s + \theta))ds\Delta W_n \right]
\]

\[
+ E_n \left[ \int_{t_n}^{t_{n+1}} g(s, \tilde{X}_n(s), \tilde{Y}_n(s), \tilde{Z}_n(s), \tilde{X}_n(s + \theta), \tilde{Y}_n(s + \theta), \tilde{Z}_n(s + \theta))dB(s)\Delta W_n \right]
\]

\[
+ E_n \left[ \tilde{Y}_n(t)\Delta W_n \right] - E_n \left[ \tilde{Y}_n(t)\Delta W_n \right].
\]

Therefore, we deduce that

\[
E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}_n(s)dW(s)\Delta W_n \right]
\]

\[
= hE_n \left[ f(t, \tilde{X}_n(t), \tilde{Y}_n(t), \tilde{Z}_n(t), \tilde{X}_n(t + \theta), \tilde{Y}_n(t + \theta), \tilde{Z}_n(t + \theta))\Delta W_n \right]
\]

\[
+ E_n \left[ g(t, \tilde{X}_n(t), \tilde{Y}_n(t), \tilde{Z}_n(t), \tilde{X}_n(t + \theta), \tilde{Y}_n(t + \theta), \tilde{Z}_n(t + \theta))\Delta B_n\Delta W_n \right]
\]

\[
+ E_n \left[ \tilde{Y}_n(t)\Delta W_n \right] - E_n \left[ \tilde{Y}_n(t)\Delta W_n \right],
\]

then

\[
E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}_n(s)dW(s)\Delta W_n \right] = E_n \left[ g(t, \tilde{X}_n(t), \tilde{Y}_n(t), \tilde{Z}_n(t), \tilde{X}_n(t + \theta), \tilde{Y}_n(t + \theta), \tilde{Z}_n(t + \theta))\Delta B_n\Delta W_n \right]
\]

\[
+ E_n \left[ \tilde{Y}_n(t)\Delta W_n \right].
\]

From the fact \( \tilde{Y}_n(t) \) and \( f(t, \tilde{X}_n(t), \tilde{Y}_n(t), \tilde{Z}_n(t), \tilde{X}_n(t + \theta), \tilde{Y}_n(t + \theta), \tilde{Z}_n(t + \theta)) \) are \( F_{t_n} \)-measurable, then

\[
E_n \left[ f(t, \tilde{X}_n(t), \tilde{Y}_n(t), \tilde{Z}_n(t), \tilde{X}_n(t + \theta), \tilde{Y}_n(t + \theta), \tilde{Z}_n(t + \theta))\Delta W_n \right] = E_n \left[ \tilde{Y}_n(t)\Delta W_n \right] = 0.
\]

Therefore, we have that

\[
E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}_n(s)dW(s)\Delta W_n \right] = h\tilde{Z}_n(t).
\]

By taking the integration, we obtain that

\[
E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}_n(s)dW(s)\Delta W_n \right] = E_n \left[ \int_{t_n}^{t_{n+1}} dW(u)\tilde{Z}_n(s)dW(s) \right]
\]
\[ + E_n \left[ \int_{t_n}^{t_{n+1}} \int_{s_n}^{s} \tilde{Z}^N(u) \, dW(u) \, dW(s) \right] \]
\[ + E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}^N(s) \, ds, \right] \]

then

\[ E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}^N(s) \, dW(s) \, \Delta W_n \right] = E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}^N(s) \, ds \right]. \]

Therefore, we have that

\[ \tilde{Z}_n^N(t) = \frac{1}{h} E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}^N(s) \, ds \right]. \]

From above equation, we get that

\[ \tilde{Z}_n^N(t) = \frac{1}{h} E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}^N(t) \, dW(s) \right] + \frac{1}{h} E_n \left[ \int_{t_n}^{t_{n+1}} \tilde{Z}^N(t) \, dW(s) \right] \Delta W_n \]

Therefore, for all \( n = 0, 1, \ldots, N - 1 \), then \( \left( \tilde{Y}_n^N(t), \tilde{Z}_n^N(t) \right) = (Y_n^N(t), Z_n^N(t)) \).

4 Main results

This section is devoted to discuss numerical convergence and rate of convergence of BDSDEs.

**Theorem 4.1** Suppose that assumption (H2) is fulfilled. Likewise, assume that \( \left\{ Y_n^N(t), Z_n^N(t) \right\} \) is a solution of equation (3), then the approximated solution \( \left\{ Y_n^N(t), Z_n^N(t) \right\} \) converges to the exact solution \( \left\{ Y(t), Z(t) \right\} \) in the sense that for all \( t \in [0, T] \), such that

\[ \lim_{n \to \infty} \| Y(t) - Y_n^N(t) \|^2 = 0 \]

and

\[ \lim_{n \to \infty} E \int_0^T \| Z(t) - Z_n^N(t) \|^2 \, dt = 0. \]

**Proof.** For all \( t \in [0, T] \), let \( \left\{ Y(t), Z(t) \right\} \) and \( \left\{ Y_n^N(t), Z_n^N(t) \right\} \) be the solution of equations (1) and (3), respectively. Therefore, we have that

\[ d(Y(t) - Y_n^N(t)) = [f(t, Y(t), Z(t), Y(t + \theta), Z(t + \theta)) - f(t, Y_n^N(t), Z_n^N(t), Y_n^N(t + \theta), Z_n^N(t + \theta))] \, dt \]
\[ + \left\{ g(t, Y(t), Z(t), Y(t + \theta), Z(t + \theta)) - g(t, Y_n^N(t), Z_n^N(t), Y_n^N(t + \theta), Z_n^N(t + \theta)) \right\} \, dB(t) \]
\[ - [Z(t) - Z_n^N(t)] \, dB(t). \]

Denote \( \langle a, b \rangle \) is inner product of two vectors \( a \) and \( b \). Now, by applying Ito’s formula to \( \| Y(t) - Y_n^N(t) \|^2 \), we get that

\[ \| Y(t) - Y_n^N(t) \|^2 = 2 \int_0^T \langle Y(s) - Y_n^N(s), f(s, Y(s), Z(s), Y(s + \theta), Z(s + \theta)) \rangle \, ds \]
\[ - \int_0^T \langle Y(s) - Y_n^N(s), g(t, Y(t), Z(t), Y(t + \theta), Z(t + \theta)) \rangle \, dB(t) \]
\[ + \left\| \int_0^T \langle Y(s) - Y_n^N(s), g(t, Y(t), Z(t), Y(t + \theta), Z(t + \theta)) \rangle \, ds \right\|^2 \]
\[ + 2 \int_0^T \langle Y(s) - Y_n^N(s), g(t, Y(t), Z(t), Y(t + \theta), Z(t + \theta)) \rangle \, dB(s) \]
\[ - \int_0^T \langle Y_n^N(t), Z_n^N(t), Y_n^N(t + \theta), Z_n^N(t + \theta) \rangle \, dB(t) \]
\[ - 2 \int_0^T \langle Y_n^N(t), Z_n^N(t), Y_n^N(t + \theta), Z_n^N(t + \theta) \rangle \, dB(s) \]
\[ - \int_0^T \langle Y_n^N(s) - Y_n^N(t), Z_n^N(t), Z_n^N(t + \theta) \rangle \, dW(s). \]

By taking the expectation, we get that

\[ E \| Y(t) - Y_n^N(t) \|^2 = 2E \int_0^T \langle Y(s) - Y_n^N(s), f(s, Y(s), Z(s), Y(s + \theta), Z(s + \theta)) \rangle \, ds \]
\[ - E \int_0^T \langle Y(s) - Y_n^N(s), g(t, Y(t), Z(t), Y(t + \theta), Z(t + \theta)) \rangle \, dB(t) \]
\[ + \left\| E \int_0^T \langle Y(s) - Y_n^N(s), g(t, Y(t), Z(t), Y(t + \theta), Z(t + \theta)) \rangle \, ds \right\|^2 \]
\[ + 2E \int_0^T \langle Y(s) - Y_n^N(s), g(t, Y(t), Z(t), Y(t + \theta), Z(t + \theta)) \rangle \, dB(s) \]
\[ - E \int_0^T \langle Y_n^N(t), Z_n^N(t), Y_n^N(t + \theta), Z_n^N(t + \theta) \rangle \, dB(t) \]
\[ - 2E \int_0^T \langle Y_n^N(t), Z_n^N(t), Y_n^N(t + \theta), Z_n^N(t + \theta) \rangle \, dB(s) \]
\[ - E \int_0^T \langle Y_n^N(s) - Y_n^N(t), Z_n^N(t), Z_n^N(t + \theta) \rangle \, dW(s). \]
\[-f(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta)))ds\]
\[+E\int_0^T | g(t,Y_i(t),Z_i(t),Y_i(t+\theta),Z_i(t+\theta)) \]
\[-g(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))|^2 ds\]
\[+2E\int_0^T \langle Y_i(s)-Y_i^n(s), g(t,Y_i(t),Z_i(t),Y_i(t+\theta),Z_i(t+\theta)) \]
\[-g(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))dB(s)\]
\[-2\int_0^T \langle Y_i(s)-Y_i^n(s), Z_i(s)-Z_i^n(s)dW(s) \rangle.\]

Making use of the Young's inequality, we derive that

\[E \left| Y_i(t) - Y_i^n(t) \right|^2 \leq 4KE\int_0^T \left| Y_i(s) - Y_i^n(s) \right|^2 ds + \frac{1}{4K} E\int_0^T \left| f(s,Y_i(s),Z_i(s),Y_i(s+\theta),Z_i(s+\theta)) \right|^2 ds\]
\[-f(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))\]
\[+E\int_0^T | g(s,Y_i(s),Z_i(s),Y_i(s+\theta),Z_i(s+\theta)) \]
\[-g(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))|^2 ds\]
\[+4KE\int_0^T \left| Y_i(s)-Y_i^n(s) \right|^2 ds + \frac{1}{4K} E\int_0^T | g(s,Y_i(s),Z_i(s),Y_i(s+\theta),Z_i(s+\theta)) \]
\[-g(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))|^2 ds - \frac{1}{4K} E\int_0^T \left| Y_i(s)-Y_i^n(s) \right|^2 ds\]
\[-4KE\int_0^T \left| Z_i(s)-Z_i^n(s) \right|^2 ds.\]

By using assumption (H2), we have that

\[E \left| Y_i(t) - Y_i^n(t) \right|^2 \leq 4KE\int_0^T \left| Y_i(s) - Y_i^n(s) \right|^2 ds + \frac{1}{4K} E\int_0^T K_2 \left| Y_i(s) - Y_i^n(s) \right|^2 ds\]
\[+E\int_0^T K_2 \left| Y_i(s) - Y_i^n(s) \right|^2 \alpha(d\theta)ds\]
\[+\frac{1}{4K} E\int_0^T K_4 \int_{-T}^T \left| Y_i(s+\theta) - Y_i^n(s+\theta) \right|^2 \alpha(d\theta)ds\]
\[+E\int_0^T K_2 \left| Y_i(s) - Y_i^n(s) \right|^2 ds\]
\[+E\int_0^T K_2 \left| Y_i(s+\theta) - Y_i^n(s+\theta) \right|^2 \alpha(d\theta)ds\]
\[+\frac{1}{4K} E\int_0^T K_2 \left| Y_i(s) - Y_i^n(s) \right|^2 ds\]
\[+4KE\int_0^T \left| Y_i(s)-Y_i^n(s) \right|^2 ds + \frac{1}{4K} E\int_0^T K_2 \left| Y_i(s) - Y_i^n(s) \right|^2 ds\]
\[+\frac{1}{4K} E\int_0^T K_2 \left| Z_i(s)-Z_i^n(s) \right|^2 ds\]

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By changing the integration order, we obtain that

\[
E \left| Y_i(t) - Y_i^n(t) \right|^2 \leq 4KE \int_0^T \left| Y_i(s) - Y_i^n(s) \right|^2 \, ds + \frac{K_2}{4K} E \int_0^T \left| Y_i(s) - Y_i^n(s) \right|^2 \, ds
\]

\[
+ K_2 \int_0^T E \int_0^T |Z_i(s) - Z_i^n(s)|^2 \, ds + \frac{K_4}{2K} E \int_{0}^{T} \left| Y_i(s) - Y_i^n(s) \right|^2 \, ds \text{ d}\theta \text{ (d}\theta) \] 

\[
+ K_4 E \int_{0}^{T} \left| Z_i(s) - Z_i^n(s) \right|^2 \, ds \text{ d}\alpha(\theta) + K_4 E \int_{0}^{T} \left| Y_i(s) - Y_i^n(s) \right|^2 \, ds \text{ d}\alpha(\theta)
\]

\[
+ K_4 E \int_{0}^{T} \left| Z_i(s) - Z_i^n(s) \right|^2 \, ds \text{ d}\alpha(\theta)
\]

\[
- \frac{1}{4K} E \int_0^T |Y_i(s) - Y_i^n(s)|^2 \, ds - 4KE \int_0^T \left| Z_i(s) - Z_i^n(s) \right|^2 \, ds.
\]

And then, we obtain that

\[
E \left| Y_i(t) - Y_i^n(t) \right|^2 \leq 4KE \int_0^T \left| Y_i(s) - Y_i^n(s) \right|^2 \, ds + \frac{K_2}{4K} E \int_0^T \left| Y_i(s) - Y_i^n(s) \right|^2 \, ds
\]

\[
+ K_2 \int_0^T E \int_0^T |Z_i(s) - Z_i^n(s)|^2 \, ds + \frac{K_4}{2K} E \int_{0}^{T} \left| Y_i(s) - Y_i^n(s) \right|^2 \, ds \text{ d}\theta \text{ (d}\theta) \]

\[
+ K_4 E \int_{0}^{T} \left| Z_i(s) - Z_i^n(s) \right|^2 \, ds \text{ d}\alpha(\theta) + K_4 E \int_{0}^{T} \left| Y_i(s) - Y_i^n(s) \right|^2 \, ds \text{ d}\alpha(\theta)
\]

\[
+ K_4 E \int_{0}^{T} \left| Z_i(s) - Z_i^n(s) \right|^2 \, ds \text{ d}\alpha(\theta)
\]

\[
- \frac{1}{4K} E \int_0^T |Y_i(s) - Y_i^n(s)|^2 \, ds - 4KE \int_0^T \left| Z_i(s) - Z_i^n(s) \right|^2 \, ds.
\]
\[-\frac{1}{4K}E\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds - 4KE\int_{0}^{T} |Z_i(s) - Z_i^n(s)|^2 \, ds.\]

And then, we have that
\[
E \left| Y_i(t) - Y_i^n(t) \right|^2 \leq 4KE\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds + \frac{\beta}{4K}E\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds
\]
\[
+ \frac{\beta}{4K}E\int_{0}^{T} |Z_i(s) - Z_i^n(s)|^2 \, ds + \frac{\beta}{4K}E\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds
\]
\[
+ \frac{\beta}{4K}E\int_{0}^{T} |Z_i(s) - Z_i^n(s)|^2 \, ds + \frac{\beta}{4K}E\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds
\]
\[
+ \frac{\beta}{4K}E\int_{0}^{T} |Z_i(s) - Z_i^n(s)|^2 \, ds + \frac{\beta}{4K}E\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds
\]
\[
+ \frac{\beta}{4K}E\int_{0}^{T} |Z_i(s) - Z_i^n(s)|^2 \, ds - \frac{1}{4K}E\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds,
\]

where \( \beta = \int_{-\theta}^{0} \alpha(d\theta), \)
\[
\int_{-\theta}^{T+\theta} |Y_i(s) - Y_i^n(s)|^2 \, ds \leq \int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds
\]
and
\[
\int_{-\theta}^{T+\theta} |Z_i(s) - Z_i^n(s)|^2 \, ds \leq \int_{0}^{T} |Z_i(s) - Z_i^n(s)|^2 \, ds.
\]

Therefore, we have that
\[
E \left| Y_i(t) - Y_i^n(t) \right|^2 \leq (8K + \frac{\beta}{2K} + \frac{\beta}{4K} + K_2 + K_4 + \frac{1}{4K})E\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds
\]
\[
+ (\frac{\beta}{2K} + \frac{\beta}{4K} + K_2 + K_4 + \frac{1}{4K})E\int_{0}^{T} |Z_i(s) - Z_i^n(s)|^2 \, ds.
\]

By choosing
\[
C_1 = 8K + \frac{\beta}{2K} + \frac{\beta}{4K} + K_2 + K_4 + \frac{1}{4K}
\]
and
\[
C_2 = \frac{\beta}{2K} + \frac{\beta}{4K} + K_2 + K_4 + \frac{1}{4K},
\]

\( C_1, C_2 > 0 \) and \( C_2 \leq K \), we deduce that
\[
E \left| Y_i(t) - Y_i^n(t) \right|^2 \leq C_1E\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds + (C_2 - 4K)E\int_{0}^{T} |Z_i(s) - Z_i^n(s)|^2 \, ds.
\]

And then, we get that
\[
E \left| Y_i(t) - Y_i^n(t) \right|^2 \leq C_1E\int_{0}^{T} |Y_i(s) - Y_i^n(s)|^2 \, ds.
\]

Now, using the Gronwall’s inequality, and for all \( i = 0, \ldots, n \) and \( t \in [0, T] \), we drive that
\[
\lim_{n \to \infty} E[\max_{i=0, \ldots, n} |Y_i(t) - Y_i^n(t)|^2] = 0.
\]
Hence \( \lim_{n \to \infty} E \left| Y(t) - Y^n(t) \right|^2 = 0 \), and consequently
\[
\lim_{n \to \infty} E \int_0^T \left| Z(t) - Z^n(t) \right|^2 \, dt = 0.
\]

**Theorem 4.2** Assume the assumptions (H1)-(H5) are fulfilled, then it holds that
\[
\sup_{0 \leq t \leq T} E \left| Y(t) - \tilde{Y}(t) \right|^2 + E \int_0^T \left| Z(s) - \tilde{Z}(s) \right|^2 \, ds \leq C \delta.
\]

**Proof.** By applying Itô’s formula \( \left| Y(t) - \tilde{Y}(t) \right|^2 \), we have that
\[
\left| Y(t) - \tilde{Y}(t) \right|^2 + \int_0^T \left| Y(s) - \tilde{Y}(s) \right|^2 \, ds + \int_0^T \left| Z(s) - \tilde{Z}(s) \right|^2 \, ds \leq \left| \xi(T) - \tilde{\xi}(T) \right|^2
\]
\[
+ 2 \int_0^T \langle Y(s) - \tilde{Y}(s), f(s, Y(s), Z(s), Y_s, Z_s) - \tilde{f}(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \rangle \, ds
\]
\[
+ 2 \int_0^T \langle g(s, Y(s), Z(s), Y_s, Z_s) - g(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \rangle \, ds
\]
\[
+ 2 \int_0^T \langle Y(s) - \tilde{Y}(s), (g(s, Y(s), Z(s), Y_s, Z_s) - g(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s)) \, dB(s) \rangle
\]
\[
+ 2 \int_0^T \langle Y(s) - \tilde{Y}(s), (Z(s) - \tilde{Z}(s)) \, dW(s) \rangle,
\]
where \( t \in [0, T] \). By Young’s inequality and assumption (H2), we get that
\[
2 \int_0^T \langle Y(s) - \tilde{Y}(s), f(s, Y(s), Z(s), Y_s, Z_s) - \tilde{f}(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \rangle \, ds \leq \gamma \int_0^T \left| Y(s) - \tilde{Y}(s) \right|^2 \, ds
\]
\[
+ \frac{1}{2} \int_0^T \left| g(s, Y(s), Z(s), Y_s, Z_s) - g(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \right|^2 \, ds
\]
\[
\leq \gamma \int_0^T \left| Y(s) - \tilde{Y}(s) \right|^2 \, ds + \frac{1}{\gamma} \int_0^T \left| K_2 \left| Y(s) - \tilde{Y}(s) \right|^2 + K_2 \left| Z(s) - \tilde{Z}(s) \right|^2
\]
\[
+ K_4 \int_0^T \left| Y(s + \theta) - \tilde{Y}(s + \theta) \right|^2 \, d\alpha(d\theta) + K_4 \int_0^T \left| Z(s + \theta) - \tilde{Z}(s + \theta) \right|^2 \, d\alpha(d\theta) \rangle \, ds
\]
\[
+ \frac{K_4}{\gamma} \int_0^T \left| Y(s + \theta) - \tilde{Y}(s + \theta) \right|^2 \, d\alpha(d\theta) + \frac{K_4}{\gamma} \int_0^T \left| Z(s + \theta) - \tilde{Z}(s + \theta) \right|^2 \, d\alpha(d\theta) \rangle \, ds,
\]
and
\[
\int_0^T \left| g(s, Y(s), Z(s), Y_s, Z_s) - g(s, \tilde{Y}(s), \tilde{Z}(s), \tilde{Y}_s, \tilde{Z}_s) \right|^2 \, ds \leq K_3 \int_0^T \left| Y(s) - \tilde{Y}(s) \right|^2 \, ds
\]
\[
+ K_3 \int_0^T \left| Z(s) - \tilde{Z}(s) \right|^2 \, ds + K_4 \int_0^T \left| Y(s + \theta) - \tilde{Y}(s + \theta) \right|^2 \, d\alpha(d\theta) \rangle \, ds
\]
\[
+ K_4 \int_0^T \left| Z(s + \theta) - \tilde{Z}(s + \theta) \right|^2 \, d\alpha(d\theta) \rangle \, ds.
\]
By changing of integration order argument, we obtain that
\[
\int_0^T \int_{t_0}^{t+\theta} \left| Y(s + \theta) - \tilde{Y}(s + \theta) \right|^2 \, d\alpha(d\theta) \, ds = \int_{t_0}^{t+\theta} \int_0^T \left| Y(s + \theta) - \tilde{Y}(s + \theta) \right|^2 \, ds \, d\alpha(d\theta)
\]
\[
= \int_{t_0}^{t+\theta} \int_{t_0}^{t+\theta} \left| Y(t) - \tilde{Y}(t) \right|^2 \, dt \, d\alpha(d\theta) \leq \beta \int_{t_0}^{t+\theta} \left| Y(t) - \tilde{Y}(t) \right|^2 \, dt,
\]
and
\[
\int_0^T \int_{t_0}^{t+\theta} \left| Z(s + \theta) - \tilde{Z}(s + \theta) \right|^2 \, d\alpha(d\theta) \, ds = \int_{t_0}^{t+\theta} \int_0^T \left| Z(s + \theta) - \tilde{Z}(s + \theta) \right|^2 \, ds \, d\alpha(d\theta)
\]
\[
= \int_{t_0}^{t+\theta} \int_{t_0}^{t+\theta} \left| Z(t) - \tilde{Z}(t) \right|^2 \, dt \, d\alpha(d\theta) \leq \beta \int_{t_0}^{t+\theta} \left| Z(t) - \tilde{Z}(t) \right|^2 \, dt,$
where \( \beta = \int_{-T}^{0} \alpha(d\theta) \). Therefore, we drive that

\[
\begin{align*}
&\left| Y(t) - \widetilde{Y}(t) \right|^2 + \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \leq \xi(T) - \widetilde{\xi}(T) \bigg| T \bigg|^2 \\
&+ \gamma \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + \frac{K_2}{\gamma} \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + \frac{K_2}{\gamma} \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \\
&+ \frac{K_3}{\gamma} \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + \frac{K_3}{\gamma} \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \\
&+ K_4 \beta \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + K_4 \beta \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \\
&+ K_4 \beta \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + K_4 \beta \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \\
&+ 2 \int_0^r \langle Y(s) - \widetilde{Y}(s), (g(s,Y(s),Z(s),Y_s,Z_s) - g(s,\widetilde{Y}(s),\widetilde{Z}(s),\widetilde{Y}_s,\widetilde{Z}_s)) dB(s) \rangle \\
&- 2 \int_0^r \langle Y(s) - \widetilde{Y}(s), (Z(s) - \widetilde{Z}(s)) dW(s) \rangle.
\end{align*}
\]

Therefore, we have that

\[
\begin{align*}
&\left| Y(t) - \widetilde{Y}(t) \right|^2 + \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \leq \xi(T) - \widetilde{\xi}(T) \bigg| T \bigg|^2 \\
&+ \gamma \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + \frac{K_2}{\gamma} \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + \frac{K_2}{\gamma} \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \\
&+ \frac{K_3}{\gamma} \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + \frac{K_3}{\gamma} \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \\
&+ K_4 \beta \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + K_4 \beta \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \\
&+ 2 \lambda_1 \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + \frac{2K_3}{\lambda_1} \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds \\
&+ \frac{2K_3}{\lambda_1} \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds + \frac{2K_4 \beta}{\lambda_1} \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds \\
&+ \frac{2K_3}{\lambda_1} \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds + \frac{2K_4 \beta}{\lambda_1} \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds \\
&+ \frac{2}{\lambda_2} \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds,
\end{align*}
\]

where \( \lambda_1, \lambda_2 > 0 \), by taking the expectation and choosing

\[
C_1 = 1 - \gamma - \frac{K_2}{\gamma} - \frac{K_4 \beta}{\gamma} - \frac{K_3}{\gamma} - \frac{K_3}{\lambda_1} - \frac{2K_3}{\lambda_1} - \frac{2K_4 \beta}{\lambda_1} - \frac{2K_4 \beta}{\lambda_1} - \frac{2\lambda_2}{\lambda_2}
\]

and

\[
C_2 = 1 - \frac{K_3}{\gamma} - \frac{K_4 \beta}{\gamma} - \frac{K_3}{\gamma} - \frac{K_4 \beta}{\lambda_1} - \frac{2K_3}{\lambda_1} - \frac{2K_4 \beta}{\lambda_1} - \frac{2\lambda_2}{\lambda_2},
\]

we have that

\[
E \left| Y(t) - \widetilde{Y}(t) \right|^2 + C_1 E \int_0^r \left| Y(s) - \widetilde{Y}(s) \right|^2 ds + C_2 E \int_0^r \left| Z(s) - \widetilde{Z}(s) \right|^2 ds \leq E \left| \xi(T) - \widetilde{\xi}(T) \right|^2.
\]
For sufficiently small $K_3$ and $K_4$ choosing $\gamma, \lambda_1, \lambda_2 > 0$ such that $C_1 > 0$ and $C_2 > 0$, then there exists a constant $C > 0$ depending on $\gamma, K_1, K_2, K_3, K_4, \beta, \lambda_1$ and $\lambda_2$ such that

$$\sup_{0 \leq s \leq T} E \left[ |Y(t) - \tilde{Y}(t)|^2 + E \int_0^T |Y(s) - \tilde{Y}(s)|^2 \, ds + E \int_0^T |Z(s) - \tilde{Z}(s)|^2 \, ds \right] \leq CE \left| \xi(T) - \tilde{\xi}(T) \right|^2.$$  By assumption (H1), we have that

$$\sup_{0 \leq s \leq T} E \left[ |Y(t) - \tilde{Y}(t)|^2 + E \int_0^T |Z(s) - \tilde{Z}(s)|^2 \, ds \right] \leq C \left| \delta \right|.$$

References

[1] V. Bally, Approximation scheme for solution of BSDE, Backward stochastic Differential Equations (N.EI Karoui and L. Mazziati, eds.), Pitman, London, 1997, 177-191.

[2] B. Brahim, V.C. Jan and N. Mrhardy, Generalized backward doubly stochastic differential equations and SPDEs with nonlinear numerical boundary conditions, Bernoulli, 2007, Vol. 13, No. 2, 433-446.

[3] Gh. Constantin, Uniqueness for backward stochastic differential equations with non-Lipschitz coefficients, Anal. Univ. Timisoara Ser. Mat-Inform. XXXIX (2001) 37-43.

[4] Gh. Constantin, On the existence and uniqueness of adapted solutions for backward stochastic differential equations, Anal. Univ. Timisoara Ser. Mat-Inform. XXXIX (2001) 15-22.

[5] L. Delong, Applications of time-delayed backward stochastic differential equations to pricing, hedging and management of insurance and financial risks. Preprint. 2010, arXiv: 1005.4417.

[6] L. Delong, Applications of time-delayed backward stochastic differential equations to pricing, hedging and portfolio management, 2011, Al. Niepodleglosiei 162, 02-554 Warsaw, Poland.

[7] L. Delong and P. Imkeller, Backward stochastic differential equations with time delayed generators-results and counterexamples. Ann. Appl. Probab. 20(2010) 1512-1536.

[8] L. Delong and P. Imkeller, On Malliavin’s differentiability of BSDE with time delayed generators driven by Brownian motions and Poisson random measures. Stochastic process. Appl. 120 (2010) 1748-1775.

[9] B. Diomande and L. Maticiuc, Multivalued backward stochastic differential equations with time delayed coefficients. preprint. 2013, arxiv: 1305.7170v1.

[10] J. Douglas and J. Ma, P. Protter, Numerical methods for forward-backward stochastic differential equations, Ann. Appl. Probab. 6, 1996, 940-968.

[11] J. P. Lepeltier and J. San Martin, Backward stochastic differential equations with continuous generator, statist. Probab. Letters, 32, 1997, 425-430.

[12] W. Lu and Y. Ren, Backward doubly stochastic differential equations with time delayed coefficients. submitted, 2012.

[13] X. Mao, Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients, Stoch. Proc. and their Appl> 58, 1995, 281-292.

[14] J. Ma, P. Protter and J. Yong, Solving forward-backward stochastic differential equations explicitly- a four step scheme, Probab. Theory Related Fields, 98, 1994, 339-359.

[15] J. Ma, P. Protter, J. San Martin and S. Torres, Numerical Method for Backward stochastic Differential Equations, The Annals of Applied Probability, 2002, Vol. 12, no. 1, 302-316.

[16] E. Pardoux and S. Peng, Adapted Solution of a backward stochastic differential equation, systems and control Letters, 14,1990, 55-61.

[17] E. Pardoux and S. Peng, Backward doubly stochastic differential equations and systems of quasilinear SPDEs probab. Theory Related Fields, 88 (1994) 209-227.

[18] Y. Wang and Z. Huang, Backward stochastic differential equations with non-Lipschitz coefficients, stat. and Prob. Letters 79, 2009, 1438-1443.

[19] L. Wen, R. Yong and H. Lanying, Multivalued backward doubly stochastic differential equations with time delayed coefficients, 13 Aug 2013 arxiv:1308.2748v1 [math. PR].

[20] W. Xiaotai and Y. Litan, Numerical solutions of doubly perturbed stochastic delay differential equations driven by Levy process, Arab J Math (2012) 1: 251-265.

[21] M. Xuerong, Y. chenggui and Z. Jiezong, Stochastic differential delay equations of population dynamics, J. Math. Anal. Appl. 304 (2005) 296-320.

[22] J. Zhang, A numerical scheme for BSDEs. The Annaals of Applied Probability, (2004) vol. 14; No 1, 459-488.