TRANSFER OF CHARACTERS IN THE THETA CORRESPONDENCE WITH ONE COMPACT MEMBER

ALLAN MERINO

Abstract. For an irreducible dual pair \((G, G') \in \text{Sp}(W)\) with one member compact and two representations \(\Pi \leftrightarrow \Pi'\) appearing in the Howe duality, we give an expression of the character \(\Theta_{\Pi'}\) of \(\Pi'\) via the character of \(\Pi\). We make computations for the dual pair \((G = U(n, \mathbb{C}), G' = U(p, q, \mathbb{C}))\), which are explicit in low dimensions. For \((G = U(1, \mathbb{C}), G' = U(1, 1, \mathbb{C}))\), we verify directly a result of H. Hecht saying that the character has the same value on both Cartan subgroups of \(G'\).

Contents

1. Introduction 1
2. Metaplectic representation and Howe’s duality theorem 4
3. Howe’s Oscillator semigroup 7
4. A general formula for \(\Theta_{\Pi'}\) 11
5. A restriction of \(\Theta\) to a maximal compact subgroup 15
6. The dual pair \((G = U(n, \mathbb{C}), G' = U(p, q, \mathbb{C}))\) 16
7. The case \(G' = U(1, 1, \mathbb{C})\) 22
8. A conjecture of T. Przebinda 24

Appendix A. The oscillator semigroup for \(U(1, 1, \mathbb{C})\) 26

References 28

1. Introduction

For a finite dimensional representation \((\Pi, V)\) of a group \(G\), the character of \(\Pi\), denote by \(\Theta_{\Pi}\), is defined by:

\[
\Theta_{\Pi} : G \ni g \rightarrow \text{tr}(\Pi(g)) \in \mathbb{C}.
\]

In general, to determine precisely the character of such representations it’s a hard problem, but in few cases, in particular for a compact connected group, the formula is explicit. Indeed, let \(G\) be a compact connected Lie group, \(T\) a Cartan subgroup of \(G\), \(\mathfrak{g}\) and \(\mathfrak{t}\) the Lie algebras of \(G\) and \(T\) respectively, \(\mathfrak{g}_\mathbb{C}\) and \(\mathfrak{t}_\mathbb{C}\) the complexifications of \(\mathfrak{g}\) and \(\mathfrak{t}\), \(\Phi(\mathfrak{g}_\mathbb{C}, t_\mathbb{C})\) (resp. \(\Phi^+(\mathfrak{g}_\mathbb{C}, t_\mathbb{C})\)) be the set of roots (resp. positive roots) of \(\mathfrak{g}_\mathbb{C}\) with respect to \(t_\mathbb{C}\) and \(\mathcal{W} = \mathcal{W}(\mathfrak{g}_\mathbb{C}, t_\mathbb{C})\) be the corresponding Weyl group. According to H. Weyl, all the irreducible representations \((\Pi, V)\) of \(G\) are finite dimensional and parametrised by a

2010 Mathematics Subject Classification. Primary: 22E45; Secondary: 22E46, 22E30.

Key words and phrases. Howe correspondence, Characters, Oscillator semigroup, Reductive dual pairs.
linear form $\lambda$ on $t_C$: this linear form is called the highest weight of $\Pi$. Moreover, the character of $\Pi$ is given by the following formula:

$$\Theta_\Pi(\exp(x)) = \sum_{\omega \in \mathcal{W}} \text{sgn}(\omega) \frac{e^{\omega(\lambda + \rho)(x)}}{\prod_{\alpha \in \Phi^+(\mathfrak{g}, t_C)} (e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}})} \quad (x \in t^\text{reg}),$$

where $\rho$ is a linear form on $t_C$ given by $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}, t_C)} \alpha$. 

In the 50’s, for a real reductive Lie group $G$, Harish-Chandra extended the concept of characters for a certain class of representation of $G$ called quasi-simple (see [12, Section 10]). More precisely, for such a representation $(\Pi, \mathcal{H})$ of $G$, he proved that the map:

$$\Theta_\Pi : \mathcal{C}_c^\text{reg}(G) \ni \Psi \rightarrow \text{tr}(\Psi(\Pi)) = \int_G \Psi(g)\Pi(g)dg \in \mathbb{C}$$

is well-defined and continuous. The map $\Theta_\Pi$ is called the global character of $\Pi$. Moreover, he proved that this distribution is given by an analytic function, still denoted by $\Theta_\Pi$, on the set of regular points of $G$, i.e.

$$\Theta_\Pi(\Psi) = \int_G \Theta_\Pi(g)\Psi(g)dg \quad (\Psi \in \mathcal{C}_c^\text{reg}(G)).$$

Again, an explicit formula for the function $\Theta_\Pi$ on $G^\text{reg}$ is hard to get. We recall briefly some well known facts on those characters. Let $G$ be reductive group, $K$ be a maximal compact subgroup of $G$ such that $\text{rk}(K) = \text{rk}(G)$ and $T$ be a Cartan subgroup of $K$ (which is also a Cartan subgroup for $G$ by our assumptions on ranks). As before, we denote by $\mathfrak{g}, \mathfrak{t}$ and $\mathfrak{t}$ their Lie algebras and by $\mathfrak{g}_C, \mathfrak{t}_C$ and $\mathfrak{t}_C$ their complexifications.

(1) If $(\Pi, \mathcal{H})$ is a discrete series representation of $G$ of Harish-Chandra parameter $\lambda \in t_C^*$, the character $\Theta_\Pi$ of $\Pi$ is given by (see [13]):

$$\Theta_\Pi(\exp(x)) = (-1)^{\dim(G/K)} \sum_{\omega \in \mathcal{W}(\mathfrak{g}_C, \mathfrak{t}_C)} \varepsilon(\omega) \frac{e^{\omega(\lambda)}}{\prod_{\alpha \in \Phi^+(\mathfrak{g}_C, \mathfrak{t}_C)} (e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}})} \quad (x \in t_C^\text{reg}),$$

where $\mathcal{W}(\mathfrak{t}_C, \mathfrak{t}_C)$ is the compact Weyl group of $K$.

(2) If $(\Pi, \mathcal{H})$ is an irreducible unitary representation of $G$ of highest weight $\lambda - \rho$, the character $\Theta_\Pi$ of $\Pi$ is given by (see [14] Corollary 2.3):

$$\prod_{\alpha \in \Phi^+(\mathfrak{g}_C, \mathfrak{t}_C)\Phi^+(\mathfrak{t}_C, \mathfrak{t}_C)} (e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}})\Theta_\Pi(\exp(x)) = \sum_{\omega \in \mathcal{W}_\lambda^\text{reg}} (-1)^{l(\omega)}\Theta(K, \Lambda(\omega, \lambda))(\exp(x)) \quad (x \in t_C^\text{reg}),$$

where $\mathcal{W}_\lambda^\text{reg}$ is defined in [14] Definition 2.1, and $\Theta(K, \Lambda(\omega, \lambda))(\exp(x))$ is the character of a $K$-representation of highest weight $\Lambda(\omega, \lambda)$, where $\Lambda(\omega, \lambda)$ is defined in [14] Corollary 2.3.

We can also mention a conjecture of A. Kirillov (see [23]), which should hold for a really general Lie group, and a paper of H. Hecht ([26]).

Let $(W, \langle \cdot, \cdot \rangle)$ be a real symplectic space, $\text{Sp}(W)$ its corresponding group of isometries, $\widetilde{\text{Sp}(W)}$ the metaplectic group and $(\omega, \mathcal{H})$ the corresponding metaplectic representation (see Section 2). For a subgroup $H \subseteq \text{Sp}(W)$, we denote by $\widetilde{H}$ its preimage in $\widetilde{\text{Sp}(W)}$ and by $\mathcal{H}(\widetilde{H}, \omega)$ the set of equivalence
classes of irreducible admissible representations of $\widetilde{H}$ which are infinitesimally equivalent to a quotient of $\omega^\infty$. For an irreducible reductive dual pair $(G, G')$ in $\text{Sp}(W)$, R. Howe proved that there exists a bijection between $\mathcal{R}(\tilde{G}, \omega)$ and $\mathcal{R}(\tilde{G}', \omega)$.

In this paper, we assume that $G$ is compact. In that case, it turns out that the situation is easier: the representations $\Pi$ and $\Pi'$ are just subrepresentations of $\omega^\infty$. Our goal here is to determine the character $\Theta_{\Pi'}$ using the character $\Theta_{\Pi}$ of $\Pi$. By projecting on the $\Pi$-isotypic component in $\mathcal{H}^\infty$, we get, for all $\Psi \in \mathcal{C}_c^\infty(\tilde{G})$, that:

$$\Theta_{\Pi'}(\Psi) = \text{tr} \int_{\tilde{G}} \int_{\tilde{G}} \Theta_{\Pi}(g') \Psi(g) \omega^\infty(gg') dg dg'.$$

So, formally, we have:

$$\Theta_{\Pi'}(g') = \int_{\tilde{G}} \Theta_{\Pi}(g) \Theta(gg') dg \quad (g' \in \tilde{G}^{\text{reg}}),$$

where the last equality is in terms of distributions on $\mathcal{C}_c^\infty(\tilde{G})$ and where $\Theta$ is the character of $\omega$ (see Section 2). To avoid the problem of non-continuity of $\Theta$, we use the Oscillator semigroup introduced by Howe (see [2] or Section 3) and denoted by $\text{Sp}(W_\omega)^{++}$. The extension of $\Theta$ on $\text{Sp}(W_\omega)^{++}$ is holomorphic and $\text{Sp}(W_\omega)^{++} \subseteq \text{Sp}(W_\omega)^{++}$. In particular, we get for $g' \in \tilde{G}^{\text{reg}}$ that:

$$\Theta_{\Pi'}(g') = \lim_{p \to 0} \int_{\tilde{G}} \Theta_{\Pi}(g) \Theta(gg') dg,$$

where $G'^{++} = G'_\omega \cap \text{Sp}(W_\omega)^{++}$. The character of the representation $\Pi$ can be obtained using Weyl's character formula (see Equation 1) together with a paper of M. Kashiwara and M. Vergne [17], where they give explicitly the weights of the representations appearing in the correspondence. Moreover, using [24] or [5], we get an explicit formula for the restriction of the character $\Theta$ on $T \cdot T'^{++}$, where $T$ (resp. $T'$) is a compact torus of $G$ (resp. $G'$) and $T'^{++} = T'_\omega \cap \text{Sp}(W_\omega)^{++}$. We focus our attention on the case $(G = U(n, \mathbb{C}), G' = U(p, q, \mathbb{C}))$. More precisely, for $n = 1$, we get the following result (see Proposition 6.4):

$$\Theta_{\Pi'}(T) = \begin{cases} 
\prod_{i=1}^{p+q} t_i^2 \sum_{h=1}^{p} \prod_{j \neq h} (t_h - t_j) t_h^{p-(k+1)} & \text{if } k \leq p - 1 \\
- \prod_{i=1}^{p+q} t_i^2 \sum_{h=p+1}^{p+q} \prod_{j \neq h} (t_h - t_j) t_h^{p-(k+1)} & \text{otherwise}
\end{cases}$$

In Section 7, we work with the pair $(G = U(1, \mathbb{C}), G' = U(1, 1, \mathbb{C}))$ and determine the value of the character $\Theta_{\Pi'}$ on the non-compact torus of $U(1, 1, \mathbb{C})$ (unique up to conjugation, see Section 7). In particular, we verify a result of H. Hecht [26], saying that the value of the character does not depend of the Cartan subgroup.

In Section 8 we recall a conjecture of T. Przebinda (see [7]) concerning the transfer of characters for a general dual pair $(G, G')$ and present in few words an ongoing project linked with recent works of T. Przebinda [6].
Acknowledgements: A part of this paper was done during my thesis at the University of Lorraine under the supervision of Angela Pasquale (University of Lorraine) and Tomasz Przebinda (University of Oklahoma). I would like to thank them for the ideas and time they shared with me. I finished this paper during my stay at the National University of Singapore, as a Research Fellow under the supervision of Hung Yean Loke. I am supported by the grant R-146-000-261-114 (Reductive dual pair correspondences and supercuspidal representations).

2. Metaplectic representation and Howe’s duality theorem

As far as I know, the first construction of this metaplectic representation used the so-called Stone-Von Neumann theorem. Briefly, let \((W, \langle \cdot, \cdot \rangle)\) be a real symplectic space and \(H(W)\) the space \(W \oplus \mathbb{R}\) with group multiplication:

\[
(w_1, \lambda_1), (w_2, \lambda_2) = (w_1 + w_2, \lambda_1 + \lambda_2 + \frac{1}{2} \langle w_1, w_2 \rangle), \quad (w_1, w_2 \in W, \lambda_1, \lambda_2 \in \mathbb{R}).
\]

Clearly, \(\mathcal{D}(H(W)) = \{(0, \lambda), \lambda \in \mathbb{R}\} \approx \mathbb{R}\). According to the Stone- Von Neumann theorem, for every character non trivial character \(\Psi\) of \(\mathcal{D}(H(W))\), there exists, up to equivalence, a unique irreducible unitary representation of \(H(W)\) with central character \(\Psi\). The group of isometries of \((W, \langle \cdot, \cdot \rangle)\), denoted by \(\text{Sp}(W)\), acts naturally on \(H(W)\) by

\[
g(w, \lambda) = (g(w), \lambda) \quad (g \in \text{Sp}(W), w, \lambda \in H(W)).
\]

By fixing an irreducible unitary representation \((\Pi, \mathcal{H})\) of \(H(W)\) with infinitesimal character \(\Psi, \lambda \in \mathbb{R}\), we get that the map:

\[
\Pi, (h) = \Pi(g^{-1}(h)) \quad (g \in \text{Sp}(W), h \in H(W)),
\]

is an irreducible unitary representation of \(H(W)\) with infinitesimal character \(\Psi, \lambda\), and then, by application of the Stone- Von Neumann theorem, there exists an operator \(\omega_{\lambda}(g)\) such that:

\[
\omega_{\lambda}(g)\Pi, (h)\omega_{\lambda}(g)^{-1} = \Pi, (g^{-1}(h)).
\]

In particular, we get a projective representation of \(\text{Sp}(W)\). One can prove that we get a representation \((\omega, \mathcal{H})\) of a non-trivial double cover of \(\text{Sp}(W)\), that we will denote by \(\tilde{\text{Sp}(W)}\) (see [11]).

In this section, we give an explicit realisation of the metaplectic representation (using a paper of A-M. Aubert and T. Przebinda [5]). In particular, we get a formula for the character of this representation (one can also check the paper of T. Thomas [24]).

Let \(\chi\) be the character of \(\mathbb{R}\) given by \(\chi(r) = e^{2\pi r}\). We denote by \(\text{sp}(W)\) the Lie algebra of \(\text{Sp}(W)\), i.e.

\[
\text{sp}(W) = \{X \in \text{End}(W), \langle X(w), w' \rangle + \langle w, X(w') \rangle = 0, (\forall w, w' \in W)\}.
\]

Let \(J\) be an element of \(\text{sp}(W)\) satisfying \(J^2 = -\text{Id}\) and such that the symmetric bilinear form \((w, w')\) defined by \(\langle w, w' \rangle = \langle J(w), w' \rangle\) is positive definite. For all \(g \in \text{Sp}(W)\), we denote by \(J_g\) the automorphism of \(W\) given by \(J_g = J^{-1}(g - 1)\). One can check easily that the adjoint \(J'_g\) of \(J_g\) with respect to the form \((\cdot, \cdot)\) is given by \(J'_g = Jg^{-1}(1 - g)\) and that the restriction of \(J_g\) to \(J_g(W)\) is well defined and invertible. The metaplectic group is defined as:

\[
(2) \quad \tilde{\text{Sp}(W)} = \left\{ g = (g, \xi) \in \text{Sp}(W) \times \mathbb{C}^*, \xi^2 = \frac{\dim_{\mathbb{R}}(W)}{\text{det}(J_g)^{1/2}} \right\}.
\]
The covering map \( \pi : \widetilde{\text{Sp}(W)} \ni (g, \xi) \to g \in \text{Sp}(W) \) is the first projection and the multiplication law is defined by:

\[
(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 C(g_1, g_2)),
\]

where the cocycle \( C : \text{Sp}(W) \times \text{Sp}(W) \to \mathbb{C} \) is defined in \([5\text{, Proposition } 4.13]\). Using \([6\text{, Equation } (3)]\), we get that the absolute value of \( C \) satisfies, for every \( g_1, g_2 \in \text{Sp}(W) \), the following equations:

\[
|C(g_1, g_2)| = \frac{\left| \det(J_{g_1}) \det(J_{g_2}) \right|}{\det(J_{g_1 g_2})},
\]

where \( \text{sgn}(q_{g_1, g_2}) \) is the signature of the form \( q_{g_1, g_2} \) defined by:

\[
q_{g_1, g_2}(u, v) = \frac{1}{2} \left( (c(g_1)u, v) + (c(g_2)u, v) \right) \quad (u, v \in (g_1 - 1)W \cap (g_2 - 1)W).
\]

To simplify the notations, for all \( g \in \text{Sp}(W) \), we denote by \( \chi_{c(g)} \) the form on \( (g - 1)W \) given by \( \chi_{c(g)}(u) = \chi \left( \frac{1}{8} \text{sgn}(c(g)u, u) \right) \).

We now construct the metaplectic representation. We denote by \( S(W) \) the Schwartz space corresponding to \( W \) and by \( \iota : \text{Sp}(W) \to S^*(W), \Theta : \text{Sp}(W) \to \mathbb{C}^* \) and \( T : \text{Sp}(W) \to S^*(W) \) defined by:

\[
\iota(g) = \chi_{c(g)} \mu_{(g-1)W}, \quad \Theta(g) = \xi, \quad T(g) = \Theta(g) \iota(g), \quad \xi = (g, \xi),
\]

where \( \mu_{(g-1)W} \in S^*(W) \) is the Lebesgue measure on the space \( (g - 1)W \) such that the volume with respect to \( (\cdot, \cdot) \) of the corresponding unit cube is 1.

We now fix a complete polarisation \( W = X \oplus Y \), i.e., a direct sum of two maximal isotropic subspaces of \( W \). The Weyl transform \( \mathcal{K} : S(W) \to S(X \times X) \) given by:

\[
\mathcal{K}(\eta)(x, x') = \int_Y \eta(x - x' + y) \chi \left( \frac{1}{2} \langle y, x + x' \rangle \right) dy
\]

is an isomorphism and the extension of \( \mathcal{K} \) to the corresponding space of tempered distributions \( \mathcal{K} : S^*(W) \to S^*(X \times X) \) is still an isomorphism. Similarly, the map \( \text{Op} : S(X \times X) \to \text{Hom}(S(X), S^*(X)) \) given by:

\[
\text{Op}(K)v(x) = \int_X K(x, x')v(x') dx'
\]

extends to isomorphism \( \text{Op} : S^*(X \times X) \to \text{Hom}(S(X), S^*(X)) \). According to \([5\text{, Section } 4.8]\), for every \( \Psi \in L^2(W) \), \( \text{Op} \circ \mathcal{K} \)(\( \Psi \)) is an Hilbert-Schmidt operator on \( L^2(X) \) and the map:

\[
\text{Op} \circ \mathcal{K} : L^2(W) \to \text{HS}(L^2(X))
\]

is an isometry. We denote by \( \omega : \widetilde{\text{Sp}(W)} \to U(L^2(X)) \) defined by:

\[
\omega = \text{Op} \circ \mathcal{K} \circ T
\]

is a unitary representation of \( \widetilde{\text{Sp}(W)} \), called metaplectic representation. Moreover, the function \( \Theta \) defined previously is the character of \( (\omega, L^2(X)) \) and the space of smooth vectors is \( S(X) \), the Schwartz space of \( X \).
Remark 2.1. We denote by $\text{Sp}(W)^c$ the subspace of $\text{Sp}(W)$ defined by $\text{Sp}(W)^c = \{g \in \text{Sp}(W), \det(g - 1) \neq 0\}$: it’s the domain of the Cayley transform of $\text{Sp}(W)$. We denote by $\text{Sp}(\tilde{W})^c$ the preimage of $\text{Sp}(W)^c$ in $\text{Sp}(\tilde{W})$. For every $\tilde{g} = (g, \xi) \in \text{Sp}(\tilde{W})^c$, we get:

$$\Theta(\tilde{g}) = \left( i^{\dim_{\mathbb{R}} W} \det(J(g - 1))^{-1} \right) = \det(i(g - 1))^{-\frac{1}{2}}.$$ 

A dual pair in $\text{Sp}(W)$ is a pair of subgroups $(G, G')$ of $\text{Sp}(W)$ which are mutually centralizer in $\text{Sp}(W)$, i.e. $C_{\text{Sp}(W)}(G) = G'$ and $C_{\text{Sp}(W)}(G') = G$. The dual pair is said to be reductive if the action of $G$ and $G'$ on $W$ is reductive. If we have a decomposition of $W$ as an orthogonal sum $W = W_1 \oplus W_2$ where $W_1$ and $W_2$ are $G \cdot G'$-invariants, then $(G_{W_i}, G'_{W_i})$ is a dual pair in $\text{Sp}(W_{W_i}), i = 1, 2$. If we cannot find such a decomposition, the dual pair is said irreducible.

The irreducible reductive dual pairs in the symplectic group had been classified by R. Howe [4]. In this paper, we assume that the group $G$ is compact. In this case, $(G, G')$ is one of the following dual pairs

1. $(U(n, \mathbb{C}), U(p, q, \mathbb{C})) \subseteq \text{Sp}(2n(p + q), \mathbb{R}),$
2. $(O(n, \mathbb{R}), \text{Sp}(2m, \mathbb{R})) \subseteq \text{Sp}(2nm, \mathbb{R}),$
3. $(U(n, \mathbb{H}), O'(m, \mathbb{H})) \subseteq \text{Sp}(4nm, \mathbb{R}).$

For the computations in the Section 6 we will focus our attention on the first one.

Remark 2.2. (1) For a dual pair $(G, G')$ in $\text{Sp}(W)$, we denote by $\tilde{G} = \pi^{-1}(G)$ and $\tilde{G}'$ the preimages of $G$ and $G'$ in $\text{Sp}(\tilde{W})$. In [4], R. Howe proved that $(\tilde{G}, \tilde{G}')$ is a dual pair in $\text{Sp}(\tilde{W})$. With the precise definition we gave for $\text{Sp}(\tilde{W})$ in Equation (2), we can see that easily. Indeed, we need to prove that for all $g \in G$ and $g' \in G'$, we have $C(g, g') = C(g', g)$. Obviously, $|C(g, g')| = |C(g', g)|$, and because $q_{g, g'} = q_{g', g}$, the result follows.

(2) If the group $G$ is compact, then $\tilde{G}$ is also compact.

From now on, we assume that $(G, G')$ is an irreducible reductive dual pair in $\text{Sp}(W)$ with $G$ compact. The Howe duality theorem can be stated in an easier way when we assume that one member is compact. As before, we consider a complete polarisation of $W$ of the form $X \oplus Y$ and we realise the metaplectic representation $\omega$ on the space $L^2(X)$. The space of smooth vector is the Schwartz space $S(X)$ and under the action of $\tilde{G}$, we get the following decomposition:

$$S(X) = \bigoplus_{(\Pi, V_{\Pi}) \in \tilde{G}_\omega} V(\Pi),$$

where $\tilde{G}_\omega$ is the set of irreducible unitary representations of $\tilde{G}$ such that $\text{Hom}_{\tilde{G}}(\Pi, \omega^{\omega}) \neq \{0\}$ and $V(\Pi)$ is the $\Pi$-isotypic component in $S(X)$, i.e. the closure with respect to the topology on $S(X)$ of the space $\{T(V_{\Pi}), T \in \text{Hom}_{\tilde{G}}(\Pi, \omega^{\omega})\}$.

Because $G'$ commute with $\tilde{G}$, the group $\tilde{G}'$ acts on $V(\Pi)$ for every $\Pi \in \tilde{G}_\omega$, and as a $\tilde{G} \times \tilde{G}'$-module, we get the following decomposition:

$$S(X) = \bigoplus_{(\Pi, V_{\Pi}) \in \tilde{G}_\omega} \Pi \otimes \Pi',$$
where $\Pi'$ is an irreducible unitary representation of $\tilde{G}'$. The map

$$\theta : \tilde{G}_\omega \ni \Pi \rightarrow \Pi' = \theta(\Pi) \rightarrow \tilde{G}_\omega'$$

is one-to-one and usually called Howe’s correspondence.

Let $(\Pi, V_\Pi) \in \hat{\tilde{G}}_\omega$ and $\Pi' = \theta(\Pi)$ the corresponding representation of $\tilde{G}'$. We denote by $\mathcal{P}_\Pi : \mathcal{S}(X) \rightarrow V(\Pi)$ the projection onto the $\Pi$-isotypic component. According to [15, Section 1.4], the map $\mathcal{P}_\Pi$ is given by the formula:

$$\mathcal{P}_\Pi = d_\Pi \int_{\tilde{G}} \Theta_{\Pi(\tilde{g})} \omega^\infty(\tilde{g}) \tilde{d}\tilde{g} = \omega^\infty(d_\Pi \Theta_{\Pi(\tilde{g})}).$$

where $d_\Pi = \dim_{\mathbb{C}}(V_\Pi)$ is the dimension of the representation $\Pi$. We get the following result for the global character of $\Pi'$.

**Proposition 2.3.** For every compactly supported function $\Psi \in \mathcal{C}_c^\infty(\tilde{G}')$, we get:

$$\Theta_{\Pi'}(\Psi) = \text{tr} \left( \int_{\tilde{G}} \left( \int_{\tilde{G}} \Theta_{\Pi(\tilde{g})} \omega^\infty(\tilde{g}\tilde{g}') \tilde{d}\tilde{g} \right) \Psi(\tilde{g}') \tilde{d}\tilde{g}' \right).$$

**Proof.** For such a function $\Psi$, we have:

$$\text{tr}(\mathcal{P}_\Pi \omega(\Psi)) = \text{tr}(\text{Id}_{V_\Pi} \otimes \Pi'(\Psi)) = d_\Pi \Theta_{\Pi'}(\Psi),$$

and then,

$$\Theta_{\Pi'}(\Psi) = \frac{1}{d_\Pi} \text{tr}(\mathcal{P}_\Pi \omega^\infty(\Psi)) = \text{tr} \left( \int_{\tilde{G}} \Psi(\tilde{g}') \mathcal{P}_\Pi \omega^\infty(\tilde{g}') \tilde{d}\tilde{g}' \right) = \text{tr} \left( \int_{\tilde{G}} \left( \int_{\tilde{G}} \Theta_{\Pi(\tilde{g})} \omega^\infty(\tilde{g}\tilde{g}') \tilde{d}\tilde{g} \right) \Psi(\tilde{g}') \tilde{d}\tilde{g}' \right).$$

Using that $\Theta$ is the character of $\omega$, we get formally:

$$\Theta_{\Pi'}(\tilde{g}') = \int_{\tilde{G}} \Theta_{\Pi(\tilde{g})} \Theta(\tilde{g}\tilde{g}') \tilde{d}\tilde{g} \quad (\tilde{g}' \in \tilde{G}').$$

Because the character $\Theta$ is not continuous, the second member of the previous equation could not make sense. To avoid this problem, we use the Oscillator semigroup introduced by R. Howe (see [21]).

### 3. Howe’s Oscillator Semigroup

Let $(W, \langle \cdot, \cdot \rangle)$ be a (finite dimensional) real symplectic vector space and $(W_\mathbb{C}, \langle \cdot, \cdot \rangle)$ its complexification. For every $w \in W_\mathbb{C}$, we consider the decomposition $w = a + ib$, $a, b \in W$ and we denote by $\overline{w} = a - ib$ the conjugate with respect to the decomposition $W_\mathbb{C} = W \oplus iW$. By extension, we get a symplectic form $\langle \cdot, \cdot \rangle$ on $W_\mathbb{C}$.

**Lemma 3.1.** The form $H : W_\mathbb{C} \times W_\mathbb{C} \rightarrow \mathbb{C}$ defined by

$$H(w, w') = i \langle w, \overline{w}' \rangle$$

is hermitian.

**Proof.** Straightforward verification.

\[ \square \]
We define now the subset $\text{Sp}(W_C)^{++}$ of $\text{Sp}(W_C)$ by:

\[(4) \quad \text{Sp}(W_C)^{++} = \{ g \in \text{Sp}(W_C) : H(w, w) > H(g(w), g(w)), (\forall w \in W_C \setminus \{0\}) \}.
\]

Similarly, we denote by $\text{sp}(W_C)^{++}$ the subset of $\text{End}(W_C)$ given by:

\[\text{sp}(W_C)^{++} = \{ z = x + iy ; x, y \in \text{sp}(W), \det(z - 1) \neq 0, \langle yw, w \rangle > 0, w \in W \setminus \{0\} \}.
\]

**Lemma 3.2.** Fix an element $z = x + iy$ with $x, y \in \text{sp}(W)$ such that $\det(z - 1) \neq 0$. Then,

\[H(w, w) > H(c(z)w, c(z)w) \quad (\forall w \in W_C \setminus \{0\})
\]

if and only if

\[\langle yw, w \rangle > 0 \quad (\forall w \in W \setminus \{0\}).
\]

Finally, we obtain $c(\text{sp}(W_C)^{++}) = \text{Sp}(W_C)^{++}$.

**Proof.** Fix $z = x + iy$, with $x, y \in \text{sp}(W)$. We have $H(c(z)w, c(z)w) = H(c(z))^{-1} c(z)w, w)$ and then $H(w, w) > H(c(z)w, c(z)w) \iff H((1 - c(z))^{-1} c(z))w, w)) > 0$. Or,

\[
1 - c(z)^{-1} c(z) = 1 - ((\bar{z} + 1)(\bar{z} - 1)^{-1})(z + 1)(z - 1)^{-1} = 1 - (\bar{z} - 1)(\bar{z} + 1)^{-1}(z + 1)(z - 1)^{-1}
\]

Then, for all $w \in W_C \setminus \{0\}$, we get:

\[
H(w, w) > H(c(z)w, c(z)w) \iff H((1 - c(z))^{-1} c(z))w, w)) > 0 \iff 4i H((\bar{z} + 1)^{-1} y(z - 1)^{-1}, w, w) > 0 \iff -4((\bar{z} + 1)^{-1} y(z - 1)^{-1}, w, w) > 0
\]

with $w' = (z - 1)^{-1} w$. Because $\langle yw', w \rangle \in \mathbb{R}_+$, by writing $w'$ as $w' = w'_1 + iw'_2$, we get:

\[
\langle yw', w \rangle = \langle y(w'_1), w'_1 \rangle + \langle y(w'_2), w'_2 \rangle.
\]

\[\square
\]

**Proposition 3.3.** The set $\text{Sp}(W_C)^{++}$ is a subsemigroup of $\text{Sp}(W_C)$, which does not contain the identity but stable under $g \rightarrow \bar{g}^{-1}$. Moreover, we have

\[(5) \quad \text{Sp}(W_C)^{++}, \text{Sp}(W) = \text{Sp}(W), \text{Sp}(W_C)^{++} \subseteq \text{Sp}(W_C)^{++}
\]

and the set $\text{Sp}(W_C)^{++} \cup \text{Sp}(W)$ is a subsemigroup of $\text{Sp}(W_C)$. To conclude, the symplectic group $\text{Sp}(W)$ is contained in the closure of $\text{Sp}(W_C)^{++}$.

**Proof.** Fix $g$ and $g'$ in $\text{Sp}(W_C)^{++}$. Obviously, $gg' \in \text{Sp}(W_C)$. For every $w \in W_C$, we have:

\[H(gg'w, gg'w) < H(g'w, g'w) < H(w, w),
\]

which imply that $gg' \in \text{Sp}(W_C)^{++}$. The subspace $\text{sp}(W_C)^{++}$ is stable under the map $z \rightarrow -\bar{z}$ and $c(z)^{-1} = c(-\bar{z})$. Then, if $g \in \text{Sp}(W_C)^{++}$, we get $\bar{g}^{-1} \in \text{Sp}(W_C)^{++}$.
Now, fix $g \in \text{Sp}(W_\mathbb{C})^{++}$ and $h \in \text{Sp}(W)$. For all $w \in W_\mathbb{C}$, we have $\overline{h(w)} = h(\bar{w})$ and then:

$$H(gh(w), gh(\bar{w})) < H(h(w), h(\bar{w})) = i(h(w), h(\bar{w})) = i(h(w), h(\bar{w})) = H(w, w).$$

In particular, $gh \in \text{Sp}(W_\mathbb{C})^{++}$. Finally, for every element $g \in \text{Sp}(W)$,

$$g = -c(0)g = \lim_{y \to 0} -c(iy)g$$

which prove that every elements of $\text{Sp}(W)$ is a limit of elements in the semigroup $\text{Sp}(W_\mathbb{C})^{++}$.

□

**Remark 3.4.** Let $z = X + iY$ with $z \in \text{sp}(W_\mathbb{C})^{++}$. Then, for all $w \in W$,

$$\chi_z(w) = e^{\frac{1}{2}w^*J(X+iy)w} = e^{\frac{1}{2}w^*JXw}e^{-\frac{1}{2}w^*JYw}.$$

The matrix $Y \in \text{sp}(W)$, the form $\langle Y, \cdot \rangle$ is positive and $JY$ is symmetric and positive definite. Then, there exists a diagonal matrix $D = \text{diag}(d_1, \ldots, d_{2n})$ and a matrix $O \in O(2n, \mathbb{R})$ such that $JY = O'DO$. So,

$$\int_w \chi_{X+iY}(w)dw = \int_w e^{-\frac{1}{2}w^*JYw}dw = \int_w e^{-\frac{1}{2}w^*O'DOw}dw = \int_w e^{-\frac{1}{2}w^*DY|\det(O)|dy}
= \int_w e^{-\frac{1}{2}w^*\sum_{k=1}^{2n}d_ky_k^2}|\det(O)|dy = \prod_{k=1}^{2n} e^{-\frac{1}{2}d_ky_k^2}dy_k = \prod_{k=1}^{2n} \frac{1}{\sqrt{d_k}} = \det^{-\frac{1}{2}}(D)$$

Using that $|\chi_{X+iY}(w)| = e^{-\frac{1}{2}w^*JYw}$, we get that the integral

$$\int_w \chi_{X+iY}(w)dw$$

is absolutely convergent. More precisely, we get:

$$\int_w \chi_{X+iY}(w)dw = \det^{-\frac{1}{2}}\left(\frac{1}{2}(X + iY)\right).$$

From now on, we denote by $\Lambda(X + iY)$ the previous determinant, i.e.

$$\begin{align}
\Lambda(X + iY) &= \det^{-\frac{1}{2}}\left(\frac{1}{2}(X + iY)\right).
\end{align}$$

(6)

Even if the complex symplectic group $\text{Sp}(W_\mathbb{C})$ is simply connected, the complex manifold $\text{Sp}(W_\mathbb{C})^{++}$ is not simply connected. We define on $\text{Sp}(W_\mathbb{C})^{++}$ a non-trivial cover, denoted by $\tilde{\text{Sp}(W_\mathbb{C})}^{++}$, by

$$\tilde{\text{Sp}(W_\mathbb{C})}^{++} = \{(g, \xi) : g \in \text{Sp}(W_\mathbb{C})^{++}, \xi^2 = \det(i(g - 1))^{-1}\},$$

and let $C : \tilde{\text{Sp}(W_\mathbb{C})}^{++} \times \tilde{\text{Sp}(W_\mathbb{C})}^{++} \to \mathbb{C}$ defined by:

$$C(g_1, g_2) = \det^{-\frac{1}{2}}\left(\frac{1}{2i}(c(g_1) + c(g_2))\right).$$

**Theorem 3.5.** The function $\Theta : \tilde{\text{Sp}(W_\mathbb{C})}^{++} \ni (g, \xi) \to \xi \in \mathbb{C}$ is holomorphic, and we have the following equality:

$$\frac{\Theta(g_1g_2)}{\Theta(g_1)\Theta(g_2)} = C(g_1, g_2) \quad \left(\tilde{g}_1, \tilde{g}_2 \in \tilde{\text{Sp}(W_\mathbb{C})}^{++} \cup \tilde{\text{Sp}(W)}\right).$$

(7)
Moreover, for every functions $\Psi \in C^\infty_c(\widehat{\Sp(W)})$, we get:

\begin{equation}
\int_{\Sp(W)} \Theta(\widehat{g})\Psi(\widehat{g})d\mu_{\Sp(W)}(\widehat{g}) = \lim_{\rho \to 1^-} \int_{\Sp(W)} \Theta(\rho \widehat{g})\Psi(\rho \widehat{g})d\mu_{\Sp(W)}(\widehat{g}).
\end{equation}

**Proof.** We assume that the support of $\Psi$ is contained in the image of $\widehat{\Theta}(0)\widehat{c}$. Then,

\begin{align*}
\int_{\Sp(W)} \Theta(\widehat{g})\Psi(\widehat{g})d\widehat{g} &= \int_{\Sp(W)} \Theta(\rho \widehat{c}(0)\widehat{g})\Psi(\widehat{c}(0)\widehat{g})d\widehat{g} = \int_{\supp \Psi} \Theta(\rho \widehat{c}(0)\widehat{g})\Psi(\widehat{c}(0)\widehat{g})d\widehat{g} \\
&= \int_{\Sp(W)} \Theta(\rho \widehat{c}(0)\widehat{c}(x))\Psi(\widehat{c}(0)\widehat{c}(x))j(x)dx = \int_{\Sp(W)} \Theta(\rho iy\widehat{c}(x))\Psi(\widehat{c}(0)\widehat{c}(x))j(x)dx
\end{align*}

where $\rho \widehat{c}(0) = \widehat{c}(iy)$ with $y \in \Sp(W)$ and $\langle y, \cdot \rangle > 0$. In particular, $y \to 0$ when $\rho \to 1$. But, according to equation (6)

$$\Theta(\rho iy\widehat{c}(x)) = \Theta(\rho iy)\Theta(\widehat{c}(x))\Lambda(x + iy).$$

We denote by $\psi$ the function of $\Sp(W)$ given by $\psi(x) = \Theta(\rho iy)\Theta(\widehat{c}(x))j(x)$ (we notice easily that $\psi \in C^\infty_c(\Sp(W))$). We get:

\begin{align*}
\int_{\Sp(W)} \Lambda(x + iy)\psi(x)dx &= \int_{\Sp(W)} \int_W \chi_{x+iy}\psi(x)dw dx = \int_W \int_{\Sp(W)} \chi_x(w)\chi_{iy}(w)\psi(x)dx dw \\
&= \int_W \chi_{iy}(w) \int_{\Sp(W)} \chi_x(w)\psi(x)dx dw
\end{align*}

and then

\begin{align*}
\int_{\Sp(W)} \chi_x(w)\psi(x)dx &= \int_{\Sp(W)} \psi(x)e^{2\pi iy\tau(w)(x)}dx = \widehat{\psi}\left(\frac{1}{4}\tau(w)\right),
\end{align*}

where $\tau : W \to \Sp(W)^*$ is the moment map and $\widehat{\psi}$ is the Fourier transform of $\psi$ on $\Sp(W)$. Then,

\begin{align*}
\int_{\Sp(W)} \Lambda(x + iy)\psi(x)dx &= \int_W \chi_{iy}(w)\widehat{\psi}\left(\frac{1}{4}\tau(w)\right)dw.
\end{align*}

For all $w \in W \setminus \{0\}$, we have

$$\chi_{iy}(w) = e^{\frac{2\pi i}{4} (iy(w), w)} = e^{-\frac{\pi}{2} (yw, w)} < 1,$$

because $\langle yw, w \rangle > 0$ for every non zero $w \in W$. Finally, we get:

$$\lim_{y \to 0^-} \int_{\Sp(W)} \Lambda(x + iy)\psi(x)dx = \int_W \widehat{\psi}\left(\frac{1}{4}\tau(w)\right)dw.$$

Using that $\lim_{y \to 0^-} \Theta(\rho iy) = \Theta(\rho iy(0))$, we get:

$$\lim_{y \to 0^-} \int_{\Sp(W)} \Theta(\rho iy)\Lambda(x + iy)\psi(x)dx = \int_W \widehat{\psi}\left(\frac{1}{4}\tau(w)\right)dw.$$

Finally, we proved that the limit we considered in Equation (8) exists. Now, we determine this limit. For every $x \in \Sp(W)$, we denote by $B$ the matrix of the bilinear form $\langle x, \cdot \rangle$. We remark that the matrix of the form $\langle J, \cdot \rangle$ is the identity matrix. For all $t > 0$, we have:

$$\Lambda(x + itJ) = \int_W \chi_{x+itJ}dw = \int_W e^{\frac{\pi i}{4} (x+itJ, w)}dw = \int_W e^{\frac{\pi i}{4} (x+itJ, w)}dw.$$
\[
= \int \frac{i}{2} e^{\frac{i}{2} w(B + it) w} dw = \int \frac{i}{2} e^{\frac{-i}{2} w(-B + it) w} dw = \text{det}^{\frac{1}{2}}(1 - iB + tI)
\]

We know that the eigenvalues of \( x \) are real numbers. So, for all \( t > 0 \),

\[
|\text{det}(-iB + tI)| > |\text{det}(iB)|
\]
i.e.

\[
|\text{det}(-iB + tI)| ^{-\frac{1}{2}} < |\text{det}(iB)| ^{-\frac{1}{2}}.
\]

Then,

\[
|\Lambda(x + itJ)| \leq |\Lambda(x)|.
\]

Using that the function \( \Lambda \) is locally integrable, we get:

\[
\lim_{t \to 0^+} \int_{\text{Sp}(W)} \Lambda(x + itJ)\psi(x) dx = \int_{\text{Sp}(W)} \Lambda(x)\psi(x) dx.
\]

\[\square\]

**Remark 3.6.** We first extend the map \( T \) on the semigroup. For every element \( g \in \text{Sp}(W_c)^{++} \), we have \( \text{det}(g - 1) \). We define the map \( T : \text{Sp}(W_c)^{++} \to \text{S}^*(W) \) by

\[
T(g) = (g, \xi) = \Theta(g)\chi_{c(g)}\mu_W.
\]

We denote by \( \text{Cont}(L^2(X)) \) the semigroup of contractions on the Hilbert space \( L^2(X) \). One can prove that:

\[
\omega = \text{Op} \circ \mathcal{K} \circ T : \text{Sp}(W_c)^{++} \to \text{Cont}(L^2(X))
\]

is a semigroup homomorphism. Moreover, for all \( \tilde{p} \in \text{Sp}(W_c)^{++} \), the operator \( \omega(\tilde{p}) \) is of trace class and

\[
\text{tr} \omega(\tilde{p}) = \Theta(\tilde{p}).
\]

4. A general formula for \( \Theta_{\Omega} \)

Let us start this section with comments concerning some particular integrals. As shown in [5, Section 4.8], for all \( \Psi \in \mathcal{C}_c^\infty(\text{Sp}(W)) \),

\[
(9) \quad \int_{\text{Sp}(W)} \Psi(g) T(g)d\bar{g}
\]

is in \( S(W) \). Briefly, for \( \Psi \in \mathcal{C}_c^\infty(\text{Sp}(W)) \) such that \( \text{supp}(\Psi) \subseteq \text{Im}(c) \) and \( \phi \in S(W) \), we have:

\[
\left( \int_{\text{Sp}(W)} \Psi(g) T(g)d\bar{g} \right)(\phi) = \int_{W} \int_{\text{Sp}(W)} \Psi(c(X))\Theta(c(X))\phi(w)j_{\text{sp}}(X)\chi_{1/4}(Xw, w) dX dw
\]

\[
= \int_{W} \left( \int_{\text{sp}(W)} \varphi(X)\chi_{1/4}(Xw, w) dX \right)\phi(w) dw = \int_{W} \tilde{\varphi} \circ \tau_{\text{sp}}(w)\phi(w) dw
\]

where \( \varphi(X) = \Psi(c(X))\Theta(c(X))j_{\text{sp}}(X) \in \mathcal{C}_c^\infty(\text{sp}(W)) \). Then, \( \tilde{\phi} \in S(\text{sp}(W)) \) and \( \lambda(w) = \tilde{\phi} \circ \tau_{\text{sp}}(w) \in S(W) \).
Similarly, for all $\tilde{p} \in \text{Sp}(W_C)^{++}$ and $\Psi \in \mathcal{C}_c^\omega(\text{Sp}(W))$, one can prove that there exists $\lambda_{\tilde{p}} \in S(W)$ such that for every $\phi \in S(W)$, we have:

$$\left( \int_{\text{Sp}(W)} \Psi(\tilde{p})T(\tilde{p}\tilde{g})d\tilde{g} \right)(\phi) = \int_{W} \lambda_{\tilde{p}}(w)\phi(w)dw. \tag{10}$$

The link between the functions $\lambda_{\tilde{p}}$ and $\lambda$ is given by the following equality:

$$\lambda_{\tilde{p}}(w) = T(\tilde{p})\lambda(w) \quad (w \in W). \tag{11}$$

**Lemma 4.1.** For all $\tilde{g} \in \text{Sp}(W)^c$ and $\tilde{h} \in \text{Sp}(W_C)^{++}$, we get:

$$C(g, h)\chi_{c(gh)}(w) = \int_{W} \chi_{c(gh)}(w)\chi\left(\frac{1}{2}\langle u, w \rangle\right)du \quad (\forall w \in W).$$

**Proof.** We have $T(\tilde{g})\chi T(\tilde{h})$, i.e. $T(\tilde{g})\chi T(\tilde{h})\phi = T(\tilde{g})\chi T(\tilde{h})\phi$ for all $\phi \in S(W)$. For all $w \in W$, we have:

$$T(\tilde{g})\chi T(\tilde{h})\phi(w) = \int_{W} \Theta(\tilde{g})\chi_{c(gh)}(w)\chi\left(\frac{1}{2}\langle u, w \rangle\right)du$$

and

$$T(\tilde{g})\chi T(\tilde{h})\phi(w) = T(\tilde{g})\chi T(\tilde{h})\phi(w) = \int_{W} \Theta(\tilde{g})\chi_{c(gh)}(w)\chi\left(\frac{1}{2}\langle u, w \rangle\right)du$$

Then, for all $v, w \in W$, we get:

$$C(g, h)\chi_{c(gh)}(w - v)\chi\left(\frac{1}{2}\langle v, w \rangle\right) = \int_{W} \chi_{c(gh)}(w - v)\chi\left(\frac{1}{2}\langle v, w - u \rangle\right)du.$$
is a Schwartz function $\Phi_{\bar{p}}$ given by:
\[
\Phi_{\bar{p}}(w) = \int_{\text{Sup}(\bar{W})} \Psi(g) \int_{\text{Sup}(\bar{W})} \Phi(h) \Theta(g\bar{h}p)\chi_{c(hp)}(w) d\bar{h}d\bar{g}.
\]

**Proof.** For all $\phi \in \mathcal{S}(W)$, we have:
\[
\left( \int_{\text{Sup}(\bar{W})} \Psi(g) \int_{\text{Sup}(\bar{W})} \Phi(h) T(g\bar{h}p) d\bar{h}d\bar{g} \right)(\phi) = \int_{\text{Sup}(\bar{W})} \Psi(g) \int_{\text{Sup}(\bar{W})} \Phi(h) T(g\bar{h}p)\phi(0) d\bar{h}d\bar{g}
\]
\[
= \int_{\text{Sup}(\bar{W})} \Psi(g) \int_{\text{Sup}(\bar{W})} \Theta(g) \chi_{c(g)}(w) \left( \int_{\text{Sup}(\bar{W})} \Phi(h) T(h\bar{p}) \phi(0) d\bar{h} \right) (-w) dwd\bar{g}
\]
\[
= \int_{\text{Sup}(\bar{W})} \Psi(g) \int_{\text{Sup}(\bar{W})} \Theta(g) \chi_{c(g)}(w) \int_{\text{Sup}(\bar{W})} \Phi(h) \left( \int_{\text{Sup}(\bar{W})} \Theta(h\bar{p}) \chi_{c(hp)}(u) \phi(-w-u) \chi \left( -\frac{1}{2}(u,w) \right) du \right) d\bar{h}dwd\bar{g}
\]
\[
= \int_{\text{Sup}(\bar{W})} \Psi(g) \int_{\text{Sup}(\bar{W})} \Theta(g) \chi_{c(g)}(w) \int_{\text{Sup}(\bar{W})} \Phi(h) \Theta(h\bar{p}) \chi_{c(hp)}(u) \chi \left( \frac{1}{2}(u,w) \right) d\bar{h}dwd\bar{g}
\]
\[
= \int_{\text{Sup}(\bar{W})} \Psi(g) \int_{\text{Sup}(\bar{W})} \Theta(g) \chi_{c(g)}(w) \int_{\text{Sup}(\bar{W})} \Phi(h) \Theta(h\bar{p}) \chi_{c(hp)}(v) d\bar{g}d\bar{h}d\bar{h}
\]
(12)
(\text{where the last equality is obtained using Lemma (4.1)).}
\]
\]

Now, we are able to prove state and prove the following theorem.

**Theorem 4.3.** For every function $\Psi \in \mathcal{C}_0^{\infty}(\bar{G}^t)$, we get:
\[
\Theta_{\bar{p}}(\Psi) = \lim_{\overline{p \rightarrow 1}} \int_{\bar{G}} \Theta_{\bar{p}}(g\bar{g}^{-1}\bar{p})\Psi(g\bar{g}^{-1}\bar{p}) d\bar{g}.
\]

Then, as a distributions on $\bar{G}$, we have:
\[
\Theta_{\bar{p}}(\bar{g}^t) = \lim_{\overline{p \rightarrow 1}} \int_{\bar{G}} \Theta_{\bar{p}}(g\bar{g}^{-1}\bar{p}) d\bar{g}.
\]

**Proof.** According to Proposition [4.2] there exists a function $\lambda_{\bar{p}} \in \mathcal{S}(W)$ such that
\[
\left( \int_{\bar{G}} \Theta_{\bar{p}}(g\bar{g}^{-1}\bar{p}) T(g\bar{g}^{-1}\bar{p})\Psi(g\bar{g}^{-1}\bar{p}) d\bar{g} \right)(\phi) = \int_{W} \lambda_{\bar{p}}(w)\phi(w) dw \quad (\phi \in \mathcal{S}(W)).
\]
Similarly, there exists $\lambda \in \mathcal{S}(W)$ such that
\[
\left( \int_{\bar{G}} \Theta_{\bar{p}}(g\bar{g}^{-1}\bar{p}) T(g\bar{g}^{-1}\bar{p})\Psi(g\bar{g}^{-1}\bar{p}) d\bar{g} \right)(\phi) = \int_{W} \lambda(w)\phi(w) dw \quad (\phi \in \mathcal{S}(W)).
\]
Using Equation (11), for all $w \in W$, we have $\lambda_{\bar{p}}(w) = T(\bar{p})\lambda(w)$.
We have that \( \delta_0 = T(\tilde{1}) = \lim_{\tilde{p} \to 1} T(\tilde{p}) \), and then, using [5, Section 4.5],
\[
\lim_{\tilde{p} \to 1} \lambda_{\tilde{p}}(0) = \lim_{\tilde{p} \to 1} T(\tilde{p})|_{\tilde{\pi}} \lambda(0) = \delta_0 |_{\tilde{\pi}} \lambda(0) = \lambda(0).
\]

Then, using [3, Theorem 3.5.4], we get:
\[
\Theta_{II}(\Psi) = \text{tr} \int_{G'} \int_{G} \Theta_{\Pi}(g) \Psi(g') \omega(gg') d\tilde{g} d\tilde{g}'
= \text{tr} \text{Op} \circ K \int_{G'} \int_{G} \Theta_{\Pi}(g) \Psi(g') T(\tilde{g}g') d\tilde{g} d\tilde{g}'
= \left( \int_{G'} \int_{G} \Theta_{\Pi}(g) \Psi(g') T(\tilde{g}g') d\tilde{g} d\tilde{g}' \right)(0) = \lambda(0)
= \lim_{\tilde{p} \to 1} \lambda_{\tilde{p}}(0) = \lim_{\tilde{p} \to 1} \int_{G} \int_{G} \Theta_{\Pi}(g) \Theta(gg' \tilde{p}) \Psi(g') d\tilde{g} d\tilde{g}'.
\]

From now on, we assume that \( G \) is connected. For every \( \tilde{p} \in \text{Sp}(\mathcal{W}_C)^{++} \) and \( \tilde{g}' \in \tilde{G}' \), we define the function \( F_{\tilde{p}, \tilde{g}'} : \tilde{G} \to \mathbb{C} \) by:
\[
F_{\tilde{p}, \tilde{g}'}(g) = \Theta_{\Pi}(g) \Theta(gg' \tilde{p}).
\]

We easily prove that for every element \( g \in G \), we have:
\[
F_{\tilde{p}, \tilde{g}'}((g, \xi)) = F_{\tilde{p}, \tilde{g}'}((g, -\xi))
\]
and by a standard result of differential geometry (see [16, Lemma A.4.2.11]), we get
\[
\int_{\tilde{G}} F_{\tilde{p}, \tilde{g}'}(g) d\tilde{g} = 2 \int_{G} H_{\tilde{p}, \tilde{g}'}(g) dg,
\]
where \( dg \) is the normalized Haar measure on \( G \) and \( H_{\tilde{p}, \tilde{g}'} : G \to \mathbb{C} \) is the function defined by:
\[
H_{\tilde{p}, \tilde{g}'}(\text{pr}(g)) = F_{\tilde{p}, \tilde{g}'}(g) \quad (g \in \tilde{G}).
\]

From now on, we assume that \( G \) is connected. By Weyl’s integration formula (see [22, Theorem 8.60]), we get:
\[
\int_{G} H_{\tilde{p}, \tilde{g}'}(g) dg = \int_{\mathcal{T}} \left( \int_{G/T} H_{\tilde{p}, \tilde{g}'}(gtg^{-1}) dg \big| D(t) d^2 t \right)
\]
where \( D \) is the Weyl denominator. We define \( G^{++} = G_C \cap \text{Sp}(\mathcal{W}_C)^{++} \) and denote by \( \tilde{G}^{++} \) the preimage in \( \text{Sp}(\mathcal{W}_C)^{++} \). For every element \( \tilde{p} \in \tilde{G}^{++} \), we prove easily that the function \( H_{\tilde{p}, \tilde{g}'} \) in invariant by conjugation. In particular, we get:
\[
\int_{\mathcal{T}} \left( \int_{G/T} H_{\tilde{p}, \tilde{g}'}(gtg^{-1}) dg \big| D(t) d^2 t \right) = \int_{\mathcal{T}} H_{\tilde{p}, \tilde{g}'}(t) \big| D(t) d^2 t
\]
Using Theorem [4.3] we get:
Lemma 4.4. For every regular element \( \tilde{g}' \in \tilde{G}' \), the character \( \Theta_{\Pi'} \) of \( \Pi' \) is given by the following formula:

\[
\Theta_{\Pi'}(\tilde{g}') = \lim_{\tilde{p} \to 1} \tilde{p} \in \tilde{G}'^+ \int_T H_{\tilde{p},\tilde{g}'}(t)|\Delta(t)|^2 dt.
\]

Using an article of M. Kashiwara and M. Vergne [17], we obtain the weights of the representations \( \Pi \in \hat{\tilde{G}}_\omega \). Then, using the Weyl character formula (Equation (1)), we get a formula for the character \( \Theta_{\Pi} \). What we need now, is an explicit realisation of the character \( \Theta \) of the metaplectic representation.

5. A restriction of \( \Theta \) to a maximal compact subgroup

We recall here the main ideas of [8, Section 2]. Here we want an explicit formula for the character \( \Theta \) of the metaplectic representation on a maximal compact subgroup of \( \text{Sp}(W) \). We know that for any positive complex structure \( J \) on \( W \), the subgroup \( \text{Sp}(W)^J \) of symplectic matrices which commute with \( J \) is a maximal compact subgroup of \( \text{Sp}(W) \). More precisely, for every compact dual pair \( (G, G') \) (with \( G \) compact), there exists a complex structure \( J \) of \( \text{Sp}(W) \) such that \( G \cdot T' \subseteq \text{Sp}(W)^J \), where \( T' \) is the maximal compact Cartan subgroup of \( G' \) (we will construct this element \( J \) explicitly for the dual pair \( (U(n, \mathbb{C}), U(p, q, \mathbb{C})) \) in Section 6).

We fix a positive complex structure \( J \) on \( W \), and we denote by \( W_\mathbb{C} \) the complexification of \( W \). With respect to the endomorphism \( J \), we get a decomposition of \( W_\mathbb{C} \) of the form

\[
W_\mathbb{C} = W_+^\mathbb{C} \oplus W_-^\mathbb{C}
\]

where \( W_+^\mathbb{C} \) (resp. \( W_-^\mathbb{C} \)) is the \( i \)-eigenspace (resp. \( -i \)-eigenspace) for \( J \). One can prove easily that the restriction of the form \( H \) defined in Equation (3) to the space \( W_+^\mathbb{C} \) is positive definite. We denote by \( U = U(W_+^\mathbb{C}, H_{|w_\mathbb{C}}) \) the subgroup of \( \text{GL}(W_+^\mathbb{C}) \) which preserve the form \( H_{|w_\mathbb{C}} \).

We define a two fold cover of \( U \), denoted by \( \tilde{U} \), as

\[
\tilde{U} = \{ (u, \xi), \xi^2 = \det(u), u \in U \} \subseteq \text{GL}(W_+^\mathbb{C}) \times \mathbb{C}^*.
\]

Then, \( \tilde{U} \) is a group (endowed with the pointwise multiplication). More precisely, it’s a connected two-fold covering of \( U \).

Proposition 5.1. The map:

\[
\text{Sp}(W)^J \ni g \to g|_{W_+^\mathbb{C}} \in U
\]

is a group isomorphism and lifts to an isomorphism

\[
\tilde{\text{Sp}}(W)^J \ni (g, \xi) \to (u, \xi \det(g - 1)|_{W_+^\mathbb{C}}) \in \tilde{U}.
\]

Then, the restriction of the metaplectic cover to \( \text{Sp}(W)^J \) is isomorphic to the covering

\[
\tilde{U} \ni (u, \xi) \to u \in U
\]

Proof. The proof of this result can be found in [8, Proposition 1].
According to Equation (12), we need a formula for $\Theta$ not only on $\text{Sp}(W)^+$, but on an analogue subset in the oscillator semigroup. Briefly, the map

$$(\text{Sp}(W_+)^+)^J \ni g \rightarrow g_{w,\tilde{c}} \in \text{GL}(W_+^\ast)$$

is well defined and bijective. We now define a subgroup $\text{GL}(W_+^\ast)^+$ of $\text{GL}(W_+^\ast)$ as

$$\text{GL}(W_+^\ast)^+ = \{ h \in \text{GL}(W_+^\ast), H_{w,\tilde{c}}(w, w) > H_{w,\tilde{c}}(hw, hw), 0 \neq w \in W_+^\ast \}$$

As in Equation (14), we define a non-trivial double cover of $\text{GL}(W_+^\ast)^+$ by

$$\widetilde{\text{GL}(W_+^\ast)^+} = \{ (h, \xi) \in \text{GL}(W_+^\ast)^+ \times \mathbb{C}^\ast, \xi^2 = \text{det}(h) \}.$$

The group structure on $\widetilde{\text{GL}(W_+^\ast)^+}$ is given by the coordinate-wise multiplication. More particularly, we get the following proposition.

**Proposition 5.2.** The set $\widetilde{\text{GL}(W_+^\ast)^+}} \cup \widetilde{U}$ is a semigroup. Moreover, the map

$$(\text{Sp}(W_+)^+)^J \cup \text{Sp}(\tilde{W})^J \ni g \rightarrow g_{w,\tilde{c}} \in \widetilde{\text{GL}(W_+^\ast)^+}} \cup \widetilde{U}$$

is a semigroup isomorphism.

The following corollary gives us the character $\Theta$ on the subsemigroup $(\text{Sp}(W_+)^+)^J \cup \text{Sp}(\tilde{W})^J$.

**Corollary 5.3.** The restriction of the character $\Theta$ on the subsemigroup $(\text{Sp}(W_+^\ast)^+)^J \cup \text{Sp}(\tilde{W})^J$ is given by

$$\Theta(\tilde{h}) = \lim_{h \rightarrow \tilde{h}} \Theta(\tilde{h} = (h, \xi)).$$

6. The dual pair $(G = U(n, \mathbb{C}), G' = U(p, q, \mathbb{C}))$

Let $(V, b)$ be a $n$-dimensional vector space over $\mathbb{C}$ endowed with a positive definite hermitian form $b$ and $B$ be a basis of $V$ such that $\text{Mat}(b, B) = \text{Id}_n$. We denote by $U(V, b)$ the group of isometries of $b$, i.e.

$$U(V, b) = \{ g \in \text{GL}(V), b(gu, gv) = b(u, v), (\forall u, v \in V) \}.$$

By writing the endomorphisms in the basis $B$, we get that the right hand side of Equation (16) can be written as:

$$\{ g \in \text{GL}(n, \mathbb{C}), g^*g = \text{Id}_n \},$$

where $g^* = g^{-1}$. We denote by $G = U(n, \mathbb{C})$ the group defined in Equation (17), by $g = u(n, \mathbb{C})$ the Lie algebra of $U(n, \mathbb{C})$. The maximal torus $T$ of $U(n, \mathbb{C})$ is given by $T = \{ \text{diag}(t_1, \ldots, t_n), t_i \in S^1 \}$ and its Lie algebra $t$ is defined as:

$$t = \bigoplus_{k=1}^{n} \mathbb{R} E_{k,k}.$$

One can check (see [22, Chapter II]) that the roots of $g_{\mathbb{C}}$ with respect to $t_{\mathbb{C}}$ are given by:

$$\Phi(g_{\mathbb{C}}, t_{\mathbb{C}}) = \{ \pm(e_i - e_j), 1 \leq i < j \leq n \}.$$
where \( e_k(\text{diag}(h_1, \ldots, h_n)) = h_k \).

Similarly, let \((V', b')\) be a \(p + q\)-dimensional vector space over \( \mathbb{C} \) endowed with a non-degenerate hermitian form \( b' \) of signature \((p, q)\) and let \( \mathcal{B}' \) be a basis of \( V' \) such that \( \text{Mat}(b', \mathcal{B}') = \text{Id}_{p, q} \). We denote by \( U(V', b') \) the group of isometries of \( b' \), i.e.

\[
U(V', b') = \{ g \in \text{GL}(V'), b'(gu, gv) = b'(u, v), (\forall u, v \in V') \},
\]

and by \( U(p, q, \mathbb{C}) \) the following group

\[
\left\{ g \in \text{GL}(p, q, \mathbb{C}), g^* \text{Id}_{p, q} g = \text{Id}_{p, q} \right\}.
\]

Let \( K' = U(p, \mathbb{C}) \times U(q, \mathbb{C}) \) be the maximal compact subgroup of \( G' \).

Using the paper of M. Kashiwara and M. Vergne [17] (we can also use the Appendix of [9]), the weights of the representations of \( \Pi \in U(n, \mathbb{C})_w \) which appears in the correspondence are given by the following formula:

\[
\lambda = \sum_{a=1}^{n} \frac{q - p}{2} e_a - \sum_{a=1}^{r} \nu_a e_{n+1-a} + \sum_{a=1}^{s} \mu_a e_a,
\]

where \( 0 \leq r \leq p, 0 \leq s \leq q, r + s \leq m \), and integers \( \nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_s \) which satisfy \( \nu_1 > \ldots > \nu_r > 0 \) and \( \mu_1 > \ldots > \mu_s > 0 \). The weights \( \lambda \) can also be written as

\[
\lambda = \sum_{a=1}^{m} \left( \frac{q - p}{2} + \lambda_a \right) e_a,
\]

where \( \lambda_i \in \mathbb{Z}, \lambda_1 \geq \ldots \geq \lambda_m \) with at most \( q \) of the integers \( \lambda_i \) are positives and \( p \) negatives.

We easily proved that, for \( G = U(n, \mathbb{C}) \), we have:

\[
\rho = \frac{1}{2} \sum_{a \in \Phi^+(\mathfrak{h}_c, \mathfrak{t}_c)} \alpha = \sum_{a=1}^{n} \frac{n - 2a + 1}{2} e_a.
\]

Using Corollary 5.3, we give a formula for the character \( \Theta \) on \( T \cdot T'^{++} \), where \( T \) and \( T' \) are diagonal Cartan subgroups of \( G \) and \( G' \) respectively and \( T'^{++} = T'_c \cap \text{Sp}(W_c)^{++} \).

**Proposition 6.1.** (1) The set \( T'^{++} \) is given by

\[
T'^{++} = \left\{ \text{diag}(t_1, \ldots, t_{p+q}) : |t_i| < 1 \text{ for } 1 \leq i \leq p, |t_i| > 1 \text{ for } p < i \leq p + q \right\}.
\]

(2) For all \( \overline{t} \in \overline{T} \) and \( \overline{t}' \in T'^{++} \), the character \( \Theta \) is given by:

\[
\Theta(\overline{t}') = \frac{(-1)^{m_0} \prod_{b=1}^{m} \prod_{a=1}^{q+q} t_b^a}{\prod_{a=1}^{p} \prod_{b=1}^{q} (t_b - t') \prod_{a=p+1}^{m} \prod_{b=1}^{q} (t_b - t')^{-a} }.
\]

**Proof.** (1) See Appendix A
Proposition 6.2. For every regular element $\tilde{t}$ in $\tilde{T}'$, we get:

$$\Theta_{\tilde{T}'}(\tilde{t}) = \frac{(-1)^{q+r} w_0}{(2\pi i)^n n!} \lim_{\varepsilon \to 0} \sum_{\alpha \in \mathcal{W}} \text{sgn}(w) \int_{S'} \cdots \int_{S'} \prod_{a=1}^{n} \prod_{b=1}^{n} (t_b - \frac{1}{r_a}) \prod_{k=1}^{n} dt_k$$

This form is symmetric and non-degenerate. Moreover, the map $J(w) = i l_{p,q}w$ is a positive definite complex structure on $W$. The maps

$$G \times W \ni (g, X) \to gX \in W \quad G' \times W \ni (g', X) = X g'^{-1}$$

give embeddings of $G$ and $G'$ into $\text{Sp}(W)$. For every matrix $E_{a,b} \in W$, we have:

$$(tt') E_{a,b} = t_a l_b^{-1} E_{a,b}$$

By definition of $J$, we get:

$$J(E_{a,b}) = \begin{cases} i E_{a,b} & \text{if } 1 \leq a \leq p \\ -i E_{a,b} & \text{if } a > p \end{cases}$$

Then, the eigenvalues of $tt'$ are of the form

$$\{t_a l_b^{-1} : 1 \leq a \leq p, 1 \leq b \leq n \} \cup \{t_p l_b^{-1} : p + 1 \leq a \leq p + q, 1 \leq b \leq n \}.$$  

Finally, using Equation (15), we get:

$$\Theta(t') = \left( \prod_{a=1}^{p} \prod_{b=1}^{n} (1 - t_a l_b^{-1}) \prod_{a=p+1}^{n} (1 - t_a l_b^{-1}) \right)^{\frac{1}{2}}$$

In particular, for every element $\tilde{t}$ of $\tilde{T}'$, we get:

$$\Theta(\tilde{t}) = \left( \prod_{a=1}^{p} (t_a')^{m_a} \prod_{a=p+1}^{n} (t_a')^{m_a} \right)^{\frac{1}{2}}$$

According to Equation (7), we get that:

$$\Theta(t') = \Theta(\tilde{t}) \Theta(\tilde{t}) \Lambda(c(t) + c(t')).$$

The rest of the proof is a straightforward computation. 

\[ \square \]
where $K'(t) = \frac{(-1)^{nq+|a_0|n} \prod_{a_1}^{p+q} \prod_{a=1}^{p+q} |t_b - t_a|}{(2\pi i)^n n!}$.

**Proof.** Using Equation (13), we get that

$$\Theta_{t}(g') = \lim_{p \to \infty} \int_{T} H_{\beta, \gamma}(t) |\Delta(t)|^2 d\mu(t).$$

The torus $T$ of $U(n, \mathbb{C})$ is isomorphic to $S^{1 \otimes n}$. Under this identification, we get:

$$d\mu(t) = \bigotimes_{i=1}^{n} d\mu_{S^1},$$

and $d\mu_{S^1}(z) = \frac{dz}{2\pi i}$. Moreover,

$$D(t = \exp(x)) = \prod_{a>0} (e^{x_a} - e^{-x_a}) = \prod_{1 \leq i < j \leq m} (t_{i}^{a_j} t_{j}^{a_i} - t_{i}^{a_i} t_{j}^{a_j}) \prod_{i=1}^{m} \left( \prod_{1 \leq j < m} (t_j - t_i) \right).$$

and by using the Vandermonde’s determinant formula, we get:

$$\prod_{1 \leq i < j \leq n} (t_j - t_i) = \sum_{\beta \in \mathcal{R}} \text{sgn}(\beta) \prod_{i=1}^{n} t_{i}^{-1}. \tag{25}$$

The rest of the proof is a straightforward computation using Equation (11), Proposition 6.1 and Equation (21).

We now give a technical lemma concerning the integrals which appears in the previous proposition (the proof is obvious using residue theorem).

**Lemma 6.3.** Let $a_1, \ldots, a_p$ be $p$-complex numbers such that $|a_i| < 1$ for all $i \in [1, p]$. Similarly, we consider $a_{p+1}, \ldots, a_{p+q} \in \mathbb{C}$ such that $|a_i| > 1$ for all $i \in [p + 1, p + q]$. Moreover, we assume that $a_i \neq a_j, i \neq j$. Then, we get:

$$\frac{1}{2\pi i} \int_{S^1} \frac{t^k}{\prod_{i=1}^{p+q} (t - a_i)} dt = \begin{cases} \sum_{h=1}^{p} \frac{a^k_{h}}{\prod_{a=1}^{p+q} (a_{h} - a_{a_i})} & \text{if } k \geq 0 \\ -\sum_{h=p+1}^{p+q} \frac{a^k_{h}}{\prod_{a=1}^{p+q} (a_{h} - a_{a_i})} & \text{otherwise} \end{cases}$$

Let’s now fix $n = 1$. In this case, the weights $\lambda$ of the representations $\Pi$ are of the form $\lambda = ke_1$ if $k$ is positive and $\lambda = ke_2$ otherwise. The reason why we voluntarily change the notations here is because
the set of irreducible genuine representations of $\hat{U}(1, \mathbb{C})$ is isomorphic to the unitary dual of $U(1, \mathbb{C})$, which is isomorphic to $\mathbb{Z}$ via the isomorphism:

$$\mathbb{Z} \ni k \to (x \to e^{2\pi ikx}) \in \hat{U}(1, \mathbb{C}).$$

Using Proposition 6.2 and Lemma 6.3 we get the following proposition.

**Proposition 6.4.** The character $\Theta_{\Pi_k}$ of the representation $\Pi_k'$ of $U(p, q, \mathbb{C})$ is given, for every $\vec{t}' \in \bar{T}'$, by:

\[
\Theta_{\Pi_k}(\vec{t}') = \begin{cases} 
\frac{p^p}{\prod_{i=1}^{p+q} t_i^p} \sum_{h=p+1}^{p+q} t_h^{p-(k+1)} & \text{if } k \leq 0 \\
-\prod_{i=1}^{p+q} t_i^{p+q} \sum_{h=p+1}^{p+q} t_h^{p-(k+1)} & \text{otherwise}
\end{cases}
\]

**Notation 6.5.** The Weyl group $\mathcal{W}$ (resp. $\mathcal{W}(t)$) of $G'$ (resp. $K'$) is isomorphic to $\mathcal{S}_{p+q}$ (resp. $\mathcal{S}_p \times \mathcal{S}_p$). For every element $h \in \{1, \ldots, p+q\}$, we denote by $\mathcal{W}^h$ the stabilizer of $h$, i.e.

$$\mathcal{W}^h = \{ \sigma \in \mathcal{W}, \sigma(h) = h \}.$$

We define similarly $\mathcal{W}(t)^h$.

**Proposition 6.6.** We get, up to a constant, the following result:

$$D(X)\Theta_{\Pi_k}(\exp(X)) = \begin{cases} 
\sum_{\mu \in \mathcal{A}^{p+1}} \text{sgn}(\mu) \sum_{\omega \in \mathcal{S}_p \times \mathcal{S}_q} \text{sgn}(\omega)e^{i\omega(\mu, \lambda_1 + \xi(x))} & \text{if } k > p - 1 \\
\sum_{\mu \in \mathcal{A}^{1}} \text{sgn}(\mu) \sum_{\omega \in \mathcal{S}_p \times \mathcal{S}_q} \text{sgn}(\omega)e^{i\omega(\mu, \lambda_2 + \xi(x))} & \text{otherwise}
\end{cases}$$

where

- $\lambda_1 = \sum_{i=1}^{p} (i-1)e_i + (p-(k+1))e_{p+1} + \sum_{i=p+2}^{p+q} (i-2)e_i$,
- $\lambda_2 = (p-(k+1))e_1 + \sum_{a=2}^{p+q} (a-2)e_a$,
- $A^1$ (resp. $A^{p+1}$) is a system of representatives of $\mathcal{W}^1/\mathcal{W}(t)^1$ (resp. $\mathcal{W}^{p+1}/\mathcal{W}(t)^{p+1}$)
- $\xi = \sum_{k=1}^{p+q} \frac{p+q-2}{2} e_k$.

**Proof.** We assume first that $k > p - 1$. According to Equation (24), we have:

$$\prod_{\alpha \in \mathcal{S}_p \times \mathcal{S}_q} (e^{\frac{2\pi i\text{sgn}(\omega)}{p+1}} - e^{-\frac{2\pi i\text{sgn}(\omega)}{p+1}}) = \prod_{i=1}^{m} \frac{p+q}{t_i^{p+q-1}} \prod_{1 \leq j \leq p+q} (t_i - t_j).$$

For all $h \in \{1, \ldots, p+q\}$, we get:

$$\prod_{1 \leq j \leq p+q} (t_i - t_j) \prod_{k \neq h} (t_h - t_k) = (-1)^{h-1} \prod_{i,j \neq h} (t_i - t_j).$$

20
and then
\[
\prod_{\alpha \in \Phi^+_{\infty}(\mathfrak{g}; \mathbb{C})} \left( e^{\frac{i}{2} \alpha} - e^{-\frac{i}{2} \alpha} \right) \Theta_{\mathbb{T}}(\exp(x)) = \prod_{i=1}^{p+q} t_i \prod_{h=p+1}^{p+q} (-1)^{h+1} t_h^{p-h} \prod_{1 \leq i < j \leq p+q} (t_i - t_j).
\]

For all \( h \in \{1, \ldots, p + q\} \), we denote by \( \tilde{t}_h \), \( 1 \leq k \leq p + q - 1 \) the following elements
\[
\tilde{t}_k = \begin{cases} t_k & \text{if } k < h \\ t_{k+1} & \text{otherwise} \end{cases}.
\]

Then, up to a \( \pm 1 \), we get:
\[
\prod_{1 \leq i < j \leq p+q} (t_i - t_j) = \prod_{1 \leq i < j \leq p+q-1} (\tilde{t}_i - \tilde{t}_j) = \sum_{\sigma \in \mathcal{S}_{p+q-1}} \text{sgn}(\sigma) \prod_{a=1}^{p+q-1} \tilde{t}_{\sigma(a)^{-1}} \prod_{a=h}^{p+q} \tilde{t}_{\sigma(a)^{-2}}.
\]

Finally, we prove that:
\[
\sum_{h=p+1}^{p+q} \sum_{\sigma \in \mathcal{S}^h_{p+q}} \text{sgn}(\sigma)(-1)^{h+1} t_h^{p-h} \prod_{a=1}^{h-1} t_{\sigma(a)}^{-1} \prod_{a=h+1}^{p+q} t_{\sigma(a)}^{-2} = (-1)^p \sum_{\mu \in A^{p+1}} \text{sgn}(\mu) \sum_{\omega \in \mathcal{F}_{p+q}} \text{sgn}(\omega) e^{\lambda_\mu(\omega^\lambda(X))}
\]

where \( \lambda_1 = \sum_{i=1}^{p} (i-1)e_i + (p-(k+1))e_{p+1} + \sum_{i=p+2}^{p+q} (i-2)e_i \). The proof is similar if \( k \leq p - 1 \).

\[ \square \]

Recall 6.7. We recall briefly some well-known facts from [10] (see also [25]) concerning the Fourier transform of co-adjoint orbits. To simplify the notations, we assume that \( G \) is a semi-simple connected Lie group such that \( \text{rk}(K) = \text{rk}(G) \), where \( K \) is a maximal compact subgroup of \( G \). We denote by \( \text{Ad}^* \) the natural co-adjoint action of \( G \) on \( \mathfrak{g}^* \). For every \( \lambda \in \mathfrak{g}^* \), we denote by \( G_\lambda \) the \( G \)-orbit associated to \( \lambda \). On the space \( G_\lambda \), we have a natural measure \( d\beta_\lambda \), usually called the Liouville measure on \( G_\lambda \) (see [25, Section 7.5]).

The Fourier transform of \( G_\lambda \), denoted by \( F_{G_\lambda} \), is the generalized function on \( \mathfrak{g} \) defined by:
\[
F_{G_\lambda}(X) = \int_{G_\lambda} e^{if(X)} d\beta_\lambda(f) \quad (X \in \mathfrak{g}).
\]

As proved in [10] Page 217, if \( \lambda \in t^{\text{reg}} \), we have, up to a constant, the following quality:
\[
\left( \prod_{\alpha \in \Phi^+_{\infty}(\mathfrak{g}; \mathbb{C})} \alpha(X) \right) F_{G_\lambda}(X) = \sum_{\omega \in \mathcal{F}_{\{\lambda\}}} \epsilon(\omega) e^{\lambda(\omega(X))} \quad (X \in t^{\text{reg}}).
\]

If the weight \( \lambda \) is not regular, see [17, Theorem 7.24]. To simplify the notations, we denote by \( \pi(X) \) the quantity \( \prod_{\alpha \in \Phi^+_{\infty}(\mathfrak{g}; \mathbb{C})} \alpha(X) \)
Corollary 6.8. Using the notations of Proposition 6.6, we get, up to a constant:

\[ \pi(X)^{-1} D(X) \Theta_{\Pi'}(\exp(X)) = \begin{cases} \sum_{\mu \in A^{p+1}} \text{sgn}(\mu) F_{G_{\mu}(\varphi, \theta)}(X) & \text{if } p < k + 1 \\ \sum_{\mu \in A^1} \text{sgn}(\mu) F_{G_{\mu}(\varphi, \theta)}(X) & \text{if } k < -q + 1 \end{cases} \]

Remark 6.9. (1) For the dual pair \((G = U(1, \mathbb{C}), G' = U(1, \mathbb{C}))\), using the Equation (26), we get:

\[ \Theta_{\Pi'}(t) = t^{r-k-\frac{1}{2}} = \Theta_{\Pi}(t^{-1}). \]

In particular, to be precise, with our method, we don’t get \(\Theta_{\Pi'}(t)\) but \(\Theta_{\Pi'}(t^{-1})\) (or \(\Theta_{\Pi}(t)\) because the representation \(\Pi'\) is unitary): in the embedding of \((G, G')\) in \(\text{Sp}(W)\), \(g' \in G'\) acts on \(w \in W\) as \(g'.w = wg'^{-1}\).

(2) The function \(t \to X \to \pi(X)^{-1} D(X) \in \mathbb{C}\) is well-known in the literature, usually denoted by \(p(x)\) (see [23] or [10]). More particularly, \(p(X)\) can be defined as:

\[ p(X) = \det \left( \frac{\sinh(\text{ad}(X/2))}{\text{ad}(X/2)} \right). \]

7. The case \(G' = U(1, 1, \mathbb{C})\)

As recalled in Equation (19), the unitary group \(U(p, q, \mathbb{C})\) is defined by

\[ U(p, q, \mathbb{C}) = \{ A \in \text{GL}(p + q, \mathbb{C}), A^* \text{Id}_{p,q} A = \text{Id}_{p,q} \}. \]

We fix the convention that \(p \leq q\). In this case, there is, up to conjugaison, \(q + 1\) Cartan subgroups in \(U(p, q, \mathbb{C})\) (see [21]). More precisely, the Cartan subgroups \(H_k, 0 \leq k \leq q\) are of the form \(H_k = H_k^+ H_k^\ast\), where \(H_k^+\) and \(H_k^\ast\) are the subgroup of \(U(p, q, \mathbb{C})\) are defined by

\[ H_k^+ = \{ H_k^+(t_1, \ldots, t_k), t_i \in \mathbb{R}, 1 \leq i \leq k \} \]

where \(H_k^+(t_1, \ldots, t_k)\) is given by

\[ H_k^+(t_1, \ldots, t_k) = \begin{pmatrix} \text{Id}_{p-k} & 0 \\ \text{ch}(t_k) & 0 \\ \text{sh}(t_k) & 0 \\ \text{sh}(t_{k-1}) & \text{ch}(t_{k-1}) \\ \vdots & \vdots \\ \text{ch}(t_1) & \text{sh}(t_1) \\ \text{sh}(t_1) & \text{ch}(t_1) \\ \vdots & \vdots \\ \text{sh}(t_1) & \text{ch}(t_1) \\ 0 & \text{Id}_{q-k} \end{pmatrix} \]

and

\[ H_k = \{ \text{diag}(e^{i\phi_1}, \ldots, e^{i\phi_{p-k}}, e^{i\theta_1}, \ldots, e^{i\theta_{p-k}}, e^{i\tau_1}, \ldots, e^{i\tau_1}), \phi_i, \theta_i, \tau_i \in \mathbb{R} \} \]
We now work with the pair \((G(V, b_V), G(V', b_{V'})) = (U(1, \mathbb{C}), U(1, \mathbb{C})) \subseteq \text{Sp}(W),\) where \(W = (\mathbb{C} \otimes \mathbb{C}^2)_\mathbb{R}\) and \(\langle \cdot, \cdot \rangle = \text{Im}(b_{V'} \otimes b_V)\). We denote by \(H_1\) and \(H_2\) the two Cartan subgroups of \(G'\) (up to conjugation), with \(H_1\) compact.

Let \(\mathcal{B}' = \{v_1, v_2\}\) be a basis of \(V'\) such that:

\[
F = \text{Mat}(\mathcal{B}', b_{V'}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Then,

\[
U(1, 1, \mathbb{C}) = \{g \in \text{GL}(2, \mathbb{C}), g^* F g = F\},
\]

and

\[
u(1, 1, \mathbb{C}) = \{A \in M(2, \mathbb{C}), A F + F A^* = 0\} = \mathbb{R} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.
\]

In particular, we have \(\nu(1, 1, \mathbb{C})_\mathbb{C} = \mathbb{C} \text{Id}_2 \oplus \text{sl}(2, \mathbb{C})\). We fix the \(\text{sl}(2, \mathbb{C})\)-triple \((h, e, f)\) defined by \(h = E_{1,1} - E_{2,2}, e = E_{1,2}, f = E_{2,1}\) and let

\[
C = \exp \left( i \frac{\pi}{4} (e + f) \right).
\]

More particularly, we have:

\[
C = \exp \left( \begin{pmatrix} 0 & \frac{\pi}{4} \\ \frac{\pi}{4} & 0 \end{pmatrix} \right) = \begin{pmatrix} \text{ch}(\frac{\pi}{4}) & \text{sh}(\frac{\pi}{4}) \\ \text{sh}(\frac{\pi}{4}) & \text{ch}(\frac{\pi}{4}) \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{4}) & i \sin(\frac{\pi}{4}) \\ i \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},
\]

so \(C \not\in \nu(1, 1, \mathbb{C})\).

**Lemma 7.1.** Let

\[
b_1 = \mathbb{R} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}
b_2 = \mathbb{R} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

Then, \(H_1 = \exp(b_1)\) and \(H_2 = \exp(b_2)\) are the two non-conjugate Cartan subgroups of \(U(1, 1, \mathbb{C})\) and \(H_1\) is compact. Moreover, we have

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = C \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} C^{-1}
\]

**Remark 7.2.** More particularly, the subgroups \(H_1\) and \(H_2\) are given by

\[
H_1 = \left\{ \begin{pmatrix} e^{i \theta_1} & 0 \\ 0 & e^{i \theta_2} \end{pmatrix}, \theta_1, \theta_2 \in \mathbb{R} \right\}
\]

and

\[
H_2 = \left\{ \begin{pmatrix} e^{i \theta_1} \text{ch}(X) & i \text{sh}(X) \\ -i \text{sh}(X) & e^{i \theta_1} \text{ch}(X) \end{pmatrix}, \theta_1, X \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} e^{i \theta_1} & 0 \\ 0 & e^{i \theta_1} \end{pmatrix}, \theta_1 \in \mathbb{R} \right\} \cdot \left\{ \begin{pmatrix} \text{ch}(X) & i \text{sh}(X) \\ -i \text{sh}(X) & \text{ch}(X) \end{pmatrix}, X \in \mathbb{R} \right\} = T_2 A_2.
\]

The set \(A_2\) is the split part of \(H_2\) (see [15, Section 2.3.6]).

**Proposition 7.3.** For all element \(\tilde{g} \in \widetilde{G}^{++}\), we get:

\[
\Theta(\tilde{g}) = \det(\tilde{g})^2 \det(\tilde{g} - 1)^{-1}.
\]
Proof. Let \( g \in U(1, 1, \mathbb{C}) \) \( \leftrightarrow \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \in \text{Sp}(4, \mathbb{R}) \). We get:

\[
\det_{W_C}(i(g - 1)) = \det_{W_C}(g - 1) = \det_{\text{det}^2}(g - 1) = \det(g - 1) \det(g^{-1} - 1) = \det(g - 1) \det(g^{-1}) \det(1 - g) = \det(g)^{-1} \det(g - 1)^2.
\]

We can now determine \( \Theta_{\Pi_k} \) on \( H_2 \) (more particularly on \( A_2 \)). According to Equation (13), we get for \( X > 0 \):

\[
\Theta_{\Pi_k} \left( \begin{pmatrix} \text{ch}(X) & i \text{sh}(X) \\ -i \text{sh}(X) & \text{ch}(X) \end{pmatrix} \right) = \Theta_{\Pi_k} \left( \exp \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix} \right) = \int_{S^1} z^{-k} \Theta \left( z \exp \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix} \right) dz
\]

\[
= \int_{S^1} z^{-k} \Theta \left( z \exp \left( \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} \right) C^{-1} \right) dz = \int_{S^1} z^{-k} \Theta \left( \begin{pmatrix} ze^{-X} & 0 \\ 0 & ze^X \end{pmatrix} \right) dz
\]

\[
= \int_{S^1} z^{-k+1} \det \left( \begin{pmatrix} ze^{-X} & 0 \\ 0 & ze^X \end{pmatrix} \right)^{-1} dz = \int_{S^1} \frac{ze^{-X}}{(ze^X - 1)(ze^{-X} - 1)} dz
\]

More generally, for all \( \theta \in \mathbb{R} \) and \( X \in \mathbb{R}^+ \), we get:

\[
\Theta_{\Pi_k} \left( \begin{pmatrix} e^{i\theta} \text{ch}(X) & i e^{i\theta} \text{sh}(X) \\ -i e^{i\theta} \text{sh}(X) & e^{i\theta} \text{ch}(X) \end{pmatrix} \right) = \Theta_{\Pi_k} \left( \exp \begin{pmatrix} i\theta & 0 \\ 0 & i\theta \end{pmatrix} + \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix} \right)
\]

\[
= \int_{S^1} z^{-k} \Theta \left( z \exp \begin{pmatrix} i\theta & 0 \\ 0 & i\theta \end{pmatrix} + \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix} \right) dz = \int_{S^1} z^{-k} \Theta \left( z \exp \left( \begin{pmatrix} i\theta & 0 \\ 0 & i\theta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \right) C^{-1} \right) dz
\]

\[
= \int_{S^1} z^{-k} \Theta \left( \begin{pmatrix} ze^{-X} e^\theta & 0 \\ 0 & ze^X e^\theta \end{pmatrix} \right) dz = \int_{S^1} z^{-k+1} e^{i\theta} \det \left( \begin{pmatrix} ze^{-X} e^\theta & 0 \\ 0 & ze^X e^\theta \end{pmatrix} \right)^{-1} dz
\]

\[
= \int_{S^1} \frac{z^{-k+1} e^{i\theta}}{(ze^{i\theta} e^X - 1)(ze^{i\theta} e^{-X} - 1)} dz = \int_{S^1} \frac{e^{i\theta} z^{-k+1}}{(z - e^{-i\theta} e^{-X})(z - e^{i\theta} e^X)} dz = \begin{cases}
\frac{e^{i\theta} e^X - e^{-X}}{e^{i\theta} e^X - e^{-X}} & \text{if } k \geq 1 \\
\frac{e^{i\theta} e^{-X} - e^X}{e^{i\theta} e^{-X} - e^X} & \text{otherwise}
\end{cases}
\]

We got similar results for \( X < 0 \).

8. A Conjecture of T. Przebinda

In [7], T. Przebinda investigated the correspondence of characters for a general dual pair. We recall here the Howe’s duality theorem in this context. Let \((W, \langle \cdot, \cdot \rangle)\) be a symplectic vector space over \( \mathbb{R} \), \( \tilde{\text{Sp}}(\tilde{W}) \) the corresponding metaplectic group, \((\omega, \mathscr{F})\) the metaplectic representation of \( \tilde{\text{Sp}}(\tilde{W}) \), \((G, G')\) be a dual pair in \( \text{Sp}(W) \) and \((\tilde{G}, \tilde{G}')\) the corresponding dual pair in \( \tilde{\text{Sp}}(\tilde{W}) \).
We denote by $\mathcal{R}(\widetilde{G}, \omega)$ the set of equivalence classes of irreducible admissible representations of $\widetilde{G}$ which are infinitisemally equivalent to a quotient of $\omega^\infty$. In [1], R. Howe proved that there exists a bijection between $\mathcal{R}(\widetilde{G}, \omega)$ and $\mathcal{R}(\widetilde{G}', \omega)$ whose graph is $\mathcal{R}(\widetilde{G}, \widetilde{G}', \omega)$. We denote by $\theta$ the following one-to-one map:

$$\theta : \mathcal{R}(\widetilde{G}, \omega) \ni \Pi \mapsto \Pi' = \theta(\Pi) \in \mathcal{R}(\widetilde{G}', \omega).$$

Let’s $(G, G')$ be an irreducible reductive dual pair in $\text{Sp}(W)$. Without loss of generality, we assume that $\text{rk}(G) \leq \text{rk}(G')$. We denote by $\{H_1, \ldots, H_n\}$ the set of conjugacy classes of Cartan subgroups of $G$. As explained in [15 Section 2.3.6], for every $1 \leq i \leq n$, there exists a decomposition of $H_i$ of the form

$$H_i = T_i A_i,$$

where $T_i$ is compact and $A_i$ is the split part of $H_i$ (by convention, $H_1$ is the compact Cartan subgroup, i.e. $A_1 = \{\text{Id}\}$).

For every $1 \leq i \leq n$, we denote by $A_i' = C_{\text{Sp}(W)}(A_i)$ and $A_i'' = C_{\text{Sp}(W)}(A_i')$ (in particular, $A_1' = \text{Sp}(W)$ and $A_1'' = Z(\text{Sp}(W)) = \{\pm 1\}$). Then, $(A_i', A_i'')$ is a reductive dual pair in $\text{Sp}(W)$ (not irreducible in general).

We define a measure $\overline{dw}$ on the quotient $A_i'' \backslash W$ given by:

$$\int_W \phi(w) dw = \int_{A_i'' \backslash W} \int_{A_i'} \phi(a'w) d\mu_{A_i'}(a') \overline{dw}.$$

In [7 Section 2], T. Przebinda define the following distribution on $A_i'$:

$$\text{Chc} = \int_{A_i'' \backslash W} \left( \int_{A_i'} \Psi(\overline{g}) T(\overline{g}) d\overline{g} \right)(w) \overline{dw} \quad (\Psi \in \mathcal{C}_c^\infty(A_i')).$$

As mentioned in Section[4], the integral

$$\int_{A_i'} \Psi(\overline{g}) T(\overline{g}) d\overline{g} \in S(W),$$

in particular, $\text{Chc}(\Psi)$ is well defined. Moreover, for all $\overline{h} \in H_i^{\text{reg}}$, the intersection of the wave front set $\text{WF}(\text{Chc})$ of the distribution $\text{Chc}$ with the conormal bundle of the embedding

$$\widetilde{G}' \ni \overline{g} \mapsto \overline{h} \overline{g} \in A_i'$$

is empty. In particular, there is a unique restriction of the distribution $\text{Chc}$ to $\widetilde{G}'$. We denote by $\text{Chc}_{\overline{h}}$ this restriction. In [7 Conjecture 2.18], T. Przebinda conjectured the following result:

**Conjecture 8.1** (T. Przebinda). We denote by $G_i'$ the connected component at identity of $G'$. We assume that $(\Theta_{\Pi'})_{G_i'} G_i' = 0$. For every $\Psi \in \mathcal{C}_c^\infty(G_i')$, the character $\Theta_{\Pi'}$ of $\Pi'$ is given by:

$$\Theta_{\Pi'}(\Psi) = \mathbf{K}_{\Pi} \sum_{i=1}^{n} \frac{1}{\mu'(H_i)} \int_{H_i^{\text{reg}}} \Theta_{\Pi}(\overline{h}) |D(\overline{h})|^2 \text{Chc}_{\overline{h}}(\Psi) d\overline{h},$$

where $\mathbf{K}_{\Pi}$ is a complex number depending of $\Pi$ (one can check [7 Definition 2.17]).
We now explain how can we get characters by double lifting starting with a compact dual pair. To simplify the notations, we will present that for the dual pair of unitary groups. Let \((G = U(1, \mathbb{C}), G' = U(1, 1, \mathbb{C}))\) in \(\text{Sp}(W_\mathbb{R})\), where \(W_\mathbb{R} = (\mathbb{C} \otimes \mathbb{C} \mathbb{C}^{-1})_\mathbb{R}\) and \((G_1, G'_1) = (U(1, 1, \mathbb{C}), U(m, m + 1))\) in \(\text{Sp}(W'_m)\), where \(W'_m = (\mathbb{C}^{1.1} \otimes \mathbb{C}^{m.1 + m})_\mathbb{R}\). We denote by \((\omega, \mathcal{H})\) the metaplectic representation of \(\text{Sp}(W_\mathbb{R})\), by 

\[
\omega, \mathcal{H}_m
\]

the metaplectic representation of \(\text{Sp}(W'_m)\) and by \(\theta\) and \(\theta_m\) the two bijections

\[
\theta : \mathcal{R}(G, \omega) \rightarrow \mathcal{R}(G', \omega) \quad \theta_m : \mathcal{R}(G_1, \omega_m) \rightarrow \mathcal{R}(G'_1, \omega_m).
\]

**Notation 8.2.** For two positive integers \(r, s\), we denote by \(\det^{r/s} - \text{cov}\) the double cover of \(U(r, s, \mathbb{C})\) given by:

\[
\{(g, s) \in U(r, s, \mathbb{C}) \times \mathbb{C}^*, s^2 = \det(g)^{r/s}\}.
\]

According to [18, Section 1.2], we have:

\[
\tilde{\mathbb{G}} \approx \det^{\frac{1}{2}} - \text{cov}, \quad \tilde{\mathbb{G}}' \approx \det^0 - \text{cov}, \quad \tilde{\mathbb{G}}_1 \approx \det^0 - \text{cov}, \quad \tilde{\mathbb{G}}'_1 \approx \det^{\frac{1}{2}} - \text{cov}.
\]

where \(\det^0 - \text{cov}\) is the trivial cover.

Let \(\Pi \in \mathcal{R}(\tilde{\mathbb{G}}, \omega)\) such that \(\theta(\Pi) \neq \{0\}\). According to a result of J-S Li [20], \(\theta(\Pi) \in \mathcal{R}(\tilde{\mathbb{G}}_1, \omega_m)\) and using the conservation of Kudla [19], \(\theta_m(\theta(\Pi)) \neq \{0\}\).

We now assume that the dual pair \((G_1, G'_1)\) is in the stable range (with \(\text{rk}(G_1) \leq \text{rk}(G'_1)\)).

As explained previously, the weights of the representations \(\Pi\) such that \(\theta(\Pi) \neq \{0\}\) are well-known (see [17]). The distribution character \(\Theta_{\theta(\Pi)}(\Psi)\) is given, for all \(\Psi \in \mathcal{C}_c^\infty(\tilde{G'_1})\), by the following formula:

\[
\Theta_{\theta(\Pi)}(\Psi) = K_\Pi \sum_{i=1}^{2} \frac{1}{|\mathcal{H}_i|} \int_{\mathbb{R}^n} \Theta_{\theta(\Pi)}(\tilde{h}_i)|D(\tilde{h}_i)|^2 \text{Chc}_{\mathbb{R}}(\Psi)d\tilde{h}_i.
\]

where \(\{H_1, H_2\}\) the set of Cartan subgroups of \(G'\) (up to equivalence). The value of the character \(\Theta_{\theta(\Pi)}\) on \(H_1\) can be obtained using the formula given in Proposition 6.4 (it can also be obtained using Enright’s formula (see [14, Corollary 2.3]). The representation \(\theta(\Pi)\) is an irreducible unitary highest weight module. According to H. Hecht’s thesis [26], the formula on the other Cartan subgroups is given by the same formula on the other Cartan subgroups. \(H_2\) can be obtained similarly.

Clearly, the same method can be applied in "higher dimensions".

**Appendix A. The oscillator semigroup for \(U(1, 1, \mathbb{C})\)**

In this appendix, we would like to prove the equality given in Equation 22 of the Proposition 6.1.

We recall here some well-known facts concerning complexifications. Let’s \(V\) be a complex dimension \(n\) endowed with an antilinear involution \(c\) (i.e. \(c^2 = 1\) and \(c(\lambda v) = \overline{\lambda} c(v), \lambda \in \mathbb{C}, v \in V\)). Then, we have the decomposition:

\[
V = \{v \in V, c(v) = v\} \oplus \{v \in V, c(v) = -v\} = \text{Re}(V, c) \oplus \text{Im}(V, c).
\]

Both \(\text{Re}(V, c)\) and \(\text{Im}(V, c)\) are vector spaces over \(\mathbb{R}\) of dimension \(n\). Moreover,

\[
\text{Re}(V, c) \ni v \rightarrow iv \in \text{Im}(V, c)
\]

is well-defined and an isomorphism of \(\mathbb{R}\) vector spaces. We denote by \(V_{\mathbb{R}}\) the vector space given in Equation 28.
Example A.1. Let $V = \mathbb{C}^n$ and $c : V \rightarrow V$ given by $c(v) = \bar{v}$ the natural conjugation on $\mathbb{C}^n$. Then,

$$\mathrm{Re}(V) = \{x \in \mathbb{C}^n, x = \bar{x}\} = \mathbb{R}^n \quad \mathrm{Im}(V) = \{x \in \mathbb{C}^n, x = -\bar{x}\} = i\mathbb{R}^n.$$

On the vector space $V \oplus V$, we define the map $\tilde{c}$ given by

$$\tilde{c} : V \oplus V \ni (u, v) \mapsto (c(v), c(u)) \in V \oplus V$$

is an antilinear involution. As in Equation (23), we define the spaces $\mathrm{Re}(V \oplus V, \tilde{c})$ and $\mathrm{Im}(V \oplus V, \tilde{c})$. In particular, we have:

$$\mathrm{Re}(V \oplus V, \tilde{c}) = \{(v, c(v)), v \in V\}.$$

Moreover, that map:

$$(29) \quad V_{\mathbb{R}} \ni v \mapsto (v, c(v)) \in \mathrm{Re}(V \oplus V, \tilde{c}) \subseteq V \oplus V$$

is an isomorphism of real vector spaces. In particular, the complexification of $V_{\mathbb{R}}$, denoted by $(V_{\mathbb{R}})_C$, is $V \oplus V$.

For all $(x, y) \in V \oplus V$, it's "conjugate" $\tilde{c}(x, y)$ is equal to $(c(y), c(x))$. Moreover, there exists $a, b \in V$ such that

$$(x, y) = (a, c(a)) + (b, -c(b)) \in \mathrm{Re}(V \oplus V, \tilde{c}) \oplus \mathrm{Im}(V \oplus V, \tilde{c}).$$

More particularly, we have $a = \frac{u + c(v)}{2}$ and $b = \frac{u - c(v)}{2}$.

Let’s now assume that the space $V$ is endowed with an hermitian form $b_V$ of signature $(p, q)$ (i.e. there exists a basis $\mathcal{B}_V$ of $V$ such that $F = \text{Mat}(b_V, \mathcal{B}_V) = \text{Id}_{p,q}$). On the space $W = V_{\mathbb{R}}$, we have a natural symplectic form $b$ given by $b = \text{Im}(b_V)$. We denote by $W_C$ the complexification of $W$ and by $b_C$ th corresponding symplectic form on $W_C$. Then, using the identification given in Equation (29), we get for all $w = (w_1, w_2) \in W_C = V \oplus V$, we get:

$$H(w, w) = \mathrm{Im}\left(b_V\left(\frac{w_1 + c(w_2)}{2}, \frac{i(w_1 - c(w_2))}{2}\right)\right)$$

$$= \frac{1}{4} \left(\mathrm{Re}(b_V(w_1, w_1)) - \mathrm{Re}(b_V(w_1, c(w_2))) + \mathrm{Re}(b_V(c(w_2), w_1)) - \mathrm{Re}(b_V(c(w_2), c(w_2))))\right)$$

$$= \frac{1}{4} \left(\mathrm{Re}(b_V(w_1, w_1)) - \mathrm{Re}(b_V(w_1, c(w_2))) + \mathrm{Re}(b_V(c(w_2), w_1)) - \mathrm{Re}(b_V(c(w_2), c(w_2))))\right)$$

$$= \frac{1}{4} \left(\mathrm{Re}(b_V(w_1, w_1)) - \mathrm{Re}(b_V(c(w_2), c(w_2)))\right)$$

For all $g \in \text{Sp}(W_C)^{++}$, we get, according to Equation (4), that:

$$g \in G^{++} \quad \iff \quad H\left(g\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, g\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) < H\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) \quad (\forall (w_1, w_2) \in V \oplus V)$$

$$\iff \quad \mathrm{Re}(b_V(w_1, w_1)) - \mathrm{Re}(b_V(w_2, w_2)) > \mathrm{Re}(b_V(gw_1, gw_1)) - \mathrm{Re}(b_V(g^*w_2, g^*w_2))$$

$$\iff \quad \begin{cases} \mathrm{Re}(b_V(w_1, w_1)) - \mathrm{Re}(b_V(gw_1, gw_1)) > 0 \\ \mathrm{Re}(b_V(w_2, w_2)) - \mathrm{Re}(b_V(g^*w_2, g^*w_2)) < 0 \end{cases}$$
Example A.2. We assume that $V = \mathbb{C}^2$ and that the signature of $b_V$ is $(1, 1)$. The compact torus $T$ is given by:

$$T = \left\{ t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, t_1, t_2 \in U(1, \mathbb{C}) \right\}.$$  

and then, using the notations of Section [7] we have:

$$F - t^2 F < 0 \iff \text{Id}_{1,1} - \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & |t_2|^2 - 1 \end{pmatrix} > 0,$$

i.e. $t \in T^+ \iff |t_1| < 1$ and $|t_2| > 1$.

References

[1] Howe, Roger. Transcending classical invariant theory. *J. Amer. Math. Soc.*, 2(3):535-552, 1989.
[2] Howe, Roger. The oscillator semigroup. In *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, volume 48 of *Proc. Sympos. Pure Math.*, pages 61-132. Amer. Math. Soc., Providence, RI, 1988.
[3] Howe, Roger. Quantum mechanics and partial differential equations. *J. Funct. Anal.*, 38(2):188-254, 1980.
[4] Howe, Roger. Preliminaries I. *Unpublished*.
[5] Aubert, Anne-Marie and Przebinda, Tomasz. A reverse engineering approach to the Weil representation. *Cent. Eur. J. Math.*, 12(10):1500-1585, 2014.
[6] Przebinda, Tomasz. The character and the wave front set correspondence in the stable range. *J. Funct. Anal.*, 274(5):1284-1305, 2018.
[7] Przebinda, Tomasz. A Cauchy Harish-Chandra integral, for a real reductive dual pair *Invent. Math.*, 141(2):299-363, 2000.
[8] T. Przebinda and A. Pasquale and M. McKee. Weyl Calculus and dual pairs. In *arXiv:1405.2431* [math-ph], pages 1-100, 2015.
[9] Przebinda, Tomasz. The duality correspondence of infinitesimal characters. *Colloq. Math.*, 70(1):93-102, 1996.
[10] Rossmann, Wulf. Kirillov’s character formula for reductive Lie groups. *Invent. Math.*, 48(3):207-220, 1978.
[11] Weil, André. Sur certains groupes d’opérateurs unitaires. *Acta Math.*, 111:143-211, 1964.
[12] Harish-Chandra. Representations of semisimple Lie groups. III. *Trans. Amer. Math. Soc.*, 76:234-253, 1954.
[13] Harish-Chandra. Discrete series for semisimple Lie groups. II. Explicit determination of the characters. *Acta Math.*, 116:1-111, 1966.
[14] Enright, Thomas J. Analogues of Kostant’s u-cohomology formulas for unitary highest weight modules. *J. Reine Angew. Math.*, 392:27-36, 1988.
[15] Wallach, Nolan R. *Real reductive groups I*. volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.
[16] Wallach, Nolan R. *Harmonic analysis on homogeneous spaces*. Marcel Dekker, Inc., New York, 1973. Pure and Applied Mathematics, No. 19.
[17] Kashiwara, M. and Vergne, M. On the Segal-Shale-Weil representations and harmonic polynomials. *Invent. Math.*, 44(1):1-47, 1978.
[18] Paul, Annegret. Howe correspondence for real unitary groups. *J. Funct. Anal.*, 159(2):384-431, 1998.
[19] Kudla, Stephen S. On the local theta-correspondence. *Invent. Math.*, 83(2):229-255, 1986.
[20] Li, Jian-Shu. Singular unitary representations of classical groups. *Invent. Math.*, 97(2):237-255, 1989.
[21] Hirai, Takeshi. The Plancherel formula for SU(p, q). *J. Math. Soc. Japan*, 22:134-179, 1970.

[22] Knapp, Anthony W. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[23] Kirillov, A. A. Characters of unitary representations of Lie groups. Reduction theorems. *unkcional. Anal. i Priložen.*, 3(1):36-47, 1969.

[24] Thomas, Teruji. The character of the Weil representation. *Lond. Math. Soc. (2)*, 77(1):221-239, 2008.

[25] Berline, Nicole and Getzler, Ezra and Vergne, Michèle. *Heat kernels and Dirac operators*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.

[26] Hecht, Henryk. The characters of Harish-Chandra representations. ProQuest LLC, Ann Arbor, MI, 1976. Thesis (Ph.D.)-Columbia University.

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, BLOCK S17, 10, LOWER KENT RIDGE ROAD, SINGAPORE 119076, REPUBLIC OF SINGAPORE

E-mail address: matafm@nus.edu.sg