OUT-OF-CORE SINGULAR VALUE DECOMPOSITION

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Abstract. Singular value decomposition (SVD) is a standard matrix factorization technique that produces optimal low-rank approximations of matrices. It has diverse applications, including machine learning, data science and signal processing. However, many common problems involve very large matrices that cannot fit in the main memory of commodity computers, making it impractical to use standard SVD algorithms that assume fast random access or large amounts of space for intermediate calculations. To address this issue, we have implemented an out-of-core (external memory) randomized SVD solution that is fully scalable and efficiently parallelizable. This solution factors both dense and sparse matrices of arbitrarily large size within arbitrarily small memory limits, efficiently using out-of-core storage as needed. It uses an innovative technique for partitioning matrices that lends itself to out-of-core and parallel processing, as well as memory and I/O use planning, automatic load balancing, performance tuning, and makes possible a number of other practical enhancements to the current state-of-the-art. Furthermore, by using persistent external storage (generally HDDs or SSDs), users can resume interrupted operations without having to recalculate previously performed steps, solving a major practical problem in factoring very large matrices.

1. Introduction

Singular value decomposition (SVD) is a widely used matrix factorization technique with broad applications. Given a matrix $A \in \mathbb{R}^{m \times n}$, singular value decomposition consists of a factorization $A = USV^T$, where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal, and $S \in \mathbb{R}^{m \times n}$ is a diagonal matrix. The non-zero values of $S$ are called singular values, while the columns of $U$ and $V$ are conventionally called the left and right singular vectors respectively. Given a matrix of rank $n \in \mathbb{N}$, its best approximation among all matrices of some fixed lower rank $r < n$ is obtained by SVD. Given the factorization $A = USV^T$, we can construct a low-rank approximation $A_r$ of rank $r$ by truncating the diagonal matrix $S$ to rank $r$ and then calculating $A_r = US_r V_r^T$.

In less formal language, SVD is a mechanism for lossy compression. It calculates a smaller matrix which preserves as much of the information in the original matrix as possible. It is often used for data compression, dimensionality reduction, and signal de-noising as well as a growing list of applications in machine learning and artificial intelligence. As SVD applications grow, so does the size of the matrices which we would like to factorize. Consequently, researchers have developed a number of methods for efficiently performing SVD and for approximating SVD factorizations. (For an overview, see Dongarra et al. [1].)

However, an increasingly important limitation on the use of SVD is more practical: Large, useful matrices do not fit into the main fast-access memory of reasonably priced computers, and the growth in the sizes of useful data sets ensures that investing in expensive supercomputers is not a robust solution. Devising efficient, implementable out-of-core algorithms for SVD has become a pressing matter with an active research community that has published various proposals. (See, inter alia, Kabir et al. [2], Haidar et al. [3], Rabani & Toledo [4], Gates et al. [5] and with a more narrow focus on natural language processing, Martin et al. [6].)

None of these approaches, however, are suited to SVD approximations. The only previous research we have identified that addresses out-of-core processing for approximate SVD is Lu, Ino & Matsushita [7], who

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present an out-of-core SVD solution based on the randomized SVD approximation algorithm developed by Halko, Martinsson & Tropp [8]. Their approach, however, uses a graphics card as a fast matrix processor, and treats the memory on the card as “core” and the main memory as “out-of-core.” It is not suited to matrices that cannot even fit into main memory.

Our approach relies on the same randomized SVD approximation from Halko, Martinsson & Tropp as Lu, Ino & Matsushita, among other techniques. However, we present a fully out-of-core solution that assumes matrices are arbitrarily large, but still fit into conventional persistent storage (like SDDs and HDDs), while main memory is arbitrarily small. The ExB SVD Library supports both full SVD factorization and randomized SVD approximation.

The core innovation of the ExB SVD Library is using block cyclic data distribution [9] to perform all matrix operations on sub-matrices. Using this technique, only a small part of any given matrix needs to be in core memory at any one time, and we can partition arbitrarily large matrices into arbitrarily small sub-matrices and move them in and out of memory as needed. Although originally deployed in ScaLAPACK for shared-memory distributed matrix processing [10], we find that block cyclic data distribution also serves well as a basis for out-of-core matrix operations.

We find previous work implementing block-wise out-of-core matrix factorization for LU, QR, and Cholesky methods, but not SVD. [3][11] All previous out-of-core solutions for SVD that we are aware of have assumed that at least individual rows and columns can fit into main memory. Our solution does not have any such limitations.

This solution implements a number of related enhancements and features designed to increase scalability and improve on the state-of-the-art, including:

1. Calculating the amount of memory required for all matrix operations before starting them, and planning memory usage based on those calculations.
2. Multiple precision support for scalar values - single, double, and in part half-precision floating point numbers - and row and column indices of either 32 or 64 bits in sparse matrices. We have implemented direct cross-precision operations for all combinations without preliminary data conversion, except for only partial half-precision support due to hardware limitations.
3. Compact formats for sparse and dense matrices and support for operations combining the two without conversion in memory.
4. Matrix data formats that readily support fast multi-threaded data format conversion. Data conversion at input and output time is particularly fast, and users can choose to maintain all out-of-core matrix data in transportable formats if they wish.
5. Automatic balancing of computational resources.
6. Support for persistence and recoverability, so that users can resume interrupted operations without having to recalculate already completed matrix operations.

This solution is also particularly portable and suited to embedded applications, since it has been designed for to work within fixed, low amounts of memory, and has only a few minor external dependencies, like the standard C++ library, POSIX pthreads, and operating system I/O.

2. Theoretical background

Given a matrix $A \in \mathbb{R}^{m \times n}$, singular value decomposition consists of a factorization $A = USV^T$, where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal, and $S \in \mathbb{R}^{m \times n}$ is a diagonal matrix. The non-zero values of $S$ are called singular values, while the columns of $U$ and $V$ are conventionally called the left and right singular vectors respectively.

2.1. Randomized SVD Approximation. The ExB SVD Library implements the randomized matrix approximation technique for SVD described in Halko, Martinsson and Tropp [8] and Halko [12]. In summary, the algorithm consists of the following steps:
(1) Generate a random Gaussian matrix \( O \in \mathbb{R}^{n \times r} \), where \( r \ll n \). Each entry is generated independently from the distribution \( N(0,1) \).

(2) Set \( Y \in \mathbb{R}^{m \times r} \) such that \( Y := AO \).

(3) Compute the QR-decomposition of \( Y \), i.e., find an orthogonal matrix \( Q \in \mathbb{R}^{m \times m} \) and upper-triangular matrix \( R \in \mathbb{R}^{m \times r} \) such that \( Y = QR \). Note that the last \( m - r \) rows of \( R \) will be zero.

(4) Let \( Q_r \in \mathbb{R}^{m \times r} \) be the sub-matrix of \( Q \) comprising of the first \( r \) columns of \( Q \), removing the zero rows of \( Q \).

(5) Set \( B := Q_r^T A \).

(6) Compute an SVD of \( B \), i.e., find \( U_B \), \( S_B \) and \( V_B \) such that \( B = U_B S_B V_B^T \).

(7) Compute the approximation \( U_r \approx U \in \mathbb{R}^{m \times r} \) by \( U_r = Q_r U_B \).

This algorithm requires us to compute an SVD for matrix \( B \) with \( r \) rows instead of matrix \( A \) with \( n \) rows, which is much easier since \( r \ll n \), and from which we can recover an approximation through a single multiplication. The mathematical basis for this algorithm and its error distribution are discussed at length in Halko, Martinsson and Tropp [8] and Halko [12]. As a brief qualitative demonstration of this algorithm’s approximations, see Section 4.2.

To compute the SVD of \( B \), we have implemented the standard QR-based SVD algorithm [13, Algorithm 8.6.2], which readily lends itself to parallelization.

2.2. Power Matrix Technique. Halko [12, 1.5.3] shows empirically that the accuracy of the randomized SVD approximation algorithm depends on the rate of decay of the singular values of the matrix to which it is applied. If this decay is sufficient, the randomized SVD approximation algorithm will yield adequate results. Otherwise, we apply the power iteration technique described in Halko [12, 1.6.2].

We apply the same randomized SVD algorithm to a modified input matrix \( A_q \in \mathbb{R}^{m \times n} \) such that:

\[
A_q := (AA^T)^q A,
\]

where \( q \) is the iteration parameter, generally a small integer value in the range \( q = 1, \ldots, 5 \).

As demonstrated in Halko [12, 1.6.2], the matrix \( A_q \) has the same singular vectors as \( A \), and furthermore, if \( \sigma \geq 0 \) is a singular value of \( A_q \), then \( \sigma \left( AA^T \right)^{-1} \) is a singular value of \( A \). The choice of iteration parameter \( q \) has a substantial impact on the accuracy – as shown visually in Section 4 – but is dependent on the rate of decay of the singular values of \( A \). We lose accuracy if we set \( q \) to low, and we cannot simply set \( q \) to a constant high value, because this will make the singular values decay too quickly, and they will be lost to the approximation. We can only calculate the optimal iteration parameter empirically for a given matrix.

One important innovation of the ExB SVD Library is that it is able to select the best value of the iteration parameter automatically. We are able to measure the rate of decay of the singular values of matrices \( A_q \in A, A_1, \ldots, A_n \) with increasing \( q \), and identify the value that maximizes accuracy. We also provide for monitoring the average \( L_2 \)-norm per matrix item so that we can stop increasing the iteration parameter when the norm falls below a pre-defined threshold.

Users may also explicitly specify the iteration parameter at run time.

Furthermore, we make significant gains in processing time using the power iteration by optimizing the order of matrix operations. Note that when performing the first step (1) of the randomized SVD algorithm, we need to compute:

\[
A_q := (AA^T)^q AO,
\]

where \( O \in \mathbb{R}^{n \times r} \) is a random Gaussian matrix. Since \( r \ll n \), it is much more efficient to compute this product from right to left, starting by calculating \( AO \) first. When appropriately optimized, computing power iterations is nearly an in-place operation and therefore requires little or no additional memory.
2.3. **Block-wise matrix operations.** Given a pair of matrices \( X, Y \) we can calculate their product \( A := XY \) by partitioning the two matrices into quadrants and calculating the products of the individual blocks:

\[
A = \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}
\begin{pmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{pmatrix}
\]

(3)

Using this procedure, we calculate a single matrix product by calculating eight products of matrices one-quarter the size of \( X \) and \( Y \). The total space in core memory needed to obtain these eight products is one-quarter as large as for the naive matrix multiplication algorithm. The number of scalar operations needed to calculate the product is identical to the naive algorithm.

This approach generalizes to more complex and recursive matrix partitions, and draws on work derived from Dongarra & Walker. \[9\] If taken to the most extreme limit, a block does not need to have more than one value in it, at which point this algorithm becomes identical to ordinary matrix product calculations. There is no other minimum block size and therefore, no specific minimum memory needed to calculate the products of arbitrarily large matrices. We do not even have to keep individual rows or columns entirely in memory.

Just as we can take a block-wise product of two large matrices without keeping more than a small part in core memory at any one time, we can also factor large matrices in a block fashion, as described in D’Azevedo and Dongarra \[11\]. We have implemented all matrix operations in the ExB SVD Library to operate over sub-matrices, so that there is no requirement for a whole matrix to ever reside in memory at any one time.

3. Description of the SVD library

The ExB SVD Library has the following essential features:

1. Support for:
   a. Approximate and exact SVD of arbitrarily large matrices, much larger than the available main memory.
   b. Both sparse and dense matrices, with all matrix operations available for both types without data conversion.
   c. Single, double and, with some limitations, half-precision floating-point matrix values.
   d. An arbitrary number of processing threads.
2. Efficient switching between in-core and out-of-core processing, so that a single library can apply to matrices of all sizes.
3. Automatic balancing of computational resources depending on the task’s scale.
4. Minimized impact of I/O operations when running in out-of-core mode.
5. Persistence and recoverability, so that users can resume interrupted operations without having to recalculate already completed matrix operations.

To the best of our knowledge, these features have never been implemented in an SVD library before.

3.1. **Distributed, parallel block-wise processing.** All matrix operations in the ExB SVD Library are performed over sub-matrices or blocks, as described in Section 2.3. Blocks are specified as ranges of row and column indices in the source matrix. Matrices are partitioned into a number of blocks calculated in function of the memory limits provided by the user.

We have implemented sparse matrices so that binary searches provide fast access to values within a range of indices and can immediately identify block operations that will produce no results because the sparse matrix contains no values in the appropriate range. For dense matrices, we use pointer arithmetic to directly access any value or range of values efficiently from its coordinates. Since accessing the values...
in matrices is thread-safe, we can easily distribute and parallelize matrix operations over as many threads as we can efficiently use, without needing to copy the matrices as long as the threads share memory access to them.

3.2. **Multiple precision arithmetic.** The ExB SVD Library has full support for both single and double precision values, including support for combinations of the two without conversion. We have additionally implemented half-precision values for in-core and out-of-core storage, but these values are converted to single precision when calculations are performed. Half-precision value support enables users to significantly reduce memory and out-of-core space requirements, but do not to save processing time in this implementation. Current commodity CPUs do not support half precision arithmetic directly at the hardware level, but CPUs with the F16C/CVT16 instruction set do support half-precision values as I/O. Our motivation for introducing half-precision in the ExB SVD Library is to enable us to investigate the numerical stability and the quality of results of randomized SVD with half-precision values, so that we can fully support them in GPUs and future CPUs. Our half-precision implementation is fully compatible with the IEEE 754-2008 standard.

3.3. **Dense and sparse matrices.** All operations in the ExB SVD Library have either matrices as arguments, or matrices and scalars. There is no specific class for vectors, which must be constructed as single row or single column matrices. However, there is explicit support for scalar values as a class. They are not treated as $1 \times 1$ matrices.

All matrices are either treated as *sparse* or *dense*, with sparse matrices represented in coordinate (COO) format [14], which is suited especially well for parallel processing. Row and column indices may be 32 or 64 bit values, to enable compact storage of arbitrarily large sparse matrices.

We have directly implemented efficient matrix multiplication for all four combinations of sparse and dense matrices (i.e., $\text{dense} \times \text{dense}$, $\text{dense} \times \text{sparse}$, $\text{sparse} \times \text{dense}$, $\text{sparse} \times \text{sparse}$) so that no matrix needs to ever be converted from sparse to dense or vice-versa. Matrices are also marked as in canonical or transposed form, with direct support for all operations in both cases, so that we do not need to perform memory-consuming transposes during processing and can, in so far as possible, ensure that matrix row and column reads are linear reads on adjacent values in memory.

By providing individual implementations for each of the four class pairs, we can use the maximally efficient algorithm for each. For example, multiplying a very large dense matrix by a very sparse one is proportionate only to the number of non-zero entries in the sparse matrix.

3.4. **Memory and processing time prediction.** Because we know before computing exactly how many non-zero scalar values there are in a sparse matrix, and can trivially calculate the number of scalar values in a dense one from its dimensions, we can estimate the number of scalar operations and memory required for each individual matrix operation in a product of multiple matrices. We can only provide an upper-bounds estimate of the memory and processing required for $\text{sparse} \times \text{sparse}$ products, but for all $\text{sparse} \times \text{dense}$ and $\text{dense} \times \text{dense}$ combinations, the calculations are exact.

Using this information, the ExB SVD Library can plan the order of operations to minimize memory usage and processing time, i.e., if a matrix product $ABC$ is more efficiently computed as $A(BC)$ than as $(AB)C$, we can readily identify this fact and proceed appropriately. We are also able to determine optimal block sizes and predict when we will need to switch to out-of-core processing without having to lose computing resources to monitoring memory usage.

We can furthermore calculate with reasonable accuracy the optimal number of threads to use to perform a matrix operation. We can tune the minimum number of scalar operations a thread must perform to compensate for the fixed computing resources required to create and run a thread, and since we have already estimated the number of scalar operations required to perform some matrix calculation, we can easily calculate the proper number of threads.

We are unaware of any other linear algebra suite that provides this kind of predictive memory and thread management.
3.5. **In-core and out-of-core matrices.** Users control the amount of main memory allocated for a matrix process by setting three parameters:

1. The amount of memory allocated for each matrix individually.
2. The amount of memory allocated for new matrices constructed during processing or as output.
3. A global memory limit for the entire process.

If the parameter in 1 is not set, then the parameter in 2 is treated as if it was also the setting for 1. Similarly, if the parameter in 2 is not specified, then the parameter in 3 is used in its place. If none of the parameters are set, the process is treated as entirely in-core by default with the ExB SVD Library choosing when to use out-of-core processing, unless the user specifies memory parameters at run-time.

This granular control of memory allocation allows users to, for example, specify that some highly used matrix must be kept in-core at all times. If at any stage in processing, any matrix does not fit in the available or specified memory, it is automatically transferred out-of-core. The matrix I/O handler is small and operates in its own low priority thread, minimizing its impact on direct matrix processes in memory.

3.6. **Storage structures.** We store sparse out-of-core matrices in a structure-of-arrays form optimized for fast parallel access. In particular, row and column indices are stored in a separate array from matrix values, as shown in Figure 1. This minimizes mutual data dependencies, which is very important for massive multi-threaded processing. It has the additional benefit that data and indices can be stored in separate files, bypassing operating system limitations on file sizes and file reads.

![Figure 1. Sparse matrix representation](image)

3.7. **Multi-threading.** ExB SVD Library uses the POSIX pthreads execution model to provide efficient process threading. Users can determine the number of threads at run-time, or at any stage during processing, including setting the number of threads used for individual operations or matrices. To avoid inefficiencies caused by threading overhead, there is an internal automatic re-balancing mechanism to prevent the generation of large numbers of threads for small tasks.

All I/O is handled via threaded streams. Using structure-of-arrays and COO formats for sparse matrices makes it possible to read the parts of an out-of-core matrix in multiple parallel threads with little or no I/O latency. There are also facilities in the ExB SVD Library for profiling I/O, and some dynamic buffer size management designed to better balance I/O time and computing time. These enhancements and optimizations enable performance to scale nearly linearly with the number of threads.
3.8. Persistence and recoverability. Among the more innovative features of the ExB SVD Library is its persistence and ability to resume computing after being interrupted. This is a natural extension of out-of-core processing, which keeps intermediate matrices and partial results in persistent stores. Every intermediate matrix, including those produced by sub-matrix operations, are stored with a header indicating their place in the execution plan and relation to other matrices. By recovering those headers, the library can identify which operations have been completed and only calculate those operations not yet completed.

This mechanism makes it possible to recover from external or internal failures, as well interrupting a running operation to re-prioritize tasks or change configurations.

4. Applications and experiments

4.1. Large Matrix Tests. To empirically verify the scaling properties of the ExB SVD Library, we chose two matrices from the SuiteSparse Matrix Collection\(^1\) (formerly the University of Florida Sparse Matrix Collection) [15]:

1. **Hamrle3**, a 1,447,360 × 1,447,360 matrix with 5,514,242 non-zero items
2. **Circuit5M**, a 5,558,326 × 5,558,326 matrix with 59,524,291 non-zero items

If encoded as a dense matrix, the Hamrle3 matrix would occupy more than 15TB of storage and the Circuit5M matrix would use roughly 225TB. Encoded in sparse matrix form, these matrices are small enough that we could fit them into the core memory of the computer we used to perform our experiments. We are therefore able to show a smooth drop in performance as the memory allocated drops below the minimum for entirely in-core processing.

Figure 2 displays the processing times calculating the approximated SVD factorizations of these two matrices, as a function of the memory limit we allocated to it. The experiments were performed on a dedicated system with dual Intel Xeon E5-2640 v4 @2.40GHz CPUs, 512GB RAM as core memory, and a 2TB NVMe-SSD as out-of-core storage. Memory limits were allocated per matrix (see Parameter 1 in Section 3.5) and not as a global memory limit.

In the slowest case - factoring the Circuit5M matrix in 128MB of core memory - the test took roughly one hour.

![Figure 2](https://sparse.tamu.edu/)

**Figure 2.** SVD factorization time as a function of memory limits. The horizontal axis is the memory limit (per matrix), and the vertical axis displays the total processing time in `hours:minutes:seconds`. Different colors represent different stages of processing.

Times in Figure 2 are divided into different parts of the factorization task:

\(^1\)https://sparse.tamu.edu/
(1) **Text Parsing**: The matrices are stored in *Matrix Market Exchange* format\(^2\) [16], which must first be parsed and stored in the *ExB SVD Library* sparse matrix representation. This did not consume a large share of total processing time.

(2) **Preparation of \(O\)**: Generating the Gaussian (random) matrix \(O\) (see Step 1 in Section 2.1). This includes allocating in- and out-of-core space for it, and writing to the out-of-core store.

(3) The next three steps – \(O \cdot A^T \cdot (O A^T)\), Orthogonalization, and \(A^T \cdot Q \cdot r\) (\(A^T Q_r\)) – encompass respectively Steps 2, 3 and 4/5 in Section 2.1.

(4) **SVD0 Preparation**: Before performing SVD calculation in Step 6, we must transpose the matrix \(B^T := A^T Q_r\) to obtain \(B\), and then compute \(B^T B\). Included in the time allotted to this part of the process are some other minor memory operations in preparation for the full SVD calculation.

(5) **SVD0**: We then perform the full SVD calculation. (Step 6 in Section 2.1.)

(6) **Postprocessing**: Finally, we compute approximate \(U, S, V^T\). (Step 7)

In Figure 2, the runs with the most memory allocated to the *ExB SVD Library* are, in both cases, instances of entirely in-core processing. The *Hamrle3* matrix can be processed with an allocation of 4GB and the *Circuit5M* matrix in 16 GB. The *ExB SVD Library* recognizes that with this large an allocation it can perform the entire factorization in memory, and never has to unload data from core memory and reload it from external stores.

In all other cases, out-of-core processing is used for at least some portion of the factorization task. However, the second bar in both charts represents cases where *in fact* the entire matrix resides in memory, but is divided unequally between two blocks in order to use out-of-core processing if needed. Block-wise processing in this case increases run-times significantly in order to switch between the blocks and reassemble the results of operations. Because the blocks are not equal in size, this is relatively inefficient.

The third bar in both charts shows slightly *less* run-time even though less memory has been allocated. In this case, matrices are partitioned into more parts with a better balance in sizes. Although in this case the input matrices remain in memory, processing remains slower than in the case where ample memory is allocated and the library does not prepare for any out-of-core processing.

As the number of matrix parts increases due to a decreasing memory limit, the time spent performing matrix multiplications grows as a share of total processing time, reflecting the degree to which time must be spent switching between blocks and, if the process is out-of-core, performing I/O. This is the principal source of the trade-off between memory allocation and processing time. Unsurprisingly, the most important bottleneck to out-of-core processing is I/O. With the use of fast SSDs, this bottleneck is less important than in the recent past. Moreover, we have made considerable efforts to streamline and optimize I/O in *ExB SVD Library* which has clearly had a positive impact.

Nonetheless, we clearly show that in our implementation, this trade-off is sub-linear. In both charts, the memory allocated to the process reported in the last bar is 128 times smaller than in the first bar, but processing time has increased by a factor of less than 10.

### 4.2. Qualitative performance of randomized SVD

Although our intended use for the *ExB SVD Library* is in artificial intelligence and natural language processing, we present here a test application in image compression to provide a qualitative assessment of *randomized SVD* as an approximation method and of the *power iteration* technique to improve the quality of the result.

We use the standard test image known as “*Lenna*”\(^3\), a 512 × 512 pixel image cropped from a scan of an image published in [17]. We treat this image as an 8-bit greyscale bitmap, structured as a matrix \(A \in \mathbb{R}^{512 \times 512}\), with pixel brightness stored as a single, double, or half precision value, depending on the test setup.

To compress the image, we perform *randomized SVD* approximate factorization on the image matrix \(A = U S V^T\), where \(U, V \in \mathbb{R}^{512 \times 50}\) and \(S \in \mathbb{R}^{50 \times 50}\) is a diagonal matrix. Multiplying the matrices out

\(^2\)[https://math.nist.gov/MatrixMarket/formats.html]

\(^3\)[For the story of this well-known image, see http://www.lenna.org]
produces a $512 \times 512$ matrix that we can convert back into an 8-bit greyscale bitmap and display as a reduced fidelity version of the original image.

These three matrices together contain 51,250 values, with the total number of bytes, and thus total compression, dependent on the precision. The choice of rank 50 is somewhat arbitrary and chosen because at higher rank there was little or no visible difference between the compressed and uncompressed images in a printed medium.

For the purposes of these demonstrations, we performed no pre- or post-processing, re-scaling, or normalization of the source image or compressed form.

![Image reconstruction via low rank SVD with and without power iterations.](image)

**Figure 3.** Image reconstruction via low rank SVD with and without power iterations.

4.2.1. **Power iteration.** Figure 3 is a demonstration of the effectiveness and stability of the randomized SVD approximation algorithm (Section 2.1) and the power iteration technique (Section 2.2) as applied to this image compression task. The quality of the compressed image is notably better using power iteration with an automatically calculated optimal iteration parameter (Figure 3c) than using no power iteration at all (Figure 3b). This result offers support for the empirical observations about this technique in Halko [12].

4.2.2. **Precision and memory usage.** The source image has dimensions of $512 \times 512$ pixels, with one byte of greyscale information per pixel. This amounts to 262,144 bytes of memory to store this as a bitmap. The total number of scalar values in the three matrices used for the rank 50 compressed version is 51,250. The number of bytes required to store these matrices depends on the precision of their values - from 102,500 bytes for half precision, to 410,000 bytes at double precision. Given that the 8-bit images are already massively less precise than even half precision values, there is no need to be concerned about this as an image compression question. However, the choice of precision does have an impact on the output of the randomized SVD approximation algorithm as well as on memory usage during processing.

Figure 4 displays the effect of using different precisions to perform the same rank 50 approximation on the Lenna image. You can see that there is little visible quality difference between the results using different precisions. We see no evidence that reduced precision undermines the randomized SVD approach, and since each reduction in precision from double to single to half reduces the memory required for processing by 50%, the gains from making fuller use of low precisions appear to be worth consideration. Stability with half-precision values indicates that there are large gains in processing speed to be made in providing GPU support for this algorithm, at very little cost in result quality. Robustness under variable precision further suggests that randomized SVD is also robust against noisy inputs.
5. Conclusions

SVD is time-honored and well-established data processing tool. Even though there are already many numerical SVD algorithms, new challenges are arising in connection with large scale machine learning and increasingly big data. Motivated by various applications in artificial intelligence and data science, we have implemented a true out-of-core scalable SVD solution – the ExB SVD Library – to handle very large datasets.

Furthermore, this library can be equally well applied to perform SVD factorization for relatively small matrices on machines with very limited core memory. It efficiently supports a number of practical resource-saving techniques, like variable precision scalar numbers, both sparse and dense matrices, and different levels of approximation in the result.

ExB SVD Library implements a sophisticated memory prediction mechanism that can decompose matrix operations into arbitrarily small tasks, preventing memory exhaustion and allowing permissive task scheduling and prioritization. Among other gains, this means that ExB SVD Library can run efficiently in a docker container\(^4\) or some other virtualized computing environment. It is also very fail-safe, since it preserves partial results and processing can be interrupted and resumed at little cost. These are, to the best of our knowledge, new features that have not previously been implemented in an SVD library.

The automatic selection of the power iterator \(q\) (Section 2.2) is an important advance in numerical methods for SVD, whose effects are visible in Figure 3 when used as an image compressor.

The ExB SVD Library is not a new algorithm for performing SVD, and does not change the fundamental computing requirements for matrix operations, but it does enable users to make efficient trade-offs between computing time, really available memory and storage, and final accuracy. This makes it possible to perform even very large matrix decomposition tasks on systems with very few resources, which has not previously been possible.

To extend the immediate usability of ExB SVD Library, we have provided a Python NumPy-like interface as well as a Java interface, in addition to the native C++ API. The architecture of ExB SVD Library makes it possible to extend it to support computational accelerators such as GPUs and FPGAs.

\(^4\)https://www.docker.com/
We have not provided such support in the current version because the main bottleneck in out-of-core computing is the I/O system. We plan to include support for hardware accelerators in future versions.

The ExB SVD Library relies on the ExB BLAS MLCore Library [18], which is used in other higher-level machine learning task such as clustering, decision tree learning, and approximate and exact kNN indexing. Because of its central role in multiple machine learning tasks, we are continuing work on its further optimization in conjunction with the ExB SVD Library.

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