Penrose Tilings, Chaotic Dynamical Systems and Algebraic $K$-Theory

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Abstract

In this article we initiate the use of noncommutative geometry in the theory of dynamical systems.

After investigating by examples the unusual and striking elementary properties of the Penrose tilings and the Arnold cat map, we associate a finite symbolic dynamics with finite grammar rules to each of them. Instead of studying these Markovian systems with the help of set-topology, which would give only pathological results, a noncommutative approximately finite $C^*$-algebra is associated to both systems. By calculating the $K$-groups of these algebras it is demonstrated that this noncommutative point of view gives a much more appropriate description of the phase space structure of these systems than the usual topological approach.

With these specific examples it is conjectured that the methods of noncommutative geometry could be successfully applied to a wider class of dynamical systems.

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# Contents

## I Introduction

## II Aperiodic tilings

### II.1 Elementary properties of the Penrose tilings

### II.2 Symbolic dynamics associated to the Penrose tilings

## III Chaotic dynamical systems

### III.1 Elementary properties of Arnold’s cat map

### III.2 Symbolic dynamics associated to the cat map

## IV Noncommutative geometrical approach

### IV.1 The $C^*$-algebra associated to factor spaces

#### IV.1.1 The AF $C^*$-algebra of the Penrose tilings

#### IV.1.2 The AF $C^*$-algebra of the cat map

### IV.2 The $K_0$ groups of the algebras

#### IV.2.1 The $K_0$ group associated to the Penrose tilings

#### IV.2.2 The $K_0$ group associated to the cat map

## V Conclusions, outlook

## Addendum

## Acknowledgement

## Appendices

### A Commutative and approximately finite dimensional $C^*$-algebras

#### A.1 $C^*$-algebras

#### A.2 Commutative $C^*$-algebras

#### A.3 Finite dimensional $C^*$-algebras

#### A.4 Approximately finite dimensional $C^*$-algebras

### B $K$-theory of approximately finite dimensional $C^*$-algebras

#### B.1 The topological $K^0$ group

#### B.2 The algebraic $K_0$ group

#### B.3 The scaled dimension group of AF algebras

## References
I Introduction

The most important aim of this work is to initiate the use of the methods of noncommutative geometry [Con94] in the theory of dynamical systems [AA68, CFS82, Wal82].

As a preliminary step, in this article we investigate the properties of two specific, well known and intensively studied systems, which hallmark the above mentioned two areas of mathematics and mathematical physics, and we highlight the deep structural similarity between these systems. The first one is the universe of ‘Penrose tilings’ [Pen74], which appears as one of the introductory examples in the book of A. Connes [Con94], and the second system is Arnold’s cat map, which is one of the simplest basic example for uniformly hyperbolic chaotic systems [AA68]. The connecting bridge between these at first sight so far from each other lying systems is the fact that the structure of both ones can be encoded with symbolic sequences of letters from a finite alphabet, obeying a finite number of grammar rules.

This Markovian property enables us to associate a noncommutative, approximately finite dimensional $C^*$-algebra to these systems, and using one of the powerful weapons from the arsenal of noncommutative geometry, namely the algebraic $K$-theory [WO93, Dav96], we demonstrate that these $C^*$-algebras, indeed, carry important information about the structure of the original systems.

In the second section we set off by exploring the surprisingly rich universe of aperiodic Penrose tilings [GS89]. First the basic properties of the tilings (as local indistinguishability) are summarized and proved with elementary methods. Then the construction of the associated symbolic sequences are explained. It turns out that the topology of the space of all Penrose tilings is rather pathologic considered from the point of view of ordinary set-topology.

In the third section similar investigations are performed for the cat map; first its elementary properties are studied, and then a Markov partition of the phase space is constructed, and the grammar rules of the symbolic dynamics are given explicitly. This symbolic coding clearly reveals the basic similarity between the cat map and the Penrose universe. Particularly, the phase space structure of the cat map, with the embedded stable and unstable manifolds determined by the dynamics, is topologically just as much pathological as the universe of Penrose tilings. This topological defect manifests itself, on the one hand, in the local isometric property of the Penrose tilings, and on the other hand, in the chaotic (uniformly hyperbolic) dynamics of the cat map.

Although the above introduced spaces are ill-behaved as ordinary topological spaces, from the point of view of noncommutative geometry they are very interesting spaces with nontrivial properties. This approach is the subject of the fourth section. First a noncommutative, approximately finite dimensional $C^*$-algebra is associated to the Penrose universe as well as to the cat map, and then an important invariant, the $K_0$ group (with its scale and order structure) is explicitly calculated for both systems. Comparing these results with the elementary properties of the systems discussed in the second and third sections, we see that in both cases these groups do reveal important topological invariants of the original systems considered.

We conclude by expressing the hope that the methods of noncommutative geometry, demonstrated here only for the simplest systems, could also be applied for more difficult cases as not uniformly hyperbolic chaotic systems or symbolic...
sequences with pruning (i.e. with infinitely many grammar rules).

Since the fourth section is technically more demanding than the previous ones, two appendices help to understand the properties of AF $C^*$-algebras as well as the basic concepts and methods of algebraic $K$-theory.
II  Aperiodic tilings

A tiling or tessellation of the Euclidean plane $\mathbb{R}^2$ of type $\mathcal{T} = \{T_1, T_2 \ldots T_n\}$ is an infinite partition of the plane into pieces congruent to one of the prototiles $\{T_i\}_{i=1}^n$ [GS89]. We stress that the set $\mathcal{T}$ of prototiles is always a finite set, the tiles must not overlap and there should not remain any uncovered area of the plane. We do not require strict congruence, i.e., the tiles must have the same shape and size as the prototiles, but they can be reflected. Sometimes it is convenient to color the vertices or to direct the edges of the prototiles and so impose matching conditions for the tessellations investigated. A given tiling is aperiodic if it does not possess translational symmetry.

In the first part of this section we summarize the basic properties of the Penrose tilings [Gar77, Pen78] of type $\mathcal{P} = \{L, S\}$, which were introduced by R. M. Robinson in 1975 (see references in [GS89]), motivated by the work of Penrose [Pen74], and then it is shown how the tessellations of type $\mathcal{P}$ can be encoded with symbolic sequences. This coding scheme is the starting point of the noncommutative geometrical investigations of Section IV.

II.1  Elementary properties of the Penrose tilings

Let us consider the tilings of type $\mathcal{P} = \{L, S\}$, the prototiles of which are two isosceles triangles depicted in Fig. 1a). (The angles are multiples of $\theta = \frac{\pi}{5}$, and the ratio of the length of the edges is the golden mean $\tau = \frac{1 + \sqrt{5}}{2}$.) As it is shown in the figure, there are matching conditions for the vertices and for some of the edges. A finite portion of a possible tiling of type $\mathcal{P}$ is pictured in Fig. 6a).

It is easy to observe that the directed edges of the tiles match only with themselves, so the prototiles $L$ and $S$ always occur in pairs, forming a Kite and
a Dart like figure (Figs. 1.a, 2.a). The matching conditions for the colors also strongly restrict the possibilities. It can easily be verified that in the immediate vicinity of a given edge resp. vertex only seven basically different arrangements occur. These edge resp. vertex neighborhoods are shown in Fig. 2.b)-c) with their fantasy names and symbols.

As a consequence of these restrictions, only the large triangle $L$ can be put beside the (nondirected) edge $\bullet \circ \cdot$ of the small triangle $S$, as it is shown in Fig. 1.b). It means that by composing the tile $S$ together with its neighboring tile $L$ everywhere in a given tiling of type $P$ (i.e., by erasing the edges $\bullet \circ \cdot$ between the tiles $S$ and $L$) a new tessellation of type $P' = \{S', L'\}$ is obtained, where the smaller triangle $S' = L$ has acute angle and the larger one $L' = S \cup L$ has obtuse angle (Fig. 1.b)).

Now the edge $\bullet \circ \cdot$ of $S'$ as well as the edge $\bullet \tau \cdot$ of $L'$ fit only with themselves forming the so called Penrose rhombs (Fig. 1.b)). Keeping this fact in mind it is simple to verify that only the triangle $L'$ can be put beside the (directed) edge $\bullet \tau \cdot$ of $S'$, so the previous composition argument can be repeated. By composing the tiles $S'$ and $L'$ along their common edge $\bullet \tau \cdot$ again a new tessellation of type $P'' = \{S'', L''\}$ is obtained (Fig. 1.c)), where the prototiles are similar to the original prototiles $S, L$ by the ratio $\tau$, and the color coding of the vertices is just reversed.

The inverse process, the decomposition can also be uniquely defined (Fig. 3). Given a tessellation of type $P''$, by dividing every tile of type $L''$ into $S'$ and

---

Figure 2:  
a) The Kite and Dart motifs.  
b) The seven edge neighborhoods.  
c) The seven vertex neighborhoods. (The thin lines denote the result of a double decomposition.)
An even more interesting and shocking property of the Penrose tilings is the fact that although we have considerable freedom in constructing different infinite tessellations of the plane.

$L'$ a new, finer tiling of $\mathcal{P}'$ is obtained, and another decomposition $L' \to S \cup L$ results in a tessellation of type $\mathcal{P}'$, where the prototiles $S$ and $L$ are $\tau$ times larger and larger tessellated patches are obtained the linear measure of which is enlarged by $\tau$ in every step (Fig. 4). Fig 4a demonstrates the result of five successive inflations applied to the prototile $L$.

It is also easy to see that no tiling of type $\mathcal{P}$ can be periodic. Suppose, on the contrary that an infinite tiling possesses a translational symmetry described by the vector $\mathbf{v}$. Then the tessellation obtained by deflation would have a periodicity of $\mathbf{v}/\tau$, since composition preserves the symmetry (edges of the same type are erased), and the reduction scales the symmetry vector by $1/\tau$. A repeated application of the deflation process would yield a tiling of the same type $\mathcal{P}$ with an arbitrarily small symmetry vector $\mathbf{v}/\tau^n$, what is nonsense, since the magnitude of the symmetry vector must be greater than the linear size of the prototiles. Thus the prototiles $\mathcal{P} = \{S, L\}$ (Fig. 4a) admit only aperiodic tessellations of the plane.
tessellations (we shall see it in detail in the next subsection), every possible finite tiling patch $P$ occurs in every infinite tessellation infinitely many times, and the ratio of appearance of $P$ is fixed. This ratio depends only on the finite patch $P$ itself, and not on the infinite tiling investigated (Statement 5). The possibly overlapping occurrences of $P$ are distinguished by the position of a preferred prototile of $P$, so the frequency of appearance is defined precisely as follows.

**Definition 1.** The number of appearance $N_P(T)$ of a finite patch $P$ (with a preferred prototile $p$ in it) in a finite tiling $T$ of type $\mathcal{P}$ is the number of prototiles $t$ in $T$, in the neighborhood of which the motif $P$ appears in such a way that the preferred tile $p$ of $P$ coincides with $t$ of $T$. The frequency or ratio of appearance $\kappa_P(T)$ of the patch $P$ in the finite tiling $T$ is the ratio $\kappa_P(T) = \frac{N_P(T)}{N(T)}$, where $N(T)$ is the number of all prototiles (of both types) in $T$. For infinite tilings $I$ the frequency of appearance of $P$ is defined by the limit $\kappa_P = \lim_{T \to I} \kappa_P(T)$, where $T$ is a finite portion of $I$ that extends to the whole tessellation $I$.

It is easy to see that the number $N_P(T)$ depends neither on the type nor on the exact choice of the preferred prototile $p$ in the patch $P$. We shall see later on that the ratio $\kappa_P$ does not depend either on the infinite tiling $I$ investigated, that is the reason why $I$ is not involved in the notation. The limit $T \to I$ can be made in any ‘reasonable way’, i.e., the finite tesselation $T$ should extend ‘uniformly in all directions’ to infinity.

As a preparation for the main Statement 1 we investigate in three lemmas the ratios of appearance of the prototiles (Fig. 1 (a)), the edge- as well as the vertex neighborhoods (Fig. 1 (b)-(c)).

**Lemma 2. (Frequency of appearance of prototiles)** In every infinite tiling of type $\mathcal{P}$ the ratio of appearance of the prototiles $L$ resp. $S$ is

\[
\kappa_L = \tau - 1, \quad \text{resp.} \quad \kappa_S = 2 - \tau. \tag{1}
\]

**Proof.** First we calculate the ratio $\lambda = \frac{\kappa_L(T)}{\kappa_S(T)} = \frac{N_L(T)}{N_S(T)}$ of the number $N_L(T)$, $N_S(T)$ of the prototiles $L$ and $S$ in finite patches obtained by repeated inflations.
of a given finite patch $T$, and then we argue that this ratio has to be the same in every infinite tiling as well. Indeed, let $N_L = N_\kappa_L = N_{\frac{1}{1+\lambda}}$ resp. $N_S = N_\kappa_S = N_{\frac{1}{1+\tau}}$ be the number of prototiles $L$ resp. $S$ in the finite patch $T$ of $N$ tiles. (The argument $T$ of $N$, $N_L$ and $N_S$ has been omitted for simplicity.) Since in every double decomposition step of the inflations the triangles $L''$ resp. $S''$ are subdivided into two samples of $L$ and one of $S$ resp. one sample of $L$ and $S$ (Fig. 3)), the number $N_L', N_S'$ of the prototiles $L, S$ after an inflation process is related to the initial values via the formulas $N_L' = 2N_L + N_S, N_S' = N_L + N_S$, which give the recursion $\lambda_{n+1} = \frac{2\lambda_n}{\lambda_n + 1}$ for the ratio $\lambda_n$ of the number of the prototiles $L$ and $S$ after the $n$-th inflation process. It is easy to see that for any nonnegative initial value $\lambda_0$ the series $\lambda_n$ converges to the limit $\lambda_{n \to \infty} = (1 + \sqrt{5})/2 = \tau$ golden mean.

This observation helps us to prove that the ratio of the two prototiles have to be the same number $\tau$ in any infinite tiling of type $P$. Indeed, by applying the double composition sufficiently many times, we obtain an (infinite) tiling of huge triangles $S''''''$ and $L''''''$, in which the ratio of the original small prototiles $S$ and $L$ can be made to be arbitrarily close to $\tau$. It means that (using any reasonable definition for the limit process $T \to I$) the ratio of the prototiles in every infinite Penrose tiling $I$ has the same value $\tau$, and $\kappa_L = \frac{1}{1+\lambda} = (-1 + \sqrt{5})/2 = \tau - 1$, $\kappa_S = \frac{1}{1+\tau} = (3 - \sqrt{5})/2 = 2 - \tau$, $\kappa_{\tau'} = -3 + 2\tau$. (2c)

In the next lemma we investigate the frequencies of appearance of the seven edge neighborhoods (Fig. 3, b).

**Lemma 3. (Frequency of appearance of edge neighborhoods)** In every infinite tiling of type $P$ the ratios of appearance of the seven edge neighborhoods $k, k', d, t, t', r, r'$ (Fig. 3, b)) are

$$
\begin{align*}
\kappa_k &= -1 + \tau, & \kappa_{k'} &= -6 + 4\tau, & \kappa_d &= 2 - \tau, \\
\kappa_t &= 5 - 3\tau, & \kappa_{t'} &= 2 - \tau, & \kappa_r &= -3 + 2\tau, \\
\text{resp.} & & \kappa_{r'} &= -3 + 2\tau.
\end{align*}
$$

It is worth noting that all these frequencies have the form $a + b\tau$, where $a, b \in \mathbb{Z}$ are integers!

**Proof.** In this proof it is more convenient to use the notions ‘number of appearance’ and ‘frequency of appearance’ in a bit modified way, i.e., referring to the occurrences of the edge neighborhoods $e$ by the position of their preferred (middle) edge, instead of using a preferred tile. These altered quantities are distinguished by a ‘hat’ (’), so $\hat{N}_e(T)$ denotes the number of inner edges in the finite tessellation $T$ with edge neighborhood $e \in \{k, k', d, t, t', r, r'\}$, and $\hat{\kappa}_e(T) = \frac{\hat{N}_e(T)}{\hat{N}(T)}$ is the ratio of appearance of $e$, in proportion to the number $\hat{N}(T)$ of all edges in $T$. (Of course, the edges lying at the boundary of the finite tessellation $T$ have no well defined edge neighborhood, but their number is negligible compared to the number of all edges, as $T \to I$ extends to infinity.)

There is, however, a simple connection between the above defined quantities $\hat{N}_e(T)$, $\hat{N}(T)$ resp. $\hat{\kappa}_e$ and the general notions $N_e(T)$, $N(T)$ resp. $\kappa_e$ of Definition 4. All prototiles have three edges, and every edge belongs to two tiles, thus $\lim_{T \to I} \frac{\hat{S}(T)}{\hat{N}(T)} = \frac{1}{2}$, i.e., in an infinite tessellation the number of edges is half as
much again as the number of tiles. For edge neighborhoods \( e \) with reflectional symmetry the preferred tile of \( e \) can be put to both sides of the preferred edge, so in this case \( N_e(T) = 2\hat{N}_e(T) \) and

\[
\kappa_e = \lim_{T \to t} \frac{N_e(T)}{N(T)} = 3 \lim_{T \to t} \frac{\hat{N}_e(T)}{N_e(T)} = 3\hat{\kappa}_e, \quad \text{if } e \in \{k, k', d, r, r'\}. \tag{3a}
\]

For the two other edge neighborhoods without symmetry \( N_e(T) = \hat{N}_e(T) \), thus

\[
\kappa_e = \lim_{T \to t} \frac{N_e(T)}{N(T)} = 3 \lim_{T \to t} \frac{\hat{N}_e(T)}{N_e(T)} = \frac{3}{2} \hat{\kappa}_e, \quad \text{if } e \in \{t, t'\}. \tag{3b}
\]

We go on likewise in the previous lemma, and investigate how the numbers \( \hat{N}_e(T) \) of the seven edge neighborhoods \( e \in \{k, k', d, t, t', r, r'\} \) change under successive inflations of the finite tessellation \( T \). For example the middle edge \( o-o \) in the \textit{kite} motif is divided into two edges with \( r \) and \( r' \) edge neighborhoods, and there are four new edges created inside the tiles with edge neighborhoods \( k, k', l, l' \) (Fig 3b). To avoid over-counting these four new edges have to be counted by one third, since they were created inside a triangular prototile, so they are considered three times with the three edges of the triangle. Thus the transformation of an edge neighborhood \( k \) under an inflation process is described schematically

\[
k \rightarrow r + r' + \frac{1}{3}(2k + 2l'). \tag{4}
\]

The transformations of the other edge neighborhoods can be similarly derived, and the final result can be expressed in a matrix equation

\[
\begin{bmatrix}
\hat{N}_k' \\
\hat{N}_{k'}' \\
\hat{N}_d' \\
\hat{N}_{d'}' \\
\hat{N}_t' \\
\hat{N}_{t'}' \\
\hat{N}_r' \\
\hat{N}_{r'}'
\end{bmatrix} = \begin{bmatrix}
2/3 & 2/3 & 1 & 1/3 & 1/3 & 0 & 2/3 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
2/3 & 2/3 & 2/3 & 2/3 & 2/3 & 2/3 & 2/3 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\hat{N}_k \\
\hat{N}_{k'} \\
\hat{N}_d \\
\hat{N}_{d'} \\
\hat{N}_t \\
\hat{N}_{t'} \\
\hat{N}_r \\
\hat{N}_{r'}
\end{bmatrix} \tag{5}
\]

relating the numbers \( \hat{N}_e' \) of the different edge neighborhoods \( e \) after the inflation to their values \( \hat{N}_e = \hat{N}_e(T) \) before the inflation. (The arguments \( T \) have been omitted for simplicity.)

The characteristic polynomial of the matrix above is \( \lambda(\lambda^2 + 1)(9\lambda^2 + 6\lambda + 2)(\lambda^2 - 3\lambda + 1) \), with roots \( \lambda_1 = 0, \lambda_{2,3} = \pm i, \lambda_{4,5} = (-1 \pm i)/3 \) and \( \lambda_{6,7} = (3 \pm \sqrt{5})/2 \). The eigenvalue with the largest magnitude is the real number \( \lambda_7 = (3 + \sqrt{5})/2 \), thus its eigenvector \( \hat{\kappa} \) describes the ratios of appearance \( \hat{\kappa}_e \) belonging to the different edge neighborhoods \( e \) in the limit of infinitely many inflations applied to the finite tessellation \( T \). The components of \( \hat{\kappa} \) are:

\[
\hat{\kappa}_k = \frac{-1 + \tau}{3}, \quad \hat{\kappa}_{k'} = \frac{-6 + 4\tau}{3}, \quad \hat{\kappa}_d = \frac{2 - \tau}{3}, \tag{6a}
\]

\[
\hat{\kappa}_r = \frac{10 - 6\tau}{3}, \quad \hat{\kappa}_{r'} = \frac{4 - 2\tau}{3}, \quad \hat{\kappa}_t = \frac{-3 + 2\tau}{3}, \quad \hat{\kappa}_{t'} = \frac{-3 + 2\tau}{3}. \tag{6b}
\]

\[
\hat{\kappa}_l = \frac{-3 + 2\tau}{3}. \tag{6c}
\]
(These ratios are normalized, i.e., \( \sum_e \hat{\kappa}_e = 1 \).)

This result can be extended to infinite tilings using the same arguments as in the previous proof. Indeed, after sufficiently many (double) compositions, the infinite tiling consists of huge triangles of type \( L'' \ldots' \), \( S'' \ldots' \), and increasing the number of compositions applied (thus the size of the huge triangles) the frequency of appearance of the different edge neighborhoods in each of the finite triangles \( L'' \ldots' \), \( S'' \ldots' \) can be brought arbitrarily close to the ideal values \( \hat{\kappa}_e \).

Using the connections \( \hat{\kappa}_e \) the statements \( \hat{\kappa}_e \) of Lemma \( \hat{\kappa}_e \) are obtained. \( \square \)

For proving similar assertions for arbitrary finite patches of tiling we need to investigate the ratio of occurrence of the seven vertex neighborhoods (Fig. \( \hat{\kappa}_e \)).

Lemma 4. (Frequency of appearance of vertex neighborhoods) In every infinite tiling of type \( P \) the ratios of appearance of the seven vertex neighborhoods \( \odot, \ast, A, D, J, Q \) resp. \( K \) (Fig. \( \hat{\kappa}_e \)) are

\[
\begin{align*}
\hat{\kappa}_\odot &= -11 + 7\tau, & \hat{\kappa}_\ast &= -29 + 18\tau, & \hat{\kappa}_A &= 2 - \tau, \\
\hat{\kappa}_D &= 3 + 2\tau, & \hat{\kappa}_J &= 5 - 3\tau, & \hat{\kappa}_Q &= -8 + 5\tau \\
\hat{\kappa}_K &= 13 - 8\tau.
\end{align*}
\]

(7a) \( \hat{\kappa}_D \) \( \hat{\kappa}_J \) \( \hat{\kappa}_Q \) \( \hat{\kappa}_K \) \( \hat{\kappa}_A \) \( \hat{\kappa}_\ast \) \( \hat{\kappa}_\odot \)

Please notice the ‘miracle’ that all these frequencies are again elements of the dense subgroup \( \mathbb{Z} + \tau \mathbb{Z} \subset \mathbb{R} \) of the additive real group!

Proof. In this proof we have to use the notions ‘number of appearance’ \( \hat{N}_V(T) \) and ‘frequency of appearance’ \( \hat{\kappa}_V(T) \) of a vertex neighborhood \( V \in \{ \odot, \ast, A, D, J, Q, K \} \) in a finite tiling \( T \) again in a bit altered way. For vertex neighborhoods it is more convenient to label their appearances with the position of a preferred vertex, namely the middle vertex, than with the place of a preferred prototile of them. To every vertex inside \( T \) there corresponds a well defined vertex neighborhood, so let \( \hat{N}_V(T) \) be the number of inner vertices of \( T \) with vertex neighborhood \( V \), and let \( \hat{\kappa}_V(T) = \frac{\hat{N}_V(T)}{N(T)} \), where \( \hat{N}(T) \) is the number of inner vertices of \( T \). (The vertex neighborhoods are not necessarily well defined for the points lying on the boundary of \( T \), but with the expansion of \( T \) the ratio of these boundary vertices becomes negligible, so this boundary effect can be ignored.)

There is, however, a straightforward connection between the above defined quantities \( \hat{N}(T) \), \( \hat{N}_V(T) \), \( \hat{\kappa}_V \) and the general notions \( N(T) \), \( N_P(T) \), \( \kappa_P \) of Definition \( \hat{\kappa}_e \). It is easy to check that \( \lim_{T \to I} N(T) = 2 \), i.e., in a tessellation \( T \) extending to infinity there are two times as many prototiles as vertices. (Indeed, the angles of all prototiles in \( T \) add up to \( \pi N(T) \) resp. to \( 2\pi \hat{N}(T) \) if they are counted by prototiles resp. by vertices.) Since the vertex neighborhoods \( A, D, J, Q, K \) has a reflectional symmetry, the preferred prototile of them can be mapped to two different tiles of \( T \), whilst the image of the middle vertex is unaltered, so in this case \( N_V(T) = 2\hat{N}_V(T) \), and

\[
\hat{\kappa}_V = \lim_{T \to I} \frac{N_V(T)}{N(T)} = \lim_{T \to I} \frac{2\hat{N}_V(T)}{2N(T)} = \hat{\kappa}_V, \quad \text{if } V \in \{ A, D, J, Q, K \}.
\]
The motifs \( \odot \) and \( \ast \) have an additional five-fold rotational symmetry, so \( N_V(T) = 10\tilde{N}_V(T) \), and

\[
\kappa_V = \lim_{T \to \infty} \frac{N_V(T)}{N(T)} = \lim_{T \to \infty} \frac{10\tilde{N}_V(T)}{2N(T)} = 5\tilde{\kappa}_V, \quad \text{if } V \in \{\odot, \ast\}. \tag{8b}
\]

We proceed similarly as in the previous proofs; first we investigate the evolution of the numbers \( \tilde{N}_V(T) \) of the different vertex neighborhoods \( V \in \{\odot, \ast, A,D,J,Q,K\} \) in a finite tiling \( T \) under successive inflations of \( T \). The closest neighborhood of every vertex determines its future under inflation, as Fig. 3(c) shows, and it is easy to check that the evolution of the vertex neighborhoods of the existing vertices takes place according to the following diagram:

\[
\begin{array}{c}
A \rightarrow J \rightarrow K \\
D \rightarrow Q
\end{array}
\]

Thus after a few inflation steps every vertex ends up its ephemeral life in a state flashing between the two celestial forms with the perfectness of five-fold rotational symmetry. It does not mean, however, that the other vertex neighborhoods would die out, since in every double decomposition process new vertices come into the world, in a state of Ace or Deuce. Indeed, the node being born on the \( 1+\tau \) edge of \( S'' \) or \( L'' \) is determined by its predecessors to start its life as an Ace, and the vertex created on the \( 1+\tau \) edge of \( L'' \) can not be but a Deuce, considering its vertex neighborhood (Fig. 3(c)).

Comprehending all these transmutations, we are capable now to summarize the evolution of the numbers of the different vertex neighborhoods during a single inflation process in a matrix equation:

\[
\begin{bmatrix}
\tilde{N}\odot \\
\tilde{N}\ast \\
\tilde{N}A \\
\tilde{N}D \\
\tilde{N}J \\
\tilde{N}Q \\
\tilde{N}K
\end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5/2 & 5/2 & 1/2 & 1/2 & 3/2 & 3/2 & 2 \\
5/2 & 0 & 0 & 1 & 3/2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{N}\odot \\
\tilde{N}\ast \\
\tilde{N}A \\
\tilde{N}D \\
\tilde{N}J \\
\tilde{N}Q \\
\tilde{N}K
\end{bmatrix} \tag{10}
\]

(The newly created nodes are counted with the two vertices of the edge they are lying on. To avoid over-counting, the matrix elements describing the new vertices have to be divided by two.)

The characteristic polynomial of the matrix above is \( \lambda(2\lambda^2 + \lambda + 1)(\lambda^2 + \lambda + 1/2)(\lambda^2 - 3\lambda + 1) \) with roots \( \lambda_1 = 0, \lambda_{2,3} = (-1 \pm i\sqrt{7})/4, \lambda_{4,5} = (-1 \pm i)/2 \) and \( \lambda_{6,7} = (3 \pm \sqrt{5})/2 \). Since only the real eigenvalue \( \lambda_7 = (3 + \sqrt{5})/3 \) has magnitude greater than unity, its eigenvector \( \tilde{\kappa} \) describes the limiting ratio of the different vertex neighborhoods after many inflations applied to the finite patch \( P \), the components of which are:

\[
\begin{align*}
\tilde{\kappa}_\odot &= \frac{-11 + 7\tau}{5}, & \tilde{\kappa}_\ast &= \frac{-29 + 18\tau}{5}, & \tilde{\kappa}_A &= 2 - \tau, \tag{11a}
\end{align*}
\]

\[
\begin{align*}
\tilde{\kappa}_D &= -3 + 2\tau, & \tilde{\kappa}_J &= 5 - 3\tau, & \tilde{\kappa}_Q &= -8 + 5\tau, \tag{11b}
\end{align*}
\]

\[
\tilde{\kappa}_K &= 13 - 8\tau. \tag{11c}
\]
These probabilities are normalized in such a way that \( \sum_V \tilde{\kappa}_V = 1 \).

This result can be extended to infinite tilings using the same arguments as in the previous two proofs, and using the connections \([8]\) the statements \([7]\) of Lemma \([5]\) are obtained.

We remark that there are certain linear relations between the results of the last three lemmas, i.e., between the ratios of appearance of the prototiles \([1]\), the edge neighborhoods \([2]\) and the vertex neighborhoods \([7]\). These connections can be deduced by considering the vertex neighborhoods as unions of edge neighborhoods, and those as unions of prototiles. We do not give the details here.

All what is needed from the above results in the followings is the fact that the ratios \([1]\), \([2]\) and \([7]\) are members of the subgroup \( \mathbb{Z} + \tau \mathbb{Z} \) of the additive group of real numbers \( \mathbb{R} \), which is dense in \( \mathbb{R} \), but does not agree with it.

Before the proof of the main Statement \([6]\) one more fact has to be observed.

**Lemma 5.** Let \( P \) be a finite patch of type \( \mathcal{P} \), and let \( \hat{P} \) denote the motif obtained by inflating \( P \). Then

\[
\kappa_{\hat{P}} = (2 - \tau) \kappa_P. \tag{12}
\]

**Proof.** Let \( T \) be a finite tiling of type \( \mathcal{P} \), and let \( \Phi : \hat{P} \to T \) be an isometry which maps the inflated motif \( \hat{P} \) onto a portion of \( T \) congruent to the motif \( \hat{P} \). Since the composition is a uniquely defined local process, \( \Phi : P'' \to T'' \) remains a congruence between the motif \( \hat{P}'' \) and the infinite tiling \( T'' \) obtained from \( \hat{P} \) resp. \( T \) by double composition. The inverse statement is also true, namely whenever there is a congruence \( \Phi : \hat{P}'' \to T'' \) mapping the (composed) motif \( \hat{P}'' \) onto one of its appearances in \( T'' \), by double decomposition a congruence \( \Phi : \hat{P} \to T \) is obtained. Thus the occurrences of a finite motif in a tiling (namely \( \hat{P} \) in \( T \)) and the (doubly) composed motif in the (doubly) composed tiling (namely the appearances of \( \hat{P}'' \) in \( T'' \)) are in one to one correspondence, so the numbers \( N_{\hat{P}}(T) = N_{\hat{P}''}(T'') \) are equal. But, by definition, \( \hat{P}'' \) is similar to the original motif \( P \), by the ratio \( \tau \), so \( \kappa_{P''} = \kappa_P \). That means that

\[
\kappa_{\hat{P}} = \lim_{T \to I} \frac{N_{\hat{P}}(T)}{N(T)} = \lim_{T \to I} \left( \frac{N_{P''}(T'')}{N(T'')} \right) = \kappa_P \lim_{T \to I} \frac{N(T'')}{N(T)}. \tag{13}
\]

We need the inverse of the factor \( \lim_{T \to I} \frac{N(T)}{N(T'')} \) by which the number of tiles increases in a double decomposition process in infinitely large tilings \( I \). Since in a double decomposition every prototile of type \( L'' \) is decomposed into three new pieces, and the tiles \( S'' \) are subdivided into two (Fig. \([4,6]\)) using the results of Lemma \([4]\) we get

\[
\lim_{T \to I} \frac{N(T)}{N(T'')} = 3\kappa_L + 2\kappa_S = 3(\tau - 1) + 2(2 - \tau) = 1 + \tau. \tag{14}
\]

So inserting its inverse \( \frac{1}{1 + \tau} = 2 - \tau \) into \((13)\) we obtain the statement \((12)\). \( \blacksquare \)

Now we turn to the main assertion of this subsection.
Statement 6. (Local isometry of Penrose tilings)

i) Every finite motif $P$ of a Penrose tiling of type $\mathcal{P}$ occurs infinitely many times in every infinite tessellation $I$.

ii) Moreover, the frequency $\kappa_P$ of occurrence of $P$ in $I$ is fixed, i.e., it does not depend on the infinite tiling $I$, it is determined solely by the motif $P$.

iii) Finally, $\kappa_P$ is an element of the subgroup $\mathbb{Z} + \tau \mathbb{Z} \subset \mathbb{R}$ of the additive group of reals.

We remark that different appearances of the finite patch $P$ are allowed to overlap in $I$, the occurrences of $P$ are distinguished by the position of a preferred tile in $P$, according to Definition 1.

Proof. Since the motif $P$ is finite, after sufficiently many (say $k \in \mathbb{N}$) double compositions applied to $I$ the size of the composed triangles $S''\ldots'$ and $L''\ldots'$ surpasses the size of $P$, what assures that any occurrence of $P$ is fully covered by a vertex neighborhood in $I''\ldots'$. (Here $I''\ldots'$ denotes the infinite tiling of type $\mathcal{P}''\ldots'$ obtained after $k$ successive compositions.) Since every vertex neighborhood appears in any infinite tessellation infinitely many times, the motif $P$ appears also infinitely many times in $I$, which is assertion i) of the Statement.

Unfortunately it is not true that a single prototile $S''\ldots'$ or $L''\ldots'$ would cover every occurrence of $P$, since the motif $P$ can appear at the vertices or edges of the composed triangles, no matter how big the triangles are (Fig. 3). But it is for sure that any occurrence of $P$ is fully covered either a) by a single prototile $S''\ldots'$, $L''\ldots'$, or b) by an edge neighborhood $e$, in such a way that $P$ intersects the middle edge of $e$, or c) by a vertex neighborhood $V$ of $I''\ldots'$, in such a way that $P$ contains the middle vertex of $V$ (Fig. 3). Moreover, the above classification of the occurrences of the motif $P$ in the composed tessellation $I''\ldots'$ is unambiguous, thus the ratio of appearance $\kappa_P$ can be expressed with the basic frequencies of the prototiles [equation (1)], edge- and vertex neighborhoods [equations (2), (7)] already determined.

More formally, let the nonnegative integers $n_L, n_S, n_e$ resp. $n_V$ denote the number of appearance of the motif $P$ in the finite tilings obtained by $k$
successive inflations of the prototiles $L, S$, the edge neighborhoods $e \in \{k, k', d,t,l',r',r\}$ resp. the vertex neighborhoods $V \in \{\odot, *, A,D,J,Q, K\}$. In $n_L, n_S$ we count only the appearances of $P$ in the interior of $L''\ldots, S''\ldots$, in $n_e$ only the occurrences of type $b)$ are considered, and in $n_V$ the occurrences of type $c)$ are counted (Fig. 3). If the edge- or vertex neighborhood $e$ or $V$ possesses a symmetry than the numbers $n_e, n_V$ should be reduced, since these symmetries had already been accounted for in the calculation of $\kappa_e$ and $\kappa_V$.

With these notations the ratio of appearance of the motif $P$ in any infinite tessellation is

$$\kappa_P = (2 - \tau)^k \left( \kappa_L n_L + \kappa_S n_S + \sum_e \kappa_e n_e + \sum_V \kappa_V n_V \right),$$  

(15)

where the result of Lemma 5 was taken into consideration with the prefactor $(2 - \tau)^k$.

This formula proves the last two assertions ii) and iii) of the Statement. Indeed, it is easy to see that $\kappa_P \in \mathbb{Z} + \tau \mathbb{Z}$, since the $n$'s are integers and the $\kappa$'s are members of the additive group $\mathbb{Z} + \tau \mathbb{Z}$ and $\tau^2 = 1 + \tau$.

We remark that the proof of the above Statement 6 without its last assertion iii) (stating that $\kappa_P \in \mathbb{Z} + \tau \mathbb{Z}$) would have been much simpler, since then the ratios $\kappa_e$ and $\kappa_V$ [equations (2), (7)] need not have been determined exactly.

II.2 Symbolic dynamics associated to the Penrose tilings

In this subsection a symbolic coding scheme is presented, with the help of which every infinite tiling of type $\mathcal{P}$ can be encoded. The coding is not unique, in the sense that different symbolic sequences describe the same (i.e., isomorphic) infinite tessellations, and this non-uniqueness can also be characterized in terms of the coding.

Given an infinite tessellation $I$ of type $\mathcal{P}$ with a preferred prototile $p \in I$ in it, the associated symbolic sequence $\bar{x}(I) = \{x_i\}_{i \in \mathbb{N}} \in \{L, S\}^\mathbb{N}$ of letters $x_i \in \{L, S\}$ is constructed in the following way. If the preferred tile $p \in I$ is of type $L$ than $x_0 = L$, otherwise (i.e., if $p$ is of type $S$) $x_0 = S$. Then make a (single) composition $I \rightarrow I'$ in the tessellation, and let $p' \in I'$ be the composed tile (of type $L'$ or $S'$) in which $p$ is contained. The next letter $x_1$ of the symbolic sequence is $L$ resp. $S$ according to the type of $p'$. This process should be repeated infinitely, i.e., the consecutive letters of the sequence $\bar{x}$ give the type of the prototile in which the preferred tile $p$ is contained after consecutive compositions. The Figures 3 and 6 demonstrate this process graphically.

In the next statement the properties of this coding are investigated.

**Statement 7.** (Symbolic coding of Penrose tilings)

i) The encoding process described above renders every infinite tessellation $I$ of type $\mathcal{P}$, with a preferred prototile $p \in I$ into a unique symbolic sequence $\bar{x} = \{x_i\} \in \{L, S\}^\mathbb{N}$ which satisfies the following ‘grammar rules’ corresponding to the transition matrix $T_P$:

$$x_i : \begin{array}{c}
L \\
\downarrow \\
S \\
\downarrow \\
L \\
\downarrow \\
S
\end{array} \quad T_P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$  

(16)
Figure 6: a) The result of five successive inflations applied to the prototile $L$. The thicker lines denote the tilings obtained after successive compositions. b) The symbolic sequences associated to the two marked prototiles $p$ and $q$.

\[ i.e., \ x_{i+1} = L \text{ whenever } x_i = S, \text{ for all } i \in \mathbb{N}. \ (\text{If the } (i, j) \in \{L, S\}^2 \text{ component of } T_P \text{ is one then the } j \rightarrow i \text{ transition is allowed.}) \]

\[ ii) \text{ If the position of the preferred tile } q \in I \text{ is changed then the new symbolic sequence } \bar{y} \in \{L, S\}^\mathbb{N} \text{ has the same infinite tail as } \bar{x}, \text{ i.e., there exists a finite } n < \infty \text{ such that } x_i = y_i \text{ for all } i > n. \]

\[ iii) \text{ Reversely, every infinite sequence } \bar{x} \in \{L, S\}^\mathbb{N} \text{ satisfying the grammar rules (16) determines a Penrose tiling.} \]

\[ iv) \text{ If two infinite sequences } \bar{x}, \bar{y} \in \{L, S\}^\mathbb{N} \text{ satisfying the grammar rules (16) have the same infinite tail (i.e., } \exists n < \infty \text{ such that } (\forall i > n) x_i = y_i) \text{ than the Penrose tilings determined by the two sequences are isometric.} \]

Thus there is a bijective correspondence between the (congruence classes of) infinite Penrose tilings and the (equivalence) classes of series in $\{L, S\}^\mathbb{N}$ satisfying (16) and having the same infinite tail.

**Proof.** The uniqueness in assertion $i)$ of the Statement is an evident consequence of the definition of the symbolic sequences. To prove the fact that a symbol $S$ can not be followed again by an $S$ we have to observe Fig 1. Indeed, in every composition step the smaller prototiles $S$ or $S'$ are always composed with $L$ or $L'$, forming a triangle of type $L'$ or $L''$, respectively.

It is also easy to see that this coding process can be reversed, i.e., given a sequence $\bar{x} \in \{L, S\}^\mathbb{N}$ satisfying the rules (16), an (up to isometry) unique infinite tessellation $I = \lim_{n \to \infty} T_n$ with preferred tile $p \in I$ can be constructed, the symbolic sequence of which agrees with $\bar{x}$. Indeed, let $T_0 = L$, if $x_0 = L$, and $T_0 = S$, if $x_0 = S$. Then given $T_n \in \{L^n, \hat{S}^n\}$, let $T_{n+1} := T_n = \hat{S}^{n+1}$, if $x_{n+1} = S$, and otherwise (if $x_{n+1} = L$) $T_{n+1} := L^{n+1} = \hat{L}^n \cup \hat{S}^n$, where $\hat{L}^n$ resp. $\hat{S}^n$ denote the finite triangular tilings of type $\mathcal{P}$, obtained by decomposing the tiles $L'' \ldots$ resp. $S'' \ldots$ with $n$ ‘primes’ of Fig. 1. Thus consecutively reading the letters of $\bar{x}$, encountering an $S$ we leave $T_n$ unaltered, and encountering an $L$ we enlarge $T_n$ by joining to it another triangle of different type. The infinite
tessellation \( I = \lim_{n \to \infty} T_n \) with preferred prototile \( p = T_0 \) has the prescribed symbolic sequence \( \bar{x} \) by construction, what proves assertion \( iii) \) of the Statement (Fig 3).

For proving assertion \( ii) \) we have to notice that after sufficiently many compositions of \( I \) the two preferred prototiles \( p, q \) of type \( \mathcal{P} \) will fall into the same composed tile of type \( \mathcal{P}'' \), and the symbolic sequences coincide from this point. Figure 4 demonstrates this phenomenon, where after seven compositions the prototiles \( p \) and \( q \) lie in the same (dark) triangle \( ABC \), and from this point on the positions of the preferred tiles \( p \) and \( q \) are indistinguishable.

Assertion \( iv) \) is a consequence of the first three statements. The first \( n \) letters of \( \bar{x} \) and \( \bar{y} \) determine isometric finite tessellations, only the position of the preferred tile is different, and from that point on the sequences coincide.

We remark that not every infinite tessellation determined by a sequence \( \bar{x} \) extends to the whole plane. There are ‘exceptional’ sequences, the tiling corresponding to which covers only a half-plane or an infinite domain bounded by an angle. We do not address here the question of classifying these ‘exceptional’ sequences.

For convenience, let the set of all possible symbolic series satisfying the grammar rules (16) be denoted with

\[
M_P = \{ \bar{x} \in \{ L, S \}^N \mid (\forall i \in \mathbb{N}) x_i = S \Rightarrow x_{i+1} = L \},
\]

and let the notation \( \bar{x} \sim \bar{y} \) resp. [\( \bar{x} \] be introduced for the equivalence resp. equivalence class of series having the same infinite tail. The ‘universe of all Penrose tilings’, i.e., the set \( X_P \) of (equivalence classes of) non-isometric infinite, type \( \mathcal{P} \) tessellations is the factor space \( X_P = M_P/\sim \).

The set \( M_P \subset \{ L, S \}^N \) with its natural subspace topology inherited from \( \{ L, S \}^N \) as well as the ambient space \( \{ L, S \}^N \) with its product topology are both totally disconnected compact spaces homeomorphic to the dyadic Cantor set. Indeed, according to the construction of the Cantor set, its points can be uniquely labeled with an infinite sequence of two letters, say \( l \) and \( s \). The space of such sequences \( \{ l, s \}^N \) with the product topology is compact by the theorem of Tychonov. The sets of sequences having prescribed values at finitely many points are open-closed sets which separate the points of \( \{ l, s \}^N \), thus the space of two-letter-sequences with the product topology is totally disconnected. The homeomorphism between the spaces \( M_P \) and \( \{ l, s \}^N \) can be established by the recoding \( l \leftrightarrow L, s \leftrightarrow SL \) of the letters in the sequences. (According to this homeomorphism the set \( M_P \) could be substituted by the space \( \{ l, s \}^N \), which has much simpler coding because of the loss of the grammar rules. The reason, however, for not doing this is the fact that the description of the equivalence \( \sim \) would be much more difficult in terms of the symbols \( l \) and \( s \).)

It is easy to see that for any symbolic sequence \( \bar{x} \in M_P \) its equivalence class \( [\bar{x}] \) is dense in \( M_P \), so the ‘Penrose universe’ \( X_P = M_P/\sim \) with its natural factor topology is pathologic from the point of view of ordinary topology. (The closure of every nonempty subset of it is the whole space \( X_P \) itself, so \( X_P \) does not differ much from the one-point topological space.) The main aim of Section 4 is to present more appropriate methods capable to grab the structure of the factor space \( X_P = M_P/\sim \).

Finally it is worth highlighting the close relationship between Penrose tilings and Markov shifts. Indeed, given an infinite tessellation \( I^0 \) of type \( \mathcal{P} \) with a
preferred prototile $p^0 \in I^0$, a `discrete time dynamics’ can be associated to it by applying successive compositions. After the $n$th step an infinite tiling $I^n = I''...'$ (with $n$ primes) of type $P''...'$ is obtained with preferred tile $p^n \in I^n$ of type $P''...'$, containing $p^0$. The Markov shift $\Phi_P$ is constructed by translating the infinite `pointed’ tessellations $(I^n, p^n)$ into binary sequences $\bar{x}^n \in M_P$. Thus its phase space is $M_P$, and the dynamics $\Phi_P : M_P \to M_P$, $\bar{x}^n \mapsto \bar{x}^{n+1}$ is the unilateral left shift $x_i^{n+1} = x_{i+1}^n$. Two tessellations $I \sim I^*$ are isometric if and only if the associated symbolic sequences $\bar{x} \sim \bar{x}^*$ have the same tail, which means that they lie on the same stable manifold of the Markov shift.

In the next section a well known dynamical system is investigated in order to point out that in chaotic systems one encounters ill-behaved topological spaces similar to $X_P$ at every turn.
Figure 7:  a) The effect of the cat map, illustrated with the traditional cat portrait; b) The stable ($v_s$), unstable ($v_u$) eigenvectors and the stable, unstable manifolds ($S_x$, $U_x$) corresponding to the point $x$.

III Chaotic dynamical systems

In this section one of the simplest uniformly hyperbolic dynamical system, Arnold’s cat map [AA68] is investigated. First the definition of the map is given and its stable and unstable manifolds are characterized. Then a generating Markov partition [CFS82] of the phase space is presented, and the dynamics as well as the stable and unstable manifolds are described in terms of the symbolic sequences. This coding scheme will be the starting point of the investigations in Section IV.

III.1 Elementary properties of Arnold’s cat map

The phase space of the cat map is the two dimensional torus $T^2 = R^2 / Z^2$ with its natural Lebesgue measure $\mu$ inherited from the covering space $R^2$. The invertible dynamics $\Phi_A : T^2 \rightarrow T^2$ is deduced from a linear map $A : R^2 \rightarrow R^2$ of the covering space, which is area preserving, i.e., $\det A = 1$, and respects the covering projection $\pi : R^2 \rightarrow T^2$, i.e., whenever two points $x, y \in R^2$ happen to be on the same fiber $[\pi(x) = \pi(y)]$ then their images are also on the same fiber $[\pi(Ax) = \pi(Ay)]$. This latter condition is satisfied if and only if the elements of the matrix $A$ are integers and in this case $\Phi_A = \pi \circ A \circ \pi^{-1}$ is a well defined continuous automorphism of the torus $T^2$ [Wal82].

In the followings we fix the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and omit the subscript of $\Phi$.

The dynamical system $(T^2, \mu, \Phi)$ is called Arnold’s cat map and its effect on the phase space $T^2$ as well as on its covering space $R^2$ is demonstrated in Fig 7.a).

The effect of the cat map can be understood via the linear map $A : R^2 \rightarrow R^2$ of the covering space. The matrix $A$ has two orthogonal eigenvectors, $v_u = \frac{1}{2} \left[ \begin{array}{c} \sqrt{5} \\ 1 \end{array} \right]$ and $v_s = \frac{1}{2} \left[ \begin{array}{c} 1 - \sqrt{5} \\ 2 \end{array} \right]$ corresponding to the eigenvalues $\lambda_u = \frac{3 + \sqrt{5}}{2}$ and $\lambda_s = \frac{3 - \sqrt{5}}{2}$. The map $A$ stretches by the factor $\lambda_u > 1$ in the unstable direction $v_u$ and contracts by the factor $\lambda_s < 1$ in the stable direction. The stable resp. unstable manifold $S_x$ resp. $U_x$ corresponding to any phase space point $x \in T^2$ is the projection by $\pi$ of the line emanating from $x$ with tangent vector $v_s$ resp. $v_u$, as shown in Fig. 7 b). As a consequence of the above facts, the cat map is a uniformly hyperbolic dynamical system.
III.2 Symbolic dynamics associated to the cat map

In this subsection a generating Markov partition of the phase space of the cat map is constructed [AW67].

We recall that a finite partition $\mathcal{P} = \{P_i\}_{i=1}^n$ of the phase space $M$ of a uniformly hyperbolic dynamical system $(M, \Phi, \mu)$ into closed parallelograms $\{P_i\}_{i=1}^n$, the edges of which are parallel to the stable resp. unstable directions, is Markovian [CFS82], if

\[
M = \bigcup_{i=1}^n P_i, \quad \text{Int}(P_i) \cap \text{Int}(P_j) = \emptyset, \quad \text{if } i \neq j, \quad \text{and} \quad (18a)
\]

\[
\Phi(\Gamma_s(\mathcal{P})) \subset \Gamma_s(\mathcal{P}), \quad \Phi^{-1}(\Gamma_u(\mathcal{P})) \subset \Gamma_u(\mathcal{P)), \quad (18b)
\]

where $\Gamma_s(\mathcal{P}) = \bigcup_{i=1}^n \Gamma_s(P_i)$ resp. $\Gamma_u(\mathcal{P}) = \bigcup_{i=1}^n \Gamma_u(P_i)$ denote the union of the stable resp. unstable edges of the parallelograms $P_i$, and $\text{Int}(P)$ is the interior of the set $P$. Moreover, a Markov partition is generating [CFS82], if the intersections $\Phi(P_i) \cap P_j$ as well as $\Phi^{-1}(P_i) \cap P_j$ (where $i, j \in \{1, 2, \ldots, n\}$) have at most one connected component (which is again a parallelogram).
Figure 9: The generating Markov partition \( P_C = \{A, B^0_0, B^1_0, B^0_1, B^1_1\} \) of the torus consisting of five rectangles. The backward \( a) \) and forward iteration \( b) \) of the partition \( P_C \) under the cat map \( \Phi \).

First let us consider the partition of the two dimensional torus \( T^2 = A \cup B \) into two disjoint squares \( A \) and \( B \) whose edges are parallel to the stable and unstable directions, as it is shown in Fig. 8. It is easy to check that this partition is Markovian, i.e., fulfills the conditions \( (18) \), but it is not generating, since the intersections \( \Phi(B) \cap B \) as well as \( \Phi^{-1}(B) \cap B \) have two disjoint components (Fig. 8). This shortcoming, however, can easily be overcome by subdividing the bigger square into four smaller rectangles \( B^0_0, B^0_1, B^1_0, B^1_1 \) shown in Fig. 9.

**Lemma 8. (Generating Markov partition of the cat map)** The partition \( P_C = \{A, B^0_0, B^1_0, B^0_1, B^1_1\} \) is a generating Markov partition of five elements. The grammar rules induced by the dynamics are depicted schematically and encoded in a transition matrix \( T_C \) as follows:

\[
x_i : A \quad & B^0_0 \quad & B^1_0 \quad & B^0_1 \quad & B^1_1 \\
x_{i+1} : A \quad & B^0_0 \quad & B^0_1 \quad & B^1_0 \quad & B^1_1
\]

\[
T_C = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]  \( (19) \)

(The \((i, j) \in P_C^2\) element of the transition matrix is \( T_{ij} = 1 \), if the interior of the parallelogram \( j \) is mapped by the dynamics into \( i \), i.e., \( \text{Int}(\Phi(j)) \cap \text{Int}(i) \neq \emptyset \).)

**Proof.** The proof of the first statement as well as the structure of the allowed transitions can be easily read off from Figure 9. \( \square \)

Almost every phase space point \( x \) is uniquely represented by an in both directions infinite series \( \bar{x} = \{x_i\}_{i \in \mathbb{Z}} \in P_C^{\mathbb{Z}} \) of letters from the finite alphabet \( P_C \) defined by the partition elements in which the successive forward and backward iterations of \( x \) fall, i.e., \( \Phi^i(x) \in x_i \). The inverse statement is also true; every symbolic sequence \( \bar{x} = \{x_i\}_{i \in \mathbb{Z}} \in P_C^{\mathbb{Z}} \), which satisfies the finite grammar rules
defines a unique point of the phase space. Thus, introducing the notation
\[ M_C = \{ \{ x_i \}_{i \in \mathbb{Z}} \in P_C^\infty \mid \{ x_i \}_{i \in \mathbb{Z}} \text{ satisfies the grammar rules (19) } \}, \]
for the (topological) space of all possible series defined by the dynamics, there is a \( \mu \)-almost everywhere defined map
\[ \pi_C : \mathbb{T}^2 \rightarrow M_C, \quad x \mapsto \bar{x} = \{ x_i \}_{i \in \mathbb{Z}}, \quad (\text{here } \Phi^i(x) \in x \in P_C), \]
which is bijective on its domain. This is another way to express that \( P_C \) (Fig. 10) is a generating Markov partition of \( \mathbb{T}^2 \). (The phase space points \( y \in \mathbb{T}^2 \) on which \( \pi_C \) is not defined are the ones lying on the forward or backward iterations of the dividing lines of the Markov partition.)

The stable (resp. unstable) manifolds can also be easily described in terms of the symbolic dynamics; two points \( x, y \in \mathbb{T}^2 \) lie on the same stable (resp. unstable) manifold if and only if their symbolic sequences \( \bar{x}, \bar{y} \) coincide after (resp. before) a threshold index \( N \in \mathbb{Z} \), i.e., \( x_i = y_i \) for all \( i > N \) (resp. for all \( i < N \)).

As we did for the Penrose tilings, let us denote the equivalence relation indicating that the points \( x, y \in \mathbb{T}^2 \) corresponding to the sequences \( \bar{x}, \bar{y} \in M_C \) are on the same stable resp. unstable manifold by \( \bar{x} \sim_s \bar{y} \) resp. \( \bar{x} \sim_u \bar{y} \). This means that \( \bar{x} \sim_s \bar{y} \) (resp. \( \bar{x} \sim_u \bar{y} \)) holds if and only if \( S_x = S_y \) (resp. \( U_x = U_y \)), and the equivalence classes \( [x]_s \) (resp. \( [x]_u \)) are the \( S_x \) stable (resp. \( U_x \) unstable) manifolds.

The topology of the factor spaces \( X^s_C := M_C / \sim_s \) (resp. \( X^u_C := M_C / \sim_u \)) are again ill behaved with their natural factor topology, since the equivalence classes of \( \sim_s \) (resp. \( \sim_u \)) are dense sets in \( M_C \).

The structural similarity between the universe of Penrose tilings and Arnold’s cat map, which is a uniformly hyperbolic dynamical system, is quite clear by now. It is also understood that this similarity is due to a Markov shift hiding in the background. To make this parallelism more tight, we give the counterpart of Statement 8 for the cat map \( (\mathbb{T}^2, \mu, \Phi) \).

**Statement 9. (Local indistinguishability of symbolic trajectories in the cat map)**

1. Every finite symbolic section \( \bar{f} = \{ f_i \}_{i=0}^n \) (where \( n < \infty \)) satisfying the grammar rules (19) (with \( f \) instead of \( x \)) appears infinitely many times in the infinite sequence \( \bar{x} \in M_C \) of \( \mu \)-almost every phase space point \( x \in \mathbb{T}^2 \).
2. Moreover, the frequency \( \kappa_f \) of occurrence of \( \bar{f} \) in \( \bar{x} \) is a fixed number depending solely on \( f \), but not on the selected point \( x \).
3. Finally, the frequency of appearance \( \kappa_f \) is an element of the dense sub-group \( \zeta \mathbb{Z} + \eta \mathbb{Z} \subset \mathbb{R} \) of the additive group of reals, where \( \zeta = \frac{1}{2} - \frac{\sqrt{5}}{2} \) and \( \eta = \frac{\sqrt{5}}{2} \).

The exclusion of a zero measure set in the statement is necessary for two reasons. First, the mapping \( \pi_C \) (given in [21]) is defined only on a subset of maximal measure of \( \mathbb{T}^2 \), and second, the ergodic theorem, which we going to utilize in the proof, is also valid only with the exclusion of a zero measure set.

**Proof.** The first two statements 1), 2) are simple consequences of the ergodicity of the cat map. Indeed, denoting with \( X(\bar{f}) \) the set \( X(\bar{f}) = \{ x \in \mathbb{T}^2 | (\pi_C)^i(x), = x_i = f_i \text{ for all } i \in \{0, 1 \ldots n\} \} \subset \mathbb{T}^2 \) of phase space points having a symbolic
sequence starting with $\bar{f}$, and applying the ergodic theorem \cite{AA68, CFS82, Wal82} for the characteristic function $\chi_{X(\bar{f})}$ of $X(\bar{f}) \subset \mathbb{T}^2$, we get

$$\kappa_{\bar{f}} = \mu(X(\bar{f})).$$

(22)

To prove assertion iii) of the statement the area $\mu(X(\bar{f}))$ has to be determined. Elementary calculations yield that

$$\mu(A) = \mu(B_0^0) = \frac{5 - \sqrt{5}}{10} = \zeta, \quad \mu(B_0^1) = \mu(B_1^0) = \frac{3\sqrt{5} - 5}{10} = 2\eta - \zeta,$$

(23a)

and

$$\mu(B_0^0) = \frac{5 - 2\sqrt{5}}{5} = 2\zeta - \eta.$$

(23b)

It is also straightforward to determine the ratio $a_{ij} = \frac{\mu(j \cap \Phi^{-1}(i))}{\mu(j)}$ of the phase space points in the parallelogram $j \in P_C$, whose first iterate falls into $i \in P_C$. In matrix notation

$$[a_{ij}] = \begin{bmatrix} \lambda_s & 0 & \lambda_s & 0 & \lambda_s \\ 1 - 2\lambda_s & 0 & 1 - 2\lambda_s & 0 & 1 - 2\lambda_s \\ \lambda_s & 0 & \lambda_s & 0 & \lambda_s \\ 0 & \lambda_s & 0 & \lambda_s & 0 \\ 0 & 1 - \lambda_s & 0 & 1 - \lambda_s & 0 \end{bmatrix},$$

(24)

where $\lambda_s = \frac{3 - \sqrt{5}}{2}$ is the stable eigenvalue of the map.

The phase space area $\mu(X(\bar{f}))$ corresponding to the finite symbolic section $\bar{f} = \{f_i\}_{i=0}^n$ is clearly

$$\mu(X(\bar{f})) = a_{f_n \cdots f_0} \cdot \mu(f_0).$$

(25)

For finishing the proof of assertion iii) of the statement we have to notice only that the multiplication with $\lambda_s$ does not lead out of the set $\zeta\mathbb{Z} + \eta\mathbb{Z}$. Indeed, elementary calculations yield that

$$\lambda_s(\zeta n + \eta m) = \zeta(2n - m) + \eta(2m - n) \quad \text{for all } n, m \in \mathbb{Z}.$$

(26)

Since the matrix elements $a_{ij}$ have the form $k + \lambda_n l$ (by equation (24)) and $\mu(f_0) = \zeta n + \eta m$ (by equation (21)), with $k, l, m, n \in \mathbb{Z}$ integers, the value (25) of $\mu(X(\bar{f}))$ is also in the group $\zeta\mathbb{Z} + \eta\mathbb{Z}$ for arbitrary finite symbolic section $\bar{f}$.

In the next section we demonstrate how the methods of noncommutative geometry can be utilized in the study of the two systems (Penrose tilings, cat map) described above.
IV Noncommutative geometrical approach

The previous two sections revealed the intimate relation between certain aperiodic tilings and chaotic dynamical systems. In both cases there naturally appears a topological space \((M, \sim)\) resp. \((M_C, \sim_s)\), and one urges to study the factor spaces \((X_P = M_P/\sim)\) resp. \((X_C^s = M_C/\sim_s, X_C^u = M_C/\sim_u)\), which are pathologic as topological spaces with their inherited factor topologies. In practice, it means that it is equally impossible to establish the equality of two Penrose tilings knowing only finite (but arbitrarily large) patches of them, as it is to make statements about the ultimate fate of phase space points in chaotic systems, if the position of the points is not known with an infinite precision.

In this section we demonstrate how the methods of noncommutative geometry \([\text{Con94}]\) can be exploited to give a more appropriate description of the factor space \(X = M/\sim\). (In the general considerations the subscripts ‘P’ as well as ‘C’ are omitted.) Since this section is mathematically more demanding than the two previous ones, two appendices have been included for the sake of better intelligibility, which summarize the basic properties of approximately finite dimensional (AF) \(C^*\)-algebras and the most important constructions of their \(K\)-theory.

In the first part of this section a noncommutative \(C^*\)-algebra \(C^*(M, \sim)\) is associated to the factor space \(X = M/\sim\), which is, due to the Markov property of the investigated systems and the finiteness of the grammar rules, an approximately finite dimensional (AF) \(C^*\)-algebra. (See Appendix A.4.) It is worth noting that the commutative \(C^*\)-algebra \(C(X)\) of continuous valued functions, which unambiguously describes well-behaved (i.e., locally compact Hausdorff) topological spaces (see Appendix A.2), constitute the center of \(C^*(M, \sim)\), thus it is obvious that the noncommutative algebra \(C^*(M, \sim)\) carries more information about the structure of \(X = M/\sim\) than the commutative one \(C(X)\). Not to mention the fact that in the case of our two systems, due to the non separable topology of \(X\), the commutative algebras \(C(X_P) \cong C(X_C) \cong \mathbb{C}\) are trivial. (Indeed, the only continuous functions on \(X_P\) or \(X_C\) are the constant ones.)

In the second part of the section, just to demonstrate the force of the noncommutative theory as opposed to the usual topology, we calculate the \(K_0\) group of the noncommutative algebras \(C^*(M_P, \sim)\) and \(C^*(M_C, \sim)\), which is a complete invariant in the case of AF algebras. (See Appendix B.3.)

A part of this section (the material concerning the Penrose tilings) is strongly motivated by (and partially overlaps with) the sections II.2 and II.3 of \([\text{Con94}]\).

IV.1 The \(C^*\)-algebra associated to factor spaces

Our first task is to define the noncommutative algebra \(C^*(M, \sim)\) associated to the space \(M\), which possesses two structures, a topology and a partition into equivalence classes. These two structures are many times, like in our concrete cases, inconsistent to each other. (The equivalence classes are everywhere dense sets; every open set contains elements from all equivalence classes.)

The most satisfactory way would be to define the operation \(C^*\) as a functor from the category of certain topological spaces with equivalence relations to the category of \(C^*\) algebras, but this approach is far beyond our present aim. We
content ourselves with the simplest definition valid for finite spaces, since the topological spaces $M_P$ resp. $M_C$ can be presented as projective limits of finite spaces.

So let us forget about the topology, more precisely, let us suppose that $M$ is a finite space with discrete topology, e.g. $M = \{m_1, m_2, m_3, m_4, m_5\}$ with equivalence classes $m_1 \sim m_2 \sim m_3$ and $m_4 \sim m_5$. One algebraic way for the description of the space $X = M/\sim$ is to consider the (continuous) complex valued functions on $M$ which are constant on the equivalence classes. This construction yields the commutative algebra $C(X)$ which is in our present example $C^2$, since $M$ has two equivalence classes.

Another algebraic way to describe the factorization $M/\sim$ is to consider the matrix algebra

$$
C^*(M, \sim) = \left\{ \begin{bmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & i & 0 & 0 \\ 0 & 0 & 0 & j & k \\ 0 & 0 & 0 & l & m \end{bmatrix} \mid a, b, \ldots, m \in C \right\} \cong M_3 \oplus M_2 \subset M_5, \quad (27)
$$

with the usual (noncommutative) matrix product and adjunction. Roughly, we associate a full matrix algebra to each equivalence class, the rank of which agrees with the number of elements in the equivalence class, and then we take the direct sum of these algebras. There is a natural inclusion

$$
C(X) \cong \mathbb{C}^2 \hookrightarrow C^*(M, \sim) \cong M_3 \oplus M_2, \quad (a, b) \mapsto \text{diag}[a, a, a, b, b], \quad (28)
$$

which is a bijection between $C(X)$ and the center of $C^*(M, \sim)$.

Another clever way of looking at the algebra $C^*(M, \sim)$ is to consider its elements as complex valued functions on the set $R_{\sim}$ of equivalent pairs in $M$, i.e.,

$$
C^*(M, \sim) = \{(\text{certain} \ R_{\sim} \rightarrow \mathbb{C} \text{ functions}) \} \quad (29a)
$$

where

$$
R_{\sim} = \{(m, n) \in M^2 \mid m \sim n\} \subset M^2. \quad (29b)
$$

(For finite $M$ all $R_{\sim} \rightarrow \mathbb{C}$ functions are allowed, but for infinite sets there are restrictions for the continuity and norm-finiteness of the functions, for details see $\text{Con94}$.) The product of two such functions, $f, g$ at $(m, n) \in R_{\sim}$ is given by

$$
(f \cdot g)(m, n) = \sum_{k} f(m, k)g(k, n). \quad (30)
$$

For finite, discrete $M$ the two definitions (27) and (29) of the algebra $C^*(M, \sim)$ are clearly equivalent, a function $f : R_{\sim} \rightarrow \mathbb{C}$ in definition (29), evaluated at the point $(i, j) \in R_{\sim}$ just gives the $(i, j)$ matrix element of the corresponding matrix in definition (27). For infinite sets with nontrivial topology the second definition can be more easily generalized $\text{Con94}$, but it is beyond our present needs.
IV.1.1 The AF \( C^* \)-algebra of the Penrose tilings

In this part the noncommutative AF algebra \( C^*(M_P, \sim) \) is constructed, by giving its Brattely diagram.

It is a simple topological fact that the totally disconnected topological space \( M_P \) is the projective limit \( M_P = \varprojlim_{i \in \mathbb{N}_+} M_i \) of the (discrete) spaces \( M_i \subset \{L,S\}^i \), which consist of finite series (of length \( i \)) satisfying the grammar rules (1). The surjections \( \pi_i : M_{i+1} \rightarrow M_i \) are given by omitting the last symbol of the sequences in \( M_{i+1} \). Recall that, by definition, the projective limit means that there exist (continuous) surjections \( \sigma_i : M_P \rightarrow M_i \) such that the upper (unprimed) part of the diagram below commutes, and \( M_P \) is universal, i.e., whenever given another candidate \( M' \) and \( \sigma'_i : M' \rightarrow M_i \) with the same properties, then there is a unique (continuous) map \( \varphi : M' \rightarrow M \) making the whole diagram commuting.

This abstract and elegant definition of \( M_P = \varprojlim_{i \in \mathbb{N}_+} M_i \) really yields the same topological space as the concrete one \([\ref{E}]\) given in Section 1.2.

The equivalence relation \( \sim \) on \( M_P \) can also be described in a similar manner. Two series \( \bar{x}, \bar{y} \in M_P \) are equivalent, \( \bar{x} \sim \bar{y} \), if their tails coincide either from the first position or from the second or from the third or \( \ldots \) from the \( n^{*th} \) position \( (n \in \mathbb{N}) \). This trivial observation, disguised in a mathematical formulation, is the statement that the equivalence relation \( \sim \) on \( M_P \) is the injective limit (increasing union) \( (M_P, \sim) = \varinjlim_{i \in \mathbb{N}_+} (M_P, \sim_i) \), where the equivalence \( \bar{x} \sim_i \bar{y} \) means that the series \( \bar{x}, \bar{y} \) coincide from the \( i^{th} \) position on, i.e., \( x_j = y_j \) for all \( j \geq i \). [The injections in the limit are the identity maps \( \text{id}_{M_P} : (M_P, \sim_i) \rightarrow (M_P, \sim_{i+1}) \), which preserve the relations, i.e., \( \bar{x} \sim_i \bar{y} \Rightarrow \bar{x} \sim_{i+1} \bar{y} \).] The diagram corresponding to the above injective limit construction is the following:

(Well, it is not really an ‘increasing’ union, since the sets \( M_P \) are the same in all terms of the sequence, so nothing really increases. The relations \( \sim_i \), however, are not the same, so the construction does make sense.) The arrows \( \rho_i \) are injective, relation preserving mappings. (They are simply the identity maps \( \rho_i = \text{id}_{M_P} : M_P \rightarrow M_P \), because of the above mentioned speciality.) The injective limit \( (M_P, \sim) \) has again the universal property, i.e., given any other candidate \( (M', \sim') \) for the injective limit and morphisms \( \rho'_i : (M_P, \sim_i) \rightarrow (M', \sim') \) (with the same properties as \( \rho_i \) have), there exists a unique relation preserving map \( \varphi : (M_P, \sim) \rightarrow (M', \sim') \) for which the whole diagram commutes.
This diagram is quite similar to the previous one \([31]\), just the arrows are reversed. Perhaps this fact is responsible for the incompatibility between the topology of \(M_p\) and the relation \(\sim\) on it.

With the help of the projections \(p_i : M_p \to M_i\), which omit the symbols after the \(i\)th position of the finite sequence \([i.e., p_i : (x_1, x_2 \ldots x_i, x_{i+1} \ldots) \to (x_1, x_2 \ldots x_i)]\), the (unprimed parts of the) two diagrams can be merged together.

The top row of the diagram describes the topology of \(M_p\) while the bottom row is related to the equivalence relation \(\sim\). With the help of the projections \(p_i\) the relations \(\sim_i\) can be ‘pushed forward’ to the finite sets \(M_i\), the resulting relations are denoted by \(\sim^i\). The relation \(\bar{x} \sim^i \bar{y}\) holds, by definition, between two finite symbolic series \(\bar{x} \sim^i \bar{y}\) (here \(\bar{x}, \bar{y} \in M_i\)) if and only if there exist two \(\sim_i\)-equivalent infinite sequences \(\bar{x}' \sim_i \bar{y}' \in M_p\) such that they project to the finite sequences, i.e., \(p_i(\bar{x}') = \bar{x}\) and \(p_i(\bar{y}') = \bar{y}\). In our case it means simply that the last symbols of \(\bar{x}\) and \(\bar{y}\) agree, i.e., \(x_i = y_i\). From this it is also clear that \(\sim^i\) is an equivalence relation on \(M_i\), too. We note that the surjections \(\pi_i\) do not preserve this equivalence.

Up to this point our investigations were purely ‘commutative’, we used only classical topological concepts, and the machinery of noncommutative geometry was not exploited. Now we associate noncommutative \(C^*\)-algebras to the spaces \((M_i, \sim^i)\) appearing in the first row of the diagram \([33]\), and substitute injective \(C^*\)-algebra homomorphisms for the continuous surjections \(\pi_i\) between them.

The noncommutative \(C^*\)-algebra \(C^*(M_i, \sim^i)\) associated to the finite, partitioned space \((M_i, \sim^i)\) is constructed according to the method demonstrated at the beginning of the section [see \([35]\) and \([29]\)]. Since there are two equivalence classes in every set \(M_i\) (labeled by the last symbols of the series), this algebra is isomorphic to the direct sum of two matrix algebras, \(C^*(M_i, \sim^i) \cong M_{f_i} \oplus M_{f_{i+1}}\), where the dimensions \(f_i\) are just the Fibonacci numbers \(1, 1, 2, 3, 5, 8, \ldots\). Indeed, let \(f_i\) be the number of series \(\bar{x} \in M_i\) with last symbol \(x_i = S\). It agrees with the number of series in \(M_{i-1}\) ending with \(L\), since every \(S\) is preceded by an \(L\), according to the grammar rules \([4]\). Thus the number of sequences in \(M_i\) with a last symbol \(L\) is \(f_{i+1} = f_{i-1} + f_i\), since it follows either an \(S\) or an \(L\) in \(M_{i-1}\).

The construction of the \(C^*\)-algebra homomorphisms \(C^*(\pi_i) : C^*(M_i, \sim^i) \to C^*(M_{i+1}, \sim^{i+1})\) is also natural, but it is a bit more complicated. First we have
to consider the mappings incorporated in the following diagram

\[
\begin{array}{cccccc}
M^2_1 & \longrightarrow & \pi_1 \times \pi_1 & \longrightarrow & \pi_2 \times \pi_2 & \longrightarrow & M^2_3 & \cdots \\
\downarrow \iota_1 & & \downarrow \iota_2 & & \downarrow \iota_3 & & \cdots \\
R^2 & & R^2 & & R^3 & & \cdots \\
\end{array}
\]  

(34)

where \( \pi_i \times \pi_i : M^2_{i+1} \rightarrow M^2_i \) is the surjection \((\bar{x}, \bar{y}) \mapsto (\pi_i(\bar{x}), \pi_i(\bar{y}))\), the symbol \( R^i = \{(\bar{x}, \bar{y}) \in M^2_i \mid \bar{x} \sim^i \bar{y}\} \subset M^2_i \) denotes the subset representing the \( \sim^i \)-equivalent pairs, and \( \iota_i : R^i \hookrightarrow M^2_i \) is the natural inclusion.

Let us apply the pull-back functor to this diagram, i.e., instead of the spaces \( M^2_i \) resp. \( R^i \) let us consider the set of (continuous) complex valued functions \( C(M^2_i) \) resp. \( C(R^i) \) with the appropriate pull-back mappings between them. (Since the spaces \( M^2_i \) and \( R^i \) are finite spaces with discrete topology, the continuity of the functions means no restriction.)

\[
\begin{array}{cccccc}
C(M^2_1) & \overset{C(\pi_1 \times \pi_1)}{\longrightarrow} & C(M^2_2) & \overset{C(\pi_2 \times \pi_2)}{\longrightarrow} & C(M^2_3) & \cdots \\
\iota_1^* & \downarrow \lambda_1 & \iota_2^* & \downarrow \lambda_2 & \iota_3^* & \downarrow \lambda_3 \\
C^*(M_1, \sim^1) & \overset{C^*(\pi_1)}{\longrightarrow} & C^*(M_2, \sim^2) & \overset{C^*(\pi_2)}{\longrightarrow} & C^*(M_3, \sim^3) & \cdots \\
\end{array}
\]

(35)

According to the definition \((29)\), the space \( C(R^i) \) is just the algebra \( C^*(M_i, \sim^i) \), as it is denoted in the diagram. The pull-back mappings act in reverse direction, and the pull-back of a surjective (resp. injective) mapping is injective (resp. surjective). The pull-back \( \iota_i^* \) of the inclusion \( \iota_i : R^i \hookrightarrow M^2_i \) is just the restriction of the functions \( M^2_i \rightarrow \mathbb{C} \) to \( R^i \subset M^2_i \). The pull-back of the surjection \( \pi_i \times \pi_i \) is the injective mapping \( (\pi_i \times \pi_i)^*: C(M^2_i) \rightarrow C(M^3_{i+1}) \), \( f \mapsto f \circ (\pi_i \times \pi_i) \), thus \( (\pi_i \times \pi_i)^* f(\bar{x}, \bar{y}) = f(\pi_i(\bar{x}), \pi_i(\bar{y})) \). In addition to these pull-back maps there are the injections \( \lambda_i \) defined by

\[
\lambda_i : C^*(M_i, \sim^i) \hookrightarrow C(M^2_i), \quad f \mapsto \tilde{f}, \quad \text{where} \quad (36)
\]

\[
\tilde{f}(\bar{x}, \bar{y}) = \begin{cases} 
    f(\bar{x}, \bar{y}), & \text{if } (\bar{x}, \bar{y}) \in R^i; \\
    0, & \text{if } (\bar{x}, \bar{y}) \notin R^i.
\end{cases}
\]

We note that \( \iota_i^* \circ \lambda_i = \text{id}_{C(R^i)} \), but \( \lambda_i \circ \iota_i^* \) is a proper projection of the algebra \( C(M^2_i) \).

A building block of the diagrams \((34)\) as well as \((35)\) is graphically represented in figure \(10\).

The direct product spaces \( M^2_i \) as well as \( M^2_{i+1} \) are represented by squares, which also resemble to the form of the matrices in the definition \((27)\). The squares are partitioned into smaller squares according to the relations \( \sim^i \), \( \sim^{i+1} \) and \( \pi_i^*(\sim^i) \). (This latter relation holds between two elements \( \bar{x}, \bar{y} \in M^2_{i+1} \) if and only if \( \pi_i(\bar{x}) \sim^i \pi_i(\bar{y}) \), i.e., if the last but one symbol of \( \bar{x} \) and \( \bar{y} \) is the same, \( x_i = y_i \).) The numbers in the small squares help us to keep track of the point-to-point as well as the pull-back mappings of the diagrams \((34)\) and \((35)\), while the shading (coloring) and the patterns designate the different equivalence relations.

The mappings \( C^*(\pi_i) : C^*(M_i, \sim^i) \rightarrow C^*(M_{i+1}, \sim^{i+1}) \) in the second row of the diagram \((35)\) are defined by the composition

\[
C^*(\pi_i) = \iota_{i+1}^* \circ (\pi_i \times \pi_i)^* \circ \lambda_i, \quad \text{for all } i \in \mathbb{N}_+ \quad (37)
\]

27
that in the first row of diagram (33) the topological space $\mathbb{M}$ (and using the technique of noncommutative geometry the ‘commutative’ spaces the projective limit of the spaces $(\mathbb{M}^i)_{i=1}^n$). According to definition (27) of $\mathbb{C}$ it preserves multiplication, since it is a mapping between block-diagonal matrices, according to definition (23) of $\mathbb{C}^\ast(M_i, \sim^i)$.

With these constructions we are almost ready with the definition of the $\mathbb{C}^\ast$-algebra $\mathbb{C}^\ast(M_P, \sim)$ associated to the universe of the Penrose tilings. Remember that in the first row of diagram (33) the topological space $M_P$ was defined as the projective limit of the spaces $(\mathbb{M}_i, \sim^i)$:

\begin{align*}
(M_1, \sim^1) & \xrightarrow{\pi_1} (M_2, \sim^2) \xrightarrow{\pi_2} (M_3, \sim^3) \ldots \\
\sigma_1 & \quad \sigma_2 & \quad \sigma_3
\end{align*}

and using the technique of noncommutative geometry the ‘commutative’ spaces $(\mathbb{M}_i, \sim^i)$ as well as the topological mappings $\pi_i$ have been changed for the noncommutative algebras $\mathbb{C}^\ast(M_i, \sim^i)$, as well as for the injective $\mathbb{C}^\ast$-homomorphisms $\mathbb{C}^\ast(\pi_i)$:

\begin{align*}
\mathbb{C}^\ast(M_1, \sim^1) & \xrightarrow{\mathbb{C}^\ast(\pi_1)} \mathbb{C}^\ast(M_2, \sim^2) \xrightarrow{\mathbb{C}^\ast(\pi_2)} \mathbb{C}^\ast(M_3, \sim^3) \ldots \tag{39}
\end{align*}

(It is the bottom row of the diagram (33).)

It is quite natural to desire that the above passage from commutative to noncommutative structures should respect the projective/injective limit constructions. In the light of this we have obtained the following result:

**Statement 10.** (The $\mathbb{C}^\ast$-algebra associated to the Penrose tilings) The approximately finite dimensional (noncommutative) algebra associated to the

\[ \text{Figure 10: The graphical representation of the building blocks of the diagrams (33) as well as (35).} \]
universe of Penrose tilings is defined as the injective limit

\[ C^*(M_P, \sim) = \lim_{i \in \mathbb{N}} C^*(M_i, \sim^i) \]  

(40a)

with the injections \( C^*\left(M_i, \sim^i\right) \hookrightarrow C^*\left(M_{i+1}, \sim^{i+1}\right) \) given in (37). (For \( i = 0 \) the map \( C^*\left(\pi_0\right) : C \hookrightarrow C^*\left(M_1, \sim^1\right) \subset M_2 \) is \( z \mapsto \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \).) The multiplicity matrices \( A_P^0 \), \( A_P^i \) (for \( i \in \mathbb{N}_+ \)) and the Bratteli diagram of the AF algebra \( C^*(M_P, \sim) \) is

\[ A_P^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_P^i = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}; \quad \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & \cdots \end{array} \]  

(40b)

(For the definition of approximately finite dimensional algebras and related notions see Appendix A.)

**Proof.** The algebra (40a) is nothing but the injective limit of the sequence (39). It is clear from figure 10 that the Bratteli diagram of this algebra is really (40b).

This algebra (40) is sometimes called the *Fibonacci algebra* since the ranks of the full matrix algebras in the injective limit construction are the Fibonacci numbers \( \text{Dav96} \).

It is worth noticing that the Bratteli diagram (40b) is built up of the blocks (13) (rotated into horizontal position), which describe the grammar rules belonging to the symbolic coding of the Penrose tilings, and the transition matrix \( T_P \) in equation (16) is the same as the multiplicity matrix \( A_P^i \) in (40b).

Thus the simplest practical way of obtaining the final result (40), the (Bratteli diagram (40b) of the) \( C^*\)-algebra \( C^*(M_P, \sim) \) would have been to draw the diagrams (13) representing the allowed transitions for the symbolic coding into a single chain.

Going through the steps of the injective limit construction (39) again, it is easy to see that the above observation is generally true for any finite Markov chain. In the next subsection we use this short cut to determine the AF algebra associated to the cat map.

**IV.1.2** The AF \( C^*\)-algebra of the cat map

In this subsection the constructions presented just before for the Penrose system are adapted to the cat map. The only relevant differences between the two systems are that in the latter case the symbolic sequences (20) are infinite in both directions, and there are two equivalence relations \( \sim_s \) and \( \sim_u \) corresponding to forward and backward iterations, respectively. These difference, however, can be easily eliminated by simple tricks.

Let \( \sim \) be the equivalence relation on the set \( M_C \) obtained by merging \( \sim_u \) and \( \sim_s \), i.e.,

\[ \bar{x} \sim \bar{y} \quad \text{if and only if} \quad (\bar{x} \sim_u \bar{y}) \land (\bar{x} \sim_s \bar{y}), \quad \bar{x}, \bar{y} \in M_C. \]  

(41)

Thus \( \bar{x} \sim \bar{y} \) means that the initial and final tail of \( \bar{x} \) and \( \bar{y} \) coincide, so the two series differ from each other only in a finite number of letters.
The other difficulty, namely the fact that the symbolic sequences associated to the cat map are infinite in both directions, can be overcome by ‘stretching the finite sequences in both directions at the same time’. More precisely, instead of the symbol set \( P_C = \{ A_0^0, B_0^0, B_1^1, B_1^0 \} \) of five elements (see Lemma 13) let us introduce the set \( \tilde{P}_C = P_C^2 \) of ordered pairs, which contains 25 elements. As phase space let us use the set

\[
\tilde{M}_C = \left\{ \tilde{X} = \{ (x_i^+, x_i^-) \}_{i \in \mathbb{N}} \in \tilde{P}_C^\mathbb{N} \right. \left| \begin{array}{c} x_0^+ = x_0^-, \text{ and for all } i \in \mathbb{N} \text{ the} \\ \text{transitions } x_i^- \rightarrow x_i^+ \text{ and } x_i^+ \rightarrow x_{i+1}^- \text{ are allowed by the grammar} \end{array} \right. \right\},
\]

which is already a set of series infinite only in one direction. There is a straightforward bijection between \( M_C \) and \( \tilde{M}_C \) given by

\[
M_C \rightarrow \tilde{M}_C, \quad \bar{x} = \{ x_i \}_{i \in \mathbb{N}} \mapsto \{ (x_{-i}, x_i) \}_{i \in \mathbb{N}}; \\
\tilde{M}_C \rightarrow M_C, \quad \tilde{X} = \{ (x_i^-, x_i^+) \}_{i \in \mathbb{N}} \mapsto \ldots x_2^-, x_1^+, x_0^+ = x_1^+, x_2^+ \ldots
\]

and the relation \( \tilde{X} \sim \tilde{Y} \) (considered on \( \tilde{M}_C \)) according to the above bijection \( M_C \rightarrow \tilde{M}_C \) holds between two elements of \( \tilde{M}_C \) if and only if their tails coincide, i.e., there exists an \( n \in \mathbb{N} \) such that \( (x_i^-, x_i^+) = (y_i^-, y_i^+) \) for all \( i \geq n \).

Instead of the set with two equivalence relations \( (M_C, \sim_u, \sim_s) \) we investigate the object \( (\tilde{M}_C, \sim) \), which is the phase space of an in one direction infinite Markov chain, with transition matrix

\[
\tilde{T}_C = T_C^T \otimes T_C, \quad \text{or in components} \quad \tilde{T}_{(i,k),(j,l)} = T_{i,j}^T \cdot T_{k,l},
\]

where \( T_C \) is defined in (19). Indeed, the backward allowed transitions are described by the transpose \( T_C^T \) of the transition matrix, and because of the tensor product \( T_C^T \otimes T_C \) acts on the first element while \( T_C \) acts on the second element of the pairs in \( M_C \). The transition \( (x^-, x^+) \rightarrow (y^-, y^+) \) in \( \tilde{P}_C \) is allowed, if and only if \( T_{(y^-, y^+),(x^-, x^+)} = T_{y^-, x^-}^T \cdot T_{x^-, y^+} = T_{x^-, y^+} \cdot T_{y^-, x^-} = 1 \) holds for the appropriate matrix element, i.e., the transitions \( y^- \rightarrow x^- \) and \( x^+ \rightarrow y^+ \) are both allowed in the original symbol space \( P_C \).

The space of equivalence classes \( X_C = M_C / \sim \) is again pathologic as topological space, what is essentially due to the fact that every open set in the phase space of the cat map contains (an infinite number of) points with arbitrarily prescribed initial and final symbolic tails.

The transition matrix \( \tilde{T}_C \) has the size of \( 25 \times 25 \), which is too big to permit a graphical representation, but still, the algebra \( C^*(\tilde{M}_C, \sim) \) can be precisely described via the matrix \( \tilde{T}_C \).

**Statement 11.** (The \( C^* \)-algebra associated to the cat map) The non-commutative \( C^* \)-algebra \( C^*(\tilde{M}_C, \sim) \) associated to the cat map is the AF algebra defined by the injective limit (see Appendix A.4)

\[
C^*(\tilde{M}_C, \sim) = \lim_{n \in \mathbb{N}} A_n, \quad \text{where} \quad A_0 = \mathbb{C}^5,
\]

and the multiplicity matrices \( A_n \) (see Appendix A.4) of the successive (unital) injections \( \Phi_n : A_n \rightarrow A_{n+1} \) are given by

\[
A_{(i,j),k}^0 = T_{i,k}^T \cdot T_{j,k}, \quad \text{and} \quad A_{(i,j),(j,l)} = T_{i,j}^T \cdot T_{k,l} \quad \text{for } n \geq 1,
\]

30
where $T_C$ is given in (13).

Proof. According to the remark at the end of the previous subsection, the multiplicity matrices of the inclusions in the injective limit are the same as the transition matrices describing the grammar rules of the symbolic sequences. For $n \geq 1$ this matrix is given in (44), independently of the value of $n$.

For $n = 0$ the algebra $A_0$ is simply $C^5$, since the zeroth letter of a sequence $X \in M_C$ can have only five different values, because $x_0^-=x_0^+$. The multiplicity matrix $A^0$ of the inclusion $\Phi_0 : A_0 \hookrightarrow A_1$ is clearly the matrix (of size $25 \times 5$) given in (51), since for a given zeroth symbol $x_0^-$ the matrix $T_C^0$ resp. $T_C$ describes the allowed backward resp. forward steps.

Although the presentation of the AF algebra $C^*(M_C, \sim)$ is not so direct as it was in the case of Penrose tilings (Statement 10), the formulas (15) still unambiguously characterize the algebra in question, and allow us to calculate its $K_0$ group, which is the subject of the next subsection.

IV.2 The $K_0$ groups of the algebras

In this subsection the $K_0$ groups corresponding to the AF algebras $C^*(M_P, \sim)$ and $C^*(M_C, \sim)$ are determined (see formulas (10) and (12)).

The $K_0$ group is a very important invariant of $C^*$-algebras, especially of AF $C^*$-algebras, for in this case it is a complete invariant ([11][7], and the subclass of commutative groups representing the possible $K_0$ groups of all AF $C^*$-algebras is also well described ([31][90]).

In our case, comparing the obtained results for $K_0(C^*(M_P, \sim))$ resp. $K_0(C^*(M_C, \sim))$ with the last assertions of the Statements 6 (on page 13) resp. 8 (on page 23) it turns out that these groups are indeed important invariants of the systems investigated.

For the sake of better intelligibility Appendix 8 gives a brief summary of the necessary part of algebraic $K$ theory. Before going into the details of the calculations let us recall that the scaled dimension group of a finite dimensional $C^*$-algebra $\bigoplus_{i=1}^k M_{n_i}$ has the form $(\mathbb{Z}^k, \mathbb{N}^k, \prod_{i=1}^k [0, n_i])$ (see also formula (B3) in Appendix 8), and the functor $K_0$ commutes with direct limit, in the sense that the group $K_0(A)$ (as an ordered, scaled group) of the approximately finite dimensional algebra $A = \lim_{i \to \infty} A_i$ (with unital inclusions $\Phi_i : A_i \hookrightarrow A_{i+1}$) is $K_0(A) = \lim_{i \to \infty} K_0(A_i)$, where the (positive unital) group homomorphisms $K_0(\Phi_i) : K_0(A_i) \to K_0(A_{i+1})$ are given by the multiplicity matrices $A_i'$ of the unital injections $\Phi_i$.

A concrete representation of the direct limit group $K_0(A)$ is given by the formulas (15) in the general case, and by the expressions (16) for the case when the matrices $A_i'$ are injective (from a threshold index).

IV.2.1 The $K_0$ group associated to the Penrose tilings

Let $f_i$ denote the series of Fibonacci numbers, i.e., $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, $f_{i+1} = f_i + f_{i-1}$. According to Statement 10, the $C^*$-algebra associated to the universe of Penrose tilings is given by the injective limit $C^*(M_P, \sim) = \lim_{i \in \mathbb{N}} C^*(M_i, \sim^i)$, where the finite dimensional algebras $C^*(M_i, \sim^i)$, as well as their $K_0$ groups (with their order and scale structure) have the form
\[ C^*(M_0, \sim^0) \cong \mathbb{C}, \quad K_0(C^*(M_0, \sim^0)) \cong \mathbb{T}, \quad (46a) \]
\[ C^*(M_i, \sim^i) \cong M_{f_i} \oplus M_{f_{i+1}}, \quad (46b) \]
\[ K_0(C^*(M_i, \sim^i)) \cong (\mathbb{Z}^2, \mathbb{N}^2, \{0 \ldots f_i\} \times \{0 \ldots f_{i+1}\}) \quad \text{for } i \in \mathbb{N}_+. \quad (46c) \]

In the followings we determine the form of \( K_0(C^*(M_P, \sim)) \), as the direct limit of the groups \( (46c) \).

**Statement 12. (The \( K_0 \) group of the Penrose universe)**  \( \) The scaled dimension group of the AF algebra \( C^*(M_P, \sim) \) has the form

\[ K_0(C^*(M_P, \sim)) \cong (\mathbb{Z}^2, K_0^+, \Gamma) \cong \cong (\mathbb{Z} + \tau \mathbb{Z}, [0, \infty) \cap (\mathbb{Z} + \tau \mathbb{Z}), [0, 1] \cap (\mathbb{Z} + \tau \mathbb{Z})), \quad (47a) \]

where \( \tau = \frac{1 + \sqrt{5}}{2} \) is the ‘golden mean’, and

\[ K_0^+ = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq \tau x + y\}, \quad (47c) \]
\[ \Gamma = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq \tau x + y \leq \tau + 1\}. \quad (47d) \]

**Proof.** First we prove the row \( (47a) \) of the statement. The homomorphisms in the direct limit construction of the group \( K_0(C^*(M_P, \sim)) \) are determined by the multiplicity matrices \( A_i^P \) given in \( (46c) \) (see also equation \( (45) \) in Appendix B.3), so they have the following form \( (i \in \mathbb{N}_+) \):

\[ A_i^P : K_0(C) \cong \mathbb{Z} \to K_0(C^2) \cong \mathbb{Z}^2, \quad z \mapsto \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, \quad (48a) \]
\[ A_i^P : K_0(M_{f_i} \oplus M_{f_{i+1}}) \cong \mathbb{Z}^2 \to K_0(M_{f_{i+3}} \oplus M_{f_{i+2}}) \cong \mathbb{Z}^2, \quad x \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x \end{bmatrix}. \quad (48b) \]

(We have used the connections \( (46c) \).)

The map \( A_0^P : \mathbb{Z} \to \mathbb{Z}^2 \) is not surjective, but for all other indices \( i \in \mathbb{N}_+ \) the linear mappings \( A_i^P \) are \( \mathbb{Z}^2 \to \mathbb{Z}^2 \) bijections, which means that the \( K_0 \) group of the injective limit is also \( \mathbb{Z}^2 \). (This case is described by the formulas \( (46c) \) of Appendix B.3.)

To determine the order and scale of the \( K_0 \) group we need to know the (“stable” and “unstable”) eigenvalues \( \lambda_s, \lambda_u \) as well as the corresponding eigenvectors \( v_s, v_u \) of the (hyperbolic) matrix \( A_0^P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \):

\[ \lambda_u = \tau = \frac{1 + \sqrt{5}}{2}, \quad v_u = \begin{bmatrix} \tau \\ 1 \end{bmatrix}, \quad (49a) \]
\[ \lambda_s = \frac{1}{\tau} = \frac{1 - \sqrt{5}}{2}, \quad v_s = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (49b) \]

These eigenvectors are illustrated in Fig. 1.

The cone \( K_0^+ \) of positive elements consists of the points of \( K_0(C^*(M_1, \sim^1)) \cong \mathbb{Z}^2 \) which after a finite (but arbitrarily large) number of bijections \( A^P : \mathbb{Z}^2 \to \mathbb{Z}^2 \)
fall (and thus remain) in the positive cone $K_0^+(\mathcal{C}^*(M_1,\sim)) \cong \mathbb{Z}^2$. (See the formulas (30) of Appendix \[\text{(3.3)}\].) These are exactly the points $u = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2$ for which the scalar product $u \cdot v_u = \tau x + y \geq 0$ is not negative, as it is stated in [47c].

Similarly, the scale $\Gamma$ consists of the points of $K_0(\mathcal{C}^*(M_1,\sim)) \cong \mathbb{Z}^2$ which after the $n$th application of the bijection $A^n : \mathbb{Z}^2 \to \mathbb{Z}^2$ fall into the scale $\Gamma_{n+1} = \{0 \ldots f_{n+1}\} \times \{0 \ldots f_{n+2}\}$ for sufficiently large $n \in \mathbb{N}$. Since $A^n \begin{pmatrix} f_i \\ f_{i+1} \end{pmatrix} = \begin{pmatrix} f_{i+1} \\ f_{i+2} \end{pmatrix}$, the scale $\Gamma$ is the set of points lying in the strip parallel to the stable direction $s$, and determined by the elements $(0,0), (1,1) \in \mathbb{Z}^2$, as it is depicted in Fig. [11] and given in (47b).

To prove assertion (47b) of the statement we have to notice that the mapping

$$\Pi : \mathbb{Z}^2 \to \mathbb{R}, \quad (x, y) \mapsto \frac{\tau x + y}{\tau + 1} = (\tau - 1)x + (2 - \tau)y = \tau(x - y) + 2y - x,$$

(50)

which is essentially (up to a scaling factor) the projection along the stable direction $s$ to the unstable direction $u$, is an injective group homomorphism. (The injectivity follows from the fact that $\tau = \frac{1+\sqrt{5}}{2}$ is irrational.) The image of the scaled dimension group $(\mathbb{Z}^2, K_0^+, \Gamma)$ under the projection $\Pi$ has the form given in (47b).

Comparing the previous result with assertion iii) of Statement 3 (page 13) we see that the noncommutative algebra $\mathcal{C}^*(\tilde{M}_0,\sim)$ associated to the universe of Penrose tilings do carry nontrivial information about the structure of this space.

**IV.2.2 The $K_0$ group associated to the cat map**

In order to determine the scaled dimension group $K_0(\mathcal{C}^*(\tilde{M}_1,\sim))$ of the (non-commutative) AF algebra $\mathcal{C}^*(\tilde{M}_1,\sim)$ associated to the cat map, we basically
follow the same steps as we did in the previous subsection, investigating the algebra of the Penrose tilings. The only difference between the two cases is rather technical; now the allowed transitions are described by the matrix $T_C$ of size $25 \times 25$, which is much bigger than the two by two matrix $A^T$, and not invertible. The analytical calculation of the $K_0$ group, however, is still possible, since the tensor product structure of $\hat{T}_C = T_C^T \otimes T_C$ permits its eigenstate decomposition.

**Statement 13. (The $K_0$ group of the cat map)** The scaled dimension group of the AF $C^*$-algebra $C^*(\hat{M}_C, \sim)$ (defined in (45)) has the form

\[
K_0(C^*(\hat{M}_C, \sim)) \cong (\mathbb{Z}^4, K_0^+, \Gamma) \cong (\zeta \mathbb{Z} + \eta \mathbb{Z}, [0, \infty) \cap (\zeta \mathbb{Z} + \eta \mathbb{Z}), [0, 1] \cap (\zeta \mathbb{Z} + \eta \mathbb{Z})) \times \mathbb{Z}^2
\]

(51a)

where \( \zeta = \frac{1}{2} - \frac{\sqrt{5}}{10}, \quad \eta = \frac{\sqrt{5}}{5}, \) and

\[
K_0^+ = \left\{ \begin{array}{l}
\left[ \begin{array}{ccc}
u \\
w \\
z
\end{array} \right] \in \mathbb{Z}^3 \mid 0 \leq u + \frac{\sqrt{5} - 1}{2}(v + w) + \frac{3 - \sqrt{5}}{2} z \\
\end{array} \right\}, \quad \Gamma = \left\{ \begin{array}{l}
\left[ \begin{array}{ccc}
u \\
w \\
z
\end{array} \right] \in \mathbb{Z}^3 \mid 0 \leq u + \frac{\sqrt{5} - 1}{2}(v + w) + \frac{3 - \sqrt{5}}{2} z \leq \frac{25 + 11\sqrt{5}}{2}
\end{array} \right\},
\]

(51c)

**Proof.** The algebra associated to the cat map is defined as the injective limit $C^*(\hat{M}_C, \sim) = \lim_{n \in \mathbb{N}} A_n$ (see Statement 11, equation (45)), and the scaled dimension groups of the finite matrix algebras $A_n$ are clearly (see Appendix B.3, equation (53)):

\[
K_0(A_0) \cong (\mathbb{Z}^5, \mathbb{N}^5, \{0, 1\}^5),
\]

(52a)

\[
K_0(A_n) \cong \left(\mathbb{Z}^{25}, \mathbb{N}^{25}, \prod_{i,j \in P_C} \{0, 1 \ldots a_{i,j}^{(n)}\}\right), \quad \text{for } n \geq 1
\]

(52b)

where $a_{i,j}^{(n)}$ is the number of symbolic sequences $\bar{x} = (x_{-n} = i, x_{-n+1} \ldots x_{n-1}, x_n = j)$ of length $2n - 1$, satisfying the grammar rules (49), starting with the symbol $x_{-n} = i \in P_C$, at the $-n^{th}$ position and ending with the symbol $x_n = j \in P_C$ at the $n^{th}$ position. It is easy to check that $a_{i,j}^{(n)} \geq 1$ holds for all these numbers $(n \geq 1, i, j \in P_C)$, i.e., in at least two steps every $i \rightarrow j$ transition is allowed. (Indeed, the matrix $T_C^2$ has no zero entries.) Moreover, since the inclusions $\Phi_n : A_n \hookrightarrow A_{n+1}$ are unital, and their multiplicity matrices are given by (45), b–c), the following recursions are valid:

\[
a_{i}^{(0)} = 1, \quad \text{for all } i \in P_C,
\]

(53a)

\[
a_{i,j}^{(1)} = \sum_{k \in P_C} T_{i,k}^T \cdot T_{j,k} : a_k^{(0)} = [(T_C \cdot T_C)^T]_{i,j} = \begin{bmatrix}
2 & 2 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix},
\]

(53b)

\[
a_{i,j}^{(n+1)} = \sum_{k,l \in P_C} \tilde{T}_{(i,j),(k,l)} : a_{k,l}^{(n)} \quad \text{for } n \geq 1.
\]

(53c)
The $K_0$ group of the algebra $C^*(\hat{M}_C, \sim)$ is the direct limit of the (scaled dimension) groups $K_0(A_n)$, given in (54), where the (positive unital) homomorphisms between the $K_0$ groups are the multiplicity matrices of the corresponding $\Phi_n : A_n \rightarrow A_{n+1}$ algebra homomorphisms [see (55c)], so

$$K_0(\Phi_0(i,j), k) = T^T_{j,k} \cdot T_{j,k}, \quad K_0(\Phi_n) = \tilde{T}_C = T^T_C \otimes T_C \quad \text{for } n \geq 1, \quad (54)$$

where the matrix $T_C$ is given in (55a).

In order to calculate the direct limit $K_0$ group, we have to know the eigenspace decomposition of the matrix $T_C$ and its transpose $T^T_C$. The eigenvalues and the corresponding right resp. left eigenvectors (eigenspaces) of $T_C$ are:

$$\lambda_0 = 0, \quad E^R_0 = \left\{ \begin{bmatrix} a & b & c \\ c & -b & -c \end{bmatrix} \Bigg| a, b, c \in \mathbb{R} \right\}, \quad E^L_0 = \left\{ \begin{bmatrix} a & b & -c \\ -a & b & -c \end{bmatrix} \Bigg| a, b, c \in \mathbb{R} \right\}; \quad (55a)$$

$$\lambda_s = \frac{3 - \sqrt{5}}{2}, \quad v^R_s = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v^L_s = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad (55b)$$

$$\lambda_u = \frac{3 + \sqrt{5}}{2}, \quad v^R_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v^L_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (55c)$$

It is clear that the rank of the matrices $T_C$ and $T^T_C$ is two, and their ranges are

$$\text{Ran } T_C = \left\{ \begin{bmatrix} a \\ a \\ b \\ b \end{bmatrix} \Bigg| a, b \in \mathbb{R} \right\} \quad \text{and} \quad \text{Ran } T^T_C = \left\{ \begin{bmatrix} a \\ b \\ a \\ b \end{bmatrix} \Bigg| a, b \in \mathbb{R} \right\}. \quad (56)$$

Consequently the rank of $\tilde{T}_C = T^T_C \otimes T_C$ is four, and its four nonzero eigenvalues with the corresponding eigenvectors are

$$v^L_s \otimes v^R_u \quad \text{and} \quad v^L_u \otimes v^R_s \quad \text{for} \quad \tilde{\lambda}_1 = \tilde{\lambda}_2 = 1, \quad (57a)$$

$$v^T_s \otimes v^R_u \quad \text{for} \quad \tilde{\lambda}_s = \lambda_s^2 < 1, \quad (57b)$$

$$v^L_u \otimes v^R_s \quad \text{for} \quad \tilde{\lambda}_u = \lambda_u^2 > 1. \quad (57c)$$

The effect of the matrix $\tilde{T}_C$ on its four dimensional range is already a bijection, which means, according to the formulas (55a-c) in Appendix B.3, that the direct limit group $K_0(C^*(\hat{M}_C, \sim))$, consisting of the series $(\lambda_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} K_0(A_i)$ with ‘predictable tails’ is isomorphic to $\mathbb{Z}^4$, as stated in (58).

A convenient way of realizing the bijection between the range of $\tilde{T}_C$ and $\mathbb{Z}^4 \cong \mathbb{Z}^2 \otimes \mathbb{Z}^2$ is to use the mapping

$$\tilde{P} = P^L \otimes P^R : \mathbb{Z}^{25} \cong \mathbb{Z}^5 \otimes \mathbb{Z}^5 \rightarrow \mathbb{Z}^4 \cong \mathbb{Z}^2 \otimes \mathbb{Z}^2,$$

where

$$P^L = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P^R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}. \quad (58)$$

Indeed, the kernel of $P^L$ resp. $P^R$ is the zero eigenspace $E^L_0$ resp. $E^R_0$ (see (55a)). It is also easy to see that the mapping $P$ preserves the natural order on $\text{Ran } T_C \otimes \text{Ran } T_C$, for the image of an element $a = (a_{i,j})_{i,j \in P_C} \in \text{Ran } T_C \otimes \text{Ran } T_C$.
Ran $T_C$ is positive (i.e., $\forall k, l \in \{1, 2\}$, $\sum_{i,j \in P_C} P_{k,i}^L \cdot P_{l,j}^R \cdot a_{i,j} \geq 0$) if and only if $a$ is positive (i.e., $\forall i, j \in P_C, a_{i,j} \geq 0$).

The effect of $T_C$ resp. $T_T C$ on Ran $T_C$ resp. on Ran $T_T C$ can be encoded by the two by two matrix $Z$, in the sense that for any integer $n \geq 1$

$$P^R \cdot T_C^n = Z^n \cdot P^R \quad \text{and} \quad P^L \cdot (T_C^n)^T = Z^n \cdot P^L,$$

thus $\tilde{P} \cdot T_{C}^{n} = (Z \otimes Z)^{n} \cdot \tilde{P} = (Z^{n} \otimes Z^{n}) \cdot \tilde{P}$, where $Z = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. (59a)

(The connections (59a) can be easily checked by direct calculations.)

The eigenvalues and the corresponding eigenvectors of the matrix $Z$ are

$$\lambda_s = \frac{3 - \sqrt{5}}{2}, \quad \omega_s = \begin{bmatrix} 1 \\ \frac{-1 + \sqrt{5}}{2} \end{bmatrix} ;$$

$$\lambda_u = \frac{3 + \sqrt{5}}{2}, \quad \omega_u = \begin{bmatrix} 1 \\ \frac{\sqrt{5} - 1}{2} \end{bmatrix} ;$$

(60a)

(cf. (55b–c)) and the eigenspace decomposition of $Z \otimes Z$ is quite similar to (57):

$$\omega_s \otimes \omega_s = \begin{bmatrix} 1 \\ \frac{-1 + \sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{5} - 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \omega_u \otimes \omega_u = \begin{bmatrix} 1 \\ \frac{\sqrt{5} - 1}{2} \end{bmatrix} \begin{bmatrix} 3 - \sqrt{5} \\ 2 \end{bmatrix}$$

$$\omega_s \otimes \omega_u = \begin{bmatrix} 1 \\ \frac{-1 + \sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{5} - 1 \\ 3 + \sqrt{5} \end{bmatrix} \quad \text{for } \tilde{\lambda}_1 = \tilde{\lambda}_2 = 1,$$

$$\omega_u \otimes \omega_s = \begin{bmatrix} 1 \\ \frac{-1 + \sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{\sqrt{5} - 1}{2} \end{bmatrix} \quad \text{for } \tilde{\lambda}_u = \lambda_u^2 < 1,$$

$$\omega_u \otimes \omega_u = \begin{bmatrix} 1 \\ \frac{\sqrt{5} - 1}{2} \end{bmatrix} \begin{bmatrix} 3 - \sqrt{5} \\ 2 \end{bmatrix} \quad \text{for } \tilde{\lambda}_u = \lambda_u^2 > 1.$$ (61a)

(61b)

(61c)

Appropriately, only the unstable eigenvector $\omega_u \otimes \omega_u$ is positive.

According to (65a), the positive cone $K^+_0(C^*(M_C, \sim)) \subset Z^2 \otimes Z^2$ consists of the vectors $x = \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} \in Z^2 \otimes Z^2$ whose entries become positive after sufficiently many iterations of the mapping $Z \otimes Z$, i.e.,

$$(Z \otimes Z)^n \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } n \text{ sufficiently large},$$

(62)

which means that the projection of $x$ to the (positive) unstable eigenvector $\omega_u \otimes \omega_u$ has to be nonnegative, so

$$x \cdot (\omega_u \otimes \omega_u) = \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{\sqrt{5} - 1}{2} \end{bmatrix} \begin{bmatrix} \frac{-1 + \sqrt{5}}{2} \\ \frac{-1 + \sqrt{5}}{2} \end{bmatrix} = u + \frac{\sqrt{5} - 1}{2} (v + w) + \frac{3 - \sqrt{5}}{2} z \geq 0,$$

(63)

as stated in (51d). (Here $x$ and $\omega_u \otimes \omega_u$ were considered as four dimensional vectors, and the operation $\cdot$ is the scalar product between them.)
Using (53b), the image of the $K_0$-group element $[\text{id}_{A_1}]$ under $P = PL \otimes PR$ is

$$\sum_{k,l \in \mathcal{P}} P_{i,k} \cdot P_{j,l} \cdot a_{k,l}^{(1)} = \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}.$$  \hfill (64)

It means, according to (B2) and (B5e), that the scale $\Gamma(C^*(\hat{M}_C, \tilde{\sim}))$ consists of the elements $x = \begin{bmatrix} u & v \\ w & z \end{bmatrix} \in K_0^+(C^*(\hat{M}_C, \sim))$, for which after a sufficiently large number $n$ of iterations

$$(Z \otimes Z)^n \cdot \begin{bmatrix} u & v \\ w & z \end{bmatrix} \leq (Z \otimes Z)^n \cdot \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} \quad \text{or} \quad (Z \otimes Z)^n \cdot \begin{bmatrix} 13 - u & 8 - v \\ 8 - w & 5 - z \end{bmatrix} \geq 0$$

(65)

holds, where the order ‘$\leq$’ is the natural one, thus it is valid to each matrix entry separately. This criterion has the same form as (62), and it gives the following inequality for the entries $u, v, w, z \in \mathbb{Z}$:

$$u + \frac{\sqrt{5} - 1}{2} (v + w) + \frac{3 - \sqrt{5}}{2} z \leq \frac{25 + 11\sqrt{5}}{2}.$$ \hfill (66)

This proves the statement (51e).

In order to obtain the form (51b) of the dimension group first let us apply the injective (scaled dimension) group homomorphism

$$\Pi : (\mathbb{Z}^4, K_0^+, \Gamma) \rightarrow (\mathbb{R}, [0, \infty), [0, 1]) \times \mathbb{Z}^2,$$  \hfill (67a)

$$\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} \mapsto \left( \frac{2u + (\sqrt{5} - 1)(v + w) + (3 - \sqrt{5})z}{25 + 11\sqrt{5}}, u, v \right) = \left( \frac{5u - 8(v + w) + 13z}{2} + \frac{\sqrt{5}}{10}( -11u + 18(v + w) - 29z), u, v \right),$$ \hfill (67b)

\hfill (67c)

to the form (51a) of the dimension group. The first component of this homomorphism is essentially (up to a scaling factor) the projection of $\mathbb{Z}^4 \cong \mathbb{Z}^2 \otimes \mathbb{Z}^2$ onto the unstable eigenvector $w_u \otimes w_u$, and the second and third trivial components of the mapping just ensure injectivity. (The order and scale preserving property of the mapping follows directly from its construction.)

The range (of the first component) of the homomorphism $\Pi$ (equation (67)) is clearly in the additive subgroup $\frac{1}{2}\mathbb{Z} + \frac{\sqrt{5}}{10}\mathbb{Z} \subset \mathbb{R}$, but it does not equal to it! Indeed, the general element $\alpha = \frac{1}{2} n + \frac{\sqrt{5}}{10} k$ for $n, k \in \mathbb{Z}$ is in the range of $\Pi$ if and only if the equations

$$n = 5u - 8(v + w) + 13z$$
$$k = -11u + 18(v + w) - 29z$$

(68)

can be satisfied for certain integer values of $u, v, w, z \in \mathbb{Z}$. Treating $u$ and $v$ as parameters, and expressing the other two variables $w$ and $z$ we obtain

$$w = 14n + 6k - u - v + \frac{n + k}{2}$$
$$z = 9n + 4k - u$$

(69)
which means that $n + k$ has to be even, say $n + k = 2m$ where $m \in \mathbb{Z}$. Then $k = 2m - n$, and $\alpha = \frac{5 - \sqrt{5}}{10} n + \frac{\sqrt{5}}{5} m$, thus the homomorphism $\Pi$ is a bijection

$$
\Pi : \mathbb{Z}^3 \longrightarrow \left( \frac{5 - \sqrt{5}}{10} \mathbb{Z} + \frac{\sqrt{5}}{5} \mathbb{Z} \right) \times \mathbb{Z}^2
$$

(70)

given explicity by (67b), and its inverse is

$$
\left( \frac{5 - \sqrt{5}}{10} n + \frac{\sqrt{5}}{5} m, u, v \right) \mapsto \begin{bmatrix}
u \\
8n + 13m - u - v \\
5n + 8m - u
\end{bmatrix}.
$$

(71)

This proves the assertion (51b) of the statement. \qed

Again, turning back to page 21, and comparing the present result with assertion iii) of Statement 8, we see that the noncommutative algebra $C^*(\tilde{M}_C, \sim)$ associated to the cat map as well as its $K_0$ group are really important invariants of this dynamical system.
V Conclusions, outlook

In the preceding sections we have successfully applied certain methods of non-
commutative geometry in the study of a specific uniformly hyperbolic chaotic
dynamical system, Arnold’s cat map. The whole investigation was motivated
by, and performed along the guiding lines of a similar but technically less so-
plicated example: the isomorphism classes of Penrose tilings [Con94].

First a Markov partition of the phase space of the cat map was con-
structed (Lemma 8, on page 20), then a noncommutative, approximately finite dimen-
sional C*-algebra has been associated to the system (Statement 11, on page 30),
and finally its scaled dimension group has been explicitly calculated (State-
ment 13, on page 34). Comparing this result with the last assertion of State-
ment 9 (on page 21), we see that the K0 group coincides with the dense additive
subgroup of reals describing the frequency of appearance of finite symbolic se-
quences in typical symbolic trajectories.

The result of this investigation clearly demonstrates that the methods of
noncommutative geometry are, indeed, adaptable for the study of chaotic dy-
amical systems, and they give better results than the usual topological meth-
ods. It, however, also poses a great amount of further questions to investigate.
First, why is this apparent coincidence between the frequency ratio of finite sym-
bolic sections and the K0 group of the noncommutative algebra? Under what
circumstances does it hold?

But there are also deeper, and from pure theoretical point of view more
interesting questions! To what extent does the noncommutative AF algebra
C*\(\tilde{M}_C, \sim\) associated to the cat map depend on the particular choice of the
Markov partition of the phase space, used in its construction? Is it possible
to generalize this procedure to a wider class of dynamical systems, which are
not necessarily Markovian? Can one do it in a functorial way? If ‘yes’, then
how? Which other invariants of the associated noncommutative C*-algebra are
important from the point of view of the original dynamical system?

There is hope to give positive answer to these questions and to build up a
‘dictionary’ between the notions and constructions in the theory of dynamical
systems and their counterparts in the language of noncommutative geometry. In
this case the noncommutative geometrical approach could shed light on new as-
ppects of the old theory of dynamical systems. According to the results presented
in this paper one row in this dictionary could be something like this:

| ratios of appearances of finite symbolic sections in typical symbolic series | scaled dimension group of the associated C*-algebra |
|---|---|

With this article we would like to encourage the research in this direction.

Addendum

After finishing the manuscript the author became aware of the fact that an even
ersimpler chaotic system, the well known baker’s map [AA68, CFS82] also nicely
fits into the frame of the present investigations. The baker’s map is a hyperbolic
dynamical system, too, possessing a Markov partition of two elements, and there
are no grammar rules at all, thus the transition matrix is simply $T_B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It means that the frequency of appearance of any finite symbolic section of length $n \in \mathbb{N}_+$ is $2^{-n}$, regardless of the actual form of the sequence.

The $C^*$-algebra related to this system is the AF algebra, the Bratteli diagram of which is built up from the block $2^n \rightarrow 2^{n+1} \rightarrow 2^n$. It is a well known algebra called Canonical Anticommutation Relations (CAR algebra) [Dav96], the dimension group of which is the set of diadic rational numbers $K_0 = \{ \frac{p}{2^n} \mid p \in \mathbb{Z}, n \in \mathbb{N} \} \subset \mathbb{R}$ with the natural ordering inherited from the set of real numbers. The scale of the group is simply $[0,1] \cap K_0$.

As a matter of fact, the possible frequencies of appearances of the finite symbolic sequences are again restricted to the elements of the $K_0$ group!

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Appendices

A Commutative and approximately finite dimensional $C^*$-algebras

In this appendix, just for the convenience of the unfamiliar reader, a few facts are summarized about $C^*$-algebras, which are particularly important for the investigations of Section [N]. All the material covered here can be found in any standard textbook on $C^*$-algebras, like e.g. [Dav96] or [WO93]. A concise summary is also contained in [Lan98] or [Lan97].

A.1 $C^*$-algebras

A $C^*$-algebra $A$ is a Banach $\ast$-algebra (in most cases over the complex number field $\mathbb{C}$) satisfying the so called $C^*$-algebra equality $\|a^\ast a\| = \|a\|^2$, for all $a \in A$. In more details, the fact that $A$ is an involutive or $\ast$-algebra, means that in addition to the linear algebraic operations there is an involution $A \to A$, $a \mapsto a^\ast$ given on $A$, which is conjugate linear [i.e., $(\alpha a + \beta b)^\ast = \bar{\alpha} a^\ast + \bar{\beta} b^\ast$], involutive [i.e., $(a^\ast)^\ast = a$], and has the property $(ab)^\ast = b^\ast a^\ast$. At the same time $A$ is a Banach algebra, thus $A$ is a Banach space (i.e., a normed complete vector space), and for the algebraic product of any two elements $a, b \in A$ the inequality $\|ab\| \leq \|a\|\|b\|$ holds. (It follows that the multiplication is separately continuous in both variables.) Two $C^*$-algebras $A$ and $B$ are isomorphic if there is a norm preserving linear bijection $\Phi : A \to B$ between them which commutes with the algebraic multiplication and with the involution, i.e., $\Phi(xy) = \Phi(x)\Phi(y)$, and $\Phi(x^\ast) = (\Phi(x))^\ast$.

The basic example for a $C^*$-algebra is the algebra $B(\mathcal{H})$ of all bounded operators acting on the Hilbert space $\mathcal{H}$, with the operator norm as $C^*$-algebra norm and the operator adjoint as $\ast$-involution. Moreover, as it was shown by Gelfand and Naimark, this example is again generic, i.e., any commutative (unital) $C^*$-algebra $C$ is isometrically $\ast$-isomorphic to a subalgebra of the concrete operator algebra $B(\mathcal{H})$ for an appropriate (not necessarily separable) Hilbert space $\mathcal{H}$.

A.2 Commutative $C^*$-algebras

One of the simplest subcategory of the $C^*$-algebras is the class of commutative $C^*$-algebras. It can be shown that if $X$ is any locally compact Hausdorff space, then the algebra $C_0(X)$ of all $X \to \mathbb{C}$ continuous (complex valued) functions vanishing at infinity, with pointwise addition and multiplication, supremum norm and complex conjugation as $\ast$-operation is a commutative $C^*$-algebra. If, in addition $X$ is compact, then the set $C(X)$ of all continuous functions (that agrees with $C_0(X)$ for compact $X$) forms a unital $C^*$-algebra, where the unit is the constant $1 \in \mathbb{C}$ function on $X$. Moreover, this example is again generic, i.e., any commutative (unital) $C^*$-algebra $C$ is isometrically $\ast$-isomorphic to the function algebra $C_0(\mathcal{H}) \cdot C(X)$ in the unital case], where the locally compact (compact) Hausdorff space $X$ can be naturally constructed from the algebra $C$. It means that the commutative function algebra $C(X)$ encodes all topological information about the space $X$. 

41
These facts can be expressed in a more abstract and concise way by stating that $C$ (resp. $C_0$) is a contravariant invertible functor from the category of compact (resp. locally compact Hausdorff) topological spaces with continuous (proper) maps to the category of unital (resp. non unital) commutative $C^*$-algebras with $*$-preserving algebra morphisms. On a continuous map $\Phi : X \to Y$ between two topological spaces the effect of the functor $C$ (or $C_0$) is defined by the pull back operation $C(\Phi) = \Phi^* : C(Y) \to C(X)$, $f \mapsto f \circ \Phi$.

Thus the topological study of locally compact (compact) Hausdorff spaces is tantamount to the algebraic study of commutative (unital) $C^*$-algebras. Given a pure topological statement, one can at will interpret it as an algebraic statement concerning commutative $C^*$-algebras and vice versa. Observations of this type constitute the basic philosophy of noncommutative geometry, which tries to generalize geometric concepts, statements and theories, interpreted in the language of commutative algebras, to noncommutative algebras. Of course, in the latter case the direct classical geometrical picture is completely missing.

A nice demonstration of these guiding principles is the emergence of algebraic $K$-theory from topological $K$-theory which is the subject of Appendix B.

### A.3 Finite dimensional $C^*$-algebras

The simplest noncommutative $C^*$-algebras are the finite dimensional ones. It can be shown that they always have the form $\bigoplus_{i=1}^k \mathcal{M}_{n_i}$, i.e., they are direct sums of full matrix algebras $\mathcal{M}_{n_i}$ of $n_i \times n_i$ complex matrices ($n_i \in \mathbb{N}^+$, $i = 1, 2, \ldots, k$). Finite dimensional $C^*$-algebras are always unital, and they are uniquely characterized (up to isomorphism) by the finite set $\{n_1, n_2, \ldots, n_k\}$ of numbers. It is also an elementary fact that given a unit preserving $*$-algebraic morphism $\Phi : A \to B$ between two finite dimensional $C^*$-algebras $A = \bigoplus_{i=1}^k \mathcal{M}_{n_i}$ and $B = \bigoplus_{j=1}^l \mathcal{M}_{m_j}$ ($n_i, m_j \in \mathbb{N}^+$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$), $\Phi$ can be uniquely decomposed into the form $\Phi = \bigoplus_{i=1}^k \Phi_i$ (of size $l \times k$) built up from the partial multiplicities $a_{ji} \in \mathbb{N}$ of the mappings $\phi_{ij}$. Since $\Phi$ is unital, $A$ satisfies the equation $m_j = \sum_{i=1}^k a_{ji} n_i$. This decomposition of $\Phi$ can be graphically visualized by the so called Bratteli diagram [Bra72]

\[
\begin{array}{c}
\mathcal{A} : n_1 & \cdots & n_k \\
| & \ddots & | \\
\vdots & \ddots & \ddots \\
| & \ddots & \ddots \\
\mathcal{B} : m_1 & \cdots & m_l
\end{array}
\]

where the two rows symbolize the direct sum decomposition of the finite dimensional algebras $\mathcal{A}$ and $\mathcal{B}$, and the arrows, labeled by the multiplicities $a_{ji}$, represent the partial embeddings $\phi_{ij}$. In practice the arrows of zero multiplicity are omitted, and the multiplicities of low degree are denoted by single, double, triple arrows instead of labels.

If, for example $\mathcal{A} = \mathcal{M}_1 \oplus \mathcal{M}_2$, $\mathcal{B} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$, and the unital homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ is given by

\[
\begin{array}{cccc}
\mathcal{A} : & n_1 & n_2 & \cdots & n_k \\
\mathcal{B} : & m_1 & m_2 & \cdots & m_l
\end{array}
\]
\[ \Phi : \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \]

then the multiplicity matrix \( A_\Phi \) and the Bratteli diagram are

\[ A_\Phi = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \]

\[ \begin{array}{c}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (2,0) {3};
  \draw[->] (1) to (2);
  \draw[->] (2) to (3);
\end{tikzpicture}
\end{array} \]

\[ \begin{array}{c}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (2,0) {3};
  \draw[->] (1) to (2);
  \draw[->] (2) to (3);
\end{tikzpicture}
\end{array} \]

\subsection{A.4 Approximately finite dimensional \( C^* \)-algebras}

Among infinite dimensional \( C^* \)-algebras in many respects the simplest ones are the (unital) approximately finite dimensional (AF) \( C^* \)-algebras, which are defined as the increasing union \( A = \lim_{\downarrow} A_i = \bigcup_{i \in \mathbb{N}} A_i \) of a (directed) sequence \( \{A_i, \Phi_i\}_{i \in \mathbb{N}} \) of finite dimensional algebras \( A_i \), where the mappings \( \Phi_i : A_i \hookrightarrow A_{i+1} \) (\( i \in \mathbb{N} \)) are unit preserving injective homomorphisms. (Most of the results easily carry over to the nonunital case, where the image \( \Phi_i(1_i) \) of the unit \( 1_i \in A_i \) is a proper projection in \( A_{i+1} \). We, however, — retaining the definition of Bratteli \cite{Bra72} — disregard this case, just for avoiding unnecessary complications.)

Because of the construction of the AF algebra \( A = \lim_{\downarrow} A_i \), every finite subset \( \{a_1, a_2, \ldots, a_n\} \subset A \) of it can be approximated in norm with elements \( \{b_1, b_2, \ldots, b_n\} \subset A_k \subset A \) from a finite dimensional subalgebra \( A_k \) in the way that \( \|a_i - b_i\| < \varepsilon \) for any arbitrarily prescribed \( \varepsilon > 0 \) and \( i \in \{1, 2, \ldots, n\} \). The above statement can also be reversed, namely if in a separable algebra \( A \) any finite set of elements can be uniformly approximated with elements from a finite dimensional subalgebra, then \( A \) is approximately finite dimensional \cite{Bra72}.

We remark that given an increasing directed sequence \( A_0 \xrightarrow{\Phi_0} A_1 \xrightarrow{\Phi_1} A_2 \xrightarrow{\Phi_2} \cdots \) of finite dimensional algebras \( A_i \) with unital inclusions \( \Phi_i, \) the AF algebra \( A = \lim_{\downarrow} A_i \) is uniquely determined, but for a given AF algebra \( A \) the defining sequence \( \{A_i, \Phi_i\}_{i \in \mathbb{N}} \) is far from being unique.

The simplest way for the graphical representation of the AF algebra \( A = \lim_{\downarrow} A_i \) is drawing the Bratteli diagrams of the subsequent mappings \( \Phi_i : A_i \hookrightarrow A_{i+1} \) in a single chain.

For example, the unital algebra \( A = \mathbb{C} 1 \oplus K \) generated by the identity \( 1 \) and the compact operators \( K \) of a separable Hilbert space \( \mathcal{H} \) is the injective limit of the sequence \( \{A_i = \mathbb{C} 1 \oplus P_i \mathcal{H} P_i, \Phi_i\}_{i \in \mathbb{N}} \), where \( P_i : \mathcal{H} \to \mathcal{H} \) is the projection onto the closed subspace of \( \mathcal{H} \) spanned by the first \( i \) vectors of a selected orthonormal basis of \( \mathcal{H} \), and \( \Phi_i \) is the natural inclusion \( A_i \hookrightarrow A_{i+1} \). Thus the multiplicity matrix \( A_0 \) of \( \Phi_0, A_i \) of \( \Phi_i \) (for all \( i \geq 1 \)) and the Bratteli diagram of this AF algebra are clearly

\[ A_0 = \begin{bmatrix} 1 \end{bmatrix}, \quad A_i = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \]

\[ \begin{array}{c}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {1};
  \node (3) at (2,0) {1};
  \node (4) at (3,0) {1};
  \node (5) at (4,0) {1};
  \node (6) at (5,0) {1};
  \draw[->] (1) to (2);
  \draw[->] (2) to (3);
  \draw[->] (3) to (4);
  \draw[->] (4) to (5);
  \draw[->] (5) to (6);
\end{tikzpicture}
\end{array} \]

(In this diagram the mappings are drawn from left to right, for convenience.)
B K-theory of approximately finite dimensional 
$C^*$-algebras

In this appendix we shall have a quick glance into the algebraic K-theory, fo-
cusing the attention on the narrow slice of the theory needed for the calculation
of the scaled dimension group of approximately finite dimensional $C^*$-algebras,
which is a complete invariant of this class of algebras, according to the result
of G. A. Elliott [Ell76]. As a general reference on K-theory of $C^*$-algebras we
recommend [WO93] and Chapter IV of [Dav96] for the case of AF $C^*$-algebras.

Algebraic K-theory has its roots in topological K-theory [Ati67], and as
the sprouting of the algebraic theory provides an excellent example how the
basic philosophy of noncommutative geometry works in practice, we start our
overview —following the chronological way— by the definition of the $K^0(X)$
group of a topological space $X$, which is the basic ingredient of topological
K-theory.

B.1 The topological $K^0$ group

Roughly speaking $K^0$ is a contravariant functor which associates to each
compact topological space $X$ an ordered Abelian group $K^0(X)$, whose elements
are equivalence classes of formal differences of vector bundles over $X$. The
group $K^0(X)$ is a homotopy invariant of $X$, and it nicely fits into certain ex-
act sequences, but these properties are beyond our present scope. The precise
definition of $K^0(X)$ is as follows.

Given a compact topological space $X$, let $\mathcal{V}(X)$ denote the set (of equivalence
classes) of all locally trivial complex vector bundles over the base space $X$. For
simplicity the notation $[\nu]|_X$ is used for the $n$-dimensional trivial bundle
$\mathbb{C}^n \times X \xrightarrow{\tau} X$, where $n \in \mathbb{N}$. From two vector bundles
$\mathcal{E} = (E \xrightarrow{\pi_E} X)$ and $\mathcal{F} = (F \xrightarrow{\pi_F} X)$ over the same base space $X$ one can construct the Whit-
ney sum $\mathcal{E} \oplus \mathcal{F} \in \mathcal{V}(X)$, which is essentially the fiberwise direct sum of the two
bundles, i.e., its fiber over the point $x \in X$ is $E_x \oplus F_x$. This operation $\oplus :$
$\mathcal{V}(x) \times \mathcal{V}(x) \to \mathcal{V}(x)$ defines a commutative semigroup structure on $\mathcal{V}(x)$, with
zero element (semigroup unit) $[0]_X$.

The topological $K^0(X)$ group is obtained from the semigroup $\mathcal{V}(X)$ by the
Grothendieck construction, i.e., basically in the same way as the additive group
of integers $\mathbb{Z}$ is obtained from the additive semigroup $\mathbb{N}$ of the natural
numbers, by considering (equivalence classes of) formal differences $[a - b] \in \mathbb{Z}$ of
natural numbers $a, b \in \mathbb{N}$. (The set $\mathbb{Q}_+$ of positive rational numbers is also a
Grothendieck group constructed from the multiplicative semigroup $\mathbb{N}_+$ of posi-
tive natural numbers by forming formal fractions $[\frac{p}{q}] \in \mathbb{Q}_+, p, q \in \mathbb{N}_+$.)

Generally the Grothendieck construction consists of two steps. Given a commu-
tative semigroup $(S, +)$ with zero element $0 \in S$, first the factor semigroup
$\tilde{S} = S/\sim$ is created, where two elements $a, b \in S$ are $\sim$-equivalent, $a \approx b$, if and
only if there is a third element $c \in S$ such that $a + c = b + c$. This factorization
ensures that $\tilde{S}$ is a cancellation semigroup. (The commutative semigroup $S$ is
called cancellation semigroup if it has the cancellation property, i.e., whenever
$a + c = b + c$ holds for arbitrary elements $a, b, c \in S$ then $a = b$.)

Second, the Grothendieck group $G$ of $S$ is defined by the formula $G = \tilde{S}_2/\sim$.
Writing the elements of $\tilde{S}^2$, for convenience, in the form of formal differences,
i.e., $x - y = (x, y) \in \tilde{S}^2$, the elements $a - b, c - d \in \tilde{S}^2$ are ~-equivalent, $(a - b) \sim (c - d)$, if and only if $a + d = c + b$ holds in $\tilde{S}$. The group addition in $G$ is defined by $[a - b] + [c - d] = [(a + c) - (b + d)]$. It is straightforward to verify that the above given group operation in $G$ is well defined on the ~-equivalence classes, and it defines a commutative group structure with unit $[0 - 0] \equiv [s - s]$ (for any $s \in \tilde{S}$). The cancellation semigroup $S$ is naturally considered as a sub-semigroup of the Grothendieck group $G$ defined by the inclusion $\tilde{S} \rightarrow G$, $s \mapsto [s - 0]$, and the subset $\tilde{S} \subset G$ is generating (by the definition of $G$), i.e., $G = \tilde{S} - \tilde{S}$ holds. It is worth remarking that the cancellation property of $\tilde{S}$ is needed for the transitivity of the relation ~.

Our previous examples of the Grothendieck group were a bit untypical, since both the additive semigroup $\mathbb{N}$ and the multiplicative semigroup $\mathbb{N}_+$ have the cancellation property, so in these cases the first step of the construction is unnecessary, $\mathbb{N} \cong \mathbb{N}, \mathbb{N}_+ \cong \mathbb{N}_+$.

In general, however, the semigroup $\mathcal{V}(X)$ of vector bundles does not have the cancellation property, so first the factor semigroup $\mathcal{V}(X)/\sim$ has to be constructed, which is denoted by $K^0(X)$, and then the group $K^0(X) = K^0+ (X) \times K^0+(X)/\sim$ is by definition the Grothendieck group of $\mathcal{V}(X)$. Thus, putting the two steps of the Grothendieck construction together, the elements of the group $K^0(X)$ are equivalence classes of formal differences of vector bundles $E_i, F_i \in \mathcal{V}(X)$ (here $i = 1, 2$), two such objects being equivalent, $(E_1 - F_1) \sim (E_2 - F_2)$, if and only if there is a third bundle $G \in \mathcal{V}(X)$ such that $E_1 + F_2 + G \cong E_2 + F_1 + G$, and the sum of two classes is $[E_1 - F_1] + [E_2 - F_2] = [(E_1 + E_2) - (F_1 + F_2)]$.

As a counter example for the cancellation property of $\mathcal{V}(X)$ consider the real tangent resp. normal bundle $TS^2 \in \mathcal{V}(\mathbb{R}(S^2))$ resp. $NS^2 \cong [1]_{S^2} \in \mathcal{V}(\mathbb{R}(S^2))$ as well as the trivial real one dimensional bundle $[1]_{S^2} \oplus \mathbb{R}$ over the two dimensional sphere $S^2 \subset \mathbb{R}^3$ as base space. Clearly, $TS^2 + NS^2 \cong [2]_{S^2} \oplus [1]_{\mathbb{R}} \cong [3]_{\mathbb{R}}$, holds, but $TS^2 \not\cong [2]_{S^2}$.

A continuous map $\Phi : X \rightarrow Y$ between two (compact) topological spaces $X$ and $Y$ induces a ‘pull back’ map $\Phi^* : \mathcal{V}(Y) \rightarrow \mathcal{V}(X)$ between the sets of vector bundles over the two spaces in reverse order, in such a way that for $E = (E \rightarrow Y) \in \mathcal{V}(Y)$ the induced pull back bundle $\Phi^* E \in \mathcal{V}(X)$ has the fiber $E_{\Phi(x)}$ at the point $x \in X$. It turns out that $\Phi^*$ commutes with the Whitney sum operation \[ i.e., \Phi^*(E \oplus F) \cong \Phi^*E \oplus \Phi^*F \text{ for } E, F \in \mathcal{V}(Y) \text{,} \] respects the equivalence classes, and thus it induces a group homomorphism $K^0(\Phi) : K^0(Y) \rightarrow K^0(X)$. It means that $K^0$ is a contravariant functor from the category of topological spaces to the category of commutative groups.

### B.2 The algebraic $K_0$ group

Now the definition of the (topological) $K_0$ group is rephrased in an algebraic language and extended from commutative to noncommutative $C^*$-algebras, according to the basic philosophy of noncommutative geometry. Again, for the sake of simplicity, we consider only unital $C^*$-algebras.

As it has been seen in Appendix A.2, the compact base space $X$ is replaced with the unital commutative $C^*$-algebra $C(X)$ of continuous $X \rightarrow \mathbb{C}$ functions. The algebraic counterpart of a given bundle $E = (E \rightarrow X)$ is the set $\Gamma(E)$ of its continuous sections (here $\Gamma(E) = \{ f : X \rightarrow E \mid f \text{ is continuous, } \pi_E \circ f = id_X \}$), which has a $C(X)$-module structure by fiberwise defined operations. Moreover, the module $\Gamma(E)$ is always projective and finitely generated, what is the algebraic
equivalence of Swan’s theorem (see [WO93, Theorem 13.1.6]) stating that for every vector bundle $E \in \mathcal{V}(X)$ over the compact base space $X$ there is an ‘orthogonal complement’ bundle $F \in \mathcal{V}(X)$ such that $E \oplus F \cong [n]_X$ is trivial. This means that the module $\Gamma(E)$ is projective and finitely generated, i.e., it can be described as the range of a $(C(X)$-linear) projection $P : \Gamma([n]_X) \to \Gamma([n]_X)$ (which acts fiberwise on the bundle $[n]_X \cong \mathcal{E} \oplus \mathcal{F}$), and since $\Gamma([n]_X) \cong \{ f : X \to C^n \mid f \text{ is continuous} \}$, the projection $P$ is a selfadjoint idempotent element of the $n \times n$ matrix algebra $M_n(C(X)) \cong M_n \otimes C(X)$ with entries in the function space $C(X)$. (Here $M_n = M_n(\mathbb{C})$ is the usual complex matrix algebra.) The converse of these statements is also true, i.e., given a projection (selfadjoint idempotent) $P \in M_n(C(X))$, there is a vector bundle $E \in \mathcal{V}(X)$ such that $\Gamma(E) \cong \text{Ran} \ P$. (These results are due to Serre and Swan [Ser58, Swa62].)

By these observations the geometric structure of a bundle $E = (E, \pi_E, X) \in \mathcal{V}(X)$ is entirely characterized in purely algebraic terms; the base space $X$ is replaced with the commutative algebra $C(X)$, and the structure of the bundle is encoded by a projection $P \in M_n(C(X))$ of the $n \times n$ matrix algebra over $C(X)$. It can also be shown that two projections $P \in M_n(C(X))$ and $Q \in M_m(C(X))$ describe isomorphic bundles if and only if they are von Neumann equivalent [MvN36] $P \sim Q$, which means that their ranges are isometric, i.e., there exists a partial isometry $U : M_n(C(X)) \to M_m(C(X))$ with support projection $P = U^*U$ and range projection $Q = UU^*$ [MvN36].

In this algebraic formulation the commutativity of the algebra $C(X)$ has never been explicitly exploited, and in principle nothing prevents the substitution of a general noncommutative $C^*$-algebra for the commutative function algebra $C(X)$. The amazing news of noncommutative topology is the astounding fact that the constructions and the main results of topological $K$-theory do survive this rather drastic change of omitting commutativity of the algebra $C(X)$, and along the guiding lines of the topological theory a similar, even richer theory can be established for noncommutative $C^*$-algebras called algebraic $K$-theory. In the rest of this appendix the construction of the $K_0(C)$ group of a general $C^*$-algebra $C$ is surveyed, and then the scaled dimension group $(K_0(A), K_0^+(A), \Gamma(A))$ is introduced for an approximately finite dimensional algebra $A$.

Given a (noncommutative) $C^*$-algebra $C$, let $M_\infty(C) = \bigcup_{n=1}^{\infty} M_n(C)$ denote the (non-complete) $*$-algebra of all finite dimensional matrix algebras with entries in $C$ [where $M_n(C) = M_n \otimes C$ is considered as a subalgebra of $M_{n+1}(C)$ by the inclusion $M_n(C) \hookrightarrow M_{n+1}(C)$ in the upper left corner $a \mapsto \text{diag}(a, 0)$], and let $\mathcal{P}(C)$ be the set of all projections in $M_\infty(C)$. The algebraic counterpart of the set (of isomorphism classes) of vector bundles is $\mathcal{V}(C) = \mathcal{P}(C)/\sim$, where $\sim$ is the von Neumann equivalence (see above). With the addition $\oplus : \mathcal{V}(C) \times \mathcal{V}(C) \to \mathcal{V}(C)$, $[P] \oplus [Q] = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ and $[0]$ as zero element $\mathcal{V}(C)$ is a commutative semigroup, and the algebraic $K_0(C)$ group is by definition its Grothendieck extension.

As in the topological theory, $\mathcal{V}(C)$ need not have the cancellation property, thus the first step of the Grothendieck construction is to introduce the cancellation semigroup $K_0^+(\mathcal{C}) = \mathcal{V}(C)/\sim$. We have again $K_0^+(\mathcal{C}) \subset K_0(\mathcal{C})$ and $K_0(\mathcal{C}) = K_0^+(\mathcal{C}) - K_0^+(\mathcal{C})$.

A unital $*$-algebraic morphism $\Phi : \mathcal{C} \to \mathcal{D}$ between two $C^*$-algebras induces morphisms $\Phi_* : M_n(\mathcal{C}) \to M_n(\mathcal{D})$ between the matrix algebras (applying $\Phi$
To overcome this difficulty, the invariant of the finite dimensional full matrix algebras with entries in \( C \) ordered dimension group of every finite dimensional matrix algebra over \( A \) and only if \( \text{Rank } P \leq n \). It means that the projections \( x, y \) to the function space over \( X \), the algebraic and topological \( K \)-groups coincide, in the sense that \( K_0(\mathcal{B}) \cong C(X) \).

B.3 The scaled dimension group of AF algebras

From this point on we restrict our attention to approximately finite dimensional \( C^* \)-algebras denoted by \( \mathcal{A} \). It can be proven that in this case \( K_0^+(\mathcal{A}) \) coincides with \( \mathcal{V}(\mathcal{A}) \) (thus \( \mathcal{V}(\mathcal{A}) \) has already the cancellation property), and \( K_0^+(\mathcal{A}) \) is generated by the (equivalence classes of) projections in \( \mathcal{A} \). Moreover, the (generating) semigroup \( K_0^+(\mathcal{A}) \subset K_0(\mathcal{A}) \) has the property \( K_0^+(\mathcal{A}) \cap ( - K_0^+(\mathcal{A}) ) = \{ 0 \} \), which means that \( K_0^+(\mathcal{A}) \) defines an order on the group \( K_0(\mathcal{A}) \) by \( x \geq y \) if and only if \( x - y \in K_0^+(\mathcal{A}) \) for all \( x, y \in K_0(\mathcal{A}) \). The ordered group \( (K_0(\mathcal{A}), K_0^+(\mathcal{A})) \) is called the dimension group of the AF algebra \( \mathcal{A} \). For the sake of brevity and for convenience, from now \( K_0(\mathcal{A}) \) refers to the ordered group of the AF algebra \( \mathcal{A} \), even if the order structure is not denoted explicitly.

As examples, we give now the dimension groups of finite dimensional \( C^* \)-algebras.

First let \( \mathcal{A} = \mathcal{M}_k \) be a full matrix algebra with entries in \( C \). [For simplicity, \( \mathcal{M}_k \) denotes \( \mathcal{M}_k(C) \).] The algebra \( \mathcal{M}_n(\mathcal{A}) \cong \mathcal{M}_{nk} \) is again a full matrix algebra (over \( C \)), thus the equivalence classes of projections in \( \mathcal{M}_\infty(\mathcal{A}) \cong \mathcal{M}_\infty(C) = \mathcal{M}_\infty \) are labeled by the rank of the projections, which can be any natural number. It means that \( K_0^+(\mathcal{M}_k) \cong \mathbb{N} \), and the dimension group of \( \mathcal{M}_k \) is the additive group of integers with its usual order, \( K_0(\mathcal{M}_k) \cong (\mathbb{Z}, \mathbb{N}) \).

Now let \( \mathcal{A} = \bigoplus_{i=1}^k \mathcal{M}_{n_i} \) be a general finite dimensional \( C^* \)-algebra over \( C \). Every finite dimensional matrix algebra over \( \mathcal{A} \) splits into the direct sum of \( k \) full matrix algebras with entries in \( C \), because

\[
\mathcal{M}_s(\mathcal{A}) \cong \mathcal{M}_s \otimes \left( \bigoplus_{i=1}^k \mathcal{M}_{n_i} \right) = \bigoplus_{i=1}^k \mathcal{M}_s \otimes \mathcal{M}_{n_i} \cong \bigoplus_{i=1}^k \mathcal{M}_{s n_i}.
\]  

(B1)

It means that the projections \( P, Q \in \mathcal{M}_s(\mathcal{A}) \) have the decomposition \( P = \bigoplus_{i=1}^k P_i, Q = \bigoplus_{i=1}^k Q_i \) where \( P_i, Q_i \) are projections in \( \mathcal{M}_{s n_i} \), and \( P \sim Q \) if and only if \( \text{Rank } P_i = \text{Rank } Q_i \) for all \( i = 1, 2 \ldots k \). Thus \( K_0^+(\mathcal{A}) \cong \mathbb{N}^k \), and the ordered dimension group of \( \mathcal{A} \) is \( K_0(\mathcal{A}) \cong (\mathbb{Z}^k, \mathbb{N}^k) \).

These two examples demonstrate that the dimension group is not a complete invariant of the finite dimensional \( C^* \)-algebra \( \mathcal{A} = \bigoplus_{i=1}^k \mathcal{M}_{n_i} \), since it does not ‘feel’ the dimensions \( n_i \) of the matrix algebras \( \mathcal{M}_{n_i} \), the algebra \( \mathcal{A} \) is built of. To overcome this difficulty, the scale \( \Gamma(\mathcal{A}) \) is introduced, which is a subset of \( K_0^+(\mathcal{A}) \), and in the case when \( \mathcal{A} \) is unital it is simply

\[
\Gamma(\mathcal{A}) = \{ [P] \in K_0^+(\mathcal{A}) | [P] \leq [1_\mathcal{A}] \},
\]  

(B2)

where \( 1_\mathcal{A} \) is the identity (largest projection) of \( \mathcal{A} \).

The scaled dimension group \( (K_0(\mathcal{A}), K_0^+(\mathcal{A}), \Gamma(\mathcal{A})) \) of the finite algebra \( \mathcal{A} =
\[ \bigoplus_{i=1}^{k} M_{n_i}, \text{ denoted again sloppily by the single symbol } K_0(A) \text{ is clearly} \]

\[ K_0(A) \cong \left( \mathbb{Z}^k, \mathbb{N}^k, [0, 0, \ldots, 0, (n_1, n_2, \ldots, n_k)] \right) \cong \left( \bigoplus_{i=1}^{k} \mathbb{Z}[1/n_i], \bigoplus_{i=1}^{k} \mathbb{N}[1/n_i], [0, 0, \ldots, 0, (1, 1, \ldots, 1)] \right), \]

(B3a)

\[ \text{where } [0, 0, \ldots, 0, (n_1, n_2, \ldots, n_k)] \text{ denotes the points } (p_1, p_2, \ldots, p_k) \in K_0^+(A) \text{ for which } 0 \leq p_i \leq n_i \text{ holds } (i = 1, 2, \ldots, k), \text{ and } \mathbb{Z}[q] \text{ (resp. } \mathbb{N}[q]) \text{ denotes the additive (semi-) group generated by } \mathbb{Z} \text{ and } q \in \mathbb{R}_+ \text{ (resp. by } \mathbb{N} \text{ and } q \in \mathbb{R}_+). \]

In (B3b), for convenience, the dimension group \( (K_0(A), K_0^+(A)) \) was ‘scaled’ so that the scale \( \Gamma(A) \) should have a standard form. It is visible that the scaled dimension group already distinguishes between different dimensions \( n_i \). Even more is true: the scaled dimension group is a complete invariant of \( \text{AF } C^* \)-algebras. \([\text{Dav96}]\).

Now, as a first step towards the description of the \( K_0 \) group of a given AF algebra let us investigate the effect of the functor \( K_0 \) on a unital \(*\)-algebra homomorphism \( \Phi : A \to B \) between two finite dimensional algebras \( A = \bigoplus_{i=1}^{k} M_{n_i} \) and \( B = \bigoplus_{j=1}^{l} M_{m_j} \) with multiplicity matrix \( A_\Phi = [a_{ji}] \). (The relation \( m_j = \sum_{i=1}^{k} a_{ji}n_i \) holds between the dimensions, because \( \Phi \) is unit preserving.)

Since in any finite dimensional (or even AF) algebra the (equivalence classes of) projections generate the \( K_0^+ \) semigroup, it is enough to investigate the effect of \( K_0(\Phi) \) on the projections. Every projection \( P \in A \) decomposes in the form \( P = \bigoplus_{i=1}^{k} P_i \) (where the projections \( P_i \in M_{n_i} \)), according to the direct sum decomposition of the algebra itself, and the element \( [P] \in K_0^+(A) \) is represented by the \( n \)-tuple \( (\text{Rank } P_1, \text{Rank } P_2, \ldots, \text{Rank } P_k) \in \mathbb{N}^k \) in the dimension group \([\text{B3a}]\). The image of \( P \) is also a projection \( Q = \Phi(P) = \bigoplus_{j=1}^{l} Q_j \) (where \( Q_j \) is a projection in \( M_{m_j} \)), and \( \text{Rank } Q_j = \sum_{i=1}^{k} a_{ji} \text{ Rank } P_i \). Since \( K_0(\Phi)[P] = [\Phi(P)] = [Q] \), the effect of \( K_0(\Phi) \) on the dimension group is simply described by the multiplication with the multiplicity matrix \( A_\Phi \), i.e.:

\[ K_0(\Phi) : K_0(A) \cong \mathbb{Z}^k \quad \to \quad K_0(B) \cong \mathbb{Z}^l, \quad \quad (B4a) \]

\[ u = (u_1, u_2, \ldots, u_k) \quad \mapsto \quad A_\Phi u = \left( \sum_{i=1}^{k} a_{ji}u_i \right)_{j=1}^{l}. \quad (B4b) \]

We remark that \( K_0(\Phi)(K_0^+(A)) \subset K_0^+(B) \), since the entries \( a_{ji} \) of the matrix \( A_\Phi \) are nonnegative numbers, \( K_0(\Phi)(\Gamma(A)) \subset \Gamma(B) \), and \( K_0(\Phi)([id_A]) = [id_B] \) since \( \Phi \) is unital, i.e., \( m_j = \sum_{i=1}^{k} a_{ji}n_i \) holds. Such kind of morphisms between scaled dimension groups are called positive unital homomorphisms.

A very nice property of the \( K_0 \) functor is that it commutes with direct limit (\([\text{Dav97}]\), Theorem IV.3.3), thus given an AF algebra \( A = \lim_i A_i \) with unital inclusions \( \Phi_i : A_i \to A_{i+1} \), the algebraic \( K_0 \) group of \( A \) is \( K_0(A) = \lim K_0(A_i) \), where the direct limit of (scaled dimension) groups on the right hand side is understood with the (positive unital) group homomorphisms \( K_0(\Phi_i) : K_0(A_i) \to K_0(A_{i+1}) \).

The direct limit group \( K_0(A) = \lim K_0(A_i) \) is defined abstractly, up to isomorphism, by its universal property (see \([\text{WO93}]\), Appendix L). For calculations, however, it is good to have a concrete realization of \( K_0(A) \). It can be defined as

\[ K_0(A) = \prod_{i \in \mathbb{N}} K_0(A_i) / \sim, \quad (B5a) \]
where the ‘primed’ product $\prod_{i \in \mathbb{N}}' K_0(\mathcal{A}_i)$ is the subset of the infinite direct product space $\prod_{i \in \mathbb{N}} K_0(\mathcal{A}_i)$ consisting of the elements with ‘predictable tails’, i.e.,

$$\prod_{i \in \mathbb{N}}' K_0(\mathcal{A}_i) = \left\{ (x_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} K_0(\mathcal{A}_i) \mid \exists n \in \mathbb{N}, \text{ such that } (\forall j > n) K_0(\Phi_j)(x_j) = x_{j+1} \right\},$$

(B5b)

and two elements $(x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}}' K_0(\mathcal{A}_i)$ are $\sim$-equivalent, if their tails coincide, i.e.,

$$(x_i)_{i \in \mathbb{N}} \sim (y_i)_{i \in \mathbb{N}} \iff \exists n \in \mathbb{N}, \text{ such that } (\forall j > n) x_j = y_j.$$  

(B5c)

The definition of the positive elements $K_0^+(\mathcal{A})$ resp. the scale $\Gamma(\mathcal{A})$ is now straightforward, they consists of the series $(x_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}}' K_0(\mathcal{A}_i)$ whose tail is positive resp. in the scale, i.e.,

$$K_0^+(\mathcal{A}) = \left\{ [(x_i)_{i \in \mathbb{N}}] \in K_0(\mathcal{A}) \mid \exists n \in \mathbb{N} \text{ such that } (\forall j > n) x_j \in K_0^+(\mathcal{A}_j) \right\},$$

(B5d)

$$\Gamma(\mathcal{A}) = \left\{ [(x_i)_{i \in \mathbb{N}}] \in K_0(\mathcal{A}) \mid \exists n \in \mathbb{N} \text{ such that } (\forall j > n) x_j \in \Gamma(\mathcal{A}_j) \right\}.$$

(B5e)

Particularly, if from a threshold index all the homomorphisms $K_0(\Phi_i) : K_0(\mathcal{A}_i) \to K_0(\mathcal{A}_{i+1})$ are injective, then

$$K_0(\mathcal{A}) = \bigcup_{i=1}^{\infty} K_0(\mathcal{A}_i), \quad K_0^+(\mathcal{A}) = \bigcup_{i=1}^{\infty} K_0^+(\mathcal{A}_i) \quad \text{and} \quad \Gamma(\mathcal{A}) = \bigcup_{i=1}^{\infty} \Gamma(\mathcal{A}_i),$$

(B6a)

where the right hand side of the equations are increasing unions defined by $K_0(\Phi_i)$, i.e.,

$$K_0(\mathcal{A}_0) \xrightarrow{K_0(\Phi_0)} K_0(\mathcal{A}_1) \xrightarrow{K_0(\Phi_1)} K_0(\mathcal{A}_2) \xrightarrow{K_0(\Phi_2)} K_0(\mathcal{A}_3) \to \cdots,$$

(B6b)

and similar inclusions hold for $K_0^+(\mathcal{A}_i)$ as well as $\Gamma(\mathcal{A}_i)$. (This injectivity property of $K_0(\Phi_i)$ holds for the algebra associated to the Penrose tilings [see Statements 10 and 12] and for the example of compact operators discussed below, but it is not true for the algebra associated to the cat map [see Statements 11 and 13].)

As a final example we present the scaled dimension group of the unital AF algebra $\mathcal{A} = \mathbb{C}1 \oplus \mathcal{K} = \varinjlim \mathcal{A}_i$ of compact operators $\mathcal{K}$ extended with the unit 1, defined by the Bratteli diagram (A4) at the end of Section A.4. According to the previous result (B3a), the scaled dimension group of the finite dimensional algebras $\mathcal{A}_i = \mathbb{C}1 \oplus P_i\mathcal{K}P_i \cong \mathcal{M}_1 \oplus \mathcal{M}_i$ (where $P_i$ is a projection of rank $i$) are

$$K_0(\mathcal{A}_0) \cong (\mathbb{Z}, \mathbb{N}, \{0,1\}),$$

(B7a)

$$K_0(\mathcal{A}_i) \cong (\mathbb{Z}^2, \mathbb{N}^2, \{0,1\} \times \{0,1 \ldots i\}), \quad \text{for } i \geq 1,$$

(B7b)

and the (scaled dimension) group homomorphisms $K_0(\Phi_i) : K_0(\mathcal{A}_i) \to K_0(\mathcal{A}_{i+1})$, which are all injective, are defined by the multiplicity matrices $A_0 = \begin{bmatrix} 1 \end{bmatrix}$,
Figure 12: The shearing transformation between the dimension groups $K_0(A_2)$ and $K_0(A_3)$.

Figure 13: The scaled dimension group $K_0(\mathbb{C}1 \oplus \mathcal{K})$ of the algebra generated by the compact operators $\mathcal{K}$ and the identity $1$.

$A_i = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}$ (for $i \geq 1$) given in (A4). Figure 12 illustrates the homomorphism $K_0(\Phi_2) : K_0(A_2) \hookrightarrow K_0(A_3)$.

Taking the injective limit $\varinjlim K_0(A_i)$, according to the formulas (B6), we get that

$$K_0(\mathbb{C}1 \oplus \mathcal{K}) \cong (\mathbb{Z}^2, K_0^+, \Gamma),$$

where

$$K_0^+ = (\{0\} \times \mathbb{N}) \cup (N_+ \times \mathbb{Z})$$

and

$$\Gamma = (\{0\} \times \mathbb{N}) \cup (\{1\} \times (1-N)),$$

as it is depicted in Figure 13. (Indeed, for $i \geq 2$ the transformations $A_i$ are $\mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ bijections, so the $K_0$ group in question is $\mathbb{Z}^2$. The positive elements $K_0^+$ are the points which are shifted into the cone $\mathbb{N}_+^2 \subset \mathbb{Z}^2$ after sufficiently many applications of the transformation $A = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}$, and the scale $\Gamma$ consists of the points $x \in \mathbb{Z}^2$ for which the $n^{th}$ iterate $A^n x$ falls into the scale $\{0,1\} \times \{0,1,\ldots,n+1\}$ of the group $K_0(A_{n+1})$.)

We remark that the $K_0$ group of the (nonunital AF) algebra $\mathcal{K}$ of compact
operators is $K_0(K) \cong (\mathbb{Z}, \mathbb{Z}, \mathbb{N})$ (see [Dav96], Example IV.3.5), so the unitization brings in an extra summand $\mathbb{Z}$, what is generally true ([WO93], Proposition 6.2.2).
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52
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