Geometry and graphics of the fractal expansion of a square and its applications

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Abstract. Geometry of the fractal expansion of simple geometric shapes, in particular the square, 5-celled cross and swastikas, is a new section of fractal geometry that correctly and clearly describes the dynamics of natural processes of growth and development of animate and inanimate cellular structures, since the iterative processes of their fractal expansion are mutually complementary and inextricably interconnected. The theoretical part of this paper requires further research, and the practical one needs for the design and engineering, and technological development of a solution to the problem for building roads with the fractal-synergetic pavement as an alternative to asphalt.

Dialectics of the evolution of modern scientific ideas about the nature of Nature and Society leads us to paradoxical concepts of chaos as a “higher form of the order, where randomness and unsystematic impulses become an organizing principle” [1]; of fractals as “geometric shapes obtained as a result of fragmentation into parts similar to the whole …” [2], describing such objects as coastlines, clouds, tree crowns, relief of the earth’s surface, etc.; of synergetics as a science that opens up “another side of the world: its instability, nonlinearity and discoverability…” [3] and describes the properties of self-organization of both natural and social formations.

The XX century entered the history of science thanks to three fundamental discoveries that completely changed our understanding of the world: the theory of relativity, quantum mechanics and theory of chaos. The latter includes computer science, cybernetics, holography, nonlinear dynamics and fractal geometry. The achievements of these sciences have revealed the content of nonlinear processes, objects, and phenomena in nature having fractal structures and being capable of self-organization.

We are witnessing a passage from the civilization based on Aristotle’s philosophy, Newton’s laws and Euclid’s geometry to a civilization grounded upon the philosophy of Heraclitus, laws of quantum mechanics and geometry of fractals. At the same time, the dialectical interpenetration of new scientific ideas and concepts into the well-developed traditional directions of both natural science and sciences about the artificial, i.e. manmade, is inevitable. The latter include all technical sciences and all applied areas of fundamental natural science. And since the manmade world is extremely diverse, but not always perfect, because it is being created by a person to satisfy his material and spiritual needs on the basis of not always perfect knowledge, it is absolutely natural for him to strive for applying new and still unconventional knowledge to eliminate shortcomings of the old and traditional solutions. The area of architecture and civil engineering is a very wide field for such application. This raises the problem of using ideas of fractal geometry and synergetics to take the most rational engineering solutions for self-
organizing designs, in particular road pavements, as well as new space-planning decisions for buildings and structures.

B. Mandelbrot’s fractal geometry arose when studying the results of computer visualization of Cantor and Julia sets, Peano curves, Weierstrass functions, etc. and determined a new approach to understanding the real nature of surrounding world, whose geometry had been considered to be Euclidian since the time of Euclid and Aristotle. The author of a new geometry has proved its fractality based on Democritus’s dialectical logic, according to which everything flows and everything changes, and that being invariable in the process of change has the force of law. This includes almost all animate and inanimate objects of organic origin, the kinetics of which is based on division of cells as structural elements of their constitution. In this case, before one mother cell is divided into two daughter ones, processes of doubling of all its structural elements take place in it, so that structures of daughter cells were isomorphic against the structure of the mother one. Therefore, the process of splitting into two or dichotomy is nothing more than the process of expansion or growth of the original cellular structure. Studying this dynamic process [4,5 et al.] showed that it is fractal in terms of quality, and described by Fibonacci series and metric features of golden proportioning in terms of quantity [6 et al].

B. Mandelbrot in his book “The Fractal Geometry of Nature” defines a fractal as “a structure consisting of parts, which are in some sense similar to the whole”, because the infinite fragmentation and similarity of the smallest particles to the whole is a principle of the “arrangement” of nature. Various mathematicians have proposed to place abstract geometric shapes having a fractal nature, in addition to real coastlines that have infinite length in finite space, tree crowns, elements of the surface relief and clouds, the dimension of which is more than two, but less than three, etc., in a class of fractal objects. These are the Koch snowflake, Harter-Heighway dragon, Sierpiński napkin and carpet, Cantor dust, etc. All of them are designed under the principle, which is to perform successive iterations of fragmentizing the elements of an original shape downwards the resulting self-similar shapes in their propensity for a fraction-dimensional limit.

In this regard, a troublesome assumption arises: if there is a real direct process of self-similar fractal reduction in the elements of an original shape, then there must be a reverse process of self-similar increase, or expansion of an original shape. If such a process exists, it becomes necessary to study and describe the dynamics of changes in shapes of successive iterations and their positional and metric properties.

Analysis of achievements and publications. In line with the logic justifying the articulation of an issue, the number of publications devoted to natural geometry and shaping in nature should include the papers of I. Sh. Shevelev [5,6 et al.], which consider the concept of the shape of an object in continuous connection with the process of its genesis. This process is dynamic as it is based on splitting of cellular structures, which causes a state of their growth or expansion. However, I. Sh. Shevelev does not reveal the fractal nature of this process, which is completely obvious both in the traditional sense of fractal structures, whose elements self-similarly decrease and by a natural assumption that these elements expectedly overgrow and expand.

Such an assumption was made in the papers [7-10], where the results of iterations of the fractal expansion of a square into cross cell structures were successively obtained, and the process itself was described analytically with an output to a numerical sequence, which turned out to be the second-order Fibonacci integer sequence. The resulting geometric shapes of the 3-rd and 4-th iterations proved to be quite complex and requiring to conduct a structural analysis.

This implies setting of a research objective: to make a graphic-analytical description focusing on the design structure of fractal shapes of successive iterations and their combinatorial connections in order to identify an analogy with the natural process of growing cellular structures of wildlife and possible practical applications of geometry for the process of the fractal expansion of a square.
**Hard core of the paper.** If we take the most technological square as an original shape and subject it to a sequential iterative expansion by adding the results of its previous transformations to the subsequent results of iterations, then shapes of the fractal nature will appear. As a result of the first iteration, the square expands or transforms conformally and topologically into a 5-celled cross; as a result of the second iteration, the central square of the 5-celled cross expands or similarly transforms into a 9-celled square, and its 4 ends expand up to the 5-celled crosses. In sum, there is a 29-celled shape inscribed in an 81-celled square with 4 axes of symmetry. Its structure includes the results of the 1-st iteration in the form of four 5-celled crosses.

As a result of the third iteration, the 29-celled shape undergoes the topological transformation into a 169-celled shape. Upon that, the central 9-celled square turns into a 25-celled central square-lack, and the 5-celled crosses of the 2-nd iteration in the middle of its sides turn into the 29-celled shapes.

![Fig.1. Three iterations of the fractal expansion of the square](image)

In doing so, the overall 81-celled squares superimpose with their 18 cells on the central 81-celled square, and each 29-celled shape seems to “sink” in the central square with its 4 cells.
The fourth iteration topologically and conformally transforms the 169-celled shape into a 985-celled one, the structure of which includes four 169-celled shapes, whose dimensions self-similarly repeat the 5-celled cross and “sink” with their 4 twenty-fourth parts in the central square (Fig.3). In terms of design, the process of this “intergrowth” occurs through the appropriate structuring of shapes of the overall square’s superimposition.

It should be especially noted that the shapes of the considered iterations are rooted in the 5-celled cross of the overall squares, inside which the “cell division” takes place. If we assume that the sizes of these squares are unchanged, then increasing the number of cells in each of them leads them to the self-similar reduction, which corresponds to the traditional assumption of the fractal nature of natural systems. However, if we consider the sizes of the original square as unchanged, the process of increasing the number of cells congruent to it in the shapes of subsequent iterations causes its fractal expansion within the accordingly increasing overall squares of the 5-celled cross. In both cases, as the shapes of subsequent
iterations become more complex, the structures of the lateral squares make a union with the central one, or there is a process of their “growing” from the central square. Likewise, the dormant buds of a plant generate the first-order branches; overtime, their buds generate the second-order branches and so on. The same processes are observed in living organisms.

The results of successive iterations of the original square cell can be represented by elements of combinatorial compositions that densely pack the plane.

The 5-celled crosses compactly pack the plane being adjacent to each other at the swastika-shaped joints (Fig. 4). As a quantitative characteristic of their relative position, one can take the ratio of the legs (1:2) of right-angled triangles, the hypotenuses of which connect their centers. The sum of their squares (1+4) is equal to 5, i.e. the number of cells of one element.

The combinatorics of four 29-celled elements compactly packs the plane owing to their entry into the lock joints (Fig. 5). The ratio of the legs of triangles, the hypotenuses of which connect their centers, is equal to 2:5. The sum of their squares (4+25) is equal to 29, i.e. the number of cells in one element (Fig. 5).

The combinatorics of combining four 169-celled elements into the lock forms their system as the integrated whole. The straight lines that connect their centers are the hypotenuses of right-angled triangles, the lengths of whose legs are related as 5/12. The sum of their squares (25+144) is equal to the square of the hypotenuse (169), i.e. the number of cells in one element.

Following the accepted technique of transforming previous iterations into subsequent ones, it is possible to obtain shapes becoming more and more complex in their geometric structure and expanding up to infinitely larger sizes.

Among the distinctive features of dense packings of the plane with various fractal shapes, there is their synergism or ability to self-organization, since the lock joints between them redistribute stresses from possible loads and force the prefabricated design, for example that of a roadway, to operate as monolithic but crack-resistant, inasmuch as the role of cracks is played by joints between the elements.

A comparative assessment of the process to change values of the leg ratio in right-angled triangles, the hypotenuses of which connect the centers of these shapes, allows determining a quantitative law of the entire iterative process (Table 1).

| Iteration No. (n) | 0 | 1 | II | III | IV | V | VI |
|-------------------|---|---|----|-----|----|---|----|
| Kb – long leg     | 1 | 2 | 5  | 12  | 29 | 70| 169 |
| Km – short leg    | 0 | 1 | 2  | 5   | 12 | 29| 70  |
| Number of cells   | 1 | 5 | 29 | 169 | 985| 5741| 33461 |
| Number of sides   | 4 | 12| 52 | 220 | 932| 3948| 16724 |

An analysis of the above Table indicates the following pattern:

\[ K_{b(n)} = 2 K_{b(n-1)} + K_{m(n-1)} \]

Since \( K_{m(n-1)} = K_{b(n-2)} \), then \( K_{b(n)} = 2 K_{b(n-1)} + K_{b(n-2)} \), whence a new numerical series follows:

\[ a_n = 2 a_{(n-1)} + a_{(n-2)} \]  \hspace{1cm} (1)

This expression is a recursion formula reflecting the iterative process of expansion of the square.

The numerical series of changing the number of cells in the shapes starting from the 2-nd iteration is described by the expression:

\[ a_n = 6 a_{(n-1)} - a_{(n-2)} \]  \hspace{1cm} (2)
The peculiarities of changes in the number of sides of the sequential results of iterations, starting from the 2-nd one, are described by the expression below:

$$a_n = 4a_{(n-1)} + a_{(n-2)}$$  \hspace{1cm} (3)

A special attention should be paid to the fact that the validity of the expressions (2) and (3) begins from the second iteration. This is explained by emergence of the process, when its regular patterns described using these expressions appear in discrete steps and do not cover the description of its beginning.

Considering the positional properties of dense packings, we draw attention to the swastika-shaped joints, as it were, clamped between the cross elements. If one mentally imagines that they expand by means of corresponding crosses up to the equality of their width, then one gets a two-element packing of crosses and swastikas, which self-closes by virtue of emerging synergy.

![Fig. 7. 5 iterations of the fractal transformation of the cross into swastikas](image)

As a result of the first iteration, the original square transforms into a 5-celled cross, and as a result of the second iteration, following the swastika structure, double squares join the lateral cells of the cross, so that a 13-celled swastika appears.
The third iteration complements the swastika cross base by 5 П-shaped cells and a 25-celled swastika is obtained.

The fourth iteration complements the cross base by 9 specially arranged cells and a 41-celled meander swastika is obtained and so on.

Fig.8. Packing of the 13-celled swastikas

Fig.9. Packing of the 25-celled swastikas

In practice, the original square, having transformed into the 5-celled cross, gives it the properties of an initiator of all subsequent iterations.

It is not hard to see that the swastika shapes of the 2-nd and 3-rd iterations tightly pack the plane (Fig.8 and 9), as well as to assume that the shapes of the 4-th, 5-th and subsequent iterations of the 5-celled cross also pack the plane compactly, and their metric characteristics easily extrapolate on the basis of available information about sizes of the legs of right-angled triangles, the hypotenuses of which connect the centers of these shapes (Table 2).

Table 2. Quantitative characteristics of the iterative process towards the fractal expansion of the 5-celled cross

| Iteration No. (n) | 0 | I | II | III | IV | V | VI | VII |
|-------------------|---|---|----|-----|----|---|----|-----|
| $K_b$ – long leg  | 1 | 2 | 3  | 4   | 5  | 6 | 7  | 8   |
| $K_m$ – short leg | 0 | 1 | 2  | 3   | 4  | 5 | 6  | 7   |
| Number of cells   | 1 | 5 | 13 | 25  | 41 | 61| 85 | 113 |

A statistical analysis of this Table shows that:

$$K_{b(n)} = 2K_{b(n-1)} - K_{m(n-1)}$$

Since $K_{m(n-1)} = K_{b(n-2)}$, then $K_{b(n)} = 2K_{b(n-1)} - K_{b(n-2)}$  

It turns out that the expression (4) describes a series of natural numbers:

$$a_{(n)} = 2a_{(n-1)} - a_{(n-2)}$$

or

$$-n, \ldots -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots n$$

As a result, it turns out that the expression (5) describes an integer sequence (3). But it is fundamentally different from the generally accepted formula of the integer sequence $a_{(n)} = a_{(n-1)} + 1$ that it follows from the quantitative analysis of features revealing the iterative process of fractal
transformations of the 5-celled cross, which spatially models it. This means that any operations with numbers indirectly induce the corresponding operations with structures of the fractal expansion of swastikas.

The comparison of the expressions (1) and (5) points to the similarity of their elements in the right side and their difference in signs between these elements. This circumstance provides evidence of not the opposite, but the mutual complementarity and inextricable connection of processes of the fractal expansions of the square and 5-celled cross into swastikas.

Based on the analysis of Figures 2 and 3, and Table 1, it follows that the longer legs of previous iterations act as the shorter legs of subsequent iterations. Therefore, it is easy to imagine the dynamics of development of the synergetic swastika structures in the form of a spiral by applying the longer leg of the subsequent iteration to the end of the previous longer leg at a right angle (Fig.10).

In this case, the ratios (2:1=2; 3:2=1,5; 4:3=1,33; 5:4 =1,25; 6:5=1,2; 8:7 =1,14; etc.) of the leg lengths in triangles of subsequent iterations are not the same, and therefore, their hypotenuses do not intersect at one point, but limit in the picture a certain region (node N), which, after each iteration, decreases in size singulating to some point O lying outside the construction triangles.

Fig.10. Visualization of the integer sequence

Thus, we can assume that this region “shuts” in the point O at the number of iterations $n = \infty$, which will divide the infinitely large hypotenuse in half. This can happen when the legs of the infinitely large triangle will become equal. It follows that the development of the spiral in Fig.10 shows the evolution of passing from the Dürer triangle (hypotenuse $\sqrt{5}$) to a right isosceles triangle.

Since fractal geometry basically visualizes natural linear and nonlinear processes in nature rather peculiarly, the proposed geometry of the fractal expansion of simple geometric shapes also models, on a geometrical and graphical basis, a process of growth of living cellular structures in the form of generating tree crowns and behaviors of climbing plants, in particular vine whiskers. Although, these models are more specific, since they are more logical and visual.
Applications: 1. One-layer synergetic pavement of the earth’s surface.

It is carried out by the effect from lateral shapes of various iterations of the fractal expansion of the square being intergrown with the central square of the overall 5-celled cross, leading to the entry of congruent shapes into the lock joints of dense packings of the plane, which forms the concept of a one-layer prefabricated synergetic pavement for footpaths (Fig.11) and large areas as an alternative to their asphalt pavement. The idea of a prefabricated pavement is widely popular in European and other countries of the world for paving footpaths, small areas, parking lots and adjacent territories. Nevertheless, as in Ukraine, in the world such paving does not have a synergetic effect of self-organization, since its tiles do not enter into the lock joints between themselves. Moreover, one should expect from this effect savings in the material of tiles due to redistributing loads between them, reducing the construction timeframe and increasing the aesthetic appeal of the entire pavement.

2. Two-layer synergetic pavement of the carriageway.

It is carried out using the positional feature, inherent in the shapes of the 3-rd and 4-th iterations just to name a few (see Fig. 1 – 3), to be located in overall squares, the vertices of which divide in half the sides of the overall square of their “5-celled cross”. A composition of “a square in a square” is obtained, when the area of the inscribed square is equal to the total area of 4 angled triangles of the overall square of the cross. This circumstance led to the idea of making a square recess in a solid square element by half its thickness that gives rise, when 4 such elements are relatively positioned, to the formation of a protruding square composed of 4 angled triangles, the area of which will be equal to the area of a recessed square. If the areas are calculated taking into account the width of joints between the elements, then the element of the upper layer with its square recess is “slipped” over the “square” prefabricated of 4 angled triangles, and thereby, “closes” 4 elements of the lower layer into a “lock”.

Fig.11. Options of one-layer synergetic pavements for footpaths
Three-layer synergetic pavement of the airport runway.

As a result, we get a two-layer prefabricated synergetic pavement of the carriageway, which operates as a monolithic but crack-resistant one, since the role of cracks is played by joints between the elements.

3. Three-layer synergetic pavement of the airport runway.

The airport runway is a specially prepared and equipped landing strip with an artificial pavement, which ensures takeoff and landing of an aircraft. It is traditionally a multi-layer design. In recent years, steel fiber concrete has been used as the material of the upper layer, being heavy-weight concrete with appropriate additives to resist the effects of high and low temperatures, reinforced with steel fiber, i.e. a mixture of steel fibers, which should be uniformly distributed in bulk concrete.

The high labor intensity of construction works to build the airport runways determines accordingly their high prime cost, and therefore, is a reason to search for alternative solutions to the problem of their building as durable, reliable and competitive facilities.

A worthwhile basis for such a search is the above-described concept of the carriageway synergetic pavement, which consists of the elements of industrial production being congruent in each layer, that is, identical both in shape and in size, which enter into the lock joints between themselves and thereby force all prefabricated pavements to be hard at work as the monolithic (Fig.13).

4. One-layer synergetic pavement of the water surface from congruent pneumatic elements
The positional property of the flat 29-celled fractal shapes (Fig. 14, a) to enter into the permanent lock joints served as the basis for the idea of giving them a third dimension in the form of 52-faced surfaces made of high-duty membrane (Fig. 14, b).

If the volumetric interpretation of this shape is a 52-faced prism, then a flat layer of space, sort of roadway paving, will be structured with these shapes. Being extremely light, it can spread on water and serves as a pontoon bridge, or an artificial island.

If the faces, which are symmetric to one of the two planes of the 52-faced prism symmetry, are given a design-level slope, it is possible to mount a cylindrical shell of any span from the obtained wedge-shaped elements. It seems that due to lightness of such shells and strength of links between them, they can overlap the large areas and entire structures, and serve as large-span hangars, warehouses and other objects for various purposes.

If we assemble from them a close cylindrical shell, capable of aeronautics, in case of its elements being filled with light gas, then we can talk about the creation of reliable airships and other aircrafts (Fig. 15).

If the 52-faced prism is transformed into a truncated pyramid, then a hemispherical or spherical shell of the large span can be assembled from these elements as an independent building or air balloon (Fig. 16, 17).
The advantages of such shells are the absence of necessity to maintain excess pressure inside the structure, their large-span and durability. Failure of one or more elements does not lead to destruction of the design, but only requires their replacement. Only the design of the air supply valves is troublesome.

Conclusions

1. Geometry of the fractal expansion of simple geometric shapes, in particular the square, 5-celled cross and swastikas, is a new section of fractal geometry that correctly and clearly describes the dynamics of natural processes of growth and development of animate and inanimate cellular structures, since the iterative processes of their fractal expansion are mutually complementary and inextricably interconnected.

2. A wide range of possible fundamentally new practical applications of geometry of the fractal expansion of the square, based on the unrestrained correspondence to their nature, determines the real opportunity of creating easy-to-manufacture, economically profitable, and competitive material objects, such as dense packings of the earth’s surface of the type of carriageways, footpaths, airport runways, earthwork structures, floating water crossing installations, artificial islands and marine terminals, large-span arch and dome buildings, self-supporting fences in a supporting framework of high-rise objects, aerostats, etc.

3. The theoretical part of this paper requires further research, and the practical one needs for the design and engineering, and technological development of a solution to the problem for building roads with the fractal-synergetic pavement as an alternative to asphalt.

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