An Anzellotti type pairing for divergence-measure fields and a notion of weakly super-1-harmonic functions

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Abstract

We study generalized products of divergence-measure fields and gradient measures of BV functions. These products depend on the choice of a representative of the BV function, and here we single out a specific choice which is suitable in order to define and investigate a notion of weak supersolutions for the 1-Laplace operator.

1 Introduction

For a positive integer $n$ and an open set $\Omega$ in $\mathbb{R}^n$, we consider the 1-Laplace equation

$$\text{div} \frac{Du}{|Du|} = 0 \quad \text{on } \Omega.$$ (1)

This equation formally arises as the Euler equation of the total variation and is naturally posed for functions $u: \Omega \to \mathbb{R}$ of locally bounded variation whose gradient $Du$ is merely a measure. In order to make sense of (1) in this setting it has become standard [12, 9, 10, 5, 13, 14] to work with a generalized product, which has been studied systematically by Anzellotti [2, 3]. The product is defined for $u \in \text{BV}_{\text{loc}}(\Omega)$ and $\sigma \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^n)$ with vanishing distributional divergence $\text{div} \sigma$ as the distribution

$$[\sigma, Du] := \text{div}(u\sigma) \in \mathcal{D}'(\Omega),$$

and in fact the pairing $[\sigma, Du]$ turns out to be a signed Radon measure on $\Omega$. By requiring $\|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)} \leq 1$ and $[\sigma, Du] = |Du|$ one can now phrase precisely what it means that $\sigma$ takes over the role of $Du$ in this setting. In order to make sense of (1), in this setting it has become standard [12, 9, 10, 5, 13, 14] to work with a generalized product, which has been studied systematically by Anzellotti [2, 3]. The product is defined for $u \in \text{BV}_{\text{loc}}(\Omega)$ and $\sigma \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^n)$ with vanishing distributional divergence $\text{div} \sigma$ as the distribution

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and in fact the pairing $[\sigma, Du]$ turns out to be a signed Radon measure on $\Omega$. By requiring $\|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)} \leq 1$ and $[\sigma, Du] = |Du|$ one can now phrase precisely what it means that $\sigma$ takes over the role of $Du$ in this setting. In a similar vein, variants of the pairing $[\sigma, Du]$ can be used to define BV solutions $u$ of $\text{div} \frac{Du}{|Du|} = f$ for right-hand sides $f \in L^1_{\text{loc}}(\Omega)$ and to explain $\text{BV} \cap L^\infty$ solutions $u$ of $\text{div} \frac{Du}{|Du|} = f$ even for arbitrary $f \in L^1_{\text{loc}}(\Omega)$.

In this note we deal with a notion of supersolutions of (1) or — this is essentially equivalent — of solutions of $\text{div} \frac{Du}{|Du|} = -\mu$ with a Radon measure $\mu$ on $\Omega$. To this end, we first collect some preliminaries in Section 2. Then, in Section 3 we consider generalized pairings, which make sense

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even for $L^\infty$ divergence-measure fields $\sigma$, but require precise evaluations of $u \in BV_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ up to sets of zero $(n-1)$-dimensional Hausdorff measure $H^{n-1}$. We mainly investigate a pairing $[\sigma, Du^+]$, which is built with a specific $H^{n-1}$-a.e. defined representative $u^+$ of $u$ and which does not seem to have been considered before (while a similar pairing with the mean-value representative $u^*$ already occurred in [7 Theorem 3.2] and [13 Appendix A]). Adapting the approach in [11 Section 5], we moreover deal with an up-to-the-boundary pairing $[\sigma, Du^+]_{u_0}$, which accounts for a boundary datum $u_0$. In Section 5 we employ the local pairing $[\sigma, Du^+]$ in order to introduce a notion of weakly super-1-harmonic functions, and we prove a compactness statement which crucially depends on the choice of the representative $u^+$. Finally, Section 4 is concerned with a refined notion of super-1-harmonicity which incorporates Dirichlet boundary values. This last notion is based on (a modification of) the pairing $[\sigma, Du^+]_{u_0}$.

We emphasize that the proofs, which are omitted in this announcement, can be found in the companion paper [15], where we also provide a more detailed study of pairings and supersolutions together with adaptations to the case of the minimal surface equation. Furthermore, in our forthcoming work [16], we will discuss connections with obstacle problems and convex duality.

2 Preliminaries

$L^\infty$ divergence-measure fields. We record two results related to the classes

$$DM_{\text{loc}}^\infty(\Omega, R^n) := \{ \sigma \in L^\infty_{\text{loc}}(\Omega, R^n) : \text{div } \sigma \text{ exists as a signed Radon measure on } \Omega \},$$

$$DM^\infty(\Omega, R^n) := \{ \sigma \in L^\infty(\Omega, R^n) : \text{div } \sigma \text{ exists as a finite signed Borel measure on } \Omega \}.$$  

**Lemma 2.1** (absolute-continuity property for divergences of $L^\infty$ vector fields). Consider $\sigma \in DM_{\text{loc}}^\infty(\Omega, R^n)$. Then, for every Borel set $A \subset \Omega$ with $H^{n-1}(A) = 0$, we have $|\text{div } \sigma|(A) = 0$.

Lemma 2.1 has been proved by Chen & Frid [7 Proposition 3.1].

**Lemma 2.2** (finiteness of divergences with a sign). If $\Omega$ is bounded with $H^{n-1}(\partial \Omega) < \infty$ and $\sigma \in L^\infty(\Omega, R^n)$ satisfies $\text{div } \sigma \leq 0$ in $\mathcal{D}'(\Omega)$, then we necessarily have $\sigma \in DM^\infty(\Omega, R^n)$. Moreover, there holds $(-\text{div } \sigma)(\Omega) \leq \frac{\mu(n)}{\omega_n} \|\sigma\|_{L^\infty(\Omega, R^n)} H^{n-1}(\partial \Omega)$ with the volume $\omega_n$ of the unit ball in $R^n$.

Indeed, it follows from the Riesz representation theorem that $\text{div } \sigma$ in Lemma 2.2 is a Radon measure. The finiteness of this measure will be established in [15] by a reasoning based on the divergence theorem.

**BV-functions.** We mostly follow the terminology of [11], but briefly comment on additional conventions and results. For $u \in BV(\Omega)$, we recall that $H^{n-1}$-a.e. point in $\Omega$ is either a Lebesgue point (also called an approximate continuity point) or an approximate jump point of $u$; compare [11 Sections 3.6, 3.7]. We write $u^+$ for the $H^{n-1}$-a.e. defined representative of $u$ which takes the Lebesgue values in the Lebesgue points and the larger of the two jump values in the approximate jump points. Correspondingly, $u^-$ takes on the lesser jump values, and we set $u^* := \frac{1}{2}(u^+ + u^-)$.

Finally, if $\Omega$ has finite perimeter in $R^n$, we write $u^\text{int}_{\partial^* \Omega}$ for the $H^{n-1}$-a.e. defined interior trace of $u$ on the reduced boundary $\partial^* \Omega$ of $\Omega$ (compare [11 Sections 3.3, 3.5, 3.7]), and in the case that $H^{n-1}(\partial^* \Omega) = 0$ we also denote the same trace by $u^\text{int}_{\partial \Omega}$.

The following two lemmas are crucial for our purposes. The first one extends [11 Proposition 3.62] and is obtained by essentially the same reasoning. The second one follows by combining [6 Theorem 2.5] and [8 Lemma 1.5, Section 6]; compare also [11 Sections 4, 10].
Lemma 2.3 (BV extension by zero). If we have \( H^{n-1}(\partial \Omega) < \infty \), then for every \( u \in BV(\Omega) \cap L^\infty(\Omega) \) \( 1\Omega u \in BV(\mathbb{R}^n) \) and \( |D(1\Omega u)|(\partial \Omega) \leq \frac{m}{\gamma} \| u \|_{L^\infty(\Omega)} H^{n-1}(\partial \Omega) \). In particular, \( \Omega \) is a set of finite perimeter in \( \mathbb{R}^n \), and \( u_{0^\Omega} \) is well-defined.

Lemma 2.4 (\( H^{n-1}\text{-a.e. approximation of a BV function from above} \)). For every \( u \in BV_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) there exist \( v_\ell \in W^{1,1}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) such that \( v_1 \geq v_\ell \geq u \) holds \( L^n\text{-a.e. on } \Omega \) for every \( \ell \in \mathbb{N} \) and such that \( v_\ell \) converges \( H^{n-1}\text{-a.e. on } \Omega \) to \( u^+ \).

3 Anzellotti type pairings for \( L^\infty \) divergence-measure fields

We first introduce a local pairing of divergence-measure fields and gradient measures.

Definition 3.1 (local pairing). Consider \( u \in BV_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) and \( \sigma \in \mathcal{DM}^\infty_{\text{loc}}(\Omega, \mathbb{R}^n) \). Then — since Lemma 2.4 guarantees that \( u^+ \) is \( |\text{div } \sigma|\) a.e. defined — we can define the distribution

\[
[\sigma, Du^+] := \text{div}(u\sigma) - u^+ \text{div } \sigma \in \mathcal{D}'(\Omega).
\]

Written out this definition means

\[
[\sigma, Du^+](\varphi) = -\int_\Omega u\sigma \cdot D\varphi \, dx - \int_\Omega \varphi u^+ \, d(\text{div } \sigma) \quad \text{for } \varphi \in \mathcal{D}(\Omega).
\]

Next we define a global pairing, which incorporates Dirichlet boundary values given by a function \( u_0 \).

Definition 3.2 (up-to-the-boundary pairing). Consider \( u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega) \), \( u \in BV(\Omega) \cap L^\infty(\Omega) \), and \( \sigma \in \mathcal{DM}^\infty(\Omega, \mathbb{R}^n) \). Then we define the distribution \( [\sigma, Du^+_u] \in \mathcal{D}'(\mathbb{R}^n) \) by setting

\[
[\sigma, Du^+_u](\varphi) := -\int_\Omega (u-u_0)\sigma \cdot D\varphi \, dx - \int_\Omega \varphi(u^+ - u_0^+) \, d(\text{div } \sigma) + \int_\Omega \varphi \sigma \cdot Du_0 \, dx
\]

for \( \varphi \in \mathcal{D}(\mathbb{R}^n) \).

We emphasize that the pairings in Definitions 3.1 and 3.2 coincide on \( \varphi \) with compact support in \( \Omega \) (since an integration-by-parts then eliminates \( u_0 \) in (3)). However, the up-to-the-boundary pairing stays well-defined even if \( \varphi \) does not vanish on \( \partial \Omega \). In addition, we remark that both pairings can be explained analogously with other representatives of \( u \).

In some of the following statements we impose a mild regularity assumption on \( \partial \Omega \), namely we require

\[
H^{n-1}(\partial \Omega) = P(\Omega) < \infty,
\]

where \( P \) stands for the perimeter. We remark that the condition \( (\text{4}) \) is equivalent to having \( P(\Omega) < \infty \) and \( H^{n-1}(\partial \Omega \setminus \partial^* \Omega) = 0 \) and also to having \( 1\Omega \in BV(\mathbb{R}^n) \) and \( |D1\Omega| = H^{n-1} \mathbb{L}\partial \Omega \).

For a more refined discussion we refer to \( [17] \), where the relevance of \( (\text{4}) \) for certain approximation results is pointed out.

Two vital properties of the pairing are recorded in the next statements. The proofs will appear in \( [15] \).
Lemma 3.3 (the pairing trivializes on \(W^{1,1}\)-functions).

\(\bullet\) (local statement) For \(u \in W^{1,1}_{\text{loc}}(\Omega) \cap L^\infty(\Omega), \sigma \in DM^\infty_{\text{loc}}(\Omega, \mathbb{R}^n),\) and \(\varphi \in \mathcal{D}(\Omega),\) we have
\[
\|\sigma, Du^+\|_\varphi = \int_\Omega \varphi \cdot D u \, dx.
\]

\(\bullet\) (global statement with traces) If \(\Omega\) is bounded with \(\mathbf{1}\), then for every \(\sigma \in DM^\infty(\Omega, \mathbb{R}^n)\) there exists a uniquely determined normal trace \(\sigma_n^* \in L^\infty(\partial \Omega; \mathcal{H}^{n-1})\) with
\[
\|\sigma_n^*\|_{L^\infty(\partial \Omega; \mathcal{H}^{n-1})} \leq \|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)}
\]
such that for all \(u, u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)\) and \(\varphi \in \mathcal{D}(\mathbb{R}^n)\) there holds
\[
\|\sigma, Du^+\|_{u_0, \varphi} = \int_\Omega \varphi(\sigma \cdot D u) \, dx + \int_{\partial \Omega} \varphi(u - u_0)^{\text{int}} \sigma_n^* \, d\mathcal{H}^{n-1}.
\]

Next we focus on bounded \(\sigma\) with \(\text{div} \, \sigma \leq 0\). By Lemma 2.2 the pairings stay well-defined in this case.

Proposition 3.4 (the pairing is a bounded measure). Fix \(\sigma \in L^\infty(\Omega, \mathbb{R}^n)\) with \(\text{div} \, \sigma \leq 0\) in \(\mathcal{D}'(\Omega)\).

\(\bullet\) (local estimate) For \(u \in \text{BV}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega),\) the distribution \([\sigma, Du^+]\) is a signed Radon measure on \(\Omega\) with
\[
\|\sigma, Du^+\| \leq \|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)} |D u| \quad \text{on} \quad \Omega.
\]

\(\bullet\) (global estimate with equality at the boundary) If \(\Omega\) is bounded with \(\mathbf{1}\), for \(u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)\) and \(u \in \text{BV}(\Omega) \cap L^\infty(\Omega)\) the pairing \([\sigma, Du^+]_{u_0}\) is a finite signed Borel measure on \(\mathbb{R}^n\) with
\[
\|\sigma, Du^+\|_{u_0} - (u - u_0)^{\text{int}} \sigma_n^* \mathcal{H}^{n-1} |\partial \Omega| \leq \|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)} |D u| L^\Omega.
\] (5)

4 Weakly super-1-harmonic functions

We now give a definition of super-1-harmonic functions, which employs the convenient notation
\[
S^\infty(\Omega, \mathbb{R}^n) := \{ \sigma \in L^\infty(\Omega, \mathbb{R}^n) : |\sigma| \leq 1 \text{ holds } \mathcal{L}^n\text{-a.e. on } \Omega \}.
\]

Definition 4.1 (weakly super-1-harmonic). We call \(u \in \text{BV}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)\) weakly super-1-harmonic on \(\Omega\) if there exists some \(\sigma \in S^\infty(\Omega, \mathbb{R}^n)\) with \(\text{div} \, \sigma \leq 0\) in \(\mathcal{D}'(\Omega)\) and \([\sigma, Du^+] = |D u|\) on \(\Omega\).

We next provide a compactness result for super-1-harmonic functions. We emphasize that this result does not hold anymore if one replaces \(u^+\) by any other representative of \(u\) in the definition. We also remark that the assumed type of convergence is very natural, and indeed the statement applies to every increasing sequence of super-1-harmonic functions which is bounded in \(\text{BV}_{\text{loc}}(\Omega)\) and \(L^\infty_{\text{loc}}(\Omega)\).

Theorem 4.2 (convergence from below preserves super-1-harmonicity). Consider a sequence of weakly super-1-harmonic functions \(u_k\) on \(\Omega\). If \(u_k\) locally weakly converges to a limit \(u\) both in \(\text{BV}_{\text{loc}}(\Omega)\) and \(L^\infty_{\text{loc}}(\Omega)\) and if \(u_k \leq u\) holds on \(\Omega\) for all \(k \in \mathbb{N}\), then \(u\) is weakly super-1-harmonic on \(\Omega\).
Proof. In view of Definition 4.1 there exist $\sigma_k \in S^\infty(\Omega, \mathbb{R}^n)$ with $\text{div} \, \sigma_k \leq 0$ in $\mathcal{D}'(\Omega)$ and $[\sigma_k, D u_k^+] = |D u_k|$ on $\Omega$. Possibly passing to a subsequence, we assume that $\sigma_k$ weak* converges in $L^\infty(\Omega, \mathbb{R}^n)$ to $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with $\text{div} \, \sigma \leq 0$ in $\mathcal{D}'(\Omega)$, and as before we regard $\text{div} \, \sigma_k$ and $\text{div} \, \sigma$ as non-positive measures on $\Omega$. We fix a non-negative $\varphi \in \mathcal{D}(\Omega)$ and finally on a lower semicontinuity property of the total variation. In this way, we deduce

\[
\begin{align*}
\varphi \cdot D u^+ \in L^1(\Omega), \\
\end{align*}
\]

Since the pairing trivializes on $v_t \in W^{1,1}_c(\Omega)$, we can exploit the local weak* convergence of $\sigma_k$ in $L^\infty_{\text{loc}}(\Omega, \mathbb{R}^n)$ and the inequalities $v_t^* \geq u^+ \geq u_k^+$ to arrive at

\[
\begin{align*}
\|\sigma, D v_t^+\|_1 &= \lim_{k \to \infty} \|\sigma_k, D v_k^+\|_1 = -\lim_{k \to \infty} \int_{\Omega} v_t \sigma_k \cdot D \varphi \, dx + \lim_{k \to \infty} \int_{\Omega} \varphi v_k^+ \text{d}(\text{div} \, \sigma_k) \\
&\geq \int_{\Omega} v_t \sigma \cdot D \varphi \, dx + \liminf_{k \to \infty} \int_{\Omega} \varphi u_k^+ \text{d}(\text{div} \, \sigma_k).
\end{align*}
\]

Next we rely in turn on the dominated convergence theorem, on the observation that $u_k \sigma_k$ locally weakly converges to $u \sigma$ in $L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$, on the definition in (2), on the coupling $[\sigma_k, D u_k^+] = |D u_k|$, and finally on a lower semicontinuity property of the total variation. In this way, we deduce

\[
\begin{align*}
\lim_{k \to \infty} \int_{\Omega} v_t \sigma_k \cdot D \varphi \, dx + \int_{\Omega} \varphi u_k^+ \text{d}(\text{div} \, \sigma_k) \\
&= -\int_{\Omega} u \sigma \cdot D \varphi \, dx + \liminf_{k \to \infty} \int_{\Omega} \varphi u_k^+ \text{d}(\text{div} \, \sigma_k) \\
&= \liminf_{k \to \infty} \int_{\Omega} u_k \sigma_k \cdot D \varphi \, dx + \int_{\Omega} \varphi u_k^+ \text{d}(\text{div} \, \sigma_k) \\
&= \liminf_{k \to \infty} \|\sigma_k, D u_k^+\|_1 = \liminf_{k \to \infty} \int_{\Omega} \varphi \text{d}|D u_k| \\
&\geq \int_{\Omega} \varphi \text{d}|D u|.
\end{align*}
\]

Collecting the estimates, we arrive at the inequality $\|\sigma, D u^+\|_1 \geq |D u|$ of measures on $\Omega$. Since the opposite inequality is generally valid by Proposition 3.4 we infer that $u$ is weakly super-1-harmonic on $\Omega$. 

5 Super-1-harmonic functions with respect to Dirichlet data

Finally, we introduce a concept of super-1-harmonic functions with respect to a generalized Dirichlet boundary datum. In 10, we will show that this notion is useful in connection with obstacle problems.

Concretely, consider a bounded $\Omega$ with $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, and $u \in \text{BV}(\Omega) \cap L^\infty(\Omega)$. We then extend the measure Du on $\Omega$ to a measure $D_{u_0} u$ on $\Omega$ which takes into account the possible
deviation of \( u^{\text{int}} \) from the boundary datum \((u_0)^{\text{int}}\). To this end, writing \( \nu \) for the inward unit normal of \( \Omega \), we set
\[
D_{u_0}u := Du\mathbb{I}_\Omega + (u-u_0)^{\text{int}}\nu \mathcal{H}^{n-1} \mathcal{L} \partial \Omega.
\]

Now, for \( \sigma \in S^\infty(\Omega, \mathbb{R}^n) \) with \( \text{div} \sigma \leq 0 \) in \( \mathcal{D}'(\Omega) \) — which is meant to potentially satisfy a coupling like \( [\sigma, Du^+]\big|_{\partial \Omega} = |D_{u_0}u| \) on \( \overline{\Omega} \) — we adopt the viewpoint that \( \sigma^*_n \) should typically equal the constant 1. If this is not the case, we compensate for this defect by extending \((-\text{div} \sigma)\) to a measure on \( \overline{\Omega} \) with
\[
(-\text{div} \sigma)\mathcal{L} \partial \Omega := (1-\sigma^*_n)\mathcal{H}^{n-1} \mathcal{L} \partial \Omega.
\]

Then we define a modified pairing \( [[\sigma, Du^+]\big|_{\partial \Omega}] \) by interpreting \( u^+ \) as \( \max\{u^{\text{int}}, (u_0)^{\text{int}}\} \) on \( \partial \Omega \) and extending the \((\text{div} \sigma)\)-integral in (6) from \( \Omega \) to \( \overline{\Omega} \). In other words, we define the measure \( [[\sigma, Du^+]\big|_{\partial \Omega}] \) by setting
\[
[[\sigma, Du^+]\big|_{\partial \Omega}] := [[\sigma, Du^+]\big|_{\partial \Omega}] + [(u-u_0)^{\text{int}}]^+ \mathcal{L} \partial \Omega.
\]

With these conventions, we now complement Definition 5.1 as follows.

**Definition 5.1** (super-1-harmonic function with respect to a Dirichlet datum). For bounded \( \Omega \) with (4) and \( u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega) \), we say that \( u \in BV(\Omega) \cap L^\infty(\Omega) \) is weakly super-1-harmonic on \( \overline{\Omega} \) with respect to \( u_0 \) if there exists some \( \sigma \in S^\infty(\Omega, \mathbb{R}^n) \) with \( \text{div} \sigma \leq 0 \) in \( \mathcal{D}'(\Omega) \) such that the equality of measures \([\sigma, Du^+]\big|_{\partial \Omega} = |D_{u_0}u| \) holds on \( \overline{\Omega} \).

For \( \sigma \in S^\infty(\Omega, \mathbb{R}^n) \) with \( \text{div} \sigma \leq 0 \) in \( \mathcal{D}'(\Omega) \) and \( u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega), u \in BV(\Omega) \cap L^\infty(\Omega) \), we get from (5), (6), (7)
\[
[\sigma, Du^+]\big|_{\partial \Omega} = \left\{ \left[(u-u_0)^{\text{int}}\right]^+ - \left[(u-u_0)^{\text{int}}\right]_n^* \right\} \mathcal{H}^{n-1} \mathcal{L} \partial \Omega.
\]

Thus, the boundary condition in Definition 5.1 is equivalent to the \( \mathcal{H}^{n-1} \)-a.e. equality \( \sigma_n^* \equiv -1 \) on the boundary portion \( \{u^{\text{int}}< (u_0)^{\text{int}}\} \cap \partial \Omega \), while no requirement is made on \( \{u^{\text{int}} \geq (u_0)^{\text{int}}\} \cap \partial \Omega \). We believe that this is very natural, in particular in the case \( n = 1 \), where super-1-harmonicity of \( u \) on an interval \([a, b]\) just means that \( u \) is increasing up to a certain point and decreasing afterwards, and where \( \sigma_n^* \) can take the value \(-1\) at most at one endpoint and only if \( u \) is monotone on the open interval \((a, b)\).

Another indication that Definition 5.1 is meaningful is provided by the next statement, which will also be proved in [15]. We emphasize that the statement does not hold anymore (not even for \( n=1, u_{0,k} = u_0 \equiv 0 \), and \( u_k \in W^{1,1}(\Omega) \)) if one replaces \([\sigma, Du^+]\big|_{\partial \Omega} \) with \([\sigma, Du^+]\big|_{\partial \Omega} \) in the definition.

**Theorem 5.2.** Suppose that \( \Omega \) is bounded with (4), and consider weakly super-1-harmonic functions \( u_k \in BV(\Omega) \cap L^\infty(\Omega) \) on \( \overline{\Omega} \) with respect to boundary data \( u_{0,k} \in W^{1,1}(\Omega) \cap L^\infty(\Omega) \). If \( u_{0,k} \) converges strongly in \( W^{1,1}(\Omega) \) and weakly* in \( L^\infty(\Omega) \) to some \( u_0 \), and if \( u_k \) weakly* converges to a limit \( u \) in \( BV(\Omega) \) and \( L^\infty(\Omega) \) such that \( u_k \leq u \) holds on \( \Omega \) for all \( k \in \mathbb{N} \), then \( u \) is weakly super-1-harmonic on \( \overline{\Omega} \) with respect to \( u_0 \).

**Remark.** In the situation of the theorem, it follows from the previously recorded reformulation of the boundary condition that \( u \) is also weakly super-1-harmonic on \( \overline{\Omega} \) with respect to every \( \bar{u}_0 \in W^{1,1}(\mathbb{R}^n) \cap L^\infty(\Omega) \) such that \( \mathcal{H}^{n-1}(\{u^* \leq u^{\text{int}} \wedge \bar{u}_0 \} \cap \partial \Omega) = 0 \). Roughly speaking, this means that the boundary values can always be decreased and that they can even be increased as long as the trace of \( u \) is not traversed. In view of the 1-dimensional case, we believe that this behavior is very reasonable.
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