CARTESIAN FIBRATIONS AND REPRESENTABILITY

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Abstract. In higher category theory, we use fibrations to model presheaves. In this paper we introduce a new method to build such fibrations. Concretely, for suitable reflective subcategories of simplicial spaces, we build fibrations that model presheaves valued in that subcategory. Using this we can build Cartesian fibrations, but we can also model presheaves valued in Segal spaces. Additionally, using this new approach, we define representable Cartesian fibrations, generalizing representable presheaves valued in spaces, and show they have similar properties.

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Introduction

0.1 Motivation. In the realm of higher category theory functoriality is very often quite complicated, which is because two maps in a $(\infty,1)$-category do not have a strict “composition map” but
rather a contractible space of such composition maps. Given this condition it seems unreasonable to expect strict functoriality to hold for functors of \((\infty,1)\)-categories. However, if we only demand functoriality "up to equivalence" then we have to keep track of all the equivalences. Thus we have to manage a lot of information, which is often an impossible task.

For that reason higher category theorists use \textit{fibrations}. Fibrations are maps over the domain with certain conditions that allow us to model such "functors up to equivalence" in a way that the necessary data is still very tractable and lends itself to computations.

Depending on the value, our fibrations need to satisfy different conditions. The first example is that of a \textit{right fibration}, which models presheaves valued in spaces. Just using the definition of right fibration, we can prove an analogue of the classical \textit{Yoneda lemma} in the context of higher categories. It has been studied quite extensively by Lurie ([Lu09, Chapter 2]) using quasicategories. Moreover, de Brito studied them using Segal spaces ([dB16]). There is also a model independent approach to right fibrations given by Riehl and Verity [RV17] (there called groupoidal Cartesian fibrations). Finally, there is also study of right fibrations over general simplicial spaces in [Ra17].

The next common example is that of a \textit{Cartesian fibration}, which models presheaves valued in \((\infty,1)\)-categories. As \((\infty,1)\)-categories are more complicated than spaces, Cartesian fibrations are also vastly more difficult to work with. In particular, although in [Lu09], Lurie defines Cartesian fibrations using simplicial sets, he has to use marked simplicial sets to get a model structure for which the fibrant objects are Cartesian fibrations. De Brito ([dB16]) studies Cartesian fibrations by defining them as certain bisimplicial spaces over a Segal space. Similar to Lurie he expanded the category he was working with to be able to get a model structure where the fibrant objects are Cartesian fibrations. On the other side, Riehl and Verity ([RV17]) take a model independent approach to Cartesian fibrations and therefore do not need to expand the category they start with, but also do not construct a model structure for Cartesian fibrations.

The goal of this work is to define Cartesian fibrations using bisimplicial spaces and show it comes with a model structure that we can understand quite well, by characterizing its fibrations and weak equivalences. Along the way we will show how the same method can be used to construct model structures where the fibrant objects model presheaves valued in objects besides \((\infty,1)\)-categories. Finally, we will use this setup to make sense of \textit{representable Cartesian fibrations}.

\textbf{0.2 Main Results}. The main results of this paper can be broken down in 2 parts.

\textit{(1)} Let \((sS)_{\text{Reef}}\) be a localization of the Reedy model structure with respect to a set of reasonable maps (Definition 6.11). Then we can build a model structure on bisimplicial spaces over \(X\), \((ssS_{/X})_{\text{ReeContra}}\), such that the fibrant objects model presheaves valued in the fibrant objects in \((sS)_{\text{Reef}}\). Moreover, we can give a good description of the fibrant objects and weak equivalences in \((ssS_{/X})_{\text{ReeContra}}\). The main results about that model structure can be summarized in the following theorem.

\textbf{Theorem 0.1}. Let \((sS)_{\text{Reef}}\) be localization of the Reedy model structure with respect to a set of maps that satisfy the conditions of Definition 6.11. Let \(X\) be a simplicial space. There is a model structure on bisimplicial spaces over \(X\), \((ssS_{/X})_{\text{ReeContra}}\), called the localized Reedy contravariant model structure such that
(1) It is left proper and simplicial (Theorem 6.6).

(2) The fibrant objects are called localized Reedy right fibrations and they model presheaves valued in fibrant simplicial spaces (fibrant in \((sS)^{Ree}_f\)) (Theorem 6.18).

(3) A map between Reedy right fibrations (not localized) is a localized Reedy contravariant equivalence if and only if it is a biReedy equivalence if and only if it is a level-wise localized Reedy equivalence if and only if it is a fiber-wise diagonal localized Reedy equivalence (Proposition 6.27).

(4) A map \(Y \to Z\) is a localized Reedy contravariant equivalence if and only if \(X_f \times_X Y \to X_f \times_X Z\) is a diagonal Reedy equivalence for every \(x : F(0) \to X\). Here \(X_f\) is a contravariant fibrant replacement of \(x : F(0) \to X\) (Theorem 6.28).

(5) A map \(g : X \to Y\) gives us a Quillen adjunction

\[
\begin{align*}
(sS/X)^{ReeContra}_f \xrightleftharpoons{g_!} (sS/Y)^{ReeContra}_f
\end{align*}
\]

which is an Quillen equivalence if \(g\) is a CSS equivalence (Theorem 6.29).

(6) A localized Reedy left fibration over \(X, p : R \to X\), gives us a Quillen adjunction (Theorem 6.31).

\[
\begin{align*}
(ssS/X)^{ReeContra}_f \xrightleftharpoons{p!p^*} (ssS/_X)_f^{ReeContra}_f
\end{align*}
\]

(7) Moreover if the maps that we are using to localize the Reedy model structure satisfy one other condition (Definition 6.34) then the localized Reedy contravariant model structure is a localization of the localized Reedy model structure (Theorem 6.38).

Then we apply this theorem to two very important cases.

**Corollary 0.2.** The Segal space model structure satisfies the necessary condition and so we get a model structure, called the Segal Cartesian model structure, where fibrant objects model presheaves valued in Segal spaces. (Theorem 7.3)

**Corollary 0.3.** The complete Segal space model structure satisfies the necessary condition and so we get a model structure, called the Cartesian model structure, where fibrant objects model presheaves valued in complete Segal spaces. (Theorem 7.16)

(2) We can use the fact that Reedy left fibrations (fibrations that model presheaves valued in Reedy fibrant simplicial spaces) are bisimplicial spaces to construct representable Reedy left fibration.

We have following main results.

**Theorem 0.4.** For each cosimplicial object \(x^* : \Delta \to X\), there is a Reedy left fibration \(X_{x^*}/\).

(Definition 5.13). If \(X\) is a Segal space then it models the functor that at point \(y\) has value the following simplicial space

\[
\begin{align*}
\text{map}(x^0, y) \xleftarrow{p_0} \text{map}(x^1, y) \xleftarrow{p_1} \text{map}(x^2, y) \xleftarrow{p_2} \cdots
\end{align*}
\]

(Proposition 5.29)

Representable Reedy left fibrations satisfy the analogue of a Yoneda embedding (Corollary 5.32)
**Corollary 0.5.** Let $X$ be a Segal space and $x, y$ two objects in $X$. Then we have an equivalence
\[ Map_{/X}(X_{/x}, X_{/y}) \cong map_X(x, y) \]

**0.3 Outline.** Before we state an outline to this work we will remind the reader how Rezk defined complete Segal spaces in his original work [Re01].

1. He started with simplicial sets with the Kan model structure.
2. Then he took simplicial objects in simplicial sets, also called simplicial spaces, with the Reedy model structure.
3. Finally he used the theory of Bousfield localizations to localize the Reedy model structure to the complete Segal space model structure, which gives us a functioning homotopy theory of $(\infty, 1)$-categories.

The general outline of this paper will exactly move along those same steps, but in a functorial manner.

We start by reviewing some basic notation and conventions in Section 1. In Section 2 we review the functorial analogue of spaces (functors in spaces), which are exactly left fibrations. For more details we advise the reader to see [Ra17] as all of the theorems and definitions regarding left fibrations is based on this source.

The next goal is then to move on to study simplicial objects in left fibrations, but before we can do that we have to do a proper analysis of simplicial objects in simplicial spaces, namely bisimplicial spaces. This is the topic of Section 3. Then we move on to work with simplicial objects in left fibrations, which we call Reedy left fibrations. We show how they model functors valued in simplicial spaces and how they are the fibrant objects in a model structure, the Reedy covariant model structure. This is is the topic of 4.

Then we do a brief excursion in Section 5 and show how this approach allows us to define representable Reedy left fibration, which model functors valued in simplicial spaces that are represented by a cosimplicial object.

Finally, we want to localize this construction, but before we can do that we need a better understanding of localizations of Reedy right fibrations. This will be the goal of Section 6. After that we can look at some worthwhile localizations in Section 7. First, we study Segal Cartesian fibrations which model presheaves valued in Segal spaces. Then we will move on to study Cartesian fibrations, which model presheaves valued in complete Segal spaces.

**0.4 Acknowledgements.** I want to thank my advisor Charles Rezk who has guided me through every step of the work.

**Basics & Conventions**

Throughout this note we use the theory of complete Segal spaces. The basic reference to CSS (complete Segal spaces) is the original paper by Charles Rezk [Re01]. Here we will only cover the basic notations.
1.1 Simplicial Sets. $S$ will denote the category of simplicial sets, which we will also call spaces. We will use the following notation with regard to spaces:

1. $\Delta$ is the indexing category with objects posets $[n] = \{0, 1, \ldots, n\}$ and mappings maps of posets.
2. $\Delta[n]$ denotes the simplicial set representing $[n]$ i.e. $\Delta[n]_k = \text{Hom}_\Delta([k], [n])$.
3. $\partial \Delta[n]$ denotes the boundary of $\Delta[n]$ i.e. the largest sub-simplicial set which does not include $id[n] : [n] \to [n]$. Similarly $\Lambda[n]_l$ denotes the largest simplicial set in $\Delta[n]$ which doesn’t have the $l^{th}$ face.
4. For a simplicial set $S$ we denote the face maps by $d_i : S_n \to S_{n-1}$ and the degeneracy maps by $s_i : S_n \to S_{n+1}$.
5. $I[l]$ be the category with $l$ objects and one unique isomorphisms between any two objects. Then we denote the nerve of $I[l]$ as $J[l]$. It is a Kan fibrant replacement of $\Delta[l]$ and comes with an inclusion $\Delta[l] \hookrightarrow J[l]$, which is a Kan equivalence.

1.2 Simplicial Spaces. $sS = \text{Map}(\Delta^{op}, S)$ denotes the category of simplicial spaces (bisimplicial sets). We have the following basic notations with regard to simplicial spaces:

1. We embed the category of spaces inside the category of simplicial spaces as constant simplicial spaces (i.e. the simplicial spaces $S$ such that, $S_n = S_0$ for all $n$).
2. Denote $F(n)$ to be the discrete simplicial space defined as $F(n)_k = \text{Hom}_\Delta([k], [n])$.
3. $\partial F[n]$ denotes the boundary of $F(n)$. Similarly $L(n)_l$ denotes the largest simplicial space in $F(n)$ which lacks the $l^{th}$ face.
4. For a simplicial space $X$ we have $X_n \cong \text{Hom}_{sS}(F(n), X)$.

1.3 Reedy Model Structure. The category of simplicial spaces has a Reedy model structure, which is defined as follows:

F A map $f : Y \to X$ is a (trivial) fibration if the following map of spaces is a (trivial) Kan fibration

$$\text{Map}_{sS}(F(n), Y) \to \text{Map}_{sS}(\partial F(n), Y) \times \text{Map}_{sS}(F(n), X).$$

W A map $f : Y \to X$ is a Reedy equivalence if it is a level-wise Kan equivalence.

C A map $f : Y \to X$ is a Reedy cofibration if it is an inclusion.

The Reedy model structure is very helpful as it enjoys many features that can help us while doing computations. In particular, it is cofibrantly generated, simplicial and proper. Moreover, it is also compatible with Cartesian closure, by which we mean that if $i : A \to B$ and $j : C \to D$ are cofibrations and $p : X \to Y$ is a fibration then the map

$$A \times D \coprod_{A \times C} B \times C \to B \times D$$

is a cofibration and the map

$$X^B \to X^A \times Y^B$$
is a fibration, which are trivial if any of the involved maps are trivial.

1.4 Diagonal & Kan Model Structure. There are two localizations of the Reedy model structure which we are going to need in the coming sections.

**Theorem 1.1.** There is a unique, cofibrantly generated, simplicial model structure on $sS$, called the Diagonal Model Structure, with the following specification.

- W A map $f : X \to Y$ is a weak equivalence if the diagonal map of spaces $\{f_{nn} : X_{nn} \to Y_{nn}\}_n$ is a Kan equivalence.
- C A map $f : X \to Y$ is a cofibration if it is an inclusion.
- F A map $f : X \to Y$ is a fibration if it satisfies the right lifting condition for trivial cofibrations.

In particular, an object $W$ is fibrant if it is Reedy fibrant and a homotopically constant simplicial space i.e. the degeneracy maps $s : W_0 \to W_n$ are weak equivalences.

**Proof.** The model structure is the localization of the Reedy model structure with respect to the maps

$$\mathcal{L} = \{F(0) \to F(n) : n \geq 0\}.$$  

A simple lifting argument shows that an object $W$ is fibrant if it is Reedy fibrant and $W_0 \to W_n$ is a weak equivalence for each $n \geq 0$. Now let $f : X \to Y$ be a map. Then $\{f_{nn} : X_{nn} \to Y_{nn}\}_n$ is a Kan equivalence if and only if $\text{Map}(Y,W) \to \text{Map}(X,W)$ is a Kan equivalence for every fibrant object $W$.  

**Remark 1.2.** A space $K$ embedded as a constant simplicial space is not fibrant in this model structure, as it is not Reedy fibrant. Rather the fibrant replacement is the simplicial space which at level $n$ is equal to $K^{\Delta[n]}$.

**Theorem 1.3.** There is a unique, cofibrantly generated, simplicial model structure on $sS$, called the Kan Model Structure, with the following specification.

- W A map $f : X \to Y$ is a weak equivalence if $f_0 : X_0 \to Y_0$ is a Kan equivalence.
- C A map $f : X \to Y$ is a cofibration if it is an inclusion.
- F A map $f : X \to Y$ is a fibration if it satisfies the right lifting condition for trivial cofibrations.

In particular, an object $W$ is fibrant if it is Reedy fibrant and the map

$$\text{Map}(F(n),W) \to \text{Map}(\partial F(n),W)$$

is a trivial Kan fibration for $n > 0$.

**Proof.** Similar to the previous theorem this model structure is a localization of the Reedy model structure with respect to maps

$$\mathcal{L} = \{\partial F(n) \to F(n) : n \geq 0\}.$$  

Basic lifting argument tells us that $W$ is fibrant if and only if it is a Reedy fibration and

$$W_n \to \text{Map}(\partial F(n),W)$$
is a trivial Kan fibration for \( n > 0 \). This also implies that \( f_0 : X_0 \to Y_0 \) is a Kan equivalence if and only if \( Map(Y, W) \to Map(X, W) \) is a Kan equivalence for every fibrant object \( W \). \( \square \)

These model structures all fit nicely into a chain of Quillen equivalences.

**Theorem 1.4.** There is the following chain of Quillen equivalences:

\[
(sS)^{Diag} \xrightarrow{Diag_\#} (sS)^{Kan} \xrightarrow{i_\#} (sS)^{Kan}
\]

Here \( Diag : \Delta \to \Delta \times \Delta \) is the diagonal map which induces an adjunction \((Diag_\#, Diag^*)\) on functor categories. Also, \( i : \Delta \to \Delta \times \Delta \) is the map that takes \([n]\) to \(([n], [0])\) which also induces an adjunction \((i_\#, i^*)\) on functor categories.

**Proof.** \((Diag_\#, Diag^*)\): By definition, a map of simplicial spaces \( f \) is a diagonal equivalence if and only if \( Diag_\#(f) \) is a Kan equivalence. Moreover, basic computation shows that the counit map \( Diag^*K \to K \) is a Kan equivalence for every Kan complex \( K \).

\((i_\#, i^*)\): By the same argument a map of simplicial spaces \( f \) is a Kan equivalence if and only if \( i^*(f) \) is a Kan equivalence. Finally, the derived unit map \( K \to i^* Ri_\#(K) \) is a Kan equivalence for every Kan complex \( K \) as \( i^* Ri_\#(K) = K \). \( \square \)

This implies that the diagonal and Kan model structure are Quillen equivalent, however, that does not mean that they are actually the same model structure.

### 1.5 Complete Segal Spaces

The Reedy model structure can be localized such that it models an \((\infty, 1)\)-category. This is done in two steps. First we define *Segal spaces*.

**Definition 1.5.** [Re01, Page 11] A Reedy fibrant simplicial space \( X \) is called a Segal space if the map

\[
X_n \xrightarrow{\sim} X_1 \times \ldots \times X_1
\]

is an equivalence for \( n \geq 2 \).

Segal spaces come with a model structure, namely the *Segal space model structure*.

**Theorem 1.6.** [Re01, Theorem 7.1] There is a simplicial closed model category structure on the category \( sS^{Seg} \) of simplicial spaces, called the Segal space model category structure, with the following properties.

1. The cofibrations are precisely the monomorphisms.
2. The fibrant objects are precisely the Segal spaces.
3. The weak equivalences are precisely the maps \( f \) such that \( Map_{sS}(f, W) \) is a weak equivalence of spaces for every Segal space \( W \).
4. A Reedy weak equivalence between any two objects is a weak equivalence in the Segal space model category structure, and if both objects are themselves Segal spaces then the converse holds.
5. The model category structure is compatible with the cartesian closed structure.
The model structure is the localization of the Reedy model structure with respect to the maps
\[ G(n) = F(1) \prod_{F(0)} ... \prod_{F(0)} F(1) \rightarrow F(n) \]
for \( n \geq 2 \).

A Segal space already has many characteristics of a category, such as objects and morphisms (as can be witnessed in [Re01, Section 5]), however, it is still does not model an actual \((\infty, 1)\)-category. For that we need complete Segal spaces.

**Definition 1.7.** Let \( J[n] \) be a fibrant replacement of \( \Delta[n] \) in the Kan model structure (as described in Subsection 1.1). We define a discrete simplicial space \( E(n) \) as
\[ E(n)_{kl} = J[n]_{k}. \]
In particular, \( E(1) \) is the free invertible arrow.

**Definition 1.8.** A Segal space \( W \) is called a complete Segal space if it satisfies one of the the following equivalent conditions.

1. The map
\[ Map(E(1), W) \rightarrow Map(F(0), W) = W_{0} \]
is a trivial Kan fibration. Here \( E(1) \) is the free invertible arrow (Definition 1.7).
2. In the following commutative rectangle
\[
\begin{array}{ccc}
W_{0} & \rightarrow & W_{3} \\
\downarrow & & \downarrow \\
W_{1} & \rightarrow & W_{1}^{s} \times_{W_{0}}^{s} W_{1}^{t} \times_{W_{0}}^{t} W_{1} \\
\downarrow & & \downarrow \\
W_{0} \times W_{0} & \rightarrow & W_{1} \times W_{1}
\end{array}
\]
the top square is a homotopy pullback square in the Kan model structure. Equivalently, the large rectangle is a homotopy pullback square in the Kan model structure.

Complete Segal spaces come with their own model structure, the complete Segal space model structure.

**Theorem 1.9.** [Re01, Theorem 7.2] There is a simplicial closed model category structure on the category \( sS \) of simplicial spaces, called the complete Segal space model category structure, with the following properties.

1. The cofibrations are precisely the monomorphisms.
2. The fibrant objects are precisely the complete Segal spaces.
3. The weak equivalences are precisely the maps \( f \) such that \( Map_{sS}(f, W) \) is a weak equivalence of spaces for every complete Segal space \( W \).
(4) A Reedy weak equivalence between any two objects is a weak equivalence in the complete Segal space model category structure, and if both objects are themselves Segal spaces then the converse holds.

(5) The model category structure is compatible with the cartesian closed structure.

(6) The model structure is the localization of the Segal space model structure with respect to the map $F(0) \to E(1)$.

A complete Segal space is a model for a $(\infty,1)$-category. For more details on this see [Re01, Sections 5,6].

A Reminder on the Covariant Model Structure

This section will serve as a short reminder on the covariant model structure and all of its relevant definitions and theorems. For more details the reader see [Ra17], where all these definitions and theorems are discussed in more detail.

**Definition 2.1.** [Ra17, Definition 3.1] Let $X$ be a simplicial space. A map $p : L \to X$ is called **left fibration** if it is a Reedy fibration and the following is a homotopy pullback square:

$$
\begin{array}{ccc}
L_n & \xrightarrow{0^*} & X_n \\
\downarrow{p_n} & & \downarrow{p_0} \\
L_0 & \xrightarrow{0^*} & X_0
\end{array}
$$

Here the map $0^*$ is the induced map we get from $0 : F(0) \to F(n)$ which sends the point to the initial vertex in $F(n)$.

Left fibrations come with a model structure.

**Theorem 2.2.** [Ra17, Theorem 3.14] Let $X$ be simplicial space. There is a unique model structure on the category $sS/X$, called the covariant model structure and denoted by $(sS/X)^{cov}$, which satisfies the following conditions:

1. It is a simplicial model category.
2. The fibrant objects are the left fibrations over $X$.
3. Cofibrations are monomorphisms.
4. A map $f : A \to B$ over $X$ is a weak equivalence if
   $$
   \text{map}_{sS/X}(B,W) \to \text{map}_{sS/X}(A,W)
   $$
   is an equivalence for every left fibration $W \to X$.
5. A weak equivalence (covariant fibration) between fibrant objects is a level-wise equivalence (Reedy fibration).

Note that the definition is not symmetric and so we have following definition.
Definition 2.3. [Ra17, Definition 3.21] Let $X$ be a simplicial space. A map $p : R \to X$ is called right fibration if it is a Reedy fibration and the following is a homotopy pullback square:

\[
\begin{array}{ccc}
R_n & \xrightarrow{n^*} & X_n \\
p_n \downarrow & & \downarrow p_0 \\
R_0 & \xrightarrow{n^*} & X_0
\end{array}
\]

Here the map $n^*$ is the induced map we get from $n : F(0) \to F(n)$ which sends the point to the final vertex in $F(n)$.

Remark 2.4. Similar to the previous case this fibration comes with its own model structure, which is called the contravariant model structure.

The model structure is defined by using the technique of Bousfield localizations. That makes it convenient to define, however it is often very difficult to recognize weak equivalences in this model structure. For that purpose we have following recognition principle for covariant equivalences.

Proposition 2.5. [Ra17, Proposition 3.28] Let $f : Y \to Z$ be a map over $X$. Then $f$ is a covariant equivalence if and only if for every map $x : F(0) \to X$, the induced map

\[ Y \times_{X} X_{/x} \to Z \times_{X} X_{/x} \]

is a diagonal equivalence. Here $X_{/x}$ is the right fibrant replacement of the map $x$ over $X$.

The proof of this result mainly relies on following theorem.

Theorem 2.6. [Ra17, Theorem 3.32] Let $p : R \to X$ be a right fibration. The following is a Quillen adjunction:

\[ (sS_{/X})^{cov} \xleftarrow{pp^*} (sS_{/X})^{cov} \]

For more details on left fibrations and it’s relevant properties see [Ra17, Chapter 3].

Left fibrations model maps into spaces. Our overall goal in this paper is it to generalize all aforementioned results to the level of presheaves into higher categories. However, before we can do so we have to expand our playing field, which leads us to the next section.

Bisimplicial Spaces

In order to generalize our results from right fibrations to fibrations that model other functors we have to first expand the underlying category. There are several ways this can be done. There is one approach, used by Lurie ([Lu09]), which adds as little extra data as possible to store the necessary information, by using marked simplicial sets. We will not follow that path and rather add a whole simplicial axis. That approach results in a lot of redundant data, however also gives us
a very convenient way to directly generalize results from right fibrations to other fibrations. First, however, we have to study the underlying objects. Thus in this section we will study bisimplicial spaces.

### 3.1 First Properties of Bisimplicial Spaces.

**Definition 3.1.** We define the category of bisimplicial spaces as $\text{Fun}(\Delta^{op}, sS)$ and denote it as $sS$.

**Remark 3.2.** Using the fact that this category is cartesian closed we have following equivalence of categories:

$$sS = \text{Fun}(\Delta^{op}, sS) = \text{Fun}(\Delta^{op} \times \Delta^{op}, S) = \text{Fun}(\Delta^{op} \times \Delta^{op} \times \Delta^{op}, \text{Set})$$

With this in mind every bisimplicial space is also simplicial simplicial space and also a tri-simplicial set. Throughout this work, however, we often ignore one axis and think about it either as a bisimplicial space, which is a collection of spaces denoted by two indices $(X_{kn})$ or as simplicial simplicial space, which is a collection of simplicial spaces denoted by one index $(X_k)$.

**Definition 3.3.** We define the discrete bisimplicial space $\varphi_k$ as

$$(\varphi_k)_n = \text{Hom}_\Delta([n], [k])$$

Thus for every $n$ the simplicial space $(\varphi_k)_n$ is just a set.

**Definition 3.4.** We define $F(m)$ and $\Delta[l]$ inside $sS$ as the bisimplicial spaces

$$F(m)_{kn} = F(m)_n$$

$$\Delta[l]_{kn} = \Delta[l]_n$$

**Remark 3.5.** By using the fact that $sS$ is a category of presheaves of sets, $sS = \text{Fun}(\Delta^{op} \times \Delta^{op} \times \Delta^{op}, \text{Set})$, and a standard application of the Yoneda lemma we conclude that $sS$ is generated by the objects $\varphi_k \times F(n) \times \Delta[l]$ (meaning it is a colimit of a diagram valued in such objects).

**Definition 3.6.** There are two ways to embed $sS$ into $sS$.

- There is a map $i_F : \Delta \times \Delta \times \Delta \to \Delta \times \Delta$ defined as $i_F(n_1, n_2, n_3) = (n_2, n_3)$. This gives us an adjunction

$$sS \cong \text{Fun}(\Delta \times \Delta \times \Delta, sS).$$

Concretely, the left adjoint is defined as

$$i_F^*(X)_{kn} = X_n$$

and the right adjoint is defined as

$$(i_F)_*(X)_{n} = X_{0n}$$

In particular, $i_F^*(F(n)) = F(n)$ (which justifies the naming) and $i_F^*(\Delta[l]) = \Delta[l]$. So it should be thought of as the standard embedding of simplicial spaces into bisimplicial spaces. Intuitively we think of this map as the “vertical embedding”.

- Second, we have a map $i_p : \Delta \times \Delta \times \Delta \to \Delta \times \Delta$ defined as $i_p(n_1, n_2, n_3) = (n_1, n_3)$. This gives us an adjunction

$$sS \cong \text{Fun}(\Delta \times \Delta \times \Delta, sS).$$
Concretely, the left adjoint is defined as
$$i^*_\varphi(X)_{kn} = X_k$$
and the right adjoint is defined as
$$(i_*\varphi)(X)_n = X_{n0}$$
This time $i_\varphi(F(n)) = \varphi_n$. We think of this embedding as the “horizontal embedding”.

**Notation 3.7.** From here on we will consider $i_F$ to be the standard embedding of simplicial spaces in bisimplicial spaces. Thus we think of any simplicial space $X$ as a bisimplicial $i^*_F(X)$.

**Notation 3.8.** We will adhere to the standard notation when it comes to $\varphi_k$. In particular:

1. **Boundaries**: $\partial \varphi_k$ is the boundary of $\varphi_k$
2. **Horns**: $\varphi^l_k$ is the $l$-Horn of $\varphi_k$.

Before we move on let us review some further basic categorical properties of $ss\mathbb{S}$.

**Mapping Objects**: $ss\mathbb{S}$ is cartesian closed. Let $X, Y \in ss\mathbb{S}$. Then we define $Y^X$ as follows:

$$Y^X = \text{Hom}_{ss\mathbb{S}}(X \times \varphi_k \times F(n) \times \Delta[l], Y)$$

This definition means that $ss\mathbb{S}$ is enriched on three different levels:

1. **Enriched over Spaces**: $ss\mathbb{S}$ is enriched over spaces as follows
   
   $$\text{Map}_{ss\mathbb{S}}(X, Y)_l = (Y^X)_0(l)$$

2. **Enriched over simplicial Spaces**: $ss\mathbb{S}$ is enriched over simplicial spaces by
   
   $$\text{Map}_{ss\mathbb{S}}(X, Y)_{nl} = (Y^X)_0(l)$$

3. **Enriched over bisimplicial Spaces**: $ss\mathbb{S}$ is enriched over bisimplicial spaces as it is cartesian closed.

**Remark 3.9.** We will use the same notation whether our maps are enriched over spaces or simplicial spaces. However, if necessary, we will use indices to specify whether our mapping object is a space or simplicial space.

By combining the enrichment and generators, the Yoneda lemma gives us the following isomorphisms:

1. $\text{Map}_{ss\mathbb{S}}(\varphi_k, X) \cong X_k$ as simplicial spaces
2. $\text{Map}_{ss\mathbb{S}}(\varphi_k \times F(n), X) \cong X_{kn}$ as spaces
3. $\text{Map}_{ss\mathbb{S}}(\varphi_k \times F(n) \times \Delta[l], X) \cong X_{knl}$ as sets.

We are now in a position to define a model structure on $ss\mathbb{S}$.
3.2 The Bisimplicial Reedy Model Structure. In this section we introduce a simplicial model structure on bisimplicial spaces. This will serve as a basis for any homotopy theory we will later discuss.

**Definition 3.10.** The category $sS$ of simplicial spaces has a model structure called the *bisimplicial Reedy model structure*. It comes from giving $Fun(\Delta^{op}, sS)$ the Reedy model structure where $sS$ has the Reedy model structure. It has following specifications.

- A map $f : X \to Y$ is a *bisimplicial Reedy cofibration* if it a level-wise inclusion of spaces.
- A map $f : X \to Y$ is a *bisimplicial weak Reedy equivalence* if it is a level-wise Kan equivalence of spaces.
- A map $f : X \to Y$ is a *bisimplicial Reedy fibration* if it satisfies the right lifting condition with respect to all trivial cofibrations.

We can give very concrete descriptions for the bisimplicial Reedy fibrations.

**Lemma 3.11.** A map $X \to Y$ is a (trivial) bisimplicial Reedy fibration if it satisfies one of the following equivalent conditions:

1. It has the right lifting condition with respect to all (trivial) cofibration.
2. The maps of simplicial spaces
   \[
   \text{Map}(\varphi_k, X) \to \text{Map}(\partial \varphi_k, X) \times_{\text{Map}(\partial \varphi_k, Y)} \text{Map}(\varphi_k, Y)
   \]
   is a (trivial) Reedy fibration for all $k$.
3. Let $P_{kn}$ and $B_{kn}$ be the following spaces:
   \[
   P_{kn} = \text{Map}(\partial \varphi_k \times \partial F(n), X) \times_{\text{Map}(\partial \varphi_k \times \partial F(n), Y)} \text{Map}(\varphi_k \times \partial F(n), Y)
   \]
   \[
   B_{kn} = \text{Map}(\partial \varphi_k \times F(n), X) \times_{\text{Map}(\varphi_k \times F(n), Y)} \text{Map}(\partial \varphi_k \times F(n), Y)
   \]
   Then, the maps of spaces:
   \[
   \text{Map}(\varphi_k \times F(n), X) \to \text{Map}(\varphi_k \times \partial F(n), X) \times_{P_{kn}} B_{kn}
   \]
   is a (trivial) Kan fibration for all $n, k$.

The bisimplicial Reedy model structure satisfies many pleasant properties, which make it easy to work with. Here we will outline the main ones.

**Compatibility with Cartesian Closure:** For any cofibrations $i : A \to B$ and $j : C \to D$ and any fibration $p : Y \to X$, $sS$ with the Reedy model structure satisfies the following equivalent conditions:

- The map
  \[
  (A \times D) \coprod_{(A \times C)} (B \times C) \to B \times D
  \]
  is a cofibration which is trivial if either of $i$ or $j$ are.
The map \( Y^B \to Y^A \times X^B \)

is a fibration, which is a trivial fibration if either one of \( i \) or \( p \) are trivial.

**Simplicial Model Structure:** Applying the compatibility above to the 00 level we see that the bisimplicial Reedy model structure is enriched over the Kan model structure. Thus, \( ssS \) is a simplicial model structure.

**Properness:** \( ssS \) with the Reedy model structure is proper, which is because \( sS \) is a proper model category and every fibration (cofibration) is in particular a level-wise fibration (cofibration).

**Cofibrantly Generated Model Category:** The last part of the previous lemma (Lemma 3.11) implies that \( ssS \) is a cofibrantly generated model category. For \( k, n, l \geq 0 \) let \( cD_{knl} \) be the colimits of the following diagram:

\[
\begin{align*}
\varphi_k & \times \partial F(n) \times \partial \Delta[l] \\
\varphi_k & \times F(n) \times \partial \Delta[l] \quad \xrightarrow{=} \quad cD_{knl} \quad \xleftarrow{=} \quad \varphi_k & \times \partial F(n) \times \Delta[l] \\
\partial \varphi_k & \times F(n) \times \partial \Delta[l] \quad \xrightarrow{=} \quad \partial \varphi_k & \times F(n) \times \Delta[l] \\
\partial \varphi_k & \times F(n) \times \partial \Delta[l] \quad \xrightarrow{=} \quad \partial \varphi_k & \times \partial F(n) \times \Delta[l]
\end{align*}
\]

Moreover, For \( k, n, l \geq 0 \) and \( 0 \leq i \leq l \) let \( tD_{knil} \) be the colimit of the following diagram:

\[
\begin{align*}
\varphi_k & \times \partial F(n) \times \Lambda[l]_i \\
\varphi_k & \times F(n) \times \Lambda[l]_i \quad \xrightarrow{=} \quad tD_{knil} \quad \xleftarrow{=} \quad \varphi_k & \times \partial F(n) \times \Delta[l] \\
\partial \varphi_k & \times F(n) \times \Lambda[l]_i \\
\partial \varphi_k & \times F(n) \times \Lambda[l]_i \quad \xrightarrow{=} \quad \partial \varphi_k & \times F(n) \times \Delta[l] \\
\partial \varphi_k & \times \partial F(n) \times \Delta[l] \\
\partial \varphi_k & \times \partial F(n) \times \Delta[l]
\end{align*}
\]

With those definitions at hand we have following lemma.

**Lemma 3.12.** The generating cofibrations are the inclusions

\[ cD_{knl} \ni \varphi_k \times F(n) \times \Delta[l] \]

and the generating trivial cofibrations are the inclusions

\[ tD_{knil} \ni \varphi_k \times F(n) \times \Delta[l] \]

Theorem A.3 gives us conditions for when a localization model structure on \( sS \) exists. The bisimplicial Reedy model structure is a cofibrantly generated simplicial model structure and thus those results extend to bisimplicial spaces as well. This means we have following theorem for bisimplicial spaces with the Reedy model structure.
Theorem 3.13. Let \( \mathcal{L} \) be a set of cofibrations in \( \text{ssS} \) with the bisimplicial Reedy model structure. There exists a cofibrantly generated, simplicial model category structure on \( \text{ssS} \) with the following properties:

1. the cofibrations are exactly the monomorphisms
2. the fibrant objects (called \( \mathcal{L} \)-local objects) are exactly the bisimplicial Reedy fibrant \( W \in \text{ssS} \) such that
   \[ \text{Map}_{\text{ssS}}(B,W) \to \text{Map}_{\text{ssS}}(A,W) \]
   is a weak equivalence of spaces
3. the weak equivalences (called \( \mathcal{L} \)-local weak equivalences) are exactly the maps \( g : X \to Y \)
   such that for every \( \mathcal{L} \)-local object \( W \), the induced map
   \[ \text{Map}_{\text{ssS}}(Y,W) \to \text{Map}_{\text{ssS}}(X,W) \]
   is a weak equivalence
4. a Reedy weak equivalence (fibration) between two objects is an \( \mathcal{L} \)-local weak equivalence (fibration), and if both objects are \( \mathcal{L} \)-local then the converse holds.

We call this model category the localization model structure.

Notation 3.14. In order to shorten the notation from now on we will call bisimplicial Reedy model structure, bisimplicial Reedy fibration and bisimplicial Reedy fibrant object simply biReedy model structure, biReedy fibration and biReedy fibrant object.

3.3 Reedy Diagonal and Reedy Model Structures. In Subsection 1.4 we discussed important localizations of the Reedy model structure on simplicial spaces that are Quillen equivalent to the Kan model structure. In a similar manner, we need localizations of the biReedy model structure that are Quillen equivalent to the Reedy model structure, so we will introduce them right here.

Theorem 3.15. There is a unique, cofibrantly generated, simplicial model structure on \( \text{ssS} \), called the diagonal Reedy Model Structure and denoted by \( \text{ssS}^{\text{DiagRee}} \), with the following specifications.

C A map \( f : X \to Y \) is a cofibration if it is an inclusion.
W A map \( f : X \to Y \) is a weak equivalence if \( \{f_{k\text{n}} : X_{k\text{n}} \to Y_{k\text{n}}\}_{k\text{n}} \) is a Reedy equivalence.
F A map \( f : X \to Y \) is a fibration if it satisfies the right lifting condition for trivial cofibrations.

In particular, an object \( W \) is fibrant if it is biReedy fibrant and the maps \( W_{k\emptyset} \to W_{k\text{n}} \) are Kan equivalences.

Proof. Here we use Theorem 3.13. The model structure is the localization model structure of the biReedy model structure with respect to the maps
\[ \mathcal{L} = \{ \varphi_k \times F(0) \to \varphi_k \times F(n) : k, n \geq 0 \} \]

In order to show this gives us the desired result, we first determine the local objects. A bisimplicial space \( W \) is local if and only if it is biReedy fibrant and the map
\[ \text{Map}(\varphi_k \times F(n),W) \to \text{Map}(\varphi_k,W) \]
is a Kan equivalence. But the map above simplifies to
\[ W_{kn} \rightarrow W_{k0}. \]
Notice \( W \) is fibrant in the diagonal Reedy model structure if and only if it is level-wise fibrant in the diagonal model structure (Theorem 1.1). Thus a map \( f \) is an equivalence if and only if it is a level-wise diagonal equivalence. Thus, \( f \) is an equivalence if and only if the map
\[ \{ f_{kn} : X_{kn} \rightarrow Y_{kn} \}_{kn} \]
is a Reedy equivalence.

**Theorem 3.16.** There is a unique, cofibrantly generated, simplicial model structure on \( \text{ssS} \), called the Reedy Model Structure, with the following specification.

- A map \( f : X \rightarrow Y \) is a weak equivalence if \((i_\varphi)_*(f) : (i_\varphi)_*(X) \rightarrow (i_\varphi)_*(Y)\) is a Reedy equivalence.
- A map \( f : X \rightarrow Y \) is a cofibration if it is an inclusion.
- A map \( f : X \rightarrow Y \) is a fibration if it satisfies the right lifting condition for trivial cofibrations.

In particular, an object \( W \) is fibrant if it is biReedy fibrant and the Reedy map \( \text{Map}(\varphi \times F(n), W) \rightarrow \text{Map}(\varphi \times \partial F(n), W) \) is a trivial Reedy fibration for \( n > 0 \).

**Proof.** Again we use Theorem 3.13, with the difference that here we are localizing with respect to the set of maps
\[ L = \{ \varphi_k \times \partial F(n) \rightarrow \varphi_k \times F(n) : n > 0 \} \]
It immediately follows that a map is local if and only if the \( \text{Map}(\varphi \times F(n), W) \rightarrow \text{Map}(\varphi \times \partial F(n), W) \) is a trivial Kan fibration. This also implies that \((i_\varphi)_*(f) : (i_\varphi)_*(X) \rightarrow (i_\varphi)_*(Y)\) is a Reedy equivalence if and only if \( \text{Map}(Y, W) \rightarrow \text{Map}(X, W) \) is Kan equivalence for every fibrant object \( W \). \( \square \)

**Definition 3.17.** We have following two diagonal maps \( \varphi_{\text{Diag}}, \Delta_{\text{Diag}} : \Delta \times \Delta \rightarrow \Delta \times \Delta \times \Delta \), defined as follows:
\[ \varphi_{\text{Diag}}(n_1, n_2) = (n_1, n_2, n_2) \]
\[ \Delta_{\text{Diag}}(n_1, n_2) = (n_1, n_1, n_2) \]

These model structures all give us following long chain of Quillen equivalences.

**Theorem 3.18.** There is the following chain of Quillen equivalences:
\[
\begin{align*}
\text{(ssS)}_{\text{DiagRee}} \cong & \quad \varphi_{\text{Diag}}^* & \quad \text{(ssS)}_{\text{Ree}} & \quad (i_\varphi)^* & \quad \text{(ssS)}_{\text{Ree}}
\end{align*}
\]

The proof is analogous to the proof of Theorem 1.4.

**Remark 3.19.** This in particular implies that the diagonal Reedy and Reedy model structures are Quillen equivalent, however, that does not mean that they are actually the same model structure.

**Remark 3.20.** For later parts it is instructive to see how the maps above act on the generators.
\[ \phi_{\text{Diag}}^*(\varphi_k \times F(n) \times \Delta[l]) = F(k) \times \Delta[n] \times \Delta[l] \]

\[ \phi_{\text{Diag}}^*(F(n) \times \Delta[l]) = \varphi_n \times F(l) \times \Delta[l] \]

\[ (i_\varphi)_\#(F(n) \times \Delta[l]) = \varphi_n \times \Delta[l] \]

\[ i_\varphi^*(\varphi_k \times F(n) \times \Delta[l]) = F(k) \times \Delta[l] \]

\[ \Delta_{\text{Diag}}^*(F(n) \times \Delta[l]) = \varphi_n \times F(n) \times \Delta[l] \]

\[ \Delta_{\text{Diag}}^*(\varphi_k \times F(n) \times \Delta[l]) = F(k) \times F(n) \times \Delta[l] \]

All of these simply follow by applying the definition of the adjunction.

In the case of bisimplicial spaces there is another Quillen adjunction that is not as obvious and will be important later on.

**Proposition 3.21.** There is a Quillen adjunction

\[
\begin{array}{c}
\text{(sS)}\rec \\
\Delta_{\text{Diag}}^* \to \Delta_{\text{Diag}} \\
\end{array}
\]

\[
\text{(ssS)}\bi\rec
\]

**Proof.** We show it is a Quillen adjunction by using Lemma A.4. Clearly the left adjoint preserves cofibrations. So, we only have to prove that the right adjoint preserves fibrations. Let \( Y \to X \) be a biReedy fibration. Then, we have to show that \( \Delta_{\text{Diag}}^*(Y) \to \Delta_{\text{Diag}}^*(X) \) is a Reedy fibration. This is equivalent to showing that

\[
\text{Map}(F(n), \Delta_{\text{Diag}}^*(Y)) \to \text{Map}(\partial F(n), \Delta_{\text{Diag}}^*(Y)) \times \text{Map}(F(n), \Delta_{\text{Diag}}^*(X))
\]

is a Kan fibrations. Using adjunction we get

\[
\text{Map}(\Delta_{\text{Diag}}^*(F(n)), Y) \to \text{Map}(\Delta_{\text{Diag}}^*(\partial F(n)), Y) \times \text{Map}(\Delta_{\text{Diag}}^*(F(n)), X)
\]

which we simplify to

\[
\text{Map}(\varphi_n \times F(n), Y) \to \text{Map}(\partial \varphi_n \times \partial F(n), Y) \times \text{Map}(\varphi_n \times F(n), X)
\]

but this is clearly a Kan fibration as the biReedy model structure is simplicial and the map

\[
\partial \varphi_n \times \partial F(n) \to \varphi_n \times F(n)
\]

is an inclusion and thus a cofibration. \( \square \)

We will generalize this adjunction when we need an in depth analysis of localizations of the Reedy contravariant model structure in Section 6 (Definition 6.36).

**The Reedy Covariant Model Structure**

In this section we generalize the covariant model structure to the category of bisimplicial spaces over a fixed simplicial space. This gives us a good model for maps valued in simplicial spaces and the room we need to further define new model structures.
Remark 4.1. **Very Important Remark: Fixing the Base** From here on until the end we assume that the base object $X$ is always a simplicial space, embedded in $ssS$ in the following way

$$X_k = X$$

This is the embedding $i^*_p(X)$ of simplicial spaces as introduced in Definition 3.6.

### 4.1 Defining the Reedy Covariant Model Structure.

In this subsection we use the biReedy model structure to define the Reedy covariant model structure and then use our knowledge of the covariant model structure to deduce some basic facts we need later on.

**Definition 4.2.** Let $X$ be a simplicial space. We say a map of bisimplicial spaces $p: Y \to X$ is a Reedy left fibration if it is a biReedy fibration and the following is a homotopy pullback square,

$$
\begin{array}{ccc}
Y_{kn} & \xrightarrow{\sigma^*} & Y_{k0} \\
\downarrow^{\gamma} & & \downarrow^{p_{k0}} \\
X_n & \xleftarrow{0^*} & X_0
\end{array}
$$

where $0^*$ is the map induced by $0 : F(0) \to F(n)$ which takes the unique point to the initial vertex.

**Remark 4.3.** This definition is equivalent to saying that a map is a Reedy left fibration if the map is a biReedy fibration and for any $k \geq 0$, $Y_k \to X$ is a left fibration.

**Remark 4.4.** Rewriting the pullback diagram above we see that a map $Y \to X$ is a Reedy left fibration if and only if for every map $\varphi_k \times F(n) \to X$, the induced map

$$Map_{/X}(\varphi_k \times F(n), Y) \to Map_{/X}(\varphi_k, Y)$$

is a trivial Kan fibration.

Using this fact it is easy to see that this fibration has features that are very similar to left fibrations. Concretely following results hold. We will state them here without proof, but will refer to the analogous proof for left fibrations.

**Lemma 4.5.** The following are true about Reedy left fibrations:

1. The pullback of Reedy left fibrations are Reedy left fibrations [Ra17, Lemma 3.5].
2. If $f$ and $g$ are Reedy left fibrations then $fg$ is also a Reedy left fibration [Ra17, Lemma 3.6].
3. If $f$ and $fg$ are Reedy left fibrations then $g$ is also a Reedy left fibration [Ra17, Lemma 3.6].
4. A map $Y \to X$ is a Reedy left fibration if and only if for every map $\varphi_k \times F(n)$, the pullback map $Y \times_X (\varphi_k \times F(n)) \to \varphi_k \times F(n)$ is a Reedy left fibration [Ra17, Lemma 3.8].

As in the case of left fibrations this construction comes with a model structure, the Reedy covariant model structure.

**Theorem 4.6.** Let $X$ be a simplicial space. There is a unique model structure on the category $ssS_{/X}$, called the Reedy covariant model structure and denoted by $(ssS_{/X})^{ReeCov}$, which satisfies the following conditions:
(1) It is a simplicial model category.
(2) The fibrant objects are the Reedy left fibrations over $X$.
(3) Cofibrations are monomorphisms.
(4) A map $f: A \to B$ over $X$ is a weak equivalence if
$$\text{Map}_{ssS/X} (B, W) \to \text{Map}_{ssS/X} (A, W)$$
is an equivalence for every Reedy left fibration $W \to X$.
(5) A weak equivalence (Reedy covariant fibration) between fibrant objects is a level-wise equivalence (biReedy fibration).

Proof. Let $\mathcal{L}$ be the collection of maps of the following form
$$\mathcal{L} = \{ \varphi_k \times F(0) \hookrightarrow \varphi_k \times F(n) \to X \}$$
Note that $\mathcal{L}$ is a set of cofibrations in $ssS/X$ with the biReedy model structure. This allows us to use the theory of Bousfield localizations with respect to $\mathcal{L}$ on the category $ssS/X$. Theorem 3.13. It results in a model structure on $ssS/X$ which automatically satisfies all the conditions we stated above except for the fact that fibrant objects are exactly the Reedy left fibrations and this we will prove here. But this follows right away from Remark 4.4. □

Note the Reedy covariant model structure behaves well with respect to base change:

**Theorem 4.7.** Let $f: X \to Y$ be map of simplicial spaces. Then the following adjunction

$$
(ssS/X)^{ReeCov} \overset{f_!}{\underset{f^*}{\rightleftarrows}} (ssS/Y)^{ReeCov}
$$
is a Quillen adjunction. Here $f_!$ is the composition map and $f^*$ is the pullback map.

**Proof.** We use lemma A.5. $f_1$ preserves inclusions. Also, the pullback of the Reedy fibration is a Reedy fibration. Finally, by Lemma 4.5, the pullback of a Reedy left fibration is a Reedy left fibration. □

For many purposes it is helpful to have a second way of thinking about this model structure. For that we need following trivial lemma:

**Lemma 4.8.** Let $X$ be a simplicial space. There is an equivalence of categories

$$
\text{Fun}(\Delta^{op}, sS/X) \overset{\text{Simp}}{\underset{\text{Func}}{\rightleftarrows}} ssS/X
$$

**Proof.** We define $\text{Simp}$ as follows. For a simplicial object $\alpha: \Delta^{op} \to sS/X$, we get a bisimplicial space defined as
$$\text{Simp}(\alpha)_k = \alpha(k)$$
where the simplicial maps follow from functoriality.

Conversely, for a bisimplicial object $Y \to X$ we define a functor as
$$\text{Func}(Y)(k) = Y_k$$
where functoriality follows from $Y$ being a bisimplicial space.

Notice that $\text{SimpFunc}(Y)_k = \text{Func}(Y)(k) = Y_k$ and $\text{FuncSimp}(\alpha)(k) = \text{Simp}(\alpha)_k = \alpha(k)$. Thus $\text{Simp}$ and $\text{Func}$ are inverses of each other and so we get an equivalence of categories. □

**Theorem 4.9.** Let $X$ be a simplicial space. Let $(\text{Fun}(\Delta^{op}, (sS/X)^{cov}))^{\text{Reedy}}$ be the category of simplicial objects in the covariant model structure over $X$, $(sS/X)^{cov}$, equipped with the Reedy model structure. Then the adjunction introduced in the lemma above,

$$(\text{Fun}(\Delta^{op}, (sS/X)^{cov}))^{\text{Reedy}} \xrightarrow{\text{Simp}} (sS/X)^{\text{ReeCov}}$$

is an isomorphism between the Reedy model structure on the covariant model structure and the Reedy covariant model structure.

**Proof.** We already know that it is an equivalence of categories. Thus it suffices to show that both sides have the same cofibrations and the same fibrant objects. Clearly on both sides cofibrations are just level-wise inclusions. So, we will show that the fibrant objects on the left hand side are the same as Reedy left fibrations.

First, recall from the Reedy model structure that a Reedy fibrant object is always in particular level-wise fibrant. But being level-wise fibrant here just means being a level-wise left fibration, which is one way to define Reedy left fibrations (see Remark 4.3). On the other hand a fibrant object on the left hand side is a simplicial object $\alpha : \Delta^{op} \to sS/X$ such that for each $k$ the restriction map

$$\alpha(k) \xrightarrow{\partial \alpha(k)} X$$

is a covariant fibration over $X$. We already know that this map is a Reedy fibration. If we proved that the two maps $\alpha(k) \to X$ and $\partial \alpha(k) \to X$ are left fibrations then we are done as covariant fibrations between left fibrations are just Reedy fibrations. So it suffices to show that both sides are left fibrations over $X$. However, we already know that for the left hand side. For the right hand side we notice that

$$\partial \alpha(k) = \lim_{\alpha(k-1)} \alpha(k-1)$$

which is a limit diagram of left fibrations in $sS/X$. But left fibrations are closed under limits and so $\partial \alpha(k) \to X$ is also a left fibration. □

**Remark 4.10.** Understanding weak equivalences in localization model structures can be very difficult. However, the theorem shows that the Reedy covariant weak equivalences are just level-wise covariant equivalences.
Remark 4.11. Intuitively, we can think of a Reedy left fibration as a “map into simplicial spaces”. In other words, if the base $X$ is a CSS then a Reedy left fibration is a model of a functor from the base $X$ into $sS$ the higher category of simplicial spaces (note this is not a definition and just a intuition). In particular, for every object $x \in X$, we think of the fiber over $x$ as the “value” of the map at the point $x$. Indeed, in Example 4.24 we will show that a Reedy left fibration over the point is just the data of a Reedy fibrant simplicial space.

Before we move on let us compare the Reedy covariant model structure to other important model structures. Clearly it is a localization of the biReedy model structure, but we also have following result.

**Theorem 4.12.** The following is a Quillen adjunction.

$$
\left( \text{ssS}/X \right)^{\text{RecCov}} \xrightarrow{id} \left( \text{ssS}/X \right)^{\text{DiagRec}}
$$

where the left hand side has the Reedy covariant model structure and the right hand side has the induced diagonal Reedy model structure. In particular, the Reedy diagonal model structure is a localization of the Reedy covariant model structure.

**Proof.** We will use A.4. Clearly it takes inclusions to inclusions. It suffices to show that if $Y \to X$ is a Reedy fibration then it is a Reedy left fibration. For that it suffices to show that the map $\varphi_k \times F(0) \to \varphi_k \times F(n)$ is a Reedy equivalence (Theorem 3.15). However, this is trivial as $\varphi\text{Diag}(\varphi_k) = F(k)$ and $\varphi\text{Diag}(F(n)) = \Delta[n]$ and the map $F(k) \times \Delta[0] \to F(k) \times \Delta[n]$ is a Reedy equivalence. □

We end this subsection with showing that similar to left fibrations, Reedy left fibrations are well behaved with respect to exponentiation.

**Lemma 4.13.** Let $L \to X$ be a Reedy left fibration. Then for any bisimplicial space $Y$, the map

$$L^Y \to X^Y$$

is a Reedy left fibration.

**Proof.** First we have to show that $X^Y$ is indeed a simplicial space, meaning that it is a homotopically constant bisimplicial space (Remark 3.7). Using adjunctions several times we get the chain of equivalences.

$$(X^Y)_k \cong \text{Map}(\varphi_k, X^Y) \cong \text{Map}(Y \times \varphi_k, X) \cong \text{Map}(Y, X^{\varphi_k}) \cong \text{Map}(\varphi_0, X^{\varphi_k}) \cong (X^Y)_0$$

Here we used the fact that $X_k = X$ as simplicial spaces and so $X^{\varphi_k}$ is equivalent to $X$.

Now we will prove the lemma. Let $L \to X$ be a Reedy left fibration. We know $L^Y \to X^Y$ is a biReedy fibration. In order to show it is a Reedy left fibration it suffices to show it is a level-wise left fibration, which means we have to show that the following is a homotopy pullback of Kan complexes
This is equivalent to the following map being a Kan equivalence

\[
\text{Map}(Y \times \varphi_k \times F(n), L) \xrightarrow{\tau} \text{Map}(Y \times \varphi_k, L)
\]

\[
\text{Map}(Y \times \varphi_k \times F(n), X) \xrightarrow{\tau} \text{Map}(Y \times \varphi_k, X)
\]

4.2 Reedy Right Fibrations. Until now we have completely focused on generalizing left fibrations to the bisimplicial setting. We can do the same thing with right fibrations (Definition 2.3). All the definitions given above will generalize in a similar fashion. We thus will just focus on several important results that come up later.

Definition 4.14. Let \( X \) be a simplicial space. We say a map of bisimplicial spaces \( p : Y \to X \) is a Reedy right fibration if it is a biReedy fibration and the following is a homotopy pullback square

\[
\begin{array}{c}
Y_{kn} \xrightarrow{n^*} Y_{k0} \\
\downarrow^{p_{kn}} \downarrow^{p_{k0}} \\
X_n \xrightarrow{n^*} X_0
\end{array}
\]

where \( n^* \) is the map induced by \( n : F(0) \to F(n) \) which takes the unique point to the final vertex.

Theorem 4.15. Let \( X \) be a simplicial space. There is a unique model structure on the category \( \text{ssS}_X \), called the Reedy contravariant model structure and denoted by \( (\text{ssS}_X)^{\text{ReeContra}} \), which satisfies the following conditions:

1. It is a simplicial model category.
2. The fibrant objects are the Reedy right fibrations over \( X \).
3. Cofibrations are monomorphisms.
4. A map \( f : A \to B \) over \( X \) is a weak equivalence if

\[
\text{Map}_{\text{ssS}_X}(B, W) \to \text{Map}_{\text{ssS}_X}(A, W)
\]

is an equivalence for every Reedy right fibration \( W \to X \).
(5) A weak equivalence (Reedy contravariant fibration) between fibrant objects is a level-wise equivalence (biReedy fibration).

**Theorem 4.16.** Let \( p : R \rightarrow X \) be a Reedy right fibration. The following is a Quillen adjunction:

\[
(\mathit{ssS}/X)^{\mathit{ReeCov}} \xrightarrow{p \cdot p^*} (\mathit{ssS}/X)^{\mathit{ReeCov}}
\]

Now that we have defined Reedy left and Reedy right fibrations we might wonder: How can we check whether a map is a Reedy right and Reedy left fibration at the same time? Unsurprisingly, we get the same result we got for right and left fibrations.

**Theorem 4.17.** A Reedy left fibration \( Y \rightarrow X \) is also a Reedy right fibration if and only if for every map \( F(1) \rightarrow X \) the induced map \( Y \times_X F(1) \rightarrow F(1) \) is a Reedy right fibration.

It suffices to proof this level-wise, which can be found in [Ra17, Theorem 5.27]

### 4.3 Recognition Principle for Reedy Covariant Equivalences.

Our goal is to find a "recognition principle" for Reedy covariant equivalences, generalizing the one for covariant equivalences (Proposition 2.5).

**Theorem 4.18.** Let \( X \) be a simplicial space (Remark 4.1). Then a map \( Y \rightarrow Z \) over \( X \) is a Reedy covariant equivalence if and only if for each map \( x : F(0) \rightarrow X \) the induced map

\[
Y \times_X X/x \rightarrow Z \times_X X/x
\]

is a diagonal Reedy equivalence. Here \( X/x \) is the contravariant fibrant replacement of \( x \) in \( \mathit{ssS}/X \) thought of as bisimplicial space (Notation 3.7).

**Proof.** Let \( Y \rightarrow Z \) be a map over \( X \). Then, based on Theorem 4.9, it is a Reedy covariant equivalence if and only if for every \( k \geq 0 \), \( Y_k \rightarrow Z_k \) is a covariant equivalence over \( X \). Based on Proposition 2.5 this is true if and only if

\[
Y_k \times_X X/x \rightarrow Z_k \times_X X/x
\]

is a diagonal equivalence for each \( k \geq 0 \). By definition of the Reedy diagonal model structure, Theorem 3.15, this is equivalent to

\[
Y \times_X X/x \rightarrow Z \times_X X/x
\]

being a diagonal Reedy equivalence. Hence we are done. \( \square \)

**Remark 4.19.** It is interesting to compare this result to the one for simplicial spaces (Proposition 2.5). In order to adjust things to the simplicial setting, we did change the equivalences we use from diagonal equivalences to diagonal Reedy equivalences. However, we still take a contravariant fibrant replacements in our pullbacks, the same as before.

The underlying reason is that for a map \( x : F(0) \rightarrow X \), contravariant fibrant replacements and Reedy contravariant fibrant replacements are the same. This follows from following chain of equivalences

\[
\text{Map}_{\mathit{ssS}/X}(F(0), R) \xrightarrow{\sim} \text{Map}_{\mathit{ssS}/X}(F(0), R_0) \xrightarrow{\sim} \text{Map}_{\mathit{ssS}/X}(X/x, R_0) \xrightarrow{\sim} \text{Map}_{\mathit{ssS}/X}(X/x, R)
\]
Here $R$ is a Reedy right fibration and so $R_0$ is a right fibration over $X$.

**Remark 4.20.** Similar to case of left fibrations it suffices to check the equivalences $Y \times_X X_{x} \to Z \times_X X_{x}$ for one point $x$ from each path component. That is because if two points $x_1$ and $x_2$ are in the same path component then the covariant fibrant replacements $X_{x_1}$ and $X_{x_2}$ are Reedy equivalent. For more details see [Ra17, Lemma 3.30].

It is helpful to state the special case of the recognition principle for the case of maps between Reedy left fibrations. The proof will follow from what we have stated above but can also easily be proven using the definition of equivalences between fibrant objects in a localization model structure.

**Theorem 4.21.** Let $L$ and $M$ be two Reedy left fibrations over $X$. Let $g : L \to M$ be a map over $X$. Then the following are equivalent.

1. $g : L \to M$ is a biReedy equivalence.
2. $(i_\varphi)_* (g) : (i_\varphi)_* (Y) \to (i_\varphi)_* (Z)$ is a Reedy equivalence.
3. For every $x : F(0) \to X$, $F(0) \times_X Y \to F(0) \times_X Z$ is a Reedy equivalence of bisimplicial spaces.
4. For every $x : F(0) \to X$, $F(0) \times_X Y \to F(0) \times_X Z$ is a diagonal Reedy equivalence of bisimplicial spaces.

### 4.4 Examples of Reedy Left Fibrations.

Before we move on it is helpful to have a set of examples to work with.

**Example 4.22.** Let $X = F(0)$. Then a Reedy left fibration $L \to F(0)$ is just a fibrant object in the diagonal Reedy model structure (Theorem 3.15). Indeed, we already know that a left fibration over the point is a homotopically constant simplicial space [Ra17, Example 3.20]. This implies that the map $Y_{k0} \to Y_{kn}$ is a Kan equivalence for every $k \geq 0$.

**Remark 4.23.** The above example in particular implies that the following adjunction, $\xymatrix{ (ssS)_{ReeCov} \ar[r]^{id} & (ssS)_{DiagRee} }$ where the left hand side has the Reedy covariant model structure and the left hand side has the diagonal Reedy model structure, is an isomorphism of model categories.

**Example 4.24.** Let us generalize this a little. Let $X = F(1)$. By [Ra17, Lemma 6.9] we realize that we can replace every left fibration $L$ with a Reedy equivalent left fibration $L^{st}$ such that it is completely determined by a map of spaces:

$\xymatrix{ L_{/01} \ar[r] & L_{/1} }$

where $L_{/01}$ is the fiber over the identity map in $F(1)_1$ and $L_{/1}$ is the fiber over the constant map in $F(1)_0$ that sends the point in $F(0)$ to 1.

But a Reedy left fibration over $L \to F(1)$ is just a level-wise left fibration $L_k \to F(1)$. Thus, which is level-wise equivalent to the data of a map of spaces

$\xymatrix{ L_{k|01} \ar[r] & L_{k|1} }$
Using the functoriality of our simplicial space, we get a map of Reedy fibrant simplicial spaces

\[ L_{\bullet|01} \to L_{\bullet|1} \]

So, the data of a Reedy left fibration over \( F(1) \) is that of a map of Reedy fibrant simplicial spaces. This is completely consistent with philosophy of Reedy left fibrations as we outlined in Remark 4.11.

**Example 4.25.** The previous example can easily be generalized to Reedy left fibrations over \( F(n) \). Again by [Ra17, Lemma 6.9], we know that a left fibration over \( F(n) \) is the data of a chain of spaces

\[ L_{0\ldots n} \to \ldots \to L_{n-1,n} \to L_n \]

and so a Reedy left fibration over \( F(n) \) is a chain of simplicial spaces

\[ L_{\bullet|0\ldots n} \to \ldots \to L_{\bullet|n-1,n} \to L_{\bullet|n} \]

Let us change the flavor of the examples a little.

**Example 4.26.** The same way that every space is a simplicial space, every left fibration \( L \to X \) can be thought of as a “constant Reedy left fibration”.

**Example 4.27.** One special instance of the previous example is that of a representable left fibration [Ra17, Subsection 5.2]. For every Segal space \( X \) and object \( x \) we can build the Segal space of object under \( x \), \( X_{x/} \), which is a Reedy left fibration, using the embedding above.

This last example might actually might make us wonder. We can build Reedy left fibrations using objects, by building representable left fibrations and embedding them into Reedy left fibrations. But that certainly does not give us very interesting Reedy left fibrations. This leads to following question: *How can we build more interesting Reedy left fibrations out of objects in our base and what kind of information do we need for that?* The next section will explore this question in further detail.

**Representable Reedy Left Fibrations**

The goal of this section is to show how we can use cosimplicial objects to construct Reedy left fibrations. We will then move on to show how it allows us to study cosimplicial objects, by proving *Yoneda Lemma for Reedy left fibrations.*

### 5.1 Cosimplicial Objects in simplicial Spaces

In this short subsection we discuss the basics of cosimplicial objects. Before we can do so, we need to clarify what we mean by the *simplex category* in the world of simplicial spaces. We have following result by Rezk.

**Theorem 5.1.** [Re01, Section 3.5, Proposition 6.1] *For every category \( C \), there is associated to it a complete Segal space \( N(C) \), called the classifying diagram of \( C \) and defined as

\[ N(C)_n = \text{nerve iso}(C^{[n]}) \]

For our purposes we will need the classifying diagram of \( \Delta \).*
Example 5.2. Using the result above we can build the classifying diagram for $\Delta$. Note that the category $\Delta$ has no non-trivial automorphisms, which implies that the simplicial set $\text{iso}(\Delta^{[n]})$ is just a set. But the nerve of a set is just the same set. Thus the definition of a the classifying diagram simplifies to the following.

$\mathcal{N}(\Delta)_n = \text{Fun}([n], \Delta)$

Notation 5.3. Henceforth we will denote $\mathcal{N}(\Delta)$ also as $\Delta$. No confusion shall arise from this as it is always clear whether we are working with categories or simplicial spaces.

Definition 5.4. Let $X$ be a simplicial space. A cosimplicial object $x^\bullet$ is a map of simplicial spaces $x^\bullet : \Delta \rightarrow X$.

Notation 5.5. We might drop the index and denote the cosimplicial object $x^\bullet$ as $x$.

Definition 5.6. Let $X$ be a simplicial space. We define the simplicial space of cosimplicial objects, $\cos X$ as $\cos X = X^\Delta$.

Remark 5.7. Note that if $X$ is a (complete) Segal space then $X^\Delta$ is also a (complete) Segal space.

We will now use this definition of cosimplicial objects in $X$ to construct Reedy left fibrations.

5.2 Defining Representable Reedy Left Fibrations. In this subsection we define representable Reedy left fibrations and study some of its properties.

Let $x^\bullet$ be a cosimplicial object. The goal is to build a Reedy left fibration which is level-wise representable, represented by the different levels of our cosimplicial object. Concretely at level $k$ the bisimplicial object should be Reedy equivalent to the representable left fibration $X_{x^k}$ for that reason we will denote the desired Reedy left fibration as $X_{x^\bullet}$ or, alternatively, as $X_{x^\bullet}/$.

Our first guess might be to define it level-wise at level $k$ as the fibrant replacement of the map $x_k : F(0) \rightarrow X$, However, all this would give us is a collection of simplicial spaces and no way to make those simplicial spaces into a bisimplicial space. We need to take a more global approach that considers objects and simplicial maps together. For that we need to find the correct analogue of a "point" in the simplicial setting, which keeps track of all the relevant simplicial data.

Construction 5.8. Let $\Delta$ be the complete Segal space of simplices. Recall that for every object $k$ we get an under-CSS $\Delta_{k^\circ}$. All these under-categories assemble into a bisimplicial space $\Delta_{\bullet^\circ}$ over $\Delta$. Indeed for every map of simplices $\delta : [m] \rightarrow [n]$ we get the obvious map $\delta^\circ : \Delta_{n^\circ} \rightarrow \Delta_{m^\circ}$ defined by pre-composition with $\delta : [m] \rightarrow [n]$. Associativity of composition implies that this construction is functorial. Indeed for two maps $\delta_1 : [m] \rightarrow [n]$ and $\delta_2 : [n] \rightarrow [k]$ we have $(\delta_1 \circ \delta_2)^\circ = (\delta_1)^\circ \circ (\delta_2)^\circ : \Delta_{k^\circ} \rightarrow \Delta_{m^\circ}$ by witnessing that for an object $f : [k] \rightarrow [k']$ in $\Delta_{k^\circ}$ we get $f \circ (\delta_2 \circ \delta_1) = (f \circ \delta_2) \circ \delta_1 : [m] \rightarrow [k']$ by associativity.
The construction above comes with a natural projection map
\[ \pi_i^\bullet : \Delta_{\bullet/} \to \Delta \]
which is a Reedy left fibration as it is a level-wise left fibration.

Because of its importance this particular Reedy left fibration deserves its own name.

**Definition 5.9.** We call the map \( \pi_i^\bullet : \Delta_{\bullet/} \to \Delta \) described above the **initial representable Reedy left fibration**.

**Remark 5.10.** The reasoning for the naming is described in Example 5.17.

**Remark 5.11.** Intuitively, \( \Delta_{\bullet/} \) is a “cosimplicial point”. A map \( \Delta_{\bullet/} \to X \) picks out all the relevant data of a cosimplicial object in a way that allows us to access every level individually.

Having our desired definition we can come back to our goal of defining a Reedy left fibration. Let \( x : \Delta \to X \) be a cosimplicial object in \( X \). We can precompose with \( \pi_i^\bullet \) to get a map of bisimplicial spaces
\[ \pi_i^\bullet \circ x^\bullet : \Delta_{\bullet/} \to X \]
The map \( \pi_i^\bullet \circ x \) is not necessarily a Reedy left fibration and so we can take a Reedy covariant fibrant replacement over \( X \).

\[ \Delta_{\bullet/} \quad \xrightarrow{\pi_i^\bullet \circ x^\bullet} \quad X_{\pi_i^\bullet \circ x^\bullet/} \]

\[ \xrightarrow{\pi} \]

**Remark 5.12.** The reason for using notation \( \pi^\ell x^\bullet \) will become clear in Definition 5.26 and Definition 5.27.

**Definition 5.13.** Let \( X \) be a simplicial space and \( x^\bullet : \Delta \to X \) be a cosimplicial object in \( X \). Then we call any Reedy covariant fibrant replacement of the map \( \pi_i^\bullet \circ x^\bullet \) a **representable Reedy left fibration represented by** \( x^\bullet \).

**Definition 5.14.** A bisimplicial space \( Y \) over \( X \) is **representable** if there exists a map
\[ i : \Delta_{\bullet/} \xrightarrow{\simeq} Y \]
over \( X \) which is an equivalence in the Reedy covariant model structure over \( X \).

**Remark 5.15.** Using the map \( \pi_i^\bullet \) we managed to build a simplicial space that at each level is still Reedy equivalent to a representable left fibration. Concretely, at level \( k \) it is Reedy equivalent to the left fibration \( X_{\pi_i^k/} \to X \). Indeed we have following diagram:
Remark 5.16. It is true that fibrant replacements can only be determined up to equivalence but the definition above is invariant under equivalences.

Let us now see some examples:

**Example 5.17.** The map \( \pi^i : \Delta_i/ \to \Delta \) is a representable Reedy left fibration represented by the cosimplicial object \( id : \Delta \to \Delta \). This also justifies the naming we have chosen in Definition 5.9.

**Example 5.18.** Let \( x : \Delta \to X \) be the constant map i.e. it maps everything to \( x \). Then the Reedy fibrant replacement is biReedy equivalent to the representable left fibration \( X_{x/} \to X \).

### 5.3 Representable Reedy Left Fibrations over Segal Spaces

When \( X \) is a Segal space then we can explicitly describe the left fibration replacement for a map \( x : F(0) \to X \) as the Segal space of objects under \( X \) [Ra17, Theorem 4.2]. In this subsection we will generalize this result to Reedy left fibrations, by giving description of representable Reedy fibrations over a Segal space.

**Remark 5.19.** For the rest of this subsection \( X \) is assumed to be a Segal space.

Let \( x^* : \Delta \to X \) be a cosimplicial object. We want to construct a bisimplicial space. Similar to the situation in the previous subsection we will run into problems with the functoriality. We could try to define \( X_{x/} = F(0) \times_X X^{F(1)} \) level-wise (for each \( k \)). Clearly, the map \( X_{x/} \to X \) is a left fibration and the map \( F(0) \to X_{x/} \) is a covariant equivalence.

However, as before this definition does not work! While it does give us everything we want levelwise, but it does not give us a bisimplicial space. For a given map \( \Delta_i/ \to \Delta_j/ \) we need to be able to define maps \( X_{x/} \to X_{x/} \) in a functorial way. The lifting conditions for fibrant objects will give us maps, but the functoriality does not follow. Therefore, some modifications are necessary. We have to enlarge the levelwise simplicial spaces \( F(0) \times_X X^{F(1)} \) such that we have clear functorial maps. In order to do that we have to consider Segal space under a diagram.

Here we rely on following definitions and results from [Ra17, Subsection 5.3].

**Theorem 5.20.** [Ra17, Definition 5.8 & Lemma 5.11] Let \( f : K \to X \) be a map of simplicial spaces. We define the Segal space of cocones under \( K \), denoted by \( X_{f/} \), as

\[
X_{f/} = \left( F(0) \times_X X^{F(1)} \right) \times_{X^K} X.
\]

This construction comes with a projection map

\[
\pi_1 : X_{f/} \to X
\]

which is a left fibration.
Notation 5.21. If the map is clear from the context we sometimes use $X_{K/}$ instead of $X_{f/}$.

We have following important lemma about cocones that allows us to enlarge representable left fibrations as much as we want

Definition 5.22. [Ra17, Definition 5.15] A map of simplicial spaces $A \to B$ is called cofinal if it is a contravariant equivalence over $B$. This is equivalent to saying it is a contravariant equivalence in any contravariant model structure.

Lemma 5.23. [Ra17, Lemma 5.20] Let $g : A \to B$ be a cofinal map. Then for any map $f : B \to X$ the induced map

$$X_{f/} \to X_{fg/}$$

is a Reedy equivalence.

Corollary 5.24. [Ra17, Corollary 5.21] Let $K$ be a simplicial space with a final object, meaning a map $v : F(0) \to K$ that is cofinal. Then by the result above, for every map $f : K \to X$ we get a Reedy equivalence

$$X_{f/} \to X_{f(v)/}$$

Remark 5.25. The upshot of this whole debate is that we can enlarge any representable left fibration as much as we want by a diagram, as long as the diagram has a final object that is mapped to our representing object.

Definition 5.26. For any $k$ we get a right fibration $\Delta_{/k} \to \Delta$. These all assemble into a cosimplicial simplicial space $\Delta_{/\cdot}$. We call the natural projection map from the cosimplicial simplicial space

$$\pi_{\cdot}^* : \Delta_{/\cdot} \to \Delta$$

the final left fibration.

Having set up all the machinery we can finally make following definition:

Definition 5.27. Let $x^* : \Delta \to X$ be a cosimplicial object. Using the previous definition we get a map of cosimplicial simplicial spaces $\pi_{\cdot}^* \circ x^* : \Delta_{/\cdot} \to X$ We define the Reedy left fibration over the Segal space $X$ represented by $x^*$ as the bisimplicial space that at level $k$ is defined as

$$X_{\pi_{\cdot}^* x^k/} = X_{\pi_{\cdot}^* x^k/} = F(0) \times^{X_{\Delta^1/k}} X_{F(1) \times \Delta^1/k} \times^{X_{\cdot}} X$$

where the map to $X$ is defined as projection on the first component.

Remark 5.28. This definition justifies the notation we introduced in 5.14.

Proposition 5.29. Let $x : \Delta \to X$ be a cosimplicial object. Then $X_{\pi_{\cdot} x^k/}$ is a left Reedy fibrant replacement of $\pi^* x^k : \Delta_{/\cdot} \to X$.

Proof. Based on Theorem 5.20, the map

$$X_{\pi_{\cdot}^* x^k/} \to X$$

is a left fibration for each $k$ and so the map $\pi_{\cdot}^* x^k$ is a Reedy left fibration. Moreover, for each $k$, we get following diagram.
The two maps out of $F(0)$ are covariant equivalences as the images are initial objects and the left horizontal map is a Reedy equivalence by Corollary 5.24. Thus the middle map is also a covariant equivalence.

Remark 5.30. Recall that in general a fibrant replacement is only defined up to equivalence. However, from here on, whenever the base is a Segal space, we will automatically assume that the chosen fibrant replacement is the specific object introduced above.

5.4 Yoneda Lemma for Reedy Left Fibrations. One of the big benefits of the representability conditions is that it helps us understand functors by studying representing objects. In particular we have following classical result with regard to representable left fibrations.

Theorem 5.31. [Ra17, Remark 4.3] Let $X$ be a Segal space and $x$ an object in $X$. Then for any left fibration $L$ over $X$ the induced map

$$\text{Map}_{/X}(X_x/ L) \to \text{Map}_{/X}(F(0), L)$$

is a Kan equivalence.

This in particular gives us the following more familiar corollary:

Corollary 5.32. Let $X$ be a Segal space and $x, y$ two objects in $X$. Then we have an equivalence

$$\text{Map}_{/X}(X_x/, X_y/) \overset{\simeq}{\to} \text{map}_X(y, x)$$

Remark 5.33. As in the last subsection $X$ is always a Segal space and we will always use our construction from the previous subsection when using representable Reedy left fibrations (Remark 5.30).

Any reasonable definition of a representable Reedy right fibration should satisfy a similar condition as the one stated above. Our goal here is to exactly prove the following analogous result:

Theorem 5.34. Let $x^\bullet$ and $y^\bullet$ be two cosimplicial objects, then we have an equivalence

$$\text{Map}_{/X}(X_{x^\bullet}, X_{y^\bullet}) \overset{\simeq}{\to} \text{map}_{cosX}(y^\bullet, x^\bullet)$$

Proof. We have following long chain of maps:
\[
\begin{align*}
\text{Map}_X(X_{\pi^*x^*/}, X_{\pi^*y^*/}) \\
\overset{(1)}{\Rightarrow} \\
\text{Map}_X(\Delta_{\pi^*/}, X_{\pi^*y^*/}) \\
\text{Map}(\Delta_{\pi^*/}, X_{\pi^*y^*/}) \times x^* \Delta[0] \\
\text{Map}(\Delta_{\pi^*/}, F(0) \times X_{\pi^*/} \times F(1)) \times X_{\pi^*/} \times \text{Map}(\Delta_{\pi^*/}, X) \times x^* \Delta[0] \\
\Delta[0] \times y^* \text{Map}(\Delta_{\pi^*/} \times \Delta_{\pi^*/} \times F(1), X) \times \text{Map}(\Delta_{\pi^*/}, X) \times x^* \Delta[0] \\
\Delta[0] \times M \text{Map}(\Delta_{\pi^*/} \times \Delta_{\pi^*/} \times F(1), X) \times \text{Map}(\Delta_{\pi^*/}, X) \times x^* \Delta[0] \\
\Delta[0] \times y^* \text{Map}(\Delta \times F(1), X) \times x^* \Delta[0] \\
\Delta[0] \times y^* \text{Map}(\Delta \times F(1), X) \times x^* [0] \\
\text{map}_{\cos X}(y^*, x^*)
\end{align*}
\]
Most maps in this chain are clearly isomorphisms of spaces (by using adjunctions or simplifying pullback diagrams). The only ones that need proof have been labeled with numbers and will be discussed here:

(1) This follows from the fact that $X_{\pi_1}^* /$ is a Reedy left fibration and our model structure is simplicial.

(2) This is true because level-wise we are looking at the mapping space of the constant objects $x^*$ and $y^*$ inside the Segal space of cosimplicial objects $cos(X_{\Delta^k}^*)$ for each $k$. As they are constant objects their mapping space is determined at the zero level.

(3) The last isomorphism follows from the fact that the following is a pushout square of bisimplicial spaces

\[
\begin{array}{ccc}
\Delta_{\pi_1}^* \times \partial F(1) & \to & \Delta_{\pi_1}^* \times F(1) \\
\downarrow & & \downarrow \quad \gamma \\
\Delta \times \partial F(1) & \to & \Delta \times F(1)
\end{array}
\]

and so the following is a pullback square of spaces

\[
\begin{array}{ccc}
Map(\Delta_{\pi_1}^* \times F(1), X) & \to & Map(\Delta_{\pi_1}^* \times \partial F(1), X) \\
\downarrow \quad \gamma & & \downarrow \\
Map(\Delta \times F(1), X) & \to & Map(\Delta \times \partial F(1), X)
\end{array}
\]

Pulling it back along the commutative square we get following diagram
In this diagram our map (3) is the pullback of the identity map and thus has to be an isomorphism of spaces.

Remark 5.35. Note that all maps in the long diagram are isomorphisms (and not merely equivalences) of spaces, except for the first map. Thus we have an actual map that is an equivalence rather than just a zig zag of equivalences.

5.5 Representable Reedy Right Fibrations. Until now we have described how we can use cosimplicial objects to build Reedy left fibration, which we call representable Reedy left fibrations. We can take a similar approach for Reedy right fibrations. As the proofs are all analogous we will just focus on the main results.

Definition 5.36. Let $X$ be a simplicial space. A simplicial object $x• : ∆^{op} → X$

(Notation 5.3). The simplicial space of simplicial objects, $sX$ is defined as

$sX = X^∆^{op}$

Recall that if $X$ is a (complete) Segal space then $sX$ is a (complete) Segal space.

Definition 5.37. We define the initial representable Reedy right fibration as the map of bisimplicial spaces

$(π^i• : (Δ^{op})^{op}/•) → Δ^{op}$

Definition 5.38. Let $X$ be a simplicial space and $x• : Δ^{op} → X$ be a simplicial object in $X$. Then we call any Reedy right fibrant replacement of the map $π^i• ∘ x•$ a representable Reedy right fibration represented by $x•$. Here

$π^i• : (Δ^{op})^{op}/• → Δ^{op}$
is the levelwise projection map of overcategories and called the initial right fibration (analogous to Definition 5.9).

Moreover, any bisimplicial space $Y$ over $X$ is representable if there exists a map

$$i : (\Delta^{op})_\bullet \to Y$$

over $X$ which is an equivalence in the Reedy contravariant model structure over $X$.

We can also give concrete description of representable Reedy right fibrations in case the base is a Segal space.

**Definition 5.39.** Let $x_\bullet : \Delta^{op} \to X$ be a simplicial object. We define the Reedy right fibration over the Segal space $X$ represented by $x_\bullet$ as the bisimplicial space that at level $k$ is defined as

$$X_{/\pi_k x_k} = X_{/\pi_k x_k} = X \times_{X^{(\Delta^{op})k/}} X^{F(1) \times (\Delta^{op})k/} \times_{X^{(\Delta^{op})k/}} F(0)$$

where the map to $X$ is defined as projection on the first component. Here

$$\pi^f_\bullet : (\Delta^{op})_k/ \to \Delta^{op}$$

is the natural projection map and called the final left fibration.

We also have the analogue of the Yoneda lemma

**Theorem 5.40.** Let $x_\bullet$ and $y_\bullet$ be two simplicial objects, then we have an equivalence

$$\text{Map}_{/X}(X_{/\pi_k x_k}, X_{/\pi_k y_k}) \simeq \text{map}_{X}(x_\bullet, y_\bullet)$$

**Remark 5.41.** Working with representable Reedy right fibrations can be at times very confusing as we are dealing with $\Delta^{op}$, the opposite simplex category. That is why in this section we have chosen to work with Reedy left fibrations instead. In the next section we follow the historical trend and mostly focus on Reedy right fibrations instead.

**LOCALIZATIONS OF REEDY RIGHT FIBRATIONS**

In Section 4 we defined fibrations which model presheaves valued in simplicial spaces. In this section we want to study presheaves valued in localizations of simplicial spaces. For example, presheaves valued in Segal spaces or complete Segal spaces. In order to achieve that we need to localize the Reedy right model structure.

Concretely, let $f : A \to B$ be an inclusion of simplicial spaces. Our goal is it to study the localization of bisimplicial spaces with respect to the image, $\Delta \text{Diag}_\#(f) : \Delta \text{Diag}_\#(A) \to \Delta \text{Diag}_\#(B)$.

**Remark 6.1.** From here on we primarily use Reedy right fibrations and the Reedy contravariant model structure.

**Remark 6.2.** As in the previous sections $X$ is a fixed simplicial space embedded in $ssS$ (Remark 4.1)

**Notation 6.3.** In most cases we will denote $\Delta \text{Diag}_\#(f)$ also as $f$ to simplify notations.

The map $f$ gives us three localization model structures, which we will need in this section.
Theorem 6.4. There is a model structure on $sS$, denoted by $sS_{Ree}^f$ and called the $f$-localized Reedy model structure, defined as follows.

- A map $Y \to Z$ is a cofibration if it is an inclusion.
- An object $Y$ is fibrant if it is Reedy fibrant and the map $$\text{Map}(B, Y) \to \text{Map}(A, Y)$$
  is a trivial Kan fibration.
- A map $Y \to Z$ is a weak equivalence if for every fibrant object $W$ the map $$\text{Map}(Z, W) \to \text{Map}(Y, W)$$
  is a Kan equivalence.

Proof. This is a special case of Theorem A.3 for the case where $\mathcal{L} = \{f\}$. $\square$

Theorem 6.5. There is a model structure on $ssS$, denoted by $ssS_{DiagRee}^f$ and called the diagonal $f$-localized Reedy model structure, defined as follows.

- A map $Y \to Z$ is a cofibration if it is an inclusion.
- A map $g : Y \to Z$ is a weak equivalence if the diagonal map $$\phi_{Diag}^*(g) : \phi_{Diag}(Y) \to \phi_{Diag}(Z)$$
  is an $f$-localized Reedy equivalence.
- A map $g : Y \to Z$ is a fibration if it satisfies the right lifting property with respect to trivial cofibrations.

In particular an object is fibrant if and only if it is biReedy fibrant, $W_{kn} \to W_{k_0}$ is a Kan equivalence and $(i\phi)_*(W)$ is fibrant in the $f$-localized Reedy model structure.

Proof. Here we have to use Theorem 3.13. The model structure is the localization of the biReedy model structure with respect to the maps $$\mathcal{L} = \{\Delta_{Diag}(f) : \Delta_{Diag}(A) \to \Delta_{Diag}(B)\} \cup \{F(0) \to F(n) : n \geq 0\}$$
This determines the cofibrations and fibrant objects. Notice that if $W$ is fibrant then $\Delta_{Diag}(W)$ is fibrant in the localized Reedy model structure. Indeed in the commutative square

\[
\begin{array}{ccc}
\text{Map}_{ssS}(\Delta_{Diag}(B), W) & \to & \text{Map}_{ssS}(\Delta_{Diag}(A), W) \\
\simeq & & \simeq \\
\text{Map}_{ssS}(B, \Delta_{Diag}(W)) & \to & \text{Map}_{ssS}(A, \Delta_{Diag}(W))
\end{array}
\]

the adjunction implies that the vertical maps are Kan equivalences and so the top map is an equivalence if and only if the bottom map is one. But we know that $\Delta_{#}(f) = (i\phi)^*(f) \times (iF)^*(f)$. But by assumption

$$i\phi(f) \to (i\phi)^*(f) \times (iF)^*(f)$$
is an equivalence. Thus $W$ is fibrant if and only if the map
\[
\text{Map}_{sS}(\phi^*(B), W) \to \text{Map}_{sS}(\phi^*(A), W)
\]
is a Kan equivalence, which by adjunction is the same as
\[
\text{Map}_{sS}(B, \phi_*(W)) \to \text{Map}_{sS}(A, \phi_*(W))
\]
being a Kan equivalence. This is equivalent to $\phi_*(W)$ being fibrant in the localized Reedy model structure. Combining this with Theorem 3.15, we see that $g$ is an equivalence if and only if $\phi^i\phi^*(g)$ is an $f$-localized Reedy equivalence.

\begin{flushright}
$\square$
\end{flushright}

**Theorem 6.6.** There is a model structure on $sS_{/X}$, denoted by $(sS_{/X})^{\text{ReeContra}_f}$ and called the $f$-localized Reedy contravariant model structure, defined as follows.

- **C** A map $Y \to Z$ over $X$ is a cofibration if it is an inclusion.
- **F** An object $Y \to X$ is fibrant if it is a Reedy right fibration and for every map $\Delta\phi^*(B) \to X$ the map
  \[
  \text{Map}_{/X}(B, Y) \to \text{Map}_{/X}(A, Y)
  \]
is a trivial Kan fibration.
- **W** A map $Y \to Z$ over $X$ is a weak equivalence if for every fibrant object $W \to X$ the map
  \[
  \text{Map}_{/X}(Z, W) \to \text{Map}_{/X}(Y, W)
  \]
is a Kan equivalence.

**Proof.** This model structure is formed by localizing the Reedy right model structure on $sS_{/X}$ with respect to the collection of maps.

\[
\{\Delta\phi^#(A) \to \Delta\phi^#(B) \to \Delta\phi^#(X) \to X\} \cup \{\phi_k \times F(0) \to \phi_k \times F(n) \to X\}
\]
where the map $\Delta\phi^#(X) \to X$ is the counit map, using the fact that $\Delta\phi^#(X) = X$. The result then follows directly from Theorem 3.13.

**Remark 6.7.** In all three cases, if the map $f$ is clear from the context we will omit it and just call the model structures localized Reedy model structure, localized diagonal Reedy model structure and localized Reedy contravariant model structure.

The first two localizations do not depend on any base but the third very much does. Therefore we have to be careful when trying to compare them to each other. In general following results hold.

**Proposition 6.8.** The following adjunction
\[
(sS)^{\phi^#\text{Ree}_f} \xrightarrow{\phi^#\phi^*} sS^{\text{Ree}_f}
\]
is a Quillen equivalence. Here the left hand side has the diagonal localized Reedy model structure and the right hand side has the localized Reedy model structure.

The proof is analogous to the argument in Theorem 3.18.
Proposition 6.9. The following adjunction

\[
\left( \text{ReeContra}_{\text{ss}S/X} \right) \cong \text{DiagRee}_{\text{ss}S/X} \\
\text{id} \quad \text{id}
\]

is a Quillen adjunction. Here the left hand side has the localized Reedy contravariant model structure and the left hand side has the induced diagonal localized Reedy model structure over the base \( X \).

The proof of this proposition is analogous to the proof of Theorem 4.12. Let us see one important example.

Example 6.10. Let \( X = F(0) \). In Example 4.22 we already showed that the Reedy contravariant model structure over the point is isomorphic to the Reedy model structure. Localizing this model structure with respect \( f \) just gives us the \( f \)-localized Reedy model structure. Thus, when the base is just a point, the adjunction

\[
\left( \text{ss}S \right)_{\text{ReeContra}} \cong \text{DiagRee}_{\text{ss}S/X} \\
\text{id} \quad \text{id}
\]

induces a Quillen equivalence between the localized Reedy model structure and the localized Reedy contravariant model structure. In fact the model structures are not just equivalent, but actually isomorphic.

We want to do a careful study of the localized Reedy contravariant model structure. In order to be able to do that we have to impose some conditions on the localizing map \( f \).

Definition 6.11. Let \( f : A \rightarrow B \) be a map of simplicial spaces. We say \( f \) is a acceptable if it satisfies the following conditions:

1. \( f \) is an inclusion.
2. \( A \) has a distinguished final vertex \( v \) such that \( f(v) \) is also a final vertex in \( B \).
3. \( A \) and \( B \) are discrete simplicial spaces (each level is just a set).
4. An object is local with respect to \( f \) if and only if it is local with respect to \( f^{op} \).
5. For each simplicial space \( Y \) the map \( A \times Y \rightarrow B \times Y \) is also a localized Reedy equivalence.

Remark 6.12. The conditions on the map \( f \) stated here are not absolute and can possibly be relaxed. However, they do apply to all cases we were interested in and so are suitable to our needs.

Remark 6.13. The last condition on an acceptable map \( f \) implies that if \( W \) is fibrant in the localized Reedy model structure then for every simplicial space \( K \), \( W^K \) is also fibrant in the localized Reedy model structure. For a proof see [Re01, Proposition 9.2].

Remark 6.14. For the rest of this section we will always assume that \( f \) is an acceptable map of simplicial spaces.

The following lemma is crucial for all other results in this section.

Lemma 6.15. Let \( p : i^+_{\varphi}(B) \rightarrow X \) be a map of bisimplicial spaces. Then there exists a map \( x : F(0) \rightarrow X \) such that \( p = qx \). Here \( q : \varphi(B) \rightarrow F(0) \) is the unique final map.
Proof. By adjunction we have an equivalence of spaces
\[ \text{Map}_{sSS}(i^*_A(B), X) \simeq \text{Map}_{sSS}(B, (i_\varphi)_*X) = \text{Map}_{sSS}(B, X_0) = \text{Map}_{SS}(B_{\bullet 0}, X_{00}) \]
In order to show that \( f \) factors through a constant map it suffices to show that the corresponding map of spaces \( \tilde{f} : B_{\bullet 0} \to X_{00} \) factors through a constant map. However \( B_{\bullet 0} \) is connected (it has a final vertex \( v \)) and \( X_{00} \) is just a set, so \( \tilde{f} \) has to be constant. \( \square \)

The Lemma has two corollaries, which we need in the next theorem.

**Corollary 6.16.** The map
\[ \text{Map}(i^*_A(B) \times i^*_F(B), X) \to \text{Map}(i^*_F(B), X) \]
is an equivalence of Kan complexes.

**Corollary 6.17.** The map
\[ \text{Map}(i^*_A(B), X) \to \text{Map}(i^*_F(A), X) \]
is an equivalence of Kan complexes.

We can now characterize the fibrant objects in the localized Reedy contravariant model structure.

**Theorem 6.18.** The following are equivalent.

1. The map \( R \to X \) of bisimplicial spaces is fibrant in the localized Reedy contravariant model structure over \( X \).
2. \( R \to X \) is a Reedy right fibration and is local with respect to the class of maps:
\[ \mathcal{L}' = \{ i^*_A(A) \xrightarrow{i^*_A(f)} i^*_F(B) \xrightarrow{p} X \} \]
3. \( R \to X \) is a Reedy right fibration and the map \( R^i_A(B) \to R^i_F(A) \) is a biReedy equivalence.
4. \( R \to X \) is a Reedy right fibration and the simplicial space \( (i_\varphi)_*R \) is fibrant in the localized Reedy model structure.
5. \( R \to X \) is a Reedy right fibration and the simplicial space \( (i_\varphi)_*(R) = R_{0\bullet} \) is fibrant in the localized Reedy model structure.
6. \( R \to X \) is a Reedy right fibration and the map \( (i_\varphi)_*(R) \to (i_\varphi)_*(X) = X_0 \) is a fibration in the localized Reedy model structure.
7. \( R \to X \) is a Reedy right fibration and for each vertex \( x : F(0) \to X \) the fiber \( (i_\varphi)_*(F(0) \times_X R) \) is fibrant in the localized Reedy model structure.
8. \( R \to X \) is Reedy right fibration and for each point \( x : F(0) \to X \) the fiber \( F(0) \times_X R \) is fibrant in the diagonal localized Reedy model structure.

**Proof.** Recall that \( f : A \to B \) is a map of simplicial spaces and we are localizing bisimplicial spaces with respect to its image \( \Delta \text{Diag}^*(f) \), which up until now we also denoted by \( f \) to simplify notation (Notation 6.3). In this proof, however, we need to be more careful about our notation which means we will distinguish between the map \( f \) and its image \( \Delta \text{Diag}^*(f) \).

1) \( \iff \) (2) As part of our proof we first have to understand the map \( f \) and its image \( \Delta \text{Diag}^*(f) \). By definition \( \Delta \text{Diag}^*(A) = i^*_A(A) \times i^*_F(A) \) (this is a consequence of how \( \Delta \text{Diag}^* \) acts on generators.
by Remark 3.20) So
\[ \Delta \text{Diag}^*(A) = i^*_\varphi(f) \times i^*_\varphi(f) : i^*_\varphi(A) \times i^*_\varphi(A) \to i^*_\varphi(B) \times i^*_\varphi(B). \]

For any map \( i^*_\varphi(B) \times i^*_\varphi(B) \to X \), we get a square of Kan complexes

\[
\begin{array}{ccc}
\text{Map}_X(i^*_\varphi(B) \times i^*_\varphi(B), R) & \xrightarrow{\simeq} & \text{Map}_X(i^*_\varphi(A) \times i^*_\varphi(A), R) \\
\downarrow & & \downarrow \\
\text{Map}_X(i^*_\varphi(B), R) & \xrightarrow{\simeq} & \text{Map}_X(i^*_\varphi(A), R)
\end{array}
\]

where the two vertical maps are equivalences by Corollary 6.16. This means the top map is an equivalence if and only if the bottom map is. This proves that these classes of maps give us the same localizations.

\((2) \Leftrightarrow (5)\) We have following diagram of Kan complexes

\[
\begin{array}{ccc}
\text{Map}(B, (i_\varphi)_*(R)) & \xrightarrow{f^*} & \text{Map}(A, (i_\varphi)_*(R)) \\
\downarrow & & \downarrow \\
\text{Map}(i^*_\varphi(B), R) & \xrightarrow{(i^*_\varphi(f))^*} & \text{Map}(i^*_\varphi(A), R) \\
\downarrow & & \downarrow \\
\text{Map}(i^*_\varphi(B), X) & \xrightarrow{(i^*_\varphi(f))^*} & \text{Map}(i^*_\varphi(A), X)
\end{array}
\]

The top vertical maps are equivalences because of the adjunction. The bottom map, \((i^*_\varphi(f))^*\), is an equivalence by Corollary 6.17. This implies that the top map, \(f^*\) is a Kan equivalence if and only if the middle map \(i^*_\varphi(f)^*\) is a Kan equivalence. But this is true if and only it is a fiberwise equivalence over the bottom map \(i^*_\varphi(f)^*\) i.e. the map

\[ \text{Map}_X(i^*_\varphi(B), R) = \text{Map}(i^*_\varphi(B), R) \times_{\text{Map}(i^*_\varphi(A), X)} \xrightarrow{\simeq} \text{Map}(i^*_\varphi(A), R) \times_{\text{Map}(i^*_\varphi(A), X)} \text{Map}_X(i^*_\varphi(A), R) \]

is an equivalence. We just proved that \((i_\varphi)_*(R)\) is fibrant in the localized Reedy model structure (which is equivalent to the top map being an equivalence) if and only if \(R\) is fibrant in the localized Reedy contravariant model structure.

\((4) \Leftrightarrow (5)\) One side is just a special case. For the other side note that we have Kan equivalences

\[ R_{kn} \to R_{k0} \times X_n \]
and so if $R \cdot 0$ is local with respect to $f$ then so is $R \cdot n$.

(3) $\iff$ (4) The map $Y \phi^*_B \to Y \phi^*_A$ is a biReedy equivalence if and only if for each $k$ and $n$ the map of Kan complexes

$$Map(i_\varphi(B) \times \varphi_k \times F(n), R) \xrightarrow{\simeq} Map(i_\varphi(A) \times \varphi_k \times F(n), R)$$

is a Kan equivalence. If we fix an $n_0$, then this is just equivalent to

$$Map(i_\varphi(B) \times \varphi_k \times F(n_0), R) \xrightarrow{\simeq} Map(i_\varphi(A) \times \varphi_k \times F(n_0), R)$$

being a Kan equivalence for every $k$, which is just saying that the row $R \cdot n_0$ is fibrant in the localized Reedy model structure. Hence, $Y_i \phi^*_B \to Y_i \phi^*_A$ is a biReedy equivalence if and only if each row is fibrant in the localized Reedy model structure.

(5) $\iff$ (6) For this part we note that the space $X_0$, thought of as a constant simplicial space, is fibrant in the localized Reedy model structure (Corollary 6.17). Thus the Reedy fibration $(i_\varphi)_*(R) \to X_0$ is a fibration in the localized Reedy model structure if and only if $(i_\varphi)_*(R)$ is fibrant in the localized Reedy model structure.

(6) $\iff$ (7) For this part we fix a map $i_\varphi(B) \to X$. As we discussed in Lemma 6.15 the map will factor $i_\varphi(B) \to F(0) \xrightarrow{\varphi} X$. Thus we get following commutative diagram

$$\begin{align*}
\xymatrix{ Map_{/X_0}(B, (i_\varphi)_*(R)) \ar[r] \ar[d]_{\simeq} & Map_{/X_0}(B, (i_\varphi)_*(R)) \ar[d]_{\simeq} \\
Map_{/X_0}(B, (i_\varphi)_*(R) \times F(0)) \ar[r] & Map_{/X_0}(A, (i_\varphi)_*(R) \times F(0))}
\end{align*}$$

The vertical maps are equivalences because of the factorization above. This implies that the top map is a Kan equivalence if and only if the bottom map is a Kan equivalence. Thus $(i_\varphi)_*(R) \to X_0$ is a fibration if and only if each fiber $(i_\varphi)_*(R \times_X F(0))$ is fibrant.

(7) $\iff$ (8) By assumption $R$ is a Reedy right fibration. Thus by Example 4.22 we know that for each map $x : F(0) \to X$, the fiber $F(0) \times_X R$ is a row-wise homotopically constant (in the sense that $i_\varphi^*(i_\varphi)_*(F(0) \times_X R) \to F(0) \times_X R$ is a biReedy equivalence) Thus the diagonal of the fiber, $\varphi \text{Diag}^n(F(0) \times_X R)$, is Reedy equivalent to the 0-row, $(i_\varphi)_*(F(0) \times_X R)$, which directly implies that the fiber is fibrant in the diagonal localized Reedy model structure if and only if $i_\varphi^*(F(0) \times_X R)$ is fibrant in the localized Reedy model structure.

Remark 6.19. This result is intuitively very reasonable. In Remark 4.11 we discussed how we can think of Reedy right fibrations as presheaves valued in simplicial spaces. The result above is basically saying that a map is fibrant in the localized Reedy contravariant model structure if the value at each point is fibrant, which means it models functors valued in fibrant simplicial spaces.

Let us see how we can use this to show exponentiability of localized Reedy right fibrations.

**Theorem 6.20.** Let $R \to X$ be a localized Reedy right fibration. Then for any bisimplicial space $Y$, $R^Y \to X^Y$ is also a localized Reedy right fibration.
We already showed this map is Reedy right fibration (Lemma 4.13). So we only have to show that $R^Y \to X^Y$ is also local. For that let $i^*_Y(B) \to X^Y$ be a map. We have to show that

$$\text{Map}_{/X^Y}(i^*_Y(B), R^Y) \to \text{Map}_{/X^Y}(i^*_Y(A), R^Y)$$

is a Kan equivalence. By definition this is equal to

$$\text{Map}(i^*_Y(B), R^Y) \times_{\text{Map}(i^*_Y(B), X^Y)} \Delta[0] \to \text{Map}(i^*_Y(A), R^Y) \times_{\text{Map}(i^*_Y(A), X^Y)} \Delta[0].$$

Using adjunctions this is map is equivalent to

$$\text{Map}(i^*_Y(B) \times Y, R) \times_{\text{Map}(i^*_Y(B) \times Y, X)} \Delta[0] \to \text{Map}(i^*_Y(A) \times Y, R) \times_{\text{Map}(i^*_Y(A) \times Y, X)} \Delta[0].$$

This map is by definition equal to the map

$$\text{Map}_{/X}(i^*_Y(B) \times Y, R) \to \text{Map}_{/X}(i^*_Y(A) \times Y, R)$$

We know that $R$ is a localized Reedy right fibration, so it suffices to show that

$$i^*_Y(A) \times Y \xrightarrow{i^*_Y(f) \times id_Y} i^*_Y(B) \times Y$$

is a localized Reedy contravariant equivalence over $X$. This follows from condition 5 of acceptable maps, which states that the product of equivalences is an equivalence (Definition 6.11).

Our definition of localized Reedy right fibration is external in the sense that we start with simplicial spaces and then extend them to bisimplicial spaces in order to be able to define localized Reedy right fibrations. However, in some situations it is helpful to have an internal definition in order to be able to compare it to fibrations in the localized Reedy model structure.

**Definition 6.21.** We say a map of simplicial spaces $S \to X$ is a localized Reedy right fibration if there exists a localized Reedy right fibration $R \to X$ such that $S = \Delta \text{Diag}^*(R)$ over $X$. Notice if such an $R$ exists then it will be unique up to biReedy equivalence, as localized Reedy right fibrations are completely determined by their diagonals.

The internal definition also has some interesting properties.

**Lemma 6.22.** Let the map of simplicial spaces $p : S \to X$ be a localized Reedy right fibration and $g : Y \to X$ be any map. Then the pullback of $g^* p : g^* S \to Y$ is a localized Reedy right fibration over $Y$. Indeed, if $S = \Delta \text{Diag}^*(R)$ then $g^* S = \Delta \text{Diag}^*(g^* R)$

**Lemma 6.23.** Let the map of simplicial spaces $p : S \to X$ be a localized Reedy right fibration and $K$ any simplicial space. Then $p^K : S^K \to X^K$ is also a localized Reedy right fibration. Indeed, if $S = \Delta \text{Diag}^*(R)$, then $S^K = \Delta \text{Diag}^*(R^K)$

**Proof.** The proof follows by straightforward application of adjunctions.

$$(S^K)_n \cong \text{Map}(F(n), S^K) \cong \text{Map}(F(n) \times K, S) = \text{Map}(F(n) \times K, \Delta \text{Diag}^*(R)) \cong \text{Map}(\Delta \text{Diag}_n(F(n) \times K), R) \cong \text{Map}(\Delta \text{Diag}_n(F(n)), \Delta \text{Diag}^*(R^K)) \cong \Delta \text{Diag}^*(R^K)$$

Here we used the fact that the $\Delta \text{Diag}_n$ also commutes with finite products.

□
Remark 6.24. Although we have given maps of simplicial spaces that we call localized Reedy right fibrations, the collection of such maps does not give us a model structure on simplicial spaces. This is one of the main motivations why we expanded simplicial spaces to bisimplicial spaces.

Being able to understand fibrant objects as locally fibrant objects also allows us to adjust the recognition principle. First of all we have following theorem for equivalences between fibrant objects.

Theorem 6.25. Let \( R \) and \( S \) be two localized Reedy right fibrations over \( X \). Let \( g : R \to S \) be a map over \( X \). Then the following are equivalent

1. \( g : R \to S \) is a biReedy equivalence
2. \((i \varphi)_*(g) : (i \varphi)_*(R) \to (i \varphi)_*(S) \) is a Reedy equivalence
3. For every \( x : F(0) \to X, F(0) \times X_0 (i \varphi)_*(R) \to F(0) \times X_0 (i \varphi)_*(S) \) is a Reedy equivalence of bisimplicial spaces.
4. For every \( x : F(0) \to X, F(0) \times X_0 (i \varphi)_*(R) \to F(0) \times X_0 (i \varphi)_*(S) \) is a diagonal Reedy equivalence of bisimplicial spaces.

The proof is analogous to Theorem 4.21.

In order to be able to now state a recognition principle for equivalences between arbitrary objects we need following technical lemma first.

Lemma 6.26. \( p : R \to X \) be a Reedy right fibration. Then there exists a fibrant replacement \( \hat{p} : \hat{R} \to X \) in the localized Reedy contravariant model structure such that the map \( R_n \to \hat{R}_n \) is a localized Reedy equivalence.

Proof. For this proof we think of the bisimplicial space \( R \) as functor \( R : \Delta^{op} \to s\mathcal{S} \), where \( R(n) = R_n \). Similarly, \( X \) is a functor \( \Delta^{op} \to s\mathcal{S} \), where \( X(n) = X_n \). Thus the map \( R \to X \) is just a natural transformation between two functors. In general, in the localized Reedy model structure we can factor each map \( R_n \to X_n \) into a trivial cofibration followed by a localized Reedy fibration. Using the fact that we can do so functorially implies that we can factor our natural transformation of functors \( R \to X \) into a natural transformations \( R \to \hat{R} \to X \) such that it satisfies following condition. \( R \to \hat{R} \) is a level-wise localized Reedy equivalence, meaning that \( R_n \to \hat{R}_n \) is a localized Reedy equivalence. \( \hat{R} \to X \) is a level-wise localized Reedy fibration, meaning that \( \hat{R}_n \to X_n \) is a localized Reedy fibration. We will show that \( \hat{R} \) is a fibrant replacement.

First it is a biReedy fibration and the zero level \( i_\varphi^*(\hat{R}) \) is fibrant in the localized Reedy model structure. Thus it suffices to show that it is a Reedy right fibration, by Theorem 6.18. For that we have following commutative square

\[
\begin{array}{ccc}
R_n & \xrightarrow{\cong} & \hat{R}_n \\
\downarrow \cong & & \downarrow \\
R_0 \times X_0 & \xrightarrow{\cong} & \hat{R}_0 \times X_0
\end{array}
\]
The horizontal maps are localized Reedy equivalences of simplicial spaces. Indeed the top map is so by definition and for the bottom map we use the fact that pulling back along spaces preserves localized Reedy equivalences. Moreover, the left vertical map is a Reedy equivalence by assumption, which means it is also a localized Reedy equivalence. This implies that the right hand vertical map is also a localized Reedy equivalence. But both simplicial spaces $\hat{R}_n$ and $\hat{R}_0 \times X_0$ are fibrant in the localized Reedy model structure and so the map is actually a Reedy equivalence. This proves that $\hat{R} \to X$ is actually a fibrant replacement.

Finally notice that the map $R_n \to \hat{R}_n$ is a localized Reedy equivalence as it is the restriction of a functorial fibrant replacement to a single point in $\Delta^{op}$ (namely the point $n$). This finishes our proof.

□

Proposition 6.27. Let $g : R \to S$ be a map between Reedy right fibrations over $X$. Then the following are equivalent:

1. $g$ is a localized Reedy contravariant equivalence.
2. The map $g_n : R_n \to S_n$ is an equivalence in the localized Reedy model structure for each $n$.
3. The map $(i_\phi)_*(g) : (i_\phi)_*(R) \to (i_\phi)_*(S)$ is an equivalence in the localized Reedy model structure.
4. For each map $x : F(0) \to X$ the induced map $(i_\phi)_*(R \times_X F(0)) \to (i_\phi)_*(S \times_X F(0))$ is an equivalence in the localized Reedy model structure.
5. For each map $x : F(0) \to X$ the induced map $R \times_X F(0) \to S \times_X F(0)$ is an equivalence in the diagonal localized Reedy model structure.

Proof. Before we start with the proof, we use the previous lemma. By the previous lemma there is a commutative square over $X$ such that $\hat{g} : \hat{R} \to \hat{S}$ is a fibrant replacement of $g$ and for each $n$ the map

\[
\begin{array}{ccc}
R & \xrightarrow{\simeq} & \hat{R} \\
\downarrow{g} & & \downarrow{\hat{g}} \\
S & \xrightarrow{\simeq} & \hat{S}
\end{array}
\]

are localized Reedy equivalences. In particular $g : R \to S$ is a localized Reedy contravariant equivalence if and only if $\hat{g} : \hat{R} \to \hat{S}$ is a biReedy equivalence. We will refer to this commutative square and the properties we just stated several times throughout this proof.

(1) $\Leftrightarrow$ (2) We restrict the square above to following square
The horizontal maps are localized Reedy equivalences. Thus \( g_{\ast n} \) is a localized Reedy equivalence if and only if \( \hat{g}_{\ast n} \) is one. But we know that \( \hat{R}_{\ast n} \) and \( \hat{S}_{\ast n} \) are fibrant, which implies that the \( \hat{g}_{\ast n} \) is a localized Reedy equivalence if and only if it is a Reedy equivalence, which itself is the same as \( \hat{g} \) being a biReedy equivalence.

\[ (2) \iff (3) \]

One side is just a special case. Thus we need to prove that if \( (i_{\varphi})_{\ast}(g) : (i_{\varphi})_{\ast}(R) \to (i_{\varphi})_{\ast}(S) \) is an equivalence in the localized Reedy model structure then \( R_{\ast n} \to S_{\ast n} \) is one as well. For that we use following commutative square

\[
\begin{array}{ccc}
R_{\ast n} & \longrightarrow & S_{\ast n} \\
\downarrow \sim & & \downarrow \sim \\
R_{\ast 0} \times X_n & \longrightarrow & S_{\ast 0} \times X_n \\
\end{array}
\]

The vertical maps are Reedy equivalences as \( R \) and \( S \) are Reedy right fibrations. The bottom map is a localized Reedy equivalence as pulling back along the map \( X_n \to X_0 \) preserves equivalences. Thus the top map \( R_{\ast n} \to S_{\ast n} \) is a localized Reedy equivalence.

\[ (3) \iff (4) \]

We can restrict the commutative square from the start of the proof to the following commutative square over \( X_0 \)

\[
\begin{array}{ccc}
i_{\varphi}^{\ast}(R) & \longrightarrow & i_{\varphi}^{\ast}(\hat{R}) \\
\downarrow & & \downarrow \\
i_{\varphi}^{\ast}(S) & \longrightarrow & i_{\varphi}^{\ast}(\hat{S}) \\
\end{array}
\]

The map
\[
i_{\varphi}^{\ast}(R) \to i_{\varphi}^{\ast}(S)
\]

is a localized Reedy equivalence if and only if
\[
i_{\varphi}^{\ast}(\hat{R}) \to i_{\varphi}^{\ast}(\hat{S})
\]

is a Reedy equivalence over \( X_0 \). This is equivalent to
\[
i_{\varphi}^{\ast}(\hat{R}) \times_{X_0} F(0) \to i_{\varphi}^{\ast}(\hat{S}) \times_{X_0} F(0)
\]
being a Reedy equivalence. Finally this is equivalent to
\[ i^*_\varphi(R) \times_{X_0} F(0) \to i^*_\varphi(S) \times_{X_0} F(0) \]
being a localized Reedy equivalence. This follows from the fact that \( i^*_\varphi(R) \times_{X_0} F(0) \to i^*_\varphi(S) \times_{X_0} F(0) \)
is still a fibrant replacement, as pulling back along the map \( F(0) \to X_0 \) preserves localized Reedy equivalences.

\((4) \Leftrightarrow (5)\) This follows directly from the fact that \( R \) and \( S \) are Reedy right fibrations, which implies that any fiber over \( X \) is a homotopically constant bisimplicial space. Thus we have Reedy equivalences
\[
\begin{align*}
i^*_\varphi(R \times_{X} F(0)) & \simeq \varphi \text{Diag}^*(R \times_{X} F(0)) \\
i^*_\varphi(S \times_{X} F(0)) & \simeq \varphi \text{Diag}^*(S \times_{X} F(0))
\end{align*}
\]
Hence, our map is a fiberwise diagonal localized Reedy equivalence if and only if the 0 level is a fiberwise localized Reedy equivalence.

**Theorem 6.28.** A map \( g : Y \to Z \) of bisimplicial spaces over \( X \) is an equivalence in the localized Reedy contravariant model structure if and only if for each map \( x : F(0) \to X \), the induced map
\[
X_{x/} \times_{X} Y \to X_{x/} \times_{X} Z
\]
is an equivalence in the diagonal localized Reedy model structure. Here \( X_{x/} \) is the left fibrant replacement of the map \( x \).

**Proof.** Let \( \hat{g} : \hat{Y} \to \hat{Z} \) be a fibrant replacement of \( g \) in the Reedy contravariant model structure (not localized). Moreover, let \( x : F(0) \to X \) be a vertex in \( X \). This gives us following zig-zag of maps:

\[
\begin{array}{ccc}
F(0) \times_{X} \hat{Y} & \longrightarrow & F(0) \times_{X} \hat{Z} \\
\downarrow \text{ReeCov} & & \downarrow \text{ReeCov} \\
X_{x/} \times_{X} \hat{Y} & \longrightarrow & X_{x/} \times_{X} \hat{Z} \\
\uparrow \text{ReeContra} & & \uparrow \text{ReeContra} \\
X_{x/} \times_{X} Y & \longrightarrow & X_{x/} \times_{X} Z
\end{array}
\]

According to Theorem 4.16 the top vertical maps are Reedy covariant equivalences and the bottom vertical maps are Reedy contravariant equivalences. By Theorem 4.12 both of these are diagonal Reedy equivalences, which itself is always a diagonal localized Reedy equivalence. Thus the top map is a diagonal localized Reedy equivalence if and only if the bottom map is one, but by Proposition 6.27 this is equivalent to \( Y \to Z \) being a localized Reedy contravariant equivalence over \( X \). \( \square \)
Another important aspect of model structures is invariance under base changes, which we can address using our knowledge of fibrant objects and weak equivalences.

**Theorem 6.29.** Let \( g : X \rightarrow Y \) be a map of simplicial spaces. Then the adjunction

\[
(ss\mathcal{S}/X)^{ReeContra} \xrightarrow{g^*} (ss\mathcal{S}/Y)^{ReeContra}
\]

is a Quillen adjunction, which is a Quillen equivalence whenever \( g \) is a CSS equivalence.

**Proof.** Clearly it is a Quillen adjunction as fibrations are stable under pullback. So, let us assume that \( g \) is a CSS equivalence. We will use Lemma A.4 to show it is a Quillen equivalence. This means we have to show the left adjoint reflects weak equivalences and the counit map is a weak equivalence.

**Reflecting Equivalences:** Before we can prove this we need following three results that we have proven before.

1. For each vertex \( y : F(0) \rightarrow Y \) there exists an \( x : F(0) \rightarrow X \) such that \( g(x) \) and \( y \) are in the same path component. This follows from the fact that the collection of maps used to build complete Segal spaces are all connected. For more details see[Ra17, Theorem 4.8].
2. According to Remark 4.20 it always suffices to check the equivalence principle for one point for each path component.
3. If \( Y_{y/} \) is the representable left fibration over \( Y \) representing the vertex \( y : F(0) \rightarrow Y \) then following is a homotopy pullback square in the Reedy model structure

\[
\begin{array}{ccc}
X_{x/} & \xrightarrow{r} & Y_{y/} \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & Y
\end{array}
\]

where \( x \) is any object such that \( g(x) \) and \( y \) are in the same path component.

Now we can start out proof. For a map of bisimplicial spaces \( Z_1 \rightarrow Z_2 \) over \( X \) we have following commutative diagram

\[
\begin{array}{ccc}
X_{x/} \times X Z_1 & \xrightarrow{\cong} & X_{x/} \times X Z_2 \\
\downarrow & & \downarrow \\
Y_{y/} \times X Z_1 & \xrightarrow{\cong} & Y_{y/} \times X Z_2 \\
\downarrow & & \downarrow \\
Y_{y/} Z_1 & \xrightarrow{\cong} & Y_{y/} Z_2
\end{array}
\]
The vertical maps are all diagonal Reedy equivalences which means the top one is a localized diagonal Reedy equivalence if and only if the bottom map is. Thus, by Theorem 6.28, $Z_1 \to Z_2$ is a localized Reedy contravariant equivalence over $X$ if and only if it one over $Y$.

Counit Map Equivalence: Let $p : R \to Y$ be a localized Reedy right fibration over $Y$. The counit map is the pullback map $p^*g : p^*X \to R$ in the diagram.

$$
\begin{array}{ccc}
p^*X & \xrightarrow{p^*g} & R \\
p \downarrow & & \downarrow \\
X & \xrightarrow{g} & Y
\end{array}
$$

However, $g$ is a CSS equivalence and $p$ is a levelwise right fibration and, by [Ra17, Theorem 5.28], pulling back along right fibrations preserves CSS equivalences. Thus the pullback map $p^*g : p^*X \to R$ is a levelwise CSS equivalence. By [Ra17, Theorem 4.12], this implies that it is also a levelwise contravariant equivalence over $Y$ (over any base actually), which is just a Reedy contravariant equivalence over $Y$. Hence, it is a localized Reedy contravariant equivalence over $Y$ and we are done. \[\square\]

Remark 6.30. As always there is a covariant version to all the definitions we have given above. We will not repeat all definitions above, but using analogous approach we can define localized Reedy left fibrations and localized Reedy covariant model structure. Then we can prove the same classification theorems for fibrant objects and weak equivalences in the localized Reedy covariant model structure. This follows from the fact that $f$ is acceptable and so $f$ and $f^{op}$ give us the same localized model structure.

Similar to the case of Reedy left and Reedy right fibrations we have following interaction between localized Reedy left and localized Reedy right fibrations.

**Theorem 6.31.** Let $p : R \to X$ be a localized Reedy right fibration over $X$. The induced adjunction

$$
(\mathcal{S}^{sS}_{/X})^{RecCov_f} \xleftarrow{p_*p^*} (\mathcal{S}^{sS}_{/X})^{RecCov_f}
$$

is a Quillen adjunction. Here both sides have the localized Reedy covariant model structure.

**Proof.** Clearly the left adjoint preserves cofibrations and the right adjoint preserves fibrations between fibrant objects (as they are just biReedy fibrations). Thus it suffices to show that the right adjoint preserves fibrant objects. So, let $L \to X$ be a localized Reedy left fibration over $X$. Then we have to show that $p_*p^*L \to X$ is also a localized Reedy left fibration over $X$. By Theorem 4.16, we already know that it is a Reedy left fibration, so all that is left is to show that it is local. By Theorem 6.28, it suffices to show that for any map $q : i^*_x(B) \to X$ the induced map

$$
Map_{/X}(i^*_x(B), p_*p^*L) \to Map_{/X}(i^*_x(A), p_*p^*L)
$$
is a Kan equivalence. By adjunction this is equivalent to
\[ \text{Map}_{/X}(p_p^*(i_p^*(B)), L) \to \text{Map}_{/X}(p_p^*(i_p^*(A)), L) \]
being a Kan equivalence. For that it suffices to show that
\[ p_p^*(i_p^*(A)) \to p_p^*(i_p^*(B)) \]
is a localized Reedy covariant equivalence over \( X \). By Lemma 6.15 the map \( q \) factors through the point. Thus \( p_p^*(i_p^*(B)) = i_p^*(B) \times_X R \cong i_p^*(B) \times (F(0) \times_X R) \) and similarly \( p_p^*(i_p^*(A)) = i_p^*(A) \times (F(0) \times_X R) \). So, we need to prove that the map
\[ i_p^*(A) \times (F(0) \times_X R) \to i_p^*(B) \times (F(0) \times_X R) \]
is a localized Reedy covariant equivalence over \( X \). However, according to Example 6.10, localized Reedy contravariant equivalences and diagonal localized Reedy equivalences are the same over \( F(0) \).

Taking diagonal we get the map
\[ A \times \varphi \text{Diag}^*(F(0) \times_X R) \to B \times \varphi \text{Diag}^*(F(0) \times_X R) \]
However, this is one is clearly a localized Reedy equivalence, as it is one of conditions that \( f \) had to satisfy in order to be acceptable (Definition 6.11).

\[ \square \]

Example 6.32. We have already discussed the localized Reedy contravariant model structure over the point. So, let us take the case where \( X = F(1) \). From Example 4.24 we already know that a Reedy right fibration over \( Y \to F(1) \) is the data of a map of simplicial spaces \( Y_{01} \to Y_0 \). By Theorem 6.18, these simplicial spaces are fibrant in the localized Reedy model structure. Thus the data of a localized Reedy right fibration over \( F(1) \) is just the data of map between fibrant objects in the localized Reedy model structure.

Example 6.33. We can generalize this to the case of localized Reedy right fibration over \( F(n) \). Combining the previous example and Example 4.25 we can easily deduce that it is just the data of a chain of \( n \) fibrant objects in the localized Reedy model structure.

In certain specific circumstances the localized Reedy contravariant model structures satisfies even stronger conditions. For that we need following additional condition.

Definition 6.34. We say an inclusion of simplicial spaces \( f : A \to B \) satisfies the right stability condition if for every right fibration \( p : R \to X \), the adjunction
\[ \left( (sS_{/X})^{Ree,f} \right) \xrightarrow{p_p^*} \left( sS_{/X} \right)^{Ree,f} \]
is a Quillen adjunction. Similarly, we can define the left stability condition.

Remark 6.35. Recall that every map \( f \) that we consider is always acceptable. This in particular implies that if \( f \) satisfies the right stability condition it also satisfies the left stability condition, as every object is local with respect to \( f \) if and only if it is local with respect to \( f^{op} \).

Having this new condition we can prove further results. However, before that we to adjust the adjunction \( (\Delta \text{Diag}_B, \Delta \text{Diag}^*) \) (Proposition 3.21) to the relative case.
Definition 6.36. There is an adjunction

\[
\begin{array}{c}
sS/X \\
\xrightarrow{\Delta \text{Diag}_X^X} \\
\xleftarrow{\Delta \text{Diag}_X^X} \\
ssS/X
\end{array}
\]

defined as

\[
\Delta \text{Diag}_X^X(p : F(n) \times \Delta[l] \to X) = p\varphi_n : \varphi_n \times F(n) \times \Delta[l] \to X
\]

and

\[
\Delta \text{Diag}_X^X(Y \to X)_{nl} = \text{Hom}_{sS/X}(F(n) \times \Delta[l], \Delta \text{Diag}_X^X(Y)) = \\
\text{Hom}_{ssS/X}(\Delta \text{Diag}_X^X(F(n) \times \Delta[l]), Y) = Y_{\text{nl}} \times \Delta[0]
\]

Remark 6.37. The right adjoint just takes an object \( Y \to X \) to \( \Delta \text{Diag}^*(Y) \to \Delta \text{Diag}^*(X) = X \), but the left adjoint differs from \( \Delta \text{Diag}^\# \) as \( \Delta \text{Diag}^\# \) does not preserve the base. In the case the base it the point, however, they are the same. Thus we can think of this adjunction is a generalization of Proposition 3.21.

Now we might wonder what would happen with the Quillen adjunction if we localize both sides with respect to a map \( f \).

Theorem 6.38. Let \( f \) satisfy the right stability condition. The adjunction

\[
(sS/X)_{\text{Ree}} \xrightarrow{\text{Diag}_X^X} (ssS/X)_{\text{ReeContra}}
\]

is a Quillen adjunction between the localized Reedy model structure over \( X \) and the localized Reedy contravariant model structure over \( X \).

Proof. Clearly the left adjoint preserves cofibrations as they are just inclusions. We will show that the left adjoint also preserves trivial cofibrations. Let \( g : C \to D \) be a trivial cofibration over \( X \) in the localized Reedy model structure. We will show that \( i^*_\varphi(C) \times i^*_\varphi(D) \to i^*_\varphi(D) \times i^*_\varphi(D) \) is a localized Reedy contravariant equivalence over \( X \). By Theorem 6.28, it suffices to show that for each map \( x : F(0) \to X \), the induced map

\[
X_{x/} \times (i^*_\varphi(C) \times i^*_\varphi(D)) \to (X_{x/} \times i^*_\varphi(D) \times i^*_\varphi(D))
\]

is a diagonal localized Reedy equivalence. Recall that we embedded simplicial spaces in bisimplicial spaces using the map \( i^*_\varphi \) (Notation 3.7). This means that \( X = i^*_\varphi(X) \) and \( X_{x/} = i^*_\varphi(X_{x/}) \), which implies that

\[
X_{x/} \times (i^*_\varphi(C) \times i^*_\varphi(D)) = i^*_\varphi(X_{x/}) \times (i^*_\varphi(C) \times i^*_\varphi(D)) = i^*_\varphi(X_{x/} \times C) \times i^*_\varphi(D)
\]

and similarly \( X_{x/} \times (i^*_\varphi(D) \times i^*_\varphi(D)) = i^*_\varphi(X_{x/} \times D) \times i^*_\varphi(D) \) So, we have to show that

\[
i^*_\varphi(X_{x/} \times C) \times i^*_\varphi(C) \to i^*_\varphi(X_{x/} \times D) \times i^*_\varphi(D)
\]
is a diagonal localized Reedy equivalence. However, $f$ is acceptable and so the product of diagonal localized Reedy equivalences is again a diagonal localized Reedy equivalence. Thus it suffices to show separately that the two maps
\[ i^*_F(X_{x/} \times C) \to i^*_F(X_{x/} \times D) \]
\[ i^*_F(C) \to i^*_F(D) \]
are diagonal localized Reedy equivalences. The second one follows by definition. For the first one, we recall that $f$ satisfies the left stability property and so the pullback of the localized Reedy equivalences $C \to D$ along the left fibration $X_{x/}$ is still a localized Reedy equivalence. Thus $i^*_F(X_{x/} \times C) \to i^*_F(X_{x/} \times D)$ is a diagonal localized Reedy equivalence. \qed

**Remark 6.39.** This adjunction does NOT necessarily hold, when we localize both sides with respect to an acceptable map $f$ that does not satisfy the right stability condition. We shall see an example in Subsection 7.6.

Theorem 6.38 gives us following corollary.

**Corollary 6.40.** Let $f$ satisfy the right stability property and $p : Y \to X$ be a map of simplicial spaces. If $p$ is a $f$-localized Reedy right fibration, then $p$ is a fibration in the $f$-localized Reedy model structure (Definition 6.21).

Before we can move on we have one very important remark regarding the localizing maps.

**Remark 6.41.** Everything we have done in this section still holds if we localize with respect to a countable set of cofibrations, as long as every map in that set is acceptable (and for the later results also satisfies the right stability property).

**Remark 6.42.** What we have done in this subsection is to give various tools which help us understand fibrant objects and weak equivalences in the localized Reedy contravariant model structure and using that understanding to study the model structure. Having these tools we can now study important cases.

**Segal Cartesian Fibrations**

We spend all of the last section (Section 6) setting up the right tools to model maps into localization of simplicial spaces. In this section we will use those tools to analyze three specific examples: presheaves valued in Segal spaces, presheaves valued in complete Segal spaces and presheaves valued in objects fibrant in the diagonal model structure.

**7.1 Segal Cartesian Fibrations.** In this subsection we will study fibrations which model presheaves valued in Segal spaces, which we call Segal Cartesian fibrations. We will use the localization techniques introduced in the past section to do so.

**Definition 7.1.** Let $k \geq 2$. We define the discrete bisimplicial space $\gamma_k$ as
\[ \gamma_k = \varphi_1 \coprod_{\varphi_0} \cdots \coprod_{\varphi_0} \varphi_1 = i^*_p(G(k)) \]
where there are $k$ summands of $\varphi_1$. It is commonly called the "spine" as there is a natural inclusion $f_k : \gamma_k \to \varphi_k$. 
Definition 7.2. We say a map $Y \to X$ over $X$ is a Segal Cartesian fibration if it is a Reedy right fibration and for $k \geq 2$ the map of simplicial spaces

$$f_k^* : \text{Map}_X(\varphi_k, Y) \to \text{Map}_X(\gamma_k, Y)$$

is an Kan equivalence of spaces for every map $\varphi_k \to X$.

Segal Cartesian fibrations come with their own model structure.

Theorem 7.3. There is a unique model structure on bisimplicial spaces over $X$, called the Segal Cartesian model structure and denoted by $(\text{ssS}/X)^{\text{SegCart}}$ such that

1. It is a simplicial model category.
2. The fibrant objects are the Segal Cartesian fibrations over $X$.
3. Cofibrations are monomorphisms.
4. A map $A \to B$ over $X$ is a weak equivalence if

$$\text{map}_{\text{ssS}/X}(B, W) \to \text{map}_{\text{ssS}/X}(A, W)$$

is an equivalence for every Segal Cartesian fibration $W \to X$.
5. A weak equivalence (Segal Cartesian fibration) between fibrant objects is a level-wise equivalence (biReedy fibration).

Proof. All of this directly follows from applying the theory of Bousfield localizations to the Reedy contravariant model structure over $X$, where the localizing set is:

$$L = \{ \gamma_k \to \varphi_k \to X : k \geq 2 \}$$

□

The maps $G(k) \to F(k)$ are acceptable and satisfy the right stability condition. Indeed, the only non-trivial part for being acceptable is condition 5, which follows from part (5) of Theorem 1.6. The right stability condition is stated in [Ra17, Remark 5.30]. Thus, we directly have the following corollaries about the Segal Cartesian model structure.

Corollary 7.4. The following are equivalent

1. The map $R \to X$ of bisimplicial spaces is a Segal Cartesian fibration over $X$.
2. $R \to X$ is a Reedy right fibration and the map of simplicial spaces

$$R_k \to R_1 \times \ldots \times R_1$$

is a trivial Reedy fibration.
3. $R \to X$ is a Reedy right fibration and the simplicial space $R_n$ is a Segal space for every $n$
4. $R \to X$ is a Reedy right fibration and the simplicial space $R_{0\bullet} = (i_{\varphi})_*(R)$ is a Segal space.
5. $R \to X$ is a Reedy right fibration and for each vertex $x : F(0) \to X$ the fiber $(i_{\varphi})_*(F(0) \times_X R)$ is a Segal space.
6. $R \to X$ is Reedy right fibration and for each point $x : F(0) \to X$ the fiber $F(0) \times_X R$ is fibrant in the diagonal Segal space model structure.
Definition 7.5. We say a map of simplicial spaces $S \to X$ is a **Segal Cartesian fibration** if there exists a Segal Cartesian fibration $R \to X$ such that $S = \Delta \text{Diag}^*(R)$ over $X$. Notice if such an $R$ exists then it will be unique up to biReedy equivalence, as Segal Cartesian fibrations are completely determined by their diagonals.

Lemma 7.6. Let the map of simplicial spaces $p : S \to X$ be a Segal Cartesian fibration and $g : Y \to X$ be any map. Then the pullback of $g^*p : g^*S \to Y$ is a Segal Cartesian fibration over $Y$. Indeed, if $S = \Delta \text{Diag}^*(R)$ then $g^*S = \Delta \text{Diag}^*(g^*R)$.

Corollary 7.7. Let the map of simplicial spaces $p : S \to X$ be a Segal Cartesian fibration and $K$ any simplicial space. Then $p^K : S^K \to X^K$ is also a Segal Cartesian fibration. Indeed, if $S = \Delta \text{Diag}^*(R)$, then $S^K = \Delta \text{Diag}^*(R^{\Delta \text{Diag}^*(K)})$.

Corollary 7.8. A map $g : Y \to Z$ of bisimplicial spaces over $X$ is an equivalence in the Segal Cartesian model structure if and only if for each map $x : F(0) \to X$, the induced map

$$X_{x/} \times_Y X \to X_{x/} \times_X Z$$

is an equivalence in the diagonal Segal space model structure. Here $X_{x/}$ is the left fibrant replacement of the map $x$.

Corollary 7.9. Let $g : X \to Y$ be a map of simplicial spaces. Then the adjunction

$$(ssS_{/X})_{\text{SegCart}} \rightleftarrows (ssS_{/Y})_{\text{SegCart}}$$

is a Quillen adjunction, which is a Quillen equivalence whenever $g : X \to Y$ is a CSS equivalence.

Corollary 7.10. The adjunction

$$(sS_{/X})_{\text{Seg}} \rightleftarrows (ssS_{/X})_{\text{SegCart}}$$

is a Quillen adjunction between the Segal space model structure over $X$ and the Segal Cartesian model structure over $X$.

Corollary 7.11. Every Segal Cartesian map is itself a fibration in the Segal space model structure.

Theorem 7.12. Let $p : S \to X$ be a Segal Cartesian fibration of simplicial spaces. Then the adjunction

$$(sS_{/X})_{\text{Seg}} \rightleftarrows (sS_{/X})_{\text{Seg}}$$

is a Quillen adjunction. Here both sides have the Segal space model structure.

This theorem has the following very useful corollary.

Corollary 7.13. If $p : S \to X$ is a Segal Cartesian fibration and $Y \to X$ is a Segal equivalence, then the induced map $S \times_X Y \to S$ is a Segal equivalence.
7.2 Cartesian Fibrations and the Cartesian Model Structure. Having set up Segal Cartesian fibrations we can now move on to Cartesian fibrations, which model presheaves valued in complete Segal spaces.

In [Re01, Page 13] the simplicial space $Z(3)$ is defined as follows

$$Z(3) = F(1) \coprod_{F(0)} F(1) \coprod_{F(0)} F(1)$$

**Definition 7.14.** We define the bisimplicial space $\zeta$ as

$$\zeta = \varphi_1 \coprod_{\varphi_0} \varphi_3 = \int^* (F(1) \coprod F(3)).$$

For the rest of the subsection we fix any map $e : \varphi_0 \to \zeta$.

**Definition 7.15.** We say a bisimplicial space $Y \to X$ over $X$ is a Cartesian fibration if it is a Segal Cartesian fibration and the map of spaces

$$e^* : Map_{/X}(\zeta, Y) \to Map_{/X}(\varphi_0, Y)$$

is an Kan equivalence of spaces for every map $\zeta \to X$.

Cartesian fibrations come with their own model structure.

**Theorem 7.16.** There is a unique model structure on bisimplicial spaces over $X$, called the Cartesian model structure and denoted by $(ssS_{/X})^{Cart}$ such that

1. It is a simplicial model category.
2. The fibrant objects are the Cartesian fibrations over $X$.
3. Cofibrations are monomorphisms.
4. A map $A \to B$ over $X$ is a weak equivalence if

$$\text{map}_{ssS_{/X}}(B, W) \to \text{map}_{ssS_{/X}}(A, W)$$

is an equivalence for every Cartesian fibration $W \to X$.
5. A weak equivalence (Cartesian fibration) between fibrant objects is a level-wise equivalence (biReedy fibration).

**Proof.** All of this directly follows from applying the theory of Bousfield localizations to the Reedy contravariant model structure over $X$, where the localizing set is:

$$L = \{ \gamma_k \to \varphi_k \to X : k \geq 2 \} \coprod_{\varphi_0} \{ \varphi_0 \to \zeta \to X \}$$

$\square$

**Remark 7.17.** This definition of a Cartesian fibration and its associated model structure have been (independently) defined by de Brito, however only for the case where the base is a Segal space [dB16, Proposition 3.4].
The map $F(0) \to Z(3)$ is acceptable and satisfies the right stability condition. Indeed, the only non-trivial condition for being acceptable is condition 5, which follows from part (5) of Theorem 1.9. The right stability condition is stated in [Ra17, Theorem 5.28]. Thus, we directly have the following corollaries about the Cartesian model structure.

**Corollary 7.18.** The following are equivalent

1. The map $R \to X$ of bisimplicial spaces is a Cartesian fibration over $X$.
2. $R \to X$ is a Reedy right fibration and the maps of simplicial spaces

   $R_k \to R_1 \times \ldots \times R_1$

   $R_3 \to R_1 \times R_1 \times R_1$

   are trivial Reedy fibrations.
3. $R \to X$ is a Reedy right fibration and the simplicial space $R_0$ is a complete Segal space.
4. $R \to X$ is a Reedy right fibration and the simplicial space $R_0 = (i_\varphi)_*(R)$ is a complete Segal space.
5. $R \to X$ is a Reedy right fibration and for each vertex $x : F(0) \to X$ the fiber $(i_\varphi)_*(F(0) \times_X R)$ is a complete Segal space.
6. $R \to X$ is Reedy right fibration and for each point $x : F(0) \to X$ the fiber $F(0) \times_X R$ is fibrant in the diagonal complete Segal space model structure.

**Definition 7.19.** We say a map of simplicial spaces $S \to X$ is a Cartesian fibration if there exists a Cartesian fibration $R \to X$ such that $S = \Delta \text{Diag}^*(R)$ over $X$. Notice if such an $R$ exists then it will be unique up to biReedy equivalence, as Cartesian fibrations are completely determined by their diagonals.

**Lemma 7.20.** Let the map of simplicial spaces $p : S \to X$ be a Cartesian fibration and $g : Y \to X$ be any map. Then the pullback of $g^*p : g^*S \to Y$ is a Cartesian fibration over $Y$. Indeed, if $S = \Delta \text{Diag}^*(R)$ then $g^*S = \Delta \text{Diag}^*(g^*R)$.

**Corollary 7.21.** Let the map of simplicial spaces $p : S \to X$ be a Cartesian fibration and $K$ any simplicial space. Then $p^K : S^K \to X^K$ is also a Cartesian fibration. Indeed, if $S = \Delta \text{Diag}^*(R)$, then $S^K = \Delta \text{Diag}^*(R^{\Delta \text{Diag}^*(K)})$.

**Corollary 7.22.** A map $g : Y \to Z$ of bisimplicial spaces over $X$ is an equivalence in the Cartesian model structure if and only if for each map $x : F(0) \to X$, the induced map

$$X_{x/} \times_X Y \to X_{x/} \times_X Z$$

is an equivalence in the diagonal complete Segal space model structure. Here $X_{x/}$ is the left fibrant replacement of the map $x$.

**Corollary 7.23.** Let $g : X \to Y$ be a map of simplicial spaces. Then the adjunction

$$(ssS/)_X \overset{g^!}{\rightleftarrows} (ssS/Y)^{\text{Cart}}$$

is a Quillen adjunction, which is a Quillen equivalence whenever $g : X \to Y$ is a CSS equivalence.

**Corollary 7.24.** The adjunction
\[(sS/X)^{CSS} \xleftrightarrow{\text{Diag}_X^*} (ssS/X)^{Cart}\]

is a Quillen adjunction between the complete Segal model structure over \(X\) and the Cartesian model structure over \(X\).

**Corollary 7.25.** Every Cartesian fibration is itself a fibration in the complete Segal space model structure.

**Theorem 7.26.** Let \(p : S \to X\) be a Cartesian fibration of simplicial spaces. Then the adjunction

\[
(sS/X)^{CSS} \xleftarrow{p^*} (ssS/X)^{CSS}
\]

is a Quillen adjunction. Here both sides have the complete Segal space model structure.

This theorem has the following very useful corollary.

**Corollary 7.27.** If \(p : S \to X\) is a Cartesian fibration and \(Y \to X\) is a complete Segal space equivalence, then the induced map \(S \times_X Y \to S\) is a complete Segal space equivalence.

### 7.3 An Alternative Approach to Cartesian Fibrations over CSS

For the specific case of Cartesian fibrations over a CSS we can give an internal criterion to determine whether a map of simplicial spaces is a Cartesian fibrations. For that we first need to discuss \(p\)-Cartesian maps. This is the original approach to define Cartesian fibrations. In particular, it was used by Lurie for quasicategories \([Lu09, \text{Section 2.4]}\) and Riehl and Verity for \(\infty\)-cosmoi \([RV17, \text{Section 4]}\).

For this section let \(p : C \to X\) be a fixed CSS fibration over a CSS \(X\). Note this implies that \(C\) is a CSS.

**Definition 7.28.** We say a morphism \(f : x \to y\) in \(C\) is a \(p\)-Cartesian morphism if the following is a homotopy pullback square in the Reedy model structure.

\[
\begin{array}{ccc}
C/f & \xrightarrow{r} & C/y \\
\downarrow & & \downarrow \\
X/p(f) & \xrightarrow{} & X/p(y)
\end{array}
\]

**Example 7.29.** We see right away that the identity map \(id : x \to x\) is always \(p\)-Cartesian.

**Example 7.30.** On the other side if \(f\) is \(p\)-Cartesian and \(p(f)\) is the identity map then \(f\) is an equivalence in \(C\). This follows from the fact that \(C/f \to C/y\) is an equivalence if and only if \(f\) is an equivalence.

We have following fact about compositions of \(p\)-Cartesian morphisms.
Lemma 7.31. Let \( p : C \to X \) be a CSS fibration and \( \sigma : F(2) \to C \) be a 2-simplex such that the boundaries \( d_0 \sigma \) and \( d_1 \sigma \) are both \( p \)-Cartesian. Then, the composition morphism \( d_1 \sigma \) is also \( p \)-Cartesian.

Proof. In order to simplify notation we denote the edges of \( \sigma \) as follows.

\[
\begin{array}{c}
\bullet & \overset{a}{\to} & \bullet & \overset{c}{\to} \\ \downarrow{g} & \\ \downarrow{f} & \\ \downarrow{b} \\ \downarrow{g f} \\ \downarrow{c} \\
\end{array}
\]

We need to show that the following is a homotopy pullback square.

\[
\begin{array}{c}
C_{/gf} & \overset{b}{\to} & C_{/c} \\ \downarrow{X_{/p(g)}(f)} \\ \downarrow{X_{/p(c)}} \\
\end{array}
\]

But we have following commutative square,

\[
\begin{array}{c}
C_{/\sigma} & \overset{\simeq}{\to} & C_{/gf} \\ \downarrow{X_{/p(\sigma)}} & \\ \downarrow{X_{/p(gf)}} \\ \downarrow{\simeq} \\
\end{array}
\]

which means it suffices to prove that the following is a homotopy pullback square.

\[
\begin{array}{c}
C_{/\sigma} & \overset{\simeq}{\to} & C_{/c} \\ \downarrow{X_{/p(\sigma)}} & \\ \downarrow{X_{/p(c)}} \\
\end{array}
\]

We can factor this map as follows,
where the right hand square is already a homotopy pullback as \( g \) is \( p \)-Cartesian and thus it suffices to prove that the left hand square is a homotopy pullback square. Now we can extend the square with following equivalences,

\[
\begin{array}{ccc}
C_{/\sigma} & \longrightarrow & C_{/g} \\
\downarrow & & \downarrow \cong \\
X_{/p(\sigma)} & \longrightarrow & X_{/p(g)}
\end{array}
\]

\[
\begin{array}{ccc}
& & C_{/c} \\
\downarrow & & \downarrow \\
& & X_{/p(c)}
\end{array}
\]

which means in order to get the desired result we can show the whole rectangle is a homotopy pullback. But this rectangle factors as,

\[
\begin{array}{ccc}
C_{/\sigma} & \longrightarrow & C_{/g} & \cong & C_{/b} \\
\downarrow & & \downarrow & & \downarrow \\
X_{/p(\sigma)} & \longrightarrow & X_{/p(g)} & \cong & X_{/p(b)}
\end{array}
\]

Definition 7.32. We say an \( n \)-simplex \( \sigma : F(n) \to C \) is \( p \)-Cartesian if for every map \( f : F(1) \to F(n) \), the restriction map \( \sigma f \) is \( p \)-Cartesian.

Definition 7.33. Let \( p : C \to X \) be a CSS fibration. We define \( RFib_X(C) \) to be the subsimplicial space of \( C \) such that \( RFib_X(C)_n \) is the subspace of \( C_n \) generated by all \( p \)-Cartesian \( n \)-simplices.

Remark 7.34. By Example 7.29, \( RFib_X(C)_0 = C_0 \).

Lemma 7.35. \( RFib_X(C) \) is a Segal space.

Proof. In order to show it is Reedy fibrant we have following diagram
The map $C \to X$ is a Reedy fibration, which means that a lift exists. However, the vertices of this lift all land in $RFib_X(C)$, which means the map will factor through $RFib_X(C)$ and give us the desired lift. Thus the map is a Reedy fibration. As the base is Reedy fibrant this implies that $RFib_X(C)$ is also Reedy fibrant.

Now we prove that it is a Segal space. We have a diagram

\[
\begin{array}{c}
\partial F(n) \times \Delta[l] \coprod_{\partial F(n) \times \Lambda[l]^i} F(n) \times \Lambda[l]^i \rightarrow RFib_X(C) \\
F(n) \times \Lambda[l]^i \downarrow \downarrow \downarrow \downarrow \\
X \rightarrow C
\end{array}
\]

Again the lift to $C$ exists as $C$ is a Segal space. All the 1-simplices which are not in the image of $\alpha$ are compositions of maps that lie in the image of $\alpha$. But by Lemma 7.31 the composition of $p$-Cartesian morphisms is again $p$-Cartesian. That means all 1-simplices in the image of $\tilde{\alpha}$ are $p$-Cartesian. Thus the map will factor through $RFib_X(C)$, which shows that $RFib_X(C)$ is a Segal space and hence we are done. □

**Lemma 7.36.** If in the following diagram $p$ is a CSS fibration, $RFib_X(C)$ is a right fibration over $X$ and $q$ is a retract of $p$, then $q$ is a CSS fibration and $RFib_A(D)$ is a right fibration over $A$.

\[
\begin{array}{c}
D \overset{j}{\longrightarrow} C \overset{s}{\longrightarrow} D \\
A \overset{i}{\longrightarrow} X \overset{r}{\longrightarrow} A
\end{array}
\]

**Proof.** CSS fibrations are fibrations in the CSS model structure and so are closed under retracts by definition, which means $q$ is a CSS fibration. Thus we only have to show that every map in $A$ has a $p$-Cartesian lift. Let $f : a \to b$ be a map in $A$, with given lift $\tilde{b}$ in $D$. Then $i(f) : i(a) \to i(b)$ is a map in $X$ with given lift $j(\tilde{b})$. By assumption $RFib(C)$ is a right fibration and so there exists a $p$-Cartesian map $\tilde{f} : \tilde{a} \to j(\tilde{b})$ such that $p(\tilde{f}) = f$ and $\tilde{f}$ is $p$-Cartesian.
Taking $s$ we get a map $s(\tilde{f}) : s(\tilde{a}) \to sj(\tilde{b}) = \tilde{b}$ which is a lift of $f : a \to b$ in $A$. All that is left is to show that this lift is $q$-Cartesian. However, that follows directly from the fact that the retract of pullback square is itself a pullback square. □

We have the following important and technical Lemma about $RFib_X(C)$.

**Lemma 7.37.** The map $RFib_X(C)_1 \to X_1 \times_{X_0} RFib_X(C)_0$ is a $(-1)$-truncated map of spaces.

**Proof.** We will prove it by showing that the homotopy fiber is either empty or contractible. In Lemma 7.35 we already showed the map is a Reedy fibration, which in particular implies that $RFib_X(C)_1 \to X_1 \times_{X_0} RFib_X(C)_0$ is a Kan fibration of spaces and so the fiber is already the homotopy fiber.

If the fiber is empty then we are done. Thus we will show that if the fiber is not empty then it is contractible. First we note that both sides are Kan fibrations over $RFib_X(C)_0 = C_0$. Thus we can pullback the triangle below

\[
\begin{array}{ccc}
RFib_X(C)_1 & \to & X_1 \times_{X_0} C_0 \\
\downarrow & & \downarrow \\
C_0 & \to & C_0
\end{array}
\]

along any map $\tilde{y} : \Delta[0] \to C_0$, to a map

\[
RFib_X(C)_1 \times_{C_0} ^\tilde{y} \Delta[0] \to X_1 \times_{X_0} ^y \Delta[0]
\]

Thus it suffices to prove that this Kan fibration has contractible fibers.

We fix a point in the codomain $\Delta[0] \to X_1 \times_{X_0} \Delta[0]$, which is a morphism $f : x \to y$. If we assume that the fiber is non-empty then there exists a lift $\tilde{f} : \tilde{x} \to \tilde{y}$ such that $p(\tilde{f}) = f$ and $\tilde{f}$ is $p$-Cartesian. Because $\tilde{f}$ is $p$-Cartesian the following square is a homotopy pullback square of spaces.

\[
\begin{array}{ccc}
RFib_X(C)_2 \times_{RFib_X(C)_1} ^\tilde{f} \Delta[0] & \to & (RFib_X(C)_1 \times_{RFib_X(C)_0} ^\tilde{y} \Delta[0]) \\
\downarrow & & \downarrow \\
X_2 \times_{X_1} ^\tilde{f} \Delta[0] & \to & X_1 \times_{X_0} ^y \Delta[0]
\end{array}
\]

Indeed, we get this pullback diagram of spaces if we restrict the pullback diagram in Definition 7.28 to level 0 and use the fact that a homotopy pullback square in the Reedy model structure is just a level-wise homotopy pullback square in the Kan model structure. Moreover, the map
$\Delta[0] \to X_1 \times^{\bar{f}}_{X_0} \Delta[0]$ factors through $X_2 \times^{f}_{X_1} \Delta[0]$ as it is just the identity cone. Thus we can take the fiber of the map

$$RFib_X(C)_2 \times^{\bar{f}}_{RFib_X(C)_1} \Delta[0] \to X_2 \times^{f}_{X_1} \Delta[0]$$

along the point in $X_2 \times^{f}_{X_1} \Delta[0]$, that is the degenerate 2-simplex of $f$ in $X_2$.

The vertex $\bar{x}$ is initial in the diagram $\bar{f}$ so, by [Ra17, Corollary 5.21], we know that in the diagram

$$\begin{array}{ccc}
RFib_X(C)_2 \times^{\bar{f}}_{RFib_X(C)_1} \Delta[0] & \xrightarrow{\simeq} & RFib_X(C)_1 \times^{\bar{x}}_{RFib_X(C)_0} \Delta[0] \\
X_2 \times^{f}_{X_1} \Delta[0] & \xrightarrow{\simeq} & X_1 \times^{x}_{X_0} \Delta[0]
\end{array}$$

the vertical arrows are Kan equivalences. By the homotopy invariance of homotopy pullbacks it thus suffices to look at the fiber of the map

$$RFib_X(C)_1 \times^{\bar{x}}_{RFib_X(C)_0} \Delta[0] \to X_1 \times^{x}_{X_0} \Delta[0]$$

over the identity map $id_x : x \to x$. However, by Example 7.30, the $p$-Cartesian lifts of the identity are always equivalences. Thus the fiber is the space $Equiv_{/\bar{x}} = Choequiv \times^{\bar{x}}_{C_0} \Delta[0]$.

But this space is always contractible as it is the pullback of the map $t : Choequiv \to C_0$, which is a homotopy equivalence by the completeness condition. Hence we are done. □

This has following obvious corollary

**Corollary 7.38.** Let $f$ be a map in $X$. Then every two $p$-Cartesian lifts of $f$ are equivalent.

The technical lemma gives us following valuable proposition.

**Proposition 7.39.** The following are equivalent:

1. The map $RFib_X(C) \to X$ is a right fibration.
2. For every morphism $f : x \to y$ in $X$ and given lift $\tilde{y}$ in $C$ there exists a $p$-Cartesian morphism in $C$, $\tilde{f} : \tilde{x} \to \tilde{y}$ such that $p(\tilde{f}) = f$.

**Proof.** In Lemma 7.35 we already showed the map is a Reedy fibration. By the same Lemma 7.35, $RFib_X(C)$ is a Segal space. Also, $X$ is a Segal space by assumption. Thus $RFib_X(C) \to X$ is a right fibration if and only if the map

$$RFib_X(C) \to X_1 \times^{C_0}_{X_0}$$
is a trivial Kan fibration. By Lemma 7.37 we know the map is always $(-1)$-truncated. Thus the map is a trivial Kan fibration if and only if it is surjective, which is exactly the condition stated above.

This result gives us a nice second way to classify right fibrations over a CSS.

**Corollary 7.40.** Let $q : R \to X$ be a CSS fibration over the CSS $X$. Then $q$ is a right fibration if and only if every arrow has a $p$-Cartesian lift and every lift is $p$-Cartesian.

**Proof.** If $R \to X$ is a right fibration then for any map $f$ in $R$ the condition

$$\begin{array}{ccc}
R/f & \rightarrow & R/y \\
\downarrow & \searrow & \downarrow \\
X_{p(f)} & \rightarrow & X_{p(y)}
\end{array}$$

is already satisfied. On other side, if every morphism has a lift then $RFib_X(R)$ is a right fibration. However, if every lift is itself $p$-Cartesian then $RFib_X(R) = R$ and so $R$ is a right fibration over $X$.

We can almost prove our main theorem, but need two more lemmas.

**Lemma 7.41.** Let $p : C \to X$ be a CSS fibration. If $R \to C$ is a subsimplicial space such that $R \to X$ is a right fibration then the map $R \to RFib_X(C)$ will factor through $R \to RFib_X(C) \to C$. Moreover, if $R_0 = C_0$ then $RFib_X(C) \to X$ is a right fibration.

**Proof.** $R \to X$ is a right fibration and so every map is $p$-Cartesian. Thus it must land in $RFib_X(C)$. Moreover, if $R_0 = C_0$, then for every map $f : x \to y$ in $X$ and object $\tilde{y}$ in $C$, the object will also be in $R$. This implies that it has a $p$-Cartesian lift, $\tilde{f}$ in $R$, which is then also in $RFib_X(C)$. Thus $RFib(C)$ is a right fibration and we are done.

**Lemma 7.42.** Let $p : C \to X$ be a CSS fibration such that $RFib_X(C)$ is a right fibration. Then the map $C^{F(n)} \to X^{F(n)}$ is also a CSS fibration and $RFib_{X^{F(n)}}(C^{F(n)}) \to X^{F(n)}$ is a right fibration.

**Proof.** First note that it suffices to prove the statement above for $n = 1$. Indeed, $F(n)$ is a retract of $F(1)^n$, which gives us following retract diagram

$$\begin{array}{ccc}
C^{F(n)} & \rightarrow & C^{F(1)^n} \\
\downarrow & & \downarrow \\
X^{F(n)} & \rightarrow & X^{F(1)^n}
\end{array} \hspace{1cm} \begin{array}{ccc}
C^{F(n)} & \rightarrow & C^{F(1)^n} \\
\downarrow & & \downarrow \\
X^{F(n)} & \rightarrow & X^{F(1)^n}
\end{array}$$
thus if we know that $C^{F(1)^n} \to X^{F(1)^n}$ satisfies the conditions stated in the lemma then so does $C^{F(n)} \to X^{F(n)}$. However, this follows from $C^{F(1)} \to X^{F(1)}$ satisfying the two conditions of our lemma. So we will prove those.

First, it is clear that the map is a CSS fibration as the CSS model structure is compatible with Cartesian closure. Thus all that is left is to show that for any map $\sigma : F(1) \to X^{F(1)}$ and initial condition $\tilde{f} : F(0) \to C^{F(1)}$ there is a Cartesian lift. We can think of $\sigma$ as a map $\sigma : F(1) \times F(1) \to X$ and depict this map as follows:

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
| & & | \\
\downarrow{g} & \downarrow{f} & \\
B & \xleftarrow{b} & Y \\
\end{array}
\]

and our given lift $\tilde{f} : F(1) \to C$ is a map

\[
\begin{array}{c}
\tilde{X} \\
\downarrow{\tilde{f}} \\
\tilde{Y}
\end{array}
\]

Our goal is it to lift it to a Cartesian square. First we use the fact that $a$ and $b$ have $p$-Cartesian lifts to build following diagram:

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{\tilde{a}} & \tilde{X} \\
| & \downarrow{\tilde{f}\tilde{a}} & | \\
\tilde{B} & \xrightarrow{\tilde{b}} & \tilde{Y}
\end{array}
\]

The last step is to find a map from $\tilde{A}$ to $\tilde{B}$. For that we use the fact that $\tilde{b}$ is $p$-Cartesian, which gives us the trivial Reedy fibration

\[
C_{/\tilde{b}} \to C_{/\tilde{Y}} \times_{X_{/\tilde{Y}}} X_{/\tilde{b}}
\]

On the right hand side we have the element $(\tilde{f}\tilde{a}, g)$. As the map is a trivial Reedy fibration this element has a lift $\tilde{g} \in C_{/\tilde{b}}$, which gives us the next diagram.
We name this whole square $\tilde{\sigma}$ as it lifts $\sigma$. We have to show that $\tilde{\sigma}$ is Cartesian in $C^F(1)$. In order to do that we have to show that the following diagram is a homotopy pullback square in the Reedy model structure

\[ \begin{array}{ccc}
C^F(1)_{/\tilde{\sigma}} & \longrightarrow & C^F(1)_{/f} \\
\downarrow & & \downarrow \\
X^F(1)_{/\sigma} & \longrightarrow & X^F(1)_{/f}
\end{array} \]

Concretely we have to show that if we have a diagram in $X$ of the form

\[ \begin{array}{ccc}
V & \xrightarrow{c} & A & \xrightarrow{a} & X \\
\downarrow b & & \downarrow g & & \downarrow f \\
W & \xrightarrow{d} & B & \xrightarrow{b} & Y
\end{array} \]

then we can lift it to in a unique manner to a diagram in $C$.

First we use the fact that $\tilde{b}$ is $p$-Cartesian to lift the map $d$ uniquely to a map $\tilde{d}$. Similarly we use the fact that $\tilde{a}$ is $p$-Cartesian to lift the map $c$ uniquely to a map $\tilde{c}$. This gives us the diagram

\[ \begin{array}{ccc}
\tilde{V} & \xrightarrow{\tilde{c}} & \tilde{A} & \xrightarrow{\tilde{a}} & \tilde{X} \\
\downarrow \tilde{g} & & \downarrow \tilde{f} & & \downarrow f \\
\tilde{W} & \xrightarrow{\tilde{d}} & \tilde{B} & \xrightarrow{\tilde{b}} & \tilde{Y}
\end{array} \]

Next we use the fact that $\tilde{b}$ is a $p$-Cartesian to lift the map $h$ uniquely to a map $\tilde{h}$. This gives us a complete lift.
which shows that that $\tilde{\sigma}$ is $p$-Cartesian.

We are now in a position to prove our new classification of Cartesian fibrations between CSS.

**Theorem 7.43.** A map $p : C \to X$ is a Cartesian fibration if and only if $RFib_X(C) \to X$ is a right fibration.

**Proof.** Let us assume $C$ is a Cartesian fibration. Then there exists a bisimplicial space $\tilde{p} : R \to X$ that is a Cartesian fibration, and such that $\Delta Diag^*(R) = C$. We prove $RFib_X(C)$ is a right fibration by showing the conditions in Lemma 7.41 are satisfied. First of all, the $0$-level of the bisimplicial space, $\tilde{p}_0 : R_0 \to X$, is a right fibration over $X$. Moreover, for every $n$ we have an inclusion map $R_0^n \to R_n = \Delta Diag^*(R)_n = C_n$. Finally, $R_0 = C_0$. Thus all the necessary conditions are satisfied and $RFib_X(C)$ is a right fibration.

For the other way around, we use Lemma 7.42. Let $C \to X$ be a CSS fibration such that $RFib_X(C) \to X$ is a right fibration. Then by the previous lemma, the map $RFib_X(C)(F(n)) \to X^F(n)$ is a right fibration as well. With this in mind we define the bisimplicial space $S$ as $S_k = RFib_X(C)(F(n))$. Then $S$ comes with a natural map $S_k \to X^{F(k)}$. So we define the bisimplicial space $R$ as

$$R_k = S_k \times_{X^{F(k)}} X.$$

We show that $R$ is a Cartesian fibration over $X$. First, $R_k \to X$ is a right fibration as it is the pullback of the right fibration $RFib_X(C)(F(n)) \to X^F(n)$. Next recall that

$$RFib_X(C)(F(n))_0 = (C^{F(n)})_0 = C_n.$$

Thus $i^*_R(R)_k = C_k \times_{X_k} X_0$, which means that $i^*_R(R) = C \times_X X_0$. But the map $C \to X$ is a CSS fibration which means $i^*_R(R) \to X_0$ is a CSS fibration. But, $X_0$ is a CSS, which implies that $i^*_R(R)$ is a CSS as well. This is one of the conditions for being a Cartesian fibration in Corollary 7.18 and hence we are done.

We can use the internal characterization to prove this corollary.

**Corollary 7.44.** The composition of two Cartesian fibrations is a Cartesian fibration.

**Proof.** Let $X$ be a simplicial space. Moreover, let $p : R \to X$ and $q : S \to R$ be two Cartesian fibrations. We can find a CSS fibrant replacement for this chain to get the following diagram
where each vertical arrow is a CSS equivalence. By Corollary 7.23, \( \hat{p} \) and \( \hat{q} \) are also Cartesian fibrations. Moreover, by the same Corollary, \( pq \) is a Cartesian fibration if and only if \( \hat{p}\hat{q} \) is one. Thus it suffices to prove the case when the base is a CSS.

Thus, henceforth we assume \( X \) is a CSS. Recall that this implies that \( R \) and \( S \) are also CSS. So, we can now use Theorem 7.43 to show that \( pq \) is also a Cartesian fibration. \( pq \) is a CSS fibration as we already know the composition of CSS fibrations is a CSS fibration. Let \( f : x \to y \) be a map in \( X \) and \( \tilde{y} \) a chosen lift of \( y \) in \( S \) \( (q(\tilde{y}) = y) \). Then, because \( p \) is a Cartesian fibration, there exists a \( p \)-Cartesian map \( f' : x' \to q(\tilde{y}) \) in \( R \). But \( q \) is also a Cartesian fibration, so there exists a \( q \)-Cartesian map \( \tilde{f} : \tilde{x} \to \tilde{y} \) in \( S \). This gives us following diagram

where both squares are pullback squares as \( f' \) is \( p \)-Cartesian and \( \tilde{f} \) is \( q \)-Cartesian. Thus the whole rectangle is a pullback square. This implies that \( \tilde{f} \) is \( pq \)-Cartesian over \( X \).

Given this result we also have locality condition.

**Theorem 7.45.** A CSS fibration \( p : C \to X \) over a CSS \( X \) is a Cartesian fibration if and only if for each map \( f : F(1) \to X \) the map \( p_f : C \times_X F(1) \to F(1) \) is a Cartesian fibration and every \( p_f \)-Cartesian map is \( p \)-Cartesian.

**Proof.** For one side clearly the pullback of every Cartesian fibration is a Cartesian fibration. On the other side, let \( f : x \to y \) be a map in \( X \) and \( \tilde{y} \) be a choice of lift of \( y \). Pulling back along
\( f : F(1) \to X \) we get a Cartesian fibration \( p_f : C \times_X F(1) \to F(1) \). Using the fact that the map is \( p_f \) is Cartesian we get a \( p_f \)-Cartesian map \( \hat{f} : \hat{x} \to \hat{y} \), which by assumption is \( p \)-Cartesian.

**Example 7.46.** Let \( p : X_{/x} \to X \) be a representable Reedy right fibration corresponding to the simplicial object \( x_\bullet : \Delta^{op} \to X \) that is also a Cartesian fibration. What are the \( p \)-Cartesian morphisms? In this case \( RFib_X(X_{/x}) = X_{/x_0} \) and so a \( p \)-Cartesian arrow is a map \( x \to x_1 \) that factors through \( x \to x_0 \). Thus in particular every map \( f : x \to x' \) in \( X \) with given lift \( g : x' \to x_0 \), lifts to a morphism \( x \to x' \xrightarrow{g \circ f} x_1 \).

### 7.4 Adjunctions

**Definition 7.47.** An adjunction is a map \( p : A \to F(1) \) that is a Cartesian and coCartesian fibration.

**Remark 7.48.** To get a more familiar picture, we denote the fiber of \( A \) over 0 as \( C \) and the fiber over 1 as \( D \). Then the fact that \( p \) is coCartesian gives us a map \( f : C \to D \) and the fact that it is Cartesian gives us a map \( g : D \to C \). The remaining data in \( A \) gives us the desired adjunction between \( f, g \).

**Notation 7.49.** From now on we will denote a choice of coCartesian lift of a point \( c \) in \( C \) as \( f(c) : c \to f(c) \). Similarly a Cartesian lift of \( d \in D \) will be denoted by \( g(d) : d \to d \). As always the choice of lift is not unique, but rather there is a contractible space of choices.

Before we prove that this definition relates to more familiar forms of adjunction, let us first clarify the remark above and get a better understanding of \( A \).

**Lemma 7.50.** Let \( c \) be an object in \( C \) and \( d \) be an object in \( D \). Then there is an equivalence

\[
map_A(c, d) \simeq map_D(f(c), d)
\]

**Proof.** We have following zig-zag of equivalences:

\[
map_A(c, d) \xrightarrow{\simeq} map_A(c, f(c), d) \times_{map_A(c, f(c))} \Delta[0] \xrightarrow{\simeq} map_A(f(c), d) = map_D(f(c), d)
\]

where the first equivalence follows from the fact that \( c \to f(c) \) is coCartesian.

**Remark 7.51.** Notice, the lemma above also has a contravariant version that gives us an equivalence

\[
map_A(c, d) \simeq map_C(c, g(d))
\]

We can put this information together to get the following diagram.
This is gives us a zig-zag of equivalences between $\text{map}_{\mathcal{D}}(f c, d)$ and $\text{map}_{\mathcal{C}}(c, g d)$, which is exactly what we expected from our notion of adjunction.

**Remark 7.52.** We can depict an object in $\text{map}_{\mathcal{A} \mathcal{F}(1)}(c \to fc, gd \to d)$ as a square

\[
\begin{array}{ccc}
  c & \xrightarrow{f} & fc \\
  \downarrow & & \downarrow \\
  gd & \xrightarrow{g} & d
\end{array}
\]

where the horizontal arrows are $f$ and $g$. The statement above is saying that any of the three vertical and diagonal maps determines the other two by using the fact that $f : c \to fc$ is coCartesian and $g : gd \to d$ is Cartesian.

Let us study some equivalent conditions for adjunctions.

**Definition 7.53.** Let $\mathcal{A} \to \mathcal{F}(1)$ be a Segal fibration. We define the *mapping Segal space* as

\[ M = F(0) \times_{F(1)^{\mathcal{F}(1)}} \mathcal{A}^{\mathcal{F}(1)} \]

$M$ has a natural Segal fibrations to $\mathcal{C}^{\text{op}} \times \mathcal{D}$ corresponding to the restriction $\mathcal{A}^{\mathcal{F}(1)} \to \mathcal{A} \times \mathcal{A}$.

**Theorem 7.54.** The map $M \to \mathcal{C}^{\text{op}} \times \mathcal{D}$ is a left fibration.

**Proof.** We already know it is a Reedy fibration between Segal spaces. Thus it suffices to show that
is a homotopy pullback square. In order words, we want to show that
\[ M_1 \rightarrow \left( C_1^{\text{op}} \times D_1 \right) \] is a trivial fibration. In order to prove that it suffices to show that each fiber is contractible.

A point on the right hand side is a choice of a map \( f : c_1 \rightarrow c_2 \) in \( C \), a map \( g : d_1 \rightarrow d_2 \) in \( D \) and a map \( h : c_2 \rightarrow d_1 \) in \( A \). The fiber over such point is the subspace of \( \text{Map}(F(1) \times F(1), A) \) of the form.

![Diagram](https://example.com/diagram.png)

However, this space is contractible by the Segal condition.

\[ \square \]

**Remark 7.55.** If \( A \rightarrow F(1) \) is a coCartesian fibration, then \( M \) models the functor that takes a point \((c, d)\) to \( \text{map}_D(f c, d) \), where \( f : C \rightarrow D \) is the map classified by the coCartesian fibration \( A \rightarrow F(1) \). Similarly, if \( A \rightarrow F(1) \) is a Cartesian fibration then \( M \) models the functor with value \( \text{map}_E(c, g d) \), where \( g : D \rightarrow E \), is the map modeled by \( A \rightarrow F(1) \).

This remark suggests following theorem.

**Theorem 7.56.** Let \( A \rightarrow F(1) \) be a coCartesian fibration. Then \( A \rightarrow F(1) \) is a Cartesian fibration if and only if for every \( d \) in \( D \), the left fibration \( M_d \) defined as the pullback

\[ \begin{array}{ccc}
M_d & \rightarrow & M \\
\downarrow & & \downarrow \\
C^{\text{op}} & \rightarrow & C^{\text{op}} \times D
\end{array} \]

is representable.

**Proof.** If \( A \rightarrow F(1) \) is a Cartesian fibration, then \( M_d \) has fiber \( \text{map}_E(c, g d) \) and so is represented by the object \( g d \). On the other hand let us assume each \( M_d \) is representable. Moreover, note that we have a map \( f : C \rightarrow D \) corresponding to the coCartesian fibration \( A \rightarrow F(1) \). Recall \( M_d \) models the left fibration which takes \( c \) to \( \text{map}_D(f c, d) \). Fix an object \( d \) in \( D \), we need to show it has a Cartesian lift.

As \( M_d \) is representable there exists an object \( c \) and a final morphism in \( \text{Map}_D(f c, d) \). However, the morphisms \( c \rightarrow f c \) in \( A \) is coCartesian and so there is an equivalence \( \text{map}_A(c, d) \simeq \text{map}_D(f c, d) \) This means there is a map \( g : c \rightarrow d \) in \( A \). Clearly this is a lift of \( d \). We will prove this lift is Cartesian. For that it suffices to prove that for each object \( c' \in C \) there is an equivalence
between \( \text{map}_C(c', c) \) and \( \text{map}_A(fc', d) \). However, this is true as \( M_d \) is represented by \( c \) and so Reedy equivalent to \( (\mathcal{E})^\text{op}_{c/} \).

The CSS \( M_d \) can be described in a more simple language, which leads to following theorem.

**Theorem 7.57.** Let \( f : \mathcal{C} \to \mathcal{D} \) be a functor of CSS. Then \( f \) is a left adjoint if and only if for each object \( d \) in \( \mathcal{D} \) the CSS \( \mathcal{C}/_d \) defined by the pullback

\[
\begin{array}{ccc}
\mathcal{C}/_d & \to & \mathcal{D}/_d \\
\downarrow \pi_d & & \downarrow \pi_d \\
\mathcal{C} & \to & \mathcal{D}
\end{array}
\]

has a final object.

**Proof.** It suffices to prove that \( \mathcal{C}/_d \) is equivalent to \( M_d \) and the result follows. \( M_d \) is a left fibration over \( \mathcal{C}^\text{op} \), which means it is right fibration over \( \mathcal{C} \). But \( \mathcal{C}/_d \) is also a right fibration over \( \mathcal{C} \) and thus to show they are the same CSS it suffices to compare them fiber wise. However, for a fixed object \( c \), they both have the value \( \text{map}_D(fc, d) \).

**Remark 7.58.** The same argument also holds for left adjoints. Thus a map \( g : \mathcal{D} \to \mathcal{C} \) has a left adjoint if and only if the CSS \( \mathcal{D}/_c \) has an initial object.

**Remark 7.59.** This proof also show that a right adjoint is homotopically unique. Concretely, if \( f \) is a left adjoint, then we can use the previous theorem to construct a coCartesian and Cartesian fibration \( A \to F(1) \).

We can use this definition to recover classical results from category theory.

The definitions given up to here are specific to CSS. We will give definition that holds for Segal spaces.

**Definition 7.60.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be two Segal spaces. Fix a section \( \text{sec} : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_2 \) of the trivial fibration \( \varphi_2 \). \( f : \mathcal{C} \to \mathcal{D} \) and \( g : \mathcal{D} \to \mathcal{C} \) be maps of Segal spaces. A unit map is a map of Segal spaces \( \mu : \mathcal{C} \times F(1) \to \mathcal{C} \) such that \( \mu_0 = \text{id}_\mathcal{C} \) and \( \mu_1 = gf \) and it satisfies following condition. For each \( c \) in \( \mathcal{C} \) and \( d \) in \( \mathcal{D} \) the composition of the following chain of maps is an equivalence of spaces.

\[
\begin{align*}
\text{map}_D(fc, d) & \xrightarrow{g} \text{map}_D(gfc, gd) \xrightarrow{\text{sec}} \text{map}_D(c, gfc, gd) \times_{\text{map}_D(c, gfc)} \Delta[0] & \xrightarrow{d_1} \text{map}_D(c, gd)
\end{align*}
\]

**Definition 7.61.** We say two maps \( f : \mathcal{C} \to \mathcal{D} \) and \( g : \mathcal{D} \to \mathcal{C} \) are unit-adjoint, if there is a unit map \( \mu : \mathcal{C} \times F(1) \to \mathcal{C} \).

Let us show for the specific case of CSS this definition using the unit map agrees with the previous definitions. But first we need an interesting lemma.
Lemma 7.62. Let $A \to F(1)$ be a coCartesian fibration and $M = F(0) \times_{F(1)} F(1)^{A^{F(1)}}$. Moreover, let $M^{\text{coCart}}$ be the sub category of $M$ generated by coCartesian morphisms in $A$. Then induced map $M^{\text{coCart}} \to M \to \mathcal{C}^{op}$ is an equivalence of CSS.

**Proof.** First notice $M^{\text{coCart}} \to \mathcal{C}^{op}$ is also a CSS fibration as it is still subcategory generated by objects that are still surjective on $\mathcal{C}$. Moreover, $M^{\text{coCart}} \to \mathcal{C}^{op}$ is a coCartesian fibration as every map has a coCartesian lift. Thus it suffices to show that each fiber is contractible to show that the map is an equivalence. But the fiber of each point $c$ is the space of coCartesian lifts of $c$ and thus is contractible (by Lemma 7.37).

**Remark 7.63.** Notice the same holds for $M^{\text{Cart}}$ for a Cartesian fibration $A \to F(1)$. In that case the map $M^{\text{Cart}} \to \mathcal{D}$ is an equivalence.

**Theorem 7.64.** Two maps of CSS $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{E}$ are adjoint if and only if there are unit-adjoint.

**Proof.** Assume that $f$ and $g$ are adjoint. Thus there is a coCartesian and Cartesian fibration $A \to F(1)$. We have following diagram

\[
\begin{array}{cccccc}
M^{\text{coCart}} \times \mathcal{D} & \to & A^{F(1)} \times_{A} A^{F(1)} & \to & A^{F(2)} & \to & A^{F(1)} \\
\approx & & \approx & & d_1 & & \\
\mathcal{C} & \to & \mathcal{C}^{F(1)} & & & &
\end{array}
\]

The map from $\mathcal{C}$ to $A^{F(1)}$ takes each point $c$ to the map $c \to gfc$ and thus factor through $\mathcal{C}^{F(1)}$. This gives us a map $\mathcal{C} \to \mathcal{C}^{F(1)}$. The fact that it satisfies the unit condition follows from the fact that $c \to fc$ is a coCartesian and $gfc \to fc$ is Cartesian.

On the other side, we have to show that if we have a unit map $\mu : \mathcal{C} \times F(1) \to \mathcal{C}$ that satisfies the stated condition, then $g$ is a right-adjoint. The fact that $f$ is a left adjoint then follows from the uniqueness of the left adjoint. In order to show that $g$ is a right adjoint we have to show that the category $\mathcal{D}_{c/}$ has a final object for any object $c$ in $\mathcal{C}$. By the unit condition we have a map $c \to gfc$ which is an object in $\mathcal{D}_{c/}$. This induces a map of left fibrations $\mathcal{D}_{fc/} \to \mathcal{D}_{c/}$. We will show this map is an equivalence. As both are left fibrations it suffices to do so fiber-wise. For each point $d$ the map of fibers is

\[map_D(fc, d) \to map_C(c, gd)\]

which is an equivalence by the unit-map condition. Thus $\mathcal{D}_{c/}$ has a final object.

We can use adjunctions to study limits and colimits.

**Theorem 7.65.** Let $I$ be a simplicial space and $\mathcal{C}$ be a CSS and let $\Delta_I : \mathcal{C} \to \mathcal{C}^I$ be the natural inclusion induced by the map $I \to F(0)$. Then $\Delta_I$ has a left adjoint if and only if each map $f : I \to \mathcal{C}$ has a colimit and has a right adjoint if each map $f : I \to \mathcal{C}$ has a limit.
Proof. \( \Delta_I : \mathcal{C} \rightarrow \mathcal{C}^I \) has a right adjoint if and only if for each map \( h : I \rightarrow \mathcal{C} \) the pullback \( \mathcal{C} \times_{\mathcal{C}^I} (\mathcal{C}^I)/h \) has a final object. However, this is just the category of cones over \( f \), namely, \( \mathcal{C}/h \), which by definition means \( h \) has a limit. The case for colimits follows similarly. \( \square \)

7.5 Path Fibrations. In this subsection we want to discuss a specific example of a Cartesian fibrations when the base is a complete Segal space. For this subsection \( \mathcal{C} \) is a fixed complete Segal space.

We want to study the map \( t : \mathcal{C}^{F(1)} \rightarrow \mathcal{C} \). As the base is a complete Segal space, we will use the approach we developed in Subsection 7.3. In particular, we want to analyze \( RFib_{\mathcal{C}}(\mathcal{C}^{F(1)}) \). For that we have to determine what kind of arrow in \( \mathcal{C}^{F(1)} \) is even \( t \)-Cartesian. An arrow in \( \mathcal{C}^{F(1)} \) is itself a map \( \sigma : F(1) \times F(1) \rightarrow \mathcal{C} \). In order to simplify things, let us depict \( \sigma \) as follows.

\[
\begin{array}{ccc}
  w & \xrightarrow{g} & z \\
  \downarrow & & \downarrow \\
  x & \xrightarrow{f} & y
\end{array}
\]

With those naming conventions, \( \sigma \) is \( t \)-Cartesian if the square

\[
\begin{array}{ccc}
  \mathcal{C}_{/\sigma}^{F(1)} & \xrightarrow{r} & \mathcal{C}_{/g}^{F(1)} \\
  \downarrow & & \downarrow \\
  \mathcal{C}_{/f} & \rightarrow & \mathcal{C}_{/y}
\end{array}
\]

is a homotopy pullback square. But this condition is just saying that \( \sigma \) is a pullback square in the CSS \( \mathcal{C} \). Thus the existence of a Cartesian lift corresponds to the existence of pullbacks. Thus, according to Theorem 7.43, we have just proven following result.

**Proposition 7.66.** The map \( t : \mathcal{C}^{F(1)} \rightarrow \mathcal{C} \) is a Cartesian fibration if and only if \( \mathcal{C} \) has pullbacks.

**Remark 7.67.** It is interesting to notice what presheaf this Cartesian fibration is modeling. Intuitively it gives us a functor which takes each point \( x \) to the over CSS \( \mathcal{C}/x \) and each map \( f : x \rightarrow y \) to the pullback map \( f^* : \mathcal{C}/y \rightarrow \mathcal{C}/x \) (here is exactly where the existence of pullbacks is important on the functorial side).

The right fibration \( RFib_{\mathcal{C}}(\mathcal{C}^{F(1)}) \rightarrow \mathcal{C} \) is very important and thus merits its own definition.

**Definition 7.68.** For each CSS \( \mathcal{C} \) with pullbacks, we define the right fibration \( O_{\mathcal{C}} = RFib_{\mathcal{C}}(\mathcal{C}^{F(1)}) \).

So, let us assume \( \mathcal{C} \) is a CSS with all finite limits. Knowing that \( t : \mathcal{C}^{F(1)} \rightarrow \mathcal{C} \) is now a Cartesian fibration, we can actually find the bisimplicial spaces that corresponded to this map. Following the construction in Theorem 7.43 we have to find \( RFib_{\mathcal{C}^{F(n)}}(\mathcal{C}^{F(n)} \times F(1)) \) over \( \mathcal{C}^{F(n)} \).

By our proposition \( RFib_{\mathcal{C}}(\mathcal{C}^{F(1)}) \rightarrow \mathcal{C} \) is the sub-CSS of \( \mathcal{C}^{F(1)} \) which has the same objects, but where all morphisms are pullback squares instead of just commutative squares. The same applies
for the higher analogues. A map in \textit{RFib}_{\mathcal{C}(n)}(\mathcal{C}^{F(n)}\times F(1)) is just a map \(F(n)\times F(1)\times F(1)\to \mathcal{C}\) such that for every map \(F(0)\to F(n)\) the restriction is a pullback square. This way we can build a bisimplicial space that is a Cartesian fibration over \(\mathcal{C}\) and corresponds to \(t: \mathcal{C}^{F(1)}\to \mathcal{C}\)

We want to use this fibration to study certain adjunctions. First we have following fact.

\textbf{Lemma 7.69.} The map \(t: \mathcal{C}^{F(1)}\to \mathcal{C}\) is a coCartesian fibration.

\textit{Proof.} We need to show that for every map \(f: x\to y\) in \(\mathcal{C}\) and chosen lift \(\hat{x}\to x\) we have a coCartesian square. However, the lift is just the degenerate square

\[
\begin{array}{ccc}
\hat{x} & \xrightarrow{f} & \hat{x} \\
\downarrow & & \downarrow \\
x & \xrightarrow{f} & y
\end{array}
\]

Thus \(t\) is coCartesian fibration. \qed

Recall that the pullback of every Cartesian and coCartesian fibration is always Cartesian and coCartesian. Thus we get following corollary.

\textbf{Corollary 7.70.} Let \(\mathcal{C}\) have finite limits and \(f: x\to y\) be a map in \(\mathcal{C}\). Then we get an adjunction \(\mathcal{C}^{F(1)}\times e F(1)\).

\textit{Remark 7.71.} This adjunction is modeling the more familiar adjunction

\[
\begin{array}{ccc}
\mathcal{C}_{/x} & \xleftarrow{f_!} & \mathcal{C}_{/y} \\
\xrightarrow{f^*} & & \\
\end{array}
\]

where \(f_!\) is post composition and \(f^*\) is pullback by \(f\).

Thus the left adjoint of \(f^*\) is \(f_!\). But the map can also have a right adjoint. This is when the discussion of Cartesian closure comes in.

\textbf{Definition 7.72.} A CSS \(\mathcal{C}\) is Cartesian closed if for each object \(x\) the composition map

\[
\mathcal{C} \xrightarrow{f_{ix}^*} \mathcal{C}_{/x} \xrightarrow{(f_{ix})_!} \mathcal{C}
\]

is a left adjoint. There \(f_{ix}: x\to \ast\) is a map to the final object.

Using Theorem \ref{7.57} we get following result.

\textbf{Corollary 7.73.} A CSS \(\mathcal{C}\) is Cartesian closed if for every object \(x\) and \(y\) the CSS with object \(x\times z\to y\) and maps \(id_x\times f: x\times z_1\to z\times z_2\) over \(y\) has a final object. This final object is often depicted as \(ev: x\times y^\ast\to y\).

We can generalize Cartesian closure to a relative case.
Definition 7.74. A CSS $\mathcal{C}$ is locally Cartesian closed if it has finite limits and for each object $x$ the over-category $\mathcal{C}_{/x}$ is Cartesian closed.

There are other ways to determine local Cartesian closure.

Theorem 7.75. A CSS $\mathcal{C}$ is locally Cartesian closed if for each map $f : x \to y$ the induced map $f^* : \mathcal{C}_{/y} \to \mathcal{C}_{/x}$ has a right adjoint.

Proof. By definition $\mathcal{C}$ is locally Cartesian closed if and only if for each map $f : x \to y$ the map
\[ \begin{array}{ccc} \mathcal{C}_{/y} & \xrightarrow{f^*} & \mathcal{C}_{/x} \\ \xrightarrow{f} & & \xrightarrow{f} \\ & & \mathcal{C}_{/y} \end{array} \]
has a right adjoint. But the right hand map has always a right adjoint and thus the composition has a right adjoint if and only if the left hand map has a right adjoint. \qed

7.6 Right Fibrations of Bisimplicial Spaces. The same way we can localize simplicial spaces to the diagonal Kan model structure on simplicial spaces, which is Quillen equivalent to the Kan model structure, we can localize bisimplicial spaces to the diagonal Reedy model structure. Taking the functorial approach gives us following definition.

Definition 7.76. We say a Reedy right fibration $R \to X$ is a right fibration if for every map $\varphi_k \to X$ the induced map
\[ k^* : \text{Map}_{/X}(\varphi_k, R) \to \text{Map}_{/X}(\varphi_0, R) \]
is Kan equivalence of spaces. Here $k : \varphi_0 \to \varphi_k$ is the map that takes the point to the final vertex in $\varphi_k$.

Remark 7.77. The definition above is equivalent to saying that there exist a simplicial space $S \to X$ over $X$ such that $i^*_R(S)$ is biReedy equivalent $R$. Thus we can think of right fibrations of bisimplicial spaces as the essential image of right fibrations under the map $i^*_R$. For a more detailed analysis see Theorem 7.79.

As always this comes with a model structure.

Theorem 7.78. There is a unique model structure on bisimplicial spaces over $X$, called the contravariant model structure and denoted by $(\text{ss}S_{/X})^{contra}$ such that

1. It is a simplicial model category.
2. The fibrant objects are the right fibration over $X$.
3. Cofibrations are monomorphisms.
4. A map $A \to B$ over $X$ is a weak equivalence if $\text{map}_{\text{ss}S_{/X}}(B, W) \to \text{map}_{\text{ss}S_{/X}}(A, W)$ is an equivalence for every Segal Cartesian fibration $W \to X$.
5. A weak equivalence (Segal Cartesian fibration) between fibrant objects is a level-wise equivalence (biReedy fibration).

Proof. All of this directly follows from applying the theory of Bousfield localizations to the Reedy contravariant model structure over $X$, where the localizing set is:
\[ \mathcal{L} = \{ \varphi_0 \to \varphi_k \to X \} \]
We have following important result about the contravariant model structure on bisimplicial spaces.

**Theorem 7.79.** The following

\[ (sS/X)^{contra} \leftrightarrow (sS/X)^{contra} \]

is a Quillen equivalence. Here both sides have the contravariant model structures.

**Proof.** The right adjoint preserves right fibrations and Reedy fibrations, so it is a Quillen adjunction. Moreover, by definition of right fibrations of bisimplicial spaces, the map \( i_F^*(i_F)_*(R) \to R \) is a biReedy equivalence for every right fibration \( R \to X \). We just showed the derived counit map is a biReedy equivalence. Finally, for a right fibration \( S \to X \) of simplicial spaces, \( i_F^*(S) \to X \) is already fibrant in the contravariant model structure on bisimplicial spaces. Thus, we only need to take biReedy fibrant replacement \( i_F^*(S) \) over \( X \) to find the derived unit map. But \( (i_F)_*(i_F)_*(S) \simeq S \), as a biReedy equivalence is a level-wise Reedy equivalence. So the derived unit map is also an equivalence. Hence, we proved this adjunction is a Quillen equivalence. □

The maps \( k : F(0) \to F(k) \) are acceptable for all \( k \) and so we could repeat everything we did for Segal Cartesian and Cartesian fibrations and their model structure. However, in light of the previous theorem, we will not do so as the contravariant model structure on simplicial spaces has already been studied in great detail. Instead we will focus on the following fact.

**Lemma 7.80.** The maps \( k : F(0) \to F(k) \) do not satisfy the right stability property.

**Proof.** There are plenty of examples. The map \( 0 : F(0) \to F(1) \) is a right fibration. The map \( 1 : F(0) \to F(1) \) is a diagonal Kan equivalence. But, the pullback of these two maps, namely the empty set, is not equivalent to \( F(0) \). □

**Remark 7.81.** One direct implication of this lemma is the fact that the adjunction

\[ (sS/X)^{Diag} \leftrightarrow (sS/X)^{Contra} \]

is not a Quillen adjunction. This shows that the right stability property that we introduced in Section 6 is necessary to prove this result and cannot be removed.

7.7 The Covariant Approach: (Segal) coCartesian Fibrations. In this section we completely focused on the contravariant case. However, all these results also have a covariant analogue. Here we will only state the important results and leave the details to the reader.

First the covariant analogue to Segal Cartesian fibrations.
Definition 7.82. We say a map $Y \to X$ over $X$ is a Segal coCartesian fibration if it is a Reedy left fibration and for $k \geq 2$ the map of simplicial spaces

$$f_k^*: \text{Map}_{/X}(\varphi_k, Y) \to \text{Map}_{/X}(\gamma_k, Y)$$

is an Kan equivalence of spaces for every map $\varphi_k \to X$.

As in the contravariant case Segal coCartesian fibrations have a model structure.

Theorem 7.83. There is a unique model structure on bisimplicial spaces over $X$, called the Segal coCartesian model structure and denoted by $(\text{ssS}_{/X})^{\text{SegcoCart}}$ such that

1. It is a simplicial model category.
2. The fibrant objects are the Segal coCartesian fibrations over $X$.
3. Cofibrations are monomorphisms.
4. A map $A \to B$ over $X$ is a weak equivalence if

$$\text{map}_{\text{ssS}_{/X}}(B, W) \to \text{map}_{\text{ssS}_{/X}}(A, W)$$

is an equivalence for every Segal Cartesian fibration $W \to X$.
5. A weak equivalence (Segal coCartesian fibration) between fibrant objects is a level-wise equivalence (biReedy fibration).

Next is the covariant case for Cartesian fibrations.

Definition 7.84. We say a map of bisimplicial spaces $Y \to X$ is a coCartesian fibration if it is a Segal coCartesian fibration and the map of simplicial spaces

$$e^* : \text{Map}_{/X}(\zeta, Y) \to \text{Map}_{/X}(\varphi_0, Y)$$

is an Kan equivalence of spaces for every map $\zeta \to X$.

Again coCartesian fibrations come with their own model structure.

Theorem 7.85. There is a unique model structure on bisimplicial spaces over $X$, called the coCartesian model structure and denoted by $(\text{ssS}_{/X})^{\text{coCart}}$ such that

1. It is a simplicial model category.
2. The fibrant objects are the coCartesian fibrations over $X$.
3. Cofibrations are monomorphisms.
4. A map $A \to B$ over $X$ is a weak equivalence if

$$\text{map}_{\text{ssS}_{/X}}(B, W) \to \text{map}_{\text{ssS}_{/X}}(A, W)$$

is an equivalence for every Cartesian fibration $W \to X$.
5. A weak equivalence (coCartesian fibration) between fibrant objects is a level-wise equivalence (biReedy fibration).

We can adjust all proofs in Section 6 to show they hold for for (Segal) coCartesian fibrations as well. We leave the details to the interested reader.
7.8 A Peak into the Future: Representable (Segal) Cartesian Fibrations. In Section 5 we discussed the notion of a representable Reedy left (and right) fibration. Now that we have introduced (Segal) Cartesian fibrations we want to consider representability for the special case of those fibrations.

Thus the key question is to understand when a representable Reedy right fibration is a (Segal) Cartesian fibration. The study of this question will lead us to the study of Segal objects and complete Segal objects. Here we will only give an overview of some of the topic and refer to [Ra18b] for more details.

First an obviously corollary of Corollary 7.4

**Corollary 7.86.** A representable right fibration $\mathcal{C}/W \to \mathcal{C}$ is a Segal Cartesian fibration if and only if for every object $D$ the fiber $F(0) \times_{\mathcal{C}/W} \mathcal{C}$ is a Segal space.

**Remark 7.87.** Recall that the simplicial space $F(0) \times_{\mathcal{C}/W} \mathcal{C}$ is level-wise equivalent to the space $\text{map}_{\mathcal{C}}(D, W_n)$. The fact that it is a Segal space tells us there is an equivalence of spaces

$$\text{map}_{\mathcal{C}}(D, W_n) \simeq \text{map}_{\mathcal{C}}(D, W_1) \times_{\text{map}_{\mathcal{C}}(D, W_0)} \text{map}_{\mathcal{C}}(D, W_1) \times_{\text{map}_{\mathcal{C}}(D, W_0)} \cdots \times_{\text{map}_{\mathcal{C}}(D, W_0)} \text{map}_{\mathcal{C}}(D, W_1)$$

This condition has to hold for every $D$.

The condition stated above suggests that the simplicial object $W$ needs to satisfy an internal version of the Segal condition, which leads to our next definition.

**Definition 7.88.** Let $x_\bullet : \Delta^{op} \to X$ be a simplicial object. We say $x_\bullet$ is a Segal object if the simplicial map

$$x_n \to x_1 \times \cdots \times x_1$$

is an equivalence inside the Segal space $X$.

**Remark 7.89.** Notice this definition is not very precise as we are not carefully using limits in a CSS.

Assuming we have a precise definition of limits and a notion of a Segal object we then get following proposition.

**Proposition 7.90.** Let $x_\bullet : \Delta^{op} \to X$ be a Segal object. Then the corresponding representable Reedy right fibration $X/\pi_{/x}$ is a Segal Cartesian Fibration.

**Proof.** By Corollary 7.4 it suffices to show that for every object $y$ in $X$ the fiber $(X_{\pi/\times X} F(0))_{y_0}$ is a Segal space, as we already know it is a Reedy right fibration. Clearly it is Reedy fibrant as $X_{\pi/\times X}$ biReedy fibrant and fibrations are preserved under pullbacks. Thus we have to show it satisfies the Segal condition. For that recall that for every $k$ we have a Reedy equivalence $X_{/\pi/_{x_k}} \to X/_{x_k}$. Thus at level $k$ the fiber over the point $y$ is Kan equivalent to the space $\text{map}_{/X}(y, x_k)$. Using the property of limits have an equivalence of spaces

$$\text{map}_{/X}(y, x_1 \times \cdots \times x_1) \simeq \text{map}_{/X}(y, x_1) \times_{\text{map}_{/X}(y, x_0)} \cdots \times_{\text{map}_{/X}(y, x_0)} \text{map}_{/X}(y, x_0)$$

Combining these two facts we see that it satisfies the Segal condition.

The discussion about Segal objects suggests a similar approach for Cartesian fibrations.
Definition 7.91. Let $x_\bullet : \Delta^{op} \to X$ be a simplicial object. We say $x_\bullet$ is a complete Segal object if it is a Segal object and the simplicial map

$$x \to x_1 \times_{x_0 \times x_1} x_3$$

is an equivalence inside the Segal space $X$.

Proposition 7.92. Let $x_\bullet : \Delta^{op} \to X$ be a complete Segal object. Then the corresponding representable Reedy right fibration $X_{/\pi f x}$ is a Cartesian Fibration.

The ideas outlined above suggest that we need to carefully study simplicial objects inside a CSS $\mathcal{C}$ with the goal of imposing Segal conditions and completeness conditions internally. Those would then give us a model of a higher category internal to $\mathcal{C}$ that come with their own notion of objects, morphisms, ... Having developed a proper theory of such complete Segal objects, we can then use them to correctly define representable Segal Cartesian fibrations and Cartesian fibrations. This is the main goal of [Ra18b], which completely focuses on complete Segal objects.

**Appendix Some Facts about Model Categories**

We primarily used the theory of model categories to tackle issues of higher category theory. Here we will only state some technical lemmas we have used throughout this note.

Lemma A.1. Let $p : S \to T$ be a Kan fibration in $\mathcal{S}$. Then $p$ is a trivial Kan fibration if and only if each fiber of $p$ is contractible.

This lemma has following important corollary

Corollary A.2. Let $p : S \to K$ and $q : T \to K$ be two Kan fibrations. A map $f : S \to T$ over $K$ is a Kan equivalence if and only if for each point $k : \Delta[0] \to K$ the fiber

$$S \times_K \Delta[0] \to T \times_K \Delta[0]$$

is a Kan equivalence.

Theorem A.3. [Re01, Proposition 9.1] Let $\mathcal{L}$ be a set of cofibrations in $s\mathcal{S}$ with the Reedy model structure. There exists a cofibrantly generated, simplicial model category structure on $s\mathcal{S}$ with the following properties:

1. the cofibrations are exactly the monomorphisms.
2. the fibrant objects (called $\mathcal{L}$-local objects) are exactly the Reedy fibrant $W \in s\mathcal{S}$ such that

$$\text{Map}_{s\mathcal{S}}(B, W) \to \text{Map}_{s\mathcal{S}}(A, W)$$

is a weak equivalence of spaces.
3. the weak equivalences (called $\mathcal{L}$-local weak equivalences) are exactly the maps $g : X \to Y$ such that for every $\mathcal{L}$-local object $W$, the induced map

$$\text{Map}_{s\mathcal{S}}(Y, W) \to \text{Map}_{s\mathcal{S}}(X, W)$$

is a weak equivalence.
4. a Reedy weak equivalence (fibration) between two objects is an $\mathcal{L}$-local weak equivalence (fibration), and if both objects are $\mathcal{L}$-local then the converse holds.
We call this model category the localization model structure.

Lemma A.4. [JT07, Proposition 7.15] Let $\mathcal{M}$ and $\mathcal{N}$ be two model categories and

\[ \begin{array}{c}
\mathcal{M} \\
F \\
\downarrow \\
G \\
\mathcal{N}
\end{array} \]

be an adjunction of model categories, then the following are equivalent:

(1) $(F, G)$ is a Quillen adjunction.
(2) $F$ takes cofibrations to cofibrations and $G$ takes fibrations between fibrant objects to fibrations.

This lemma has following useful corollary:

Corollary A.5. Let $X$ be a simplicial space and let $(\mathcal{sS}/X, \mathcal{M})$ and $(\mathcal{sS}/X, \mathcal{N})$ be two localizations of the Reedy model structure. Then an adjunction

\[ \begin{array}{c}
(\mathcal{sS}/X)^{\mathcal{M}} \\
F \\
\downarrow \\
G \\
(\mathcal{sS}/X)^{\mathcal{N}}
\end{array} \]

is a Quillen adjunction if it satisfies following conditions:

(1) $F$ takes cofibrations to cofibrations.
(2) $G$ takes fibrants to fibrants.
(3) $G$ takes Reedy fibrations to Reedy fibrations.

Lemma A.6. [JT07, Proposition 7.22] Let

\[ \begin{array}{c}
\mathcal{M} \\
F \\
\downarrow \\
G \\
\mathcal{N}
\end{array} \]

be a Quillen adjunction of model categories. Then the following are equivalent:

(1) $(F, G)$ is a Quillen equivalence.
(2) $F$ reflects weak equivalences between cofibrant objects and the derived counit map $F LG(n) \to n$ is an equivalence for every fibrant-cofibrant object $n \in \mathcal{N}$ (Here $LG(n)$ is a cofibrant replacement of $G(n)$ inside $\mathcal{M}$).
(3) $G$ reflects weak equivalences between fibrant objects and the derived unit map $m \to G RF(m)$ is an equivalence for every fibrant-cofibrant object $m \in \mathcal{M}$ (Here $RF(m)$ is a fibrant replacement of $F(m)$ inside $\mathcal{N}$).

There is only one lemma that we will actually prove here and that will allow us to compare relative and absolute model structures. Before we do so we will have to review two different model structures: the induced model structures and the relative localized model structure.

Definition A.7. Let $\mathcal{M}$ be a model structure on $\mathcal{sS}$. Let $X$ be a simplicial space. There is a simplicial model structure on $(\mathcal{sS}/X)$, which we call the induced model structure and denote by $(\mathcal{sS}/X)^{\mathcal{M}}$, and which satisfies following conditions:
A map \( f : Y \to Z \) over \( X \) is a (trivial) fibration if \( Y \to Z \) is a (trivial) fibration.

A map \( f : Y \to Z \) over \( X \) is an equivalence if \( Y \to Z \) is an equivalence.

A map \( f : Y \to Z \) over \( X \) (trivial) cofibration if \( Y \to Z \) is a (trivial) cofibration.

**Remark A.8.** This model structure can be defined for any model category and not just for model structures on \( s\mathcal{S} \), but for our work there was no need for further generality.

**Definition A.9.** Let \( \mathcal{M} \) be a model structure on \( s\mathcal{S} \), which is the localization of the Reedy model structure with respect to the cofibration \( A \to B \). Let \( X \) be simplicial space. There is a simplicial model structure on \( (s\mathcal{S}/X) \), which we call the relative localized model structure and denote by \( (s\mathcal{S}/X)^{\text{loc}\mathcal{M}} \). It is the localization of the induced Reedy model structure on \( s\mathcal{S}/X \) with respect to all map \( A \to B \to X \).

**Remark A.10.** Note that the two model structures constructed above are generally not the same. However, there is a special case where they coincide.

**Lemma A.11.** Let \( \mathcal{M} \) be a localization model structure on \( s\mathcal{S} \) with respect to the map \( A \to B \). Let \( W \) be a fibrant object in that model structure. The following adjunction

\[
\begin{array}{ccc}
(s\mathcal{S}/W)^\mathcal{M} & \xleftarrow{id} & (s\mathcal{S}/W)^{\text{loc}\mathcal{M}} \\
\downarrow id & & \downarrow \equiv \\
(\mathcal{S}/W)^\mathcal{M} & \xrightarrow{id} & (\mathcal{S}/W)^{\text{loc}\mathcal{M}}
\end{array}
\]

is a Quillen equivalence. In fact, the two model structures are isomorphic.

**Proof.** Clearly, both model structures have the same set of cofibrations. We will show that they have the same set of weak equivalences and the rest will follow. Both model structures are simplicial and so the weak equivalences are determined by the set of fibrant objects. So, it suffices to show that they have the same set of fibrant objects. Let \( Y \to W \) be a map. We have the following commutative square:

\[
\begin{array}{ccc}
\text{Map}_{s\mathcal{S}}(B,Y) & \to & \text{Map}_{s\mathcal{S}}(B,W) \\
\downarrow & & \downarrow \equiv \\
\text{Map}_{s\mathcal{S}}(A,Y) & \to & \text{Map}_{s\mathcal{S}}(A,W)
\end{array}
\]

The right-hand map is always a trivial Kan fibration (because \( W \) is fibrant). So, this square is homotopy pullback square if and only if the left-hand map is a trivial Kan fibration. But being homotopy pullback square by definition means being fibrant in the relative localized model structure, whereas being trivial Kan fibration means being a fibration in our model structure as a Reedy fibration between two fibrant objects is a fibration. \( \square \)

**Appendix Comparison with Quasi-Categories**

As we already pointed out in Subsection 7.3, the notion of a Cartesian fibration is already established in the literature. Lurie introduced Cartesian fibrations using quasicategories [Lu09].
Moreover, it has been studied by Riehl and Verity using ∞-cosmoi and its corresponding homotopy 2-category [RV17].

The goal of this appendix is to show that the definition of Cartesian fibration introduced here agrees with those previous definitions. We will do so by using the framework of ∞-cosmoi introduced by Riehl and Verity. In [RV17], Riehl and Verity develop a theory of Cartesian fibrations that works for every ∞-cosmos, which can be thought of as a model independent approach to higher category theory. In particular, they prove following corollary

**Corollary B.1.** [RV17, Corollary 4.1.15] Let \( p : E \to B \) be an isofibration in \( K \). Then \( p \) is a cartesian fibration if and only if for every cofibrant object \( X \in K \), the isofibration map(\( X, p \)) : \( \text{map}(X, E) \to \text{map}(X, B) \) is a cartesian fibration of quasi-categories.

The goal of this appendix is to show that the definition of Cartesian fibration given here agrees with this general definition given by Riehl and Verity. First, notice that their framework works only when the base is a complete Segal space. However, this should not be an issue. Using Theorem 6.29 we know that a map of simplicial spaces \( R \to X \) is a Cartesian fibration if and only if the CSS fibrant replacement of the map \( R \to X \) is a Cartesian fibration. Thus if we can prove that the definitions of Cartesian fibration agree over a CSS, then they agree in general, which means we can use the framework of ∞-cosmoi to compare the definitions. Before that, however, we show how ∞-cosmoi compare to complete Segal spaces.

According to [RV17, Example 2.2.5] the simplicial category of complete Segal spaces forms an ∞-cosmos. Here we have to be careful. The enrichment is not the one we have been using before. Rather, by the work of Joyal and Tierney, there is a Quillen equivalence

\[
\text{sSet}^{\text{Joy}} \xrightarrow{i_!} \text{sCSS} \xleftarrow{p_*} \text{sCSS}
\]

between the Joyal model structure on simplicial sets and the CSS model structure on simplicial spaces ([JT07, Theorem 4.11]). In particular the right adjoint \( i_! \) is a functor that takes a simplicial space \( X_* \) and gives us the 0-th row \( X_0 \). As it is a Quillen adjunction, if \( X \) is a CSS then \( i_!(X) \) is a quasicategory. Thus for every two CSS, \( X \) and \( Y \), we get a quasicategory \( i_!(Y^X) \). This is the simplicial enrichment that makes the category of complete Segal spaces into a ∞-cosmos. Notice in particular that an isofibration of complete Segal spaces is just a fibration in the complete Segal space model structure. Also, in this particular ∞-cosmos every object is cofibrant (this follows from the fact that every simplicial space is cofibrant in the complete Segal space model structure). So we get an ∞-cosmos \( \text{CSS} \), in which the objects are complete Segal spaces.

Translating the corollary above into this more concrete language we now have to prove following result.

**Corollary B.2.** ([RV17, Corollary 4.1.15] for complete Segal spaces) Let \( p : E \to B \) be a CSS fibration in \( \text{CSS} \). Then \( p \) is a Cartesian fibration if and only if for every object \( X \in \text{CSS} \), the CSS fibration \( i_!(p^X) : i_!(E^X) \to i_!(B^X) \) is a Cartesian fibration of quasi-categories.
In Corollary 7.21 we showed that if \( C \to X \) is a Cartesian fibration and \( Y \) is any simplicial space then the map \( C^Y \to X^Y \) is also a Cartesian fibration. Thus all that is left is to prove the following statement.

**Theorem B.3.** Let \( X \) be a CSS and \( p : E \to B \) be a CSS fibration. The following two are equivalent

1. \( p : E \to B \) is a Cartesian fibration in the sense of 7.19.
2. The map \( i_1^*(p) : i_1^*E \to i_1^*B \) is a Cartesian fibration of quasicategories in the sense [Lu09, Definition 2.4.2.1].

**Proof.** The proof follows right away by using our alternative characterization of Cartesian fibrations between CSS (Theorem 7.43). \( E \to B \) is a fibration in the complete Segal space model structure if and only if \( i_1^*E \to i_1^*B \) is a fibration in the Joyal model structure. Moreover, the adjunction preserves over categories and homotopy pullback squares. In other words, \( i_1^*(E)_x \cong i_1^*(E)_{/x} \) (recall that both \( E \) and \( i_1^*(E) \) have the same set of objects). Thus a map \( f \) in \( E \) is \( p \)-Cartesian if and only if \( f \) in \( i_1^*(E) \) is \( i_1^*(p) \)-Cartesian. This implies that \( B \) has \( p \)-Cartesian lifts if and only if \( i_1^*(B) \) has \( i_1^*(p) \)-Cartesian lifts and hence we are done. \( \square \)

**References**

[dB16] P. B. de Brito, *Segal objects and the Grothendieck construction*, arXiv preprint arXiv:1605.00706 (2016).

[JT07] A. Joyal, M. Tierney, *Quasi-categories vs Segal spaces*, Contemp. Math 431 (2007): pp. 277-326

[Lu09] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies 170, Princeton University Press, Princeton, NJ, 2009, xviii+925 pp. A

[Re01] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math.Soc., 353(2001), no. 3, 973-1007.

[RV17] E. Riehl, D. Verity, *Fibrations and Yoneda’s lemma in an ∞-cosmos*, Journal of Pure and Applied Algebra 221.3 (2017): 499-564.