Lattice QCD Equation of State for Nonvanishing Chemical Potential by Resumming Taylor Expansion

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Taylor expansion in powers of baryon chemical potential ($\mu_B$) is an oft-used method in lattice QCD to compute QCD thermodynamics for $\mu_B > 0$. Based only upon the few known lowest order Taylor coefficients, it is difficult to discern the range of $\mu_B$ where such an expansion around $\mu_B = 0$ can be trusted. We introduce a resummation scheme for the Taylor expansion of the QCD equation of state in $\mu_B$ that is based on the $n$-point correlation functions of the conserved current ($D_n$).

The method resums the contributions of the first $N$ correlation function $D_1, \ldots, D_N$ to the Taylor expansion of the QCD partition function to all orders in $\mu_B$. We show that the resummed partition function is an approximation to the reweighted partition function at $\mu_B \neq 0$. We apply the proposed approach to high-statistics lattice QCD calculations using 2+1 flavors of Highly Improved Staggered Quarks with physical quark masses on $32^3 \times 8$ lattices and for temperatures $T \approx 145 – 176$ MeV. We demonstrate that, as opposed to the Taylor expansion, the resummed version not only leads to improved convergence but also reflects the zeros of the resummed partition function and severity of the sign problem, leading to its eventual breakdown. We also provide a generalization of our scheme to include resummation of powers of temperature and quark masses in addition to $\mu_B$, and show that the alternative expansion scheme of [S. Borsányi et al., Phys. Rev. Lett. 126, 232001 (2021).] is a special case of this generalized resummation.

Introduction.— Lattice Quantum Chromodynamics (QCD) results for the QCD equation of state (EoS) plays a critical role in the dynamical modeling of heavy-ion collisions [1–4] and, thereby, in the experimental explorations of the QCD phase diagram in the $T$-$\mu_B$ plane. Due to the fermion sign problem it is difficult to carry out lattice QCD computations directly at $\mu_B \neq 0$. Despite some recent progress [5–10], direct lattice computations of the QCD EoS $\mu_B \neq 0$ with physical quark masses, fine lattice spacings and large lattice volumes have remained elusive. Instead, the present state-of-the-art lattice QCD EoS at $\mu_B > 0$ has been obtained using the Taylor expansion [11, 12] and the analytic continuation [13, 14] methods. In the Taylor expansion method one expands the pressure in powers of $\mu_B$ around $\mu_B = 0$ and directly computes the Taylor coefficients at $\mu_B = 0$. For the analytic continuation, one avoids the fermion sign problem using simulations at purely imaginary values $\mu_B$, fits these results with a power series in $\mu_B$ to determine the Taylor coefficients at $\mu_B = 0$ and then provides the EoS at real $\mu_B > 0$ based on these Taylor coefficients. On the other hand, it is well-known that the applicability of the Taylor expansion as well as the analytic continuation should be limited by the zeros, nearest to $\mu_B = 0$, of the partition function in the entire complex-$\mu_B$ plane [15–17]. In principle, it is possible to gain some knowledge about the locations of the zeros of the partition function by re-expressing the power series in real or imaginary $\mu_B$ in terms of Padé approximants [12] or in a power series of the fugacity [18–20]. Armed, in reality, with only the few lowest order Taylor coefficients, this becomes a very difficult task and, in practice, one just restricts the EoS to $\{T, \mu_B\}$ that avoids any pathological nonmonotonicity in the truncated Taylor series [11, 14]. Furthermore, these methods provide very little guidance on the severity of the fermion problem, i.e. how rapidly the phase of the partition function fluctuates as $\mu_B$ is increased. It is possible to determine the zeros of the partition function as well as its average phase by reweighting the fermion determinant to $\mu_B \neq 0$ [21–25]. However, due to the computational cost associated with exact evaluation of the fermion determinant, at present this method is restrained within coarse lattice spacings and small lattice volumes.

In this work, we introduce a method for the calculation of the lattice QCD EoS that genuinely resums the truncated Taylor series to all orders in $\mu_B$ and whose breakdown encodes the severity of the sign problem and zeros of the resummed partition function.

The resummation method.— The Taylor expansion to $O(\mu_B^n)$ of the excess pressure, $\Delta P(T, \mu_B) \equiv P(T, \mu_B) - P(T, 0)$, is given by

$$\frac{\Delta P_N^E}{T^4} = \sum_{n=1}^{N} \frac{\chi_n^B(T, \mu_B)}{n!} \left( \frac{\mu_B}{T} \right)^n,$$

where the Taylor coefficients are defined as

$$\chi_n^B(T) = \frac{1}{VT^3} \left. \frac{\partial^n \ln Z(T, \mu_B)}{\partial (\mu_B/T)^n} \right|_{\mu_B=0}.$$

Here, the QCD partition function is denoted as $Z = \int e^{-S} \det[M] \, dU$, $V$ is the spatial volume, $U$ is the $SU(3)$ gauge fields, $S$ is the pure gauge action and $M$ is the fermion matrix. Each $\chi_n^B$ consists of sum of terms like...
be resummed into exponential forms. For example, lowest order Taylor coefficients are sketched in FIG. 1. If the physical interpretation of \(\langle \cdot \rangle\) and the \(\langle \cdot \rangle\) net baryon-number densities can be straightforwardly obtained as a single \(\mu_B\)-
derivative of \(\Delta P^E\) and \(\Delta P^R\) in Eq. 1 and Eq. 4, respectively.

The resummed version in Eq. 4 also highlights the connection between the Taylor expansion and the reweighting method. In the reweighting method \(Z(T,\mu_B)/Z(T,0) = \langle \det[M(T,\mu_B)]/\det[M(T,0)] \rangle\) can be calculated, if computationally feasible, by exactly evaluating the ratio of the fermion matrix determinants on the gauge fields generated at \(\mu_B = 0\). In more realistic lattice calculations with large volumes, exact evaluations of the determinant ratios might not be computationally feasible and one may consider evaluating \(\det[M(T,\mu_B)]\) within some approximation scheme to obtain approximate partition function \(Z^R_N(T,\mu_B) \approx Z(T,\mu_B)\). Following the spirit of the Taylor expansion, one such approximation scheme can be expansion of \(\det[M(T,\mu_B)]\) in powers of \(\mu_B/T\). Keeping in mind \(\det[M] = \exp[\text{Tr} \ln M]\) and Eq. 3, one can immediately recognize

\[
\frac{Z^R_N(T,\mu_B)}{Z(T,0)} = \left\langle \exp \left[ \sum_{n=1}^{N} \bar{D}_n \left( \frac{\mu_B}{T} \right)^n \right] \right\rangle. \tag{5}
\]

Since CP symmetry dictates that the even(odd) \(D_n\) are purely real(imaginary) and the partition function must be real, a measure of the severity of the sign problem is given by the average phase factor for \(Z^R_N\) (with \(\mu_B\) real),

\[
\langle \cos \Theta^R_N \rangle = \left\langle \cos \left( \sum_{n=1}^{N/2} \text{Im}[\bar{D}_{2n-1}] \left( \frac{\mu_B}{T} \right)^{2n-1} \right) \right\rangle. \tag{6}
\]

An expansion of \(\langle \cos \Theta^R_N \rangle\) in \(\mu_B/T\) leads to the Taylor expanded average of the average phase of the partition function \(Z^R_N\), which we will denote by \(\Theta^E_N\). As the sign problem becomes more severe the average phase \(\langle \cos \Theta^R_N \rangle\) \(\approx 0\) and resummed results will also show signs of breakdown. Furthermore, although \(\Delta P^E_N\) can be evaluated for any complex value of \(\mu_B\), \(\Delta P^R_N\) becomes undefined when \(\text{Re}[Z^R_N] \leq 0\) for a given \(N\) and statistics, leading to a natural breakdown of the resummed results. The location of the zeros of \(Z^R_N\) in the complex-\(\mu_B\) plane will indicate the \(\mu_B\) region where such resummation can be applicable. Obviously, for any given \(N\) the region of applicability of \(\Delta P^R_N\) cannot exceed the same for \(\Delta P^E_N\).

Lattice QCD computations. For this work, we used the data for \(\chi^B_N\) and \(D_n\) generated by the HotQCD collaboration for calculations of the QCD EoS [11] and the chiral crossover temperature [28] at \(\mu_B > 0\) using the Taylor expansion method. The HotQCD ensembles were generated with 2+1-flavors of Highly Improved Staggered Quarks.

\[
\Delta P^R_N \quad \text{in powers of } \mu_B/T \quad \text{yields an infinite series in } \mu_B/T, \quad \text{in addition to the truncated Taylor series:}
\]

\[
\Delta P^R_N + \sum_{i,j=0}^{\infty} \langle \bar{D}_1 \cdots \bar{D}_N \rangle \left( \frac{\mu_B}{T} \right)^n, \quad \text{where } i,j = 0, \ldots, N \quad \text{satisfying } 1 \cdot i + \cdots + N \cdot j = n. \quad \text{The Taylor expanded } \langle N^R_N \rangle \quad \text{and the resummed } \langle N^R_R \rangle \quad \text{baryon-number densities can be straightforwardly obtained as a single } \mu_B-\text{derivative of } \Delta P^E \quad \text{and } \Delta P^R \quad \text{in Eq. 1 and Eq. 4, respectively.}
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and the tree-level improved Symanzik gauge action [29–31]. Bare quark masses were chosen to reproduce, within a few percent, the physical value of the kaon mass and a pseudo-Goldstone pion mass of 138 MeV in the continuum limit at $T = \mu_B = 0$ and the lattice spacing were calibrated against the physical value of the kaon decay constant [32]. We present lattice QCD results from a single lattice size $32^3 \times 8$ and for 6 temperatures $T = 145, 151, 157, 166, 171, 176$ MeV. About 475K, 520K, 716K, 522K, 232K and 152K gauge field configurations were used to measure $D_n$ at these temperatures respectively. The gauge field configurations were separated by 10 Rational Hybrid Monte Carlo trajectories of unit length. The $D_n$ were calculated within the formalism adopted in Refs. [11, 28], i.e., using the exponential-$\mu$ formalism [33] for $n \leq 4$ and the linear-$\mu$ formalism [34, 35] for $n > 4$. The expressions for $D_n$ in terms of the traces involving the inverse of the staggered fermion matrix and its $\mu_B$-derivatives are well-known [26, 36]. Each trace was calculated stochastically for each configuration by employing 2000 random Gaussian volume sources for the trace $D_4$ and 500 random sources for the rest [36].

Results. – To demonstrate the superiority of the resummation method over the Taylor expansion, we chose the temperature where we had the largest statistics, i.e., $T = 157$ MeV, which is also closest to the QCD crossover temperature [28]. In Fig. 2, we compare $\Delta P^E_N$ with $\Delta P^R_N$ (top) and $N^E_N$ with $N^R_N$ for different orders $N$. Comparisons are shown both for real as well as imaginary values of $\mu_B$, corresponding to positive and negative values of $(\mu_B/T)^2$, respectively. The $\Delta P^R_N$ and $N^R_N$ show very good convergence between different orders $N = 2, 4, 6, 8$. The Taylor-expanded results seem to approach their respective resummed results as contributions from higher orders in $\mu_B$ are included; however the convergence of the Taylor-expanded results is slow due the alternating signs of the higher order $x^B_n$ near the QCD crossover [11–13]. The resummation method overcomes this problem by including contributions from all orders in $\mu_B$ and shows markedly improved convergence. In contrast to the Taylor expansion, the resummed results break down for $|\mu_B/T| \gtrsim 1.5$. For $|\mu_B/T| \gtrsim 1.5$, $\text{Re}[Z_n] \leq 0$ and $N^R_N$ becomes divergingly large. We checked that such a breakdown is not a mere statistical issue by repeating the calculations using only parts of the gauge configurations available at this temperature. Similar breakdown for $\mu_B/T \gtrsim 1.5$ was also observed in Refs. [12, 37, 38] when the EqS was reconstructed from the Padé approximants of the Taylor series in $\mu_B$. While Padé-based continuations of the QCD crossover temperature from imaginary values $\mu_B$ did not encounter such breakdowns [39, 40], the same in the case of the EqS seemed to break down due to singularities in the complex-$\mu_B$ plane [41].

To investigate the origin of this breakdown, we computed the average phase as a function of real $\mu_B$, c.f.
The results are shown in FIG. 3 (top). Also, \( \langle \cos \theta_N^R \rangle \approx 0 \) for \( \mu_B/T \gtrsim 1.5 \), which shows that the sign problem is uncontrollably severe where the EoS calculations broke down. The resummation method thus faithfully captures the severity of the sign problem, as opposed to the Taylor expansion. The phase factor cannot be calculated exactly within the Taylor series approach. Its Taylor series expansion too converges very slowly, as the bands plotted in FIG. 3 (top) show. Further, we searched for the zeros of resummed partition function, \( \text{c.f. Eq. 5, in the complex-} \mu_B \text{ plane. We solved for } Z_N^R = 0 \text{ using the Newton-Raphson algorithm with initial guesses chosen from a uniform distribution over a grid } 0 \leq \{ \text{Re(} \mu_B/T), \text{Im(} \mu_B/T) \} \leq 2.5 \). The results are shown in FIG. 3 (bottom). The zeros of \( Z_N^R \) and \( Z_N^R \) are more or less consistent with each other and appears only for \( |\mu_B/T| \gtrsim 1.5 \). The exact nature of the singularity responsible for breakdown of the resummation method is certainly of great interest, \( \text{i.e. whether it is associated with the Yang-Lee edge singularity of the QCD chiral transition [15, 17] or the QCD critical point and approaches the real axis [12, 21, 22, 37, 38] etc. This will need detailed quantitative studies involving careful finite-volume scaling analyses using more sophisticated techniques [18, 20, 42] and is beyond the scope of the present work. But our results demonstrate that the breakdown of the resummation method reflects the associated singularities of the partition function, at least qualitatively.

Finally, in FIG. 4 we summarize results for all \( T = 145 - 176 \text{ MeV} \) by showing comparisons between \( \Delta P^R_6 \) and \( N^R_6 \) with the corresponding \( \Delta P^E_6 \) and \( N^E_6 \). As in the case of \( T = 157 \text{ MeV} \), \( \Delta P^R_6 \) and \( N^R_6 \) show improved convergence over \( \Delta P^E_6 \) and \( N^E_6 \) at all temperatures. Again, in contrast to the Taylor expansion the resummation method shows signs of breakdown for \( \mu_B \gtrsim 200 - 250 \text{ MeV} \), depending on the temperature. As before, we checked that in all cases, these breakdowns reflect the severity of the sign problem and the singularities of the partition function in the complex-\( \mu_B \) plane.

Generalization to multi-parameter and joint expansion in \( T, \mu_B \).– Akin to multi-parameter reweighting [21–24] in bare gauge coupling, \( \Delta \beta = \beta - \beta_0 \), and quark mass, \( \Delta m = m - m_0 \), our resummation scheme also can be extended to obtain \( Z_N^R(T, \mu_B) \) starting from a different temperature \( T_0(\beta_0) \) and bare quark mass \( m_0 \),

\[
\frac{Z_N^R(T, \mu_B)}{Z(T_0, 0)} = e^{-S_G \Delta \beta + \sum_{i,j=1}^n \tilde{G}_{ij}(\tilde{\mu}^r) \langle \tilde{\mu}^m \rangle},
\]

where the expectation value is taken over gauge fields associated with \( \{ \beta_0, m_0, 0 \} \). Here, \( S_G \) is the pure gauge action and

\[
\tilde{G}_{ij}(\beta_0, m_0) = \frac{\partial^i \partial^j \ln \det[M(m, \mu_B)]}{\partial^i \partial^j \tilde{\mu}^r(\mu_B/T)^i \partial^m(\mu_B/T)^j}_{|_{m_0, 0}}.
\]

Note, \( \tilde{G}_{ij} \) are the chiral condensate and higher order chiral susceptibilities, and general \( \tilde{G}_{ij} \) are \( \mu_B \)-derivatives of various chiral observables [26, 36, 43, 44]. This generalization can possibly mitigate the overlap problem that one might encounter while resumming only in \( \mu_B \). Further, a systematic expansion of the logarithm of Eq. 7 in powers of \( \Delta \beta \), \( \Delta m \) and \( \mu_B \) yields the expansion of the pressure difference, \( P(T, \mu_B)/T^4 - P(T_0, \mu_B)/T_0^4 \), in powers of \( \Delta T = T - T_0 \) and \( \mu_B \); particular choice of \( T_0(\mu_B) \) defined by a line of constant physics in the \( T-\mu_B \)-plane reproduces the expansion scheme used in Ref. [45] by resumming up to \( N \)-point baryon-current correlations to all orders in \( \mu_B \) and \( \Delta T \) [46]. Thus, our method also generalizes the alternative expansion scheme of Ref. [45].

Conclusions.– We have introduced a new method to compute lattice QCD EoS by resumming contributions of up to \( N \)-point baryon-current correlations to all orders in \( \mu_B \). When expanded in powers of \( \mu_B \) this resummed partition function exactly reproduces the Taylor expansion up to \( O(\mu_B^0) \), plus an infinite series in \( \mu_B \) capturing all possible contributions involving only the \( n \leq N \)-point baryon-current correlations. This resummation method also amounts to an approximate reweighting method, thereby bridging two traditional lattice QCD techniques for \( \mu_B \neq 0 \). With illustrative high-statistics lattice QCD computations we have demonstrated that the resummation method show improved convergence over the Taylor
Eq. 7 in Ref. [45] is a special case of this—Taylor expansion of the resummation, Eq. 7, and shown that the method of also introduced a generalized multi-parameter version of using only the power series of \( \mu \). To avoid singularities in the complex-\( \mu \) plane, we have also introduced a generalized multi-parameter version of the resummation, Eq. 7, and shown that the method of Ref. [45] is a special case of this—Taylor expansion of Eq. 7 in \( T \) and \( \mu_B \) along a specific line in the \( T-\mu_B \)-plane.

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explicit demonstration of the generalized resummation method resums the alternative expansion scheme presented here to resummation in both $\beta$, $\hat{m}$, and $m$. The generalizes method reweights the alternative expansion scheme presented in S. Borsányi et al. \cite{45} to all orders in $\beta$, $\hat{m}$, and $m$. 

Generalization to multi-parameter and joint expansion in $T$ and $\mu_B$

Our starting point for the multi-parameter expansion is Eq. \ref{eq:7}. There, the expectation value is taken over gauge fields associated with $\{\beta_0, m_0, \mu_B = 0\}$, where $\beta$ is the QCD gauge coupling, $S_G$ is the pure gauge action and the $\tilde{G}_{ij}$ are defined by Eq. \ref{eq:8}.

For brevity, in this section, we will use notations $\hat{\mu}_B \equiv \mu_B/T$, $\hat{m} = m/T$ and $\Delta \hat{m} \equiv \hat{m} - \hat{m}_0$, and provide explicit demonstration of the generalized resummation by keeping only leading order terms, i.e. $O(\Delta \beta)$, $O(\Delta \hat{m})$ and $O(\hat{\mu}_B^2)$, in all expansions. Extensions to higher orders are straightforward. We consider the case where the temporal extent ($N_s$) of the lattice is kept fixed. In this case, $T$ is changed by varying the bare gauge coupling $\beta$, and the bare quark mass $m$ must be tuned with $\beta$ to keep vacuum hadron masses constant. Thus, $T(\beta, m)$ and $T_0(\beta_0, m_0)$. Applying chain rule for derivatives as well as expanding $T(\beta, m)$ around $(\beta_0, m_0)$ one gets

$$\Delta T \frac{\partial \bar{Z}(\beta, m)}{\partial T} \bigg|_{T_0} = \Delta \beta \frac{\partial \bar{Z}(\beta, m)}{\partial \beta} \bigg|_{\beta_0} + \Delta \hat{m} \frac{\partial \bar{Z}(\beta, m)}{\partial \hat{m}} \bigg|_{\hat{m}_0} + \ldots. \quad (9)$$

With

$$\langle \bar{D}_i \rangle = \frac{1}{Z(\beta, m)} \int dU \bar{D}_i e^{-\beta S_G[U] + \ln \det M(m)} , \quad (10)$$

to the leading order

$$\Delta T \frac{d\langle \bar{D}_i \rangle}{dT} = - \langle [S_G \bar{D}_i] - \langle S_G \rangle \langle \bar{D}_i \rangle \rangle \Delta \beta
+ \langle \tilde{G}_{01} \bar{D}_i \rangle - \langle \bar{G}_{01} \rangle \langle \bar{D}_i \rangle + j \langle \bar{D}_i^{-1} \bar{G}_{11} \rangle \Delta \hat{m}
+ \ldots. \quad (11)$$

The goal is to obtain $Z^R_B(T, \mu_B)$ by expanding around $Z(T_0, 0)$, while resumming contributions of up to $N$-point baryon-current correlations to all orders in $\mu_B$ and $\Delta T$. Following multi-parameter reweighting technique

$$Z(T, \mu_B) = \left\langle e^{-\Delta \beta S_G \det M(m, \mu_B)} / \det M(m_0, 0) \right\rangle , \quad (12)$$

where the expectation value is with respect to a gauge field ensemble generated for $\{\beta_0, m_0, \mu_B = 0\}$. Simultaneously expanding in $\mu_B$ and $\Delta m$ the determinant ratio in Eq. \ref{eq:12} can be written as

$$\det M(m, \mu_B) / \det M(m_0, 0) = \exp \left[ \sum_{i+j=1}^{\infty} \tilde{G}_{ij} \hat{\mu}_B \hat{m}^j \right] , \quad (13)$$

where $\tilde{G}_{ij}$ are defined through Eq. \ref{eq:8}. By plugging Eq. \ref{eq:13} back into Eq. \ref{eq:12} and truncating the sum at $i+j = N$, we obtain Eq. \ref{eq:7}.

Next, by Taylor expanding Eq. \ref{eq:7} in powers of $\Delta \beta$, $\Delta \hat{m}$ and $\hat{\mu}_B$ we get

$$Z(T, \mu_B) / Z(T_0, 0) = 1 - \langle S_G \Delta \beta \rangle \Delta \hat{m} + \langle \tilde{G}_{01} \Delta \hat{m} \rangle + \langle \bar{D}_1 \rangle \hat{\mu}_B
- \left[ \langle S_G \bar{D}_1 \rangle \Delta \beta - \langle \tilde{G}_{01} \bar{D}_1 + \tilde{G}_{11} \Delta \hat{m} \rangle \hat{\mu}_B
+ \langle \bar{D}_2 + \bar{D}_1^2/2 \rangle \hat{\mu}_B^2
- \langle S_G (\bar{D}_2 + \bar{D}_1^2/2) \rangle \Delta \beta - \langle \tilde{G}_{01} (\bar{D}_2 + \bar{D}_1^2/2) \rangle \Delta \hat{m}
- \langle \tilde{G}_{21} + \tilde{G}_{11} \Delta \hat{m} \rangle \Delta \hat{m} \hat{\mu}_B^2 + \ldots. \quad (14)$$

The pressure difference is given by

$$\Delta \left[ \frac{P}{T^4} \right] = \frac{P(T, \mu_B)}{T^4} - \frac{P(T_0, 0)}{T_0^4} = \frac{N^3}{N^3_s} \ln \left[ \frac{Z(T, \mu_B)}{Z(T_0, 0)} \right] .$$
where \( N_s \) is the spatial extent of the lattice. Using Eq. 14, expanding the logarithm in powers of \( \Delta \beta, \Delta \hat{m}, \hat{\mu}_B \) and keeping only the real part one obtains

\[
\frac{N^3_s}{N^3_s} \Delta \left[ \frac{P}{T^4} \right] = \langle \bar{G}_{01} \rangle \Delta \hat{m} - \langle S_G \rangle \Delta \beta + \langle \bar{D}_2 + \bar{D}_1^2/2 \rangle \hat{\mu}_B^2
\]

\[-\left[ \langle S_G (\bar{D}_2 + \bar{D}_1^2/2) \rangle - \langle S_G \rangle \langle \bar{D}_2 + \bar{D}_1^2/2 \rangle \right] \hat{\mu}_B^2 \Delta \beta
\]

\[+ \left[ \langle G_{01} (\bar{D}_2 + \bar{D}_1^2/2) \rangle - \langle G_{01} \rangle \langle \bar{D}_2 + \bar{D}_1^2/2 \rangle \right]
\]

\[+ \langle \bar{G}_{21} + \bar{G}_{11} \hat{D}_1 \rangle \hat{\mu}_B^2 \Delta \hat{m} + \ldots . \]  

(15)

Noting that

\[
\frac{d[P(T,0)/T^4]}{dT} \bigg|_{T_0} \Delta T = \langle \bar{G}_{01} \rangle \Delta \hat{m} - \langle S_G \rangle \Delta \beta ,
\]

(16)

and using Eq. 9 of the main paper, it is easy to identify that Eq. 15 is nothing but a joint Taylor expansion of \( P(T,\mu_B) \) in \( T \) and \( \mu_B \) around \((T_0,0)\),

\[
\Delta \left[ \frac{P}{T^4} \right] = \frac{d[P(T,0)/T^4]}{dT} \bigg|_{T_0} \Delta T + \frac{1}{2!} \chi^B_2(T_0,0) \hat{\mu}_B^2
\]

\[+ \frac{1}{2!} \frac{d\chi^B_2(T)}{dT} \bigg|_{T_0} \hat{\mu}_B^2 \Delta T + O((\hat{\mu}_B^2, (\Delta T)^2)^2) . \]  

(17)

Thus, the generalized version given by Eq. 7 genuinely resums contributions of up to \( N \)-point baryon current in the Taylor expansion of EoS to all orders in \( T, \mu_B \).

Following Ref. [45], the generalized resummation of Eq. 7 can be made even more powerful by choosing the expansion point \( T_0 \) along some physically motivated line in the \( T-\mu_B \)-plane, \( i.e. \) by choosing some physically motivated \( \beta_0(\mu_B) \) and \( m_0(\mu_B) \). The one-to-one correspondence between the Taylor expansion of Eq. 7 and alternative expansion scheme presented in Ref. [45] can be readily observed. By including the \( O(\hat{\mu}_B^4) \) in Eq. 17 and taking a \( \hat{\mu}_B \)-derivative we get

\[
\chi^B_1(T,\mu_B) = \hat{\mu}_B \chi^B_2(T_0,0) + \frac{d\chi^B_2(T_0,0)}{dT} \bigg|_{T_0} \hat{\mu}_B \Delta T
\]

\[+ \frac{1}{6} \chi^B_4(T_0,0) \hat{\mu}_B^3 + \ldots . \]  

(18)

If one chooses

\[
T_0(\hat{\mu}_B) = T - \frac{1}{6} \frac{\chi^B_4(T_0,0)}{(d\chi^B_2(T,0)/dT)_{T_0}} \hat{\mu}_B^2 , \]

(19)

in Eq. 18 then one arrives at the starting point of Ref. [45], namely \( \chi^B_1(T_0,\hat{\mu}_B) = \hat{\mu}_B \chi^B_2(T_0,0) \). Hence, the method used in Ref. [45] is a special Taylor-expanded case of the generalized resummation Eq. 7.