Strange Attractors in the Vannimenus Model on an Arbitrary Order Cayley Tree

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Abstract. We consider the Vannimenus model on a Cayley tree of arbitrary order $k$ with competing nearest-neighbour interactions $J_1$ and next-nearest-neighbour interactions $J_2$ and $J_3$ in the presence of an external magnetic field $h$. In this paper we study the phase diagram of the model using an iterative scheme for a renormalized effective nearest-neighbour coupling $K_r$ and effective field per site $X_r$ for spins on the $r$th level; it recovers, as particular cases, previous works by Vannimenus, Inawashiro \textit{et al}, Mariz \textit{et al} and Ganikhodjaev and Uğuz. Each phase is characterized by a particular attractor and the phase diagram is obtained by following the evolution and detecting the qualitative changements of these attractors. These changements can be either continuous or abrupt, respectively characterizing second- or first- order phase transitions. We present a few typical attractors and at finite temperatures, several interesting features (evolution of reentrances, separation of the modulated region into few disconnected pieces, etc) are exhibited for typical values of parameters.

1. Introduction

The anisotropic next-nearest-neighbour Ising (ANNNI) model, which consists of an Ising model with nearest-neighbour interactions augmented by competing next-nearest-neighbour couplings acting parallel to a single axis direction, is one of the simplest model displaying a rich phase diagram with a Lifshitz point and many spatially modulated phases [1]-[8]. There has been a considerable theoretical effort to obtain the structure of the global phase diagram of the ANNNI model in the $T - p$ space, where $T$ is temperature and $p = -J_2/J_1$ is the ratio between the competing exchange interactions. On the basis of numerical mean-field calculations, Bak and von Boehm [7] suggested the existence of an infinite succession of commensurate phases, the so-called devil's staircase, at low temperatures. This mean-field picture has been supported by low-temperature series expansions performed by Fisher and Selke [8]. An Ising model with competing interactions on the Cayley tree has recently been studied extensively because of the appearance of nontrivial magnetic orderings (see [9] - [19] and references therein).

The Cayley tree is not a realistic lattice; however, its amazing topology makes the exact calculation of various quantities possible. For many problems the solution on a tree is much simpler than on a regular lattice and is equivalent to the standard Bethe-Peierls theory [20].
On the Cayley tree one can consider two type of next-nearest-neighbours: prolonged and one-level next-next-nearest-neighbours (definitions see below). In the case of the Ising model with competing nearest-neighbour interactions $J$ and prolonged next-nearest-neighbour interactions $J_p$ Vannimenus [9] was able to find new modulated phases, in addition to the expected paramagnetic and ferromagnetic ones. From this result follows that Ising model with competing interactions on a Cayley tree is real interest since it has many similarities with models on periodic lattices. In fact, it has many common features with them, in particular the existence of a modulated phase, and shows no sign of pathological behavior - at least no more than mean-field theories of similar systems [9]. Moreover a detailed study of its properties was carried out with essentially exact results, using rather simple numerical methods.

Later Mariz et al [10] extended this results assuming existence also interaction $J_3$ of the one-level nearest-next-neighbours with $k = 2$. C.R. da Silva and S. Coutinho [16] studied the Ising model on a general Cayley tree of arbitrary order with competing interactions between the first-, second-, and third–next-nearest-neighbour spins belonging to the same branch, and in the presence of an external magnetic field. Note that da Silva and Coutinho [16] have generalized the approach used by Thompson [11] for the Ising model on the Cayley tree with only nearest-neighbour interactions and an external field. Yokoi et al [14] extended the calculations of Vannimenus [9] for a tree of arbitrary order and gave strong numerical evidence for the existence of chaotic phases associated with strange attractors.

In this paper we extend the model considered by Mariz et al [10] for a tree of arbitrary order using the procedures of Inawashiro, Thompson, and Honda [12] to write first-order recursion relations and give strong numerical evidence for the existence of chaotic phases associated with strange attractors. We consider the Vannimenus model on a Cayley tree of arbitrary order $k$ with competing nearest-neighbour interactions $J_1$, prolonged next-nearest-neighbor interactions $J_2$ and one-level $k$-tuple neighbours interaction $J_3$ in the presence of magnetic field $h$. Note that the inclusion of the $k$- tuple neighbours competing interaction $J_3$ is essential for the presence of different stable modulated phases at $T = 0$. Apparently the one-level $k$ tuple neighbour interactions represent interest not only for physical and biological models but also for models in sociology [22],[23].

This paper has been organized in the following way. In Section 2 the model Hamiltonian is discussed. In Section 3 the recursion equations are defined. Section 4 is devoted to the discussion of the attractors of this dynamical system and finally, the conclusions are given in Section 5.

2. The Vannimenus model

Cayley Tree. A Cayley tree $\Gamma^k$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles with exactly $(k + 1)$ edges issuing from each vertex. Let denote the Cayley tree as $\Gamma^k = (V, \Lambda)$, where $V$ is the set of vertices of $\Gamma^k$, $\Lambda$ is the set of edges of $\Gamma^k$. Two vertices $x$ and $y$, $x, y \in V$ are called nearest-neighbors if there exists an edge $l \in \Lambda$ connecting them, which is denoted by $l =< x, y >$. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree $\Gamma^k$, is the number of edges in the shortest path from $x$ to $y$. For a fixed $x^0 \in V$ we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \{x \in V | d(x, x^0) \leq n\}$$

and $L_n$ denotes the set of edges in $V_n$. The fixed vertex $x^0$ is called the 0-th level and the vertices in $W_n$ are called the $n$-th level. For the sake of simplicity we put $|x| = d(x, x^0)$, $x \in V$. Two vertices $x, y \in V$ are called the next-nearest-neighbours if $d(x, y) = 2$. The next-nearest-neighbour vertices $x$ and $y$ are called prolonged next-nearest-neighbours if $|x| \neq |y|$ and is denoted by $> x, y <$. The next-nearest-neighbour vertices $x, y \in V$ that are not prolonged are called one-level next-nearest-neighbours since $|x| = |y|$ and are denoted by $> x, y <$. Two vertices $x, y \in V$ are called the third-next-nearest-neighbour [16] if $d(x, y) = 3$ with $x \in W_n, y \in W_{n+3}$ for some $n$. 

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We write \( x \prec y \) if the path from \( x^0 \) to \( y \) goes through \( x \). We call the vertex \( y \) a direct successor of \( x \), if \( y \succ x \) and \( x, y \) are nearest neighbours. The set of the direct successors of \( x \) is denoted by \( S(x) \), i.e., if \( x \in W_n \) , then 

\[
S(x) = \{ y_i \in W_{n+1} | d(x, y_i) = 1, i = 1, \ldots, k \}.
\]

We observe that for any vertex \( x \neq x^0 \), \( x \) has \( k \) direct successors and \( x^0 \) has \( k + 1 \).

The collection \( S(x) = \{ y_1, \ldots, y_k \} \) we will call one-level \( k \)-tuple of neighbours. Note that if \( k = 2 \) then \( S(x) \) is a one-level next-nearest-neighbours.

Below we will consider a semi-infinite Cayley tree \( \Gamma_k^\infty \) of \( k \)-th order, i.e. an infinite graph without cycles with \( (k+1) \) edges issuing from each vertex except for \( x^0 \) which has only \( k \) edges. In this case \( |S(x)| = k \) for any \( x \in V \).

**The Model** For the Vannimenus model with spin values in \( \Phi = \{-1,1\} \), the relevant Hamiltonian with competing nearest-neighbour and prolonged next-nearest-neighbour binary interactions has the form

\[
H(\sigma) = -J_1 \sum_{<x,y>} \sigma(x)\sigma(y) - J_2 \sum_{>x,y<} \sigma(x)\sigma(y) - J_3 \sum_x \prod_{y \in S(x)} \sigma(y) - h \sum_x \sigma(x),
\]

where the sum in the first term ranges over all prolonged next-nearest-neighbours, the second sum ranges over all one-level \( k \)-tuple neighbours, the third sum ranges over all nearest neighbours and the spin variables \( \sigma(x) \) assume the values \( \pm 1 \). Here \( J_1, J_2, J_3, h \in R \) are coupling constants. This model recovers that of Vannimenus for \( k = 2, J_3 = 0 \), that of Inawashiro *et al* for \( k = 2, J_2 = J_3 \), that Mariz *et al* for \( k = 2 \) and that Ganikhodjaev *et al* for \( h = 0 \). As noted above the one-level \( k \) tuple neighbour interaction \( J_3 \) represent interest not only for physical and biological models but also for models in sociology [22], [23].

3. Recurrence Relations

In order to set up the iterative scheme we sum successively over spins as shown schematically in Fig.1.

![Fig. 1](image)

**Fig. 1.** (a) Three successive generations of a Cayley tree; (b) Schematic illustration of selective summation of spins

Simple but tedious algebra gives

\[
\sum_{\sigma_{A_i} = \pm 1} \exp \left[ (K_1 \sigma_1 + K_2 \sigma_2) \cdot \sum_{i=1}^k \sigma_{A_i} + K_3 \sum_{i=1<j}^k \sigma_{A_i} \sigma_{A_j} + X \sum_{i=1}^k \sigma_{A_i} \right] = C \exp[W \sigma_1 \sigma_2 + U \sigma_2 + V \sigma_1]
\]

(2)
where $K_i \equiv J_i/k_BT$, $i = 1, 2, 3$ and

\[ U(X, K_1, K_2, K_3) \equiv U = \frac{1}{4} \ln \frac{\omega(1, 1)\omega(1, -1)}{\omega(-1, 1)\omega(-1, -1)} \]  

(3)

\[ V(X, K_1, K_2, K_3) \equiv V = \frac{1}{4} \ln \frac{\omega(1, 1)\omega(-1, 1)}{\omega(1, -1)\omega(-1, -1)} \]  

(4)

\[ W(X, K_1, K_2, K_3) \equiv W = \frac{1}{4} \ln \frac{\omega(1, 1)\omega(-1, -1)}{\omega(-1, 1)\omega(-1, -1)} \]  

(5)

\[ C(X, K_1, K_2, K_3) \equiv C = [\omega(1, 1)\omega(-1, -1)\omega(1, 1)\omega(-1, 1)]^\frac{1}{4} \]  

(6)

where the value $\omega$ depends from evenness of $k$.

If $k$ is odd, then

\[ \omega(\sigma_1, \sigma_2) = 2 \sum_{i=0}^{k-1} \binom{k}{i} \exp \left[ \frac{k^2 - 4ki - k + i^2 + 3i}{2} \cdot K_3 \right] \cosh[(k - 2i)(K_1\sigma_1 + K_2\sigma_2 + X)] \]  

(7)

and if $k$ is even, then

\[ \omega(\sigma_1, \sigma_2) = 2 \sum_{i=0}^{k-1} \binom{k}{i} \exp \left[ \frac{k^2 - 4ki - k + i^2 + 5i}{2} \cdot K_3 \right] \cosh[(k - 2i)(K_1\sigma_1 + K_2\sigma_2 + X)] \]  

\[ + \binom{k}{k/2} \exp \left[ \frac{6k-3k^2}{8} \cdot K_3 \right] \]  

(8)

Beginning with $X^{(1)} = B = h/k_BT$ and $K_1^{(1)} = K_1 = J_1/k_BT$ where $h$ is the initial applied magnetic field per site and $J_1$ is the initial nearest-neighbour coupling, we obtain from the above equalities the iteration scheme, for $r = 2, 3, \cdots, N$,

\[ X^{(r)} = B + k \cdot U(X^{(r-2)}, K_1^{(r-2)}, K_2, K_3) + V(X^{(r-1)}, K_1^{(r-1)}, K_2, K_3) \]  

(9)

and

\[ K_1^{(r)} = K_1 + W(X^{(r-1)}, K_1^{(r-1)}, K_2, K_3) \]  

(10)

Equations (9) and (10) with functions $U, V$, and $W$ defined by equations (3–5) and either (7) or (8) constitute basic recursion relations with initial conditions $X^{(0)} = K_1^{(0)} = 0$ and $X^{(1)} = B$, $K_1^{(1)} = K_1$.

4. The Phase Diagram and Attractors

It is convenient to know the broad features of the phase diagram before discussing the different transitions in more detail. This can be achieved numerically in a straightforward fashion. The recursion relations (9-10) provide us the numerically exact phase diagram in $(T/J_1, -J_2/J_1, J_3/J_1, h/J_1)$ space. Let $k_BT/J_1 = \alpha, -J_2/J_1 = \beta, J_3/J_1 = \gamma, h/J_1 = \delta$ and respectively $K_1 = 1/\alpha; K_2 = -\beta/\alpha; K_3 = \gamma/\alpha$ and $B = \delta/\alpha$. By iterating equations (9) and (10) numerically for selected values of $\alpha, \beta, \gamma, \delta$ we consider limiting behaviour of the sequence of points $\{(X^{(n)}; K_1^{(n)}) : n = 2, 3, \cdots\}$ in the $X - K_1$ plane and describe phases. Each phase: Paramagnetic (P), Ferromagnetic (F), Modulated (M), Antiphase ($<2>$), Antiferromagnetic (AF) is characterized by a particular attractor in the $X - K_1$ plane, where $X^{(n)}$ plays the role of an effective field. The phase diagram is obtained by following the evolution and detecting the qualitative changes of these attractors. These changes can be either continuous
or abrupt, respectively, characterizing second- or first-order transitions. We find firstly that for any $k$ when $J_1 > 0, J_2 > 0$ and $J_3$ is arbitrary, $X^{(n)}$ and $K_1^{(n)}$ approach a stable fixed point $(\tilde{X}, \tilde{K}_1)$ corresponding to a ferromagnetic phase when $\tilde{X} > 0$ and paramagnetic phase when $\tilde{X} = 0$. The same phases result when $J_2$ is negative and small in magnitude relative to $J_1 > 0$ where this magnitude is smaller and smaller with increasing $k$. On the other hand when $J_2$ is negative and sufficiently large in magnitude relative to $J_1 > 0$, $X^{(n)}$ and $K_1^{(n)}$ iterate to a four cycle of points in the $X - K_1$ plane yielding a modulated antiphase <2> of the same form discussed previously by Katsura and Takizawa [20] and later on by Vannimenus [9].

The interesting region occurs for intermediate negative values of $J_2$ and both positive and negative values of $J_1$ where frustration takes hold and the system cannot decide whether it wants to be ferromagnetic or antiferromagnetic. Below we plot the attractors associated with the mappings (9)-(10) for some selected values of $\alpha, \beta, \gamma, \delta$ with $\delta = +0.1$, $\delta = 0$, and $\delta = -0.1$ to clarify the role of external magnetic field. The first $10^4$ iterations are discarded; the subsequent $10^4$ iterations are plotted.

![Figure 2](image2.png)

**Figure 2.** The attractors in $X - K_1$ plane for $k = 2$, $\gamma = -0.35$, $\beta = 0.35$, and $\alpha = 0.11$ with external magnetic for (a) $\delta = +0.1$, (b) $\delta = 0$ and (c) $\delta = -0.1$

The phase corresponding to the attractors in Fig. 2 is a periodic with period 6 and one can see that under external magnetic field the shape of attractor consisting from 6 points a little bit deformed preserving the number of points.

![Figure 3](image3.png)

**Figure 3.** The attractors in $X - K_1$ plane for $k = 2$, $\gamma = -0.475$, $\beta = 0.475$, and $\alpha = 0.13$ with external magnetic for (a) $\delta = +0.1$, (b) $\delta = 0$ and (c) $\delta = -0.1$

The phase corresponding to the attractors in Fig.3 also is a periodic, however one can see that under external magnetic field the shape and cardinality, i.e., the number of points, of attractor consisting from 14 points for $h = 0$ a little bit changed. For $d = \pm 0.1$ the cardinality of corresponding attractor is 19.
The attractors in Fig.4 are finite set, and one can see that under external magnetic field the cardinality of corresponding attractor essentially increase.

![Figure 4](image-url) The attractors in $X - K_1$ plane for $k = 2$, $\gamma = -0.4$, $\beta = 0.4$, and $\alpha = 0.13$ with external magnetic for (a) $\delta = +0.1$, (b) $\delta = 0$ and (c) $\delta = -0.1$

In Fig. 5, the attractors are "strange" and one can see that the shape of attractor preserved under external magnetic field.

![Figure 5](image-url) The attractors in $X - K_1$ plane for $k = 2$, $\gamma = -1.35$, $\beta = 0.9$, and $\alpha = 0.48$ with external magnetic for (a) $\delta = +0.1$, (b) $\delta = 0$ and (c) $\delta = -0.1$

Let us consider the attractors in Fig. 6. Here for $h = 0$ the corresponding phase is incommensurate, however in the presence of external magnetic field the corresponding phase transforms into commensurate phase.

![Figure 6](image-url) The attractors in $X - K_1$ plane for $k = 2$, $\gamma = -0.38$, $\beta = 0.38$, and $\alpha = 0.11$ with external magnetic for (a) $\delta = +0.1$, (b) $\delta = 0$ and (c) $\delta = -0.1
The study of additional aspects of this model is planned to be the subject of forthcoming publications.

5. Conclusions
In this paper we have studied the Vannimenus model on an arbitrary order Cayley tree with competing nearest-neighbour and next-nearest-neighbour interactions in the presence of external magnetic field. We obtain recursion relations for effective fields $X_r$ and nearest-neighbour interactions $K_r$ which are rigorous, but which must in general be analyzed numerically. We clarify the role of external magnetic field.

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