HÖLDER ESTIMATES FOR SOLUTIONS OF THE CAUCHY PROBLEM FOR THE POROUS MEDIUM EQUATION WITH EXTERNAL FORCES

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ABSTRACT. We study the interior Hölder regularity problem for weak solutions of the porous medium equation with external forces. Since the porous medium equation is the typical example of degenerate parabolic equations, Hölder regularity is a delicate matter and does not follow by classical methods. Caffarelli-Friedman, and Caffarelli-Vázquez-Wolansky showed Hölder regularity for the model equation without external forces. DiBenedetto and Friedman showed the Hölder continuity of weak solutions with some integrability conditions of the external forces but they did not obtain the quantitative estimates. The quantitative estimates are important for studying the perturbation problem of the porous medium equation. We obtain the scale invariant Hölder estimates for weak solutions of the porous medium equations with the external forces. As a particular case, we recover the well known Hölder estimates for the linear heat equation.

1. INTRODUCTION

We consider the following degenerate parabolic equation:

\[
\begin{aligned}
\partial_t u - \Delta u^m &= \text{div} f + g, & t > 0, & x \in \mathbb{R}^n, \\
\end{aligned}
\]

where \( m > 1 \) is a constant, \( u = u(t, x) : (0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) is unknown, \( u_0 = u_0(x) : \mathbb{R}^n \to [0, \infty) \), \( f = f(t, x) : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n \) and \( g = g(t, x) : (0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) are given. For \( f, g \equiv 0 \), the equation (1.1) is called the porous medium equation. The equation (1.1) is a degenerate parabolic equation since the diffusion coefficient \( mu^{m-1} \) may vanish. It is well-known that solutions of the degenerate parabolic equation (1.1) are not generally smooth even if the initial datum \( u_0 \) is smooth enough. We now introduce the notion of weak solutions.

Definition 1.1. For \( u_0 \in L^1(\mathbb{R}^n) \) and for \( f, g \in L^1((0, \infty) \times \mathbb{R}^n) \), we call \( u \) a weak solution of (1.1) if there exists \( T > 0 \) such that

1. \( u(t, x) \geq 0 \) for almost all \( (t, x) \in [0, T] \times \mathbb{R}^n \);
2. \( u \in L^\infty(0, T; L^1(\mathbb{R}^n) \cap L^{m+1}(\mathbb{R}^n)) \) with \( \nabla u^m \in L^2((0, T) \times \mathbb{R}^n) \);
3. \( u \) satisfies (1.1), namely for all \( \varphi \in C^1(0, T; C_0^1(\mathbb{R}^n)) \) and for almost all \( 0 < t < T \),

\[
\int_{\mathbb{R}^n} u(t)\varphi(t) \, dx - \int_0^t \int_{\mathbb{R}^n} u\partial_t \varphi \, dx \, dt + \int_0^t \int_{\mathbb{R}^n} \nabla u^m \cdot \nabla \varphi \, dx \, dt \\
= \int_{\mathbb{R}^n} u_0\varphi(0) \, dx - \int_0^t \int_{\mathbb{R}^n} f \cdot \nabla \varphi \, dx \, dt + \int_0^t \int_{\mathbb{R}^n} g \varphi \, dx \, dt.
\]

We remark that the existence of weak solutions is shown by Oleinik-Kalašnikov-Čzhou [19] and J. L. Lions [16] (cf. Ōtani [22]). Our aim in this paper is to obtain a priori Hölder estimates for weak solutions of (1.1).

Caffarelli-Friedman [5] and Caffarelli-Vázquez-Wolanski [6] showed Hölder continuity for solutions of the porous medium equation. They essentially use a pointwise estimates for the derivative of solutions given by Aronson-Benilan [2] and the comparison principle for the porous medium equation. For the general case with the external force (1.1), the Aronson-Benilan type estimate is not known. In addition, if the equation involves non-local effect such as the system with other equations, the comparison principle does not generally hold. For instance, we consider the following degenerate Keller-Segel system:

\[
\begin{aligned}
\partial_t u - \Delta u^m + \text{div}(u \nabla \psi) &= 0, & t > 0, & x \in \mathbb{R}^n, \\
-\Delta \psi + \psi &= u, & t > 0, & x \in \mathbb{R}^n, \\
u(0, x) &= u_0(x), & x \in \mathbb{R}^n.
\end{aligned}
\]

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It is known by Sugiyama-Kunii [24] that there exists a time global bounded weak solution $(u, \psi)$ of (1.2) in the case of $1 \leq m \leq 2 - \frac{2}{n}$ and $n \geq 3$ for small initial data. Regularity estimates of solutions of (1.2) are closely related to the large time asymptotic behavior of solutions of (1.2) (cf. Luckhaus-Sugiyama [17], Ogawa [20], Ogawa-Mizuno [21]), however comparison principles do not hold for (1.2). Therefore it is an worth to derive the regularity of the weak solution of (1.1) without using the comparison principle.

On the other hand, DiBenedetto-Friedman [9], Wiegner [26] considered the $p$-Laplace evolution equation:

\begin{equation}
\tag{1.3}
\begin{cases}
\partial_t v - \text{div}(\sqrt[2-p]{\nabla v}) = 0, & t > 0, \ x \in \mathbb{R}^n, \\
v(0, x) = v_0(x), & x \in \mathbb{R}^n.
\end{cases}
\end{equation}

The $p$-Laplace evolution equation is a typical example of degenerate parabolic equations. They showed the Hölder continuity for the gradient of the solutions of (1.3) by using the method of alternative and intrinsic rescaling. Misawa [18] showed the gradient Hölder estimates for more general $p$-Laplace evolution equations. We remark that they do not rely on the comparison principle for the $p$-Laplace evolution equation (1.3). Roughly speaking, the gradient of the solution may be regarded to satisfy (1.1) with $f, g \equiv 0$ and it seems possible to apply their methods for solutions of (1.1). In fact, DiBenedetto-Friedman [9] showed Hölder continuity for solutions of (1.1) with $f, g \equiv 0$ and $m > 1$. They also mentioned the Hölder continuity of the weak solution of (1.1) involving the external forces $f \in L^1(0, \infty; L^p(\mathbb{R}^n))$, $g \in L^\frac{n}{m}(0, \infty; L^\frac{n}{m}(\mathbb{R}^n))$ with $\frac{2}{q} + \frac{n}{m} < 1$. In this paper we extend the above mentioned results to the case of general external forces, more specifically, we prove Hölder continuity for bounded weak solutions of (1.1). In addition, we obtain Hölder estimates with explicit dependence on the external forces $f$ and $g$.

It is well-known that Harnack estimates are closely related to Hölder continuity of solutions (cf. Aronson-Caffarelli [3], DiBenedetto [7], [8] and DiBenedetto-Gianazza-Vespri [10], [11]). We remark that the porous medium equation (1.1) is not additive, in particular for a solution $u$ of (1.1) and a constant $k \in \mathbb{R}$, both $u - k$ and $k - u$ do not satisfy (1.1). Thus the Harnack inequality does not imply Hölder continuity of solutions of (1.1) directly. DiBenedetto-Gianazza-Vespri [10], [11] pointed out this fact and considered Hölder estimates for the singular porous medium equation, namely (1.1) with $0 < m < 1$. It is also well-known that regularity of the gradient of solutions imply Hölder continuity. Gradient estimates for $p$-Laplace evolution equations are recently studied by Kinnunen-Lewis [13], Acerbi-Mingione [1], Duzaar-Mingione [12] and Kuusi-Mingione [14].

Before stating our main theorem, we introduce weak $L^p$ spaces.

**Definition 1.2.** For a domain $\Omega \subset \mathbb{R}^n$ and an exponent $p > 1$, a function $f \in L^p_{\text{loc}}(\Omega)$ belongs to $L^p_w(\Omega)$ if

$$
\|f\|_{L^p_w(\Omega)} := \sup_{K \subset \Omega \text{ compact}} \frac{1}{|K|^{1/p}} \int_K |f| \, dx < \infty.
$$

**Remark 1.3.** By Hölder inequality, we find $L^p(\Omega) \subset L^p_w(\Omega)$. In fact, $L^p_w(\Omega)$ is strictly larger than $L^p(\Omega)$ since $|x|^{-\frac{2}{p}} \notin L^p(\mathbb{R}^n)$ but belonging to $L^p_w(\mathbb{R}^n)$.

Now, we state our main theorem.

**Theorem 1.4.** Let $m > 1$ and let $u$ be a bounded weak solution of (1.1). Assume $f \in L^q(0, \infty; L^p_w(\mathbb{R}^n))$ and $g \in L^{\frac{n}{m}}(0, \infty; L^{\frac{n}{m}}(\mathbb{R}^n))$ for some $p, q > 2$ satisfying $\frac{2}{q} + \frac{n}{m} < 1$. Then, for all $\varepsilon > 0$, the solution $u$ is uniform Hölder continuous with respect to $(t, x)$ in $(\varepsilon, \infty) \times \mathbb{R}^n$. Precisely, there exist constants $C, \sigma > 0$ such that

\begin{equation}
\tag{1.4}
|u(t, x) - u(s, y)| \leq C (\|u\|_{L^\infty((0, \infty) \times \mathbb{R}^n)} + \|u\|_{L^\infty((0, \infty) \times \mathbb{R}^n)}^{(1-\frac{4}{n})} \|f\|_{L^{\frac{n}{m}}(0, \infty; L^\frac{n}{m}(\mathbb{R}^n))} + \|u\|_{L^\infty((0, \infty) \times \mathbb{R}^n)}^{\frac{2}{m} - \frac{n}{m}} \|g\|_{L^{\frac{n}{m}}(0, \infty; L^{\frac{n}{m}}(\mathbb{R}^n))})
\end{equation}

for all $(t, x), (s, y) \in (\varepsilon, \infty) \times \mathbb{R}^n$, where $\sigma > 0$ depends only on $n, m, p, q$ and $C > 0$ depends only on $n, m, p, q, \varepsilon$.

When the initial datum $u_0$ is bounded positive and the external force $\text{div} f + g$ is bounded, then solutions of (1.1) is bounded on some time interval $(0, T)$ by the maximum principle. Sugiyama-Kunii [24] and Ogawa [20] showed the boundedness for the solution and rescaled solution of (1.2) hence we may apply our results for (1.2).
We emphasize that our estimate (1.4) is scale invariant. In fact, for the parameter $M > 0$, we consider the scale transform

$$
(1.5) \quad t = \frac{1}{M^{m-1}} s, \quad u_M(s, x) = \frac{1}{M} u(t, x),
$$

Then $u_M$ satisfies

$$
\partial_s u_M - \Delta u_M^m = \text{div} \left( \frac{1}{M^m} f_M \right) + \left( \frac{1}{M^m} g_M \right)
$$

and Theorem 1.4 implies

$$
|u_M(s, y) - u_M(s', y')| \leq C \left< u_M^{L^\infty((0, \infty) \times \mathbb{R}^n)} \right. + \left. \left( \frac{1}{M^m} \right)^{\frac{s}{1-s}} \left< u_M^{L^\infty((0, \infty) \times \mathbb{R}^n)} \right> \right. \left. \left( f_M \right)^{\frac{1}{1-s}} \right. \left. \left( g_M \right)^{\frac{1}{1-s}} \right. \left. \left. \times \left| s - s' \right|^{\sigma} + |y - y'|^\sigma \right).
$$

By the scale transform (1.5), we have (1.4) hence we find that the Hölder estimates (1.4) is invariant for the scale transform (1.5).

Our Hölder estimates (1.4) is some generalization for the case of the heat equation. Actually, letting $m \to 1$, we find that the constant $C > 0$ is bounded and the Hölder exponent $\sigma > 0$ is away from 0. Therefore the estimates (1.4) implies the well-known Hölder estimates for the case of the heat equation.

The basic strategy to prove Theorem 1.4 is to use the method of alternative and intrinsic scaling by DiBenedetto-Friedman [21]. Since they use the local oscillation of solutions as the intrinsic scaling, it seems difficult to obtain Hölder estimates of solutions. On the other hand, we use the local maximum of solutions as the intrinsic scaling and we make the more exact Caccioppoli estimate. The Caccioppoli estimate plays an important role to show the method of alternative. Reconstructing the iteration argument, we obtain the Hölder estimates of solutions.

For an application, we may consider the external force as the perturbation of solutions (cf. Ogawa-Mizuno [21]). Applying our theorem, we do not need $L^p$ integrability of the external force, but growth order of $L^2$ integral. Therefore, it is useful to study $L^2$ theory of non-linear degenerate parabolic equations. Furthermore, we can exactly estimate the Hölder norm of solutions by the external force and the maximum of solutions.

The paper is organized as follows. In section 2, we first give an alternative lemma and show Theorem 1.4 using the alternative lemma. The alternative lemma gives either the better lower bounds or the better upper bounds of solutions. We show the lower bounds of the solution in section 3 and the upper bounds of solution in section 4. In appendix, we give some fundamental results of calculus which are necessary for the proof of the main theorem.

At the end of this section, we introduce some notations. For $\rho, \sigma, \theta_0 > 0$ and $t_0 \in \mathbb{R}$, we let open intervals

$$
I_\rho(t_0) = (t_0 - \rho^2, t_0), \quad I_\rho^{0,\rho}(t_0) = \left( t_0 - \frac{\rho^2}{2}, t_0 \right)
$$

and

$$
I_{\rho,\sigma}(t_0) = \left( t_0 - \frac{\rho^2}{M^{1-\frac{\sigma}{2}}}, t_0 \right), \quad I_{\rho,\sigma,\rho}(t_0) = \left( t_0 - \frac{\rho^2}{2 M^{1-\frac{\sigma}{2}}}, t_0 \right).
$$

For $\rho > 0$ and $x_0 \in \mathbb{R}^n$, we denote the $n$-dimensional open ball with radius $\rho$ and center $x_0$ by $B_{\rho}(x_0)$. We define parabolic cylinders $Q_\rho(t_0, x_0)$, $Q_\rho^{0,\rho}(t_0, x_0)$ and modified parabolic cylinders $Q_{\rho,\sigma}(t_0, x_0)$, $Q_{\rho,\sigma,\rho}(t_0, x_0)$ by

$$
Q_\rho(t_0, x_0) = I_\rho(t_0) \times B_{\rho}(x_0), \quad Q_\rho^{0,\rho}(t_0, x_0) = I_\rho^{0,\rho}(t_0) \times B_{\rho}(x_0)
$$

and

$$
Q_{\rho,\sigma}(t_0, x_0) = I_{\rho,\sigma}(t_0) \times B_{\rho}(x_0), \quad Q_{\rho,\sigma,\rho}(t_0, x_0) = I_{\rho,\sigma,\rho}(t_0) \times B_{\rho}(x_0).
$$

We often abbreviate the center of parabolic cylinders $(t_0, x_0)$. We denote the $n$-dimensional Lebesgue measure by $m_n$. A function space $L^q(I_{\rho,\sigma}; L^p_{\rho,\sigma}(B_{\rho}))$ is abbreviated to $L^q(L^p_{\rho,\sigma}(B_{\rho}))$ and another function spaces are also same. For a function $f$ on a set $A$, we denote the oscillation of $f$ in $A$ by $\text{osc}_A f := \sup_A f - \inf_A f$. We denote the positive part of $f$ and the negative part of $f$ by $f_+ := \max\{0, f\}$ and $f_- := \max\{0, -f\}$, respectively. We
remark that a superscript plus or minus are different of the positive part or the negative part. For a constant $k \in \mathbb{R}$ and a function $f$ on a set $\Omega$, we let
$$
\{ f > k \} := \{ x \in \Omega : f(x) > k \}
$$
and other level sets such as $\{ f < k \}$ are defined in a similar manner. We put $\sigma_0 = 1 - \frac{2}{p} - \frac{2}{q}$ and $h(\rho, M, \omega) := \| f \|_{L^n(L^q(Q_{\rho, M})} + \omega \| g \|_{L^\infty(L^\infty(Q_{\rho, M})}$. We denote a constant depending on $m, \beta, \ldots$ by $C(m, \beta, \ldots)$. The same letter $C$ will be used to denote different constants. We use subscript numbers if we consider the relation between the constants. For a open interval $(a, b) \subset \mathbb{R}$ and a open ball $B_{\rho}(x_0) \subset \mathbb{R}^n$, we call $\eta = \eta(t, x)$ a cut-off function in $Q = (a, b) \times B_{\rho}(x_0)$ if $\eta \in C^\infty(Q)$ satisfies
$$
\eta(t, x) = 0 \quad a \leq t \leq b, \quad x \in \partial B_{\rho}(x_0) \quad \text{and} \quad \eta(a, x) = 0 \quad x \in B_{\rho}(x_0).
$$

2. Alternative lemma and proof of the main theorem

We hereafter replace $u^m$ by $u$ and we consider the following equation:

$$
\partial_t u + \Delta u = - \text{div} f + g.
$$

Let $M$ and $\omega$ be an approximated supremum and oscillation of the weak solution $u$ of (2.1), namely

$$
\sup_{Q_{\rho, M}(t_0, x_0)} u \leq M \leq 3 \sup_{Q_{\rho, M}(t_0, x_0)} u,
$$

and

$$
\frac{3}{4} \omega \leq \text{osc}_{Q_{\rho, M}(t_0, x_0)} u \leq \omega.
$$

**Lemma 2.1** (alternative lemma). *Let us assume (2.2) and (2.3). Then there exist constants $0 < \theta_0, \eta_0 < 1$ and $\delta_0 > 0$ depending only on $n, m, p, q$ such that for all $\rho > 0$ satisfying $\rho^{\sigma_0} \leq \delta_0 \omega M^{-\frac{1}{2}}(1 - \frac{1}{p}) h(\rho, M, \omega)^{-\frac{1}{2}}$, we obtain the following estimates:

(i) **Lower bounds.** If

$$
m_{n+1} \left( Q_{\rho, M}(t_0, x_0) \cap \left\{ u < \inf_{Q_{\rho, M}(t_0, x_0)} u + \frac{\omega}{2} \right\} \right) \leq \theta_0 m_{n+1} \left( Q_{\rho, M}(t_0, x_0) \right),
$$

then

$$
u(t, x) \geq \inf_{Q_{\rho, M}(t_0, x_0)} u + \eta_0 \omega \quad \text{for} \ (t, x) \in Q_{\frac{\rho}{2}, M}(t_0, x_0);
$$

(ii) **Upper bounds.** If

$$
m_{n+1} \left( Q_{\rho, M}(t_0, x_0) \cap \left\{ u < \inf_{Q_{\rho, M}(t_0, x_0)} u + \frac{\omega}{2} \right\} \right) > \theta_0 m_{n+1} \left( Q_{\rho, M}(t_0, x_0) \right),
$$

then

$$
u(t, x) \leq \sup_{Q_{\rho, M}(t_0, x_0)} u - \eta_0 \omega \quad \text{for} \ (t, x) \in Q_{\frac{\rho}{2}, M}(t_0, x_0).
We will prove part (i), which is Proposition [5.1] in Section 3 and part (ii), which is Proposition [4.1] in Section 4. According to Lemma 2.1, we obtain

\[ (2.4) \quad \text{osc}_{Q_0}^0 \rho, M(t_0, x_0) \leq \text{osc}_{Q_0}^0 \rho, M(t_0, x_0) \leq (1 - \eta_0)\omega \]

provided \( \rho > \delta_0 M^{-\frac{\theta}{2}}(1 - \frac{\theta}{2}) h(\rho, M, \omega)^{-\frac{1}{2}} \). We remark that we may take \( \eta_0 \) as small as we want since we obtain by (2.4)

\[ \text{osc}_{Q_0}^0 \rho, M(t_0, x_0) \leq (1 - \eta_0)\omega \leq (1 - \eta)\omega \]

for any \( 0 < \eta < \eta_0 \).

**Remark 2.2.** We explain an advantage to use the modified parabolic cylinder. For \( \rho \ll 1 \) and \( M > 0 \), we consider

\[ \partial_t u^\rho - \Delta u = - \text{div} f + g \quad \text{in} \ Q_{\rho, M}. \]

Introducing the scale transform

\[ t = \frac{\rho^2}{M^{1-\frac{\theta}{2}}} s, \quad x = \rho y, \]

\[ u_{\rho, M}(s, y) = \frac{1}{M} u(t, x), \quad f_{\rho, M}(s, y) = f(t, x), \quad g_{\rho, M}(s, y) = g(t, x), \]

we obtain

\[ (2.5) \quad \partial_s u_{\rho, M}^\rho - \Delta y u_{\rho, M} = - \text{div} \left( \frac{\rho^2}{M} f_{\rho, M} \right) + \frac{\rho^2}{M} g_{\rho, M} \quad \text{in} \ Q_1. \]

Since \( M \) can be regarded as the supremum of \( u \) on \( Q_{\rho, M} \) by the assumption (2.2), we may consider (2.5) as the uniformly parabolic equation. Furthermore, in view of

\[ \left\| \frac{\rho^2}{M} f_{\rho, M} \right\|_{L^2(L^\theta}(Q_1) = M^{2(1-\frac{\theta}{2})} \left\| f \right\|_{L^2(L^\theta}(Q_{\rho, M}), \]

\[ \left\| \frac{\rho^2}{M} g_{\rho, M} \right\|_{L^2(L^\theta}(Q_1) = M^{2(1-\frac{\theta}{2})} \left\| g \right\|_{L^2(L^\theta}(Q_{\rho, M}), \]

the inequality \( 1 - \frac{2}{\theta} - \frac{n}{p} > 0 \) is the sufficient condition to ignore the external force.

We now show Theorem [1.4] by temporary admitting Lemma [2.1]. We put \( Q = (0, \infty) \times \mathbb{R}^n \), \( M_0 = \text{sup}_Q u \) and \( \omega_0 = \text{osc}_Q u \). Let \( \theta_0, \delta_0 \) and \( \eta_0 \) be as in Lemma [2.1]. We choose \( 0 < \rho_0 < \varepsilon \) satisfying

\[ \rho_0^\sigma \leq \delta_0 \omega_0 M_0^{-\frac{\theta}{2} \left(1 - \frac{\theta}{2} \right) h(\rho_0, M_0, \omega_0)^{-\frac{1}{2}} \]}.

For \( (t_0, x_0) \in (0, \infty) \times \mathbb{R}^n \), we denote \( Q_0 = Q_{\rho_0, M_0}(t_0, x_0) \), \( \mu_0^+ = \text{sup}_Q u \) and \( \mu_0^- = \text{inf}_Q u \). Then, we find

\[ \begin{cases} \text{osc}_{Q_0} u \leq \omega_0, \\
\sup_{Q_0} u \leq \text{sup}_Q u \leq M_0, \\
\rho_0^\sigma \leq \delta_0 \omega_0 M_0^{-\frac{\theta}{2} \left(1 - \frac{\theta}{2} \right) h(\rho_0, M_0, \omega_0)^{-\frac{1}{2}} \]}.
\]

We choose

\[ r_0 := \min \left\{ (1 - \eta_0)\omega_0, \frac{1}{2} (1 - \frac{\theta}{2}) \left( \frac{\theta_0}{2} \right)^{\frac{1}{2}} \right\} \]

and choose sequences as follows: For \( j \in \mathbb{N} \),

\[ \omega_j := (1 - \eta_0)\omega_{j-1}, \quad \rho_j := r_0 \rho_{j-1}, \]

\[ M_j := \max\{\mu_{j-1}^+, \omega_j\}, \quad Q_j := Q_{\rho_j, M_j}(t_0, x_0), \]

\[ \mu_j^+ := \sup_{Q_j} u, \quad \mu_j^- := \inf_{Q_j} u. \]

Using Lemma [2.1] we obtain the following oscillation estimates.
Lemma 2.3. Let $\{\omega_j, \rho_j, M_j, Q_j\}_{j=0}^\infty$ is defined by the above (2.6). Then for $0 < \delta_0 < 1$ defined in Lemma 2.1 and for $j \in \mathbb{N}$, we obtain

\[
\begin{align*}
\text{osc}_Q u & \leq \omega_j, \\
\sup_{Q_j} u & \leq \sup_{Q_{j-1}} u \leq M_j, \\
\rho_j^{\sigma_0} & \leq \delta_0 \omega_j M_j^{-\frac{1}{4}(1-\frac{1}{m})} h(\rho_j, M_j, \omega_j)^{-\frac{1}{4}},
\end{align*}
\]

(2.7)

Proof of Lemma 2.3. By the definition of $M_j$, we obtain $\sup_{Q_j} u \leq \sup_{Q_{j-1}} u \leq M_j$. Since $r_0 \leq (1 - \eta_0) \omega_0$ and definition of $\omega_j$, we find

\[
\rho_j^{\sigma_0} \leq \delta_0 \omega_j M_j^{-\frac{1}{4}(1-\frac{1}{m})} h(\rho_j, M_j, \omega_j)^{-\frac{1}{4}}.
\]

We show $\text{osc}_Q u \leq \omega_j$.

To show $\text{osc}_Q u \leq \omega_j$, we make induction. First we consider the case $j = 1$. Either if $\text{osc}_{Q_0} u \leq \frac{4}{3} \omega_0$, then we find $Q_1 \subset Q_0$ since $r_0 \leq (\frac{3}{4})^{\frac{1}{4}(1-\frac{1}{m})}$ and

\[
M_1 \geq \frac{\omega_1}{M_0} = (1 - \eta_0) \geq \frac{3}{4}.
\]

For this reason, we obtain

\[
\text{osc}_Q u \leq \text{osc}_{Q_0} u \leq \frac{3}{4} \omega_0 \leq (1 - \eta_0) \omega_0 = \omega_1.
\]

Otherwise, if $\frac{4}{3} \omega_0 \leq \text{osc}_{Q_0} u \leq \omega_0$, we obtain $M_0 = \omega_0 \leq \frac{4}{3} \mu_0^+$. Applying Lemma 2.1 we find

\[
\text{osc}_{Q_0^{\mu_0^+}} \sup_{Q_0^{\mu_0^+}} (t_0, x_0) u \leq (1 - \eta_0) \omega_0.
\]

Since $r_0 \leq \frac{1}{3} \left(\frac{1}{3}\right)^{\frac{1}{4}(1-\frac{1}{m})} \left(\frac{\omega_0}{M_0}\right)^{\frac{1}{4}}$, we have $Q_1 \subset Q_0^{\mu_0^+}$ and hence

\[
\text{osc}_{Q_1} u \leq \text{osc}_{Q_0^{\mu_0^+}} u \leq (1 - \eta_0) \omega_0 = \omega_1.
\]

In either case, we obtain (2.7) for $j = 1$. Next we assume (2.7) for $j \leq k$ and we show for $j = k + 1$ using the following inequality:

\[
\mu_{k-1}^+ \leq \max \left\{ \frac{3}{2(1 - \eta_0)} \omega_k, 3 \mu_k^+ \right\}.
\]

(2.8)

To show (2.8), we consider the case $\mu_{k-1}^- \leq \frac{2}{3} \mu_{k-1}^+$ first. Then

\[
\mu_{k-1}^+ \leq \text{osc}_{Q_{k-1}} u + \mu_{k-1}^- \leq \omega_{k-1} + \frac{1}{3} \mu_{k-1}^+
\]

and hence $\mu_{k-1}^+ \leq \frac{3}{2} \omega_{k-1} = \frac{3}{2(1 - \eta_0)} \omega_k$. For the other case, namely if $\mu_{k-1}^- > \frac{2}{3} \mu_{k-1}^+$, then we have $\mu_{k-1}^+ < 3 \mu_{k-1}^-$ and we obtain (2.8).

We show (2.7) for $j = k + 1$. First we consider the case $\text{osc}_{Q_k} u \leq \frac{4}{3} \omega_k$ and we show $Q_{k+1} \subset Q_k$. Either if $M_k = \omega_k$, then

\[
\frac{M_{k+1}}{M_k} = \frac{M_{k+1}}{\omega_k} \geq \frac{(1 - \eta_0) \omega_k}{\omega_k} = (1 - \eta_0) \geq \frac{3}{4},
\]

Since $r_0 \leq (\frac{4}{3})^{\frac{1}{4}(1-\frac{1}{m})}$, we obtain $Q_{k+1} \subset Q_k$. Otherwise, if $M_k = \mu_{k-1}^+$, we obtain by (2.8)

\[
\frac{M_{k+1}}{M_k} = \mu_{k-1}^+ \geq \max \left\{ \frac{3}{2(1 - \eta_0)} \omega_k, 3 \mu_k^+ \right\}
\]

\[
\geq \max \left\{ \frac{3}{2(1 - \eta_0)} \omega_k, 3 \mu_k^+ \right\} \geq \frac{1}{3}.
\]
Since \( r_0 \leq \left( \frac{3}{4} \right)^{(1-\frac{1}{q})} \), we have \( Q_{k+1} \subset Q_k \). In either case, we have \( Q_{k+1} \subset Q_k \) and hence

\[
\text{osc}_{Q_{k+1}} u \leq \text{osc}_{Q_k} u \leq \frac{3}{4} \omega_k \leq \omega_{k+1}.
\]

Second we consider the case \( \frac{3}{4} \omega_k \leq \text{osc}_{Q_k} u \leq \omega_k \). Since \( \omega_k \leq \frac{4}{3} \mu_k^+ \), we obtain

\[
\mu_{k-1}^+ \leq \max \left\{ \frac{3}{2(1-\eta_0)} \omega_k, 3\mu_k^+ \right\} \leq \max \left\{ \frac{2}{1-\eta_0} \mu_k^+, 3\mu_k^+ \right\} \leq 3\mu_k^+
\]

and hence

\[
M_k \leq \max \left\{ \frac{4}{3} \mu_k^+, 3\mu_k^+ \right\} \leq 3\mu_k^+.
\]

Hence we may apply Lemma 2.1 and we obtain

\[
\text{osc}_{Q_{k+1}} u \leq (1-\eta_0)\omega_k = \omega_{k+1}.
\]

Since \( r_0 \leq \frac{1}{2} \left( \frac{3}{4} \right)^{(1-\frac{1}{q})} \left( \frac{2}{3} \right)^{\frac{1}{q}} \) and \( \frac{M_{k+1}}{M_k} \geq \frac{\mu^+}{\mu_k} = \frac{1}{3} \), we have \( Q_{k+1} \subset Q_{\frac{1}{3} \mu_k, M_k} \subset Q_{\frac{1}{3} \mu_k, x_0} \) and hence

\[
\text{osc}_{Q_{k+1}} u \leq \text{osc}_{Q_{\frac{1}{3} \mu_k, x_0}} u \leq \omega_{k+1}.
\]

\[\square\]

**Proof of Theorem 1.2** Remarking that \( M_j \geq M_{j+1} \) for \( j \in \mathbb{N} \), we have by Lemma 2.8

\[
\text{osc}_{Q_{\rho_j, M_j(x_0, t_0)}} u \leq \text{osc}_{Q_j} u \leq \omega_j.
\]

We choose \( 0 < \sigma < 1 \) satisfying \( r_0^\sigma \geq 1 - \eta_0 \). Then we obtain

\[
\text{osc}_{Q_{\rho_j, M_j(x_0, t_0)}} u \leq (1-\eta_0)^j \omega_0 = \omega_0 \left( \frac{\rho_k}{\rho_0} \right)^\sigma.
\]

For \( \rho \leq \rho_0 \), there exists \( k \in \mathbb{N} \) such that \( \rho_k \leq \rho \leq \rho_{k-1} \) and hence

\[
\text{osc}_{Q_{\rho, M_0(x_0, t_0)}} u \leq \omega_0 \left( \frac{\rho_k}{\rho_0} \right)^\sigma \leq M_0 r_0^\sigma \left( \frac{\rho_k}{\rho_0} \right)^\sigma.
\]

Taking \( \rho_0 > 0 \) as

\[
\rho_0^\sigma = \delta_0 \omega_0 M_0^{-\frac{1}{q} \left( 1 - \frac{1}{m} \right)} \left( \|f\|_{L^q(L^p)} + \omega_0 \|g\|_{L^q(L^p)} \right)^{\frac{1}{q} \left( 1 - \frac{1}{m} \right)}
\]

we find

\[
(2.9) \quad \text{osc}_{Q_{\rho, M_0(x_0, t_0)}} u \leq C M_0^{-\frac{1}{q} \left( 1 - \frac{1}{m} \right)} \left( \|f\|_{L^q(L^p)} + \omega_0 \|g\|_{L^q(L^p)} \right) \frac{1}{r_0^\sigma} \rho^\sigma
\]

for \( \rho \leq \rho_0 \) where the constant \( C \) depends only on \( n, m, p \) and \( q \). Furthermore, if \( \rho > \rho_0 \), then

\[
\text{osc}_{Q_{\rho, M_0(x_0, t_0)}} u \leq M_0 \left( \frac{\rho}{\rho_0} \right)^\sigma \leq C M_0^{-\frac{1}{q} \left( 1 - \frac{1}{m} \right)} \left( \|f\|_{L^q(L^p)} + \omega_0 \|g\|_{L^q(L^p)} \right) \frac{1}{r_0^\sigma} \rho^\sigma.
\]

Therefore, we find

\[
\text{osc}_{Q_{\rho, M_0(x_0, t_0)}} u \leq C \left( M_0 + M_0^{\frac{1}{q} \left( 1 - \frac{1}{m} \right)} \right) \left( \|f\|_{L^q(L^p)} + \omega_0 \|g\|_{L^q(L^p)} \right) \frac{1}{r_0^\sigma} \rho^\sigma
\]

and proof of Theorem 1.4 is complete. \[\square\]
Proposition 3.1. We hereafter write

\[ m_n + 1 \left( Q_{\rho, M} \cap \left\{ u < \mu^- + \frac{\omega}{2} \right\} \right) \leq \theta_0 m_n + 1 (Q_{\rho, M}), \]

then

\[ u(t, x) \geq \mu^- + \frac{\omega}{4} \text{ for } (t, x) \in Q_{\rho, M}. \]

To show the lower bounds, the following Caccioppoli estimate plays an important role.

Lemma 3.2 (the Caccioppoli estimate for sub-level sets). Let \( \eta = \eta(t, x) \) be a cut-off function in \( Q_{\rho, M} \). For \( \mu^- < k < \mu^- + \frac{\omega}{4} \), there exists a constant \( C > 0 \) depending only on \( m \) such that

\[ \sup_{t \in Q_{\rho, M}} \int_{B_{\rho}} (u(t) - k)^2 \eta^2 \, dx + (\mu^+)^{1 - \frac{1}{\omega}} \int_{Q_{\rho, M}} |\nabla (u - k)| \eta^2 \, dt \, dx \]

\[ \leq C \left\{ \omega \int_{Q_{\rho, M}} (u - k)^2 \eta \partial_t \eta \, dt \, dx + (\mu^+)^{1 - \frac{1}{\omega}} \int_{Q_{\rho, M}} (u - k)^2 |\nabla \eta|^2 \, dt \, dx \right. \]

\[ + (\mu^+)^{1 - \frac{1}{\omega}} h(\rho, M, \omega) \left( \int_{Q_{\rho, M} \cap \{ u(t) < k \} } m_n (B_{\rho} \cap \{ u(t) < k \} ) \right) \left( \frac{1}{2} \right)^{\frac{1}{2}}, \]

where \( \frac{1}{2} = \frac{1}{q} + \frac{1}{q} \).

Proof. Testing a function \(- (u - k)^{- \eta^2} \) in (2.1), we obtain

\[ \frac{1}{m} \int_{Q_{\rho, M}} \int_{0}^{(u - k)^2 (k - \xi)^{1 - \frac{1}{2}} \eta^2 \, dx \, dt} \]

\[ + \int_{Q_{\rho, M}} \nabla (u - k)^2 \cdot \nabla \{ (u - k)^{- \eta^2} \} \, dt \, dx \]

\[ = - \int_{Q_{\rho, M}} f \cdot \nabla \{ (u - k)^{- \eta^2} \} \, dt \, dx - \int_{Q_{\rho, M}} g(u - k)^{- \eta^2} \, dt \, dx. \]

By the integration by parts and the Young inequality, we obtain

\[ \frac{1}{m} \sup_{t \in Q_{\rho, M}} \int_{B_{\rho}} \int_{0}^{(u(t) - k)^2} (k - \xi)^{1 - \frac{1}{2}} \eta^2 \, dx \, dt + \int_{Q_{\rho, M}} |\nabla (u - k)|^{- \eta^2} \, dt \, dx \]

\[ \leq \frac{1}{m} \int_{Q_{\rho, M}} \int_{0}^{(u(t) - k)^2} (k - \xi)^{1 - \frac{1}{2}} \eta^2 \, dx \, dt \]

\[ + 3 \int_{Q_{\rho, M}} (u - k)^2 |\nabla \eta|^2 \, dt \, dx \]

\[ + 2 \int_{Q_{\rho, M} \cap \{ u < k \} } |f|^2 \eta^2 \, dt \, dx + \int_{Q_{\rho, M} \cap \{ u < k \} } |g(u - k)^{- \eta^2} \, dt \, dx. \]

We estimate the 1st term of the left-hand side of (3.2). Since (2.3) and \( k \leq \mu^- + \frac{\omega}{4} \leq \mu^- + \text{osc}_{Q_{\rho, M}} u + \frac{\omega}{4} \leq \mu^+ \), we have

\[ (k - \xi)^{1 - \frac{1}{2}} \geq k^{1 - \frac{1}{2}} \geq (\mu^+)^{1 - \frac{1}{2}} \text{ for } \xi \geq 0. \]
and hence
\[
\frac{1}{2m} \sup_{t \in I_{\rho,M}} \int_{B_{\rho}} (u(t) - k)^2 \eta^2(t) \, dx + \frac{1}{4} (\mu^+)^{1 - \frac{\omega}{\mu}} \int_{Q_{\rho,M}} |\nabla (u - k)|^2 \eta^2 \, dt \, dx
\]
\[
\leq \frac{1}{m} (\mu^+)^{1 - \frac{\omega}{\mu}} \int_{Q_{\rho,M}} \left( \int_0^{(u-k)_-} (k - \xi)^{\frac{\mu}{\omega} - 1} \xi \, d\xi \right) \partial_t \eta^2 \, dt \, dx
\]
\[
+ 3 (\mu^+)^{1 - \frac{\omega}{\mu}} \int_{Q_{\rho,M}} (u - k)^2 |\nabla \eta|^2 \, dt \, dx + 2 (\mu^+)^{1 - \frac{\omega}{\mu}} \int_{Q_{\rho,M} \cap \{u < k\}} |f|^2 \eta^2 \, dt \, dx
\]
\[
+ (\mu^+)^{1 - \frac{\omega}{\mu}} \int_{Q_{\rho,M} \cap \{u < k\}} |g|(u - k)_- \eta^2 \, dt \, dx
\]
\[=: I_1 + I_2 + I_3 + I_4.\]

We estimate \(I_3\) and \(I_4\). By the definition of the weak \(L^p\) space and by the Hölder inequality, we have
\[
\int_{Q_{\rho,M} \cap \{u < k\}} |f|^2 \eta^2 \, dt \, dx \leq \int_{I_{\rho,M}} \int_{B_{\rho} \cap \{u(t) < k\}} |f|^2 \, dx \, dt
\]
\[
\leq \left\| |f(t)| \right\|_{L^p(B_{\rho})} m_n \left( \int_{I_{\rho,M}} \left( B_{\rho} \cap \{u(t) < k\} \right)^{1 - \frac{\omega}{\mu}} \, dt \right)^{\frac{\omega}{\mu}}
\]
\[
\leq \left\| |f|^2 \right\|_{L^p(\rho,M)} \left( \int_{I_{\rho,M}} m_n \left( B_{\rho} \cap \{u(t) < k\} \right)^{\frac{\omega}{\mu}} \, dt \right)^{\frac{\omega}{\mu}},
\]
and
\[
\int_{Q_{\rho,M} \cap \{u < k\}} |g|(u - k)_- \eta^2 \, dt \, dx \leq \frac{\omega}{2} \int_{I_{\rho,M}} \int_{B_{\rho} \cap \{u(t) < k\}} |g| \, dx \, dt
\]
\[
\leq \frac{\omega}{2} \int_{I_{\rho,M}} \left\| g(t) \right\|_{L^q(B_{\rho})} m_n \left( \int_{I_{\rho,M}} \left( B_{\rho} \cap \{u(t) < k\} \right)^{1 - \frac{\omega}{\mu}} \, dt \right)^{\frac{\omega}{\mu}}
\]
\[
\leq \frac{\omega}{2} \left\| g \right\|_{L^q(\rho,M)} \left( \int_{I_{\rho,M}} m_n \left( B_{\rho} \cap \{u(t) < k\} \right)^{\frac{\omega}{\mu}} \, dt \right)^{\frac{\omega}{\mu}}.
\]
Therefore
\[
\int_{Q_{\rho,M} \cap \{u < k\}} |f|^2 \eta^2 \, dt \, dx \leq \frac{\omega}{2} \int_{I_{\rho,M}} \int_{B_{\rho} \cap \{u(t) < k\}} |g| \, dx \, dt
\]
\[
\leq \frac{\omega}{2} \left\| g \right\|_{L^q(\rho,M)} \left( \int_{I_{\rho,M}} m_n \left( B_{\rho} \cap \{u(t) < k\} \right)^{\frac{\omega}{\mu}} \, dt \right)^{\frac{\omega}{\mu}},
\]
and
\[
(3.4) \quad I_3 + I_4 \leq 2 (\mu^+)^{1 - \frac{\omega}{\mu}} h(\rho,M,\omega) \left( \int_{I_{\rho,M}} m_n \left( B_{\rho} \cap \{u(t) < k\} \right)^{\frac{\omega}{\mu}} \, dt \right)^{\frac{\omega}{\mu}}.
\]

We estimate \(I_1\). Since
\[
\int_0^{(u-k)_-} (k - \xi)^{\frac{\mu}{\omega} - 1} \xi \, d\xi \leq -m(u - k)_- \int_0^{(u-k)_-} \frac{\partial}{\partial \xi} (k - \xi)^{\frac{\mu}{\omega}} \, d\xi
\]
\[
= m(u - k)_- [k^{\frac{\mu}{\omega}} - (k - (u - k)_-)^{\frac{\mu}{\omega}}],
\]
we have
\[
I_1 \leq (\mu^+)^{1 - \frac{\omega}{\mu}} \int_{Q_{\rho,M}} [k^{\frac{\mu}{\omega}} - (u - k)_-^{\frac{\mu}{\omega}}] (u - k)_- \partial_t \eta^2 \, dt \, dx
\]
\[
\leq (\mu^+)^{1 - \frac{\omega}{\mu}} \int_{Q_{\rho,M}} \left[ \left( \mu^+ + \frac{\omega}{2} \right)^{\frac{\mu}{\omega}} - (\mu^-)^{\frac{\mu}{\omega}} \right] (u - k)_- \partial_t \eta^2 \, dt \, dx.
\]
Either if \(\mu^- \leq \frac{1}{2} \mu^+\), then \(\mu^+ \leq \omega + \mu^-\) and hence \(\mu^+ \leq 2\omega\). Therefore
\[
(\mu^+)^{1 - \frac{\omega}{\mu}} \left[ \left( \mu^+ + \frac{\omega}{2} \right)^{\frac{\mu}{\omega}} - (\mu^-)^{\frac{\mu}{\omega}} \right] \leq (2\omega)^{1 - \frac{\omega}{\mu}} \left( \frac{\omega}{2} \right)^{\frac{\mu}{\omega}} \leq 2^{1 - \frac{\omega}{\mu}} \omega
\]
and hence
\[
I_1 \leq C(m,\omega) \int_{Q_{\rho,M}} (u - k)_- \partial_t \eta^2 \, dt \, dx.
Otherwise, if \( \mu^- > \frac{1}{2} \mu^+ \), then

\[
\left( \mu^- + \frac{\omega}{2} \right)^{\frac{1}{2m}} - (\mu^-)^{\frac{1}{2m}} = \int_0^1 \frac{d}{ds} \left( \mu^- + \frac{\omega}{2} s \right)^{\frac{1}{2m}} ds
\]

\[
= \frac{\omega}{2m} \int_0^1 \left( \mu^- + \frac{\omega}{2} s \right)^{\frac{1}{2m} - 1} ds \leq \frac{\omega}{2m} \left( \frac{1}{2} \mu^+ \right)^{\frac{1}{2m} - 1},
\]

and hence

\[
(\mu^+)^{\frac{1}{2m}} \left[ \left( \mu^- + \frac{\omega}{2} \right)^{\frac{1}{2m}} - (\mu^-)^{\frac{1}{2m}} \right] \leq \frac{\omega}{2m} \left( \frac{1}{2} \mu^+ \right)^{\frac{1}{2m} - 1} \leq C(m) \omega.
\]

In either case, we obtain

(3.5) \[ I_1 \leq C(m) \omega \int_{Q_{\rho, M}} (u - k) \partial_i \eta^2 \, dt \, dx. \]

Substituting (3.4) and (3.5) for (3.3) we obtain (3.1). \( \square \)

**Proof of Proposition 3.1.** We consider the scale transform

\[
s = M^{1-\frac{1}{4m}} t, \quad \tilde{\varphi}(s, x) = u(t, x), \quad \tilde{\eta}(s, x) = \eta(t, x),
\]

\[
\tilde{f}(s, x) = f(t, x), \quad g(s, x) = g(t, x).
\]

and we put \( \tilde{h}(\rho, \omega) := \| \tilde{f} \|_{L^s(L^\infty)(Q_\rho)}^2 + \omega \| \tilde{g} \|_{L^s(L^\infty)(Q_\rho)}^2 \). We rewrite the Caccioppoli estimate (3.1) as follows:

(3.6) \[ \sup_{s \in I_\rho} \int_{B_\rho} (\tilde{\varphi}(s) - k)^2 \tilde{\eta}^2 \, ds + \frac{(\mu^+)^{1-\frac{1}{2m}}}{M^{1-\frac{1}{2m}}} \int_{Q_\rho} |\nabla (\tilde{\varphi} - k) - \partial_s \tilde{\eta}^2| \, ds \, dx \]

\[ \leq C(m) \left\{ \omega \int_{Q_\rho} (\tilde{\varphi} - k) \partial_s \tilde{\eta}^2 \, ds \, dx + \frac{(\mu^+)^{1-\frac{1}{2m}}}{M^{1-\frac{1}{2m}}} \int_{Q_\rho} (\tilde{\varphi} - k)^2 |\nabla \tilde{\eta}|^2 \, ds \, dx \right. \]

\[ + \left. \frac{(\mu^+)^{1-\frac{1}{2m}}}{M^{1-\frac{1}{2m}}} \tilde{h}(\rho, \omega) \left( \int m_n (B_\rho \cap \{ \tilde{\varphi}(s) < k \})^{q' \left( \frac{1}{2} - \frac{1}{p} \right)} \, ds \right)^{\frac{1}{q'}} \right\}. \]

We take \( p_*, q_* > 0 \) as

\[
\frac{2}{q'} = \frac{2}{q_*} \left( 1 + \frac{2\sigma_0}{n} \right), \quad q' \left( \frac{1}{2} - \frac{1}{p} \right) = \frac{q_*}{p_*}.
\]

We remark that \( \frac{2}{q_*} + \frac{1}{p_*} = \frac{1}{2} \). For \( i \in \mathbb{N} \), we take \( \rho = \rho_i, k = k_i, \tilde{\eta} = \tilde{\eta}_i \) satisfying \( \tilde{\eta}_i \equiv 1 \) on \( Q_{\rho_{i+1}} \) and

\[
k_i = \mu^- + \frac{1}{4} \omega + \frac{1}{2i+1} \omega, \quad \rho_i = \frac{1}{2} \rho + \frac{1}{2i+1} \rho_i,
\]

\[
Y_i := \frac{m_{n+1}(Q_\rho \cap \{ \tilde{\varphi} < k_i \})}{m_n(Q_\rho)},
\]

\[
Z_i = \frac{\rho^2}{m_{n+1}(Q_\rho)} \left( \int_{I_{\rho_i}} m_n \left( B_{\rho_i} \cap \{ \tilde{\varphi}(s) < k_i \} \right)^{\frac{2q_*}{p_*}} \, ds \right)^{\frac{1}{q_*}},
\]

\[ |\nabla \tilde{\eta}_i| \leq \frac{2}{\rho_i - \rho_{i+1}} \leq \frac{8 \cdot 2^{2i}}{\rho}, \quad \partial_s \tilde{\eta}_i \leq \frac{2}{\rho_i - \rho_{i+1}} \leq 16 \cdot 2^{2i} \frac{1}{3 \rho^2}. \]
Then, by using (2.2) and \((\tilde{u} - k_i) - \tilde{n} \leq \frac{\tilde{u}}{\tilde{\rho}}\), we rewrite (3.6) as

\[
\|(\tilde{u} - k_i) - \tilde{n}\|_{L^2(L^2) \cap L^2(H^1)(Q_{\rho})}^2
\leq C(m) \left\{ \omega \int_{Q_{\rho_i}} (\tilde{u} - k_i)_- \partial_\nu \tilde{n}_i^2 \, dx \, dt + \int_{Q_{\rho_i}} (\tilde{u} - k_i)^2 \left| \nabla \tilde{n}_i \right|^2 \, dx \, dt \\
+ \tilde{h}(\rho, \omega) \left( \int_{I_{\rho_i}} m_n \left( B_{\rho_i} \cap \{ \tilde{u}(s) < k_i \} \right) \frac{2^n}{\rho^2} \, ds \right)^{\frac{1}{2n}(1 + \frac{2n\alpha}{n})} \right\}
\]

\[
\leq C(m) \left\{ \frac{2^{2i} \omega^2}{\rho^2} m_{n+1} \left( Q_{\rho_i} \cap \{ u < k_i \} \right) \right. \\
+ \tilde{h}(\rho, \omega) \left( \int_{I_{\rho_i}} m_{n+1} \left( B_{\rho_i} \cap \{ \tilde{u}(s) < k_i \} \right) \frac{2^n}{\rho^2} \, ds \right)^{\frac{1}{2n}(1 + \frac{2n\alpha}{n})} \right\}
\]

\[
\leq C(m) \frac{\omega^2 m_{n+1}(Q_{\rho})}{\rho^2} \left\{ 2^{2i} Y_i + \tilde{h}(\rho, \omega) \omega^{-2} \left( \frac{m_{n+1}(Q_{\rho})}{\rho^2} \right)^{\frac{2n\alpha}{n}} Z_i^{1 + \frac{2n\alpha}{n}} \right\}.
\]

Using the Ladyženskaja inequality (cf. Proposition A.1) and the Hölder inequality, we have

\[
\|(\tilde{u} - k_i) - \tilde{n}\|_{L^2(Q_{\rho})}^2 \leq \|(\tilde{u} - k_i) - \tilde{n}\|_{L^2(Q_{\rho})}^2 \leq C(m, n) \omega^2 m_{n+1}(Q_{\rho}) Y_i^{\frac{1}{2n}} \times \left\{ 2^{2i} Y_i + \tilde{h}(\rho, \omega) \omega^{-2} \left( \frac{m_{n+1}(Q_{\rho})}{\rho^2} \right)^{\frac{2n\alpha}{n}} Z_i^{1 + \frac{2n\alpha}{n}} \right\}
\]

and

\[
\|(\tilde{u} - k_i) - \tilde{n}\|_{L^2(Q_{\rho})}^2 \leq C(m, n) \frac{\omega^2 m_{n+1}(Q_{\rho})}{\rho^2} \left\{ 2^{2i} Y_i + \tilde{h}(\rho, \omega) \omega^{-2} \left( \frac{m_{n+1}(Q_{\rho})}{\rho^2} \right)^{\frac{2n\alpha}{n}} Z_i^{1 + \frac{2n\alpha}{n}} \right\}.
\]

Since

\[
\|(\tilde{u} - k_i) - \tilde{n}\|_{L^2(Q_{\rho})}^2 \geq \|(\tilde{u} - k_i) - \tilde{n}\|_{L^2(Q_{\rho})}^2 \frac{2^{2i} m_{n+1}(Q_{\rho}) Y_{i+1}}{64 \cdot 2^{2i}} \]

and

\[
\|(\tilde{u} - k_i) - \tilde{n}\|_{L^2(Q_{\rho})}^2 \geq \|(\tilde{u} - k_i) - \tilde{n}\|_{L^2(Q_{\rho})}^2 \frac{2^{2i} m_{n+1}(Q_{\rho})}{64 \cdot 2^{2i}} Z_{i+1},
\]

we obtain

\[
Y_{i+1} \leq C(m, n) \left\{ 2^{2i} Y_i^{1 + \frac{2n\alpha}{n}} + \frac{2^{2i} \tilde{h}(\rho, \omega) \omega^{-2} \left( \frac{m_{n+1}(Q_{\rho})}{\rho^2} \right)^{\frac{2n\alpha}{n}} Y_i^{\frac{2n\alpha}{n}} Z_i^{1 + \frac{2n\alpha}{n}} \right\}
\]

and

\[
Z_{i+1} \leq C(m, n) \left\{ 2^{2i} Y_i + 2^{2i} \tilde{h}(\rho, \omega) \omega^{-2} \left( \frac{m_{n+1}(Q_{\rho})}{\rho^2} \right)^{\frac{2n\alpha}{n}} Z_i^{1 + \frac{2n\alpha}{n}} \right\}.
\]
Either if \( q \geq p \), then \( \frac{\rho^2}{m} \leq 1 \) and we obtain
\[
Z_0 = \frac{\rho^2}{m_{n+1}(Q_p)} \left( \int_{I_p} m_n \left( B_{p_0} \cap \{ \hat{u}(s) < k_0 \} \right)^{\frac{2}{p^*}} ds \right)^{\frac{1}{2}}
\]
by the H"older inequality. Otherwise, if \( q < p \), then
\[
Z_0 = \frac{\rho^2}{m_{n+1}(Q_p)} \left( \int_{I_p} m_n \left( B_{p_0} \cap \{ \hat{u}(s) < k_0 \} \right) \left( B_{p_0} \cap \{ \hat{u}(s) < k_0 \} \right)^{1-\frac{q}{p^*}} ds \right)^{\frac{1}{2}}
\]
\[
\leq C(n, p, q) Y \frac{2}{\rho}
\]
Therefore, by using \( \rho^2 \leq \omega \hat{h}(\rho, \omega)^{-\frac{1}{2}} \) and Lemma \[A.4\], there exists \( 0 < \theta_0 = \theta_0(n, m, p, q) < 1 \) such that if \( Y \leq \theta_0 \), then \( Y_i \to 0 \) as \( i \to \infty \), i.e.
\[
\hat{u}(s, x) > \mu^* + \frac{\omega}{2} \quad \text{a.a.} \ (s, x) \in Q^\frac{4}{2}.
\]
\[
\square
\]

4. PROOF OF UPPER BOUNDS \[\text{II} \] OF LEMMA \[2.1\]

In this section, we prove Upper bounds in Lemma \[2.1\]. More precisely we show the following proposition:

**Proposition 4.1.** Let \( 0 < \theta_0 < 1 \). Assume inequalities \[2.2\] and \[2.3\]. Then, there exist \( \eta_1, \delta_1 > 0 \) depending only on \( n, m, p, q \) and \( \theta_0 \) such that if
\[
\rho^{\sigma_0} \leq \delta_1 \omega M^{-\frac{1}{2}} (1 - \frac{1}{p}) h(\rho, M, \omega)^{-\frac{1}{2}}
\]
and
\[
m_{n+1} \left( Q_{p, M} \cap \left\{ u < \inf_{Q_{p, M}} u + \frac{\omega}{2} \right\} \right) > \theta_0 m_{n+1} \left( Q_{p, M} \right),
\]
then
\[
u(t, x) \leq \sup_{Q_{p, M}} u - \eta_1 \omega \quad \text{for} \ (t, x) \in Q^\theta_{\frac{4}{2}, M}.
\]

Taking \( \theta_0 \) as in Proposition \[3.1\], \( \delta_1, \eta_1 > 0 \) as in Proposition \[4.1\] and
\[
\delta_0 = \min \{ 1, \delta_1 \}, \quad \eta_0 = \min \left\{ \frac{1}{4}, \eta_1 \right\},
\]
we obtain Lemma \[2.1\].

To prove Proposition \[4.1\] we first show measure estimates of sub level sets of some time slice.

**Lemma 4.2.** Let \( 0 < \theta_0 < 1 \). If
\[
m_{n+1} \left( Q_{p, M} \cap \left\{ u < \mu^- + \frac{\omega}{2} \right\} \right) > \theta_0 m_{n+1} \left( Q_{p, M} \right),
\]
then for all \( 0 < \theta < \theta_0 \), there exists \( -\frac{\rho^2}{M^1 - \frac{1}{p}} < \tau_0 < -\theta \frac{\rho^2}{M^1 - \frac{1}{p}} \) depending only on \( \theta \) and \( \theta_0 \) such that
\[
m_n \left( B_{\rho} \cap \left\{ u(\tau_0) > \mu^- + \frac{\omega}{2} \right\} \right) \leq \frac{1 - \theta_0}{1 - \theta} m_n \left( B_{\rho} \right).\]
Proof. By the change of variable $t = \frac{\sigma^2}{M^2 + m}s$, $\tilde{u}(s, x) = u(t, x)$ and (4.1), we obtain

$$\int_{-1}^{0} m_{n}(B_{\rho} \cap \{ \tilde{u}(s) > \mu^{-} + \frac{\omega}{2} \}) \, ds = \frac{M^{1-\frac{\pi}{\rho^2}}}{\rho^2} m_{n+1} \left( Q_{\rho, M} \cap \{ u > \mu^{-} + \frac{\omega}{2} \} \right)$$

$$\leq \frac{M^{1-\frac{\pi}{\rho^2}}}{\rho^2} \left( (m_{n+1}(Q_{\rho, M}) - m_{n+1} \left( Q_{\rho, M} \cap \{ u < \mu^{-} + \frac{\omega}{2} \} \right)) \right)$$

$$< \frac{M^{1-\frac{\pi}{\rho^2}}}{\rho^2} (1 - \theta_{0}) m_{n+1}(Q_{\rho, M}) = (1 - \theta_{0}) m_{n}(B_{\rho}).$$

If $m_{n}(B_{\rho} \cap \{ u(s) > \mu^{-} + \frac{\omega}{2} \}) > \frac{1 - \theta_{0}}{1 - \theta_{0}} m_{n}(B_{\rho})$ for all $-1 < s < -\theta$, then

$$\int_{-1}^{0} m_{n}(B_{\rho} \cap \{ \tilde{u}(s) > \mu^{-} + \frac{\omega}{2} \}) \, ds \geq \int_{-1}^{0} m_{n}(B_{\rho} \cap \{ \tilde{u}(s) > \mu^{-} + \frac{\omega}{2} \}) \, ds$$

$$\geq (1 - \theta_{0}) m_{n}(B_{\rho}),$$

which is contradiction. \hfill \Box

We next show Bernstein type estimates for the positive part of solutions.

**Lemma 4.3.** There exist $\tau_{0}, \delta_{2} > 0$ depending only on $n, m, p, q$ and $\theta_{0}$ such that

$$m_{n}(B_{\rho} \cap \{ u(t) > \mu^{+} - \frac{\omega}{2} \}) \leq 1 - \left( \frac{\theta_{0}}{2} \right)^{2} m_{n}(B_{\rho})$$

for $t \in I_{\rho, M}^{k_{0}}$, provided $\rho^{n_{0}} \leq \delta_{2} \omega M^{-\frac{\pi}{2} \left( 1 - \frac{\pi}{\rho} \right)} h(\rho, M, \omega)^{-\frac{1}{2}}$.

**Proof.** We rewrite (2.1) as

$$\partial_{t} u - m u^{\frac{n-1}{m}} \Delta u = -m u^{\frac{n-1}{m}} \text{div} f + m u^{\frac{n-1}{m}} g.$$ 

Let

$$\psi(\xi) := \log_{+} \left( \frac{H}{H - (\xi - k_{0})} \right),$$

where $k = \mu^{-} + \frac{\omega}{2}$, $H = \mu^{+} - k = \text{osc}_{Q_{\rho, M}} u - \frac{\omega}{2}$, $c = \frac{\omega}{2}$, and $\tau_{0} > 2$ be chosen later. We remark that $\psi, \psi', \psi'' = (\psi')^{2} \geq 0$, where $f' = \frac{df}{d\xi}$. We take the cut-off function $\eta = \eta(x)$ as

$$\eta \in C_{\infty}^{0}(B_{\rho}), \eta \equiv 1 \text{ on } B_{(1-\sigma)}\rho \text{ and } |\nabla \eta| \leq \frac{2}{\sigma \rho},$$

where $\sigma > 0$ will be chosen later. Putting $w = \psi(u)$ and taking the test function $(\psi^{2})'(u)\eta^{2}$ in $(\tau_{0}, t) \times B_{\rho}$, where $\tau_{0}$ will be chosen later, we have

$$\frac{1}{2} \left[ \int_{B_{\rho}} \frac{w^{2}}{\eta^{2}} \, dx \right]^{t}_{\tau_{0}} + m \int_{\tau_{0}}^{t} \int_{B_{\rho}} \left( \nabla u \cdot \nabla (u^{\frac{n-1}{m}}(\psi^{2})'\eta^{2}) \right) \, dt \, dx$$

$$= m \int_{\tau_{0}}^{t} \int_{B_{\rho}} \left( f \cdot \nabla (u^{\frac{n-1}{m}}(\psi^{2})'\eta^{2}) \right) \, dt \, dx + m \int_{\tau_{0}}^{t} \int_{B_{\rho}} u^{\frac{n-1}{m}} g(\psi^{2})'\eta^{2} \, dt \, dx.$$

Since

$$\nabla (u^{\frac{n-1}{m}}(\psi^{2})'\eta^{2}) = \left( 1 - \frac{1}{m} \right) u^{-\frac{n-1}{m}}(\psi^{2})'\eta^{2} \nabla u + u^{\frac{n-1}{m}}(\psi^{2})''\eta^{2} \nabla u + u^{\frac{n-1}{m}}(\psi^{2})'\eta^{2} \nabla \eta,$$
we obtain

\[
\frac{1}{2} \int_{B_p} w^2 \eta^2 \, dx \bigg|_{t_0}^t + (m - 1) \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)' \| \nabla u \|^2 \eta^2 \, dt \, dx + m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)' \| \nabla u \|^2 \eta^2 \, dt \, dx \]

\[
= -m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)' \| \nabla u \cdot \nabla \eta \|^2 \, dt \, dx + (m - 1) \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (g^2)' (f \cdot \nabla u) \eta^2 \, dt \, dx \\
+ m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)'' (f \cdot \nabla u) \eta^2 \, dt \, dx + m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)' (f \cdot \nabla \eta^2) \, dt \, dx \\
+ m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)' g \eta^2 \, dt \, dx \\
=: I_1 + I_2 + I_3 + I_4 + I_5.
\]

Using the property \((\psi^2)' \nabla u = 2w \nabla w\) and the Young inequality, we have

\[
I_1 \leq m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} w \| \nabla w \|^2 \eta^2 \, dt \, dx + 4m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} w \| \nabla \eta \|^2 \, dt \, dx,
\]

\[
I_2 \leq \frac{m - 1}{2} \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)' \| \nabla u \|^2 \eta^2 \, dt \, dx + \frac{m - 1}{2} \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)' \| f \|^2 \eta^2 \, dt \, dx,
\]

\[
I_3 \leq \frac{m}{4} \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)'' \| \nabla u \|^2 \eta^2 \, dt \, dx + m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)'' \| f \|^2 \eta^2 \, dt \, dx,
\]

\[
I_4 \leq 4m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} w \psi' \| f \| \| \nabla \eta \| \eta \, dt \, dx \\
\leq 2m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} w \| \nabla \eta \|^2 \, dt \, dx + 2m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (g')^2 \| f \|^2 \eta^2 \, dt \, dx,
\]

\[
I_5 \leq 2m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} w \psi' \| g \| \eta^2 \, dt \, dx.
\]

Since \(\psi'' = (\psi')^2, (\psi^2)'' = 2(\psi')^2(1 + \psi)\), we have

\[
m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)'' \| \nabla u \|^2 \eta^2 \, dt \, dx \\
= 2m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} \| \nabla w \|^2 \eta^2 \, dt \, dx + 2m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} w \| \nabla w \|^2 \eta^2 \, dt \, dx.
\]

Combining the above estimates, we have

\[
\frac{1}{2} \int_{B_p} w^2(t) \eta^2(t) \, dx + \frac{m - 1}{2} \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi^2)' \| \nabla u \|^2 \, dt \, dx \\
+ \frac{3}{2} m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} \| \nabla w \|^2 \eta^2 \, dt \, dx + \frac{m}{2} \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} w \| \nabla w \|^2 \eta^2 \, dt \, dx \\
\leq \frac{1}{2} \int_{B_p} w^2(\tau_0) \eta^2(\tau_0) \, dx + 6m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} \| \nabla \eta \|^2 \, dt \, dx \\
+ \frac{m - 1}{2} \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} \| f \|^2 \eta^2 \, dt \, dx + 2m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} (\psi')^2 (1 + w) \| f \|^2 \eta^2 \, dt \, dx \\
+ 2m \int_{\tau_0}^t \int_{B_p} u^1 - \frac{\delta}{\partial t} w \psi' \| g \| \eta^2 \, dt \, dx \\
=: I_6 + I_7 + I_8 + I_9 + I_{10}.
\]
For simplicity, we put $k' = \mu^+ - c = \mu^+ - \frac{2\rho}{\omega}$. First, we estimate the left-hand side of (4.2). Since $k' > k$, we have

\[
\frac{1}{2} \int_{B_p} w^2(t) \eta^2(t) dx \geq \frac{1}{2} \int_{B_{(1-\sigma)p} \cap \{u(t) > k\}} w^2(t) dx \\
\geq \frac{1}{2} \int_{B_{(1-\sigma)p} \cap \{u(t) > k\}} \log^2 \left( \frac{H}{H - (k' - k) + c} \right) dx \\
\geq \frac{1}{2} \log^2 \left( \frac{r_0}{\omega} \frac{\sigma}{2^0} \right) m_n \left( B_{(1-\sigma)p} \cap \{u(t) > k'\} \right) \\
= \frac{1}{2} (r_0 - 3)^2 \log^2 2 m_n \left( B_{(1-\sigma)p} \cap \{u(t) > k'\} \right).
\]

Second, we estimate $I_6$. Taking $\tau_0$ as in Lemma 4.2 with $\theta = \frac{\theta_0}{2}$, we obtain

\[
w = \log_+ \left( \frac{H}{H - (u - k) + c} \right) \leq \log \left( \frac{\frac{r_0}{\omega}}{2^0} \right) = (r_0 - 1) \log 2
\]

and hence

\[
I_6 \leq \frac{1}{2} \int_{B_p \cap \{u(t_0) > k\}} \omega^2 (\tau_0) dx \\
\leq \frac{1}{2} (r_0 - 1)^2 \log^2 2 |B_p \cap \{u(t_0) > k\}| \leq \frac{1}{2} \cdot \frac{1 - \theta_0}{1 - \frac{\theta_0}{2}} (r_0 - 1)^2 \log^2 2 m_n (B_p).
\]

We estimate $I_7$. From $t - \tau_0 \leq \frac{\theta^2}{M^1 - \frac{1}{m}}$ and (2.2), we have

\[
I_7 \leq 6m(\mu^+)^{1-\frac{1}{m}} (t - \tau_0)(r_0 - 1) \log 2 \left( \frac{2}{\sigma\rho} \right)^2 m_n (B_p) \\
\leq C(m) \left( \frac{r_0}{2} - 1 \right) m_n (B_p).
\]

We estimate $I_8$. Since

\[
\psi' \leq \frac{1}{H - (u - k) + c} \leq \frac{1}{c} = \frac{2r_0}{\omega}, \quad (\psi^2)' = 2\psi \psi' \leq \frac{2r_0 + 1}{\omega} (r_0 - 1) \log 2
\]

and

\[
u^{-\frac{1}{m}} \leq k^{-\frac{1}{m}} \leq \left( \frac{\omega}{2} \right)^{-\frac{1}{m}} \quad \text{for } u \geq k,
\]

we have

\[
I_8 \leq C(m) (r_0 - 1)^2 r_0 \omega^{-1 - \frac{1}{m}} \int_{\tau_0}^t \int_{B_{(1-\sigma)p} \cap \{u(s) > k\}} |f|^2 t dx.
\]

By the definition of the weak $L^p$ space and by the Hölder inequality, we have

\[
\int_{\tau_0}^t \int_{B_{(1-\sigma)p} \cap \{u(s) > k\}} |f|^2 t dx \leq \int_{\tau_0}^t \|f(s)|^2\|_{L^\frac{p}{2}(B_p)} \, m_n \left( B_p \cap \{u(s) > k\} \right)^{\frac{1}{p}} ds \\
\leq C(n, p) M^\frac{\frac{1}{p}(1 - \frac{1}{m})}{p} \|f\|^\frac{\frac{1}{p}}{L^\frac{p}{2}(L^\frac{p}{2})(Q_{\rho, M})} \rho^{2\sigma_0} m_n (B_p)
\]

Using (2.3), we obtain

\[
I_8 \leq C(n, m, p) \left( \frac{r_0 - 1}{2} \right) \left( \frac{2r_0 + 1}{\omega} \right) (r_0 - 1)^2 \log^2 2 m_n (B_p) \\
\times \left( \frac{\omega}{M} \right)^{1 - \frac{1}{m}} \left( \frac{2r_0 + 1}{\omega} \right) (r_0 - 1)^2 \log^2 2 m_n (B_p)
\]

(4.6)

\[
\leq C(n, m, p) \left( \frac{r_0 - 1}{2} \right) \left( \frac{2r_0 + 1}{\omega} \right) (r_0 - 1)^2 \log^2 2 m_n (B_p).
\]
We estimate $I_9$ and $I_{10}$. Considering the same calculation for $I_8$, we have

$$I_9 \leq C(n, m, p) \left( \frac{\rho^{2 \sigma_0}}{\omega^2} M^{\frac{1}{2}} \left\| f \right\|_{L^2(\mathbb{T}^n)} \right) 2^{2 \sigma_0} (1 + 2 (r_0 - 1) \log 2) m_n(B_\rho)$$

and

$$I_{10} \leq C(n, m, p) \left( \frac{\rho^{2 \sigma_0}}{\omega^2} M^{\frac{1}{2}} \left\| g \right\|_{L^2(\mathbb{T}^n)} \right) 2^{\sigma_0} (r_0 - 1) m_n(B_\rho).$$

Combining estimates (4.3)–(4.8), we have

$$m_n\left( B_{(1 - \sigma) \rho} \cap \{ u(t) > k' \} \right) \leq \left\{ 1 - \theta_0 \left( \frac{r_0 - 1}{r_0 - 3} \right)^2 + \frac{C_1(m)}{\sigma^2} \frac{r_0 - 1}{(r_0 - 3)^2} \right. + C_2(n, m, p) \left( \frac{\rho^{2 \sigma_0}}{\omega^2} M^{\frac{1}{2}} \left\| f \right\|_{L^2(\mathbb{T}^n)} \right) 2^{\sigma_0} (r_0 - 1) \left( \frac{r_0 - 1}{(r_0 - 3)^2} \right) \left. + C_3(n, m, p) \left( \frac{\rho^{2 \sigma_0}}{\omega^2} M^{\frac{1}{2}} \left\| f \right\|_{L^2(\mathbb{T}^n)} \right) 2^{\sigma_0} (1 + 2 (r_0 - 1) \log 2) \right\} m_n(B_\rho).$$

Since

$$m_n\left( B_\rho \cap \{ u(t) > k' \} \right) = m_n\left( (B_\rho \setminus B_{(1 - \sigma) \rho}) \cap \{ u(t) > k' \} \right) + m_n\left( B_{(1 - \sigma) \rho} \cap \{ u(t) > k' \} \right)$$

$$\leq (1 - (1 - \sigma)^n) m_n(B_\rho) + m_n\left( B_{(1 - \sigma) \rho} \cap \{ u(t) > k' \} \right),$$

we have

$$m_n\left( B_{(1 - \sigma) \rho} \cap \{ u(t) > k' \} \right) \leq \left\{ 1 - \theta_0 \left( \frac{r_0 - 1}{r_0 - 3} \right)^2 + \frac{C_1(m)}{\sigma^2} \frac{r_0 - 1}{(r_0 - 3)^2} + (1 - (1 - \sigma)^n) \right. + \left. \max\{C_2, C_3, C_4\} \frac{\rho^{2 \sigma_0}}{\omega^2} M^{\frac{1}{2}} \left\| f \right\|_{L^2(\mathbb{T}^n)} \right\} m_n(B_\rho),$$

where

$$C_5(r_0) = \max\left\{ \frac{2^{\sigma_0} (r_0 - 1)}{(r_0 - 3)^2}, \frac{2^{\sigma_0} (1 + 2 (r_0 - 1) \log 2)}{(r_0 - 3)^2} \right\}.$$  

We choose parameters $r_0, \sigma$ and $\delta_2$. First we choose $\sigma = \sigma(n, \theta_0)$ satisfying $1 - (1 - \sigma)^n \leq \frac{1}{8} \theta_0^2$. Second, we choose $r_0 = r_0(n, m, \theta_0)$ satisfying

$$\left( \frac{r_0 - 1}{r_0 - 3} \right)^2 \leq 1 - \theta_0 \text{ and } \frac{C_1(m)}{\sigma^2} \frac{r_0 - 1}{(r_0 - 3)^2} \leq \frac{1}{8} \theta_0^2.$$  

Finally, we choose $\delta_2 = \delta_2(n, m, \rho, \theta_0) > 0$ sufficiently small such that

$$\max\{C_2, C_3, C_4\} C_5(r_0) \delta_2 \leq \frac{1}{2} \theta_0^2.$$  

Then, if $\rho^{2 \sigma_0} \leq \delta_2 \omega^2 M^{-\frac{1}{2}}(1 + \delta_0)\rho(\rho, M, \omega)^{-1}$, we have

$$m_n\left( B_\rho \cap \{ u(t) > k' \} \right) \leq \left( 1 - \frac{\theta_0}{2} \right)^2 m_n(B_\rho).$$

In the proof of Proposition 4.1 we need to show the Caccioppoli estimate.
Lemma 4.4 (the Caccioppoli estimate for super level sets). Let \( \eta = \eta(t, x) \) be a cut-off function in \( Q_{\rho, M}^0 \). For \( k \geq \mu^+ - \frac{\omega}{2} \), there exists a constant \( C > 0 \) depending only on \( m \) such that

\[
\sup_{t \in I^0_{\rho, M}} \int_{B_\rho} (u(t) - k)^+ \eta^2(t) \, dx + M^{1 - \frac{1}{m}} \int_{Q_{\rho, M}^0} |\nabla (u - k)|^2 \eta^2 \, dt \, dx \\
\leq C \left\{ \left( \frac{M}{\mu^+} \right)^{1 - \frac{1}{m}} \int_{Q_{\rho, M}^0} (u - k)^2 \partial_\eta \eta^2 \, dt \, dx + M^{1 - \frac{1}{m}} \int_{Q_{\rho, M}^0} (u - k)^2 |\nabla \eta|^2 \, dt \, dx \\
+ M^{1 - \frac{1}{m}} h(\rho, M, \omega) \left( \int_{t \in I^0_{\rho, M}} m_n (B_\rho \cap \{ u(t) > k \}) \right)^{\frac{1}{2}} \right\}.
\]

where \( \frac{1}{2} = \frac{1}{q} + \frac{1}{q} \).

Proof. Testing a function \((u - k)^+ \eta^2\) to (2.1), we have

\[
\frac{1}{m} \sup_{t \in I^0_{\rho, M}} \int_{B_\rho} \left( \int_0^{(u(t) - k)_+} (k + \xi)^{\frac{1}{m} - 1} \xi \, d\xi \right) \eta^2(t) \, dx + \int_{Q_{\rho, M}^0} |\nabla (u - k)|^2 \eta^2 \, dt \, dx
\]

\[
\leq \frac{1}{m} \int_{Q_{\rho, M}^0} \left( \int_0^{(u - k)_+} (k + \xi)^{\frac{1}{m} - 1} \xi \, d\xi \right) \partial_\eta \eta^2 \, dt \, dx - \int_{Q_{\rho, M}^0} (\nabla (u - k)_+ \cdot \nabla \eta^2)(u - k)_+ \, dt \, dx
\]

\[
+ \int_{Q_{\rho, M}^0} f \cdot \nabla (u - k)_+ \eta^2 \, dt \, dx + \int_{Q_{\rho, M}^0} (f \cdot \nabla \eta^2)(u - k)_+ \, dt \, dx
\]

\[
=: I_1 + I_2 + I_3 + I_4 + I_5.
\]

By the Young inequality and since \( k > \mu^+ - \frac{\omega}{2} \), we have

\[
I_2 \leq \frac{1}{2} \int_{Q_{\rho, M}^0} |\nabla (u - k)|^2 \eta^2 \, dt \, dx + 2 \int_{Q_{\rho, M}^0} (u - k)^2 |\nabla \eta|^2 \, dt \, dx,
\]

\[
I_3 \leq \frac{1}{4} \int_{Q_{\rho, M}^0} |\nabla (u - k)|^2 \eta^2 \, dt \, dx + \int_{Q_{\rho, M}^0 \cap \{ u > k \}} |f|^2 \eta^2 \, dt \, dx,
\]

\[
I_4 \leq \int_{Q_{\rho, M}^0} (u - k)^2 |\nabla \eta|^2 \, dt \, dx + \int_{Q_{\rho, M}^0 \cap \{ u > k \}} |f|^2 \eta^2 \, dt \, dx,
\]

\[
I_5 \leq \frac{\omega}{2} \int_{Q_{\rho, M}^0 \cap \{ u > k \}} |g| \eta^2 \, dt \, dx.
\]

We estimate the first term of the left-hand side in (4.10). Since

\[
(k + \xi)^{\frac{1}{m} - 1} \geq u^{\frac{1}{m} - 1} \geq (\mu^+)^{\frac{1}{m} - 1} \geq M^{\frac{1}{m} - 1} \text{ for } 0 \leq \xi \leq (u - k)_+,
\]

we have

\[
(4.12) \int_0^{(u(t) - k)_+} (k + \xi)^{\frac{1}{m} - 1} \xi \, d\xi \geq \frac{1}{2} M^{\frac{1}{m} - 1} (u(t) - k)_+^2.
\]

Finally, we estimate \( I_1 \). By (2.3), we have

\[
(4.13) I_1 \leq \frac{1}{2} \left( \frac{1}{3} \mu^+ \right)^{\frac{1}{m} - 1} \int_{Q_{\rho, M}^0} (u - k)^2 \partial_\eta \eta^2 \, dt \, dx.
\]
Combining estimates (4.11), (4.12) and (4.13), we obtain

\begin{equation}
 M^{-1} \sup_{t \in I_{p,M}} \int_{B_{r_0}} (u(t) - k) \eta^2(t) \, dx + \int_{Q_{p,M}} \|
abla (u - k) \|^2 \eta^2 \, dt \, dx \\
 \leq C(m) \left\{ \left( \mu^+ \right)^{\frac{\nu}{2}} \int_{Q_{p,M}} (u - k)^2 \rho \eta^2 \, dt \, dx + \int_{Q_{p,M}} (u - k)^2 |\nabla \eta|^2 \, dt \, dx \\
 + \int_{Q_{p,M} \cap \{ u > k \}} |f|^2 \eta^2 \, dt \, dx + \frac{\omega}{2} \int_{Q_{p,M} \cap \{ u > k \}} |g| \eta^2 \, dt \, dx \right\}.
\end{equation}

Using the same argument of the proof of Lemma 4.2, we have

\begin{align*}
 \int_{Q_{p,M} \cap \{ u > k \}} |f|^2 \eta^2 \, dt \, dx & \leq \| f \|_{L^{2\frac{n}{n-1}}(Q_{p,M})} \left( \int_{Q_{p,M}} m_n \left( B_{r_0} \cap \{ u(t) > k \} \right) \frac{q'(\frac{1}{2} - \frac{1}{p})}{\theta_0^{\frac{1}{2} - \frac{1}{p}}} \right) \frac{1}{\theta_0^{\frac{1}{2} - \frac{1}{p}}} \\
 \int_{Q_{p,M} \cap \{ u > k \}} |g| \eta^2 \, dt \, dx & \leq \| g \|_{L^{2\frac{n}{n-1}}(Q_{p,M})} \left( \int_{Q_{p,M}} m_n \left( B_{r_0} \cap \{ u(t) > k \} \right) \frac{q'(\frac{1}{2} - \frac{1}{p})}{\theta_0^{\frac{1}{2} - \frac{1}{p}}} \right) \frac{1}{\theta_0^{\frac{1}{2} - \frac{1}{p}}}
\end{align*}

hence we obtain (4.14) from (4.11).

Using Bernstein type estimates, the Caccioppoli estimate and the hole filling argument, we may prove the smallness of measures of super level sets.

Lemma 4.5. Let $\rho_0 = \frac{\omega}{2} \rho$. For $0 < \nu < 1$, there exist $q_0$, $\delta_1 > 0$ depending only on $n$, $m$, $p$, $q$, $\theta_0$ and $\nu$ such that

\[ m_{n+1} \left( Q_{\rho_0,M}^0 \cap \left\{ u > \mu^+ - \frac{\omega}{2} \right\} \right) \leq \nu m_{n+1} \left( Q_{\rho_0,M}^0 \right) \]

provided $\rho_0 \leq \delta_2 \omega M^{-\frac{1}{4}(1-\frac{1}{p})} h(\rho, M, \omega)$.

Remark 4.6. We obtain the estimate of $\delta_1$ as

\[ \delta_1 \leq \frac{1}{\theta_0^{\frac{1}{2} - \frac{1}{p}}} \frac{1}{2} - q_0. \]

Proof of Lemma 4.5 We fix $t \in I_{p,M}$ and set

\[ l := \mu^+ - \frac{\omega}{2^j+1}, \quad k := \mu^+ - \frac{\omega}{2^j}, \]

where $j \geq r_0$ and the constant $r_0$ is given by Lemma 4.3 By the Poincaré type inequality (cf. Proposition A.2), we have

\[ \frac{\omega}{2^j+1} m_n \left( B_{\rho_0} \cap \{ u(t) > l \} \right) \leq \frac{C(n)\rho_0^{\frac{1}{2}+1}}{m_n \left( B_{\rho_0} \cap \{ u(t) \leq k \} \right)} \int_{B_{\rho_0} \cap \{ k < u(t) \leq l \}} |\nabla u(t)| \, dx. \]

Since $k > \mu^+ - \frac{\omega}{2^j}$ and Lemma 4.3 we have

\[ m_n \left( B_{\rho_0} \cap \{ u(t) \leq k \} \right) = m_n \left( B_{\rho_0} \right) - m_n \left( B_{\rho_0} \cap \{ u(t) > k \} \right) \geq \left( \frac{\theta_0}{2} \right)^2 m_n \left( B_{\rho_0} \right) \]

and hence

\begin{equation}
 \frac{\omega}{2^j+1} m_n \left( B_{\rho_0} \cap \{ u(t) > l \} \right) \leq \frac{C(n)\rho_0}{\theta_0^2} \int_{B_{\rho_0} \cap \{ k < u(t) \leq l \}} |\nabla u(t)| \, dx.
\end{equation}

Integrating over $I_{p,M}$ for (4.15), we obtain

\begin{align*}
 \omega \frac{1}{2^j+1} m_{n+1} \left( Q_{\rho_0,M}^0 \cap \{ u > l \} \right) & \leq \frac{C(n)\rho_0}{\theta_0^2} \int_{I_{p,M}} \int_{B_{\rho_0} \cap \{ k < u \leq l \}} |\nabla u(t)| \, dt \, dx \\
 & \leq \frac{C(n)\rho_0}{\theta_0^2} \| (u - k)^+ \|_{L^2(Q_{\rho_0,M}^0)} m_{n+1} \left( Q_{\rho_0,M}^0 \cap \{ k < u \leq l \} \right) \frac{1}{\theta_0^{\frac{1}{2} - \frac{1}{p}}}. \end{align*}
We estimate \( \| \nabla (u - k) \|_{L^2(Q_{\rho_0, M}^0)} \). Let \( \eta = \eta(t, x) \) be a cut-off function in \( Q_{\rho, M}^0 \) satisfying

\[
\eta \equiv 1 \text{ on } Q_{\rho_0, M}^0, \quad \| \nabla \eta \| \leq \frac{8}{\rho} \quad \text{and} \quad \partial_t \eta \leq \frac{10M^{1-\frac{1}{q}}}{\theta_0 \rho^2}.
\]

Then, by the Caccioppoli estimate (Lemma 4.4), we have

\[
\| \nabla (u - k) \|_{L^2(Q_{\rho_0, M}^0)} \leq \| \nabla (u - k) + \eta \|_{L^2(Q_{\rho_0, M}^0)}
\]

\[
\leq C(m) \left\{ \int_{Q_{\rho_0, M}^0} (u - k)^2 \left( \| \nabla \eta \|^2 + (\mu^+) \frac{1}{\mu^+} \partial_t \eta^2 \right) dt dx 
+ h(\rho, M, \omega) \left( \int_{Q_{\rho_0, M}^0} m_n(B_{\rho} \cap \{ u(t) > k \}) q' \left( \frac{x}{\rho} \right) dt \right)^{\frac{1}{q'}} \right\}
=: I_1 + I_2.
\]

First we estimate \( I_1 \). By the inequality (2.53), we have

\[
I_1 \leq C(m)(\mu^+ - k)^{\frac{2}{\rho_0} + \frac{M^{1-\frac{1}{q}}}{\theta_0 \rho^2}(\mu^+) \frac{1}{\mu^+ - k}) m_{n+1}(Q_{\rho_0, M}^0)
\]

\[
 \leq C(m) \left( \frac{\mu^+}{2} \right)^2 \frac{1}{\theta_0 \rho^2} \left( \frac{M}{\mu^+} \right)^{1-\frac{1}{q}} m_{n+1}(Q_{\rho_0, M}^0)
\]

\[
 \leq C(m) \left( \frac{\mu^+}{2} \right)^2 \frac{1}{\theta_0 \rho^2} m_{n+1}(Q_{\rho_0, M}^0).
\]

We estimate \( I_2 \). Since

\[
\left( \int_{Q_{\rho_0, M}^0} m_n(B_{\rho} \cap \{ u(t) > k \}) q' \left( \frac{x}{\rho} \right) dt \right)^{\frac{1}{q'}}
\]

\[
\leq m_n(B_{\rho})^{\frac{1}{q'}} \left( \frac{\theta_0}{2} \frac{\rho^2}{M^{1-\frac{1}{q}}} \right)^{\frac{1}{q'}}
\]

\[
\leq C(q) m_n(B_{\rho})^{\frac{1}{q'}} \left( \frac{\theta_0}{2} \frac{\rho^2}{M^{1-\frac{1}{q}}} \right)^{\frac{1}{q'}} m_{n+1}(Q_{\rho_0, M}^0)
\]

\[
\leq C(q) m_n(B_{\rho})^{\frac{1}{q'}} \left( \frac{\theta_0}{2} \frac{\rho^2}{M^{1-\frac{1}{q}}} \right)^{\frac{1}{q'}} m_{n+1}(Q_{\rho_0, M}^0)
\]

we obtain

\[
I_2 \leq C(n, m, p, q) \left( \rho^{2\sigma_0} M^{\frac{2}{q}(1 - \frac{1}{q})} \left( \frac{2j}{\omega} \right)^{\frac{2}{q} - \frac{1}{q'}} \theta_0^\frac{1}{q} \right) \frac{1}{\theta_0 \rho^2} \left( \frac{\mu^+}{2} \right)^2 m_{n+1}(Q_{\rho_0, M}^0).
\]

Combining estimates (4.16), (4.17) and (4.18), we obtain

\[
\| \nabla (u - k) \|_{L^2(Q_{\rho_0, M}^0)}^2 \leq C(n, m, p, q) \left( 1 + \rho^{2\sigma_0} M^{\frac{2}{q}(1 - \frac{1}{q})} \left( \frac{2j}{\omega} \right)^{\frac{2}{q} - \frac{1}{q'}} \theta_0^\frac{1}{q} \right) \frac{1}{\theta_0 \rho^2} \left( \frac{\mu^+}{2} \right)^2 m_{n+1}(Q_{\rho_0, M}^0)
\]

and hence

\[
\left( \frac{\omega}{2j} \right)^2 \frac{1}{\theta_0^2} \left( \rho^{2\sigma_0} M^{\frac{2}{q}(1 - \frac{1}{q})} \left( \frac{2j}{\omega} \right)^{\frac{2}{q} - \frac{1}{q'}} \theta_0^\frac{1}{q} \right) \frac{1}{\theta_0 \rho^2} \left( \frac{\mu^+}{2} \right)^2 m_{n+1}(Q_{\rho_0, M}^0)
\]

\[
\times m_{n+1}(Q_{\rho_0, M}^0) \cdot m_{n+1}(Q_{\rho_0, M}^0 \cap \{ k < u \leq l \}).
\]
Summing over \( i = r_0 + 1, \ldots, q_0 \), we have
\[
\sum_{i=r_0+1}^{q_0} m_{n+1} \left( Q^0_{\rho_0,M} \cap \left\{ u > \mu^+ - \frac{\omega}{2^{i+1}} \right\} \right)^2 \\
\leq C(n, m, p, q) \frac{r_0}{\rho_0^2} \left( Q^0_{\rho_0,M} \right) \\
\times \sum_{i=r_0+1}^{q_0} \left( 1 + \rho^{2\alpha_0} M^{\frac{2}{q} \left(1 - \frac{1}{q'}\right)} \left( \frac{2r_0}{\omega} \right)^{\frac{2}{q} \rho M} h(\rho, M, \omega) \right)^2 m_{n+1} \left( Q^0_{\rho_0,M} \cap \left\{ \mu^+ - \frac{\omega}{2^i} < u \leq \mu^+ - \frac{\omega}{2^{i+1}} \right\} \right) \\
\leq C(n, m, p, q) \frac{r_0^2}{\rho_0^2} \left( Q^0_{\rho_0,M} \right)^2 \left( 1 + \rho^{2\alpha_0} M^{\frac{2}{q} \left(1 - \frac{1}{q'}\right)} \left( \frac{2r_0}{\omega} \right)^{\frac{2}{q} \rho M} h(\rho, M, \omega) \right)^2 \\
\times \sum_{i=r_0+1}^{\infty} m_{n+1} \left( Q^0_{\rho_0,M} \cap \left\{ \mu^+ - \frac{\omega}{2^i} < u \leq \mu^+ - \frac{\omega}{2^{i+1}} \right\} \right) \\
\leq C(n, m, p, q) \frac{r_0^2}{\rho_0^2} \left( Q^0_{\rho_0,M} \right)^2 \left( 1 + \rho^{2\alpha_0} M^{\frac{2}{q} \left(1 - \frac{1}{q'}\right)} \left( \frac{2r_0}{\omega} \right)^{\frac{2}{q} \rho M} h(\rho, M, \omega) \right) \\
\end{equation}

We take \( q_0 > 0 \) enough large such that
\[
2C(n, m, p, q) \frac{r_0}{\rho_0^2} (q_0 - r_0) \leq \nu^2. 
\]

Since
\[
\sum_{i=r_0+1}^{q_0} m_{n+1} \left( Q^0_{\rho_0,M} \cap \left\{ u > \mu^+ - \frac{\omega}{2^{i+1}} \right\} \right)^2 \geq (q_0 - r_0) m_{n+1} \left( Q^0_{\rho_0,M} \cap \left\{ u > \mu^+ - \frac{\omega}{2^{q_0+1}} \right\} \right)^2 , 
\]
we have
\[
m_{n+1} \left( Q^0_{\rho_0,M} \cap \left\{ u > \mu^+ - \frac{\omega}{2^{q_0+1}} \right\} \right)^2 \leq 2C(n, m, p, q) \frac{r_0}{\rho_0^2} \left( Q^0_{\rho_0,M} \right)^2 \\
\leq \nu^2 m_{n+1} \left( Q^0_{\rho_0,M} \right)^2 
\]
provided \( \rho^{2\alpha_0} \leq \min \left\{ \frac{2}{\rho M^{\frac{2}{q} \left(1 - \frac{1}{q'}\right)}}, \delta_2^2 \right\} \omega^2 M^{\frac{2}{q} \left(1 - \frac{1}{q'}\right)} h(\rho, M, \omega)^{-1} \), where \( \delta_2 > 0 \) is given by Lemma 4.3. Taking \( \delta_2^2 := \min \left\{ \frac{2}{\rho M^{\frac{2}{q} \left(1 - \frac{1}{q'}\right)}}, \delta_2^2 \right\} \), we obtain Lemma 4.5. \( \square \)

**Proof of Proposition 4.7** Let \( 0 < \nu < 1 \) be chosen later. We take \( \delta_1 > 0 \) and \( q_0 \) as in Lemma 4.5. We introduce the following scale transform
\[
s = M^{1 - \frac{1}{q'}} t, \quad \tilde{u}(s, x) = u(t, x), \quad \tilde{\eta}(s, x) = \eta(t, x), \\
\tilde{f}(s, x) = f(t, x), \quad \tilde{g}(s, x) = g(t, x).
\]

Then, using (22), we may rewrite the Caccioppoli estimate (4.9) as follows:
\[
\sup_{s \in I^0} \int_{B^s_\rho} (\tilde{u}(s) - k)^2 \tilde{\eta}^2(s) \, dx + \int_{I^0} \int_{\rho_M^0} |\nabla (\tilde{u}(s) - k) + |\tilde{\eta}(s)|^2 \, ds \, dx \\
\leq C(m) \left\{ \int_{I^0} \int_{\rho_M^0} (\tilde{u} - k)^2 \left\{ \frac{M}{\mu^+} \left(1 - \frac{1}{q'}\right) \rho M + |\tilde{\eta}(s)|^2 \right\} \, ds \, dx \right. \\
+ \int_{I^0} \int_{\rho_M^0} (\tilde{u} - k)^2 \left\{ \rho \tilde{\eta}^2 + |\nabla \tilde{\eta}|^2 \right\} \, ds \, dx \\
\right. \\
\left. \left. \leq C(m) \left\{ \int_{I^0} \int_{\rho_M^0} (\tilde{u} - k)^2 \left\{ \rho \tilde{\eta}^2 + |\nabla \tilde{\eta}|^2 \right\} \, ds \, dx \right. \\
+ \int_{I^0} \int_{\rho_M^0} m_n (B^s_\rho \cap \{ \tilde{u}(s) > k \}) \, \tilde{\eta}' \left( \frac{1}{2} - \frac{1}{2} \right) \, ds \, dx \right\} \right\} \right\} \\
\end{equation}
where \( \tilde{h}(\rho, \omega) := \| \tilde{f} \|_{L^2(H^1(Q_\rho))}^2 + \omega \| \tilde{y} \|_{L^2(L^\infty(Q_\rho))}^2 \).

We take \( \rho_0, q_0 > 0 \) as in the proof of Proposition 3.1 and for \( i \in \mathbb{N} \) we take \( \rho = \rho_i, k = k_i, \tilde{\eta} = \tilde{\eta}_i \) satisfying \( \tilde{\eta}_i \equiv 1 \) on \( Q_{\rho_{i+1}} \) and

\[
\begin{align*}
k_i &= \rho_i + \frac{\omega}{2\rho_0} + \frac{1}{2\rho_0 + 1} \rho_i, \\
m_i &= m_{n+1} \left( Q_{\rho_i}^0 \cap \{ \tilde{u} > k_i \} \right), \\
Y_i &= \frac{m_{n+1} \left( Q_{\rho_i}^0 \right)}{m_{n+1} \left( Q_{\rho_0}^0 \right)}, \\
Z_i &= \frac{\rho_0^2}{m_{n+1} \left( Q_{\rho_0}^0 \right)} \left( \int_{Q_{\rho_i}^0} m_i \left( B_{\rho_i} \cap \{ \tilde{u}(s) > k_i \} \right) \frac{\omega}{\rho_i} \, ds \right)^{\frac{\omega}{\rho_i}}, \\
|\nabla \tilde{\eta}_i| &\leq \frac{2}{\rho_i - \rho_i + 1}, \quad \partial_s \tilde{\eta}_i \leq \frac{4}{\theta_0 \rho_i^2 - \rho_i^2 + 1} \leq \frac{48 \cdot 2^{2i}}{\theta_0 \rho_i^2}.
\end{align*}
\]

From (4.19) and \( (\tilde{u} - k_i)_+ \leq \frac{\omega}{2\rho_0 + 1} \), we obtain

\[
\begin{align*}
\| (\tilde{u} - k_i)_+ \tilde{\eta}_i \|_{L^\infty(L^2(Q_{\rho_0}^0))}^2 &+ \tilde{h}(\rho, \omega) \left( \int_{Q_{\rho_0}^0} m_i \left( B_{\rho_i} \cap \{ \tilde{u}(s) > k_i \} \right) \frac{\omega}{\rho_i} \, ds \right)^{\frac{\omega}{\rho_i}} \\
&\leq C(m) \left\{ (\frac{\omega}{2\rho_0})^2 \left( \frac{1}{\theta_0} + 1 \right)^{\frac{2^{2i} \rho_i}{\rho_0}} m_{n+1} \left( Q_{\rho_0}^0 \cap \{ \tilde{u} > k_i \} \right) \\
&\quad + \tilde{h}(\rho, \omega) \left( \int_{Q_{\rho_0}^0} m_i \left( B_{\rho_i} \cap \{ \tilde{u}(s) > k_i \} \right) \frac{\omega}{\rho_i} \, ds \right)^{\frac{\omega}{\rho_i}} \right\}^{\frac{2^{2i}}{\rho_i}} \\
&\leq C(m, \theta_0) \frac{m_{n+1} \left( Q_{\rho_0}^0 \right)}{\rho_0^2} \left( \frac{\omega}{2\rho_0} \right)^2 \left( \frac{m_{n+1} \left( Q_{\rho_0}^0 \right)}{\rho_0} \right)^{\frac{2\omega}{\rho_0}} Z_i^{1 + \frac{2\omega}{\rho_0}} \\
&\times \left\{ 2^{2i} Y_i + \tilde{h}(\rho, \omega) \left( \frac{2^{m_0} \omega}{\rho_0} \right)^{\frac{2\omega}{\rho_0}} (\frac{m_{n+1} \left( Q_{\rho_0}^0 \right)}{\rho_0})^{\frac{2\omega}{\rho_0}} Z_i^{1 + \frac{2\omega}{\rho_0}} \right\}.
\end{align*}
\]

Since \( \delta_1 \leq \theta_0^{-\frac{2^{2i}}{\rho_0} - 2^{-\rho_0}} \), we have

\[
\tilde{h}(\rho, \omega) \left( \frac{2^{2i} \rho_i}{\rho_0} \right)^{\frac{\omega}{\rho_0}} \left( \frac{m_{n+1} \left( Q_{\rho_0}^0 \right)}{\rho_0} \right)^{\frac{2\omega}{\rho_0}} \leq C(n, p, q, \theta_0)
\]

and hence

\[
\| (\tilde{u} - k_i)_+ \tilde{\eta}_i \|_{L^\infty(L^2(Q_{\rho_0}^0))} \leq C(n, m, p, q, \theta_0) \left( \frac{\omega}{2\rho_0} \right)^{\frac{2m_{n+1} \left( Q_{\rho_0}^0 \right)}{\rho_0}} \left\{ 2^{2i} Y_i + Z_i^{1 + \frac{2\omega}{\rho_0}} \right\}.
\]

By the Ladyženskaja inequality (cf. Proposition A.1) and the Hölder inequality, we have

\[
\| (\tilde{u} - k_i)_+ \tilde{\eta}_i \|_{L^2(Q_{\rho_0}^0)} \leq \| (\tilde{u} - k_i)_+ \tilde{\eta}_i \|_{L^2(Q_{\rho_0}^0)} \| \chi \{ \tilde{u} > k_i \} \|_{L^{n+2}(Q_{\rho_0}^0)} \leq C(n, m, p, q, \theta_0) \left( \frac{\omega}{2\rho_0} \right)^{\frac{2m_{n+1} \left( Q_{\rho_0}^0 \right)}{\rho_0}} \left\{ 2^{2i} Y_i + Z_i^{1 + \frac{2\omega}{\rho_0}} \right\}
\]

and

\[
\| (\tilde{u} - k_i)_+ \tilde{\eta}_i \|_{L^2(Q_{\rho_0}^0)} \leq C(n, m, p, q, \theta_0) \left( \frac{\omega}{2\rho_0} \right)^{\frac{2m_{n+1} \left( Q_{\rho_0}^0 \right)}{\rho_0}} \left\{ 2^{2i} Y_i + Z_i^{1 + \frac{2\omega}{\rho_0}} \right\}.
\]

Since

\[
\| (\tilde{u} - k_i)_+ \tilde{\eta}_i \|_{L^2(Q_{\rho_0}^0)} \geq \| (\tilde{u} - k_i)_+ \|_{L^2(Q_{\rho_{i+1}}^0 \cap \{ \tilde{u} > k_{i+1} \})}
\]

\[
\quad \geq (k_{i+1} - k_i)^2 m_{n+1} \left( Q_{\rho_{i+1}}^0 \cap \{ \tilde{u} > k_{i+1} \} \right)
\]

\[
\quad = \left( \frac{\omega}{2\rho_0 + 1} \right)^{\frac{2^{2i} \rho_i}{\rho_0}} m_{n+1} \left( Q_{\rho_0}^0 \right) Y_{i+1}
\]

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and
\[ \| (\tilde{u} - k_i) + \eta_i \|_{L^p(L^q(Q_{r_i}^{0})}^2 \geq \| (\tilde{u} - k_i) + \|_{L^p(L^q(Q_{r_i}^{0}) \cap \{ \tilde{u} > k_i + 1 \})}^2 \geq \frac{\omega}{2n+1} \frac{2 m_{n+1} (\rho_0)}{\rho_{n+1}} Z_{i+1}, \]
we obtain
\[ Y_{i+1} \leq C(n, m, p, q, \theta_0) \left\{ 2^{4i} Y_i^{1+\frac{n+2}{n}} + 2^{2i} Y_i^{1+\frac{2n}{n}} Z_i^{1+\frac{2n}{n}} \right\} \]
and
\[ Z_{i+1} \leq C(n, m, p, q, \theta_0) \left\{ 2^{4i} Y_i + 2^{2i} Z_i^{1+\frac{2n}{n}} \right\}. \]
Considering the same calculation of (3.7) and (3.8), we obtain
\[ Z_0 \leq \left\{ \begin{array}{ll} C(n, p, q, \theta_0) Y_0^{\frac{n}{p}} & \text{if } q \geq p, \\ C(n, p, q, \theta_0) Y_0^{\frac{n}{p}} & \text{if } q < p. \end{array} \right. \]
Therefore, by Lemma A.4 there exists \( 0 < \nu = \nu(n, m, p, q, \theta_0) < 1 \) such that if \( Y_0 \leq \nu \), then \( Y_i \to 0 \) as \( i \to \infty \), i.e.
\[ \tilde{u}(s, x) < \mu + \frac{\omega}{2n+2} \text{ a.a. } (s, x) \in Q_{r_0}^{0}. \]
By Lemma A.3 we obtain the upper bounds of \( u \). \( \square \)

APPENDIX A. APPENDIX

Their results are well-known, however we give the proof here for reader's convenience.

A.1. Some Sobolev type inequality.

**Proposition A.1** (Ladyženskaja-Solonnikov-Ural'ceva [15, p.74]). Let \( I \subset \mathbb{R} \) be an open interval and let \( \Omega \subset \mathbb{R}^n \) be a domain. Then for \( f \in L^\infty(I; L^2(\Omega)) \cap L^2(I; H_0^1(\Omega)) \) and \( p, q \geq 2 \) satisfying
\[ \frac{2}{q} + \frac{n}{p} = \frac{n}{2} \text{ if } n \neq 2, \]
\[ \frac{2}{q} + \frac{n}{p} = \frac{n}{2} \text{ without } q = 2, p = \infty \text{ if } n = 2, \]
we obtain
\[ \| f \|_{L^p(I; L^p(\Omega))} \leq C(n, p, q)(\| f \|_{L^\infty(I; L^2(\Omega))} + \| \nabla f \|_{L^2(I; \Omega))}. \]

*Proof.* By the Gagliardo-Nirenberg-Sobolev inequality, we have
\[ \| f(t) \|_{L^p(\Omega)} \leq C(n, p) \| \nabla f(t) \|_{L^2(\Omega)}^{\frac{n}{2}} \| f(t) \|_{L^2(\Omega)}^{1 - \frac{n}{2}} \text{ a.a. } t \in I. \]
Taking \( L^n(I) \) norm on both side, we obtain (A.1). \( \square \)

**Proposition A.2** (Ladyženskaja-Solonnikov-Ural’ceva [15, p.91]). Let \( f \) be a non-negative function belonging to \( W^{1,1}(B_r) \) and let \( l > k \). Then there exists a constant \( C > 0 \) depending on \( n \) only such that
\[ (l - k)m_n(\{ f > l \}) \leq \frac{C \rho^{n+1}}{m_n(B_r) - m_n(\{ f > k \})} \int_{ \{ k < f \leq l \} } | \nabla f | dx. \]

For the proof of Proposition A.2 we need the following Poincaré inequalities:

**Lemma A.3** (Ladyženskaja-Solonnikov-Ural’ceva [15, Lemma 5.1 in p.89]). Let \( g \in W^{1,1}(B_r) \) be a non-negative function and let \( N_0 := \{ g = 0 \} \). Let \( \eta(x) = \eta(|x|) \) be a decreasing function of \( |x| \) satisfying \( 0 \leq \eta \leq 1 \) and \( \eta|_{N_0} \equiv 1 \). Then for measurable set \( N \subset B_r \), we have
\[ \int_N g(x) \eta(x) dx \leq \frac{C_n \rho^n}{m_n(N_0)} m_n(N) \frac{1}{2} \int_{B_r} | \nabla g(x) | \eta(x) dx. \]
We now show the following estimate:

\[ (A.2) \quad \eta(x) \leq \eta(x + r\omega) \quad \text{for} \quad 0 < r \leq |x'| - x. \]

Either if \(|x| \leq |x'|\), then \(x + r\omega \in B_{|x'|}\) by the convexity of \(B_{|x'|}\). By the monotonicity of \(\eta\), we have \(\eta(x + r\omega) \geq \eta(x') = 1\). Otherwise, if \(|x| > |x'|\), then \(x + r\omega \in B_{|x|}\). Since \(\eta(x + r\omega) \geq \eta(x)\), we obtain \((A.2)\).

By \((A.2)\), we have

\[ g(x)\eta(x) \leq \int_0^{|x'| - x} |\nabla g(x + r\omega)| \eta(x + r\omega) \, dr. \]

Integrating over \(x \in N\) and \(x' \in N_0\), we have

\[ m_n(N_0) \int_N g(x)\eta(x) \, dx \leq \int_N dx \int_{N_0} dx' \int_0^{|x' - x|} |\nabla g(x + r\omega)| \eta(x + r\omega) \, dr. \]

Let \(g(x) = \eta(x) = 0\) on \(x \in \mathbb{R}^n \setminus B_\rho\). Introducing the polar coordinate, we obtain

\[
\int_{N_0} dx' \int_0^{|x' - x|} |\nabla g(x + r\omega)| \eta(x + r\omega) \, dr \\
\leq \int_{B_{2\rho}(x)} dx' \int_0^{|x' - x|} |\nabla g(x + r\omega)| \eta(x + r\omega) \, dr \\
\leq \int_{B_{2\rho}(x')} dx' \int_0^{|x' - x|} |\nabla g(x + r\omega)| \eta(x + r\omega) \, dr \\
= \int_0^{2\rho} s^{n-1} ds \int_{S^{n-1}} d\sigma \int_0^s |\nabla g(x + r\omega)| \eta(x + r\omega) r^{n-1} dr \\
= \int_0^{2\rho} s^{n-1} ds \int_{B_{\rho}(x)} \frac{|\nabla g(y)|\eta(y)}{|x - y|^{n-1}} dy \\
\leq \frac{(2\rho)^n}{n} \int_{B_\rho} |\nabla g(y)|\eta(y) \, dy.
\]

Therefore,

\[ m_n(N_0) \int_N g(x)\eta(x) \, dx \leq \frac{(2\rho)^n}{n} \int_N dx \int_{B_\rho} |\nabla g(y)|\eta(y) \, dy \\
= \frac{(2\rho)^n}{n} \int_{B_\rho} |\nabla g(y)|\eta(y) \, dy \int_N \frac{1}{|x - y|^{n-1}} dx. \]

We now show the following estimate:

\[ (A.3) \quad \int_N \frac{1}{|x - y|^{n-1}} dx \leq (1 + \mathcal{H}^{n-1}(S^{n-1}))m_n(N)^{\frac{1}{n}}, \]

where \(\mathcal{H}^{n-1}(S^{n-1})\) is the \((n - 1)\)-dimensional Hausdorff measure of the \((n - 1)\)-dimensional unit sphere. To show \((A.3)\), let \(\delta > 0\) to be chosen later. We split the integral

\[
\int_N \frac{1}{|x - y|^{n-1}} dx \\
\leq \int_{N \cap \{|x - y| \leq \delta\}} \frac{1}{|x - y|^{n-1}} dx + \int_{N \cap \{|x - y| \geq \delta\}} \frac{1}{|x - y|^{n-1}} dx =: I_1 + I_2.
\]
By a simple calculation, we obtain
\[ I_1 \leq \int_0^\delta \frac{\nu^{n-1}}{r^{n-1}} \, dr \int_{S_n} \, d\sigma = \delta \mathcal{H}^{n-1}(S^{n-1}), \]
\[ I_2 \leq \int_N \frac{1}{\delta^{n-1}} \, dx \leq \delta^{1-n} m_n(N). \]
Taking \( \delta = m_n(N)^\frac{1}{n} \), we have \( I_1 + I_2 \leq (1 + \mathcal{H}^{n-1}(S^{n-1}))m_n(N)^\frac{1}{n} \) and we obtain (A.3).

Using (A.3), we have
\[ m_n(N_0) \int_N g(x) \eta(x) \, dx \leq \frac{2n(1 + \mathcal{H}^{n-1}(S^{n-1}))}{n} \rho^m m_n(N)^\frac{1}{n} \int_{B_\rho} |\nabla g(y)| \eta(y) \, dy. \]
We consider the case \( n = 1 \). For \( x \in N \) and \( x' \in N_0 \), we have
\[ g(x) = g(x) - g(x') = \int_{x'}^x \frac{d}{dy} g(y) \, dy \leq \int_{x'}^1 \left| \frac{d}{dy} g(y) \right| \, dy. \]
Since
\[ g(x) \eta(x) \leq \left| \int_{x'}^x \frac{d}{dy} g(y) \eta(y) \right| \, dy \leq \int_1^1 \left| \frac{d}{dy} g(y) \eta(y) \right| \, dy, \]
we obtain
\[ \int_N g(x) \eta(x) \, dx \leq m_n(N) \int_{-1}^1 \left| \nabla g(x) \right| \eta(x) \, dx. \]
\[ \square \]

Proof of Proposition A.2

Let
\[ g(x) := \max\{1 - k, (f - k)_+\} \in W^{1,1}(B_\rho), \quad N_0 := \{f < k\}, \quad \eta(x) = 1, \quad N := \{f > k\}. \]
Then, by the Lemma A.3 we have
\[ \int_N g(x) \, dx \leq \frac{C_n \rho^m m_n(N)^\frac{1}{n}}{m_n(N_0)} \int_{B_\rho} |\nabla g(x)| \, dx, \]
hence
\[ (l - k)m_n(\{f > k\}) \leq \frac{C_n \rho^m m_n(\{f > k\})^\frac{1}{n}}{m_n(\{f < k\})} \int_{\{k < f \leq l\}} |\nabla f(x)| \, dx. \]
\[ \square \]

A.2. The recursive inequalities.

Lemma A.4 (Ladyženskaja-Solonnikov-Ural’ceva [15] Lemma 5.7 in p.96). Let \( C, \varepsilon, \delta > 0 \) and \( b \geq 1 \). Assume that sequences \( \{Y_n\}_{n=0}^\infty, \{Z_n\}_{n=0}^\infty \subset (0, \infty) \) satisfy
\[ Y_{n+1} \leq C b^n (Y_n^{1+\delta} + Y_n^{1+\varepsilon} Z_n^{1+\varepsilon}), \]
\[ Z_{n+1} \leq C b^n (Y_n + Z_n^{1+\varepsilon}). \]
Let
\[ d := \min \left\{ \delta, \frac{\varepsilon}{1 + \varepsilon} \right\}, \quad \lambda = \min \left\{ (2C)^{-\frac{1}{d}} b^{-\frac{1}{d}}, (2C)^{-\frac{1 + \varepsilon}{d}} b^{-\frac{1 + \varepsilon}{d}} \right\}. \]
Then, if \( Y_0 \leq \lambda \) and \( Z_0 \leq \lambda^{1+d} \), we obtain
\[ Y_n \leq \lambda b^{-\frac{1}{d}}, \quad Z_n \leq (\lambda b^{-\frac{1}{d}})^{1+d}. \]

Proof. Inequalities (A.5) are valid for \( n = 0 \). We prove (A.5) by induction. If (A.5) hold for \( n \), then by (A.4), we have
\[ Y_{n+1} \leq 2C \lambda^{1+\delta} b^{n(1-\frac{1+\varepsilon}{d})}, \quad Z_{n+1} \leq 2C \lambda b^{n(1-\frac{1}{d})}. \]
Since \( \lambda \leq (2C)^{-\frac{1}{d}} b^{-\frac{1}{d}} \) and \( d \leq \delta \), we have
\[ 2C \lambda^{1+\delta} b^{n(1-\frac{1+\varepsilon}{d})} \leq \lambda b^{-\frac{1}{d}} b^{-\frac{1+\varepsilon}{d}} b^{n(1-\frac{1}{d})} \leq \lambda b^{-\frac{n+\varepsilon}{d}}. \]
Similarly, since \( \lambda \leq (2C)^{-\frac{1}{n+1}} b^{-\frac{1}{b}} \), we obtain
\[
2C \lambda b^{n(1-\frac{1}{b})} = 2C \lambda \frac{1}{b^{1+b}} b^{n(1-\frac{1}{b})} + \frac{n+b}{(n+1)\lambda} \\
\leq (\lambda b^{-\frac{1}{b+b}})^{\frac{1}{1+b}} b^{n(1-\frac{1}{b+b})}.
\]
Since \( d \leq \frac{1}{2} \), we find \( 1 - \frac{d}{\sqrt{\lambda}} \leq 0 \) and hence we have \((\mathbb{L}, \mathcal{S})\) for \( n+1 \).

\[ \square \]

A.3. The weak \( L^p \) spaces and the Lorentz spaces. Let \( \Omega \subset \mathbb{R}^n \) be a domain (not necessarily bounded).

**Definition A.5** (The Lorentz spaces). For \( 1 \leq p < \infty \), we define the Lorentz space \( L^{p,\infty}(\Omega) \) by
\[
L^{p,\infty}(\Omega) := \{ f \in L^1_{\text{loc}}(\Omega) : \lambda^p \mu_{|f|}(\lambda) \text{ is bounded for all } \lambda > 0 \}
\]
where \( \mu_{|f|}(\lambda) := m_n(\{|f| > \lambda\}) \).

**Proposition A.6** (cf. Benilan-Brezis-Crandall [4, pp.548]). For \( 1 < p < \infty \), we have
\[
\frac{p-1}{p} \|f\|_{L^p(\Omega)} \leq \sup_{\lambda > 0} \lambda \mu_{|f|}(\lambda)^{\frac{1}{p}} \leq \|f\|_{L^{p,\infty}(\Omega)}.
\]

**Proof.** First, we show \( \sup_{\lambda > 0} \lambda \mu_{|f|}(\lambda)^{\frac{1}{p}} \leq \|f\|_{L^p(\Omega)} \). For \( \rho, \lambda > 0 \), we take \( K = \{|f| > \lambda\} \cap B_\rho \). Then we have
\[
\|f\|_{L^p(\Omega)} \geq m_n(\{x \in \Omega \cap B_\rho : |f(x)| > \lambda\})^{\frac{1}{p}-1} \int_{\{|f| > \lambda\} \cap B_\rho} |f(x)| \, dx \\
\geq \lambda m_n(\{x \in \Omega \cap B_\rho : |f(x)| > \lambda\})^{\frac{1}{p}}.
\]
Letting \( \rho \to \infty \), we find
\[
\lambda \mu_{|f|}(\lambda)^{\frac{1}{p}} \leq \|f\|_{L^p(\Omega)}.
\]
Second, we show \( \frac{p-1}{p} \|f\|_{L^p(\Omega)} \leq \sup_{\lambda > 0} \lambda \mu_{|f|}(\lambda)^{\frac{1}{p}} \). We fix \( \lambda_0 > 0 \). For measurable set \( K \subset \Omega \), we have
\[
\int_K |f(x)| \, dx \leq \lambda_0 m_n(K) + \int_{\{|f| > \lambda_0\}} |f(x)| \, dx.
\]
By the above inequality, we have
\[
\int_{\{|f| > \lambda_0\}} |f(x)| \, dx = \int_0^{\lambda_0} m_n(\{|f| > \lambda_0\}) \, d\lambda \\
= \int_0^{\lambda_0} m_n(\{|f| > \lambda\}) \, d\lambda + \int_{\lambda_0}^{\lambda_\infty} m_n(\{|f| > \lambda\}) \, d\lambda \\
= \lambda_0 \mu_{|f|}(\lambda_0) + \int_{\lambda_0}^{\lambda_\infty} \mu_{|f|}(\lambda) \, d\lambda \\
\leq \lambda_0^{1-p} \sup_{\lambda > 0} \lambda^p \mu_{|f|}(\lambda) + \lambda_0^p \mu_{|f|}(\lambda) \int_{\lambda_0}^{\lambda_\infty} \lambda^{-p} \, d\lambda \\
= \frac{p}{p-1} \lambda_0^{1-p} \sup_{\lambda > 0} \lambda^p \mu_{|f|}(\lambda).
\]
Taking \( \lambda_0^p m_n(K) = p \sup_{\lambda > 0} \lambda^p \mu_{|f|}(\lambda) \), we find
\[
\int_K |f(x)| \, dx \leq \frac{p}{p-1} (p \sup_{\lambda > 0} \lambda^p \mu_{|f|}(\lambda))^{\frac{1}{p}} |K|^{1-\frac{1}{p}}
\]
or
\[
\|f\|_{L^{p,\infty}(\Omega)} \leq \frac{p^{\frac{1}{p}}}{p-1} \sup_{\lambda > 0} \lambda \mu_{|f|}(\lambda)^{\frac{1}{p}}.
\]

\[ \square \]

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