New Approach to Hermitian q–Differential Operators on $\mathbb{R}_q^N$

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New approach to Hermitian $q$-differential operators on $\mathbb{R}^N_q$

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Abstract

We report on our recent breakthrough [9] in the construction for $q > 0$ of Hermitean and “tractable” differential operators out of the $U_q so(N)$-covariant differential calculus on the noncommutative manifolds $\mathbb{R}^N_q$ (the so-called “quantum Euclidean spaces”).

1 Introduction

Noncommutative geometry for non-compact noncommutative manifolds is usually much more difficult than for compact ones, especially when trying to proceed from an algebraic to a functional-analytical treatment (see e.g. [12]). Major sources of complications are $*$-structures and $*$-representations of the involved algebras. As a noncommutative space here we adopt the so-called $N$-dimensional quantum Euclidean space $\mathbb{R}^N_q$ ($N \geq 3$), a deformation of $\mathbb{R}^N$ characterized by its covariance under the quantum group $U_q so(N)$ (with $q$ real so that the latter is compact; for $*$-representation purposes we shall later need and choose $q > 0$) and shall denote by $F$ the $*$-algebra “of functions on $\mathbb{R}^N_q$”. The $*$-structure [13] associated to the corresponding $U_q so(N)$-covariant differential calculus [1]

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on $\mathbb{R}^N_q$ [6] is however characterized by an unpleasant nonlinear action on the differentials, the partial derivatives and the exterior derivative. This at the origin of a host of formal and substantial complications: it seems hardly possible to construct physically relevant differential operators (e.g. momentum components, Laplacian, kinetic term in a would-be field theory on $\mathbb{R}^N_q$) which are at the same time Hermitean and “tractable”.

Here we report on a recent breakthrough [9] to the approach to these problems. This is based on our result (Thm 1 in [9]) that the transformation laws [13] of the $N$ partial derivatives $\partial^\alpha$ and of the exterior derivative $d$ of the differential calculus [1] on $\mathbb{R}^N_q$ under the $\star$-structure can be expressed as the similarity transformations

$$\partial^\alpha \star = -\tilde{\nu}'^{-2} \tilde{\nu}^2 \partial^\alpha \tilde{\nu}^2, \quad d \star = -\tilde{\nu}'^2 d\tilde{\nu}'^{-2}, \quad (1)$$

where $\tilde{\nu}'$ is a $U_q so(N)$-invariant positive-definite pseudodifferential operator, more precisely the realization of the fourth root of the ribbon element of the extension of $U_q so(N)$ with a central element generating dilatations of $\mathbb{R}^N_q$.

The situation reminds us of what one encounters in functional analysis on the real line when taking the Hermitean conjugate of a differential operator like

$$\partial := \tilde{\nu}'(x) \frac{d}{dx} \frac{1}{\tilde{\nu}'(x)},$$

where $\tilde{\nu}'(x)$ is a positive smooth function; as an element of the Heisenberg algebra $\partial$ is not imaginary (excluding the trivial case $\tilde{\nu}' \equiv 1$) w.r.t. the $\star$-structure

$$x^* = x, \quad \left( \frac{d}{dx} \right)^* = -\frac{d}{dx},$$

but fulfills the similarity transformation (1). This corresponds to the fact that it is not antihermitean as an operator on $L^2(\mathbb{R})$. $\partial$ is however (formally) antihermitean on $L^2(\mathbb{R}, \tilde{\nu}'^{-2} dx)$, the Hilbert space of square-integrable functions over $\mathbb{R}$ endowed with measure $\tilde{\nu}'^{-2} dx$. In other words, if we introduce the scalar product

$$\langle \phi, \psi \rangle = \int \phi^\star(x) \tilde{\nu}'^{-2}(x) \psi(x) dx, \quad (2)$$

[as one does when setting the Sturm-Liouville problem for $\partial^2$], $\partial$ becomes antihermitean under the corresponding Hermitean conjugation $\dagger$:

$$\langle A^\dagger \phi, \psi \rangle := \langle \phi, A \psi \rangle \quad \Rightarrow \quad \partial^\dagger = -\partial.$$

The idea is to adopt the analog of (2), where now $\nu'$ is a pseudodifferential operator rather than a function, as a scalar product also in our noncommutative space.

\footnote{The Hermitean conjugation $\dagger$ is the representation of the following modified $\star$-structure $\star'$ of the Heisenberg algebra: $a^\star' = (\tilde{\nu}'^{-2} a \tilde{\nu}^2)^\star = \tilde{\nu}'^2 a^\star \tilde{\nu}'^{-2}$.}
2 Preliminaries and notation

The algebras.

$F$ is essentially the unital associative algebra over $\mathbb{C}[[h]]$ generated by $N$ elements $x^\alpha$ (the cartesian “coordinates”) modulo the relations (3) given below, and is extended to include formal power series in the generators; out of $F$ one can extract subspaces consisting of elements that can be considered integrable or square-integrable functions. The $U_q so(N)$-covariant differential calculus on $\mathbb{R}_q^N$ [1] is defined introducing the invariant exterior derivative $d$, satisfying nilpotency and the Leibniz rule

$$d(fg) = dfg + fdg,$$

and imposing the covariant commutation relations (4) between the $x^\alpha$ and the differentials $\alpha := dx^\alpha$. Partial derivatives are introduced through the decomposition $d =: \alpha \partial$. All the other commutation relations are derived by consistency. The complete list is given by

$$P_{\alpha\beta} x^\gamma x^\delta = 0,$$  \hfill (3)

$$x^\gamma \xi^\alpha = q \hat{R}^{\gamma\alpha}_{\beta\delta} \xi^\beta x^\delta,$$  \hfill (4)

$$(P_s + P_t)_{\alpha\beta} \xi^\gamma \xi^\delta = 0,$$  \hfill (5)

$$P_{\alpha\beta} \partial_\beta \partial_\alpha = 0,$$  \hfill (6)

$$\partial_\alpha x^\beta = \delta^\beta_\alpha + q \hat{R}^{\beta\gamma}_{\alpha\delta} x^\gamma \partial_\gamma,$$  \hfill (7)

$$\partial^2 \xi^\alpha = q^{-1} \hat{R}^{\alpha\gamma}_{\beta\delta} \xi^\beta \partial^\gamma.$$  \hfill (8)

Here the $N^2 \times N^2$ matrix $\hat{R}$ is the braid matrix of $SO_q(N)$ [6]. The matrices $P_s, P_a, P_t$ are $SO_q(N)$-covariant deformations of the symmetric trace-free, antisymmetric and trace projectors respectively, which appear in the orthogonal projector decomposition of $\hat{R}$

$$\hat{R} = qP_s - q^{-1}P_a + q^{1-N}P_t.$$  \hfill (9)

Thus they satisfy the equations (with $A, B = s, a, t$)

$$P_A P_B = P_A \delta_{AB}, \quad P_s + P_a + P_t = 1_{N^2}.$$  \hfill (10)

The $P_t$ projects on a one-dimensional sub-space and can be written in the form

$$P^{\alpha\beta}_{t\gamma\delta} \sim g^{\alpha\beta} g_{\gamma\delta};$$  \hfill (11)

here the $N \times N$ matrix $g_{\alpha\beta}$ is a $SO_q(N)$-isotropic tensor, deformation of the ordinary Euclidean metric. The elements

$$r^2 \equiv x \cdot x := x^\alpha g_{\alpha\beta} x^\beta, \quad \partial \cdot \partial := g^{\alpha\beta} \partial_\beta \partial_\alpha$$

are $U_q so(N)$-invariant and respectively generate the centers of $F, F'$. $r^2$ is real and positive-definite w.r.t. the $*$-structure in $F$ for real $q$ introduced in [6]. $\partial \cdot \partial$ is a deformation of the Laplacian on $\mathbb{R}^N$. We shall adopt
here mainly a set \( \{x^a\} \) of real generators \( x^a = V_i^a x^i \), whereas by \( \{x_i\} \) we mean the standard non-real basis adopted since the works \([6, 1]\), in which \( r^2 \) takes the form \( r^2 = \sum x^i x_i \) and the relations (3-8) take the simplest explicit form, and \( V \) is the (complex) linear transformation relating the two bases.\(^2\)

We shall slightly extend \( F \) by introducing the square root \( r \) of \( r^2 \) and its inverse \( r^{-1} \) as new (central) generators; \( r \) is “the deformed Euclidean distance of the generic point of coordinates \((x^i)\) of \( \mathbb{R}_q^N \) from the origin”. The elements

\[ t^\alpha := r^{-1} x^\alpha \]

fulfill \( t \cdot t = 1 \) and thus generate the subalgebra \( F(S_q^{N-1}) \) of “functions on the unit quantum Euclidean sphere”. \( F(S_q^{N-1}) \) can be completely decomposed into eigenspaces \( V_l \) of the quadratic Casimir of \( U_q so(N) \), or equivalently of the Casimir \( w \) defined in (2) with eigenvalues \( w_l := q^{-l(l+N-2)} \), implying a corresponding decomposition for \( F \):

\[
F(S_q^{N-1}) = \bigoplus_{l=0}^{\infty} V_l, \quad F = \bigoplus_{l=0}^{\infty} (V_l \otimes \mathbb{C}[\left \{ r, r^{-1}\right \}]).
\]

(12)

An orthonormal basis \( \{S^l_f\} \) (consisting of ‘spherical harmonics’) of \( V_l \) can be extracted out of the set of homogeneous, completely symmetric and trace-free polynomials of degree \( l \), suitably normalized. Therefore for the generic \( f \in F \)

\[
f = \sum_{l=0}^{\infty} f_l = \sum_{l=0}^{\infty} \sum_{l=1}^{l} S^l_l f_l, l(r).
\]

(13)

We shall call \( DC^* \) (differential calculus algebra on \( \mathbb{R}_q^N \)) the unital associative algebra over \( \mathbb{C}[\left \{ h \right \}] \) generated by \( x^a, \xi^a, \partial_a \) modulo relations (3-8). We shall denote by \( \wedge^* \) (exterior algebra, or algebra of exterior forms) the graded unital subalgebra generated by the \( \xi^a \) alone, with grading \( \xi = \) the degree in \( \xi^a \), and by \( \wedge^p \) (vector space of exterior \( p \)-forms) the component with grading \( \xi = p, p = 0, 1, 2 \ldots \). Each \( \wedge^p \) carries an irreducible representation of \( U_q so(N) \), and its dimension is the binomial coefficient \( \binom{N}{p} \) \([10]\), exactly as in the \( q = 1 \) (i.e. undeformed) case. We shall endow \( DC^* \) with the same grading \( \xi \), and call \( DC^p \) its component with grading \( \xi = p \). The elements of \( DC^p \) can be considered differential-operator-valued \( p \)-forms. We

\( ^2\)For instance, relations (3), (5) for \( \mathbb{R}_q^2 \) in the basis \( \{x^a\} = \{x^-, x^0, x^+\} \) \([6, 1]\) amount to:

\[
\begin{align*}
x^- x^0 &= qx^0 x^- \quad x^0 x^+ = qx^+ x^0, \quad [x^-, x^+] = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) x^0 x^0, \\
q \xi^- \xi^0 + \xi^0 \xi^- &= q \xi^0 \xi^+ + \xi^+ \xi^0 = \xi^0 \xi^+ + \xi^+ \xi^- = (\xi^0)^2 = (\xi^0)^2 - (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \xi^0 \xi^+ = 0.
\end{align*}
\]

The \( *- \)structure acts on the coordinates as follows:

\[
(x^-)^* = q^{\frac{1}{2}} x^+ , \quad (x^0)^* = x^0 , \quad (x^+)^* = q^{-\frac{1}{2}} x^- .
\]
shall denote by $H$ (Heisenberg algebra on $\mathbb{R}^N_q$) the unital subalgebra generated by the $x^\alpha, \partial_\alpha$. Note that by definition $DC^0 = H$, and that both $DC^*$ and $DC^p$ are $H$-bimodules. We shall denote by $\Omega^p$ (algebra of differential $p$-forms) the graded unital subalgebra generated by the $\xi^\alpha, x^\alpha$, with grading $\sharp$, and by $\Omega^p$ (space of differential $p$-forms) its component with grading $p$; by definition $\Omega^0 = F$ itself. Clearly both $\Omega^*$ and $\Omega^p$ are $F$-bimodules. We shall denote by $\algebraslash{\Omega^*}$ (algebra of differential forms) the graded unital subalgebra generated by the $\omega^\alpha, x^\alpha$, with grading $\sharp$, and by $\algebraslash{\Omega^p}$ its component with grading $p$; by definition $\algebraslash{\Omega^0} = F$ itself. Clearly both $\algebraslash{\Omega^*}$ and $\algebraslash{\Omega^p}$ are $F$-bimodules.

Finally, we recall that by replacing $q \rightarrow q^{-1}, \tilde{R} \rightarrow \tilde{R}^{-1}$ in (3-8) one gets an alternative $U_q so(N)$-covariant calculus, whose exterior/partial derivative and differentials $\tilde{d}, i\tilde{\partial}^\alpha, \xi^\alpha$ are the $\ast$-conjugates of $d, i\partial^\alpha, \xi^\alpha$ and are related to the latter by a rather complicated rational transformation [14].

**Hodge duality.**

The “Hodge map” [10] is a $U_q so(N)$-covariant, $H$-bilinear map

$$* : DC^p \rightarrow DC^{N-p}$$

(14)

such that $*1 = dV$ and on each $DC^p$ (hence on the whole $DC^*$)

$$*^2 \equiv * \circ * = \text{id},$$

(15)

defined by setting on the monomials in the $\xi^\alpha$

$$*(\xi^{\alpha_1} \cdots \xi^{\alpha_p}) = q^{-N(p-N/2)} c_p \xi^{\alpha_{p+1}} \cdots \xi^\alpha N \cdot \alpha_{p+1} \cdots \alpha_p \Lambda^{2p-N}.$$  

(16)

Here $c_p$ is a suitable normalization factor [going to 1/(N−p)! in the commutative limit $q \rightarrow 1$], $\xi^{\alpha_1 \cdots \alpha_N}$ is the $q$-deformed $\varepsilon$-tensor and $\Lambda^{\pm 1}$ are the square root and the inverse square root of

$$\Lambda^{-2} := 1 + (q^2 - 1)x^\alpha \partial_\alpha + \frac{q^{N-2}(q^2 - 1)^2}{(1 + q^{N-2})^2} \cdot \partial \cdot \partial \equiv 1 + O(h);$$

(17)

as a pseudodifferential operator $\Lambda$ acts as a dilatation:

$$\Lambda x^\alpha = q^{-1} x^\alpha \Lambda, \quad \Lambda \partial^\alpha = q \partial^\alpha \Lambda, \quad \Lambda \xi^\alpha = \xi^\alpha \Lambda, \quad \Lambda 1 = 1.$$  

(18)

In $DC^*$ one can introduce [9, 2] as an alternative basis of 1-forms a “frame” $\{\theta^\alpha\}$, in the sense that $[\theta^\alpha, H] = 0$ (implying in particular $[\theta^\alpha, F] = 0$). In the latter (16) takes the form

$$*(\theta^{\alpha_1} \theta^{\alpha_2} \cdots \theta^{\alpha_p}) = c_p \theta^{\alpha_{p+1}} \cdots \theta^{\alpha_N} \varepsilon_{\alpha_{N} \cdots \alpha_{p+1}} \alpha_{p+1} \cdots \alpha_p,$$  

(19)

Restricting the domain of $*$ to the unital subalgebra $\Theta^* \subset DC^*$ generated by $x^\alpha, \xi^\alpha, \Lambda^\pm 1$ one obtains a $U_q so(N)$-covariant, $\tilde{F}$-bilinear map (with $\tilde{F} \equiv \tilde{\Omega}^0$),

$$* : \Theta^p \rightarrow \tilde{\Omega}^{N-p}$$

(20)

fulfilling again $*1 = dV$ and (15). Finally, introducing the exterior coderivatives

$$\delta := -* d^* \quad \delta := -* \tilde{d}^*$$

(21)
one finds that on all of $\mathcal{D}C^*$, and in particular on all of $\Omega^*$, the Laplacians
\begin{align}
\hat{\Delta} &:= \hat{d} \hat{\delta} + \hat{\delta} \hat{d} = -q^{2\hat{\partial}} \hat{\partial} \Lambda^2 = -q^{-N} \hat{\partial} \cdot \hat{\partial}, \\
\Delta &:= \hat{\partial} \hat{\partial} = q^{-2} \hat{\partial} \cdot \hat{\partial} = q^{N} \hat{\partial} \cdot \hat{\partial}
\end{align}
(22)
are respectively quadratic in the $\hat{\partial}_\alpha, \hat{\partial}_\alpha$ (note: not in the $\hat{\partial}_\alpha, \hat{\partial}_\alpha$).

The restriction (20) is the notion closest to the conventional notion of a Hodge map on $\mathbb{R}^N_q$; as a matter of fact, there is no $F$-bilinear restriction of $\cdot$ to $\Omega^*$. However $\Omega^*$ is not closed under the $\cdot$-structure which we shall recall below.

\section*{$U_q so(N)$ and its realization by (pseudo)differential operators.}

We extend [11] the Hopf algebra $U_q so(N)$ by adding a central, primitive generator $\eta$
\[ \phi(\eta) = 1 \otimes \eta + \eta \otimes 1, \quad \epsilon(\eta) = 0, \quad S\eta = -\eta \]
(here $\phi, \epsilon, S$ respectively denote the coproduct, counit, antipode), and we endow the resulting Hopf algebra $\hat{U}_q so(N)$ by the quasitriangular structure
\[ \hat{R} := R q^{0 \otimes \eta}, \]
(24)
where $R \equiv R^{(1)} \otimes R^{(2)}$ (in a Sweedler notation with upper indices and suppressed summation index) denotes the quasitriangular structure of $U_q so(N)$.

The action (which here and in [9] we choose to be right) on $\mathcal{D}C^*$ is completely specified by the transformation laws of the generators $\sigma^\alpha = x^\alpha, \xi^\alpha, \partial^\alpha$, which read
\[ \rho^\beta(g) \sigma^\beta \quad \text{and} \quad x^\alpha \prec \eta = x^\alpha, \quad \xi^\alpha \prec \eta = \xi^\alpha, \quad \partial^\alpha \prec \eta = -\partial^\alpha; \]

here $\rho$ denotes the $N$-dimensional representation of $U_q so(N)$. The elements
\[ Z_\alpha := T^{(1)} \rho^\alpha(T^{(2)}), \quad T := R_{21} R \equiv T^{(1)} \otimes T^{(2)}, \quad R_{21} \equiv R^{(2)} \otimes R^{(1)} \]
are generators of $U_q so(N)$, and make up the “$SO_q(N)$ vector field matrix” $Z$ [17, 15]. We recall that the ribbon element $w \in U_q so(N)$ is the central element such that
\[ w^2 = u_1 S(u_1), \quad u_1 := (S R^{(2)}) R^{(1)}. \]
It is well-known [5] that there exist isomorphisms $U_h so(N)[[h]] \simeq U so(N))[[h]]$ of $*$-algebras over $C[[h]]$. This essentially means that it is possible to express the elements of either algebra as power series in $h = \ln q$ with coefficients in the other. In particular $w$ has an extremely simple expression in terms of the quadratic Casimir $C$ of $so(N)$:
\[ w = q^{-C} = e^{-hC} = 1 + O(h), \quad C := X^\alpha X^\alpha =: L(L + N - 2). \]
\[ X^a \text{ is a basis of } \mathfrak{so}(N). \text{ We denote by } \nu \equiv w^{1/4} \text{ and by } \tilde{w}, \tilde{\nu}, \tilde{T}, \tilde{Z} \text{ the analogs of } w, \nu, T, Z \text{ obtained by replacing } R \text{ by } \tilde{R}. \text{ As an immediate consequence}

\[ \tilde{w} = q^{-C} q^{-\frac{\eta^2}{2}}, \quad \tilde{\nu} = q^{-C/4} q^{-\frac{\eta^2}{4}}, \quad \tilde{T} = T q^{2\nu \bar{\nu}}. \]

As shown in [7, 3, 10], for real \( q \) there exists a \( \ast \)-algebra homomorphism\(^3\)

\[ \varphi : \mathcal{H} \rightarrow \tilde{U}_q \mathfrak{so}(N) \rightarrow \mathcal{H}, \] \hspace{1cm} (26)

acting as the identity on \( \mathcal{H} \) itself,

\[ \varphi(a) = a \quad a \in \mathcal{H}. \] \hspace{1cm} (27)

(This requires introducing an additional generator \( \eta' = \varphi(\eta) \in \mathcal{D} \mathcal{C}^* \) subject to the condition \( \varphi(q^\theta) = q^\theta = q^{-N/2} \Lambda \), so that \( [\eta', \xi^\alpha] = 0, [\eta', x^\alpha] = -x^\alpha, [\eta', \partial^\alpha] = \partial^\alpha \). In the sequel we shall often use the shorthand notation

\[ \varphi(g) =: g', \quad g \in \tilde{U}_q \mathfrak{so}(N). \]

One finds [9] on the spherical harmonics of level \( l \) (with \( l = 0, 1, 2, \ldots \))

\[ \nu' S^l_i = q^{-l(l+N-2)/4} S^l_i \prec \nu. \] \hspace{1cm} (28)

\( \ast \)-Structure and Hermiticity in configuration space representation

**Theorem 2.1** [9] For \( q > 0 \) the \( \ast \)-structure of \( \mathcal{D} \mathcal{C}^* \) given in [6, 14] can be expressed in the form

\[ x^{\alpha*} = x^\alpha, \] \hspace{1cm} (29)

\[ \xi^\alpha* = q^N \xi^\beta Z^\alpha_\beta \Lambda^{-2}, \] \hspace{1cm} (30)

\[ \partial^{\alpha*} = -\tilde{\nu}'^{-2} \partial^\alpha \tilde{\nu}'^2 = -q^{1-N/2} \tilde{\nu}'^{-2} \partial^\alpha \tilde{\nu}'^2 \Lambda, \] \hspace{1cm} (31)

\[ d^* = -\tilde{\nu}'^2 d\tilde{\nu}'^{-2}. \] \hspace{1cm} (32)

\(^3\)This is the \( q \)-deformed analog of the realization in the \( q = 1 \) case of the Cartan-Weyl \( L^{\alpha\beta} \) generators of \( U \mathfrak{so}(N) \) through vector fields (i.e. 1st-order differential operators):

\[ \varphi(L^{\alpha\beta}) = x^\alpha \partial^\beta - x^\beta \partial^\alpha, \quad \varphi(x^\alpha) = x^\alpha \quad \varphi(\partial^\alpha) = \partial^\alpha, \quad \varphi(\eta) = -x^\alpha \partial^\alpha. \]
Integration over $\mathbb{R}^N_q$.

Up to normalization this is defined by [16]

$$
\int_q f(x) \, d^N x = \int_0^\infty dr \, m(r) \, r^{N-1} \int_{S^q_{N-1}} d^{N-1} t \, f(t, r) = \\
\int_0^\infty dr \, m(r) \, r^{N-1} \, f_0(r),
$$

(33)

where $f_0$ denotes the $l = 0$ component in (13). The ‘weight’ $m$ along the radial direction $r$ has to fulfill, beside $m(r) > 0$, the $q$-scaling condition

$$
m(r) = m(qr).
$$

(34)

Among the weights fulfilling the latter we can single out two “extreme” cases:

1. $m(r) \equiv 1$;
2. $m(r) \equiv m_J,_{r_0}(r) := |q - 1| \sum_{n=-\infty}^\infty r^\delta(r - r_0 q^n)$ (‘Jackson’ integral).

Integration over $\mathbb{R}^N_q$ fulfills the standard properties of reality, positivity, $U_q so(N)$-invariance, and a slightly modified cyclicity property [16]. Moreover, if $f$ is a regular function decreasing faster than $1/r^{N-1}$ as $r \to \infty$ the Stokes theorem holds

$$
\int_q \partial_\alpha f(x) \, d^N x = 0, \quad \int_q \partial_{\alpha^*} f(x) \, d^N x = 0.
$$

(35)

Integration of functions immediately leads to integration of $N$-forms $\omega_N$: upon moving all the $\xi$’s to the right of the $x$’s we can express $\omega_N$ in the form $\omega_N = f \, d^N x$ ($d^N x$ denotes the unique independent exterior $N$-form), and just have to set

$$
\int_q \omega_N = \int_q f \, d^N x.
$$

(36)

We introduce the scalar product of two “wave-functions” $\phi, \psi \in F$ and more generally of two “wave-forms” $\alpha_p, \beta_p \in \Omega^p$ by

$$
(\phi, \psi) := \int_q \phi^* \psi^\prime_{\prime} \, d^N x, \quad (\alpha_p, \beta_p) := \int_q \alpha_p^* \psi_{\prime}^{\prime-2} \beta_p.
$$

(37)

Setting $U^{-1}_\gamma \delta := g^{\lambda \gamma} g_{\lambda \gamma}$, the second can be easily expressed in terms of integrals

$$
(\alpha_p, \beta_p) = \frac{1}{c_{N-p}} \int_q \alpha_p^* \, U^{-1}_\gamma U^{-1}_\delta \, d^N x
$$

(38)

involving the form components of the generic $\omega_p \in \Omega^p$ in the frame basis:

$$
\omega_p = \theta_{\gamma_1} \ldots \theta_{\gamma_p} \omega_{\gamma_p \ldots \gamma_1}.
$$
[Note that for \( q \neq 1 \) \( \omega_{\gamma_p \gamma_1} \in \mathcal{H} \setminus F \), because \( \theta^a \in \mathcal{D} \Gamma^* \setminus \Omega^* \). As an integrand function at the rhs (38) we mean what one obtains after letting all \( \partial_n \) present under the integral sign act on functions on their right, using the derivation rule (7)].

As a consequence of Theorem 2.1 we obtain

\[
(\alpha_p, \hat{d} \beta_{p-1}) = (\hat{d} \alpha_p, \beta_{p-1}), \quad (\hat{d} \beta_{p-1}, \alpha_p) = (\beta_{p-1}, \hat{d} \alpha_p). \tag{39}
\]

and the (formal) hermiticity of both the momenta \( p^a = i \partial^a \) and the Laplacian \( \Delta \):

\[
(\phi, p^a \psi) = (p^a \phi, \psi), \quad (\alpha_p, \Delta \beta_p) = (\Delta \alpha_p, \beta_p). \tag{40}
\]

The kinetic term in the action for a (real) \( p \)-form (i.e. an antisymmetric tensor with \( p \)-indices) field theory of mass \( M \) can be now introduced as

\[
S_k = (\Delta \alpha_p, \alpha_p) + M^2 (\alpha_p, \alpha_p). \tag{41}
\]

Spectral analysis for \( p^a, \Delta \) is affordable (see e.g. [8]), because of the relatively simple derivation rules (7), compared e.g. to those of \( p_R^a := p^a + p'^a \).

The above considerations are still formal unless we make sense out of

\[
\hat{p}' = q^{-C'/4} q^{-\eta'^2/4}
\]

and its inverse as positive-definite (pseudodifferential) operators on suitable Hilbert subspaces of \( F, \Omega^* \). Now \( q^{-C'/4} \) is well-defined and positive-definite by (12), (28). To define the action of \( q^{-\eta'^2/4} \) on functions \( \phi(r) \) we change variables \( r \to y = \ln r \), whereby \( \eta' = -\partial_y - N/2 \) and \( r^{N-1} dr = e^{N y} dy \), for any function \( \phi(r) \) denote \( \phi(y) := \phi(r) \), and express \( e^{y N/2} \phi(y) = r^{N/2} \phi(r) \) in terms of its Fourier transform,

\[
e^{N \pi} \hat{y} \phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\omega) e^{i\omega y} d\omega. \tag{42}\]

One finds that \( q^{-\eta'^2/4} \) acts as multiplication by \( q^{\eta'^2/4} \) on the Fourier transform:

\[
e^{N \pi y} q^{-\eta'^2/4} \phi(y) = q^{-\partial_y^2/4} e^{N \pi y} \phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \omega \phi(\omega) e^{i\omega y} q^{\eta'^2/4}.
\]

For \( q < 1 \) the UV convergence of scalar products is drastically improved by \( q^{\eta'^2/4} \). Introducing tentatively the ‘Hilbert space of square integrable functions on \( \mathbb{R}^N \):

\[
L_{2^m}^m := \left\{ \mathbf{f}(x) \equiv \sum_{l=0}^{\infty} \sum_l S_l^m f_{l,l}(r) \in F \mid (\mathbf{f}, \mathbf{f}) < \infty \right\} \tag{43}
\]

one finds [9] that all works automatically with \( L_{2^m}^{m+1} \) (i.e. if we choose \( m(r) \equiv 1 \)). Otherwise, we have to
1. further require that $m$ is invariant under inversion, $m(r^{-1}) = m(r)$, and fulfills condition (A.26) in [9] in order that the scalar product $(\cdot, \cdot)$ be positive-definite;

2. at rhs(43) replace $F$ with the subspace $E_{m,\beta,n}$ whose elements $f_{1,\ell}(r)$’s admit each an analytic continuation in the complex $r$-plane with poles possibly only in

$$r_{j,k} := q^{j\beta} e^{i(2k+1)n},$$  \hspace{1cm} (44)

where $\beta \in \{0, 1/2\}$, $n$ is an integer submultiple of $N$, $k = 0, 1, ..., n-1$, and $j \in \mathbb{Z}$, in order that the hermiticity (40) is actually implemented.

Condition 1. is fulfilled by a large class of weights, including the ‘Jackson’ ones $m_{\lambda,\rho}$. Condition 2. selects interesting subclasses of functions, including $q$-special functions with “quantized parameters”, among the simplest one e.g. $1/[1+(q^{-1}r)^n]$.

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