Fréchet frames, general definition and expansions

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Abstract

We define an \((X_1, \Theta, X_2)\)-frame with Banach spaces \(X_2 \subseteq X_1\), \(\lVert \cdot \rVert_1 \leq \lVert \cdot \rVert_2\), and a BK-space \(\langle \Theta, \lVert \cdot \rVert \rangle\). Then by the use of decreasing sequences of Banach spaces \(\{X_s\}_{s=0}^\infty\) and of sequence spaces \(\{\Theta_s\}_{s=0}^\infty\), we define a general Fréchet frame on the Fréchet space \(X_F = \bigcap_{s=0}^\infty X_s\). We give frame expansions of elements of \(X_F\) and its dual \(X_F^*\), as well of some of the generating spaces of \(X_F\) with convergence in appropriate norms. Moreover, we give necessary and sufficient conditions for a general pre-Fréchet frame to be a general Fréchet frame, as well as for the complementedness of the range of the analysis operator \(U : X_F \to \Theta_F\).

Keywords: general pre-Fréchet frame, general Fréchet frame, \((X_1, \Theta, X_2)\)-frame

MSC 2000: 42C15, 46A13

1 Introduction

For given Fréchet spaces \(X_F = \bigcap_{s=0}^\infty X_s\) and \(\Theta_F = \bigcap_{s=0}^\infty \Theta_s\) (\(X_s\) and \(\Theta_s\) are Banach space and Banach sequence space with the norms \(\lVert \cdot \rVert_s\) and \(\lVert |\cdot| \rVert_s\) respectively), in [6], we have determined conditions on a sequence \(\{g_i\}_{i=1}^\infty\), \(g_i \in X_F^*\), which imply the existence of \(\{f_i\}_{i=1}^\infty\), \(f_i \in X_F\), such that every \(f \in X_F\) and every \(g \in X_F^*\) can be written as \(f = \sum_{i=1}^\infty g_i(f)f_i\) and \(g = \sum_{i=1}^\infty g(f_i)g_i\). These conditions are related to the frame inequalities

\[
\{g_i(f)\}_{i=1}^\infty \in \Theta_F \quad \text{and} \quad A_s\|f\|_s \leq \|\{g_i(f)\}_{i=1}^\infty\|_s \leq B_k\|f\|_s, \quad f \in X_F, s \in \mathbb{N}_0.
\]

In the present paper we are concerned with the series expansions via more general sequences in Fréchet spaces allowing different norms in the inequalities given above, namely,

\[
\{g_i(f)\}_{i=1}^\infty \in \Theta_F \quad \text{and} \quad A_k\|f\|_{s_k} \leq \|\{g_i(f)\}_{i=1}^\infty\|_k \leq B_k\|f\|_{s_k}, \quad f \in X_F, k \in \mathbb{N}_0,
\]

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where \( \{s_k\}_{k \in \mathbb{N}_0} \) and \( \{\tilde{s}_k\}_{k \in \mathbb{N}_0} \) are increasing subsequences of \( \mathbb{N}_0 \). The statements in [6] Theorem 5.3(b)(c) give sufficient conditions for an operator defined on \( \Theta_F \) to imply series expansions in some of the generating Banach spaces \( X_s \). In our main theorems we extend these results and determine conditions on \( V : \Theta_F \rightarrow X_F \) which are necessary and sufficient for the aim of expansions. In particular, in Theorem 3.1 we prove that the existence of a sequence \( \{f_i\}_{i=1}^{\infty} \subseteq (X_F)^{\mathbb{N}} \) which is a \( \Theta_F \)-Bessel sequence for \( X_{s_k}^* \) for every \( k \in \mathbb{N}_0 \) and gives series expansions in \( X_{s_k} \) with convergence in \( \|\cdot\|_{s_k} \)-norm is equivalent to the existence of an operator \( V : \Theta_F \rightarrow X_F \) so that \( V\{g_i(f)\}_{i=1}^{\infty} = f, \forall f \in X_F, \) and \( \|Vd\|_{s_k} \leq C_k \|d\|_{s_k} \), \( \forall d \in \Theta_F, \forall k \in \mathbb{N}_0 \) for some constants \( C_k \). While the continuity of \( V \) implies series expansions in \( X_F \) and the above boundedness properties of \( V \) imply series expansions in all the spaces \( X_{s_k}, k \in \mathbb{N}_0, \) with convergence in \( \|\cdot\|_{s_k} \)-norm (see Theorem 3.1), in Theorem 3.2 we prove that the continuity property of \( V \) is enough to imply the existence of subsequence of \( \{X_{s_k}\}_{k=1}^{\infty} \) with series expansions. In general, it is not easy to show that a general pre-Fréchet frame is a general Fréchet frame. We devote Section 4 (Theorem 4.1) to this problem. Several examples in Sections 3 and 4 illustrate our investigations.

The paper is organized as follows. The notation used in the paper are recalled in Subsection 1.1. In Section 2 we give the definition of a general pre-Fréchet frame. Further, we give some statements needed for the main theorems of Section 3. Section 3 concerns series expansions via general pre-Fréchet frames. We determine sufficient conditions for a general pre-Fréchet frame to imply series expansions in a Fréchet space and its dual, as well as necessary and sufficient conditions for a general pre-Fréchet frame to imply series expansions in spaces generating the Fréchet space via a sequence with the Bessel properties. Moreover, in Section 4 we define a general Fréchet frame and give an example of a general Fréchet frame for \( X_F \) with respect to \( \Theta_F \) which is not a Fréchet frame for \( X_F \) with respect to \( \Theta_F \) (according to the definition in [6]). In Section 4 we give necessary and sufficient conditions for a general pre-Fréchet frame to be a general Fréchet frame.

Concerning the list of references, one can find more information about papers, related to Banach frame expansions, in the bibliography of [6].

1.1 Preliminaries

Throughout the paper, \((X, \|\cdot\|)\) is a Banach space and \((X^*, \|\cdot\|_{X^*})\) is its dual, \((\Theta, \|\cdot\|)\) is a Banach sequence space and \((\Theta^*, \|\cdot\|_{\Theta^*})\) is the dual of \(\Theta\). Recall that \(\Theta\) is called a \(BK\)-space if the coordinate functionals on \(\Theta\) are continuous. The symbol \(\delta_i\) denotes the \(i\)-th canonical vector \(\{\delta_k\}_{k=1}^{\infty}, i \in \mathbb{N}\). A \(BK\)-space \(\Theta\) is called a \(\lambda-BK\)-space \((\lambda \geq 1)\) if it contains all the canonical vectors and

\[
\|\{(c_i)_{i=1}^{n}\}_{\Theta}\| \leq \lambda \|\{(c_i)_{i=1}^{\infty}\}_{\Theta}\|, \quad n \in \mathbb{N}, \quad \{c_i\}_{i=1}^{\infty} \subseteq \Theta.
\]

A \(BK\)-space for which the canonical vectors form a Schauder basis, is called a \(CB\)-space. If \(\Theta\) is a \(CB\)-space, then the space \(\Theta^\oplus := \{\{g(c_i)\}_{i=1}^{\infty}: g \in \Theta^*\}\) with the norm \(\|\{(g(c_i)\}_{i=1}^{\infty}\}_{\Theta^\oplus} := \|g\|_{\Theta^*}\) is a \(BK\)-space, isometrically isomorphic to...
Definition 2.1 Let \((X_i, \| \cdot \|_i), i = 1, 2,\) be Banach spaces such that \(X_2 \subseteq X_1, \| \cdot \|_1 \leq \| \cdot \|_2,\) and let \((\Theta, \| \cdot \|)\) be a BK-space. The sequence \(\{g_i\}_{i=1}^{\infty} \in (X_2^*)^N\) is called an \((X_1, \Theta, X_2)\)-frame with bounds \(A, B\) if \(0 < A \leq B < \infty\) and for every \(f \in X_2,\)

\[
\{g_i(f)\}_{i=1}^{\infty} \in \Theta \quad \text{and} \quad A\|f\|_1 \leq \|\{g_i(f)\}_{i=1}^{\infty}\| \leq B\|f\|_2. \tag{2}
\]

Note that when \(\Theta\) is a BK-space, the validity of the upper inequality in (2) for every \(f \in X_2\) (i.e., \(\{g_i\}_{i=1}^{\infty}\) being a \(\Theta\)-Bessel sequence for \(X_2\)) implies that \(g_i\) must be bounded on \(X_2\). When \(X_1 = X_2 = X\), then an \((X_1, \Theta, X_2)\)-frame becomes a \(\Theta\)-frame for \(X\).

We give a generalization of [7] Theorem 3.3(ii)].

Proposition 2.2 Let \((X_i, \| \cdot \|_i), i = 1, 2,\) be Banach spaces such that \(X_2 \subseteq X_1\) and \(\| \cdot \|_1 \leq \| \cdot \|_2\). Let \((\Theta, \| \cdot \|)\) be a \(\lambda\)-BK-space, \(W\) be a dense subset of \(X_2\) and \(\{g_i\}_{i=1}^{\infty} \in (X_2^*)^N\). If (3) holds for all \(f \in W\), then \(\{g_i\}_{i=1}^{\infty}\) is an \((X_1, \Theta, X_2)\)-frame with bounds \(A, \lambda B\).

Proof. By [7] Theorem 3.3(i)], it follows that

\[
\{g_i(f)\}_{i=1}^{\infty} \in \Theta \quad \text{and} \quad \|\{g_i(f)\}_{i=1}^{\infty}\|_\Theta \leq \lambda B\|f\|_2, \quad \forall x \in X_2.
\]

For the lower inequality, take \(f \in X_2 \setminus W\) and a sequence \(\{f_n\} \in W^N\) such that \(f_n \to f\) when \(n \to \infty\) in \(\| \cdot \|_2\)-norm (and hence, in \(\| \cdot \|_1\)-norm). Since

\[
\lim_{n \to \infty} \|\{g_i(f_n)\}_{i=1}^{\infty}\| = \|\{g_i(f)\}_{i=1}^{\infty}\|,
\]

it follows that \(A\|f\|_1 \leq \|\{g_i(f)\}_{i=1}^{\infty}\|\).

Now we define a general pre-Fréchet frame. Let \(\{Y_s, | \cdot |_s\}_{s \in \mathbb{N}_0}\) be a sequence of separable Banach spaces such that

\[
\{0\} \neq \cap_{s \in \mathbb{N}_0} Y_s \subseteq \cdots \subseteq Y_2 \subseteq Y_1 \subseteq Y_0 \tag{3}
\]

\[
| \cdot |_0 \leq | \cdot |_1 \leq | \cdot |_2 \leq \cdots \tag{4}
\]

\[
Y_F := \cap_{s \in \mathbb{N}_0} Y_s \quad \text{is dense in} \quad Y_s, \quad s \in \mathbb{N}_0. \tag{5}
\]

Then \(Y_F\) is a Fréchet space with the sequence of norms \(| \cdot |_s, s \in \mathbb{N}_0\). We will use such sequences in two cases:

1. \(Y_s = X_s\) with norm \(| \cdot |_s, s \in \mathbb{N}_0;\)
2. \(Y_s = \Theta_s\) with norm \(| \cdot |_s, s \in \mathbb{N}_0.\)
Remark 2.3 Let \( X_F \) be a Fréchet space determined by the separable Banach spaces \( X_s, s \in \mathbb{N}_0 \), satisfying \([3]-[4]\), and let \( \Theta_F \) be a Fréchet space determined by the BK-spaces \( \Theta_s, s \in \mathbb{N}_0 \), satisfying \([3]-[2]\). A sequence \( \{g_i\}_{i=1}^{\infty} \in (X_F)^N \) is called a general pre-Fréchet frame (in short, general pre-F-frame) for \( X_F \) with respect to \( \Theta_F \) if there exist sequences \( \{\tilde{s}_k\}_{k \in \mathbb{N}_0} \subseteq \mathbb{N}_0, \{s_k\}_{k \in \mathbb{N}_0} \subseteq \mathbb{N}_0 \) which increase to \( \infty \) with the property \( s_k \leq \tilde{s}_k, k \in \mathbb{N}_0 \), and there exist constants \( B_k, A_k > 0, k \in \mathbb{N}_0 \), satisfying

\[
\{g_i(f)\}_{i=1}^{\infty} \in \Theta_F \quad \text{and} \quad A_k \|f\|_{s_k} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_k \leq B_k \|f\|_{\tilde{s}_k}, \ f \in X_F. \tag{6}
\]

The above definition reduces to the definition of a pre-F-frame in [3] Def. 2.3 if the sequences \( \{s_k\} \) and \( \{\tilde{s}_k\} \) coincide. We give the definition of a general F-frame after Theorem 3.1. We will use the names strict pre-F-frame and strict F-frame in the cases considered in [6] (when \( \{s_k\} \) and \( \{\tilde{s}_k\} \) coincide).

Remark 2.4 Let \( \{g_i\}_{i=1}^{\infty} \) be a general pre-F-frame for \( X_F \) with respect to \( \Theta_F \) according to Definition 2.3. One can see that every subsequences \( \{X_{p_k}\}_{k=1}^{\infty} \) of \( \{X_j\}_{j=1}^{\infty} \), and \( \{\Theta_{q_k}\}_{k=1}^{\infty} \) of \( \{\Theta_j\}_{j=1}^{\infty} \) have suitable sub-subsequences so that (6) holds with the same \( \{g_i\}_{i=1}^{\infty} \) and corresponding sub-subsequences of norms.

In the sequel, when we consider a general pre-F-frame \( \{g_i\}_{i=1}^{\infty} \) for \( X_F \) with respect to \( \Theta_F \), we always assume that \( X_F \) is determined by the sequence \( \{X_s, \|\cdot\|_s\}_{s \in \mathbb{N}_0} \) of Banach spaces satisfying \([3]-[4]\). \( \Theta_F \) is determined by a sequence \( \{\Theta_s, \|\cdot\|_s\}_{s \in \mathbb{N}_0} \) of BK-spaces satisfying \([3]-[5]\), and \( \{g_i\}_{i=1}^{\infty} \) fulfills (6).

Remark 2.5 Let \( \{g_i\}_{i=1}^{\infty} \) be a general pre-F-frame for \( X_F \) with respect to \( \Theta_F \). For every \( i \in \mathbb{N} \) and every \( k \in \mathbb{N}_0 \), the functional \( g_i \) has a unique continues extension on \( X_{\tilde{s}_k} \) which will be denoted by \( g^\tilde{s}_k_i \). By Proposition 2.2 for every \( k \in \mathbb{N}_0 \), the sequence \( \{g^\tilde{s}_k_i\}_{i=1}^{\infty} \) is an \((X_{\tilde{s}_k}, \Theta_k, X_{\tilde{s}_k})\)-frame. Thus, we can consider operators

\[
U_k : X_{\tilde{s}_k} \to \Theta_k, \quad U_k f = \{g^\tilde{s}_k_i(f)\}_{i=1}^{\infty}, \ k \in \mathbb{N}_0, \tag{7}
\]

\[
U : X_F \to \Theta_F, \quad U f = \{g_i(f)\}_{i=1}^{\infty}. \tag{8}
\]

Clearly, they are injective and continuous.

Proposition 2.6 Let \( \{g_i\}_{i=1}^{\infty} \in (X_F^*)^N \) be a general pre-F-frame for \( X_F \) with respect to \( \Theta_F \). Then the following holds.

(i) The range \( R(U) \) of the operator \( U \), defined by (8), is closed in \( \Theta_F \) and the inverse operator \( U^{-1} : R(U) \to X_F \) is continuous.

(ii) The existence of a continuous projection \( \mathcal{P} \) from \( \Theta_F \) onto \( R(U) \) (i.e. \( R(U) \) being complemented in \( \Theta_F \)) is equivalent to the existence of a continuous operator \( V : \Theta_F \to X_F \) so that \( V\{g_i(f)\}_{i=1}^{\infty} = f \) for all \( f \in X_F \).

Proof. (i) Let \( f_n \in X_F, n \in \mathbb{N} \), and let \( \{U f_n\}_{n=1}^{\infty} \) converge to some \( b = (b_n)_{n=1}^{\infty} \in \Theta_F \) in \( \Theta_F \) when \( n \to \infty \). Fix an arbitrary \( k \in \mathbb{N}_0 \). The lower
inequality in (5) implies that \( \{f_n\}_{n=1}^{\infty} \) converges in \( X_k \) when \( n \to \infty \) and thus, \( \{f_n\}_{n=1}^{\infty} \) converges in \( X_F \) to some element \( a \in X_F \). Furthermore, the upper inequality in (5) implies that \( \{Uf_n\}_{n=1}^{\infty} \) converges to \( Ua \in R(U) \) in \( \Theta_F \). Therefore, \( R(U) \) is closed in \( \Theta_F \). The continuity of \( U^{-1} \) is easy to see.

(ii) Let \( \mathcal{P} \) be a continues projection from \( \Theta_F \) onto \( R(U) \). This implies that the operator \( V \) defined by \( V = U^{-1}\mathcal{P} : \Theta_F \to X_F \) is also continues. Clearly, \( V \) is an extension of \( U^{-1} \).

Conversely, let \( V : \Theta_F \to X_F \) be continuous and such that \( V \{g_i(f)\}_{i=1}^{\infty} = f \), \( \forall f \in X_F \). Then the operator \( \mathcal{P} := UV \) is a continuous projection from \( \Theta_F \) onto \( R(U) \). \( \Box \)

**Remark 2.7** Note that the assumption \( s_k \to \infty \) is essentially used in Proposition 2.6 to prove that \( R(U) \) is closed in \( \Theta_F \). If \( \{g_i\}_{i=1}^{\infty} \in (X_2)^N \) is an \( (X_1, \Theta, X_2) \)-frame, then the range of the operator \( \tilde{U} : X_2 \to \Theta \hat{f} := \{g_i(f)\}_{i=1}^{\infty} \), is not necessarily closed in \( \Theta \). For example, consider \( X_1 = \ell^q, \Theta = \ell^2, X_2 = \ell^p \) for some \( 1 < p < 2 < q < \infty \). Let \( g_i \) be the \( i \)-th coordinate functional on \( \ell^p \), \( i \in \mathbb{N} \). For every \( c = \{c_i\}_{i=1}^{\infty} \in \ell^p \), \( \|c\|_{\ell^p} \leq \|\|g_i(c)\|\|_{\ell^2} \leq \|c\|_{\ell^p} \), and thus \( \{g_i\}_{i=1}^{\infty} \) is an \( (\ell^q, \ell^2, \ell^p) \)-frame. Furthermore, \( R(\tilde{U}) \) coincides with \( \ell^p \) as sets and thus, \( R(\tilde{U}) \) is not closed in \( \ell^2 \). Note that if \( \{g_i\}_{i=1}^{\infty} \in (X_2)^N \) is an \( (X_1, \Theta, X_2) \)-frame and \( R(\tilde{U}) \) is closed in \( \Theta \), then \( \{g_i\}_{i=1}^{\infty} \) must satisfy the lower \( \Theta \)-frame inequality for \( X_2 \) and thus \( \{g_i\}_{i=1}^{\infty} \) must be a \( \Theta \)-frame for \( X_2 \).

3 Expansions

In this section we are interested in series expansions via general pre-\( F \)-frames. First note that if \( \{g_i\}_{i=1}^{\infty} \) is a general pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \), such that every \( f \in X_F \) can be written as \( f = \sum_{i=1}^{\infty} g_i(f) \), convergence in \( X_F \), then clearly one can define the operator \( V : \Theta_F \to X_F \) by \( V \{g_i(f)\}_{i=1}^{\infty} = f \) and \( V \) must be continuous. Below we continue with sufficient (resp. necessary and sufficient) conditions for the existence of series expansions in \( X_F \) and in the generating Banach spaces \( X_s \).

**Theorem 3.1** Let \( \{g_i\}_{i=1}^{\infty} \) be a general pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \).

(a) Let \( \Theta_s, s \in \mathbb{N}_0 \), be \( CB \)-spaces, and let there exist a continuous operator \( V : \Theta_F \to X_F \) so that \( V \{g_i(f)\}_{i=1}^{\infty} = f \) for all \( f \in X_F \). Then there exists a sequence \( \{f_i\}_{i=1}^{\infty} \in (X_F)^N \) such that

\[
f = \sum_{i=1}^{\infty} g_i(f) \quad f \in X_F, \quad (in \ X_F), \quad (9)
\]

\[
g = \sum_{i=1}^{\infty} g(f_i) \quad g \in X_F^*, \quad (in \ X_F). \quad (10)
\]

(b) Let \( \Theta_s, s \in \mathbb{N}_0 \), be \( CB \)-spaces. Then the following three statements are equivalent:
\[ \mathcal{A}_1: \text{There exists an operator } V : \Theta_F \to X_F \text{ so that } V\{g_i(f)\}_{i=1}^{\infty} = f, \forall f \in X_F, \text{ and for every } k \in \mathbb{N}_0 \text{ there is a constant } C_k > 0 \text{ satisfying } \|Vd\|_{s_k} \leq C_k\|d\|_k \text{ for all } d \in \Theta_F. \]

\[ \mathcal{A}_2: \text{There exists } \{f_i\}_{i=1}^{\infty} \in (X_F)^{\mathbb{N}} \text{ such that for every } k \in \mathbb{N}_0, \{f_i\}_{i=1}^{\infty} \text{ is a } \Theta_k^* \text{-Bessel sequence for } X_{s_k}^*, \text{ and } [\|] \text{ holds.} \]

\[ \mathcal{A}_3: \text{There exists } \{f_i\}_{i=1}^{\infty} \in (X_F)^{\mathbb{N}} \text{ such that for every } k \in \mathbb{N}_0, \{f_i\}_{i=1}^{\infty} \text{ is a } \Theta_k^* \text{-Bessel sequence for } X_{s_k}^*, \text{ and} \]

\[ f = \sum_{i=1}^{\infty} g_i^k(f_i) f_i \text{ in } \|\cdot\|_{s_k} \text{-norm, } f \in X_{s_k}. \] (11)

In particular, the equivalent conditions \( \mathcal{A}_1-\mathcal{A}_3 \) imply validity of [4] and \( \mathcal{A}_1 \) with a same sequence \( \{f_i\}_{i=1}^{\infty} = \{V\alpha \}_{i=1}^{\infty} \).

(c) Let \( \Theta_s \text{ and } \Theta_s^*, s \in \mathbb{N}_0, \text{ be CB-spaces. Then } \mathcal{A}_1 \text{ is equivalent to} \]

\[ \mathcal{A}_4: \text{There exists } \{f_i\}_{i=1}^{\infty} \in (X_F)^{\mathbb{N}} \text{ such that for every } k \in \mathbb{N}_0, \{f_i\}_{i=1}^{\infty} \text{ is an } (X_{s_k}^*, \Theta_k^*, X_{s_k}^*) \text{-frame, and} \]

\[ g|_{X_{s_k}} = \sum_{i=1}^{\infty} g(f_i)g_i^k \text{ in } \|\cdot\|_{X_{s_k}} \text{-norm, } g \in X_{s_k}^*. \] (12)

In particular, the equivalent conditions \( \mathcal{A}_1-\mathcal{A}_4 \) imply validity of [4] and [12] with a same sequence \( \{f_i\}_{i=1}^{\infty} = \{V\alpha \}_{i=1}^{\infty} \).

**Proof.** (a) Let \( f_i = V\alpha_i, i \in \mathbb{N}. \) By the assumptions, one can write, in \( \Theta_F, \)

\[ Uf = \sum_{i=1}^{\infty} g_i(f)e_i, f \in X_F. \]

For every \( f \in X_F, \) the continuity of \( V \) implies that \( V(\sum_{i=1}^{\infty} g_i(f)e_i) \to VF = f \text{ in } X_F \text{ as } n \to \infty, \text{ and this gives } (3) \text{ and } (\|). \]

(b) \( \mathcal{A}_1 \Rightarrow \mathcal{A}_2, \mathcal{A}_3: \) Assume that \( \mathcal{A}_1 \) holds. Let \( f_i = V\alpha_i, i \in \mathbb{N}. \) First observe that \( V \) is continuous and hence, by (a), the representations (3) and (\|) hold. Fix an arbitrary \( k \in \mathbb{N}_0. \) The operator \( V \) has a bounded linear extension \( V_k : \Theta_k \to X_{s_k} \) and \( F \) can consider the bounded operator \( V_kU_k : X_{s_k} \to X_{s_k}. \)

Since \( V_kU_kf = f \text{ for every } f \in X_F, X_F \text{ is dense in } X_{s_k} \text{ and } \|\cdot\|_{s_k} \leq \|\cdot\|_{s_k}, \text{ it follows that } V_kU_kf = f \text{ for every } f \in X_{s_k}. \text{ Let } f \in X_{s_k}. \text{ Then } \{g_i^k(f)\}_{i=1}^{\infty} \in \Theta_k \text{ (see Remark 2.5) and} \]

\[ \sum_{i=1}^{n} g_i^k(f) f_i = V_k \left( \sum_{i=1}^{n} g_i^k(f) e_i \right) \overset{n \to \infty}{\to} V_kU_kf = f \text{ in } \|\cdot\|_{s_k}. \]

It is clear that \( \{f_i\}_{i=1}^{\infty} \) is a \( \Theta_k^* \)-Bessel sequence for \( X_{s_k}^*. \)

\( \mathcal{A}_2 \Rightarrow \mathcal{A}_1: \) Assume that \( \mathcal{A}_2 \) holds. Fix \( k \in \mathbb{N}_0. \) Since \( \{f_i\}_{i=1}^{\infty} \) is a \( \Theta_k^* \)-Bessel sequence for \( X_{s_k}^*, \) it follows that the synthesis operator \( T_k \) given by \( T_k(d)_{i=1}^{\infty} = \sum_{i=1}^{\infty} d_i f_i \) is well defined (and bounded) from \( \Theta_k \) into \( X_{s_k}. \) [1]. For \( \{d_i\}_{i=1}^{\infty} \in \Theta_F, \) the series \( \sum_{i=1}^{\infty} d_i f_i \) converges in \( X_{s_k} \) for every \( k \in \mathbb{N}_0, \) and thus,
it converges in $X_F$. Then we can consider the operator $V : \Theta_F \to X_F$ defined by $V\{d_i\}_{i=1}^\infty = \sum_{i=1}^\infty d_i f_i$. For every $d \in \Theta_F$,

$$\|V d\|_{s_k} = \|T_k d\|_{s_k} \leq \|T_k\| \cdot \|d\|_k, \ \forall k \in \mathbb{N}_0.$$ Further, the validity of (11) implies that $V \{g_i(f)\}_{i=1}^\infty = f$ for all $f \in X_F$. Clearly, $V e_i = f_i$, $i \in \mathbb{N}$.

$A_3 \Rightarrow A_2$: Assume that $A_3$ holds. The representations in (11) imply that $f = \sum_{i=1}^\infty g_i(f) f_i$ in $\|\cdot\|_{s_k}$-norm for every $k \in \mathbb{N}_0$ and every $f \in X_F$, which implies that (11) holds.

(c) $A_1 \Rightarrow A_4$: Assume that $A_1$ holds. Let $\{f_i\}_{i=1}^\infty$ be given as in (b) and fix $k \in \mathbb{N}_0$. Then $\{f_i\}_{i=1}^\infty$ is a $\Theta_k^*$-Bessel sequence for $X_{s_k}^*$. Therefore the synthesis operator $\tilde{T}_k$ given by $\tilde{T}_k \{d_i\}_{i=1}^\infty = \sum_{i=1}^\infty d_i g_i^F$ is well defined and bounded from $\Theta_k^*$ into $X_{s_k}^*$ (see (11)). Let $g \in X_{s_k}^*$. For every $f \in X_{s_k}$, it follows by (b) that $\sum_{i=1}^\infty g_i^F(f) f_i \to f$ in $\|\cdot\|_{s_k}$-norm when $n \to \infty$, which implies that $g(\sum_{i=1}^n g_i^F(f) f_i) \to g(f)$ when $n \to \infty$. Furthermore,

$$\|g\|_{X_{s_k}^*} = \sup_{f \in X_{s_k}, \|f\|_{s_k} \leq 1} \left| \sum_{i=1}^\infty g_i^F(f) g(f_i) \right| = \sup_{f \in X_{s_k}, \|f\|_{s_k} \leq 1} \left| \tilde{T}_k \{g(f_i)\}_{i=1}^\infty(f) \right| \leq \|\tilde{T}_k\| \|\{g(f_i)\}_{i=1}^\infty\|_{\Theta_k^*}.$$

Therefore, $\{f_i\}_{i=1}^\infty \in (X_F)^\mathbb{N}$ is an $(X_{s_k}^*, \Theta_k^*, X_{s_k}^*)$-frame.

To prove (11), denote the canonical basis of $\Theta_k^*$ by $\{\delta_i\}_{i=1}^\infty$. Let $g \in X_{s_k}^*$. Then $g|_{X_{s_k}^*} \in X_{s_k}^*$ and

$$\|g|_{X_{s_k}^*} - \sum_{i=1}^n g(f_i) g_i^F \|_{X_{s_k}^*} = \sup_{f \in X_{s_k}, \|f\|_{s_k} \leq 1} \left| \sum_{i=1}^\infty g_i^F(f) g(f_i) - \sum_{i=1}^n g(f_i) g_i^F(f) \right| \leq \|\tilde{T}_k\| \left| \sum_{i=n+1}^\infty g(f_i) \delta_i \right| \|\delta_i\|_{\Theta_k^*} \xrightarrow{n \to \infty} 0.$$
\[ \mathcal{A}_4 \Rightarrow \mathcal{A}_3: \text{ Assume that } \mathcal{A}_4 \text{ holds. For } k \in \mathbb{N}, \text{ let } B_k \text{ denote a } \Theta^*_k\text{-Bessel bound for } \{f_i\}_{i=1}^{\infty}. \text{ Fix an arbitrary } k \in \mathbb{N}. \text{ For every } f \in X_{\tilde{s}_k}, \]

\[
\|f - \sum_{i=1}^{n} g_i^k(f) f_i\|_{\tilde{s}_k} = \sup_{g \in X_{\tilde{s}_k}, \|g\|_{X_{\tilde{s}_k}} = 1} |g(f) - \sum_{i=1}^{n} g(f_i) g_i(f)| \\
= \sup_{g \in X_{\tilde{s}_k}, \|g\|_{X_{\tilde{s}_k}} = 1} |\sum_{i=n+1}^{\infty} g(f_i) g_i(f)| \\
\leq \sup_{g \in X_{\tilde{s}_k}, \|g\|_{X_{\tilde{s}_k}} = 1} \|\{g(f_i)\}_{i=1}^{\infty}\|_{\Theta^*_k} \|\sum_{i=n+1}^{\infty} g_i(f) e_i\|_k \\
\leq B_k \|\sum_{i=n+1}^{\infty} g_i(f) e_i\|_k \to 0 \text{ as } n \to \infty. \]

Therefore, (11) holds. \(\square\)

Theorem 3.1 extends [6, Theorem 5.3] and improves the formulation of [6, Theorem 5.3](a), where it is silently assumed that \(\Theta_s, s \in \mathbb{N}_0, \) are CB-spaces.

As one can see in Theorem 3.1, the continuity property of the operator \(V\) implies series expansions in \(X_f\), while some boundedness properties of \(V\) imply series expansions in all the spaces \(X_{s_k}, k \in \mathbb{N}_0,\) with convergence in \(\|\cdot\|_{s_k}\)-norm. Below we prove that the continuity property of \(V\) is enough to imply the existence of a subsequence \(\{X_{\tilde{s}_k}\}_{k=0}^{\infty}\) of \(\{X_{s_k}\}_{k=0}^{\infty}\) such that one has series expansions in \(X_{\tilde{s}_k}, j \in \mathbb{N}_0,\) with convergence in appropriate norms.

**Theorem 3.2** Let \(\{g_i\}_{i=1}^{\infty}\) be a general pre-
\(F\)-frame for \(X_F\) with respect to \(\Theta_F\) and let \(\Theta_s, s \in \mathbb{N}_0,\) be CB-spaces. Assume that there exists a continuous operator \(V : \Theta_F \to X_F\) so that \(V(g_i(f))_{i=1}^{\infty} = f\) for all \(f \in X_F.\) Then there exist sequences \(\{w_j\}_{j \in \mathbb{N}_0}, \{r_j\}_{j \in \mathbb{N}_0}, \{\tilde{w}_j\}_{j \in \mathbb{N}_0}\) which increase to \(\infty\) and there exist constants \(A_j, B_j, j \in \mathbb{N}_0,\) such that for every \(j \in \mathbb{N}_0,\)

\[ A_j \|f\|_{w_j} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{r_j} \leq B_j \|f\|_{\tilde{w}_j}, \forall f \in X_F. \]

Moreover, there exists a sequence \(\{f_i\}_{i=1}^{\infty} \in (X_F)^\mathbb{N}\) such that for every \(j \in \mathbb{N}_0, \{f_i\}_{i=1}^{\infty}\) is a \(\Theta^*_j\)-Bessel sequence for \(X_{w_j}\) and

\[ f = \sum_{i=1}^{\infty} g_i^\tilde{w}_j(f) f_i \text{ in } \|\cdot\|_{w_j}\text{-norm}, \; f \in X_{\tilde{w}_j}. \]

**Proof.** Assume that \(V : \Theta_F \to X_F\) is a continuous operator satisfying \(V\{g_i(f)\}_{i=1}^{\infty} = f, \forall f \in X_F.\) Then

\[ \forall k \in \mathbb{N}_0, \exists p_k \in \mathbb{N}_0 \text{ and } \exists C_k \text{ so that } \|Vd\|_{s_k} \leq C_k \|d\|_{p_k}, \forall d \in \Theta_F. \quad (13) \]
Consider the sequence \( \{n_k\}_{k \in \mathbb{N}_0} \) defined by \( n_k = \max(k, p_k) \), \( k \in \mathbb{N}_0 \). Clearly, the sequence \( \{n_k\}_{k \in \mathbb{N}_0} \) is not bounded. Take a strictly increasing subsequence \( \{n_{k_j}\}_{j \in \mathbb{N}_0} \) of \( \{n_k\}_{k \in \mathbb{N}_0} \). Let \( q \in \{k_j\}_{j=1}^{\infty} \). Since \( n_q \geq q \), (6) implies

\[
A_q\|f\|_{s_q} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{n_q} \leq B_n\|f\|_{\tilde{s}_n}, \quad \forall f \in X_F.
\]

Since \( n_q \geq p_q \), (13) implies that \( \|Vd\|_{s_q} \leq C_q \|d\|_{n_q}, \quad \forall d \in \Theta_F \). Now Theorem 3.1(b) implies that there exists \( \{f_i\}_{i=1}^{\infty} \in (X_F)^{\mathbb{N}} \) such that for every \( q \in \{k_j\}_{j=1}^{\infty} \), \( \{f_i\}_{i=1}^{\infty} \) is a \( \Theta_{n_q}^s \)-Bessel sequence for \( X_{s_q}^* \) and

\[
f = \sum_{i=1}^{\infty} g_i(f) f_j \quad \text{in } \|\|_{s_q}-\text{norm}, \quad f \in X_{\tilde{s}_n}.
\]

For \( j \in \mathbb{N}_0 \), take \( \tilde{A}_j = A_{k_j}, \tilde{B}_j = B_{n_{k_j}}, w_j = s_{k_j}, r_j = n_{k_j}, w_j = \tilde{s}_{n_{k_j}}. \)

Motivated by Theorems 3.1 and 3.2, we give the definition of a general \( F \)-frame:

**Definition 3.3** The sequence \( \{g_i\}_{i=1}^{\infty} \in (X_F^*)^{\mathbb{N}} \) is called a general Fréchet frame (in short, general \( F \)-frame) for \( X_F \) with respect to \( \Theta_F \) if \( \{g_i\}_{i=1}^{\infty} \) is a general \( F \)-frame for \( X_F \) with respect to \( \Theta_F \) and there exists a continuous operator

\[
V : \Theta_F \to X_F
\]

so that \( V(g_i(f))_{i=1}^{\infty} = f \) for all \( f \in X_F \).

We end the section with an example of a general \( F \)-frame for \( X_F \) with respect to \( \Theta_F \) which is not a strict \( F \)-frame for \( X_F \) with respect to \( \Theta_F \).

**Example 3.4** Let \( A \) be a self-adjoint differential operator (for example, one dimensional normalized harmonic oscillator \(( -d^2/dx^2 + 1)/2 \)) with eigenvalues \( \lambda_j = j, j \in \mathbb{N} \), and eigenfunctions \( \psi_j, j \in \mathbb{N} \) (Hermite functions) which make an orthonormal basis of \( X_0 = L^2(\mathbb{R}) \). For \( s \in \mathbb{N} \), let \( X_s \) be the Hilbert space consisting of \( L^2 \)-functions \( \phi = \sum_{j=1}^{\infty} a_j \psi_j, a_j \in \mathbb{C}, j \in \mathbb{N} \), with the property \( \sum_{j=1}^{\infty} |a_j|^2 j^{2s} < \infty \) and with the inner product

\[
\langle \phi_1, \phi_2 \rangle_s = \sum_{j=1}^{\infty} a_{1,j} \overline{a_{2,j}} j^{2s}.
\]

Then \( X_F \) is the Fréchet space \( \mathcal{S}(\mathbb{R}) \), the Schwartz class of rapidly decreasing functions, and its dual is \( X_F^* = \mathcal{S}'(\mathbb{R}) \), the space of tempered distributions. For the sequence spaces \( \Theta_s, s \in \mathbb{N}_0 \), we take

\[
\{d_j\}_{j=1}^{\infty} \in \Theta_s \text{ if and only if } \sum_{j=1}^{\infty} |d_j|^2 j^{2s} < \infty,
\]

with the usual inner product; \( \Theta_F \) is the space of rapidly decreasing sequences.

Note, the space \( \Theta_F \) defined above is actually the space of the type

\[
\Lambda_\infty(\alpha) = \left\{ \{x_j\}_{j=1}^{\infty} : \|\|\{x_j\}_{j=1}^{\infty}\|_s := \left( \sum_{j=1}^{\infty} |x_j|^2 e^{2\alpha s j} \right)^{1/2} < \infty, \ \forall s \in \mathbb{N} \right\},
\]
with \( \alpha_j = \log j, \ j \in \mathbb{N} \). For more information about the spaces \( \Lambda_\infty(\alpha) \) we refer to Sect. 29.

Let \( r \in \mathbb{N} \) be given and let \( \{b_j\}_{j=1}^{\infty} \) be a sequence of complex numbers such that

\[
|b_j| = \begin{cases} 
1, & j = 1, 3, 5, \ldots; \\
\frac{1}{r}, & j = 2, 4, 6, \ldots.
\end{cases}
\]

Let \( g_j = b_j \psi_j, \ j \in \mathbb{N} \), and \( \phi = \sum_{j=1}^{\infty} a_j \psi_j \in X_F \). Then \( \{g_j(\phi)\}_{j=1}^{\infty} \in \Theta_F \) and

\[
|||\phi||| \leq |||\{g_j(\phi)\}_{j=1}^{\infty}||| = \sqrt{\sum_{j=1}^{\infty} |a_j b_j|^2 j^{2s}} \leq |||\phi|||_{s+r}, \ s \in \mathbb{N}_0.
\]

Thus, \( \{g_j\}_{j=1}^{\infty} \) is a general pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \). Let the operator \( U \) be given by \( \mathbb{S} \). Observe that \( R(U) = \Theta_F \). Therefore, \( \{g_j\}_{j=1}^{\infty} \) is a general \( F \)-frame for \( X_F \) with respect to \( \Theta_F \).

Furthermore, we will show that \( \{g_j\}_{j=1}^{\infty} \) is not a strict pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \). Conversely, assume that there exist constants \( A_\infty \in (0, \infty) \), \( B_s \in (0, \infty) \), \( s \in \mathbb{N}_0 \), and a sequence \( \{n_s\}_{s=0}^{\infty} \), satisfying

\[
A_s |||\phi|||_{n_s} \leq |||\{g_j(\phi)\}_{j=1}^{\infty}||| \leq B_s |||\phi|||_{n_s}, \forall \phi \in X_F. \tag{14}
\]

Fix an arbitrary \( s \in \mathbb{N}_0 \).

If \( n_s \leq s \), then (14) applied to \( \psi_j, j \in \mathbb{N} \), implies that \( |b_j| \leq B_s j^{n_s-s} \leq B_s \) for all \( j \in \mathbb{N} \), which leads to a contradiction.

If \( n_s > s \), then (14) applied to \( \psi_j, j \) - odd, implies that \( A_s \leq j^{s-n_s} \) for all odd \( j \), which leads to a contradiction.

Therefore, (14) can not hold.

Note that if the sequence \( \{b_j\}_{j=1}^{\infty} \) is defined by \( |b_j| = j^r, j \in \mathbb{N} \), then \( |||\{g_j(\phi)\}_{j=1}^{\infty}||| = |||\phi|||_{s+r}, s \in \mathbb{N}_0, \phi \in X_F \), and the sequence \( \{g_j\}_{j=1}^{\infty} \) is a strict pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \).

4 On the existence of a continuous projection from \( \Theta_F \) onto \( R(U) \)

Let \( \{g_i\}_{i=1}^{\infty} \) be a general pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \). As it is shown in Section 3 the existence of a continuous operator \( V : \Theta_F \to X_F \) such that \( V \{g_i(f)\}_{i=1}^{\infty} = f \) for all \( f \in X_F \) is important for series expansions in \( X_F \) and in some of the generating Banach spaces (see Theorems 3.1 and 3.2). Here we consider equivalences of this condition. Thus, we give necessary and sufficient conditions for a general pre-\( F \)-frame to be a general Fréchet frame.

Theorem 4.1 Let \( \{g_i\}_{i=1}^{\infty} \) be a general pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \) and let \( \Theta_s, s \in \mathbb{N}_0 \), be \( CB \)-spaces. Then the following statements are equivalent.

\( \Box \)
(i) There exists a continuous operator \( V : \Theta_F \to X_F \) so that \( V \{ g_i(f) \}_{i=1}^{\infty} = f \) for all \( f \in X_F \).

(ii) There exists \( \{ f_i \}_{i=1}^{\infty} \in (X_F)^N \) such that \( \sum_{i=1}^{\infty} c_i f_i \) converges in \( X_F \) for every \( \{ c_i \}_{i=1}^{\infty} \in \Theta_F \) and \( \{ g_i \}_{i=1}^{\infty} \) holds.

(iii) There exist \( \{ f_i \}_{i=1}^{\infty} \in (X_F)^N \) and sequences \( \{ w_j \}_{j \in \mathbb{N}_0}, \{ r_j \}_{j \in \mathbb{N}_0} \), such that \( \{ f_i \}_{i=1}^{\infty} \) is a \( \Theta^* \)-Bessel sequence for \( X_{w_j} \) for every \( j \in \mathbb{N}_0 \) and \( \{ g_i \}_{i=1}^{\infty} \) holds.

**Proof.** The proof is similar to the one of [1] Prop. 3.4, extending it to the Fréchet case.

(i) \( \Rightarrow \) (ii): For \( i \in \mathbb{N} \), define \( f_i = V e_i \). For every \( \{ c_i \}_{i=1}^{\infty} \in \Theta_F \), \( V(\sum_{i=1}^{\infty} c_i e_i) \to V(\{ c_i \}_{i=1}^{\infty}) \) in \( X_F \) as \( n \to \infty \), which implies that \( \sum_{i=1}^{\infty} c_i f_i \) is convergent in \( X_F \). Furthermore, for every \( f \in X_F \), \( f = V \{ g_i(f) \}_{i=1}^{\infty} = \sum_{i=1}^{\infty} g_i(f) f_i \).

(ii) \( \Rightarrow \) (i): Assume that (ii) holds and consider the operator \( V : \Theta_F \to X_F \) defined by \( V(\{ c_i \}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i \), \( \{ c_i \}_{i=1}^{\infty} \in \Theta_F \). Fix an arbitrary \( N \in \mathbb{N} \) and consider the operator \( V_N : \Theta_F \to X_F \) defined by \( V_N(\{ c_i \}_{i=1}^{\infty}) = \sum_{i=1}^{N} c_i f_i \), \( \{ c_i \}_{i=1}^{\infty} \in \Theta_F \). For every \( k \in \mathbb{N} \), denote the \( i \)-th coordinate functional on \( \Theta_k \) by \( E^k_i \) and observe that for every \( \{ c_i \}_{i=1}^{\infty} \in \Theta_F \) one has

\[
\| V_N(\{ c_i \}_{i=1}^{\infty}) \|_{s_k} = \| \sum_{i=1}^{N} c_i f_i \|_{s_k} \leq \sum_{i=1}^{N} |c_i| \cdot \| f_i \|_{s_k} \leq \sum_{i=1}^{N} \| E^k_i \| \cdot \| \{ c_i \}_{i=1}^{\infty} \|_k \cdot \| f_i \|_{s_k} = \left( \sum_{i=1}^{N} \| E^k_i \| \cdot \| f_i \|_{s_k} \right) \| \{ c_i \}_{i=1}^{\infty} \|_k ,
\]

which implies that \( V_N \) is continuous on \( \Theta_F \). Now the Principle of Uniform Boundedness (see [2] II.1.17) implies that \( V \) is continuous. Furthermore, for every \( f \in X_F \), \( \{ g_i(f) \}_{i=1}^{\infty} = \sum_{i=1}^{\infty} g_i(f) f_i = f \).

(i) \( \Rightarrow \) (iii): By the proofs of Theorems 3.2 and 3.1(a)(b), it follows that the sequence \( f_i = V e_i \), \( i \in \mathbb{N} \), fulfills the required properties.

(iii) \( \Rightarrow \) (ii): Let \( \{ d_i \}_{i=1}^{\infty} \in \Theta_F \). Similar to the proof of (A2 \( \Rightarrow \) A1) in Theorem 3.1 the series \( \sum_{i=1}^{\infty} d_i f_i \) converges in \( X_{w_j} \) for every \( j \in \mathbb{N}_0 \), and thus, it converges in \( X_F \).

Recall that Proposition 2.3(ii) contains one more equivalent condition of Theorem 3.1, namely, the existence of a continuous projection of \( \Theta_F \) onto \( R(U) \). In Example 3.1 we constructed a general pre-\( F \)-frame with \( R(U) = \Theta_F \) and thus it was automatically a general \( F \)-frame. In Example 4.2 below we construct a general pre-\( F \)-frame with \( R(U) \not\subseteq \Theta_F \) and show the existence of a continuous projection of \( \Theta_F \) onto \( R(U) \).

**Example 4.2** Let \( \psi_i, i \in \mathbb{N}, \) and \( \Theta_s, s \in \mathbb{N}_0 \), be defined as in Example 3.1. Let \( X_0 \) be the closed linear span of the functions \( \psi_{2k}, k \in \mathbb{N}, \) in \( L^2(\mathbb{R}) \). For
Let \( \phi \) be defined by (8). Clearly, \( \phi \) is a general pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \).

Thus, \( \{g_j\}_{j=1}^\infty \) is a general pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \).

Let the operator \( U \) be given by (3). Clearly, \( R(U) \subseteq \Theta_F \). We will prove that \( R(U) \) is complemented on \( \Theta_F \). Consider the operator \( P \) defined on \( \Theta_F \) by

\[ P(\{d_j\}_{j=1}^\infty) := \{d_2, d_4, d_6, d_8, \ldots\}, \{d_j\}_{j=1}^\infty \in \Theta_F. \]

Fix an arbitrary \( \{d_j\}_{j=1}^\infty \in \Theta_F \). Let

\[ \{a_j\}_{j=1}^\infty := \left\{ \frac{d_2}{4^2}, \frac{d_4}{4^4}, \frac{d_6}{8^2}, \frac{d_8}{8^4}, \ldots \right\} \text{ and } \phi = \sum_{j=1}^\infty a_j \psi_{2j} \text{ in } L^2. \]

Clearly, \( \phi \in X_F \) and \( P(\{d_j\}_{j=1}^\infty) = \{g_j(\phi)\}_{j=1}^\infty \). Hence, \( R(P) \subseteq R(U) \). For every \( \phi \in X_F \), one has \( U \phi \in \Theta_F \) and \( \phi = P(U \phi) \). Therefore, \( R(U) = R(P) \) and \( P(= P^2) \) is a projection of \( \Theta_F \) onto \( R(U) \). Further on, for every \( \{d_j\}_{j=1}^\infty \in \Theta_F \),

\[ \|P(\{d_j\}_{j=1}^\infty)\|_2^2 \leq 2 \sum_{j=1}^\infty |d_{2j}|^2 (2j)^{2s} \leq 2 \|\{d_j\}_{j=1}^\infty\|_2. \]

Thus, \( P \) is continuous and \( \{g_j\}_{j=1}^\infty \) is a general \( F \)-frame for \( X_F \) with respect to \( \Theta_F \).

Furthermore, in a similar way as in Example 5.4 one can show that \( \{g_j\}_{j=1}^\infty \) is not a strict pre-\( F \)-frame for \( X_F \) with respect to \( \Theta_F \).

In Example 4.2, the space \( \Theta_F \) is of the type \( \Lambda_\infty(\alpha) \) and \( R(U) \) is closed in \( \Theta_F \) by Proposition 2.6. In this example we prove that \( R(U) \) is complemented in \( \Theta_F \) by a direct construction of a continuous projection of \( \Theta_F \) onto \( R(U) \).
For another necessary and sufficient condition for \( R(U) \) to be complemented in \( \Lambda_{\infty}(\alpha) \), we refer to [5, Section 30, Exercise 2]:

“A closed subspace \( E \) of \( \Lambda_{\infty}(\alpha) \) is complemented in \( \Lambda_{\infty}(\alpha) \) if and only if \( E \) has the property (\( \Omega \)) and \( \Lambda_{\infty}(\alpha)/E \) has the property (DN).”

For the definitions of the properties (\( \Omega \)) and (DN), see [5] Sect. 29.

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