Fractional Riccati equation to model the dynamics of COVID-19 coronavirus infection

D A Tverdyi\textsuperscript{1,3} and R I Parovik\textsuperscript{1,2}

\textsuperscript{1} Mathematics and Physics Faculty, Vitus Bering Kamchatka State University, Kamchatskiy Kray, Petropavlovsk-Kamchatskiy, 4, Pogranichnaya str., 683032, Russia
\textsuperscript{2} Laboratory of Physical Processes Modeling, Institute of Cosmophysical Research and Radio Wave Propagation, Far Eastern Branch of the Russian Academy of Sciences, Kamchatskiy Kray, Paratunka, 7, Mirnaya str., 684034, Russia
\textsuperscript{3} E-mail: dimsolid95@gmail.com

Abstract. The article proposes a mathematical model based on the fractional Riccati equation to describe the dynamics of COVID-19 coronavirus infection in the Republic of Uzbekistan and the Russian Federation. The model fractional Riccati equation is an equation with variable coefficients and a derivative of a fractional variable order of the Gerasimov-Caputo type. The solution to the model Riccati equation is given using the modified Newton method. The obtained model curves are compared with the experimental data of COVID-19 coronavirus infection in the Republic of Uzbekistan and the Russian Federation. It is shown that with a suitable choice of parameters in the mathematical model, the calculated curves give results close to real experimental data.

1. Introduction

According to the WHO (World Health Organization), the Chinese authorities of the city of Wuhan on January 7, 2020, identified a new virus (2019-nCoV) causing pneumonia of unknown etiology. The virus has been named Severe Acute Respiratory Syndrome Coronavirus-2 (SARS-CoV-2).

On February 11, 2020, by the international committee on the taxonomy of viruses, the study [1] was aimed at developing a mathematical model for calculating the transmissibility of the virus. In this study, the authors developed a model to investigate the potential transmission of infection from a source of infection (probably bats) to a human infection. The transmissibility of SARS-CoV-2 was shown to be higher than MERS (Middle East Respiratory Syndrome) in the Middle East, but lower than MERS in the Republic of Korea.

In article [2], the authors propose graphical modeling of the spread of COVID-19 infection. A conceptual model of the COVID-19 epidemic is proposed, taking into account the many factors affecting the increase in the number of cases of infection, although the study could not take into account all factors. The advantage of this approach is justified by the fact that it allows backward propagation analysis as a result of dynamic recording of detected infections according to the model. This approach allows the identification of undetected cases of infection.

In a study [3], based on the Caputo-Fabrizio fractional derivative, a model was developed for the transmission of COVID-19 in Wuhan, China. For the numerical simulation of the fractional derivative,
the Adomian-Bashforth scheme was used. A comparative analysis of the classical model and the fractional model and experimental data is carried out.

In article [4], the authors studied a model of the coronavirus epidemic with experimental data from India using the Predictor-Corrector scheme. For the proposed Covid-19 model, numerical and graphical modeling is performed within the framework of a new generalized derivative of non-integer order in the sense of Caputo. The applicability of the new generalized version of the non-integer operator of the Caputo type in mathematical epidemiology is shown.

Currently, the further development of mathematical models of the dynamics of the spread of COVID-19 is of wide interest. One of the directions of research is based on the use of the mathematical apparatus of fractional calculus to take into account the memory effects in the model of the dynamics of the coronavirus [3] - [10]. Memory effects characterize the property of a dynamic system to remember its previous states. Memory effects can be described using derived fractional orders, with the orders accounting for the intensity of the dynamic process. For the dynamics of the spread of coronavirus, the memory effect means that the symptoms of the disease in a person may not appear immediately, but after a certain period of time.

The application of fractional calculus to the study of the spread of the coronavirus epidemic is reflected in the works [3] - [10]. Basically, these are theoretical works that consider the existence and uniqueness of the solution to the Cauchy problem for a model equation or a system of equations with various operators of fractional constant order, and also investigate methods for its solution. Unfortunately, most of these works do not compare the simulation results with experimental data on the epidemic situation of COVID-19 in various countries, and the proposed models are quite complex for numerical implementation.

From the review of studies, it can be concluded that the application of fractional calculus to the tasks of epidemiology is reasonable, which in turn determines the relevance of this study.

In this paper, we propose a fairly simple mathematical model based on the Riccati equation with a fractional variable order derivative with variable coefficients. The Riccati equation well describes the processes that obey the logistic law, which is true for the dynamics of COVID-19, and the arbitrary order of the fractional derivative gives a wide range for refining the mathematical model and takes into account the effect of the variable memory of the dynamic system.

2. Mathematical model
As a model equation, consider the Riccati equation with variable coefficients and a fractional derivative of variable order [12] - [14]:

$$\frac{\partial^{\alpha(t)} u(\sigma)}{\partial \sigma} + a(t)u^{2}(t) - b(t)u(t) - c(t) = 0, \quad u(0) = u_{0}, \quad (1)$$

where $u(t) \in C[0,T]$ - monotonically increasing function (number of infected), $t \in [0,T]$ - time, $T$ - model time, $u_{0}$ - given constant (number of infected at the initial moment of time), $0 < a(t), b(t), c(t) < 1$ - continuous functions on a segment $[0,T]$. The model was considered in [11].

The fractional operator $\frac{\partial^{\alpha(t)} u(\sigma)}{\partial \sigma}$ of variable order $0 < \alpha(t) < 1$ of the Gerasimov-Caputo type acting on the function $u(t) \in C[0,T]$ has the form:

$$\frac{\partial^{\alpha(t)} u(\sigma)}{\partial \sigma} = \frac{1}{\Gamma(1-\alpha(t))} \int_{0}^{\sigma} \frac{u(t) d\sigma}{(t-\sigma)^{1-\alpha(t)}}, \quad \Gamma(\cdot) - \text{Euler's gamma function,} \quad \dot{u}(t) = du/dt.$$  \hspace{1cm} (2)

where $\Gamma(\cdot) - \text{Euler's gamma function},$ $\dot{u}(t) = du/dt.$ Some properties of the fractional operator (2) can be found in [15, 16].

The Cauchy problem (1) describes a wide class of dynamic processes with variable memory in saturated media [12], and the model equation in (1) will be called the fractional Riccati equation.
The solution to the Cauchy problem (1) will be sought using numerical methods. Take a segment $[0,T]$ on a uniform grid, with $N$ equal parts - grid nodes with a step $h = T/N$. We assume that the solution function $u(t)$ is smooth enough to construct a finite-difference scheme. Then the functions: $u(t), a(t), b(t), c(t) \quad (t)$ will pass into them discrete analogs: $u_k, a_k, b_k, c_k$, where $k = 1, ..., N$. We write the approximation of the derivative of a fractional variable order of the Gerasimov-Caputo type in equation (1) according to [13]. Then we turn to a discrete analogue of the Cauchy problem (1):

$$\xi_j \sum_{j=1}^{\infty} \omega_k \left(u_{j-\xi_j} - u_{j-1}\right) + a_j u_k ^2 - a_j u_k - a_k = 0, \quad u_0 = \text{const},$$

$$(3)$$

We will seek a numerical solution to system (3) using the modified Newton method (MNM) [14].

3. Simulation results
The purpose of this study is to describe trends: both in new cases of COVID-19 infection and in the total number of infected, in different countries of the world. The data [17] are provided by the Our World in Data project with the support of the Center for Systems Science and Engineering (CSSE) at Johns Hopkins University (JHU).

The data is a multi-parameter measurement in one day increments. JHU updates its data several times every day, and makes it publicly available. This data comes from governments, national and subnational agencies around the world - a complete list of data sources for each country is published on the ourworldindata.org [17] site.

For the input data and the simulation results, the number of infected people was normalized to the maximum, therefore, the experimental data and the data of the simulation results are indicated in relative units. Calculations according to the model (3), the MNM method, processing of the results and visualization of the simulation results are implemented in the environment of symbolic computer mathematics Maple 2021.

3.1. Example 1. New cases of infection in the Republic of Uzbekistan
Let us compare the simulation result with the data on new cases of infection in the Republic of Uzbekistan, from March 15, 2020 to September 16, 2021, with a step of 1 day (Figure 1, solid). We will approximate the input by the model (3), with the parameters: $N = 551, T = 551, u_0 = 0.0001$. Other parameters: $a_k = 0.98 \sin \left(\frac{1.64 \pi}{kh/T}\right)^2, \quad b_k = kh/T^2, \quad c_k = 0.25, \quad e^\frac{7kh}{T}$. Other parameters: $a_k = 0.98 \sin \left(\frac{1.64 \pi}{kh/T}\right)^2, \quad b_k = kh/T^2, \quad c_k = 0.25, \quad e^\frac{7kh}{T}$.

Figure 1. Republic of Uzbekistan. Model curve (solid) with correlation coefficient: 82.5% - for model (3). Model curve (dash) - prediction until December 31, 2021.
Let’s carry out the proposed model (3), prediction (figure 1, dash) for new cases of infection in the Republic of Uzbekistan from September 16 to December 31, 2021. Main parameters: $N = 657, T = 657, u_0 = 0.0001$. Other parameters: $a_1 = 0.98 \sin((1.96 \pi) kh/ T)^2, a_4 = (1.1923) kh/ T$, $b_4 = 0.25, c_4 = 0.5 \sin(e^\frac{(8.78) kh}{T}) kh/ T^2$.

3.2. Example 2. Total cases of infection in the Republic of Uzbekistan

Let us compare the simulation result with the data on the total number of infected people in the Republic of Uzbekistan, from March 15, 2020 to September 16, 2021, with a step of 1 day (figure 2, solid). We will approximate the input by the model (3), with the parameters: $N = 551, T = 551, u_0 = 6.000816111\times10^{-6}$. Other parameters: $a_1 = 0.15, a_4 = 0.4 \sin((2 \pi) kh/ T)^2, b_4 = 0.15, c_4 = 0.25 \sin((0.5 \pi) kh/ T)^2$.

Figure 2. Republic of Uzbekistan. Model curve (solid) with correlation coefficient: 98.3% - for model (3). Model curve (dash) - prediction until December 31, 2021.

Let us carry out the proposed model (3), prediction (figure 2, dash) for new cases of infection in the Republic of Uzbekistan from September 16 to December 31, 2021. Main parameters: $N = 657, T = 657, u_0 = 0.0001$. Other parameters: $a_1 = 0.98 \sin((1.96 \pi) kh/ T)^2, a_4 = (1.1923) kh/ T$, $b_4 = 0.25, c_4 = 0.5 \sin(e^\frac{(8.78) kh}{T}) kh/ T^2$.

3.3. Example 3. New cases of infection in the Russian Federation

Let us compare the simulation result with data on new infections in the Russian Federation, from March 15, 2020 to September 16, 2021, with a step of 1 day (figure 3, solid). We will approximate the input by the model (3), with the parameters: $N = 595, T = 595, u_0 = 0.00006$. Other parameters: $a_1 = 0.15, a_4 = 0.4 \sin((2.385 \pi) kh/ T)^2, b_4 = 0.15, c_4 = 0.25 \sin((0.595 \pi) kh/ T)^2$.

Let us carry out the proposed model (3), prediction (figure 3, dash) for new cases of infection in the Russian Federation from September 16 to December 31, 2021. Main parameters: $N = 701, T = 701, u_0 = 0.00006$. Other parameters: $a_1 = 0.98 \sin((2.7 \pi) kh/ T)^2, a_4 = 0.8 \sin((2 \pi) kh/ T)^2 - 0.1, b_4 = 0.2, c_4 = 0.5 \sin((1.82 \pi) kh/ T)^2 + 0.25$.

Let us carry out the proposed model (3), prediction (figure 3, dash) for new cases of infection in the Russian Federation from September 16 to December 31, 2021. Main parameters: $N = 701, T = 701, u_0 = 0.00006$. Other parameters: $a_1 = 0.98 \sin((3.181 \pi) kh/ T)^2, a_4 = 0.8 \sin((2.356 \pi) kh/ T)^2 - 0.1, b_4 = 0.2, c_4 = 0.5 \sin((2.14 \pi) kh/ T)^2 + 0.25$. 
3.4. Example 4. Total cases of infection in the Russian federation
Let’s compare the simulation result with the data on the total number of infected people in the Russian Federation, from January 31, 2020 to September 16, 2021, with a step of 1 day. We will approximate the input by the model (3), with the parameters: \( N = 595, T = 595, u_0 = 2.812680012 \times 10^{-7} \). Other parameters: \( \alpha = 0.2, a_i = 0.2\sin((3\pi)kh/T)^2, b_i = 0.05, c_i = 0.25\sin\left((0.33\pi)kh/T\right)^2 \).

![Figure 3](image1.png)

**Figure 3.** Russian Federation. Model curve (solid) with correlation coefficient: 76.2\% for model (3). Model curve (dash) - prediction until December 31, 2021.

![Figure 4](image2.png)

**Figure 4.** Russian Federation. Model curve (solid) with correlation coefficient: 99.4\% for model (3). Model curve (dash) - prediction until December 31, 2021.

Let us carry out the proposed model (3), prediction (figure 3, dash) for total number of infected in the Russian Federation from September 16 to December 31, 2021. Main parameters: \( N = 701, T = 701, u_0 = 2.812680012 \times 10^{-7} \). Other parameters: \( \alpha = 0.2, a_i = 0.2\sin((3.534\pi)kh/T)^2, b_i = 0.05, c_i = 0.25\sin\left((0.3887\pi)kh/T\right)^2 \).

4. Conclusion
As seen from figure 1 – figure 4 mathematical model (3) in the selection of appropriate simulation parameters: and is capable of giving results close to real data. This indicates the potential for using fractional equations to describe processes of this type.

Further continuation of the work may consist in refining the parameters of the model as a result of solving the corresponding inverse problem, as well as in their further understanding.
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