THE CONCAVITY OF THE GAUSSIAN CURVATURE OF THE
CONVEX LEVEL SETS OF MINIMAL SURFACE WITH RESPECT TO
THE HEIGHT

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ABSTRACT. For the minimal graph with strict convex level sets, we find an auxiliary
function to study the Gaussian curvature of the level sets. We prove that this curvature
function is a concave function with respect to the height of the minimal surface while
this auxiliary function is almost sharp when the minimal surface is the Catenoid.

1. Introduction

In this paper, for minimal graph with strictly convex level sets, we shall explore the
relation of its Gaussian curvature of the level sets and the height of the function.

The convexity of the level sets of the solutions of elliptic partial differential equations
has been studied for a long time. For instance Alfhors [1] contains the well-known result
that level curves of Green function on simply connected convex domain in the plane are
the convex Jordan curves. In 1956, Shiffman [26] studied the minimal annulus in $\mathbb{R}^3$
whose boundary consists of two closed convex curves in parallel planes $P_1, P_2,$ he proved that
the intersection of the surface with any parallel plane $P,$ between $P_1$ and $P_2,$ is a convex
Jordan curve. In 1957, Gabriel [10] proved that the level sets of the Green function on a
3-dimensional bounded convex domain are strictly convex, Lewis [15] extended Gabriel’s
result to $p$-harmonic functions in higher dimensions. Makar-Limanov [22] and Brascamp-
Lieb [4] got the beautiful results on the Poisson equation and first eigenfunction with
Dirichlet boundary value problem on bounded convex domain. Caffarelli-Spruck [7]
gen-eralized the Lewis [15] results to a class of semilinear elliptic partial differential equations.
Motivated by the result of Caffarelli-Friedman [5], Korevaar [14] gave a new proof on the
results of Gabriel and Lewis ([10], [15]) using the deformation process and the constant
rank theorem of the second fundamental form of the convex level sets of $p$-harmonic func-
tion. A survey of this subject is given by Kawohl [13]. For more recent related extensions,
please see the papers by Bianchini-Longinetti-Salani [3] and Bian-Guan-Ma-Xu [2].

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foundation of QFNU.
Now we turn to the curvature estimates of the level sets of the solutions of elliptic partial differential equations. For 2-dimensional harmonic function and minimal surfaces with convex level curves, Ortel-Schneider [23], Longinetti [16] and [17] proved that the curvature of the level curves attains its minimum on the boundary (see Talenti [27] for related results). Longinetti also studied the precisely relation between the curvature of the convex level curves and the height of minimal graph in [17]. Jost-Ma-Ou [12] and Ma-Ye-Ye [20] proved that the Gaussian curvature and the principal curvature of the convex level sets of three dimensional harmonic functions attains its minimum on the boundary. Recently, Ma-Ou-Zhang [19] got the Gaussian curvature estimates of the convex level sets on high dimension harmonic function with the Gaussian curvature of the boundary and the norm of the gradient on the boundary, Ma-Zhang [21] also found the concavity of Gaussian curvature of the level sets of harmonic functions with respect to the height. Also recently, Wang-Zhang [28] got the Gaussian curvature estimates of the convex level sets of minimal surface, Poisson equations and a class of semilinear elliptic partial differential equations which have been studied by Caffarelli-Spruck [7].

In this paper, utilizing the support function of the strictly convex level sets and the maximum principle, we obtain the concavity of the Gaussian curvature of the convex level sets of minimal graph with respect to the height of the minimal graph.

Now we state our main theorems.

**Theorem 1.1.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2$ and $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$, $t_0 \leq u(x) \leq t_1$ be a p-harmonic function in $\Omega$, i.e.

\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in} \quad \Omega.
\]

Assume $|\nabla u| \neq 0$ in $\overline{\Omega}$. Let

$\Gamma_t = \{x \in \Omega | u(x) = t\}$ for $t_0 < t < t_1$,

and $K$ be the Gaussian curvature of the level sets. For

$\displaystyle f(t) = \min\{[(\frac{|\nabla u|^2}{1 + |\nabla u|^2})^{\frac{n-3}{2}}K]^{\frac{1}{n-1}}(x) | x \in \Gamma_t\},$

If the level sets of $u$ are strictly convex with respect to normal $\nabla u$, we have the following differential inequality:

$D^2f(t) \leq 0, \quad \text{in} \quad (t_0, t_1).$

Under the same assumption in Theorem 1.1, Wang-Zhang [28] proved the following statement: for $n \geq 2$, the function $(\frac{|\nabla u|^2}{1 + |\nabla u|^2})^\theta K$ attains its minimum on the boundary, where $\theta = -\frac{1}{2}$ or $\theta \geq \frac{n-3}{2}$, from this fact they got the lower bound estimates for the Gaussian curvature of the level sets.
Now we give a corollary.

**Corollary 1.1.** Let $u$ satisfy

\[
\begin{cases} 
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\
u = 0 & \text{on } \partial \Omega_0, \\
u = 1 & \text{on } \partial \Omega_1,
\end{cases}
\]

where $\Omega_0$ and $\Omega_1$ are bounded smooth convex domains in $\mathbb{R}^n$, $n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$. Assume $|\nabla u| \neq 0$ in $\Omega$ and the level sets of $u$ are strictly convex with respect to normal $\nabla u$. Let $K$ be the Gaussian curvature of the level sets. For any point $x \in \Gamma_t$, $0 < t < 1$, we have the following estimates.

**Case 1:** for $n = 3$, we have

\[
K(x) \geq (1 - t) \left( \min_{\partial \Omega_0} K \right) + t \left( \min_{\partial \Omega_1} K \right).
\]

**Case 2:** for $n \neq 3$, we have

\[
\left( \frac{|\nabla u|^2}{1 + |\nabla u|^2} \right)^{\frac{n-3}{2}} K \geq (1 - t) \min_{\partial \Omega_0} \left( \left( \frac{|\nabla u|^2}{1 + |\nabla u|^2} \right)^{\frac{n-3}{2}} K \right) + t \min_{\partial \Omega_1} \left( \left( \frac{|\nabla u|^2}{1 + |\nabla u|^2} \right)^{\frac{n-3}{2}} K \right).
\]

Now we give an example to show our estimates are almost sharp in some sense.

**Remark 1.2.** Let $u(r, \theta)(r > 2)$ be the $n$–dimensional catenoid, i.e.

\[
u(r, \theta) = \int_{-2}^{-r} \frac{1}{\sqrt{r^2(s-1)}} \, ds.
\]

Then

\[
|\nabla u| = \frac{1}{\sqrt{r^2(s-1) - 1}},
\]

and the Gaussian curvature of the level set at $x$ is

\[
K(x) = r^{1-n}.
\]

Hence,

\[
\varphi = \left( \frac{|\nabla u|^2}{1 + |\nabla u|^2} \right)^{\frac{n-3}{2}} K \right)^{\frac{1}{n-1}} = r^{2-n}.
\]

For $n = 2$, it is easily to see that our estimate is sharp. Now we turn to the case $n > 2$. Denote by

\[
R = \int_{-\infty}^{-2} \frac{1}{\sqrt{s^2(n-1) - 1}} \, ds.
\]

Then we have
\begin{equation}
-u + R = \int_{-\infty}^{-r} \frac{1}{s^{n-1}} ds + \int_{-\infty}^{-r} \frac{1}{s^{n-1}} \left(1 - \frac{1}{\sqrt{1 - s^{-2(n-1)}}}\right) ds
\end{equation}

\begin{equation}
= \frac{(-1)^n}{2 - n} r^{2-n} + \mathcal{O}(r^{4-3n}).
\end{equation}

It means that
\begin{equation}
\varphi = (-1)^n (2 - n)(R - t) + \mathcal{O}(r^{4-3n})
\end{equation}
which shows the “almost sharpness” of our estimate in the higher dimensional cases.

To prove these theorems, let $K$ be the Gaussian curvature of the convex level sets, and let $\varphi = \log K(x) + \rho(|\nabla u|^2)$. For suitable choice of $\rho$ and $\beta$ we shall show the following elliptic differential inequality:
\begin{equation}
L(e^{\beta \varphi}) \leq 0 \mod \nabla \theta \varphi \text{ in } \Omega,
\end{equation}
where $L$ is the elliptic operator associated with the equation we discussed and here we have suppressed the terms involving $\nabla \theta \varphi$ (See the notations below) with locally bounded coefficients, then we apply the strong minimum principle to obtain the main results.

In Section 2, we first give brief definitions on the support function of the level sets, then obtain the equation of minimal graph in terms of support function. We prove the Theorem 1.1 in Section 3 by the formal calculations. The main technique in the proof of these theorems consists of rearranging the second and third derivative terms using the equation and the first derivative condition for $\varphi$. The key idea is the Pogorelov’s method in a priori estimates for fully nonlinear elliptic equations.

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2. Notations and preliminaries

Firstly, we start by introducing some basic concepts and notations. Let $\Omega_0$ and $\Omega_1$ be two bounded smooth open convex subsets of $\mathbb{R}^n$ such that $\tilde{\Omega}_1 \subset \Omega_0$ and let $\Omega = \Omega_0 \setminus \tilde{\Omega}_1$. Let $u : \tilde{\Omega} \to \mathbb{R}$ be a smooth function with $|Du| > 0$ in $\Omega$ and its level sets are strictly convex with respect to the normal direction $Du$.

For simplicity reasons, we will assume that
\[
u = 0 \text{ on } \partial \Omega_0, \quad u = 1 \text{ on } \partial \Omega_1.
\]
For $0 \leq t \leq 1$, we set
\[
\tilde{\Omega}_t = \{ x \in \tilde{\Omega}_0 : u \geq t \};
\]
note that, throughout this paper, we systematically extend \( u = 1 \) in \( \Omega_1 \). Then every \( x \in \Omega \) belongs to the boundary of \( \Omega_{u(x)} \).

Under these assumptions, it is then possible to define a function
\[
H : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}
\]
as follows: for each \( t \in [0, 1] \), \( H(\cdot, t) \) is the support function of the convex body \( \Omega_t \), i.e.
\[
H(X, t) = H_{\Omega_t} (X) \quad \forall X \in \mathbb{R}^n, \ t \in [0, 1].
\]
We call \( H \) the support function of \( u \). For more details, see [18].

The rest of this section is devoted to derive the minimal graph by means of support function. Before doing this, we should reformulate first and second derivatives of \( u \) in support function \( h \) which is the restriction of \( H(\cdot, t) \) to the unit sphere \( S^{n-1} \), see [8, 18]. But for convenience of the reader, we report the main steps here.

Recall that \( h \) is the restriction of \( H \) to \( S^{n-1} \times [0, 1] \), so \( h(\theta, t) = H(Y(\theta), t) = h_{\Omega_t}(Y(\theta)), \ t \in [0, 1] \). Since the level sets of \( u \) are strictly convex and it is well defined the map \( x(\theta, t) = x_{\Omega_t}(Y(\theta)) \),

which to every \( (X, t) \in \mathbb{R}^n \setminus \{0\} \times (0, 1) \) assigns the unique point \( x \in \Omega \) on the level surface \( \{u = t\} \) where the gradient of \( u \) is parallel to \( X \) (and orientation reversed).

Let
\[
T_i = \frac{\partial Y}{\partial \theta_i},
\]
so that \( \{T_1, \cdots, T_{n-1}\} \) is a tangent frame field on \( S^{n-1} \), and let
\[
x(\theta, t) = x_{\Omega_t}(Y(\theta));
\]
we denote its inverse map by
\[
\nu : (x_1, \cdots, x_n) \rightarrow (\theta_1, \cdots, \theta_{n-1}, t).
\]
Notice that all these maps (\( h, x \) and \( \nu \)) depend on the considered function \( u \) (like \( H \)), even if we do not adopt any explicit notation to stress this fact.

For \( h(\theta, t) = \langle x(\theta, t), Y(\theta) \rangle \), since \( Y \) is orthogonal to \( \partial \Omega_t \) at \( x(\theta, t) \), deriving the previous equation we obtain
\[
h_i = \langle x, T_i \rangle.
\]
In order to simplify some computations, we can also assume that \( \theta_1, \cdots, \theta_{n-1}, Y \) is an orthonormal frame positively oriented. Hence, from the previous two equalities, we have
\[
x = hY + \sum_i h_i T_i,
\]
and

\[ \frac{\partial T_i}{\partial \theta_j} = -\delta_{ij} Y \quad \text{at } x, \]

where the summation index runs from 1 to \( n - 1 \) if no extra explanation, and \( \delta_{ij} \) is the standard Kronecker symbol. Following [8], we obtain at the considered point \( x \),

\[ \frac{\partial x}{\partial t} = h_t Y + \sum_i h_t T_i; \]

\[ \frac{\partial x}{\partial \theta_j} = h T_j + \sum_i h_{ij} T_i, \quad j = 1, \cdots, n - 1. \]

The inverse of the above Jacobian matrix is

\[ \frac{\partial t}{\partial x_\alpha} = h_t^{-1}[Y]_\alpha, \quad \alpha = 1, \cdots, n; \]

\[ \frac{\partial \theta_i}{\partial x_\alpha} = \sum_j b^{ij} [T_j - h_t^{-1} h_{ij} Y]_\alpha, \quad \alpha = 1, \cdots, n, \]

where \([i]_i\) denotes the \( i \)-coordinate of the vector in the bracket and \( b^{ij} \) denotes the inverse tensor of the second fundamental form

\[ b_{ij} = \langle \frac{\partial x}{\partial \theta_i}, \frac{\partial Y}{\partial \theta_j} \rangle = h_t \delta_{ij} + h_{ij} \]

of the level surface \( \partial \Omega_t \) at \( x(\theta, t) \). The eigenvalue of the tensor \( b^{ij} \) are the principal curvatures \( \kappa_1, \cdots, \kappa_{n-1} \) of \( \partial \Omega_t \) at \( x(\theta, t) \) (see [25]).

The first equation of (2.1) can be rewritten as

\[ Du = \frac{Y}{h_t}, \]

where the left hand side is computed at \( x(\theta, t) \) while the right hand side is computed at \( (\theta, t) \), it follows that

\[ |Du| = -\frac{1}{h_t}. \]

By chain rule and (2.1), the second derivatives of \( u \) in terms of \( h \) can be computed as

\[ u_{\alpha\beta} = \sum_{i,j} [ -h_t^{-2} h_{ti} Y + h_t^{-1} T_i ]_\alpha b^{ij} [T_j - h_t^{-1} h_{ij} Y]_\beta - h_t^{-3} h_t [Y]_\alpha [Y]_\beta, \]

for \( \alpha, \beta = 1, \cdots, n \).

So the minimal graph equation, \( \text{div}(\sqrt{1 + |\nabla u|^2}) = 0 \), reads under this new coordinates

\[ h_{tt} = \sum_{i,j} [(1 + h_t^2) \delta_{ij} + h_{ti} h_{tj}] b^{ij} \]
and the associated linear elliptic operator is

\[
L = \sum_{i,j,p,q} \left[ (1 + h_t^2)\delta_{pq} + h_{tp}h_{tq} \right] b^{jp} b^{jq} \frac{\partial^2}{\partial \theta_i \partial \theta_j} - 2 \sum_{i,j} h_{ij} b^{ij} \frac{\partial^2}{\partial \theta_i \partial t} + \frac{\partial^2}{\partial t^2}.
\]

(2.5)

Now we recall the well-known commutation formulas for the covariant derivatives of a smooth function \(u \in C^4(S^n)\).

\[
u_{ij} - u_{ikj} = -u_k\delta_{ij} + u_j\delta_{ik},
\]

(2.6)

\[
u_{ijkl} - u_{ijlk} = u_{ik}\delta_{jl} - u_{il}\delta_{jk} + u_{kj}\delta_{il} - u_{lj}\delta_{ik},
\]

(2.7)

they will be used during the calculations in the next section. By the definition of \(b_{ij}\) and the above commutation formulas, we can easily get the following Codazzi’s type formula

\[
b_{ij,k} = b_{ik,j}.
\]

(2.8)

3. Gauss curvature of the level sets of minimal graph

We first state the following lemma from [17], then we prove Theorem 1.1.

For a continuous function \(f(t)\) on \([0, 1]\) we define its generalized second order derivative at any point \(t\) in \((0, 1)\) as

\[
D^2 f(t) = \limsup_{h \to 0} \frac{f(t + h) + f(t - h) - 2f(t)}{h^2}.
\]

Let \(B\) be the quotient set \(B \equiv \mathbb{R}^n/2\pi\mathbb{Z}^n\) and let \(Q \equiv B \times (0, 1)\). Let \(G(\theta, t)\) be a regular function in \(Q\) such that

\[
\mathcal{L}(G(\theta, t)) \geq 0 \quad \text{for } (\theta, t) \in Q,
\]

where \(\mathcal{L}\) is an elliptic operator of the form

\[
\mathcal{L} = \sum_{i,j} a^{ij} \frac{\partial^2}{\partial \theta_i \partial \theta_j} + \sum_i b^i \frac{\partial^2}{\partial \theta_i \partial t} + \frac{\partial^2}{\partial t^2} + \sum_i c^i \frac{\partial}{\partial \theta_i}
\]

with regular coefficients \(a^{ij}, b^i, c^i\).

**lemma 3.1.** (See [17]) Set

\[
\phi(t) = \max\{G(\theta, t) | \theta \in B\}.
\]

Then \(\phi(t)\) satisfies the following differential equality:

\[
D^2 \phi(t) \geq 0.
\]

Moreover, \(\phi(t)\) is a convex function with respect to \(t\).
Since the level sets of $u$ are strictly convex with respect to the normal $Du$, the matrix of second fundamental form $(b_{ij})$ is positive definite in $\Omega$. Set

$$\varphi = \rho(h_t^2) - \log K(x),$$

where $K = \det(b_{ij})$ is the Gaussian curvature of the level sets and $\rho(t)$ is a smooth function defined on $(0, +\infty)$. For suitable choice of $\rho$ and $\beta$, we will derive the following differential inequality

$$L(e^{\beta\varphi}) \leq 0 \mod \nabla_\theta \varphi \quad \text{in } \Omega,$$

where the elliptic operator $L$ is given in (2.5) and we have modified the terms involving $\nabla_\theta \varphi$ with locally bounded coefficients. Then by applying a maximum principle argument in Lemma 3.1, we can obtain the desired result.

In order to prove (3.1) at an arbitrary point $x_o \in \Omega$, we may assume the matrix $(b_{ij}(x_o))$ is diagonal by rotating the coordinate system suitably. From now on, all the calculation will be done at the fixed point $x_o$. In the following, we shall prove the theorem in three steps.

**Step1:** we first compute $L(\varphi)$.

Taking first derivative of $\varphi$, we get

$$\frac{\partial \varphi}{\partial \theta_j} = 2\rho'h_i h_{ij} + \sum_{k,l} b_{kl} b_{kl,j},$$

(3.2)

$$\frac{\partial \varphi}{\partial t} = 2\rho'h_t h_{tt} + \sum_{k,l} b_{kl} b_{kl,t}.$$  

(3.3)

Taking derivative of equation (3.2) and (3.3) once more, we have

$$\frac{\partial^2 \varphi}{\partial \theta_i \partial \theta_j} = (2\rho' + 4\rho'' h_t^2) h_{ii} h_{ij} + 2\rho'h_i h_{jj} - \sum_{k,l,r,s} b_{kr} b_{rs,i} b_{sl} b_{kl,j} + \sum_{k,l} b_{kl} b_{kl,ij},$$

$$\frac{\partial^2 \varphi}{\partial \theta_i \partial t} = (2\rho' + 4\rho'' h_t^2) h_{ii} h_{tt} + 2\rho'h_i h_{tt} - \sum_{k,l,r,s} b_{kr} b_{rs,i} b_{sl} b_{kl,t} + \sum_{k,l} b_{kl} b_{kl,ti},$$

$$\frac{\partial^2 \varphi}{\partial t^2} = (2\rho' + 4\rho'' h_t^2) h_{tt}^2 + 2\rho'h_t h_{tt} - \sum_{k,l,r,s} b_{kr} b_{rs,i} b_{sl} b_{kl,t} + \sum_{k,l} b_{kl} b_{kl,tt}.$$
So

\[ L(\varphi) = (2\rho' + 4\rho''h^2_t) \left[ \sum_{i,j} [(1 + h^2_t)\delta_{ij} + h_t h_j]b^{ij}b^{ij}\sigma_t - 2\sum_i h_i^2b^{ii}\sigma_t + h_t^2 \right] \]

\[ + 2\rho' h_t \left[ \sum_{i,j} [(1 + h^2_t)\delta_{ij} + h_t h_j]b^{ij}b^{ij}h_{ij} - 2\sum_i h_t b^{ii}h_{tii} + h_{tt} \right] \]

\[ - \sum_{k,l} b^{kk}b^{ll} \left[ \sum_{i,j} [(1 + h^2_t)\delta_{ij} + h_t h_j]b^{ij}b^{ij}b_{kl,i}b_{kl,j} - 2\sum_i h_t b^{ii}b_{kl,i}b_{kl,t} + b_{kl,t}^2 \right] \]

\[ + \sum_k b^{kk}L(b_{kk}) \]

\[ = I_1 + I_2 + I_3 + I_4. \]

In the rest of this section, we will deal with the four terms above respectively. For the term \( I_1 \), by recalling our equation, i.e.

\[ (3.5) \quad h_{tt} = \sum_{i,j} [(1 + h^2_t)\delta_{ij} + h_t h_j]b^{ij}, \]

we have by recalling that \((b^{ij})\) is diagonal at \( x_0 \)

\[ (3.6) \quad I_1 = (2\rho' + 4\rho''h^2_t) \left[ \sum_{i,j} [(1 + h^2_t)\delta_{ij} + h_t h_j]b^{ij}b^{ij}\sigma_t - 2\sum_i h_i^2b^{ii}\sigma_t + h_t^2 \right] \]

\[ = (2\rho' + 4\rho''h^2_t) \left[ (1 + h^2_t)\sum_i (h_t b^{ii})^2 + \left( \sum_i h_i^2b^{ii} - h_{tt} \right)^2 \right] \]

\[ = (2\rho' + 4\rho''h^2_t)(1 + h^2_t)\sum_i (h_t b^{ii})^2 + (2\rho' + 4\rho''h^2_t)(1 + h^2_t)^2\sigma_1^2, \]

where \( \sigma_1 = \sum_i b^{ii} \) is the mean curvature.

Now we treat the term \( I_2 \). Differentiating \((3.5)\) with respect to \( t \), we have

\[ (3.7) \quad h_{ttt} = 2h_t h_{ttt}\sigma_t + 2\sum_{i,j} h_{ttt} h_{ij} b^{ij} - \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_t h_j]b^{ij}b^{ij}b_{ij,t}. \]

By inserting \((3.7)\) into \( I_2 \), we can get

\[ I_2 = 2\rho' h_t \left[ \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_t h_j]b^{ij}b^{ij}h_{ij} - 2\sum_i h_t b^{ii}h_{tii} + h_{ttt} \right] \]

\[ = 2\rho' h_t \left[ \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_t h_j]b^{ij}b^{ij}(h_{ij} - b_{ij,t}) + 2h_t h_{ttt}\sigma_1 \right]. \]
Recalling the definition of the second fundamental form, i.e. (2.2), together with the equation (3.5), we obtain

$$I_2 = 2\rho h_t \left[ \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_{ti}h_{tj}]b^{ij2}(-h_t\delta_{ij}) + 2h_t h_{tt}\sigma_1] \right]$$

(3.8)

$$I_2 = -2\rho h_t^2 (1 + h_t^2) \sum_i (b^{ii})^2 - 2\rho h_t^2 \sum_i (h_{ti}b^{ii})^2 + 4\rho h_t^2 (1 + h_t^2)\sigma_1^2$$

$$+ 4\rho h_t^2 \sigma_1 \sum_i h_{ti}^2 b^{ii}$$

Combining (3.6) and (3.8),

$$I_1 + I_2 = 4\rho h_t^2 \sigma_1 \sum_i h_{ti}^2 b^{ii} + \left[ 4\rho h_t^2 (1 + h_t^2) + (2\rho' + 4\rho'' h_t^2) (1 + h_t^2) \right] \sigma_1^2$$

$$+ \left[ (2\rho' + 4\rho'' h_t^2) (1 + h_t^2) - 2\rho h_t^2 \right] \sum_i (h_{ti}b^{ii})^2 - 2\rho h_t^2 (1 + h_t^2) \sum_i (b^{ii})^2.$$

(3.9)

In order to deal with the last two terms, we shall compute $L(b_{kk})$ in advance. By differentiate (3.5) twice with respect to $\theta_k$, we have

$$h_{ttkk} = \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_{ti}h_{tj}]kkb^{ij} + 2 \sum_{i,j,p,q} [(1 + h_t^2)\delta_{ij} + h_{ti}h_{tj}]k(-b^{ip}b_{pq,k}b^{ij})$$

$$+ \sum_{i,j,p,q,r,s} [(1 + h_t^2)\delta_{ij} + h_{ti}h_{tj}](2b^{ir}b_{rs,k}b^{sp}b_{pq,k}b^{gj})$$

$$+ \sum_{i,j,p,q} [(1 + h_t^2)\delta_{ij} + h_{ti}h_{tj}](-b^{ip}b_{pq,k}b^{gj})$$

(3.10)

$$= J_1 + J_2 + J_3 + J_4.$$

For the term $J_1$, we have

$$J_1 = \sum_{i,j} (2h_t h_{tk} \delta_{ij} + h_{tik}h_{tj} + h_{titj})kkb^{ij}$$

$$= 2h_{tk}^2 \sigma_1 + 2h_t h_{tik} \sigma_1 + 2 \sum_i h_{tik} h_{ti} b^{ii} + 2 \sum_i h_{tik}^2 b^{ii}.$$

Noticing that

$$h_{tik} = h_{kit} = b_{ki,t} - h_t \delta_{ki},$$

$$h_{tikk} = h_{ikkt} = b_{ik,kt} - h_{kt} \delta_{ik} = b_{kk,kt} - h_{kt} \delta_{ik},$$
hence we obtain

\begin{equation}
J_1 = 2h_{tk}^2\sigma_1 + 2h_t b_{kk,t}\sigma_1 - 2h_t^2\sigma_1 + 2 \sum_i b_{kk,i} h_{ti} b^{ij} \tag{3.11}
- 2h_{tk}^2 b^{kk} + 2 \sum_l b_{kl,t} b^{ll} - 4h_t b_{kk,t} b^{kk} + 2h_t^2 b^{kk}.
\end{equation}

For the term \( J_2 \), we have

\begin{equation}
J_2 = 2 \sum_{i,j} (2h_t h_{tk}\delta_{ij} + h_{tk} h_{tj} + h_{ti} h_{tjk})(-b^{ij} b_{ij,k} b^{jj}) \tag{3.12}
= - 4h_t h_{tk} \sum_i (b^{ij})^2 b_{ii,k} - 4 \sum_{i,j} h_{tk} h_{tj} b^{ij} b^{jj} b_{ij,k}
= - 4h_t h_{tk} \sum_i (b^{ij})^2 b_{ii,k} - 4 \sum_{i,l} h_{tj} b^{ij} b^{ll} b_{kl,t} b_{kl,t} + 4h_t \sum_{j} h_{tj} b^{kk} b^{jj} b_{kk,j}.
\end{equation}

Also we have

\begin{equation}
J_3 = 2 \sum_{i,j,l} [(1 + h_t^2)\delta_{ij} + h_{ti} h_{tj}] b^{ij} b^{ll} b_{kl,j} b_{kl,j}. \tag{3.13}
\end{equation}

Applying the commutation rule

\[ b_{ij,kl} - b_{ij,kl} = b_{jk}\delta_{il} - b_{jl}\delta_{ik} + b_{ik}\delta_{jl} - b_{il}\delta_{jk}, \]

for the term \( J_4 \), we have

\begin{equation}
J_4 = - \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_{ti} h_{tj}] b^{ij} b^{jj} b_{ij,kk} \tag{3.14}
= - \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_{ti} h_{tj}] b^{ij} b^{jj} (b_{kk,ij} + b_{ij} - b_{kk}\delta_{ij}).
\end{equation}

On the other hand,

\begin{equation}
h_{ttkk} = h_{kttk} = b_{kk,tt} - h_{tt} = b_{kk,tt} - \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_{ti} h_{tj}] b^{jj}. \tag{3.15}
\end{equation}

By putting (3.11) – (3.13) into (3.10), recalling the definition of operator \( L \), we obtain

\[ L(b_{kk}) = \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_{ti} h_{tj}] b^{ij} + 2h_{tk}^2 \sigma_1 + 2h_t b_{kk,t}\sigma_1 - 2h_t^2 \sigma_1 \]
\[ - 2h_{tk}^2 b^{kk} + 2 \sum_l b_{kl,t} b^{ll} - 4h_t b^{kk} b_{kk,t} + 2h_t^2 b^{kk} - 4h_t h_{tk} \sum_i (b^{ij})^2 b_{ii,k} \]
\[ - 4 \sum_{i,l} h_{tj} b^{ij} b^{ll} b_{kl,t} b_{kl,t} + 2 \sum_{i,j,l} [(1 + h_t^2)\delta_{ij} + h_{ti} h_{tj}] b^{ij} b^{jj} b_{kl,t} b_{kl,j} \]
\[ + 4h_t \sum_i h_{ti} b^{kk} b^{ij} b_{kk,i} - \sum_{i,j} [(1 + h_t^2)\delta_{ij} + h_{ti} h_{tj}] b^{ij} (b_{ij} - b_{kk}\delta_{ij}).\]
Therefore,

\[
I_4 = 2 \sum_{i,j,k,l} [(1 + h_{ii}^2) \delta_{ij} + h_{ii} h_{ij}] b_{ii} b_{ij} b_{kk} b_{kl} b_{kl,j} b_{kl,t} - 4 \sum_{i,k,l} h_{ii} b_{ii} b_{kk} b_{kk} b_{kl} b_{kl,t} \\
+ 2 h_t \sigma_1 \sum_k b_{kk} b_{kk,t} - 4 h_t (b_{kk}^2 b_{kk,t} - 2 h_{ii}^2 \sigma_1^2 + 2 \sum_{k,l} b_{kk} b_{kk,t}^2) \\
+ [(n - 1)(1 + h_{ii}^2) + 2 h_{ii}^2] \sum_i (b_{ii}^2) + 2 \sigma_1 \sum_i h_{ii} b_{ii} + (n - 3) \sum_i (h_{ii} b_{ii})^2.
\]

(3.16)

By substituting (3.9) and (3.16) in (3.4), we obtain

(3.17)

\[
L(\varphi) = \sum_{i,j,k,l} [(1 + h_{ii}^2) \delta_{ij} + h_{ii} h_{ij}] b_{ii} b_{ij} b_{kk} b_{kl} b_{kl,i} b_{kl,j} - 2 \sum_{i,k,l} h_{ii} b_{ii} b_{kk} b_{kk} b_{kl} b_{kl,t} + \sum_{k,l} b_{kk} b_{kk,t}^2 \\
+ 2 h_t \sigma_1 \sum_k b_{kk} b_{kk,t} - 4 h_t (b_{kk}^2 b_{kk,t} + (2 + 4 \rho h_{ii}^2) \sigma_1 \sum_i h_{ii}^2 b_{ii} \\
+ [(n - 1)(1 + h_{ii}^2) + 2 h_{ii}^2] \sum_i (b_{ii}^2) + 2 \sigma_1 \sum_i h_{ii} b_{ii} + (n - 3) \sum_i (h_{ii} b_{ii})^2.
\]

Step 2: In this step we shall calculate \(L(e^{\beta \varphi})\) and deal with the third order derivatives involving \(b_{kk,t}\).

Notice that

\[
L(e^{\beta \varphi}) = \beta e^{\beta \varphi} L(\varphi) + \beta \varphi^2 + \beta^2 e^{\beta \varphi} \sum_{i,j,p,q} [(1 + h_{ii}^2) \delta_{pq} + h_{tp} h_{tq}] b_{ij} b_{pq} \frac{\partial \varphi}{\partial \theta_i} \frac{\partial \varphi}{\partial \theta_j} \\
- 2 \beta^2 e^{\beta \varphi} \sum_{i,j} h_{ij} b_{ij} \frac{\partial \varphi}{\partial \theta_i} \frac{\partial \varphi}{\partial \theta_j}.
\]

To reach (3.11), we only need to prove

\[
L(\varphi) + \beta \varphi^2 \geq 0 \mod \nabla_\theta \varphi
\]

and we now come to compute \(\beta \varphi^2\).
By (3.3), we have

\begin{equation}
\varphi_t^2 = 4(\rho')^2 h_t^2 \rho_t^2 + 4 \rho' h_t \rho_t \sum_k b^{kk}b_{kk,t} + \left( \sum_k b^{kk} b_{kk,t,t} \right)^2
\end{equation}

\begin{equation}
= 4(\rho')^2 h_t^2 (1 + h_t^2)^2 \rho_t^2 + 8(\rho')^2 h_t^2 (1 + h_t^2) \rho_t \sum_i h_{ti}^2 b^{ii} + 4(\rho')^2 h_t^2 \left( \sum_i h_{ti}^2 b^{ii} \right)^2
\end{equation}

\begin{equation}
+ 4 \rho' h_t (1 + h_t^2) \rho_t \sum_i h_{ti}^2 b^{ii} + 4 \rho' h_t (\sum_i h_{ti}^2 b^{ii}) (\sum_k b^{kk} b_{kk,t}) + \left( \sum_k b^{kk} b_{kk,t} \right)^2
\end{equation}

Jointing (3.17) with (3.18), we regroup the terms in $L(\varphi) + \beta \varphi_t^2$ as follows

\[L(\varphi) + \beta \varphi_t^2 = P_1 + P_2 + P_3,
\]

where

\[P_1 = \sum_{k \neq l} \left( \sum_{i,j} h_{ti} h_{tj} b^{ii} b^{jj} b^{kk} b_{kk,l} b_{kl,j} + 2 \sum_i h_{ti} b^{ii} b^{kk} b_{kl,l} b_{kl,t} + b^{kk} b_{kl,t} \right),\]

\[P_2 = \sum_k \left( b^{kk} b_{kk,t} \right)^2 + \beta \left( \sum_k b^{kk} b_{kk,t} \right)^2 + 2 \sum_k \left[ (1 + 2 \beta \rho' (1 + h_t^2)) h_t \rho_t + 2 \beta \rho' h_t (\sum_i h_{ti}^2 b^{ii}) - \sum_i h_{ti} b^{ii} b^{kk} b_{kk,i} - 2 h_t b^{kk} \right],\]

\[P_3 = (1 + h_t^2) \sum_{i,k,l} \left( b^{ii} \right)^2 b^{kk} b_{kl,t}^2 + \sum_{i,j,k} h_{ti} h_{tj} b^{ii} b^{jj} b^{kk} b_{kk,i} b_{kk,j} + \left[ 2 + 4 \rho' h_t^2 + 8 \beta (\rho')^2 h_t^2 (1 + h_t^2) \right] \rho_t \sum_i h_{ti}^2 b^{ii}
\]

\[+ \left[ (n - 1)(1 + h_t^2) + 2 h_t^2 - 2 \rho' h_t^2 (1 + h_t^2) \right] \sum_i (b^{ii})^2
\]

\[+ \left[ 4 \rho' h_t^2 (1 + h_t^2) + (2 \rho' + 4 \rho'' h_t^2)(1 + h_t^2)^2 - 2 h_t^2 + 4 \beta (\rho')^2 h_t^2 (1 + h_t^2)^2 \right] \rho_t^2
\]

\[+ \left[ (2 \rho' + 4 \rho'' h_t^2) (1 + h_t^2) - 2 \rho' h_t^2 + (n - 3) \right] \sum_i (h_{ti} b^{ii})^2 + 4 \beta (\rho')^2 h_t^2 \left( \sum_i h_{ti}^2 b^{ii} \right)^2.
\]

In the rest of this step, we will deal with the term $P_2$. Let $X_k = b^{kk} b_{kk,t}$ ($k = 1, 2, \cdots, n - 1$). Then $P_2$ can be rewritten as

\[P_2(X_1, X_2, \cdots, X_{n-1}) = \sum_k X_k^2 + \beta (\sum_k X_k)^2 + 2 \sum_k c_k X_k,
\]

where

\[c_k = [1 + 2 \beta \rho' (1 + h_t^2)] h_t \rho_t + 2 \beta \rho' h_t (\sum_i h_{ti}^2 b^{ii}) - \sum_i h_{ti} b^{ii} b^{kk} b_{kk,i} - 2 h_t b^{kk}.
\]
Denote $P_2$ the matrix
\[
\begin{pmatrix}
1 + \beta & \beta & \cdots & \beta \\
\beta & 1 + \beta & \cdots & \beta \\
\vdots & \vdots & \ddots & \vdots \\
\beta & \beta & \cdots & 1 + \beta
\end{pmatrix}.
\]

In a word, we want to bound $P_2(X_1, X_2, \cdots, X_{n-1})$ from below, thus the semi-definitive of $P_2$ is necessary and this requires
\[
\beta \geq -\frac{1}{n - 1}.
\]

For convenience, let us choose the degenerate case, i.e. $\beta = -\frac{1}{n - 1}$. By setting $\tau = (1, 1, \cdots, 1)$, the null eigenvector of matrix $P_2$, we then have by (3.2)
\[
P_2(1, 1, \cdots, 1) = 2 \sum_k c_k = 2[n - 3 - 2\rho'(1 + h_t^2)]h_t\sigma_1 - 2 \sum_i h_t b^{ij} \frac{\partial \varphi}{\partial \theta_i},
\]
which suggests that the simple selection should be $\rho(t) = \frac{n - 3}{2} \log(1 + t)$.

From now on, let us fix $\rho(t) = \frac{n - 3}{2} \log(1 + t)$ and $\beta = -\frac{1}{n - 1}$. But for simplicity, we do not always substitute for the values of $\rho$ and $\beta$.

By straightforward computation, we have
\[
\sum_k \left(X_k + \beta \sum_i X_i + c_k\right)^2 = P_2(X_1, X_2, \cdots, X_{n-1}) + \sum_k c_k^2 + P_2(\nabla \varphi),
\]
where
\[
P_2(\nabla \varphi) = 2\beta(\sum_i X_i) \sum_k c_k = 2\beta(\sum_j X_j) \sum_i h_t b^{ij} \frac{\partial \varphi}{\partial \theta_i}.
\]

Putting $\rho$ and $\beta$ into some terms in $c_k$, we derive that
\[
c_k = \frac{2}{n - 1} h_t \sigma_1 - \frac{2}{n - 1} \rho' h_t (\sum_i h_t^2 b^{ii}) - \sum_i h_t b^{ij} b^{kk} b_{kk,i} - 2 h_t b^{kk},
\]
therefore, joint with (3.2),
\[
P_2(X_1, X_2, \cdots, X_{n-1}) \geq -\sum_k c_k^2 - P_2(\nabla \varphi)
\]
\[
= -\sum_{i,j,k} h_t b^{ij} b^{jj} b^{kk} b_{kk,i} b^{kk,j} b_{kk,j} - 4 h_t \sum_{i,k} h_t b^{ii} (b^{kk})^2 b_{kk,i}
\]
\[
- 4 h_t^2 \sum_k (b^{kk})^2 + \frac{4}{n - 1} h_t^2 \sigma_1^2 - \frac{8}{n - 1} \rho' h_t^2 \sigma_1 \sum_i h_t^2 b^{ii}
\]
\[
+ \frac{4}{n - 1} h_t^2 (\rho')^2 (\sum_i h_t^2 b^{ii})^2 + \tilde{P}_2(\nabla \varphi).
\]
Observing that $P_1 \geq 0$, hence

\begin{equation}
L(\varphi) + \beta \varphi_t^2 \geq (1 + h_t^2) \sum_{i,k,l} (b^{ii})^2 b^{kk} b^{ll} b_{kl,i}^2 - 4h_t \sum_{i,k} h_{ti} b^{ii} (b^{kk})^2 b_{kk,i} + \frac{4}{n - 1} h_t \left[ \sigma_1 - \rho' \sum_j h_{tj}^2 b^{jj} \right] \sum_i h_{ti} b^{ii} \frac{\partial \varphi}{\partial \theta_t} \tag{3.19}
\end{equation}

In the next step we will concentrate on the following two terms

\begin{equation}
R = (1 + h_t^2) \sum_{i,k,l} (b^{ii})^2 b^{kk} b^{ll} b_{kl,i}^2 - 4h_t \sum_{i,k} h_{ti} b^{ii} (b^{kk})^2 b_{kk,i}.
\end{equation}

**Step 3:** Let us complete the proof of (3.1).

Recalling our first order condition (3.2), we have

\begin{equation}
b^{11} b_{11,j} = \frac{\partial \varphi}{\partial \theta_j} - \sum_{k \geq 2} b^{kk} b_{kk,j} - 2 \rho' h_t h_{t,j}, \quad \text{for } j = 1, 2, \ldots, n - 1.
\end{equation}

For the term $R$, we have

\begin{align*}
R &= (1 + h_t^2) \left[ \sum_i \sum_{k \neq l} (b^{ii})^2 b^{kk} b^{ll} b_{kl,i}^2 + \sum_{i,k} (b^{ii})^2 (b^{kk} b_{kk,i})^2 \right] - 4 \sum_{i,k} h_{ti} h_{ti} b^{ii} (b^{kk})^2 b_{kk,i} \\
&= (1 + h_t^2) \left[ 2 \sum_{k \geq 2} (b^{11})^2 b^{kk} b_{11,k}^2 + 2 \sum_{i,k} (b^{ii})^2 b^{kk} b_{11,i}^2 + \sum_{i,k,l \geq 2} (b^{ii})^2 b^{kk} b^{ll} b_{kl,i}^2 \\
&\quad + \sum_i (b^{ii})^2 (b^{11} b_{11,i})^2 + \sum_{i,k \geq 2} (b^{ii})^2 (b^{kk} b_{kk,i})^2 \right] - 4 \sum_i h_{ti} h_{ti} b^{ii} (b^{11})^2 b_{11,i} \\
&\quad - 4 \sum_i \sum_{k \geq 2} h_{ti} h_{ti} b^{ii} (b^{kk})^2 b_{kk,i} \\
&= R_1 + R_2 + R_3,
\end{align*}
where

\[
R_1 = (1 + \tilde{h}_t^2) \left[ 2 \sum_{k \geq 2} (b^{11})_k^2 b_{k,1}^1 b_{1,1}^2 + \sum_i (b^{ii})_i^2 (b^{11})_b_{1,1}^2 \right] - 4 \sum_i h_t h_t (b^{ii})_i^2 (b^{11})_b_{1,1}^2,
\]

\[
R_2 = 2 \sum_{i,k \geq 2} (1 + \tilde{h}_t^2) (b^{ii})_i^2 b^{kk} b_{kk,i}^2 + \sum_{i,k,l \geq 2} (1 + \tilde{h}_t^2) (b^{ii})_i^2 b^{kk} b_{kl,i}^2,
\]

\[
R_3 = \sum_i \sum_{k \geq 2} (1 + \tilde{h}_t^2) (b^{ii})_i^2 (b^{kk} b_{kk,i})^2 - 4 \sum_i \sum_{k \geq 2} h_t h_t (b^{ii})_i^2 (b^{kk} b_{kk,i})^2.
\]

By (3.20), one has

\[
R_1 = (1 + \tilde{h}_t^2) \left[ 2b^{11} \sum_{i,k,l \geq 2} b^{kk} b_{kk,i} b_{ll,i} + 8\rho h_t b^{11} \sum_i h_t b^{ii} b^{kk} b_{kk,i} + 8(\rho')^2 h^2 b^{11} \sum_i h^2 b^{ii} \right]
+ \sum_i \sum_{k,l \geq 2} (b^{ii})_i^2 b^{kk} b_{kk,i} b_{ll,i} + 4\rho' h_t \sum_{i,k \geq 2} h_t (b^{ii})_i^2 b^{kk} b_{kk,i} + 4(\rho')^2 h^2 \sum_{i,k} (h_t b^{ii})^2
\]

\[+ 4h_t \sum_{i,k \geq 2} h_t b^{ii} b^{11} b^{kk} b_{kk,i} + 8\rho h^2 b^{11} \sum_i h^2 b^{ii} + R(\nabla \theta \varphi),\]

where

\[
R(\nabla \theta \varphi) = (1 + \tilde{h}_t^2) \left[ 2b^{11} \sum_{k \geq 2} b^{kk} (\partial \varphi / \partial k)^2 - 4b^{11} \sum_{k,l \geq 2} b^{kk} b^{ll} b_{kl,k} \partial \varphi / \partial k - 8\rho h_t b^{11} \sum_{k \geq 2} b^{kk} h_t \partial \varphi / \partial k \right]
+ \sum_i (b^{ii})_i^2 (\partial \varphi / \partial i)^2 - 2 \sum_i \sum_{k \geq 2} (b^{ii})_i^2 b^{kk} b_{kk,i} \partial \varphi / \partial i - 4\rho' h_t \sum_{i,k \geq 2} (b^{ii})_i^2 h_t \partial \varphi / \partial i
\]

\[+ 4h_t b^{11} \sum_i b^{ii} h_t \partial \varphi / \partial i.
\]

On the other hand,

\[
R_2 = (1 + \tilde{h}_t^2) \left[ 2b^{11} \sum_{k \geq 2} (b^{kk})_k^2 b_{kk,1}^1 + 2 \sum_{i,k \geq 2} (b^{ii})_i^2 b^{kk} b_{1,1}^2 b_{kk,i} + 2 \sum_{i,k \geq 2} b^{ii} (b^{kk})_k^2 b_{kk,i}^2 \right]
+ \sum_i \sum_{k,l \geq 2} (b^{ii})_i^2 b^{kk} b_{kk,i}^2 \sum_{k,l \neq i, i \neq k, k \neq l}.
\]

Recall that \(2\rho' (1 + \tilde{h}_t^2) = n - 3\) which will be denoted by \(\alpha\) for simplicity in the following calculations. Now we are at a stage to rewrite the terms in \(R\) in a natural way: \((T_1)\) the terms involving \(b_{kk,1}(k \geq 2)\); \((T_2)\) the terms involving \(b_{kk,i}(k, i \geq 2)\); \((T_3)\) all of the rest.
terms. More precisely,

\[
T_1 = \sum_{k \geq 2} (1 + 2b_{11}b^{kk}) \cdot ((1 + h_t^2)^{\frac{1}{2}}b_{11}b^{kk}b_{kk,1})^2 + \left( \sum_{k \geq 2} (1 + h_t^2)^{\frac{1}{2}}b_{11}b^{kk}b_{kk,1} \right)^2 \\
+ 4h_th_t b_{11} (1 + h_t^2)^{\frac{1}{2}} \sum_{k \geq 2} (1 + \frac{\alpha}{2} - b_{11}b^{kk}) \cdot ((1 + h_t^2)^{\frac{1}{2}}b_{11}b^{kk}b_{kk,1}),
\]

and

\[
T_2 = (1 + h_t^2) \sum_{i \geq 2} \left\{ (1 + 2b_{ii}b^{11}) \cdot \left( \sum_{k \geq 2} b^{ii}b^{kk}b_{kk,i} \right) \right\}^2 + \sum_{k \geq 2} (1 + 2b_{ii}b^{kk}) \cdot (b^{ii}b^{kk}b_{kk,i})^2 \\
+ \sum_{k \geq 2} (b^{ii}b^{kk}b_{kk,i})^2 + 4h_th_t b^{ii} (1 + h_t^2)^{\frac{1}{2}} \sum_{k \geq 2} \left[ -b_{ii}b^{kk} + \frac{\alpha}{2} + (1 + \alpha)b_{ii}b^{11} \right] \cdot (b^{ii}b^{kk}b_{kk,i}),
\]

and the rest terms

\[
T_3 = h_t^2 (1 + h_t^2)^{\frac{1}{2}} \left[ 2\alpha^2 b^{11} \sum_{i \geq 2} h_{ti}^2 b^{ii} + \alpha^2 \sum_{i} (h_{ti}b^{ii})^2 + 4\alpha^2 \sum_{i} b_{ii}b^{kk} \right] \\
+ (1 + h_t^2) \left[ 2 \sum_{i,k \geq 2} (b^{ii})^2 b_{kk}b_{kk,i}^2 + \sum_{i} \sum_{k,l \geq 2} (b^{ii})^2 b_{kk}b_{kl,i}^2 \right] + R(\nabla \theta \varphi).
\]

We shall maximize the terms \(T_1\) and \(T_2\) via Lemma \ref{lemma:3.2} for different choice of parameters.

At first let us examine the term \(T_1\). set \(X_k = (1 + h_t^2)^{\frac{1}{2}}b_{11}b^{kk}b_{kk,1}, \lambda = 1, \mu = h_t b_{11}b_t (1 + h_t^2)^{\frac{1}{2}}, b_k = 1 + 2b_{11}b^{kk} \) and \(c_k = b_{11}b^{kk} - (1 + \frac{\alpha}{2})\) where \(k \geq 2\). By Lemma \ref{lemma:3.2} we have

\[
T_1 \geq -4h_t^2 (1 + h_t^2)^{-1} (h_t b_{11}^2)^{2} \Gamma_1,
\]

where

\[
\Gamma_1 = \sum_{k \geq 2} \frac{c_k^2}{b_k} - \left( 1 + \sum_{k \geq 2} \frac{\lambda}{b_k} \right)^{-1} \left( \sum_{k \geq 2} \frac{c_k}{b_k} \right)^2.
\]

Next we shall simplify \(\Gamma_1\). By denoting

\[
\beta_k = \frac{1}{b_k},
\]

we have

\[
b_{11}b^{kk} = \frac{1}{2\beta_k} - \frac{1}{2}, \quad c_k = \frac{1}{2\beta_k} - \frac{3 + \alpha}{2}.
\]
Hence,
\[
\Gamma_1 = \sum_{k \geq 2} \beta_k \left( \frac{1}{2\beta_k} - \frac{3 + \alpha}{2} \right)^2 - \left( 1 + \sum_{k \geq 2} \beta_k \right)^{-1} \left[ \sum_{k \geq 2} \beta_k \left( \frac{1}{2\beta_k} - \frac{3 + \alpha}{2} \right) \right]^2 \\
= \frac{1}{4} \sum_{k \geq 2} \frac{1}{\beta_k} \left( \frac{(n + 1 + \alpha)^2}{4} + \frac{(3 + \alpha)^2}{4} \right),
\]
since
\[
1 \leq 1 + \sum_{k \geq 2} \beta_k \leq n - 1,
\]
it follows that
\[
\Gamma_1 \leq \frac{1}{4} \sum_{k \geq 2} \frac{1}{\beta_k} \frac{(n + 1 + \alpha)^2}{4(n - 1)} + \frac{(3 + \alpha)^2}{4} \\
= \frac{n - 2}{4(n - 1)} (2 + \alpha)^2 + \frac{1}{4} (2\sigma_1 b_{11} - 2).
\]
Therefore,
\[
(3.21) \quad T_1 \geq -\frac{(n - 2)}{n - 1} (2 + \alpha)^2 + 2\sigma_1 b_{11} - 2 \left[ h_i^2 (1 + h_i^2)^{-1} (h_{11} b_{11})^2 \right].
\]

Now we will deal with term $T_2$. For every $i \geq 2$ fixed, set $X_k = b_{ii} b_{kk} b_{kk,i}$, $\lambda = 1 + 2b_{ii} b_{11}$, $\mu = -h_{ti} b_{ii} h_t (1 + h_t^2)^{-1}$, $b_k = 1 + 2b_{ii} b_{kk} (k \neq i)$, $b_i = 1$ and $c_k = b_{ii} b_{kk} - \frac{1}{2} \alpha - (1 + \alpha) b_{ii} b_{11}$.

By Lemma 3.2 we have
\[
T_2 \geq -4(1 + h_t^2) \sum_{i \geq 2} (h_{ti} b_{ii})^2 \Gamma_i,
\]
where
\[
\Gamma_i = c_i^2 + \sum_{\substack{k \geq 2 \\ k \neq i}} \frac{c_k^2}{b_k} - \left( \frac{1}{\lambda} + 1 + \sum_{\substack{k \geq 2 \\ k \neq i}} \frac{1}{b_k} \right)^{-1} \left( c_i + \sum_{\substack{k \geq 2 \\ k \neq i}} \frac{c_k}{b_k} \right)^2.
\]
For $k \neq i$, denoting
\[
\beta_k = \frac{1}{b_k},
\]
we have
\[
b_{ii} b_{kk} = \frac{1}{2\beta_k} - \frac{1}{2}, \quad c_k = \frac{1}{2\beta_k} - \delta,
\]
where
\[
\delta = \frac{1 + \alpha}{2} + (1 + \alpha) b_{ii} b_{11}.
\]
Noticed that
\[
c_i = \frac{3}{2} - \delta, \quad \delta = \frac{1 + \alpha}{2},
\]
we obtain
\[ \Gamma_i = c_i^2 + \sum_{k \geq 2, k \neq i} \beta_k \left( \frac{1}{2\beta_k} - \delta \right)^2 - \left( \frac{1}{\lambda} + 1 + \sum_{k \geq 2, k \neq i} \beta_k \right)^{-1} \left[ c_i + \sum_{k \geq 2, k \neq i} \beta_k \left( \frac{1}{2\beta_k} - \delta \right) \right]^2 \]
\[ = \frac{1}{4} \sum_{k \geq 2, k \neq i} \frac{1}{\beta_k} - \left( \frac{1}{\lambda} + 1 + \sum_{k \geq 2, k \neq i} \beta_k \right)^{-1} \left( \frac{n}{2} + \delta \right)^2 + \frac{9}{4} + \frac{\delta^2}{\lambda} \]
\[ = \frac{1}{4} \sum_{k \geq 2, k \neq i} \frac{1}{\beta_k} - \left( \frac{1}{\lambda} + 1 + \sum_{k \geq 2, k \neq i} \beta_k \right)^{-1} \left( \frac{n + 1 + \alpha}{4} \right)^2 + \frac{9}{4} + \frac{1 + \alpha}{2} \delta. \]

Obviously,
\[ 1 \leq \frac{1}{\lambda} + 1 + \sum_{k \geq 2, k \neq i} \beta_k \leq n - 1, \]
hence
\[ \Gamma_i \leq \frac{1}{4} \sum_{k \geq 2, k \neq i} \frac{1}{\beta_k} - \left( \frac{n + 1 + \alpha}{4(n - 1)} \right)^2 + \frac{9}{4} + \frac{1 + \alpha}{2} \delta \]
\[ = \frac{n - 2}{4(n - 1)} \alpha^2 - \frac{1}{n - 1} \alpha + \frac{n - 3}{2(n - 1)} + \frac{1}{2} \sigma_1 b_{ii} + \frac{1}{2} \alpha^2 b_{ii} b_{11} + \alpha b_{ii} b_{11}. \]

Therefore, we have
\[ (3.22) \]
\[ T_2 \geq - \frac{h_i^2}{1 + h_i^2} \sum_{i \geq 2} \left( \frac{n - 2}{n - 1} \alpha^2 - \frac{4}{n - 1} \alpha + \frac{2n - 6}{n - 1} - 2\sigma_1 b_{ii} + 2\alpha^2 b_{ii} b_{11} + 4\alpha b_{ii} b_{11} \right) (h_{ti} b_{ii})^2. \]

Up to now, combine (3.21) and (3.22) we obtain
\[ (3.23) \]
\[ R \geq \frac{h_i^2}{1 + h_i^2} \sum_i \left( \frac{1}{n - 1} \alpha^2 + \frac{4}{n - 1} \alpha - \frac{2n - 6}{n - 1} - 2\sigma_1 b_{ii} \right) (h_{ti} b_{ii})^2 + R(\nabla \theta \varphi). \]

For choice of \( \rho \) and \( \beta \), by (3.19) and (3.23) we have for \( n \geq 2 \)
\[ L(\varphi) - \frac{1}{n - 1} \varphi_i^2 \geq \frac{2\sigma_1}{1 + h_i^2} \sum_i h_{ti}^2 b_{ii} + (n - 1) \sum_i (b_{ii})^2 + (n - 3) \sigma_1^2 \]
\[ + \frac{2(n - 3)}{1 + h_i^2} \sum_i (h_{ti} b_{ii})^2 + \tilde{P}_2(\nabla \theta \varphi) + R(\nabla \theta \varphi) \geq 0 \mod \nabla \theta \varphi. \]

The proof of (3.1) is completed. \( \square \)

Now we state the following elementary calculus lemma which has appeared in [19].
Lemmas 3.2. (See [19]) Let $\lambda \geq 0$, $\mu \in \mathbb{R}$, $b_k > 0$ and $c_k \in \mathbb{R}$ for $2 \leq k \leq n - 1$. Define the quadratic polynomial
\[
Q(X_2, \cdots, X_{n-1}) = -\sum_{2 \leq k \leq n-1} b_k X_k^2 - \lambda \left( \sum_{2 \leq k \leq n-1} X_k \right)^2 + 4\mu \sum_{2 \leq k \leq n-1} c_k X_k.
\]
Then we have
\[
Q(X_2, \cdots, X_{n-1}) \leq 4\mu^2 \Gamma,
\]
where
\[
\Gamma = \sum_{2 \leq k \leq n-1} \frac{c_k^2}{b_k} - \lambda \left( 1 + \lambda \sum_{2 \leq k \leq n-1} \frac{1}{b_k} \right)^{-1} \left( \sum_{2 \leq k \leq n-1} \frac{c_k}{b_k} \right)^2.
\]

Now we give a remark on Theorem 1.1.

Remark 3.1. In the proof of Theorem 1.1, if we restrict to the case $n = 2$ and just set $\rho = 0$, then (3.2) shows that $b_{11,1} = 0 \mod \nabla \theta \varphi$.

Applying this into the expression of $L(\varphi)$ in (3.17) will give
\[
L(\varphi) = (b_{11} b_{11,t})^2 - 2 h_t (b_{11})^2 b_{11,t} + (b_{11})^2 h_{t1}^2 + (1 + h_t^2)(b_{11})^2
\]
\[
= \left[ b_{11}^2 b_{11,t}^2 - h_t b_{11} \right] + (b_{11})^2 h_{t1}^2 + (b_{11})^2 \geq 0 \mod \nabla \theta \varphi,
\]
and this means that for any point $x \in \Gamma_t, 0 < t < 1$,
\[
\log K(x) \geq (1 - t) \min \limits_{\partial \Omega_0} \log K + t \min \limits_{\partial \Omega_1} \log K,
\]
which has been already proved by Longinetti in [17]. Also, by Remark 1.2 we know that this is not the sharp estimates in the 2-dimensional case.

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