Reliable Covariance Estimation

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Abstract

Covariance or scatter matrix estimation is ubiquitous in most modern statistical and machine learning applications. The task becomes especially challenging since most real-world datasets are essentially non-Gaussian. The data is often contaminated by outliers and/or has heavy-tailed distribution causing the sample covariance to behave very poorly and calling for robust estimation methodology. The natural framework for the robust scatter matrix estimation is based on elliptical populations. Here, Tyler’s estimator stands out by being distribution-free within the elliptical family and easy to compute. The existing works thoroughly study the performance of Tyler’s estimator assuming ellipticity but without providing any tools to verify this assumption when the covariance is unknown in advance. We address the following open question: Given the sampled data and having no prior on the data generating process, how to assess the quality of the scatter matrix estimator? In this work we show that this question can be reformulated as an asymptotic uniformity test for certain sequences of exchangeable variables. We develop a consistent and easily applicable hypothesis test against all alternatives to ellipticity when the scatter matrix is unknown.

I. INTRODUCTION

Parameter estimation from the observed data is one of the main focuses of statistics and data science. All models used for parameter inference rely on various assumptions such as independence of the samples, number of samples, certain parametric family of possible distributions, etc. Very rarely these assumptions are verified on the observed data and even if such attempt is made the data almost never agrees with the assumptions. This leads to a poor estimation, or even to scenarios where the researcher does not know the quality of the achieved estimator. The main reason for the lack of such tests is the technical complexity of their analysis especially when the data is far from being Gaussian. In this paper we focus on the covariance scatter matrix estimation in multivariate populations under quite week assumptions. We develop a consistent and easy to apply hypothesis test reliably validating the exploited assumptions and thus quantitatively assessing the quality of the estimator based on the data.
A. Covariance Estimation

Covariance estimation is a fundamental problem in multivariate statistical analysis. It arises in diverse applications such as signal processing, where knowledge of the covariance matrix is unavoidable in constructing optimal detectors [1], genomics, where it is widely used to measure correlations between gene expression values [2–4], and functional MRI [5]. Most of the modern algorithms analyzing social networks are based on Gaussian Graphical Models [6], where the independences between the graph nodes are completely determined by the sparsity structure of the inverse covariance matrix [7]. In empirical finance, knowledge of the covariance matrix of stock returns is a fundamental question with implications for portfolio selection and for tests of asset pricing models such as the CAPM [8, 9]. Application of structured covariance matrices instead of Bayesian classifiers based on Gaussian mixture densities or kernel densities proved to be very efficient for many pattern recognition tasks, among them speech recognition, machine translation and object recognition in images [10]. In geometric functional analysis and computational geometry [11] the exact estimation of covariance matrix is necessary to efficiently compute volume of a body in high dimension. The classical problems of clustering and Discriminant Analysis are entirely based on precise knowledge of covariance matrices of the involved populations [12], etc.

Most practically important covariance matrix estimators are formulated as Maximum Likelihood (ML) solutions making the choice of the parametric model essential. For example, the sample covariance is the ML of the Gaussian population when the number of samples is at least the dimension of the ambient space. However, in many real world applications the underlying multivariate distribution is non-Gaussian and robust covariance estimation methods are required. This occurs whenever the distribution of the measurements is heavy-tailed or a small proportion of the samples exhibits outlier behavior [13, 14]. Probably the most common extension of the Gaussian family of distributions allowing for treating heavy-tailed populations is the class of elliptically shaped distributions [15]. Elliptical populations served as the basis for defining a family of the so-called scatter matrix $M$-estimators [14], of which we focus on Tyler’s estimator [16, 17]. Given $n$ samples $x_1, \ldots, x_n \in \mathbb{R}^p$, $i = 1, \ldots, n$, Tyler’s scatter matrix estimator is defined as a solution to the fixed point equation

$$T = \frac{p}{n} \sum_{i=1}^{n} \frac{x_ix_i^\top}{x_i^\top T^{-1}x_i},$$

(1)
satisfying some condition to avoid the apparent scaling ambiguity (for a solution $T$ to (1), $c \cdot T$ is also a solution when $c > 0$), e.g. $\text{Tr}(T) = p$. Note that in elliptical populations the scatter matrix is equal to a positive multiple of the covariance matrix when the latter exists. This scaling factor is unimportant in most applications therefore we focus on the scatter matrix estimation instead of the covariance without loss of generality.

When $\{x_i\}$ are i.i.d. (independent and identically distributed) elliptical [15], their true scatter matrix $\Omega$ is positive definite and $n > p$, Tyler’s estimator exists with probability one and is a consistent estimator of $\Omega$. In [16] Tyler also demonstrated that his estimator can be viewed as a ML estimator of a certain distribution over a unit sphere.

The behavior of Tyler’s estimator had been thoroughly investigated in various asymptotic regimes and multiple high-probability performance bounds have been developed for its analysis [18–26]. However, all of these results only hold if the sample is elliptically distributed, which is never known in applications. Therefore a much more practical question can be formulated as follows: Given the data, verify that Tyler’s estimator indeed provides a reliable estimator of the scatter matrix. This is the question we address in our work.

B. Approach

In this article, we develop an asymptotically consistent hypothesis test against all alternatives to the ellipticity of the sample when the scatter matrix is unknown. To enable analytical treatment of this hypothesis test, we reformulate it as an asymptotic uniformity test for a certain stochastically dependent sequence of unit random vectors. The main tool used in the construction and analysis of the uniformity tests for i.i.d. samples is the Central Limit Theorem (CLT) [27–29] which is clearly not applicable when the sample is not independent. For our case, we develop a novel toolbox that allows verification of the null hypothesis by resorting to the concept of exchangeability. Exchangeable random variables were first introduced by de Finetti [30, 31] as a direct and natural generalization of i.i.d. sequences. Interestingly, exchangeable random variables serve as one of the fundamental building blocks of the Bayesian statistics [32]. Unlike the i.i.d. case, the behavior of exchangeable sequences is much harder to analyze. We exploit certain versions of CLT and the Strong Law of Large Numbers (SLLN) for exchangeable variables to demonstrate asymptotic consistency of our test statistics built analogously to generalized Ajne
and Giné statistics [27, 28, 33] developed for the i.i.d. case. However, unlike the i.i.d. scenario our statistics are calculated using a random subset of the sample and not all of it. We discuss below that this fundamental distinction cannot be avoided due to the nature of the non-extendable exchangeability phenomenon. We also explain how our statistics can be easily used in practice.

The rest of the article is organized as follows. In Section II we introduce the setup and the main notation. The problem is formulated in Section III where we also present the existing hypothesis tests for the known scatter case. In Section IV we reformulate the problem and introduce necessary background on exchangeable variables. Section V provides some additional notation an auxiliary results. In Section VI we formulate the main result and in Section VII we provide the conclusion. Some of the proofs are postponed to the Appendix.

II. NOTATION AND SETUP

Definition 1 ([35]). A vector $\mathbf{y} \in \mathbb{R}^p$ is elliptically distributed with scatter matrix $\mathbf{\Omega} \succ 0$ and mean $\mathbf{\mu}$ if there exists a random vector $\mathbf{w} \in \mathcal{S}^{p-1}$ uniformly distributed over the unit $p$-dimensional sphere and an independent random variable $r \geq 0$, such that

$$
\mathbf{y} = \mathbf{\mu} + r \cdot \mathbf{\Omega}^{1/2} \mathbf{w}.
$$

(2)

For example, if $r \sim \sqrt{\chi^2_p}$, then $\mathbf{y} \sim \mathcal{N} (\mathbf{\mu}, \mathbf{\Omega})$. In what follows we always assume that the data is centered, $\mathbf{\mu} = 0$. Let us consider the normalized vector,

$$
\mathbf{x} = \frac{\mathbf{y}}{\|\mathbf{y}\|} = \frac{\mathbf{\Omega}^{1/2} \mathbf{w}}{\|\mathbf{\Omega}^{1/2} \mathbf{w}\|^2},
$$

(3)

which can be equivalently viewed as disregarding the information stored in the scalar variable $r$ but keeping the information provided by the scatter matrix. As we see below, the distribution of $\mathbf{x}$ contains all the information about the scatter matrix $\mathbf{\Omega}$. We are going to recover the scatter matrix by sampling from the distribution of $\mathbf{x}$. Denote by $\mathbf{I} = \mathbf{I}_p$ the $p$-dimensional identity matrix.

Definition 2 ([16]). The family of real Angular Central Gaussian (ACG) distributions on $\mathcal{S}^{p-1}$ is defined by the densities of the form

$$
p(\mathbf{x}; \mathbf{\Omega}) = \frac{\Gamma(p/2)}{2^{p/2} \|\mathbf{\Omega}\|^{p/2} \|\mathbf{x}^\top \mathbf{\Omega}^{-1} \mathbf{x}\|^{p/2}} \frac{1}{\|\mathbf{\Omega}^{1/2} \mathbf{w}\|^2}, \quad \mathbf{x} \in \mathcal{S}^{p-1},
$$

(4)

$^1$The Ajne statistic was originally introduced for distributions on a circle [33], the idea was extended by [34] for the 2-dimensional unit sphere and later generalized by [28] for the $p-1$-dimensional spheres. Similarly, Giné’s statistic was originally defined for 1- and 2-dimensional spheres and later extended by [28] for the general dimension.
for $\Omega \succ 0$ which is called the scatter matrix.

When $x$ is ACG distributed with the scatter matrix $\Omega$, we write

$$x \sim \mathcal{U}(\Omega),$$  \hspace{1cm} (5)

in particular when $\Omega = I$ we get the uniform distribution over the unit sphere $\mathcal{U}(I)$. Note that ACG is not a member of the elliptical family but actually belongs to a wider class of generalized elliptical populations whose definition is identical to Definition 1 except for weakened assumptions on $r$ [35]. In generalized elliptical population, $r$ does not have to be stochastically independent of $w$ and does not have to be non-negative. The following result allows us to reduce estimation of the scatter matrices of elliptical populations to the estimation of the scatter matrices of ACG vectors.

**Lemma 1** ([35]). For a random vector $y$ sampled from a centered elliptical population with scatter matrix $\Omega$, $x$ defined in (3) is ACG distributed with the scatter matrix $\Omega$.

Now assume $n > p$ i.i.d. random vectors $x_1, \ldots, x_n \in S^{p-1}$ are sampled from $\mathcal{U}(\Omega)$, then as shown in [16] the ML estimator of the scatter matrix exists almost surely and is given by the fixed point equation (1). The solutions to this equation form a ray since the latter is invariant under multiplication of the matrix $T$ by a positive constant. To resolve the ambiguity we choose $T$ to satisfy $\text{Tr}(T) = p$, however, we note that the specific choice of the scaling does not affect any of the results presented below.

### III. **Problem Formulation and State of the Art**

#### A. Main Goal

The problem considered in this article can be formulated as follows. Given a sequence of vectors $\{x_i\}_{i=1}^{n} \subset S^{p-1}$ sampled independently, we want to test two alternative hypotheses,

$$\mathcal{H}_0 : x_1, \ldots, x_n \sim \mathcal{U}(\Omega), \text{ for some } \Omega,$$  \hspace{1cm} (6)

$$\mathcal{H}_1 : x_1, \ldots, x_n \sim \mathcal{U}(\Omega), \text{ for any } \Omega,$$  \hspace{1cm} (7)

and in the case of $\mathcal{H}_0$ we want to estimate the scatter matrix $\Omega$, as well.

A remarkable feature of the hypothesis test (6)-(7) is that the scatter matrix under $\mathcal{H}_0$ is unknown. When it is known, the problem can equivalently be reformulated as a uniformity test on the sphere as shown below.
B. Uniformity Tests on $\mathbb{S}^{p-1}$

Assume the scatter matrix $\Omega$ in the hypothesis test (6)-(7) is known and introduce a derived i.i.d. sequence,

$$w_i = \frac{\Omega^{-1/2}x_i}{\|\Omega^{-1/2}x_i\|}, \quad i = 1, \ldots, n.$$ (8)

Under $H_0$, $w_1, \ldots, w_n \sim U(I)$ and therefore the test (6)-(7) becomes actually a uniformity test on the unit sphere,

$$G_0 : w_1, \ldots, w_n \overset{i.i.d.}{\sim} U(I),$$ (9)

$$G_1 : w_1, \ldots, w_n \overset{i.i.d.}{\sim} U(I).$$ (10)

Next, we summarize a number of asymptotically consistent uniformity tests on $\mathbb{S}^{p-1}$ concluding this section with Proposition 3 providing a uniformity test consistent against all alternatives. Based on it later we will develop an analogous test for (6)-(7) with unknown scatter matrix. Denote by

$$V_{p-1} = \int_{x \in \mathbb{S}^{p-1}} dx = \frac{2\pi}{\Gamma\left(\frac{p}{2}\right)}$$ (11)

the area of the unit sphere. In addition, by

$$\psi_{ij} = \arccos(x_i^T x_j)$$ (12)

we denote the angular separation (great circle distance) between $x_i$ and $x_j$ and by

$$N(y) = |\{x_i \mid y^T x_i \geq 0\}|, \quad y \in \mathbb{S}^{p-1},$$ (13)

the number of points falling into the hemisphere with the pole at $y$. Denote also

$$\alpha = \frac{p}{2} - 1,$$ (14)

$$\nu(a, b) = \left[\frac{a + b - 2}{a - 1}\right] + \left[\frac{a + b - 1}{a - 1}\right].$$ (15)

The following two popular statistics and detailed investigation of their behavior can be found in [27, 33]. These results were later generalized in [28] and summarized in [29].

Proposition 1 (Generalized Ajne Test, [28, 33]). Under the uniformity hypothesis, the Ajne statistic

$$t_A = \frac{1}{nV_{p-1}} \int_{y \in \mathbb{S}^{p-1}} \left( N(y) - \frac{n}{2} \right)^2 dy = \frac{n}{4} - \frac{1}{\pi n} \sum_{i<j} \psi_{ij}$$ (16)
is asymptotically distributed as $L \left( \sum_{q=1}^{\infty} a_{2q-1}^2 K_{\nu(p-1,2q-1)} \right)$, where $K_\xi$ are independent random variables distributed as $\chi^2_\xi$ and

$$a_{2q-1} = \frac{(-1)^{q-1} 2^{p-2} \Gamma(\alpha+1) \Gamma(q+\alpha)(2q-2)}{\pi(q-1)!(2q+p-3)!}.$$  \hspace{1cm} (17)

**Proposition 2** (Generalized Giné Test, [27, 28]). *Under the uniformity hypothesis, the Giné statistic*

$$t_G = \frac{n}{2} - \frac{p-1}{2n} \left( \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha+1)} \right)^2 \sum_{i<j} \sin(\psi_{ij})$$ \hspace{1cm} (18)

*is asymptotically distributed as $L \left( \sum_{q=1}^{\infty} a_{2q}^2 K_{\nu(p-1,2q)} \right)$, where $K_\xi$ are independent random variables distributed as $\chi^2_\xi$ and

$$a_{2q} = \frac{(p-1)(2q-1)}{8\pi(2q+p-1)} \left( \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(q + \frac{1}{2})}{\Gamma(q + \alpha + \frac{1}{2})} \right)^2.$$  \hspace{1cm} (19)

The following statement provides a concise and directly applicable test for uniformity under the assumption that the random vectors are sampled i.i.d. from $\mathcal{U}(I)$.

**Proposition 3** (Uniformity test, [27, 28]). *Any weighted sum of $t_A$ and $t_G$ is consistent against all alternatives to uniformity on $\mathbb{S}^{p-1}$.*

In practice, one way to make the decision about accepting or rejecting $H_0$ is as follows. The statistician truncates the series mentioned in the last two propositions in a data-driven manner and compares the sample values of $t_A$ and $t_G$ with the tables (or explicit numerical approximations) of the corresponding distributions. Another more general approach consists in replacing $t_A$ and $t_G$ by statistics whose expansions only have finite number of non-zero coefficients $a_k$ (see [27] for more details). An efficient data-driven approach to the design of the uniformity tests based on a modification of the Bayesian Information Criterion was developed by [36].

In this paper we are interested in the case of unknown scatter matrix in (6)-(7). As we see below this makes the hypothesis test much more involved. In the next sections we develop analogs of generalized Ajne and Giné uniformity tests for this scenario.

### IV. Problem Reformulation and Exchangeability

#### A. Methodology

From Theorem 3.1 from [17] we know that under $H_0$ Tyler’s estimator converges almost surely to the true scatter matrix when $n \to \infty$. This idea motivated our study of a new sequence
of vectors, defined as follows. Under $\mathcal{H}_0$ introduced in (6), we now consider the sequence

$$t_i = \frac{T^{-1/2}x_i}{\|T^{-1/2}x_i\|} \in S^{p-1}, \quad i = 1, \ldots, n,$$

where $T$ is defined in (1). The main challenge we face in the study of $\{t_i\}$ is the lack of stochastic independence unlike the case of $\{w_i\}$ defined in (8). Indeed, most existing convergence results explicitly rely on independence in their derivations in such a way that any deviation from this assumption ruins the performance analysis. For example, all the results of Ajne, Giné, and Prentice utilize the CLT and thus require independence as the most crucial assumption [27–29, 33].

Next we include a very brief summary of the exchangeability concept and the related toolbox. We then use it in Section V to overcome the loss of independence in our analysis of the consistency of $\{t_i\}$ and their statistics.

**B. Exchangeable Random Variables**

**Definition 3.** Given a sequence $\{X_i\}$ (finite or infinite) of random variables, we say that it is exchangeable if the joint distribution of any finite subset of variables is invariant under arbitrary permutations of the variables.

In other words, exchangeability is our indifference to the order of the measurements. This is clearly a much weaker hypothesis than independence, as any i.i.d. sequence is obviously exchangeable. In his seminal works de Finetti [30, 31] demonstrated that in certain sense every (infinite) exchangeable sequence can be represented as a composition of sequences of i.i.d. variables. This result can be viewed as the analog of Fourier decomposition in analysis, as it allows one to represent a more complicated exchangeable sequence as a superposition of basic building blocks - independent sequences - objects much easier accessible for analysis and reasoning.

De Finetti [30, 31] and some of his followers focused on infinite exchangeable sequences. There exist, however, finite sets of exchangeable random variables which cannot be embedded into infinite sequences, these are called finitely exchangeable or non-extendable. The analysis of extendable sequences can be reduced to the analysis of infinite sequences. On the other hand, the non-extendable sequences require quite different approaches. Our sequence of samples $\{t_i\}_{i=1}^n$ is an example of a non-extendable exchangeable sequence of random vectors. Indeed, their order obviously does not matter since $T$ is not affected by permutations of the measurements $\{x_i\}_{i=1}^n$. 
We can also see that this sequence is non-extendable, since addition of new random vectors \( x_j \) without an amendment of \( T \) will turn the sequence into non-exchangeable. For a detailed study of non-extendability we refer the reader to [37] and references therein.

The main result of our paper can be briefly summarized as follows. We demonstrate that the limiting behavior of the samples \( \{t_i\}_{i=1}^n \) is in certain sense analogous to the behavior of the vectors uniformly distributed over the unit sphere and therefore, we can apply similar tools for the hypotheses analysis. Below we show how to overcome the technical challenges on this way.

C. Limit Theorems for Exchangeable Variables

To illustrate the previous section and better describe the nature of the exchangeability phenomenon and its relation to stochastic independence, in this section we present analogs of the SLLN and CLT for triangular arrays of exchangeable variables.

**Lemma 2** (Strong Law of Large Numbers for Exchangeable Arrays). Let \( \{X_{ni}\}_{n,i=1}^{\infty,n} \) be a triangular array of row-wise exchangeable random variables and \( \{X_{\infty i}\}_{i=1}^{\infty} \) be an sequence of exchangeable random variables of bounded second moment such that

1) \( X_{n1} \overset{a.s.}\rightarrow X_{\infty 1}, \ n \rightarrow \infty, \)
2) \( \text{var} (X_{n1} - X_{\infty 1}) \rightarrow 0, \ n \rightarrow \infty, \)
3) \( E [X_{n1}X_{n2}] \rightarrow 0, \ n \rightarrow \infty. \)

Then

\[
\frac{1}{n} \sum_{i=1}^{n} X_{ni} \overset{a.s.}\rightarrow 0, \ n \rightarrow \infty. \tag{21}
\]

**Proof.** Our proof is based on an analogous result in [38]. Both Lemmas 1 and 2 from [38] can be easily restated for our setup after replacing the Banach space \( E \) by \( \mathbb{R} \) and linear functionals by scalar multiplication. In addition, note that our condition 1) immediately implies requirement (2.5) from [38]. Now, the reasoning from the proof of Theorem 1 from [38] applies verbatim. \( \square \)

Let \( m_n < n \) be two sequences of natural numbers such that

\[
\frac{m_n}{n} \rightarrow \gamma \in [0, 1). \tag{22}
\]

**Lemma 3** (Central Limit Theorem for Exchangeable Arrays, Theorem 3 from [39]²). Let \( \{X_{ni}\}_{n,i=1}^{\infty,n} \) be a triangular array of row-wise exchangeable random variables such that

²To simplify the notation we assume the number of the elements in the \( n \)-th row to be \( n \) unlike the seemingly more general case of \( k_n \) variables considered in [39].
1) \( \sum_{i=1}^{n} X_{ni} = 0, \quad \forall n \),

2) \( \max_{1 \leq i \leq n} \frac{|X_{ni}|}{\sqrt{n}} \xrightarrow{P} 0, \quad \forall n \),

3) \( \sum_{i=1}^{m_n} X_{ni}^2 \xrightarrow{P} 1, \quad n \to \infty \).

Then

\[
\frac{1}{m_n} \sum_{k=1}^{m_n} X_{ni} \xrightarrow{L} N(0, 1 - \gamma), \quad n \to \infty.
\]

Remark 1. As mentioned earlier this result provides an analog of the CLT for exchangeable sequences. However, it is important to stress its distinction from the classical CLT-type claims for i.i.d. variables. Indeed, Lemma 3 only allows us to consider a subset of the sample of cardinality \( m_n \) smaller than the number of variables \( n \) in the row so that even their ratio must not approach one. This is a reflection of the essential difference between non-extendable exchangeable sequences and their extendable counterparts that include i.i.d. sequences as a particular case. We also emphasize that the restriction of the proposition to subsamples only is not due to limitations of the technical tools used for the proof but a deep phenomenon [37] which can be easily verified empirically, e.g. on the sequences we study below.

V. ADDITIONAL NOTATION AND AUXILIARY RESULTS

Assume that an infinite i.i.d. sequence \( \{x_i\}_{i=1}^{\infty} \) is sampled under \( H_0 \). For every \( n > p \), let the sequence of corresponding Tyler’s estimators be

\[
T_n = \frac{p}{n} \sum_{i=1}^{n} \frac{x_i x_i^\top}{T_n^{-1/2} x_i}, \quad n = p + 1, \ldots,
\]

which exist almost surely for a random sample [40, 41]. Consider a triangular array of row-wise exchangeable random vectors

\[
t_{ni} = \frac{T_n^{-1/2} x_i}{\|T_n^{-1/2} x_i\|} \in S^{p-1}, \quad i = 1, \ldots, n, \quad n = p + 1, \ldots.
\]

Introduce also their row-wise sample averages,

\[
\hat{t}_n = \frac{1}{n} \sum_{i=1}^{n} t_{ni}.
\]

Note that by Definition 1, the sequence \( \{x_i\}_{i=1}^{\infty} \) can equivalently be defined as follows. Given a sequence \( \{w_i\}_{i=1}^{\infty} \sim \mathcal{U}(I_p) \) of uniform i.i.d. random vectors, we look at their transforms

\[
x_i = \frac{\Omega^{1/2} w_i}{\|\Omega^{1/2} w_i\|},
\]

where

\[
\Omega = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top.
\]
for some fixed but unknown $\Omega > 0$. Define also an auxiliary sequence

$$t_{\infty i} = w_i.$$  \hspace{1cm} (28)

**Lemma 4.** With the notation introduced above,

$$t_{ni} \overset{a.s.}{\rightarrow} t_{\infty i}, \ n \rightarrow \infty.$$ \hspace{1cm} (29)

**Proof.** The proof can be found in the Appendix.

We are now interested in the empirical distributions of the rows of the obtained triangular array, which are the finite sets $\{t_{ni}\}_{i=1}^n$ for every fixed $n > p$. As above we assume a sequence $\{m_n\}_{n=p+1}^\infty$ is given satisfying (22) then the following CLT-type result holds in our scenario.

**Proposition 4.** For the triangular array of vectors $\{t_{ni}\}_{n=p,i=1}^{\infty,n}$ defined above,

$$\sqrt{p} \cdot \frac{1}{m_n} \sum_{i=1}^{m_n} t_{ni} - \hat{t}_n \xrightarrow{L} \mathcal{N}(0, (1-\gamma)I_p), \ n \rightarrow \infty.$$ \hspace{1cm} (30)

**Proof.** The proof can be found in the Appendix.

**Corollary 1.** Under $H_0$, for any differentiable function $f : \mathcal{S}^{p-1} \rightarrow \mathbb{R}$,

$$\sqrt{p} \cdot \frac{1}{m_n} \sum_{i=1}^{m_n} f(t_{ni} - \hat{t}_n) \xrightarrow{L} \mathcal{N}(0, (1-\gamma)\|\nabla f(0)\|^2), \ n \rightarrow \infty.$$ \hspace{1cm} (31)

**Proof.** The proof follows the i.i.d. case verbatim using the Maclaurin expansion of $f$.

**VI. ASYMPTOTIC UNIFORMITY TESTS FOR EXCHANGEABLE VECTORS**

In Section III-B we introduced statistics $t_A$ and $t_G$ to test the null hypothesis of uniformity for independent samples over the unit sphere $\mathcal{S}^{p-1}$. Our next statements constitute analogs of those result for the row-wise exchangeable array $\{t_{ni}\}$. Mind the contrast with the i.i.d. case in that we only consider subsamples of the rows due to non-extendability as explained in Remark 1. Note that this restriction cannot be lifted.

Let $\{t_{ni}\}_{n=p,i=1}^{\infty,n}$ be a triangular array as in Proposition 4 and a sequence $m_n < n$ satisfying condition (22) be given. For every $n$ uniformly pick $m_n$ elements of the $n$-th row $\{t_{ni_k}\}_{k=1}^{m_n}$ and consider an additional triangular array

$$\hat{t}_{nk} = t_{ni_k} - \hat{t}_n, \ k = 1, \ldots, m_k, \ n = p + 1, \ldots.$$ \hspace{1cm} (32)
Proposition 5 (Generalized Ajne Test for $\bar{t}_{nk}$). Under $\mathcal{H}_0$, the scaled Ajne statistic
\[
\frac{1}{\sqrt{1-\gamma}} t_A \left( \{\bar{t}_{nk}\} \right) = \frac{1}{\sqrt{1-\gamma}} \left[ \frac{m_n}{2} - \frac{1}{\pi m_n} \sum_{k<l} \arccos(\bar{t}_{nk}^\top \bar{t}_{nl}) \right]
\] (33)
is asymptotically distributed as $L \left( \sum_{q=1}^\infty a_{2q-1}^2 K_{2q-1} \right) \) as $n \to \infty$, where $K_\xi$ are independent random variables distributed as $\chi_\xi^2$ and
\[
a_{2q-1} = \frac{(-1)^{q-1}2^{p-2}\Gamma(\alpha+1)\Gamma(q+\alpha)(2q-2)}{\pi(q-1)!(2q+p-3)!}.
\] (34)

Proof. The proof follows [27] and [28] verbatim using Corollary 1.

Proposition 6 (Generalized Giné Test for $\bar{t}_{nk}$). Under $\mathcal{H}_0$, the scaled Giné statistic
\[
\frac{1}{\sqrt{1-\gamma}} t_G \left( \{\bar{t}_{nk}\} \right) = \frac{1}{\sqrt{1-\gamma}} \left[ \frac{m_n}{2} - \frac{p-1}{2m_n} \left( \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)} \right)^2 \sum_{k<l} \sin(\bar{t}_{nk}^\top \bar{t}_{nl}) \right],
\] (35)
is asymptotically distributed as $L \left( \sum_{q=1}^\infty a_{2q}^2 K_{2q} \right)$, where $K_\xi$ are independent random variables distributed as $\chi_\xi^2$ and
\[
a_{2q} = \frac{(p-1)(2q-1)}{8\pi(2q+p-1)} \left( \frac{\Gamma(\alpha+\frac{1}{2}) \Gamma(q-\frac{1}{2})}{\Gamma(q+\alpha+\frac{1}{2})} \right)^2.
\] (36)

Proof. The proof follows [27] and [28] verbatim using Corollary 1.

Theorem 1 (Uniformity Test for $t_{ni}$). Under $\mathcal{H}_0$, any weighted sum of $t_A \left( \{\bar{t}_{nk}\} \right)$ and $t_G \left( \{\bar{t}_{nk}\} \right)$ is consistent against all alternatives to the asymptotic uniformity of $\{t_{ni}\}$ on $S^{p-1}$.

Proof. Note that we choose the $m_n$ elements of $\bar{t}_{nk}$ uniformly randomly from the $n$ elements of each row. Now the proof follows [27] and [28] verbatim using Propositions 5 and 6.

Similarly to the i.i.d. case discussed in Section III-B, the application of the derived hypothesis test is straightforward. Given a finite sample of $n$ vectors, the statistician computes Tyler’s estimator for them, chooses $\gamma \in (0,1)$ and calculates $m_k = \lceil \gamma n \rceil$. Then they compute the values of the statistics $t_A \left( \{\bar{t}_{nk}\} \right)$ and $t_G \left( \{\bar{t}_{nk}\} \right)$ for a randomly chosen subset of $m_k$ samples and decided on accepting or rejecting $\mathcal{H}_0$ following e.g. the procedure described in Section III-B.

VII. Conclusion

A very common question arising is almost any multi-dimensional statistical application can be briefly formulated as: Is the empirically estimated covariance (scatter) matrix close to the true...
covariance of the population? This natural question has been addressed by numerous publications since the very inception of statistical science. However, all the existing performance bounds clearly rely on numerous assumptions such as normality or any other parametric family of distributions which do not verify in real-world data and are rarely even checked in practice mostly due to the complexity of such tests. In this article, we focus on the family of elliptical distributions leading to ubiquitous robust scatter M-estimators and specifically on the distribution-free within this family Tyler’s estimator. Given the data and making no assumptions on the unknown scatter matrix, we develop a hypothesis test consistent against all alternatives to the ellipticity assumption. On the way to this result we also introduce a novel general framework based on the theory of exchangeable random variables for the analysis of such non-Gaussian cases that can be applied much broadly than covariance estimation.

APPENDIX

Proof of Lemma 4. As shown in Theorem 3.1 from [17],

\[ T_n \xrightarrow{a.s.} \Omega \succ 0, \quad n \to \infty, \]  

(37)

therefore, starting from some \( n_0 \), \( T_n \) is almost surely invertible for \( n \geq n_0 \) and

\[ T_n^{-1/2} \Omega^{1/2} \xrightarrow{a.s.} I_p, \quad n \to \infty. \]  

(38)

Now the claim follows from the definition of the sequence \( \{t_{ni}\}_n \),

\[ t_{ni} = \frac{T_n^{-1/2}x_i}{\|T_n^{-1/2}x_i\|} = \frac{T_n^{-1/2}\Omega^{1/2}w_i}{\|T_n^{-1/2}\Omega^{1/2}w_i\|} \xrightarrow{a.s.} w_i, \quad n \to \infty. \]  

(39)

\hfill \square

Proof of Proposition 4. As above, we can equivalently rewrite \( t_{ni} \) as

\[ t_{ni} = \frac{T_n^{-1/2}\Omega^{1/2}w_i}{\|T_n^{-1/2}\Omega^{1/2}w_i\|}, \quad i = 1, \ldots, n, \quad n = p + 1, \ldots, \]  

(40)

which is just a useful representation as clearly \( \Omega \) is not revealed to the researcher. Fix a vector \( a_1 \in \mathbb{R}^p \) of unit norm \( \|a_1\| = 1 \) and consider the following triangular array of random variables,

\[ 1X_{ni} = \sqrt{p} \cdot a_1^T \left( t_{ni} - \hat{t}_n \right), \quad i = 1, \ldots, n, \quad n = p + 1, \ldots. \]  

(41)

Clearly,

\[ \sum_{i=1}^{n} 1X_{ni} = \sqrt{p} a_1^T \left( \sum_{i=1}^{n} (t_{ni} - \hat{t}_n) \right) = \sqrt{p} a_1^T \left( \sum_{i=1}^{n} t_{ni} - \sum_{i=1}^{n} \hat{t}_n \right) = 0, \]  

(42)
fulfilling requirement 1) of Theorem 3. Note that
\[
\frac{|X_n|}{\sqrt{n}} = \sqrt{\frac{p}{n}} |a_1^\top (t_n - \hat{t}_n)| \leq \sqrt{\frac{p}{n}} \left\| t_n - \hat{t}_n \right\| \leq \sqrt{\frac{p}{n}} \left\| t_n \right\| + \left\| \hat{t}_n \right\| \leq 2 \sqrt{\frac{p}{n}} \to 0,
\] (43)
meaning that condition 2) from Lemma 3 is satisfied as well. To apply Lemma 3 to the array \( \{X_{ni}\}_{ni} \) we need to show that
\[
\frac{1}{m_n} \sum_{i=1}^{m_n} X_{ni}^2 \overset{P}{\to} 1, \quad n \to \infty.
\] (44)
Indeed,
\[
\frac{1}{m_n} \sum_{i=1}^{m_n} X_{ni}^2 = p \frac{1}{m_n} \sum_{i=1}^{m_n} a_1^\top (t_n - \hat{t}_n)(t_n - \hat{t}_n)^\top a_1
\]
\[
= p \operatorname{Tr} \left( a_1 a_1^\top \left[ \frac{1}{m_n} \sum_{i=1}^{m_n} (t_n - \hat{t}_n)(t_n - \hat{t}_n)^\top \right] \right). \tag{45}
\]
Now let us complete vector \( a_1 \) to an orthonormal basis \( \{a_1, \ldots, a_p\} \) of \( \mathbb{R}^p \) and define \( p - 1 \) new arrays of real random variables,
\[
X_{ni} = \sqrt{p} \cdot a_j^\top (t_n - \hat{t}_n), \quad i = 1, \ldots, n, \quad n = p + 1, \ldots, \quad j = 2, \ldots, p, \tag{46}
\]
and the corresponding limits,
\[
q_j = \mathbb{P}\text{-}\lim_{n \to \infty} \frac{1}{m_n} \sum_{i=1}^{m_n} X_{ni}^2, \quad j = 1, \ldots, p. \tag{47}
\]
By the symmetry of the problem,
\[
q_1 = \cdots = q_p, \tag{48}
\]
therefore,
\[
q_1 = \frac{1}{p} \sum_{j=1}^{p} q_j = \mathbb{P}\text{-}\lim_{n \to \infty} \sum_{j=1}^{p} \operatorname{Tr} \left( a_j a_j^\top \left[ \frac{1}{m_n} \sum_{i=1}^{m_n} (t_n - \hat{t}_n)(t_n - \hat{t}_n)^\top \right] \right)
\]
\[
= \mathbb{P}\text{-}\lim_{n \to \infty} \operatorname{Tr} \left( \sum_{j=1}^{p} a_j a_j^\top \left[ \frac{1}{m_n} \sum_{i=1}^{m_n} (t_n - \hat{t}_n)(t_n - \hat{t}_n)^\top \right] \right). \tag{49}
\]
Since the chosen basis of \( \{a_j\} \) is orthonormal,
\[
\sum_{j=1}^{p} a_j a_j^\top = I, \tag{50}
\]
and we conclude,

\[ q_1 = \lim_{n \to \infty} \text{Tr} \left( \frac{1}{m_n} \sum_{i=1}^{m_n} (t_{ni} - \hat{t}_n)(t_{ni} - \hat{t}_n)^\top \right) = \lim_{n \to \infty} \text{Tr} \left( \frac{1}{m_n} \sum_{i=1}^{m_n} t_{ni}t_{ni}^\top - \hat{t}_n\hat{t}_n^\top \right) \]

\[ = \lim_{n \to \infty} \frac{1}{m_n} \sum_{i=1}^{m_n} \|t_{ni}\|^2 - \|\hat{t}_n\|^2 = 1 - \lim_{n \to \infty} \|\hat{t}_n\|^2 = 1, \] (51)

where the last equality follows from Lemma 4.

Now all the conditions are satisfied and the claim follows. \(\square\)

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