Closed similarity Lorentzian affine manifolds.

Abstract.

A Sim(n − 1, 1) affine manifold is a n−dimensional affine manifold whose linear holonomy lies in the similarity Lorentzian group but not in the Lorentzian group. In this paper, we show that a compact Sim(n − 1, 1) affine manifold is incomplete. Let \(<, >_L\) be the Lorentz form, and \(q\) the map on \(\mathbb{R}^n\) defined by \(q(x) = < x, x >_L\). We show that for a compact radiant Sim(n − 1, 1) affine manifold \(M\) whose developing map is injective, if a connected component \(C\) of \(\mathbb{R}^n - q^{-1}(0)\) intersects the image of the universal cover of \(M\) by the developing map, then either \(C\) or a connected component of \(C - H\), where \(H\) is an hyperplane is contained in this image.

Introduction.

An \(n\)-dimensional affine manifold \(M\), is an \(n\)-dimensional differentiable manifold endowed with an atlas whose coordinate changes are locally affine maps. The affine structure of \(M\) pulls back to its universal cover \(\hat{M}\), and defines on it an affine structure determined by a local diffeomorphism \(D: \hat{M} \to \mathbb{R}^n\), called the developing map. The developing map gives rise to a representation \(h: \pi_1(M) \to Aff(\mathbb{R}^n)\), called the holonomy of the affine manifold. Its linear part \(L(h)\), is called the linear holonomy of the affine manifold. We will say that the affine manifold is complete, if and only if the developing map is a diffeomorphism. An \(n\)-affine manifold is said to be radiant if its holonomy fixes an element of \(\mathbb{R}^n\).

We denote by \(O(p, q)\), the subgroup of linear automorphisms of \(\mathbb{R}^n\) which preserve a bilinear symmetric form of type \(p, q\), and by \(Sim(p, q)\) the group generated by \(O(p, q)\) and the homotheties. An \(O(p, q)\) affine manifold \(M\) is an affine manifold \(M\) such that the image of its linear holonomy \(L(h)\) is a subgroup of \(O(p, q)\). An \(Sim(p, q)\) affine manifold \(M\) is an affine manifold \(M\) such that the image of its linear holonomy \(L(h)\) is a subgroup of \(Sim(p, q)\), and contains an element which is not in \(O(p, q)\).

Let consider the flat riemannian torus \(T^n\), Bieberbach has shown that closed \(O(n, 0)\) affine manifolds are finitely covered by \(T^n\). Using the notion of
discompacity, Yves Carrière has shown that closed $O(n - 1, 1)$ affine manifolds are complete. It is obvious that a $\text{Sim}(n, 0)$ affine manifold is incomplete, since an element of its holonomy which doesn’t lie in $O(n, 0)$ fixes an element of $\mathbb{R}^n$. There exist examples of complete $\text{Sim}(n - 1, 1)$ affine manifolds. Let’s give one:

Endow $\mathbb{R}^n$ with its basis $(e_1, ..., e_n)$ and with the lorentzian product defined by

\[ < e_i, e_i >_{L} = 1; 0 < i < n; < e_i, e_j >_{L} = 0; i \neq j; < e_n, e_n >_{L} = -1. \]

We restrict this product to $\mathbb{R}^2$. The affine map whose linear part is

\[
\begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}
\]

in the basis $(e_1 + e_2, e_1 - e_2)$, and whose translation part is $e_1 - e_2$ generates a group which acts properly and freely on $\mathbb{R}^2$.

The goal of this paper is to study closed $\text{Sim}(n - 1, 1)$ affine manifolds. First we show:

**Theorem 1.** A compact $\text{Sim}(n - 1, 1)$ affine manifold is incomplete.

After, using the notion of discompacity, we show

**Theorem 2.** Let $M$ be a closed radiant $\text{Sim}(n - 1, 1)$ affine manifold whose developing map is injective, if a connected component $C$ of $\mathbb{R}^n - q^{-1}(0)$ intersects $D(M)$, then either $C$ is contained in $D(M)$ or a connected component of $C - H$, where $H$ is an hyperplane.

Interesting structures of $\text{Sim}(n - 1, 1)$ affine manifolds can be constructed using the work of Goldman on projective structures on surfaces see [Gol]. For instance a $\text{Sim}(2, 1)$ structure which linear holonomy is Zariski dense in $\text{Gl}(3, \mathbb{R})$ is given in this paper.

1. **Closed $\text{Sim}(n - 1, 1)$ affine manifolds are incomplete.**

The main goal of this part is to show that a closed $\text{Sim}(n - 1, 1)$ affine manifold cannot be complete.

Let suppose that there exists a complete closed $\text{Sim}(n - 1, 1)$ affine manifold $M$; $M$ is the quotient of $\mathbb{R}^n$ by a subgroup of affine transformations $\Gamma$, whose linear part is contained in $\text{Sim}(n - 1, 1)$.

**Lemma 1.1.** Let $\gamma$ be an element of $\Gamma$ whose linear part has a determinant < 1. Then there exists a basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$ such that the linear part of $\gamma$ in this basis has the following form:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\lambda} & 0 \\
0 & 0 & \frac{1}{\lambda} B^n
\end{pmatrix}
\]

where $\lambda$ is a real number strictly superior to 1 in absolute value, and $B^n$ is a matrix which preserves the restriction of an euclidean product to the sub vector space generated by $e_3, ..., e_n$. 

2
Proof.
We have supposed that the determinant of the linear part $L(\gamma)$ of $\gamma$, is strictly inferior to 1 in absolute value. This implies that there exists a real number $\lambda > 1$ such that $\lambda L(\gamma) = L(\gamma)'$, where $L(\gamma)'$ is an element of $O(n-1,1)$. The linear map $L(\gamma)$ has 1 as eigenvalue, since $\gamma$ acts freely. We deduce that $\lambda$ is an eigenvalue of $L(\gamma)'$. We remark that $L(\gamma)'$ has another eigenvalue $\alpha$ which module is different from 1 and the module of $\lambda$ since the absolute value of its determinant is 1. If $\alpha$ is not a real number, then $\alpha$ and its complex conjugated $\bar{\alpha}$ are eigenvalues associated to the complex eigenvectors $u_1$ and $u_2$. In this case the restriction of $L(\gamma)'$ to the plane generated by $u_1 + u_2$ and $i(u_1 - u_2)$ is an euclidean similitude whose ratio is different from 1. This is impossible since $L(\gamma)'$ lies in $O(n-1,1)$. Let $v_1$ and $v_2$ be the eigenvectors associated to $\lambda$ and $\alpha$ and $\langle , \rangle_L$ the lorentzian product preserved by the linear holonomy. We have:

$$\langle v_1, v_1 \rangle_L = \langle v_2, v_2 \rangle_L = 0.$$  

We deduce that the restriction of $\langle , \rangle_L$ to the plane $P$ generated by $v_1$ and $v_2$ is nondegenerate and has signature $(1,1)$. This implies that the restriction of $\langle , \rangle_L$ to the orthogonal $W$ of $P$ with respect to itself is a scalar product. The restriction $B^\circ$ of $L(\gamma)'$ to $W$ is an orthogonal linear map. We can suppose that its determinant is 1. We deduce that $\alpha = \frac{1}{\lambda}$.

Up to a change of origin, we can suppose that $\gamma(0) = (a_1, 0, ..., 0)$ where $a_1$ is a real number. The restriction $B$ of $L(\gamma)$ to the linear subspace generated by $(e_2, ..., e_n)$ is strictly contracting. It is easy to show that the group generated by $\gamma$ is not cocompact, so $\Gamma$ contains another element $\gamma_1$ different from $\gamma$.

Lemma 1.2. Let $C$ be the linear part of $\gamma_1$, then $C(e_1) = e_1 + b$ where $b$ lies in the linear subspace generated by $e_2, ..., e_n$.

Proof.
Let $k$ be an element of $I$. Consider the element $\gamma^k \gamma_1$. Its linear part has 1 as eigenvalue. The matrix of this linear part in the basis $(e_1, ..., e_n)$ is $A_k C$, where $A_k$ is the matrix of the linear part of $\gamma$. Let $u_k$ be an eigenvector of $A_k C$ associated to 1. We assume that the norm of $u_k$ with respect to the euclidean scalar product defined by $\langle e_i, e_j \rangle = \delta_{ij}$ is 1. Let $u_k = (u_k^1, u_k^n)$. We have $C(u_k) = (v_k^1, v_k^n)$, where $u_k^1$ and $v_k^1$ are elements of $R$, and $u_k^n$ and $v_k^n$ are elements of the vector space $F$ generated by $e_2, ..., e_n$. We have $v_k^1 = u_k^1$ since $A_k(e_1) = e_1$, and $A_k$ preserves $F$. Since $B$ is strictly contracting the norm of $A_k(0, v_k^n)$ goes to 0 with respect to the euclidean norm. So $u_k$ goes to $e_1$ and $C(e_1)$, which is the limit of $C(u_k) = A^{-k}(u_k)$ is $e_1 + b$ where $b$ is an element of $F$.

First proof of the theorem 1.
Let $c$ be the translational part of $\gamma_1$ in the basis $(c_1, ..., c_n)$. Put $c = (c_1, ..., c_n)$. We have $\gamma^k \circ \gamma_1 \circ \gamma^{-k}(0) = (c_1, B^k(-ka_1 b + (c_2, ..., c_n)))$. Since $B$ is contracting and the action of $\Gamma$ is proper, we deduce that $(c_2, ..., c_n) = b = 0$. This implies that $\gamma^m \gamma_1^\circ(0) = (na_1 + mc_1)c_1$. Since the action of $\Gamma$ on $\mathbb{R}^n$ is
proper and free we deduce that the subgroup \( \{ n, m \in \mathbb{Z}, na_1 + mc_1 \} \) is discrete. We deduce from this fact that there exist \( p, q \in \mathbb{Z} \) such that \( pa_1 + qc_1 = 0 \). This implies that \( \gamma^p = \gamma_1^{-q} \).

Let \( K \) be a fundamental domain of the action of \( \Gamma \), recall that it is a compact such that for each \( x \in \mathbb{R}^n \), there exists an element \( \gamma_0 \) of \( \Gamma \) such that \( \gamma_0(x) \in K \), and for each element \( \gamma \) of \( \Gamma \), \( \gamma(K^0) \cap K^0 \) is empty, where \( K^0 \) is the interior of \( K \). Let \( ||, || \) be a norm associated to \( <, > \), the scalar product for which \((e_1, \ldots, e_n)\) is an orthonormal basis. There exists a real \( A > 0 \) and an integer \( l \) such that \( || K || < A \) and \( l \mid a_1 \mid > A \). Consider the element \( u = (0, y_0) \) of \( \mathbb{R}^n \) where \( y_0 \) is an element in the vector subspace generated by \((e_3, \ldots, e_n)\) if \( n \geq 3 \), otherwise \( y_0 \) is an element of \( \text{Vect}(e_2) \) such that \( ||B^l(y_0)|| > A \). There must exist an element \( \gamma_0 \) in \( \Gamma \) such that \( \gamma_0(u) \) is in \( K \). We have shown that there exist elements \( p, q \) in \( \mathbb{Z} \) such that \( \gamma_0^p = \gamma_1^{-q} \). Denote by \( D \) the restriction of \( \gamma_0 \) to \( \text{Vect}(e_2, \ldots, e_n) \), and \( d_1 \) the real number such that \( \gamma_0(0) = d_1e_1 \). We have \( D^p = B^q \) and \( pd_1 = qa_1 \). Since \( || y_0 || > A \), \( D \) is contractant and \( q > p \). It results from lemma 1.1 that the restriction of \( D \) and \( B \) to \( \text{Vect}(e_3, \ldots, e_n) \) are similarities of respective ratio \( r_d \) and \( r_b \) (If the dimension is 2, we consider the restriction of \( D \) and \( B \) to \( \text{Vect}(e_2) \) which is a similarity). We have \( (r_d)^p = (r_b)^q \). Moreover \( ||B^l(y_0)|| = (r_b)^l \mid y_0 \mid > A \), and \( A > ||D(y_0)|| = r_d \mid y_0 \mid \). We deduce that \( (r_b)^l > r_d \). This implies that \( (r_b)^q > (r_d)^q \), which is equivalent to saying that \( (r_b)^q > (r_d)^p \). This implies that \( q > lp \).

\( p \) and \( q \) are elements of \( \mathbb{N} \). We have that \( \gamma_0(u) = (d_1, D(y_0)) \in K \). This implies that \( A > \mid d_1 \mid \). But we have \( qA > lpA > lp \mid d_1 \mid = lq \mid a_1 \mid \). Which is a contradiction since we have supposed that \( l \mid a_1 \mid > A \).

We can also deduce the proof of Theorem 1, from this deep result

**Theorem [F-G-H].**

Let \( M \) a compact affine manifold, which is the quotient of \( \mathbb{R}^n \) by the group \( \Gamma \) which acts properly and freely; then \( \Gamma \) does not preserve proper affine subspaces.

**Proof.** Up to a finite cover, we can assume that \( M \) is oriented. We denote by \((C(M), d)\) the simplicial complex we define the simplicial homology of \( M \). We can lift the simplicial decomposition to \( \mathbb{R}^n \), and thus lift the simplicial complex \((C(M), d)\) to the simplicial complex \((C(\mathbb{R}^n, d'))\). This last complex has a structure of a \( \mathbb{Z} \Gamma \) module. Its homology is trivial since \( \mathbb{R}^n \) is contractible. We deduce that it is a resolution of \( \Gamma \). We can use this resolution to calculate the real cohomology of \( \Gamma \). The cohomology obtained is also the real cohomology of \( M \). The De Rham theorem implies that \( H^{DR}(M, \mathbb{R}) = H^n(\Gamma, \mathbb{R}) = H^n(\Gamma, \mathbb{R}) = \mathbb{R} \) since we have supposed that \( M \) is oriented. Suppose that \( \Gamma \) preserves an \( l \)-affine subspace \( F \) of \( \mathbb{R}^n \). We denote by \( N \) the quotient of \( F \) by \( \Gamma \). We have also that \( H^D_{DR}(N, \mathbb{R}) = H^n(\Gamma, \mathbb{R}) = \mathbb{R} \). Since \( N \) is compact, its implies that \( l \geq n \), we deduce that \( l = n \).

**Second proof of theorem 1.**

Let \( c \) be the translational part of \( \gamma_1 \) as in the first proof, we can remark that in the basis \((e_1, \ldots, e_n)\), we have \( c = (c_1, 0, \ldots, 0) \) this implies that \( \Gamma \) preserves the line \( \mathbb{R}e_1 \). This fact contradicts the previous theorem.
Remark.

The both proofs of theorem 1 are related. While proving theorem 1, we have shown that for every element \( \gamma_1 \) in \( \Gamma \), there exists \( p, q \) in \( \mathbb{Z} \) such that \( \gamma_1^p = \gamma_q \). This implies that the quotient of \( \Gamma \) by the group generated by \( \Gamma \) is a torsion group.

We deduce that the real cohomological dimension of \( \Gamma \) is one which is contrary to the fact that up to a finite cover \( H^n(M, \mathbb{R}) = H^n(\Gamma, \mathbb{R}) = \mathbb{R} \).

In contrast to the \( \text{Sim}(n, 0) \) affine manifolds (See [Fr] theorem 1), there exist compact \( \text{Sim}(n - 1, 1) \) affine manifolds which are not radiant. Here is an example.

Endow \( \mathbb{R}^2 \) with the Lorentzian product \( (, ) \) such that \( (e_1, e_1) = (e_2, e_2) = 0 \), and \( (e_1, e_2) = 1 \).

Consider the subgroup \( \Gamma \) of \( \text{Aff}(\mathbb{R}^2) \) generated by the following transformations:

\[
\gamma_1(x, y) = (x + 1, y), \\
\gamma_2(x + y) = (x, 2y),
\]

the quotient of \( \mathbb{R} \times (\mathbb{R} - \{0\}) \) by \( \Gamma \) is a compact \( \text{Sim}(n - 1, 1) \) affine manifold.

2. On the universal cover of compact \( \text{Sim}(n - 1, 1) \) affine manifolds.

In this part we are going to find properties of the universal cover of a closed radiant \( \text{Sim}(n - 1, 1) \) affine manifold. We use the notion of discompacity defined by Carrière [Car] 2.2.1. Let us recall it.

We consider in \( \mathbb{R}^n \) the unit ball \( B_n \). The euclidean metric induces on closed subsets of \( \mathbb{R}^n \) the Hausdorff distance. Let \( G \) be a subgroup of \( GL(n, \mathbb{R}) \), and \( (g_p)_{p \in \mathbb{N}} \) a sequence of elements of \( G \). The limit of the family \( (g_p(B_n) \cap B_n)_{p \in \mathbb{N}} \) converges in \( B_n \). It is a degenerated ellipsoid (see [Car]). The codimension of this ellipsoid is the discompacity \( d \), of the family \( (g_p)_{p \in \mathbb{N}} \), the discompacity of the group with respect to the euclidean metric is the smallest \( d \).

Obviously we cannot use the notion of discompacity in this form since the linear holonomy of our manifold may contain homotheties. Denote \( q : \mathbb{R}^n \to \mathbb{R} \) \( x \to < x, x >_L \). We can define in \( \mathbb{R}^n - q^{-1}(0) \) the metric

\[
(u, v) \to < u, v >_c = \frac{< u, v >_\text{euc}}{q(x)}
\]

where \( u \) and \( v \) are vectors of the tangent space at \( x \) and \( <, >_\text{euc} \) is the euclidean scalar product.

Theorem 2.1.

Let \( \tilde{x} \) be an element of \( \tilde{M} \), and \( u \) and \( v \), elements of \( T_{\tilde{x}}\tilde{M} \), such that the geodesics \( c_1 : [0, 1] \to \tilde{M}, t \to \exp_{\tilde{x}}(tu) \), and the one \( c_2 : [0, 1] \to \tilde{M}, t \to \exp_{\tilde{x}}(tv) \) are defined. Suppose that the elements \( \exp_{\tilde{x}}(u) \) and \( \exp_{\tilde{x}}(v) \) can't be joined by a geodesic, but for every \( t, t' < 1 \), there is a geodesic between \( \exp_{\tilde{x}}(tu) \) and \( \exp_{\tilde{x}}(t'v) \). Let \( c : [0, 1] \to \mathbb{R}^n, t \to \exp_D(\exp_{\tilde{x}}(u))(tw) \) be the geodesic between \( \exp_{\tilde{x}}(tu) \) and \( \exp_{\tilde{x}}(tv) \), and let \( U_{\tilde{x}} \) be the domain of definition of \( \exp_{\tilde{x}} \). Consider the element \( t_0 \in [0, 1] \) such that for every \( t < t_0 \), \( \exp_D(\exp_{\tilde{x}}(u))(tw) \)
is an element of $D(exp_{\hat{z}}(U_{\hat{z}}))$ but not is $exp_{D(exp_{\hat{z}}(u))}(t_0w) = y$. Then $y$ is an element of $q^{-1}(0)$.

**Proof.**

There is a geodesic $\hat{c}_3 : [0,1] \rightarrow \hat{M}$, $t \rightarrow exp_{\hat{z}}(tb)$ such that $y$ is an element of the adherence of $D(\hat{c}_3([0,1]))$ and such that $D(\hat{c}_3)$ is contained in the convex hull of $D(\hat{c}_1)$ and $D(\hat{c}_2)$, where $\hat{c}_1$, and $\hat{c}_2$ are geodesics of $\hat{M}$ respectively above $c_1$ and $c_2$. Set $p(\hat{x}) = x$, the image $c_3$ of $p(\hat{c}_3)$ is a maximal incomplete geodesic of $M$. Since $M$ is compact, there exists an element $z$ of $M$ such that the geodesic $c_3$ is recurrent in an affine chart $U$ which contains $z$. We deduce as Carrière, the existence of a family of ellipsoids $s_p$ of $R^n$ whose centers are elements of $D(\hat{c}_3)$, such that for each $p,p'$, there is an element $\gamma_{p,p'}$ of the holonomy such that $\gamma_{p,p'}(s_p) = s_{p'}$ and the centers $x_p$ of $s_p$ goes to $y$.

Suppose that $y$ is not an element of $q^{-1}(0)$.

Let $z_p$ be an element of an ellipsoid $s_p$, and $u_p$, $v_p$ two vectors in its tangent space. Put $\gamma_{p,p'} = \lambda_{p,p'}g_{p,p'}$ where $g_{p,p'}$ is an element of $O(n-1,1)$. We have:

$$\frac{<\gamma_{p,p'}(u_p),\gamma_{p,p'}(v_p)>_{euc}}{q(\gamma_{p,p'}(x))} = \frac{<g_{p,p'}(u_p),g_{p,p'}(v_p)>_{euc}}{q(x)}.$$

Since the holonomy of $M$ is supposed to be radiant.

The metrics $<,>_{euc}$ and $<,>'$ are equivalent in a neighborhood of $y$ since $q(y)$ is different from 0. We know that the discompacity of the family of $g_p$ in respect to the riemannian metric $<,>_{euc}$ is 1. The family of ellipsoids $s_p$ goes to an ellipsoid, or a codimension 1 degenerated ellipsoid centered in $y$. We conclude as in Carrière that $y$ must be an element of $D(exp_{\hat{z}}(U_{\hat{z}}))$. This is not possible, so $q(y) = 0$.

A similar result is given in [Gol].

**Corollary 2.2.** Let $M$ be a compact radiant $Sim(n−1,1)$ affine manifold, let $\hat{z}$, $u$, and $v$ be respectively elements of $\hat{M}$ and $T_{\hat{z}}\hat{M}$, such that $exp_{\hat{z}}(u)$ and $exp_{\hat{z}}(v)$ are defined. If the convex hull $E$ of $(D(\hat{z}), D(exp_{\hat{z}}(u)), D(exp_{\hat{z}}(v)))$ is contained in a connected component of $R^n - q^{-1}(0)$, then it is contained in $D(exp_{\hat{z}}(U_{\hat{z}}))$.

**Proof.**

Suppose that $E$ is not contained in $D(exp_{\hat{z}}(U_{\hat{z}}))$. Let $y$ and $z$ be two elements of $D(\cap D(exp_{\hat{z}}(U_{\hat{z}})))$ such that $y = D(exp_{\hat{z}}(u_1))$, $z = D(exp_{\hat{z}}(u_2))$, and for every $t_1,t_2 < 1$, $exp_{\hat{z}}$ is defined on the convex hull of $0, tu_1, tu_2$, but the elements $exp_{\hat{z}}(u_1)$ and $exp_{\hat{z}}(u_2)$ cannot be joined by a geodesic. Consider the geodesic $c : [0,1] \rightarrow R^n$, $t \rightarrow exp_{\hat{z}}(tw)$ between $y$ and $z$. There exists a real number $0 < t_0 < 1$, such that for $0 < t < t_0$, $exp_{\hat{z}}(tw)$ lies in $D(exp_{\hat{z}}(U_{\hat{z}}))$, but not $exp_{\hat{z}}(t_0w)$. We deduce from the theorem 2.1. that $exp_{\hat{z}}(t_0w)$ must lie in $q^{-1}(0)$. This is contrary to the hypothesis.

More generally, we can determine, how the boundary of the image of the developing map of a compact radiant $Sim(n−1,1)$ affine manifold is: more precisely, we have the following proposition which implies theorem 2:
Proposition 2.3. Let \( M \) be a compact radiant \( \text{Sim}(n-1,1) \) affine manifold whose developing map is injective, the boundary of \( D(\hat{M}) \), is contained in the union of \( q^{-1}(0) \) and an hyperplane.

Proof. As in [Car] p. 625, one can remark that elements of the boundary of \( D(\hat{M}) \) which are not elements of \( q^{-1}(0) \) are limits of \( (\gamma_n e)_n \in \mathbb{N} \), where \( \gamma_n \) is an element of the holonomy and \( e \) is an ellipsoid. We conclude that those elements are contained in at most two hyperplane \( H_1, H_2 \). The case of two hyperplane is impossible, since those hyperplane are stable by the holonomy, the affine function \( \alpha \) such that \( \alpha(H_1) = 0 \) and \( \alpha(H_2) = 1 \), will be invariant by the holonomy and so define a differentiable function on \( M \) without maximal. (It is the same argument used in [Car]).

Proposition 2.4. Let \( M \) be a compact radiant affine manifold, if the image of the developing map is a convex set contained in an open set of \( \mathbb{R}^n - q^{-1}(0) \), then the developing map is injective.

Proof. Let \( \hat{x} \) be an element of \( \hat{M} \). For every elements, \( u \) and \( v \) of \( U_{\hat{x}} \), the convex hull of \( D(\hat{x}), y = D(exp_{\hat{x}}(u)) \) and \( z = D(exp_{\hat{x}}(v)) \) is a subset of \( D(\hat{M}) \cap (\mathbb{R}^n - q^{-1}(0)) \). We deduce from the corollary 2.2 that \( y \) and \( z \) are elements of \( D(U_{\hat{x}}) \). This implies that \( U_{\hat{x}} \) is a convex set. We can conclude by using [Kos].

A particular case of the situation of corollary 2.4 is the following: endow a compact oriented surface \( S \) of genus \( \geq 2 \), with an hyperbolic structure, and consider \( q \) the Lorentzian form defined on \( \mathbb{R}^3 \) by \( q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 \). The hyperbolic structure can be defined by a representation of the fundamental group of \( S \), \( \pi_1(S) \rightarrow O(2,1) \) such that the quotient of \( H = q^{-1}(-1) \) by \( \pi_1(S) \) is \( S \). The quotient of \( W = \{ x : q(x) < 0, x_3 > 0 \} \) by the group generated by \( \pi_1(S) \), and a homothetic of ratio \( 0 < \lambda < 1 \), is a compact \( \text{Sim}(n-1,1) \) affine manifold whose universal cover is \( W \).

More generally we have

Corollary 2.5. Let \( M \) be a radiant compact affine manifold such that the image of its developing map is contained in \( W = \{ x : q(x) < 0, z > 0 \} \), \( M \) is the quotient of a connected component of \( W - H \) by a discrete group of \( \text{Sim}(n-1,1) \), where \( H \) is an hyperplane of \( \mathbb{R}^n \).

Proof. We remark that the interior of a connected component of \( W - H \) is convex. This implies the image of the developing map is a convex set. The result follows using 2.3 and 2.4.

Let \( M \) be a compact radiant \( \text{Sim}(n-1,1) \) affine manifold, the foliation \( D(\tilde{F}_q) \) of \( \mathbb{R}^n - \{0\} \) whose leaves are the sub manifolds defined by \( q = \text{constant} \) is invariant by the holonomy of \( M \). Its pull back on \( \hat{M} \) defines a foliation \( \tilde{F}_q \) of \( \hat{M} \), which gives rise to a foliation \( F_q \) of \( M \). If \( D(\tilde{N}) = D(\hat{M}) \cap q^{-1}(0) \) is not empty, then \( N = p(D^{-1}(D(\tilde{N}))) \) is a compact submanifold of \( M \). Note that the 1–parameter group \( \phi_t \) generated by the radiant vector field \( X_0 \), preserves
the foliation $\mathcal{F}_q$, and is transverse to all the leaves but not to the connected components of $N$.

**Proposition 2.6.** If $N$ is empty, then $M$ is the total space of a bundle over $S^1$.

**Proof.**
If $N$ is empty, then $\phi_t$ is transverse to the foliation $\mathcal{F}_q$. This implies that this foliation is a Lie foliation. We conclude using [God] Corollary 2.6 p. 154.

**Remark.**
Recall that a riemannian foliation on a manifold $M$, is a foliation $\mathcal{F}$ of $M$ such that there exists a riemannian metric of $M$ which projects locally along the leaves of $M$, this is equivalent to saying that locally the distance between the leaves is well-defined, or the foliation is defined by locally submersions $M \to N$ which transitions functions preserve a riemannian metric of $N$.

Suppose that the image of the developing map is included in the upper cone $C = \{ x/q(x) < 0, x_n > 0 \}$, one can define a map $f : C \to q^{-1}(-1)$ such that $f(x)$ is the element of $C$ colinear to $x$ such that $q(f(x)) = -1$. Let $(,)_L$ be the flat Lorentzian product of $\mathbb{R}^n$. The restriction of the lorentzian product defined by $(u,v)_x = \frac{(u,v)_L}{(x,x)_L}$ to $q^{-1}(-1)$ is an hyperbolic metric. This endows the radial flow $\phi_t$ of $C$ with a transverse riemannian structure. Since the holonomy of the manifold is included in $\mathbb{RO}(n-1,1)$, the riemannian structure of $\phi_t$ pushes forward to a riemannian structure of its radial flow $\phi_t$. In fact the flow $\phi_t$ is also a transversally $(O(n-1,1),q^{-1}(-1))$ homogeneous foliation where $O(n-1,1)$. This enables one to define the global holonomy $h_{\phi} : \pi_1(M) \to O(n-1,1)$. It assigns to any element $\gamma$ of $\pi_1(M)$ (identified in this case as a subgroup of $\mathbb{RO}(n-1,1)$), the element $t\gamma(t \in \mathbb{R})$ such that $t\gamma$ is an element of $O(n-1,1)$ and it preserves $C$.

The riemannian foliation have been intensively studied, It has been shown by Molino that the adherence of a leaf of a riemannian foliation defined on a compact manifold is a submanifold. Carriere and Carron have shown that in the case of riemannian flows, the adherence of leaves are torus.

More precisely, for a riemannian foliation, let denote by $(M_1,\mathcal{F}_1)$ the bundle of transverse orthogonal frames of $M$ endowed with the pulls back $\mathcal{F}_1$ of $\mathcal{F}$, one can define the sheaf of local vector fields of $M_1$ which commute with the global foliated vector fields of $\mathcal{F}_1$ This sheaf pushes forward to a sheaf $C(M,\mathcal{F})$ of $M$. It is called the commuting sheaf of $\mathcal{F}$. The second structure theorem of riemannian foliations says that the adherences of the leaves of $\mathcal{F}$ are orbits of the pseudogroup defined by $C(M,\mathcal{F})$. In the case of a riemannian flow, the Lie algebra $C(M,\mathcal{F})$ is commutative.

In the other hand consider the adherence $L$ of the image of $h_{\phi}$ in $O(n-1,1)$.

If the image of $h_{\phi}$ is discrete, it implies that the orbit of $\phi_t$ are closed, the holonomy of a leaf of $\phi_t$ is finite, this implies that up to a finite cover we can consider that the foliation $\phi_t$ does not have holonomy. This finite cover of $M$ is the total space of a bundle whose typical fiber is an hyperbolic manifold. This manifold is the quotient of $q^{-1}(-1)$ by the image of $h_{\phi}$.
If $L$ is not discrete, then its lie algebra $l$ is isomorphic to $C(M, \mathcal{F})$ see [W], we deduce that this Lie algebra is commutative. This imply that the connected component $L_0$ of $L$ is a non trivial commutative group. Since $\pi_1(M)$ normalizes $L_0$, using [G-K] 1.3 one can conclude that $\pi_1(M)$ is a solvable group. We have shown:

**Theorem.**

Suppose that the image of the developing map of a $\text{Sim}(n-1, 1)$ compact radiant affine manifold is contained in $C = \{x \in \mathbb{R}^n, 0 > q(x), \text{and} x_n > 0\}$, then if $\pi_1(M)$ is not solvable, the leaves of the radiant flow are compact.

**Remark.**

The previous result is a particular case of a result due to Epstein for transversely hyperbolic foliation. Moreover if Epstein shows that if the leaves of the radiant flow are not compact, then the dimension of $M$ is 3 or 4, and $M$ is a the quotient of a solvable group $G$ endowed with a left symmetric structure.

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