Point particle in general background fields vs.
free gauge theories of traceless symmetric tensors

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Abstract

Point particle may interact to traceless symmetric tensors of arbitrary rank. Free gauge theories of traceless symmetric tensors are constructed, that provides a possibility for a new type of interactions, when particles exchange by those gauge fields. The gauge theories are parameterized by the particle’s mass $m$ and otherwise are unique for each rank $s$. For $m = 0$, they are local gauge models with actions of $2s$-th order in derivatives, known in $d = 4$ as ”pure spin”, or ”conformal higher spin” actions by Fradkin and Tseytlin. For $m \neq 0$, each rank-$s$ model undergoes a unique nonlocal deformation which entangles fields of all ranks, starting from $s$. There exists a nonlocal transform which maps $m \neq 0$ theories onto $m = 0$ ones, however, this map degenerates at some $m \neq 0$ fields whose polarizations are determined by zeros of Bessel functions. Conformal covariance properties of the $m = 0$ models are analyzed, the space of gauge fields is shown to admit an action of an infinite-dimensional ”conformal higher spin” Lie algebra which leaves gauge transformations intact.
1 Introduction and background.

The study of point particles living in background of electromagnetic and gravitational fields represented by rank-1 and rank-2 tensor fields, is basic in General Relativity. On the other hand, point particles may experience an influence of higher rank symmetric tensors.

In 1980, De Wit and Freedman had addressed the generalization of the point particle dynamics in gravitational and electromagnetic background to the case when higher rank symmetric tensors are switched on [1]. Specifically, they had considered the action

\[ S [x(\tau) \mid h_{m(k)}] = - \int d\tau \sqrt{-m^2 \dot{x}^2} \left( 1 + \frac{\epsilon_k}{m^2} h_{m_1...m_k}(x) \dot{x}^{m_1}...\dot{x}^{m_k} \left( -\frac{m^2}{\dot{x}^2} \right)^{\frac{k}{2}} \right), \]  

(1)

where \( x^m(\tau), m = 0, 1, ...d-1 \) represent the particle’s world-line, \( h_{m_1...m_k}(x) \) are symmetric tensor fields and \( \epsilon_k \) are corresponding coupling constants, while \( \dot{x}^2 = g_{mn}(x) \dot{x}^m \dot{x}^n \) with a general metric \( g_{mn}(x) \). The action is clearly invariant under world-line reparametrizations and thereby governs some good point particle’s space-time evolution.

It was demonstrated also that the action possesses the first-order invariance w.r.t following simultaneous transformations of background fields \( h_{m_1...m_k}(x) \) and particle’s world line\(^1\)

\[ S [x + \delta x \mid h_{m(k)} + \delta h_{m(k)}] = S [x \mid h_{m(k)}] + o(\epsilon_k^2) \]

(2)

\[ \delta h_{m(k)}(x) = \epsilon_{m(k-1); m} \]

\[ \delta x^m(\tau) = (k-1) \epsilon_k \epsilon_{n_1...n_{k-2}} \dot{x}^{n_1}...\dot{x}^{n_{k-2}} \left( -\frac{\dot{x}^2}{m^2} \right)^{1-k/2}, \]

where \( \epsilon_{m(k-1)}(x)\(^2\) are arbitrary functions of \( x^m \), ’’; ’’ denotes the covariant derivative compatible with the metric \( g_{mn} \).

For \( k = 1, 2 \), one gets the standard gauge transformations for the fluctuations of Maxwell and gravitational fields

\[ \delta h_m(x) = \epsilon_{,m} \quad \delta h_{m(2)}(x) = \epsilon_{m,m}, \]

(3)

so, if the higher \( (k > 2) \) fields are set zero, the action \(^1\) may be viewed as that describing the first-order interaction of the particle to general fluctuations of gravitational and Maxwell fields.

Clearly, the \( k = 1, 2 \) transformations for background fields \(^3\) present linearization of full \( U(1) \) and general coordinate ones. A natural question is then what is a nonlinear

\(^1\)On our notation: we use signature \((- + +...+)\), alternative as compared to DeWit and Freedman’s paper and re-introduce the mass parameter \( m \) explicitly. The action \(^1\) coincides with the De Wit-Freedman \((DW-F)\) one \(^1\) after the identification \( \varphi_{m_1...m_k}^{DW-F} = -h_{m_1...m_k} \) and setting \( m^2 = -1 \) (the negative sign of \( m^2 \) just accounts the difference in metric’s signature, \( \sqrt{x^2} \mid_{DW-F} = \sqrt{-m^2 x^2} \mid_{our} \)).

\(^2\)Whenever indices denoted by the same letter appear in our paper, their full symmetrization is implied, the symbols like \( m(s) \) stand for \( m_1...m_s \).

\(^3\)The parameters \( \epsilon_{m(k-1)} \) correspond to gauge parameters \( \xi_{m(k-1)} \) of \(^1\) as \( \epsilon_{m(k-1)} = k\xi_{m(k-1)}, k = 1, 2... \)
generalization of linearized higher-rank transformations \( (2) \). This question has recently been answered in the author’s paper \[2\]. It appears that, if one passes to Hamiltonian formalism, the action \( (1) \) turns out to present the first order approximation to the general Hamiltonian action of the point particle (without higher derivatives), while the symmetries \( (2) \) are nothing but canonical transformations of the particle’s phase space, with higher order terms thrown away, this automatically insures the invariance of the action \( (1) \). The nonlinear invariance identified in \[2\] is a semidirect product of all canonical transformations to an abelian ideal of “hyperWeyl” transformations. The origin of these “hyperWeyl” transformations is due to the fact being exposed already in the first order action \( (1) \): the trace-part of every rank-\( k \) tensor provides exactly the same contribution to the particle action as a rank-(\( k - 2 \)) tensor. So, if one considers interaction of the particle to all symmetric tensors altogether, only traceless parts of rank-\( k \) tensors are actually involved. Below we will provide, following \[2\], a brief derivation of all these matters in the Hamiltonian formalism, which, in our opinion, most simply incorporates all essential properties of the model and, besides, includes naturally the massless case \( m = 0 \).

The second natural question is whether there exist some dynamical equations, consistent with linearized gauge invariance, for higher rank background fields (that is something analogous to Einstein and Maxwell equations for low rank tensors). If such equations do exist \textit{at least at the first-order level} then it becomes possible to consider first order processes like “the particle ”A” emits a higher rank field ”F” which propagates through the space-time according to its linearized field equations and then hits the particle ”B””, and thereby point particle would interact to each other by means of higher rank symmetric tensors.

\textbf{In this paper}, we address this second question, in arbitrary space-time dimension \( d \). Our starting point is the first-order gauge transformations (with the \textit{flat} metric background) for the infinite system of symmetric traceless tensors derived in \[2\]. Our target is a gauge invariant and Poincaré invariant quadratic action for these fields. We show that, for \( m = 0 \), for each rank \( s \), there exists a local action of \( 2s \)-th order in derivatives, which is \textit{unique} (modulo fields redefinitions of a trivial type) and scales homogeneously under dilations. We call it spin-\( s \) \textit{traceless higher spin theory}. In \( d = 4 \), this model coincides with ”pure spin-\( s \) model”, described by Fradkin and Tseytlin and conjectured to be conformally invariant \[3\]. These higher derivative models should not be confused with second order Fronsdal higher spin theories \[8\] described in terms of \textit{double-traceless} tensors.

For \( m \neq 0 \), each traceless higher spin-\( s \) theory undergoes a deformation to a \textit{unique} and \textit{nonlocal} one (with the nonlocality governed by the \( m^2 \) operator), which mixes fields of all ranks, starting from \( s \), and reduces to corresponding local traceless higher spin theory at the point \( m = 0 \). We have found a nonlocal transform that maps \( m \neq 0 \) models to \( m = 0 \) ones, however it degenerates at some \( m \neq 0 \) fields which may be of importance. Therefore, it may be worth studying these theories in the original basis, where they are nonlocal, and name them differently from \( m = 0 \) models. We will call these deformed models ”\textit{deformed traceless higher spin theories}”. According to the above reasoning, all these theories are of interest, as they can govern unique first order interactions of either massless point particles via traceless higher spin fields or massive point particles via double-traceless tensors.\footnote{Also, the Hamiltonian treatment automatically implies equality of all the coupling constants, \( e_k = e \).}
deformed traceless higher spin fields.

The paper is organized as follows. In Section 2, we briefly re-derive, following [2], linearized gauge transformations for the infinite collection of symmetric tensors, governing first-order dynamics of the point particle. In Section 3, we derive the quadratic gauge invariant actions both for \( m = 0 \) and \( m \neq 0 \) case and analyze the nonlocal map from \( m \neq 0 \) to \( m = 0 \). In Section 4, we analyze some conformal covariance properties of \( m = 0 \) models, specifically, we show that the space of gauge fields may be assigned with an action of an infinite-dimensional "conformal higher spin" Lie algebra (which contains the conformal algebra as its maximal finite-dimensional subalgebra) in such a way that gauge transformations remain intact. In Conclusion, we discuss the results and outline a possibility of studying the nonlinear action.

2 Point particle and gauge transformations for symmetric traceless tensors.

The Hamiltonian action of a point particle in general background fields reads

\[
S_H[x(\tau), p(\tau), \lambda(\tau)] = \int d\tau \{p_m \dot{x}^m - \lambda H(p, q)\},
\]

where \( x^m(\tau), m = 0, 1...d-1 \), are the coordinates of the particle’s world line, \( p_m(\tau) \) are the momenta and \( \lambda \) is a Lagrange multiplier to the unique first class constraint \( H(x^m, p_m) \approx 0 \) which we shall call Hamiltonian. The Hamiltonian is supposed to be a power series in momenta,

\[
H = \sum_{k=0}^{\infty} H^{m_1...m_k}(x)p_{m_1}...p_{m_k} = \sum_{k=0}^{\infty} H_k
\]

where \( H_k \) denotes the homogeneous polynomial of \( k \)-th degree in momenta. When \( H_k = 0 \) for \( k > 2 \), the model describes a particle in general gravitational + Maxwell background, while otherwise the particle experiences the influence of higher rank symmetric tensors \( H^{m_1...m_k}(x) \). In [2], the gauge transformations for \( H^{m_1...m_k}(x) \) were derived by postulating that Hamiltonians \( H \) and \( H' \) are gauge equivalent if they describe equivalent particle’s dynamics. Specifically, if one makes an infinitesimal canonical transformation \( x'(x, p) = x + \delta x, \; p'(x, p) = p + \delta p \),

\[
\delta x^m = \{x^m, \epsilon\}, \quad \delta p_m = \{p_m, \epsilon\},
\]

with generating function \( \epsilon(x, p) \) (\( \{,\} \) stands for the canonical Poisson bracket, \( \{x^m, p_n\} = \delta^m_n, \; \{x^m, x^n\} = \{p_m, p_n\} = 0 \), the dynamics in \( x', p' \) variables is determined by the canonically transformed Hamiltonian

\[
H'(x, p) = H(x, p) + \delta H(x, p)
\]

\[
\delta H(x, p) = \{\epsilon, H(x, p)\},
\]

\footnote{It should be noted that the notion of physical equivalence is not straightforward, however, the "generalized equivalence principle" which underlies our derivation [2], is relevant at least because it automatically provides one with closed gauge transformations as forming the covariance algebra of some physical system (a particle).}
which is, by definition, equivalent to $H$. Provided $\epsilon(x, p)$ is also a power series in momenta,

$$\epsilon = \sum_{k=0}^{\infty} \epsilon^{m_1 \ldots m_k} p_{m_1} \ldots p_{m_k} = \sum_{k=0}^{\infty} \epsilon_k,$$

these transformations provide the action of the canonical transformations algebra on the infinite collection of symmetric tensor fields $H^{m_1 \ldots m_k}(x)$ comprising the power series $H(x, p)$. Also, the Hamiltonians differing by a factor which never comes to zero,

$$H'(x, p) = A(x, p) H(x, p), A(x, p) \neq 0$$

are by definition equivalent (they do determine the same dynamics of the particle, as its dynamics is localized on the constraint surface $H \approx 0$). Representing $A(x, p)$ as $A = e^{a(x, p)}$, one may write down the infinitesimal form of (9)

$$\delta H(x, p) = a(x, p) H(x, p),$$

where $a(x, p)$ is also a power series in momenta:

$$a = \sum_{k=0}^{\infty} a^{m_1 \ldots m_k} p_{m_1} \ldots p_{m_k} = \sum_{k=0}^{\infty} a_k.$$  

As a result, one gets the action of some huge gauge algebra,

$$\delta_{(a, \epsilon)} H(x, p) = a(x, p) H(x, p) + \{\epsilon, H(x, p)\}$$

on the infinite collection of symmetric tensor fields $H^{m_1 \ldots m_k}(x)$. This algebra clearly contains $U(1)$ "phase" transformations and x-diffeomorphisms, generated by $\epsilon = \epsilon(x) + \xi^m(x)p_m$ ($\xi$ generates $U(1)$ and $\xi^m$ the diffeomorphisms), and Weyl dilations, generated by $p$-independent $a(x, p) = a_0$, and contains much more.

Expand $H(x, p)$ around the natural vacuum

$$H = H_v + h(x, p) \equiv \frac{1}{2} (\eta^{mn} p_m p_n + m^2) + h(x, p),$$

with Minkowski background for metric $g_{mn} = \eta_{mn}$. It is worth noting that after passing to the Lagrangian formulation and throwing away the higher (except zero and first) orders in $h$ in the particle’s Lagrangian, one arrives exactly at the sum of the actions \(1\) \(2\).

Now rewrite the gauge transformations \(12\) in terms of $h(x, p)$ and make the linearization, i.e. extract the lowest order in $h(x, p)$. On obtains

$$\delta h(x, p) = a(x, p) H_v + \{\epsilon, H_v\} \equiv \frac{1}{2} a(x, p) (p^2 + m^2) + p_m \eta^{mn} \partial_n \epsilon(x, p),$$

where $\partial_m$ is the derivative w.r.t. $x^m$. In the component form, the gauge transformations read

$$\delta h^m(s) = \frac{1}{2} (\partial^m(2) a^{m(s-2)} + m^2 a^m(s+1)) + \partial^m \epsilon^m(s-1).$$

Our program is to look for quadratic theories, invariant w.r.t these gauge transformations. Before dwelling on details let us note that these theories will describe dynamics of infinite
collection of traceless tensors, as the trace parts of $h^{m(s)}$ are gauged away by purely algebraic $a(x, p)$-transformations. It is worth extracting the invariants of $a$-transformations and looking how these invariants transform under $\epsilon$-gauge symmetries.

The simplest case is $m^2 = 0$ one. Representing $h^{m(s)}$, $\epsilon^{m(s-1)}$ as a sum of their traceless and trace parts,

$$
h^{m(s)} = \varphi^{m(s)} + \eta^{m(2)} \chi^{m(s-2)}; \quad \varphi^{l m(s-2)} = 0,
$$

one observes that all the traces $\chi^{m(s-2)}$ are gauged away by $a$-transformations, while for the traceless parts one derives the gauge transformations

$$
\delta \varphi^{m(s)} = \text{Traceless part of } \partial^{m} \varepsilon^{m(s-1)}. \tag{17}
$$

The invariant action will be shown below to present scale-covariant theory of $2s$-th order in derivatives, formulated in terms of the traceless tensor of rank $s$.

In the $m \neq 0$ case, the situation is more complicated. To describe it, we repeat the derivation from [2] in the Appendix. The result is that, in the $m^2 \neq 0$ case, $a$-invariants are traceless tensors $\varphi^{m(s)}$ built out of $h^{m(k)}$, which possess gauge transformations

$$
\delta \varphi^{m(s)} = \text{(Traceless part of } \partial^{m} \varepsilon^{m(s-1)}) - m^2 \frac{s + 1}{2s + d} \partial_n \varepsilon^{nm(s)}. \tag{18}
$$

These gauge transformations entangle components of all ranks, and in this case the invariant actions will be shown to be nonlocal and involve fields of all ranks from $s$ to $\infty$.

## 3 Invariant actions.

Now let us look for a Poincaré- and gauge-invariant action. The most general Poincaré-invariant quadratic action is

$$
A_P[h] = \sum_{k=0,k'=0}^{\infty} \int d^d x h^{m_1...m_k}(x) P_{\{m_1...m_k\vert n_1...n_{k'}\}}(\partial_l) h^{n_1...n_{k'}}(x), \tag{19}
$$

where $P_{\{m_1...m_k\vert n_1...n_{k'}\}}(\partial_l)$ are some (pseudo)differential operators constructed from the partial derivative $\partial_m$ and the Minkowski metric, they are also allowed to contain any function of $\Box$. The operator $P_{\{m(k)\vert n(k)\}}(\partial_l)$ is supposed to be symmetric, therefore

$$
P_{\{m(k)\vert n(k')\}}(-\partial_l) = P_{\{n(k')\vert m(k)\}}(\partial_l). \tag{20}
$$

The gauge invariance of the quadratic action is equivalent to the gauge invariance of the equations of motion

$$
\sum_{k'=0}^{\infty} P_{\{m_1...m_k\vert n_1...n_{k'}\}}(\partial_l) h^{n_1...n_{k'}}(x) = 0; \forall k. \tag{21}
$$

To find out the solution for $P$’s one has to substitute the transformations (15) into the last equation and require all the identities associated with a given parameter ($a^{m(k)}$ or $\epsilon^{m(k)}$)
to hold. It is easy to see $P$'s have to be $\partial_m$-transversal and satisfy certain tracelessness constraints:

$$P\{l_1, l_2, \ldots l_k|n_1 \ldots n_{k-1}\} \partial r = 0;$$

(22)

$$\eta^{ab} P\{l_1 \ldots l_k|n_1 \ldots n_{k-2}|a b\} + m^2 P\{l_1 \ldots l_k|n_1 \ldots n_{k-2}\} = 0; \forall k, k',$$

and analogously for $n \leftrightarrow l$.

It turns out this infinite system of identities may be studied in a simple generating framework. Introduce the power series of two variables $q^m, q'^m$

$$P(q, q') = \sum_{k=0, k'=0}^{\infty} \frac{1}{k!k'!} q^{n_1} \ldots q^{n_k'} q'^{m_1} \ldots q'^{m_k'} P\{m_1 \ldots m_k|n_1 \ldots n_k\}(\partial_l).$$

(23)

Then the infinite system of identities (21) is equivalent to two equations on $P(q, q')$:

$$\nabla P(q, q') = 0; \nabla \equiv \eta^{mn} \frac{\partial}{\partial q^m} \frac{\partial}{\partial x^n}$$

(24)

and

$$(\diamond + m^2) P(q, q') = 0; \diamond \equiv \eta^{mn} \frac{\partial}{\partial q^m} \frac{\partial}{\partial q^n}.$$  

(25)

The equation (21) turns into

$$P(q, q')(\partial) = P(q', q)(-\partial),$$

(26)

and therefore one gets also

$$\nabla' P(q, q') = 0; \nabla' \equiv \eta^{mn} \frac{\partial}{\partial q'^m} \frac{\partial}{\partial x^n}$$

(27)

and

$$(\diamond' + m^2) P(q, q') = 0; \diamond' \equiv \eta^{mn} \frac{\partial}{\partial q'^m} \frac{\partial}{\partial q'^n}.$$  

(28)

The equations (24)-(28) constitute the full set of conditions for gauge invariance of the action (19).

Before starting to solve the equations (24)-(28), it is worth noting the important property of the formalism. If one "dresses" $P(q, q')$ like

$$P(q, q') = U(q)U(q')P_U(q, q') \Leftrightarrow P_U(q, q') = U^{-1}(q)U(q')^{-1} P(q, q'),$$

(29)

with some $U(q) \neq 0$, then new operator $P_U(q, q')$ already does not satisfy the same equations for $P(q, q')$. However, if this change is accompanied by the following "dressing" of gauge fields

$$h(x, p) = U(\frac{\partial}{\partial p})h_{U}(x, p) \Leftrightarrow h_{U}(x, p) = U^{-1}(\frac{\partial}{\partial p})h(x, p)$$

(30)

then it results in the same theory, i.e.

$$A_{P_U}[h_U] = A_P[h].$$

(31)
This property of "dressing" will play an important role below.

Now let us find the solution to these equations. Note that the partial derivative w.r.t. $x^m$ enters all these equations just as some constant vector, to emphasize this we will denote it below as some constant vector

$$\frac{\partial}{\partial x^n} \equiv d_n.$$

Below, we will also denote $d_m d^m \equiv \Box$ and use the self-evident notation like $(AB) \equiv A_k B^k, A^2 \equiv A_k A^k$.

First, we study the equations (24) and (27). Their (Poincaré-co variant) general solution is easily obtained as they are just first order differential equations w.r.t. $q, q'$: $P(q, q')$ should depend on the $d^n$-transversal combinations:

$$P = P(q_\perp, q'_\perp)$$

$$q^m_\perp = q^m - \frac{(qd^m)}{\Box};$$

$$q'^m_\perp = q'^m - \frac{(q'd^m)}{\Box}.$$  \hspace{1cm} (32)

As the solution has to be Poincaré invariant, $P$ should be a function of three Poincaré invariants built out of $q^m_\perp, q'^m_\perp$:

$$P = Q(\sigma, \sigma', \tau),$$

$$\sigma = \Box q^2_\perp = \Box q^2 - (qd)^2; \sigma' = \Box q'^2_\perp = \Box q'^2 - (q'd)^2 \hspace{1cm} (33)$$

$$\tau = \Box (q_\perp q'_\perp) = \Box (qq') - (qd)(q'd).$$

Now we have to determine the function of three variables $Q$. Note that $\sigma, \sigma', \tau$ should enter the solution only analytically, otherwise the solution could not be interpreted as power series in $q, q'$. By virtue of (26) $Q$ satisfies

$$Q(\sigma, \sigma', \tau) = Q(\sigma', \sigma, \tau). \hspace{1cm} (34)$$

Implement the equation (25). We get (commas in subscripts denote derivatives w.r.t. $q^m$)

$$0 = (\Box + m^2) Q = m^2 Q +$$

$$+ \frac{\partial^2 Q}{\partial \sigma \partial \sigma' m} \sigma, m \sigma, m + \frac{\partial^2 Q}{\partial \sigma \partial \sigma' m} \sigma', m \sigma', m + \frac{\partial^2 Q}{\partial \tau \partial \tau m} \tau, m \tau, m +$$

$$+ 2 \frac{\partial^2 Q}{\partial \sigma \partial \sigma' \partial \tau} \sigma, m \tau, m + 2 \frac{\partial^2 Q}{\partial \sigma' \partial \tau \partial \tau} \sigma', m \tau, m + 2 \frac{\partial^2 Q}{\partial \sigma \partial \sigma' \partial \tau} \sigma, m \sigma', m +$$

$$+ \frac{\partial Q}{\partial \sigma} \sigma, m + \frac{\partial Q}{\partial \sigma'} \sigma', m + \frac{\partial Q}{\partial \tau} \tau, m.$$  \hspace{1cm} (35)

After employing the identities

$$\sigma, m \sigma, m = 4 \Box \sigma; \tau, m \tau, m = \Box \sigma'; \sigma, m \tau, m = 2 \Box \tau$$

$$\sigma, m = 2 \Box (d - 1); \tau, m = 0; \sigma', m = 0,$$  \hspace{1cm} (36)
the equation (35) is rewritten as
\[ 0 = m^2 Q + \Box (4\sigma \frac{\partial^2 Q}{\partial \sigma^2} + \sigma' \frac{\partial^2 Q}{\partial \tau^2} + 4\tau \frac{\partial^2 Q}{\partial \sigma \partial \tau} + 2(d - 1) \frac{\partial Q}{\partial \sigma}). \] (37)

It is convenient to represent \( Q \) as a power series in \( \tau \),
\[ Q = \sum_{l=0}^{\infty} Q_l(\sigma, \sigma') \tau^l, \] (38)
then the equation (37) reads
\[ (4\sigma \frac{\partial^2}{\partial \sigma^2} + 2(2l + d - 1) \frac{\partial}{\partial \sigma} + \mu^2)Q_l + (l + 1)(l + 2)Q_{l+2} \sigma' = 0, \] (39)
where the "nonlocality" parameter \( \mu^2 \equiv \frac{m^2}{\Box} \) is introduced. The \( \sigma' \) counterpart of this equation holds either due to (28):
\[ (4\sigma' \frac{\partial^2}{\partial \sigma'^2} + 2(2l + d - 1) \frac{\partial}{\partial \sigma'} + \mu^2)Q_l + (l + 1)(l + 2)Q_{l+2} \sigma = 0. \] (40)

Making change of variables
\[ \rho = \sqrt{\sigma}, \; \rho' = \sqrt{\sigma'}, \] (41)
and introducing new function \( R(\rho, \rho', \tau) \) by the rule
\[ Q(\sigma, \sigma', \tau) = (\rho \rho')^{-\frac{d-3}{2}} R(\mu \rho, \mu \rho', \frac{\tau}{\rho \rho'}) \equiv (\rho \rho')^{-\frac{d-3}{2}} \sum_{l=0}^{\infty} R_l(\mu \rho, \mu \rho')(\frac{\tau}{\rho \rho'})^l, \] (42)
one rewrites the equations (39) and (40) in the form
\[ (B_\rho - (l + \frac{d-3}{2})^2)R_l = -(l + 1)(l + 2)R_{l+2} \]
\[ (B_{\rho'} - (l + \frac{d-3}{2})^2)R_l = -(l + 1)(l + 2)R_{l+2}, \] (43)
where
\[ B_\rho \equiv \rho^2 \frac{\partial^2}{\partial \rho^2} + \rho \frac{\partial}{\partial \rho} + \rho^2 \] (44)
is the operator governing Bessel’s equation (and \( B_{\rho'} \) is the same operator acting on \( \rho' \)). Represent \( R_l(\rho, \rho') \) as a series
\[ R_l(\rho, \rho') = \sum_{\nu, \nu' \in \Omega} R_{\nu, \nu', l} C_\nu(\rho) C_{\nu'}(\rho'), \] (45)
where \( \Omega \) is some set of points in the complex plane, \( R_{\nu, \nu', l} \) are constants, and \( C_\nu(\rho) \) is a solution of Bessel’s equation with index \( \nu \)
\[ (B_\rho - \nu^2)C_\nu(\rho) \equiv (\rho^2 \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} + \rho^2 - \nu^2)C_\nu(\rho) = 0. \] (46)
Then the equations (43) take the form

\[
(\nu^2 - (l + \frac{d-3}{2})^2)R_{\nu,\nu',l} = -(l + 1)(l + 2)R_{\nu,\nu',l+2} \tag{47}
\]

\[
(\nu^2 - (l + \frac{d-3}{2})^2)R_{\nu,\nu',l} = -(l + 1)(l + 2)R_{\nu,\nu',l+2}.
\]

It is seen that these equations are split into independent chains of simple iterative equations w.r.t. \(l\), where each chain corresponds to particular values of Bessel indices \(\nu, \nu'\). It is readily seen that only solutions with \(\nu^2 = \nu'^2\) are nonzero, so one has to study just one family of iterative chains

\[
r_{\nu,l} \equiv R_{\nu,\nu,l},
\]

\[
(\nu^2 - (l + \frac{d-3}{2})^2)r_{\nu,l} = -(l + 1)(l + 2)r_{\nu,l+2}.
\]

(48)

Now recall that we are looking for solutions (42) which are real and analytic at the origin \(\sigma, \sigma'\) and \(\tau\). Hence we have to choose only those solutions for \(R\) which are real and such that \(\rho^{\frac{d-3}{2}}C_\nu(\rho)\) possesses analytic decomposition at the origin. This particularly implies (see (48)) \(\nu^2\) is real. Then, \(\nu^2\) should be nonnegative as otherwise the iteration from \(r_l\) to \(r_{l+2}\) governed by the equations (48), never stops, which will result (see (42, 45)) in arbitrarily large negative powers of \(\rho \rho'\).

Denote

\[
\gamma \equiv \frac{d-3}{2}.
\]

(49)

Below we consider the case \(d \geq 3\) and hence \(\gamma \geq 0\) and after discussing this general case we return to \(d = 2\), while \(d = 1\) case is obviously a trivial one (in \(d = 1\), \(\rho = \rho' = \tau \equiv 0\)). As \(\gamma \geq 0\) while \(\rho^{-\gamma}C_\nu(\rho)\) should be analytic, the solution \(C_\nu(\rho)\) should be regular at the origin. For nonnegative \(\nu^2\), among the solutions to Bessel’s equation, only the Bessel’s functions of the first kind and of nonnegative index \(\nu\) are known to possess this property, therefore one has to make the choice

\[
\nu \geq 0; \ C_\nu(\rho) = J_\nu(\rho) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{\rho}{2}\right)^{\nu + 2k}, \tag{50}
\]

where \(J_\nu(\rho)\) is the Bessel’s function of the first kind, these solutions to Bessel’s equations are known to possess regular behaviour at \(\rho = 0\):

\[
J_\nu(\rho) \approx_{\rho \to 0} \frac{1}{\Gamma(\nu+1)} \left(\frac{\rho}{2}\right)^\nu, \tag{51}
\]

Then, analyzing the behaviour of solutions (42) at the origin and applying this analyticity requirement along with (43) and (51), one gets that \(Q\) is analytic in \(\sigma, \sigma'\) iff \(\nu - \gamma - l\) is even positive integer or zero, therefore

\[
\nu - l - \gamma = 2t \geq 0, \ t = 0, 1, 2, \ldots \tag{52}
\]

which implies

\[
\nu \geq \gamma, \ 0 \leq l \leq \nu - \gamma. \tag{53}
\]
Now one has to study the chain (18) and to find the solutions consistent with (52), (53). Their existence is not a priori guaranteed, as in accordance with (53) solutions should contain finite number of terms in \( l \). Nevertheless, the solutions do exist for each integer and half-integer \( \nu \). Denote
\[
s = \nu - \frac{d - 3}{2} = 0, 1, 2, 3, \ldots = 2n + \zeta \quad n = 0, 1, 2, 3, \ldots \quad \zeta = 0, 1
\]
so \( \zeta \) accounts whether \( s \) is even or odd.

Then the only solutions satisfying (18) and (52) are
\[
r_{2k+\zeta,\zeta+2n+\zeta} = 0, \quad k < 0 \text{ or } k > n;
\]
\[
r_{2k+\zeta,\zeta+2n+\zeta} = \frac{(-1)^k (2n)!!(2n+2\gamma+2\zeta+2k-2)!!}{(2k+\zeta)!!(2n-2k)!!(2n+2\gamma+2\zeta-2)!!} r_{\zeta,\zeta+2n+\zeta} \equiv
\]
\[
\equiv c_{2k+\zeta,\zeta+2n+\zeta} r_{\zeta,\zeta+2n+\zeta};
\]
\[
k = 0, 1, 2, \ldots, n - 1, n,
\]
where
\[
a!! \equiv a(a - 2)(a - 4)\ldots 2 \text{ or } 1.
\]
\( r_{\zeta,\zeta+2n+\zeta} \) is an arbitrary ”constant”, i.e. arbitrary function of \( \Box \), which is convenient to redefine as follows
\[
r_{\zeta,\zeta+2n+\zeta} = \left(\frac{\mu}{2}\right)^{2(\gamma+2n+\zeta)} r_{n,\zeta}
\]
The solutions (55) exist because the coefficients of recurrent relations (18) become zero at the three lines in the \((l, \nu)\) plane, \( L_{right} : \nu = l + \gamma \) and \( L_1, L_2 : l = -1, -2 \), then the chain of iterations along the line \( L_{iterate} : \nu = s + \gamma \) terminates at its right edge as it crosses \( L_{right} \) and at its left edge as it crosses one of \( L_1, L_2 \). It may be easily verified that, if \( \nu \geq 0 \), these are the only solutions that contain finite number of points along \( L_{iterate} \). And they are exactly within the domain prescribed by the condition (53).

On the other hand, the equations (18) are invariant w.r.t. change \( \nu \rightarrow -\nu \). It means that there is one more branch of solutions which contains finite number of points along \( L_{iterate} \), it is obtained from (18) by substitution \( J_\nu \rightarrow J_{-\nu} \). These solutions are already non-regular at the origin, so they are not connected with our main task, however they could be of some interest as they are still polynomial in \( \tau \) and so may correspond to some ”not so wild” models. For integer \( \nu \), the map \( J_\nu \rightarrow J_{-\nu} \) does not produce new solutions as \( J_n = (-\nu)^n J_{-n} \), but if one has already given up regularity at the origin, one can choose the second solution to the Bessel’s equation, the Hankel function \( C_n = Y_n \).

Thus, we have the general solution for the equations (24)-(28), which is parameterized by \( \mu, n = 0, 1, 2, 3, \ldots \) and \( \zeta = 0, 1:
\]
\[
P_{\{\mu, n, \zeta\}}(q, q') = (\rho \rho')^{-\gamma} J_{2n+\gamma+\zeta}(\mu \rho) J_{2n+\gamma+\zeta}(\mu \rho') \sum_{k=0}^{n} \left(\frac{\tau}{\rho \rho'}\right)^{2k+\zeta} r_{2k+\zeta, 2n+\gamma+\zeta}
\]

\[6\]All other solutions contain infinite number of terms in \( l \), either infinite both to the left and to the right or starting from some finite point in the \((l, \tau)\) plane and then going to infinity. All these solutions are inappropriate for our considerations in the paper.
Note that the solution contains only even powers of $\rho, \rho'$ as it should.

Let us analyze the theories we have obtained and clarify the meaning of the parameters $n$ and $\varsigma$. To this purpose it is worth taking the limit $m^2 \to 0$, or, what is the same, $\mu \to 0$. In this case, only first term of the Bessel’s series survives, so one gets, according to (54)-(55),

$$P_{(0,n,\varsigma)}(q,q') = \frac{\tilde{f}_{n,\varsigma}}{(\Gamma(2n + \varsigma + \gamma + 1))^2} \sum_{k=0}^{n} \tau^{2k+\varsigma} (\rho \rho')^{2(n-k)} c_{2k+\varsigma,\gamma+2n+\varsigma}. \quad (59)$$

As it is clear from the very definition (23), a term $I_{a,b,c} \sim \tau^a \rho^{2b} \rho'^{2c}$ in the solution leads to the corresponding operator in the action (15), which contains $2a + 2b + 2c$ $x$-derivatives, and involve tensor fields of ranks $a + 2b$ and $a + 2c$. Applying this account to the last equation we see that the corresponding theory contains terms $I_{2k+\varsigma,n-k,n-k}$ for $k = 0, ..., n$ and thereby describes theory of symmetric tensor fields of rank-$(2n + \varsigma)$ only, with operators of only $2(2n + \varsigma))$ order in $x$-derivatives (for $n = \varsigma = 0$, the solution is just a constant, i.e. arbitrary function of $\square$). Therefore, these theories are covariant under scale transformations $x'^m = e^\lambda x^m$. We call these models \textit{traceless higher spin theories of spin $s$}. In $d = 4$, they were described by Fradkin and Tseytlin [3] ("pure spin" models), and were conjectured to be invariant w.r.t. full conformal algebra $so(4,2)$. They possess supersymmetric extensions, studied by Fradkin and Linetsky [4, 5].

In the next section, we will analyze some properties of traceless higher spin models w.r.t. full conformal algebra in $d$ dimensions. We will show that if one considers all these models altogether, the space of fields $h(x,p)$ may be assigned with an action of an infinite-dimensional ”conformal higher spin” Lie algebra (which contains the conformal algebra as its maximal finite-dimensional subalgebra) in such a way that gauge transformations remain intact.

As $\mu \neq 0$, the solution presents a deformation of traceless higher spin theories. These deformed models are \textit{nonlocal} as the solution contains all the powers of $\mu^2 = m^2$ inside Bessel’s functions. Moreover, as the product of Bessel’s functions of $\rho, \rho'$ is not the product $\rho \rho'$, the corresponding theory seems to mix tensors of all ranks from $2n + \varsigma$ to $\infty$. However, it appears all these peculiar properties may be cancelled after a change of variables. Indeed, the structure of solution (58) exhibits ”dressing” (23):

$$P_{(\mu,n,\varsigma)}(\rho,\rho') = U_{2n+\varsigma+\gamma}(\mu \rho) U_{2n+\varsigma+\gamma}(\mu \rho') P_{(0,n,\varsigma)}(\rho,\rho'), \quad (60)$$

where

$$U_\nu(z) \equiv z^{-\nu} J_\nu(z), \quad U(0) = (\frac{1}{2})^{\nu} \frac{1}{\Gamma(\nu + 1)}, \quad (61)$$

and thus $U_\nu$ is invertible at least in the small vicinity of origin. On the other hand, $U_\nu$ definitely has zeros which coincide with zeros $z_{(\nu)k}$ of Bessel function $J_\nu(z_{(\nu)k}) = 0$, except $z = 0$ point. According to (23,[31], this brings the following interpretation of $m^2 \neq 0$ deformations of traceless higher spin models. Consider such fields that operator $\mu \rho(q \to \frac{\partial}{\partial \rho}, d)$ (corresponding to the argument of Bessel functions) is not equal to any of Bessel functions zeros $z_{(s+\gamma)k}$. Then the $m^2 \neq 0$ theory is identified with $m^2 = 0$ one by means of invertible nonlocal change of variables of the form (30) with dressing $U(\rho)$ given by (60,[61]).
On the other hand, the dressing becomes non-invertible if the dressing operator is zero on \( h, \ U_h(x, p) = 0 \), and it is clear that zeros of \( U \) are simultaneously the solutions of full equations of motions. Therefore, the map from \( m^2 \neq 0 \) to \( m^2 = 0 \) models may be non-bijective, it degenerates e.g. on the subspace of solutions for

\[
D h(x, p) \equiv \mu p(q \mapsto \frac{\partial}{\partial p}, d) h(x, p) = z_{(s+\gamma)k} h(x, p) , \quad J_{s+\gamma}(z_{(s+\gamma)k}) = 0,
\]

(62)
corresponding to zeros of Bessels functions. To illustrate that this class of solutions should not be neglected let us look for them in the form of "plane waves"

\[
h_{r,b}(x, p) = \exp(ix^a r_a + p_a b^a), \quad r_a = \text{const}, \quad b^a = \text{const},
\]

(63)
the operator \( D (62) \) acts on this function by multiplying it by \((- \frac{m^2}{r^2})^{\frac{1}{2}} \rho(b, ir)\) and therefore \( (62) \) is satisfied provided

\[
(- \frac{m^2}{r^2})^{\frac{1}{2}} \rho(b, ir) = \left[- \frac{m^2}{r^2} ((rb)^2 - r^2 b^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} = z_{(s+\gamma)k},
\]

(64)"Massive" momenta \( r \) correspond to \( r^2 = -M^2 \). For massive momenta, one may consider \( r \)-transversal space-like "polarizations" \( b \), then the formula \( (64) \) determines the "spin spectrum"

\[
(r b) = 0, \quad b^2 = m^{-2} S_k^2, \quad S_k \in \mathbb{R}
\]

(55)
governed by real zeros of Bessel function \( J_{s+\gamma} \).

We see \( m \neq 0 \) models may possess new phenomena as compared to their \( m = 0 \) cousins, therefore it may be worth keeping these theories in their original basis, where they are nonlocal. Anyway, \( m \neq 0 \) models are as unique as their \( m = 0 \) limits and are worth studying. As argued in the introduction, corresponding gauge fields may transfer first-order interactions of massive point particles. We will call the deformed model, corresponding to \( \{n, \varsigma\} \) "deformed traceless spin-\( (2n + \varsigma) \) theory".

Now analyze \( d = 2 \) case. The results are quite the same, but the analysis is slightly different. The point is that now \( \gamma = -\frac{1}{2} < 0 \) and thus (see (12,15)) solutions to the Bessel’s equation with negative index are allowed. But the only new solution which is not covered by the formula \( (58) \) contains a single point along \( L_{\text{iterate}}, \ l = 0, \nu = -\frac{1}{2} \) and may be described by general formula \( (58) \) as well. This is the solution describing the lowest spin, \( s = 0 \), and thereby it reduces to a constant (i.e., arbitrary function of \( \Box \)) in the \( m = 0 \) limit. As in general case, all other solutions which contain finite number of terms in \( l \), are obtained from our main family \( (58) \) by \( \nu \mapsto -\nu \) change, but they are not analytic in \( \rho^2, \rho' \).

Let us make a technical remark. Due to invariance w.r.t. \( a \)-transformations, the actions \( (19),(58) \) depend only upon the special traceless combinations of \( h^{m(k)} \), described in the Appendix (one traceless tensor for each rank \( s \)). Specifically, \( h(x, p) \) may be represented as

\[
h(x, p) = \varphi(x, p) + (p^2 + m^2) \chi(x, p),
\]

(66)
where \( \chi(x, p) \) is arbitrary power series in \( p_m \) while \( \varphi \) is a traceless power series:

\[
\varphi(x, p) = \sum_{s=0}^{\infty} \varphi^{m(s)}(x)p_{m_1}...p_{m_s} \ ; \ \varphi^{nm(s-2)} = 0. \tag{67}
\]

Then it is clear the action may be written in terms of \( \varphi \) by making substitution \( h(x, p) \mapsto \varphi(x, p) \) in the action (19), after the substitution, the terms \( q^2, q'^2 \) in the generating function \( P(q, q') \) (58) may be dropped as they give vanishing contributions. This simplify the form of \( \sigma \mapsto -(qd)^2, \sigma' \mapsto -(q'd)^2 \). In this basis, gauge transformations of \( \varphi(x)^{m(s)} \) appear to depend only upon the special traceless parts of \( \epsilon \) (18,95,97) and entangle \( \varphi \)-components of all ranks, with \( m^2 \) playing the role of entanglement magnitude. Formally, they may be disentangled by ”undressing” the \( m^2 \neq 0 \) fields according to (30,60,61), but this transformation is highly nonlocal and is not well-defined always, e.g., it degenerates on the massive fields with ”spin spectrum” governed by zeros of Bessel functions (63-65).

4 Conformal invariance of gauge transformations at \( m = 0 \).

Here we demonstrate the invariance, in the \( m = 0 \) case, of the gauge transformations (14) w.r.t infinite-dimensional algebra which contains the conformal algebra of \( d \)-dimensional Minkowski space \( so(d,2) \) as its maximal finite-dimensional subalgebra. The conformal algebra transforms every fixed rank tensor into itself, while general infinite-dimensional transformation mix all ranks.

The proof employs essentially the origin of gauge fields as background fields of the particle’s theory, as their gauge transformations are formulated directly in terms of the particle phase space (12).

The proof will be simple, but to exhibit the simplicity we have to start from rather general facts. Suppose we have a Lie algebra \( g \) of a group \( G \) acting on the manifold \( M \):

\[ Z = Z + g(Z) + O(g) \ ; \ Z \in M. \]

Let \( Z_v \in M \) be some point (”vacuum”) and \( g_v \subset g \) its stability subalgebra,

\[ g_v(Z_v) = 0. \tag{68} \]

It is well known that there is representation \( T_{g_v} \) of the stability subalgebra in the tangent space to the point \( Z_v, T(Z_v) \):

\[ T_{g_v}Y = \left( \frac{\partial g_v}{\partial Z} \right|_{Z=Z_v}) Y; Y \in T(Z_v). \tag{69} \]

Consider the shift of the vacuum \( Z_v \) w.r.t. general element of the algebra \( g \), as a function of \( g \) with values in \( T(Z_v) \): \( \delta_g Z_v = g(Z_v) \equiv R_v(g) \). Then as it is clear that \( \{[,]\} \) is Lie algebra commutator

\[ T_{g_v}g(Z_v) = [g_v, g](Z_v), \tag{70} \]

\( R_v(g) \equiv g(Z_v) \) satisfies the equation

\[ T_{g_v}R_v(g) = R_v([g_v, g]). \tag{71} \]
This means that general variation of the vacuum $R_v(g)$ is covariant w.r.t. stability subalgebra transformations in the tangent space.

Now specify the "manifold" $M$ and the algebra of transformations $g$. The "manifold" is the space of all Hamiltonians $H(x, p)$ (understood as power series in momenta). Recall the full gauge transformations in the space of all Hamiltonians (12):

$$\delta(\epsilon, a)H(x, p) = a(x, p)H(x, p) + \{\epsilon, H(x, p)\}. \quad (72)$$

This transformations are easily seen to form an infinite-dimensional algebra $g$, isomorphic to the semidirect product of all canonical transformations $\epsilon$ to an abelian ideal of "hyperWeyl" transformations $a$:

$$[\delta(\epsilon_1, a_1), \delta(\epsilon_2, a_2)]H = \delta(\epsilon_3, a_3)H \quad (73)$$

$$\epsilon_3 = \{\epsilon_1, \epsilon_2\}, \quad a_3 = \{\epsilon_1, a_2\} - \{\epsilon_2, a_1\}.$$

The stability subalgebra $g_v$ of a point $H_v$ consists of all parameters $\epsilon_v, a_v$, satisfying the equation

$$a_v(x, p)H_v + \{\epsilon_v, H_v(x, p)\} = 0. \quad (74)$$

It is worth pointing out that the stability subalgebra has direct physical interpretation of global symmetries algebra of the point particle with the Hamiltonian $H_v$. Indeed, it is easy to see every canonical transformation $\epsilon_v$ maps the equations of motion of the particle with Hamiltonian $H_v$ into themselves (remember that particle’s dynamics is bound to the constraint surface $H = 0$).

Now specify the function $R_v(g)$. It is given by general variation of the vacuum Hamiltonian,

$$R_v(g) \equiv R_v(\epsilon, a) = a(x, p)H_v + \{\epsilon(x, p), H_v\} \quad (75)$$

Of our main concern is the covariance property (71). As the Lie algebra action in the space of all $H$ is linear, the $T_v$ representation coincides with $g$ action (provided tangent space is canonically identified with original linear space of $H$), so we obtain

$$\delta(\epsilon, a)\delta(\epsilon, a)R_v(\epsilon, a) = \delta(\epsilon, a)\delta(\epsilon, a)R_v(\epsilon, a)$$

$$= R_v(\{\epsilon_v, \epsilon\}, \{\epsilon_v, a\} - \{\epsilon, a_v\}), \quad (76)$$

which of course may be checked by direct calculation.

The next step is the appreciation of the fact that general gauge variation of the vacuum $R_v(g)$ is nothing but the linearized gauge transformation for the fluctuation $h$ of general Hamiltonian $H = H_v + h$ around the vacuum $H_v$,

$$\delta(\epsilon, a)h(x, p) = \delta(\epsilon, a)H_v = R_v(\epsilon, a). \quad (77)$$

Thus the linearized gauge transformations (77) possess covariance w.r.t. global symmetry group $g_v$:

$$\delta(\epsilon, a)h(x, p) = \delta(\epsilon, a)(a + \{\epsilon_v, \cdot\})h(x, p). \quad (78)$$
This means that the infinitesimal global symmetry transformations
\[ \delta_{(\epsilon_v, a_v)} h = (a_v + \{\epsilon_v, \cdot\}) h \]  
result in new \( h \) that obeys \textit{exactly the same} gauge laws but with transformed gauge parameters
\[ \delta_{(\epsilon_v, a_v)} \epsilon = \{\epsilon_v, \epsilon\} ; \quad \delta_{(\epsilon_v, a_v)} a = \{\epsilon_v, a\} - \{\epsilon, a_v\}. \]  
As we are always able to redefine the gauge parameters, all this is equivalent to the statement that \textit{while} \( h \) \textit{changes}, the gauge transformations do not change.

One more important property concerns the action on \( h \) of \textit{trivial global symmetries}  
\[ (\epsilon_v, a_v) \in g_{\text{triv}} \) of the form
\[ \epsilon_v^{(\text{triv})} = \mu(x, p) H_v, \quad a_v^{(\text{triv})} = -\{\mu(x, p), H_v\} \]  
with \( \epsilon_v \) vanishing on the constraint surface, where \( \mu \) is an arbitrary power series in \( p \). One may check that
\[ \delta_{(\epsilon_v^{(\text{triv})}, a_v^{(\text{triv})})} h = R_v(-\mu h, \{\mu, h\}) \]  
and thereby trivial global symmetries act on \( h \) as some \( h \)-dependent gauge transformations. It is easy to check that \( g_v^{\text{triv}} \) form an ideal in \( g_v \). So, the space of gauge invariants of \( h \) (and, needless to say, the physical phase space of the particle) acquires action of the algebra of observables \( g_o \) which is defined as a factor-algebra
\[ g_o \equiv g_v / g_{\text{triv}}. \]

Now let us apply all these matters to the Hamiltonian of the massless particle on Minkowski space
\[ H_v = \frac{1}{2} p^2 \]  
It is well-known the massless particle’s theory possesses conformal invariance, and our derivation allows one to transfer this invariance to the gauge transformations of traceless higher spin theories.

Consider the canonical generators of conformal transformations of \( d \)-dimensional Minkowski metric on the particle’s phase space:
\[ \epsilon = k^{ab} M_{ab} + b^a P_a + f^a K_a + f D; \]
\[ M_{ab} = x_a p_b - x_b p_a ; \quad P_a = p_a ; \quad K_a = x^2 p_a - 2(x p) x_a ; \quad D = (x p). \]  
Here \( k^{ab}, b^a, f^a, f \) are parameters for infinitesimal Lorentz transformations, translations, special conformal transformations and dilations, respectively. By their very definition, all these generators either leave invariant the Hamiltonian of the massless particle \( H_v = \frac{1}{2} p^2 \) (Poincare generators \( M_{ab}, P_a \)) or scale it by a function of \( x \) (special conformal \( K_a \) and dilations \( D \)):  
\[ \{\epsilon, p^2\} = (2D - 4f^a x_a)p^2 \equiv -a_c p^2. \]  
Note that any product of \( \epsilon \) possesses this property either:
\[ \Upsilon_t = \epsilon^{(1)c} \epsilon^{(2)c} \cdots \epsilon^{(t)c} \Rightarrow \]
\[ \{\Upsilon_t, p^2\} = -(a^{(1)c} \epsilon^{(2)c} \cdots \epsilon^{(t)c} + \cdots + \epsilon^{(1)c} \cdots \epsilon^{(t-1)c} a^{(t)c}) \equiv -A_t p^2. \]
Comparing the last equation with (74) we see that pairs $\Upsilon_t, A_t$ are the elements of the stability subalgebra of the vacuum, or the global symmetries (of the particle). Moreover, the linear space of all $\Upsilon_t, A_t, t = 1, 2, 3, ...$ is the infinite-dimensional Lie algebra w.r.t. to the composition law (73), while $\Upsilon_1, A_1$ is the original conformal algebra. Thus, at least $\Upsilon_t, A_t, t = 1, 2, 3, ...$ form a subalgebra $\bar{g}_v$ of the total symmetry algebra $g_v$, but actually one can show that any global symmetry (which has a form of power series in momenta) is represented as a combination of $\Upsilon_t, A_t$, so $\bar{g}_v = g_v$. Among global symmetries, there are trivial ones ($\epsilon_v, a_v \in g_{triv}$) of the form (81)

$$
\epsilon_v = \frac{1}{2} p^2 \mu(x, p), \quad a_v = -p \partial_x \mu(x, p),
$$

(88)

The algebra of observables defined as (83) is an infinite dimensional algebra isomorphic to a contraction of some conformal higher spin algebra of the type proposed by Fradkin and Linetsky in $d = 3, 4$ [6, 7]. It may be shown that, if one considers the quantum massless particle, its algebra of observables deforms to non-contracted conformal higher spin algebra [8].

According to the above reasoning, all transformations from $g_v$ (79) present a symmetry of the gauge transformations (14) in the case $m^2 = 0$. The finite-dimensional conformal subalgebra preserves the subspace of every $s$-th degree in momenta and thereby acts separately on each spin-$s$ model, while higher transformations mix all spins.

Now if the gauge transformations were determined the wave equation for $h$ uniquely, then one could state that while $h$ changes, the wave equation does not change, so then transformations (74) would transform the space of solutions of the wave equation into itself, i.e. present a symmetry of the free wave equation for $h$. However, our solution for wave equation were unique only under requirements of Poincaré invariance, while special conformal transformations $K_a$ may break it, so for analyzing the conformal invariance of our wave equations more information is required.

### 5 Conclusion.

Starting from gauge transformations, induced by the first-order point particle-symmetric tensors interactions, we have constructed Poincaré- and gauge-invariant free actions for traceless symmetric tensor fields (which should not be confused with unitary double-traceless higher spin theories [8]). These actions set some dynamics for these fields which thereby may mediate point particles interactions. The typical processes may look just like those in classical electrodynamics: there exists a free field "F" which propagates through space-time according to its equations of motion, and there are sources localized on point particle world lines. To study interaction of two particles, "A" and "B", one has to solve the equations for free gauge fields "F" with a source formed by "A" and then to study the motion of "B" in the field "F", and vice versa. This is one of interesting tasks to be studied in future.

One of the manifestations of importance of these models is their uniqueness. For each spin $s$, there is just one family of theories parameterized by the particle’s mass $m$. At the point $m = 0$, the theories reduce to local higher derivative scale-covariant models.
with actions of 2s-th order in derivatives. In $d = 4$, these models have been described by Fradkin and Tseytlin ("pure spin" models) \[3\] and their supersymmetric cubic interactions were elaborated by Fradkin and Linetsky \[4, 5\]. It may be interesting to note that the gauge transformations (3.5) in \[4\] look identically to our (15) (at $m = 0$), but there they are not linked to any first-order interaction.

The "pure spin" models were claimed to be conformally invariant \[3, 4, 5\]. Yet, we have not found a simple way to analyze the covariance properties of traceless higher spin theories w.r.t conformal group. Perhaps, the most appropriate way to study conformal invariance in the models is to reformulate them in $2T$-physics framework advocated by Bars \[11\]. Besides, we have presented the action of an infinite dimensional Lie algebra, (which contains full conformal algebra $so(d, 2)$ of $d$-dimensional Minkowski space as a maximal finite-dimensional subalgebra) on the gauge fields, this action appears to leave gauge transformation intact. This algebra may be deformed to the "conformal higher spin algebra in $d$-dimensions" analogous to that studied in \[6, 7\], which is the same as "higher spin algebra" in $d + 1$ dimensions \[12\], that may present interest in view of $AdS/CFT$ correspondence. The origin of gauge fields as background fields in the particle’s theory is basic in our derivation.

As $m^2 \neq 0$, the theories become essentially nonlocal as it is manifested by inverse powers of $\Box$ up to an infinite order in the generating function (58), and entangle traceless tensors of all spins. Although these peculiar properties may be cured by a nonlocal "undressing" change of variables (30, 60, 61) which maps $m \neq 0$ models to their $m = 0$ counterparts of the same spin, it may be worth dealing with $m \neq 0$ case in the original basis (where they are nonlocal) as otherwise one may throw away some potentially important solutions. We have presented a class of solutions of this type, with "spin spectrum" governed by zeros of Bessel functions (63-65). Anyway, "deformed traceless higher spin theories" are interesting to study as they are, in a sense, unique and able to mediate interactions of massive point particles.

Besides studying new interactions of point particles, another immediate task is a non-linear theory of traceless fields. Indeed, as the quadratic actions do exist and the nonlinear gauge transformations are known (12), it is worth examining the perturbative solution for the nonlinear action \[10\]. If the solution exists it may present a new gauge theory generalizing conformal gravity and describing interactions of infinite number of traceless tensor fields.

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**Appendix. Gauge transformations for traceless tensors.**
Here we rewrite gauge transformations (14)

\[ \delta h(x, p) = \frac{1}{2} a(x, p)(p^2 + m^2) + p_m \eta^m n \partial_n \epsilon(x, p) \]  

(89)
in terms of \( a \)-invariant traceless tensors.

We need some simple tools to handle traces of tensor coefficients of arbitrary functions. Let us note that, given any function

\[ F(x, p) = \sum_{k=0}^{\infty} F^{(k)}(x)p_{m_1} \cdots p_{m_k}, \]

one can unambiguously represent it in the form

\[ F(x, p) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} F^{(l)}_{(k)} (p^2)^l p_{m_1} \cdots p_{m_k}, \]  

(90)

where \( F^{(l)}_{(k)} \) are traceless, \( F_{(l)n} n^{(k-2)} = 0 \). This is easily done by decomposing each \( F^{(k)} \) to its traceless part and the traces \( F^{(k)} = F_{(0)}^{(k)} + \eta^{m(2)} F_{(1)}^{(m(k-2))} + \eta^{m(2)} \eta^{m(2)} F_{(2)}^{(m(k-4))} + \cdots \), then summing up the power series by momenta and noting that the trace parts give the powers of \( p^2 \). The decomposition (91) is then rewritten as

\[ F(x, p) = \sum_{k=0}^{\infty} F^{(k)}(p^2)p_{m_1} \cdots p_{m_k}, \]  

(91)

where \( F^{(k)}(p^2) = \sum_{l=0}^{\infty} F^{(l)}_{(k)} (p^2)^l \). Decomposing the power series \( F^{(k)}(\sigma) \) at the point \( \sigma = -m^2 \) one gets

\[ F(x, p) = \sum_{k=0}^{\infty} F^{(k)}_{[-m^2]} (p^2 + m^2)p_{m_1} \cdots p_{m_k} = \]

\[ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} F^{(k)}_{[-m^2]} (l) (p^2 + m^2)^l p_{m_1} \cdots p_{m_k} = \sum_{l=0}^{\infty} F^{(l)}_{[-m^2]} (p^2 + m^2)^l, \]  

(92)

where the power series \( F^{(l)}_{[-m^2]} \) contain only traceless coefficients. Given \( m^2 \), we will say that the \( F^{(l)}_{[-m^2]} \) term is the traceless part of the function \( F(x, p) \) and the first, second and further traces of \( F \) are represented by \( F^{(l)}_{[-m^2]} (p^2 + m^2)^l \) with \( l = 1, 2, \ldots \) forming altogether the traceful part of \( F \). The function is traceless if it is equal to its traceless part. In this sense each coefficient \( F^{(l)}_{[-m^2]} \) is a traceless function.

Now represent all the entries of the gauge transformation laws (89) in the form (92) to get

\[ \delta \sum_{l=0}^{\infty} h^{(l)}_{[-m^2]} (p^2 + m^2)^l = \sum_{l=0}^{\infty} \left\{ \frac{1}{2} a^{(l)}_{[-m^2]} (p^2 + m^2)^{l+1} + p^m \partial_m a^{(l)}_{[-m^2]} (p^2 + m^2)^l \right\} \]  

(93)

wherefrom it is seen that all the traces of \( h \) may be gauged away by \( a \)-transformations. In fact, the very destination of \( a \) is to gauge away the traces of \( h \). It is worth noting that
the traceful part of $\epsilon$ is already contained in $a$ as the gauge transformations (93) do not change if one redefines $\epsilon, a$ according to

$$
\delta \epsilon = \frac{1}{2} (p^2 + m^2) \nu, \quad \delta a = -p^m \partial_m \nu.
$$

(94)

Therefore without loosing a generality one may set $\epsilon$ traceless

$$
\epsilon = \epsilon_{[-m^2]0} \equiv \epsilon = \sum_{k=0}^{\infty} \epsilon^{m(k)} \prod_{k} p_{m_k} \ ; \ \epsilon^{nm(k-2)} = 0.
$$

(95)

For any action $\mathcal{A}[h]$ invariant w.r.t. gauge transformations (89), $h$ should enter $\mathcal{A}[h]$ in $a$- and $c$-invariant combinations only as far as $a$ transformations are purely algebraic. It is easy to see that the only $a$-invariant is the traceless function

$$
\varphi = h_{[-m^2]0}.
$$

(96)

It is easy to derive the $\epsilon$ transformation laws for the coefficients of $\varphi$ which read

$$
\delta \varphi^{m(s)} = (\text{Traceless part of } \partial^m \epsilon^{m(s-1)}) - m^2 \frac{s+1}{2s+d} \partial_n \epsilon^{nm(s)}.
$$

(97)

For $m^2 = 0$, these are the gauge transformations of conformal higher spin theories, which are seen to decay in independent subsystems described in terms of rank-$s$ traceless tensor and rank-$(s-1)$ traceless parameter. For $m^2 \neq 0$, as it is proved in the main text, for each spin $s$ there exists the deformed traceless higher spin theory, which reduces to traceless spin-$s$ model in the limit $m^2 \rightarrow 0$. The deformed theory is nonlocal, with nonlocality being measured by $\frac{m^2}{2}$.

We now see from (97) that traceless tensors of all ranks get entangled by gauge transformations in $m^2 \neq 0$ case. In this sense, $m^2$ exhibits itself also as an entanglement magnitude.

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