Abstract. Let \( p_1 = 2, p_2 = 3, \ldots \) be the sequence of all primes. Let \( \epsilon \) be an arbitrarily small but fixed positive number, and fix a coprime pair of integers \( q \geq 3 \) and \( a \). We will establish a lower bound for the number of primes \( p_r \), up to \( X \), such that both \( p_{r+1} - p_r < \epsilon \log p_r \) and \( p_r \equiv p_{r+1} \equiv a \mod q \) simultaneously hold. As a lower bound for the number of primes satisfying the latter condition, the bound we obtain improves upon a bound obtained by D. Shiu.

1. Introduction

Let \( p_1 = 2, p_2 = 3, \ldots \) be the sequence of all primes, and let \( \epsilon \) be an arbitrarily small but fixed positive number. In 2005 [3, 5], Goldston, Pintz, and Yıldırım made a significant breakthrough by proving that \( p_{r+1} - p_r < \epsilon \log p_r \) for infinitely many pairs \( p_r, p_{r+1} \) of primes. That is, for infinitely many \( r \), the \( r \)th prime gap, \( p_{r+1} - p_r \), is arbitrarily small compared to the ‘expected’ gap of \( \log p_r \). In 2006 [4] they extended their method to prove an analogous result for primes in arithmetic progressions. Thus, given a coprime pair of integers \( q \geq 3 \) and \( a \), if \( p_1' < p_2' < \cdots \) is the sequence of all primes congruent to \( a \mod q \), then for infinitely many pairs \( p_m', p_{m+1}' \), we have \( p_{m+1}' - p_m' < \epsilon \log p_m \).

Given any such pair \( p_m', p_{m+1}' \), there may or may not be a third prime \( p \), not congruent to \( a \mod q \), such that \( p_m' < p < p_{m+1}' \). Thus, either there are infinitely many triples of primes \( p_r, p_{r+1}, p_{r+2} \), not necessarily in the same arithmetic progression mod \( q \), such that \( p_{r+2} - p_r < \epsilon \log p_r \); or there are infinitely many pairs of consecutive primes \( p_r, p_{r+1} \) such that both \( p_{r+1} - p_r < \epsilon \log p_r \) and \( p_r \equiv p_{r+1} \equiv a \mod q \) simultaneously hold. Presumably both statements are true, but one can only deduce that one of them is true, and one does not know which one, from the result in [4].

Although we would like to prove the first statement, unfortunately it seems beyond reach of the method of Goldston, Pintz, and Yıldırım, at least at present. (See [5, §1, Question 3].) It is natural, then, to ask whether one can at least prove the second statement. In so doing, one would establish a conjecture of Chowla that there are infinitely many pairs of consecutive primes \( p_r, p_{r+1} \) such that \( p_r \equiv p_{r+1} \equiv a \mod q \). This conjecture was in fact already proved by D. Shiu in 2000 [10].

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As it turns out, the ideas of Shiu can be combined with those of Goldston, Pintz, and Yıldırım to prove that there are indeed infinitely many pairs of consecutive primes $p_r, p_{r+1}$ such that both $p_{r+1} - p_r < \epsilon \log p_r$ and $p_r \equiv p_{r+1} \equiv a \mod q$ simultaneously hold. We did this in [2], where we also obtained a very weak quantitative result [2, §7]: there is a positive constant $A = A(q)$, depending only on $q$, such that for all sufficiently large $X$,

$$\sum_{p_r \leq X \atop p_{r+1} - p_r < \epsilon \log p_r \atop p_r \equiv p_{r+1} \equiv a \mod q} 1 \geq X^{1/3(\log \log X)^4}.$$  

Our purpose here is to improve this lower bound to the following:

**Theorem 1.1.** Let $p_1 = 2, p_2 = 3, \ldots$ be the sequence of all primes. Fix any positive number $\epsilon$, and fix a pair of coprime integers $q \geq 3$ and $a$. There is an absolute positive constant $c$ such that, for all sufficiently large $X$,

$$\sum_{p_r \leq X \atop p_{r+1} - p_r < \epsilon \log p_r \atop p_r \equiv p_{r+1} \equiv a \mod q} 1 \geq X^{1-c/\log \log X}.$$  

As a lower bound for the number of primes $p_r$ up to $X$ for which $p_r \equiv p_{r+1} \equiv a \mod q$, (1.2) is, once $X$ is sufficiently large, greater than that obtained by Shiu [10, Theorem 2], namely $X^{1-\varepsilon(X)}$, where

$$\varepsilon(X) = C_1(q) \left( \frac{\log \log \log X}{\log \log X} \right)^{1/\phi(q)}$$

if $a \equiv \pm 1 \mod q$, and

$$\varepsilon(X) = C_2(q) \left( \frac{(\log \log X)^2}{(\log \log \log X)(\log \log \log \log X)} \right)^{1/\phi(q)}$$

otherwise. (Here, $C_1(q)$ and $C_2(q)$ are constants depending only on $q$.)

2. Discussion

The way to incorporate the ideas of Shiu into the work of Goldston, Pintz, and Yıldırım is explained in [2, §2]. Basically, Goldston, Pintz, and Yıldırım [4] proved that for all sufficiently large $N$, there is at least one integer\(^1\) $n \in (N, 2N]$ such that there are at least two primes of the form $Qn + h$, where: $Q$ is a multiple of $q$ such that $\log QN \sim \log N$; $h$ is in the set

$$S = S(H) := \{1 \leq h \leq H : (Q, h) = 1 \text{ and } h \equiv a \mod q\};$$

\(^1\)In fact, a lower bound for the number of such integers $n$, of the form $N/(\log N)^c$, $c$ a positive constant, is implicit in the work of Goldston, Pintz, and Yıldırım.
and $H = \epsilon \log N$. Goldston, Pintz, and Yildrim $[4, \text{(2.1) -- (2.4)}]$ took

$$Q = Q(H) := q \prod_{p \in \mathcal{P}} p,$$

where

$$\mathcal{P} = \mathcal{P}(H) := \{p \leq H/(\log H)^2\},$$

but if we remove from $\mathcal{P}$ any subset of the primes in the interval $(\log H, H/(\log H)^2]$, the key estimates $[2, \text{Proposition 3.2}]$ still hold, with one exception — namely, we do not necessarily have $[2, \text{(2.2)}]: |S| \gg_q H\phi(Q)/Q$.

Our goal is to remove primes from $\mathcal{P}$ in such a way that we have the following for the resulting $Q$: almost all of the integers $h \in [1, H]$ that are coprime with $Q$ are congruent to $a \bmod q$, in the sense that if

$$T = T(H) := \{1 \leq h \leq H : (Q, h) = 1 \text{ and } h \neq a \bmod q\},$$

then $|T| = o(|S|)$ as $H \to \infty$; and $|S| \gg_q H\phi(Q)/Q$ for all sufficiently large $H$. Since $Qn + h$ is prime only if $(Q, h) = 1$, we could deduce from this that, for infinitely many of those $n$ for which $(Qn, Qn + H)$ contains at least two primes congruent to $a \bmod q$, among those primes is a pair of consecutive primes. Indeed, we would be able to establish (1.2). (See $[2, \text{§4, §7}]$ for details.)

Based on a construction used by Shiu in $[10]$, we defined such a set $\mathcal{P}$ in $[2, \text{§6.2}]$ (also see (3.1) — (3.4) below). In fact, denoting by $\mathcal{P}' = \mathcal{P}'(H)$ the set considered by Shiu, we have $\mathcal{P} = \mathcal{P}' \cup \{p \leq \log H : p \equiv 1 \bmod q\}$. Since $\mathcal{P}'$ is defined in such a way that it consists only of primes up to $H/(\log H)^2$, and contains all primes $p \not\equiv 1 \bmod q$ up to $\log H$, $\mathcal{P}$ consists only of primes up to $H/(\log H)^2$ and, in particular$^3$, all primes up to $\log H$.

However, in $[2]$, we were only able to establish the following: $|T| \ll H/\log H$ for all sufficiently large $H$; $H/\log H = o(H\phi(Q)/Q)$; and there is a positive constant $A$, depending only on $q$, such that for all sufficiently large $Y$, there is some $H \in [Y/(\log Y)^4, Y]$ for which $|S| \gg_q H\phi(Q)/Q$. From this we deduced (1.1) in $[2, \text{§7}]$.

The reason we were not able to establish that $|S| \gg_q H\phi(Q)/Q$ for all sufficiently large $H$ in $[2]$ is that we used $[10, \text{Lemma 2}]$ (Lemma 6.2 in $[2]$): an asymptotic for the number of integers up to $H$ that are composed only of primes congruent to $1 \bmod q$. Defining $Q' =

\begin{itemize}
\item[2]Actually, if there happens to be an exceptional modulus $q_0 \leq N^{1/(\log \log N)^2}$, and if $p_0$ is its greatest prime factor, we remove $p_0$ from the product defining $Q$, so that $(Q, p_0) = 1$. See $[4, \text{Lemma 2}]$ and $[2, \text{§5}]$ for details. We overlook this technical complication for the purposes of simplifying the present discussion.
\item[3]The fact that $\mathcal{P}$ contains all primes up to $\log H$ is used to show that, for a given $k$-tuple of linear forms $\mathcal{H} = \{Qx + h_1, \ldots, Qx + h_k\}$, $h_i \in [1, H]$, $(Q, h_1 \cdots h_k) = 1$, we have $S(\mathcal{H}) \sim (Q/\phi(Q))^k$ as $H \to \infty$, where $S(\mathcal{H})$ is the singular series for $\mathcal{H}$. See $[2, \text{Lemma 5.1}]$ for details.
\end{itemize}
Q′(H) and S′ = S′(H) analogously to Q and S, but with \( \mathcal{P}' \) in place of \( \mathcal{P} \), Shiu used this asymptotic to show that \( |S'| \gg_q H\phi(Q')/Q' \) for all sufficiently large \( H \). In [2, §6], we took this as our starting point, and then dealt with the extra primes \( \{ p \leq \log H : p \equiv 1 \mod q \} \) in \( \mathcal{P} \).

What we need is an asymptotic for the number of integers, up to \( H \), that are composed only of primes both congruent to 1 mod \( q \) and greater than \( \log H \). Much of this note is devoted to establishing such a result (Lemma 3.3 below). Using this we are able to show that \( |S| \gg_q H\phi(Q)/Q \) for all sufficiently large \( H \). Indeed, using Lemma 3.4 (below) instead of [2, Lemma 6.5] in [2, §7], we are able to establish Theorem 1.1.

We will show that the inequalities in Lemma 3.4 hold for \( q \) in a certain range depending on \( H \). This uniformity is not needed to prove Theorem 1.1, but it can be used to prove a version of Theorem 1.1 in which \( q \) is allowed to tend very slowly to infinity with \( X \). It is hoped to publish an account of this, in which we will also consider ‘strings’ of more than 2 congruent primes — in longer intervals.

3. Proof of Theorem 1.1

Throughout this section, at each and every occurrence of \( O \) and \( \ll \), the implied constant is absolute. The letter \( c \), by itself, always denotes an absolute positive constant, possibly a different constant at each occurrence.

Theorem 1.1 will follow from Lemma 3.4, below. Lemma 3.4 is a corollary of: Theorem 3.1, which is a version of the Siegel-Walfisz theorem; Lemma 3.2, which is a version of Mertens’ theorem in which the primes are restricted to the arithmetic progression 1 mod \( q \); and Lemma 3.3, which gives an asymptotic for the number of integers, up to \( X \), composed only of primes that are both congruent to 1 mod \( q \) and greater than a power of \( \log X \).

In each of the lemmas below, the estimates are shown to hold uniformly for \( q \) in a certain range. We do not need this uniformity to prove Theorem 1.1 — it would suffice to use the prime number theorem for arithmetic progressions instead of Theorem 3.1, and versions of Lemmas 3.2, 3.3, and 3.4 in which \( q \) is arbitrary but bounded.

We use the Siegel-Walfisz theorem, in the following form, in the proofs of Lemmas 3.2 and 3.4:

**Theorem 3.1** (Siegel-Walfisz). Fix a positive number \( A \). For all sufficiently large \( X \) we have, uniformly for integers \( q \) satisfying \( 1 \leq q \leq (\log X)^A \), the following estimate:

\[
\sum_{\substack{p \leq X \\ p \equiv 1 \mod q}} 1 = \left(1 + O\left(\frac{1}{\log X}\right)\right) \frac{X}{\phi(q) \log X}.
\]
Proof. Indeed, we have \([9, \S 11.3, \text{Corollary 11.20}]:\)

\[
\sum_{\substack{p < X \\ p \equiv a \mod q}} 1 = \frac{\text{li}(X)}{\varphi(q)} + O \left( X \exp \left( -C_A \sqrt{\log X} \right) \right),
\]

uniformly for \(1 \leq q \leq (\log X)^A\) and integers \(a\) coprime with \(q\), where \(C_A\) is a positive constant depending only on \(A\). The less precise and less general statement of Theorem 3.1, which follows since \(\text{li}(X) = X/\log X + O(X/(\log X)^2)\), is sufficient for our purposes. \(\square\)

We will use the following version of Mertens’ theorem in the proof of Lemma 3.4:

**Lemma 3.2.** Fix a positive number \(A\). For all sufficiently large \(X\) we have, uniformly for integers \(q\) satisfying \(1 \leq q \leq (\log X)^A\), the following estimate:

\[
\prod_{\substack{p < X \\ p \equiv 1 \mod q}} \left( 1 - \frac{1}{p} \right)^{-1} = \left( 1 + O \left( \frac{1}{\log X} \right) \right) e^{\gamma/\varphi(q)} c(q)(\log X)^{1/\varphi(q)},
\]

where \(\gamma = 0.57721\ldots\) is the Euler-Mascheroni constant, and \(c(q)\) is a positive constant depending only on \(q\). We have \(c(1) = 1\) and \(c(2) = 1/2\).

**Proof.** The case \(q = 1\) is Mertens’ theorem, and the case \(q = 2\) follows at once from this. We prove the result for \(3 \leq q \leq (\log X)^A\) in \(\S 4\), where \(c(q)\) is given explicitly. \(\square\)

The following result, which reduces to \([10, \text{Lemma 2}]\) in the case \(Y = 1\) (and \(q\) fixed), is the key that allows us to establish the inequalities in Lemma 3.4 for all sufficiently large \(H\), rather than just for a certain sequence of \(H\) tending to infinity as in \([2, \S 6]\).

**Lemma 3.3.** Fix a positive number \(A\) and a number \(\alpha \in (0, \frac{1}{2})\). For all sufficiently large \(X\) we have, uniformly for \(Y\) satisfying \(1 \leq Y \leq (\log X)^A\) and integers \(q\) satisfying \(3 \leq q \leq (\log X)^\alpha\), the following estimate:

\[
\sum_{\substack{n \leq X \\ p | n \Rightarrow p \equiv 1 \mod q \quad \text{and} \quad p > Y}} 1 = \left( 1 + O \left( \frac{(\log \log X)^c}{(\log X)^{1-2\alpha}} \right) \right) \frac{c(q)}{\Gamma(1/\varphi(q))} \cdot \frac{X(\log X)^{1/\varphi(q)}}{\log X} \prod_{\substack{p < Y \\ p \equiv 1 \mod q}} \left( 1 - \frac{1}{p} \right),
\]

where \(c(q)\) is the positive constant, depending only on \(q\), in the statement of Lemma 3.2.

**Proof.** See \(\S 4\). \(\square\)

Before stating Lemma 3.4, we need some definitions. Let a sufficiently large number \(H\), and a coprime pair of integers \(q \geq 3\) and \(a\), be given. If \(a \equiv 1 \mod q\), let

\[
\mathcal{P}(H) := \{p \leq \log H : p \equiv 1 \mod q\} \cup \{p \leq H/(\log H)^2 : p \not\equiv 1 \mod q\}. \quad (3.1)
\]
If \( a \not\equiv 1 \mod q \), define
\[
    t(H) := \exp \left( \frac{(\log H)(\log \log H)}{2 \log \log H} \right),
\]
and, noting that \( \log H < t(H) < H/t(H) < H/(\log H)^2 \) for all sufficiently large \( H \), let
\[
    \mathcal{P}(H) := \{ p \leq \log H : p \equiv 1 \mod q \}
    \cup \{ p \leq H/(\log H)^2 : p \not\equiv 1 \mod q \text{ and } p \not\equiv a \mod q \}
    \cup \{ t(H) < p \leq H/(\log H)^2 : p \equiv 1 \mod q \}
    \cup \{ p \leq H/t(H) : p \equiv a \mod q \}.
\]
In other words, \( \mathcal{P}(H) \) consists of all primes up to \( H/(\log H)^2 \), except for the primes
\[
    \{ \log H < p \leq t(H) : p \equiv 1 \mod q \} \cup \{ H/t(H) < p \leq H/(\log H)^2 : p \equiv a \mod q \}.
\]
In either case, set
\[
    \hat{Q} = \hat{Q}(H; q, a) := q \prod_{p \in \mathcal{P}(H)} p, \quad Q = Q(H; q, a) := q \prod_{\substack{p \in \mathcal{P}(H) \setminus \{ p_0 \} \setminus \{ p \} \not\equiv q \not\equiv a \mod q}} p,
\]
where
\[
    p_0 = 1 \text{ or } p_0 \text{ is a prime satisfying } p_0 > \log H.
\]
(The minor technical complication of \( p_0 \) has to be accounted for in the proof of [4, Theorem 1], and consequently in the proof of [2, Theorem 1.1]. It arises when taking into consideration the possible existence of Siegel zeros — see [4] for details.) Finally, set
\[
    S = S(H; q, a) := \{ 1 \leq h \leq H : (Q, h) = 1 \text{ and } h \equiv a \mod q \};
    T = T(H; q, a) := \{ 1 \leq h \leq H : (Q, h) = 1 \text{ and } h \not\equiv a \mod q \}.
\]

**Lemma 3.4.** Given a sufficiently large number \( H \), and a coprime pair of integers \( q \geq 3 \) and \( a \), let \( Q = Q(H; q, a) \), \( S = S(H; q, a) \), and \( T = T(H; q, a) \) be as defined in (3.1) — (3.6). (a) For all sufficiently large \( H \) we have, for integers \( q \) satisfying
\[
    3 \leq q \leq \frac{\log \log H}{\log \log \log H}
\]
and \( a \equiv 1 \mod q \), the inequality
\[
    |S| - |T| \geq \frac{H}{\Gamma(1/\phi(q))} \left( \frac{\phi(Q)}{Q} \right).
\]
(b) For all sufficiently large $H$ we have, for integers $q$ satisfying

$$3 \leq q \leq \frac{\log \log H}{2 \log \log \log H},$$

and $a \not\equiv 1 \mod q$ coprime with $q$, the inequality

$$|S| - |T| \geq \frac{2}{5} \frac{H}{(1 + \phi(q)) \Gamma(1/\phi(q))} \left( \frac{\phi(Q)}{Q} \right). \quad (3.8)$$

**Proof of Theorem 1.1.** Fix an integer $q \geq 3$, arbitrary but bounded, and an integer $a$ that is coprime with $q$. Let $Q = Q(H; q, a)$, $S = S(H; q, a)$, and $T = T(H; q, a)$ be as defined in (3.1) — (3.6). In [2, §6.2], we showed that there is a positive constant $A$, depending only on $q$, such that $|S| - |T| \gg_q H\phi(Q)/Q$ for some $H \in [Y/(\log Y)^4, Y]$ and all sufficiently large $Y$. Using this, inter alia, we established the lower bound (1.1) in [2, §7]. We also showed that if $|S| - |T| \gg_q H\phi(Q)/Q$ for all sufficiently large $H$, then (1.2) holds. Thus, Theorem 1.1 follows from Lemma 3.4, in the way described in [2, §7]. (Here, the constant implied by $\gg_q$ depends only on $q$.) \hfill \Box

**Proof of Lemma 3.4.** Let $H$ be a sufficiently large number, and let a coprime pair of integers $q \geq 3$ and $a$ be given. Let $\mathcal{P}(H)$, $t(H)$, $\tilde{Q} = \tilde{Q}(H; q, a)$, $Q = Q(H; q, a)$, $p_0$, $S = S(H; q, a)$, and $T = T(H; q, a)$ be as defined in (3.1) — (3.6). We have

$$|T| \ll \frac{H}{\log H}. \quad (3.9)$$

This was shown in [2, §6.2], where $q \geq 3$ was arbitrary but bounded. However, the larger $q$ is, the more primes there are that divide $Q$, hence the smaller the size of $T$. In [2, §6.2], we actually bounded the size of $T$ by counting: the primes up to $H$; the integers of the form $pp'$, where $p \in (H/(\log H)^2, H]$ and $p' \in (\log H, (\log H)^2]$; the integers up to $H$ composed only of primes $p \leq t(H)$ (using a result of de Bruijn on smooth numbers); and, in the case $p_0 \neq 1$, so that $p_0 > \log H$ by (3.5), the multiples of $p_0$ up to $H$. Thus, (3.9) indeed holds uniformly for $q \geq 3$.

Note that by definition of $\tilde{Q}$ and $Q$ ((3.4), (3.5)),

$$|S| \geq \sum_{\substack{1 \leq h \leq H \\ h \equiv 1 \mod q \\ (Q, h) = 1}} 1, \quad (3.10)$$

and

$$\phi(\tilde{Q})/Q \geq \left(1 - \frac{1}{\log H}\right) \phi(Q)/Q. \quad (3.11)$$

We will work mainly with $\tilde{Q}$. 
Now we suppose that $q$ satisfies $3 \leq q \leq (\log H)^\alpha$, $\alpha \in (0, \frac{1}{2})$ given, and that $a \equiv 1 \mod q$. Note that
\[
\log \left( \frac{H}{(\log H)^2} \right) = (\log H) \left( 1 + O \left( \frac{\log \log H}{\log H} \right) \right).
\]
Thus, by definition of $\tilde{Q}$ ((3.1), (3.4)), and two applications of Lemma 3.2,
\[
\phi(\tilde{Q})/\tilde{Q} = \prod_{p \leq H/(\log H)^2} \left( 1 - \frac{1}{p} \right) \prod_{p \leq H/(\log H)^2 \atop p \equiv 1 \mod q} \left( 1 - \frac{1}{p} \right)^{-1} \prod_{p \leq H \atop p \equiv 1 \mod q} \left( 1 - \frac{1}{p} \right)
\]
\[
= \left( 1 + O \left( \frac{\log \log H}{\log H} \right) \right) e^{-\gamma(1-1/\phi(q))} c(q) \frac{(\log H)^{\frac{1}{\phi(q)}}}{\log H} \prod_{p \leq H \atop p \equiv 1 \mod q} \left( 1 - \frac{1}{p} \right). \tag{3.12}
\]
and so, by Lemma 3.3,
\[
\sum_{1 \leq h \leq H \atop h \equiv 1 \mod q} 1 \geq \sum_{1 \leq h \leq H \atop (Q,h)=1} 1 \cdot \frac{\phi(\tilde{Q})/\tilde{Q}}{\Gamma(1/\phi(q)) c(q)} \frac{H(\log H)^{\frac{1}{\phi(q)}}}{\log H} \prod_{p \leq H \atop p \equiv 1 \mod q} \left( 1 - \frac{1}{p} \right). \tag{3.13}
\]
The left-hand side here is a lower bound for $|S|$ (3.10), so using the second line of (3.13) with the bound $|T| \ll H/\log H$ (3.9), we obtain
\[
\frac{|T|}{|S|} \ll \frac{\Gamma(1/\phi(q))}{c(q)(\log H)^{\frac{1}{\phi(q)}}} \prod_{p \leq H \atop p \equiv 1 \mod q} \left( 1 - \frac{1}{p} \right)^{-1}.
\]
At this point we suppose that $q$ is in the rather smaller range
\[
3 \leq q \leq \frac{\log \log H}{\log \log \log H}.
\]
Then we may apply Lemma 3.2 to this last product to obtain
\[
\frac{|T|}{|S|} \ll \frac{\Gamma(1/\phi(q))}{c(q)(\log H)^{\frac{1}{\phi(q)}}} \ll \phi(q) \left( \frac{\log H}{\log H} \right)^{\frac{1}{\phi(q)}}.
\]
and so
\[
\log(|T|/|S|) \leq O(1) + \log \phi(q) + \frac{1}{\phi(q)} (\log \log H - \log \log H)
\]
\[
\leq O(1) + \log q + \frac{1}{q} (\log \log H - \log \log H)
\]
\[
\leq O(1) - \log \log \log \log H.
\]
Hence \(|T|/|S| \ll 1/\log \log \log H\), and combining this with (3.10), (3.11), and (3.13), we obtain
\[
|S| - |T| \geq \left(1 + O\left(\frac{1}{\log \log \log H}\right)\right) e^{\gamma(1-1/\phi(q))} \frac{\phi(Q)}{Q} H^{\gamma(\phi(q))} \frac{1}{\phi(q)} \prod_{p \leq \log H \leq \frac{H}{\log H}} \left(1 - \frac{1}{p}\right).
\]
Noting that \(e^{\gamma(1-1/\phi(q))} \geq e^{\gamma/2} > 1\) we obtain (3.7) (for all sufficiently large \(H\)), and the proof of part (a) is complete.

Now we suppose \(a \not\equiv 1 \pmod q\). Once again we suppose that \(3 \leq q \leq (\log H)^{\alpha}, \alpha \in (0, \frac{1}{2})\) given, until we want to show that \(|T|/|S| = o(1)\). Let us first of all show that
\[
\frac{\phi(\tilde{Q})}{\tilde{Q}} = \left(1 + O\left(\frac{\log \log \log H}{\log \log H}\right)\right) e^{-\gamma(1-1/\phi(q))} \frac{1}{\phi(q)} \cdot \frac{(\log t(H))^{\gamma(\phi(q))}}{\log H} \prod_{p \leq \log H \leq \frac{H}{\log H}} \left(1 - \frac{1}{p}\right).
\]
(3.14)

For by definition of \(\tilde{Q}\) ((3.2) — (3.4)),
\[
\frac{\phi(\tilde{Q})}{\tilde{Q}} = \prod_{p \leq H/(\log H)^2} \left(1 - \frac{1}{p}\right) \prod_{\log H < p \leq t(H)} \left(1 - \frac{1}{p}\right)^{-1} \prod_{H/t(H) < p \leq H/(\log H)^2} \left(1 - \frac{1}{p}\right)^{-1}.
\]
(3.15)

By Mertens’ theorem (the case \(q = 1\) in Lemma 3.2),
\[
\prod_{p \leq H/(\log H)^2} \left(1 - \frac{1}{p}\right) = \left(1 + O\left(\frac{\log \log H}{\log H}\right)\right) \frac{e^{-\gamma}}{\log H}.
\]
(3.16)
Since \(\log t(H) = (\log H)(\log \log \log H)/(2 \log \log H)\) by definition (3.2) of \(t(H)\), we certainly have \(3 \leq q \leq (\log H)^{\frac{1}{2}} \leq \log t(H)\) for all sufficiently large \(H\), so applying Lemma 3.2 with
\( A = 1 \), we obtain
\[
\prod_{\log H < p \leqslant t(H)} \left(1 - \frac{1}{p}\right)^{-1} = \left(1 + O\left(\frac{\log \log H}{(\log H)(\log \log \log H)}\right)\right) \\
\times e^{\gamma/\phi(q)} c(q)(\log t(H))^{\frac{1}{\phi(q)}} \prod_{p \leqslant \log H} \left(1 - \frac{1}{p}\right). \quad (3.17)
\]

As for the third product on the right-hand side of (3.15), we have
\[
1 \leqslant \prod_{H/t(H) < p \leqslant H/(\log H)^2} \left(1 - \frac{1}{p}\right)^{-1} \leqslant \prod_{H/t(H) < p \leqslant H/(\log H)^2} \left(1 - \frac{1}{p}\right)^{-1},
\]
and so two further applications of Mertens’ theorem, plus a short calculation using the fact that \( \log t(H) = (\log H)(\log \log \log H)/(2\log \log H) \), reveal that
\[
1 \leqslant \prod_{H/t(H) < p \leqslant H/(\log H)^2} \left(1 - \frac{1}{p}\right)^{-1} \leqslant 1 + O\left(\frac{\log \log \log H}{\log \log H}\right). \quad (3.18)
\]

Combining (3.18), (3.17), (3.16), and (3.15) gives (3.14).

Next, we will show that
\[
\sum_{1 \leqslant h \leqslant H \atop h \equiv a \mod q \atop (Q, h) = 1} 1 \geqslant \left(1 + O\left(\frac{\log \log \log H}{\log \log H}\right)\right) \\
\times \frac{1}{2} \frac{(1 - \frac{1}{\phi})}{1 + \phi(q)} \cdot \frac{c(q)}{\Gamma(1/\phi(q))} \cdot \frac{H(\log t(H))^{\phi(q)}}{\log H} \prod_{p \leqslant \log H} \left(1 - \frac{1}{p}\right). \quad (3.19)
\]

To this end we note, from the definition of \( \tilde{Q} \) ((3.3), (3.4)), that if \( h = pm \), where \( p > H/t(H) \) is a prime congruent to \( a \mod q \), and \( m \leqslant H/p < t(H) \) is composed only of primes that are greater than \( \log H \) and congruent to \( 1 \mod q \), then \( h \equiv a \mod q \) and \( (\tilde{Q}, h) = 1 \). We partition \( (H/t(H), H] \) into sub-intervals
\[
I_l := \left(e^{l-1}H/t(H), e^lH/t(H]\right), \quad 1 \leqslant l \leqslant \log t(H),
\]
and deduce that
\[
\sum_{1 \leqslant h \leqslant H \atop h \equiv a \mod q \atop (Q, h) = 1} 1 \geqslant \sum_{1 \leqslant l \leqslant \log t(H)} \sum_{p \in I_l} \sum_{p \equiv a \mod q \atop p | m} \sum_{m | H/t(\tilde{Q})/e^l \atop p | m} 1. \quad (3.20)
\]
Now, for $0 \leq l \leq \log t(H)$, we have
\[
\log \left( \frac{e^l H}{t(H)} \right) = (\log H) \left( 1 + O \left( \frac{\log t(H)}{\log H} \right) \right) = (\log H) \left( 1 + O \left( \frac{\log \log \log H}{\log \log H} \right) \right),
\]
because $\log t(H) = (\log H)(\log \log \log H)/(2 \log \log H)$ by definition (3.2) of $t(H)$. In particular, since $q \leq (\log H)^\alpha$, $\alpha < \frac{1}{2}$, we certainly have $q \leq \log(e^l H/t(H))$ for all sufficiently large $H$. Therefore we may apply Theorem 3.1 (Siegel-Walfisz), with $A = 1$, to obtain, for
\[
1 \leq l \leq \log t(H),
\]
\[
\sum_{\substack{p \leq l \leq (e^l H)/t(H) \\ p \equiv a \mod q}} 1 = \sum_{\substack{p \leq l \leq (e^l - 1) H/t(H) \\ p \equiv a \mod q}} 1
= \left( 1 + O \left( \frac{\log \log \log H}{\log \log H} \right) \right) \frac{1}{\phi(q)} \frac{H}{t(H) \log H} \left( 1 - \frac{1}{e} \right) e^l.
\]
Also, since $\log t(H) = (\log H)(\log \log \log H)/(2 \log \log H)$ by definition (3.2) of $t(H)$, we have, for $1 \leq l \leq \frac{1}{2} \log t(H)$, that $H = (\log(t(H)/e^l))^{1+o(1)}$, where $o(1)$ is shorthand for $O(\log \log H/\log \log H)$. Thus, for $1 \leq l \leq \frac{1}{2} \log t(H)$ and all sufficiently large $H$, we have
\[
3 \leq q \leq (\log H)^\alpha \leq (\log \left( t(H)/e^l \right))^{\beta}, \quad \beta := \frac{1}{2}(\alpha + \frac{1}{2}) \in (0, \frac{1}{2}).
\]
Therefore, for $1 \leq l \leq \frac{1}{2} \log t(H)$, we may apply Lemma 3.3, with $\beta$ in place of $\alpha$, and $A = 2$, say, to obtain
\[
\sum_{\substack{m \leq t(H)/e^l \\ p \mid m \Rightarrow p \equiv 1 \mod q \\ p > \log H}} 1 = \left( 1 + O \left( \frac{(\log t(H))^c}{(\log t(H))^{1-2\beta}} \right) \right) \frac{c(q)}{\Gamma(1/\phi(q))}
\times \frac{t(H)}{e^l} \frac{(\log (t(H)/e^l))^{1/\phi(q)}}{(\log t(H)/e^l)} \prod_{\substack{p \leq \log H \\ p \equiv 1 \mod q}} \left( 1 - \frac{1}{p} \right)^{\frac{1}{\phi(q)}}
\geq \left( 1 + O \left( \frac{(\log t(H))^c}{(\log t(H))^{1-2\beta}} \right) \right) \frac{c(q)}{\Gamma(1/\phi(q))}
\times \frac{t(H)}{e^l} \frac{(\log (t(H))^{1/\phi(q)}}{(\log t(H))} \left( 1 - \frac{l}{\log t(H)} \right)^{\frac{1}{\phi(q)}} \prod_{\substack{p \leq \log H \\ p \equiv 1 \mod q}} \left( 1 - \frac{1}{p} \right).}
\]
Note that, since $\log t(H) = (\log H)(\log \log \log H)/(2 \log \log H)$ by definition (3.2),
\[
\frac{(\log t(H))^c}{(\log t(H))^{1-2\beta}} \ll \frac{\log \log H}{\log H}.
\]
for all sufficiently large $H$. Thus, combining (3.22) and (3.21) with (3.20), we obtain

$$
\sum_{1 \leq h \leq H} 1 \geq \sum_{1 \leq h \leq \frac{1}{2} \log t(H)} \sum_{p \equiv a \mod q} \frac{1}{\phi(q)} \sum_{p \mid m \Rightarrow p \equiv 1 \mod q} \sum_{m \leq t(H)/e'} \sum_{p \leq \log H} \frac{1}{1 - \frac{1}{p}} \prod_{p \leq \log H} \left(1 - \frac{1}{p}\right)
$$

\[\geq \left(1 + O\left(\frac{\log \log \log H}{\log \log H}\right)\right) \frac{c(q)}{\Gamma(1/\phi(q))} \cdot \frac{H(\log t(H))^{1/\phi(q)}}{\log H} \prod_{p \equiv 1 \mod q} \left(1 - \frac{1}{p}\right)
\]

\[\times \left(1 - \frac{1}{e}\right) \frac{1}{\phi(q)} \cdot \frac{1}{\log t(H)} \sum_{1 \leq l \leq \frac{1}{2} \log t(H)} \left(1 - \frac{l}{\log t(H)}\right)^{\frac{1}{\phi(q)}}.
\]

Finally,

$$
\sum_{1 \leq l \leq \frac{1}{2} \log t(H)} \left(1 - \frac{l}{\log t(H)}\right)^{\frac{1}{\phi(q)}} \geq \int_{1}^{\frac{1}{2} \log t(H)} \left(1 - \frac{u}{\log t(H)}\right)^{\frac{1}{\phi(q)}} \, du
$$

\[= \frac{\log t(H)}{1 + \frac{1}{\phi(q)}} \left(\left(1 - \frac{1}{\log t(H)}\right)^{1 + \frac{1}{\phi(q)}} - (\frac{1}{2})^{1 + \frac{1}{\phi(q)}}\right)
\]

\[\geq \frac{\log t(H)}{1 + \frac{1}{\phi(q)}} \left(\left(1 - \frac{1}{\log t(H)}\right)^{2} - \frac{1}{2}\right)
\]

\[= \frac{\log t(H)}{1 + \frac{1}{\phi(q)}} \cdot \frac{1}{2} \left(1 + O\left(\frac{1}{\log t(H)}\right)\right).
\]

and combining this with (3.23) gives (3.24).

Comparing (3.14) with (3.19), then using (3.10) and (3.11), we see that

$$
|S| \geq \sum_{1 \leq h \leq H} 1
$$

\[\geq \left(1 + O\left(\frac{\log \log H}{\log \log H}\right)\right) \frac{1}{2} \left(1 - \frac{1}{e}\right) \cdot \frac{e^{\gamma(1-1/\phi(q))} H(\log t(H))^{1/\phi(q)}}{\Gamma(1/\phi(q))} \cdot \frac{\phi(q) \Gamma(1/\phi(q))}{c(q)(\log t(H))^{\phi(q)}} \prod_{p \equiv 1 \mod q} \left(1 - \frac{1}{p}\right)^{-1}.
\]

Also, using the bound $|T| \ll H/\log H$ (3.9), and combining (3.10) with (3.19), we obtain

$$
\frac{|T|}{|S|} \ll \frac{\phi(q) \Gamma(1/\phi(q))}{c(q)(\log t(H))^{\phi(q)}} \prod_{p \equiv 1 \mod q} \left(1 - \frac{1}{p}\right)^{-1}.
\]
At this point we suppose that $q$ is in the rather smaller range
\[ 3 \leq q \leq \frac{1}{2} \cdot \frac{\log \log H}{\log \log \log H}. \]

Then we may apply Lemma 3.2 to this last product to obtain
\[ \left| \frac{T}{S} \right| \ll \phi(q) \Gamma(1/\phi(q)) \left( \frac{\log \log H}{\log t(H)} \right)^{1/\phi(q)} \ll (\log \log H)^{1/\phi(q)}, \]
and so, since $\log t(H) = (\log H)(\log \log \log H)/(2 \log \log H)$ by (3.2),
\[ \log(|T|/|S|) \leq O(1) + 2 \log \phi(q) + \frac{1}{\phi(q)} (2 \log \log \log H - \log \log t(H)) \leq O(1) + 2 \log q \]
\[ + \frac{1}{q} (2 \log \log \log H - \log \log H - \log \log \log \log H + O(1)) \leq O(1) - 2 \log \log \log \log H. \]

Hence $|T|/|S| \ll 1/((\log \log H)^2)$, and, combining this with the last inequality in (3.24), we obtain
\[ |S| - |T| \geq \left( 1 + O\left( \frac{1}{(\log \log \log H)^2} \right) \right)^{1/2} \left( 1 - \frac{1}{e} \right) \cdot e^{\gamma(1-1/\phi(q))} \cdot \frac{\Gamma(1/\phi(q))H^{\phi(q)}}{\phi(q)}. \]

Noting that $\frac{1}{2} (1 - \frac{1}{e}) e^{\gamma(1-1/\phi(q))} \geq \frac{1}{2} (1 - \frac{1}{e}) e^{\gamma/2} = 0.42 \ldots > \frac{2}{5}$, we obtain (3.8) (for all sufficiently large $H$), and the proof of part (b) is complete. \(\square\)

4. Proof of Lemmas 3.2 and 3.3

Throughout this section, at each and every occurrence of $O$, $\ll$, and $\gg$, the implied constant is absolute. When we write $A \asymp B$, we mean $A \ll B$ and $B \ll A$ both hold, the implied constants being absolute. The letter $c$, by itself, always denotes an absolute positive constant, possibly a different constant at each occurrence. Also, $s = \sigma + i\tau$ denotes a complex variable, $\sigma$ and $\tau$ being real. We often use $s$ and $\sigma + i\tau$ interchangeably: for example if we write $|X^s| = X^\sigma$, or $\zeta(s) \neq 0$ for $|\tau| \geq 2$, $\sigma \geq 1 - 1/\log |\tau|$, it is to be understood that $s = \sigma + i\tau$. We always assume $q$ is an integer and that $q \geq 3$. A Dirichlet character $\chi$ is always to be taken as a character to the modulus $q$, with corresponding Dirichlet $L$-function $L(s, \chi)$. Finally, $\chi_0$ always denotes the principal character to the modulus $q$.

The following proof of Lemma 3.2 is, mutatis mutandis, a proof due to Hardy [6] of Mertens’ theorem (the case $q = 1$), in which the following propositions are used:
**Proposition 4.1.** If $a > 0$, $\delta > 0$, and $\delta \to 0$, then
\[ \int_{a}^{\infty} e^{-\delta t} t \, dt - \log \frac{1}{\delta} \to - \log a - \gamma \]
when $\delta \to 0$, where $\gamma = 0.57721 \ldots$ is the Euler-Mascheroni constant.

*Proof.* This is Proposition (A) in [6], where it is not proved, but is said to be “familiar in the theory of the Gamma function”. 

**Proposition 4.2.** If
\[ J(\delta) = \int_{a}^{\infty} f(t) t^{-\delta} \, dt \]
where $a > 0$, is convergent for $\delta > 0$ and tends to a limit $l$ when $\delta \to 0$, and
\[ f(t) = O \left( \frac{1}{t \log t} \right), \]
then $J(0)$ is convergent and has the value $l$.

*Proof.* This result of Landau [8] is Proposition (D) in [6].

In addition to Propositions 4.1 and 4.2, we will use some basic properties of Dirichlet $L$-series. In particular, for $\chi \neq \chi_0$, $L(s, \chi)$ is analytic and converges for $\sigma > 0$; it is absolutely convergent for $\sigma > 1$. We have $L(1, \chi) > 0$ for $\chi \neq \chi_0$ (see Theorem 4.4 below). We have
\[ L(s, \chi_0) = \zeta(s) \prod_{\nu|q} \left( 1 - \frac{1}{p^s} \right), \quad \sigma > 1, \tag{4.1} \]
where $\zeta(s)$ is the Riemann zeta function, analytic throughout the complex plane except for a simple pole at $s = 1$, where the residue is 1; in fact [12, (2.1.16)]
\[ \zeta(s) = \frac{1}{s-1} + \gamma + O (|s-1|), \quad s \to 1. \tag{4.2} \]

Because of the orthogonality relation [11, II §8.1, Theorem 2 (a)]:
\[ \sum_{\chi \mod q} \chi(n) = \begin{cases} \phi(q) & \text{if } n \equiv 1 \mod q, \\ 0 & \text{otherwise}, \end{cases} \tag{4.3} \]
we have, for $\sigma > 1$,
\[
\log \prod_{\chi \bmod q} L(s, \chi) = \sum_{\chi \bmod q} \log \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}
\]
\[
= \sum_{\chi \bmod q} \sum_p \sum_{m=1}^{\infty} \chi(p^m) mp^{ms}
\]
\[
= \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \sum_{\chi \bmod q} \chi(p^m)
\]
\[
= \phi(q) \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}.
\]
Exponentiating, we deduce that for $\sigma > 1$, $\prod_{\chi \bmod q} L(\sigma, \chi)$ is a real number greater than or equal to 1. Also, we see that for $\sigma > 1$,
\[
\sum_{p \equiv 1 \bmod q} \log \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{p \equiv 1 \bmod q} \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}
\]
\[
= \log \Theta(s) + \frac{1}{\phi(q)} \log \prod_{\chi \bmod q} L(s, \chi),
\]
where
\[
\Theta(s) := \exp \left( - \sum_{p \equiv 1 \bmod q} \sum_{m=2}^{\infty} \frac{1}{mp^{ms}} \right)
\]
is absolutely convergent and analytic for $\sigma > 1/2$, and $\Theta(1) \asymp 1$ (see the proof of Lemma 4.5 (a), below).

We will also use Theorem 3.1 (Siegel-Walfisz), and therefore, implicitly, certain properties and results concerning $L$-functions that are used in its proof, in the proof of Lemma 3.2.

Proof of Lemma 3.2. Let $X \geq 3$ be a number and let $q \geq 3$ be an integer. We use the notation
\[
\pi(t; q, 1) := \sum_{p \leq t \bmod q} 1.
\]
We begin by noting that for $\sigma \geq 1$, partial summation gives
\[
\sum_{p \leq X} \log \left(1 - \frac{1}{p^s}\right)^{-1} = \pi(X; q, 1) \log \left(1 - \frac{1}{X^s}\right)^{-1} + \sigma \int_e^X \frac{\pi(t; q, 1)}{t(t^s - 1)} \, dt.
\]
Now \( \pi(X; q, 1) \log (1 - X^{-\sigma})^{-1} \ll X^{1-\sigma} \), and so, for \( \sigma > 1 \), letting \( X \to \infty \) in (4.7) gives

\[
\sum_{p \equiv 1 \mod q} \log \left( 1 - \frac{1}{p^\sigma} \right)^{-1} = \sigma \int_e^\infty \frac{\pi(t; q, 1)}{t(t^\sigma - 1)} \, dt
\]

\[
= \sigma \int_e^\infty \left( \pi(t; q, 1) - \frac{t}{\phi(q) \log t} \right) t^{-1-\sigma} \, dt
+ \sigma \int_e^\infty \frac{\pi(t; q, 1)}{t^1+\sigma(t^\sigma - 1)} \, dt + \int_e^\infty \frac{t^{-\sigma}}{\phi(q) \log t} \, dt
\]

\[
= J_1(\sigma) + J_2(\sigma) + J_3(\sigma).
\]

On the other hand, with \( \Theta(\sigma) \) as defined in (4.6) we have, by (4.5) and (4.1),

\[
\sum_{p \equiv 1 \mod q} \log \left( 1 - \frac{1}{p^\sigma} \right)^{-1} = \log \Theta(\sigma) + \frac{1}{\phi(q)} \log \zeta(\sigma)
+ \frac{1}{\phi(q)} \log \left( \prod_{p \mid q} \left( 1 - \frac{1}{p^\sigma} \right) \prod_{\chi \neq \chi_0} L(\sigma, \chi) \right).
\]

Comparing the last two expressions for \( \sum_{p \equiv 1 \mod q} \log (1 - p^{-\sigma})^{-1} \), we obtain, for \( \sigma > 1 \),

\[
J_1(\sigma) = -J_2(\sigma) - J_3(\sigma) + \frac{1}{\phi(q)} \log \zeta(\sigma)
+ \log \Theta(\sigma) + \frac{1}{\phi(q)} \log \left( \prod_{p \mid q} \left( 1 - \frac{1}{p^\sigma} \right) \prod_{\chi \neq \chi_0} L(\sigma, \chi) \right).
\]

As \( \sigma \to 1^+ \), \( J_2(\sigma) \to J_2(1) \) by uniform convergence;

\[
J_3(\sigma) = \frac{\sigma}{\phi(q)} \int_1^\infty e^{-(\sigma-1)u} \frac{du}{u} = \frac{1}{\phi(q)} \left( \log \left( \frac{1}{\sigma-1} \right) - \gamma + o(1) \right),
\]

by Proposition 4.1; and

\[
\frac{1}{\phi(q)} \log \zeta(\sigma) = \frac{1}{\phi(q)} \left( \log \left( \frac{1 + o(1)}{\sigma-1} \right) \right) = \frac{1}{\phi(q)} \left( \log \left( \frac{1}{\sigma-1} \right) + o(1) \right),
\]

by (4.2). Combining all of this with (4.8), we see that, as \( \sigma \to 1^+ \),

\[
J_1(\sigma) \to l := -J_2(1) + \frac{\gamma}{\phi(q)} + \log \Theta(1) + \frac{1}{\phi(q)} \log \left( (\phi(q)/q) \prod_{\chi \neq \chi_0} L(1, \chi) \right).
\]

Applying Proposition 4.2, with \( \delta = \sigma - 1 \), and

\[
f(t) = \left( \pi(t; q, 1) - \frac{t}{\phi(q) \log t} \right) t^{-2} \ll \left( \sum_{p \leq t} 1 + \frac{t}{\log t} \right) t^{-2} \ll \frac{1}{t \log t},
\]
(by the prime number theorem), we see that $J_1(1)$ is convergent and has value $l$. Thus

$$J_1(1) + J_2(1) = \frac{\gamma}{\phi(q)} + \log \Theta(1) + \frac{1}{\phi(q)} \log \left( \frac{\phi(q)/q}{\prod_{\chi \neq \chi_0} L(1, \chi)} \right).$$

(4.9)

Now, supposing $3 \leq q \leq (\log X)^A$ for some positive number $A$, Theorem 3.1 (Siegel-Walfisz) implies that

$$\left( \pi(t; q, 1) - \frac{t}{\phi(q) \log t} \right) t^{-2} \ll \frac{1}{t(\log t)^2}$$

for all $t \geq X$, and so

$$\int_e^X \left( \pi(t; q, 1) - \frac{t}{\phi(q) \log t} \right) t^{-2} dt = J_1(1) - \int_X^\infty \left( \pi(t; q, 1) - \frac{t}{\phi(q) \log t} \right) t^{-2} dt$$

$$= J_1(1) + O \left( \frac{1}{\log X} \right).$$

(4.10)

Also, since, by the prime number theorem, $\pi(t; q, 1)/(t^2(t-1)) \ll 1/(t^2 \log t)$ for all $q \geq 3$ and $t \geq e$, we have

$$\int_e^X \frac{\pi(t; q, 1)}{t^2(t-1)} dt = J_2(1) - \int_X^\infty \frac{\pi(t; q, 1)}{t^2(t-1)} dt = J_2(1) + O \left( \frac{1}{X \log X} \right).$$

(4.11)

We return at last to (4.7), in which we now take $\sigma = 1$, $X$ a sufficiently large number, and $q$ an integer in the range $3 \leq q \leq (\log X)^A$. Using the fact that $\pi(X; q, 1) \ll X/\log X$ by the prime number theorem, followed by (4.10), and (4.11), we obtain

$$\sum_{\substack{p \leq X \\text{mod } q}} \log \left( 1 - \frac{1}{p} \right)^{-1} = \pi(X; q, 1) \log \left( 1 - \frac{1}{X} \right)^{-1} + \int_e^X \frac{\pi(t; q, 1)}{t(t-1)} dt$$

$$= O \left( \frac{1}{\log X} \right) + \int_e^X \left( \pi(t; q, 1) - \frac{t}{\phi(q) \log t} \right) t^{-2} dt$$

$$+ \int_e^X \frac{\pi(t; q, 1)}{t^2(t-1)} dt + \frac{1}{\phi(q)} \int_e^X \frac{dt}{t \log t}$$

$$= O \left( \frac{1}{\log X} \right) + J_1(1) + J_2(1) + \frac{1}{\phi(q)} \log \log X.$$
with
\[ c(q) := \Theta(1) \left( \frac{\phi(q)}{q} \prod_{\chi \neq \chi_0} L(1, \chi) \right)^{1/\phi(q)}. \] (4.12)

By the observations made before the proof, \( \Theta(1) \) is positive, and the product \( \prod_{\chi \neq \chi_0} L(1, \chi) \) is real and positive, so we may indeed take the real, positive \( \phi(q) \)th root here. Hence \( c(q) \) is real and positive. \( \square \)

We now proceed with the proof of Lemma 3.3. We will use the following estimate:

**Theorem 4.3** (Effective Perron formula). Let \( F(s) := \sum_{n=1}^{\infty} a_n n^{-s} \) be a Dirichlet series with abscissa of absolute convergence \( \sigma_a \). For \( \kappa > \max(0, \sigma_a) \), \( T \geq 1 \), and \( X \geq 1 \), we have
\[
\sum_{n \leq X} a_n = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} F(s) \frac{X^s}{s} \, ds + O \left( \left| a_n \right| n^{\kappa(1 + T|\log(X/n)|)} \right).
\] (4.13)

*Proof.* The first estimate is the effective Perron formula (see [11, II §2.1, Theorem 2]). From this, assuming \( X \geq 2 \), \( 0 < \kappa - 1 \ll 1/\log X \), and \( a_n \ll 1 \) for every \( n \), then
\[
\sum_{n \leq X} a_n = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} F(s) \frac{X^s}{s} \, ds + O \left( (X \log X)/T \right).
\] (4.13)

For \( X/2 < n \leq 2X \), we use
\[
|\log(X/n)| = \left| \log \left( 1 + \frac{X-n}{n} \right) \right| > \begin{cases} \frac{1}{2} \left( \frac{X-n}{n} \right) & \text{if } n < X, \\ \frac{n-X}{n} & \text{if } n > X. \end{cases}
\]
We have
\[
X^\kappa \sum_{X/2 < n \leq 2X} \frac{1}{n^\kappa (1 + T |\log(X/n)|)} \leq \frac{X^\kappa}{(X - 1/2)^\kappa} + \frac{X^\kappa}{T} \sum_{X/2 < n \leq 2X} \frac{n}{|n - X|} \\
\ll 1 + \frac{X}{T} \sum_{X/2 < n \leq 2X} \frac{1}{n - X} \\
\ll 1 + \frac{X}{T} \sum_{n \leq 2X} \frac{1}{n} \\
\ll 1 + (X \log X)/T.
\]

Combining, we obtain the error term in (4.13). \qed

Let us now gather some more properties of Dirichlet characters and $L$-series that will be used in the proof of Lemma 3.3.

**Theorem 4.4.** There is an effectively computable positive constant $\bar{c}$ such that the following holds for any given integer $q \geq 3$. The product $\prod_{\chi \bmod q} L(s, \chi)$ has at most one zero in the region
\[
D := \left\{ \sigma + i\tau : \sigma \geq 1 - \frac{\bar{c}}{\log \max(q, q|\tau|)} \right\}.
\]
Such a zero, if it exists, is real and simple, and corresponds to a non-principal real character.

**Proof.** See [1, Chapter 14]. As far as explicit constants $\bar{c}$ go, the best result is due to Kadiri [7, Theorem 1.1], who has shown that $\bar{c} = 1/6.41$ is admissible. \qed

As noted earlier, the $L$-series $L(s, \chi), \chi \neq \chi_0$, converge for $\sigma > 0$. Indeed, the Pólya-Vinogradov inequality [1, Chapter 23]:
\[
\max_{t \geq 1} \left| \sum_{n \leq t} \chi(n) \right| \ll q^{1/2} \log q, \quad \chi \neq \chi_0,
\]
together with partial summation, gives
\[
L(s, \chi) \ll (q^{1/2} \log q) \left( 1 + |s| \int_1^\infty \frac{dt}{t^{\sigma + 1}} \right) \ll (q^{1/2} \log q) \left( 1 + |s|/\sigma \right), \quad \sigma > 0, \quad \chi \neq \chi_0. \tag{4.14}
\]

Thus, the product $\prod_{\chi \bmod q} L(s, \chi)$ is analytic at every point in the region $D$ of Theorem 4.4, except for a simple pole at $s = 1$. This is due to the Riemann zeta function, which is analytic except for a simple pole at $s = 1$, where the residue is one. More precisely, we have (4.2). Let us also recall here that
\[
\zeta(\sigma + i\tau) \ll \log |\tau|, \quad |\tau| \geq 2, \quad \sigma \geq 1 - c/\log |\tau|. \tag{4.15}
\]
By (4.1), we have
\[ \prod_{\chi \bmod q} L(s, \chi) = \zeta(s) \prod_{p \bmod q} \left( 1 - \frac{1}{p^s} \right) \prod_{\chi \neq \chi_0} L(s, \chi), \]
and, by Theorem 4.4, \( L(1, \chi) \neq 0 \) for \( \chi \neq \chi_0 \). In fact, we have
\[ L(1, \chi) \gg \begin{cases} q^{-\frac{1}{2}} \chi^2 = \chi_0, \\ (\log q)^{-7} \chi^2 \neq \chi_0. \end{cases} \tag{4.16} \]
(See [9, §11.3, Theorem 11.11] and [11, §II.8, Theorem 8] respectively.) Furthermore, as we have already observed, as a consequence of (4.4), we have \( \prod_{\chi \bmod q} L(\sigma, \chi) \geq 1 \) for \( \sigma > 1 \).
Thus, if we define a simply connected domain
\[ \mathcal{D}^* := \left\{ \sigma + i \tau : \sigma \geq 1 - \frac{\bar{c}}{\log \max(q, q|\tau|)} \right\} \setminus \left[ 1 - \frac{\bar{c}}{\log q}, 1 \right], \tag{4.17} \]
we see that the function
\[ \left( (s - 1) \prod_{\chi \bmod q} L(s, \chi) \right)^{1/\phi(q)} \]
is analytic throughout \( \mathcal{D}^* \cup \{1\} \), whereas the function
\[ \left( \prod_{\chi \bmod q} L(s, \chi) \right)^{1/\phi(q)} = (s - 1)^{-1/\phi(q)} \left( (s - 1) \prod_{\chi \bmod q} L(s, \chi) \right)^{1/\phi(q)} \]
has a branch point at \( s = 1 \), but is analytic throughout \( \mathcal{D}^* \). We always choose the principal value of the complex logarithm, so that \( \lim_{s \to 1} ((s - 1) \zeta(s))^{1/\phi(q)} = 1 \), for instance.
These functions, slightly modified, feature in Lemma 4.5 below. Given an integer \( q \geq 3 \) and a number \( Y \geq 1 \), we define \( \Theta(s) \), for \( \sigma > 1/2 \), as in (4.6); then we define
\[ G(s) := \Theta(s) \left( (s - 1) \prod_{\chi \bmod q} L(s, \chi) \right)^{1/\phi(q)} \prod_{p \leq Y} \left( 1 - \frac{1}{p^s} \right), \quad s \in \mathcal{D}^* \cup \{1\}; \tag{4.18} \]
and
\[ F(s) := (s - 1)^{-1/\phi(q)} G(s), \quad s \in \mathcal{D}^*. \tag{4.19} \]
Also, given \( \sigma > 0 \), we define
\[ \Pi_1(\sigma; q) := \prod_{p \mid q} \left( 1 + \frac{1}{p^\sigma} \right), \quad \Pi_2(\sigma; Y) := \prod_{p \leq Y} \left( 1 + \frac{1}{p^\sigma} \right). \tag{4.20} \]
Note that if \( q \ll 1 \) then \( \Pi_1(\sigma; q) \ll 1 \). Otherwise, since \( \log(1 + x) \leq x \) for \( x \geq 0 \), we have

\[
\Pi_1(\sigma; q) \leq \exp \left( \sum_{p \mid q} \frac{1}{p^\sigma} \right) \leq \exp \left( q^{1-\sigma} \sum_{p \mid q} \frac{1}{p} \right) \leq \exp \left( cq^{1-\sigma} \log \log q \right) \quad (4.21)
\]

by a standard estimate. Similarly, if \( Y \ll 1 \), or if \( Y \ll q \), then \( \Pi_2(\sigma; Y) \ll 1 \), and otherwise

\[
\Pi_2(\sigma; Y) \leq \exp \left( \sum_{p \leq Y} \frac{1}{p^\sigma} \right) \leq \exp \left( Y^{1-\sigma} \sum_{p \leq Y} \frac{1}{p} \right) \leq \exp \left( cY^{1-\sigma} \log \log Y \right). \quad (4.22)
\]

**Lemma 4.5.** Fix an integer \( q \geq 3 \) and a number \( Y \geq 1 \), and let \( D^* \), \( \Theta(s) \), \( G(s) \), \( F(s) \), \( \Pi_1(\sigma; q) \), and \( \Pi_2(\sigma; Y) \) be as defined in (4.6), (4.17) — (4.20). (a) The function \( G(s) \) is analytic throughout \( D^* \cup \{1\} \). For \( s \in D^* \) with \( |\tau| \geq 2 \), we have

\[
G(s) \ll (\log |\tau|)^{1/\phi(q)}(1 + |\tau|)^{1-1/\phi(q)}(q^{1/2} \log q)\Pi_1(\sigma; q)\Pi_2(\sigma; Y). \quad (4.23)
\]

For \( s \in D^* \cup \{1\} \) with \( s - 1 \ll 1 \), we have

\[
G(s) \ll (q^{1/2} \log q)\Pi_1(\sigma; q)\Pi_2(\sigma; Y). \quad (4.24)
\]

Also,

\[
G(s) = G(1) + O \left( (s - 1)(q^{1/2} \log q)\Pi_1(\sigma; q)\Pi_2(\sigma; Y) \right). \quad (4.25)
\]

(b) The function \( F(s) \) is analytic at every point in \( D^* \). We have

\[
F(s) = \prod_{p \equiv 1 \mod q} \left( 1 - \frac{1}{p^\sigma} \right)^{-1} \prod_{p \equiv 1 \mod q} \prod_{p \leq Y} \left( 1 - \frac{1}{p^\sigma} \right), \quad \sigma > 1. \quad (4.26)
\]

**Proof.** (a) For any \( s \) with \( \sigma \geq 1/2 + \delta > 1/2 \), we have, by definition (4.6) of \( \Theta(s) \), the bound

\[
|\log \Theta(s)| \leq \sum_p \sum_{m=2}^{\infty} \frac{1}{p^{m\sigma}} \leq \sum_p \sum_{m=2}^{\infty} \frac{1}{p^{m(1/2 + \delta)}} = \sum_p \frac{1/p^{1+2\delta}}{1 - 1/p^{1/2 + \delta}}
\]

\[
\leq \frac{1}{1 - 1/2^{1+2\delta}} \sum_{n=1}^{\infty} \frac{1}{n^{1+2\delta}} \leq \frac{1}{1 - 1/2^{1/2 + \delta}} \left( 1 + \int_1^\infty \frac{dt}{t^{1+2\delta}} \right)
\]

\[
= \frac{1}{1 - 1/2^{1/2 + \delta}} \left( 1 + \frac{1}{2\delta} \right).
\]

Denoting this last expression by \( c(\delta) \), we have \( |\Theta(s)| \leq e^{\log \Theta(s)} \leq e^{c(\delta)} \), and likewise \( 1/|\Theta(s)| \leq e^{-\log \Theta(s)} \leq e^{c(\delta)} \). Thus, uniformly for \( \sigma \geq 1/2 + \delta \), we have

\[
e^{-c(\delta)} \leq |\Theta(s)| \leq e^{c(\delta)}, \quad c(\delta) := \frac{1}{1 - 1/2^{1/2 + \delta}} \left( 1 + \frac{1}{2\delta} \right), \quad (4.27)
\]
and so we can differentiate the series for \( \log \Theta(s) \) term by term by the uniform convergence theorem. Indeed,

\[
\frac{d^k}{ds^k} \log \Theta(s) = - \sum_{p \not\equiv 1 \mod q} \sum_{m=2}^{\infty} \frac{(- \log p^m)^k}{mp^{ms}}, \quad \sigma > 1/2, \quad k \geq 1.
\]

We see that \( \log \Theta(s) \), and therefore \( \Theta(s) = \exp (\log \Theta(s)) \), is analytic throughout the half-plane \( \sigma > 1/2 \).

Note in particular that (4.27) implies \( \Theta(s) \ll 1 \) on \( D^* \cup \{1\} \). By our earlier discussion, the term \( (\cdots)^{1/\phi(q)} \) in the definition (4.18) of \( G(s) \) is analytic on \( D^* \cup \{1\} \). The final product in (4.18), being finite, is non-zero and uniformly bounded by \( \prod_{p \leq Y} (1 + p^{-1/2}) \), say, for \( s \in D^* \cup \{1\} \).

Hence \( G(s) \) is analytic on \( D^* \cup \{1\} \), where by the Pólya-Vinogradov inequality (4.14), and since \( \sigma \gg 1 \) throughout \( D^* \), we have (recalling the definition (4.20) of \( \Pi_2(\sigma, Y) \)):

\[
G(s) \ll \left( (s-1) \prod_{\chi \mod q} L(s, \chi) \right)^{1/\phi(q)} \Pi_2(\sigma; Y)
\]

\[
\ll \left( (s-1)\zeta(s) \prod_{p \mid q} \left( 1 - \frac{1}{p^s} \right) \right)^{1/\phi(q)} (q^{1/2} \log q)(1 + |\tau|)^{1-1/\phi(q)} \Pi_2(\sigma; Y).
\]

For \( s - 1 \ll 1 \), we can use (4.2) to obtain (recalling the definition (4.20) of \( \Pi_1(\sigma, q) \)):

\[
\left( (s-1)\zeta(s) \prod_{p \mid q} \left( 1 - \frac{1}{p^s} \right) \right)^{1/\phi(q)} \ll \Pi_1(\sigma; q).
\]

For \( s \in D^* \) with \( |\tau| \geq 2 \), we use (4.15) to obtain

\[
\left( (s-1)\zeta(s) \prod_{p \mid q} \left( 1 - \frac{1}{p^s} \right) \right)^{1/\phi(q)} \ll (\log |\tau|)^{1/\phi(q)} \Pi_1(\sigma; q).
\]

Combining gives (4.23) (upon noting that \( 1 + |\tau| \ll 1 \) if \( s - 1 \ll 1 \)), and (4.24).

We may express \( G(s) \) as a Taylor series in a neighbourhood of 1. Let \( s \in D^* \) with \( s - 1 \ll 1 \) be given, and choose a positive number \( r \ll 1 \) so that \( r - |s - 1| \gg 1 \), and so that the circle \( \gamma(r) := \{ 1 + re^{i\theta} : -\pi < \theta < \pi \} \), without the point \( 1 - r \), is contained in \( D^* \). By the Cauchy integral formulae, we have

\[
G(s) = G(1) + \sum_{j=1}^{\infty} \frac{G^{(j)}(1)}{j!} (s-1)^j
\]

\[
= G(1) + \sum_{j=1}^{\infty} \frac{(s-1)^j}{2\pi i} \int_{\gamma(r)} \frac{G(w)}{(w-1)^{j+1}} \, dw
\]

\[
= G(1) + O \left( \max_{|w-1|=r} |G(w)| \sum_{j=1}^{\infty} \frac{|s-1|^j}{r^j} \right).
\]
By our choice of $r$, we have
\[ \sum_{j=1}^{\infty} \frac{|s - 1|^j}{r^j} = \frac{|s - 1|}{r - |s - 1|} \ll |s - 1|. \]

Using (4.24) for $\max_{|w-1|=r} |G(w)|$ and combining gives (4.25).

(b) By definition (4.19) of $F(s)$ and analyticity of $G(s)$ throughout $\mathcal{D}^*$, we conclude that $F(s)$ is also analytic throughout $\mathcal{D}^*$. It has a branch point at $s = 1$. Now, for $\sigma > 1$, we can write
\[ F(s) = \Theta(s) \left( \prod_{\chi \mod q} L(s, \chi) \right)^{1/\phi(q)} \prod_{p \equiv 1 \mod q} \left( 1 - \frac{1}{p^s} \right), \]
and as in (4.3) — (4.5), we see that, for $\sigma > 1$,
\[ \log \Theta(s) + \frac{1}{\phi(q)} \log \left( \prod_{\chi \mod q} L(s, \chi) \right) = \sum_{p \equiv 1 \mod q} \log \left( 1 - \frac{1}{p^s} \right)^{-1}. \]
Exponentiating and combining gives (4.26).

We are finally ready to prove Lemma 3.3. The proof is an adaptation of the Selberg-Delange method, as presented in [11, §II.5].

**Proof of Lemma 3.3.** Fix numbers $A > 0$ and $\alpha \in (0, \frac{1}{2})$. Let $X$ be a sufficiently large number, fix an integer $q$ satisfying $3 \leq q \leq (\log X)^\alpha$, and fix a number $Y$ satisfying $1 \leq Y \leq (\log X)^A$.

Let
\[ a_n := \begin{cases} 1 & \text{of } p \mid n \Rightarrow p \equiv 1 \mod q \text{ and } p > Y, \\ 0 & \text{otherwise}, \end{cases} \]
be the characteristic function of the integers composed only of primes that are both congruent to 1 mod $q$ and greater than $Y$, and let
\[ F(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p \equiv 1 \mod q} \left( 1 - \frac{1}{p^s} \right)^{-1} \prod_{p \equiv 1 \mod q} \left( 1 - \frac{1}{p^s} \right), \quad \sigma > 1, \]
be its associated Dirichlet series and Euler product. Let $\kappa$ and $T$ be parameters, to be determined later, but satisfying $1 < \kappa \leq 1 + 1/\log X$ and $T > 1$. By equation (4.13) of Theorem 4.3, we have
\[ \sum_{n \leq X \atop p \mid n \Rightarrow p \equiv 1 \mod q \text{ and } p > Y} a_n = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} F(s) \frac{X^s}{s} ds + O(1) + O((X \log X)/T). \quad (4.28) \]
By Lemma 4.5 (b), $F(s)$ admits an analytic continuation to the simply connected domain $\mathcal{D}^*$ (defined in (4.17)), namely (4.19):

$$F(s) = (s - 1)^{-1/\phi(q)} G(s) = \Theta(s) \left( \zeta(s) \prod_{p \mid q} \left( 1 - \frac{1}{p^s} \right) \prod_{\chi \neq \chi_0} L(s, \chi) \right)^{1/\phi(q)} \prod_{p \leq Y \atop p \equiv 1 \mod q} \left( 1 - \frac{1}{p^s} \right), \quad s \in \mathcal{D}^*.$$

Here, $G(s)$ is as in (4.18) (analytic throughout $\mathcal{D}^* \cup \{1\}$), and $\Theta(s)$ is as in (4.6) (analytic throughout the half-plane $\sigma > 1/2$). The Cauchy integral theorem then allows us to deform the segment of integration $[\kappa - iT, \kappa + iT]$ in (4.28) into a closed, rectifiable path $\mathcal{C}$, joining its end-points and lying inside $\mathcal{D}^*$.

Let us define our contour $\mathcal{C}$. Let $\kappa$, $T$, and $\eta$ be such that the rectangle with corners $(1 - \eta, \pm iT)$ and $(\kappa, \pm iT)$, with the point $1 - \eta$ deleted, lies inside the simply connected domain $\mathcal{D}^*$. The contour $\mathcal{C}$ is this same rectangle, with the point $1 - \eta$ deleted, traversed clockwise, and with a detour taken around $s = 1$ via the truncated ‘Hankel’ contour, $\mathcal{H}$. The contour $\mathcal{H}$ consists of the circle $|s - 1| = r$ (the only condition on $r$ is that $0 < r < \kappa - 1$), excluding the point $s = 1 - r$, together with the line segment $[1 - \eta, 1 - r]$, traced out twice, with respective arguments $-\pi$ and $+\pi$. That is,

$$\mathcal{H} := [1 + \eta e^{-\pi i}, 1 + re^{-\pi i}] \cup \{1 + re^{i\theta} : -\pi < \theta < \pi\} \cup [1 + re^{\pi i}, 1 + \eta e^{\pi i}].$$

We denote the left vertical line segments of $\mathcal{C}$ by $C_1$ and the horizontal line segments by $C_2$. That is,

$$C_1 := [1 - \eta - iT, 1 - \eta) \cup (1 - \eta, 1 - \eta + iT];$$

$$C_2 := [\kappa - iT, 1 - \eta - iT] \cup [1 - \eta + iT, \kappa + iT].$$

Thus,

$$\mathcal{C} = C_1 \cup C_1 \cup \mathcal{H} \cup [\kappa + iT, \kappa - iT].$$

By Cauchy’s integral theorem we have

$$\left\{ \int_{C_1 + C_2 + \mathcal{H}} - \int_{C_1 - iT}^{\kappa + iT} \right\} F(s) \frac{X^s}{s} ds = \int_{\mathcal{C}} F(s) \frac{X^s}{s} ds = 0,$$

and so

$$\frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} F(s) \frac{X^s}{s} ds = \frac{1}{2\pi i} \int_{\mathcal{H}} F(s) \frac{X^s}{s} ds + O \left( \int_{C_1 + C_2} F(s) \frac{X^s}{s} ds \right). \quad (4.29)$$
Let us return to the $O$-term later. We will now show that
\begin{equation}
\frac{1}{2\pi i} \int_{\mathcal{H}} F(s) \frac{X^s}{s} \, ds = \frac{X(\log X)^{\frac{1}{\phi(q)}}}{\log X} \left( \frac{G(1)}{\Gamma(1/\phi(q))} + O(\varepsilon_1) + O(\varepsilon_2) \right), \tag{4.30}
\end{equation}
where
\[ \varepsilon_1 := \frac{(q^{\frac{1}{2}} \log q)\Pi_{1}(1-\eta;q)\Pi_{2}(1-\eta;Y)}{\log X}, \quad \varepsilon_2 := G(1)X^{-\eta}. \]

On the Hankel contour $\mathcal{H}$ we have $s \asymp 1$, that is $1/s = 1 - (s-1)/s = 1 + O(|s-1|)$. Therefore, by (4.24) and (4.25) of Lemma 4.5 (a), for $s \in \mathcal{H}$, we have
\[ \frac{G(s)}{s} = G(s) + O(|s-1|G(s)) = G(1) + O \left( |s-1|(q^{\frac{1}{2}} \log q)\Pi_{1}(\sigma;q)\Pi_{2}(\sigma;Y) \right). \]

Hence, since $F(s) = (s-1)^{-1/\phi(q)}G(s)$, and since $\Pi_{1}(\sigma;q)$ and $\Pi_{2}(\sigma;Y)$ are bounded respectively by $\Pi_{1}(1-\eta;q)$ and $\Pi_{2}(1-\eta;Y)$ on $\mathcal{H}$ (recall the definitions (4.20)), we have
\begin{equation}
\frac{1}{2\pi i} \int_{\mathcal{H}} F(s) \frac{X^s}{s} \, ds = \frac{G(1)}{2\pi i} \int_{\mathcal{H}} \frac{X^s}{(s-1)^{1/\phi(q)}} \, ds
\end{equation}
\[ \quad + O \left( (q^{\frac{1}{2}} \log q)\Pi_{1}(1-\eta;q)\Pi_{2}(1-\eta;Y) \int_{\mathcal{H}} X^{\sigma} |s-1|^{1-1/\phi(q)} \, ds \right). \tag{4.31}
\]

Via the substitution $\sigma = 1 - s$, we see that the straight line segments of $\mathcal{H}$ contribute
\begin{equation}
\frac{G(1)}{2\pi i} \left( - \int_{\eta}^{\rho} \frac{X^{1-\sigma}}{(\sigma e^{-i\pi})^{1/\phi(q)}} \, d\sigma + \int_{\rho}^{\eta} \frac{X^{1-\sigma}}{(\sigma e^{i\pi})^{1/\phi(q)}} \, d\sigma \right)
\end{equation}
\[ = \frac{G(1)\sin(\pi/\phi(q))}{\pi} \int_{\rho}^{\eta} X^{1-\sigma} \sigma^{-1/\phi(q)} \, d\sigma \]
to the main term on the right-hand side of (4.31). The circle contributes
\begin{equation}
\frac{G(1)}{2\pi i} \int_{-\pi}^{\pi} \frac{X^{1+r e^{i\theta}}}{(r e^{i\theta})^{1/\phi(q)}} \, i r e^{i\theta} \, d\theta \leq G(1)X^{1+r} r^{1-1/\phi(q)} \ll G(1)X r^{1-1/\phi(q)},
\end{equation}
because $r < \kappa - 1 < 1/\log X$. As all of this holds for arbitrarily small $r$, we conclude that
\begin{equation}
\frac{G(1)}{2\pi i} \int_{\mathcal{H}} \frac{X^s}{(s-1)^{1/\phi(q)}} \, ds = \frac{G(1)\sin(\pi/\phi(q))}{\pi} \int_{0^+}^{\eta} X^{1-\sigma} \sigma^{-1/\phi(q)} \, d\sigma
\end{equation}
\[ = \frac{G(1)\sin(\pi/\phi(q))}{\pi} \cdot \frac{X(\log X)^{\frac{1}{\phi(q)}}}{\log X} \int_{0^+}^{\eta \log X} e^{-u} u^{1/\phi(q)} \, du,
\]
after the substitution $u = \sigma \log X$. We can approximate the last integral by $\Gamma(1-1/\phi(q))$:
\begin{equation}
\int_{0^+}^{\eta \log X} e^{-u} u^{-1/\phi(q)} \, du = \left\{ \int_{0^+}^{\infty} - \int_{\eta \log X}^{\infty} \right\} e^{-u} u^{-1/\phi(q)} \, du
\end{equation}
\[ = \left( \Gamma \left( 1 - \frac{1}{\phi(q)} \right) + O \left( X^{-\eta} \right) \right). \]
Combining and using the identity \( \Gamma(\theta)\Gamma(1 - \theta) = \pi / \sin(\pi \theta) \), we obtain
\[
\frac{G(1)}{2\pi i} \int_{\gamma} \frac{X^s}{(s - 1)^{1/\phi(q)}} \, ds = \frac{X(\log X)^{\frac{1}{\phi(q)}}}{\log X} \left( \frac{G(1)}{\Gamma(1/\phi(q))} + O\left( G(1)X^{-\eta}\right) \right).
\] (4.32)

Similarly, to the integral over \( \gamma \) in the \( O \)-term of (4.31), each straight line segment contributes
\[
\int_{1-\eta}^{1-r} X^\sigma |s - 1|^{1-1/\phi(q)} \, ds \lesssim \int_{1-\eta}^1 X^\sigma (1 - \sigma)^{1-1/\phi(q)} \, d\sigma
\]
\[
= \frac{X(\log X)^{\frac{1}{\phi(q)}}}{(\log X)^2} \int_{0}^{\eta \log X} e^{-u} u^{1-1/\phi(q)} \, du
\]
\[
= \frac{X(\log X)^{\frac{1}{\phi(q)}}}{(\log X)^2} (\Gamma(2 - 1/\phi(q)) + X^{-\eta} \eta^{1-1/\phi(q)}).
\]
(We used the substitution \( u = (1 - \sigma) \log X \) and approximated the resulting integral by \( \Gamma(2 - 1/\phi(q)) \).) To the integral over \( \gamma \) in the \( O \)-term of (4.31), the circle \( |s - 1| = r \) contributes at most
\[
X^{1+\sigma-1} \int_{|s-1|=r} r^{1-1/\phi(q)} |ds| \lesssim 2\pi X^{1+r} r^{2-1/\phi(q)} \ll X^{r^{2-1/\phi(q)}}.
\]
Letting \( r \) tend to zero and combining gives
\[
\int_{\gamma} X^\sigma |s - 1|^{1-1/\phi(q)} \, ds \ll \frac{X(\log X)^{\frac{1}{\phi(q)}}}{\log X} \cdot \frac{1}{\log X}.
\] (4.33)

Putting (4.33) and (4.32) into (4.31) gives (4.30).

Let us turn now to the integral over \( C_1 + C_2 \) in the \( O \)-term of (4.29). First note that \( \Pi_1(\sigma; q) \) and \( \Pi_2(\sigma; Y) \) (defined in (4.20)), attain their maximums on \( \gamma \) when \( \sigma = 1 - \eta \). On \( C_1 \), we use (4.23) to bound the integrand for \( |\tau| \geq 2 \), and (4.24), plus the fact that \( |s - 1| \geq \eta \) on \( C_1 \), to bound the integrand for \( |\tau| \leq 2 \) (that is \( s - 1 \ll 1 \)). Thus,
\[
F(s) = (s - 1)^{-1/\phi(q)} G(s) \ll \begin{cases} T(\log T)(q^2 \log q)\Pi_1(1 - \eta; q)\Pi_2(1 - \eta; Y) & \text{if } |\tau| \geq 2, \\ \eta^{-1}(q^{3/2} \log q)\Pi_1(1 - \eta; q)\Pi_2(1 - \eta; Y) & \text{if } |\tau| \leq 2. \end{cases}
\]

We see that
\[
\int_{C_1} F(s) \frac{X^s}{s} \, ds
\ll X^{1-\eta} \left( \eta^{-1} + T \log T \int_{2}^{T} \frac{d\tau}{\tau} \right) (q^2 \log q)\Pi_1(1 - \eta; q)\Pi_2(1 - \eta; Y)
\ll X^{1-\eta} \left( \eta^{-1} + T(\log T)^2 \right) (q^{3/2} \log q)\Pi_1(1 - \eta; q)\Pi_2(1 - \eta; Y).
\] (4.34)
On $C_2$ we have $|s| \asymp T$, so we just use (4.23) and the fact that $|X^s| \leq X^\kappa \ll X$ ($\kappa \leq 1 + 1/\log X$):

\[
\int_{C_2} F(s) \frac{X^s}{s} \, ds
\ll (\eta + \kappa) \frac{X}{T} (\log T)^{1/\phi(q)} T^{1-1/\phi(q)} (q^{1/2} \log q) \Pi_1(1-\eta; q) \Pi_2(1-\eta; Y) \quad (4.35)
\]

\[
\ll (\eta + \kappa - 1) X \left( \frac{\log T}{T} \right)^{1/\phi(q)} (q^{1/2} \log q) \Pi_1(1-\eta; q) \Pi_2(1-\eta; Y).
\]

We are now ready to choose our parameters $\kappa$, $T$, and $\eta$. With $\bar{c}$ as in Theorem 4.4, and as in the definition (4.17) of $D^*$, we set

\[
\kappa := 1 + \frac{1}{\log X}, \quad T := \exp \left( \frac{1}{4} (\bar{c} \log X)^{1/2} \right), \quad \eta := \frac{\bar{c}}{2 \log T} = \frac{2\bar{c}}{(\bar{c} \log X)^{1/2}}.
\]

Straightforward calculations reveal that, for all sufficiently large $X$,

\[
X^{1-\eta} (\eta^{-1} + T (\log T)^2) = \exp \left( -\frac{7}{4} (\bar{c} \log X)^{1/2} + \log \log X + O(1) \right) \leq X \exp \left( -\frac{3}{2} (\bar{c} \log X)^{1/2} \right), \quad (4.36)
\]

and

\[
(\eta + \kappa - 1) X \left( \frac{\log T}{T} \right)^{1/\phi(q)} \ll \eta X \exp \left( -\frac{(\bar{c} \log X)^{1/2}}{4\phi(q)} + \frac{\log \log X}{\phi(q)} + O(1) \right) \quad (4.37)
\]

\[
\ll X \exp \left( -\frac{(\bar{c} \log X)^{1/2}}{5\phi(q)} \right).
\]

Putting (4.37) into (4.35), and (4.36) into (4.34), then putting the resulting bounds, as well as (4.30), into (4.29), we obtain

\[
\frac{1}{2\pi i} \int_{C} F(s) \frac{X^s}{s} \, ds
= \frac{X (\log X)^{\frac{1}{\phi(q)}}}{\log X} \left( \frac{G(1)}{\Gamma(1/\phi(q))} + O(\mathcal{E}_1) + O(\mathcal{E}_2) + O(\mathcal{E}_3) + O(\mathcal{E}_4) \right), \quad (4.38)
\]
where
\[
\begin{align*}
\mathcal{E}_1 &= \frac{(q^{\frac{1}{2}} \log q)\Pi_1(1-\eta; q)\Pi_2(1-\eta; q)}{\log X}, \\
\mathcal{E}_2 &= G(1)X^{-\eta} \ll (q^{\frac{1}{2}} \log q)\Pi_1(1-\eta; q)\Pi_2(1-\eta; q); \\
\mathcal{E}_3 &= (\log X)^{1-1/\phi(q)}(q^{\frac{3}{2}} \log q)\Pi_1(1-\eta; q)\Pi_2(1-\eta; q) \exp \left(-\frac{3}{2} \bar{c} \log X\right) \\
&\ll (q^{\frac{1}{2}} \log q)\Pi_1(1-\eta; q)\Pi_2(1-\eta; q) \exp \left(-\frac{\bar{c} \log X}{5\phi(q)}\right); \\
\mathcal{E}_4 &= (\log X)^{1-1/\phi(q)}(q^{\frac{3}{2}} \log q)\Pi_1(1-\eta; q)\Pi_2(1-\eta; q) \exp \left(-\frac{\bar{c} \log X}{6\phi(q)}\right). \end{align*}
\]

We used (4.24) to bound \(\mathcal{E}_2\).

Now \(\phi(q)\) may be of the same order as \(q\), for instance if \(q\) is prime, but as \(q \leq (\log X)^\alpha\) for some \(\alpha \in (0, \frac{1}{2})\), the largest error term here, for all sufficiently large \(X\), is \(O(\mathcal{E}_1)\). Thus

\[
O(\mathcal{E}_1) + O(\mathcal{E}_2) + O(\mathcal{E}_3) + O(\mathcal{E}_4) = O(\mathcal{E}_1) = O \left(\frac{q^{\frac{1}{2}}(\log \log X)^c}{\log X}\right). 
\tag{4.39}
\]

This bound holds uniformly for \(1 \leq q \leq (\log X)^\alpha\) and \(1 \leq Y \leq (\log \log X)^A\) (\(A > 0\) given), by (4.21) and (4.22).

Note that

\[
G(1) = c(q) \prod_{\substack{p \leq Y \\ p \equiv 1 \mod q}} \left(1 - \frac{1}{p}\right), 
\tag{4.40}
\]

where

\[
c(q) := \Theta(1) \left(\frac{\phi(q)}{q} \prod_{\chi \mod q} L(1, \chi)\right)^{1/\phi(q)}
\]

is the constant of Lemma 3.2, defined in (4.12). We have \(\Theta(1) \ll 1\) by (4.27); we have \(\Gamma(1/\phi(q)) \ll \phi(q)\); we also have the lower bounds for \(L(1, \chi)\) in (4.16); and so

\[
\frac{q^{\frac{1}{2}} \Gamma(1/\phi(q))}{c(q)} \ll q^{\frac{1}{2}} \phi(q) \left(\frac{\phi(q)}{q} q^{\frac{1}{2}(\phi(q)-1)}\right)^{1/\phi(q)} \ll \phi(q)^{1+1/\phi(q)} q^{1-1/\phi(q)} \ll q^2.
\]

Since, for \(Y \leq (\log X)^A\), we also have

\[
\prod_{p \leq Y} \left(1 - \frac{1}{p}\right)^{-1} \ll \log Y \ll A \log \log X \ll (\log \log X)^c,
\]

for some
combining (4.40) and (4.39) with (4.38) gives

\[
\frac{1}{2\pi i} \int_{\mathcal{H}} F(s) \frac{X^s}{s} ds = \left( \frac{c(q)}{\Gamma(1/\phi(q))} + O\left( \frac{q^2 (\log \log X)^c}{\log X} \right) \right) X(\log X)^{\frac{1}{\phi(q)}} \prod_{p \equiv 1 \mod q} \left( 1 - \frac{1}{p} \right)
\]

(4.41)

Putting (4.41) into (4.28), where the \(O\)-term on the right-hand side is

\[
O(1) + O(\frac{(X \log X)/T}{T}) = O\left( X e^{-c\sqrt{\log X}} \right)
\]

we obtain

\[
\sum_{n \leq X \atop \text{prime } p \mid n} 1 = \left( 1 + O\left( \frac{q^2 (\log \log X)^c}{\log X} \right) \right)
\]

\[
\times \frac{c(q)}{\Gamma(1/\phi(q))} \cdot \frac{X(\log X)^{\frac{1}{\phi(q)}}}{\log X} \prod_{p \leq Y \atop p \equiv 1 \mod q} \left( 1 - \frac{1}{p} \right).
\]

Noting that \(q^2/\log X \leq (\log X)^{2\alpha}/\log X\), Lemma 3.3 is proved.

\[\square\]

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