BI-WARPED PRODUCT SUBMANIFOLDS OF NEARLY KAehler MANIFOLDS

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Abstract. We study bi-warped product submanifolds of nearly Kaehler manifolds which are the natural extension of warped products. We prove that every bi-warped product submanifold of the form

\[ M = M_T \times f_1 M_\perp \times f_2 M_\theta \]

in a nearly Kaehler manifold satisfies the following sharp inequality:

\[ \|h\|^2 \geq 2p\|\nabla (\ln f_1)\|^2 + 4q \left( 1 + \frac{10}{9} \cot^2 \theta \right) \|\nabla (\ln f_2)\|^2, \]

where \( p = \dim M_\perp, q = \frac{1}{2} \dim M_\theta \), and \( f_1, f_2 \) are smooth positive functions on \( M_T \). We also investigate the equality case of this inequality. Further, some applications of this inequality are also given.

1. Introduction

Bi-warped product manifolds are natural extensions of (ordinary) warped product and Riemannian product manifolds. Let \( M_0, M_1 \) and \( M_2 \) be Riemannian manifolds and \( M = M_0 \times M_1 \times M_2 \) be the Cartesian product of \( M_0, M_1 \) and \( M_2 \). For each \( i = 0, 1, 2 \), we denote by \( \pi_i : M \to M_i \) the canonical projection of \( M \) onto \( M_i \). For each \( \pi_i : M \to M_i \), let \( \pi_i^* \) denote the corresponding tangent map \( \pi_i^* : TM \to TM_i \). Denote by \( \Gamma(TM) \) the Lie algebra of vector fields of \( M \).

If \( f_1, f_2 \) are positive real valued functions on \( M_0 \), then

\[ g(X, Y) = g(\pi_0^* X, \pi_0^* Y) + (f_1 \circ \pi_1)^2 g(\pi_1^* X, \pi_1^* Y) + (f_2 \circ \pi_2)^2 g(\pi_2^* X, \pi_2^* Y), \]

for \( X, Y \in \Gamma(TM) \),

defines a Riemannian metric on \( M_0 \times M_1 \times M_2 \), called a bi-warped product metric. The product manifold \( M = M_0 \times M_1 \times M_2 \) endowed with this warped product metric \( g \), denoted by \( M_1 \times f_2 M_2 \times f_3 M_3 \), is called a bi-warped product manifold. The functions \( f_1, f_2 \) are called the warping functions. Obviously, if \( f_1, f_2 \) are both constant, \( M \) is simply a Riemannian product; and if exactly one of \( f_1, f_2 \) is constant, then \( M \) is an (ordinary) warped product manifold. Further, if none of \( f_1, f_2 \) is constant, then \( M \) is called a proper bi-warped product manifold.

Let \( M = M_0 \times f_1 M_1 \times f_2 M_2 \) be a bi-warped product submanifold. We put

\[ D = TM_T, \quad D^\perp = TM_\perp, \quad D^\theta = TM_\theta, \quad N = f_1 M_1 \times f_2 M_2. \]
Then we have (cf. [12] and [22])

$$\nabla X Z = \sum_{i=1}^{2} (X(\ln f_i)) Z^i,$$

for $X \in \mathcal{D}_0$ and $Z \in \Gamma(TN)$, where $\nabla$ is the Levi-Civita connection on $M$ and $Z^i$ ($i=1,2$) is the $M_i$-component of $Z$.

Nearly Kaehler manifolds, also known as almost Tachibana manifolds, were first studied in 1959 by S. Tachibana [19] and then in 1970 by A. Gray [15]. Obviously, Kaehler manifolds are nearly Kaehler, but the converse is not true. Non-Kaehlerian nearly Kaehler manifolds are called strict nearly Kaehler manifolds.

The best known example of a strict nearly Kaehler manifold is the unit 6-sphere $S^6$. More general examples are homogeneous spaces $G/K$, where $G$ is a compact semisimple Lie group and $K$ is the fixed point set of an automorphism of $G$ of order 3 (cf. [24]). In 1985, T. Friedrich and R. Grunewald proved in [14] that a Riemannian 6-manifold is nearly Kaehler if and only if admits a Riemannian Killing spinor. After then, strict nearly Kaehler manifolds obtained a lot of attentions due to their relation to Killing spinors.

The notion of warped products plays very important roles not only in geometry but also in mathematical physics, especially in general relativity. The term of “warped product” was introduced by R. L. Bishop and B. O’Neill in [2], who used it to construct a large class of complete manifolds of negative curvature. Inspired by Bishop and O’Neill’s article, many important works on warped products from intrinsic point of view were done during the last fifty years.

On the other hand, the study of warped product submanifolds from extrinsic point of view was initiated around the beginning of this century in [5, 6, 7]. Since then warped product submanifolds have became an active research subject (see, e.g., [9, 10, 11, 12, 13, 17, 18, 21]). For instance, B. Sahin studied in [18] warped product pointwise semi-slant submanifolds in Kaehler manifolds. H. M. Tastan [20] extended this study to bi-warped product submanifolds in Kaehler manifolds by considering that one of the fiber of warped product is a pointwise slant submanifold.

In this article, we study bi-warped product submanifolds in nearly Kaehler manifolds. In section 2, we give basic definitions and formulas. In section 3, we prove some useful results for the proof of our main result. In section 4, we prove a sharp inequality for bi-warped product submanifolds in nearly Kaehler manifolds. We also discuss the equality case of the inequality. In the last section, we provide some applications of our main result.

2. Preliminaries

An even-dimensional differentiable manifold $N_K$ with Riemannian metric $g$ and almost complex structure $J$ is called a nearly Kaehler manifold if (cf. [5, 15])

$$g(JX, JY) = g(X, Y), \quad (\nabla_X J)Y + (\nabla_Y J)X = 0,$$

for any vector fields $X, Y \in \Gamma(TN_K)$. 
Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$ with induced metric $g$. Let $\Gamma(T^1M)$ denote the set of all vector fields normal to $M$. Then the Gauss and Weingarten formulas are given respectively by (see, for instance, [3, 10])

\begin{align}
\nabla_X Y &= \nabla_X Y + h(X, Y), \\
\nabla_X \xi &= -A_{\xi}X + \nabla_{\xi}X,
\end{align}

for vector fields $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^1M)$, where $\nabla$ and $\nabla^\perp$ denote the induced connections on the tangent and normal bundles of $M$, respectively, and $h$ is the second fundamental form, $A$ is the shape operator of the submanifold. The second fundamental form $h$ and the shape operator $A$ are related by

\begin{align}
\langle h(X, Y), N \rangle &= g(A_N X, Y).
\end{align}

For an $n$-dimensional submanifold $M$ of an almost Hermitian $2m$-manifold $\tilde{M}$, we choose a local orthonormal frame field $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{2m}\}$ such that, restricted to $M$, $e_1, \cdots, e_n$ are tangent to $M$ and $e_{n+1}, \cdots, e_{2m}$ are normal to $M$.

Let $\{h^r_{ij}\}, 1 \leq i, j \leq n; n + 1 \leq r \leq 2m$, denote the coefficients of the second fundamental form $h$ with respect to the local frame field. Then, we have

\begin{align}
\langle h(e_i, e_j), e_r \rangle &= g(A_{e_r} e_i, e_j), \\
||h||^2 &= \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).
\end{align}

For any $X \in \Gamma(TM)$, we put

\begin{align}
JX &= TX + FX,
\end{align}

where $TX$ and $FX$ are the tangential and normal components of $JX$, respectively. A submanifold $M$ of an almost Hermitian manifold $\tilde{M}$ is said to be holomorphic (resp. totally real) if $J(T_pM) = T_pM$ (resp. $J(T_pM) \subseteq T^\perp_pM$) $\forall p \in M$.

There are other important classes of submanifolds determined by the behaviour of almost complex structure $J$ acting on the tangent space of $M$: For a nonzero vector $X \in T_pM, p \in M$, the angle $\theta(X)$ between $JX$ and $T_pM$ is called the Wirtinger angle of $X$. A submanifold $M$ is said to be slant (cf. [3, 4]) if the Wirtinger angle $\theta(X)$ is constant on $M$, i.e., it is independent of the choice of $X \in T_pM$ and $p \in M$. In this case, $\theta$ is called the slant angle of $M$. Holomorphic and totally real submanifolds are slant submanifolds with slant angles $0$ and $\pi$, respectively. A slant submanifold is called proper slant if it is neither holomorphic nor totally real. More generally, a distribution $\mathfrak{D}$ on $M$ is called a slant distribution if the angle $\theta(X)$ between $JX$ and $\mathfrak{D}_p$ is independent of the choice of $p \in M$ and of $0 \neq X \in \mathfrak{D}_p$.

It is well-known from [3] that a submanifold $M$ of an almost Hermitian manifold $\tilde{M}$ is slant if and only if we have

\begin{align}
T^2X &= -(\cos^2 \theta)X, \quad X \in \Gamma(TM).
\end{align}

From (2.7) we have the following.

\begin{align}
g(TX, TY) &= (\cos^2 \theta)g(X, Y),
\end{align}
for any vector fields $X, Y$ tangent to $M$.

3. Bi-warped product submanifolds

Now, we study bi-warped product submanifolds in a nearly Kaehler manifold $\tilde{M}$ which are of the form $M = M_T \times f_1 M_\perp \times f_2 M_\theta$, where $M_T$, $M_\perp$, $M_\theta$ are holomorphic, totally real and proper slant submanifolds of $\tilde{M}$, respectively. If we put

$$\mathcal{D} = TM_T, \quad \mathcal{D}^\perp = TM_\perp, \quad \mathcal{D}^\theta = TM_\theta,$$

then the tangent and normal bundles of $M$ are decomposed as

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta, \quad T^\perp M = J\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \mu$$

where $\mu$ is an $j$-invariant normal subbundle of the normal bundle $T^\perp M$. From now one, we use the following conventions: $X_1, Y_1, \ldots$ are vector fields in $\Gamma(\mathcal{D})$ and $X_2, Y_2, \ldots$ are vector fields in $\Gamma(\mathcal{D}^\theta)$, while $Z, W, \ldots$ are vector fields in $\Gamma(\mathcal{D}^\perp)$.

We present the following useful results for later use.

**Lemma 3.1.** Let $M = M_T \times f_1 M_\perp \times f_2 M_\theta$ be a bi-warped product submanifold of a nearly Kaehler manifold $M$. Then we have

(i) $g(h(X_1, Y_1), JZ) = 0$,

(ii) $g(h(X_1, Y_1), FX_2) = 0$,

(iii) $g(h(X_1, Z), JW) = -JX_1(ln f_1)g(Z, W)$,

for any $X_1, Y_1 \in \Gamma(\mathcal{D})$, $Z, W \in \Gamma(\mathcal{D}^\perp)$ and $X_2 \in \Gamma(\mathcal{D}^\theta)$.

**Proof.** For any $X_1, Y_1 \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^\perp)$, we have

$$g(h(X_1, Y_1), JZ) = g(\nabla_{X_1} Y_1, JZ) = g((\nabla_{X_1} J) Y_1, Z) - g(\nabla_{X_1} J Y_1, Z).$$

Using (1.1), we find

$$g(h(X_1, Y_1), JZ) = g((\nabla_{X_1} J) Y_1, Z) + X_1(ln f_1)g(JY_1, Z).$$

By the orthogonality of vector fields, we have

$$g(h(X_1, Y_1), JZ) = g((\nabla_{X_1} J) Y_1, Z).$$

Interchanging $X_1$ by $Y_1$ in (3.1), we get

$$g(h(X_1, Y_1), JZ) = g((\nabla_{Y_1} J) X_1, Z).$$

Then, first part follows from (3.1) and (3.2) by using (2.1). In a similar fashion, we can prove (ii). For the third part, we have

$$g(h(X_1, Z), JW) = g(\nabla_{X_1} J W, Z) = g((\nabla_{X_1} J) W, Z) - g(\nabla_{X_1} J W, Z),$$

for any $X_1 \in \Gamma(\mathcal{D})$ and $Z, W \in \Gamma(\mathcal{D}^\perp)$. Again, using (1.1) and (2.1), we derive

$$g(h(X_1, Z), JW) = -g((\nabla_{X_1} J) Z, W) = JX_1(ln f_1)g(Z, W) = -g(\nabla_{X_1} J Z, W) + g(JZ, W) - JX_1(ln f_1)g(Z, W).$$
Again, using (1.1), (2.2)-(2.4) and the orthogonality of vector field $s$, we obtain

\begin{equation}
2g(h(X_1, Z), JW) = g(h(X_1, W), JZ) - JX_1(ln f_1)g(Z, W).
\end{equation}

Interchanging $Z$ by $W$ in (3.3), we obtain

\begin{equation}
2g(h(X_1, W), JZ) = g(h(X_1, Z), JW) - JX_1(ln f_1)g(Z, W).
\end{equation}

Hence, the third part follows from (3.3) and (3.4), which proves the lemma. \(\square\)

A bi-warped product submanifold $M = M_T \times_{f_1} M_\perp \times_{f_2} M_\theta$ in a nearly Kaehler manifold $M$ is said to be $\mathcal{D} \oplus \mathcal{D}^\perp$-mixed totally geodesic (resp., $\mathcal{D} \oplus \mathcal{D}^\theta$-mixed totally geodesic) if its second fundamental $h$ satisfies

\[ h(X_1, Z) = 0 \quad \forall X_1 \in \Gamma(\mathcal{D}), \quad \forall Z \in \Gamma(\mathcal{D}^\perp) \]

(resp., $h(X_1, X_2) = 0 \quad \forall X_1 \in \Gamma(\mathcal{D}), \quad \forall X_2 \in \Gamma(\mathcal{D}^\theta)$).

**Lemma 3.2.** Let $M = M_T \times_{f_1} M_\perp \times_{f_2} M_\theta$ be a bi-warped product submanifold of a nearly Kaehler manifold $M$. Then we have

(i) \[ g(h(X_1, Z), FX_2) = \frac{1}{2} g(h(X_1, X_2), JZ) = 0, \]

(ii) \[ g(h(X_1, X_2), FY_2) = \frac{1}{2} X_1(ln f_2) g(TX_2, Y_2) - JX_1(ln f_2)g(X_2, Y_2), \]

for any $X_1 \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^\perp)$ and $X_2, Y_2 \in \Gamma(\mathcal{D}^\theta)$.

**Proof.** For any $X_1 \in \Gamma(\mathcal{D})$, $Z \in \Gamma(\mathcal{D}^\perp)$, and $X_2 \in \Gamma(\mathcal{D}^\theta)$, we have

\[ g(h(X_1, Z), FX_2) = g(\nabla_Z X_1, JX_2 - TX_2) = g((\nabla_Z J)X_1, X_2) - g(\nabla_Z JX_1, X_2) - g(\nabla_Z X_1, TX_2). \]

Using (2.1), (1.1) and the orthogonality of vector fields, we derive

\[ g(h(X_1, Z), FX_2) = -g((\nabla_X J)Z, X_2) = -g(\nabla_X JZ, X_2) + g(J\nabla_X Z, X_2). \]

Then, from (2.1)-(2.4), we obtain

\begin{equation}
2g(h(X_1, Z), FX_2) = \frac{1}{2} g(h(X_1, X_2), JZ),
\end{equation}

which is the first equality of (i).

On the other hand, we have

\[ g(h(X_1, X_2), JZ) = g(\nabla_{X_2} X_1, JZ) = g((\nabla_X J)X_1, Z) - g(\nabla_X JX_1, Z). \]

Using (2.1), (1.1) and the orthogonality of vector fields, we find

\[ g(h(X_1, X_2), JZ) = -g((\nabla_X J)X_2, Z) = -g(\nabla_X JX_2, Z) + g(J\nabla_X X_2, Z). \]

Then it follows from (2.4) and (2.6) that

\[ g(h(X_1, X_2), JZ) = -g(\nabla_X TX_2, Z) - g(\nabla_X FX_2, Z) - g(\nabla_X X_2, FZ). \]

Again, using (1.1), (2.2)-(2.4) and the orthogonality of vector fields, we obtain

\begin{equation}
2g(h(X_1, X_2), JZ) = \frac{1}{2} g(h(X_1, Z), FX_2).
\end{equation}
Hence, the second equality of (i) follows from (3.5) and (3.6). For, the second part of the lemma, we have

\[
g(h(X_1, X_2), FY_2) = g(\nabla X_2 X_1, JY_2 - TY_2)
\]

\[
= g((\nabla X_2 J)X_1, Y_2) - g(\nabla X_2 JX_1, Y_2) - g(\nabla X_2 X_1, TY_2)
\]

\[
= -g((\nabla X_1 J)X_2, Y_2) - JX_1(\ln f_2)g(X_2, Y_2) - X_1(\ln f_2)g(X_2, TY_2)
\]

\[
= -g(\nabla X_1 JX_2, Y_2) - g(\nabla X_1 FX_2, Y_2) - g(\nabla X_1 X_2, TY_2)
\]

\[
= -g(\nabla X_1 X_2, FY_2) - JX_1(\ln f_2)g(X_2, Y_2) - X_1(\ln f_2)g(X_2, TY_2).
\]

Using (2.2)-(2.4) and (1.1), we find

\[
2g(h(X_1, X_2), FY_2) = g(h(X_1, Y_2), FX_2) - X_1(\ln f_2)g(X_2, TY_2)
\]

\[
- JX_1(\ln f_2)g(X_2, Y_2).
\]

Using (2.2)-(2.4) and (1.1), we find

\[
2g(h(X_1, Y_2), FX_2) = g(h(X_1, X_2), FY_2) + X_1(\ln f_2)g(X_2, TY_2)
\]

\[
- JX_1(\ln f_2)g(X_2, Y_2).
\]

The second part follows from (3.7) and (3.8). Hence the proof is complete.

The following relations are easily obtained by interchanging \(X_1\) by \(JX_1\) and \(X_2\) and \(Y_2\) by \(TX_2\) and \(TY_2\), respectively.

\[
g(h(X_1, X_2), FTY_2) = \frac{1}{3}X_1(\ln f_2) \cos^2 \theta g(X_2, Y_2) - JX_1(\ln f_2)g(X_2, TY_2),
\]

\[
g(h(JX_1, X_2), FTY_2) = \frac{1}{3}JX_1(\ln f_2) \cos^2 \theta g(X_2, Y_2)
\]

\[
+ X_1(\ln f_2)g(X_2, TY_2),
\]

\[
g(h(X_1, TX_2), FY_2) = -\frac{1}{3}X_1(\ln f_2) \cos^2 \theta g(X_2, Y_2)
\]

\[
- JX_1(\ln f_2)g(TX_2, Y_2),
\]

\[
g(h(JX_1, TX_2), FY_2) = -\frac{1}{3}JX_1(\ln f_2) \cos^2 \theta g(X_2, Y_2)
\]

\[
+ X_1(\ln f_2)g(TX_2, Y_2),
\]

\[
g(h(X_1, TX_2), FTY_2) = -\frac{1}{3}X_1(\ln f_2) \cos^2 \theta g(X_2, TY_2)
\]

\[
- JX_1(\ln f_2) \cos^2 \theta g(X_2, Y_2).
\]

From Lemma 3.1(iii) we obtain immediately the following.

**Theorem 3.1.** Let \(M = M_\perp \times f_1 M_\perp \times f_2 M_\perp\) be a bi-warped product submanifold of a nearly Kaehler manifold \(M\). If \(M\) is \(\mathfrak{D} \oplus \mathfrak{D}^\perp\)-mixed totally geodesic, then \(f_1\) is constant, and hence \(M\) is an ordinary warped product manifold.

Similarly, from Lemma 3.2(ii), we may obtain the following.
Theorem 3.2. Let $M = M_T \times f_1 M_\perp \times f_2 M_\theta$ be a proper bi-warped product submanifold of a nearly Kaehler manifold $M$. If $M$ is $\mathcal{D} \oplus \mathcal{D}^\theta$–mixed totally geodesic, then $f_2$ is constant on $M$.

Proof. From Lemma 3.2 (ii) and (3.10), we have

$$\left(\cos^2 \theta - 9\right) J X_1 (\ln f_2) g(X_2, Y_2)$$

(3.14)

$$= 9g(h(X_1, X_2), FY_2) + 3g(h(JX_1, X_2), FT Y_2).$$

If $M$ is $\mathcal{D} \oplus \mathcal{D}^\theta$–mixed totally geodesic, then we find from (3.14) that

$$\left(\cos^2 \theta - 9\right) J X_1 (\ln f_2) = 0,$$

which implies that either $\cos \theta = \pm 3$, which is not possible or $J X_1 (\ln f_2) = 0$, i.e., $f_2$ is constant. This completes the proof. □

Remark 3.1. Theorems 3.1 and 3.2 imply that a proper bi-warped product submanifold $M = M_T \times f_1 M_\perp \times f_2 M_\theta$ in a nearly Kaehler manifold is neither $\mathcal{D} \oplus \mathcal{D}^\perp$–mixed totally geodesic nor $\mathcal{D} \oplus \mathcal{D}^\theta$–mixed totally geodesic.

4. Inequality for the Second Fundamental Form

Let $M = M_T \times f_1 M_\perp \times f_2 M_\theta$ be an $n$-dimensional proper bi-warped product submanifold of a nearly Kaehler manifold $M^{2m}$. We consider a local orthonormal frame field $\{e_1, \ldots, e_n\}$ of $TM$ such that

$$\mathcal{D} = \text{Span}\{e_1, \ldots, e_t, e_{t+1} = J e_1, \ldots, e_{2t} = Je_t\},$$

$$\mathcal{D}^\perp = \text{Span}\{e_{2t+1} = \hat{e}_1, \ldots, e_{2t+p} = \hat{e}_p\},$$

$$\mathcal{D}^\theta = \text{Span}\{e_{2t+p+1} = e_1^*, \ldots, e_{2t+p+q} = e_t^*, \ldots, e_{2t+p+q+1} = \sec \theta e_1^*, \ldots, e_n = \sec \theta e_n^*\}.$$

Then $\dim M_T = 2t$, $\dim M_\perp = p$ and $\dim M_\theta = 2q$. Moreover, the orthonormal frame fields $E_1, \ldots, E_{2m-n-p-2q}$ of the normal subbundle $T^\perp M$ are given by

$$J \mathcal{D}^\perp = \text{Span}\{E_1 = J \hat{e}_1, \ldots, E_p = J \hat{e}_p\},$$

$$F \mathcal{D}^\theta = \text{Span}\{E_{p+1} = \csc \theta F e_1^*, \ldots, E_{p+q} = \csc \theta F e_p^*, \ldots, E_{p+q+1} = \csc \theta \sec \theta F e_1^*, \ldots, E_{p+2q} = \csc \theta \sec \theta F e_q^*\},$$

$$\mu = \text{Span}\{E_{p+2q+1}, \ldots, E_{2m-n-p-2q}\}.$$

The main result of this article is the following sharp inequality for bi-warped product submanifolds in a nearly Kaehler manifold.

Theorem 4.1. Let $M = M_T \times f_1 M_\perp \times f_2 M_\theta$ be a bi-warped product submanifold of a nearly Kaehler manifold $M$, where $M_T$, $M_\perp$ and $M_\theta$ are holomorphic, totally real and proper slant submanifolds of $M$, respectively. Then we have:

(i) The second fundamental form $h$ and the warping functions $f_1$, $f_2$ satisfy

$$\|h\|^2 \geq 2p\|\nabla(\ln f_1)\|^2 + 4q \left(1 + \frac{10}{9} \cot^2 \theta\right) \|\nabla(\ln f_2)\|^2$$

(4.1)

where $p = \dim M_\perp$, $q = \frac{1}{2} \dim M_\theta$ and $\nabla(\ln f_i)$ is the gradient of $\ln f_i$. 
Proof. From the definition of $h$, we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=1}^{p} \sum_{i,j=1}^{2} g^2(h(e_i, e_j), E_r).$$

Then we decompose the above relation for the normal subbundles as follows

$$\|h\|^2 = \sum_{r=1}^{p} \sum_{i,j=1}^{2} g^2(h(e_i, e_j), J\hat{\epsilon}_r) + \sum_{r=p+1}^{p+2q} \sum_{i,j=1}^{n} g^2(h(e_i, e_j), E_r)$$

(4.2)

$$+ \sum_{r=p+2q+1}^{p+m-n-p-2q} \sum_{i,j=1}^{n} g^2(h(e_i, e_j), E_r).$$

Leaving the last $\mu$-components term in (4.2) and using the frame fields of tangent and normal subbundles of $M$, we derive

$$\|h\|^2 \geq \sum_{r=1}^{p} \sum_{i,j=1}^{2t} g^2(h(e_i, e_j), J\hat{\epsilon}_r) + \sum_{r=1}^{2q} \sum_{i,j=1}^{p} g^2(h(e_i, e_j), J\hat{\epsilon}_r)$$

$$+ \sum_{r=1}^{p} \sum_{i,j=1}^{2q} g^2(h(e_i, e_j), J\hat{\epsilon}_r) + \sum_{r=1}^{2q} \sum_{i,j=1}^{p} g^2(h(e_i, e_j), J\hat{\epsilon}_r)$$

(4.3)

$$+ \csc^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{2t} \left[ g^2(h(e_i, e_j), F\epsilon_r^*) + \sec^2 \theta g^2(h(e_i, e_j), FTE_r^*) \right]$$

$$+ \csc^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{2q} \left[ g^2(h(e_i, e_j), F\epsilon_r^*) + \sec^2 \theta g^2(h(e_i, e_j), FTE_r^*) \right]$$

$$+ \csc^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{2q} \left[ g^2(h(e_i, e_j), F\epsilon_r^*) + \sec^2 \theta g^2(h(e_i, e_j), FTE_r^*) \right]$$

$$+ \csc^2 \theta \sum_{r=1}^{q} \sum_{i,j=1}^{2q} \left[ g^2(h(e_i, e_j), F\epsilon_r^*) + \sec^2 \theta g^2(h(e_i, e_j), FTE_r^*) \right].$$

We have no relation for warped products for the third, fifth, sixth, ninth, tenth and eleventh terms in (4.3), therefore, we leave these positive terms. Moreover, by
using Lemma 3.1 and Lemma 3.2 with the relations (3.9)-(3.13), we find that
\[
\|h\|^2 \geq 2p \sum_{i=1}^{t} \left[ (-Je_i(ln f_1))^2 + (e_i(ln f_1))^2 \right] \\
+ 4q \csc^2 \theta \sum_{i=1}^{t} \left[ (-Je_i(ln f_2))^2 + (e_i(ln f_2))^2 \right] \\
+ \frac{4q}{9} \cot^2 \theta \sum_{i=1}^{t} \left[ (-Je_i(ln f_2))^2 + (e_i(ln f_2))^2 \right] \\
= 2p \sum_{i=1}^{2t} (e_i(ln f_1))^2 + 4q \left( \csc^2 \theta + \frac{1}{9} \cot^2 \theta \right) \sum_{i=1}^{2t} (e_i(ln f_2))^2.
\]
Then we find the required inequality from the definition of gradient.

For the equality case, we have from the leaving third term in (4.2) that
\[
h(TM, TM) \perp \mu
\] (4.4)
From the vanishing first term and leaving seventh term in (4.3), we find
\[
h(\mathcal{D}, \mathcal{D}) \perp J\mathcal{D}^\perp \quad \text{and} \quad h(\mathcal{D}, \mathcal{D}) \perp F\mathcal{D}^\theta.
\] (4.5)
Then we find from (4.4) and (4.5) that
\[
h(\mathcal{D}, \mathcal{D}) = 0.
\] (4.6)

On the other hand, from the leaving third and ninth terms in (4.3), we get
\[
h(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp J\mathcal{D}^\perp \quad \text{and} \quad h(\mathcal{D}^\perp, \mathcal{D}^\perp) \perp F\mathcal{D}^\theta.
\] (4.7)
Again, we conclude from (4.4) and (4.7) that
\[
h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0.
\] (4.8)
Also, from the leaving fifth and eleventh terms in the right hand side of (4.3), we have
\[
h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp J\mathcal{D}^\perp \quad \text{and} \quad h(\mathcal{D}^\theta, \mathcal{D}^\theta) \perp F\mathcal{D}^\theta.
\] (4.9)
Then we obtain from (4.4) and (4.9) that
\[
h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0.
\] (4.10)
Moreover, from the leaving sixth and tenth terms in (4.3), we get
\[
h(\mathcal{D}^\perp, \mathcal{D}^\theta) \perp J\mathcal{D}^\perp \quad \text{and} \quad h(\mathcal{D}^\perp, \mathcal{D}^\theta) \perp F\mathcal{D}^\theta.
\] (4.11)
Therefore, from (4.4) and (4.11) we obtain
\[
h(\mathcal{D}^\perp, \mathcal{D}^\theta) = 0.
\] (4.12)
On the other hand, from the vanishing eighth term in (4.3) with (4.4), we have
\[
h(\mathcal{D}, \mathcal{D}^\perp) \subset J\mathcal{D}^\perp.
\] (4.13)
Similarly, from the vanishing forth term in (4.3) with (4.4), we get
\[
h(\mathcal{D}, \mathcal{D}^\theta) \subset F\mathcal{D}^\theta.
\] (4.14)
Since \( M_T \) is totally geodesic in \( \tilde{M} \) (see, e.g., \([5, 8]\)), using this fact together with (4.6), (4.7) and (4.12), we know \( M_T \) is totally geodesic in \( \tilde{M} \). Also, since \( M_{\perp} \) and \( M_{\theta} \) are totally umbilical in \( M \), using this fact together with (4.8), (4.10), (4.13) and (4.14), we conclude that \( M_{\perp} \) and \( M_{\theta} \) are both totally umbilical in \( \tilde{M} \). Further, it follows from Remark 3.1, (4.13) and (4.14) that \( M \) is neither \( D \oplus D_{\perp} \)-mixed totally geodesic nor \( D \oplus D_{\theta} \)-mixed totally geodesic in \( \tilde{M} \). Consequently, the theorem is proved completely. \( \square \)

5. Some applications

Theorem 4.1 implies the following.

**Theorem 5.1.** \([5]\) Let \( M = M_T \times f M_{\perp} \) be a CR-warped product in a Kaehler manifold \( \tilde{M} \). Then the second fundamental form \( h \) of \( M \) satisfies

\[
||h||^2 \geq 2p||\nabla (\ln f)||^2,
\]

where \( p = \dim M \). Moreover, if the equality sign of (5.1) holds identically, then \( M_T \) is totally geodesic and \( M_{\perp} \) is totally umbilical in \( M \).

A warped submanifold of the form \( M = M_T \times f M_{\theta} \) in a nearly Kaehler manifold \( \tilde{M} \) is called semi-slant if \( M_T \) is a holomorphic submanifold and \( M_{\theta} \) is a proper slant submanifold in \( \tilde{M} \).

The next result was proved in \([10]\).

**Theorem 5.2.** Let \( M_T \times f M_{\theta} \) be a semi-slant warped product of a nearly Kähler manifold \( \tilde{M} \). Then the second fundamental form \( h \) of \( M \) satisfies

\[
||h||^2 \geq 4q \csc^2 \theta \left\{ 1 + \frac{9}{9} \cos^4 \theta \right\} ||\nabla (\ln f)||^2.
\]

On the other hand, Theorem 4.1 implies the following.

**Theorem 5.3.** \([1]\) Let \( M = M_T \times f M_{\theta} \) be a semi-slant warped product submanifold of a nearly Kaehler manifold \( \tilde{M} \). Then second fundamental form \( h \) and the warping function \( f \) satisfy

\[
||h||^2 \geq 4q \left\{ 1 + \frac{10}{9} \cot^2 \theta \right\} ||\nabla (\ln f)||^2.
\]

Moreover, if the equality sign in (4.1) holds identically, then \( M_T \) is totally geodesic and \( M_{\theta} \) are totally umbilical in \( \tilde{M} \).

**Remark 5.1.** Theorem 5.3 improves Theorem 5.2 since

\[
9 + 10 \cot^2 \theta > \csc^2 \theta (9 + \cos^4 \theta)
\]

holds for every \( \theta \in (0, \frac{\pi}{2}) \). Furthermore, Theorem 5.3 shows that inequality (5.2) in Theorem 5.2 is not sharp.

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