Pavelka-style completeness in expansions of Łukasiewicz logic

HECTOR FREYTES *

University of Cagliari,
Dipartimento di Scienze Pedagogiche e Filosofiche,
Via Is Mirrionis 1,09123, Cagliari-Italia
e-mail: hfreytes@gmail.com

Abstract

An algebraic setting for the validity of Pavelka style completeness for some natural expansions of Łukasiewicz logic by new connectives and rational constants is given. This algebraic approach is based on the fact that the standard MV-algebra on the real segment \([0, 1]\) is an injective MV-algebra. In particular the logics associated with MV-algebras with product and with divisible MV-algebras are considered.

Keywords: Injectives, MV-algebras, Pavelka completeness

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Introduction

The completeness theorem for fuzzy propositional logic was proved by J. Pavelka in [13], who built a propositional many valued logical systems over Łukasiewicz logic adding to the language a truth constant \(\tau\) for each \(r \in [0, 1]\), together with additional axioms. Later, Hájek gave a proof of the Pavelka completeness of Łukasiewicz logic with rational constants, based on

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*Permanent address: Instituto Argentino de Matemática (IAM), Saavedra 15 - 3er Piso - 1083 Buenos Aires, Argentina, CONICET.
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the continuity of Lukasiewicz implication on the real segment \([0,1]\) (see [8, §3.3]). Following the ideas of Hájek’s proof, analogous results were obtained for some expansions of Lukasiewicz logic (see [5], [7] and [10]).

The aim of this paper is to offer a new method for proving the validity of Pavelka style completeness of Lukasiewicz logic with rational constants, based on the crucial fact that the standard MV-algebra \([0,1]_{MV}\) is injective in the category of MV-algebras. This method can also be applied for a special class of expansions of Lukasiewicz logic, called compatible expansions. Such expansions guarantee the injectivity of the standard MV-algebra \([0,1]_{MV}\) with the corresponding additional operations.

The paper is organized as follows. In section 1 we recall some basic definitions and properties of injectives, algebraizable logics and compatible expansions. Section 2 contains the main results of the paper. It is shown that Pavelka completeness can be established under some mild hypothesis on compatible expansions of Lukasiewicz logic. A type compactness is also proved. The results of section 2 are applied in section 3 to obtain a new proof of Pavelka completeness for Lukasiewicz logic, as well as Pavelka style completeness for product Lukasiewicz logic and for divisible Lukasiewicz logic.

1 Basic notions

We use freely all basic notions of universal algebra that came be found in [3]. Let \(\mathcal{A}\) be a class of algebras of type \(\tau\). For all algebras \(A,B\) in \(\mathcal{A}\), \([A,B]_\mathcal{A}\) will denote the set of all homomorphism \(g : A \to B\).

An algebra \(A\) in \(\mathcal{A}\) is injective iff for every monomorphism \(f \in [B,C]_\mathcal{A}\) and every \(g \in [B,A]_\mathcal{A}\) there exists \(h \in [C,A]_\mathcal{A}\) such that \(hf = g\). \(A\) is self-injective iff every homomorphism from a subalgebra of \(A\) into \(A\), extends to an endomorphism of \(A\).

Recall from [6] that a simple algebra \(I_M\) is said to be maximum simple iff for each simple algebra \(I\), \(I\) can be embedded into \(I_M\). Two constant terms \(0, 1\) of the language of \(\mathcal{A}\) are called distinguished constants iff \(A \models 0 \neq 1\) for each nontrivial algebra \(A\) in \(\mathcal{A}\).

**Theorem 1.1** [6, Theorem 3.4] Let \(\mathcal{A}\) be a variety satisfying the congruence extension property, with distinguished constants \(0,1\). If \(I\) is a self-injective maximum simple algebra in \(\mathcal{A}\) then \(I\) is injective.

If \(\tau\) is a family of finitary function symbols, by a logic \(\mathcal{L}\) of type \(\tau\) we
will understand a structural finitary consequence relation \( \vdash_{\mathcal{L}} \) in the absolutely free algebra \( Fm_\tau \) of type \( \tau \) generated by the propositional variables \( p_1, p_2, \ldots \). Each function symbol of \( \tau \) is called a connective, and the terms of \( Fm_\tau \) are called formulas. Usually, \( \vdash_{\mathcal{L}} \) is specified by a set of Hilbert style axiom schemes and inference rules. A set \( T \) of formulas is called a theory of \( \mathcal{L} \) if \( T \vdash_{\mathcal{L}} \varphi \) implies \( \varphi \in T \) for every \( \varphi \in Fm_\tau \).\(^1\) We denote by \( \text{Th}\mathcal{L} \) the lattice of theories associated to \( \mathcal{L} \) ([2, 1.1]). If \( S \) is a set of formulas, we denote by \( T(S) \) the theory generated by \( S \). Recall [2, Definition 2.8, Corollary 2.9] that a logic \( \mathcal{L} \) is algebraizable by a quasivariety \( K \) of type \( \tau \) via a finite set of binary formulas \( p \leftrightarrow q = (p \leftrightarrow_i q)_i \) (called the equivalence formulas of \( \mathcal{L} \)) and a finite set of identities \( \delta(p) \approx \epsilon(q) = (\delta_j(p) \approx \epsilon_j(q))_j \) (the defining equation of \( \mathcal{L} \)) if and only if the following conditions are met:

\[
\begin{align*}
\text{i} & \quad \phi_1, \ldots, \phi_n \vdash_{\mathcal{L}} \phi \text{ iff } \{\delta(\phi_i) \approx \epsilon(\phi_i)\}_{i=1}^n \models_K \delta(\phi) \approx \epsilon(\phi) \\
\text{ii} & \quad \varphi \approx \psi \models_K \delta(\varphi \leftrightarrow \psi) \approx \epsilon(\varphi \leftrightarrow \psi)
\end{align*}
\]

we are using the abbreviations \( p \leftrightarrow q \) for \( \{p \leftrightarrow_i q\}_i \), \( \delta(p \leftrightarrow q) \approx \epsilon(p \leftrightarrow q) \) for \( \{\delta_j(p \leftrightarrow_i q) \approx \epsilon_j(p \leftrightarrow_i q)\}_{i,j} \). The quasivariety \( K \), is uniquely determined by \( \mathcal{L} \), when it exists and the equivalence formulas and defining equations are also unique in the sense that for any other system \( p \leftrightarrow q \), \( \delta'(p) \approx \epsilon'(q) \) satisfying \( \text{i}, \text{ii} \), \( \vdash_{\mathcal{L}} p \leftrightarrow q \text{ iff } \vdash \varphi \leftrightarrow \psi \) and \( \models_K \delta'(p) \approx \epsilon'(q) \) iff \( \models_K \delta(p) \approx \epsilon(q) \). Will be denote this quasivariety as \( K_\mathcal{L} \). From [2, Theorem 4.7 and Theorem 4.10], if we consider for each \( T \in \text{Th}\mathcal{L} \)

\[
\Omega T = \{ \varphi \approx \psi : \varphi \leftrightarrow \psi \in T \}
\]

then \( \Omega T \) is a congruence in \( Fm_\tau \) and the quasivariety generated by the set \( \{Fm_\tau/\Omega T : T \in \text{Th}\mathcal{L}\} \) is \( K_\mathcal{L} \). In what follows, for each \( \alpha \in Fm_\tau \) will be denoted by \([\alpha]\) the equivalence class of \( \alpha \) with respect \( Fm_\tau/\Omega T \). Let \( A \in K \), a valuation of formulas on \( A \) is an homomorphism \( v : Fm_\tau \rightarrow A \). If \( T \in \text{Th}\mathcal{L} \) and \( a \in A \), \( v(T) = a \) means, as usual, that \( v(\alpha) = a \) for each \( \alpha \in T \).

Let \( E \) be a set of equations in the vocabulary \( \tau \) that gives an axiomatization of a variety \( A \). If \( \sigma = (f_i)_{i \in I} \) is a family of operation symbols such that \( \sigma \cap \tau = \emptyset \) and \( E(\sigma) \) a set of equation in the expanded vocabulary \( \tau \cup \sigma \), we denote by \( A_{E(\sigma)} \) the variety of type \( \tau \cup \sigma \) defined by the equations in the set \( E \cup E(\sigma) \). We will refer to \( A_{E(\sigma)} \) as the \( E(\sigma) \)-expansion of \( A \). For each

\(^1\)Notice that our notion of a theory differs from the one used in [8], where a theory is just a set of formulas.
A \in \mathcal{A}_{E(\sigma)}$, we denote by $\text{Con}_A(A)$ the lattice of $A$-congruences of $A$ and by $\text{Con}_{\mathcal{A}_{E(\sigma)}}(A)$ the lattice of $\mathcal{A}_{E(\sigma)}$-congruences of $A$. We say that $\mathcal{A}_{E(\sigma)}$ is a compatible expansion of $\mathcal{A}$ iff for each $A \in \mathcal{A}_{E(\sigma)}$, $\text{Con}_{\mathcal{A}_{E(\sigma)}}(A) = \text{Con}_A(A)$.

Note that an algebra is simple in the compatible expansion $\mathcal{A}_{E(\sigma)}$ iff it is simple in $\mathcal{A}$.

Let $A \in \mathcal{A}$, then $A$ admits an $E(\sigma)$-expansion iff there is a family of $A$-operation $\sigma_A = (f_i^A)_{i \in I}$ making $(A, \sigma_A)$ into a $\mathcal{A}_\sigma$-algebra. This $E(\sigma)$-expansion on $A$ is canonical iff $\sigma_A$ is unique and each sub algebra of $A$ admits at most one $E(\sigma)$-expansion.

**Proposition 1.2** Let $\mathcal{A}$ be a variety and $\mathcal{A}_{E(\sigma)}$ be a compatible expansion. If $I$ is maximum simple algebra in $\mathcal{A}$ admitting a canonical $E(\sigma)$-expansion, then $I$ is maximum simple in $\mathcal{A}_{E(\sigma)}$.

**Proof:** Let $(I, \sigma_I)$ be canonical $E(\sigma)$-expansion of $I$ and $A$ be a simple algebra in $\mathcal{A}_{E(\sigma)}$. Since $A$ is simple in $\mathcal{A}$, there exists a sub $\mathcal{A}$-algebra $A_0$ of $I$ $\mathcal{A}$-isomorphic to $A$. An $\mathcal{A}$-isomorphism $i : A \to A_0$ induces $\sigma$-operations on $A_0$ making $(A_0, \sigma_{A_0})$ into a $\mathcal{A}_{\sigma}$-algebra. Since the expansion $(I, \sigma_I)$ is canonical, $\sigma_{A_0}$ is the set of restrictions to $A_0$ of the $\sigma_I$-operations. Thus $i$ is a $\mathcal{A}_\sigma$-monomorphism, proving that $I$ is maximum simple algebra in $\mathcal{A}_{E(\sigma)}$.

\[\Box\]

## 2 MV-expansions and Pavelka-style completeness

**Definition 2.1** An MV-algebra [4], [8] is an algebra $\langle A, \land, \lor, \circ, \to, 0, 1 \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ satisfying the following axioms:

1. $\langle A, \circ, 1 \rangle$ is an abelian monoid,
2. $L(A) = \langle A, \lor, \land, 0, 1 \rangle$ is a bounded lattice,
3. $(x \circ y) \to z = x \to (y \to z)$,
4. $((x \to y) \circ x) \land y = (x \to y) \circ x$,
5. $(x \land y) \to y = 1$,
6. $x \circ (x \to y) = x \land y$,
7. $(x \to y) \lor (y \to x) = 1$,
8. $(x \to 0) \to 0 = x$
We denote by $\mathcal{M}V$ the variety of MV-algebras. In agreement with the usual MV-algebraic operations we define the negation as the unary operation $\neg x = x \rightarrow 0$, and the binary operation $x \oplus y = \neg (\neg x \odot \neg y)$, $x \rightarrow y = \neg x \odot y$. For element $x$ in a MV-algebra and $n \in \mathbb{N}$, we denote $nx$ the element inductively defined by $0x = 0$, $(n+1)x = (nx) \oplus x$. $\mathcal{M}V$ satisfies congruence extension property [1, Theorem 1.8] and then each compatible expansion also satisfies. An important example is $[0,1]_{\mathcal{M}V} = \langle [0,1], \odot, \rightarrow, \land, \lor, 0, 1 \rangle$ such that $[0,1]$ is the real unit segment, $\land, \lor$ are the natural meet and join on $[0,1]$ and $\odot$ and $\rightarrow$ are defined as follows: $x \odot y := \max(0, x + y - 1)$, $x \rightarrow y := \min(1, 1 - x + y)$. $[0,1]_{\mathcal{M}V}$ is the maximum simple algebra in $\mathcal{M}V$ (see [4, Theorem 3.5.1]). Moreover $[0,1]_{\mathcal{M}V}$ is a rigid algebra (see [4, Corollary 7.2.6]), hence self-injective resulting injective in the variety $\mathcal{M}V$ in view of Theorem 1.1.

**Proposition 2.2** Let $\mathcal{M}V_{E(\sigma)}$ be a compatible expansion of $\mathcal{M}V$ such that $[0,1]_{\mathcal{M}V}$ admits canonical $E(\sigma)$-expansion. Then the $E(\sigma)$-expansion of $[0,1]_{\mathcal{M}V}$ is injective in $\mathcal{M}V_{E(\sigma)}$.

**Proof:** Let $[0,1]_{E(\sigma)}$ be the $E(\sigma)$-expansion of $[0,1]_{\mathcal{M}V}$. Since $\mathcal{M}V_{E(\sigma)}$ is a compatible expansion of $\mathcal{M}V$, it satisfies the congruence extension property, and clearly 0, 1 are distinguished constant. Then by Proposition 1.2 $[0,1]_{E(\sigma)}$ is a maximum simple algebra. Moreover $[0,1]_{E(\sigma)}$ is self injective since the expansion is canonical. Thus by Theorem 1.1 $[0,1]_{E(\sigma)}$ is injective.

**Definition 2.3** A compatible expansion $\mathcal{M}V_{E(\sigma)}$ of $\mathcal{M}V$ is said to be admissible if

1. $[0,1]_{\mathcal{M}V}$ admits canonical $E(\sigma)$-expansion,
2. $Q_{[0,1]}$ can be $E(\sigma)$-expanded, were $Q_{[0,1]}$ denotes the subalgebra of rational numbers in $[0,1]_{\mathcal{M}V}$.

**Definition 2.4** We can define the Lukasiewicz propositional calculus $L$ [4] from $\tau = \{\neg, 0\}$ where $\neg$ is a binary connective and 0 is constant. Further connectives are defined as follows: $\neg \alpha$ is $\alpha \rightarrow 0$, 1 is $\neg 0$ and $\alpha \odot \beta$ is $\neg (\alpha \rightarrow \neg \beta)$. The following formulas are axioms

$L1 \quad \alpha \rightarrow (\beta \rightarrow \alpha)$

$L2 \quad (\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$
L3 \((-\alpha \rightarrow \neg \beta) \rightarrow (\beta \rightarrow \alpha)\)

L4 \(((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha)\)

The unique deduction rule is the Modus Ponens.

This calculus is algebraizable in the variety \(\mathcal{MV}\) via the system \(p \leftrightarrow q = \{p \rightarrow q, q \rightarrow p\}\) and \(\delta(p) \approx e(p) = \{p \approx p \rightarrow p\}\) (see [14]).

**Definition 2.5** By an admissible expansion of Lukasiewicz logic \(L\) we understand a logic \(L_\sigma\) involving a set \(\sigma\) of new connectives and new axioms, algebraizable in an admissible expansion \(\mathcal{MV}_{E(\sigma)}\) of \(\mathcal{MV}\).

**Proposition 2.6** Given an admissible expansion \(L'\) of \(L\), an \(L'-\)formula \(\alpha\), and \(T \in \text{Th}_{L'}\) such that \(\alpha \notin T\), there exists \(T' \in \text{Th}_{L'}\) such that \(T \subseteq T'\), \(\alpha \notin T'\) and \(\text{Fm}_{L'}/\Omega T'\) is totally ordered.

**Proof:** Follows from [8, Lemma 2.4.2] and the fact that compatible expansions do not involve new inference rules.

**Definition 2.7** [8, §3.3] A \(\sigma\)-Pavelka expansion of Lukasiewicz logic is obtained by adding into the language of an admissible expansion \(L_\sigma\) of \(L\), truth constant \(\top\) for each \(r \in Q \cap [0, 1]\), together with the following additional book-keeping axioms:

1. \(\top \iff 0\),
2. \(\top \rightarrow \top \iff \top \rightarrow \top\),
3. \(f(\top_1 \ldots \top_n) \iff f(\top_1 \ldots \top_n)\) for each \(n\)-ary connective \(f\) in \(\sigma\) and \(r_1 \ldots r_n \in Q\).

Moreover, for each theory \(T\) and each formula \(\alpha\) in the \(\sigma\)-Pavelka expansion we define:

The truth degree of \(\alpha\) over \(T\) is \(\|\alpha\|_T = \bigwedge\{v(\alpha) : v(T) = 1 \text{ in } [0, 1]\}\).

The proof degree of \(\alpha\) is \(|\alpha|_T = \bigvee\{r \in Q : \top \rightarrow \alpha \in T\}\).
Observe that $\sigma$-Pavelka expansion is an admissible expansion $L_{\sigma \cup \{r, r \in Q\}}$ of $L$.

We denote by $Fm_\sigma(Q)$ the set of the formulas of the $\sigma$-Pavelka expansion. The following theorem is a generalization of Pavelka completeness for Lukasiewicz logic (see [8, §3.3]):

**Theorem 2.8** Let $L_\alpha$ be a admissible expansion of $L$. If $T$ is a theory and $\varphi$ is a formula, both in the $\sigma$-Pavelka expansion of $L$, then

$$|\varphi|_T = ||\varphi||_T.$$ 

**Proof:** Let $[0,1]_{E(\sigma)}$ the $E(\sigma)$-expansion of $[0,1]_{MV}$ and the $Q_{E(\sigma)}$ the $E(\sigma)$-expansion of $Q_{[0,1]}$. Assume that $T \neq Fm_\sigma(Q)$. We first prove that $|\varphi|_T$ is a lower bound of $\{v(\varphi) : v(T) = 1\}$. Let $v$ be a valuation such that $v(T) = 1$. If $r \in Q$ is such that $\tau \rightarrow \varphi \in T$ then, we have that $1 = v(\tau \rightarrow \varphi) = r \rightarrow v(\varphi)$. Thus $r \leq v(\varphi)$ resulting $|\varphi|_T \leq v(\varphi)$. We proceed now to prove that $|\varphi|_T$ is the greatest lower bound of $\{v(\varphi) : v(T) = 1\}$. In fact, let $b$ be a lower bound of $\{v(\varphi) : v(T) = 1\}$. Suppose that $|\varphi|_T < b$. Then there exists $r_0 \in Q$ such that $|\varphi|_T < r_0 < b$. Thus $T$ does not prove $\overline{r_0} \rightarrow \varphi$. By Proposition 2.6 there exists a theory $T'$ such that $T \subseteq T'$, $T'$ does not prove $\overline{r_0} \rightarrow \varphi$ and $Fm_\sigma(Q)/\Omega T'$ is a totaly ordered algebra in $M\mathcal{V}_{E(\sigma)}$ containing $\{[\tau] \}_{r \in Q}$ as sub algebra isomorphic to $Q_{E(\sigma)}$. Since $T'$ does not prove $\overline{r_0} \rightarrow \varphi$, $[\tau_0] \rightarrow [\varphi] < 1$ in $Fm_\sigma(Q)/\Omega T'$ resulting $[\varphi] < [\tau_0]$ since it is a totaly order. Let $i_1$ the canonical embedding $Q_{E(\sigma)} \rightarrow [0,1]_{E(\sigma)}$ and $i_2$ the canonical embedding $Q_{E(\sigma)} \rightarrow Fm_\sigma(Q)/\Omega T'$. Since $[0,1]_{E(\sigma)}$ is injective in the $M\mathcal{V}_{E(\sigma)}$, there exist an homomorphism $f : Fm_\sigma(Q)/\Omega T' \rightarrow [0,1]_{E(\sigma)}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
Q_{E(\sigma)} & \xrightarrow{i_1} & [0,1]_{E(\sigma)} \\
\downarrow{i_2} & \equiv & \downarrow{f} \\
Fm_\sigma(Q)/\Omega T' & \\
\end{array}
$$

By the commutativity $f([\varphi]) \leq r_0 < b$. If we consider the valuation $\pi : Fm_\sigma(Q) \rightarrow Fm_\sigma(Q)/\Omega T'$ such that $\alpha \rightarrow [\alpha]$ then the composition $fp$ is a valuation over $[0,1]_{E(\sigma)}$ such that $fp(T') = 1$ and $fp(T) = 1$ since $T \subseteq T'$ resulting $fp(\varphi) \in \{v(\varphi) : v(T) = 1\}$. But $fp(\varphi) \leq r_0 < b$ which is a contradiction since $b$ is a lower bound of $\{v(\varphi) : v(T) = 1\}$. Therefore $b \leq |\varphi|_T$ resulting $|\varphi|_T = ||\varphi||_T$. 

\[\square\]
From the above completeness theorem, we can establish a kind of compactness theorem.

**Theorem 2.9** Let $S$ be a set of formulas and $\alpha$ be a formula in a $\sigma$-Pavelka expansion. Then we have:

If $r \leq \|\alpha\|_{T(S)}$ then $\exists S_0 \subseteq S$ finite such that $r \leq \|\alpha\|_{T(S_0)}$

**Proof:** If $r \leq \|\alpha\|_{T(S)}$ then by Theorem 2.8, $\overline{r} \rightarrow \alpha \in T(S)$. If $\alpha_1, \ldots, \alpha_n, \overline{r} \rightarrow \alpha$ is a proof of $\overline{r} \rightarrow \alpha$ from $S$, we can consider the finite set $S_0 = \{\alpha_1, \ldots, \alpha_n\}$. Using again Theorem 2.8 we have $r \leq |\alpha|_{T_{S_0}} = \|\alpha\|_{T_{S_0}}$. \qed

**3 Applications**

**3.1 Łukasiewicz logic**

Since Łukasiewicz logic $L$, is an admissible expansion of itself (with $\sigma = \emptyset$), Theorem 2.8 provides a new proof of Pavelka completeness for Łukasiewicz logic with rational constants (cf. [8, §3.3]).

**3.2 Product Łukasiewicz logic**

**Definition 3.1** A product MV-algebra [11] (for short: PMV-algebra) is an algebra $\langle A, \bullet \rangle$ satisfying the following

1. $A$ is an MV-algebra
2. $\langle A, \bullet, 1 \rangle$ is an abelian monoid
3. $x \bullet (y \odot \neg z) = (x \bullet y) \odot \neg (x \bullet z)$

We denote by $PMV$ the variety of PMV-algebras. If $A$ is a PMV-algebra then [11, Lemma 2.11] $Con_{PMV}(A) = Con_{MV}(A)$, resulting $PMV$ a compatible expansion of $MV$. $[0,1]_{MV}$ equipped with the usual multiplication in the unitary interval is a PMV-algebra denoted by $[0,1]_{PMV}$. It contain $Q[0,1]$ equipped with multiplication as sub PMV-algebra. It is clear that $[0,1]_{PMV}$ is a hereditarily simple algebra. If $A$ is a semisimple MV-algebra then is at most one operation $\bullet$ making $\langle A, \bullet \rangle$ into a PMV-algebra [12, Lemma 3.1.14]. Thus $[0,1]_{PMV}$ is the canonical PMV-expansion of $[0,1]_{MV}$.  

8
Definition 3.2 We define the product Lukasiewicz propositional calculus \( PL \) adding into the language of \( L \) the binary connective \( \bullet \) and considering the following formulas as axioms

- **PL0** Lukasiewicz axioms
  - \( (\alpha \bullet \beta) \rightarrow (\beta \bullet \alpha) \) (PL1)
  - \( (\top \bullet \alpha) \iff \alpha \) (PL2)
  - \( (\alpha \bullet \beta) \rightarrow \beta \) (PL3)
  - \( (\alpha \bullet \beta) \bullet \gamma \iff \alpha \bullet (\beta \bullet \gamma) \) (PL4)
  - \( x \bullet (y \odot \neg z) \iff (x \bullet y) \odot (x \bullet z) \) (PL5)

The unique deduction rule is the **Modus Ponens**.

Proposition 3.3 \( PL \) calculus is algebraizable in \( PMV \).

Proof: We only need to prove that \( \alpha \iff \beta \vdash_{PL} \gamma \bullet \alpha \iff \gamma \bullet \beta \). For this we prove that \( \vdash (\alpha \rightarrow \beta) \rightarrow ((\gamma \bullet \alpha) \rightarrow (\gamma \bullet \beta)) \).

1. \( \vdash \gamma \bullet (\alpha \odot \neg \beta) \rightarrow ((\alpha \odot \neg \beta)) \) by Ax PL3
2. \( \vdash ((\gamma \bullet \alpha) \odot (\neg (\gamma \bullet \beta))) \rightarrow \gamma \bullet (\alpha \odot \neg \beta) \) by Ax PL5
3. \( \vdash ((\gamma \bullet \alpha) \odot (\neg (\gamma \bullet \beta))) \rightarrow (\alpha \odot \neg \beta) \) by 1, 2 L2
4. \( \vdash (((\gamma \bullet \alpha) \odot (\neg (\gamma \bullet \beta))) \rightarrow (\alpha \odot \neg \beta) \rightarrow (\neg (\alpha \odot \neg \beta) \rightarrow (\neg ((\gamma \bullet \alpha) \odot (\neg (\gamma \bullet \beta)))) \) by L3
5. \( \vdash (\neg (\alpha \odot \neg \beta) \rightarrow (\neg ((\gamma \bullet \alpha) \odot (\neg (\gamma \bullet \beta)))) \) by MP 3, 4
6. \( \vdash (\alpha \rightarrow \beta) \rightarrow (\neg (\alpha \odot \neg \beta) \rightarrow (\neg (\alpha \odot \neg \beta) \rightarrow (\neg ((\gamma \bullet \alpha) \odot (\neg (\gamma \bullet \beta)))) \) by [4, Lemma 3.1.1]
7. \( \vdash (\alpha \rightarrow \beta) \rightarrow (\neg (\gamma \bullet \alpha) \odot (\neg (\gamma \bullet \beta))) \) by 5, 6 , L2
8. \( \vdash (\neg ((\gamma \bullet \alpha) \odot (\neg (\gamma \bullet \beta))) \rightarrow ((\gamma \bullet \alpha) \rightarrow ((\gamma \bullet \alpha) \rightarrow (\gamma \bullet \beta)) \) by [4, Lemma 3.1.1]
9. \( \vdash (\alpha \rightarrow \beta) \rightarrow ((\gamma \bullet \alpha) \rightarrow (\gamma \bullet \beta)) \) by MP 3, 4

From the above follows that \( PL \) is an admissible expansion of \( L \). Consequently we can apply Theorem 2.8 to obtain a Pavelka style completeness for product Lukasiewicz logic. The Pavelka style completeness for product Lukasiewicz logic was obtained in a different form in [10].
3.3 Division Lukasiewicz logic

Definition 3.4 A divisible MV-algebra [7] (for short: DMV-algebra) is an algebra \( \langle A, (\delta_n)_{n \in \mathbb{N}} \rangle \) with \( \delta_n \) unary operation satisfying the following for each \( x \in A, n \in \mathbb{N} \):

1. \( A \) is an MV-algebra
2. \( n\delta_n x = x \)
3. \( \delta_n x \odot (n - 1)\delta_n x = 0 \)

We denote by \( D\text{MV} \) the variety of DMV-algebras. If \( A \) is a DMV-algebra then \([7, \text{Proposition 5.1.7}]\) \( \text{Con}_{D\text{MV}}(A) = \text{Con}_{\text{MV}}(A) \), resulting \( D\text{MV} \) a compatible expansion of \( \text{MV} \). An important example of DMV-algebra is \([0, 1]_{\text{MV}} \) equipped with the unary operations \( \delta_n x = x/n \), the division by \( n \). This algebra is denoted by \([0, 1]_{D\text{MV}} \) containing \( \mathbb{Q} \cap [0, 1]_{\text{MV}} \) as subalgebra. It is clear that \([0, 1]_{D\text{MV}} \) is hereditarily simple algebra. By \([7, \text{Proposition 5.1.3}]\) \( (\delta_n = x/n)_{n \in \mathbb{N}} \) are the unique operations making \([0, 1]_{\text{MV}} \) and \( \mathbb{Q} \cap [0, 1]_{\text{MV}} \) into a DMV-algebra resulting canonical expansions.

Definition 3.5 We define the division Lukasiewicz propositional calculus \( DL \) adding into the language of \( \text{L} \) unary connectives \( (\delta_n)_{n \in \mathbb{N}} \) plus defining inductively \( 2\alpha = \neg \alpha \rightarrow \alpha \), \( (k + 1)\alpha = \neg \alpha \rightarrow k\alpha \) and considering the following formulas as axioms

\[
\begin{align*}
\text{DL0} & \quad \text{Lukasiewicz axioms} \\
\text{DL1} & \quad \alpha \iff k(\delta_k \alpha) \\
\text{DL2} & \quad (\alpha \rightarrow k\beta) \rightarrow (\delta_k \alpha \rightarrow \beta)
\end{align*}
\]

The unique deduction rule is the \textit{Modus Ponens}.

Proposition 3.6 \([7, \text{Proposition 5.2.1}]\) \( DL \) calculus is algebraizable in \( D\text{MV} \).

From the above follows that \( DL \) is an admissible expansion of \( \text{L} \). Consequently we can apply Theorem 2.8 to obtain a Pavelka style completeness for this logic.
3.4 Łukasiewicz logics with the negation fixpoint

Let $A$ be anyone of the following varieties: $MV$, $PMV$ or $DMV$ and $L(A)$ be the corresponding propositional calculus. Let $A_{1/2}$ be the variety built from $A$, adjoining the new constant symbol $k$ and the equation $\neg k = k$. It is clear that if $A$ is a $A_k$-algebra then, $Con_{A_k}(A) = Con_{MV}(A)$, resulting $A_k$ a compatible expansion of $MV$. $[0, 1]_A$ equipped with the constant $k = 1/2$ is a $A_k$-algebra noted $[0, 1]_{A_k}$. It contains $Q[0,1]$ equipped with the $A_k$-operations as sub $A_k$-algebra. It is clear that $[0, 1]_{A_k}$ is hereditarily simple. By [9, Lemma 2.10], $k = 1/2$ is the unique fixpoint of the negation of $[0, 1]_A$. Thus $[0, 1]_{A_k}$ is the $A_k$-canonical expansion of $[0, 1]_{MV}$. We define the propositional calculus $L(A_k)$, adding into the language of $L(A)$ the constant symbol $k$, and consider the following formulas as axioms:

1. $L(A)$-axioms
2. $\neg k \iff k$

and modus ponens as unique deduction rule. It is clear that $L(A_k)$ is algebraizable in $A_k$. From this, follows that $L(A_k)$ is an admissible expansion of $L$. Consequently we can apply Theorem 2.8 to obtain a Pavelka style completeness for these logics (see also [5]).

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e-mail: hfreytes@gmail.com