Semi-equivelar toroidal maps and their $k$-semiregular covers

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Abstract

If the face-cycles at all the vertices in a map are of same type then the map is called semi-equivelar. In particular, it is called equivelar if the face-cycles contain same type of faces. A map is semiregular (or almost regular) if it has as few flag orbits as possible for its type. A map is $k$-regular if it is equivelar and the number of flag orbits of the map $k$ under the automorphism group. In particular, if $k = 1$, its called regular. A map is $k$-semiregular if it contains more number of flags as compared to its type with the number of flags orbits $k$. Drach et al. [7] have proved that every semi-equivelar toroidal map has a finite unique minimal semiregular cover. In this article, we show the bounds of flag orbits of semi-equivelar toroidal maps, i.e., there exists $k$ for each type such that every semi-equivelar toroidal map is $\ell$-uniform for some $\ell \leq k$. We show that none of the Archimedean types on the torus is semiregular, i.e., for each type, there exists a map whose number of flag orbits is more than its type. We also prove that if a semi-equivelar map is $m$-semiregular then it has a finite index $t$-semiregular minimal cover for $t \leq m$.

We also show the existence and classification of $n$ sheeted $k$-semiregular maps for some $k$ of semi-equivelar toroidal maps for each $n \in \mathbb{N}$.

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1 Introduction

A map $M$ is an embedding of a graph $G$ on a surface $S$ such that the closure of components of $S \setminus G$, called the faces of $M$, are homeomorphic to 2-discs. A map $M$ is said to be a polyhedral map if the intersection of any two distinct faces is either empty, a common vertex, or a common edge. Here map means a polyhedral map.

The face-cycle $C_u$ of a vertex $u$ (also called the vertex-figure at $u$) in a map is the ordered sequence of faces incident to $u$. So, $C_u$ is of the form $(F_{i,1} \cdots F_{i,n_i}) \cdots (F_{k,1} \cdots F_{k,n_k})F_{1,1}$, where $F_{i,\ell}$ is a $p_{i,\ell}$-gon for $1 \leq \ell \leq n_i$, $1 \leq i \leq k$, $p_r \neq p_{r+1}$ for $1 \leq r \leq k-1$ and $p_n \neq p_1$. The types of the faces in $C_u$ defines the type of $C_u$. In this case, the type of face-cycle$(u)$ is $[p_1^{n_1}, \ldots, p_k^{n_k}]$, is called vertex type of $u$. A map $M$ is called semi-equivelar ([4], we are including the same definition for the sake of completeness) if $C_u$ and $C_v$ are of same type for all $u,v \in V(X)$. More precisely, there exist integers $p_1, \ldots, p_k \geq 3$ and $n_1, \ldots, n_k \geq 1$, $p_i \neq p_{i+1}$ (addition in the suffix is modulo $k$) such that $C_u$ is of the form as above for all $u \in V(X)$. In such a case, $X$ is called a semi-equivelar map of type (or vertex type) $[p_1^{n_1}, \ldots, p_k^{n_k}]$ (or, a map of type $[p_1^{n_1}, \ldots, p_k^{n_k}]$).
Two maps of fixed type on the torus are isomorphic if there exists a homeomorphism
of the torus which maps vertices to vertices, edges to edges, faces to faces and preserves
incidents. More precisely, if we consider two polyhedral complexes $M_1$ and $M_2$ then an
isomorphism to be a map $f : M_1 \rightarrow M_2$ such that $f|_{V(M_1)} : V(M_1) \rightarrow V(M_2)$ is a bijection
and $f(\sigma)$ is a cell in $M_2$ if and only if $\sigma$ is a cell in $M_1$. In particular, if $M_1 = M_2$, then $f$ is
called an automorphism. The automorphism group $\text{Aut}(M)$ of $M$ is the group consisting of
automorphisms of $M$.

Throughout the last few decades there have been many results about maps and semi-
equivelar maps that are highly symmetric. In particular, there has been recent interest in
the study of discrete objects using combinatorial, geometric, and algebraic approaches, with
the topic of symmetries of maps receiving a lot of interest. There is a great history of work
surrounding maps on the Euclidean plane $\mathbb{R}^2$ and the 2-dimensional torus.

An Archimedean tiling of the plane $\mathbb{R}^2$ is a tiling of $\mathbb{R}^2$ by regular polygons such that all
the vertices of the tiling are of same type. Grünbaum and Shephard [3] showed that there
are exactly eleven types of Archimedean tilings on the plane (see Section 2). These types are
$[3^6], [4^4], [6^3], [3^4, 6^1], [3^2, 4^2], [3^2, 4^4, 3^1, 4^1], [3^1, 6^1, 3^1, 6^1], [3^1, 4^1, 6^1, 4^1], [3^1, 12^2], [4^1, 6^1, 12^1],$
$[4^1, 8^2]$. Clearly, these tilings are also semi-equivelar on $\mathbb{R}^2$. But, there are semi-equivelar
maps on $\mathbb{R}^2$ which are not (not isomorphic to) Archimedean tilings. In fact, there exists $[p^q]
$equivelar maps on $\mathbb{R}^2$ whenever $1/p + 1/q < 1/2$ (e.g., $[2], [3]$). We know from $[5, 3, 4, 4]
$that the Archimedean tilings $E_i$ ($1 \leq i \leq 11$) (in Section 2) are unique as semi-equivelar maps.
That is, we have the following.

**Proposition 1.1.** Let $E_1, \ldots, E_{11}$ be the Archimedean tilings on the plane given in Section
2 Let $X$ be a semi-equivelar map on the plane. If the type of $X$ is same as the type of $E_i$, for some $i \leq 11$, then $X \cong E_i$. In particular, $X$ is vertex-transitive.

As a consequence of Proposition 1.1, we have

**Proposition 1.2.** All semi-equivelar maps on the torus are the quotient of an Archimedean
tiling on the plane by a discrete subgroup of the automorphism group of the tiling.

A map is regular if its automorphism group acts regularly on flags (which, in nondegenerate
cases, may be identified with mutually incident vertex-edge-face triples). In general,
a map is semiregular (or almost regular) if it has as few flag orbits as possible for its type.
A map is $k$-regular if it is equivelar and the number of flag orbits of the map $k$ under the
automorphism group. In particular, if $k = 1$, its called regular. Similarly, a map is called
$k$-semiregular if it contains more number of flags as compared to its type and the number of
flag orbits $k$. The study of regular maps on compact surfaces has a long and rich history.
Its early stages go back to the ancient Greeks’ interest in highly symmetric solids and (much
later) to Kepler’s discovery of stellated polyhedra. A new dimension to the combinatorial
and group-theoretic nature of the study of highly symmetric maps was added in the late 19th
century in the work of Klein and Poincaré by revealing facts that relate the theory of maps
to hyperbolic geometry and automorphic functions.

A systematic approach to classification of regular maps on a given surface was initiated
by Brahana in the early 20th century. In the span of the following 70 years this was gradually
extended by contributions of numerous authors, resulting by the end of 1980’s in a classification
of all chiral and regular maps on orientable surfaces of genus up to 7, and regular maps
on nonorientable surfaces of genus at most 8. Details of this development are summarized in
the survey paper [17]. In 2000, the classification was extended with the help of computing
power to orientable and nonorientable surfaces of genus up to 101 and 202, respectively [14].
Nevertheless, by the end of 20th century, classification of regular maps was available only for a finite number of surfaces.

Many ideas of the discrete symmetric structures on torus follow from the concepts introduced by Coxeter and Moser in [2]. A surjective mapping $\eta: X \to Y$ from a map $X$ to a map $Y$ is called a covering if it preserves adjacency and sends vertices, edges, faces of $X$ to vertices, edges, faces of $Y$ respectively. That is, let $G \leq \text{Aut}(X)$ be a discrete group acting on a map $X$ properly discontinuously ([14 Chapter 2]). This means that each element $g$ of $G$ is associated with an automorphism $h_g$ of $X$ onto itself, in such a way that $h_{gh}$ is always equal to $h_g h_h$ for any two elements $g$ and $h$ of $G$, and $G$-orbit of any vertex $u \in V(X)$ is locally finite. Then, there exists $\Gamma \leq \text{Aut}(X)$ such that $Y = X/\Gamma$. In such a case, $X$ is called a cover of $Y$. A map $X$ is called regular if the automorphism group of $X$ acts transitively on the set of flags of $X$. Clearly, if a semi-equivelar map is not equivelar then it cannot be regular.

A natural question then is:

**Question 1.3.** Let $X$ be a semi-equivelar map on the torus. Let $X$ be $k$-semiregular. Does there exist any cover $Y(\neq X)$ of some $m$-semiregular map? Does this cover exist for every sheet, if so, how many? How the flag orbits of $X$ and $Y$ are related?

In this context, there is also much interest in finding minimal regular covers of different families of maps and polytopes (see [10] [15] [16]). In [6], Drach et al. constructed the minimal rotary cover of any equivelar toroidal map. Then, they have extended their idea to toroidal maps that are no longer equivelar, and constructed minimal toroidal covers of the Archimedean toroidal maps with maximal symmetry (see in [7]), called these covers almost regular; they will no longer be regular (or chiral), but instead will have the same number of flag orbits as their associated tessellation of the Euclidean plane. Here, we have the following.

**Theorem 1.4.** Let $X$ be a semi-equivelar map on the torus. Let the flags of $X$ form $m \text{Aut}(X)$-orbits.

(a) If the type of $X$ is $[3^6]$ or $[6^3]$ then $m \leq 6$.

(b) If the type of $X$ is $[4^4]$ then $m \leq 4$.

(c) If the type of $X$ is $[3^3, 4^2]$ or $[3^2, 4^1, 3^1, 4^1]$ then $m \leq 10$.

(d) If the type of $X$ is $[4^1, 8^2]$ or $[3^1, 6^1, 3^1, 6^1]$ then $m \leq 12$.

(e) If the type of $X$ is $[3^1, 12^2]$ then $m \leq 18$.

(f) If the type of $X$ is $[3^1, 4^1, 6^1, 4^1]$ then $m \leq 24$.

(g) If the type of $X$ is $[3^4, 6^1]$ then $m \leq 30$.

(h) If the type of $X$ is $[4^1, 6^1, 12^1]$ then $m \leq 36$. These bounds are also sharp.

**Proposition 1.5.** ([16] [7]) Let $E$ be an Archimedean tiling of type $Z$ and $k$-semiregular. If $X$ is a semi-equivelar toroidal map of type $Z$ then there exists a covering $\eta: Y \to X$ where $Y$ is $k$-semiregular and unique.

In this context of Prop. 1.5, we prove the following.

**Theorem 1.6.** (a) If $X_1$ is a $m_1$-semiregular toroidal map of type $[3^6]$ or $[6^3]$, then there exists a covering $\eta_{k_1}: Y_{k_1} \to X_1$ where $Y_{k_1}$ is $k_1$-semiregular for each $k_1 \leq m_1$ except $k_1 = 4, 5$.

(b) If $X_2$ is a $m_2$-semiregular toroidal map of type $[4^4]$, then there exists a covering $\eta_{k_2}: Y_{k_2} \to X_2$ where $Y_{k_2}$ is $k_2$-semiregular for each $k_2 \leq m_2$ except $k_2 = 3$.

(c) If $X_3$ is a $m_3$-semiregular toroidal map of type $[3^1, 4^1, 6^1, 4^1]$. Then, there exists a covering $\eta_{k_3}: Y_{k_3} \to X_3$ where $Y_{k_3}$ is $k_3$-semiregular for each $(k_3, m_3) = (4, 8), (8, 24), (4, 12), (12, 24), (4, 24)$.

(d) If $X_7$ is a $m_7$-semiregular toroidal map of type $[3^1, 6^1, 3^1, 6^1]$ then there exists a covering
η_{k_7}: Y_{k_7} \rightarrow X_7 \text{ where } Y_{k_7} \text{ is } k_7\text{-semiregular for each } k_7 \leq m_7 \text{ for } (k_7, m_7) = (4, 8), (8, 24), (4, 12), (12, 24), (4, 24)

(e) If \(X_8\) is a \(m_8\)-semiregular toroidal map of type \([3^1, 12^2]\) then there exists a covering \(\eta_{k_8}: Y_{k_8} \rightarrow X_8\) where \(Y_{k_8}\) is \(k_8\)-semiregular for each \((k_8, m_8) = (3, 6), (3, 9), (3, 18), (6, 18), (9, 18).\)

(f) If \(X_{11}\) is a \(m_{11}\)-semiregular toroidal map of type \([4^1, 6^1, 12^1]\), then there exists a covering \(\eta_{k_{11}}: Y_{k_{11}} \rightarrow X_{11}\) where \(Y_{k_{11}}\) is \(k_{11}\)-semiregular for each \(k_{11} \leq m_{11}\) for \((k_{11}, m_{11}) = (6, 12), (6, 18), (6, 36), (12, 36), (18, 36).\)

(g) If \(X_6\) is a \(m_6\)-semiregular toroidal map of type \([4^1, 8^2]\) then there exists a covering \(\eta_{k_6}: Y_{k_6} \rightarrow X_6\) where \(Y_{k_6}\) is \(k_6\)-semiregular for each \((k_6, m_6) = (6, 12), (3, 6), (3, 12).\)

(h) If \(X_4\) is a \(m_4\)-semiregular toroidal map of type \([3^3, 4^2]\), then there exists a covering \(\eta_{k_4}: Y_{k_4} \rightarrow X_4\) where \(Y_{k_4}\) is \(k_4\)-semiregular for \((k_4, m_4) = (5, 10).\)

(i) If \(X_5\) is a \(m_5\)-semiregular toroidal map of type \([3^2, 4^1, 3^1, 4^1]\), then there exists a covering \(\eta_{k_5}: Y_{k_5} \rightarrow X_5\) where \(Y_{k_5}\) is \(k_5\)-semiregular for each \((k_5, m_5) = (5, 10), (5, 20), (10, 20).\)

(j) If \(X_{10}\) is a \(m_{10}\)-semiregular toroidal map of type \([3^4, 6^1]\), then there exists a covering \(\eta_{k_{10}}: Y_{k_{10}} \rightarrow X_{10}\) where \(Y_{k_{10}}\) is \(k_{10}\)-semiregular for \((k_{10}, m_{10}) = (10, 30).\)

**Theorem 1.7.** Let \(X\) be a semi-equivelar toroidal map and \(k\)-semiregular. Then, there exists a \(n\) sheeted covering \(\eta: Y \rightarrow X\) for each \(n \in \mathbb{N}\) where \(Y\) is \(m\)-semiregular for some \(m \leq k.\)

**Theorem 1.8.** Let \(X\) be a \(n\) sheeted semi-equivelar \(k\)-semiregular toroidal map and \(\sigma(n) = \sum_{d|n} d.\) Then, there exists different \(n\) sheeted \(m\)-semiregular covering \(\eta_{\ell}: Y_{\ell} \rightarrow X\) for \(\ell \in \{1, 2, \ldots, \sigma(n)\}, i.e., Y_1, Y_2, \ldots, Y_{\sigma(n)}\) are \(n\) sheeted \(m\)-semiregular covers of \(X\) and different upto isomorphism for some \(m \leq k.\)

**Theorem 1.9.** Let \(X\) be a \(m\)-semiregular semi-equivelar toroidal map and \(Y\) be a \(k\)-semiregular covers of \(X.\) Then, there exists a \(k\)-semiregular covering map \(\eta: Z \rightarrow X\) such that \(Z\) is minimal.

## 2 Examples

We first present eleven Archimedean tilings on the plane. We need these examples for the proofs of our results in Section 3.

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**Figure 1:** \(E_1 ([3^6])\)

**Figure 2:** \(E_2 ([4^4])\)
3 Classification of $k$-semiregular covers of semi-equivelar maps

Before going to the proofs of main theorems we need following series of results. From [7, Proposition 3.2-3.7] we get

**Proposition 3.1.** Let $E$ be a semi-equivelar tiling on the plane. Suppose $E$ has $m$ flag-orbits. Then

(a) If the type of $E$ is $[3^6]$, $[4^4]$ or $[6^3]$ then $m = 1$.

(b) If the type of $E$ is $[3^1, 6^1, 3^1, 6^1]$ then $m = 2$.

(c) If the type of $E$ is $[3^1, 12^2]$ or $[4^1, 8^2]$ then $m = 3$.

(d) If the type of $E$ is $[3^1, 4^1, 6^1, 4^1]$ then $m = 4$.

(e) If the type of $E$ is $[3^2, 4^1, 3^1, 4^1]$ then $m = 5$.

(f) If the type of $E$ is $[4^1, 6^1, 12^1]$ then $m = 6$.

(g) If the type of $E$ is $[3^4, 6^1]$ then $m = 10$.

(h) If the type of $E$ is $[3^3, 4^2]$ then $m = 5$.

**Proof of Theorem 1.4.** Let for $i = 1, 2, \ldots, 11$ $E_i$ be the Archimedean tiling of the plane as in Section 2. Consider $\alpha_i$ and $\beta_i$ be the fundamental translations of $E_i$. $\alpha_i : z \mapsto z + A_i$ and $\beta_i : z \mapsto z + B_i$. Let $X$ be a semi-equivelar map of type $[p_1^{r_1}, \ldots, p_k^{r_k}]$. Then there exists
a discrete subgroup $K_i$ of $\text{Aut}(E_i)$ with out any fixed element such that $X = E_i/K_i$. Let $p_i : E_i \to X$ be the polyhedral covering map. By above description of $K_i$, it contains only translations and glide reflections. Since, $X$ is orientable so $K_i$ does not contain any glide reflections. Thus $K_i \leq H_i$. Suppose $K_i = \langle \gamma_i, \delta_i \rangle$. Let $\chi_i$ denotes the reflection about origin in $E_i$. Then $\chi_i \in \text{Aut}(E_i)$. Consider the group $G_i = \langle \alpha_i, \beta_i, \chi_i \rangle \leq \text{Aut}(E_i)$.

**Claim 3.1.** $K_i \leq G_i$.

To prove this it is enough to show that $\chi_i \circ \gamma_i \circ \chi_i^{-1}$ and $\chi_i \circ \delta_i \circ \chi_i^{-1} \in K_i$. We know that conjugation of a translation by reflection is translation by the reflected vector. Let $\gamma_i$ and $\delta_i$ are translation by vectors $C_i$ and $D_i$ respectively. Then $\chi_i \circ \gamma_i \circ \chi_i^{-1}$ and $\chi_i \circ \delta_i \circ \chi_i^{-1}$ are translation by $-C_i$ and $-D_i$. Clearly these vectors are belongs to lattice of $K_i$. Our claim follows from this.

Case 1. Let $X$ is of type $[3^6], [6^3]$. Suppose $X = E_i/K_i$. $E_1$ has 12 flag orbits by action of $H_1$. Under action of $G_1$, $E_1$ has 6 flag orbits. Hence action of $G_i/K_i$ on flags of $X$ also gives same number of orbits. $G_i/K_i \leq \text{Aut}(X)$. Thus number of flag orbits of $X$ is less that or equals to 6 for maps of type $[3^6]$ and $[6^3]$. This proves part (a) of Theorem 1.4

Case 2. Let $X$ be a semi-equivelar map of type $[4^4]$. Then by Proposition 1.2 we can assume $X = E_2/K_2$ for some subgroup $K_2$ of $\text{Aut}(E_2)$. Now $F(E_2)$ has 4 $G_2$ orbits. Hence $X$ also has 4 $G_2/K_2$-orbits. As $G_2/K_2 \leq \text{Aut}(X)$. Therefore number of $\text{Aut}(X)$-orbits of $F(X)$ is less than or equals to 4.

Case 3. Let $X$ be a semi-equivelar map of type $[3^3, 4^2]$ or $[3^2, 4^1, 3^1, 4^1]$. Then by Proposition 1.2 we can assume $X = E_11/K_11$ or $E_5/K_5$ for some subgroup $K_11$ of $\text{Aut}(E_11)$ and $K_5$ of $\text{Aut}(E_5)$. Now $F(E_11)$ and $F(E_5)$ has 10 $G_{11}$ and $G_5$ orbits respectively. Hence $X$ also has 10 $G_i/K_i$-orbits for $i = 11, 5$. As $G_i/K_i \leq \text{Aut}(X)$. Therefore number of $\text{Aut}(X)$-orbits of $F(X)$ is less than or equals to 10.

Case 4. Let $X$ be a semi-equivelar map of type $[3^1, 6^1, 3^1, 6^1]$ or $[4^1, 8^2]$. Then by Proposition 1.2 we can assume $X = E_i/K_i$ for some subgroup $K_i$ of $\text{Aut}(E_i)$ for $i = 6, 7$. Now $F(E_i)$ has 12 $G_i$ orbits. Hence $X$ also has 12 $G_i/K_i$-orbits. As $G_i/K_i \leq \text{Aut}(X)$. Therefore number of $\text{Aut}(X)$-orbits of $F(X)$ is less than or equals to 12.

Case 5. Let $X$ be a semi-equivelar map of type $[3^1, 12^2]$. Then by Proposition 1.2 we can assume $X = E_8/K_8$ for some subgroup $K_8$ of $\text{Aut}(E_8)$. Now $F(E_8)$ has 18 $G_8$ orbits. Hence $X$ also has 18 $G_8/K_8$-orbits. As $G_8/K_8 \leq \text{Aut}(X)$. Therefore number of $\text{Aut}(X)$-orbits of $F(X)$ is less than or equals to 18.

Case 6. Let $X$ be a semi-equivelar map of type $[3^4, 6^1]$. Then by Proposition 1.2 we can assume $X = E_9/K_9$ for some subgroup $K_9$ of $\text{Aut}(E_9)$. Now $F(E_9)$ has 24 $G_9$ orbits. Hence $X$ also has 24 $G_9/K_9$-orbits. As $G_9/K_9 \leq \text{Aut}(X)$. Therefore number of $\text{Aut}(X)$-orbits of $F(X)$ is less than or equals to 24.

Case 7. Let $X$ be a semi-equivelar map of type $[3^4, 6^1]$. Then by Proposition 1.2 we can assume $X = E_{10}/K_{10}$ for some subgroup $K_{10}$ of $\text{Aut}(E_{10})$. Now $F(E_{10})$ has 30 $G_{10}$ orbits. Hence $X$ also has 30 $G_{10}/K_{10}$-orbits. As $G_{10}/K_{10} \leq \text{Aut}(X)$. Therefore number of $\text{Aut}(X)$-orbits of $F(X)$ is less than or equals to 30.

Case 8. Let $X$ be a semi-equivelar map of type $[4^1, 6^1, 12^1]$. Then by Proposition 1.2 we can assume $X = E_{11}/K_{11}$ for some subgroup $K_{11}$ of $\text{Aut}(E_{11})$. Now $F(E_{11})$ has 36 $G_{11}$ orbits. Hence $X$ also has 36 $G_{11}/K_{11}$-orbits. As $G_{11}/K_{11} \leq \text{Aut}(X)$. Therefore number of $\text{Aut}(X)$-orbits of $F(X)$ is less than or equals to 36.

**Proof of Theorem 1.6.** Let $X_1$ be a semi-equivelar toroidal map of type $[3^6]$. Then by Proposition 1.2 we can assume that $X_1 = E_1/K_1$ for some fixed point free subgroup $K_1$ of $\text{Aut}(X_1)$.
Thus $K_1$ consist of only translations and glide reflections. Since $X_1$ is orientable so $K_1$ contains only translations. Consider $H_1$, $K_1$ and $G_1$ as in proof of Theorem 1.3. Now $F(X_1)$ has 6 $G_1$-orbits. Consider the group $G_1' = \langle \alpha_1, \beta_1, \chi_1, \rho_{OA} \rangle$. Clearly $F(E_1)$ has 3 $G_1'$ orbits. Let $K_1 = \langle \gamma, \delta \rangle$. Now to get a cover of $X$ we need $L_1 \leq \text{Aut}(E_1)$ such that $G_1'/L_1$ is defined and $E_1/L_1$ has 3 $G_1'/L_1$ orbits. For that we made the following.

**Claim 3.2.** There exists $m \in \mathbb{Z}$ such that $L_1 := \langle \gamma^m, \delta^m \rangle \leq G_1'$.

Suppose $L_1 = \langle \gamma^m_1, \delta^m_2 \rangle$. We show that there exists suitable $m_1, m_2$ such that $L_1 \leq G_1'$. It turns out that we can take $m_1 = m_2$. To satisfy $L_1 \leq G_1'$ it is enough to show that $\rho_1 \gamma_1^m \rho_1^{-1}, \rho_1 \delta_2^m \rho_1^{-1} \in L_1$. It is known that conjugation of a translation by rotation or reflection is also a translation by rotated or reflected vector. Since $\gamma$ and $\delta$ are translation by vectors $C$ and $D$ respectively so $\gamma^m_1$ and $\delta^m_2$ are translation by vectors $m_1 C$ and $m_2 D$ respectively. Hence $\rho_1 \gamma_1^m \rho_1^{-1}$ and $\rho_1 \delta_2^m \rho_1^{-1}$ are translation by the vectors $C'$ and $D'$ respectively.

Then

$$C' = p(m_1 C) + q(m_2 D), \quad D' = s(m_1 C) + t(m_2 D).$$

Putting expressions of $C', D', C, D$ in above equations we get,

$$(bm_1 - pam_1 - qcm_2)A_1 + (am_1 - pbm_2 - qdm_2)B_1 = 0$$

$$(cm_2 - sbm_1 - tdm_2)A_1 + (dm_2 - sam_1 - tcm_2)B_1 = 0.$$ 

Since, $\{A_1, B_1\}$ is a linearly independent set we have,

$$pam_1 + qcm_2 = bm_1, \quad pbm_1 + qdm_2 = am_1, \quad sbm_1 + tdm_2 = cm_2, \quad sam_1 + tcm_2 = dm_2.$$ 

Now as $\text{rank}(L_1) = 2$ so $m_1, m_2 \neq 0$. Dividing the above system by $m_1 m_2$ we get,

$$\frac{pa}{m_2} + \frac{qc}{m_1} = \frac{b}{m_2}, \quad \frac{pb}{m_2} + \frac{qd}{m_1} = \frac{a}{m_2}, \quad \frac{sb}{m_2} + \frac{td}{m_1} = \frac{c}{m_2}, \quad \frac{sa}{m_2} + \frac{tc}{m_1} = \frac{d}{m_1}.$$ 

Now consider $p, q, s, t$ as variables. We can treat above system as a system of linear equations. We can write this system in matrix form as follows.

$$\begin{bmatrix}
a/m_2 & c/m_1 & 0 & 0 
b/m_2 & d/m_1 & 0 & 0 
0 & 0 & a/m_2 & c/m_1 
0 & 0 & b/m_2 & d/m_1
\end{bmatrix}\begin{bmatrix}p 
q 
s 
t\end{bmatrix} \begin{bmatrix}b/m_2 
a/m_2 
d/m_1 
c/m_1\end{bmatrix}$$

Now $C, D$ are linearly independent thus $ad - bc \neq 0$. Hence the coefficient matrix of the above system has non zero determinant. Therefore the system has an unique solution. After solving we get,

$$p = \frac{m_1^2 m_2 (d - a)b}{(ad - bc)^2}, \quad q = \frac{m_1^2 m_2 (a^2 - bc)}{(ad - bc)^2}, \quad s = \frac{m_1 m_2^2 (d^2 - bc)}{(ad - bc)^2}, \quad t = \frac{m_1 m_2^2 (a - dc)}{(ad - bc)^2}.$$ 

Now if we take $m_1 = m_2 = |ad - bc| = m$(say) then $p, q, s, t \in \mathbb{Z}$. Let $L_1 := \langle \gamma^m, \delta^m \rangle$. Then we have $L_1 \leq G_1'$. Hence our Claim 3.2 proved.

**Claim 3.3.** $K_1/L_1 \leq \text{Aut}(M_1/L_1)$. 

8
Let \( \rho \in \text{Nor}_{\text{Aut}(M_1)}(L_1) \). Then \( \rho \gamma \rho^{-1}, \rho \delta \rho^{-1} \in L_1 \). \( \rho \gamma \rho^{-1} \) is translation by the vector \((\rho \gamma \rho^{-1})(0)\). That is \((\rho \gamma \rho^{-1})(0)\) is lattice of \( L_1 \). Thus there exists \( n_1, n_2 \) such that \((\rho \gamma \rho^{-1})(0) = n_1 \gamma(0) + n_2 \delta(0) \). As \( K_1 \) is generated by \( \gamma \) and \( \delta \) so \( \rho \gamma \rho^{-1}(0) = n_1 \gamma(0) + n_2 \delta(0) \). Thus \( \rho \gamma \rho^{-1} \in K_1 \). Similarly, \( \rho \delta \rho^{-1} \in K_1 \). Therefore \( \rho \in \text{Aut}(K_1) \Rightarrow \text{Aut}(M_1) \leq \text{Aut}(K_1) \Rightarrow K_1 \leq \text{Aut}(M_1)(L_1) \Rightarrow K_1/L_1 \leq \text{Aut}(M_1)(L_1)/L_1 \).

From \([7]\) we know \( \text{Aut}(M_1/L_1) = \text{Aut}(M_1)(L_1)/L_1 \). This proves Claim 3.3.

Now by Claim 3.2 \( G_1^1/L_1 \) is a group and acts on \( E(M_1/L_1) \). Clearly \( E(M_1/L_1) \) has 2 \( G_1^1/L_1 \)-orbits. Since \( L_1 \) contains two independent vectors, it follows that \( Y_1 := M_1/L_1 \) is a toroidal map and \( v + L_1 \mapsto v + K_1 \) is a covering \( \eta : Y_1 \to X \). Our next aim is to show that \( Y \) is a 2-orbital map. For that we need the following.

**Result 3.1.** Let \( L \leq K \) and \( K \) acts on a topological space \( E \). Then \( \frac{E/L}{K/L} \) is homeomorphic to \( E/K \) and \( \phi : \frac{E/L}{K/L} \to E/K \) defined by \( \left( \frac{K}{L} \right)(Lv) \mapsto Kv \forall v \in E \) is a homeomorphism.

Let \( p : M_1/L_1 \to \frac{M_1/L_1}{K_1/L_1} = M_1/K_1 \) be the quotient map.

**Claim 3.4.** Given \( \alpha \in \text{Aut}(M_1/L_1) = \text{Nor}_{\text{Aut}(M_1)}(L_1)/L_1 \) there exists \( \tilde{\alpha} \in \text{Aut}(M_1/K_1) \) such that \( p \circ \alpha = \tilde{\alpha} \circ p \).

By Result 3.1 we can think \( M_1/K_1 \) as \( \frac{M_1/L_1}{K_1/L_1} \). We show that \( \alpha \) takes orbits to orbits for the action of \( K_1/L_1 \) on \( M_1/L_1 \). Let \( \mathcal{O}(\overline{\sigma}) \) denotes \( K_1/L_1 \)-orbit of \( \overline{\sigma} \in M_1/L_1 \). Then

\[
\alpha(\mathcal{O}(\overline{\sigma})) = \alpha \left( \frac{K_1}{L_1}(\overline{\sigma}) \right) = \frac{K_1}{L_1}(\alpha(\overline{\sigma})) \quad \text{[since } \frac{K_1}{L_1} = \frac{K_1}{L_1} \alpha \text{ because } K_1/L_1 \leq \text{Aut}(M_1/L_1)]
\]

\[
= \mathcal{O}(\alpha(\overline{\sigma}))
\]

Therefore by universal property of quotient there exists \( \tilde{\alpha} : M_1/K_1 \to M_1/K_1 \) such that the following diagram commutes.

\[
\begin{array}{ccc}
M_1/L_1 & \xrightarrow{\alpha} & M_1/L_1 \\
\downarrow{p} & & \downarrow{p} \\
\frac{M_1/L_1}{K_1/L_1} & \xrightarrow{\tilde{\alpha}} & \frac{M_1/L_1}{K_1/L_1}
\end{array}
\]

Figure 12: Diagram

Now we have to show that \( \tilde{\alpha} \) is an automorphism. Clearly \( \tilde{\alpha} \) is onto. Let \( \overline{\sigma_1}, \overline{\sigma_2} \in Y \). Suppose \( \mathcal{O}(\overline{\sigma_1}) \) and \( \mathcal{O}(\overline{\sigma_2}) \) be \( K_1/L_1 \) orbits of \( Y \). Now,

\[
\tilde{\alpha}(\mathcal{O}(\overline{\sigma_1})) = \tilde{\alpha}(\mathcal{O}(\overline{\sigma_2})) \implies \mathcal{O}(\alpha(\overline{\sigma_1})) = \mathcal{O}(\alpha(\overline{\sigma_2})) \\
\implies \exists \omega \in K_1/L_1 \text{ such that } \omega(\overline{\sigma_1}) = \alpha(\overline{\sigma_2}) \\
\implies \alpha^{-1}(\omega(\overline{\sigma_1})) = \overline{\sigma_2} \\
\implies \mathcal{O}(\overline{\sigma_1}) = \mathcal{O}(\overline{\sigma_2}) \quad [\alpha^{-1}\omega \in K_1/L_1 \text{ since } K_1/L_1 \leq \text{Aut}(M_1/L_1)]
\]
Therefore, $\bar{\alpha}$ is one-one. Now, by the commutativity of the diagram and using the fact that $p$ is a covering map one can see that $\bar{\alpha}$ takes vertices to vertices, edges to edges, faces to faces. It also preserves incidence relations. Let $v \in \text{Domain of } \bar{\alpha}$. Since $p$ is a covering map there exists a neighbourhood $N$ of $v$ which is evenly covered by $p$. Let $U$ be a component of $p^{-1}(N)$. Then $p: U \to N$ is a homeomorphism. Therefore $(p \circ \alpha)|_U = \bar{\alpha}|_N$. As $p$ and $\alpha$ both are continuous so is $\bar{\alpha}|_N$. Thus $\bar{\alpha}$ is continuous. Now, replacing $\alpha$ by $\alpha^{-1}$ we get $\bar{\beta}$ in place of $\bar{\alpha}$. $\bar{\beta}$ has same properties as of $\bar{\alpha}$. Now, $\bar{\alpha} \circ \bar{\beta} = \text{id}_{M_1/K_1} = \bar{\beta} \circ \bar{\alpha}$. Therefore $\bar{\alpha}^{-1} = \bar{\beta}$. So $\bar{\alpha}$ is a homeomorphism. Hence $\bar{\alpha}$ is an automorphism of $M_1/K_1$.

**Claim 3.5.** If $\alpha \in \text{Aut}(M_1/L_1) \setminus \frac{G_1}{L_1}$ then $\alpha(\mathcal{O}) = \mathcal{O}$ for all $\frac{G_1}{L_1}$-orbits $\mathcal{O}$ of $M_1/L_1$.

Let $\alpha \in \text{Aut}(M_1/L_1) \setminus \frac{G_1}{L_1}$ and $\bar{\alpha}$ be the induced automorphism on $M_1/K_1$ as in Fig. 12. Suppose, $O_1$ and $O_2$ be two $G_1'/L_1'$-orbits of $M_1/L_1$. Let $a_1, a_2 \in M_1$ be such that $L_1a_1 \in O_1$ and $L_1a_2 \in O_2$ and $\alpha(L_1a_1) = L_1a_2$. Since, $p(L_1a_1) = K_1a_1$ by commutativity of the diagram in Claim 3.4 we get $\bar{\alpha}(K_1a_1) = K_1a_2$. As $\bar{\alpha}$ does not take an element of $G_1/K_1$-orbit to an element of any other orbit so $K_1a_1$ belongs to the same $G_1/K_1$-orbit of $M_1/K_1$. Therefore, there exists $gK_1 \in G_1/K_1$ such that $(gK_1)(K_1a_1) = K_1a_2$. Now, Since $(gK_1)(K_1a_1) = K_1a_2 \implies K_1(\bar{\alpha}(g_1)a_1) = K_1a_2 \implies \exists k \in K_1$ such that $(k \circ g)(a_1) = a_2$.

Let $g' := k \circ g \in G_1'$ then $g'(a_1) = a_2$. Consider $g'L_1 \in G_1'/L_1$. Then $(g'L_1)(L_1a_1) = L_1a_2$. This contradicts our assumption that $L_1a_1$ and $L_1a_2$ belong to two different $G_1'/L_1$-orbit of $M_1/L_1$. This proves Claim 3.5.

Let $R_1$ and $R_2$ denote the reflections of $E_1$ about $OA$ and $OA$. Now if the given map $X_1$ is 6-semiregular then consider the group $G_4 = \langle \alpha_1, \beta_1, \chi_1, R_1 \circ R_2 \rangle$ instead of $G_1'$. Then proceeding as above we get $Y_1 := E_1/L_3$ is a 2-semiregular cover of $X_1$, where $L_3 = \langle \gamma_1^{m_2}, \delta_1^{m_2} \rangle \leq G_4$ for some $m_2 \in \mathbb{Z}$. Now if the given map $X_1$ is 3 or 2 semiregular then consider the group $G_3 = \langle \alpha_1, \beta_1, \chi_1, R_1, R_2 \rangle$. Then proceeding as above we get $Y_2 := E_1/L_2$ is a 1-semiregular cover of $X_1$, where $L_2 = \langle \gamma_1^{m_1}, \delta_1^{m_1} \rangle \leq G_3$ for some $m_1 \in \mathbb{Z}$.

Now if two conjugate subgroups of $\text{Aut}(E_1)$ acts on $F(E_1)$ then they give same number of orbits. So to find how many different orbital maps are there for a given type we need to check number of orbits under action of a group taken from each conjugacy class of $\text{Aut}(E_1)$. Now we know that automorphism groups on the plane are of the form $T \times S$ where $T$ is the translation group and $S$ is stabilizer of origin for the action of $\text{Aut}(E_1)$ on $E_1$. If $X = E_1/K$ is a toroidal map then $\text{Aut}(X) = \text{Nor}(K)/K$. Since $K$ contains only translations so $\text{Nor}(K)$ always contains $T \times \langle \chi \rangle$. Thus one needs to determine which symmetries in $S$ normalizes $K$. Now $S = \langle R_1, R_2, \chi \rangle$. For type $[3^6]$ there are 4 subgroups of $S$ up to conjugates. They are $\langle \chi \rangle, \langle \chi, R_2 \rangle, \langle \chi, R_1 \circ R_2 \rangle, S$. Hence $\text{Aut}(X)$ is of the form $(T \times K')/K$ where $K'$ is conjugate to one of the above groups. Thus it is enough to see number of orbits under action of $T \times K'$ on $F(E_1)$. In above proof the groups $G_1', G_2, G_3$ are nothing but $T \times K'$ for different $K'$. we did not get 4 or 5 orbits under action of these groups. Thus there does not exist 4 or 5-semiregular toroidal map of type $[3^6]$. This completes the prove of part (a) of Theorem 1.6.

Now let $X_2$ be a semi-equivelar toroidal map of type $[4^4]$. Then by Proposition 1.2 we can assume that $X_2 = E_2/K_2$ for some fixed point free subgroup $K_2$ of $\text{Aut}(X_2)$. Thus $K_2$ consist of only translations and glide reflections. Since $X_2$ is orientable so $K_2$ contains only translations. Consider $H_2, K_2$ and $G_2$ as in proof of Theorem 1.4. Now $F(X_2)$ has 4 $G_2$-orbits. Consider the group $G_3 = \langle \alpha_2, \beta_2, \chi_2, \rho_{OA} \rangle$. Clearly $F(E_2)$ has 2 $G_2$-orbits. Now proceeding in same way as in previous case we can prove that $Y_2 := E_2/L_2$ is a 2-semiregular cover of $X_2$, where $L_2 = \langle \gamma_2^{m_2}, \delta_2^{m_2} \rangle \leq G_2$ for some $m_2 \in \mathbb{Z}$. Now if the given map $X_2$ is 2-semiregular then consider the group $G_3 = \langle \alpha_2, \beta_2, \chi_2, \rho_{OA}, \rho_{OB_1} \rangle$. Then proceeding as
above we get \( Y_1 := E_2/L_3 \) is a 1-semiregular cover of \( X_2 \), where \( L_3 = \langle \gamma_2^{m_1}, \delta_2^{m_1} \rangle \leq G_3 \) for some \( m_1 \in \mathbb{Z} \). There are 5 subgroups of point stabilizer \( S \) up to conjugates. They are \( \langle \chi \rangle, \langle \chi, R_1 \rangle, \langle \chi, R_2 \rangle, \langle \chi, R_1 \circ R_2 \rangle, S \). Now by the same type of argument as in previous case we can see that there does not exists a 3-semiregular map of type \([4^4]\). This completes the proof of part (b) in Theorem 1.6.

Now we proof a series of lemmas and use them to proof other parts of Theorems 1.4 and 1.6.

For a given semi-equivelar map \( X = E/K \) consider \( \hat{X} \) be the associated equivelar map defined by \( \hat{X} = \hat{E}/K \), where \( \hat{E} \) be the associated equivelar tessellation obtained from \( E \) as shown in Figures 1 to 11.

**Lemma 3.2.** Let \( X_9 = E_9/G_9 \) is semiregular toroidal map of type \([3^1, 4^1, 6^1, 4^1]\). Then \( \hat{X}_9 \) is \( m_9 \)-semiregular if and only if \( X_9 \) is \( 4m_9 \)-semiregular.

**Proof.** Here by Theorem 1.4 and 1.6 we can conclude that \( m_4 \in \{1, 2, 3, 6\} \). The case \( m_4 = 1 \) discussed in [7]. Here we discuss \( m_4 = 2, 3 \) and 6. Let \( G_9 = \langle \alpha_9, \beta_9, \chi_9 \rangle \). Where \( \alpha_9 : z \mapsto z + A_9, \beta_9 : z \mapsto z + B_9 \) and \( \chi_9 \) be the 180 degree rotation about origin, see Figure 3. \( \hat{E}_9 \) is of type \([3^6]\). First suppose \( m_9 = 6 \). Let \( \hat{X}_9 \) is 6-semiregular. \( \text{Aut}(\hat{X}_9) = \text{Nor}_{\text{Aut}(\hat{E}_9)}(K_9)/\hat{K}_9 = \text{Nor}_{\text{Aut}(E_9)}(K_9)/K_9 = \text{Aut}(X_9) \). Now, \( G_9 \leq \text{Nor}_{\text{Aut}(\hat{E}_9)}(K_9) \). Action of \( G_9 \) on \( E(\hat{E}_9) \) also gives 6 flag orbits. Hence \( \hat{X}_9 \) to be 6-semiregular we must have \( \text{Nor}_{\text{Aut}(\hat{E}_9)}(K_9) = G_9 \) or some conjugate of \( G_9 \). Now under the action of \( G_9 \), \( F(\hat{E}_9) \) has 24 orbits. Symmetries of \( E_9 \) which fixes origin are also symmetries of \( \hat{E}_9 \). Hence \( F(\hat{E}_9) \) has 24 \( \text{Nor}_{\text{Aut}(\hat{E}_9)}(K_9) \)-orbits. Thus \( X_9 \) is 24-semiregular.

Now, let \( m_9 = 3 \). Then \( \text{Aut}(\hat{X}_9) \) is of the form \( (H_9 \rtimes K')/K_9 \) where \( K' \) is conjugate to \( \langle \chi_9, R_2 \rangle \) or \( \langle \chi_9, R_1 R_2 \rangle \). Since \( m_9 = 3 \) \( K' \) is conjugate to \( \langle \chi_9, R_2 \rangle \). One can see that under action of this group \( F(X_9) \) has 12 flag orbits. Thus \( X_9 \) is 12-semiregular.

Now, let \( m_9 = 2 \). Then \( \text{Aut}(\hat{X}_9) \) is of the form \( (H_9 \rtimes K')/K_9 \) where \( K' \) is conjugate to \( \langle \chi_9, R_2 \circ R_1 \rangle \). One can see that under action of this group \( F(\hat{E}_9) \) has 8 orbits. Thus \( X_9 \) is 8-semiregular.

Conversely, let \( X_9 \) is 24 orbital. Then \( G_9/K_9 \leq \text{Aut}(X_9) \). These symmetries are also present in \( \text{Aut}(\hat{E}_9) \) and the group \( G_9 \) gives 6 orbits on \( F(\hat{E}_9) \). Since, \( \text{Aut}(X_9) \) does not change \( G_9/K_9 \)-orbits of \( F(X_9) \) so \( \text{Aut}(\hat{X}_9) \) will also not change \( G_9/K_9 \)-orbits of \( F(\hat{X}_9) \). Thus \( \hat{X}_9 \) is 6-orbital. Now suppose \( X_9 \) is 12 orbital. Then its automorphism group will contain either \( R_1 \) or \( R_2 \) along with \( G_9 \). With these symmetries \( F(\hat{X}_9) \) will have 3 orbits. Hence \( \hat{X}_9 \) is 3-semiregular.

Now suppose \( X_9 \) is 8 orbital. Then its automorphism group will contain either \( R_1 \) and \( R_2 \) along with \( G_9 \). With these symmetries \( F(\hat{X}_9) \) will have 2 orbits. Hence \( \hat{X}_9 \) is 2-semiregular. This completes the proof of Lemma 3.2.

**Lemma 3.3.** Let \( X_8 = E_8/G_8 \) is semiregular toroidal map of type \([3^1, 12^2]\). Then \( \hat{X}_8 \) is \( m_8 \)-semiregular if and only if \( X_8 \) is \( 3m_8 \)-semiregular.

**Proof.** Here by Theorem 1.4 and 1.6 we can conclude that \( m_8 \in \{1, 2, 3, 6\} \). The case \( m_8 = 1 \) discussed in [7]. Here we discuss \( m_8 = 2, 3 \) and 6. Let \( G_8 = \langle \alpha_8, \beta_8, \chi_8 \rangle \). Where \( \alpha_8 : z \mapsto z + A_8, \beta_8 : z \mapsto z + B_8 \) and \( \chi_8 \) be the 180 degree rotation about origin, see Figure 4. \( \hat{E}_8 \) is of type \([3^6]\). First suppose \( m_8 = 6 \). Let \( \hat{X}_8 \) is 6-semiregular. Then by similar reason as in Lemma 3.2, we must have \( \text{Nor}_{\text{Aut}(\hat{E}_8)}(K_8) = G_8 \) or some conjugate of \( G_8 \). Now under the action of \( G_8 \), \( F(\hat{E}_8) \) has 18 orbits. Symmetries of \( E_8 \) which fixes origin are also symmetries
Now suppose \( \hat{X}_8 \) is 3-semiregular. This completes the proof of Lemma 3.3.

Now, let \( m_8 = 3 \). Then \( \text{Aut}(\hat{X}_8) \) is of the form \( (H_8 \rtimes K')/K_8 \) where \( K' \) is conjugate to \( \langle \chi_8, R_2 \rangle \) or \( \langle \chi_8, R_1 R_2 \rangle \). Since \( m_8 = 3 \) \( K' \) is conjugate to \( \langle \chi_8, R_2 \rangle \). One can see that under action of this group \( F(X_8) \) has 9 flag orbits. Thus \( X_8 \) is 9-semiregular map.

Now, let \( m_8 = 2 \). Then \( \text{Aut}(\hat{X}_8) \) is of the form \( (H_8 \rtimes K')/K_8 \) where \( K' \) is conjugate to \( \langle \chi_8, R_2 \circ R_1 \rangle \). One can see that under action of this group \( F(\hat{X}_8) \) has 6 orbits. Thus \( X_8 \) is 6-semiregular map.

Conversely, let \( X_8 \) is 18-semiregular. Then \( G_8/K_8 \leq \text{Aut}(X_8) \). These symmetries are also present in \( \text{Aut}(\hat{E}_8) \) and the group \( G_8 \) gives 6 orbits on \( F(\hat{E}_8) \). Since, \( \text{Aut}(X_8) \) does not change \( G_8/K_8 \)-orbits of \( F(X_8) \) so \( \text{Aut}(\hat{X}_8) \) will also not change \( G_8/K_8 \)-orbits of \( F(\hat{X}_8) \). Thus \( \hat{X}_8 \) is 6-semiregular. Now suppose \( X_8 \) is 9-semiregular. Then its automorphism group will contain either \( R_1 \) or \( R_2 \) along with \( G_8 \). With these symmetries \( F(\hat{X}_8) \) will have 3 orbits. Hence \( \hat{X}_8 \) is 3-semiregular.

Now suppose \( X_8 \) is 6-semiregular. Then its automorphism group will contain either \( R_1 \) and \( R_2 \) along with \( G_8 \). With these symmetries \( F(\hat{X}_8) \) will have 2 orbits. Hence \( \hat{X}_8 \) is 2-semiregular.

This completes the proof of Lemma 3.3

**Lemma 3.4.** Let \( X_{11} = E_{11}/G_{11} \) is semiregular toroidal map of type \([4^1, 6^1, 12^1]\). Then \( \hat{X}_{11} \) is \( m_{11} \)-semiregular if and only if \( X_{11} \) is \( 6m_{11} \)-semiregular.

**Proof.** Here by Theorem 3.3 and 3.6 we can conclude that \( m_{11} \in \{1, 2, 3, 6\} \). The case \( m_{11} = 1 \) discussed in 7. Here we discuss \( m_{11} = 2, 3 \) and 6. Let \( G_{11} = \langle \alpha_{11}, \beta_{11}, \chi_{11} \rangle \). Where \( \alpha_{11} : z \mapsto z + A_{11}, \beta_{11} : z \mapsto z + B_{11} \) and \( \chi_{11} \) be the 180 degree rotation about origin, see Figure 8. \( \hat{E}_{11} \) is of type \([3^6]\). First suppose \( m_{11} = 6 \). Let \( \hat{X}_{11} \) be 6-semiregular. Then by similar reason as in Lemma 3.2 we must have \( \text{Nor}_{\text{Aut}(\hat{E}_{11})}(K_{11}) = G_{11} \) or some conjugate of \( G_{11} \). Now under the action of \( G_{11} \), \( F(\hat{E}_{11}) \) has 36 orbits. Symmetries of \( E_{11} \) which fixes origin are also symmetries of \( \hat{E}_{11} \). Hence \( F(E_{11}) \) has 36 \( \text{Nor}_{\text{Aut}(E_{11})}(K_{11}) \)-orbits. Thus \( X_{11} \) is 36-semiregular.

Now, let \( m_{11} = 3 \). Then \( \text{Aut}(\hat{X}_{11}) \) is of the form \( (H_{11} \rtimes K')/K_{11} \) where \( K' \) is conjugate to \( \langle \chi_{11}, R_2 \rangle \) or \( \langle \chi_{11}, R_1 R_2 \rangle \). Since \( m_{11} = 3 \) \( K' \) is conjugate to \( \langle \chi_{11}, R_2 \rangle \). One can see that under action of this group \( F(X_{11}) \) has 18 orbit. Thus \( X_{11} \) is 18-semiregular map.

Now, let \( m_{11} = 2 \). Then \( \text{Aut}(\hat{X}_{11}) \) is of the form \( (H_{11} \rtimes K')/K_{11} \) where \( K' \) is conjugate to \( \langle \chi_{11}, R_2 \circ R_1 \rangle \). One can see that under action of this group \( F(E_{11}) \) has 12 orbits. Thus \( X_{11} \) is 12-semiregular map.

Conversely, let \( X_{11} \) is 36-semiregular. Then \( G_{11}/K_{11} \leq \text{Aut}(X_{11}) \). These symmetries are also present in \( \text{Aut}(\hat{E}_{11}) \) and the group \( G_{11} \) gives 6 orbits on \( F(\hat{E}_{11}) \). By similar reason as above \( \hat{X}_{11} \) is 6-semiregular. Now suppose \( X_{11} \) is 18-semiregular. Then its automorphism group will contain either \( R_1 \) or \( R_2 \) along with \( G_{11} \). With these symmetries \( F(\hat{X}_{11}) \) will have 3 orbits. Hence \( \hat{X}_{11} \) is 3-semiregular.

Now suppose \( X_{11} \) is 12-semiregular. Then its automorphism group will contain either \( R_1 \) and \( R_2 \) along with \( G_{11} \). With these symmetries \( F(\hat{X}_{11}) \) will have 2 orbits. Hence \( \hat{X}_{11} \) is 2-semiregular. This completes the proof of Lemma 3.3

**Lemma 3.5.** Let \( X_5 = E_5/G_5 \) is semiregular toroidal map of type \([3^2, 4^1, 3^1, 4^1]\). If \( \hat{X}_5 \) is 4-semiregular then \( X_5 \) is 20-semiregular.

**Proof.** Let \( G_5 = \langle \alpha_5, \beta_5, \chi_5 \rangle \). Where \( \alpha_5 : z \mapsto z + A_5, \beta_5 : z \mapsto z + B_5 \) and \( \chi_5 \) be the 180 degree rotation about origin, see Figure 5. \( \hat{E}_5 \) is of type \([4^4]\). Let \( \hat{X}_5 \) is 4-semiregular. Then by similar reason as in Lemma 3.2 we must have \( \text{Nor}_{\text{Aut}(\hat{E}_5)}(K_5) = G_5 \) or some conjugate
of $G_5$. Now under the action of $G_5$, $F(E_5)$ has 20 orbits. Symmetries of $E_5$ which fixes origin are also symmetries of $\hat{E}_5$. Hence $F(E_5)$ has 20 $\text{Nor}_{\text{Aut}(E_5)}(K_5)$-orbits. Thus $X_5$ is 20-semiregular.

**Lemma 3.6.** Let $X_6 = E_6/G_6$ is semiregular toroidal map of type $[4^1, 8^2]$. Then $\hat{X}_6$ is 4-semiregular then $X_6$ is 12-semiregular.

**Proof.** Let $G_6 = \langle \alpha_6, \beta_6, \chi_6 \rangle$. Where $\alpha_6 : z \mapsto z + A_6$, $\beta_6 : z \mapsto z + B_6$ and $\chi_6$ be the 180 degree rotation about origin, see Figure 6 $\hat{E}_6$ is of type $[4^1]$. Let $\hat{X}_6$ is 4-semiregular. Then by similar reason as in Lemma 3.2 we must have $\text{Nor}_{\text{Aut}(E_6)}(K_6) = G_6$ or some conjugate of $G_6$. Now under the action of $G_6$, $F(E_6)$ has 12 orbits. Symmetries of $E_6$ which fixes origin are also symmetries of $\hat{E}_6$. Hence $F(E_6)$ has 12 $\text{Nor}_{\text{Aut}(E_6)}(K_6)$-orbits. Thus $X_6$ is 12-semiregular.

**Lemma 3.7.** Let $X_7 = E_7/G_7$ is semiregular toroidal map of type $[3^1, 6^1, 3^1, 6^1]$. Then $\hat{X}_7$ is $m_7$-semiregular if and only if $X_7$ is $4m_7$-semiregular.

**Proof.** Here by Theorem 1.4 and 1.6 we can conclude that $m_7 \in \{1, 2, 3, 6\}$. The case $m_7 = 1$ discussed in [7]. Here we discuss $m_7 = 2, 3$ and 6. Let $G_7 = \langle \alpha_7, \beta_7, \chi_7 \rangle$. Where $\alpha_7 : z \mapsto z + A_7$, $\beta_7 : z \mapsto z + B_7$ and $\chi_7$ be the 180 degree rotation about origin, see Figure 6 $\hat{E}_7$ is of type $[3^6]$. First suppose $m_7 = 6$. Let $\hat{X}_7$ is 6-semiregular. Then by similar reason as in Lemma 3.2 we must have $\text{Nor}_{\text{Aut}(E_7)}(K_7) = G_7$ or some conjugate of $G_7$. Now under the action of $G_7$, $F(E_7)$ has 24 orbits. Symmetries of $E_7$ which fixes origin are also symmetries of $\hat{E}_7$. Hence $F(E_7)$ has 24 $\text{Nor}_{\text{Aut}(E_7)}(K_7)$-orbits. Thus $X_7$ is 24-semiregular.

Now, let $m_7 = 3$. Then $\text{Aut}(\hat{X}_7)$ is of the form $(H_7 \rtimes K')/K_7$ where $K'$ is conjugate to $\langle \chi_7, R_2 \rangle$ or $\langle \chi_7, R_1 R_2 \rangle$. Since $m_7 = 3$ $K'$ is conjugate to $\langle \chi_7, R_2 \rangle$. One can see that under action of this group $F(X_7)$ has 12 flag orbits. Thus $X_7$ is 12-semiregular map.

Now, let $m_7 = 2$. Then $\text{Aut}(\hat{X}_7)$ is of the form $(H_7 \rtimes K')/K_7$ where $K'$ is conjugate to $\langle \chi_7, R_2 \circ R_1 \rangle$. One can see that under action of this group $F(E_7)$ has 8 orbits. Thus $X_7$ is 8-semiregular map.

Conversely, let $X_7$ is 24-semiregular. Then $G_7/K_7 \leq \text{Aut}(X_7)$. These symmetries are also present in $\text{Aut}(\hat{E}_7)$ and the group $G_7$ gives 6 orbits on $F(\hat{E}_7)$. By similar reason as above $\hat{X}_7$ is 6-semiregular. Now suppose $X_7$ is 12-semiregular. Then its automorphism group will contain either $R_1$ or $R_2$ along with $G_7$. With these symmetries $F(\hat{X}_7)$ will have 3 orbits. Hence $\hat{X}_7$ is 3-semiregular.

Now suppose $X_7$ is 8-semiregular. Then its automorphism group will contain either $R_1$ and $R_2$ along with $G_7$. With these symmetries $F(\hat{X}_7)$ will have 2 orbits. Hence $\hat{X}_7$ is 2-semiregular. This completes the proof of Lemma 3.7.

**Lemma 3.8.** Let $X_{10} = E_{10}/G_{10}$ is semiregular toroidal map of type $[3^4, 6^1]$. Then $\hat{X}_{10}$ is 6-semiregular if and only if $X_9$ is 30-semiregular.

**Proof.** Let $G_{10} = \langle \alpha_{10}, \beta_{10}, \chi_{10} \rangle$. Where $\alpha_{10} : z \mapsto z + A_{10}$, $\beta_{10} : z \mapsto z + B_{10}$ and $\chi_{10}$ be the 180 degree rotation about origin, see Figure 7 $\hat{E}_{10}$ is of type $[3^6]$. Let $\hat{X}_{10}$ is 6-semiregular. Then by similar reason as in Lemma 3.2 we must have $\text{Nor}_{\text{Aut}(E_{10})}(K_{10}) = G_{10}$ or some conjugate of $G_{10}$. Now under the action of $G_{10}$, $F(E_{10})$ has 30 orbits. Symmetries of $E_{10}$ which fixes origin are also symmetries of $\hat{E}_{10}$. Hence $F(E_{10})$ has 30 $\text{Nor}_{\text{Aut}(E_{10})}(K_{10})$-orbits. Thus $X_{10}$ is 30-semiregular.

Conversely, let $X_{10}$ is 30-semiregular. Then $G_{10}/K_{10} \leq \text{Aut}(X_{10})$. These symmetries are
also present in $\text{Aut}(\hat{E}_{10})$ and the group $G_{10}$ gives 6 orbits on $F(\hat{E}_{10})$. By similar reason as above $\hat{X}_{10}$ is 6-semiregular.

**Proof of Theorem 1.4 continued.** Here we will show that the bounds in Theorem 1.4 are strict. Now we show that there exist a 6-semiregular toroidal map of type $[3^6]$. Let $X$ be a equivelar map of type $[3^6]$. Then $X = E_1/K$ for some discrete fixed point free subgroup $K$ of $\text{Aut}(E_1)$. $\text{Aut}(X) = \text{Nor}(K)/K$. Now $F(X)$ has 6 $G_1/K$-orbits. If we can show that there exists some $K \leq H_1$ such that $\text{Nor}(K) = G_1$ then we are done.

Consider $K = \langle \alpha_1^5, \beta_1^7 \rangle$. $\alpha_1^5$ and $\beta_1^7$ are translations by the vectors $5A_1$ and $3B_1$ respectively. $\text{Nor}(K) = \{ \gamma \in \text{Aut}(E_1) \mid \gamma \alpha_1^5 \gamma^{-1}, \gamma \beta_1^7 \gamma^{-1} \in K \} = \{ \gamma \in \text{Aut}(E_1) \mid \gamma(5A_1), \gamma(3B_1) \in Z5A_1 + Z3B_1 \}$. Clearly $G_1 \leq \text{Nor}(K)$. But 60 and 120 degree rotations and reflection about a line does not belongs to $\text{Nor}(K)$. Hence $\text{Nor}(K) = G_1$. The same process will work for equivelar maps of type $[4^4]$. For other semi-equivelar maps we use above lemmas to conclude the bounds are sharp. The argument will go as following. Let $X$ be a semi-eqivular map of type $[p_1^{r_1}, p_2^{r_2}, \ldots, p_k^{r_k}]$ such that $\hat{X}$ is of type $[3^6]$. Now by above discussion there exists a 6 semiregular toroidal map of type $[3^6]$. Now using above lemmas depending on type of $X$ it follows that the bounds of flag orbits are strict. Similarly we can do for maps whose corresponding equivellar map is of type $[4^4]$. This completes the proof of Theorem 1.4.

**Proof of Theorem 1.6 continued.** Let $X_9$ be a $m_9$-semiregular toroidal map of type $[3^1, 4^1, 6^1, 4^1]$. Let $\hat{X}_9$ be the associated equivelar map of type $[3^6]$. By Lemma 3.2 we get $\hat{X}_3$ has $m_9 := m_9/2$ many edge orbits. Now by Theorem 1.6 we have covering $\eta_{k_9} : \hat{Y}_{k_9} \to \hat{X}_9$ where $\hat{Y}_{k_9}$ is $k_9$-semiregular for each $(k_9, m_9) = (1, 2), (2, 6), (1, 3), (3, 6), (1, 6)$. Now, if we consider the map of type $[3^1, 4^1, 6^1, 4^1]$ corresponding to the equivelar map $\hat{Y}_{k_9}$, say $Y_{k_9}$, then by Lemma 3.2 it will be a $(4 \times k_9)$-edge orbital map. Clearly $Y_{k_9}$ is a cover of $X_9$. Hence for given $m_9$-semiregular map of type $[3^1, 4^1, 6^1, 4^1]$ there exists a $k_9$ orbital cover of it for each $(k_9, m_9) = (4, 8), (8, 24), (4, 12), (12, 24), (4, 24).

Proceeding in exactly similar way we can conclude the followings also. Given $m_7$-semiregular map of type $[3^1, 6^1, 3^1, 6^1]$ there exists a $k_7$-semiregular cover of it for each $(k_7, m_7) = (4, 8), (8, 24), (4, 12), (12, 24), (4, 24).

Given $m_8$-semiregular map of type $[3^1, 12^2]$ there exists a $k_8$-semiregular cover of it for each $(k_8, m_8) = (3, 6), (3, 9), (3, 18), (6, 18), (9, 18).

Given $m_{11}$-semiregular map of type $[4^1, 6^1, 12^1]$ there exists a $k_{11}$-semiregular cover of it for each $(k_{11}, m_{11}) = (6, 12), (6, 18), (6, 36), (12, 36), (18, 36).

Given $m_6$-semiregular map of type $[4^1, 8^2]$ there exists a $k_6$-semiregular cover of it for each $(k_6, m_6) = (6, 12), (3, 6), (3, 12).

This completes the proof of parts (c),(d),(e),(f),(g) of Theorem 1.6.

Let $X_4$ be a 10-semiregular map. We can take $X_4 = E_4/K_4$ for some $K_4 \leq H_4 \leq \text{Aut}(E_4)$. Let $G_4$ be as in the proof of Theorem 1.3 Then $F(E_4)$ has 10 flag orbits. Consider $G_4' = \langle G_4, R_1 \rangle$ where $R_1$ is the map obtained by taking reflection of $E_4$ about the line passing through $O$ and $A$ (see Figure 11). Observe that $F(E_4)$ has 5 $G_4'$-orbits. Now proceeding in similar way as in part (a) of this theorem we get existence of a 5-semiregular cover of $X_4$. This proves part (h) of Theorem 1.6.

Let $X_5$ be a 20-semiregular map of type $[3^2, 4^1, 3^1, 4^1]$. We can take $X_5 = E_5/K_5$ for some $K_5 \leq H_5 \leq \text{Aut}(E_5)$. Let $G_5$ be as in the proof of Theorem 1.3 Then $F(E_5)$ has 20 flag orbits. Consider $G_5'' = \langle \alpha_5, \beta_5, \chi_5, R_1 \rangle$ and $G_5'' = \langle \alpha_5, \beta_5, \chi_5, R_1, R_2 \rangle$ where $R_1$ and $R_2$ is the map obtained by taking reflection of $E_5$ about the line passing through $O, A$ and $A_5, B_5$ respectively (see Figure 5). Observe that $F(E_5)$ has 10 $G_4'$-orbits and 5 $G_5'$-orbits. Now proceeding in similar way as in part (a) of this theorem we get existence of a 10 and 5-semiregular
cover of \( X_5 \). This proves part (i) of Theorem 1.6

Let \( X_{10} \) be a 30-semiregular map. We can take \( X_{10} = E_{10}/K_{10} \) for some \( K_{10} \leq H_{10} \leq \text{Aut}(E_{10}) \). Let \( G_{10} \) be as in the proof of Theorem 1.4. Then \( F(E_{10}) \) has 30 flag orbits. Consider \( G'_{10} = \langle \alpha_{10}, \beta_{10}, \chi_{10}, \rho_{10} \rangle \) where \( R_1 \) is the map obtained by taking rotation of \( E_{10} \) about origin (see Figure 7). Observe that \( F(E_{10}) \) has 10 \( G'_{10} \)-orbits. Now proceeding in similar way as in part (a) of this theorem we get existence of a 10-semiregular cover of \( X_{10} \). This proves part (j) of Theorem 1.6.

Now we are moving to see number of sheets of the covers obtained above. For that we make,

**Claim 3.6.** The cover \( Y \) in Theorem 1.6 is a \( m^2 \) sheeted covering of \( X \).

To do this we need following two results from the theory of covering spaces.

**Result 3.2.** (12) Let \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) be a path-connected covering space of the path-connected, locally path-connected space \( X \), and let \( H \) be the subgroup \( p_*\langle \pi_1(\tilde{X}, \tilde{x}_0) \rangle \subset \pi_1(X, x_0) \). Then,

1. This covering space is normal if and only if \( H \) is a normal subgroup of \( \pi_1(X, x_0) \)

2. \( G(\tilde{X}) \) (the group of deck transformation of the covering \( \tilde{X} \to X \)) is isomorphic to \( N(H)/H \) where \( N(H) \) is the normalizer of \( H \) in \( \pi_1(X, x_0) \).

In particular, \( G(\tilde{X}) \) is isomorphic to \( \pi_1(X, x_0)/H \) if \( \tilde{X} \) is a normal covering. Hence for universal cover \( \tilde{X} \to X \) we have \( G(\tilde{X}) \simeq \pi_1(X) \).

**Result 3.3.** (12) The number of sheets of a covering space \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) with \( X \) and \( \tilde{X} \) path-connected equals the index of \( p_*\langle \pi_1(\tilde{X}, \tilde{x}_0) \rangle \) in \( \pi_1(X, x_0) \).

In our situation applying Result 3.2 for the covering \( E_i \to E_i/K_i \) we get \( \pi_1(E_i/K_i) = K_i \). For the covering \( E_i \to E_i/L_i \) we get \( \pi_1(E_i/L_i) = L_i \). Thus applying Result 3.3 we get number of sheets of \( Y \) over \( X \) is \( n = [K_i : L_i] = m^2 \) for all \( i = 3, 4, 5, 6, 7 \). This proves our Claim 3.6.

**Proof of Theorem 1.7** Let \( X \) be an semiregular map of type \((m, \ell, u, v)\). Then form Prop. 1.12 we get \( X = M_i/K \) for some discrete subgroup \( K \) of \( \text{Aut}(M_i) \). Now \( Y \) covers \( X \) if and only if \( Y = M_i/L \) for some subgroup \( L \) of \( K \) generated by 2 translations corresponding to 2 independent vectors. Let \( K = \langle \gamma, \delta \rangle \). Now consider \( L_n = \langle \gamma^n, \delta \rangle \) and \( Y_n = M_i/L_n \). then \( Y_n \) covers \( X \). Number of sheets of the cover \( Y_n \to X \) is equal to \( [K : L_n] = n \). Hence \( Y_n \) is our required \( n \) sheeted cover of \( X \).

**Proof of Theorem 1.8** Here two maps are isomorphic if they are isomorphic as maps. Two maps are equal if the orbits of \( \mathbb{R}^2 \) under the action of corresponding groups are equal as sets. Suppose \( X \) and \( K \) be as in the proof of Theorem 1.7. Let \( n \in \mathbb{N} \). Let \( Y = E/L \) be \( n \) sheeted cover of \( X \). Let \( L = \langle \omega_1, \omega_2 \rangle \). Where \( \omega_1, \omega_2 \in K = \langle \gamma, \delta \rangle \). Suppose \( \omega_1 = \gamma^a \circ \delta^b \) and \( \omega_2 = \gamma^c \circ \delta^d \) where \( a, b, c, d \in \mathbb{Z} \). Define \( M_Y = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \). We represent \( Y \) by the associated matrix \( M_Y \). This matrix representation corresponding to a map is unique as \( \gamma \) and \( \delta \) are translations along two linearly independent vectors. Denote area of the torus \( Y \) by \( \Delta_Y \). As \( Y \) is \( n \) sheeted covering of \( X \) so \( \Delta_Y = n \Delta_X \implies \) area of the parallelogram spanned by \( w_1 \)
and \(w_2 = n \times \text{area of the parallelogram spanned by } \gamma \text{ and } \delta\). That means \(|\det(M_Y)| = n\). Therefore for each \(n\) sheeted covering, the associated matrix belongs to

\[
S := \{M \in GL(2, \mathbb{Z}) : |\det(M)| = n\}.
\]

Conversely for every element of \(S\) we get a \(n\) sheeted covering \(Y\) of \(X\) by associating \[
\begin{bmatrix}
a & c \\
b & d
\end{bmatrix}
\]
to \(E/(a\gamma + b\delta, c\gamma + d\delta)\). So there is an one to one correspondence to \(n\)-sheeted covers of \(X\) and \(S\). To proceed further we need following two lemmas.

**Lemma 3.9.** Let \(Y_1\) and \(Y_2\) be maps and \(M_1\) and \(M_2\) be associated matrix of them respectively. Then \(Y_1 = Y_2\) if and only if there exists an unimodular matrix \((\text{an integer matrix with determinant } 1 \text{ or } -1)\) \(U\) such that \(M_1U = M_2\).

**Proof.** Let \(Y_1 = Y_2\). Let \(i : Y_1 \to Y_2\) be an isomorphism. We can extend \(i\) to \(\tilde{i} \in \text{Aut}(E)\). Then \(\tilde{i}\) will take fundamental parallelogram of \(Y_1\) to that of \(Y_2\). Hence the lattices formed by \(L_1\) and \(L_2\) are same say \(\Lambda\). \(\tilde{i}\) transforms \(\Lambda\) to itself. Therefore from \((\text{Theorem 32, Chapter 3})\) we get matrix of the transformation is unimodular. Our lemma follows from this.

Conversely suppose \(M_1U = M_2\) where \(U\) is an unimodular matrix. Let \(M_1 = (w_1 \ w_2)\), \(M_2 = (w'_1 \ w'_2)\) and \(U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) where \(w_i, w'_i\) are column vectors for \(i = 1, 2\). Therefore

\[
M_2 = M_1U \implies (w'_1 \ w'_2) = (w_1 \ w_2) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (aw_1 + cw_2 \ bw_1 + dw_2).
\]

Now suppose \(L_1 = \langle \alpha_1, \beta_1 \rangle\) and \(L_2 = \langle \alpha_2, \beta_2 \rangle\) and \(A_i, B_i\) be the vectors by which \(\alpha_i\) and \(\beta_i\) translating the plane for \(i = 1, 2\) and let \(C\) and \(D\) be the vectors corresponding to \(\gamma\) and \(\delta\). Let

\[
A_1 = p_1C + q_1D, B_1 = s_1C + t_1D, A_2 = p_2C + q_2D, B_2 = s_2C + t_2D.
\]

Now \(w'_1 = \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = a \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} + c \begin{bmatrix} s_1 \\ t_1 \end{bmatrix} = \begin{bmatrix} ap_1 + cs_1 \\ aq_1 + ct_1 \end{bmatrix}\). Therefore

\[
A_2 = (ap_1 + cs_1)C + (aq_1 + ct_1)D = a(p_1C + q_1D) + c(s_1C + t_1D) = aA_1 + cB_1
\]

Hence \(\alpha_2 \in L_1\). Similarly \(\beta_2 \in L_1\). Therefore \(L_2 \leq L_1\). Proceeding in the similar way and using the fact that \(\det(U) = \pm 1\) we get \(L_1 \leq L_2\). Therefore \(L_1 = L_2\). Thus \(Y_1 = E/L_1 = E/L_2 = Y_2\). This completes the proof of Lemma 3.9.

**Lemma 3.10.** Let \(Y_1\) and \(Y_2\) be two toroidal maps with associated matrix \(M_1\) and \(M_2\) respectively. Then \(Y_1 \simeq Y_2\) if and only if there exists \(A \in G_0\) and \(B \in GL(2, \mathbb{Z})\) such that \(M_1 = AM_2B\) where \(G_0\) is group of rotations and reflections fixing the origin in \(E\).

**Proof.** Let \(Y_1 \simeq Y_2\) and \(\alpha : Y_1 \to Y_2\) be an isomorphism. Now \(\alpha\) can be extended to an automorphism of the covering plane \(E\), call that extension be \(\tilde{\alpha}\). Clearly \(\tilde{\alpha}\) will take fundamental parallelogram of \(Y_1\) to that of \(Y_2\). Now the only ways to transform one fundamental region to another are rotation, reflection and change of basis of \(E\). Multiplication by an element of \(GL(2, \mathbb{Z})\) will take care of base change. Rotation, reflection or their composition will take care by multiplication by \(A \in G_0\). Hence we get \(M_1 = AM_2B\).

Conversely let \(M_1 \neq AM_2B\). \(A \in G_0\) so the combinatorial type of the torus associated
to the matrix $AM_2$ and $M_2$ are same. Geometrically multiplying by elements of $GL(2,\mathbb{Z})$ corresponds to modifying the fundamental domain by changing the basis. Hence this will not change the combinatorial type of the torus. Thus $Y_1 \simeq Y_2$. This completes the proof of Lemma 3.10.

Now define a relation on $S$ by $P \sim Q \iff P = QU$ for some unimodular matrix $U$. Clearly this is an equivalence relation. Consider $S' = S/\sim$. So by Lemma 3.9 we can conclude that there are $\#S'$ many distinct $n$ sheeted cover of $X$ exists. Let’s find this cardinality. Now for every $m \times n$ matrix $P$ with integer entries has an unique $m \times n$ matrix $H$, called hermite normal form of $P$, such that $H = PU$ for some unimodular matrix $U$. All elements of an equivalence class of $S'$ has same hermite normal form and we take this matrix in hermite normal form as representative of that equivalence class. Thus to find cardinality of $S'$ it is enough to find number of distinct matrices $M$ which are in hermite normal form and has determinant $n$. We do not take the matrices with determinant $-n$ because by multiplying by the unimodular matrix \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\] changes sign of the determinant. As $M$ is in lower triangular form so take $M = \begin{bmatrix}a & 0 \\ b & d \end{bmatrix}$. Then $det(M) = ad = n \implies a = n/d$. By definition of hermite normal form $b \geq 0$ and $b < d$ so $b$ has $d$ choices for each $d/n$. Hence there are precisely $\sigma(n) := \sum_{d|n} d$ many distinct $M$ possible. Thus $\#S' = \sigma(n)$. Let $S_1 = \{M| M$ is a representative of an equivalence class of $S'$ which is in hermite normal form\} Clearly $\#S_1 = \sigma(n)$. Now define a relation on $S_1$ by $M_1 \sim M_2 \iff \exists A \in G_0$ such that $M_1 = AM_2$. Clearly this is an equivalence relation. Consider $S_2 := S_1/\sim$. By Lemma 3.10 it follows that there are $\#S_2$ many $n$ sheeted covers upto isomorphism. Because here all matrices $M_i$ has same determinant so $M_1 = AM_2B \implies det(A)det(B) = 1$. As $det(A)$ and $det(B)$ both are integer so they belongs to $\{1, -1\}$ i.e. they are unimodular matrices. Now we have to find $\#S_2$. Observe that the matrix representation of elements of $G_0$ with respect to the basis $\{z, \beta(0)\}$ have integer entries because lattice points must go to lattice points by a symmetry of the plane where $\alpha : z \mapsto z + A_i$ and $\beta : z \mapsto z + B_i$ are two translations of $E_i$. Suppose $M_1$ and $M_2 \in S_2$ such that $M_1 \sim M_2$. So there exists $A \in G_0$ such that $M_1 = AM_2$.

Let $M_1 = \begin{bmatrix}a & 0 \\ c_1 & d_1 \end{bmatrix}, M_2 = \begin{bmatrix}n & 0 \\ c_2 & d_2 \end{bmatrix}$ and $A = \begin{bmatrix}p & q \\ r & s \end{bmatrix}$. Then

\[
M_1 = AM_2 \implies \begin{bmatrix}a & 0 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix}p & q \\ r & s \end{bmatrix} \begin{bmatrix}n & 0 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix}np/d_2 + qc & qd \\ nc_2 + sd & sd_2 \end{bmatrix}
\]

(1)

Therefore $A = \begin{bmatrix}p & 0 \\ r & s \end{bmatrix}$. $det(A) = 1 \implies ps = 1 \implies s = 1/p$. Again from equation (1) we get

\[
np/d_2 = n/d_1 \implies p = d_2/d_1
\]

and

\[
rn/d_2 + c_2/p = c_1 \implies r = (d_2c_1 - d_1c_2)/n
\]

Therefore

\[
A = \begin{bmatrix}d_2/d_1 & 0 \\ (d_2c_1 - d_1c_2)/n & d_1/d_2 \end{bmatrix}
\]
As $A$ has integer entries and $d_1, d_2$ are positive so $d_1 = d_2 = d$ (say) and $n|d(c_1 - c_2)$. Hence

$$A = \begin{bmatrix} 0 & 1 \\ d(c_1 - c_2)/n & 0 \\ 1 & 1 \end{bmatrix}.$$ 

Now $A \in G_0$ and $G_0 = D_6$ for maps of type $[3^6], [6^3], [3^4, 6^1, 3^1, 6^1], [3^1, 4^1, 6^1, 4^1], [3^1, 12^2], [4^1, 6^1, 12^1]$; $G_0 = D_4$ for maps of type $[4^4], [3^2, 4^1, 3^1, 4^1], [4^1, 8^2]$; $G_0 = \mathbb{Z}_6$ for $[3^1, 6^1]$; $G_0 = \mathbb{Z}_2^2$ for maps of type $[3^3, 4^2]$. Here $D_6$ is generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. $D_4$ is generated by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. One can check that only matrices in $G_0$ having diagonal entries 1 is identity matrix. Hence $A = I_2$. Therefore $c_1 = c_2 \implies M_1 = M_2$. Each equivalence class of $S_2$ is singleton. Therefore $\#S_2 = \#S_1 = \sigma(n)$. This proves Theorem 1.8.

**Proof of Theorem 1.8.** Let $X$ be a $m$-orbital map of vertex type $(m, \ell; u, v)$. Let $Y_1$ be a $k$-orbital cover of $X$. Consider number of sheets of the cover $Y_1 \longrightarrow X$ be $n_1$. Let the set $C_1$ containing all $n$ sheeted covering of $X$ for $n \leq n_1 - 1$. Now check that does there exists a $k$-orbital cover or not in $C_1$. If there does not exists one, then $Y_1$ be a minimal $k$-orbital cover otherwise take $Y_2$ be a $k$-orbital cover in $C_1$. Let number of sheets for the covering $Y_2 \longrightarrow X$ be $n_2$. Then consider $C_2$ be the collection of all $s$ sheeted cover of $X$ for $s \leq n_2$. Again check if there exists a $k$-orbital cover in $C_2$. If not then $Y_2$ minimal $k$-orbital cover of $X$. Otherwise proceed similarly to more lower sheeted covering. Since there are only finitely many covers of each sheeted so the process will terminate. This proves Theorem 1.9.

Now to answer the last part of Question 1.8 we prove the following,

**Claim 3.7.** Let $X$ be a $m$-orbital map. Then there exists a group $\tilde{G} \leq \text{Aut}(E)$ such that $E(E)$ has $m \tilde{G}$-orbits.

**Proof.** Let $X$ be a semi-equivelar toroidal map of type $[p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k}]$. By proposition 1.12 we get $X = E_j/K$ for some discrete subgroup $K$ of $\text{Aut}(E_j)$ where $E_j$ is semi-equivelar tiling of $\mathbb{R}^2$. Let $O_1, O_2, \ldots, O_m$ be $G$-orbits of $E(X)$. Let $\eta : E_j \rightarrow X$ be the covering map. Then $\{\eta^{-1}(O_i)|i = 1, 2, \ldots, m\}$ be a partition of $E(E_j)$. $\text{Aut}(X)\cong \text{Nor}(K)/K$. Now consider $\tilde{G} = \text{Nor}(K)$. Then $E(E_j)$ forms $m \tilde{G}$-orbits. This proves Claim 3.7.

**Lemma 3.11.** Let $X$ be a $m$-orbital semiregular toroidal map and $Y$ be a $k$-orbital cover of $X$. Then $k \leq m$.

**Proof.** Let $k \geq m + 1$. Let $O_1, O_2, \ldots, O_{m+1}$ be distinct $\text{Aut}(Y)$-orbits of $E(Y)$. Let $\eta$ be the covering map. Suppose $a_i \in O_i$ for $i = 1, 2, \ldots, m + 1$. Then $\eta(a_i) \in E(X) \forall i$. Since $E(X)$ has $m$ orbits so by pigeon hole principle there exists $i, j \in \{1, 2, 3, \ldots, m + 1\}$ such that $\eta(a_i), \eta(a_j)$ are in same $\text{Aut}(X)$ orbits of $E(X)$. Therefore there exists $\Upsilon \in \text{Aut}(X)$ such that $\Upsilon(\eta(a_i)) = \eta(a_j)$. Let $\tilde{\Upsilon} \in \text{Aut}(Y)$ be the preimage of $\Upsilon$ under the projection $\pi : \text{Aut}(Y) \rightarrow \text{Aut}(X)$. If $a_i$ and $a_j$ belongs to same sheet of the covering $Y \rightarrow X$ then $\tilde{\Upsilon}(a_i) = a_j$. If $a_i$ and $a_j$ belongs to two different sheet then apply a suitable translation on $a_j$ and get an element $a'_j \in O(a_j)$ such that $a_i$ and $a'_j$ belongs to same sheet. Therefore in both cases $\exists \tilde{\Upsilon} \in \text{Aut}(Y)$ such that $\tilde{\Upsilon}(a_i) = a_j$. This is a contradiction to $a_i$ and $a_j$ are in different orbits. This proves Lemma 3.11.
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