ACTIVE NOISE CONTROL WITH SAMPLED-DATA FILTERED-x
ADAPTIVE ALGORITHM

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ABSTRACT. Analysis and design of filtered-x adaptive algorithms are conventionally done by assuming that the transfer function in the secondary path is a discrete-time system. However, in real systems such as active noise control, the secondary path is a continuous-time system. Therefore, such a system should be analyzed and designed as a hybrid system including discrete- and continuous-time systems and AD/DA devices. In this article, we propose a hybrid design taking account of continuous-time behavior of the secondary path via lifting (continuous-time polyphase decomposition) technique in sampled-data control theory.

1. INTRODUCTION

Recent development of digital technology enables us to make digital signal processing (DSP) systems much more robust, flexible, and cheaper than analog systems. Owing to the recent digital technology, advanced adaptive algorithms with fast DSP devices are used in active noise control (ANC) systems [2, 8], air conditioning ducts [5], noise canceling headphones [6], and automotive applications [12], to name a few.

Fig. 1 shows a standard active noise control system. In this system, \( x(t) \) represents continuous-time noise which we want to eliminate during it goes through the duct. Precisely, we aim at diminishing the noise at the point \( C \). For this purpose, we set a loudspeaker near the point \( C \) which emits anti-phase sound signals to cancel the noise. Since the noise is unknown in many cases, it is almost impossible to determine anti-phase signals a priori. Hence, we set a microphone at the point \( A \) to measure the continuous-time noise, and adopt a digital filter \( K(z) \) with AD (analog-to-digital) and DA (digital-to-analog) devices. Namely, the continuous-time signal \( x(t) \) is discretized to produce a discrete-time signal \( x_d \), which is processed by the digital filter \( K(z) \) to produce another discrete-time signal \( y_d \). Then a DA converter and a loudspeaker at the point \( B \) are used to emit anti-phase signals to cancel the noise in the duct.

In active noise control, it is important to compensate the distortion by the transfer characteristic of the secondary path (from \( B \) to \( C \)). To compensate this, a standard adaptive algorithm uses a filtered signal of the noise \( x \), and is called filtered-x algorithm [9]. This filter is usually chosen by a discrete-time model of the secondary path [9, 2]. Consequently, the adaptive filter \( K(z) \) optimizes the norm (or the variance in the stochastic setup) of the discretized signal \( e(nh) \), \( n = 0, 1, 2, \ldots \) where \( h \) is the sampling period of AD and DA device. This is proper if the secondary path is also a discrete-time system. However, in reality, the path is a continuous-time system, and hence the optimization should be executed taking account of the behavior of the continuous-time error signal \( e(t) \). Such an optimization may seem to be difficult because the system is a hybrid system containing both continuous- and discrete-time signals.

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Recently, several articles have been devoted to the design considering a continuous-time behavior. In [13], a hybrid controller containing an analog filter and a digital adaptive filter has been proposed. Owing to the analog filter, a robust performance is attained against the variance of the secondary path. However, an analog filter is often unwelcome because of its poor reliability or maintenance cost. Another approach has been proposed in [8]. In this paper, they assume that the noise $x(t)$ is a linear combination of a finite number of sinusoidal waves. Then the adaptive algorithm is executed in the frequency domain based on the frequency response of the continuous-time secondary path. This method is very effective if we a priori know the frequencies of the noise. However, unknown signal with other frequencies cannot be eliminated. If we prepare adaptive filters considering many frequencies to avoid such a situation, the complexity of the controller will be very high.

The same situation has been considered in control systems theory. The modern sampled-data control theory [1] has been developed in 90's [15], which gives an exact design/analysis method for hybrid systems containing continuous-time plants and discrete-time controllers. The key idea is lifting. Lifting is a transformation of continuous-time signals to an infinite-dimensional (i.e., function-valued) discrete-time signals. The operation can be interpreted as a continuous-time polyphase decomposition. In multirate signal processing, the (discrete-time) polyphase decomposition enables the designer to perform all computations at the lowest rate [14]. In the same way, by lifting, continuous-time signals or systems can be represented in the discrete-time domain with no errors.

The lifting approach is recently applied to digital signal processing [4, 10, 16], and proved to provide an effective method for digital filter design. Motivated these works, this article focuses on a new scheme of filtered-x adaptive algorithm which takes account of the continuous-time behavior. More precisely, we define the problem of active noise control as design of the digital filter which minimizes a continuous-time cost function. By using the lifting technique, we derive the Wiener solution for this problem, and a steepest descent algorithm based on the Wiener solution. Then we propose an LMS (least mean square) type algorithm to obtain a causal system. The LMS algorithm involves an integral computation on a finite interval, and we adopt an approximation based on lifting representation. The approximated algorithm can be easily executed by a (linear, time-invariant, and finite dimensional) digital filter.

The paper is organized as follows: Section 2 formulates the problem of active noise control. Section 3 gives the Wiener solution, the steepest descent algorithm, and the LMS-type algorithm with convergence theorems. Section 4 proposes an approximation method for computing an integral of signals for the LMS-type algorithm. Section 5 shows simulation results to illustrate the effectiveness of the proposed method. Section 6 concludes the paper.
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\[ P(s) \]

\[ \mathcal{J}_h \]

\[ x_d \]

\[ K(z) \]

\[ y_d \]

\[ H_h \]

\[ y \]

\[ F(s) \]

\[ e \]

\[ d \]

**Figure 2.** Block diagram of active noise control system

### Notation.
- \( \mathbb{R}, \mathbb{R}_+ \): the sets of real numbers and non-negative real numbers, respectively.
- \( \mathbb{Z}, \mathbb{Z}_+ \): the sets of integers and non-negative integers, respectively.
- \( \mathbb{R}^n, \mathbb{R}^{n \times m} \): the sets of \( n \)-dimensional vectors and \( n \times m \) matrices over \( \mathbb{R} \), respectively.
- \( L^2, L^2[0, h) \): the sets of all square integrable functions on \( \mathbb{R}_+ \) and \([0, h)\), respectively.
- \( M^\top \): transpose of a matrix \( M \).
- \( \overline{a} \): the complex conjugate of a complex number \( a \)
- \( s \): the symbol for Laplace transform
- \( z \): the symbol for Z transform

### 2. Problem Formulation

In this section, we formulate the design problem of active noise control. Let us consider the block diagram shown in Fig. 2 which is a model of the active noise control system shown in Fig. 1. In this diagram, \( P(s) \) is the transfer function of the primary path from \( A \) to \( C \) in Fig. 1. The transfer function of the secondary path from \( B \) to \( C \) is represented by \( F(s) \). Note that \( P(s) \) and \( F(s) \) are continuous-time systems. We model the AD device by the ideal sampler \( \mathcal{J}_h \) with a sampling period \( h \) defined by

\[ (\mathcal{J}_h x)[n] := x(nh), \quad n \in \mathbb{Z}_+. \]

That is, the ideal sampler \( \mathcal{J}_h \) converts continuous-time signals to discrete-time signals. Then, the DA device is modeled by the zero-order hold \( H_h \) with the same period \( h \) defined by

\[ (H_h y)(t) := \sum_{n=0}^{\infty} \phi_0(t - nh)y[n], \quad t \in [0, \infty), \]

where \( \phi_0(t) \) is the zero-order hold function or the box function defined by

\[ \phi_0(t) := \begin{cases} 
1, & t \in [0, h), \\
0, & \text{otherwise.} 
\end{cases} \]

That is, the zero-order hold \( H_h \) converts discrete-time signals to continuous-time signals.

With the setup, we formulate the design problem as follows:

**Problem 1.** Find the optimal FIR (finite impulse response) filter

\[ K(z) = \sum_{k=0}^{N-1} a_k z^{-k} \]
which minimizes the continuous-time cost function
\[ J = \int_0^\infty e(t)^2 \, dt. \]  

Instead of the conventional adaptive filter design, this problem deals with the continuous-time behavior of the error signal \( e(t) \). To solve such a hybrid problem (i.e., a problem for a mixed continuous- and discrete-time system), we introduce the lifting approach based on the sampled-data control theory.

In what follows, we assume the following:

**Assumption 2.** The following properties hold:

1. The noise \( x \) is unknown but causal, that is, \( x(t) = 0 \) if \( t < 0 \), and belongs to \( L^2 \).
2. The primary path \( P(s) \) is unknown, but proper and stable.
3. The secondary path \( F(s) \) is known, proper and stable.

### 3. Sampled-Data Filtered-\( x \) Algorithm

In this section, we discretize the continuous-time cost function without any approximation, and derive optimal filters. We also give convergence theorems for the proposed adaptive filters. The key idea to derive the results in this section is the lifting technique.

#### 3.1. Wiener Solution

In this subsection, we derive the optimal filter coefficients \( a_0, a_1, \ldots, a_{n-1} \) which minimize the cost function \( J \) in (1).

First, we split the time domain \([0, \infty)\) into the union of sampling intervals \([nh,(n+1)h)\), \( n \in \mathbb{Z}_+ \), as

\[ [0, \infty) = [0, h) \cup [h, 2h) \cup [2h, 3h) \cup \cdots. \]

By this, the cost function (1) is transformed into the sum of the \( L^2[0,h) \)-norm of \( e(t) \) on the intervals:

\[ J = \int_0^\infty e(t)^2 \, dt = \sum_{n=0}^{\infty} \int_0^h e(nh + \theta)^2 \, d\theta = \sum_{n=0}^{\infty} \int_0^h e_n(\theta)^2 \, d\theta; \]

where \( e_n(\theta) = e(nh + \theta), \theta \in [0,h), n \in \mathbb{Z}_+ \). The sequence \( \{ e_n \} \) of functions \( e_1, e_2, \ldots \) on \([0,h)\) is called the lifted signal of the continuous-time signal \( e \in L^2 \), and we denote the lifting operator by \( \mathcal{L} \), that is, \( \{ e_n \} = \mathcal{L} e \). In what follows, we use the notion of lifting to derive the optimal coefficients.

Next, we assume that a state space realization is given for \( F(s) \) as

\[ F: \begin{cases} \dot{z}(t) = Az(t) + By(t), \\ w(t) = Cz(t), \quad t \in \mathbb{R}_+ \end{cases} \]

where \( z(0) = 0, A \in \mathbb{R}^{v \times v}, B \in \mathbb{R}^{v \times 1}, \) and \( C \in \mathbb{R}^{1 \times v} \). By Fig. 2, the continuous-time signal \( w \) is given by

\[ w = Fy = \mathcal{F}_h y_d \]

where \( y_d \) is a discrete-time signal \( \{ y_d[n] \} \) which is produced by the filter \( K(z) \). Let \( w_n(\theta) := w(nh + \theta), \theta \in [0,h), n \in \mathbb{Z}_+ \) (i.e., \( \{ w_n \} := \mathcal{L} w \)). Then, the sequence of functions \( \{ w_n \} \) is obtained as

\[ \{ w_n \} = \mathcal{L} F \mathcal{H}_h y_d. \]

Let \( \mathcal{F}_h := \mathcal{L} F \mathcal{H}_h \). Then the system \( \mathcal{F}_h \) is a discrete-time system as shown in the following lemma [1] Sec. 10.2:
Lemma 3. \( \mathcal{F}_h \) is a linear time-invariant discrete-time (infinite-dimensional) system with the following state-space representation:

\[
\mathcal{F}_h : \begin{cases}
\xi[n+1] = A_h \xi[n] + B_h y_d[n], \\
w_n = C_h \xi[n] + D_h y_d[n],
\end{cases} \quad n \in \mathbb{Z}_+,
\]

(3)

where

\[
A_h := e^{Ah} \in \mathbb{R}^{v \times v}, \quad B_h := \int_0^h e^{A\theta} B d\theta \in \mathbb{R}^{v \times 1},
\]

(4)

The LTI property of \( \mathcal{F}_h \) in Lemma 3 gives

\[
\{w_n\} = \mathcal{F}_h \{y_d[n]\}
\]

(5)

where \( \{u_n\} := \mathcal{F}_h \{x_d[n]\} \). Note that \( \{u_n\} \) is the lifted signal of the continuous-time signal \( u = F \mathcal{H}_h x_d \), that is,

\[
\{u_n\} = \mathcal{L}(F \mathcal{H}_h x_d) = \mathcal{L}u.
\]

The relation (5) gives the continuous-time relation as

\[
w(t) = \sum_{k=0}^{N-1} \alpha_k u(t - kh), \quad t \in \mathbb{R}_+.
\]

By using this relation, we obtain the following theorem for the optimal filter.

**Theorem 4** (Wiener solution). Let \( u := (F \mathcal{H}_h)x_d \). Define a matrix \( \Phi \) and a vector \( \beta \) as

\[
\Phi := [\Phi_{k,l}]_{k,l=0,1,\ldots,N-1} \in \mathbb{R}^{N \times N}, \quad \beta := [\beta_k]_{k=0,1,\ldots,N-1} \in \mathbb{R}^N,
\]

where for \( k,l = 0,1,\ldots,N-1 \),

\[
\Phi_{k,l} := \int_0^\infty u(t - kh)u(t - lh)dt, \quad \beta_k := \int_0^\infty d(t)u(t - kh)dt.
\]

Assume the matrix \( \Phi \) is nonsingular. Then the gradient of \( J \) defined in (1) is given by

\[

\nabla_{\alpha} J = 2 (\Phi \alpha - \beta), \quad \alpha := [\alpha_0, \alpha_1, \ldots, \alpha_{N-1}]^T,
\]

(6)

and the optimal FIR parameter \( \alpha_{\text{opt}} = [\alpha_{0,\text{opt}}, \alpha_{1,\text{opt}}, \ldots, \alpha_{N-1,\text{opt}}]^T \) which minimizes \( J \) is given by

\[
\alpha_{\text{opt}} = \Phi^{-1} \beta.
\]

(7)
3.2. Steepest Descent Algorithm. In this subsection, we derive the steepest descent algorithm (SD algorithm) \cite{3} for the Wiener solution obtained in Theorem 4. This algorithm is a base for adaptation of the ANC system discussed in the next subsection.

According to the identity (6) in Theorem 4, for the gradient of $J$, the steepest descent algorithm is described by

$$
\alpha[n + 1] = \alpha[n] - \frac{\mu}{2} \nabla_{\alpha[n]} J,
$$

$$
= \alpha[n] + \mu (\beta - \Phi \alpha[n]), \quad n \in \mathbb{Z}_+,
$$

(9)

where $\mu > 0$ is the step-size parameter.

We then analyze the stability of the above recursive algorithm. Before deriving the stability condition, we give an upper bound of the eigenvalues of the matrix $\Phi$.

Lemma 5. Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of the matrix $\Phi$. Let $\hat{u}$ denote the Fourier transform of $u = F \mathcal{K}_b k_d$, and define

$$
S(j\omega) := \frac{1}{h} \sum_{n=-\infty}^{\infty} \Big| \hat{u} \left( j\omega + \frac{2n\pi}{h} \right) \Big|^2.
$$

Then we have

$$
0 \leq \lambda_i \leq ||S||_{\infty} = \sup \{ S(j\omega) \mid \omega \in (-\pi/h, \pi/h) \},
$$

(10)

for $i = 1, 2, \ldots, N$.

Proof: See [A]

By this lemma, we derive a sufficient condition on the step size $\mu$ for convergence.

Theorem 6 (Stability of SD algorithm). Suppose that $\Phi > 0$ and the step size $\mu$ satisfies

$$
0 < \mu < 2 ||S||_{\infty}^{-1}.
$$

(11)

Then the sequence $\{\alpha[n]\}$ produced by the iteration (9) converges to the Wiener solution $\alpha_{\text{opt}}$ for any initial vector $\alpha[0] \in \mathbb{R}^N$.

Proof: The iteration (9) is rewritten as

$$
\alpha[n + 1] = (I - \mu \Phi) \alpha[n] + \mu \beta.
$$

Suppose $\Phi > 0$. Let $\lambda_{\text{max}}$ denote the maximum eigenvalue of $\Phi$. Then $\lambda_{\text{max}} > 0$ since $\Phi > 0$. The condition (11) and the inequality (10) in Lemma 5 give $0 < \mu < 2 \lambda_{\text{max}}^{-1}$, which is equivalent to $|1 - \mu \lambda_i| < 1$, $i = 1, 2, \ldots, N$. It follows that the eigenvalues of the matrix $I - \mu \Phi$ lie in the open unit disk in the complex plane, and hence the iteration (9) is asymptotically stable. The final value

$$
\alpha_{\infty} := \lim_{n \to \infty} \alpha[n]
$$
of the iteration is clearly given by the solution of the equation $\Phi_\infty = \beta$. Thus, since $\Phi > 0$, we have $\alpha_\infty = \Phi^{-1}\beta = \alpha_{\text{opt}}$. □

3.3. LMS-type Algorithm. The steepest descent algorithm assumes that the matrix $\Phi$ and the vector $\beta$ are known apriori. That is, the noise $\{x(t)\}_{t \in \mathbb{R}_+}$ and the primary path $P(s)$ are assumed to be known. However, in practice, the noise $\{x(t)\}_{t \in \mathbb{R}_+}$ cannot be fixed before we run the ANC system. In other words, the ANC system should be noncausal for running the steepest descent algorithm. Moreover, we cannot produce arbitrarily noise $\{x(t)\}_{t \in \mathbb{R}_+}$ (this is why $x$ is noise), we cannot identify the primary path $P(s)$. Hence, the assumption is difficult to be satisfied.

In the sequel, we can only use data up to the present time for causality and we cannot use the model of $P(s)$. Under this limitation, we propose to use an LMS-type adaptive algorithm using the filtered noise $u = F \mathcal{H}_h x_d$ and the error $e$ up to the present time.

First, by the equation (5) and the relation $e = d - w$, we have

$$\frac{\partial J}{\partial \alpha_k} = -2 \left( \beta_k - \sum_{j=0}^{N-1} \Phi_{kj} \alpha_j \right) = -2 \int_0^\infty e(t)u(t-kh)dt, \quad k = 0, 1, \ldots, N - 1.$$ 

Based on this, we propose the following adaptive algorithm:

$$\alpha[n+1] = \alpha[n] + \mu \delta[n], \quad n \in \mathbb{Z}_+,$$  

(12)

where $\delta[n] = [\delta_0[n], \delta_1[n], \ldots, \delta_{N-1}[n]]^\top$ with

$$\delta_k[n] := \int_0^{nh} e(t)u(t-kh)dt, \quad k = 0, 1, \ldots, N - 1.$$ 

The update direction vector $\delta[n]$ can be recursively computed by

$$\delta[n+1] = \delta[n] + \int_{nh}^{(n+1)h} e(t)u(t)dt, \quad n \in \mathbb{Z}_+,$$  

(13)

where

$$u(t) := \left[ u(t), u(t-h), \ldots, u(t-(N-1)h) \right]^\top.$$ 

This means that to obtain the vector $\delta[n]$ one needs to measure the error $e$ and the signal $u = F \mathcal{H}_h x_d$ on the interval $[(n-1)h, nh)$ and compute the integral in (13). We call this scheme the sampled-data filtered-x adaptive algorithm. The term “sampled-data” comes from the use of sampled-data $x_d$ of the continuous-time signal $x$. The sampled-data filtered-x adaptive algorithm is illustrated in Fig. 3. As shown in this figure, in order to run the
adaptive algorithm, we should use the signal \( u \) which is "filtered" \( x_d \) by \( F \), and also use the error signal \( e \).

To analyze the convergence of the iteration, we consider the following autonomous system:

\[
\alpha[n + 1] = (I - \mu \Phi[n])\alpha[n], \quad n \in \mathbb{Z}_+,
\]

where \( \Phi[n] = [\Phi_{kl}[n]]_{k,l=0,1,...,N-1} \) with

\[
\Phi_{kl}[n] := \int_0^{nh} u(t-kh)u(t-lh)dt.
\]

Then we have the following lemma:

**Lemma 7.** Suppose the following conditions:

(1) The sequence \( \{\Phi[n]\} \) is uniformly bounded, that is, there exists \( \gamma > 0 \) such that

\[
\|\Phi[n]\| \leq \gamma, \quad \forall n \in \mathbb{Z}_+.
\]

(2) The step-size parameter \( \mu \) satisfies

\[
0 < \mu < 2 \left( \max_{n \in \mathbb{Z}_+} \lambda_{\text{max}}(\Phi[n]) \right)^{-1},
\]

where \( \lambda_{\text{max}}(\Phi[n]) \) is the maximum eigenvalue of \( \Phi[n] \).

(3) The sequence \( \{\mu \Phi[n]\} \) is slowly-varying, that is, there exists a sufficiently small \( \varepsilon > 0 \) such that

\[
\|\mu(\Phi[n] - \Phi[n-1])\| \leq \varepsilon, \quad \forall n \in \mathbb{Z}_+.
\]

Then the autonomous system (14) is uniformly exponentially stable.\(^1\)

**Proof:** See [8].

By Lemma 7 we have the following theorem:

**Theorem 8** (Stability of LMS algorithm). Suppose the conditions 1–3 in Lemma 7. Then the sequence \( \{\alpha[n]\} \) converges to the Wiener solution \( \alpha^{\text{opt}} \).

**Proof:** Let \( \beta[n] := [\beta_k[n]]_{k=0,1,...,N-1} \in \mathbb{R}^N \) with

\[
\beta_k[n] := \int_0^{nh} d(t)u(t-kh)dt.
\]

Put \( c[n] := \alpha[n] - \alpha^{\text{opt}} \) and \( q[n] := \beta[n] - \Phi[n]\alpha^{\text{opt}} \). Then, \( \Phi[n] \to \Phi \) and \( \beta[n] \to \beta \) as \( n \to \infty \), and hence

\[
q[n] \to \infty \quad \text{as} \quad n \to \infty.
\]

By Lemma 7, the autonomous system (14) is uniformly exponentially stable and from (15) it follows that \( c[n] \to 0 \) as \( n \to \infty \). Thus, we have \( \alpha[n] \to \alpha^{\text{opt}} \) as \( n \to \infty \).\(\square\)

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\(^1\) The system (14) is said to be uniformly exponentially stable if there exist a finite positive constant \( c \) and a constant \( 0 \leq r < 1 \) such that for any \( n_0 \) and \( \alpha_0 = \alpha[0] \in \mathbb{R}^N \), the corresponding solution satisfies \( \|\alpha[n]\| \leq ce^{rn_0}\|\alpha_0\| \) for all \( n \geq n_0 \).
4. Approximation Method

To run the algorithm (12) with (13), we have to calculate the integral in (13). It is usual that the error signal $e$ is given as sampled data, and hence the exact value of this integral is difficult to obtain in practice. Therefore, we introduce an approximation method for this computation.

First, we split the interval $[0,h]$ into $L$ short intervals as

$$[0,h) = [0,h/L) \cup [h/L,2h/L) \cup \cdots \cup [h-1h/L,h).$$

Assume that the error $e$ is constant on each short interval. Then we have,

$$\int_{nh}^{(n+1)h} e(t)u(t-kh)dt = \sum_{l=0}^{L-1} \int_{lh/L+nh}^{(l+1)h/L+nh} e(t)u(t-kh)dt = e[n] \top U[n-k],$$

where

$$e[n] := \begin{bmatrix} e(0) \\ e(h/L) \\ \vdots \\ e(h-h/L+nh) \end{bmatrix}, \quad U[n] := \begin{bmatrix} \int_0^{h/L} u(\theta + nh)d\theta \\ \int_{h/L}^{2h/L} u(\theta + nh)d\theta \\ \vdots \\ \int_{(L-1)h/L}^h u(\theta + nh)d\theta \end{bmatrix}.$$

Then the integral in $U[n]$ can be computed via the state-space representation of $\mathcal{F}_h$ given in (3). In fact, $U[n]$ can be computed by the following digital filter:

$$F_h \left\{ \begin{array}{l} \eta[n+1] = A_h \eta[n] + B_h x_d[n], \\ U[n] = C_h \eta[n] + D_h x_d[n], \quad n \in \mathbb{Z}_+ \end{array} \right.$$ 

where $A_h$ and $B_h$ are given in (4). $C_h$ and $D_h$ are matrices defined by

$$C_h := \begin{bmatrix} \int_0^{h/L} \tau \theta d\theta \\ \int_{h/L}^{2h/L} \tau \theta d\theta \\ \vdots \\ \int_{(L-1)h/L}^h \tau \theta d\theta \end{bmatrix}, \quad D_h := \begin{bmatrix} \int_0^{h/L} \tau \theta e^{A\tau} d\theta \\ \int_{h/L}^{2h/L} \tau \theta e^{A\tau} d\theta \\ \vdots \\ \int_{(L-1)h/L}^h \tau \theta e^{A\tau} d\theta \end{bmatrix}.$$

Note that the integrals in $B_h$, $C_h$, and $D_h$ can be effectively computed by using matrix exponentials [21].

Let us summarize the proposed adaptive algorithm. The continuous-time error $e(t)$ is sampled with the fast sampling period $h/L$ and blocked to become the discrete-time signal $e[n]$, and the signal $x(t)$ is sampled with the sampling period $h$ to become $x_d[n]$. Then the sampled signal $x_d$ is filtered by $F_h(z)$ and the signal $U[n]$ is obtained. By using $e[n]$ and \{ $U[n],U[n-1],\ldots,U[n-N+1]$ \}, we update the filter coefficient $\alpha[n]$ by (12) and (13) with

$$\int_{nh}^{(n+1)h} e(t)u(t)dt \approx \begin{bmatrix} e[n] \top U[n] \\ e[n] \top U[n-1] \\ \vdots \\ e[n] \top U[n-N+1] \end{bmatrix}.$$

We show the proposed adaptive scheme in Fig. [4].
5. Simulation

In this section, we show simulation results of active noise control. The analog systems $F(s)$ and $P(s)$ are given by

\[
F(s) = \frac{1}{s+1.1} \cdot \frac{1}{20} \sum_{k=1}^{4} \frac{k^2}{s^2 + 2\zeta ks + k^2},
\]

\[
P(s) = \frac{1.2 \times 1.3}{(s+1.2)(s+1.3)} \cdot \frac{1}{20} \sum_{k=1}^{4} \frac{(1.2k)^2}{s^2 + 2\zeta(1.2k)s + (1.2k)^2}.
\]

The Bode gain plots of these systems are shown in Fig. 5. The gain $|F(j\omega)|$ has peaks at $\omega = 1, 2, 3, 4$ (rad/sec) and $|P(j\omega)|$ has peaks at $\omega = 1.2, 2.4, 3.6, 4.8$ (rad/sec). We set the sampling period $h = 1$ (sec) and the fast-sampling ratio $L = 8$. Note that the systems $F(s)$ and $P(s)$ are stable and have peaks beyond the Nyquist frequency $\omega = \pi$ (rad/sec).

Then we run a simulation of active noise control by the proposed method with the input signal $x(t)$ shown in Fig. 6. Note that the input $x(t)$ belongs $L^2$ and satisfies our assumption.

To compare with the proposed method, we also run a simulation by a standard discrete-time LMS algorithm [2], which is obtained by setting the fast-sampling parameter $L$ to be 1. The step-size parameter $\mu$ in the coefficient update in (12) is set to be 0.1. Fig. 7 shows the absolute values of error signal $e(t)$ (see Fig. 1 or Fig. 2). The errors by the conventional design is much larger than that by the proposed method. In fact, the $L^2$ norm
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Figure 6. Input signal $x(t)$ with $0 \leq t \leq 100$ (sec).

Figure 7. Absolute values of error signal $e(t)$: conventional (dash) proposed (solid).

of the error signal $e(t)$, $0 \leq t \leq 100$ (sec) is $2.805$ for the conventional method and $1.392$ for the proposed one, which is improved by about $49.6\%$. The result shows the effectiveness of our method.

Fig. 8 shows the $L^2$ norm of the error $e(t)$, $0 \leq t \leq 100$ (sec) with some values of the step-size parameter $\mu$. Fig. 8 shows that the error by the proposed method is equal to or smaller than that by the conventional method for almost all values of $\mu$. Moreover, the error by the proposed method can be small for much wider interval than that by the conventional method. In fact, the $L^2$ norm of the error $\|e\|_2 < 10$ if $\mu \in (0, 0.7257)$ by the proposed method, while $\|e\|_2 < 10$ if $\mu \in (0, 0.4051)$ by the conventional method. That is, the interval by the proposed method is about $1.8$ times wider than that by the conventional method.
In summary, the simulation results show that the proposed method gives better performance for wider interval of the step-size parameter $\mu$ on which the adaptive system is stable than the conventional method.

6. Conclusion

In this article, we have proposed a hybrid design of filtered-x adaptive algorithm via lifting method in sampled-data control theory. The proposed algorithm can take account of the continuous-time behavior of the error signal. We have also proposed an approximation of the algorithm, which can be easily implemented in DSP. Simulation results have shown the effectiveness of the proposed method.

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Appendix A. Proof of Lemma 5

First, we prove $\lambda_i \geq 0$ for $i = 1, 2, \ldots, N$. Let

$$U(t) = [u(t), u(t-h), \ldots, u(t-Nh + h)]^\top.$$

Then, for non-zero vector $v \in \mathbb{R}^N$, we have

$$v^\top \Phi v = v^\top \left( \int_0^\infty U(t)U(t)^\top \, dt \right) v = \int_0^\infty |v^\top U(t)|^2 \, dt \geq 0.$$

Thus $\Phi \succeq 0$ and hence $\lambda_i \geq 0$ for $i = 1, 2, \ldots, N$. Next, since $u(t) = 0$ for $t < 0$, we have

$$\Phi_U = \int_0^\infty u(t-kh)u(t-lh) \, dt = \int_0^\infty u(t-(k-l)h)u(t) \, dt.$$
By Parseval’s identity,

$$
\Phi_{kl} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(j\omega) \hat{u}(j\omega) e^{j\omega(k-l)h} d\omega
$$

$$
= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left( \frac{1}{h} \int_{-\frac{h}{\pi}}^{\frac{h}{\pi}} u(j\omega + \frac{2n\pi}{h}) \right)^2 e^{j\omega(k-l)h} d\omega
$$

$$
= \frac{h}{2\pi} \int_{-\frac{h}{\pi}}^{\frac{h}{\pi}} S(j\omega) e^{j\omega(k-l)h} d\omega.
$$

Then, let $v = [v_0, v_1, \ldots, v_{N-1}]^T$ be a nonzero vector in $\mathbb{R}^N$. Let $\hat{v}$ denote the discrete Fourier transform of $v$, that is,

$$
\hat{v}(j\omega) := \sum_{k=0}^{N-1} v_k e^{-j\omega kh}, \quad \omega \in (-\pi/h, \pi/h).
$$

Perseval’s identity again gives

$$
v^T v = \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{v}(j\omega) \hat{v}(j\omega) d\omega.
$$

Then we have

$$
v^T \Phi v = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} v_k v_l \Phi_{kl}
$$

$$
= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} v_k v_l \frac{h}{2\pi} \int_{-\frac{h}{\pi}}^{\frac{h}{\pi}} S(j\omega) e^{j\omega(k-l)h} d\omega
$$

$$
= \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} S(j\omega) \hat{v}(j\omega) \hat{v}(j\omega) d\omega
$$

$$
\leq \|S\|_{\infty} \cdot v^T v.
$$

It follows that

$$
\max_{1 \leq i \leq N} \lambda_i = \max \{ v^T \Phi v \mid v \in \mathbb{R}^N, \ v^T v = 1 \}
$$

$$
\leq \|S\|_{\infty}.
$$

**APPENDIX B. PROOF OF LEMMA [7]**

Let $\Psi[n] := I - \mu \Phi[n]$, $n \in \mathbb{Z}_+$. By the assumption 1, we have

$$
\|\Psi[n]\| = \|I - \mu \Phi[n]\| \leq N + \mu \|\Phi[n]\| \leq N + \mu \gamma.
$$

Thus, the sequence $\{\Psi[n]\}$ is uniformly bounded. By the assumption 2, we have

$$
|\lambda_{\text{max}}(\Psi[n])| < 1, \quad \forall n \in \mathbb{Z}_+.
$$

Also, by the assumption 3, we have

$$
\|\Psi[n] - \Psi[n-1]\| \leq \varepsilon,
$$

that is, the sequence $\{\Psi[n]\}$ is slowly varying. With these inequalities, the uniform exponential stability of the system (14) follows from Theorem 24.8 in [11].
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