UNITARIZABILITY IN GENERALIZED RANK THREE FOR CLASSICAL $p$-ADIC GROUPS

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Abstract. In [78] we propose an approach to the unitarizability problem in the case of classical groups over a $p$-adic field of characteristic zero based on cuspidal reducibility points. The unitarizability for these groups is reduced to the case of so called weakly real representations in [73]. Following C. Jantzen, to an irreducible weakly real representation $\pi$ of a classical group one can attach a sequence $(\pi_1, \ldots, \pi_k)$ of irreducible representations of classical groups, each of them supported by a line of cuspidal representations $X_\rho$ of general linear groups containing a selfcontragredient representation $\rho$, and an irreducible cuspidal representation $\sigma$ of a classical group ([25]). The first question is if $\pi$ is unitarizable if and only if all $\pi_i$ are unitarizable.

Further, the pair $\rho, \sigma$ determines the non-negative reducibility exponent $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}$ among $\rho$ and $\alpha$. The following question is if the unitarizability of irreducible representations supported by $X_\rho \cup \sigma$ can be described solely in terms of the reducibility point $\alpha_{\rho,\sigma}$ (see [78] for precise statement). If the answer to the above two questions is positive, then the unitarizability problem for classical $p$-adic groups would be reduced to a problem of a systems of real numbers.

Following the above proposed strategy, in this paper we solve the unitarizability problem for irreducible subquotients of representations $\text{Ind}_G^P(\tau)$, where $G$ is a classical group over a $p$-adic field of characteristic zero, $P$ is a parabolic subgroup of $G$ of the generalized rank (at most) 3 and $\tau$ is an irreducible cuspidal representation of a Levi factor $M$ of $P$. As a consequence, this gives also a solution of the unitarizability problem for classical $p$-adic groups of the split rank (at most) three. This paper also provides some very limited support for the possibility of the above approach to the unitarizability could work in general.

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Date: March 6, 2018.
2000 Mathematics Subject Classification. Primary: 22E50.
Key words and phrases. non-archimedean local fields, classical groups, unitarizability.
This work has been supported by Croatian Science Foundation under the project 9364.
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1. Introduction

The unitarizability for general linear group in the $p$-adic case was solved in [63]. Recall that the solution in the archimedean case is the same (see [60] and [72]). Although the unitarizability was solved practically in two steps ([63] relies essentially only on [11] regarding unitarizability), the history of the development of the ideas which led to [63] was long (some basic steps in developing of the ideas were already in [16], [17] and [29]). After [63] was published, a considerable amount of work was done in the following two decades to solve the unitarizability problem for general linear groups over $p$-adic division algebras ([5] and [55]; see also [64]). A kind of surprise was the discovery of E. Lapid and A. Mínguez that we can get unitarizability in a pretty simple way, starting only from the knowledge of the exponents of the reducibility among irreducible cuspidal representations of general linear groups, which in the case of split general linear groups is always

$$\pm 1.$$  

This is a consequence of the paper [32] (and also [3]). Further, the non-unitary theory also can be obtained starting only from the cuspidal reducibility points, and the representation

\footnote{Note that these papers have been published much before [38], which is usually considered as the beginning of the representation theory of reductive $p$-adic groups.}
theory of general reductive $p$-adic groups developed mainly in 1970-es (see appendix of \cite{32} and also \cite{77}).

Not much is known about solution of the unitarizability problem for classical groups in the general case (there exist explicit classifications of some important subclasses of unitary representations, like generic representation in \cite{33} or spherical representations in \cite{50}).

In a recent fundamental work \cite{1} of J. Arthur (completed by J.-L. Waldspurger), he classified irreducible tempered representations of classical $p$-adic groups. Further, C. Mœglin classified the parameters corresponding to the cuspidal representations. She also got a very simple formula for the exponents of the reducibility points among irreducible cuspidal representations of general linear groups and classical groups. In other words, one now has a classification of irreducible cuspidal representations of classical groups and this classification gives also the cuspidal reducibility points.

A natural question is if we can solve the unitarizability problem for the classical groups based only on the cuspidal reducibility points here (as is the case for the general linear groups). One needs to be aware that several new phenomenons arise here. Unlike (1.1), in this case we have infinitely many such cuspidal reducibility points:

\begin{equation}
0, \pm \frac{1}{2}, 1, \pm \frac{3}{2}, \pm 2, \ldots
\end{equation}

The second difference is that the parabolic induction does not always carry irreducible unitarizable representations to the irreducible ones.

We propose in \cite{78} an approach to the unitarizability problem in the case of classical groups over a $p$-adic field $F$ of characteristic zero, based on cuspidal reducibility points. The main aim of this paper is to classify, following the proposed strategy in \cite{78}, irreducible unitarizable subquotients of representations $\text{Ind}_P^G(\tau)$, where $G$ is a classical group over a $p$-adic field of characteristic zero, $P$ is a parabolic subgroup of $G$ of generalized rank (at most) 3 and $\tau$ is an irreducible cuspidal representation of a Levi factor $M$ of $P$.

Before we give a precise description of the results of the paper, we shall write some general comments about unitarizability and comments about results and proofs of this paper.

For a reductive $p$-adic group $G$, denote by $\hat{G}$ the set of all equivalence classes of its irreducible smooth representations, and by $\hat{G}$ the subset of the unitarizable classes ($\hat{G}$ is called the non-unitary dual of $G$, while $\hat{G}$ is called the unitary dual of $G$). Unitarizability problem is determination of the subset $\hat{G}$ of $\hat{G}$. One usually breaks it into two parts. The first is construction of representations of $\hat{G}$, while the second is showing that the constructed representations exhaust $\hat{G}$ (which we call exhaustion or the completeness argument). The exhaustion is usually achieved by showing that all the classes in $\hat{G}\setminus\hat{G}$ are non-unitarizable. We may call such an approach to the exhaustion indirect.
In the construction of the representations of \( \hat{G} \), the hardest part is the construction of representations that are isolated in the natural topology of \( \hat{G} \) (there are also other very hard problems like explicit understanding of the reducibility of the unitary parabolic induction, or explicit construction of complementary series, for example). Our expectation is that for (at least split) classical \( p \)-adic groups, all the isolated representations will be automorphic (this holds in the spherical case by \([50]\)). Here C. Mœglin results on the construction of Arthur packets seem to provide a powerful tool for construction of isolated representations.

Regarding exhaustion, indirect exhaustion (which is used in this paper) becomes less satisfactory for (simple) groups of larger (split) rank (as this paper shows for rank 3). Namely, it requires a very detailed knowledge of the structure of the representations in \( \hat{G} \) and for higher ranks \( \hat{G} \) is much, much larger than \( \hat{G} \). From the other side, the indirect strategy is not completely satisfactory from the point of view of the unitarizability, since our central interest are the non-unitarizable representations, while the unitarizable representations which are our main interest have very sporadic role (so in a sense, we are wasting our time on the ”wrong” class of representations).

Unfortunately, the prospect of finding a direct approach to the exhaustion for classical groups does not seem to be on the horizon in the moment. This is not surprising having in mind the history of the unitarizability in the case of general linear groups. Namely, the only successful direct approach to the exhaustion for groups with large enough ranks seems to be that in \([63]\), \([60]\). It is for the case of general linear groups (see also \([72]\) and \([65]\)). After these papers, we know that the unitarizability problem for the general linear groups has a surprisingly simple solution (see Theorem 2.9 in this paper), and relatively simple proof\(^2\). Note that before that papers, the situation did not look that way (i.e. simple), although the first lists of candidates for unitary duals of closely related groups \( SL(n, \mathbb{C}) \) existed already in 1947 in \([16]\), and these lists were very simple (they were not too far from the actual unitary duals; only the complementary series in higher ranks constructed in \([58]\) were missing). It took almost four decades after \([16]\) to get a direct approach to the exhaustion in the case of general linear groups, which finally enabled the completion of the vision of I. M. Gelfand and M. A. Naimark. In the moment, for the classical \( p \)-adic groups we do not have even a candidate for a list of representations which would form the unitary duals of classical \( p \)-adic groups. It is obvious that the situation regarding unitarizability is much more complicated here than in the case of general linear groups. Nevertheless, our hope is that we shall wait much shorter for the final answer here than in the case of general linear groups.

\(^2\)There is also D. Vogan’s classification of unitary duals of \( GL(n, \mathbb{C}), GL(n, \mathbb{R}) \) and \( GL(n, \mathbb{H}) \) (Theorem 6.18 of \([79]\)). One can find at the end of the seventh section of \([6]\) remarks about relation between our approach and that of Vogan. We shall quote here only a part which indicates the main difference between these two approaches: ”Vogan’s classification is conceptually very different from Tadić’s classification. It has its own merits, but the final result is quite difficult to state and to understand, since it uses sophisticated concepts and techniques of the theory of real reductive groups.”
Although this paper is about unitarizability, most of it deals with the non-unitarizability because of the indirect exhaustion argument. A very small part of the non-unitary dual is unitarizable, and their unitarizability, excluding only the representations (1.4), is very natural to expect (and not too hard to prove). In the analysis of the non-unitarizability of representations, the most delicate ones are those whose support is contained in a segment of cuspidal representations which contains the reducibility point, and which are not fully induced (non-unitarizability of the other representations is obtained by deformation to these representations or reducing to the non-unitarizability in the case of general linear groups). In the third section we settle the non-unitarizability of the most delicate cases for all reducibilities except five ones. The remaining five cases of reducibility we settle in a similar way, but since they are technically different, we handle them separately (this is topic of the sections 4 - 8).

The way how we show the non-unitarizability of such a most delicate representation $\pi$ is that we tensor it by a suitable irreducible unitarizable representation $\tau$, then parabolically induce the tensor product and show that the length of the induced representation is bigger then the multiplicity of $\tau \otimes \pi$ in the Jacquet module of the induced representation. This implies that $\pi$ cannot be unitarizable (in one case of reducibility exponent $\frac{1}{2}$ we needed a slightly more subtle analysis).

We already noted that in the construction of new irreducible unitarizable representations, the most difficult cases are isolated representations. The simplest examples of representations which are relatively often isolated in the unitary duals are square integrable representations (whose unitarizability is obvious) and their dual representation with respect to the Aubert-Schneider-Stuhler involution (whose unitarizability is not obvious, except for the trivial representation which is isolated by [27] if the split rank of the simple group is not one). Interesting question is to know which is the lowest rank when we have isolated representation which are out of this picture (i.e. not square integrable and its duals). In the case of $p$-adic general linear groups, the first such example is for the generalized rank 8 (we get it for $GL(9, F)$). In the case of classical groups, the first such isolated representation shows up in the generalized rank 3, when the reducibility point is $> 1$. These are the representations $[14]$ whose unitarizability was proved by C. Mœglin (the first group where we can have such representation is split $SO(11, F)$).

We shall now briefly describe some parts of the strategy in [18] proposed to handle the unitarizability problem in the case of classical $p$-adic groups. We have first one simple reduction. Namely, the unitarizability for these groups is easy to reduce to the case of representations supported by selfcontragredient irreducible cuspidal representations of general

\footnote{These are representations $u(\delta(\rho, 3), 3)$ (the representations $\delta(\rho, n)$ are defined later in the introduction, and representations $u(\delta(\rho, n), m)$ in [2.17]. Note that in the case of general linear groups we consider representations isolated modulo center, since the center is not compact.}
linear groups and irreducible cuspidal representations of classical groups\footnote{In the case of unitary groups we need to consider $F'/F$-contragredients, whose definition is recalled in the second section of the paper. In this introduction we shall follow only the case of symplectic and orthogonal groups.} (see 2.17 in the paper). Because of this, we shall consider in the sequel only the unitarizability of such representations, which we call weakly real representations.

Jantzen decomposition attaches to an irreducible (weakly real) representation $\pi$, irreducible representations supported by single cuspidal lines
\begin{equation}
\pi \to (\pi_1, \ldots, \pi_k)
\end{equation}
(see [25] or the eighth section of [78] for more details\footnote{Let $\pi$ be an irreducible square integrable representation of a classical group. Then in [17] is described its construction from cuspidal representation. If one fixes a cuspidal line $X_\rho$ of representations of general linear groups containing an irreducible selfcontragredient cuspidal representation $\rho$, and in the construction in [17] first performs all the steps including representations supported by this cuspidal line, one will get Jantzen component corresponding to this line. Using this, it is obvious how to describe Jantzen component of an irreducible tempered representation (since we get them by inducing irreducible square integrable representations). Using the Jantzen decomposition of the irreducible tempered representations, one can now define Jantzen component of a general irreducible representation using the Langlands parameter of the representation (and a simple very well-known fact that an irreducible representation of a general linear group is a product of irreducible representations supported by different cuspidal lines).}). A very important question is if this decomposition preserves unitarizability in both directions, i.e. is $\pi$ unitarizable if and only if all $\pi_i$ are unitarizable (for some very limited support for this see [78]). If this is the case, then we would have reduction of the general case to the unitarizabilities related to single cuspidal reducibilities.

Consider now an irreducible representation $\pi$ of a classical group supported by a cuspidal line $X_\rho$ along a selfcontragredient irreducible cuspidal representation $\rho$ of a general linear group, and an irreducible cuspidal representation $\sigma$ of a classical group. The pair $\rho, \sigma$ determines the non-negative reducibility exponent $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}$ among $\rho$ and $\alpha$. The following question is if the unitarizability of $\pi$ can be describe in terms of the reducibility exponent $\alpha_{\rho,\sigma}$ only (one can find a precise formulation in [78]). In the case of positive answers to the above two questions, the unitarizability problem for classical $p$-adic groups would be reduced to a problem of a systems of real numbers.

We shall now recall of some basic notation that we shall use it the rest of the introduction. J. Bernstein and A. V. Zelevinsky used $\times$ to describe parabolic induction for general linear groups: for representations $\pi_i$ of $GL(n_i, F)$, $i = 1, 2$, they denoted by
\[ \pi_1 \times \pi_2 \]
a representation of $GL(n_1 + n_2, F)$ parabolically induced by $\pi_1 \otimes \pi_2$ from suitable (standard) parabolic subgroup (see [80]). A natural generalization of this notation is multiplication
\[ \pi \rtimes \tau \]
between representations $\pi$ and $\tau$ of a general linear group and a classical group respectively, defined again in terms of the parabolic induction (see the second section for more details). We denote by $| \cdot |_F$ the normalized absolute value on $F$, and by $\nu$ the character $g \mapsto | \det(g)F |$ of $GL(n,F)$.

The main aim of this paper is to classify the irreducible unitarizable (weakly real) subquotients of the representations

$$\theta_1 \times \ldots \times \theta_k \rtimes \sigma, \quad k \leq 3,$$

where $\theta_i$, $1 \leq i \leq k$, and $\sigma$ are irreducible cuspidal representations of general linear groups and of a classical group respectively, following the above proposed strategy. In this way we get also the solution of the unitarizability problem for classical $p$-adic groups of the split rank (at most) three. This gives some very limited support for the possibility of the above approach to the unitarizability to work in general.

In the last section of the paper we prove that the Jantzen decomposition preserves unitarizability in both directions for the cases that we need in this paper. More precisely, we prove the following

**Proposition 1.1.** Let $\pi$ be a weakly real irreducible subquotient of $\theta_1 \times \ldots \times \theta_k \rtimes \sigma$, where $\theta_i$ are irreducible cuspidal representations of general linear groups and $k \leq 3$. Then $\pi$ is unitarizable if and only if all $\pi_i$ in the Jantzen decomposition of $\pi$ are unitarizable.

The following result, which is much harder to prove then the previous one, we do not need for the classification of irreducible unitarizable subquotients in the generalized rank up to three. It gives some additional (very limited) support in the direction of preservation of unitarizability by the Jantzen decomposition.

**Proposition 1.2.** Let $\pi$ be a weakly real irreducible representation of a classical group. Suppose that some $\pi_i$ is a non-unitarizable subquotient of $\theta_1 \times \ldots \times \theta_k \rtimes \sigma$, where $\theta_i$ are irreducible cuspidal representations of general linear groups and $k \leq 3$. Then $\pi$ is not unitarizable.

As a direct consequence of the above proposition we get the following fact. Let $\pi$ be a weakly real irreducible unitarizable representation of a classical group such that for each $\pi_i$ from its Jantzen decomposition there exist irreducible cuspidal representations $\theta_i$ with $k \leq 3$ of general linear groups, such that $\pi_i$ is a subquotient of $\theta_1 \times \ldots \times \theta_k \rtimes \sigma$. Then all $\pi_i$ are unitarizable.

For a general connected reductive group over $F$ there is a natural involution $\pi \mapsto D_G(\pi)$ established in [2] and [54], which carries an irreducible representations of $G$ to an irreducible representation of $G$, up to a sign (called Aubert-Schneider-Stuhler involution, or duality). Take $\epsilon_\pi \in \{ \pm 1 \}$ such that $\epsilon_\pi D_G(\pi)$ is a representation. We denote then $\epsilon_\pi D_G(\pi)$ by $\pi^t$ and call it the ASS involution of $\pi$. 
Now we shall describe unitarizability in the generalized rank up to three. We shall express the classification of irreducible subquotients in the shortest way.

By Proposition 1.1, it is enough to consider representations supported by single cuspidal lines. It means that we fix an irreducible selfcontragredient cuspidal representation $\rho$ of a general linear group and an irreducible cuspidal representation $\sigma$ of classical group. Then there exist a unique non-negative $\alpha_{\rho,\sigma} \in \frac{1}{2} \mathbb{Z}$ such that

$$\nu^{\alpha_{\rho,\sigma}} \rho \times \sigma$$

reduces. Then to simplify notation, we denote

$$\alpha := \alpha_{\rho,\sigma}.$$  

Suppose that $\alpha > 0$. Let $k$ be a non-negative integer. Then the representation

$$\nu^{k+\alpha} \rho \times \nu^{k-1+\alpha} \rho \times \cdots \nu^{\alpha} \rho \times \sigma$$

has a unique irreducible subrepresentation, which will be denoted by

$$\delta([\alpha, \alpha + k]^{(\rho)}; \sigma)$$

and called generalized Steinberg representation ($\delta([\alpha, \alpha]^{(\rho)}; \sigma)$ will be denoted simply by $\delta([\alpha]^{(\rho)}; \sigma)$). Generalized Steinberg representations are square integrable.

**Remark 1.3.** Unitarizability in the generalized rank 1 is very simply to describe: an irreducible subquotient $\pi$ of $\nu^{x} \rho \times \sigma$, $x \in \mathbb{R}_{\geq 0}$, is unitarizable $\iff x \leq \alpha$.

The following answer to the unitarizability problem in the generalized rank 2 is more or less already well known (although we do not know that is somewhere written for all the $\alpha$’s).

In the following propositions we shall always assume

$$0 \leq x_{1} \leq \cdots \leq x_{k}.$$  

**Proposition 1.4.** Let $\pi$ be an irreducible unitarizable subquotient of $\nu^{x_{1}} \rho \times \nu^{x_{2}} \rho \times \sigma$, $x_{i} \in \mathbb{R}_{\geq 0}$.

(1) Assume

$$\alpha \geq 1.$$  

Then $\pi$ is one of the following irreducible unitarizable representations:

(a) $\delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)$ or $\delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)^{t}$.

(b) Irreducible subquotients for

$$x_{1} + x_{2} \leq 1$$

or

$$x_{1} + 1 \leq x_{2} \leq \alpha.$$
(2) Assume \( \alpha = \frac{1}{2} \).

Then \( \pi \) is one of the following irreducible unitarizable representations:

(a) \( \delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma) \) or \( \delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)^t \).

(b) Irreducible subquotients for

\[ x_1 \leq \frac{1}{2}, \quad x_2 \leq \frac{1}{2}. \]

(3) Assume \( \alpha = 0 \).

Then \( \pi \) is an irreducible subquotient of the following region

\[ x_1 + x_2 \leq 1, \]

where each irreducible subquotient is unitarizable.

Suppose \( \alpha > 1 \) then the representation \( \nu^\alpha \rho \times \nu^{\alpha-1} \rho \times \delta([\alpha]^{(\rho)}; \sigma) \) has a unique irreducible (Langlands) quotient, which is denoted by

\[ L(\nu^\alpha \rho, \nu^{\alpha-1} \rho; \delta([\alpha]^{(\rho)}; \sigma)^6). \]

The answer of unitarizability for generalized rank 3 is given by the following four propositions (corresponding to different reducibility points):

**Proposition 1.5.** Let \( \pi \) be an irreducible unitarizable subquotient of \( \nu^{x_1} \rho \times \nu^{x_2} \rho \times \nu^{x_3} \rho \times \sigma \), \( x_i \in \mathbb{R}_{\geq 0} \). Assume

\[ \alpha \geq \frac{3}{2}. \]

Then \( \pi \) is one of the following irreducible unitarizable representations:

1. \( \delta([\alpha, \alpha + 2]^{(\rho)}; \sigma) \) or \( \delta([\alpha, \alpha + 2]^{(\rho)}; \sigma)^t \).

2. Irreducible subquotient of \([x_1]^{(\rho)} \rtimes \theta\), where

\[ \theta \in \{ \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma), \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)^t \} \text{ and } 0 \leq x_1 \leq \alpha - 1. \]

3. \( L(\nu^\alpha \rho, \nu^{\alpha-1} \rho; \delta([\alpha]^{(\rho)}; \sigma)). \)

4. Irreducible subquotients for

   (a) \[ x_2 + x_3 \leq 1, \]

   (b) \[ x_1 + x_2 \leq 1, \quad x_2 + 1 \leq x_3 \leq \alpha, \]

   (c) \[ x_1 + x_2 \leq 1, \quad 1 - x_1 \leq x_3 \leq 1 + x_1, \]

   (d) \[ x_1 + 1 \leq x_2, \quad x_2 + 1 \leq x_3 \leq \alpha \]

   (the last region does not show up if \( \alpha = \frac{3}{2} \)).

\( ^6 \)This representation is self dual for the ASS involution.
Proposition 1.6. Let $\pi$ be an irreducible unitarizable subquotient of $\nu^{x_1}\rho \times \nu^{x_2}\rho \times \nu^{x_3}\rho \times \sigma$, $x_i \in \mathbb{R}_{\geq 0}$. Assume
\[
\alpha = 1.
\]
Then $\pi$ is one of the following irreducible unitarizable representations:

1. $\delta([1, 3](\rho); \sigma)$ or $\delta([1, 3](\rho); \sigma)^t$.
2. Irreducible subquotient of $[0](\rho) \rtimes \delta([1, 2](\rho); \sigma)$ or $[0](\rho) \rtimes \delta([1, 2](\rho); \sigma)^t$.
3. Irreducible subquotients for
   - $(a)$ $x_2 + x_3 \leq 1$,
   - $(b)$ $x_1 + x_2 \leq 1$, $1 - x_1 \leq x_3 \leq 1$.

For a positive integer $n$, the representation $\nu^{n-1}\rho \times \nu^{n-3}\rho \times \nu^{n-1}\rho \times \ldots \times \nu^{n-1}\rho$ contains a unique irreducible subrepresentation. We denote it by $\delta(\rho, n)$.

This representation of a general linear group is square integrable modulo center.

Suppose that $\alpha = \frac{1}{2}$. Then the representation $\delta(\rho, 2) \rtimes \sigma$ contains a unique irreducible subquotient $\pi$ which is not a subquotient of $[\frac{1}{2}](\rho) \rtimes \delta([\frac{1}{2}](\rho); \sigma)$. We denote $\pi$ by $\delta([-\frac{1}{2}, \frac{1}{2}](\rho); \sigma)$.

Now we have the following classification in generalized rank 3 for $\alpha = \frac{1}{2}$:

Proposition 1.7. Let $\pi$ be an irreducible unitarizable subquotient of $\nu^{x_1}\rho \times \nu^{x_2}\rho \times \nu^{x_3}\rho \times \sigma$, $x_i \in \mathbb{R}_{\geq 0}$. Assume
\[
\alpha = \frac{1}{2}.
\]
Then $\pi$ is one of the following irreducible unitarizable representations:

1. $\delta([\frac{1}{2}, \frac{5}{2}](\rho); \sigma)$ or $\delta([\frac{1}{2}, \frac{5}{2}](\rho); \sigma)^t$.
2. Irreducible subquotient of $[x_1](\rho) \rtimes \delta([\frac{1}{2}, \frac{3}{2}](\rho); \sigma)$,
   \[
   [x_1](\rho) \rtimes \delta([\frac{1}{2}, \frac{3}{2}](\rho); \sigma)^t, \quad 0 \leq x_1 \leq \frac{3}{2}.
   \]
3. Irreducible subquotient of $\nu^x\delta(\rho, 2) \rtimes \delta([\frac{1}{2}](\rho); \sigma)$,
   \[
   \nu^x\delta(\rho, 2)^t \rtimes \delta([\frac{1}{2}](\rho); \sigma)^t, \quad 0 \leq x \leq 1.
   \]
4. Irreducible subquotient of $[x_3](\rho) \rtimes \delta([-\frac{1}{2}, \frac{1}{2}](\rho); \sigma)$,
   \[
   [x_3](\rho) \rtimes \delta([-\frac{1}{2}, \frac{1}{2}](\rho); \sigma)^t, \quad 0 \leq x_3 \leq \frac{3}{2}.
   \]
(5) Irreducible subquotient of \( \nu^x \delta(\rho, 3) \rtimes \sigma \),
\[
\nu^x \delta(\rho, 3)^t \rtimes \sigma, \quad 0 \leq x \leq \frac{1}{2}.
\]
(6) Irreducible subquotient for \( x_i \leq \frac{1}{7} \), \( i = 1, 2, 3 \).

Suppose \( \alpha = 0 \). For a positive integer \( k \) the representation \( \nu^k \delta(\rho, k + 1) \rtimes \sigma \) has precisely two irreducible subrepresentations, which are denoted by
\[
\delta([0, k]_{\pm}^{(\rho)}; \sigma) \text{ and } \delta([0, k]_{-}^{(\rho)}; \sigma).
\]
They are square integrable. Now we can describe the unitarizability in the case \( \alpha = 0 \) for generalized rank 3.

**Proposition 1.8.** Let \( \pi \) be an irreducible unitarizable subquotient of \( \nu^{x_1} \rho \times \nu^{x_2} \rho \times \nu^{x_3} \rho \rtimes \sigma \), \( x_i \in \mathbb{R}_{\geq 0} \). Assume
\[
\alpha = 0.
\]
Then \( \pi \) is one of the following irreducible unitarizable representations:

1. \( \delta([0, 2]_{\pm}^{(\rho)}; \sigma) \) or \( \delta([0, 2]_{\pm}^{(\rho)}; \sigma)^t \).
2. \( [x]^{(\rho)} \rtimes \delta([0, 1]_{\pm}^{(\rho)}; \sigma) \), \( [x]^{(\rho)} \rtimes \delta([0, 1]_{\pm}^{(\rho)}; \sigma)^t \), \( 0 \leq x \leq 1 \).
3. Irreducible subquotients for \( x_1 = 0 \), \( x_2 + x_3 \leq 1 \).

Note that the above explicit classifications directly imply that the ASS involution preserves unitarizability for subquotients in the generalized rank at most three.

Further, note that the in the above classifications only the reducibility point \( \alpha \) is what plays role in determining exponents of the representations that are unitarizable (not \( \rho \) and \( \sigma \) itself).

We are thankful to M. Hanzer, E. Lapid and G. Muić for useful discussion during writing of this paper. The motivation for writing this paper came from a discussion with C. Mœglin at Simons Symposium on Geometric Aspects of the Trace Formula in Schloss Elmau in Germany (2016). Some of the results of this paper were presented in a minicourse at the Special Trimester on Representation Theory of Reductive Groups Over Local Fields and Applications to Automorphic forms, which was held at the Weizmann Institute. We are thankful to the Simons Foundation and the Weizmann Institute.

We are particularly thankful to C. Mœglin, who has written for us in [45] the proof that the representation (1.4) is in an Arthur packet, which proves the unitarizability of that representation.

Note that the content of the paper is the following. The second section introduces notation that we use in the paper and recalls of already known results that we use often in the paper. From the third until the eighth section the unitarizability is solved for representations supported
by segments of cuspidal representations which contain the reducibility point. The most important part of the paper is the third section where we solve the cases of the reducibility points which are $> 2$. From the fourth until seventh section we address special cases which are not covered by the third section. They correspond to the reducibilities $2, \frac{3}{2}, 1, \frac{1}{2}$ respectively. The cases of reducibilities $1$ and $\frac{1}{2}$ require considerable additional work. The eighth section is devoted to the reducibility at $0$. All the cases here are completely new. The ninth section recalls some estimates where the unitarizability can show up in the parabolically induced representations. There is also description of the unitarizability in the generalized rank two case. The following section describes the unitarizability in the generalized rank three when some exponent is greater then the exponent of the reducibility point, except for reducibility at $0$, where we show that if we have unitarizability, that at least one exponent must be $0$. The tenth section brings the solution of the unitarizability for representations supported by a single cuspidal line in the generalized rank three. The last section shows that we can reduce unitarizability in the generalized rank three to the same question for the representations supported by single cuspidal lines.

2. Notation and preliminary results

We fix a local non-archimedean field $F$ of characteristic zero. Let $G$ be the group of rational points of a reductive group defined over $F$. By a representation of $G$ in this paper we shall mean smooth representations of $G$. The Grothendieck group of the category $\text{Alg}_{f.l.}(G)$ of all the representations of $G$ of finite length is denoted by $\mathcal{R}(G)$. It carries a natural ordering $\leq$. We denote by $\text{s.s.}(\tau)$ the semi simplification of $\tau \in \text{Alg}_{f.l.}(G)$. For $\pi_1, \pi_2 \in \text{Alg}_{f.l.}(G)$, the fact $\text{s.s.}(\pi_1) \leq \text{s.s.}(\pi_2)$ we write shorter as $\pi_1 \leq \pi_2$.

The contragredient representation of $\pi$ of $G$ is denoted by $\tilde{\pi}$, while the complex conjugate representation is denoted by $\bar{\pi}$. We call $\tilde{\pi}$ the hermitian contragredient of $\pi$, which we denote by $\pi^+$. Then $\pi \mapsto \pi^+$ is an (exact) contravariant functor. It is well known that if $\pi$ is unitarizable, then $\pi^+ \cong \pi$.

2.1. General linear groups. After we introduce below notation for general groups, we shall consider symplectic, orthogonal and unitary groups (and call them classical groups). When we work with a series of unitary groups, then $F'$ will denote a separable quadratic extension of $F$ which enters the definition of the unitary groups. Otherwise, $F'$ denotes $F$. If $F' \neq F$, then $\Theta$ denotes the non-trivial $F$-automorphism of $F'$. Otherwise, $\Theta$ denotes the identity mapping on $F$. Now the representation

$$(2.5) \quad \gamma \mapsto \tilde{\pi}(\Theta(g))$$

will be called $F'/F$-contragredient of $\pi$, and it will be denoted by $\tilde{\pi}$.

\footnote{We can drop the assumption on the characteristic, but the we would need an additional assumption (like in \cite{[17]})}.
The representation \( g \mapsto \pi(\Theta(g)) \) will be denoted by \( \pi^\Theta \).

We shall now recall notation for the general linear groups (following mainly [80]). The modulus character of \( F' \) is denoted by \( | \cdot |_{F'} \). The character \(|\det|_{F'} \) of \( GL(n, F') \) will be denoted by \( \nu \).

For \( 0 \leq k \leq n \), there is a unique standard parabolic subgroup \( P_{(k,n-k)} = M_{(k,n-k)}N_{(k,n-k)} \) of \( GL(n, F') \) whose Levi factor \( M_{(k,n-k)} \) is naturally isomorphic to \( GL(k, F') \times GL(n-k, F') \) (standard parabolic subgroups are subgroups that contain the subgroup of upper triangular matrices in \( GL(n, F') \)). For \( \pi_1 \in \text{Alg}_{f.l.}(GL(k, F')) \) and \( \pi_2 \in \text{Alg}_{f.l.}(GL(n_i, F')) \), \( \pi_1 \times \pi_2 \in \text{Alg}_{f.l.}(GL(n_1+n_2, F')) \) is defined to be the representation parabolically induced by \( \pi_1 \otimes \pi_2 \) from \( P_{(k,n-k)} \). Let \( R = \oplus_{n \geq 0} \mathfrak{N}(GL(n, F')) \). Now \( \times \) defines in a natural way a \( \mathbb{Z} \)-bilinear mapping \( R \times R \to R \), which will be also denoted by \( \times \). Further, we factor \( \pi : R \times R \to R \) through \( R \otimes R \) by a map denoted by \( m : R \otimes R \to R \).

The normalized Jacquet module of \( \pi \in \text{Alg}_{f.l.}(GL(n, F')) \) with respect to \( P_{(k,n-k)} \) is denoted by \( r_{(k,n-k)}(\pi) \). The comultiplication \( m^*(\pi) \) of \( \pi \) is defined by

\[
m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}(r_{(k,n-k)}(\pi)) \in R \otimes R.
\]

One extends \( m^* \) additively to a mapping \( m^* : R \to R \otimes R \) in a natural way. With \( m \) and \( m^* \), \( R \) is a graded Hopf algebra.

Denote by \( \mathcal{C} \) the set of all equivalence classes of all irreducible cuspidal representations of all \( GL(n, F') \), \( n \geq 1 \).

By \( \mathbb{Z} \)-segment in \( \mathbb{R} \) we shall call the set of form \( \{x, x+1, \ldots, x+n\} \), where \( x \in \mathbb{R} \) and \( n \in \mathbb{Z}_{\geq 0} \). We shall denote the above set by \( [x, x+n]_\mathbb{Z} \). For a \( \mathbb{Z} \)-segment \( \Delta = [x, y]_\mathbb{Z} \) in \( \mathbb{R} \) and \( \rho \in \mathcal{C} \). Denote

\[
\Delta^\rho = [x, y]^\rho = [\nu^x \rho, \nu^y \rho] := \{\nu^z \rho; z \in \Delta\}.
\]

The set \( \Delta^\rho \) is called a segment in \( \mathcal{C} \). The set of all segments in \( \mathcal{C} \) is denoted by \( \mathcal{S}(\mathcal{C}) \) (we take \( \emptyset^\rho = \emptyset \)).

We say that segments \( \Delta_1, \Delta_2 \in \mathcal{S}(\mathcal{C}) \) are linked if \( \Delta_1 \cup \Delta_2 \in \mathcal{S}(\mathcal{C}) \) and \( \Delta_1 \cup \Delta_2 \notin \{\Delta_1, \Delta_2\} \). We say that \( \Delta_1 \) precedes \( \Delta_2 \), and write

\[
\Delta_1 \rightarrow \Delta_2.
\]

For a set \( X \), the set of all finite multisets in \( X \) will be denoted by \( M(X) \) (we can view them as all functions \( X \to \mathbb{Z}_{\geq 0} \) with finite support; note that finite subsets correspond to all functions \( X \to \{0, 1\} \) with finite support). Elements of \( M(X) \) will be denoted by \( (x_1, \ldots, x_n) \) (repetitions of elements can occur, and we get the same element if we permute \( x_i \)’s). The set \( M(X) \) has a natural structure of a commutative associative semi group with
For $\Delta \in \mathcal{S}(\mathcal{C})$ we define $\text{supp}(\Delta)$ to be $\Delta$, but considered as an element of $M(\mathcal{C})$. For $a = (\Delta_1, \ldots, \Delta_n) \in M(\mathcal{S}(\mathcal{C}))$ we define

$$\text{supp}(a) = \sum_{i=0}^{n} \text{supp}(\Delta_i) \in M(\mathcal{C}).$$

2.2. Classifications of non-unitary duals of general linear groups. Fix some $\Delta = \{\rho, \nu^\rho, \ldots, \nu^n \rho\} \in \mathcal{S}(\mathcal{C})$. Then the representation

$$\rho \times \nu^\rho \times \ldots \times \nu^n \rho$$

has the unique irreducible subrepresentation, which is denoted by $\mathfrak{z}(\Delta)$, and the unique irreducible quotient, which is denoted by $\delta(\Delta)$. Then

(2.6) $m^*(\delta([\rho, \nu^n \rho])) = \sum_{i=-1}^{n} \delta([\nu^{i+1} \rho, \nu^n \rho]) \otimes \delta([\rho, \nu^i \rho])$,

(2.7) $m^*(\mathfrak{s}([\rho, \nu^n \rho])) = \sum_{i=-1}^{n} \mathfrak{s}([\rho, \nu^i \rho]) \otimes \mathfrak{s}([\nu^{i+1} \rho, \nu^n \rho])$.

Let $a = (\Delta_1, \ldots, \Delta_n) \in M(\mathcal{S}(\mathcal{C}))$. We can chose an enumeration satisfying

$$\text{if } \Delta_i \rightarrow \Delta_j \text{ for some } 1 \leq i, j \leq n, \text{ then } i > j.$$

Then the representations

$$\zeta(a) := \mathfrak{z}(\Delta_1) \times \mathfrak{z}(\Delta_2) \times \ldots \times \mathfrak{z}(\Delta_n),$$

$$\lambda(a) := \delta(\Delta_1) \times \delta(\Delta_2) \times \ldots \times \delta(\Delta_n)$$

are determined by $a$ up to an isomorphism. The representation $\zeta(a)$ has the unique irreducible subrepresentation, which is denoted by $Z(a)$, while the representation $\lambda(a)$ has the unique irreducible quotient, which is denoted by $L(a)$. Now $Z$ (resp. $L$) is called Zelevinsky (resp. Langlands) classification of irreducible representations of general linear groups over $F'$ (we follow the presentation of these classifications by F. Rodier in [Rod-Bourb]).

Denote by $D$ the set of all essentially square integrable modulo center classes of irreducible representations of general linear groups over $F'$, and by $D_u$ the subset of all unitarizable classes in $D$ (i.e. those ones having the unitary central character). The mapping

(2.8) $(\rho, n) \mapsto \delta([-\frac{n-1}{2}, \frac{n-1}{2}]^{(\rho)}), \quad \mathcal{C} \times \mathbb{Z}_{\geq 1} \rightarrow D$

is a bijection. We denote $\delta(\rho, n) = \delta([-\frac{n-1}{2}, \frac{n-1}{2}]^{(\rho)})$ for $n \in \mathbb{Z}_{>0}$. 
For $\delta \in D$ define $\delta^u \in D_u$ and $e(\delta) \in \mathbb{R}$ by the following requirement:

$$\delta = e(\delta) \delta^u.$$

Let $d = (\delta_1, \ldots, \delta_n) \in M(D)$. After a renumeration of elements of $d$, we can assume

$$e(\delta_1) \geq e(\delta_2) \geq \cdots \geq e(\delta_n).$$

Let

$$\lambda(d) = \delta_1 \times \delta_2 \times \cdots \times \delta_n.$$

Then the representation $\lambda(d)$ has the unique irreducible quotient, denoted by $L(d)$. Again $d \mapsto L(d)$ is a version of Langlands classification for general linear groups (irreducible representations are parameterized by elements of $M(D)$).

For $d = (\delta_1, \ldots, \delta_n) \in M(D)$ denote $\tilde{d} = (\tilde{\delta}_1, \ldots, \tilde{\delta}_n) \in M(D)$, $\check{d} = (\check{\delta}_1, \ldots, \check{\delta}_n)$, $d^+ = (\delta_1^+, \ldots, \delta_n^+)$ and $d^\Theta = (\delta_1^\Theta, \ldots, \delta_n^\Theta)$. Then $L(d)^- = L(\tilde{d})$, $L(d)^- = L(\check{d})$, $L(d)^+ = L(d^+)$ and $L(d)^\Theta = L(d^\Theta)$.

Define a mapping $^t$ on irreducible representations of general linear groups over $F'$ by $Z(a)^t = L(a), a \in M(S(C))$. Extend $^t$ additively to $R$. Clearly, $^t$ is positive mapping, i.e. satisfies: $r_1 \leq r_2 \implies t(r_1) \leq t(r_2)$. A non-trivial fact is that $^t$ is also a ring homomorphism (see [2] and [54]). Further, $^t$ is an involution, called Zelevinsky involution.

For $a \in M(S(C))$, define $a^t \in M(S(C))$ by the requirement

$$L(a)^t = L(a^t).$$

### 2.3. Classical groups - basic definitions

Now we recall of the notation for the classical $p$-adic groups following mainly [MT] (the main difference is that indexing of classical groups is different here). First we shall recall of the case of symplectic and orthogonal groups.

Fix a Witt tower $V \in \mathcal{V}$ of symplectic of orthogonal vector spaces over $F$, starting with an anisotropic space $V_0$ of the same type\(^8\) (see sections III.1 and III.2 of [31] for details). Let $V \in \mathcal{V}$ be the space whose biggest isotropic subspace has dimension $n$. Denote by $S_n$ the group of isometries of $V$ if $V$ is even-dimensional orthogonal space. Otherwise, denote by $S_n$ the group of isometries of $V$ of determinant one\(^9\). For $0 \leq k \leq n$, one chooses a parabolic subgroup $P(k)$ whose Levi factor $M(k)$ is naturally isomorphic to $GL(k, F) \times S_{n-k}$ (the group $P(k)$ is the stabilizer of an isotropic space of dimension $k$ - see [31], III.2.\(^{10}\)). Moreover, for any partition $\beta$ of $\ell \leq n$ we can in a natural way define parabolic subgroup

---

\(^8\) In the symplectic case, $V_0 = \{0\}$.

\(^9\) For some purposes a different indexing of groups $S_n$ may be more convenient - see [47].

\(^{10}\) One can find in [54] matrix realizations of the symplectic and split odd-orthogonal groups. In a similar way one can make matrix realizations also for other orthogonal groups (and for unitary groups which we introduce below).
$P_\beta$ and its Levi subgroup $M_\beta$ (for $M_\beta$ first consider $M(\ell)$, and then apply the construction from the case of general linear groups).

We do not follow the case of split even orthogonal groups in this paper, although we expect that the results of this paper hold also in this case, with the same proofs (split even orthogonal groups are not connected, which requires some additional checkings that we have not done).

Now we shall recall of the case of unitary groups. Let $F'$ be a quadratic extension of $F$, and denote by $\Theta$ the non-trivial element of the Galois group. In analogous way one defines the Witt tower of unitary spaces over $F'$, starting with an anisotropic hermitian space $V_0$. One considers the isometry groups and denotes by $S_n$ the group of $F'$-split rank $n$. Again one defines parabolic subgroups $P(k)$ and their Levi subgroups $M(k) \cong GL(k,F') \times S_{n-k}$ in a similar way as in the previous case. Further, one can for any partition $\beta$ of $\ell \leq n$ in a natural way define parabolic subgroup $P_\beta$ and its Levi subgroup $M_\beta$.

A minimal parabolic subgroup in $S_n$, which is the intersection of all $P(k)$’s, will be fixed (only standard parabolic subgroups with respect to the fixed minimal parabolic subgroup will be considered in this paper).

Fix one of the series $\{S_n\}_n$ as above.

2.4. Twisted Hopf algebra structure. For $\pi \in \operatorname{Alg}_{f.l.}(GL(k,F'))$ and $\sigma \in \operatorname{Alg}_{f.l.}(S_{n-k})$, the representation parabolically induced by $\pi \otimes \sigma$ is denoted by

$$\pi \rtimes \sigma.$$ 

We shall often use that

$$\pi_1 \rtimes (\pi_2 \rtimes \sigma) \cong (\pi_1 \rtimes \pi_2) \rtimes \sigma. \tag{2.9}$$

For $\pi$ as above holds

$$\operatorname{ss} (\pi \rtimes \sigma) = \operatorname{ss} (\tilde{\pi} \rtimes \sigma). \tag{2.10}$$

Therefore, if $\pi \rtimes \sigma$ is irreducible, then $\pi \rtimes \sigma \cong \tilde{\pi} \rtimes \sigma$. We say that a representation $\pi$ of a general linear group over $F'$ is $F'/F$-selfcontragredient if $\pi \cong \tilde{\pi}$.

The normalized Jacquet module of $\tau \in \operatorname{Alg}_{f.l.}(S_n)$ with respect to $P(k)$ is denoted by $s_{(k)}(\tau)$. Let $\tau$ and $\omega$ be irreducible representations of $GL(p,F)$ and $S_q$, respectively, and let $\pi$ be an admissible representation of $S_{p+q}$. Then a special case of the Frobenius reciprocity tells us

$$\operatorname{Hom}_{S_{p+q}}(\pi, \tau \rtimes \omega) \cong \operatorname{Hom}_{GL(p,F) \times S_q}(s(p)(\pi), \tau \otimes \omega),$$

while the second second adjointness implies

$$\operatorname{Hom}_{S_{p+q}}(\tau \rtimes \omega, \pi) \cong \operatorname{Hom}_{GL(p,F) \times S_q}(\tilde{\tau} \otimes \omega, s(p)(\pi)).$$
Denote
\[ R(S) = \bigoplus_{n \geq 0} \mathcal{R}(S_n). \]

Now \( \rtimes \) induces in a natural way a mapping \( R \times R(S) \to R(S) \), which is denoted again by \( \rtimes \). For \( \tau \in \text{Alg}_{\text{f.l.}}(S_n) \), denote
\[ \mu^*(\tau) = \sum_{k=0}^{n} \text{s.s.}(s_{(k)}(\tau)). \]

We extend \( \mu^* \) additively to \( \mu^*: R(S) \to R \otimes R(S) \). Denote
\[ (2.11) \quad M^* = (m \otimes 1) \circ (\tilde{\rtimes} \otimes m^*) \circ \kappa \circ m^*: R \to R \otimes R, \]
where \( \tilde{\rtimes}: R \to R \) is a group homomorphism determined by the requirement that \( \pi \mapsto \tilde{\pi} \) for all irreducible \( \pi \), and \( \kappa: R \times R \to R \times R \) maps \( \sum x_i \otimes y_i \) to \( \sum y_i \otimes x_i \). The action \( \rtimes \) of \( R \otimes R \) on \( R \otimes R(S) \) is defined in a natural way. Then
\[ (2.12) \quad \mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma) \]
holds for \( \pi \in R \) and \( \sigma \in R(S) \).

For a finite length representation \( \pi \) of \( GL(k, F') \), the component of \( M^*(\pi) \) which is in \( \mathcal{R}(GL(k, F')) \otimes \mathcal{R}(GL(0, F')) \), will be denoted by
\[ M^*_{GL}(\pi) \otimes 1. \]

For a finite length representation \( \tau \) of \( S_n \), \( \mu^*(\tau) \) will denote \( \mu^*(\text{s.s.}(\tau)) \). The similar convention we will be used for \( M^* \) and \( M^*_{GL} \).

Let \( \pi \) be a representation of \( GL(k, F') \) of finite length, and let \( \sigma \) be an irreducible cuspidal representation of \( S_n \). Suppose that \( \tau \) is a subquotient of \( \pi \rtimes \sigma \). Then we shall denote \( s_{(k)}(\tau) \) also by \( s_{GL}(\tau) \).

If \( \tau \) is additionally irreducible, then we shall say that \( \sigma \) is the partial cuspidal support of \( \tau \). We say that \( \theta \in \mathcal{C} \) is a factor of \( \tau \) if there exist an irreducible subquotient \( \beta \otimes \sigma \) of \( s_{GL}(\tau) \) such that \( \theta \) is in the support of \( \beta \).

Let \( \pi \) be a finite length representation of a general linear group, and let \( \tau \) be a similar representation of \( S_n \). Then (1-1) implies
\[ (2.13) \quad \text{s.s.}(s_{GL}(\pi \rtimes \tau)) = M^*_{GL}(\pi) \times \text{s.s.}(s_{GL}(\tau)) \]
(\( \times \) in the above formula denotes multiplication in \( R \) of \( M^*(\pi) \) with the factors on the left hand side of \( \otimes \) in \( \text{s.s.}(s_{GL}(\tau)) \)).

Let \( \tau \) be a representation of some \( GL(m, F) \) and let \( m^*(\tau) = \sum x \otimes y \). Then, the formula (2.11) implies directly
\[ (2.14) \quad M^*_{GL}(\tau) = \sum x \times \bar{y}. \]
Further, the sum of the irreducible subquotients of the form $1 \otimes *$ in $M^*(\tau)$ is

\[(2.15)\quad 1 \otimes \tau.\]

2.5. **Some formulas for $M^*$**. Let $\rho$ be an irreducible $F'/F$-selfcontragredient cuspidal representation of a general linear group. Suppose that $x, y \in \mathbb{R}$ satisfy $y - x \in \mathbb{Z}_{\geq 0}$. Then one directly gets from (2.6) and (2.11)

\[(2.16)\quad M^*(\delta([x, y]^{(\rho)})) = \sum_{i=x-1}^{y} \sum_{j=i}^{y} \delta([-i, -x]^{(\rho)}) \times \delta([j+1, y]^{(\rho)}) \otimes \delta([i+1, j]^{(\rho)}),\]

where $y - i, y - j \in \mathbb{Z}_{\geq 0}$ in the above sums. In particular

\[(2.17)\quad M^*_{GL}(\delta([x, y]^{(\rho)})) = \sum_{i=x-1}^{y} \delta([-i, -x]^{(\rho)}) \times \delta([i+1, y]^{(\rho)}).\]

Let $\pi = L(\Delta_1, \ldots, \Delta_k)$ be a ladder representations, i.e. we can write $\Delta_i = [a_i, b_i]^{(\rho)}$ where $a_k < \cdots < a_1$ and $b_k < \cdots < b_1$ (we continue to assume below $\rho \cong \hat{\rho}$). Then using [28] we get

\[(2.18)\quad M^*_{GL}(\pi) = \sum_{a_1-1 \leq x_1 \leq b_1, x_2 < \cdots < x_k} L([-x_1, -a_1]^{(\rho)}_{1 \leq i \leq k}) \times L([x_1 + 1, b_1]^{(\rho)}_{1 \leq i \leq k}).\]

In a similar way one gets for Zelevinsky segment representations

\[M^*(s([x, y]^{(\rho)})) = \sum_{x-1 \leq i \leq y} \sum_{x-1 \leq j \leq i} s([-y, -i - 1]^{(\rho)}) \times s([x, j]^{(\rho)}) \otimes s([j + 1, i]^{(\rho)}).\]

2.6. **Langlands classification for classical groups** ([57], [12], [30], [51], [80]). Denote

\[D_+ = \{ \delta \in D; e(\delta) > 0 \}.\]

Let $T$ denotes the set of all equivalence classes of irreducible tempered representations of $S_n$, for all $n \geq 0$. For $t = (\delta_1, \delta_2, \ldots, \delta_k, \tau) \in M(D_+) \times T$ take a permutation $p$ of $\{1, \ldots, k\}$ such that

\[(2.19)\quad \delta_{p(1)} \geq \delta_{p(2)} \geq \cdots \geq \delta_{p(k)}.\]

Then the representation

\[\lambda(t) := \delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(k)} \times \tau\]

has a unique irreducible quotient, denoted by

\[L(t).\]

The mapping

\[t \mapsto L(t)\]
defines a bijection from the set $M(D_+) \times T$ onto the set of all equivalence classes of the irreducible representations of all $S_n$, $n \geq 0$. This is the Langlands classification for classical groups. The multiplicity of $L(t)$ in $\lambda(t)$ is one.

Write $t = (d; \tau)$. Then $L(d; \tau)^{-} \cong L(\bar{d}; \bar{\tau})$ and $L(d; \tau)^{-} \cong L(d^{\Theta}; \bar{\tau})$.

Let $t = ((\delta_1, \delta_2, \ldots, \delta_k), \tau) \in M(D_+) \times T$ and suppose that a permutation $\rho$ satisfies (2.19). Let $\delta_{\rho}(i)$ be a representation of $GL(n_i, F)$ and $L(t)$ a representation of $S_n$. Denote by

$$e_* (t) = \left( \frac{\delta_{\rho(1)}, \ldots, \delta_{\rho(1)}}, \ldots, \frac{\delta_{\rho(k)}, \ldots, \delta_{\rho(k)}}{n_1 \text{ times}, \ldots, n_k \text{ times}} \right)$$

where $n' = n - n_1 - \cdots - n_k$. Consider a partial ordering on $\mathbb{R}^n$ given by $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if and only if

$$\sum_{i=1}^j x_i \leq \sum_{i=1}^j y_i, \quad 1 \leq j \leq n.$$ 

Suppose $t, t' \in M(D_+) \times T$ and $L(t')$ is a subquotient of $\lambda(t)$. Then

$$\epsilon_*(t') \leq \epsilon_*(t), \quad \text{and the equality holds in the previous relation } \iff t' = t$$

(see section 6. of [64] for the symplectic groups - this holds in the same form for the other classical groups different from the split even orthogonal groups).

For $\Delta \in S$ define $e(\Delta)$ to be $e(\delta(\Delta))$. Let

$$S(C)_+ = \{ \Delta \in S(C); e(\Delta) > 0 \}.$$ 

In this way we can define in a natural way the Langlands classification $(a, \tau) \mapsto L(a; \tau)$ using $M(S(C)_+) \times T$ for the parameters.

### 2.7. An irreducible subquotients of induced representations of classical groups.

Here we shall recall of a formula from [73] which we shall use very often in this paper.

For $d = (\delta_1, \ldots, \delta_k) \in M(D)$ denote by

$$d'$$

the element of $M(D_+)$ which we get from $d$ by removing all unitarizable $\delta_i$'s, and changing all $\delta_i$'s for which $e(\delta_i) < 0$, by $\delta_i$. Denote by

$$d_u$$

the multiset in $M(D)$ which we get from $d$ removing all $\delta_i$'s which are not unitarizable.

**Proposition 2.1.** Let $d \in M(D)$ and $t = (d', \tau) \in M(D_+) \times T(S)$. Denote by

$$T_{d, \tau}$$

the set of all (equivalence classes of) irreducible subrepresentations of $\lambda(d_u) \times \tau$. Then each of the representations

$$L(d^i + d'; \tau'), \quad \tau' \in T_{d, \tau}$$
is a subquotient of \( L(d) \rtimes L(d'; \tau) \).

The multiplicity of each of these representations in \( L(d) \rtimes L(d'; \tau) \) is one.

2.8. Involution. Zelevinsky involution is a special case of an involution \( D_G \) which exists on the irreducible representations of general connected reductive \( p \)-adic group. This involution is constructed in \([2]\) and \([54]\). It carries irreducible representations to the irreducible ones up to a sign. For an irreducible representation \( \pi \), we chose \( \epsilon \in \{ \pm 1 \} \) such that \( \epsilon D_G(\pi) \) is an irreducible representation, and denote \( \epsilon D_G(\pi) \) by \( \pi^t \). We call \( \pi^t \) the ASS involution of \( \pi \), or ASS dual of \( \pi \).

Regarding the parabolic induction (and classical groups), for the involution holds
\[
(\pi \rtimes \tau)^t = \pi^t \rtimes \tau^t
\]
(on the level of Grothendieck groups).

Further, for Jacquet modules, the mapping
\[
\pi_1 \otimes \ldots \pi_l \otimes \mu \mapsto \tilde{\pi}_1^t \otimes \ldots \tilde{\pi}_l^t \otimes \mu^t,
\]
is a bijection from the semi simplification of \( s_\beta(\pi) \) onto the semisimplification of \( s_\beta(\pi^t) \) (\( \beta \) is the partition which parametrizes the corresponding parabolic subgroup).

2.9. Reducibility point and generalized Steinberg representations. Fix irreducible cuspidal representations \( \rho \) and \( \sigma \) of \( GL(p, F) \) and \( S_q \) respectively. We assume that \( \rho \) is \( F'/F \)-selfcontragredient. Then
\[
(2.21) \quad \nu^{\alpha+n} \rho \rtimes \sigma
\]
reduces for unique \( \alpha_{\rho,\sigma} \geq 0 \) \((57)\). C. Mœglin has proved that In this paper we shall assume that \( \alpha_{\rho,\sigma} \in \frac{1}{2} \mathbb{Z} \). We shall denote the reducibility point \( \alpha_{\rho,\sigma} \) simply by \( \alpha \).

The representation \( \nu^{\alpha+n} \rho \rtimes \nu^{\alpha+n-1} \rho \times \cdots \times \nu^{\alpha+1} \rho \times \nu^\alpha \rho \rtimes \sigma \) contains a unique irreducible subrepresentation, which is denoted by \( \delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \sigma) \) \((n \geq 0) \). This subrepresentation is square integrable and it is called a generalized Steinberg representation. We have
\[
\mu^* \left( \delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \sigma) \right) = \sum_{k=1}^{n} \delta([\nu^{\alpha+k} \rho, \nu^{\alpha+n} \rho]) \otimes \delta([\nu^\alpha \rho, \nu^{\alpha+k} \rho]; \sigma),
\]
\[
\delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \sigma) \cong \delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \tilde{\sigma}).
\]

Applying the ASS involution, we get
\[
\mu^* \left( L(\nu^{\alpha+n} \rho, \ldots, \nu^{\alpha+1} \rho, \nu^\alpha \rho; \sigma) \right) =
\sum_{k=1}^{n} L(\nu^{-(\alpha+n)} \rho, \ldots, \nu^{-(\alpha+k+2)} \rho, \nu^{-(\alpha+k+1)} \rho) \otimes L(\nu^{\alpha+k} \rho, \ldots, \nu^{\alpha+1} \rho, \nu^\alpha \rho; \sigma).
\]
The generalized Steinberg representation and its ASS dual are the only irreducible subquotients of $\nu^{a+n}\rho \times \nu^{a+n-1}\rho \times \cdots \times \nu^{a+1}\rho \times \nu^a\rho \rtimes \sigma$ which are unitarizable (\cite{21}, \cite{20}; see also section 13. of \cite{18}).

2.10. **Representations of segment type.** We shall now recall of the formulas for Jacquet modules obtained in \cite{37}. We fix an irreducible $F'/F$-selfcongradient cuspidal representations $\rho$ of a general linear group and an irreducible cuspidal representations $\sigma$ of a classical group. We shall consider irreducible subquotients of $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$, where $c, d \in \frac{1}{2}\mathbb{Z}$, $c + d \in \mathbb{Z}_{\geq 0}$ and $d - c \geq 0$. As above, $\alpha \in (1/2)\mathbb{Z}_{\geq 0}$ denotes the reducibility exponent (2.21). We assume $d - \alpha \in \mathbb{Z}$.

The length of $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ is at most three. This is a multiplicity one representation. It is reducible if and only if $[-c, d] \cap \{-\alpha, \alpha\} \neq \emptyset$. It has length three if and only if $\{-\alpha, \alpha\} \subseteq [-c, d]$ and $c \neq d$.

Now we define terms $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma), \delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma)$ and $L_\alpha([\nu^{-c}\rho, \nu^d\rho]; \sigma)$, which of them is either irreducible representation or zero. They always satisfy

$$\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma = \delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma) + \delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma) + L_\alpha([\nu^{-c}\rho, \nu^d\rho]; \sigma)$$

in the corresponding Grothendieck group.

Suppose first that $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ is irreducible. Then we take $\delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma) = 0$. Furthermore, in this case we require $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma) \neq 0$ if and only if $[-c, d] \subseteq [-\alpha + 1, \alpha - 1]$. For irreducible $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$, this requirement and (2.22) obviously determine $L_\alpha([\nu^{-c}\rho, \nu^d\rho]; \sigma)$.

Suppose now that $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ reduces. If $c = d$, we take $L_\alpha([\nu^{-c}\rho, \nu^d\rho]; \sigma) = 0$. Otherwise, $L_\alpha([\nu^{-c}\rho, \nu^d\rho]; \sigma) = L([\nu^{-c}\rho, \nu^d\rho]; \sigma)$.

If $\alpha > 0$, then there is the unique irreducible subquotient $\gamma$ of $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ which has in $s_{GL}(\gamma)$ an irreducible subquotient $\tau \otimes \sigma$ such that $\tau$ is generic, and $e(\theta) \geq 0$ for all $\theta$ in supp$(\tau)$. We denote this $\gamma$ by $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma)$.

If $\alpha = 0$, we write $\rho \rtimes \sigma$ as a sum of irreducible subrepresentations $\tau_+ \otimes \tau_-$. We denote also $\tau_\pm$ by $\delta([\rho]_\pm, \sigma)$. Then there exists the unique irreducible subquotient of $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ that contains an irreducible representation of the form $\tau \otimes \tau_\pm$ in Jacquet module with respect to appropriate standard parabolic subgroup, and we denote it by $\delta([\nu^{-c}\rho, \nu^d\rho]_\pm; \sigma)$.

If $c = d$ or the length of $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ is three, then $\delta([\nu^{-c}\rho, \nu^d\rho]) \rtimes \sigma$ contains the unique irreducible subrepresentation different from $\delta([\nu^{-c}\rho, \nu^d\rho]_+; \sigma)$ and we denote it by $\delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma)$. Otherwise, we take $\delta([\nu^{-c}\rho, \nu^d\rho]_-; \sigma) = 0$.

\footnote{The results of \cite{37} are proved for symplectic and split odd-orthogonal groups. One easily extends them to other classical groups (without use of classification of irreducible square integrable representations of classical groups modulo cuspidal data).}

\footnote{We denoted this term in \cite{37} by $L_\alpha(\delta([\nu^{-c}\rho, \nu^d\rho]); \sigma)$.}
The representation $\delta([\nu^{-c}\rho, \nu^{d}\rho]_+; \sigma)$ is square integrable if and only if $c \neq d$, $\{-\alpha, \alpha\} \subseteq [-c, d]$ or $\alpha = -c$. If $\delta([\nu^{-c}\rho, \nu^{d}\rho]_+; \sigma)$ is square integrable, then $\delta([\nu^{-c}\rho, \nu^{d}\rho]_-; \sigma)$ is also square integrable if it is non-zero. Furthermore, if $\delta([\nu^{-c}\rho, \nu^{d}\rho]_-; \sigma)$ is square integrable, then $\delta([\nu^{-c}\rho, \nu^{d}\rho]_+; \sigma)$ is square integrable.

For the two formulas below we symmetrize notation in the following way. We define

$$
(2.23) \quad \mu^*\left(\delta([-c,d]^{(\rho)}_\pm; \sigma)\right)
$$

$$
(2.24) \quad = \sum_{i=-c-1}^{c} \sum_{j=i+1}^{d} \delta([-i, c]^{(\rho)}) \times \delta([j+1, d]^{(\rho)}_\pm) \otimes \delta([i+1, j]^{(\rho)}_\pm; \sigma) + \\
(2.25) \quad + \sum_{-c-1 \leq i \leq d} \sum_{i+1 \leq j \leq c} \delta([-i, c]^{(\rho)}) \times \delta([j+1, d]^{(\rho)}_\pm) \otimes L_\alpha([i+1, j]^{(\rho)}_\pm; \sigma) + \\
(2.26) \quad + \sum_{i=-c-1}^{c} \delta([-i, c]^{(\rho)}) \times \delta([i+1, d]^{(\rho)}_\pm) \otimes \sigma.
$$

If additionally $c \neq d$, and $c < \alpha$ or $\alpha \leq c < d$, then we have

$$
(2.27) \quad \mu^*\left(\mu([\nu^{-c}\rho, \nu^{d}\rho]^{(\rho)}_\pm; \sigma)\right)
$$

$$
(2.28) \quad = \sum_{-c-1 \leq i \leq d} \sum_{i+1 \leq j \leq d} L([-i, c]^{(\rho)}_\pm, [j+1, d]^{(\rho)}_\pm) \otimes L_\alpha([i+1, j]^{(\rho)}_\pm; \sigma) + \\
(2.29) \quad + \sum_{i=\alpha}^{d} L([-i, c]^{(\rho)}_\pm, [i+1, d]^{(\rho)}_\pm) \otimes \sigma.
$$

### 2.11. Jordan blocks

Now we shall recall the definition of the Jordan blocks $\text{Jord}(\pi)$ of an irreducible square integrable representation $\pi$ of $S_n$.

**Definition 2.2.** $\text{Jord}(\pi)$ is the set of all square integrable representations $\delta(\rho, a)$ where $\rho$ is an irreducible $F'/F$-selfcontragredient cuspidal representation of a general linear group and $a \in \mathbb{Z}_{>0}$, which satisfy that $\delta(\rho, a) \rtimes \pi$ is irreducible and that $\delta(\rho, a') \rtimes \pi$ is reducible for some $a'$ of the same parity as $a$ (i.e. $a - a' \in 2\mathbb{Z}$).

For an irreducible $F'/F$-selfcontragredient cuspidal representation of a general linear group $\rho$, we denote $\text{Jord}_\rho(\pi) = \{a; (\rho, a) \in \text{Jord}(\pi)\}$. 
The irreducible square integrable representations of classical groups are classified by admissible triples (see [47] for details). Such a representation \( \pi \) is parametrized by a triple \((\text{Jord}(\pi), \epsilon_{\pi}, \pi_{\text{cusp}})\), where \( \epsilon_{\pi} \) is a function defined odd a subset of \( \text{Jord}(\pi) \cup \text{Jord}(\pi) \times \text{Jord}(\pi) \) and \( \pi_{\text{cusp}} \) is the partial cuspidal support (which we have earlier defined).

The construction of irreducible square integrable representations in [47] starts with strongly positive representations. The simples example of such representations are generalized Steinberg representations. We shall give one more example of strongly positive representations.

Assume that the reducibility point \( \alpha = \alpha_{\rho,\sigma} \) is strictly positive. Take \( k \in \mathbb{Z}_{\geq 0} \) such that \( k < \alpha \). Then the representation \( \nu^{\alpha-k}\rho \times \nu^{\alpha-k+1}\rho \times \ldots \times \nu^{\alpha}\rho \rtimes \sigma \) has a unique irreducible subrepresentation, which we denote by \( \delta([\nu^{\alpha-k}\rho], [\nu^{\alpha-k+1}\rho], \ldots, [\nu^{\alpha}\rho]; \sigma) \).

This is an example of strongly positive (square integrable) representations.

Sometimes when we deal with strongly positive representations, to stress this we shall add subscript \( s.p. \) (we shall not do this for the generalized Steinberg representations). Therefore, the above representations we shall also denote by \( \delta_{s.p.}([\nu^{\alpha-k}\rho], [\nu^{\alpha-k+1}\rho], \ldots, [\nu^{\alpha}\rho]; \sigma) \).

2.12. Induction of \( GL \)-type. Here we shall recall of the results of [34], except that we shall formulate results in terms of the Langlands classification. As above, \( \alpha = \alpha_{\rho,\sigma} \) denotes the reducibility point (then \( \rho \cong \rho^* \)). Let \( \pi \) be an irreducible representation of a general linear group.

If \( \text{supp}(\pi) \) contains \( \nu^{\alpha}\rho \) or \( \nu^{-\alpha}\rho \), then \( \pi \rtimes \sigma \) reduces ([34]).

Suppose now that \( \text{supp}(\pi) \) does not contain \( \nu^{\alpha}\rho \) or \( \nu^{-\alpha}\rho \). Assume that all members of \( \text{supp}(\pi) \) are contained in \( \{ \nu^{k+x}\rho; k \in \mathbb{Z} \} \), for some fixed \( x \in \frac{1}{2}\mathbb{Z} \). Write \( \pi = L(d) \), for some \( d \in M(D) \). Denote by \( d_{>0} \) (resp. \( d_{<0} \)) the multiset consisting of all \( \delta \) in \( d \) such that \( e(\delta) > 0 \) (resp. \( e(\delta) < 0 \)), counted with multiplicities. Then if \( \pi \) is a ladder representation or if \( \alpha \leq 1 \), then holds

\[
L(d) \rtimes \sigma \text{ reduces } \iff L(d_{>0}) \times L(d_{<0})^{-} \text{ reduces}
\]

Observe that the above discussion imply a very old and very useful result proved in [69]. For \( \Delta \in \mathcal{S} \) holds:

\[
\delta(\Delta) \rtimes \sigma \text{ reduces } \iff \theta \rtimes \sigma \text{ reduces for some } \theta \in \Delta.
\]

Remark 2.3.
We shall often use the following simple consequence of Proposition 3.2 of [73]. Let \( \rho \cong \rho^* \) and assume that \( \pi \) is an irreducible representation of a general linear group supported by
\[
\{ \nu^{x+z} \rho; z \in \mathbb{Z} \} \quad \text{for some fixed} \quad 0 < x < 1.
\]
Then \( \pi \otimes \sigma \) is irreducible.\(^{14}\)

We can combine the above fact with the Jantzen decomposition (see section 8 of [78]) to get further irreducibilities.

2.13. Technical lemma on irreducibility.

**Lemma 2.4.** Let \( d_1, d_2, d_3 \in M(D_+) \) and \( \tau \in T \). Write \( d_i = (\delta_1^{(i)}, \ldots, \delta_{k_i}^{(i)}) \), \( i = 1, 2, 3 \). Suppose

1. \( L(d_1) \times L(d_2) \) is irreducible;
2. \( L(d_1) \times L(d_2)^* \) is irreducible;
3. \( L(d_1) \times L(d_3; \tau) \) is irreducible;
4. \( e(\delta_j^{(i)}) \geq e(\delta_1^{(i)}) \) for all \( i = 1, 2, 3 \), \( 1 \leq j \leq k_i \), \( 1 \leq l \leq k_3 \);
5. \( d_i \cong d_i^\Theta \), \( i = 1, 2, 3 \).

Then
\[
L(d_1) \times L(d_2 + d_3; \tau)
\]
is irreducible.

**Proof.** First \( L(d_2 + d_3; \tau) \) is the unique irreducible quotient of \( \lambda(d_2 + d_3; \tau) \). Condition (3) implies that \( L(d_2) \times L(d_3; \tau) \) is also a quotient of \( \lambda(d_2 + d_3; \tau) \). Therefore, \( L(d_2 + d_3; \tau) \) is (the unique irreducible) quotient of \( L(d_2) \times L(d_3; \tau) \). This implies that \( L(d_1) \times L(d_2 + d_3; \tau) \) is a quotient of \( L(d_1) \times L(d_2) \times L(d_3; \tau) \). Since \( L(d_1) \times L(d_2) \times L(d_3; \tau) = L(d_1 + d_2) \times L(d_3; \tau) \), condition (4) implies that the last representation is a quotient of \( \lambda(d_1 + d_2 + d_3; \tau) \). This implies that \( L(d_1) \times L(d_2 + d_3; \tau) \) has a unique irreducible quotient, which is \( L(d_1 + d_2 + d_3; \tau) \), and that this quotient has multiplicity one. Observe that (1) - (3) imply \( L(d_1) \times L(d_2) \times L(d_3; \tau) \cong L(d_1) \times L(d_2 + d_3; \tau) \cong L(d_1)^* \times L(d_2) \times L(d_3; \tau) \). Therefore, \( L(d_1) \times L(d_2 + d_3; \tau) \) is a quotient of \( L(d_1)^* \times L(d_2) \times L(d_3; \tau) \).

Obviously, \( L(d_1)^* \times L(d_2 + d_3; \tau) \) is a quotient of \( L(d_1)^* \times L(d_2) \times L(d_3; \tau) \), which implies that \( L(d_1 + d_2 + d_3; \tau) \) is a quotient of \( L(d_1)^* \times L(d_2 + d_3; \tau) \). Now observe that (5) implies \( L(d_1 + d_2 + d_3; \tau)^+ \cong L(d_1 + d_2 + d_3; \tau) \) and \( L(d_2 + d_3; \tau)^+ \cong L(d_2 + d_3; \tau) \). Therefore,
\[
L(d_1 + d_2 + d_3; \tau) \hookrightarrow (L(d_1)^\vee)^* \times L(d_2 + d_3; \tau) \cong L(d_1) \times L(d_2 + d_3; \tau),
\]

\(^{14}\)One can combine the above fact with the Jantzen decomposition (see section 8 of [78]) to get further irreducibilities. We shall do it later in the paper. One can get these irreducibilities also directly from Proposition 3.2 of [73].
since \((\Lambda(d_1)^+)^+ \cong (\Lambda(d_1)^+)^\dagger\) by (5) and (2.6). This implies the irreducibility of \(\Lambda(d_1) \times \Lambda(d_2 + d_3; \tau)\) since \(\Lambda(d_1 + d_2 + d_3; \tau)\) is a unique irreducible quotient of \(\Lambda(d_1) \times \Lambda(d_2 + d_3; \tau)\), and it has multiplicity one in \(\Lambda(d_1) \times \Lambda(d_2 + d_3; \tau)\).

We shall most often use the following special case of the above lemma:

**Corollary 2.5.** Let \(\rho\) be an irreducible \(F^* F\)-selfcontragredient cuspidal representations of a general linear group and \(\tau \in T\). Suppose that \(d_1, d_2 \in M(D)\) such that all elements in their supports are contained in \(\{\nu^{k+\frac{1}{2}} \rho; k \in \mathbb{Z}\}\), or that they are contained in \(\{\nu^k \rho; k \in \mathbb{Z}\}\).

If the following three representations
\[
\Lambda(d_1) \times \Lambda(d_2), \quad \Lambda(d_1) \times \Lambda(d_2^\dagger), \quad \Lambda(d_1) \times \tau
\]
are irreducible, then
\[
\Lambda(d_1) \times \Lambda(d_2; \tau)
\]
is irreducible.

### 2.14. Distinguished irreducible subquotient in induced representation.

Fix an irreducible \(F^* F\)-selfcontragredient cuspidal representations \(\rho\) of a general linear group and an irreducible cuspidal representations \(\sigma\) of a classical group.

Let \(c\) be a multiset of elements of \(\{\nu^{k+\frac{1}{2}} \rho; k \in \mathbb{Z}\}\) (\(\subseteq M(\mathcal{C}) \subseteq M(D)\)). Then \(\Lambda(c^\dagger)\) has a unique generic irreducible subquotient (which has multiplicity one in \(\Lambda(c^\dagger)\)). Denote it by \(\Lambda(c^\dagger)_{\text{gen}}\). Now the formula (2.12) directly implies that the multiplicity of \(\Lambda(c^\dagger)_{\text{gen}} \otimes \sigma\) in \(s_{GL}(\Lambda(c) \otimes \sigma)\) is one. This implies that \(\Lambda(c) \otimes \sigma\) has a unique irreducible subquotient \(\pi\) which contains \(\Lambda(c^\dagger)_{\text{gen}} \otimes \sigma\) in \(s_{GL}(\pi)\) as a subquotient. We denote this \(\pi\) by
\[
\Lambda(c; \rho)_{\dagger}.
\]

Let now \(c\) be a (finite) multiset of elements of \(\{\nu^{k} \rho; k \in \mathbb{Z}\}\). Then \(\Lambda(c^\dagger + c_u)\) has a unique generic irreducible subquotient (which has multiplicity one in \(\Lambda(c^\dagger + c_u)\)). Denote it by \(\Lambda(c^\dagger + c_u)_{\text{gen}}\). Again the formula (2.12) directly implies that the multiplicity of \(\Lambda(c^\dagger + c_u)_{\text{gen}} \otimes \sigma\) in \(s_{GL}(\Lambda(c) \otimes \sigma)\) is \(2m(\rho, c)\), where \(m(\rho, c)\) is the multiplicity of \(\rho\) in \(c\).

**Lemma 2.6.** Suppose that \(c\) does not contain the reducibility point \(\nu^{\alpha} \rho\), or that \(\alpha > 0\). Then \(\Lambda(c) \otimes \sigma\) has a unique irreducible subquotient \(\pi\) which contains \(\Lambda(c^\dagger + c_u)_{\text{gen}} \otimes \sigma\) in \(s_{GL}(\pi)\) as a subquotient. We denote this \(\pi\) by
\[
\Lambda(c; \rho)_{\dagger}.
\]

**Proof.** First consider the case when \(c = \sum_{i=1}^{n} \Delta_i\), for some \(\Delta_i \in \mathcal{S}(\mathcal{C})\) such that \(\Delta_i^\dagger = \Delta_i\) for all \(i\). We shall see by induction that in this case the lemma holds, and we shall show that \(\Lambda(c; \rho)_{\dagger}\) is a subrepresentation of \((\prod_{i=1}^{n} \delta(\Delta_i)) \otimes \sigma\). From the theory of \(R\)-groups easily follows that to prove the above claim, it is enough to prove the claim when all \(\Delta_i\) are different, and all \(\Delta_i\) contain \(\nu^{\alpha} \rho\).
Let \( i = 1 \). Denote \( \Delta_{1, \alpha \leq} := \{ \nu^\beta \rho \in \Delta_i ; \beta \geq \alpha \} \). Consider \( \delta(\Delta_1 \setminus \Delta_{1, \alpha \leq}) \times \delta(\Delta_{1, \alpha \leq} ; \sigma) \) and \( \delta(\Delta_1) \times \sigma \). The last representation has length two. In Jacquet module of both representations, \( \lambda(c^\dagger + c_u)_{gen} \otimes \sigma \) has multiplicity 2. From Jacquet module easily follows that \( \delta(\Delta_1) \times \sigma \nleq \delta(\Delta_1 \setminus \Delta_{1, \alpha \leq}) \times \delta(\Delta_{1, \alpha \leq} ; \sigma) \). This together with the above multiplicities of \( \lambda(c^\dagger + c_u)_{gen} \otimes \sigma \), imply the claim.

For \( i = 2 \), we consider \( \delta(\Delta_1) \times \lambda(\Delta_2 ; \rho)^+ \) and \( \delta(\Delta_2) \times \lambda(\Delta_1 ; \rho)^+ \). We conclude in a similar way. Multiplicity of \( \lambda(c^\dagger + c_u)_{gen} \otimes \sigma \) is now 4 in both Jacquet modules.

For general step, we consider \( \delta(\Delta_1) \times \lambda(\Delta_2 + \cdots + \Delta_n ; \rho)^+ \) and \( \delta(\Delta_n) \times \lambda(\Delta_1 + \cdots + \Delta_{n-1} ; \rho)^+ \) (see also Proposition 5.1 of [75], and its proof).

Now we go to the proof of the general case. The first observation is that one can easily show that there exists \( c' \in M(\mathcal{C}) \) such that

1. \( \text{s.s.}(\lambda(c) \times \sigma) = \text{s.s.}(\lambda(c') \times \sigma) \)
2. there exist \( \Delta_1, \ldots, \Delta_k, \Gamma_1, \ldots, \Gamma_l \in \mathcal{S}(\mathcal{C}) \) such that
   a. \( c' = \Delta_1 + \cdots + \Delta_k + \Gamma_1 + \cdots + \Gamma_l \);
   b. \( c(\Delta_i) \geq 0 \) and \( \rho \in \Delta_i, i = 1, \ldots, k \);
   c. \( \Delta_{i+1} \cup \Delta_{i+1}^- \subseteq \Delta_i \cap \Delta_i^-, i = 1, \ldots, k - 1 \);
   d. \( c(\Gamma_j) > 0 \) and \( \rho \not\in \Gamma_j, j = 1, \ldots, l \);
   e. \( \Gamma_j \) is not linked to any other \( \Gamma_j' \), or any \( \Delta_i, j = 1, \ldots, l \);
   f. \( \Gamma_j^- \) is not linked to any \( \Delta_i, j = 1, \ldots, l \).

Suppose that the lemma does not hold for this \( c \) (and \( \sigma \)). This implies that in \( \lambda(c) \times \sigma \) exists an irreducible subquotient \( \pi \) such that the multiplicity \( m \) of \( \lambda(c^\dagger + c_u)_{gen} \otimes \sigma \) in \( s_{GL}(\pi) \) satisfies \( 0 < m < 2^k \). We know

\[
\pi \leq (\prod_{i=1}^k \delta(\Delta_i)) \times (\prod_{j=1}^l \delta(\Gamma_j)) \times \sigma,
\]

since the multiplicity of \( \lambda(c^\dagger + c_u)_{gen} \otimes \sigma \) in the Jacquet module of the right hand side is \( 2^k \), which is the same as it is in \( \lambda(c') \times \sigma \) (we shall use this argument also below, without repeating this explanation). The above inequality implies

\[
(\prod_{i=1}^k \delta(\Delta_i \setminus \Delta_i)) \times \pi \leq (\prod_{i=1}^k \delta(\Delta_i \setminus \Delta_i)) \times (\prod_{i=1}^l \delta(\Delta_i)) \times (\prod_{j=1}^k \delta(\Gamma_j)) \times \sigma.
\]

\[\text{To get these segments, one consider } \nu^x \rho \in c \text{ with maximal } |x| \text{. Then } \nu^{|x|} \rho \text{ is the right end of } \Delta_1 \text{ or } \Gamma_1 \text{ (it depends on the fact if by the process that follows one will reach } \rho \text{ or not). Then one looks if } \nu^{|x|} \rho \in c \text{ or } \nu^{-(|x|)} \rho \in c \text{ (if there is no such a member, then the first segment consists of } \nu^{|x|} \rho \text{ and we repeat above search with } c - (\nu^x \rho) \text{. If yes, one has the next point of the segment of cuspidal representations and we continue the above procedure (looking for an exponent which is smaller for one then the previous exponent) as long as we can, in forming the first segment of cuspidal representations. After we cannot continue the above procedure, we have got the first segment (which is } \Delta_1 \text{ or } \Gamma_1 \text{, depending if } \rho \text{ is in it, or not). Now we repeat the above procedure with } c \text{ from which we have removed terms used in the above process. We repeat these steps as long as there are remaining members of } c \text{. In this way one gets segments in (2).} \]
Denote $\Delta_i'' = \Delta_i \cup \Delta_i^-$ and $c'' = \Delta_i'' + \cdots + \Delta_k'' + \Gamma_1 + \cdots + \Gamma_l$. Considering how on the right hand side we get $\lambda((c''\uparrow + c''_{\text{gen}}) \otimes \sigma)$ in the Jacquet module (all of them we get from terms of $\lambda(c\uparrow + c_{\text{gen}}) \otimes \sigma$ multiplying with $\delta(\Delta\setminus \Delta_i^-)$’s and taking appropriate subquotient), we conclude that its multiplicity is $m$. Therefore, there is an irreducible subquotient $\pi''$ of the left hand side which has $\lambda((c''\uparrow + c''_{\text{gen}}) \otimes \sigma)$ in its Jacquet module with multiplicity $m$. Now in the same way as in the case of (2.30), we conclude

\begin{equation}
\pi'' \leq (\prod_{j=1}^l \delta(\Gamma_j)) \times (\prod_{i=1}^k \delta(\Delta_i'')) \times \sigma.
\end{equation}

Write $\Gamma_j = [\nu^{g_j c_j}\rho, \nu^{g_j c_j^-}\rho]$. Denote $\Gamma'_j = [\nu^{-g_j c_j}\rho, \nu^{g_j c_j^-}\rho]$. $\Gamma''_j = \Gamma_j \cup \Gamma'_j$, $c'' = \Delta_i'' + \cdots + \Delta_k'' + \Gamma''_1 + \cdots + \Gamma''_l$. Then

\begin{equation}
(\prod_{j=1}^l \delta(\Gamma'_j)) \times \pi'' \leq (\prod_{j=1}^l \delta(\Gamma_j)) \times (\prod_{i=1}^k \delta(\Delta_i'')) \times \sigma.
\end{equation}

Considering how on the right hand side we get $\lambda((c''\uparrow + c''_{\text{gen}}) \otimes \sigma)$ in the Jacquet module (all of them we get from terms of $\lambda(c''\uparrow + c''_{\text{gen}}) \otimes \sigma$ multiplying with the following two $\delta([\rho, \nu^{g_j c_j}]) \times \delta([\nu \rho, \nu^{g_j c_j^-}\rho])$, $\delta([\nu \rho, \nu^{g_j c_j^-}\rho]) \times \delta([\rho, \nu^{g_j c_j^-}\rho])$ subquotients of $M^*(\delta(G'_j))$’s (and taking appropriate irreducible subquotient), we conclude that its multiplicity is $2^l m$ in the left hand side, which is strictly smaller than $2^{k+l}$. Directly follows that this multiplicity is positive. This is a contradiction with the first part of the proof. The proof is now complete.

\textbf{Remark 2.7.} (1) If $\sigma$ is generic, then $\lambda(c; \rho)_+$ is generic (and $\alpha \in \{1/2, 1\}$).

(2) Since $C \subseteq D$, then $M(C) \subseteq M(D)$. For an irreducible representation $\pi$ of a general linear group we say that it is cogenous if $\pi = L(d)$ for some $d \in M(C)$.

Let $c$ be a multiset of elements of $\{\nu^{\gamma_{k+i}}\rho; k \in \mathbb{Z}\}$. Then $\lambda(c\uparrow^-)$ has a unique cogenous irreducible subquotient (which has multiplicity one in $\lambda(c\uparrow^-)$). Denote it by $\lambda(c\downarrow)_{\text{cogen}}$. Now the formula (2.12) directly implies that the multiplicity of $\lambda(c\downarrow)_{\text{cogen}} \otimes \sigma$ in $s_{GL}(\lambda(c \times \sigma))$ is one. This implies that $\lambda(c \times \sigma)$ has a unique irreducible subquotient $\pi$ which contains $\lambda(c\downarrow)_{\text{cogen}} \otimes \sigma$ in $s_{GL}(\pi)$ as a subquotient.

We denote this $\pi$ by

$$\lambda(c; \rho)_-.$$

We define $\lambda(c; \rho)_-$ analogously for a multiset $c$ of elements of $\{\nu^{k+i}\rho; k \in \mathbb{Z}\}$ (in this case we consider $\lambda(c\uparrow^- + c_{\text{gen}}) \otimes \sigma$). Then analogous lemma to the above one holds for $\lambda(c; \rho)_-$. Further,

\begin{equation}
\lambda(c; \rho)_+^t = \lambda(c; \rho)_-.
\end{equation}

Suppose that $\pi = \lambda(c; \rho)_+$ is square integrable, and $\Delta$ is a segment consisting of elements of type $\{\nu^{k+i}\rho; k \in \mathbb{Z}\}$, or consisting of elements of type $\{\nu^k \rho; k \in \mathbb{Z}\}$, such that $\Delta = \Delta^-$. Then we denote $\pi = \lambda(c + \Delta; \rho)_+$ by

$$\tau(\Delta_+; \pi).$$
One directly sees that $\tau(\Delta_+; \pi) \leq \delta(\Delta) \times \pi$. Further, if $\delta(\Delta) \times \pi$ reduces, then it reduces into a direct sum of two nonequivalent irreducible (tempered) representations. The other one we denote by

$$\tau(\Delta_-; \pi).$$

2.15. Irreducible subrepresentations, quotients and filtrations. Let $\pi$ be a representation of a group $H$ on a vector space $W$, and let $(\tau, W')$ be an irreducible quotient of $\pi$. Suppose that $\{0\} = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_m = W$ is a filtration of $W$ (i.e. $W_i$ are $H$-invariant). Then for some $i \in \{1, \ldots, m\}$, $\sigma$ is a quotient of $W_i/W_{i-1}$. One sees this considering minimal $i$ such that projection $W_i \to W'$ is non-zero; then obviously we have epimorphism $W_i/W_{i-1} \twoheadrightarrow W'$.

Analogous conclusion holds if we replace above quotient by subrepresentation, i.e. if irreducible $W''$ embeds at representation $W$ with filtration as above. One considers minimal $i$ such that $W'' \hookrightarrow W_i$. Then $W'' \hookrightarrow W_i/W_{i-1}$.

Moreover, suppose that $W''_1, \ldots, W''_l$ are different irreducible subrepresentations, and that they all are isomorphic to some irreducible representation $W''$. Therefore, we have an embedding $\varphi : W''_1 \oplus \cdots \oplus W''_l \hookrightarrow W$. Chose minimal $i$ such that $\text{Im}(\varphi) \cap W_i \neq \{0\}$. Therefore, we have an embedding (denoted again by) $\varphi : W''_1 \oplus \cdots \oplus W''_l \hookrightarrow W/W_{i-1}$. Remove from the sequence $W''_1, \ldots, W''_l$ all the representations $W''_j$ such that $\varphi(W''_j) \subseteq W_i$. Then the restriction of $\varphi$ to all the remaining representations embeds them into $W/W_i$. Continuing the process, we get that all $W''_i$ are subrepresentations of the elements of grading (i.e. the gradings have $l$ different subrepresentations isomorphic to $W''$).

Analogous observation holds for quotients. Let $\varphi : W \twoheadrightarrow W''_1 \oplus \cdots \oplus W''_l$ be an epimorphism. Chose maximal $i$ such that the restriction $\varphi|W_i$ is still epimorphism. Remove from the sequence $W''_1, \ldots, W''_l$ all the representations which are not in the image of $\varphi|W_{i-1}$. Now the sum of the removed representations is a quotient of $W_i/W_{i-1}$. Further, the sum of the remaining representations is the image of $\varphi|W_{i-1}$. Continuing procedure, we get that all $W''_i$ are quotients of the elements of grading (i.e. the gradings have $l$ different quotients isomorphic to $W''$).

Now let $\pi$ be a representation of $GL(d, F)$, and $\sigma$ of $Sp(2m, F)$ (or more generally, of a classical group). Let $\pi_1 \otimes \pi_2 \otimes \pi_3$ be an irreducible subquotient of some $r_{(n_1, n_2, n_3)}(\pi)$ ($n_1 + n_2 + n_3 = n$) and let $\pi_4 \otimes \sigma_0$ be an irreducible subquotient of some $s_{(m_1)}(\sigma)$ ($m_1 \leq m$). Then

$$\pi_1 \times \pi_4 \times \pi_3 \otimes \pi_2 \rtimes \sigma_0$$

is a subquotient of the corresponding Jacquet module (see Lemma 5.1 of [Tad-Str] and the part preceding the lemma).

\[\text{Note that } \tau(\Delta_-; \pi) \text{ is not related to the representations of type } \lambda(c; \rho)_-. \text{ Further, note that if } \pi \text{ is a cuspidal representation } \sigma, \text{ then } \tau(\Delta_-; \pi) = \delta(\Delta_-; \pi).\]
In other words, if we get
\[ \pi' \times \pi'' \times \pi''' \otimes \pi'''' \rtimes \sigma' \]
in \( M^*(\pi) \times \mu^*(\sigma) \) (applying \( \times : R \otimes R(S) \to R(S) \) exactly as it is defined), where \( \pi', \pi'', \pi''', \pi'''' \) and \( \sigma' \) are irreducible, then
\[ \pi'' \times \pi''' \times \pi' \otimes \pi'''' \rtimes \sigma' \]
is a \textbf{subquotient} of corresponding Jacquet module, i.e. they are subquotients after shifting the \( GL \)-tensor factors circularly to the left for one place (not only in the Grothendieck group).

Recall that for irreducible \( u \) and \( \pi \),
\[ u \otimes \pi \]
is always a quotient of a Jacquet module of \( u \rtimes \pi \), and
\[ \tilde{u} \otimes \pi \]
is always a subrepresentation of that Jacquet module.

Suppose that \( u \) is an irreducible unitarizable \( F'/F \)-selfcontragredient representation, and \( \pi \) is an irreducible unitarizable representation of a classical group. Let \( \tau_1, \ldots, \tau_k \) be the composition series of \( u \rtimes \pi \). Since each \( \tau_i \leftrightarrow u \rtimes \pi \), we have an epimorphism from the Jacquet module of \( \tau_i \) onto \( u \otimes \pi \). Since Jacquet module is exact functor, it carries direct sums to direct sums. Therefore, the Jacquet module of \( u \rtimes \pi \) has quotient which is isomorphic to a direct sum of \( k \) copies of \( u \otimes \pi \).

Further, \( u \rtimes \pi \to \tau_i \), which implies that \( u \otimes \pi \) embeds into the Jacquet module of \( \tau_i \). Therefore, the Jacquet module of \( u \rtimes \pi \) contains \( u \otimes \pi \) \( k \) times as a subrepresentation (up to an isomorphism), i.e. it contains \( k \) (different) irreducible representations, each of them isomorphic to \( u \otimes \pi \).

Moreover, the Jacquet module of each \( \tau_i \) contains a quotient and a subrepresentation isomorphic to \( u \otimes \pi \).

2.16. **Some well known ways of obtaining unitarizability.** One very well know way of getting unitarizable representations is parabolic induction of unitarizable representations (which is called the unitary parabolic induction).

One can get unitarizability in opposite direction: if \( \theta \) is an irreducible hermitian representation of Levi subgroup \( M \) of a parabolic subgroup \( P \) of a reductive group \( G \), and if \( \text{Ind}_P^G(\theta) \) is irreducible, then \( \theta \) is unitarizable. This method of proving unitarizability will be called the unitary parabolic reduction.

Third way of proving reduction are limits of irreducible unitarizable representations: if \( \pi_n \) is a sequence of irreducible unitarizable representations of a reductive group \( G \), \( \tau_i \) irreducible
representations of $G$ and $m_i \in \mathbb{Z}_{>0}$ such that distribution characters $\Theta_{\pi_n}$ of $\pi_n$ converge pointwise to $\sum_i m_i \Theta_{\tau_i}$, then all $\tau_i$ are unitarizable ([39]).

Fourth way is a continuous family of irreducible hermitian representation of a reductive groups $G$, which contains at least one unitarizable representation. Then all representations in the family are unitarizable (for a definition of continuous family of representation see (b) in section 3 of [65]).

The above methods of proving unitarizability can be easily modified for proving non-unitarizability.

2.17. **Reduction of unitarizability to the weakly real case.** An irreducible representation $\pi$ of a classical group will be called weakly real if it is a subquotient of a representation of the form

$$\nu^{r_1} \rho_1 \times \ldots \times \nu^{r_k} \rho_k \rtimes \sigma,$$

where $\rho_i \in \mathcal{C}$, they satisfy $\rho_i \cong \rho_i^\ast$, $r_i \in \mathbb{R}$ and $\sigma$ is an irreducible cuspidal representation of a classical group. Now we recall of (i) of Theorem 4.2 of [73]:

**Theorem 2.8.** If $\pi$ is an irreducible unitarizable representation of some $S_q$, then there exist an irreducible unitarizable representation $\theta$ of a general linear group and a weakly real irreducible unitarizable representation $\pi'$ of some $S_{q'}$ such that

$$\pi \cong \theta \rtimes \pi'.$$

Note that the claim (ii) of Theorem 4.2 of [73] gives a more precise reduction then the above theorem. Since Theorem 7.5 of [63] (which we recall below) gives a classification of unitary duals of general linear groups, the above theorem reduces the unitarizability problem for classical $p$-adic groups to the weakly real case.

For $\delta \in D_u$ and $m \geq 1$ denote by $u(\delta,m)$ the unique irreducible quotient of $\nu^{(m-1)/2}\delta \times \nu^{(m-1)/2-1}\delta \times \ldots \times \nu^{-(m-1)/2}\delta$, which is called a Speh representation. Let $B_{\text{rigid}}$ be the set of all Speh representations, and

$$B = B(F) = B_{\text{rigid}} \cup \{\nu^\alpha \sigma \times \nu^{-\alpha} \sigma; \sigma \in B_{\text{rigid}}, 0 < \alpha < 1/2\}.$$

Now the following simple theorem solves the unitarizability for archimedean and non-archimedean general linear groups in the uniform way:

**Theorem 2.9.** The mapping $(\sigma_1, \ldots, \sigma_k) \mapsto \sigma_1 \times \ldots \times \sigma_k$ defined on $M(B)$ goes into $\cup_{n \geq 0} GL(n,F)^\Gamma$, and it is a bijection.
3. Unitarizability for integral exponents in generalized rank at most 3, cases covering reducibility $\alpha \geq \frac{5}{2}$

In the sequel $\rho$ will be an irreducible $F'/F$-selfcontragredient cuspidal representations of a general linear group and $\sigma$ an irreducible cuspidal representation of a classical groups such that

$$\begin{align*}
[\alpha]^{(\rho)} \rtimes \sigma \quad (= \nu^\alpha \rho \rtimes \sigma)
\end{align*}$$

reduces for fixed $\alpha \in \frac{1}{2} \mathbb{Z}$, $\alpha \geq 0$.

In the following few paragraphs we shall determine unitarizability of irreducible subquotients of

$$\begin{align*}
[x_1]^{(\rho)} \times \ldots \times [x_k]^{(\rho)} \rtimes \sigma
\end{align*}$$

when $k \leq 3$ and $\{\nu^x \rho, \ldots, \nu^y \rho\}$ forms a segment of cuspidal representations which contains the reducibility point $\nu^\alpha \rho$.

Since $[x_1]^{(\rho)} \times \ldots \times [x_k]^{(\rho)} \rtimes \sigma$ and $[\epsilon_1 x_1]^{(\rho)} \times \ldots \times [\epsilon_k x_k]^{(\rho)} \rtimes \sigma$, $\epsilon_i \in \{\pm 1\}$, have the same composition series, we can always switch to the case

$$0 \leq x_1 \leq \ldots \leq x_k.$$

We shall assume this in the sequel. Since $\rho$ and $\sigma$ are fixed, the representations $[x_1]^{(\rho)} \times \ldots \times [x_k]^{(\rho)} \rtimes \sigma$ is determined with the multiset

$$(x_1, \ldots, x_k).$$

The exponents satisfying $(x_1, \ldots, x_k)$, the above condition we shall call integral exponents with respect to the reducibility point, or shorter integral exponents.

Suppose that $(x_1, \ldots, x_k)$ has $k$ different members. If $x_1 = \alpha > 0$, then we know the answer to the unitarizability question (see E.1 in section 3). If $x_1 > 0$ and $x_k = \alpha$, then one directly sees that all irreducible subquotients are unitarizable (since they are limits of complementary series).

Although in this section we consider cases of integral exponents which covering reducibility $\alpha \geq \frac{5}{2}$ (for generalized rank at most 3), number of cases will apply also to some smaller values of the reducibility point.
3.1. Generalized rank one.

A.1. The case of exponent \((α)\) and \(α ≥ \frac{1}{2}\).

**Proposition 3.1.** Suppose \(α ≥ \frac{1}{2}\).

Then we have the following decomposition into irreducible subquotients

\[ [α]^{(ρ)} × σ = L([α]^{(ρ)}; σ) + δ([α]^{(ρ)}; σ), \]

and both representations on the right hand side are unitarizable. The ASS involution switches them. We have

\[ µ^*(δ([α]^{(ρ)}; σ)) = 1 ⊗ δ([α]^{(ρ)}; σ) + [α]^{(ρ)} ⊗ σ. \]

\[ µ^*(L([α]^{(ρ)}; σ)) = 1 ⊗ L([α]^{(ρ)}; σ) + [−α]^{(ρ)} ⊗ σ. \] □

Here

\[ Jord_ρ(δ([α]^{(ρ)}; σ)) = \{2(α − [α]) + 1, 2(α − [α]) + 3, \ldots, 2α − 3, 2α + 1\}, \]

where \([α]\) denotes \(\max\{k ∈ \mathbb{Z}; k ≤ α\}\) (use Proposition 2.1 of [47]17).

3.2. Generalized rank two.

B.1. The case of exponents \((α, α + 1)\) and \(α ≥ \frac{1}{2}\).

We have the following decomposition into irreducible subquotients

\[ [α + 1]^{(ρ)} × [α]^{(ρ)} × σ = \]

\[ L([α + 1]^{(ρ)}, [α]^{(ρ)}; σ) + L([α + 1]^{(ρ)}; δ([α]^{(ρ)}; σ)) + L([α, α + 1]^{(ρ)}; σ) + δ([α, α + 1]^{(ρ)}; σ). \]

**Proposition 3.2.** Suppose \(α ≥ \frac{1}{2}\).

1. The representations

\[ L([α + 1]^{(ρ)}, [α]^{(ρ)}; σ), \quad δ([α, α + 1]^{(ρ)}; σ) \]

are unitarizable, and the ASS involution switches them.

2. The representations

\[ L([α + 1]^{(ρ)}, δ([α]^{(ρ)}; σ)), \quad L([α, α + 1]^{(ρ)}; σ) \]

are not unitarizable, and the ASS involution switches them. □

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17 In the sequel we shall conclude Jordan blocks from Proposition 2.1 of [47] if it is not indicated that we conclude in a different way.
Recall that for \( \mu^*(L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)) \) and \( \mu^*(\delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)) \) we have already written a more general formula which implies the above two formulas.

Further
\[
\mu^*(L([\alpha + 1]^{(\rho)}, \delta_{s.p.}([\alpha]^{(\rho)}; \sigma))) = 1 \otimes L([\alpha + 1]^{(\rho)}, \delta_{s.p.}([\alpha]^{(\rho)}; \sigma)) + \\
[-\alpha - 1]^{(\rho)} \otimes \delta_{s.p.}([\alpha]^{(\rho)}; \sigma) + [\alpha]^{(\rho)} \otimes [\alpha + 1]^{(\rho)} \times \sigma \\
+ [\alpha]^{(\rho)} \times [-\alpha - 1]^{(\rho)} \otimes \sigma + L([\alpha]^{(\rho)}, [\alpha + 1]^{(\rho)}) \otimes \sigma
\]
and
\[
\mu^*(L([\alpha, \alpha + 1]^{(\rho)}; \sigma)) = 1 \otimes L([\alpha, \alpha + 1]^{(\rho)}; \sigma) \\
[\alpha]^{(\rho)} \otimes [\alpha + 1]^{(\rho)} \times \sigma + [\alpha + 1]^{(\rho)} \otimes L([\alpha]^{(\rho)}, \sigma) + \\
\delta([-\alpha - 1, -\alpha]^{(\rho)} \otimes \sigma + [-\alpha]^{(\rho)} \times [\alpha + 1]^{(\rho)} \otimes \sigma.
\]

One easily gets that
\[
Jord_{\rho}(\delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)) = \{2(\alpha - [\alpha]) + 1, 2(\alpha - [\alpha]) + 3, \ldots, 2\alpha - 3, 2\alpha + 3\}.
\]

C.1. The case of exponents \((\alpha, \alpha)\) and \(\alpha \geq 1\)

First note that \([\alpha]^{(\rho)} \times \delta([\alpha]^{(\rho)}; \sigma)\) is irreducible by part (iv) of Proposition 6.1 from [73]. Irreducible subquotients of the induced representation in this case are
\[
[\alpha]^{(\rho)} \times [\alpha]^{(\rho)} \times \sigma = [\alpha]^{(\rho)} \times L([\alpha]^{(\rho)}; \sigma) + [\alpha]^{(\rho)} \times \delta([\alpha]^{(\rho)}; \sigma),
\]
\[
= L([\alpha]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) + L([\alpha]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)).
\]
Both above irreducible representations are non-unitarizable. One sees this deforming them to the point \((\alpha, \alpha + 1)\). There at limits we have at both cases one non-unitarizable subquotient (one gets the last fact using Lemma 3.1 of [21]).

D.1. The case of exponents \((\alpha - 1, \alpha)\) and \(\alpha \geq \frac{3}{2}\)

Now the representation is regular and we have the following two decompositions
\[
[\alpha]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times \sigma = [\alpha - 1]^{(\rho)} \times L([\alpha]^{(\rho)}; \sigma) + [\alpha - 1]^{(\rho)} \times \delta([\alpha]^{(\rho)}; \sigma),
\]
\[
= \delta([\alpha - 1, \alpha]^{(\rho)} \times \sigma + L([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}) \times \sigma.
\]
Now we easily deduce the decomposition into irreducible representations of the representations in the second row
\[
\delta([\alpha - 1, \alpha]^{(\rho)} \times \sigma = L([\alpha - 1, \alpha]^{(\rho)}; \sigma) + L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)),
\]
\[
L([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)} \times \sigma = L([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) + \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma).
\]
This further implies
\[
[\alpha - 1]^{(\rho)} \times L([\alpha]^{(\rho)}; \sigma) = L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) + L([\alpha - 1, \alpha]^{(\rho)}; \sigma),
\]
\footnote{We shall often use this way of arguing non-unitarizability using Lemma 3.1 of [21].}
Now we directly get

**Proposition 3.3.** For
\[
\alpha \geq \frac{3}{2}
\]
all the irreducible subquotients of
\[
[\alpha - 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma
\]
are unitarizable and
\[
\delta_{s.p.}( [\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) = L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) + \delta_{s.p.}( [\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma))
\]

Further,
\[
Jord_{\rho}(\delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)) = \{ 2(\alpha - |\alpha|) + 1, 2(\alpha - |\alpha|) + 3, \ldots, 2\alpha - 5, 2\alpha - 1, 2\alpha + 1 \},
\]
where in the last case partially defined function \( \epsilon \) attached to the square integrable representation is different on \( 2\alpha - 1 \) and \( 2\alpha + 1 \). \( \square \)

Further,
\[
\mu^*(\delta_{s.p.}( [\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma))) = 1 \otimes \delta_{s.p.}( [\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) +
\]
\[
+ [\alpha - 1]^{(\rho)} \otimes \delta([\alpha]^{(\rho)}; \sigma) + L([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}) \otimes \sigma,
\]

\[
\mu^*(L([\alpha - 1, \alpha]^{(\rho)}; \sigma)) = 1 \otimes L([\alpha - 1, \alpha]^{(\rho)}; \sigma) +
\]
\[
+ [-\alpha + 1]^{(\rho)} \otimes L([\alpha]^{(\rho)}; \sigma) +
\]
\[
\delta([-\alpha, -\alpha + 1]^{(\rho)}) \otimes \sigma,
\]

\[
\mu^*(L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))) = 1 \otimes L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) +
\]
\[
+ [\alpha]^{(\rho)} \otimes [\alpha - 1]^{(\rho)} \rtimes \sigma + [-\alpha + 1]^{(\rho)} \otimes \delta([\alpha]^{(\rho)}; \sigma) +
\]
\[
+ [-\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \otimes \sigma + \delta([\alpha - 1, \alpha]^{(\rho)}) \otimes \sigma,
\]

\[
\mu^*(L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma)) = 1 \otimes L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) +
\]
\[
+ [-\alpha]^{(\rho)} \otimes [\alpha - 1]^{(\rho)} \rtimes \sigma + [\alpha - 1]^{(\rho)} \otimes L([\alpha]^{(\rho)}; \sigma) +
\]
\[
+ [\alpha - 1]^{(\rho)} \times [-\alpha]^{(\rho)} \otimes \sigma + L([-\alpha]^{(\rho)}, [-\alpha + 1]^{(\rho)}) \otimes \sigma.
\]
3.3. **Generalized rank 3.**

**E.1. The case of exponents** $(\alpha, \alpha + 1, \alpha + 2)$ and $\alpha \geq \frac{1}{2}$

The representation

$$[\alpha + 2]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma$$

has length 8, and this is a multiplicity one representation. Of the irreducible subquotients, precisely two are unitarizable. They are generalized Steinberg representation and its dual, i.e.

$$\delta_{s.p.}([\alpha, \alpha + 2]^{(\rho)}; \sigma), \quad L([\alpha + 2]^{(\rho)}, [\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma),$$

are unitarizable.

**F.1. The case of exponents** $(\alpha, \alpha + 1, \alpha + 1)$ and $\alpha \geq \frac{1}{2}$

Consider

$$[\alpha + 1]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \times \sigma = [\alpha + 1]^{(\rho)} \times L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) + [\alpha + 1]^{(\rho)} \times L([\alpha + 1]^{(\rho)}, \delta([\alpha]^{(\rho)}; \sigma)) + [\alpha + 1]^{(\rho)} \times L([\alpha, \alpha + 1]^{(\rho)}; \sigma) + [\alpha + 1]^{(\rho)} \times \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma).$$

Recall that

$$\text{Jord}_{\rho}(\delta_{s.p.}([\alpha, \alpha + 1]^{(\rho)}; \sigma)) = \{2(\alpha - |\alpha|) + 1, 2(\alpha - |\alpha|) + 3, \ldots, 2\alpha - 3, 2\alpha + 3\}.$$

Now (iv) of Proposition 6.1, \(\text{[3]}\) implies that the last representation in the above decomposition of $[\alpha + 1]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \times \sigma$ is irreducible. Therefore, also the first representation on the right hand side of the above equality is irreducible (by duality).

Further, the third representation in the above decomposition is irreducible by Corollary 2.5. Now the duality implies that also the second representation is irreducible.

We deform the exponent $\alpha + 1$ in above four irreducible representations. Recall

$$[\alpha + 2]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \times \sigma = [\alpha + 2]^{(\rho)} \times L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) + [\alpha + 2]^{(\rho)} \times L([\alpha + 1]^{(\rho)}, \delta([\alpha]^{(\rho)}; \sigma)) + [\alpha + 2]^{(\rho)} \times L([\alpha, \alpha + 1]^{(\rho)}; \sigma) + [\alpha + 2]^{(\rho)} \times \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma).$$

All above four terms on the right hand side are reducible. Since the representation $L([\alpha + 2]^{(\rho)}, [\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)$ is a subquotient of the first term of the right hand side of the above equation, and $\delta([\alpha, \alpha + 2]^{(\rho)}; \sigma)$ is a subquotient of the last term, we conclude that all irreducible subquotient here are not unitarizable.

**G.1. The case of exponents** $(\alpha, \alpha, \alpha + 1)$ and $\alpha \geq 1$

**Lemma 3.4.** Let $\alpha \geq 1$. Then
(1) The following irreducible representations
\[ L([\alpha]^{(\rho)}, [\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) \]
\[ L([\alpha]^{(\rho)}, [\alpha + 1]^{(\rho)}, \delta([\alpha]^{(\rho)}; \sigma)) \]
\[ L([\alpha]^{(\rho)}, [\alpha, \alpha + 1]^{(\rho)}; \sigma) \]
\[ L([\alpha, \alpha + 1]^{(\rho)}, \delta([\alpha]^{(\rho)}; \sigma)) \]
are all the possible irreducible subquotients of
\[ [\alpha]^{(\rho)} \times [\alpha]^{(\rho)} \times [\alpha + 1]^{(\rho)} \rtimes \sigma. \]

(2) The involution switches the representations in the first two rows in (1), and fixes the representation in the third row of (1).

(3) We have
\[ L([\alpha]^{(\rho)}, [\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) = [\alpha]^{(\rho)} \rtimes L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma), \]
\[ L([\alpha]^{(\rho)}; \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)) = [\alpha]^{(\rho)} \rtimes \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma), \]
\[ L([\alpha]^{(\rho)}, [\alpha + 1]^{(\rho)}, \delta([\alpha]^{(\rho)}; \sigma)) = L([\alpha]^{(\rho)}, [\alpha + 1]^{(\rho)}) \rtimes \delta([\alpha]^{(\rho)}; \sigma), \]
\[ L([\alpha]^{(\rho)}, [\alpha, \alpha + 1]^{(\rho)}; \sigma) = \delta([\alpha, \alpha + 1]^{(\rho)}) \rtimes L([\alpha]^{(\rho)}; \sigma). \]

Proof. The classification of irreducible square integrable representations modulo cuspidal data of classical groups implies (1).

Recall that the formula for \( Jord_\rho(\delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)) \) (see above) and (iv) of Proposition 6.1 of [73] imply that \([\alpha]^{(\rho)} \rtimes \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)\) is irreducible. Therefore, also its ASS dual is irreducible. This implies the equalities in the first two rows in (3) (using [2,7]). These equivalences imply the involution switches the representations in the first row of (1).

Further, both representations in the first row have multiplicity one in the whole induced representation.

Consider now \( \delta([\alpha, \alpha + 1]^{(\rho)}) \rtimes L([\alpha]^{(\rho)}; \sigma) \). Properties of the Langlands classification imply the that no one of the first representations in the first two rows in (1) can be subquotients of it.

Suppose that \( L([\alpha, \alpha + 1]^{(\rho)}, \delta([\alpha]^{(\rho)}; \sigma)) \). This would imply
\[ \delta([-\alpha - 1, -\alpha]^{(\rho)} \times [\alpha]^{(\rho)}) \leq \delta([-\alpha - 1, -\alpha]^{(\rho)}) + [-\alpha]^{(\rho)} \times [\alpha + 1]^{(\rho)} + \delta([\alpha, \alpha + 1]^{(\rho)}) \times [-\alpha]^{(\rho)}, \]
which obviously does not hold. Therefore \( L([\alpha, \alpha + 1]^{(\rho)}, \delta([\alpha]^{(\rho)}; \sigma)) \) is not a subquotient of \( \delta([\alpha, \alpha + 1]^{(\rho)}) \rtimes L([\alpha]^{(\rho)}; \sigma). \)

It remains to see if \( L([\alpha]^{(\rho)}; \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)) = [\alpha]^{(\rho)} \rtimes \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma) \) is. Suppose that it is. Then
\[ ([\alpha]^{(\rho)} + [-\alpha]^{(\rho)}) \times \delta([\alpha, \alpha + 1]^{(\rho)}) \leq \]
\[ \delta([-\alpha - 1, -\alpha]^{(\rho)}) + [-\alpha]^{(\rho)} \times [\alpha + 1]^{(\rho)} + \delta([\alpha, \alpha + 1]^{(\rho)}) \times [-\alpha]^{(\rho)}, \]
which obviously does not hold. Thus, $\delta([\alpha, \alpha+1]^{(\rho)} \rtimes L([\alpha]^{(\rho)}; \sigma)$ is irreducible. Therefore, the equality in the last row in (3) holds. Now ASS involution implies that the equality in the last third in (3) holds.

This implies that the involution switches the representations in the second row of (1), which further implies that the representation in the last row of (1) is fixed by the involution. □

Since $L([\alpha]^{(\rho)}, [\alpha+1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)$ is a subquotient of $L([\alpha]^{(\rho)}, [\alpha+1]^{(\rho)} \rtimes L([\alpha]^{(\rho)}; \sigma)$, applying ASS involution and using the properties of the standard modules in the Langlands classification we get

$$\delta([\alpha, \alpha+1]^{(\rho)} \rtimes \delta([\alpha]^{(\rho)}; \sigma) = L([\alpha, \alpha+1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) + L([\alpha]^{(\rho)}; \delta([\alpha, \alpha+1]^{(\rho)}; \sigma)).$$

**Lemma 3.5.** For $\alpha \geq 1$, all the irreducible subquotients of $[\alpha+1]^{(\rho)} \rtimes [\alpha]^{(\rho)} \rtimes [\alpha]^{(\rho)} \rtimes \sigma$ are not unitarizable.

**Proof.** In a similar way as in the previous case (using also the non-unitarizability proved in that case), we get that the irreducible subquotients of the whole induced representation different from $L([\alpha, \alpha+1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))$ are not unitarizable.

For completing the proof of the above lemma, it remains to show that the representation $L([\alpha, \alpha+1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))$ is not unitarizable. Consider

$$\delta([-(\alpha-1), \alpha-1]^{(\rho)} \rtimes L([\alpha, \alpha+1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))$$

Then have here two irreducible subquotients by (2.7)

$$L([\alpha, \alpha+1]^{(\rho)}; \tau([-(\alpha-1), \alpha-1]^{(\rho)} \rtimes \delta([\alpha]^{(\rho)}; \sigma)).$$

Further, (3.33) implies

$$\delta([-(\alpha-1), \alpha-1]^{(\rho)} \rtimes \delta([\alpha, \alpha+1]^{(\rho)} \rtimes \delta([\alpha]^{(\rho)}; \sigma) =$$

$$\delta([-(\alpha-1), \alpha-1]^{(\rho)} \rtimes L([\alpha, \alpha+1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) +$$

$$\delta([-(\alpha-1), \alpha-1]^{(\rho)} \rtimes L([\alpha]^{(\rho)}; \delta([\alpha, \alpha+1]^{(\rho)}; \sigma)).$$

Observe that

$$L([-\alpha-1), \alpha+1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) \leq \delta([-\alpha-1), \alpha+1]^{(\rho)} \rtimes \delta([\alpha]^{(\rho)}; \sigma) \leq$$

$$\delta([-\alpha-1), \alpha+1]^{(\rho)} \rtimes \delta([\alpha, \alpha+1]^{(\rho)} \rtimes \delta([\alpha]^{(\rho)}; \sigma)$$

Suppose

$$L([-\alpha-1), \alpha+1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) \leq \delta([-\alpha-1), \alpha-1]^{(\rho)} \rtimes L([\alpha]^{(\rho)}; \delta([\alpha, \alpha+1]^{(\rho)}; \sigma)).$$

Then

$$L([-\alpha-1), \alpha+1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) \leq \delta([-\alpha-1), \alpha-1]^{(\rho)} \rtimes \delta([\alpha, \alpha+1]^{(\rho)}; \sigma).$$
This cannot be by the properties of the standard modules in the Langlands classification. Therefore

\[ L([-\alpha, 1), \alpha + 1; \delta([\alpha]^{\rho}; \sigma)] \leq \delta([-\alpha, 1), \alpha - 1]^{\rho}) \times L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma)). \]

This implies that \( \delta([-\alpha, 1), \alpha - 1]^{\rho}) \times L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma)) \) is a representation of length at least three.

Suppose that \( L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma)) \) is unitarizable. This implies that the multiplicity of \( \delta([-\alpha, 1), \alpha - 1]^{\rho}) \otimes L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma)) \) in \( \mu^*(\delta([-\alpha, 1), \alpha - 1]^{\rho}) \times L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma))) \) is at least three. Recall

\[ \mu^*(\delta([-\alpha, 1), \alpha - 1]^{\rho}) \times L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma))) = \]

\[ M^*(\delta([-\alpha, 1), \alpha - 1]^{\rho})) \times \mu^*(L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma))). \]

Now \( \delta([-\alpha, 1), \alpha - 1]^{\rho}) \otimes 1 \) has multiplicity two in the first factor on the left hand side of the above equation. If we take any other summand from the first factor, we get on the right hand side of \( \otimes \) for a factor at least one representation from the segment \( \delta([-\alpha, 1), \alpha - 1]^{\rho}) \). Since \( L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma)) \) does not have such factors, we conclude that the multiplicity of \( \delta([-\alpha, 1), \alpha - 1]^{\rho}) \otimes L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma)) \) in \( \mu^*(\delta([-\alpha, 1), \alpha - 1]^{\rho}) \times L([\alpha, \alpha + 1]^{\rho}; \delta([\alpha]^{\rho}; \sigma))) \) is is two. This contradiction completes the proof of the lemma.

\[ \square \]

**H.1. The case of exponents \( (\alpha, \alpha, \alpha) \) and \( \alpha \geq 1 \)**

Consider

\[ [\alpha]^{\rho} \times [\alpha]^{\rho} \times [\alpha]^{\rho} \times \sigma = [\alpha]^{\rho} \times [\alpha]^{\rho} \times L([\alpha]^{\rho}; \sigma) + [\alpha]^{\rho} \times [\alpha]^{\rho} \times \delta([\alpha]^{\rho}; \sigma). \]

The last representation on the right hand side is irreducible by (iv) of Proposition 6.1 in [73] and the factorization of the long intertwining operator in the Langlands classification. Applying the Aubert involution to this representation, we get that the other representation on the right hand side is irreducible.

Here no irreducible subquotient is unitarizable (we see this applying unitary parabolic reduction and using \( [\alpha]^{\rho} \times [\alpha]^{\rho} \times \delta([\alpha]^{\rho}; \sigma) \cong [\alpha]^{\rho} \times [-\alpha]^{\rho} \times \delta([\alpha]^{\rho}; \sigma)) \).

**I.1. The case of exponents \( (\alpha - 1, \alpha, \alpha + 1) \) and \( \alpha \geq \frac{3}{2} \).**

The representation \( [\alpha - 1]^{\rho} \times \delta([\alpha, \alpha + 1]^{\rho}) \times \sigma \) has a unique irreducible subrepresentation. It is a strongly positive square integrable representation (see [71] for details of the description of the classification of strongly positive representations convenient for this paper). We denote this representation by

\[ \delta_{s.p.}([\alpha - 1]^{\rho}, [\alpha, \alpha + 1]^{\rho}; \sigma). \]

**Proposition 3.6. Suppose**

\[ \alpha \geq \frac{3}{2}. \]
Consider the following decomposition of this representation (in the Grothendieck group)

\[ \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha, \alpha + 1]^{(\rho)}; \sigma), \quad L([\alpha + 1]^{(\rho)}, [\alpha - 1, \alpha]^{(\rho)}; \sigma), \quad L([\alpha - 1]^{(\rho)}; \delta_{s.p.}([\alpha, \alpha + 1]^{(\rho)}; \sigma)), \quad L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma), \]

are unitarizable.

(2) The representations

\[ L([\alpha + 1]^{(\rho)}; \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)), \quad L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma), \]

\[ L([\alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \delta_{s.p.}([\alpha]^{(\rho)}; \sigma)), \quad L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma), \]

are not unitarizable.

(3) Representations of (1) and (2) form the complete Jordan-Hölder composition series of

\[ [\alpha - 1]^{(\rho)} \times [\alpha]^{(\rho)} \times [\alpha + 1]^{(\rho)} \rtimes \sigma. \]

(4) In (1) and (2), the Aubert involution switches the representations in the same rows.

Proof of (1), (3) and (4). Note that the representation \([\alpha - 1]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma\) is regular (i.e. all the Jacquet modules of the representation are multiplicity free, including the representation itself).

Consider the following decomposition of this representation (in the Grothendieck group)

\[ [\alpha - 1]^{(\rho)} \rtimes [\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma = \]

\[ [\alpha - 1]^{(\rho)} \rtimes L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) \times [\alpha - 1]^{(\rho)} \rtimes L([\alpha + 1]^{(\rho)}, \delta([\alpha]^{(\rho)}; \sigma)) + \]

\[ [\alpha - 1]^{(\rho)} \rtimes L([\alpha, \alpha + 1]^{(\rho)}; \sigma) + [\alpha - 1]^{(\rho)} \rtimes \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma). \]

(3.34)

Note that all the irreducible subquotients of the first and the last representations on the right hand side of the above equality are unitarizable (since they are irreducible subquotients at the ends of complementary series). Further, the last induced representation on the right hand side of the above equation is reducible (use (v) of Proposition 6.1 of [73]). The ASS duality implies that also the first representation on the right hand side of the above equation is reducible.

We shall now list analyze all the possible irreducible square integrable subquotients of \([\alpha - 1]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma\). First observe that the square integrable subquotient (if exists) must be strongly positive (since the exponent 0 is not in the cuspidal support of the representation, and we are in the case of integral exponents). Since \(\alpha - 1\) is the exponent of a factor, and \(\alpha\) appears with multiplicity one in the supports of irreducible subquotients of \(s_{GL}([\alpha - 1]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma)\), we directly get that strongly positive subquotient must be \(\delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha, \alpha + 1]^{(\rho)}; \sigma)\). We also directly see that there are no other new irreducible tempered subquotients here (again since the exponent 0 is not in the cuspidal support of the whole induced representation, and we are in the case of integral exponents).

Now we easily see that (3) holds.
Using the properties of the Langlands classification (described in \(2.6\) and \(3\)), we get
\[
(3.35) \quad [\alpha - 1]^{(\rho)} \times \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma) =
L([\alpha - 1]^{(\rho)}; \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)) + \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha, \alpha + 1]^{(\rho)}; \sigma).
\]
Applying the Aubert involution, we get
\[
(3.36) \quad [\alpha - 1]^{(\rho)} \times L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) =
L([\alpha - 1]^{(\rho)}; \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma))^t + \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha, \alpha + 1]^{(\rho)}; \sigma)^t.
\]
These irreducible representations are unitarizable (since they are subquotients at the end of complementary series).

We easily see that in \(1\) and \(2\) each representation except possibly \(L([\alpha + 1]^{(\rho)}, [\alpha - 1, \alpha]^{(\rho)}; \sigma)\) has at least one factor with positive exponent. This implies
\[
\delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha, \alpha + 1]^{(\rho)}; \sigma)^t = L([\alpha + 1]^{(\rho)}, [\alpha - 1, \alpha]^{(\rho)}; \sigma).
\]
We know by \(2.7\) that
\[
L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) \leq [\alpha - 1]^{(\rho)} \times L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma).
\]
The above two facts imply
\[
(3.37) \quad [\alpha - 1]^{(\rho)} \times L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) =
L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) + L([\alpha + 1]^{(\rho)}, [\alpha - 1, \alpha]^{(\rho)}; \sigma),
\]
which further implies
\[
L([\alpha - 1]^{(\rho)}; \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma))^t = L([\alpha + 1]^{(\rho)}, [\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma).
\]
Therefore, we have proved \((1)\), and that the Aubert involution switches the representations in rows in \((1)\).

We know that representations \([\alpha - 1]^{(\rho)} \times L([\alpha, \alpha + 1]^{(\rho)}; \sigma)\) and \([\alpha - 1]^{(\rho)} \times L([\alpha + 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))\) are dual. Now \((3.3)\), \((3.36)\), \((3.37)\) and \(3\) imply that both these representations are of length two.

Consider now \([\alpha - 1]^{(\rho)} \times L([\alpha + 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))\). Observe that \((3.3)\) implies that representations in \((1)\) cannot be subquotients here. Now properties of standard modules (see \(2.6)\) and the multiplicity one of the whole induced representation imply
\[
(3.38) \quad [\alpha - 1]^{(\rho)} \times L([\alpha + 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) =
L([\alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) + L([\alpha + 1]^{(\rho)}; \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)).
\]
This further implies
\[
(3.39) \quad [\alpha - 1]^{(\rho)} \times L([\alpha, \alpha + 1]^{(\rho)}; \sigma) = L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) + L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma).
\]
Observe that in the Jacquet module of $L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)^t$ is $\delta([\alpha - 1, \alpha + 1]^{(\rho)})^t \otimes \sigma$, and that the last representation has multiplicity one in the whole Jacquet module. Further consider

$$L([\alpha + 1]^{(\rho)}; \delta, p, ([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)) \leftrightarrow [\alpha - 1]^{(\rho)} \otimes \delta, s, p, ([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)$$

$$\leftrightarrow [\alpha - 1]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times [\alpha]^{(\rho)} \times \sigma \cong [\alpha - 1]^{(\rho)} \times [\alpha]^{(\rho)} 
\times [\alpha + 1]^{(\rho)} \otimes \sigma.$$  

Therefore, we have in the Jacquet module of $L([\alpha + 1]^{(\rho)}; \delta, s, p, ([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma))$ the representation $[\alpha - 1]^{(\rho)} \otimes [\alpha]^{(\rho)} \otimes [\alpha + 1]^{(\rho)} \otimes \sigma$, which implies (by the transitivity of Jacquet modules) that in the Jacquet module of this representation is also $\delta([\alpha - 1, \alpha + 1]^{(\rho)})^t \otimes \sigma$. Therefore,

$$L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)^t = L([\alpha + 1]^{(\rho)}; \delta, s, p, ([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)),$$

which further implies

$$L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma)^t = L([\alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)).$$

Therefore, we have proved that the Aubert involution switches also representations in rows in (2).

In the following several lemma we shall prove that the representations in (2) are not unitarizable.

**Lemma 3.7.** For $\alpha \geq \frac{3}{2}$, the representations $L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)$ and $L([\alpha + 1]^{(\rho)}; \delta, s, p, ([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma))$ are not unitarizable.

**Proof.** First we shall prove the non-unitarizability of $L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)$. We know from (A2) from Theorem 4.1 of [37] that

$$\delta([\alpha - 1, \alpha + 1]^{(\rho)}) \times \sigma = L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma) + L([\alpha - 1]^{(\rho)}; \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)).$$

Now consider

$$\delta([-\alpha, \alpha]^{(\rho)}) \times \delta([\alpha - 1, \alpha + 1]^{(\rho)}) \times \sigma \geq \delta([-\alpha, \alpha + 1]^{(\rho)}) \times \delta([\alpha - 1, \alpha]^{(\rho)}) \times \sigma.$$

The left hand side representation above has (among others) the following irreducible subquotients

$$L([\alpha - 1, \alpha + 1]^{(\rho)}; \delta([-\alpha, \alpha]^{(\rho)}; \sigma)),$$

$$L([\alpha - 1, \alpha]^{(\rho)}; \delta([-\alpha, \alpha + 1]^{(\rho)}; \sigma)),$$

$$L([-\alpha, \alpha + 1]^{(\rho)}; [\alpha - 1, \alpha]^{(\rho)}; \sigma),$$

$$L([-\alpha, \alpha + 1]^{(\rho)}; [\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)).$$

For these irreducible subquotients we know that they are subquotients of $\delta([-\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)$, since they cannot be subquotients of

$$\delta([-\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1]^{(\rho)}; \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma)).$$
Therefore, \( \delta([\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma) \) is a representation of length at least 6.

We shall show that the multiplicity of
\[
\tau := \delta([\alpha, \alpha]^{(\rho)}) \otimes L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)
\]
in the Jacquet module of
\[
\pi := \delta([\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)
\]
is strictly smaller then 6. This will imply that \( L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma) \) is not unitarizable.

We shall use obvious relation
\[
\delta([\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma) \leq \delta([\alpha, \alpha]^{(\rho)}) \times \delta([\alpha - 1, \alpha + 1]^{(\rho)}) \times \sigma.
\]
Now
\[
\mu^*(\delta([\alpha, \alpha]^{(\rho)}) \times \delta([\alpha - 1, \alpha + 1]^{(\rho)}) \times \sigma) = \\
\left( \sum_{x=1}^{\alpha} \sum_{y=1}^{\alpha} \delta([-x, \alpha]^{(\rho)}) \times \delta([y + 1, \alpha]^{(\rho)}) \otimes \delta([x + 1, y]^{(\rho)}) \right) \times \\
\left( \sum_{i=1}^{\alpha + 1} \sum_{j=1}^{\alpha + 1} \delta([-i, -\alpha + 1]^{(\rho)}) \times \delta([j + 1, \alpha + 1]^{(\rho)}) \otimes \delta([i + 1, j]^{(\rho)}) \right) \times (1 \otimes \sigma).
\]
To be able to get \( \tau \) as a subquotient, we must take \( j = \alpha + 1 \), and what remains of that factor is \( \sum_{i=1}^{\alpha + 1} \delta([-i, -\alpha + 1]^{(\rho)}) \otimes \delta([i + 1, \alpha + 1]^{(\rho)}) \). One possibility to get \( \tau \) is to take \( i = \alpha - 2 \). Then we must take \( x = -\alpha - 1 = y \) or \( x = \alpha = y \). Each of these two cases gives multiplicity one of \( \tau \) (we are in the regular situation).

Suppose now that \( i > \alpha - 2 \). Then the formula for \( M^*(\delta([\alpha, \alpha]^{(\rho)}) \) implies that \( i = \alpha \).

Now we have two possibilities: \( x = -\alpha + 1, y = -\alpha + 1 \), or \( x = \alpha - 2, y = \alpha \). Again, each of these two cases gives multiplicity one of \( \tau \).

Therefore, the multiplicity of \( \tau \) in the Jacquet module of \( \delta([\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma) \) is at most 4. Now we conclude non-unitarizability of \( \delta([\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma) \) in a usual way.

Now applying the Aubert involution we get easily that \( L([\alpha + 1]^{(\rho)}; \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)) \) is not unitarizable.

**Lemma 3.8.** For \( \alpha \geq \frac{3}{2}, L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) \) is not unitarizable.
Proof. From (2.7) follows
\[ L(\alpha, \alpha + 1)(\rho), [\alpha - 1]^{(\rho)}; \delta([-\alpha, \alpha]^{(\rho)}); \sigma) \leq \delta([-\alpha, \alpha]^{(\rho)}) \times L(\alpha, \alpha + 1)(\rho), [\alpha - 1]^{(\rho)}; \sigma). \]

Recall (3.39)
\[ [\alpha - 1]^{(\rho)} \times L(\alpha, \alpha + 1)(\rho); \sigma) = L(\alpha, \alpha + 1)(\rho), [\alpha - 1]^{(\rho)}; \sigma) + L(\alpha - 1, \alpha + 1)^{(\rho)}; \sigma), \]
and also
\[ [\alpha - 1]^{(\rho)} \times [\alpha, \alpha + 1]^{(\rho)}; \sigma) = [\alpha - 1]^{(\rho)} \times L(\alpha, \alpha + 1)^{(\rho)}; \sigma) + [\alpha - 1]^{(\rho)} \times [\alpha, \alpha + 1]^{(\rho)}; \sigma). \]

Therefore
\[ [\alpha - 1]^{(\rho)} \times [\alpha, \alpha + 1]^{(\rho)}; \sigma) = L(\alpha, \alpha + 1)^{(\rho)}; [\alpha - 1]^{(\rho)}; \sigma) + [\alpha - 1]^{(\rho)} \times [\alpha, \alpha + 1]^{(\rho)}; \sigma). \]

which implies
\[ \delta([-\alpha, \alpha + 1]^{(\rho)}) \times [\alpha - 1]^{(\rho)} \times [\alpha]^{(\rho)}; \sigma) \leq \delta([-\alpha, \alpha]^{(\rho)}) \times [\alpha - 1]^{(\rho)} \times [\alpha, \alpha + 1]^{(\rho)}; \sigma) = \delta([-\alpha, \alpha]^{(\rho)}) \times L(\alpha, \alpha + 1)^{(\rho)}; [\alpha - 1]^{(\rho)}; \sigma) + \delta([-\alpha, \alpha]^{(\rho)}) \times L(\alpha - 1, \alpha + 1)^{(\rho)}; \sigma) + \delta([-\alpha, \alpha]^{(\rho)}) \times [\alpha - 1]^{(\rho)} \times [\alpha, \alpha + 1]^{(\rho)}; \sigma). \]

We have in (3.41) subquotients
\[ L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}, \delta([-\alpha, \alpha + 1]^{(\rho)}); \sigma), \]
\[ L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}, [-\alpha, \alpha + 1]^{(\rho)}; \sigma), \]
\[ L([-\alpha, \alpha + 1]^{(\rho)}; \delta([\alpha - 1]^{(\rho)}; [\alpha]^{(\rho)}; \sigma)). \]

By the properties of the Langlands classification 2.6, no one of the above four representations can be a subquotient of the representation in the last row of (3.40).

Suppose
\[ L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \delta([-\alpha, \alpha + 1]^{(\rho)}); \sigma) \leq \delta([-\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma). \]

Observe that
\[ L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \delta([-\alpha, \alpha + 1]^{(\rho)}); \sigma) \rightarrow L([-\alpha]^{(\rho)}, [-\alpha + 1]^{(\rho)} \times [\alpha - 1]^{(\rho)}; \sigma) \]
implies
\[ L([-\alpha]^{(\rho)}, [-\alpha + 1]^{(\rho)} \times [\alpha - 1]^{(\rho)}; \sigma) \leq \delta([-\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma). \]

Therefore
\[ (3.41) \quad L([-\alpha]^{(\rho)} + b) \otimes \sigma \leq \delta([-\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma) \]
for some multisegment b. In the same way we get the above conclusion if we suppose
\[ L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}, [-\alpha, \alpha + 1]^{(\rho)}; \sigma) \leq \delta([-\alpha, \alpha]^{(\rho)}) \times L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma). \]
If we suppose
\[ L([-\alpha, \alpha + 1]^{(\rho)}; \delta([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)) \leq \delta([-\alpha, \alpha]^{(\rho)}) \rtimes L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma), \]
then we get that
\[ L([\alpha - 1]^{(\rho)} + b) \otimes \sigma \leq s_{GL}(\delta([-\alpha, \alpha]^{(\rho)}) \rtimes L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)) \]
for some multisegment \( b \).

Consider the formula
\[
s_{GL}(\delta([-\alpha, \alpha]^{(\rho)}) \rtimes L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)) \leq \left( \sum_{x=-\alpha-1}^{\alpha} \sum_{y=x}^{\alpha} \delta([-x, \alpha]^{(\rho)}) \times \delta([x + 1, \alpha]^{(\rho)}) \right) \times \left( \sum_{i=\alpha-2}^{\alpha+1} \delta([-i, -\alpha + 1]^{(\rho)}) \times \delta([i + 1, \alpha + 1]^{(\rho)}) \otimes \sigma. \right)
\]
Obviously we have above neither subquotients of the form \( L([\alpha - 1]^{(\rho)} + b) \otimes \sigma \) nor of \( L([-\alpha]^{(\rho)} + b) \otimes \sigma \).

From this follows that \( \delta([-\alpha, \alpha]^{(\rho)}) \rtimes L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) \) is a representation of length at least 6. Now we complete the proof of non-unitarizability of \( L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) \) directly from the following lemma and [2.15] \( \square \)

**Lemma 3.9.** For \( \alpha \geq \frac{3}{2} \), the multiplicity of \( \tau := \delta([-\alpha, \alpha]^{(\rho)}) \otimes L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) \) in \( \mu^*(\pi) \) where
\[ \pi := \delta([-\alpha, \alpha]^{(\rho)}) \rtimes L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma), \]
is at most 4.

**Proof.** We have proved earlier
\[ L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) = [\alpha - 1]^{(\rho)} \rtimes L([\alpha, \alpha + 1]^{(\rho)}; \sigma) - L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma). \]

We shall now analyze which term of \( \mu^*(L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma)) \) can yield to \( \tau \) as a subquotient. If we take the term \( 1 \otimes L([\alpha, \alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma) \), this will result with multiplicity two of \( \tau \). It remains to consider the terms of the form \( \gamma \otimes - \) where \( \gamma \) is a representation of \( GL(k) \) with \( k > 0 \).

First write the following formulas
\[
\mu^*([\alpha - 1]^{(\rho)} \rtimes L([\alpha, \alpha + 1]^{(\rho)}; \sigma)) = (1 \otimes [\alpha - 1]^{(\rho)} + [\alpha - 1]^{(\rho)} \otimes 1 + [-\alpha + 1]^{(\rho)} \otimes 1) \times \left( 1 \otimes L([\alpha, \alpha + 1]^{(\rho)}; \sigma) \right) \]
\[
- \alpha]^{(\rho)} \otimes [\alpha + 1]^{(\rho)} \rtimes \sigma + [\alpha + 1]^{(\rho)} \otimes L([\alpha]^{(\rho)}, \sigma) + \]

A short analyses implies that the only possibility to get \( \tau \). Now consider the term in the second sum corresponding to \( i \). Then Lemma 3.10. Let

\[
\mu^*(L([\alpha - 1, \alpha + 1]^{(\rho)}; \sigma)) = \\
\left( \sum_{a-2 \leq i \leq a+1} \sum_{i+1 \leq j \leq a+1} L([-i, -\alpha + 1]^{(\rho)}, [j + 1, \alpha + 1]^{(\rho)}) \otimes L_{\alpha}([i + 1, j]^{(\rho)}; \sigma) + \\
+ \sum_{i=\alpha}^{a+1} L([-i, -\alpha + 1]^{(\rho)}, [i + 1, \alpha + 1]^{(\rho)}) \otimes \sigma \right).
\]

Recall the formula for \( M^*(\delta([-\alpha, \alpha]^{(\rho)})) \) (which we already wrote before)

\[
M^*(\delta([-\alpha, \alpha]^{(\rho)})) = \sum_{x=-\alpha-1}^{\alpha-1} \sum_{y=x}^{\alpha-1} \delta([-x, \alpha]^{(\rho)}) \times \delta([y + 1, \alpha]^{(\rho)}) \otimes \delta([x + 1, y]^{(\rho)}).
\]

A short analyses implies that the only possibility to get \( \tau \) is to take from \( \mu^*([\alpha - 1]^{(\rho)} \times L([\alpha, \alpha + 1]^{(\rho)}; \sigma)) \) the following terms

\[
[-\alpha]^{(\rho)} \otimes [\alpha - 1]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times \sigma, \quad [-\alpha]^{(\rho)} \times [-\alpha + 1]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times \sigma.
\]

Now consider the term in the second sum corresponding to \( i = \alpha, j = \alpha + 1: \delta([-\alpha, -\alpha + 1]^{(\rho)} \otimes [\alpha + 1]^{(\rho)} \times \sigma. \) After subtraction the last term from the second term above, we get

\[
L([-\alpha]^{(\rho)}, [-\alpha + 1]^{(\rho)}) \otimes [\alpha + 1]^{(\rho)} \times \sigma \text{ which cannot yield } \tau
\]

Therefore, we are left with \( [-\alpha]^{(\rho)} \otimes [\alpha - 1]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times \sigma. \) Now we have precisely two possibilities to take terms from \( M^*(\delta([-\alpha, \alpha]^{(\rho)})) \) which can give \( \tau: x = \alpha - 1, y = \alpha \text{ and } x = -\alpha - 1, y = -\alpha. \) In both cases we get multiplicity one of \( \tau. \) This completes the proof of total multiplicity 4 of \( \tau. \)

Applying the Aubert involution, we now get, similarly as before when we have proved non-unitarizability for the dual representation, that \( L([\alpha + 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \delta_{s.p.}([\alpha]^{(\rho)}; \sigma)) \) is not unitarizable.

**J.1. The case of exponents \((\alpha - 1, \alpha, \alpha)\) and \(\alpha \geq \frac{3}{2}\).**

**Lemma 3.10.** Let

\[
\alpha \geq \frac{3}{2}.
\]

Then

1. The following irreducible representations

\[
L([\alpha]^{(\rho)}; \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)), \quad L([\alpha]^{(\rho)}, [\alpha - 1, \alpha]^{(\rho)}; \sigma), \\
L([\alpha - 1, \alpha]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)), \quad L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma), \\
L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))
\]
are all the possible irreducible subquotients of
\([\alpha - 1](\rho) \times [\alpha](\rho) \times [\alpha](\rho) \rtimes \sigma\).

(2) The involution switches the representations in the first two rows in (1), and fixes the representation in the third row of (1).

(3) We have
\[
L([\alpha](\rho), [\alpha - 1, \alpha](\rho); \sigma) = [\alpha](\rho) \rtimes L([\alpha - 1, \alpha](\rho); \sigma),
\]
\[
L([\alpha](\rho), \delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma)) = [\alpha](\rho) \times \delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma),
\]
\[
L([\alpha - 1, \alpha](\rho); \delta([\alpha](\rho); \sigma)) = \delta([\alpha - 1, \alpha](\rho)) \rtimes \delta([\alpha](\rho); \sigma),
\]
\[
L([\alpha - 1](\rho), [\alpha](\rho), [\alpha](\rho); \sigma) \cong L([\alpha - 1](\rho), [\alpha](\rho)) \rtimes L([\alpha](\rho); \sigma).
\]

Proof. The classification of irreducible square integrable representations modulo cuspidal data implies (1).

Recall
\[\text{Jord}_\rho(\delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma)) = \{2(\alpha - |\alpha|) + 1, 2(\alpha - |\alpha|) + 3, \ldots, 2\alpha - 5, 2\alpha - 1, 2\alpha + 1\},\]
and that the partially defined function \(\epsilon\) attached to the above square integrable representation is different on \(2\alpha - 1\) and \(2\alpha + 1\). Now
\[[\alpha](\rho) \times \delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma)\]
is irreducible by (vi) of Proposition 6.1 from [73]. Further, the ASS involution implies that
\[[\alpha](\rho) \rtimes L([\alpha - 1, \alpha](\rho); \sigma)\]
is irreducible. This implies that the first two equalities in (2) hold (using 2.47).

This also implies that the involution switches the representations in the first row of (1) (since \(\delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma)^t = L([\alpha - 1, \alpha](\rho); \sigma))

Looking at the GL-type of Jacquet module of the whole induced representation, we get that the multiplicity of \(L([\alpha](\rho), [\alpha - 1, \alpha](\rho); \sigma)\) in the whole induced representation is one. Therefore, also its ASS dual has multiplicity one in the whole induced representation.

Considering the properties of the standard module in the Langlands classification, we see that \(\delta([\alpha - 1, \alpha](\rho)) \rtimes \delta([\alpha](\rho); \sigma)\) is irreducible. Therefore, the third equality in (3) holds.

Applying the ASS involution we get also that the fourth equality holds.

This implies that the ASS involution switches representations in the second row of (1), and that fixes the one in the third row. \(\square\)
Lemma 3.11. For \( \alpha \geq \frac{3}{2} \), the representations
\[
L([\alpha](\rho); \delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma)), \quad L([\alpha](\rho), [\alpha - 1, \alpha](\rho); \sigma),
\]
\[
L([\alpha - 1, \alpha](\rho); \delta([\alpha](\rho); \sigma)), \quad L([\alpha](\rho), [\alpha - 1](\rho), [\alpha](\rho); \sigma),
\]
are not unitarizable.

Proof. Since \( L([\alpha + 1](\rho); \delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma)) \) is not unitarizable, we get that
\[
L([\alpha](\rho); \delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma)) = [\alpha](\rho) \times \delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma)
\]
is not unitarizable (because of the ends of complementary series).

Note that \(\alpha + 1\) and \(\alpha - 1\) are irreducible subquotients. Thus
\[
L([\alpha + 1](\rho); \delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma)) \leq [\alpha + 1] \times \delta_{s.p.}([\alpha - 1](\rho), [\alpha](\rho); \sigma) = ([\alpha + 1] \times L([\alpha - 1, \alpha](\rho); \sigma))^t.
\]

Applying the ASS involution we get
\[
L([\alpha - 1, \alpha + 1](\rho); \sigma) \leq [\alpha + 1] \times L([\alpha - 1, \alpha](\rho); \sigma).
\]
Now \(L([\alpha](\rho), [\alpha - 1, \alpha](\rho); \sigma) = [\alpha](\rho) \times L([\alpha - 1, \alpha](\rho); \sigma)\) is not unitarizable (since \(L([\alpha - 1, \alpha + 1](\rho); \sigma)\) is not).

For
\[
\delta([\alpha - 1, \alpha](\rho)) \times \delta([\alpha](\rho); \sigma) = L([\alpha - 1, \alpha](\rho); \sigma)
\]
and
\[
L([\alpha - 1](\rho), [\alpha](\rho)) \times L([\alpha](\rho); \sigma) = L([\alpha - 1](\rho), [\alpha][\alpha](\rho); \sigma).
\]
on-unitarizability follows from the fact that for exponents we do not have unitarizable irreducible subquotients. \(\square\)

Remark 3.12. (1) Observe that (2.7) implies
\[
[\alpha - 1](\rho) \times [\alpha](\rho) \times [\delta([\alpha](\rho); \sigma) \geq L([\alpha - 1, \alpha](\rho); \sigma). \]

Now the previous lemma implies that the representation on the left hand side of the above inequality contains an irreducible subquotient which is not unitarizable. Further, the previous lemma and (2) of Lemma 3.10 imply that the same claim holds for ASS involution of the left hand side of the above inequality.

(2) Let \( \theta \) and \( \tau \) be any irreducible subquotients of \([\alpha - 1](\rho) \times [\alpha](\rho) \) and \([\alpha](\rho) \times [\sigma] \) respectively. Then similarly as above one gets that \( \theta \times [\sigma] \) contains an irreducible subquotient which is not unitarizable.

We are very thankful to C. Moeglin, who has informed us that the representation
\[
L([\alpha](\rho), [\alpha - 1](\rho), [\delta([\alpha](\rho); \sigma))
\]
is in an Arthur packet (15). Therefore, it is unitarizable.
Appendix to J.1 For the purpose of this paper, we do not need to study the representation further, and one can skip now directly to K.1. Nevertheless, since this is pretty distinguished representation, we shall do some additional analysis of this it (the Jacquet modules of this representation) which might be of interest in some calculations in the future.

Consider

$$[\alpha]^p \times [\alpha - 1]^p \times [\alpha]^p \times \sigma = L([\alpha]^p, [\alpha - 1]^p) \times L([\alpha]^p; \sigma) + \delta([\alpha - 1, \alpha]^p) \times L([\alpha]^p; \sigma)$$

$$+ L([\alpha]^p, [\alpha - 1]^p) \times \delta([\alpha]^p; \sigma) + \delta([\alpha - 1, \alpha]^p) \times \delta([\alpha]^p; \sigma).$$

Looking at $s_{GL}$ of the whole induced representation, we get that the multiplicity of the representation $L([\alpha]^p, [\alpha - 1]^p; \delta([\alpha]^p; \sigma))$ in the whole induced representation is at most two.

All the above discussion implies

$$\delta([\alpha - 1, \alpha]^p) \times L([\alpha]^p; \sigma) = L([\alpha - 1, \alpha]^p, [\alpha]^p; \sigma) + L([\alpha]^p, [\alpha - 1]^p; \delta([\alpha]^p; \sigma))$$

$$(\alpha^p, [\alpha - 1]^p) \times \delta([\alpha]^p; \sigma) = L([\alpha]^p, [\alpha - 1]^p; \delta([\alpha]^p; \sigma) + L([\alpha]^p; \delta_{s.p.}([\alpha - 1]^p, [\alpha]^p; \sigma)).$$

Observe that

$$L([\alpha]^p, [\alpha - 1]^p; [\alpha]^p; \sigma) \leq \delta([\alpha]^p; \sigma)) = ([\alpha]^p \times L([\alpha - 1]^p; [\alpha]^p; \sigma))$$

Passing to the ASS dual, we get

$$L([\alpha - 1, \alpha]^p; \delta([\alpha]^p; \sigma)) \leq [\alpha]^p \times L([\alpha - 1]^p; \delta([\alpha]^p; \sigma)).$$

From this we conclude

$$(3.44) \quad [\alpha]^p \times L([\alpha - 1]^p; \delta([\alpha]^p; \sigma)) =$$

$$L([\alpha]^p, [\alpha - 1]^p; \delta([\alpha]^p; \sigma)) + L([\alpha - 1, \alpha]^p; \delta([\alpha]^p; \sigma))$$

$$+ L([\alpha]^p, [\alpha - 1]^p, [\alpha]^p; \sigma) =$$

$$L([\alpha]^p, [\alpha - 1]^p; [\alpha]^p; \sigma) + L([\alpha]^p, [\alpha - 1]^p; \delta([\alpha]^p; \sigma)).$$

Now we shall start the computation of the Jacquet module of the representation $$(3.43).$$

From (3.44), we know that $[\alpha]^p \times L([\alpha - 1]^p; \delta([\alpha]^p; \sigma))$ reduces. Further we know that $L([\alpha]^p, [\alpha - 1]^p; \delta([\alpha]^p; \sigma))$ is a subquotient. We have\(^{19}\)

$^{19}$We shall explain later what means boxed and dash boxed terms below.
\[
\mu^*([\alpha]^{(\rho)} \times L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))) = (1 \otimes [\alpha]^{(\rho)} + [\alpha]^{(\rho)} \otimes 1 + [-\alpha]^{(\rho)} \otimes 1) \times \\
\left(1 \otimes L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) + [\alpha]^{(\rho)} \otimes [\alpha - 1]^{(\rho)} \times \sigma + [-\alpha + 1]^{(\rho)} \otimes \delta([\alpha]^{(\rho)}; \sigma) + [-\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \otimes \sigma + \delta([\alpha - 1, \alpha]^{(\rho)}) \otimes \sigma \right) = \\
[\alpha]^{(\rho)} \times L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) + [-\alpha]^{(\rho)} \otimes L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) + \\
[\alpha]^{(\rho)} \otimes [\alpha - 1]^{(\rho)} \times \sigma + [\alpha]^{(\rho)} \times [-\alpha + 1]^{(\rho)} \otimes \delta([\alpha]^{(\rho)}; \sigma) + \\
[-\alpha]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times \sigma + [-\alpha]^{(\rho)} \times [-\alpha + 1]^{(\rho)} \otimes \delta([\alpha]^{(\rho)}; \sigma) + \\
[-\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \times \sigma + \delta([\alpha - 1, \alpha]^{(\rho)}) \otimes [\alpha]^{(\rho)} \times \sigma \\
= 1 \otimes [\alpha]^{(\rho)} \times L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) + \\
[\alpha]^{(\rho)} \otimes L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) + [-\alpha]^{(\rho)} \otimes L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) + \\
[\alpha]^{(\rho)} \otimes L([\alpha - 1]^{(\rho)}; [\alpha]^{(\rho)}; \sigma) + [\alpha]^{(\rho)} \otimes \delta_{s,p}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) + \\
[-\alpha + 1]^{(\rho)} \otimes [\alpha]^{(\rho)} \times \sigma \right)^{[2+]} + \\
[\alpha]^{(\rho)} \times [\alpha]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times \sigma + [\alpha]^{(\rho)} \times [-\alpha + 1]^{(\rho)} \otimes \delta([\alpha]^{(\rho)}; \sigma) + \\
[-\alpha]^{(\rho)} \times [\alpha]^{(\rho)} \otimes [\alpha - 1]^{(\rho)} \times \sigma + \\
L([-\alpha]^{(\rho)}, [-\alpha + 1]^{(\rho)}) \otimes \delta([\alpha]^{(\rho)}; \sigma) + \delta([-\alpha, -\alpha + 1]^{(\rho)}) \otimes \delta([\alpha]^{(\rho)}; \sigma) + \\
[-\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \otimes \delta([\alpha]^{(\rho)}; \sigma) + [-\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \otimes L([\alpha]^{(\rho)}; \sigma) + \\
\delta([\alpha - 1, \alpha]^{(\rho)}) \otimes \delta([\alpha]^{(\rho)}; \sigma) + \delta([\alpha - 1, \alpha]^{(\rho)}) \otimes L([\alpha]^{(\rho)}; \sigma) + 
\]
Let now \(\pi\) module of 50 MARKO TADIĆ gives that Now a simple analysis using the above composition series (which we shall not write here) non-trivial Jacquet modules (and transitivity of them) imply that its Jacquet module the boxed term with super script \([1]\). An easy analysis of the minimal \([2+]\) in its Jacquet module. Since \([2]\) has \([\alpha-1]^{(\rho)}\) has in the Jacquet module of \(\pi\) have in the Jacquet module of \(\pi\). The above formula implies that \(\pi\) module. The above formula implies that \(\pi\) module. The above formula implies that \(\pi\) module.

\[
L([-\alpha]^{(\rho)}, [-\alpha + 1]^{(\rho)}) \times \sigma + \delta([-\alpha, -\alpha + 1]^{(\rho)}) \times \sigma 
\]

\[
[-\alpha]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma
\]

\[
\delta((\alpha - 1, \alpha)\rho) \times \sigma
\]

We start with an irreducible subquotient \(\pi_1\) of \([\alpha]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma\) which has in its Jacquet module the boxed term with super script \([1]\). An easy analysis of the minimal non-trivial Jacquet modules (and transitivity of them) imply that \(\pi_2\) must have \([2]\) and \([2+]\) in its Jacquet module. Since \([2]\) has \([\alpha-1]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma\) in its Jacquet module, \([3]\) must be also in. From \([2+]\) we conclude that \([3+]\) must be in the Jacquet module of \(\pi_1\).

Let now \(\pi_2\) be the irreducible subquotient which contains dash boxed term (1) in its Jacquet module. The above formula implies that \(\pi_2 = L((\alpha)\rho, [\alpha-1]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma\). From (1) and previous formulas for Jacquet modules of \(L([\alpha-1]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma\) we see that we must have in the Jacquet module of \(\pi_2\) the terms

\[
[-\alpha]^{(\rho)} \times [-\alpha + 1]^{(\rho)} \times \sigma, [-\alpha]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times \alpha + 1]^{(\rho)} \times \sigma, [-\alpha]^{(\rho)} \times [\alpha]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times \sigma.
\]

From this we see that \([2]\) and \([2+]\) must be in the Jacquet module of \(\pi_2\).

We have got \(s_{GL}(\pi_2)\). Observe that the semi-simplification of the minimal non-trivial Jacquet modules of \(\pi_2\) is

\[
\begin{align*}
[\alpha]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times [-\alpha]^{(\rho)} \times \sigma, \\
[\alpha]^{(\rho)} \times [-\alpha]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times \sigma, \\
[\alpha]^{(\rho)} \times [-\alpha]^{(\rho)} \times [-\alpha + 1]^{(\rho)} \times \sigma, \\
[-\alpha]^{(\rho)} \times [\alpha]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times \sigma, \\
[-\alpha]^{(\rho)} \times [\alpha]^{(\rho)} \times [-\alpha + 1]^{(\rho)} \times \sigma, \\
[-\alpha]^{(\rho)} \times [-\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \times \sigma.
\end{align*}
\]

Now a simple analysis using the above composition series (which we shall not write here) gives that

\[
\mu^*(L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma)) =
\]

\[
1 \times L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma) + \\
[-\alpha]^{(\rho)} \times L([\alpha - 1]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma) + [\alpha]^{(\rho)} \times L([\alpha - 1]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma) + \\
[-\alpha]^{(\rho)} \times [\alpha]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times \sigma + L([-\alpha]^{(\rho)}, [-\alpha + 1]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma) + \\
\delta((\alpha - 1, \alpha)\rho) \times L([\alpha]^{(\rho)} \times \sigma) + \\
L([-\alpha]^{(\rho)}, [-\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \times \sigma + [-\alpha]^{(\rho)} \times \delta((\alpha - 1, \alpha)\rho) \times \sigma.
\]
K.1. The case of exponents \((\alpha - 1, \alpha - 1, \alpha)\) and \(\alpha \geq 2\).

Consider

\[
[\alpha - 1]^{(\rho)} \times [\alpha - 1]^{(\rho)} \rtimes L([\alpha]^{(\rho)}; \sigma) = [\alpha - 1]^{(\rho)} \rtimes L([\alpha]^{(\rho)}; [\alpha - 1]^{(\rho)}; \sigma)
\]

\[
+ [\alpha - 1]^{(\rho)} \rtimes L([\alpha - 1, \alpha]^{(\rho)}; \sigma),
\]

\[
[\alpha - 1]^{(\rho)} \times [\alpha - 1]^{(\rho)} \rtimes \delta([\alpha]^{(\rho)}; \sigma) = [\alpha - 1]^{(\rho)} \rtimes L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))
\]

\[
+ [\alpha - 1]^{(\rho)} \rtimes \delta_{s.p.}([\alpha - 1]^{(\rho)}; [\alpha]^{(\rho)}; \sigma).
\]

Recall again

\[
\delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)^t = L([\alpha - 1, \alpha]^{(\rho)}; \sigma),
\]

\[
L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma)^t = L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)),
\]

and

\[
Jord_{\rho}(\delta_{s.p.}([\alpha]^{(\rho)}; \sigma))) = \{2(\alpha - [\alpha]) + 1, 2(\alpha - [\alpha]) + 3, \ldots, 2\alpha - 3, 2\alpha + 1\},
\]

\[
Jord_{\rho}(\delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma))) = \{2(\alpha - [\alpha]) + 1, 2(\alpha - [\alpha]) + 3, \ldots, 2\alpha - 5, 2\alpha - 1, 2\alpha + 1\},
\]

where in the last case partially defined function \(\epsilon\) attached to the square integrable representation is different on \(2\alpha - 1\) and \(2\alpha + 1\).

Now \([\alpha - 1]^{(\rho)} \rtimes \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)\) is irreducible by (iv) of Proposition 6.1 from 73. Further, the ASS involution implies that \([\alpha - 1]^{(\rho)} \rtimes L([\alpha - 1, \alpha]^{(\rho)}; \sigma)\) is irreducible.

All the possible irreducible subquotients here are

\[
L([\alpha - 1]^{(\rho)}; \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)),
\]

\[
L([\alpha - 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)),
\]

\[
L([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma).
\]

Consider \([\alpha - 1]^{(\rho)} \times L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))\). If this is not irreducible, the properties of the Langlands classification would imply that

\[
[\alpha - 1]^{(\rho)} \rtimes \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) \leq [\alpha - 1]^{(\rho)} \rtimes L([\alpha - 1]^{(\rho)}; \delta_{s.p.}([\alpha]^{(\rho)}; \sigma)),
\]

which would imply further

\[
([\alpha - 1]^{(\rho)} + [-\alpha + 1]^{(\rho)}) \times L([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)} \otimes \sigma \leq
\]

\[
([\alpha - 1]^{(\rho)} + [-\alpha + 1]^{(\rho)}) \times \left([-\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} + \delta([\alpha - 1, \alpha]^{(\rho)})\right) \otimes \sigma.
\]

This is impossible (consider \([\alpha - 1]^{(\rho)} \times L([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)} \otimes \sigma)\).

Therefore, \([\alpha - 1]^{(\rho)} \times L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))\) is irreducible, which implies by ASS involution that \([\alpha - 1]^{(\rho)} \rtimes L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \sigma)\) is irreducible.
Lemma 3.13. Let \( \alpha \geq 2 \).

Then no irreducible subquotient of

\[ [\alpha - 1]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma \]

is unitarizable.

Proof. Recall that each irreducible subquotient above can be written as \([\alpha - 1]^{(\rho)} \rtimes \tau\), where \(\tau\) is an irreducible subquotient of \([\alpha - 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma\).

We have seen that the following two representations

\[
L([\alpha]^{(\rho)}; \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)) = [\alpha]^{(\rho)} \rtimes \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma),
\]

\[
L([\alpha]^{(\rho)}, [\alpha - 1, \alpha]^{(\rho)}; \sigma) = [\alpha]^{(\rho)} \rtimes L([\alpha - 1, \alpha]^{(\rho)}; \sigma)
\]

are not unitarizable. This implies that also the following representations

\[
L([\alpha - 1]^{(\rho)}, \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)) = [\alpha - 1]^{(\rho)} \rtimes \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma),
\]

\[
L([\alpha - 1]^{(\rho)}, [\alpha - 1, \alpha]^{(\rho)}; \sigma) = [\alpha - 1]^{(\rho)} \rtimes L([\alpha - 1, \alpha]^{(\rho)}; \sigma)
\]

are not unitarizable.

It remains to consider

\[
L([\alpha - 1]^{(\rho)}, [\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)) \cong [\alpha - 1]^{(\rho)} \rtimes L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)),
\]

\[
L([\alpha - 1]^{(\rho)}, [\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma) \cong [\alpha - 1]^{(\rho)} \rtimes L([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma).
\]

Now we can deform \(a - 1\) to \(a\) in both representations. If the staring representation is unitarizable, then all irreducible subquotients of deformed representation must be unitarizable. For the second representation, this is not the case (since \(L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)\) is subquotient there by 2.7, and we have seen already that this representation is not unitarizable).

We shall now see also that the first representation has an irreducible subquotient which is not unitarizable. Recall that we have proved that

\[
L([\alpha - 1, \alpha]^{(\rho); \sigma}) \leq [\alpha]^{(\rho)} \rtimes L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma)).
\]

Since we have seen that \(L([\alpha - 1, \alpha]^{(\rho); \sigma})\) is not unitarizable, we conclude that \([\alpha - 1]^{(\rho)} \times L([\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))\) is not unitarizable. This completes the proof of non-unitarizability. \(\square\)
L.1. The case of exponents \((\alpha - 2, \alpha - 1, \alpha)\) and \(\alpha \geq \frac{5}{2}\).

All the irreducible subquotients of the induced representation

\[
[\alpha - 2]^{(\rho)} \times [\alpha - 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma.
\]

are unitarizable (since they are subquotients of the ends of complementary series). They are:

\[
L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}, [\alpha - 2]^{(\rho)}; \sigma), \quad L([\alpha]^{(\rho)}, [\alpha - 2, \alpha - 1]^{(\rho)}; \sigma),
\]

\[
L([\alpha - 1, \alpha]^{(\rho)}, [\alpha - 2]^{(\rho)}; \sigma), \quad L([\alpha - 2, \alpha]^{(\rho)}; \sigma),
\]

\[
L([\alpha - 1]^{(\rho)}, [\alpha - 2]^{(\rho)} \delta([\alpha]^{(\rho)}; \sigma)), \quad L([\alpha - 2, \alpha - 1]^{(\rho)} \delta([\alpha]^{(\rho)}; \sigma)),
\]

\[
L([\alpha - 2]; \delta_{s.p.}([\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma)), \quad \delta_{s.p.}([\alpha - 2], [\alpha - 1]^{(\rho)}, [\alpha]^{(\rho)}; \sigma).
\]

4. Unitarizability for \(\alpha = 2\) not covered by section 3.

All the cases A.1 – K.1 apply also to this reducibility. The case L.1 does not apply to this reducibility. Here we have instead of L.1 we have L.2:

4.1. Generalized rank three case.

L.2. The case of exponents \((0, 1, 2)\) and \(\alpha = 2\).

All the irreducible subquotients of the induced representation

\[
[0]^{(\rho)} \times [1]^{(\rho)} \times [2]^{(\rho)} \rtimes \sigma.
\]

are unitarizable (since they are subquotients of the ends of complementary series). They are:

\[
L([2]^{(\rho)}, [0, 1]^{(\rho)}; \sigma), \quad L([0, 2]^{(\rho)}; \sigma),
\]

\[
L([2]^{(\rho)}, [1]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma), \quad L([1, 2]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma),
\]

\[
L([1]^{(\rho)}; [0]^{(\rho)} \rtimes \delta([2]^{(\rho)}; \sigma)), \quad L([0, 1]^{(\rho)}; \delta([2]^{(\rho)}; \sigma)),
\]

\[
\tau([0]^{(\rho)}_{\pm}; \delta_{s.p.}([1]^{(\rho)}, [2]^{(\rho)}; \sigma)).
\]

5. Unitarizability for \(\alpha = \frac{3}{2}\) not covered by section 3.

All the cases A.1 – J.1 apply also to this reducibility. Here K and L cases are equivalent (therefore we drop L case). Therefore, we shall consider now K case only, and instead of K.1 we shall have K.3.
5.1. **Generalized rank three case.**

**K.3. The case of exponents** \( \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right) \) **and** \( \alpha = \frac{3}{2} \).

All the irreducible subquotients of the induced representation

\[ \left[ \frac{1}{2} \right]^{(\rho)} \times \left[ \frac{1}{2} \right]^{(\rho)} \times \left[ \frac{3}{2} \right]^{(\rho)} \rtimes \sigma \]

are unitarizable (since they are at the ends of the complementary series, or since they are all subrepresentations of unitarily induced representations). They are

\[
L\left(\left[ \frac{1}{2} \right]^{(\rho)}, \left[ \frac{1}{2} \right]^{(\rho)}, \left[ \frac{3}{2} \right]^{(\rho)}; \sigma\right), \quad L\left(\left[-\frac{1}{2}, \frac{3}{2}\right]^{(\rho)}; \sigma\right), \quad L\left(\left[ \frac{1}{2}, \frac{3}{2} \right]^{(\rho)}, \left[ \frac{1}{2} \right]^{(\rho)}; \sigma\right), \quad L\left(\left[ \frac{3}{2} \right]^{(\rho)}; \delta\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^{(\rho)}\right) \rtimes \sigma\right),
\]

\[
L\left(\left[ \frac{1}{2} \right]^{(\rho)}, \delta_{s.p.}\left(\left[ \frac{1}{2} \right]^{(\rho)}, \left[ \frac{3}{2} \right]^{(\rho)}; \sigma\right)\right), \quad L\left(\left[ \frac{1}{2} \right]^{(\rho)}, \delta_{s.p.}\left(\left[ \frac{3}{2} \right]^{(\rho)}; \sigma\right)\right), \quad \tau\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^{(\rho)}; \delta_{s.p.}\left(\left[ \frac{3}{2} \right]^{(\rho)}; \sigma\right)\right).
\]

### 6. Unitarizability for \( \alpha = 1 \) not covered by section [3]

In the rank one case A.1 applies for \( \alpha = 1 \). Now we shall write differences in ranks 2 and 3.

#### 6.1. **Generalized rank two case.**

Here B.1 and C.1 apply to \( \alpha = 1 \). Instead of D.1 we have

**D.4. The case of exponents** \( (0, 1) \) **and** \( \alpha = 1 \).

Now

\[
\left[0\right]^{(\rho)} \times \left[1\right]^{(\rho)} \rtimes \sigma = \left[0\right]^{(\rho)} \times L\left(\left[1\right]^{(\rho)}; \sigma\right) + \left[0\right]^{(\rho)} \times \delta\left(\left[1\right]^{(\rho)}; \sigma\right),
\]

where

\[
\left[0\right]^{(\rho)} \times L\left(\left[1\right]^{(\rho)}; \sigma\right) = L\left(\left[1\right]^{(\rho)}; \left[0\right]^{(\rho)} \rtimes \sigma\right) + L\left(\left[0, 1\right]^{(\rho)}; \sigma\right),
\]

\[
\left[0\right]^{(\rho)} \times \delta\left(\left[1\right]^{(\rho)}; \sigma\right) = \tau\left(\left[0\right]^{(\rho)}; \delta\left(\left[1\right]^{(\rho)}; \sigma\right)\right) + \tau\left(\left[0\right]^{(\rho)}; \delta\left(\left[1\right]^{(\rho)}; \sigma\right)\right).
\]

Further

\[
\delta\left(\left[0, 1\right]^{(\rho)} \rtimes \sigma\right) = \tau\left(\left[0\right]^{(\rho)}; \delta\left(\left[1\right]^{(\rho)}; \sigma\right)\right) + L\left(\left[0, 1\right]^{(\rho)}; \sigma\right),
\]

\[
L\left(\left[0\right]^{(\rho)}, \left[1\right]^{(\rho)} \rtimes \sigma\right) = L\left(\left[1\right]^{(\rho)}; \left[0\right]^{(\rho)} \rtimes \sigma\right) + \tau\left(\left[0\right]^{(\rho)}; \delta\left(\left[1\right]^{(\rho)}; \sigma\right)\right).
\]

All the irreducible subquotients here are unitarizable (since they are at the end of complementary series).

In the above decompositions, the first representations on the right hand side are dual, and also the second representations are dual, i.e.

\[
\tau\left(\left[0\right]^{(\rho)}; \delta\left(\left[1\right]^{(\rho)}; \sigma\right)\right)^t = L\left(\left[1\right]^{(\rho)}; \left[0\right]^{(\rho)} \rtimes \sigma\right),
\]

\[
L\left(\left[0, 1\right]^{(\rho)}; \sigma\right)^t = \tau\left(\left[0\right]^{(\rho)}; \delta\left(\left[1\right]^{(\rho)}; \sigma\right)\right).
\]
Further,
\[
\mu^*(\tau([0]_+^{[\rho]}; \delta([1]^{[\rho]}; \sigma))) = 1 \otimes \tau([0]_+^{[\rho]}; \delta([1]^{[\rho]}; \sigma)) + [1]^{[\rho]} \otimes [0]^{[\rho]} \otimes \sigma + [0]^{[\rho]} \otimes \delta([1]^{[\rho]}; \sigma) + 2\delta([0,1]^{[\rho]} \otimes \sigma + L([0]^{[\rho]}, [1]^{[\rho]} \otimes \sigma,
\]
\[
\mu^*(\tau([0]_-^{[\rho]}; \delta([1]^{[\rho]}; \sigma))) = 1 \otimes \tau([0]_-^{[\rho]}; \delta([1]^{[\rho]}; \sigma)) + [0]^{[\rho]} \otimes \delta([1]^{[\rho]}; \sigma) + L([0]^{[\rho]}, [1]^{[\rho]} \otimes \sigma,
\]
\[
\mu^*(L([1]^{[\rho]}; [0]^{[\rho]} \otimes \sigma)) = 1 \otimes L([1]^{[\rho]}; [0]^{[\rho]} \otimes \sigma) + [-1]^{[\rho]} \otimes [0]^{[\rho]} \otimes \sigma + [0]^{[\rho]} \otimes L([1]^{[\rho]}; \sigma) + 2L([-1]^{[\rho]}, [0]^{[\rho]} \otimes \sigma + \delta([-1,0]^{[\rho]} \otimes \sigma,
\]
\[
\mu^*(L([0,1]^{[\rho]}; \sigma)) = 1 \otimes L([0,1]^{[\rho]}; \sigma) + [0]^{[\rho]} \otimes L([1]^{[\rho]}; \sigma) + \delta([-1,0]^{[\rho]} \otimes \sigma.
\]

6.2. Rank three case.

In the rank 3, E.1 - H.1 apply also to this situation. It remains to consider I - K cases (L case is here equivalent to the J case, and therefore we drop it).

I.4. The case of exponents \((0,1,2)\) and \(\alpha = 1\).

Proposition 6.1. Let 
\[
\alpha = 1.
\]

(1) The following irreducible representations
\[
L([2]^{[\rho]}, [1]^{[\rho]}; [0]^{[\rho]} \otimes \sigma), \quad \tau([0]_+^{[\rho]}; \delta([1,2]^{[\rho]}; \sigma)),
\]
\[
L([2]^{[\rho]}, [0,1]^{[\rho]}; \sigma), \quad \tau([0]_-^{[\rho]}; \delta([1,2]^{[\rho]}; \sigma))
\]
are unitarizable.

(2) The representations
\[
L([0,2]^{[\rho]}; \sigma), \quad L([2]^{[\rho]}; \tau([0]_-^{[\rho]}; \delta([1]^{[\rho]}; \sigma)))
\]
\[
L([1,2]^{[\rho]}; [0]^{[\rho]} \otimes \sigma), \quad L([2]^{[\rho]}; \tau([0]_+^{[\rho]}; \delta([1]^{[\rho]}; \sigma)))
\]
are not unitarizable.
(3) The Aubert involution switches the representations in the same rows in (1) and (2).
(4) All the irreducible subquotients of the induced representation
\[ [0]^{(\rho)} \times [1]^{(\rho)} \times [2]^{(\rho)} \rtimes \sigma \]
are precisely the representations of (1) and (2).

Proof of (1), (3) and (4). We have
\[ (6.45) \quad [0]^{(\rho)} \rtimes \delta([1, 2]^{(\rho)}; \sigma) = \tau([0]^{(\rho)}_{+} \delta([1, 2]^{(\rho)}; \sigma)) \oplus \tau([0]^{(\rho)}_{-} \delta([1, 2]^{(\rho)}; \sigma)), \]
and these representations are unitarizable. Further, \([0]^{(\rho)} \rtimes L([2]^{(\rho)}, [1]^{(\rho)}; \sigma)\) is a sum of two irreducible unitarizable representations, and one of them is \(L([2]^{(\rho)}, [1]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma)\).

Observe
\[ [2]^{(\rho)} \times L([0]^{(\rho)}, [1]^{(\rho)}) \rtimes \sigma \leftrightarrow [2]^{(\rho)} \times [0]^{(\rho)} \times [1]^{(\rho)} \rtimes \sigma \]
\[ \cong [0]^{(\rho)} \times [2]^{(\rho)} \times [1]^{(\rho)} \rtimes \sigma \rightarrow [0]^{(\rho)} \rtimes L([2]^{(\rho)}, [1]^{(\rho)}; \sigma). \]
Suppose
\[ [0]^{(\rho)} \rtimes L([2]^{(\rho)}, [1]^{(\rho)}; \sigma) \leq [2]^{(\rho)} \times L([0]^{(\rho)}, [1]^{(\rho)}) \rtimes \sigma. \]
This implies
\[ 2[0]^{(\rho)} \rtimes L([-2]^{(\rho)}, [1]^{(\rho)}) \leq \]
\[ ([2]^{(\rho)} + [-2]^{(\rho)}) \rtimes (L([0]^{(\rho)}, [1]^{(\rho)}) + [0]^{(\rho)} \times [-1]^{(\rho)}) + L([0]^{(\rho)}, [-1]^{(\rho)}). \]
Then we must have also inequality of terms whose exponents in the support are not positive. This gives
\[ 2[0]^{(\rho)} \rtimes L([-2]^{(\rho)}, [1]^{(\rho)}) \leq [-2]^{(\rho)} \times [0]^{(\rho)} \times [-1]^{(\rho)} + [-2]^{(\rho)} \times L([0]^{(\rho)}, [-1]^{(\rho)}). \]
Now the multiplicity of \(L([-2]^{(\rho)}, [-1, 0]^{(\rho)})\) in the left hand side is two (compute directly, or use Zelevinsky involution), while on the right hand side is one (since \(L([-2]^{(\rho)}, [-1, 0]^{(\rho)})\) is not a subquotient of \([2]^{(\rho)} \rtimes [0]^{(\rho)} \times [1]^{(\rho)} \rtimes \sigma \rtimes [2]^{(\rho)} \times L([0]^{(\rho)}, [1]^{(\rho)}; \sigma), \]
and the multiplicity of \(L([-2]^{(\rho)}, [-1, 0]^{(\rho)})\) in \([-2]^{(\rho)} \times [0]^{(\rho)} \times [-1]^{(\rho)}\) is one).

Since \([2]^{(\rho)} \times [0]^{(\rho)} \times [1]^{(\rho)} \rtimes \sigma / [2]^{(\rho)} \times L([0]^{(\rho)}, [1]^{(\rho)}) \rtimes \sigma \cong [2]^{(\rho)} \times \delta([0, 1]^{(\rho)}) \rtimes \sigma, \]
this implies \(L([2]^{(\rho)}, [0, 1]^{(\rho)}; \sigma) \leq [0]^{(\rho)} \times L([2]^{(\rho)}, [1]^{(\rho)}; \sigma), \)
which further implies
\[ (6.46) \quad [0]^{(\rho)} \times L([2]^{(\rho)}, [1]^{(\rho)}; \sigma) = L([2]^{(\rho)}, [1]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma) + L([2]^{(\rho)}, [0, 1]^{(\rho)}; \sigma). \]
Therefore, the representations in (1) are unitarizable.

From (2.32) we get
\[ \tau([0]^{(\rho)}_{+} \delta([1, 2]^{(\rho)}; \sigma)) = L([2]^{(\rho)}, [1]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma), \]
which further implies (using (6.45) and (6.46))
\[ \tau([0]^{(\rho)}_{-} \delta([1, 2]^{(\rho)}; \sigma)) = L([2]^{(\rho)}, [0, 1]^{(\rho)}; \sigma). \]
Therefore, we have shown that the Aubert involution in (1) switches the representations in the same rows.

Consider now \([0]^{(\rho)} \times L([1,2]^{(\rho)}; \sigma)] and [0]^{(\rho)} \times L([2]^{(\rho)}; \delta([1]^{(\rho)}; \sigma)). We know that these two induced representations contain all the remaining irreducible subquotients. Therefore, they contain irreducible representations listed in (2). These two induced representations are dual by involution. Therefore, both are either irreducible or reducible. Since there are 4 irreducible subquotients in (2), both representations are reducible (note that the second induced representation is reducible by 2.7).

A short analysis implies that (4) holds.

In \([0]^{(\rho)} \times L([1,2]^{(\rho)}; \sigma)\) is \(L([1,2]^{(\rho)}; [0]^{(\rho)} \times \sigma).\) Observe that from B.1 we get that the GL-type Jacquet module of \([0]^{(\rho)} \times L([1,2]^{(\rho)}; \sigma)\) is
\[
2 \cdot [0]^{(\rho)} \times \delta([-2, -1]^{(\rho)}) \otimes \sigma + 2 \cdot [0]^{(\rho)} \times [-1]^{(\rho)} \times [2]^{(\rho)} \otimes \sigma =
2 \cdot \delta([-2, 0]^{(\rho)}) \otimes \sigma + 2 \cdot L([0]^{(\rho)}; [-2, -1]^{(\rho)}) \otimes \sigma + 2 \cdot L([0]^{(\rho)}; [-1]^{(\rho)}) \times [2]^{(\rho)} \otimes \sigma.
\]

From this follows that \([0]^{(\rho)} \times L([1,2]^{(\rho)}; \sigma)\) has no tempered subquotients. This, reducibility of \([0]^{(\rho)} \times L([1,2]^{(\rho)}; \sigma)\) and the properties of the Langlands classification 2.6 imply that we must have in this representation \(L([0,2]^{(\rho)}; \sigma)\) for a subquotient. Further, it implies that we have at most two different subquotients (up to an isomorphism). This holds also for the dual representation \([0]^{(\rho)} \times L([2]^{(\rho)}; \delta([1]^{(\rho)}; \sigma)).\) Now 2.7 implies
\[
(6.47) \quad [0]^{(\rho)} \times L([2]^{(\rho)}; \delta([1]^{(\rho)}; \sigma)) =
L([2]^{(\rho)}; \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma))) + L([2]^{(\rho)}; \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma))).
\]

Therefore
\[
(6.48) \quad [0]^{(\rho)} \times L([1,2]^{(\rho)}; \sigma) = L([1,2]^{(\rho)}; [0]^{(\rho)} \times \sigma) + L([0,2]^{(\rho)}; \sigma).
\]

From this we see that whole induced representation is a multiplicity one representation of length 8.

Suppose \(L([1,2]^{(\rho)}; [0]^{(\rho)} \times \sigma)^t = L([2]^{(\rho)}; \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma))).\) Since \(L([-2, -1]^{(\rho)}, [0]^{(\rho)}) \otimes \sigma\) is in the Jacquet module of \(L([1,2]^{(\rho)}; [0]^{(\rho)} \times \sigma), L([0,1]^{(\rho)}, [2]^{(\rho)}) \otimes \sigma\) is in the Jacquet module of \(L([1,2]^{(\rho)}; [0]^{(\rho)} \times \sigma)^t.\) Therefore
\[
L([0,1]^{(\rho)}, [2]^{(\rho)}) \otimes \sigma \leq ([2]^{(\rho)} + [-2]^{(\rho)}) \times L([0]^{(\rho)}, [1]^{(\rho)}) \otimes \sigma,
\]
which is impossible. Therefore
\[
L([1,2]^{(\rho)}; [0]^{(\rho)} \times \sigma)^t = L([2]^{(\rho)}; \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma))).
\]

which further implies
\[
L([0,2]^{(\rho)}; \sigma)^t = L([2]^{(\rho)}; \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma))),
\]
In this way we have proved that the Aubert involution switches representations in the rows also in (2), which implies that (3) holds.

It remains to prove (3). We shall do it in several following lemma.

**Lemma 6.2.** For $\alpha = 1$, the representation $L([0, 2]^{(\rho)}; \sigma)$ is not unitarizable.

**Proof.** By (A1) of Theorem 4.1 in [37] we know $\delta([0, 2]^{(\rho)}) \rtimes \sigma = L([0, 2]^{(\rho)}; \sigma) + \delta([0, 2]^{(\rho)}; \sigma)$, where the last representation is tempered. Now we shall consider

$$\delta([-1, 1]^{(\rho)}) \times \delta([0, 2]^{(\rho)}) \rtimes \sigma = \delta([-1, 1]^{(\rho)}) \times (L([0, 2]^{(\rho)}; \sigma) + \delta([0, 2]^{(\rho)}; \sigma)).$$

Using the fact that $\delta([-1, 1]^{(\rho)})$ and $\delta([0, 2]^{(\rho)}; \sigma)$ are tempered and

$$\delta([-1, 2]^{(\rho)}) \times \delta([0, 1]^{(\rho)}) \rtimes \sigma \leq \delta([-1, 1]^{(\rho)}) \times \delta([0, 2]^{(\rho)}) \rtimes \sigma$$

we get that the following six representations

$$L([0, 2]^{(\rho)}; \tau([-1, 1]^{(\rho)}; \sigma)), \quad L([-1, 2]^{(\rho)}; \tau([0]^{(\rho)}; \delta([1]^{(\rho)}); \sigma))),$$

$$L([0, 1]^{(\rho)}; \delta([-1, 2]^{(\rho)}; \sigma)), \quad L([0, 1]^{(\rho)}; [-1, 2]^{(\rho)}; \sigma)$$

are subquotients of $\delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma)$.

Suppose that $L([0, 2]^{(\rho)}; \sigma)$ is unitarizable. This implies that the $\delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma)$ has $\delta([-1, 1]^{(\rho)}) \otimes L([0, 2]^{(\rho)}; \sigma)$ in its Jacquet module at least with multiplicity six. Now we complete the proof of non-unitarizability in usual way using the following lemma.

**Lemma 6.3.** For $\alpha = 1$, the multiplicity of $\delta([-1, 1]^{(\rho)}) \otimes L([0, 2]^{(\rho)}; \sigma)$ in the Jacquet module of $\delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma)$ is $\leq 4$.

**Proof.** We start to analyze the multiplicity of $\delta([-1, 1]^{(\rho)}) \otimes L([0, 2]^{(\rho)}; \sigma)$ in the Jacquet module of $\delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma)$. For this recall

$$M^*(\delta([-1, 1]^{(\rho)})) = \sum_{-2 \leq x \leq 1} \delta([-x, 1]^{(\rho)}) \times \sum_{x \leq y \leq 1} \delta([y + 1, 1]^{(\rho)}) \otimes \delta([x + 1, y]^{(\rho)}),$$

$$\mu^*(L([0, 2]^{(\rho)}; \sigma))$$

$$= \sum_{-1 \leq i \leq 2} \sum_{0 \leq i + j \leq 2} L(\delta([-i, 0]^{(\rho)}), \delta([j + 1, 2]^{(\rho)})) \otimes L_\alpha([i + 1, j]^{(\rho)}; \sigma) +$$

$$+ \sum_{i' = 1}^2 L(\delta([-i', 0]^{(\rho)}), \delta([i' + 1, 2]^{(\rho)})) \otimes \sigma.$$

From the above two formulas we see that the only terms from the Jacquet module of $L([0, 2]^{(\rho)}; \sigma)$ which can lead to $\delta([-1, 1]^{(\rho)}) \otimes L([0, 2]^{(\rho)}; \sigma)$ as a subquotient are $1 \otimes L([0, 2]^{(\rho)}; \sigma)$ and $\delta([-1, 0]^{(\rho)}) \otimes [2]^{(\rho)} \rtimes \sigma.$
The first possibility will give it with multiplicity two (for \( x = y = -2 \) and \( x = y = 1 \)).

The second possibility implies that we need to consider terms \([1]^{(\rho)} \otimes \delta([0, 1]^{(\rho)})\) \((x = -1, y = 1)\) and \([1]^{(\rho)} \otimes \delta([-1, 0]^{(\rho)})\) \((x = -2, y = 0)\). In the first case we get

\[
[1]^{(\rho)} \otimes \delta([-1, 0]^{(\rho)}) \otimes \delta([0, 1]^{(\rho)}) \cong [2]^{(\rho)} \rtimes \sigma
\]

Observe that in the second case we get a term which is in the Grothendieck group equal to the above one.

We know that the multiplicity of \(\delta([-1, 1]^{(\rho)})\) in the left tensor factor is one.

We also know that the multiplicity of \(L([0, 2]^{(\rho)}; \sigma)\) in \(\delta([0, 2]^{(\rho)}) \rtimes \sigma\) is one.

Suppose that \(L([0, 2]^{(\rho)}; \sigma) \leq L([0, 1]^{(\rho)}, [2]^{(\rho)}) \rtimes \sigma\). In the Jacquet module of the left hand side is \(\delta([-2, 0]^{(\rho)}) \otimes \sigma\). One can easily see that this term is not on the left hand side. One gets this directly from (2.17) (one can also read this from (2.18)). This implies that the inequality cannot hold.

This implies that the multiplicity of \(\delta([-1, 1]^{(\rho)}) \otimes L([0, 2]^{(\rho)}; \sigma)\) in the Jacquet module of \(\delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma)\) is four.

**Lemma 6.4.** For \(\alpha = 1\), the representation \(L([2]^{(\rho)}; \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma)))\) is not unitarizable.

**Proof.** Denote

\[
\pi := L([2]^{(\rho)}; \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma))).
\]

From the previous case we know that \(\delta([-1, 1]^{(\rho)})^t \rtimes \pi\) has length six (pass to the ASS duals). Suppose that \(\pi\) is unitarizable. These two facts imply that the multiplicity of \(\delta([-1, 1]^{(\rho)})^t \otimes \pi\) in the Jacquet module of \(\delta([-1, 1]^{(\rho)})^t \rtimes \pi\) is at least six. Passing to the ASS duals, we get that the multiplicity of \(\delta([-1, 1]^{(\rho)}) \otimes \pi^t = \delta([-1, 1]^{(\rho)}) \otimes L([0, 2]^{(\rho)}; \sigma)\) in the Jacquet module of \(\delta([-1, 1]^{(\rho)}) \rtimes \pi^t = \delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma)\) is at least six. By previous lemma this multiplicity is four. This contradiction completes our proof of non-unitarizability.

We now consider the dual of \(L([2]^{(\rho)}; \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma)))\):

**Lemma 6.5.** For \(\alpha = 1\), the representation \(L([1, 2]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma)\) is not unitarizable.

**Proof.** Consider

\[
\delta([-1, 1]^{(\rho)}) \rtimes L([1, 2]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma).
\]

First observe that by (2.7) we have here two irreducible subquotients

\[
L([1, 2]^{(\rho)}; [0]^{(\rho)} \rtimes \delta([-1, 1]^{(\rho)}; \sigma)).
\]
In the proof of the previous proposition we have proved that we have in the Grothendieck group

\[ [0]^{(\rho)} \ltimes L([1, 2]^{(\rho)}; \sigma) = L([1, 2]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma) + L([0, 2]^{(\rho)}; \sigma). \]

Since

\[ [0]^{(\rho)} \times \delta([1, 2]^{(\rho)}) \rtimes \sigma = [0]^{(\rho)} \times L([1, 2]^{(\rho)}; \sigma) + [0]^{(\rho)} \times \delta([1, 2]^{(\rho)}; \sigma), \]

we get that in the representation on the left hand side of the equality the only non-tempered irreducible subquotients that can appear are

\[ L([1, 2]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma), \quad L([0, 2]^{(\rho)}; \sigma). \]

Therefore, if an irreducible non-tempered representation is a subquotient of \( \delta([-1, 1]^{(\rho)}) \rtimes [0]^{(\rho)} \times \delta([1, 2]^{(\rho)}) \rtimes \sigma \), then it must be a subquotient of \( \delta([-1, 1]^{(\rho)}) \rtimes L([1, 2]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma) \) or of \( \delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma) \). Observe

\[ \delta([-1, 2]^{(\rho)}) \times [0]^{(\rho)} \times [1]^{(\rho)} \rtimes \sigma \leq \delta([-1, 1]^{(\rho)}) \times [0]^{(\rho)} \times \delta([1, 2]^{(\rho)}) \rtimes \sigma. \]

From this we see that we have here the following non-tempered irreducible subquotients:

\[ L([1]^{(\rho)}; [0]^{(\rho)} \rtimes \delta([-1, 2]^{(\rho)}; \sigma)^\boxempty), \quad L([1]^{(\rho)}; [-1, 2]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma), \]

\[ L([-1, 2]^{(\rho)}; \tau([0]^{(\rho)} \rtimes (1 \otimes \sigma)). \]

From the Frobenius reciprocity one gets that each of the three representations in the first row above have in the Jacquet module an irreducible subquotient of the following form

\[ L([-1]^{(\rho)}; [0]^{(\rho)} \rtimes \tau \otimes \sigma. \]

Therefore, they have for a subquotient a representation of the form

\[ L([-1]^{(\rho)} + b) \otimes \sigma \]

for some multisegment \( b \). Therefore, if any representation from the first row above is a subquotient of \( \delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma) \), then

\[ L([-1]^{(\rho)} + b) \otimes \sigma \leq M^*(\delta([-1, 1]^{(\rho)})) \times M^*(\delta([0, 2]^{(\rho)})) \times (1 \otimes \sigma). \]

One directly sees that this is not possible.

In this way we have proved that \( \delta([-1, 1]^{(\rho)}) \rtimes L([1, 2]^{(\rho)}; [0]^{(\rho)} \rtimes \sigma) \) is a representation of length \( \geq 5 \).

We shall get one more irreducible subquotient (but we do not need it for the proof, so one can skip this). Suppose

\[ L([-1, 2]^{(\rho)}; \tau([0]^{(\rho)} \rtimes \delta([-1]^{(\rho)}; \sigma))) \leq \delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma). \]

This implies

\[ \delta([-2, 1]^{(\rho)} \otimes \tau([0]^{(\rho)} \rtimes \delta([-1]^{(\rho)}; \sigma))) \leq \mu^*(\delta([-1, 1]^{(\rho)}) \rtimes L([0, 2]^{(\rho)}; \sigma)). \]

Recall an earlier formulas for \( \mu^*(L([0, 2]^{(\rho)}; \sigma)) \) and \( M^*(\delta([-1, 1]^{(\rho)}) \). If we want to get \( \delta([-2, 1]^{(\rho)}) \otimes \tau([0]^{(\rho)} \rtimes \delta([-1]^{(\rho)}; \sigma)) \), we must take from \( \mu^*(L([0, 2]^{(\rho)}; \sigma)) \) obviously the term \( \delta([-2, 0]^{(\rho)} \otimes \sigma \) (this is the only possibility to get exponent -2 on the left side of \( \otimes \)).

\[ \text{One gets directly from Proposition 2.1 of [47] that } Jord_{\mu}(\delta([-2, 1]^{(\rho)} \otimes \tau([0]^{(\rho)} \rtimes \delta([-1]^{(\rho)}; \sigma))) = \{1, 3, 5\}. \]
from the formula for $M^*(\delta([-1,1])$) we see that the other factor must be $[1] \otimes \delta([-1,0])$ or $[1] \otimes \delta([-1,0])$. But in $\delta([-1,1]) \otimes \sigma$ and $\delta([-1,0]) \otimes \sigma$, $\tau([0];\delta([1];\sigma))$ is not a subquotient (see D.4).

Therefore, we have proved that

$$L([-1,2];\tau([0];\delta([1];\sigma))) \leq \delta([-1,1]) \otimes L([1,2];[0] \otimes \sigma).$$

Therefore, we have proved that the length of the right hand side representation is at least 6. For us is enough to know that it is at least 5. Now we complete the proof of non-unitarizability of $L([1,2];[0] \otimes \sigma)$ in usual way using the following lemma.

**Lemma 6.6.** For $\alpha = 1$ the multiplicity of $\tau := \delta([-1,1]) \otimes L([1,2];[0] \otimes \sigma)$ in the Jacquet module of $\pi := \delta([-1,1]) \otimes L([1,2];[0] \otimes \sigma)$ is $\leq 4$.

**Proof.** We use the formula $\mu^*(\pi) = M^*(\delta([-1,1])) \otimes \mu^*(L([1,2];[0] \otimes \sigma))$. If we take from $\mu^*(L([1,2];[0] \otimes \sigma))$ a term of the form $1 \otimes -$, then one directly sees that in this way we shall get $\tau$ two times. It remains to consider the case when we take from $\mu^*(L([1,2];[0] \otimes \sigma))$ a term which is not of the form $1 \otimes -$. We shall analyze this below.

Recall that we have proved that

$$L([1,2];[0] \otimes \sigma) = [0] \otimes L([1,2];\sigma) - L([0,2];\sigma).$$

First we have

$$\mu^*(0 \otimes L([1,2];\sigma)) = (1 \otimes [0] \otimes 2 \cdot [0] \otimes 1)$$

$$\times \left(1 \otimes L([1,2];\sigma) + [-1] \otimes [2] \otimes \sigma + [2] \otimes L([1];\sigma) + \delta([-2,-1]) \otimes \sigma + [-1] \otimes [2] \otimes \sigma \right).$$

The formula for $M^*(\delta([-1,1]))$ implies (considering the cuspidal support) that we can consider only two terms from the above formula: $2 \cdot [-1] \otimes [0] \otimes [2] \otimes \sigma$ and $[-1] \otimes [0] \otimes [2] \otimes \sigma$.

Further, the term for $i = 1, j = 2$ in the formula \[2\] for $\mu^*(L([0,2];\sigma))$ will reduce the possibilities to $[-1] \otimes [0] \otimes [2] \otimes \sigma$ and $[-1] \otimes [0] \otimes [2] \otimes \sigma$.

\[2\] This formula written differently for this case is

$$\mu^*(L([0,2];\sigma)) = 1 \otimes L([0,2];\sigma) + ([2] \otimes L([0,1];\sigma) + [0] \otimes L([1,2];\sigma)) + ([0] \otimes [2] \otimes L([1];\sigma) + \delta([-1,0]) \otimes [2] \otimes + \delta([-1,0]) \otimes [2] \otimes \sigma + \delta([-2,0]) \otimes \sigma).$$
Now for $[-1] \otimes [0] \times [2] \rtimes \sigma$ we must take from $M^* (\delta([-1, 1]))$ the term $\delta([0, 1]) \otimes ((-1) + [1])$. Now using that $[\pm 1] \otimes [0] \times [2] \rtimes \sigma$ is multiplicity one representation, we get multiplicity two of $\mathcal{L}$ in this case.

Further, for $[-1] \otimes [0] \otimes [2] \rtimes \sigma$ we must take from $M^* (\delta([-1, 1]))$ the term $[1] \otimes (\delta([0, 1]) + \delta([-1, 0]))$. The multiplicity of $\tau$ that we shall get from this case is equal to the multiplicity of $L([1, 2]; [0] \rtimes \sigma)$ in $2 \cdot \delta([0, 1]) \times [2] \rtimes \sigma$.

Observe that by \[L([1, 2]; [0] \rtimes \sigma) \leq L([1, 2], [0] \rtimes \sigma \leq L([1], [0] \times [2] \rtimes \sigma.

This implies that $L([1, 2]; [0] \rtimes \sigma)$ is not a subquotient of $\delta([0, 1]) \times [2] \rtimes \sigma$ (since the whole induced representation is multiplicity one).

We shall show that $L([1, 2]; [0] \rtimes \sigma)$ is not a subquotient of $\delta([0, 1]) \times [2] \rtimes \sigma$ (which will complete the proof of the lemma). To prove the last fact, it is enough to prove that $L([1, 2]; [0] \rtimes \sigma)$ is a subquotient of $L([0], [1] \rtimes [2] \rtimes \sigma$ (since the whole induced representation is multiplicity one).

Therefore, we have proved the multiplicity 4, and the proof is complete now.

From the above computations, applying duality (in usual way) we get non-unitarizability of $L([2]; \tau([0], \delta([1]; \sigma))$.

### J.4. The Case of Exponents $(0, 1, 1)$ and $\alpha = 1$.

In the Grothendieck group we have

$$[0] \times [1] \times [1] \rtimes \sigma = L([0], [1] \rtimes \delta([1]); \sigma) + L([0], [1] \rtimes L([1]; \sigma) + \delta([0, 1]) \times L([1]; \sigma) + \delta([0, 1]) \rtimes \delta([1]; \sigma).$$

All the irreducible subquotients of the above representation are (either) in complementary series, or its ends. So all the irreducible subquotients are unitarizable. Observe also that $[0] \times [1] \times [-1] \rtimes \sigma = \delta([-1, 1]) \times \sigma + \delta([-1, 1]) \times \sigma + 2L([-1], [0, 1] \rtimes \sigma.$

We get here the following irreducible unitarizable representations

$$L([0, 1]; [1]; \sigma), \quad L([1], [1]; [0] \rtimes \sigma), \quad L([0, 1]; \delta([1]; \sigma)),$$

$$L([1]; \tau([0], \delta([1]; \sigma))), \quad \delta([-1, 1]) \rtimes \sigma.$$

Further $[1] \times [0] \times [-1] \rtimes \sigma \rightarrow L([1], [0], [-1] \rtimes \sigma$ implies

$$L([1], [0]) \rtimes L([1]; \sigma) \rightarrow L([1], [0]) \times [-1] \rtimes \sigma \rightarrow L([1], [0], [-1] \rtimes \sigma.$$
We see that
\[ L([1]^{(\rho)}, [0]^{(\rho)}, [-1]^{(\rho)}) \rtimes \sigma \not\leq L([1]^{(\rho)}, [0]^{(\rho)}) \rtimes L([1]^{(\rho)}; \sigma) \]
from the Jacquet modules (consider multiplicity of \( L([1]^{(\rho)}, [0]^{(\rho)}, [-1]^{(\rho)}) \otimes \sigma \) in both sides, on the left is two and one on the right). Therefore we have a non-zero intertwining
\[ [1]^{(\rho)} \times [0]^{(\rho)} \rtimes \delta([1]^{(\rho)}; \sigma) \to L([1]^{(\rho)}, [0]^{(\rho)}) \rtimes \delta([1]^{(\rho)}; \sigma) \to L([1]^{(\rho)}, [0]^{(\rho)}, [-1]^{(\rho)}) \rtimes \sigma. \]
Note
\[ [1]^{(\rho)} \times [0]^{(\rho)} \rtimes \delta([1]^{(\rho)}; \sigma) = [1]^{(\rho)} \rtimes \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma)) \oplus [1]^{(\rho)} \rtimes \tau([0]^{(\rho)}_+; \delta([1]^{(\rho)}; \sigma)). \]
Therefore at least one of \( L([1]^{(\rho)}, \tau([0]^{(\rho)}, \delta([1]^{(\rho)}; \sigma)) \)) is a subquotient of \( L([1]^{(\rho)}, [0]^{(\rho)}, [-1]^{(\rho)}) \rtimes \sigma. \) Suppose
\[ [1]^{(\rho)} \times \tau([0]^{(\rho)}_+; \delta([1]^{(\rho)}; \sigma)) \leq L([1]^{(\rho)}, [0]^{(\rho)}, [-1]^{(\rho)}) \rtimes \sigma. \]
Since in the Jacquet module of the right hand side is \( 2(L([-1]^{(\rho)}, [0]^{(\rho)}, [1]^{(\rho)}) \otimes \sigma + L([-1]^{(\rho)}, [0]^{(\rho)} \times [-1]^{(\rho)} \otimes \sigma), \) we immediately see that this cannot hold. Namely on the left hand side shows up among others \( [1]^{(\rho)} \times [0, 1]^{(\rho)} \otimes \sigma. \) This implies
\[ L([1]^{(\rho)}, \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma))) \leq L([1]^{(\rho)}, [0]^{(\rho)}, [-1]^{(\rho)}) \rtimes \sigma. \]
This implies
\[ \delta([-1, 1]^{(\rho)}; \sigma^t) = L([1]^{(\rho)}, \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma))). \]
Therefore
\[ (6.53) \quad L([1]^{(\rho)}, [0]^{(\rho)}, [-1]^{(\rho)}) \rtimes \sigma = L([1]^{(\rho)}, [1]^{(\rho)}, [0]^{(\rho)} \rtimes \sigma) + L([1]^{(\rho)}, \tau([0]^{(\rho)}; \delta([1]^{(\rho)}; \sigma))). \]

K.4. The case of exponents \((0, 0, 1)\) and \(\alpha = 1.\)

All the irreducible subquotients of the induced representation
\[ [0]^{(\rho)} \times [0]^{(\rho)} \times [1]^{(\rho)} \rtimes \sigma \]
\[ = [0]^{(\rho)} \times [0]^{(\rho)} \times L([1]^{(\rho)}; \sigma) + [0]^{(\rho)} \times [0]^{(\rho)} \times \delta([1]^{(\rho)}; \sigma) \]
are unitarizable. Here we get four irreducible pieces, and all of them are unitarizable (irreducible subrepresentations of representations induced by unitarizable once). They are
\[ [0]^{(\rho)} \rtimes L([0, 1]^{(\rho)}; \sigma), \quad L([1]^{(\rho)}; [0]^{(\rho)} \times [0]^{(\rho)} \rtimes \sigma), \quad [0]^{(\rho)} \rtimes \tau([0]^{(\rho)}_\pm; \times \delta([1]^{(\rho)}; \sigma)). \]

7. Unitarizability for \(\alpha = \frac{1}{2}\) not covered by section 3.

We shall write differences which we have in this case. In rank 1, A.1 applies here.
7.1. Generalized rank two case. The case B.1 applies here also. The case C and D are equivalent. Therefore we need in generalized rank two only to settle

C.5. The case of exponents \((\frac{1}{2}, \frac{1}{2})\) and \(\alpha = \frac{1}{2}\).

Here all irreducible subquotients are unitarizable. We have

\[
\delta([-\frac{1}{2}, \frac{1}{2})] \otimes \sigma = \delta([-\frac{1}{2}, \frac{1}{2}]^0; \sigma) + \delta([-\frac{1}{2}, \frac{1}{2}]^+; \sigma),
\]

\[
L([-\frac{1}{2}]^0, \frac{1}{2}]^0) \otimes \sigma = L([-\frac{1}{2}]^0, \frac{1}{2}]^0; \sigma) + L([-\frac{1}{2}]^0, \delta([\frac{1}{2}]^0; \sigma)),
\]

\[
\frac{1}{2}]^0 \otimes \delta([\frac{1}{2}]^0; \sigma) = L([-\frac{1}{2}]^0; \delta([\frac{1}{2}]^0; \sigma)) + \delta([-\frac{1}{2}, \frac{1}{2}]^0; \sigma),
\]

\[
\frac{1}{2}]^0 \otimes L([\frac{1}{2}]^0; \sigma) = L([-\frac{1}{2}]^0, \frac{1}{2}]^0; \sigma) + \delta([-\frac{1}{2}, \frac{1}{2}]^0; \sigma).
\]

This implies

\[
\delta([-\frac{1}{2}, \frac{1}{2}]^0; \sigma)^t = L([\frac{1}{2}]^0, \frac{1}{2}]^0; \sigma),
\]

\[
\delta([-\frac{1}{2}, \frac{1}{2}]^0; \sigma)^t = L([-\frac{1}{2}]^0, \delta([\frac{1}{2}]^0; \sigma)).
\]

We have

\[
\mu^*(\delta([-\frac{1}{2}, \frac{1}{2}]^0; \sigma)) = 1 \otimes \delta([-\frac{1}{2}, \frac{1}{2}]^0; \sigma) + \frac{1}{2}]^0 \otimes \frac{1}{2}]^0 \otimes \delta([\frac{1}{2}]^0; \sigma) + \delta([-\frac{1}{2}, \frac{1}{2}]^0) \otimes \sigma + \frac{1}{2}]^0 \times \frac{1}{2}]^0 \otimes \sigma,
\]

\[
\mu^*(\delta([-\frac{1}{2}, \frac{1}{2}]^0; \sigma)) = 1 \otimes \delta([-\frac{1}{2}, \frac{1}{2}]^0; \sigma) + \frac{1}{2}]^0 \otimes L([\frac{1}{2}]^0; \sigma) + \delta([-\frac{1}{2}, \frac{1}{2}]^0) \otimes \sigma,
\]

\[
\mu^*(L([\frac{1}{2}]^0, \frac{1}{2}]^0; \sigma)) = 1 \otimes L([\frac{1}{2}]^0, \frac{1}{2}]^0; \sigma) + [-\frac{1}{2}]^0 \otimes \frac{1}{2}]^0 \otimes \delta([\frac{1}{2}]^0; \sigma) + L([-\frac{1}{2}]^0, \frac{1}{2}]^0) \otimes \sigma + [-\frac{1}{2}]^0 \times [-\frac{1}{2}]^0 \otimes \sigma,
\]

\[
\mu^*(L([\frac{1}{2}]^0; \delta([\frac{1}{2}]^0; \sigma))) = 1 \otimes L([\frac{1}{2}]^0; \delta([\frac{1}{2}]^0; \sigma)) + [-\frac{1}{2}]^0 \otimes \delta([\frac{1}{2}]^0; \sigma) + L([-\frac{1}{2}]^0, \frac{1}{2}]^0) \otimes \sigma.
\]
7.2. Generalized rank three.

In the rank 3, E.1 and F.1 apply also to this situation. It remains to consider cases G and H (cases G, I and L are equivalent, and cases H, J and K are equivalent).

G.5. The case of exponents \((\frac{1}{2}, \frac{1}{2}, \frac{3}{2})\) and \(\alpha = \frac{1}{2}\).

**Proposition 7.1.** Let

\(\alpha = \frac{1}{2}\).

1. The representations

\[
L(\left[\frac{3}{2}\right]^\rho, \left[\frac{1}{2}\right]^\rho, \frac{1}{2})^\rho; \sigma), \quad \delta(\left[-\frac{1}{2}, \frac{3}{2}\right]^\rho; \sigma)
\]

are unitarizable.

2. The representations

\[
L\left(\left[\frac{1}{2}\right]^\rho, \frac{3}{2})^\rho; \sigma\right), \quad L\left(\left[\frac{3}{2}\right]^\rho, \frac{1}{2})^\rho; \delta(\left[\frac{1}{2}\right]^\rho; \sigma)\right)
\]

are unitarizable.

3. The Aubert involution switches the representations in the same rows in (1) and (2).

4. The representations in (1) and (2) are the Jordan-Hölder composition series for

\[
\frac{1}{2})^\rho \times \frac{3}{2})^\rho \rtimes \sigma\]

are not unitarizable.

Remark 7.2. The unitarizability of all the representations in (1) will be obtained using complementary series (they are irreducible subquotients of ends of complementary series).

A easy analysis gives that (4) holds. We shall prove (1) - (3) through a number of steps.

(1) First observe that the representations

\[
\delta(\left[-\frac{1}{2}, \frac{3}{2}\right]^\rho; \sigma)
\]

are unitarizable (since they are square integrable). Further

\[
L\left(\left[-\frac{1}{2}, \frac{3}{2}\right]^\rho; \sigma\right), \quad L\left(\left[\frac{3}{2}\right]^\rho, \frac{1}{2})^\rho; \delta(\left[\frac{1}{2}\right]^\rho; \sigma)\right)
\]

are unitarizable, since they are at the ends of the complementary series (starting with \(\delta([-1, 1])^\rho \rtimes \sigma\) and \(L([-1])^\rho, [0]^\rho, [1]^\rho \rtimes \sigma\) respectively). Analogously,

\[
L\left(\left[\frac{1}{2}\right]^\rho, \delta(\left[\frac{1}{2}\right]^\rho; \sigma)\right)
\]

(and also \(L\left(\left[\frac{3}{2}\right]^\rho, \frac{1}{2})^\rho; \delta(\left[\frac{1}{2}\right]^\rho; \sigma)\right)\) is unitarizable since it is at the ends of the complementary series starting with \(\delta([-\frac{1}{2}, \frac{1}{2})^\rho) \rtimes \delta([\frac{1}{2})^\rho; \sigma)\) (and with \(L([\frac{1}{2})^\rho, [-\frac{1}{2})^\rho) \rtimes L([\frac{1}{2})^\rho; \sigma)\)).

Further

\[
L\left(\left[\frac{1}{2}\right]^\rho, \delta(\left[\frac{1}{2}, \frac{3}{2}\right]^\rho; \sigma)\right)
\]
is unitarizable (since it is at the end of the complementary series which start with $[0]^{(\rho)} \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma)$).

(2) Observe that $[2.32]$ implies
\[ \delta([-\frac{1}{2}, \frac{3}{2}]_+^{(\rho)}; \sigma)^t = L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma). \]

Another way to get it is to use the fact that $[-\frac{1}{2}]^{(\rho)} \times \delta([-\frac{3}{2}, -\frac{1}{2}]^{(\rho)}) \otimes \sigma$ is in the Jacquet module of the right hand side. This follows from the fact that $L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) \hookrightarrow [-\frac{3}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \times \sigma$ and $\delta([-\frac{3}{2}, -\frac{1}{2}]^{(\rho)})^t \times [-\frac{3}{2}]^{(\rho)} \otimes \sigma \hookrightarrow [-\frac{3}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \times \sigma$. The uniqueness of the irreducible subrepresentation implies $L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) \hookrightarrow \delta([-\frac{3}{2}, -\frac{1}{2}]^{(\rho)})^t \times [-\frac{3}{2}]^{(\rho)} \times \sigma$ and now the Frobenius reciprocity implies the above claim.

(3) We know that $\delta([-\frac{1}{2}, \frac{3}{2}]_+^{(\rho)}; \sigma)$ has multiplicity one in the whole induced representation. Now the properties of the Langlands classification imply
\[ ([\frac{1}{2}]^{(\rho)} \times \delta([\frac{1}{2}, \frac{3}{2}]_+^{(\rho)}; \sigma) = L([\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}, \frac{3}{2}]_+^{(\rho)}; \sigma)) + \delta([-\frac{1}{2}, \frac{3}{2}]_+^{(\rho)}; \sigma) \]
(one easily checks that $\delta([-\frac{1}{2}, \frac{3}{2}]_+^{(\rho)}; \sigma)$ is not in the left hand side). This implies
\[ ([\frac{1}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}, [\frac{3}{2}]^{(\rho)}; \sigma) = L([\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}, \frac{3}{2}]_+^{(\rho)}; \sigma)) + L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma). \]

(4) Consider
\[ [\frac{3}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) \hookrightarrow [\frac{3}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}; \sigma) \cong [-\frac{1}{2}]^{(\rho)} \times [\frac{3}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}; \sigma). \]

Suppose
\[ [-\frac{1}{2}]^{(\rho)} \times L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) \leq [\frac{3}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma). \]

Then
\[ ([\frac{1}{2}]^{(\rho)} + [-\frac{1}{2}]^{(\rho)}) \times L([-\frac{1}{2}]^{(\rho)}, [-\frac{3}{2}]^{(\rho)}) \leq ([\frac{3}{2}]^{(\rho)} + [-\frac{3}{2}]^{(\rho)}) \times (L([-\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}) + [-\frac{1}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)}). \]

Consider the part supported by $[-\frac{3}{2}]^{(\rho)}$, $[-\frac{1}{2}]^{(\rho)}$, $[\frac{1}{2}]^{(\rho)}$ we get
\[ [\frac{1}{2}]^{(\rho)} \times L([-\frac{1}{2}]^{(\rho)}, [-\frac{3}{2}]^{(\rho)}) \leq [-\frac{3}{2}]^{(\rho)} \times L([-\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}), \]
which is impossible (on the left hand side is $L([-\frac{3}{2}]^{(\rho)}, [-\frac{1}{2}, \frac{1}{2}]^{(\rho)})$, which is not on the right hand side).

Denote by $\phi$ the above composition of embedding and epimorphism. We have just shown that $\phi$ is not an epimorphism. This implies that we have a non-trivial intertwining from $[\frac{3}{2}]^{(\rho)} \times ([\frac{1}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}; \sigma)/L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma)) \cong [\frac{3}{2}]^{(\rho)} \times \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)$ into a quotient of $[-\frac{3}{2}]^{(\rho)} \times L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma)$ (precisely, into $[-\frac{1}{2}]^{(\rho)} \times L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma)/Im(\varphi)$). Therefore
\[ L([\frac{3}{2}]^{(\rho)}, \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)) \leq [-\frac{1}{2}]^{(\rho)} \times L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma)). \]
This implies that
\[ L([\frac{3}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]_{-}; \sigma)) \]
is unitarizable. Further, we get
\[ [\frac{1}{2}]^{(\rho)} \rtimes L([\frac{1}{2}]^{(\rho)}; [\frac{3}{2}]^{(\rho)}; \sigma) = L([\frac{3}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]_{-}; \sigma)) + L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}; \sigma). \]
and
\[ L([\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma)) = L([\frac{3}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]_{-}; \sigma)). \]
Recall
\[ \delta([-\frac{1}{2}, \frac{3}{2}]^{(\rho)} \rtimes \sigma = \delta([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma) + \delta([-\frac{1}{2}, \frac{1}{2}]_{-}; \sigma) + L([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma). \]

(5) Consider
\[ L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}) \rtimes L([\frac{1}{2}]^{(\rho)}; \sigma) \hookrightarrow L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}) \times [-\frac{1}{2}]^{(\rho)} \times \sigma \rightarrow L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}, [-\frac{1}{2}]^{(\rho)}) \rtimes \sigma. \]
Suppose
\[ L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}, [-\frac{1}{2}]^{(\rho)}) \rtimes \sigma \leq L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}) \rtimes \sigma \cdot L([\frac{1}{2}]^{(\rho)}; \sigma). \]
Observe that in the left hand side of GL-type Jacquet module is 2 \cdot L([-\frac{3}{2}]^{(\rho)}, [-\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}) \rtimes \sigma.
Consider now the right hand side Jacquet module:
\[ (L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}) + [-\frac{3}{2}]^{(\rho)} \times [\frac{1}{2}]^{(\rho)} + L([-\frac{3}{2}]^{(\rho)}, [-\frac{1}{2}]^{(\rho)})) \times [-\frac{1}{2}]^{(\rho)} \otimes \sigma. \]
Here the multiplicity of \( L([-\frac{3}{2}]^{(\rho)}, [-\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}) \rtimes \sigma \) is one.
Denote by \( \phi \) the above composition of embedding and epimorphism. We have just shown that \( \phi \) is not an epimorphism. Therefore we have a non-trivial intertwining
\[ L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}) \rtimes \delta([\frac{1}{2}]^{(\rho)}; \sigma) \rightarrow (L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}, [-\frac{1}{2}]^{(\rho)}) \rtimes \sigma)) / \text{Im} (\phi). \]
This implies
\[ L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}]^{(\rho)}; \sigma)) \leq L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}, [-\frac{1}{2}]^{(\rho)}) \rtimes \sigma. \]
This also implies that
\[ L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}]^{(\rho)}; \sigma)) \]
is unitarizable. Note that we have now completed the proof of (1).

(6) Further, we know that
\[ L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}]^{(\rho)}; \sigma)) \subseteq L([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma), \delta([-\frac{1}{2}, \frac{3}{2}]_{-}; \sigma)). \]
Observe \( L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}]^{(\rho)}; \sigma)) \) is a quotient of \( L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}) \rtimes \delta([\frac{1}{2}]^{(\rho)}; \sigma). \) Therefore
\[ L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}]^{(\rho)}; \sigma)) \subseteq \delta([\frac{1}{2}]^{(\rho)}; \sigma) \times L([\frac{1}{2}]^{(\rho)}; \sigma). \]
Note that in the Jacquet module of the last representation is not \( \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)} \otimes \sigma. \) This implies
\[ L([\frac{3}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}]^{(\rho)}; \sigma)) \subseteq \delta([-\frac{1}{2}, \frac{3}{2}]_{-}; \sigma). \]
(7) Consider $L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma)$. It is a subquotient in
\[
delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times [\frac{1}{2}]^{(\rho)} \times \sigma = \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma) + \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma).
\]
Observe that in the left hand side we have obviously both $\delta([-\frac{3}{2}, \frac{3}{2}]^{(\rho)}; \sigma)$. Further since $\delta([-\frac{3}{2}, \frac{3}{2}]^{(\rho)}; \sigma) \leq \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma)$ and the multiplicity of $\delta([-\frac{3}{2}, \frac{3}{2}]^{(\rho)}) \otimes \sigma$ in the Jacquet module of $\delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma)$ is one, we get
\[
\delta([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma) \leq \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma).
\]

(8) Now we shall analyze the representation $\delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma)$ (below will be explained what dash boxed terms mean). Write
\[
\mu^*(\delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma)) =
\]
\[
1 \otimes \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)})
\]
\[
+[-\frac{1}{2}]^{(\rho)} \otimes [\frac{3}{2}]^{(\rho)} + [\frac{3}{2}]^{(\rho)} \otimes [\frac{1}{2}]^{(\rho)}
\]
\[
+\delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \otimes 1 + [-\frac{1}{2}]^{(\rho)} \times [\frac{3}{2}]^{(\rho)} \otimes 1 + \delta([-\frac{3}{2}, -\frac{1}{2}]^{(\rho)}) \otimes 1
\]
\[
\times (1 \otimes L([\frac{1}{2}]^{(\rho)}; \sigma) + [-\frac{1}{2}]^{(\rho)} \otimes \sigma) =
\]
\[
1 \otimes \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma)
\]
\[
+[-\frac{1}{2}]^{(\rho)} \otimes [\frac{3}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}; \sigma)
\]
\[
([\frac{3}{2}]^{(\rho)} \otimes L([\frac{1}{2}]^{(\rho)}; [\frac{1}{2}]^{(\rho)}; \sigma); [\frac{2}{2}]^{(\rho)} \otimes \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma))
\]
\[
+[-\frac{1}{2}]^{(\rho)} \otimes \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \otimes \sigma
\]
\[
+\delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \otimes L([\frac{1}{2}]^{(\rho)}; \sigma) + \delta([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \otimes L([\frac{1}{2}]^{(\rho)}; \sigma)
\]
\[
+[-\frac{1}{2}]^{(\rho)} \times [\frac{3}{2}]^{(\rho)} \otimes \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma))
\]
\[
+([\frac{1}{2}]^{(\rho)} \times [\frac{3}{2}]^{(\rho)} \otimes \sigma)^{2+} + [+\frac{1}{2}]^{(\rho)} \times [\frac{3}{2}]^{(\rho)} \otimes L([\frac{1}{2}]^{(\rho)}; \sigma)
\]
\[
+[-\frac{1}{2}]^{(\rho)} \times [\frac{3}{2}]^{(\rho)} \otimes \delta([\frac{1}{2}]^{(\rho)}; \sigma))
\]
\[
+([\frac{1}{2}]^{(\rho)} \times [\frac{3}{2}]^{(\rho)} \otimes \sigma)^{3+} +
\]
\[
+([-\frac{1}{2}]^{(\rho)} \times [\frac{3}{2}]^{(\rho)} \otimes \sigma)^{4+} + [\frac{1}{2}]^{(\rho)} \times \delta([-\frac{3}{2}, -\frac{1}{2}]^{(\rho)}; \sigma)^{3+}
\]
We consider in $\delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma)$ an irreducible subquotient $\pi$ which has dash boxed term with super script [1]. Now using the transitivity of Jacquet modules one gets that $\pi$ must have in Jacquet module terms [2], [2+] and [2++], and further [3] and [3+]. Now
s_{GL}(\delta([1/2, 3/2]^{(\rho)}) \times L([1/2]^{(\rho)}; \sigma)) \) implies that \( \delta([1/2, 3/2]^{(\rho)}) \times L([1/2]^{(\rho)}; \sigma) \) is a multiplicity one representation of length two. Therefore

\[
\delta([1/2, 3/2]^{(\rho)}) \times L([1/2]^{(\rho)}; \sigma) = L([1/2, 3/2]^{(\rho)}, [1/2]^{(\rho)}; \sigma) + \delta([-1/2, 3/2]^{(\rho)}; \sigma).
\]

Further, a simple analysis using the transitivity of Jacquet modules\(^{22}\) gives

\[
\mu^*(L([1/2, 3/2]^{(\rho)}, [1/2]^{(\rho)}; \sigma)) =
\]

\[
1 \otimes L([1/2, 3/2]^{(\rho)}, [1/2]^{(\rho)}; \sigma) +
\]

\[
+ [-1/2]^{(\rho)} \otimes [3/2]^{(\rho)} \times L([1/2]^{(\rho)}; \sigma) + [3/2]^{(\rho)} \otimes L([1/2]^{(\rho)}, [1/2]^{(\rho)}; \sigma) + [-1/2]^{(\rho)} \otimes \delta([1/2, 3/2]^{(\rho)}) \times \sigma
\]

\[
+ [-1/2]^{(\rho)} \times [3/2]^{(\rho)} \otimes L([1/2]^{(\rho)}; \sigma) + \delta([-3/2, -1/2]^{(\rho)}) \otimes L([1/2]^{(\rho)}; \sigma)
\]

\[
+ [-1/2]^{(\rho)} \times [-1/2]^{(\rho)} \otimes [3/2]^{(\rho)} \otimes \sigma + [-1/2]^{(\rho)} \times [3/2]^{(\rho)} \otimes L([1/2]^{(\rho)}; \sigma) + [-1/2]^{(\rho)} \times [3/2]^{(\rho)} \otimes \delta([1/2]^{(\rho)}; \sigma))
\]

\[
+ L([-1/2]^{(\rho)}, [1/2]^{(\rho)}; \sigma) + [-1/2]^{(\rho)} \times [-1/2]^{(\rho)} \times [3/2]^{(\rho)} \otimes \sigma + [-1/2]^{(\rho)} \times [-1/2]^{(\rho)} \times \delta([-3/2, -1/2]^{(\rho)}) \otimes \sigma.
\]

(9) We know from the above formula that the dual representation of \( L([1/2, 3/2]^{(\rho)}, [1/2]^{(\rho)}; \sigma) \) is not tempered (look at the term \([-1/2]^{(\rho)} \times [-1/2]^{(\rho)} \times [3/2]^{(\rho)} \otimes \sigma \) which gives in the Jacquet module of the dual representation \([1/2]^{(\rho)} \times [1/2]^{(\rho)} \times [-3/2]^{(\rho)} \otimes \sigma \). Looking at the Jacquet module, we get that the Langlands parameter of the dual representation must come from a Jacquet module of

\[
[-3/2]^{(\rho)} \otimes L([1/2]^{(\rho)}, [1/2]^{(\rho)}; \sigma) = [-3/2]^{(\rho)} \otimes \delta([-1/2, 1/2]_+^{(\rho)}; \sigma).
\]

Since \( \delta([-1/2, 1/2]_+^{(\rho)}; \sigma) \) is tempered, this implies

\[
L([1/2, 3/2]^{(\rho)}, [1/2]^{(\rho)}; \sigma)^t = L([3/2]^{(\rho)}; \delta([-1/2, 1/2]_+^{(\rho)}; \sigma)).
\]

Now observe that

\[
L([-1/2, 3/2]^{(\rho)}; \sigma)^t \neq L([-1/2, 3/2]^{(\rho)}; \sigma).
\]

Namely, in the Jacquet module of \( L([-1/2, 3/2]^{(\rho)}; \sigma)^t \) is \( \delta([-1/2, 3/2]^{(\rho)})^t \otimes \sigma \). One directly sees that this term is not in the Jacquet module of \( \delta([-1/2, 3/2]^{(\rho)}) \times \sigma \). This and the formulas for the remaining involutions imply

\[
L([-1/2, 3/2]^{(\rho)}; \sigma)^t = L([1/2, 3/2]^{(\rho)}; \delta_{s,p}([1/2]^{(\rho)}; \sigma)).
\]

In this way we have completed the proof of (3). It remains to prove (2).

**Lemma 7.3.** Let \( \alpha = 1/2 \). The length of the representation

\[
\delta([-1/2, 1/2]^{(\rho)}) \times L([1/2, 3/2]^{(\rho)}, [1/2]^{(\rho)}; \sigma).
\]

is at least 6.

\(^{22}\)Recall that we know also \( \mu^*(\delta([-1/2, 3/2]^{(\rho)}; \sigma)) \).
Proof. Here we have first two irreducible subquotients by \textcolor{red}{[2.7]} They are \[ L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}, \frac{1}{2}] \pm \sigma)). \]

Now using \textcolor{red}{[7.54]} we get
\[
\delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma) =
\delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma) + \delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{1}{2}] - \sigma).
\]

Therefore, any non-tempered irreducible subquotient of the left hand side of the equality must be a subquotient of \[ \delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma). \]

Observe
\[
\delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times [\frac{1}{2}]^{(\sigma)} \times L([\frac{1}{2}]^{(\rho)}; \sigma) \leq \delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma).
\]

Non-tempered irreducible subquotients of the left hand side are
\[
L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma), \quad L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)).
\]

Therefore, we have a representation of length at least four.

Consider now
(7.56)
\[
\delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times [\frac{1}{2}]^{(\rho)} \times \sigma = \delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma) + \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma).
\]

We have
\[
\delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times [\frac{1}{2}]^{(\sigma)} \times [\frac{1}{2}]^{(\rho)} \times \sigma \leq \delta([\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times [\frac{1}{2}]^{(\rho)} \times \sigma.
\]

Irreducible subquotients of the left hand side of the above equality include
\[ L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma)), \quad L([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)). \]

Suppose that
(7.57)
\[ L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma)) \leq \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma). \]

Observe that
\[ L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma)) \leftrightarrow [-\frac{1}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \times \delta([-\frac{1}{2}, \frac{3}{2}]^{(\rho)}; \sigma)). \]

Therefore, we have \([-\frac{1}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \otimes - \) for a subquotient of the left hand side of \textcolor{red}{[7.57]}. One easily sees that one cannot get a term of the form \([-\frac{1}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \otimes - \) in the Jacquet module of the right hand side of \textcolor{red}{[7.57]}. This implies that the following two representations
\[ L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)) \]
are subquotients of \[ \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma). \]

Therefore, the induced representation \[ \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) \] is of length 6 at least.

\[ \square \]

Lemma 7.4. For \( \alpha = \frac{1}{2} \), the multiplicity of \( \tau := \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \otimes L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) \) in the Jacquet module of \( \pi := \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) \) is 6.
Proof. Recall that $M^* (\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)})) =$
$1 \otimes \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) + [\frac{1}{2}]^{(\rho)} \otimes [\frac{1}{2}]^{(\rho)} + [\frac{1}{2}]^{(\rho)} \otimes [-\frac{1}{2}]^{(\rho)} + 2 \cdot \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \otimes 1 + [\frac{1}{2}]^{(\rho)} \times [\frac{1}{2}]^{(\rho)} \otimes 1.$
Now from the formula for $\mu^*(L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma))$ we get that we need to find multiplicity of $\tau$ in the following terms:
$2 \cdot \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \otimes L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma), \quad 2 \cdot [\frac{1}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \otimes [\frac{1}{2}]^{(\rho)} \times [\frac{3}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}; \sigma),$
$2 \cdot [\frac{1}{2}]^{(\rho)} \times [-\frac{1}{2}]^{(\rho)} \otimes [\frac{1}{2}]^{(\rho)} \times \delta([\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \times \sigma.$
To show multiplicity 6, it is enough to prove that
$L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) \not\leq L([\frac{1}{2}]^{(\rho)}, [\frac{3}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma).$
For this, it is enough to prove that
$\delta([-\frac{3}{2}, -\frac{1}{2}]^{(\rho)}) \times [-\frac{1}{2}]^{(\rho)} \not\leq (L([\frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}) + [\frac{1}{2}]^{(\rho)} \times [-\frac{3}{2}]^{(\rho)} + L([-\frac{3}{2}]^{(\rho)}, [-\frac{1}{2}]^{(\rho)})) \times [-\frac{1}{2}]^{(\rho)},$
which obviously holds. \hfill \Box

Lemma 7.5. The representations
$\pi_1 := L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma), \quad \pi_2 = L([\frac{3}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma))$
are not unitarizable.

Proof. Suppose that $\pi_1$ is unitarizable. Then if $\gamma$ is an irreducible subquotient of $\Pi_1 = \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \pi$, then $\gamma$ is actually a quotient, and now Frobenius reciprocity implies that
$\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \otimes \pi$ embeds the Jacquet module of $\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \pi$. Therefore the Jacquet module of $\Pi_1$ contains at least 6 different representations isomorphic to $\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \otimes \pi$ as subrepresentations. Similarly, $\gamma$ is a subrepresentation, and therefore $\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \otimes \pi$ is a quotient of the Jacquet module of $\gamma$, so of $\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \pi$, and we have at least 6 different representations isomorphic to $\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \otimes \pi$ as quotients. The above lemma now implies that if we find a subquotient of the filtration from Geometric lemma, where $\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \otimes \pi$ is not a quotient and subrepresentation, the non-unitarizability will be proved. Look at
$[\frac{1}{2}]^{(\rho)} \times 1 \otimes [\frac{1}{2}]^{(\rho)}, \quad 1 \times [\frac{1}{2}]^{(\rho)} \otimes [-\frac{1}{2}]^{(\rho)}$
from $M^*(\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}))$. Now multiplying with corresponding terms from $\mu^*(\pi_1)$, we get on the left hand side a regular reducible representation, which therefore do not have $\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)})$ for a quotient and subrepresentation in the same time. This implies this case.

The same argument (applying duality) gives non-unitarizability for the dual representation. \hfill \Box

Now the proof of the proposition is complete.

H.5. The case of exponents $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\alpha = \frac{1}{2}$. 

Here all the irreducible subquotients are unitarizable. They are

\[ \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma), \quad L([\frac{1}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)), \]

\[ L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}]^{(\rho)}; \sigma)), \quad L([-\frac{1}{2}]^{(\rho)}), [\frac{1}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}; \sigma) \]

(above \( \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma) \) is irreducible, and therefore its dual is also irreducible). Obviously

\[ (\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma))^{t} = L([-\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma). \]

From the other side

\[ \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times L([\frac{1}{2}]^{(\rho)}; \sigma) \]

is by 2.47 of length at least two. Therefore, its dual is also of length at least two.

Further \( \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma) \not\subset [\frac{1}{2}]^{(\rho)} \times \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma) \) (consider the most positive term in the GL-type Jacquet module, which is on the left hand side, but not on the right hand side). Now the properties of the Langlands classification imply that

\[ (7.58) \quad [\frac{1}{2}]^{(\rho)} \times \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma) \]

is irreducible. Applying duality, we get that

\[ [\frac{1}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}]^{(\rho)}; \sigma)) \]

is also irreducible. Therefore

\[ ([\frac{1}{2}]^{(\rho)} \times \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma))^{t} = [\frac{1}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}; \delta([\frac{1}{2}]^{(\rho)}; \sigma)), \]

which implies

\[ L([\frac{1}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma))^{t} = L([\frac{1}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)). \]

Note that \([\frac{1}{2}]^{(\rho)} \times \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)\) is reducible (since besides the Langlands quotient inside of it is \( \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma) \)). Obviously \([\frac{1}{2}]^{(\rho)} \times \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma) \not\subset [\frac{1}{2}]^{(\rho)} \times \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)\).

Now 2.6 implies

\[ [\frac{1}{2}]^{(\rho)} \times d([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma) = L([\frac{1}{2}]^{(\rho)}; \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)) + \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma) \times \delta([\frac{1}{2}]^{(\rho)}; \sigma). \]

Therefore

\[ [\frac{1}{2}]^{(\rho)} \times L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) = L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma) + L([\frac{1}{2}]^{(\rho)}, \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)). \]

The multiplicity of \( L([\frac{1}{2}]^{(\rho)}, \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)) \) in the whole induced representation two. All the other irreducible subquotients have multiplicity one.
8. Unitarizability for integral exponents in generalized rank at most 3 for reducibility $\alpha = 0$

For reducibility $\alpha = 0$ we denote the cuspidal representation of GL by $\psi$ (not by $\rho$, as before), i.e. $\psi \rtimes \sigma$ reduces and $\psi^\vee \cong \psi$. No one case of A.1 – L.1 applies for the reducibility $\alpha = 0$.

8.1. Generalized rank one case.

A.6. The case of exponents $(0)$ and $\alpha = 0$.

Here all irreducible subquotients are unitarizable and $[0]^{(\psi)} \rtimes \sigma = \delta([0]^{(\psi)}_+;\sigma) \oplus \delta([0]^{(\psi)}_-;\sigma)$ is a sum of two irreducible tempered representations. We have

$$\mu^*(\delta([0]^{(\psi)}_\pm;\sigma)) = 1 \otimes \delta([0]^{(\psi)}_\pm;\sigma) + [0]^{(\psi)} \otimes \sigma.$$ 

Note that involution permutes irreducible tempered pieces, i.e.

$$\delta([0]^{(\psi)}_+;\sigma)^t = \delta([0]^{(\psi)}_-;\sigma).$$

8.2. Generalized rank two case.

Here we have only cases B and C (D is equivalent to B).

B.6. The case of exponents $(0, 1)$ and $\alpha = 0$.

We have the following decomposition

$$[1]^{(\psi)} \times [0]^{(\psi)} \rtimes \sigma = L([1]^{(\psi)},\delta([0]^{(\psi)}_+;\sigma)) + L([1]^{(\psi)},\delta([0]^{(\psi)}_-;\sigma)) + 2L([0,1]^{(\psi)};\sigma) + \delta([0,1]^{(\psi)}_+;\sigma) + \delta([0,1]^{(\psi)}_-;\sigma)$$

in the Grothendieck group (see section 5 of [53]). The irreducible representations on the right hand side of equality are unitarizable representations.

We have

$$\mu^*(\delta([0,1]^{(\psi)}_\pm;\sigma)) = 1 \otimes \delta([0,1]^{(\psi)}_\pm;\sigma)$$

(8.59)

$$+ [1]^{(\psi)} \otimes \delta([0]^{(\psi)}_\pm;\sigma)$$

$$+ \delta([0,1]^{(\psi)}_\pm) \otimes \sigma.$$

$$\mu^*(L([1]^{(\psi)},\delta([0]^{(\psi)}_\pm;\sigma)))) = 1 \otimes L([1]^{(\psi)},\delta([0]^{(\psi)}_\pm;\sigma)) +$$

(8.60)

$$+ [-1]^{(\psi)} \otimes \delta([0]^{(\psi)}_\pm;\sigma) +$$
\[
L([-1]^{(\psi)}, [0]^{(\psi)}) \otimes \sigma.
\]

The involution acts as follows
\[
\delta([0, 1]^{(\psi)}; \sigma)^t = L([0, 2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma)), \quad \epsilon \in \{\pm\},
\]
\[
L([1]^{(\psi)}; \sigma) = L([0, 1]^{(\psi)}; \sigma).
\]

C.6. The case of exponents (0, 0) and \(\alpha = 0\).

The representation
\[
[0]^{(\psi)} \times [0]^{(\psi)} \rtimes \sigma = [0]^{(\psi)} \rtimes \delta([0]^{(\psi)}_+; \sigma) \oplus [0]^{(\psi)} \rtimes \delta([0]^{(\psi)}_-; \sigma)
\]
is a sum of two nonequivalent tempered representations. The involution switches them, i.e.
\[
([0]^{(\psi)} \rtimes \delta([0]^{(\psi)}_+; \sigma))^t = [0]^{(\psi)} \rtimes \delta([0]^{(\psi)}_-; \sigma).
\]

8.3. Generalized rank three case.

Here we have only cases E – H (E is equivalent to L; F, I and K are equivalent; G and J are equivalent).

E.6. The case of exponents (0, 1, 2) and \(\alpha = 0\).

Proposition 8.1. Let
\(\alpha = 0\).

(1) The representations
\[
\delta([0, 2]^{(\psi)}; \sigma), \quad L([2]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)), \quad \epsilon \in \{\pm\},
\]
are unitarizable.

(2) The representations
\[
L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma)), \quad L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}_+; \sigma)), \quad \epsilon \in \{\pm\},
\]
\[
L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma), \quad L([0, 2]^{(\psi)}; \sigma).
\]
are not unitarizable.

(3) The Aubert involution switches the representations in the same rows in (1) and (2).

(4) The representations in (1) and (2) are the Jordan-Hölder composition series for
\([0]^{(\psi)} \times [1]^{(\psi)} \times [2]^{(\psi)} \rtimes \sigma\).
We shall prove the above proposition through a number of steps.

(1) One easily sees (from the classification of irreducible square integrable representations of classical groups for example) that \((1)\) and \((2)\) form the Jordan-Hölder series of \([0]^{(\psi)} \times [1]^{(\psi)} \times [2]^{(\psi)} \rtimes \sigma\). Therefore, \((4)\) holds.

\[(2)\] Consider 
\[
[0]^{(\psi)} \times [1]^{(\psi)} \times [2]^{(\psi)} \rtimes \sigma \\
= [2]^{(\psi)} \rtimes \left( L([1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)) + L([1]^{(\psi)}, \delta([0]^{-}\sigma); \sigma)) + 2L([0, 1]^{(\psi)}; \sigma) \\
+ \delta([0, 1]^{(\psi)}; \sigma) + \delta([0, 1]^{-}\sigma); \sigma) \right).
\]

On the right hand side of the above equation, the first two and the last two products are reducible ((\(v\)) of Proposition 6.1 from \([73]\) gives reducibility for the last two products, while the reducibility of the first two products follow from the application of the involution to the previous two products).

In the Jacquet module of \(\delta([0, 2]^{(\psi)}; \sigma)\) is \([2]^{(\psi)} \otimes [1]^{(\psi)} \otimes \tau([0]^{(\psi)}; \sigma)\), and this term characterizes this subquotient. Therefore, the ”dual” term in the Jacquet module is \([-2]^{(\psi)} \otimes [-1]^{(\psi)} \otimes \tau([0]^{(\psi)}; \sigma)\) is in the dual of \(\delta([0, 2]^{-}\sigma); \sigma)\), and it characterizes it. Since it is in the Jacquet module of \(L([2]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))\), we get 
\[
\delta([0, 2]^{(\psi)}; \sigma)^{t} = L([2]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)), \quad \epsilon \in \{\pm 1\}.
\]

The representations on the left hand side are unitarizable since they are square integrable, while the representations on the right hand side are unitarizable by \([19]^{23}\). Therefore, \((1)\) holds.

(3) We shall in this section use several times the following special case of a formula from \([2,10]\)
\[
\mu^{*}(\delta([0, d]^{(\psi)}; \sigma)) = \sum_{j=1}^{d} \delta([j + 1, d]^{(\psi)}; \sigma) \otimes \delta([0, j]^{(\psi)}; \sigma),
\]

where we take formally \(\delta(\emptyset; \sigma) = \sigma\).

We have an epimorphism \([2]^{(\psi)} \rtimes \delta([0, 1]^{(\psi)}; \sigma) \rightarrow L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma))). One directly sees from \((8.62)\)
\[
\delta([0, 2]^{(\psi)}; \sigma) \leq [2]^{(\psi)} \rtimes \delta([0, 1]^{(\psi)}; \sigma),
\]

\(^{23}\)This reference does not cover the case of unitary groups. These groups are covered by results of C. Mœglin (see the section 13 of \([78]\) for more details). She has shown that ASS dual of a general irreducible square integrable representation of a classical group over field of characteristic zero is unitarizable.
and that the multiplicity of this subrepresentation is one (since it is one in the whole induced representation from the cuspidal one). Actually we have embedding above (it follows from the formula and the Frobenius reciprocity).

The above discussion, the fact that (1) and (2) form composition series of the whole induced representation by the cuspidal one, and the properties of the Langlands classification imply

\[(8.63) \quad [2]^{(\psi)} \rtimes \delta([0, 1]^{(\psi)}; \sigma) = L([2]^{(\psi)}; \delta([0, 1]^{(\psi)}; \sigma)) + \delta([0, 2]^{(\psi)}; \sigma), \quad \epsilon \in \{\pm\}\]

(we do not have \(\delta([0, 2]^{(\psi)}; \sigma)\), above, since it has multiplicity one in the whole induced representation from the cuspidal one, and it is in \( [2]^{(\psi)} \rtimes \delta([0, 1]^{(\psi)}; \sigma)\)).

Applying duality we get

\[ [2]^{(\psi)} \rtimes L([1]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma)) = L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma))^t + L([2]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)). \]

Observe that

\[ \delta([1, 2]^{(\psi)} \rtimes \delta([0]^{(\psi)}; \sigma) \mapsto [2]^{(\psi)} \times [1]^{(\psi)} \rtimes \delta([0]^{(\psi)}; \sigma) \mapsto [2]^{(\psi)} \rtimes L([1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)). \]

Suppose

\[ [2]^{(\psi)} \rtimes L([1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)) \leq L([1]^{(\psi)}, [2]^{(\psi)}) \times \delta([0]^{(\psi)}; \sigma). \]

On the level of the GL-Jacquet module this implies

\[ ([2]^{(\psi)} + [-2]^{(\psi)}) \times L([-1]^{(\psi)}, [0]^{(\psi)}) \otimes \sigma \]

\[ \leq (L([1]^{(\psi)}, [2]^{(\psi)}) + [1]^{(\psi)} \times [-2]^{(\psi)} + L([-2]^{(\psi)}, [-1]^{(\psi)}) \times [0]^{(\psi)} \otimes \sigma. \]

Observe that \([2]^{(\psi)} \times L([-1]^{(\psi)}, [0]^{(\psi)}) \otimes \sigma \) shows up on the left hand side, but not on the right hand side. Therefore, the inequality is false, which implies that \(L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)\) is a subquotient of \([2]^{(\psi)} \rtimes L([1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)). \)

Thus

\[ [2]^{(\psi)} \rtimes L([1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)) = L([2]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)) + L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)), \quad \epsilon \in \{\pm\}. \]

The above two composition series imply

\[ L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma))^t = L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)). \]

(4) The last case regarding the duality is to decide how it acts on \(L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma)\) and \(L([0, 2]^{(\psi)}; \sigma)\). These are the only two remaining irreducible subquotients of the whole induced representation. Since the set of other subquotients is invariant for duality, the duality preserves the set \(\{L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma), L([0, 2]^{(\psi)}; \sigma)\}\). Therefore, it acts as identity, or switch them. Observe that \(\delta([0, 2]^{(\psi)})^t \otimes \sigma\) is in the Jacquet module of \(L([0, 2]^{(\psi)}; \sigma)\). One easily sees that \(\delta([0, 2]^{(\psi)})^t \otimes \sigma\) is not in the Jacquet module of \(L([0, 2]^{(\psi)}; \sigma)\) (since it is not in \(\delta([0, 2]^{(\psi)}) \times \sigma\). This implies that the last representation is not self dual, which further implies

\[ L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma)^t = L([0, 2]^{(\psi)}; \sigma). \]
Thus, (3) holds. It remains to prove (2).

(5) Consider now \([2]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)\), which is a self-dual part of the Grothendieck group. Then here is \(L([2]^{(\psi)}; L([0, 1]^{(\psi)}; \sigma)\) a, subquotient, with multiplicity one. Applying duality, we see that here is also \(L([0, 2]^{(\psi)}; \sigma)\) a subquotient, with multiplicity one.

Other possible irreducible subquotients (by the properties of the Langlands classification \(^{2.6}\)) are \(\delta([0, 2]^{(\psi)}; \sigma), L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma))\) and \(L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))\) with \(\epsilon \in \{\pm\}\). Applying the involution, we see that no representation of the form \(\delta([0, 2]^{(\psi)}; \sigma)\) can be a subquotient of \([2]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)\) (this would contradict the properties of the Langlands classification \(^{2.6}\)). Further, one would have some \(L([2]^{(\psi)}; \delta([0, 1]^{(\psi)}; \sigma))\) for a subquotient if and only if one would have \(L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))\) for a subquotient. Consider

\[
\mu^*(\tau) \times L([0, 1]^{(\psi)}; \sigma) = (1 \otimes [2]^{(\psi)} + [2]^{(\psi)} \otimes 1 + [-2]^{(\psi)} \otimes 1) \\
\times \left( 1 \otimes L([0, 1]^{(\psi)}; \sigma) + [0]^{(\psi)} \otimes [1]^{(\psi)} \times \sigma + \delta([-1, 0]^{(\psi)}) \otimes \sigma + L([0]^{(\psi)}, [1]^{(\psi)} \otimes \sigma) \right)
\]

This gives the top Jacquet module to be

\[
[2]^{(\psi)} \otimes L([0, 1]^{(\psi)}; \sigma) + [-2]^{(\psi)} \otimes L([0, 1]^{(\psi)}; \sigma) + [0]^{(\psi)} \otimes [2]^{(\psi)} \times [1]^{(\psi)} \times \sigma.
\]

Obviously, here is not a subquotient a term of the form \([-2]^{(\psi)} \otimes \delta([0, 1]^{(\psi)}; \sigma)\). This and the above discussion imply

\[(8.64) \quad \tau \times L([0, 1]^{(\psi)}; \sigma) = L([2]^{(\psi)}; L([0, 1]^{(\psi)}; \sigma)) + L([0, 2]^{(\psi)}; \sigma).
\]

Therefore, the whole induced representation is of length 12.

**Lemma 8.2.** Let \(\alpha = 0\). The representation \(L([0, 2]^{(\psi)}; \sigma)\) is not unitarizable.

**Proof.** We shall consider

\[
\delta([-1, 1]^{(\psi)}) \times L([0, 2]^{(\psi)}; \sigma).
\]

From \(^{2.7}\), we know that we have two irreducible subquotients:

\[
L([0, 2]^{(\psi)}; \delta([-1, 1]^{(\psi)}; \sigma)).
\]

The equality \(\delta([-1, 2]^{(\psi)}) \times \sigma = L([0, 2]^{(\psi)}; \sigma) + \delta([0, 2]^{(\psi)}; \sigma) + \delta([0, 2]_{\pm}^{(\psi)}; \sigma)\) (see \(^{2.10}\)) implies that if we have an irreducible non-tempered subquotient of

\[
\delta([-1, 1]^{(\psi)}) \times \delta([0, 2]^{(\psi)}) \not\times \sigma,
\]

then it is a subquotient of \(\delta([-1, 1]^{(\psi)}) \times L([0, 2]^{(\psi)}; \sigma)\).

Now we shall list some non-tempered subquotient of \(\delta([-1, 1]^{(\psi)}) \times \delta([0, 2]^{(\psi)}) \not\times \sigma\). Observe that

\[
\delta([-1, 1]^{(\psi)}) \times \delta([0, 2]^{(\psi)}) \not\times \sigma \geq \delta([-1, 2]^{(\psi)}) \times \delta([0, 1]^{(\psi)}) \not\times \sigma
\]

\[
= \delta([-1, 2]^{(\psi)}) \times \left( L([0, 1]^{(\psi)}; \sigma) + \delta([0, 1]_{\pm}^{(\psi)}; \sigma) + \delta([0, 1]^{(\psi)}; \sigma) \right)
\]
\[ = \delta([0, 1]^{(\psi)}) \times \left( L([-1, 2]^{(\psi)}; \sigma) + \delta([-1, 2]_{-}^{(\psi)}; \sigma) + \delta([-1, 2]_{+}^{(\psi)}; \sigma) \right). \]

From this we conclude that we have the following non-tempered subquotients of the representation \( \delta([-1, 1]^{(\psi)}) \times L([0, 2]^{(\psi)}; \sigma) \):

\[
L([-1, 2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma), \quad L([-1, 2]^{(\psi)}, d([0, 1]_{\pm}^{(\psi)}; \sigma)), \quad L([0, 1]^{(\psi)}, d([-1, 2]_{\pm}^{(\psi)}; \sigma)).
\]

Therefore, the length of \( \delta([-1, 1]^{(\psi)}) \times L([0, 2]^{(\psi)}; \sigma) \) is at least 7.

Let \( \pi \) be an irreducible subrepresentation of \( \delta([-1, 1]^{(\psi)}) \times L([0, 2]^{(\psi)}; \sigma) \). Then passing to the hermitian contragredient we get that it is a quotient. Now the second adjointness implies that it has in its Jacquet module

\[ \delta([-1, 1]^{(\psi)}) \otimes L([0, 2]^{(\psi)}; \sigma) \]

for a subrepresentation.

Suppose that \( L([0, 2]^{(\psi)}; \sigma) \) is unitarizable.

Therefore, the Jacquet module of \( \delta([-1, 1]^{(\psi)}) \times L([0, 2]^{(\psi)}; \sigma) \) has at least 7 copies of \( \tau := \delta([-1, 1]^{(\psi)}) \otimes L([0, 2]^{(\psi)}; \sigma) \) as subrepresentations.

Consider now

\[ \mu^*(\delta([-1, 1]^{(\psi)}) \times L([0, 2]^{(\psi)}; \sigma)). \]

Recall

\[ M^*(\delta([-1, 1]^{(\psi)})) = \sum_{-2 \leq x \leq 1} \delta([-x, 1]^{(\psi)}) \times \sum_{x \leq y \leq 1} \delta([y + 1, 1]^{(\psi)}) \otimes \delta([x + 1, y]^{(\psi)}), \]

\[ \mu^*(L([0, 2]^{(\psi)}; \sigma)) \]

\[ = \sum_{-1 \leq i \leq 2} \sum_{0 \leq i+j \leq 2} L([-i, 0]^{(\psi)}, [j + 1, 2]^{(\psi)}) \otimes L([i + 1, j]^{(\psi)}; \sigma) + \]

\[ + \sum_{i=0}^{2} L([-i, 0]^{(\psi)}, [i + 1, 2]^{(\psi)}) \otimes \sigma, \]

i.e.

\[ (8.65) \quad \mu^*(L([0, 2]^{(\psi)}; \sigma)) = 1 \otimes L([0, 2]^{(\psi)}; \sigma) \]

\[ + [2]^{(\psi)} \otimes L([0, 1]^{(\psi)}; \sigma) + [0]^{(\psi)} \otimes \delta([1, 2]^{(\psi)}) \times \sigma \]

\[ + [0]^{(\psi)} \times [2]^{(\psi)} \otimes [1]^{(\psi)} \times \sigma + \delta([-1, 0]^{(\psi)}) \times [1]^{(\psi)} \times [2]^{(\psi)} \times \sigma \]

\[ + L([0]^{(\psi)}, [1, 2]^{(\psi)}) \otimes \sigma + \delta([-1, 0]^{(\psi)}) \times [2]^{(\psi)} \otimes \sigma + \delta([-2, 0]^{(\psi)}) \otimes \sigma. \]

We have precisely two terms that can play role from the last formula to get \( \tau \) for a subquotient.
The first is $1 \otimes L([0, 2]^{(\psi)}; \sigma)$, and this will give with $2 \cdot \delta([-1, 1]^{(\psi)}) \otimes 1$ from the first formula, multiplicity two of $\tau$.

The remaining term is $\delta([-1, 0]^{(\psi)}) \otimes [2]^{(\psi)} \rtimes \sigma$. From the first formula we must take $[1]^{(\psi)} \otimes [0, 1]^{(\psi)}$ or $[1]^{(\psi)} \otimes [-1, 0]^{(\psi)}$.

So, on the right hand side of $\otimes$ we get in the Grothendieck group two times

$$[0, 1]^{(\psi)} \rtimes [2]^{(\psi)} \rtimes \sigma = [2]^{(\psi)} \rtimes \left( L([0, 1]^{(\psi)}; \sigma) + \delta([0, 1]^{(\psi)}; \sigma) + \delta([0, 1]^{(\psi)}; \sigma) \right).$$

Observe that $L([0, 2]^{(\psi)}; \sigma)$ can be subquotient only of $[2]^{(\psi)} \rtimes L([0, 1]^{(\psi)}; \sigma)$ (because of the exponent $-1$). Now from (8.6), we see that the multiplicity is one.

So, the total multiplicity is 4. This is a contradiction, which completes the proof of non-unitarizability of $L([0, 2]^{(\psi)}; \sigma)$. □

**Corollary 8.3.** For $\alpha = 0$, the representations $L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma)$ is not unitarizable.

**Proof.** Consider first

$$\left( \delta([-1, 1]^{(\psi)}) \rtimes L([0, 2]^{(\psi)}; \sigma) \right)^t = \delta([-1, 1]^{(\psi)})^t \rtimes L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma).$$

From the previous case we know that this representation has length at least seven.

Suppose that

$$L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma)$$

is unitarizable. Then in the Jacquet module of $\delta([-1, 1]^{(\psi)})^t \rtimes L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma)$ we would have $\delta([-1, 1]^{(\psi)})^t \otimes L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma)$ with multiplicity at least seven. Passing to the dual picture, we would get that in the Jacquet module of

$$\delta([-1, 1]^{(\psi)}) \rtimes L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma)^t = \delta([-1, 1]^{(\psi)}) \rtimes L([0, 2]^{(\psi)}; \sigma)$$

the representation

$$((\delta([-1, 1]^{(\psi)})^t)^t) \otimes L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma)^t = \delta([-1, 1]^{(\psi)} \otimes L([0, 2]^{(\psi)}; \sigma)$$

has multiplicity at least seven. We have seen that this is not the case.

This completes the proof of non-unitarizability of $L([2]^{(\psi)}, [0, 1]^{(\psi)}; \sigma)$. □

(6) The rest of the proof of the proposition is mainly the proof that $L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma))$ is not unitarizable (from this proof will easily follow that the dual representation is not unitarizable).

We shall now compute $L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma))$. First we compute

$$\mu^*([2]^{(\psi)} \rtimes \delta([0, 1]^{(\psi)}; \sigma)) =$$

$$(1 \otimes [2]^{(\psi)} + [2]^{(\psi)} \otimes 1 + [-2]^{(\psi)} \otimes 1) \rtimes \sigma.$$
\[
\left( 1 \otimes \delta(0, 1]_\epsilon^{(v)}; \sigma) + [1]^{(v)} \otimes \delta([0]_\epsilon^{(v)}; \sigma) + \delta([0, 1]^{(v)} \otimes 1) \right)
\]

\[
= 1 \otimes [2]^{(v)} \times \delta(0, 1]_\epsilon^{(v)}; \sigma) + [2]^{(v)} \otimes \delta([0, 1]_\epsilon^{(v)}; \sigma) + [1]^{(v)} \otimes [2]^{(v)} \times \delta([0]_\epsilon^{(v)}; \sigma)
\]

\[
+ [2]^{(v)} \times [1]^{(v)} \otimes \delta([0]_\epsilon^{(v)}; \sigma) + [2]^{(v)} \times [1]^{(v)} \otimes \delta([0]_\epsilon^{(v)}; \sigma) + \delta([0, 1]^{(v)} \otimes [2]^{(v)} \times \sigma + (\delta([0, 2]^{(v)} \otimes 1 + L([2]^{(v)}; [0, 1]^{(v)}) \otimes 1) + [-2]^{(v)} \times \delta([0, 1]^{(v)} \otimes 1.
\]

Now the above formula, \((8.63)\) and \((8.62)\) imply
\[
(8.67)
\]

\[
\mu^*(L([2]^{(v)}; \delta([0, 1]_\epsilon^{(v)}; \sigma)) =
\]

\[
= 1 \otimes L(2]^{(v)}; \delta([0, 1]_\epsilon^{(v)}; \sigma)) + [2]^{(v)} \otimes \delta([0, 1]_\epsilon^{(v)}; \sigma)
\]

\[
+ L([1]^{(v)}; [2]^{(v)} \otimes \delta([0]_\epsilon^{(v)}; \sigma) + [2]^{(v)} \times [1]^{(v)} \otimes \delta([0]_\epsilon^{(v)}; \sigma) + \delta([0, 1]^{(v)} \otimes [2]^{(v)} \times \sigma + L([2]^{(v)}; [0, 1]^{(v)}) \otimes \sigma + [-2]^{(v)} \times \delta([0, 1]^{(v)} \otimes \sigma.
\]

Applying duality to this and changing \(-\epsilon\) by \(\epsilon\), we get
\[
(8.67)
\]

\[
\mu^*(L([1, 2]^{(v)}; \delta([0]_\epsilon^{(v)}; \sigma)) =
\]

\[
= 1 \otimes L([1, 2]^{(v)}; \delta([0]_\epsilon^{(v)}; \sigma)) + [2]^{(v)} \times \delta([0]_\epsilon^{(v)}; \sigma)
\]

\[
+ \delta([2]^{(v)} \otimes \delta([0, 1]^{(v)}; \sigma) + [-1]^{(v)} \otimes [2]^{(v)} \times \delta([0]_\epsilon^{(v)}; \sigma)
\]

\[
+ L([1]^{(v)}; [-1]^{(v)} \otimes \delta([0]_\epsilon^{(v)}; \sigma) + [2]^{(v)} \times [-1]^{(v)} \otimes \delta([0]_\epsilon^{(v)}; \sigma) + L([-1]^{(v)}; [0]^{(v)}) \otimes [2]^{(v)} \times \sigma + L([-2, -1]^{(v)}; [0]^{(v)}) \otimes \sigma + [2]^{(v)} \times L([-1]^{(v)}; [0]^{(v)}) \otimes \sigma^{24}.
\]

(7) We shall now describe the composition series of the representation
\[
\delta([1, 2]^{(v)} \times \delta([0]_\epsilon^{(v)}; \sigma).
\]

Observe
\[
\delta([0, 2]^{(v)}; \sigma) + L([1, 2]^{(v)}; \tau([0]_\epsilon^{(v)}; \sigma)) \leq \delta([1, 2]^{(v)} \times \delta([0]_\epsilon^{(v)}; \sigma).
\]

We have
\[
\delta([0, 2]^{(v)} \times \sigma \leftrightarrow \delta([1, 2]^{(v)} \times [0]^{(v)} \times \sigma \rightarrow \delta([1, 2]^{(v)} \times \delta([0]_\epsilon^{(v)}; \sigma).
\]

Suppose
\[
\delta([1, 2]^{(v)} \times \delta([0]_\epsilon^{(v)}; \sigma) \leq L([1, 2]^{(v)}; [0]^{(v)}) \times \sigma.
\]

For the Jacquet of GL-type modules this implies when we consider the terms whose exponents are non-negative, that holds
\[
\delta([1, 2]^{(v)} \times [0]^{(v)} \times \sigma \leq L([1, 2]^{(v)}; [0]^{(v)}) \otimes \sigma.
\]

This is a contradiction. This implies directly the following subquotient claim
\[
L([0, 2]^{(v)}; \sigma) \leq \delta([1, 2]^{(v)} \times \delta([0]_\epsilon^{(v)}; \sigma).
\]

\[24\] We could use also \([2]^{(v)} \times L([1]^{(v)}; \delta([0]_\epsilon^{(v)}; \sigma) = L([1, 2]^{(v)}; \delta([0]_\epsilon^{(v)}; \sigma) + L([2]^{(v)}; [1]^{(v)}; \delta([0]_\epsilon^{(v)}; \sigma))\) to compute the above Jacquet module formula.
This implies that $\delta([1,2]^{(\psi)}_\epsilon) \times \delta([0]^{(\psi)}_\epsilon; \sigma)$ has length $\geq 3$, $\epsilon \in \{\pm\}$. Therefore, the dual representation has length $\geq 3$, $\epsilon \in \{\pm\}$. Now the fact that the length of the whole induced representation is 12 implies
\begin{equation}
\delta([1,2]^{(\psi)}_\epsilon) \times \delta([0]^{(\psi)}_\epsilon; \sigma) = L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\epsilon; \sigma)) + L([0,2]^{(\psi)}_\epsilon; \sigma) + \delta([0,2]^{(\psi)}_\epsilon; \sigma).
\end{equation}

For proving that $L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\pm; \sigma))$ is not unitarizable, we shall consider
\[\delta([-1,1]^{(\psi)}_\epsilon) \times L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\epsilon; \sigma)).\]

First we shall determine the following multiplicity:

**Lemma 8.4.** Let $\alpha = 0$. The multiplicity of
\[
\tau := \delta([-1,1]^{(\psi)}_\epsilon) \otimes L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\epsilon; \sigma))
\]
in
\[\mu^*(\delta([-1,1]^{(\psi)}_\epsilon) \times L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\epsilon; \sigma))).\]
is four.

**Proof.** Recall
\[\mu^*(\delta([-1,1]^{(\psi)}_\epsilon) \times L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\epsilon; \sigma))) =
M^*(\delta([-1,1]^{(\psi)}_\epsilon)) \times \mu^*(L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\epsilon; \sigma))),
\]
\[M^*(\delta([-1,1]^{(\psi)}_\epsilon)) = \sum_{-2 \leq x \leq 1} \delta([-x,1]^{(\psi)}_\epsilon) \times \sum_{x \leq y \leq 1} \delta([y + 1,1]^{(\psi)}_\epsilon) \otimes \delta([x + 1, y]^{(\psi)}_\epsilon).
\]

There are two possibilities to get $\tau$ from $\mu^*(L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\epsilon; \sigma)))$ (see (8.67) for the formula for this). The first is to take from it $1 \otimes L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\epsilon; \sigma))$. For this term one gets multiplicity two of $\tau$ (for $x = 1, y = 1$ and for $x = y = -2$ each time one gets one $\tau$).

The second possibility is to take $[-1]^{(\psi)}_\epsilon \otimes [2]^{(\psi)}_\epsilon \times \delta([0]^{(\psi)}_\epsilon; \sigma)$. Then we have two possibilities for the term from $M^*(\delta([-1,1]^{(\psi)}_\epsilon))$. The first possibility is $x = 0, y = 1$. This gives the term $\delta([0,1]^{(\psi)}_\epsilon) \times [1]^{(\psi)}_\epsilon$, which after multiplication gives
\[\delta([0,1]^{(\psi)}_\epsilon) \times [1] \otimes [-1]^{(\psi)}_\epsilon \otimes [1]^{(\psi)}_\epsilon \times [2]^{(\psi)}_\epsilon \times \delta([0]^{(\psi)}_\epsilon; \sigma).
\]
The multiplicity of $\delta([-1,1]^{(\psi)}_\epsilon)$ in the first tensor factor above is one. Regarding the second factor observe
\[\delta([1]^{(\psi)}_\epsilon) \times [2]^{(\psi)}_\epsilon \times \delta([0]^{(\psi)}_\epsilon; \sigma) = \delta([1,2]^{(\psi)}_\epsilon) \times \delta([0]^{(\psi)}_\epsilon; \sigma) + L([1]^{(\psi)}_\epsilon, [2]^{(\psi)}_\epsilon) \times \delta([0]^{(\psi)}_\epsilon; \sigma).
\]
The multiplicity of $L([1,2]^{(\psi)}_\epsilon; \delta([0]^{(\psi)}_\epsilon; \sigma))$ in the first summand on the right hand side is one. From (8.68) and (3) of the proposition (which we have already proved) we get that the multiplicity in the other summand is zero.

The second possibility for the term from $M^*(\delta([-1,1]^{(\psi)}_\epsilon))$ we get for $x = -2, y = -1$, which gives the term $1 \times \delta([0,1]^{(\psi)}_\epsilon) \otimes [-1]^{(\psi)}_\epsilon$. The above analysis gives multiplicity one also in this case.
In the same way as above we get multiplicity one of \( \tau \) also here.

(9) Now we shall determine several irreducible subquotients of
\[
\pi := \delta([-1,1]^{(\psi)} \times L([-1,2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)))
\]

Before we need to introduce a notation which we shall use in description of that irreducible subquotients. Let \( \theta \) be a an tempered representation of a classical group and \( \Delta \) a segment of cuspidal representations such that \( \Delta = \Delta \) and that \( \delta(\Delta) \times \theta \) reduces. Then it reduces into a sum of two nonequivalent irreducible tempered subrepresentations which will be denoted by
\[
\tau(\Delta \pm; \theta)
\]

First from 2.7 we get that the following two irreducible representations
\[
L([-1,2]^{(\psi)}; \tau([-1,1]^{(\psi)}; \delta([0]^{(\psi)}; \sigma))), \quad \mu \in \{\pm\}.
\]
are subquotients of \( \pi \).

It will require much more work to determine additional irreducible subquotients of \( \pi \). First we shall list some natural candidates for subquotients.

Summing the identity (8.68) for \( \epsilon = 1 \) and \(-1\) we get
\[
(8.69) \quad \delta([-1,2]^{(\psi)} \times [0]^{(\psi)} \times \sigma = \delta([0,2]^{(\psi)}; \sigma) + \delta([0,2]^{(\psi)}; \sigma) +
L([-1,2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)) + L([-1,2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)) + 2 \cdot L([0,2]^{(\psi)}; \sigma).
\]

Observe that
\[
\delta([-1,1]^{(\psi)} \times [0]^{(\psi)} \times \sigma \geq \delta([-1,2]^{(\psi)} \times [1]^{(\psi)} \times [0]^{(\psi)} \times \sigma.
\]

We know that the left hand side of the above inequality has among others the following non-tempered irreducible subquotients:
\[
L([0,2]^{(\psi)}; \delta([-1,1]^{(\psi)}; \sigma)),
L([-1,2]^{(\psi)}; [1]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)),
L([-1,2]^{(\psi)}; \delta([0,1]^{(\psi)}; \sigma)),
L([-1,2]^{(\psi)}; [0,1]^{(\psi)}; \sigma),
L([1]^{(\psi)}; \tau([0]^{(\psi)}; \delta([-1,2]^{(\psi)}; \sigma))), \quad \epsilon_1, \epsilon_2 \in \{\pm\}.
\]

Multiplying (8.68) by \( \delta([-1,1]^{(\psi)} \times \delta([1,2]^{(\psi)} \times [0]^{(\psi)}; \sigma) =
\[
\delta([-1,1]^{(\psi)} \times \left( L([1,2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)) + L([0,2]^{(\psi)}; \sigma) + \delta([0,2]^{(\psi)}; \sigma) \right)
\]

\[25\]One can be more specific in description of these representations, like in [75], but we do not need these details for the purpose of this paper.
Then (8.70) has among others the following irreducible subquotients:

\[ L([0, 2]^{(\psi)}; \delta([1, -1, 1]^{(\psi)}; \mu)) \]
\[ L([-1, 2]^{(\psi)}, [1]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)) \]

\[ \mu \in \{ \pm \} \]

Now we prove

**Lemma 8.5.** For \( \alpha = 0 \) holds

\[ L([-1, 2]^{(\psi)}, [1]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)) \leq \delta([-1, 1]^{(\psi)}) \times L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)) \]

**Proof.** Observe

\[ L([-1, 1]^{(\psi)}, [1, 2]^{(\psi)} \times \delta([0]^{(\psi)}; \sigma)) \rightarrow \]
\[ \delta([-1, 1]^{(\psi)}) \times \delta([1, 2]^{(\psi)}) \times \delta([0]^{(\psi)}; \sigma)) \rightarrow \delta([-1, 1]^{(\psi)}) \times L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)) \]

Suppose

\[ \delta([-1, 1]^{(\psi)}) \times L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)) \leq \delta([-1, 1]^{(\psi)}, [1, 2]^{(\psi)} \times \delta([0]^{(\psi)}; \sigma)) \]

Use the formula (8.67) to get that that in the Jacquet module of the left hand side (among others) we have

\[ 2 \cdot \delta([0, 1]^{(\psi)}) \times [1]^{(\psi)} \times [2]^{(\psi)} \times L([-1]^{(\psi)}, [0]^{(\psi)}) \otimes \sigma \]

Obviously, here shows up

\[ 2 \cdot \delta([0, 1]^{(\psi)}) \times [1]^{(\psi)} \times [2]^{(\psi)} \times L([-1]^{(\psi)}, [0]^{(\psi)}) \otimes \sigma \]

In particular,

\[ 2 \cdot L([-1]^{(\psi)}, [0]^{(\psi)}, [0, 1]^{(\psi)}, [1, 2]^{(\psi)} \otimes \sigma \]

is in the Jacquet module of the left hand side of (8.71).

Now in the Jacquet module of the right hand side of (8.71), considering the cuspidal support, we see that the only terms that can dominate the above term are

\[ [0]^{(\psi)} \times \left( L([-1, 1]^{(\psi)}, [1, 2]^{(\psi)}) + L([0, 1]^{(\psi)}, [2]^{(\psi)} \times [-1]^{(\psi)} \times [1]^{(\psi)} + L([0, 1]^{(\psi)}, [-1]^{(\psi)} \times L([1, 2]^{(\psi)}) \right) \]

(one gets the above upper bound by direct computation, or using the formula (2.18)).

Obviously the term \([0]^{(\psi)} \times L([-1, 1]^{(\psi)}, [1, 2]^{(\psi)}) \) has disjoint composition series with the left hand side (it has a segment of length three). Therefore

\[ 2 \cdot L([-1]^{(\psi)}, [0]^{(\psi)}, [0, 1]^{(\psi)}, [1, 2]^{(\psi)}) \leq \]
\[ [0]^{(\psi)} \times \left( L([0, 1]^{(\psi)}, [2]^{(\psi)} \times [-1]^{(\psi)} \times [1]^{(\psi)} + L([0, 1]^{(\psi)}, [-1]^{(\psi)} \times L([1, 2]^{(\psi)}) \right) \]
Suppose
\[ L([-1]^{(ψ)}, [0]^{(ψ)}, [0, 1]^{(ψ)}, [1, 2]^{(ψ)}) \leq [0]^{(ψ)} \times L([0, 1]^{(ψ)}, [-1]^{(ψ)}) \times L([1], [2]^{(ψ)}). \]

Then applying the Zelevinsky involution we get
\[ Z([-1]^{(ψ)}, [0]^{(ψ)}, [0, 1]^{(ψ)}, [1, 2]^{(ψ)}) \leq [0]^{(ψ)} \times Z([0, 1]^{(ψ)}, [-1]^{(ψ)}) \times Z([1], [2]^{(ψ)}). \]

Now the highest derivative of the left hand side is less than or equal to the derivative of the right hand side, i.e.
\[ Z([0]^{(ψ)}, [1]^{(ψ)}) \leq ([0]^{(ψ)} + 1) \times \\
(\frac{(Z([0, 1]^{(ψ)}, [-1]^{(ψ)}) + Z([0]^{(ψ)}, [1]^{(ψ)}) + [0]^{(ψ)} \times \\
(\frac{(Z([1], [2]^{(ψ)}) + [2] + Z([0]))}{
}

Obviously, we can not get this inequality (if we want to have \([1]^{(ψ)})\), we will have also \([2]^{(ψ)}\), which implies that we cannot get \(Z([0]^{(ψ)}, [1]^{(ψ)})\) as a subquotient of the right hand side). This implies
\[ 2 \cdot L([-1]^{(ψ)}, [0]^{(ψ)}, [0, 1]^{(ψ)}, [1, 2]^{(ψ)}) \leq [0]^{(ψ)} \times L([0, 1]^{(ψ)}, [2]^{(ψ)}) \times [-1]^{(ψ)} \times [1]^{(ψ)}. \]

Applying Zelevinsky involution, we would get
\[ 2 \cdot Z([-1]^{(ψ)}, [0]^{(ψ)}, [0, 1]^{(ψ)}, [1, 2]^{(ψ)}) \leq [0]^{(ψ)} \times Z([0, 1]^{(ψ)}, [2]^{(ψ)}) \times [-1]^{(ψ)} \times [1]^{(ψ)}. \]

Again the highest derivative of the left hand side is \(\leq\) then the derivative of the right hand side, i.e.
\[ 2 \cdot Z([0]^{(ψ)}, [1]^{(ψ)}) \leq ([0]^{(ψ)} + 1) \times \\
(\frac{(Z([0, 1]^{(ψ)}, [2]^{(ψ)}) + [0]^{(ψ)} \times [2]^{(ψ)} + [0]^{(ψ)} \times \\
(\frac{([-1]^{(ψ)} + 1) \times ([1]^{(ψ)} + 1)}{
}

Now if we want to get \(Z([0]^{(ψ)}, [1]^{(ψ)})\) from the above derivative on the right hand side of he above inequality, we obviously must take from the first factor 1 (otherwise, we would have \([0]^{(ψ)}\) with multiplicity two in the cuspidal support), from the second factor \([0]^{(ψ)}\) (because of the cuspidal support), 1 from the third factor, and \([1]^{(ψ)}\) from the last factor. This gives \([0]^{(ψ)} \times [1]^{(ψ)}\), and the multiplicity of \(Z([0]^{(ψ)}, [1]^{(ψ)})\) here is one. So we get contradiction. Therefore, the above inequality cannot hold.

This easily implies that we have a non-zero morphism
\[ \delta([-1, 2]^{(ψ)} \times [1]^{(ψ)} \times \delta([0]^{(ψ)}; \sigma)) \rightarrow \delta([-1, 1]^{(ψ)} \times L([1], [2]^{(ψ)}; \delta([0]^{(ψ)}; \sigma). \]

Thus
\[ L([-1, 2]^{(ψ)}, [1]^{(ψ)}; d([0]^{(ψ)}; \sigma)) \leq \delta([-1, 1]^{(ψ)} \times L([1], [2]^{(ψ)}; \delta([0]^{(ψ)}; \sigma)), \]
what we needed to prove.
After this lemma we know that \( \delta([-1, 1]^{(\psi)}) \rtimes L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)) \) has length at least three.

(10) At the end we shall prove that there are two more irreducible subquotients of \( \delta([-1, 1]^{(\psi)}) \rtimes L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}; \sigma)) \). Recall

\[
(8.72) \quad \sum_{\epsilon_1, \epsilon_2 \in \{\pm\}} L([1]^{(\psi)}; \tau([0]^{(\psi)}_{\epsilon_1}; \delta([-1, 2]^{(\psi)}_{\epsilon_2}; \sigma))) \leq \delta([-1, 1]^{(\psi)}) \times \delta([1, 2]^{(\psi)}) \times [0]^{(\psi)} \rtimes \sigma.
\]

We shall show next that no one of \( L([1]^{(\psi)}; \tau([0]^{(\psi)}_{\epsilon_1}; \delta([-1, 2]^{(\psi)}_{\epsilon_2}; \sigma))) \) is a subquotient of \( L([-1, 1]^{(\psi)}) \rtimes L([0, 2]^{(\psi)}; \sigma) \), i.e.

**Lemma 8.6.** Let \( \alpha = 0 \). For each \( \epsilon_1, \epsilon_2 \in \{\pm\} \) holds

\[
L([1]^{(\psi)}; \tau([0]^{(\psi)}_{\epsilon_1}; \delta([-1, 2]^{(\psi)}_{\epsilon_2}; \sigma))) \not\leq L([-1, 1]^{(\psi)}) \rtimes L([0, 2]^{(\psi)}; \sigma).
\]

**Proof.** To prove the lemma, it will be enough to show that in the Jacquet module of \( \delta([-1, 1]^{(\psi)}) \rtimes L([0, 2]^{(\psi)}; \sigma) \) we do not have terms of the form \( [-1]^{(\psi)} \otimes \sigma \). For this, it is enough to show that we have neither terms of such form in \( M^*(\delta([-1, 1]^{(\psi)})) \) nor in \( \mu^*(L([0, 2]^{(\psi)}; \sigma)) \). From the formula

\[
(8.73) \quad M^*(\delta([-1, 1]^{(\psi)})) = \sum_{-2 \leq x \leq 1} \delta([-x, 1]^{(\psi)}) \times \sum_{x \leq y \leq 1} \delta([y + 1, 1]^{(\psi)}) \otimes \delta([x + 1, y]^{(\psi)})
\]

and the formula (8.65) for \( \mu^*(L([0, 2]^{(\psi)}; \sigma)) \) we immediately see that there are no (non-zero) terms of the form \( 1 \otimes \sigma \). This completes the proof of the lemma. \( \square \)

Now (8.69) and the above lemma imply

\[
(8.74) \quad \sum_{\epsilon_1, \epsilon_2 \in \{\pm\}} L([1]^{(\psi)}; \tau([0]^{(\psi)}_{\epsilon_1}; \delta([-1, 2]^{(\psi)}_{\epsilon_2}; \sigma))) \leq \sum_{\epsilon \in \{\pm\}} \delta([-1, 1]^{(\psi)}) \rtimes (L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}_{\epsilon}; \sigma))
\]

We shall now prove that each term on the right hand side has two terms from the left hand side as a subquotient.

We analyze in the Jacquet module of \( \delta([-1, 1]^{(\psi)}) \rtimes L([1, 2]^{(\psi)}; \delta([0]^{(\psi)}_{\epsilon}; \sigma)) \) terms of the form \( [-1]^{(\psi)} \otimes \tau([0]^{(\psi)}_{\epsilon_1}; \delta([-1, 2]^{(\psi)}_{\epsilon_2}; \sigma)) \).

Now the formula (8.67) and (8.73) imply we need to find how many times show up representations \( [-1]^{(\psi)} \otimes \tau([0]^{(\psi)}_{\epsilon_1}; \delta([-1, 2]^{(\psi)}_{\epsilon_2}; \sigma)) \) in \( [-1]^{(\psi)} \otimes \delta([-1, 1]^{(\psi)}) \times [2]^{(\psi)} \rtimes \delta([0]^{(\psi)}_{\epsilon}; \sigma) \).

Equivalently, we need to consider how many times is \( \tau([0]^{(\psi)}_{\epsilon_1}; \delta([-1, 2]^{(\psi)}_{\epsilon_2}; \sigma)) \) in the representation \( \delta([-1, 1]^{(\psi)}) \times [2]^{(\psi)} \rtimes \delta([0]^{(\psi)}_{\epsilon}; \sigma) \).

Observe that in the Jacquet module of each four representations \( \tau([0]^{(\psi)}_{\epsilon_1}; \delta([-1, 2]^{(\psi)}_{\epsilon_2}; \sigma)) \) we have always \( [0]^{(\psi)} \times \delta([-1, 2]^{(\psi)}) \otimes \sigma \) (since each of the four representations embeds into \( [0]^{(\psi)} \times \delta([-1, 2]^{(\psi)}) \rtimes \sigma \)). The Jacquet module of \( \delta([-1, 1]^{(\psi)}) \times [2]^{(\psi)} \rtimes \delta([0]^{(\psi)}_{\epsilon}; \sigma) \),
which can give the above part of the Jacquet module for a subquotient, is $2 \cdot \delta([-1, 1]^{(\psi)}) \times [2]^{(\psi)} \times [0]^{(\psi)} \otimes \sigma$. Here the multiplicity of $[0]^{(\psi)} \times \delta([-1, 2]^{(\psi)}) \otimes \sigma$ is two. Therefore, at most two of the above four representations can show up in $\delta([-1, 1]^{(\psi)}) \times L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))$. Since they show up four times in the sum of $\delta([-1, 1]^{(\psi)}) \times L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))$ and $\delta([-1, 1]^{(\psi)}) \times L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))$, this implies that they show up two times in each of these representations.

Therefore, we have proved that $\delta([-1, 1]^{(\psi)}) \times L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))$ is a representation of length at least five.

Now directly follows the following

**Corollary 8.7.** For $\alpha = 0$, the representation $L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))$ is not unitarizable.

*Proof.* Suppose that it is unitarizable, and consider $\delta([-1, 1]^{(\psi)}) \times L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))$. This representation is of length at least five, while the multiplicity of $\delta([-1, 1]^{(\psi)}) \otimes L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))$ in the Jacquet module of $\delta([-1, 1]^{(\psi)}) \times L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma))$ is four. This implies contradiction and completes the proof. \(\square\)

(11) Now we shall prove that $L([2]^{(\psi)}; \delta([1, 0]^{(\psi)}; \sigma))$ is not unitarizable. Recall that we have $L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma))^t = L([1, 2]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)), \ \epsilon \in \{\pm\}$.

**Corollary 8.8.** The representation $L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma))$ is not unitarizable for $\alpha = 0$.

*Proof.* Suppose that it is unitarizable, and consider $\delta([-1, 1]^{(\psi)})^t \times L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma))$. From the dual case follows that this is a representation of length at least five. Further, the multiplicity of $\delta([-1, 1]^{(\psi)})^t \otimes L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma))$ in the Jacquet module of $\delta([-1, 1]^{(\psi)})^t \times L([2]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma))$ is four (which one concludes from the dual case). This implies contradiction (in the same was as before). This completes the proof. \(\square\)

**F.6. The case of exponents $(0, 1, 1)$ and $\alpha = 0$.**

**Proposition 8.9.** Let $\alpha = 0$.

(1) The representations

$$L([1]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}; \sigma)), \ \delta([-1, 1]^{(\psi)}; \sigma), \ \epsilon \in \{\pm\},$$

$$L([1]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma)), \ \ L([1]^{(\psi)}, \delta([0, 1]^{(\psi)}; \sigma)),$$

are unitarizable.
We have further

\[ L([1]^{(\psi)}; [0, 1]^{(\psi)}; \sigma) \]

is not unitarizable.

(3) The Aubert involution switches the representations in the same rows in (1) and fixes the representation in (2).

(4) The representations in (1) and (2) are the Jordan-Hölder composition series for

\[ [0]^{(\psi)} \times [1]^{(\psi)} \times [1]^{(\psi)} \times \sigma. \]

We shall prove the proposition in several steps. One directly sees that (4) holds.

(1) Write

\[ [0]^{(\psi)} \times [1]^{(\psi)} \times [1]^{(\psi)} \times \sigma = [1]^{(\psi)} \times \left( L([1]^{(\psi)}, \delta([0]^{(\psi)}_+; \sigma)) + L([1]^{(\psi)}, \delta([0]^{(\psi)}_-; \sigma)) + 2L([0, 1]^{(\psi)}; \sigma) + \delta([0, 1]^{(\psi)}; \sigma) \right). \]

The last two products on the right hand side of the above equality are reducible by (vi) of Proposition 6.1 from [73]. Applying the ASS involution of the involution to them, we get that also the first two product on the right hand side of the above equality are reducible. These four representations are in the ends of the complementary series. Therefore, all the subquotients there are unitarizable. Now [2.7] gives the following irreducible (unitarizable) subquotients

\[ L([1]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}_\pm; \sigma)), \quad L([1]^{(\psi)}, \delta([0, 1]^{(\psi)}_\pm; \sigma)). \]

We have further

\[ \delta([-1, 1]^{(\psi)} \times \sigma = \delta([-1, 1]^{(\psi)}_+; \sigma) \oplus \delta([-1, 1]^{(\psi)}_-; \sigma), \]

\[ \delta([-1, 1]^{(\psi)} \times \sigma = L([1]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}_+; \sigma)) \oplus L([1]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}_-; \sigma)). \]

Therefore, we have also the following additional irreducible unitarizable subquotients

\[ \delta([-1, 1]^{(\psi)}_\pm; \sigma). \]

This completes the proof of (1).

(2) From [2.23] we know that \([1]^{(\psi)} \times [1]^{(\psi)} \otimes \delta([0]^{(\psi)}_\epsilon; \sigma)\) is a direct summand in the Jacquet module of \(\delta([-1, 1]^{(\psi)}_\epsilon; \sigma)\). This implies that \([-1]^{(\psi)} \times [-1]^{(\psi)} \otimes \delta([0]^{(\psi)}_\epsilon; \sigma)\) is in the Jacquet module of \(\delta([-1, 1]^{(\psi)}_\epsilon; \sigma)^t\). From this follows

\[ \delta([-1, 1]^{(\psi)}_\epsilon; \sigma)^t = L([1]^{(\psi)}, [1]^{(\psi)}, \delta([0]^{(\psi)}_\epsilon; \sigma)), \quad \epsilon \in \{\pm\}. \]

This implies that the multiplicity of each \(\delta([-1, 1]^{(\psi)}_\epsilon; \sigma)\) and \(\delta([-1, 1]^{(\psi)}_\epsilon; \sigma)^t\) in the whole induced representation (from the cuspidal one) is one. Further, (2.23) implies that \([1]^{(\psi)} \otimes \delta([0, 1]^{(\psi)}_\epsilon; \sigma)\) is in the Jacquet module of \(\delta([-1, 1]^{(\psi)}_\epsilon; \sigma)\).

Consider

\[ \mu^*([1]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)) = \]
Now the top Jacquet module is
\[
(1 \otimes [1]^{(\psi)} + [1]^{(\psi)} \otimes 1 + [-1]^{(\psi)} \otimes 1) \times 
\left(1 \otimes L([0, 1]^{(\psi)}; \sigma) + [0]^{(\psi)} \otimes [1]^{(\psi)} \times \sigma + \delta([-1, 0]^{(\psi)}) \otimes \sigma + L([0]^{(\psi)}, [1]^{(\psi)} \otimes \sigma) \right).
\]

Now we can conclude
\[
[1]^{(\psi)} \otimes L([0, 1]^{(\psi)}; \sigma) + [-1]^{(\psi)} \otimes L([0, 1]^{(\psi)}; \sigma) + [0]^{(\psi)} \otimes [1]^{(\psi)} \times [1]^{(\psi)} \times \sigma.
\]

Further, no one of $[1]^{(\psi)} \otimes \delta([0, 1]^{(\psi)}_\pm; \sigma)$ is a subquotient of $[1]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)$. This implies that no one of $\delta([-1, 1]^{(\psi)}_\pm; \sigma)$ is a subquotient of $[1]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)$.

Further, no one of $[-1]^{(\psi)} \otimes \delta([0, 1]^{(\psi)}_\pm; \sigma)$ is a subquotient of the Jacquet module of $[1]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)$. This implies that no one of $L([1]^{(\psi)}_\pm, \delta([0, 1]^{(\psi)}_\pm; \sigma))$ is a subquotient of $[1]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)$. These two observations and 2.6 imply that $[1]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)$ is irreducible. This further implies that
\[
([1]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma))^t = [1]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma).
\]

Now we can conclude $L([1]^{(\psi)}_\pm, \delta([0, 1]^{(\psi)}_\pm; \sigma))^t = L([1]^{(\psi)}_\pm, \delta([0, 1]^{(\psi)}_\pm; \sigma))$ for some sign. The formula 8.62 implies
\[
L([1]^{(\psi)}_\pm, \delta([0, 1]^{(\psi)}_\pm; \sigma))^t = L([1]^{(\psi)}_\pm, \delta([0, 1]^{(\psi)}_\pm; \sigma)).
\]

Therefore (3) holds.

(3) The non-unitarizability of $[1]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)$ follows from a non-unitarizability at the following reducibility point $[2]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)$ (after $L([1]^{(\psi)}_\pm, [0, 1]^{(\psi)}_\pm; \sigma)$ 26). Since $L([2]^{(\psi)}_\pm, [0, 1]^{(\psi)}_\pm; \sigma) \leq [2]^{(\psi)}_\pm \times L([0, 1]^{(\psi)}_\pm; \sigma)$ and we have proved that $L([2]^{(\psi)}_\pm, [0, 1]^{(\psi)}_\pm; \sigma)$ is not unitarizable, this implies non-unitarizability of $L([1]^{(\psi)}_\pm, [0, 1]^{(\psi)}_\pm; \sigma)$. Therefore, (2) holds, and the proof of the proposition is now complete.

G.6. The case of exponents $(0, 0, 1)$ and $\alpha = 0$.

All irreducible subquotients of the induced representation
\[
[0]^{(\psi)} \times [0]^{(\psi)} \times [1]^{(\psi)} \times \sigma
\]
are unitarizable (complementary series in rank 2, and then unitary induction implies the unitarizability).

We shall now analyze this more precisely:
\[
[0]^{(\psi)} \times [0]^{(\psi)} \times [1]^{(\psi)} \times \sigma = [0]^{(\psi)} \times 
\left(L([1]^{(\psi)}_\pm, \delta([0]^{(\psi)}_\pm; \sigma)) + L([1]^{(\psi)}_\pm, \delta([0]^{(\psi)}_-; \sigma)) + 2L([0, 1]^{(\psi)}_\pm; \sigma) + \delta([0, 1]^{(\psi)}_\pm; \sigma) + \delta([0, 1]^{(\psi)}_-; \sigma) \right).
\]

\[26\] That $[2]^{(\psi)} \times L([0, 1]^{(\psi)}; \sigma)$ is the next reducibility point follows using section 8 of \[78\].
First the product of the first two and the last two representation on the right hand side is irreducible (we conclude this from Jordan blocks for the last two representations, and get the irreducibility for the first two applying the duality).

Now consider $[0]^{(\psi)} \times L([0,1]^{(\psi)};\sigma)$. Observe that the Jacquet module of GL-type of this induced representation is

$$2 \cdot [0]^{(\psi)} \times (\delta([-1,0]^{(\psi)}) \otimes \sigma + L([0]^{(\psi)}, [1]^{(\psi)})) \otimes \sigma,$$

which is a length four representation. By 2.7 both representations $L([0,1]^{(\psi)};\delta([0]^{(\psi)};\sigma))$ are irreducible subquotients of $[0]^{(\psi)} \times L([0,1]^{(\psi)};\sigma)$. Observe that

$$L([0,1]^{(\psi)};\delta([0]^{(\psi)};\sigma)) \hookrightarrow \delta([-1,0]^{(\psi)}) \times \delta([0]^{(\psi)};\sigma) \hookrightarrow \delta([-1,0]^{(\psi)}) \times [0]^{(\psi)} \times \sigma \cong [0]^{(\psi)} \times \delta([-1,0]^{(\psi)}) \times [0]^{(\psi)} \times [1]^{(\psi)} \times \sigma.$$

This implies that the Jacquet module of GL-type of these two representations has length at least two. Therefore, we have

$$[0]^{(\psi)} \times L([0,1]^{(\psi)};\sigma) = L([0,1]^{(\psi)};\delta([0]^{(\psi)};\sigma)) + L([0,1]^{(\psi)};\delta([0]^{(\psi)};\sigma))$$

in the Grothendieck group. Now we conclude that

$$[0]^{(\psi)} \times [0]^{(\psi)} \times [1]^{(\psi)} \times \sigma = L([1]^{(\psi)}, [0]^{(\psi)} \times \delta([0]^{(\psi)};\sigma)) + L([1]^{(\psi)}, [0]^{(\psi)} \times \delta([0]^{(\psi)};\sigma)) +
2L([0,1]^{(\psi)};\delta([0]^{(\psi)};\sigma)) + 2L([0,1]^{(\psi)};\delta([0]^{(\psi)};\sigma)) + [0]^{(\psi)} \times \delta([0,1]^{(\psi)};\sigma) + [0]^{(\psi)} \times \delta([0,1]^{(\psi)};\sigma)$$

is a decomposition into irreducible representations.

**H.6. The case of exponents $(0,0,0)$ and $\alpha = 0$.**

The following representation

$$[0]^{(\psi)} \times [0]^{(\psi)} \times [0]^{(\psi)} \times \sigma = [0]^{(\psi)} \times [0]^{(\psi)} \times \delta([0]^{(\psi)};\sigma) \oplus [0]^{(\psi)} \times [0]^{(\psi)} \times \delta([0]^{(\psi)};\sigma)$$

is a sum of two tempered representations. Both irreducible subquotients are unitarizable (and moreover tempered). The involution switches them.
9.1. **General estimate for unitarizability.** First we recall of Proposition 3.3 from the [76], which gives an upper bound where unitarizability can show up in parabolically induced representations. We shall use these estimates in the sequel.

**Proposition 9.1.** Let $\pi$ be an irreducible unitarizable representation of a classical group $S_q$. Let $\rho$ be a factor of $\pi$. Suppose that $\rho_1, \ldots, \rho_n$ are all the factors $\tau$ of $\pi$ such that $\tau^u \cong \rho^u$.

1. Let $\rho^u \not\cong (\rho^u)^\ast$. Renumerate $\rho_1, \ldots, \rho_n$, $n \geq 1$, in a way that $|e(\rho_1)| \leq |e(\rho_2)| \leq \cdots \leq |e(\rho_n)|$. Then $|e(\rho_i)| \leq \frac{i}{2}$, $1 \leq i \leq n$

2. Suppose $\rho^u \cong (\rho^u)^\ast$. Write the set of all $|e(\rho_i)| > \alpha_{\rho^u, \pi_{cusp}}$, $1 \leq i \leq n$, as
   $$\{\alpha_1, \ldots, \alpha_\ell\},$$
   where $\ell \geq 0$ and $\alpha_1 < a_2 < \cdots < \alpha_\ell$. Then
   $$\alpha_i - \alpha_{i-1} \leq 1$$
   for each $i = 2, 3, \ldots, \ell$

   if $\ell \geq 2$. Further

   (i) If $\alpha_{\rho^u, \pi_{cusp}} = 0$, then
   $$\alpha_i \leq i - \frac{1}{2}; \quad i = 1, \ldots, \ell.$$

   (ii) If $\alpha_{\rho^u, \pi_{cusp}} \geq \frac{1}{2}$, then there exists index $i$ such that
   $$|e(\rho_i)| \leq \alpha_{\rho^u, \pi_{cusp}}.$$

Denote by
$$\alpha_{\rho^u, \pi_{cusp}}^{(\pi)} = \max\{|e(\rho_i)|; |e(\rho_i)| \leq \alpha_{\rho^u, \pi_{cusp}} \& 1 \leq i \leq n\}.$$ 

Then holds

(a) $\alpha_1 - \alpha_{\rho^u, \pi_{cusp}}^{(\pi)} \leq 1$ if $\ell \geq 1$.

(b) $\alpha_i \leq \alpha_{\rho^u, \pi_{cusp}}^{(\pi)} + i; \quad i = 1, \ldots, \ell.$

In this and the following section, $\rho$ will be an irreducible $F'/F$-selfcontragredient cuspidal representations of a general linear group and $\sigma$ an irreducible cuspidal representation of a classical groups such that
$$[\alpha]^{(\rho)} \rtimes \sigma$$

reduces for some (fixed) $\alpha \in \frac{1}{2}\mathbb{Z}$, $\alpha \geq 0$.

We shall determine unitarizability of irreducible subquotients of
$$[x_1]^{(\rho)} \times \cdots \times [x_k]^{(\rho)} \rtimes \sigma$$
when \( k \leq 3 \) and
\[
0 \leq x_1 \leq \cdots \leq x_k.
\]

Let \( \pi \) be an irreducible subquotient of \([x_1]^{(\rho)} \times \cdots \times [x_k]^{(\rho)} \rtimes \sigma\).

If \( k = 0 \), then \( \pi = \sigma \), and it is obviously unitarizable.

Let \( k = 1 \). Then \( \pi \) is unitarizable if and only if
\[
x_1 \leq \alpha.
\]
For \( 0 \leq x_1 < \alpha \), \( \pi = [x_1]^{(\rho)} \rtimes \sigma \), while for \( x_1 = \alpha \) we have two non-equivalent irreducible (unitarizable) subquotients. In the case \( \alpha > 0 \), they are \( \delta([\alpha]^{(\rho)}; \sigma) \) and \( L([\alpha]^{(\rho)}; \sigma) \), while for \( \alpha = 0 \), they are \( \delta([0]^+; \sigma) \) and \( \delta([0]^−; \sigma) \).

These two representations are the only subquotients which are unitarizable for \( \alpha = 0 \).

9.2. **Generalized rank 2.** The following proposition is more or less well known, but we do not know for written proof in this generality. Therefore, we present the proof below.

**Proposition 9.2.** Let \( \pi \) be an irreducible unitarizable subquotient of \([x_2]^{(\rho)} \times [x_1]^{(\rho)} \rtimes \sigma\)

(1) Assume
\[
\alpha \geq 1.
\]
Then \( \pi \) is one of the following irreducible unitarizable representations:
(a) An irreducible unitarizable subquotient of \([\alpha + 1]^{(\rho)} \times [\alpha]^{(\rho)} \rtimes \sigma\)
(i.e. the generalized Steinberg representation or its ASS dual).
(b) An irreducible subquotient of representation with exponents satisfying
\[
x_1 + x_2 \leq 1
\]
or
\[
x_1 + 1 \leq x_2 \leq \alpha.
\]

(2) Assume
\[
\alpha = \frac{1}{2}.
\]
Then \( \pi \) is one of the following irreducible unitarizable representations:
(a) An irreducible unitarizable subquotient of \(\left[\frac{3}{2}\right]^{(\rho)} \times \left[\frac{1}{2}\right]^{(\rho)} \rtimes \sigma\)
(i.e. the generalized Steinberg representation or its ASS dual).
(b) An irreducible subquotient of representation with exponents satisfying
\[
x_1 \leq \frac{1}{2}, \quad x_2 \leq \frac{1}{2}.
\]
(3) Assume 
\[ \alpha = 0. \]
Then \( \pi \) is an irreducible subquotient of a representation with exponents satisfying 
\[ x_1 + x_2 \leq 1, \]
and each irreducible subquotient of the above representation is unitarizable.

Proof. (1) First we shall see unitarizability of the representations in (b). The representations for \( x_1 + x_2 < 1 \) form continuous family of irreducible Hermitian representations, and at \((0,0)\) we have unitarizability. Therefore, all irreducible subquotients for 
\[ x_1 + x_2 \leq 1 \]
are unitarizable.

Similarly, constructing complementary series in two steps (or in other words, constructing complementary series from the complementary series in generalized rank one), we get the following family of irreducible unitarizable representations: \( x_1 + 1 < x_2 < \alpha \) (for this to be non-empty, we need to have \( \alpha > 1 \)). Therefore, all irreducible subquotients for 
\[ x_1 + 1 \leq x_2 \leq \alpha \]
are unitarizable (for \( \alpha = 1 \) see D.4).

Now we shall show exhaustion. Suppose that \( \pi \) is an irreducible unitarizable subquotient. First consider the case 
\[ x_2 > \alpha. \]
We know \( x_1 \leq \alpha \) and \( x_2 - x_1 \leq 1 \), which implies \( x_2 \leq \alpha + 1 \). If \( x_2 = \alpha + 1 \), then \( x_1 = \alpha \), and \( \pi \) is in (a). Therefore, we can assume \( x_2 < \alpha + 1 \). Consider now the case \( \alpha < x_2 \).

If \( x_1 = \alpha \), then \( \pi = [x_2]^{(\rho)} \times \theta \) where \( \theta \in \{ \delta([\alpha]^{(\rho)}; \sigma), L([\alpha]^{(\rho)}; \sigma) \} \) (see 2.3). Now we can deform \( x_2 \) to \( \alpha + 1 \) (again using 2.3) and get a contradiction (one directly sees that we get a non-unitarizable subquotient listed in B.1 at the limit when one deforms \( x_2 \) to \( \alpha \)). Therefore, it remains to consider the case \( x_1 < \alpha \).

If \( x_2 - x_1 = 1 \), then \( \pi \) is fully induced from a proper parabolic subgroup and we can deform to the representation with exponents \((\alpha, \alpha + 1)\) (similarly as above) and get there in the limit a not unitarizable subquotient (listed in B.1), which is a contradiction.

If \( x_2 - x_1 < 1 \), then we can deform (increase) \( x_2 \) to the previous case, consider a limit, and the repeat the above argument. Therefor, we get again contradiction.

Consider now the case 
\[ x_2 = \alpha. \]
For \( 0 \leq x_1 \leq \alpha - 1 \) we have unitarizability of irreducible subquotients and these representations are in (b). If \( x_1 > \alpha \), we can deform \( x_1 \) to \( \alpha \), and get that there is an irreducible subquotient, which is not the case (see C.1).
It remains to consider the case
\[ x_2 < \alpha. \]

It remains to see what happens in the region
\[ 1 - x_1 < x_2 < x_1 + 1. \]

This is a continuous family of irreducible Hermitian representations. Consider the part \( \frac{1}{2} < x_1 = x_2 < \alpha \) of the above region. After switching \( x_1 \) to \(-x_1\), we can apply the unitary parabolic reduction and get that no representation in this region is unitarizable.

\[(2) \text{ First we shall see unitarizability of the representations in (b). The representations for } x_1, x_2 < \frac{1}{2} \text{ are form a continuous family of irreducible Hermitian representations, and at } (0,0) \text{ we have unitarizability. Therefore, all irreducible subquotients for } x_1 \leq 1, x_2 \leq \frac{1}{2} \text{ are unitarizable.} \]

Now we go to exhaustion. Suppose that \( \pi \) is an irreducible unitarizable subquotient. First consider the case \( x_2 > 1 \).

We know \( x_1 \leq \frac{1}{2} \) and \( x_2 - x_1 \leq 1 \). If \( x_2 = \frac{3}{2} \), then \( x_1 = \frac{1}{2} \), and then \( \pi \) is in (a). Therefore, we can assume \( x_2 < \frac{3}{2} \).

If \( x_1 = \frac{1}{2} \), then \( \pi = [x_2]^{(\rho)} \rtimes \theta; \theta \in \{ \delta([\frac{1}{2}^{(\rho)}]; \sigma), L([\frac{1}{2}^{(\rho)}]; \sigma) \} \). Now we can deform \( x_2 \) to \( \frac{3}{2} \) and get a contradiction (a not unitarizable subquotient from B.1 at the limit). Therefore, it remains to consider the case \( x_1 < \frac{1}{2} \).

If \( x_2 - x_1 = 1 \), then \( \pi \) is fully induced from a proper parabolic subgroup, we can deform \( \pi \) to exponents \( (\frac{1}{2}, \frac{3}{2}) \) (similarly as in (1)) and get in the limit a not unitarizable subquotient (from B.1), which is a contradiction.

If \( x_2 - x_1 < 1 \), then we can deform \( x_2 \) to the previous case, consider a limit, repeat the above argument and get a contradiction (with B.1).

Consider now the case \( x_2 = 1 \).

If \( x_1 = 0 \), then \( \pi \) is fully induced. We can deform to \( (\frac{1}{2}, \frac{3}{2}) \) and get a contradiction (with B.1).

For \( 0 < x_1 \) we deform first \( x_1 \) to \( \frac{1}{2} \), and then \( x_2 \) to \( \frac{3}{2} \). We get a contradiction (with B.1).

It remains to consider the case \( \frac{1}{2} < x_2 < 1 \).

If \( x_1 = \frac{1}{2} \), we can deform \( x_3 \) to \( \frac{3}{2} \) and get not unitarizable subquotient there (from B.1), which is a contradiction. Consider now \( x_1 < \frac{1}{2} \). If \( x_1 + x_2 = 1 \), we can deform the fully induced representation from a proper parabolic subgroup to \( (\frac{1}{2}, \frac{3}{2}) \), and get a contradiction.
If $x_1 + x_2 \neq 1$, we can deform $x_1$ to get $x_1 + x_2 = 1$ and get a contradiction as in previous case. This completes the proof of (2).

(3) First we shall see unitarizability of the representations in (3). The representations for $x_1 + x_2 < 1, x_1 > 0$ form a continuous family of irreducible Hermitian representations. It contains unitarizable representation (complementary series) induced from proper parabolic subgroup (for example $\left[ \frac{1}{2} \right]^{(\rho)} \times \left[ \frac{1}{2} \right]^{(\rho)} \times \sigma \approx \left[ \frac{1}{2} \right]^{(\rho)} \times \left[ -\frac{1}{2} \right]^{(\rho)} \times \sigma$). Therefore, all irreducible subquotients for $x_1 + x_2 \leq 1$ are unitarizable.

Now we go to exhaustion. Let $\pi$ be an irreducible unitarizable subquotient. First consider the case

$$x_2 > 1.$$  

We know $x_1 \leq \frac{1}{2}$ and $x_2 - x_1 \leq 1$.

If $x_2 - x_1 = 1$, then representation is fully induced from proper parabolic subgroup. We can deform the representation to exponents $(x, x + 1)$ as far to the right as we want and get there in the limit a not unitarizable subquotient, which is a contradiction (we have unitarizable subquotients only at bounded regions).

If $x_2 - x_1 < 1$, then we can deform $x_2$ to the previous case, take an irreducible subquotient at the limit, and then repeat the above argument (which gives a contradiction). Therefore $x_2 \leq 1$.

Consider now the case

$$x_2 = 1.$$  

If $x_1 = 0$, we have unitarizability. Suppose $x_1 > 0$. Then we deform $x_1$ to 1, and use parabolic reduction (switching before exponent 1 to -1). We would get complementary series which go to 1, which is contradiction.

It remains to consider the case

$$x_2 < 1.$$  

It remains to see what happens in the region

$$x_1 + x_2 > 1.$$  

This is continuous family of irreducible Hermitian representations. Consider the part $\frac{1}{2} < x_1 = x_2$ of the above region. Letting them to tend to go to 1, we would get a contradiction in the same way as before. This completes the proof of (3).
10. Unitarizability for Generalized Rank 3 - Some Particular Results

**Lemma 10.1.** Let \( \alpha \geq 0 \), and let \( \gamma \) be an irreducible subquotient of \([\alpha]^{(\rho)} \times [\alpha + 1]^{(\rho)} \times [\alpha + 2]^{(\rho)}\).

Then \( \gamma \ltimes \sigma \) contains an irreducible subquotient which is not unitarizable.

**Proof.** If \( \alpha > 0 \), this follows from the fact that the generalized Steinberg representation and its dual cannot be subquotients of the same \( \gamma \ltimes \sigma \) (which one directly proves).

Let \( \alpha = 0 \). Then we have proved the non-unitarizability of \( L([0, 2]^{(\rho)}; \sigma), L([2]^{(\rho)}, [0, 1]^{(\rho)}; \sigma), L([1, 2]^{(\rho)}; \delta([0]^{(\rho)}; \sigma)) \) and that \( L([0, 2]^{(\rho)}; \sigma)^t = L([2]^{(\rho)}, [0, 1]^{(\rho)}; \sigma) \) (see Proposition 8.1). The last relation implies \( L([2]^{(\rho)}, [0, 1]^{(\rho)}; \sigma) \leq L([2]^{(\rho)}, [1]^{(\rho)}, [0]^{(\rho)}) \ltimes \sigma \). This and 2.7 imply the claim in this case.

**Lemma 10.2.** Let \( \alpha \geq 1 \), and let \( \gamma \) be an irreducible subquotient of \([\alpha - 1]^{(\rho)} \times [\alpha]^{(\rho)} \times [\alpha + 1]^{(\rho)}\).

Then \( \gamma \ltimes \sigma \) contains an irreducible subquotient which is not unitarizable.

**Remark 10.3.** Below is what happens in the remaining two reducibilities.

1. Consider the case \( \alpha = \frac{1}{2} \). Now both following representations

\[ L([-\frac{1}{2}]^{(\rho)}, [\frac{1}{2}, \frac{3}{2}]^{(\rho)}) \ltimes \sigma, \quad L([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}, [\frac{3}{2}]^{(\rho)}) \ltimes \sigma \]

contain a not unitarizable subquotient (use (2) of Proposition 7.1 and 2.4). In the remaining two cases, all the subquotients are unitarizable (since all they are subquotients of the ends of complementary series).

2. Consider the case \( \alpha = 0 \). Then for exponents \((-1, 0, 1)\), only \( L([1]^{(\psi)}, [0, 1]^{(\psi)}; \sigma) \) is not unitarizable by Proposition 8.9. Therefore both

\[ L([-1]^{(\psi)}, [0, 1]^{(\psi)}) \ltimes \sigma, \quad L([-1, 0]^{(\psi)}, [1]^{(\psi)}) \ltimes \sigma \]

contain a non-unitarizable subquotient. In remaining two cases all the subquotients are obviously unitarizable (since we are dealing with unitarily induced representations).
Proof. Consider first the proof $\alpha > 1$. If $\gamma = \delta([\alpha-1, \alpha+1])$ or $\gamma = L([\alpha-1], [\alpha, \alpha+1])$, then this follows the claim follows from \ref{subsection} and (2) of Proposition \ref{prop:irreducible}

Let $\gamma = L([\alpha-1], [\alpha], [\alpha+1])$. Then by \ref{subsection} $(\gamma \rtimes \sigma)^t = \gamma^t \rtimes \sigma$ contains $L([\alpha-1], [\alpha, \alpha+1], \sigma)$ as a subquotient. Therefore, $\gamma \rtimes \sigma$ contains $L([\alpha-1], [\alpha, \alpha+1], \sigma)^t$ for which we know from (2) and (4) of Proposition \ref{prop:irreducible} that it is not unitarizable.

Let $\gamma = L([\alpha-1, \alpha], [\alpha+1])$. Then by \ref{subsection} $(\gamma \rtimes \sigma)^t = \gamma^t \rtimes \sigma$ contains $L([\alpha-1], [\alpha, \alpha+1], \sigma)$ as a subquotient. Therefore, $\gamma \rtimes \sigma$ contains $L([\alpha-1], [\alpha, \alpha+1], \sigma)^t$ for which we know from (2) and (4) of Proposition \ref{prop:irreducible} that it is not unitarizable.

Consider now the case $\alpha = 1$ (which goes almost the same way as the previous case). If $\gamma = \delta([0, 2])$ or $\gamma = L([0], [1, 2])$, then this follows the claim follows from \ref{subsection} and (2) of Proposition \ref{prop:irreducible}

Let $\gamma = L([0], [1], [2])$. Then by \ref{subsection} $(\gamma \rtimes \sigma)^t = \gamma^t \rtimes \sigma$ contains $L([0], [1], \sigma)$ as a subquotient. Therefore, $\gamma \rtimes \sigma$ contains $L([0], [1], \sigma)^t$ for which we know by (2) and (3) of Proposition \ref{prop:irreducible} that it is not unitarizable.

Let $\gamma = L([0, 1], [2])$. Then by \ref{subsection} $(\gamma \rtimes \sigma)^t = \gamma^t \rtimes \sigma$ contains $L([1, 2], [0], \sigma)$ as a subquotient. Therefore, $\gamma \rtimes \sigma$ contains $L([1, 2], [0], \sigma)^t$ for which we know by (2) and (3) of Proposition \ref{prop:irreducible} that it is not unitarizable.

Lemma 10.4. Let

$\alpha \geq 1$.

Suppose that

$x_3 > \alpha$,

and that $\pi$ is an irreducible unitarizable subquotient corresponding to exponents $(x_1, x_2, x_3)$. Then $\pi$ is an irreducible unitarizable subquotient at a point

$(\alpha, \alpha + 1, \alpha + 2)$

(i.e. the generalized Steinberg representation there, or its ASS dual), or a an irreducible subquotient of one of the two following sets

$[x_1] \times \delta([\alpha, \alpha+1], \sigma), \quad [x_1] \times L([\alpha+1], [\alpha], \sigma), \quad 0 \leq x_1 < \alpha - 1$

(for $0 \leq x_1 < \alpha - 1$ these are complementary series, while for $x_1 = \alpha - 1$, the irreducible subquotients are described by Propositions \ref{prop:irreducible} and \ref{prop:complementary}).

Proof. We shall prove the lemma in several steps. Suppose that $\pi$ is an irreducible unitarizable subquotient.

(1) We first analyze the case

$x_3 = \alpha + 1, \quad x_2 = \alpha$.

Suppose $x_1 > \alpha - 1$. If $x_1 < \alpha$, then $\pi \cong [x_1] \times \theta$ for an irreducible subquotient $\theta$ of $[\alpha] \times [\alpha+1] \rtimes \sigma$ (see Remark \ref{remark} for this; we shall use it often below without further
referring to it). Now we can deform $x_1$ to $\alpha$ and get an irreducible unitarizable subquotient for exponents $(\alpha, \alpha, \alpha + 1)$, which is a contradiction (see G.1). Clearly, the claim of the lemma holds for $x_1 = \alpha$. If $x_1 = \alpha - 1$, then Proposition 3.6, (3.35) and (3.37) imply that the lemma holds in this case.

It remains to consider the case $x_1 < \alpha - 1$. First observe that short discussion implies that we have above complementary series if $\alpha > 1$ (for one complementary series consider Jordan blocks of $\delta([\alpha, \alpha + 1]; \sigma)$ and apply Proposition 6.1 of [73] to know that there is a Hermitian family of irreducible representations, which is unitary for $x_1 = 1$; one gets the irreducibility necessary for constructing the other complementary series applying the involution).

Suppose that $\pi$ is a subquotient of the whole induced representation where the above complementary series are subquotients, but that $\pi$ is not a member of the complementary series. Then Jantzen decomposition implies that $\pi \cong [x_1]^{(\rho)} \rtimes \theta$, where $\theta$ is a non-unitarizable hermitian representation (then $\alpha > 1$). Now we (can) deform $x_1$ to 0 (irreducibly). Applying the unitary parabolic reduction we get a contradiction.

We have settled the case of $x_3 = \alpha + 1, x_2 = \alpha$. Also if $x_3 = \alpha + 2$, the lemma holds. We need to consider $x_3 < \alpha + 2$.

We know that reducibility hyperplanes are precisely

$$x_j + x_i = 1; \quad (x_i)_i \in \mathbb{R}^3, \quad 1 \leq i < j \leq 3,$$

$$x_j - x_i = 1; \quad (x_i)_i \in \mathbb{R}^3, \quad 1 \leq i < j \leq 3,$$

$$x_i = \alpha; \quad (x_i)_i \in \mathbb{R}^3, \quad 1 \leq i \leq 3.$$

(2) Consider the case

$$\alpha + 1 < x_3 < \alpha + 2.$$

Now $x_2 > \alpha$ and $x_1 \leq \alpha$ among others (Proposition 9.1). If $x_2 = \alpha + 1$, then $x_1 = \alpha$. Now $\pi \cong [x_3]^{(\rho)} \rtimes \theta$. We deform $x_3$ to $\alpha + 2$ and get a contradiction. Therefore, we need to consider the case $\alpha < x_2 < \alpha + 1$. Now we can deform $x_3$ to get $x_3 - x_2 = 1$ if it is not already the case, and pass to an irreducible subquotient (which is unitarizable).

Suppose $x_1 = \alpha$. Then $\pi \cong \theta \rtimes \tau$, where $\theta$ is an irreducible subquotient of $[x_2]^{(\rho)} \times [x_3]^{(\rho)}$ and $\tau$ is an irreducible subquotient of $[\alpha]^{(\rho)} \rtimes \sigma$. Now we can deform $\theta$ to exponents $\alpha + 1, \alpha + 2$ and get a contradiction.

It remains to consider the case $x_1 < \alpha$. Now we can deform $x_1$ to get $x_2 - x_1 = 1$ if it is not already the case, and pass to an irreducible subquotient. Then obtained unitarizable representation is of form $\theta \rtimes \sigma$, where $\theta$ is an irreducible subquotient of $[x_1]^{(\rho)} \times [x_1 + 1]^{(\rho)} \times [x_1 + 2]^{(\rho)} \rtimes \sigma$. Now we can deform $\theta$ to exponents $\alpha, \alpha + 1, \alpha + 2$. Lemma 10.1 gives a contradiction. Therefore, we need to consider $x_3 \leq \alpha + 1$. 
Consider first
\[ x_3 = \alpha + 1. \]
Then \( x_2 \geq \alpha \), and \( x_2 - x_1 \leq 1 \) if \( x_2 > \alpha \). The case \( x_2 = \alpha \) we have settled. Therefore we need to consider \( x_2 > \alpha \). If \( x_1 = \alpha \), we can deform \( x_2 \) to \( \alpha + 1 \) if it is not already there, and get a contradiction (see F.1). Therefore \( x_1 < \alpha \). Now we deform \( x_3 \) to get \( x_3 - x_2 = 1 \). Previous case implies that we cannot have unitarizability here.

It remains to consider
\[ \alpha < x_3 < \alpha + 1. \]
Suppose \( x_1 = \alpha \). Then we can deform \( x_2 \) to \( \alpha \) if it is not there already, and then deform \( x_3 \) to \( \alpha \) (or \( \alpha + 1 \)), and get a contradiction (with H.1). Therefore, we need to consider \( x_1 < \alpha \).

Consider the case
\[ x_2 = \alpha. \]
Suppose \( x_3 - x_1 \leq 1 \). Then we can deform \( x_3 \) to get \( x_3 - x_1 = 1 \) if it is not already the case. Now we can deform the pair \( x_1, x_3 \) to \( \alpha, \alpha + 1 \) and get a contradiction (see G.1). It remains to consider \( x_3 - x_1 > 1 \). Now we can deform \( x_1 \) to \( \alpha - 1 \). After this, we can deform \( x_3 \) to \( \alpha + 1 \).

This implies that there is an irreducible subquotient \( \theta \) of \( [\alpha][(\rho)] \times [\alpha - 1][(\rho)] \times \sigma \) such that all the irreducible subquotients of \( \pi := [\alpha + 1][(\rho)] \times \theta \) are unitarizable.

Consider first the case \( \alpha > 1 \). Then for \( \theta \) square integrable, this cannot happen by Proposition 3.6 and 2.7. Also for \( \theta = L([\alpha - 1][(\rho)]; \delta([\alpha][(\rho)]; \sigma)) \) this cannot happen again by Proposition 3.6 and 2.7. Two remaining \( \pi \)'s we get applying involution. We have there non-unitarizable subquotients since the involution preserves unitarizability for exponents \( \alpha - 1, \alpha, \alpha + 1 \) by Proposition 3.6. Therefore, we get a contradiction.

Consider now the case \( \alpha = 1 \). Then for \( \theta = \tau([0][(\rho)]; \delta([1][(\rho)]; \sigma)) \) this cannot happen by Proposition 6.1 and 2.7. Two remaining \( \pi \)'s we get applying involution. We have there non-unitarizable subquotients since the involution preserves unitarizability for exponents \( 0, 1, 2 \) by Proposition 6.1. Again we get a contradiction.

Therefore, we need to consider the case
\[ x_2 \neq \alpha. \]
Recall \( \alpha < x_3 < \alpha + 1 \) and \( x_1 < \alpha \).

Consider first the case
\[ x_2 > \alpha. \]
If \( x_3 - x_1 > 1 \), we can deform \( x_3 \) to \( \alpha + 1 \). We have seen that in this case we do not have unitarizability, so we get a contradiction (recall \( \alpha < x_2 < \alpha + 1 \)). It remains to consider
the case $x_3 - x_1 \leq 1$. Now we can deform $x_2$ to get $x_3 - x_1 = 1$ if it is not already. We can now in two steps deform to exponents $\alpha, \alpha, \alpha + 1$ to get unitarizability there, which contradicts to G.1.

(4-c) It remains to consider the case

$$x_2 < \alpha.$$ 

If $x_1 < x_2 - 1$, then deform $x_1$ to 0 if it is not already there, use the unitary parabolic reduction and get a contradiction with the rank two case. Therefore, $x_2 - x_1 \leq 1$. We know from Proposition 9.1 that $x_3 - x_2 \leq 1$.

Now deform $x_3$ to the right to get $x_3 - x_1 = 1$ or $x_3 - x_2 = 1$. Suppose that we are in the first case, i.e. $x_3 - x_1 = 1$. If $x_1 + x_2 < 1$, we can deform $x_1$ to 0, use the unitary parabolic reduction and get a contradiction with the rank two case. Therefore $x_1 + x_2 \geq 1$. Suppose $x_1 + x_2 = 1$. Then $\pi \cong \theta \times \sigma$, where $\theta$ is an irreducible subquotient of $[-x_1]^{(\rho)} \times [x_2]^{(\rho)} \times [x_2 + 1]^{(\rho)} \times \sigma$. Now we can deform $\theta$ to exponents $\alpha - 1, \alpha, \alpha + 1$ and get contradiction with Lemma 10.2. It remains to consider the case $x_1 + x_2 > 1$. Now we can deform $x_2$ to $\alpha$ (since $x_1 \leq x_2 \leq x_3$) and get a contradiction with a previous case.

We need now to consider the second case, i.e. $x_3 - x_2 = 1$. Recall $x_1 + x_2 \leq 1$. In the case $x_1 + x_2 < 1$, we get by deformation of $x_1$ to 0 a contradiction with rank two case as before. Therefore, we need to consider $x_1 + x_2 \geq 1$. If we have the equality, we get a contradiction in a way that we have already applied above (using Lemma 10.2). We are left with the case $x_1 + x_2 > 1$. If $x_2 - x_1 = 1$, we get a contradiction as in a previous case (using Lemma 10.2). If $x_2 - x_1 < 1$, we can deform $x_1$ to $\alpha$ and get a contradiction with a previous case. □

**Lemma 10.5.** Let

$$\alpha = \frac{1}{2}$$

Suppose that

$$x_3 > \frac{1}{2},$$

and that $\pi$ is an irreducible unitarizable subquotient corresponding to exponents $0 \leq x_1 \leq x_2 \leq x_3$. Then $\pi$ is a subquotient at a point

$$\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right)$$

(i.e. the generalized Steinberg representation there, or its ASS dual), or it is at one of eight complementary series

$$\begin{align*}
[x_1]^{(\rho)} \times \delta_{s.p.}(\frac{1}{2}, \frac{3}{2})^{(\rho)}; \sigma), & \quad [x_1]^{(\rho)} \times L((\frac{1}{2})^{(\rho)}, [\frac{3}{2}]^{(\rho)}; \sigma), & \quad 0 \leq x_1 < \frac{1}{2}, \\
\delta([-\frac{1}{2} + x, \frac{1}{2} + x]^{(\rho)}) \times \delta(\frac{1}{2})^{(\rho)}; \sigma), & \quad L([-\frac{1}{2} + x]^{(\rho)}, [\frac{1}{2} + x]^{(\rho)}) \times L((\frac{1}{2})^{(\rho)}; \sigma), & \quad 0 < x < 1, \\
x_3]^{(\rho)} \times \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma), & \quad x_3]^{(\rho)} \times L((\frac{1}{2})^{(\rho)}; \delta((\frac{1}{2})^{(\rho)}; \sigma)), & \quad \frac{1}{2} < x_3 < \frac{3}{2}, \\
\delta([-1 + x, 1 + x]^{(\rho)}) \times \sigma), & \quad L([-1 + x]^{(\rho)}, x]^{(\rho)}, [1 + x]^{(\rho)} \times \sigma), & \quad 0 \leq x < \frac{1}{2},
\end{align*}$$

or at their ends (i.e. an irreducible subquotient for exponents $(\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$, which are described by Proposition 7.1).
Proof. One easily sees that the above complementary series exist (for the third complementary series use the irreducibility of \((7.58)\)). Observe that the second and the third complementary series start also from 0 (as well as the first and the last ones).

It remains to see the exhaustion. We shall suppose that \(\pi\) is an irreducible unitarizable subquotient corresponding to exponents \(x_1, x_2, x_3\). First we shall consider the whole induced representations where some member of the above the complementary series shows up.

Consider some of the above four pairs of complementary series, and suppose that \(\pi\) is a subquotient of the whole induced representation where some member of that complementary series shows up, but that \(\pi\) is not a member of these two complementary series.

Suppose that we consider the first pair of complementary series. Then we have \(\pi \cong [x_1]^{(\rho)} \times \theta\), where \(\theta\) is not unitarizable hermitian representation. Now we (can) deform \(x_1\) to 0 irreducibly. At 0 we have irreducible induced representation. Applying the unitary parabolic reduction, we get contradiction.

Consider now the second pair of complementary series. Then \(\pi \cong \delta([-\frac{1}{2} + x, \frac{1}{2} + x]^{(\rho)}) \rtimes L([\frac{1}{2}]^{(\rho)}; \sigma)\) or \(L([-\frac{1}{2} + x]^{(\rho)}, [\frac{1}{2} + x]^{(\rho)}) \rtimes \delta([\frac{1}{2}]^{(\rho)}; \sigma)\). If we at the first case, deform \(x\) to 1 and Proposition 7.1 implies that we have in the limit a non-unitarizable representation (use also 2.7). This is contradiction. In the case of ASS dual, one gets contradiction in a similar way (using that \(L([\frac{1}{2}, \frac{3}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma)\) is not unitarizable).

Consider now the third pair of the complementary series. Then \(\pi = [x_3]^{(\rho)} \rtimes \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}; \sigma)\) or \(\pi = [x_3]^{(\rho)} \rtimes L([\frac{1}{2}]^{(\rho)}, [\frac{1}{2}]^{(\rho)}; \sigma), \frac{1}{2} < x_3 < \frac{3}{2}\). Now letting \(x_3\) to go to \(\frac{3}{2}\) in the first case, we conclude the non-unitarizability of the whole family (use (2) of Proposition 7.1 and 2.7). In the second case, again take \(x_3 = \frac{3}{2}\) and use that unitarizability is preserved by ASS involution for irreducible subquotients considered in Proposition 7.1.

In the case of last complementary series, the non-unitarizability of \(\pi\) follows letting \(x\) to go to \(\frac{1}{2}\), and applying (1) of Remark 11.3.

Therefore in the rest of the proof we do not need to consider unitarizability for exponents \(x_3 = \frac{3}{2}, x_2 = \frac{1}{2}, x_1 \leq \frac{1}{2}\). Neither we need to consider unitarizability for exponents \((1 - \frac{1}{2} + x, \frac{1}{2} + x), 0 < x < 1\). Neither we need to consider unitarizability for exponents \((\frac{1}{2}, \frac{1}{2}, x_3), \frac{1}{2} < x_3 < \frac{3}{2}\). Neither we need to consider unitarizability for exponents \((-1 + x, 1 + x), 0 \leq x < \frac{1}{2}\). Now we analyze remaining cases.

By Proposition 9.1 \(x_3 \leq \frac{5}{2}\). Suppose \(x_3 = \frac{5}{2}\). Then the same proposition implies \(x_2 = \frac{3}{2}\) and \(x_1 = \frac{1}{2}\). Therefore, the claim of the proposition holds here. It remains to consider the case \(x_3 < \frac{5}{2}\). We shall divide the analysis into several steps.

\(^{27}\)Note that for \(x_3 = \frac{1}{2}\) all irreducible subquotients are unitarizable.
(1) We consider first the case
\[ \frac{3}{2} < x_3 < \frac{5}{2}. \]
Recall that we must have
\[ x_3 - x_2 \leq 1, \quad x_2 - x_1 < 1, \quad x_1 \leq \frac{1}{2}, \]
which further implies
\[ \frac{1}{2} < x_2 \leq \frac{3}{2}. \]
Suppose \( x_1 = \frac{1}{2} \). Now a short discussion gives that if \( x_2 = \frac{3}{2} \), that we can deform \( x_3 \) to \( \frac{5}{2} \) and get a contradiction. If \( x_2 < \frac{3}{2} \), then we can deform \( x_2 \) to the left to get \( x_3 - x_2 = 1 \) if it is not already. Now we can deform an irreducible unitarizable subquotient to \((\frac{1}{2}, \frac{3}{2}, \frac{5}{2})\) and get a contradiction. It remains to consider the following case.

\[ x_1 < \frac{1}{2}. \]

Therefore we assume below that
\[ x_1 < \frac{1}{2} < x_2 \leq \frac{3}{2} < x_3 < \frac{5}{2}. \]

First we deform (increase) \( x_3 \) to get
\[ x_3 - x_2 = 1, \]
if it is not already. If \( x_1 + x_2 < 1 \), then we can deform (increase) \( x_3 - x_2 = 1 \) to get \( x_1 + x_2 = 1 \). Now switch \( x_1 \) to \( -x_1 \), pass to an irreducible subquotient of the induced representation of a general linear group corresponding to exponents \((-x_1, x_2, x_3)\), and deform it to \((\frac{1}{2}, \frac{3}{2}, \frac{5}{2})\). We get a contradiction with Lemma [10.1]. This settles also the case \( x_1 + x_2 = 1 \).

It remains to consider the case \( x_1 + x_2 > 1 \). If \( x_2 \geq 1 \), then we can deform (decrease) \( x_1 \) to get \( x_2 - x_1 = 1 \) if it is not already. Now we repeat the previous argument (we do not need switching). Let \( x_2 < 1 \) (recall \( \frac{1}{2} < x_2 \) and \( \frac{3}{2} < x_3 \)). Then we can deform (decrease) \( x_1 \) to get \( x_2 + x_1 = 1 \) if it is not already. Now we repeat the previous argument using Lemma [10.1] (we need switching \( x_1 \) to \(-x_1 \) in this case).

(2) Now consider the case
\[ x_3 = \frac{3}{2}. \]
If \( x_2 = \frac{3}{2} \), then \( x_1 = \frac{1}{2} \). Here we do not have irreducible unitarizable subquotients (see F.1). Therefore, we need to consider \( \frac{1}{2} \leq x_2 < \frac{3}{2} \). Suppose \( x_2 = \frac{1}{2} \). We need to consider \( x_1 < \frac{1}{2} \). In this case we have seen that we need to have unitarizability as it is claimed in the lemma (this is the place of the first group of the complementary series). Also for \( x_1 = \frac{1}{2} \).

Therefore we need to consider
\[ \frac{1}{2} < x_2 < \frac{3}{2}. \]
Suppose $x_1 = \frac{1}{2}$. Then there is an irreducible subquotient $\theta$ of $\begin{bmatrix} 1 \end{bmatrix}^{(\rho)} \times \begin{bmatrix} 3 \end{bmatrix}^{(\rho)} \rtimes \sigma$ such that after deformation $x_2$ to $\frac{3}{2}$, all the irreducible subquotients of $\begin{bmatrix} 3 \end{bmatrix}^{(\rho)} \rtimes \theta$ are unitarizable. This is a contradiction (see F.1). Therefore, we need to consider the case

$$x_1 < \frac{1}{2}.$$  

In other words, we need to analyze the case

$$x_1 < \frac{1}{2} < x_2 < \frac{3}{2} = x_3.$$  

Let $x_2 \geq 1$. Then we can deform $x_1$ decreasingly to get $x_2 - x_1 = 1$ if it is not already the case. Now we can deforming an irreducible subquotient of a representation of a general linear group corresponding to exponents $x_1, x_2$ to $\frac{1}{2}, \frac{3}{2}$ and get a contradiction with F.1.

In the case $x_2 < 1$ we can deform $x_1$ to get $x_2 + x_1 = 1$ if it is not already. We switch $x_1$ to $-x_1$, and get a contradiction like in previous case.

(3) Now we consider the case

$$\frac{1}{2} < x_3 < \frac{3}{2}.$$  

First assume that $x_1 = \frac{1}{2}$. Suppose $x_2 > \frac{1}{2}$. Then we can deform $x_2$ to get $x_2 = x_3$, and get a contradiction using the unitary parabolic reduction. Therefore we need to consider $x_2 = \frac{1}{2}$ in this case. Now we are in the case of the third complementary series, when $\frac{1}{2} < x_3 < \frac{3}{2}$ (in the case $x_3 \leq \frac{1}{2}$ in the notations there, all the irreducible subquotients at that place are unitarizable). We have seen that in this case the claim of the lemma holds. Therefore, it remains to consider the case $x_1 < \frac{1}{2}$, i. e.

$$x_1 < \frac{1}{2} < x_3 < \frac{3}{2}.$$  

Now we consider three possibilities regarding $x_2$.

(3-a) First consider the case

$$\frac{1}{2} < x_2.$$  

We now analyze several possibilities:

Suppose $x_3 - x_1 > 1$. Now we can deform (increase) $x_3$ to $x_3'$ to get $x_3' - x_2 = 1$. Then $\frac{3}{2} < x_3' < \frac{5}{2}$. We know by (1) that we do not have here unitarizable subquotients, so we have got a contradiction. Therefore, we need to assume

$$x_3 - x_1 \leq 1.$$  

Suppose first $x_3 - x_1 = 1$.

If $x_1 + x_2 = 1$, then we are in the case of the last group of complementary series, and we know that the claim of the lemma holds in this case.
If $x_1 + x_2 < 1$, then we first deform $x_2$ to $\frac{1}{2}$, chose there the irreducible subquotient of opposite type then the one that we have for exponents $x_1, x_2$, deform to the right the irreducible subquotient corresponding to $x_1, x_3$ until reducibility. We get a contradiction (a non-unitarizable subquotient).

If $x_1 + x_2 > 1$, then we first deform the irreducible subquotient corresponding to $x_1, x_3$ to $(\frac{1}{2}, \frac{3}{2})$, and then deform $x_2$ to $\frac{1}{2}$. We easily conclude the non-unitarizability of at least one irreducible subquotient after these deformations (if the irreducible subquotient is essentially square integrable, then the non-unitarizable subquotient is the first representation of (2) in Proposition 7.1; otherwise it is the second one).

At the end suppose

$$x_3 - x_1 < 1.$$ 

Let $x_1 + x_2 < 1$.

If $x_3 \geq 1$, we can deform (decrease) $x_1$ to get $x_3 - x_1 = 1$ if it is not already. Now we handle situation in a usual way to get a contradiction (performing two deformations, and choosing appropriate subquotients - first deforming $x_2$ to $\frac{1}{2}$ and then deforming $x_3 - x_1 = 1$ to the right to the first reducibility point).

If $x_3 < 1$, then we again deform $x_1$ to get $x_3 + x_1 = 1$ if it is not already (if $x_3 + x_1$ was $> 1$, then decrease, in the other case increase, which we can since $x_2 + x_1 \leq x_3 + x_2$). We complete as in the previous case.

Suppose now $x_1 + x_2 = 1$. Now $\pi \cong \theta \times [x_3]^{(\rho)} \rtimes \sigma$ for $\theta$ an irreducible subquotient of $[-x_1]^{(\rho)} \times [x_2]^{(\rho)}$. Now one easily sees that can we deform (irreducibly) $x_3$ to $\frac{1}{2}$ (since $x_3 - x_2 \leq x_3 - x_1 < 1$ and $x_3 - x_1 < 1$). In the limit chose an irreducible subquotient of opposite type then $\theta$ (in the same way as before). Then deform $\theta$ to exponents $\frac{1}{2}, \frac{3}{2}$. One easily gets that in the limit is an irreducible subquotient which is not unitarizable (use Proposition 7.1).

It remains to consider $x_1 + x_2 > 1$. Then also $x_1 + x_3 > 1$. Now we can deform (increase) $x_3$ to get $x_3 - x_1 = 1$. Further we deform $x_2$ to $\frac{1}{2}$. We finish similarly as in the previous cases.

(3-b) Now we consider the case

$$x_2 = \frac{1}{2}.$$ 

We analyze three possibilities.

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28This means that if we have an essentially square integrable representation corresponding to exponents $x_1, x_3$, then we chose the Langlands quotient of $[\frac{1}{2}]^{(\rho)} \rtimes \sigma$, and vice versa.
If $x_3 + x_1 = 1$, then this is the place where we have two complementary series (this is the second group of the complementary series). Here we have seen that the claim of the lemma holds.

If $x_3 + x_1 > 1$, then the representation is of the form $[x_1]^{(\rho)} \times [x_3]^{(\rho)} \rtimes \theta$, where $\theta = \delta([1/2]^{(\rho)}; \sigma)$ or $L([1/2]^{(\rho)}; \sigma)$. Further if $x_3 \geq 1$, then we can deform (decrease) $x_1$ to get $x_3 - 1 = 1$, while if $x_3 < 1$, then we deform (irreducibly) $x_1$ to get $x_1 + x_3 = 1$. In both cases we chose an irreducible subquotient which is opposite type then $\theta$. Now from the case of second group of complementary series, we know that we do not have unitarizability here. So we have got a contradiction.

If $x_1 + x_3 < 1$, then we can deform (increase) $x_1$ to get $x_1 + x_3 = 1$. Now we get a contradiction in the same way as in the previous case.

(3-c) It remains to consider the case $x_2 < \frac{1}{2}$.

Suppose $x_2 + x_3 < 1$. Now we can deform (decrease) $x_2$ to get $x_1 = x_2$, apply the unitary parabolic reduction and get a contradiction with the rank one case. Therefore

$$x_2 + x_3 \geq 1.$$

Suppose $x_2 + x_3 = 1$. Now $\pi \cong \theta \times [x_1]^{(\rho)} \rtimes \sigma$ for $\theta$ an irreducible subquotient of $[-x_2]^{(\rho)} \times [x_3]^{(\rho)}$. One can deform $x_1$ to $\frac{1}{2}$, chose there (in generalized rank case one of classical group) an irreducible subquotient of opposite type then $\theta$. Now the case which we have settled of induced representations where second complementary series show up gives a non-unitarizability, which is a contradiction.

It remains to consider $x_2 + x_3 > 1$.

Suppose $x_1 + x_3 < 1$. Now we can deform (increase) $x_1$ to get $x_1 + x_3 = 1$. Here we have two choices of irreducible subquotients (of general linear group - both give unitarizable subquotients). Chose any one of them, and denote it by $\theta$. Now we can deform $x_2$ to $\frac{1}{2}$. Here we chose (in generalized rank case one of classical group) an irreducible subquotient of opposite type then in previous step. Now we deform $\theta$ to $\frac{1}{2}, \frac{5}{2}$ and get a contradiction in as previously (we are in the setting of the second group of complementary series).

Consider now the case $x_1 + x_3 = 1$. Deform $x_2$ to $\frac{1}{2}$ and chose here different type of the subquotient then we have in the case of $x_1, x_3$ (see previous cases for more details). Further deform $x_1, x_3$ to $(\frac{1}{2}, \frac{3}{2})$. We get a contradiction with the case of the second group of complementary series which we have settled earlier.

It remains to consider the case $x_1 + x_3 > 1$. Now we deform (increase) $x_3$ to get $x_3 - x_1 = 1$. Now we finish as in the previous case (deforming first $x_2$ to $\frac{1}{2}$). \qed
Lemma 10.6. Let 
\[ \alpha = 0. \]
Suppose that \( \pi \) is an irreducible unitarizable subquotient corresponding to \((x_1, x_2, x_3)\). Then 
\[ x_1 = 0. \]

Proof. Suppose that the lemma does not hold, i.e. \( x_1 > 0 \), and let \( \pi \) be an irreducible unitarizable subquotient. Then 
\[ 0 < x_1 \leq \frac{1}{2}. \]
We know 
\[ x_2 \leq x_1 + 1, \quad x_3 \leq x_2 + 1. \]
Observe that \( \pi \leq \theta \rtimes \sigma \) for some irreducible subquotient \( \theta \) of \([x_1]^{(\rho)} \times [x_2]^{(\rho)} \times [x_3]^{(\rho)}\). Now Jantzen decomposition and 2.12 imply that \( \pi \cong \theta \rtimes \sigma \).

Suppose that \( \theta \) is not fully induced representation. Then \( \theta \) is an irreducible subquotient of \([x_1]^{(\rho)} \times [x_1 + 1]^{(\rho)} \times [x_1 + 2]^{(\rho)}\). Now we deform \( \theta \) in a way that \( x_1 \) comes to 0. Then we get a contradiction to Lemma 10.1.

It remains to consider the case when \( \theta \) is fully induced. We have two possibilities.

The first possibility is that \( \theta \cong [x_i]^{(\rho)} \times \tau \rtimes \sigma \), where \( \tau \) is an irreducible subquotient of some \([x_j]^{(\rho)} \times [x_j + 1]^{(\rho)}\), for some different \( 1 \leq i \neq j \leq 3 \). Now deform \( x_i \) to the right to the first reducibility point, and take an irreducible subquotient there. It is of form \( \theta' \rtimes \sigma \). Now \( \theta' \) is not fully induced. We get a contradiction in the same way as in the previous case (using Lemma 10.1).

At the end suppose \( \theta \cong [x_1]^{(\rho)} \times [x_2]^{(\rho)} \times [x_3]^{(\rho)} \). Then we can deform \( x_3 \) to the right until we reach the first reducibility point. We would get a new irreducible unitarizable subquotient of the form 
\[ [x_i]^{(\rho)} \times \tau \rtimes \sigma, \]
where \( \tau \) is an irreducible subquotient of some \([x_j]^{(\rho)} \times [x_j + 1]^{(\rho)}\), for some different \( 1 \leq i \neq j \leq 3 \). Now the previous case implies that we cannot have here unitarizability. \( \square \)

11. Unitarizability for generalized rank 3

Proposition 11.1. Assume 
\[ \alpha \geq \frac{3}{2}. \]
Let \( \pi \) be an irreducible unitarizable subquotient of \([x_3]^{(\rho)} \times [x_2]^{(\rho)} \times [x_1]^{(\rho)} \rtimes \sigma \). Then \( \pi \) is one of the following irreducible unitarizable representations:

(1) \( \delta([\alpha, \alpha + 2]^{(\rho)}; \sigma) \) or its ASS dual.
(2) Irreducible subquotient of \([x_1]^{(\rho)} \rtimes \theta\), where
\[ \theta \in \{ \delta([\alpha, \alpha + 1]^{(\rho)}; \sigma), L([\alpha + 1]^{(\rho)}; [\alpha]^{(\rho)}; \sigma) \} \] and \(0 \leq x_1 \leq \alpha - 1\).

(3) \(L([\alpha]^{(\rho)}, [\alpha - 1]^{(\rho)}; \delta([\alpha]^{(\rho)}); \sigma)\).

(4) Irreducible subquotient for
(a) \(x_2 + x_3 \leq 1\),
(b) \(x_1 + x_2 \leq 1, \ x_2 + 1 \leq x_3 \leq \alpha\),
(c) \(x_1 + x_2 \leq 1, \ 1 - x_1 \leq x_3 \leq 1 + x_1\),
(d) \(x_1 + 1 \leq x_2, \ x_2 + 1 \leq x_3 \leq \alpha\). 29

Proof. (1) First we shall see unitarizability of the representations whose unitarizability is claimed in the proposition. The unitarizability of representations in (1) we have already explained in E.1. The unitarizability of the representations in (2) is explained in the proof of lemma 10.4. The unitarizability of the representation in (3) follows from the fact that it is in an Arthur packet. This was communicated to us by C. Mœglin, who also wrote me the explanation how it follows from [40].

Now we shall explained how one constructs complementary series in cases (a) - (d) of (4) (we shall explain unitarizability for the interior of regions, while the unitarizability for the closure follows from the fact that limits of complementary series have unitarizable irreducible subquotients).

The unitarizability of representations in (a) is obvious (if \(x_2 + x_3 < 1\), then no other reducibility can happen, and we have unitarizability at the origin \((0, 0, 0)\)). These complementary series exist for \(\alpha \geq 1\).

For unitarizability in (b), we construct first complementary series in rank one for with \(x_3 < \alpha\), and then construct further complementary series for \(x_1 + x_2 < 1, \ x_2 + 1 < x_3\). These complementary series exist for \(\alpha \geq \frac{3}{2}\).

Similarly goes the construction in the case (d) (but here we have three steps in construction). These complementary series exist for \(\alpha > 2\) (one easily sees that for \(\alpha = 2\) the point which shows up then have unitarizable irreducible subquotients, since it is at the end of complementary series (b)).

Irreducibility in these three cases when we are in not in the boundary of the region is obvious since reducibility hyperplanes are
\[ x_i = \alpha, \quad 1 \leq i \leq 3, \]

29This region is empty if \(\alpha = \frac{3}{2}\), while for \(\alpha = 2\) it has only one point.
\[ x_i + x_j = 1, \quad 1 \leq i \neq j \leq 3, \]
\[ x_i - x_j = 1, \quad 1 \leq i \neq j \leq 3 \]

We assume \( 0 \leq x_1 \leq x_2 \leq x_3 \).

For unitarizability of the family (c), we consider the region
\[ x_1 + x_2 < 1, \quad 1 - x_1 < x_3 < 1 + x_1. \]

One easily checks that the induced representations corresponding to above exponents are irreducible (observe that then in particular \( x_1 < \frac{1}{2} \) and \( \frac{1}{2} < x_3 < \frac{3}{2} \)). Fix any \( x_1 \) and \( x_3 \) satisfying \( 0 < x_1 < \frac{1}{2} \), and \( 1 - x_1 < x_3 < 1 + x_1 \). Now the representation \([x_3]^{(\rho)} \times [x_1]^{(\rho)} \times [1 - x_1]^{(\rho)} \times \sigma \cong ([x_3]^{(\rho)} \times [-x_1]^{(\rho)}) \times ([x_3]^{(\rho)} \times \sigma)\) is in the region and it is unitarizable (since it is unitarily induced). This implies the unitarizability of representations in (c).

(2) Now we shall prove the exhaustion claimed in the proposition. We shall assume that \( \pi \) is an irreducible unitarizable subquotient. Lemma 10.4 implies that it is enough to settle the case \( x_3 \leq \alpha \). We shall now consider the case of equality and the case of strict inequality below. First consider the case
\[ x_3 = \alpha. \]

Suppose \( x_2 > \alpha - 1 \). We cannot have \( x_1 = x_2 = \alpha \) by H.1. If \( x_2 = \alpha \) and \( x_1 < \alpha, x_1 \neq \alpha - 1 \), we can deform \( x_1 \) to \( \alpha - 1 \). Now Remark 3.12 implies contradiction. For \( x_1 = \alpha - 1 \), Lemmas 3.10 and 3.11 imply that \( \pi \) cannot be here.

Therefore, (when \( x_3 = \alpha \)) it remains to consider the case \( x_2 < \alpha \). Suppose \( x_2 > \alpha - 1 \). If \( x_1 \geq \alpha - 1 \), we deform \( x_1 \) and \( x_2 \) to \( \alpha - 1 \) and get a contradiction with K.1. Thus \( x_1 < \alpha - 1 \). Now we can deform \( x_1 \) to get \( x_2 - x_1 = 1 \) or \( x_2 + x_1 = 1 \) (and pass to an irreducible subquotient at the limit). Next step is to deform to \((\alpha - 1, \alpha)\). In this way we get a contradiction with Remark 3.12.

Therefore, it remains to consider \( x_2 \leq \alpha - 1 \). Suppose that we have equality, i.e. \( x_2 = \alpha - 1 \). If \( x_1 > \alpha - 2 \), then we can deform \( x_1 \) to \( \alpha - 1 \) and get contradiction with K.1 if \( \alpha \geq 2 \) (all subquotients there are non-unitarizable). If \( \alpha = \frac{3}{2} \), then we are in the complementary series (b) of (4). Therefore, we need to consider the case
\[ x_2 < \alpha - 1. \]

It remains to consider the case when we are not in any of the complementary series in (4). This implies that \( x_1 + x_2 > 1 \) and \( x_2 < x_1 + 1 \), i.e.
\[ 1 - x_1 < x_2 < 1 + x_1. \]

One easily sees that this is a family of irreducible Hermitian representations.

Suppose \( \alpha \geq 2 \). Then we consider here \( \frac{1}{2} < x_1 = x_2 < \alpha - 1 \), apply parabolic reduction, and get contradiction with the unitarizability in the case of general linear groups. For
\( \alpha = \frac{3}{2}, \quad x_2 \leq \frac{1}{2}, \) and therefore \( x_1 + x_2 \leq 1. \) Therefore, this is empty set for \( \alpha = \frac{3}{2}. \) This completes the proof also for \( \alpha = \frac{3}{2}. \)

(3) It remains to consider the case

\[ x_3 < \alpha, \]

when we are not in any of the complementary series in (4). We need to assume

\[ x_2 + x_3 > 1, \]

since otherwise, we are at the complementary series (or its ends). Then obviously

\[ x_3 > \frac{1}{2}. \]

(3-a) Suppose \( x_3 - x_2 < 1 \) and \( x_1 + 1 \leq x_2. \) Then we can deform \( x_3 \) to \( x_2, \) keeping distance the same all the time between \( x_1 \) and \( x_2. \) If this distance is 1, then we deform to \( (\alpha - 1, \alpha, \alpha) \) and get a contradiction with Remark [3.12]. If the distance is \( < 1, \) we deform to \( (\alpha, \alpha, \alpha) \) and get contradiction with H.1. Therefore, we do not have unitarizability here. It remains to consider the case when

\[ x_2 + 1 \leq x_3 \quad \text{or} \quad x_2 < x_1 + 1. \]

(3-b) Consider the possibility

\[ x_2 + 1 \leq x_3. \]

Since we assume that we are in neither of the complementary series (a) - (d), we get \( 1 < x_1 + x_2 \) and \( x_2 < x_1 + 1. \)

Now we can deform (decrease) \( x_3 \) to get \( x_3 - x_2 = 1 \) if it is not already. Then since \( x_1 + x_2 > 1 \) and \( x_2 < x_1 + 1, \) we can deform \( x_1 \) to \( x_3. \) Now we can deform all this to \( (\alpha - 1, \alpha, \alpha) \) and get a contradiction (see Remark [3.12]).

(3-c) Therefore, it remains to consider the case when

\[ x_2 < x_1 + 1 \quad \text{and} \quad x_3 < x_2 + 1. \]

Since we are not in the complementary series (a) of (4), we get know that holds

\[ 1 < x_2 + x_3, \quad \text{and} \quad 1 < x_1 + x_2 \text{ or } x_3 \notin [1 - x_1, 1 + x_1]. \]

(3-d) Suppose \( x_1 > \frac{1}{2}. \) If \( x_3 - x_1 < 1, \) then we can deform to \( (\alpha, \alpha, \alpha) \) and get a contradiction. In the case \( x_3 - x_1 \geq 1, \) we can deform \( x_1 \) to get \( x_3 - x_1 = 1 \) (possibly also deforming \( x_2 \) suitable that it is between \( x_1 \) and \( x_3). \) Now we can deform \( x_2 \) to \( x_3, \) and then we can deform to \( (\alpha - 1, \alpha, \alpha) \) and get a contradiction (with Remark [3.12]).

If \( x_1 = \frac{1}{2} \) and \( x_2 > \frac{1}{2}, \) we get a contradiction in a similar way (first deform \( x_3 \) to get \( x_3 - x_2 = 1 \) or \( x_3 - x_1 = 1 \) and then act as in the previous case).

In the case \( x_1 = x_2 = \frac{1}{2} \) we have unitarizability (it is in complementary series (a)).
Therefore, it remains to consider the case
\[ x_1 < \frac{1}{2}. \]

(3-e) Suppose that
\[ x_1 + x_2 > 1. \]
Now we can deform \( x_1 \) increasing it to get \( x_3 - x_1 = 1 \) or \( x_1 = x_3 \). In both cases we can now deform to the right and get a contradiction (with the unitarizability at \((\alpha - 1, \alpha, \alpha)\) or \((\alpha, \alpha, \alpha)\)).

Therefore, we need to consider the case
\[ x_1 + x_2 \leq 1. \]
We have also \( x_2 + x_3 > 1 \) and \( x_3 < x_2 + 1 \), i.e
\[ 1 - x_2 < x_3 < x_2 + 1. \]
If \( 1 - x_1 \leq x_3 \leq 1 + x_1 \), then we are in the complementary series (c). Consider the remaining two cases

(3-f) The first case is
\[ 1 + x_1 < x_3 < 1 + x_2, \]
i.e.
\[ x_1 < x_3 - 1 < x_2. \]
Now we can deform \( x_2 \) to the right to get \( x_1 + x_2 = 1 \) and consider an irreducible subquotient at \( x_2 - x_1 = 1 \) of the limit. Further we can deform to \((\alpha - 1, \alpha, \alpha)\), and get a contradiction.

(3-g) Consider now the remaining case
\[ 1 - x_2 < x_3 < 1 - x_1. \]
This is a region of irreducible representations. Then we take in this region \( x_2 = x_3 > \frac{1}{2} \) (for example \( x_1 = \frac{1}{5}, x_2 = \frac{2}{5}, x_3 = \frac{3}{5} \)). Now using the unitary parabolic reduction we get a contradiction with the complementary series in the case of general linear groups. \(\square\)

**Proposition 11.2.** Let \( \pi \) be an irreducible unitarizable subquotient of \([3]^{(\rho)} \times [2]^{(\rho)} \times [1]^{(\rho)} \rtimes \sigma \). Assume
\[ \alpha = 1. \]
Then \( \pi \) is one of the following irreducible unitarizable representations:

1. Two irreducible unitarizable subquotients of \([3]^{(\rho)} \times [2]^{(\rho)} \times [1]^{(\rho)} \rtimes \sigma \) (the Steinberg representation of its ASS dual).
2. Irreducible subquotient of \([0]^{(\rho)} \rtimes \delta([1, 2]^{(\rho)}; \sigma), [0]^{(\rho)} \rtimes L([2]^{(\rho)}, [1]^{(\rho)}; \sigma). \]
3. Irreducible subquotients for one of the following two regions
(a) \[ x_2 + x_3 \leq 1, \]
(b) \[ x_1 + x_2 \leq 1, \quad 1 - x_1 \leq x_3 \leq 1. \]

**Proof.** First we shall see unitarizability of the representations. Unitarizability of representations in (1) and (2) is clear, as well as complementary series and its ends in (a) of (3).

Recall that the reducibility hyperplanes are
\[
x_i = 1, \quad 1 \leq i \leq 3,
\]
\[
x_i + x_j = 1, \quad 1 \leq i < j \leq 3,
\]
\[
x_i - x_j = 1, \quad 1 \leq i \neq j \leq 3
\]
(we also assume \( 0 \leq x_1 \leq x_2 \leq x_3 \)). The irreducibility of the interior of the region in (b) is also obvious. Take any \( x_1 < \frac{1}{2}, x_3 < 1 \) such that \( x_1 + x_3 > 1 \). Then \( (x_1, x_1, x_3) \) is in the interior of the region and the corresponding irreducible representation \([x_3]^{(\rho)} \times [x_1]^{(\rho)} \times [x_1]^{(\rho)} \rtimes \sigma \cong ([x_1]^{(\rho)} \times [-x_1]^{(\rho)}) \rtimes ([x_3]^{(\rho)} \rtimes \sigma)\) is unitarizable. This implies the unitarizability of representations in (b).

Now we come to the exhaustion. We shall suppose that \( \pi \) is an irreducible unitarizable subquotient, and that it is not any of the irreducible unitarizable representations listed in (1) - (3). Lemma 10.4 implies that it is enough to consider the case \( x_3 \leq 1 \).

Consider first the case \( x_3 = 1 \). If \( x_1 = 0 \), then we are in the region (b) of (3). Therefore, we need to consider the case \( x_1 > 0 \). If \( x_2 = 1 \), we can deform \( x_1 \) to 1 and we are in the point where we do not have unitarizability (H.1). It remains to consider \( x_2 < 1 \). Since we are not in the region (b), we have \( x_1 + x_2 > 1 \). Now we can deform \( x_2 \) to 1 and get a contradiction with a previous case.

It remains to consider the case of \( x_3 < 1 \).

We need to assume \( x_2 + x_3 > 1 \), and \( x_1 + x_2 > 1 \) or \( x_1 + x_3 < 1 \), since otherwise we are in the regions in (3). This implies \( x_2 + x_3 > 1 \), and \( x_1 + x_3 < 1 \).

This defines a region of irreducible representations. Take \( x_1 = 0 \) and any \( \frac{1}{2} < x_2 = x_3 < 1 \). Then \( (x_1, x_2, x_3) \) is in the above region. Corresponding representation is \([x_2]^{(\rho)} \times [x_2]^{(\rho)} \times [0]^{(\rho)} \rtimes \sigma \cong ([x_2]^{(\rho)} \times [-x_2]^{(\rho)}) \rtimes ([0]^{(\rho)} \rtimes \sigma)\). Using the unitary parabolic reduction, we get that this representation is not unitarizable (since complementary series for general linear groups end at \( \frac{1}{2} \)). This completes the proof of the lemma. \( \square \)
A direct consequence of Lemma 10.5 is the following

**Proposition 11.3.** Let be an irreducible unitarizable subquotient of \([x_3]^{(\alpha)} \times [x_2]^{(\alpha)} \times [x_1]^{(\alpha)} \rtimes \sigma\). Assume
\[
\alpha = \frac{1}{2}.
\]
Then \(\pi\) is one of the following irreducible unitarizable representations:

1. Two irreducible unitarizable subquotient of \([\frac{5}{2}]^{(\alpha)} \times [\frac{3}{2}]^{(\alpha)} \times [\frac{1}{2}]^{(\alpha)} \rtimes \sigma\).
2. Irreducible subquotient of \(\pi_1 \rtimes \delta_{s.p.}(\frac{1}{2}, \frac{3}{2}, \sigma)\)
   \[
   [\pi_1]^{(\alpha)} \rtimes L(\frac{1}{2}, \frac{3}{2}, \sigma), \quad 0 \leq \pi_1 \leq \frac{1}{2}.
   \]
3. Irreducible subquotient of \(\delta([-\frac{1}{2} + x, \frac{1}{2} + x]^{(\alpha)}) \rtimes \delta_{s.p.}(\frac{1}{2}^{(\alpha)} ; \sigma)\),
   \[
   L([-\frac{1}{2} + x, \frac{1}{2} + x]^{(\alpha)}) \rtimes L(\frac{1}{2}; \sigma), \quad 0 \leq \pi_1 \leq 1.
   \]
4. Irreducible subquotient of \(\pi_3 \rtimes \delta([-\frac{1}{2}, \frac{1}{2}]^{(\alpha)} ; \sigma)\),
   \[
   [\pi_3]^{(\alpha)} \rtimes L([1]^{(\alpha)}; \delta_{s.p.}(\frac{1}{2}^{(\alpha)} ; \sigma)) \rtimes \delta_{s.p.}(\frac{1}{2}^{(\alpha)} ; \sigma)) \quad 0 \leq \pi_3 \leq \frac{3}{2}.
   \]
5. Irreducible subquotient of \(\delta([-1 + x, 1 + x]^{(\alpha)}) \rtimes \sigma\),
   \[
   L([-1 + x]^{(\alpha)}, [x]^{(\alpha)}, [1 + x]^{(\alpha)}) \rtimes \sigma \quad 0 \leq \pi_1 \leq \frac{1}{2}.
   \]
6. Irreducible subquotients for
   \[
   x_i \leq \frac{1}{2}, \quad i = 1, 2, 3.
   \]
\(\square\)

**Proposition 11.4.** Let be an irreducible unitarizable subquotient of \([x_3]^{(\alpha)} \times [x_2]^{(\alpha)} \times [x_1]^{(\alpha)} \rtimes \sigma\) Assume
\[
\alpha = 0.
\]
Then \(\pi\) is one of the following irreducible unitarizable representations:

1. Four irreducible unitarizable subquotients of \([2]^{(\alpha)} \times [1]^{(\alpha)} \times [0]^{(\alpha)} \rtimes \sigma\) (i.e. two irreducible square integrable representations and their ASS duals)
2. \([x]^{(\alpha)} \rtimes \delta([0, 1]^{(\alpha)} ; \sigma), \quad [x]^{(\alpha)} \rtimes L([1]^{(\alpha)}; \delta([0]^{(\alpha)} ; \sigma)) \quad 0 \leq \pi_1 \leq 1.
3. Irreducible subquotients for
   \[
   x_1 = 0, \quad x_2 + x_3 \leq 1,
   \]

**Proof.** We know the unitarizability of the representations claimed in the proposition. We consider now the exhaustion. We shall suppose that \(\pi\) is an irreducible unitarizable subquotient, and that it is not any of the irreducible unitarizable representations listed in the above proposition. Lemma 10.6 implies that it is enough to settle the case
\[
\pi_1 = 0.
\]
This implies $x_2 \leq 1$. If $x_3 = 2$, then $x_2 = 1$ and Proposition 8.1 implies the claim of the proposition in this case. Therefore we need to consider

$$x_3 < 2.$$ 

Let

$$1 < x_3 < 2.$$ 

Suppose $x_2 = 1$. Now deforming $x_3$ to 2, we would get that there is an irreducible subquotient $\theta$ of $[0](\rho) \times [1](\rho) \rtimes \sigma$ such that all irreducible subquotients of $[2](\rho) \rtimes \theta$ are unitarizable. From Proposition 8.1 and 2.7 we know that this is not the case if $\theta = \delta([0,1]_{\pm};\sigma)$ and if $\theta = L([1](\rho);\tau([0]_{\pm};\sigma))$. For $\theta = L([1](\rho);\tau([0]_{\pm};\sigma))$, apply two times Aubert involution, which will give a contradiction.

For $x_3 = 1$, all the irreducible subquotients are in unitarizable, and they are in (2) of the proposition. Therefore, we need to consider the case

$$x_3 < 1.$$ 

Not to be in the region (3), we need to assume

$$x_2 + x_3 > 1.$$ 

Obviously, $x_3 > \frac{1}{2}$. Then $\pi \cong [x_2](\rho) \times [x_3](\rho) \rtimes \delta([0]_{\pm};\sigma)$. Now we can deform $x_2$ to $x_3$, apply the unitary parabolic reduction and get a contradiction with the unitarizability in the case of general linear groups. This completes the proof. 

12. Unitarizability in mixed case for generalized rank $\leq 3$

In this section we shall use notation and terms introduced in sections 8 and 9 of [78] regarding Jantzen decomposition of an irreducible representation of a classical $p$-adic group. We shall recall some of the most basic definitions. One needs to go to section 8 of [78] for more details.

Let $X \subseteq \mathcal{C}$ and suppose that $X$ is $F'/F$-selfcontragredient, i.e. that

$$\tilde{X} = X,$$

where $\tilde{X} = \{\tilde{\rho}; \rho \in X\}$, and let $\sigma$ be an irreducible cuspidal representation of a classical group. An irreducible representation $\gamma$ of a classical group will be called supported by $X \cup \{\sigma\}$ if there exist $\rho_1, \ldots, \rho_k$ from $X$ such that

$$\gamma \leq \rho_1 \times \ldots \times \rho_k \rtimes \sigma.$$ 

For not-necessarily irreducible representation $\pi$ of a classical group, one says that it is supported by $X \cup \{\sigma\}$ if each irreducible subquotient of it is supported by that set.

Let

$$X = X_1 \cup X_2$$
be a partition of an \( F'/F \)-selfcontragredient \( X \subseteq C \). We shall say that this partition is regular if \( X_1 \) is \( F'/F \)-selfcontragredient, and if among \( X_1 \) and \( X_2 \) there is no reducibility, i.e. if \( \rho_1 \times \rho_2 \) is irreducible for all \( \rho_1 \in X_1 \) and \( \rho_2 \in X_2 \).

Let \( \pi \) be an irreducible representation of a classical group supported in \( X \cup \{ \sigma \} \), where \( X \) is \( F'/F \)-selfcontragredient, and let \( X = X_1 \cup X_2 \) be a regular partition of \( X \). Fix \( i \in \{1, 2\} \). Then there exists an irreducible representation \( \beta \) of a general linear group supported by \( X_{3-i} \) and an irreducible representation \( \gamma \) of a classical group supported on \( X_i \cup \{ \sigma \} \) such that

\[
\pi \mapsto \beta \times \gamma.
\]

The representation \( \gamma \) is uniquely determined by the above requirement. It is denoted by

\[
X_i(\pi)
\]

and called the Jantzen component of \( \pi \) corresponding to the member \( X_i \) in the regular partition \( X = X_1 \cup X_2 \).

Let \( X \subseteq \mathcal{C} \) such that \( \mathcal{C} = X \cup (\mathcal{C} \setminus X) \) is a regular partition of \( \mathcal{C} \). Further, let \( \pi \) be any irreducible representation of a classical group. Then we denote

\[
X(\pi)
\]

and call it the Jantzen component of \( \pi \) corresponding to \( X \).

For \( \rho \in \mathcal{C} \) which is \( F'/F \)-selfcontragredient, denote \( X_\rho = \{ \nu^x \rho; x \in \mathbb{R} \} \). Let \( \pi \) be an irreducible weakly real representation of a classical \( p \)-adic group. One can find finitely many different \( F'/F \)-selfcontragredient representations \( \rho_1, \ldots, \rho_k \in \mathcal{C} \) such that the support of \( \pi \) is in \( X_{\rho_1} \cup \cdots \cup X_{\rho_k} \cup \{ \sigma \} \), where \( \sigma \) is an irreducible cuspidal representation of a classical group. Then representations

\[
(X_{\rho_1}(\pi), \ldots, X_{\rho_k}(\pi))
\]

determine \( \pi \), and it is a bijection from the set of irreducible representations supported by \( X_{\rho_1} \cup \cdots \cup X_{\rho_k} \cup \{ \sigma \} \) onto the direct product of irreducible representations supported by \( X_{\rho_i} \cup \{ \sigma \} \), \( i = 1, \ldots, k \). The inverse map is denoted by

\[
\Psi_{X_{\rho_1}, \ldots, X_{\rho_k}}.
\]

The correspondence \( \pi \mapsto (X_{\rho_1}(\pi), \ldots, X_{\rho_k}(\pi)) \) have a number of very nice properties (see [25] or section 8 of [78]). We shall now prove one additional very simple property.

**Lemma 12.1.** Let \( X \) be an \( F'/F \)-selfcontragredient subset of \( \mathcal{C} \), and let \( X = X_1 \cup X_2 \) be a regular partition of \( X \). Let \( \theta_i \) be an irreducible representation of a general linear group supported in \( X_i \) and \( \pi_i \) be an irreducible representation of a classical group supported in \( X_i \cup \{ \sigma \} \), \( i = 1, 2 \). Suppose that both \( \theta_i \times \pi_i \) are irreducible (i.e. for \( i = 1, 2 \)). Then

\[
(12.75) \quad \Psi_{X_1, X_2}(\theta_1 \times \pi_1, \theta_2 \times \pi_2) \cong \theta_1 \times \theta_2 \times \Psi_{X_1, X_2}(\pi_1, \pi_2).
\]
Proof. Note that \( \theta_1 \times \theta_2 \times \Psi_{X_1, X_2}(\pi_1, \pi_2) \) is irreducible by (1) of Remark 8.9 of \( \mathcal{L} \).

By the definition of \( \Psi_{X_1, X_2}(\pi_1, \pi_2) \), we know that \( \Psi_{X_1, X_2}(\pi_1, \pi_2) \) is irreducible and supported by \( X_2 \). This implies \( \tau \times \pi_1 \), where \( \tau \) is irreducible and \( x \) unitarizable. Here we give a very limited support to the possibility of positive answer to this question. We start with several simple lemmas.

Lemma 12.2. Let \( \pi \) be a weakly real irreducible subquotient of \( \theta_1 \times \ldots \times \theta_k \times \sigma \), where \( \theta_i \in \mathcal{C} \) and

\[ k \leq 3. \]

Suppose that all \( X_{\rho_i}(\pi) \) in the Jantzen decomposition of \( \pi \) are unitarizable. Then \( \pi \) is unitarizable.

Proof. For \( k = 1 \), there is nothing to prove. The case \( k = 2 \) is almost obvious (it goes the same way as the case \( \ell = 3 \) below). We shall now prove the case \( k = 3 \).

Let \( \pi \mapsto (X_{\rho_1}(\pi), \ldots, X_{\rho_k}(\pi)) \) be the Jantzen decomposition of \( \pi \). For the proof, we shall consider only those \( \rho_i \) for which \( X_{\rho_i}(\pi) \neq \sigma \). We shall assume this in the rest of the proof. Denote

\[ \alpha_i = \alpha_{\rho_i, \sigma}. \]

If \( \ell = 1 \), the claim obviously holds (since the in general \( \pi = X_{\rho_1}(\pi) \)).

Consider now the case \( \ell = 3 \). Then \( \pi \) is a subquotient of \( [x_1]^{(\rho_1)} \times [x_2]^{(\rho_2)} \times [x_3]^{(\rho_3)} \times \sigma \), where \( x_i \geq 0 \). Then \( X_{\rho_i}(\pi) \) are irreducible subquotients of \( [x_i]^{(\rho_i)} \times \sigma \). Since \( X_{\rho_i}(\pi) \) are unitarizable, then we know \( x_i \leq \alpha_i, 1 \leq i \leq 3 \). But then each irreducible subquotient of \( [x_1]^{(\rho_1)} \times [x_2]^{(\rho_2)} \times [x_3]^{(\rho_3)} \times \sigma \) is unitarizable (since we are in complementary series or its ends). Therefore, \( \pi \) is unitarizable.

Suppose \( \ell = 2 \). Then we may assume that \( \pi \) is a subquotient of \( [x_1]^{(\rho_1)} \times [x_2]^{(\rho_1)} \times [x_3]^{(\rho_2)} \times \sigma \). Then \( X_{\rho_2}(\pi) \) is an irreducible unitarizable subquotient of \( [x_3]^{(\rho_2)} \times \sigma \). This implies \( x_3 \leq \alpha_2 \). Further, \( \pi \) is an irreducible subquotient of \( [x_3]^{(\rho_2)} \times X_{\rho_1}(\pi) \). Since \( X_{\rho_1}(\pi) \) is unitarizable and \( x_3 \leq \alpha_2 \), this implies that \( \pi \) is unitarizable (again we are in complementary series or its ends). \( \square \)
Lemma 12.3. Let $\pi$ be a weakly real irreducible representation of a classical group. Suppose that some $X_{\rho_i}(\pi)$ is a non-unitarizable subquotient of $\theta_1 \times \ldots \times \theta_k \rtimes \sigma$, where $\theta_i \in \mathcal{C}$ and $k \leq 2$.

Then $\pi$ is not unitarizable.

Proof. Suppose opposite, i.e. that $\pi$ is unitarizable. Denote $\rho_i$ simply by $\rho$ and let $X^c_\rho = \mathcal{C} \setminus X_\rho$. We also denote $\alpha = \alpha_{\rho,\sigma}$. Now $X_\rho(\pi)$ is a subquotient of $[x_1]^{(\rho)} \times \ldots \times [x_k]^{(\rho)} \rtimes \sigma$, where $x_i \geq 0$ and $k \leq 2$. Denote $\pi_\rho = X_\rho(\pi)$ and $\pi^c_\rho = X^c_\rho(\pi)$. Clearly

$$
\pi = \Psi_{X_\rho, X^c_\rho}(\pi_\rho, \pi^c_\rho).
$$

If $k = 1$, then non-unitarizability of $\pi_\rho$ implies $\pi_\rho \cong [x_1]^{(\rho)} \rtimes \sigma$ where $x_1 > \alpha$. Now Lemma 12.1 implies

$$
\pi \cong [x_1]^{(\rho)} \rtimes \pi^c_\rho.
$$

This cannot be unitarizable, since we can deform $x_1$ to the right as far as we want. We get a contradiction (with the fact that unitarizability can show up only in bounded domains - see [59] for more details).

Consider now the case $k = 2$. We shall suppose as usually $0 \leq x_1 \leq x_2$. Recall that $\pi_\rho$ is a subquotient of

$$
[x_1]^{(\rho)} \times [x_2]^{(\rho)} \rtimes \sigma.
$$

We consider several cases. The first is

$$
\alpha = 0.
$$

The non-unitarizability of $\pi_\rho$ implies that $x_1 + x_2 > 1$. This implies $\pi_\rho \cong [x_1]^{(\rho)} \times [x_2]^{(\rho)} \rtimes \sigma$. Now Lemma 12.1 implies

$$
\pi \cong [x_1]^{(\rho)} \rtimes \pi^c_\rho.
$$

Now we can deform $x_1$ to $x_2$, use the unitary parabolic reduction and get a contradiction with the unitarizability in the case of general linear group (more precisely, with the complementary series there).

Suppose now

$$
\alpha > 0.
$$

First recall that Theorem 1.2 of [78] implies that

(12.76)

$$
\Psi_{X_\rho, X^c_\rho}(\tau, \pi^c_\rho).
$$

is not unitarizable if $\tau$ is a non-unitarizable irreducible subquotient of $[\alpha]^{(\rho)} \times [\alpha + 1]^{(\rho)} \rtimes \sigma$. 

We continue to consider the case $\alpha > 0$. By the remark about (12.76), it is enough to consider the case

$$(x_1, x_2) \neq (\alpha, \alpha + 1),$$

what we shall assume in the sequel.

Consider first the case $\alpha = \frac{1}{2}$.

Since $\pi_\rho$ is not unitarizable, then $x_1 > \frac{1}{2}$ or $x_2 > \frac{1}{2}$.

Suppose $x_i \neq \frac{1}{2}$ for $i = 1, 2$.

Let $x_2 \pm x_1 \neq 1$. Then $\pi \cong [x_1]^{(\rho)} \times [x_2]^{(\rho)} \rtimes \sigma$. Now Lemma 12.1 implies

$$\pi \cong [x_1]^{(\rho)} \times [x_2]^{(\rho)} \rtimes \pi_\rho^c.$$

If $x_1 > \frac{1}{2}$, then we can deform $x_1$ to $x_2$, switch one $x_2$ to $-x_2$, use the unitary parabolic reduction and get a contradiction (with existence of complementary series for general linear group), i.e. that $\pi$ cannot be unitarizable.

If $x_1 < \frac{1}{2}$. Then we can deform $x_2$ to get $x_2 + \epsilon x_1 = 1$ for some $\epsilon \in \{\pm 1\}$, and take there an irreducible subquotient denoted again by $\pi$ (which must be unitarizable). Now $\pi_\rho \cong \tau \times \sigma$ for some irreducible subquotient of reducible $[\epsilon x_1]^{(\rho_1)} \times [x_2]^{(\rho_1)}$. Lemma 12.1 implies

$$\pi \cong \tau \times \pi_\rho^c.$$

Now we can deform $\tau$ to exponents $(\frac{1}{2}, \frac{3}{2})$. The properties of the Jantzen decomposition and the fact that we mentioned above about (12.76) imply that $\pi$ is not unitarizable (since in the limit we have a non-unitarizable subquotient).

Let now $x_2 + \epsilon x_1 = 1$ for some $\epsilon \in \{\pm 1\}$. Then in the same way as above we get

$$\pi \cong \tau \times \pi_\rho^c$$

for some irreducible subquotient of reducible $[\epsilon x_1]^{(\rho)} \times [x_2]^{(\rho)}$. Now we finish this case as the previous one.

It remains to consider the case $x_1 = \frac{1}{2}$. Then $\pi_\rho \cong [x_2]^{(\rho)} \rtimes \theta$ where $\theta$ is an irreducible subquotient of $[\frac{1}{2}]^{(\rho_2)} \rtimes \sigma$ (recall $x_2 \neq \frac{2}{3}$). Now Lemma 12.1 implies

$$\pi \cong [x_2]^{(\rho)} \rtimes \psi_{X_\rho, X_\rho^c}(\theta, \pi_\rho^c).$$

We now deform $x_2$ to $\frac{3}{2}$, and in a similar way as before, we get a contradiction.

It remains to consider the case

$$\alpha \geq 1.$$
First assume 
\[ x_2 > \alpha. \]
We know from Proposition 9.1 that \( x_1 \leq \alpha \) and \( x_2 - x_1 \leq 1 \), which implies \( x_2 \leq \alpha + 1 \). Therefore if \( x_2 = \alpha + 1 \), then \( x_1 = \alpha \). Then we know that \( \pi \) is not unitarizable by above remark about (12.76). It remains to consider the case 
\[ x_2 < \alpha + 1. \]

Let 
\[ x_2 > \alpha. \]
Consider first the case \( x_1 = \alpha \). Now \( \pi_{\rho} = [x_2]^{(\rho)} \times \theta \) where \( \theta \in \{ \delta([\alpha]^{(\rho)}; \sigma), L([\alpha]^{(\rho)}; \sigma) \} \) (see 2.3). Now one directly gets that 
\[ \pi \cong [x_2]^{(\rho)} \times \Psi_{X_{\rho},X_{\rho}^c}(\theta, \pi_{\rho}^c). \]
We now deform \( x_2 \) to \( \alpha + 1 \), and in a similar way as before, we get a contradiction.

Therefore, we need to consider the case 
\[ x_1 < \alpha. \]

Assume first that \( x_2 - x_1 = 1 \), then a short analysis using Lemma 12.1 implies that \( \pi \cong \tau \times \pi_{\rho}^c \), where \( \tau \) is an irreducible subquotient of \([x_1]^{(\rho_1)} \times [x_2]^{(\rho_2)}\). Now deforming \( \tau \) to exponents \( \alpha, \alpha + 1 \), and using properties of the Jantzen decomposition would give a contradiction.

If \( x_2 - x_1 < 1 \), then similarly we get \( \pi \cong [x_1]^{(\rho_1)} \times [x_2]^{(\rho_2)} \times \pi_{\rho}^c \). Now we can deform (increase) \( x_2 \) to the previous case, consider a limit, and the repeat the above argument. Therefore, we get again contradiction.

Consider now the case 
\[ x_2 = \alpha. \]
We need to consider the case \( \alpha - 1 < x_1 \leq \alpha \). Then we know that \( \pi_{\rho} \cong [x_1]^{(\rho)} \times \theta \) for some irreducible subquotient \( \theta \) of \([\alpha]^{(\rho)} \times \sigma\), which implies by Lemma 12.1 
\[ \pi \cong [x_1]^{(\rho)} \times \Psi_{X_{\rho},X_{\rho}^c}(\theta, \pi_{\rho}^c). \]
Now we can deform \( x_1 \) to \( \alpha + 1 \), and get a contradiction as in previous cases.

It remains to consider the case 
\[ x_2 < \alpha \]
and what happens in the region 
\[ 1 - x_1 < x_2 < x_1 + 1. \]
Then representations \( \pi \cong [x_1]^{(\rho)} \times [x_2]^{(\rho)} \times \pi_{\rho}^c \), and moreover, for the exponents satisfying above relations, \([x_1]^{(\rho)} \times [x_2]^{(\rho)} \times \pi_{\rho}^c\) form a continuous family of irreducible Hermitian representations. Consider the point \( \frac{1}{2} < x_1 = x_2 < \alpha \) of the above region. After switching
$x_1$ to $-x_1$, the unitary parabolic reduction implies that this representation is not unitary. Therefore, the whole family is non-unitary. This completes the proof of the lemma \(\square\)

Now directly follows the following

**Corollary 12.4.** Let $\pi$ be a weakly real irreducible subquotient of $\theta_1 \times \ldots \times \theta_k \rtimes \sigma$, where $\theta_i \in \mathcal{C}, k \leq 3$. Then $\pi$ is unitarizable if and only if all $X_{\rho_i}(\pi)$ in the Jantzen decomposition of $\pi$ are unitarizable. \(\square\)

Our following aim will be to prove Lemma 12.3 for the case $k = 3$. We shall start with the following

**Lemma 12.5.** Let $\pi$ be a weakly real irreducible representation of a classical group. Suppose that some $X_{\rho_i}(\pi)$ is a representation belonging the following list some of representations (which are not unitarizable):

1. Representation $L([\alpha, \alpha + 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))$ from (1) of Lemma 3.4 (in Lemma 3.2 is proved that this representation is not unitarizable).
2. Representations in (2) of Proposition 3.6
3. Representations in (2) of Proposition 6.1
4. Representations in (2) of Proposition 7.1
5. Representations in (2) of Proposition 6.1

Then $\pi$ is not unitarizable.

**Proof.** Denote

$$\pi_{\rho} = X_{\rho}(\pi), \quad \pi_{\rho}^{c} = X_{\rho}^{c}(\pi).$$

Now

$$\pi = \Psi_{X_{\rho},X_{\rho}^{c}}(\pi_{\rho}, \pi_{\rho}^{c}).$$

Suppose that $\pi$ is a representation from (1) - (5), and that it is unitarizable.

Now proofs of the propositions and the lemma mentioned in (1) - (5) imply that that there exists an irreducible $F'/F$-selfcontragredient unitarizable representation $\tau$ of a general linear group supported in $X_{\rho}$ such that the length of

$$\tau \rtimes \pi_{\rho}$$

is at least $K_{\tau,\pi_{\rho}}$, and that the multiplicity of $\tau \otimes \pi_{\rho}$ in the Jacquet module of $\tau \rtimes \pi_{\rho}$ is at most $k_{\tau,\pi_{\rho}}$, where

$$k_{\tau,\pi_{\rho}} < K_{\tau,\pi_{\rho}}$$

if $\pi_{\rho}$ is not from (4), and

$$k_{\tau,\pi_{\rho}} = K_{\tau,\pi_{\rho}} = 6$$

if $\pi_{\rho}$ is from (4).
Now
\[ \tau \times \pi = \tau \times \Psi_{X_\rho,X^c_\rho}(\pi_\rho, \pi_\rho^c) \]
is a representation of length \( \geq K_{\tau,\pi_\rho} \) by (5) of Jantzen theorem 8.8 in [78] (take in that theorem \( \beta(X_\rho) = \tau, \beta(X^c_\rho) = 1 \), and multiply it by the representation \( \pi = \Psi_{X_\rho,X^c_\rho}(\pi_\rho, \pi_\rho^c) \)).

Since \( \pi \) is unitarizable and the length of \( \tau \times \pi \) is at least \( K_{\tau,\pi_\rho} \), the Frobenius reciprocity and the exactness of the Jacquet module functor imply that the multiplicity of \( \tau \otimes \pi \) in \( \mu^*(\tau \times \pi) \) is at least \( K_{\tau,\pi_\rho} \).

The definition of \( \pi_\rho \) implies that there exists an irreducible representation \( \varphi \) of a general linear group supported in \( X^c_\rho \) such that
\[ \pi \hookrightarrow \varphi \times \pi_\rho. \]

By the Frobenius reciprocity, \( \varphi \otimes \pi_\rho \) is a subrepresentation of the Jacquet module of \( \pi \). Denote by \( n \) the multiplicity of \( \varphi \otimes \pi_\rho \) in the Jacquet module of the representation \( \pi \).

Now we shall analyze the multiplicity of \( \varphi \otimes \tau \otimes \pi_\rho \) in the Jacquet module of \( \tau \times \pi \). Observe that \( \varphi \otimes \tau \otimes \pi_\rho \) must be a subquotient of a Jacquet module of the following part
\[ \mu^*_{X_\rho}(\tau \otimes \Psi_{X_\rho,X^c_\rho}(\pi_\rho, \pi_\rho^c)) = (1 \otimes \tau) \times \mu^*_{X^c_\rho}(\Psi_{X_\rho,X^c_\rho}(\pi_\rho, \pi_\rho^c)) \]
of \( \mu^*(\tau \times \Psi_{X_\rho,X^c_\rho}(\pi_\rho, \pi_\rho^c)) \). Recall that by (8.1) of [T8], \( \mu^*_{X_\rho}(\Psi_{X_\rho,X^c_\rho}(\pi_\rho, \pi_\rho^c)) \) is of the form
\[ * \otimes \pi_\rho. \]

If we want to get \( \varphi \otimes \tau \otimes \pi_\rho \) from a term from here, it must be from \( \varphi \otimes \pi_\rho \). Recall that we have this term with multiplicity \( n \) here. Because of this, we need to see the multiplicity of \( \varphi \otimes \tau \otimes \pi_\rho \) in the Jacquet module of \( n \cdot (1 \otimes \tau) \times (\varphi \otimes \pi_\rho) = n \cdot (\varphi \otimes \tau \times \pi_\rho) \).

Therefore the multiplicity of \( \varphi \otimes \tau \otimes \pi_\rho \) in the Jacquet module of \( \tau \times \pi \) is at most \( n \cdot k_{\tau,\pi_\rho} \).

The fact that the support of \( \tau \) is in \( X_\rho \) and the support of \( \varphi \) is in \( X^c_\rho \), and among \( X_\rho \) and \( X^c_\rho \) there is no reducibility, implies that if \( \Pi \) is an irreducible representation of a general linear group which has in its Jacquet module \( \tau \otimes \varphi \), then \( \Pi \cong \tau \times \varphi \). Further, \( \Pi \) contains \( \tau \otimes \varphi \) and \( \varphi \otimes \tau \) with multiplicity one.

Using the above observation and the transitivity of Jacquet modules, we get that both multiplicities of \( \varphi \otimes \tau \otimes \pi_\rho \) and of \( \tau \otimes \varphi \otimes \pi_\rho \) in the Jacquet module of \( \tau \times \pi \) are equal.

Therefore the multiplicity of \( \tau \otimes \varphi \otimes \pi_\rho \) in the Jacquet module of \( \tau \times \pi \) is at most \( n \cdot k_{\tau,\pi_\rho} \).

This implies that the multiplicity of \( \tau \otimes \pi \) in the Jacquet module of \( \tau \times \pi \) is at most \( k_{\tau,\pi_\rho} \), which further implies
\[ K_{\tau,\pi_\rho} \leq k_{\tau,\pi_\rho}. \]

Now if \( \pi_\rho \) is not a representation from (4), we got a contradiction, and the proof is complete in this case.

It remains to consider the case when \( \pi_\rho \) is from (4) (we know \( K_{\tau,\pi_\rho} = k_{\tau,\pi_\rho} = 6 \)). Further, such \( \pi \) is equivalent to one of the following two representations
\[ L(\frac{1}{2}, [\frac{3}{2}]^\rho; \frac{1}{2}; \sigma), \quad L([\frac{3}{2}]^\rho; \frac{1}{2}; \sigma). \]
If $\pi_\rho$ is the first representation above, then we take $\tau := \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)})$. In the case of the second representation above, we multiply with $\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)})^t$.

We shall follow the case of the first representation above (the other case goes analogously).

Since the assumption is that $\pi$ is unitarizable, we get that $\tau \ltimes \pi$ has $\tau \otimes \pi$ for a subrepresentation in its Jacquet module at least 6 times, as well as at least 6 times it has $\tau \ltimes \pi$ as a quotient.

For the completion of the proof, it is enough to show that $\tau \otimes \pi$ cannot be 6 times a subrepresentation and also 6 times a quotient of the Jacquet module of $\tau \ltimes \pi$. For this, it is enough to show that there is at least one subquotient of some filtration of Jacquet module where $\tau \otimes \pi$ is a subquotient, but it is not both a subrepresentation and a quotient. Now we shall show this.

First recall $\pi \hookrightarrow \varphi \times \pi_\rho$. This and the formula (7.55) imply that

$$\tau \otimes \xi \not\leq \mu^*(\pi)$$

for any irreducible $\xi$. Recall now that $M^*(\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)})) = 1 \otimes \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)}) + \left[\frac{1}{2}\right]^{(\rho)} \otimes \left[\frac{1}{2}\right]^{(\rho)} + \left[-\frac{1}{2}\right]^{(\rho)} \otimes \left[-\frac{1}{2}\right]^{(\rho)} + 2 \cdot \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)} \otimes 1 + \left[\frac{1}{2}\right]^{(\rho)} \times \left[\frac{1}{2}\right]^{(\rho)} \otimes 1.

Observe that we get precisely two times $\delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)} \otimes \pi$ using $2 \cdot \delta([-\frac{1}{2}, \frac{1}{2}]^{(\rho)} \otimes 1$ above. Therefore, any other $\tau \otimes \pi$ (and there are 4 of them) we must get from a term of the form $\left[\frac{1}{2}\right]^{(\rho)} \times \left[-\frac{1}{2}\right]^{(\rho)} \otimes$. Obviously, the first tensor factor implies that all $\tau \otimes \pi$ that show up here as subquotients, cannot be in the same time subrepresentations and quotients. This completes the proof.

**Corollary 12.6.** Let $\pi$ be a weakly real irreducible representation of a classical group. Suppose that some $X_{\rho_i}(\pi)$ is a non-unitarizable subquotient of $\theta_1 \times \ldots \times \theta_k \ltimes \sigma$, where $\theta_i \in C$ and

$$k \leq 3.$$

Then $\pi$ is not unitarizable.

**Proof.** For $k = 1$ and 2 we have seen that the claim of the corollary holds (Lemma 12.3). It remains to prove the case $k = 3$.

First observe that by methods of proof of Lemma 12.3 we get from Lemma 12.5 using Lemma 12.1 that if $X_{\rho_i}(\pi)$ belongs to a representation from the following list

(1) all the representations from (1) of Lemma 3.3 except $L([\alpha, \alpha + 1]^{(\rho)}; \delta([\alpha]^{(\rho)}; \sigma))$ (in Lemma 3.3 is proved that these representations are not unitarizable);

(2) representations in H.1;

(3) representations in Lemma 3.11;

(4) representations in Lemma 3.13.
(5) representations in (2) of Proposition 8.9, then \( \pi \) is not unitarizable.

Now using the above fact, Lemma 12.5 and the unitary parabolic reduction, we get the above corollary from the proof or non-unitarizability in section 11, applying Lemma 12.1 in the same way as we got Lemma 12.3.

Now we have the following obvious consequence of the above corollary.

**Corollary 12.7.** Let \( \pi \) be a weakly real irreducible unitarizable representation of a classical group with the Jantzen decomposition \( (X_{\rho_1}(\pi), \ldots, X_{\rho_\ell}(\pi)) \). Suppose that for each index \( i \) these exist \( \theta_i \in \mathbb{C} \) with \( k \leq 3 \) such that \( X_{\rho_i}(\pi) \) is a subquotient of \( \theta_1 \times \ldots \times \theta_k \rtimes \sigma \). Then all \( X_{\rho_i}(\pi) \) are unitarizable.

We end the paper with some come comments. There is another point of view to the classification of irreducible unitarizable subquotients that we have obtained. We shall only briefly indicate this point of view.

Fix a series of classical groups \( S_n, n \geq 0 \). Denote

\[
Irr_S = \bigcup_{n \geq 0} S_n, \quad Irr_u^S = \bigcup_{n \geq 0} \hat{S}_n.
\]

Fix \( m \geq 0 \) and denote by

\[
Irr^m_S
\]

the set of all classes \( \pi \in Irr_S \) for which there exist irreducible cuspidal representations \( \theta_1, \ldots, \theta_m \in \mathbb{C} \) and an irreducible cuspidal representation \( \sigma \) of classical group such that \( \pi \) is a subquotient of \( \theta_1 \times \ldots \times \theta_m \times \sigma \). Note that

\[
Irr^0_S
\]

is the set of all irreducible cuspidate representations of the groups that we consider.

We shall bellow consider only weakly real representations (even if this is not stressed).

**Definition 12.8.**

1. Denote by \( B_{\text{rigid}} \) the set of all weakly real classes \( \pi \in Irr_u^S \) for which we cannot find \( \beta > 0 \), a Speh representation \( \gamma \) and \( \pi' \in Irr_u^S \) such that \( \pi \cong \nu^\beta \gamma \rtimes \pi' \).

2. Denote by \( B_{\text{rigid}}^{\text{no-ind}} \) the set of all classes \( \pi \in B_{\text{rigid}} \) for which we cannot find a Speh representation \( \gamma \) and \( \pi' \in B_{\text{rigid}} \) such that \( \pi \cong \gamma \rtimes \pi' \).

3. Denote by \( B_{\text{rigid}}^{\text{st}} \) the set of all classes \( \pi \in B_{\text{rigid}}^{\text{no-ind}} \) for which we cannot find a Speh representation \( \gamma \) and \( \pi' \in B_{\text{rigid}}^{\text{no-ind}} \) such that \( \pi \hookrightarrow \gamma \rtimes \pi' \).
Importance of the above three classes is obvious for the classification of the unitarizable representations of classical groups. Namely, each $\pi \in \text{Irr}^u_S$ is equivalent to some $\nu^{\beta_1} \gamma_1 \times \ldots \times \nu^{\beta_k} \gamma_k \rtimes \tau$ where $\gamma_i$ are Speh representations, $\beta_i > 0$ and $\tau \in \mathcal{B}_{\text{rigid}}$ (we expect that this must be the complementary series starting from $\gamma_1 \times \ldots \times \gamma_k \rtimes \tau$). One gets representations $\tau \in \mathcal{B}_{\text{rigid}}$ as $\tau \cong \gamma_1 \times \ldots \times \gamma_k \times \tau$ where $\gamma_i$ are Speh representations and $\tau \in \mathcal{B}_{\text{no-ind}}$. Further, one gets representations $\tau \in \mathcal{B}_{\text{no-ind}}$ as irreducible subrepresentations of $\gamma_1 \times \ldots \times \gamma_k \times \tau$ where $\gamma_i$ are Speh representations and $\tau \in \mathcal{B}_{\text{str rigid}}$.

Definitely, the most important of these classes is $\mathcal{B}_{\text{str rigid}}$ (it contains all the isolated representations, but also may contain representations that are not isolated in the unitary dual, like non-isolated square integrable representations).

Now we have the following

**Question:** Are all the representations in $\mathcal{B}_{\text{str rigid}}$ automorphic? If this is the case, we have the following question: is the set $\mathcal{B}_{\text{str rigid}}$ equal to the union of all the isolated representations in automorphic duals of groups $S_n, n \geq 0$ (it is shown in [74] that this follows in the spherical case from a conjecture ”Arthur + $\epsilon$” of L. Clozel from [14]).

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30 This is the case in [33] if $\pi$ is generic or in [50] if $\pi$ is spherical.
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