SIMPLICITY AND SIMILARITY OF KIRILLOV-RESHETIKHIN CRYSTALS

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ABSTRACT. We show that the Kirillov-Reshetikhin crystal $B^{r,s}$ for nonexceptional affine types is simple and have the similarity property. As a corollary of the first fact we can derive that $B^{r_1,s_1} \otimes \cdots \otimes B^{r_l,s_l}$ is connected. Variations of the second property are also given.

1. INTRODUCTION

It is widely recognized that Kirillov-Reshetikhin modules [20, 2, 3] have distinguished properties among finite-dimensional modules of quantum affine algebras. One of such properties is the so called $T$-systems [21], certain algebraic relations among their $q$-characters [7, 24, 10]. Another property is the conjectural existence of crystal basis [9, 8]. We call it Kirillov-Reshetikhin (KR) crystal, and denote it by $B^{r,s}$ since it is parametrized by $(r, s) \in I \setminus \{0\} \times \mathbb{Z}_{>0}$, where $I$ is the index set of the Dynkin diagram of the affine algebra and 0 is the index as prescribed in [12]. In [14] many KR crystals were constructed through the study of perfect crystals [13]. The existence of $B^{r,1}$ for any affine type was shown in [18] and its combinatorial structure was clarified in [23, 11]. Recently, the existence of KR crystals was settled in [26] for all affine algebras of nonexceptional types. The crystal structure of these KR crystals was clarified in [4]. Using this result the conjecture on the perfectness [9, 8] was solved for nonexceptional types in [5].

In this paper, we show two more important and naturally expected properties of KR crystals. The first one is the simplicity (Definition 3.1, Theorem 3.2) introduced in [1]. As a consequence, one can show the tensor product of KR crystals $B^{r_1,s_1} \otimes \cdots \otimes B^{r_l,s_l}$ is connected. The second one is the similarity (Theorem 4.2), which was first observed in [17] for the crystal $B(\lambda)$ of the irreducible highest weight module of highest weight $\lambda$. We also consider variations of the similarity in §5. Combining (2-i) and (1-iii) (or (1-v)) one can settle Conjecture 6.6 in [27] on the alignedness of the virtual crystal.

Two remarks are in order. In September 2009 Sagaki informed the author that the simplicity can also be shown using the method in [22]. The connectedness of the two fold tensor product of perfect KR crystals was shown in [6] under certain assumptions, which were settled later in [29].

2. REVIEW ON KR CRYSTALS

2.1. Crystals. We do not review the notion of crystal bases or crystals but refer the reader to [15, 16]. In this paper crystal operators are denoted by $e_i, f_i$ instead
of \(\hat{e}_i, \hat{f}_i\) and we only consider seminormal crystals satisfying
\[
\varepsilon_i(b) = \max \{k \in \mathbb{Z}_{\geq 0} \mid e_i^k b \neq 0\}, \quad \varphi_i(b) = \max \{k \in \mathbb{Z}_{\geq 0} \mid f_i^k b \neq 0\}
\]
for an element \(b\) of a crystal \(B\). Throughout the paper \(g\) stands for an affine Kac-Moody Lie algebra \([12]\) whose weight lattice, simple roots, simple coroots, fundamental weights are denoted by \(P, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I}, \{\Lambda_i\}_{i \in I}\) with index set \(I\). \(g_0\) is the underlying finite-dimensional simple Lie algebra of \(g\) whose weight lattice is \(I_0 := I \setminus \{0\}\) following the convention of \([12]\). \(\mathfrak{T}, \{\alpha_i\}_{i \in I_0}, \{h_i\}_{i \in I_0}, \{\Lambda_i\}_{i \in I_0}\) are the weight lattice, simple roots, simple coroots, fundamental weights of \(g_0\). \(U_q(g), U_q(g_0)\) are the quantized enveloping algebras of \(g, g_0\) and \(U_q^0(g)\) is the subalgebra of \(U_q(g)\) without the degree operator \(q^d\).

A crystal \(B\) with index set \(I\) is said to be regular if, for any subset \(J \subset I\), \(B\) is isomorphic to the crystal bases associated to an integrable \(U_q(g_J)\)-module, where \(U_q(g_J)\) is the subalgebra of \(U_q(g)\) generated by the Chevalley generators with index set \(J\). It is known \([16]\) that the Weyl group \(W\) acts on any regular crystal. Let \(\{s_i\}_{i \in I}\) be the simple reflections of \(W\). The action \(S\) is given by
\[
S_{s_i} b = \begin{cases} 
\hat{f}_i^{(h_i, \wt(b))} b & \text{if } (h_i, \wt(b)) \geq 0, \\
\hat{e}_i^{-(h_i, \wt(b))} b & \text{if } (h_i, \wt(b)) \leq 0.
\end{cases}
\]

Let a crystal \(B\) with index set \(I\) be given and \(J \subset I\). Then we can regard \(B\) as a \(J\)-crystal by only considering \(e_i, f_i\) for \(i \in J\). We also say an element \(b\) is \(J\)-highest (resp. \(J\)-lowest) if \(e_i b = 0\) (resp. \(f_i b = 0\)) for all \(i \in J\).

2.2. \(X_n \to X_{n-1}\) branching and \(\pm\)-diagrams. Let \(X_n = B_n, C_n,\) or \(D_n\). In this subsection we review from \([4, \S 3.2]\) the branching rule for \(X_n \to X_{n-1}\) involving \(\pm\)-diagrams introduced in \([25]\). A \(\pm\)-diagram \(P\) of shape \(\Lambda/\lambda\) is a sequence of partitions \(\lambda \subset \mu \subset \Lambda\) such that \(\Lambda/\mu\) and \(\mu/\lambda\) are horizontal strips (i.e. every column contains at most one box). We depict this \(\pm\)-diagram by the skew tableau of shape \(\Lambda/\lambda\) in which the cells of \(\mu/\lambda\) are filled with the symbol \(+\) and those of \(\Lambda/\mu\) are filled with the symbol \(−\). Write \(\Lambda = \text{outer}(P)\) and \(\lambda = \text{inner}(P)\) for the outer and inner shapes of the \(\pm\)-diagram \(P\). When drawing partitions or tableaux, we use the French convention where the parts are drawn in increasing order from top to bottom.

There are a couple further type-specific requirements:

1. For type \(C_n\) the outer shape \(\Lambda\) contains columns of height at most \(n\), but the inner shape \(\lambda\) is not allowed to be of height \(n\) (hence there are no empty columns of height \(n\)).

2. For type \(B_n\) the outer shape \(\Lambda\) contains columns of height at most \(n\); for the columns of height \(n\), the \(\pm\)-diagram can contain at most one 0 between \(+\) and \(−\) at height \(n\) and no empty columns are allowed; furthermore there may be a spin column of height \(n\) and width 1/2 containing \(+\) or \(−\).

3. For type \(D_n\) suppose \(\Lambda = k_1 \Lambda_1 + \cdots + k_n \Lambda_n\). If \(k_n \geq k_{n-1}\) we depict this weight by \((k_n - k_{n-1})/2\) columns of height \(n\) colored 1 (where we interpret a 1/2 column as a \(\Lambda_n\) spin column if \(k_n - k_{n-1}\) is odd), \(k_{n-1}\) columns of height \(n-1\), and as usual \(k_i\) columns of height \(i\) for \(1 \leq i \leq n-2\). If \(k_n < k_{n-1}\) we depict this weight by \((k_{n-1} - k_n)/2\) columns of height \(n\) colored 2 (where we interpret a 1/2 column as a \(\Lambda_n\) spin column if \(k_{n-1} - k_n\) is odd), \(k_n\) columns of height \(n-1\), and as usual \(k_i\) columns of height \(i\) for \(1 \leq i \leq n-2\). We require that columns of height \(n\) are
colored, contain $+$, $-$, or $\mp$, but cannot simultaneously contain $+$ and $-$; spin columns can only contain $+$ or $-$.

Then, for an $X_n$-dominant weight $\Lambda$, there is an isomorphism of $X_{n-1}$-crystals

$$B_{X_n}(\Lambda) \cong \bigoplus_{\pm \text{-diagrams } P, \text{outer}(P)=\Lambda} B_{X_{n-1}}(\text{inner}(P)).$$

In fact, there is a bijection $\Phi : P \mapsto b$ from $\pm$-diagrams $P$ of shape $\Lambda/\lambda$ to the set of $X_{n-1}$-highest elements $b$ of $X_{n-1}$-weight $\lambda$ as given below. We use Kashiwara-Nakashima (KN) tableaux [19] in French convention to represent elements of $B_{X_n}(\Lambda)$. Suppose $\Lambda$ is a dominant weight; we require that $\Lambda$ does not contain any columns of height $n$ for type $D_n$. For any columns of height $n$ containing $+$, place a column $12\ldots n$ (this includes spin columns for type $B_n$). Otherwise, place $1$ in all positions in $P$ that contain a $-$, place a $0$ in the position containing $0$, and fill the remainder of all columns by strings of the form $23\ldots k$. We move through the columns of $b$ from top to bottom, left to right. Each $+$ in $P$ (starting with the leftmost moving to the right ignoring $+$ at height $n$) will alter $b$ as we move through the columns. Suppose the $+$ is at height $h$ in $P$. If one encounters a spin column of type $B_n$, replace it by a column $12\ldots h h + 2 \ldots n$ (read from bottom to top). Otherwise, if one encounters a $1$, replace $1$ by $h + 1$. If one encounters a $2$, replace the string $23\ldots k$ by $12\ldots h h + 2 \ldots k$.

2.3. KR crystals. KR crystals $B^{r,s}$ for nonexceptional types have the following general features: As an $I_0$-crystal $B^{r,s}$ decomposes into

$$B^{r,s} \cong \bigoplus_{\lambda} B(\lambda),$$

where $B(\lambda)$ stands for the crystal of the highest weight $U_q(\mathfrak{g}_0)$-module of highest weight $\lambda$ and the $\lambda$ runs over all partitions that can be obtained from the $r \times s$ (or $r \times (s/2)$ only when $\mathfrak{g} = B^{(1)}_n$ and $r = n$) rectangle by removing pieces of shape $\nu$ ($\nu, \mathfrak{g}_0$) are given in Table 1. However, there are some exceptions, where $B^{r,s}$ is connected and does not decomposes as in (2.1). We call these nodes exceptional and they are the filled nodes in Table 1.

In this subsection we review from [4] all KR crystals for nonexceptional affine types by dividing into cases according to the shape $\nu$ first and then treat the exceptional nodes.

2.3.1. Type $A^{(1)}_{n-1}$. Since

$$B^{r,s} \cong B(s\overline{A}_r)$$

as an $I_0$-crystal $B^{r,s}$ is identified with the set of all semistandard Young tableaux of rectangular shape $(s')$ over the alphabet $1 \prec 2 \prec \cdots \prec n$. The Dynkin diagram of $A^{(1)}_{n-1}$ has a cyclic automorphism $i \mapsto i + 1 \pmod n$. The action of $e_0$ and $f_0$ is given by

$$e_0 = pr^{-1} \circ e_1 \circ pr \quad \text{and} \quad f_0 = pr^{-1} \circ f_1 \circ pr,$$

where $pr$ is Schützenberger’s promotion operator [30], which is the cyclic Dynkin diagram automorphism on the level of crystals [31]. On a rectangular tableau $b \in B^{r,s}$, $pr$ is obtained from $b$ by removing all letters $n$, adding one to each letter in the remaining tableau, using jeu-de-taquin to slide all letters up, and finally filling the holes with 1s.
2.3.2. Type $B^{(1)}_n$, $D^{(1)}_n$, $A^{(2)}_{2n-1}$. The cases when $r = n-1$, $n$ for $g = D^{(1)}_n$ are excluded from here since they are exceptional nodes. As an $I_0$-crystal

$$B^{r,s} \simeq \bigoplus_{\lambda} B(\lambda),$$

where $\lambda$ runs over all partitions obtained from the $r \times s$ (or $r \times (s/2)$ only when $g = B^{(1)}_n$ and $r = n$) rectangle by removing $\boxtimes$.

The Dynkin diagrams in this case all have an automorphism interchanging nodes 0 and 1. The corresponding automorphism $\sigma$ of on the level of crystals exists. By construction $\sigma$ commutes with $e_i$ and $f_i$ for $i = 2, 3, \ldots, n$. Hence it suffices to define $\sigma$ on $\{2, 3, \ldots, n\}$-highest elements.

Because of the bijection $\Phi$ described in §2.2, it remains to define $\sigma$ on $\pm$-diagrams. For the following description of the map $\mathcal{G}$, we further assume $r \neq n$ for $B^{(1)}_n$. Let $P$ be a $\pm$-diagram of shape $\Lambda/\lambda$. Let $c_i = c_i(\lambda)$ be the number of columns of height $i$ in $\lambda$ for all $1 \leq i < r$ with $c_0 = s - \lambda_1$. If $i \equiv r - 1 \pmod{2}$, then in $P$, above each column of $\lambda$ of height $i$, there must be a + or a -. Interchange the number of such + and − symbols. If $i \equiv r \pmod{2}$, then in $P$, above each column of $\lambda$ of height $i$, either there are no signs or a $\mp$ pair. Suppose there are $p_i \mp$ pairs above the columns of height $i$. Change this to $(c_i - p_i) \mp$ pairs. The result is $\mathcal{G}(P)$, which has the same inner shape $\lambda$ as $P$ but a possibly different outer shape.

The affine crystal operators $e_0$ and $f_0$ are defined as

$$e_0 = \sigma \circ e_1 \circ \sigma \quad \text{and} \quad f_0 = \sigma \circ f_1 \circ \sigma.$$

The remaining KR crystal $B^{n,s}$ for $B^{(1)}_n$ is constructed as follows. Let $B^{n,s}$ be the $A^{(2)}_{2n-1}$-KR crystal. Then there exists a regular $B^{(1)}_n$-crystal $B^{n,s}$ and a unique

| $A^{(1)}_{n-1}$ | $B^{(1)}_n$ | $C^{(1)}_n$ | $D^{(1)}_n$ | $A^{(2)}_{2n}$ | $A^{(2)}_{2n-1}$ | $D^{(2)}_{n+1}$ |
|-----------------|-------------|------------|------------|-------------|-------------|-------------|
| (φ, $A_{n-1}$) | (φ $B_n$)   | (φ $C_n$)  | (φ $D_n$)  | (φ $C_n$)   | (φ $C_n$)   | (φ $B_n$)   |

Table 1. Dynkin diagrams

The affine crystal operators $e_0$ and $f_0$ are defined as

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The remaining KR crystal $B^{n,s}$ for $B^{(1)}_n$ is constructed as follows. Let $B^{n,s}$ be the $A^{(2)}_{2n-1}$-KR crystal. Then there exists a regular $B^{(1)}_n$-crystal $B^{n,s}$ and a unique
injective map \( S : B^{r,s} \to \hat{B}^{r,s} \) such that
\[
S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b), \\
\varepsilon_i(S(b)) = m_i \varepsilon_i(b), \quad \varphi_i(S(b)) = m_i \varphi_i(b)
\]
for \( i \in I \), where \((m_i)_{0 \leq i \leq n} = (2, 2, \ldots, 2, 1)\).

2.3.3. Type \( C_n^{(1)} \). The case when \( r = n \) is excluded from here since it is an exceptional node. We realize \( B^{r,s} \) inside the ambient crystal \( \hat{B}^{r,s} \) of type \( A_n^{(2)} \). Let \( I = \{0, 1, \ldots, n\} \) be the index set for type \( C_n^{(1)} \) and \( \hat{I} = \{0, 1, \ldots, n+1\} \) be that for \( A_{2n+1}^{(2)} \). Denote the crystal operators of \( B^{r,s} \) by \( \hat{e}_i \) and \( \hat{f}_i \). Then the crystal with crystal operators defined by
\[
e_i = \begin{cases} \hat{e}_0 \hat{e}_1 & \text{if } i = 0 \\ \hat{e}_{i+1} & \text{if } 1 \leq i \leq n \end{cases} \quad \text{and} \quad f_i = \begin{cases} \hat{f}_0 \hat{f}_1 & \text{if } i = 0 \\ \hat{f}_{i+1} & \text{if } 1 \leq i \leq n. \end{cases}
\]

turns out a regular \( C_n^{(1)} \)-crystal \( B^{r,s} \).

2.3.4. Type \( A_{2n}^{(2)}, D_{n+1}^{(2)} \). The case when \( r = n \) for is excluded from here since it is an exceptional node. Let \( \hat{B}^{r,s} \) stand for the \( C_n^{(1)} \)-KR crystal (The \( r = n \) case is treated in 2.4.1). According to types \( g = A_{2n}^{(2)}, D_{n+1}^{(2)} \) define positive integers \( m_i \) for \( i \in I \) as
\[
(m_0, m_1, \ldots, m_{n-1}, m_n) = \begin{cases} (1, 2, \ldots, 2, 2) & \text{for } A_{2n}^{(2)}, \\ (1, 2, \ldots, 2, 1) & \text{for } D_{n+1}^{(2)}. \end{cases}
\]

Then there exists a regular \( g \)-crystal \( B^{r,s} \) and a unique injective map \( S : B^{r,s} \to \hat{B}^{r,s} \) such that
\[
S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b), \\
\varepsilon_i(S(b)) = m_i \varepsilon_i(b), \quad \varphi_i(S(b)) = m_i \varphi_i(b)
\]
for \( i \in I \).

2.4. KR crystals for exceptional nodes. The KR crystal \( B^{r,s} \) corresponding to an exceptional node \( r \) is isomorphic to \( B(s\Lambda_r) \) as an \( I_0 \)-crystal.

2.4.1. \( B^{r,s} \) of type \( C_n^{(1)}, D_{n+1}^{(2)} \). First consider type \( C_n^{(1)} \). The elements in \( B(s\Lambda_n) \) are KN-tableaux of shape \( (s^n) \). Recall from 2.2.2 that the \( J \)-highest elements \((J = \{2, 3, \ldots, n\}) \) of shape \( (s^n) \) are in bijection with \( \pm \)-diagrams. Since all columns are of height \( n \), each column is either filled with \( + \), \(- \), or \( \mp \). Hence, if there are \( \ell_1 \) columns containing \(+ \), \( \ell_2 \) columns containing \(- \), and \( \ell_3 \) columns containing \( \mp \), we may identify \( \pm \)-diagrams \( P \) with triples \((\ell_1, \ell_2, \ell_3)\) such that \( \ell_1 + \ell_2 + \ell_3 = s \) and \( \ell_1, \ell_2, \ell_3 \geq 0 \).

In order to describe the affine structure, it suffices to define \( e_0, f_0 \) on such triples, since they commute with \( e_i, f_i \) for \( i = 2, \ldots, n \). The rule is given by
\[
e_0(\ell_1, \ell_2, \ell_3) = \begin{cases} (\ell_1 - 1, \ell_2 + 1, \ell_3) & \text{if } \ell_1 > 0, \\ 0 & \text{otherwise}, \end{cases} \\
f_0(\ell_1, \ell_2, \ell_3) = \begin{cases} (\ell_1 + 1, \ell_2 - 1, \ell_3) & \text{if } \ell_2 > 0, \\ 0 & \text{otherwise}. \end{cases}
\]
Next consider type $D_{n+1}^{(2)}$ whose classical subalgebra is of type $B_n$. The elements in $B(s\overline{X}_n)$ are KN-tableaux of shape $((s/2)^n)$ when $s$ is even and of shape $(((s−1)/2)^n)$ plus an extra spin column when $s$ is odd. By [22], the $J$-highest elements are in bijection with $±$-diagrams, where columns of height $n$ can contain $+$, $−$, $\mp$ and at most one $0$; the spin column of half width can contain $+$ or $−$.

We again encode a $±$-diagram $P$ as a triple $(\ell_1, \ell_2, \ell_3)$, where $\ell_1$ is twice the number of columns containing a single $+$ sign, $\ell_2$ is twice the number of columns containing a single $−$ sign (where spin column are counted as $1/2$ columns), and $\ell_3$ is twice the number of columns containing $\mp$. If $P$ contains a 0-column, then $\ell_1 + \ell_2 + \ell_3 = s − 2$, otherwise $\ell_1 + \ell_2 + \ell_3 = s$.

The action of $e_0, f_0$ on $J$-highest elements in this case is given by

\[
e_0(\ell_1, \ell_2, \ell_3) = \begin{cases} 
(\ell_1, \ell_2 + 2, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 < s, \\
(\ell_1 - 2, \ell_2, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_1 > 1, \\
(0, \ell_2 + 1, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_1 = 1, \\
0 & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_1 = 0,
\end{cases}
\]

\[
f_0(\ell_1, \ell_2, \ell_3) = \begin{cases} 
(\ell_1 + 2, \ell_2, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 < s, \\
(\ell_1 - 2, \ell_2, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_2 > 1, \\
(\ell_1, \ell_2 - 2, \ell_3) & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_2 = 1, \\
0 & \text{if } \ell_1 + \ell_2 + \ell_3 = s \text{ and } \ell_2 = 0.
\end{cases}
\]

2.4.2. $B^{n,s}$ and $B^{n−1,s}$ of type $D_n^{(1)}$. We first introduce an involution $\sigma : B^{n,s} \leftrightarrow B^{n−1,s}$ corresponding to the Dynkin diagram automorphism that interchanges the nodes $0$ and $1$. Under this involution, $J$-components ($J = \{2, 3, \ldots, n\}$) need to be mapped to $J$-components. Hence it suffices to define $\sigma$ on $J$-highest elements or equivalently $±$-diagrams. Recall from [22] that for weights $\Lambda = \overline{sX}_n$ or $\overline{sX}_{n−1}$, the $±$-diagram can contain columns with $+$ and $\mp$ or with $−$ and $\mp$ (but not a mix of $−$ and $+$ columns). The involution $\sigma : B^{n,s}$ maps a $±$-diagram $P$ to a $±$-diagram $P'$ of opposite color where columns containing $+$ are interchanged with columns containing $−$ and vice versa. Then the action of $e_0$ and $f_0$ is given by

\[
e_0 = \sigma \circ e_1 \circ \sigma \quad \text{and} \quad f_0 = \sigma \circ f_1 \circ \sigma.
\]

3. Simplicity

In this section we review the notion of simple crystal from [1] and show the KR crystal $B^{r,s}$ simple. Recall $W$ is the Weyl group. We say that an element $b$ of a regular crystal $B$ is extremal if it satisfies the following conditions: we can find elements $\{b_w\}_{w \in W}$ such that

- $b_w = b$ for $w = e$;
- if $\langle h_i, w\Lambda \rangle \geq 0$, then $e_i b_w = 0$ and $f_i^{(h_i, w\Lambda)} b_w = b_{s_i w}$;
- if $\langle h_i, w\Lambda \rangle \leq 0$, then $f_i b_w = 0$ and $e_i^{−(h_i, w\Lambda)} b_w = b_{s_i w}$.

**Definition 3.1.** We say a finite regular crystal $B$ is simple if $B$ satisfies the following:

1. there exists $\lambda \in P$ such that the weight of any extremal element of $B$ is contained in $W\lambda$;
2. $\sharp(B_\lambda) = 1$. 
Then it was shown [11] that simple crystals have the following properties.

**Proposition 3.1.**

1. A simple crystal is connected.
2. The tensor product of simple crystals is also simple.

We show the first main theorem of this paper.

**Theorem 3.2.** The KR crystal $B^{r,s}$ of nonexceptional affine types is simple.

*Proof.* We can assume $\lambda$ in Definition 3.1 is classically dominant, namely, $\lambda \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \Lambda_i$. Then an extremal element of weight $\lambda$ is necessarily $I_0$-highest. Hence, the cases when $g = A_{n-1}^{(1)}$ or $r$ is an exceptional node are done. We assume $r$ is a nonexceptional node and prove the theorem by showing any $I_0$-highest element $b$ of weight $\lambda \neq s\Lambda_r$ is not extremal.

Type $B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$: We assume $(g, r) \neq (B_n^{(1)}, n)$ first. Let $\lambda = \sum_{j \equiv r \mod 2} s_j \Lambda_j$. When $r$ is even, set $s_0 = s - \sum_{j \neq r} s_j$. Let $k = \min \{0 \leq j \leq r \mid j \equiv r \mod 2, s_j > 0\}$. Consider the element $b' = S_2 S_1 \cdots S_{k+1} b$. In the KN tableau representation $b'$ differs from $b$ in that the rightmost $s_k$ columns have entries $2, 3, \ldots, k + 1$. By the rule of $\sigma$ given in (3.1), $\sigma(b')$ is given as follows. The shape of $\sigma(b')$ differs from $b'$ in that the height of the rightmost $s_k$ columns is $k + 2$. From the left there are $[k/2]$ columns with entries $1, 3, \ldots, h, \overrightarrow{3}$, there is a column with entries $2, 3, \ldots, h, \overrightarrow{3}$ if $s_k$ is odd, and in the other columns, entries are $2, 3, \ldots, h, \overrightarrow{3}$, where $h$ is the height of the column. From this description of $\sigma(b')$ one finds that $\varepsilon_0(b') = \varepsilon_1(\sigma(b')) = 2s - s_k > 0$ and $\varepsilon_0(b') = \varepsilon_1(\sigma(b')) = s_k > 0$, thereby showing that $b$ is not extremal.

The remaining case when $g = B_n^{(1)}$ and $r = n$ is clear by construction.

Type $C_n^{(1)}$: Let $\lambda = \sum_{j} s_j \Lambda_j$ and set $s_0 = s - \sum_{j \neq r} s_j$. Let $k = \min \{0 \leq j \leq r \mid s_j > 0\}$. Consider the element $b' = S_1 S_2 \cdots S_{k+1} b$. In the KN tableau representation $b'$ differs from $b$ in that the rightmost $s_k$ columns have entries $2, 3, \ldots, k + 1$. One calculates

\[(\langle b_0, \text{wt}(b) \rangle) = s_k - s.\]

Now recall the inclusion $\iota: B^{r,s} \hookrightarrow \hat{B}^{r,s}$ in (2.3.3) where $\hat{B}^{r,s}$ is the ambient KR crystal of type $A_{2n+1}^{(2)}$. Since $\iota(b)$ is $\{2, \ldots, n + 1\}$-highest, it corresponds to a ±-diagram described as follows: By inner height we mean the height of the corresponding column of the inner shape. There are $s_k/2$ columns of inner height $h$ with $\mp$, each if $0 \leq h < r$ and $r - h$ is even, and with $+, -$ each if $r - h$ is odd. In the KN tableau representation, viewing from left, there are columns with entries $1, 2, \ldots, h, h + 2, h + 3, \ldots$ and possibly $h' + 1$ on top for some $h, h' > k$. After such columns we encounter a column with $\overrightarrow{k + 1}$ on top and/or with entries $1, 2, \ldots, k, k + 2, k + 3, \ldots$. Then the KN tableau representation of $\iota(b)$ differs from $\iota(b)$ by replacing the last description with $\overrightarrow{k}$ on top and/or with entries $2, 3, \ldots$. Hence we obtain $\varphi_0(b') = \varphi_1(\iota(b')) = s_k/2$, and from (3.1), $\varepsilon_0(b') = s - s_k/2$. Since $\varepsilon_0(b'), \varphi_0(b') > 0$, $b$ is not extremal.

Type $A_{2n}^{(2)}, D_n^{(2)}$: It is clear from the previous case by construction. \qed

4. **Similarity**

Let $B(\lambda)$ be the crystal basis for the irreducible highest weight module with highest weight $\lambda$ and let $m$ be a positive integer. In [17] Kashiwara showed the following.
Theorem 4.1. ([17]) There exists a unique injective map $S_m : B(\lambda) \rightarrow B(m\lambda)$ satisfying

\begin{align}
(4.1) \quad S_m(e_i b) = e_i^m S_m(b), \quad S_m(f_i b) = f_i^m S_m(b), \\
(4.2) \quad \varepsilon_i(S_m(b)) = m\varepsilon_i(b), \quad \varphi_i(S_m(b)) = m\varphi_i(b)
\end{align}

for $i \in I$ and $b \in B(\lambda)$. Here $S_m(0)$ is understood to be 0.

Note that (4.2) implies wt($S_m(b)$) = mwt(b). Our second main theorem states that similar properties hold also for KR crystals.

Theorem 4.2. There exists a unique injective map $S_m : B^{r,s} \rightarrow B^{r,ms}$ satisfying (4.1) and (4.2) for $i \in I$ and $b \in B^{r,s}$.

Proof. Thanks to Theorem 4.1 the map $S_m$ is uniquely determined by (4.1) and (4.2) for $i \in I_0$, since $B^{r,s}$ is multiplicity free as $I_0$-crystal and $I_0$-highest elements should be mapped to $I_0$-highest ones again. Hence it remains to show that the map $S_m$ so determined satisfies (4.1) and (4.2) for $i \in I$. We check it case by case, treating the exceptional node case separately.

Type $A^{(1)}_n$: In the semistandard tableau representation the map $S_m$ is given by replacing each node having entry $a$ with $m$ nodes having entry $a$ concatenated horizontally. By the explicit combinatorial procedure of jeu de taquin, one can show pr commutes with $S_m$. Then we have

\[ S_m(e_\sigma b) = S_m((\text{pr}^{-1} \circ e_1 \circ \text{pr})(b)) = (\text{pr}^{-1} \circ e_1^m \circ \text{pr})(S_m(b)) = e_\sigma^m S_m(b), \]
\[ \varepsilon_\sigma(S_m(b)) = \varepsilon_1(S_m(\text{pr}(b))) = \varepsilon_1(S_m(\text{pr}(b))) = m\varepsilon_1(b) = m\varepsilon_0(b). \]

The other relations are shown similarly.

Type $B^{(1)}_n, D^{(1)}_n, A^{(2)}_{2n-1}$: Similarly to $A^{(1)}_{n-1}$ case, it is enough to show that $S_m$ commutes with the involution $\sigma$. Since $\sigma$ commutes with $e_i$ and $f_i$ ($i = 2, \ldots, n$), it is reduced to showing that $S_m$ commutes with $\sigma$ for any $\{2, \ldots, n\}$-highest element $b$. Let $b$ correspond to a $\pm$-diagram $P$. Then, by Proposition 2.2 of [25], $S_m(b)$ corresponds to the $\pm$-diagram $P'$, where the number of $\mp$, $\pm$, $-$, $\cdot$ on the columns of the inner shape of the same height is multiplied by $m$. Hence $S_m$ commutes with $\sigma$. Note that it is valid also for $B^{\alpha,s}$ of $B^{(n)}_\alpha$.

Type $C^{(1)}_n$: We consider the inclusion $i : B^{r,s} \hookrightarrow B^{r,s}$ in (2.3.3) where $B^{r,s}$ is the ambient KR crystal of type $A^{(2)}_{2n+1}$. Then we have

\[ B^{r,s} \xrightarrow{i} B^{r,s} \xrightarrow{S_m} B^{r,ms}. \]

Since $\sigma$ commutes with $S_m$ on $B^{r,s}$, the image of the above composition is invariant under $\sigma$. Hence it belongs to $B^{r,ms}$, thereby defining $S_m$ for $B^{r,s}$. Then we have

\[ S_m(e_\sigma b) = S_m(e_\sigma e_1 b) = e_\sigma^m e_1^m S_m(b) = e_\sigma^m b, \]
\[ \varepsilon_\sigma(S_m(b)) = \varepsilon_\sigma S_m(b) = m\varepsilon_0(b). \]

Calculations for $f_0$ and $\varphi_0$ are similar.

Type $A^{(2)}_{2n}, D^{(2)}_{n+1}$: Since $e_0$ ($f_0$) commutes with $e_i$ ($f_i$) ($i = 2, \ldots, n$), and similar relations for $f_0, \varphi_0$, for any $\{2, \ldots, n\}$-highest element $b$. Recall the construction in
and consider the following diagram

\[
\begin{array}{ccc}
B^{r,s} & S & \hat{B}^{r,2s} \\
\downarrow & & \downarrow \\
B^{r,ms} & S' & \hat{B}^{r,2ms}
\end{array}
\]

where \(\hat{B}^{r,s}\) is the ambient \(C_n^{(1)}\)-KR crystal, \(S, S'\) are the injective maps in \([2,3,4]\) and \(\hat{S}_m\) is the map just constructed for type \(C_n^{(1)}\). For \(b \in B^{r,s}\) the \(\pm\)-diagrams corresponding to \(S(b)\) and \(S'(b)\) both have even number of \(\pm, +, -\) or \(\cdot\) on the columns of the inner shape of the same height. Hence it is clear that there exists a map \(S_m\) (broken line in the diagram) that makes the diagram commutative. Therefore the assertion follows from the properties of \(\hat{S}_m\).

Exceptional nodes: Similarly to the previous case, it is enough to show the desired properties for any \(\{2, \ldots, n\}\)-highest element. However, it is clear from the formulas given in \([2,3]\) \(\square\)

5. Variations

We give variations of Theorem 4.2. Since we treat KR crystals of different affine types, we signify the type \(g\) as \(B_n^{g,s}\).

**Theorem 5.1.** (1) For each case below there is a unique injective map \(S\) satisfying

\[
S(e_i b) = e_i^m S(b), \quad S(f_i b) = f_i^m S(b),
\]

\[
\varepsilon_i(S(b)) = m_i \varepsilon_i(b), \quad \varphi_i(S(b)) = m_i \varphi_i(b)
\]

for \(i \in I\). We set \(c_r = 1 (r \neq n), = 2 (r = n)\).

(i) \(S : B_{B_1^{(1)}}^{r,s} \rightarrow B_{A_2(2)_n-1}^{2r,2s} \) with \((m_i)_{i \in I} = (2, \ldots, 2, 1)\).

(ii) \(S : B_{C_n^{(1)}}^{r,s} \rightarrow B_{A_2}^{r,s} \) with \((m_i)_{i \in I} = (2, 1, \ldots, 1)\).

(iii) \(S : B_{C_n^{(1)}}^{r,s} \rightarrow B_{D_n(2)_{n+1}}^{r,s} \) with \((m_i)_{i \in I} = (2, 1, \ldots, 1, 2)\).

(iv) \(S : B_{C_n^{(2)}}^{r,s} \rightarrow B_{C_n^{(2)}}^{r,2s} \) (\(r \neq n\)) with \((m_i)_{i \in I} = (1, 2, \ldots, 2)\).

(v) \(S : B_{A_2^{(2)}}^{r,s} \rightarrow B_{A_2^{(2)}}^{r,s} \) (\(r \neq n\)) with \((m_i)_{i \in I} = (1, \ldots, 1, 2)\).

(vi) \(S : B_{A_2^{(2)}}^{r,s} \rightarrow B_{A_2^{(2)}}^{r,s} \) (\(r \neq n\)) with \((m_i)_{i \in I} = (1, \ldots, 1, 2)\).

(vii) \(S : B_{D_n(2)_{n+1}}^{r,s} \rightarrow B_{C_n^{(1)}}^{r,2s} \) with \((m_i)_{i \in I} = (1, 2, \ldots, 2, 1)\).

(viii) \(S : B_{D_n(2)_{n+1}}^{r,s} \rightarrow B_{A_2^{(2)}}^{r,2s} \) with \((m_i)_{i \in I} = (2, \ldots, 2, 1)\).

(2) Let \(I, \hat{I}\) be the index set of the Dynkin diagram of \(g, \hat{g}\). Let \(\xi\) be a map from \(\hat{I}\) to \(I\). For each case below then there is a unique injective map \(S\) satisfying

\[
S(e_i b) = (\prod_{j \in \xi^{-1}(i)} \hat{e}_j) S(b), \quad S(f_i b) = (\prod_{j \in \xi^{-1}(i)} \hat{f}_j) S(b),
\]

\[
\varepsilon_i(b) = \hat{\varepsilon}_j(S(b)), \quad \varphi_i(b) = \hat{\varphi}_j(S(b)) \quad \text{for any } j \in \xi^{-1}(i)
\]

for \(i \in I\).

(i) \(S : B_{B_1^{(1)}}^{r,s} \rightarrow B_{A_2^{(2)}_{2n+1}}^{r,2s} \) (\(r \neq n\), \(\xi(j) = \begin{cases} 0 & (j = 0, 1) \\ j - 1 & (2 \leq j \leq n + 1) \end{cases}\)).
For instance, (1-ii) can be shown by considering them as \( \{0,1,\ldots,n-1\} \)-crystals and applying \([17] \) Theorem 5.1. Proof of (2-iii) is given in \([27,22]\). Other cases are shown similarly and left to the reader. We can also consider an inclusion from \( A_{2n-1}^{(2)} \) or \( A_{2n}^{(2)} \) to \( A_{2n-1}^{(2)} \)-KR crystals, whose labeling of Dynkin nodes is opposite from \( A_{2n}^{(2)} \) or \( A_{2n-1}^{(2)} \), but we do not list them here.

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