THE BERNSTEIN PROJECTOR DETERMINED BY A WEAK ASSOCIATE CLASS OF GOOD COSETS

YEANSU KIM, LOREN SPICE, AND SANDEEP VARMA

Abstract. Let $G$ be a reductive group over a $p$-adic field $F$ of characteristic zero, with $p \gg 0$. In [Kim04], J.-L. Kim studied an equivalence relation called weak associativity on the set of unrefined minimal $K$-types for $G$ in the sense of A. Moy and G. Prasad. Following [Kim04], we attach to the set $\mathfrak{s}$ of good $K$-types in a weak associate class of positive-depth unrefined minimal $K$-types a $G$-invariant open and closed subset $\mathfrak{g}_s$ of the Lie algebra $\mathfrak{g}$ of $G(\mathbb{F})$, and a subset $\tilde{G}_s$ of the admissible dual $\tilde{G}$ of $G$ consisting of those representations containing an unrefined minimal $K$-type that belongs to $\mathfrak{s}$. Then $\tilde{G}_s$ is the union of finitely many Bernstein components for $G$, so that we can consider the Bernstein projector $E_s$ that it determines.

We show that $E_s$ vanishes outside the Moy–Prasad $G$-domain $\mathcal{G}_r \subset G$, and reformulate a result of Kim as saying that the restriction of $E_s$ to $\mathcal{G}_r$, pushed forward via the logarithm to the Moy–Prasad $G$-domain $\mathfrak{g}_r \subset \mathfrak{g}$, agrees on $\mathfrak{g}_r$ with the inverse Fourier transform of the characteristic function of $\mathfrak{g}_s$. This is a variant of one of the descriptions given by R. Bezrukavnikov, D. Kazhdan and Y. Varshavsky in [BKV16] for the depth-$r$ Bernstein projector.

1. Introduction

Let $G$ be a connected reductive group over a $p$-adic field $F$ of characteristic zero, and put $\mathfrak{g} = \text{Lie}(G)(\mathbb{F})$. We use similar notation for other groups, writing, for example, $H = H(\mathbb{F})$ (once we define $H$). Let $\mathcal{G}$ (respectively, $\tilde{G}$) denote the set of isomorphism classes of irreducible admissible (respectively, irreducible tempered) representations of $G$. Throughout, we will use Fraktur letters to denote the rational points of Lie algebras — $\mathfrak{g} = \text{Lie}(G)(\mathbb{F})$, etc. The terms undefined in the introduction are either standard and can be found in the references we cite, or are reviewed in Section 2.

Recall that the set of $G$-conjugacy classes of pairs $(M, \sigma)$ consisting of a Levi subgroup $M$ of $G$ and a (not necessarily unitary) irreducible supercuspidal representation $\sigma$ of $M$, is (the set of $\mathbb{C}$-points of) a complex variety $\Omega(G)$, called the Bernstein variety of $G$. The Zariski-connected components of $\Omega(G)$ are also its Hausdorff-connected components, and by a Bernstein component of $G$, we will refer to the preimage of such a component in $\mathcal{G}$ under the infinitesimal character map $\text{inf} : \mathcal{G} \to \Omega(G)$ that sends any $\pi \in \mathcal{G}$ to the well defined $G$-conjugacy class of its supercuspidal support. We refer to [BKV16, Section 1.3] for a review of general facts concerning the Bernstein center $Z(G)$ of $G$; in Remark 2.9 we will further briefly review a few that are particularly important for us.

Recall from [MP96] Section 3.4 the notion of an (positive-depth) unrefined minimal $K$-type for $G$. If $r > 0$ and $x \in \mathcal{B}(G)$, and if $\chi$ is a character of $G_{x,r}$ that is trivial on $G_{x,r+}$, then combining Pontrjagin duality with a Moy–Prasad isomorphism $G_{x,r}/G_{x,r+} \cong \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ considered in [MP94] Sections 3.7 and 3.8 lets us attach to $\chi$ a certain subset $Y = \mathfrak{g}_{x,(-r)+}^* \subset \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the $F$-vector space dual to $\mathfrak{g}$. Following [Kim04] Definition 2.1.2], we will call this subset $Y = \mathfrak{g}_{x,(-r)+}^*$ the dual blob of the pair $\mathfrak{s} = (G_{x,r}, \chi)$; and we call $\mathfrak{s}$ an unrefined minimal $K$-type if its dual blob contains no nilpotent elements. Further, an (positive-depth) unrefined minimal $K$-type of the form $(G_{y,s}, \xi)$ is called associate to $\mathfrak{s}$ if the dual blobs of $(G_{x,r}, \chi)$ and $(G_{y,s}, \xi)$ have $G$-conjugates that intersect non-trivially [MP96 Section 3.4], in which case automatically

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Thus, it is natural to ask if one could refine the above description of the Fourier transform of the characteristic function $1_{\mathcal{F}}(G)$ defined in the introduction. See Subsubsection 2.4.1. Accordingly, we write \( \mathcal{F} \) for the set of good unrefined minimal \( K \)-types that are weakly associate to \( \mathfrak{s} \), so that the body of the paper will call \( \mathfrak{s} \) a weak associate class.

Write \( \tilde{G}_s \) for the subset of \( \tilde{G} \) formed of (the isomorphism classes of) all the irreducible admissible representations of \( G \) containing a (not necessarily good) weak associate of \( \mathfrak{s} \), and \( \tilde{G}_\mathcal{F} \) for the analogous subset of representations containing an element of \( \mathcal{F} \). Remark 2.10 uses [Kim04, Theorem 4.5.1] to show that these sets are equal. Lemma 2.10 shows that \( \tilde{G}_s = \tilde{G}_\mathcal{F} \) is a union of Bernstein components of \( G \), so one can consider the element \( E_s = E_{\mathcal{F}} \in \mathcal{Z}(G) \) that acts as the identity on elements of \( \tilde{G}_\mathcal{F} \), and as zero on the elements of \( \tilde{G} \setminus \tilde{G}_\mathcal{F} \).

In [BKV16], for a nonnegative real number \( s \), R. Bezrukavnikov, D. Kazhdan and Y. Varshavsky study the depth-\( s \) Bernstein projector for \( G \). This is the element \( E_s \in \mathcal{Z}(G) \) that acts as the identity on every representation \( \pi \in \mathcal{G} \) of depth at most \( s \), and as zero on every representation \( \pi \in \mathcal{G} \) of depth greater than \( s \). They give two descriptions for \( E_s \). The first is an Euler–Poincaré formula for \( E_s \), [BKV16, Theorem 1.6]. The second involves the Lie algebra \( \mathfrak{g} \); they show [BKVI0, Corollary 1.22] that \( E_s \) is supported on the Moy–Prasad \( G \)-domain \( G_{s+} \), and that, if there is an ‘\( s \)-logarithm’ map \( \mathcal{L}: G_{s+} \to \mathfrak{g}_{s+} \), then the push-forward \( \mathcal{L}_!(E_{s+}) \) of \( E_{s+} \), which is a distribution on \( \mathfrak{g} \) supported in \( \mathfrak{g}_{s+} \), equals the (suitably normalized) inverse Fourier transform of the characteristic function \( 1_{\mathfrak{g}_{s+}} \) of the Moy–Prasad \( G \)-domain \( \mathfrak{g}^* \subset \mathfrak{g}^* \).

Thus, it is natural to ask if one could refine the above description of \( E_s \) into a description of \( E_{\mathcal{F}} \). We do not give an Euler–Poincaré description of \( E_{\mathcal{F}} \). However, under Hypotheses 2.1, 4.2, and 4.10 we do describe \( E_{\mathcal{F}} \) in terms of an inverse Fourier transform of a characteristic function \( 1_{\mathfrak{g}_{\mathcal{F}}} \), where \( \mathfrak{g}_{\mathcal{F}} \subset \mathfrak{g} \) is a subset defined by Ju-Lee Kim; see Subsubsection 2.4.2 for this notation. (There appears to be a difference here, since [BKV10] deals with a subset of \( \mathfrak{g}^* \) but we deal with a subset of \( \mathfrak{g} \); but our Hypothesis 2.1 affords an identification of \( \mathfrak{g} \) with \( \mathfrak{g}^* \), so really this is just a cosmetic difference.)

In Subsection 2.4 we impose Hypothesis 2.1(i) which, among other things, guarantees that, for every \( s > 0 \), the logarithm map is a \( G \)-equivariant, \( p \)-adic analytic isomorphism \( \log: G_{s+} \to \mathfrak{g}_{s+} \). Thus, we can speak of the push-forward \( \log_!(E_{\mathcal{F}}|_{\mathfrak{g}_{\mathcal{F}}}) \) of \( E_{\mathcal{F}} \) along log as an invariant distribution on the \( G \)-domain \( \mathfrak{g}_{\mathcal{F}} \subset \mathfrak{g} \).

Our main result is a description of the projectors \( \mathcal{E}_{\mathcal{F}} \).

**Theorem 1.1.** Let \( \mathcal{F} \) be the set of good unrefined minimal \( K \)-types in a weak associate class of unrefined minimal \( K \)-types of positive depth \( r > 0 \), and let \( \Gamma \in -\mathfrak{g}_{\mathcal{F}} \) be a good semisimple element. Assume Hypothesis 2.7 for the positive real number \( r \), as well as Hypotheses 4.2 and 4.10 for \( \mathcal{F} \) and \( \Gamma \in -\mathfrak{g}_{\mathcal{F}} \). Then:

(i) \( \mathcal{E}_{\mathcal{F}} \) is supported on \( G_r \).

(ii) The transfer \( \log_!(E_{\mathcal{F}}|_{\mathfrak{g}_{\mathcal{F}}}) \) of \( E_{\mathcal{F}}|_{\mathfrak{g}_{\mathcal{F}}} \) to \( \mathfrak{g}_{\mathcal{F}} \) coincides with the restriction to \( \mathfrak{g}_{\mathcal{F}} \) of the inverse Fourier transform of \( 1_{\mathfrak{g}_{\mathcal{F}}} \), where the Fourier transform is normalized as in Subsection 2.2.

It turns out that the strategy for our proof of Theorem 1.1(ii) is a modification of the strategy of the proof of [Kim04, Theorem 3.3.1]: while that proof studies certain invariant distributions using the character expansion of [KM03], which is valid near the identity element, our proof of Theorem 1.1(ii) uses [Spi18, Theorem 4.4.11], which gives a character expansion in the spirit of [KM03] about semisimple elements of \( G \) other than the identity.

As we were finishing up our work, we noticed the recent preprint [MS20] by A. Moy and G. Savin, which gives an Euler–Poincaré formula for a Bernstein projector that is very closely related to (possibly the same as or a bit finer than) the one that we are considering. Their work does not require the strong assumptions on \( p \) that we are imposing. However, apart from the fact that the description that we offer is different from
theirs, our work also introduces the idea of using character expansions to study Bernstein projectors, making our proof different from the one in [MS20], shedding light on these projectors from a different angle. Our work also offers a glimpse of a possible version of Theorem 1.1 where good unrefined minimal K-types are replaced by the ‘slightly refined’ minimal K-types \((K_+^T, \phi_\Sigma)\) from [KM06] Section 4.2.2, that were studied by by J.-L. Kim and F. Murnaghan in [KM06] and [Kim07]. In fact, using [KM06] in place of [KM03] and [Kim07] Equation (0.2) in place of [Kim04] Theorem 3.3.1 should allow one to generalize Theorem 1.1(ii) to the setting of the \((K_+^T, \phi_\Sigma)\), provided one also proves the analogue of Lemma 2.10 in this ‘slightly refined’ situation. Generalizing Theorem 1.1(ii) to the \((K_+^T, \phi_\Sigma)\) would take more work, since the results from [Spi18] that we use haven’t yet been adapted to deal with the \((K_+^T, \phi_\Sigma)\). Such a generalization would be a nice way to think of J.-L. Kim’s approach in [Kim07] to the exhaustiveness of J.-K. Yu’s construction of supercuspidal representations. We hope to return to this question in future work.

2. Some notation and preliminaries

We fix an algebraic closure \(\bar{F}\) of \(F\), and let \(\text{val} : \bar{F} \to \mathbb{Q} \cup \{\infty\}\) be the usual extension to \(\bar{F}\) of the normalized discrete valuation on \(F\). Throughout the rest of this paper, Hypothesis 2.1, which includes [Kim04] Hypothesis 1.3.6], will be in force.

2.1. Bruhat–Tits buildings and Moy–Prasad filtrations. Let \(B(G)\) denote the enlarged Bruhat–Tits building of \(G\). For \(x \in B(G)\) and \(r \geq 0\) (respectively, \(r \in \mathbb{R}\)), we have the Moy–Prasad subgroups \(G_{x,r}, G_{x,r^+} \subset G\) (respectively, the Moy–Prasad lattices \(\mathfrak{g}_{x,r}, \mathfrak{g}_{x,r^+} \subset \mathfrak{g}\)). For \(r \geq 0\) (respectively, \(r \in \mathbb{R}\)), we have Moy–Prasad \(G\)-domains \(G_r, G_{r^+} \subset G\) (respectively, \(\mathfrak{g}_r, \mathfrak{g}_{r^+} \subset \mathfrak{g}\)) defined by:

\[
G_r := \bigcup_{x \in B(G)} G_{x,r}, \quad G_{r^+} := \bigcup_{x \in B(G)} G_{x,r^+}, \quad \mathfrak{g}_r := \bigcup_{x \in B(G)} \mathfrak{g}_{x,r}, \quad \mathfrak{g}_{r^+} := \bigcup_{x \in B(G)} \mathfrak{g}_{x,r^+}.
\]

Obvious modifications of this notation will apply to other groups: thus, if \(H\) is a reductive group, we will talk of \(H_r, h_{r^+}\), etc. Let \(G'\) be a reductive subgroup of \(G\) containing a maximal torus \(T\) of \(G\) that splits over a tame extension \(L\) of \(F\) (which every torus in \(G\) will, once we impose Hypothesis 2.1.1). Then [AS08 Proposition 4.6] constructs a family of embeddings \(B(G') \to B(G)\) of the Bruhat–Tits building of \(G'\) into that of \(G\), with the property that for every such embedding \(\iota\), every \(x \in B(G')\) and every \(r > 0\), we have \(G_{x,r} = G' \cap G(\iota(x), r)\).

For this subsection alone, we informally refer to the elements of such a family of embeddings \(B(G') \to B(G)\) as ‘canonical’. In fact, the canonical embedding constructed by [AS08 Proposition 4.6] is part of a larger collection of analogous embeddings over discretely valued, separable extensions of \(F\), but these extra data will not concern us, except that we use below the embedding \(B(G', L) \to B(G, L)\) corresponding to the splitting field \(L\) of \(T\) to show that our notion of canonical embedding agrees with that of [Kim04 Section 1.3].

These embeddings are compatible in the sense that the precomposition of a canonical embedding \(B(G') \to B(G)\) with conjugation by an element \(g \in G\) is a canonical embedding \(B(\iota^{-1}g^{-1}G') \to B(G)\); and, if \(H'\) is a reductive subgroup of \(G'\) containing a maximal torus of \(G'\), hence also a reductive subgroup of \(G\) containing a maximal torus of \(G\), then the composition of any of the canonical embeddings \(B(H') \to B(G')\) and any of the canonical embeddings \(B(G') \to B(G)\) is one of the canonical embeddings \(B(H') \to B(G)\).

Any two canonical embeddings differ in a well understood way [AS08 Remark 4.7]. In particular, they all have the same image, so we may, and do, consider \(B(G')\) as a subset of \(B(G)\).

When \(G'\) is a twisted Levi subgroup of \(G\), these embeddings are also discussed by [Kim04 Section 1.3], which references [Yu01 Remark 2.11]. Since the explicit description of the embeddings in [Yu01 Remark 2.11] and [AS08 Proposition 4.6] work by tame descent from the splitting field of \(T\), it suffices to work with \(B(G', L) \to B(G, L)\), and so assume that \(T\) is actually a split maximal torus. Then the explicit apartment-by-apartment descriptions in the two references show that the notions of canonical embedding agree.
2.2. **An additive character and the Fourier transform.** For the remainder of the paper, we fix a continuous character \( \Lambda : F \to \mathbb{C}^\times \) that is nontrivial on the ring \( \mathcal{O} \) of integers of \( F \), but trivial on the maximal ideal of \( \mathcal{O} \). Let \( V \) be any \( F \)-vector space. Then our choice of \( \Lambda \) gives an isomorphism between the Pontryagin dual \( \text{Hom}_{\mathcal{O}}(V, \mathbb{C}^\times) \) and the \( \mathbb{C}^\times \) valued dual \( V^\times = \text{Hom}_F(V, F) \), so further choosing a Haar measure \( d\mu \) on \( V \) defines a Fourier transform \( \hat{C}_{c}^\infty(V) \to \hat{C}_{c}^\infty(V^\times) \) by \( f \mapsto \hat{f} \), where

\[
\hat{f}(Y) = \int_{V} f(X) \Lambda(\langle Y, X \rangle) \, d\mu(X).
\]

If \( T \) is a distribution on \( V^\times \) (i.e., a linear map \( C_{c}^\infty(V^\times) \to \mathbb{C} \)), then we define its Fourier transform \( \hat{T} : C_{c}^\infty(V) \to \mathbb{C} \) to be the distribution given by \( \hat{T}(f) = \hat{f} \).

When we apply this definition, \( V = \mathfrak{h} \) will be the space of \( F \)-rational points of the Lie algebra of an algebraic \( F \)-group \( H \) for which we can use Hypothesis \[2.1(1)\] to identify \( \mathfrak{h} = \mathfrak{h}^\ast \), and so to regard the Fourier transform \( \hat{f} \) of a function \( f \) on \( \mathfrak{h} \) as again a function on \( \mathfrak{h} \), and hence to define the Fourier transform of a distribution on \( \mathfrak{h} \) as again a distribution on \( \mathfrak{h} \).

2.3. **Good cosets, weak associativity and a partition of \( \mathfrak{g} \).**

2.3.1. **Good elements.** Recall from \[Kim04\] Definition 1.2.1 that for \( s \in \mathbb{R} \), a good element of depth \( s \) is a tame element \( \Gamma \in \mathfrak{g} \) (i.e., an element that belongs to the Lie algebra of some tamely ramified maximal torus — which, once we impose Hypothesis \[2.1(1)\] will be any semisimple element) such that, for some (equivalently, every) maximal torus \( T \) with \( \Gamma \in \mathfrak{t} \), we have that \( \Gamma \in t_s \setminus t_{s+} \), and such that for every absolute root \( \alpha \) of \( T \) in \( G \), \( da(\Gamma) \) is either zero or has valuation \( s \). (The latter condition *almost* implies the former, in the sense that, if it is satisfied, then \( \Gamma \) does not belong to \( t_{s+} \) unless \( \Gamma \) is centralized by \( G \), and \( \Gamma \) belongs to \( t_s \) if \( G \) is adjoint.)

2.3.2. **Good cosets and their weak associativity.** Let \( r > 0 \). By a good coset of depth \(-r\), we mean a set of the form \( \Gamma + \mathfrak{g}_{x,(-r)+} \), where \( \Gamma \in \mathfrak{g} \) is a good element of depth \(-r\), and \( x \in B(G') \subset B(G) \), where \( G' \) is the centralizer of \( \Gamma \) in \( G \). By \[KM03\] Theorem 2.3.1], this agrees with the marginally different formulation of the same notion in \[Kim04\] Definition 1.2.2].

Two good cosets are said to be associate to each other if they have \( G \)-conjugates that intersect non-trivially, in which case their depths are the same \[KM03\] Remark 2.4.2(1)]. The transitive closure of the relation of associativity is an equivalence relation, called weak associativity. (The definition in \[Kim04\] Definition 1.4.1] is priori coarser, in the sense that it might call more good cosets weakly associate, because the chain of associate non-degenerate cosets linking them is not required to consist only of good cosets. Actually, this does not happen, thanks to \[Kim04\] Lemma 1.4.2], which implies that two good cosets that are weakly associate in the more general sense of \[Kim04\] Definition 1.4.1] are associate to a common good coset, hence weakly associate in our sense.)

The notions of good coset, and of weak associativity for good cosets, are also defined in \[Kim04\] for \( r = 0 \), but we will avoid referring to these notions.

2.4. **Unrefined minimal cosets and dual blobs.** We now need to impose Hypothesis \[2.1\] in order to speak of dual blobs. This hypothesis depends on a positive real number \( r \), which we will take to be the depth of a weak associate class \( \mathfrak{f} \) of good unrefined minimal \( K \)-types in Subsection 2.6. We will later add Hypotheses \[4.2\] and \[1.10\].

**Hypothesis 2.1.**

(i) The group \( G \) satisfies \[Kim04\] Hypothesis 1.3.6] (consisting of hypotheses labeled (HB), (HGT) and (Hk) there), where the reference to the adjoint representation in (Hk) is replaced by a faithful representation of \( G \), as in \[Kim99\] Section 3.1.0 and \[DR09\] Appendix B]. This (together with the fact that there are finitely many rational conjugacy classes of maximal tori in \( G \)) guarantees the
existence of a finite, tame, Galois extension $L_G$ of $F$ that splits every maximal torus in $G$, and we require also that the analogue of \cite{Kim04} (Hk) is satisfied for the base change of $G$ to $L_G$, with respect to the same faithful representation defined over $F$.

(ii) For all $x$ belonging to the Bruhat–Tits building $\mathcal{B}(G)$ of $G$, the logarithm map with respect to our chosen faithful representation of $G$ (which, from \cite{Yu01}, restricts to an analytic (but not group-theoretic) isomorphism $G(L_G)_{x,r} \to \text{Lie}(G)(L_G)_{x,r}$) carries cosets of $G(L_G)_{x,r+}$ to cosets of $\text{Lie}(G)(L_G)_{x,r+}$, and the induced map

$$\log: G(L_G)_{x,r}/G(L_G)_{x,r+} \to \text{Lie}(G)(L_G)_{x,r}/\text{Lie}(G)(L_G)_{x,r+}$$

is an isomorphism of abelian groups.

(iii) $p > e(L_G/\mathbb{Q}_p)/(p-1)$, where $L_G$ is as in \cite{Yu15} and $e(L_G/\mathbb{Q}_p)$ denotes its ramification degree over $\mathbb{Q}_p$.

**Remark 2.2.**

- Hypothesis 2.1 is satisfied whenever the residue characteristic $p$ is greater than a constant that depends only on the absolute root datum of $G$ and the degree of ramification of $F$ over $\mathbb{Q}_p$. To avoid a lengthy digression here, we do not go into details; but, for some conditions that imply parts of Hypothesis 2.1, see \cite[Proposition 4.1]{AR00}, \cite[Corollary 2.6 and Theorem 3.3]{Kim99}, \cite[Proposition 3.1.1]{Yu01}, and \cite[Lemma B.5.4 and Lemma B.6.12]{DR09} (noting that the proof of \cite[Lemma B.7.2]{DR09} also shows that, in the notation of that result, $Z_r \times G'_r \to G_r$ is a bijection for all $r > 0$).

- One way that we use Hypothesis 2.1 is to ensure the existence of Moy–Prasad isomorphisms, which do not exist in general. See, for example, \cite[remark following Corollary 5.6]{Yu15}.

- Hypothesis 2.4[1] says that there exists a symmetric nondegenerate $\text{Ad}G$-invariant bilinear form $B$ on $\mathfrak{g}$ such that the resulting isomorphism from $\mathfrak{g}$ to its dual vector space $\mathfrak{g}^*$ identifies, for each $x \in \mathcal{B}(G)$ and each $r \in \mathbb{R}$, the Moy–Prasad lattice $\mathfrak{g}_{x,r}$ with

$$\mathfrak{g}_{x,r}^* := \{ Y \in \mathfrak{g}^* \mid \langle Y, \mathfrak{g}_{x,(-r)+} \rangle \text{ is contained in the maximal ideal of } \mathcal{D} \}.$$  

If $H$ is a reductive subgroup of $G$ containing a maximal torus in $G$ (for example, $H = G$), then the restriction of $B$ to $\mathfrak{h}$ is non-degenerate, hence also furnishes an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$. This allows us to view the Fourier transform as a map $C^\infty_c(\mathfrak{h}) \to C^\infty_c(\mathfrak{h}^*)$, and hence, in particular, to iterate it. We will always equip $\mathfrak{h}$ with the unique Haar measure, called self dual, for which $\hat{f}(Y) = f(-Y)$ for all $f \in C^\infty_c(\mathfrak{h})$ and all $Y \in \mathfrak{h}$.

- By \cite[Corollary 2.3]{Yu01} and the fact that log is $\text{Gal}(L_G/F)$-equivariant, Hypothesis 2.4[1] which is a statement about $L_G$-rational points, implies the analogous statement about $F$-rational points.

Compare Lemma 2.3 and Corollary 2.4 to \cite[Hypothesis 8.3]{AD02}. Corollary 2.4 is a strictly stronger result, but, since its proof, which uses the exponential map, is different in spirit from that of Lemma 2.3, we have separated them.

**Lemma 2.3.** If $H$ is a connected, reductive subgroup of $G$, then $\mathfrak{h} \cap \mathfrak{g}_s$ equals $\mathfrak{h}_s$ for all real numbers $s$, and $H \cap G_s$ equals $H_s$ for all $s > 0$.

**Proof.** By \cite[Lemma 3.3.8 and Lemma 3.7.18]{AD02}, it suffices to show that the sets of semisimple elements in $H \cap G_s$ and $H_s$, and in $\mathfrak{h} \cap \mathfrak{g}_s$ and $\mathfrak{h}_s$, are the same. Let us first prove the ‘group case’, i.e., fixing a semisimple element $h \in H$, let us see that $h \in G_s$ if and only if $h \in H_s$. Since $H$ is connected, we can choose a maximal torus $T_H$ in $H$ containing $h$. Let $T$ be a maximal torus in $G$ containing $T_H$. Note that $T \cap H$ is contained in, hence equals, $C_H(T_H) = T_H$. By Hypothesis 2.4[1] we have that $T$ splits over a tame extension of $F$. By \cite[Lemma 2.2.3]{AD04}, we may, and do, replace $F$ by $L_G$, so that $T$, hence also $T_H$, is split. Now \cite[Lemma 2.2.9]{AD04} gives that $T_H \cap H_s$ equals $(T_H)_s$, and $T \cap G_s$ equals $T_s$; and, since $T_H$ and $T$ are split
and every (algebraic) character of $T_H$ extends to one of $T$, we see that

$$(T_H)_s = \{ t \in T_H \mid \chi(t) - 1 \text{ has valuation at least } s \text{ for all characters } \chi \text{ of } T_H \}$$

is the intersection with $H$ of

$$T_s = \{ t \in T \mid \chi(t) - 1 \text{ has valuation at least } s \text{ for all characters } \chi \text{ of } T \}.$$ 

The result for the group follows. The proof in the Lie-algebra case is similar, replacing [AD04, Lemma 2.2.3 and Lemma 2.2.9] by [AD04, Lemma 2.2.5 and Corollary 2.2.7].

**Corollary 2.4.** If $H$ is a possibly disconnected reductive subgroup of $G$, then we have for all real numbers $s$ that $\mathfrak{h} \cap \mathfrak{g}_s = \mathfrak{h}_s$; and for all $s > 0$ that $H \cap G_s = H_s$ and $\log : G_s \to \mathfrak{g}_s$ and $\exp : \mathfrak{g}_s \to G_s$ restrict to mutually inverse $p$-adic analytic isomorphisms $H_s \leftrightarrow \mathfrak{h}_s$.

**Proof.** By Lemma 2.3 applied with the identity component $H^0$ of $H$ in place of $H$, we have that $\mathfrak{h} \cap \mathfrak{g}_s = \text{Lie}(H)(F) \cap \mathfrak{g}_s = \text{Lie}(H^0)(F) \cap \mathfrak{g}_s = \mathfrak{h}_s$ for all $s \in \mathbb{R}$ and $H^0(F) \cap G_s = (H^0)(F)_s = H_s$ for all $s > 0$. Thus, it remains to show that the inclusion $H^0(F) \cap G_s \subset H \cap G_s$ is an equality for all $s > 0$. We slightly abuse notation by writing $H^0$ for $H^0(F)$.

We recall that Hypothesis 2.1(iii) does not depend on $r$, and gives us mutually inverse, $G$-equivariant, $p$-adic analytic maps $\exp : \mathfrak{g}_s \to G_s$ and $\log : G_s \to \mathfrak{g}_s$. By [Bor91, Corollary 3.8], the subset $\log(H \cap G_s) \subset \mathfrak{g}_s$ lies in $\mathfrak{h}$, hence in $\mathfrak{h} \cap \mathfrak{g}_s = \mathfrak{h}_s$. Thus, $\log^{-1}(\mathfrak{h}_s)$ contains $H \cap G_s$. It now suffices to show that $\exp(\mathfrak{h}_s)$ is contained in $H^0$, so that $\log^{-1}(\mathfrak{h}_s)$ is contained in $H^0$, and hence $H \cap G_s \subset \log^{-1}(\mathfrak{h}_s)$ is contained in $H^0 \cap G_s = H_s$. Fix an element $X$ of $\mathfrak{h}_s$. By [AD02, Theorem 3.1.2 and Lemma 3.3.8], the semisimple parts $X_{ss}$ and $X_{sl}$ of $X$ also belong to $\mathfrak{g}_s$. Then $\exp(X) = \exp(X_{ss})\exp(X_{sl})$. Because $\exp(X_{sl}) \in H$ is unipotent, it is contained in $H^0$ (because $F$ has characteristic 0). We thus may, and do, replace $X$ by its semisimple part.

Then there is some maximal torus $T_H$ in $H^0$ such that $X$ lies in $t_{H}$. By [Bor91, Corollary 3.8] again, we have that $\exp(X)$ lies in $T_H$, which is contained in $H^0$.

**Remark 2.5.** Let $T$ be a maximal torus in $G$ such that $x \in \mathcal{B}(T)$. We claim that the map $\exp = \log^{-1}$ agrees with the map $\varphi_{T,x,r,r+}$ of [Adl98, Section 1.5, pages 12–13]. (Strictly speaking, for consistency with [Adl98, we should instead write $\varphi_{T,(\varepsilon)x,r,r+}$ for $\varepsilon > 0$ sufficiently small, but we will not do so.) In fact, this property is part of why those maps are called mock exponential.

Since $\varphi$ is defined by (tame) descent, it suffices by [Yu01, Corollary 2.3] to replace $F$ by $L_G$ and $G$ by its base change, and so assume that $T$ is split. For each absolute root $b$ of $T$ in $\text{Lie}(G)$, we have that $\exp$ is actually a homomorphism of algebraic groups $\text{Lie}(G)_{b} \to G$. It is $T$-equivariant, and hence has image in the corresponding root subgroup. Since its derivative is the identity, it agrees with the maps $\exp_b$ of [Adl98, Section 1.5, page 11]. Further, Hypothesis 2.1(iii) implies that, for all $t$ of positive valuation in $F$, the series $\tau = \sum \frac{t^i}{i!}$ converges, and the valuation of $\tau - 1 - t$ is greater than that of $t$. Thus, the restriction of $\exp$ to $t_r$ agrees with the map $\varphi_{T,r,r+} : t_r \to G/T_{r}$ of [Adl98, Section 1.5, page 11]. Since $\exp$ and $\varphi_{T,x,r,r+}$ are both homomorphisms, and since they agree on a set of generators for their domain, it follows that they agree everywhere.

**Notation 2.6.** Fix $X \in \mathfrak{g}$.

(i) Let $\Lambda_X$ be the character $g \to \mathbb{C}^\times$ given by $z \mapsto \Lambda(B(X,z))$.

(ii) Given $x \in \mathcal{B}(G)$ and $r > 0$, let $\Lambda_{X,x,r}$ be the function $G_{x,r} \to \mathbb{C}^\times$ given by $g \mapsto \Lambda(B(X, \log g))$. If $X \in \mathfrak{g}_{x,-r}$, then Hypothesis 2.1(iii) shows that $\Lambda_{X,x,r}$ is a character, and allows us to think of $\Lambda_{X,x,r}$ as the map $g \mapsto \Lambda(B(X, \log |g|))$ on $G_{x,r}/G_{x,r+}$.

**Notation 2.7.** Suppose $K \subset G$ is a compact open subgroup of the form $\exp(L)$, for a lattice $L \subset \mathfrak{g}$. Let $\chi : K \to \mathbb{C}^\times$ be a continuous character. Then the dual blob of $\chi$ or of $(K, \chi)$ is defined to be the set of all $Y \in \mathfrak{g}$ such that $\chi$ equals $\Lambda_Y \circ \log |K|$, provided this set is nonempty.
Note that if the dual blob of \( \chi \) exists, then it is a single coset in \( g \) for the kernel \( L^\perp \) of the homomorphism \( Y \mapsto \Lambda_{Y,|L} \in \text{Hom}(L, \mathbb{C}^\times) \), so this notion is in agreement with [Kim07, Definition 3.10].

Recall that, if \( r > 0 \), then an unrefined minimal \( K \)-type of depth \( r \) for \( G \) in the sense of [MP96, Section 3.4] is a pair \((G_{x,r}, \chi)\), where \( \chi \) is a character of \( G_{x,r}/G_{x,r+} \) (or, by inflation, of \( G_{x,r} \)) of the form \( \Lambda_{Y,x,r} \) as in Notation 2.4(iii) for some \( x \in \mathcal{B}(G) \) and \( Y \in g_{x,-r} \), with the property that the coset \( Y + g_{x,(-r)+} \) contains no nilpotent element (i.e., is non-degenerate in the sense of [MP96, Section 3.4]). In this case, the coset \( Y + g_{x,(-r)+} \) is equal to the dual blob of \((G_{x,r}, \chi)\) (and in particular depends on the triple \((G, x, r)\) only through the subgroup \( G_{x,r} \)).

2.4.1. \textit{Good unrefined minimal \( K \)-types and their weak associativity.} We now recall notions for unrefined minimal \( K \)-types parallel to the notions for cosets introduced in Subsubsection 2.3.2. Recall that there is a depth-reversing bijection between positive-depth unrefined minimal \( K \)-types and negative-depth, non-degenerate Moy–Prasad cosets that sends each type to its dual blob, i.e., \((G_{x,r}, \Lambda_{x,x,r}) \mapsto X + g_{x,(-r)+} \). We say (following [Kim04, Definition 3.2.1]) that an unrefined minimal \( K \)-type of positive depth is good if its dual blob is a good coset, so that good (positive-depth) unrefined minimal \( K \)-types are in bijection with good (negative-depth) cosets, and then use the bijection to transport the notion of weak associativity from the latter to the former. (In particular, since weakly associate good cosets have the same (negative) depth, weakly associate good \( K \)-types have the same (positive) depth.) As in Subsubsection 2.3.2, although our definition might a priori distinguish good \( K \)-types that are equivalent according to [Kim04, Definition 2.2.1] (which allows a chain of associated \( K \)-types that includes non-good \( K \)-types), actually this does not happen. The reason we find it convenient to restrict to good unrefined minimal cosets, unlike [Kim04], is that we are able to appeal to one of the culminating results of that paper, [Kim04, Theorem 4.5.1] (or [KM03, Theorem 2.4.10]), which guarantees the existence of a good unrefined minimal \( K \)-type in every element of \( \hat{G} \).

2.4.2. \textit{Remarks on the sets of weak associate classes of good unrefined minimal \( K \)-types and good cosets.} Let \( \mathcal{S}_K \) be the disjoint union of a singleton set \( \{*\} \) and the set of weak associate classes of good positive-depth unrefined minimal \( K \)-types. Let \( \mathcal{S} \) be the disjoint union of a singleton set \( \{s'\} \) and the set of all weak associate classes of good positive-depth \( G \)-cosets. One thinks of \( * \) as standing for the collection of all depth-0 \( K \)-types, in the sense of [MP96, Section 3.4], and of \( s' \) as standing for \( g_0 \); this is in accord with the depth-0 case of [Kim04, Definition 2.1.1]. Passage to weak associate classes affords a well defined bijection \( \iota: \mathcal{S}_K \rightarrow \mathcal{S} \) such that \( \iota(\{*\}) = s' \) and \( \iota(\mathcal{S}) = \mathcal{S}' \) whenever there are \( s \in \mathcal{S} \) and \( S \in \mathcal{S}' \) such that the dual blob of \( s \) is \( S \) (in which case the dual blob of every element of \( S \) belongs to \( \mathcal{S}' \)). Recall that \( \iota \) is depth-reversing, in the sense that, if some \( s \in \mathcal{S} \) has depth \( r \), then, for every coset \( S \in \iota(\mathcal{S}) \), every element of \( S \) has depth \( -r \). For each weak associate class \( \mathcal{S} \in \mathcal{S}_K \setminus \{s'\} \) of negative-depth good cosets, [Kim04, Definition 1.4.3] defines the subset \( g_{\mathcal{S}} \subset g \) to be the union of all \( \text{Ad} \, G(S) \), as \( S \) ranges over \( \mathcal{S} \). If \( \mathcal{S} = s' \in \mathcal{S}_K \), define \( g_{\mathcal{S}} \) to be the Moy–Prasad \( G \)-domain \( g_0 \subset g \). Given \( s \in \mathcal{S}_K \), we define \( g_s \) to be \( g_{\mathcal{S}} \), where \( \mathcal{S} = \iota(s) \).

Note that, given \( s \in \mathcal{S}_K \setminus \{\{*\} \}, \) the set \( \mathcal{S}' = \{s' \in \mathcal{S} \mid \iota(s) \in \iota(\mathcal{S}) \} \). Moreover

\[
\iota(\mathcal{S}') = -\iota(\mathcal{S}) = \{ -S \mid S \in \iota(\mathcal{S}) \}.
\]

We conclude that \( g_{\mathcal{S}} = -g_{\mathcal{S}} \) for every weak associate class \( s \) of good positive-depth unrefined minimal \( K \)-types.

2.4.3. \textit{Subsets of \( g \), \( \tilde{G} \) and \( \hat{G} \) determined by a good unrefined minimal \( K \)-type.} By [Kim04, Lemma 1.4.5], we have that

\[
g = \bigcup_{\mathcal{S} \in \mathcal{S}} g_{\mathcal{S}}.
\]

and, since each \( g_{\mathcal{S}} \) is clearly open in \( g \) and since they are pairwise disjoint, each is closed as well. Thus, each \( g_{\mathcal{S}} \) with \( \mathcal{S} \in \mathcal{S} \), and hence also each \( g_s \subset g \) with \( s \in \mathcal{S}_K \), is a \( G \)-domain (i.e., a subset of \( g \) that is open, closed and invariant under \( G \)-conjugation).
Following [Kim04] Remark 2.2.2, for \( \mathfrak{f} \in \mathcal{S}_K \setminus \{ \ast \} \), let \( \hat{G}_\mathfrak{f} \) (respectively, \( \hat{G}_\mathfrak{f} \)) be the set of \( \pi \in \hat{G} \) (respectively, \( \pi \in \hat{G} \)) such that \( \pi \) contains an element of \( \mathfrak{f} \); and then, following [Kim04] Definition 3.2.3, for \( \mathfrak{f} \in \mathcal{S} \setminus \{ \ast \} \), let \( \hat{G}_\mathfrak{f} \) (respectively, \( \hat{G}_\mathfrak{f} \)) denote the set of all elements of \( \hat{G} \) (respectively, \( \hat{G} \)) containing an unrefined minimal \( K \)-type whose dual blob belongs to \( \mathfrak{f} \) (hence is good). Let \( \hat{G}_\pi = \hat{G}_{\pi'} \) (respectively, \( \hat{G}_\pi = \hat{G}_{\pi'} \)) denote the set of depth-zero representations in \( \hat{G} \) (respectively, \( \hat{G} \)). Note that \( \hat{G}_\pi \) equals \( \hat{G}_{\pi'} \) and \( \hat{G}_\pi \) equals \( \hat{G}_{\pi'} \) whenever \( \mathfrak{f} = \mathfrak{f}' \).

Remark 2.8. As we recalled in the introduction, [Kim04] Remark 2.2.2] defines a set \( \hat{G}_\pi \) for every (not necessarily good) positive-depth unrefined minimal \( K \)-type \( \pi \). Given such an \( \pi \), we have by [Kim04] Lemma 1.4.5] that there exist good, positive-depth unrefined minimal \( K \)-types that are weakly associate to \( \pi \) in the wider sense of [Kim04] Definition 2.2.1], and by our prior discussion that the set of such good unrefined minimal \( K \)-types forms a weak associate class \( \mathfrak{f} \) in our sense. By [Kim04] Theorem 4.5.1] (or [KM03] Theorem 2.4.10]) and the associativity of the unrefined minimal \( K \)-types contained in a fixed element of \( \hat{G} \) [MP96] Theorem 3.5], we have that \( \hat{G}_\pi \) equals the set \( \hat{G}_{\pi'} \) that we have just defined. Moreover, \( \hat{G} \) and \( \hat{G} \) are respectively the disjoint unions of the sets \( \hat{G}_\mathfrak{f} \) and the \( \hat{G}_\mathfrak{f} \), as \( \mathfrak{f} \) runs over \( \mathcal{S}_K \).

2.5. The Bernstein center and unrefined minimal \( K \)-types.

Remark 2.9. We briefly summarize some facts concerning the Bernstein center \( Z(G) \) of \( G \), referring the reader to [BK16] Section 1.3] or to [Ber84] for more details. The ring \( Z(G) \) can be realized as the ring \( \mathbb{C}[\Omega(G)] \) of regular functions on \( \Omega(G) \). Giving an element \( z \in Z(G) \) is equivalent to giving an endomorphism \( \pi(z) \) of each object \( \pi \) in the category of smooth representations of \( G \), such that \( \pi \mapsto \pi(z) \) is an endomorphism of the identity functor of that category (i.e., such that it respects the morphisms in the category). One thinks of \( \pi(z) \) as specifying a \( G \)-equivariant action of \( z \) on the space of \( \pi \). Note that, when \( \pi \) is irreducible, the only such action is by scalar multiplication. Recall that every \( \pi \in \hat{G} \) determines an infinitesimal character \( \inf(\pi) \in \Omega(G) \). We can match our two perspectives on the Bernstein center by requiring that, for every \( z \in Z(G) \) and \( \pi \in \hat{G} \), the action \( \pi(z) \) is by multiplication by \( z(\inf(\pi)) \). One can also realize \( Z(G) \) as the set of invariant distributions \( z \) on \( G \) such that \( z \ast f \in C^\infty_c(G) \) for all \( f \in C^\infty_c(G) \); this connects to the earlier description by thinking of \( z \ast f \) as \( \pi(z)(f) \), where \( \pi = \ell \) is the left-regular representation of \( G \) on \( C^\infty_c(G) \).

Lemma 2.10. For every weak associate class \( \mathfrak{f} \) of positive-depth, good unrefined minimal \( K \)-types for \( G \), the subset \( \hat{G}_\mathfrak{f} \subset \hat{G} \) is a union of Bernstein components of \( G \).

Remark 2.11. By Remark 2.8 Lemma 2.10] is equivalent to the claim that, for every positive-depth unrefined minimal \( K \)-type \( \mathfrak{f} \) for \( G \), the subset \( \hat{G}_\mathfrak{f} \subset \hat{G} \) defined in the introduction is a union of Bernstein components of \( G \). It is this that we prove.

Proof. Our justification will be along the lines of the second proof of [Kim04] Theorem 4.5.1(2)]. We first claim that, for every positive-depth Bernstein component \( \mathfrak{b} \), there is a positive-depth unrefined minimal \( K \)-type \( \mathfrak{s}_\mathfrak{b} \) that is contained in every representation in \( \mathfrak{b} \). Indeed, consider the Bernstein component indexed by the inertial equivalence class of \( (M, \sigma) \), where \( \sigma \) has positive depth \( r \). By [MP96] Theorem 3.5(2)(ii)], there is a depth-\( r \), unrefined minimal \( K \)-type \( \mathfrak{s}_M \) contained in \( \sigma \). Write \( \mathcal{S}_M = Y + m^* \) for the dual blob of \( \mathfrak{s}_M \), and put \( \mathfrak{s}_\mathfrak{b} = (G_{x,r}, \Lambda_{Y,x,r}) \). If \( \pi \) lies in the chosen Bernstein component, then there is some unramified character \( \lambda_M \) of \( M \) such that \( \pi \) has \( (M, \sigma \otimes \lambda_M) \) as a supercuspidal support. Note that \( \lambda_M \) is trivial on every compact subgroup of \( M \), so that \( \sigma \otimes \lambda_M \) also contains \( \mathfrak{s}_M \). We have by [MP96] Theorem 2.5] that there is some parabolic subgroup \( P \) of \( G \) with Levi component \( M \) such that \( \pi \) is a subrepresentation of the representation \( \text{Ind}^G_M(\sigma) \) of \( G \) obtained by unnormalized parabolic induction from \( \sigma \). Hence by Frobenius reciprocity (e.g., in the form recalled at the end of [MP96] Section 2.3]), the Jacquet module of \( \pi \) surjects to \( \sigma \). Therefore, we have by [MP96] Theorem 4.5] that \( \pi \) contains a positive-depth, unrefined \( K \)-type \( \mathfrak{s}_\pi = (G_{x,r}, \xi) \) that restricts to \( \mathfrak{s}_M \), and whose dual blob \( \mathcal{S}_\pi \) intersects \( m^* \). Since, in this case, the dual blob
Definition 2.12. Let $\mathfrak{g}_\pi$ be a weak associate class of good unrefined minimal $K$-types. Write $r$ for the depth of every $K$-type in $\mathfrak{g}$. As in the introduction, we let $E_\pi$ be the element of $Z(G)$ that acts as the identity on the set $\widehat{G}_r \subset \widehat{G}$ and is zero on all the elements of $\widehat{G} \setminus \widehat{G}_r$.

(1) We call $E_\pi$ the $\mathfrak{g}$-projector or the Bernstein $\mathfrak{g}$-projector.
(2) Let $E_\pi$ be the distribution on $g$ obtained as the inverse Fourier transform of the distribution represented by the characteristic function $1_{\mathfrak{g}_\pi}$ of $\mathfrak{g}_\pi$ (which is a locally constant function, as $\mathfrak{g}_\pi$ is open and closed in $\mathfrak{g}$).

Remark 2.13. Since we have imposed the Haar measure on $g$ that is self-dual with respect to $A$ and $B$ (see Remark 2.2), we have that $E_\pi$ is also the Fourier transform of the distribution on $g$ represented by the function $X \mapsto 1_{\mathfrak{g}_\pi}(-X)$, i.e., $E_\pi$ is the Fourier transform of the distribution represented by the characteristic function $1_{\mathfrak{g}_\pi^\vee}$ of $\mathfrak{g}_\pi^\vee$.

3. The proof of Theorem 1.1(ii) — the description of $E_\pi$ near the identity

We now explain the sense in which Theorem 1.1(ii) is a reformulation of [Kim04 Theorem 3.3.1].

3.1. The Bernstein center and the Plancherel formula. Recall that $\hat{G} \subset \widehat{G}$ is the tempered dual of $G$. For every $f \in C_c^\infty(G)$, the Plancherel formula gives an equation of the following form:

$$f(1) = \int_{\hat{G}} \Theta_\pi(f) \, d\pi = \int_{\hat{G}} \Theta_{\pi^\vee}(f) \, d\pi,$$

where $\pi^\vee$ denotes the contragredient of $\pi$, and where $d\pi$ refers to the Plancherel measure. Here, we note that [Kim04, p. 56, Equation (P1)] gives the equality of the first two terms, and that this automatically gives the equality of the first term and the third. Indeed, if $f' \in C_c^\infty(G)$ is defined by requiring that $f'(g) = f(g^{-1})$ for all $g \in G$, then $f(1) = f'(1)$, and it is easy to see that for all $\pi \in \hat{G}$, the operator $\pi^\vee$ on the space of $\pi^\vee$ is the transpose of the operator $\pi(f')$ on the space of $\pi$, so that $\Theta_{\pi^\vee}(f) = \Theta_\pi(f')$.

With this notation, for all $z \in Z(G)$ and $f \in C_c^\infty(G)$, one can use the equality of the first and second terms of Equation (3.1) to write in the spirit of [MT02 Equation (2.4.3)]:

$$z(f) = (z * f')(1) = \int_{\hat{G}} \Theta_\pi(z * f') \, d\pi = \int_{\hat{G}} z(\pi)\Theta_\pi(f') \, d\pi = \int_{\hat{G}} z(\pi)\Theta_{\pi^\vee}(f) \, d\pi,$$

where in the left-most term, $z$ is thought of as a distribution on $G$ and evaluated at $f \in C_c^\infty(G)$, while in the fourth and the fifth terms, we have written $z(\pi)$ for the value $z(\inf(\pi))$ of $z \in Z(G) = \mathbb{C}[\Omega(G)]$ at the
image \inf(\pi) of \pi in \Omega(G), i.e., for the scalar by which \pi(z) acts on the space of \pi; or the equality of the first and the third terms in Equation (3.1) to write

\begin{equation}
\int_G \frac{\Theta_{\pi}}{\pi}(z \ast f^\vee) d\pi = \int_G \frac{\Theta_{\pi}}{\pi} \Theta_{\pi}(f) d\pi.
\end{equation}

3.2. Theorem 1.1(ii) as a reformulation of a result of Kim.

Proof of Theorem 1.1(ii). For every \( f \in C_c(G) \), we obtain a function \( f \circ \log \in C_c(G) \), which we view, after extending by zero, as an element of \( C_c(G) \). Using [Kim04] Theorem 3.3.1, we have for every \( f \in C_c(G) \) that

\begin{equation}
\frac{\Theta_{\pi}}{\pi}(f) = \frac{\Theta_{\pi}}{\pi}(f \circ \log).
\end{equation}

On the left-hand side of Equation (3.4), we have taken \( f \circ \log \) to be the Haar measure on \( g \) that is self-dual with respect to \( \Lambda \) and \( B \) (see Remark 2.2). The condition \( \text{vol}_g(g_{x,r}) \text{vol}_g(g_{x,-r}) = 1 \) that appears below [Kim04] Equation (4) (and the fact that \( g_{x,r} \) is the set of those elements \( Y \in g \) for which \( Y; g_{x,-r} \) is contained in the maximal ideal of \( \mathcal{O} \)) shows that [Kim04] is also using this self-dual measure. The left-hand side of Equation (3.4) equals \( \mathcal{E}_\pi(f) \) (see Remark 2.13). Thus, it suffices to show that its right-hand side equals \( E_\pi(f \circ \log) \). Since \( \pi \in \mathcal{G}_\pi \) if and only if \( \pi^\vee \in \mathcal{G}_\pi \), this follows from Equation (3.3).

Remark 3.1. Thus, Theorem 1.1(ii) is merely a restatement of [Kim04] Theorem 3.3.1, using the spectral description of elements of the Bernstein center and the Plancherel formula. In fact, the proof of Theorem 1.1(ii) that we give in Section 4 amounts to adapting the strategy of the proof of [Kim04] Theorem 3.3.1 to neighborhoods of suitable non-identity elements of \( G \), via semisimple descent. Just as the proof of [Kim04] Theorem 3.3.1 uses the character expansion of [KM03], the proof of Theorem 1.1(ii) uses the second author’s generalization of the character expansion of [KM03] to an expansion about non-identity semisimple elements of \( G \), namely, [Spi18] Theorem 4.4.11.

4. A Proof of Theorem 1.1(i) — \( E_\pi \) Vanishes Away from the Identity.

Let us begin with some preparation. We keep the fixed weak associate class \( \mathfrak{g} \), and its depth \( r \), of Subsection 2.6. We now also fix as a good element \( \Gamma \in -\mathfrak{g}_\pi = \mathfrak{g}_{\pi^\vee} \), and write \( G' \) for the centralizer of \( \Gamma \) in \( G \). Note that \( \Gamma \) has depth \( -r \) (see Subsubsection 2.4.2), and that \( (G_{x,r}, \Lambda_{G,x,r}) \) belongs to \( \mathfrak{g} \) for every \( x \in \mathcal{B}(G') \).

4.1. Some functions on which \( E_\pi \) acts as the identity.

Lemma 4.1. Let \( x \in \mathcal{B}(G') \subset \mathcal{B}(G) \), let \( X' \in \mathfrak{g}_{x,-r} \cap \mathfrak{g}_{(-r)} \), and consider \( \Lambda_{G,X',x,r} \subset C_c(G_{x,r}) \subset C_c(G) \). Then \( E_\pi \ast \Lambda_{G,X',x,r} = \Lambda_{G,X',x,r} \).

Proof: If \( f_1, f_2 \in C_c(G) \), then, to show that \( f_1 \ast f_2 \) is equal to \( \Lambda_{G,X',x,r} \), we must verify that the Plancherel formula shows that \( \pi(f_1) \ast \pi(f_2) \) for each reducible admissible representation \( \pi \) of \( G \) (note that \( \pi(f_1) \ast \pi(f_2) \) refers to an equality of two linear operators, and not just of their traces). Thus, let \( \pi \in \mathcal{G} \). We need to show that \( \pi(E_\pi \ast \Lambda_{G,X',x,r}) = \pi(\Lambda_{G,X',x,r}) \). Since \( \pi(E_\pi \ast \Lambda_{G,X',x,r}) = \pi(\mathcal{E}_\pi) \pi(\Lambda_{G,X',x,r}) \), we need to show that if \( \pi(\Lambda_{G,X',x,r}) = 0 \), then \( \pi(\mathcal{E}_\pi) \) is the identity. In other words, assuming that \( \pi \) contains \( (G_{x,r}, \Lambda_{G,X',x,r}) \), we need to show that \( \pi \) contains an unrefined minimal \( K \)-type which belongs to the weak associate class \( \mathfrak{g}_\pi \). Since \( \Gamma \) is a good element in \( \mathfrak{g}_{\pi^\vee} = -\mathfrak{g}_\pi \), any unrefined minimal \( K \)-type of \( G_{y,r}, \Lambda_{G,X',x,r} \), with \( y \in \mathcal{B}(G') \), belongs to \( \mathfrak{g}_\pi \). Thus, it suffices to show that, whenever \( \pi \) contains \( (G_{x,r}, \Lambda_{G,X',x,r}) \), it also
contains \((G_{y,r}, A_{-\Gamma,y,r})\) for some \(y \in \mathcal{B}(G')\). Since we are imposing Hypothesis \([2.1(i)] which includes \([KM03]\) Hypothesis (HB)], the desired result follows from \([KM03]\) Lemma 2.4.11 (with our \(-\Gamma, -X'\) and \(r\) as the \(\Gamma, X'\) and \(g\), respectively, of that lemma).

\[\square\]

4.2. Review of asymptotic expansions around semisimple elements. In \([KM03]\) Theorem 5.3.1], Kim and Murnaghan proved that if \(\pi\) is an irreducible admissible representation of \(G\) of depth \(r\), then a variant of the Howe–Harish-Chandra local character expansion for \(\pi\) at the identity element of \(G\) is valid on \(g_r\), which is bigger than the region \(g_{r+}\) on which the validity of the Howe–Harish-Chandra expansion was proved by S. DeBacker (following J.-L. Waldspurger). J. Adler and J. Korman used semisimple descent to generalize the work of DeBacker to give a range of validity for the Howe–Harish-Chandra expansion around a non-identity semisimple element \([AK07]\) Corollary 12.10]. Similarly, the second author generalized the work of \([KM03]\) to give an asymptotic expansion in the spirit of \([KM03]\) around many semisimple elements of \(G\), as well as an explicit region for its validity that is in general bigger than the analogous region in \([AK07]\).

All these statements are only true under appropriate technical hypotheses. We will impose two hypotheses — Hypotheses \([4.2] and [4.10] — that allow us to use some results, particularly Theorem 4.4.11, from \([Spi18]\). Of these, we now state Hypothesis \([4.2] and proceed to make a few observations towards setting the stage for the statement of Hypothesis \([4.10] in Subsection 4.3\).

**Hypothesis 4.2.** There is a collection \(\{\gamma\}\) of semisimple elements of \(G\), containing \(\gamma = 1\), with the following properties.

(a) For each \(\gamma\) in the collection, each eigenvalue \(\lambda \in \hat{F}\) of \(\text{Ad}\ \gamma\) on \(g \otimes_F \hat{F}\) satisfies \(\text{val}(\lambda - 1) < r\) or \(\lambda = 1\).

(b) For each \(\gamma\) in the collection, write \(C_G(\gamma)^c\) for the connected centralizer of \(\gamma\), and \(U_r\) for the union of all \(G\)-conjugates of elements of \(\gamma\) in \(C_G(\gamma)^c\). Then \(G = \bigcup U_r\).

Let \(\gamma' \in G \setminus G_r\). We need to show that the distribution \(E_{\gamma'}\) is zero on a neighborhood of \(\gamma'\). Hypothesis \([4.2] gives us a semisimple element \(\gamma\) and a \(G\)-conjugation invariant subset \(U = U_r \subset G\) containing \(\gamma'\). We will show in Remark \([4.5] that \(U\) is open and, once we have imposed Hypothesis \([4.10] in further Subsection 4.5\), that \(E_{\gamma'}\) vanishes on \(U\), hence around \(\gamma'\). Set \(H = C_G(\gamma)\), and write \(H^o\) for the identity component of \(H\). We slightly abuse notation by writing \(H^o\) for \(H_{\gamma}(F)\).

**Remark 4.3.** We show that \(\gamma\) does not belong to \(G_r\). Suppose it does, and let \(T\) be a maximal torus of \(G\) (necessarily split over a tame extension of \(F\), by Hypothesis \([2.1]\) containing \(\gamma\). Then we have that \(\gamma\) (which is central in \(H\)) lies in \(T\), hence in \(H^o\), hence in the center \(Z(H^o)\) of \(H^o\). By Corollary \([2.4] we have that \(\gamma\) lies in \(H_r\). By \([AS08] Corollary 3.14\), it follows that \(U = G(\gamma H_r)\) is contained in \(G_r\), hence that \(\gamma' \in U_r\) belongs to \(G_{r}\), which is contrary to our assumption.

In \([Spi18] Theorem 4.4.11\), it is shown that, under some hypotheses, there is a Kim–Murnaghan-type asymptotic expansion for the Harish-Chandra character \(\Theta_{\pi}\) of \(\pi\) about \(\gamma\) that is valid on \(U\).

Before reviewing this expansion, we make some informal remarks to help the reader think of the relationship between \(\gamma'\) and \(\gamma\). These notions will be easier to relate to if one is familiar with the notion of singular depth from \([AK07] Definition 4.1\) and the relevance of this notion to the main ‘range of validity’ result of that paper, \([AK07] Corollary 12.10\). The reason for considering \(\gamma\) is that the singular depth of the semisimple part of \(\gamma'\) may be strictly bigger than \(r\), yielding a range of validity that is not large enough for our purposes. (Remember that larger real numbers parameterize smaller sets in the Moy–Prasad filtration.) To explain the relation between \(\gamma'\) and \(\gamma\), recall that in ‘good situations’ we have a decomposition \(\gamma' = \gamma'_{<r} \gamma'_{\geq r}\) of \(\gamma'\) into a product of commuting elements (see \([AS08] Definition 6.8 ff\)), where \(\gamma'_{<r}\) is semisimple with singular depth less than \(r\), and \(\gamma'_{\geq r}\) belongs to \(C_G(\gamma'_{<r})^c(F)\). Then \(\gamma\) can be taken to be \(\gamma'_{<r}\).

Let \(T\) be an invariant distribution on \(U = G(\gamma H_r)\) (i.e., a linear functional \(C_c(\mathcal{U}) \to \mathbb{C}\) that is invariant under precomposition with \(G\)-conjugation). Then the usual process of semisimple descent \([AK07] Definition 7.3\) gives an \(Ad H^o\)-invariant distribution on \(\gamma H_r\), in the notation of \([AK07] page 387\).
Lemma 4.4. The set $H'_s$ of [AK07] page 387] equals $H_r$. 

Proof. Let us temporarily write $V = (g/h) \otimes F \tilde{F}$. Recall that, by definition, $H'_s$ is the set of $h \in H_r$ such that $\text{Ad}(\gamma h) - 1$ is invertible on $g/h$, or, equivalently, on $V$. It suffices to show that, for every $h \in H_r$, no eigenvalue $\mu$ of $\text{Ad}(h)$ on $V$ equals an eigenvalue of $\text{Ad}(\gamma)$ there. By Hypothesis 4.4(a) which says that the eigenvalues of $\text{Ad}(\gamma)$ on $V$ all have valuation less than $r$, it suffices to show that $\mu - 1$ has valuation at least $r$.

Remark 4.5. By the comment on submersivity just before [AK07, Theorem 7.1], and Lemma 4.4, we have $\alpha h$ which is such a root value, we are done.

Remark 4.6. Given any invariant distribution $T$ on $U$, we will denote by $T_{\gamma}$ its semisimple descent to $\gamma H_r$ (see [AK07] Definition 7.3) and Lemma 4.4.

Notation 4.6. We remark that the process of semisimple descent involves certain choices of measures, which we make arbitrarily and fix for the rest of this section. Up to a scalar whose value is of no concern to us (and that could be absorbed into the choices of measures if desired), this is also the distribution denoted $T_{\gamma}$ in [Spi18 Lemma 4.4.3] (see the proof of that lemma and the comment just before it).

Remark 4.7. The distribution $T_{\gamma}$ determines $T$, thanks to the surjectivity assertion of [AK07] Theorem 7.1.

Following the notation of [Spi18 page 2311], write $O^{H^s}(\hat{G}\Gamma)$ for the set of $H^s$-orbits in $h$ whose closure intersects the $G$-orbit $\hat{G}\Gamma$ of $\hat{G}$. In other words, these are the $\text{Ad} H^s$-orbits of elements in $h$ whose semisimple parts are $G$-conjugate to $\Gamma$.

Remark 4.8. We claim that $O^{H^s}(\hat{G}\Gamma)$ is finite. Since $F$ has characteristic 0, so that every connected, reductive group has finitely many rational orbits of nilpotent elements, it suffices to show that there are finitely many $H^s$-orbits in $\hat{G}\Gamma \cap h$. This is [Spi18 Lemma 4.4.10], but, to avoid assuming the hypotheses of [Spi18], we outline a proof in a similar spirit. The proof simplifies because, by [Ser97 Chapter III, Example 4.2(d) and Theorem 4.4.5], every $H^s(F)$-orbit in $\text{Lie}(H)(F)$ intersects $h$ in finitely many $H^s$-orbits.

It thus suffices to show that there are only finitely many $H^s(F)$-orbits in $\text{Ad}(G(F))(\hat{G}\Gamma \cap \text{Lie}(H)(F))$. Fix a maximal torus $T$ in $H$. Then every $H^s(F)$-orbit of semisimple elements in $\text{Lie}(H)(F)$ intersects $\text{Lie}(T)(F)$, so it suffices to show that $\text{Ad}(G(F))(\hat{G}\Gamma \cap \text{Lie}(T)(F))$ is finite. Since $T$ is also a maximal torus in $G$, this last set is contained in the orbit of $G$ under the Weyl group of $T$ in $G$, hence is finite.

For each $O \in O^{H^s}(\hat{G}\Gamma)$, the choice of an $H^s$-invariant measure on it gives us a distribution $\nu_O$ on $h$ supported on $O$ [RR72 Theorem 1], and thus its Fourier transform $\hat{\nu}_O$, as in Subsection 2.2. Recall from Corollary 2.4 that $\log : G_r \to g_r$ restricts to a homeomorphism $H_r \to h_r$ (which we also call $\log$).

Definition 4.9. Let $T$ be an invariant distribution on $U$. Let $\theta_{\gamma}$ be the distribution on $h_r$ obtained by pushing forward the semisimple descent $T_{\gamma}$ of $T$ from $\gamma H_r$ to $h_r$ via $\gamma h \mapsto \log h$. We say that $T$ has a $\Gamma$-asymptotic expansion about $\gamma$ if there exists a tuple $(b_O)_{O \in O^{H^s}(\hat{G}\Gamma)}$ of complex numbers indexed by the elements of $O^{H^s}(\hat{G}\Gamma)$, such that we have the following equality of distributions on $h_r$:

$$\theta_{\gamma} = \sum_{O \in O^{H^s}(\hat{G}\Gamma)} b_O \nu_O|_{h_r}.$$
In such a situation, the distribution \( \sum_{\mathcal{O} \in \mathcal{O}^\mu (g_T)} b_\mathcal{O} \hat{\nu}_\mathcal{O} \) on \( \mathfrak{h} \) will be referred to as a \( \Gamma \)-asymptotic expansion of \( T \) about \( \gamma \) on \( \mathcal{U} \).

4.3. Hypotheses guaranteeing asymptotic expansions. Note that Hypothesis \[4.10\] below involves not only \( G \) but also our fixed weak associate class \( \mathcal{G} \) (introduced in Subsection 2.6) and the element \( \Gamma \in \mathfrak{g}_T = -\mathfrak{g}_\mathcal{G} \) (introduced at the beginning of Section 4).

**Hypothesis 4.10.** Every element \( \gamma \) of the collection of semisimple elements from Hypothesis \[4.2\] satisfies the following properties. Let \( \mathcal{U} = \mathcal{U}_\gamma \) be as in Hypothesis \[4.2(b)\]

(a) For every \( \pi \) in \( \tilde{G}_\mathcal{G} \), the Harish-Chandra character \( \Theta_{\pi^\vee} \big|_\mathcal{U} \) has a \( \Gamma \)-asymptotic character expansion about \( \gamma \) in the sense of Definition \[4.9\]

(b) Suppose that we have a distribution

\[
\theta = \sum_{\mathcal{O} \in \mathcal{O}^\mu (g_T)} b_\mathcal{O} \hat{\nu}_\mathcal{O}
\]

on \( \mathfrak{h} \). Then \( \theta = 0 \) on \( \mathfrak{h}_\gamma \), if and only if \( \theta(\Lambda_{-X}\big|_{\mathfrak{h}_{x,r}}) = 0 \) for all \( X, \mathfrak{h}_{x,r} \), and \( g \in G \) such that

- \( \text{Ad}(g^{-1})\Gamma \in \mathfrak{h} \);
- \( x \in \mathcal{B}(H'_g) \), where \( H'_g := H \cap g^{-1}G'g \); and
- \( X \in \text{Ad}(g^{-1})((\Gamma + \mathfrak{g}_{(r)}^-)_+) \).

Here, \( \Lambda_{-X}\big|_{\mathfrak{h}_{x,r}} \) is viewed as an element of \( C^\infty_c(\mathfrak{h}) \) that is zero outside \( \mathfrak{h}_{x,r} \). The element \( X \) is denoted \( X^* \) in [Spi18, page 2368]. We have also used that \( H'_g \), being the centralizer in \( H \) of the semisimple element \( \text{Ad}(g^{-1})\Gamma \) of \( \mathfrak{h} \), contains a maximal torus of \( H \) and hence of \( G \), which Hypothesis \[2.1(i)\] shows is tame, so that Subsection 2.1 applies to let us view \( \mathcal{B}(H'_g) \) as a subset of \( \mathcal{B}(G) \).

(c) Whenever \( x \in \mathcal{B}(H) \) and \( x \in \mathfrak{h}_{x,-r} \), we have that \( \Lambda_{-X_{x,r}}|H_{x,r} \) and \( \Lambda_{-X_{x,r}} \) are ‘related by semisimple descent’, by which we mean that for all invariant distributions \( T \) on \( \mathcal{U} \) with semisimple descent \( T_\gamma \) onto \( \gamma H_{x,r} \), we have:

\[
\text{vol}_G(G_{x,r})^{-1}T(\ell_\gamma(\Lambda_{-X_{x,r}})) = \text{vol}_H(H_{x,r})^{-1}T_\gamma(\ell_\gamma(\Lambda_{-X_{x,r}}|H_{x,r}))
\]

where \( \ell \) is the left regular representation. We have written

- \( \ell_\gamma(\Lambda_{-X_{x,r}}) \) for the element of \( C^\infty(G) \) that is supported on \( \gamma G_{x,r} \), and given on it by \( \gamma g \mapsto \Lambda_{-X_{x,r}}(g) \); and
- \( \ell_\gamma(\Lambda_{-X_{x,r}}|H_{x,r}) \) for the element of \( C^\infty(H) \) that is supported on \( \gamma H_{x,r} \), and given on it by \( \gamma h \mapsto \Lambda_{-X_{x,r}}(h) \).

**Remark 4.11.** Hypothesis \[4.10\] consists of statements that follow from [Spi18]. The reason that we do not cite the results of that paper directly is that we do not wish to recapitulate the lengthy list of hypotheses on which it depends. (For the interested reader, these are [Spi18, Hypotheses 3.2.2, 3.2.8, 4.1.5, 4.3.4, 4.4.2, 5.1.6], all of which must hold for all the elements to which we apply them.)

- If \( \pi \) belongs to \( \tilde{G}_\mathcal{G} \), then \( \pi^\vee \) belongs to \( \tilde{G}'_\mathcal{G} \), hence contains a good unrefined minimal \( K \)-type that belongs to \( \tilde{\mathfrak{g}}' \). Upon applying [Kim04, Lemma 1.4.2] to the dual blob of this \( K \)-type, we see that \( \pi^\vee \) contains a good unrefined minimal \( K \)-type whose dual blob \( \Gamma' + \mathfrak{g}_{\pi^\vee}(-r)_+ \) is contained in \( \Gamma + \mathfrak{g}_{\pi^\vee}(-r)_+ \) for some \( y \in \mathcal{B}(G') \). First, this gives that \( \mathfrak{g}_{\pi^\vee}(-r)_+ \subset \mathfrak{g}_{\pi^\vee}(-r)_+ \), hence, by taking duals, that \( \mathfrak{g}_{\pi^\vee}(-r)_+ \) is contained in \( \mathfrak{g}_{\pi^\vee} \); so \( G_{y,r} \), \( G_{\pi^\vee} \), \( G_{\pi^\vee} \), and \( G_{\pi^\vee} \), so that \( \pi \) contains the good unrefined minimal \( K \)-type \( (G_{y,r}, \Gamma_{y,r}) \). We now see that Hypothesis \[4.10(a)\] is a consequence of [Spi18, Theorem 4.4.11] (with \( \Gamma \) in place of \( Z'_H \)).

- Hypothesis \[4.10(b)\] is a weaker statement than the one in [Spi18, Lemma 4.4.14]. To see this, use that the set \( \mathcal{O}^\mu (U^*) \) from that lemma is, by definition, the set of \( H^\circ \)-orbits in \( \tilde{G}(\Gamma + \mathfrak{g}_{(r)}^-)_+ \cap \mathfrak{h} \), and hence contains the set \( \mathcal{O}^H (U^*) \).
• Hypothesis 4.10(c) is a consequence of [Spi18] Lemma 4.4.4, since the facts that \( x \in \mathcal{B}(H) \) and \( X \in \mathfrak{h} \) mean that the assignment \( \phi \mapsto \hat{\phi} \) of that corollary takes \( \Lambda_{X,x,r}|_{H_{x,r}} \) on \( H_{x,r} \) to \( \Lambda_{X,x,r} \) on \( G_{x,r} \).

4.4. \textit{E}_\pi \textit{has} \( \Gamma \)-asymptotic expansions. \textit{Proof of Theorem 1.1(i).} In Subsection 4.2, we picked out a semisimple element \( \gamma \in G \setminus G_r \), and showed that \( E_{\pi} \) vanishes on the set \( U = U_{c} \) defined there. By Lemma 4.12 \( E_{\pi} \) has a \( \Gamma \)-asymptotic expansion on \( U \) (Definition 4.9). Namely, write \( E_{\pi,\gamma} \) for the semisimple descent of \( E_{\pi} \) to \( \gamma H_r \) (Notation 4.6), and \( \theta \) for its push-forward to \( \mathfrak{h} \), via \( \gamma h \mapsto \gamma \log h \). Then \( \theta \) is given by an expression \( \sum_{\ell \in \mathcal{O}_{H}^{\mathcal{H}}(\mathcal{G})} b_{\ell} \ell \phi_{\gamma} \). It is enough to show that \( \theta = 0 \) on \( \mathfrak{h}_r \).

Now we apply Hypothesis 4.10(b) which says that, to check the desired equality, it is enough to check that \( \theta(\Lambda_{X,x,r}) = 0 \) for all \( x, X \) and \( g \) as in that hypothesis. We recall the notations \( \ell_{\gamma}(\Lambda_{X,x,r}|_{H_{x,r}}) \) and \( \ell_{\gamma}(\Lambda_{X,x,r}) \) from Hypothesis 4.10(c), noting specifically that we regard them as functions on \( H \) and \( \mathfrak{h} \) by extension by 0. By the definitions of \( \theta \) and \( E_{\pi,\gamma} \), we have \( \theta(\Lambda_{X,x,r}) = E_{\pi,\gamma}(\ell_{\gamma}(\Lambda_{X,x,r}|_{H_{x,r}})) \).

By using Hypothesis 4.10(c) (in the second of the two equalities below), we get:

\[
\text{vol}_{H}(H_{x,r})^{-1}\theta(\Lambda_{X,x,r}) = \text{vol}_{H}(H_{x,r})^{-1}E_{\pi,\gamma}(\ell_{\gamma}(\Lambda_{X,x,r}|_{H_{x,r}})) = \text{vol}_{G}(G_{x,r})^{-1}E_{\pi}(\ell_{\gamma}(\Lambda_{X,x,r})).
\]

Thus, it is enough to show that \( E_{\pi}(\ell_{\gamma}(\Lambda_{X,x,r})) = 0 \). Now, recalling that \( z(f) = (z \ast f')(1) \) for \( z \in Z(G) \) and \( f \in C_{c}^{\infty}(G) \), where \( f' \in C_{c}^{\infty}(G) \) is given by \( g \mapsto f(g^{-1}) \), we have:

\[
E_{\pi}(\ell_{\gamma}(\Lambda_{X,x,r})) = (E_{\pi} \ast \rho_{\gamma}(\Lambda_{X,x,r}))(1),
\]
where \( \rho_\gamma(\Lambda_{X,x,r}) \) is the element of \( \mathcal{C}_r \) that is supported on \( G_{x,r} \mathcal{L} \), and given on it by \( g^{-1} \mapsto \Lambda_{X,x,r}(g) \). Since \( \gamma \) does not lie in \( G_{x,r} \), which is where \( \Lambda_{X,x,r} \) is supported, we have that \( \rho_\gamma(\Lambda_{X,x,r}) \) vanishes at 1. It thus remains to show that \( E \mathcal{L} \rho_\gamma(\Lambda_{X,x,r}) = \rho_\gamma(\Lambda_{X,x,r}) \). For this, it suffices to show that \( E \mathcal{L} \Lambda_{X,x,r} = \Lambda_{X,x,r} \) (because \( f \mapsto z \cdot f \) is \( G \times G \)-equivariant), which in turn, by the same \( G \times G \)-equivariance, follows if we show that \( E \mathcal{L} \Lambda_{Adg(x),g,x,r} = \Lambda_{Adg(x),g,x,r} \). We claim that this last equality follows from Lemma 4.1. To see this, note (using Subsection 2.1) that, since \( x \in B(H_y) \subseteq B(\mathcal{L}^{-1} G') \) we have that \( g \cdot x \in B(G') \), and
\[
\text{Ad} g(X) \in g_{g,x,-r} \cap (\Gamma + g'_{(-r)+}) = (g_{g,x,-r} \cap g') \cap (\Gamma + g'_{(-r)+}) = g'_{g,x,-r} \cap (\Gamma + g'_{(-r)+}),
\]
where the second equality follows from Corollary 2.4. We have that \( \Gamma \) is centralized by \( G' \), so belongs to the Lie algebra of every maximal torus in \( G' \). In particular, we may choose a maximal torus \( T \) in \( G' \) (even maximally split in \( G' \)) such that \( g \cdot x \) belongs to \( B(T) \). Since \( \Gamma \) is of depth \(-r \) in \( G \), we have by Corollary 2.4 again that it belongs to \( t \cap g_{-r} = t_{-r} \subseteq g'_{g,x,-r} \). Thus we can apply Lemma 4.1 with \( g \cdot x \) being the \( X \) of that lemma and with \( \text{Ad} g(X) = \Gamma \in g_{g,x,-r} \cap g'_{(-r)+} \) being the \( X' \) of that lemma.

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