SPECTRAL FLOW INSIDE ESSENTIAL SPECTRUM IV:
$F^*F$ IS A REGULAR DIRECTION

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Abstract. Let $H_0$ and $V$ be self-adjoint operators such that $V$ admits a
factorisation $V = F^*JF$ with bounded self-adjoint $J$ and $|H_0|^{1/2}$ compact $F$.
Flow of singular spectrum of the path of self-adjoint operators $H_0 + rV$, $r \in \mathbb{R}$,
— also called spectral flow, through a point $\lambda$ outside the essential spectrum
of $H_0$ is well studied, and appears in such diverse areas as differential geometry
and condensed matter physics.

Inside the essential spectrum the spectral flow through $\lambda$ for such a path is
well-defined if the norm limit
$$\lim_{y \to 0^+} F(H_0 + rV - \lambda - iy)^{-1} F^*$$
exists for at least one value of the coupling variable $r \in \mathbb{R}$. This raises the
question: given a self-adjoint operator $H_0$ and $|H_0|^{1/2}$ compact operator $F$,
for which real numbers $\lambda$ there exists a bounded self-adjoint operator $J$ such
that the limit above exists? Real numbers $\lambda$ for which this statement is true
we call essentially regular or semi-regular and the operator $V = F^*JF$ we call
a regular direction for $H_0$ at $\lambda$.

In this paper we prove that $\lambda$ is semi-regular for $H_0$ if and only if the
direction $F^*F$ is regular.

1. Introduction

Flow of eigenvalues of a norm-continuous path of self-adjoint operators,
$$H_r = H_0 + rV,$$
which share the common essential spectrum, $\sigma_{ess}$, through a point $\lambda$ outside the
essential spectrum is well studied. The resulting integer number is also called spectral
flow, which has independent origins in operator theory [Kr] and differential geometry [APS]
and since then appeared in such areas as index theory and condensed matter physics, see e.g. [C].

For $\lambda$ inside the essential spectrum the spectral flow, whether it is flow of eigen-
values or more generally flow of singular spectrum, is not well-defined, due to well-
known extreme volatility of singular spectrum embedded in the essential spectrum.
The spectral shift function (SSF) $\xi(\lambda)$ could have been considered as an analogue
of spectral flow, which has independent origins in operator theory [Kr] and differential geometry [APS]
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principle.
total resonance index (TRI), see [Az2] and [AD]. TRI is integer-valued for a.e. \( \lambda \), and coincides with the classical spectral flow outside \( \sigma_{ess} \), see [Az3].

In order to define TRI one does not need a trace class condition, — it suffices to assume the limiting absorption principle (LAP), see e.g. [AMG] and [Y] for more information on LAP. LAP admits many interpretations. We shall outline one which we will use.

Let \( H_0 \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \) and \( F: \mathcal{H} \to \mathcal{K} \) be a closed \( |H_0|^{1/2} \)-compact operator, which we call a \textit{rigging} in \( \mathcal{H} \). Then the pair \( H_0 \) and \( F \) is said to obey LAP if there exists the norm limit, \( T_{\lambda + i0}(H_0) \), of the operator

\[
T_{\lambda + iy}(H_0) := FR_{\lambda + iy}(H_0)F^* := F(H_0 - \lambda - iy)^{-1}F^*
\]

for a.e. \( \lambda \in \mathbb{R} \). TRI is well-defined at \( \lambda \) for a pair of operators \( H_0 \) and \( V = F^*JF \), where \( J \in B_{sa}(\mathcal{K}) \), if the norm limit \( T_{\lambda + i0}(H_r) \) exists for at least one value of the coupling variable \( r \), in which case it automatically exists for all \( r \) except a discrete set. We say that a real number \( \lambda \) is \textit{essentially regular} or \textit{semi-regular} for \( H_0 \) if there exists at least one \( J \in B_{sa}(\mathcal{K}) \) such that the norm limit \( T_{\lambda + i0}(H_0 + rF^*JF) \) exists for at least one \( r \). In this case we also say that \( V = F^*JF \) is a \textit{regular direction} for \( H_0 \) at \( \lambda \).

For more motivation for this paper I refer to papers [Az2] [Az3] [AD] [AD2] and their introductions.

The main result of this paper is the following theorem.

**Theorem 1.1.** Let \( H_0 \) be a self-adjoint operator and \( F \) be a closed \( |H_0|^{1/2} \)-compact operator. If \( \lambda \) is semi-regular for \( H_0 \), then \( F^*F \) is a regular direction.

Theorem [1.1] is simple but important, as it gives a natural choice of a regular direction. It also allows to simplify the definition of a semi-simple point \( \lambda \) of a s.a. operator \( H_0 \) as follows: if \( F^*F \) is a regular direction for \( H_0 \) at \( \lambda \) then \( \lambda \) is semi-simple. Finally, Theorem [1.1] shows that semi-simplicity of a point \( \lambda \) depends on a relationship between a self-adjoint operator \( H_0 \) and a rigging \( F \) only.

## 2. Proof of Theorem [1.1]

The premise means by definition that there exists a regular direction \( V = F^*JF \) at \( H_0 \), that is, \( T_{\lambda + i0}(H_0 + rV) \) exists for all real numbers \( r \) except a discrete set. We need to show that for some real number \( r \) the norm limit

\[
T_{\lambda + i0}(H_0 + rF^*F) =: T_{\lambda + i0}(\tilde{H}_r)
\]

also exists. The second resolvent identity applied to the operator

\[
\tilde{H}_{sr} = H_r + r(sF^*F - V),
\]

where \( s \in \mathbb{R} \), gives

\[
T_z(\tilde{H}_{sr}) = T_z(H_r + rF^*(s - J)F) = \left[ 1 + rT_z(H_r)(s - J) \right]^{-1}T_z(H_r).
\]

Thus, for some real number \( s \) the norm limit \( T_{\lambda + i0}(\tilde{H}_{sr}) \) exists if and only if the operator

\[
1 + rT_{\lambda + i0}(H_r)(s - J)
\]
is invertible and this is what we will prove. Assume the contrary. Then, since
\( T_{\lambda+0}(H_r)(s-J) \) is compact, by Fredholm alternative for some non-zero analytic
vector-valued function \( \varphi_s \) we have for all real numbers \( s \)
\[
|1 + r T_{\lambda+0}(H_r)(s-J)| \varphi_s = 0.
\]
We can assume that \( s > \|J\| \). Thus, the equality above means that \(-1\) is an
eigenvalue of \( r \sqrt{s-J} T_{\lambda+0}(H_r) \sqrt{s-J} \) for all real \( s > \|J\| \), so, for non-zero vector
function \( \psi_s \) we have
\[
r \sqrt{s-J} T_{\lambda+0}(H_r) \sqrt{s-J} \psi_s = -\psi_s.
\]
Taking the scalar product of both sides of this equality by \( \psi_s \) and then taking the
imaginary part of both sides we get
\[
\langle \psi_s, \sqrt{s-J} \Im T_{\lambda+0}(H_r) \sqrt{s-J} \psi_s \rangle = 0,
\]
and since \( \Im T_{\lambda+0}(H_r) \geq 0 \) from this we find
\[
\Im T_{\lambda+0}(H_r) \sqrt{s-J} \psi_s = 0.
\]
Therefore, the equality (1) turns into
\[
(\sqrt{s-J} \Re r T_{\lambda+0}(H_r) \sqrt{s-J} \psi_s, \psi_s) = 0.
\]
Recall that \( s \) is large enough for the operator \( s-J \) to be invertible. Now we use a
well-known lemma: for an analytic path of self-adjoint operators \( N_s \) the eigenvalue equation
\[
N_s \varphi_s = \lambda(s) \varphi_s
\]
implies
\[
(N_s^* \varphi_s, \varphi_s) = \lambda(s) (\varphi_s, \varphi_s).
\]
Applying this lemma to (2) gives
\[
(\sqrt{s-J} \Re r T_{\lambda+0}(H_r) \sqrt{s-J} \psi_s, \psi_s) + (\sqrt{s-J} \Re T_{\lambda+0}(H_r) \sqrt{s-J}^{-1} \psi_s, \psi_s) = 0.
\]
Combining this with (2) implies
\[
(\psi_s, (s-J)^{-1} \psi_s) = 0.
\]
Now since for large \( s \) the operator \( (s-J)^{-1} \) is positive definite, it follows that for
such \( s \) we have \( \psi_s = 0 \). Which is clearly impossible. This contradiction completes
the proof.

**Corollary 2.1.** If \( V = F^* J F \) is a regular direction then so is \( F^* |J| F \).

**Proof.** This corollary is a consequence of the proof of Theorem 1.1. In the proof
we need to replace \( s-J \) by \( s |J| - J \) and assume that \( s > 1 \). There is one slight
difficulty, as the operator \( \sqrt{s |J| - J} \) is not necessarily invertible, but it can be
easily overcome: clearly \( \psi_s \) belongs to the closure of the range of \( |J| \) and so we can
restrict the eigenvalue equation to this subspace on which the operator \( \sqrt{s |J| - J} \) is
invertible.

An argument used in the proof of Theorem 1.1 also allows to prove

**Corollary 2.2.** Suppose \( \lambda \) is a semi-regular point for \( H_0 \). If \( J \geq 0 \) and the direction \( V = F^* J F \) is \( \lambda \)-regular at \( H_0 \) then so is the direction \( F^* \tilde{J} F \) for any \( \tilde{J} \geq J \).

**Proof.** Proof follows verbatim that of Theorem 1.1 with some obvious changes such
as replacing \( s-J \) by \( s \tilde{J} - J \).
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