The total helicity of electromagnetic fields and matter

Ivan Fernandez-Corbaton

1 Institute of Nanotechnology, Karlsruhe Institute of Technology, 76021 Karlsruhe, Germany

The electromagnetic helicity of the free electromagnetic field and the static magnetic helicity are shown to be two different embodiments of the same physical quantity, the total helicity. The total helicity is the sum of two terms that measure the difference between the number of left-handed and right-handed photons of the free field, and the screwiness of the static magnetization density in matter, respectively. This unification provides the theoretical basis for studying the conversion between the two embodiments of total helicity upon light-matter interaction.

I. INTRODUCTION AND SUMMARY

The electromagnetic helicity \( H_\text{EM} \) is a property of the free electromagnetic field that extends the concept of circular polarization handedness from individual plane waves to general Maxwell fields. Its integrated value is a pseudo-scalar proportional to the difference between the number of left- and right-handed photons contained in the field. A recently renewed interest in electromagnetic helicity \( H_\text{EM} \) is revealing and exploiting its effectiveness for understanding and engineering light-matter interactions, in particular at the challenging micro and nano scales. Such effectiveness is, to some extent, due to the connection between electromagnetic helicity and the electromagnetic duality symmetry of free fields \( \Pi_\text{EM} \), which greatly facilitates the use of symmetries and conservation laws in the analysis of light-matter interactions. Electromagnetic helicity is, in many ways, at the same level of generality as electromagnetic energy, momentum and angular momentum: It is a measurable property of the field which is connected to a fundamental symmetry transformation. But there is an important difference: While energy, momentum, and angular momentum are also defined for material systems, and the possibility and effects of the exchange of such properties between fields and matter are theoretically understood and practically exploited, the same is not true for electromagnetic helicity. It is so far unclear whether a material system can have electromagnetic helicity, which means that we lack the theoretical basis for considering an exchange of electromagnetic helicity between fields and matter. This is an unsatisfactory state of affairs, in particular because the integrated electromagnetic helicity of the field is typically different before and after a light-matter interaction.

In this article, we extend the definition of the integrated electromagnetic helicity of the free field to include an additive contribution from static material sources. Such contribution turns out to be the magnetic helicity \( H_\text{MAG} \), which quantifies the screwiness of the static magnetic field lines. The resulting total helicity is equal to the sum of the screwiness of the static magnetization density in matter plus the difference between the number of left- and right-handed photons in the free electromagnetic field. The unification provides the theoretical basis for studying the conversion between the two embodiments of total helicity upon light-matter interaction.

The rest of the article is organized as follows. In Sec. II, we start with the question can the electromagnetic free field and a material system exchange electromagnetic helicity? which is motivated by the fact that the integrated electromagnetic helicity of the field is typically different before and after the light-matter interaction. The question forces upon us the need for defining, for a material system, the counterpart of the electromagnetic helicity for the free field. In Sec. III, we follow the example of electrostatic energy and consider the static electromagnetic sources or, alternatively and equivalently, the static fields that they generate, as the potential reservoirs of electromagnetic helicity. More precisely, we consider sources in static equilibrium where the time derivatives of macroscopic quantities vanish. We then show that the static equilibrium version of the most commonly assumed Maxwell sources does not allow the sought after helicity storage. This roadblock is then removed by assuming electric charge and magnetic spin as the primordial sources, instead of electric charge and magnetic charge, or electric charge only. Then, we show in Sec. IV that the use of electric charge and magnetic spin enables us to extend the definition of the integrated electromagnetic helicity of the free field so that it includes a static contribution coming from the transverse(divergence-free) part of the static spin magnetization density. This contribution turns out to be the magnetic helicity, albeit in different units. We then show in Sec. V that, for fixed magnetostatic self-energy and total magnetization squared, the most efficient magnetization modes for storing helicity in matter are maximally helical, that is, they are eigenstates of the helicity operator with eigenvalue +1 or −1. We derive an upper bound for the absolute value of helicity that can be stored in a system with a given total magnetization squared. The bound scales with the fourth power of the linear dimension \( L \) of the system.

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*ivan.fernandez-corbaton@kit.edu
The study of electromagnetic helicity in the presence of fundamental sources has been based in the following microscopic Maxwell equations, either including \[2\] or [3]...
excluding the magnetic sources:

\[
\nabla \cdot \mathbf{B}(t, r) = \frac{\mu_0 \rho_m(t, r)}{\epsilon_0}, \quad \nabla \times [\epsilon_0 \mathbf{E}(t, r)] = \frac{\partial B(t, r)}{\mu_0} + J_m(t, r),
\]

\[
\nabla \cdot \mathbf{E}(t, r) = \frac{\rho_e(t, r)}{\epsilon_0}, \quad \nabla \times \mathbf{B}(t, r) = \mu_0 \left[ J_e(t, r) + \epsilon_0 \partial_t \mathbf{E}(t, r) \right],
\]

(4)

where \( \rho_e(t, r) \) and \( J_e(t, r) \) are the electric/magnetic charge and current densities, respectively. In particular, Zwanziger used Eq. (4) to show that electrodynamics is invariant when, besides applying Eq. (2) to the fields, the duality transformation is also applied to the sources:

\[
\rho_e^\theta(t, r) = \rho_e(t, r) \cos \theta - c_0 \rho_m(t, r) \sin \theta, \quad c_0 \rho_m^\theta(t, r) = \rho_e(t, r) \sin \theta + c_0 \rho_m(t, r) \cos \theta,
\]

\[
J_e^\theta(t, r) = J_e(t, r) \cos \theta - c_0 J_m(t, r) \sin \theta, \quad c_0 J_m^\theta(t, r) = J_e(t, r) \sin \theta + c_0 J_m(t, r) \cos \theta.
\]

(5)

We note that, in this article, we are not concerned with the possibility of an exact conservation law for the electromagnetic helicity including both fields and sources. The questions that we are addressing, discussed in Sec. II, aim at establishing whether the exchange of electromagnetic helicity between fields and sources is at all meaningful. A positive answer is apparently a pre-requisite for considering a joint light-matter conservation law for electromagnetic helicity.

Let us examine the static version of Eqs. (4). To such end, it is important to examine the arguments underlying Eqs. (5). Namely, that there exist primordial electric and magnetic elementary charges which result in electric and magnetic charge densities inside material systems, and that the electric and magnetic current densities are due to the movement of the electric and magnetic charge densities, respectively. The charge and current densities together transform as four-vectors under the Poincaré group of special relativity:

\[
j_e(t, r) = \begin{bmatrix} \rho_e(t, r) \\ J_e(t, r) \end{bmatrix}, \quad j_m(t, r) = \begin{bmatrix} \rho_m(t, r) \\ J_m(t, r) \end{bmatrix}.
\]

(6)

The static sources are hence

\[
j_e(r) = \begin{bmatrix} \rho_e(r) \\ 0 \end{bmatrix}, \quad j_m(r) = \begin{bmatrix} \rho_m(r) \\ 0 \end{bmatrix},
\]

(7)

where the vanishing of the 3-vector currents \( J_{e/m}(r) = 0 \) is due to the vanishing of net macroscopic movement of the static charge densities \( \rho_e(r) \) and \( \rho_m(r) \) inside the material system. While there will generally be some microscopical dynamics, like e.g. due to thermal fluctuations, we will assume that the material system before the light-matter interaction in Fig. (a), and after it in Fig. (b) is in a state of static equilibrium where the time derivatives of macroscopic quantities vanish. This includes the vanishing of the macroscopic time derivative of the position of the electric and magnetic charge densities, and hence the vanishing of \( J_{e/m}(r) \). The static equilibrium version of Eq. (4) is hence obtained by eliminating all the terms containing time derivatives, and using the sources from Eq. (7):

\[
\nabla \cdot \mathbf{E}(r) = \frac{\rho_e(r)}{\epsilon_0}, \quad \nabla \cdot \mathbf{B}(r) = \mu_0 \rho_m(r),
\]

\[
\nabla \times \mathbf{E}(r) = 0, \quad \nabla \times \mathbf{B}(r) = 0.
\]

(8)

According to our previous discussion, the electromagnetic helicity of the material system before and after the light-matter interaction is contained in the configuration of the static sources in panels (a) and (b), respectively, or, equivalently, in the static fields produced by them. We now show that the static electromagnetic fields in Eq. (8) cannot store helicity because both \( \mathbf{E}(r) \) and \( \mathbf{B}(r) \) have vanishing curl (are longitudinal).

To such end, we consider the Fourier-transformed momentum space version of Eq. (8), which is obtained using the correspondences between the time-space \((t, r)\)-domain and the frequency-momentum \((\omega, p)\)-domain contained in Tab. (1) for \( X(\omega, p) \) functions, particularized for the time independent \( \omega = 0 \) case:

\[
i \mathbf{p} \cdot \mathbf{E}(0, \mathbf{p}) = \frac{\rho_{e}(0, \mathbf{p})}{\epsilon_0}, \quad i \mathbf{p} \cdot \mathbf{B}(0, \mathbf{p}) = \mu_0 \rho_{m}(0, \mathbf{p}),
\]

\[
-\epsilon_0^2 i \mathbf{p} \times [\epsilon_0 \mathbf{E}(0, \mathbf{p})] = 0, \quad i \mathbf{p} \times \mathbf{B}(0, \mathbf{p}) = 0.
\]

(9)

The two \( i \mathbf{p} \times \) equations in Eq. (9) imply that both \( \mathbf{E}(0, \mathbf{p}) \) and \( \mathbf{B}(0, \mathbf{p}) \) are purely longitudinal, that is, \( \mathbf{E}(0, \mathbf{p}) \) and \( \mathbf{B}(0, \mathbf{p}) \) are parallel to the momentum vector \( \mathbf{p} \), having zero components that are transverse (perpendicular) to \( \mathbf{p} \). Purely longitudinal (transverse) fields in the \( \mathbf{p} \)-domain correspond to fields with zero curl (divergence) in the \( r \)-domain. Let us now consider the \( \mathbf{p} \)-domain representation of the helicity operator, which follows from the general definition of helicity as the projection of the angular momentum operator \( \mathbf{J} \) onto the linear momentum \( \mathbf{P} \) direction:

\[
\Lambda = \frac{\mathbf{J} \cdot \mathbf{P}}{|\mathbf{P}|} = \frac{i \mathbf{p} \times |\mathbf{P}|}{|\mathbf{P}|} = i \mathbf{p} \times .
\]

(10)

Applying the helicity operator to the static electric(magnetic) fields from Eq. (9) results in the zero field:

\[
i \mathbf{p} \times \mathbf{E}(0, \mathbf{p}) = i \mathbf{p} \times \mathbf{B}(0, \mathbf{p}) \quad \text{Eq. (8) 0}.
\]

(11)

\[\Lambda = \frac{\mathbf{J} \cdot \mathbf{P}}{|\mathbf{P}|^2} = \frac{\mathbf{S} \cdot \mathbf{P}}{|\mathbf{P}|^2} \equiv i \mathbf{p} \times, \quad \text{where for electromagnetism, \( \mathbf{S} \) is the vector of spin-1 matrices. The second equality in the previous equation can be seen to follow, for example, from considering the coordinate representation of the angular momentum and linear momentum operator vectors.}[24,25] \quad \mathbf{J} = -i \mathbf{r} \times \nabla + \mathbf{S}, \quad \mathbf{P} = -i \nabla. \quad \text{Their inner product then reads} \quad \mathbf{J} \cdot \mathbf{P} = -i (\mathbf{r} \times \nabla) \cdot \nabla - i \mathbf{r} \cdot \nabla \mathbf{S}. \quad \text{The first term vanishes since it is the divergence of a curl. Finally, the equivalence} \quad \frac{\mathbf{S} \cdot \mathbf{P}}{|\mathbf{P}|^2} \equiv i \mathbf{p} \times \text{follows from applying Eq. (2.2)} \text{in momentum space where} \quad \mathbf{P} \rightarrow \mathbf{p} \implies \mathbf{P}/|\mathbf{P}| \rightarrow \hat{\mathbf{p}}.\]
showing that the static electromagnetic fields in Eq. (9) cannot store any electromagnetic helicity. The conclusion is the same if the magnetic sources in Eq. (11) are removed, and the vanishing of \( J_e(t, \mathbf{r}) \) is maintained. In such a case, \( \mathbf{E}(0, \mathbf{p}) \) is longitudinal and \( \mathbf{B}(0, \mathbf{p}) = 0 \).

We have reached the conclusion that the model underly-

ing Eq. (4) implies that material systems in static equilib-

rium cannot store electromagnetic helicity. This issue must be added to the lack of experimental evidence for isolated magnetic charges (magnetic monopoles).

It turns out that the two issues have the same origin and can be overcome with the same solution: Adopting a different set of primordial electromagnetic sources for Maxwell equations. Supported by the fact that the existence of magnetic spin is beyond doubt, we will now consider electric charges and magnetic spins as the primary sources, instead of electric and magnetic charges, or electric charges only. The static equilibrium sources that we assume from now on are

\[
j_e(r) = \begin{bmatrix} \rho_e(r) \\ 0 \end{bmatrix}, \quad \Sigma(r) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -M_z(r) & M_y(r) \\ 0 & M_y(r) & 0 & -M_z(r) \\ 0 & -M_y(r) & M_z(r) & 0 \end{bmatrix},
\]

where \( j_e(r) \) transforms as a four-vector, and \( \Sigma(r) \) is an antisymmetric tensor which transforms like the electromagnetic tensor \( F \). The three distinct components of \( \Sigma(r) \) constitute the static spin magnetization density \( \mathbf{M}(r) \). The spatial integral of \( \mathbf{M}(r) \) over the volume of the material system defines the intrinsic magnetic moment of the system in static equilibrium. With these assumptions, and for our purposes, the question of whether to model magnetic effects by microscopic electric current loops or microscopic magnetic dipoles [34, Chap. 2, § 1] is decided in favor of the later.

The movement of charge and spin result in the dynamic sources

\[
\begin{align*}
j_e(t, \mathbf{r}) &= \begin{bmatrix} \rho_e(t, \mathbf{r}) \\ \mathbf{J}_e(t, \mathbf{r}) \end{bmatrix}, \\
\Sigma(t, \mathbf{r}) &= \begin{bmatrix} 0 & -P_x(t, \mathbf{r}) & -P_y(t, \mathbf{r}) & -P_z(t, \mathbf{r}) \\ -P_x(t, \mathbf{r}) & 0 & -M_z(t, \mathbf{r}) & M_y(t, \mathbf{r}) \\ -P_y(t, \mathbf{r}) & M_z(t, \mathbf{r}) & 0 & -M_x(t, \mathbf{r}) \\ -P_z(t, \mathbf{r}) & -M_y(t, \mathbf{r}) & M_x(t, \mathbf{r}) & 0 \end{bmatrix},
\end{align*}
\]

where, as before, \( \mathbf{J}_e(t, \mathbf{r}) \) appears due to the movement of \( \rho_e(t, \mathbf{r}) \). Additionally, the movement of \( \Sigma(t, \mathbf{r}) \) produces a dynamic \( \Sigma(t, \mathbf{r}) \) which contains both magnetic spin density \( \mathbf{M}(t, \mathbf{r}) \), and electric spin density \( \mathbf{P}(t, \mathbf{r}) \).

In here, we will use Eqs. (12,13) for an extended ma-

terial system. Their point-particle versions have a long history in the study of relativistic electrodynamics [35 Chap. II, Sec. 4], including the effect of the electron spin on the atomic nucleus [36], and the relativistic spin pre-

cession [37]. In that context, the spatial integral of \( \Sigma(t, \mathbf{r}) \) is often called dipole moment tensor, moment tensor, or polarization tensor.

The sources in Eq. (13) result in a version of Maxwell’s equations [35 Sec. 5] that is quite different from Eq. (4):

\[
\begin{align*}
\nabla \cdot \mathbf{B}(t, \mathbf{r}) &= 0, \\
\n\nabla \times \mathbf{E}(t, \mathbf{r}) + \partial_t \mathbf{B}(t, \mathbf{r}) &= 0, \\
\n\n\nabla \cdot \mathbf{E}(t, \mathbf{r}) &= \frac{\rho_e(t, \mathbf{r})}{\epsilon_0} - \nabla \cdot \mathbf{P}(t, \mathbf{r}), \\
\n\n\nabla \times \mathbf{B}(t, \mathbf{r}) - \partial_t \mathbf{E}(t, \mathbf{r}) &= \frac{1}{\epsilon_0} [\mathbf{J}_e(t, \mathbf{r}) + \partial_t \mathbf{P}(t, \mathbf{r}) + \nabla \times \mathbf{M}(t, \mathbf{r})],
\end{align*}
\]

where the magnetic sources are of a different kind, and appear in a different position with respect to Eq. (4). In particular, the homogeneous equations in Eq. (14) contain the statement that there are no magnetic monopoles, and the divergence of \( \mathbf{B}(t, \mathbf{r}) \) is always zero, making it a purely transverse field in all cases. This difference is crucial for enabling static sources to store electromagnetic helicity.

Let us now consider the static equilibrium limit of Eqs. (14,15). Noting that both \( \mathbf{J}_e(t, \mathbf{r}) \) and \( \mathbf{P}(t, \mathbf{r}) \) vanish, we obtain:

\[
\begin{align*}
\nabla \cdot \mathbf{E}(t, \mathbf{r}) &= \frac{\rho_e(t, \mathbf{r})}{\epsilon_0}, \\
\n\nabla \times \mathbf{B}(t, \mathbf{r}) &= 0, \\
\n\n\nabla \cdot \mathbf{B}(t, \mathbf{r}) &= 0, \\
\n\n\nabla \times \mathbf{E}(t, \mathbf{r}) &= \frac{1}{\epsilon_0} \mathbf{J}_e(t, \mathbf{r}), \\
\n\n\n\nabla \cdot \mathbf{M}(t, \mathbf{r}) &= \frac{1}{\epsilon_0} \mathbf{P}(t, \mathbf{r}),
\end{align*}
\]

The first line in Eq. (16) are the common equations that define the electrostatic field \( \mathbf{E}(\mathbf{r}) \) [33 Chap. 4]. The second line in Eq. (16) coincides with the common equations that define the magnetostatic field \( \mathbf{B}(\mathbf{r}) \), if, according to our previous discussion, we assume the vanishing of the electric current density \( \mathbf{J}_e(t, \mathbf{r}) \) that appears in those common equations (see e.g. [34, Eq. (2.40)], or [33 Eqs. (5.80,5.82)]).

IV. THE TOTAL HELICITY OF FIELDS AND MATTER

We now proceed to show that, when the fundamental static sources from Eq. (12) are assumed, the typical definition of integrated dynamic electromagnetic helicity can be extended to include a static contribution. This contribution turns out to be, essentially, the static magnetic helicity [26,29].

It is now convenient to change from the \((t, \mathbf{r})\)-domain to the \((\omega, \mathbf{p})\)-domain by means of the 4D-Fourier decom-

position

\[
\mathbf{X}(t, \mathbf{r}) = \int_{\omega \geq 0} \frac{d\omega}{\sqrt{2\pi}} \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{\sqrt{(2\pi)^3}} \mathbf{X}(\omega, \mathbf{p}) \exp(-i\omega t + i\mathbf{p} \cdot \mathbf{r}),
\]

(17)
\[ X(\omega, p) = X^\| (\omega, p) + X^\perp (\omega, p), \quad X^\| (\omega, p) = \hat{p} \cdot X(\omega, p), \quad X^\perp (\omega, p) X^\| (\omega, p) = 0 \]

\[ i\hat{p} \times X_\lambda(\omega, p) = \lambda X_\lambda(\omega, p) \quad \text{for} \quad \lambda \in [-1, 0, +1], \quad X_\lambda(\omega, p) = x_\lambda(\omega, p) \hat{e}_\lambda(\hat{p}) \]

\[ X(\omega, p) = X_-(\omega, p) + X_0(\omega, p) + X_+(\omega, p) \]

\[ X_0(\omega, p) = X^\| (\omega, p), \quad X^\perp (\omega, p) = X_+(\omega, p) + X_-(\omega, p) \]

\[ (i\hat{p} \times)^2 X(\omega, p) = i\hat{p} \times i\hat{p} \times X(\omega, p) = X^\perp (\omega, p) = X(\omega, p) - X^\| (\omega, p) \]

\[ \text{TABLE I. Various identities involving the decomposition of a vectorial } X(\omega, p) \text{ function in terms of its longitudinal } (\|) \text{ and transverse parts } (\perp) \text{ (Helmholtz decomposition), and in terms of the eigenvectors of the helicity operator } \Lambda, \text{ which are collected in Tab. II and the correspondences between operators in } (t, r) \text{ and operators in } (\omega, p) \text{ collected in Tab. II.} \]

\[ \begin{align*}
& (t, r) : \quad X(t, r) \quad \partial_t X(t, r) \quad \nabla \cdot X(t, r) \quad \nabla \times X(t, r) \\
& (\omega, p) : \quad X(\omega, p) \quad -i\omega X(\omega, p) \quad i\hat{p} \cdot X(\omega, p) \quad i\hat{p} \times X(\omega, p)
\end{align*} \]

where only frequencies \( \omega \geq 0 \) are included: \( \omega = 0 \) corresponds to the static fields and \( \omega > 0 \) to the dynamic fields. Excluding \( \omega < 0 \) amounts to considering dyadic fields with only positive energy. This is possible in electromagnetism because the photon is its own anti-particle. Then, both sides of the spectrum contain the same information [5, §], and only one sign of the energy(frequency) is needed.

In the following, we will often use properties of the decomposition of a \( X(\omega, p) \) function in terms of its longitudinal \( (\|) \) and transverse \( (\perp) \) parts, and in terms of the eigenvectors of the helicity operator \( \Lambda \), which are collected in Tab. II and the correspondences between operators in \( (t, r) \) and operators in \( (\omega, p) \) collected in Tab. II.

\[
\begin{align*}
&D(t, r) = \pm i \frac{\mathbf{B}(t, r)}{\sqrt{2c_0}} = \sqrt{\frac{c_0}{2}} [\mathbf{E}(t, r) \pm i c_0 \mathbf{B}(t, r)] = \mathbf{F}_\pm (t, r) \\
&= \int_{\mathbb{R}^3} \frac{dp}{\sqrt{(2\pi)^3}} \mathbf{F}_\pm (c_0|\mathbf{p}|, \mathbf{p}) \exp(i\mathbf{p} \cdot \mathbf{r} - ic_0|\mathbf{p}|t),
\end{align*}
\]

where \( \omega \) is restricted to be equal to \( c_0|\mathbf{p}| \) because, for \( \omega > 0 \), the dynamic electromagnetic fields \( \mathbf{E}(\omega, p), \mathbf{B}(\omega, p) \) are constrained to the domain \( \omega = c_0|\mathbf{p}| \). This is the well-known constraint to the positive energy light-cone, which may be seen as a consequence of the massless photon dispersion relations in vacuum \( \omega^2 = c_0^2 (p_x^2 + p_y^2 + p_z^2) \), together with the \( \omega > 0 \) choice.

Reference 40 contains the proof of the equivalence between Eq. 18 and the most common expression for integrated electromagnetic helicity 11 13 15 16 18 22 23:

\[
\langle A_{\omega>0} \rangle = \frac{1}{2} \int_{\mathbb{R}^3} \mathcal{E}(t, r) \cdot \mathcal{A}(t, r) - \mathcal{E}(t, r) \cdot \mathcal{C}(t, r),
\]

where \( \mathcal{E}(t, r), \mathcal{C}(t, r) \) and \( \mathcal{B}(t, r), \mathcal{A}(t, r) \) are the real-valued electric and magnetic fields[potentials], respectively.

We want to extend the definition of integrated electromagnetic helicity in Eq. 18 so that static electromagnetic fields with \( \omega = 0 \) are included in the same footing as the dynamic \( \omega > 0 \) fields. We achieve this by exploiting the fact that the action of the helicity operator \( i\hat{p} \times \) is defined for both dynamic fields, whose domain is \( \omega = c_0|\mathbf{p}|, \) and static fields, whose domain is \( \omega = 0, \mathbf{p} \). We can then complete the definition in Eq. 18 as follows. For each value of \( \mathbf{p} \) in the integral of Eq. 18, we include two different branches for the fields: one branch corresponds to the dynamic fields with \( \omega = c_0|\mathbf{p}| \), and the other corresponds to static fields with \( \omega = 0 \), thereby covering all the domain of definition.
of electric and magnetic fields. We then obtain:

\[
\langle A \rangle = \langle A_{w>0} \rangle + \langle A_{w=0} \rangle = \\
\int_{\mathbb{R}^3} \frac{dp}{c_0|p|} [F_+ (c_0|p|, p) |i\hat{p} \times F_+ (c_0|p|, p) \rangle + F_- (c_0|p|, p) |i\hat{p} \times F_- (c_0|p|, p) \rangle + \\
\int_{\mathbb{R}^3} \frac{dp}{c_0|p|} F_+ (0, p) |i\hat{p} \times F_+ (0, p) \rangle + F_- (0, p) |i\hat{p} \times F_- (0, p) \rangle.
\]

(21)

We can use the well-known fact that, since \(i\hat{p} \times E(c_0|p|, p) = i\epsilon_0 B(c_0|p|, p)\), and \(i\hat{p} \times c_0 B(c_0|p|, p) = -iE(c_0|p|, p)\), the \(F_{\pm} (c_0|p|, p)\) are eigenstates of the helicity operator with eigenvalue \(\pm 1\)

\[
i\hat{p} \times F_{\pm} (c_0|p|, p) = \pm F_{\pm} (c_0|p|, p),
\]

(22)
to rewrite \(\langle A_{w>0} \rangle\) in Eq. \((21)\):

\[
\langle A \rangle = \langle A_{w>0} \rangle + \langle A_{w=0} \rangle = \\
\int_{\mathbb{R}^3} \frac{dp}{c_0|p|} [|F_+ (c_0|p|, p)|^2 - |F_- (c_0|p|, p)|^2 + \\
\int_{\mathbb{R}^3} \frac{dp}{c_0|p|} F_+ (0, p) |i\hat{p} \times F_+ (0, p) \rangle + F_- (0, p) |i\hat{p} \times F_- (0, p) \rangle.
\]

(23)

and set out to work on \(\langle A_{w=0} \rangle\) by elucidating the action of \(i\hat{p} \times\) on \(F_{\pm} (0, p)\). To such end, we will use the momentum space version of Eq. \((16)\):

\[
i\hat{p} \cdot E(0, p) = \frac{\rho(0, p)}{\epsilon_0}, \quad i\hat{p} \cdot E(0, p) = 0, \\
i\hat{p} \cdot B(0, p) = 0, \quad i\hat{p} \times B(0, p) = \mu_0 i\hat{p} \times M(0, p).
\]

(24)
The longitudinal character of the E and the transverse character of the B are manifest in Eqs. \((16)\): \(\nabla \times E(\mathbf{r}) = i\hat{p} \times E(0, p) = 0, \nabla \cdot B(\mathbf{r}) = i\hat{p} \cdot B(0, p) = 0\). Therefore, when applying the helicity operator \(i\hat{p} \times\) to \(F_{\pm} (0, p)\), the electric field \(E(0, p)\) vanishes (see Tab. \([1]\))

\[
i\hat{p} \times F_{\pm} (0, p) = i\hat{p} \times \sqrt{\frac{\epsilon_0}{2}} [E(0, p) \pm i\epsilon_0 B(0, p)]
\]

(25)

and we see that the static \(F_{\pm} (0, p)\) are not helicity eigenstates, in contrast to the dynamic case. Now, the purely transverse \(B(0, p)\) in Eq. \((25)\) can be decomposed into two pieces of well-defined and opposite helicity \(\lambda = \pm 1\), with an obvious action of \(i\hat{p} \times\) on each of them (Tab. \([1]\))

\[
B(0, p) = B_+(0, p) + B_-(0, p) \quad \Rightarrow \\
i\hat{p} \times B(0, p) = B_+(0, p) - B_-(0, p),
\]

(26)

with which Eq. \((25)\) changes into

\[
i\hat{p} \times F_{\pm} (0, p) = \pm i \sqrt{\frac{1}{2\mu_0}} [B_+(0, p) - B_-(0, p)].
\]

(27)

Using Eq. \((25)\), Eq. \((27)\), and Tab. \([1]\) we can readily see that

\[
F_{\pm} (0, p) |i\hat{p} \times F_{\pm} (0, p) \rangle = \\
= \frac{\sqrt{\epsilon_0}}{2} B(0, p) \sqrt{\frac{1}{2\mu_0}} i\hat{p} \times B(0, p)
\]

(28)

which we can substitute in Eq. \((23)\)

\[
\langle A \rangle = \langle A_{w>0} \rangle + \langle A_{w=0} \rangle = \\
\int_{\mathbb{R}^3} \frac{dp}{c_0|p|} [|F_+ (c_0|p|, p)|^2 - |F_- (c_0|p|, p)|^2 + \\
\int_{\mathbb{R}^3} \frac{dp}{c_0|p|} [B_+(0, p) |i\hat{p} \times B(0, p) \rangle
\]

(29)

The newly introduced contribution of the static field is added to the integrated value of the dynamic electromagnetic helicity. We will now show that \(\langle A_{w=0} \rangle\) is nothing but the magnetic helicity in different units. Let us use the second and third lines in Eq. \((28)\) to write

\[
\langle A_{w=0} \rangle = \\
\int_{\mathbb{R}^3} \frac{dp}{c_0|p|} \frac{1}{2\mu_0} [B_+(0, p) |i\hat{p} \times B(0, p) \rangle
\]

(30)

and work on the expression inside the box. We consider the relationship between the magnetic field and the magnetic vector potential \(B(\mathbf{r}) = \nabla \times A(\mathbf{r})\) in p-domain: \(B(0, p) = i\hat{p} \times A(0, p)\), and operate on both its sides with \(\frac{i\hat{p} \times}{|p|}\) from the left:

\[
i\hat{p} \times B(0, p) = i\hat{p} \times i\hat{p} \times A(0, p) = (i\hat{p} \times)^2 A(0, p)
\]

Tab. \([1]\)

\[
A^\pm (0, p) = A(0, p) - A^\pm (p) = A(0, p) - \hat{p} [\hat{p} \cdot A(0, p)]
\]

(31)

5

\[
F_{\pm} (0, p) |i\hat{p} \times F_{\pm} (0, p) \rangle
eq\frac{1}{2\mu_0} [B_+(0, p) + B_-(0, p)] |i\hat{p} \times B(0, p) \rangle
\]

(23)

\[
\sqrt{\frac{\epsilon_0}{2}} |E(0, p) \pm \i\epsilon_0 B(0, p)| |i\hat{p} \times B(0, p) \rangle
\]

= \sqrt{\frac{\epsilon_0}{2}} |E(0, p)| |i\hat{p} \times B(0, p) \rangle

+ \frac{\sqrt{\epsilon_0}}{2} c_0 B(0, p) \sqrt{\frac{1}{2\mu_0}} i\hat{p} \times B(0, p)
\]

Tab. \([1]\)

\[
\frac{1}{2\mu_0} [B_+(0, p) + B_-(0, p)] [B_+(0, p) - B_-(0, p)]
\]

Tab. \([1]\)

\[
\frac{1}{2\mu_0} [B_+(0, p)^2 - |B_-(0, p)|^2]
\]
We substitute the last expression in Eq. (31) into the box in Eq. (30) to get:

\[
\langle \Lambda_{\omega=0} \rangle = \int_{\mathbb{R}^3} \frac{dp}{2\pi} \mathbf{B}(0, p) \cdot \{ \mathbf{A}(0, p) - \hat{p} \cdot \mathbf{A}(0, p) \}
\]

\[
= \int_{\mathbb{R}^3} \frac{dp}{2\pi} \mathbf{B}(0, p) \cdot \mathbf{A}(0, p),
\]

(32)

where the equality follows because the longitudinal field of the static magnetic field is canceled by the projection with the transverse field of the static magnetization. It readily follows from Eq. (24) that the longitudinal part of the static magnetization is given by

\[
\mathbf{B}(0, p) = \mu_0 \mathbf{M}^z(0, p),
\]

showing that helicity can be stored in the transverse part of the magnetization. We can rewrite Eq. (29) as

\[
\langle \Lambda \rangle = \int_{\mathbb{R}^3} \frac{dp}{2\pi} |\mathbf{F}_+(c_0 |p|, p)|^2 - |\mathbf{F}_-(c_0 |p|, p)|^2
\]

\[+
\int_{\mathbb{R}^3} \frac{dp}{2\mu_0} |\mathbf{M}_+(0, p)|^2 - |\mathbf{M}_-(0, p)|^2
\]

(36)

The definition of helicity in Eqs. (23, 29, 36) unifies the static magnetic helicity and the dynamic electromagnetic helicity into a single quantity. This total helicity is the sum of two terms that measure the difference between the number of left-handed and right-handed photons of the free field, and the screwiness of the static magnetization, respectively. While the magnetic and electromagnetic helicities have previously been discussed together [6, 7, 23], they have, as far as I know, not been unified into a single physical property until now. According to this unification, the static and dynamic helicities are two manifestations of the same fundamental quantity. As such, they are susceptible to change into each other, giving a positive answer to the question of whether helicity can be exchanged between light and matter. Regarding the question of the effects of such exchange: Equation (36) indicates that, systems able to sustain static magnetization states with some degree of screwiness, have the potential for storing electromagnetic helicity coming from the free dynamic field. Such storage implies a modification of the transverse part of the initial static magnetization. Conversely, such systems have the potential for returning the stored helicity to the free field by means of electromagnetic radiation, with a corresponding change in their static magnetization state.

V. AN UPPER BOUND ON HELICITY STORAGE IN LONG-RANGE MAXIMALLY-HELICAL MAGNETIZATION DISTRIBUTIONS

To finalize, we will obtain an upper bound for the amount of helicity that a given system can store. To such end, we consider the following integral involving the polarization tensor in Eq. (12):

\[
S^2 = \int_{\mathbb{R}^3} d\mathbf{r} \Sigma(\mathbf{r}, \nu) \Sigma(\mathbf{r}, \nu) = \int_{\mathbb{R}^3} d\mathbf{r} |\mathbf{M}(\mathbf{r})|^2 = \int_{\mathbb{R}^3} dp \ |\mathbf{M}(0, p)|^2
\]

which is a relativistic invariant of \(\Sigma\), and reflects the total spin square of the material system. The available \(S^2\) budget is split between the longitudinal and transverse parts of \(\mathbf{M}(0, p)\). Considering that, according to Eq. (36), the longitudinal part does not contribute to the stored helicity, which is all contained in the transverse part, the question arises regarding the role of the longitudinal part...
of the static magnetization density. The answer is that it stores the magnetostatic self-energy. To see this, let us consider the definition of magnetostatic self-energy ([34 Ch. 3, § 2.2, [35 Problem 5.21])

\[ W_m = -\frac{\mu_0}{2} \int_{\mathbb{R}^3} dr \, \mathbf{M}(r) \cdot \mathbf{H}(r), \quad (38) \]

and show that such energy depends only on \( \mathbf{M} \). In static equilibrium, the \( \mathbf{H}(r) \) field in Eq. (41) is defined as in [34 Eq. (2.41)], or [35 Eqs. (5.81,5.82)], albeit setting \( J_c(r) = 0 \)

\[ \mathbf{H}(\mathbf{r}) = \frac{\mathbf{B}(\mathbf{r})}{\mu_0} - \mathbf{M}(\mathbf{r}), \quad \nabla \cdot \mathbf{H}(\mathbf{r}) = -\nabla \cdot \mathbf{M}(\mathbf{r}), \quad \nabla \times \mathbf{H}(\mathbf{r}) = 0, \quad (39) \]

which, in \( \mathbf{p} \)-domain, read

\[ \mathbf{H}(0, \mathbf{p}) = \frac{\mathbf{B}(0, \mathbf{p})}{\mu_0} - \mathbf{M}(0, \mathbf{p}), \quad (40) \]

\[ i\mathbf{p} \cdot \mathbf{H}(0, \mathbf{p}) = -i\mathbf{p} \cdot \mathbf{M}(0, \mathbf{p}), \quad i\mathbf{p} \times \mathbf{H}(0, \mathbf{p}) = 0, \]

which can be readily shown to imply\(^6\) that \( \mathbf{H}(0, \mathbf{p}) = -\mathbf{M}^\parallel(0, \mathbf{p}) \). We can use this to work on Eq. (38) and show that \( W_m \) depends only on \( \mathbf{M}^\parallel(0, \mathbf{p}) \):

\[ W_m = -\frac{\mu_0}{2} \int_{\mathbb{R}^3} dr \, \mathbf{M}(r) \cdot \mathbf{H}(r) = -\frac{\mu_0}{2} \int_{\mathbb{R}^3} d\mathbf{p} \, \mathbf{M}^\parallel(0, \mathbf{p}) \mathbf{H}(0, \mathbf{p}) = \frac{\mu_0}{2} \int_{\mathbb{R}^3} d\mathbf{p} \, \mathbf{M}^\parallel(0, \mathbf{p}) \mathbf{M}^\parallel(0, \mathbf{p}) \quad \text{Tab. I}, \quad (41) \]

We now use Eq. (41) and the last line of Eq. (37) to obtain an equation for the total size of the transverse magnetization involving the self-energy, which we write together with the stored helicity

\[ \frac{\mu_0}{2} S^2 - W_m = \frac{\mu_0}{2} \int_{\mathbb{R}^3} d\mathbf{p} \, |\mathbf{M}_+(0, \mathbf{p})|^2 + |\mathbf{M}_-(0, \mathbf{p})|^2, \]

\[ \langle \Lambda_{\omega=0} \rangle = \frac{\mu_0}{2} \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{c_0|\mathbf{p}|} \, |\mathbf{M}_+(0, \mathbf{p})|^2 - |\mathbf{M}_-(0, \mathbf{p})|^2. \quad (42) \]

According to Eq. (42), fixing \( S^2 \) and \( W_m \) fixes the size of the transverse magnetization. For a fixed size of the transverse magnetization, the system will be able to store a larger absolute value of helicity when either \( \mathbf{M}_+(0, \mathbf{p}) \) or \( \mathbf{M}_-(0, \mathbf{p}) \) vanishes, and the non-vanishing component is contained in transverse modes with the smallest possible \( |\mathbf{p}| \). That is, long-range maximally helical magnetization distributions in static equilibrium. If we consider a hypothetical single mode with fixed \( |\mathbf{p}| \), the ultimate bound for the absolute value of stored helicity \( |\langle \Lambda \rangle| \) is set by \( S^2 \) when \( W_m = 0 \):

\[ |\langle \Lambda \rangle| = \frac{\mu_0}{2} \frac{\mathbf{M}^\parallel(0, \mathbf{p})|^2}{c_0|\mathbf{p}|} = \frac{\mu_0}{2} S^2 - W_m \leq \frac{\mu_0}{2} S^2 - \frac{\mu_0 S^2}{2c_0|\mathbf{p}|}. \quad (43) \]

Given a linear size of the system \( L \), a \( L^4 \) scaling of the bound in Eq. (43) can be argued as follows. The total \( S^2 \) can be expected to scale proportionally to the total number of spins, which should grow as \( L^3 \) in 3D systems. The additional factor of \( L \) comes from the value of the smallest possible \( |\mathbf{p}| \), which should scale as \( L^{-1} \).

VI. CONCLUSIONS AND DISCUSSION

In conclusion, this article shows that the electromagnetic helicity of the free electromagnetic field, and the magnetic helicity of the static magnetization are two different parts of the same physical quantity, the total helicity. The total helicity is the sum of two terms. One term quantifies the screwiness of the static magnetization in matter, and the other quantifies the difference between the number of left- and right-handed photons in the free electromagnetic field. The unification provides the theoretical basis for studying the conversion between these two embodiments of helicity upon light-matter interaction.

Both kinds of helicity are separately relevant in quite diverse areas of physics. Electromagnetic helicity is particularly relevant in chiral light-matter interactions, and magnetic helicity is relevant in areas like cosmology [41, 42], solar physics [43], fusion physics [44, 45], magneto-hydrodynamics [46–48], and condensed matter [49], in particular regarding helical magnets and skyrmions [50, 51]. Consequently, their unified understanding has the potential for impacting several different fields. For example, the new link between optics and magnetism is apparently relevant for the physics of all optical switching of magnetization with circularly polarized radiation [29], and for the optical control of helical magnets and skyrmions [30, 31, 32]. I believe that fusion physics, where the injection of magnetic helicity is considered for controlling the plasma [47, 48], and cosmology, where helical magnetic fields with galactic-scale coherent lengths are considered [41, 45] for explaining parity violation and matter-antimatter imbalance in the universe, are other areas where the results of this paper could be useful.
