A Geometric Criterion for the Existence of Chaos Based on Periodic Orbits in Continuous-Time Autonomous Systems

Xu Zhang1 · Guanrong Chen2

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Abstract
A new geometric criterion is derived for the existence of chaos in continuous-time autonomous systems in three-dimensional Euclidean spaces, where a type of Smale horseshoe in a subshift of finite type exists, but the intersection of stable and unstable manifolds of two points on a hyperbolic periodic orbit does not imply the existence of a Smale horseshoe of the same type on any cross section of these two points. This criterion is based on the existence of a hyperbolic periodic orbit, differing from the classical equilibrium-based Shilnikov criterion and the condition of transversal homoclinic or heteroclinic orbit of a Poincaré map.

Keywords Chaos · Hyperbolic periodic orbit · Smale horseshoe · Stable/Unstable manifolds · Subshift of finite type

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1 Introduction

Consider a continuous-time autonomous system described by an ordinary differential equation \( \dot{x} = \Phi(x), \ x \in \mathbb{R}^3 \), where \( \Phi : U \rightarrow \mathbb{R}^3 \) is \( C^r \) on some open set \( U \subseteq \mathbb{R}^3 \). An equilibrium point \( q \) is the state satisfying \( \Phi(q) = 0 \), that is, \( x = q \) is a solution for all \( t \). If the eigenvalues of the Jacobian matrix of the system at the equilibrium have non-zero real parts, namely there is no center manifold, then the equilibrium is called hyperbolic, which can be classified as node, saddle, node-focus, and saddle-focus. Typical hyperbolic chaotic
systems include the Lorenz system [13] and the Chen system [3] which, with the typical parameter values, have two saddle-foci and one unstable saddle. Another typical example is the generalized Lorenz system with multi-stability, where two stable equilibria could exist [11]. For hyperbolic equilibria, there are many well-known criteria on the existence of chaos. In the study of continuous-time autonomous systems, they could be simplified so as to study suitably defined Poincaré maps on cross sections. Since a Poincaré map represents a discrete dynamical system, many powerful tools could be utilized, such as the Smale horseshoe [26], the Smale-Birkhoff Theorem (the existence of a transversal homoclinic orbit) [22], and the existence of transversal heteroclinic orbits [1], to show the existence of chaos in the system. Besides, the Shilnikov criterion [23–25, 30] and the Melnikov method [15] are useful tools for proving the existence of chaos in a continuous-time autonomous system.

On the other hand, there are some autonomous systems without hyperbolic equilibrium points in three-dimensional spaces, but these systems have chaotic attractors discovered by numerical experiments. In climate systems, ecosystems, financial markets, engineering applications, and mechanical and electromechanical systems, there often exist more than one attractor, which is referred to as multi-stability. The multi-stability is a typical property of systems without hyperbolic equilibrium points [4]. For example, a mechanical system, discovered by Sommerfeld [5, 27], has oscillations caused by a motor driving an unbalanced weight and resonance capture (Sommerfeld effect), which captures the failure of the rotating system due to the resonant interactions. Other examples include a double-mass mathematical model of the drilling system studied in [16] and the Rabinovic system describing the interactions of three resonantly coupled plasma waves [20, 21].

There are some other interesting mathematical models without hyperbolic equilibrium points: a chaotic Chua’s circuit [10], some rare flows with chaotic attractors but no equilibrium [28], a chaotic autonomous system with a line of equilibria [17], a chaotic system with a surface of equilibria [8], a chaotic system with one and only one stable equilibrium [29], and some others [7]. For these systems with chaotic attractors, their equilibria might be stable, or may not even exist; therefore, many classical tools such as the Shilnikov criterion are not applicable to describe their chaotic dynamics. The classical Smale-Birkhoff Theorem, or the existence of a transversal heteroclinic orbit, requires the strong assumption of transversal dynamics, which is difficult to verify in real applications (see Section 3.1 below for more detailed discussions).

An interesting problem is the mechanism for the existence of chaotic attractors in continuous-time autonomous systems without hyperbolic equilibrium points. In this paper, a geometric criterion is derived to describe the existence of chaos in such systems, revealing the chaos-forming mechanism. Specifically, some chaotic dynamics are shown to have a Smale horseshoe in a subshift of finite type, and the classical intersection mechanism of stable and unstable manifolds of two points on a hyperbolic periodic orbit does not imply the existence of a Smale horseshoe of the same type on cross sections of these two points (see Remark 3.4 and Theorem 3.4 in Section 3).

The rest of the paper is organized as follows. In Section 2, some basic concepts and useful preliminaries are introduced. In Section 3, the complexity of the dynamics of autonomous systems without hyperbolic equilibrium points is studied. This section is divided into three parts. In the first subsection, the classical transversal homoclinic or heteroclinic orbits are applied to explain the complexity of the dynamics. In the second subsection, a topological model is established for the Smale horseshoe in a subshift of finite type with a particular transition matrix. In the third subsection, a geometric criterion is derived, where some new dynamics are observed with a Smale horseshoe in a subshift of finite type.
2 Basic Concepts and Preliminaries

First, recall the symbolic dynamics [22]. Let \( m \geq 2 \) be an integer, \( S_0 = \{1, 2, ..., m\} \), and

\[
\sum_m := \{ \alpha = (..., a_{-2}, a_{-1}, a_0, a_1, ...) : a_i \in S_0, \ i \in \mathbb{Z} \}
\]

be the two-sided sequence space. For any \( \alpha = (..., a_{-1}, a_0, a_1, ...) \) and \( \beta = (..., b_{-1}, b_0, b_1, ...) \in \sum_m \), the distance between them is

\[
d(\alpha, \beta) = \sum_{i=-\infty}^{\infty} \frac{d(a_i, b_i)}{2^{|i|}}, \ d(a_i, b_i) = \left\{ \begin{array}{ll} 1, & \text{if } a_i \neq b_i \\ 0, & \text{if } a_i = b_i, \ i \in \mathbb{Z}. \end{array} \right.
\]

The shift map \( \sigma : \sum_m \to \sum_m \) is defined by \( \sigma(\alpha) = (..., b_{-2}, b_{-1}, b_0, b_1, ...) \), where \( \alpha = (..., a_{-2}, a_{-1}, a_0, a_1, a_2...,) \in \sum_m \) and \( b_i = a_{i+1}, \ i \in \mathbb{Z} \). The system \((\sum_m, \sigma)\) is called a two-sided symbolic dynamical system on \( m \) symbols, or simply two-sided fullshift on \( m \) symbols. A matrix \( A = (a_{ij})_{m \times m} \) is called a transition matrix if \( a_{ij} = 0 \) or 1 for all \( 1 \leq i, j \leq m \). For a transition matrix \( A \), define

\[
\sum_m(A) := \{ \beta = (..., b_{-2}, b_{-1}, b_0, b_1, ...) \in \sum_m : a_{b_ib_{i+1}} = 1, \ i \in \mathbb{Z} \}.
\]

The map \( \sigma_A := \sigma \mid \sum_m(A) : \sum_m(A) \to \sum_m(A) \) is called the two-sided subshift of finite type with matrix \( A \).

Lemma 2.1 [32, Lemma 3.1] The topological entropy for the subshift map \( \sigma_A : \sum_4(A) \to \sum_4(A) \) is \( \log 2 > 0 \), where

\[
A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.
\]

Definition 2.1 [30, Definition 11.0.1] Consider two \( C^r \) diffeomorphisms \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \to \mathbb{R}^n \). \( f \) and \( g \) are said to be \( C^k \) conjugate \((k \leq r)\) if there exists a \( C^k \) diffeomorphism \( h : \mathbb{R}^n \to \mathbb{R}^n \) such that \( g \circ h = h \circ f \). If \( k = 0 \), \( f \) and \( g \) are said to be topologically conjugate.

Remark 2.1 The underlying space \( \mathbb{R}^n \) in the above definition of conjugacy can be extended to subsets of \( \mathbb{R}^n \), manifolds, or general metric spaces.

Next, recall the classical Smale horseshoe map. Consider a square, denoted by \( U \), which is a compact subset on a two-dimensional manifold. A horseshoe map \( F \) is constructed as follows. The action of the map is defined geometrically by contracting the square along one direction, then stretching the result into a long strip along the perpendicular direction, and finally folding the strip into the shape of a horseshoe, where \( F(U) \cap U \neq \emptyset \). This operation is repeated infinitely many times. An invariant set is formed by \( \Lambda = \bigcap_{i \in \mathbb{Z}} F^i(U) \), and the dynamical behavior on this invariant set is described by the topological conjugacy of two-sided fullshift on two symbols [22].
Now, introduce the Smale horseshoe in a subshift of finite type [22]. For brevity, only a special case is discussed, which will be used in the sequel.

Consider two squares on a two-dimensional manifold, denoted by $U_1$ and $U_2$, respectively, with empty intersection. The horseshoe map $F$ defined on $U_1 \cup U_2$ is obtained as follows. The action of the map is defined geometrically by contracting the two squares along the same direction, then stretching the results into two long strips along the perpendicular direction, and finally folding the two strips into the shape of two horseshoes. $F(U_1)$ and $U_2$ contribute to a horseshoe, and $F(U_2)$ and $U_1$ to another horseshoe. Figure 1 illustrates the Smale horseshoe in a subshift of finite type with the matrix $A$ defined in Eq. 2.1, where $U_1$ and $F(U_1)$ are represented by green colors, and $U_2$ and $F(U_2)$ by yellow colors. Note that $F$ is contracting along the horizontal direction and expanding along the vertical direction, where $F(U_1)$ and $U_2$ form a horseshoe, and $F(U_2)$ and $U_1$ form another horseshoe. The set defined by $\Lambda = \cap_{i \in \mathbb{Z}} F^i (U_1 \cup U_2)$ is invariant under the map, and the dynamics on this invariant set are described by the topological conjugacy with the two-sided subshift of finite type with matrix $A$.

3 Chaotic Dynamics of Autonomous Systems Without Hyperbolic Equilibria

In this section, the complexity of the dynamics of autonomous systems without hyperbolic equilibria is investigated in three parts. For convenience, consider only systems in three-dimensional Euclidean spaces, but higher-dimensional cases and even differential equations defined on smooth manifolds can be similarly discussed.
Consider an ordinary differential equation, \( \dot{x} = \Phi_1(x) \), \( x \in \mathbb{R}^3 \), where \( \Phi : U \to \mathbb{R}^3 \) is \( C^r \) on some open set \( U \subset \mathbb{R}^3 \). Let \( \phi(t, \cdot) \) be a flow generated by this differential equation. For \( x_0 \in \mathbb{R}^3 \), the flow \( \phi(t, x_0) \) is the solution to the initial value problem \( \dot{x} = \Phi_1(x) \) with \( x(0) = x_0 \). Suppose that this equation has a periodic solution of period \( T > 0 \), denoted also by \( \phi(t, x_0) \), where \( x_0 \) is now any point through which this periodic solution passes, namely \( \phi(t + T, x_0) = \phi(t, x_0) \). Consider moreover a two-dimensional surface \( \Sigma \) transversal to the vector field at \( x_0 \), where “transversal” means that \( \Phi_1(x) \cdot n(x) \neq 0 \) with \( n(x) \) being the normal to \( \Sigma \) and “\( \cdot \)” denoting the vector inner product. The surface \( \Sigma \) is called a cross section to the vector field.

It is noted that, if \( \Phi_1(x) \) is \( C^r \), then \( \phi(t, x) \) is \( C^r \) (Theorem 7.1.1 in [30]). Thus, there is an open subset \( V \subset \Sigma \) such that the orbits starting in \( V \) will return to \( \Sigma \) in a time close to \( T \). The associate Poincaré map is the image of the points in \( V \) with their first returns to \( \Sigma \), namely,

\[
P : V \to \Sigma
x \to \phi(\tau(x), x).
\] (3.1)

It is clear that \( \tau(x_0) = T \) and \( P(x_0) = x_0 \).

**Remark 3.1** There are several different definitions of chaos, for example, Li-Yorke chaos, Devaney chaos, positive entropy, or positive Lyapunov exponents. In this paper, a continuous system is said to be chaotic if there is a cross section on which the Poincaré map has positive topological entropy.

### 3.1 Transversal Homoclinic/Heteroclinic Orbits

In this subsection, the classical homoclinic or heteroclinic orbits are applied to explain the existence of complexity of the dynamics in continuous-time autonomous systems with hidden attractors. Here, the assumption of the existence of transversal homoclinic or heteroclinic orbits is needed.

Consider a periodic orbit of the system and a cross section of a Poincaré map containing two points \( p \) and \( q \) on this periodic orbit. Thus, these two points correspond to a periodic orbit with period 2 on the Poincaré map denoted by \( P \). Furthermore, if there is a transversal homoclinic orbit corresponding to this periodic orbit, then the following Smale-Birkhoff Theorem could be applied to show the existence of chaos in the system.

**Theorem 3.1** [22, Smale-Birkhoff Theorem] *Suppose that \( q \) is a transversal homoclinic point corresponding to a hyperbolic periodic point \( p \) of a diffeomorphism \( f \). For each neighborhood \( U \) of \( \{ p, q \} \), there is a positive integer \( n \) such that \( f^n \) has a hyperbolic invariant set \( \Lambda \subset U \), with \( p, q \in \Lambda \), on which \( f^n \) is topologically conjugate to the two-sided fullshift map on two symbols.*

Now, assume that there are \( m \) periodic orbits in the autonomous system, and there is a cross section for a Poincaré map containing one point from each of these \( m \) periodic orbits. Clearly, these points are fixed points of the Poincaré map. If there are transversal heteroclinic orbits with these fixed points, then the following results on the transversal heteroclinic orbits could be applied.
Theorem 3.2 [1, Theorem 2.3.1] If a diffeomorphism \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) possesses \( m \) fixed points, \( p_1, ..., p_m \), which are non-degenerate hyperbolic saddle points, and if there exist points \( q_i \) at which the unstable manifold \( W^u(p_i) \) intersects the stable manifold \( W^s(p_{i+1} \mod m) \) transversally for all \( i \), then \( f \) possesses an invariant set on which some iteration \( f^k \) is topologically conjugate to the fullshift on \( m \) symbols.

Remark 3.2 The classical Shilnikov criterion [23–25, 30] does not need to consider the Poincaré map, but it requires the existence of a saddle-focus fixed point for the continuous-time autonomous system, which means that this criterion works only for self-excited systems but not for systems with hidden attractors [10].

3.2 A Topological Criterion for the Existence of a Smale Horseshoe in a Subshift of Finite Type

In this subsection, a topological criterion is established for the existence of Smale horseshoe in a subshift of finite type with matrix \( A \), which is a transition matrix introduced in Eq. 2.1. A similar criterion could be derived for other transition matrices. This is a direct extension of the classical Conley-Moser condition [18, 30].

Definition 3.1 [30, Definition 25.1.1] Consider a region \([a, b] \times [c, d] \subset \mathbb{R}^2\), where \( b - a = 1 \) and \( d - c = 1 \). A \( \mu_v \)-vertical curve is the graph of a function \( v(y) \) that satisfies
\[
a \leq v(y) \leq b, \quad |v(y_1) - v(y_2)| \leq \mu_v|y_1 - y_2| \quad \text{for} \quad c \leq y_1, y_2 \leq d.
\]
Similarly, a \( \mu_h \)-horizontal curve is the graph of a function \( h(x) \) that satisfies
\[
c \leq h(x) \leq d, \quad |h(x_1) - h(x_2)| \leq \mu_h|x_1 - x_2| \quad \text{for} \quad a \leq x_1, x_2 \leq b.
\]

Definition 3.2 [30, Definition 25.1.2] Given two non-intersecting \( \mu_v \)-vertical curves, \( v_1(y) < v_2(y) \) and \( y \in [c, d] \), define a \( \mu_v \)-vertical strip by
\[
V = \{(x, y) \in [a, b] \times [c, d] \subset \mathbb{R}^2 : x \in [v_1(y), v_2(y)], y \in [c, d]\}.
\]
Similarly, given two non-intersecting \( \mu_h \)-horizontal curves, \( h_1(x) < h_2(x) \) and \( x \in [a, b] \), define a \( \mu_h \)-horizontal strip by
\[
H = \{(x, y) \in [a, b] \times [c, d] \subset \mathbb{R}^2 : y \in [h_1(x), h_2(x)], x \in [a, b]\}.
\]
The widths of the horizontal and vertical strips are defined respectively as
\[
d(V) = \max_{y \in [c, d]} |v_2(y) - v_1(y)|.
\]
\[
d(H) = \max_{x \in [a, b]} |h_2(x) - h_1(x)|.
\]

Lemma 3.1 [30, Lemma 25.1.3]

(i) If \( V^1 \supseteq V^2 \supseteq \cdots \supseteq V^k \supseteq \cdots \) is a nested sequence of \( \mu_v \)-vertical strips, with \( d(V^k) \to 0 \) as \( k \to \infty \), then \( V^\infty := \cap_k V^k \) is a \( \mu_v \)-vertical curve.

(ii) If \( H^1 \supseteq H^2 \supseteq \cdots \supseteq H^k \supseteq \cdots \) is a nested sequence of \( \mu_h \)-horizontal strips, with \( d(H^k) \to 0 \) as \( k \to \infty \), then \( H^\infty := \cap_k H^k \) is a \( \mu_h \)-horizontal curve.

Lemma 3.2 [30, Lemma 25.1.4] Suppose that \( 0 \leq \mu_v \mu_h < 1 \). Then, a \( \mu_v \)-vertical curve and a \( \mu_h \)-horizontal curve intersect at a unique point.
Now, consider a map \( f : D_1 \cup D_2 \to \mathbb{R}^2 \), where
\[
D_1 = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq -1, \ 0 \leq y \leq 1\}, \\
D_2 = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2, \ 0 \leq y \leq 1\}.
\]
Consider, also, a finite set \( S = \{1, 2, 3, 4\} \), four \( \mu_h \)-horizontal strips, \( H_i^j \subset D_j \), \( 1 \leq i, j \leq 2 \), and four \( \mu_v \)-vertical strips, \( V_i^j \subset D_j \), \( 1 \leq i, j \leq 2 \).

Suppose that \( f \) maps \( H_1^i \) homeomorphically onto \( V_2^i \), and maps \( H_2^i \) homeomorphically onto \( V_1^i \), \( 1 \leq i \leq 2 \). Suppose, moreover, that \( f \) satisfies the following two assumptions.

**Assumption 1** With \( 0 \leq \mu_v \mu_h < 1 \), the horizontal boundaries of \( H_1^i \) are mapped to the horizontal boundaries of \( V_2^i \) and the vertical boundaries of \( H_1^i \) are mapped to the vertical boundaries of \( V_2^i \); the horizontal boundaries of \( H_2^i \) are mapped to the horizontal boundaries of \( V_1^i \) and the vertical boundaries of \( H_2^i \) are mapped to the vertical boundaries of \( V_1^i \).

**Assumption 2** Suppose that \( H \) is a \( \mu_h \)-horizontal strip contained in \( H_1^2 \cup H_2^2 \subset D_2 \), and that
\[
f^{-1}(H) \cap H_1^i = \tilde{H}_1^i, \ 1 \leq i \leq 2,
\]
is a \( \mu_h \)-horizontal strip. Moreover,
\[
d(\tilde{H}_1^i) \leq v_h d(H) \text{ for some } 0 < v_h < 1.
\]
Similarly, suppose that \( V \) is a \( \mu_v \)-vertical strip contained in \( V_1^1 \cup V_2^1 \subset D_1 \). Then,
\[
f(V) \cap V_2^i = \tilde{V}_2^i, \ 1 \leq i \leq 2,
\]
is a \( \mu_v \)-vertical strip. Moreover,
\[
d(\tilde{V}_2^i) \leq v_v d(V) \text{ for some } 0 < v_v < 1.
\]
Similar assumptions apply to \( H \), which is a \( \mu_h \)-horizontal strip contained in \( H_1^1 \cup H_2^1 \subset D_1 \), and \( V \) is a \( \mu_v \)-vertical strip, which is contained in \( V_1^2 \cup V_2^2 \subset D_2 \).

An illustrative diagram is given in Fig. 2, where \( H_1^1 \) and \( H_2^1 \) are in green color in \( D_1 \), and \( V_1^1 \) and \( V_2^1 \) are in yellow color in \( D_1 \); \( H_1^2 \) and \( H_2^2 \) are in yellow color in \( D_2 \), and \( V_1^2 \) and \( V_2^2 \) are in green color in \( D_2 \).

Now, the following result can be established.

**Theorem 3.3** Suppose that \( f \) satisfies Assumptions 1 and 2. Then, \( f \) has an invariant Cantor set \( \Lambda \), on which \( f \) is topologically conjugate to a subshift of finite type with the matrix \( A \) specified in Eq. 2.1, such that the following relations hold:

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{f} & \Lambda \\
\psi & \downarrow & \psi \\
\Sigma_4(A) & \xrightarrow{\sigma_A} & \Sigma_4(A)
\end{array}
\]

where \( \psi \) is a homeomorphism mapping \( \Lambda \) onto \( \Sigma_4(A) \).

**Proof** It follows from arguments similar to the proof of [30, Theorem 25.1.5].
Fig. 2 Illustrative diagram for the horseshoe in a subshift of finite type with matrix $A$, where $H_1^1$ and $H_2^1$ are in green color in $D_1$, and $V_1^1$ and $V_2^1$ are in yellow color in $D_1$; $H_1^2$ and $H_2^2$ are in yellow color in $D_2$, and $V_1^2$ and $V_2^2$ are in green color in $D_2$.

For clarity, an outline of the arguments is provided.

**Step 1. Constructing the invariant set $\Lambda$.**

Let

$$\tilde{H}_1 := H_1^1, \quad \tilde{H}_2 := H_2^1, \quad \tilde{H}_3 := H_1^2, \quad \tilde{H}_4 := H_2^2,$$

and

$$\tilde{V}_1 := V_1^1, \quad \tilde{V}_2 := V_2^1, \quad \tilde{V}_3 := V_1^2, \quad \tilde{V}_4 := V_2^2.$$

The index set is $S = \{1, 2, 3, 4\}$. The following set

$$\bigcap_{i \geq 0} f^i \left( \bigcup_{i \in S} V_i \right)$$

is a collection of an uncountably infinitely many $\nu_v$-vertical curves; the following set

$$\bigcap_{i \leq 0} f^i \left( \bigcup_{i \in S} H_i \right)$$
is a collection of an uncountably infinitely many $\nu_h$-horizontal curves. The intersection of these two sets is the invariant set

$$\Lambda = \left( \bigcap_{i \geq 0} f_i \left( \bigcup_{i \in S} V_i \right) \right) \cap \left( \bigcap_{i \leq 0} f_i \left( \bigcup_{i \in S} H_i \right) \right).$$

**Step 2.** Introducing the conjugate map $\psi : \Lambda \to \Sigma_4(A)$.

For any point $p \in \Lambda$, there are two and only two infinite sequences

$$(s_0, s_1, \cdots, s_k, \cdots), \ s_k \in S, \ k \geq 0,$$

and

$$(\cdots, s_k, s_{k+1}, \cdots, s_{-2}, s_{-1}), \ s_k \in S, \ k < 0,$$

satisfying that $a_{s_k s_{k+1}} = 1$, where $a_{s_k s_{k+1}}$ is an element in the matrix $A$. For these two sequences, introduce two sets

$$V(\cdots, s_k, s_{k+1}, \cdots, s_{-2}, s_{-1}) = \{ z : f^k(z) \in V_{s_k}, \ k < 0 \},$$

and

$$H(s_0, s_1, \cdots) = \{ z : f^k(z) \in H_{s_k}, \ k \geq 0 \}.$$ 

Note that the point $p \in \Lambda$ has the following property:

$$\{ p \} = V(\cdots, s_{-3}, s_{-2}, s_{-1}) \cap H(s_0, s_1, \cdots).$$

The map $\psi$ is defined as $p \mapsto (\cdots, s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3, s_4, s_5, \cdots)$.

**Step 3.** Showing that the map $\psi$ is a homeomorphism.

It suffices to show that $\psi$ is one-to-one, onto, and continuous, implying that the conjugacy map $\psi$ is a homeomorphism.

The arguments of these three properties are based on the construction of the map and the properties of the $\nu_h$-horizontal and the $\nu_v$-vertical curves.

**Step 4.** Verifying the relation $\psi \circ f = \sigma_A \circ \psi$.

This could be derived from the construction of the conjugate map, or the orbit of the systems.

\[ \square \]

### 3.3 A Geometric Criterion for the Existence of Chaos

Consider an ordinary differential equation, $\dot{x} = \Phi(x)$, $x \in \mathbb{R}^3$, where $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is differentiable. For any initial point $x_0 \in \mathbb{R}^3$, let the solution to the corresponding initial value problem be $\phi(t, x_0)$, called a flow and denoted by $\phi^t$ for simplicity.

**Definition 3.3** [22] An invariant set $\Lambda$ for the flow $\phi^t$ defined on a smooth manifold $M$ has a hyperbolic structure, namely $\Lambda$ is a hyperbolic invariant set, provided that

(i) at each point $p$ in $\Lambda$, the tangent space to $M$ can be split as the direct sum of $E^u_p$, $E^s_p$, and $\text{span}(\Phi(p))$:

$$T_p(M) = E^u_p \oplus E^s_p \oplus \text{span}(\Phi(p));$$

(ii) the above splitting is invariant under the action of the derivative, in the sense that

$$D(\phi^t)_p E^u_p = E^u_{\phi^t(p)}, \ D(\phi^t)_p E^s_p = E^s_{\phi^t(p)}, \ D(\phi^t)_p \Phi(p) = \Phi(\phi^t(p));$$

(iii) $E^u_p$ and $E^s_p$ vary continuously with $p$;
(iv) there exist $\mu > 0$ and $C \geq 1$ such that, for any $t \geq 0$,
$$
|D\phi_t v^s| \leq C e^{-\mu t} |v^s| \quad \text{for } v^s \in \mathbb{R}^s,
$$
$$
|D\phi_t v^u| \leq C e^{-\mu t} |v^u| \quad \text{for } v^u \in \mathbb{R}^u.
$$

Furthermore, assume that there is a hyperbolic periodic orbit with period $T > 0$; that is, for any point on the periodic orbit, there exists a cross section passing through this point. Any point on the periodic orbit is a saddle fixed point of the Poincaré map. For illustration of the Poincaré map defined near the hyperbolic periodic orbit, see Fig. 3. (For similar illustration of the Poincaré map defined near the hyperbolic periodic orbit, see Figures 10.1.2 and 10.1.3 in [30].)

Now, take any point $x_0$ on this periodic orbit. Its stable and unstable manifolds are defined as follows:
$$
W^s(x_0) = \left\{ x \in \mathbb{R}^3 : \lim_{t \to +\infty} d(\phi(t, x), \phi(t, x_0)) = 0 \right\}
$$
and
$$
W^u(x_0) = \left\{ x \in \mathbb{R}^3 : \lim_{t \to -\infty} d(\phi(t, x), \phi(t, x_0)) = 0 \right\},
$$
where $d(\cdot, \cdot)$ is a metric induced by the Euclidean norm on $\mathbb{R}^3$. Similarly, its local stable and unstable manifolds are defined by
$$
W^s_{loc}(x_0) = \left\{ x \in \mathbb{R}^3 : d(x, x_0) < r \text{ and } \lim_{t \to +\infty} d(\phi(t, x), \phi(t, x_0)) = 0 \right\}
$$
and
$$
W^u_{loc}(x_0) = \left\{ x \in \mathbb{R}^3 : d(x, x_0) < r \text{ and } \lim_{t \to -\infty} d(\phi(t, x), \phi(t, x_0)) = 0 \right\},
$$
where $r > 0$ is a positive constant. For convenience, $W^s_{loc}(x_0)$ and $W^u_{loc}(x_0)$ represent the local stable and unstable manifolds, both with sufficiently small radii.

![Fig. 3](image-url) Illustrative diagram for the Poincaré map defined near the hyperbolic periodic orbit, where the periodic orbit $\gamma$ is in blue color, the rectangle with red boundary represents the unstable manifold of the periodic orbit, denoted by $W^u(\gamma)$, the rectangle with green boundary represents the stable manifold of the periodic orbit, denoted by $W^s(\gamma)$, the region bounded by the purple curve is the Poincaré section, the origin curve is the orbit starting from the Poincaré section and first returning to the section, the point $x_0$ is the intersection of the periodic orbit $\gamma$ and the Poincaré section, and the local unstable manifold $W^u(x_0)$ and the local stable manifold $W^s(x_0)$ are contained in $W^u(\gamma)$ and $W^s(\gamma)$, respectively.
Actually, the stable and unstable manifolds can be defined by another method. Consider a discrete map induced by the flow \( g = \phi(T, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3 \). It is evident that, for any point \( x_0 \) on this periodic orbit, one has \( \phi(T, x_0) = x_0 \); that is, any point on this periodic orbit is a fixed point. By the assumption of the hyperbolic periodic orbit, for the discrete map \( g \) and any point \( x_0 \) on this periodic orbit, there exists a decomposition of the tangent space, \( E^s_{x_0} \oplus E^u_{x_0} \oplus E^c_{x_0} \), where \( E^s_{x_0} \) is the contraction direction, \( E^u_{x_0} \) is the expansion direction, and \( E^c_{x_0} = \text{span}\Phi(x_0) \) is the center direction (flow direction), and the derivative of the \( g \) along this center direction \( E^c_{x_0} \) is 1. The local stable and unstable manifolds with respect to \( g \) are denoted by \( W^s_{loc}(x_0, g) \) and \( W^u_{loc}(x_0, g) \), respectively, for which the existence of the local stable and unstable manifolds is guaranteed by the Stable Manifold Theorem [22, 30]. The stable and unstable manifolds of \( x_0 \) with respect to the flow \( \phi_t \) can be expressed by

\[
W^s(x_0) = \bigcup_{k \in \mathbb{Z}, k \leq 0} \phi(kT, W^s_{loc}(x_0, g)),
\]

\[
W^u(x_0) = \bigcup_{k \in \mathbb{Z}, k \geq 0} \phi(kT, W^u_{loc}(x_0, g)).
\]

Remark 3.3 For any point \( x_0 \) on the periodic orbit and any cross section containing the point \( x_0 \), suppose that \( x_0 \) is a saddle fixed point of the Poincaré map, denoted by \( P \). Then, there exist local stable and unstable manifolds of \( x_0 \) with respect to the Poincaré map, denoted by \( W^s_{loc}(x_0, P) \) and \( W^u_{loc}(x_0, P) \), respectively. However, \( W^s_{loc}(x_0, P) \) and \( W^u_{loc}(x_0, P) \) cannot be used to define the stable and unstable manifolds of \( x_0 \) with respect to the flow \( \phi^t \), because in the present case, in the Poincaré map defined by Eq. 3.1, \( \tau(x) \) might not be equal to \( T \). Yet, to define the stable and unstable manifolds of \( x_0 \) with respect to the flow \( \phi^t \), it is required that \( \tau(x) = T \); that is, the return time should be equal for all points on the same cross section.

Now, the following main result is obtained.

Theorem 3.4 Consider an ordinary differential equation, \( \dot{x} = \Phi(x), x \in \mathbb{R}^3 \), where \( \Phi : \mathbb{R}^3 \to \mathbb{R}^3 \) is differentiable. Assume that

- There exist a hyperbolic periodic orbit of period \( T > 0 \), and two points \( p \) and \( q \) on this periodic orbit with \( \phi \left( \frac{T}{2}, p \right) = q \) and \( \phi \left( \frac{T}{2}, q \right) = p \);
- there are an open subset \( \Upsilon_1 \subset W^u_{loc}(p) \times W^s_{loc}(p) \) containing a line segment of \( W^u_{loc}(p) \), an open subset \( \Upsilon_2 \subset W^u_{loc}(q) \times W^s_{loc}(q) \) containing a line segment of \( W^u_{loc}(q) \), and two positive integers \( m_p \) and \( m_q \) such that \( \phi \left( \frac{T}{2} + m_p T, \Upsilon_1 \right) \) contains a segment of \( W^s_{loc}(q) \) with \( \phi \left( \frac{T}{2} + m_p T, \Upsilon_1 \right) \subset W^u_{loc}(q) \times W^s_{loc}(q) \), and \( \phi \left( \frac{T}{2} + m_q T, \Upsilon_2 \right) \) contains a segment of \( W^s_{loc}(p) \) with \( \phi \left( \frac{T}{2} + m_q T, \Upsilon_2 \right) \subset W^u_{loc}(p) \times W^s_{loc}(p) \).

Then, there exist a positive integer \( m \) and an invariant set \( \Lambda \subset \mathbb{R}^3 \) such that the following relations hold:

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{\phi \left( \frac{T}{2} + mT, \cdot \right)} & \Lambda \\
\Psi & \downarrow & \Psi \\
\sum_4(A) & \xrightarrow{\sigma_A} & \sum_4(A)
\end{array}
\]

where \( \Psi \) is a homeomorphism from \( \Lambda \) to \( \sum_4(A) \), which is a topological conjugacy.
Remark 3.4 The conventional assumption that $W^u(p) \cap W^s(q) \neq \emptyset$ and $W^s(p) \cap W^u(q) \neq \emptyset$ does not imply the existence of Smale horseshoe in this situation.

Suppose that there is $p^* \in W^u(p) \cap W^s(q)$. Since $p$ and $q$ are on the periodic orbit of period $T$, one has $\lim_{k \to -\infty} d(\phi(kT, p), \phi(kT, p^*)) = 0$. This, together with $\phi(kT, p) = p$, implies that $\lim_{k \to -\infty} \phi(kT, p^*) = p$. Similarly, $\lim_{k \to +\infty} \phi(kT, p^*) = q$. Therefore, this horseshoe structure could not be obtained by the conventional assumptions. On the other hand, the conventional assumptions might imply the existence of the classical Smale horseshoes.

Remark 3.5 A simplified model for a Smale horseshoe is illustrated by Fig. 1. For a particular discrete system with a Smale horseshoe in a subshift of finite type with matrix $A$, see [32].

Remark 3.6 In Theorem 3.4, a Smale horseshoe in a subshift of finite type with a larger topological entropy might be obtained under further assumptions. For example, Figs. 4, 5, and 6 provide possible horseshoe structures with larger entropy, for which the corresponding transition matrix is represented in the following form:

$$
\begin{pmatrix}
m & n \\
m & 0_{m \times m} & 1_{m \times n} \\
n & 1_{n \times m} & 0_{n \times n}
\end{pmatrix},
$$

where $0_{m \times m}$ is an $m \times m$ zero matrix, and $1_{m \times n}$ is an $m \times n$ matrix with all the element 1.

For chaotic dynamics of discrete systems with irreducible transition matrices with one row-sum no less than 2, see [33, 34].

Remark 3.7 Continuous-time autonomous dynamical systems can be classified into two types according to their attractors: self-excited systems and hidden-attraction systems. For a system, if its basin of attraction intersects arbitrarily small neighborhoods of an existing equilibrium, it is called self-excited; otherwise, namely if its basin of attraction does not intersect a small neighborhood of an existing equilibrium, it is called hidden [12]. For example, the chaotic Lorenz system [13] and Chen system [3] with the typical parameter values are self-excited systems, and there are some other systems with hidden attractors [4, 7].
Maps with hidden dynamics could be similarly defined. Some hidden attractors in one-dimensional maps were obtained in [6] by extending the Logistic map. A class of two-dimensional quadratic maps with hidden dynamics was studied in [9]. A one-dimensional and a two-dimensional generalized Hénon map with hidden dynamics were studied in [32].

For systems with hidden attractors, there are some without equilibria, and there are some with stable equilibrium, as summarized in the survey [4]. For these systems, the classical chaos criteria based on equilibrium cannot be applied. Since many systems might have periodic orbits, Theorem 3.4 above can be used to study such chaotic dynamics of autonomous systems with hidden chaotic attractors.

The following detailed analysis of chaotic dynamics is divided into two steps:

1. illustrating the vector field on a neighborhood of the hyperbolic periodic orbit;
2. analyzing the existence of a Smale horseshoe in a subshift of finite type with matrix $A$ (this part is in the proof of Theorem 3.4).
Step 1. Illustrating the vector field on a neighborhood of the hyperbolic periodic orbit (not an equilibrium).

First, two examples with periodic orbits are provided to illustrate a system with a saddle-focus periodic orbit.

Example 1 Consider the following local representation near a periodic orbit:

\[
\begin{align*}
\dot{x} &= -y + (x^2 + y^2 - 1)F_1(x, y, z) := G_1(x, y, z), \\
\dot{y} &= x + (x^2 + y^2 - 1)F_2(x, y, z) := G_2(x, y, z), \\
\dot{z} &= (z + x^2 + y^2 - 1)F_3(x, y, z) := G_3(x, y, z).
\end{align*}
\]

It is evident that \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\} is a periodic solution to this system. Next, it is shown that under certain conditions it has a saddle-focus near this periodic orbit.

By direct calculation, one has

\[
\frac{\partial G_1}{\partial x} = 2xF_1+(x^2+y^2-1)\frac{\partial F_1}{\partial x}, \quad \frac{\partial G_1}{\partial y} = -1+2yF_1+(x^2+y^2-1)\frac{\partial F_1}{\partial y}, \quad \frac{\partial G_1}{\partial z} = (x^2+y^2-1)\frac{\partial F_1}{\partial z},
\]

\[
\frac{\partial G_2}{\partial x} = 1+2xF_2+(x^2+y^2-1)\frac{\partial F_2}{\partial x}, \quad \frac{\partial G_2}{\partial y} = 2yF_2+(x^2+y^2-1)\frac{\partial F_2}{\partial y}, \quad \frac{\partial G_2}{\partial z} = (x^2+y^2-1)\frac{\partial F_2}{\partial z},
\]

\[
\frac{\partial G_3}{\partial x} = 2xF_3+(z+x^2+y^2-1)\frac{\partial F_3}{\partial x}, \quad \frac{\partial G_3}{\partial y} = 2yF_3+(z+x^2+y^2-1)\frac{\partial F_3}{\partial y}, \quad \frac{\partial G_3}{\partial z} = (z+x^2+y^2-1)\frac{\partial F_3}{\partial z}.
\]

So, the Jacobian on this periodic orbit is

\[
\begin{pmatrix}
\frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y} & \frac{\partial G_1}{\partial z} \\
\frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} & \frac{\partial G_2}{\partial z} \\
\frac{\partial G_3}{\partial x} & \frac{\partial G_3}{\partial y} & \frac{\partial G_3}{\partial z}
\end{pmatrix} = \begin{pmatrix}
2xF_1 & -1+2yF_1 & 0 \\
1+2xF_2 & 2yF_2 & 0 \\
2xF_3 & 2yF_3 & F_3
\end{pmatrix}.
\]

So, the eigenvalues are the solutions to the equation

\[
\begin{vmatrix}
2xF_1 - \lambda & -1+2yF_1 & 0 \\
1+2xF_2 & 2yF_2 - \lambda & 0 \\
2xF_3 & 2yF_3 & F_3 - \lambda
\end{vmatrix} = (\lambda^2 + (-2yF_2 - 2xF_1)\lambda + (1+2xF_2 - 2yF_1))(F_3 - \lambda) = 0.
\]

It is obvious that one eigenvalue is \(F_3\), and the other two eigenvalues are the solutions to the quadratic polynomial \(\lambda^2 + (-2yF_2 - 2xF_1)\lambda + (1+2xF_2 - 2yF_1) = 0\). If the following inequalities hold (along the periodic orbit):

- \(F_3\) is real and \(F_3 > 0\),
- \(-2yF_2 - 2xF_1 > 0\),
- \((-2yF_2 - 2xF_1)^2 - 4(1+2xF_2 - 2yF_1) < 0\),

then this periodic orbit is a saddle-focus, where one eigenvalue is positive and the other two eigenvalues are conjugate complex numbers with negative real parts.

For example, choose \(F_3 = \gamma > 0\), \(F_1 = -ax(x^2 + y^2)\), and \(F_2 = -ay(x^2 + y^2)\), where \(\gamma\) and \(\alpha\) \in (0, 1) are two constants. By direct calculation, we have \(-2yF_2 - 2xF_1 = 2\alpha > 0\), \((-2yF_2 - 2xF_1)^2 - 4(1+2xF_2 - 2yF_1) = 4\alpha^2 - 4 = 4(\alpha^2 - 1) < 0\),
the eigenvalues are solutions to the equation \( \lambda^2 + 2\alpha \lambda + 1 = 0 \), the solutions are
\[
\lambda = -\alpha \pm \sqrt{1 - \alpha^2}i.
\]

**Example 2** Consider another example with non-constant eigenvalues:
\[
\begin{cases}
\dot{x} = -y + (z + x^2 + y^2 - 1)F_1(x, y, z), \\
\dot{y} = x + (z + x^2 + y^2 - 1)F_2(x, y, z), \\
\dot{z} = (z + x^2 + y^2 - 1)F_3(x, y, z).
\end{cases}
\]

It is evident that \( (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0 \) is a periodic solution to this system. Next, it is shown that under certain conditions it has a saddle-focus near this periodic orbit.

The Jacobian on the periodic orbit is
\[
\begin{pmatrix}
2xF_1 & -1 + 2yF_1 & F_1 \\
1 + 2xF_2 & 2yF_2 & F_2 \\
2xF_3 & 2yF_3 & F_3
\end{pmatrix}
\]

The characteristic function is
\[
\begin{vmatrix}
2xF_1 - \lambda & -1 + 2yF_1 & F_1 \\
1 + 2xF_2 & 2yF_2 - \lambda & F_2 \\
2xF_3 & 2yF_3 & F_3 - \lambda
\end{vmatrix} = -\lambda^3 + \lambda^2(2xF_1 + 2yF_2 + F_3) + \lambda(1 + 2xF_2 - 2yF_1) - F_3 = 0.
\]

Let \( F_1 = -\alpha x(x^2 + y^2) \) and \( F_2 = -\alpha y(x^2 + y^2) \), where \( \alpha > 0 \) is a constant. The eigenvalue function is simplified as follows
\[
\lambda^3 + \lambda^2(2\alpha - F_3) - \lambda + F_3 = 0.
\]

For this cubic equation, if it has a positive solution, and two conjugate complex solutions, then the corresponding periodic orbit is saddle-focus.

Suppose \( \lambda = t - \frac{2\alpha - F_3}{3} \). Then, one has
\[
\lambda^3 + (2\alpha - F_3)\lambda^2 - \lambda + F_3 = t^3 + \left( \frac{(2\alpha - F_3)^2}{3} - 1 \right) t + \frac{2}{27}(2\alpha - F_3)^3 + \frac{2\alpha - F_3}{3} + F_3.
\]

For the cubic polynomial, if
\[
\left( \frac{1}{27}(2\alpha - F_3)^3 + \frac{2\alpha - F_3}{6} + \frac{F_3}{2} \right)^2 + \left( -\frac{(2\alpha - F_3)^2}{9} - \frac{1}{3} \right)^3 > 0,
\]
then there are a positive solution, and two conjugate complex solutions.

It is evident that if \( F_3 \gg \alpha > 0 \) is sufficiently large restricted to the periodic orbit \( \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0 \} \), then the expanding direction is almost parallel to the \( z \)-axis.

Suppose that \( \lambda_0 \) is the eigenvalue in the unstable subspace. Hence, the direction of the unstable subspace is parallel to the eigenvector, which is the solution to the following equation:
\[
\begin{pmatrix}
2xF_1 - \lambda_0 & -1 + 2yF_1 & F_1 \\
1 + 2xF_2 & 2yF_2 - \lambda_0 & F_2 \\
2xF_3 & 2yF_3 & F_3 - \lambda_0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

Since \( \lambda_0 \) is a single root, and \( F_3 \) is sufficiently large, one has
\[
\begin{vmatrix}
2xF_1 - \lambda_0 & -1 + 2yF_1 \\
1 + 2xF_2 & 2yF_2 - \lambda_0
\end{vmatrix} \neq 0.
\]
Hence, the eigenvector can be chosen as a solution to
the following equation:
\[
\begin{pmatrix}
2xF_1 - \lambda_0 - 1 + 2yF_1 \\
1 + 2xF_2 
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 
\end{pmatrix}
= \begin{pmatrix}
-F_1 \\
-F_2 
\end{pmatrix}.
\]
Hence, the eigenvector is
\[
\begin{pmatrix}
\frac{F_1\lambda_0 - F_2}{\lambda_0^2 + 2\alpha\lambda_0 + 1} \\
\frac{1 + \lambda_0 F_2}{\lambda_0^2 + 2\alpha\lambda_0 + 1}
\end{pmatrix}.
\]
For sufficiently large \(\lambda_0\), this vector is parallel to the z-axis.

Next, a geometric description of the general vector field near the periodic orbit is provided.

Let \(\gamma = \{\phi(t, x_0) : t \in [0, T]\}\) be the periodic orbit, where \(x_0\) is any point on this periodic orbit. Consider a cross section passing through this point \(x_0\) and the corresponding Poincaré map \(P\). The stable and unstable manifolds of the periodic orbit are
\[
W^s(\gamma) = \bigcup_{t \leq 0} \phi(t, W^s_{loc}(x_0, P)) \quad \text{and} \quad W^u(\gamma) = \bigcup_{t \geq 0} \phi(t, W^u_{loc}(x_0, P)),
\]
where \(W^s_{loc}(x_0, P)\) and \(W^u_{loc}(x_0, P)\) are the local stable and unstable manifolds at the point \(x_0\) with respect to the Poincaré map. In Eq. 3.2, \(W^s(\gamma)\) and \(W^u(\gamma)\) are two-dimensional surfaces, which intersect on the closed curve \(\gamma\). For an illustrative diagram, see Figure 10.1.3 in [30].

Note that a periodic orbit might be homeomorphic to a knot [2, 31]. For illustration, consider a periodic orbit that is homeomorphic to a circle. Even for this simple case, the vector field near the periodic orbit might not be simple, since there might exist Möbius bands (or Möbius strips) contained in \(W^s(\gamma)\) and \(W^u(\gamma)\), respectively. For example, construct a simple vector field defined in a neighborhood of a periodic orbit as follows: start from a region \(V = [-1, 1] \times [-1, 1] \times [1, 2] \subset \mathbb{R}^3\) with the vector field described by
\[
\vec{r} = -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad w = (x, y, z) \in V.
\]
For this set \(V\), consider the quotient space given by the following equivalent relationship:
\[
\tilde{V} = \{(x, y, z) \in V : (x, y, 1) \sim (-x, -y, 2)\},
\]
where \((x, y, 1)\) and \((-x, -y, 2)\) are regarded as the same point in the quotient space \(\tilde{V}\). By the definition of the vector field \(\vec{r}\), this induces a natural continuous vector field on \(\tilde{V}\), denoted by \(\tilde{R}\). For convenience, use the coordinates on \(V\) to represent the point on \(\tilde{V}\). For the vector field \(\tilde{R}\), it is evident that there exists a periodic orbit, \(\{(0, 0, t) : t \in [0, 1]\}\). This periodic orbit is hyperbolic, which is contracting along the \(x\)-direction and expanding along the \(y\)-direction. There exist Möbius bands (or Möbius strips) contained in \(W^s(\gamma)\) and \(W^u(\gamma)\), respectively. For more discussions on the existence of Möbius bands for vector fields and their corresponding dynamics, see [30, Section 27.2].

Now, the proof of Theorem 3.4 is provided.

**Step 2. (Proof of Theorem 3.4)** The discussions are divided into four parts.

(i) Consider the local coordinates on the cross sections at the points \(p\) and \(q\).

Since the periodic orbit is hyperbolic, there are local stable and unstable manifolds for both the points \(p\) and \(q\). Take two cross sections, \(\Pi_p\) and \(\Pi_q\) at the points \(p\) and \(q\), respectively, where these two cross sections are generated by the product of the local stable and unstable manifolds. And, the local coordinates on the cross sections are taken as \(W^s_{loc}(w)\) and \(W^u_{loc}(w)\), respectively, denoted by \(W^s(w)\) and \(W^u(w)\) for simplicity.
Figure 7 shows an illustrative diagram for the hypotheses of the system. Consider a periodic orbit (in blue color) with period $T > 0$, and two points $p$ and $q$ on the periodic orbit satisfying $\phi\left(\frac{T}{2}, p\right) = q$ and $\phi\left(\frac{T}{2}, q\right) = p$. By the assumption of $\Upsilon_1$, the red line represents the orbit from $W^u(q)$ to $W^s(p)$, and the green line refers to the orbit from $W^u(p)$ to $W^s(q)$. Note that $W^s_{loc}(p) \setminus \{p\}$ and $W^s_{loc}(q) \setminus \{q\}$ contain two disjoint curves, respectively. As seen from Remark 3.3, for any cross section of the point $w$ on the periodic orbit, the local stable and unstable manifolds of $w$ might not be contained in the cross section.

Then, by identifying $E^s_{\sigma p}$ and $E^s_{\sigma q}$ with a subspace for $\sigma = s, u$, one can take coordinates so that a neighborhood can be considered a subset of $E^s_p \times E^u_p$ or $E^s_q \times E^u_q$, and the local stable and unstable manifolds are disks in the subspaces given by the splitting, $W^s_{loc}(p) \subset E^s_p \times \{0\}$, $W^u_{loc}(p) \subset \{0\} \times E^u_p$, $W^s_{loc}(q) \subset E^s_q \times \{0\}$, and $W^u_{loc}(q) \subset \{0\} \times E^u_q$.

Choose positive constants $\delta_{s p}, \delta_{u p}, \delta_{s q}, \delta_{u q}$, and set $D^s_p := W^s_{\delta_{s p}}(p)$, $D^u_p := W^u_{\delta_{u p}}(p)$, $D^s_q := W^s_{\delta_{s q}}(q)$, and $D^u_q := W^u_{\delta_{u q}}(q)$.

For convenience, suppose that $\Pi_p = D^s_p \times D^u_p$ and $\Pi_q = D^s_q \times D^u_q$.

(ii) Consider the complexity of the dynamics on the generalized “heteroclinic” orbit joining the points $p$ and $q$.

By the assumptions on $\Upsilon_1$ and $\Upsilon_2$, there exist $p_0 \in D^u_p \setminus \{p\}$, $q_0 \in D^u_q \setminus \{q\}$, a positive integer $k_1 \geq \max\{m_p, m_q\}$, and two constants $0 < \eta^u_p < \delta^u_p$ and $0 < \eta^u_q < \delta^u_q$, with $\hat{H}^1_{1} := D^s_p \times (p_0 - \eta^u_p, p_0 + \eta^u_p)$ and $\hat{H}^2_{1} := D^s_q \times (q_0 - \eta^u_q, q_0 + \eta^u_q)$.

**Fig. 7** Illustrative diagram for the hypotheses of the system, where the periodic orbit with period $T$ is in blue color, the red line represents the orbit from $W^u(q)$ to $W^s(p)$, and the green line indicates the orbit from $W^u(p)$ to $W^s(q)$.
where \((p_0 - \eta_p^n, p_0 + \eta_p^n) \subset D_p^u \setminus \{p\}\) and \((q_0 - \eta_q^n, q_0 + \eta_q^n) \subset D_q^u \setminus \{q\}\), such that

\[
\phi \left( \frac{T}{2} + k_1 T, \hat{H}_1^1 \right) \subset \Pi_q \text{ and } \phi \left( \frac{T}{2} + k_1 T, \hat{H}_1^2 \right) \subset \Pi_p.
\]

It follows from the \(\lambda\)-Lemma or the Inclination Lemma [19, Lemma 7.1] that there is an integer \(k_2 \geq 0\) such that

\[
D_q^u \subset \text{Proj}_{W^u(q)} \left( \phi \left( \frac{T}{2} + (k_1 + k_2) T, \hat{H}_1^1 \right) \right),
\]

\[
D_p^u \subset \text{Proj}_{W^u(p)} \left( \phi \left( \frac{T}{2} + (k_1 + k_2) T, \hat{H}_1^2 \right) \right),
\]

where \(\text{Proj}_{W^u(p)}\) and \(\text{Proj}_{W^u(q)}\) are the projections onto \(W^u(p)\) and \(W^u(q)\), respectively, and the assumptions that \(\phi \left( \frac{T}{2} + m_p T, \Upsilon_1 \right)\) contains a segment of \(W^s_{\text{loc}}(q)\) and \(\phi \left( \frac{T}{2} + m_q T, \Upsilon_2 \right)\) contains a segment of \(W^s_{\text{loc}}(p)\) are used here. Figure 8 shows an illustrative diagram of the map from \(\hat{H}_1^1 \subset \Pi_p\) to \(\Pi_q\).

(iii) Consider the complexity of the dynamics near the horizontal neighborhoods containing the points \(p\) and \(q\).

---

**Fig. 8** Illustrative diagram of the map from \(\hat{H}_1^1 \subset \Pi_p\) to \(\Pi_q\)
Recall that the periodic orbit is hyperbolic. It follows from the $\lambda$-Lemma or Inclination Lemma [19, Lemma 7.1] that there exists an integer $k_3 \geq 0$ such that, for any $k \geq k_3$, there exist positive constants $\epsilon_{p,k}^u$ and $\epsilon_{q,k}^u$, denoting

$$H_{2,k}^1 := D_p^s \times (p - \epsilon_{p,k}^u, p + \epsilon_{p,k}^u) \text{ and } H_{2,k}^2 := D_q^s \times (q - \epsilon_{q,k}^u, q + \epsilon_{q,k}^u),$$

where $(p - \epsilon_{p,k}^u, p + \epsilon_{p,k}^u) \subset D_p^u$ and $(p_0 - \eta_p^u, p_0 + \eta_p^u) \cap (p - \epsilon_{p,k}^u, p + \epsilon_{p,k}^u) = \emptyset$, and $(q - \epsilon_{q,k}^u, q + \epsilon_{q,k}^u) \subset D_q^u$ and $(q_0 - \eta_q^u, q_0 + \eta_q^u) \cap (q - \epsilon_{q,k}^u, q + \epsilon_{q,k}^u) = \emptyset$, it follows that

$$D_p^u \subset \text{Proj}_{W^u(p)} \left( \phi \left( \frac{T}{2} + kT, H_{2,k}^2 \right) \right),$$

$$D_q^u \subset \text{Proj}_{W^u(q)} \left( \phi \left( \frac{T}{2} + kT, H_{2,k}^2 \right) \right).$$

In the above discussion, one may assume that $\lim_{k \to +\infty} \epsilon_{p,k}^u = 0$ and $\lim_{k \to +\infty} \epsilon_{q,k}^u = 0$. An illustrative diagram is shown in Fig. 9. In this figure, in subgraph (a), the region bounded by red lines in $\Pi_p$ is $H_{2,k_3+1}$, the region bounded by blue lines in $\Pi_p$ is $H_{2,k_3+1}^1$; also in subgraph (a), the region bounded by red lines in $\Pi_q$ is $\phi \left( \frac{T}{2} + (k_3 + 1)T, H_{2,k_3+1}^1 \right)$, the region bounded by blue lines in $\Pi_q$ is $\phi \left( \frac{T}{2} + (k_3 + 2)T, H_{2,k_3+2}^1 \right)$.

(iv) Now, the task is to show the existence of a Smale horseshoe in a subshift of finite type with matrix $A$.  

![Illustrative diagram for the complexity of the dynamics near the horizontal neighborhoods containing the points $p$ and $q$. In subgraph (a), the region bounded by red lines in $\Pi_p$ is $H_{2,k_3}^1$, the region bounded by green lines in $\Pi_p$ is $H_{2,k_3+1}^1$, the region bounded by blue lines in $\Pi_p$ is $H_{2,k_3+2}^1$; also in subgraph (a), the region bounded by red lines in $\Pi_q$ is $\phi \left( \frac{T}{2} + (k_3 + 1)T, H_{2,k_3+1}^1 \right)$, the region bounded by blue lines in $\Pi_q$ is $\phi \left( \frac{T}{2} + (k_3 + 2)T, H_{2,k_3+2}^1 \right)$. In subgraph (b), the red rectangle in $\Pi_p$ is $H_{2,k_3}^1$, the green rectangle in $\Pi_p$ is $H_{2,k_3+1}^1$, the blue rectangle in $\Pi_p$ is $H_{2,k_3+2}^1$; also in subgraph (b), the red rectangle in $\Pi_q$ is $\phi \left( \frac{T}{2} + (k_3 + 1)T, H_{2,k_3+1}^1 \right)$, the green rectangle in $\Pi_q$ is $\phi \left( \frac{T}{2} + (k_3 + 2)T, H_{2,k_3+2}^1 \right)$.](image)
Take a sufficiently large $m \geq \max\{k_3, k_1+k_2\}$. Following the above discussions, by modifying some constants one can obtain: $0 < \tilde{\eta}_p^u, \hat{\eta}_p^u \leq \eta_p^u$, $0 < \tilde{\eta}_q^u, \hat{\eta}_q^u \leq \eta_q^u$, $0 < \tilde{\epsilon}_p^u, \hat{\epsilon}_p^u \leq \epsilon_p^u$, and $0 < \tilde{\epsilon}_q^u, \hat{\epsilon}_q^u \leq \epsilon_q^u$. Set

$$H_1^1 := D_p^s \times (p_0 - \tilde{\eta}_p^u, p_0 + \hat{\eta}_p^u)$$
and

$$H_1^2 := D_q^s \times (q_0 - \tilde{\eta}_q^u, q_0 + \hat{\eta}_q^u),$$

and

$$H_2^1 := D_p^s \times (p - \tilde{\epsilon}_p^u, p + \hat{\epsilon}_p^u)$$
and

$$H_2^2 := D_q^s \times (q - \tilde{\epsilon}_q^u, q + \hat{\epsilon}_q^u),$$

and

$$V_1^2 := \phi \left( \frac{T}{2} + mT, H_1^1 \right) \subset \Pi_q, V_2^2 := \phi \left( \frac{T}{2} + mT, H_2^1 \right) \subset \Pi_q,$$

$$V_1^1 := \phi \left( \frac{T}{2} + mT, H_1^2 \right) \subset \Pi_p, V_2^1 := \phi \left( \frac{T}{2} + mT, H_2^2 \right) \subset \Pi_p.$$

By the $\lambda$-lemma or Inclination Lemma in [19, Lemma 7.1], for sufficiently large $m$, $H_j^i$ are $\mu_h$-horizontal strips, $1 \leq i, j \leq 2$, and $V_j^i$ are $\mu_v$-vertical strips, $1 \leq i, j \leq 2$, with $0 \leq \mu_h \mu_v < 1$. This, together with Theorem 3.3, proves Theorem 3.4.

An illustrative diagram for the map $\phi \left( \frac{T}{2} + mT, \cdot \right)$, which generates a Smale horseshoe in a subshift of finite type with matrix $A$, is shown in Fig. 10.

This completes the proof of Theorem 3.4.

![Illustrative diagram for the map $\phi \left( \frac{T}{2} + mT, \cdot \right)$](image-url)

**Fig. 10** Illustrative diagram for the map $\phi \left( \frac{T}{2} + mT, \cdot \right)$, which generates a Smale horseshoe in a subshift of finite type with matrix $A$.
Question 3.1 In the above discussions, different assumptions bring various types of Smale horseshoes with respect to the Poincaré maps. An interesting question is if it is possible to obtain a classification of the continuous systems depending on the characterization of the chaotic dynamics by the existence of different types of Smale horseshoes with respect to different transition matrices? Figures 4, 5, and 6 provide possible horseshoe structures.

Similar results could be obtained if there exist several periodic orbits, which might be used in the explanation of the complexity of the dynamics of the multiscroll attractors [14].

For simplicity, consider the situation with only two periodic orbits.

**Theorem 3.5** Consider an ordinary differential equation, \( \dot{x} = \Phi(x), x \in \mathbb{R}^3 \), where \( \Phi: \mathbb{R}^3 \to \mathbb{R}^3 \) is differentiable. Assume that

- there exist two hyperbolic periodic orbits of the same period \( T > 0 \), denoted by \( \Gamma_1 \) and \( \Gamma_2 \) respectively, and points \( p_i, q_i \in \Gamma_i, i = 1, 2 \), satisfying \( \phi\left(\frac{T}{2}, p_i\right) = q_i \) and \( \phi\left(\frac{T}{2}, q_i\right) = p_i, i = 1, 2 \);  
- there are an open subset \( \Upsilon_1 \subset W^u_{loc}(p_1) \times W^s_{loc}(p_1) \) containing a line segment of \( W^u_{loc}(p_1) \), an open subset \( \Upsilon_2 \subset W^u_{loc}(q_2) \times W^s_{loc}(q_2) \) containing a line segment of \( W^u_{loc}(q_2) \), and two positive integers \( m_{p_1} \) and \( m_{q_2} \) such that \( \phi\left(\frac{T}{2} + m_{p_1} T, \Upsilon_1\right) \) contains a segment of \( W^s_{loc}(p_2) \) with \( \phi\left(\frac{T}{2} + m_{p_1} T, \Upsilon_1\right) \subset W^u_{loc}(p_2) \times W^s_{loc}(p_2) \), and \( \phi\left(\frac{T}{2} + m_{q_2} T, \Upsilon_2\right) \) contains a segment of \( W^s_{loc}(q_1) \) with \( \phi\left(\frac{T}{2} + m_{q_2} T, \Upsilon_2\right) \subset W^u_{loc}(q_1) \times W^s_{loc}(q_1) \).

Then, there exist a positive integer \( m \) and an invariant set \( \Lambda \subset \mathbb{R}^3 \) such that the following relations hold:

\[
\begin{align*}
\Lambda & \xrightarrow{\phi \left(\frac{T}{2} + mT, \cdot\right)} \Lambda \\
\Psi & \downarrow \quad \downarrow \Psi \\
\sum_8(B) & \xrightarrow{\sigma} \sum_8(B)
\end{align*}
\]

where \( \Psi \) is a homeomorphism from \( \Lambda \) to \( \sum_8(B) \), which is a topological conjugacy, and \( B \) is a transition matrix:

\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

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