Understanding Permutation Symmetry.

STEVEN FRENCH & DEAN RICKLES

If a system in atomic physics contains a number of particles of the same kind, e.g. a number of electrons, the particles are absolutely indistinguishable one from another. No observable change is made when two of them are interchanged ... A satisfactory theory ought, of course, to count any two observationally indistinguishable states as the same state and to deny that any transition does occur when two similar particles exchange places. (Dirac 1958, 207.)

1. Introduction.

In our contribution to this volume we deal with discrete symmetries: these are symmetries based upon groups with a discrete set of elements (generally a set of elements that can be enumerated by the positive integers). In physics we find that discrete symmetries frequently arise as ‘internal’, non-spacetime symmetries. Permutation symmetry is such a discrete symmetry arising as the mathematical basis underlying the statistical behaviour of ensembles of certain types of indistinguishable quantum particle (e.g., fermions and bosons). Roughly speaking, if such an ensemble is invariant under a permutation of its constituent particles (i.e., permutation symmetric) then one doesn’t ‘count’ those permutations which merely ‘exchange’ indistinguishable particles; rather, the exchanged state is identified with the original state.

This principle of invariance is generally called the ‘indistinguishability postulate’ [IP], but we prefer to use the term ‘permutation invariance’ [PI]. It is this symmetry principle that is typically taken to underpin and explain the nature of (fermionic and bosonic) quantum statistics (although, as we shall see, this characterisation is not uncontroversial), and it is this principle that has important consequences regarding the metaphysics of identity and individuality for particles exhibiting such statistical behaviour.
In this paper we will largely be dealing with the following two types of problem:

(1) **How are we to understand the metaphysics of PI?**

For instance, do we follow the ‘received view’ and say that permutation invariance shows us that quantum particles are not individuals? Do we maintain that they are individuated by their spatiotemporal location, or perhaps by some extra-theoretical property (e.g., the ‘primitive thisness’ of the object)? Given this individuation how are we to understand PI? Maybe we can resolve the issue in some completely different way, with ‘structures’ replacing ‘objects’ perhaps? It is clear that such questions readily relate to ‘traditional’ metaphysical issues connected to identity and individuality.

(2) **How are we to understand the status of PI, theoretically and empirically?**

For example, should PI be considered as an axiom of quantum mechanics? Or should it be taken as justified empirically? Why do there appear to be only bosons and fermions in the world when PI allows the possibility of many more types? This is usually resolved by postulating, ad hoc, some ‘super-selection rule’, called the “symmetrisation postulate” [SP], restricting the state vector to the fermionic and bosonic subspaces of the systems’ Hilbert space. However, rather than resolving the difficulty, this simply moves the explanatory task one step backwards (i.e., how are we then to understand SP?). Alternatively, the extra, possibly redundant, mathematical structure responsible for the extra possibilities regarding symmetry types of particles can be understood as ‘surplus structure’ (in the sense of Redhead 1975). One often finds such surplus structure in theories possessing lots of symmetry, and it frequently points to the existence of ‘gauge freedom’ in a theory (e.g., in general relativity, Yang-Mills theory, and electromagnetism). It is here that a possible relation of permutation invariance to diffeomorphism invariance (the symmetry underlying the general covariance of general relativity) becomes apparent.

In this paper we survey a number of these issues and their consequences, introducing the reader to the various schools of thought regarding the status and interpretation of PI (and , likewise, though to a lesser extent, SP). Let us begin with a brief introduction to the formal aspects of PI and relevant related topics in group theory and classical/quantum statistical mechanics.
2. The Mathematics & Physics of Permutation Symmetry.

Permutation symmetry is a discrete symmetry supported by the permutation group $\text{Perm}(\mathcal{X})$ of bijective maps (the permutation operators, $\hat{P}$) of a set $\mathcal{X}$ onto itself.\footnote{The fact that the set $\text{Perm}(\mathcal{X})$ has the structure of a group simply means that: (1) we can combine any two elements ($\hat{P}_1, \hat{P}_2 \in \text{Perm}(\mathcal{X})$) in the set to produce another element ($\hat{P}_3 = \hat{P}_1 \cdot \hat{P}_2$) that is also contained within that set ($\hat{P}_3 \in \text{Perm}(\mathcal{X})$); and (2) each element $\hat{P} \in \text{Perm}(\mathcal{X})$ also has an inverse $\hat{P}^{-1} \in \text{Perm}(\mathcal{X})$.} When $\mathcal{X}$ is of finite dimension $\text{Perm}(\mathcal{X})$ is known as the symmetric group $S_n$ (where the $n$ refers to the dimension of the group). For instance, $\mathcal{X}$ might be the set consisting of the labels of the two sides of a coin: heads ‘H’ and tails ‘T’. Or perhaps the ‘names’ of $n$ particles making up some quantum mechanical system, an He$^4$ atom for example. If we take the coin as our example, then $\mathcal{X} = \{H, T\}$ and $\text{Perm}(\mathcal{X})$ is an order two group, $S_2$, consisting of two elements (computed as having $2!$ elements via the dimension, $n = 2$, of the group): (1) the identity map, $id_{\mathcal{X}}$, which maps $H$ to $H$ and $T$ to $T$; and (2) the ‘flip’ map (or ‘exchange’ operator), $\hat{P}_{HT}$, which maps $H$ to $T$ and $T$ to $H$.

Now, to say that some object (i.e. a set or the total state vector of a system of particles) is ‘permutation symmetric’ means that it is invariant under the action of $\text{Perm}(\mathcal{X})$: it remains unchanged (in some relevant sense) when it is operated upon by the elements (i.e., the permutation operators) of $\text{Perm}(\mathcal{X})$, including (for $n \geq 2$) the elements that ‘exchange’ the components of the object (in this case the labels of the sides of the coin or the labels of the particles in a quantum system).

The coin clearly is not permutation symmetric (i.e., does not satisfy PI), since we must distinguish ‘heads’ from ‘tails’; that is, there is an observable difference between these two states of a coin. However, when we consider systems containing several indistinguishable particles\footnote{Particles are said to be indistinguishable in that they possess the same state independent (intrinsic) properties, such as rest mass, charge, and spin. Since these quantities have a continuous spectrum in classical mechanics we can still individuate particles by their variations with respect to these properties. If it were the case that we had a classical system containing particles that exactly matched in these properties, then we could still distinguish the subsystems by their spatiotemporal location. Such luxuries are not available in quantum theory because of discrete spectra and the absence of definite trajectories.}, each with several possible states (particles such as electrons, neutrons, and photons), we find that they are indeed permutation symmetric, and that this symmetry ‘shrinks’ the number of possible states of the total system, thus altering the statistical
behaviour of the ensemble. In this way PI is generally taken to explain the divergence of quantum statistics from classical statistics.\footnote{Note, however, that this explanatory link has been contested by Huggett (1999a).}

To see how these ‘altered statistics’ follow from PI, and what they look like, let us compare classical and (bosonic/fermionic) quantum statistics using a simple example.

Consider the distribution of a system of $n$ indistinguishable objects (e.g., free particles) over $m$ microstates. It is helpful at this stage to view the objects as balls and the microstates as the two halves of a box (making each side big enough to accommodate all $n$ balls).\footnote{The separators in the diagrams are there as an aid to visualization rather than as a part of the system we are considering.} Statistical mechanics is, very loosely, the study of the number of ways one can redistribute the objects over the microstates without altering the macrostate. Let us consider the simple case where we have two objects (balls) and two microstates (boxes). Let us label the balls by ‘$a$’ and ‘$b$’, and the sides of the box by ‘L’ (left) and ‘R’ (right). Let ‘L($a$)’ be the state where ball $a$ is in the left hand side (LHS) of the box; let ‘L(ab)’ be the state where both balls are in the LHS; and let ‘L(0)’ mean that the LHS is empty (similarly, \textit{mutatis mutandis}, for the right hand side (RHS) and ball $b$). Classically, we have four possible distributions:

| L(a) | R(b) |
|------|------|
| L(b) | R(a) |
| L(ab)| R(0) |
| L(0) | R(ab) |

Each possible permutation of the balls is counted in the statistics and, if we assume equiprobability, each configuration has a probability of 1/4 of being realized. Such a distribution is known as a Maxwell-Boltzmann distribution, and it follows the corresponding statistics for such distributions.\footnote{Huggett (op. cit.) has argued that Maxwell-Boltzmann statistics do not necessarily imply that we must count permutations as distinct: when there are many states available to each particle the rule breaks down. In the case of the present example we are dealing with many particles per state, and so the relation between Maxwell-Boltzmann statistics and counting permutations as distinct still holds.}

The situation is different when we consider quantum particles because, in addition to being indistinguishable, they are subject to PI. There are two
types of statistical behaviour for particles in quantum mechanics having to

do with the ways in which they can combine in ensembles.\(^6\) Firstly, we have

bosons (particles with integer spins; e.g., photons) behaving according to the

Bose-Einstein statistics: meaning, \textit{inter alia}, that these particles can occupy

the same state in a quantum system (the balls can reside the same side of

the box). Secondly, we have fermions (particles with half-integer spins; e.g.,

electrons) behaving according to the Fermi-Dirac statistics: meaning, \textit{inter

alia}, that these particles \textit{cannot} occupy the same state in a quantum system

(the balls cannot reside in the same side of the box). This latter principle - not

directly connected to PI - is generally known as Pauli’s Exclusion Principle.

These two points have an impact on the possible configurations we can

count in the statistics. For instance, in the case of bosons we identify those

configurations which differ only by an exchange of identical particles (i.e., the

first and second configurations from the classical statistics above), but we can

allow those configurations in which two objects occupy the same state. So if

we consider the balls as bosons we get the following three distinct possibilities

(where ‘L(1)’ means that ‘some’ particle is in the LHS - similarly, \textit{mutatis

mutandis} for the RHS):

\begin{tabular}{|c|c|}
\hline
L(1) & R(1) \\
\hline
L(ab) & R(0) \\
\hline
L(0) & R(ab) \\
\hline
\end{tabular}

So we have removed a classically possible state by identifying ‘exchanged

states’.\(^7\) This has the consequence that the probabilities for finding a system

in a certain state (still assuming equiprobability) each go from 1/4 to 1/3.

Following a similar procedure with fermions, and then applying the exclusion

principle, we get just one possible state:

\begin{tabular}{|c|c|}
\hline
L(1) & R(1) \\
\hline
\end{tabular}

\(^6\)Of course, we are, for the moment, ignoring the case of para-statistics; namely, types

of quantum statistics that violate SP, on which see §3 and §4.

\(^7\)Note that we have simplified the first configuration here, since what we actually have,

formally, is the state: \([\text{L(a) & R(b)}] + \text{L(b) & R(a)})\]. In the fermionic case we find a

similar superposition only with a change in sign (when the permutations are odd): \([\text{L(a)}

\& R(b)) - \text{L(b) & R(a)}\]). Note that the change of sign has no effect on the observable

properties (expectation values) of the system.
Which, of course, has a probability of 1 of being realized. All we have done here is to identify those configurations which differ only in which ball occupies which side of the box (following Dirac’s intuition expressed in the opening quote) and then we have forbidden two balls to occupy the same side. In both the quantum cases the systems (or, more formally, their state vectors) are invariant under the action of the permutation group: when we apply the permutation operators to the state vectors they continue to describe the same physical state; following Dirac’s intuition we identify the states. Hence, the quantum systems satisfy PI, unlike the classical system. Let us now make some of these ideas more exact by introducing some elementary quantum theory.

States of quantum systems (single or many-particle) are represented by rays $\Psi$ in a Hilbert space $\mathcal{H}$. For many particle systems the Hilbert space is the joint space constructed by tensoring together the component particles’ Hilbert spaces. The observables $\hat{O}$ of a quantum system are represented by Hermitian operators acting upon that system’s Hilbert space.

Now consider a system consisting of two indistinguishable particles. The Hilbert space for this system is: $\mathcal{H}_\text{total} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where the subscripts ‘1’ and ‘2’ label the composite particles, and $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. If the particles are in the pure states $\phi$ and $\psi$ respectively, then the composite system is in the (pure) state $\Psi = \phi \otimes \psi$. The permutation operators act upon $\Psi$ as follows: (1) $\hat{P}_{\text{id}}(\Psi) = (\phi \otimes \psi)$ and (2) $\hat{P}_{\phi\psi}(\Psi) = (\psi \otimes \phi)$.

The Hamiltonian, $\hat{H}_\Psi = \hat{H}(\phi \otimes \psi)$, of the composite system is symmetric with respect to $\phi$ and $\psi$. Hence, $\hat{H}_\Psi$ is invariant under the action of the permutation group of permutations of the composite particles’ labels: $[\hat{H}, \hat{P}] = 0, \forall \hat{P}$. By an invariance of a quantum state under the action of the permutation group (i.e., PI) we then mean that every physical observable $\hat{O}$ commutes with every permutation operator $\hat{P}$: $[\hat{O}, \hat{P}] = 0, \forall \hat{O} \forall \hat{P}$ - the physical interpretation of this being that there is no measurement that we could perform which would result in a discernible difference between permuted (final) and unpermuted (initial) states. This has the consequence that expectation values for unpermuted states are equal to expectation values for permutations of that state. Or, more formally, for any arbitrary state $\psi$, Hermitian operator $\hat{O}$, and permutation operator $\hat{P}$:

$$\langle \psi \mid \hat{O} \mid \psi \rangle = \langle \hat{P}\psi \mid \hat{O} \mid \hat{P}\psi \rangle = \langle \psi \mid \hat{P}^{-1}\hat{O}\hat{P} \mid \psi \rangle \quad (1)$$

It is this result - basically a formal expression of PI - which motivates the
claim that PI can be understood as a restriction on the possible observables of a system given its state and, as such, it can be viewed as a superselection rule determining which observables are physically relevant. We shall return to this claim and, more generally, the status of PI in later sections.

Finally, let us turn to the mathematical representation of particle types. For this we need the concept of an ‘irreducible representation’. Firstly, a representation $\rho$ of a group $G$ on a linear space $V$ is simply a map that assigns to each element of the group $g \in G$ a linear operator $\hat{O}(V)$ on the space. When the linear space is the (joint) Hilbert space $\mathcal{H}$ spanned by the states, $\{\phi \otimes \phi, \phi \otimes \psi, \psi \otimes \phi, \psi \otimes \psi\}$, of two indistinguishable particles, and the group is the permutation group, the representation will associate a unitary operator acting on $\mathcal{H}$ (i.e., on the state vector $\Psi \in \mathcal{H}$) to each permutation operator $\hat{P} \in \text{Perm}(\mathcal{X})$. We can represent this schematically as follows (beginning with the group element, then the representation of that element, and finally the physical operation)$^8$:

$$(1) \quad \hat{P}_{\phi\psi} \Psi \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi \Rightarrow \text{‘exchanging the particles’}.$$  

$$(2) \quad \hat{P}_{\text{id}} \Psi \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Psi \Rightarrow \text{‘leaving them alone’}.$$  

This pair of matrices gives a unitary representation of $\text{Perm}(\mathcal{X})$ on $\mathcal{H}$. The only matrix that commutes with both of them is the unit matrix or some scalar multiple of the unit matrix. Representations of this kind are said to be ‘irreducible’. Alternatively, a representation is said to be irreducible if the only invariant subspaces it possesses are $\{0\}$ and $\mathcal{H}$ (i.e., the zero vector and the whole space) - where a subspace $\mathcal{H}'$ of $\mathcal{H}$ is invariant if $\Psi \in \mathcal{H}'$ implies $\rho(\hat{P})\Psi \in \mathcal{H}'$, $\forall \hat{P}$.

We are interested in the irreducible representations of the permutation group because each such representation is ‘carried’ by an irreducible subspace of the Hilbert space, where each such subspace is invariant under the action of the permutation operators. Thus, the subspaces represent symmetry sectors corresponding to the possible types of permutation symmetry possessed...

$^8$We should point out that this way of doing things is an oversimplification in the following respects: firstly, since the joint states lie in a four dimensional Hilbert space they are represented by 4-vectors, but here we are assuming that they are 2-vectors. Also, the permutation operators should properly be $4 \times 4$ matrices, here we write them as $2 \times 2$ matrices. However, since nothing of import depends on this, we prefer to keep things simple in this way to facilitate understanding.
by the particles whose state vectors lie in that subspace. In the case we
are considering we find that the total Hilbert space is partitioned into two
subspaces invariant under the permutation group: (1) a (three-dimensional)
symmetric subspace (spanned by three vectors: \( \{ \phi \otimes \phi, \psi \otimes \phi, \phi \otimes \phi + \psi \otimes \phi \} \))
corresponding to bosons; and (2) a (one-dimensional) anti-symmetric sub-
space (spanned by one vector: \( \{ \phi \otimes \phi - \psi \otimes \phi \} \)) corresponding to fermions.\(^9\)
The symmetric subspace is quite clearly reducible, but the three subspaces
spanning it are one-dimensional and, therefore, irreducible: they contain no
permutation invariant proper subspaces. Hence, the irreducible representa-
tions correspond to \textit{types} of particle.\(^{10}\)

However, when we consider more than two particles (giving a non-abelian
permutation group) we find that we get more than the two symmetry types
that we observe in nature. For instance, for three indistinguishable partic-
les we have \(3!\) irreducible representations of the permutation group in the
joint Hilbert space. In addition to the standard symmetry types (bosons and
fermions), we also obtain ‘parabosons’ and ‘parafermions’, transforming dif-
ferently under the action of the permutation group, and leading to alternative
kinds of statistical behaviour known as ‘parastatistics’.\(^{11}\)

Thus, with the framework we have built up so far we can see that there is
more ‘mathematical structure’ than there is ‘physical structure’: nature has
shown us (so far) that there are bosons and fermions, whilst the theory allows
for particles with different symmetry types (with potentially observable dif-
fences). In order to overcome this problem Messiah (1962, 595) introduced
a postulate, the “symmetrisation postulate” [SP], which served to restrict the
possible particles to those two classes that we have so far found the world
to be grouped into. This postulate can be stated simply: States of identi-
cal particle systems must be either symmetrical or anti-symmetrical under
the action of permutation operators. We can now turn to the philosophical
implications of these ideas.

\(^9\)These ‘spanning’ vectors correspond, of course, to the possible outcomes in the quan-
tum statistics.

\(^{10}\)The symmetry properties mentioned earlier mean also that symmetry type is con-
served; that is, state vectors remain in one or the other subspace over time: once a boson
(fermion) always a boson (fermion)! However, this rule breaks down in supersymmetric
theories, since such theories possess a symmetry relating Fermi (matter) to Bose (force); however, we shall ignore this complication here.

\(^{11}\)The statistical behaviour of these ‘higher-dimensional’ irreducible representations is
best modelled by the ‘braid group’ rather than the ‘permutation group’. 

3. The Relationship Between Permutation Invariance and the Symmetrisation Postulate

Let us consider in a little more detail the relationship between SP and PI. The former is obviously a restriction on the states of the assembly. If the latter is likewise understood, then it is easy to see that SP is sufficient but not necessary for PI. Understood in this way, the fact that PI is implied by SP means it picks up indirect empirical support from the latter (assuming that all known particles are either bosons or fermions - an assumption which has been questioned (see the papers in Hilborn and Tino 2000)).

However, as Greenberg and Messiah argued, PI should be interpreted, not as a restriction on the states, but rather as a restriction on the possible observables for the assembly (Greenberg and Messiah 1964). On this view, PI dictates that any permitted observable must commute with any permutation operator and this in turn implies that the observable must be a symmetric function of the particle labels. The difference between SP and PI can thus be expressed as follows: SP expresses a restriction on the states for all observables, Q; whereas PI expresses a restriction on the observables, Q, for all states. From this perspective what PI does is restrict the accessibility of certain states, such that once in a certain set of states, whether Bose-Einstein, Fermi-Dirac or parastatistical of a given order, the particles cannot move into a different set. However, the question as to its status now becomes acute. Before we discuss this question in more detail, we shall consider one further aspect of the formal representation of quantum statistics and PI.

4. Permutation Invariance and the Topological Approach to Particle Identity.

As is well known, Schrödinger’s early attempt to give a broadly classical interpretation of the new quantum mechanics foundered on the point that the appropriate space for a many-particle wave function had to be multidimensional. Even then, as Einstein pointed out, use of the full configuration space formed by the N-fold Cartesian product of three-dimensional Euclidean space appeared to conflict with the new quantum statistics insofar as within this full space, configurations related by a particle permutation are regarded as distinct. The standard resolution of this problem, of course, is to move to the reduced quotient space formed by the action of the permutation group on the full configuration space, in which points corresponding to a permutation
of the particles are identified, and then apply appropriate quantum conditions (see Leinaas and Myrheim 1977). In this context PI is effectively coded into the topology of configuration space itself, and the different statistical types then correspond to different choices of boundary conditions on the wave function (see, for example, Bourdeau and Sorkin 1992).

This reduced configuration space is not in general a smooth manifold since it possesses singular points where two or more particles coincide. This leads to two technical difficulties: first, it is not clear how one might define the relevant Hamiltonian at such singularities (ibid., 687) and secondly, the existence of these singular points is not compatible with Fermi-Dirac statistics. The obvious, and now standard, solution is to simply remove from the configuration space the subcomplex consisting of all such coincidence points, yielding a smooth manifold. The relevant group for n particles is then the n-string braid group as we noted above, and the irreducible unitary representations of this group can be used to label the different statistics that are possible. Imbo, Shah Imbo and Sudarshan have provided a definition of the ‘statistical equivalence’ of two such representations in terms of which they obtain not merely ordinary statistics, parastatistics and fractional or anyon statistics but more exotic forms which they call ‘ambistatistics’ and ‘fractional ambistatistics’ (Imbo, Shah Imbo and Sudarshan 1990). The deployment of the braid group in this manner may appear to conflict with the suggestion that one of the advantages of the configuration space approach is that it actually excludes the possibility of non-standard statistics. This claim is based on the work of Leinaas and Myrheim (op. cit.) which apparently demonstrates that for a space of dimension 3 or greater only the standard statistics are possible. The conclusion drawn is that “... the (anti-) symmetrisation condition on the wave function is now seen to be related to the dimensionality of space, in contrast to the Messiah and Greenberg analysis wherein the (anti-) symmetrisation condition receives the status of a postulate” (Brown et al. 1992, 230).

It turns out, however, that this treatment assumes the standard quantisation procedure which incorporates one-dimensional Hilbert spaces only. From the group-theoretical perspective this amounts to allowing only one-dimensional representations and so it should come as no surprise that para- and ambi-statistics cannot arise. Effectively what Leinaas and Myrheim have done is to ignore the ‘kinematical ambiguity’ inherent in the quantisation procedure which derives from the (mathematical) fact that the set of irreducible representations of the permutation group contains not just the trivial repre-
sentation manifested above but also others corresponding to exotic statistics (Imbo et al. 1990, 103-104). In what follows we shall occasionally return to the topological approach to see if it can shed any light on the issue of the status of PI.

5. Permutation Invariance and the Metaphysics of Individuality.

5.1 The Received View: Quantum Particles as Non-Individuals.

One well known approach to this issue takes PI to be profoundly related to the peculiar metaphysical character of quantum particles, namely that they are ‘identical’ or ‘non-individual’, in some sense. Referring back to our illustration of the difference between classical and quantum statistics above, the argument for such a view goes like this: in classical Maxwell-Boltzmann statistics, a permutation of the particles is taken to give rise to a new, countable arrangement. Since the particles are indistinguishable, in the sense of possessing all intrinsic or state-independent properties in common (that is, properties such as rest mass, charge, spin, etc.), this generation of new arrangements must reflect something about the particles which goes beyond their intrinsic properties, something which allows us to treat them as distinct individuals. In the quantum case - whether Fermi-Dirac or Bose-Einstein - the distribution is permutation invariant and a permutation of the particles does not yield a new arrangement. Hence the statistical weight in quantum statistics - of either form - is appropriately reduced. Since the particles are regarded as indistinguishable in the same sense as their classical counterparts, this reduction in the count, due to PI, must reflect the fact that the particles can no longer be regarded as individuals - they are, in some sense, ‘non-individuals’. In other words, according to this argument, PI implies non-individuality.

We shall call this view - that quantum particles are, in some sense, not individuals the Received View. It became fixed in place almost immediately after the development of quantum statistics itself (and in its modern incarnation it can be found in Dieks 1990, for example). Thus at the famous Solvay Conference of 1927, Langevin noted that quantum particles could apparently no longer be identified as individuals and that same year, both Born and Heisenberg insisted that quantum statistics implied that the “individuality of the corpuscle is lost” (Born 1926; see Miller 1987, 310). Some years
later, in 1936, Pauli wrote to Heisenberg that he considered this loss of individuality to be “... something much more fundamental than the space-time concept” (see von Meyenn 1987, 339).

5.2 Challenges to the Received View.

The Received View has been challenged on a variety of grounds over the past fifteen years or so. These challenges come at the issue from two directions, and in both cases it is denied that there exists some fundamental metaphysical difference between quantum and classical particles. The first challenge insists that classical particles, like their quantum counterparts, are not only ‘indistinguishable’, in the above sense, but should also be understood as subject to PI. The grounds for this claim rest on a positivistic understanding of the meaning of ‘non-individuality’ which takes the latter notion to be determined experimentally (Hestenes 1970). The idea is that non-individuality follows from the requirement that in order for the entropy to be extensive, the relevant expression must be divided by $N!$, where $N$ is the number of particles in the assembly. It is this extensivity of entropy which, it is claimed, resolves the infamous Gibbs’ paradox: if like gases at the same pressure and temperature are mixed, then there is no change in the experimental entropy. This is in disagreement with the result obtained from Maxwell-Boltzmann statistics, incorporating the considerations of particle permutations sketched above. Excluding such permutations from the calculation of the statistical entropy by dividing by $N!$ is then understood as resolving the ‘paradox’. If this were correct, then it is claimed - classical statistical mechanics would have to be regarded as permutation invariant also (see, for example, Saunders forthcoming, 22, fn. 13) and the contrast with quantum physics would have to be sought elsewhere.

Historically, however, the failure of extensivity and the Gibbs Paradox were seen as revealing a fundamental flaw in the Maxwell-Boltzmann definition of entropy and one which is corrected by shifting to an understanding of the particles as quantum in nature.\textsuperscript{12} In other words, the force of the argument can be turned around: what it shows is that the world is actually quantum in nature, as one would expect. What the exclusion of the

\textsuperscript{12}Here we are following Post who writes, “... the flaw in classical statistical mechanics represented by Gibbs’ paradox points to a radical theory of non-individuality such as Bose’s” (1971, 23, fn. 50). By ‘flaw’ here Post means a ‘neuralgic point’ which can act as the heuristic stimulus for new theoretical development.
permutations (by dividing by N!) is a manifestation of is precisely that the particles are not just indistinguishable in the classical sense. If this aspect is incorporated into the analysis from the word go, the so-called ‘paradox’ simply does not arise\textsuperscript{13}.

The second challenge suggests that, contrary to the Received View quantum particles, just like their classical counterparts, can also be regarded as individuals. The question immediately arises: if this were the case, how would one account for the different treatment of permutations; that is, how would one account for the difference between classical and quantum statistics? We recall that this difference lies in the drop in statistical weight assigned to the relevant arrangements. This can be accounted for, without appealing to the supposed ‘non-individuality’ of the particles, by focussing explicitly on the role of PI, understood as a kind of initial ‘accessibility’ condition (French 1989a).

To see this, let us recall that, understood as a restriction on the observables, PI acts as a super-selection rule which divides up the relevant Hilbert space into a number of irreducible sub-spaces, corresponding to irreducible representations of the symmetric group. It can be shown that transitions between such subspaces are (generally) forbidden (at least in non-supersymmetric theories). Hence PI imposes a restriction on the states of the assembly such that once a particle is in a given subspace, the other - corresponding to other symmetry types - are inaccessible to it. Returning to the argument above, the reduction in statistical weight is now explained by the inaccessibility of certain states, rather than by a change in the metaphysical nature of the particles. In the simple case of two particles distributed over two one-particle states, the only subspaces available are the symmetric and antisymmetric and so only one of the two possible states formed by a permutation is ever available to the system. Thus the statistical weight corresponding to the distribution of one particle in each such state is half the classical value.

We shall examine the two components of this alternative to the Received View in a little more detail. First of all, with regard to the role of PI, the idea of restrictions on the set of states accessible to a system can also be found in classical statistical mechanics, of course. There it is the energy

\textsuperscript{13}Historically, the N! division was the subject of a vigorous dispute between Planck, who defended the move, and Ehrenfest, who argued that it was ad hoc as it stood and hence required further justification.
integral which imposes the most important restriction as it determines which regions of the relevant phase space are accessible. Other uniform integrals of the motion may also exist for a particular assembly but these are not generally thermodynamically significant. What PI represents is an additional constraint or initial condition, imposed on the situation. In particular, the symmetry type of any suitably specified set of states is an absolute constant of motion equivalent to an exact uniform integral in classical terms (see Dirac op. cit., 213-216). Of course, some may wonder whether this actually sheds much light on the status of PI, since it appears to leave it standing as a kind of ‘brute fact’ but at least it is no more brutish than the other, classical, constraints.

Secondly, there is the issue of how we are to understand the individuality of the particles. As is well known, we have a range of options to choose from, some more attractive than the others:

1. Haecceity or primitive thisness (Adams 1979).
2. Some form of Lockean substance.
3. Spatiotemporal location.
4. Some subset of properties.

The first two are perhaps the least attractive as far as broadly ‘empiricist’ philosophers are concerned, since they appeal to factors which are utterly empirically superfluous (see French and Redhead 1988; Redhead and Teller 1991 and 1992; van Fraassen 1991). However, they share what some would see as the advantage of making manifest the conceptual distinction between individuality and distinguishability, where the latter is to be understood in terms of some difference in properties. The third and fourth collapse this distinction by taking that which renders the entity distinguishable as that which also ‘confers’ individuality. Each requires some extra postulate, however, in order to effect this collapse. In the case of option (3), this extra ‘something’ is the postulate that the particles are impenetrable, since if they were not, their spatiotemporal locations could not serve to distinguish and hence individuate them. As is well known, however, this option is problematic in the quantum context, as standardly understood, since it can be shown that the family of observables corresponding to the positions of single particles cannot provide distinguishing spatiotemporal trajectories (as Huggett and Imbo 2000 emphasise, one can prove the more general result that no family
of observables can provide such trajectories)\textsuperscript{14}.

Nevertheless, considerations of impenetrability do feature within the topological approach. We recall that, according to this perspective, we must move to the reduced configuration space formed by removing the points where the particles would coincide. An obvious justification for this adverts to the impenetrability of the particles, understood, in turn, as due to certain repulsive forces holding between them. Such a conjecture has been made in the case of anyons (see Aitchison and Mavromatos 1991) and more generally this has been taken to confer a further advantage on the configuration space approach, in the sense that

... it allows particle statistics to be understood as a kind of ‘force’ in essence similar to other interactions with a topological character, like the interaction between an electric and magnetic charge in three spatial dimensions, or the type of interaction in two dimensions which is responsible for the Bohm-Aharonov effect and fractional statistics. (Bourdeau and Sorkin op. cit., 687).\textsuperscript{15}

Some have regarded such an understanding as suspiciously ad hoc (see Brown et al., op. cit.). One way of eliminating this ‘ad hocness’ is to shift to the framework of de Broglie-Bohm pilot wave theory\textsuperscript{16}. Here, as is well known, there is a dual ontology of point particles plus pilot wave, where the

\textsuperscript{14}They also stress that this lack of trajectories does not imply anything about the status of PI.

\textsuperscript{15}Again, we can’t help but recall some relevant history here. The suggestion that the non-classical aspects of quantum statistics reflects a lack of statistical independence and hence a kind of correlation between the particles can be traced as far back as Ehrenfest’s early reflections on Planck’s work and crops up again and again in the literature. On the philosophical side, Reichenbach (1956, 234-235) argued that such correlations - taken realistically in this sense - represent causal anomalies in the behaviour of the particles: for bosons these anomalies consist in a mutual dependence in the motions of the particles which could be characterised as a form of action-at-a-distance; for fermions, the anomaly is expressed in the Exclusion Principle if this is interpreted in terms of an interparticle force. As far as Reichenbach himself was concerned, such acausal interactions should be rejected and thus he preferred the account of quantum statistics which emphasises the metaphysical lack of individuality of the particles and in which the correlations are not regarded in force-like terms.

\textsuperscript{16}Of course, adopting such a framework means abandoning the standard eigenvalue-eigenstate link and the latter is precisely what is assumed in the above proofs that spatiotemporal trajectories cannot serve to distinguish.
role of the latter is to determine the instantaneous velocities of the former through the so-called ‘guidance equations’ (ibid.). Since these equations are first-order, the trajectories of two particles which are non-coincident to begin with will never coincide. In effect the impenetrability of the particles is built into the guidance equations and the singularity points remain inaccessible. Hence the conclusion that “... within the topological approach to identical particles the removal of the set ... of coincidence points from the reduced configuration space ... follows naturally from de Broglie-Bohm dynamics as it is defined in the full space ...” (ibid., 233).17

It is part of the attraction of this framework that it retains, or appears to retain, a form of classical ontology which meshes well with the metaphysical view of particles as individuals18. However, it is also important to recognise that the topological approach can also accommodate the Received View of particles as non-individuals. Indeed, one of the motivations given by Leinaas and Myrheim is that it allows one to dispense with the whole business of introducing particle labels and then effectively emasculating their ontological force by imposing appropriate symmetry constraints (op. cit., 2; cf. also Bourdeau and Sorkin op. cit., 687). Of course, if one is going to insist that the particles are non-individuals, then some alternative justification for the removal of the coincidence points must be sought for. One possibility is to tackle the problem of collisions directly. Bourdeau and Sorkin, for example (op. cit.), show that for fermions, the self-adjoint extension of the Hamiltonian to cover the singularities is unique, at least in the two-dimensional case, so that collisions are strictly forbidden, whereas in the case of both Bose-Einstein and fractional statistics there are a range of alternative extensions, some of which allow collisions but some which do not19. By requiring that the wave-function remains finite at the coincident point they argue that a...
unique choice of Hamiltonian can then be made and it turns out that collisions are allowed only in the case of Bose-Einstein statistics. Thus, whereas for fermions it doesn’t really matter whether the singular points are retained or not, for bosons and anyons, on this account, it does, since these points are either the locations of collisions in the boson case or the locations of vanishing $\Psi$ for anyons.

6. Individuality and the Identity of Indiscernibles.

Let us return to our list of options for understanding quantum individuality. Option (4) attempts to ground it in some subset of properties of the particles. This also requires a supplementary principle in order to block the possibility of two individuals sharing the same subset of properties, and this is provided, of course, by the Principle of Identity of Indiscernibles (PII). In terms of second-order logic with equality, PII can be written as

$$\forall \Gamma \{ \Gamma(a) \equiv \Gamma(b) \} \rightarrow a = b \hspace{2cm} (2)$$

where ‘a’ and ‘b’ are individual constants designating the entities concerned and $\Gamma$ is a variable ranging over the possible attributes of these entities. Different forms of PII then arise depending on what sort of attributes feature in the range of $\Gamma$. The logically weakest form, PII(1), states that it is not possible for two individuals to possess all properties and relations in common; PII(2) excludes properties and relations which can be described as spatiotemporal; while the strongest form PII(3), includes only monadic, non-relational properties. Before we consider the status of these forms of PII in quantum physics, it is worth noting, first of all, that PII(1) has often been taken as necessarily true on the grounds that no two individuals can possess exactly the same spatiotemporal properties or enter into exactly the same spatiotemporal relations (see, for example, Quinton 1973, 25). This obviously assumes that the individuals concerned are impenetrable and amounts to a form of option (3) above. Both PII(1) and PII(2) allow for the possibility that relations might be capable of distinguishing entities and hence confer individuality (see for example Casullo 1984, who argues that the view of entities as nothing more than bundles of properties and relations is only

---

20We exclude the attribute of ‘being identical with a’, since PII would then simply be a theorem of second-order logic. Furthermore, Adams identifies haecceity or ‘primitive thisness’ with precisely this attribute (op. cit.) and hence admitting it here would be tantamount to adopting option (1).
plausible if based on PII(1) with relations given the capacity to individuate). However, such a possibility has been vigorously disputed on the grounds that since relations presuppose numerical diversity, they cannot account for it (see Russell 1956 and Armstrong 1978, 94-95). We shall return to this possibility shortly.

When it comes to the status of PII in quantum physics, if the non-intrinsic, state-dependent properties are identified with all the monadic or relational properties which can be expressed in terms of physical magnitudes associated with self-adjoint operators that can be defined for the particles, then it can be shown that two bosons or two fermions in a joint symmetric or anti-symmetric state respectively have the same monadic properties and the same relational properties one to another (French and Redhead 1988; see also Butterfield 1993). On the basis of such an identification, even the weakest form of the Principle, PII(1), fails for both bosons and fermions (French and Redhead op. cit.; French 1989b). Hence the Principle of Identity of Indiscernibles cannot be used to effectively guarantee individuation via the state-dependent properties and option (4) fails, leaving Lockean substance or primitive thisness as the only alternatives.

However, there may still be hope for this option. Saunders (ibid., 10-11) has recently revived Quine's proposal for the analysis of identity, which he understands as yielding a version of PII. Roughly speaking this is the condition that $x = y$ (where $x, y, u_1, u_2, \ldots$ are variables) if and only if, for all unary predicates $A$, binary predicates $B$, \ldots, $n$-ary predicates $P$, we have:

\begin{itemize}
  \item $A(x) \equiv A(y)$
  \item $B(x, u_1) \equiv B(y, u_1); B(u_1, x) \equiv B(u_1, y)$
  \item $P(x, u_1, \ldots, u_{n_1}) \equiv B(y, u_1, \ldots, u_{n_1})$ and permutations
\end{itemize}

\begin{itemize}
  \item together with all universal quantifications over the free variables $u_1, \ldots, u_{n_1}$ other than $x$ and $y$ (ibid., 11). If the relevant language contains monadic
\end{itemize}

\begin{itemize}
\item Margenau had earlier concluded that PII(3) fails, since the same reduced state can be assigned to the fermions in an antisymmetric state and hence they possess the same monadic properties (Margenau 1944). This conclusion has been criticised on the grounds that such reduced states cannot be regarded either as ontologically separate or as encoding genuinely monadic properties (see Mittelstaedt and Castellani 2000; Massimi 2001).
\item For alternative discussions see Cortes 1976; Barnette 1978; Ginsberg 1981; Teller 1983; and van Fraassen 1985 and 1991.
\end{itemize}
predicates only, then this principle amounts to the claim that two entities are identical if and only if they have all properties in common. Two entities are said to be absolutely discernible if there is a formula with only one free variable which applies to one entity but not the other. With only monadic predicates allowed, the principle states that numerically distinct entities are absolutely discernible. If relations are admitted, however, one can have entities which are not identified by the principle yet are not absolutely discernible. Two entities are said to be relatively discernible if there is a formula in two free variables which applies to them in any order. But there is a further category: If the admitted relations include some which are irreflexive, then one can have entities which are counted as distinct according to the principle but are not even relatively discernible. These are said to be weakly discernible, and the principle excludes the possibility of entities neither absolutely, relatively, or weakly discernible.

This, it is claimed, is more natural from a logical point of view (being immune to the standard counter examples - such as the infamous two globes - which beset PII; see Saunders (ibid., 7), and is also better suited to quantum mechanics in that, unlike traditional versions of PII, it is not violated by fermions at least, since an irreflexive relation always exists between them. Consider for example, two fermions in a spherically-symmetric singlet state. The fermions are not only indistinguishable but also have exactly the same spatiotemporal properties and relations in themselves and everything else. However, each satisfies the symmetric but irreflexive relation of having opposite direction of spin to and so are weakly discernible. Thus for fermions, at least, we have the possibility of grounding their individuality via a version of PII, without having to appeal to anything like primitive thisness.\textsuperscript{23} However, there is an obvious concern one might have here, which reprises the worry hinted at above, regarding the individuating power of relations: doesn’t the appeal to irreflexive relations in order to ground the individuality of the objects which bear such relations involve a circularity? Such concerns are rooted in the - apparently plausible - view that relata have ontologically priority over relations, such that the former can be said to ‘bear’ the latter. Suppose we were to drop such a view. Of course, in order to describe

\textsuperscript{23}No such possibility exists for bosons; however, Saunders adopts the Redhead and Teller option of regarding them as non-individual field quanta. He takes this metaphysical difference as tracking the physical one between the ‘stable constituents of ordinary matter’ (fermions) and gauge quanta (bosons), although it is not clear why the metaphysics should follow the physics in this particular way, or at all.
the relation - either informally as above or set-theoretically in terms of an ordered tuple \((x, y)\) - we have to introduce some form of label, as in the example above, but description should not dictate conceptualisation. The label can be understood as a kind of place-holder and instead of talking of relata ‘bearing’ relations, one can talk of the intersection of relations as constituting relata, as Cassirer did, or of relata as unifying relations, as Eddington did. The names here give the game away - what this amounts to is some kind of structuralist ontology which allows for individuation via relations.

7. PI is Neither Sufficient nor Necessary for Non-Individuality.

Returning now to the issue of the status of PI, the point we want to emphasise is that something further needs to be added to get from it to the Received View of particles as non-individuals. In other words, PI is not sufficient for non-individuality. The question remains, is it necessary? van Fraassen, for example, has answered that it is not (1991, 375), while acknowledging that the claim - that “when identity is properly understood, it entails Permutation Invariance tautologically” (ibid.) - nevertheless contains a ‘core’ of truth. Butterfield has disagreed, however, insisting that this claim merely summarises the motivation for PI, namely that expectation values for the composite system cannot be sensitive to the differences between \(\phi\) and \(P\phi\) (1993, 457). At issue here, of course, is what is meant by ‘identity’, or non-individuality, being ‘properly understood’. Butterfield’s understanding appears to correspond to Dirac’s above, but if one were to reject this as broadly positivistic, how do the alternatives stand up?

In order to examine this question, we need to consider what we mean by non-individuality in this context. All of the options considered here are constructed through a combination of indistinguishability - of course - together with the denial of some ‘principle’ of individuality, whether that be substance, primitive thisness, spatiotemporal trajectories, or PII. The question then is whether any of these alternatives can provide the restriction on observables that PI demands. Clearly neither substance nor primitive thisness can, since by their very nature, neither can be expressed in terms of observables! What about indistinguishability plus the denial of spatiotemporal trajectories? Huggett and Imbo (2000) have recently considered this case and have argued that here too PI does not follow. Their argument considers each conjunct separately: first of all, the absence of spatiotemporal trajectories follows from the dynamics of QM together with the general way
observables are treated and again this imposes no restrictions on these observables. Hence, the lack of spatiotemporal trajectories is simply irrelevant in this case. Furthermore, there exists the possibility of particles, such as first quantised versions of Greenberg’s quons (Greenberg 1991)\textsuperscript{24} which are indistinguishable but for which every Hermitian operator is an observable – hence some of the observables are non-symmetric and PI is violated.

Of course, if PII were to hold for quons, then the implication would be restored for that particular understanding of non-individuality; that is, if quons could not be considered as non-individuals in this sense, then the violation of PI would not be indicative of the failure of the entailment. However, what would have to be shown is that there are no possible quon states for which PII is violated in the same ways as for bosons and fermions. In the case of paraparticles, French and Redhead 1988 showed that although there do exist states for which the monadic properties of all the separate particles are the same, there also exist possible paraparticle states for which PII is violated. So far as we know, nothing equivalent has been demonstrated for quons, although in what little philosophical discussion there has been about them, it appears to be assumed that such PII violating states exist (Hilborn and Yuca forthcoming).

The choice, then, is stark: either adopt the Dirac/Butterfield understanding of non-individuality, in which case PI is indeed an expression of it but not in a metaphysically interesting way, or accept that non-individuality does not imply permutation invariance. The latter option leaves PI metaphysically ungrounded.

8. Underdetermination and the Structuralist View of Particles

This metaphysical ‘detachment’ of PI has been expressed in terms of a kind of underdetermination which holds between the Received View, in which PI is tied up with the non-individuality of the particles and the alternative account of particles as individuals, with PI taken as some sort of initial condition (French and Redhead op. cit.). As we have indicated, PI does not discriminate between these conceptual possibilities (French 1989a; 1998; van Fraassen 1991; Huggett 1995; Balousek 2000) and hence any argument for one

\textsuperscript{24}This possibility is not unproblematic, of course. The q-mutator formalism apparently requires some observables to be non-local and there is the further issue as to whether this possibility is ruled out on experimental grounds (again see the papers in Hilborn and Tino 2000).
or the other is going to have to proceed on different grounds. Most famously, perhaps, Redhead and Teller have elaborated a methodological argument to the effect that the Received View meshes better with the metaphysics of Quantum Field Theory where - it is claimed - individuality is not assumed from the word go (this argument goes back to Post 1963). This last claim has been disputed (by de Muynck 1975 and van Fraasssen 1991; for criticisms of the latter, see Butterfield 1993) and Balousek has insisted that to appeal to methodology to break the underdetermination is to accede to a form of conventionalism (Balousek 2000; for a response see Teller and Redhead 2001).

We shall not pursue the ins and outs of this debate here. The alternative is to accept the underdetermination and explore its implications. As far as the particles themselves are concerned, it motivates a shift - some might say retreat - to a structuralist view of entities which eschews talk of individuality or non-individuality entirely (see Ladyman 1998). According to such a view, the particles are nothing but ‘nodes’ or ‘intersections’ in some kind of physical structure, which now bears all the ontological weight. We shall consider how PI looks from such a position shortly.

9. The Experimental and Theoretical Status of PI.

Returning to the issue of the status of PI, let us consider whether there could there be direct evidence for the principle. We can examine this question in a broader context: what is required for any symmetry principle to be observable? Kosso has argued that the distinction between observable and unobservable symmetries matches that between global and local. Thus Lorentz invariance, for example, is directly observable whereas general covariance is not, since a specific dynamical principle such as the principle of equivalence must be assumed in order to infer the symmetry from the observation (Kosso 2000). Now, a problem arises when it comes to symmetries such as PI: in order to observe whether a symmetry holds or not, one must be able to observe that a) the specified transformation has taken place; and b) that the specified invariant property remains the same under the transformation (ibid., 86). The first condition requires there to be a fixed point of reference with respect to which the transformation can be measured. In the case of the permutation of protons and neutrons underlying isospin symmetry, for example, there exists a ‘fixed standard’ of what it is to be a neutron and what it is to be a proton with respect to which the permutation makes sense (ibid.). If, however, the symmetry is ‘observationally complete’, in the
sense that all of the observable properties of the system are invariant under the transformation, then the symmetry will be unobservable in principle (ibid., 88). Kosso does not consider PI in his discussion but it is observationally complete in precisely this sense: in order to test whether it holds or not, one would have to be able to experimentally distinguish the states represented by $|\Psi\rangle$ and $\hat{P}|\Psi\rangle$ to begin with. However, that is precisely what the principle itself denies (Balousek 1999, 20).

If there can be no direct evidence for PI, is it demanded by the theory of quantum mechanics itself? It is often claimed that PI is not logically required by the axioms of the theory (Balousek ibid., 20), but of course, this depends on what the latter are taken to include. Adopting a crude historical perspective, the work of Weyl, Dirac and von Neumann can be seen as an attempt to impose order upon what was, in the late 1920s, a bit of a hodgepodge of laws and principles, some of them ‘phenomenological’ in nature, such as the Exclusion Principle. Weyl’s framework was explicitly group-theoretical and here, as we have indicated, PI can be incorporated within this framework as an expression of the metaphysical nature of quantum objects. Dirac, on the other hand, eschewed both group theory and metaphysics, referring his own ‘bra’ and ‘ket’ framework in which PI is seen as nothing more than an expression of the observational indistinguishability of $|\Psi\rangle$ and $\hat{P}|\Psi\rangle$, from a rather crude verificationist perspective. In the context of von Neumann’s Hilbert space formalism PI does appear to be an extra postulate reflecting either the metaphysical nature of quantum particles or some kind of ‘initial condition’ as we noted above but of course one could make the case that it should be added to the standard axioms of QM whatever they are in order to extend the theory to give a quantum statistics. Whatever framework one chooses, claims that PI is, in some sense, ‘ad hoc’ must be treated with caution. Where does all this leave the status of PI? It appears to be a kind of ‘free-floating’ principle, one that is required neither experimentally nor metaphysically. Huggett has proposed that it be regarded straightforwardly as a symmetry on a par with rotational symmetry, for example (Huggett 1999). Now, of course, as Huggett acknowledges, the two symmetries are very different\textsuperscript{25}, but, nevertheless, PI is implied by the conjunction of a further symmetry principle which space-time symmetries also obey together.

\textsuperscript{25}A quantum system of the kind we have been considering is not just \textit{covariant} with respect to permutations but \textit{invariant}: permutations are not just indistinguishable to appropriately transformed observers but to \textit{all} observers.
with the formal structure of the permutation group. This further principle is what he calls ‘global Hamiltonian symmetry’ which implies that the relevant symmetry operator commutes with the relevant Hamiltonian. With regard to the permutation group, of course, permutations of a sub-system are permutations of the whole system and this ‘global Hamiltonian symmetry’ very straightforwardly implies PI, without any additional assumptions concerning the structure of state space (ibid., 344-345).

Hence, Huggett concludes, ... we should view permutations in a similar light to rotations: we should not take [permutation invariance] as a fundamental symmetry principle in order to explain quantum statistics. Instead we should recognize that it is a particular consequence of global Hamiltonian symmetry given the group structure of the permutations. Further, if we accept the similarity of permutation and rotation symmetry, it becomes natural to see quantum statistics as a natural result of the role symmetries play in nature. (ibid., 346).

However, as Huggett acknowledges, permutation invariance only follows from his general symmetry principle given the particular structure of the permutation group. So the issue of the status of PI is pushed back a step: what is the status of the structure of the permutation group? Or, to put it another way, why should that particular group structure be applicable? At one extreme we have the view that it is a priori. As is well known, Weyl insisted that “all a priori statements in physics have their origin in symmetry” (1952, 126). Not surprisingly, empiricists such as van Fraassen have tended to resist this line (van Fraassen 1989) and move to the other end of the spectrum, offering a broadly pragmatic answer to our question. From this perspective, PI comes to be seen as nothing more than a problem

---

26 What we take the relevant Hamiltonian to cover is crucial here because, again as Huggett acknowledges, the principle would appear to be violated in the case where, for example, we have a noncentral potential term in the Hamiltonian of an atomic system, but, he insists, the symmetry is restored if we consider the ‘full’ Hamiltonian of system plus field, which does commute with the operators of the rotation group. As he points out (ibid., 345), if observers are taken to be systems too, this symmetry principle is equivalent to covariance for space-time symmetries.

27 It does, however, assume that the system being measured and the measurement apparatus are composed of the same indistinguishable particles, otherwise the Hamiltonian will not remain unchanged. Thanks to Nick Huggett for pointing this out.
solving device (see Bueno 2001). Occupying the middle ground we have the following alternative answers to our question:

1. It is just a brute fact. We have already encountered this option in our discussion of PI as an ‘initial condition’ imposed on the situation.

2. It is to be understood as reflecting the metaphysically peculiar nature of the particles themselves. However, given that the particles can also be described in a metaphysically straightforward way - as individuals - this option is always going to require some further principle whose status may be less well grounded than that of PI itself.

3. It is to be understood as reflecting a structural aspect of the world. From this perspective, that the permutation group is applicable is neither a simple ‘brute fact’ nor metaphysically derivative, in the sense of mathematically representing the nature of the particles, but rather it represents something profoundly structural about the world.

Huggett rejects the first two options but then leaves the metaphysical status of PI hanging. We shall pursue the last option a little further in another context, namely the connection between permutations of particles and diffeomorphisms of space-time points.

10 Permutation Symmetry, Structuralism and Diffeomorphism Invariance.

Structuralism has a long and interesting history which is intimately bound up with developments in physics. Both General Relativity and Quantum Mechanics had a profound impact on the work of early structuralists such as Cassirer and Eddington (for discussion, see French 2001). Putting it crudely, the central idea of this programme is to effectively deconstruct the ‘object of knowledge’ - whether space-time or quantum particles - into a web of relations bound together by symmetry principles represented group theoretically. If we focus on the group-theoretic representation of invariant properties such as mass and spin, what this ‘deconstruction’ yields are kinds of particles (see Castellani 1993 and 1998). Similarly, PI can be seen as embodying a form of structuralist representation of ‘broader’ kinds, namely bosons, fermions, 

\[\text{This is, in essence, the heart of the dispute between Balousek and Redhead and Teller.}\]
parabosons, parafermions and so on. In other words, the status of PI, from this perspective, is that of one of the fundamental symmetry principles which effectively binds the ‘web of relations’ constituting the structure of the world into these broad kinds.

**10.1 Permutation Symmetry and Structural Realism.**

It is this kind of structuralist deconstruction of objects which is incorporated into Ladyman’s ‘ontic’ form of structural realism, alluded to above. As we indicated, this attempts to avoid the metaphysical underdetermination that PI yields by reconceptualising quantum objects entirely in structural terms (see French and Ladyman 2002 and Saunders 2002). However, the following objection has been raised to such a move: if structure is understood in ‘relational’ terms - as it typically is - then there need to be ‘relata’, and these cannot be relational themselves. The force of the objection is clearly seen in the case of PI: we began by considering the distribution of objects over states and the effects of permutations on such objects. How can PI play a part in the ‘deconstruction’ of objects into structures when its very articulation is based on the assumption that there are objects to begin with? In responding to this objection, structuralists typically appeal to the following manoeuvre (it can be found in Eddington and, before him, Poincaré, for example): we recall that we begin by introducing particle labels and it is upon these that the particle permutation operator acts. We then assume that these labels denote objects - an assumption that may be supported by the observation of the individual flashes on a scintillation screen, for example - and this allows us to apply the mathematics of group theory (with its underlying standard set theory). However, once we have obtained the relevant structure, we can dispense with our original assumption, regarding it as no more than a heuristic crutch and the labels as simply convenient place holders which serve only to help us focus attention on what is metaphysically fundamental - PI in this case. To use a famous metaphor, the object is a kind of ‘ladder’ which we use to reach the structure but which we can then ‘throw away’, or ‘deconstruct’. Of course, there are other objections to structural realism which must be addressed (see, for example, Bueno op. cit., and Chakravartty 2001) but our intention here is just to indicate what may be a natural home for the structuralist understanding of PI. Furthermore, this sort of picture can accommodate a structuralist conception of space-time as well.
10.2 Diffeomorphisms, Permutations, and the Structuralist Conception of Space-Time.

Stachel (2002) has recently explored the connections between the interpretation of general covariance and permutation invariance on the one hand and the metaphysical analyses of space-time points and quantum particles on the other. He begins by abstracting the differentiability and continuity properties of manifolds leaving a bare set of points. The (continuous) principle of diffeomorphism invariance then becomes (discrete) permutation invariance. A version of the hole argument can then be seen to apply to this set (see Norton 1988 for an elementary account of this argument). We have already seen how such a set, along with PI, models the statistical behaviour of ensembles of indistinguishable quantum particles. Thus the analogy is complete and extends, mutatis mutandis, to any theory which “demands the complete indistinguishability of its fundamental objects” (ibid., 18).

On this basis the choice between ‘substantivalist’ and ‘relationist’ conceptions of space-time is rendered as that between the ‘individual’ and ‘non-individual’ metaphysics of points. Stachel himself opts for the latter package (applying the result to both the space-time and particle cases). The ‘reduced phase space’ method of solving the hole argument is then understood as applying to the permutation case (where the gauge orbits are equivalence classes of permuted states). Since that solution is seen by Stachel as corresponding to a relational solution, the particle case is understood similarly. The idea is that the objects in a set (be they the points of a manifold or the members of a quantum ensemble) are individuated only by the relations of that set: indistinguishable objects are not individuals intrinsically but derive that property from relations. We have been here before, of course, with Saunders’ Quinean approach to PII above and, indeed, Saunders also applies this approach to the space-time case.

The central idea here, then, is that any theory that demands the complete indistinguishability of its fundamental objects requires invariance under the full permutation group for discrete symmetries or the diffeomorphism group for continuous symmetries. Stachel explicitly draws the analogy between substantivalism in the space-time case and assuming individuality for quantum particles, and relationism and non-individuality, respectively. Moving in the other direction both Maidens (1993) and Hoefer (1996) have explored the idea

---

29 This solution takes the equivalence classes of diffeomorphic states (i.e., the gauge orbits) as the points of a new ‘reduced’ or ‘physical’ phase space.
of regarding space-time points as ‘non-individuals’ in some sense (Saunders op. cit. endorses Hoefer’s conclusions). What then becomes crucial is the sense in which the points are regarded as non-individuals, just as we have discussed for particles. Stachel, in particular, understands the non-individuality of particles as their being individuated “entirely in terms of the relational structures in which they are embedded” (hence the analogy with relationism on the space-time side). But then it is not clear what metaphysical work the notion of ‘non-individuality’ is doing, when we still have ‘objects’ which are represented by standard set theory (and this is precisely the criticism that can be levelled against attempts to import non-individuality into the space-time context). What one needs to do to flesh out such an account is to apply to space-time points the kind of ‘quasi-set’ theory that has been applied to non-individual quantum particles (Krause 1992). Of course, Stachel could still maintain individuality in both cases but at the price of introducing inaccessible states in quantum mechanics and indeterminism in spacetime theory. In both cases we seem to have a kind of metaphysical underdetermination.

Again the alternative, ‘middle way’, is to drop objects out of the ontology entirely, regarding both space-time and particles in structural terms. Indeed, this appears to be the more appropriate way of understanding both Stachel’s talk of individuating objects “entirely in terms of the relational structures in which they are embedded” (ibid., 2) and, as we have seen, Saunders’ account of “weakly discernible” entities. However, rather than thinking of the objects being individuated, we suggest they should be thought of as being structurally constituted in the first place. In other words, it is the relational structures which are regarded as metaphysically primary and the objects as secondary or ‘emergent’. The labels that appear in both cases, assigned to space-time points and particles respectively, are just mathematical devices which allow us to apply our set-theoretical resources. And in both cases, the relevant symmetries, encoded in diffeomorphism invariance and PI, respectively, will be seen as essential components of this structural metaphysics.

There is, for sure, plenty of work to be done here; but this structural perspective places symmetry at the heart of our metaphysics, and that surely makes the task a tantalising one!

References.

Adams, J. (1979), ‘Primitive Thisness and Primitive Identity’, Journal of Philosophy 76, 5-26.
Aitchison, I.J.R. and Mavromatos, (1991), *Contemporary Physics*, 32, 219.

Armstrong, D. (1978), *Nominalism and Realism*, Vols. I and II. Cambridge University Press.

Balousek, D. (1999, preprint), ‘Indistinguishability, Individuality and the Identity of Indiscernibles in Quantum Mechanics’.

Balousek, D. (2000), ‘Statistics, Symmetry and the Conventionality of Indistinguishability in Quantum Mechanics’, *Foundations of Physics* 30, 1-34.

Barnette, R. L. (1978), ‘Does Quantum Mechanics Disprove the Principle of the Identity of Indiscernibles?’, *Philosophy of Science* 45, 466-470.

Bedard, K. (1999), ‘Material Objects in Bohm’s Interpretation’, *Philosophy of Science* 66, 221-242.

Born, M. (1926), ‘Quantenmechanik der Stobvurgnge’, *Zeitschrift für Physik* 38, 803-827.

Bourdeau, M. and Sorkin, R.D. (1992), ‘When Can Identical Particles Collide?’, *Physical Review* D 45, 687-696.

Brown, H., Sjöqvist, E. and Bacciagaluppi, G. (1999), ‘Remarks on Identical Particles in de Broglie-Bohm Theory’, *Physics Letters* A 251, 229-235.

Bueno, O. (2001), ‘Weyl and von Neumann: Symmetry, Group Theory, and Quantum Mechanics’, [http://philsci-archive.pitt.edu/documents/disk0/00/00/04/09/index.html](http://philsci-archive.pitt.edu/documents/disk0/00/00/04/09/index.html).

Butterfield, J. (1993), ‘Interpretation and Identity in Quantum Theory’, *Studies in History and Philosophy of Science* 24, 443-476.

Casullo, A. (1984), ‘The Contingent Identity of Particulars and Universals’, *Mind* 123, 527-554.

Castellani, E. (1993), ‘Quantum Mechanics, Objects and Objectivity’, in *The Foundations of Quantum Mechanics - Historical Analysis and Open Questions*, C. Garola and A. Rossi. Kluwer, 105-114.

Castellani, E. (1989), ‘Galilean Particles: An Example of Constitution of Objects’ in *Interpreting Bodies: Classical and Quantum Objects in Modern Physics*, E. Castellani (ed.). Princeton: Princeton University Press, 181-194.

Chakravartty, A. (forthcoming), ‘Science, Metaphysics and Structural Realism’, Cambridge University Preprint.

Cortes, A. (1983), ‘Leibniz’s Principle of the Identity of Indiscernibles: A
False Principle’, *Philosophy of Science* 43, 491-505.

de Muynck, W. (1975), ‘Distinguishable and Indistinguishable-Particle Descriptions of Systems of Identical Particles’, *International Journal of Theoretical Physics* 14, 327-346.

Dickson, M. (2000), ‘Discussion: Are There Material Objects in Bohm’s Theory?’, *Philosophy of Science* 67, 704-710.

Dieks, D. (1990), ‘Quantum Statistics, Identical Particles and Correlations’, *Synthese* 82, 127-155.

Dirac, P. A. M. (1958), *The Principles of Quantum Mechanics*. Clarendon Press: Oxford.

French, S. (1989a), ‘Identity and individuality in classical and quantum physics’, *Australasian Journal of Philosophy* 67, 432-446.

French, S. (1989b), ‘Why the Principle of Identity of Indiscernibles is not Contingently True Either’, *Synthese* 78, 141-166.

French, S. (1998), ‘On the Withering Away of Physical Objects’, in *Interpreting Bodies*, E. Castellani (ed.). Princeton University Press, 93-113.

French, S. (2000), ‘Putting a New Spin on Particle Identity’, in *Spin-Statistics Connection and Commutation Relations: Experimental Tests & Theoretical Implications*, R. Hilborn and G.M. Tino (eds.). American Institute of Physics, 305-317.

French, S. (2001), ‘Symmetry, Structure and the Constitution of Objects’, [http://philsci-archive.pitt.edu/documents/disk0/00/00/03/27/index.html](http://philsci-archive.pitt.edu/documents/disk0/00/00/03/27/index.html).

French, S. and Ladyman, J. ‘Remodelling Structural Realism: Quantum Physics and the Metaphysics of Structure’, forthcoming in *Synthese*.

French, S. and Redhead, M. (1988), ‘Quantum Physics and the Identity of Indiscernibles’, *British Journal for the Philosophy of Science* 39, 233-246.

French, S. and Krause, D. (Forthcoming) *Identity and Individuality in Quantum Physics*.

Ginsberg, A. (1981), ‘Quantum Theory and the Identity of Indiscernibles Revisited’, *Philosophy of Science* 48, 487-491.

Greenberg, O.W. (1991), ‘Interactions of Particles having Small Violations of Statistics’, *Physica* 180, 419-427.
Greenberg, O.W. and Messiah, A.M.L. (1964), ‘Symmetrization Postulate and Its Experimental Foundation’, *Physical Review* 136B, 248-267.

Hestenes, D. (1970), ‘Entropy and Indistinguishability’, *American Journal of Physics* 38, 840-845.

R. C. Hilborn & G. M. Tino. (2000), *Spin-Statistics Connection and Commutation Relations: Experimental Tests & Theoretical Implications*. American Institute of Physics.

Hoefer, C. (1996), ‘The Metaphysics of Spacetime Substantivalism’, *Journal of Philosophy* Vol.93, No.1, 5-27.

Hilborn, R.C. and Yuca, C.L. (forthcoming), ‘Identical Particles in Quantum Mechanics Revisited’.

Huggett, N. (1994), ‘What are Quanta, and Why Does it Matter?’, *PSA 1994*, Vol. 2, Philosophy of Science Association, 69-76.

Huggett, N. (1997), ‘Identity, Quantum Mechanics and Common Sense’, *The Monist* 80, 118-130.

Huggett, N. (1999a), ‘Atomic Metaphysics’, *J. Phil.*, 96, 5-24.

Huggett, N. (1999b), ‘On The Significance of the Permutation Symmetry’, *British Journal for the Philosophy of Science* 50, 325-347.

Huggett, N. & Imbo, T. D. (2000), ‘What is an Elementary Quarticle?’, University of Chicago Preprint.

Imbo, T.D., Shah Imbo, C. and Sudarshan, E.C.G. (1990), ‘Identical Particles, Exotic Statistics and Braid Groups’, *Physics Letters* B 234, 103-107.

Kosso, P. (2000), ‘The Empirical Status of Symmetries in Physics’, *British Journal for the Philosophy of Science* 51, 81-98.

Krause, D. (1992), ‘On a Quasi-set Theory’, *Notre Dame Journal of Formal Logic* 33, 402-411.

Ladyman, J. (1998), ‘What is Structural Realism?’, *Studies in History and Philosophy of Science* 29, 409-424.

Leinaas, J.M. and Myrheim, J. (1997), ‘On the Theory of Identical Particles’, *Nuovo Cimento* 37B, 1-23.

Maidens, A. V. (1993), *The Hole Argument: Substantivalism and Determinism in General Relativity*. Ph.D Thesis, University of Cambridge.
Maidens, A. V. (1998), ‘Symmetry Groups, Absolute Objects and Action Principles in General Relativity’, *Stud. Hist. Phil. Mod. Phys.*, Vol. 29, No.2, 245-272.

Margenau, H. (1944), ‘The Exclusion Principle and its Philosophical Importance’, *Philosophy of Science* 11, 187-208.

Massimi, M. (2001), ‘Exclusion Principle and the Identity of Indiscernibles: A Response to Margenau’s Argument’, *British Journal for the Philosophy of Science* 52, 303-330.

Messiah, A. M. L. (1962), *Quantum Mechanics, Vol. II*. Amsterdam: North Holland.

Miller, A. (1987), ‘Symmetry and Imagery in the Physics of Bohr, Einstein and Heisenberg’, in *Symmetries in Physics (1600-1980)*, M.G. Doncel et. al. (eds.). Bellaterra, 300-325.

Mittelstaedt, P. and Castellani, E. (2000), ‘Leibniz’s Principle, Physics and the Language of Physics’, *Foundations of Physics* 30, 1585-1604.

Norton, J. (1988), ‘The Hole Argument’, *PSA 1988*, Vol. 2, Philosophy of Science Association, 56-64.

Post, H. (1963), ‘Individuality and Physics’, *The Listener* 70, 534-537.

Post, H. (1971), ‘Correspondence, Invariance and Heuristics’, *Studies in History and Philosophy of Science* 2, 213-255.

Quinton, A. (1973), *The Nature of Things*. Routledge and Kegan Paul.

Redhead, M. L. G. (1975), ‘Symmetry in Intertheory Relations’, *Synthese*, 32, 77-112.

Redhead, M. and Teller, P. (1991), ‘Particles, Particle Labels, and Quanta: The Toll of Unacknowledged Metaphysics’, *Foundations of Physics* 21, 43-62.

Redhead, M. and Teller, P. (1992), ‘Particle Labels and the Theory of Indistinguishable Particles in Quantum Mechanics’, *British Journal for the Philosophy of Science* 43, 201-218.

Reichenbach, H. (1956), *The Direction of Time*. University of California Press.

Russell, B. (1956), ‘On the Relations of Universals and Particulars’, in *Logic and Knowledge*, R.C. Marsh (ed.). New York, 105-124.
Saunders, S. (2002) ‘Scientific Realism, Again’, forthcoming in Synthese.
Saunders, S. (forthcoming), ‘Physics and Leibniz’s Law’, Oxford University Preprint.
Stachel, J. (2002), “‘The Relations Between Things’ versus ‘The Things Between Relations’: The Deeper Meaning of the Hole Argument’”, in Reading Natural Philosophy/ Essays in the History and Philosophy of Science and Mathematics, David Malament (ed.). Open Court, Chicago and LaSalle, Illinois, 231-266.
Teller, P. (1983), ‘Quantum Physics, The Identity of Indiscernibles and Some Unanswered Questions’, Philosophy of Science 50, 309-319.
Teller, P. and Redhead, M. (2001), ‘Is Indistinguishability in Quantum Mechanics Conventional?’, Foundations of Physics 30, 951-957.
van Fraassen, B. (1984), ‘The Problem of Indistinguishable Particles’, in Science and Reality: Recent Work in the Philosophy of Science, J.T. Cushing, C.F. Delaney and G.M. Gutting (eds.). University of Notre Dame Press, 153-172.
Van Fraassen, B. (1989), Laws and Symmetry. Oxford University Press.
van Fraassen, B. (1991), Quantum mechanics: An Empiricist View. Oxford University Press.
von Meyenn, K. (1987), ‘Pauli’s Belief in Exact Symmetries’, in Symmetries in Physics (1600-1980), M.G. Doncel et. al. (eds.). Bellaterra.
Weyl, H. (1928), The Theory of Groups and Quantum Mechanics. Methuen and Co.; English trans. 2nd ed..
H, Weyl. (1952), Symmetry. Princeton University Press.