Manipulating Edge Majorana Fermions in Vortex State

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(Dated: December 30, 2011)

A vortex in a model spinless $p_x + ip_y$ superconductor induces two Majorana fermions (MFs), one in the core and the other at the sample edge. In the present work, we show that edge MF can be generated, fused, transported, and braided easily by tuning gate voltages at point-like constriction junctions. Solving the time-dependent Bogoliubov-de Gennes equation, we demonstrate that the braiding of edge MFs obeys the non-Abelian statistics. The present setup is therefore a promising implementation for topological quantum computation, and has the advantage of easy manipulation and simple device structure.

PACS numbers:

Topological quantum computation is attracting considerable interests due to the unique feature of fault tolerance, where quantum information is stored non-locally and robust against decoherence caused by interaction with environment.1,2 Ground state degeneracy and rotations within the degenerate subspace with unitary non-Abelian transformation are the two important ingredients for its implementation. Systems with Majorana fermions (MFs)3–24, particles equivalent to their antiparticles, as zero-energy excitations are promising, since they are half of conventional fermions and can form non-local qubits. Superconducting state provides a natural host for these zero-energy MFs where Bogoliubov quasi-particles are composed by both electrons and holes.

It was illustrated that a vortex in spinless $p_x + ip_y$ superconductor can accommodate a zero-energy MF at its core, and the braiding of vortices obeys the non-Abelian statistics. A hetero structure of $s$-wave superconductor (S) and topological insulator (TI)25–27 was then proposed to behave similarly to the spinless $p$-wave superconductor due to strong spin-orbit coupling, and circuits of S/TI/S junctions can achieve creation, manipulation and fusion of MFs by tuning the superconductivity phases.11 Very recently, it was demonstrated that a spin-orbit coupling semiconductor (SM) in proximity to a ferromagnetic insulator (FI) plays a similar role as TI, and thus a S/SM/FI hetero structure can be a generic platform to provide topological phase.15 It was also illustrated clearly by a toy model that a one-dimensional (1D) spinless superconductor carries two zero-energy MFs at its two ends, which can be realized by a 1D spin-orbit coupling semiconductor under a magnetic field and in proximity to an $s$-wave superconductor.18,19 The non-Abelian braiding of end MFs has been demonstrated very recently in a network of these nanowires with voltage applications.20

In the present work, we notice that the edge MF realized in a finite sample with a superconducting vortex is very useful. Since MFs always come as pairs, the zero-energy MF bounded at the vortex core has its counterpart, which appears at the sample edge (see Fig. 1(a)).

The edge MFs have important advantages in manipulation, namely they can be created, fused, transported and braided easily by applying gate voltages on constriction junctions between finite samples as revealed in the present work. In contrast, motion of core MFs should always be accompanied by vortices, which is not an easy task practically. Comparing with previous proposals based on control of superconductivity phases in circuit of S/TI/S junctions11 and manipulation of end MFs in network of 1D nanowires20, gate voltages are applied at the point-like constriction junctions in the present setup, which makes the device structure and operations simpler. As a drawback of the present device, the energy
gap between edge zero-energy MFs and finite-energy excitations is smaller compared with the core counterparts, thus requiring manipulation at lower temperature. It is expected, however, that this difficulty can be overcome relatively easier by using superconductor with large energy gap.

Explicitly we consider a finite sample of the S/SM/FI hetero structure with a superconducting vortex at the sample center \[15, 18\], which we call a brick. Bricks are connected by constriction junctions, where gate voltages can be tuned to connect and disconnect the constriction junctions adiabatically. We show that edge MFs can be created, fused, transported, and braided with sequences of switching on and off the gate voltages in the system. Solving the time-dependent Bogoliubov-de Gennes (BdG) equation, we monitor the time evolution of the wave functions, and demonstrate clearly that braiding of edge MFs obeys the non-Abelian statistics.

**Bogoliubov-de Gennes Hamiltonian**

Our elementary system is schematically depicted in Fig. 1(b). The Hamiltonian of a spin-orbit coupling semiconductor in proximity to a ferromagnetic insulator is given by

\[
H_0 = \int d\vec{r} \tilde{c}_\sigma^\dagger(\vec{r}) \left[ \frac{\vec{p}^2}{2m} - \mu + \alpha_R(\vec{\sigma} \times \vec{p}) \cdot \hat{z} + V_\perp z \right] \tilde{c}(\vec{r})
\]

with \(m^*, \mu, \alpha_R\) and \(V_\perp\) being the effective electron mass, chemical potential, strength of the Rashba spin-orbit coupling, and Zeeman field, respectively, and \(\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\) the Pauli matrices and \(\tilde{c} = (\tilde{c}_\uparrow, \tilde{c}_\downarrow)^T\) the electron annihilation operators. The proximity effect from the s-wave superconductor is described by

\[
H_{sc} = \int d\vec{r} [\Delta(\vec{r}) \tilde{c}_\uparrow(\vec{r})^\dagger \tilde{c}_\downarrow(\vec{r}) + h.c.],
\]

where \(\Delta(\vec{r})\) is the effective s-wave pairing potential. The total Hamiltonian is \(H_{tot} = H_0 + H_{sc}\). For chemical potential \(\sqrt{\Delta^2 + \mu^2} < V_\perp\), the system behaves effectively as a 2D spinless \(p_x + ip_y\) superconductor with the dispersion and spin configuration given in Fig. 1(c) \[2, 10, 28, 33\].

The quasiparticle excitations in the system are described by the following Bogoliubov-de-Gennes (BdG) equation

\[
\begin{pmatrix}
H_0 & \Delta \\
\Delta^\dagger & -\sigma_y H_0^* \sigma_y
\end{pmatrix}
\begin{pmatrix}
\Psi(\vec{r})
\end{pmatrix} = E \begin{pmatrix}
\Psi(\vec{r})
\end{pmatrix},
\]

where the Nambu spinor notation \(\Psi(\vec{r}) = [u_\uparrow(\vec{r}), u_\downarrow(\vec{r}), v_\downarrow(\vec{r}), -v_\uparrow(\vec{r})]^T\) is adopted. Since \(H_0\) is Hermitian, any eigen vector \(\Psi\) of Eq.(3) with energy \(E\) has its twin \(\tilde{\sigma}_y \tau_y \Psi^*\) with energy \(-E\), where \(\tau_y\) is the Pauli matrix in Nambu spinor space. Because the BdG equation is defined by an even dimensional Hamiltonian, zero-energy eigen modes, if any, should appear in pairs. By recombining these zero-energy eigen functions, one can always achieve \(\Psi = \tilde{\sigma}_y \tau_y \Psi^*\), for which the quasiparticle operator defined by

\[
\tilde{\gamma}^\dagger = \int d\vec{r} \sum_{\sigma} u_\sigma(\vec{r}) \hat{c}_\sigma(\vec{r}) + v_\sigma(\vec{r})\hat{c}_\sigma(\vec{r})
\]

satisfies the relation \(\tilde{\gamma}^\dagger = \tilde{\gamma}\). Therefore, the zero-energy excitations of the system are actually pairs of MFs.

It has been shown \[10, 29, 30\] that the system Eq. (3) with a vortex in the superconductor accommodates a MF at the vortex core. For the case that there is only one vortex in the system, its twin should appear at the edge of the system \[28, 31\]. While the core MF has been investigated in many previous studies, little attention has been paid on the edge MF (see \[11\]), which is the main focus of the present work.

In order to explore the edge MF, we resort to numerical analysis. For this purpose, we derive the tight-binding version of \(H_0\) on a square grid,

\[
\tilde{H}_0 = -t_0 \sum_{i,j,\sigma} c_{i+1,j,\sigma}^\dagger c_{i,j,\sigma} - \mu \sum_{i,\sigma} c_{i,\sigma}^\dagger c_{i,\sigma} + \sum_{i} V_\perp (c_{i+1,\uparrow}^\dagger c_{i,\uparrow} - c_{i,\uparrow}^\dagger c_{i+1,\uparrow})
\]

\[
+it_\alpha \sum_{i,\sigma} \left[ c_{i,\sigma}^\dagger \tilde{\sigma}_y \hat{c}_1 - c_{i,\sigma}^\dagger \tilde{\sigma}_y \hat{c}_1 + h.c. \right],
\]

where both spin-reserved hopping \(t_0 = h^2/2m^*a^2\) and spin-flipped hopping \(t_\alpha = \alpha g_\perp/2a\) are between nearest neighbors with \(a\) the grid spacing, and \(\mu\) is measured from band bottom.

The tight-binding BdG equation has the same form of Eq.(3), and becomes 4\(N\) coupled linear equations with \(N\) being the total number of grid sites. The energy spectrum and all wave functions of excitations including those of MFs can be obtained by diagonalizing the 4\(N\) \(\times 4N\) matrix. In the present study, the square sample is divided into a grid of 100 \(\times 100\) sites. Although the size of the Hamiltonian matrix can be large typically in the order of \(10^3\), in most cases only several states near \(E = 0\) are relevant, for which a powerful numerical technique is available \[32\].

**Edge Majorana Fermions**

We first study a square sample (see Fig. 1(b)) with a vortex at the center \(\Delta(\vec{r}) = \Delta_0 [1 - e^{-(r/p)^2}] e^{i\phi}\) with \(s = 4a\), namely the brick. As shown in the energy spectrum in Fig. 2(a), we found a pair of zero-energy excitations in the system. By recombination, we get two wave functions, one at the vortex core and the other on the sample edge with their norms plotted in the inset of Fig. 2(a), both satisfying the generic relation for MF. We define them as core MF and edge MF. This treatment of wave functions is understood for all discussions through the present work.

Next we study two bricks connected by a constriction junction. As shown in Fig. 2(b), we find again two zero-energy MFs, each bounded by one of the two vortices.
FIG. 2: Energy spectrum of several lowest excitations and distribution of wave-function norms of zero-energy MFs, (a) for one brick with two MFs, one at vortex core and the other at brick edge, and (b) for two connected bricks with two core MFs. Results are for $\Delta_0 = 0.5 t_0$, $V_z = 0.8 t_0$, $\mu = 0$ and $t_\alpha = 0.9 t_0$, with 100 $\times$ 100 sites for the brick and 10 $\times$ 10 sites for the constriction junction.

There is no edge MF this time (see inset of Fig. 2(b)), since the two edge MFs in the two bricks meet each other at the constriction junction and fuse into a Bogoliubov quasiparticle with finite energy.

We proceed to investigate a system of three bricks, and introduce dynamic process of switching on and off constriction junctions. This is implemented by tuning the hopping integrals at the constriction junctions among bricks, which can be realized practically by adjusting gate voltage. At the initial state, the left brick is isolated while the other two are connected. There are three core and one edge MFs as revealed above. As confirmed by calculation, the core MFs remain unchanged (in an exponential accuracy) during the switching process, we omit them in the following discussions for simplicity. The constriction junction between the left and central bricks is then turned on adiabatically. As displayed in Fig. 3(a), it is interesting to observe that, the edge MF on the left brick spreads its wave function to the central and right bricks, since now there is only one unified edge of the three bricks. We emphasize that this is one of the clearest manifestations of the mobility of the edge state.

We then turn off the constriction junction between the central brick and the right one adiabatically. As shown in Fig. 3(a), the wave function of edge MF now shrinks to the right brick. As the result of these two switchings, the edge MF initially at the left brick is transported to the right brick.

During the whole process, we monitor the energy spectrum and make sure that the gap between the zero-energy MFs and excitations of finite energy remains open (see Fig. 3(b)), which protects the topological phase of the system. Compared with previous prescriptions [11, 20], the present manipulations on edge MFs are performed by applications of point-like gate voltages, which makes the device simpler.

**Braiding with Non-Abelian Statistics**

Based on the above results, we can exchange edge MFs pairwisely. The simplest structure for this purpose consists of four bricks, as depicted in Fig. 1(a), and the diagram of exchange process of two edge MFs are demonstrated in Fig. 4. As the initial state (step-0), there are two edge MFs located at the left and right bricks, whereas the central and top bricks are connected (thus no edge MFs). Following the process for transportation revealed above, we first transport the red edge MF to the top brick (step-1, -2). We then transport the green edge MF from the right brick to left brick (step-3, -4). Finally, we bring the red edge MF tentatively stored at the top brick to the right brick (step-5, -6). After these operations, the setup of the system, or the Hamiltonian, returns to the initial one, but leaving the two edge MFs exchanged.

Now let us reveal the impact of this exchange to the wave function of system. For this purpose, we evaluate the time evolution of wave function upon adiabatic switchings by using the time-dependent BdG (TDBdG) method [33–35]. We first derive the wave functions of edge MFs at the left and right bricks by diagonalizing the
shown in Fig. 5, the order parameter $O$ bricks (step-3), edge MF spreads its wave function to the central and right junction is then turned on and thus the green is totally transported to the top brick (step-2). When $1$ in Fig. 4); it drops to zero when the red edge MF is readily. When the red edge MF stored tentatively at the top brick spreads to the right brick (step-5 and -6), $O_R$ becomes positive and then achieves $O_R = +1$ by the time the whole process is finished.

Therefore, during the process of exchanging the two edge MFs illustrated in Fig. 4, the wave function of the system evolves as

$$|\Psi(t = 0)\rangle = |\phi_L\rangle + |\phi_R\rangle \Rightarrow |\Psi(t = 6T)\rangle = -|\phi_L\rangle + |\phi_R\rangle,$$

or in terms of operators,

$$\gamma_L \rightarrow \gamma_R, \quad \gamma_R \rightarrow -\gamma_L,$$

which can be presented by a unitary transformation $U_{LR} = e^{\gamma_L\gamma_R}$, the same form observed first for core MFs [6] (see also [20]).

The above exchange rule of edge MFs can be understood in the following way. Let us start from the MF transport in Fig. 3. An effective Hamiltonian for the low-energy physics of the adiabatic transport is given by

$$\tilde{H}_{\text{eff}}(t) = i\lambda_1(t)\Gamma_{LC}\tilde{\gamma}_L\tilde{\gamma}_C + i\lambda_2(t)\Gamma_{CR}\tilde{\gamma}_R\tilde{\gamma}_C,$$

where $\Gamma_{ij} = -\Gamma_{ji}$ denotes interaction between MFs, and for edge MF transport $\lambda_1(t)$ increases from 0 to 1 from $t = 0$ to $t = T$ while $\lambda_2$ remains unity, and from $t = T$ to $t = 2T$ $\lambda_2(t)$ decreases from 1 to 0 while $\lambda_1$ remains unity. A zero-energy edge MF can be composed from the three MF creation operators at any moment provided the topological phase is protected by the adiabatic processes

$$\dot{\gamma}(t) = \frac{\gamma_L\lambda_2(t)}{\sqrt{[\lambda_1(t)\Gamma_{LC}/\Gamma_{CR}]^2 + \lambda_2^2(t)}} + \frac{\gamma_R\lambda_1(t)\Gamma_{LC}/\Gamma_{CR}}{\sqrt{[\lambda_1(t)\Gamma_{LC}/\Gamma_{CR}]^2 + \lambda_2^2(t)}},$$

with $\dot{\gamma}(t = 0) = \dot{\gamma}_L$ and $\dot{\gamma}(t = 2T) = \text{sgn}(\Gamma_{LC}/\Gamma_{CR})\dot{\gamma}_R$. An interaction between $\gamma_L$ and $\gamma_R$ can be included, but it is easy to see that the result of transport remains the same.

The exchanging process which consists of three transport processes with two edge MFs as shown in Fig. 4 is then described in the following way: $\dot{\gamma}_1(t = 2T) = \text{sgn}(\Gamma_{LC}/\Gamma_{CT})\dot{\gamma}_T$ and $\dot{\gamma}_2(t = 2T) = \dot{\gamma}_R$, $\dot{\gamma}_1(t = 4T) = \dot{\gamma}_L$, $\dot{\gamma}_2(t = 4T) = \dot{\gamma}_R$. The evolution of wave function can be monitored by its projections to the initial wave functions of the two edge MFs, $O_L \equiv \langle \phi_L | \Psi(t) \rangle$ and $O_R \equiv \langle \phi_R | \Psi(t) \rangle$. At the initial state (step-0), one has $O_L = O_R = +1$ by definition. As shown in Fig. 5, the order parameter $O_L$ drops gradually since the wave function of the red edge MF spreads to the red edge MF spreads to the two edge MFs prepared at the initial stage. The time for a transport process is taken as $T = 40000\hbar/\tau_0$.

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$$\tilde{H}_{\text{eff}}(t) = i\lambda_1(t)\Gamma_{LC}\tilde{\gamma}_L\tilde{\gamma}_C + i\lambda_2(t)\Gamma_{CR}\tilde{\gamma}_R\tilde{\gamma}_C,$$
responds directly to the inequality of braiding generate two different final states, which correspond the same initial state, the two processes of different orders in Fig. 6 with five bricks and three edge MFs, starting from MFs obeys the non-Abelian statistics. As illustrated in Fig. 5.

Now we are ready to exhibit that the braiding of edge MFs have been simulated by TDBdG method, and the non-Abelian statistics is proved. The present proposal therefore provides a promising way for implementing quantum topological computation based on zero-energy MFs, and has the advantage of easy operation and simple device structure.

**Methods**

In this work, we use TDBdG method to simulate the adiabatic braiding of edge MFs upon switching on and off the constriction junctions. The main task of the simulation is to solve the TDBdG equation (6). The formal solution of the equation is \( |\Psi(t)\rangle = \exp(-i\hbar \int H(t)dt)|\Psi(0)\rangle \). To solve it numerically, we first divide the total simulation time into small steps of \( \Delta t \), in which the Hamiltonian can be considered as time independent, and thus \( |\Psi(t + \Delta t)\rangle \approx \exp(-i\hbar H(t)\Delta t)|\Psi(t)\rangle \). The exponential can be expanded by the Chebyshev polynomials \( T_m(H) \)

\[
e^{-i\hbar H(t)\Delta t} = \sum_{m=0}^{\infty} c_m(\Delta t)T_m(H). \tag{11}
\]

Because the coefficients \( c_m \) decrease exponentially only a finite number (\( M_{\text{max}} \)) of terms in the above expansion are needed for sufficient accuracy. Using the recursive relations of the Chebyshev polynomials \( T_m(H) = 2HT_{m-1}(H) - T_{m-2}(H) \), the summation in Eq.(11) can

\[
\]

FIG. 6: Diagrams for two braiding processes of three edge MFs (red, green and blue for eye-guide) and time evolution of the wave function of edge MFs, in terms of its projections \( O_L, O_R \) and \( O_T \) to the three edge MFs prepared at the initial stage.

\[
\dot{\gamma}_1(t = 2T) = \text{sgn}(\Gamma_{RC}/\Gamma_{CL})\dot{\gamma}_L, \quad \dot{\gamma}_1(t = 6T) = \text{sgn}(\Gamma_{TC}/\Gamma_{CR})\text{sgn}(\Gamma_{LC}/\Gamma_{CT})\dot{\gamma}_R = -\text{sgn}(\Gamma_{LC}/\Gamma_{CR})\dot{\gamma}_R \quad \text{and} \quad \dot{\gamma}_2(t = 6T) = \dot{\gamma}_2(t = 4T).
\]

It is then clear that after the braiding, the two MFs at the left and right bricks acquire opposite signs, as revealed in Fig. 5.

Now we are ready to exhibit that the braiding of edge MFs obeys the non-Abelian statistics. As illustrated in Fig. 6 with five bricks and three edge MFs, starting from the same initial state, the two processes of different orders of braiding generate two different final states, which corresponds directly to the inequality \( U_{LT}U_{TR} \neq U_{TR}U_{LT} \), namely the non-Abelian feature of braiding.

**Discussions**

In an implementation of topological quantum computation based on the zero-energy MF modes, the working temperature is limited by the energy gap to the lowest excitation. The excitation gap is smaller in the present setup based on edge MFs than an implementation using core MFs simply because the edge MFs are distributed over the sample edge. Quantitatively, the excitation gap is about 0.01\( \Delta_0 \) for parameters \( \Delta_0 = 0.5t_0 \), \( V_z = 0.8t_0 \) and \( \alpha_R = 0.9t_0 \). For a superconductivity gap \( \Delta_0 \sim 1\text{meV} \), the corresponding temperature of the excitation gap will be \( \sim 100\text{mK} \), which is not hard for low-temperature technologies in these days.

While not appearing apparently in the criterion for chemical potential \( V_z > \sqrt{\Delta_0^2 + \mu^2} \) supporting the topological phase, the spin-orbit coupling \( \alpha_R \) does play an important role in determining the magnitude of effective \( p \)-wave pairing and thus the excitation gap. Besides InAs [15], the layered polar semiconductor BiTeI with giant spin-orbit coupling can be a promising material for our device [37]. The Zeeman splitting \( V_z \) of order of several meV can be realized in the thin film of strong ferromagnetic insulator [38] according to a recent work [29].

The present setup has good scalability, and an array of the units in Fig. 1(a) supports a ground state wave function consisted of a linear combination of the many-body Majorana edge states. The ground state can be rotated in the degenerate subspace by exchanging pairwise the Majorana edge states, which is given by a multidimensional unitary matrix representation of the 2D braid group, and governed by the non-Abelian statistics.

To conclude, we have demonstrated that the edge MFs induced by vortices in topological superconductor can be manipulated easily by application of gate voltages at point-like constriction junctions. Adiabatic braidings of edge MFs have been simulated by TDBdG method, and the non-Abelian statistics is proved. The present proposal therefore provides a promising way for implementing quantum topological computation based on zero-energy MFs, and has the advantage of easy operation and simple device structure.
be performed by $M_{\text{max}}$ times matrix-multiplication. For a sparse matrix $H$ in our system, the computation time is of $O(N)$ [23].

The time-dependent part of the Hamiltonian is the hopping integrals at the sites on constriction junctions between bricks. In the present study, we use a parameter $0 \leq \lambda(t) \leq 1$ to scale the hopping integrals between zero to $t_0$. In order to simulate adiabatic processes, the function $\lambda(t)$ should vary slowly with $t$, and thus the total evolution time $T$ is taken sufficiently long as compared with the inverse of the lowest excitation energy. Furthermore, we use $\lambda(t) = (t/T)^2$ rather than a linear function of $t$, which suppresses efficiently excitations from zero-energy MFs to states with finite energy.

Acknowledgements

This work was supported by WPI Initiative on Materials Nanoarchitectonics, MEXT of Japan, and Grants-in-Aid for Scientific Research (No.22540377), JSPS, and partially by CREST, JST. Q.F.L. is also supported by NSFC under grants 10904092.

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