WELL-POSEDNESS OF A FULLY-COUPLED NAVIER-STOKES/Q-TENSOR SYSTEM WITH INHOMOGENEOUS BOUNDARY DATA

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Abstract. We prove short-time well-posedness and existence of global weak solutions of the Beris–Edwards model for nematic liquid crystals in the case of a bounded domain with inhomogeneous mixed Dirichlet and Neumann boundary conditions. The system consists of the Navier-Stokes equations coupled with an evolution equation for the $Q$-tensor. The solutions possess higher regularity in time of order one compared to the class of weak solutions with finite energy. Therefore the well-posedness is shown with the aid of the contraction mapping principle using that the linearized system is an isomorphism between the associated function spaces.

1. Introduction

We study the well-posedness of a model for the instationary flow of a nematic liquid crystal described by a model due to Beris and Edwards, cf. [2]. In this model the orientation and degree of ordering of the liquid crystal is described by a symmetric, traceless $d \times d$ tensor $Q$. This description goes back to Landau and DeGennes, cf. [4]. In the case that the tensor is uniaxial, i.e., it has two equal non-zero eigenvalues, it can be represented as

$$Q = s \left( n \otimes n - \frac{1}{d} I_d \right),$$

where the scalar order parameter $s \in [-\frac{1}{2}, 1]$ measures the degree of orientational ordering and $n$ is a unit vector and describes the direction of orientation. The Beris-Edwards models leads to a system, which couples the incompressible Navier-Stokes equations with a second order parabolic equation for the evolution of the tensor $Q$. More precisely, we consider

$$\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla p &= \text{div} (\nu(Q) D(u)) + \text{div}(\sigma(Q, H) + \tau(Q, H)), \\
\text{div} u &= 0, \\
\partial_t Q + (u \cdot \nabla) Q - S(\nabla u, Q) &= \Gamma H(Q)
\end{align*}$$

(1.1)

in $\Omega_T = \Omega \times (0, T)$ for a sufficiently smooth bounded domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, $T > 0$. Here $\sigma$ is a skew-symmetric tensor and $H$, $\tau$, and $S$ are symmetric tensors given by

$$
\begin{align*}
H &= \lambda \Delta Q - aQ + b(Q^2 - \frac{1}{d} \text{tr}(Q^2) I_d) - c \text{tr}(Q^2) Q, \\
\sigma(Q, H) &= QH - HQ = Q\Delta Q - \Delta QQ, \\
\tau(Q, H) &= -\lambda \nabla Q \otimes \nabla Q - \xi (Q + \frac{1}{d} I_d) H - \xi H (Q + \frac{1}{d} I_d) + 2\xi (Q + \frac{1}{d} I_d) \text{tr}(QH), \\
S(\nabla u, Q) &= (\xi D(u) + W(u)) \left( (Q + \frac{1}{d} I_d) (Q + \frac{1}{d} I_d) \left( \xi D(u) - W(u) \right) - 2\xi (Q + \frac{1}{d} I_d) \text{tr}(Q \nabla u) \right).
\end{align*}

(1.2)

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where we used the notation
\[ D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \quad W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T) \]
for the stretch and the vorticity tensor, respectively. Moreover, \( \Gamma, \lambda, a, b, \) and \( c \) are positive constants. We note that \( S(\nabla u, Q) \) is introduced to describe how the flow gradient rotates and stretches the director field.

Here \( H \) relates to the variational derivative of the free energy functional which uses the one-constant approximation for the Oseen-Frank energy of liquid crystals together with a Landau-DeGennes expression for the bulk energy
\[
F(Q) = \int_{\Omega} \left( \frac{\lambda}{2} |\nabla Q|^2 + f_B(Q) \right) \, dx, \tag{1.3}
\]
where the bulk energy \( f_B \) is given by
\[
f_B(Q) = \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}(Q^4). \]
Hence \( H = H(Q) \) can be rewritten as
\[
H(Q) = \lambda \Delta Q + L, \quad L = -aQ + b(Q^2 - \frac{1}{d} \text{tr}(Q^2) I_d) - c \text{tr}(Q^2)Q, \tag{1.4}
\]
where \( L = -Df_B(Q) \) consists of lower-order terms in the equation.

We complement this system (1.1)-(1.2) by the initial condition
\[
(u, Q)|_{t=0} = (u_0, Q_0) \quad \text{in } \Omega \tag{1.5}
\]
and the Dirichlet-Neumann boundary conditions of mixed type,
\[
u = 0 \quad \text{on } (0,T) \times \partial \Omega, \tag{1.6}
\]
\[
Q = Q_D \quad \text{on } (0,T) \times \Gamma_D, \quad \partial_n Q = Q_N \quad \text{on } (0,T) \times \Gamma_N,
\]
where \( \partial \Omega = \Gamma_D \cup \Gamma_N \) and \( \Gamma_D, \Gamma_N \) are closed, disjoint subsets of \( \mathbb{R}^d \) and \( (Q_D, Q_N) \) will be independent of \( t \in (0,T) \) in the following.

So far there are only a few results on the mathematical analysis of this system. First contributions were given by Paicu and Zarnescu. In [13] the authors consider the case \( \xi = 0, \Omega = \mathbb{R}^d \).

They prove existence of weak solutions for \( d = 2,3 \) as well as higher regularity of weak solutions and weak-strong uniqueness if \( d = 2 \). In [12] existence of weak solutions is proved provided that \( \xi \) is sufficiently close to 0 and \( \Omega = \mathbb{R}^d, d = 2,3 \). Wilkinson studied in [20] the system (1.1)-(1.2) under periodic boundary condition in the case that \( f_B \) is replaced by a certain singular potential. The potential guarantees that \( Q \) attains only physically reasonable values. He established existence of weak solutions for a general \( \xi \) and higher regularity in the case of two space dimensions and \( \xi = 0 \). Finally, Feireisl et al. [5] derived a non-isothermal variant of the Beris-Edwards system and proved existence of weak solutions for this system in the case of a singular potential and for periodic boundary conditions. Recently, Wang et al. establish in [18] a rigorous derivation from Beris-Edwards system to the Ericksen-Leslie system, which is widely investigated in the literature. Here we refer to recent works [8], [9], [19], [22] and the references therein for more details.

In the present paper we discuss existence of weak solutions in a bounded domain with mixed Dirichlet-Neumann boundary conditions as well as well-posedness of the system in a class of solutions, which possess higher regularity in time than the class of weak solutions. These solutions are not necessarily more regular with respect to the space variable. We note that in the case without boundary, one could establish higher regularity in space for these solutions by using e.g. standard difference quotient techniques. But in the present case with boundary conditions we do not have an appropriate regularity result for the principal part of the linearized system,
which is a Stokes system coupled with an elliptic equation for $Q$ through the terms $S(\nabla u, \tilde{Q})$ and $\text{div} \sigma(Q, H)$ for a suitable $\tilde{Q}$. These coupling terms cancel in the standard energy argument. However, they give rise to extra boundary integrals, when testing with higher order spatial derivatives of the solution, which cannot be absorbed. Fortunately, for higher order temporal derivatives these boundary terms vanish again. The main novelty in the paper is to use this observation together with the fact that one more temporal derivative (compared to the regularity class of weak solutions) is enough to prove Lipschitz continuity of the non-linear terms in the associated function spaces. Therefore we are able to prove existence of unique solutions in this regularity class for sufficiently short times. Let us note that we expect that our solutions also possess the natural higher regularity with respect to the spatial variables. This might be a future work. But to obtain well-posedness of the system locally in time such a regularity result is not needed.

In order to formulate our main results, we have to introduce some assumptions and notation. In the sequel, we shall assume that $\Gamma = \lambda = a = b = c = 1$ to simplify the notation. But all results hold true for general values of these constants if $c > 0$. In the following we assume that $\Omega$ is a bounded domain with $C^3$-boundary and

$$\nu \in C^2(\mathbb{R}^{d \times d}), \quad 0 < c_0 \leq \nu(\cdot) \leq c_1 < \infty$$

for some constants $c_0, c_1$. In the following $S_0$ denotes the vector space of all symmetric and trace free $d \times d$ matrices. More details on the notation are given in Section 2.1 below. We use the following notion of weak solution.

**Definition 1.1.** Suppose that $T > 0$, $u_0 \in L^2_\sigma(\Omega)$, $Q_0 \in H^1(\Omega; S_0)$, $Q_D \in H^2(\Gamma_D; S_0)$, and $Q_N \in H^\frac{3}{2}(\Gamma_N; S_0)$. A pair $(u, Q)$ with

$$u \in BC_w([0, T]; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_{0,\sigma}(\Omega)), \quad Q \in BC_w([0, T]; H^1(\Omega; S_0)) \cap L^2(0, T; H^2(\Omega; S_0))$$

is called a weak solution of the system (1.1) in $\Omega_T$ with initial conditions (1.5) and boundary conditions (1.6) if the following holds:

1. For any $v \in C^1([0, T]; H^1_{0,\sigma}(\Omega)) \cap H^1(\Omega; \mathbb{R}^d)$ and $\Psi \in C^1([0, T]; H^1(\Omega; S_0))$ with $v|_{t=T} = \Psi|_{t=T} = 0$, it holds that

$$\int_{\Omega_T} \left( (-u \cdot \partial_t v + (u \cdot \nabla u) \cdot v + \nu(Q)D(u) : D(v)) \, d(x, t) \right)$$

$$+ \int_{\Omega_T} ((\sigma + \tau)(Q, H(Q))) : \nabla v \, d(x, t) = \int_\Omega u_0 v|_{t=0} \, dx \quad (1.8)$$

and

$$- \int_{\Omega_T} Q : \partial_t \Psi \, d(x, t) + \int_{\Omega_T} u \cdot \nabla Q : \Psi \, d(x, t) - \int_{\Omega_T} S(\nabla u, Q) : \Psi \, d(x, t)$$

$$= \int_{\Omega_T} H(Q) : \Psi \, d(x, t) + \int_{\Omega} Q_0 : \Psi|_{t=0} \, dx. \quad (1.9)$$

2. For almost every $t \in (0, T)$ the following energy inequality holds:

$$\frac{1}{2} \int_\Omega |u(t, x)|^2 \, dx + \mathcal{F}(Q(t, \cdot)) + \int_{\Omega_T} (\nu(Q(\tau, x))|Du(\tau, x)|^2 + |H(Q(\tau, x))|^2) \, dx, \tau$$

$$\leq \frac{1}{2} \int_\Omega |u_0(x)|^2 \, dx + \mathcal{F}(Q_0) .$$

3. For almost every $t \in [0, T]$, $Q|_{\Gamma_D} = Q_D$ and $\frac{\partial Q}{\partial n}|_{\Gamma_N} = Q_N$. 


Throughout the paper $\Omega$ is a bounded domain with $C^3$-boundary. Our first result is a result on global existence of weak solutions in the case of homogeneous Neumann boundary conditions for the director field $Q$.

**Theorem 1.2** (Existence of weak solutions). Let $T, Q_D, Q_N, u_0$ be as in Definition 1.1. Then the system (1.1) has a global weak solution for any $T > 0$.

Our second result concerns regularity in time for weak solutions of the system (1.1). This result requires a subtle compatibility condition related to the initial data for $Q$. As (1.1) is an evolution equation, it can be written in the abstract form

$$\frac{d}{dt}(u, Q) = \mathcal{E}(u, Q)$$

where $\mathcal{E} : H^1_{0,\sigma}(\Omega) \times H^2(\Omega) \to H^{-1}_\sigma(\Omega) \times L^2(\Omega)$ is defined by

$$\langle \mathcal{E}(u, Q), (\varphi, \Psi) \rangle = -\int_\Omega (-u \otimes u + \nu(Q)D(u) + (\tau + \sigma)(Q, H)) : \nabla \varphi \, dx$$

$$+ \int_\Omega ((u \cdot \nabla)Q + S + \Gamma H) : \Psi \, dx$$

for all $(\varphi, \Psi) \in H^1_{0,\sigma}(\Omega) \times H^2(\Omega; S_0)$. Since (1.6) specifies a time-independent Dirichlet boundary condition, it follows that $\partial_t Q|_{\Gamma_D} = 0$ and this observation leads to the compatibility condition that the trace of the second component on the right-hand side of (1.10) vanishes on $\Gamma_D$. Consequently we define the phase space,

$$Z = \{(u, Q) \in H^1_{0,\sigma}(\Omega) \times H^2(\Omega) : \mathcal{E}(u, Q) \in L^2_\sigma \times H^1_{\Gamma_D}, Q = Q_D \text{ on } \Gamma_D, \partial_n Q = Q_N \text{ on } \Gamma_N \},$$

where

$$H^1_{\Gamma_D} = H^1_{\Gamma_D}(\Omega; S_0) := \{Q \in H^1(\Omega; S_0) : Q|_{\Gamma_D} = 0 \}.$$

Note that the phase space defined above is non-empty. For instance, if we choose $u_0 \in H^2(\Omega) \cap H^1_{0,\sigma}(\Omega)$ and $Q_0 \in H^3(\Omega)$ satisfying the boundary condition in (1.6) such that $\Delta Q_0|_{\Gamma_D} = 0$, then $(u_0, Q_0) \in Z$.

**Theorem 1.3** (Local existence and uniqueness of solutions with regularity in time). Suppose that

$$(Q_D, Q_N) \in H^2(\Gamma_D; S_0) \times H^2(\Gamma_N; S_0)$$

and that the initial data satisfy $(u_0, Q_0) \in Z$. Then there exists some $T > 0$ such that the system (1.1) together with (1.5) and (1.6) has a unique solution $(u, Q)$ with

$$u \in H^2(0, T; H^{-1}_\sigma(\Omega)) \cap H^1(0, T; H^1_{0,\sigma}(\Omega)),$$

$$Q \in H^2(0, T; L^2(\Omega; S_0)) \cap H^1(0, T; H^2(\Omega; S_0)).$$

**Remark 1.4.** Theorems 1.2 and 1.3 are also valid for $\Gamma_D = \emptyset$ or $\Gamma_N = \emptyset$.

Finally, we note that the idea to use higher regularity in time to obtain a unique solution of (1.1)-(1.6) (in the case $\xi = 0$) has also been used by Guillén-González and Rodríguez-Bellido [7].

2. **Preliminaries**

2.1. **Notation.** For two vectors $a, b \in \mathbb{R}^d$ we set $a \cdot b = \sum_{i=1}^d a_i b_i$ and $a \otimes b = ab^T = (a_i b_j)_{1 \leq i, j \leq d}$ and for two matrices $A, B \in \mathbb{R}^d$ we set $A : B = \sum_{i,j=1}^d A_{ij} B_{ij} = \text{tr}(A^T B)$. Then

$$(AB) : C = B : (A^T C) = A : (CB^T) \quad \text{for all } A, B, C \in \mathbb{R}^{d \times d}$$

(2.1)

and we omit the parentheses for simplicity in the following if they are clear from the context. Einstein’s summation convention is applied throughout the paper if repeated indices are written.
We define the norm of a matrix $A \in \mathbb{R}^{d \times d}$ by $|A|^2 = \text{tr}(A^T A) = A : A$. For a differentiable matrix-valued function $F : \Omega \to \mathbb{R}^{d \times d}$, we denote by $\text{div} F = (\partial_\beta F_{\alpha \beta})_{1 \leq \alpha, \beta \leq d}$ the vector whose $\alpha$th component is the divergence of the $\alpha$th row in $F$. Moreover, if $A, B : \Omega \to \mathbb{R}^{d \times d}$ are differentiable, we introduce the contraction $\odot$ by

$$\nabla A \odot \nabla B = (\partial_i A_{\alpha \beta} \partial_j B_{\alpha \beta})_{1 \leq i, j \leq d} = (\partial_i A : \partial_j B)_{1 \leq i, j \leq d}$$

and $Q : \nabla A = (Q : \partial_j A)_{1 \leq j \leq d}$. Finally, $I_d$ denotes the matrix, which represents the identity on $\mathbb{R}^d$.

For the following it is convenient to rewrite the definitions of $\tau$ and $S$ as

$$\tau(Q, H) = \tau_1(Q) + \xi \tau_2(Q, H) - \frac{2\xi}{d} H,$$

$$S(\nabla u, Q) = S_1(\nabla u, Q) + \xi S_2(\nabla u, Q) + \frac{2\xi}{d} D(u),$$

with

$$\tau_1(Q) = -\nabla Q \odot \nabla Q - \frac{1}{d} I_d \text{ tr} Q^2,$$

$$\tau_2(Q, H) = -QH - HQ + 2(Q + \frac{1}{d} I_d) \text{ tr}(QH),$$

$$S_1(\nabla u, Q) = W(u)Q - QW(u),$$

$$S_2(\nabla u, Q) = D(u)Q + QD(u) - 2(Q + \frac{1}{d} I_d) \text{ tr}(Q \nabla u).$$

We note that

$$-\text{div} \tau_1(Q) = \Delta Q \odot \nabla Q + \nabla \left( \frac{1}{d} \text{ tr} Q^2 + \sum_{j=1}^n \frac{1}{2} |\partial_j Q|^2 \right)$$

$$= H(Q) : \nabla Q + \nabla \left( \frac{1}{d} \text{ tr} Q^2 + \sum_{j=1}^n \frac{1}{2} |\partial_j Q|^2 + f_B(Q) \right).$$

Therefore

$$\int_{\Omega_T} \tau_1(Q) : \nabla v \, d(x, t) = \int_{\Omega_T} (H(Q) : \nabla Q) \cdot v \, d(x, t)$$

for all $Q$ and $v$ as in Definition 1.1.

Finally, $(x', x)_{X', X}$ denotes the duality product of $x' \in X'$ and $x \in X$ for a Banach space $X$ and $(., .)_H$ denotes the inner product of a Hilbert space $H$.

**2.2. Function spaces.** We use standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{k,p}(\Omega)$ as well as $L^p(\Omega; M)$ and $W^{k,p}(\Omega; M)$ for the corresponding spaces for $M$-valued functions. Sometimes we omit the domain and the range if they are clear from the context. The $L^2$-based Sobolev spaces are denoted by $H^k(\Omega)$ and $H^k(\Omega; M)$. The usual spaces of divergence free vector fields are introduced by

$$L^2_\sigma(\Omega) = \{ u \in L^2(\Omega; \mathbb{R}^d), \text{ div } u = 0, \gamma(u) = 0 \},$$

$$H^1_{0, \sigma}(\Omega) = \{ u \in H^1(\Omega; \mathbb{R}^d), \text{ div } u = 0 \}, \quad H^{-1}_\sigma(\Omega) = (H^1_{0, \sigma}(\Omega))'.$$

where $\gamma(u) = u \cdot n \in H^{-\frac{1}{2}}(\partial \Omega)$ is defined a generalized trace sense, cf. e.g. [16]. Note that $L^2(\Omega; \mathbb{R}^d) = L^2_\sigma(\Omega) \oplus (L^2_\sigma(\Omega))^\perp$ with $(L^2_\sigma(\Omega))^\perp = \{ u \in L^2(\Omega; \mathbb{R}^d), u = \nabla q \text{ for some } q \in H^1(\Omega) \}$. 
Lemma 2.1. Let $Q_1, Q_2 \in L^2(\Omega; \mathbb{S}_0)$ and $u \in H^1_{0,\nu}(\Omega)$. Then
\begin{equation}
S(\nabla u, Q_1) : Q_2 + \left( \sigma(Q_1, Q_2) + \xi \tau_2(Q_1, Q_2) - \frac{2\xi}{d} Q_2 \right) : \nabla u = 0.
\end{equation}

Proof. We are going to prove the following two identities
\[
\sigma(Q_1, Q_2) : \nabla u + S_1(\nabla u, Q_1) : Q_2 = 0, \quad \tau_2(Q_1, Q_2) : \nabla u + S_2(\nabla u, Q_1) : Q_2 = 0.
\]
These identities together with $-\frac{2\xi}{d} Q_2 : \nabla u + \frac{2\xi}{d} D(u) : Q_2 = 0$ imply the assertion. We use (2.1) and the symmetries to obtain
\begin{align*}
2S_1(\nabla u, Q_1) : Q_2 &= \nabla u Q_1 : Q_2 - (\nabla u)^T Q_1 : Q_2 - Q_1 \nabla u : Q_2 + Q_1 (\nabla u)^T : Q_2 \\
&= 2(\nabla u : Q_2 Q_1 - \nabla u : Q_1 Q_2) = -2\nabla u : \sigma(Q_1, Q_2).
\end{align*}
Similarly, using the symmetry of $Q_1$ and $Q_2$,
\begin{align*}
\tau_2(Q_1, Q_2) : \nabla u + S_2(\nabla u, Q_1) & : Q_2 \\
&= (-Q_1 Q_2 - Q_2 Q_1 + 2(Q_1 + \frac{1}{d} I_d) (Q_1 : Q_2)) : \nabla u \\
&\quad + (D(u) Q_1 + Q_1 D(u) - 2(Q_1 + \frac{1}{d} I_d) (Q_1 : \nabla u)) : Q_2 = 0,
\end{align*}
where we use that $Q_2$ and $\nabla u$ are trace free. The proof is complete. \qed

2.3. An algebraic identity. The following cancellation property plays an important role in the sequel.

Lemma 2.2. There exists an orthonormal basis $(e_n)_{n \in \mathbb{N}} \subset H^2(\Omega; \mathbb{S}_0)$ of $L^2(\Omega; \mathbb{S})$ and a non-decreasing sequence of corresponding eigenvalues $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{n \to \infty} \lambda_n = \infty$ such that
\begin{align*}
-\Delta e_n &= \lambda_n e_n & \text{in } \Omega, \\
e_n &= 0 & \text{on } \Gamma_D, \\
\frac{\partial e_n}{\partial n} &= 0 & \text{on } \Gamma_N.
\end{align*}
A similar result holds for Stokes operator $A := -P_\sigma \Delta$.

Lemma 2.3. There exists an orthonormal basis $(\nu_n)_{n \in \mathbb{N}} \subset H^1_{0,\nu}(\Omega) \cap W^{1,\infty}(\Omega; \mathbb{R}^d)$ of $L^2(\Omega; \mathbb{R}^d)$ of eigenfunctions and a non-decreasing sequence $(\omega_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ of corresponding eigenvalues with $\lim_{n \to \infty} \omega_n = \infty$ and $A \nu_i = \omega_i \nu_i$ for all $i \geq 1$.
The regularity result \( v_n \in W^{1,\infty}(\Omega; \mathbb{R}^d) \) follows from the standard regularity theory for the Stokes system provided that \( \partial \Omega \in C^3 \), cf. e.g. [6].

This result allows us to define the fractional Stokes operator \( A_T : \mathcal{D}(A_T) \to L^2(\Omega; \mathbb{R}^d) \) for \( m \in \mathbb{N}, m \geq 1 \). The following lemma gives the regularity of functions in \( \mathcal{D}(A_T) \) and the proof can be found in [3, Proposition 4.12].

**Lemma 2.4.** Let \( \Omega \subset \mathbb{R}^d \) be an open and bounded set of class \( C^\ell \) with \( \ell \geq 1 \). Then \( \mathcal{D}(A_T) \) is contained in \( H^m(\Omega; \mathbb{R}^d) \cap H^1_{0,\sigma}(\Omega) \) provided \( 1 \leq m \leq \ell \).

We recall a compactness result of Aubin-Lions type, see [15] for the proof.

**Lemma 2.5.** Suppose that \( p_1, p_2 \in (1, \infty) \). Assume that \( X_1, X_0 \) and \( X_1 \) are three separable and reflexive Banach spaces such that the inclusion \( X_1 \hookrightarrow X_0 \) is compact and the inclusion \( X_0 \hookrightarrow X_1 \) is continuous. If \((u^{(m)})_{m \in \mathbb{N}}\) is a bounded sequence in \( L^{p_1}(0, T; X_1) \) such that \((\partial_t u^{(m)})_{m \in \mathbb{N}}\) is bounded in \( L^{p_2}(0, T; X_1) \), then there exists a subsequence \( u^{(m')} \) which converges in \( L^{p_1}(0, T; X_0) \). Furthermore, if \( X_0 = (X_1, X_1)[1/2] \), where \( (\cdot, \cdot)_{[\sigma]} \) denotes the complex interpolation functor, and \( p_1 = \infty \), then there exists a subsequence \( u^{(m')} \) which converges in \( C([0, T]; X_0) \).

The next interpolation result is stated in the three-dimensional situation which is the main focus of this paper.

**Lemma 2.6** (Interpolation). There is some \( C > 0 \) such that for all \( f \in H^2(\Omega) \) the estimates

\[
\| f \|_{L^\infty(\Omega)} \leq C \| f \|_{H^1(\Omega)}^{\frac{2}{3}} \| f \|_{H^2(\Omega)}^{\frac{1}{3}},
\]

and

\[
\| f \|_{L^\infty(\Omega)} \leq C \| f \|_{L^2(\Omega)}^{\frac{1}{2}} \| f \|_{H^2(\Omega)}^{\frac{3}{2}},
\]

hold. If, additionally, \( H_0 \hookrightarrow H \hookrightarrow H_1 \) is a Gelfand-triple, then

\[
\| f \|_{L^2([0, T]; H)}^2 \leq 2(\| f \|_{H^1(0, T; H_1')} \| f \|_{L^2(0, T; H_1)} + \| f(0, \cdot) \|_{H_1}^2).
\]

Proofs of these statements can be found in [1, Section 2.1].

**Lemma 2.7.** Let \( X \) and \( Y \) be two Banach spaces such that \( X \subset Y \) with a continuous injection. If \( f \in L^\infty(0, T; X) \) is weakly continuous with values in \( Y \), then \( f \) is weakly continuous with values in \( X \).

The proof can be found in [17, pp.263].

### 3. Existence of weak solutions and proof of Theorem 1.2

This section is devoted to the proof of the existence of global weak solutions via a modified Galerkin method introduced in [10]. In view of Lemma 2.2 and Lemma 2.3, we define the finite-dimensional Banach spaces

\[
E_n = \text{Span}\{e_1, \ldots, e_n\} \subset H^2(\Omega; S_0) \subset L^2(\Omega; S_0),
\]

\[
V_n = \text{Span}\{v_1, \ldots, v_n\} \subset H^1_{0,\sigma}(\Omega) \cap W^{1,\infty}(\Omega; \mathbb{R}^d) \subset L^2_\sigma(\Omega),
\]

along with the orthogonal projection operators

\[
\pi_n : L^2(\Omega; S_0) \to E_n \quad \text{and} \quad \mathcal{P}_n : L^2_\sigma(\Omega) \to V_n.
\]

These two orthogonal projection operators are bounded linear operators with norms bounded by one, a fact which will be used in the calculations in this section without being mentioned. To simplify notation we use \( (\cdot, \cdot)_\Omega \) to denote the inner product in \( L^2(\Omega; \mathbb{R}^N) \), \( N \geq 1 \), and in \( L^2_\sigma(\Omega) \). Since \( QD \) coincides with some element of \( H^{\frac{3}{2}}(\partial\Omega) \) on \( \Gamma_D \), by standard results on
elliptic boundary value problems, there exists an harmonic extension \( \tilde{Q}(x) \in H^2(\Omega; S_0) \) such that \( \tilde{Q}|_{\Gamma_D} = Q_D \) and \( \partial_n \tilde{Q}|_{\Gamma_N} = 0 \).

With this notation in place, we seek approximations of the solutions of the system (1.1) of the form

\[
u^{(n)}(x,t) = \sum_{i=1}^{n} d_i(t) v_i(x), \quad Q^{(n)}(x,t) = \tilde{Q}(x) + \sum_{i=1}^{n} h_i(t) e_i(x)
\]

(3.1)
such that \((u^{(n)}, Q^{(n)})\) satisfies the generalized Navier-Stokes equations on \( V_n \), i.e.,

\[
(\partial_t u^{(n)}, v_k)_{\Omega} + ((u^{(n)} \cdot \nabla) u^{(n)}, v_k)_{\Omega} + (\nu (Q^{(n)}), D(u^{(n)}), D(v_k))_{\Omega} + ((\pi_n H(Q^{(n)})) : \nabla Q^{(n)}, v_k)_{\Omega} + \left( (\sigma + \xi \tau) (Q^{(n)}, \pi_n H(Q^{(n)})) - \frac{2\kappa}{d} \pi_n H(Q^{(n)}), \nabla v_k \right)_{\Omega} = 0
\]

(3.2)
in \((0, T)\) for all \( k \in \{1, \ldots, n\} \), the evolution equation for the director field on \( E_n \), i.e.,

\[
(\partial_t Q^{(n)}, e_\ell)_{\Omega} + ((u^{(n)} \cdot \nabla) Q^{(n)}, e_\ell)_{\Omega} - (S(\nabla u^{(n)}, Q^{(n)}), e_\ell)_{\Omega} = (H(Q^{(n)}), e_\ell)_{\Omega}
\]

(3.3)
for all \( \ell \in \{1, \ldots, n\} \), and the initial conditions

\[
u^{(n)}|_{t=0} = \mathcal{P}_n u_0, \quad Q^{(n)}|_{t=0} = \tilde{Q} + \pi_n (Q_0 - \tilde{Q})
\]

(3.4)
(3.5)
in \( \Omega \). Note that we have replaced the term \(- \text{div} \tau_1(Q, H(Q))\) in the approximate system by \(H(Q) : \nabla Q\) because of (2.3). In the following we will use that \(\partial_t Q^{(n)} \Delta Q^{(n)} \in E_n\) since \(\tilde{Q}\) is independent of \( t \) and harmonic. Hence \(\pi_n \partial_t Q^{(n)} = \partial_t Q^{(n)}\) and \(\pi_n \Delta Q^{(n)} = \Delta Q^{(n)}\).

By Lemma 2.2 and 2.3, the above system is well defined and can be regarded as a finite-dimensional system of ordinary differential equations which has a solution on a maximal time interval \([0, T_n]\) with \(T_n > 0\) for each \( n \geq 1 \). The following proposition implies that \(T_n = \infty\).

Moreover, the following a priori bounds will be essential to pass to the limit \( n \to \infty \) in the proof of Theorem 1.2.

**Proposition 3.1** (Lyapunov functional).

Let \( n \geq 1 \). Then the system (3.2)–(3.5) has the Lyapunov functional

\[
E(u^{(n)}(t, \cdot), Q^{(n)}(t, \cdot)) = \frac{1}{2} \int_{\Omega} |u^{(n)}(t, x)|^2 dx + F(Q^{(n)}(t, \cdot))
\]

which satisfies

\[
\frac{d}{dt} E(u^{(n)}(t, \cdot), Q^{(n)}(t, \cdot)) + \int_{\Omega} \nu(Q^{(n)}) |D(u^{(n)})|^2 dx + \int_{\Omega} |\pi_n H(Q^{(n)})|^2 dx = 0
\]

(3.6)
for all \( t \in [0, T_n] \). Consequently \( T_n = \infty \) for any \( n \in \mathbb{N} \).

**Proof.** We multiply (3.2) by \( d_k(t) \), integrate in space, and sum over \( k = 1, \ldots, n \). This is equivalent to replacing \( v_k \) by \( u^{(n)}(t) \) and an integration by parts leads to

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u^{(n)}|^2 dx + \int_{\Omega} \nu(Q^{(n)}) |D(u^{(n)})|^2 dx + (\pi_n H(Q^{(n)}), \nabla Q^{(n)}(t))_{\Omega}
\]

(3.7)
\[
+ \left( (\sigma + \xi \tau) (Q^{(n)}, \pi_n H(Q^{(n)})) - \frac{2\kappa}{d} \pi_n H(Q^{(n)}), \nabla u^{(n)} \right)_{\Omega} = 0
\]
in \((0, T_n)\). Note that by (3.3) the boundary conditions for \( Q^{(n)} \) are time-independent and therefore

\[
\partial_t Q^{(n)} : \partial_n Q^{(n)} = 0 \quad \text{on} \quad \partial \Omega.
\]

This fact shows in combination with the chain rule and an integration by parts that

\[
(\partial_t Q^{(n)}, \pi_n H(Q^{(n)}))_{\Omega} = (\partial_t Q^{(n)}, H(Q^{(n)}))_{\Omega} = -\frac{d}{dt} F(Q^{(n)}).
\]
Consequently we may replace \( e_\ell \) in (3.3) by \( \pi_n H(Q^{(n)}) \) and an integration by parts leads to
\[
\frac{d}{dt} F(Q^{(n)}) - ((u^{(n)} \cdot \nabla) Q^{(n)}, \pi_n H(Q^{(n)}))_\Omega \\
+ (S(\nabla u^{(n)}, Q^{(n)}), \pi_n H(Q^{(n)}))_\Omega + \int_\Omega |\pi_n H(Q^{(n)})|^2 \, dx = 0
\] (3.8)
in \([0, T_n]\). Moreover, recall that by the algebraic identity (2.4)
\[
(S(\nabla u^{(n)}, Q^{(n)}), \pi_n H(Q^{(n)}))_\Omega + ((\sigma + \xi \tau_2)(Q^{(n)}, \pi_n H(Q^{(n)})) - \frac{2\xi}{d} \pi_n H(Q^{(n)}), \nabla u^{(n)})_\Omega = 0
\]
and this identity implies the assertion of the proposition together with (3.7) and (3.8).

Finally, (3.6) implies that the norm of the solution \((u^{(n)}, Q^{(n)})\) cannot blow up in finite time. Hence the characterization of the maximal existence time for solutions of ordinary differential equations yields \( T_n = \infty \).

To construct the solution of the system (1.1) as a weak limit of approximations we need stronger a priori estimates concerning regularity in space and time. The following results hold for all \( T > 0 \). Note that, unless otherwise indicated, all constants are generic constant which may depend on \( \Omega, T, \xi, \nu \) and its derivatives, and other parameters of the system (1.1) but are independent of \( t \in [0, T] \) and the index \( n \) in the approximating system (3.2)–(3.5).

**Proposition 3.2** (Regularity in space). Let \( n \in \mathbb{N} \) and let \((u^{(n)}, Q^{(n)})\) be the solution of the system (3.2)–(3.5). Then
\[
u^{(n)} \in L^2(0, T; H^1_0(\Omega)) \cap L^\infty(0, T; L^2_\sigma(\Omega)), \quad Q^{(n)} \in L^2(0, T; H^2(\Omega))
\]
and we have the a priori estimates
\[
\|u^{(n)}\|_{L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2_\sigma(\Omega))} + \|Q^{(n)}\|_{L^2(0,T;H^2(\Omega))} \leq C_1(E_0)
\]
where the constant \( C_1(E_0) \) is independent of \( n \) but depends on \( E_0 = E(u_0, Q_0) \).

**Proof.** Proposition 3.1 implies that the ODE system (3.2)–(3.5) has a solution for all times \( T > 0 \) and that the solution satisfies the a priori estimates
\[
\sup_{t \in [0,T]} F(Q^{(n)}(t, \cdot)) + \|\pi_n H(Q^{(n)})\|_{L^2(\Omega_T)} \leq C(E_0),
\]
(3.9a)
\[
\|D(u^{(n)})\|_{L^2(\Omega_T)} + \|u^{(n)}\|_{L^\infty(0,T;L^2_\sigma(\Omega))} \leq C(E_0)
\]
(3.9b)
with a constant \( C \) independent of \( n \). Korn’s inequality implies the bound on \( u^{(n)} \) of the proposition.

To prove the bound on \( Q^{(n)} \), we need to improve the bound for \( \pi_n H(Q^{(n)}) \) to a uniform bound for \( H(Q^{(n)}) \). Since
\[
\|\pi_n\|_{L^2(\Omega)} \leq 1 \quad \text{and} \quad \pi_n H(Q^{(n)}) = \Delta Q^{(n)} + \pi_n L(Q^{(n)})
\]
these bounds can be obtained from \( \Delta Q^{(n)} \) and \( L(Q^{(n)}) \), respectively. We obtain from (1.3), and Young’s inequality for almost all \( t \in [0,T] \) that
\[
\int_\Omega \left( |\nabla Q^{(n)}(x,t)|^2 + |Q^{(n)}(x,t)|^2 \right) \, dx \leq C \left( F(Q^{(n)}(t, \cdot)) + 1 \right).
\]
As a result
\[
\|Q^{(n)}\|_{L^\infty(0,T;H^1(\Omega))} \leq C.
\]
(3.10)
The definition of \( H \) in (1.4) and (3.9a) imply
\[
\|\Delta Q^{(n)}\|_{L^2(\Omega_T)} \leq C_1 + \|\pi_n L(Q^{(n)})\|_{L^2(\Omega_T)}.
\]
(3.11)
Since \( L(Q^{(n)}) \) contains at most cubic terms in \( Q \), we infer from Sobolev’s inequality and (3.10) that \( \| L(Q^{(n)}) \|_{L^2(\Omega_T)} \leq C \) and the combination of this bound with (3.11) leads to

\[
\| \Delta Q^{(n)} \|_{L^2(\Omega_T)} \leq C
\]

and thus

\[
\| H(Q^{(n)}) \|_{L^2(\Omega_T)} \leq C.
\]

Note that the leading part of \( H(Q^{(n)}) \) is \( \Delta Q^{(n)} \) and therefore we obtain an \( H^2 \)-estimate for \( Q^{(n)} \).

In fact, for all \( t \in [0, T] \),

\[
\| Q^{(n)}(\cdot, t) \|_{H^2(\Omega)} \leq C \left( \| \Delta Q^{(n)}(\cdot, t) \|_{L^2(\Omega)} + \| Q^{(n)}(\cdot, t) \|_{L^2(\Omega)} \right).
\]

This estimate combined with (3.12) gives the second assertion in the proposition. The proof is now complete. \( \square \)

**Proposition 3.3 (Regularity in time).** Let \((u^{(n)}, Q^{(n)})\) be the solution of the system (3.2)–(3.5) for some \( n \in \mathbb{N} \). Then we have the a priori estimate

\[
\| \partial_t u^{(n)} \|_{L^2(0, T; \mathcal{D}(A^{3/2}))} + \| \partial_t Q^{(n)} \|_{L^2(0, T; H^{-1}(\Omega))} \leq C(E_0),
\]

where \( C_2(E_0) \) is independent of \( n \).

**Proof.** We begin with the estimate for \( \partial_t u^{(n)} \). By Lemma 2.4, we have the embedding \( \mathcal{D}(A) \hookrightarrow L^\infty(\Omega) \). Thus

\[
\left| \left( (u^{(n)} \cdot \nabla) u^{(n)}(t), v_k \right)_\Omega \right| \leq \| u^{(n)}(t) \|_{L^2(\Omega)} \| \nabla u^{(n)}(t) \|_{L^2(\Omega)} \| v_k \|_{L^\infty(\Omega)} \leq C \| u^{(n)}(t) \|_{L^2(\Omega)} \| \nabla u^{(n)}(t) \|_{L^2(\Omega)} \| v_k \|_{\mathcal{D}(A)},
\]

for all \( t \in [0, T] \) as well as

\[
\left| \left( (\pi_n H(Q^{(n)})) \cdot \nabla Q^{(n)}(t), v_k \right)_\Omega \right| \leq C \| H(Q^{(n)}(t)) \|_{L^2(\Omega)} \| \nabla Q^{(n)}(t) \|_{L^2(\Omega)} \| v_k \|_{\mathcal{D}(A)},
\]

and by (1.7)

\[
\left| \left( (\nu(Q^{(n)})) \nabla Q^{(n)}(t), \nabla v_k \right)_\Omega \right| \leq C \| \nabla u^{(n)}(t) \|_{L^2(\Omega)} \| v_k(t) \|_{\mathcal{D}(A^{1/2})}.
\]

Moreover, Lemma 2.4 implies the embedding \( \mathcal{D}(A^{3/2}) \hookrightarrow W^{1, \infty}(\Omega) \) and hence

\[
\left| \left( (\sigma(Q^{(n)}, \pi_n H(Q^{(n)}))(t) - \frac{2\varepsilon}{d} (\pi_n H(Q^{(n)}))(t), \nabla v_k \right)_\Omega \right| \leq C \left( 1 + \| Q^{(n)}(t) \|_{L^2(\Omega)} \right) \| H(Q^{(n)}(t)) \|_{L^2(\Omega)} \| v_k \|_{\mathcal{D}(A^{3/2})}
\]

and

\[
\left| \left( (\xi_2(Q^{(n)}, \pi_n H(Q^{(n)}))(t), \nabla v_k \right)_\Omega \right| \leq C \| Q^{(n)}(t) \|_{L^4(\Omega)} \| v_k(t) \|_{\mathcal{D}(A^{3/2})}.
\]

The combination of (3.15)–(3.18) together with (3.2) yields for all \( k \in \{1, \ldots, n\} \) and almost all \( t \in [0, T] \) that

\[
\left| \left( \partial_t u^{(n)}(t), v_k \right)_\Omega \right| \leq C b_n(t) \| v_k(t) \|_{\mathcal{D}(A^{3/2})},
\]

where \( b_n(t) \) is defined by

\[
b_n(t) = \left( 1 + \| u^{(n)}(t) \|_{L^2(\Omega)} \right) \| \nabla u^{(n)}(t) \|_{L^2(\Omega)} + \| H(Q^{(n)}(t)) \|_{L^2(\Omega)} \| \nabla Q^{(n)}(t) \|_{L^2(\Omega)} + \left( 1 + \| Q^{(n)}(t) \|_{L^4(\Omega)}^2 \right) \| H(Q^{(n)}(t)) \|_{L^2(\Omega)}.
\]

In view of \( \left( \partial_t u^{(n)}(t), v \right)_\Omega = 0 \) for \( v \perp V_n \) one obtains

\[
\left| \left( \partial_t u^{(n)}(t), v \right)_\Omega \right| \leq C b_n(t) \| v \|_{\mathcal{D}(A^{3/2})} \quad \text{for all } v \in \mathcal{D}(A^{3/2}) \text{ and for almost all } t \in [0, T].
\]
Thus
\[ \|\partial_t u^{(n)}(t)\|_{D(A^{3/2})} \leq C b_n(t) \quad \text{for almost all } t \in [0, T]. \] (3.21)
The above estimate implies the a priori bound for \( \partial_t u \) since \( b_n(t) \) is in \( L^2(0, T) \) due to (3.9) and (3.10).

Now we turn to the estimate for \( \partial_t Q^{(n)} \). By Hölder’s inequality, Sobolev’s embedding \( H^1(\Omega) \hookrightarrow L^6(\Omega) \) and (3.10) we find for almost all \( t \in [0, T] \) that
\[ \left|(u^{(n)}(t), \nabla Q^{(n)}(t), e, \ell)_{\Omega} \right| \leq \|u^{(n)}(t)\|_{L^3(\Omega)} \|\nabla Q^{(n)}(t)\|_{L^2(\Omega)} \|e\|_{L^6(\Omega)} \leq C \|u^{(n)}(t)\|_{H^1(\Omega)} \|\nabla Q^{(n)}(t)\|_{L^2(\Omega)} \|e\|_{H^1(\Omega)}, \]
\[ \] and
\[ \left|(S(u^{(n)}, Q^{(n)}))(t), e, \ell)_{\Omega} \right| \leq C \|\nabla u^{(n)}(t)\|_{L^2(\Omega)} \left(\|Q^{(n)}(t)\|_{L^6(\Omega)} + \|Q^{(n)}(t)\|_{L^3(\Omega)}\right) \|e\|_{L^6(\Omega)} \leq C \|\nabla u^{(n)}(t)\|_{L^2(\Omega)} \left(\|\nabla Q^{(n)}(t)\|_{L^2(\Omega)}^2 + 1\right) \|e\|_{H^1(\Omega)}, \]
as well as
\[ \left|(H(Q^{(n)}(t)), e, \ell)_{\Omega} \right| \leq \|H(Q^{(n)}(t))\|_{L^2(\Omega)} \|e\|_{L^6(\Omega)}. \] These estimates imply together with (3.3) that for almost every \( t \in [0, T] \),
\[ \left|(\partial_t Q^{(n)}(t), e)_{\Omega} \right| \leq C y_n(t) \|e\|_{H^1(\Omega)}, \quad \forall e \in E_n, \] (3.22)
where \( y_n(t) \) is defined by
\[ y_n(t) = \|u^{(n)}\|_{H^1(\Omega)} \left(\|\nabla Q^{(n)}(t)\|_{L^2(\Omega)}^2 + 1\right) + \|H(Q^{(n)}(t))\|_{L^2(\Omega)}. \] (3.23)
By the orthogonality of the eigenvectors, \( \langle \partial_t Q^{(n)}, e \rangle = 0 \) for all \( e \perp E_n \) and consequently
\[ \left|(\partial_t Q^{(n)}(t), e)_{\Omega} \right| \leq C y_n(t) \|e\|_{H^1(\Omega)} \quad \text{for all } e \in H^1(\Omega; S_0) \text{ and for almost all } t \in [0, T], \] (3.24)
which leads to
\[ \|\partial_t Q^{(n)}(t)\|_{H^{-1}(\Omega)} \leq C y_n(t). \] (3.25)
The assertion of the proposition follows now since \( y_n(t) \) is integrable in \( L^2(0, T) \) in view of the estimates in Proposition 3.2 and (3.23), (3.10) and (3.13).

After these preparations we are in a position to give the proof of Theorem 1.2.

Proof of Theorem 1.2. We divide the proof into several steps.

Step 1: Compactness and construction of weak limits. We conclude from Proposition 3.2 and 3.3 the following bounds on the solutions \( (u^{(n)}, Q^{(n)}) \) of the Galerkin approximation
\[ \|u^{(n)}\|_{L^2(0,T;H^1_{0,0})} + \|\Delta Q^{(n)}\|_{L^2(0,T;L^2)} + \|\partial_t Q^{(n)}\|_{L^2(0,T;H^{-1})} \leq C, \] (3.26)
where the constant \( C \) is independent of \( n \). Moreover,
\[ \|Q^{(n)}\|_{L^2(0,T;H^2)} \leq C(\Omega). \]
By the weak compactness of reflexive Banach spaces and the weak compactness of the dual spaces of separable spaces we can extract a subsequence of \( (u^{(n)}, Q^{(n)}) \), which we denote again by \( (u^{(n)}, Q^{(n)}) \), such that the weak convergences
\[ u^{(n)} \rightharpoonup_{n \to \infty} u \quad \text{in } L^2(0, T; H^1_{0,0}(\Omega)), \]
\[ Q^{(n)} \rightharpoonup_{n \to \infty} Q \quad \text{in } L^2(0, T; H^2(\Omega; S_0)), \]
\[ \Delta Q^{(n)} \rightharpoonup_{n \to \infty} \Delta Q \quad \text{in } L^2(\Omega_T; S_0), \] (3.27)
and the weak-*-convergences
\[
Q^{(n)} \rightharpoonup_{n \to \infty} Q \quad \text{in } L^\infty(0,T; H^1(\Omega; S_0)), \\
u^{(n)} \rightharpoonup_{n \to \infty} u \quad \text{in } L^\infty(0,T; L^2(\Omega))
\] (3.28)
hold. Moreover, for fixed $\epsilon > 0$ we may choose the subsequence in view of Lemma 2.5 and (3.26) in such a way that additionally the strong convergences
\[
Q^{(n)} \to_{n \to \infty} Q \quad \text{in } L^2(0,T; H^{2-\epsilon}(\Omega)) \cap L^p(\Omega_T), \ \forall p \in (1,6),
\]
\[
u^{(n)} \to_{n \to \infty} u \quad \text{in } L^2(0,T; L^2(\Omega)),
\] (3.29)
and
\[
Q^{(n)} \rightharpoonup_{n \to \infty} Q \quad \text{in } C([0,T]; L^2(\Omega)), \\
u^{(n)} \rightharpoonup_{n \to \infty} u \quad \text{in } C([0,T]; H^{-1}_\sigma(\Omega))
\] (3.30) hold. The estimates (3.30), (3.28) and Lemma 2.7 imply that
\[
u(Q^{(n)}) D(u^{(n)}) \to_{n \to \infty} \nu(Q) D(u) \quad \text{in } L^2(\Omega_T; \mathbb{R}^d), \\
H(Q^{(n)}) \to_{n \to \infty} H(Q) \quad \text{in } L^2(\Omega_T; S_0).
\] (3.32)
In fact, for all $\varphi \in L^2(\Omega_T)$ we infer from Lebesgue’s dominated convergence theorem, (3.29) and the properties of the viscosity coefficient that $\varphi \nu(Q^{(n)})$ converges strongly to $\varphi \nu(Q)$ in $L^2(\Omega_T)$ and the conclusion follows from the weak convergence of $D(u^{(n)})$ to $D(u)$ in $L^2(\Omega_T)$.

For the second assertion, note that by the strong convergence of $\pi_n$ to the identity map on $L^2(\Omega; S_0)$, we deduce from the third convergence in (3.27) that $\pi_n \Delta Q^{(n)} \rightharpoonup \Delta Q$. Since $H(Q) = \Delta Q + L(Q)$, where $L(Q)$ is a polynomial of degree less than or equal to three in $Q$, cf. (1.4), we only need to show that $L(Q^{(n)}) \to_{n \to \infty} L(Q)$ weakly in $L^2(\Omega_T)$. However, since $(L(Q^{(n)}))_{n \in \mathbb{N}}$ is bounded in $L^\infty(0,T; L^2(\Omega))$ due to $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the $L^\infty(0,T; H^1(\Omega))$-bound for $Q^{(n)}$, $(L(Q^{(n)}))_{n \in \mathbb{N}}$ possesses a weak limit in $L^2(\Omega_T)$ (for a suitable subsequence). This weak limit coincides with $L(Q)$ since $L(Q^{(n)}) \to_{n \to \infty} L(Q)$ in $L^1(\Omega_T)$ because of (3.29) with $p = 3$.

**Step 2: Derivation of the equation for $u$.** We replace $v_k$ in (3.2) by $v \in C^1([0,T]; W^{1,\infty}(\Omega))$ of the form
\[
v(t) = \sum_{k=1}^N d^k(t)v_k
\] (3.33)
and obtain the following equation which holds pointwise for $t \in [0,T]$:
\[
(\partial_t u^{(n)}, v)_\Omega + ((u^{(n)} \cdot \nabla) u^{(n)}, v)_\Omega + (\nu(Q^{(n)}) D(u^{(n)}), D(v))_\Omega + ((\pi_n H(Q^{(n)})) : \nabla Q^{(n)}, v)_\Omega \\
+ ((\sigma + \xi \tau_2)(Q^{(n)}, \pi_n H(Q^{(n)})) - \frac{2\xi}{d} \pi_n H(Q^{(n)}), \nabla v)_\Omega = 0.
\]
If we choose \( d^k(t) \) such that \( v_{\mid t=T} = 0 \) and integrate this equation in time, then an integration by parts for the first term yields

\[
\int_0^T \left( -\left( u^{(n)} , \partial_t v \right)_\Omega + \left( (u^{(n)} \cdot \nabla) u^{(n)} , v \right)_\Omega + \left( \nu(Q^{(n)}) D(u^{(n)}) , D(v) \right)_\Omega \right) dt \\
+ \int_0^T \left( \left( \sigma + \xi \tau_2 \right)(Q^{(n)} , \pi_n H(Q^{(n)})) - \frac{2\xi}{d} \pi_n H(Q^{(n)}) , \nabla v \right)_\Omega \right) dt \\
+ \int_0^T \left( \left( \pi_n H(Q^{(n)}) \right) : \nabla Q^{(n)} , v \right)_\Omega dt = (u^{(n)} , v)_{\mid t=0}.
\]

(3.34)

By the convergences (3.27), (3.32) and (3.29), one can pass to the limit as \( n \to \infty \) in the first integral in (3.34). It remains to show

\[
\int_{\Omega_T} (\sigma + \xi \tau_2)(Q^{(n)} , \pi_n H(Q^{(n)})) : \nabla v \, d(x,t) \to \int_{\Omega_T} (\sigma + \xi \tau_2)(Q , H(Q)) : \nabla v \, d(x,t),
\]

\[
\int_{\Omega_T} (\pi_n H(Q^{(n)})) : \nabla Q^{(n)} \cdot v \, d(x,t) \to \int_{\Omega_T} H(Q) : \nabla Q \cdot v \, d(x,t)
\]

as \( n \to \infty \). To prove the second assertion, we use (3.27), (3.29) and (3.32) to obtain

\[
\nabla Q^{(n)} \cdot v \to_{n \to \infty} \nabla Q \cdot v \quad \text{in} \quad L^2(\Omega_T),
\]

\[
\pi_n H(Q^{(n)}) \to_{n \to \infty} H(Q) \quad \text{in} \quad L^2(\Omega_T).
\]

Using the strong convergence of \((Q^{(n)})_{n \in \mathbb{N}}\) in \( L^1(\Omega_T) \) and the weak convergence of \( H(Q^{(n)}) \) in \( L^2(\Omega_T) \) one can easily prove the first assertion since all terms in \( \tau_2(Q,H) \) and \( \sigma(Q,H) \) are linear with respect to \( H \) and at most quadratic with respect to \( Q \). Hence we conclude

\[
\int_{\Omega_T} (-u \cdot \partial_t v + (u \cdot \nabla) u \cdot v + \nu(Q) D(u) : D(v) + H(Q) : \nabla Q \cdot v) \, d(x,t) \\
+ \int_{\Omega_T} \left( (\sigma + \xi \tau_2)(Q , H(Q)) - \frac{2\xi}{d} H(Q) \right) : \nabla v \, d(x,t) = \int_{\Omega_T} u_0(x) \cdot v(0,x) \, dx
\]

for any \( v \) of the form (3.33) with \( v(T,\cdot) = 0 \). By a density argument, the above equation also holds for any \( v \in C_0^1([0,T); V(\Omega)) \). This equation together with (2.3) implies the weak formulation (1.8).

**Step 3:** **Derivation of the equation for \( Q \).** We replace \( e_\ell \) in (3.3) by \( \Psi \in C^1([0,T]; H^1(\Omega; \mathbb{S}_0)) \) of the form \( \Psi(t) = \sum_{\ell=1}^N d^\ell(t) e_\ell \), integrate in time on \([0,T]\) and integrate by parts in the first term. This yields

\[
- \int_{\Omega_T} Q^{(n)} : \partial_t \Psi \, d(x,t) + \int_{\Omega_T} (u^{(n)} \cdot \nabla) Q^{(n)} : \Psi \, d(x,t) - \int_{\Omega_T} S(\nabla u^{(n)} , Q^{(n)}) : \Psi \, d(x,t)
\]

\[
= \int_{\Omega_T} H(Q^{(n)}) : \Psi \, d(x,t) + (Q^{(n)} , \Psi)_{\Omega t=0}.
\]

Employing (3.27) and (3.29) we conclude

\[
\int_{\Omega_T} S(\nabla u^{(n)} , Q^{(n)}) : \Psi \, d(x,t) \to_{n \to \infty} \int_{\Omega_T} S(\nabla u , Q) : \Psi \, d(x,t).
\]

Hence we can pass to the limit in the equation above. Through a density argument we obtain the weak formulation (1.9). Finally, the boundary conditions (1.6) (for almost every \( t \)) follow from the fact that \((u^{(n)},Q^{(n)})\) satisfy these boundary conditions and the (weak) continuity of the Dirichlet and Neumann trace operators on \( H^1(\Omega), H^2(\Omega) \), respectively.
4. Regularity in time and proof of Theorem 1.3

The proof of a unique local solution with additional regularity in time is obtained by Banach’s fixed-point theorem. In Section 4.1 we define the function spaces and the operators to which we will apply the fixed-point theorem, in Section 4.2 we prove that the linear operator $\mathcal{L} : X_0 \to Y_0$ defined in (4.4) is bounded, onto and one-to-one, in Section 4.3 we verify that the nonlinear operator $\mathcal{N}_0$ in (4.6) is locally Lipschitz continuous with small Lipschitz constant for $T$ sufficiently small, and in Section 4.4 we give the proof of Theorem 1.3. In this section we assume that $(u_0, Q_0) \in Z$. As usual, we formulate the first equation in (1.1) weakly by testing with divergence free vector fields. Then we obtain

$$\partial_t u - P_\sigma \text{div}(\nu(Q) D(u)) = P_\sigma \text{div}(\tau(Q, H(Q)) + \sigma(Q, H(Q)) - u \otimes u), \quad (4.1)$$

$$\partial_t Q - \Delta Q = -((u \cdot \nabla)Q + S(\nabla u, Q) + L(Q), \quad (4.2)$$

where $P_\sigma : H^{-1}(\Omega; \mathbb{R}^d) \to H^{-1}(\Omega; \mathbb{R}^d)$ and $\text{div} : L^2(\Omega; \mathbb{R}^{d \times d}) \to H^{-1}(\Omega; \mathbb{R}^d)$ are defined as in Section 2.1.

4.1. Function spaces and operators. The idea is to rewrite the nonlinear system (1.1) as an operator equation between suitable Banach spaces. We begin with the definition of the linear and the nonlinear operator in this fixed-point formulation and use these definitions together with the regularity in time asserted in Theorem 1.3 as motivation for the definition of the function spaces for the domain and the range of the operators. We linearize the system about the constant trajectory $Q_0$ of the $Q$-tensor. Then the principal part of the linear system is given by $\mathcal{S}$ and $\mathcal{L}$, where

$$\mathcal{S}(Q_0) \begin{pmatrix} u \\ Q \end{pmatrix} = \left( P_\sigma \text{div}[\nu(Q_0) D(u) + (\sigma + \xi \tau_2)(Q_0, \Delta Q) - \frac{2\xi}{\sigma} \Delta Q] \right), \quad (4.3)$$

and

$$\mathcal{L}(Q_0) \begin{pmatrix} u \\ Q \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} u \\ Q \end{pmatrix} - \mathcal{S}(Q_0) \begin{pmatrix} u \\ Q \end{pmatrix}, \quad (4.4)$$

respectively.

As a result, we can consider all the terms in (4.1) as a functional over $H^{1}_{0,\sigma}(\Omega)$ and once we obtain a solution to (4.1), we can disregard the $P_\sigma$ in (4.1) by adding a pressure term $\nabla p$ due to standard results. The nonlinear operator $\mathcal{N}$ in the reformulation of the system of partial differential equations as the operator equation $\mathcal{L}(Q_0)(u, Q) = \mathcal{N}(Q_0)(u, Q)$ is given by

$$\mathcal{N}(Q_0) \begin{pmatrix} u \\ Q \end{pmatrix} = \left( P_\sigma \text{div}[(\nu(Q) - \nu(Q_0)) D(u) + \tau_1(Q) - u \otimes u] \right)$$

$$\left( - (u \cdot \nabla)Q - L(Q) \right)$$

$$+ \left( \begin{array}{c} P_\sigma \text{div}[(\sigma + \xi \tau_2)(Q, \Delta Q) - (\sigma + \xi \tau_2)(Q_0, \Delta Q) + (\sigma + \xi \tau_2)(Q, L(Q)) - \frac{2\xi}{\sigma} L(Q)] \\ S_1(\nabla u, Q) - S_1(\nabla u, Q_0) + \xi S_2(\nabla u, Q) - \xi S_2(\nabla u, Q_0) \end{array} \right).$$

It is also useful to pass to a formulation with homogeneous initial and boundary conditions. Note that (1.1) together with the initial and boundary conditions can be formulated by the operator equation

$$\mathcal{L}(Q_0) \begin{pmatrix} u_h + u_0 \\ Q_h + Q_0 \end{pmatrix} = \mathcal{N}(Q_0) \begin{pmatrix} u_h + u_0 \\ Q_h + Q_0 \end{pmatrix},$$

where $(u_h, Q_h)$ satisfies the corresponding homogeneous initial and boundary conditions. By the definition of the linear operator $\mathcal{L}$ in (4.4), the above identity is equivalent to

$$\mathcal{L}(Q_0) \begin{pmatrix} u_h \\ Q_h \end{pmatrix} = \mathcal{N}(Q_0) \begin{pmatrix} u_h + u_0 \\ Q_h + Q_0 \end{pmatrix} + \mathcal{S}(Q_0) \begin{pmatrix} u_h \\ Q_h \end{pmatrix} \quad (4.5)$$

and the right-hand side defines a nonlinear operator

$$\mathcal{N}_0(Q_0) \begin{pmatrix} u_h \\ Q_h \end{pmatrix} = \mathcal{N}(Q_0) \begin{pmatrix} u_h + u_0 \\ Q_h + Q_0 \end{pmatrix} + \mathcal{S}(Q_0) \begin{pmatrix} u_h \\ Q_h \end{pmatrix}. \quad (4.6)$$
We now turn to the definition of functions spaces $X_0$ and $Y_0$ such that $\mathcal{L}, \mathcal{N}_0 : X_0 \to Y_0$ with $\mathcal{L}$ an isomorphism. Motivated by the idea to construct solutions which are twice differentiable in time and the precise assertions in Theorem 1.3, we define the function space for the range of the operators by

$$
Y_u = H^1(0, T; H^{-1}_\sigma(\Omega)), \quad Y_Q = H^1(0, T; L^2(\Omega; S_0)).
$$

In particular, we need to prove regularity of solutions of the linear equation $\mathcal{L}(Q_0)(u_h, Q_h) = (f, g)$ with right-hand side $(f, g) \in Y_0$ subject to homogeneous initial data. The general linear theory requires a compatibility condition which is taken care of by the definition of $Y_0$ as

$$
Y_0 = \left\{ (f, g) \in Y_u \times Y_Q : (f, g)|_{t=0} \in L^2(\Omega) \times H^1(\Omega) \right\}.
$$

These spaces are equipped with the usual norms in product spaces and for spaces of functions of one variable with values in a Banach space together with the correct norm of the initial data. More precisely, the norm of $Y_0$ is given by

$$
\|(f, g)\|_{Y_0} = \left(\|(f, g)\|^2_{Y_u \times Y_Q} + \|(f, g)\|_{L^2(\Omega) \times H^1(\Omega)}^2\right)^{\frac{1}{2}}.
$$

Note that the second part of the norm is not controlled by trace theorems applied to $Y_u \times Y_Q$. The domains of the operators are given by the Banach spaces

$$
X^1_u = H^2(0, T; H^{-1}_\sigma(\Omega)), \quad X^2_u = H^1(0, T; H^1_{0, \sigma}(\Omega)), \quad X_u = X^1_u \cap X^2_u,
$$

$$
X^1_Q = H^2(0, T; L^2(\Omega; S_0)), \quad X^2_Q = H^1(0, T; H^2(\Omega; S_0)), \quad X_Q = X^1_Q \cap X^2_Q
$$

together with the norms

$$
\|u\|_{X_u} = \left(\|u\|_{X^1_u}^2 + \|u\|_{X^2_u}^2 + \|u\|_{H^1_{0, \sigma}(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}},
$$

$$
\|Q\|_{X_Q} = \left(\|Q\|_{X^1_Q}^2 + \|Q\|_{X^2_Q}^2 + \|Q\|_{H^1(\Omega)}^2 + \|\partial_t Q\|_{H^1(\Omega)}^2\right)^{\frac{1}{2}}.
$$

Note that the last two terms in the norms are important to obtain in the subsequent constants that are uniformly bounded as $T \to 0$, cf. e.g. (2.7). The corresponding subspaces related to the homogeneous initial and boundary conditions in the formulation of the problem are defined by

$$
X_0 = \{(u, Q) \in X_u \times X_Q : \mathcal{T}(Q) = (0, 0), (u, Q)|_{t=0} = (0, 0)\}.
$$

Here the trace operator $\mathcal{T}(Q)$ is given by

$$
\mathcal{T}(Q) = (Q|_{(0,T) \times \Gamma_D}, \partial_n Q|_{(0,T) \times \Gamma_N}),
$$

and $X_0$ is equipped with the product norm

$$
\|(u, Q)\|_{X_0} = \|(u, Q)\|_{X_u \times X_Q}.
$$

Together with these norms the space $X_0$ and $Y_0$ are closed subspaces of the spaces $X_u \times X_Q$ and $Y_u \times Y_Q$, respectively.

One can check the compatibility condition in $Z$ that the right-hand side of (4.5) belongs to $Y_0$ if $(u_h, Q_h) \in X_0$, cf. the proof of Proposition 4.3 (i) below.

### 4.2. Existence and uniqueness for the linear system.

The key point in the proof of the local existence of solutions with additional regularity in time is the verification of global solvability of the linear system and of its regularity properties. This is achieved based on results on abstract parabolic evolution equations which we recall for the convenience of the reader.

Suppose that $\mathbb{V}$ and $\mathbb{H}$ are two separable Hilbert spaces such that the embedding $\mathbb{V} \hookrightarrow \mathbb{H}$ is injective, continuous, and dense. Fix $T \in (0, \infty)$. Suppose that for all $t \in [0, T]$ a bilinear form $a(t; \cdot, \cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is given which satisfies for all $\phi, \psi \in \mathbb{V}$ the following assumptions:

(a) $a(\cdot; \phi, \psi)$ is measurable on $[0, T]$;

(b) $a(t; \cdot, \cdot)$ is continuous in $t$;

(c) $a(t; \cdot, \cdot)$ is measurable in $\phi$ and $\psi$;

(d) $a(t; \cdot, \cdot)$ is continuous in $\phi$ and $\psi$;

(e) $a(t; \cdot, \cdot)$ is Lipschitz continuous in $\phi$ and $\psi$;

(f) $a(t; \cdot, \cdot)$ is Lipschitz continuous in $\phi$ and $\psi$;

(g) $a(t; \cdot, \cdot)$ is Lipschitz continuous in $\phi$ and $\psi$;

(h) $a(t; \cdot, \cdot)$ is Lipschitz continuous in $\phi$ and $\psi$.

We now show that there exists a unique solution $(u, Q) \in X_0$ of the linear system (4.5) satisfying the compatibility condition in $Z$.
(b) there exists a constant $c > 0$, independent of $t, \phi$ and $\psi$, with
\[ |a(t; \phi, \psi)| \leq c\|\phi\|\|\psi\|_V \quad \text{for all } t \in [0, T]; \]
(c) there exist $k_0, \alpha \geq 0$ independent of $t$ and $\phi$, with
\[ a(t; \phi, \phi) + k_0\|\phi\|^2_H \geq \alpha\|\phi\|^2_V \quad \text{for all } t \in [0, T]; \]
(d) $a(\cdot; \phi, \psi)$ is differentiable, $a(\cdot; \phi, \psi)$ is continuous in $[0, T]$ and $\partial_t a(t; \phi, \psi)$ is measurable with $|\partial^2_t a(t; \phi, \psi)| \leq c\|\phi\|\|\psi\|_V$ for $j = 0, 1$ with $c$ independent of $t$.

**Theorem 4.1.** Suppose that (a)–(c) hold. Then there exists a representation operator $L(t) : V \to V'$ with $a(t; \phi, \psi) = \langle L(t)\phi, \psi \rangle_{V', V}$, which is continuous and linear for fixed $t$. Moreover, for all $f \in L^2((0, T); V')$ and $y_0 \in \mathbb{H}$, there exists a unique solution
\[ y \in \{ v : [0, T] \to V \text{ with } v \in L^2(0, T; V), \partial_t v \in L^2(0, T; \mathbb{V}') \} \]
which solves the equation
\[ \partial_t y + L(t)y = f \quad \text{in } \mathbb{V}' \text{ for a.e. } t \in (0, T), \]
subject to the initial condition $y(0) = y_0$. Finally, assume additionally that (d) holds and that $y_0 \in \mathbb{V}$. Then $L : H^1((0, T); V) \to H^1((0, T); \mathbb{V}')$ is continuous and for all $f \in H^1((0, T); \mathbb{V}')$ which satisfy the compatibility condition $f(0) - L(0)y_0 \in \mathbb{H}$ the solution $y$ satisfies
\[ y \in H^1((0, T); V) \quad \text{and} \quad \partial_t y \in L^2((0, T); \mathbb{V}'). \]

The proof of this theorem can be found in [21, Lemma 26.1 and Theorem 27.2].

The following result establishes the invertibility of the linear operator equation. Note that we are seeking a solution of the linear equation in $X_0$, i.e., a solution with homogeneous initial and boundary conditions.

**Proposition 4.2** (Homogeneous linear system). Let $T \in (0, 1]$. Then $L : X_0 \to Y_0$, and for every $(f, g) \in Y_0$, the operator equation
\[ L(Q_0)(u, Q) = (f, g) \]
has a unique solution $(u, Q) \in X_0$ satisfying
\[ \|L^{-1}(Q_0)(f, g)\|_{X_0} = \|(u, Q)\|_{X_0} \leq C_L\|(f, g)\|_{Y_0} \quad (4.12) \]
where $C_L$ is independent of $T \in (0, 1]$ and $(f, g) \in Y_0$. In particular $L(Q_0) : X_0 \to Y_0$ is invertible and $L^{-1}(Q_0)$ is a bounded linear operator with norm independent of $T \in (0, 1]$.

**Proof.** The idea is to apply Theorem 4.1 and we carry out this program in the subsequent steps.

**Step 1: Function spaces.** Since the regularity is in time, we only need to incorporate the regularity in space into the spaces $\mathbb{V}$ and $\mathbb{H}$. We define the Hilbert spaces
\[ \mathbb{H} = \mathbb{H}_1 \times \mathbb{H}_2 = L^2_\chi(\Omega) \times H^1_D(\Omega; S_0), \]
\[ \mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2 = H^1_{0, \sigma}(\Omega) \times \{ Q \in H^2(\Omega; S_0) \cap H^1_D(\Omega; S_0), \partial_n Q|_{\Gamma_N} = 0 \} \]
and equip them with the inner products
\[ ((u, Q), (v, P))_\mathbb{H} = (u, v)_{L^2(\Omega)} + (Q, P)_{H^1(\Omega)} \quad \text{for all } (u, Q), (v, P) \in \mathbb{H}, \]
\[ ((u, Q), (v, P))_\mathbb{V} = (u, v)_{H^1(\Omega)} + (Q, P)_{H^2(\Omega)} \quad \text{for all } (u, Q), (v, P) \in \mathbb{V}. \]

Here the inner product in the Sobolev spaces $H^k$, $k \geq 1$, is the usual inner product in these spaces. The spaces $\mathbb{H}$ and $\mathbb{V}$ are Hilbert spaces.
Recall that $V_2 \hookrightarrow H_2 \cong H_2' \rightarrow V_2'$, where $H_2$ is identified with $H_2'$ via the Riesz isomorphism $P \mapsto (\nabla P, \nabla \cdot)_{L^2(\Omega)} + (P, \cdot)_{L^2(\Omega)}$. This implies that for all $P, \Phi \in V_2$
\[\langle P, \Phi \rangle_{V_2', V_2} = (P, \Phi)_{H_2} = \int_{\Omega} \nabla P : \nabla \Phi \, dx + \int_{\Omega} P : \Phi \, dx = \int_{\Omega} P : (I - \Delta)\Phi \, dx. \tag{4.13}\]

**Step 2: Operators and bilinear forms.** The most subtle point is the correct definition of the bilinear form since the natural bilinear form associated with $S$ given by
\[\langle L(v, P), (\varphi, \Phi) \rangle_{V', V} = \int_{\Omega} \nu(Q_0) D(v) : D(\varphi) \, dx + \int_{\Omega} ((\sigma + \xi \tau_2)(Q_0, \Delta P) - \frac{2\xi}{\alpha} \Delta P) : \nabla \varphi \, dx - \int_{\Omega} (\Delta P + S(\nabla v, Q_0)) : (I - \Delta)\Phi \, dx\]
does not lead to a bilinear form which is coercive on $V$. This is achieved by taking advantage of the cancellation property in (2.4) and by defining a bilinear form which is independent of time
\[\langle \tilde{L}(v, P), (\varphi, \Phi) \rangle_{V', V} = \int_{\Omega} \nu(Q_0) D(v) : D(\varphi) \, dx + \int_{\Omega} ((\sigma + \xi \tau_2)(Q_0, \Delta P) - \frac{2\xi}{\alpha} \Delta P) : \nabla \varphi \, dx - \int_{\Omega} (\Delta P + S(\nabla v, Q_0)) : (I - \Delta)\Phi \, dx\]
for all $(v, P), (\varphi, \Phi) \in V$. The additional term in the equation has to be compensated for on the right-hand side of the linear system and therefore we associate to $(f, g) \in Y_0$ the element $(F, G) \in L^2((0, T); V')$ by
\[\langle (F(t), G(t)), (\phi, \Phi) \rangle_{V', V} = \int_{\Omega} (f(t) \cdot \phi + g(t) : (I - \Delta)\Phi) \, dx\]
for all $(\phi, \Phi) \in V$ and almost all $t \in (0, T)$. We now assert that the solution of the abstract evolution equation
\[\langle (\partial_t u, \partial_t Q), (\phi, \Phi) \rangle_{V', V} + \langle \tilde{L}(u, Q), (\varphi, \Phi) \rangle_{V', V} = \langle (F, G), (\phi, \Phi) \rangle_{V', V} \tag{4.14}\]
for all $(\varphi, \Phi) \in V$ subject to the initial condition $(u(0), Q(0)) = (0, 0) \in H$ is indeed a weak solution of the linear evolution equation. The choice of $(\phi, 0) \in V$ implies the correct equation for $u$. To identify the equation for $Q$, choose $(0, \Phi) \in V$ as test function and obtain by (4.13)
\[\int_{\Omega} g(x, t) : (I - \Delta)\Phi(x) \, dx = \langle \partial_t Q, \Phi \rangle_{V'_2, V_2} - \int_{\Omega} (\Delta Q + S(\nabla u, Q_0)) : (I - \Delta)\Phi \, dx\]
\[= \int_{\Omega} (\partial_t Q - \Delta Q - S(\nabla u, Q_0)) : (I - \Delta)\Phi \, dx.\]
Since $(I - \Delta) V_2 \rightarrow L^2(\Omega; S_0)$ is bijective, cf. e.g. [11, Theorems 4.10 and 4.18], we conclude
\[\partial_t Q - \Delta Q - S(\nabla u, Q_0) = g \quad \text{a.e. in } \Omega \times (0, T).\]

**Step 3: Existence of time-regular solutions.** The existence of time-regular solutions follows from Theorem 4.1 once we have verified the assumptions (a)–(d) on the bilinear form and the regularity assumptions on the right-hand side. Since $a$ does not depend on time, (a) and (d) are immediate. By Sobolev’s embedding theorem, $H^2 \hookrightarrow C^0$, hence $Q_0 \in L^\infty$ and (b) follows from (1.7) and Hölder’s inequality. Moreover, in view of the cancellation property (2.4), Korn’s inequality and Young’s inequality,
\[\langle L(v, P), (v, P) \rangle_{V', V} = \int_{\Omega} \nu(Q_0) D(v) : D(v) \, dx + \int_{\Omega} |\Delta P|^2 \, dx - \int_{\Omega} (\Delta P + S(\nabla v, Q_0)) : P \, dx \geq c_0 \|(v, P)\|_{V}^2 - C \|(v, P)\|_{H}^2,
\]
for all \((v, P) \in \mathbb{V}, t \in [0, T]\), and suitable constants \(c_0, C > 0\). Therefore (c) is satisfied. Finally we obtain by the definition of \(Y_0\) that \((f, g) \in Y_0\) is equivalent to \((F, G) \in H^1(0, T; \mathbb{V}')\) and \((F(0), G(0)) \in \mathbb{H}' \cong \mathbb{H}\). Hence there exists a unique solution \((u, Q) \in H^2(0, T; \mathbb{V}') \cap H^1(0, T; \mathbb{V})\) of the abstract evolution equation and therefore for the linear equation.

**Step 4:** \(\mathcal{L}\) is a bounded isomorphism. The only regularity statement which does not follow from the regularity of the solution in Step 3 is the assertion \(Q \in H^2(L^2)\). Note that the right-hand side in the equation for \(Q\) belongs to \(H^1(L^2)\). Therefore \(\partial_t Q \in H^1(L^2)\) and \(Q \in H^2(L^2)\).

Altogether, we have proven that \(\mathcal{L}(Q_0) : X_0 \to Y_0\) is an isomorphism. The boundedness of the operator norm of \(\mathcal{L}(Q_0)^{-1} : Y_0 \to X_0\) uniformly in \(0 < T \leq 1\) can be shown as follows.

By a standard energy estimate, i.e., by taking the duality product of (4.14) and \((u, Q)^T\) and integration in time, we derive

\[
\sup_{0 \leq t \leq T} \| (u(t), Q(t)) \|_{\mathbb{H}}^2 + c_0 \int_0^T \| (u(t), Q(t)) \|_{\mathbb{V}}^2 \, dt \leq C \| (F, G) \|_{L^2(0, T; \mathbb{V}')}^2
\]

with constants \(c_0, C\) independent of \(T > 0\). Moreover, if we differentiate (4.14) with respect to \(t\) and take the duality product with \((\partial_t u, \partial_t Q)^T\), then we discover

\[
\begin{align*}
&\sup_{0 \leq t \leq T} \| (\partial_t u(t), \partial_t Q(t)) \|_{\mathbb{H}}^2 + c_0 \int_0^T \| (\partial_t u(t), \partial_t Q(t)) \|_{\mathbb{V}}^2 \, dt \\
&\quad \leq C \left( \| (\partial_t F, \partial_t G) \|_{L^2(0, T; \mathbb{V}')}^2 + \| (\partial_t u(0), \partial_t Q(0)) \|_{\mathbb{H}}^2 \right).
\end{align*}
\]

By the previous estimate, Young’s inequality, and (4.14) for \(t = 0\) we conclude

\[
\begin{align*}
&\sup_{0 \leq t \leq T} \| (\partial_t u(t), \partial_t Q(t)) \|_{\mathbb{H}}^2 + c_0 \int_0^T \| (\partial_t u(t), \partial_t Q(t)) \|_{\mathbb{V}}^2 \, dt \\
&\quad \leq C \left( \| (F, G) \|_{H^1(0, T; \mathbb{V}')}^2 + \| (F(0), G(0)) \|_{\mathbb{H}}^2 \right) + C \| (F, G) \|_{Y_0}^2
\end{align*}
\]

for all \(0 < T \leq 1\). Finally, second order time derivatives of (4.14) imply the same estimate for \((\partial^2_t u, \partial^2_t Q) \in L^2(0, T; \mathbb{V}')\). The foregoing estimates can be summarize by

\[
\| (u, Q) \|_{X_0} \leq C \| (F, G) \|_{Y_0}
\]

for all \(T \in (0, 1]\). \(\square\)

### 4.3. Local Lipschitz continuity of the nonlinear operator

In this section we analyze the nonlinear terms. The fundamental properties of the nonlinear operator are summarized in the following proposition.

**Proposition 4.3.** Fix \(0 < T \leq 1, R > 0\), \((u_0, Q_0) \in Z\), let \(\mathcal{N}_0(Q_0)\) be the nonlinear operator defined in (4.6), and recall that \(B_{X_0}(0, R) = \{(v, P) \in X_0, \| (v, P) \|_{X_0} \leq R\}\). Then the following assertions are true for all \((u_{h_i}, Q_{h_i}) \in B_{X_0}(0, R), i = 1, 2:\)

(i) \(\mathcal{N}_0(Q_0)\) maps \(X_0\) to \(Y_0\).

(ii) **Local Lipschitz continuity:** There exists a constant \(C_{\mathcal{N}_0}(T, R, Q_0, u_0) > 0\) such that

\[
\| \mathcal{N}_0(Q_0)(u_{h_1}, Q_{h_1}) - \mathcal{N}_0(Q_0)(u_{h_2}, Q_{h_2}) \|_{Y_0} \leq C_{\mathcal{N}_0}(T, R, Q_0, u_0) \| (u_{h_1} - u_{h_2}, Q_{h_1} - Q_{h_2}) \|_{X_0}.
\]

iii) **Local boundedness:** There exists a constant \(C_{\mathcal{R}}(u_0, Q_0) > 0\) independent of \(T\) and \(R\) such that

\[
\| \mathcal{N}_0(Q_0)(u_{h_1}, Q_{h_1}) \|_{Y_0} \leq C_{\mathcal{N}_0}(T, R, Q_0, u_0) \| (u_{h_1}, Q_{h_1}) \|_{X_0} + \| \mathcal{E}(u_0, Q_0) \|_{Y_0}.
\]

(iv) For \(R > 0\) fixed we have \(\lim_{T \to 0} C_{\mathcal{N}_0}(T, R, Q_0, u_0) = 0\).
We define \( \mathcal{N}_\ell \) as the \( \mathcal{N}_\ell \) component in \( \mathcal{N} \). Proof of (ii): Local Lipschitz continuity of \( \mathcal{N}_\ell \). Moreover, \( \mathcal{N}_\ell \) will skip the time interval \( (0, T) \) and domain \( \Omega \) in the vector-valued functions spaces for better readability, e.g., we denote \( L^p(L^q) = L^p(0, T; L^q(\Omega)) \). For any function \( F : \mathbb{R}^k \to \mathbb{R}^\ell \) with \( k, \ell \in \mathbb{N} \), and any points \( a_1, a_2 \in \mathbb{R}^k \) we define
\[
\|F(a)\| = F(a_1) - F(a_2).
\]
Note that by the definitions of \( P_\ell : H^{-1}(\Omega; \mathbb{R}^d) \to H_T^{-1}(\Omega) \) and \( \text{div} : L^2(\Omega; \mathbb{R}^{d \times d}) \to H^{-1}(\Omega; \mathbb{R}^d) \), we can estimate the \( H^1(H_T^{-1}) \) norm of the difference of the fields in the first component in \( \mathcal{N}_0 \) in \( Y_u \) by their \( H^1(L^2) \)-norm. Therefore all estimates in the proof of the Lipschitz continuity involve the \( H^1(L^2) \)-norm and will be accomplished based on the fact that most of the expressions are bilinear or trilinear.

Proof of (i): The range of \( \mathcal{N}_0 \) lies in \( Y_0 \). Fix \( (u_h, Q_h) \in X_0 \). The compatibility condition for the initial conditions of elements in \( \mathcal{N} \) follows from \( \mathcal{N}(u, 0) \in Z \), (4.10) and (4.6) since for all \( (u_h, Q_h) \in Z_0 \),
\[
\mathcal{N}_0(Q_0) \left( \begin{array}{c} u_h \\ Q_h \end{array} \right) \big|_{t=0} = \mathcal{N}(Q_0) \left( \begin{array}{c} u_h + u_0 \\ Q_h + Q_0 \end{array} \right) \big|_{t=0} + \mathcal{S}(Q_0) \left( \begin{array}{c} u_0 \\ Q_0 \end{array} \right).
\]
Moreover, \( \mathcal{N}_0(Q_0)(u_h, Q_h) \in Y_u \times Y_Q \) follows by inspection of all terms in the definition of \( \mathcal{N}_0 \) in the same way as in the proof of (ii).

Proof of (ii): Local Lipschitz continuity of \( \mathcal{N}_0 \). Let \( X_T = \{(u, Q) \in X_u \times X_Q : T(Q) = (Q_D, Q_N), (u, Q) \big|_{t=0} = (u_0, Q_0)\} \). We define \( (u_i, Q_i) = (u_{hi}, u_0, Q_{hi} + Q_0) \in X_T \) and \( (\overline{u}, \overline{Q}) = (u_1 - u_2, Q_1 - Q_2) \in X_0 \), where we identify as usual a function independent of \( t \) with its extension to \( (0, T) \) as a constant function. By definition,
\[
[\mathcal{N}_0(Q_0)(u_h, Q_h)] = [\mathcal{N}(Q_0)(u, Q)]
\]
and since \( \mathcal{N} \) involves only spatial derivatives, we infer
\[
\mathcal{N}(Q_0)(u_1, Q_1) \big|_{t=0} - \mathcal{N}(Q_0)(u_2, Q_2) \big|_{t=0} = 0.
\]
Hence (4.15) is equivalent to
\[
\|\mathcal{N}(Q_0)(u_1, Q_1) - \mathcal{N}(Q_0)(u_2, Q_2)\|_{Y_u \times Y_Q} \leq C_{\mathcal{N}_0}(T, R, Q_0, u_0)\|\overline{u}, \overline{Q}\|_{X_0}
\]
for all \( (u_i, Q_i) \in X_T \) such that
\[
\|(u_i, Q_i) - (u_0, Q_0)\|_{X_0} \leq R.
\]
The proof of the Lipschitz continuity requires additional estimates and is therefore divided into several steps in which we estimate the differences between the various terms in the operators.

Step 1: Uniform bounds. We have for \( i = 1, 2 \) uniform bounds in space–time,
\[
\|Q_i - Q_0\|_{L^\infty(T, \Omega)}^2 + \|\partial_t Q_i - \partial_t Q_0\|_{L^2(L^\infty)}^2 \leq C T^\frac{1}{2} R^2,
\]
\[
\|\overline{Q}\|_{L^\infty(T, \Omega)}^2 + \|\partial_t \overline{Q}\|_{L^2(L^\infty)}^2 \leq C T^\frac{1}{2} \overline{\|Q\|}_{X_0}^2.
\]
as well as uniform bounds in time for higher-order norms in space,
\[
\|Q_i\|_{L^\infty(H^k)} \leq C(R + \|Q_0\|_{H^k}), \quad k \in \{0, 1, 2\},
\]
\[
\|u_i\|_{L^\infty(H^k)} \leq C(R + \|u_0\|_{H^k}), \quad k \in \{0, 1\},
\]
\[
\|\mathbf{Q}\|_{L^\infty(H^1)} + \|\mathbf{Q}\|_{L^\infty(H^2)} \leq C\left(\|\mathbf{Q}\|_{X_Q} + \|\mathbf{Q}\|_{X_Q}\right).
\]

Note carefully, that the constants are independent of \(T\). More precisely, by the interpolation result (2.5),
\[
\|Q_i - Q_0\|_{L^\infty(\Omega_T)} \leq C\|Q_i - Q_0\|^\frac{1}{2}_{L^\infty(H^1)}\|Q_i - Q_0\|^\frac{1}{2}_{L^\infty(H^2)}.
\]

We apply (2.7) with \(H = H_1 = H^k(\Omega), 0 \leq k \leq 2\) to \(Q_i - Q_0\), observe that the term related to the initial conditions vanishes since \(Q_i|_{t=0} = Q_0\) and obtain for \(i = 1, 2\)
\[
\|Q_i - Q_0\|_{L^\infty(H^1)}\|Q_i - Q_0\|_{L^\infty(H^2)}
\]
\[
\leq C\|Q_i - Q_0\|_{L^2(H^1)}\|Q_i - Q_0\|_{L^2(H^2)}\|Q_i - Q_0\|^\frac{1}{2}_{H^1(H^1)}\|Q_i - Q_0\|^\frac{1}{2}_{H^1(H^2)}
\]
\[
\leq CT^\frac{1}{2}\|Q_i - Q_0\|_{L^2(H^1)}\|Q_i - Q_0\|_{L^2(H^2)}\|Q_i - Q_0\|^\frac{1}{2}_{H^1(H^1)}\|Q_i - Q_0\|^\frac{1}{2}_{H^1(H^2)}
\]
\[
\leq CT^\frac{1}{2}\|Q_i - Q_0\|^2_{H^1(H^2)} \leq CT^\frac{1}{2} R^2,
\]

where we used (4.18) in the last step. The uniform bound for \(Q_i - Q_0\) in (4.19) follows immediately. To estimate the time derivatives of \(Q_i - Q_0\), we use Hölder’s inequality and (2.6) and find for \(i = 1, 2\) that
\[
\|
\partial_t Q_i - \partial_t Q_0\|_{L^2(L^\infty)}^2
\]
\[
\leq C\|
\partial_t Q_i - \partial_t Q_0\|_{L^2(L^2)}^2\|
\partial_t Q_i - \partial_t Q_0\|_{L^2(H^2)}^2
\]
\[
\leq CT^\frac{1}{2}\|
\partial_t Q_i - \partial_t Q_0\|_{L^\infty(L^2)}\|
\partial_t Q_i - \partial_t Q_0\|_{L^2(H^2)}^2
\]
\[
\leq CT^\frac{1}{2}\left(\|
\partial_t Q_i - \partial_t Q_0\|_{H^1(L^2)}^2 + \|
\partial_t Q_i - \partial_t Q_0\|_{t=0}\|\n\partial_t Q_i - \partial_t Q_0\|_{H^1(H^2)}^2\right)
\]
\[
\leq CT^\frac{1}{2}\|Q_i - Q_0\|^2_{X_Q} \leq CT^\frac{1}{2} R^2.
\]

In the last step, we used the definition (4.9) of the norm in \(X_Q\). The estimates for \(\mathbf{Q}\) are analogous and the proof of (4.19) is complete. To verify (4.20) we employ the triangle inequality, (2.7) and (4.18) and find for \(i = 1, 2\) and \(k = 0, 1, 2\) that
\[
\|Q_i\|_{H^k} \leq \|Q_i - Q_0\|_{H^k} + \|Q_0\|_{H^k}
\]
\[
\leq C\left(\|Q_i - Q_0\|_{H^k} + \|Q_i - Q_0\|_{t=0}\|H^k\right)
\]
\[
\leq C(R + \|Q_0\|_{H^k})
\]

and
\[
\|\mathbf{Q}\|_{H^k} \leq C(\|\mathbf{Q}\|_{H^k} + \|\mathbf{Q}\|_{t=0}\|H^k\}) = C\|\mathbf{Q}\|_{X_Q}.
\]

The estimates for \(\mathbf{Q}\) and \(u_i\) are similar and therefore (4.20) has been established.

**Step 2: Estimates for differences of viscosities.** Note that by the fundamental theorem of calculus,
\[
\nu(Q_1) - \nu(Q_2) = \int_0^1 \frac{d}{d\tau} \nu(\tau Q_1 + (1-\tau)Q_2) \, d\tau = \int_0^1 (\nabla \nu)(\tau Q_1 + (1-\tau)Q_2) : \mathbf{Q} \, d\tau,
\]

and by (4.19) and for \(i = 1, 2\),
\[
\|\nu(Q_1) - \nu(Q_2)\|_{L^\infty(\Omega_T)} \leq C(R, \nu, Q_0)\|\mathbf{Q}\|_{L^\infty(\Omega_T)} \leq C(R, \nu, Q_0)T^\frac{1}{2}\|\mathbf{Q}\|_{X_Q},
\]
\[
\|\nu(Q_i) - \nu(Q_0)\|_{L^\infty(\Omega_T)} \leq C(R, \nu, Q_0)\|Q_i - Q_0\|_{L^\infty(\Omega_T)} \leq C(R, \nu, Q_0)T^\frac{1}{2}.
\]
If one differentiates the integral representation, then one finds
\[
\partial_t (\nu(Q_1) - \nu(Q_2)) = \int_0^t \left\{ (\nabla^2 \nu)(\tau Q_1 + (1 - \tau)Q_2) \nabla \partial_t Q_1 + (1 - \tau)\partial_t Q_2 \right\} \, d\tau.
\]
We deduce from (1.7), (4.19), (4.20) and the foregoing formula that for \(i = 1, 2\) and a.e. in \(\Omega_T\),
\[
|\partial_t (\nu(Q_i))|, |\partial_t (\nu(Q_0))| \leq C(R)(|Q_i| + |\partial_t Q_i|),
\]
\[
|\partial_t (\nu(Q_i)) - \partial_t (\nu(Q_0))| \leq C(R)(|Q_i - Q_0| + |\partial_t Q_i - \partial_t Q_0|).
\]
Finally note that
\[
|\partial_t Q_i|_{L^\infty(\Omega_T)} \leq C||Q_i||_{X_0}.
\]

**Step 3: Estimates for differences of viscous stress tensor.** We verify the estimate
\[
\left\| P \operatorname{div}(\nu(Q) - \nu(Q_0))D(u) \right\|_{H^1_H^{-1}} \leq C(\nu, R) T^{\frac{1}{2}} \left\| (\mathbf{\nabla}, Q) \right\|_{X_u \times X_Q}.
\]
To this end we rewrite this expression as
\[
\left\| P \operatorname{div}(\nu(Q_1) - \nu(Q_0))D(u_1) - (\nu(Q_2) - \nu(Q_0))D(u_2) \right\|_{H^1_H^{-1}}
\leq \left\| \operatorname{div}(\nu(Q_1) - \nu(Q_2))D(u_1) + (\nu(Q_2) - \nu(Q_0))D(\mathbf{\nabla}) \right\|_{H^1_H^{-1}}
\leq \left\| (\nu(Q_1) - \nu(Q_2))D(u_1) \right\|_{H^1_L^2} + \left\| (\nu(Q_2) - \nu(Q_0))D(\mathbf{\nabla}) \right\|_{H^1_L^2},
\]
Expressing these norms as \(L^2(L^2)\)-norms of the functions and their first order derivative in time leads to four higher-order and two lower-order terms. For the higher-order terms we find by (4.27) and (4.19)
\[
\left\| \partial_t (\nu(Q_1) - \nu(Q_2))D(u_1) \right\|_{L^2(\Omega_T)} \leq C(R) \left\| (\mathbf{\nabla}) + |\partial_t \mathbf{\nabla}| \right\|_{L^2(\Omega_T)}
\leq C(R) \left\| (\mathbf{\nabla}) \right\|_{H^1_L^2} \left\| D(u_1) \right\|_{L^\infty_L^2} \leq C(R) T^{\frac{1}{2}} \left\| \mathbf{\nabla} \right\|_{X_u},
\]
and analogously
\[
\left\| \partial_t (\nu(Q_i) - \nu(Q_0))D(\mathbf{\nabla}) \right\|_{L^2(\Omega_T)} \leq C(R) \left\| (\mathbf{\nabla}) \right\|_{H^1_L^2} \left\| D(\mathbf{\nabla}) \right\|_{L^\infty_L^2} \leq C(R) T^{\frac{1}{2}} \left\| \mathbf{\nabla} \right\|_{X_u}.
\]
We obtain for the remaining two higher-order terms by (4.19), (4.20) and (4.26)
\[
\left\| (\nu(Q_1) - \nu(Q_2))\partial_t D(u_1) \right\|_{L^2(\Omega_T)} \leq C(R) \left\| (\mathbf{\nabla}) \right\|_{L^\infty_L(\Omega_T)} \left\| \partial_t D(u_1) \right\|_{L^2(\Omega_T)}
\leq C(R) \left\| (\mathbf{\nabla}) \right\|_{L^\infty_L(\Omega_T)} \left\| (\mathbf{\nabla}) \right\|_{L^2(\Omega_T)},
\]
and
\[
\left\| (\nu(Q_2) - \nu(Q_0))\partial_t D(\mathbf{\nabla}) \right\|_{L^2(\Omega_T)} \leq C(R) T^{\frac{1}{2}} \left\| \mathbf{\nabla} \right\|_{X_u}.
\]
To estimate the lower-order terms in the \(H^1_L^2\)-norm we use (4.26) and obtain
\[
\left\| (\nu(Q_1) - \nu(Q_2))D(u_1) \right\|_{L^2_L(\Omega_T)} + \left\| (\nu(Q_2) - \nu(Q_0))D(\mathbf{\nabla}) \right\|_{L^2_L(\Omega_T)}
\leq C(R) T^{\frac{1}{2}} \left\| (\mathbf{\nabla}) \right\|_{X_u \times X_Q}.
\]
The combination of the foregoing estimates implies the assertion of this step.
Step 4: Fundamental estimates for bilinear forms. Suppose that $P_1, P_2$ are time dependent tensor fields with initial value $P_0$, that $\mathcal{T} = P_1 - P_2$, and that $B : \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d}$ is a bilinear form with constant coefficients. Then
\[
\|B(Q - Q_0, P)\|_{H^1(L^2)} \\
\leq C T \frac{1}{2} R(\|\mathcal{T}\|_{L^\infty(L^2)} + \|\mathcal{T}\|_{H^1(L^2)}) + C \frac{1}{2} \|Q\|_{X_0} \left(\|P_2\|_{L^\infty(L^2)} + \|P_2\|_{H^1(L^2)}\right)
\]
where we assume that all norms on the right-hand side are finite. In particular,
\[
\|\|B(Q - Q_0, P)\|_{H^1(L^2)} \leq C(R) T \frac{1}{2} \|Q\|_{X_0}
\]
for $P \in \{\nabla u, D(u), W(u), \Delta Q\}$. In fact, by the triangle inequality and the product rule for bilinear forms,
\[
\|B(Q_1 - Q_0, P_1) - B(Q_2 - Q_0, P_2)\|_{H^1(L^2)} \\
\leq \|B(Q_1 - Q_0, \mathcal{T})\|_{H^1(L^2)} + \|B(Q_2 - Q_0, \mathcal{T})\|_{H^1(L^2)} \\
\leq \|B(\partial_t Q_1 - \partial_t Q_0, \mathcal{T})\|_{L^2(L^2)} + \|B(Q_1 - Q_0, \partial_t \mathcal{T})\|_{L^2(L^2)} + \|B(Q_2 - Q_0, \partial_t \mathcal{T})\|_{L^2(L^2)} \\
\leq \|Q_1 - Q_0\|_{L^\infty(L^\infty)} \|\mathcal{T}\|_{H^1(L^2)} + \|Q_1 - Q_0\|_{L^\infty(\Omega_T)} \|\mathcal{T}\|_{H^1(L^2)} \\
\]
The assertion follows from (4.19) and (4.20).

Step 5: Fundamental estimates for trilinear forms. Suppose that $P_1, P_2$ are time dependent tensor fields with initial values $P_0$ and that $E : \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d}$ is a trilinear form with constant coefficients. Then
\[
\|\|E(Q, Q, P) - E(Q_0, Q_0, P)\|_{H^1(L^2)} \leq C(R) T \frac{1}{2} \|Q\|_{X_0}
\]
for $P \in \{\nabla u, D(u), W(u), \Delta Q\}$. To see this, note that
\[
E(Q_1, Q_1, P_1) - E(Q_0, Q_0, P_1) - E(Q_2, Q_0, P_2) + E(Q_0, Q_0, P_2) \\
= E(Q_1, Q_1, \mathcal{T}) - E(Q_0, Q_0, \mathcal{T}) + E(Q_1, Q_1, P_2) - E(Q_2, Q_0, P_2) \\
= E(Q_1 - Q_0, Q_1, \mathcal{T}) + E(Q_0, Q_1 - Q_0, \mathcal{T}) + E(Q_1 - Q_2, Q_0, P_2) + E(Q_2, Q_1 - Q_2, P_2) .
\]
We need to estimate this sum in the $H^1(L^2)$-norm. By Hölder’s inequality and the product rule for trilinear forms, each term leads to four terms that need to be estimated in $L^2(L^2)$. For the first term we find
\[
\|E(Q_1 - Q_0, Q_1, \mathcal{T})\|_{H^1(L^2)} \\
\leq \|E(\partial_t Q_1 - \partial_t Q_0, Q_1, \mathcal{T}) + E(Q_1 - Q_0, \partial_t Q_1, \mathcal{T}) + E(Q_1 - Q_0, Q_1, \partial_t \mathcal{T})\|_{L^2(L^2)} \\
\leq \|\|E(\partial_t Q_1 - \partial_t Q_0, Q_1, \mathcal{T})\|_{L^2(L^2)} + \|E(Q_1 - Q_0, \partial_t Q_1, \mathcal{T})\|_{L^2(L^2)} + \|E(Q_1 - Q_0, Q_1, \partial_t \mathcal{T})\|_{L^2(L^2)}\}
\]
for $P \in \{\nabla u, D(u), W(u), \Delta Q\}$. To see this, note that
\[
E(Q_1, Q_1, P_1) - E(Q_0, Q_0, P_1) - E(Q_2, Q_0, P_2) + E(Q_0, Q_0, P_2) \\
= E(Q_1, Q_1, \mathcal{T}) - E(Q_0, Q_0, \mathcal{T}) + E(Q_1, Q_1, P_2) - E(Q_2, Q_0, P_2) \\
= E(Q_1 - Q_0, Q_1, \mathcal{T}) + E(Q_0, Q_1 - Q_0, \mathcal{T}) + E(Q_1 - Q_2, Q_0, P_2) + E(Q_2, Q_1 - Q_2, P_2) .
\]
The second term can be estimated the same way. For the third term one finds
\[ \|E(Q_1 - Q_2, Q_1, P_2)\|_{H^1(L^2)} \]
\[ \leq \|E(\partial_t Q_1 - \partial_t Q_2, Q_1, P_2) + E(Q_1 - Q_2, \partial_t Q_1, P_2) + E(Q_1 - Q_2, Q_1, \partial_t P_2)\|_{L^2(L^2)} \]
\[ + \|E(Q_1 - Q_2, Q_1, P_2)\|_{L^2(L^2)} \]
\[ \leq C\{\|
\partial_t Q_1 - \partial_t Q_2\|_{L^2(L^2)}\|Q_1\|_{L^\infty(\Omega_T)}\|P_2\|_{L^\infty(L^2)} + \|Q_1 - Q_2\|_{L^\infty(\Omega_T)}\|
\partial_t Q_1\|_{L^2(L^\infty)}\|P_2\|_{L^\infty(L^2)} \]
\[ + \|Q_1 - Q_2\|_{L^\infty(\Omega_T)}\|Q_1\|_{L^\infty(\Omega_T)}\|P_2\|_{H^1(L^2)} \} \].

The fourth term can be estimated as before.

**Step 6:** Estimates for additional stress tensors in the fluid equation. We have
\[ \|P_\sigma \text{div} \{\sigma(Q - Q_0, \Delta Q) + \xi(\tau_2(Q, \Delta Q) - \tau_2(Q_0, \Delta Q))\}\|_{H^1(\Omega_T)} \leq C(R)T^\frac{1}{2}\|\mathcal{Q}\|_{X_Q} . \]

This follows for \( Q(Q - Q_0, \Delta Q) \) and the bilinear part in \( \tau_2(Q, \Delta Q) - \tau_2(Q_0, \Delta Q) \) from Step 4 and for the trilinear part \( Q \text{tr}(Q \Delta Q) - Q_0 \text{tr}(Q_0 \Delta Q) \) from Step 5.

**Step 7:** The coupling term in the evolution of the tensor field. We have
\[ \|\|S_1(\nabla u, Q) + \xi S_2(\nabla u, Q)\|_{H^1(\Omega_T)} \leq C(R)T^\frac{1}{2}\|\mathcal{Q}\|_{X_u \times X_Q} . \]

This follows for the bilinear part in \( S_1(\nabla u, Q) + \xi S_2(\nabla u, Q) \) from Step 4 and for the trilinear part \( Q \text{tr}(Q \nabla u) - Q_0 \text{tr}(Q_0 \nabla u) \) from Step 5.

**Step 8:** Additional lower-order terms. The terms \( u \otimes u, (u \cdot \nabla)Q + L(Q) \) and
\[ J(Q) = P_\sigma \text{div} [\tau_1(Q) + \sigma(Q, L(Q)) + \xi \tau_2(Q, L(Q)) - \frac{2\xi}{d} L(Q)] \]
are of lower-order and lead to the estimates
\[ \|J(Q_1) - J(Q_2)\|_{Y_u} \leq T^\frac{1}{2}C(R)\|\mathcal{Q}\|_{X_Q} , \]
\[ \|\text{div} (u_1 \otimes u_1 - u_2 \otimes u_2)\|_{Y_u} \leq T^\frac{1}{2}C(R)\|\mathcal{Q}\|_{X_u} , \]
\[ \|(u_1 \cdot \nabla)Q_1 + L(Q_1) - (u_2 \cdot \nabla)Q_2 - L(Q_2)\|_{Y_Q} \leq T^\frac{1}{2}C(R)\|\mathcal{Q}\|_{X_u \times X_Q} . \]

These estimates can be done the same way as in Step 4 and 5.

**Proof of (iii): Boundedness of \( N_0 \).**

If suffices to show that (4.16) is a consequence of (4.15). In fact, the choice of \( (u_2, Q_2) = 0 \) in (4.15) implies
\[ \|N_0(Q_0)(u_1, Q_1) - N_0(Q_0)(0, 0)\|_{Y_0} \leq C_{N_0}(T, R)\|(u_1, Q_1)\|_{Y_0} \] (4.29)
and the assertion follows by the triangle inequality since \( N_0(Q_0)(0, 0) = E(u_0, Q_0) \), see the proof of (i).

**Proof of (iv): Asymptotic behaviour of the constant.** This assertion follows from the scaling of the constants in step (ii) in \( T \).

\[ \square \]

4.4. **Proof of Theorem 1.3.** By (4.5) and (4.6), the proof of Theorem 1.3 can be reduced to the statement that the nonlinear mapping
\[ \mathcal{L}'(Q_0) := \mathcal{L}^{-1}(Q_0)N_0(Q_0) : X_0 \to X_0 \] (4.30)
has a unique fixed-point. By (4.12) and (4.15) we find for all \( (u_{hi}, Q_{hi}) \in B_{X_0}(0, R) \) that
\[ \|\mathcal{L}^{-1}(Q_0)N_0(Q_0)(u_{h1}, Q_{h1}) - \mathcal{L}^{-1}(Q_0)N_0(Q_0)(u_{h2}, Q_{h2})\|_{X_0} \]
\[ \leq C_{\mathcal{E}}\|N_0(Q_0)(u_{h1}, Q_{h1}) - N_0(Q_0)(u_{h2}, Q_{h2})\|_{Y_0} \]
\[ \leq C_{\mathcal{E}}C_{N_0}(T, R, Q_0)\|(u_{h1} - u_{h2}, Q_{h1} - Q_{h2})\|_{X_0} . \]
Therefore $\mathcal{L}(Q_0)$ is a contraction mapping for $T \ll 1$. A similar argument shows that $\mathcal{L}$ maps $B_{X_0}(0, R)$ into itself. In fact, by (4.16)
\begin{align*}
\|\mathcal{L}(Q_0)(u_{h1}, Q_{h1})\|_{X_0} &\leq C_L \|N_0(Q_0)(u_{h1}, Q_{h1})\|_{Y_0} \\
&\leq C_L (C_N(T, R, Q_0)\|((u_{h1}, Q_{h1})\|_{X_0} + \|\mathcal{E}(u_0, Q_0)\|_{Y_0})
\end{align*}
and this estimate allows us to fix $R \gg 1$ large enough and $T \ll 1$ small enough in such a way that
\begin{align*}
\|\mathcal{L}(Q_0)(u_{h1}, Q_{h1})\|_{X_0} &\leq C_L C_N(T, R)((u_{h1}, Q_{h1})\|_{X_0} + \frac{R}{2} \leq R.
\end{align*}
We conclude from Banach’s fixed-point theorem that $\mathcal{L}$ possess a unique fixed-point $(u_h, Q_h) \in X_0$ and this fixed-point is a solution of the system (1.1) subject to (1.5) and (1.6).

The argument implies the uniqueness as well. Suppose that there was another solution $(\tilde{u}_h, \tilde{Q}_h)$ in $B_{X_0}(0, R_1)$ with $R_1 > R$. Choose $\tilde{T} \ll T$ and repeat the above argument to show the uniqueness of fixed-points of $\mathcal{L}$, which implies $(u_h, Q_h) = (\tilde{u}_h, \tilde{Q}_h)$ on $(0, \tilde{T}) \times \Omega$. Then the uniqueness follows by the continuity argument.

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