An Integer Approximation Method for Discrete Sinusoidal Transforms

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Abstract

Approximate methods have been considered as a means to the evaluation of discrete transforms. In this work, we propose and analyze a class of integer transforms for the discrete Fourier, Hartley, and cosine transforms (DFT, DHT, and DCT), based on simple dyadic rational approximation methods. The introduced method is general, applicable to several block-lengths, whereas existing approaches are usually dedicated to specific transform sizes. The suggested approximate transforms enjoy low multiplicative complexity and the orthogonality property is achievable via matrix polar decomposition. We show that the obtained transforms are competitive with archived methods in literature. New 8-point square wave approximate transforms for the DFT, DHT, and DCT are also introduced as particular cases of the introduced methodology.

Keywords

Approximate transforms discrete sinusoidal transforms, low-complexity transforms, nonorthogonal transforms, orthogonalization

1 Introduction

Discrete transforms play a significant role in digital signal processing. Among the possible discrete transforms, those based on sinusoidal transformation kernels occupy a prominent position. Examples of discrete sinusoidal transforms include the discrete Fourier transform (DFT), the discrete Hartley transform (DHT), and the discrete cosine transform (DCT) [1,2].

Mathematically, discrete sinusoidal transforms relate two $n$-dimensional vectors $\mathbf{v}$ and $\mathbf{v}'$ possibly defined over the complex numbers field according to the formalism below:

$$\mathbf{v}'_k = \sum_{i=0}^{n-1} v_i \cdot \text{ker}(i,k,n), \quad k = 0,1,\ldots,n-1,$$

$$v_i = \sum_{k=0}^{n-1} \mathbf{v}'_k \cdot \text{ker}^{-1}(i,k,n), \quad i = 0,1,\ldots,n-1,$$

where $\text{ker}(\cdot,\cdot,\cdot)$ and $\text{ker}^{-1}(\cdot,\cdot,\cdot)$ are known as the forward and inverse transformation kernels, respectively. Table 1 lists some possible kernel functions; the Kronecker delta was employed.

Although a variety of factors contribute to the computational complexity of a numerical method [1], the number of required arithmetical operations is frequently utilized as a measure of complexity. When computed directly according to Equations (1) and (2), discrete sinusoidal transforms require a number of multiplications and additions in $O(n^2)$. Thus, direct computation may not be practical.

Additionally, kernels can be complex-valued, which require further arithmetical considerations. The DFT kernel is a notable example. Moreover, the necessary values are often irrational numbers. Therefore, computations described in Equations (1) and (2) not only require a significant amount of operations, but they are expected to handle floating-point representation over a possibly complex field.

Fast algorithms constitute a collection of methods aiming at dramatically reducing arithmetical complexity figures. For discrete transforms, classical methods usually attain such minimization by means of (i) divide-and-conquer strategies [3]; (ii) matrix factorization schemes [4]; and (iii) convolution methods [5].

Efforts have been directed to the reduction of the multiplicative complexity. This is explained in part due to the well-developed multiplicative complexity theory, championed by Winograd [6] and Heideman [7]. Such developments allowed the prediction of theoretical lower bounds for the number of multiplications required by some discrete transforms [7,8]. Much of the research in this field is concerned the design of algorithms that can be considered “optimal” in a multiplicative complexity measurement.

However, bounded by theoretical constraints, exact methods could only achieve the prognosticated complexity minima at best. Even in such an optimal scenario, floating-point operations are involved. In particular, floating-point multiplications are known to possess relatively slow implementations, even in hardware [9].

One way to circumvent a possibly significant multiplica-

| Transform kernel | $\text{ker}(i,k,n)$ |
|------------------|-------------------|
| Fourier          | $\frac{1}{\sqrt{n}} \exp(-2\pi j k/n)$ |
| Cosine           | $(1 - (1 - 1/\sqrt{2})\delta_{0,k}) \sqrt{2/n} \cos(\frac{\pi(i+1/2)k}{n})$ |
| Hartley          | $\frac{1}{\sqrt{n}} \cos(2\pi i k/n)$ |
tive complexity is to consider, not exact, but approximate computations. In this case, the theoretical limits on the multiplicative complexity do not apply. A trade-off between complexity and accuracy may take place.

Several approximation methods have been proposed in literature. Arithmetic transform procedures explore nonuniform sampling and number-theoretic functions to devise multiplication-free algorithms for the DFT [10], the DHT [11], and the DCT [12]. Approximations for the DHT based on Ramanujan numbers [13] and on wavelets [14] were also suggested. The DCT computation was shown to be approximated in many ways. In particular, integer approximations are a significant category of methods [15–21]. In [22], the DFT was submitted to an integer approximation study as well.

Integer approximation procedures constitute a class of practical interest. Essentially, these methods take advantage of the fast computation of the integer arithmetic, when compared to floating-point manipulations. With the use of dyadic rational approximations [2] and the canonical signed digit representation [23], integer multiplications can be elegantly converted into combinations of additions and bit shifting operations. As a consequence, multiplication counts are virtually zeroed and, in its place, the number of additions and shifts are often quantified.

Another aspect of this discussion concerns usual requirements of orthogonality and perfect reconstruction. Much emphasis has been put in these properties, which frequently impose challenging design constraints for integer approximation algorithms [2]. In particular, orthogonal integer approximations for large blocklengths can be difficult to be obtained. Several existing design procedures require the solution of large constrained non-linear optimization problems in integer domain. Even for small blocklengths and considering exhaustive search, solutions are not trivial [2,22].

On the other hand, nonorthogonal methods are becoming increasingly popular [24]. Classes of nonorthogonal transforms have been defined [25] and algorithms for designing nonorthogonal basis have been considered [26]. Nonorthogonal, but closely orthogonal, matrices have found applications in soft clustering analysis [27]. Recently, blind source separation procedures were given a comprehensive treatment with nonorthogonal matrices [28]. Nonorthogonal basis images were also explored as a means to provide better representation methods for compressed images [29].

In this context, the goal of the present work is the proposal of an integer approximation method for discrete sinusoidal transforms based on dyadic rational approximations. In this study, we initially relax, but not neglect, orthogonality and perfect reconstruction constraints. Subsequently, we submit the proposed approximate transforms to a convenient orthogonalization method based on the matrix polar decomposition [30]. Related fast algorithms are suggested. We also aim at introducing new square wave transforms in an integer domain. Even for small blocklengths and considering large constrained non-linear optimization problems, approximations for large blocklengths can be difficult to be obtained.

In this context, the goal of the present work is the proposal of an integer approximation method for discrete sinusoidal transforms based on dyadic rational approximations. In this study, we initially relax, but not neglect, orthogonality and perfect reconstruction constraints. Subsequently, we submit the proposed approximate transforms to a convenient orthogonalization method based on the matrix polar decomposition [30]. Related fast algorithms are suggested. We also aim at introducing new square wave transforms in a comparable fashion as studied in [2,31,33].

The paper is organized as follows. In Section 2, a dyadic rational approximation of the cosine function is examined. Afterwards, integer approximations for the DFT, DHT, and DCT matrices are proposed; and an optimized global scaling factor is considered. In Section 3, the inverse transformation and orthogonality issues are discussed; an error analysis is also derived. Section 4 suggests some potential applications for the proposed approximations. Finally, Section 5 concludes the paper.

2 Approximating Procedure for Discrete Sinusoidal Transforms

2.1 Dyadic rational approximations

The nearest integer function offers a possible venue to map the values of the transformation kernel into dyadic rational numbers. This function simply returns integer values according to the following construction:

$$[x] \triangleq \text{sgn}(x) \left| x + \frac{1}{2} \right|,$$

where $[\cdot]$ is the floor function, $\text{sgn}(\cdot)$ is the sign function, and $\lfloor \cdot \rfloor$ returns the absolute value. This definition is in agreement to the implementation of the rounding algorithm available in the standard mathematical library of C language. A complex number $z = x + jy$, where $x, y \in \mathbb{R}$, is rounded off according to $[z] \triangleq [x] + j[y]$. More generally, let the $m$th order dyadic rational approximating function be defined as

$$[x]_m \triangleq \frac{[2^m x]}{2^m},$$

where $x$ is a real number and $m$ is a nonnegative integer. When $m = 0$, the approximating function $[\cdot]_0$ is equal to the nearest integer function. Intuitively, as $m \to \infty$, we have that $[\cdot]_\infty$ becomes the identity function.

Thus, the $m$th order dyadic rational approximation of a given kernel function can be obtained according to

$$\text{ker}_m(i, k, n) \triangleq [\text{ker}(i, k, n)]_m.$$  \hspace{1cm} (5)

As a consequence, transform vector $\hat{v}'$ can be approximated as $\hat{v}$, whose components are expressed as follows:

$$\hat{v}'_k = \sum_{i=0}^{n-1} v_i \cdot \text{ker}_m(i, k, n), \quad k = 0, 1, \ldots, n - 1.$$  \hspace{1cm} (6)

For $m = 0$, the approximation is derived via the usual rounding off operation. In this case, the values of $\text{ker}_0(\cdot, \cdot, \cdot)$ are $0$ or $\pm 1$. Although this mapping can be regarded as a coarse approximation for $\text{ker}(\cdot, \cdot, \cdot)$, it has zero multiplicative complexity, and only addition operations are necessary to render the approximate transformed signal. Another particularly interesting case occurs when $m = 1$. In this situation, $\text{ker}_1(\cdot, \cdot, \cdot)$ returns only $0, \pm 1$, or $\pm 2^{-1}$. Thus, only additions and simple bitwise shift operations are employed to obtain the approximate transform. For higher values of $m$, canonical signed digit representation could be applied to furnish multiplierless computations for the quantities displayed in Equation (6). Clearly, as $m \to \infty$, $\text{ker}_m(\cdot, \cdot, \cdot) \to$
Table 2: Optimal scaling

| $m$ | $\alpha$ |
|-----|----------|
| 0   | 1.1455   |
| 1   | 1.0754   |
| 2   | 1.0385   |
| 3   | 1.0196   |
| 4   | 1.0098   |
| $\infty$ | 1        |

ker(·, ·), where the limiting process indicates pointwise convergence [34].

Obviously, different functions can possess the same nearest integer approximation. Thus we may examine how good the approximations given by the function $\lfloor \cdot \rfloor$ are. Due to its ubiquity in discrete transform theory, let us consider the cosine function to be approximated. A multiplicative scaling factor $\alpha$ is also included for an additional degree of freedom. Adopting the mean square error as the objective function to be minimized, we can set up the following unconstrained optimization problem:

$$
\min_{\alpha} \int_{-\pi}^{\pi} (\alpha \cos(x) - \lfloor \alpha \cos(x) \rfloor)_m)^2 dx, \tag{7}
$$

for a fixed $m$.

Routine manipulations, which employ standard optimization methods, show that the optimal scaling factor is a root of the following non-linear equation:

$$
\frac{\pi}{2} \alpha^2 = \frac{1}{4^m} \sum_{k=0}^{2^m-1} \sqrt{4^{m+1} \alpha^2 - (2k+1)^2}. \tag{8}
$$

For $m = 0$, we have that the optimal scaling factor is exactly $(\frac{\sqrt{2}}{\pi}) \sqrt{4 + \sqrt{16 - \pi^2}} \approx 1.1455$. Since, for larger values of $m$, analytical computations are beyond purpose, Table 2 lists optimal values obtained by numerical methods. When $m \to \infty$, the optimal scaling factor collapses to 1, as expected. As a byproduct, in this case, Equation (8) furnishes an infinite summation formula for the value of $\pi$ when $\alpha = 1$.

Despite of optimality issues, for several practical short blocklengths and small approximation orders, we have that ker$_m(\cdot, \cdot, n) = \lfloor \alpha \cdot \text{ker}(\cdot, \cdot, n) \rfloor_m$, for an optimal value $\alpha$. Thus, for operational purposes, we could admit $\alpha = 1$. Figure 1(a) displays the plots of $\lfloor \cos(t) \rfloor_m$, for $m = 0, 1$, compared to $\cos(t)$, over the interval $[-\pi, \pi]$.

In this case, the mean square error due to approximating the cosine function as $[\cos(t)]_m$, $-\pi < t \leq \pi$, has a closed formula given by:

$$
\text{MSE}(\lfloor \cos(t) \rfloor_m, \cos(t)) = \frac{\pi}{4} \frac{1}{4^m} \sum_{k=0}^{2^m-1} \sqrt{4^{m+1} - (2k+1)^2} + \frac{1}{4^m} \sum_{k=0}^{2^m-1} (2k+1)\cos^{-1}\left(\frac{2k+1}{2^{m+1}}\right)^2. \tag{9}
$$

Figure 1(b) depicts the above mean square error calculated for $0 \leq m \leq 6$.

Henceforth, we adopt matrix notation. Thus, let the Fourier, Hartley, and cosine transformation matrices be defined, in their unitary form, as

$$
F_n = \frac{1}{\sqrt{n}} \left\{ \exp\left(\frac{-2\pi i j k}{n}\right) \right\}^{n-1}_{(i,k)=0}, \tag{10}
$$

$$
H_n = \frac{1}{\sqrt{n}} \left\{ \cos\left(\frac{2\pi i j k}{n}\right) \right\}^{n-1}_{(i,k)=0}, \tag{11}
$$

$$
C_n = \sqrt{\frac{2}{n}} \left\{ 1 - (1 - 1/\sqrt{2})\delta_{0,k} \cos\left(\frac{\pi(k+1/2)}{n}\right) \right\}^{n-1}_{(i,k)=0}, \tag{12}
$$

respectively.

In account of the previous discussion, we define the $m$th order dyadic rational matrices associated to $F_n$, $H_n$, and $C_n$ according to

$$
F_n^{(m)} \triangleq \left[ \sqrt{n}F_n \right]_m, \tag{13}
$$

$$
H_n^{(m)} \triangleq \left[ \sqrt{n}H_n \right]_m, \tag{14}
$$

$$
C_n^{(m)} \triangleq \left[ \sqrt{n}C_n \right]_m, \tag{15}
$$

where the operator $[\cdot]_m$ when applied to matrices acts componentwise. Prior to the application of the dyadic rational approximation procedure, the elements of original transform matrices were subject to a normalization by $\sqrt{n}$ or $\sqrt{n/2}$. Moreover, when scaled by $2^m$, the resulting matrices are constituted of integer numbers only. For instance, when $m = 0$, each entry of the dyadic rational matrices is $-1, 0,$ or $+1$.

### 2.2 Optimization

Indeed, the proposed dyadic rational matrices can furnish approximations for $F_n$, $H_n$, and $C_n$. A possible way to obtain such approximations is by the inclusion of a scaling factor, as suggested in [2, p. 275]. Scaling factors are introduced in such a way to minimize a chosen error measure between an original transformation matrix and its approximation.

In matrix terms, the dyadic rational matrices link a vector $\mathbf{v}$ to its approximate transform $\hat{\mathbf{v}}$, according to the following expression

$$
\hat{\mathbf{v}} = \beta \cdot K_n^{(m)} \mathbf{v}, \tag{16}
$$

where $K_n^{(m)} \in \{F_n^{(m)}, H_n^{(m)}, C_n^{(m)}\}$, for a fixed $m$, and $\beta$ is a real number. The quantity $\beta$ provides a global adjustment in such a way that $\beta K_n^{(m)}$ satisfactorily approximates $K_n \in \{F_n, H_n, C_n\}$, respectively.

Thus, we have set the following unconstrained optimization problem:

$$
\min_{\beta} \text{Error}(K_n, \beta K_n^{(m)}), \tag{17}
$$

where Error(·, ·) quantifies the dissimilarity between its arguments, according to a selected measure. Adopting the Frobenius norm of the matrix difference $K_n - \beta K_n^{(m)}$ as the objective function to be minimized [35, p. 523], we obtain...
optimal values for the scaling factor \( \beta \) using conventional optimization procedures. Tables 3, 4, and 5 show the numerically computed optimal values of \( \beta \) for selected practical blocklengths at various approximation orders.

Other error measures, such as the spectral norm or the \( p \)-norm for \( p \neq 2 \), could be considered instead of the Frobenius norm. In that case, different, but comparable, values of \( \beta \) would be found. Since \( \beta \) is an overall scaling factor, it does not affect further mathematical considerations on the nature of the approximate matrices \( K_n^{(m)} \).

For illustrative purposes, consider a pure sinusoidal signal \( v_i = \sin \left(2\pi \frac{i}{128}\right), i = 0, \ldots, 127 \). Figure 2 shows the DHT of \( v \), compared to the approximations given by the discussed method for \( m = 0, 1, \ldots, 4 \). As \( m \) increases, approximations become more accurate. Approximations for the DFT and the DCT showed similar behavior.

### 3 Inverse Transformation

#### 3.1 Matrix invertibility and nonorthogonality

Although in several applications, such as pattern classification based on transform domain feature extraction and transform adaptive filtering, only the forward transform is required, we investigate the inverse transformation of the proposed approximate method.

Function \( \lceil \cdot \rceil_m \) imposes inherent analytical difficulties. Consequently, it may not be obvious to establish whether, for any given value of \( n \) and \( m \), the inverse matrix \( K_n^{(m)} \) does exist. Then, we resort to exhaustive computational search. For \( n \leq 1024 \) and \( m \leq 6 \), the inverse of \( K_n^{(m)} \) was always found to exist. The evaluation of the condition number is adopted as a means to assess how well-conditioned these matrices are in terms of matrix inversion. Indeed, the condition number measures the sensitivity of matrix inversion. Thus, for the considered search space, the 2-norm condition number of \( K_n^{(m)} \) is small, never exceeding 2.5295, 2.5295, and 2.9432, for the Fourier, Hartley, and cosine approximate matrices, respectively. Such low values of the condition number indicate well-conditioned matrices. Figure 3 depicts the values of the condition number of \( F_n^{(m)} \) as a function of the transform size \( n \). Approximate matrices associated to the Fourier and Hartley transforms had identical condition number and matrices \( C_n^{(m)} \) presented similar values. For comparison, notice that since the exact Fourier, Hartley, and cosine matrices are unitary, their condition numbers are equal to one for any blocklength.

Thus, for practical purposes, assuming that the inverse of \( K_n^{(m)} \) is well-defined, the following manipulation holds true:

\[
(K_n^{(m)})^H(K_n^{(m)}(K_n^{(m)})^H)^{-1}K_n^{(m)} = I_n,
\]

where the superscript \( H \) indicates the Hermitian transposition and \( I_n \) is the identity matrix of size \( n \). Consequently, we conclude that

\[
(K_n^{(m)})^{-1} = (K_n^{(m)})^H(K_n^{(m)}(K_n^{(m)})^H)^{-1}.
\]

Strictly, the proposed approximate matrices lack unitary property, since \( (K_n^{(m)})^{-1} \neq (K_n^{(m)})^H \). An extra multiplicative term \( D^{-1} \Delta \) is necessary to furnish the matrix inversion, which enables perfect signal reconstruction. We have then obtained the following set of relations:

\[
\hat{v}' = \beta \cdot K_n^{(m)}v,
\]

\[
v = \frac{1}{\beta} \cdot (K_n^{(m)})^HD^{-1} \hat{v}'.
\]
Table 3: Optimal $\beta$ for the approximate DFT

| $m$ | $n$ | $0$     | $1$    | $2$    | $3$    | $4$    | $\infty$ |
|-----|-----|---------|--------|--------|--------|--------|----------|
| 4   | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 1/2   |
| 6   | 0.3513 | 0.3864 | 0.4255 | 0.4068 | 0.4068 | 1/$\sqrt{3}$ |
| 8   | 0.3121 | 0.3745 | 0.3480 | 0.3480 | 0.3560 | 1/$\sqrt{3}$ |
| 12  | 0.2478 | 0.2732 | 0.3008 | 0.2877 | 0.2877 | 1/$\sqrt{12}$ |
| 16  | 0.2169 | 0.2548 | 0.2386 | 0.2488 | 0.2511 | 1/4   |
| 24  | 0.1706 | 0.1977 | 0.2076 | 0.2011 | 0.2050 | 1/$\sqrt{24}$ |
| 32  | 0.1488 | 0.1741 | 0.1704 | 0.1757 | 0.1771 | 1/$\sqrt{32}$ |
| 64  | 0.1046 | 0.1195 | 0.1202 | 0.1242 | 0.1252 | 1/8   |
| 128 | 0.0734 | 0.0838 | 0.0854 | 0.0879 | 0.0883 | 1/$\sqrt{128}$ |

Table 4: Optimal $\beta$ for the approximate DHT

| $m$ | $n$ | $0$     | $1$    | $2$    | $3$    | $4$    | $\infty$ |
|-----|-----|---------|--------|--------|--------|--------|----------|
| 4   | 0.5000 | 1.0000 | 0.6666 | 0.6666 | 0.7273 | 1/$\sqrt{2}$ |
| 6   | 0.4509 | 0.6095 | 0.5511 | 0.5511 | 0.5944 | 1/$\sqrt{3}$ |
| 8   | 0.3745 | 0.6243 | 0.4780 | 0.4779 | 0.5105 | 1/2   |
| 12  | 0.3189 | 0.4310 | 0.3897 | 0.3897 | 0.4203 | 1/$\sqrt{5}$ |
| 16  | 0.2800 | 0.3839 | 0.3328 | 0.3465 | 0.3575 | 1/$\sqrt{8}$ |
| 24  | 0.2300 | 0.2950 | 0.2811 | 0.2784 | 0.2947 | 1/$\sqrt{12}$ |
| 32  | 0.2011 | 0.2529 | 0.2392 | 0.2466 | 0.2513 | 1/4   |
| 64  | 0.1445 | 0.1709 | 0.1694 | 0.1749 | 0.1774 | 1/$\sqrt{32}$ |
| 128 | 0.1026 | 0.1192 | 0.1206 | 0.1240 | 0.1250 | 1/8   |

Table 5: Optimal $\beta$ for the approximate DCT

| $m$ | $n$ | $0$   | $1$  | $2$  | $3$  | $4$  | $\infty$ |
|-----|-----|-------|------|------|------|------|----------|
| 4   | 0.5511 | 0.7363 | 0.6478 | 0.7006 | 0.7133 | 1/$\sqrt{2}$ |
| 6   | 0.4462 | 0.5955 | 0.5682 | 0.5572 | 0.5873 | 1/$\sqrt{3}$ |
| 8   | 0.3922 | 0.4891 | 0.4831 | 0.4925 | 0.5014 | 1/2   |
| 12  | 0.3303 | 0.3920 | 0.3929 | 0.4065 | 0.4090 | 1/$\sqrt{5}$ |
| 16  | 0.2876 | 0.3320 | 0.3380 | 0.3497 | 0.3548 | 1/$\sqrt{8}$ |
| 24  | 0.2342 | 0.2709 | 0.2816 | 0.2853 | 0.2889 | 1/$\sqrt{12}$ |
| 32  | 0.2038 | 0.2366 | 0.2424 | 0.2481 | 0.2495 | 1/4   |
| 64  | 0.1456 | 0.1657 | 0.1728 | 0.1750 | 0.1762 | 1/$\sqrt{32}$ |
| 128 | 0.1030 | 0.1172 | 0.1223 | 0.1241 | 0.1247 | 1/8   |
Figure 2: Discrete Hartley transform of a pure sinusoidal signal: (a) exact computation, (b) 0th order approximation, (c) 1st order approximation, (d) 2nd order approximation, (e) 3rd order approximation, and (f) 4th order approximation.

Figure 3: Condition number of approximate Fourier matrices for $m = 0$ (–o–), $m = 1$ (⋯o⋯), $m = 2$ (⋯–o⋯), $m = 3$ (⋯o⋯), $m = 4$ (⋯–x⋯), $m = 5$ (⋯⋯⋯), and $m = 6$ (⋯⋯⋯).
Except for the presence of $D^{-1}$, both forward and inverse approximate transformations share the same computational complexity. This is because their matrices are related by a simple transposition.

However, if $m \to \infty$, then $K_n^{(m)}$ converges to $K_n \in \{F_n, H_n, C_n\}$. Being $K_n$ a unitary matrix, we obtain that

$$\lim_{m \to \infty} K_n^{(m)}(K_n^{(m)})^H = K_n K_n^H = I_n.$$  \hspace{1cm} (22)

Thus, $K_n^{(m)}$ is asymptotically unitary in terms of $m$. Therefore, for sufficiently large $m$, Equations (20) and (21) can be recast as:

$$\hat{v}' = \beta \cdot K_n^{(m)} v,$$ \hspace{1cm} (23)

$$v \approx \hat{v} = \frac{1}{\beta} \cdot (K_n^{(m)})^H \hat{v'},$$ \hspace{1cm} (24)

where $\hat{v}$ is an approximation due to the nonorthogonality of $K_n^{(m)}$.

### 3.2 Orthogonalization

In view of the above discussion, we can re-examine the optimization procedure performed in the previous section. There, the approximate matrices were corrected by an overall constant factor termed $\beta$. In contrast to that, we can consider a more refined correction term. Notice that the term $K_n^{(m)}(K_n^{(m)})^H$ is a Gram matrix, which is Hermitian, non-negative definite. Then, $D^{-1}$ is also Hermitian [35 p. 82] and its matrix square root is well-defined, Hermitian [35, p. 82], and unique [40 p. 89]. Consequently, Equation (18) becomes:

$$(K_n^{(m)})^H (D^{-1})^{1/2} (D^{-1})^{1/2} K_n^{(m)} = I_n.$$ \hspace{1cm} (25)

Taking into account that the inverse of a Hermitian matrix is also Hermitian [35 p. 82], we have that

$$(D^{-1})^{1/2} K_n^{(m)} (D^{-1})^{1/2} K_n^{(m)} = I_n.$$ \hspace{1cm} (26)

It follows that $(D^{-1})^{1/2} K_n^{(m)}$ is a unitary matrix; and the term $S^{-1} \equiv (D^{-1})^{1/2}$ adjusts the approximate matrix. In fact, this adjustment term is exactly the inverse of the unique Hermitian non-negative definite matrix obtained when $K_n^{(m)}$ is factorized according to the matrix polar decomposition procedure [35 p. 348].

Being the unique unitary matrix of a polar decomposition, the term $S^{-1} K_n^{(m)}$ possesses some optimal approximation properties. In particular, when the Frobenius norm is utilized as a distance measure, $S^{-1} K_n^{(m)}$ is the nearest unitary matrix to $K_n^{(m)}$ [30].

Thus, Equations (20) and (21) can be modified into the following form:

$$\hat{v}' = S^{-1} K_n^{(m)} v,$$ \hspace{1cm} (27)

$$v = (K_n^{(m)})^H S^{-1} \hat{v'},$$ \hspace{1cm} (28)

where the scaling factor $\beta$ was suppressed and its role is taken by $S^{-1}$.

Regarding the closeness of these approximations to the exact matrices, computational calculations for $n \leq 1024$ and $m = 0,1,\ldots,4$ confirm that

$$\|K_n - S^{-1} K_n^{(m)}\|_F < \|K_n - \beta K_n^{(m)}\|_F,$$ \hspace{1cm} (29)

where $\|\cdot\|_F$ denotes the Frobenius norm. In a sense, this could be expected, since the adjustment offered by the optimal scaling factor $\beta$ corresponds simply to the matrix $\beta I_n$.

On the other hand, the adjustment by $S^{-1}$ possesses a higher arithmetic complexity, which allows a better approximation.

Therefore, a significant part of the computational cost of using $S^{-1} K_n^{(m)}$ as an approximation for $K_n$ relies on the complexity of $S^{-1}$. This observation prompts us to examine the behavior of $S^{-1}$.

### 3.3 Error analysis

Being $K_n$ already unitary matrices, when $\eta K_n$ are submitted to a polar decomposition, one obtains that

$$\eta \cdot K_n = (\eta \cdot I_n) \cdot K_n,$$ \hspace{1cm} (30)

where $\eta$ is a normalizing factor equal to $\sqrt{n}$ for the DFT and to $\sqrt{n}/2$ for the DHT or DCT. Introducing a perturbation matrix $\Delta E_1$, we can represent the discussed dyadic approximation according to $K_n^{(m)} = \eta \cdot K_n + \Delta E_1$. Therefore, we have that polar decomposition of $K_n^{(m)}$ is given by

$$\eta \cdot K_n + \Delta E_1 = \eta \cdot (I_n + \Delta S) \cdot (K_n + \Delta E_2),$$ \hspace{1cm} (31)

where $\Delta S$ and $\Delta E_2$ are induced perturbation matrices in the Hermitian and orthogonal polar decomposition factors, respectively. The Hermitian matrix is related to the adjustment matrix as $S = \eta \cdot (I_n + \Delta S)$. The matrix $K_n + \Delta E_2$ is recognized as an orthogonal approximation to $K_n$ [30, 35].

More explicitly, we have that

$$K_n + \Delta E_2 = S^{-1} \cdot K_n^{(m)} = \eta^{-1} \cdot (I_n + \Delta S)^{-1} \cdot (\eta K_n + \Delta E_1),$$ \hspace{1cm} (32)

where $\Delta E_2$ has minimum Frobenius norm [30]. Thus, if $\Delta S$ approaches a null matrix, then $\eta \cdot S^{-1}$ is close to an identity matrix. This is desirable, since the complexity of an identity matrix is null.

Aiming to determine an upper bound for the distance between $\eta \cdot S^{-1}$ and $I_n$, we first analyze the distance between its inverse $\eta^{-1} \cdot S$ and $I_n$. This latter distance is quantified by the following approximation error

$$\epsilon = \frac{\|\eta^{-1} S - I_n\|_F}{\|I_n\|_F}$$ \hspace{1cm} (33)

and

$$\frac{\|\Delta S\|_F}{\|I_n\|_F}. \hspace{1cm} (34)

In a similar manner, it is reasonable to quantify the perturbation error induced by $\Delta E_1$ as

$$\epsilon_1 = \frac{\|\Delta E_1\|_F}{\|\eta K_n\|_F}.$$ \hspace{1cm} (35)
By construction, the elements of $\Delta E_1$ are bounded by $1/2^{m+1}$. Thus, in the worst possible scenario, we have that

$$\Delta E_1 = \frac{1}{2^{m+1}} J_n,$$

where $J_n$ is a square matrix of ones with dimension $n$. This type of matrix and its properties are discussed in [41, p. 2]. Therefore, we obtain that $\|\Delta E_1\|_F = \frac{n}{2^{m+1}}$. Additionally, for all considered discrete sinusoidal transforms, we have that $\|K_n\|_F = \sqrt{n}$. Thus, we have that

$$\epsilon_1 \leq \frac{1}{2^{m+1}} \sqrt{n} = \frac{1}{2^{m+1}} \times \left\{ \begin{array}{lcl} 1, & \text{for the DFT,} \\ \sqrt{2}, & \text{for the DHT or DCT} \end{array} \right\}. \quad (37)$$

In [30], Higham submitted the polar decomposition procedure to a comprehensive error analysis. It was demonstrated that error measures $\epsilon$ and $\epsilon_1$ could be related according to

$$\epsilon \leq \sqrt{2} \epsilon_1 + O(\epsilon^2). \quad (38)$$

Performing the necessary substitutions and observing that $\eta/\sqrt{n}$ is a constant, we obtain:

$$\epsilon \leq \frac{1}{2^m} \times \left\{ \begin{array}{lcl} 1/\sqrt{2}, & \text{for the DFT,} \\ 1, & \text{for the DHT or DCT} \end{array} \right\} + O\left(\frac{1}{4^m}\right). \quad (39)$$

Therefore, $\epsilon = O\left(\frac{1}{2^m}\right)$.

Now we return to the error analysis of $\eta S^{-1}$. Analogously, the sought approximation error is given by

$$\epsilon' = \frac{\|\eta S^{-1} - I_n\|_F}{\|I_n\|_F}. \quad (40)$$

Invoking results on the stability of matrix inversion [38-42], we conclude that $\epsilon'$ has the same asymptotic behavior as $\epsilon$. Additionally, the asymptotic behaviors of $\epsilon$ and $\epsilon'$ are independent of the blocklength and are related only to the approximation order $m$. For a fixed $m$, we have that $\epsilon, \epsilon' = O(1)$. On the other hand, fixing the blocklength size, we also obtain that $\epsilon, \epsilon' = O(1/2^{m})$.

4 Applications

4.1 8-point approximate DFT

Usual 8-point DFT is a fundamental building block of several signal processing methods [343]. The 1st order 8-point approximate DFT possesses the following matrix transformation:

$$F_8^{(1)} = 1/2 \begin{pmatrix} \begin{array}{cccccccc} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 1-j & -2j & -1-j & -2-1-j & -2 & 2 & 1+j \\ 2 & -2j & -2 & 2 & 2 & -2 & 2 & 2 \\ 2 & -1-j & -2 & 2 & 2 & 1-j & -2 & -2 \\ 2 & 2j & -2 & 2 & 2 & 1+j & -2 & -2 \\ 2 & 1-j & -2j & -2-1-j & -2 & 2 & 2 & 1+j \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \end{pmatrix}. \quad (41)$$

Applying usual methods for matrix factorization [3], one can derive the following construction:

$$F_8^{(1)} = P \cdot A_4 \cdot A_3 \cdot T \cdot A_2 \cdot A_1, \quad (42)$$

where each factor is defined according to

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (43)$$

$$A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -j & -j & -j \end{pmatrix}, \quad (44)$$

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -j & -j & -j \\ 1 & -j & -j & -j \end{pmatrix}, \quad (45)$$

$$\text{where} \ "-" \ \text{represents} \ -1 \ \text{and blank spaces are zeroes. With this sparse matrix factorization, the transformation has its computational complexity reduced to 26 additions, and only two bit shifts.}$$

To illustrate one of the proposed approximation methods, let us consider the nonorthogonal approach as indicated in Equation (21). The exact computation of the inverse transformation $F_8^{(1)}$ furnishes the following relation:

$$(F_8^{(1)})^{-1} = (F_8^{(1)})^H D^{-1}, \quad (46)$$

where

$$16 \cdot D^{-1} = \begin{pmatrix} \begin{array}{ccc} 3 & 2 & -1 \\ 2 & 3 & -2 \\ -3 & 2 & 3 \end{array} \end{pmatrix} = 2 \cdot I_8 + \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (47)$$

Notice the extremely simple expression for $D^{-1}$ as well as its low computational complexity, which requires only additions and bit shifts.

Surprisingly, the inverse matrix $F_8^{(1)}$ is related to the zeroth order approximate matrix $F_8^{(0)}$:

$$(F_8^{(1)})^{-1} = \frac{1}{8} (F_8^{(0)})^H. \quad (48)$$

Per se, this relation constitutes the basis for a new complex-valued square wave transform, equipped with meaningful transform domain (cf. the SDCT by Haweel [31]). It is important to recognize that this relation guarantees the perfect reconstruction property.

Additionally, whenever a scaled version of the spectral components is admissible, the quantity $\beta$ (Equation (21)) can be set to one. Possible scenarios are Fourier descriptors evaluation [44], transform domain threshold-based detectors [45], and signal classification based on transform efficient features [46].
Figure 4: Flow graph for the fast algorithm of 0th order 16-point approximate DHT. Dashed boxes denote shorter transforms embedded in 16-point algorithm.

### 4.2 8- and 16-point approximate DHT

Besides being an elementary short blocklength transform employed as a means to evaluate larger size transforms [47], the 8-point DHT has found applications as an operator for edge detection [48]. The 1st order 8-point approximate DHT also implies another simple square wave transform, according to:

\[
(H^{(1)})^{-1} = \frac{1}{4} (H^{(0)}). \tag{49}
\]

This fortunate relation allows the definition of a real-valued transformation, whose spectrum approximates the 8-point DHT.

Now let us examine a fast algorithm for the 0th order 16-point approximate DHT. The obtained transformation matrix \( H^{(0)}_{16} \) is given by

\[
H^{(0)}_{16} = \begin{bmatrix}
    1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
    -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
    -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
    -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
    -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
    -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
    -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
    -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}. \tag{50}
\]

Using classical methods described in [43], the implementation diagram of a fast algorithm for \( H^{(0)}_{16} \) was found and is displayed in Figure 4. The resulting algorithm turned out to possess embedding properties. In the 16-point algorithm, the presence of fast algorithms for the 0th order 2-, 4-, and 8-point approximate DHT is noticeable. In Figure 4, such smaller transformations are separated in dashed boxes.

### 4.3 8-point approximate DCT

The 8-point DCT has attracted considerable research effort. This particular blocklength is widely adopted in several image and video coding standards, such as JPEG, MPEG-1, MPEG-2, H.261, and H.263 [49]. The 8-point DCT is also subject to an extensive analysis in [2]. Using Equations (20) and (27), we propose a new approximation for the 8-point DCT.

In order to evaluate the suggested approximations, we consider an input vector modelled after a first-order stationary Markov process with zero mean and unity variance. Additionally, it is assumed that adjacent vector components possess a correlation coefficient of 0.95 [2]. Considering these assumptions, commonly employed evaluation criteria, such as (i) mean square error; (ii) transform coding gain \( C_g \); and (iii) transform efficiency \( \eta \), can be computed deterministically [9].

Tables 6 and 7 list evaluation data for the proposed approximate 8-point DCT, when Equation (20) and (27) are considered, respectively. High values for the coding gain in Table 6 are due to the nonorthogonality of the considered transformations. This phenomenon was already reported in [50]. In [51, p. 18], Goyal gives a comprehensive account on how nonorthogonal transforms could outperform the Kahunen-Loève transform, for instance.

Existing approximation methods for the DCT include (i) the C-matrix transform (CMT) [52]; (ii) the integer cosine transform (ICT) [53]; (iii) the generalized Chen transform.
(GCT) \cite{54}; and (iv) the binDCT algorithm \cite{55}. The C-matrix transform consists of an approximation for the 8-point DCT transform using the Walsh-Hadamard transform as a pre-processing stage, followed by a conversion matrix of integer entries \cite{52}. The integer cosine transform adopts a somewhat different approach. It directly approximates the DCT matrix by integer elements without any pre-processing.

The GCT takes advantage of a parametrization of the DCT matrix replacing exact parameter values by rational approximations, such as 3/16, 3/8, 11/16, 91/128 \cite{2} p. 211. Being the parameters multiplicatively combined, the GCT scheme can provide a final approximate DCT matrix with elements of large integer representation (e.g., 1729/2048 or 1183/2048). In its turn, the binDCT employs an approach based on lifting schemes \cite{9}. Although the individual multiplicative elements of the binDCT lifting structure are relatively small (e.g., 13/16 or 15/16), they are multiplied in cascade. The resulting basis vectors that approximate the DCT matrix possess elements such as 7823/8192 or 3217/4096 \cite{2, p. 211}. So the elements of the final effective transformation matrix have a significantly larger dynamic range when compared to that of the individual constants employed by GCT parametrization or by the binDCT lifting scheme.

Therefore, the final approximate quantities could be taken into consideration when deriving a comparison between approximation methods. Since the accuracy of approximation is closely related to the dynamic range of the utilized integer numbers, a fair comparison of performance could limit the size of the considered bit representation in the final approximate matrix. Table 8 brings a quantitative comparison of the suggested methodology with the referred alternative methods described in literature. For each method, the dynamic range of the final approximation matrix is shown in parenthesis; design parameters are also indicated for the proposed methodology.

The low complexity of the proposed 0th order approximate DCT can be of additional practical interest; C\textsubscript{8}(0) requires only 24 additions. Moreover, its orthogonalizing adjustment matrix is a simple diagonal matrix given by:

\[
S^{-1} = \text{diag}\left(\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right).
\]  

(51)

Considering the adjustment offered by S\textsuperscript{-1}, the final approximation offers a MSE of 9 × 10\textsuperscript{-3}. For comparison, several versions of the ICT\textsubscript{8-II} could only exhibit a similar MSE performance, ranging from 6 × 10\textsuperscript{-3} to 8 × 10\textsuperscript{-3} \cite{2} p. 177, at the expense of using 2- or 3-bit arithmetic.

If the proposed nonorthogonal formulation that requires D\textsuperscript{-1} is chosen (Equations (20) and (21)), we obtain that

\[
(C\textsubscript{8}(0))^{-1} = (C\textsubscript{8}(0))^H D^{-1},
\]  

(52)

where D\textsuperscript{-1} = \text{diag}(1/8, 1/6, 1/4, 1/6, 1/8, 1/6, 1/4, 1/6). The diagonal elements are small and impose low computational requirements. This can be interpreted as the basis relation for the definition of another new square wave transform.

Additionally, in any case, when considering the DCT as a pre-processing step for a subsequent coefficient quantization procedure for image compression, the elements of D\textsuperscript{-1} can be included into the quantization step. This procedure is suggested and adopted in several works \cite{32, 33}. As a consequence, the computational complexity of the approximation is totally confined to that of C\textsubscript{8}(0).

Additionally, Figure 5(a) illustrates the absence of DC leakage of C\textsubscript{8}(0). All frequency response curves vanish at null frequency, except, of course, the first curve, which corresponds to the moving average filter associated to the first row of the transformation matrix. Figure 5(b) shows the frequency response of the exact DCT for comparison \cite{56}.

5 Conclusions

Using a simple and straightforward approach, we demonstrate that several approximate transforms can be conveniently obtained. It is important to emphasize that the proposed methods are general approaches, encompassing distinct transforms and various blocklengths. This fact contrasts with the highly specialized procedures archived in literature. Nevertheless, we could still derive meaningful performance comparisons.

The proposed integer approximations are well suited for architectures that take advantage of the dyadic rationals and canonical signed digit representation. Overall, low computational complexities are obtained. Further dedicated optimization methods could enhance the proposed methods for selected blocklengths and kernels. Possible venues include the elaboration of specially designed algorithms for the computation of particular adjustment matrices. Additionally, new 8-point square wave transforms equipped with perfect reconstruction and meaningful spectra were suggested for the DFT, DHT, and DCT.

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References

[1] W. L. Briggs and V. E. Henson, The DFT: an owner’s manual for the discrete Fourier transform. SIAM, 1995.

[2] V. Britanak, P. Yip, and K. R. Rao, Discrete Cosine and Sine Transforms. Academic Press, 2007.

[3] R. E. Blahut, Fast Algorithms for Digital Signal Processing. Addison-Wesley, 1985.

[4] C. Van Loan, Computational Frameworks for the Fast Fourier Transform. SIAM, 1992.

[5] C. S. Burrus and T. Parks, DFT/FFT and Convolution Algorithms. New York: John Wiley & Sons, 1985.

[6] S. Winograd, Arithmetic Complexity of Computations, vol. 33 of CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, 1980.
Table 8: Comparison with some existing methods

| Method                  | MSE      | $C_g$  | $\eta$  |
|-------------------------|----------|--------|---------|
| ICT$_8$-II (3-bit)      | 2.7217e-3 | 8.6513 | 91.1212 |
| Proposed (3-bit, $m=2$, $\beta=1/2$) | 4.3366e-3 | 8.8755 | 92.0474 |
| Proposed (3-bit, $m=2$, $\beta=0.4831$) | 1.6963e-3 | 9.0248 | 92.0474 |
| CMT$_8$ (4-bit)         | 3.3003e-3 | 8.8259 | 93.9912 |
| ICT$_8$-II (4-bit)      | 2.0607e-4 | 8.8141 | 94.0945 |
| Proposed (4-bit, $m=3$, $\beta=1/2$) | 3.6916e-3 | 9.0840 | 93.4396 |
| Proposed (4-bit, $m=3$, $\beta=0.4925$) | 2.2113e-3 | 9.1496 | 93.4396 |
| ICT$_8$-II (5-bit)      | 1.3029e-4 | 8.8144 | 93.9799 |
| binDCT-IIIC (13-bit)    | 2.7190e-4 | 8.8160 | 93.0667 |
| Proposed (5-bit, $m=5$, $\beta=1/2$) | 2.6404e-4 | 8.8664 | 94.1521 |
| Proposed (5-bit, $m=5$, $\beta=0.4966$) | 1.0624e-4 | 8.8961 | 94.1522 |
| CMT$_8$ (6-bit)         | 2.2680e-4 | 8.8070 | 93.1832 |
| Proposed (6-bit, $m=6$, $\beta=1/2$) | 3.5499e-5 | 8.8053 | 93.8968 |
| Proposed (6-bit, $m=6$, $\beta=0.5009$) | 2.0968e-5 | 8.7975 | 93.8968 |
| GCT$_8$-II (11-bit)     | 4.2336e-5 | 8.8226 | 93.9912 |
| Proposed (7-bit, $m=7$, $\beta=1/2$) | 2.7188e-5 | 8.8539 | 93.9519 |
| Proposed (7-bit, $m=7$, $\beta=0.4995$) | 1.8626e-5 | 8.8582 | 93.9519 |

Figure 5: Fourier transform (magnitude) of the basis vectors of (a) $C_8^{(0)}$ and (b) the exact DCT.
[7] M. T. Heideman, *Multiplicative Complexity, Convolution, and the DFT*. Springer-Verlag, 1988.

[8] E. Feig and S. Winograd, “On the multiplicative complexity of discrete cosine transforms,” *IEEE Transactions on Information Theory*, vol. 38, pp. 1387–1391, July 1992.

[9] J. Liang and T. D. Tran, “Fast multiplierless approximations of the DCT with the lifting scheme,” *IEEE Transactions on Signal Processing*, vol. 49, pp. 3032–3044, Dec. 2001.

[10] I. S. Reed, M.-T. Shih, T. K. Truong, E. Hendon, and D. W. Tufts, “A VLSI architecture for simplified arithmetic Fourier transform algorithm,” *IEEE Transactions on Signal Processing*, vol. 40, pp. 1122–1133, May 1992.

[11] R. J. Cintra and H. M. Oliveira, “How to interpolate in arithmetic transform algorithms,” in *Proceedings of the IEEE 27th International Conference on Acoustics, Speech, and Signal Processing*, vol. 4, (Orlando, FL), p. IV, May 2002.

[12] R. J. Cintra and V. S. Dimitrov, “The arithmetic cosine transform: Exact and approximate algorithms,” *IEEE Transactions on Signal Processing*, vol. 58, no. 6, pp. 3076–3085, 2010.

[13] N. Bhatnagar, “A binary friendly algorithm for computing discrete Hartley transform,” in *Proceedings of the 13th International Conference on Digital Signal Processing*, (Santorini, Greece), pp. 353–356, July 1997.

[14] H. S. Dee and V. Jeoti, “Computing DFT using approximate fast Hartley transform,” in *Proceedings of the International Symposium on Signal Processing and its Applications (ISSPA)*, (Kuala Lumpur, Malaysia), pp. 100–103, Aug. 2001.

[15] N. Merhav and B. Vasudev, “A multiplication-free approximate algorithm for the inverse discrete cosine transform,” in *Proceedings of the 1999 International Conference on Image Processing*, vol. 2, pp. 759–763, 1999.

[16] K. Lengwehasatit and A. Ortega, “DCT computation based on variable complexity fast approximations,” in *Proceedings of the 1998 International Conference on Image Processing*, vol. 3, pp. 95–99, Oct. 1998.

[17] A. Hossen and U. Heute, “Fast approximate DCT: basic-idea, error analysis, applications,” in *Proceedings of the 1997 IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 3, pp. 2005–2008, Apr. 1997.

[18] B. K. Natarajan and B. Vasudev, “A fast approximate algorithm for scaling down digital images in the DCT domain,” in *Proceedings of the International Conference on Image Processing*, vol. 2, pp. 241–243, Oct. 1995.

[19] R. K. W. Chan and M.-C. Lee, “Multiplierless fast DCT algorithms with minimal approximation errors,” in *International Conference on Pattern Recognition*, vol. 3, (Los Alamitos, CA, USA), pp. 921–925, IEEE Computer Society, 2006.

[20] H. S. Malvar, A. Hallapuro, M. Karczewicz, and L. Kerofsky, “Low-complexity transform and quantization in H.264/AVC,” *IEEE Transactions on Circuits and Systems for Video Technology*, vol. 13, pp. 598–603, July 2003.

[21] K. A. Wahid, V. S. Dimitrov, W. Badawy, and G. A. Jullien, “Error-free arithmetic and architecture for H.264,” in *Conference Record of the Thirty-Ninth Asilomar Conference on Signals, Systems and Computers*, pp. 703–707, 2005.

[22] S. Orailtara, Y.-J. Chen, and T. Q. Nguyen, “Integer fast Fourier transform,” *IEEE Transactions on Signal Processing*, vol. 50, pp. 607–618, Mar. 2002.

[23] F. Xu, C.-H. Chang, and C.-C. Jong, “Hamming weight pyramid - a new insight into canonical signed digit representation and its applications,” *Computers & Electrical Engineering*, vol. 33, no. 3, pp. 195–207, 2007.

[24] M. Koyuturk, A. Grama, and N. Ramakrishnan, “Nonorthogonal decomposition of binary matrices for bounded-error data compression and analysis,” *ACM Transactions on Mathematical Software*, vol. 32, pp. 33–69, Mar. 2006.

[25] R. C. French and R. F. Mitchell, “Class of nonorthogonal transformations for signal processing,” *Electronics Letters*, vol. 10, pp. 78–79, 21 1974.

[26] X. Yu, J. Wang, N. K. Loh, G. A. Jullien, and W. C. Miller, “Method for generating a new optimal nonorthogonal base in signal representation,” *Electronics Letters*, vol. 28, pp. 2191–2193, Nov. 1992.

[27] C. Ding, X. He, and H. D. Simon, “On the equivalence of nonnegative matrix factorization and spectral clustering,” in *Proceedings of SIAM Data Mining Conference*, pp. 606–610, 2005.

[28] E. M. Fadaili, N. T. Moreau, and E. Moreau, “Nonorthogonal joint diagonalization/zero diagonalization for source separation based on time-frequency distributions,” *IEEE Transactions on Signal Processing*, vol. 55, pp. 1673–1687, May 2007.

[29] W. B. Mikhal and A. P. Berg, “Image representation using nonorthogonal basis images with adaptive weight optimization,” *IEEE Signal Processing Letters*, vol. 3, pp. 165–167, June 1996.

[30] N. J. Higham, “Computing the polar decomposition— with applications,” *SIAM Journal on Scientific and Statistical Computing*, vol. 7, pp. 1160–1174, Oct. 1986.

[31] T. I. Haweel, “A new square wave transform based on the DCT,” *Signal Processing*, vol. 82, pp. 2309–2319, 2001.

[32] S. Bouguezel, M. O. Ahmad, and M. N. S. Swamy, “Low-complexity 8×8 transform for image compression,” *Electronics Letters*, vol. 44, pp. 1249–1250, Sept. 2008.

[33] K. Lengwehasatit and A. Ortega, “Scalable variable complexity approximate forward DCT,” *IEEE Transactions on Circuits and Systems for Video Technology*, vol. 14, pp. 1236–1248, Nov. 2004.

[34] S. G. Krantz, *Real Analysis and Foundations*. Chapman & Hall/CRC, 2005.

[35] G. A. F. Seber, *A Matrix Handbook for Statisticians*. John Wiley & Sons, Inc., 2007.

[36] X.-Y. Jing and D. Zhang, “A face and palmprint recognition approach based on discriminant DCT feature extraction,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 34, pp. 2405–2415, Dec. 2004.

[37] P. S. R. Diniz, *Adaptive Filtering: Algorithms and Practical Implementation*. Springer, 3 ed., 2008.

[38] J. J. Du Croz and N. J. Higham, “Stability of methods for matrix inversion,” *IMA Journal of Numerical Analysis*, vol. 12, pp. 1–19, 1992.

[39] D. J. Higham, “Condition numbers and their condition numbers,” *Linear Algebra and its Applications*, vol. 214, pp. 193–213, 1995.

[40] C. R. Johnson, *Matrix theory and applications*. American Mathematical Society, 1990.
[41] B. K. Moser, *Linear models: a mean model approach*. Probability and Mathematical Statistics, Academic Press, 1996.

[42] G. H. Golub and C. F. Van Loan, *Matrix Computations*. JHU Press, 3 ed., 1996.

[43] Y.-W. Lin and C.-Y. Lee, “Design of an FFT/IFFT processor for MIMO OFDM systems,” *IEEE Transactions on Circuits and Systems-I: Regular Papers*, vol. 54, pp. 807–815, Apr. 2007.

[44] W.-T. Wonga, F. Y. Shihb, and J. Liua, “Shape-based image retrieval using support vector machines, Fourier descriptors and self-organizing maps,” *Information Sciences*, vol. 177, pp. 1878–1891, Apr. 2007.

[45] D. Bellan, “DFT-based detection of sinusoids in real-time applications,” in *Proceedings of the IEEE International Symposium on Intelligent Signal Processing*, pp. 104–108, Aug. 2009.

[46] G. J. Miao and M. A. Clements, *Digital Signal Processing and Statistical Classification*. Artech House, 2002.

[47] R. N. Bracewell, *The Hartley Transform*. Oxford, 1986.

[48] R.-H. Park, K. S. Yoon, and W. Y. Choi, “Eight-point discrete hartley transform as an edge operator and its interpretation in the frequency domain,” *Pattern Recognition Letters*, vol. 19, pp. 569–574, May 1998.

[49] N. Roma and L. Sousa, “Efficient hybrid DCT-domain algorithm for video spatial downscaling,” *EURASIP Journal on Advances in Signal Processing*, vol. 2007, no. 2, pp. 30–30, 2007.

[50] S. K. Raikar and A. Makur, “Noise feedback structure for non-orthogonal transform coding,” in *Proceedings of the 2003 IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 6, pp. 497–500, Apr. 2003.

[51] V. K. Goyal, “Theoretical foundations of transform coding,” *IEEE Signal Processing Magazine*, vol. 18, pp. 9–21, Sept. 2001.

[52] H. W. Jones, D. N. Hein, and S. C. Knauer, “The Karhunen-Loève, discrete cosine and related transforms obtained via the Hadamard transform,” in *Proceedings of the International Telemetering Conference*, (Los Angeles, CA), pp. 14–16, Nov. 1978.

[53] W. K. Cham and Y. T. Chan, “Integer discrete cosine transforms,” in *Proceedings of the International Symposium on Signal Processing, Theories, Implementations, and Applications*, (Brisbane, Australia), pp. 674–676, 1987.

[54] J. D. Allen and S. M. Blonstein, “The multiply-free Chen transform — a rational approach to JPEG,” in *Proceedings of the Picture Coding Symposium*, (Tokyo, Japan), pp. 237–240, 1991.

[55] T. D. Tran, “The binDCT: fast multiplierless approximation of the DCT,” *IEEE Signal Processing Letters*, vol. 7, pp. 141–144, June 2000.

[56] G. Strang, “The discrete cosine transform,” *SIAM Review*, vol. 41, pp. 135–147, Mar. 1999.