Addition theorems for spin spherical harmonics: I. Preliminaries

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Abstract
We develop a systematic approach to deriving addition theorems for, and some other bilocal sums of, spin spherical harmonics. In this first part we establish some necessary technical results. We discuss the factorization of orbital and spin degrees of freedom in certain products of Clebsch–Gordan coefficients, and obtain general explicit results for the matrix elements in configuration space of tensor products of arbitrary rank of the position and angular-momentum operators. These results are the basis of the addition theorems for spin spherical harmonics obtained in part II (2011 J. Phys. A: Math. Theor. 44 165302).

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1. Introduction

The importance of the representation theory of the three-dimensional rotation group \([1–7]\) in the study of all natural quantum systems hardly needs to be mentioned. That central role is due to the fact that the four fundamental interactions of nature, and their effective interactions relevant to nuclear, atomic and molecular physics, are all invariant under the rotation group. It is also for that reason that partial-wave expansions are essential tools in the analysis of both classical and quantum scattering processes.

The algebraic aspects of the computation of partial-wave expansions with spin states are best codified by the addition theorems for spin spherical harmonics. Spin-\(s\) spherical harmonics \(Y^{(s)}_{lj}(\mathbf{r})\) \([4, 6]\) are the angular-momentum eigenfunctions relevant to the description of spin-\(s\) particles subject to spin-dependent central interactions\(^1\). As such, they are of interest in their own right beyond their above-mentioned specific application to partial-wave expansions that constitutes our main motivation for their study. In this paper, together with its second part \([10]\)

\(^1\) Familiar examples of spin-\(s\) spherical harmonics are ordinary scalar spherical harmonics, \(Y^{(0)}_{lj}(\mathbf{r}) = Y_{lj}(\mathbf{r})\), and vector spherical harmonics \([4, 6, 8, 9]\), \(Y^{(s)}_{lj}(\mathbf{r})\).
(hereafter referred to as II), we develop a systematic framework to derive addition theorems for spin spherical harmonics and employ it to obtain those addition theorems for low spins, $0 < s', s \leq 3/2$, and for arbitrary integer spin $s'$ if $s = 0$.

In this first part, we establish the preliminary results that will be used as basic building blocks for constructing addition theorems for spin spherical harmonics in II. We consider first the factorization of orbital and spin degrees of freedom in products of Clebsch–Gordan (henceforth CG) coefficients of the form $\langle \ell', \ell_z'; s', s_z' | j, j_z \rangle \langle \ell, \ell_z; s, s_z | j, j_z \rangle$. We obtain a general factorization result, make its tensor and spinor structure explicit, and discuss the particular cases relevant to spins $1/2$, $1$ and $3/2$. Such products of CG coefficients occur in the computation of physical quantities, such as scattering amplitudes or other matrix elements, related to transitions among states with definite orbital and spin quantum numbers. Thus, these results may be of interest independently of the addition theorems considered here and in II. We also study the matrix elements of arbitrary tensor products of orbital operators between angular-momentum projections of position eigenstates, for which we obtain completely general results. These are the other main ingredients in our derivation of addition theorems for spin spherical harmonics in II. Those matrix elements are closely related to bilocal spherical harmonics and other bilocal sums of ordinary spherical harmonics, as discussed in detail in II, so we expect that these results and the approach used to derive them should be applicable in other contexts as well.

The outline of the paper is as follows. In the next section, we discuss the factorization of orbital and spin degrees of freedom in products of CG coefficients. In section 3, we introduce the angular-momentum projector operator and obtain general expressions for its matrix elements with orbital tensor operators. In appendix A, we state our notation and conventions. In appendix B, we give the definitions and main properties of the standard bases of irreducible tensors and spinors, of arbitrary spin, used as spin wavefunctions throughout the paper. A detailed list of general results for reduced matrix elements of arbitrary tensor products of orbital and spin operators, needed in the main sections of the paper, are given in appendix C. Finally, appendix D gathers some ancillary calculations needed in section 3.

2. Factorization of orbital and spin dependence

We begin by considering products of CG coefficients of the form

$$S(j, \ell', \ell, s'; \ell_z', \ell_z, s_z) = \sum_{j, \ell = -j}^{j} \langle \ell', \ell_z'; s', s_z' | j, j_z \rangle \langle \ell, \ell_z; s, s_z | j, j_z \rangle$$

with $\ell'$, $\ell$ integer and $s'$, $s$ integer or half-integer. We will usually write $S(j, \ell', \ell, s', s)$ omitting the last arguments for brevity. For the purpose of obtaining addition theorems for spin spherical harmonics we need to rewrite $S(j, \ell', \ell, s', s)$ as a sum of terms with a completely factorized dependence on orbital and spin angular-momentum projections and to make explicit the tensor structure of those terms. The first goal is achieved by making use of relation (A.1) to write (see appendix A for our notation and conventions)

$$S(j, \ell', \ell, s', s) = (-1)^{\ell + \ell_z} (-1)^{s' - s} \frac{2j + 1}{\sqrt{(2\ell' + 1)(2s' + 1)}} \sum_{\Delta = \min \Delta}^{\Delta = \max \Delta} \left\{ \begin{array}{ccc} \ell' & \Delta & \ell \\ j & s & s' \end{array} \right\} \times \langle \ell, \ell_z; j, \Delta \Delta \ell_z | \ell', \ell_z' \rangle |s, s_z; j, \Delta s_z s_z' |s', s_z' \rangle,$$

$$\Delta_{\text{min}} = \max \{|\Delta \ell|, |\Delta s|\}, \quad \Delta_{\text{max}} = \min \{\ell' + \ell, s' + s\}, \quad \Delta \ell_z + \Delta s_z = 0. \quad (2)$$
In order to make the tensor structure of each term on the rhs explicit, we use the Wigner–Eckart (WE) theorem [1–6, 8] to write the CG coefficients in terms of appropriate irreducible tensor operators

\[
S(j', \ell', \ell, s', s) = \sum_{\Delta m_{\text{min}}} C_{\ell' \ell j}^{\ell' \ell m} (\ell', \ell' \ell \ell m) \frac{2 j + 1}{(2 j' + 1)(2 j + 1)} \frac{(2 j' + 1)}{2} \times \langle \ell' | \hat{\chi}^A(s' \ell \ell) \hat{\gamma}^A(s' \ell \ell) \rangle \langle \ell' | \hat{\gamma}^B(s' \ell \ell) \hat{\chi}^B(s' \ell \ell) \rangle \]

where irreducible tensor operators are defined in appendix B. The operators appearing in (3) are the orbital $\hat{L}$ and spin $\hat{S}$ angular momenta, the position versor $\hat{r} = \hat{r} / r$, and the spin-transition operator $\hat{T}$ defined here as the spin analog of $\hat{r}$ (see section 2.4 for a detailed discussion of this operator). The reduced matrix elements appearing in (3) are explicitly given for arbitrary values of their parameters in appendix C.

Whereas (3) is completely general, it is not yet explicit enough for the purpose of deriving addition theorems for spin spherical harmonics. From a practical point of view, we only need to consider low values of $s'$, $s$ since quantum states with high values of spin occur infrequently in nature. In this paper we restrict ourselves to $s', s = 0, 1/2, 1, 3/2$, and $|\Delta s| \leq 1$. For those spins the coefficients $C_{\ell' \ell j}^{\ell' \ell m}$ can be drastically simplified by exploiting the fact that $j - \ell$ and $j + \ell$ can take only a small set of values. The spin matrix elements in (3) can be compactly evaluated in terms of standard tensors and spinors, and spin matrices. (The orbital matrix elements could, in principle, also be expressed in that way, but it would be impractical because $\ell'$, $\ell$ are not bounded above thus requiring tensor wavefunctions of arbitrarily large rank. We evaluate those matrix elements with a completely different technique in section 3.) Clearly, we must discuss separately the different values of $s'$ and $s$, as those will determine the kind of spin spherical harmonics involved in the addition theorem. For fixed $s'$ and $s$, different values of $|\Delta \ell|$ correspond to different addition theorems, so we must consider those cases separately as well. Furthermore, both in addition theorems for spin spherical harmonics and in partial wave expansions, terms with different $|\Delta \ell|$ have different tensor and spin structure.

2.1. $s' = 1/2 = s$

For $s' = 1/2 = s$ there are two possible values of $|\Delta \ell| = 0, 1$.

$\Delta \ell = 0$. In the case $s = 1/2, j$ can take only two values and (3) reduces to

\[
S(j, \ell, 1/2, 1/2) = C_{\ell \ell j}^{1/2} \delta_{\ell', \ell} \delta_{s', s} + C_{\ell \ell j}^{1/2} (\ell, \ell' \ell) \hat{\gamma}^A(s' \ell \ell) \hat{\chi}^A(s' \ell \ell) \hat{\gamma}^B(s' \ell \ell) \hat{\chi}^B(s' \ell \ell)
\]

which is a well-known result [see, e.g., [11]]. Note that due to the conservation of $j_A$, the product $\hat{\gamma}^A(s' \ell \ell) \hat{\chi}^A(s' \ell \ell)$ in (4) can be replaced by $\hat{\gamma}^A(s' \ell \ell)$. We quote here also the simplified

\[
C_{\ell' \ell j}^{1/2} = \frac{j + 1/2}{2j + 1}, \quad C_{\ell' \ell j}^{1/2} = \frac{2(j - \ell)}{\ell + 1/2},
\]
form of (2) in this case:
\[
S(j, \ell, 1/2, 1/2) = C_{1lj}^{113} \delta_{j1} \delta_{r1},
\]
\[
+ \frac{\sqrt{3}}{2} \sqrt{\ell(\ell + 1)} C_{1lj}^{114} \langle \ell, \ell_1; 1, \Delta \ell; \ell, \ell_1 \rangle(1/2, s_1; 1, \Delta \ell_1; 1/2, s_2),
\]
which results from (4) by applying the WE theorem.

\(|\Delta \ell| = 1\). In this case \(S(j, \ell', \ell, 1/2, 1/2)\) can be non-vanishing only if \(\ell' = \ell + 1\) and 
\(j = \ell + 1/2\), or if \(\ell' = \ell - 1\) and \(j = \ell - 1/2\). Equation (3) takes the simplified form
\[
S(j, \ell_1, 1, 1, 1/2, 1/2) = C_{1lj}^{113} \langle \ell_1, \ell; 1, \Delta \ell; \ell, \ell_1 \rangle(1/2, s_1; 1, \Delta \ell_1; 1/2, s_2),
\]
\[
C_{1lj}^{113} = -2,
\]
which is appropriate to derive an addition theorem for spin-1/2 spherical harmonics. As in the previous case, the product of standard versors can be substituted by \(\delta^{kh}\). Similarly, (2) reduces to
\[
S(j, \ell, 1, 1, 1/2, 1/2) = -\Delta \ell \sqrt{\frac{3}{2}} \sqrt{\frac{2(\ell) + 1}{2\ell + 1}} \langle \ell, \ell_1; 1, \Delta \ell; \ell_1, \ell \rangle(1/2, s_1; 1, \Delta \ell_1; 1/2, s_2).
\]

As a side remark, we note that in this case \((2\ell + 1)/(2\ell_1 + 1) = (2j + 1)/(2j + 1 + \Delta \ell)\), which provides an alternate form for (7).

2.2. \(s' = 1 = s\)

The general result (3) is greatly simplified in the case of \(S(j, \ell', \ell, 1, 1)\) because, for fixed \(\ell, j - \ell\) can take only the three values \(\pm 1, 0\). The spin matrix elements are given by
\[
\langle 1, s'_1|\hat{\sigma}^h(\Delta \ell_1)^*|S^h|1, s_z\rangle = -i\hat{\sigma}^h(\Delta \ell_1)^*i(\hat{\sigma}(s'_1)^* \wedge \hat{\sigma}(s_z))^h,
\]
\[
\langle 1, s'_1|\hat{\sigma}^{h_1h_2}(\Delta \ell_1)^*|S^{h_1}\hat{S}^{h_2}|1, s_z\rangle = -\hat{\sigma}^{h_1h_2}(\Delta \ell_1)^*\hat{\sigma}^{h_1}(s'_1)^*\hat{\sigma}^{h_2}(s_z).
\]

There are three possible values for \(|\Delta \ell| = 0, 1, 2\).

\(|\Delta \ell| = 0\). In this case, (3) can be written as
\[
S(j, \ell, 1, 1) = C_{1lj}^{110} \delta_{j1} \delta_{r1},
\]
\[
- C_{1lj}^{111} \langle \ell, \ell_1; 1, \Delta \ell; \ell, \ell_1 \rangle(1/2, s_1; 1, \Delta \ell_1; 1/2, s_2),
\]
\[
C_{1lj}^{110} = 1, \quad C_{1lj}^{111} = \frac{2}{D_{ij}}(2j + 1)((j - \ell)^2 + 1)(j + \ell + 2)(j + \ell + 1)(j + \ell),
\]
\[
D_{ij} = (-1)^{j-\ell+1}((j - \ell)^2 + 1)(j + \ell + 2)(j + \ell + 1)(j + \ell).
\]

This form of (3) is needed in the derivation of addition theorems for vector spherical harmonics. Note that in the term multiplied by \(C_{1lj}^{111}\) we can replace \(\hat{\sigma}^h(\Delta \ell_1)^*\hat{\sigma}^{h_1}(\Delta \ell_1)^*\delta^{kh}\), and in the term multiplied by \(C_{1lj}^{112}\) we can rewrite the matrix elements as
\[
\langle \ell, \ell_1; \hat{\sigma}^{h_1h_2}(\Delta \ell_1)\hat{L}^{h_1} L^{h_2}|\ell, \ell_1; \hat{\sigma}^{h_1h_2}(\Delta \ell_1)\hat{L}^{h_1} L^{h_2}|\ell, \ell_1; \hat{\sigma}^{h_1h_2}(\Delta \ell_1)\hat{L}^{h_1} L^{h_2}|\ell, \ell_1; \hat{\sigma}^{h_1}(s'_1)^*\hat{\sigma}^{h_2}(s_z)
\]
\[
= \langle \ell, \ell_1; \hat{L}^{h_1} L^{h_2}|\ell, \ell_1; \hat{L}^{h_1} L^{h_2}|\ell, \ell_1; \hat{L}^{h_1} L^{h_2}|\ell, \ell_1; \hat{L}^{h_1} L^{h_2}|\ell, \ell_1; \hat{L}^{h_1} L^{h_2}|\ell, \ell_1; \hat{L}^{h_1} L^{h_2}|\ell, \ell_1; \hat{L}^{h_1} L^{h_2}\rangle.
\]
The relation (2) in this case reduces to
\[
S(j, \ell, 1) = \frac{C_{ij}^{111}}{2}\delta_{ij}\delta_{\ell j}\delta_{s j} + \sqrt{2}\sqrt{(\ell + 1)}C_{ij}^{111}\langle \ell, \ell; 1, \Delta \ell|\ell, \ell; 1, \Delta \ell|1, s \rangle
+ \frac{\sqrt{10}}{6}\sqrt{(\ell + 1)}\sqrt{2(\ell + 1)(2\ell + 3)}C_{ij}^{112}\times\langle \ell, \ell; 2, \Delta \ell|\ell, \ell; 1, \Delta \ell|1, s \rangle.
\]
with \(C_{ij}^{111}\) as in (9).

\(|\Delta \ell| = 1\). In this case, \(S(j, \ell', \ell, 1, 1)\) can be non-vanishing only if \(\ell' = \ell + 1\) and \(j = \ell + 1\), \(\ell\) or if \(\ell' = \ell - 1\) and \(j = \ell - 1\). With that discrete set of values for \(j - \ell\), the spin matrix elements (8), and the reduced matrix elements in appendix C, from (3) we obtain
\[
S(j, \ell_1, \ell, 1, 1) = -C_{ij}^{111}\langle \ell_1, \ell; 1, \Delta \ell|\ell, \ell; 1, \Delta \ell|1, s \rangle
- C_{ij}^{112}\langle \ell_1, \ell; 1, \Delta \ell|\ell, \ell; 1, \Delta \ell|1, s \rangle
\]
\[
C_{ij}^{111} = -\frac{1}{2(\ell + 1)}\sqrt{2j + 1}\sqrt{2j - \langle \ell \rangle + 1/2}, \quad C_{ij}^{112} = -\frac{4(\ell - \langle \ell \rangle)}{2(\ell + 1)}\sqrt{2j + 1}\sqrt{2j - \langle \ell \rangle + 1/2}.
\]
(12)

As before, on the first line of (12) we can replace \(\hat{r}^k(\Delta \ell_2)|\hat{r}^b(\Delta \ell_2)^*\) by \(\delta^{kb}\), and on the second line we can rewrite the matrix elements as
\[
\langle \ell', \ell'_j|\hat{r}^{kb}(\Delta \ell_2)|\hat{r}^b(\Delta \ell_2)^*\rangle(\Delta \ell_2)^*|\hat{r}^b(\Delta \ell_2)^*(s_j^*)^\ell, \ell; 1, \Delta \ell|1, s \rangle
= \langle \ell', \ell'_j|\hat{r}^{kb}(\Delta \ell_2)|\hat{r}^b(\Delta \ell_2)^*(s_j^*)^\ell, \ell; 1, \Delta \ell|1, s \rangle.
\]
(13)
The form of (2) in this case is
\[
S(j, \ell_1, \ell, 1, 1) = \Delta \ell_1\sqrt{2(\ell_1 + 1)}\left(\frac{C_{ij}^{111}}{2}\langle \ell, \ell_1; 1, \Delta \ell|\ell_1, \ell'_j; 1, \Delta \ell|1, s \rangle
+ \frac{1}{4}\frac{\sqrt{\alpha}}{3}\sqrt{2(\ell_1) - 1}(2\ell_1 + 3)C_{ij}^{112}\langle \ell, \ell_1; 2, \Delta \ell|\ell_1, \ell'_j; 1, s \rangle\right),
\]
(14)
with the coefficients (12).

\(|\Delta \ell| = 2\). In this case, \(S(j, \ell', \ell, 1, 1)\) can be non-vanishing only if \(\ell' = \ell + 2\) and \(j = \ell + 1\), or if \(\ell' = \ell - 2\) and \(j = \ell - 1\). Using (8), (C.5) and (C.3), from (3) and (2) we obtain
\[
S(j, \ell_2, \ell, 1, 1) = -C_{ij}^{112}\langle \ell_2, \ell; 1, \Delta \ell|\ell, \ell; 1, \Delta \ell|1, s \rangle
\]
\[
C_{ij}^{112} = \frac{2j + 1}{\sqrt{\alpha}}.
\]
(15)
The first line is the form needed for an addition theorem for vector spherical harmonics. Note that the matrix element on that line is traceless, because \(\ell' \neq \ell\), and symmetric, so we can replace \(\hat{r}^{kb}(\Delta \ell_2)|\hat{r}^b(\Delta \ell_2)^*\) there by \(\delta^{kb}\). The second line is just (2).
2.3. $s = 3/2 = s'$

The spin matrix elements entering (3) when $s = 3/2 = s'$ are given by

$$\tilde{\mathcal{C}}^{\ell \delta \ell \ell} (3/2, s') = \frac{3}{2} \tilde{\mathcal{C}}^{\ell \delta \ell \ell} (3/2, s')\tilde{\mathcal{C}}^{\ell (-\delta) \ell \ell} (3/2, s'),$$

$$\tilde{\mathcal{C}}^{\ell \ell j} (3/2, s') [S S'] (3/2, s') = -\frac{3}{2} \tilde{\mathcal{C}}^{\ell \ell j} (3/2, s')\tilde{\mathcal{C}}^{\ell \ell j} (3/2, s').$$

(16)

The orbital angular momentum change is 0 ≤ $\Delta \ell ≤ 3$.

$|\Delta \ell| = 0$. Substituting (16) in (3) we obtain

$$S(j, \ell, 3/2, 3/2) = C_{i j}^{\ell \ell \ell} \delta_{\ell \ell \ell} \delta_{\ell \ell \ell},$$

$$+ \frac{3}{2} C_{i j}^{\ell \ell j \ell} \delta_{\ell \ell \ell} \delta_{\ell \ell \ell},$$

$$- \frac{3}{2} C_{i j}^{\ell \ell j \ell} \delta_{\ell \ell \ell} \delta_{\ell \ell \ell},$$

$$\times \tilde{\mathcal{C}}^{\ell \ell j} (3/2, s')\tilde{\mathcal{C}}^{\ell \ell j} (3/2, s'),$$

(17a)

where the coefficients $C_{i j}^{\ell \ell \ell}$ given in (3) in this case take the form

$$C_{i j}^{\ell \ell \ell} = \frac{2 j + 1}{4 \ell (\ell + 1)},$$

$$C_{i j}^{\ell \ell j \ell} = \frac{2 j - \ell}{4 \ell (\ell + 1)} \left( j + \frac{1}{2} + 4 \left( j - \ell - \frac{3}{2} \right) (j - \ell + \frac{3}{2}) \right) \frac{1}{D_{i j}},$$

$$C_{i j}^{\ell \ell j \ell} = -\frac{1}{2} \frac{1}{\ell (\ell + 1)} \left( \frac{5}{4} (j - \ell)^2 \right) \left( j + \frac{1}{2} - 2(j - \ell)^2 + \frac{9}{2} (j - \ell) \right) \frac{1}{D_{i j}},$$

$$C_{i j}^{\ell \ell j \ell} = -\frac{1}{2} \frac{(j - \ell)(j + \ell + 1)}{\ell (\ell + 1)} \frac{1}{D_{i j}}.$$

$$\tilde{D}_{i j} = \left( \ell + 1 + \frac{2}{3} \left( j - \ell - \frac{3}{2} \right) \left( j - \ell + \frac{1}{2} \right) \right) \left( j - \ell + \frac{1}{2} \right) \left( j - \ell + \frac{1}{2} \right).$$

$$D_{i j} = \tilde{D}_{i j} \times \left( j - \ell - \frac{3}{2} \right) \left( j - \ell + \frac{1}{2} \right).$$

(17b)

As above, we note that the matrix elements in (17a) can also be written as

$$\tilde{\mathcal{C}}^{\ell j} (3/2, s') = (j, \ell, 3/2)\tilde{\mathcal{C}}^{\ell j} (3/2, s')\tilde{\mathcal{C}}^{\ell j} (3/2, s').$$

$$\tilde{\mathcal{C}}^{\ell j} (3/2, s') = \frac{3}{2} \tilde{\mathcal{C}}^{\ell j} (3/2, s')\tilde{\mathcal{C}}^{\ell j} (3/2, s').$$

$$\times \tilde{\mathcal{C}}^{\ell j} (3/2, s')\tilde{\mathcal{C}}^{\ell j} (3/2, s').$$

(18)

With the matrix elements written as in (17a), $S(j, \ell, 3/2, 3/2)$ takes the form needed in the derivation of addition theorems for spin-3/2 spherical harmonics.
Similarly, (2) can be rewritten as

\[
S(\ell, \ell', \ell, 3/2, 3/2) = \kappa_{\ell \ell}^{\Delta} \delta_{\ell' \ell} \delta_{\ell' \ell} + \kappa_{\ell \ell}^{\Delta 1} \langle \ell, \ell' | \Delta \ell \ell | \ell, \ell' \rangle (3/2, 3/2) + \kappa_{\ell \ell}^{\Delta 2} \langle \ell, \ell' | \Delta \ell \ell | \ell, \ell' \rangle (3/2, 3/2) + \kappa_{\ell \ell}^{\Delta 3} \langle \ell, \ell' | \Delta \ell \ell | \ell, \ell' \rangle (3/2, 3/2),
\]

\[
\kappa_{\ell \ell}^{\Delta} = \sqrt{\frac{15}{4}} \sqrt(\ell + 1)C_{\ell \ell}^{\Delta 1}, \quad \kappa_{\ell \ell}^{\Delta 1} = \frac{5}{4} \sqrt(\ell + 1) \sqrt(2\ell - 1)(2\ell + 3)C_{\ell \ell}^{\Delta 2},
\]

\[
\kappa_{\ell \ell}^{\Delta 2} = \frac{3}{2} \sqrt(\ell - 1)(\ell + 1)(\ell + 2)(\ell - 1)(2\ell + 3)C_{\ell \ell}^{\Delta 3}.
\]

Equations (17) and (19) are of course related by the WE theorem, with the reduced matrix elements in appendix C.

\( |\Delta \ell| = 1 \).

In this case, \( S(\ell, \ell', \ell, 3/2, 3/2) \) can be non-vanishing only if \( \ell' = \ell + 1 \) and \( j = \ell + 3/2, \ell + 1/2, \ell - 3/2, \) or if \( \ell' = \ell - 1 \) and \( j = \ell + 1/2, \ell - 1/2, \ell - 3/2 \). Evaluating the spin matrix elements with (16) we find

\[
S(\ell, \ell, \ell, 3/2, 3/2) = \frac{3}{2} C_{\ell \ell}^{\Delta 1} \langle \ell, \ell' | \delta(\Delta \ell \ell) \delta^{h}(\ell, \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \sigma^{h}_{AB} \delta^{\ell}_{B}(s_{B}) \\
- 3 \kappa_{\ell \ell}^{\Delta 2} \langle \ell, \ell' | \delta(\Delta \ell \ell) \delta^{h}(\ell, \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \sigma^{h}_{AB} \delta^{\ell}_{B}(s_{B}) \\
- 3 \kappa_{\ell \ell}^{\Delta 3} \langle \ell, \ell' | \delta(\Delta \ell \ell) \delta^{h}(\ell, \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \sigma^{h}_{AB} \delta^{\ell}_{B}(s_{B}).
\]

(20a)

with the coefficients \( C_{\ell \ell}^{\Delta} \) given by

\[
C_{\ell \ell}^{\Delta 1} = \frac{1}{5} \frac{j + 1/2}{2(\ell + 1)} \left( 1 - \sqrt(\ell + 1) - \sqrt(1 - \ell^2) \right),
\]

\[
C_{\ell \ell}^{\Delta 2} = - \frac{2}{(\ell + 1)^2} \frac{(3\ell - j + 1)}{\left( 1 - \sqrt(1 - \ell^2) \right)} \left( \ell + 1 \right) \left( j + 1/2 \right) \left( \ell^2 - 4 \right) ^2 \left( j - \ell \right) ^2. \tag{20b}
\]

\[
C_{\ell \ell}^{\Delta 3} = - \frac{4}{(\ell + 1)^2} \frac{\sqrt(\ell + 1)}{\left( 1 - \sqrt(1 - \ell^2) \right)} \left( 3\ell - j + 1 \right) \left( 3/2(j - \ell) \right) ^2 - 4 \left( 1 - \ell \right)^2.
\]

This expression in terms of matrix elements of irreducible tensor operators is needed in the derivation of addition theorems for spin-3/2 spherical harmonics. We also note that the matrix elements in (20a) can be written without standard tensors as

\[
\langle \ell, \ell' | \delta^{\ell}(\Delta \ell \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \sigma^{h}_{AB} \delta^{\ell}_{B}(s_{B}) \rangle = \langle \ell, \ell' | \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \sigma^{h}_{AB} \delta^{\ell}_{B}(s_{B}) \rangle
\]

\[
\langle \ell, \ell' | \delta^{h}(\Delta \ell \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \sigma^{h}_{AB} \delta^{\ell}_{B}(s_{B}) \rangle = \frac{1}{2} \langle \ell, \ell' | \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \sigma^{h}_{AB} \delta^{\ell}_{B}(s_{B}) \rangle
\]

\[
\langle \ell, \ell' | \delta^{h}(\Delta \ell \ell) \delta^{h}(\ell, \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \sigma^{h}_{AB} \delta^{\ell}_{B}(s_{B}) \rangle = \frac{1}{6} \langle \ell, \ell' | \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \delta^{\ell}(\ell, \ell) \sigma^{h}_{AB} \delta^{\ell}_{B}(s_{B}) \rangle.
\]

(21)
Analogously, (2) can be written as

\[ S(j, \ell_1, \ell, 3/2, 3/2) = \kappa_{ij}^{3/2} \langle \ell, \ell; 1, \Delta \ell \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle + \kappa_{ij}^{3/2} \langle \ell, \ell; 2, \Delta \ell \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle + \kappa_{ij}^{3/2} \langle \ell, \ell; 3, \Delta \ell \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle, \tag{22a} \]

with

\[ \kappa_{ij}^{3/2} \Delta \ell = \frac{1}{2} \sqrt{\frac{15}{2} \frac{2}{\sqrt{2\ell_1 + 1}}} \sqrt{2j + 1} C_{ij}^{3/2} \]

\[ \kappa_{ij}^{3/2} \Delta \ell = \frac{1}{2} \sqrt{\frac{15}{2} \frac{2}{\sqrt{2\ell_1 + 1}}} \sqrt{2(2\ell - 1)(2\ell + 1)(2\ell + 3)} C_{ij}^{3/2} \tag{22b} \]

\[ \kappa_{ij}^{3/2} \Delta \ell = \frac{1}{8} \sqrt{\frac{21}{5} \frac{2}{\sqrt{2\ell_1 + 1}}} \sqrt{(2\ell - 2)(2\ell - 1)(2\ell + 1)(2\ell + 3)(2\ell + 4)} C_{ij}^{3/2} \]

where the coefficients \( C_{ij}^{3/2} \) are given in (20b).

\(|\Delta \ell| = 2\). In this case, \( S(j, \ell', \ell, 3/2) \) can be non-vanishing only if \( \ell' = \ell + 2 \) and \( j = \ell + 3/2, \ell + 1/2 \) or \( \ell' = \ell - 2 \) and \( j = \ell - 1/2, \ell - 3/2 \). Taking into account this reduced set of \( j \) values, and the spin matrix elements (16), equation (3) takes the form

\[ S(j, \ell_2, \ell, 3/2, 3/2) = -3C_{ij}^{3/2} \langle \ell_2, \ell; 2, (\Delta \ell) \rangle \langle \ell, \ell \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle \]

\[ + \frac{3}{2} C_{ij}^{3/2} \langle \ell_2, \ell; 2, (\Delta \ell) \rangle \langle \ell, \ell \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle \]

\[ C_{ij}^{3/2} \Delta \ell = \frac{2}{\sqrt{3} \sqrt{(2j - 1)(2j + 3)}} \]

\[ C_{ij}^{3/2} \Delta \ell = \frac{8}{\sqrt{(2j - 1)(2j + 3)}} \]

\[ (23) \]

We note here that the matrix elements appearing in this equation can also be written as

\[ \langle \ell', \ell'| (\Delta \ell) \rangle \langle \ell, \ell \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle \]

\[ \langle \ell', \ell'| (\Delta \ell) \rangle \langle \ell, \ell \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle \]

\[ 1 = \frac{1}{6} \langle \ell', \ell'| (\Delta \ell) \rangle \langle \ell, \ell \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle \]

\[ (24) \]

where the matrix element on the rhs of the first equality is traceless because \( \ell' \neq \ell \). Similarly, from equation (2) we obtain

\[ S(j, \ell_2, \ell, 3/2, 3/2) = \kappa_{ij}^{3/2} \langle \ell, \ell; 2, (\Delta \ell) \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle + \kappa_{ij}^{3/2} \langle \ell, \ell; 3, (\Delta \ell) \rangle \langle \ell_1, \ell_1; 3/2, 3/2 \rangle \]

\[ \kappa_{ij}^{3/2} \Delta \ell = \frac{1}{2} \sqrt{\frac{21}{2} \frac{2}{\sqrt{2\ell_1 + 1}}} \frac{1}{\sqrt{(2\ell - 1)(2\ell + 1)(2\ell + 3)}} C_{ij}^{3/2} \]

\[ \kappa_{ij}^{3/2} \Delta \ell = \frac{1}{2} \sqrt{\frac{21}{2} \frac{2}{\sqrt{2\ell_1 + 1}}} \frac{1}{\sqrt{(2\ell - 1)(2\ell + 1)(2\ell + 3)}} C_{ij}^{3/2} \]

\[ (25) \]

with the coefficients \( C_{ij}^{3/2} \) of (23).
We define the spin transition operator \( \vec{T} \).

2.4. The spin transition operator

We define the spin transition operator \( \vec{T} \) as a spin vector operator, commuting with all orbital operators, and satisfying

\[
[S', T^j] = i\epsilon^{ijk}T^k, \quad \langle s'|\vec{T}^j||s\rangle = \frac{s' - s}{\sqrt{2}} \sqrt{\frac{2(s) + 1}{2s' + 1}} \left( \delta_{s'(s+1)} + \delta_{s'(s-1)} \right).
\]

From (29) we have

\[
\langle s', s||\vec{T}^2||s, s\rangle = \begin{cases} 
\delta_{s,s'} & \text{if } s' > 1/2 \text{ or } s > 1/2 \\
3/4 \delta_{s,s'} & \text{if } s' = s = 1/2 
\end{cases}.
\]

From (29) and (30) the operator \( \vec{T} \) is seen to be the spin-space analog of the orbital operator \( \vec{T} \). In particular, the reduced matrix elements of tensor products of \( \vec{T} \) and \( \vec{L} \) given in appendix C apply without changes to tensor products of \( \vec{T} \) and \( \vec{S} \).

2.5. \( s', s = 1, 0 \) or 0, 1

The relevant spin matrix element in this case is

\[
\langle 1, s'|\vec{T}^k|0, 0 \rangle = \frac{1}{\sqrt{3}} \varepsilon^k(s')^*.
\]

\(|\Delta \ell|\) can take the values 0 and 1.
$|\Delta \ell| = 0$. \(S(j, \ell, \ell, 1, 0)\) (resp. \(S(j, \ell, \ell, 0, 1)\)) can be non-vanishing only if \(j = \ell\) (resp. \(j = \ell'\)). Then, (3) and (2) take the form

\[
S(\ell, \ell, \ell, 1, 0) = \frac{1}{\sqrt{3}} C_{\ell \ell \ell}^{101} \langle \ell, \ell', | \ell \ell \ell \rangle \bar{\epsilon}^k(s')^* \\
= -\sqrt{3}(\ell, \ell; 1, \Delta \ell, | \ell, \ell' \rangle (1, s' \ell, 1, \Delta \ell, | 0, 0),
\]

\[C_{\ell \ell \ell}^{101} = \sqrt{\frac{3}{\ell(\ell + 1)}}. \tag{32}\]

Similarly,

\[
S(\ell, \ell, \ell, 0, 1) = \frac{1}{\sqrt{3}} C_{\ell \ell \ell}^{101} \langle \ell, \ell', | \ell \ell \ell \rangle \bar{\epsilon}^k(s) = \langle \ell, \ell, 1, \Delta \ell, | \ell, \ell' \rangle (0, 0; 1, \Delta \ell, | 1, s),
\]

with \(C_{\ell \ell \ell}^{101} = C_{\ell \ell \ell}^{101}\).

$|\Delta \ell| = 1$. In this case \(S(j, \ell', \ell, s', s)\) can be non-vanishing only if \(j = \ell\) (if \(s = 0\) or \(\ell'\) (if \(s' = 0\)). Thus, with the spin matrix element (31) we obtain, from (3) and (2),

\[
S(\ell, \ell, \ell, 1, 0) = \frac{C_{\ell \ell \ell}^{101}}{\sqrt{3}} \langle \ell, \ell', | \ell \ell \ell \rangle \bar{\epsilon}^k(s')^* \\
= \sqrt{\frac{3}{2\ell + 1}}(\ell, \ell; 1, \Delta \ell, | \ell, \ell' \rangle (1, s' \ell, 1, \Delta \ell, | 0, 0),
\]

\[C_{\ell \ell \ell}^{101} = -\sqrt{6\Delta \ell} \sqrt{\frac{3}{2\ell + 1}}. \tag{34}\]

Analogously,

\[
S(\ell, \ell, \ell, 0, 1) = \frac{C_{\ell \ell \ell}^{101}}{\sqrt{3}} \langle \ell, \ell', | \ell \ell \ell \rangle \bar{\epsilon}^k(s) \\
= \langle \ell, \ell, 1, \Delta \ell, | \ell, \ell' \rangle (1, s \ell, 1, \Delta \ell, | 0, 0),
\]

\[C_{\ell \ell \ell}^{101} = \sqrt{6\Delta \ell} \sqrt{\frac{3}{2\ell + 1}}. \tag{35}\]

2.6. \(s', s = 3/2, 1/2\) or \(1/2, 3/2\)

The spin matrix elements appearing in (3) in this case are

\[
\langle 3/2, s' \ell | \bar{\epsilon}^h (\Delta \ell z)^* T^{h 1/2} | 1/2, s \rangle = \frac{1}{2} \sqrt{\frac{3}{2}} \bar{\epsilon}^h (\Delta \ell z)^* \bar{\sigma}_A^{h b} (s')^* \bar{\chi}_A (s),
\]

\[
\langle 3/2, s' \ell | \bar{\epsilon}^h b z (\Delta \ell z)^* T^{h 1/2} | 1/2, s \rangle = \frac{1}{4} \sqrt{\frac{3}{2}} \bar{\epsilon}^h b z (\Delta \ell z)^* \bar{\sigma}_A^{h b} (s')^* \bar{\chi}_A b (s). \tag{36}\]

There are three possible values for \(0 \leq |\Delta \ell| \leq 2\).

$|\Delta \ell| = 0$. If \(s' = 3/2, s = 1/2, \) and \(\ell' = \ell\), then \(S(j, \ell, \ell, 3/2, 1/2)\) can be non-vanishing only if \(j = \ell \pm 1/2\). In the case \(s' = 1/2, s = 3/2\), it must be \(j = \ell' \pm 1/2\). Substituting (36) in (3) we obtain

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Expressions (37) and (40) will be used later to derive addition theorems involving one spin-3/2 and one spin-1/2 spherical harmonic.
\(|\Delta \ell| = 1\). We consider the case \(s' = 3/2\), \(s = 1/2\) first. In this case \(S(j, \ell', \ell, 3/2, 1/2)\) can be non-vanishing only if \(\ell' = \ell + 1\) and \(j = \ell \pm 1/2\) or if \(\ell' = \ell - 1\) and \(j = \ell \pm 1/2\). With the spin matrix elements in (36), from (3) we obtain

\[
S(j, \ell_1, \ell, 3/2, 1/2) = \frac{1}{2} \sqrt{\frac{3}{2}} C_{\ell_1, \ell}^{1/2+1} (\ell_1 \ell_1') \langle \ell_1 \ell_1' | e^{i k_1} (\Delta \ell_z) e^{i k_1} (\Delta \ell_z)^* \mathcal{A}_A (s_1) \mathcal{A}_B (s_2) 
+ 1 \langle 3/2 \rangle C_{\ell_1, \ell}^{1/2+2} (\ell_1 \ell_1') \langle \ell_1 \ell_1' | e^{i k_1} (\Delta \ell_z) e^{i k_1} (\Delta \ell_z)^* \mathcal{A}_A (s_1) \mathcal{A}_B (s_2). 
\]

(42a)

with the coefficients

\[
C_{\ell_1, \ell}^{1/2+1} = -\sqrt{2} \Delta s \Delta \ell \frac{2 j + 1}{2(\ell + 1)} \frac{2 j - (\ell) + \Delta s \Delta \ell + \frac{1}{2}}{2(\ell) - \Delta s \Delta \ell + \frac{1}{2}} \sqrt{1 - 2 \Delta s \Delta \ell (j - (\ell))},
\]

\[
C_{\ell_1, \ell}^{1/2+2} = -8 \sqrt{\frac{3}{2}} \Delta \ell \frac{2 j + 1}{2(\ell) - \Delta s \Delta \ell + 1} \frac{2 j - (\ell) + \Delta s \Delta \ell + 1}{2(\ell) + \frac{1}{2}} \sqrt{1 - 2 \Delta s \Delta \ell (j - (\ell))},
\]

(42b)

where in this case \(\Delta s = 1\). On the first line of (42a), we can replace \(e^{i k_1} (\Delta \ell_z) e^{i k_1} (\Delta \ell_z)^*\) with \(\delta^{k_1}\), and on the second line,

\[
\langle \ell' \ell_1' | e^{i k_1} (\Delta \ell_z) e^{i k_1} (\Delta \ell_z)^* \mathcal{A}_A (s_1) \mathcal{A}_B (s_2) \rangle = \frac{1}{2} \langle \ell' \ell_1' | e^{i k_1} (\Delta \ell_z) e^{i k_1} (\Delta \ell_z)^* \mathcal{A}_A (s_1) \mathcal{A}_B (s_2).
\]

(43)

Similarly, from (2) we obtain

\[
S(j, \ell_1, \ell, 3/2, 1/2) = -\frac{1}{2} \sqrt{\frac{3}{2}} \Delta \ell \frac{2 j + 1}{2(\ell_1 + 1)} \frac{2 j - (\ell) + \Delta s \Delta \ell + \frac{1}{2}}{2(\ell) - \Delta s \Delta \ell + \frac{1}{2}} \sqrt{1 - 2 \Delta s \Delta \ell (j - (\ell))},
\]

\[
\times \langle \ell, \ell_2, 1, \Delta \ell_z | \ell_1, \ell_1' (3/2, s_1') 1, \Delta \ell_z | 1/2, s_2 \rangle \Delta \ell \frac{30 j + 1}{2 \ell_1 + 1} \frac{2 j - (\ell) + \Delta s \Delta \ell + 1}{2(\ell + 1)} \sqrt{1 - 2 \Delta s \Delta \ell (j - (\ell))},
\]

\[
\times \langle \ell, \ell_2, 2, \Delta \ell_z | \ell_1, \ell_1' (3/2, s_1') 2, \Delta \ell_z | 1/2, s_2 \rangle \Delta \ell \frac{30 j + 1}{2 \ell_1 + 1} \frac{2 j - (\ell) + \Delta s \Delta \ell + 1}{2(\ell + 1)} \sqrt{1 - 2 \Delta s \Delta \ell (j - (\ell))},
\]

(44)

with the coefficients (42b).

In the case \(s' = 1/2, s = 3/2\), \(S(j, \ell', \ell, 1/2, 3/2)\) can be non-vanishing only if \(\ell' = \ell + 1\) and \(j = \ell + 1/2\), or if \(\ell' = \ell - 1\) and \(j = \ell - 1/2\). The treatment of this case is completely analogous to the previous one. In terms of matrix elements, from (3) we obtain

\[
S(j, \ell_1, \ell, 1/2, 3/2) = \frac{1}{2} \sqrt{\frac{3}{2}} C_{\ell_1, \ell}^{1/2+1} (\ell_1 \ell_1') \langle \ell_1 \ell_1' | e^{i k_1} (\Delta \ell_z) e^{i k_1} (\Delta \ell_z)^* \mathcal{A}_A (s_1) \mathcal{A}_B (s_2) 
+ 1 \langle 3/2 \rangle C_{\ell_1, \ell}^{1/2+2} (\ell_1 \ell_1') \langle \ell_1 \ell_1' | e^{i k_1} (\Delta \ell_z) e^{i k_1} (\Delta \ell_z)^* \mathcal{A}_A (s_1) \mathcal{A}_B (s_2).
\]

(45)

with the coefficients \(C_{\ell_1, \ell}^{1/2+1}\) given by (42b) with \(\Delta s = -1\). From (2) we obtain, analogously,

\[
S(j, \ell_1, \ell, 1/2, 3/2) = \frac{1}{2} \sqrt{\frac{3}{2}} \Delta \ell \frac{2 j + 1}{2 \ell_1 + 1} \frac{2 j - (\ell) + \Delta s \Delta \ell + \frac{1}{2}}{2(\ell) - \Delta s \Delta \ell + \frac{1}{2}} \sqrt{1 - 2 \Delta s \Delta \ell (j - (\ell))},
\]

\[
\times \langle \ell, \ell_2, 1, \Delta \ell_z | \ell_1, \ell_1' (1/2, s_1') 1, \Delta \ell_z | 1/2, s_2 \rangle \Delta \ell \frac{30 j + 1}{2 \ell_1 + 1} \frac{2 j - (\ell) + \Delta s \Delta \ell + 1}{2(\ell + 1)} \sqrt{1 - 2 \Delta s \Delta \ell (j - (\ell))},
\]

\[
\times \langle \ell, \ell_2, 2, \Delta \ell_z | \ell_1, \ell_1' (1/2, s_1') 2, \Delta \ell_z | 1/2, s_2 \rangle \Delta \ell \frac{30 j + 1}{2 \ell_1 + 1} \frac{2 j - (\ell) + \Delta s \Delta \ell + 1}{2(\ell + 1)} \sqrt{1 - 2 \Delta s \Delta \ell (j - (\ell))},
\]

(46)
We define the angular momentum projection operators as

$$
\mathcal{P}_\ell = \sum_{m=-\ell}^{\ell} |\ell, m\rangle \langle \ell, m|, \quad 0 \leq \ell < \infty.
$$

(52)

3. Angular momentum projection operators

We define the angular momentum projection operators as

$$
|\Delta \ell| = 2. \quad \text{As in the previous paragraph, we consider the case } s' = 3/2, s = 1/2 \text{ first. In this case } S(j, \ell', 3/2, 1/2) \text{ can be non-vanishing only if } \ell' = \ell + 2 \text{ and } j = \ell + 1/2, \text{ or if } \ell' = \ell - 2 \text{ and } j = \ell - 1/2. \quad \text{Using the spin matrix elements (36) and the range of } j, (3) \text{ reduces to}

$$
S(j, \ell_2, \ell, 3/2, 1/2) = \frac{1}{4} \sqrt{3} \frac{\Delta \ell \Delta \ell}{\sqrt{2}} \mathcal{C}_{\ell, \ell_2}^{\frac{\ell + 1}{2}} \langle \ell_2 | \mathcal{E}^h \mathcal{E}^k | \Delta \ell \mathcal{E}_z \mathcal{E}^h \mathcal{E}^k | \ell, \ell \rangle
$$

(47a)

where $\Delta \ell = 1$. The matrix elements in (47a) can be written without standard tensors as

$$
S(j, \ell_2, \ell, 3/2, 1/2) = \frac{1}{4} \sqrt{3} \frac{\Delta \ell \Delta \ell}{\sqrt{2}} \mathcal{C}_{\ell, \ell_2}^{\frac{\ell + 1}{2}} \langle \ell_2 | \mathcal{E}^h \mathcal{E}^k | \Delta \ell \mathcal{E}_z \mathcal{E}^h \mathcal{E}^k | \ell, \ell \rangle
$$

(48)

Similarly, in this case (2) takes the simplified form

$$
S(j, \ell_2, \ell, 3/2, 1/2) = \frac{1}{2} \sqrt{2} \sqrt{2} \mathcal{C}_{\ell, \ell_2} \Delta \ell \Delta \ell |\ell_2\rangle \langle \ell_2 | \mathcal{E}^h \mathcal{E}^k | \Delta \ell \mathcal{E}_z \mathcal{E}^h \mathcal{E}^k | \ell, \ell \rangle
$$

(49)

In the case $s' = 1/2, s = 3/2$, $S(j, \ell', \ell, 1/2, 3/2)$ can be non-vanishing only if $\ell' = \ell + 2$ and $j = \ell - 1/2$, or if $\ell' = \ell - 2$ and $j = \ell + 1/2$. In this case (3) reduces to

$$
S(j, \ell_2, \ell, 1/2, 3/2) = \frac{1}{4} \sqrt{3} \frac{\Delta \ell \Delta \ell}{\sqrt{2}} \mathcal{C}_{\ell, \ell_2} \Delta \ell \Delta \ell |\ell_2\rangle \langle \ell_2 | \mathcal{E}^h \mathcal{E}^k | \Delta \ell \mathcal{E}_z \mathcal{E}^h \mathcal{E}^k | \ell, \ell \rangle
$$

(50)

with $\mathcal{C}_{\ell, \ell_2}$ given by the same expression (47b) as $\mathcal{C}_{\ell, \ell_2}^{\frac{\ell + 1}{2}}$, with $\Delta \ell = -1$. Equality (2) takes the simplified form

$$
S(j, \ell_2, \ell, 1/2, 3/2) = \frac{1}{4} \sqrt{3} \frac{\Delta \ell \Delta \ell}{\sqrt{2}} \mathcal{C}_{\ell, \ell_2} \Delta \ell \Delta \ell |\ell_2\rangle \langle \ell_2 | \mathcal{E}^h \mathcal{E}^k | \Delta \ell \mathcal{E}_z \mathcal{E}^h \mathcal{E}^k | \ell, \ell \rangle
$$

(51)
It is immediate that \( \mathcal{P}_\ell \) are a sequence of orthogonal projectors, which commute with \( \hat{L} \) and resolve the identity

\[
\mathcal{P}_\ell \mathcal{P}_\ell = \mathcal{P}_\ell \delta_{\ell,\ell}, \quad \sum_{\ell=0}^\infty \mathcal{P}_\ell = 1, \quad [\hat{L}, \mathcal{P}_\ell] = 0.
\] (53)

The matrix elements of \( \mathcal{P}_\ell \) in configuration space follow from their definition (52) and the addition theorem for spherical harmonics

\[
\mathcal{P}_\ell(\vec{r}) = \sum_{m=-\ell}^{\ell} Y_{\ell m}(\vec{r})^* |\ell, m\rangle, \quad \langle \vec{r} | \mathcal{P}_\ell | \vec{r}' \rangle = \frac{2\ell + 1}{4\pi} P_{\ell} (\vec{r} \cdot \vec{r}').
\] (54)

From (52)–(54) the matrix elements of \( \mathcal{P}_\ell \) with other orbital operators can be computed. Those matrix elements, together with the results on products of CG coefficients of the previous section, are the basic building blocks needed in our derivation of addition theorems for spin spherical harmonics. As will be discussed in II, however, the results for \( \mathcal{P}_\ell \) matrix elements constitute by themselves addition theorems for spherical harmonics.

### 3.1. Matrix elements with tensor powers of \( \hat{L} \)

The matrix elements of \( \mathcal{P}_\ell \) with tensor powers of \( \hat{L} \) will be used in the derivation of addition theorems for spin spherical harmonics. For the irreducible components, from which the full tensor matrix elements can be reconstructed, we have

\[
\langle \vec{r}' | L^{[h_1]} \ldots L^{[h_p]} \mathcal{P}_\ell | \vec{r} \rangle = \frac{2\ell + 1}{4\pi} i^p \left[ \mathcal{P} (\vec{r} \wedge \nabla)^{[h_p]} \ldots (\vec{r} \wedge \nabla)^{[h_1]} \right] P_{\ell}(x)
\]

\[
= i \left( \mathcal{P} \wedge \nabla \right)^{[h_p]} \ldots \left( \mathcal{P} \wedge \nabla \right)^{[h_1]} \mathcal{P}_\ell \langle \vec{r}' | \mathcal{P}_\ell | \vec{r} \rangle = \frac{1}{(p - 1)!} \left( \mathcal{P} \wedge \nabla \right)^{[h_p]} \ldots \left( \mathcal{P} \wedge \nabla \right)^{[h_1]} \mathcal{P}_\ell \langle \vec{r}' | \mathcal{P}_\ell | \vec{r} \rangle.
\] (55)

Thus, we can obtain these irreducible matrix elements recursively, with the result

\[
\langle \vec{r}' | L^{[h_1]} \ldots L^{[h_p]} \mathcal{P}_\ell | \vec{r} \rangle = \frac{2\ell + 1}{4\pi} i^p \sum_{q=0}^{[p/2]} C_{p,q} Z^{[h_1 \ldots h_q \ldots h_{p-q+1}]} v_1^{h_{p-q+1}} \ldots v_q^{h_{p+1}} P_{\ell}^{(p-q)}(x),
\] (56a)

with \( P_{\ell}^{(k)} \) the \( k \)th derivative of \( P_{\ell} \) and

\[
x = \vec{r} \cdot \vec{r}', \quad v = \vec{r} \wedge \vec{r}', \quad C_{p,q} = (2q - 1)!! \binom{p}{2q},
\]

\[
C_{p,0} = 1, \quad Z^{[1 \ldots h_q \ldots h_{p-q+1}]} = \vec{r}_1^{h_{p+1}} \ldots \vec{r}_q^{h_{p+1}}.
\] (56b)

For notational simplicity, in what follows we adopt the following useful conventions. In \( Z^{[h_1 \ldots h_q \ldots h_{p-q+1}]} \), one or both sets of indices may be empty:

\[
Z^{[h_1 \ldots h_q \ldots h_{p-q}]} = \vec{r}_1^{h_{p+1}} \ldots \vec{r}_q^{h_{p+1}} = \vec{r}_1^{h_{p+1}} \ldots \vec{r}_q^{h_{p+1}}, \quad Z^* = 1.
\] (56c)

In the term \( q = 0 \) in (56a) the index sets are empty, \( Z^{[h_1 \ldots h_q \ldots h_{p-q}]} = Z^* \). Similarly, in the term \( q = [p/2] \), when \( p \) is even, we set \( v_1^{h_{p+1}} \ldots v_{p-q}^{h_{p+1}} = v_1^{h_{p+1}} \ldots v_{p-q}^{h_{p+1}} \equiv 1 \).

The cases \( n = 1, 2, 3 \) will be needed for the derivation of addition theorems for spherical harmonics of spin 1/2, 1 and 3/2 so we quote them here explicitly:

\[
\langle \vec{r}' | \mathcal{P}_\ell | \vec{r} \rangle = \frac{2\ell + 1}{4\pi} v P_{\ell} (\vec{r} \cdot \vec{r}'),
\] (57a)
The matrix element with \( k \) is irreducible. The irreducible matrix elements will be used in the derivation of addition theorems for spin spherical harmonics. They satisfy the recursion relation

\[
\langle \mathbf{p} | \mathbf{p}^{(i)} \cdots \mathbf{p}^{(i_n)} | \mathbf{p} \rangle = -\frac{l - l'i}{4\pi} \left[ \left( \ell_{n+1} - \ell_n \right) \mathbf{W}_{p'}^{l} - \frac{1}{2} \ell_{n+1} + \ell_n + 1 \right] \mathbf{p}^{(i)}
\]

with

\[
\langle \mathbf{p} | \mathbf{p}^{(i)} \mathbf{p} | \mathbf{p} \rangle = \frac{\ell - l'i}{4\pi} \left( \mathbf{p}^{(i)}\mathbf{p}'(x) - \mathbf{p}^{(i)}\mathbf{p}'(x) \right).
\]

In these equations we used the notation \( \ell_k = \ell \pm k \), see appendix A. Equations (58) and (59) are established in appendix D. The recursion relation (58) can be solved with the initial condition (59) to give

\[
\langle \mathbf{p} | \mathbf{p}^{(i)} \cdots \mathbf{p}^{(i_n)} | \mathbf{p} \rangle = A_n \sum_{k_1, k_2 \geq 0} \frac{(-1)^{k_1}}{k_1! k_2!} Z^{(i_1 \cdots i_n)} Z^{(i_1 \cdots i_n)} P^{(i)}(x),
\]

\[
A_n = \frac{\ell - l'i}{4\pi} \frac{2\ell + 1}{2\ell_1 + 1} \frac{2\ell + 1}{2\ell_2 + 1} \frac{1}{2\ell + 1} \frac{1}{2\ell + 1} \left( \frac{\ell_{n+1} + \ell_n + 1}{\ell} \right)^n \left( \frac{\ell_{n+1} + \ell_n + 1}{\ell} \right)^n \left( \frac{\ell_{n+1} + \ell_n + 1}{\ell} \right)^n \left( \frac{\ell_{n+1} + \ell_n + 1}{\ell} \right)^n \left( \frac{\ell_{n+1} + \ell_n + 1}{\ell} \right)^n.
\]

With \( Z^{(i_1 \cdots i_n)} \) defined in (56b) and (56c). As before, we adopt the convention that in the term with \( k_1 = n, k_2 = 0 \) in (60) the first index set in \( Z \) is empty \( Z^{(i_1 \cdots i_n)} \), and in the term with \( k_1 = 0, k_2 = n \), the second index set is empty \( Z^{(i_1 \cdots i_n)} \).

The particular cases \( n = 1, 2, 3 \) will be needed for the derivation of addition theorems for spherical harmonics of spin 1/2, 1 and 3/2, so we give them explicitly here:

\[
\langle \mathbf{p} | \mathbf{p}^{(i)} | \mathbf{p} \rangle = \frac{\ell - l'i}{4\pi} \left( -\mathbf{p}^{(i)}\mathbf{p}'(x) + \mathbf{p}^{(i)}\mathbf{p}'(x) \right),
\]

\[
\langle \mathbf{p} | \mathbf{p}^{(i)} \mathbf{p}^{(j)} | \mathbf{p} \rangle = \frac{1}{4\pi} \frac{1}{2\ell + 1} \left( \frac{1}{2} \mathbf{p}^{(i)}\mathbf{p}^{(j)}\mathbf{p}'(x) - \mathbf{p}^{(i)}\mathbf{p}^{(j)}\mathbf{p}'(x) + \frac{1}{2} \mathbf{p}^{(i)}\mathbf{p}^{(j)}\mathbf{p}'(x) + \frac{1}{2} \mathbf{p}^{(i)}\mathbf{p}^{(j)}\mathbf{p}'(x) \right),
\]

\[
\langle \mathbf{p} | \mathbf{p}^{(i)} \mathbf{p}^{(j)} \mathbf{p}^{(k)} | \mathbf{p} \rangle = \frac{\ell - l'i}{4\pi} \frac{1}{2\ell + 1} \left( \frac{1}{2} \mathbf{p}^{(i)}\mathbf{p}^{(j)}\mathbf{p}'(x) - \mathbf{p}^{(i)}\mathbf{p}^{(j)}\mathbf{p}'(x) + \frac{1}{2} \mathbf{p}^{(i)}\mathbf{p}^{(j)}\mathbf{p}'(x) + \frac{1}{2} \mathbf{p}^{(i)}\mathbf{p}^{(j)}\mathbf{p}'(x) \right).
\]
3.3. Mixed matrix elements

We consider now matrix elements of $P_\ell$ with tensor products of both $\vec{L}$ and $\vec{F}$. The matrix elements of the irreducible tensor operator $\vec{\tau}^{ij} \ldots \vec{\tau}^{k_1} L^{k_1} \ldots L^{k_0}$ are of the form

$$
\langle \vec{p}|P_\ell \vec{\tau}^{ij} \ldots \vec{\tau}^{k_1} L^{k_1} \ldots L^{k_0}| \vec{p}\rangle = i^\ell \sum_{k_1, k_2=0}^n (-1)^{k_2} Z_{k_1, k_2}^{i,\ldots,j} P_{k_1}^{(n+1)}(x).
$$

From this equation and (60), for $t = 1$ we obtain

$$
\langle \vec{p}|P_\ell \vec{\tau}^{ij} \ldots \vec{\tau}^{k_1} L^{k_1} \ldots L^{k_0}| \vec{p}\rangle = i A_n \sum_{k_1, k_2=0}^n (-1)^{k_2} \left( \sum_{k_1, k_2=0}^n \langle \vec{p}|P_\ell \vec{\tau}^{ij} \ldots \vec{\tau}^{k_1} L^{k_1} \ldots L^{k_0}| \vec{p}\rangle \right). 
$$

For $s > 1$, equation (62) leads to the recursion relation

$$
\langle \vec{p}|P_\ell \vec{\tau}^{ij} \ldots \vec{\tau}^{k_1} L^{k_1} \ldots L^{k_0}| \vec{p}\rangle = \frac{i}{n+1} \sum_{k_1, k_2=0}^n \langle \vec{p}|P_\ell \vec{\tau}^{ij} \ldots \vec{\tau}^{k_1} L^{k_1} \ldots L^{k_0}| \vec{p}\rangle,
$$

which can be solved with the initial condition (63) to yield

$$
\langle \vec{p}|P_\ell \vec{\tau}^{ij} \ldots \vec{\tau}^{k_1} L^{k_1} \ldots L^{k_0}| \vec{p}\rangle = i^\ell A_n \sum_{k_1, k_2=0}^n (-1)^{k_2} \left( \sum_{k_1, k_2=0}^n \langle \vec{p}|P_\ell \vec{\tau}^{ij} \ldots \vec{\tau}^{k_1} L^{k_1} \ldots L^{k_0}| \vec{p}\rangle \right) 
$$

with $[s/2]$ in the inner summation denoting the integer part. $A_n$ as defined in (60), and $C_{r,q}, r, v, Z^{k_i,\ldots,j}$ as defined in (56b1), with $\vec{\tau}$ instead of $\vec{\tau}$. As before, for notational simplicity, we have not separated from the sum the terms with $(k_1, k_2) = (n, 0), (0, n)$, those with $q = 0$ and, for even $s$, those with $q = [s/2]$. In those cases we apply the same conventions as explained after equations (56) and (60).

Particular cases of importance for the derivation of addition theorems for spherical harmonics of spin 1/2, 1 and 3/2 are

$$
\langle \vec{p}|P_\ell \vec{\tau}^{ij} L^k| \vec{p}\rangle = i \frac{\ell - \ell - 1}{4\pi} \left( \vec{p}|P_\ell P_{1,\ell}^{(\nu)}(x) - \vec{p}|P_{1,\ell}^{(\nu)}(x) \right),
$$

$$
\langle \vec{p}|P_\ell \vec{\tau}^{ij} L^k| \vec{p}\rangle = \frac{i}{2\ell + 1} \left( \vec{p}|P_\ell P_{1,\ell}^{(\nu)}(x) - 2\vec{p}|P_{1,\ell}^{(\nu)}(x) + \vec{p}|P_{1,\ell}^{(\nu)}(x) \right),
$$

$$
\langle \vec{p}|P_\ell \vec{\tau}^{ij} L^k L^k| \vec{p}\rangle = -i \frac{\ell - \ell - 1}{4\pi} \left( \vec{p}|P_{1,\ell}^{(\nu)}(x) - \vec{p}|P_{1,\ell}^{(\nu)}(x) + \vec{p}|P_{1,\ell}^{(\nu)}(x) \right).
$$
4. Final remarks

We have presented in the foregoing sections preliminary results needed for the systematic derivation of addition theorems for spin spherical harmonics in II. In section 2, we obtained the factorization of orbital and spin degrees of freedom in products of CG coefficients of the form (1) in the general form, and discussed the particular cases with $0 \leq s', s \leq 3/2$. In those cases the coefficients $C_{s'}^{s, \Delta \ell}$, given in full generality in (3), were reduced to much smaller forms, and the tensor and spinor structures of each term in the expansion given explicitly.

In section 3, the matrix elements of the angular-momentum projector operator with the irreducible components of arbitrary tensor products of $\hat{r}$ and $\vec{L}$ are given in the general form for all values of their parameters. Those matrix elements will be used in II to obtain general expressions for bilocal spherical harmonics and to derive addition theorems for spin spherical harmonics. Their applicability is even wider, however, when they are appropriately combined to obtain matrix elements of reducible tensor operators. Some examples of those applications will also be considered in II.

As a side remark we point out that an unexpected byproduct of the results of section 2 is an improvement in computational efficiency, at least in the specific case of infinite-precision computation [13] in which the results are given as a rational number times the square root of a rational number. Consider, for example, the computation of both sides of (19), at fixed $\ell$ and $j$, for all possible values of $-\ell \leq \ell' z, \ell z \leq \ell$ and $-3/2 \leq s', s z \leq 3/2$. If the rhs of (19) is computed as written on the second line of (1), we find that the ratio of CPU times $\tau_{\text{rhs}} / \tau_{\text{lhs}}$ begins at $\sim 2$ at $\ell = 1$, monotonically decreasing with $\ell$ to reach 1 at $\ell \sim 20$, $\sim 0.5$ at $\ell \sim 100$, and $\sim 0.25$ at $\ell \sim 200$. We remark that those numbers are subject to statistical fluctuations.

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Appendix A. Notation and conventions

Throughout the paper we set $\hbar = 1$. We denote tensor product states of orbital and spin angular momentum by $|\ell, \ell z; s, s z \rangle = |\ell, \ell z \rangle \otimes |s, s z \rangle$, and states coupled to total angular momentum $j, j z$ by $|\ell, s, j, j z \rangle$. We follow the notation of [7] for CG coefficients

$$
\langle \ell, \ell z; s, s z | j, j z \rangle \equiv \langle \ell, \ell z; s, s z | \ell, s, j, j z \rangle.
$$

(Different notations are used, e.g., in [1, 6, 8, 12].) We adopt the usual convention that $|\ell, \ell z; s, s z; j, j z \rangle = 0$ if $\ell z + s z \neq j z$, or $j < |\ell - s|$, or $j > \ell + s$. For Wigner $6j$-symbols [1, 4–6] we use the definition and notation of [1, 6]. The most important property of $6j$-symbols for our purposes is the relation [1]

$$
\langle j_1, m_1; j_2, m_2 | j', m_1 + m_2; j_3, m_3 | j, m \rangle = \sum_{j''=|j_1-j_3|}^{j_2+j_3} \sqrt{(2j' + 1)(2j'' + 1)(-1)^{j_2+j_3+j_1}} \times \binom{j_1}{j_2 j_3} \binom{j_2}{j' j''} \binom{j_3}{j''} \langle j_1, m_1; j', m_2 + m_3 | j, m \rangle \langle j_2, m_2; j, m_3 | j'', m_2 + m_3 \rangle.
$$

(A.1)
This relation is satisfied by the implementation of CG coefficients and 6j-symbols in the software system [13]. Our notation for the reduced matrix elements entering the WE theorem is explained in appendix B.1. For spherical harmonics $Y_{\ell m}$ and Legendre polynomials $P_{\ell}$ we use the standard definitions [1, 6, 8]. Furthermore, we adopt the convention, usual in numerical computations [13], $Y_{\ell m}(\hat{r}) \equiv 0$ if $|m| > \ell$. When states with different angular momenta or spins are considered, we use the notations
\[
\langle \ell \rangle = \ell + \ell', \quad \Delta \ell = \ell' - \ell, \quad \Delta \ell_z = \ell'_z - \ell_z,
\] (A.2)
and similarly $\langle s \rangle$, $\Delta s$, $\Delta s_z$. We also find useful the notation $\ell_k = \ell \pm k$ for orbital angular-momentum quantum numbers, with the convention that when several $\ell_k$, $\ldots$, $\ell_k$, appear in the same equation, then either the upper or the lower sign is chosen for all of them simultaneously. Thus, $\ell_{n+1} - \ell_n = \ell_{p+1} - \ell_p = \pm 1$ for any $n$ and $p$.

We denote tensor indices by lowercase Latin superindices, $A^{i_1 \ldots i_n}$, and spinor indices by uppercase Latin subindices, $\chi_A$. Since we consider only tensors and spinors of the su(2) algebra (as opposed to the sl(2) algebra), there is no need to raise spinor indices or lower tensor ones. We denote normal spinors and tensors by a caret, e.g. $\hat{r} = \vec{r}/r$. Our choice of orthonormal bases for spinors and tensors is explained in appendix B.

Given a numeric or operator tensor $A^{i_1 \ldots i_n}$, we denote its associated totally symmetrized and antisymmetrized tensors as
\[
A^{[i_1 \ldots i_n]} = \sum_{\sigma} A^{i_{\sigma_1} \ldots i_{\sigma_n}}, \quad A^{(i_1 \ldots i_n)} = \sum_{\sigma} \text{sgn}(\sigma) A^{i_{\sigma_1} \ldots i_{\sigma_n}},
\] (A.3)
where the sums extend over all permutations $\sigma$ of $i_1, \ldots, i_n$. The same notations apply to tensor products, e.g. $\hat{r}^i L^j = \hat{r}^i L^j + \hat{r}^j L^i$. We denote the traceless part of $A^{i_1 \ldots i_n}$ by $A^{(i_1 \ldots i_n)}$. The cases of importance in this paper are rank-2 and -3 tensors:
\[
A^{ij} = A^{ij} - \frac{1}{3} A^{kk} \delta^{ij},
\]
\[
A^{pqr} = A^{pqr} - \frac{2}{5} (A^{p|j} \delta^{qr} + A^{q|p} \delta^{jr} + A^{i|r} \delta^{pq})
+ \frac{1}{10} (A^{q|j} \delta^{pr} + A^{r|j} \delta^{pq} + A^{p|q} \delta^{jr} + A^{i|r} \delta^{pq} + A^{i|q} \delta^{pr} + A^{i|p} \delta^{qr}).
\] (A.4)
The same notation applies to tensor products, such as $\hat{r}^{ij} \hat{r}^{ji} = \hat{r}^{ij} \hat{r}^{ji} - 1/3 \delta^{ij}$. We denote by $A^{(i_1 \ldots i_n)}$ the traceless part of $A^{i_1 \ldots i_n}$. Thus, the irreducible component of $A^{i_1 \ldots i_n}$ is $1/n! A^{(i_1 \ldots i_n)}$. A more systematic approach to irreducible tensors is given in the following section.

Appendix B. Standard tensor and spinor bases

In this appendix, we introduce the standard bases for irreducible spinors and tensors of the su(2) algebra used as spin wavefunctions throughout the paper. We use spin wavefunctions of the Rarita–Schwinger type [14], which carry only vector indices varying from 1 to 3 and spinor indices from 1 to 2, and whose components are linearly related by symmetry and tracelessness and, for spinors, by transversality conditions. In fact, the spinors discussed in appendix B.2 are the non-relativistic version of Rarita–Schwinger spinors. The standard bases are defined so that they are orthonormal, have definite complex-conjugation properties and satisfy the Condon–Shortley phase conventions [1, 2, 4, 8].

Our starting point is the usual basis for Pauli spinors
\[
\hat{\chi}_A(1/2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\chi}_A(-1/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\] (B.1)
The triplet component of basis-spinors tensor products yields a basis for the representation space of the adjoint representation

\[ \hat{\chi}_{AB}(m) = \frac{1}{2} \sum_{s_z} \frac{1}{2} \langle \frac{1}{2}, s_z | m \rangle \hat{\chi}(s_z)_{AB} \hat{\chi}(s_z), \quad m = 1, 2, 3, \]  

(B.2)

whose associated unit vectors are

\[ \hat{\mathbf{e}}^k(1) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad \hat{\mathbf{e}}^k(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{\mathbf{e}}^k(-1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}. \]  

(B.3)

These are the standard basis vectors, satisfying

\[ \hat{\mathbf{e}}^k(s_z)^* = (-1)^s \hat{\mathbf{e}}^k(-s_z), \quad \hat{\mathbf{e}}^k(s_z)^* \hat{\mathbf{e}}^k(s_z') = \delta_{s_z s_z'}, \quad \sum_{s_z} \hat{\mathbf{e}}^k(s_z) \hat{\mathbf{e}}^k(s_z') = 0. \]  

(B.4)

Furthermore, from the action of the spin operator on basis spinors we get

\[ \hat{\mathbf{e}}^k(m) \sigma^k_{AB} \hat{\mathbf{e}}^k(s_z) = \frac{2 s_z - m}{\sqrt{1 + m^2}} \hat{\mathbf{e}}^k((-1)^m s_z), \quad m = -1, 0, 1, s_z = \pm 1/2, \]  

(B.5)

a relation that will be needed in the discussion of spinors later.

### B.1. Standard irreducible tensor bases

Starting with the standard vectors (B.3) we define the standard basis of irreducible rank-n tensors recursively as

\[ \hat{\mathbf{e}}^{i_1 \ldots i_n}(m) = \sum_{s_z=-n}^{n-1} \sum_{m=-n}^{n+1} (n-1, m'; 1, s_z | n, m) \hat{\mathbf{e}}^{i_1 \ldots i_{n-1}}(m') \hat{\mathbf{e}}^{s_z}(s_z), \quad -n \leq m \leq n. \]  

(B.6)

This recursion can be solved to yield

\[ \hat{\mathbf{e}}^{i_1 \ldots i_n}(m) = \sum_{s_1, \ldots, s_n=-1}^{1} f_n(s_1, \ldots, s_n) \hat{\mathbf{e}}^{i_1}(s_1) \cdots \hat{\mathbf{e}}^{i_n}(s_n), \]  

(B.7a)

with

\[ f_n(s_1, \ldots, s_n) = \prod_{j=2}^{n} \left( 1 + s_j - \sum_{i=1}^{j-1} s_i | j; \sum_{h=1}^{j} s_h \right) \]

\[ = \left( \frac{2^n}{(2n)!} \prod_{h=1}^{n} (1 + s_h)! \right) \left( \frac{n!}{\prod_{h=1}^{n} (1 + s_h)!} \right)^{\frac{1}{2}}. \]  

(B.7b)

The second equality in (B.7b) follows from the first one by using the explicit expression for CG coefficients coupling angular momenta differing in one unit [7, 8]. From (B.1) it is clear that \( \hat{\mathbf{e}}^{i_1 \ldots i_n}(m) \) is totally symmetric. Explicit evaluation shows that \( \hat{\mathbf{e}}^{i_1 \ldots i_n}(m) \), \( -2 \leq m \leq 2 \), are traceless and therefore it follows from (B.6) by induction that \( \hat{\mathbf{e}}^{i_1 \ldots i_n}(m) \) are totally traceless for all \( n \geq 2, -n \leq m \leq n \). Thus, (B.6) defines a set of \( 2n + 1 \) irreducible tensors of rank.
where in the last equality 
which is useful to evaluate reduced matrix elements. We adopt the convention

showing that the maximal-rank coupling of two standard tensors is again a standard basis tensor. From (B.7) we obtain the equality

which is useful to evaluate reduced matrix elements. We adopt the convention

The matrix \( \tilde{S}_{(n)} \) of the spin operator \( \tilde{S} \) in the basis of spin-\( n \) states with \( n \) integer is given by

If \( A^{i_1 \ldots i_n} \) is a rank-\( n \) real tensor, or a self-adjoint tensor operator, its irreducible component is

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Furthermore, for self-adjoint tensor operators, the WE theorem [8] holds:

$$\langle \ell', \ell' \rangle = \langle \ell | \mathcal{R}^{\ell_{1} \cdots \ell_{k}} (s) | \ell \rangle = \langle \ell | \mathcal{R}^{\ell_{1} \cdots \ell_{k}} (s) | \ell \rangle \langle \ell, n, s, \ell | \ell', \ell' \rangle. \quad (B.15)$$

For a normal vector \( \vec{\tau} \), let \( \mathcal{Y}_{nm}(\vec{\tau}) = \mathcal{R}^{\ell_{1} \cdots \ell_{k}} (m) \mathcal{R}^{\ell_{1} \cdots \ell_{k}} \). It follows from (B.6) by induction that

$$\vec{\tau}^{i_{1}} \cdots \vec{\tau}^{i_{k}} = \sum_{m=-\ell}^{\ell} \hat{\mathcal{R}}^{i_{1} \cdots i_{k}} (m) \mathcal{R}^{i_{1} \cdots i_{k}} \mathcal{R}^{i_{1} \cdots i_{k}} \mathcal{R}^{i_{1} \cdots i_{k}}.$$  

We omit the proofs for brevity. We must then have \( \mathcal{Y}_{nm}(\vec{\tau}) \propto \mathcal{Y}_{nm}(\vec{\tau}) \). Evaluation of both sides of the proportionality relation at \( \vec{\tau} = \hat{\mathcal{Y}} \) yields

$$Y_{nm}(\vec{\tau}) = \frac{1}{2} \sqrt{\ell!} \mathcal{R}^{i_{1} \cdots i_{k}} (m) \mathcal{R}^{i_{1} \cdots i_{k}} \mathcal{R}^{i_{1} \cdots i_{k}}.$$  

(B.17)

Relation (B.17) can be inverted, for each \( \ell \), by extending the basis (B.6) with appropriate reducible tensors to an orthogonal basis of the entire space of rank-\( n \) tensors.

### B.2. Standard spinor bases

We define the standard spin-(\( n + 1/2 \)) spinors with \( n \geq 1 \) as

$$\hat{\mathcal{N}}^{i_{1} \cdots i_{k}} (s) = \frac{1}{\sqrt{2}} \sum_{m=-n}^{n} \langle n, m; s \rangle | m \rangle | m \rangle \mathcal{Y}_{nm} (m) \mathcal{R}_{nm} (s),$$

(B.19)

with \( 1 \leq q \leq n - 1 \). From definition (B.19) we see that \( \hat{\mathcal{N}}^{i_{1} \cdots i_{k}} \) is completely symmetric and traceless in its tensor indices. Thus, (B.20) remains valid after an arbitrary permutation of tensor indices on its rhs. From (B.19) with \( n = 1 \) and (B.5) we find, for the spin-3/2 spinor, \( \hat{\mathcal{N}}^{i_{1} \cdots i_{k}} (s) = 0 \), \( -3/2 \leq s \leq 3/2 \). Thus, from (B.20) with \( q = 1 \) we immediately obtain

$$\hat{\mathcal{N}}^{i_{1} \cdots i_{k}} (s) = 0 \quad 1 \leq k \leq n, \quad -n - 1/2 \leq s \leq n + 1/2.$$  

(B.21)

From (B.19) and (B.21), the number of independent components in the spinor basis is \( 2n + 2 \) as it should. The orthonormality and complex-conjugation relations read

$$\hat{\mathcal{N}}^{i_{1} \cdots i_{k}} (s) \hat{\mathcal{N}}^{i_{1} \cdots i_{k}} (s') = \delta_{s,s'}, \quad i \sigma_{AB}^{i_{1} \cdots i_{k}} (s) = (-1)^{1/2 + n} \hat{\mathcal{N}}^{i_{1} \cdots i_{k}} (s) \hat{\mathcal{N}}^{i_{1} \cdots i_{k}} (s).$$  

(B.22)

The completeness relation for spin-\( n \) spinors, with \( n = 1 + 1/2, n \geq 1 \), defines the orthogonal projector \( X_{AB}^{i_{1} \cdots i_{k} j} \) onto the subspace of spin-(\( n + 1/2 \)) spinors of \( \mathbb{C}^{(2n+1)2} \), which has the same dimension.

\(^2\) For notational simplicity we identify the space of complex rank-\( n \) completely symmetric and traceless tensors with \( \mathbb{C}^{(2n+1)2} \), which has the same dimension.
where $X^{i_1...i_k; j_1...j_k}$ is the projector defined in (B.1). The orthogonal complement of the spin-$(n + 1/2)$ subspace in $\mathbb{C}^{(2n+1)\times 2}$ is $\ker(X)$, which can be parameterized as $\sigma^{i_1...i_k; j_1...j_k}_{AB}$ with $\chi^i = k_i^{j_{i-1}...j_1}$, a spin-$(n - 1/2)$ spinor. This is the spin-$(n - 1/2)$ subspace of $\mathbb{C}^{(2n+1)\times 2}$.

The spin operator for $s = (n + 1/2)$ spinors is given by the operator tensor product of the spin-$n$ and spin-$1/2$ operators, its matrix being

$$
(S^{i_1...i_k}_{(n+1/2)})_{AB} = (S^i_{(n)})^{i_1...i_k}_{AB} \delta_{i_1...i_k} + \frac{1}{2} \delta_i^{i_1...i_k} \sigma^j_{AB},
$$

with $S^i_{(n)}$ defined in (B.13). From (B.21) and (B.24) we obtain the following useful identities:

$$
ie i k h \sigma^k_{AB} \chi^h = i k i h \chi^i_{AB} (s_z) = \tilde{\chi}^{-i...-i}_{AB} (s_z),
$$

$$\sigma^k_{AB} \chi^h = \sigma^{i...i}_{AB} \chi^i (s_z) = 0,
$$

$$\tilde{\chi}^{-i...-i}_{AB} (s_z) \sigma^{i...i}_{AB} \chi^i (s_z) = \tilde{\chi}^{-i...-i}_{AB} (s_z) \sigma^{i...i}_{AB} \chi^i (s_z) = \frac{1}{2} \sigma^j_{AB} \chi^j (s_z),
$$

which remain valid after permutation of $i_1...i_k$, etc.

**Appendix C. Reduced matrix elements**

In this appendix, we gather expressions for matrix elements of certain irreducible tensor operators needed in the foregoing. The notation used is the same as in (B.15). We begin from the known results [8]

$$
\langle j' | \vec{e} \cdot \vec{j} | j \rangle = \sqrt{j(j+1)} \delta_{j'j},
$$

$$
\langle \ell' | | Y_{L} | | \ell \rangle = \frac{1}{2L + 1} \sqrt{\frac{2L + 1}{2\ell' + 1}} \delta_{\ell\ell'}.
$$

In (C.1) $J'$ can of course be any angular momentum operator: orbital $L'$, spin $S'$ or total $J'$. For tensor products of angular momentum operators, we have, using (B.11) and (C.1),

$$
\langle j' | \vec{e}^{i...i} j^{i...i} | j \rangle = \frac{n!}{\sqrt{2^n (2n)!}} \sqrt{\frac{2j(j + 1)!}{2j' + 1}} \delta_{j'j}.
$$

For convenience, we quote here the explicit form of the CG coefficient appearing in (C.2). If $t$ is odd the CG coefficient vanishes, and if $t \geq 0$ is even

$$
\langle \ell, 0; n + t, 0|\ell_a, 0 \rangle = (-1)^{t/2}(\ell_1 - \ell)^n \sqrt{\frac{(2n + t - 1)!!(t - 1)!!}{(n + t/2)!(n/2)!!}}
$$

$$
\times \sqrt{2\ell_a + 1} \sqrt{\frac{(\ell_a + \ell + n + t)!!}{(\ell_a + \ell - n - t)!!}} \sqrt{\frac{(\ell_a + \ell - n + t)!!}{(\ell_a + \ell + n + t)!!}}
$$

$\ell \geq 0$ even.
Next, we consider tensor products of the position versor \( \hat{r} = r/r \). From (B.17) and (C.2)

\[
\langle \ell' | \vec{r}^{\ell_1 \ldots \ell_n}, \ldots, \vec{r}^{\ell_s} | \ell \rangle = \frac{n!}{(2n - 1)!!} \sqrt{\frac{2\ell + 1}{2\ell' + 1}} \langle \ell, 0; n, 0 | \ell', 0 \rangle.
\]

In the case \( |\Delta \ell| = n \), which is of interest to us, a more explicit expression is given by

\[
\langle \ell_n | \vec{r}^{\ell_1 \ldots \ell_n}, \ldots, \vec{r}^{\ell_s} | \ell \rangle = \left( -1 \right)^{\frac{n - |\Delta \ell|}{2}} \frac{2\ell + 1}{2\ell' + 1} \frac{\Gamma (\ell) - \frac{n}{2} + \frac{1}{2}}{\Gamma (\ell - \frac{n}{2} + \frac{1}{2})} 
\]

\[
\times \frac{\Gamma (|\Delta \ell| + \frac{n}{2} + 1)}{\Gamma (|\Delta \ell| - \frac{n}{2} + 1)}.
\]

The cases \( n = 1, 2, 3 \) are of particular importance to us. Noting that \( (-1)^{\frac{n - |\Delta \ell|}{2}} = \frac{n}{\sqrt{2|\Delta \ell|}} \),

we obtain

\[
\langle \ell | \vec{r}^{\ell} | \ell \rangle = \frac{\Delta \ell}{\sqrt{2^\ell \times 2\ell' + 1}} \]

\[
\langle \ell_2 | \vec{r}^{\ell_2} | \ell \rangle = \frac{\Delta \ell}{\sqrt{2^\ell + 1}} \sqrt{\frac{\langle \ell | (\ell + 1) \rangle}{(2\ell - 1)(2\ell + 1)(2\ell + 3)}}.
\]

\[
\langle \ell_3 | \vec{r}^{\ell_3} | \ell \rangle = \frac{\Delta \ell}{24\sqrt{2}} \sqrt{\frac{2^\ell + 1}{2\ell + 1}} \sqrt{\frac{\langle 2\ell - 1 \rangle(2\ell + 1)(2\ell + 3)}{(2\ell - 1)(2\ell + 1)(2\ell + 3)}}
\]

Equations (C.6) and (C.7) remain valid under the replacements \( \ell \rightarrow s, \vec{r} \rightarrow T \), that give the reduced matrix elements \( \langle s \pm n | \vec{r}^{\ell_1 \ldots \ell_n} \ldots, \vec{r}^{\ell_s} | s \rangle \) of tensor products of the spin transition operator for integer or half-integer \( s \).

For tensor products of \( \vec{r} \) and \( \vec{L} \), using (C.3), (C.6) and (B.11) we obtain

\[
\langle \ell + \Delta \ell | \vec{r}^{\ell_1 \ldots \ell_n} \ldots, \vec{L}^{\Delta \ell=1} \ldots, \vec{L}^{\Delta \ell=1} | \ell \rangle = \left( \frac{\Delta \ell}{|\Delta \ell|} \right)^{\frac{\Delta \ell}{2n/2|\Delta \ell|}} \left( \frac{n + |\Delta \ell|!(n - |\Delta \ell|)!}{(2n)!} \right).
\]

This equation reduces to (C.3) for \( \Delta \ell = 0 \) and to (C.6) for \( |\Delta \ell| = n \). The particular cases \( n = 2, 3 \) of (C.8) are important in the following:

\[
\langle \ell_1 | \vec{r}^{\ell_1} \vec{L}^{\ell_1} | \ell \rangle = \frac{\Delta \ell}{4} \sqrt{2^\ell + 1} \frac{\langle 2\ell - 1 \rangle(2\ell + 1)(2\ell + 3)}{2\ell + 1},
\]

\[
\langle \ell_1 | \vec{L}^{\ell_1} \vec{L}^{\ell_1} | \ell \rangle = \frac{\Delta \ell}{2\sqrt{3} \sqrt{2^\ell + 1}} \frac{\langle 2\ell - 2 \rangle(2\ell - 1)(2\ell + 1)(2\ell + 3)(2\ell + 4)}{2\ell + 1},
\]

\[
\langle \ell_2 | \vec{r}^{\ell_2} \vec{L}^{\ell_2} | \ell \rangle = \frac{\Delta \ell}{\sqrt{3}} \sqrt{2^\ell + 1} \frac{\langle (2\ell - 1)(2\ell + 1)(2\ell + 2) \rangle(2\ell - 1)(2\ell + 1)(2\ell + 3)}{2\ell + 1}.
\]
The following mixed reduced matrix elements are also useful when dealing with tensor decompositions in irreducible components

\[
\langle \ell_1 | \tilde{\mathbf{r}} (\tilde{\mathbf{r}} \wedge \tilde{\mathbf{L}}') | | \ell \rangle = -\frac{i}{2\sqrt{2}} ((\ell_1 + \ell + 1) - 2\Delta \ell) \sqrt{\ell_1 + \ell + 1},
\]

\[
\langle \ell_1 | \tilde{\mathbf{r}} (\tilde{\mathbf{r}} \wedge \tilde{\mathbf{L}}') L' | | \ell \rangle = -\frac{i}{8} ((\ell_1 + \ell + 1) - 2\Delta \ell) \sqrt{\frac{(\ell_1 + \ell - 1)(\ell_1 + \ell + 1)(\ell_1 + \ell + 3)}{2\ell_1 + 1}},
\]

\[
\langle \ell_2 | \tilde{\mathbf{r}} (\tilde{\mathbf{r}} \wedge \tilde{\mathbf{L}}') | | \ell \rangle = -\frac{i}{2} ((\ell_1 - \ell)(2\ell_1 + 1) - 3) \sqrt{\frac{\ell_1(\ell_1 + 1)}{(2\ell_1 + 1)(2\ell_2 + 1)}}.
\]

With the replacements \( \ell \to s \), \( \Delta \ell \to \Delta s \), \( \tilde{\mathbf{r}}' \to T', \langle \tilde{\mathbf{r}}' \wedge \tilde{\mathbf{L}}' \rangle \to S', \) (C.8) and (C.9) give the reduced matrix elements \( \langle s \pm n || \tilde{\mathbf{r}}^{s_1} ... \tilde{\mathbf{r}}^{s_n} | P_{\ell_1} | \tilde{\mathbf{p}} \rangle \) for integer or half-integer \( s \), and similarly for (C.10).

**Appendix D. The matrix elements \( \langle \tilde{\mathbf{p}}' | P_{\ell_1} \tilde{\mathbf{r}}^{s_1} ... \tilde{\mathbf{r}}^{s_n} | \tilde{\mathbf{p}} \rangle \)**

In this appendix we derive (58) and (59). First, we consider the matrix element

\[
\langle \tilde{\mathbf{p}}' | P_{\ell_1} \tilde{\mathbf{r}}^{s_1} ... \tilde{\mathbf{r}}^{s_n} | \tilde{\mathbf{p}} \rangle = \int d^2q \langle \tilde{\mathbf{p}}' | P_{\ell_1} \tilde{\mathbf{r}}^{s_1} ... \tilde{\mathbf{r}}^{s_n} \tilde{\mathbf{q}} | \tilde{\mathbf{p}} \rangle
\]

\[
= \frac{2\ell_1 + 1}{4\pi} \int d^2q P_{\ell_1} \tilde{\mathbf{r}}^{s_1} ... \tilde{\mathbf{r}}^{s_n} \tilde{\mathbf{q}} P_{\ell_1}(\tilde{\mathbf{p}} \cdot \tilde{\mathbf{q}})
\]

\[
= \frac{\ell_1 + 1}{2\ell_1 + 3} \langle \tilde{\mathbf{p}}' | P_{\ell_1} \tilde{\mathbf{p}} \rangle + \frac{\ell_1}{2\ell_1 + 1} \langle \tilde{\mathbf{p}}' | P_{\ell_1} \tilde{\mathbf{p}} \rangle,
\]

where the integrals extend over the unit sphere, and in the last equality we used the relation

\[
P_{\ell_1}(x) = \frac{\ell_1 + 1}{2\ell_1 + 1} P_{\ell_1+1}(x) + \frac{\ell_1}{2\ell_1 + 1} P_{\ell_1-1}(x).
\]

Applying the projector property of \( P_{\ell_1} \) in the last line of (D.1) we obtain

\[
\langle \tilde{\mathbf{p}}' | P_{\ell_1} \tilde{\mathbf{r}}^{s_1} ... \tilde{\mathbf{r}}^{s_n} | \tilde{\mathbf{p}} \rangle = \frac{1}{2} \ell_1 + 1 \langle \tilde{\mathbf{p}}' | P_{\ell_1+1} \tilde{\mathbf{p}} \rangle.
\]

Similarly, using

\[
\nabla_{\tilde{\mathbf{r}}'} \tilde{\mathbf{r}}^{s_1} ... \tilde{\mathbf{r}}^{s_n} \tilde{\mathbf{q}} \wedge \tilde{\mathbf{p}}' = \tilde{\mathbf{r}}^{s_1} ... \tilde{\mathbf{r}}^{s_n} \tilde{\mathbf{q}} \wedge \tilde{\mathbf{p}}',
\]

we obtain

\[
\langle \tilde{\mathbf{p}}' | P_{\ell_1} \tilde{\mathbf{r}}^{s_1} ... \tilde{\mathbf{r}}^{s_n} \tilde{\mathbf{q}} | \tilde{\mathbf{p}} \rangle = \nabla_{\tilde{\mathbf{r}}'} \left( \frac{\ell_1 - \ell}{4\pi} P_{\ell_1}(\tilde{\mathbf{p}} \cdot \tilde{\mathbf{r}}') \right) = \frac{\ell_1 - \ell}{4\pi} \tilde{\mathbf{r}}' \wedge \tilde{\mathbf{p}}' P_{\ell_1}(\tilde{\mathbf{p}} \cdot \tilde{\mathbf{r}}').
\]

Summing (D.2) and (D.3) we obtain (59) with the help of the relation

\[
x P_{\ell_1}(x) = (n - k) P_{\ell_1}(x) + \frac{1}{2}(\ell_{n+1} - \ell_n)(\ell_{n+1} + 1) P_{\ell_1} = P_{\ell_{n+1}}, \quad 0 \leq k \leq n.
\]
The derivation of (58) runs along the same lines
\[
\langle \hat{p}' | P_{\ell n} \hat{r}^{l_1} \cdots \hat{r}^{l_{i-1}} | \hat{p} \rangle = \int d^2q \langle \hat{p}' | P_{\ell n} \hat{r}^{l_1} \cdots \hat{r}^{l_{i-1}} | \hat{q} \rangle \langle \hat{q} | P_{\ell n} \hat{r}^{l_1} \cdots \hat{r}^{l_{i-1}} | \hat{p} \rangle
\]
\[
= \int d^2q \langle \hat{p}' | P_{\ell n} \hat{r}^{l_1} \cdots \hat{r}^{l_{i-1}} \hat{p} \cdot \hat{r} | P_{\ell n} | \hat{q} \rangle \langle \hat{q} | P_{\ell n} \hat{r}^{l_1} \cdots \hat{r}^{l_{i-1}} | \hat{p} \rangle
\]
\[
+ \int d^2q \langle \hat{p}' | P_{\ell n} (\hat{r}^{l_1} \cdots \hat{r}^{l_{i-1}} | \hat{q} \rangle \langle \hat{q} | P_{\ell n} \hat{r}^{l_1} \cdots \hat{r}^{l_{i-1}} | \hat{p} \rangle .
\] (D.5)

Applying now (D.2) to the first integral and (D.3) to the second one in the last equality, we obtain (58).

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