Coordinate-Descent Diffusion Learning by Networked Agents

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Abstract—This work examines the mean-square error performance of diffusion stochastic algorithms under a generalized coordinate-descent scheme. In this setting, the adaptation step by each agent is limited to a random subset of the coordinates of its stochastic gradient vector. The selection of coordinates varies randomly from iteration to iteration and from agent to agent across the network. Such schemes are useful in reducing computational complexity in power-intensive large data applications. Interestingly, the results show that the steady-state performance of the learning strategy is not affected, while the convergence rate suffers some degradation. The results provide yet another indication of the resilience and robustness of adaptive distributed strategies.

Index Terms—Coordinate descent, stochastic partial update, computational complexity, diffusion strategies, stochastic gradient algorithms, strongly-convex cost.

I. INTRODUCTION AND RELATED WORK

Consider a strongly-connected network of \( N \) agents, where information can flow in either direction between any two connected agents and, moreover, there is at least one self-loop in the topology \([11, p. 436]\). We associate a strongly convex differentiable risk, \( J_k(w) \), with each agent \( k \) and assume in this work that all these costs share a common minimizer, \( w^* \in \mathbb{R}^M \). This case models important situations where agents work cooperatively towards the same goal. The objective of the network is to determine the unique minimizer of the following aggregate cost, assumed to be strongly-convex:

\[
J^\text{glob}(w) = \sum_{k=1}^{N} J_k(w) \quad (1)
\]

It is also assumed that the individual cost functions, \( J_k(w) \), are each twice-differentiable and satisfy

\[
0 < \nu_d I_M \leq \nabla^2_{w} J_k(w) \leq \delta_d I_M \quad (2)
\]

where \( \nabla^2_{w} J_k(w) \) denotes the \( M \times M \) Hessian matrix of \( J_k(w) \) with respect to \( w \), \( \nu_d \leq \delta_d \) are positive parameters, and \( I_M \) is the \( M \times M \) identity matrix. This condition is automatically satisfied by important cases of interest, such as logistic regression or mean-square-error designs \([11, 22]\).

The agents work cooperatively in an adaptive manner to seek the minimizer \( w^* \) of problem (1) by applying the following diffusion strategy \([11, 23]\):

\[
\begin{align}
\phi_{k, i-1} &= \sum_{\ell \in N_k} a_{1, \ell k} w_{\ell, i-1} \\
\psi_{k, i} &= \phi_{k, i-1} - \mu_k \nabla w_{\ell, i} J_k(\phi_{k, i-1}) \\
w_{k, i} &= \sum_{\ell \in N_k} a_{2, \ell k} \psi_{l, i}
\end{align} \quad (3)
\]

where the \( M \)-vector \( w_{k, i} \) denotes the estimate by agent \( k \) at iteration \( i \) for \( w^* \), while \( \psi_{k, i} \) and \( \phi_{k, i-1} \) are intermediate estimates. Moreover, an approximation for the true gradient vector of \( J_k(\cdot) \) is used \([27]\) since it is generally the case that the true gradient vector is not available (e.g., when \( J_k(w) \) is defined as the expectation of some loss function and the probability distribution of the data is not known beforehand to enable computation of \( J_k(\cdot) \) or its gradient vector). The symbol \( N_k \) in (3) refers to the neighborhood of agent \( k \). The \( N \times N \) combination matrices \( A_1 = [a_{1, \ell k}] \) and \( A_2 = [a_{2, \ell k}] \) are left-stochastic matrices consisting of nonnegative convex combination coefficients that satisfy:

\[
\begin{align}
a_{j, k} \geq 0, \quad \sum_{\ell=1}^{N} a_{j, \ell k} &= 1, \quad a_{j, \ell k} = 0, \quad \text{if } \ell \notin N_k
\end{align} \quad (4)
\]

for \( j = 1, 2 \). Either of these two matrices can be chosen as the identity matrix, in which case algorithm \([33]\) reduces to one of two common forms for diffusion adaptation: the adapt-then-combine (ATC) form when \( A_1 = I \) and the combine-then-adapt (CTA) form when \( A_2 = I \). We continue to work with the general formulation \([33]\) in order to treat both algorithms, and other cases as well, in a unified manner. The parameter \( \mu_k > 0 \) is a constant step-size factor used to drive the adaptation process. Its value is set to a constant in order to enable continuous adaptation in response to streaming data or drifting minimizers. We could also consider a distributed implementation of the useful consensus-type \([11, 33–100]\). However, it has been shown in \([11, 11]\) that when constant step-sizes are used to drive adaptation, the diffusion networks have wider stability ranges and superior performance. This is because consensus implementations have an inherent asymmetry in their updates, which can cause network graphs to behave in an unstable manner when the step-size is constant. This problem does not occur over diffusion networks. Since adaptation is a core element of the proposed strategies in this work, we therefore focus on diffusion learning mechanisms.
The main distinction in this work relative to prior studies on diffusion or consensus adaptive networks is that we now assume that, at each iteration $i$, the adaptation step in (3) has only access to a random subset of the entries of the approximate gradient vector. This situation may arise due to missing data or a purposeful desire to reduce the computational burden of the update step. We model this scenario by replacing the approximate gradient vector by

$$\nabla w^T J^\text{miss}_k (\phi_{k,i-1}) = \Gamma_{k,i} \nabla w^T J_k (\phi_{k,i-1})$$

(5)

where the random matrix $\Gamma_{k,i}$ is diagonal and consists of Bernoulli random variables $\{r_{k,i}(m)\}$; each of these variables is either zero or one with probability

$$\text{Prob}(r_{k,i}(m) = 0) \equiv r_k$$

(6)

where $0 \leq r_k < 1$ and

$$\Gamma_{k,i} = \text{diag}\{r_{k,i}(1), r_{k,i}(2), \ldots, r_{k,i}(M)\}$$

(7)

In the case when $r_{k,i}(m) = 0$, the $m$-th entry of the gradient vector is missing, and then the $m$-th entry of $\psi_{k,i}$ in (3b) is not updated. Observe that we are attaching two subscripts to $r$: $k$ and $i$, which means that we are allowing the randomness in the update to vary across agents and also over time.

### A. Relation to Block-Coordinate Descent Methods

Our formulation provides a nontrivial generalization of the powerful random coordinate-descent technique studied, for example, in the context of deterministic optimization in [12]–[14] and the references therein. Random coordinate-descent has been primarily applied in the literature to single-agent convex optimization, namely, to problems of the form:

$$w^* = \arg \min_w J(w)$$

(8)

where $J(w)$ is assumed to be known beforehand. The traditional gradient descent algorithm for seeking the minimizer of $J(w)$, assumed differentiable, takes the form

$$w_i = w_{i-1} - \mu \nabla w^T J(w_{i-1})$$

(9)

where the full gradient vector is used at every iteration to update $w_{i-1}$ to $w_i$. In a coordinate-descent implementation, on the other hand, at every iteration $i$, only a subset of the entries of the gradient vector is used to perform the update. These subsets are usually chosen as follows. First, a collection of $K$ partitions of the parameter space $w$ is defined. These partitions are defined by diagonal matrices, $\{\Omega_k\}$. Each matrix has ones and zeros on the diagonal and the matrices add up to the identity matrix:

$$\sum_{k=1}^K \Omega_k = I_M$$

(10)

Multiplying $w$ by any $\Omega_k$ results in a vector of similar size, albeit one where the only nontrivial entries are those extracted by the unit locations in $\Omega_k$. At every iteration $i$, one of the partitions is selected randomly, say, with probability

$$\text{Prob}(\Gamma_i = \Omega_k) = \omega_k$$

(11)

where the $\{\omega_k\}$ add up to one. Subsequently, the gradient descent iteration is replaced by

$$w_i = w_{i-1} - \mu \Gamma_i \nabla w^T J(w_{i-1})$$

(12)

This formulation is known as the randomized block-coordinate descent (RBCD) algorithm [12]–[14]. At each iteration, the gradient descent step employs only a collection of coordinates represented by the selected entries from the gradient vector.

If we reduce our formulation (3) to the single agent case, it will become similar to (12) in that the desired cost function is optimized only along a subset of the coordinates at each iteration. However, our algorithm offers more randomness in generating the coordinate blocks than the RBCD algorithm, by allowing more random combinations of the coordinates at each time index. In particular, we do not limit the selection of the coordinates to a collection of $K$ possibilities predetermined by the $\{\Omega_k\}$. Moreover, in our work we use a random subset of the stochastic gradient vector instead of the true gradient vector to update the estimate, which is necessary for adaptation and online learning when the true risk function itself is not known (since the statistical distribution of the data is not known). Also, our results consider a general multi-agent scenario involving distributed optimization where each individual agent employs random coordinates for its own gradient direction, and these coordinates are generally different from the coordinates used by other agents. In other words, the networked scenario adds significant flexibility into the operation of the agents under model (5).

### B. Relation to Partial Updating Schemes

It is also important to clarify the differences between our formulation and other works in the literature, which rely on other useful notions of partial information updates. To begin with, our formulation (5) is different from the models used in [15]–[17] where the step-size parameter was modeled as a random Bernoulli variable, $\mu_k(i)$, which could assume the values $\mu_k$ or zero with certain probability. In that case, when the step-size is zero, all entries of $w_{k,i-1}$ will not be updated and adaptation is turned off completely. This is in contrast to the current scenario where only a subset of the entries are left without update and, moreover, this subset varies randomly from one iteration to another.

Likewise, the useful works [18], [19] employ a different notion of partial sharing of information by focusing on the exchange of partial entries of the weight estimates themselves rather than on partial entries of the gradient directions. In other words, the partial information used in these works relate to the combination steps (5a) and (5c) rather than to the adaptation step (5b). They also focus on the special case in which the risks $J_k(w)$ are quadratic in $w$. In [18], it is assumed that only a subset of the weight entries are shared (diffused) among neighbors and that the estimate itself is still updated fully. In comparison, the formulation we are considering diffuses all entries of the weight estimates. Similarly, in [19] it is assumed that some entries of the regression vector are missing, which cause changes to the gradient vectors. In order to undo these changes, an estimation scheme is proposed in [19] to estimate
the missing data. In our formulation, more generally, a random subset of the entries of the gradient vector are set to zero at each iteration, while the remaining entries remain unchanged and do not need to be estimated.

There are also other criteria that have been used in the literature to motivate partial updating. For example, in [20], the periodic and sequential least-mean-squares (LMS) algorithms are proposed, where the former scheme updates the whole coefficient vector every $N$-th iteration, with $N > 1$, and the latter updates only a fraction of the coefficients, which are predetermined, at each iteration. In [21], [22] the weight vectors are partially updated by following a set-membership approach, where updates occur only when the innovation obtained from the data exceeds a predetermined threshold. In [22] – [24], only entries corresponding to the largest magnitudes in the regression vector or the gradient vector at each agent are updated. However, such scheduled updating techniques may suffer from non-convergence in the presence of nonstationary signals [25].

Partial update schemes can also be based on dimensionality reduction policies using Krylov subspace concepts [26] – [28]. There are also techniques that rely on energy considerations for the spectral radius of a matrix, e.g., [29].

The objective of the analysis that follows is to examine the effect of random partial gradient information on the learning performance and convergence rate of adaptive networks for general risk functions. We clarify these questions by adapting the framework described in [1], [2].

Notation: We use lowercase letters to denote vectors, uppercase letters for matrices, plain letters for deterministic variables, and boldface letters for random variables. We also use $(\cdot)^T$ to denote transposition, $(\cdot)^{-1}$ for matrix inversion, $\text{Tr}(\cdot)$ for the trace of a matrix, $\text{diag}(\cdot)$ for a diagonal matrix, $\text{col}(\cdot)$ for a column vector, $\lambda(\cdot)$ for the eigenvalues of a matrix, $\rho(\cdot)$ for the spectral radius of a matrix, $\| \cdot \|$ for the 2-norm of a matrix or the Euclidean norm of a vector. Besides, we use $A \preceq B$ to denote that $A - B$ is positive semi-definite, and $p \succ 0$ to denote that all entries of vector $p$ are positive. Moreover, $\alpha = O(\mu)$ signifies that $|\alpha| \leq c|\mu|$ for some constant $c > 0$, and $o(\mu)$ signifies that $\alpha/\mu \to 0$ as $\mu \to 0$.

## II. DATA MODEL AND ASSUMPTIONS

Let $\mathcal{F}_{i-1}$ represent the filtration (collection) of all random events generated by the processes $\{w_{k,j}\}$ and $\{\Gamma_{k,j}\}$ at all agents up to time $i - 1$. In effect, the notation $\mathcal{F}_{i-1}$ refers to the collection of all past $\{w_{k,j}, \Gamma_{k,j}\}$ for all $j \leq i - 1$ and all agents.

Assumption 1: (Conditions on indicator variables). It is assumed that the indicator variables $r_{k,i}(m)$ and $r_{\ell,i}(m)$ are independent of each other, for all $k, \ell, m, n$. In addition, the variables $\{r_{k,i}(m)\}$ are independent of $\mathcal{F}_{i-1}$ and $\nabla_{w^T} J_k(w)$ for any iterates $w \in \mathcal{F}_{i-1}$ and for all agents $k$.

Let

$$s_{k,i}(\phi_{k,i-1}) \triangleq \nabla_{w^T} J_k(\phi_{k,i-1}) - \nabla_{w^T} J_k(\phi_{k,i-1})$$

II denote the gradient noise at agent $k$ at iteration $i$, based on the complete approximate gradient vector, $\nabla_{w^T} J_k(w)$. We introduce its conditional second-order moment

$$R_{s,k,i}(w) \triangleq \mathbb{E}[s_{k,i}(w)s_{k,i}^T(w)|\mathcal{F}_{i-1}].$$

The following assumptions are standard and are satisfied by important cases of interest, such as logistic regression risks or mean-square-error risks, as already shown in [1], [2]. These references also motivate these conditions and explain why they are reasonable.

Assumption 2: (Conditions on gradient noise) [1] pp. 496–497). It is assumed that the first and fourth-order conditional moments of the individual gradient noise processes satisfy the following conditions for any iterates $w \in \mathcal{F}_{i-1}$ and for all $k, \ell = 1, 2, \ldots, N$:

$$\mathbb{E}[s_{k,i}(w)|\mathcal{F}_{i-1}] = 0$$

$$\mathbb{E}[s_{k,i}(w)s_{k,i}^T(w)|\mathcal{F}_{i-1}] = 0, k \neq \ell$$

$$\mathbb{E}[\|s_{k,i}(w)\|^4|\mathcal{F}_{i-1}] \leq \beta_k^4 \|w\|^4 + \sigma_{s,k}^4$$

almost surely, for some nonnegative scalars $\beta_k^4$ and $\sigma_{s,k}^4$.

Assumption 3: (Smoothness conditions) [1] pp. 552,576).

It is assumed that the Hessian matrix of each individual cost function, $J_k(w)$, and the covariance matrix of each individual gradient noise process are locally Lipschitz continuous in a small neighborhood around $w = w^o$ in the following manner:

$$\|\nabla_w^2 J_k(w^o + \Delta w) - \nabla_w^2 J_k(w^o)\| \leq \kappa_c \|\Delta w\|$$

$$\|R_{s,k,i}(w^o + \Delta w) - R_{s,k,i}(w^o)\| \leq \kappa_d \|\Delta w\|^2$$

for any small perturbations $\|\Delta w\| \leq \varepsilon$ and for some $\kappa_c \geq 0$, $\kappa_d \geq 0$, and parameter $0 < \gamma \leq 4$. ☐

## III. MAIN RESULTS: STABILITY AND PERFORMANCE

For each agent $k$, we introduce the error vectors:

$$\tilde{w}_{k,i} \triangleq w^o - w_{k,i}$$

$$\tilde{\phi}_{k,i} \triangleq w^o - \phi_{k,i}$$

$$\tilde{\psi}_{k,i} \triangleq w^o - \psi_{k,i}$$

We also collect all errors, along with the gradient noise processes, from across the network into block vectors:

$$\tilde{w}_i \triangleq \text{col}\{\tilde{w}_{1,i}, \tilde{w}_{2,i}, \ldots, \tilde{w}_{N,i}\}$$

$$\tilde{\phi}_i \triangleq \text{col}\{\tilde{\phi}_{1,i}, \tilde{\phi}_{2,i}, \ldots, \tilde{\phi}_{N,i}\}$$

$$\tilde{\psi}_i \triangleq \text{col}\{\tilde{\psi}_{1,i}, \tilde{\psi}_{2,i}, \ldots, \tilde{\psi}_{N,i}\}$$

$$s_i \triangleq \text{col}\{s_{1,i}, s_{2,i}, \ldots, s_{N,i}\}$$

We further introduce the extended matrices:

$$\mathcal{M} \triangleq \text{diag}\{\mu_1, \mu_2, \ldots, \mu_N\} \otimes I_M$$

$$A_1 \triangleq A_1 \otimes I_M, A_2 \triangleq A_2 \otimes I_M$$

$$\Gamma_i \triangleq \text{diag}\{\Gamma_{1,i}, \Gamma_{2,i}, \ldots, \Gamma_{N,i}\}$$

In the results that follow, we will refer whenever convenient to derivations or results already established in [1] so that more
emphasis is given in this manuscript to the new arguments and results that are necessary for the coordinate-descent scenario.

**Lemma 1:** (Network error dynamics). Consider a network of $N$ interacting agents running the diffusion strategy (3) with the gradient vector replaced by (5). The evolution of the error dynamics across the network relative to the reference vector $w^o$ is described by the following recursion:

$$\tilde{w}_1 = B_i \tilde{w}_{i-1} + A_i^T A \tilde{\Gamma}_i s_i$$

where

$$B_i \triangleq A_i^T (I - M \tilde{\Gamma}_i \mathcal{H}_i) A_i$$
$$\mathcal{H}_i \triangleq \text{diag}\{H_1, H_2, \ldots, H_{N,i}\}$$
$$H_{k,i} \triangleq \int_0^1 \nabla^2_w J_k(w^o - t \phi_{k,i-1}) dt.$$ (30)

Proof: Refer to (11) pp. 498–504, which is still applicable to the current context. We only need to set in that derivation the matrix $A_o$ to $A_o = I$, and the vector $b$ to $b = 0_{MN}$. These quantities were defined in (8.131) and (8.136) of (11). The same derivation will lead to (30)–(33), with the main difference being the appearance now of the random matrix $\tilde{\Gamma}_i$ in (30) and (31). □

**Theorem 1:** (Network stability). Consider a strongly-connected network of $N$ interacting agents running the diffusion strategy (3) with the gradient vector replaced by (5). Assume that the individual cost functions, $J_k(w)$, satisfy the condition in (2) and that Assumptions [1] hold. Then, the second and fourth-order moments of the network error vectors are stable for sufficiently small step-sizes, namely, it holds that

$$\limsup_{i \to \infty} E\|\tilde{w}_{k,i}\|^2 = O(\mu_{max})$$
$$\limsup_{i \to \infty} E\|\tilde{w}_{k,i}\|^4 = O(\mu^2_{max})$$

for any $\mu_{max} < \mu_o$, for some small enough $\mu_o$, where

$$\mu_{max} \triangleq \max\{\mu_1, \mu_2, \ldots, \mu_N\}. \quad (36)$$

Proof: The argument requires some effort and is given in Appendix A □

**Lemma 2:** (Long-term network dynamics). Consider a strongly-connected network of $N$ interacting agents running the diffusion strategy (3) under (5). Assume that the individual cost functions satisfy (2) and that Assumptions [1] through [3] hold. After sufficient iterations, $i \gg 1$, the error dynamics of the network relative to the reference vector $w^o$ is well-approximated by the following model:

$$\tilde{w}'_i = B'_i \tilde{w}'_{i-1} + A_i^T A \tilde{\Gamma}_i s_i, \quad i \gg 1 \quad (37)$$

where

$$B'_i \triangleq A_i^T (I - M \tilde{\Gamma}_i \mathcal{H}_i) A_i$$
$$\mathcal{H} \triangleq \text{diag}\{H_1, H_2, \ldots, H_N\}$$
$$H_k \triangleq \nabla^2_w J_k(w^o)$$

More specifically, it holds for sufficiently small step-sizes that

$$\limsup_{i \to \infty} E\|\tilde{w}'_{k,i}\|^2 = O(\mu_{max})$$
$$\limsup_{i \to \infty} E\|\tilde{w}'_{k,i}\|^4 = O(\mu^2_{max})$$

$$\limsup_{i \to \infty} E\|\tilde{w}'_{i}\|^2 = \limsup_{i \to \infty} E\|\tilde{w}_{i}\|^2 + O(\mu_{max}^3). \quad (43)$$

Proof: To establish (37), we refer to the derivation in (11) pp. 553–555, and note that, in our case, $\|\tilde{\Gamma}_i\| \leq 1$ and $b = 0_{MN}$ (which appeared in (10.2) of (11)). With regards to result (43), we refer to the argument in [2, pp. 557–560] and note again that $\|\Gamma_i\| \leq 1$. □

Result (43) ensures that the mean-square-error (MSE) performance of the network is in the order of $\mu_{max}$. Using the long-term model (37), we can be more explicit and derive the proportionality constant that describes the value of the network mean-square-error to first-order in $\mu_{max}$. To do so, we first introduce some useful variables. We assume that the matrix product $P = A_1 A_2$ is primitive. This condition is guaranteed automatically, for example, for ATC and CTA scenarios when the network is strongly-connected. This means, in view of the Perron-Frobenius Theorem [1], [2], that $P$ has a single eigenvalue at one. We denote the corresponding eigenvector by $p$, and normalize the entries of $p$ to add up to one. It follows from the same theorem that the entries of $p$ are strictly positive, written as

$$Pp = p, \quad p^T p = 1, \quad p > 0$$ (44)

with $1$ being the vector of size $N$ with all its entries equal to one. We introduce the quantity

$$q \triangleq \text{diag}\{\mu_1, \mu_2, \ldots, \mu_N\} A_2 p$$ (45)

and the gradient-noise covariance matrices:

$$G_k \triangleq \lim_{i \to \infty} R_{k,k,i}(w^o)$$
$$G'_k \triangleq \mathbb{E}[\Gamma_{k,i} G_k \Gamma_{k,i}].$$ (47)

Observe that $G_k$ is the limiting covariance matrix of the gradient noise process, while $G'_k$ is a weighted version of it. It follows by direct inspection that the entries of $G'_k$ are given by:

$$G'_k(m,n) = \begin{cases} (1 - r_k)^2 G_k(m,n), & m \neq n \\ (1 - r_k) G_k(m,m), & m = n. \end{cases}$$ (48)

We also define the mean-square-deviation (MSD) for each agent $k$, and the average MSD across the network to first-order in $\mu_{max}$. We also define the mean-square-deviation (MSD) for each agent $k$, and the average MSD across the network to first-order in $\mu_{max}$ — see (11) for further clarifications on these expressions where it is explained, for example, that MSD$_k$ provides the steady-state value of the error variance $E\|\tilde{w}_{k,i}\|^2$ to first-order in $\mu_{max}$:

$$\text{MSD}_k \triangleq \mu_{max} \left(\lim_{\mu_{max} \to 0} \limsup_{i \to \infty} \frac{1}{\mu_{max}} E\|\tilde{w}_{k,i}\|^2\right)$$ (49)

$$\text{MSD}_{av} \triangleq \frac{1}{N} \sum_{k=1}^N \text{MSD}_k.$$ (50)
Likewise, we define the excess-risk (ER) for each agent $k$ as the average fluctuation of the normalized aggregate cost

$$
\tilde{J}^{\text{glob}}(w) \triangleq \left( \sum_{k=1}^{N} q_k \right)^{-1} \sum_{k=1}^{N} q_k J_k(w)
$$

(51)

with \( \{q_k\} \) defined by (45), around its minimum value \( \tilde{J}^{\text{glob}}(w^o) \), namely \([1, p. 581]\):

$$
\text{ER}_k \triangleq \mu_{\max} \left( \lim_{\mu_{\max} \to 0} \lim_{i \to \infty} \frac{1}{\mu_{\max}} \mathbb{E}[\tilde{J}^{\text{glob}}(w_{k,i}) - \tilde{J}^{\text{glob}}(w^o)] \right).
$$

(52)

The average ER across the network is defined by

$$
\text{ER}_{av} \triangleq \frac{1}{N} \sum_{k=1}^{N} \text{ER}_k
$$

(53)

By following similar arguments to \([1, p. 582]\), it can be verified that the excess risk can also be evaluated by computing a weighted mean-square-error variance:

$$
\text{ER}_k \triangleq \mu_{\max} \left( \lim_{\mu_{\max} \to 0} \lim_{i \to \infty} \frac{1}{\mu_{\max}} \mathbb{E}[\tilde{J}^{\text{glob}}(w_{k,i}) - \tilde{J}^{\text{glob}}(w^o)] \right).
$$

(54)

where \( \tilde{H} \) denotes the Hessian matrix of the normalized aggregate cost, \( \tilde{J}^{\text{glob}}(w) \), evaluated at the minimizer \( w = w^o \):

$$
\tilde{H} \triangleq \left( \sum_{k=1}^{N} q_k \right)^{-1} \sum_{k=1}^{N} q_k H_k
$$

(55)

with \( H_k \) defined by (44). The following conclusion is one of the main results in this work. It shows how the coordinate descent construction influences performance in comparison to the standard diffusion strategy where all entries of the gradient vector are used at each iteration. Following the statement of the result, we illustrate its implications by considering several important cases.

**Theorem 2:** (**MSD and ER performance**). Under the same setting of Theorem\([1]\) it holds that, for sufficiently small stepsizes:

$$
\text{MSD}_{\text{coor}, k} = \text{MSD}_{\text{coor}, av}
$$

$$
= \frac{1}{2} \text{Tr} \left( \left( \sum_{k=1}^{N} q_k(1 - r_k)H_k \right)^{-1} \sum_{k=1}^{N} q_k^2 G_k \right)
$$

(56)

$$
\text{ER}_{\text{coor}, k} = \text{ER}_{\text{coor}, av} = \frac{1}{2} \text{Tr} \left( X \sum_{k=1}^{N} q_k^2 G_k' \right)
$$

(57)

where the subscript “coor” denotes the stochastic coordinate-descent diffusion implementation, and matrix \( X \) is the unique solution to the following Lyapunov equation:

$$
X \left( \sum_{k=1}^{N} q_k(1 - r_k)H_k \right) + \left( \sum_{k=1}^{N} q_k(1 - r_k)H_k \right) X = \tilde{H}.
$$

(58)

Moreover, for large enough \( i \), the convergence rate of the error variances, \( \mathbb{E}[\tilde{w}_{k,i}]^2 \), towards the steady-state region \( \Delta \) is given by

$$
\alpha_{\text{coor}} = 1 - 2\lambda_{\min} \left( \sum_{k=1}^{N} q_k(1 - r_k)H_k \right) + O \left( \mu_{\max}^{(N+1)/N} \right)
$$

(59)

**Proof:** See Appendix \([3]\)

\(\Box\)

**IV. IMPLICATIONS AND USEFUL CASES**

**A. Uniform missing probabilities**

Consider the case when the missing probabilities are identical across the agents, i.e., \( \{r_k \equiv r\} \).

1) **MSD performance**. The matrix \( G'_k \) defined by (48) can be written as

$$
G'_k = (1 - r)^2 G_k + \left( (1 - r) - (1 - r)^2 \right) \text{diag}\{G_k\}
$$

$$
= (1 - r)^2 \left( G_k + \frac{r}{1 - r} \text{diag}\{G_k\} \right)
$$

(60)

where the term \( \text{diag}\{G_k\} \) is a diagonal matrix that consists of the diagonal entries of \( G_k \). Then, the MSD expression (56) gives

$$
\text{MSD}_{\text{coor}, k} = \frac{1}{2} (1 - r) \text{Tr} \left( \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} \sum_{k=1}^{N} q_k^2 G_k \right)
$$

$$
= \frac{1}{2} (1 - r) \text{Tr} \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} \sum_{k=1}^{N} q_k^2 G_k \right) +
$$

$$
\frac{r}{2} \text{Tr} \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} \sum_{k=1}^{N} q_k^2 \text{diag}\{G_k\} \right) +
$$

$$
\frac{r}{2} \text{Tr} \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} \sum_{k=1}^{N} q_k^2 \text{diag}\{G_k\} \right) -
$$

$$
\frac{r}{2} \text{Tr} \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} \sum_{k=1}^{N} q_k^2 G_k \right).
$$

(61)

By recognizing that the first item in (61) is exactly the MSD expression for the stochastic full-gradient diffusion case \([1, p.\)
which is denoted by “MSD$_{\text{grad},k}$”, we get

$$\text{MSD}_{\text{coor},k} - \text{MSD}_{\text{grad},k}$$

$$= \frac{r}{2} \text{Tr} \left( \sum_{k=1}^{N} q \, H_k \right) - \sum_{k=1}^{N} q_k^2 \text{diag} \{ G_k \}$$

$$= \frac{r}{2} \text{Tr} \left( \sum_{k=1}^{N} q \, H_k \right) - \sum_{k=1}^{N} q_k^2 \tilde{G}_k$$

where

$$\tilde{G}_k \triangleq \text{diag} \{ G_k \} - G_k.$$  

Consider now MSE networks where the risk function that is associated with each agent $k$ is the mean-square-error:

$$J_k(w) = \mathbb{E}(d_k(i) - u_{k,i}w^o)^2$$

where $d_k(i)$ denotes the desired signal, and $u_{k,i}$ is a (row) regression vector. In these networks, the data $\{d_k(i), u_{k,i}\}$ are assumed to be related via the linear regression model

$$d_k(i) = u_{k,i}w^o + v_k(i)$$

where $v_k(i)$ is zero-mean white measurement noise with variance $\sigma^2_w$, and assumed to be independent of all other random variables. Assume that the regression data $\{u_{k,i}\}$ are white over time and space with

$$\mathbb{E} u_{k,i}u_{l,j} \equiv R_{u,k}\delta_{i,j}$$

where $R_{u,k} > 0$, and $\delta_{i,j}$ denotes the Kronecker delta sequence. Consider the case when the covariance matrices of the regressors are identical across the network, i.e., $\{R_{u,k} = R_u > 0\}$. Then, it holds that [1 p. 598]

$$H_k \equiv 2R_u, \quad G_k = 4\sigma^2_w R_u.$$  

Substituting into (62) we have

$$\text{MSD}_{\text{coor},k} - \text{MSD}_{\text{grad},k}$$

$$= \frac{r}{2} \text{Tr} \left( \sum_{k=1}^{N} 2q \right) - \sum_{k=1}^{N} 4q_k^2 \sigma^2_w R_u$$

$$\equiv \frac{r}{2} \text{Tr} \left( \sum_{k=1}^{N} 2q \right) - \sum_{k=1}^{N} 4q_k^2 \sigma^2_w R_u.$$  

$$\leq 0$$

where (73) holds because $\text{Tr} \{ R_u^{-1} \text{diag} \{ R_u \} \} \geq M$, which can be shown by using the property that $\text{Tr} \{ X \} \text{Tr} \{ X^{-1} \} \geq M^2$ for any $M \times M$ symmetric positive-definite matrix $X$ [30 p. 317], and choosing $X = \text{diag}^\frac{1}{2} \{ R_u \} R_u^{-1} \text{diag}^\frac{1}{2} \{ R_u \}$. In the case of MSE networks, by exploiting the special relation between the matrices $\{H_k\}$ and $\{G_k\}$ in (72), we are able to show that the MSD in the stochastic coordinate-descent case is always larger (i.e., worse) than or equal to that in the stochastic full-gradient diffusion case (although by not more than $O(\mu_{\text{max}})$, as indicated by (65). We are also able to provide a general upper bound on the difference between these two MSDs.

Corollary 4: (MSE networks). Under the same conditions of Corollary 2 and for MSE networks with uniform covariance matrices, i.e., $\{R_{u,k} \equiv R_u > 0\}$, it holds that

$$0 \leq \text{MSD}_{\text{coor},k} - \text{MSD}_{\text{grad},k} \leq$$

$$\frac{r}{2} \mu \left( \frac{1}{\nu_d} - \frac{1}{\delta_d} \right) \sum_{k=1}^{N} \nu_k^2 \text{Tr} \{ G_k \}.$$  

Moreover, it holds that $\text{MSD}_{\text{coor},k} = \text{MSD}_{\text{grad},k}$ if, and only if, $R_u$ is diagonal.

Proof: See Appendix [E]
2) ER performance: Consider the scenario when the missing probabilities are identical across the agents, i.e., \( \{r_k \equiv r\} \). Then, expression (58) simplifies to
\[
(1 - r) \left( \sum_{k=1}^{N} q_k \right) X \tilde{H} + (1 - r) \left( \sum_{k=1}^{N} q_k \right) \tilde{H} X = \tilde{H}. \tag{75}
\]
where we used the equality \( \sum_{k=1}^{N} q_k H_k = \left( \sum_{k=1}^{N} q_k \right) \tilde{H} \), it follows that
\[
X = \frac{1}{2} (1 - r)^{-1} \left( \sum_{k=1}^{N} q_k \right)^{-1} I_M. \tag{76}
\]
Thus, the ER expression in (57) can be rewritten as:
\[
\text{ER}_{\text{coord},k} = \frac{1}{4} (1 - r)^{-1} \left( \sum_{k=1}^{N} q_k \right)^{-1} \text{Tr} \left( \sum_{k=1}^{N} q_k^2 G_k \right) \tag{77}
\]
which is exactly the same result for the full gradient case from [1, p. 608], and where the equality (a) holds because \( \text{Tr} (G_k') = (1 - r) \text{Tr} (G_k) \) according to the definition in (48).

B. Uniform individual costs

Consider the case when the individual costs, \( J_k(w) \), are identical across the network, namely, [1, p. 610]
\[
J_k(w) \equiv J(w) \triangleq \mathbb{E} Q(w; x_k,i) \tag{78}
\]
where \( Q(w; x_k,i) \) denotes the loss function. In this case, it will hold that the matrices \( \{H_k, G_k\} \) are uniform across the agents, i.e.,
\[
H_k = \nabla^2_{w,i} J(w_o) \equiv H \tag{79}
\]
\[
G_k = \mathbb{E} \nabla_{w,i} Q(w_o; x_k,i) [\nabla_{w,i} Q(w_o; x_k,i)]^\top \equiv G \tag{80}
\]
which ensures the matrix \( \tilde{H} = H \) according to the definition in (55). By referring to (56), we have
\[
X = \frac{1}{2} \left( \sum_{k=1}^{N} q_k (1 - r_k) \right)^{-1} I_M. \tag{81}
\]
Then, expressions (56) and (57) reduce to
\[
\text{MSD}_{\text{coord},k} = \text{MSD}_{\text{coord},\text{av}} \nonumber = \frac{1}{2} \left( \sum_{k=1}^{N} q_k (1 - r_k) \right)^{-1} \sum_{k=1}^{N} q_k^2 \text{Tr} (H^{-1} G_k') \tag{82}
\]
\[
\text{ER}_{\text{coord},k} = \text{ER}_{\text{coord},\text{av}} \nonumber = \frac{1}{4} \left( \sum_{k=1}^{N} q_k (1 - r_k) \right)^{-1} \sum_{k=1}^{N} q_k^2 (1 - r_k) \text{Tr} (G). \tag{83}
\]
We proceed to compare the MSE and ER performance in the stochastic full-gradient and coordinate-descent cases. Let
\[
\alpha \triangleq \frac{\sum_{k=1}^{N} q_k^2 (1 - r_k)^2}{\sum_{k=1}^{N} q_k (1 - r_k)} - \frac{\sum_{k=1}^{N} q_k^2}{\sum_{k=1}^{N} q_k}, \quad \beta \triangleq \frac{\sum_{k=1}^{N} q_k^2 (1 - r_k)^2}{\sum_{k=1}^{N} q_k (1 - r_k)} - \frac{\sum_{k=1}^{N} q_k^2}{\sum_{k=1}^{N} q_k} \tag{84}
\]
and note that \( \alpha \leq \theta \), with equality if, and only if, \( \{r_k \equiv r\} \).

**Corollary 5:** (Performance comparison) Under the same conditions of Theorem 2 when the individual costs \( J_k(w) \) are identical across the agents, it holds that:

a) if \( \alpha \geq 0 \):
\[
0 \leq \text{MSD}_{\text{coord},k} - \text{MSD}_{\text{grad},k} \leq \frac{\theta}{2} \text{Tr} (G). \tag{86}
\]
b) if \( \alpha < 0 \), and \( \theta \geq (1 - \delta_d/\nu_d) \alpha \geq 0 \):
\[
0 \leq \text{MSD}_{\text{coord},k} - \text{MSD}_{\text{grad},k} \leq \frac{1}{2} \left( \frac{\theta}{\nu_d} + \frac{1}{\nu_d} - \frac{1}{\delta_d} \right) \text{Tr} (G). \tag{87}
\]
c) if \( \alpha < 0 \), and \( \theta \leq (1 - \delta_d/\nu_d) \alpha \leq 0 \):
\[
\frac{1}{2} \left( \frac{\theta}{\nu_d} + \frac{1}{\nu_d} - \frac{1}{\delta_d} \right) \text{Tr} (G) \leq \text{MSD}_{\text{coord},k} - \text{MSD}_{\text{grad},k} \leq 0. \tag{88}
\]
Likewise, it holds that
\[
\text{ER}_{\text{coord},k} - \text{ER}_{\text{grad},k} = \frac{\theta}{4} \text{Tr} (G). \tag{89}
\]
Then, in the case when either the missing probabilities or the quantities \( \{q_k\} \) are uniform across the agents, namely, \( \{r_k \equiv r\} \) or \( \{q_k \equiv q\} \), it follows that
\[
\text{ER}_{\text{coord},k} = \text{ER}_{\text{grad},k}. \tag{90}
\]

**Proof:** See Appendix [4]

Note that for the other choices of parameter \( \theta \) that are not indicated in Corollary 5 there is no consistent conclusion on which MSD (between MSD\(_{\text{coord},k}\) and MSD\(_{\text{grad},k}\)) is lower.

V. SIMULATION RESULTS

In this section, we illustrate the results by considering MSE networks and logistic regression networks; both settings satisfy condition 2 and Assumptions 1 through 3.

A. MSE Networks

In the following examples, we will test performance of the associated algorithms in the case when uniform missing probabilities are utilized across the agents. We adopt the ATC formulation, and set the combination matrices \( A_1 = I \), and \( A_2 \) according to the Metropolis rule in [1, p. 664]. In the first example, we revisit the two-agent MSE network discussed in Appendix [C] i.e., \( N = 2 \). We randomly generate \( w_o \) of size \( M = 2 \). The step-sizes \( \mu_1 = \mu_2 = 5 \times 10^{-3} \) are uniform across the agents, which gives \( q_1 = q_2 = 2.5 \times 10^{-3} \). The missing probabilities \( r_1 = r_2 = 0.5 \). The noises \( \{v_1(i), v_2(i)\} \) are zero-mean white Gaussian sequences with the variances
\[ \{\sigma_{v,1}^2 = 0.5, \sigma_{v,2}^2 = 5 \times 10^{-4}\}. \]

The regressors, uncorrelated with the noise sequences, are scaled such that the covariance matrices are of the form

\[ R_{u,1} = \begin{bmatrix} |\pi_1| & \pi_1 \\ \pi_1 & 1 \end{bmatrix}, \quad R_{u,2} = \begin{bmatrix} |\pi_2| & \pi_2 \\ \pi_2 & 1 \end{bmatrix}. \quad (91) \]

with \(|\pi_1| < 1, |\pi_2| < 1\). Now we select parameters \(\{\pi_1 = -0.34, \pi_2 = 0.99\}\), which satisfy condition (187), and \(\{\pi_1 = 0.34, \pi_2 = 0.99\}\) to illustrate the cases of \(\text{MSD}_{\text{cor},k} < \text{MSD}_{\text{grad},k}\) and \(\text{MSD}_{\text{cor},k} > \text{MSD}_{\text{grad},k}\) respectively. Fig. 1 shows the simulation results with the parameters \(\{\pi_1 = -0.34, \pi_2 = 0.99\}\). Figures 2 and 3 show the simulation results with the parameters \(\{\pi_1 = 0.34, \pi_2 = 0.99\}\). All results are averaged over 10000 independent runs. It is clear from the figures that the simulation results match well with the theoretical results from Theorem 2. In Fig. 1, the steady-state MSD of the stochastic coordinate-descent case is slightly lower than that of the full-gradient diffusion case, by about 0.32dB, which is close to the theoretical MSD difference of 0.41dB from (62). The MSD performance is better in the full-gradient diffusion case in Fig. 2 and the difference between these two MSDs at steady state is 1.71dB, which is close to the theoretical difference of 1.49dB from (62). The ER performance for both the stochastic coordinate-descent and full-gradient diffusion cases are the same as illustrated in Fig. 3 which verifies the theoretical result in (77).

In the second example, we test the case when the gradient vectors are missing with small probabilities across the agents. Figure 4 shows a network topology with \(N = 20\) agents. The parameter vector \(w^0\) is randomly generated with \(M = 10\). The regressors are generated by the first-order autoregressive model

\[ u_{k,i}(m) = \pi_k u_{k,i}(m-1) + \sqrt{1-\pi_k^2} t_{k,i}(m), \quad 1 \leq m < M \]

and the variances are scaled to be 1. The processes \(\{t_{k,i}\}\) are zero-mean, unit-variance, and independent and identically distributed (i.i.d) Gaussian sequences. The \(\{\pi_k\}\) are generated from a uniform distribution on the interval \((-1,1)\). The noises, uncorrelated with the regression vectors, are zero-mean white Gaussian sequences with the variances uniformly distributed over \((0.001,0.1)\). The step-sizes \(\{\mu_k\}\) across the agents are generated from a uniform distribution on the interval \((1 \times 10^{-4}, 5 \times 10^{-4})\). We choose a small missing probability \(\{r_k = 0.1\}\). Figure 5 shows the simulation results, which are averaged over 1000 independent runs. It is clear from the figure that, when the gradient information is missing with small probabilities, the performance of the coordinate-descent case is close to that of the full-gradient diffusion case.

In the third example, we test the case when the regressors are white across the agents. We randomly generate \(w^0\) of size \(M = 6\). The white regressors are generated from zero-mean white Gaussian sequences, and the powers, which vary from entry to entry, and from agent to agent, are uniformly distributed over \((0.5,1.5)\). The noises \(\{v_{k,i}\}\), uncorrelated with the regressors, are zero-mean white Gaussian sequences, with the variances \(\{\sigma^2_k\}\) generated from uniform distribution on the interval \((0.001,0.1)\). The step-sizes are uniformly distributed over \((1 \times 10^{-4}, 8 \times 10^{-4})\). The results, including the theoretical MSD value from (56) in Theorem 2, the simulated
MSD learning curves, and the convergence rates from (59), are illustrated by Fig. 6 where the results are averaged over 1000 independent runs. It is clear from the figure that, when white regressors are utilized in MSE networks, the stochastic coordinate-descent case converges to the same MSD level as the full-gradient diffusion case under this setting, which verifies (67). Besides, the learning curves match well with the convergence rates formulated in (59).

B. Logistic Networks

We now consider an application in the context of pattern classification. We assign with each agent the logistic risk

$$J_k(w) = \frac{\rho}{2} ||w||^2 + \mathbb{E} \left\{ \ln \left[ 1 + e^{-\gamma_k(i) h_k^T w} \right] \right\}$$

(93)

with regularization parameter $\rho > 0$, and where the labels $\{\gamma_k(i) = \pm 1\}$ are binary random and the $\{h_{k,i}\}$ are feature vectors. The objective is for the agents to determine a parameter vector $w^o$ to enable classification by estimating the class labels via $\hat{\gamma}_k(i) = h_{k,i}^T w^o$.

We proceed to test the theoretical findings in Corollary 5. Consider the same network topology from Fig. 4. We still adopt the ATC formulation, and set the combination matrices $A_1 = I$, and $A_2$ according to the Metropolis rule in [11] p. 664]. The feature vectors and the unknown parameter vector are randomly generated from uncorrelated zero-mean unit-variance i.i.d Gaussian sequences, both of size $M = 10$. The parameter $\rho$ in (93) is set to 0.01. To generate the trajectories for the experiments, the optimal solution to (93), $w^o$, the Hessian matrix $H$, and the gradient noise covariance matrix, $G$, are first estimated off-line by applying a batch algorithm to all data points.

In the first example, we consider the case when a uniform step-size $\{\mu_k = 0.005\}$ is utilized across the agents. All entries of the stochastic gradient vectors are missing completely at random with positive probabilities that are uniformly distributed over $(0, 1)$. Figure 7 shows the transient ER curves for the diffusion strategies with complete and partial gradients, where the results are averaged over 1000 independent runs. The figure also shows the theoretical result calculated from (83). It is clear from Figure 7 that the same ER performance is obtained in the stochastic coordinate-descent and full-gradient diffusion cases, by utilizing a uniform step-size and a doubly-stochastic combination matrix across the agents (in which case the parameters $\{q_k\}$ in (83) are identical across the agents), which is in agreement with the theoretical analysis in (90).

In the second and third examples, we randomly generate the step-sizes $\{\mu_k\}$ and missing probabilities $\{r_k\}$ by following uniform distributions on the intervals $[0.001, 0.01]$ and $(0, 1)$ respectively. In Figure 8 the parameters $\{\mu_k\}$ and $\{r_k\}$ are scaled to get a negative value for $\theta$ in (83), and in Fig. 9 those parameters are scaled to make $\theta$ positive. Figures 8 and 9 show respectively the transient ER learning curves in these two cases for the diffusion strategies with complete and partial gradients, where the results are averaged over 1000 independent runs. The figures also show the theoretical results calculated from (83). It is clear from Figs. 8 and 9 that these learning curves converge to their theoretical results at steady state. In Fig. 8 where $\theta < 0$, the stochastic coordinate-descent case converges to a lower ER level than the full-gradient diffusion case, and the difference between these two ERs is 0.637dB, which is close to the theoretical difference of 0.640dB from (59). In Fig. 9 where $\theta > 0$, the steady-state ER in the full-gradient diffusion case is lower than that of the stochastic coordinate-descent case, by about 0.726dB, which
The simulation result shows that the full-gradient diffusion case converges to a lower ER level than the stochastic coordinate-descent case under this setting, which verifies (89).

Fig. 7. ER learning curves for diffusion learning over a logistic network with full or partial updates in the case of Corollary 5, where the step-sizes and missing probabilities are scaled so that $\theta$ is positive in (85). The simulation result shows that the stochastic coordinate-descent and full-gradient diffusion cases converge to the same ER level under this setting, which verifies (89).

Fig. 8. ER learning curves for diffusion learning over a logistic network with full or partial updates in the case of Corollary 5 when a uniform step-size is defined by (44), and the matrix $A$ appearing in the first lower diagonal rather than unit entries, e.g., one typical sub-block in $J_e$ with eigenvalue $\lambda$ would look like:

$$
\begin{bmatrix}
\lambda & \epsilon \\
\epsilon & \lambda
\end{bmatrix}.
$$

All eigenvalues of $J_e$ are strictly inside the unit circle. Then, $P \triangleq P \otimes I_M \triangleq V_e J V_e^{-1}$

where $V_e \triangleq V_L \otimes I_M$, $J \triangleq J \otimes I_M$. Using (99), we can rewrite $B_i$ from (31) as

$$
B_i \triangleq (V_e^{-1})^T (J^T - D_i^T) V_e^T
$$

where

$$
D_i^T \triangleq V_i^T A_i^T M \Gamma_i H_{i-1} A_i^T (V_e^{-1})^T
$$

with block entries given by

$$
D_{11,i} = \sum_{k=1}^N q_k H_{k,i-1} \Gamma_{k,i}
$$

$$
D_{12,i} = (I^T \otimes I_M) \ H_{i-1} \Gamma_i M A_2 (V_R \otimes I_M)
$$

$$
D_{21,i} = (V_L^T \otimes I_M) A_i \ H_{i-1} \Gamma_i (q \otimes I_M)
$$

$$
D_{22,i} = (V_L^T \otimes I_M) A_i \ H_{i-1} \Gamma_i M A_2 (V_R \otimes I_M)
$$

and the vector $q$ is defined by (45) with entries $\{q_k\}$. Using the fact that $\|\Gamma_i\| \leq 1$, and following the same argument from [1] p. 513, we can verify that

$$
\|D_{12,i}\| \leq \sigma_{12}\mu_{\text{max}}
$$

$$
\|D_{21,i}\| \leq \sigma_{21}\mu_{\text{max}}
$$

$$
\|D_{22,i}\| \leq \sigma_{22}\mu_{\text{max}}
$$

for some constants $\{\sigma_{12}, \sigma_{21}, \sigma_{22}\}$. With regards to the norm of $D_{11,i}$, we observe that contrary to the arguments in [1] p.
for some positive constants $\lambda$ and $\sigma$. Then, matrices $\tilde{D}_{11,i}$ and $R_{D_{11,i}}$ are symmetric positive-definite. Following similar arguments to those in [2], pp. 511–512, we have

$$\|I_M - \tilde{D}_{11,i}\| \leq 1 - \sigma_{11}\mu_{\text{max}}$$

and

$$\|R_{D_{11,i}}\| \leq \beta_{11}\mu_{\text{max}}^2$$

for some positive constants $\sigma_{11}$ and $\beta_{11}^2$.

Now, multiplying both sides of (50) by $V_i^T$, we have

$$V_i^T \tilde{w}_i = (J_i^T - D_{11,i}^T) V_i^T \tilde{w}_i - V_i^T D_{12,i} M \Gamma_i s_i$$  \hspace{1cm} (117)$$

Let

$$V_i^T \tilde{w}_i = \left[ \begin{array}{c} (p^T \otimes I_M) \tilde{w}_i \\ (V_i^T \otimes I_M) \tilde{w}_i \end{array} \right] \triangleq \left[ \begin{array}{c} \tilde{w}_i \\ \tilde{s}_i \end{array} \right]$$

and

$$V_i^T A_{2} M \Gamma_i s_i = \left[ \begin{array}{c} (q^T \otimes I_M) \Gamma_i s_i \\ (V_i^T \otimes I_M) A_{2} M \Gamma_i s_i \end{array} \right] \triangleq \left[ \begin{array}{c} s_i \\ \tilde{s}_i \end{array} \right].$$

We then rewrite (117) as

$$\left[ \begin{array}{c} \tilde{w}_i \\ \tilde{s}_i \end{array} \right] = \left[ \begin{array}{c} I_M - D_{11,i}^T \\ -D_{12,i}^T \\ J_i^T - D_{22,i}^T \end{array} \right] \left[ \begin{array}{c} \tilde{w}_i - \tilde{D}_{11,i}^T \tilde{w}_i \\ \tilde{s}_i \end{array} \right].$$

where the asymmetry of the matrix $D_{11,i}$ in this case leads to a difference in the first row, compared to the arguments in [2] pp. 514–515. We adjust the arguments as follows. Using Jensen’s inequality, we have [2] p. 515:

$$\mathbb{E}[\|\tilde{w}_i\|^2 | \mathcal{F}_{i-1}] \leq \frac{1}{1 - t} \mathbb{E}[\| (I_M - D_{11,i}) \tilde{w}_i - \tilde{D}_{11,i} \|^2 | \mathcal{F}_{i-1}] + \frac{1}{t} \mathbb{E}[\| D_{21,i} \tilde{w}_i - \tilde{D}_{21,i} \|^2 | \mathcal{F}_{i-1}] + \mathbb{E}[\| \tilde{s}_i \|^2 | \mathcal{F}_{i-1}]$$

(121)

for any $0 < t < 1$, where the first term can be bounded by

\[
\mathbb{E}[\| (I_M - D_{11,i}) \tilde{w}_i - \tilde{D}_{11,i} \|^2 | \mathcal{F}_{i-1}]
\]

(118)

\[
\leq \lambda_{\text{max}}^2 (\mathbb{E}[\| (I_M - D_{11,i}) \tilde{w}_i - \tilde{D}_{11,i} \|^2 | \mathcal{F}_{i-1}]) \mathbb{E}[\| \tilde{w}_i \|^2 | \mathcal{F}_{i-1}]
\]

(119)

\[
\leq \lambda_{\text{max}}^2 \mathbb{E}[\| (I_M - D_{11,i}) \tilde{w}_i - \tilde{D}_{11,i} \|^2 | \mathcal{F}_{i-1}]
\]

(122)

where in step (a) we called upon the Rayleigh-Ritz characterization of eigenvalues [31, 32], and (b), (c) hold because $\|A\| = \lambda_{\text{max}}(A)$ for any symmetric positive-semidefinite matrix $A$, and $\|A\|^2 = \|A\|^2$ for any symmetric matrix $A$.

Computing the expectations again on both sides of (121), we have

$$\|D_{21,i}\|^2 \leq \frac{1}{1 - t} \mathbb{E}[\| (I_M - D_{11,i}) \tilde{w}_i - \tilde{D}_{11,i} \|^2 | \mathcal{F}_{i-1}]
\]

(119)

\[
+ \frac{1}{t} \mathbb{E}[\| D_{21,i} \tilde{w}_i - \tilde{D}_{21,i} \|^2 | \mathcal{F}_{i-1}]
\]

(121)

\[
+ \mathbb{E}[\| \tilde{s}_i \|^2 | \mathcal{F}_{i-1}]
\]

(122)

where we used the expressions in (122) and (108). Assume we select $t$ as

$$t \triangleq \sigma_{11}\mu_{\text{max}}^2$$

(124)
in (121). Then, inequality (123) simplifies to
\[
E[\|\tilde{w}_i\|^2] \leq \left(1 - \sigma_{11} \mu_{\max} + (1 - \sigma_{11} \mu_{\max})^{-1} \beta_{11}^2 \mu_{\max}^2\right) \times \\
E[\|\tilde{w}_{i-1}\|^2] + \left(1 - \sigma_{11} \mu_{\max}\right)E[\|\tilde{w}_{i-1}\|^2] + E[\|\bar{s}_i\|^2]
\]
\[
\leq (1 - \sigma_{11} \mu_{\max})E[\|\tilde{w}_{i-1}\|^2] + \frac{\sigma_{21}}{\sigma_{11}} \mu_{\max} E[\|\tilde{w}_{i-1}\|^2] + E[\|\bar{s}_i\|^2]
\]
(125)
for some positive number \(\sigma_{11}' < \sigma_{11}\), and small enough \(\mu_{\max}\), such that
\[
\sigma_{11}' \leq \sigma_{11} - (1 - \sigma_{11} \mu_{\max})^{-1} \beta_{11}^2 \mu_{\max}.
\]
(126)
We can now establish (34) by completing the argument starting from Eq. (9.69) in the proof of Theorem 9.1 in [1] pp. 516–521, where the quantity \(b = 0_{MN}\) (appeared in (9.54) of [1]).

We next establish (35). Compared to the proof for Theorem 9.2 in [1], the main difference, apart from the second-order moments evaluated in (123), is the term
\[
\frac{1}{(1 - t)^3} E \left[ \left( (I - D_{11}^T) \tilde{w}_{i-1}\right)^4 \right] 
\]
(127)
for any \(0 < t < 1\), which appeared in (9.117) of [1]. Let
\[
K_i \triangleq (I - D_{11, i}) \left( (I - D_{11, i})^T \tilde{w}_{i-1}\right)^T \times (I - D_{11, i})^T (I - D_{11, i})^T
\]
and
\[
L_i \triangleq \left( (I - D_{11, i}) \left( (I - D_{11, i})^T \tilde{w}_{i-1}\right) \right)^2.
\]
(129)
Then, both matrices \(K_i\) and \(L_i\) are symmetric positive semi-definite. Thus, we have
\[
E \left[ \left( (I - D_{11, i}) \tilde{w}_{i-1}\right)^4 \right] | \mathcal{F}_{i-1}
\]
\[
= (\tilde{w}_{i-1})^T E \left[ K_i | \mathcal{F}_{i-1}\right] \tilde{w}_{i-1}
\]
\[
\leq \lambda_{\max} \left( E \left[ K_i | \mathcal{F}_{i-1}\right] \right) \| \tilde{w}_{i-1} \|^2
\]
(\sigma)
\[
\leq \lambda_{\max} \left( E \left[ L_i | \mathcal{F}_{i-1}\right] \right) \| \tilde{w}_{i-1} \|^4
\]
\[
= \lambda_{\max}(E \left[ L_i | \mathcal{F}_{i-1}\right]) \| \tilde{w}_{i-1} \|^4
\]
(130)
where the inequality (a) holds because \(\lambda_{\max}(\Sigma) \leq \text{Tr}(\Sigma)\) for any symmetric positive semi-definite matrix \(\Sigma\). We proceed to deal with the term \(E \left[ L_i | \mathcal{F}_{i-1}\right]\). Note that
\[
L_i = I_{M} - L_{1,i} + L_{2,i} - L_{3,i} + L_{4,i}
\]
(131)
where
\[
L_{1,i} \triangleq 2D_{11,i} + 2D_{11,i}^T
\]
(132)
\[
L_{2,i} \triangleq 3D_{11,i}D_{11,i}^T + D_{11,i}^T D_{11,i} + (D_{11,i})^2 + (D_{11,i}^T)^2
\]
(133)
\[
L_{3,i} \triangleq (D_{11,i})^2 D_{11,i}^T + D_{11,i} (D_{11,i}^T)^2 + D_{11,i}^T D_{11,i} + D_{11,i}^T D_{11,i}^T
\]
(134)
\[
L_{4,i} \triangleq (D_{11,i} D_{11,i}^T)^2
\]
(135)
and we have
\[
E [L_{1,i} | \mathcal{F}_{i-1}] = 4D_{11,i}
\]
(136)
according to (110).

Let \(X\) be a constant matrix of size \(M \times M\). Then,
\[
E [\Gamma_{k,i} X \Gamma_{j,i}] = \begin{cases} (1 - r_k) (1 - r_j) X', & k \neq j \\ (1 - r_k)^2 X, & k = j \end{cases}
\]
(137)
where the \((m,n)\)-th entry of \(X'\) is defined by
\[
X'(m,n) = \begin{cases} (1 - r_k)^2 X(m,n), & m \neq n \\ (1 - r_k) X(m,m), & m = n. \end{cases}
\]
(138)
Thus, \(X'\) can be split into
\[
X' = (1 - r_k)^2 X + ((1 - r_k) - (1 - r_k)^2) \text{diag} \{X\}
\]
\[
= (1 - r_k)^2 X + (1 - r_k) r_k \text{diag} \{X\}
\]
(139)
with \(\text{diag} \{X\}\) being a diagonal matrix that consists of the diagonal entries of \(X\). It follows that
\[
E \left[ D_{11,i}^T D_{11,i} | \mathcal{F}_{i-1} \right] - (\bar{D}_{11,i})^2
\]
\[
= \sum_{k=1}^{N} \sum_{j=1}^{N} q_k q_j E \left[ \Gamma_{k,i} H_{k,i-1} H_{j,i-1} \Gamma_{j,i} | \mathcal{F}_{i-1} \right]
\]
\[
= \sum_{k=1}^{N} \sum_{j=1}^{N} q_k q_j (1 - r_k)(1 - r_j) H_{k,i-1} H_{j,i-1}
\]
\[
= \sum_{k=1}^{N} q_k^2 (1 - r_k)^2 H_{k,i-1} \leq \sum_{k=1}^{N} q_k^2 (1 - r_k)^2 \text{diag} \{H_{k,i-1}^2\}
\]
\[
= \sum_{k=1}^{N} q_k^2 (1 - r_k) r_k \text{diag} \{H_{k,i-1}^2\}
\]
(140)
Recall from (14) that \(\{H_{k,i-1} > 0\}\). Then, \(\{H_{k,i-1}^2 > 0\}\) and \(\{\text{diag} \{H_{k,i-1}^2\} > 0\}\). Computing Euclidean norms on both sides of (140), we have
\[
\|E \left[ D_{11,i}^T D_{11,i} | \mathcal{F}_{i-1} \right] - (\bar{D}_{11,i})^2\|
\]
(\sigma)
\[
\leq \sum_{k=1}^{N} q_k^2 (1 - r_k) r_k \\|\text{diag} \{H_{k,i-1}^2\}\|
\]
(\sigma)
\[
\leq \sum_{k=1}^{N} q_k^2 (1 - r_k) r_k \\|\text{diag} \{H_{k,i-1}^2\}\|
\]
\[
= O(\mu_{\max}^2)
\]
(141)
where in step (a) we used the property that \(\|A + B\| \leq \|A\| + \|B\|\) and (b) holds because \(\|X\| \leq \text{Tr}(X)\),
for any symmetric positive semi-definite matrix $X$, and that $\text{Tr} [\text{diag} \{ X \}] = \text{Tr} [X]$. Likewise, it follows that
\[
\begin{align*}
\| \mathbb{E} \left[ (D_{11}, t)^2 | \mathcal{F}_{-1} \right] - (\tilde{D}_{11}, t)^2 \| &= O(\mu_{\max}^2) \quad (142) \\
\| \mathbb{E} \left[ (D_{11}^T, t)^2 | \mathcal{F}_{-1} \right] - (\tilde{D}_{11}, t)^2 \| &= O(\mu_{\max}^2). \quad (143)
\end{align*}
\]
Recall from (116) that
\[
\| \mathbb{E} [D_{11}, t] \mathcal{F}_{-1}^T - (\tilde{D}_{11}, t)^2 \| = O(\mu_{\max}^2). \quad (144)
\]
Substituting (141)–(144) into (133), we obtain
\[
\| \mathbb{E} [L_{2, t} | \mathcal{F}_{-1} ] - 6 (\tilde{D}_{11}, t)^2 \| = O(\mu_{\max}^2). \quad (145)
\]
Similarly, it can be verified that
\[
\begin{align*}
\| \mathbb{E} [L_{3, t} | \mathcal{F}_{-1} ] - 4 (\tilde{D}_{11}, t)^3 \| &= O(\mu_{\max}^3) \quad (146) \\
\| \mathbb{E} [L_{4, t} | \mathcal{F}_{-1} ] - (\tilde{D}_{11}, t)^4 \| &= O(\mu_{\max}^4). \quad (147)
\end{align*}
\]
It follows that
\[
\begin{align*}
\| \mathbb{E} [L_{1, t} | \mathcal{F}_{-1} ] \| &= \| I - \mathbb{E} [L_{1, t} | \mathcal{F}_{-1} ] - \mathbb{E} [L_{2, t} | \mathcal{F}_{-1} ] + \mathbb{E} [L_{3, t} | \mathcal{F}_{-1} ] - \mathbb{E} [L_{4, t} | \mathcal{F}_{-1} ] \| \\
&= \| I - 4 \tilde{D}_{11, t} + 6 (\tilde{D}_{11, t})^2 - 4 (\tilde{D}_{11, t})^3 + (\tilde{D}_{11, t})^4 + (\mathbb{E} [L_{2, t} | \mathcal{F}_{-1} ] - 6 (\tilde{D}_{11, t})^2) - (\mathbb{E} [L_{3, t} | \mathcal{F}_{-1} ] - 4 (\tilde{D}_{11, t})^3) \| \\
&+ (\mathbb{E} [L_{4, t} | \mathcal{F}_{-1} ] - (\tilde{D}_{11, t})^4) \| \\
&\leq \| I - 4 \tilde{D}_{11, t} + 6 (\tilde{D}_{11, t})^2 - 4 (\tilde{D}_{11, t})^3 + (\tilde{D}_{11, t})^4 \| + \| \mathbb{E} [L_{2, t} | \mathcal{F}_{-1} ] - 6 (\tilde{D}_{11, t})^2 \| + \| \mathbb{E} [L_{3, t} | \mathcal{F}_{-1} ] - 4 (\tilde{D}_{11, t})^3 \| + \| \mathbb{E} [L_{4, t} | \mathcal{F}_{-1} ] - (\tilde{D}_{11, t})^4 \| \\
&= \| I - \tilde{D}_{11, t} \|^4 + O(\mu_{\max}^2) \\
&\leq (1 - \sigma_{11, \mu_{\max}})^4 + O(\mu_{\max}^2). \quad (148)
\end{align*}
\]
Substituting into (130), we have
\[
\begin{align*}
\mathbb{E} \| (I_{M} - D_{11}^T, t) \mathcal{W}_{-1} \|^4 | \mathcal{F}_{-1} \| &\leq (1 - \sigma_{11, \mu_{\max}})^4 + O(\mu_{\max}^2) \| \mathcal{W}_{-1} \|^4. \quad (149)
\end{align*}
\]
Let $t = \sigma_{11, \mu_{\max}}$ in (127). Thus,
\[
\begin{align*}
\frac{1}{(1 - t)^3} \mathbb{E} \| (I_{M} - D_{11}^T, t) \mathcal{W}_{-1} \|^4 \\
&\leq (1 - \sigma_{11, \mu_{\max}}^4 + O(\mu_{\max}^2)) \mathbb{E} \| \mathcal{W}_{-1} \|^4 \\
&\leq (1 - \sigma_{11, \mu_{\max}}^4) \mathbb{E} \| \mathcal{W}_{-1} \|^4 \quad (150)
\end{align*}
\]
for some positive constant $\sigma_{11}^4 < \sigma_{11}$, and for small enough $\mu_{\max}$. Then, the result in (55) can be obtained by continuing from Eq. (9.117) (by choosing $t = \sigma_{11, \mu_{\max}}$) in the proof of Theorem 9.2 in [pp. 523–530].

\section*{APPENDIX B}

\section*{PROOF OF THEOREM 2}

Define
\[
F \triangleq \mathbb{E} [B_1^T \otimes b B_1^T]^T \quad (151)
\]
Then, by following similar techniques shown in the proof of Lemma 9.5 in [pp. 542–546], we have
\[
(I - F)^{-1} = [(p \otimes p) (I \otimes I)]^T \otimes Z^{-1} + O(1) \quad (152)
\]
where
\[
Z \triangleq \sum_{k=1}^{N} q_k (1 - r_k) \left[ (H_k \otimes I_M) + (I_M \otimes H_k) \right]. \quad (153)
\]
The desired results (56) and (57) in Theorem 2 now follow by referring to the proofs of Theorem 11.2 and Lemma 11.3 in [pp. 583–596], and Theorem 11.4 in [pp. 608–609].

Evaluating the squared Euclidean norms on both sides of (57) and taking expectations conditioned on the past history gives:
\[
\mathbb{E} \left[ \| \mathcal{W}_{-1} \|^2 | \mathcal{F}_{-1} \right] = \mathbb{E} \left[ \left\| \mathcal{W}_{-1} \right\|^2 \right] + \mathbb{E} \left\{ \left| \mathcal{W}_{-1} \right|^2 \left| \mathcal{F}_{-1} \right. \right\} \quad (154)
\]
Taking expectations again removes the conditioning on $\mathcal{F}_{-1}$ and we get
\[
\mathbb{E} \left[ \| \mathcal{W}_{-1} \|^2 \right] = \mathbb{E} \left\{ \left\| \mathcal{W}_{-1} \right\|^2 \right\} + \mathbb{E} \left\{ \left| \mathcal{W}_{-1} \right|^2 \right\} \quad (155)
\]
equivalently, using the weighted vector notation $||x||_2^2 = \sum_{k=1}^{N} q_k (1 - r_k) \left[ (H_k \otimes I_M) + (I_M \otimes H_k) \right]. \quad (156)
\]
Iterating the relation we get
\[
\mathbb{E} \left[ \| \mathcal{W}_{-1} \|^2 \right] \leq \mathbb{E} \left\{ \left\| \mathcal{W}_{-1} \right\|^2 \right\} + \sum_{n=0}^{i} \mathbb{E} \left\{ \left| \mathcal{W}_{-1} \right|^2 \right\} \quad (157)
\]
where the first-term corresponds to a transient component that dies out with time, and the convergence rate of $\mathbb{E} \| \mathcal{W}_{k, t} \|^2$ towards the steady-state regime is seen to be dictated by $\rho (F)$ [p. 592]. Now, let
\[
\begin{align*}
\Gamma \triangleq \mathbb{E} [\Gamma_{-1} | \mathcal{F} ] = \text{diag} \{(1 - r_k)\}_{k=1}^{N} \otimes I_{M} \quad (158) \\
\mathcal{M} \triangleq \mathcal{M} \Gamma = \text{diag} \{\mu_k (1 - r_k)\}_{k=1}^{N} \otimes I_{M} \quad (159) \\
\mathcal{B}^T \triangleq \mathbb{E} [\mathcal{B}^T | \mathcal{F} ] = \mathcal{A}^T \Gamma \mathcal{H} \mathcal{A}^T \quad (160)
\end{align*}
\]
We now rewrite (151) in terms of $\mathcal{B}^T$ as
\[
\begin{align*}
\mathcal{F} \triangleq \mathbb{E} \left\{ \left[ (A_1^T \otimes \mathcal{H} \mathcal{A}^T) \otimes_b (A_2^T \otimes \mathcal{H} \mathcal{A}^T) \right]^T \right\} \\
= \left\{ (A_1 \otimes_b A_1) (I - I \otimes (\mathcal{H} \mathcal{M} \mathcal{A}^T \otimes_b (\mathcal{H} \mathcal{A}^T) + (\mathcal{H} \mathcal{M} \otimes (\mathcal{H} \mathcal{M} \otimes (A_2 \otimes_b A_2) \right\} \quad (161)
\end{align*}
\]
where $\Delta_F(\mu_{\max}^2)$ is a matrix whose entries are in the order of $O(\mu_{\max}^2)$. Following similar techniques to the proof of Theorem 9.3 [1] pp. 535–540], we make the same Jordan canonical decomposition for the matrix $P = A_1 A_2$ as (99), then substituting into (166) we get

$$B' = (V_e^{-1})^T (J^T - D'T) V_e^T \quad (162)$$

where

$$D'T \triangleq V_e^T A_2^T M'H A_1^T (V_e^{-1})^T = \begin{bmatrix} D'_{11} & D'_{12} \\ D'_{12} & D_{22} \end{bmatrix} \quad (163)$$

and

$$D'_{11} = \sum_{k=1}^N q_k (1 - r_k) H_k \quad (164)$$

$$D'_{21} = O(\mu_{\max}) \quad (165)$$

We now introduce the following eigen-decomposition for the symmetric positive-definite matrix $D'_{11}$:

$$D'_{11} \triangleq U \Lambda U^T \quad (166)$$

where $U$ is unitary, and $\Lambda$ is a diagonal matrix composed of the eigenvalues of $D'_{11}$. Let

$$T = \text{diag}\{\mu_{\max}^{1/N} U, \mu_{\max}^{2/N} I_M, \ldots, \mu_{\max}^{(N-1)/N} I_M, \mu_{\max} I_M\} \quad (167)$$

then we have

$$T^{-1} V_e^T B' (V_e^{-1})^T T = \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B_{22} \end{bmatrix} \quad (168)$$

with

$$B'_{11} \triangleq I_M - \Lambda. \quad (169)$$

It follows that [1 p. 538]

$$B'_{12} = O(\mu_{(N+1)/N}^2) \quad (170)$$

where

$$\mu'_{\max} \triangleq \max_{1 \leq k \leq N} \{\mu_k(1 - r_k)\} \quad (171)$$

is in the same order of $\mu_{\max}$ because

$$\mu_{\max}(1 - r_{\max}) \leq \mu'_{\max} \leq \mu_{\max}(1 - r_{\min}) \quad (172)$$

for small step-sizes $\mu_{\max} \ll 1 - r_{\max}$, where $r_{\max}$ and $r_{\min}$ are the maximum and minimum values respectively of $\{r_k\}_{k=1}^N$. The matrix $B'$ has the same eigenvalues as the block matrix on the right hand side of (168). By referring to Gershgorin’s Theorem [31, 32], it is shown in [1] pp. 539–540] that the union of the $M$ Gershgorin discs, each centered at an eigenvalue of $B'_{11}$ with radius $O(\mu_{\max}^{(N+1)/N})$, is disjoint from that of the other $M(N-1)$ Gershgorin discs, centered at the diagonal entries of $B'_{22}$, and therefore

$$\rho(B') = \rho(B'_{11}) + O(\mu_{\max}^{(N+1)/N}). \quad (173)$$

Let

$$\hat{\Delta}_F \triangleq (T^T V_e^{-1}) \otimes_b (T^T V_e^{-1}) \Delta_F(\mu_{\max}^2) \times \left( \begin{bmatrix} (V_e(T^{-1}))^T \otimes_b (V_e(T^{-1})^T) \\ O(\mu_{\max}^{1/N}) \end{bmatrix} \left[ \begin{array}{cccc} O(\mu_{\max}^2) & O(\mu_{\max}^2) & \cdots \\ O(\mu_{\max}^{2/N}) \end{array} \right] \right). \quad (174)$$

It follows from (172) that all the diagonal blocks of $\hat{\Delta}_F$ are in the order of $O(\mu_{\max}^2)$, the remaining block matrices in the first row are in the order of $O(\mu_{\max}^{1/N})$, the remaining block matrices in the first column are in the order of $O(\mu_{\max}^2)$, and the upper and lower triangular blocks in the $(2,2)$th block of $\Delta_F$ are in the order of $O(\mu_{\max}^{2/N})$ and $O(\mu_{\max}^{1+1/N})$ respectively. Then, substituting (168) and (174) into (161), we have

$$(T^T V_e^{-1}) \otimes_b (T^T V_e^{-1}) \mathcal{F} (T_e^{-1}) \otimes_b (T_e^{-1}) \right) + \hat{\Delta}_F$$

$$= \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{bmatrix} \otimes_b \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{bmatrix}^T + \hat{\Delta}_F$$

$$= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}^T \quad (175)$$

where

$$F_{11} = B'_{11} \otimes B'_{11} + O(\mu_{\max}^2) \quad (176)$$

$$F_{12} = O(\mu_{(N+1)/N}^{(N+1)/N}) \quad (177)$$

Recall that $B'_{11}$ is a diagonal matrix, so is $B'_{11} \otimes B'_{11}$, then we have

$$\text{diag}\{F_{11}\} = \text{diag}\{\lambda(B'_{11} \otimes B'_{11})\} + O(\mu_{\max}^2) \quad (178)$$

which means that the diagonal entries of $F_{11}$ are the eigenvalues of $B'_{11} \otimes B'_{11}$ perturbed by a second-order term, $O(\mu_{\max}^2)$. Referring to Gershgorin’s Theorem, the union of the $M^2$ Gershgorin discs, centered at the diagonal entries of $F_{11}$ with radius $O(\mu_{\max}^{(N+1)/N})$, is disjoint from the union of the other $M^2(N^2 - 1)$ Gershgorin discs, centered at the diagonal entries of $F_{22}$. Note that $\mathcal{F}$ has the same eigenvalues as the block matrix on the right hand side of (175), that eigenvalues are invariant under a transposition operation. It follows from (173) that

$$\rho(\mathcal{F}) = \rho(B'_{11} \otimes B'_{11}) + O(\mu_{\max}^{(N+1)/N}). \quad (179)$$

Using the fact that

$$\rho(B'_{11} \otimes B'_{11}) = [\rho(B'_{11})]^2 \quad (180)$$

we arrive at the desired result [59].

**APPENDIX C**

**EXAMINING THE DIFFERENCE IN (62)**

We revisit the MSE networks discussed under Case 3. Assume that there are only two agents in the network as shown in Fig. 10, namely, $N = 2$, with $M = 2$. Then, we have
with \( \{ \sigma_{v,k}^2 \} \) and \( \{ R_{u,k} \} \) defined in (70) and (71) respectively. Assume that
\[ \sigma_{v,1}^2 > \sigma_{v,2}^2. \]
For simplicity, uniform parameters \( \{ q_k \equiv q \} \) are used across the agents (which may occur, for example, in the ATC or CTA forms when the step-sizes are uniform across the agents, i.e., \( \{ \mu_k \equiv \mu \} \), and doubly-stochastic combination matrices are adopted. In this case, we get \( \{ q_k \equiv \mu/N \} \) (11 pp. 493–494).
Let
\[ R_{u,1} = \begin{bmatrix} \pi_1 & \pi_1 \\ \pi_1 & 1 \end{bmatrix}, \quad R_{u,2} = \begin{bmatrix} \pi_2 & \pi_2 \\ \pi_2 & 1 \end{bmatrix} \]
where the numbers \( |\pi_1| < 1, |\pi_2| < 1 \) ensure that \( R_{u,1} > 0, R_{u,2} > 0 \). Then, expression (62) can be rewritten as
\[
\text{MSD}_{\text{coor},k} - \text{MSD}_{\text{grad},k} = r q \text{Tr} \left( (R_{u,1} + R_{u,2})^{-1} (\sigma_{v,1}^2 \text{diag}(R_{u,1}) - R_{u,1}) + \sigma_{v,2}^2 \text{diag}(R_{u,2}) - R_{u,2}) \right) 
\]
\[ \overset{(184)}{=} r q \text{Tr} \left( \begin{bmatrix} |\pi_1| + |\pi_2| & \pi_1 + \pi_2 \\ \pi_1 + \pi_2 & 2 \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & -\pi_1 \sigma_{v,1}^2 - \pi_2 \sigma_{v,2}^2 \\ -\pi_1 \sigma_{v,1}^2 - \pi_2 \sigma_{v,2}^2 & 0 \end{bmatrix} \right) 
\]
\[ = \frac{2 r q}{2(|\pi_1| + |\pi_2|) - (\pi_1 + \pi_2)^2} (\pi_1 + \pi_2)(\pi_1 \sigma_{v,1}^2 + \pi_2 \sigma_{v,2}^2) \]
\[ \overset{(185)}{=} \frac{2 r q}{2(|\pi_1| + |\pi_2|) - (\pi_1 + \pi_2)^2} (\pi_1 + \pi_2)(\pi_1 \sigma_{v,1}^2 + \pi_2 \sigma_{v,2}^2) \]
Note that
\[ \frac{2 r q}{2(|\pi_1| + |\pi_2|) - (\pi_1 + \pi_2)^2} > 0 \]
for all \( |\pi_1| < 1, |\pi_2| < 1 \). Then, MSD_{\text{coor},k} < MSD_{\text{grad},k} if, and only if
\[ (\pi_1 + \pi_2)(\pi_1 \sigma_{v,1}^2 + \pi_2 \sigma_{v,2}^2) < 0 \]
which implies that
\[ 0 < \pi_2 < 1, -\pi_2 < \pi_1 < -\left( \sigma_{v,2}^2/\sigma_{v,1}^2 \right) \pi_2 \]
or
\[ -1 < \pi_2 < 0, -\left( \sigma_{v,2}^2/\sigma_{v,1}^2 \right) \pi_2 < \pi_1 < -\pi_2. \]
Otherwise, MSD_{\text{coor},k} \geq MSD_{\text{grad},k}.

**APPENDIX D**

**PROOF OF COROLLARY 2**

We proceed to evaluate the last two items on the right hand side of (61). Note that in the case when the matrices \( \{ H_k \} \) or \( \{ G_k \} \) are diagonal, the difference between these two items will be zero.

More generally, let
\[
\Delta = \frac{r}{2} \text{Tr} \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} - \sum_{k=1}^{N} q_k^2 \text{diag}(G_k) < 0 \]
\[
\frac{r}{2} \text{Tr} \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} - \sum_{k=1}^{N} q_k^2 \text{diag}(G_k) \geq 0 \]
Then, according to (2) and (40), we have
\[
\sum_{k=1}^{N} q_k H_k > 0, \quad \sum_{k=1}^{N} q_k^2 G_k \geq 0, \quad \sum_{k=1}^{N} q_k^2 \text{diag}(G_k) \geq 0 \]
Then, applying the inequality (33):
\[
\lambda_{\text{min}}(A) \text{Tr}(B) \leq \text{Tr}(AB) \leq \lambda_{\text{max}}(A) \text{Tr}(B) \quad (190) \]
for any symmetric positive semi-definite matrices \( A \) and \( B \), where \( \lambda_{\text{min}}(A) \) and \( \lambda_{\text{max}}(A) \) represent respectively the largest and smallest eigenvalues of \( A \), we obtain
\[
\Delta \leq \frac{r}{2} \left\{ \lambda_{\text{max}} \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} \right\} - \sum_{k=1}^{N} q_k^2 \text{Tr}(G_k) \]
\[
\lambda_{\text{min}} \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} \sum_{k=1}^{N} q_k^2 \text{Tr}(G_k) \quad (191) \]
where the inequality holds because \( \text{Tr}(G_k) = \text{Tr}(\text{diag}(G_k)) \).
Then, noting that
\[
0 < \sum_{k=1}^{N} q_k \lambda_{\text{min}}(H_k) \leq \lambda \left( \sum_{k=1}^{N} q_k H_k \right) \leq \sum_{k=1}^{N} q_k \lambda_{\text{max}}(H_k). \]
(192)
we have
\[
1/ \left( \frac{\delta_d \sum_{k=1}^{N} q_k}{(a)} \right) \leq 1/ \left( \sum_{k=1}^{N} q_k \lambda_{\text{max}}(H_k) \right) \leq \lambda \left( \sum_{k=1}^{N} q_k H_k \right)^{-1} \leq 1/ \left( \sum_{k=1}^{N} q_k \lambda_{\text{min}}(H_k) \right) \leq 1/ \left( \sum_{k=1}^{N} q_k \lambda_{\text{max}}(H_k) \right) \quad (193) \]
where the inequalities (a) and (b) hold due to (2). Substituting (193) into (191), we obtain
\[
\Delta \leq \frac{r}{2} \left( \sum_{k=1}^{N} q_k \right)^{-1} \left( \frac{1}{\delta_d} - \frac{1}{\delta_d} \sum_{k=1}^{N} q_k^2 \text{Tr}(G_k) \right) \quad (194) \]
Similarly,
\[
\Delta \geq \frac{r}{2} \left( \sum_{k=1}^{N} q_k \right)^{-1} \left( \frac{1}{\delta_d} - \frac{1}{\delta_d} \sum_{k=1}^{N} q_k^2 \text{Tr}(G_k) \right) \quad (195) \]
Then, applying the upper and lower bounds in (194) and (195) to (61), we arrive at the desired result in Corollary 2.

**APPENDIX E**

**PROOF OF COROLLARY 4**

Recalling (2) and (72), we know that
\[ \nu_d / 2 \leq \lambda (R_u) \leq \delta d / 2. \] (196)

Let \( \{G_k = 4 \sigma^2 v, k R_u\} \). Then, it follows from Corollary 2 and (73) that
\[
\text{MSD}_{\text{coor}, k} - \text{MSD}_{\text{grad}, k} \\
\leq r \left( \sum_{k=1}^{N} q_k \right)^{-1} \left( \frac{1}{\nu_d} - \frac{1}{\delta d} \right) \left( \sum_{k=1}^{N} 4 q_k^2 \sigma_{v, k}^2 \text{Tr}(R_u) \right)^{-1} \left\{ \sum_{k=1}^{N} q_k^2 \sigma_{v, k}^2 \right\} \left( \frac{\delta d}{\nu_d} - 1 \right) M. \] (197)

**APPENDIX F**

**PROOF OF COROLLARY 5**

We start from the MSD expression in (82) and note first that
\[ G'_k = (1 - r_k)^2 \left( G + \frac{r_k}{1 - r_k} \text{diag}(G) \right) \] (198)

Substituting into (82) we have:
\[
\text{MSD}_{\text{coor}, k} = \frac{1}{2} \left( \sum_{k=1}^{N} q_k (1 - r_k) \right)^{-1} \left( \sum_{k=1}^{N} q_k^2 (1 - r_k)^2 \right) \times \text{Tr} \left( H^{-1} \left( G + \frac{r_k}{1 - r_k} \text{diag}(G) \right) \right) \\
\leq \frac{1}{2} \left( \sum_{k=1}^{N} q_k (1 - r_k) \right)^{-1} \left( \sum_{k=1}^{N} q_k^2 (1 - r_k)^2 \right) \times \text{Tr} \left( H^{-1} G \right) + \frac{1}{2} \left( \sum_{k=1}^{N} q_k (1 - r_k) \right)^{-1} \left( \sum_{k=1}^{N} q_k^2 (1 - r_k)^2 \right) \times \text{Tr} \left( H^{-1} \text{diag}(G) \right) \\
= \frac{1}{2} \left( \sum_{k=1}^{N} q_k (1 - r_k) \right)^{-1} \left( \sum_{k=1}^{N} q_k^2 (1 - r_k)^2 \right) \times \text{Tr} \left( H^{-1} \text{diag}(G) \right) + \frac{1}{2} (\theta - \alpha) \text{Tr} \left( H^{-1} \text{diag}(G) \right). \] (199)

where (199) holds because
\[ \theta - \alpha = \frac{\sum_{k=1}^{N} q_k^2 (1 - r_k)}{\sum_{k=1}^{N} q_k (1 - r_k)} - \frac{\sum_{k=1}^{N} q_k^2 (1 - r_k)^2}{\sum_{k=1}^{N} q_k (1 - r_k) r_k} \]

\[ = \frac{\sum_{k=1}^{N} q_k^2 (1 - r_k) r_k}{\sum_{k=1}^{N} q_k (1 - r_k)} \] (200)

with the numbers \( \alpha \) and \( \theta \) being defined in (84) and (85), respectively. Recall that
\[
\text{MSD}_{\text{grad}, k} = \frac{1}{2} \left( \sum_{k=1}^{N} q_k \right)^{-1} \left( \sum_{k=1}^{N} q_k^2 \text{Tr} (H^{-1} G) \right) \] (201)

is the MSD performance for the full-gradient case. Thus,
\[
\text{MSD}_{\text{coor}, k} - \text{MSD}_{\text{grad}, k} \\
= \frac{1}{2} \left( \sum_{k=1}^{N} q_k^2 (1 - r_k)^2 \right) \text{Tr} (H^{-1} G) + \frac{1}{2} (\theta - \alpha) \text{Tr} (H^{-1} \text{diag}(G)). \] (202)

Applying (190) and (2) to (202), we obtain the desired results for the MSD performance shown in Corollary 5. The result for the ER performance in Corollary 5 can be shown by adding and subtracting the ER expression, \( \text{ER}_{\text{grad}, k} \), on the both sides of (83).

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