High-order curvilinear mesh in the numerical solution of PDEs with moving frames on the sphere

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Q: How much does the quality of mesh contribute to the accuracy of the numerical scheme and the conservation of mass and energy in the numerical solution of PDEs on the sphere?

Rephrasing: What should be the maximal mesh error* in order to guarantee that mesh error does not affect the overall accuracy and conservational properties for (p>1)*?

* Mesh error: measure on how much the grid points are deviated from the original design
* p>2 : order of polynomial of the numerical solution is more than 1.

\[
\text{Mesh error}(L_2) \equiv \sum_{i=1}^{N} \sqrt{\frac{r_{\text{exact}}^2 - r_i^2}{N}}
\]

High-order curvilinear mesh in the numerical solution of PDEs with moving frames on a (sphere) curved surface, S. Chun*, J. Marcon, J. Peiro, S. J. Sherwin**, submitted and under revision, 2019
**Curvilinear mesh**: Jacobian of the element is not constant.

**Triangular mesh**: specific to the method of discretization on the sphere (Method of moving frames to solve PDEs on curved surfaces). But, it may apply to the different type of mesh for different discretization methods.
Curvilinear meshes considered in this research

1. **ProjMesh**: Starting from a unit cubic cell, every uniformly divided grid points of edges are projected on the sphere *(Low integration accuracy, High mesh error)*

2. **Gmsh**: Popular open-source curvilinear mesh using Frontal algorithm for optimal distribution of grid points. *(High integration accuracy, High mesh error)*.

3. **NekMesh** *(High integration accuracy, Low mesh error, p-dependent generation of mesh)*
NekMesh

• A high-order mesh generation and modification problem which generates high-order meshes from a CAD definition by Joaquim Peiro and Spencer J. Sherwin at Imperial college London.

• Set-up in pipeline style: a series of modules are constructed and executed in order to arrive at a high-order mesh.

• Using Opencascade, a large open-source library encapsulating a vast range of CAD operations and manipulations.

• Automation: Use octree structure for the spatial decomposition of the domain to allow non-expert users to obtain a linear mesh which has suitable resolution in high curvature.

• Linear Mesh generation by NekMesh CAD engine.
1) First issue on curvilinear mesh for $p>2$

- For objects with curved boundary, **curvilinear elements** can be effective in time step size and stability.

- The primary strategy is to align one boundary of the element with the boundary of the object.

- Then what should happen for the **grid points inside** the element?

- Best accuracy in integration and differentiation, the **Fekete distribution**, or **equipotential distribution**, of grid points are used in views of
Minimizing spring energy

The, optimize the locations of the nodes to reduce distortion by modeling the mesh entities as sprint networks and minimizing the spring energy.
2) Second issue on curvilinear mesh for $p>2$

Curving process can frequently create elements which **self-intersect** (one face impinges on another face) or having **near-tangent vertices**.

![Diagram](image.png)

(a) Valid linear element.
(b) Valid high-order element.
(c) Invalid high-order element.
(d) Invalid high-order element.
3) Third issue on curvilinear mesh for $p>2$

Vertices always lie on the surface, but for $p > 1$, there is no guarantee that the interpolated grid point lies on the surface.

**Interpolated grid point:** Not appearing on mesh, but the computational scheme introduces them to generate a solution of polynomial order of $p>1$.

Vertices **always** lie on the surface, but for $p > 1$, there is no guarantee that the interpolated grid point lies on the surface.
On traditional mesh, mesh error not converging for $p>1$

Mesh error vs. $h$ (length of edge) for $p=4$

Mesh error vs. $p$ (polynomial order) for $h=0.4$

- Only way to reduce mesh error is to reduce $h$ (size of elements).
- But, then discretization error is also very low, thus the proportion of mesh error vs. total error remains fixed.
- For a high order mesh like Nekmesh, mesh error should converge as $p$ increases
Test conditions

- **High-order methods** using spectral hp and discontinuous Galerkin methods
- **Method of moving frames** for the solution of PDEs on the sphere.
- **Curved triangular mesh**.
- Tests are published at the open-source code **Nektar++ (version more than 4.5.0)** available of GitHub at [www.nektar.info](http://www.nektar.info).
Tests on
Static differential operators

Moving frames
Zonal flow
Rossby flow
Divergence

Zonal Direct

Rossby Direct

Zonal Weak

Rossby Weak

ProjMesh: Dashed and Square, Gmsh: Dashed Dot and Triangle, NekMesh: Solid and filled circle
Curl

Zonal
Direct

Zonal
Weak

Rossby
Direct

Rossby
Weak

ProjMesh: Dashed and Square, Gmsh: Dashed Dot and Triangle, NekMesh: Solid and filled circle
Gradient

Zonal Direct

Zonal Weak

Rossby Direct

Rossby Weak

ProjMesh: Dashed and Square, Gmsh: Dashed Dot and Triangle, NekMesh: Solid and filled circle
Laplacian

Zonal Direct

Rossby Direct

ProjMesh: Dashed and Square, Gmsh: Dashed Dot and Triangle, NekMesh: Solid and filled circle
Summary of tests on static differential operators

• Nekmesh shows better accuracy for a high p, but we cannot say that Nekmesh always performs better than other two meshes.

• Numerical accuracy rather seems to depend on the optimal distribution of collocation grid points for and corresponding mesh error for integration and differentiation,
Tests on
Time-dependent PDEs

* Gmsh testing is not included to avoid controversies, but the results are the same or worse than those ProjMesh
Conservational Laws

Conditions

L2 & Linf

Mass error

ProjMesh: Dashed and Square, NekMesh: Solid and filled circle
Diffusion

Initial Conditions

L2 & Linf

ProjMesh: Dashed and Square, NekMesh: Solid and filled circle
Maxwell’s equations

Initial conditions
Fields
Energy

Ez at antipod

Energy error

ProjMesh: Dashed and Square, NekMesh: Solid and filled circle
SWE: Steady Zonal Flow

Conditions

L2 & Linf

Mass error

ProjMesh: Dashed and Square, NekMesh: Solid and filled circle

Bernard et al (2009) (p=2)
LAUTER et al (2009) (p=2)
Giraldo and Warburton (2005) (p=2)

MMF (ProjMesh) (p=2)
MMF (NekMesh) (p=2)

Bernard et al (2009) (p=4)
LAUTER et al (2009) (p=4)
Giraldo and Warburton (2005) (p=4)
Nair (2005) (p=4)

MMF (ProjMesh) (p=4)
MMF (NekMesh) (p=4)
SWE: Unsteady Zonal Flow

Conditions

L2 & Linf

Mass error

ProjMesh: Dashed and Square, NekMesh: Solid and filled circle
SWE: Rossby-Haurwitz Flow

Conditions

L2 & Linf

Mass error

Comparisons with other schemes

ProjMesh: Dashed and Square, NekMesh: Solid and filled circle
Conclusion

• In time-dependent PDEs, a relatively low mesh error can **deteriorate** the overall **accuracy** and especially **conservational** properties of mass and energy on the sphere.

• Q: Overall accuracy is around $10^{-3}$. How much should small **mesh error** be? Even with a very small mesh, **conservational** properties are deteriorated by that mesh error.

• Reducing h (size of element) is always **not** the solution.

• Making a **large difference** between the **discretization error** depending $h/p$ and **mesh error** could be a solution.
References

• High-order curvilinear mesh in the numerical solution of PDEs with moving frames on a (sphere) curved surface, S. Chun, J. Marcon, J. Peiro, S. J. Sherwin, submitted and under revision, 2019.

• More info about Nekmesh and Nektar++, www.nektar.info. Spencer J. Sherwin, Mike Kirby, et. al.

• Sehun Chun, Method of moving frames to solve the time-dependent Maxwell's equations on anisotropic curved surfaces: Applications to invisible cloak and ELF propagation, J. Comput. Phys., 340, 85-104, 2017.

• S. Chun and C. Eskilsson, Method of moving frames to solve the shallow water equations on arbitrary rotating curved surfaces, J. Compt. Phys., 333, 1-23, 2017.

• Sehun Chun, Method of moving frames to solve (an)isotropic diffusion equations on curved surfaces, J. Sci. Compt., 59(3), 626-666, 2014.

• Sehun Chun, Method of moving frames to solve conservation laws on curved surfaces, J. Sci. Compt., 53(2), 268-294, 2012.