SUPERFORMS, TROPICAL COHOMOLOGY, AND POINCARÉ DUALITY

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Abstract. We establish a canonical isomorphism between two bigraded cohomology theories for polyhedral spaces: Dolbeault cohomology of superforms and tropical cohomology. Furthermore, we prove Poincaré duality for cohomology of tropical manifolds, which are polyhedral spaces locally given by Bergman fans of matroids.

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1. Introduction

Superforms on $\mathbb{R}^r$ are bigraded real-valued differential forms introduced by Lagerberg [Lag12]. They have differential operators $d'$, $d''$, and $d$ analogous to the differential operators $\partial$, $\overline{\partial}$, and $d$ on complex differential forms. Recently, superforms restricted to tropicalizations were used by Chambert-Loir and Ducros to construct real-valued differential forms on analytic spaces in the sense of Berkovich [CLD12]. Superforms have also been used to provide a non-Archimedean analytic description of heights by Gubler and Künnemann [GK14].

A Poincaré lemma with respect to the differential operators $d'$ and $d''$ for superforms on polyhedral complexes in $\mathbb{R}^r$ and Berkovich spaces was proven by the first author [Jel16b]. Here we consider the cohomology with respect to the operator $d''$. We call this the Dolbeault cohomology of superforms since the operator $d''$ behaves analogously to the operator $\overline{\partial}$ for complex differential forms.

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Tropical cohomology as introduced by Itenberg, Katzarkov, Mikhalkin, and Zharkov [IKMZ16], is the cohomology of singular cochains of a polyhedral complex with non-constant coefficients. The coefficient systems are determined by the geometry of the complex (see Definition 3.1). Via the tropicalization procedure, this cohomology theory can sometimes be related to the Hodge theory of projective varieties. For example, under suitable conditions on the tropicalization of a family of non-singular complex projective varieties, the dimensions of the tropical cohomology groups are equal to the Hodge numbers of a generic member of the family [IKMZ16, Corollary 2].

Our first goal is to prove that Dolbeault cohomology of superforms and tropical cohomology of a family of non-singular complex projective varieties, the dimensions of the tropical cohomology groups are equal to the Hodge numbers of a generic member of the family [IKMZ16, Corollary 2].

In Subsection 3.1, we recall the definition of the tropical cohomology groups $H^p_q(X)$ of a polyhedral space equipped with a face structure (see Definition 3.2). We also define the tropical cohomology groups with compact support $H^{p,q}_c(X)$. The Dolbeault cohomology of superforms is the cohomology of the complexes of global sections $(A^p_X, d')$. We denote these groups by $H^p_q(X) = H^q((A^p_X(X), d'))$. We also write $H^p_q(d',c)(X)$ for the cohomology of global sections with compact support (see Definition 2.27).

The first theorem relates the Dolbeault cohomology of superforms and tropical cohomology.

**Theorem 1.** Let $X$ be a polyhedral space equipped with a face structure. Then there are canonical isomorphisms

$$H^p_q(X) \cong H^p_q(X) \quad \text{and} \quad H^p_q(X) \cong H^p_q(d',c)(X).$$

To prove Theorem 1, we first show that for every $p$, the complex $A^p_X$ is an acyclic resolution of certain sheaves denoted $L^p_X$ on $X$. Tropical cohomology was already shown to be equivalent to the cohomology of constructible sheaves, denoted $F^p_X$ [MZ14, Proposition 2.8]. Comparing explicit descriptions of these sheaves on a basis of the topology, we show that $L^p_X$ and $F^p_X$ are isomorphic, which implies the above theorem. In fact, the sheaves $F^p_X$ are defined for a polyhedral space $X$ even in the absence of a face structure. This relates the Dolbeault cohomology of superforms with the cohomology of the sheaves $F^p_X$ for general polyhedral spaces (see Remark 3.23).

Secondly, we prove a version of Poincaré duality for tropical manifolds. For $X$ an $n$-dimensional tropical space (see Definition 4.8), there is a map

$$PD: H^p_q(X) \to H^{n-p,n-q}(X),$$

which we call the Poincaré duality map. This map is induced by integration of superforms (see Definition 4.11), and thus is similar to the integration pairing on the cohomology of a complex manifold. The fact that the Poincaré duality map on spaces of superforms descends to cohomology when $X$ is a tropical space follows from an analogue of Stokes’ theorem (see Theorem 4.9).

Tropical manifolds are tropical spaces with the extra condition that they are locally modeled on matroidal tropical cycles [MR, Sha11]. A matroidal tropical cycle is supported on the Bergman fan of a matroid and equipped with weight one. Some matroidal cycles arise as tropicalizations of linear spaces, however they are much more general and may even have no algebraic counterpart [Stu02]. Despite perhaps being far from smooth objects in the algebraic or differentiable sense, tropical manifolds exhibit many properties analogous to smooth spaces [Sha11]. Establishing Poincaré duality for the tropical cohomology of these spaces provides another instance of this phenomenon.
Theorem 2. If $X$ is an $n$-dimensional tropical manifold then the Poincaré duality map is an isomorphism for all $p$ and $q$.

As in the proof of Poincaré duality for smooth manifolds, the statement is first established for the local models, which in our case are matroidal cycles. This is done in Propositions 4.27 and 4.30. The main ingredient in the proof of the local case is a recursive description of matroidal cycles using tropical modifications (see Definition 4.18). We restrict to tropical modifications of matroidal cycles which are induced by deletion and contraction operations on the underlying matroids [Sha13b]. Poincaré duality for general tropical manifolds is then established from the local situation via standard methods.

In recent work, Adiprasito, Huh and, Katz consider an intersection ring associated to a matroid [AHK15]. For a matroid $M$, the graded ring $A^\ast(M)$ is shown to satisfy many striking properties in line with the cohomology rings of compact Kähler manifolds, such as Poincaré duality, the Hard Lefschetz theorem, and an analogue of the Hodge-Riemann bilinear relations. We expect that this ring is related to the cohomology groups presented here in the following way: For a matroid $M$ and $V$ its associated matroidal cycle, there is a suitable compactification $\overline{V}$ of $V$ for which $H^{k,k}(\overline{V}) \cong A^k(M) \otimes \mathbb{R}$. Moreover, the product structures on the tropical cohomology of $\overline{V}$ and $A^\ast(M)$ should also be isomorphic.

In addition to the Poincaré duality relation established here, there is a lot of interest in other properties of the tropical cohomology groups. For instance, there are analogues of Lefschetz hyperplane section theorems for tropical cohomology [AB15]. It was already shown that the tropical homology of tropical manifolds does not in general satisfy a direct translation of the Hodge-Riemann bilinear relations [Sha13a]. Furthermore, an interesting open question is to establish the appropriate condition on a tropical manifold $X$ so that $H^{p,q}(X) \cong H^{q,p}(X)$ [MZ14, Section 5].

It is also worthwhile to mention that tropical varieties can be used to construct currents on smooth complex projective varieties. This was recently used to construct a counter-example to the strongly positive Hodge conjecture [BH15]. Although their construction does not use the theory of superforms, it points to the power of connections between tropical geometry and complex differential forms.

We now outline the presentation of this paper. Section 2 reviews superforms on $\mathbb{R}^r$ and extends their definition to superforms on $\mathbb{T}^r$. For an open subset of the support of a polyhedral complex in $\mathbb{T}^r$ we define the space of $(p, q)$-superforms and show that this produces a sheaf. This construction is also extended to produce sheaves of superforms on polyhedral spaces. Section 3 recalls the definitions of tropical cohomology and calculates the cohomology of basic open sets (see Definition 3.7). It also establishes a Poincaré lemma for the complexes of superforms on a polyhedral space and furthermore computes the sections of $\mathcal{L}_X^p$ over basic open sets. Following this, we show that the Dolbeault cohomology of superforms and tropical cohomology are isomorphic (see Theorem 3.22). Subsection 4.1 introduces integration and proves Stokes’ theorem mentioned above. Finally, Subsection 4.2 is devoted to the proof of Poincaré duality for tropical manifolds (see Theorem 4.33).

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2. Superforms

2.1. Superforms on polyhedral subspaces of tropical affine space. In this subsection we define bigraded sheaves of superforms on polyhedral complexes in tropical affine space $\mathbb{T}^r$. We start by recalling the definitions for open subsets of $\mathbb{R}^r$ due to Lagerberg [Lag12]. After that we extend these to open subsets of $\mathbb{T}^r$ and to open subsets of polyhedral complexes in $\mathbb{T}^r$.

Definition 2.1. Let $U \subset \mathbb{R}^r$ be an open subset. Denote by $A^q(U)$ the space of differential forms of degree $q$ on $U$. The space of $(p,q)$-superforms on $U$ is defined as

$$A^{p,q}(U) := A^p(U) \otimes_{C^\infty(U)} A^q(U) = \bigwedge^p \mathbb{R}^r \otimes_{\mathbb{R}} A^q(U),$$

where $\bigwedge^p$ denotes the $p$-th exterior power.

If we choose a basis $x_1, \ldots, x_r$ of $\mathbb{R}^r$, following [CLD12] and [Gub16], we formally write a superform $\alpha \in A^{p,q}(U)$ as

$$\alpha = \sum_{|K|=p,|L|=q} \alpha_{KL} dx_K \wedge d''x_L,$$

where $K = \{i_1, \ldots, i_p\}$ and $L = \{j_1, \ldots, j_q\}$ are ordered subsets of $\{1, \ldots, r\}$, the coefficients $\alpha_{KL} \in C^\infty(U)$ are smooth functions and

$$d'x_K \wedge d''x_L := (dx_{i_1} \wedge \ldots \wedge dx_{i_p}) \otimes_{\mathbb{R}} (dx_{j_1} \wedge \ldots \wedge dx_{j_q}).$$

There is a differential operator

$$d'' : A^{p,q}(U) = \bigwedge^p \mathbb{R}^r \otimes_{\mathbb{R}} A^q(U) \to A^{p,q+1}(U) = \bigwedge^p \mathbb{R}^r \otimes_{\mathbb{R}} A^{q+1}(U),$$

given by $(-1)^p id \otimes D$, where $D$ is the usual differential operator on forms. In coordinates we have

$$d'' \left( \sum_{K,L} \alpha_{KL} dx_K \wedge d''x_L \right) = \sum_{K,L=1}^r \frac{\partial \alpha_{KL}}{\partial x_{i_1}} d''x_{i_1} \wedge d'x_K \wedge d''x_L.$$

Remark 2.2. There are also differential operators $d' := D \otimes id$ and $d := d' + d''$, which are not considered in this paper. It is easy to see that the theories for $d'$ and $d''$ are symmetric up to sign. We choose to consider the operator $d''$, since it produces the same cohomology as tropical cohomology. The cohomology of the operator $d'$ is isomorphic to that of $d''$ up to switching the bigrading.

There is also a wedge product of superforms

$$\wedge : A^{p,q}(U) \times A^{p',q'}(U) \to A^{p+p',q+q'}(U)$$

$$(\alpha, \beta) \mapsto \alpha \wedge \beta,$$

which is, up to sign, induced by the usual wedge product. In coordinates the wedge product is given by

$$(\alpha_{KL} dx_K \wedge d''x_L) \wedge (\beta_{K'L'} dx_{K'} \wedge d''x_{L'}) := \alpha_{KL} \beta_{K'L'} dx_K \wedge d'x_K \wedge d''x_L \wedge d'x_{K'} \wedge d''x_{L'} := (-1)^{p'q'} \alpha_{KL} \beta_{K'L'} dx_K \wedge d'x_{K'} \wedge d''x_L \wedge d''x_{L'}.$$

If one of $\alpha, \beta$ has compact support then so does $\alpha \wedge \beta$. Note that we have the usual Leibniz formula

$$d''(\alpha \wedge \beta) = d'' \alpha \wedge \beta + (-1)^{p+q} \alpha \wedge d'' \beta.$$

Let $\mathbb{T} = [-\infty, \infty)$ and equip it with the topology of a half open interval. Then $\mathbb{T}^r$ is equipped with the product topology. We write $[r] := \{1, \ldots, r\}$. 

Definition 2.3. The *sedentarity* of a point \( x \in \mathbb{T}^r \) is the subset \( \text{sed}(x) \subset [r] \) consisting of coordinates of \( x \) which are \(-\infty\).

The space \( \mathbb{T}^r \) is naturally stratified by the sedentarity of points. For \( I \subset [r] \) set
\[
\mathbb{R}_I^r := \{ x \in \mathbb{T}^r \mid x_i = -\infty \text{ if and only if } i \in I \}.
\]

Clearly, \( \mathbb{R}_I^r \cong \mathbb{R}^{-|I|} \). As a convention throughout, for a subset \( S \subset \mathbb{T}^r \) we denote \( S_I := S \cap \mathbb{R}_I^r \).

Moreover, for \( J \subset I \) there is a canonical projection \( \pi_{IJ} : \mathbb{R}_J^r \to \mathbb{R}_I^r \). Coordinate-wise the map \( \pi_{IJ} \) sends \( x_i \to -\infty \) if \( i \in I \) and \( x_i \) otherwise.

**Definition 2.4.** Let \( U \subset \mathbb{T}^r \) be an open subset. A \((p,q)\)-superform \( \alpha \) on \( U \) is given by a collection of superforms \( (\alpha_I)_{I \subset [r]} \) such that,
1. \( \alpha_I \in \mathcal{A}^{p,q}(U_I) \) for all \( I \),
2. for each point \( x \in U \subset \mathbb{T}^r \) of sedentarity \( I \), there exists a neighborhood \( U_x \) of \( x \) contained in \( U \) such that for each \( J \subset I \) the projection satisfies \( \pi_{IJ}(U_{x,J}) = U_{x,I} \) and \( \pi_{IJ}^* \alpha_I|_{U_{x,I}} = \alpha_J|_{U_{x,J}} \).

We denote the space of \((p,q)\)-superforms on an open subset \( U \) by \( \mathcal{A}^{p,q}(U) \). Note that a superform in \( \mathcal{A}^{0,0}(U) \) defines a collection of smooth functions on the subsets \( U_I \) which give a continuous function on \( U \). Therefore, we sometimes refer to \((0,0)\)-superforms as smooth functions.

**Condition ii** of Definition 2.4 will be referred to as the *condition of compatibility* of superforms along strata. Let \( U \subset \mathbb{T}^r \) be an open subset and \( \alpha \in \mathcal{A}^{p,q}(U) \). Suppose that the points in \( U \) have a unique maximal sedentarity and denote this by \( I \). If for each \( J \subset I \) we have \( \pi_{IJ}^* \alpha_I = \alpha_J \), then we say that \( \alpha \) is determined by \( \alpha_I \) on \( U \). Notice that the condition of compatibility along strata implies that each \( x \in U \) has an open neighborhood \( U_x \) such that \( \alpha_I|_{U_x} \) is determined by \( (\alpha_I|_{U_x})_{\text{sed}(x)} \) on \( U_x \).

If \( U \subset \mathbb{T}^r \) is an open subset and \( \alpha = (\alpha_I)_I \in \mathcal{A}^{p,q}(U) \) is a superform, define \( d' \alpha \) to be given by the collection \( (d' \alpha_I)_I \). Pullbacks along the projections \( \pi_{IJ} \) commute with \( d' \), therefore \( d'' \alpha \) is a superform in \( \mathcal{A}^{p,q+1}(U) \). If also \( \beta = (\beta_I)_I \in \mathcal{A}^{p',q'}(U) \), then we define the wedge product \( \alpha \wedge \beta := (\alpha_I \wedge \beta_I)_I \in \mathcal{A}^{p+p',q+q'}(U) \). This is indeed a superform on \( U \), since the pullbacks along the projections commute with the wedge product.

**Remark 2.5.** Notice that there is a natural isomorphism \( J : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{q,p}(U) \), which, up to sign, maps \( d'x_K \otimes d'x_L \) to \( d'' x_K \otimes d'' x_L \) for \( U \subset \mathbb{T}^r \) [CLD12, Section (1.2.5)]. This is clear when \( U \subset \mathbb{R}^r \). When \( U \) contains points of non-empty sedentarity the map \( J \) preserves the condition of compatibility on the boundary strata. This involution is still well-defined for the spaces of superforms on polyhedral subspaces and polyhedral spaces defined in Subsection 2.2. In the theory of tropical cohomology, outlined in Subsection 3.1, such an involution does not exist on the chain level.

**Example 2.6.** Consider an open neighborhood \( U \) of \(-\infty \in \mathbb{T} \). For a \((p,q)\)-superform \( \alpha \in \mathcal{A}^{p,q}(U) \) with \( \max(p,q) = 1 \), by the condition of compatibility of superforms along strata, there must exist a smaller neighborhood \( U' \subset U \) of \(-\infty \) such that \( \alpha \) is zero on \( U' \).

Similarly, a \((0,0)\)-superform on \( U \) must be a constant function in some neighborhood of \(-\infty \).

In the next lemma we use upper indexing of open sets to avoid confusion with the notation for the sedentarity.

**Lemma 2.7.** Let \( U \subset \mathbb{T}^r \) be an open subset and \( (U^I)_{I \in L} \) an open cover of \( U \). Then there exist a countable, locally finite cover \( (V^k)_{k \in K} \) of \( U \), a collection of non-negative smooth functions \( (f^k : V^k \to \mathbb{R})_{k \in K} \) with compact support, and a map \( s : K \to L \) such that \( V^k \subset U^{s(k)} \) for every \( k \in K \), and \( \sum_{k \in K} f^k \equiv 1 \).

Such a family of functions is called a *partition of unity* subordinate to the cover \((U^I)_{I \in L}\).
Proof. We first show that for any \( x = (x_1, \ldots, x_r) \in \mathbb{T}^r \) and any open neighborhood \( x \in V \) there exists a non-negative function \( f \in A^{0,1}(\mathbb{T}^r) \) and a neighborhood \( V' \) of \( x \) such that \( f|_{V'} \equiv 1 \) and \( \text{supp}(f) \subset V \). This is clear if \( r = 1 \). Otherwise, a basis of open neighborhoods of \( x \) is given by products of open sets in \( \mathbb{T} \), thus we may assume \( V \) to be of this form. Then taking functions \( f^i \) on neighborhoods of \( x_i \), in \( \mathbb{T} \) with the above property for every \( i \in [r] \) and defining \( f(x_1, \ldots, x_r) = \prod f^i(x_i) \) gives the desired function.

The general statement of the lemma now follows from standard arguments, see for instance the proof in [War63, Theorem 1.11]. \( \square \)

**Definition 2.8.** A polyhedron in \( \mathbb{R}^r \) is a subset defined by a finite system of affine (non-strict) inequalities. A face of a polyhedron \( \sigma \) is a polyhedron which is obtained by turning some of the defining inequalities of \( \sigma \) into equalities. For conventions of convex geometry we follow [Gub13, Appendix A].

A polyhedron in \( \mathbb{T}^r \) is the closure of a polyhedron in \( \mathbb{R}^r \). A face of a polyhedron \( \sigma \) in \( \mathbb{T}^r \) is the closure of a face of \( \sigma \cap \mathbb{R}^J \) for some \( J \subset \sigma \). A polyhedral complex \( \mathcal{C} \) in \( \mathbb{T}^r \) is a finite set of polyhedra in \( \mathbb{T}^r \), satisfying the following properties:

i) For a polyhedron \( \sigma \in \mathcal{C} \), if \( \tau \) is a face of \( \sigma \) (denoted \( \tau \prec \sigma \)) we have \( \tau \in \mathcal{C} \).

ii) For two polyhedra \( \sigma, \tau \in \mathcal{C} \) the intersection \( \sigma \cap \tau \) is a face of both \( \sigma \) and \( \tau \).

The maximal polyhedra, with respect to inclusion, are called facets. The support of a polyhedral complex \( \mathcal{C} \) is the union of all its polyhedra and is denoted \( |\mathcal{C}| \). If \( X = |\mathcal{C}| \), then \( X \) is called a polyhedral subspace of \( \mathbb{T}^r \) and \( \mathcal{C} \) is called a polyhedral structure on \( X \).

The relative interior of a polyhedron \( \sigma \) in \( \mathbb{T}^r \) is denoted \( \text{int}(\sigma) \). Given a polyhedral complex \( \mathcal{C} \) in \( \mathbb{T}^r \) let \( \mathcal{C}_I \) denote the union of polyhedra \( \sigma \in \mathcal{C} \) for which \( \text{int}(\sigma) \) is contained in \( \mathbb{R}_f^I \). By the definition of polyhedral complexes in \( \mathbb{T}^r \), the collection \( \mathcal{C}_I \) is a polyhedral complex in \( \mathbb{R}_f^I \). Notice that \( |\mathcal{C}_I| = |\mathcal{C}| \).

**Definition 2.9.** Let \( \mathcal{C} \) be a polyhedral complex in \( \mathbb{T}^r \) and \( \sigma \in \mathcal{C} \). Let \( x \in \sigma \) be of sedentarity \( I \). Define the tangent space of \( \sigma \) at \( x \) to be \( L(\sigma, x) := L(\sigma_I) \subset \mathbb{R}^I_f \), where \( L(\sigma_I) \) is the tangent space to \( \sigma_I \) at any point in its relative interior.

For \( U \subset \mathbb{R}^r \) an open subset containing \( x \), \( \alpha \in A^{p-q}(U) \), and \( s \in [p] \) the contraction of \( \alpha \) by \( v \in \mathbb{R}^r \) in the \( s \)-th component is a \( (p-1,q) \) superform denoted \( \langle \alpha; v \rangle_s \in A^{p-1,q}(U) \). The form \( \langle \alpha; v \rangle_s \) evaluated at a collection of vectors \( v_1, \ldots, v_{p-1}, w_1, \ldots, w_q \in L(\sigma, x) \) is

\[
\langle \alpha(x); v_1, \ldots, v, \ldots, v_{p-1}, w_1, \ldots, w_q, \rangle
\]

where the vector \( v \) is in the \( s \)-th position.

For \( U \subset \mathbb{T}^r \) an open subset containing \( x \), let \( \alpha = (\alpha_I)_{I} \in A^{p,q}(U) \), the contraction of \( \alpha \) by \( v \in \mathbb{R}^r \) in the \( s \)-th component is the superform \( \langle \alpha; v \rangle_s \in A^{p-1,q}(U) \) given by the collection \( \langle \langle \alpha_I, \pi_{I,\delta}(v) \rangle \rangle_s \).

Let \( U \) be an open subset containing \( x \). Then the evaluation of a superform \( \alpha \in A^{p,q}(U) \) at a collection of vectors \( v_1, \ldots, v_p, w_1, \ldots, w_q \in L(\sigma, x) \) is denoted \( \langle \alpha_I(x); v_1, \ldots, v_p, w_1, \ldots, w_q \rangle \).

Next we consider the restriction of bigraded superforms to polyhedral complexes in \( \mathbb{T}^r \).

**Definition 2.10.** Let \( \mathcal{C} \) be a polyhedral complex in \( \mathbb{T}^r \) and \( \Omega \subset |\mathcal{C}| \) an open subset. Then a \( (p,q) \)-superform on \( \Omega \) is given by a superform \( \alpha \in A^{p,q}(U) \) such that \( U \subset \mathbb{T}^r \) is an open subset satisfying \( \Omega = U \cap |\mathcal{C}| \). Two such pairs \( (U, \alpha) \) and \( (U', \alpha') \) are equivalent if for any \( \sigma \in \mathcal{C} \), any \( x \in \Omega \cap \sigma \) and all tangent vectors \( v_1, \ldots, v_p, w_1, \ldots, w_q \in L(\sigma, x) \) we have

\[
\langle \alpha_I(x); v_1, \ldots, v_p, w_1, \ldots, w_q \rangle = \langle \alpha'_I(x); v_1, \ldots, v_p, w_1, \ldots, w_q \rangle.
\]

Let \( A^{p,q}(\Omega) \) denote the set of equivalence classes of pairs \( (U, \alpha) \) as above.

**Example 2.11.** Consider the standard tropical line \( L \subset \mathbb{R}^2 \). The space \( L \) is the support of the one dimensional fan consisting of three rays in directions \((-1,0), (0,-1) \) and \((1,1) \). Let \( \Omega \) be an open connected neighborhood of the origin in \( L \). Since \( L \) is one dimensional \( A^{p,q}(\Omega) = 0 \) if
max(p, q) > 1. The space \( A_{\Omega}^{0,0} \) is the space of maps \( f: \Omega \to \mathbb{R} \) which extend to a smooth function \( \overline{f}: \mathbb{R}^2 \to \mathbb{R} \) for some open neighborhood \( U \) of \( \Omega \) in \( \mathbb{R}^2 \).

By construction \( A_{\Omega}^{p,q} \) is a \( A_{\Omega}^{0,0} \) module via the wedge product. The space \( A_{\Omega}^{1,0} \) is spanned by \( d'x \) and \( d'y \) over \( A_{\Omega}^{0,0} \), where \( x \) and \( y \) are the coordinates on \( \mathbb{R}^2 \). Note that these two forms each vanish on one of the rays of \( L \) and agree on the ray in direction \((1, 1)\). The space \( A_{\Omega}^{0,1} \) is analogous.

The space of superforms \( A_{\Omega}^{1,1} \) is spanned by
\[
d'x \wedge d''x, d'x \wedge d''y, d'y \wedge d''x, \text{ and } d'y \wedge d''y.
\]
The forms \( d'x \wedge d''y \) and \( d'y \wedge d''x \) both vanish on the half rays of \( L \) that are in directions \((-1, 0)\) and \((0, -1)\). On the ray in direction \((1, 1)\) we have \( d'x \wedge d''y = d'y \wedge d''x \). This shows that \( d'x \wedge d''y \) and \( d'y \wedge d''x \) holds on \( \Omega \). Furthermore, we find that in the stalk of \( A_{\Omega}^{1,1} \) at the vertex of \( L \), the forms \( d'x \wedge d''x, d'x \wedge d''y = d'y \wedge d''x, \) and \( d'y \wedge d''y \) are linearly independent over the stalk of \( A_{\Omega}^{0,0} \) at the same point. This differs from the situation over the complex numbers, where the space of top dimensional forms is always a free module of rank one over the space of smooth functions.

**Remark 2.12.** By definition we have that \( \alpha \) and \( \alpha' \) define the same superform on \( \Omega \) if and only if for all \( I \subseteq [r] \) the superforms \( \alpha_I \) and \( \alpha'_I \) define the same superform on \( \Omega_I \). Moreover, to determine if two superforms are equivalent when restricted to \( \Omega \), by continuity, it is enough to consider only points in the relative interior of facets.

The differential map \( d'' \) and the wedge product both descend to forms in \( A_{\Omega}^{p,q} \) in the sense that if superforms \( \alpha, \beta \in A_{\Omega}^{p,q} \) are given by \( \alpha' \in A_{\Omega}^{p,q}(U) \) and \( \beta' \in A_{\Omega}^{p,q}(U') \) then defining \( d''\alpha \) to be given by \( d''\alpha' \) and \( \alpha \wedge \beta \) to be given by \( \alpha'|_{U \cap U'} \wedge \beta'|_{U \cap U'} \) is independent of the choices of \( \alpha' \) and \( \beta' \).

For an open subset \( \Omega \) of a polyhedral space \( X \subset \mathbb{T}' \), the space of superforms \( A_{\Omega}^{p,q} \) does not depend on the underlying polyhedral complex \( C \). To see this we introduce the multi-(co)tangent spaces. These spaces will appear again in Section 3.1 in relation to tropical (co)homology.

**Definition 2.13.** Let \( C \) be a polyhedral complex in \( \mathbb{T}' \) and \( x \in |C| \). Then the \( p \)-th multi-tangent space multi-cotangent spaces at \( x \) are defined respectively by
\[
F_p(x) = \sum_{\tau \in C_x, x \in \tau} \mathbb{L}(\tau, x) \subset \mathbb{R}_+^\tau \quad \text{and} \quad F^p(x) = \left( \sum_{\tau \in C_x, x \in \tau} \mathbb{L}(\tau, x) \right)^*.
\]

**Lemma 2.14.** Let \( \Omega \) be an open subset of a polyhedral space \(|C| \subset \mathbb{T}'\). Then the space of superforms \( A_{\Omega}^{p,q} \) only depends on \( \Omega \).

**Proof.** For \( x \in \Omega \), we claim that the vector space \( F_p(x) \) only depends on \( \Omega \) and \( x \). To see this, consider a refinement \( C' \) of the polyhedral complex \( C \). If \( \sigma \in C \) and \( \sigma' \in C' \) are both maximal faces containing \( x \) such that \( \sigma' \) is contained in \( \sigma \), then \( L(\sigma, x) = L(\sigma', x) \). Furthermore, for each facet \( \sigma \in C \) there exists at least one \( \sigma' \in C' \) with the above property. This shows that \( L(\sigma, x) = \sum_{\sigma' \in C', x \in \sigma' \cap \sigma} L(\sigma', x) \) which in turn implies that the definition of \( F_p(x) \) is the same for polyhedral structures \( C \) and \( C' \).

Now given another polyhedral complex \( C'' \) such that \( \Omega \) is an open subset of \(|C''| \), we can find a polyhedral complex \( C' \) which is a common refinement of both \( C \) and \( C' \) when restricted to \( \Omega \). It follows from the statement proved above that the vector space \( F_p(x) \) depends only on \( x \) and \( \Omega \).

Now \( \alpha \in A_{\Omega}^{p,q} \) equals zero if and only if \( \langle \alpha(x); v, w \rangle = 0 \) for all \( x \in \Omega, v \in F_p(x), \) and \( w \in F^p(x) \). Finally, since \( F_p(x) \) is independent of the polyhedral structure on \( \Omega \) so is \( A_{\Omega}^{p,q} \). This completes the proof of the lemma. \( \square \)

For a polyhedral subspace \( X \) in \( \mathbb{T}' \), the functor on open subsets of \( X \) given by \( \Omega \mapsto A_{\Omega}^{p,q} \) will be denoted by \( A_{X}^{p,q} \) or simply \( A^{p,q} \) if the space \( X \) is clear. The next lemma shows that this is
an acyclic sheaf, where by acyclicity we always mean with respect to both the functor of global sections and the functor of global sections with compact support.

**Lemma 2.15.** For a polyhedral subspace $X$ in $\mathbb{T}^r$, the presheaf

$$
\Omega \mapsto \mathcal{A}_X^{p,q}(\Omega)
$$

is a sheaf on $X$. Furthermore, this sheaf is fine, hence soft and acyclic.

**Proof.** We start with the case $X = \mathbb{T}^r$. In this case, all of the sheaf axioms are clearly satisfied except for the gluing property. Given a collection of superforms agreeing on intersections, we can glue on each $\mathbb{R}^r$ getting a collection of superforms $\alpha_I$. The condition of compatibility along the boundary strata is respected for the glued superforms since it is local and was respected for the superforms before gluing.

For the general case we rely on the existence of partitions of unity. Let $X$ be a polyhedral subspace of $\mathbb{T}^r$ and suppose that we have superforms $\alpha_I \in \mathcal{A}^{p,q}(\Omega_I)$ which agree on the intersections $\Omega_I \cap \Omega_J$ for $I, J \subseteq \mathcal{L}$ and are the restrictions to $X$ of superforms $\beta_I \in \mathcal{A}^{p,q}(U_I)$ for $\Omega_I = U_I \cap X$.

We take a partition of unity $(f^k)_{k \in K}$ subordinate to the cover $(U_I)_{I \in \mathcal{L}}$. By definition there is a map $s: K \to \mathcal{L}$, so that if $s(k) = I$, then $f^k$ is supported on $U_I$. Thus $\beta = \sum_{I \in \mathcal{L}} \sum_{k: s(k) = I} f^k \beta_I$ is a superform on the union $\bigcup_I \Omega_I$. Moreover for a fixed $l_0$ we have

$$
\beta|_{\Omega_{l_0}} = \sum_{I \in \mathcal{L}} \sum_{k: s(k) = I} f^k|_{\Omega_{l_0}} \beta_I|_{\Omega_{l_0}} = \sum_{I \in \mathcal{L}} \sum_{k: s(k) = I} f^k|_{\Omega_{l_0}} \alpha_I|_{\Omega_{l_0}}
$$

$$
= \sum_{I \in \mathcal{L}} \sum_{k: s(k) = I} f^k|_{\Omega_{l_0}} \alpha_I|_{\Omega_{l_0}} = (\sum_{k \in K} f^k|_{\Omega_{l_0}}) \alpha_{l_0} = \alpha_{l_0}.
$$

Therefore the superform given by $\beta$ restricted to $\bigcup_I \Omega_I$ gives the gluing of the superforms $\alpha_I$ above. This shows that $\mathcal{A}_X^{p,q}$ is a sheaf on $X$.

The fact that $\mathcal{A}^{0,0}$ is fine follows from Lemma 2.7. Then the sheaves $\mathcal{A}^{p,q}$ are also fine since they are $\mathcal{A}^{0,0}$-modules via the wedge product. Softness and acyclicity for global sections follows from [Wel80, Chapter II, Proposition 3.5 & Theorem 3.11] respectively and acyclicity for sections with compact support follows from [Ive86, III, Theorem 2.7].

**Definition 2.16.** Let $X$ be a polyhedral subspace of $\mathbb{T}^r$ and $\Omega$ an open subset. The **support** of a superform $\alpha \in \mathcal{A}^{p,q}(\Omega)$ is its support in the sense of sheaves, thus it consists of the points $x$ which do not have a neighborhood $\Omega_x$ such that $\alpha|_{\Omega_x} = 0$. The space of $(p, q)$-superforms with compact support on $\Omega$ is denoted $\mathcal{A}_X^{p,q}(\Omega)$.

**Lemma 2.17.** Let $X$ be a polyhedral subspace of $\mathbb{T}^r$ and $\Omega$ an open subset. Let $\alpha = (\alpha_I)_I \in \mathcal{A}^{p,q}(\Omega)$. Then we have $\text{supp } \alpha = \bigcup_I \text{supp } \alpha_I$.

**Proof.** Consider $x \in \Omega_I$. If $x \notin \text{supp } \alpha_I$, then there exists a neighborhood $U$ of $x$ in $\Omega_I$ such that $\alpha_I|_U = 0$. By the condition of compatibility, we may find a neighborhood $V$ of $x$ in $X$ such that $\alpha|_V$ is determined by $\alpha|_U$ on $V$ and where $V_I \subseteq U$. Therefore, $\alpha|_V = 0$. This shows $\text{supp } (\alpha) \subseteq \bigcup_I \text{supp } \alpha_I$. The other inclusion is immediate, thus we have equality. \qed

2.2 **Polyhedral spaces.** This subsection defines superforms on polyhedral spaces. These are spaces equipped with an atlas of charts to polyhedral subspaces in $\mathbb{T}^r$, with coordinate changes given by extensions of affine maps. First we establish pullbacks of superforms along extended affine maps, which permit the gluing of the sheaves $\mathcal{A}^{p,q}$ defined in the last subsection.

Let $F: \mathbb{R}^{r'} \to \mathbb{R}^r$ be an affine map and let $M_F$ denote the matrix representing the linear part of $F$. Let $I$ be the set of $i \in \{r\}$ such that the $i$-th column of $M_F$ has only non-negative entries. Then $F$ can be extended to a map

$$
F: \left( \bigcup_{J \subseteq I} \mathbb{R}_{\bar{J}}^{r'} \right) \to \mathbb{T}^r
$$
by continuity, (equivalently, using the usual $-\infty$-conventions for arithmetic). The extended map is also denoted by $F$.

**Definition 2.18.** Let $U' \subset \mathbb{T}'$ be an open subset, then a map $F: U' \to \mathbb{T}'$, which is the restriction to $U'$ of a map arising as above is called an extended affine map. Note that this only makes sense once we have $\text{sed}(x) \subset I$ for all $x \in U'$. Similarly, for a polyhedral subspace $X'$ and an open subset $I'$ of $X'$ an extended affine map $F: \Omega' \to \mathbb{T}'$ is given by the restriction of an extended affine map to $I'$. An extended affine map is called an integral extended affine map, if it is the extension of an integer affine map $\mathbb{R}' \to \mathbb{R}'$, i.e. its linear part is induced by a map of the standard lattices $\mathbb{Z}' \to \mathbb{Z}'$.

**Definition 2.19 (Pullback).** Let $U' \subset \mathbb{T}'$ be an open subset and $F : U' \to \mathbb{T}'$ be an extended affine map. Let $U \subset \mathbb{T}'$ be an open subset such that $F(U') \subset U$. Define

$$F : \{\text{sedentaries of points in } U'\} \to 2[^r]$$

$$I' \mapsto \text{sed}(F(x))$$

for some and then every $x \in \mathbb{R}'_x$. Notice that this map respects inclusions. $F$ induces an affine map $F_U : \mathbb{R}'_{U'} \to \mathbb{R}'_{F(U')}$ with $F_U(U) \subset F(U')$. The pullback of the superform $\alpha = (\alpha_I)_{I} \in A^{p,q}(U)$ along $F$ is the collection of superforms $F^*(\alpha) := (F^p_\alpha(\alpha_{F(U')}))_\nu$, where $F^p_\alpha(\alpha_{F(U')}) \in A^{p,q}(U')$. The next lemma shows that this collection satisfies the compatibility condition, and hence defines a superform on $U'$. Thus we have a pullback map $F^* : A^{p,q}(U) \to A^{p,q}(U')$.

**Lemma 2.20.** The pullback of a $(p,q)$-superform $\alpha$ on $U \subset \mathbb{T}'$ along an extended affine map $F : U' \to U$ is a $(p,q)$-superform on $U' \subset \mathbb{T}'$.

**Proof.** We have to verify the condition of compatibility of superforms along the strata. For $I' \subset I'$ we have $F(I') \subset F(I)$ and $F(I') \circ \pi_{I',I} = \pi_{F(I'),F(I)} \circ F(I')$. Thus if $\alpha \in A^{p,q}(U)$ is determined by $\alpha_{F(I')} = (\alpha_I)_{I'}$ on $U_{x'}$, then we have

$$
\begin{align*}
\pi^*_{I',I'}(F^*(\alpha)) &= \pi^*_{I',I'}(F^p_\alpha(\alpha_I)) \\
&= F^p_{I'}(\pi^*_{F(I'),F(I)}(\alpha_{F(I)})) \\
&= F^p_{I'}(\alpha_{F(I')}) \\
&= (F^*(\alpha))_{I'},
\end{align*}
$$

which shows that $F^*(\alpha)$ is determined by $F^*(\alpha)_{I'}$ on $F^{-1}(U')$. This shows the required compatibility. \hfill $\square$

**Lemma 2.21.** Let $X \subset \mathbb{T}'$ and $X' \subset \mathbb{T}'$ be polyhedral subspaces and let $\Omega \subset X$ and $\Omega' \subset X'$ be open subsets. If $F : \Omega' \to \Omega$ is an extended affine map, then there exists a well defined pullback $F^* : A^{p,q}(\Omega) \to A^{p,q}(\Omega')$, which is induced by the pullback in Definition 2.19. Moreover, the pullback is functorial and commutes with the differential $d^r$ and the wedge product.

**Proof.** Let $\alpha \in A^{p,q}(\Omega)$, then there exist open subsets $U' \subset \mathbb{T}'$ and $U \subset \mathbb{T}'$ such that $\alpha$ is defined by some $\beta \in A^{p,q}(U)$, $F(U') \subset U$, and $U \cap X = \Omega'$. Now the pullback $F^*(\beta) \in A^{p,q}(U')$ defines a superform on $\Omega'$. Set this to be $F^*(\alpha)$. To see that this is independent of the choice of $\beta$ we suppose that $\gamma$ is another superform on an open set defining $\alpha$ on $\Omega$. After intersecting the domains of definition of $\beta$ and $\gamma$, we may assume that $\beta$ and $\gamma$ are defined on the same open set $U$. Since $\beta|_U = \gamma|_U$ we have that $\beta|_{U_{F(U')}} = \gamma|_{U_{F(U')}}$ for all $I' \subset [r']$. Since the pullback via affine maps between vector spaces is well defined on polyhedral complexes [Gub16, 3.2], we have $F^p_\beta(\gamma)|_{\Omega_{F(U')}} = F^p_\gamma(\gamma)|_{\Omega_{F(U')}}$ for all $I' \subset [r']$ and therefore $F^*(\beta)|_{\Omega'} = F^*(\gamma)|_{\Omega'}$, so that the pullback is well defined. The last two statements of the lemma are direct consequences of the definition of pullbacks of forms along extended affine maps and the fact that the pullback by affine maps is functorial and commutes with $d^r$ and the wedge product. \hfill $\square$
We can now consider spaces equipped with an atlas of charts to polyhedral subspaces in $\mathbb{T}^r$. The following definition is a generalization of the definition of tropical spaces given in [Mik06, MZ14, BIMS15]. We do not require our polyhedral subspaces to be rational, also the transition maps are required only to be extended affine maps, not integral affine. We also remove the finite type condition on the charts in [MZ14, Definition 1.2].

**Definition 2.22.** A polyhedral space $X$ is a paracompact, second countable Hausdorff topological space with an atlas of charts $(\varphi_i: U_i \to \Omega_i \subset X_i)_{i \in I}$ such that:

i) The $U_i$ are open subsets of $X$, the $\Omega_i$ are open subsets of polyhedral subspaces $X_i \subset \mathbb{T}^r$, and $\varphi_i: U_i \to \Omega_i$ is a homeomorphism for all $i$;

ii) For all $i, j \in I$ the transition map

$$\varphi_i \circ \varphi_j^{-1}: \varphi_j(U_i \cap U_j) \to X_i$$

is an extended affine map.

As in usual manifold theory, two atlases on $X$ are considered equivalent if their union is an atlas on $X$.

The dimension of $X$ is the maximal dimension among polyhedra which intersect the $\Omega_i$. The polyhedral complex is pure dimensional if the dimension of the maximal, with respect to inclusion, polyhedra intersecting the open sets $\Omega_i \subset X_i$ is constant.

**Example 2.23.** The tropical projective space $\mathbb{T}P^r$ is the space

$$(\mathbb{T}^{r+1} \setminus \{(-\infty, \ldots, -\infty)\})/\sim, \text{ where } x \sim y \text{ if there exists } \lambda \in \mathbb{R} \text{ s.t. } x + (\lambda, \ldots, \lambda) = y.$$ 

For $i \in [r+1]$ the space $U_i = \{[x] \in \mathbb{T}P^r \mid x_i \neq -\infty\}$ is homeomorphic to $\mathbb{T}^r$ via the maps

$$\varphi_i: U_i \to \mathbb{T}^r; [x] \mapsto (x_j - x_i)_{j \in [r+1] \setminus \{i\}}$$

and

$$\varphi_i^{-1}: \mathbb{T}^r \to U_i; x \mapsto (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_r).$$

The transition maps are given by

$$\varphi_j \circ \varphi_i^{-1}: \mathbb{T}^r \setminus \mathbb{R}_{\{i\}} \to \mathbb{T}^r, x \mapsto (x_1 - x_j, \ldots, x_{i-1} - x_j, -x_j, x_{i+1} - x_j, \ldots, x_r - x_j),$$

which is an extended affine map. Thus $\mathbb{T}P^r$ together with the atlas $(\varphi_i: U_i \to \mathbb{T}^r)_{i \in [r+1]}$ is a polyhedral space.

**Definition 2.24.** Let $X$ be a polyhedral space with atlas $(\varphi_i: U_i \to \Omega_i \subset X_i)_{i \in I}$. The sheaf of superforms $\mathcal{A}_U^{p,q}$ is given by the pullback of the sheaves $\mathcal{A}_{\Omega_i}^{p,q}$ via $\varphi_i$. Then the sheaf $\mathcal{A}_X^{p,q}$ of $(p,q)$-superforms on $X$ is defined by gluing of the sheaves $\mathcal{A}_U^{p,q}$. The pullback of forms along the charts $\varphi_i$ is well defined and functorial, so this gives a well defined sheaf of superforms on $X$.

We also again denote the sections with compact support by $\mathcal{A}_c^{p,q}(X)$.

**Example 2.25.** Let $[0,1]$ be the closed unit interval and define the following charts:

$$\varphi_0: [0,1] \cong \mathbb{T}^1 \ni x \mapsto \tan((x - 1/2)\pi) \quad \varphi_1: [0,1] \cong \mathbb{T}^1 \ni x \mapsto -\tan((x - 1/2)\pi).$$

The interval $[0,1]$ equipped with these two charts defines a polyhedral space, denoted by $X$. The single transition map for this atlas is $\varphi_0 \circ \varphi_1^{-1}: \mathbb{R} \to \mathbb{R} = (x \mapsto -x)$. In Example 2.6, we saw that $(0,0)$-superforms on $\mathbb{T}$ are functions which are locally constant around $-\infty$. Thus $(0,0)$-superforms on $X$ are locally constant around both 0 and 1. Furthermore, similar to Example 2.6, superforms of positive degree vanish locally at the two boundary points of $X$.

The space $[0,1]$ can also be equipped with an atlas consisting of a single chart which is just the inclusion $[0,1] \hookrightarrow \mathbb{R}$. Denote this polyhedral space by $\hat{X}$. Then superforms in $\mathcal{A}^{0,0}(\hat{X})$ are just smooth functions on $[0,1]$ in the usual sense, since superforms are not required to satisfy any compatibility conditions. Also the superforms $d'x, d''x$, and $d'x \wedge d''x$ are nowhere vanishing superforms of positive degree.
Proposition 2.26. Let $X$ be a polyhedral space, then the differential $d''$ and the wedge product of superforms on $X$ are well defined. Thus, for each $p \in \mathbb{N}$ we have a complex
\[ 0 \to A^{p,0}_X \xrightarrow{d''} A^{p,1}_X \to \cdots. \]
If $X$ is n-dimensional, then $A^{p,q}_X = 0$ for $\max(p,q) > n$. The sheaves $A^{p,q}_X$ are fine, hence soft and acyclic.

Proof. Let $(\varphi_i: U_i \to \Omega_i \subset X_i)_{i \in I}$ be an atlas for $X$. Thus $A^{p,q}_X = 0$ if $\max(p,q) > \dim(X)$, since $A^{p,q}_{\Omega_i} = 0$. Since $d''$ and the wedge product are compatible with pullbacks along extended affine maps, both of these maps are well defined on the glued sheaves $A^{p,q}_X$. Since $X$ is paracompact and $A^{p,q}_X$ is glued from the fine sheaves $A^{p,q}_{\Omega_i}$, the sheaf $A^{p,q}_X$ is fine as well. Softness and acyclicity for global sections follows from [Wel80, Chapter II, Proposition 3.5 & Theorem 3.11] respectively and acyclicity for sections with compact support follows from [Ive86, III, Theorem 2.7].

Definition 2.27. Let $X$ be a polyhedral space, then the Dolbeault cohomology of superforms is defined as $H^{p,q}_{d''}(X) := H^q(A^{p,\cdot}_X(X), d'')$ and the Dolbeault cohomology of superforms with compact support is defined as $H^{p,\cdot,c}_{d''}(X) := H^q(A^{p,\cdot,c}_X(X), d'')$.

3. Comparison of Cohomologies

In this section we show that the Dolbeault cohomology of superforms agrees with tropical cohomology on polyhedral spaces. Subsection 3.1 recalls the definition of tropical cohomology using singular cochains. We then give another description of tropical cohomology in terms of sheaves [MZ14]. In Subsection 3.2, we show that Dolbeault cohomology of superforms is also equivalent to the cohomology of certain sheaves. We then calculate sections of these sheaves and deduce from this that the sheaves defining tropical and Dolbeault cohomologies agree.

3.1. Tropical Cohomology. This subsection describes tropical cohomology from [IKMZ16]. Recall the definitions of the multi-(co)tangent spaces from Definition 2.13. We now extend this definition to faces of a polyhedral complex.

Definition 3.1. Let $C$ be a polyhedral complex in $\mathbb{T}^r$. For $\sigma \in C$, let $x \in \text{int}(\sigma)$ and $I \subset [r]$ be such that $\text{int}(\sigma) \subset \mathbb{R}^I_f$. The $p$-th multi-tangent and multi-cotangent space of $C$ at $\sigma$ are the vector subspaces
\[ F_p(\sigma) = \sum_{\tau \in C_I; \sigma \prec \tau} \bigwedge^p L(\tau) \subset \bigwedge^p \mathbb{R}^I_f \quad \text{and} \quad F^p(\sigma) = \left( \sum_{\tau \in C_I; \sigma \prec \tau} \bigwedge^p L(\tau) \right)^*, \]
respectively.

For $\sigma \prec \tau$ for every $p$ there is a map $i_{\sigma\tau} : F_p(\tau) \to F_p(\sigma)$, which is an inclusion of vector spaces if $\sigma$ and $\tau$ are of the same sedentarity. Otherwise, if $\sigma$ is of sedentarity $J$ and $\tau$ of sedentarity $I$, the map $i_{\sigma\tau}$ is given by the composition of the projection $\pi_{IJ}$ and the above inclusion. On the dual spaces $F^p(\sigma)$, the maps are reversed $r_{\tau\sigma} : F^p(\sigma) \to F^p(\tau)$.

To define the maps $r_{\tau\sigma} : F^p(\sigma) \to F^p(\tau)$ for a polyhedral space $X$ we impose an additional condition on $X$, which we call here a face structure [MZ14, Definition 1.10].

Definition 3.2. Let $X$ be a polyhedral space with atlas $(\varphi_i : U_i \to \Omega_i \subset X_i)_{i \in I}$. A face structure on $X$, consists of fixed polyhedral structures $C_i$ on $X_i$ for each $i$ and a finite number of closed sets $\{\sigma_k\}$, called facets, which cover $X$, such that
\begin{enumerate}
  \item each facet $\sigma_k$ is contained in some chart $U_i$ for some $i$ such that $\varphi_i(\sigma_k)$ is the intersection of $\Omega_i$ with a facet $\tau_{ik}$ of the polyhedral complex $C_i$;
  \item for any collection of facets $S$ and $\sigma_l \in S$ the image of the intersection $\cap_{\sigma_k \in S} \sigma_k$ in the chart $\varphi_i : U_i \to X_i$ containing $\sigma_l$ is the intersection of $\Omega_i$ with a face of $\tau_{il}$.
\end{enumerate}
Given a face structure of $X$ a face of $X$ is an intersection of facets.

Note that every open subset of the support of a polyhedral complex in $\mathbb{T}^r$ is a polyhedral space with a face structure. For example, one can take the facets to be the intersections of maximal polyhedra of the polyhedral complex with the open subset.

Two face structures on $X$ are equivalent if there exists a common refinement, i.e. a face structure $\{\sigma_k\}$ on $X$ such that every facet $\sigma_k$ is contained in a facet of each of the two original face structures.

Given a face structure on $X$, for faces $\sigma \prec \tau$ of $X$ there are canonical maps $i_{\tau\sigma}$ and $r_{\tau\sigma}$ between the multi-tangent and multi-cotangent spaces respectively. These maps are induced by the maps between the multi-tangent and multi-cotangent spaces of the images of the faces under a chart of the polyhedral space.

**Example 3.3.** We again consider the polyhedral space given by equipping the space $X = [0, 1]$ with two charts to $\mathbb{T}^1$ as in Example 2.25. Notice that $[0, 1]$ cannot be the only facet of a face structure on $X$ since it is not contained in a single chart. Choose in both charts the polyhedral structure on $\mathbb{T}$ with facets $[-\infty, 0]$ and $[0, \infty)$. Then we can take as a face structure on $X$ consisting of the facets $\sigma_1 = [0, 1/2]$ and $\sigma_2 = [1/2, 1]$.

Denote by $\Delta_q$ the standard $q$-simplex.

**Definition 3.4.** Let $X$ be a polyhedral space together with a face structure $\mathcal{C}$ on $X$.

i) For every face $\tau \in \mathcal{C}$, we write $C_q(\tau)$ for the free $\mathbb{R}$-vector space generated by continuous maps $\delta: \Delta_q \to \tau$ such that the image of $\text{int}(\Delta_q)$ is contained in $\text{int}(\tau)$ and in addition the image of each open face of $\Delta_q$ is contained in the relative interior of a face of $\tau$. The space of tropical $(p, q)$-chains on $X$ with respect to $\mathcal{C}$ is

$$C_{p,q}(X) := \bigoplus_{\tau \in \mathcal{C}} F_p(\tau) \otimes C_q(\tau).$$

ii) For $\delta \in C_q(\tau)$ write $\partial \delta = \sum_{k=0}^{q} (-1)^k \delta^k$ for the usual boundary map, considered as a map $C_q(\tau) \to \bigoplus_{\tau \prec \sigma} C_{q-1}(\sigma)$. For every $\sigma \prec \tau$ in $\mathcal{C}$ we have the map of multi-tangent spaces, $i_{\tau\sigma}: F_p(\tau) \to F_p(\sigma)$. For $v \otimes \delta \in F_p(\tau) \otimes C_q(\tau)$ we define the boundary operator by

$$\partial(v \otimes \delta) := \sum_{k=0}^{q} (-1)^k v^k \otimes \delta^k \in C_{p,q-1}(X),$$

where $v^k := i_{\tau\sigma}(v)$ when $\delta^k(\Delta_{q-1}) \subset \text{int}(\sigma)$. We obtain complexes $(C_{p,*}(X), \partial)$ of real vector spaces.

iii) We define the tropical homology groups to be

$$H_{p,q}^{\text{trop}}(X) := H_q(C_{p,*}(X), \partial).$$

Dually, we define tropical cochains by $C^{p,q}(X) := \text{Hom}(C_{p,q}(X), \mathbb{R})$ and the tropical cohomology of $X$ as the cohomology of the dual complex

$$H_{p,q}^{\text{trop}}(X) := H^q(C_{p,*}(X), \partial^*).$$

iv) We say that $\alpha \in C^{p,q}(X)$ has compact support if there exists a compact subset $K_\alpha \subset X$ such that $\alpha(v \otimes \delta) \neq 0$ implies $\delta(\Delta_q) \cap K_\alpha \neq \emptyset$. The cochains with compact support form a complex $C^p_{*,c}(X)$ and we define tropical cohomology with compact support by

$$H_{p,q}^{\text{trop,c}}(X) := H^q(C^p_{*,c}(X), \partial^*).$$

**Remark 3.5.** There are also cellular versions of tropical homology and cohomology [MZ14, Section 2.2]. The advantage of the cellular versions is that they the (co)homology groups of finitely generated complexes.
Let \( C \) be a polyhedral complex in \( \mathbb{T}^r \). From the vector spaces \( F^p(\sigma) \), it is possible to construct a sheaf on \( |C| \subset \mathbb{T}^r \) following the lines of [MZ14, Section 2.3]. For each open set \( \Omega \subset |C| \), consider the poset \( P(\Omega) \) whose elements are the connected components \( \sigma \) of faces of \( C \) intersecting with \( \Omega \). The elements of \( P(\Omega) \) are ordered by inclusion and if \( \sigma \prec \tau \) recall there are maps \( r_{\tau \sigma} : F^p(\sigma) \to F^p(\tau) \).

**Definition 3.6.** For an open set \( \Omega \subset |C| \) define the vector space

\[
F^p(\Omega) := \lim_{\sigma \in P(\Omega)} F^p(\sigma).
\]

The above defines a constructible sheaf of vector spaces on \( |C| \) [MZ14]. These sheaves do not depend on the polyhedral structure \( C \) and thus are well defined for polyhedral subspaces. For a polyhedral space \( X \), the sheaves \( F^p_X \) are defined by gluing along charts. Note that this definition does not require a face structure on \( X \).

**Definition 3.7.** A subset \( \Delta \subset \mathbb{T}^r \) is an open cube if it is a product of intervals which are either \((a_i, b_i)\) or \([−\infty, c_i]\) for \( a_i \in \mathbb{T}, b_i, c_i \in \mathbb{R} \cup \{\infty\} \).

For a polyhedral complex \( C \) in \( \mathbb{T}^r \), an open subset \( \Omega \) of \( |C| \) is called a basic open subset if there exists an open cube \( \Delta \subset \mathbb{T}^r \) such that \( \Omega = |C| \cap \Delta \) and such that the set of polyhedra of \( C \) intersecting \( \Omega \) has a unique minimal element. Note that the sedimentarity of the minimal polyhedron of \( \Omega \) is the maximal sedimentarity among points in \( \Omega \).

Let \( X \) be a polyhedral space with atlas \((\varphi_i : U_i \to X_i)_{i \in I}\), such that for each \( i \) we have a fixed polyhedral structure \( C_i \) on \( X_i \). Then we say that an open subset \( U \) is a basic open subset (with respect to these structures) if there exists a chart \( \varphi_i : U_i \to X_i \) such that \( U \subset U_i \) and \( \varphi_i(U) \) is a basic open subset of \( |C_i| \).

**Lemma 3.8.** Let \( C \) be a polyhedral complex in \( \mathbb{T}^r \). Then the basic open sets form a basis of the topology on \( |C| \). Furthermore, if \( \Omega \) is a basic open subset of \( |C| \) of sedimentarity \( I \), then \( \Omega_I \) is a basic open subset of \( |C_I| \) in \( \mathbb{R}^r_I \).

**Proof.** Basic open sets form a basis of the topology of \( |C| \) since open cubes form a basis of the topology of \( \mathbb{T}^r \). For the second statement, we have that \( \Omega_I = |C_I| \cap \Delta_I \) and the minimal polyhedron of \( \Omega_I \) is the same as the one of \( \Omega \), so the lemma is proven.

**Lemma 3.9.** Let \( C \) be a polyhedral complex in \( \mathbb{T}^r \) and \( \Omega \) a basic open subset of \( |C| \). Then

\[
F^p(\Omega) = F^p(\sigma),
\]

where \( \sigma \) is the minimal polyhedron of \( C \).

**Proof.** Let \( \Delta \) be an open cube such that \( \Omega = \Delta \cap |C| \) and suppose that \( I \) is such that \( \text{int}(\sigma) \subset \mathbb{R}^r_I \). Then \( \Omega \cap \sigma = \Delta \cap \sigma = (\Delta \cap \mathbb{R}^r_I) \cap \text{int}(\sigma) \) is connected, since it is the intersection of two convex sets. Thus the poset \( P(\Omega) \) has \( \Omega \cap \sigma \) as its unique minimal element and the lemma follows.

**Example 3.10.** Recall the definition of tropical projective space \( \mathbb{T}P^r \) from Example 2.23. The sets \( U_i \) are identified with \( \mathbb{T}^r \) via the charts \( \varphi_i \). Since \( \mathbb{T}^r \) is a basic open with minimal stratum \( \sigma_{\infty} = \{−\infty, \ldots, \infty\} \), we have \( F^p(U_i) = F^p(\mathbb{T}^r) = F^p(\sigma_{\infty}) \). By definition we have that \( F^p(\sigma_{\infty}) = 0 \) for \( p > 0 \) and \( F^0(\sigma_{\infty}) = \mathbb{R} \). Therefore, \( F^p(\mathbb{T}P^r) = 0 \) for \( p > 0 \) and \( F^0(\mathbb{T}P^r) = \mathbb{R} \).

Recall the tropical line \( L \) from Example 2.11. The entire line \( L \) satisfies the conditions to be a basic open subset. Its minimal polyhedron is the vertex, which we denote by \( \sigma \). By Lemma 3.9, we have \( F^p(L) = F^p(\sigma) \). Therefore,

\[
F^0(L) = \mathbb{R} \quad \text{and} \quad F^1(L) = \langle −e_1, −e_2, e_1 + e_2 \rangle = \mathbb{R}^2.
\]

An open edge \( \tau \) of \( L \) is also a basic open subset. If \( v \) is the direction of \( \tau \), then \( F^p(\tau) = \mathbb{R} \) for \( p = 0 \), \( F^p(\tau) = \langle v \rangle \) for \( p = 1 \), and \( F^p(\tau) = 0 \) otherwise.

Next we compute \( H^{p,q}_{\text{trop}}(\Omega) \) for a basic open set \( \Omega \).
Proposition 3.11. Let $\Omega$ be a basic open subset of a polyhedral subspace $|C| \subset \mathbb{T}^r$, for a polyhedral complex $C$ in $\mathbb{T}^r$. Then

$$H^q_{\text{trop}}(\Omega) = 0$$

for $q > 0$. Furthermore, we have canonical isomorphisms

$$H^0_{\text{trop}}(\Omega) = F^q(\sigma),$$

where $\sigma$ is the minimal polyhedron of $\Omega$.

Proof. Suppose first that the minimal polyhedron of $\Omega$ is of sedentarity 0 and denote it by $\sigma$. Choose a point $x_0 \in \Omega$ which is in the relative interior of $\sigma$. By performing a translation of $\Omega$ we may assume that $x_0 = 0$.

Since the complex $C^\bullet_{\text{sing}}(x_0, F^q(\sigma))$ is a subcomplex of $C^\bullet_{\text{trop}}(X)$ there is a canonical projection $\pi: C^\bullet_{\text{trop}}(\Omega) \to C^\bullet_{\text{sing}}(x_0, F^q(\sigma))$. There is also a map $\iota: C^\bullet_{\text{sing}}(x_0, F^q(\sigma)) \to C^\bullet_{\text{trop}}(\Omega)$ that is dual to the map which pushes forward the simplicies to $x_0$ and preserves the coefficients.

Define $f: \Omega \times [0, 1] \to \Omega$ by $f(x, t) = (1 - t)x$, and let $f_1 := f(\cdot, t)$. Notice that $f_1$ is the contraction of $\Omega$ to the origin. For all $t$ there is a map $f_t^*: C^\bullet_{\text{trop}}(\Omega) \to C^\bullet_{\text{trop}}(\Omega)$ which is dual to the map which pushes forward the simplicies along $f_t$ and preserves the coefficients. This is possible since $f$ preserves the polyhedral structure of $\Omega$. Notice that for $t = 1$, we have $f_1^* = \iota \circ \pi: C^\bullet_{\text{trop}}(\Omega) \to C^\bullet_{\text{trop}}(\Omega)$. It is clear that $\pi \circ \iota = \text{id}$. We claim that $\iota \circ \pi$ is homotopic to the identity.

Attached to $f$ there is a prism operator $P_f: C^\bullet_{q-1}(\Omega) \to C^\bullet_q(\Omega)$, which provides a homotopy between $f_0, \ldots = \text{id}$ and $f_1, \ldots$. Using the prism operator we can construct a map $P_f^q: C^\bullet_{\text{trop}}(\Omega) \to C^\bullet_{\text{trop}}(\Omega)$ on the tropical cochains groups given by $(P_f^q)(\alpha)(v \otimes \delta) = \alpha(v \otimes P_f^\delta)$. It can be checked by following the argument for the case of constant coefficients that $P_f^q$ provides a homotopy between $\text{id}$ and $f_1^*$ for $\pi$. Therefore, $H^p_{\text{trop}}(\Omega) \cong H^q_{\text{sing}}(x_0, F^q(\sigma))$ and the statement of the proposition follows.

When the minimal polyhedron of $\Omega$ is of sedentarity $I \neq \emptyset$ Lemma 3.12 constructs a deformation retraction of $\Omega$ onto $\Omega_I$ which preserves the underlying polyhedral structure. Using this retraction we can apply the argument above and obtain the canonical isomorphism in the claim. This proves the proposition.

Lemma 3.12. Let $C$ be a polyhedral complex and $\Omega$ a basic open subset with maximal sedentarity $I$. Then there exists a continuous map

$$g: \Omega \times [0, 1] \to \Omega$$

such that

i) $g(x, 0) = x$ for all $x \in \Omega$;

ii) $g(x, t) = x$ for all $x \in \Omega_I$ and all $t \in [0, 1]$;

iii) $g(x, 1) \in \Omega_I$ for all $x \in \Omega$;

iv) For $\sigma \in C$ and $x \in \text{int}(\sigma)$ we have $g(x, t) \in \text{int}(\sigma)$ for all $t \in [0, 1]$.

Proof. We will define a deformation retraction

$$g(x, t) = x - \log(1 - t) \cdot w(x)$$

where $w: \Omega \to \mathbb{R}^r$ is a continuous map. Notice that then property i) from the statement of the lemma is satisfied.

The map $w$ will be constructed so that for $x \in \text{int}(\sigma)$ and of sedentarity $J$ we have

2) $w(x) = 0$ if $x \in \Omega_I$;

3) $w(x) \in S_{J \setminus I} := \{y \in \mathbb{R}^r \mid y_i < 0 \text{ if } i \in I \setminus J \text{ and } y_i = 0 \text{ else}\}$;

4) $w(x) \in \text{cone}(\sigma) := \{v \in \mathbb{R}^r \mid v_i = 0 \text{ for } i \in J \text{ and } y + n \cdot v \in \sigma \text{ for all } n \in \mathbb{N} \text{ and all } y \in \sigma\}$. 
Proposition 3.15. The next proposition is stated in [MZ14, Proposition 2.8]. Here we provide the details of its proof.

Let $\Omega \rightarrow C^{p,q}(\Omega)$.

Lemma 3.14. The sheaves $C^{p,q}$ are flasque. Furthermore, the canonical maps $(C^{p,q}(\Omega), \partial) \rightarrow (C^{p,q}(\Omega), \partial)$ and $(C^{\bullet,q}(\Omega), \partial) \rightarrow (C^{p,q}(\Omega), \partial)$ are quasi-isomorphisms.

Proof. The presheaves $\Omega \rightarrow C^{p,q}(\Omega)$ are flasque. Given an open subset $U$, an open cover $(U_i)_{i \in I}$ and elements $c_i \in C^{p,q}(U_i)$ which agree when restricted to intersections, we can define a cochain $c \in C^{p,q}(U)$ by assigning to every element $v \otimes \delta \in C_{p,q}(U)$ the value $c(v \otimes \delta) := c_i(v \otimes \delta)$ if $\delta(\Delta) \subset U_i$ and $c(v \otimes \delta) := 0$ else. Thus the presheaves $C^{p,q}(\Omega)$ satisfy the glueing axiom. This implies that the map $C^{p,q}(\Omega) \rightarrow C^{p,q}(\Omega)$ is surjective for all $\Omega$. This implies flasqueness of $C^{p,q}$.

As in [Ram05, p.110] for a $q$-simplex $\delta$, let $\tilde{b}(\delta)$ denote the collection of simplicies which is the barycentric subdivision of $\delta$. For $\alpha \in C^{p,q}(\Omega)$ define the cochain $b(\alpha)$ by $b(\alpha)(v \otimes \delta) := \alpha(v \otimes \tilde{b}(\delta))$. For an open cover $\mathcal{U} = (U_i)_{i \in I}$ of $\Omega$ we define $C^{p,q}(\mathcal{U})$ to be the subcomplex of $C^{p,q}(\Omega)$ given by cochains supported on simplicies which are contained in one of the open sets $U_i$. As in [Ram05, Chapter 4, Proposition 4.10 i)], we can use $b$ to give a canonical map $C^{p,q}(\Omega) \rightarrow C^{p,q}(\mathcal{U})$ which is a quasi-isomorphism. Following the proof of [Ram05, Chapter 4, Proposition 4.10 ii)], the claim follows from the fact that the map $C^{p,q}(\Omega) \rightarrow C^{p,q}(\mathcal{U})$ factors through $C^{p,q}(\mathcal{U})$ for any cover $\mathcal{U}$ and also that a cochain $\alpha \in C^{p,q}(\Omega)$ which vanishes in $C^{p,q}(\mathcal{U})$ must also vanish in $C^{p,q}(\mathcal{U})$ for some cover $\mathcal{U}$.

Since the operator $b$ does not change the support of a cochain, the same arguments work when considering cochains with compact support.

The next proposition is stated in [MZ14, Proposition 2.8]. Here we provide the details of its proof.

Proposition 3.15. For a polyhedral space $X$ equipped with a face structure, there are canonical isomorphisms

$$H^{p,q}_{\operatorname{trop}}(X) \cong H^q(X, \mathcal{F}^p) \quad \text{and} \quad H^{p,q}_{\operatorname{trop},c}(X) \cong H^q(X, \mathcal{F}^p).$$

Proof. Once again we adopt the proof for constant coefficients to our situation. Let $\Omega$ be a basic open subset of $X$. Then $0 \rightarrow \mathcal{F}^p(\Omega) \rightarrow C_{p,0}(\Omega) \rightarrow \ldots$ is exact by Lemma 3.9 and Proposition 3.11. Thus by Lemma 3.14 the complex $0 \rightarrow \mathcal{F}^p(\Omega) \rightarrow C_{p,0}(\Omega) \rightarrow \ldots$ is also exact. Since basic open subsets form a basis of the topology, this means that $0 \rightarrow \mathcal{F}^p \rightarrow C_{p,0} \rightarrow \ldots$ is an exact sequence of sheaves, and therefore an acyclic resolution of $\mathcal{F}^p$ by Lemma 3.14. Thus,

$$H^q(X, \mathcal{F}^p) = H^q(C^{p,q}_0(X), \partial) = H^q(C^{p,q}(X), \partial) = H^{p,q}_{\operatorname{trop}}(X).$$
and
\[ H^q_d(X, \mathcal{F}^p) = H^q_d(\mathcal{C}^{p,\bullet}(X), \partial) = H^q(\mathcal{C}^{p,\bullet}(X), \partial) = H^p_{\text{top},c}(X). \]
Where we again used Lemma 3.14 at both middle equalities. This completes the proof of the proposition.

3.2. Dolbeault cohomology of superforms. In this subsection we prove a local exactness result for superforms on polyhedral spaces, called the Poincaré lemma. This extends the Poincaré Lemma for superforms on polyhedral complexes in \( \mathbb{R}^r \) from [Jel16b, Theorem 2.16].

**Theorem 3.16** (Poincaré lemma). Let \( X \) be a polyhedral space and \( U \subset X \) an open subset. Let \( \alpha \in \mathcal{A}^{p,q}(U) \) with \( q > 0 \) and \( d'\alpha = 0 \). Then for every \( x \in U \) there exists an open subset \( V \subset X \) with \( x \in V \) and a superform \( \beta \in \mathcal{A}^{p,q-1}(V) \) such that \( d''\beta = \alpha|_V \).

**Proof.** After shrinking \( U \), we may assume that there is a chart \( \varphi: U \to \Omega \) for \( \Omega \) an open subset of the support of a polyhedral complex \( C \) in \( \mathbb{T}^r \). Since this question is purely local, we may prove the statement for \( x \in \Omega \), where \( \Omega \) is an open subset of \( |C| \) for a polyhedral complex \( C \) in \( \mathbb{T}^r \).

For a polyhedral complex in \( \mathbb{R}^r \), the statement of the theorem is proven in [Jel16b, Theorem 2.16]. By replacing \( C \) by \( C_{\emptyset} \) and \( \Omega \) by \( \Omega_{\emptyset} \), we conclude that if \( \text{sed}(x) = \emptyset \) the theorem holds.

For the general case, let \( I = \text{sed}(x) \) and after possibly shrinking \( \Omega \) we may assume that \( I \) is the unique maximal sedentarity among points in \( \Omega \) and \( \alpha \) is determined by \( \alpha_I \) on \( \Omega \). After possibly shrinking \( \Omega \) again, by the case \( I = \emptyset \), there is a superform \( \beta_I \in \mathcal{A}^{p,q}(\Omega_I) \) such that \( d'\beta_I = \alpha_I \). For each \( J \subset I \), set \( \beta_J = \pi_I^J \beta_I \). Then this determines a superform \( \beta \in \mathcal{A}^{p,q}(\Omega) \) and since the affine pullback commutes with \( d' \), we have \( d''\beta_J = \alpha_J \), hence \( \beta \) has the required property and the theorem is proven. \( \square \)

**Definition 3.17.** For \( X \) a polyhedral space and \( p \in \mathbb{N} \) we define the sheaf
\[ \mathcal{L}^p_X := \ker(d''': \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X). \]
Again we omit the subscript \( X \) on \( \mathcal{L}^p \) if the space \( X \) is clear from the context.

**Corollary 3.18.** For a polyhedral space \( X \) and all \( p \in \mathbb{N} \), the complex
\[ 0 \to \mathcal{L}^0 \to \mathcal{A}^{0,0} \xrightarrow{d'} \mathcal{A}^{0,1} \xrightarrow{d'} \mathcal{A}^{0,2} \to \ldots \]
of sheaves on \( X \) is exact. Furthermore it is an acyclic resolution, we thus have canonical isomorphisms
\[ H^q(X, \mathcal{L}^p) \cong H^{p,q}_d(X) \quad \text{and} \quad H^q_c(X, \mathcal{L}^p) \cong H^{p,q}_d(X). \]

**Proof.** Exactness is a direct consequence of Theorem 3.16 and Definition 3.17. Acyclicity follows from Proposition 2.26. \( \square \)

**Example 3.19.** We calculate the dimensions of the Dolbeault cohomology for the polyhedral spaces from Example 2.25. Let \( h^{p,q}(X) := \dim H^{p,q}_d(X) \) for all \( p, q \).

It is easy to see that, for any polyhedral space, \( \mathcal{L}^0 \) is the constant sheaf with stalk \( \mathbb{R} \). By Corollary 3.18 and comparison with singular cohomology, we obtain \( h^{0,0}(X) = 1 \) and \( h^{0,1}(X) = 0 \). This argument shows that, in general, the cohomology groups \( H^{p,q}_d(X) \) do not depend on the atlas of \( X \).

For the polyhedral space \( X \) from Example 2.25, recall that the compatibility condition for superforms along the boundary strata implies that all smooth functions are locally constant at points \( 0, 1 \in X \). Also all superforms of positive degree have support away from the boundary points. Thus \((1, 0)\) and \((1, 1)\)-superforms on \( X \) are simply forms on \( \mathbb{R} \) with compact support. Fix a coordinate \( x \) on \( \mathbb{R} \). Then \( \alpha \in \mathcal{A}^{1,0}_c(\mathbb{R}) \) is of the form \( \alpha = f dx \) with \( f \in C^\infty_c(\mathbb{R}) \) and is closed precisely if \( \frac{df}{dx} = 0 \). This means \( f = 0 \) and hence \( \alpha = 0 \), thus \( h^{1,0}(X) = 0 \). For \( h^{1,1}(X) \) note that a superform \( f dx \wedge d'x \) with \( f \in C^\infty_c(\mathbb{R}) \) is exact precisely if \( f \) has an antiderivative with compact support.
We want to show that then define with the inclusion are We will revisit this in Example 4.13.

\( \alpha \)

The closedness of \((1)\) and \(h\) \(\tau\) and a polyhedron. For a fixed \(v\) \(\sigma\) minimal polyhedron \(\sigma\) of \(\Omega\) set \(\sigma\) of sedentarity \(\sigma\). Recall that \(\sigma\) is the minimal polyhedron of the basic open set \(\Omega\). Given a \((p, 0)\)-superform in the kernel of \(d''\), the strategy is to construct a superform whose coefficient functions are all constant and to show that this superform agrees with the original superform on \(\Omega\).

Recall that \(\sigma\) is the minimal polyhedron of the basic open set \(\Omega\). Set \(V = \bigwedge^p \Lambda(\tau)\). There is a natural map \(V^* : \mathcal{L}^p(\Omega) \subset \mathcal{A}^{p, 0}(\Omega)\) and this is clearly injective. To show surjectivity choose \(v_1, \ldots, v_k\) such that each \(v_i \in \bigwedge^p \Lambda(\tau)\) for some \(\tau\) and \(v_1, \ldots, v_k\) is a basis of \(V\) and extend to a basis \(v_1, \ldots, v_k, v_{k+1}, \ldots, v_s\) of \(\bigwedge^p \mathbb{R}^r\). Write \(\alpha \in \mathcal{L}^p(\Omega)\) as

\[
\alpha = \sum_{i=1}^{s} f_i d' v_i
\]

for \(f_i\) smooth functions on open subsets of \(\mathbb{R}^r\). Here \(d' v_1, \ldots, d' v_k\) is the dual to the fixed basis of \(\bigwedge^p \mathbb{R}^r\). By definition we have \(f_i = \langle \alpha, v_i \rangle\).

Notice that for any \(\sigma \prec \tau\), the set \(\Omega \cap \tau\) is connected, since it is the intersection of an open cube and a polyhedron. For a fixed \(\tau\) such that \(\sigma \prec \tau\) and a fixed vector \(w_\tau \in \bigwedge^p \Lambda(\tau)\) define the function

\[
\langle \alpha, w_\tau \rangle : \tau \cap \Omega \to \mathbb{R}.
\]

The closedness of \(\alpha\) implies that this function is constant over all \(x \in \tau \cap \Omega\). Fix a point \(x \in \sigma \cap \Omega\), then define \(c_i := f_i(x)\) and \(\alpha' := \sum_{i=1}^{k} c_i d' v_i\).

We want to show that \(\alpha\) and \(\alpha'\) are equivalent when restricted to \(\Omega\). Then we are done because \(\alpha'\) is certainly in the image of \(V^*\). For any \(\tau \in \mathcal{C}\) and any \(w_\tau \in \bigwedge^p \Lambda(\tau)\) write \(w_\tau = \sum_{i=1}^{k} \lambda_i v_i\).
Then for any \( y \in \Omega \) such that \( y \in \text{int}(\tau) \) we have
\[
\langle \alpha, w_\tau \rangle(y) = \langle \alpha, w_\tau \rangle(x) = \sum_{i=1}^{k} \lambda_i \langle \alpha, v_i \rangle(x) = \sum_{i=1}^{k} \lambda_i f_i(x)
\]
\[
= \sum_{i=1}^{k} \lambda_i c_i = \sum_{i=1}^{k} \lambda_i \langle \alpha', v_i \rangle(y) = \langle \alpha', w_\tau \rangle(y).
\]
The first equality follows because the function defined in (1) is constant. The second equality follows by the definition of \( f_i \). The third equality follows from the fact that \( f_i = \langle \alpha, v_i \rangle \). The final equality also follows by definition. Therefore, \( \alpha \) and \( \alpha' \) are equivalent when restricted to \( \Omega \).

For the general case \( I \neq \emptyset \), first we apply the above argument to \( \Omega_I \) which is a basic open subset of the polyhedral complex \( C_I \) by Lemma 3.8. Writing \( X = \vert C \vert \) and \( X_I = \vert C_I \vert \) we obtain
\[
\mathcal{L}^p_{X_I}(\Omega_I) = \left( \sum_{\tau \in \mathcal{C}_I : \sigma < \tau} \mathcal{L}^p(\tau) \right)^*.
\]
Thus we only have to show
\[
\mathcal{L}^p_{X_I}(\Omega_I) \cong \mathcal{L}^p_X(\Omega).
\]
Using the pullbacks of the projection maps define
\[
\mathcal{L}^p_{X_I}(\Omega_I) \rightarrow \mathcal{L}^p_X(\Omega)
\]
\[
\alpha_I \mapsto \left( \pi^*_I \alpha_I \right)_{J \subseteq I}.
\]
This is clearly well defined and injective, we thus have to show surjectivity. More precisely, for \( \alpha \in \mathcal{L}^p_X(\Omega) \), it remains to show that \( \alpha_I|_{\Omega_{J \cap \tau}} = \pi^*_I \alpha_I|_{\Omega_{J \cap \tau}} \) for all \( J \subseteq I \) and \( \tau \) such that \( \sigma < \tau \).

By the condition of compatibility for \( \alpha \) there exists a neighborhood \( \Omega_x \) of \( x \) such that
\[
\alpha_j|_{\Omega_{x,J}} = \pi^*_J \alpha_J|_{\Omega_{x,J}},
\]
hence in particular
\[
\alpha_j|_{\Omega_{x,J \cap \tau}} = \pi^*_J \alpha_J|_{\Omega_{x,J \cap \tau}}.
\]
Since \( \Omega_I \cap \tau \) is connected, the restriction \( \mathcal{L}^p_{X_I}(\Omega_I \cap \tau) \rightarrow \mathcal{L}^p_{X_I}(\Omega_{x,I \cap \tau}) \) is injective and similarly if we replace \( I \) with \( J \). Thus we have
\[
\alpha_J|_{\Omega_{x,J \cap \tau}} = (\pi^*_J \alpha_J|_{\Omega_{x,J \cap \tau}}),
\]
proving that
\[
\mathcal{L}^p(\Omega) = \left( \sum_{\tau \in \mathcal{C}_J : \sigma < \tau} \mathcal{L}^p(\tau) \right)^*.
\]

For \( \Omega_\prime \subseteq \Omega \), the claim concerning the restriction maps is clear if the minimal polyhedra of \( \Omega \) and \( \Omega_\prime \) are of the same sedentarity. If the minimal polyhedron \( \sigma' \) of \( \Omega_\prime \) is of sedentarity \( J \), then the restriction \( \mathcal{L}^p_X(\Omega) \rightarrow \mathcal{L}^p_{X_\prime}(\Omega_\prime) \) is given by restriction on each stratum. By identifying \( \mathcal{L}^p_X(\Omega) \cong \mathcal{L}^p_{X_I}(\Omega_I) \) and \( \mathcal{L}^p_{X_\prime}(\Omega_\prime) \cong \mathcal{L}^p_{X_J}(\Omega_J) \) we obtain the claimed restriction maps. \( \square \)

3.3. Equivalence of cohomologies. We are now ready to prove that tropical cohomology and Dolbeault cohomology of superforms are isomorphic. We will use the results established in the previous two subsections.

**Lemma 3.21.** Let \( X \) be a polyhedral space. Then there is a canonical isomorphism of sheaves \( \mathcal{L}^p_X \cong \mathcal{F}_X^p \).
Proposition 3.24. Let \((\phi_i : U_i \to \Omega_i \subset X_i)_{i \in I}\) be an atlas for \(X\) and choose a polyhedral structure \(\mathcal{C}\) on \(X_i\) for all \(i\). For \(\Omega \subset X\) a basic open subset there is an isomorphism \(\mathcal{L}^p(\Omega) \to \mathcal{F}^p(\Omega)\) by Proposition 3.20 and Lemma 3.9. Also for a basic open subset \(\Omega'\) contained in \(\Omega\), the restriction maps form the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{L}^p(\Omega) & \to & \mathcal{L}^p(\Omega') \\
\mathcal{F}^p(\Omega) & \to & \mathcal{F}^p(\Omega)
\end{array}
\]

Thus \(\mathcal{L}^p\) and \(\mathcal{F}^p\) agree on a basis of the topology of \(X\), and by [EH00, Proposition I-12, i)] the two sheaves agree. \(\square\)

Now we arrive at Theorem 1 from the introduction.

Theorem 3.22. Let \(X\) be a polyhedral space equipped with a face structure. Then there are canonical isomorphisms

\[
H_{\text{trop}}^{p,q}(X) \cong H_{d^p}^{p,q}(X) \quad \text{and} \quad H_{\text{trop},c}^{p,q}(X) \cong H_{d^p}^{p,q,c}(X).
\]

Proof. Corollary 3.18 relates the Dolbeault cohomology of superforms on \(X\) with the cohomology of the sheaf \(\mathcal{L}^p\). Proposition 3.15 does the same with tropical cohomology and the cohomology of \(\mathcal{F}^p\). Combining this with Lemma 3.21 proves the isomorphisms. \(\square\)

Remark 3.23. Notice that in the absence of a face structure on \(X\), the sheaf cohomology of \(\mathcal{F}^p\) and the Dolbeault cohomology of superforms are still isomorphic by applying Corollary 3.18 and Lemma 3.21.

For a polyhedral space \(X\) we have \(\mathcal{L}^0_X = \mathbb{R} = \mathcal{F}^0_X\), where \(\mathbb{R}\) is the constant sheaf with stalks \(\mathbb{R}\).

Thus we have \(H_{d^p}^{0,q}(X) \cong H_{\text{trop}}^{0,q}(X) \cong H_{\text{sing}}^{0,q}(X)\) by Proposition 3.15, Corollary 3.18 and [Bre97, Chapter III, Theorem 1.1].

The tropical cohomology groups and Dolbeault cohomology groups of superforms for \(p > 0\) do however depend heavily on the equivalence class of the chosen atlas, and not just on the topological space underlying a polyhedral space (see Example 3.19).

Proposition 3.24. Let \(\mathcal{C}\) and \(\mathcal{D}\) be polyhedral complexes in \(\mathbb{T}^r\) and \(\mathbb{T}^s\) respectively and let \(\delta : |\mathcal{D}| \to |\mathcal{C}|\) be a map induced by an extended affine map \(\delta : \mathbb{T}^r \to \mathbb{T}^s\) such that the image of every face of \(\mathcal{D}\) is a face of \(\mathcal{C}\). Let \(X \subset |\mathcal{C}|\) and \(Y \subset |\mathcal{D}|\) be open subsets such that \(\delta(Y) \subset X\). Then there are maps \(\delta_{d^p} : H_{d^p}^{p,q}(X) \to H_{d^p}^{p,q}(Y)\) and \(\delta_{\text{trop}} : H_{\text{trop}}^{p,q}(X) \to H_{\text{trop}}^{p,q}(Y)\). Moreover the following diagram commutes:

\[
\begin{array}{ccc}
H_{d^p}^{p,q}(X) & \to & H_{d^p}^{p,q}(X) \\
\delta_{\text{trop}}^* & & \delta_{\text{trop}}^* \\
H_{d^p}^{p,q}(Y) & \to & H_{d^p}^{p,q}(Y).
\end{array}
\]

If \(\delta\) is a proper map, then the same holds for cohomology with compact support.

Proof. We have the following commutative diagram:

\[
\begin{array}{cccccccc}
\delta^{-1} \mathcal{A}^p_X & \to & \delta^{-1} \mathcal{L}^p_X & \to & \delta^{-1} \mathcal{F}^p_X & \to & \delta^{-1} \mathcal{C}^p_X \\
\delta_{d^p}^* & \to & \delta^* & \to & \delta_{\text{trop}}^* & \to & \delta_{\text{trop}}^*
\end{array}
\]

All horizontal maps are quasi-isomorphisms. Taking hypercohomology of the functor of global sections, we get the cohomology of the complexes of global sections on both the far left and the
far right as well as the cohomology of the sheaves in the middle. This shows the commutativity of the left square in the diagram

$$
\begin{array}{ccc}
H^{p,q}_d(X) & \longleftarrow & H^q(C\mathcal{P}^\bullet(X),\partial^*) \\
\delta_{trop}' & \downarrow & \delta_{trop}' \\
H^{p,q}_d(Y) & \longleftarrow & H^q(C\mathcal{P}^\bullet(Y),\partial^*)
\end{array}
$$

The commutativity of the right square follows by definition of the complexes \((C\mathcal{P}^\bullet(X),\partial^*)\) and tropical cohomology. This proves the claim for usual cohomology.

If \(\delta\) is proper then the pullbacks are well defined for sections with compact support and the above arguments can be applied directly. This completes the proof of the proposition. \(\square\)

Remark 3.25. Another technique to prove Theorem 3.22 could be to use a map similar to the de Rham map, which provides an isomorphism between the de Rham and singular cohomologies in the classical theory. This de Rham map is given explicitly by

\[
\text{de Rham map, which provides an isomorphism between the de Rham and singular cohomologies}
\]

In this section we prove Poincaré duality for a class of polyhedral spaces, known as tropical manifolds. By Poincaré duality we mean an explicit isomorphism between the de Rham and singular cohomologies \(\text{PD}: H^q_d(X) \rightarrow H^q_{\text{sing}}(X)\), where \([\alpha] \mapsto \left(\delta : \Delta_q \rightarrow X \mapsto \int_{\Delta} \delta^*(\alpha)\right)\).

There is a similar map from spaces of superforms to tropical cochains given by contracting a \((p,q)\)-superform by the coefficient of a singular tropical cell. This produces a \((0,q)\)-superform which can be integrated over the simplex following Section 4.1. Some care needs to be taken to allow only smooth simplicies (as in the classical case) and also to ensure that the integrals are well defined when passing to cohomology. We will not do this since it is not required for our considerations, but we point out that this could approach could be used to identify the wedge product on Dolbeault cohomology of superforms with the cup product on tropical cohomology.

4. Poincaré duality

In this section we prove Poincaré duality for a class of polyhedral spaces, known as tropical manifolds. By Poincaré duality we mean an explicit isomorphism \(\text{PD}: H^{p,q}(X) \rightarrow H^{n-p,n-q}(X)^*\). Just as for standard differential forms, this map is defined using a pairing given by integration of superforms. In Subsection 4.1, we show that the pairing given by integration descends to a pairing on the Dolbeault cohomology of superforms on tropical spaces. Finally we show Poincaré duality for tropical manifolds in Subsection 4.2.

4.1. Integration of superforms. Throughout the next sections we consider the standard lattice \(\mathbb{Z}^r \subset \mathbb{R}^r\). Notice that in \(\mathbb{T}^r\) there is an induced lattice in each stratum \(\mathbb{R}^r_I\). We define integration of superforms on rational polyhedral complexes in \(\mathbb{T}^r\) by extending the theory already developed in \(\mathbb{R}^r\). Then we use partitions of unity to define the integration of superforms on rational polyhedral complexes. We prove a version of Stokes’ theorem for superforms on tropical spaces which ensures that this integration descends to Dolbeault cohomology.

Lemma 4.1. Let \(X\) be a polyhedral subspace of dimension \(n\) in \(\mathbb{T}^r\). Suppose that there exists \(I \subset [r]\) such that \(X_I\) is dense in \(X\). If \(\alpha \in A^{p,q}_n(X)\) is such that \(\max(p,q) = n\), then \(\alpha_I \in A^{p,q}(X_I)\) has compact support and \(\alpha_J = 0\) for each \(J \supseteq I\).

Proof. For \(J \neq I\) we have \(\dim X_J < n\), so for dimension reasons \(\text{supp}(\alpha_J) = \emptyset\). Then Lemma 2.17 shows \(\text{supp}(\alpha) = \text{supp}(\alpha_I)\). \(\square\)

Definition 4.2. A polyhedral complex \(C\) in \(\mathbb{T}^r\) is called rational if every polyhedron \(\sigma\) is parallel to a subspace of \(\mathbb{R}^r_{\text{sed}(\sigma)}\) defined over \(\mathbb{Z}\).
A polyhedral space $X$ with atlas $(\varphi_i: U_i \to X_i \subset \mathbb{T}^r)_{i \in I}$ is called rational if every $X_i$ is the support of a rational polyhedral complex and the transition functions $\varphi_i \circ \varphi_j^{-1}$ are integral extended affine maps.

By definition, for any polyhedron $\sigma$ in a rational polyhedral complex there is a canonical lattice of full rank $\mathbb{Z}(\sigma) \subset \mathbb{L}(\sigma)$.

**Definition 4.3.** Let $\mathcal{C}$ be a polyhedral complex of pure dimension $n$. We write $\mathcal{C}_n$ for the set of $n$-dimensional polyhedra. Then $\mathcal{C}$ is weighted if it is equipped with a weight function $w: \mathcal{C}_n \to \mathbb{Z}$.

A polyhedral space $X$ is weighted if it is equipped with a continuous weight function $w: X^0 \to \mathbb{Z}$, which is defined on a dense open subset $X^0$ of $X$. Furthermore, we require that for every chart $\varphi_i: U_i \to X_i \subset \mathbb{T}^r$ of $X$ and every connected component $U$ of $X^0$ the space $\varphi_i(U \cap U_i)$ is the intersection of $\varphi_i(U_i)$ with the relative interior of a polyhedron in $\mathbb{T}^r$.

For another open dense subset $Y^0 \subset X$ and weight function $w': Y^0 \to \mathbb{Z}$ we say $w$ is equivalent to $w'$ if $w'|_{Y^0 \cap X^0} = w|_{Y^0 \cap X^0}$ and $Y^0 \cap X^0$ is dense in $X$.

Note that the continuity of a weight function $w$ ensures that it is constant on any connected component of $X^0$.

**Definition 4.4.** Let $X$ be a rational weighted polyhedral space with weight function $w: X^0 \to \mathbb{Z}$ and charts $(\varphi_i: U_i \to |C_i|)_{i \in I}$, where the $C_i$ are weighted rational polyhedral complex. We say that $\{C_i\}_{i \in I}$ is a weighted rational polyhedral structure on $X$ if for every connected component $U$ of $X^0$ the image $\varphi_i(U \cap U_i) = \varphi_i(U_i) \cap \text{int}(\sigma)$ for a polyhedron $\sigma$ in $C_i$. Moreover, we require that $w|_U \equiv m_\sigma$, where $m_\sigma$ is the weight of $\sigma$ in $C_i$.

For a concrete choice of a weighted rational polyhedral structure on $X$, the weights of faces of $C_i$ outside of $\varphi_i(U_i)$ will not matter for any of our constructions.

Now we recall the definition of integration of superforms on weighted polyhedral complexes in $\mathbb{R}^r$ from Chamber-Loir and Ducros [CLD12] but follow the notation of Gubler [Gub16]. We also extend the definition of integration to polyhedral complexes in $\mathbb{T}^r$.

**Definition 4.5.** Let $\mathcal{C}$ be a pure $n$-dimensional weighted rational polyhedral complex in $\mathbb{R}^r$.

i) Let $\alpha \in \mathcal{A}_{c}^{n,n}(|\mathcal{C}|)$. For $\sigma \in \mathcal{C}_n$, choose a basis $x_1, \ldots, x_n$ of $\mathbb{Z}(\sigma)$. Then $\alpha|_\sigma$ can be written as

$$f_\alpha d^r_1 \wedge d^r_2 \wedge \ldots \wedge d^r_n = (-1)^{n(n-1)/2} f_\alpha d^r_1 \wedge d^r_2 \wedge \ldots \wedge d^r_{n-1} \wedge d^r_n$$

for $f_\alpha \in \mathcal{A}_c^{0,0}(\sigma)$. Since this is an integral basis, $f_\alpha$ is independent of the choice of $x_1, \ldots, x_n$. Then the integral of $\alpha$ over $\sigma$ is

$$\int_\sigma \alpha := \int_\sigma f_\alpha,$$

where the integral on the right is taken with respect to the volume defined by the lattice $\mathbb{Z}(\sigma) \subset \mathbb{L}(\sigma)$. The integral over the weighted rational polyhedral complex $\mathcal{C}$ is

$$\int_\mathcal{C} \alpha := \sum_{\sigma \in \mathcal{C}_n} m_\sigma \int_\sigma \alpha,$$

where $m_\sigma$ is the weight of $\sigma$.

ii) Let $\tau \prec \sigma$ be a face of $\sigma$ of codimension one. Denote by $\nu_{\tau,\sigma} \in \mathbb{Z}(\sigma)$ a representative of the unique generator of $\mathbb{Z}(\sigma)/\mathbb{Z}(\tau)$ which points inside of $\sigma$. Then for $\beta \in \mathcal{A}_c^{n,n-1}(|\mathcal{C}|)$ the boundary integral of $\beta$ over $\partial \sigma$ is

$$\int_{\partial \sigma} \beta = \sum_{\tau \prec \sigma} \langle \beta; \nu_{\tau,\sigma} \rangle_n,$$
where on the right hand side we use the integral of the \((n-1, n-1)\)-form \(\langle \beta; \nu_{\tau, \sigma} \rangle_n\) over the \((n-1)\)-dimensional polyhedron \(\tau\) as defined in \(i\). The integral over the boundary of the weighted rational polyhedral complex \(\mathcal{C}\) is

\[
\int_{\partial \mathcal{C}} \beta := \sum_{\sigma \in \mathcal{C}_n} m_{\sigma} \int_{\partial \sigma} \beta,
\]

where \(m_{\sigma}\) is the weight of \(\sigma\).

iii) If \(\mathcal{C}\) is a weighted rational polyhedral complex in \(\mathbb{T}^r\), then the definitions from \(i\) and \(ii\) can be extended. Note that if \(\alpha \in \mathcal{A}_{\mathcal{C}}^{n,n}(\mathcal{C})\) and \(\sigma \in \mathcal{C}_n\) is the closure of \(\sigma' \in \mathcal{C}_{I,n}\), then the support of \(\alpha|_{\sigma}\) is contained in \(\sigma'\) by Lemma 4.1 and we define

\[
\int_{\sigma} \alpha := \int_{\sigma'} \alpha|_{\sigma'}.
\]

The same works for integrals of \((n, n-1)\)-forms with compact support over codimension one faces. Note that if the codimension one face \(\tau\) of \(\sigma\) is of a higher sedentarity than \(\sigma\), then by Lemma 4.1, for \(\beta \in \mathcal{A}_{\mathcal{C}}^{n,n-1}(\mathcal{C})\) the restriction \(\beta|_{\sigma}\) has support away from \(\tau\). Thus for \(\int_{\partial \sigma} \beta\) we only integrate over codimension one faces of \(\sigma\) which have the same sedentarity as \(\sigma\).

Now integration on polyhedral spaces can be defined using the integration on polyhedral subspaces and partitions of unity just as in manifold theory.

**Definition 4.6.** Let \(X\) be a pure \(n\)-dimensional weighted rational polyhedral space with weight function \(w\) and atlas \((\varphi_i : U_i \to |\mathcal{C}_i|)_{i \in I}\) where \(\{\mathcal{C}_i\}_{i \in I}\) is a weighted rational polyhedral structure on \(X\). Let \(\alpha \in \mathcal{A}_{\mathcal{C}}^{n,n}(X)\) and \((f_j)_{j \in J}\) be a partition of unity with functions in \(\mathcal{A}^{0,0}\) subordinate to the cover \((U_i)\) as in Lemma 2.7. Define the integral of \(\alpha\) over \(X\) by

\[
\int_X \alpha := \sum_{j \in J} \int_{\mathcal{C}_i} \alpha_j,
\]

where \(\alpha_j \in \mathcal{A}_{\mathcal{C}}^{n,n}(\mathcal{C}_i)\) is the superform corresponding to \(f_j \alpha \in \mathcal{A}_{\mathcal{C}}^{n,n}(U_i)\). Since \(\alpha\) has compact support the sum on the right hand side is finite. The integral on the right is defined in Definition 4.5.

Notice that the above definition is dependent on the choice of weight function \(w\). However, the following lemma ensures the above defined integral is independent of the choice of charts and partition of unity on the polyhedral space.

**Lemma 4.7.** Let \(X\) be a pure \(n\)-dimensional weighted rational polyhedral space. Then the integral from Definition 4.6 is independent of the choice of atlas, weighted rational polyhedral structure, and partition of unity.

**Proof.** Consider an atlas \((\varphi_i : U_i \to |\mathcal{C}_i|)_{i \in I}\), where \(\{\mathcal{C}_i\}\) is a weighted rational polyhedral structure on \(X\). For \(i, j \in I\) let \(F := \varphi_j \circ \varphi_i^{-1}, U := U_i \cap U_j\) and \(\alpha \in \mathcal{A}_{\mathcal{C}}^{p,q}(U)\). Denote by \(\alpha_i\) and \(\alpha_j\) the forms corresponding to \(\alpha\) on \(\varphi_i(U)\) and \(\varphi_j(U)\) respectively. We claim that \(\int_{\mathcal{C}_i} \alpha_i = \int_{\mathcal{C}_j} \alpha_j\).

We may now assume that \(\mathcal{C}_i\) is a polyhedral complex in \(\mathbb{R}^r\). This is because the pullback of a superform \(\alpha\) on a polyhedral complex in \(\mathbb{T}^r\) is defined by pulling back the components \(\alpha_j\) along affine maps and integration is also defined by considering the intersection of the polyhedral subspace with the vector spaces \(\mathbb{R}^r\).

Let \(F_* \mathcal{C}_i\) be the pushforward of \(\mathcal{C}_i\) in the sense of weighted polyhedral complexes [Gub16, 3.9]. Since \(F\) is an isomorphism between \(\varphi_i(U)\) and \(\varphi_j(U)\), the weights are constructed from the weight function of \(X\), and \(\text{supp}(\alpha_j)\) is contained in \(\varphi_j(U)\), we have \(\int_{F_* \mathcal{C}_i} \alpha_j = \int_{\mathcal{C}_j} F^* \alpha_j\). By the projection formua [Gub16, Proposition 3.10] we have \(\int_{F_* \mathcal{C}_i} \alpha_j = \int_{\mathcal{C}_i} F^* \alpha_j\). By definition we have \(\alpha_i = F^* \alpha_j\). This shows \(\int_{\mathcal{C}_i} \alpha_i = \int_{\mathcal{C}_j} \alpha_j\).
The lemma now follows by the standard argument for classical manifolds. Given two atlases, two weighted rational polyhedral structures and two partitions of unity subordinate to their respective covers we can consider the union of the atlases and weighted rational polyhedral structures. We can also form a partition of unity for this new atlas by multiplying the given partitions of unity. The argument above then shows independence of the integral. □

**Definition 4.8.** Let $\mathcal{C}$ be a weighted rational polyhedral complex in $\mathbb{T}^r$ which is pure $n$-dimensional. Let $\tau$ be a face of $\mathcal{C}$ of dimension $n-1$. We say that $\mathcal{C}$ is balanced at $\tau$ if
\[
\sum_{\sigma \in \mathcal{C}^n: \tau - \sigma; \text{sed} (\sigma) = \text{sed} (\tau)} m_\sigma \nu_{\sigma, \tau} \in \mathbb{Z} (\tau),
\]
with $\nu_{\sigma, \tau}$ as introduced in Definition 4.5 ii). Note that the above sum is well defined since we sum only over faces $\sigma$ having the same sedentarity as $\tau$. We say that $\mathcal{C}$ satisfies the balancing condition if it is balanced at every face of dimension $n-1$.

Let $X$ be a weighted rational polyhedral space with atlas $(\varphi_i : U_i \to |\mathcal{C}_i|)_{i \in I}$ where $\{\mathcal{C}_i\}_{i \in I}$ is a weighted rational polyhedral structure on $X$. Then $X$ is a tropical space if for all $i$, the weighted polyhedral complex $\mathcal{C}_i$ is balanced at each face which intersects $\varphi_i (U_i)$.

A tropical space equipped with a single chart will also be called a tropical cycle.

It follows from the next theorem that whether or not a weighted polyhedral space is a tropical space does not depend on the choice of a weighted rational polyhedral structure on $X$.

**Theorem 4.9** (Stokes’ theorem for tropical spaces). Let $X$ be an $n$-dimensional weighted rational polyhedral space. Then $X$ is a tropical space if and only if for all $\beta \in \mathcal{A}_{c,n-1}^c (X)$ we have
\[
\int_X d'' \beta = 0.
\]

*Proof.* The analogous statement for a weighted rational polyhedral complex in $\mathbb{R}^r$ is true [Gub16, Proposition 3.8]. If $\mathcal{C}$ is a polyhedral complex in $\mathbb{T}^r$, denote by $I_1, \ldots, I_k$ the subsets of $[r]$ such that there exist maximal faces of sedentarity $I_j$. Then $\mathcal{C}$ is balanced if and only if each $\mathcal{C}_{I_j}$ is balanced and integration over $\mathcal{C}$ is just the sum of the integration over the $\mathcal{C}_{I_j}$ (for details see [Jel16a, Lemma 1.2.30 & Remark 2.1.48]). By considering each $\mathcal{C}_{I_j}$ separately, we may assume that each facet of $\mathcal{C}$ is of the same sedentarity. Without loss of generality, we may also suppose that this sedentarity is the empty set. Then the balancing condition is a condition on codimension one faces of $\mathcal{C}_\emptyset \subset \mathbb{T}^r$. Once again by [Gub16, Proposition 3.8] the balancing condition is equivalent to the vanishing of the integral $\int_{\mathcal{C}_\emptyset} d'' \beta$ for all $\beta \in \mathcal{A}_{c,n-1}^c (|\mathcal{C}_\emptyset|)$. Given $\beta \in \mathcal{A}_{c,n-1}^c (|\mathcal{C}|)$, the form $\beta_\emptyset$ also has compact support by Lemma 4.1. The equality of the integrals
\[
\int_{\mathcal{C}} d'' \beta = \int_{\mathcal{C}_\emptyset} d'' \beta_\emptyset
\]
follows from part iii) of Definition 4.5.

The statement for polyhedral spaces follows from the linearity of both $d''$ and integration. □

**Remark 4.10.** Let $X$ be a tropical space of dimension $n$. There is a product
\[
\mathcal{A}^{p,q} (X) \times \mathcal{A}_{c,n-p,n-q}^c (X) \to \mathbb{R},
\]
\[\begin{array}{c}
(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.
\end{array}\]

By Stokes’ theorem 4.9, given $\alpha \in \mathcal{A}^{p,q} (X)$ and $\beta \in \mathcal{A}_{c,n-p,n-q-1}^c (X)$ we have
\[
0 = \int_X d'' (\alpha \wedge \beta) = \int_X d'' \alpha \wedge \beta + \int_X (-1)^{p+q} \alpha \wedge d'' \beta,
\]
so that
\[ \int_X d^n \alpha \wedge \beta = (-1)^{p+q+1} \int_X \alpha \wedge d^n \beta. \]

**Definition 4.11.** Let \( X \) be a tropical space of dimension \( n \). We define
\[ PD: \mathcal{A}^{p,q}(X) \to \mathcal{A}_c^{n-p,n-q}(X)^*, \]
where \( \mathcal{A}_c^{n-p,n-q}(X)^* := \text{Hom}_\mathbb{R}(\mathcal{A}_c^{n-p,n-q}(X), \mathbb{R}) \) denotes the (non-topological) dual vector space of \( \mathcal{A}_c^{n-p,n-q}(X) \) and \( \varepsilon = (-1)^{p+q/2} \) if \( q \) is even and \( \varepsilon = (-1)^{(q+1)/2} \) if \( q \) is odd. Our choice of \( \varepsilon \) together with the Leibniz rule and Stokes’ theorem implies that we have a morphism of complexes
\[ PD: \mathcal{A}^{p,*}(X) \to \mathcal{A}_c^{n-p,n-*}(X)^*, \]

We will show that the map PD is an isomorphism for tropical manifolds in the next section.

### 4.2. Poincaré duality for tropical manifolds.

In this section we will let \( H^{p,q}(X) \) denote \( H^q(X, \mathcal{L}^p) \). By Corollary 3.18, we have \( H^{p,q}(X) = H^{p,q}_c(X) \). If \( X \) has a face structure, the tropical cohomology groups of \( X \) are defined and Theorem 3.22 provides canonical isomorphisms
\[ H^{p,q}(X) \cong H^q(X, \mathcal{L}^p) \cong H^q(X, \mathcal{L}^p) \cong H^{p,q}_c(X). \]

Similarly, in the setting of cohomology with compact support, let \( H^{p,q}(X) \) denote \( H^q_c(X, \mathcal{L}^p) \).

We will show that the Poincaré duality map given in Definition 4.11 is an isomorphism for tropical manifolds.

**Definition 4.12.** An \( n \)-dimensional tropical space \( X \) has Poincaré duality (PD) if the Poincaré duality map
\[ PD: H^{p,q}(X) \to H^{n-p,n-q}_c(X)^* \]
is an isomorphism for all \( p, q \).

**Example 4.13.** Consider again the polyhedral spaces from Example 3.19. In these examples, the underlying topological space is \([0, 1]\), hence compact. Therefore, cohomology and cohomology with compact support are isomorphic.

First take the charts for \( X \) in such a way that \([0, 1]\) is the gluing of two copies of \( T \) as in Example 2.25. Letting \( h^{p,q}_c \) denote \( \dim H^{p,q}_c(X) \), then by Example 3.19 we have the equalities \( h^{0,0}(X) = 1 = h^{1,1}(X) \) and \( h^{1,0}(X) = 0 = h^{0,1}(X) \). If we equip \( X \) with the weight function equal to one everywhere, then \( X \) is a tropical space. It is easy to see that the integration pairing \( H^{0,0}(X) \times H^{1,1}(X) \to \mathbb{R} \) is non-degenerate and so \( X \) has PD.

Alternatively, we can consider the weighted polyhedral space \( X \) defined by taking a single chart on \([0, 1]\) given by the inclusion of the interval into \( \mathbb{R} \) and again equip \( X \) with weight one. However, this does not yield a tropical space since it does not satisfy Stokes’ theorem. Thus the PD map is not defined on cohomology. We already saw in Example 3.19 that the dimensions of the respective cohomology groups do not agree.

There are examples of tropical spaces which do not satisfy PD. Take for example \( Y \) to be the union of the coordinate axes in \( \mathbb{R}^2 \), again with weight one on each facet. Then it is clear that...
The rest of this section is devoted to proving Theorem 2, which states that tropical manifolds have Poincaré duality. Tropical manifolds are tropical spaces locally modeled on matroidal fans (see Definition 4.15).

Matroids are a combinatorial abstraction of the notion of independence in mathematics [Oxl11] and every matroid has a representation as a fan tropical variety [Stu02]. Given a matroid \( M \) there are explicit constructions of different polyhedral structures for this fan coming from the matroid [FS05, AK06]. In what follows, matroidal fans are always considered to be weighted polyhedral complexes, whose weights are equal to one on all facets.

**Definition 4.14.** A tropical cycle in \( \mathbb{R}^r \) is **matroidal** if it is the support of a matroidal fan \( \Sigma \) in \( \mathbb{R}^r \) and its weight function is equal to one.

**Definition 4.15.** A **tropical manifold** is a tropical space \( X \) of dimension \( n \) whose weight function is equal to one and has an atlas \( (\varphi_i: U_i \rightarrow \Omega_i \subset X_i)_{i \in I} \) such that for all \( i \) the spaces \( X_i = \mathbb{T}^n \times V_i \) where \( V_i \) are matroidal tropical cycles of dimension \( n - r_i \) in \( \mathbb{R}^{s_i} \).

**Example 4.16.** Tropical projective space \( TP^r \) is a tropical manifold using the atlas constructed in Example 2.23.

Consider the tropical line \( L \subset \mathbb{R}^2 \) from Example 2.11 and equip each edge with weight equal to one. The resulting weighted polyhedral complex defines a tropical manifold since it is the support of the matroidal fan associated with the uniform matroid \( U_{2,3} \) of rank 2 on 3 elements. We can also consider the closure of the tropical line in \( \mathbb{R}^2 \) in tropical projective space \( TP^2 \). The result is again a tropical manifold, with charts given by restrictions of the charts for \( TP^2 \).

We begin by showing that matroidal cycles in \( \mathbb{R}^r \) have Poincaré duality. To do this we use an alternative recursive description of matroidal cycles via an operation known as tropical modification [BIMS15]. In the language of matroids, this operation is related to deletions and contractions.

**Construction 4.17.** We now describe tropical modifications. Let \( W \subset \mathbb{R}^{r-1} \) be a tropical cycle and \( P: \mathbb{R}^{r-1} \rightarrow \mathbb{R} \) a piecewise integer affine function. The graph \( \Gamma_P(W) \subset \mathbb{R}^r \) is the support of a weighted rational polyhedral complex. The weight function on \( \Gamma_P(W) \) is inherited from the weights of \( W \). In general, this graph does not satisfy the balancing condition because \( P \) is only a piecewise integer affine function. However, the graph \( \Gamma_P(W) \) can be completed to a tropical cycle \( V \) in a canonical way. At a codimension one face \( E \) of \( \Gamma_P(W) \) that does not satisfy the balancing condition, we can attach a facet to \( E \) generated by the direction \(-e_r\). This facet can be equipped with a unique integer weight so that the resulting polyhedral complex is now balanced at \( E \). Applying this procedure at all codimension one faces of \( \Gamma_P(W) \) produces a tropical cycle \( V \). Notice that there is a map \( \delta: V \rightarrow W \) induced by the linear projection.

**Definition 4.18.** Let \( W \subset \mathbb{R}^{r-1} \) be a tropical space and \( P: \mathbb{R}^{r-1} \rightarrow \mathbb{R} \) a piecewise integer affine function, then the **open tropical modification** of \( W \) along \( P \) is the map \( \delta: V \rightarrow W \) where \( V \subset \mathbb{R}^r \) is the tropical cycle described above. The **divisor** of a piecewise integer affine function \( P \) restricted to \( W \) is the tropical space \( \text{div}_W(P) \subset W \) supported on the points \( w \in W \) such that \( \delta^{-1}(w) \) is a half-line. The weight function on \( \text{div}_W(P) \) is inherited from the tropical cycle \( V \). We also say that \( \text{div}_W(P) \) is the divisor of the modification \( \delta \).

A **closed tropical modification** is a map \( \delta: V \rightarrow W \) where \( V \subset \mathbb{R}^{r-1} \times T \) is the closure of \( V \) and \( \delta \) is the extension of an open tropical modification \( \delta: V \rightarrow W \).

A **matroidal tropical modification** is a modification where \( V, W, \) and \( \text{div}_W(P) \) are all matroidal.

**Remark 4.19.** Suppose the underlying matroid of \( V \) is \( M \). Given a matroidal tropical modification, the matroid of \( W \) is the deletion matroid \( M \setminus e \) for some element \( e \). Moreover, the matroid of \( \text{div}_W(P) \) is the contraction \( M \setminus e \).
Note that for a closed tropical modification $\bar{\delta}: \overrightarrow{V} \to W$ with divisor $D$, the map $\overrightarrow{\Omega}_{V_r}: \overrightarrow{V}_r \to D$ identifies the subspace $\overrightarrow{V}_r = \{ x \in V \mid x_r = -\infty \}$ with $D$. We thus may also consider $D$ as a subspace of $\overrightarrow{V}$.

**Example 4.20.** Let $W = \mathbb{R}$ and equip this space with weight function equal to one. Consider the piecewise integer affine function $P: \mathbb{R} \to \mathbb{R}$ defined by $P(x) = \max(0, x)$. The graph $\Gamma_P(W) \subset \mathbb{R}^2$ consists of two half lines meeting at the origin in directions $(0, -1)$ and $(1, 1)$. The weight on each of the half lines when inherited from $W$ is one. To balance the graph, we must attach the half line in direction $(0, -1)$ at the origin in $\mathbb{R}^2$ and equip this half line with weight one. The resulting space is the tropical line $L$ from Example 2.11. The open tropical modification $\delta: L \to W$ of $W$ along $P$ is the map induced by the linear projection with kernel generated by $e_2$. The divisor of the modification is the origin in $W = \mathbb{R}$ equipped with weight one. When equipped with weight function equal to one, the spaces $L$, and the origin are all matroidal. Therefore $\delta: L \to W$ is a matroidal tropical modification.

It follows from the next proposition that for any matroidal cycle $V \subset \mathbb{R}^r$ of dimension $n$ there is a sequence of open matroidal tropical modifications $V \to W_1 \to \cdots \to W_{r-n} = \mathbb{R}^n$.

**Proposition 4.21.** [Sha13b, Proposition 2.25] Let $V \subsetneq \mathbb{R}^r$ be a matroidal cycle, then there is a coordinate direction $e_i$ such that the linear projection $\delta: \mathbb{R}^r \to \mathbb{R}^{r-1}$ with kernel generated by $e_i$ is a matroidal tropical modification $\delta: V \to W$ along a piecewise integer affine function $P$, i.e., $W \subset \mathbb{R}^{r-1}$ and $D = \text{div}_W(P) \subset \mathbb{R}^{r-1}$ are matroidal cycles.

Tropical cohomology is invariant under closed tropical modifications [Sha15, Theorem 4.13]. The next lemma checks that this isomorphism also applies to cohomology with compact support and that it is compatible with the PD map.

**Proposition 4.22.** Let $\delta : \overrightarrow{V} \to W$ be a closed matroidal tropical modification for $W \subset \mathbb{R}^{r-1}$ and $\overrightarrow{V} \subset \mathbb{R}^{r-1} \times \mathbb{T}$. Then there are isomorphisms $\delta^*: H^{p,q}(W) \to H^{p,q}(\overrightarrow{V})$ and $\delta^*: H^{p,q}_c(W) \to H^{p,q}_c(\overrightarrow{V})$ which are induced by the pullback of superforms and are compatible with the Poincaré duality map.

**Proof.** The fact that $\delta^*$ is an isomorphism for tropical cohomology is shown in [Sha15, Theorem 4.13]. By Proposition 3.24 this also applies to $H^{p,q}$. The same arguments used in [Sha15, Theorem 4.13] work for cohomology with compact support, since $\delta$ and the homotopy used in the proof are proper maps. Thus by Proposition 3.24 the argument also applies to $H^{p,q}_c$.

To show that the isomorphism $\delta^*$ is compatible with the Poincaré duality map, if suffices to show that for $\omega \in A^{n-q}_c(W)$ we have

$$\int_W \omega = \int_{\overrightarrow{V}} \delta^*(\omega).$$

Showing this is sufficient since the wedge product is compatible with the pullback. We can choose polyhedral structures on $W$ and $\overrightarrow{V} \cap \mathbb{R}^r$ so that $W$ is the pushforward of $\overrightarrow{V} \cap \mathbb{R}^r$ along $\delta$ in the sense of polyhedral complexes. Then the result follows from the projection formula [Gub16, Proposition 3.10] since the support of $\delta^{-1}(\omega)$ is contained in $\overrightarrow{V} \cap \mathbb{R}^r$ by Lemma 4.1. This completes the proof. \hfill $\Box$

The next statement relates the cohomology with compact support of the matroidal cycles of an open tropical modification by an exact sequence.

**Proposition 4.23.** Let $\Omega$ be an open subset of a polyhedral subspace in $\mathbb{T}^r$. For $i \in [r]$, let $D = \Omega \cap \mathbb{T}^r_i$ and $U := \Omega \setminus D$. Then there exists a long exact sequence in cohomology with compact support

$$\cdots \to H^{p,q}_{c-i}(D) \to H^{p,q}_{c-i}(U) \to H^{p,q}_c(\Omega) \to H^{p,q}_c(D) \to H^{p,q}_{c-i+1}(U) \to \cdots$$
Proof. Note first that $U$ and $\Omega$ are polyhedral spaces via the inclusion into $\mathbb{T}^r$. Furthermore, $D$ is a polyhedral space via the inclusion into $\mathbb{T}^r$. We claim that the natural sequence of complexes

$$0 \to \mathcal{A}^{\bullet}_{\Omega, c}(U) \to \mathcal{A}^{\bullet}_{\Omega, c}(\Omega) \to \mathcal{A}^{\bullet}_{D, c}(D) \to 0$$

is exact. By the condition of compatibility for superforms along strata, if a superform restricts to 0 on $D \subset \mathbb{T}^r$, then it must be 0 on a neighborhood of $D$ in $\mathbb{T}^r$. This shows exactness in the middle of the short exact sequence. Both surjectivity of the last map and injectivity of the first map are clear. The statement of the proposition follows by passing to the long exact cohomology sequence.

We can now prove a vanishing lemma for the cohomology of compact support of a matroidal cycle.

**Lemma 4.24.** Let $V \subset \mathbb{R}^r$ be a matroidal cycle of dimension $n$. Then for all $p$

$$H^p_{c\alpha}(V) = 0 \text{ if } q \neq n.$$

**Proof.** The lemma is proven by induction on $r$, which is the dimension of the ambient space. When $r = 0$ the space $V$ is a point and the assertion holds. We now proceed by induction. If $r = n$, then $V = \mathbb{R}^n$ and we have $H^p_{c\alpha}(\mathbb{R}^n) = \Lambda^p_{\mathbb{R}} \mathbb{R}^n \otimes H^q_{\mathbb{R}}(\mathbb{R}^n)$, where $H^q_{\mathbb{R}}(\mathbb{R}^n)$ denotes the usual de Rham cohomology with compact support of $\mathbb{R}^n$. We have $H^q_{\mathbb{R}}(\mathbb{R}^n) = 0$ unless $q = n$, so the statement holds in this case.

Otherwise $r > n$ and we can apply Proposition 4.21 to obtain a tropical modification $\delta : V \to W$ with divisor $D \subset W$ such that $D$ and $W$ are matroidal cycles in $\mathbb{R}^{r-1}$. By the induction assumption, $H^p_{c\alpha}(D) = 0$ unless $q = n - 1$ and $H^p_{c\alpha}(W) = 0$ unless $q = n$. Applying the long exact sequence from Proposition 4.23 and replacing $H^p_{c\alpha}(V)$ with $H^p_{c\alpha}(W)$ we obtain the sequence

$$\ldots \to H^p_{c\alpha}-1(D) \to H^p_{c\alpha}(V) \to H^p_{c\alpha}(W) \to \ldots$$

by Proposition 4.22. By the vanishing of cohomologies for $W$ and $D$ we obtain that $H^p_{c\alpha}(V) = 0$ if $q \neq n$. This proves the lemma.

The next lemma gives a short exact sequence for the $(p,0)$-cohomology groups of matroidal cycles related by an open tropical modification. The statement of the lemma uses the contraction of superforms from Definition 2.9.

**Lemma 4.25.** Let $\delta : V \to W$ be an open tropical modification along a matroidal fan divisor $D \subset W$. Suppose also that $\delta$ is the restriction of a linear map on $\mathbb{R}^r \to \mathbb{R}^{r-1}$ whose kernel is generated by $e_i$, then

$$0 \to H^{p,0}(W) \xrightarrow{i^*} H^{p,0}(V) \xrightarrow{\langle \cdot, e_i \rangle_p} H^{p-1,0}(D) \to 0$$

is an exact sequence.

**Proof.** For a matroidal fan $V$ let $F^p(0)$ denote the value of the cellular sheaf $F^p$ from Definition 2.13 on $V$ at the origin of the fan. Then $H^{p,0}(V) = F^p(0)$ and this group is isomorphic to the $p$-th graded piece of the Orlik-Solomon algebra of the associated matroid [Sha11, Lemma 2.2.7], [Zha13, Theorem 4]. The pieces of the Orlik-Solomon algebras of the deletion and contraction of a matroid satisfy a short exact sequence similar to the one above [OT92, Theorem 3.65]. The first part of Remark 4.19 explains that $W$ is the matroidal cycle of a deletion of the matroid for $V$ and $D$ is the matroidal cycle of the contracted matroid. A direct comparison with this short exact sequence shows that the last map of the sequence above is given by the contraction by $e_i$ in the $p$-th component (see the proof of [Sha11, Lemma 2.2.7]). This proves the statement of the lemma.
The contraction map in the above proof is not induced by a map on the level of forms. However, for a closed \((p,0)\)-form \(\alpha\), the form \(\langle \alpha; e_i \rangle_p \in \mathcal{L}^{p-1,0}(V)\) is the restriction of a unique form \(\beta \in \mathcal{L}^{p-1,0}(V)\). We can restrict \(\beta\) to \(D\), since we can identify \(D\) with a subset of \(\overline{V}\) as in Remark 4.19. This is done using the identifications in Proposition 3.20.

**Lemma 4.26.** Let \(\delta: V \to W\) be an open matroidal tropical modification along a matroidal divisor \(D \subset W\). Then the diagram

\[
\begin{array}{cccccc}
0 & \to & H^{p,0}(W) & \overset{\partial}{\to} & H^{p,0}(V) & \overset{\langle \cdot; e_i \rangle_p}{\to} & H^{p-1,0}(D) & \to & 0 \\
0 & \to & H_c^{n-p,n}(W)^* & \overset{\partial D}{\to} & H_c^{n-p,n}(V)^* & \overset{g^*}{\to} & H_c^{n-p,n-1}(D)^* & \to & 0,
\end{array}
\]

which is obtained by the exact sequences in Proposition 4.23 and Lemma 4.25, is commutative.

**Proof.** Note that by Proposition 4.22 the diagram is equivalent to the one obtained by replacing \(W\) by \(\overline{V}\). Then the fact that the first square commutes is immediate. The map \(g: H_c^{n-p,n-1}(D) \to H_c^{n-p,n}(V)\) is the boundary operator in a long exact cohomology sequence. We recall its construction. For a closed superform \(\beta \in \mathcal{A}{^n}_c\), take any lift \(l(\beta) \in \mathcal{A}{^n}_c\) such that \(l(\beta)|_D = \beta\). Then \(d'(l(\beta))\) restricts to 0 on \(D\) and so it is a superform with compact support on \(V\). Then \(g(\beta)\) is given by the class of \(d'(l(\beta))\) in \(H_c^{n-p,n}(V)\). As usual this does not depend on the choice of \(l(\beta)\). We have to show that for all closed forms \(\alpha \in \mathcal{A}^p(\overline{V})\), \(\beta \in \mathcal{A}_c^{n-p,n}(V)\), and a lift \(l(\beta) \in \mathcal{A}_c^{n-p,n-1}(\overline{V})\) that we have

\[(-1)^p \int_V \alpha \wedge d''(l(\beta)) = (-1)^{n+p} \int_D \langle \alpha; e_i \rangle_p \wedge \beta.\]

In the above integral \(e_i\) is the coordinate direction of the modification. Let \(P\) be the piecewise linear function of the modification and \(P' = P - 1\). The graph of \(P'\) divides \(V\) into two subsets, one living above the graph and the other one below. Both of these subsets are the support of some polyhedral complexes, which we denote by \(C_1\) and \(C_2\), respectively. Equip all facets of both polyhedral complexes \(C_1\) and \(C_2\) with weight one. Note that \(\delta(\{C_2\}) \subset D\). We find a lift \(l(\beta) \in \mathcal{A}_c^{n-p,n-1}(\overline{V})\) such that \(l(\beta)|_{C_1} = (\delta|_{C_1})^*(\beta)\). Then we have

\[
\int_{C_1} \alpha \wedge d''(l(\beta)) = \int_{\hat{C}_1} \alpha \wedge d''(l(\beta)) + \int_{\hat{C}_2} \alpha \wedge d''(l(\beta)) = \int_{\hat{C}_1} \alpha \wedge d''(l(\beta)).
\]

By Stokes’ theorem in the version [Gub16, Proposition 3.5] and the Leibniz rule we have

\[
\int_{\hat{C}_1} \alpha \wedge d''(l(\beta)) = (-1)^p \int_{\partial \hat{C}_1} \alpha \wedge l(\beta).
\]

It follows from the proof of [Gub16, Theorem 3.8] that the boundary integral of \(\alpha \wedge l(\beta)\) over balanced codimension one faces vanishes. We further have that the unbalanced faces of \(C_1\) are precisely the ones in the polyhedral subspace \(D' := C_1 \cap \Gamma_{P'}(W) = C_2 \cap \Gamma_{P'}(W)\). The facets of the polyhedral subspace \(D'\) are equipped with weight one. Thus we obtain

\[
\int_{\partial \hat{C}_1} \alpha \wedge l(\beta) = \int_{D'} \langle \alpha \wedge l(\beta); e_i \rangle_n.
\]

Since \(l(\beta)|_{D'} = (\delta|_{D'})^*(\beta)\) and \(\delta(e_i) = 0\), we have that \(\langle l(\beta); e_i \rangle_{D'} = 0\) and therefore

\[
\int_{D'} \langle \alpha \wedge l(\beta); e_i \rangle_n = (-1)^{n-p} \int_{D'} \langle \alpha \wedge l(\beta); e_i \rangle_p = (-1)^{n-p} \int_{D'} \langle \alpha; e_i \rangle_p \wedge l(\beta).
\]

Altogether, we obtain

\[
\int_V \alpha \wedge d''(l(\beta)) = (-1)^n \int_{D'} \langle \alpha; e_i \rangle_p \wedge l(\beta).
\]
Denote by $F: W \to \Gamma^{p'}(W)$ the map into the graph of the function $P'$. Then there exist polyhedral structures $\mathcal{D}$ on $D$ and $\mathcal{D}'$ on $D'$, such that for each facet $\sigma$ of $\mathcal{D}$ the restriction $F|_{\sigma}$ is linear, the image $F(\sigma)$ is a facet of $\mathcal{D}'$, and each facet of $\mathcal{D}'$ is of this form. Then the inverse of $F|_{\sigma}$ is given by $\delta|_{\sigma'}$. Thus $\delta|_{\sigma'}$ is an isomorphism of rational polyhedra for each $\sigma' \in \mathcal{D}'$. Since we have that $\delta^*$ preserves $\langle \alpha; e_i \rangle$ and $\langle \delta|_{\mathcal{D}'} \rangle^* \beta = \langle \beta \rangle|_{\mathcal{D}'}$, we obtain

$$\int_{\mathcal{D}'} \langle \alpha; e_i \rangle_p \wedge l(\beta) = \int_{\mathcal{D}} \langle \alpha; e_i \rangle_p \wedge \beta,$$

which concludes the proof.

**Proposition 4.27.** Let $V \subset \mathbb{R}^r$ be a matroidal cycle. Then $V$ has Poincaré duality.

**Proof.** Let $n$ be the dimension of $V$. We perform induction on $r$. The base case $r = 0$ is obvious. For the induction step, we have two cases, these being $n = r$ and $n < r$. If $n = r$, then $V = \mathbb{R}^n$ and we have

$$H^{p,q}(\mathbb{R}^n) = \bigwedge^p \mathbb{R}^{n*} \otimes H^q(\mathbb{R}^n) \quad \text{and} \quad H^{p,q}_c(\mathbb{R}^n) = \bigwedge^p \mathbb{R}^{n*} \otimes H^q_c(\mathbb{R}^n),$$

where $H^q$, respectively $H^q_c$, denote the usual de Rham cohomology. Therefore, $H^{p,0}(\mathbb{R}^n) = 0$ and $H^{n-p,0}(\mathbb{R}^n) = 0$ unless $q = 0$. Otherwise $H^{p,0}(\mathbb{R}^n) = \bigwedge^p \mathbb{R}^{n*}$, $H^{n-p,0}(\mathbb{R}^n) = \bigwedge^{n-p} \mathbb{R}^{n*}$, and the PD map is just $(-1)^p$ times the map induced by the canonical pairing $\bigwedge^p \mathbb{R}^{n*} \times \bigwedge^{n-p} \mathbb{R}^{n*} \to \bigwedge^n \mathbb{R}^{n*} \cong \mathbb{R}$. Since this pairing is non-degenerate the PD map is an isomorphism.

If $n < r$, then by Proposition 3.11 and Lemma 4.24 the only non-trivial case to check is when $q = 0$. In other words, that PD: $H^{p,0}(V) \to H^{n-p,0}(V)^*$ is an isomorphism. Consider an open matroidal tropical modification $\delta: V \to W$ along a matroidal divisor $D$. Now $D$ and $W$ have PD by the induction hypothesis, so that in the commutative diagram from Lemma 4.26 the vertical arrows on the left and right are isomorphisms. By the five lemma we obtain PD for $V \subset \mathbb{R}^r$ and the proposition is proven. □

The next two lemmas help to prove Proposition 4.30, which is analogous to Proposition 4.27 but for spaces of the form $V \times \mathbb{T}^r$ where $V$ is a matroidal cycle. We first relate the cohomologies of $V \times \mathbb{T}^r$, $V \times \mathbb{R}^r$, and $V$ by way of an exact sequence.

**Lemma 4.28.** Let $Y = V \times \mathbb{T}^r$ where $V$ is the support of a polyhedral fan. Then we have a short exact sequence

$$0 \longrightarrow H^{p,0}(Y \times \mathbb{T}) \longrightarrow H^{p,0}(Y \times \mathbb{R}) \longrightarrow \bigwedge^p L(Y) \longrightarrow 0$$

where $e_i$ is the coordinate of $\mathbb{R}$ in $Y \times \mathbb{R}$.

**Proof.** We use the explicit calculation in Proposition 3.20. First this shows that none of the cohomology groups in the statement change when we replace $Y$ by $V$, thus we assume $Y = V$. We begin by showing that the sequence

$$0 \longrightarrow \sum_{\sigma \in Y} \bigwedge^{p-1} L(\sigma) \longrightarrow \sum_{\sigma \in Y} \bigwedge^p L(\sigma \times \mathbb{R}) \longrightarrow \sum_{\sigma \in Y} \bigwedge^p L(\sigma) \longrightarrow 0$$

is exact. The first map is clearly injective and the last map is clearly surjective. For exactness in the middle notice the composition of the maps is certainly zero and that any element $v \in \sum_{\sigma \in Y} \bigwedge^p L(\sigma \times \mathbb{R})$ can be written as $v = \sum_{\sigma \in Y} v_\sigma + \left( \sum_{\sigma \in Y} v'_\sigma \right) \wedge e_i$ for $v_\sigma \in \bigwedge^p L(\sigma)$ and $v'_\sigma \in \bigwedge^{p-1} L(\sigma)$. If $v$ maps to zero then it is of the form $\sum_{\sigma \in Y} v'_\sigma \wedge e_i$ and thus in the image of $\bigwedge^p e_i$. This proves exactness of that sequence.

Identifying $L^p$ with $H^{p,0}$ and dualizing the sequence (2) completes the proof. □
Lemma 4.29. Let \( Y = V \times \mathbb{T}^r \) be pure \( n \)-dimensional, where \( V \subset \mathbb{R}^s \) is a matroidal cycle, then the following diagram

\[
\begin{array}{ccc}
H^{p,0}(Y \times \mathbb{T}) & \xrightarrow{\cdot e_i} & H^{p-1,0}(Y) \\
\xrightarrow{\text{PD}} & & \xrightarrow{\text{PD}} \\
H^{n-p+1,n+1}_c(Y \times \mathbb{T})^* & \xrightarrow{g^*} & H^{n-p+1,n+1}(Y \times \mathbb{T})^*,
\end{array}
\]

which is obtained by the sequences in Proposition 4.23 and Lemma 4.28, commutes.

Proof. The proof follows exactly along the lines of the proof of the commutativity of the diagram in Lemma 4.26 for tropical modifications with \( Y \) replacing \( D \), \( Y \times \mathbb{R} \) replacing \( V \), \( Y \times \mathbb{T} \) replacing \( V \) and \( P^r \) being any constant function. \( \square \)

Proposition 4.30. Let \( Y = V \times \mathbb{T}^r \) for a matroidal cycle \( V \subset \mathbb{R}^s \). Then \( Y \) has Poincaré duality.

Proof. We perform an induction on \( r \). The base case \( r = 0 \) follows from Proposition 4.27. For the induction step we have to show that if \( Y \) has PD then \( Y \times \mathbb{T} \) also has PD. Since \( Y \) is a basic open subset, \( H^{p,q}(Y) = 0 \) unless \( q = 0 \) by Proposition 3.11. Since \( Y \) has PD, we have \( H^{p,q}(Y) = 0 \) unless \( q = n = \dim(Y) \). Note also that \( Y \times \mathbb{R} = V \times \mathbb{R} \times \mathbb{T}^r \) and so this space has PD. Therefore \( H^{p,q}(Y \times \mathbb{R}) = 0 \) unless \( q = n+1 = \dim(Y \times \mathbb{R}) \). Then the sequence from Proposition 4.23 yields that \( H^{p,q}_c(Y \times \mathbb{T}) = 0 \) if \( q \neq n, n+1 \) and that the sequence

\[
0 \to H^{0,n}_c(Y \times \mathbb{T}) \to H^{0,n}_c(Y) \xrightarrow{f} H^{0,n+1}(Y \times \mathbb{R}) \to H^{0,n+1}_c(Y \times \mathbb{T}) \to 0
\]

is exact. By the commutativity of the second square of the diagram in Lemma 4.29, the map \( f \) is, up to sign, the dual map to \( \langle \cdot : e_i \rangle_p \). This can be seen once we apply PD for \( Y \times \mathbb{T} \) to identify \( H^{p,0}(Y \times \mathbb{T}) \cong H^{n-p+1,n+1}_c(Y \times \mathbb{T})^* \) and \( H^{p-1,0}(Y) \cong H^{n-p+1,n}_c(Y)^* \). Now \( \langle \cdot : e_i \rangle_p \) is surjective by Lemma 4.28, thus \( f \) is injective and we have \( H^{p,q}_c(Y \times \mathbb{T}) = 0 \) unless \( q = n+1 \). Since \( Y \times \mathbb{T} \) is a basic open subset, we also have \( H^{p,q}(Y \times \mathbb{T}) = 0 \) unless \( q = 0 \) by Proposition 3.11. Therefore, it remains to consider PD: \( H^{p,0}(Y \times \mathbb{T}) \to H^{n+1-p,n+1}_c(Y \times \mathbb{T}) \). Note that this is precisely the first vertical map in the diagram in Lemma 4.29. The respective first horizontal maps are injective by Lemma 4.28 and the sequence (3). Since the other vertical maps are isomorphisms, this shows that \( Y \times \mathbb{T} \) has PD. \( \square \)

The following technical lemma allows us to deduce Poincaré duality for basic open subsets of matroidal fans.

Lemma 4.31. Let \( V \subset \mathbb{R}^s \) be a matroidal cycle, \( Y = V \times \mathbb{T}^r \), and \( \Omega \) a basic open neighborhood of \((0, \ldots, 0, -\infty, \ldots, -\infty)\). Then there are canonical isomorphisms

\[
H^{p,q}(Y) \cong H^{p,q}(\Omega) \quad \text{and} \quad H^{p,q}_c(\Omega) \cong H^{p,q}_c(Y)
\]

which are induced by restriction and inclusion of superforms. In particular \( \Omega \) has PD.

Proof. For \( H^{p,q} \) the statement follows immediately from the explicit calculation done in Proposition 3.20. For cohomology with compact support, we first see that there is a homeomorphism between \( \Omega \) and \( Y \) which respects strata of \( \mathbb{T}^r \) and the polyhedra for any fan polyhedral structure on \( Y \). This homeomorphism induces an isomorphism of tropical cohomology with compact support and therefore \( H^{p,q}_c(\Omega) \cong H^{p,q}_c(Y) \) by Theorem 3.22. By PD for \( Y \) these cohomology groups with compact support are finite dimensional, thus it is sufficient to show that inclusion of superforms induces a surjective map on cohomology. Again by PD for \( Y \) this is trivial if \( q \neq n = \dim(Y) \).

For \( q = n \), choose a basis \( \alpha_1, \ldots, \alpha_k \) of \( H^{0,0}_c(Y) \). By PD for \( Y \) there exist \( \omega_1, \ldots, \omega_k \in H^{n-p,n}_c(Y) \) such that \( f_Y \alpha_i \wedge \omega_j = \delta_{ij} \). For surjectivity of \( H^{n-p,n}_c(\Omega) \to H^{n-p,n}_c(Y) \) it is sufficient to show that there exist \( \beta_1, \ldots, \beta_k \in H^{n-p,n}_c(\Omega) \) such that \( f_Y \alpha_i \wedge \beta_j = 0 \) if and only if \( i = j \). Let \( B \) be the union of the supports of all \( \omega_i \). Take \( C \in \mathbb{R}_{>0} \) and \( v \in \mathbb{R}^r \) such that \( B \subset C \cdot \Omega + v \). Define

\[
f_B: f_Y \alpha_i \wedge \omega_j \rightarrow f_Y \alpha_i \wedge \beta_j
\]
$F$ to be the extended affine map given by $w \mapsto C \cdot w + v$ and set $\beta_i := F^*(\omega_i) \in \mathcal{A}^p_q(\Omega)$ for all $i$. Since $\alpha_i \in \mathcal{L}^p(Y)$ we have $F^*(\alpha_i) = C^p\alpha_i$ and

$$\int_Y \alpha_i \wedge \beta_j = \int_Y \alpha_i \wedge F^*(\omega_j) = C^{-p} \int_Y F^*(\alpha_i \wedge \omega_j) = C^{n-p}\delta_{ij},$$

where the last equality is given by the transformation formula [Gub16, 2.4]. This proves surjectivity and thus the lemma.

**Lemma 4.32.** Let $V$ be a matroidal cycle in $\mathbb{R}^s$, $Y = V \times \mathbb{T}^r$, and $\Omega$ a basic open subset for some polyhedral structure on $Y$. Then $\Omega$ has PD.

**Proof.** If $I \subset \{v\}$ is the maximal sedentarity among points of $\Omega$, then $\Omega$ is a basic open subset of a polyhedral structure on $V \times \mathbb{T}^{|I|} \times \mathbb{R}^{r-|I|}$ of maximal sedentarity. Let $x$ be a point in the relative interior of the minimal face of the basic open set $\Omega$. The star of any point in a matroidal fan is again a matroidal fan, see [AK06, Proposition 2]. Applying this fact to $V \times \mathbb{R}^{r-|I|}$ we obtain that, after translation of $x$ to the origin, the basic open set $\Omega$ is a neighborhood of $(0, \ldots, 0, -\infty, \ldots, -\infty)$ in the matroidal cycle $|\text{Star}_x(\Sigma)| \times \mathbb{T}^{|I|}$ for a matroidal fan $\Sigma$ with support $V$. Therefore, we can apply Lemma 4.31, and it follows that $\Omega$ has PD.

Finally, we obtain Poincaré duality for tropical manifolds:

**Theorem 4.33.** Let $X$ be an $n$-dimensional tropical manifold. Then the Poincaré duality map is an isomorphism for all $p, q$.

**Proof.** We write $A^{p,q}$ for the sheaf $U \mapsto \text{Hom}(A^{p,q}(U), \mathbb{R})$. Then $A^{p,q}$ is a sheaf, since $A^p$ is fine. Furthermore $A^{p,q}_c$ is a flasque sheaf, since for $U' \subset U$ the inclusion $A^{p,q}_c(U') \rightarrow A^{p,q}_c(U)$ is injective. We then obtain the commutative diagram

$$\begin{array}{c}
0 \rightarrow \mathcal{L}^p \rightarrow \mathcal{A}^{0,0} \rightarrow \mathcal{A}^{1,0} \rightarrow \cdots \\
\downarrow \text{id} \quad \quad \downarrow \text{PD} \quad \quad \downarrow \text{PD} \\
0 \rightarrow \mathcal{L}^p \rightarrow \mathcal{A}^{n-p,n} \rightarrow \mathcal{A}^{n-p,n-1} \rightarrow \cdots
\end{array}$$

and we have

$$H^q(A^{n-p,n}_c(U), d''^*) = (H_q(A^{n-p,n}_c(U), d''))^* = H^{p,q}_c(U)^*. $$

If we consider the sections of this diagram over a basic open subset, then the first row is exact by Proposition 3.11. By Lemma 4.32 the second row is also exact. This shows that both rows are exact sequences of sheaves on $X$. Thus we have a commutative diagram of acyclic resolutions of $\mathcal{L}^p$, thus PD induces isomorphisms on the cohomology of the complexes of global sections. This precisely means that $X$ has PD.

When $X$ is a compact tropical manifold, the above theorem immediately implies the following.

**Corollary 4.34.** Let $X$ be a compact tropical manifold of dimension $n$. Then

$$\text{PD}: H^{p,q}(X) \rightarrow H^{n-p,n-q}(X)^*$$

is an isomorphism for all $p, q$.

**References**

[AB15] Karim Alexander Adiprasito and Anders Björner. Filtered geometric lattices and lefschetz section theorems over the tropical semiring. 2015. http://arxiv.org/abs/1401.7301.

[AHK15] Karim Alexander Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries. 2015. http://arxiv.org/abs/1511.02888.

[AK06] Federico Ardila and Caroline Klivans. The Bergman complex of a matroid and phylogenetic trees. J. Comb. Theory Ser. B, 96(1):38–49, 2006.
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