Kneser-Poulsen conjecture for a small number of intersections\

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Abstract

The Kneser-Poulsen conjecture says that if a finite collection of balls in the Euclidean space $E^d$ is rearranged so that the distance between each pair of centers does not get smaller, then the volume of the union of these balls also does not get smaller. In this paper we prove that if in the beginning configuration the intersection of any two balls has common points with no more than $d+1$ other balls, then the conjecture holds.

1 Introduction

Let $|| \cdot ||$ be the Euclidean norm. Let $p = (p_1, \ldots, p_N)$ and $q = (q_1, \ldots, q_N)$ be two configurations of $N$ points, where each $p_i \in E^d$ and each $q_i \in E^d$. If for all $1 \leq i < j \leq N$, $|p_i - p_j| \leq |q_i - q_j|$, we say that $q$ is an expansion of $p$ and $p$ is a contraction of $q$. We denote by $B_d(p_i, r_i)$ the closed $d$-dimensional ball of radius $r_i \geq 0$ in $E^d$ about the point $p_i$, and by $S_{d-1}(p_i, r_i)$ we denote the boundary of $B_d(p_i, r_i)$. Let $Vol_d$ represent the $d$-dimensional volume.

The following is a longstanding conjecture independently stated by Kneser [Kne55] in 1955 and Poulsen [Pou54] in 1954 for the case when $r_1 = \cdots = r_N$:

Conjecture 1.1. If $q = (q_1, \ldots, q_N)$ is an expansion of $p = (p_1, \ldots, p_N)$ in $E^d$, then for any vector of radii $r = (r_1, \ldots, r_N)$,

$$Vol_d \left( \bigcup_{i=1}^{N} B_d(p_i, r_i) \right) \leq Vol_d \left( \bigcup_{i=1}^{N} B_d(q_i, r_i) \right). \quad (1)$$

This conjecture is proved by K. Bezdek and R. Connelly in [BC02] only for the case when $d = 2$. References to related results as well as the history of this conjecture can be found in the same paper [BC02]. In the current paper we prove the following theorem:

Theorem 1.2. If in the initial configuration of closed balls, determined by their centers $p = (p_1, \ldots, p_N)$ in $E^d$ and radii $r = (r_1, \ldots, r_N)$, each pair of balls has common points with no more than $d+1$ other balls, then for any expansion $q = (q_1, \ldots, q_N) \subset E^d$ of the centers $p$ inequality (1) holds.

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As the limiting case of this theorem, we get the following corollary:

**Corollary 1.3.** If in the initial configuration of closed balls, determined by their centers \( p = (p_1, \ldots, p_N) \) in \( \mathbb{E}^d \) and radii \( r = (r_1, \ldots, r_N) \), each pair of balls has common interior points with no more than \( d + 1 \) other balls, then for any expansion \( q = (q_1, \ldots, q_N) \subset \mathbb{E}^d \) of the centers \( p \) inequality (1) holds.

## 2 Voronoi decomposition

Let \( p = (p_1, \ldots, p_N) \) be a configuration of points in \( \mathbb{E}^d \) with balls of radii \( r = (r_1, \ldots, r_N) \) centered at corresponding points of the configuration. Consider a one parameter family \( r(s) = (\sqrt{r^2_1 + s}, \ldots, \sqrt{r^2_k + s}, \ldots, \sqrt{r^2_N + s}) \) which coincides with the initial vector of radii, when \( s = 0 \).

The following sets are called (extended) nearest point Voronoi regions:

\[
C_{d,i}(p) = \{ p_0 \in \mathbb{E}^d \mid \text{for all } j, |p_0 - p_i|^2 - r_i^2(s) \leq |p_0 - p_j|^2 - r_j^2(s) \}.
\]

The next remark can be easily verified.

**Remark 2.1.** The sets \( C_{d,i}(p) \) do not depend on the parameter \( s \).

Now we consider so called truncated Voronoi regions \( C_{d,i}(p, s) = B_d(p_i, r_i(s)) \cap C_{d,i}(p) \), and for each \( i \neq j \), define the wall between two truncated Voronoi regions: \( W_{d-1,ij}(p, s) = C_{d,i}(p, s) \cap C_{d,j}(p, s) \).

Further we will be interested in the behavior of the function

\[
V_d(p, s) = \text{Vol}_d \left[ \bigcup_{i=1}^N B_d(p_i, r_i(s)) \right] = \sum_{i=1}^N \text{Vol}_d [C_{d,i}(p, s)].
\]

Consider a smooth (infinitely many times differentiable) motion \( p(t) = (p_1(t), \ldots, p_N(t)) \) of some configuration of \( N \) points in \( \mathbb{E}^d \). Let \( d_{ij} = |p_i(t) - p_j(t)| \), and let \( d'_{ij} \) be the \( t \)-derivative of \( d_{ij} \). The following is Csikós’s formula [Csí98] for the \( t \)-derivative of the function \( V_d(p(t), s) \).

**Theorem 2.2.** Let \( d \geq 2 \) and let \( p(t) \) be a smooth motion of a configuration of points in \( \mathbb{E}^d \) such that for each \( t \), all the points are pairwise distinct. Then the function \( V_d(p(t), s) \) is differentiable with respect to \( t \) and,

\[
\frac{d}{dt} V_d(p(t), s) = \sum_{1 \leq i < j \leq N} d'_{ij} \text{Vol}_{d-1} [W_{d-1,ij}(p(t), s)].
\]

## 3 Truncated polytopes

Let \( M \) be an \( n \)-dimensional Euclidian space where \( n \geq 2 \), and let \( p_0 \in M \) be a point in it.

**Definition 3.1.** We will call a local one parametric family of sets \( P_{M, p_0}(s) \subset M \) a convex truncated polytope in \( M \), if there exist \( r_0 > 0 \) and finitely many halfspaces \( H_1, \ldots, H_m \subset M \), such that
for each real parameter $s$ from a sufficiently small neighborhood of zero, the set $P_{M, p_0}(s)$ is the intersection of these halfspaces with the ball $B_n(p_0, \sqrt{r_0^2 + s}) \subset M$:

$$P_{M, p_0}(s) = B_n(p_0, \sqrt{r_0^2 + s}) \cap \left( \bigcap_{i=1}^{m} H_i \right).$$

Note that truncated Voronoi regions $C_{d,i}(p, s)$ are truncated polytopes in $\mathbb{E}^d$, and each wall $W_{d-1,i,j}(p, s)$ is a truncated polytope in the hyperplane $L_{ij}$ that contains the intersection of the spheres $S_{d-1}(p_i, r_i) \cap S_{d-1}(p_j, r_j)$. Indeed, $W_{d-1,i,j}(p, s)$ is the intersection of finitely many halfspaces in $L_{ij}$ intersected with a ball of radius $\sqrt{r_i^2 - h_{ij}^2 + s}$, where $h_{ij}$ is the distance from the point $p_i$ to the hyperplane $L_{ij}$.

The next lemma is slightly reformulated Corollary 6 from [BC02].

**Lemma 3.2.** Let $P_{M, p_0}(s) = B_{n+2}(p_0, \sqrt{r_0^2 + s}) \cap (\bigcap_{i=1}^{m} H_i)$ be a truncated polytope in an $(n+2)$-dimensional Euclidian space $M$, such that all the boundary hyperplanes $\partial H_1, \ldots, \partial H_m$ are orthogonal to some $n$-dimensional affine subspace $S \subset M$, and $p_0 \in S$. Then the following derivative exists and

$$\frac{d}{ds} \text{Vol}_{n+2} [P_{M, p_0}(s)] = \pi \text{Vol}_n [S \cap P_{M, p_0}(s)].$$

The set $S \cap P_{n, p_0}(r(s))$ is a truncated polytope in $S$, so by induction we get the following corollary:

**Corollary 3.3.** Let $P_{M, p_0}(s) = B_{n+2k}(p_0, \sqrt{r_0^2 + s}) \cap (\bigcap_{i=1}^{m} H_i)$ be a truncated polytope in an $(n + 2k)$-dimensional Euclidian space $M$, such that all the boundary hyperplanes $\partial H_1, \ldots, \partial H_m$ are orthogonal to some $n$-dimensional affine subspace $S \subset M$, and $p_0 \in S$. Then the following derivative exists and

$$\frac{d^k}{ds^k} \text{Vol}_{n+2k} [P_{M, p_0}(s)] = \pi^k \text{Vol}_n [S \cap P_{M, p_0}(s)].$$

Let $p$ and $q$ be configurations of centers that together with the vector of radii $r$ satisfy the conditions of Theorem 1.2. For each positive integer $k$ we can regard $\mathbb{E}^d$ as the subset $\mathbb{E}^d = \mathbb{E}^d \times \{0\} \subset \mathbb{E}^d \times \mathbb{E}^{2k} = \mathbb{E}^{d+2k}$. We chose $k$ to be sufficiently large, so that there exists a smooth motion $p(t) = (p_1(t), \ldots, p_N(t))$ defined for $t \in [0, 1]$ where $p_i(t) \in \mathbb{E}^{d+2k}$, $p(0) = p$, $p(1) = q$, and for each pair of indices $i \neq j$ the distance $|p_i(t) - p_j(t)|$ is non-decreasing as a function of $t$.

**Lemma 3.4.** Under conditions of Theorem 1.2, for each pair of indices $i \neq j$ and $l = 0, 1, \ldots, k$ the functions

$$\frac{\partial^l}{\partial s^l} \text{Vol}_{d+2k-1} [W_{d+2k-1, i,j}(p(t), s)]$$

are continuous and nonnegative in some fixed closed rectangle $(t, s) \in [0, 1] \times [-\epsilon, \epsilon]$.

**Proof.** The lemma is obvious for $l = 0$, since the volume of the wall is nonnegative and depends continuously on the variables $s$ and $t$.

By $L_{ij}(t)$ denote the hyperplane that contains the intersection of the spheres $S_{d+2k-1}(p_i(t), r_i) \cap S_{d+2k-1}(p_j(t), r_j)$. By assumption of Theorem 1.2 for $t = 0$ in the beginning ball configuration there are no more than $d + 1$ triple intersections that involve balls $B_{d+2k}(p_i(0), r_i)$ and
It is easy to check that since for each \( t > 0 \), \( p(t) \) is an expansion of \( p(0) \), new triple intersections cannot appear as we increase \( t \). Thus, for each \( t \) the wall \( W_{d+2k-1,ij}(p(t), s) \) is a truncated polytope in \( L_{ij}(t) \) such that it can be defined as a ball intersected with no more than \( d + 1 \) halfspaces. Since \( L_{ij}(t) \) is a \( (d + 2k - 1) \)-dimensional Euclidean space, then for each \( l = 1, \ldots, k \) there exists a \( (d + 2k - 2l + 1) \)-dimensional plane \( S_{ij,l}(t) \subseteq L_{ij}(t) \) that is orthogonal to all of the boundary hyperplanes of the wall \( W_{d+2k-1,ij}(p(t), s) \). We can choose \( S_{ij,l}(t) \) to depend continuously on \( t \).

By Corollary 3.3

\[
\frac{\partial^l}{\partial s^l} \text{Vol}_{d+2k-1} [W_{d+2k-1,ij}(p(t), s)] = \pi^{l-1} \frac{\partial}{\partial s} \text{Vol}_{d+2k-2l+1} [S_{ij,l}(t) \cap W_{d+2k-1,ij}(p(t), s)] .
\]

Let \( h_{ij}(t) \) denote the distance from the point \( p_i(t) \) to the hyperplane \( L_{ij}(t) \). Then \( r_{ij}(t, s) = \sqrt{r_i^2 - h_{ij}^2(t)} + s \) is the radius of the ball that intersects the halfspaces to form the truncated polytope \( W_{d+2k-1,ij}(p(t), s) \) in \( L_{ij}(t) \). Thus,

\[
\frac{\partial}{\partial s} \text{Vol}_{d+2k-2l+1} [S_{ij,l}(t) \cap W_{d+2k-1,ij}(p(t), s)] = \frac{\partial}{\partial r_{ij}} \text{Vol}_{d+2k-2l+1} [S_{ij,l}(t) \cap W_{d+2k-1,ij}(p(t), s)] \cdot \frac{\partial r_{ij}}{\partial s} = \frac{1}{2r_{ij}(t, s)} \frac{\partial}{\partial r_{ij}} \text{Vol}_{d+2k-2l+1} [S_{ij,l}(t) \cap W_{d+2k-1,ij}(p(t), s)].
\]

Note that \( \frac{\partial}{\partial r_{ij}} \text{Vol}_{d+2k-2l+1} [S_{ij,l}(t) \cap W_{d+2k-1,ij}(p(t), s)] \) is the perimeter of the spherical part of the truncated polytope \( S_{ij,l}(t) \cap W_{d+2k-1,ij}(p(t), s) \) in \( S_{ij,l}(t) \):

\[
\frac{\partial}{\partial r_{ij}} \text{Vol}_{d+2k-2l+1} [S_{ij,l}(t) \cap W_{d+2k-1,ij}(p(t), s)] = \text{Vol}_{d+2k-2l} [S_{ij,l}(t) \cap W_{d+2k-1,ij}(p(t), s) \cap S_{d+2k-1}(p_{ij}(t), r_{ij}(t, s))],
\]

where \( p_{ij}(t) \) is the orthogonal projection of the point \( p_i(t) \) onto the hyperplane \( L_{ij}(t) \). Since the perimeter is always nonnegative, this shows that the functions in (3) are nonnegative. Since the plane \( S_{ij,l}(t) \) and the wall \( W_{d+2k-1,ij}(p(t), s) \) depend continuously on \( s \) and \( t \), the functions in (3) are continuous as well.

## 4 Proof of Theorem 1.2

Note that the sets \( \bigcup_{i=1}^N B_d(p_i, r_i) \) and \( \bigcup_{i=1}^N B_d(q_i, r_i) \) are disjoint unions of truncated Voronoi regions, so we can apply Corollary 3.3 to them:

\[
\pi^k(V_d(q, s) - V_d(p, s)) = \frac{\partial^k}{\partial s^k} V_{d+2k}(q, s) - \frac{\partial^k}{\partial s^k} V_{d+2k}(p, s) = \frac{\partial^k}{\partial s^k} \int_0^1 \frac{\partial}{\partial t} V_{d+2k}(p(t), s) dt.
\]

Now by applying Csikós’s formula (Theorem 2.2) we get that

\[
\pi^k(V_d(q, s) - V_d(p, s)) = \frac{\partial^k}{\partial s^k} \int_0^1 \sum_{1 \leq i < j \leq N} d_{ij}^* \text{Vol}_{d+2k-1} [W_{d+2k-1,ij}(p(t), s)] dt.
\]
Since the functions in (3) are continuous as shown in Lemma 3.4, the last expression can be written as
\[ \pi^k (V_d(q, s) - V_d(p, s)) = \int_0^1 \sum_{1 \leq i < j \leq N} d'_{ij} \frac{\partial^k}{\partial s^k} \text{Vol}_{d+2k-1} [W_{d+2k-1,ij}(p(t), s)] \, dt. \]
According to Lemma 3.4, the function inside the integral is nonnegative, so the whole expression is nonnegative. This proves the theorem.

References

[BC02] Károly Bezdek and Robert Connelly. Pushing disks apart—the Kneser-Poulsen conjecture in the plane. J. Reine Angew. Math., 553:221–236, 2002.

[Csi98] B. Csikós. On the volume of the union of balls. Discrete Comput. Geom., 20(4):449–461, 1998.

[Kne55] Martin Kneser. Einige Bemerkungen über das Minkowskische Flächenmass. Arch. Math. (Basel), 6:382–390, 1955.

[Pou54] Ebbe Thue Poulsen. Problem 10. Math. Scand., 2:346, 1954.