THE DIMENSION OF VECTOR-VALUED MODULAR FORMS OF INTEGER WEIGHT

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Abstract. We present a dimension formula for spaces of vector-valued modular forms of integer weight in case the associated multiplier system has finite image, and discuss the weight distribution of the module generators of holomorphic and cusp forms, as well as the duality relation between cusp forms and holomorphic forms for the contragredient.

1. Introduction

The classical theory of scalar modular forms [1, 16, 17] has been a major theme of mathematics in the last centuries. Its applications are numerous, ranging from number theory to topology and mathematical physics, a showpiece being the mathematics involved in the proof of Fermat’s Last Theorem [7]. An important tool in the applications of the theory is the explicit description of the different spaces of modular forms, which allows to identify highly transcendental functions via their analytic and transformation properties. In particular, a major result is the dimension formula for spaces of holomorphic and cusp forms, which allows to determine explicit bases for these spaces, and describe the involved algebraic structures very precisely [18, 24].

While the need for a theory of vector-valued forms, i.e. holomorphic maps from the complex upper half-plane into a linear space that transform according to some nontrivial (projective) representation of the modular group $SL_2(\mathbb{Z})$, has been recognized long ago, its systematic development has begun only recently [15, 4, 21, 5]. The importance of vector-valued modular forms for mathematics lies, besides the intrinsic interest of the subject, in the fact that important classical problems may be reduced to the study of suitable vector-valued forms, like the theory of Jacobi forms [9] or of scalar modular forms for finite index subgroups [23]; from a modern perspective, trace functions of vertex operator algebras [12, 14] satisfying suitable restrictions also provide important examples of vector-valued modular forms [26]. From the point of view of theoretical physics, vector-valued modular forms play an important role in string theory [13, 22] and two-dimensional conformal field theory [11], as the basic ingredients (chiral blocks) of torus partition functions and other correlators.

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The above connections justify amply the interest in obtaining explicit expressions for the dimension of spaces of vector-valued modular forms. Such results do indeed exist in the literature [8, 19], mostly based on the pioneering work [25]; while the latter, relying on the Eichler-Selberg trace formula, provides a closed expression for the dimensions, it doesn’t give a constructive procedure for determining explicit bases, which is a serious drawback from the point of view of many applications. The present paper offers an alternative approach, based on the results of [4, 5], for computing the dimension of various spaces of vector-valued modular forms, which is conceptually simpler, and can be modified easily to provide effective procedures for computing explicit bases.

2. Vector-valued modular forms

Let $V$ denote a finite dimensional linear space, $\rho : \Gamma \rightarrow \text{GL}(V)$ a representation of $\Gamma = \text{SL}_2(\mathbb{Z})$ on $V$, and $w$ an integer. A (vector-valued) modular form of weight $w$ with multiplier $\rho$ is a map $X : \mathcal{H} \rightarrow V$ that is holomorphic everywhere in the upper half-plane $\mathcal{H} = \{ \tau \mid \text{Im}\tau > 0 \}$, and transforms according to the rule

$$(2.1) \quad X \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^w \rho \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) X(\tau)$$

for $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$.

A form is called weakly holomorphic if it has at worst finite order poles in the limit $\tau \rightarrow i\infty$, i.e. its Puiseux-expansion in terms of the local uniformizing parameter $q = \exp(2\pi i\tau)$ has only finitely many terms with negative exponents; it is holomorphic, respectively a cusp form if it is bounded (resp. vanishes) as $\tau \rightarrow i\infty$, meaning that its Puiseux-expansion contains only non-negative (resp. positive) powers of $q$. We’ll denote by $M_w(\rho)$ the (in general infinite dimensional) linear space of weakly holomorphic forms, and by $M_w(\rho)$ and $S_w(\rho)$ the subspaces of holomorphic and cusp forms; clearly, we have the inclusions $S_w(\rho) < M_w(\rho) < M_w(\rho)$. An obvious but important observation is that

$$(2.2) \quad \dim M_w(\rho_1 \oplus \rho_2) = \dim M_w(\rho_1) + \dim M_w(\rho_2)$$

for any two representations $\rho_1$ and $\rho_2$, and a similar decomposition holds for the spaces of holomorphic and cusp forms, which implies that

$$(2.3) \quad \dim M_w(\rho_1 \oplus \rho_1) = \sum_i \dim M_k(\rho_i) , \quad \dim S_w(\rho_1 \oplus \rho_1) = \sum_i \dim S_k(\rho_i) .$$

Note that we recover the classical theory of (scalar) modular forms of $\text{SL}_2(\mathbb{Z})$ when $\rho = \rho_0$ is the trivial (identity) representation. In this case the weight should be an even integer for nontrivial forms to exist, which is non-negative (resp. positive) for holomorphic (resp. cusp

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1Except for [3], which anticipates the results of the present paper.
forms). By a well known result [1, 24], the spaces $M_{2k}(\rho_0)$ and $S_{2k}(\rho_0)$ are all finite dimensional: $M_0(\rho_0)$ consists of constants; $M_2(\rho_0)$ is empty, while $M_4(\rho_0)$ and $M_6(\rho_0)$, each having dimension 1, are spanned by the Eisenstein series

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

and

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

where $\sigma_k(n) = \sum_{d \mid n} d^k$ is the $k^{th}$ power sum of the divisors of $n$. What is more, any holomorphic form may be expressed uniquely as a bivariate polynomial in the Eisenstein series $E_4(q)$ and $E_6(q)$, in other words

$$M = \bigoplus_{k=0}^{\infty} M_k(\rho_0) = \mathbb{C}[E_4, E_6]$$

as graded rings. On the other hand, there are no cusp forms of weight less than 12, while $S_{12}(\rho_0)$ is spanned by the discriminant form

$$\Delta(q) = \frac{1}{1728} (E_4(q)^3 - E_6(q)^2) = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

and any cusp form of weight $k \geq 12$ is the product of $\Delta(q)$ with a holomorphic form of weight $k-12$. Finally, the ring $M_0(\rho_0)$ of scalar weakly-holomorphic forms of weight 0, which we shall denote simply by $M_0$ in the sequel, coincides with the univariate polynomial algebra $\mathbb{C}[J]$ generated by the Hauptmodul (the trace function of the Moonshine module [6, 12])

$$J(q) = \frac{E_4(q)^3}{\Delta(q)} - 744 = q^{-1} + 196884q + \cdots,$$

and each $M_{2k}(\rho_0)$ is a module over $M_0$ generated by a single element.

Since multiplying a holomorphic form $X(\tau) \in M_k(\rho)$ with a scalar holomorphic form $f(\tau) \in M_{2n}(\rho_0)$ results in a new holomorphic form $f(\tau)X(\tau) \in M_{k+2n}(\rho)$, and the same is true for cusp forms, the direct sums $M(\rho) = \bigoplus_{k=0}^{\infty} M_k(\rho)$ and $S(\rho) = \bigoplus_{k=0}^{\infty} S_k(\rho)$ are (graded) modules over the ring $M$ of holomorphic scalar modular forms. An important result [20] states that these are free modules of rank $d$. An interesting question in this respect is to determine the weight distribution of a set of free generators, which may be answered by considering the Hilbert-Poincaré series of these modules [10], i.e. the generating functions $M_\rho(z) = \sum_k \dim M_k(\rho) z^k$ and $S_\rho(z) = \sum_k \dim S_k(\rho) z^k$: the number of independent generators of weight $k$ equals the coefficient of $z^k$ in the finite polynomials $(1 - z^4)(1 - z^6) M_\rho(z)$ and $(1 - z^4)(1 - z^6) S_\rho(z)$.
We’ll call a representation \( \rho : \Gamma \to \text{GL}(V) \) even in case \( \rho \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \text{id}_V \), and odd if \( \rho \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\text{id}_V \). Any representation \( \rho \) may be decomposed uniquely into a direct sum \( \rho = \rho_+ \oplus \rho_- \) of even and odd representations, and any irreducible representation is either even or odd. It follows from Eq. (2.1) that for an even (resp. odd) representation \( \rho \) there are no nontrivial forms of odd (resp. even) weight. Combining this result with Eq. (2.2), one gets at once that

\[
\dim M_k(\rho) = \begin{cases} 
\dim M_k(\rho_+) & \text{if } k \text{ is even,} \\
\dim M_k(\rho_-) & \text{if } k \text{ is odd,} 
\end{cases}
\]

and a similar result for cusp forms. This result shows that it is enough to treat separately purely even and odd representations, the general case can be reduced to these.

Since the discriminant form \( \Delta(\tau) \) does not vanish on the upper half-plane \( \mathbb{H} \), its 12th root \( \varsigma(\tau) = q^{1/12} \prod_{n=1} \left(1 - q^n\right)^2 \) (the square of Dedekind’s eta function) is well-defined and holomorphic on \( \mathbb{H} \), with an algebraic branch point at the cusp \( \tau = i\infty \). Moreover, \( \varsigma(\tau) \) is a weight 1 cusp form with multiplier \( \kappa \), where \( \kappa \) denotes the one dimensional representation of \( \text{SL}_2(\mathbb{Z}) \) for which

\[
\kappa \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i \\
\kappa \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \exp\left(\frac{4\pi i}{3}\right)
\]

It does follow that, for any representation \( \rho \) and any form \( \mathcal{X} \in M_w(\rho) \), one has \( \varsigma(\tau)^k \mathcal{X}(\tau) \in M_{w+k}(\rho \otimes \kappa^k) \) for all integers \( k \in \mathbb{Z} \); in other words, one has an injective map

\[
\varpi_k : M_w(\rho) \to M_{w+k}(\rho \otimes \kappa^k) \\
\mathcal{X}(\tau) \mapsto \varsigma(\tau)^k \mathcal{X}(\tau)
\]

The map \( \varpi_k \) relates forms of different weights with a slightly different multiplier. Note that, since \( \varsigma(\tau) \in S_1(\kappa) \) is a cusp form, multiplication by a positive power of \( \varsigma(\tau) \) takes a holomorphic form into a cusp form, i.e. \( \varpi_k(M_w(\rho)) < S_{w+k}(\rho \otimes \kappa^k) \) for \( k > 0 \), in particular

\[
\dim M_w(\rho) \leq \dim S_{w+k}(\rho \otimes \kappa^k).
\]

The idea underlying most of what follows is that the injectivity of the weight-shifting map \( \varpi_k \) allows to reduce the study of forms of arbitrary weights to that of forms of weight 0. For example, the space \( M_w(\rho) \) of weight \( w \) holomorphic forms may be characterized through

\[\text{$^2$}$\kappa$ generates the group of linear characters of $\text{SL}_2(\mathbb{Z})$, which is cyclic of order 12; moreover, $\kappa$ is an odd representation, and tensoring with $\kappa$ takes an even representation into an odd one and vice versa.}
its image $\mathcal{M}_w^\rho (\rho \otimes \tau^w) = \mathcal{M}_w^\rho (\mathcal{M}_w (\rho)) < \mathcal{M}_0 (\rho \otimes \tau^w)$. This means that the basic objects of study are the spaces $\mathcal{M}_0 (\rho)$, together with their various subspaces. Fortunately, the structure of $\mathcal{M}_0 (\rho)$ is pretty well understood and under control; let’s shortly review the relevant results.

To start with, we make an important restriction on the representation $\rho$: from now on, we require that $\rho$ has finite image (equivalently, that its kernel be of finite index). While this requirement could seem too restrictive at first sight, it is satisfied in most cases of interest: just to cite an example relevant to physics, the representation describing the modular properties of the chiral characters of a Rational Conformal Field Theory has a kernel of finite index $[2]$. Moreover, while one may develop a theory for more general representations $\rho$, both the formulation and the proof of the relevant results becomes much more cumbersome, and lacks the elegance of the results to be presented.

The benefits of requiring the image of $\rho$ to be finite are numerous, the most important being:

1. $\rho$ is completely reducible (by Maschke’s theorem), i.e. it can be decomposed into a direct sum of irreducible representations;
2. the operator $\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ can be diagonalized, and its eigenvalues are roots of unity (since it is of finite order);
3. the kernel of $\rho$ uniformizes a finite sheeted cover of the modular curve $H/\Gamma \cong \mathbb{C}P^1$.

Let’s now turn to the properties of $\mathcal{M}_0 (\rho)$ (recall that $\rho$ is supposed to be even and of finite image). The basic observation is that the product $f(\tau) X(\tau)$ of a weakly holomorphic form $X(\tau) \in \mathcal{M}_0 (\rho)$ with a scalar form $f(\tau) \in \mathcal{M}_0$ is again a weakly holomorphic form belonging to $\mathcal{M}_0 (\rho)$: in other words, $\mathcal{M}_0 (\rho)$ is an $\mathcal{M}_0$-module, what is more, it is a torsion free module. Taking into account the fact that $\mathcal{M}_0$ is the univariate polynomial algebra $\mathbb{C}[J]$ generated by the Hauptmodul, this means that actually $\mathcal{M}_0 (\rho)$ is a free module $[10]$, whose rank equals the dimension $d$ of the representation $\rho$. This means that there exists forms $X_1, \ldots, X_d \in \mathcal{M}_0 (\rho)$ that freely generate $\mathcal{M}_0 (\rho)$ as an $\mathcal{M}_0$-module, i.e. any weakly holomorphic form $X \in \mathcal{M}_0 (\rho)$ may be decomposed uniquely into a sum

$$X(\tau) = \sum_{i=1}^d \varphi_i (\tau) X_i (\tau) ,$$

where the coefficients $\varphi_1, \ldots, \varphi_d \in \mathcal{M}_0$ are weight 0 weakly holomorphic scalar forms: since $\mathcal{M}_0 = \mathbb{C}[J]$, the coefficients may be considered as univariate polynomials in the Hauptmodul $J(\tau)$.

Since, by assumption, $\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ can be diagonalized, there exists a diagonalizable operator $\Lambda$ (called the exponent matrix), such that

$$\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp (2\pi i \Lambda) .$$
Note that $\Lambda$ is far from unique, its eigenvalues being only determined up to integers. Taking into account the transformation rule Eq. \((2.1)\), it is clear that for all $X \in M_0(\rho)$ the expression $\exp(-2\pi i \Lambda \tau) X(\tau)$ is periodic in $\tau$ with period 1, hence

\[
q^{-\Lambda} X(q) = \sum_n X[n] q^n
\]

for some coefficients $X[n] \in V$, with only finitely many negative powers of $q$ on the right hand side of Eq. \((2.15)\). The sum

\[
P_{\Lambda} X = \sum_{n<0} X[n] q^n
\]

of these negative powers is the ($\Lambda$-)principal part of the form $X(\tau)$; clearly, it depends on the actual choice of $\Lambda$.

An important result is that one may always choose $\Lambda$ such that the corresponding principal part map $P_{\Lambda}$ is bijective \[5\], i.e. any form is uniquely determined by its principal part, and any sum $\sum_{n<0} X[n] q^n$ is the principal part of some form $X \in M_0(\rho)$. A necessary condition for the bijectivity of $P_{\Lambda}$ is the relation

\[
\text{Tr} \, \Lambda = d - \frac{\alpha}{2} - \frac{\beta_1 + 2 \beta_2}{3},
\]

where the integers $\alpha$, $\beta_1$ and $\beta_2$ are important numerical characteristics of the representation $\rho$, termed collectively its signature: $\alpha$ denotes the multiplicity of $-1$ as an eigenvalue of $\rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (which is an involution, since $\rho$ is even), while $\beta_1$ and $\beta_2$ denote the multiplicities of $\exp \left( \frac{2 \pi i}{3} \right)$ and $\exp \left( \frac{4 \pi i}{3} \right)$ as eigenvalues of $\rho \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. Note that the signature can be determined through the relations

\[
\text{Tr} \, \rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = d - 2\alpha,
\]

\[
\text{Tr} \, \rho \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = d - \frac{3}{2} (\beta_1 + \beta_2) + i \frac{\sqrt{3}}{2} (\beta_1 - \beta_2).
\]

3. **The general dimension formula**

Let’s consider an even irreducible representation $\rho: \text{SL}_2(\mathbb{Z}) \to \text{GL}(V)$ having finite image, an integer $k$ and an exponent matrix $\Lambda$ for which the principal part map $P_{\Lambda}$ is bijective; in particular, $\Lambda$ has to satisfy the trace formula Eq. \((2.17)\). Let $[x]$ denote the integer part of $x \in \mathbb{R}$, i.e. the largest integer not exceeding $x$, and for a (diagonalizable) operator $A$, let $\text{Tr} \, [A]$ denote the sum of the integer parts of its eigenvalues. The

\[\text{Recall that } q = \exp(2\pi i \tau) \text{ is the uniformizing parameter at } \tau = i \infty.\]
overdetermined, having no nontrivial solutions at all, and
dimensions than nonvanishing coefficients, then the resulting linear system is

dimensions than variables, and
Putting all this together, we get Eq. (3.1)

\[
\dim M_k(\rho \otimes \mathcal{X}^k) = \max \left(0, \text{Tr} \left[ \Lambda + \frac{k}{12} \right] \right)
\]

\[
\dim S_k(\rho \otimes \mathcal{X}^k) = \max \left(0, -\text{Tr} \left[ 1 - \Lambda - \frac{k}{12} \right] \right).
\]

Let's see how the above result comes about. The first observation is that, thanks to the injectivity of the weight shifting map \( w_{-k} \), one has \( \dim M_k(\rho \otimes \mathcal{X}^k) = \dim M_k^{\circ}(\rho) \) and \( \dim S_k(\rho \otimes \mathcal{X}^k) = \dim S_k^{\circ}(\rho) \). By definition, \( M_k^{\circ}(\rho) \) (resp. \( S_k^{\circ}(\rho) \)) consists of those weakly holomorphic forms \( \mathcal{X}(\tau) \in \mathcal{M}_0(\rho) \) for which \( \zeta(\tau)^k \mathcal{X}(\tau) \) remains bounded (resp. vanishes) as \( \tau \to \infty \), i.e. for which

\[
q^{\frac{k}{12} + \Lambda + n} \mathcal{X}[n]
\]
tends to a finite limit (resp. vanishes) for each \( n \) as \( q \to 0 \), cf. Eq. (2.15).

Since the principal part map \( \mathcal{P}_\Lambda \) is bijective by assumption, any form \( \mathcal{X} \in \mathcal{M}_0(\rho) \) is completely determined by its expansion coefficients \( \mathcal{X}[n] \) with \( n < 0 \); in particular, for \( n \geq 0 \) the expansion coefficients \( \mathcal{X}[n] \) are linear expressions in the coefficients \( \mathcal{X}[m] \) with \( m < 0 \). Choosing a basis in which \( \Lambda \) is diagonal (this is always possible, thanks to our assumptions on \( \rho \)), and denoting by \( \mathcal{X}_i \) the component of \( \mathcal{X} \) corresponding to the eigenvalue \( \Lambda_i \), the condition for \( \mathcal{X} \) belonging to \( M_k^{\circ}(\rho) \) (resp. \( S_k^{\circ}(\rho) \)) is that \( \mathcal{X}_i[n] = 0 \) provided \( \frac{k}{12} + \Lambda_i + n \) is negative (resp. non-positive).

Let's observe that these conditions constitute a linear system of equations in the variables \( \mathcal{X}_j[m] \) with \( m < 0 \), and holomorphic (resp. cusp) forms are in one-to-one correspondence with solutions of this system.

Let's consider the quantity \( \mu_i = \frac{k}{12} + \Lambda_i \). If \( \mu_i > 0 \), there are exactly \( [\mu_i] \) (resp. \( -[1-\mu_i] \)) negative integers \( n \) for which \( \mu_i + n \) is non-negative (resp. positive), and for these values of \( n \) the corresponding \( \mathcal{X}_i[n] \) may be nonvanishing according to the above. On the other hand, for \( \mu_i \leq 0 \) not only the \( \mathcal{X}_i[n] \)-s with \( n < 0 \), but also the first \( -[\mu_i] \) (resp. \( [1-\mu_i] \)) components with \( n \geq 0 \) have to vanish: the later, being linear expressions in the \( \mathcal{X}_j[m] \)-s with \( m < 0 \), supply us with \( -[\mu_i] \) (resp. \( [1-\mu_i] \)) linear relations on the coefficients of a holomorphic (resp. cusp) form. Subtracting the total number \( r \) of relations from the number \( m \) of possible nonvanishing coefficients, we get a total of \( m-r = \sum_i \left[ \frac{k}{12} + \Lambda_i \right] = \text{Tr} \left[ \Lambda + \frac{k}{12} \right] \) (resp. \( m-r = -\text{Tr} \left[ 1 - \Lambda - \frac{k}{12} \right] \)) free coefficients by the above reasoning. If \( m < r \), i.e. there are more relations than nonvanishing coefficients, then the resulting linear system is overdetermined, having no nontrivial solutions at all, and \( \dim M_k^{\circ}(\rho) = 0 \) (resp. \( \dim S_k^{\circ}(\rho) = 0 \)). On the other hand, for \( m \geq r \) there are less relations than variables, and \( \dim M_k^{\circ}(\rho) = m-r \) (resp. \( \dim S_k^{\circ}(\rho) = m-r \)).

Putting all this together, we get Eq. (3.1). Note that the restriction to irreducible \( \rho \) is important, since the above argument assumes that the representation space \( V \) is a minimal (nontrivial) invariant subspace for
ρ; it could very well happen for a reducible representation that \( m \geq r \), but for some subrepresentation there are more relations than nonvanishing coefficients, with the result that the relevant dimension is strictly less than \( m - r \).

While Eq.\((3.1)\) solves the original problem, it still needs some elaboration. Indeed, one would like a formula expressing \( \dim M_k(\rho) \) and \( \dim S_k(\rho) \) as a function of \( k \) for fixed \( \rho \). Of course, it is trivial to arrive to such an expression from Eq.\((3.1)\), by simply replacing the representation \( \rho \) with \( \rho \otimes \mathbb{Z}^{-k} \) (note that, since we have defined exponent matrices for even representations only, this makes sense for even \( k \) only in case \( \rho \) is even, and for odd \( k \) only if \( \rho \) is odd), giving

\[
\begin{align*}
\dim M_k(\rho) &= \max \left( 0, \operatorname{Tr} \left[ \Lambda^{(k)} + \frac{k}{12} \right] \right), \\
\dim S_k(\rho) &= \max \left( 0, -\operatorname{Tr} \left[ 1 - \Lambda^{(k)} - \frac{k}{12} \right] \right),
\end{align*}
\]

with \( \Lambda^{(k)} \) denoting the exponent matrix of \( \rho \otimes \mathbb{Z}^{-k} \).

The problem with Eq.\((3.3)\) is twofold. First, it makes reference to the exponent matrix \( \Lambda^{(k)} \) of the representation \( \rho \otimes \mathbb{Z}^{-k} \), but one would like to dispense of the need to compute these quantities, and express the relevant traces solely in terms of some simple numerical characteristics of \( \rho \). The second problem with Eq.\((3.3)\) is that it is only valid for irreducible representations, and one would like a general result valid for any representation. While there is an obvious solution to this, exploiting the additivity Eq.\((2.3)\) of dimensions and the fact that representations with finite image are completely reducible, this approach requires the knowledge of the irreducible decomposition of \( \rho \), while one would like an explicit expression for the dimensions in terms of some global characteristics of the representation. Since the way to achieve the above goals differs slightly for even and odd representations, we shall treat these cases separately in the subsequent sections, starting with the even case.

### 4. Even representations

Let’s fix an even representation \( \rho \), and recall that in this case there are no forms of odd weight. Our starting point is the observation that, by the very definition of \( \Lambda^{(k)} \),

\[
\exp(2\pi i \Lambda^{(2k)}) = (\rho \otimes \mathbb{Z}^{-2k}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \mathbb{Z} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-2k} \rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp(2\pi i (\Lambda - \frac{k}{6})),
\]

from which one concludes that all the eigenvalues of
\( \Gamma_k = \Lambda^{(2k)} - \Lambda + \frac{k}{6} \)

are necessarily integers, consequently

\[
\begin{align*}
\text{Tr} \left[ \Lambda^{(2k)} + \frac{k}{6} \right] &= \text{Tr}[\Lambda + \Gamma_k] = \text{Tr}[\Lambda] + \text{Tr}[\Gamma_k], \\
\text{Tr} \left[ 1 - \Lambda^{(2k)} - \frac{k}{6} \right] &= \text{Tr}[1 - \Lambda - \Gamma_k] = \text{Tr}[1 - \Lambda] - \text{Tr}[\Gamma_k].
\end{align*}
\]

Upon introducing (for arbitrary even \( \rho \)) the notations

\[
\lambda_+ = \text{Tr}[\Lambda], \\
\lambda_- = -\text{Tr}[1 - \Lambda]
\]

and

\[
\gamma_k = \text{Tr}[\Gamma_k],
\]

the formula for even irreducible \( \rho \) takes on the form

\[
\begin{align*}
\dim M_{2k}(\rho) &= \max(0, \lambda_+ + \gamma_k), \\
\dim S_{2k}(\rho) &= \max(0, \lambda_- + \gamma_k).
\end{align*}
\]

A few comments are in order at this point. First, let’s note that the difference \( \lambda_+ - \lambda_- \) equals the number of integer eigenvalues\(^4\) of \( \Lambda \), i.e. the number of invariant vectors of the operator \( \rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \); in particular, one has \( \lambda_- \leq \lambda_+ \), in complete accord with the inclusion \( S_{2k}(\rho) < M_{2k}(\rho) \). Moreover, the integer sequence of \( \gamma_k \)-s follows a simple repetitive pattern, namely (recall that \( d \) denotes the dimension of \( \rho \))

\[
\gamma_{k+6} = \gamma_k + d
\]

for all \( k \), as a consequence of the fact that the 12th power of \( \kappa \) is the identity representation \( \rho_0 \), hence \( \Lambda^{(k+12)} = \Lambda^{(k)} \). This means that, since \( \gamma_0 = 0 \), the whole sequence is determined by \( \gamma_1, \ldots, \gamma_5 \), and one has \( \dim M_{2k+12}(\rho) = \dim M_{2k}(\rho) + d \) provided \( \dim M_{2k}(\rho) > 0 \), with a similar result for cusp forms. Finally, since \( \Gamma_k(\rho \otimes \kappa^{2n}) = \Lambda^{(k+n)} - \Lambda^{(n)} + \frac{k}{6} = \Gamma_{k+n}(\rho) - \Gamma_n(\rho) \), one concludes that

\[
\gamma_k(\rho \otimes \kappa^{-2n}) = \gamma_{k+n}(\rho) - \gamma_n(\rho).
\]

For Eq.\((4.6)\) to be effective, it remains to give a practical method to determine the \( \gamma_k \)-s. This is based on the observation that, because the eigenvalues of \( \Gamma_k \) are integers, one has

\[
\text{Tr}[\Gamma_k] = \text{Tr}(\Gamma_k) = \text{Tr}\Lambda^{(2k)} - \text{Tr}\Lambda + \frac{kd}{6}.
\]

\(^4\)This follows from the observation that, for any \( x \in \mathbb{R} \), the sum \( [x] + [1-x] \) equals 1 if \( x \) is an integer, and 0 otherwise.
But the traces appearing in this expression may be expressed, thanks to the trace formula Eq. (2.17), in terms of the signatures of the representations $\rho$ and $\rho \otimes \kappa^{-2k}$. Because $\dim \kappa = 1$, the latter may be determined by counting the eigenvalue multiplicities of the matrices $(-1)^k \rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\exp \left( \frac{-2\pi i}{3} \right)^k \rho \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, which is pretty straightforward, leading to the result summarized in Table 1.

**Table 1.** The signature of $\rho \otimes \kappa^{-2k}$

| $k \mod 6$ | $\alpha(\rho \otimes \kappa^{-2k})$ | $\beta_1(\rho \otimes \kappa^{-2k})$ | $\beta_2(\rho \otimes \kappa^{-2k})$ |
|------------|-------------------------------------|-------------------------------------|-------------------------------------|
| 0          | $\alpha$                            | $\beta_1$                           | $\beta_2$                           |
| 1          | $d - \alpha$                        | $\beta_2$                           | $d - \beta_1 - \beta_2$            |
| 2          | $\alpha$                            | $d - \beta_1 - \beta_2$            | $\beta_1$                           |
| 3          | $d - \alpha$                        | $\beta_1$                           | $\beta_2$                           |
| 4          | $\alpha$                            | $\beta_2$                           | $d - \beta_1 - \beta_2$            |
| 5          | $d - \alpha$                        | $d - \beta_1 - \beta_2$            | $\beta_1$                           |

It follows that the sequence of $\gamma_k$-s is completely determined by the signature of $\rho$, the first few values being summarized in Table 2.

**Table 2.** The values of $\gamma_k$ for $0 \leq k < 6$

| $k$ | $\gamma_k$ |
|-----|------------|
| 0   | 0          |
| 1   | $\alpha + \beta_1 + \beta_2 - d$ |
| 2   | $\beta_2$  |
| 3   | $\alpha$   |
| 4   | $\beta_1 + \beta_2$ |
| 5   | $\alpha + \beta_2$ |

Inspection of Table 2 reveals that $\gamma_k \geq 0$ for $k > 1$ (this can fail for $k = 1$, a prime example being the trivial representation $\rho_0$, for which $d = 1$ and $\alpha = \beta_1 = \beta_2 = 0$, hence $\gamma_1 = -1$), and the relations $\gamma_7 = \gamma_1 + d = \gamma_3 + \gamma_4$ and $\gamma_5 = \gamma_2 + \gamma_3$. Thanks to Eq. (4.8), these results generalize to $\gamma_{k+n} \geq \gamma_k$ for $n > 1$, and

\begin{align}
\gamma_{k+7} + \gamma_k &= \gamma_{k+3} + \gamma_{k+4} \\
\gamma_{k+5} + \gamma_k &= \gamma_{k+3} + \gamma_{k+2} .
\end{align}

The next important relation follows by considering the contragredient representation $\rho^\ast$, which assigns to each $\gamma \in \Gamma$ the transposed inverse of its representation operator $\rho(\gamma)$:

\begin{align}
\rho^\ast(\gamma) &= \rho(\gamma^{-1}) .
\end{align}

Since transposition does not change eigenvalues, while inversion inverts them, it follows that the signature of $\rho^\ast$ is given by

\begin{align}
d' &= d, \quad \alpha' = \alpha, \quad \beta_1' = \beta_2, \quad \beta_2' = \beta_1 .
\end{align}
Combining this with Table 2 and the periodicity relation Eq. (4.7), one arrives at the duality relation

\[(4.14) \quad \gamma_k (\rho^\vee) = \gamma_1 (\rho) - \gamma_{1-k} (\rho) .\]

On the other hand, by the definition of the contragredient

\[(4.15) \quad \exp (2\pi i \Lambda^\vee) = \rho^\vee \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) = \exp (-2\pi i \Lambda) ,\]

if one denotes by $\Lambda^\vee$ the exponent matrix of $\rho^\vee$; consequently, the sum $t\Lambda + \Lambda^\vee$ has integer eigenvalues. As a result,

\[(4.16) \quad \text{Tr} \left[ t\Lambda + \Lambda^\vee - 1 \right] = \text{Tr} \left[ t\Lambda + \Lambda^\vee - 1 \right] = \text{Tr} (t\Lambda) + \text{Tr} \Lambda^\vee - d = d - \alpha - \beta_1 - \beta_2 = -\gamma_1 ,\]

according to Eqs. (4.13) and (2.17). But

\[(4.17) \quad \lambda_+ (\rho^\vee) = -\text{Tr} [1 - \Lambda^\vee] = -\text{Tr} \left[ t\Lambda - (t\Lambda + \Lambda^\vee - 1) \right] = -\text{Tr} [\Lambda] + \text{Tr} [t\Lambda + \Lambda^\vee - 1] = -\text{Tr} [\Lambda] - \gamma_1 (\rho) . \]

proving the following supplement to Eq. (4.14)

\[(4.18) \quad \lambda_+ (\rho^\vee) + \lambda_- (\rho) = -\gamma_1 = \lambda_+ (\rho^\vee) + \lambda_- (\rho) .\]

The last major ingredient that we shall need is the observation that, for a not necessarily irreducible even representation $\rho : \text{SL}_2 (\mathbb{Z}) \rightarrow \text{GL}(V)$ with finite image, a weight 0 holomorphic form $X \in M_0 (\rho)$ is a constant vector invariant under $\rho$; as a consequence, $\dim M_0 (\rho)$ equals the multiplicity $h_0$ of the trivial representation in $\rho$, and $\dim S_0 (\rho)$ is always zero:

\[(4.19) \quad \dim M_0 (\rho) = h_0 , \quad \dim S_0 (\rho) = 0 .\]

To see this, note that any form $X \in M_0 (\rho)$ is invariant under the kernel $\ker \rho$ of $\rho$. This implies that, if $X (\rho)$ denotes the surface uniformized by $\ker \rho$ and $\pi : \mathcal{H} \rightarrow X(\rho)$ the associated natural projection, there exists a single valued map $\hat{X} : X(\rho) \rightarrow V$ such that $X = \hat{X} \circ \pi$. Since $X$ is holomorphic on $\mathcal{H}$ and bounded at the cusp $\tau = i\infty$, the map $\hat{X}$ is holomorphic on all of $X(\rho)$ (a finite sheeted cover of the Riemann sphere $\mathbb{C}P^1$), and bounded at all its cusps: by Liouville’s theorem, it should be a constant map. But this means that $X \in M_0 (\rho)$ should be independent of $\tau$, and this constant vector $X \in V$ has to satisfy $X = \rho(\gamma) X$ for all $\gamma \in \Gamma$ because of Eq. (2.1), hence it should be an invariant vector of the representation $\rho$. Should $X$ be a cusp form, it should vanish as $\tau \rightarrow i\infty$, hence it should vanish identically.
An immediate consequence of the above result is that, for \( n > 0 \), \( \dim M_{-2n}(\rho) \leq \dim S_0(\rho \otimes \mathbb{R}^{2n}) = 0 \) by Eq. (2.12), i.e. there are no holomorphic forms of negative weight\(^5\). What is more, \( \lambda_+ \leq 0 \) for irreducible and nontrivial \( \rho \) (since such a \( \rho \) has no invariant vectors), and \( \lambda_- \leq 0 \) for every \( \rho \); combining the above with the reciprocity relation Eq. (4.18), and noting that the contragredient \( \rho^\vee \) is irreducible and nontrivial whenever \( \rho \) is, one gets the important result that \( \gamma_1 + \lambda_+ \geq 0 \) for all \( \rho \), and \( \gamma_1 + \lambda_- \geq 0 \) for irreducible and nontrivial \( \rho \).

Let’s now suppose that \( \rho \) is even irreducible of dimension \( d > 1 \). In this case \( \gamma_1 \geq -\lambda_- \geq 0 = \gamma_0 \) by the above. But if \( \rho \) is irreducible of dimension \( d > 1 \), then the same is true of its tensor product with any representation of dimension 1, hence \( \gamma_1(\rho \otimes \mathbb{R}^{2k}) \geq 0 \) for all \( k \). But \( \gamma_1(\rho \otimes \mathbb{R}^{2k}) = \gamma_{k+1} - \gamma_k \) according to Eq. (4.8), leading to the conclusion that in this case the \( \gamma_k \)-s form an increasing sequence:

\[
\ldots \leq \gamma_{-1} \leq \gamma_0 = 0 \leq \gamma_1 \leq \gamma_2 \leq \ldots
\]

(note that this fails for \( d=1 \)). Combining this result with \( \gamma_1 + \lambda_- \geq 0 \), one gets that for an irreducible \( \rho \) with \( d > 1 \) the inequality \( \lambda_+ + \gamma_k \geq 0 \) holds for all \( k > 0 \); a simple case by case check shows that it does also hold for all nontrivial \( \rho \) with \( d=1 \).

Finally, putting everything together, and taking into account Eq. (2.3), we get that for an even, not necessarily irreducible representation \( \rho \)

\[
\text{dim } M_{2k}(\rho) = \begin{cases} 
0 & \text{if } k < 0; \\
h_0 & \text{if } k = 0; \\
\lambda_+ + \gamma_k & \text{if } k > 0,
\end{cases}
\]

and

\[
\text{dim } S_{2k}(\rho) = \begin{cases} 
0 & \text{if } k \leq 0; \\
\lambda_- + \gamma_1 + h_0 & \text{if } k = 1; \\
\lambda_- + \gamma_k & \text{if } k > 1.
\end{cases}
\]

It follows from Eqs. (4.21) and (4.22), combined with the duality relations Eqs. (4.14) and (4.18), that

\[
\text{dim } M_{2k}(\rho) + \text{dim } S_{12n+2-2k}(\rho^\vee) = nd
\]

for positive integers \( k \) and \( n \) such that \( k < 6n \), expressing the duality between holomorphic and cusp forms, and

\[
\begin{align*}
\lambda_+(\rho) &= \text{dim } M_0(\rho) - \text{dim } S_2(\rho^\vee) \\
\lambda_- (\rho) &= \text{dim } S_0(\rho) - \text{dim } M_2(\rho^\vee).
\end{align*}
\]

Let’s take a look at the classical case, when \( \rho \) is the identity representation \( \rho_0 \). In this case \( \lambda_+ = h_0 = 1, \lambda_- = \alpha = \beta_1 = \beta_2 = 0 \), and for

\(^5\)Recall that we assume \( \rho \) to have finite image.
\( k < 6 \) all \( \gamma_k \)'s are zero except for \( \gamma_1 = -1 \). This leads to the well-known result

\[
\dim M_{2k} = \left\{ \begin{array}{ll}
\left\lceil \frac{k}{6} \right\rceil & \text{if } k \equiv 1 \pmod{6} , \\
\left\lceil \frac{k}{6} \right\rceil + 1 & \text{otherwise}
\end{array} \right.
\]

and \( \dim S_{2k} = \max(0, \dim M_{2k} - 1) \). As we can see, the classical case is somewhat exceptional because of the existence of an invariant vector for \( \rho_0 \).

To finish, let’s consider the weight distribution of the generators of \( \mathcal{M}(\rho) = \oplus_k M_k(\rho) \) and \( \mathcal{S}(\rho) = \oplus_k S_k(\rho) \), considered as free modules over the ring \( \mathbb{C}[E_4, E_6] \) of holomorphic scalar modular forms, cf. Eq. (4.26).

The first step is to compute the Hilbert-Poincaré series

\[
\mathcal{M}_\rho(z) = \sum_{k=0}^{\infty} \dim M_{2k}(\rho) z^{2k} = h_0 + \sum_{k=2}^{\infty} (\lambda_+ + \gamma_k) z^{2k} = h_0 + \frac{\lambda_+ z^2}{1 - z^2} + \frac{\gamma_1 z^2 + \gamma_2 z^4 + (\gamma_3 - \gamma_1) z^6 + (\gamma_4 - \gamma_2 - \gamma_1) z^8}{(1 - z^4)(1 - z^6)}
\]

and

\[
\mathcal{S}_\rho(z) = \sum_{k=0}^{\infty} \dim S_{2k}(\rho) z^{2k} = (\gamma_1 + \lambda_- + h_0) z^2 + \sum_{k=2}^{\infty} (\lambda_- + \gamma_k) z^{2k} = h_0 z^2 + \frac{\lambda_- z^2}{1 - z^2} + \frac{\gamma_1 z^2 + \gamma_2 z^4 + (\gamma_3 - \gamma_1) z^6 + (\gamma_4 - \gamma_2 - \gamma_1) z^8}{(1 - z^4)(1 - z^6)}
\]

where we have used the relations Eqs. (4.10) and (4.11) to sum up the power series \( \sum_{k=0}^{\infty} \gamma_k z^{2k} \). From Eqs. (4.26) and (4.27) one reads off the weight distribution of the generators as the coefficients of the polynomials \( (1 - z^4)(1 - z^6) \mathcal{M}_\rho(z) \) and \( (1 - z^4)(1 - z^6) \mathcal{S}_\rho(z) \), the results being tabulated in Table 3.

**Table 3.** Weight distribution of free generators for even \( \rho \).

| Weight | \( \mathcal{M}(\rho) \) | \( \mathcal{S}(\rho) \) |
|--------|----------------|----------------|
| 0      | \( h_0 \)       | 0              |
| 2      | \( \gamma_1 + \lambda_+ \) | \( \gamma_1 + \lambda_- + h_0 \) |
| 4      | \( \gamma_2 + \lambda_+ - h_0 \) | \( \gamma_2 + \lambda_- \) |
| 6      | \( \gamma_3 - \gamma_1 - h_0 \) | \( \gamma_3 - \gamma_1 - h_0 \) |
| 8      | \( \gamma_6 - \gamma_5 - \lambda_+ \) | \( \gamma_6 - \gamma_5 - \lambda_- - h_0 \) |
| 10     | \( h_0 - \lambda_+ \) | \( -\lambda \) |
| 12     | 0              | \( h_0 \) |

Note that in both cases we have a total of \( \gamma_6 = d \) generators, in accord with the fact that these are free modules of rank \( d \). Finally, the underlying duality Eq. (4.23) between cusps forms for \( \rho \) and holomorphic
forms for its contragredient is elegantly expressed by the relation
\[
S_{\rho^\vee}(z) = z^2 M_{\rho}(z^{-1})
\]
between the respective Hilbert-Poincaré series\(^6\).

5. Odd representations

Let’s consider an odd representation \( \rho : \text{SL}_2(\mathbb{Z}) \to \text{GL}(V) \) with finite image. In this case, nontrivial forms have odd weight, and the representation \( \hat{\rho} = \rho \otimes \varpi^{-1} \) is even, and still has finite image: we shall denote its exponent matrix by \( \Lambda \), and introduce the notations
\[
\hat{\gamma}_k(\rho) = \gamma_k(\hat{\rho})
\]
and
\[
\hat{\lambda}_\pm(\rho) = \text{Tr} \left[ \Lambda \pm \frac{1}{12} \right]
\]
(5.1)
(5.2)

Note that comments similar to those after Eq.(4.6) apply in this case as well, e.g. \( \hat{\gamma}_{k+6} = \hat{\gamma}_k + d \). Moreover, since \([y-x] + [x] \leq 0\) for all \( x \in \mathbb{R} \) and \( y < 1 \), one has
\[
\lambda_+(\rho) \leq \hat{\lambda}_-(\rho) \leq \hat{\lambda}_+(\rho),
\]
the second inequality following from the observation that \( \hat{\lambda}_+ - \hat{\lambda}_- \) equals the number of invariant vectors of \( \rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), hence it can’t be negative.

Recalling from the discussion following Eq.(4.20) that \( \gamma_k + \lambda_+ \geq 0 \) for even representations and positive \( k \), we get from Eq.(5.3) the following inequality for odd \( \rho \) and \( k > 0 \):
\[
\hat{\lambda}_+(\rho) + \hat{\gamma}_k(\rho) \geq \hat{\lambda}_-(\rho) + \hat{\gamma}_k(\rho) \geq \lambda_+(\hat{\rho}) + \gamma_k(\hat{\rho}) \geq 0.
\]

There is, however, an important difference with respect to the case of even \( \rho \); namely, since \( (\rho^\vee) = (\hat{\rho})^\vee \otimes \varpi^{-2} \) as a consequence of \( \varpi^\vee = \varpi^{-1} \), for odd representations the duality relations take the form
\[
\hat{\gamma}_k(\rho^\vee) = -\hat{\gamma}_{-k}(\rho)
\]
and
\[
\hat{\lambda}_+(\rho^\vee) = -\hat{\lambda}_-(\rho),
\]
(5.5)
(5.6)
to be compared with Eqs.(4.14) and (4.18).

The next question concerns the minimal possible weight of a holomorphic or cusp form for odd \( \rho \). The case of \( \rho = \varpi \) shows that there could exist weight 1 cusp forms, the prime example being the form \( \varsigma(\tau) \in S_1(\varpi) \) entering the weight-shifting map from Section 2. On the

\(^6\)This has to be interpreted as a relation between the corresponding rational expressions to which these power series sum up.
other hand, there can be no nontrivial holomorphic forms of negative weights, for Eq. (2.12) and Eq. (4.19) imply that for $k \geq 0$

\[(5.7) \quad \dim S_{-(2k+1)}(\rho) \leq \dim M_{-(2k+1)}(\rho) \leq \dim S_0(\rho \otimes \mathcal{X}^{2k+1}) = 0.\]

For irreducible $\rho$, an argument paralleling the one leading to Eq. (4.6) gives at once

\[(5.8) \quad \dim M_{2k+1}(\rho) = \max \left(0, \hat{\lambda}_+ + \hat{\gamma}_k\right),\]

\[(5.9) \quad \dim S_{2k+1}(\rho) = \max \left(0, \hat{\lambda}_- + \hat{\gamma}_k\right).\]

Taking into account the above discussed inequalities, one concludes that for odd irreducible $\rho$

\[(5.10) \quad \dim M_{2k+1}(\rho) = \begin{cases} 0 & \text{if } k < 0; \\ \hat{\lambda}_+ + \hat{\gamma}_k & \text{if } k > 0, \end{cases}\]

\[(5.11) \quad \dim S_{2k+1}(\rho) = \begin{cases} 0 & \text{if } k < 0; \\ \hat{\lambda}_- + \hat{\gamma}_k & \text{if } k > 0, \end{cases}\]

which should be supplemented with the relations $\dim M_1(\rho) = \max \left(0, \hat{\lambda}_+\right)$ and $\dim S_1(\rho) = \max \left(0, \hat{\lambda}_-\right)$. Combining the above results with Eqs. (5.5) and (5.6), one obtains the duality relation

\[(5.12) \quad \hat{\lambda}_+ = \dim M_1(\rho) - \dim S_1(\rho^\vee),\]

\[(5.13) \quad \hat{\lambda}_- = \dim S_1(\rho) - \dim M_1(\rho^\vee),\]

explaining the meaning of the parameters $\hat{\lambda}_+$ and $\hat{\lambda}_-$. Note that, taking into account Eq. (2.3), this last result holds for arbitrary, not necessarily irreducible odd representations, because both sides of the equalities hold for each irreducible constituent separately and the relevant quantities are additive. What is more, by an analogue argument the same is true of Eqs. (5.9) and (5.10), which should be regarded as the final form of the dimension formula for odd representations.

As an example, let’s consider the odd representation $\mathcal{X}$. Since $\mathcal{X} = \rho_0$, one has $\hat{\lambda}_+(\mathcal{X}) = \hat{\lambda}_-(\mathcal{X}) = 1$, and $\hat{\gamma}_k = 0$ for $0 \leq k < 7$, except for $\hat{\gamma}_1 = -1$. It follows that $\dim M_1(\mathcal{X}) = \dim S_1(\mathcal{X}) = 1$ (and indeed, $\varsigma(\tau) \in S_1(\mathcal{X})$ as

\[\text{Note that this allows us to dispense with irreducible decomposition of } \rho, \text{ since the dimensions can be expressed solely in terms of some global parameters characterizing } \rho. \text{ This is to be contrasted with the weight 1 case, where one needs the explicit knowledge of the irreducible decomposition to be able to compute the relevant dimensions.}\]
remarked before), while $\text{dim } M_3(\varpi) = 0$. From Eq.(5.12), we see that $\text{dim } M_1(\varpi') = 0$.

Finally, let’s discuss the weight distribution of the generators of $M(\rho) = \oplus M_k(\rho)$ and $S(\rho) = \oplus S_k(\rho)$, considered as modules over the ring $M = \mathbb{C}[E_4, E_6]$ of scalar holomorphic forms. An argument completely parallel to that leading to Eq.(4.26), but based on the results relevant to odd representations, gives Table 4 for the weight distribution of the generators (note that we again have a total of $\gamma_6 = d$ independent generators). The duality relation Eq.(4.28) goes over verbatim to the odd case.

**Table 4.** Weight distribution of free generators for odd $\rho$.

| weight | $M(\rho)$ | $S(\rho)$ |
|--------|------------|------------|
| 1      | $\dim M_1(\rho)$ | $\dim S_1(\rho)$ |
| 3      | $\gamma_1 + \lambda_+$ | $\gamma_1 + \lambda_+$ |
| 5      | $\gamma_2 + \lambda_1 - \dim M_1(\rho)$ | $\gamma_2 + \lambda_1 - \dim S_1(\rho)$ |
| 7      | $\gamma_3 - \gamma_1 - \dim M_1(\rho)$ | $\gamma_3 - \gamma_1 - \dim S_1(\rho)$ |
| 9      | $\gamma_6 - \gamma_5 - \lambda_+$ | $\gamma_6 - \gamma_5 - \lambda_+$ |
| 11     | $\dim M_1(\rho) - \lambda_+$ | $\dim S_1(\rho) - \lambda_+$ |

6. **Outlook**

We have investigated spaces of vector-valued holomorphic and cusp forms of integer weight for finite dimensional representations of the modular group $\Gamma = \text{SL}_2(\mathbb{Z})$ having finite image, and have obtained explicit expressions, see Eqs.(4.21), (4.22), (5.9) and (5.10), for the dimension of these spaces. Based on these results, we have described the weight distribution of the generators of the module of holomorphic and cusp forms, and the duality, most elegantly expressed by Eq.(4.28), relating cusp forms with holomorphic forms for the contragredient.

It goes without saying that the results presented here agree completely with those of [25]: to see this, one has to rewrite the quantities appearing in [25] in terms of those of the present paper, which is a straightforward job using Eq.(2.18). The advantage of our approach is that it doesn’t only give us the actual dimensions, but it does also provide an effective procedure for computing explicit bases, by solving the relevant system of linear relations, as described in the argument leading to Eq.(3.1). As an extra bonus, we get a much better control over the quantities involved, making it easier to recognize relations like Eq.(4.28).

Several possible generalizations offer themselves at once. First, one could try to generalize the theory from integer weight to half-integer, or even arbitrary real weights: this would necessitate the consideration of suitable projective representations of $\text{SL}_2(\mathbb{Z})$, making the whole
story a bit more complicated than in the integer weight case. Next, one could contemplate the possibility to dispense of the finite image requirement: this could result in severe difficulties, both analytic (the surface uniformized by the kernel would not be anymore a finite sheeted cover of the modular curve, allowing for holomorphic forms of negative weight) and algebraic (the possibility of reducible but indecomposable representations could lead to non-semisimple exponent matrices), but is certainly a most interesting issue to be dealt with, since this is the case relevant for logarithmic conformal theories (to be contrasted with the finite image case relevant for rational theories). Finally, an obvious generalization, making direct contact with classical knowledge, would be to consider forms for (finite index) subgroups of $\text{SL}_2(\mathbb{Z})$. To sum up, there are many interesting questions left open for future investigations.

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