Fourth order perturbative expansion for the Casimir energy with a slightly deformed plate

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(Dated: today)

We apply a perturbative approach to evaluate the Casimir energy for a massless real scalar field in 3 + 1 dimensions, subject to Dirichlet boundary conditions on two surfaces. One of the surfaces is assumed to be flat, while the other corresponds to a small deformation, described by a single function $\eta$, of a flat mirror. The perturbative expansion is carried out up to the fourth order in the deformation $\eta$, and the results are applied to the calculation of the Casimir energy for corrugated mirrors in front of a plane. We also reconsider the proximity force approximation within the context of this expansion.

PACS numbers: 12.20.Ds, 03.70.+k, 11.10.-z

I. INTRODUCTION

Among the different approximate analytical approaches to the calculation of the Casimir energy and related objects\cite{1}, a particularly interesting one has been developed some time ago\cite{2}, for cases where the mirrors’ configurations can be described as a small geometrical deformation of the simplest possible case: two infinite, flat, parallel mirrors. We shall focus here, in particular, on cases where just one of the surfaces defined by the (zero width) mirrors is deformed. Since one of the surfaces is, indeed, flat, it is always possible to adopt Cartesian coordinates $x_1, x_2, x_3$ in such a way that surface coincides with the $x_3 = 0$ plane. The other mirror’s surface will, in turn, be assumed to be a small deformation of the $x_3 = a$ ($a > 0$) plane, namely: $x_3 = a + \eta(x_\parallel)$. Here, $\eta$ is a scalar function of $x_\parallel \equiv (x_1, x_2)$, coordinates which are then parallel to the flat mirror surface. Hence, $\eta$ represents the (assumed) small departure from a configuration corresponding to two flat parallel mirrors (i.e., $x_3 = 0, a$).

As already mentioned, for this, and similar situations, a perturbative expansion in powers of $\eta(x_\parallel)$ has been developed, in order to compute the Casimir energy, under the assumption that $|\eta(x_\parallel)| < < a$. This program has been carried out up to the second order in $\eta$\cite{2}, and applied, for example, to the evaluation of the effect of corrugations on the Casimir energy\cite{2,4}. The perturbative calculations of Refs.\cite{2,4} are based on a functional approach, that we will also follow here. On the other hand, in Ref.\cite{3} a scattering approach is used, and a general formula for the Casimir energy is given in terms of the reflection coefficients associated to the mirrors. In this approach, one can incorporate finite temperature, finite conductivity and roughness corrections. However, as far as we know, all previous calculations for the normal force\cite{2,4} have been performed up to second order in $\eta$.

In recent years, accurate numerical evaluations of the Casimir energy for several geometries, including corrugated surfaces of arbitrary amplitude, have become available. In spite of that, approximate analytical results are always welcome, for a variety of reasons. Among them, to improve the physical understanding of the dependence of the Casimir force with the geometry, as well as to use them as benchmarks of complex numerical calculations.

It is the aim of this paper to present new analytical results for the perturbative expansion, which had been implemented in previous works up to the second order in $\eta$, by calculating the two subsequent corrections, of order three and fourth in the same expansion (note that a similar program has been carried out in Ref.\cite{5}, to compute the lateral force between corrugated surfaces up to the fourth order in the amplitude).

The structure of this paper is as follows: in Sec.\cite{1} we introduce the system considered and derive the (formal) expression for the Casimir energy, in terms of an “effective action”, using a functional approach. In Sec.\cite{2} we introduce the perturbative expansion for the effective action, focusing on the expressions corresponding to terms of up to the fourth order. In Sec.\cite{3} we present the explicit results up to the fourth order term. In Sec.\cite{4} we analyze further approximations and resummations of the perturbative expansion. In particular, we will see that the proximity force approximation (PFA) can be understood as a resummation of the perturbative expansion when the form factors involved in the expansion are replaced by their zero-momentum values. In Sec.\cite{5} we apply the results to compute the correction to the Casimir energy up to the fourth order for a sinusoidally corrugated surface, and up to the third order for a corrugation described by a combination of two sinusoidal functions of different wavelengths. We summarize our findings in Sec.\cite{6}.
II. THE SYSTEM

We adopt Euclidean conventions, whereby spacetime coordinates are denoted by $x^{\mu} \equiv x_{\mu}$ ($\mu = 0, 1, 2, 3$), $x_0$ being the imaginary time and $x_i$, ($i = 1, 2, 3$) the spatial coordinates. We shall also use the notation $x_{\parallel} \equiv (x_{0}, x_{1}, x_{2}) \equiv (x_{0}, x_{i})$, which shall be used as spacetime coordinates on each mirror.

As mentioned above, the scalar field $\varphi$ satisfies Dirichlet boundary conditions on two surfaces: $L$, which is the $x_{3} = 0$ plane, while the other, $R$, can be represented by a single function of $x_{i}$:

- $L$) $x_{3} = 0$
- $R$) $x_{3} = a + \eta(x_{i})$.

Following the functional approach to the Casimir effect, we introduce $Z$, the zero temperature limit of the partition function for $\varphi$ in the presence of the two mirrors:

$$Z \equiv \int D\varphi \delta_{L}(\varphi) \delta_{R}(\varphi) e^{-S_{0}(\varphi)},$$

where $S_{0}$ is the massless real scalar field Euclidean action

$$S_{0}(\varphi) = \frac{1}{2} \int d^{4}x (\partial \varphi)^{2},$$

while $\delta_{L}$ ($\delta_{R}$) imposes Dirichlet boundary conditions on the $L$ ($R$) surface. Because of several reasons, it becomes convenient to generalize, at this point, the boundary condition on the $R$ mirror, by assuming that $\eta$ may also depend on $x_{0}$, the Euclidean time. Namely, $\eta(x_{\parallel}) \to \eta(x_{0})$.

One of the reasons for doing this is that the results thus obtained become more symmetrical, in the sense that the coordinates $x_{\parallel}$ shall appear on an equal footing. Besides, as a byproduct, one could make use of those results in dynamical Casimir effect situations. Note that, strictly speaking, $Z$ will not be a quantum partition function when $\eta$ depends on time. We do keep, however, the same notation ($Z$) for the resulting object. Static Casimir effect results will be obtained by simply dropping the time dependence of $\eta$, at the end of the calculation.

Finally, note that $Z$, evaluated with a time dependent $\eta$, may also be interpreted as the classical partition function for a static system in 4 spatial dimensions, in the presence of two (static) boundaries.

To proceed, note that the two functional delta functions may be exponentiated by introducing two auxiliary fields, $\lambda_{L}$ and $\lambda_{R}$, to obtain for $Z$ the equivalent expression:

$$Z = \int D\varphi D\lambda_{L} D\lambda_{R} e^{-S(\varphi; \lambda_{L}, \lambda_{R})},$$

with

$$S(\varphi; \lambda_{L}, \lambda_{R}) = S_{0}(\varphi) - i \int d^{4}x [\lambda_{L}(x_{\parallel}) \delta(x_{3}) + \lambda_{R}(x_{\parallel}) \delta(x_{3} - \psi(x_{\parallel}))] \varphi(x),$$

where $\psi(x_{\parallel}) \equiv a + \eta(x_{\parallel})$.

Integrating out $\varphi$, we see that a $Z_0$ factor, the partition function for the field $\varphi$ in the absence of boundary conditions, factors out, while the rest is an integral over the auxiliary fields:

$$Z = Z_{0} \int D\lambda_{L} D\lambda_{R} e^{-\frac{1}{2} \int d^{4}x_{i} \int d^{4}y_{\parallel} \sum_{\alpha, \beta} \lambda_{\alpha}(x_{\parallel}) \tau_{\alpha, \beta}(x_{\parallel}, y_{\parallel}) \lambda_{\beta}(y_{\parallel})},$$

where $\alpha, \beta = L, R$ and we have introduced the object:

$$T_{LL}(x_{\parallel}, y_{\parallel}) = \langle x_{\parallel}, 0 | (-\partial^{2})^{-1} | y_{\parallel}, 0 \rangle$$

$$T_{LR}(x_{\parallel}, y_{\parallel}) = \langle x_{\parallel}, 0 | (-\partial^{2})^{-1} | y_{\parallel}, \psi(y_{\parallel}) \rangle$$

$$T_{RL}(x_{\parallel}, y_{\parallel}) = \langle x_{\parallel}, \psi(x_{\parallel}) | (-\partial^{2})^{-1} | y_{\parallel}, 0 \rangle$$

$$T_{RR}(x_{\parallel}, y_{\parallel}) = \langle x_{\parallel}, \psi(x_{\parallel}) | (-\partial^{2})^{-1} | y_{\parallel}, \psi(y_{\parallel}) \rangle. $$

We have used a ‘bra-ket’ notation as a convenient way to denote matrix elements of operators. $\partial^{2}$ is the four-dimensional Laplacian. Thus,

$$\langle x_{\parallel} | (-\partial^{2})^{-1} | y_{\parallel} \rangle = \int \frac{d^{4}k}{(2\pi)^{4}} \frac{e^{ik(x-y)}}{k^{2}}. $$

The four kernels $T_{\alpha, \beta}$ can be naturally regarded as matrix elements of a $2 \times 2$ kernel matrix: $T = (T_{\alpha, \beta})_{\alpha, \beta = L, R}$.

The vacuum energy of the system, $E_{\text{vac}}$, obtained by subtracting the zero-point energy of a free field with trivial boundary conditions at infinity, may then be obtained as follows:

$$E_{\text{vac}} = \lim_{T \to \infty} \left[ \frac{\Gamma(\eta)}{T} \right]_{\eta = \psi(x_{\parallel})},$$

where $T$ is the extent of the time dimension, and $\Gamma$ is given by:

$$\Gamma \equiv - \log \left( \frac{Z}{Z_{0}} \right) = \frac{1}{2} \text{Tr} \left( \log T \right),$$

wich resembles an effective action in the presence of a background field $\eta(x_{\parallel})$. Note that the trace above is meant to act on both discrete and continuous indices.

The Casimir energy is then obtained from the vacuum energy $E_{\text{vac}}$, by simply discarding the zero-point contributions due to the vacuum field distortion produced when the mirrors are infinitely far apart. This ‘self-energy’ piece, is irrelevant for the kind of physical observable we have in mind here, and shall therefore be subtracted.

In the next section, we construct the expansion of $\Gamma$ in powers of $\eta$, whence the information about the Casimir energy will be extracted from $[12]$, after discarding self-energy terms.

III. PERTURBATIVE EXPANSION

The procedure to obtain the formal expression for the expansion of $\Gamma$ is rather straightforward. Indeed, assuming that $T^{(i)}$ denotes the term of order $i$ in the expansion.
of $T$ in powers of $\eta$,

$$\Gamma = T^{(0)} + T^{(1)} + T^{(2)} + T^{(3)} + T^{(4)} + \ldots,$$  \hspace{1cm} (14)

we obtain an expansion for $\Gamma$ of the form

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} + \Gamma^{(3)} + \Gamma^{(4)} + \ldots$$  \hspace{1cm} (15)

Terms of up to the second order for $\Gamma$ are then given by

$$\Gamma^{(0)} = \frac{1}{2} \text{Tr} \log (T^{(0)}), \quad \Gamma^{(1)} = \frac{1}{2} \text{Tr} \left[ (T^{(0)})^{-1} T^{(1)} \right]$$  \hspace{1cm} (16)

and

$$\Gamma^{(2)} = \Gamma^{(2,1)} + \Gamma^{(2,2)}$$  \hspace{1cm} (17)

where:

$$\Gamma^{(2,1)} = \frac{1}{2} \text{Tr} \left[ (T^{(0)})^{-1} T^{(2)} \right]$$

$$\Gamma^{(2,2)} = -\frac{1}{4} \text{Tr} \left[ (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(1)} \right].$$  \hspace{1cm} (18)

Collecting all the terms of third order in $\eta$, we see that $\Gamma^{(3)}$ may be expressed as the sum of three contributions:

$$\Gamma^{(3)} = \Gamma^{(3,1)} + \Gamma^{(3,2)} + \Gamma^{(3,3)},$$  \hspace{1cm} (19)

where

$$\Gamma^{(3,1)} = \frac{1}{2} \text{Tr} \left[ (T^{(0)})^{-1} T^{(3)} \right],$$  \hspace{1cm} (20)

$$\Gamma^{(3,2)} = -\frac{1}{2} \text{Tr} \left[ (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(2)} \right],$$  \hspace{1cm} (21)

and

$$\Gamma^{(3,3)} = \frac{1}{6} \text{Tr} \left[ (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(1)} \right].$$  \hspace{1cm} (22)

Finally, $\Gamma^{(4)}$ is the sum of five independent terms, which in turn result from collecting all the terms of fourth order in the expansion for $\Gamma$:

$$\Gamma^{(4)} = \sum_{i=1}^{5} \Gamma^{(4,i)};$$  \hspace{1cm} (23)

with:

$$\Gamma^{(4,1)} = \frac{1}{2} \text{Tr} \left[ (T^{(0)})^{-1} T^{(4)} \right],$$  \hspace{1cm} (24)

$$\Gamma^{(4,2)} = -\frac{1}{2} \text{Tr} \left[ (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(3)} \right],$$  \hspace{1cm} (25)

$$\Gamma^{(4,3)} = -\frac{1}{4} \text{Tr} \left[ (T^{(0)})^{-1} T^{(2)} (T^{(0)})^{-1} T^{(2)} \right],$$  \hspace{1cm} (26)

$$\Gamma^{(4,4)} = \frac{1}{2} \text{Tr} \left[ (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(2)} \right],$$  \hspace{1cm} (27)

and

$$\Gamma^{(4,5)} = -\frac{1}{8} \times \text{Tr} \left[ (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(1)} (T^{(0)})^{-1} T^{(1)} \right].$$  \hspace{1cm} (28)

To proceed to the explicit form for each term, it is convenient to introduce $\hat{\eta}(k_{||})$, the Fourier transform of $\eta(x_{||})$.

$$\hat{\eta}(k_{||}) = \int d^{3}x_{||} e^{-ik_{||} \cdot x_{||}} \eta(x_{||}).$$  \hspace{1cm} (29)

Also, we note that $(T^{(0)})^{-1}$, which is translation invariant, may be written as follows:

$$(T^{(0)})^{-1}(x_{||}, x'_{||}) = (T^{(0)})^{-1}(x_{||} - x'_{||}) = \int \frac{d^{3}k_{||}}{(2\pi)^{3}} e^{ik_{||} \cdot (x_{||} - x'_{||})} \tilde{\mathbb{D}}(k_{||}),$$  \hspace{1cm} (30)

with:

$$\tilde{\mathbb{D}}(k_{||}) = \frac{2|k_{||}|}{1 - e^{-2|k_{||}|a}} \left( \frac{1}{1 - e^{-|k_{||}|a}} - e^{-|k_{||}|a} \right).$$  \hspace{1cm} (31)

Another important ingredient is the form of the matrix elements of $T$ to the desired order. Regarding the diagonal elements, one sees that $T^{(j)}_{LL} = 0$ for $j > 0$, while $T^{(j)}_{RR} = 0$ for odd $j$. Thus, for the diagonal elements, we need:

$$T^{(2)}_{RR}(x_{||}, x'_{||}) = \int \frac{d^{3}k_{||}}{(2\pi)^{3}} e^{ik_{||} \cdot (x'_{||} - x_{||})} \frac{|k_{||}|}{4} \left[ \eta(x_{||}) - \eta(x'_{||}) \right]^{2},$$  \hspace{1cm} (32)

and

$$T^{(4)}_{RR}(x_{||}, x'_{||}) = \int \frac{d^{3}k_{||}}{(2\pi)^{3}} e^{ik_{||} \cdot (x'_{||} - x_{||})} \frac{|k_{||}|^{3}}{48} \left[ \eta(x_{||}) - \eta(x'_{||}) \right]^{4}. $$  \hspace{1cm} (33)

For the off-diagonal matrix elements, their general form at an arbitrary order may be written in a rather compact form. Indeed,

$$T^{(j)}_{LR}(x_{||}, x'_{||}) = \frac{(-1)^{j}}{2j!} \int d^{3}k_{||} e^{ik_{||} \cdot (x'_{||} - x_{||})} e^{-|k_{||}|a} |k_{||}|^{j-1} \times \left[ \eta(x'_{||}) \right]^{j},$$  \hspace{1cm} (34)

$$T^{(j)}_{RL}(x_{||}, x'_{||}) = T^{(j)}_{LR}(x'_{||}, x_{||}).$$  \hspace{1cm} (35)

### IV. Results

Although the terms of order zero, one and two are already known, we present, for the sake of completeness, the results corresponding to those orders as well.
A. Zeroth order

The zeroth-order term is simply obtained by recalling the form of $\Gamma^{(0)}$, from which one just needs to subtract its $a \to \infty$ limit. After that subtraction, one obtains a result that may be written in the following way:

$$\lim_{T \to \infty} \left[ \frac{\Gamma^{(0)}}{T} \right] = \frac{L^2}{2} \int \frac{d^3p_\parallel}{(2\pi)^3} \log (1 - e^{-2p_\parallel a})$$

(36)

where $L$ is a length that measures the size of the mirror along the parallel directions.

Thus, the Casimir energy due to this term, $E^{(0)}_{\text{vac}}$, becomes:

$$E^{(0)}_{\text{vac}} = \frac{L^2}{2} \int \frac{d^3p_\parallel}{(2\pi)^3} \log (1 - e^{-2p_\parallel a}) = -\frac{\pi^2 L^2}{1440 a^3},$$

(37)

as expected.

B. First order

It is quite straightforward to compute the first order term $\Gamma^{(1)}$. Introducing a function $B$, defined as:

$$B(q_\parallel) = \frac{1}{e^{2\langle q_\parallel \rangle a} - 1},$$

(38)

where $q_\parallel$ is a 3-vector, we obtain

$$\Gamma^{(1)} = \int \frac{d^3p_\parallel}{(2\pi)^3} B(p_\parallel) |p_\parallel| \times \int d^3x_\parallel \eta(x_\parallel)$$

$$= \frac{\pi^2}{480a^5} \int d^3x_\parallel \eta(x_\parallel).$$

(39)

Since, by definition, $\eta$ measures the departure from a flat parallel mirrors configuration, one could impose on it the condition that the integral $\int d^3x_\parallel \eta(x_\parallel)$ equals zero. Were it not the case, this condition could nevertheless have been achieved by subtracting a constant (which has to be added to $a$) from $\eta$. Thus, in this case $\Gamma^{(1)} = 0$.

C. Second order

The second order result has been known for quite some time. In Fourier space, it may be written as follows:

$$\Gamma^{(2)} = \frac{1}{2} \int \frac{d^3k_\parallel}{(2\pi)^3} f^{(2)}(k_\parallel) |\tilde{\eta}(k_\parallel)|^2,$$

(40)

with:

$$f^{(2)}(k_\parallel) = -2 \int \frac{d^3p_\parallel}{(2\pi)^3} \left[ B(p_\parallel)B(p_\parallel + k_\parallel) + B(p_\parallel + k_\parallel) \right]$$

$$\times |p_\parallel| |p_\parallel + k_\parallel|.$$

(41)

From the structure of (40), it seems that, for the perturbative expansion to make sense, an extra necessary condition (besides $|\eta| << a$) is that the integral over $k_\parallel$ has to be well-defined. However, we can, and will, consider cases where $\tilde{\eta}(k_\parallel)$ is a generalized function. In particular, $\delta$-like, concentrated around a particular value of $k_\parallel$. In this case, all the correction terms (not just the second order one) become proportional to the size of the system, due to the periodicity of the perturbation. Of course, the same is always true of the zeroth order term. Thus, even for this singular case, one could make sense of the perturbative corrections, at least if there is a region where the corrections are reasonably small, in comparison with the zeroth order term, after factorizing the (common) spatial size factor.

On the other hand, when $\tilde{\eta}(k_\parallel)$ is a regular function, a nontrivial condition emerging from (40) proceeds from requiring its convergence at large momentum. It may be seen that, at second order, this amounts to:

$$\int \frac{d^3k_\parallel}{(2\pi)^3} |k_\parallel| |\tilde{\eta}(k_\parallel)|^2 < \infty.$$

(42)

In a realistic situation, the Fourier spectrum of the deformation $\eta$ should have a cutoff; for instance, a continuous description of the mirrors should, at some point, be replaced by a discrete (lattice) one, which introduces a large momentum cutoff of the order of the lattice spacing.

D. Third order

The contribution denoted by $\Gamma^{(3,1)}$ is ultralocal, understanding by that an integral of a term involving (three) $\eta$’s at a single point (without derivatives). The first example of an ultralocal contribution was the first order term, but it could, as we have seen, be assumed to be equal to zero. On the contrary, $\Gamma^{(3,1)}$, given explicitly by:

$$\Gamma^{(3,1)} = \frac{1}{3!} \int \frac{d^3p_\parallel}{(2\pi)^3} B(p_\parallel) |p_\parallel|^3 \times \int d^3x_\parallel [\eta(x_\parallel)]^3 \equiv \Gamma^{(3)}_I$$

(43)

does not necessarily vanish. We have used $\Gamma^{(3)}_I$, to denote this contribution, the only ultralocal term at the third order.

On the other hand, $\Gamma^{(3,2)}$ is ‘bilocal’, i.e., it is cubic in $\eta$ and involves products of $\eta$ at two (eventually equal) points. It is the only bilocal piece of $\Gamma^{(3)}$. Its explicit form is

$$\Gamma^{(3,2)} = \frac{1}{2} \int \frac{d^3k_\parallel}{(2\pi)^3} \tilde{f}^{(3)}(k_\parallel) \tilde{\eta}(k_\parallel) \tilde{\eta}^2(-k_\parallel) \equiv \Gamma^{(3)}_b,$$

(44)

(note the Fourier transform of $\eta^2$), where we introduced $\tilde{f}^{(3)}_b$, the bilocal kernel at the third order:

$$\tilde{f}^{(3)}_b(k_\parallel) = -\int \frac{d^3p_\parallel}{(2\pi)^3} B(p_\parallel) |p_\parallel| |p_\parallel + k_\parallel|^2.$$

(45)

Finally, $\Gamma^{(3,3)}$ is trilocal, since the three $\eta$’s appear at
(generally) different spacetime points:

\[
\Gamma^{(3,3)} = \frac{1}{3!} \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} f_3^{(3)}(k||,q||) \tilde{\eta}(k||) \tilde{\eta}(q||) \\
\times \tilde{\eta}(-k|| - q||) = \Gamma^{(3)}_i ,
\]

(46)

where the \( f^{(3)} \) kernel is given by:

\[
f_3^{(3)}(k||,l||) = 2 \int \frac{d^3p}{(2\pi)^3} |l|| + p|| |l|| + p|| |p|| \\
\times \left[ 4B(k|| + l|| + p||)B(l|| + p||)B(p||) \\
+ B(k|| + l|| + p||)B(l|| + p||) \\
+ 5B(k|| + l|| + p||)B(p||) + 3B(k|| + l|| + p||) \right].
\]

(47)

The full third order term is then given by the sum of the previously derived local, bilocal and trilocal contributions:

\[
\Gamma^{(3)} = \Gamma^{(3)}_i + \Gamma^{(3)}_b + \Gamma^{(3)}_t .
\]

(48)

E. Fourth order

As we have just done for the third order contribution, we present the partial results corresponding to what we denoted as \( \Gamma^{(4,1)} \), for the different values of \( l \), which run from 1 to 5.

The \( \Gamma^{(4,1)} \) term, given by \( [24] \), contains both local and bilocal parts, the local one being \( a \)-independent. After discarding the \( a \)-independent part, the explicit form of the term becomes:

\[
\Gamma^{(4,1)} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} f^{(4,1)}(k||) \left[ \tilde{\eta}^2(k||) \tilde{\eta}^2(-k||) \\
- \frac{4}{3} \eta^3(-k||) \tilde{\eta}(k||) \right]
\]

(49)

with

\[
f^{(4,1)}(k||) = \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} B(p||) |p|| |p|| + k||^3 .
\]

(50)

The next term, \( \Gamma^{(4,2)} \) may be written as follows:

\[
\Gamma^{(4,2)} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} f^{(4,2)}(k||) \tilde{\eta}^3(-k||) \tilde{\eta}(k||) .
\]

(51)

with

\[
f^{(4,2)}(k||) = \frac{4}{3} f^{(4,1)}(k||) \\
- \frac{2}{3} \int \frac{d^3p}{(2\pi)^3} B(p|| + k||)B(p||) |p|| |p|| + k||^3 .
\]

(52)

We now write the result corresponding to \( \Gamma^{(4,3)} \), which involves three different kernels:

\[
\Gamma^{(4,3)} = \frac{1}{4!} \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} f_1^{(4,3)}(k||,q||,l||) \tilde{\eta}(k||) \\
\times \tilde{\eta}(q||) \tilde{\eta}(l||) \tilde{\eta}(-k|| - q|| - l||) \\
+ \frac{1}{3!} \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} f_2^{(4,3)}(k||,q||) \tilde{\eta}(k||) \tilde{\eta}(q||) \\
\times \tilde{\eta}^2(-k|| - q||) \\
+ \frac{1}{2!} \int \frac{d^3k}{(2\pi)^3} f_3^{(4,3)}(k||) \tilde{\eta}^2(k||) \tilde{\eta}^2(-k||) .
\]

(53)

After some rather lengthy algebraic manipulations, we see that the form of the kernels introduced above is the following (again, discarding \( a \)-independent terms):

\[
f_1^{(4,3)}(k||,q||,l||) = -6 \int \frac{d^3p}{(2\pi)^3} \left[ B(k|| + q|| + l|| + p||)B(l|| + p||) \\
+ B(k|| + q|| + l|| + p||) + B(l|| + p||) \right] \\
\times |k|| + q|| + l|| + p|| |q|| + l|| + p|| |l|| + p|| |p|| ,
\]

(54)

\[
f_2^{(4,3)}(k||,q||) = 3 \int \frac{d^3p}{(2\pi)^3} B(p||) |k|| + q|| + p||^2 |q|| + p|| |p||
\]

(55)

and

\[
f_3^{(4,3)}(k||) = -f^{(4,1)}(k||) .
\]

(56)

Let us present now the results corresponding to the \( \Gamma^{(4,4)} \) contribution. We have found that it contains both 4-local and 3-local terms:

\[
\Gamma^{(4,4)} = \frac{1}{4!} \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3l}{(2\pi)^3} f_1^{(4,4)}(k||,q||,l||) \tilde{\eta}(k||) \\
\times \tilde{\eta}(q||) \tilde{\eta}(l||) \tilde{\eta}(-k|| - q|| - l||) \\
+ \frac{1}{3!} \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} f_2^{(4,4)}(k||,q||) \tilde{\eta}(k||) \tilde{\eta}(q||) \\
\times \tilde{\eta}^2(-k|| - q||) \\
+ \frac{1}{2!} \int \frac{d^3k}{(2\pi)^3} f_3^{(4,4)}(k||) \tilde{\eta}^2(k||) \tilde{\eta}^2(-k||) ,
\]

(57)

with the kernels:

\[
f_1^{(4,4)}(k||,q||,l||) = -12 \int \frac{d^3p}{(2\pi)^3} |k|| + q|| + l|| + p|| \\
\times |q|| + l|| + p|| |l|| + p|| |p|| \\
\times \left[ 4B(k|| + q|| + l|| + p||)B(q|| + l|| + p||)B(l|| + p||) \\
+ 3B(k|| + q|| + l|| + p||)B(q|| + l|| + p||) \\
+ B(q|| + l|| + p||)B(l|| + p||) \\
+ B(k|| + q|| + l|| + p||)B(l|| + p||) + B(q|| + l|| + p||) \right] ,
\]

(58)

and

\[
f_2^{(4,4)}(k||,q||) = 3 \int \frac{d^3p}{(2\pi)^3} |k|| + q|| + p|| |q|| + p|| |p||^2 \\
\times \left[ 3B(k|| + q|| + p||)B(q|| + p||) \\
- B(q|| + p||)B(p||) + B(q|| + p||) \right].
\]

(59)
The $\Gamma^{(4,5)}$ term is 4-local:

$$\Gamma^{(4,5)} = \frac{1}{4!} \int \frac{d^3 k_\parallel}{(2\pi)^3} \frac{d^3 q_\parallel}{(2\pi)^3} \frac{d^3 l_\parallel}{(2\pi)^3} f^{(4,5)}(k_\parallel, q_\parallel, l_\parallel, \bar{\eta}(k_\parallel)) \times \bar{\eta}(q_\parallel)\bar{\eta}(l_\parallel)\bar{\eta}(-k_\parallel - q_\parallel - l_\parallel), \quad (60)$$

with:

$$f^{(4,5)}(k_\parallel, q_\parallel, l_\parallel) = -6 \int \frac{d^3 p_\parallel}{(2\pi)^3} |k_\parallel + q_\parallel + l_\parallel + p_\parallel|$$

$$\times |q_\parallel + l_\parallel + p_\parallel| |l_\parallel + p_\parallel|$$

$$\times \left[ 8B(k_\parallel + q_\parallel + l_\parallel + p_\parallel)B(q_\parallel + l_\parallel + p_\parallel)B(l_\parallel + p_\parallel)B(p_\parallel) 
+ 5B(k_\parallel + q_\parallel + l_\parallel + p_\parallel)B(q_\parallel + l_\parallel + p_\parallel)B(l_\parallel + p_\parallel)B(p_\parallel) 
+ 2B(k_\parallel + q_\parallel + l_\parallel + p_\parallel)B(q_\parallel + l_\parallel + p_\parallel)B(l_\parallel + p_\parallel)B(p_\parallel) 
+ B(k_\parallel + q_\parallel + l_\parallel + p_\parallel)B(l_\parallel + p_\parallel)B(p_\parallel) \right]. \quad (61)$$

V. LOW AND HIGH MOMENTUM EXPANSIONS: CONNECTION WITH THE PFA

In order to discuss the forthcoming approximations, we find it convenient to first write the previously considered perturbative expansion as follows:

$$E_{\text{vac}} = -\frac{\pi^2 L^2}{1440a^3} + \frac{1}{T} \sum_{n \geq 1} \frac{1}{a^{3+n}} \int \frac{d^3 k_\parallel^{(1)}}{(2\pi)^3} \frac{d^3 k_\parallel^{(n)}}{(2\pi)^3} \times \delta(k_\parallel^{(1)} + \ldots + k_\parallel^{(n)})h^{(n)}(ak_\parallel^{(1)}, \ldots, ak_\parallel^{(n)}) \times \bar{\eta}(k_\parallel^{(1)}) \ldots \bar{\eta}(k_\parallel^{(n)}). \quad (62)$$

The explicit form of the form factors $h^{(n)}$, up to $n = 4$, is completely determined by the results obtained in the previous section. Indeed, one just needs to include the proper $\delta$ function factors in some of the terms we calculated, before adding them. In the equation above we have made explicit the fact that the (dimensionless) functions $h^{(n)}$ depend on the dimensionless variables $ak_\parallel^{(j)}$, $j = 1, \ldots, n$.

Note that (62) has the structure of the general Taylor expansion of a functional in terms of its argument, an expression entirely analogous to the one we could use for the expansion of the effective action in a $2 + 1$ dimensional quantum field theory, in terms of the ‘field’ $\eta$, the form factors $h^{(n)}$ playing the role of the $n$-point vertex functions.

A. PFA and the low momentum expansion

Let us assume here that the function $\eta(x_\parallel)$ is slowly varying. In terms of its Fourier transform, it will be peaked at zero momentum; therefore we can approximate

$$h^{(n)}(ak_\parallel^{(1)}, \ldots, ak_\parallel^{(n)}) \approx h^{(n)}(0, \ldots, 0).$$

As a consequence:

$$E_{\text{vac}} \approx \frac{\pi^2 L^2}{1440a^3} + \frac{1}{T} \sum_{n \geq 1} \frac{1}{a^{3+n}} h^{(n)}(0, \ldots, 0) \int d^2 x_\parallel \eta(x_\parallel)^n \quad (63)$$

In principle, one could evaluate the first terms by using the results of Sec. [IV]. However, there is also a shortcut: for a constant $\eta(x_\parallel) = \eta_0$, the vacuum energy is simply given by Eq. (63) with the replacement $\int d^2 x_\parallel \eta(x_\parallel)^n \rightarrow L^2 \eta_0^n$. But for this particular case we know, of course, that the answer is the Casimir energy between parallel plates separated by a distance $a + \eta_0$. Therefore, in the low momentum approximation the perturbative series can be summed up, the result being

$$E_{\text{vac}} \approx -\frac{\pi^2 L^2}{1440a^3} \int \frac{d^2 x_\parallel}{(a + \eta(x_\parallel))^3}, \quad (64)$$

which agrees with the PFA.

Note also that the above suggests a nontrivial consistency check for our calculations: indeed, one can compute the first terms of the series in Eq. (63) using the explicit expressions given in Sec. [IV] for the different contributions to the effective action $\Gamma^{(i,j)}$. We present some details of that calculation in the Appendix. The result is

$$E_{\text{vac}} \approx -\frac{\pi^2 L^2}{1440a^3} \int d^2 x_\parallel \left( 1 - 3\frac{\eta}{a} + 6\left(\frac{\eta}{a}\right)^2 
- 10\left(\frac{\eta}{a}\right)^3 + 15\left(\frac{\eta}{a}\right)^4 \right), \quad (65)$$

that agrees with the expansion in powers of $\eta$ of the PFA result.

One could also go beyond the PFA, by expanding the form factors around $k_\parallel = 0$. In this way, one should be able to recover the derivative expansion for the Casimir energy that we introduced in Ref. [6]. Once more, it is useful to have in mind the analogy with the usual expansions of the effective action in quantum field theory: the expansion in powers of the field corresponds to the perturbative expansion in powers of $\eta$, while the derivative expansion corresponds to an expansion in derivatives of $\psi = a + \eta$, i.e. the improved PFA [5].

B. High momentum expansion

We shall now consider the opposite limit, namely, $|k_\parallel|a \gg 1$, in which the scale of variation of the shape of the surface is much shorter than the mean separation between surfaces.

Let us begin by considering the second order term. In order to obtain the large-momentum behavior of the form factor $f^{(2)}$ in Eq. (41), we note that the first term is exponentially suppressed, i.e., it can be written as $|k_\parallel|e^{-2\eta|k_\parallel|}$
times a convergent integral. After a shift in the integration variables, the second term can be approximated by
\[
f^{(2)}(k_\parallel) \simeq -2 \int \frac{d^3p_\parallel}{(2\pi)^3} B(p_\parallel) \left| p_\parallel \right| \left| p_\parallel - k_\parallel \right|
\]
\[
\simeq -2|k_\parallel| \int \frac{d^3p_\parallel}{(2\pi)^3} B(p_\parallel) |p_\parallel|
\]
\[
= -\frac{\pi^2}{240} \frac{|k_\parallel|^3}{a^4}.
\]
Inserting this result into Eq.(40) we obtain
\[
\Gamma^{(2)} \simeq -\frac{\pi^2}{480a^4} \int d^3k_\parallel |k_\parallel| |\tilde{\eta}(k_\parallel)|^2.
\]
Unlike the low momentum expansion, the result is a non-local functional of the shape of the surface. It can be rewritten in configuration space in terms of the nonlocal operator \((-\nabla^2)^{1/2}\) as follows:
\[
\Gamma^{(2)} \simeq -\frac{\pi^2}{480a^4} \int d^3x_\parallel \eta(x_\parallel)(-\nabla^2)^{1/2}\eta(x_\parallel).
\]
We now consider the third order terms. From Eq.(45) it is quite straightforward to see that
\[
f^{(3)}_b(k_\parallel) \simeq -\frac{\pi^2}{480a^4} |k_\parallel|^2.
\]
Note that this contribution grows quadratically with the momentum and becomes local in configuration space:
\[
\Gamma^{(3,2)} \simeq \frac{\pi^2}{960a^4} \int d^3x_\parallel \eta^2(x_\parallel)\nabla^2\eta(x_\parallel).
\]
The trilocal contribution is determined by the form factor given in Eq.(48). For high momenta, the leading contribution comes from the last term, which is linear in \(B\). We obtain
\[
f^{(3)}_t(k_\parallel, l_\parallel) \simeq \frac{\pi^2}{80} \frac{|k_\parallel| (k_\parallel + l_\parallel)|}{a^4},
\]
which produces a nonlocal contribution to the vacuum energy
\[
\Gamma^{(3,3)} \simeq \frac{\pi^2}{480a^4} \int d^3x_\parallel \eta(x_\parallel)(-\nabla^2)^{1/2}\eta(x_\parallel)
\]
\[
\times (-\nabla^2)^{1/2}\eta(x_\parallel).
\]
The fourth order terms can be treated in a similar way. One can show that, to leading order, they grow cubically with the momenta.

VI. EXAMPLES

In this section we present some applications of the perturbative results obtained so far. As we will consider surfaces with periodic perturbations, the vacuum energy will be proportional to the size of the system. Therefore, we will compute the vacuum energy per unit area \(E_{\text{vac}} = E_{\text{vac}}/L^2\).

![FIG. 1. Plot of the function \(g_2\) defined in Eq.(73), as a function of the dimensionless variable \(x = q_1a\). Linear behavior of \(g_2\) for large \(x\) is shown.](image)

A. Sinusoidally corrugated surface

We will first consider the simplest case of a sinusoidally corrugated surface \(\eta(x_\parallel) = \epsilon \sin(q_1x_1)\), with \(\epsilon \ll a\). For this particular corrugation, the first and third order corrections do vanish, so on general grounds we expect
\[
E_{\text{vac}} \simeq -\frac{\pi^2}{1440a^3} \left[ 1 + g_2(q_1a) \left(\frac{\epsilon}{a}\right)^2 + g_4(q_1a) \left(\frac{\epsilon}{a}\right)^4 + \ldots \right]
\]
for some functions \(g_2\) and \(g_4\). The function \(g_2\) has already been calculated in Ref.\([2]\). For the sake of completeness, we present a plot of its numerical evaluation in Fig.1. In the zero momentum limit, the numerical results reproduce the expected value, given in Eq.(65). Moreover, the behavior for low momentum is quadratic \([2, 6]\), which is consistent with the existence of a derivative expansion for the Casimir energy \([6]\). On the other hand, in the limit \(q_1a \gg 1\), the plot reproduces the linear behavior \(g_2 \sim q_1a\) with the appropriate coefficient predicted in Eq.(67).

The behavior of \(g_4\) is shown in Fig.2. Once more, in the zero momentum limit the numerical result is consistent with Eq.(65) (see inset in Fig.2 for the small momentum behavior). On the other hand, in the high momentum limit, \(g_4 \sim (q_1a)^3\). It is worth to note that the perturbative expansion breaks down for very high momenta (short wavelength of the corrugations). Indeed, the ratio of the fourth to second order corrections is proportional to \((\epsilon q_1)^2\). Therefore, the fourth order correction becomes more relevant when approaching the short wavelength limit. In Fig.3 we plot the ratio between the fourth and second order correction coefficients, \(g_4/g_2\), as a function of \(x = q_1a\).
As a side point of the perturbative calculations, we have also discussed the relation between different approximations usually considered to compute Casimir forces. The results could be generalized, for example, to the case of the electromagnetic field in a geometry with two deformed mirrors. As a side point of the perturbative calculations, we have also discussed the relation between different approximations usually considered to compute Casimir forces. The results could be generalized, for example, to the case of the electromagnetic field in a geometry with two deformed mirrors.

VII. CONCLUSIONS

In this paper we have extended the perturbative results for the Casimir energy between slightly deformed mirrors up to the fourth order in the amplitude of the deformations. For simplicity we considered the case of a deformed mirror in front of a plane mirror, for a scalar field satisfying Dirichlet boundary conditions. The third and fourth order results are important to improve the accuracy of the (almost) analytic perturbative calculations, to have an explicit way to evaluate the validity of the rather simpler second order results, and to provide a benchmark for complex numerical calculations. The results could be generalized, for example, to the case of the electromagnetic field in a geometry with two deformed mirrors.

As a side point of the perturbative calculations, we have also discussed the relation between different approximations usually considered to compute Casimir forces. For the geometry considered in this paper, the Casimir...
energy is a functional of the shape of the surface \( \psi = a + \eta \). This functional can be expanded in powers of \( \eta/a \), as we did here, assuming small deviations from a flat surface. When the additional assumption that the function \( \eta \) is slowly varying is reliable, a resummation of the perturbative series is possible, and approximating the form factors by their zero-momentum values, the final result coincides with the PFA. This is a non perturbative result, valid for arbitrary amplitudes as long as the surface is gently curved. The PFA can be improved by expanding the form factors around \( k_\parallel = 0 \). If this expansion only involves even powers of the momentum, the higher order corrections can be written in terms of derivatives of the shape of the surface. This is the derivative expansion for the Casimir energy we proposed in previous papers \[6\]. On the other hand, when the form factors contain nonanalytic terms, the corrections to the PFA will be nonlocal, as it indeed happens for scalar field satisfying Neumann boundary conditions for nonzero temperature \[7\]. We have also obtained explicit expressions for the Casimir energy under the opposite assumption, i.e. strongly varying surfaces. In this case, the final result could be written in terms of nonlocal operators, as \((-\nabla^2)^{1/2}\).

Finally, we have presented some explicit examples. On the one hand, we have evaluated numerically the results, up to the fourth order, for the Casimir energy for the case of a sinusoidal corrugation. The numerical results reproduce the analytic results expected in the limits of low and high wavelengths. The fourth order term is particularly relevant when evaluating the Casimir energy in the limit of short wavelengths, where it becomes dominant. The results presented in the paper allowed us to assess the validity of the second order calculations obtained in previous works, without the necessity of a full numerical evaluation of the Casimir energy.

On the other hand, we have shown the case in which the corrugation consists of a combination of two sinusoidal functions of different wavelengths, one being twice the other. The distinctive characteristic of the result for this case is that, when relation between the two wavelengths holds, the Casimir energy bears a dependence on the relative phase of the sinusoidal functions appearing in the combination. This ‘interference effect was absent in the second order term, namely, the relative phases of the components are irrelevant.

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**APPENDIX**

In this Appendix we prove that the first terms in the perturbative expansion reproduce the PFA approximation when the form factors are evaluated at zero momentum. The first term in the series of Eq.\[65\] is given in Eq.\[39\]. Indeed, comparing both equations we obtain

\[
\frac{h^{(1)}(0)}{a^4} = \int \frac{d^3p_\parallel}{(2\pi)^3} B(p_\parallel) |p_\parallel|^2 = \frac{\eta^2}{480},
\]

that reproduces the linear term in \( \eta \) in Eq.\[65\].

In order to evaluate the term quadratic in \( \eta \), we compare Eqs.\[63, 40\]. We get

\[
h^{(2)}(0,0) = a^5 \int \frac{d^3p_\parallel}{(2\pi)^3} B(p_\parallel)(1+B(p_\parallel)) |p_\parallel|^2 = \frac{\eta^2}{240},
\]

giving the quadratic term in Eq.\[63\].

The zero momentum contribution to the third order can be read from Eqs.\[43, 44, 46\].

\[
h^{(3)}(0,0,0) = a^6 \int \frac{d^3p_\parallel}{(2\pi)^3} |p_\parallel|^3
\]

\[
\times \left[ \frac{B}{6} - \frac{B}{2} + \frac{B}{3}(4B^3 + 6B + 3) \right],
\]

where \( B = B(p_\parallel) \). The three terms in the integrals are the contributions coming from \( \Gamma^{(3,j)}, j = 1, 2, 3 \), respectively. The evaluation of the integral gives \( h^{(3)}(0,0,0) = \pi^2/144 \), reproducing the third term in Eq.\[65\].

The evaluation of the fourth order is more cumbersome but straightforward. The form factor \( h^{(4)}(0,0,0) \) can be written as a linear combination of \( f^{(4,i)}, i = 1,..,5 \), all of them evaluated at zero momenta. Adding all contributions we obtain

\[
h^{(4)}(0,0,0) = a^7 \int \frac{d^3p_\parallel}{(2\pi)^3} |p_\parallel|^4
\]

\[
\times \left[ \frac{1}{3}B + \frac{7}{3}B^2 + 4B^3 + 2B^4 \right].
\]

Evaluating the integral we obtain \( h^{(4)}(0,0,0) = -\pi^2/96 \), in agreement with the fourth order term in Eq.\[65\].

**ACKNOWLEDGEMENTS**

This work was supported by ANPCyt, CONICET, UBA and UNCuyo.

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