Optimal large deviation estimates and Hölder regularity of the Lyapunov exponents for quasi-periodic Schrödinger cocycles

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Abstract

We consider one-dimensional quasi-periodic Schrödinger operators with analytic potentials. In the positive Lyapunov exponent regime, we prove large deviation estimates which lead to optimal Hölder continuity of the Lyapunov exponents and the integrated density of states, in both small Lyapunov exponent and large coupling regimes. Our results cover all the Diophantine frequencies and some Liouville frequencies.

1 Introduction and the Main Results

In this paper, we study the following one dimensional discrete quasi-periodic operators on $\ell^2(\mathbb{Z})$:

\[(H(x)\varphi)(n) = \varphi(n-1) + \varphi(n+1) + v(x+n\omega)\varphi(n), \quad n \in \mathbb{Z},\]

where $x \in \mathbb{T} := [0, 1]$ is called phase, $\omega \in \mathbb{T} \setminus \mathbb{Q}$ is called frequency and the real valued analytic function $v : \mathbb{T} \rightarrow \mathbb{R}$ is called potential.

For an energy $E \in \mathbb{R}$, the Schrödinger equation

\[\varphi(n-1) + \varphi(n+1) + v(x+n\omega)\varphi(n) = E\varphi(n)\]

can be rewritten as the following skew-product:

\[
\begin{pmatrix}
\varphi(n+1) \\
\varphi(n)
\end{pmatrix} = M_n(\omega, E; x)
\begin{pmatrix}
\varphi(1) \\
\varphi(0)
\end{pmatrix},
\]

where $M_n$ is called $n$-step transfer matrix defined as follows:

\[
M_n(\omega, E; x) = \prod_{j=n}^{1} A_j(\omega, E; x), \quad A_j(\omega, E; x) = \begin{pmatrix} E-v(x+j\omega) & -1 \\ 1 & 0 \end{pmatrix}.
\]

Let

\[
u_n(\omega, E; x) := \frac{1}{n} \log \|M_n(\omega, E; x)\|, \quad L_n(\omega, E) := \int_{\mathbb{T}} \nu_n(\omega, E; x)dx.
\]

For any irrational $\omega \in [0, 1]$, the shift $x \rightarrow x+\omega$ is ergodic. The Früstenberg-Kesten theorem implies that the following limit exists for a.e. $x$:

\[
limes_{n \to \infty} \nu_n(\omega, E; x) = \lim_{n \to \infty} L_n(\omega, E) := L(\omega, E).
\]
The limit \( L(\omega, E) \) is called the \textit{Lyapunov exponent}.

Note that for any fixed \( \kappa > 0 \), \( \epsilon, \omega, \) the a.e. convergence in (1.6) implies

\[
\text{mes}\{ x \in \mathbb{T} : |u_n(\omega, E; x) - L_n(\omega, E)| > \kappa \} \to 0 \quad \text{as} \quad n \to \infty.
\] (1.7)

Thus the question lies in the convergent rate w.r.t. \( n \) and the dependence on \( \omega, E \) and \( \epsilon \). Such estimate is in general known as the \textit{Large Deviation Theory/Principle (LDT/LDP)} in probability theory. Roughly speaking, a \textit{law of large numbers (LLN)} in probability theory describes the most frequently visited states in a large system. To go beyond LLN, we may examine the states that deviate by large amount. This will lead to a LDP of the system if: (i) the probability of visiting a non-typical state is exponentially small and (ii) the exponential rate of convergence can be formulated with a precise formula as the size of the system. We will not go any further into the general LDP theory but will focus on LDT for the norms of the monodromy matrices as in form (1.7). For the general LDP theory in probability theory, we refer readers to [33, 20]. Some interesting results of LDT type estimates for random cocycles in a similar spirit to our paper can be found in [19, 13].

Another important quantity in the study of spectral theory is the \textit{integrated density of states (I.D.S.)}, defined as follows: let \( \Lambda = [a, b] \subset \mathbb{Z} \) be an interval, and \( H_\Lambda(x) \) be the restriction of \( H(x) \), see (1.1), onto \( \Lambda \) with zero boundary condition, in the sense that \( \varphi(a-1) = \varphi(b+1) = 0 \). Let \( E_{\Lambda,j}(x), j = 1, 2, \ldots, b - a + 1 = |\Lambda| \) be the eigenvalues of \( H_\Lambda(x) \). Consider the density function

\[
N_\Lambda(E, x) = \frac{1}{|\Lambda|} \sum_j \chi_{(-\infty, E)}(E_{\Lambda,j}(x)).
\]

It is well-known that the weak limit

\[
\lim_{a \to -\infty, b \to +\infty} dN_\Lambda(\cdot, x) =: dN(\cdot)
\]

exists and does not depend on \( x \) (up to a zero measure set). The distribution function \( N(\cdot) \) is called the \textit{integrated density of states}. It is connected with the Lyapunov exponent via the Thouless formula, see e.g. [16]:

\[
L(E) = \int \log |E - E'| dN(E').
\]

Large deviation type estimates were introduced to study quasi-periodic Schrödinger operators in the late 1990s in a series of papers by Bourgain, Goldstein and Schlag, [10, 24]. Their method has been well developed ever since and has shown to be sufficiently robust in the super-critical regime to deal with the following questions (not only restricted to the one dimensional quasi-periodic Schrödinger case):

1. Regularity of the \( L(E) \) and \( N(E) \) in energy \( E \), (e.g. [7, 24, 12, 31, 25]),

2. Localization of the eigenfunctions, (e.g. [10, 11, 31, 27]),

3. Eigenvalue separation and topological structure of the spectrum, (e.g. [24, 21, 28]).

In this paper, we will focus on Problem 1. For more details about rest of the problems, we refer readers to [23, 9, 29, 20] and references therein. By virtue of the Hilbert-transform, Hölder regularities of \( N(E) \) and \( L(E) \) pass from one to the other, for a proof of this fact, see [24]. Therefore we shall focus on the Lyapunov exponent in the rest of this paper.
Proving regularity of $L(E)$ and $N(E)$ (in $E$) is considered difficult for any type of sequence of potentials, see [15]. Some weak regularity for general ergodic families were first proved in cf [16]. For quasi-periodic Schrödinger operators, the first breakthrough was made by Goldstein and Schlag in [24]. They proved Hölder regularity of $L(E)$ and $N(E)$ for typical frequencies in $T$, assuming analyticity of the potential and positive Lyapunov exponents. Some weaker Hölder regularity was also obtained in the same paper for higher dimensional torus. Later Bourgain and Jitomirskaya proved in [12] that $L(\omega, E)$ is jointly-continuous in $(\omega, E)$ at any irrational frequency for analytic potentials. More delicate estimates on the sharp Hölder regularity were obtained in [25].

In the sub-critical regime with highly smooth potential, regularity results were proved more often by reducibility method, cf [1, 3, 4]. In the finitely smooth potential regime, fewer results were obtained with more restrictions on potential and frequency, see for example [31, 1, 36, 14].

In this paper we follow the scheme developed by Goldstein and Schlag [24], namely by combining LDT and the Avalanche Principle (see Theorem C.1) to obtain the Hölder continuity of $L(E)$ and $N(E)$:

$$|L(E) - L(E')| + |N(E) - N(E')| \leq |E - E'|^\tau, \quad |E - E'| \ll 1. \quad (1.8)$$

One of their key estimate for the one-dimensional case is

$$\text{mes}\{x \in \mathbb{T} : |u_n(\omega, E; x) - L_n(\omega, E)| > \kappa L(\omega, E)\} \leq e^{-c(\omega, v, \kappa)L^2(\omega, E)n}, \quad (1.9)$$

under the positive Lyapunov exponent condition $L(\omega, E) > \gamma > 0$, for $\omega$ satisfying the strong Diophantine condition, see (1.11). However, due to the $L^2(\omega, E)$ in the exponential estimate on the right hand side of (1.9), the Hölder exponent $\tau$ in (1.8) will tend to 0 as the lower bound $\gamma$ approaches 0.

In [9], the LDT estimate (1.9) was improved to be

$$\text{mes}\{x \in \mathbb{T} : |u_n(\omega, E; x) - L_n(\omega, E)| > \kappa L(\omega, E)\} \leq e^{-c(\omega, v, \kappa)L(\omega, E)n}, \quad (1.10)$$

in the small Lyapunov exponent regime, under the same assumption on $\omega$. The improvement implies that the local Hölder exponent is independent of the lower bound $\gamma$.

As we mentioned above, both (1.9) and (1.10) were established for $\omega$ satisfying S.D.C.. Going beyond S.D.C. is considered difficult for establishing LDT and Hölder continuity in general. Our first achievement of this paper is to extend the LDT estimates to more frequencies in the best possible regime, see (1.11). Indeed, Hölder continuity fails for generic $\omega$, see [31, 5]. Thus, the exponential decay (1.9) or (1.10) can not hold for all frequencies. Our extension is optimal in this sense.

In both (1.9) and (1.10), the dependence of $c(\omega, v, \kappa)$ on $v$ are not written down explicitly. In our paper, we incorporate a refined Riesz-representation of subharmonic functions of [25] into the proof of the LDT estimates. This leads to an explicit dependence of $c$ on $v$. It turns out the constant depends on the potential $v$ in a “sup–sup” form, see (2.5). If $v = \lambda f$, the “sup–sup” leads to a magical cancellation of $\lambda$. This leads to the second achievement of our paper, see Corollary 1.3. Combining with the Avalanche Principle, we obtain, for the first time, a $\lambda$-independent Hölder exponent in the large coupling regime for general non-trivial analytic potentials, see Theorem 1.10. Such kind of result was previously only known for trigonometric polynomials.

\footnotetext[1]{Note that we use the same symbol $c(\omega, v, \kappa)$ in both (1.9) and (1.10), but they are not the same constants.}

\footnotetext[2]{See the paragraph below Theorem 1.2 of [3]: for $v = \lambda \cos$ with $\lambda \neq 0$, Lyapunov exponent is discontinuous at rational $\omega$’s, thus it is not Hölder for $\omega$’s that are well approximated by rationals.
In order to formulate our results, we introduce the following notations: for any $x \in \mathbb{R}$, let $\|x\|_\mathbb{T} := \inf_{n \in \mathbb{Z}} |x - n|$. For any $\omega \in [0, 1] \setminus \mathbb{Q}$, let $[a_1, a_2, a_3, \ldots]$ be its continued fraction expansion. Let $\{p_s/q_s\}_{s=1}^\infty$ be its continued fraction approximants, defined by $p_s/q_s = [a_1, a_2, \ldots, a_s]$. It is well known that $\|q_s \omega\|_\mathbb{T} \leq q_s^{-1}$. We say $\omega$ satisfies the strong Diophantine condition (S.D.C.) (or $\omega$ is strong Diophantine)\(^3\), if for some constants $a > 1$, $c > 0$, the following holds for any $n \geq 1$,

$$\|n \omega\|_\mathbb{T} \geq \frac{c}{n (\log n)^a}. \quad (1.11)$$

Note that for any $a > 1$, a.e. $\omega$ satisfies S.D.C. for some $c = c(\omega) > 0$. It is also clear from the definition of S.D.C. that for strong Diophantine $\omega$,

$$q_{s+1} \leq e^{-1} q_s (\log q_s)^a. \quad (1.12)$$

Next we introduce an exponential growth exponent $\beta$ defined as follows:

$$\beta(\omega) := \limsup_{s \to +\infty} \frac{\log q_{s+1}}{q_s} \in [0, \infty]. \quad (1.13)$$

It is then clear from (1.12) that S.D.C. $\subseteq \{ \omega : \beta(\omega) = 0 \}$. Those $\omega$ with $\beta(\omega) > 0$ are usually called Liouville numbers.

Since our potential $v(x)$ is a real analytic function, it has a bounded extension to a strip $|\text{Im} z| < \rho$ with width denoted by $\rho > 0$. Let $\mathcal{N}_v = [-2 - \|v\|_\infty, 2 + \|v\|_\infty]$ be the numerical range of the Schrödinger operator $H$. It is well known that $\sigma(H) \subset \mathcal{N}_v$ and $L(E)$ is $C^\infty$ functions outside of the spectrum. Hence we will only consider $E \in \mathcal{N}_v$ throughout the paper.

**Theorem 1.1.** Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$. There exist constants $c(v, \rho), \tilde{c}(v, \rho) \in (0, 1)$ such that, if

$$0 \leq \beta(\omega) < c(v, \rho) \inf_{E \in [a, b]} L(\omega, E), \quad (1.14)$$

then there is $N = N(\omega, \inf_{E \in [a, b]} L(\omega, E), v, \rho) \in \mathbb{N}$, we have for any $n \geq N$ the following large deviation estimates hold uniformly in $E \in [a, b],$

(a) If $0 < L(\omega, E) < 1$, then

$$\text{mes} \left\{ x \in \mathbb{T} : |u_n(\omega, E; x) - L_n(\omega, E)| > \frac{1}{20} L(\omega, E) \right\} \leq e^{-\tilde{c}(v, \rho)L(\omega, E)n}. \quad (1.15)$$

(b) If $L(\omega, E) \geq 1$, then

$$\text{mes} \left\{ x \in \mathbb{T} : |u_n(\omega, E; x) - L_n(\omega, E)| > \frac{1}{20} L(\omega, E) \right\} \leq e^{-\tilde{c}(v, \rho)L(\omega, E)n}. \quad (1.16)$$

**Remark 1.2.** The parameter $1/20$ in Theorem 1.1 can be replaced by any $0 < \kappa < 1$. The new constants $c(\kappa(v, \rho), \tilde{c}(v, \rho))$ only differ from $c(v, \rho), \tilde{c}(v, \rho)$ by a constant multiple of $\kappa^{-2}$. However, in order to apply Avalanche Principle to obtain Hölder continuity, $\kappa$ at most can be taken to be $1/9$ due to technique reasons (see (C.10)). We do not intend to improve the Hölder exponents in the paper by getting the best possible $\kappa$, thus we take $\kappa = 1/20$ for simplicity. See more discussions about the sharp Hölder exponents after Theorem 1.5.

\(^3\) We say $\omega$ satisfies Diophantine condition (D.C.) if $\|n \omega\|_\mathbb{T} \geq \frac{c}{n}$ for all $n > 1$ and some $a > 1, c > 0$. Note that for any $a > 1$, a.e. $\omega$ satisfies D.C. with some $c = c(\omega) > 0$. 
Corollary 1.3. Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $v(x)$ in (1.1) is given by $v(x) = \lambda f(x)$, where $\lambda$ is a positive constant. There exist constants $0 < b = b(f, \rho) < 1$, $B = B(f, \rho) > 1$ and $\lambda = \lambda(f, \rho) > 0$ with the following properties: for any irrational $\omega$ with $0 \leq \beta(\omega) < \infty$, suppose

$$\lambda > \max(\tilde{\lambda}, e^{B\beta(\omega)}),$$

then there is $N(\omega; \lambda, f, \rho) \in \mathbb{N}$ such that for any $n \geq N(\omega; \lambda, f, \rho)$, the following holds

$$\text{mes}\left\{x \in \mathbb{T} : |u_n(\omega, E; x) - L_n(\omega, E)| > \frac{1}{19} \log \lambda\right\} \leq e^{-n B \log \lambda}. \quad (1.17)$$

Remark 1.4. The above exponential decay of the measure estimate w.r.t. $\log \lambda$ for large coupling $\lambda$ is known for the first time even for $\beta(\omega) = 0$ or S.D.C. $\omega$ to the authors’ knowledge.

As mentioned previously in (1.8), a direct consequence of the above large deviation estimates is the Hölder regularity of the Lyapunov exponents. With the refined parameters in the LDT estimates (1.15)-(1.17), we have the following optimal Hölder continuity of the Lyapunov exponents.

Theorem 1.5. Let $c = c(v, \rho), \tilde{c} = \tilde{c}(v, \rho)$ be the constants in Theorem 1.7. There exists a constant $\tau > 0$ depending explicitly (and only) on $\tilde{c}(v, \rho)$ that satisfies the following property: if $(\omega_0, E_0) \in (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{N}_+$ is a point with $L(\omega_0, E_0) = \gamma > 0$, and $U \times I$ is a neighborhood of $(\omega_0, E_0)$ such that $L(\omega, E) \in [\frac{18}{19} \gamma, \frac{20}{19} \gamma]$, then for any $\omega \in U$ with

$$0 \leq \beta(\omega) < \frac{1}{2} c \gamma,$$

there is $\eta = \eta(\omega, I, \gamma, v)$ such that the following holds for any $E, E' \in I$ and $|E - E'| < \eta$,

$$|L(\omega, E) - L(\omega, E')| \leq |E - E'|^\tau. \quad (1.18)$$

Remark 1.6. By (1.2), $L(\omega, E)$ is jointly-continuous in $(\omega, E)$ at $(\omega_0, E_0)$. Hence the neighborhood $U \times I$ always exists.

Remark 1.7. Theorem 1.5 is optimal in the sense that the exponent $\tau$ is independent of the lower bound of the Lyapunov exponent, $\gamma$. This generalizes the result in [3] for general analytic potentials from $\omega$ satisfying S.D.C. to $0 \leq \beta(\omega) \lesssim \gamma$.

Remark 1.8. For trigonometric polynomial potentials, there are results on sharp Hölder exponents that only depend on the degree of the polynomial: $\frac{1}{2}$-Hölder if $v = \lambda \cos$, $\lambda \neq 0, 1$, [7, 3]; and $(\frac{1}{2k} - \epsilon)$-Hölder if $v$ is a small $C^\infty$ perturbation of trigonometric polynomial of degree $k$ [25]. Our current approach does not lead to such kind of sharp exponent for general analytic potentials, even for S.D.C. $\omega$.

Remark 1.9. If $v$ is of the form $\lambda f$, with a general analytic $f$, in the small coupling regime $\lambda < \lambda_0(f)$, $\frac{1}{2}$-Hölder exponents were obtained in [3] using reducibility method. However there is no such kind of result for the large coupling regime. In general, from a dynamical systems perspective, it would be natural to expect bad behavior in the positive Lyapunov exponent regime, see [6]. Our Theorem 1.10 is a breakthrough in this regime, by giving a $\lambda$-independent Hölder exponent for general analytic $f$.

\footnote{For general analytic potential $v = \lambda f$, if one applies the LDT in [24] and check all the constants explicitly, the Hölder exponent behaves like $O((\log \lambda)^{-1})$ for large $\lambda$ even for S.D.C. $\omega$, see more explanation in [27].}
If \( v = \lambda f \), we have the following:

**Theorem 1.10.** Under the same condition of Corollary 1.3, let constants \( \tilde{\lambda}, b, B \) be given there. There exists constant \( \tilde{\tau} > 0 \) depending explicitly (and only) on \( b \) (hence independent of \( \lambda \)) such that for any irrational \( \omega \) with \( 0 \leq \beta(\omega) < \infty \), if \( \lambda > \max(\tilde{\lambda}, e^{B\beta(\omega)}) \), then there exists \( \tilde{\eta} = \tilde{\eta}(\omega, \lambda, f, \rho) > 0 \), such that for any \( E, E' \in \mathcal{N}_f \) and \( |E - E'| \leq \tilde{\eta} \), we have

\[
|L(E) - L(E')| \leq |E - E'|^{\tilde{\tau}}. \tag{1.19}
\]

The rest of the paper is organized as follows: in section 2, we state all the important technique lemmas. In section 3, we prove the three large deviation estimates by the useful lemmas in Section 2. Our Hölder continuity follows directly from LDT and a standard argument combined with Avalanche Principle. For sake of completeness, we sketch the proof in section 5. Many details of this part are included in the Appendix for reader’s convenience.

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## 2 Useful lemmas

Let \( \mathcal{N}_v = [-2 - \|v\|_\infty, 2 + \|v\|_\infty] \), as we mentioned before, we will only consider \( E \in \mathcal{N}_v \) throughout the paper. Recall that \( u_n(\omega, E; x) \) is defined as in (1.5).

This section contains lemmas that will be used in the proofs of Theorems 1.1 and 1.3. The proofs of these lemmas will be included in Sec. 4. Let

\[
\Lambda_v := \log(3 + 2\|v\|_{L^\infty(\mathbb{T})}).
\]

Simple computations yield

\[
\sup_{E \in \mathcal{N}_v} \|u_n(\omega, E; \cdot)\|_{L^\infty(\mathbb{T})} \leq \Lambda_v,
\]

holds uniformly in \( \omega \in \mathbb{T} \) and \( 1 \leq n \in \mathbb{N} \).

Since in our model, \( v \) is assumed to have bounded analytic extension to \( \mathbb{T}_\rho := \{ z : |\text{Im}z| < \rho \} \), \( u_n \) has subharmonic extension on \( \mathbb{T}_\rho \) with a uniform upper bound

\[
\sup_{E \in \mathcal{N}_v} \sup_{n \in \mathbb{N}} \|u_n(\omega, E; \cdot)\|_{L^\infty(\mathbb{T}_\rho)} \leq \log(3 + 2\|v\|_{L^\infty(\mathbb{T}_\rho)}) < \infty.
\]

### 2.1 Estimates of the Fourier coefficients \( \hat{u}_n(\omega, E; k) \)

\( u_n(\omega, E; x) \) is a 1-periodic function on \( \mathbb{R} \), we denote its Fourier coefficients by

\[
\hat{u}_n(\omega, E; k) = \int_{\mathbb{T}} u_n(\omega, E; x)e^{-2\pi ikx} dx.
\]

The following estimate of the Fourier coefficient is well-known and crucial to establishing our LDT, see e.g. Bourgain’s monograph [9, Corollary 4.7]. For a version of this estimate written precisely in the “sup — sup” form below, see [13, Lemma 2.8]. To obtain this “sup — sup” estimate, one need to invoke a refined Riesz-representation theorem [23, Lemma 2.2]. See details in Sec. 4.1.
Lemma 2.1. There is a constant $\alpha(\rho) > 0$ depending on $\rho$ only, such that for any $k \neq 0$,

$$|\hat{u}_n(\omega; E; k)| \leq \frac{\alpha(\rho)}{|k|} \left( \sup_{|\text{Im} z| < \rho} u_n(\omega; E; z) - \sup_{|\text{Im} z| < \rho/2} u_n(\omega; E; z) \right).$$

(2.4)

Corollary 2.2. Let

$$C(v, \rho) := \alpha(\rho) \sup_{E \in \mathcal{N}_v} \left( \sup_{|\text{Im} z| < \rho} u_n(\omega; E; z) - \sup_{|\text{Im} z| < \rho/2} u_n(\omega; E; z) \right) < \infty.$$  

(2.5)

We then have that for any $k \neq 0$ and $E \in \mathcal{N}_v$,

$$|\hat{u}_n(\omega; E; k)| \leq \frac{C(v, \rho)}{|k|}.$$  

(2.6)

When $v$ is given as $\lambda f$, we can bound the above constant $C(\lambda f, \rho)$ by a constant independent of $\lambda$. This turns out to be crucial to our proof of Corollary 1.3.

Lemma 2.3. Let $C(v, \rho)$ be the constant defined as in (2.5). Suppose $v = \lambda f$. Then there is $C_0(f, \rho) > 0$, independent of $\lambda$, such that for any $\lambda > 0$,

$$C(\lambda f, \rho) \leq C_0(f, \rho).$$

(2.7)

Besides the Fourier decay estimate in Lemma 2.1, we also prove a new estimate as follows. This estimate improves that of Lemma 2.1 for small $|k|$ when $n$ is large. It will play a crucial role in our proof of part (a) of Theorem 1.1.

Lemma 2.4. Let $\Lambda_v$ be the constant defined in (2.1). We have the following bounds of the Fourier coefficients, for any $k \neq 0$,

$$|\hat{u}_n(\omega; E; k)| \leq \frac{\Lambda_v}{2n|k\omega|^T}.$$  

2.2 $\|u_n(\omega; E; \cdot)\|_{L^\infty(T)}$ under small Lyapunov exponent condition

We present an upper bound of $\|u_n(\omega; E; x)\|$, see Lemma 2.6 below. This can be viewed as a generalization of [9, Lemma 8.18], where a similar bound was proved for Diophantine $\omega$. Compared to a trivial bound $\|u_n(\omega; E; x)\| \leq \Lambda_v$, the new bound is much more effective when the Lyapunov exponent is small.

Compared to [9, Lemma 8.18], our improvement lies in the fact that we can relax the Diophantine condition on $\omega$. Indeed we give explicit dependence of the upper bound on the continued fraction approximants of $\omega$, through the $\log q_s+1/q_s$ term. This improvement enables us to cover Liouville frequencies.

For $R \in \mathbb{N}$, let $u_n^{(R)}$ be the average of $u_n$ along a trajectory with length $\sim R$, defined as below:

$$u_n^{(R)}(\omega; E; x) := \sum_{|j| < R} \frac{R - |j|}{R^2} u_n(\omega; E; x + j\omega).$$

(2.8)
Lemma 2.5. Let $C(v, \rho), C_3$ be the constants as in (2.5), (4.21). Suppose $0 < L(\omega, E) < 1$, we have the following upper bound of $u_n^{(R)}(\omega, E; x)$,

$$
\|u_n^{(R)}(\omega, E; \cdot)\|_{L^\infty(T)} \leq L_n(\omega, E) + (2 + 8C(v, \rho) + 4\pi C_3 C(v, \rho)) L(\omega, E) + 120C(v, \rho) \frac{\log q_s + 1}{q_s},
$$

(2.9)

holds for

$$
n \geq 2\Lambda_v L(\omega, E)^{-2} \sup_{1 \leq |k| \leq L(\omega, E)^{-1}} \frac{1}{\|k\omega\|_T},
$$

and

$$
R \geq 144L(\omega, E)^{-5}.
$$

Lemma 2.5 leads to the following

Lemma 2.6. Let $C(v, \rho), C_3$ be the constants as in (2.5), (4.21). Suppose $0 < L(\omega, E) < 1$, we have the following upper bound of $u_n(\omega, E; x)$,

$$
\|u_n(\omega, E; \cdot)\|_{L^\infty(T)} \leq L_n(\omega, E) + C_1 L(\omega, E) + 120C(v, \rho) \frac{\log q_s + 1}{q_s},
$$

(2.10)

holds for $n \geq N_0(\omega, L(\omega, E), v, \rho)$, where $C_1$ explicitly depends on $C(v, \rho), \Lambda_v$ as

$$
C_1 := 2 + \Lambda_v + 8C(v, \rho) + 4\pi C_3 C(v, \rho)
$$

(2.11)

and

$$
N_0(\omega, L(\omega, E), v, \rho) := L(\omega, E)^{-2} \max \left( \frac{2\Lambda_v}{4\Lambda_v}, \frac{1}{\|k\omega\|_T}, 49L(\omega, E)^{-4} \right).
$$

(2.12)

2.3 Two estimates of $\|u_n(\omega, E; \cdot) - u_n^{(R)}(\omega, E; \cdot)\|_{L^\infty(T)}$

The following lemmas give upper bounds of $\|u_n - u_n^{(R)}\|_{L^\infty(T)}$ under different conditions.

Lemma 2.7. Let $\Lambda_v$ be the constant defined in (2.1). For any $n, R, \omega$, we have

$$
\left\|u_n(\omega, E; \cdot) - u_n^{(R)}(\omega, E; \cdot)\right\|_{L^\infty(T)} \leq 2\Lambda_v \frac{R}{n},
$$

Recall the following uniform convergence in [12].

Lemma 2.8. [12] Corollary 3] Suppose $v$ is analytic. Then

$$
\limsup_{n \to \infty} u_n(\omega, E; x) \leq L(\omega, E)
$$

(2.13)

uniformly in $x$ and $E$ in a compact set.

A direct consequence is
Lemma 2.9. Suppose $L(\omega, E) > 0$ for all $E \in [a, b]$. There exists $\tilde{N}_0(\omega, [a, b], v)$ such that for any $n > \tilde{N}_0(\omega, [a, b], v)$, any $x \in \mathbb{T}$ and $E \in [a, b]$, we have

$$u_n(\omega, E; x) \leq \left(1 + \frac{1}{20}\right) L(\omega, E)$$

and

$$L_n(\omega, E) \leq \left(1 + \frac{1}{20}\right) L(\omega, E).$$

A more delicate upper bound of the difference $u_n - u_n^{(R)}$, when $L(\omega, E)$ is small, is given as follows. This upper bound will be the key to Theorem 1.1, part (a). Let $N_0$ be as in (2.12) and $\tilde{N}_0$ be as in Lemma 2.9. Define

$$N_1(\omega, [a, b], L(\omega, E), v, \rho) := \max(N_0(\omega, L(\omega, E), v, \rho), \tilde{N}_0(\omega, [a, b], v) + 1).$$

Using Lemma 2.6 and Lemma 2.9 we obtain the following:

Lemma 2.10. Let $C_1, N_1$ be as in (2.11), (2.16) and $C(v, \rho), \Lambda_v$ be as in (2.5), (2.1) respectively. Suppose $0 < L(\omega, E) < 1$. For $R = \lfloor(400(C_1 + 2))^{-1} n\rfloor + 1$, we have

$$\left\|u_n(\omega, E; \cdot) - u_n^{(R)}(\omega, E; \cdot)\right\|_{L^\infty(\mathbb{T})} \leq \frac{1}{100} L(\omega, E) + \frac{1}{5} C(v, \rho) \log \frac{q_{s+1}}{q_s},$$

holds for $n \geq N_2(\omega, [a, b], L(\omega, E), v, \rho)$, where

$$N_2(\omega, [a, b], L(\omega, E), v, \rho) := \max(150\Lambda_v N_1 L(\omega, E)^{-1}, 400(C_1 + 2)N_1 + 1).$$

Remark 2.11. We point out $N_1(\omega, [a, b], L(\omega, E), v, \rho)$ is a deceasing function in the third parameter $L(\omega, E)$, so is $N_2(\omega, [a, b], L(\omega, E), v, \rho)$. This is clear from the definitions (2.12), (2.16) and (2.17).

3 Large deviation estimates.

For simplicity, from this point on, when there is no ambiguity, we will sometimes write $u_n(x) = u_n(\omega, E; x)$, $L_n = L_n(\omega, E)$ and $L = L(\omega, E)$.

3.1 Preparation

Let $\hat{u}_n(k)$ and $u_n^{(R)}(x)$ be defined as in (2.3), (2.8). Let

$$F_R(k) := \sum_{|j| < R} \frac{R - |j|}{R^2} e^{2\pi ikj\omega}.$$ 

Let us recall the following estimates of $F_R(k)$ in [12, 37], whose proofs are included in the Appendix E.

$$0 \leq F_R(k) \leq \min\left(1, \frac{2}{1 + R^2\|k\omega\|_T^2}\right),$$

where
\[
\sum_{1 \leq |k| < q/4} \frac{1}{1 + R^2 \|k\omega\|^2_T} \leq 2 \frac{q}{R}, \quad (3.3)
\]
\[
\sum_{|k| \in ([q/4], (q+1)/4)} \frac{1}{1 + R^2 \|k\omega\|^2_T} \leq 2 + 4 \frac{q}{R}, \quad \forall \ell \in \mathbb{N}, \quad (3.4)
\]

in which \(p/q\) is any continued fraction approximant of \(\omega\).

Direct computation shows that
\[
u_n^{(R)}(x) = L_n + \sum_{k \in \mathbb{Z}, k \neq 0} \hat{u}_n(k) F_R(k) e^{2\pi ikx} \quad (3.5)
\]

Let \(p_s/q_s, p_{s+1}/q_{s+1}\) be any two consecutive continued fraction approximants of \(\omega\). For \(0 < \delta \leq 1\), let us consider
\[
\nu_n(x) - L_n = \nu_n(x) - \nu_n^{(R)}(x) + \nu_n^{(R)}(x) - L_n
\]
\[
= \nu_n(x) - \nu_n^{(R)}(x) \quad (=: U_1(x))
\]
\[
+ \sum_{1 \leq |k| < \delta^{-1}} \hat{u}_n(k) F_R(k) e^{2\pi ikx} \quad (=: U_2(x))
\]
\[
+ \sum_{\delta^{-1} \leq |k| < q_s/4} \hat{u}_n(k) F_R(k) e^{2\pi ikx} \quad (=: U_3(x))
\]
\[
+ \sum_{q_s/4 \leq |k| < q_{s+1}/4} \hat{u}_n(k) F_R(k) e^{2\pi ikx} \quad (=: U_4(x))
\]
\[
+ \sum_{q_{s+1}/4 \leq |k| < K} \hat{u}_n(k) F_R(k) e^{2\pi ikx} \quad (=: U_5(x))
\]
\[
+ \sum_{|k| \geq K} \hat{u}_n(k) F_R(k) e^{2\pi ikx} \quad (=: U_6(x)).
\]

By Lemma 2.4, we have some refined estimates of \(U_2(x)\) and \(U_3(x)\):

**Proposition 3.1.** Let \(\Lambda_v, C(v, \rho)\) be given as in (2.1), (2.5). For any \(n \geq 1\) and \(R \in [q_s, q_{s+1})\), we have
\[
\|U_2(\cdot)\|_{L^{\infty}(\mathbb{T})} \leq \Lambda_v \frac{\delta n}{\delta} \sup_{1 \leq |k| \leq \delta^{-1}} \|k\omega\|_T \quad (3.7)
\]
and
\[
\|U_3(\cdot)\|_{L^{\infty}(\mathbb{T})} \leq 4\pi \delta C(v, \rho) \quad (3.8)
\]

**Proof:** By Lemma 2.4 and (3.2), we have
\[
\|U_2(\cdot)\|_{L^{\infty}(\mathbb{T})} \leq \sum_{1 \leq |k| < \delta^{-1}} |\hat{u}_n(k)| \leq \Lambda_v \frac{\delta n}{\delta} \sum_{1 \leq |k| < \delta^{-1}} \|k\omega\|_T \leq \Lambda_v \frac{\delta n}{\delta} \sup_{1 \leq |k| \leq \delta^{-1}} \|k\omega\|_T. \quad (3.9)
\]
By Lemma 2.1, (3.2), (3.3) and $q_s \leq R$, we obtain

$$
\|U_3(\cdot)\|_{L^\infty(T)} \leq 2 \sum_{\delta^{-1} \leq |k| < q_s / 4} |\beta_n(k)| \frac{1}{1 + R^2 \|k\omega\|^2_T}
$$

$$
\leq 2 \sum_{\delta^{-1} \leq |k| < q_s / 4} \frac{C(v, \rho)}{\delta^{-1}} \frac{1}{1 + R^2 \|k\omega\|^2_T}
$$

$$
\leq 2C(v, \rho) \cdot \delta \cdot \sum_{1 \leq |k| < q_s / 4} \frac{1}{1 + R^2 \|k\omega\|^2_T}
$$

$$
\leq 4\pi C(v, \rho) \cdot \delta \cdot \frac{q_s}{R}
$$

$$
\leq 4\pi C(v, \rho),
$$

as desired. \hfill\square

We have some general estimates for $U_4(x) + U_5(x)$ and $U_6(x)$.

**Proposition 3.2.** Let $C(v, \rho)$ be given as in (2.5). For any $n \geq 1$, and $q_s \leq R < q_{s+1} \leq K$, we have

$$
\|U_4(\cdot) + U_5(\cdot)\|_{L^\infty(T)} \leq 120C(v, \rho) \left( \frac{\log q_{s+1}}{q_s} + \frac{\log K}{R} \right),
$$

(3.11)

and

$$
\|U_6(\cdot)\|_{L^2(T)} \leq C^2(v, \rho) \frac{2}{K}.
$$

(3.12)

This part has been proved in [37], but we sketch the proof below for reader’s convenience.

**Proof:** By Lemma 2.1, (3.2), (3.4) and the choice of $R \in [q_s, q_{s+1})$, we have

$$
\|U_4(\cdot)\|_{L^\infty(T)} \leq 2 \sum_{q_s / 4 \leq |k| < q_{s+1} / 4} |\beta_n(k)| \frac{1}{1 + R^2 \|k\omega\|^2_T}
$$

$$
\leq 2 \sum_{\ell=1}^{q_{s+1} / q_s} \sum_{|k| \in [(\ell\cdot q_s, (\ell+1)q_s) / 4]} |\beta_n(k)| \frac{1}{1 + R^2 \|k\omega\|^2_T}
$$

$$
\leq 8C(v, \rho) \sum_{\ell=1}^{q_{s+1} / q_s} \sum_{|k| \in [(\ell\cdot q_s, (\ell+1)q_s) / 4]} \frac{1}{\ell q_s} \cdot \frac{1}{1 + R^2 \|k\omega\|^2_T}
$$

(3.13)

$$
\leq 8C(v, \rho) \sum_{\ell=1}^{q_{s+1} / q_s} \frac{1}{\ell q_s} \left( 2 + 4\pi \frac{q_s}{R} \right)
$$

$$
= 16C(v, \rho) \left( 1 + 2\pi \frac{q_s}{R} \right) \frac{1}{q_s} \sum_{\ell=1}^{q_{s+1} / q_s} \frac{1}{\ell}
$$

$$
\leq 16C(v, \rho) (1 + 2\pi) \frac{\log q_{s+1}}{q_s}.
$$

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In view of $\mathcal{U}_5$, we have by Lemma 2.1 and (3.4) that
\[
\| \mathcal{U}_5(\cdot) \|_{L^\infty(T)} \leq 2 \sum_{q_{s+1}/4 \leq |k| \leq K} |\hat{u}_n(k)| \frac{1}{1 + R^2 \|k\omega\|_T^2}
\leq 8C(v, \rho) \sum_{\ell=1}^{[4K/q_{s+1}]+1} \sum_{|k| \in [\ell q_{s+1}/4, (\ell+1)q_{s+1}/4)} \frac{1}{\ell q_{s+1}} \cdot \frac{1}{1 + R^2 \|k\omega\|_T^2}
= 8C(v, \rho) \left( 2 + 4\pi \frac{q_{s+1}}{R} \right) \frac{1}{q_{s+1}} \sum_{\ell=1}^{[4K/q_{s+1}]+1} \frac{1}{\ell}
\leq 16C(v, \rho) (1 + 2\pi) \log \frac{K}{R}.
\] (3.14)
Combining (3.13) with (3.14), and use that $16(1 + 2\pi) < 120$, we prove (3.11).

For $\mathcal{U}_6$, we have that by Lemma 2.1,
\[
\| \mathcal{U}_6(\cdot) \|_{L^2(T)} \leq \sum_{|k| > K} |\hat{u}_n(k)|^2 \leq C^2(v, \rho) \sum_{|k| > K} \frac{1}{k^2} \leq C^2(v, \rho) \frac{2}{K},
\] (3.15)
as claimed.

### 3.2 Proof of Theorem 1.1

Let
\[
L(\omega, [a, b]) = \inf_{E \in [a, b]} L(\omega, E), \text{ and } \tilde{L}(\omega, [a, b]) = \min(L(\omega, [a, b]), 1).
\] (3.16)

For simplicity, we will sometimes omit the dependence on $\omega$ and $[a, b]$ and write $L$ and $\tilde{L}$ instead.

Recall our notations: $N_2$ as in (2.17), and $\Lambda_v, C(v, \rho), C_1$ as in (2.1), (2.5) and (2.11).

We choose $c$ and $\tilde{c}$ in the statement of the theorem as follows:
\[
c(v, \rho) = (36000C(v, \rho))^{-1}, \quad \tilde{c}(v, \rho) = (2 \times 10^7 (C_1 + 2) C(v, \rho))^{-1}.
\] (3.17)

By our condition:
\[
\beta(\omega) = \limsup_{k \to \infty} \frac{\log q_{k+1}}{q_k} \leq c(v, \rho) L(\omega).
\]

Hence there exists $s_0 = s_0(\omega, [a, b], v, \rho)$ such that
\[
\frac{\log q_{k+1}}{q_k} \leq 2c(v, \rho) L(\omega, [a, b]),
\] (3.18)
for any $k \geq s_0$.
Let \( n \geq N \), with \( N \) defined as follows:

\[
N(\omega, L, v, \rho) := \max \left\{ \begin{array}{l}
(i). \ 400(C_1 + 2)q_{s_0}, \\
(ii). \ N_2(\omega, [a, b], L, v, \rho), \\
(iii). \ 1.6 \times 10^5 \pi \Lambda v \rho \| \omega \|_T^{-2} \sup_{1 \leq k \leq 800} C(v, \rho) L^{-1} \frac{1}{\| k \omega \|_T}, \\
(iv). \ 2 \times 10^7 (C_1 + 2) C(v, \rho) L^{-1} \log \left( 2 \times 10^4 C^2(v, \rho) L^{-2} + e \right). \end{array} \right. 
\]

This gives four lower bounds of \( n \).

**Remark 3.3.** By Remark 2.11, \( N_2 \) is decreasing in \( L \). It is also clear that both (iii) and (iv) are decreasing in \( L \). Hence \( N \) is non-increasing in \( L \).

### 3.2.1 Parameters for part (a)

In this case, \( L < 1 \), hence

\[
\tilde{L} = L. 
\]

In our decomposition of \( u_n(x) - L_n \) in (3.6), we choose the following parameters:

\[
\delta = \frac{L}{800 \pi C(v, \rho)}, \\
R = \left[ \frac{n}{400(C_1 + 2)} \right] + 1, \\
K = \left[ \exp \left( \frac{RL}{1.2 \times 10^4 C(v, \rho)} \right) \right], \\
s = \max \{ s \in \mathbb{N} : q_s \leq R \}. 
\]

It is clear from the choice of \( s \) that \( q_s \leq R < q_{s+1} \). Let us also note that with \( \delta \) defined above, the lower bound (iii) in (3.19) becomes

\[
\left( \frac{200 \Lambda v}{\delta L} \right) \sup_{1 \leq k \leq \delta^{-1}} \frac{1}{\| k \omega \|_T}. 
\]

Indeed, by (i) of (3.19), we have

\[
R > \left( 400(C_1 + 2) \right)^{-1} n \geq q_{s_0}. 
\]

By our definition of \( s \), see (3.21), we clearly have \( s \geq s_0 \). This, by (3.18), implies

\[
\log q_{s+1} \leq 2c(v, \rho) L. 
\]

An upper bound of \( q_{s+1} \) could be derived from (3.23). Indeed,

\[
q_{s+1} \leq \exp \left( 2c(v, \rho) L \right) \leq \exp \left( 2c(v, \rho) L R \right) \leq \exp \left( \frac{LR}{1.8 \times 10^3 C(v, \rho)} \right). 
\]
By (iv) of (3.19),
\[
\begin{align*}
n &\geq 2 \times 10^7 (C_1 + 2) C(v, \rho) \frac{L}{L} - 1 \log \left( 2 \times 10^4 C(v, \rho) L(\omega)^{-2} + e \right) \\
&\geq 2 \times 10^7 (C_1 + 2) C(v, \rho) \frac{L}{L} - 1 \log e \\
&= 2 \times 10^7 (C_1 + 2) C(v, \rho) \frac{L}{L} - 1,
\end{align*}
\]
hence, we have
\[
\exp \left( \frac{LR}{1.8 \times 10^4 C(v, \rho)} \right) \geq \exp \left( \frac{Ln}{7.2 \times 10^6 (C_1 + 2) C(v, \rho)} \right) \geq \exp \left( \frac{2 \times 10^7}{7.2 \times 10^8} \right) > 16.
\]
Using the fact that
\[
x < x^\frac{3}{2} - 1, \text{ for } x > 3,
\]
we have
\[
\exp \left( \frac{LR}{1.8 \times 10^4 C(v, \rho)} \right) < \exp \left( \frac{LR}{1.2 \times 10^4 C(v, \rho)} \right) - 1 \leq K. \quad (3.25)
\]
Combining (3.24) with (3.25), we arrive at
\[
q_{s+1} \leq K. \quad (3.26)
\]

### 3.2.2 Proof of part (a)

By (ii) of (3.19) and Remark 2.11, we have
\[
n \geq N \geq N_2(\omega, [a, b], L(\omega), v, \rho) \geq N_2(\omega, [a, b], L(\omega, E), v, \rho).
\]
Hence by Lemma 2.10 and (3.23), we have,
\[
\|U_1(\cdot)\|_{L^\infty(T)} \leq \frac{1}{100} L + \frac{1}{2} C(v, \rho) \frac{\log q_{s+1}}{q_s} \\
\leq \frac{1}{100} L + \frac{2}{5} C(v, \rho) c(v, \rho) L \quad (3.27)
\]
\[
= \left( \frac{1}{100} + \frac{1}{9 \times 10^4} \right) L.
\]
By Proposition 3.1 and our choice of \( \delta \), we have
\[
\|U_2(\cdot) + U_3(\cdot)\|_{L^\infty(T)} \leq \frac{A_v}{\delta n} \sup_{1 \leq k \leq s-1} \frac{1}{\|k\omega\|_T} + 4\pi \delta C(v, \rho) \\
= \frac{A_v}{\delta n} \sup_{1 \leq k \leq s-1} \frac{1}{\|k\omega\|_T} + \frac{1}{200} L \quad (3.28)
\]
\[
\leq \frac{1}{100} L.
\]
in which we used (iii) of (3.19), see also (3.22),
\[ n \geq N \geq \frac{200\Lambda_v}{\delta L} \sup_{1 \leq k \leq \delta - 1} \| k\omega \|_T \geq \frac{200\Lambda_v}{\delta L} \sup_{1 \leq k \leq \delta - 1} \| k\omega \|_T. \]

Note that (3.26) verifies the condition \( q_{s+1} \leq K \) of Proposition 3.2. Hence Proposition 3.2 implies that,
\[ \| U_4(\cdot) + U_5(\cdot) \|_{L^\infty(T)} \leq 120C(v, \rho) \left( \frac{\log q_{s+1}}{q_s} + \frac{\log K}{R} \right) \leq 120C(v, \rho) \frac{\log q_{s+1}}{q_s} + \frac{1}{100} L. \]

Taking (3.23) into account, we have
\[ \| U_4(\cdot) + U_5(\cdot) \|_{L^\infty(T)} \leq 240C(v, \rho)c(v, \rho) + \frac{1}{100} L = \frac{1}{60} L. \] (3.29)

Combining (3.27), (3.28), (3.29) with our choice of \( \rho(v, \rho) \), see (3.17), we have
\[ \| \sum_{j=1}^{5} U_j(\cdot) \|_{L^\infty(T)} \leq \frac{1}{25} L. \] (3.30)

By (3.12) and (3.25),
\[ \| U_6(\cdot) \|_{L^2(T)}^2 \leq C^2(v, \rho) \frac{2}{K} \leq 2C^2(v, \rho) \exp \left( -\frac{RL}{1.8 \times 10^4 C(v, \rho)} \right) \leq 2C^2(v, \rho) \exp \left( -\frac{nL}{7.2 \times 10^6 (C_1 + 2) C(v, \rho)} \right) \leq 2C^2(v, \rho) \exp \left( -\frac{nL}{10^7 (C_1 + 2) C(v, \rho)} \right). \] (3.31)

Combining (3.26), (3.30) with (3.31), and using Markov’s inequality, we obtain
\[
\begin{align*}
\mes \left\{ x \in T : |u_n(x) - L_n| > \frac{1}{20} L \right\} \\
\leq \mes \left\{ x \in T : |U_6(x)| > \frac{1}{100} L \right\} \\
\leq 2 \times 10^4 C^2(v, \rho)L^{-2} \exp \left( -\frac{nL}{10^7 (C_1 + 2) C(v, \rho)} \right) \\
\leq \exp \left( -\frac{nL}{2 \times 10^7 (C_1 + 2) C(v, \rho)} \right)
\end{align*}
\]

in which we used (iv) of (3.19),
\[ n \geq 2 \times 10^7 (C_1 + 2) C(v, \rho)L^{-1} \log \left( 2 \times 10^4 C^2(v, \rho)L^{-2} + \epsilon \right) \geq 2 \times 10^7 (C_1 + 2) C(v, \rho)L^{-1} \log \left( 2 \times 10^4 C^2(v, \rho)L^{-2} \right) \geq 2 \times 10^7 (C_1 + 2) C(v, \rho)L^{-1} \log \left( 2 \times 10^4 C^2(v, \rho)L^{-2} \right). \]

This proves part (a) of Theorem 1.1.
3.2.3 Parameters for part (b)

In our decomposition of \( u_n(x) - L_n \) in (3.6), we choose parameters as follows:

\[
\delta = \frac{1}{800\pi C(v, \rho)}, \\
R = \left\lfloor \frac{nL}{400\Lambda_v} \right\rfloor + 1, \\
K = \left\lfloor \exp \left( \frac{RL}{1.2 \times 10^4 C(v, \rho)} \right) \right\rfloor, \\
s = \max\{s \in \mathbb{N} : q_s \leq R\}.
\]  

(3.32)

It is clear that \( q_s \leq R < q_{s+1} \).

Use the fact that \( C_1 > \Lambda_v \), see (2.11), and \( \tilde{L} \leq 1 \), (3.19) implies

\[
\left\{ \begin{array}{l}
\text{(i').} \quad 400(\Lambda_v + 1)q_{s_0}, \\
\text{(ii').} \quad \frac{200\Lambda_v}{\delta} \sup_{1 \leq \delta \leq 1/\|\omega\|_2}, \\
\text{(iii').} \quad 2 \times 10^7 \Lambda_v C(v, \rho) \log \left( 2 \times 10^4 \Lambda_v C(v, \rho) + e \right). 
\end{array} \right.
\]

(3.33)

Note that (i') implies that

\[
R > (400\Lambda_v)^{-1}nL \geq (400\Lambda_v)^{-1}n \geq q_{s_0}. 
\]

(3.34)

By our definition of \( s \), we have \( s \geq s_0 \). This, by (3.18), implies

\[
q_{s+1} \leq \exp \left( 2c(v, \rho)L_q s \right) \leq \exp \left( 2c(v, \rho)Lq_s \right) \leq \exp \left( \frac{LR}{1.8 \times 10^4 C(v, \rho)} \right). 
\]

(3.35)

By (iv') of (3.33),

\[
n \geq 2 \times 10^7 \Lambda_v C(v, \rho) \log \left( 2 \times 10^4 \Lambda_v C(v, \rho) + e \right) \geq 2 \times 10^7 \Lambda_v C(v, \rho),
\]

hence

\[
\exp \left( \frac{RL}{1.8 \times 10^4 C(v, \rho)} \right) \geq \exp \left( \frac{nL^2}{7.2 \times 10^6 \Lambda_v C(v, \rho)} \right) \geq \exp \left( \frac{n}{7.2 \times 10^6 \Lambda_v C(v, \rho)} \right) \geq \exp \left( \frac{2 \times 10^7}{7.2 \times 10^6} \right) > 16.
\]

Thus, similar to (3.25), using the fact

\[
x < x^2 - 1, \quad \text{for} \ x > 3,
\]

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we have
\[
\exp\left(\frac{RL}{1.8 \times 10^4 C(v, \rho)}\right) \leq \exp\left(\frac{RL}{1.2 \times 10^4 C(v, \rho)}\right) - 1 \leq K. \tag{3.36}
\]
Combining (3.35) with (3.36), we obtain, similar to (3.26), that
\[
q_{s+1} \leq K. \tag{3.37}
\]

### 3.2.4 Proof of part (b)

We use the trivial upper bound in Lemma 2.7 for \( U_1 \),
\[
\|U_1(\cdot)\|_{L^\infty(T)} \leq 2\Lambda_v \frac{R}{n} \leq \frac{1}{200} L + \frac{2\Lambda_v}{n} \leq \frac{1}{100} L, \tag{3.38}
\]
in which we used, see (i’) of (3.33), that
\[
n \geq 400(\Lambda_v + 1)q_{s_0} \geq 400\Lambda_v L^{-1}.
\]
Proposition 3.11 yields that
\[
\|U_4(\cdot) + U_5(\cdot)\|_{L^\infty(T)} \leq 120 C(v, \rho) 2^{\log q_{s+1}} q_s + 100 C(v, \rho) \log 100 L.
\]
By (3.23), we then have
\[
\|U_4(\cdot) + U_5(\cdot)\|_{L^\infty(T)} \leq 240 C(v, \rho) \log q_{s+1} \geq \frac{1}{100} L + \frac{1}{100} L = \frac{1}{60} L. \tag{3.41}
\]
In view of \( U_6 \), (3.2) and (3.30) yield that
\[
\|U_6(\cdot)\|_{L^2(T)}^2 \leq C'(v, \rho) \frac{2}{K} \leq 2C'(v, \rho) \exp\left(\frac{RL}{1.8 \times 10^4 C(v, \rho)}\right) \leq 2C'(v, \rho) \exp\left(\frac{nL^2}{7.2 \times 10^6 \Lambda_v C(v, \rho)}\right). \tag{3.42}
\]

Combining (3.38), (3.39) (3.41) with (3.42), we get that by Markov’s inequality,

\[
\begin{align*}
\text{mes} \left\{ x \in T : |u_n(x) - L_n| > \frac{1}{20} L \right\} \\
\leq & \text{mes} \left\{ x \in T : |U_0(x)| > \frac{1}{100} L \right\} \\
\leq & 2 \times 10^4 C^2(v, \rho)L^{-2} \exp \left( -\frac{nL^2}{10^7 \Lambda_v C(v, \rho)} \right) \\
\leq & \exp \left( -\frac{nL^2}{2 \times 10^4 \Lambda_v C(v, \rho)} \right),
\end{align*}
\]

in which we used (iv') of (3.33). Using that \( C_1 > \Lambda_v \), we obtain

\[- (2 \times 10^7 \Lambda_v C(v, \rho))^{-1} < - (2 \times 10^7 (C_1 + 2)C(v, \rho))^{-1} = -\tilde{c}(v, \rho).\]

Hence

\[
\begin{align*}
\text{mes} \left\{ x \in T : |u_n(x) - L_n| > \frac{1}{20} L \right\} \\
\leq & \exp \left( -\tilde{c}(v, \rho)nL^2 \right),
\end{align*}
\]

as claimed. \(\square\)

### 3.3 Proof of Corollary 1.3

In general, a large \( \|v\|_\infty \) norm does not guarantee positive Lyapunov exponent. However if the potential function \( v \) is of the form \( \lambda f \), then the following well-known result by Sorets-Spencer \[34\] gives a lower bound of the Lyapunov exponent in the large coupling regime.

**Theorem 3.4.** For any non-constant real analytic potential \( f \) with an analytic extension on \( \{|\text{Im} z| < \rho\} \), there exist constants \( \lambda_0 = \lambda_0(f) > 0 \) and \( h_0 = h_0(f) \) depending only on \( f \), such that for all \( E, \omega \) and \( \lambda > \lambda_0 \), the Lyapunov exponent \( L(\omega, E) \geq \log \lambda + h_0 \).

Let \( \lambda_0 = \lambda_0(f) \) be given as in Theorem 3.4. For \( \lambda > \lambda_1(f) := \max(e^{-19h_0}, 3, \lambda_0) \), we have

\[
L(\omega, E) > \log \lambda + h_0 > \frac{18}{19} \log \lambda > 1,
\]

holds uniformly in \( \omega \) and \( E \). Let \( \Lambda_v = \Lambda_{\lambda f} \) be defined as in (2.1), we have

\[
L(\omega, E) \leq \Lambda_{\lambda f} = \log (3 + 2\lambda \|f\|_{L^\infty(T)}) \leq \frac{20}{19} \log \lambda,
\]

provided that \( \lambda \geq \lambda_2(\|f\|_{L^\infty(T)}) \).

Let \( C(\lambda f, \rho), c(\lambda f, \rho) \) and \( \tilde{c}(\lambda f, \rho) \) be defined as in (2.4), (3.17). With the help of Lemma 2.3 we can make the dependence of the three constants on \( \lambda \) more explicit.

First, Lemma 2.3 yields that there exists \( C_0 = C_0(f, \rho) \) such that

\[
C(\lambda f, \rho) \leq C_0(f, \rho),
\]

\[\footnote{\( \lambda_0 \) is in general large, however for some concrete examples, e.g. \( f = \cos, \lambda_0 = 2 \), cf [12].} \]
for any $\lambda \geq 0$.

Second, plugging (3.44) and (3.45) into our definition of $C_1$, see (2.11), we have,

$$C_1 + 2 = 4 + \Lambda f + (8 + 4C_3)C(\lambda f, \rho) \leq 4 + \frac{20}{19}\log \lambda + (8 + 4C_3)C_0 \leq 2\log \lambda,$$

(3.46)

provided that $\lambda \geq \lambda_3(f, \rho) := \max (\lambda_2, \exp \left(\frac{18}{19}(4 + (8 + 4C_3)C_0)\right))$. Thus putting (3.45) and (3.46) together, we have that for $\lambda \geq \lambda_3$,

$$\hat{c}(\lambda f, \rho) = (2 \times 10^7(C_1 + 2)C(v, \rho))^{-1} \geq (4 \times 10^7C_0 \log \lambda)^{-1}. \quad (3.47)$$

Third, note that (3.45) also yields

$$c(\lambda f, \rho) = (36000 C(\lambda f, \rho))^{-1} \geq (36000 C_0)^{-1}. \quad (3.48)$$

Let us take

$$\hat{\lambda}(f, \rho) := \max (\lambda_1, \lambda_3),$$

and $\lambda > \hat{\lambda}$. We are in the place to apply Theorem 1.1. Let us note that by (3.43), we always have $L(\omega, E) > 1$, hence we will only apply part (b). One condition of the theorem is $0 \leq \beta(\omega) < c(\lambda f, \rho)L(\omega, E)$. In view of (3.48) and $L(\omega, E) > \frac{18}{19}\log \lambda$, this condition will always be satisfied if

$$\beta(\omega) < (36000 C_0)^{-1} \frac{18}{19}\log \lambda = (38000 C_0)^{-1} \log \lambda =: B^{-1} \log \lambda. \quad (3.49)$$

Therefore, for $\lambda > \max (\hat{\lambda}, \exp (B\beta(\omega)))$, part (b) of Theorem 1.1 implies

$$\text{mes} \left\{ x \in T : | u_n(\omega, E; x) - L_n(\omega, E) | > \frac{1}{20}L(\omega, E) \right\} \leq \exp (-\hat{c}(\lambda f, \rho)L^2(\omega, E)n). \quad (3.50)$$

Using upper and lower bounds of $L(\omega, E)$ in (3.44) and (3.43), we obtain from (3.50) that

$$\text{mes} \left\{ x \in T : | u_n(\omega, E; x) - L_n(\omega, E) | > \frac{1}{19} \log \lambda \right\} \leq \exp (-\hat{c}(\lambda f, \rho)L^2(\omega, E)n) \leq \exp \left(-\hat{c}(\lambda f, \rho)\frac{18^2(\log \lambda)^2}{19^2} n\right) \leq \exp \left(-n\frac{\log \lambda}{5 \times 10^7 C_0}\right) =: \exp (-nb \log \lambda),$$

(3.51)

in which we used (3.47) in the last inequality.
4 Proof of the lemmas

4.1 Proof of Lemma 2.1

We need the following lemma:

Lemma 4.1. [25, Lemma 2.2] Let \( u : \Omega \to \mathbb{R} \) be a subharmonic function on a domain \( \Omega \subset \mathbb{C} \). Suppose that \( \partial \Omega \) consists of finitely many piece-wise \( C^1 \) curves. There exists a positive measure \( \mu \) on \( \Omega \) such that for any \( \Omega_1 \subset \subset \Omega \) (i.e., \( \Omega_1 \) is a compactly contained sub-region of \( \Omega \))

\[
\begin{align*}
  u(z) &= \int_{\Omega_1} \log |z - \zeta| \, d\mu(\zeta) + h(z) \\
  \mu(\Omega_1) &\leq C(\Omega, \Omega_1) (\sup_{\Omega_1} u - \sup_{\Omega} u) \\
  \|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_2)} &\leq C(\Omega, \Omega_1, \Omega_2) (\sup_{\Omega_1} u - \sup_{\Omega} u)
\end{align*}
\]

where \( h \) is harmonic on \( \Omega_1 \) and \( \mu \) is unique with this property. Moreover, \( \mu \) and \( h \) satisfy the bounds

\[
\mu(\Omega_1) \leq C(\Omega, \Omega_1) (\sup_{\Omega_1} u - \sup_{\Omega} u)
\]

for any \( \Omega_2 \subset \subset \Omega_1 \).

Note that \( u_n(z) \) is a bounded subharmonic function on \( \Omega := \{ z : |\text{Re} z| < 1, |\text{Im} z| < \rho \} \). We consider the following nested domains \( \Omega_0 \subset \subset \Omega_2 \subset \subset \Omega_1 \subset \subset \Omega \), where

\[
\begin{align*}
  \Omega_1 &= \{ z : |\text{Re} z| \leq \frac{5}{6}, |\text{Im} z| < \frac{\rho}{2} \} \\
  \Omega_2 &= \{ z : |\text{Re} z| \leq \frac{4}{5}, |\text{Im} z| < \frac{\rho}{4} \} \\
  \Omega_0 &= \{ z : |\text{Re} z| \leq \frac{3}{4}, |\text{Im} z| = 0 \} = \left[ -\frac{3}{4}, \frac{3}{4} \right].
\end{align*}
\]

Now we apply Lemma 4.1 to \( u(z) = u_n(z) \) on \( \Omega \). We have then a positive measure \( \mu \) and a harmonic function \( h \) on \( \Omega_1 \) satisfying (4.1), (4.2) and (4.3).

Since \( h - \sup_{\Omega_1} u \) is a harmonic function, by the Poisson integral formula and (4.3), we have

\[
\max (\|\partial_x h\|_{L^\infty(\Omega_0)}, \|\partial^2_x h\|_{L^\infty(\Omega_0)}) \leq C(\Omega, \Omega_1, \Omega_2, \Omega_0) (\sup_{\Omega_1} u - \sup_{\Omega} u).
\]

Now we only need to the bound for \( \partial_x h \) here, we will use the one for \( \partial^2_x h \) in Sec. 4.3.

Combine (1.1) with the technique in [10], one can then that for some absolute constant \( C_2 > 0 \), the following holds for any \( k \neq 0 \):

\[
|\hat{u}_n(k)| \leq \frac{C_2}{|k|} \left( \mu(\Omega_1) + \|\partial_x h\|_{L^\infty(\Omega_0)} + \|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_0)} \right).
\]

Clearly, (2.2), (2.5) follow directly from (1.2)-(1.6) by setting

\[
\alpha(\rho) := C_2 \max (C(\Omega, \Omega_1), C(\Omega, \Omega_1, \Omega_2), C(\Omega, \Omega_1, \Omega_2, \Omega_0)).
\]

This finishes the proof of Lemma 2.1. We will include the proof of (4.0) in Appendix A.
4.2 Proof of Lemma 2.3

On one hand, for any $E \in N$, trivially we have

$$\sup_{|\text{Im} z|<\rho} \|A_j(E, z)\| \leq 2\lambda \|f\|_\rho + 2 \leq 3\lambda \|f\|_\rho,$$ provided $\lambda > 2\|f\|^{-1}_\rho$

and

$$\sup_{|\text{Im} z|<\rho} u_n(z) \leq \log \left(3\lambda \|f\|_\rho\right)$$

On the other hand, since $f$ is non-constant analytic on $|\text{Im} z| < \rho$, for $\delta = \rho/2$, there exists $\epsilon_0 = \epsilon_0(f) > 0$ such that

$$\inf_{E_i} \sup_{y \in (\delta/2, \delta)} \inf_x |f(x + iy) - E_1| > \epsilon_0$$

This implies that for any $\lambda, E$, there is $y_0 \in (\delta/2, \delta)$ such that $\forall x$

$$|f(x + iy_0) - E/\lambda| > \epsilon_0$$

The computation contained in [10, Appendix] shows that

$$\|M_n(iy_0, E)\| \geq \prod_{j=1}^n \left(|\lambda f(j\omega + iy_0) - E| - 1\right) \geq \left(\lambda \epsilon_0 - 1\right)^n \geq \left(\frac{1}{2} \lambda \epsilon_0\right)^n, \tag{4.9} \tag{4.10}$$

holds for $\lambda > 2\epsilon_0^{-1}$. Therefore,

$$\sup_{|\text{Im} z|<\rho/2} u_n(z) \geq u_n(iy_0) = \frac{1}{n} \log \|M_n(iy_0, E)\| \geq \log \left(\frac{1}{2} \lambda \epsilon_0\right)$$

Clearly, we have that for $\lambda > \max\{2\|f\|^{-1}_\rho, 3\epsilon_0^{-1}\}$,

$$\sup_{|\text{Im} z|<\rho} u_n(z) - \sup_{|\text{Im} z|<\rho/2} u_n(z) \leq \log \left(3\lambda \|f\|_\rho\right) - \log \left(\frac{1}{2} \lambda \epsilon_0\right) = \log \left(\frac{6\|f\|_\rho}{\epsilon_0}\right)$$

Therefore by 2.5,

$$C(\lambda f, \rho) \leq \alpha(\rho) \log \left(\frac{6\|f\|_\rho}{\epsilon_0}\right) =: C_0(f, \rho) \text{ independent of } \lambda,$$

as desired. \qed
4.3 Proof of Lemma 2.4

We have that by (2.2),
\[
\|u_n(\cdot + \omega) - u_n(\cdot)\|_{L^\infty(T)} = \frac{1}{n} \|\log \|M_n(\cdot + \omega)\| - \log \|M_n(\cdot)\|\|_{L^\infty(T)} \\
\leq \frac{1}{n} \|\log \|M_1(\cdot + n\omega)\| + \log \|M_{n-1}(\cdot + \omega)\| + \log \|M_1(\cdot)\| - \log \|M_{n-1}(\cdot + \omega)\|\|_{L^\infty(T)} \\
\leq \frac{2\Lambda_v}{n}.
\]

This implies
\[
2|\hat{u}_n(k)e^{2\pi ik\omega} - \hat{u}_n(k)| = \left|\int_T u_n(x + \omega)e^{-2\pi ikx} \, dx - \int_T u(x)e^{-2\pi ix} \, dx\right| \\
\leq\|u_n(\cdot + \omega) - u_n(\cdot)\|_{L^\infty(T)} \\
\leq \frac{2\Lambda_v}{n}.
\] (4.11) implies
\[
2|\hat{u}_n(k)|\sin (\pi \|k\omega\|_{T}) \leq \frac{2\Lambda_v}{n},
\]
hence by \(\sin (\pi x) \geq 2x\) for \(0 \leq x \leq \frac{1}{2}\), we get that for \(k \neq 0\),
\[
|\hat{u}_n(k)| \leq \frac{\Lambda_v}{2\pi k\|\omega\|_{T}},
\]
as stated.

Before we move on, let us mention a simple consequence of (4.11):
\[
\|u_n(\cdot + \omega) - u_n(\cdot)\|_{L^\infty(T)} \leq \frac{2\Lambda_v |j|}{n},
\] (4.13)
this estimate will be used at several places.

4.4 Proof of Lemma 2.5

Let \(R \geq R_0(L)\) and \(n \geq N_3(v, \omega, L)\), where
\[
R_0 := 144L^{-5},
\] (4.14)
and
\[
N_3 := 2\Lambda_v L^{-2} \sup_{1 \leq |k| \leq L^{-1}} \frac{1}{\|k\omega\|_{T}}.
\] (4.15)

Lemma 4.1 implies that \(u_n\) has a Riesz-representation with a positive measure \(\mu\) and a harmonic function \(h\). Let us take
\[
\delta = (LR)^{-1},
\] (4.16)
and
\[ u_{n,\delta}(x) = \int_{\Omega} \log (|x - w| + \delta) \mu(dw) + h(x), \quad (4.17) \]

We then have, point-wisely,
\[ u_n(x) \leq u_{n,\delta}(x). \quad (4.18) \]

It is clear from our definitions of \( R_0 \) and \( \delta \) that,
\[ \delta \leq \frac{L^4}{144} < \frac{1}{144}. \quad (4.19) \]

### 4.4.1 Fourier decay of \( u_{n,\delta} \)

The following two inequalities \((4.20)\) and \((4.21)\) are \((2.4)\) and \((2.3)\) of \([10]\) (see also \((8.12)\) of \([9]\)). We include its proof in Appendix B.

**Lemma 4.2.** Let \( C(v, \rho) \) be defined as in \((2.5)\). There exists an absolute constant \( C_3 \) such that for any \( k \in \mathbb{Z} \), we have
\[ |\hat{u}_{n,\delta}(k)| \leq |\hat{u}_n(k)| + 3\delta \log \delta^{-1}, \quad (4.20) \]

and for any \( k \neq 0 \),
\[ |\hat{u}_{n,\delta}(k)| \leq C_3 C(v, \rho) \min \left( \frac{1}{|k|}, \frac{1}{k^2 \delta} \right), \quad (4.21) \]

holds for \( k \neq 0 \).

Note that \((4.20)\) together with Lemma 2.4 leads to the following corollary.

**Corollary 4.3.** For \( k \neq 0 \), we have
\[ |\hat{u}_{n,\delta}(k)| \leq \frac{\Lambda_v}{2 \pi k \omega_T} + 3\delta \log \delta^{-1}. \quad (4.22) \]

### 4.4.2 Proof of Lemma 2.5

Let \( s \in \mathbb{N} \) be such that \( q_s \leq R < q_{s+1} \). Recall our definition of \( u^{(R)}_n \), see \((2.8)\). \((4.18)\) clearly yields
\[ 0 \leq u^{(R)}_n(x) \leq u^{(R)}_{n,\delta}(x) \]

Let \( F_R(k) \) be as in \((3.1)\), invoking \((3.5)\), we have
\[ 0 \leq u^{(R)}_n(x) \leq u^{(R)}_{n,\delta}(x) = \hat{u}_{n,\delta}(0) + \sum_{k \neq 0} \hat{u}_{n,\delta}(k) F_R(k). \quad (4.23) \]

We now split the Fourier series in \((4.23)\) into low/high-frequency parts,
\[ u^{(R)}_{n,\delta}(x) = \hat{u}_{n,\delta}(0) + \sum_{1 \leq |k| \leq q_{s+1}/4} \hat{u}_{n,\delta}(k) F_R(k) + \sum_{|k| > q_{s+1}/4} \hat{u}_{n,\delta}(k) F_R(k) \]
\[ =: \hat{u}_{n,\delta}(0) + S_1 + S_2. \quad (4.24) \]
Using the $\left( k^2 \delta \right) \delta^{-1}$ bound of $|\hat{u}_{n,\delta}(k)|$ in (4.21) and $|F_R(k)| \leq 1$ in (3.2), we have

$$| \mathcal{S}_2 | \leq \sum_{|k| > q_{s+1}/4} |\hat{u}_{n,\delta}(k)| \leq \sum_{|k| > q_{s+1}/4} \frac{C(v, \rho)}{k^2 \delta} \leq \frac{8C(v, \rho)}{q_{s+1} \delta} \leq \frac{8C(v, \rho)}{\delta R} = 8C(v, \rho)L, \quad (4.25)$$

in which we used $R < q_{s+1}$ and our choice of $\delta$, see (4.16).

We further decompose $\mathcal{S}_1$ into

$$| \mathcal{S}_1 | \leq \left( \sum_{1 \leq |k| \leq L^{-1} \text{ if } 1 \leq |k| \leq q_{s}/4} + \sum_{L^{-1} < |k| \leq q_{s}/4} + \sum_{q_{s}/4 \leq |k| \leq q_{s+1}/4} \right) |\hat{u}_{n,\delta}(k)| F_R(k) \quad (4.26)$$

By (4.22) and $|F_R(k)| \leq 1$, see (3.2), we have

$$S_{1,1} \leq \sum_{1 \leq |k| \leq L^{-1}} \left( \frac{\Lambda_v}{2n \|k\omega\|_T} + 3\delta \log \delta^{-1} \right) \leq 2 \left( \frac{\Lambda_v}{L} \sup_{1 \leq |k| \leq L^{-1}} \frac{1}{\|k\omega\|_T} + \frac{6}{RL \log (RL)} \right).$$

Using a trivial estimate: $\log x \leq \sqrt{x}$ that holds for any $x > 0$, we obtain

$$S_{1,1} \leq \left( \frac{\Lambda_v}{nL} \sup_{1 \leq |k| \leq L^{-1}} \frac{1}{\|k\omega\|_T} + \frac{6}{\sqrt{RL}^3} \right) \leq L, \quad (4.27)$$

in the last step we used $R \geq R_0 = 144L^{-5}$ and $n \geq N_3$, see (4.14) and (4.15).

Using the $|k|^{-1}$ bound of $|\hat{u}_{n,\delta}(k)|$ in (4.21), and non-trivial bound of $|F_R(k)|$ in (3.2), we have

$$S_{1,2} \leq 2C_3 C(v, \rho) L \sum_{L^{-1} < |k| < q_{s}/4} \frac{1}{1 + R^2 \|k\omega\|^2_\beta} \leq 2C_3 C(v, \rho) L \sum_{L^{-1} < |k| < q_{s}/4} \frac{1}{1 + R^2 \|k\omega\|^2_\beta}. \quad (4.28)$$

Applying (3.3), we obtain

$$S_{1,2} \leq 4\pi C_3 C(v, \rho) \frac{L R_0}{R} \leq 4\pi C_3 C(v, \rho) L,$$

in which we used $q_{s} \leq R$.

The estimate of $S_{1,3}$ is similar to that of $S_{1,2}$, except that we use (3.4) instead of (3.3). Indeed,
by (4.21) and (3.4), we have

\[ S_{1,3} \leq \sum_{\ell=1}^{[q_{s+1}/q_s]} \sum_{|k| \in \ell q_s/(\ell+1) q_s/4} |\hat{u}_{n,\delta}(k) F_R(k)| \]

\[ \leq \sum_{\ell=1}^{[q_{s+1}/q_s]} \sum_{|k| \in \ell q_s/(\ell+1) q_s/4} \frac{2|\hat{u}_{n,\delta}(k)|}{1 + R^2 \|k\omega\|_q^2} \]

\[ \leq \sum_{\ell=1}^{[q_{s+1}/q_s]} \frac{8 C(v, \rho)}{\ell q_s} (2 + 4\pi q_s R) \]

\[ \leq 16(1 + 2\pi)C(v, \rho) \frac{\log q_{s+1}}{q_s} \]

\[ \leq 120 C(v, \rho) \frac{\log q_{s+1}}{q_s} \]  

(4.29)

Note that by (4.20) with \( k = 0 \), we have

\[ \hat{u}_{n,\delta}(0) \leq L_n + \frac{1}{RL} \log (RL). \]

Trivial estimate \( \log x \leq \sqrt{x} \) for \( x > 0 \) implies

\[ \hat{u}_{n,\delta}(0) \leq L_n + \frac{1}{\sqrt{RL}} \leq L_n + \frac{L^2}{12} < L_n + L, \]  

(4.30)

in which we used \( R \geq R_0 \geq 144L^{-5} \) and \( 0 < L < 1 \).

Combining (4.23), (4.24), (4.25), (4.26), (4.27), (4.28), (4.29) with (4.30), we arrive at

\[ 0 \leq u_n(R)(x) \leq L_n + (2 + 8C(v, \rho) + 4\pi C_3 C(v, \rho))L + 120 C(v, \rho) \frac{\log q_{s+1}}{q_s}, \]

holds uniformly in \( x \).

\[ \Box \]

4.5 Proof of Lemma 2.6

We apply Lemma 2.5 to \( R = [3Ln] \). The conditions \( R \geq R_0 \) and \( n \geq N_3 \), see (4.14) and (4.15), can be reduced to

\[ n \geq N_0(\omega, L, v, \rho) := L^{-2} \max \left( 2\Lambda_{\omega} \sup_{1 \leq |k| \leq L^{-1}} \frac{1}{\|k\omega\|_q^2}, 49L^{-4} \right). \]  

(4.31)

Indeed, due to \( 0 < L < 1 \), we have

\[ R \geq 3Ln - 1 \geq 147L^{-5} - 1 > 144L^{-5}. \]

Now for \( n \geq N_0 \), Lemma 2.6 implies

\[ 0 \leq u_n(x) \leq |u_n(x) - u_n(R)(x)| + u_n(R)(x) \]

\[ \leq |u_n(x) - u_n(R)(x)| + L_n + (2 + 8C(v, \rho) + 4\pi C_3 C(v, \rho))L + 120 C(v, \rho) \frac{\log q_{s+1}}{q_s}. \]  

(4.32)

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By (4.11), we have
\[
|u_n(x) - u_n^{(R)}(x)| \leq \sum_{|j| < R} \frac{R - |j|}{R^2} \cdot \frac{2\Lambda_v|j|}{n} = \frac{(R^2 - 1)\Lambda_v}{3Rn} < \Lambda_v L. \tag{4.33}
\]

Hence combining (4.32) with (4.33), we get
\[
0 \leq u_n(x) \leq L_n + (2 + \Lambda_v + 8C(v, \rho) + 4\pi C_3 C(v, \rho))L + 120C(v, \rho)\frac{\log q_{s+1}}{q_s},
\]
holds uniformly in \(x\). \(\square\)

4.6 Proof of Lemma 2.10

Let
\[
N_2 = \max (150\Lambda_v N_1 L^{-1}, 400(C_1 + 2)N_1 + 1),
\]
be as in (2.17). Let \(n \geq N_2\) and \(R = \lfloor 400(C_1 + 2) - 1 \rfloor n \rfloor + 1\).

By (4.13), we have
\[
\left\| u_n(\cdot) - u_n^{(R)}(\cdot) \right\|_{L^\infty(T)} \leq \sum_{|j| < R} \frac{R - |j|}{R^2} \left\| u_n(\cdot) - u_n(\cdot + j\omega) \right\|_{L^\infty(T)}
\]
\[
\leq \sum_{|j| < R} \frac{|j|(R - |j|)}{nR^2} \left\| u_j(\cdot + n\omega) + u_j(\cdot) \right\|_{L^\infty(T)} \leq \sum_{|j| < R} \frac{2|j|(R - |j|)}{nR^2} \left\| u_j(\cdot) \right\|_{L^\infty(T)}. \tag{4.34}
\]

By our choice of \(R\) and \(n \geq N_2 \geq 400(C_1 + 2)N_1 + 1\), we have
\[
R \geq \frac{n}{400(C_1 + 2)} > N_1. \tag{4.35}
\]

We could split the sum in (4.34) into:
\[
\left\| u_n(\cdot) - u_n^{(R)}(\cdot) \right\|_{L^\infty(T)} \leq \sum_{|j| < N_1} \frac{2|j|(R - |j|)}{nR^2} \left\| u_j(\cdot) \right\|_{L^\infty(T)} + \sum_{N_1 \leq |j| < R} \frac{2|j|(R - |j|)}{nR^2} \left\| u_j(\cdot) \right\|_{L^\infty(T)}. \tag{4.36}
\]

We will use trivial upper bound \(\left\| u_j(\cdot) \right\|_{L^\infty(T)} \leq \Lambda_v\), see (2.2), in the first summation of (4.36). Note that \(j \geq N_1 \geq N_0\), hence we can apply Lemma 2.6 to \(u_j\) in the second sum. We have
\[
\left\| u_n(\cdot) - u_n^{(R)}(\cdot) \right\|_{L^\infty(T)} \leq \sum_{|j| < N_1} \frac{2\Lambda_v|j|(R - |j|)}{nR^2} + \sum_{N_1 \leq |j| < R} \frac{2|j|(R - |j|)}{nR^2} \left( L_j + C_1 L + 120C(v, \rho)\frac{\log q_{s+1}}{q_s} \right). \tag{4.37}
\]
For $j \geq N_1 \geq \tilde{N}_0 + 1$, Lemma 2.9 implies $L_j \leq 21L/20 < 2L$, hence
\[
\left\| u_n(\cdot) - u_n^{(R)}(\cdot) \right\|_{L^\infty(\mathbb{T})} \leq \sum_{|j|<N_1} \frac{2\Lambda_n |j|}{nR^2} + \sum_{N_1 \leq |j|<R} \frac{2|R - |j||}{nR^2} \left( (C_1 + 1)L + 120C(v, \rho) \frac{\log q_{s+1}}{q_s} \right). \tag{4.38}
\]
Use that
\[
\sum_{|j|<N_1} \frac{2|R - |j||}{R^2} = N_1 \frac{2(N_1 - 1)(3R - 2N_1 + 1)}{3R^2} \leq \frac{3}{4} N_1,
\]
and
\[
\sum_{N_1 \leq |j|<R} \frac{2|R - |j||}{R^2} = (R + 1 - N_1) \frac{2\left(R(R - 1) + (R + 1)N_1 - 2N_1^2\right)}{3R^2} \leq \frac{R(2R^2 - R)}{3R^2} = \frac{2}{3}(R - 1).
\]
We could control (4.38) by
\[
\left\| u_n(\cdot) - u_n^{(R)}(\cdot) \right\|_{L^\infty(\mathbb{T})} \leq \frac{3\Lambda_n N_1}{4n} + \frac{2(R - 1)}{3n} \left( (C_1 + 2)L + 120C(v, \rho) \frac{\log q_{s+1}}{q_s} \right). \tag{4.39}
\]
For the first term in (4.39), note that $n \geq N_2 \geq 150\Lambda_n N_1 L^{-1}$ implies
\[
\frac{3\Lambda_n N_1}{4n} \leq \frac{1}{200} L. \tag{4.40}
\]
For the second term, we plug in $R = \lfloor 400^{-1}(C_1 + 2)^{-1}n \rfloor + 1$, then we have
\[
\frac{2(C_1 + 2)(R - 1)}{3n} L < \frac{1}{200} L, \quad \text{and} \quad \frac{2(R - 1)}{3n} \cdot 120C(v, \rho) \frac{\log q_{s+1}}{q_s} \leq \frac{4C(v, \rho)}{15(C_1 + 2)} \frac{\log q_{s+1}}{q_s} \leq \frac{1}{5} C(v, \rho) \frac{\log q_{s+1}}{q_s}. \tag{4.41}
\]
Incorporating the estimates in (4.40) and (4.41) into (4.39), we have
\[
\left\| u_n(\cdot) - u_n^{(R)}(\cdot) \right\|_{L^\infty(\mathbb{T})} \leq \frac{1}{100} L + \frac{1}{5} C(v, \rho) \frac{\log q_{s+1}}{q_s},
\]
as stated. \qed

5 Optimal H"{o}lder continuity

5.1 Proof of Theorem 1.5

Fix $(\omega_0, E_0) \in (\mathbb{R} \setminus \mathbb{Q}) \times \mathcal{N}_v$ with $L(\omega_0, E_0) = \gamma > 0$. As we explained in Remark 1.6 that the neighborhood $U \times I$ as in Theorem 1.3 always exists. For any $(\omega, E) \in U \times I$:
\[
\frac{18}{19} \gamma \leq L(\omega, E) \leq \frac{20}{19} \gamma. \tag{5.1}
\]
Let $c(v, \rho)$ and $\tilde{c} = \tilde{c}(v, \rho)$ be the constants in Theorem 1.1. Define a subset $\tilde{U}$ of $U$ as follows

$$\tilde{U} := \{ \omega \in \mathbb{R} \setminus \mathbb{Q} : 0 \leq \beta(\omega) < c(v, \rho)\gamma/2 \} \cap U.$$  

(5.2)

In particular, $\tilde{U}$ contains all the Diophantine numbers in $U$, thus $\text{mes}(U \setminus \tilde{U}) = 0$.

We are going to apply Theorem 1.1 on interval $[a, b] = I$. Note that for any $\omega \in \tilde{U}$, by (5.1), we have

$$0 \leq \beta(\omega) < \frac{1}{2} c(v, \rho)\gamma < c(v, \rho) \inf_{E \in I} L(\omega, E).$$

(5.3)

Hence the condition of Theorem 1.1 is verified. Let $\tilde{N} = N(\omega, \inf_{E \in I} L(\omega, E), v, \rho)$ be as in (3.19), which is the constant in Theorem 1.1. Let $\tilde{N} = N(\omega, \tilde{c}(v, \rho), v, \rho)$ be the constant defined in (3.19) with $L = \frac{18}{19}\gamma$. Then by (5.1) and Remark 3.3, we have $\tilde{N} \geq N$. Let

$$\Omega_n(\omega, E) := \left\{ x \in \mathbb{T} : u_n(\omega, E; x) - L_n(\omega, E) > \frac{1}{20} L(\omega, E) \right\}.$$  

Theorem 1.1 implies that for $n \geq \tilde{N} \geq N$ and any $(\omega, E) \in \tilde{U} \times I$, we have

$$\text{mes}((\omega, E) \in \Omega_n(\omega, E) \leq e^{-\tilde{c}nL(\omega, E)} \leq e^{-\tilde{c}n\gamma/2}.$$  

(5.4)

in which we used $L(\omega, E) \geq \frac{18}{19}\gamma > \frac{1}{2}\gamma$, see (5.1).

In the rest of the section, we will fix $\omega \in \tilde{U}$ and denote $L(E) = L(\omega, E)$, $L_n(E) = L_n(\omega, E)$ for simplicity whenever it is clear. Apply Lemma 2.9 to the interval $I$. Let $\tilde{N}_0(\omega, I, v)$ be given as in Lemma 2.9. Then for any $n > \tilde{N}_0$ and $E \in I$, we have

$$L(E) \leq L_n(E) < \left(1 + \frac{1}{20}\right) L(E).$$  

(5.5)

Combining (5.5) with the fact that $L_{2n}(E) \leq L_n(E)$, we have for all $n > \tilde{N}_0$ and $E \in I$,

$$0 \leq L_n(E) - L_{2n}(E) < \frac{1}{20} L(E).$$  

(5.6)

After combining the large deviation estimate (5.4), the initial scale estimate (5.6), and the Avalanche Principle (Theorem 1.1), we obtain the following convergence rate of $L_n(E)$ to $L(E)$:

**Proposition 5.1.** There exists $N_4 \in \mathbb{N}$ explicitly depends on $\tilde{N}, \tilde{N}_0, \Lambda, \tilde{c}(v, \rho)$ and $\gamma$. For any $n > N_4$ and $(\omega, E) \in \tilde{U} \times I$,

$$|L(E) + L_n(E) - 2L_{2n}(E)| < e^{-\tilde{c}(v, \rho)n\gamma/5}. $$

(5.7)

Proposition 5.1 can be derived from an induction method developed by Goldstein and Schlag in [24] (see also in [9], [37]). For sake of completeness, we include the proof in Appendix C.

Another key ingredient for the proof of Theorem 1.3 is the following control on $\partial_E L_n(\omega, E)$ with respect to $\gamma$.

**Proposition 5.2.** There exists $N_5 \in \mathbb{N}$ explicitly depends on $\tilde{N}_0, \Lambda, \tilde{c}(v, \rho)$ and $\gamma$. For any $n > N_5$ and $(\omega, E) \in \tilde{U} \times I$,

$$|\partial_E L_n(E)| \leq 2c^{2n\gamma}.$$  

(5.8)
Proposition 5.2 is essentially contained in [9], we include the proof in Appendix D with these specific parameters.

Now we are in the place to complete the proof of Theorem 1.5 by using (5.7) and (5.8). For short hand we will write \(\tilde{c}(c, \rho)\) as \(\tilde{c}\), and denote
\[
\tilde{c}_0 := \tilde{c} + 20. \tag{5.9}
\]
Let
\[
N_6 = \max\{N_4, N_5\}
\]
and
\[
\eta := \min\left(e^{-2\gamma N_6 \tilde{c}_0 / 5}, 8^{-4\tilde{c}_0 / \tilde{c}} \right) < 1. \tag{5.10}
\]
Now for any \(E, E' \in I\) such that \(|E - E'| < \eta\), let
\[
n = \left\lfloor -\frac{5 \log |E - E'|}{\gamma \tilde{c}_0} \right\rfloor. \tag{5.11}
\]
Using the first term in (5.10), it is easy to check that
\[
-\frac{5 \log |E - E'|}{\gamma \tilde{c}_0} \geq n \geq -\frac{5 \log |E - E'|}{2\gamma \tilde{c}_0} \geq N_6 = \max(N_4, N_5). \tag{5.12}
\]
Now we can apply Proposition 5.1 and Proposition 5.2 to the above \(n, E, E'\) to obtain
\[
|L(E) - L(E')| \leq \frac{2 \left( e^{-\tilde{c} \eta \gamma / 5} + 4 e^{2n \eta} |E - E'| \right)}{\tilde{c}_0}. \tag{5.13}
\]
In view of the upper and lower bound of \(n\) in (5.12), we have
\[
e^{n \gamma} < |E - E'|^{-5/\tilde{c}_0}, \tag{5.14}
\]
and
\[
e^{-n \gamma} < |E - E'|^{5/(2\tilde{c}_0)}. \tag{5.15}
\]
By (5.13), (5.14) and (5.15), we have that for all \(\omega \in \tilde{U}, E, E' \in I\) and \(|E - E'| < \eta < 1,\)
\[
|L(E) - L(E')| \leq 2 |E - E'|^{5/(2\tilde{c}_0)} + 6 |E - E'|^{1-20/\tilde{c}_0}
\]
\[
= 2 |E - E'|^{5/(2\tilde{c}_0)} + 6 |E - E'|^{\tilde{c}/\tilde{c}_0} \tag{5.16}
\]
\[
\leq 8 |E - E'|^{5/(2\tilde{c}_0)}.
\]
Using the second term in (5.10), we have
\[
8 \leq \eta^{-\tilde{c}/(4\tilde{c}_0)} < |E - E'|^{-\tilde{c}/(4\tilde{c}_0)}. \tag{5.17}
\]
Plugging it into (5.10), we obtain
\[
|L(E) - L(E')| \leq |E - E'|^{\tilde{c}/(4\tilde{c}_0)} =: |E - E'|^\gamma. \tag{5.17}
\]
This proves Theorem 1.5.
5.2 Proof of Theorem 1.10

Let $\hat{\lambda}, b, B$ and $N = N(\omega, \lambda, f, \rho)$ be given as in Corollary 1.3. Assume that $\lambda > \max\{\hat{\lambda}, e^{B_\beta(\omega)}\}$, Corollary 1.3 implies that for any $n \geq N$, we have

$$\text{mes}\left\{x \in T : |u_n(\omega; E, x) - L_n(\omega; E)| > \frac{1}{19} \log \lambda\right\} \leq e^{-n b \log \lambda}. \quad (5.18)$$

In view of (3.43) and (3.44), we have that for $n \geq N$

$$\frac{18}{19} \log \lambda \leq L_n(E) \leq \frac{20}{19} \log \lambda, \quad 0 \leq L_n(E) - L_{2n}(E) \leq \frac{2}{19} \log \lambda. \quad (5.19)$$

By (5.18), (5.19) and the same reasoning for Proposition 5.1, we have Proposition 5.3. Assume that $\beta(\omega) < \infty$ and $\lambda > \max\{\hat{\lambda}, e^{B_\beta(\omega)}\}$. There exists $N_7 \in \mathbb{N}$ explicitly depends on $\lambda$ and $b$ such that for any $n > N_7$ and $E \in N_{\lambda f}$

$$|L(E) + L_n(E) - 2L_{2n}(E)| < e^{-\frac{1}{3}n b \log \lambda} \quad (5.20)$$

By the trivial bound $\sup_{n \in \mathbb{N}} \sup_{x \in T} \sup_{E \in N_{\lambda f}} u_n(x) \leq \Lambda_v \leq 2 \log \lambda$, we have for any $n, x$ and $E \in N_{\lambda f}$

$$\left|\partial_E \log \|M_n(\omega, E; x)\|\right| \leq \left|\partial_E M_n(\omega, E; x)\right| \leq \sum_{j=1}^n \|M_{n-j}(x + j\omega; E)\| \cdot \|M_{j-1}(\omega, E; x)\| \leq n e^{2n \log \lambda},$$

which implies

$$\left|\partial_E L_n(\omega, E)\right| \leq e^{2n \log \lambda}. \quad (5.21)$$

Clearly, by (5.20), (5.21) and the same argument from (5.10) to (5.17), we can prove (1.19). More precisely, for all $E, E' \in N_{\lambda f}$ satisfying

$$|E - E'| < \tilde{\eta} := \min\{e^{-2(12 + b)N_7(\log \lambda)/3}, 5^{-4(12 + b)/b}\}, \quad (5.22)$$

set $n = \left\lfloor \frac{3 \log |E - E'|^{-1}}{\log (12 + b)} \right\rfloor$. Then we have

$$|L(E) - L(E')| < 2e^{-\frac{1}{3}n b \log \lambda} + 3e^{4n \log \lambda} |E - E'| \leq 5|E - E'| \frac{e^{-4n \log \lambda}}{N(12 + b)} \leq |E - E'| \frac{e^{-4n \log \lambda}}{N(12 + b)} =: |E - E'|^\ast. \quad (5.23)$$

This completes the proof of Theorem 1.10.

A Proof of (4.6)

The proof is essentially contained in [10, Section II], we include a proof here for completeness.
Proof of (4.6)

Let us pick a bump function \( \eta(x) \) defined as follows:

\[
\eta(x) = \begin{cases} 
32(x + \frac{3}{4})^3, & -\frac{3}{4} \leq x < -\frac{1}{4}, \\
1 - 32(x + \frac{1}{4})^3, & -\frac{1}{4} \leq x < -\frac{1}{4}, \\
1, & \frac{1}{4} \leq x < \frac{3}{4}, \\
1 - 32(x - \frac{1}{4})^3, & \frac{1}{4} \leq x < \frac{3}{4}, \\
32(x - \frac{3}{4})^3, & \frac{3}{4} \leq x < \frac{1}{4}.
\end{cases}
\]

(A.1)

Then it is easy to see that

\[
\text{supp}\eta \subset \left[ -\frac{3}{4}, \frac{3}{4} \right], \quad \sum_{s \in \mathbb{Z}} \eta(x + s) = 1, \quad \text{and}
\]

\[
0 \leq \eta(x) \leq 1, \quad |\eta'(x)| \leq 6, \quad |\eta''(x)| \leq 48 \quad \text{for all } x \in \mathbb{R}
\]

(A.2)

Let \( w(x) := \int_{\Omega} \log |x - \zeta| \, d\mu(\zeta) \) and \( t := \sup_{\Omega} u_n(z) \). Since \( u_n(x) \) is 1-periodic on \( \mathbb{R} \), we have

\[
\hat{u}_n(k) = \langle \hat{u}_n - t \rangle(k)
\]

\[
= \int_{-\frac{1}{2}}^{1} (u_n(x) - t) e^{-2\pi i k x} \, dx
\]

\[
= \int_{\mathbb{R}} (u_n(x) - t) \eta(x) e^{-2\pi i k x} \, dx
\]

\[
= \frac{i}{2\pi k} \int_{\mathbb{R}} \partial_x \left( (w(x) + h(x) - t) \eta(x) \right) e^{-2\pi i k x} \, dx
\]

\[
= \frac{i}{2\pi k} \int_{\mathbb{R}} \partial_x (w(x) \eta(x)) e^{-2\pi i k x} \, dx + \frac{i}{2\pi k} \int_{\mathbb{R}} \partial_x (h(x) \eta(x)) e^{-2\pi i k x} \, dx
\]

(A.3)

\[
= \frac{i}{2\pi k} \int_{\mathbb{R}} \eta(x) \partial_x w(x) e^{-2\pi i k x} \, dx + \frac{i}{2\pi k} \int_{\mathbb{R}} w(x) \partial_x \eta(x) e^{-2\pi i k x} \, dx
\]

(A.4)

\[
+ \frac{i}{2\pi k} \int_{\mathbb{R}} \eta(x) \partial_x h(x) e^{-2\pi i k x} \, dx
\]

(A.5)

\[
+ \frac{i}{2\pi k} \int_{\mathbb{R}} (h(x) - t) \partial_x \eta(x) e^{-2\pi i k x} \, dx
\]

(A.6)

Clearly, (A.5), (A.6) can be bounded by

\[
|\langle A.5 \rangle| + |\langle A.6 \rangle| \leq \frac{1}{2\pi |k|} \left( \|\partial_x h\|_{L^\infty(\Omega_0)} + 6 \|h - \sup_{\Omega_i} u_n\|_{L^\infty(\Omega_0)} \right)
\]

(A.7)

It is enough to estimate (A.3) and (A.4) by (4.1). The bound for (A.4) is trivial since

\[
|\int_{\mathbb{R}} w(x) \partial_x \eta(x) e^{-2\pi i k x} \, dx| \leq 6 \int_{\Omega} \int_{-1}^{1} \log |x - \zeta| \, dx \, d\mu(\zeta)
\]

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\[
\leq 6\int_{\Omega_1} d\mu(\zeta) \sup_{\zeta \in \Omega_1} \int_{-1}^{1} \left| \log |x - \zeta| \right| dx \\
\leq 6\mu(\Omega_1) \int_{-2}^{2} |\log |x|| dx \\
= (24 \log 2)\mu(\Omega_1)
\]

The bound for (A.4) follows from the direct computation in [10]:

\[
\left| \int_{\mathbb{R}} \eta(x) \partial_x w(x)e^{-2\pi ikx} \, dx \right| = \left| \int_{\Omega_1} \int_{\mathbb{R}} \frac{x - \Re \zeta}{|x - \zeta|^2} e^{-2\pi ikx} \eta(x) \, dx \, d\mu(\zeta) \right| \\
\leq \int_{\Omega_1} \left| \int_{\mathbb{R}} \frac{x - \Re \zeta}{|x - \zeta|^2} e^{-2\pi ikx} \eta(x) \, dx \right| d\mu(\zeta) \\
\leq \mu(\Omega_1) \sup_{\zeta \in \Omega_1} \left| \int_{\mathbb{R}} \frac{x - \Re \zeta}{|x - \zeta|^2} e^{-2\pi ikx} \eta(x) \, dx \right| \leq C_4\mu(\Omega_1),
\]

where \( C_4 > 0 \) is some absolute constant given as in [10] such that

\[
\sup_{\zeta \in \Omega_1} \left| \int_{\mathbb{R}} \frac{x - \Re \zeta}{|x - \zeta|^2} e^{-2\pi ikx} \eta(x) \, dx \right| \leq C_4.
\]

This finishes the proof. \( \square \)

\section*{B Proof of Lemma 4.2}

Let \( \eta(x) \) be the bump function defined as in (A.1). Then

\[
|\hat{u}_{n,\delta}(k) - \hat{u}_{n}(k)| = \left| \int_{\mathbb{R}} \int_{\Omega_1} \log \left( \frac{|x - w| + \delta}{|x - w|} \right) e^{-2\pi ikx} \eta(x) \, \mu(dw) \, dx \right| \\
\leq \int_{\Omega_1} \left| \int_{\mathbb{R}} \log \left( \frac{|x - w| + \delta}{|x - w|} \right) e^{-2\pi ikx} \eta(x) \, dx \right| \mu(dw) \\
\leq \mu(\Omega_1) \sup_{w \in \Omega_1} \left| \int_{\mathbb{R}} \log \left( \frac{|x - w| + \delta}{|x - w|} \right) e^{-2\pi ikx} \eta(x) \, dx \right|.
\]

By Lemma 4.1 we already have control of \( \mu(\Omega_1) \), thus it suffices to estimate the following term for \( w = w_1 + iw_2 \in \Omega_1 \):

\[
\left| \int_{\mathbb{R}} \log \left( \frac{|x - w| + \delta}{|x - w|} \right) e^{-2\pi ikx} \eta(x) \, dx \right| \\
= \int_{-3/4 + w_1}^{3/4 + w_1} \log \left( 1 + \frac{\delta}{\sqrt{x^2 + w_2^2}} \right) e^{-2\pi ikx} \eta(x + w_1) \, dx,
\]

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in which we used \( \text{supp}(\eta) \subset [-3/4, 3/4] \). Next use the fact that \(|\eta(x)| \leq 1\) for any \( x \in \mathbb{R} \) and the integrand is monotone decreasing in \( x \), we have

\[
\left| \int_{-3/4+w_1}^{3/4+w_1} \log \left( 1 + \frac{\delta}{\sqrt{x^2 + w_2^2}} \right) e^{-2\pi ikx} \eta(x + w_1) \, dx \right| \\
\leq \int_{-3/4}^{3/4} \log \left( 1 + \frac{\delta}{\sqrt{x^2 + w_2^2}} \right) \, dx \\
\leq 2 \int_{0}^{3/4} \log \left( 1 + \frac{\delta}{x} \right) \, dx \\
= -2\delta \log \delta^{-1} + \frac{3}{2} \log \left( 1 + \frac{4\delta}{3} \right) + 2\delta \log \left( \frac{3}{4} + \delta \right).
\]

Use that \( \delta < \frac{1}{144} \), see (4.19), and that the following holds,

\[
\frac{3}{2} \log \left( 1 + \frac{4\delta}{3} \right) + 2\delta \log \left( \frac{3}{4} + \delta \right) < \delta \log \delta^{-1}, \text{ for } 0 < \delta < 0.15,
\]

we obtain that

\[
\left| \int_{-3/4+w_1}^{3/4+w_1} \log \left( 1 + \frac{\delta}{\sqrt{x^2 + w_2^2}} \right) e^{-2\pi ikx} \eta(x + w_1) \, dx \right| \leq 3\delta \log \delta^{-1}. \quad \text{(B.3)}
\]

(4.20) follows from combining (B.1), (B.2) with (B.3).

The proof of (4.21) follows from a similar idea to that of (2.1), the difference is that we need to do integration by parts twice in order to get \((k^2 \delta)^{-1}\) Fourier decay. Let us mention that one needs the control of \( \|\partial^2 h\|_{L^\infty(\Omega_0)} \), which is provided in (4.5), as well as \(|\eta''(x)| \leq 48\) as in (A.2).

\section{Proof of Proposition 5.1}

\textbf{Theorem C.1} (Avalanche Principle, [24]). Let \( B_1, \ldots, B_m \) be a sequence of unimodular \( 2 \times 2 \)-matrices. Suppose that

\[
\min_{1 \leq j \leq m} \|B_j\| \geq \mu > m \quad \text{and} \quad \max_{1 \leq j < m} \left| \log \|B_{j+1}\| + \log \|B_j\| - \log \|B_{j+1}B_j\| \right| < \frac{1}{2} \log \mu. \quad \text{(C.1)}
\]

Then

\[
\left| \log \|B_m \cdots B_1\| + \sum_{j=2}^{m-1} \log \|B_j\| - \sum_{j=1}^{m-1} \log \|B_{j+1}B_j\| \right| < C_A \frac{m}{\mu}, \quad \text{(C.3)}
\]

where \( C_A \) is an absolute constant.

For any \( n \geq N(\omega, \frac{18}{19} \gamma, v, \rho) \) and \( E \in I \), set

\[
\Omega_n(j) = \{ x \in T : \|u_n\left(x + (j-1)n\omega\right) - L_n(E)\| > \frac{1}{20} L(E) \}
\]

(33)
\[\Omega_{2n}(j) = \{ x \in \mathbb{T} \mid u_{2n}(x + (j-1)n\omega) - L_{2n}(E) > \frac{1}{20} L(E) \}\]

\[\Omega = \bigcup_{j=1}^{m} \Omega_n(j) \bigcup_{j=1}^{m-1} \Omega_{2n}(j),\]

(5.4) implies that \(\operatorname{mes}\Omega_n(j) < e^{-\frac{1}{4}\tilde{c}(v,\rho)n\gamma}, \operatorname{mes}\Omega_{2n}(j) \leq e^{-\frac{1}{4}\tilde{c}(v,\rho)n\gamma}\). Take \(m = [n^{-1}\exp(\frac{1}{4}\tilde{c}(v,\rho)n\gamma)]\) and \(n_1 = mn\), then \((2n)^{-1}\exp(\frac{1}{4}\tilde{c}(v,\rho)n\gamma) < m < n_1 < e^{-\frac{1}{4}\tilde{c}(v,\rho)n\gamma}\). Therefore, \(\operatorname{mes}\Omega < 2me^{-\frac{1}{4}\tilde{c}(v,\rho)n\gamma} < 2e^{-\frac{1}{4}\tilde{c}(v,\rho)n\gamma}\) provided \(\exp(\frac{1}{4}\tilde{c}(v,\rho)n\gamma) > 2n\).

For any \(x \notin \Omega\),

\[|u_n(x + (j-1)n\omega) - L_n(E)| < \frac{1}{20} L(E) < \frac{1}{20} L_n(E), \quad j = 1, \ldots, m,\]  

(5.5)

\[|u_{2n}(x + (j-1)n\omega) - L_{2n}(E)| < \frac{1}{20} L(E) < \frac{1}{20} L_{2n}(E), \quad j = 1, \ldots, m - 1.\]  

(5.6)

Thus

\[
\begin{align*}
\frac{19}{20} L_n(E) &< u_n(x + (j-1)n\omega) < \frac{21}{20} L_n(E), \\
\frac{19}{20} L_{2n}(E) &< u_{2n}(x + (j-1)n\omega) < \frac{21}{20} L_{2n}(E).
\end{align*}
\]

(5.7)

(5.8)

Denote \(B_j = M_n(x + (j-1)n\omega)\), then

\[u_n(x + (j-1)n\omega) = \frac{1}{n} \log \| M_n(x + (j-1)n\omega) \| = \frac{1}{n} \log \| B_j \|,\]

\[u_{2n}(x + (j-1)n\omega) = \frac{1}{2n} \log \| M_{2n}(x + (j-1)n\omega) \| = \frac{1}{2n} \log \| B_{j+1}B_j \|.\]

Notice that \(\tilde{c}(v,\rho) < 1\), by (5.7) and the choice of \(m\),

\[\| B_j \| > e^{\frac{19}{20} nL_n(E)} > e^{\frac{19}{20} nL(E)} := \mu > e^{\frac{19}{20} n\gamma} > e^{\frac{1}{4}\tilde{c}(v,\rho)n\gamma} > m, \quad j = 1, \ldots, m.\]

(5.9)

By (5.6), (5.5) and (5.6),

\[
\frac{\| \log B_{j+1} \| + \log \| B_j \| - \log \| B_{j+1}B_j \|}{\| B_{j+1} \| + \log \| B_j \| - \log \| B_{j+1}B_j \|} < \frac{n \| L(E) \| + \| 2nL_n(E) - 2nL_{2n}(E) \|}{\| L(E) \| + \| 2nL_n(E) \| + \| 2nL_{2n}(E) - \log \| B_{j+1}B_j \| \|}
\]

\[
\begin{align*}
&< \frac{\| L(E) \| + \| 2nL_n(E) - 2nL_{2n}(E) \|}{\| L(E) \| + \| 2nL_n(E) \| + \| 2nL_{2n}(E) \|}
\end{align*}
\]

(5.10)

Now \((C.1), (C.2)\) required by Avalanche Principle are fully filled. Apply Theorem (C.2) to \(B_j, j = 1, \ldots, m\), we have

\[
\| \log B_{m \cdots B_1} \| + \sum_{j=2}^{m-1} \log \| B_j \| - \sum_{j=1}^{m-1} \log \| B_{j+1}B_j \| < C \frac{m}{\mu}\]
Recall $n_1 = mn$, clearly
\[
\left| \frac{1}{n_1} \log \| M_{n_1} (x + (j - 1)n) \| + \frac{1}{m} \sum_{j=2}^{m-1} \frac{1}{n} \log \| M_n (x + (j - 1)n) \| \right| - \frac{2}{m} \sum_{j=1}^{m-1} \frac{1}{2n} \log \| M_{2n} (x + (j - 1)n) \| < C_A \frac{m}{n_1 \mu} < \frac{C_A}{\mu}.
\] (C.11)

Denote the sum of the left side of (C.11) by $F(x)$, we have got the above bound of $|F(x)|$ outside the set $\Omega$. For those $x \in \Omega$, we use the upper bound (2.2) such that
\[
\sup_{\Omega} |F(x)| < 4\Lambda v.
\] (C.12)

Integrate $F(x)$ over $\mathbb{T}$, by (C.4) and (C.9), for $n > \max\{N_0(\omega, I, v), N(\omega, \gamma/2, v, \rho)\}$ and $E \in I$, we have
\[
| L_{n_1}(E) + \frac{m-2}{m} L_n(E) - \frac{2(m-1)}{m} L_{2n}(E) | = | \int_{\mathbb{T}} F(x) dx |
\]
\[
< \frac{C_A}{\mu} + 4\Lambda v \cdot \text{mes} \Omega < C_A e^{-\frac{\hat{c} v n\gamma}{\gamma}} + 8\Lambda v \cdot e^{-\frac{\hat{c}(v, \rho) n\gamma}{\gamma}} < \frac{1}{20} e^{-\frac{\hat{c}(v, \rho) n\gamma}{\gamma}},
\] (C.13)

provided
\[
n > \frac{10}{\hat{c}(v, \rho) \gamma} \log(40C_A) + \frac{20}{\hat{c}(v, \rho) \gamma} \log(320\Lambda v).
\]

By (C.13), (5.5), (5.6) and the choice of $m$,
\[
| L_{n_1}(E) + L_n(E) - 2L_{2n}(E) | < \frac{2}{m} | L_n(E) - L_{2n}(E) | + \frac{1}{20} e^{-\frac{\hat{c}(v, \rho) n\gamma}{\gamma}} < 4n e^{-\frac{\hat{c}(v, \rho) n\gamma}{\gamma}} \cdot \frac{1}{20} 2\gamma + \frac{1}{20} e^{-\frac{\hat{c}(v, \rho) n\gamma}{\gamma}} < \frac{1}{10} e^{-\frac{\hat{c}(v, \rho) n\gamma}{\gamma}}
\] (C.14)

provided
\[
\hat{c}(v, \rho) n\gamma > 20 \log(80n\gamma).
\]

Take $n = 2n_1 = 2mn$, the above argument also shows that
\[
| L_{2n_1}(E) + L_n(E) - 2L_{2n}(E) | < \frac{1}{10} e^{-\frac{\hat{c}(v, \rho) n\gamma}{\gamma}}.
\] (C.15)

Therefore,
\[
| L_{2n_1}(E) - L_n(E) | < \frac{2}{10} e^{-\frac{\hat{c}(v, \rho) n\gamma}{\gamma}} < \frac{1}{40} \gamma < \frac{1}{20} L(E),
\] (C.16)
provided \( n > 5(\tilde{c}(v, \rho)\gamma)^{-1} \log(8\gamma^{-1}) \).

Let \( n_0 = n \) and for \( s = 0, 1, \cdots \), let
\[
 n_{s+1} = n_s [n_s^{-1} e^{\frac{1}{5} \tilde{c}(v, \rho)n_s \gamma}] .
\]  
(C.17)
Inductively, we can prove that

**Proposition C.2 (Iteration of \( L_n(E) \)).**

1\(^s\)
\[
| L_{n+1}(E) + L_n(E) - 2L_{2n}(E) | < \frac{1}{10} e^{-\frac{1}{5} \tilde{c}(v, \rho)n \gamma} ,
\]
(C.18)
\[
| L_{2n+1}(E) + L_n(E) - 2L_{2n}(E) | < \frac{1}{10} e^{-\frac{1}{5} \tilde{c}(v, \rho)n \gamma} .
\]
(C.19)
2\(^s\)
\[
| L_{2n+1}(E) - L_{n+1}(E) | < \frac{2}{10} e^{-\frac{1}{5} \tilde{c}(v, \rho)n \gamma} < \frac{1}{40} \gamma < \frac{1}{20} L(E) .
\]
(C.20)

Once we have 1\(^{s-1}\), 2\(^{s-1}\), we prove 1\(^s\) first as (C.14), (C.15). Then 2\(^s\) directly follows from 1\(^s\) as (C.16). By 1\(^s\) and 2\(^{s-1}\), we get 3\(^s\) as follows:
\[
| L_{n+1}(E) - L_n(E) | < 
\]
\[
< \frac{1}{10} e^{-\frac{1}{5} \tilde{c}(v, \rho)n \gamma} + \frac{4}{10} e^{-\frac{1}{5} \tilde{c}(v, \rho)n_{s-1} \gamma} 
\]
\[
< \frac{1}{2} e^{-\frac{1}{5} \tilde{c}(v, \rho)n_{s-1} \gamma} .
\]
(C.21)

□

When the iteration is established for all \( s \geq 1 \), it is easy to check \( n_{s-1} > sn \) by (C.17), we have then
\[
| L(E) - L_n(E) | \leq \sum_{s=1}^{\infty} | L_{n+1}(E) - L_n(E) | 
\]
\[
\leq \frac{1}{2} \sum_{s=1}^{\infty} e^{-\frac{1}{5} \tilde{c}(v, \rho)n_{s-1} \gamma} 
\]
\[
\leq \frac{1}{2} e^{-\frac{1}{5} \tilde{c}(v, \rho)n \gamma} \frac{2}{1 - e^{-\frac{1}{5} \tilde{c}(v, \rho)n \gamma}} 
\]
\[
\leq \frac{9}{10} e^{-\frac{1}{5} \tilde{c}(v, \rho)n \gamma} ,
\]
provided \( e^{-\frac{1}{5} \tilde{c}(v, \rho)n \gamma} < \frac{4}{9} \).

By (C.14), we have
\[
| L(E) + L_n(E) - 2L_{2n}(E) | < e^{-\frac{1}{5} \tilde{c}(v, \rho)n \gamma} .
\]
(C.22)
D Proof Proposition 5.2

It is enough to show that for $n$ large

$$\sup_{x \in \mathbb{T}} \left| \partial_E \log \| M_n(\omega, E; x) \| \right| \leq 2ne^{2n\gamma} \quad \text{(D.1)}$$

Lemma 2.9 and (5.1) imply that for $n > \tilde{N}_0$, for any $x \in \mathbb{T}$ and $E \in I$

$$u_n(\omega, E; x) \leq 2\gamma \quad \text{(D.2)}$$

i.e., $\| M_j(\omega, E; x) \| \leq e^{2\gamma}$ for $j > \tilde{N}_0$. For $j \leq \tilde{N}_0$, we use the trivial bound

$$\| M_j(\omega, E; x) \| \leq e^{2\gamma} \quad \text{(D.3)}$$

Direct computation shows that for any $x \in \mathbb{T}$

$$\left| \partial_E \log \| M_n(\omega, E; x) \| \right| = \left| \frac{\partial E}{\| M_n(\omega, E; x) \|} \right| \| M_n(\omega, E; x) \| \leq \| \partial_E M_n(\omega, E; x) \| \leq \sum_{j=1}^{n} \| M_{n-j}(x + j\omega; E) \| \cdot \| M_{j-1}(\omega, E; x) \|$$

$$= \sum_{j=1}^{\tilde{N}_0} + \sum_{j=\tilde{N}_0+1}^{n} \| M_{n-j}(x + j\omega; E) \| \cdot \| M_{j-1}(\omega, E; x) \|$$

$$\leq \sum_{j=1}^{\tilde{N}_0} C_5 e^{2(n-j)\gamma} + \sum_{j=\tilde{N}_0+1}^{n} e^{2(n-j)\gamma} e^{2(j-1)\gamma} + \sum_{j=n-\tilde{N}_0+1}^{n} C_5 e^{2(j-1)\gamma}$$

$$\leq 2C_5 \tilde{N}_0 e^{2n\gamma} + (n - 2\tilde{N}_0) e^{2n\gamma}$$

$$\leq (n + 2C_5 \tilde{N}_0) e^{2n\gamma}$$

$$\leq 2ne^{2n\gamma}$$

provided

$$n > 2C_5 \tilde{N}_0 > 2\tilde{N}_0$$

E Proofs of (3.2), (3.3), (3.4)

Proof of (3.2):

First, trivially we have $F_R(k) \leq 1$. Direct computation shows:

$$0 \leq F_R(k) = \frac{\sin^2(\pi R k \omega)}{R^2 \sin^2(\pi k \omega)} = \frac{\sin^2(\pi R \|k \omega\|_\mathbb{T})}{R^2 \sin^2(\pi \|k \omega\|_\mathbb{T})} \leq \frac{\sin^2(\pi R \|k \omega\|_\mathbb{T})}{4R^2 \|k \omega\|^2}.$$

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in which we used \( \sin(\pi x) \geq 2x \) for \( 0 \leq x \leq 1/2 \).

If \( R\|k\omega\|_T \geq 1 \), then using \( 2R^2\|k\omega\|_T^2 \geq 1 + R^2\|k\omega\|_T^2 \), we obtain
\[
F_R(k) \leq \frac{1}{2} \cdot \frac{1}{1 + R^2\|k\omega\|_T^2}.
\]

Suppose \( R\|k\omega\|_T < 1 \), we have
\[
F_R(k) \leq \frac{2}{1 + R^2\|k\omega\|_T^2}.
\]

Therefore by combining the two estimates above, we have
\[
0 \leq F_R(k) \leq \min\left(1, \frac{2}{1 + R^2\|k\omega\|_T^2}\right),
\]
as claimed.

**Proofs of (3.3) and (3.4):**

Since \( \frac{\ell}{q} \) is a continued fraction approximant of \( \omega \), we have \( |\omega - \frac{\ell}{q}| < \frac{1}{2q} \). This implies that for any \( 0 \neq |k| < \frac{q}{2} \), \( |k\omega - \frac{kp}{q}| < \frac{k}{2}\), and hence
\[
\|k\omega\|_T \geq \|k\omega\|_T - |k\omega - \frac{kp}{q}| \geq \frac{1}{2q}. \tag{E.1}
\]

If we take \( j_1 \neq j_2 \in (0, \frac{2}{4}) \subset \mathbb{Z} \), then clearly \( |j_1 - j_2| < \frac{1}{2q} \). Thus by (E.1),
\[
\|j_1\omega\|_T - \|j_2\omega\|_T \geq \min\{\|(j_1 + j_2)\omega\|_T, \|(j_1 + j_2)\omega\|_T\} \geq \frac{1}{2q}.
\]
This implies that \( \{\|k\omega\|_T\}_{k=1}^{[q/4]} \) are \( \frac{1}{2q} \) departed, and by (E.1) the smallest one is \( \frac{1}{2q} \). If we rearrange them in the increasing order and label them as \( \|1\omega\|_T < \|2\omega\|_T < \cdots < \|k_{[q/4]}\omega\|_T \), then \( \|k_{s}\omega\|_T \geq \frac{1}{2q} \). Hence
\[
\sum_{1 \leq k < \frac{q}{4}} \frac{1}{1 + R^2\|k\omega\|_T^2} = 2 \sum_{1 \leq k < \frac{q}{4}} \frac{1}{1 + R^2\|k\omega\|_T^2} \leq 2 \sum_{s=1}^{[q/4]} \frac{1}{1 + R^2\left(\frac{2q}{2q}\right)^2} \leq \frac{4q}{R} \int_{0}^{\infty} \frac{dx}{1 + x^2} = \frac{2\pi q}{R},
\]
this proved (3.3).

For \( \ell \geq 1 \), let \( I_\ell := \left[\frac{\ell}{4}, \frac{\ell}{4}(\ell + 1)\right) \cap \mathbb{Z}, \ell \geq 1 \). We divide \( I_\ell \) into two disjoint sets, \( S_1 = \{k \in I_\ell, |k\omega - \lfloor k\omega \rfloor| \leq 0.5 \}, S_2 = \{k \in I_\ell, |k\omega - \lfloor k\omega \rfloor| > 0.5 \}. \) Then for \( j_1 \neq j_2 \in I_\ell \) belonging to the same subset (either \( S_1 \) or \( S_2 \)), we have \( \|j_1\omega\|_T - \|j_2\omega\|_T \geq \|\omega\|_T \). Since clearly \( |j_1 - j_2| < \frac{q}{2} \), by (E.1), we have \( \|\omega\|_T \geq \frac{1}{2q} \). This implies that \( \{\|k\omega\|_T\}_{k \in S_1} \) are \( \frac{1}{2q} \) apart from each other, and the same holds for \( S_2 \). Thus we could arrange the terms \( \{\|k\omega\|_T\}_{k \in S_1 (or \ S_2)} \) in the increasing order and label them as \( \|1\omega\|_T < \|2\omega\|_T < \cdots \|k_{[q/4]}\omega\|_T \), and we have \( \|k\omega\|_T \geq \frac{1}{2q} \). Hence
\[
\sum_{\lfloor q/4 \rfloor \leq k \leq \lceil q/4 \rceil} \frac{1}{1 + R^2\|k\omega\|_T^2} = 2 \sum_{k \in I_\ell} \frac{1}{1 + R^2\|k\omega\|_T^2} \leq 2 \sum_{k \in S_1} \frac{1}{1 + R^2\|k\omega\|_T^2} + 2 \sum_{k \in S_2} \frac{1}{1 + R^2\|k\omega\|_T^2} \leq 2 \sum_{s=1}^{[q/4]} \frac{1}{1 + R^2\left(\frac{2q}{2q}\right)^2} \leq 2 + \frac{4q}{R} \int_{0}^{\infty} \frac{dx}{1 + x^2} = 2 + \frac{4\pi q}{R},
\]
this proves (3.4).
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