Affine convex body semigroups and Buchsbaum rings

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Abstract

In this work, new families of Buchsbaum rings, developed by means of convex body semigroups, are presented. We characterize Buchsbaum circle and convex polygonal semigroups and describe algorithmic methods to check these characterizations.

Keywords: Affine semigroup, Buchsbaum ring, Cohen-Macaulay ring, convex body semigroup.

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Introduction

Buchsbaum rings were introduced in the last half of the twentieth century and it has been treated from different points of view. Two good introductions to Buchsbaum rings can be found in [7] and [12]. Given a field $k$, $r$ indeterminates over it, $t_1, \ldots, t_r$, and an affine cancellative commutative semigroup $S$, the semigroup ring $k[S]$ can be defined to be the subring of $k[t_1, \ldots, t_r]$ generated by $t^s = t_1^{s_1} \cdots t_r^{s_r}$ with $s = (s_1, \ldots, s_r)$ $\in$ $S$. We say $S$ is Buchsbaum if its associated semigroup ring $k[S]$ is a Buchsbaum ring. There are many works devoted to the study of Buchsbaum affine semigroup rings (see for example [1] [2], [6], [8], [11], [12] and the references therein) and a proposed problem in some of these works is to find a criteria, expressed in terms of the affine semigroup $S$, to know if $k[S]$ is Buchsbaum (see [2]).

In this work we focus our attention on the study of Buchsbaum convex body semigroup rings. Given $F \subset \mathbb{R}_+^r$ a non-empty convex body, we consider the so-called convex body semigroup $\mathcal{F} = \bigcup_{i=0}^{\infty} F_i \cap \mathbb{N}^r$ where $F_i = i \cdot F$ with $i \in \mathbb{N}$ (see [3] for further details). This class of semigroups are useful to obtain examples...
of different kinds of rings. For instance, in [5], the authors characterize some Cohen-Macaulay and Gorenstein convex body semigroup rings and they give computational methods to get examples of them. The straightforward method to check whether an element of \( \mathbb{N}^p \) belongs or not to a convex body semigroup makes easy to work with this kind of semigroups. In this work we characterize Buchsbaum convex body semigroups when the convex body is a circle or a compact convex polygon (Proposition 4 and Theorem 6, respectively). We see that these characterizations can be checked by using basic tools of Linear Algebra and Basic Geometry. These tools are also used for the construction of Buchsbaum semigroup rings. Besides, in Corollary 8 and 9 we give explicitly families of Buchsbaum semigroups by mean of their initial polygons. We introduce the program \texttt{PolygonalSG} ([10]), which is used to compute the minimal generating set of a convex polygonal semigroup given by a rational polygon.

The contents of this paper are organized as follows. In Section 1, we provide some basic tools and definitions that are used in the rest of the work. In Section 2, Buchsbaum affine circle semigroups are characterized. Finally, Section 3 is devoted to prove the properties that Buchsbaum affine convex polygonal semigroups must satisfy (Theorem 6) and to give explicit families of Buchsbaum affine convex polygonal semigroups.

1 Preliminaries

For any \( L \) subset of \( \mathbb{R}^r \), denote by \( L_\geq \) the set \( \{(x_1, \ldots, x_r) \in L | x_i \geq 0, i = 1, \ldots, r\} \). Let \( G \) be a non-empty subset of \( \mathbb{R}^r_\geq \), denote by \( L_{\mathbb{Q} \geq}(G) \) the rational cone \( \{\sum_{i=1}^p q_i f_i | p \in \mathbb{N}, q_i \in \mathbb{Q} \geq, f_i \in G\} \). Let \( Fr(L_{\mathbb{Q} \geq}(G)) \) be the boundary of \( L_{\mathbb{Q} \geq}(G) \) considering the usual topology of \( \mathbb{R}^r \) and define the interior of \( G \) as \( G \setminus Fr(L_{\mathbb{Q} \geq}(G)) \), denote it by \( \text{int}(G) \). We use \( d(P) \) to represent the Euclidean distance from a point \( P \) to the origin \( O \).

Let \( S \subset \mathbb{N}^r \) be the affine semigroup generated by \( \{n_1, \ldots, n_r, n_{r+1}, \ldots, n_{r+m}\} \). A semigroup \( S \) is called simplicial if \( L_{\mathbb{Q} \geq}(S) \cap \mathbb{N}^r \) is an affine semigroup which is denoted by \( C \).

Let \( S \) be the semigroup \( \{a \in \mathbb{N}^r | a + n_i \in S, \forall i = 1, \ldots, r + m\} \). It is straightforward to prove that \( \overline{S} \subset C \). The following result shows a characterization of Buchsbaum rings in terms of their associated semigroups.

**Theorem 1.** ([6] Theorem 5) The following conditions are equivalent:

1. \( S \) is Buchsbaum.
2. \( \overline{S} \) is Cohen-Macaulay.

In [6] Theorem 9 it is given a method to check if a simplicial semigroup is Buchsbaum. To apply such method it is necessary to compute the intersection of the Apéry sets of the generators of the rational cone of \( S \) (the elements \( n_1, \ldots, n_r \)). That intersection is computed using the method presented in [7] which uses some bounds to describe a region where the elements of the Apéry set are. The problem is that, in many practical cases, the high value of the bound
obtained makes the algorithm impractical. Thus, to know if $S$ is Buchsbaum, it is necessary to check if the semigroup $\overline{S}$ is Cohen-Macaulay. In this work we focus to solve algorithmically this problem in some kinds of subsemigroups of $\mathbb{N}^2$.

Given $S \subseteq \mathbb{N}^2$ an affine semigroup, denote by $\tau_1$ and $\tau_2$ the extremal rays of $C = L_{\mathbb{Q}_2}(S) \cap \mathbb{N}^2$ with $\tau_1$ the ray with greater slope and by $n_1 \in \tau_1$ the element of $S \cap \tau_1$ with less module, similarly define $n_2 \in \tau_2$. Note that $C \cap \tau_j = \mathbb{N}^2 \cap \tau_j$ (with $j = 1, 2$) is a subsemigroup of $\mathbb{N}^2$ and that it is generated only by an element.

**Corollary 2.** Let $S \subseteq \mathbb{N}^2$, the following conditions are equivalent:

1. $S$ is Cohen-Macaulay.
2. For all $a \in C \setminus S$, $a + n_1$ or $a + n_2$ does not belong to $S$.

**Lemma 3.** Let $S \subseteq \mathbb{N}^2$ be a simplicial affine semigroup such that $\text{int}(C) \setminus \text{int}(S)$ is a non-empty finite set, then $S$ is not Cohen-Macaulay.

From now on, we consider only semigroups associated to convex bodies. Let $F \subset \mathbb{R}^2$ be a non-empty convex body, the convex body semigroup generated by $F$ is the semigroup $F = \bigcup_{i=0}^{\infty} F_i \cap \mathbb{N}^2$. In general, these semigroups are not finitely generated. An interesting property of them is that it is easy to check whether an element $P$ belongs to a given semigroup. Just proceed as follows: take $\tau$ the ray defined by $P$ and the segment $\tau \cap F = AB$ with $d(A) \leq d(B)$; the element $P$ belongs to $F$ if and only if the set \( \{ k \in \mathbb{N} | \frac{d(P)}{d(A)} \leq k \leq \frac{d(P)}{d(B)} \} \) is non-empty.

In $[3]$, affine convex body semigroups are characterized when the initial convex body is a circle or a convex polygon. In both cases, a convex body semigroup is affine if and only if the intersection of the initial convex body with each extremal ray of its associated positive integer cone contains at least a rational point. Besides, the minimal system of generators of these convex body semigroups can be computed algorithmically (see Theorem 14 and Theorem 18 in $[3]$ for further details). Let $F \subset \mathbb{R}^2$ be a convex body, in this case the positive integer cone $C$ is equal to $L_{\mathbb{Q}_2}(F \cap \mathbb{R}^2_+) \cap \mathbb{N}^2$ and $\text{int}(C) = C \setminus \{ \tau_1, \tau_2 \}$.

### 2 Buchsbaum affine circle semigroups

Let $C \subset \mathbb{R}^2$ be the circle with center $(a, b)$ and radius $r > 0$ with $a, b, r \in \mathbb{R}$; define $C',$ the circle with center $(ia, ib)$ and radius $ir$ and $S = \bigcup_{i=0}^{\infty} C_i \cap \mathbb{N}^2$ the so-called circle semigroup associated to $C$. Note that when $C \cap \mathbb{R}^2_+$ has at least two points the circle semigroup is simplicial. In this section, we consider that $S$ is always a simplicial affine circle semigroup. Let $\overline{S}$ be the semigroup \( \{ a \in \mathbb{N}^2 \mid a + n_i \in S, \forall i = 1, \ldots, m \} \) with \( \{ n_1, n_2, \ldots, n_m \} \) the minimal system of generators of $S$.

**Proposition 4.** Let $S \subset \mathbb{N}^2$ be an affine circle semigroup. The semigroup $S$ is Buchsbaum if and only if $\text{int}(C) = \text{int}(\overline{S})$ and $\overline{S} \cap \tau_j$ is generated only by one element for $j = 1, 2$.  

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Proof. By Theorem 1, $S$ is Buchsbaum if and only if $S$ is Cohen-Macaulay. We prove that $S$ is Cohen-Macaulay if and only if $\text{int}(C) = \text{int}(S)$ and $S \cap \tau_j$ is generated by only one element for $j = 1, 2$.

Assume that $S$ is Cohen-Macaulay and suppose that $\text{int}(C) \setminus \text{int}(S) \neq \emptyset$. Let $n'_j$ be one element of the minimal system of generators of $S \cap \tau_j$ with $j = 1, 2$. Since there exists a real number $d > 0$ such that $\{a \in \text{int}(C)|d(a) > d\} \subset S$ (see [3, Lemma 17]), the set $\text{int}(C) \setminus \text{int}(S)$ is finite, and thus $\text{int}(C) \setminus \text{int}(S) = \{d(a)|a \in \text{int}(C) \setminus \text{int}(S)\}$.

Take $a \in \text{int}(C) \setminus \text{int}(S)$ verifying that $d(a) = \max\{d(a')|a' \in \text{int}(C) \setminus \text{int}(S)\}$. The elements $a + n'_j$ and $a + n'_2$ are in $S$ and by Corollary 2 the semigroup $S$ is not Cohen-Macaulay which is a contradiction. Let us prove now that $S \cap \tau_j$ is generated by only one element for $j = 1, 2$. We consider two different cases: $S \cap \tau_j$ is generated by only one element or not. If there exist $n_j \in \mathbb{N}^2$ such that $S \cap \tau_j = \langle n_j \rangle$ for $j \in \{1, 2\}$, then for every $a \in (\tau_j \setminus S) \cap \mathbb{N}^2$ we have that $a + n_j \in \tau_j \setminus S$ and hence $S \cap \tau_j = S \cap \tau_j$. We consider now the case that $S \cap \tau_j$ is minimally generated by two or more elements with $j \in \{1, 2\}$. We have that $C \cap \tau_j$ is a segment and that $(C \setminus S) \cap \tau_j$ is a finite non-empty set. This implies that $(C \setminus S) \cap \tau_j$ is a finite non-empty set too. Take the element $a \in (C \setminus S) \cap \tau_j$ such that $d(a) = \max\{d(a')|a' \in (C \setminus S) \cap \tau_j\}$. It verifies that $a + n'_1$ and $a + n'_2$ belong to $S$ and therefore $S$ is not Cohen-Macaulay (Corollary 2).

Assume now that $\text{int}(C) = \text{int}(S)$ and that $S \cap \tau_j$ is generated by only one element for $j = 1, 2$. In this case, it is straightforward to prove that for all $a \in C \setminus S$, $a + n_1 \not\in S$ or $a + n_2 \not\in S$. By Corollary 2, $S$ is Cohen-Macaulay. □

Using the above proof, the conditions of Proposition 4 can be determined from the initial circle. To check whether $\text{int}(C) = \text{int}(S)$, we only have to compute the finite set $\text{int}(C) \setminus \text{int}(S)$ by using the bound provided by [3, Lemma 17]. The second condition is satisfied whether $C \cap \tau_j$ is a point or, in case $C \cap \tau_j$ is a segment, if the generator of $C \cap \tau_j$ belongs to $S$. Both conditions can be checked algorithmically.

Example 5. Let $C$ be the circle with center $(7/5, 4/5)$ and radius $1/5$. Computing with the program CircleSG (see [4]), we obtain that the affine circle semigroup $S$ associated to $C$ is minimally generated by the set

\[
\{(4, 2), (5, 3), (6, 3), (6, 4), (7, 3), (7, 4), (7, 5), (8, 5), (9, 4), (9, 6), (10, 7),
(11, 8), (15, 11), (19, 8), (19, 14), (23, 17), (27, 20), (31, 13), (31, 23), (32, 24),
(35, 26), (43, 18), (55, 23), (67, 28), (79, 33), (91, 38), (96, 40)\}
\]

and $\text{int}(C) \setminus \text{int}(S) = \{(2, 1), (3, 2)\}$ (see Figure 1). It is easy to check that the points $(2, 1)$ and $(3, 2)$ belong to $S$. Thus, we obtain $\text{int}(C) = \text{int}(S)$. Besides, $S \cap \tau_1 = \langle(32, 24)\rangle$ and $S \cap \tau_2 = \langle(96, 40)\rangle$. By Proposition 4, the affine circle semigroup $S$ is Buchsbaum.

For any affine semigroup, a problem of a high computational complexity is the problem of determining whether an element belongs to it. As stated above, in the particular case of circle semigroups this problem is simple. This makes that the previous example can be computed quickly.

\footnote{Note that $C \cap \tau_1 = \langle(32/25, 24/25)\rangle$ and $C \cap \tau_2 = \langle(96/65, 8/13)\rangle$.}
3 Buschbaum affine convex polygonal semigroups

Denote by $F \subset \mathbb{R}^2$ a compact convex polygon (not equal to a segment) with vertex set $P = \{P_1, \ldots, P_t\}$ arranged in counterclockwise direction and let $\mathcal{P} = \bigcup_{i=0}^{\infty} F_i \cap \mathbb{N}^2$ be its associated semigroup. We assume that $\mathcal{P}$ is a simplicial affine convex polygonal semigroup. As in previous sections, let $\tau_1$ and $\tau_2$ be the extremal rays of $\mathcal{C}$ assuming $\tau_1$ with a slope greater than the slope of $\tau_2$. Let $\mathcal{P}_1$ be the semigroup $\{a \in \mathbb{N}^2 | a + n_i \in P, \forall i = 1, \ldots, m\}$ with $\{n_1, n_2, \ldots, n_m\}$ the minimal system of generators of $\mathcal{P}$ and let $n_j'$ be a minimal generator of $\mathcal{P} \cap \tau_j$ with $j = 1, 2$.

In order to prove the results of this section, we consider different special subsets of the cone $\mathcal{C}$ and some points and lines in $L_{\mathbb{Q}} (F \cap \mathbb{R}^2)$. We distinguish two cases, $F \cap \tau_1$ is a point or it is formed by more than one point.

Assume $F \cap \tau_1 = \{P_1\} \subset P$, let $j$ be the least positive integer such that $jP_1P_2 \cap (j+1)P_1P_2$ is not empty. Since $P_1P_2$ and $P_1P_2$ are not parallel, there exists a point $\{V_1\} = jP_1P_2 \cap (j+1)P_1P_2$ (using [3, Lemma 11], $V_1$ can be easily computed). Denote by $T_1$ the triangle with vertex set $\{O, P_1, V_1\}$, and by $\hat{T}_1$ its topological interior. By [3, Lemma 11], for every $h \in \mathbb{N}$ with $h \geq j$ the points $hP_1P_2 \cap (h+1)P_1P_2$ are in the same straight line, which we denote by $\nu_1$. Note that $((T_1 \cup (O\hat{T}_1 \setminus \{O, P_1\})) + \mu P_1) \cap \mathcal{P} = \emptyset$ for all $\mu \in \mathbb{Z}_{\geq}$. This construction allows us to define the set

$$B_1 = \{D + \lambda n_1 | D \in (jP_1)V_1 \text{ and } \lambda \in \mathbb{Q}_{\geq}\} \cap \mathcal{C}$$

whose elements are in $\mathcal{P}$ or they are in $\bigcup_{\mu \in \mathbb{N}, \mu \geq j} ((T_1 \cup (O\hat{T}_1 \setminus \{O, P_1\})) + \mu P_1)$. The elements of $B_1$ verify that if $P \in B_1 \setminus \mathcal{P}$ then $P + n_1 \notin \mathcal{P}$ and thus $P \notin \mathcal{P}$; this implies that $\mathcal{P} \cap B_1 = \mathcal{P} \cap B_1$. Denote by $\mathcal{T}_1$ the finite set ConvexHull(\{O, jP_1, V_1, \nu_1 \cap \mathcal{C}\}) \cap \mathbb{N}^2. Analogously, if the set $F \cap \tau_2 = \{P_2\} \subset P$, we call it again $P_2$ for sake of simplicity, there exists the least integer $j$ such that $jP_1P_2 \cap (j+1)P_1P_2$ is equal to $\{V_2\}$. Let $T_2$ be the triangle with vertex set $\{O, P_1, V_2 - jP_1\}$, and denote by $\nu_2$ the line containing the
points \( \{hP_iP_j \cap (h+1)P_iP_j | h \geq j, h \in \mathbb{N} \} \) and by \( B_2 \) the set \( \{D + \lambda_{v_2} | D \in \{jP_i \} \cap \mathbb{C} \} \) and \( \lambda \in \mathbb{Q} \). All of the properties of these sets are analogous to the properties of the sets defined previously for \( \tau_1 \). Denote by \( T_2 \) the finite set \( \text{ConvexHull} \{O, jP_i, v_2, v_2 \cap \tau_1 \} \cap \mathbb{N}^2 \). In case \( F \cap \tau_1 \) is a segment for some \( i \), we take \( \nu_i = \tau_i \) and \( T_i = \{O\} \).

We define the set \( \Upsilon = (Q + L_{\mathbb{Q}}(F)) \cap \mathbb{N}^2 \subset \mathbb{C} \) with \( \{Q\} = \nu_1 \cap \nu_2 \subset L_{\mathbb{Q}}(F) \). Note that the boundary of the set \( \Upsilon \) intersects with two different sides of the polygon \( i_0F \) when \( i_0 \gg 0 \) and therefore the sets \( \Upsilon \setminus \mathbb{P} \) and \( \Upsilon \setminus \mathbb{P}^{\prime} \) are finite. The last set we define is the finite set \( \Upsilon' = \{a \in (\Upsilon_1 \cup \Upsilon_2) \setminus \overline{\mathbb{P}} | a + n_1', a + n_2' \subset \mathbb{P} \} \). It is straightforward to prove that the cone \( \mathbb{C} \) is the union of \( B_1 \), \( B_2 \), \( T_1 \), \( T_2 \) and \( \Upsilon \).

**Theorem 6.** Let \( \mathbb{P} \) be a simplicial affine convex polygonal semigroup. Then

1. if \( \text{int}(\mathbb{C}) = \text{int}(\overline{\mathbb{P}}) \), the semigroup \( \mathbb{P} \) is Buchsbaum if and only if \( \mathbb{P} \cap \tau_j \) is generated by only one element for \( j = 1, 2 \),

2. if \( \text{int}(\mathbb{C}) \neq \text{int}(\overline{\mathbb{P}}) \), the semigroup \( \mathbb{P} \) is Buchsbaum if and only if \( \Upsilon' = \emptyset \) and \( \Upsilon \subset \overline{\mathbb{P}} \).

**Proof.** We prove that \( \mathbb{P} \) is Cohen-Macaulay if and only if the conditions of the theorem are fulfilled. Due to its similarity with the proof of Proposition \( 4 \) case \( 1 \) is left to the reader.

Assume that \( \text{int}(\mathbb{C}) \neq \text{int}(\overline{\mathbb{P}}) \) and that \( \overline{\mathbb{P}} \) is Cohen-Macaulay. By Corollary \( 2 \) the set \( \Upsilon' \) has to be empty. If \( \Upsilon \not\subset \overline{\mathbb{P}} \), choose \( a \in \Upsilon \setminus \overline{\mathbb{P}} \) such that \( d(a') = \max \{d(a') | a' \in \Upsilon \setminus \overline{\mathbb{P}} \} \). Then \( a + n_1' \) and \( a + n_2' \) belong to \( \overline{\mathbb{P}} \) which implies that \( \overline{\mathbb{P}} \) is not Cohen-Macaulay. Thus \( \Upsilon \subset \overline{\mathbb{P}} \).

Conversely, let \( a \) be an element of \( \mathbb{C} \setminus \overline{\mathbb{P}} \) (note that \( a \not\in \Upsilon \subset \overline{\mathbb{P}} \)). We discuss the possibilities we have. If \( F \cap \tau_1 \) is a point and \( a \) belongs to the strip bounded by the parallel lines \( \tau_1 \) and \( \nu_1 \), we have that \( \mathbb{P} \cap \tau_1 = \overline{\mathbb{P}} \cap \tau_1 \) and \( n_1 = n_1' \). Besides, the element \( a \) belongs to \( \Upsilon_1 \setminus \overline{\mathbb{P}} \) or it belongs to \( B_1 \setminus \overline{\mathbb{P}} \). Since \( \Upsilon' = \emptyset \), if \( a \in \Upsilon_1 \setminus \overline{\mathbb{P}} \), the element \( a + n_1' \) or \( a + n_2' \) does not belong to \( \overline{\mathbb{P}} \) and if \( a \in B_1 \setminus \overline{\mathbb{P}} \) then \( a + n_1' \not\in \overline{\mathbb{P}} \). We proceed similarly in case of \( F \cap \tau_2 \) is a point and \( a \) belongs to the strip bounded by the parallel lines \( \tau_2 \) and \( \nu_2 \), obtaining that the element \( a + n_1' \) or \( a + n_2' \) does not belong to \( \overline{\mathbb{P}} \). If \( F \cap \tau_2 \) is a segment, since \( \Upsilon \subset \overline{\mathbb{P}} \), we have that \( \mathbb{C} \setminus \overline{\mathbb{P}} \subset \Upsilon_1 \cup B_1 \), and thus every \( a \in \mathbb{C} \setminus \overline{\mathbb{P}} \) verifies \( a + n_1' \) or \( a + n_2' \) is not in \( \overline{\mathbb{P}} \). Similarly, when \( F \cap \tau_1 \) is a segment and \( F \cap \tau_2 \) is a single point, for every \( a \in \mathbb{C} \setminus \overline{\mathbb{P}} \) we obtain again that \( a + n_1' \) or \( a + n_2' \) is not in \( \overline{\mathbb{P}} \). In any of the above cases, every element \( a \in \mathbb{C} \setminus \overline{\mathbb{P}} \) fulfills that at least one element of the set \( \{a + n_1', a + n_2'\} \) does not belong to \( \overline{\mathbb{P}} \) and hence \( \overline{\mathbb{P}} \) is Cohen-Macaulay (Corollary \( 2 \)). Finally, if \( F \cap \tau_1 \) and \( F \cap \tau_2 \) are both segments, \( \mathbb{C} = \Upsilon \) and therefore the semigroup \( \overline{\mathbb{P}} \) is Cohen-Macaulay. \( \square \)

As in the circle semigroup case, to apply the above result it is necessary to check whether \( \text{int}(\mathbb{C}) = \text{int}(\overline{\mathbb{P}}) \). The different situations are the following:

1. If \( F \cap \tau_1 \) is a segment \( \overline{\mathbb{P}} \cap \overline{\mathbb{P}} \) and \( F \cap \tau_2 \) is a segment \( \overline{\mathbb{P}} \cap \overline{\mathbb{P}} \), the set \( \Upsilon \) is equal to the positive integer cone \( \mathbb{C} \) and the sets \( \mathbb{C} \setminus \overline{\mathbb{P}} \) and \( \mathbb{C} \setminus \overline{\mathbb{P}} \) are finite. Let \( j \in \mathbb{N} \) be the least integer such that \( \overline{\mathbb{P}} \cap (j+1) \overline{\mathbb{P}} \neq \emptyset \) and \( \overline{\mathbb{P}} \cap (j+1) \overline{\mathbb{P}} \neq \emptyset \), and let \( T \) be the triangle with vertex set \( \{O, jP_i_j, P_i_j \} \). Clearly, \( T \cap \mathbb{N}^2 \) is finite and \( \text{int}(\mathbb{C}) \setminus \text{int}(\overline{\mathbb{P}}) \subset T \cap \mathbb{N}^2 \). This is illustrated in Figure \( 2 \).
2. Let us suppose that $P \cap \tau_1$ is a point $P_1$ and $P \cap \tau_2$ is a point $P_2$. If $P \in (\text{int}(C) \cap (B_1 \cup B_2)) \setminus \text{int}(P)$, the element $P + n_1$ or $P + n_2$ does not belong to $P$ and thus $P \notin P$. This implies that $P \cap (B_1 \cup B_2) = \overline{P} \cap (B_1 \cup B_2)$. Let $j \in \mathbb{N}$ such that $jP_1P_2 \cap (j+1)P_1P_2 = \{V_1\}$ and let $t \in \mathbb{N}$ satisfying $tP_1 = n_1$. For every $r, k \in \mathbb{Z}_2$ there exists $h \in \{0, \ldots, t-1\}$ such that $(T_1 + (r+j)P_1) \cap \mathbb{N}^2 = (T_1 + (h+j)P_1) \cap \mathbb{N}^2 + kn_1$. Note that this construction is for $B_1$ and that for $B_2$ we must proceed similarly with the triangle $T_2$. So to compare $\text{int}(C) \cap (B_1 \cup B_2)$ with $\text{int}(\overline{P}) \cap (B_1 \cup B_2)$ it is only necessary to check if there are nonnegative integer points in the sets $T_1 + (h+j)P_1$ (with $h \in \{0, \ldots, t-1\}$) and, analogously, in some translations of $T_2$ in the direction of $P_2$. Since $T_1$ and $T_2$ are parallelograms, $(T_1 \cup T_2) \cap \mathbb{N}^2$ is a finite set and therefore $(\text{int}(C) \cap (T_1 \cup T_2)) \setminus \text{int}(\overline{P})$ can be computed.

In order to compute $(\text{int}(C) \cap \mathcal{T}) \setminus \text{int}(\overline{P})$, just take $j \in \mathbb{N}$ the least integer such that both sets $jP_1P_2 \cap (j+1)P_1P_2 = \{V\}$ and $jP_2P_{t+1} \cap (j+1)P_{t+1}P_{t+2} = \{V'\}$ are formed by only one point and let $T$ be the triangle with vertex set $\{Q, V, V'\}$. By construction, the sets $(\text{int}(C) \cap \mathcal{T}) \setminus \mathcal{T}$ and $(\text{int}(\overline{P}) \cap \mathcal{T}) \setminus \mathcal{T}$ are equal. Therefore $\text{int}(C) \cap \mathcal{T} = \text{int}(\overline{P}) \cap \mathcal{T}$ if and only if the finite sets $\text{int}(C) \cap \mathcal{T}$ and $\{a \in \text{int}(\overline{P}) \cap \mathcal{T} | a + n_1 \in \mathcal{P}, \forall t = 1, \ldots, m\}$ are equal. This case is illustrate in Example 7 (see Figure 3).

3. If $F \cap \tau_1 = \{P_1\}$ and $F \cap \tau_2$ is a segment $P_{d-1}P_d$, to compare the sets $\text{int}(C \setminus \mathcal{T})$ and $\text{int}(\overline{P}) \setminus \mathcal{T}$, just proceed as in the second case with the sets $B_1$ and $T_1$. Let now $j \in \mathbb{N}$ be the least integer such that $jP_1P_2 \cap (j+1)P_1P_2$ is a point $V$ and $jP_{d-1}P_d \cap (j+1)P_{d-1}P_d \neq \emptyset$, and let $T$ be the triangle with vertex set $\{Q, V, P_d\}$. Then $\text{int}(C \cap \mathcal{T}) = \text{int}(\overline{P}) \cap \mathcal{T}$ if and only if the finite sets $\text{int}(C \setminus \mathcal{T})$ and $\text{int}(\overline{P}) \setminus \mathcal{T}$ are equal.

4. Finally, the case $F \cap \tau_2$ is a point and $F \cap \tau_1$ is a segment is analogous to the above case.

In any case, all the necessary sets to compare $\text{int}(C)$ with $\text{int}(\overline{P})$ are finite and they can be obtained algorithmically. Besides, the conditions $\mathcal{T}' = \emptyset$ and
\( \Upsilon \subset \overline{P} \) can be checked algorithmically and “\( \overline{P} \cap \tau_j \) is generated by only one element” can be tested in a similar way to the case of circle semigroup.

**Example 7.** Let \( F \) be the polygon determined by the rational points \((3.6, 1.8), (3.6, 0.6), (3.3, 1.05), (4.2, 1.5), (4.14, 0.99)\) and \( P \) its associated affine convex polygonal semigroup (the dark grey region in Figure 3). The minimal system of generators of \( P \) can be computed with the program `PolygonalSG` (see [10]).

\[
\text{In}[1]:= \text{PolygonalSG}[[3.6, 1.8], [3.6, 0.6], [3.3, 1.05], [4.2, 1.5], [4.14, 0.99]]
\]
\[
\text{Out}[1]= \{[4, 1], [7, 2], [7, 3], [8, 3], [10, 3], [11, 2], [11, 5], [14, 3], [18, 3], [18, 9], [20, 8], [23, 10]\}
\]

We obtain that \( P \) is minimally generated by

\[
G = \{(18, 9), (18, 3), (4, 1), (20, 8), (23, 10), (8, 3), (11, 5), (11, 2), (10, 3), (14, 3), (7, 2), (7, 3)\}
\]

Using basic tools of Linear Algebra we compute the sets \( \Upsilon_1, \Upsilon_2 \), the triangle \( T \) and the necessary translations of \( T_1 \) and \( T_2 \) (the above sets are needed to check the conditions of Theorem 6). Those translations are the lighter grey triangles in Figure 3. We also have \((13, 4) + n \in P \) for all \( n \in G \). Thus \( (13, 4) \in \overline{P} \) and therefore \( \Upsilon \subset \overline{P} \). This can be checked with the function `BelongToSG` of [10]. For example,

\[
\text{In}[2]:= \text{BelongToSG}[\{13, 4\} + (18, 9), [3.6, 1.8], [3.6, 0.6], [3.3, 1.05], [4.2, 1.5], [4.14, 0.99]]
\]
\[
\text{Out}[2]= \text{True}
\]

The set \( (\text{int}(C) \setminus \text{int}(P)) \cap (\Upsilon_1 \cup \Upsilon_2) \) is equal to

\[
D = \{(3, 1), (5, 1), (5, 2), (6, 2), (9, 2), (10, 2), (9, 3), (13, 3), (16, 3), (17, 3), (9, 4), (10, 4), (17, 4), (12, 5), (13, 5), (13, 6)\},
\]

but none of these points are in \( \overline{P} \). Besides, for all \( a \in D \), \( a + n_1 \) or \( a + n_2 \) does not belong to \( \overline{P} \). Therefore \( \Upsilon' \) is the empty set. By Theorem 6 we conclude that \( P \) is a non-Cohen-Macaulay Buchsbaum affine semigroup.

If we use the method of Theorem 9 in [6], it is necessary to compute the intersection of the Apéry set of \( n_1 \) and the Apéry set of \( n_2 \) by checking if \( 2 \times 7771556800000 \) elements belong to \( P \); this causes such method to be inefficient.

As indicated before, the problem of determining whether an element belongs to a convex polygonal semigroup is straightforward; this implies a reduction of the time of computation.

In Example 7, it is used only Elementary Algebra, but Buchsbaum semigroups can be generated using an even simpler approach. The following results provide two user-friendly properties which allow us to obtain easily Buchsbaum rings.
Figure 3: Affine polygonal semigroup $\mathcal{P}$ associated to the polygon \{(3.6, 1.8), (3.6, 0.6), (3.3, 1.05), (4.2, 1.5), (4.14, 0.99)\}.

Corollary 8. Every affine convex polygonal semigroup associated to a triangle with rational vertices is Buchsbaum.

Proof. Note that if $F$ is a triangle, $\mathcal{P}$ and $\overline{\mathcal{P}}$ are equal. Corollary 12 in [5] proves that every affine convex polygonal semigroup associated to a triangle with rational vertices is Cohen-Macaulay. Thus, $\overline{\mathcal{P}} = \mathcal{P}$ is Cohen-Macaulay and therefore $\mathcal{P}$ is Buchsbaum (Theorem 1).

Corollary 9. Let $F$ be a convex polygon with vertices $P_1, \ldots, P_4 \in \mathbb{Q}_2$ and let $\mathcal{P}$ be its associated affine convex polygonal semigroup. If $P_1 \in \mathcal{P} \cap \tau_1, P_4 \in \mathcal{P} \cap \tau_2$ and the points $O, P_2$ and $P_3$ are aligned, $\mathcal{P}$ is Buchsbaum.

Proof. Let $C_1$ be the positive integer cone delimited by the ray $\tau_1$ and the line $OP_2$, and let $C_2$ be the cone delimited by the ray $\tau_2$ and the line $OP_2$. Trivially $C = L_{Q_2}(F \cap \mathbb{R}_2^n) \cap N^2$ is the union of $C_1$ and $C_2$, and the semigroup $\mathcal{P}$ is the union of the affine convex polygonal semigroups, $\mathcal{P}_1$ and $\mathcal{P}_2$, associated to the triangles with vertex sets $\{P_1, P_2, P_3\}$ and $\{P_2, P_3, P_4\}$, respectively. With that decomposition of the affine convex polygonal semigroup $\mathcal{P}$ and from the hypothesis, we can assert $\mathcal{P}$ is equal to $\overline{\mathcal{P}}$, $\mathcal{Y} \subset \mathcal{P}$ and $\mathcal{P} \cap \tau_1$ and $\mathcal{P} \cap \tau_2$ are generated by only one element each (Figure 4 illustrates this situation). Under such conditions, let $a$ be an element belonging to $C \setminus \mathcal{P}$. Note that if $a \in C_1 \setminus \mathcal{P}_1$ then $a + n_1 \notin \mathcal{P}$, otherwise, if $a \in C_2 \setminus \mathcal{P}_2$ then $a + n_2 \notin \mathcal{P}$. In any case, $a + n_1$

Figure 4: Affine polygonal semigroup $\mathcal{P}$ associated to the polygon \{(3.6, 1.2), (4.8, 1.6), (4, 2), (4, 1)\}.
or \( a + n_2 \) does not belong to \( \mathcal{P} \). Thus \( \overline{\mathcal{P}} (= \mathcal{P}) \) is Cohen-Macaulay (Corollary 2) and then \( \mathcal{P} \) is Buchsbaum.

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