A stock loan is a loan, secured by a stock, which gives the borrower the right to redeem the stock at any time before or on the loan maturity. The way of dividends distribution has a significant effect on the pricing of stock loans and the optimal redeeming strategy adopted by the borrower. We present the pricing models of the finite maturity stock loans subject to various ways of dividend distribution. Because closed-form price formulas are generally not available, we provide a thorough analysis to examine the optimal redeeming strategy. Numerical results are presented as well.

**KEY WORDS:** stock loans, finite maturity, optimal strategy, optimal stopping.

1. **INTRODUCTION**

Optimal stopping problems in finance have gained growing interest due to their close linkage with various optimal strategies. A typical and well-known example is the American vanilla option pricing model which has been extensively studied in the existing literature. Many researchers have also considered plenty of more sophisticated models. For example, Gerber and Shiu (1996) and Broadie and Detemple (1997) analyzed the optimal exercise strategy for American options on multiassets, which was further addressed by Villeneuve (1999), Jiang (2002), Detemple, Feng, and Tian (2003), etc. Cheuk and Vorst (1997), Windcliff, Forsyth, and Vetzal (2001), and Dai, Kwok, and Wu (2004) investigated the optimal shouting strategy for shout options. Dai and Kwok (2006) characterized the optimal exercise strategy for American-style Asian options and lookback options. Hu and Oksendal (1998) studied the optimal strategy of an investment problem. Other studies along this line include the pricing of game options, swing options and convertible bonds, and the multiple-stopping problems in finance, see Kifer (2000), Carmona and Touzi (2008), Ibanez (2004), Meinshausen and Hambly (2004), Dai and Kwok (2005), Dai and Kwok (2008), etc.

In this paper, we take into consideration another optimal stopping problem arising from the pricing of a financial product: **stock loan**, which is a contract between a client (borrower) and a bank (lender). The borrower, who owns one share of a stock, obtains...
a loan from the lender with the share as collateral. The borrower may redeem the stock at any time before or on the loan maturity by repaying the lender the principal and a predetermined loan interest rate, or surrender the stock instead of repaying the loan. The accumulative dividends may be gained by the borrower or the lender, subject to the provision of the loan.

A natural pricing problem arises for both the borrower and the lender: given the principal $K$ and the loan interest rate $\gamma$, what is the fair fee charged by the lender (referred to as the price of the stock loan)? Moreover, the borrower is facing another problem: what is the optimal redeeming strategy? Xia and Zhou (2007) initiated the study of the above problems under the Black–Scholes framework. Assuming that the loan is of infinite maturity and the dividends accrued are gained by the lender until the borrower redeems the stock, they revealed that the stock loan is essentially an American call option with a possibly negative interest rate. Moreover, they obtained a closed-form price formula as well as the analytical optimal redeeming strategy. Zhang and Zhou (2009) further extended it to a regime switching market.

In this paper, we are concerned with finite maturity stock loans with various ways of dividend distribution. Except for special cases, closed-form price formulas are no longer available. As in Xia and Zhou (2007), the pricing model of a stock loan resembles that of (finite maturity) American vanilla options if the dividends are gained by the lender before redemption. However, other ways of dividends distribution may significantly alter the pricing model as well as the optimal redeeming strategy. In particular, if the accumulative dividends are assumed to be returned to the borrower on redemption, the pricing model will get one path-dependent variable involved, which leads the study of the associated redeeming strategy to be challenging. We will provide an analytical method to analyze the optimal redeeming strategy.

Throughout the paper, we assume that the risk neutral stock price follows a geometric Brownian motion:

$$dS_t = (r - \delta)S_t \, dt + \sigma S_t \, dW_t,$$

where constants $r > 0$, $\delta \geq 0$ and $\sigma > 0$ are the riskless interest rate, continuous dividend yield of and volatility of the stock, respectively, and $\{W_t; t \geq 0\}$ is a standard one-dimension Brownian motion on a filtered probability space $(\mathcal{S}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with $W_0 = 0$ almost surely. We denote by $K > 0$ the stock loan’s principal, $\gamma$ the (continuous compounding) loan interest rate, and $T > 0$ the maturity date. For later use, we let $C_E(\cdot, \cdot; r, \delta, X)$ (or $P_E(\cdot, \cdot; r, \delta, X)$) be the price of European vanilla call (or put) option with riskless rate $r$, dividend yield $\delta$, and strike price $X$.

The rest of the paper is organized as follows. In Section 2, we consider a relatively simple case: the dividends are gained by the lender before redemption. In Section 3, we assume that the dividends are reinvested in the stock and will be returned to the borrower on redemption. We will see that this reduces to a special case in Section 2, and the resulting optimal strategy can be linked to those presented in the subsequent two sections. Section 4 is devoted to the scenario that the cash dividends are delivered to the borrower immediately, no matter whether the borrower redeems the stock. It is worth pointing out that the pricing models in Sections 2–4 involve one state variable only and are then relatively easy to analyze. In Section 5, we will take into consideration the most

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1 The assumption of continuous dividend payments is valid when the collateralized security is foreign currency. If dividends are paid discretely, a jump condition is needed at the dividend payment date (see, e.g., Wilmott, Dewynne, and Howison 1993, section 8.3 and 11.3).
challenging and interesting case: the accumulative cash dividends are delivered to the borrower on redemption. The pricing model proves to be a two-dimensional parabolic variational inequality. Section 6 discusses some extensions with conclusive remarks.

2. DIVIDENDS GAINED BY THE LENDER BEFORE REDEMPTION

Assume that the dividends are gained by the lender before redemption. Let $V_1 = V_1(S, t)$ be the value of the stock loan at time $t$ with stock price $S$. Then

$$V_1(S, t) = \sup_{v \in \mathcal{T}[t, T]} \mathbb{E}_t \left[ e^{-r(t-t)} (S_v - Ke^{\gamma t})^+ \right].$$

where $\mathbb{E}_t$ is the risk neutral expectation conditionally on $\mathcal{F}_t$ and $\mathcal{T}[t, T]$ denotes the set of all $\{F_s\}_{t \leq s \leq T}$-stopping times with values in $[t, T]$. It turns out that $V_1$ satisfies the following variational inequality (cf. Karatzas and Shreve 1998)

$$\begin{cases}
\min \{-L_{t, \delta} V_1, V_1(S, t) - (S - Ke^{\gamma t})\} = 0,
V_1(S, T) = (S - Ke^{\gamma t})^+, \quad (S, t) \in Q,
\end{cases}$$

where $Q \equiv (0, \infty) \times [0, T)$, and

$$L_{t, \delta} = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - \delta) S \frac{\partial}{\partial S} - r.$$

Let us define the redemption region

$$E_1 \equiv \{(S, t) \in Q : V_1(S, t) = S - Ke^{\gamma t}\}.$$

The following proposition characterizes the properties of $E_1$.

PROPOSITION 2.1. Assume that the dividends are gained by the lender before redemption.

(i) If $r \geq \gamma$ and $\delta = 0$, then $E_1 = \emptyset$. This indicates that early redemption should never happen. In this scenario the stock loan is equivalent to a European call option with strike price $Ke^{\gamma T}$, namely, $V_1(S, t) = C_{\mathbb{P}}(S, t; r, 0, Ke^{\gamma T})$.

(ii) If $r \geq \gamma$ and $\delta > 0$, or $r < \gamma$, then there is an optimal redeeming boundary $S^*_1(t)$: $[0, T) \rightarrow (0, \infty)$ such that

$$E_1 = \{(S, t) \in Q : S \geq S^*_1(t)\}.$$

In addition, $e^{-\gamma t} S^*_1(t)$ is monotonically decreasing with $t$,

$$S^*_1(T) = \lim_{t \rightarrow T-} S^*_1(t) = \begin{cases} e^{\nu T} \max \left( K, \frac{r - \gamma}{\delta} K \right) & \text{if } r \geq \gamma \text{ and } \delta > 0, \\
 e^{\nu T} K & \text{if } r < \gamma. \end{cases}$$

Proof. By similarity reduction

$$f_1(x, t) = e^{-\gamma t} V_1(S, t), \quad x = e^{-\gamma t} S,$$

we get
\[
\begin{align*}
\min \left\{ -L^\tau f_1, f_1(x, t) - (x - K) \right\} &= 0, \\
f_1(x, T) &= (x - K)^+, \quad (x, t) \in Q,
\end{align*}
\]  

where \( \tau = r - \gamma \). Then, \( f_1(x, t) \) can be regarded as the value of an American call option with riskless rate \( \tau \) that is likely to be nonpositive. Hence, part (i) is a well-known result when \( \tau \geq 0 \) and \( \delta = 0 \).

When \( \tau \geq 0 \) and \( \delta > 0 \), we know from equation (2.4) (cf. Jiang 2005) that there is a decreasing function \( x_1^*(t) : [0, T) \rightarrow (0, +\infty) \), such that

\[
\{ (x, t) \in Q : f_1(x, t) = x - K \} = \{ (x, t) \in Q : x \geq x_1^*(t), \ t \in [0, T) \}
\]

and

\[
x_1^*(T) \equiv \lim_{t \to T^-} x_1^*(t) = \max(K, \frac{\tau}{\delta} K).
\]

Using a similar argument as in Jiang (2005), we are able to show that equations (2.5) and (2.6) are still true when \( \tau < 0 \). Notice that \( \max(K, \frac{\tau}{\delta} K) = K \) for \( \tau < 0 \), part (ii) then follows.

**Remark 2.2.** The collateral is a stock that does not incur storage costs. If the collateral is assumed to be a tradable investment asset with proportional storage costs and no dividends, then the redemption region (if exists) may not take the form as given in (2.2). We refer interested readers to Battauz, De Donno, and Sbuelz (2008, 2009).

Now we examine the asymptotic behavior of \( S_1^*(t) \) as the time to maturity goes to infinity.

**Proposition 2.3.** Assume \( r \geq \gamma \) and \( \delta > 0 \), or \( r < \gamma \). Let \( S_1^*(t) \) be as given in part (ii) of Proposition 2.1. Then

\[
S_{1, \infty}^*(t) \equiv \lim_{T \to +\infty} S_1^*(t) = e^{\tau t} x_{1, \infty}^*,
\]

where

\[
x_{1, \infty}^* = \begin{cases} 
\frac{\alpha_+}{\alpha_+ - 1} K, & \text{if } \delta > 0, \ or \ \delta = 0 \ and \ \ r < \gamma - \frac{1}{2} \sigma^2 \\
+\infty, & \text{if } \delta = 0 \ and \ \gamma - \frac{1}{2} \sigma^2 \leq r < \gamma
\end{cases}
\]

and

\[
\alpha_+ = \frac{-\left( r - \gamma - \delta - \frac{1}{2} \sigma^2 \right) + \sqrt{\left( r - \gamma - \delta + \frac{1}{2} \sigma^2 \right)^2 + 2 \delta \sigma^2}}{\sigma^2}.
\]

To prove the above proposition, we only need to consider a perpetual stock loan and \( S_{1, \infty}^* \) is the corresponding optimal redeeming boundary. This has been done by Xia and Zhou (2007) in terms of a probabilistic approach. We would like to provide in the Appendix a simpler partial differential equation (PDE) argument in combination with the continuous dependence on parameter \( \delta \) of solutions. It is worth pointing out that
the explicit solution of \( \gamma_{1,\infty} \) is nothing but the optimal exercise boundary of a perpetual American call option when \( r \geq \gamma \) and \( \delta > 0 \).

3. REINVESTED DIVIDENDS RETURNED TO THE BORROWER
ON REDEMPTION

Assume that the dividends are immediately reinvested in the stock and will be returned to the borrower on redemption. The intrinsic value (i.e., the redemption payoff) of the stock loan at time \( t \) becomes

\[
(e^{rt} S_t - Ke^{\gamma t})^+, \ t \in [0, T].
\]

Let \( V_2 = V_2(S, t) \) be the price function of the stock loan. Then

\[
V_2(S_t, t) = \sup_{\nu \in [0, T]} \mathbb{E}_{\nu}[e^{-\nu(t-s)}(e^{\delta t} S_t - Ke^{\gamma \nu})^+].
\]

Denote

\[
\widehat{V}_2(S, t) = V_2(S_t, t),
\]

where

\[
\widehat{S}_t \equiv e^{rt} S_t = S_0 \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right).
\]

As a result, the stock loan can be regarded as the one written on the nondividend paying stock \( \widehat{S} \), which has been studied in Section 2.

For later use, we present the PDE model for \( \widehat{V}_2 \)

\[
\begin{align*}
\min \left\{ \frac{-L_v^0 \widehat{V}_2, \ \widehat{V}_2(\widehat{S}, t) - (\widehat{S} - Ke^{\gamma \nu})} \right\} &= 0, \\
\widehat{V}_2(\widehat{S}, T) &= (\widehat{S} - Ke^{\gamma T})^+, \ (\widehat{S}, t) \in Q
\end{align*}
\]

and the redemption region associated with \( \widehat{V}_2 \) as

\[
\widehat{E}_2 = \{(\widehat{S}, t) \in Q : \widehat{V}_2(\widehat{S}, t) = \widehat{S} - Ke^{\gamma T}\}.
\]

Later we will see that \( \widehat{E}_2 \) has a close link with the redemption regions addressed in the subsequent two sections. By Proposition 2.1, \( \widehat{E}_2 = \emptyset \) when \( r \geq \gamma \), and if \( r < \gamma \), then there is a monotonically decreasing function \( \widehat{S}^*_2(t) : [0, T] \rightarrow (0, \infty) \) such that

\[
\widehat{E}_2 = \{(\widehat{S}, t) \in Q : \widehat{S} \geq \widehat{S}^*_2(t), \ t \in [0, T]\},
\]

\[
\widehat{S}^*_2(T) = \lim_{t \rightarrow T^-} \widehat{S}^*_2(t) = K, \text{ and}
\]

\[
\widehat{S}^*_2 \equiv \lim_{t \rightarrow +\infty} \widehat{S}^*_2(t) \text{ is finite if } r < \gamma - \frac{1}{2} \sigma^2 \text{, and infinite if } \gamma - \frac{1}{2} \sigma^2 \leq r < \gamma.
\]

Remark 3.1. It is easy to show that \( \widehat{V}_2(S, t) \geq V_1(S, t) \) for all \((S, t) \in Q\). Then for any \((S, t) \in \widehat{E}_2\), we have \( S - Ke^{\gamma T} \geq \widehat{V}_2(S, t) \geq V_1(S, t) \geq S - Ke^{\gamma T} \), namely, \( V_1(S, t) = S - Ke^{\gamma T} \). This indicates \( \widehat{E}_2 \subset E_1 \) and \( \widehat{S}^*_2(t) \geq S_t(t) \) for all \( t \in [0, T] \).
4. DIVIDENDS ALWAYS DELIVERED TO THE BORROWER

Assume that the dividends are always delivered to the borrower during the lifetime of the stock loan. The intrinsic value of the stock loan becomes

\[(S_t - Ke^{\gamma t})^+ + \int_0^t \delta e^{r(t-u)} S_u \, du, \quad t \in [0, T].\] (4.1)

By introducing a path-dependent variable

\[I_t = \int_0^t \delta e^{r(t-u)} S_u \, du,\] (4.2)

the value of the stock loan can be expressed as

\[V_3(S_t, I_t, t) = \sup_{v \in T_{t, T}} \mathbb{E}_t (e^{-r(T-t)} [(S_v - Ke^{\gamma v})^+ + I_v])\]

\[= I_t + \sup_{v \in T_{t, T}} \mathbb{E}_t \left( e^{-r(T-t)} [(S_v - Ke^{\gamma v})^+] + \int_t^v e^{-r(u-t)} S_u \, du \right).\] (4.3)

Observe that \(V_3(S_t, I_t, t) - I_t\) is independent of \(I_t\). Then we can write \(H(S, t) \equiv V_3(S_t, I_t, t) - I_t\), which satisfies

\[
\begin{aligned}
\min \left\{ -L_S^\delta H - \delta S, H(S, t) - (S - Ke^{\gamma t}) \right\} &= 0, \\
H(S, T) &= (S - Ke^{\gamma T})^+, \quad (S, t) \in Q.
\end{aligned}
\] (4.4)

Here \(H(S, t)\) can be regarded as the value of the stock loan excluding the accumulative dividends. Compared with equation (2.1) in Section 2, equation (4.4) has a source term \(\delta S\) due to the dividends delivered.

Now we define the redemption region

\[E_3 \equiv \{(S, t) \in Q : H(S, t) = S - Ke^{\gamma t}\}.

We proceed with a lemma.

**Lemma 4.1.** Let \(\hat{V}_2(\cdot, \cdot)\) and \(\hat{E}_2\) be as given in (3.2) and (3.4), respectively. Then

\[\hat{V}_2(S_t, t) \leq H(S, t) \quad \text{and} \quad E_3 \subset \hat{E}_2.

**Proof.** First we prove \(\hat{V}_2(S, t) \leq H(S, t)\). Compared with equation (3.3), equation (4.4) can be rewritten as

\[
\begin{aligned}
\min \left\{ -L_S^\delta H - \delta S \left( 1 - \frac{\partial H}{\partial S} \right), H(S, t) - (S - Ke^{\gamma t}) \right\} &= 0, \\
H(S, T) &= (S - Ke^{\gamma T})^+, \quad (S, t) \in Q.
\end{aligned}
\]

Since both \(\hat{V}_2(S, t)\) and \(H(S, t)\) grow at most linearly in \(S\) as \(S\) goes to infinity\(^2\), \(\hat{V}_2|_{S=0} = H|_{S=0} = 0\), and \(\hat{V}_2|_{t=T} = H|_{t=T}\), by the maximum principle (cf. Crandall, Ishii, and Lions 1992), it suffices to show \(\frac{\partial H}{\partial S} \leq 1\), namely,

\(^2\) In fact, it is apparent that \(\hat{V}_2(S, t) \leq S\) and \(H(S, t) \leq S\).
\[
\frac{\partial}{\partial S}(H - S) \leq 0.
\]

Notice that for any stopping time \(v \in \mathcal{T}_{[t, T]}\),

\[
\mathbb{E}_t \left( \int_t^v \delta e^{-r(u-t)} S_u du - S_t \right) = \mathbb{E}_t \left[ \int_t^v e^{\left( -\delta - \frac{\sigma^2}{2}(u-t) + \sigma (B_u - B_t) \right)} du - S_t \right]
\]

\[
= \mathbb{E}_t \left[ \int_t^v e^{-\delta(u-t)} du - S_t \right]
\]

\[
= -S_t e^{-\delta(v-t)} = -\mathbb{E}_t(e^{-r(v-t)} S_v).
\]

Combining equations (4.3) and (4.6), we have

\[
H(S_t, t) - S_t = \sup_{v \in \mathcal{T}_{[t, T]}} \mathbb{E}_t(e^{-r(v-t)} [(S_v - Ke^{\gamma v})^+ - S_v])
\]

\[
= \sup_{v \in \mathcal{T}_{[t, T]}} \mathbb{E}_t(e^{-r(v-t)} \max(-Ke^{\gamma v}, -S_v]),
\]

which gives equation (4.5), then \(\hat{V}_2(S, t) \leq H(S, t)\) follows.

Now let us show \(E_3 \subset \hat{E}_2\). For any \((S, t) \in E_3\), we have

\[
S - Ke^{\gamma t} \leq \hat{V}_2(S, t) \leq H(S, t) = S - Ke^{\gamma t},
\]

which implies \(\hat{V}_2(S, t) = S - Ke^{\gamma t}\), i.e., \((S, t) \in \hat{E}_2\). This completes the proof. \(\square\)

The following proposition characterizes the shape of \(E_3\).

**Proposition 4.2.** Assume \(\delta > 0\) and the dividends are always delivered to the borrower during the lifetime of the stock loan.

(i) If \(r \geq \gamma\), then \(E_3 = \emptyset\). That is, early redemption should never happen. In addition, we have

\[
H(S, t) = CE(S, t; r, \delta, Ke^{\gamma T}) + (1 - e^{-\delta(T-t)})S.
\]

(ii) If \(r < \gamma\), then there is an optimal redeeming boundary \(S_3^*(t) : [0, T) \rightarrow (0, +\infty)\) such that

\[
E_3 = \{(S, t) \in Q : S \geq S_3^*(t)\}.
\]

In addition, \(e^{-\gamma t} S_3^*(t)\) is monotonically decreasing in \(t\),

\[
S_3^*(T) = \lim_{t \to T^-} S_3^*(t) = e^{\gamma T} K,
\]

and

\[
S_3^*(t) \geq \hat{S}_2^*(t),
\]

where \(\hat{S}_2^*(t)\) is as given in equation (3.5).
The argument is standard and is stated as follows. Obviously

\[ H(S, t) \geq C_E(S, t; r, \delta, Ke^{rT}) + (1 - e^{-\delta(T-t)})S, \]

where the right-hand side is the price of the corresponding stock loan without early redemption right. Because \( C_E(S, t; r, \delta, Ke^{rT}) > Se^{-\delta(T-t)} - Ke^{rT}e^{-r(T-t)} \) (see, for example, Hull 2003), it follows:

\[
H(S, t) \geq S - Ke^{rT}e^{-r(T-t)}
\]

which implies part (i).

To show part (ii), as before, we make a transformation: \( f_3(x, t) = e^{-\gamma t} H(S, t) \) and \( x = e^{-\gamma t} S \). It follows:

\[
\begin{cases}
\min \{-L^\gamma_3 f_3 - \delta x, f_3(x, t) - (x - K)\} = 0,
\end{cases}
\]

By equation (4.5), we have \( \partial f_3 / \partial x (f_3 - x) \leq 0 \), from which we infer that there is a single-value function \( x^*_3(t) : [0, T) \to (0, +\infty) \cup +\infty \) such that

\[
\{(x, t) \in Q : f_3(x, t) = x - K\} = \{(x, t) \in Q : x \geq x^*_3(t)\}.
\]

Since

\[
-L^\gamma_3(x - K) - \delta x = -\gamma K > 0, \quad \text{for} \quad r < \gamma,
\]

using a similar analysis as in Brezis and Friedman (1976) or Dai, Kwok, and Wu (2004), we can deduce \( f_3(x, t) - (x - K) \) has a compact support for all \( t \). This indicates \( x^*(t) < +\infty \) for all \( t \). So, we can take \( S^*_3(t) = e^{\gamma t} x^*_3(t) \). Equation (4.8) is a corollary of Lemma 4.1.

Apparently

\[
\frac{\partial f_3}{\partial t} \leq 0,
\]

which yields the monotonicity of \( x^*_3(t) \). It remains to show \( x^*_3(T) = \lim_{t \to T-} x^*_3(t) \).

By virtue of equations (3.6) and (4.8), we immediately obtain that \( S^*_3(t) \to +\infty \) as \( T \to +\infty \) in the case of \( \gamma - \frac{1}{2} \sigma^2 \leq r < \gamma \). We will show that it is also true for \( r < \gamma - \frac{\sigma^2}{2} \). Again, we need to study the corresponding perpetual stock loan. Denote \( H_\infty(S) = \lim_{T \to +\infty} H(S, t) \). We assert

\[
H_\infty(S) = S.
\]
Indeed, from equation (4.9), we have
\[ H(S, t) \geq S - Se^{-\delta(T-t)} \quad \text{for } \delta > 0, \]
which yields \( H_{\infty}(S) \geq S \) by letting \( T \to \infty \). The converse inequality is apparent. So, we get equation (4.12), which implies that one should never redeem the perpetual stock loan. We summarize the above result as follows:

**Proposition 4.3.** Assume \( \delta > 0 \) and the dividends are always delivered to the borrower during the lifetime of the stock loan. Then the perpetual stock loan is equivalent to the stock, i.e., \( H_{\infty}(S) = \lim_{T \to +\infty} H(S, t) = S \). In addition, \( S^*_{\infty} \equiv \lim_{T \to +\infty} S^*_{3}(t) = +\infty \), for \( r < \gamma \), where \( S^*_{3}(t) \) is the optimal redeeming boundary in part (ii) of Proposition 4.2.

### 5. Dividends Returned to the Borrower on Redemption

In this section, we assume that the accumulative dividends will be returned to the borrower on redemption. In contrast to equation (4.1), the intrinsic value of the stock loan is now
\[ \left( S_t - Ke^{\gamma T} + \int_0^t \delta e^{\gamma(t-u)} S_u \, du \right)^+, \quad t \in [0, T]. \]

Again we introduce the path-dependent variable \( I_t \) as given in equation (4.2), then the value of the stock loan can be expressed as
\[ V_4(S_t, I_t, t) = \sup_{v \in \Omega_1} \mathbb{E}_t[e^{-r(t-T)}(S_v - Ke^{\gamma T})^+]. \]

Note that
\[ dI_t = (\delta S_t + r I_t) \, dt. \]

It follows that \( V_4(S, I, t) \) satisfies
\[ \begin{cases} 
\min \left\{ -L_S^V V_4 - (\delta S + r I) \frac{\partial V_4}{\partial I}, \ V_4(S, I, t) - \left( S + I - Ke^{\gamma T} \right) \right\} = 0, \\
V_4(S, I, T) = (S + I - Ke^{\gamma T})^+, \quad (S, I, t) \in \Omega, 
\end{cases} \tag{5.1} \]
where \( \Omega = (0, +\infty) \times (0, +\infty) \times [0, T] \). Equation (5.1) is analogous to the pricing model for arithmetic Asian options (Wilmott, Dewynne, and Howison 1993; Dai and Kwok 2006; Pascucci 2008; and Monti and Pascucci 2009). We emphasize that equation (5.1) is indeed a two-dimensional time-dependent problem and does not permit dimension reduction.

As before, we define the redemption region by
\[ E_4 = \{(S, I, t) \in \Omega : V_4(S, I, t) = S + I - Ke^{\gamma T}\}. \]

Apparently \( V_4(S, I, t) \leq V_3(S, I, t) = H(S, t) + I \), from which we immediately get
\[ \{(S, I, t) \in \Omega : (S, t) \in E_3\} \subset E_4. \]
The following lemma presents a stronger result which plays an important role in analysis of the shape of $E_4$:

**Lemma 5.1.** Let $\widehat{V}_2(\cdot, \cdot)$ and $\widehat{E}_2$ be as defined in equations (3.2) and (3.4), respectively. Then we have

$$(5.2) \quad V_4(S, I, t) \leq \widehat{V}_2(S + I, t)$$

and thus

$$(5.3) \quad \{(S, I, t) \in \Omega : (S + I, t) \in \widehat{E}_2\} \subset E_4.$$

**Proof.** We only need to prove equation (5.2). When $I = 0$, it is clearly true, namely,

$$(5.4) \quad V_4(S, 0, t) \leq \widehat{V}_2(S, t).$$

To deal with the case of $I > 0$, let us adopt $y \equiv S + I$ as a new state variable in place of $S$, and denote

$$U(y, I, t) = V_4(S, I, t),$$

which satisfies

$$\begin{cases}
\min\{-LU, U(y, I, t) - (y - Ke^\nu t)\} = 0, \\
U(y, I, T) = (y - Ke^\nu T)^+, \quad (y, I, t) \in \Omega_y,
\end{cases}$$

where $\Omega_y \equiv \{(y, I, t) : 0 < I < y < +\infty, \ 0 \leq t < T\}$,

$$L = \frac{\partial}{\partial t} + (\delta y + (r - \delta) I) \frac{\partial}{\partial I} + \frac{1}{2} \sigma^2(y - I)^2 \frac{\partial^2}{\partial y^2} + ry \frac{\partial}{\partial y} - r.$$

Note that equation (3.3) can be rewritten as

$$\begin{cases}
\min\{-L\widehat{V}_2 - \frac{1}{2} \sigma^2[y^2 - (y - I)^2] \frac{\partial^2}{\partial y^2}, \widehat{V}_2(y, t) - (y - Ke^\nu t)\} = 0, \\
\widehat{V}_2(y, T) = (y - Ke^\nu T)^+, \quad (y, I, t) \in \Omega_y.
\end{cases}$$

Due to the convexity of $(y - Ke^\nu t)^+$ in $y$, we can deduce $\frac{\partial^2 \widehat{V}_2}{\partial y^2} \geq 0$, then

$$\frac{1}{2} \sigma^2[y^2 - (y - I)^2] \frac{\partial^2}{\partial y^2} \geq 0, \quad \text{for any} \ y > I > 0, \ t \in [0, T).$$

Clearly $0 \leq U(y, I, t) \leq y, \ 0 \leq \widehat{V}_2(y, t) \leq y$, and $U|_{t=T} = \widehat{V}_2|_{t=T}$. From equation (5.4), we have $U(y, 0, t) \leq \widehat{V}_2(y, t)$. When $y = I$, i.e., $S = 0$, it is easy to infer $U|_{y=I} = V_4(0, I, t) = (I - Ke^\nu t)^+ \leq \widehat{V}_2(I, t)$. Applying the maximum principle then gives

$$U(y, I, t) \leq \widehat{V}_2(y, t),$$

which is desired. \qed
We would like to give a financial interpretation to equation (5.2). \( \hat{V}_2(\cdot, \cdot) \) and \( V_4(\cdot, \cdot, \cdot, \cdot) \) represent the prices for the stock loans respectively with reinvested dividends and with cash dividends, both of which will be returned to the borrowers on redemption. In the risk-neutral world, the return rate of reinvested dividends is the same as that of cash dividends. As we can regard the combination of the dividends and the stock as an imaginary underlying asset (i.e., \( y \)), the cash dividends essentially decrease the volatility of the underlying asset, which leads to a lower price of the corresponding stock loan.

5.1. The Case of \( r \geq \gamma \)

Now let us investigate the case of \( r \geq \gamma \).

**Proposition 5.2.** Assume \( \delta > 0 \) and the dividends are gained by the borrower on redemption.

(i) If \( r > \gamma \), then \( E_4 = \emptyset \). That is, early redemption should never happen.

(ii) If \( r = \gamma \), then it is optimal to hold the option before expiry.

**Proof.** We adopt an arbitrage argument. If the borrower redeems the loan at time \( t < T \), then he or she will get

one stock \( S_t \) + cash \( I_t - Ke^{\gamma T} \),

which amounts to at expiry

\[
S_T + I_T - Ke^{\gamma (T-t)} < S_T + I_T - Ke^{\gamma T} \leq (S_T + I_T - Ke^{\gamma T})^+,
\]

This indicates that early redemption is not optimal. Part (i) then follows.

To prove part (ii), the argument is similar and the unique difference lies in that “\(<\)” in equation (5.5) should be replaced by “\(=\).” \( \square \)

It is worth distinguishing two statements of part (i) and (ii) in Proposition 5.2. For the latter case, early redemption may be optimal on some occasions. For example, when \( I \geq Ke^{\gamma T} \) and \( r = \gamma \), it is not hard to show that \( V_4(S, I, t) = S + I - Ke^{\gamma T} \), which implies that the redemption at any time is optimal.

Owing to Proposition 5.2, equation (5.1) is reduced to a linear problem:

\[
\begin{cases}
-L^i \delta \frac{\partial V_4}{\partial S} - (\delta S + r I) \frac{\partial V_4}{\partial T} = 0, \\
V_4(S, I, T) = (S + I - Ke^{\gamma T})^+, \quad (S, I, t) \in \Omega.
\end{cases}
\]

5.2. The Case of \( r < \gamma \)

By Lemma 5.1 and the properties of \( \hat{E}_2 \), we know that \( E_4 \) is always nonempty when \( r < \gamma \). Furthermore, we have

**Proposition 5.3.** Assume \( \delta > 0 \) and the dividends are gained by the borrower on redemption. If \( r < \gamma \), then \( \{ (S, I, t) \in \Omega: I \geq Ke^{\gamma T} \} \subset E_4 \).
Proof. Given \((S_t, I_t, t) \in \Omega\) with \(I_t \geq Ke^{\gamma t}\), we claim that the loan should be redeemed immediately at time \(t\). Indeed, if it is redeemed at time \(t\), then we have a payoff: stock \(S_t + \text{nonnegative cash } I_t - Ke^{\gamma t}\), which becomes at a later time \(t' \in (t, T]\)
\[
\text{stock } S_t + \text{cash } I_t - Ke^{\gamma t} e^{\gamma(t'-t)} > (S_t + I_t - Ke^{\gamma t})^+ \text{ if } r < \gamma.
\]
This implies the conclusion. \(\square\)

We stress that Proposition 5.2 and Lemma 5.3 only rely on the no-arbitrage principle, and therefore are independent of the geometric Brownian motion assumption of stock price. Such a remark also applies to some of previous results.

Denote \(\widehat{\Omega} = \{(S, I, t) \in \Omega : I < Ke^{\gamma t}\}\). Due to equation (5.3), we only need to study
\[
\widehat{E}_4 \equiv E_4 \cap \widehat{\Omega}.
\]

**Proposition 5.4.** Assume that the dividends are gained by the borrower on redemption and \(r < \gamma\). Then, there is an optimal redeeming boundary \(S^*_4(I, t) : (0, Ke^{\gamma t}) \times [0, T) \rightarrow (0, \infty)\) such that
\[
\widehat{E}_4 = \{(S, I, t) \in \widehat{\Omega} : S \geq S^*_4(I, t)\}.
\]
Moreover, \(e^{-\gamma t} S^*_4(I, t)\) is monotonically decreasing in \(I\) and \(t\),
\[
S^*_4(I, T) \equiv \lim_{t \to T} S^*_4(I, t) = e^{\gamma T} K - I, \quad \text{for all } I,
\]
and
\[
S^*_4(I, t) \leq \widehat{S}^*_2(t) - I, \quad \text{for all } t,
\]
where \(\widehat{S}^*_2(t)\) is as given in equation (3.5). In particular, \(S^*_4(0, t) \equiv \lim_{T \to 0} S^*_4(I, t) \leq \widehat{S}^*_2(t)\) for all \(t\).

Proof. Since \((S_t + I_t - Ke^{\gamma t})^+\) is convex in \(S\), we infer that \(V_4(S_t, I_t, t)\) is also convex in \(S\), namely
\[
V_4(aS_1 + (1-a)S_2, I_t, t) \leq aV_4(S_1, I_t, t) + (1-a) V_4(S_2, I_t, t),
\]
which implies the convexity of \(\widehat{E}_4\) in \(S\). On other hand, due to Lemma 5.1, we have \((S, I, t) \in \widehat{E}_4\) for \(S > \widehat{S}^*_2(t) - I\). This indicates the existence of \(S^*_4(I, t)\) and \(S^*_4(I, t) \leq \widehat{S}^*_2(t) - I\).

Using the similarity reduction \(V_4(S, I, t) = e^{-\gamma t} f_4(x, A, t), x = e^{-\gamma t} S\) and \(X = e^{-\gamma t} I\), we get
\[
f_4(x_t, A_t, t) = \sup_{x \in [T_0, T]} E_t[e^{-\gamma(v-t)}(x_v - K + A_v)^+],
\]
where \(dx_t = \left((r - \delta)x_t dt + \sigma x_t dW_t\right)\) and \(A_t = \int_0^t \delta e^{\gamma(t-u)} x_u du\). Let \(x^*_4(A, t) = e^{-\gamma t} X^*_4(I, t)\). It suffices to show that \(x^*_4(A, t)\) is monotonically decreasing in \(A\) and \(t\), and
\[
x^*_4(A, T) \equiv \lim_{t \to T} x^*_4(A, t) = K - A, \quad \text{for all } A.
\]
By equation (5.7), it is easy to see

\[ f_4(x, A, t_1) \leq f_4(x, A, t_2) \quad \text{for any } t_2 \leq t_1 \leq T, \]

and

\[ f_4(x, A_2, t) \leq f_4(x, A_1, t) + A_2 - A_1 \quad \text{for any } A_1 \leq A_2, \]

where the second inequality is due to the convexity of the early exercise payoff w.r.t. \( A \).

Then the monotonicity of \( x^*_4(A, t) \) in \( A \) and \( t \) follows. Using a similar argument as in Proposition 4.2, we are able to obtain equation (5.8).

6. NUMERICAL EXAMPLES

In this section we present numerical results to verify our theoretical results. Let us first look at the pricing models in Sections 2–4 which belong to standard one-dimensional parabolic variational inequalities. They can be numerically solved using many sophisticated numerical methods such as the projected successive over relaxation (cf. Wilmott, Dewynne, and Howison 1993), the recursive integration method (cf. Huang, Subrahmanyam, and Yu 1996) and the penalty approach (cf. Forsyth and Vetzal 2002). We simply make use of the binomial tree method (cf. Hull 2003) which is easy to implement. The default data used are \( r - \gamma = -0.04, \delta = 0.03, \gamma = 0.1, \) and \( K = 0.7 \). Figures 6.1 and

![Figure 6.1](image_url)

**Figure 6.1.** Optimal redeeming boundaries (Parameters: \( r - \gamma = -0.04, \delta = 0.03, \sigma = 0.4, \) and \( K = 0.7 \)).
6.2 plot the optimal redeeming boundaries against the time to maturity $\tau = T - t$ with $\sigma = 0.4$ and with $\sigma = 0.15$, respectively. Here $x_1^*(\cdot) = e^{-\gamma_1 t} S_1^\tau(t)$, $x_2^*(\cdot) = e^{-\gamma_2 t} \hat{S}_2^\tau(t)$, and $x_3^*(\cdot) = e^{-\gamma_3 t} S_3^\tau(t)$. Observe that $x_1^* \leq x_2^* \leq x_3^*$ and these boundaries are monotonically increasing in $\tau$ (so, decreasing in $t$). Since $r < \gamma$, we can see that all boundaries go to $x = K \equiv 0.7$ at $\tau \to 0$. Theoretical results indicate that $x_1^*(\cdot)$ has an asymptotic line ($x = 1.97$ for Figure 6.1 and $x = 0.82$ for Figure 6.2) whereas $x_3^*(\cdot)$ does not as $\tau \to +\infty$, which can be seen from the figures. In addition, $x_2^*(\cdot)$ has an asymptotic line ($x = 0.97$) as $\tau \to +\infty$ in Figure 6.2 due to $r < \gamma - \frac{\sigma^2}{2}$, whereas not in Figure 6.1 ($\gamma - \frac{\sigma^2}{2} \leq r < \gamma$). This is also consistent with our theoretical analysis.

Now let us move on to the case in Section 5, where the pricing model, resembling that of American-style Asian options, is a degenerate parabolic variational inequality. We then employ the forward shooting grid method (cf. Barraquand and Pudet 1996), which is widely used to deal with this type of degenerate problems. Figure 6.3 shows the optimal redeeming boundary $x_4^*(\cdot, \cdot)$ in $x-A, \tau$ plane, where $x = e^{-\gamma_1 t} S$, $A = e^{-\gamma_1 t} I$, and $\tau = T - t$. The data used are $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.4$, and $K = 0.7$. Time snapshots of the boundary are depicted in Figure 6.4. It can be seen that the boundary, as a function of $A$ and $\tau$, is decreasing in $A$, increasing in $\tau$ (i.e., decreasing in $t$). At maturity, the boundary is a straight line $x + A = K \equiv 0.7$. Nevertheless, theoretical analysis indicates that $x_4^*(\cdot, \tau) \leq x_2^*(\tau)$. Figure 6.1 shows that approximately $x_2^*$ equals 1.35 when $\tau = 1$, and 1.9 when $\tau = 3$. It can be observed from Figure 6.4 that $x_2^*(\cdot, \tau)$ is indeed bounded from above by $x_2^*(\tau)$. All these verify Proposition 5.4.
FIGURE 6.3. Optimal redeeming boundary $x^*_4(A, \tau) = e^{-\gamma t} S^*_4(I, t)$ (Parameters: $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.4$ and $K = 0.7$).

FIGURE 6.4. Time snapshots of $x^*_4(\cdot, \cdot)$ (Parameters: $r - \gamma = -0.04$, $\delta = 0.03$, $\sigma = 0.4$ and $K = 0.7$).
In the Black–Scholes framework, we formulate the pricing of stock loans as an optimal stopping problem, or equivalently, a variational inequality. Since closed-form price formulas are generally not available, we provide an analytic approach to analyze the optimal redeeming strategy. It turns out that the way of dividends distribution significantly alters the pricing model and the optimal redeeming strategy. Numerical results are presented as well.

To conclude the paper, we briefly discuss some possible extensions of the pricing models. For illustration, we assume that the dividends are gained by the lender and let $V = V(S, t)$ be the price function of the stock loan.

### 7.1. Amortized Loan

Assume that the loan is amortized. Let $C$ be the amortized rate defined as follows:

$$\int_0^T C e^{\gamma (T - t)} \, dt = Ke^{\gamma T},$$

which yields $C = \frac{\gamma}{1 - e^{-\gamma T}} K$. At time $t$, the amount to be repaid is

$$Ke^{\gamma t} - \int_0^t C e^{\gamma (t - u)} \, du = \frac{C}{\gamma} (1 - e^{-\gamma (T - t)}),$$

So, the early redemption payoff is $S_t - \frac{C}{\gamma} (1 - e^{-\gamma (T - t)})$, $t \in [0, T)$. Then the pricing model becomes

$$\begin{cases} 
\min \left\{ -L_{S}^{\gamma, \delta} V + C, \ V - \left[ S - \frac{C}{\gamma} (1 - e^{-\gamma (T - t)}) \right] \right\} = 0, \\
V(S, T) = S, \quad (S, t) \in Q.
\end{cases}$$

### 7.2. Withdrawal Feature

The withdrawal feature aims to protect the lender from the extremely downside risk of the collateralized stock. It allows the lender to withdraw the loan at any time by requesting the borrower to redeem the stock with the price $L$, where $L < K$. This implies

$$V(S, t) \leq L.$$

Like the pricing of game options, the model is described as a double obstacle problem (cf. Kifer 2000; Dai and Kwok 2005)

$$\begin{cases} 
\max \left\{ \min \left\{ -L_{S}^{\gamma, \delta} V, \ V - (S - Ke^{\gamma t}) \right\}, \ V(S, t) - L \right\} = 0, \\
V(S, T) = \min \{(S - Ke^{\gamma T})^+, L\}, \quad (S, t) \in Q.
\end{cases}$$
7.3. Renewal Feature

If we assume that the borrower has right to renew the loan, this will lead to multiple stopping problems. See, for example, Carmona and Touzi (2008), Ibanez (2004), Meinshausen and Hambly (2004), and Dai and Kwok (2008). From the point of view of PDEs, the pricing model is described by a series of variational inequalities (cf. Dai and Kwok 2008). We refer interested readers to the references.

**APPENDIX: THE PROOF OF PROPOSITION 2.3**

It is easy to see that the solution to problem (2.4), denoted by \( f(x, t; T) \), possesses the following properties:

\[
0 \leq f(x, t; T) \leq x \quad \text{and} \quad 0 \leq \frac{\partial f}{\partial x} \leq 1.
\]

(A.1)

We can rewrite \( f(x, t; T) = \bar{f}(x, T - t) \). Denote

\[
f_\infty(x) = \lim_{(T - t) \to +\infty} \bar{f}(x, T - t).
\]

Then \( f_\infty(x) \) satisfies the stationary counterpart of equation (2.4), which is equivalent to a free boundary problem

\[
-\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 f_\infty}{\partial x^2} - (\bar{r} - \delta)x \frac{\partial f_\infty}{\partial x} + \bar{r} f_\infty = 0, \quad x \leq x_\infty^*.
\]

(A.2)

\[
f_\infty(x_\infty^*) = x - K, \quad \text{and} \quad f_\infty'(x_\infty^*) = 1.
\]

(A.3)

Here \( x_\infty^* \) is the free boundary to be determined. Due to equation (A.1), we are only concerned with the solution to equations (A.2) and (A.3) under the restrictions

\[
0 \leq f_\infty(x) \leq x \quad \text{and} \quad 0 \leq f_\infty'(x) \leq 1.
\]

(A.4)

As is well known, the general solution to equation (A.2) is

\[
f_\infty(x) = C_1 x^{\alpha_+} + C_2 x^{\alpha_-},
\]

where \( C_1 \) and \( C_2 \) are to be determined, and \( \alpha_+ \) (as given in equation (2.9)) and \( \alpha_- = 1 - \frac{2(r - \gamma - \delta)}{\sigma^2} - \alpha_+ \) are two roots of the algebraic equation \( \frac{\alpha_+^2}{2} \alpha^2 + (\bar{r} - \delta - \frac{\sigma^2}{2}) \alpha - \bar{r} = 0 \). Let us first assume \( \delta > 0 \). It is easy to check that \( \alpha_- < 1 < \alpha_+ \), which yields \( C_2 = 0 \) or

\[
f_\infty(x) = C_1 x^{\alpha_+}.
\]

Otherwise we would have \( f_\infty(x) = C_1 \alpha_+ x^{\alpha_+ - 1} + C_2 \alpha_- x^{\alpha_- - 1} \to \infty \) as \( x \to 0 \), which contradicts equation (A.4).

From equation (A.5), we can make use of equation (A.3) to get

\[
C_1 = \frac{1}{\alpha_+} \left( \frac{\alpha_+ - 1}{\alpha_+ K} \right)^{\alpha_+ - 1} \quad \text{and} \quad x_\infty^* = \frac{\alpha_+}{\alpha_- - 1} K
\]

for \( \delta > 0 \).
Now let us look at the case of $\delta = 0$. If $\bar{r} + \frac{1}{2}\sigma^2 < 0$, then $\alpha_- = 1$ and $\alpha_+ = -\frac{2\bar{r}}{\sigma^2} > 1$, and the general solution takes the form of $f_{\infty}(x) = C_1x^{-2\bar{r}/\sigma^2} + C_2x$. However, equation (A.4) is not sufficient to deduce $C_2 = 0$, we then resort to the continuous dependence on $\delta$ of $f_{\infty}(x)$ and $x^*_\infty$. Indeed, we let $\delta$ go to 0 in equation (A.6). It is easy to see that if $\bar{r} + \frac{1}{2}\sigma^2 < 0$ and $\delta \to 0$, then $\alpha_+ > 1$ and the expressions of $x^*_\infty$ and $f_{\infty}(x)$ are the same as given in equations (A.6) and (A.5). If $\bar{r} + \frac{1}{2}\sigma^2 \geq 0$ and $\delta \to 0$, then $\alpha_+ \to 1$, which yields $x^*_\infty = +\infty$ and $f(x) = x$. The proof is complete. \hfill $\Box$

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