THE WEAK MIN-MAX PROPERTY IN BANACH SPACES

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Abstract. In this paper, we investigate the relationship between the weak min-max property and the diameter uniformity of domains in Banach spaces with dimensions at least 2. As an application, we show that diameter uniform domains are invariant under relatively quasimöbius mappings.

1. INTRODUCTION AND MAIN RESULTS

Uniform domains in $\mathbb{R}^n$ were introduced by Martio and Sarvas in 1978 [10], who used them in studying injectivity theorems in $\mathbb{R}^n$. Since then, uniform domains have played an important role in the classical function theory, quasiconformal mappings, and many other fields of modern mathematical analysis (see e.g. [3, 5, 11, 12, 15]). Several different characterizations of uniform domains have been established, see for example [1, 2, 6, 7, 8, 9].

In particular, Martio [9] showed that both the class of diameter uniform domains and the class of $\delta$-uniform domains are equivalent to the class of uniform domains in $\mathbb{R}^n$. Gehring and Hag [1] introduced the min-max property of a domain $G \subseteq \mathbb{R}^n$ involving the properties of hyperbolic geodesics in $B^n$. By using this property, they showed other characteristics of uniform domains in $\mathbb{R}^n$. Note that the above classes of domains in $\mathbb{R}^n$ are equivalent. The definitions of these classes of domains are given in Section 2.

Through this paper, let $E$ and $E'$ be real Banach spaces with dimension at least 2. For domains in the setting of Banach spaces, Väisälä [16] showed that the class of distance uniform domains is equivalent to the diameter uniform domains. Furthermore, it is known that uniform domains are diameter uniform. For the converse, Rasila and Zhou [18] showed that a diameter uniform domain is uniform if the domain satisfies a natural condition, which is as follows:

**Theorem 1.1.** ([18 Theorem 1.2]) Suppose that $G \subseteq E$ is a domain. Then $G$ is $A$-uniform if and only if $G$ is diameter $A_1$-uniform and $\psi$-natural, where $A$, $A_1$ and $\psi$ depend on each other.

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Concerning on uniform domains in Banach spaces $E$, Rasila and Zhou studied the following implications:

**Theorem 1.2.** ([18, Theorem 1.1]) Suppose that $G \subset E$ is a domain. If $G$ is $A$-uniform, then $G$ satisfies the $c$-min-max property, $G$ is a diameter $A_1$-uniform domain, where $c = c(A, A_1, A)$ and $c = c(A)$ means that $c$ depends only on $A$. Furthermore, $G$ is a diameter $A_1$-uniform domain if and only if $G$ is $\delta$-uniform for some $0 < \delta < 1$, where $A_1$ and $\delta$ depend on each other.

It is reasonable to consider the question whether diameter uniform domains satisfy the min-max property. To deal with this question, we introduce a condition called weak min-max property, and use it to characterize diameter domains in Banach spaces.

**Definition 1.1.** Let $G \subset E$ be a domain. We say that $G$ has the weak min-max property if there exists a family of curves $\Gamma$ in $G$ and a constant $c \geq 1$ such that any $x_1$ and $x_2$ in $G$ can be joined by a curve $\gamma \in \Gamma$ satisfying

\[
    c^{-1} \min_{j=1,2} |x_j - y| \leq |x - y| \leq c \max_{j=1,2} |x_j - y|,
\]

for all $x \in \gamma$ and all $y \in \partial G$.

**Theorem 1.3.** Suppose that $G \subset E$ is a domain. Then $G$ has the weak $c$-min-max property if and only if $G$ is diameter $A$-uniform, where $c$ and $A$ depend only on each other.

It follows from [13, Theorem 6.26] that uniform domains are preserved under quasimöbius mappings. In [18, Theorem 1.3], Rasila and Zhou recently proved that relatively quasimöbius mappings map uniform domains to diameter uniform domains. As an application of Theorem 1.3 we study the relative quasimöbius invariance of diameter uniform domains.

**Theorem 1.4.** Suppose that $G \subset E$ and $G' \subset E'$ are domains. Suppose that $G$ is diameter $A$-uniform, and that a homeomorphism $f : \overline{G} \to \overline{G'}$ is $\theta$-quasimöbius relative to $\partial G$ and maps $G$ onto $G'$. Then $G'$ is diameter $A_1$-uniform, where $A_1 = A_1(A, \theta)$.

Using Theorems 1.3 and 1.4 we see that the weak min-max property is invariant under relatively quasimöbius mappings.

**Corollary 1.5.** Suppose that $G \subset E$ and $G' \subset E'$ are domains. Assume that $G$ has the weak $c$-min-max property, and that a homeomorphism $f : \overline{G} \to \overline{G'}$ is $\theta$-quasimöbius relative to $\partial G$ and maps $G$ onto $G'$. Then $G'$ satisfies the weak $c_1$-min-max property, where $c_1 = c_1(c, \theta)$.

In [18, Theorem 1.3], it was shown that relatively quasimöbius mappings map uniform domains onto uniform domains if the target domains are natural. By Corollary 1.5 and Theorem 1.1 we have the following corollary.

**Corollary 1.6.** Suppose that $G \subset E$ and $G' \subset E'$ are domains. Suppose that $G$ is diameter $A$-uniform, $G'$ is $\psi$-natural, and that $f : G \to G'$ is $\theta$-quasimöbius relative to $\partial G$. Then $G$ is $A_1$-uniform, where $A_1 = A_1(A, \theta, \psi)$.
The rest of this paper is organized as follows. In Section 2, we recall necessary definitions and preliminary results. The proof of Theorem 1.3 is given in Section 3. Section 4 is devoted to the proofs of Theorem 1.4 and Corollary 1.5.

2. Preliminaries

2.1. Notation. Following the notation and terminology of [4, 15], the norm of a vector \( x \) in \( E \) is written as \( |x| \), and for every pair of points \( x, y \) in \( E \), the distance between them is denoted by \( |x - y| \), the closed line segment with endpoints \( x \) and \( y \) by \([x, y]\).

For a set \( A \) in \( E \), we use \( \overline{A} \) to denote the completion of \( A \) and \( \partial A = \overline{A} \setminus A \) to be its norm boundary. For a bounded set \( A \) in \( E \), \( \text{diam}(A) \) is the diameter of \( A \). A curve is a continuous function \( \gamma : [a, b] \to E \). The length of \( \gamma \) is defined by

\[
\ell(\gamma) = \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \right\},
\]

where the supremum is taken over all partitions \( a = t_0 < t_1 < t_2 < \ldots < t_n = b \).

2.2. Uniform domains.

Definition 2.1. Let \( G \subsetneq E \) be a domain and let \( A \geq 1 \). Then \( G \) is called diameter \( A \)-uniform, if each pair of points \( x_1, x_2 \) in \( G \) can be joined by a curve \( \alpha \) in \( G \) satisfying:

1. \( \min_{j=1,2} \text{diam}(\alpha[x_j, x]) \leq A d_G(x) \) for all \( x \in \alpha \), and
2. \( \text{diam}(\alpha) \leq A |x_1 - x_2| \),

where \( d_G(x) = \text{dist}(x, \partial G) \). Moreover, the curve \( \alpha \) satisfying the above conditions is said to be a diameter uniform curve.

The domain \( G \subsetneq E \) is called \( A \)-uniform if the diameter conditions in Definition 2.1 are replaced by the length conditions. We say that \( G \) is distance \( A \)-uniform if the first condition in Definition 2.1 is replaced by

\[
\min_{j=1,2} |x_j - x| \leq A d_G(x),
\]

for all \( x \in \alpha \).

Remark 2.1. It is not difficult to see from the definitions that an \( A \)-uniform domain is diameter \( A \)-uniform and a diameter \( A \)-uniform domain is distance \( A \)-uniform. For the converse, diameter uniform domains are equivalent to distance uniform domain in Banach spaces [16, Page 10]. It follows from [9, Theorem 4.5] that a diameter \( A \)-uniform domain in \( \mathbb{R}^n \) is \( A_1 \)-uniform, where \( A_1 \) depends only on \( A \) and \( n \). Note that in infinite dimensional Hilbert spaces, Väisälä [16] constructed a so-called broken tube domain, which is diameter \( A \)-uniform but not \( A_1 \)-uniform for any constant \( A_1 \geq 1 \).

Definition 2.2. Let \( G \subsetneq E \) be a domain and let \( 0 < \delta < 1 \). Then \( G \) is called \( \delta \)-uniform if each pair of points \( x_1, x_2 \) in \( G \) can be joined by a curve \( \alpha \) in \( G \) such
that the cross ratio
\begin{equation}
\tau(x, x_i, y, x_j) = \frac{|x - y|}{|x - x_i|} \cdot \frac{|x_i - x_j|}{|x_j - y|} \geq \delta, \quad i \neq j \in \{1, 2\},
\end{equation}
for all \( x \in \alpha \setminus \{x_1, x_2\} \) and \( y \in E \setminus G \).

**Definition 2.3.** Let \( \psi : [0, \infty) \to [0, \infty) \) be an increasing function. A domain \( G \subseteq E \) is called \( \psi \)-natural if
\[
k_G(A) \leq \psi(r_G(A))
\]
for every nonempty connected set \( A \subseteq G \) with \( r_G(A) < \infty \), where
\[
r_G(A) = \sup \left\{ \frac{|x - y|}{\min\{d_G(x), d_G(y)\}} \mid x, y \in A \right\},
\]
and \( k_G(A) \) is the diameter of \( A \) in the quasihyperbolic metric.

**Remark 2.3.** The class of natural domains is fairly large, for example all proper domains in \( \mathbb{R}^n \) is \( \psi \)-natural with \( \psi = \psi(n) \), see [17, Corollary 2.18]. In an infinite dimensional Hilbert space the broken tube construction given in [16] provides an example of a domain, which is not natural.

2.3. **Min-max property.**

**Definition 2.4.** Let \( G \subseteq E \) be a domain. We say that \( G \) has the min-max property if
\[
k_G(A) \leq \psi(r_G(A))
\]
for every nonempty connected set \( A \subseteq G \) with \( r_G(A) < \infty \), where
\[
r_G(A) = \sup \left\{ \frac{|x - y|}{\min\{d_G(x), d_G(y)\}} \mid x, y \in A \right\},
\]
and \( k_G(A) \) is the diameter of \( A \) in the quasihyperbolic metric.

2.4. **Quasisymmetric mappings and quasimöbius mappings.** By a triple in \( X \) we mean an ordered sequence \( T = (x, y, z) \) of three distinct points in \( X \). The ratio of \( T \) is the number
\[
\rho(T) = \frac{|y - x|}{|z - x|}.
\]
If \( f : X \to Y \) is an injective map, then the image of a triple \( T = (x, y, z) \) is the triple \( f(T) = (f(x), f(y), f(z)) \).

Suppose that \( A \subseteq X \). A triple \( T = (x, y, z) \) in \( X \) is said to be a triple in the pair (\( X, A \)) if \( x \in A \) or if \( \{y, z\} \subseteq A \). Equivalently, both \( |y - x| \) and \( |z - x| \) are distances from a point in \( A \).

**Definition 2.5.** Let \( X \) and \( Y \) be two metric spaces, and let \( \eta : [0, \infty) \to [0, \infty) \) be a homeomorphism. Suppose \( A \subseteq X \). A homeomorphism \( f : X \to Y \) is said to be \( \eta \)-quasisymmetric relative to \( A \), if \( \rho(f(T)) \leq \eta(\rho(T)) \) for each triple \( T \) in \( (X, A) \).

Note that a homeomorphism \( f : X \to Y \) is \( \eta \)-quasisymmetric relative to \( A \) if and only if \( \rho(T) \leq t \) implies that \( \rho(f(T)) \leq \eta(t) \) for each triple \( T \) in \( (X, A) \) and \( t \geq 0 \) (cf. [15]). Obviously, a quasisymmetric mapping relative to \( X \) is equivalent to usual quasisymmetry.
Observe that the definition of the cross ratio can be extended in the usual manner to the case where one of the points is \( \infty \). For example,

\[
\tau(x, y, z, \infty) = \frac{|x - z|}{|x - y|}.
\]

Let \( X \) be a metric space and \( \hat{X} = X \cup \{ \infty \} \). If \( X_0 \subseteq \hat{X} \) and \( f : X_0 \to \hat{Y} \) is an injective map, then the image of a quadruple \( Q = (x, y, z, w) \) in \( X_0 \) is the quadruple \( f(Q) = (f(x), f(y), f(z), f(w)) \). Suppose that \( A \subseteq X_0 \). We say that a quadruple \( Q = (x, y, z, w) \) in \( X_0 \) is a quadruple in the pair \((X_0, A)\) if \( \{x, w\} \subseteq A \) or \( \{y, z\} \subseteq A \). Equivalently, all four distances in the definition of \( \tau(Q) \) are distances from a point in \( A \).

**Definition 2.6.** Let \( X \) and \( Y \) be two metric spaces, \( X_1 \subset \hat{X} \), and \( Y_1 \subset \hat{Y} \), and let \( \eta : [0, \infty) \to [0, \infty) \) be a homeomorphism. Suppose \( A \subseteq \hat{X} \). A homeomorphism \( f : X_1 \to Y_1 \) is said to be \( \eta \)-quasimöbius relative to \( A \), if the inequality \( \tau(f(Q)) \leq \eta(\tau(Q)) \) holds for each quadruple in \((X, A)\).

Apparently, \( \eta \)-quasimöbius relative to \( X \) is equivalent to \( \eta \)-quasimöbius.

### 3. Proof of Theorem 1.3

Assume that \( G \subseteq E \) is a domain. First, we show the sufficiency part and with the assumption that \( G \) is diameter \( A \)-uniform.

Fix \( x_1, x_2 \in G \). By the assumption, there is a diameter uniform curve \( \alpha \) joining \( x_1 \) and \( x_2 \) in \( G \). For all \( x \in \gamma \) and for all \( y \in \partial G \), we have

\[
\min_{j=1,2} |x_j - x| \leq \min_{j=1,2} \text{diam}(\alpha[x, x_j]) \leq Ad_G(x) \leq A|x - y|,
\]

which implies

\[
(3.1) \quad \min_{j=1,2} |x_j - y| \leq \min_{j=1,2} |x_j - x| + |x - y| \leq (A + 1)|x - y|.
\]

Moreover, we see from the diameter uniformity of \( \alpha \) that

\[
|x - y| \leq \text{diam}(\alpha) + |x_1 - y| \leq A|x_1 - x_2| + |x_1 - y| \leq (2A + 1) \max_{j=1,2} |x_j - y|.
\]

This, together with \((3.1)\), shows that

\[
c^{-1} \min_{j=1,2} |x_j - y| \leq |x - y| \leq c \max_{j=1,2} |x_j - y|,
\]

where \( c = 2A + 1 \).

Now, we prove the necessity. Assume that \( G \) has the weak \( c \)-min-max property. Fix \( x_1, x_2 \in G \). Without loss of generality, we may assume that

\[
(3.2) |x_1 - x_2| > \frac{1}{2}d_G(x_1).
\]

Indeed, if \( |x_1 - x_2| \leq 1/2d_G(x_1) \), then it is easy to see that the line segment \([x_1, x_2]\) is the desired diameter uniform curve.
Join $x_1$ and $x_2$ by a curve $\gamma$ satisfying the weak min-max property, and fix $y \in \partial G$ with $|x_1 - y| \leq 2d_G(x_1)$. Then it follows from (3.2) that for all $x \in \gamma$, we have

$$|x - y| \leq c \max_{j=1,2} |x_j - y| \leq c(|x_1 - x_2| + |x_1 - y|) \leq 5c|x_1 - x_2|,$$

which yields

$$\text{(3.3)} \quad \text{diam}(\gamma) \leq 10c|x_1 - x_2|.$$

For $x \in \gamma$, choose $z \in \partial G$ with $|x - z| \leq 2d_G(x)$. Thus by the weak min-max property, we have

$$\text{(3.4)} \quad \min_{j=1,2} |x_j - x| \leq \min_{j=1,2} |x_j - z| + |z - x| \leq (c + 1)|z - x| \leq 2(c + 1)d_G(x).$$

By (3.3) and (3.4), we obtain that $G$ is distance $A$-uniform with $A = 10c$. Therefore, it follows from [16, Page 10] that $G$ is diameter $A_1$-uniform, where $A_1 = A_1(c)$.

4. Proofs of Theorem 1.4 and Corollary 1.6

Before giving the proof of Theorem 1.4, we show that the weak min-max property is invariant under relatively quasisymmetric mappings.

**Lemma 4.1.** Let $G \subseteq E$ and $G' \subseteq E'$ be domains. Suppose that $G$ has the weak $c$-min-max property, and that a homeomorphism $f : G \to G'$ is $\eta$-quasisymmetric relative to $\partial G$ and maps $G$ onto $G'$, then $G'$ satisfies the weak $c_1$-min-max property, where $c_1 = c_1(c, \eta)$.

**Proof.** Fix $x'_1, x'_2 \in G'$. Then we find two points $x_1, x_2 \in G$ such that $x'_j = f(x_j)$ for $j = 1, 2$. By the min-max property of $G$, there is a curve $\gamma$ in $G$ joining $x_1$ and $x_2$ such that for all $x \in \gamma$ and for all $y \in \partial G$,

$$\text{(4.1)} \quad c^{-1} \min_{j=1,2} |x_j - y| \leq |x - y| \leq c \max_{j=1,2} |x_j - y|$$

Without loss of generality, we may assume that $|x_1 - y| \leq |x_2 - y|$. Hence, by (4.1), and by the relative quasisymmetry of $f$, we have

$$|x'_1 - f(y)| = |f(x_1) - f(y)| \leq \eta(c)|f(x) - f(y)|$$

and

$$|f(x) - f(y)| \leq \eta(c)|f(x_2) - f(y)| = \eta(c)|x'_2 - f(y)|,$$

which imply

$$\eta(c)^{-1} \min_{j=1,2} |x'_j - f(y)| \leq |f(x) - f(y)| \leq \eta(c) \max_{j=1,2} |x'_j - f(y)|.$$

It follows that $G'$ has the weak $c_1$-min-max property with $c_1 = \eta(c)$.

Väisälä showed that the inversions are quasimöbius and the relationship between quasisymmetric mappings and quasimöbius mappings, which are as follows:
Lemma 4.2. ([11] Page 220] Let $u(x)$ be the inversion in the unit sphere $S$, where
\[ u(x) = \frac{x}{|x|^2} \]
for $x \in E \setminus \{o\}$ and $o$ is the origin. Then $u^{-1} = u$, and $u$ is $\theta_0$-quasimöbius with the control function $\theta_0(t) = 81t$ for $t \in [0, \infty)$.

Theorem 4.3. ([11] Theorem 3.10]) Let $X$ and $Y$ be two metric spaces. Suppose that $X$ is unbounded and that $f : X \to Y$ is $\theta$-quasimöbius. Then $f$ is quasimöbius if and only if $f(x) \to \infty$ as $x \to \infty$. In this case, $f$ is $\theta$-quasimöbius.

Now by Theorem 1.3 and Lemma 4.1, we know that diameter uniform domains are preserved under relatively quasisymmetric mappings.

Corollary 4.4. Let $G \subset E$ and $G' \subset E'$ be domains. Suppose that $G$ is diameter $A$-uniform, and that $f : \overline{G} \to \overline{G'}$ is $\eta$-quasisymmetric relative to $\partial G$ homeomorphism maps $G$ onto $G'$. Then $G'$ is diameter $A_2$-uniform, where $A_2 = A_2(A, \eta)$.

Let $u(x)$ be the inversion in the unit sphere $S$ as defined in Lemma 4.2. It follows from Lemma 4.2 that $u$ and $u^{-1}$ are $\theta_0$-quasimöbius with the control function $\theta_0(t) = 81t$, $t \in [0, \infty)$.

Next, we show that diameter uniformity is invariant under an inversion transformation.

Lemma 4.5. Suppose that $G$ is a diameter $A$-uniform domain of $E$. Then $u(G)$ is diameter $A_2$-uniform with $A_3 = A_3(A)$.

Proof. By Theorem 1.3 the diameter uniformity of $G$ gives that $G$ is $\delta$-uniform with $\delta = \delta(A)$. Because $u$ is $\theta_0$-quasimöbius with the control function $\theta_0(t) = 81t$, $t \in [0, \infty)$, a simple computation shows that $u(G)$ is $\delta_1$-uniform with $\delta_1 = \delta_1(A)$. Using Theorem 1.3 we know that $u(G)$ is diameter $A_3$-uniform with $A_3 = A_3(A)$. \qed

Proof of Theorem 1.4. By composing translations if necessary, we may assume that $o \in \partial G$ and $o \in \partial G'$. Denote
\[ g := u \circ f \circ u^{-1} : u(G) \to u(G'). \]
It follows from Lemma 4.2 that $u$ and $u^{-1}$ are $\theta_0$-quasimöbius with the control function $\theta_0(t) = 81t$, $t \in [0, \infty)$. Hence, we obtain that $g$ is $\theta_1$-quasimöbius relative to $u(\partial G)$ with $g(x) \to \infty$ as $x \to \infty$, where the control function $\theta_1(t) = 81\theta(81t)$, $t \in [0, \infty)$. By Theorem 1.3, $g$ is $\theta_1$-quasisymmetric relative to $u(\partial G)$ with $\theta_1 = \theta_1(\theta)$.

On the other hand, by the diameter uniformity of $G$, Lemma 4.5 yields that $u(G)$ is diameter $A_3$-uniform. Therefore, we see form Corollary 1.4 that $u(G')$ is diameter $A_4$-uniform with $A_4 = A_4(A, \theta)$. Because $u^{-1} = u$, Lemma 4.5 yields that $G'$ is diameter $A_1$-uniform with $A_1 = A_1(A, \theta)$. \qed

Proof of Corollary 1.6. By Theorem 1.4, we obtain that $G'$ is diameter $A'$-uniform, where $A' = A'(A, \theta)$. It follows from Theorem 1.1 that $G'$ is $A_2$-uniform, where $A_2 = A_2(A, \theta, \psi)$.
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