Spline-oriented inter/extrapolation-based multirate schemes of higher order

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Abstract. Multirate integration uses different time step sizes for different components of the solution based on the respective transient behavior. For inter/extrapolation-based multirate schemes, we construct a new subclass of schemes by using clamped cubic splines to obtain multirate schemes up to order 4. Numerical results for a n-mass-oscillator demonstrate that 4th order of convergence can be achieved for this class of schemes.

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1 Introduction

Many technical applications, e.g. electric circuits [4, 11], can be modeled as coupled systems of ordinary differential equations (ODEs). Often, the transient behavior of these applications is characterized by different time constants. At a given time point \( t_n \), there are many slowly evolving components \( y^{(S)} \in \mathbb{R}^{d_S} \) and a few fast components \( y^{(F)} \in \mathbb{R}^{d_F} \) (\( d_S \gg d_F \)). To exploit this, multirate integration schemes are developed following the pioneer work [6]. Thereby, the fast part \( y^{(F)} \) is integrated with a small step size \( h \) (micro step) and the slow part \( y^{(S)} \) using a larger step size \( H = m \cdot h \) (macro step). The appropriate choice of the coupling variables is the challenging part. Here, we focus on inter/extrapolation-based multirate schemes to describe the novel construction of spline-oriented multirate schemes of higher order. Numerical results demonstrate an order four convergence based on cubic spline coupling.
2 Inter/extrapolation-based multirate schemes

We consider the component-wise partitioned initial value problem (IVP)

\[
\dot{y}^{(S)} = f^{(S)} \left( t, y^{(S)}, y^{(F)} \right), \quad y^{(S)}(t_0) = y_0^{(S)} \in \mathbb{R}^{d_S},
\]

\[
\dot{y}^{(F)} = f^{(F)} \left( t, y^{(S)}, y^{(F)} \right), \quad y^{(F)}(t_0) = y_0^{(F)} \in \mathbb{R}^{d_F},
\]

of coupled ODEs on \([t_0, T]\), where \(f^{(S)}\) and \(f^{(F)}\) are sufficiently smooth.

Depending on the sequence of computation, one can distinguish five versions of inter/extrapolation-based multirate schemes: 

- fully-decoupled approach \([1]\),
- decoupled slowest-first approach \([4]\),
- decoupled fastest-first approach \([4]\),
- coupled first-step approach \([5]\) and
- the coupled slowest-first approach, which is a special case of the time-stepping strategy introduced in \([7]\) and traces back to Rice \([6]\). In this work, we focus on the decoupled slowest-first (DSF) approach.

**Definition 2.1 (DSF approach)** We advance the solution of (1) from \(t_n\) to \(t_n + H\). Firstly, the slow subsystem is numerically integrated by a scheme \(\Phi_H\) and an extrapolated waveform \(\tilde{y}^{(F)}\) for \(y^{(F)}\). A scheme \(\hat{\Phi}_h\) and an interpolated waveform \(\hat{y}^{(S)}\) for the slow components based on information from the current time window \([t_n, t_n + H]\).

**Remark 2.2** All five inter/extrapolation-based multirate schemes have convergence order \(p\), if the basic integration schemes have convergence order \(p\) and the inter/extrapolation schemes are of approximation order \(p - 1\) \([2]\).

3 Multirate schemes using spline-oriented inter/extrapolation

As derivative information is provided by the ODE, we consider clamped cubic splines of order three \([3]\). This enables multirate schemes of order \(p = 4\).

**DSF clamped-spline approach** We have two parts:

(i) After a numerical approximation is obtained for a macro step \([t_{n-1}, t_n]\), compute a clamped cubic spline \(S^{(F)}_n(t)\) for the fast components based on computed micro step values \(y^{(F)}_{n-1+i/m}, i = 0, \ldots, m\), and the derivative information

\[
\dot{y}^{(F)}_{n-1} = f^{(F)} \left( t_{n-1}, y^{(S)}_{n-1}, y^{(F)}_{n-1} \right), \quad \dot{y}^{(F)}_n = f^{(F)} \left( t_n, y^{(S)}_n, y^{(F)}_n \right).
\]

(ii) After the integration of the slow subsystem, compute a cubic polynomial \(S^{(S)}_{n+1}(t)\) for the slow components by using the idea of clamped cubic splines based on the computed values \(y^{(S)}_n\) and \(y^{(S)}_{n+1}\), and the derivative information

\[
y^{(S)}_n = f^{(S)} \left( t_n, y^{(S)}_n, y^{(F)}_n \right) \quad \text{and} \quad y^{(S)}_{n+1} = f^{(S)} \left( t_{n+1}, y^{(S)}_{n+1}, y^{(F)}_{n+1} \right).
\]
Note that we have to use the extrapolated value \( \tilde{y}_{n+1}^{(F)} \) as \( y_{n+1}^{(F)} \) is unknown. Fortunately (except of some special cases [1]), the multirate setting implies that \( \left| \frac{\partial f^{(S)}}{\partial y^{(F)}} \right| \) is small such that the use of the extrapolated value is not problematic.

**The first macro step** We need a special treatment for the first macro step as there exists no spline \( S_0^{(F)}(t) \). Thus, for the first macro step \([t_0, t_0 + H]\), we apply \( m \) micro steps of a singlerate integration scheme. At the end of this macro step, one has to compute the clamped cubic spline \( S_1^{(F)}(t) \) on \([t_0, t_1]\).

**The general procedure** Consider the time window \([t_n, t_{n+1}], n \geq 1\).

(i) integrate the slow subsystem for \( y^{(S)} \) with step size \( H \) using the extrapolated waveform \( \tilde{y}^{(F)} = S_n^{(F)} \big|_{[t_{n-h}, t_n]} \),

(ii) compute the cubic polynomial \( S_n^{(S)}(t) \) on \([t_n, t_{n+1}]\),

(iii) perform the \( m \) micro steps for the fast subsystem for \( y^{(F)} \) with step size \( h \) using the interpolated waveform \( \hat{y}^{(S)} = S_n^{(S)} \),

(iv) compute the clamped cubic spline \( S_n^{(F)}(t) \) on \([t_n, t_{n+1}]\) with nodes \( t_n + ih, i = 0, \ldots, m \).

These methods are referred to as **spline-oriented multirate schemes**.

**Convergence Analysis** Let the solution be computed on \([t_0, t_n]\). Then, one computes a cubic polynomial \( \tilde{y}^{(F)} \) based on the fast data

\[
y_{n-1+i/m}^{(F)}, i = 0, \ldots, m, \quad f^{(F)} \left( t_{n-1}, y_{n-1}^{(S)}, y_{n-1}^{(F)} \right), \quad f^{(F)} \left( t_n, y_n^{(S)}, y_n^{(F)} \right)
\]

of the last macro step \([t_{n-1}, t_n]\). Thus it holds \( |y^{(F)}(t) - \tilde{y}^{(F)}(t)| = O(H^4) \) for \( t \in [t_n, t_{n+1}] \). For the computation of the next macro step \([t_n, t_{n+1}]\), the use of the extrapolated waveform in step (i) of the general procedure results in a perturbed, decoupled ODE system for the slow subsystem. This introduces a model error to the slow part

\[
\hat{y}^{(S)} = f^{(S)} \left( t, \hat{y}^{(S)}, \tilde{y}^{(F)}(t) \right) = f^{(S)} \left( t, \hat{y}^{(S)}, y^{(F)} \right) + \Psi_n^{(S)}(t) \tag{2}
\]

with \( \Psi_n^{(S)} \in O(H^4) \). Subtracting this from the original ODE for \( y^{(S)} \) and using Gronwall’s lemma, we deduce

\[
\left| y^{(S)}(t) - \hat{y}^{(S)}(t) \right| = O(H^4), \quad t \in [t_n, t_{n+1}].
\]

An integration scheme of order \( p \geq 4 \) yields an approximation \( y_{n+1}^{(S)} \) with error

\[
\left| y^{(S)}(t_{n+1}) - y_{n+1}^{(S)} \right| \leq \left| y^{(S)}(t_{n+1}) - \hat{y}^{(S)}(t_{n+1}) \right| + \left| \hat{y}^{(S)}(t_{n+1}) - y_{n+1}^{(S)} \right|, \tag{3}
\]
i.e., the error is, like the approximation and model error, in $\mathcal{O}(H^4)$. To update the fast part, step (ii) computes a cubic polynomial $\hat{y}^{(S)}$ using

$$y^{(S)}_n, \ y^{(S)}_{n+1}, \ f^{(S)} \left(t_n, y^{(S)}_n, y^{(S)}_{n+1}\right), \ f^{(S)} \left(t_n+1, y^{(S)}_{n+1}, y^{(S)}_{n+1}\right).$$

As $\hat{y}^{(F)} = y^{(F)} + \mathcal{O}(H^4)$, it holds $|y^{(S)}(t) - \hat{y}^{(S)}(t)| = \mathcal{O}(H^4)$ for $t \in [t_n, t_{n+1}]$. This leads to the perturbed, decoupled fast ODE system ($\Psi^{(F)}(t) \in \mathcal{O}(H^4)$)

$$\hat{y}^{(S)} = f^{(S)} \left(t, \hat{y}^{(S)}, \hat{y}^{(F)}\right) = f^{(S)} \left(t, y^{(S)}, y^{(F)}\right) + \Psi^{(F)}(t) \quad (4)$$

In step (iii), computing $m$ micro steps for (4) with a numerical integration scheme of order $p \geq 4$ yields a fourth order update as it holds

$$\left|y^{(F)}(t_{n+1}) - y^{(F)}_{n+1}\right| \leq \left|y^{(F)}(t_{n+1}) - \hat{y}^{(F)}(t_{n+1})\right| + \left|\hat{y}^{(S)}(t_{n+1}) - y^{(F)}_{n+1}\right| \quad (5)$$

In the end, the spline-oriented approximation at the final time $t = T$ can be viewed as a numerical approximation of the decoupled ODE system, where the coupling variables are represented by inter- and extrapolated functions of $t$. This yields modifications of the right hand sides: $\Psi^{(S)}$ and $\Psi^{(F)}$. Both functions are successively being build up during the integration and result in a modeling error of order 4. This gives:

**Theorem 3.1 (Convergence)** The spline-oriented multirate scheme is a convergent procedure. The order depends on the employed basic schemes and spline approximation. In the case of clamped cubic splines and a numerical integration scheme of order $p \geq 4$, we obtain a method of order four.

**Remark 3.2** The spline-oriented multirate scheme has a predictor-corrector type of structure.

**Definition 3.3 (Spline-oriented multirate RK scheme)** We consider a $s$-staged Runge–Kutta (RK) scheme $[3]$ given by the Butcher tableau $(A, b, c)$. Based on the DSF-approach, one macro step of a spline-oriented multirate RK scheme with $m$ micro steps, applied to (1), advances the solution $(y^{(S)}_n, y^{(F)}_n)$ at $t_n = t_0 + n \cdot H$ to the solution $(y^{(S)}_{n+1}, y^{(F)}_{n+1})$ at $t_{n+1} = t_n + H$ as follows:

1.) macro step

$$y^{(S)}_{n+1} = y^{(S)}_n + H \sum_{i=1}^s b_i k_i^{(S)},$$

$$k_i^{(S)} = f^{(S)} \left(t_n + c_i H, \ y^{(S)}_n + H \sum_{j=1}^s a_{ij} k_j^{(S)}, \ y^{(S)}_{n+1}\right),$$

$$y^{(F)}_{n+1} = y^{(F)}_n + H \sum_{i=1}^s b_i \hat{k}_i^{(F)},$$

$$\hat{k}_i^{(F)} = \hat{f}^{(F)} \left(t_n + c_i H, \ y^{(F)}_n + H \sum_{j=1}^s a_{ij} \hat{k}_j^{(F)}, \ y^{(F)}_{n+1}\right).$$

...
2.) micro steps (for $i = 1, \ldots, s$ and $\lambda = 0, \ldots, m - 1$)

$$y_{n+\frac{\lambda+1}{m}}^{(F)} = y_{n+\frac{\lambda}{m}}^{(F)} + h \sum_{i=1}^{s} b_i k_i^{(F,\lambda)},$$

$$k_i^{(F,\lambda)} = f_i^{(F)} \left( t_n + (\lambda + c_i)h, \ y_n^{(S)}, y_n^{(F)} + h \sum_{j=1}^{s} a_{i,j} k_j^{(F,\lambda)} \right).$$

(6b)

4 Numerical results

![Diagram of a system with n masses and n+1 springs](image)

Figure 1: System with $n$ masses: one light mass $m_1$, $n-1$ heavy masses; $n+1$ springs, one strong spring $k_1$, $n$ light springs $k_2$. The system is attached to fixed walls.

We consider the line configuration of $n$ masses and $n+1$ springs as given in Fig. 1. The equation of motion reads ($x_i$ position of $i$th mass):

$$m_1 \ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2,$$

$$m_2 \ddot{x}_2 = k_2x_1 - 2k_2x_2 + k_2x_3,$$

$$\vdots$$

$$m_2 \ddot{x}_{n-1} = k_2x_{n-2} - 2k_2x_{n-1} + k_2x_n,$$

$$m_2 \ddot{x}_n = k_2x_{n-1} - 2k_2x_n.$$  

(7)

To demonstrate the construction of an order four multirate scheme, we use as underlining schemes the classical Runge-Kutta (RK4) scheme which we refer to as MR-RK4. With multirate factor $m = 20$, we solve the ODE system (7) for $n = 10$ numerically on the time interval $[0, 40]$ with masses $(m_1, m_2) = (1, 20)$, spring constants $(k_1, k_2) = (20, 1)$ and initial values

$$(x_1(0), \dot{x}_1(0)) = (-0.005, 0), \quad (x_i(0), \dot{x}_i(0)) = (0.1, 0), \quad \forall i = 2, \ldots, n.$$  

Numerical results are shown in Figure 2. Order 4 of the MR-RK4 is observed.
Figure 2: Numerical results for a non-stiff 10-mass-oscillator \(^{(7)}\) for MR-RK4. Left: numerical solutions of slow (red) and fast (blue) components. Right: convergence order for fixed multirate factor \(m = 20\) \((H = mh)\) as absolute error at \(t = 40\).

We discuss the achieved accuracy of MR-RK4 \((m = 20)\) compared to the singlerate case (RK4 with micro step size \(h\)) in the above test \((7)\): i) The error of the fast part \(y_F\) is above three orders of magnitude less than the absolute error of the singlerate RK4 scheme. ii) The error of the slow part \(y_S\) is above 1.5 orders of magnitude less than the absolute error of the singlerate RK4 scheme. On the other hand, multirate saves right-hand evaluations. For the ODE system \((7)\), for one macro step, the number of scalar function evaluations is:

\[
\text{singlerate: } 4 \cdot m \cdot n \quad \text{vs.} \quad \text{multirate: } 4 \cdot (m \cdot 1 + 1 \cdot (n - 1)) .
\]

For \(m = 20\) and \(n = 10\), we thus have 800 vs. 116 evaluations.

5 Conclusion and outlook

We combined successfully inter/extrapolation-based multirate schemes with a clamped cubic spline. Thereby, we preserve the order of the basic integration scheme. Next steps are some further stability investigations and generalizations of this class of multirate schemes. Apart from this, we then plan to extend this strategy to the application of DAEs. Of course, the interpolation of the algebraic variables will move us from the manifold. This needs to be treated.

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