Resolving Gödel's Incompleteness Myth:
Polynomial Equations and Dynamical Systems for Algebraic Logic

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December 23, 2011

ABSTRACT

A new computational method that uses polynomial equations and dynamical systems to evaluate logical propositions is introduced and applied to Gödel's incompleteness theorems. The truth value of a logical formula subject to a set of axioms is computed from the solution to the corresponding system of polynomial equations. A reference by a formula to its own provability is shown to be a recurrence relation, which can be either interpreted as such to generate a discrete dynamical system, or interpreted in a static way to create an additional simultaneous equation. In this framework the truth values of logical formulas and other polynomial objectives have complex data structures: sets of elementary values, or dynamical systems that generate sets of infinite sequences of such solution-value sets. Besides the routine result that a formula has a definite elementary value, these data structures encode several exceptions: formulas that are ambiguous, unsatisfiable, unsteady, or contingent. These exceptions represent several semantically different types of undecidability; none causes any fundamental problem for mathematics. It is simple to calculate that Gödel's formula, which asserts that it cannot be proven, is exceptional in specific ways: interpreted statically, the formula defines an inconsistent system of equations (thus it is called unsatisfiable); interpreted dynamically, it defines a dynamical system that has a periodic orbit and no fixed point (thus it is called unsteady). These exceptions are not catastrophic failures of logic; they are accurate mathematical descriptions of Gödel's self-referential construction. Gödel's analysis does not reveal any essential incompleteness in formal reasoning systems, nor any barrier to proving the consistency of such systems by ordinary mathematical means.
1 INTRODUCTION

In the last century, Kurt Gödel’s incompleteness theorems [15] sent shockwaves through the world of mathematical logic. The conventional wisdom is that Gödel’s theorems and his interpretations thereof are correct; the prevalent discussion concerns what these results mean for logic, mathematics, computer science, and philosophy [26, 20, 16]. But as I shall demonstrate here, Gödel’s theorems are profoundly misleading and his interpretations were incorrect: his analysis was corrupted by the simplistic and flawed notions of truth value and proof that have troubled logic since antiquity, compounded by his misapplication of a static definition of consistency to a dynamical system. Exposing these errors reveals that reports of logic’s demise have been greatly exaggerated; we may yet realize the rationalist ideals of Leibniz and complete the logicist and formalist programs of Frege, Russell, and Hilbert.

There are two key principles here: first, that proof in formal reasoning systems is an exercise in solving systems of polynomial equations, yielding solutions that are sets of elementary truth values; and second, that certain self-referential formula definitions are recurrence relations that define discrete dynamical systems (and in turn infinite sequences of basic solution sets). It is a corollary to these principles that all of the various syntactic results from such calculations make semantic sense as the truth values of formulas, including solution sets that are empty or have multiple members, and dynamic solutions that change with each iteration and depend on initial conditions.

These principles are familiar and uncontroversial in the contexts of elementary algebra and dynamical systems; they apply just as well when the basic mathematical objects are logical truth values instead of ordinary numbers. If you understand how to do arithmetic in different number systems, what it means to solve equations, and how to deal with recursive constructions like the Fibonacci sequence, then you can understand Gödel’s mistakes. You will find the powerful new paradigm of dynamic polynomial logic, which is a continuation of the pioneering 19th-century work of George Boole [4, 5]. Dynamic polynomial logic is grounded in the intrinsic unity of logic and mathematics.

Relative to classical logic, dynamic polynomial logic is paraconsistent, paracomplete, and modal. In this algebraic framework the misguided principle of explosion is corrected: inconsistent axioms are shown to prove nothing instead of everything. Moreover the principle of the excluded middle is clarified; in classical logic this idea is applied incorrectly, reflecting confusion between arithmetical and algebraic systems. Dynamic polynomial logic computes precise solutions that can be interpreted as alethic and temporal modalities.

1.1 Gödel’s Argument

Gödel considered formal reasoning systems as described in Whitehead and Russell’s Principia Mathematica [31], which was an epic attempt to formalize the whole of mathematics. Gödel’s basic argument was that every formal reasoning system powerful enough to describe logical formulas, proof, and natural numbers (like PM) must allow the construction of a special formula that is semantically correct but syntactically undecidable: true by metalevel consideration of its content, but impossible to prove or disprove by mathematical calculation within the formal system itself. This special formula, denoted both \([R(q); q]\) and 17 Gen \(r\) in Gödel’s paper, asserts that the formula itself cannot be proven within the system. Thus follows the apparent paradox, which Gödel described in this way (as translated in [30]):

From the remark that \([R(q); q]\) says about itself that it is not provable it follows at once that \([R(q); q]\) is true, for \([R(q); q]\) is indeed unprovable (being undecidable). Thus, the proposition that is undecidable in the system PM still was decided by metamathematical considerations. The precise analysis of this curious situation leads to surprising results concerning consistency proofs for formal systems, results that will be discussed in more detail in Section 4 (Theorem XI).

Gödel’s Theorem VI states that there must exist (in a formal system like PM) a formula such as his special \([R(q); q]\) which can neither be proven nor disproven within the formal system that contains it. His claim that his special formula \([R(q); q]\) is semantically true is presented in the text of his paper but is not called out as a theorem. Gödel’s Theorem XI states that the existence of such an undecidable formula renders the consistency of the enclosing formal system itself an undecidable proposition.
1.2 The Truth (Value) Is Complicated

The surprising result from my analysis is that Gödel’s special formula is neither semantically correct nor semantically incorrect; instead it is *exceptional* in a particular way, relative to the expectation that a formula should have a definite elementary value. Such exceptions, which appeared to Gödel as ‘undecidability,’ are features not bugs in formal reasoning systems: it is appropriate that some logical formulas cannot be proven to be simply true or false, because they are in fact neither. Evaluating Gödel’s special formula is analogous to taking the square root of a negative number: such a square root cannot be proven to be any real number, because it is not any real number. More elaborate data structures are needed to describe the values of some logical formulas, just as complex numbers are needed to describe the values of some arithmetic formulas.

We shall consider four different exceptions that can be raised by logical formulas: the static exceptions of *unsatisfiability* and *ambiguity* and the dynamic exceptions of *unsteadiness* and *contingency*. These exceptions represent four distinct types of ‘undecidability,’ and each type has a particular semantic meaning and gives characteristic syntactic results. Ignorance of these exceptions has plagued logic since ancient times. One can already find in the riddles of Aristotle’s adversary Eubulides, who originated the liar paradox in the 4th century B.C., perfect demonstrations of unsatisfiable and ambiguous logical propositions [24]. Yet the same exceptions occur in algebra with ordinary numbers, and in this context they are well-understood and not at all controversial. With the appropriate data structures and algorithms, these exceptional results become no more problematic for logic than $\sqrt{-1}$ is for algebra. These four exceptions are consequences of the two key principles discussed next.

1.3 Logical Equations Give Sets of Elementary Solutions

The first key principle is that the axioms in a formal system (including the definitions of formulas) constitute a system of simultaneous equations; hence the proper way to describe the truth value of a formula is to give the *solution-value set* to its defining system of equations—the possible elementary values of the formula when all the equations are satisfied.

As in general algebra, a system of logical equations can have zero, one, or more solutions: thus the solution-value set can be empty, have one member, or have many members.

Using binary logic with the elementary values true and false (T and F), there are four possible solution-value sets (hence four different truth values): the set \{T\}; the set \{F\}; the set \{T, F\}; and the empty set \{\}. The first two sets are the definite or ‘unexceptional’ results: if the feasible set of values for some formula is \{T\}, then that formula is *necessarily true* (equivalently, it is a *theorem*); if the feasible set is \{F\}, then the formula is *necessarily false* (its negation is a theorem). The last two sets are exceptional: if the solution-value set is \{T, F\}, then the formula is *ambiguous*; if the solution-value set is \{\}, then the formula is *unsatisfiable* (and the underlying axioms are inconsistent). This style of categorization extends easily to sets of elementary values with more than two values, including infinite sets (like the natural numbers) and uncountable sets (like the real numbers). In general we consider truth values in the *power set* of the set of elementary values.

As Boole explained in his *Laws of Thought* [5], logical formulas with binary truth values can be translated into polynomial expressions: the coefficients 1 and 0 represent the elementary values true and false; symbolic variables represent basic propositions, with each variable $x$ subject to the constraint $x^2 = x$ to ensure that its only feasible values are 0 and 1; logical conjunction translates as multiplication; logical negation translates as the difference from 1; and logical disjunction translates as a certain combination of addition and multiplication. Boole presented a complete algorithm to translate any logical formula into a polynomial with ordinary integer coefficients.

For example, using Boole’s original method the logical formula $x \rightarrow y$ translates as the polynomial $xy - y + 1$ and the logical formula $y \rightarrow x$ translates as the polynomial $yx - y + 1$. Using these translations, constraining each formula to be true (i.e. to equal the polynomial 1), and separately constraining each variable to be either 0 or 1 yields the following system of polynomial equations:

\[
\begin{align*}
xy - x + 1 &= 1 \\
yx - y + 1 &= 1 \\
x^2 &= x \\
y^2 &= y
\end{align*}
\]

(1)

The only solutions to these equations for the variables $(x, y)$ are $(0, 0)$ and $(1, 1)$. You can see that $x$ and $y$ have the same value in each solution, so you might expect the biconditional formula $x \leftrightarrow y$ to be a theorem given the axioms $x \rightarrow y$
and \( y \to x \). Using Boole’s original method, the logical formula \( x \leftrightarrow y \) is translated into the polynomial \( 2xy - x - y + 1 \); substituting either solution for \( (x, y) \) the value of this polynomial is 1. Thus the solution-value set for the polynomial \( 2xy - x - y + 1 \), subject to the constraints in Equation 11 is the set \( \{1\} \).

Translating back into logical notation, the solution-value set for the formula \( x \leftrightarrow y \) given the axioms \( x \to y \) and \( y \to x \) is \( \{T\} \); therefore \( x \leftrightarrow y \) is called a theorem relative to these axioms. However, neither the formula \( x \) nor the formula \( y \) has a definite value given these equations; each formula has the solution-value set \( \{0, 1\} \) in polynomial notation, which is \( \{F, T\} \) in logical notation. Therefore each formula \( x \) and \( y \) is called ambiguous given the axioms \( x \to y \) and \( y \to x \). Note that neither \( x \) nor \( \neg x \) is a theorem given these axioms; likewise neither \( y \) nor \( \neg y \) is a theorem. This ‘undecidability’ for the formulas \( x \) and \( y \) is a correct interpretation of the relevant equations, not a sign of pathological incompleteness in formal reasoning.

Infeasible equations have empty solution sets and render unsatisfiable all formulas that are subject to them. For example the simultaneous equations:

\[
\begin{align*}
z &= 1 - z, \\
 z^2 &= z 
\end{align*}
\]  

have no solution; the first constraint is violated for both values \( z = 0 \) and \( z = 1 \) that satisfy the second constraint. Therefore the formula \( z \) is unsatisfiable subject to these equations, as are the formulas 0, 1, and \( 1 - z \); all share the empty solution-value set \( \{\} \). The equation \( z = 1 - z \) is a polynomial translation of the logical axiom \( z = \neg z \) which states that the formula \( z \) is defined to be true exactly if it is not true. This is one way to model the liar paradox, and in this version the problem is no more paradoxical than the unsatisfiable equation \( 1 = 0 \).

1.4 Self-Reference Gives Dynamical Systems

The second key principle is that, if it is permitted for a formula definition (or any other axiom or equation) to refer to the solution generated by evaluating its own system of equations, then the value of every formula may gain a more complex data structure. Such a solution self-reference is a recurrence relation that can be interpreted in two distinct ways: in the dynamic interpretation the recurrence is used as such to define infinite sequences of solutions governed by a discrete dynamical system; and in the static interpretation the recurrence is used to generate an additional simultaneous equation. Both interpretations can be reasonable and useful, but it is important not to confuse them during analysis.

To illustrate, let us consider the quadratic equation \( 2x^2 + 3x + c = 0 \) in which we define the coefficient \( c \) to be the number of real solutions (for \( x \)) to the equation in which it appears. In other words we have the following specification for a system of equations, using real-valued variables \( x, c \in \mathbb{R} \):

\[
\begin{align*}
2x^2 + 3x + c &= 0, \\
c &= \text{the number of real solutions for } x \text{ to Equation } 3
\end{align*}
\]  

In this specification the value of \( c \) depends on itself. To model this dependence, let us introduce an evolution function denoted \( F(c) \) that gives the number of real solutions to Equation 3 when \( c \) takes the value supplied as the function’s argument. In the dynamic interpretation we take the specification to define a system that changes over time according to this evolution function \( F(c) \), with the state \( c_{t+1} \) at the next time \( t + 1 \) given by the value \( F(c_t) \) of the evolution function applied to the current state \( c_t \). Thus we have:

\[
\begin{align*}
2x^2 + 3x + c_t &= 0, \\
c_{t+1} &= F(c_t); \quad t \in \{0, 1, 2, \ldots\}
\end{align*}
\]  

Alternatively, in the static interpretation we take the problem specification to mean that the input value \( c \) and the output value \( F(c) \) must agree simultaneously, as in:

\[
2x^2 + 3x + c = 0, \quad c = F(c)
\]

We must now determine what this evolution function \( F(c) \) is and which particular values of \( c \) need to be considered in solving the equations. For this problem the quadratic formula serves both needs; a more general approach will be presented later. As you may recall, the quadratic formula states that the number of distinct real roots of an equation \( ax^2 + bx + c = 0 \) depends on its determinant \( b^2 - 4ac \): there are no real solutions if the determinant is negative, one if
it is zero, and two if it is positive. Therefore, the possible values of $c$ are $\{0, 1, 2\}$; and using the quadratic-equation coefficients $a = 2$ and $b = 3$ from Equation (3) the evolution function $F(c)$ must satisfy:

$$F(c) = \begin{cases} 
0, & 9 - 8c < 0 \\
1, & 9 - 8c = 0 \\
2, & 9 - 8c > 0 
\end{cases}$$

(7)

Therefore $F(0) = 2$ since $9 - 8 \cdot 0 > 0$; $F(1) = 2$ since $9 - 8 \cdot 1 > 0$; and $F(2) = 0$ since $9 - 8 \cdot 2 < 0$. Using polynomial interpolation, a closed-form function $F(c)$ can be constructed that performs exactly these mappings $0 \mapsto 2$, $1 \mapsto 2$, and $2 \mapsto 0$:

$$F(c) : -c^2 + c + 2$$

(8)

For the static interpretation, substituting the evolution function from Equation (8) into the system in Equation (6) and then making explicit the domain of each variable produces the following system of equations:

$$2x^2 + 3x + c = 0, \quad c = -c^2 + c + 2; \quad x \in \mathbb{R}, \quad c \in \{0, 1, 2\} \subset \mathbb{R}$$

(9)

After rearranging the second equation it is evident that this system has no solution: $c^2 = 2$ implies $c = \sqrt{2}$ but $c$ is required to be 0, 1, or 2. Thus in the static interpretation, the formula $x$ is unsatisfiable and the whole system of equations is inconsistent.

For the dynamic interpretation, substituting the evolution function from Equation (8) into the system in Equation (5) and leads to the following system of equations, which includes the recurrence $c_{t+1} = F(c_t)$:

$$2x_t^2 + 3x_t + c_t = 0, \quad c_{t+1} \equiv -c_t^2 + c_t + 2; \quad x_t \in \mathbb{R}, \quad c_t \in \{0, 1, 2\} \subset \mathbb{R}, \quad t \in \{0, 1, 2, \ldots\}$$

(10)

In this interpretation we consider $c_t$ to be the state of the dynamical system at time $t$. Each initial state $c_0 \in \{0, 1, 2\}$ generates an infinite sequence $(c_0, c_1, \ldots)$ of states at successive times $t$:

$$\begin{bmatrix}
c_0 = 0 & \mapsto & (0, 2, 0, 2, \ldots) \\
c_0 = 1 & \mapsto & (1, 2, 0, 2, \ldots) \\
c_0 = 2 & \mapsto & (2, 0, 2, 0, \ldots)
\end{bmatrix}$$

(11)

In the state $c_t = 0$ the main equation $2x_t^2 + 3x_t + c_t = 0$ specializes to $2x_t^2 + 3x_t + 0 = 0$ which has the two solutions $x_t = -\frac{3}{2}$ and $x_t = 0$. Then given $c_t = 1$ the main equation becomes $2x_t^2 + 3x_t + 1 = 0$ which has two solutions $x_t = -1$ and $x_t = -\frac{3}{2}$. Finally, given $c_t = 2$ the main equation becomes $2x_t^2 + 3x_t + 2 = 0$ which has no real solutions. Therefore Equation (10) also generates a collection of sequences of solution-value sets for $x$, again depending on the initial condition $c_0$:

$$\begin{bmatrix}
c_0 = 0 & \mapsto & \{(\{-\frac{3}{2}, 0\}, \{\}, \{-\frac{3}{2}, 0\}, \{\}, \ldots)\} \\
c_0 = 1 & \mapsto & \{(\{-1, -\frac{3}{2}\}, \{\}, \{-\frac{3}{2}, 0\}, \{\}, \ldots)\} \\
c_0 = 2 & \mapsto & \{(\{\}, \{-\frac{3}{2}, 0\}, \{\}, \{-\frac{3}{2}, 0\}, \ldots)\}
\end{bmatrix}$$

(12)

These collections of infinite sequences are governed by a discrete dynamical system. This system can be displayed compactly as a graph in which each node indicates a state of the parameter $c$ and each edge shows the solution-value set for the formula $x$ that is generated from assuming the state corresponding to the originating node:

$$\begin{array}{c}
0 \\
\xrightarrow{\{-\frac{3}{2}, 0\}} \\
1 \xrightarrow{(\{-1, -\frac{3}{2}\}, \{\}, \{-\frac{3}{2}, 0\}, \{\}, \ldots)\} \quad (13) \\
\xrightarrow{(\{\}, \{-\frac{3}{2}, 0\}, \{\}, \{-\frac{3}{2}, 0\}, \ldots)\}) \\
2 \\
\xrightarrow{\{\}}
\end{array}$$

This graph, constructed from Equation (10) offers some explanation as to why the static system in Equation (9) is unsatisfiable: the matching dynamical system has no fixed points, only a periodic orbit that never reaches a steady state.
You can read off the sequences of states or solution-value sets generated by this dynamical system by following the edges in its graph.

In general many different patterns could be discerned in the behaviors of dynamical systems; but for the purpose of classifying formulas we consider only the number of fixed points. If a dynamical system has exactly one fixed point then that system is called stable; if it has no fixed points then it is called unstable; and if it has more than one fixed point then it is called contingent. In these terms, the dynamical system defined by Equation [10] and graphed in Equation [13] is unstable. Unstable dynamical systems correspond to unsatisfiable static equations, and contingent dynamical systems correspond to ambiguous static equations.

1.5 Anticipating Gödel’s Error

We shall see that Gödel’s special formula \([R(q); q]\) behaves just like the self-referential quadratic equation above. Using \(x\) to represent Gödel’s formula, it turns out that his definition specifies the evolution function \(F(x) = 1 - x\). When Gödel’s reference to provability is interpreted statically as the constraint \(x = F(x)\), his self-denying formula \([R(q); q]\) becomes an unsatisfiable system of equations:

\[
x = 1 - x, \quad x \in \{0, 1\}
\]

And when interpreted dynamically as \(x_{t+1} = F(x_t)\), Gödel’s formula becomes the recurrence:

\[
x_{t+1} = 1 - x_t, \quad x \in \{0, 1\}
\]

This specifies a simple dynamical system with one periodic orbit and no fixed points. This dynamical system generates an alternating sequence of values for its state \(x_t\) at successive times \(t\), for each initial condition \(x_0\):

\[
\begin{align*}
  x_0 = 0 & \quad \mapsto \quad (0, 1, 0, 1, \ldots) \\
  x_0 = 1 & \quad \mapsto \quad (1, 0, 1, 0, \ldots)
\end{align*}
\]

By generating these results, dynamic polynomial analysis will show that Gödel’s ‘formula \([R(q); q]\) that is true if and only if is not provable’ is exceptional in precisely the same way as is ‘the quadratic equation \(2x^2 + 3x + c = 0\) that has exactly \(c\) solutions’: interpreted as static systems of equations, neither specification can be satisfied; and interpreted as dynamical systems, both specifications lead to sequences that oscillate infinitely and never converge to a fixed value.

These types of ‘undecidability’ are not syntactic aberrations; they are semantically appropriate descriptions of the mathematical objects specified by their respective self-referential definitions. In fact, these complex results only seem exceptional because of the misguided expectation that the specified mathematical objects should have simple elementary values. Imagine the confusion that would result if we were to speak of ‘the Fibonacci formula’ (expecting it to have a definite numeric value) or ‘the Fibonacci number’ (expecting there to be just one) instead of ‘the Fibonacci sequence’ and ‘a Fibonacci number.’ Although Gödel’s self-denying formula may seem puzzling, Gödel’s oscillating sequence and Gödel’s inconsistent equation are rather less so.

It remains to be demonstrated that systems of logical axioms are indeed equivalent to systems of polynomial equations, and that self-reference of the type that Gödel described is accurately translated using evolution functions, recurrence relations, and dynamical systems. But before discussing the details of translating logic to algebra, let us review several useful kinds of results that can be calculated by general methods of algebra from systems of polynomial equations and from discrete dynamical systems.

2 GENERAL ANALYSIS OF POLYNOMIAL EQUATIONS

Polynomials are an important class of formulas in elementary and abstract algebra. In this section we introduce terminology and notation to describe systems of polynomial equations and their solutions (for the task of formula evaluation), and we consider ways to construct and to count the members of a polynomial ring that meet certain desirable criteria (for the task of formula discovery). We develop simple algorithms to perform these tasks by hand for small problems, and make reference to more general and efficient methods from computational algebraic geometry. For the moment we are just discussing polynomials in the context of general algebra, without any mention of their provenance in logic and axiomatic formal reasoning.
2.1 Formula Evaluation

We consider systems of polynomial equations whose variables and coefficients take values from an algebraic field \( K \) such as the real number system \( \mathbb{R} \), the rational number system \( \mathbb{Q} \), or a finite field \( \mathbb{F}_d \) of order \( d \) (with \( d \) a prime number). Note that in a finite field \( \mathbb{F}_d \) we must use integer arithmetic modulo \( d \) for calculation; for example in the binary finite field \( \mathbb{F}_2 \) the sum \( 1 + 1 = 0 \) since \( 2 \equiv 0 \pmod{2} \). \[11\] provides a good general reference for polynomials and algebraic geometry; \[10\] and \[12\] describe fields and other structures in abstract algebra; modular arithmetic is discussed in \[17\].

**Definition 1 (System of Polynomial Equations)** Given an algebraic field \( K \) and a vector \( \mathbf{x} := (x_1, x_2, \ldots, x_n) \) of variables, the polynomial ring over the variables \( \mathbf{x} \) with coefficients in \( K \) is denoted \( K[\mathbf{x}] \). We consider a set \( A \) of simultaneous equations:

\[
A := \{ q_1 = 0, q_2 = 0, \ldots, q_m = 0 \}
\]

in which each polynomial \( q_j \) is a member of the ring \( K[\mathbf{x}] \) and also constrained to equal zero. The solution set to this system of equations, denoted \( \mathcal{V}(A) \) or \( \mathcal{V}(q_1 = 0, q_2 = 0, \ldots, q_m = 0) \), is the set of values of \( \mathbf{x} \) for which these equality constraints are satisfied, assuming that each variable \( x_i \) takes a value in the field \( K \):

\[
\mathcal{V}(A) := \{ \mathbf{x} \in K^n : q_1(\mathbf{x}) = 0, q_2(\mathbf{x}) = 0, \ldots, q_m(\mathbf{x}) = 0 \}
\]

By its construction the solution set \( \mathcal{V}(A) \subseteq K^n \) must be a subset of the affine space \( K^n \).

In the extreme case that the equations in \( A \) are inconsistent then \( \mathcal{V}(A) \) is the empty set; conversely if the equations are tautological (or if there are no equations) then \( \mathcal{V}(A) \) is the entire set \( K^n \). In algebraic geometry, the solution set \( \mathcal{V}(A) \) is called the affine variety defined by the polynomials \( \{q_1, q_2, \ldots, q_m\} \) used in the equations. Note that in the special case \( n = 0 \) that there are no variables, we imagine that the affine space \( K^0 \) has one member (that is the unique zero-length tuple. Thus with \( n = 0 \) the solution set \( \mathcal{V}(0 = 0) \Rightarrow \{()\} \) has one member since the equation \( 0 = 0 \) holds; however \( \mathcal{V}(1 = 0) \Rightarrow \{\} \) is the empty set since the equation \( 1 = 0 \) does not hold. Additionally, we take the polynomial ring \( K[\mathbf{x}] \) with no variables to be the same as the original field \( K \).

We next consider the evaluation of a polynomial function subject to the constraints in a system of polynomial equations. Recall that the power set \( 2^s \) of any set \( s \) is the set of all possible subsets of \( s \) (including by necessity the empty set and the original set \( s \) itself).

**Definition 2 (Solution-Value Set)** Consider a system of polynomial equations as described in Definition \[17\] along with an objective formula \( p \in K[\mathbf{x}] \) from the same polynomial ring. The solution-value set \( \mathcal{S}_A(p) \) of the objective \( p \) subject to the equations \( A \) is defined as the set of its feasible values when the equations in the system are satisfied:

\[
\mathcal{S}_A(p) := \{ p(\mathbf{x}) \in K : \mathbf{x} \in \mathcal{V}(A) \}
\]

in other words the image under \( p \) of the solution set \( \mathcal{V}(A) \). By its construction the solution-value set \( \mathcal{S}_A(p) \subseteq K \) must be a subset of the set of elementary values in the original field \( K \). Equivalently the solution-value set must be a member of the power set of \( K \):

\[
\mathcal{S}_A(p) \in 2^K
\]

Here using \( K \) to stand for the set of elementary values in the field as well as the field itself.

If the solution-value set \( \mathcal{S}_A(p) \) for a polynomial \( p \) is a singleton \( \{k\} \) that contains just one member \( k \in K \), then we describe \( p \) as necessarily \( k \) subject to \( A \); this is the unexceptional result. If the solution-value set is empty, then we describe \( p \) as unsatisfiable subject to \( A \); this means that the equations \( A \) in the system are inconsistent. Otherwise the solution-value set must have more than one member (including the special cases that it is infinite or uncountable) and we describe \( p \) as ambiguous subject to \( A \).

At the extremes the solution-value set \( \mathcal{S}_A(p) \) may be the empty set or the entire set of elementary values from the underlying algebraic structure \( K \).

Moving on, when using coefficients from the binary finite field \( \mathbb{F}_2 \) it is possible to simplify systems of polynomial equations in the following way.
Lemma 3 (Conjunction of Binary Constraints) Any system of polynomial equations with coefficients in the binary finite field $\mathbb{F}_2$ or with each polynomial constrained to the values 0 and 1 can be simplified to a single equation. For any $a, b \in \{0, 1\}$ the product $ab = 1$ if and only if both factors $a = 1$ and $b = 1$. From this it follows that any set $A$ of equations:

$$\{ q_1(x) = 0, q_2(x) = 0, \ldots, q_m(x) = 0 \}$$

with each $q_j \in \mathbb{F}_2[x]$ or each $q_j(x) \in \{0, 1\}$ for all feasible values of $x$, must have the same solution set as the single equation:

$$(q_1(x) + 1)(q_2(x) + 1) \cdots (q_m(x) + 1) = 1$$

In this case the set $A$ of equations given in Definition 1 can be replaced with the singleton $\{ q^* = 0 \}$ or the equivalent $\{ q^* + 1 = 1 \}$ whose solitary member uses the conjunction polynomial $q^*$ derived from the original equations:

$$q^* := (q_1(x) + 1)(q_2(x) + 1) \cdots (q_m(x) + 1) - 1$$

The conjunction polynomial produces the same solution set as the original equations:

$$V(q^* = 0) = V(q^* + 1 = 1) = V(q_1 = 0, q_2 = 0, \ldots, q_m = 0)$$

and consequently identical solution-value sets for any objective formula.

Note that in the special case $m = 0$ that there are no constraints we take the product of zero factors to be the multiplicative identity. Hence from $A = \{ \}$ we derive the conjunction polynomial $q^* = 1 - 1$ which gives the tautological constraint $0 = 0$.

Example 4 To illustrate these definitions, consider the following system of two simultaneous equations in the variables $x$ and $y$ (thus $n = 2$, $x_1 = x$, $x_2 = y$, $x = (x, y)$, and $m = 2$):

$$x(y + 1) = 0 \quad (17)$$
$$y(x + 1) = 0 \quad (18)$$

Expanding these polynomials yields the constraint set:

$$A := \{ xy + x = 0, xy + y = 0 \} \quad (19)$$

Let us proceed to evaluate this system of polynomial equations using two different algebraic structures: the real numbers $\mathbb{R}$ and the binary finite field $\mathbb{F}_2$.

First we assume that the variables $x$ and $y$ and their coefficients take real-number values. Let us use $V(A)_{\mathbb{R}}$ to denote the solution set to Equations (17) and (18) using the algebraic structure $\mathbb{R}$. According to Definition 1

$$V(A)_{\mathbb{R}} := \{ (x, y) \in \mathbb{R}^2 : \begin{cases} xy + x = 0 \\ xy + y = 0 \end{cases} \} \quad (20)$$

You can see from inspection that only two pairs $(x, y)$ of real numbers satisfy the equations, namely $(-1, -1)$ and $(0, 0)$. Thus we have the solution set:

$$V(A)_{\mathbb{R}} \Rightarrow \{ (-1, -1), (0, 0) \} \quad (21)$$

This variety $V(A)_{\mathbb{R}}$ is a subset of the affine space $\mathbb{R}^2$ of all pairs of real numbers. Given this solution set, it is straightforward to identify the solution-value sets for the objective formulas $x$, $y$, and $x - y + 1$ by substituting the values from $V(A)_{\mathbb{R}}$ into each objective:

$$S_A(x)_{\mathbb{R}} \Rightarrow \{ -1, 0 \} \quad (22)$$
$$S_A(y)_{\mathbb{R}} \Rightarrow \{ -1, 0 \} \quad (23)$$
$$S_A(x - y + 1)_{\mathbb{R}} \Rightarrow \{ 1 \} \quad (24)$$
Next if we use the binary finite field \( \mathbb{F}_2 \) as the algebraic structure for the variables \( x \) and \( y \) and their coefficients, then the polynomial system defined by Equations\(^1\) and \(^2\) has the solution set:

\[
\mathcal{V}(A)_{\mathbb{F}_2} := \left\{ (x,y) \in (\mathbb{F}_2)^2 : \frac{xy + x = 0}{xy + y = 0} \right\} \\
= \{(0,0), (1,1)\}  \quad (25)
\]

Recall that in the finite field \( \mathbb{F}_2 \) the sum \( 1 + 1 = 0 \) using modular arithmetic. Based on this solution set \( \mathcal{V}(A)_{\mathbb{F}_2} \), the solution-value sets for the objectives \( x, y \), and \( x - y + 1 \) are:

\[
\mathcal{S}_A(x)_{\mathbb{F}_2} \Rightarrow \{0,1\}  \quad (26)
\]
\[
\mathcal{S}_A(y)_{\mathbb{F}_2} \Rightarrow \{0,1\}  \quad (27)
\]
\[
\mathcal{S}_A(x - y + 1)_{\mathbb{F}_2} \Rightarrow \{1\}  \quad (28)
\]

In either polynomial ring \( \mathbb{R}[x,y] \) or \( \mathbb{F}_2[x,y] \) the formula \( x - y + 1 \) has only one feasible value given the constraints in Equations\(^1\) and \(^2\); it is necessarily 1. However, given these same constraints each formula \( x \) and \( y \) is ambiguous in either polynomial ring: the solution-value sets \{\(-1,0\)\} and \{\(0,1\)\} each have two members.

### 2.2 Formula Discovery

We now consider the inverse problem: subject to a set of polynomial equations, instead of computing the solution-value set for some polynomial objective formula, we must find the set of polynomials that yield a given solution-value set.

**Definition 5 (Inverse-Value Set)** Consider a system of polynomial equations as specified in Definition\(^1\) along with a query set \( s \subseteq K \) of elementary values from the algebraic field \( K \). The inverse-value set \( \mathcal{S}_A^{-1}(s) \) is the set of polynomials in the ring \( K[x] \) whose solution-value set is exactly \( s \):

\[
\mathcal{S}_A^{-1}(s) := \{ p \in K[x] : \mathcal{S}_A(p) = s \}
\]

When working with coefficients in the finite field \( \mathbb{F}_2 \), replacing the original set \( A \) of equations with the set \( \{q^* = 0\} \) using the conjunction polynomial from Lemma\(^3\) does not change the inverse-value sets. In that case, for every possible query \( s \subseteq K \):

\[
\mathcal{S}_A^{-1}_{{\{q^* = 0\}}}(s) = \mathcal{S}_A^{-1}_{{\{q_1=0, q_2=0, ..., q_m=0\}}}(s)
\]

If the set \( A \) of equations is inconsistent then according to Definition\(^2\) the solution-value set \( \mathcal{S}_A(p) \) for every polynomial \( p \in K[x] \) must be the empty set. Hence when \( A \) is inconsistent the inverse-value set \( \mathcal{S}_A^{-1}(\{\}) \) for the empty-set query yields the entire polynomial ring \( K[x] \), and the inverse-value set \( \mathcal{S}_A^{-1}(s) \) with any non-empty query \( s \) yields the empty set of polynomials. In other words, given infeasible constraints, every polynomial has no feasible value and no polynomial has any feasible value.

Considering a fixed set of equations \( A \), the inverse-value-set function \( \mathcal{S}_A^{-1} \) is the inverse image or preimage of the solution-value-set function \( \mathcal{S}_A \). Thus if \( \mathcal{S}_A^{-1}(s) = \{p_1, p_2, \ldots\} \) then for every polynomial \( p_i \) in this inverse-value set it holds that \( \mathcal{S}_A(p_i) = s \); and furthermore the members \( \{p_i\} \) of this inverse-value set are the only polynomials in the ring \( K[x] \) that have the solution-value set \( s \). In algebraic geometry, the inverse-value set \( \mathcal{S}_A^{-1}(\{0\}) \) with the query \( \{0\} \) is called the ideal of the affine variety \( \mathcal{V}(A) \): the set of every polynomial that vanishes on \( \mathcal{V}(A) \). The characteristics of ideals\(^1\) lead to the following special case.

**Lemma 6 (The Ideal from a Single Equation)** In the special case that the set \( A \) of equations contains a single equation \( q = 0 \) that is satisfiable, then every member of the inverse-value set \( \mathcal{S}_A^{-1}_{{\{q = 0\}}}(\{0\}) \) must be the product of some polynomial \( p \) from \( K[x] \) with the polynomial \( q \) from the equation; conversely every such product \( p \times q \) must be a member of the inverse-value set:

\[
\mathcal{S}_A^{-1}_{{\{q = 0\}}}(\{0\}) = \{p \times q : p \in K[x]\}
\]
The products \( p \times q \) are not necessarily distinct for different values of \( p \). Recall from Definition 5 that infeasible equations yield empty inverse-value sets for non-empty queries: thus \( \mathcal{S}_A^{-1}(\{0\}) = \{\} \) if the equation \( q = 0 \) is infeasible.

Since there are dedicated methods in algebraic geometry to compute the ideal generated by a set of polynomials (in my terminology the inverse-value set for the query \( \{0\} \)), it is useful to define inverse-value sets for queries other than \( \{0\} \) in terms of the ideal.

**Corollary 7 (Incrementing Inverse Polynomials)** For any query \( \{k\} \) that contains a solitary value \( k \in \mathcal{S} \), the inverse-value set \( \mathcal{S}_A^{-1}(\{k\}) \) can be calculated by adding that value \( k \) to every polynomial in the inverse-value set for the query \( \{0\} \):

\[
\mathcal{S}_A^{-1}(\{k\}) = \{ p + k : p \in \mathcal{S}_A^{-1}(\{0\}) \}
\]

In the case that the inverse-value set \( \mathcal{S}_A^{-1}(\{0\}) \) was generated from a single constraint as in Lemma 6, then the same incremented inverse-value set is also given by:

\[
\mathcal{S}_A^{-1}(\{q=0\}) = \{ p \times q + k : p \in \mathcal{S}[\mathcal{S}] \}
\]

This corollary allows closed-form description of inverse-value sets for singleton queries.

Continuing on, it is useful to appreciate that the set of polynomials over a finite number of variables with coefficients in a finite field is itself finite.

**Lemma 8 (Counting Polynomials in Finite Fields)** The polynomial ring \( \mathcal{S}_d[x] \) over \( n \) variables \( x := (x_1,x_2,\ldots,x_n) \) with coefficients in a finite field \( \mathcal{S}_d \) of order \( d \) (with \( d \) a prime number) contains exactly \( d^n \)

distinct polynomials, each of which can be expressed as a unique sum \( \sum_{i} c \cdot t \) of coefficients and monomials. In this sum each coefficient \( c \) is a member of the field \( \mathcal{S}_d \) and each monomial (power product) \( t \) is a product of the variables \( x_1 \) through \( x_n \) in which each variable is raised to a power between 0 and \( d - 1 \). There are \( d^n \) possible monomials.

Since (by Fermat’s Little Theorem) the identity \( a^d = a \) holds for every element \( a \in \mathcal{S}_d \) of a finite field of order \( d \) (with \( d \) a prime number), exponents higher than \( d - 1 \) are not required in monomials that use \( \mathcal{S}_d \). For example, using the finite field of order 2 and two variables \( x \) and \( y \), the resulting polynomial ring \( \mathcal{S}_2[x,y] \) has a set of \( 2^4 = 4 \) possible monomials:

\[
\{ x^1 y^1, x^1 y^0, x^0 y^1, x^0 y^0 \} \Rightarrow \{ xy, x, y, 1 \}
\]

(29)

In any polynomial in this ring, each of these 4 monomials must be assigned one of the 2 coefficients in \( \{0,1\} \). Thus there are \( 2^4 = 16 \) possible polynomials in \( \mathcal{S}_2[x,y] \); they are listed in Table 1. Alternatively, using coefficients from the finite field of order 3 there would be \( 3^4 = 19,683 \) possible polynomials in the corresponding ring \( \mathcal{S}_3[x,y] \).

**Example 9** We can enumerate all sixteen polynomials in the ring \( \mathcal{S}_2[x,y] \) with binary finite-field coefficients, and partition them into the four possible inverse-value sets (for the queries \( s = \{\} \), \( s = \{0\} \), \( s = \{1\} \), and \( s = \{0,1\} \)) given the constraints \( \mathcal{A} = \{ x+y = 0, xy = 0 \} \) from Equations 17 and 18. Table 1 shows these sixteen polynomials and their solution-value sets; Definition 10 below explains how the table was created. From the table you can see that there are four polynomials whose only feasible value is zero:

\[
\mathcal{S}_A^{-1}(\{0\}) \Rightarrow \{ p_1, p_7, p_{11}, p_{13} \} \Rightarrow \{ 0, x+y, xy+y, xy+x \}
\]

(30)

There are also four polynomials whose only feasible value is one:

\[
\mathcal{S}_A^{-1}(\{1\}) \Rightarrow \{ p_2, p_8, p_{12}, p_{14} \} \Rightarrow \{ 1, x+y+1, xy+y+1, xy+x+1 \}
\]

(31)
There are no polynomials whose solution-value set is empty (because the constraints are consistent):

$$\mathcal{S}_A^{-1}(\{\}) \Rightarrow \{\}$$  \hfill (32)

The remaining eight polynomials share the solution-value set \{0, 1\} of both binary finite-field values:

$$\mathcal{S}_A^{-1}(\{0, 1\}) \Rightarrow \{ p_3, p_4, p_5, p_6, p_9, p_{10}, p_{15}, p_{16} \}$$  \hfill (33)

Thus, using the constraints in Equations [17] and [18], every polynomial \( p_i \in F_2[x, y] \) is assigned to one of the four possible inverse-value sets for polynomials with coefficients in the binary finite field.

Some of these inverse-value sets can be derived in another way. Since we are using polynomials with coefficients in the binary finite field \( F_2 \), Lemma 5 states that the original constraints \( xy + x = 0 \) and \( xy + y = 0 \) in Equations [17] and [18] are equivalent to the single constraint:

\[
(xy + x + 1)(xy + y + 1) = 1
\]  \hfill (34)

Thus the conjunction polynomial \( q^* \) is given by:

\[
q^* := (xy + x + 1)(xy + y + 1) - 1 \Rightarrow x + y
\]  \hfill (35)

using modular arithmetic. Definition 5 assures us that for any query \( s \) the inverse-value set computed from the conjunction polynomial is the same as the original:

\[
\mathcal{S}_A^{-1}_{x+y=0}(s) = \mathcal{S}_A^{-1}_{xy+x=0, xy+y=0}(s)
\]  \hfill (36)

Now, Lemma 6 says that every polynomial in the inverse-value set \( \mathcal{S}_A^{-1}_{xy+y=0}(\{0\}) \) must be the product of some polynomial \( p \) in \( F_2[x, y] \) with the conjunction polynomial \( q^* = x + y \) just computed:

\[
\mathcal{S}_A^{-1}_{xy+y=0}(\{0\}) = \{ p \times (x + y) : p \in F_2[x, y] \}
\]  \hfill (37)

And indeed, multiplying any polynomial in the ring \( F_2[x, y] \) by the sum \( x + y \) will yield one of the four polynomials in \( \mathcal{S}_A^{-1}(\{0\}) \) listed in Equation [30]. For example, we have the following products (keeping in mind that the finite field \( F_2 \) uses integer arithmetic modulo 2 in which \( x^2 = x, 1 + 1 = 0 \), etc.):

\[
(y + 1)(x + y) \Rightarrow xy + x
\]  \hfill (38)

\[
(xy)(x + y) \Rightarrow 0
\]  \hfill (39)

\[
(xy + y)(x + y) \Rightarrow xy + y
\]  \hfill (40)

\[
(xy + x + y)(x + y) \Rightarrow x + y
\]  \hfill (41)

Furthermore, Corollary 7 states that for every polynomial \( p \) in the inverse-value set \( \mathcal{S}_A^{-1}_{xy+y=0}(\{0\}) \), the sum \( p + 1 \) must be a member of the related inverse-value set \( \mathcal{S}_A^{-1}_{xy-y=0}(\{1\}) \). You can see this illustrated by comparing Equations [30] and [31]

2.3 How to Solve It

The calculations required for formula evaluation and formula discovery using systems of polynomial equations with coefficients in finite fields can be performed by an extension of the truth-table methods of [21] and [32]. It is practical to carry out these calculations by hand with pen and paper for small problems. Alternatively, there are sophisticated algebraic geometry methods implemented in several widely-available computer algebra systems that accomplish the necessary calculations in a more efficient, robust, and scalable manner (using coefficients from finite fields as easily as rational or real coefficients). These computational methods are derived from the Gröbner-basis algorithms for solving polynomial equations that were invented by Buchberger in the 1970s [6]. Commands to perform the requisite calculations in a computer algebra system are included in Section 4. Table-based inference is explained presently.
For a system of $m$ polynomial equations with $n$ variables and coefficients in a finite field with $d$ elements, naive table-based inference requires the construction of a table with $d^n$ rows and $d^n$ columns. The resulting $d^{2n}$ entries must be exhaustively enumerated. The following definition introduces the value worksheet data structure and an inference algorithm based on it. Each row of the value worksheet contains the same information as a traditional logical truth table, but in a flattened form.

**Definition 10 (Table-Based Inference with Finite Fields)** For a system of polynomial equations with coefficients in a finite field $\mathbb{F}_d$ of some prime order $d$, the solution set, solution-value sets, and inverse-value sets described in Definitions 7 and 8 can be computed using the following algorithm.

(a) Construct the value worksheet $W(A)$:

(i) Make a table $W(A)$ with a row for every polynomial $p_i \in \mathbb{F}_d[x]$ and a column for every point $x_i \in (\mathbb{F}_d)^n$. Following Lemma 5 this table $W(A)$ will have $d^n$ rows and $d^n$ columns (excepting the row and column labels).

(ii) For every polynomial $p_i \in \mathbb{F}_d[x]$ and every point $x_i \in (\mathbb{F}_d)^n$, use the value $p_i(x_i)$ of that polynomial evaluated at that point as the entry at row $i$ and column $k$ of the table.

(b) Compute the solution set $\mathcal{S}(A)$ from the constraint polynomials $A$:

(i) If $d = 2$ and $m \neq 1$, replace the set $A$ of equations with the singleton $\{q^* = 0\}$ using the conjunction polynomial described in Lemma 5 (this step may be omitted for pedagogic purposes).

(ii) For each constraint $q_j = 0 \in A$ find the matching polynomial $p_i = q_j$ at row $i$ in the table $W(A)$. Examining this row $i$ of the table, mark as infeasible every column $k$ for which the value $p_i(x_i)$ at row $i$ and column $k$ is not zero. If there are no constraint polynomials then all columns remain unmarked.

(iii) The unmarked columns identify the feasible points that constitute the solution set $\mathcal{S}(A)$. If all columns are marked as infeasible then the solution set is empty.

(c) Compute the solution-value set $\mathcal{T}_A(p_i)$ for every possible polynomial objective $p_i \in \mathbb{F}_d[x]$:

(i) Add a column labeled $\mathcal{T}_A(p_i)$ to the table $W(A)$ for solution-value sets.

(ii) For each polynomial $p_i$ at row $i$, the solution-value set is the set of table entries at the unmarked columns. Thus the solution-value set for each polynomial $p_i$ is the set of every value $p_i(x_i)$ taken by that polynomial at a feasible point $x_i$ indicated as an unmarked column $k$.

(iii) If there are no unmarked columns then every solution-value set is empty.

(d) Compute the inverse-value set $\mathcal{T}_A^{-1}(s)$ for every possible solution-value-set query $s \subseteq \mathbb{F}_d$:

(i) Examine the entries in the solution-value-set column $\mathcal{T}_A(p_i)$. For every unique entry $s$ in that column, select the rows $i_1, i_2, \ldots$ such that $\mathcal{T}_A(p_{i_1}) = s, \mathcal{T}_A(p_{i_2}) = s$, and so on. The corresponding polynomials $p_{i_1}, p_{i_2}, \ldots$ constitute the inverse set $\mathcal{T}_A^{-1}(s)$.

(ii) The inverse-value set for every query $s \subseteq \mathbb{F}_d$ that does not appear as an entry in the solution-value-set column $\mathcal{T}_A(p_i)$ is empty.

**Example 11** Table 1 shows the value worksheet $W(A)$ constructed according to Definition 10 from the system of constraints $A = \{xy + x = 0, xy + y = 0\}$ in Equation 10 the polynomial ring $\mathbb{F}_2[x, y]$ with binary finite-field coefficients is used. There are two ways to mark infeasible points. Using the original constraint polynomials we note that $xy + x$ appears as $p_{13}$ and $xy + y$ appears as $p_{11}$ in $W(A)$. Examining row 11 we see that $p_{11}(0, 1) \neq 0$, therefore we mark the column for $(0, 1)$ infeasible. In row 13 the entry $p_{13}(1, 0) \neq 0$ hence we mark the column for $(1, 0)$ infeasible. Alternatively, using the conjunction polynomial $q^* = x + y$ gives equivalent results. The polynomial $x + y$ appears at row 7 in $W(A)$ and in this row both entries $p_7(0, 1)$ and $p_7(1, 0)$ have nonzero values; hence the corresponding columns are marked infeasible.
Table 1  The value worksheet \( W(A) \) for the equations \( A = \{ xy+x=0, xy+y=0 \} \) using the polynomial ring \( \mathbb{F}_2[x,y] \). The points \((0,1)\) and \((1,0)\) are marked infeasible; the solution set \( y(A)_{\mathbb{F}_2} \) contains the remaining points \((0,0)\) and \((1,1)\).

| \( i \) | \( p_i(x,y) \) | \( p_i(0,0) \) | \( p_i(0,1) \) | \( p_i(1,0) \) | \( p_i(1,1) \) | \( S_{xy+x=0,xy+y=0}(p_i) \) |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | \( \{0\} \) |
| 2 | 1 | 1 | 1 | 1 | 1 | \( \{1\} \) |
| 3 | \( y \) | 0 | 1 | 0 | 1 | \( \{0,1\} \) |
| 4 | \( y+1 \) | 1 | 0 | 1 | 0 | \( \{0,1\} \) |
| 5 | \( x \) | 0 | 0 | 1 | 1 | \( \{0,1\} \) |
| 6 | \( x+1 \) | 1 | 1 | 0 | 0 | \( \{0,1\} \) |
| 7 | \( x+y \) | 0 | 1 | 1 | 0 | \( \{0\} \) |
| 8 | \( x+y+1 \) | 1 | 0 | 0 | 1 | \( \{1\} \) |
| 9 | \( xy \) | 0 | 0 | 0 | 1 | \( \{0,1\} \) |
| 10 | \( xy+1 \) | 1 | 1 | 0 | 0 | \( \{0,1\} \) |
| 11 | \( xy+y \) | 0 | 1 | 0 | 0 | \( \{0\} \) |
| 12 | \( xy+y+1 \) | 1 | 0 | 1 | 1 | \( \{1\} \) |
| 13 | \( xy+x \) | 0 | 0 | 1 | 0 | \( \{0\} \) |
| 14 | \( xy+x+1 \) | 1 | 1 | 0 | 1 | \( \{1\} \) |
| 15 | \( xy+x+y \) | 0 | 1 | 1 | 1 | \( \{0,1\} \) |
| 16 | \( xy+x+y+1 \) | 1 | 0 | 0 | 0 | \( \{0,1\} \) |

Infeasible? \( \square \quad \times \quad \times \quad \square \)

Each solution-value set \( S_A(p_i) \) at row \( i \) is the set \( \{ p_i(0,0), p_i(1,1) \} \) of values taken by the polynomial \( p_i \) at the feasible points (the unmarked columns in \( W(A) \)). For example in row 8 we have \( p_8 = x+y+1 \) and both \( p_8(0,0) = 1 \) and \( p_8(1,1) = 1 \); thus the solution-value set \( S_A(x+y+1) \Rightarrow \{1\} \). The inverse-value sets are given by selected rows: for example rows 1, 7, 11, and 13 share the solution-value set \( \{0\} \) so the inverse-value set \( S_A^{-1}(\{0\}) \) is \( \{ p_1, p_7, p_{11}, p_{13} \} \). The inverse-value sets for the queries \( \{1\} \) and \( \{0,1\} \) are constructed in a similar way. However, since no polynomial has an empty solution-value set (the entry \( \{\} \) does not appear in the last column of the value worksheet), the inverse-value set of the empty set is itself empty: \( S_A^{-1}(\{\}) \Rightarrow \{\} \).

3  DYNAMICAL SYSTEMS FROM REFERENCES TO SOLUTIONS

Now let us consider the complexity added to a system of equations when it is permitted to refer to the solution of that system of equations within the system itself. This type of self-reference is an elaborate recurrence relation, which can either be interpreted as such to define a discrete dynamical system, or interpreted in a static way to provide an additional simultaneous equation. Although much of the terminology and notation for ‘dynamical systems’ is relatively new (following a resurgence of interest in the 1970s, especially in nonlinear and chaotic dynamical systems), the mathematical treatment of recursion is quite old. For example, what we know as the Fibonacci sequence has been studied in various guises since at least the Middle Ages; and methods for finite differences and difference equations have been developed since the work of Newton and then Taylor around the turn of the 18th century [3]. The modern treatment of dynamical systems [14] [18] dates from Poincaré’s work at the end of the 19th century. It happens that the dynamical systems that we will encounter in the study of logic are quite simple: discrete time, finite phase-space, first-order, autonomous, and usually linear.

3.1 Extended Systems of Polynomial Equations

In order to develop a computable representation of solution self-reference, we introduce several new features to the systems of polynomial equations described in Section2 parameters, equation templates, iterative assignments, and solution references. The idea is that each constraint may be specified as a template instead of as a simple polynomial equation; the parameters of the templates are allowed to refer to solution sets and solution-value sets from the systems of equations in which they reside.
Definition 12 (Parametric System of Polynomial Equations) Consider a system of polynomial equations as specified in Definition [1] with variables \( x := (x_1, x_2, \ldots, x_n) \) and coefficients in an algebraic field \( K \); the polynomials in the system are members of the ring \( K[x] \). We introduce a tuple \( \Theta := (\theta_1, \theta_2, \ldots, \theta_j) \) of parameters with the requirement that each parameter \( \theta_j \) must take a value in some specified set \( U_j \). Thus the set \( U \) of possible values for \( \Theta \) is given by the Cartesian product:

\[
U := U_1 \otimes U_2 \otimes \cdots \otimes U_\ell
\]

For each variable \( x_i \) and each parameter \( \theta_j \) we add a data-type constraint \( \tau \) to identify the appropriate set of possible values:

\[
\tau_{x_i} : x_i \in K
\]
\[
\tau_{\theta_j} : \theta_j \in U_j
\]

Instead of using a simple set \( A \) of polynomial equations as described in Definition [1], an extended polynomial system is specified using a set \( A(\Theta) \) of parametric constraint templates and a set \( D \) of assignment templates, in addition to the above data-type constraints. In the set \( A(\Theta) := \{ \alpha_1, \alpha_2, \ldots, \alpha_m \} \) each constraint template \( \alpha_i \) is a function of the parameters \( \Theta \):

\[
\alpha_i : q_i(\Theta) = 0
\]

such that for any instantiation of the parameter values \( \Theta \) the template \( \alpha_i \) reduces to an ordinary polynomial equation with \( q_i(\Theta) \in K[x] \). In the set \( D := \{ \delta_{\theta_1}, \delta_{\theta_2}, \ldots, \delta_{\theta_j} \} \) each assignment template \( \delta_j \) specifies the value to be assigned to the corresponding parameter \( \theta_j \) at every iteration, using some parameter-updating function \( \lambda_j \) whose arguments may include the solution set \( \forall(A(\Theta)) \) to the extended system of equations being defined, as well as any parameter including \( \theta_j \) itself (here the double left arrow \( \leftarrow \) denotes assignment):

\[
\delta_{\theta_j} : \theta_j \leftarrow \lambda_j(\theta_1, \theta_2, \ldots, \theta_l, \forall(A(\Theta)))
\]

In particular a parameter-updating function \( \lambda_j \) may use the solution-value set \( \forall(\alpha_i(\Theta)) \) for some objective formula \( p \) or the cardinality \( |\forall(\alpha_i(\Theta))| \) of such a solution-value set. Although every parameter-updating function \( \lambda_j \) that assigns a value to a parameter \( \theta_j \) must return a value in the designated set \( U_j \) of possible values for that parameter, the updating functions are not required to be polynomial. If an explicit updating function for any parameter \( \theta_j \) is omitted, then the identity function \( \lambda_j(\Theta) : \Theta \) is used as a default.

According to Definition [12] the self-referential quadratic system in Equations [3] and [4] can be specified as the following constraint and assignment templates:

\[
\tau_x : x \in \mathbb{R}
\]
\[
\tau_c : c \in \{0, 1, 2\} \subset \mathbb{R}
\]
\[
\alpha_1 : 2x^2 + 3x + c = 0
\]
\[
\delta_c : c \leftarrow |\forall(\alpha_1(x))|
\]

(42)

Here the variable \( x \) takes real values and the possible values of the single parameter \( c \) are the set \( U := \{0, 1, 2\} \) of integers between 0 and 2. This parameter \( c \) is assigned the cardinality of the solution-value set for the objective \( x \) (subject to the equation \( \alpha_1 \)) by the updating function \( \lambda_1(c) \) specified by the assignment template \( \delta_c \).

For a different example, a system to generate Fibonacci-like sequences can be specified with two parameters and a pair of assignment templates, without the use of any conventional variables or any constraint templates:

\[
\tau_a : a \in \mathbb{Z}
\]
\[
\tau_b : b \in \mathbb{Z}
\]
\[
\delta_a : a \leftarrow b
\]
\[
\delta_b : b \leftarrow a + b
\]

(43)

We next develop an algorithm to make explicit the discrete dynamical system implied by a parametric system of polynomial equations.
The evolution function \( F \) is given by the tuple of parameter-updating functions:

\[
F(\Theta) = (\lambda_1(\Theta), \lambda_2(\Theta), \ldots, \lambda_c(\Theta))
\]

This is the usual situation with recurrence relations in general algebra.

Second, in the case that the parameter domain \( U \) is countable and finite then we can proceed with hypothetico-deductive analysis to derive \( F \) by computing a mapping \( \Theta_i \mapsto \Theta'_i \) for every parameter value \( \Theta_i \in U \). To express the hypothesis that the parameters \( \Theta \) have some particular value \( \Theta_i \), each parameter \( \Theta_j \) is assigned a constant value \( \hat{\Theta}_j \) from its domain \( U_j \). Using these values the constraint templates are instantiated into an ordinary set \( A(\Theta_i) \) of polynomial equations:

\[
A(\Theta_i) := \{ q_1(\Theta_i) = 0, q_2(\Theta_i) = 0, \ldots, q_m(\Theta_i) = 0 \}
\]

The solution set \( \mathcal{V}(A(\Theta_i)) \) for these instantiated equations is computed by the table-based algorithm in Definition 10 or by a general algebraic geometry method as appropriate. Following this solution subroutine the assignment templates in the set \( D \) are processed: an updated value \( \hat{\Theta}_i \) is computed for every parameter via its parameter-updating function, using as arguments the solution-value set \( \mathcal{V}(A(\Theta_i)) \) just computed along with the input values of the parameters:

\[
\hat{\Theta}_i = \lambda_j(\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_i, \mathcal{V}(A(\Theta_i)))
\]

The desired evolution function \( F \) must map the hypothesized parameter values to their updated counterparts:

\[
(\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_i) \mapsto (\theta'_1, \theta'_2, \ldots, \theta'_i)
\]

The set of mappings \( \Theta_i \mapsto \Theta'_i \) for every parameter value \( \Theta_i \in U \) completely defines the evolution function \( F \). When the parameter domain \( U \) is a subset of the field \( K \) used as polynomial coefficients, then the evolution function \( F \) can be expressed in closed form using polynomial interpolation.

To illustrate the first case in Definition 13 the evolution function for the Fibonacci-like parametric system in Equation 43 is simply:

\[
F(a, b) = (b, a + b)
\]

using the parameter-updating functions \( \lambda_1(a) : b \) and \( \lambda_2(b) : a + b \) specified by the assignment templates \( \hat{\delta}_a \) and \( \hat{\delta}_b \) in Equation 43.

The requisite calculations for the second case in Definition 13 can be organized in a state-transition worksheet constructed as follows. In each row of the state-transition worksheet we record an index \( i \), some value \( \Theta_i \) from the set \( U \), the instantiation \( A(\Theta_i) \) of the constraint templates at that value, the relevant feature of the solution-value set
\(Y(A(\Theta_t))\) given those instantiated equations, and the value \(F(\Theta_t)\) of the successor that was computed from \(Y(A(\Theta_t))\) and \(\Theta_t\) by processing the assignment templates according to Definition 13. Each row of the state-transition worksheet gives a specific value \(F(\Theta_t)\) of the evolution function for the argument \(\Theta_t\). The complete evolution function \(F(\Theta)\) can be represented as a simple transition matrix or as a state-transition table; or using the method described in Lemma 19 below, \(F(\Theta)\) can be specified as a polynomial with coefficients in a finite field.

Table 2 shows the state-transition worksheet for the parametric system in Equation 42 which uses a single parameter \(c\) with domain \(U = \{0, 1, 2\}\). In this case the relevant feature of the solution set is the solution-value set \(S_{\mathrm{A(c)}}(x)\) for the objective \(x\). This worksheet constructs the following evolution function:

\[
F : \{0, 1, 2\} \rightarrow \{0, 1, 2\} \\
0 \rightarrow 2 \\
1 \rightarrow 2 \\
2 \rightarrow 0
\] (45)

The function-development method of Lemma 19 below enables us to construct a polynomial function that matches any transition function \(F : U \rightarrow U\) for a finite phase space \(U\) (after assigning integers to identify the states if they are not already numeric); it happens that \(F(c) : = -c^2 + c + 2\) agrees with the mappings in Equation 45 (using the polynomial ring \(\mathbb{R}[c]\) with real coefficients).

**Definition 14 (The Derived Dynamical System)** A parametric system of polynomial equations as described in Definition 13 in turn defines a discrete-time dynamical system with state \(\Theta\) whose phase space is the set \(U\) of parameter values and whose evolution function is \(F(\Theta)\) from Definition 13. The set \(\{0, 1, 2, \ldots\}\) of nonnegative integers is used as the domain for the evolution (time) parameter \(t\). The state \(\Theta_t\) of the dynamical system at any time \(t\) is the result of the \(t\)-fold composition of the evolution function \(F\) applied to the initial state \(\Theta_0\):

\[\Theta_t \leftarrow F^{(t)}(\Theta_0)\]

For example, given state \(\Theta_0\) at time \(t = 0\) the state of the system at time \(t = 3\) is the composed value \(F^{(3)}(\Theta_0) = F(F(F(\Theta_0)))\). Following the usual conventions for discrete dynamical systems, a sequence of successive states \(\Theta_t, F(\Theta_t), F(F(\Theta_t)), \ldots\) constitutes an orbit. Furthermore a state \(\Theta_{t*}\) is a fixed point in the dynamical system exactly if it is its own successor: \(\Theta_{t*} = F(\Theta_{t*})\). We categorize a dynamical system by the number of fixed points it has. Let us say that a dynamical system is steady if it has one fixed point; unsteady if it has no fixed points; and contingent if it has more than one fixed point.

Note that this categorization concerns the state \(\Theta\) of the dynamical system indicated by the parameters \((\theta_1, \theta_2, \ldots, \theta_t)\) rather than the solution-value set for any objective formula per se; it could happen in an unsteady system that the sequences of solution-value sets for some formulas are nonetheless monotonous. Note also that if a system of equations has no parameters, then the dynamical system extracted according to Definition 14 will have the phase space \(U = \{\}\) containing one state which is the empty tuple (\(\)\). In this case the state-transition function \(F() : ()\) is the identity function and the solitary state must be a fixed point; the system is trivially steady.

By the categorisation scheme in Definition 14 the dynamical system derived from Equation 42 for the quadratic equation \(2x^2 + 3x + c = 0\) is unsteady; as the graph in Equation 15 shows there is a periodic cycle and there are no fixed points. In contrast the Fibonacci-like system derived from Equation 43 is steady; as we shall see there is one fixed point at \((a, b) = (0, 0)\) even though the orbits through the other points do not converge.

### 3.2 Static and Dynamic Interpretations

A dynamical system from Definition 14 can be interpreted in two different ways to evaluate an objective formula. In the dynamic interpretation, the orbits in the dynamical system are used to generate infinite sequences of solution-value sets for the objective (a sequential report). In the static interpretation, the evolution function is used to provide an additional static constraint limiting attention to the fixed points; what is reported is the union of solution-value sets for the objective from these fixed points (a stationary report). Such static and dynamic interpretations of a self-referential system of equations reflect subtly different views; it is fine to try either or both for any given problem.
Definition 15 (Collected Sequences of Solution-Value Sets) Consider a dynamical system derived as in Definitions 13 and 14 from a parametric polynomial system of equations as described in Definition 12. For any given objective formula \( p \in K[x] \) or \( p \in \Theta \), the dynamical system specified by \( U \) and \( F(\Theta) \) encodes an infinite sequence of solution-value sets for \( p \) using each state \( \Theta_j \in U \) as an initial condition. Each infinite sequence \( \mathcal{S} \) of solution-value sets results from solving the systems of equations \( A(\Theta_j) \) instantiated from the constraint templates at successive states \( \Theta_0, \Theta_1, \Theta_2, \ldots \):

\[
\mathcal{S}_{A(\Theta)}(p | \Theta_j) := \left\{ A_{A(\Theta_0)}(p), A_{A(\Theta_1)}(p), A_{A(\Theta_2)}(p), \ldots \right\}
\]

With reference to Definition 13 when the objective \( p \) refers to a parameter \( \theta_j \) then the initial (hypothesized) value \( \hat{\theta}_j \) for the current state should be used to evaluate that objective, instead of the updated parameter value \( \theta'_j \).

The collection \( \mathcal{S}^* \) of solution-value set sequences is defined as a relation between the states in \( U \) and the infinite sequences that arise from them. For every state \( \Theta_j \in U \) the collection contains a mapping to the sequence with the respective initial condition:

\[
\mathcal{S}_{A(\Theta)}^*(p) := \left\{ \Theta_j \mapsto \mathcal{S}_{A(\Theta)}(p | \Theta_j) : \Theta_j \in U^* \right\}
\]

Such a collection is usually written in square brackets as illustrated in the examples. If every solution-value set in a sequence has exactly one member (which is always the case when the objective is a parameter), then the braces around those sets may be omitted to simplify notation: thus the sequence \( \{\{k_0\}, \{k_1\}, \{k_2\}, \ldots\} \) could instead be written \( (k_0, k_1, k_2, \ldots) \).

For the parametric system in Equation 42, Equation 11 in the introduction already illustrated the collection \( \mathcal{S}^*_{A(\alpha_1)}(x) \) of solution-value-set sequences for the objective formula \( x \) (with the set brackets omitted since each solution-value set is a singleton). Likewise Equation 12 shows the collection \( \mathcal{S}^*_{A(\alpha_1)}(c) \) of solution-set sequences for the objective \( c \).

Note that you can read the sequences in either collection from the graph in Equation 13.

Considering the Fibonacci example, the parametric system in Equation 43 specifies a dynamical system with phase space \( U = \mathbb{Z}^2 \), state \( \Theta = (a, b) \), and evolution function \( F(a, b) : (a, b + a) \). Using this dynamical system the collection of solution-value-set sequences for the objective parameter \( a \) is given by:

\[
\mathcal{S}_{A(\alpha_1)}^*(a) = \begin{bmatrix}
(a, b)_0 = (0, 0) & \mapsto & (0, 0, 0, 0, 0, 0, \ldots) \\
(a, b)_0 = (0, 1) & \mapsto & (0, 1, 1, 2, 3, 5, 8, \ldots) \\
(a, b)_0 = (2, 1) & \mapsto & (2, 1, 3, 4, 7, 11, 18, \ldots) \\
& \vdots & 
\end{bmatrix} \quad (46)
\]

The initial condition \((a, b)_0 = (0, 1)\) yields the familiar Fibonacci sequence. Again since each solution-value set is a singleton the braces around sequence elements have been omitted.

The dynamical system can instead be interpreted in a static way to examine its fixed points.

Definition 16 (Simultaneous Solution for Fixed Points) Consider a parametric system of polynomial equations as described in Definition 12 and its extracted evolution function \( F(\Theta) \) derived by Definition 13 in the special case that the domain \( U_j \) for each parameter \( \theta_j \) in \( \Theta := (\theta_1, \theta_2, \ldots, \theta_s) \) is a subset of the algebraic structure \( K \) used for the conventional variables and polynomial coefficients in the problem. In this special case the solutions from the fixed points in the dynamical system can be computed in an alternative way by solving an extended system of static equations in which each parameter \( \theta_j \) is treated as an indeterminate along with the conventional variables in \( \mathbf{x} := (x_1, x_2, \ldots, x_n) \). The extended static system includes the data-type constraints \( \tau \) and a polynomial equation transcribed from each constraint template \( \alpha \) and from each assignment template \( \delta \):

\[
\begin{align*}
\tau_{x_j} : & \quad x_j \in K \\
\tau_{\theta_j} : & \quad \theta_j \in U_j \subseteq K \\
\alpha_i : & \quad q_i = 0 \\
\delta_{\theta_j} : & \quad \theta_j = \lambda_j(\Theta, \varphi(A))
\end{align*}
\]
Each parametric constraint polynomial \( q_i \) is now treated as a member of the extended polynomial ring:

\[
K[x_1, x_2, \ldots, x_n, \theta_1, \theta_2, \ldots, \theta_\ell]
\]

Note that in this alternative formulation, fixed points that give empty solution-value sets are not reported.

By Definition 14 the parametric quadratic system in Equation (42) yields the following system of static equations:

\[
\begin{align*}
\tau_\alpha &: \quad a \in \mathbb{Z} \\
\tau_b &: \quad b \in \mathbb{Z} \\
\delta_\alpha &: \quad a = b \\
\delta_b &: \quad b = a + b
\end{align*}
\]

using the polynomial interpolation of the extracted evolution function \( F(c) \) shown in Equation (45). This system of equations is infeasible; thus the solution-value set \( \mathcal{S}_{\tau_a, \tau_b, \alpha, \beta} (x) \) evaluates to the empty set, as does the solution-value set for any other objective formula subject to these unsatisfiable constraints.

For the Fibonacci-like system in Equation (43), transcribing the assignment templates into static equations yields the following set \( A \) of equations (using the extended polynomial ring \( \mathbb{Z}[a, b] \) with integer coefficients):

\[
\begin{align*}
\tau_\alpha &: \quad a \in \mathbb{Z} \\
\tau_b &: \quad b \in \mathbb{Z} \\
\delta_\alpha &: \quad a = b \\
\delta_b &: \quad b = a + b
\end{align*}
\]

The unique solution \((a, b) = (0, 0)\) identifies the fixed point noted earlier. Consequently for the objective \( a \) the solution-value set \( \mathcal{S}_{\tau_a, \tau_b, \alpha, \beta} (a) \Rightarrow \{0\} \). Interpreting its familiar recurrence as a static constraint, it is appropriate to say that ‘the Fibonacci number’ is zero.

**Example 17** Consider a new quadratic equation \( \frac{1}{2}x^2 + 3bx + \frac{11}{2} = 0 \) in which the coefficient \( b \) is defined to be the number of real solutions to the equation in which it appears. This problem is specified as the following parametric system, using the conventional variable \( x \) and the parameter \( b \):

\[
\begin{align*}
\tau_\alpha &: \quad x \in \mathbb{R} \\
\tau_b &: \quad b \in \{0, 1, 2\} \subset \mathbb{R} \\
\alpha_1 &: \quad x^2 + 3bx + \frac{11}{2} = 0 \\
\delta_b &: \quad b \leq |\mathcal{S}_a (x)|
\end{align*}
\]

By Definition 14 these templates yield a dynamical system with the phase space \( U = \{0, 1, 2\} \) and an evolution function \( F(b) \) that must satisfy the following criteria:

\[
F(0) = 0, \quad F(1) = 0, \quad F(2) = 2
\]

The polynomial \( F(b) : b^2 - b \) (constructed by Lemma 19) meets these criteria for \( b \in \{0, 1, 2\} \). The derived dynamical system, labeled with the solution-value sets discussed next, is displayed in this graph:

\[
\begin{align*}
\{\} & \quad \circlearrowright \quad 0 \\
\{\} & \quad \circlearrowright \quad 1 \\
\{−11, −1\} & \quad \circlearrowleft \quad 2
\end{align*}
\]

There are fixed points at \( b = 0 \) and \( b = 2 \) and no nonconvergent orbits; in the terminology of Definition 14 this dynamical system contingent. By Definition 15 the dynamical system yields the following collection of solution-value sets for the objective \( x \):

\[
\mathcal{S}_{\alpha_1} (x) \Rightarrow \begin{bmatrix}
b_0 = 0 & \mapsto & \{\}, \{\}, \ldots \\
b_0 = 1 & \mapsto & \{\}, \{\}, \ldots \\
b_0 = 2 & \mapsto & \{−11, −1\}, \{−11, −1\}, \ldots
\end{bmatrix}
\]

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Reviewing the fixed points and their associated solution-value sets shown in the graph in Equation 51, the dynamic interpretation shows that there are two cases in which the equation \( \frac{1}{2}x^2 + 3bx + \frac{11}{2} = 0 \) has exactly one real root: when \( b = 0 \) and there are no solutions, and when \( b = 2 \) and there are two solutions (namely \(-11\) and \(-1\)).

Finally, using Definition 16 and the polynomial interpolation \( F(b) : b^2 - b \) of the evolution function, the templates in Equation 49 can be transcribed into a static set of equations using polynomials in the extended ring \( \mathbb{R}[x, b] \):

\[
\begin{align*}
\tau_x : & \quad x \in \mathbb{R} \\
\tau_b : & \quad b \in \{0, 1, 2\} \subset \mathbb{R} \\
\alpha_1 : & \quad \frac{1}{2}x^2 + 3bx + \frac{11}{2} = 0 \\
\delta_b : & \quad b = b^2 - b
\end{align*}
\]

(53)

The static polynomial system in Equation 53 has two solutions \((x, b) = (-11, 2)\) and \((x, b) = (-1, 2)\); these identify the fixed-point solutions. Consequently in this static formulation the solution-value set for \( x \) is given by \( \mathcal{S}_{\{\tau, \tau, \alpha_1, \delta_0\}}(x) \Rightarrow \{-11, -1\} \). Note that this static interpretation does not reveal that \( b = 0 \) gives a valid instantiation of the template equation \( \alpha_1 \), since in that instantiation there is no real solution for \( x \).

4 TRANSLATION FROM LOGIC INTO ALGEBRA

Let us now visit the foundations of logic and explore the relationship between logical reasoning with truth values and mathematical reasoning with ordinary numbers. What we call logic—reasoning with binary truth values, formulas, axioms, theorems, and proof—is nothing more than algebra presented in peculiar notation. This basic equivalence was asserted by Boole in his innovative mathematical treatment of logic [4, 5]. Indeed, it seems apparent that in both logic and algebra we find common concepts of number, operation and formula; variable, function, equation, and solution to equations; recursion and infinite sequence. However, through a cascade of historical misunderstanding and obfuscation (much of it associated with Gödel’s incompleteness argument), the intrinsic unity of logic and mathematics has not achieved universal acceptance. Especially, recurrence relations and dynamical systems have not been recognized as such in the contexts of logic and set theory.

Considering these things I have extended Boole’s ideas, and the two key principles presented in the introduction of this essay, into several Articles of Algebraic Translation for Logic:

I. Logical formula \( \leadsto \) Polynomial formula

II. Axiom \( \leadsto \) Polynomial equation

III. Solution-value set \( \leadsto \) Truth value (when self-reference is forbidden)

IV. Polynomial with solution-value set \( \{1\} \leadsto \) Theorem

V. Reference to provability \( \leadsto \) Recurrence relation

VI. Features of discrete dynamical system \( \leadsto \) Truth value (when self-reference is allowed)

Here the wavy arrow \( \leadsto \) should be read ‘translates as,’ with the idea that these general principles subsume many specific functional mappings. By analogy the general principle:

Roman number \( \leadsto \) Hindu–Arabic number

includes the specific mappings III \( \leadsto 4 \) and IV \( \leadsto 4 \). Gödel’s formula \([R(q); q]\), which denies its own provability, is analyzed during the presentation of Articles V and VI. For concreteness, the examples in this section are accompanied by commands to perform their calculations in the computer algebra system Mathematica [33]. Carnielli has studied Boole’s polynomial formulation of mathematical logic and uses related ideas in his Polynomial Ring Calculus [7, 11].

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### 4.1 Logical Formulas as Polynomials

Let us consider two methods to represent logical formulas as polynomial expressions: the original method created by Boole, and a revised method using finite fields and modular arithmetic. In either case we start with a logical formula in the propositional calculus built from some atomic formulas and it is truth tables that are directly translated into polynomial functions; in fact, each translated polynomial can be considered a closed-form representation of a truth table. Many different logical formulas might share a given truth table; hence many different logical formulas might have the same polynomial translation. Translation is not limited to 2-valued logic; in general any finite number of elementary truth values can be accommodated.

Although I discuss polynomial translation, for Boole polynomials were not used to translate from some other symbolic notation for logical formulas; they were his only mathematical representation. Notably, Frege’s *Begriffsschrift* was published many years after Boole’s lifetime, as were the subsequent works from Peano and Hilbert in which the ideography of contemporary logical notation was developed.

The detailed translation methods described below have been used to generate a table of polynomials representing common logical formulas, including the elementary truth values, the unary negation of a formula, and the common binary operations applied to a pair of formulas. These translations, which are often called the Stone isomorphisms after [28], appear in Table 3. A practical way to translate a simple logical formula is to apply the substitutions listed in the table recursively, until all traditional logical operations have been converted into polynomial form. Using Table 3 it is possible to translate any formula from the propositional calculus into a polynomial with either real-number coefficients, or coefficients in the binary finite field \( \mathbb{F}_2 \).

For example the logical formula \( y \wedge (z \oplus w) \), which is the logical conjunction of the atomic proposition \( y \) with the exclusive disjunction (XOR) of the atomic propositions \( z \) and \( w \), can be translated to a polynomial in the ring \( \mathbb{R}[w,y,z] \) by first substituting the polynomial form of the inner disjunction, then substituting the outer conjunction, and finally simplifying the whole expression using standard algebra:

\[
\begin{align*}
y \wedge (z \oplus w) & \quad \rightarrow \quad y \wedge (z + w - 2zw) \\
& \rightarrow (z + w - 2zw) \\
& \rightarrow yz + yw - 2yzw
\end{align*}
\]

Besides translating the entire formula \( y \wedge (z \oplus w) \), each variable in it must be constrained to limit its possible values to 0 and 1. This is accomplished by the equations \( w^2 = w \), \( y^2 = y \), and \( z^2 = z \) (which would be tautological if we were using constraints in \( \mathbb{F}_2 \)).

| Logical | Polynomial in \( \mathbb{R}[x] \) | Polynomial in \( \mathbb{F}_2[x] \) | Description |
|---------|----------------------------------|----------------------------------|-------------|
| \( T \) | 1 | 1 | True |
| \( F \) | 0 | 0 | False |
| \( \neg p \) | \( 1 - p \) | \( 1 + p \) | Logical negation (NOT) |
| \( p \land q \) | \( pq \) | \( pq \) | Conjunction (AND) |
| \( p \lor q \) | \( p + q - 2pq \) | \( p + q \) | Exclusive disjunction (XOR) |
| \( p \Rightarrow q \) | \( 1 - p + pq \) | \( 1 + p + pq \) | Material implication |
| \( p \leftrightarrow q \) | \( 1 - p + pq \) | \( 1 + p + pq \) | Biconditional (XNOR) |
| \( p \uparrow q \) | \( 1 - pq \) | \( 1 + pq \) | Nonconjunction (NAND) |
| \( p \downarrow q \) | \( 1 - p + pq \) | \( 1 + p + q + pq \) | Nondisjunction (NOR) |
4.1.1 Boole’s Original Representation Scheme

Boole’s method to represent logical formulas as polynomial formulas was presented in preliminary form in *Mathematical Analysis of Logic* and in more complete form in *Laws of Thought*. This method produces polynomials with integer coefficients that can be manipulated using ordinary arithmetic. Boole used the number 1 to represent the logical value true, the number 0 to represent the logical value false, and symbolic variables such as \( x, y, \) and \( z \) to represent atomic logical propositions. Boole noted that if a variable \( x \) is limited to the values 0 and 1, then the equation \( x^2 = x \) must hold; he identified this equation as a ‘special law.’ Based on this equation and its rearranged form \( x(1 - x) = 0 \), Boole devised a clever method to develop polynomial functions from arbitrary truth tables in two-valued logic. Each developed polynomial has the property that it agrees in value with its logical predecessor for any combination of truth values of the atomic propositions.

Boole’s function-development method is illustrated here for functions of two variables, which is the most useful case since traditional logical notation uses unary and binary operations (negation, conjunction, modal implication, and so on). Suppose that we have two real-valued variables \( x \) and \( y \), each restricted to values in the set \( \{0, 1\} \). We require a polynomial function \( p(x, y) \) that yields some specified value \( z_1 \) (also either 0 or 1) when \( x = 1 \) and \( y = 1 \); likewise we require \( p(1, 0) = z_2 \), \( p(0, 1) = z_3 \), and \( p(0, 0) = z_4 \), with each value \( z_i \in \{0, 1\} \). These required values can be arranged as illustrated on page 76 of [5], perhaps the earliest specimen of a logical truth table:

| \( x \) | \( y \) | \( p(x, y) \) |
|-------|-------|-------------|
| 1     | 1     | \( z_1 \)   |
| 1     | 0     | \( z_2 \)   |
| 0     | 1     | \( z_3 \)   |
| 0     | 0     | \( z_4 \)   |

(57)

Boole proved that the requisite function \( p(x, y) \) can always be calculated as the following polynomial, using the required values \( z_1 \) through \( z_4 \) as coefficients:

\[
p(x, y) := z_1xy + z_2x(1-y) + z_3(1-x)y + z_4(1-x)(1-y)
\]

(58)

Following Boole’s special law we add the constraints \( x^2 = x \) and \( y^2 = y \) to require that each variable must be either 0 or 1. Note that for any \( x \in \{0, 1\} \) and \( y \in \{0, 1\} \) only one of the four terms Equation 58 attains a nonzero value. This feature, which holds for functions of any number of binary variables, is the basis of Boole’s polynomial translation method.

For example, the translation of the logical exclusive disjunction \( x \oplus y \) uses its truth table:

| \( x \) | \( y \) | \( x \oplus y \) |
|-------|-------|----------------|
| 1     | 1     | 0              |
| 1     | 0     | 1              |
| 0     | 1     | 1              |
| 0     | 0     | 0              |

(59)

which produces the following polynomial according to Equation 58:

\[
p(x, y) := x(1-y) + (1-x)y
\]

(60)

\[
\Rightarrow x + y - 2xy
\]

(61)

along with the constraints \( x^2 = x \) and \( y^2 = y \). It can be verified by simple calculations that \( p(x, y) \) satisfies the stated criteria: \( p(1, 1) = 0 \); \( p(1, 0) = 1 \); \( p(0, 1) = 1 \); and \( p(0, 0) = 0 \). Note that in Boole’s original method there is no cause to invoke unusual rules of arithmetic such as \( 1 + 1 = 1 \). Boole’s arithmetic was emphatically the standard fare, not what we now call ‘Boolean algebra’. However, as we shall soon see it is helpful to use instead of real numbers coefficients in the finite field \( \mathbb{F}_2 \) and thus adopt integer arithmetic modulo 2 (in which \( 1 + 1 = 0 \)).

Table 3 shows the polynomial representations of several logical formulas using Boole’s original method, alongside their conventional forms. In general, for any formula of the propositional calculus in a logical system with atomic formulas \( \mathbf{x} := \langle x_1, x_2, \ldots, x_n \rangle \), Boole’s representation method yields a polynomial in the ring \( \mathbb{R}[\mathbf{x}] \) with real-number
coefficients. The coefficients are more specifically always integers; hence the Boolean polynomials can also be considered members of the ring \( \mathbb{Z}[x] \) with integer coefficients or the ring \( \mathbb{Q}[x] \) with rational coefficients as convenient.

Note that there are some polynomials in the ring \( \mathbb{Z}[x] \) (hence also \( \mathbb{Q}[x] \) and \( \mathbb{R}[x] \)) that do not correspond to any well-formed logical formulas at all using Boole’s representation; Boole recognized this fact and called such polynomials ‘not interpretable.’ Specifically, in Boole’s original scheme it is not the case that polynomial addition corresponds directly to either kind of logical disjunction (exclusive or nonexclusive): the polynomial \( x + y \) is neither the translation of \( x \oplus y \) (using \( \text{XOR} \)) nor the translation of \( x \lor y \) (using \( \text{OR} \)). The polynomial difference \( x - y \) is similarly not interpretable as a logical formula in Boole’s original translation scheme.

There were some minor flaws in Boole’s presentation of his polynomial representation method. Although the general algorithm presented in Chapter V of his Laws of Thought is correct, Boole did not always apply his own algorithm correctly. For example on page 95 of [5] he translated \( y \land (z \oplus w) \) as \( yz + yw \), having omitted the last term \(-2yw\). It is also confusing that Boole used the signs for addition and subtraction in two different senses (sometimes as the elementary arithmetic operators and sometimes to signify set union and set difference); either usage is fine but it takes careful reading to disambiguate the overloading. Finally, Boole did not have very robust methods to solve the multivariate polynomial equations that he had formulated.

4.1.2 Revised Translation Using Linear Algebra and Finite Fields

Instead of using an ordinary number system such as the real numbers \( \mathbb{R} \), logical formulas can be translated into polynomials with coefficients and variable values in a finite field \( \mathbb{F}_d \) of (prime) order \( d \). At the same time the range of acceptable input formulas can be widened from those using binary logic to those using multivalued logic (with \( d \geq 2 \) elementary truth values). There are three main benefits to using coefficients in a finite field \( \mathbb{F}_d \) instead of the real numbers \( \mathbb{R} \) (or the integers, rationals, or complex numbers). First, the polynomial ring \( \mathbb{F}_d[x] \) over a finite set \( x := (x_1, x_2, \ldots, x_n) \) of variables is itself countable and finite. Having the set of possible polynomials thus limited allows a simple tabular approach to solving systems of equations. In contrast there are infinitely many polynomials in any ring \( \mathbb{R}[x] \) (as well as \( \mathbb{Z}[x] \) etc.). Second, polynomial translations of logical formulas are a bit simpler and more intuitive using finite-field coefficients; for example the operation of addition in a polynomial ring \( \mathbb{F}_2[x] \) maps directly to exclusive disjunction in 2-valued logic. Third, every polynomial in a ring \( \mathbb{F}_d[x] \) corresponds to a well-formed formula in \( d \)-valued logic; there are no longer any uninterpretable polynomials.

First let us generalize the notion of a logical truth table into a finite-integer function.

Definition 18 (Finite-Integer Function) A finite-integer function \( T : (\mathbb{Z}_d)^n \rightarrow \mathbb{Z}_d \) of order \( d \) and arity \( n \) maps from the \( n \)-dimensional affine space \( (\mathbb{Z}_d)^n \) to the set \( \mathbb{Z}_d \), where \( \mathbb{Z}_d := \{0, 1, 2, \ldots, d - 1\} \) is the ring of integers modulo \( d \) (with \( d \geq 2 \)). Such a function \( T \) can be visualized as a table in which each row \( i \) describes the corresponding mapping \( (a_{i,1}, a_{i,2}, \ldots, a_{i,n}) \mapsto z_i \), such that \( T(x_1, x_2, \ldots, x_n) = z_i \) when each \( x_j = a_{i,j} \).

\[
\begin{array}{c|cccc|c}
   i & x_1 & x_2 & \cdots & x_n & T(x_i) \\
  \hline
   1 & a_{1,1} & a_{1,2} & \cdots & a_{1,n} & z_1 \\
   2 & a_{2,1} & a_{2,2} & \cdots & a_{2,n} & z_2 \\
   \vdots & \vdots & \ddots & \vdots & \vdots \\
   d^n & a_{d^n,1} & a_{d^n,2} & \cdots & a_{d^n,n} & z_{d^n}
\end{array}
\]

(62)

Lemma 19 (Polynomial Encoding by Linear Algebra) Consider a finite-integer function \( T : (\mathbb{Z}_d)^n \rightarrow \mathbb{Z}_d \) of arity \( n \) and order \( d \) as described in Definition 18 with the additional restriction that \( d \) is prime. This function \( T \) can be represented in closed form as a polynomial \( p \) over the variables \( x := (x_1, x_2, \ldots, x_n) \), with coefficients that are either rational numbers or integers in the modular ring \( \mathbb{Z}_d \), such that \( p(x) = T(x) \) for every point \( x \in (\mathbb{Z}_d)^n \) (using modular arithmetic to evaluate \( p \) if its coefficients are in \( \mathbb{Z}_d \); otherwise ordinary arithmetic). The developed polynomial \( p \) requires \( d^n \) coefficients, which can be computed as the solution of a system of \( d^n \) linear equations in \( d^n \) variables.

Referring to Definition 18 let us designate a point \( \mathbf{a}_i := (a_{i,1}, a_{i,2}, \ldots, a_{i,n}) \in (\mathbb{F}_d)^n \) as an index vector; we consider the list \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{d^n} \) of all \( d^n \) possible index vectors to be arranged in some (arbitrary) order which is used throughout this computation. By Lemma 58 there are \( d^n \) possible monomials in the polynomial ring \( \mathbb{F}_d[x_1, x_2, \ldots, x_n] \); each of
them can be generated by using an index vector to supply exponents for the variables in the tuple \( \mathbf{x} := (x_1, x_2, \ldots, x_n) \). From each index vector \( \mathbf{a}_j \) we generate the corresponding monomial \( t_j \) as the following product:

\[
   t_j := (x_1)^{a_{j,1}} (x_2)^{a_{j,2}} \cdots (x_n)^{a_{j,n}}
\]

The vector \( \mathbf{t} := (t_1, t_2, \ldots, t_{d^n}) \) contains all the monomials for the polynomial ring. An index vector \( \mathbf{a}_i \) can also be used as coordinates for some point \( \mathbf{x}_i \in (\mathbb{Z}_d)^n \), as given by:

\[
   \mathbf{x}_i := (a_{i,1}, a_{i,2}, \ldots, a_{i,n})
\]

Combining these uses, we construct a matrix \( \mathbf{M} \) of size \( d^n \times d^n \) in which each row \( \mathbf{M}_i \) is the monomial vector \( \mathbf{t} \) evaluated at the point \( \mathbf{x}_i = \mathbf{a}_i \) defined by the index vector \( \mathbf{a}_i \):

\[
   \mathbf{M}_i := \mathbf{t}(\mathbf{a}_i)
\]

Equivalently the element at each row \( i \) and each column \( j \) of the matrix \( \mathbf{M} \) is the value of the monomial \( t_j \) defined by the exponents \( \mathbf{a}_j \) evaluated at the point \( \mathbf{x}_i \) defined by the coordinates \( \mathbf{a}_i \) (taking \( 0^0 = 1 \) for this calculation):

\[
   \mathbf{M}_{i,j} := (a_{i,1})^{a_{j,1}} (a_{i,2})^{a_{j,2}} \cdots (a_{i,n})^{a_{j,n}}
\]

Each matrix element is roughly \( (\mathbf{a}_i)^{\mathbf{a}_j} \), where exponentiation works element-wise and the resulting products are themselves multiplied together.

Next we introduce the column vector \( \mathbf{z} := (z_1, z_2, \ldots, z_{d^n}) \) in which each \( z_i := T(\mathbf{x}_i) \) is the value of the finite-integer function \( T \) evaluated at the corresponding point. Finally we use the column vector \( \mathbf{c} := (c_1, c_2, \ldots, c_{d^n}) \) for the coefficients of the polynomial \( p \) which are to be determined. The system of linear equations \( \mathbf{M} \mathbf{c} = \mathbf{z} \) is a restatement of the original function \( T \), amened to include the computed monomial values. The desired coefficients \( \mathbf{c} \) of the new polynomial \( p \) are given by the solution \( \mathbf{c} = \mathbf{M}^{-1} \mathbf{z} \) to this linear system. The resulting polynomial \( p = \mathbf{c} \cdot \mathbf{t} = (\mathbf{M}^{-1} \mathbf{z}) \cdot \mathbf{t} \) is the scalar product of the solved coefficient vector and the monomials. The matrix inversion and scalar product can be computed using ordinary arithmetic, in which case the inferred polynomial \( p \) will be a member of the ring \( \mathbb{Q}[\mathbf{x}] \) with rational coefficients (thus also a member of \( \mathbb{R}[\mathbf{x}] \) with real coefficients); alternatively, the requisite calculations can be performed using integer arithmetic modulo \( d \) in which case the inferred polynomial \( p \) will be a member of the polynomial ring \( \mathbb{Z}_d[\mathbf{x}] \) with coefficients in the ring of integers modulo \( d \) (which is the finite field \( \mathbb{F}_d \) when \( d \) is prime).

Lemma[19] assumes that the matrix \( \mathbf{M} \) constructed by its directions is invertible.

**Corollary 20 (Encoding Indeterminate Polynomials)** The polynomial construction in Lemma[19] works essentially unchanged if the values in the finite-integer function \( T \) are left indeterminate. Using the elements of the vector \( \mathbf{x} := (z_1, z_2, \ldots, z_{d^n}) \) of function values as symbolic variables, the result \( p = (\mathbf{M}^{-1} \mathbf{z}) \cdot \mathbf{t} \) inferred by Lemma[19] as a member of the extended polynomial ring \( \mathbb{F}_d[x_1, x_2, \ldots, x_n; z_1, z_2, \ldots, z_{d^n}] \) that includes these \( d^n \) new variables. This allows a fully-parametric closed-form representation of any finite-integer function \( T \) of order \( d \) and arity \( n \) as described in Definition[18] at the cost of introducing \( d^n \) new variables.

**Example 21** Using 2-valued logic (thus \( d = 2 \)) and the variables \( x \) and \( y \) (thus \( \mathbf{x} = (x, y) \) and \( n = 2 \)) leads to the following \( d^n = 4 \) index vectors according to Lemma[19]

\[
   \mathbf{a}_1 = (1, 1), \quad \mathbf{a}_2 = (1, 0), \quad \mathbf{a}_3 = (0, 1), \quad \mathbf{a}_4 = (0, 0)
\]

Using these index vectors as exponents generates the following monomials:

\[
   t_1 = x^1 y^1 = x y, \quad t_2 = x^1 y^0 = x, \quad t_3 = x^0 y^1 = y, \quad t_4 = x^0 y^0 = 1
\]

and thus the monomial vector \( \mathbf{t} = (x y, x, y, 1) \). The following *Mathematica* commands compute these index vectors and monomials:

```mathematica
d = 2; xs = {x, y}; n = Length[xs]; (* variables *)
as = Tuples[Reverse[Range[0, d - 1]], n]; (* index vectors *)
t = Table[Apply[Times, xs^as[[ii]]], {ii, 1, d^n}]; (* monomials *)
```

24
Continuing on, we construct the $4 \times 4$ matrix $M$ according to Lemma [19]

\[
M = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\] (65)

For example the second row $M_2 = (0, 1, 0, 1)$ is the value of the monomials $t = (xy, x, y, 1)$ at the point $(x, y) = (1, 0)$ defined by the second index vector $a_2$. In Mathematica the matrix $M$ is computed by:

\[
M = \text{Table[ReplaceAll[t, Table[xs[[j]] -> as[[i]][[j]], {j, 1, n}], {i, 1, d^n}];}
\]

Now we add a vector $z = (0, 1, 1, 0)$ of truth values for the logical formula $x \oplus y$ arranged in the same order as the index vectors in Equation [63]. These correspond to the truth table shown in Equation [59]. The desired coefficients in the inferred polynomial are given by the linear system $c = \text{LinearSolve}[M, z]$. Using ordinary arithmetic, this system yields the solution coefficients $c = (-2, 1, 1, 0)$. Combining this solution $c$ with the generated monomials $t$, the inferred polynomial $p$ in the ring $\mathbb{R}[x, y]$ is given by $c \cdot t$; thus $p(x, y) : x + y - 2xy$. In Mathematica this same solution is accomplished by:

\[
z = \{0, 1, 1, 0\};
c = \text{LinearSolve}[M, z];
p = c \cdot t;
\]

For completeness we add the constraints $x^2 = x$ and $y^2 = y$ to limit the values of these variables appropriately. This first linear-algebra solution agrees with the result from Boole’s original translation method.

Alternatively, solving $Mc = z$ using integer arithmetic modulo 2 yields the solution vector $c = (0, 1, 1, 0)$ and consequently the inferred polynomial $p(x, y) : x + y$ in the ring $\mathbb{F}_2[x, y]$. No additional constraints are necessary in this case. In Mathematica this same modular-arithmetic solution is computed by:

\[
c2 = \text{LinearSolve}[M, z, \text{Modulus} \rightarrow d];
p2 = \text{PolynomialMod}[c2 \cdot t, d];
\]

Leaving the desired values $(z_1, z_2, z_3, z_4)$ indeterminate and computing $(M^{-1}z) \cdot t$ produces the following parametric polynomial $p_1$ in the ring $\mathbb{R}[x, y, z_1, z_2, z_3, z_4]$:

\[
p_1 : xz_1 - xz_2 - xz_3 - xz_4 + xz_5 - yz_1 + yz_2 + z_1
\] (66)

In factored form this is exactly Boole’s polynomial shown in Equation [58]. Using modular arithmetic instead yields a similar parametric polynomial $p_2$ in the ring $\mathbb{F}_2[x, y, z_1, z_2, z_3, z_4]$:

\[
p_2 : xz_1 + xz_2 + xz_3 + xz_4 + xz_5 + yz_1 + yz_2 + z_1
\] (67)

In Mathematica these parametric solutions are produced by:

\[
\text{Clear}[z]; \quad (* \text{remove old numeric values} *)
zs = \text{Table}[\text{Subscript}[z, i], \{i, 1, d^n\}]; \quad (* \text{list of variables} *)
p1 = \text{Expand}[(\text{Inverse}[M] \cdot zs) \cdot t];
p2 = \text{PolynomialMod}[(\text{Inverse}[M, \text{Modulus} \rightarrow d] \cdot zs) \cdot t, d];
\]

Using Lemma [19] and modular arithmetic, the familiar values and operations of 2-valued logic can be rewritten as polynomial formulas with coefficients in the binary finite field $\mathbb{F}_2$: the results are summarized in Table [3]. The basic values true (T) and false (F) map to the respective elementary values 1 and 0. Other logical functions of atomic formulas $x$ and $y$ are translated as follows. Logical conjunction (AND) translates directly as multiplication: for $x \land y$ we substitute the product $x \cdot y$ (abbreviated $xy$ or $x \cdot y$ in the usual algebraic fashion). Exclusive logical disjunction (XOR) translates directly as addition: for $x \oplus y$ we substitute the sum $x + y$. Logical negation translates as incrementation by 1 (or equivalently the difference from 1): for the negation $\neg x$ of any formula $x$ we substitute $1 + x$ (which is the
same as \(x - 1\) or \(x + 1\) or \(1 - x\) using integer arithmetic modulo 2; but not the same as \(-x\) with a unary minus, which is just \(x\) itself). Nonexclusive disjunction (the usual OR) is a polynomial sum: for \(x \lor y\) we substitute \(x + y + xy\). For the material implication \(x \rightarrow y\) we substitute \(1 + x + xy\). The biconditional \(x \leftrightarrow y\) (also written \(x \equiv y\) or designated XNOR) is translated into the polynomial \(1 + x + y\). Finally, the nonconjunction \(x \uparrow y\) (alternative denial, NAND, Sheffer stroke, \(x\langle y\rangle\)) is translated into the polynomial \(1 + xy\), and the nondisjunction \(x \downarrow y\) (joint denial, NOR, Pierce arrow, Quine dagger) is translated into the polynomial \(1 + x + y + xy\).

There is already a BooleanTable command in Mathematica to generate truth tables for binary-logical functions, and a Boole function to convert the logical values True and False to the numbers 0 and 1. The Tuples[] statement above was written so that the index vectors are generated in the same order that the associated polynomial values are computed. For example the following commands:

\[
\begin{align*}
z &= \text{Boole[BooleanTable[Implies[x,y]]];} \quad (* \text{ truth table } \{1,0,1\} *) \\
p1 &= (\text{Inverse}[M] . z) . t; \\
p2 &= \text{PolynomialMod}[(\text{Inverse}[M, \text{Modulus} \to d] . z) . t, d];
\end{align*}
\]

produce the values \(1 - x + xy\) for \(p1\) and \(1 + x + xy\) for \(p2\) which you can see listed in Table 3 as the translations for \(x \rightarrow y\) using real and finite-field coefficients.

4.1.3 Translation as a Function

The translation from a logical formula (in the propositional calculus) to a polynomial can be considered a function; every logical formula maps to a unique polynomial. But in general there are infinitely many distinct well-formed logical formulas that map to any given polynomial in a ring \(\mathbb{F}_2[x]\) with finite-field coefficients; in other words the translation function is surjective but not injective. By comparison, translation into \(\mathbb{Z}[x]\) (hence also \(\mathbb{Q}[x]\) or \(\mathbb{R}[x]\)) is neither injective nor surjective—some polynomials with integer (hence also rational or real) coefficients correspond to many well-formed logical formulas and others correspond to none.

**Definition 22 (Logical Preimage of a Polynomial Function) Let us designate the set of well-formed logical formulas that map to a given polynomial as the logical preimage of that polynomial; these formulas constitute an equivalence class in the sense that they always have the same value as one another (their truth tables are identical).**

For example the logical formulas \(x \rightarrow y\) and \(-x \lor y\) and \((x \rightarrow y) \land (-x \lor y)\) are each translated into the polynomial \(1 + x + xy\) in the ring \(\mathbb{F}_2[x,y]\); therefore they are members of its logical preimage.

4.2 Equations from Axioms

An axiom in a logical system has the same meaning as an equation in an algebraic system: each is an assertion made for the purpose of deducing what follows. Equations and axioms are *constraints* rather than *commandments*: it is generally possible to write systems of axioms or other equations that cannot be satisfied. The essential nature of an equation does not change because its formulas originated in logical instead of polynomial notation.

Logical axioms translate directly as polynomial equations, when those axioms are simple formulas from the propositional calculus (not involving references to provability, quantifiers, or indeterminate predicates over infinite domains; any of which complicate matters as addressed later). Using the turnstile \(\vdash\) to mark an axiom as in Frege’s Begriffsschrift \([13]\), an axiom \(\vdash q\) is the assertion that the included formula \(q\) is true; in Frege’s terminology the whole axiom is a ‘judgment’ and the included formula is its ‘content.’ Let us say that the logical system of interest has a set \(A := \{\vdash q_1, \vdash q_2, \ldots, \vdash q_m\}\) of axioms, in which each content \(q_i\) is a formula built from some atomic formulas \(x := (x_1, x_2, \ldots, x_n)\) and the usual logical operations. For concreteness we assume 2-valued logic, though this is not required for algebraic analysis.

Using Table 3 or the general method in Lemma 19 each axiom \(\vdash q_i\) can be translated into a polynomial equation with either real-number coefficients or binary finite-field coefficients. Each content \(q_i\) is first translated from logical to polynomial notation using \(\mathbb{R}[x]\) or \(\mathbb{F}_2[x]\) as desired. Then each judgment is translated to a polynomial constraint in the
obvious way: the axiom \( \vdash q_i \) becomes the equation \( q_i = 1 \). The set \( A \) of translated axiom-equations has a conjunction polynomial according to Lemma 3:

\[
q^* := (q_1)(q_2) \cdots (q_m) - 1
\]

(68)

with a slightly different form since each right-hand-side is 1 instead of 0. The constraint \( q^* = 0 \) using this conjunction polynomial corresponds to the assertion that the logical conjunction \( q_1 \land q_2 \land \cdots \land q_m \) of the axiom contents is true. Therefore using binary finite-field coefficients a set \( A \) of axioms translates to a single equation featuring its conjunction polynomial:

\[
A \leadsto \{ q^* = 0 \}, \quad q^* \in \mathbb{F}_2[x]
\]

(69)

Example 23 Lewis Carroll [8] provided this succinct version of his delightful logic puzzle about the barbers Allen, Brown, and Carr:

There are two Propositions, \( A \) and \( B \).

It is given that

1. If \( C \) is true, then, if \( A \) is true, \( B \) is not true;
2. If \( A \) is true, \( B \) is true.

The question is, can \( C \) be true?

Using the variables \( a \), \( b \), and \( c \) for the respective propositions \( A \) (that Allen is out of the shop), \( B \) (that Brown is out), and \( C \) (that Carr is out), we model the problem as the following axioms:

\[
\vdash c \rightarrow (a \rightarrow \neg b)
\]

(70)

\[
\vdash a \rightarrow b
\]

(71)

Translating the content of each axiom into a polynomial in the ring \( \mathbb{F}_2[a,b,c] \) with binary finite-field coefficients according to Table 3 and Lemma 19 yields the following polynomial equations:

\[
\vdash c \rightarrow (a \rightarrow \neg b) \leadsto abc + 1 = 1
\]

(72)

\[
\vdash a \rightarrow b \leadsto ab + a + 1 = 1
\]

(73)

By Lemma 3 the conjunction polynomial for this pair of equations is given by the following expression (calculated by integer arithmetic modulo 2 since the polynomials use coefficients in \( \mathbb{F}_2 \)):

\[
q^* := (abc + 1)(ab + a + 1) - 1 \Rightarrow abc + ab + a
\]

(74)

Thus the axioms \( A \) in Equations 70 and 71 are translated into the following set of a solitary polynomial equation:

\[
A \leadsto \{ abc + ab + a = 0 \}
\]

(75)

using polynomial coefficients in \( \mathbb{F}_2 \).

4.3 Solution-Value Sets as Truth Values

Translating axioms from logical to polynomial notation is only the first step; next it is necessary to solve these equations and interpret their solution. The truth value of a logical formula \( p \) subject to a set \( A \) of axioms is given by the solution-value set \( \mathcal{S}_A(p) \) of that formula subject to those axiom-equations. In order to compute the solution-value set as specified in Definition 2 it is first necessary to compute the solution set \( \mathcal{V}(A) \) to the equations \( A \) that is specified in Definition 1 using either a general method from computational algebraic geometry or the manual table-based method given in Definition 10. Calculations can be performed with real or finite-field coefficients.
According to Definition \[2\] the solution-value set \( \mathcal{S}_A(p) \subseteq K \) is a subset of the set of elementary values in the algebraic field \( K \) that contains the value of the objective formula \( p \); therefore the possible values of \( \mathcal{S}_A(p) \) are the members of the power set \( 2^K \). Thus using the binary finite field \( \mathbb{F}_2 \) with elementary values \( \{0,1\} \) there are four possible solution-value sets:

\[
\{\{\}, \{0\}, \{1\}, \{0,1\}\} \quad (76)
\]

It happens that these are the only four solution-value sets encountered even when logical formulas have been translated into the polynomial ring \( \mathbb{F}[x] \) with real coefficients, since the range of each polynomial function is still limited to \( \{0,1\} \) by its construction in Lemma \[19\]. Any way each set of elementary values in Equation \[76\] makes sense as the truth value members of the power set \( 2^K \).

Example 24 Returning to Carroll’s barbershop, it was established in Example \[23\] that axioms about Allen, Brown, and Carr translate as the equation \( abc + ab + a = 0 \) using polynomials in the binary finite field \( \mathbb{F}_2 \). The solution set \( \mathcal{V}(abc + ab + a = 0) \) is easily calculated using the appropriate modular arithmetic in Mathematica:

\[
\text{In}[4] := \text{Solve}\{a \ b \ c + a \ b + a == 0\}, \{a, b, c\}, \text{Modulus} \to 2\]

\[
\text{Out}[4] = \{\{a \to 0, b \to 0, c \to 0\}, \{a \to 0, b \to 0, c \to 1\}, \{a \to 0, b \to 1, c \to 0\}, \{a \to 0, b \to 1, c \to 1\}\}
\]

The result contains five solutions for \((a,b,c)\):

\[\mathcal{V}(abc + ab + a = 0) \Rightarrow \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,1,0)\} \quad (77)\]

Using the constraints in Equations \[72\] and \[73\] directly (instead of their conjunction polynomial) gives the same solution set:

\[
\text{In}[5] := \text{Solve}\{a \ b \ c + 1 == 1, a \ b + a + 1 == 1\}, \{a, b, c\}, \text{Modulus} \to 2\]

\[
\text{Out}[5] = \{\{a \to 0, b \to 0, c \to 0\}, \{a \to 0, b \to 0, c \to 1\}, \{a \to 0, b \to 1, c \to 0\}, \{a \to 0, b \to 1, c \to 1\}\}
\]

You can verify that the logical back-translation of each solution satisfies the axioms in Carroll’s problem stated in Equations \[70\] and \[71\]. For example with \( a = T, b = T, \) and \( c = F \) as in the last solution \((a,b,c) = (1,1,0)\): the material implication \( a \to b \) is true; the inner material implication \( a \to \neg b \) is false; but since \( c \) is false the outer material implication \( c \to (a \to \neg b) \) is nonetheless true.

Carroll’s problem statement requests the truth value of the logical formula \( c \). It is evident from the solution set \( \mathcal{V}(abc + ab + a = 0) \) shown in Equation \[77\] that both 0 and 1 are feasible solutions for \( c \). Thus we have:

\[\mathcal{S}_{(abc + ab + a = 0)}(c) \Rightarrow \{0,1\} \quad (78)\]

and it is neither the case that \( c \) is necessarily 0 nor the case that \( c \) is necessarily 1. The formula \( c \) is ambiguous—given Carroll’s axioms the barber Carr could be either in or out of the shop.
Let us also calculate the truth value of the proposition $abc$. You can see by inspection of Equation 77 that every solution in $Y(A)$ contains the value zero for at least one variable in $(a, b, c)$. Thus the product $abc$ must always be zero using these solutions; in other words the solution-value set for the objective formula $abc$ is given by:

$$\mathcal{S}_{(abc+ab+a=0)}(abc) \Rightarrow \{0\}$$

(79)

This result indicates that the polynomial $abc$ is necessarily 0, and therefore that the corresponding logical formula is necessarily false—given Carroll’s axioms all three barbers cannot be out of the shop simultaneously.

4.4 Discovering All Theorems

In logic there is a special name for a formula that is necessarily true: it is a theorem. Therefore a logical formula $p$ is a theorem given some set $A$ of axioms exactly if its solution-value set $\mathcal{S}_A(p)$ has the value $\{1\}$ (that is, the set containing exactly the number one). Furthermore the set of all theorems entailed by the set $A$ of axioms (using some finite list $x := (x_1, x_2, \ldots, x_n)$ of propositional variables) is given by the inverse-value set $\mathcal{S}_A^{-1}(\{1\})$ described in Definition 5, Lemma 6 and Corollary 7. Every logical formula (whose propositional variables are $x$) that is a theorem given these axioms must translate as one of the polynomials in this inverse-value set (which is empty if the axioms are unsatisfiable). Using polynomials with finite-field coefficients, this algebraic formulation transforms the discovery of theorems from a search through an infinite set of logical formulas to a search among a finite set of polynomials (each of which corresponds to a unique truth table).

It is left as a separate exercise to choose which member of the logical preimage of each polynomial theorem should be used to represent it; this is properly an optimization problem in the area of logic synthesis. For a polynomial with coefficients in the binary field $\mathbb{F}_2$, a reasonable default choice for its representative logical formula is the transliteration of the polynomial that maps multiplication back to logical conjunction ($\land$, AND) and that maps addition back to logical exclusive disjunction ($\oplus$, XOR). But note that it is not appropriate to use what is commonly called ‘disjunctive normal form’ since as Table 3 indicates, non-exclusive disjunction ($\lor$, OR) does not map directly to polynomial addition (using either real or finite-field coefficients). Note also that an exclusive disjunction $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ in 2-valued logic is true if and only if an odd number of its arguments are true; otherwise the disjunction is false. This corresponds to integer arithmetic modulo 2 which is appropriate for polynomials with coefficients in the binary finite field $\mathbb{F}_2$.

The concept underlying theorems is not exclusive to logic; the more primitive notion is that a formula might have a definite solution value given some set of equations or other constraints. In general algebra the definite solution value zero is given special status. As mentioned after Definition 5, what is defined herein as the inverse-value set is in algebraic geometry called the ideal generated by the polynomials in the equations $A$. In this sense a theorem is almost ideal! Others have made the connection between logical theorems and polynomial ideals [19, 22].

Example 25 Consider the axioms $\vdash x$ and $\vdash x \rightarrow y$. Using Table 3 each axiom translates as an equation using polynomials with binary finite-field coefficients in the ring $\mathbb{F}_2[x, y]$:  

$$\vdash x \iff x = 1 \quad \text{(80)}$$

$$\vdash x \rightarrow y \iff 1 + x + xy = 1 \quad \text{(81)}$$

By Lemma 8 these two equations yield the conjunction polynomial $q^* = (x)(1 + x + xy) − 1$ which evaluates to $1 + xy$ using modular arithmetic. Thus the axioms are translated as the set $A = \{1 + xy = 0\}$ containing one polynomial equation. Using the table-based inference method in Definition 10 or the Mathematica command:  

`Solve[{1 + x y == 0}, {x, y}, Modulus -> 2]`

reveals that his set of equations has the solution set $\mathcal{Y}(A) = \{(1, 1)\}$ for $(x, y)$. Evaluating each of the 16 polynomials in the ring $\mathbb{F}_2[x, y]$ at this unique solution reveals 8 polynomials whose only feasible value is 1 (those whose listed value is 1 in the column labeled $p_i(1, 1)$ in Table 1 namely $p_2, p_3, p_5$, and so on). These 8 polynomials, which comprise the inverse-value set $\mathcal{S}_{(xy+1=0)}^{-1}(\{1\})$, are the translations of all the theorems entailed by the stated axioms:

$$1, \ y, \ x \ 1 + x + y, \ xy, \ 1 + y + xy, \ 1 + x + xy, \ x + y + xy$$

(82)
Following Lemma 6 and Corollary 7 this inverse-value set can also be described in closed form as:

\[ S_{xy+1=0}^{-1}(\{1\}) \Rightarrow \{ p \times (1 + xy) + 1 : p \in \mathbb{F}_2[x, y] \} \]

(83)

where modular arithmetic is to be used to evaluate the included polynomial expression. Each polynomial theorem has infinitely many formulas in its logical preimage; for example the last polynomial of \( (x \land y) \lor x \oplus y \) as well as to the logical formula \( x \lor y \). Using the matching formulas in Table 3 provides one choice of logical back-translation for each of the 8 theorems entailed by the axioms \( \vdash x \) and \( \vdash x \rightarrow y \):

\[
T, \quad y, \quad x, \quad x \leftrightarrow y, \quad x \land y, \quad y \rightarrow x, \quad x \rightarrow y, \quad x \lor y
\]

(84)

Included in this complete set of theorems is the sole formula \( y \) that a direct application of modus ponens would prove from the axioms \( \vdash x \) and \( \vdash x \rightarrow y \). You can see that algebraic analysis provides more comprehensive results.

**Example 26** Returning once again to Carroll’s barbershop, we can use algebraic analysis to discover the set of all theorems entailed by the axioms \( \vdash c \rightarrow (a \rightarrow \neg b) \) and \( \vdash a \rightarrow b \). The conjunction polynomial \( q^* = a + ab + abc \) in Equation 7 is a translation of these logical axioms. It follows from Definition 5, Lemma 6, and Corollary 7 that the set of theorem polynomials is given by:

\[ S_{a+ab+abc=0}^{-1}(\{1\}) \Rightarrow \{ p \times (a + ab + abc) + 1 : p \in \mathbb{F}_2[a, b, c] \} \]

(85)

By Lemma 8 there are \( 2^3 = 256 \) distinct polynomials in the ring \( \mathbb{F}_2[a, b, c] \). Evaluating all of these polynomials with Mathematica reveals that there are exactly 8 theorems in the polynomial ring \( \mathbb{F}_2[a, b, c] \) entailed by Carroll’s barbershop axioms:

\[ S_{a+ab+abc=0}^{-1}(\{1\}) \Rightarrow \begin{bmatrix}
1 \\
1 + a + ab + abc \\
1 + ac \\
1 + a + ab + ac + abc \\
1 + abc \\
1 + a + ab \\
1 + ac + abc \\
1 + a + ab + ac
\end{bmatrix} \]

(86)

The logical negation of each of these 8 theorems has the solution-value set \( \{0\} \); thus each negated theorem is necessarily false. The remaining 240 polynomials in \( \mathbb{F}_2[a, b, c] \) are ambiguous, with the common solution-value set \( \{0, 1\} \). Note that Carroll’s original query \( c \) and its negation \( c + 1 \) are both members of this set \( S_{a+ab+abc=0}^{-1}(\{0, 1\}) \) of ambiguous polynomials.

For concreteness, the following Mathematica commands were used to generate the members of the inverse-value set \( S_{a+ab+abc=0}^{-1}(\{1\}) \) shown above:

```
d = 2; xs = {a, b, c}; n = Length[xs]; (* variables *)
as = Tuples[Reverse[Range[0, d - 1]], n]; (* index vectors *)
t = Table[Apply[Times, xs^as[[i]]], {i, 1, d^n}]; (* monomials *)
all = Tuples[Range[0, d - 1], d^n] . t; (* all polys in ring *)
(* construct inverse-value set from closed-form expression: *)
theorems = Table[all[[i]] * (a + a b + a b c) + 1, {i, 1, Length[all]}]; (* substitute to remove squares by Fermat’s little theorem: *)
theorems = Expand[theorems] /. {a^2 -> a, b^2 -> b, c^2 -> c};
theorems = DeleteDuplicates[PolynomialMod[theorems, 2]] (* modulo 2 *)
```

The last command gives the output displayed above:

\{1, 1 + a + a b + a b c, 1 + a c, 1 + a + a b + a c + a b c, 1 + a b c, 1 + a + a b, 1 + a c + a b c, 1 + a + a b + a c\}
4.5 Dynamical Systems from References to Provability

A logical formula that refers to its own provability can be translated into algebraic form using a parametric system of polynomial equations as described in Definition 12. The parameters and parameter-updating functions introduced in the definition are expressly permitted to use features of solution sets and solution-value sets; this allows references to provability to be modeled. From the specified parametric system of polynomial equations is extracted by Definition 14 an evolution function \( F(\Theta) \) that describes how the parameters change as a function of themselves. This evolution function can be used in one of two ways: either as a static constraint \( \Theta = F(\Theta) \) that is added to the system of equations (which is then instantiated from parametric form into an ordinary system of equations; or as a recurrence relation \( \Theta_{t+1} \leftarrow F(\Theta_t) \) that is used to extend the static equations into a dynamical system.

**Example 27** Let us proceed now with the analysis of Gödel’s formula \([R(q); q]\) that asserts its own unprovability. Using \( x \) to denote the formula and \( \text{Bew}(x) \) to denote the proposition that \( x \) is provable (‘beweisbar’ in German), we begin with the declaration \( x \in \{T, F\} \) that the possible values of the formula \( x \) are the elementary logical values true and false. We define the formula \( x \) using the axiom:

\[
\vdash x = \neg \text{Bew}(x) \tag{87}
\]

Using Definition 12 and Table 6 the type declaration for \( x \) and the axiom in Equation 87 translate as the following parametric system of polynomial equations:

\[
\begin{align*}
\tau_x & : x \in \{0, 1\} \subset \mathbb{R} \\
\delta_x & : x \leftarrow 1 - (\mathcal{S}_1(x) = \{1\})
\end{align*}
\]

For concreteness Equation 88 uses real-number coefficients; however for this problem the calculations would be identical using coefficients in \( \mathbb{F}_2 \) instead.

Hypothetico-deductive analysis according to Definition 14 now reveals the evolution function \( F(x) \) for the parametric system in Equation 88. Since the parameter \( x \) has the domain \( \{0, 1\} \), we note that the function \( F \) must map from the set \( \{0, 1\} \) back to itself. We consider the two possible values of the parameter. In the case \( x = 0 \) the parameter-updating function is instantiated as \( \lambda_1(0) : 1 - (\mathcal{S}_1(0) = \{1\}) \). Absent any constraints, the only feasible value of the objective formula \( 0 \) is 0; in general for any constant \( k \) the solution-value set \( \mathcal{S}_1(k) \) simply yields \( \{k\} \). Hence the value of \( \lambda_1(0) \) is \( 1 - (\{0\} = \{1\}) \), which is 1 since the false comparison statement yields the value 0. Similarly, in the case \( x = 1 \), the parameter-updating function is instantiated as \( \lambda_1(1) : 1 - (\mathcal{S}_1(1) = \{1\}) \). The solution-value set \( \mathcal{S}_1(1) \Rightarrow \{1\} \) gives \( \lambda_1(1) \Rightarrow 1 - (\{1\} = \{1\}) \) which evaluates to 0. These mappings \( 0 \mapsto 1 \) and \( 1 \mapsto 0 \) define the evolution function \( F(x) \) encoded by the parametric system in Equation 88. It is evident from inspection that the interpolated polynomial function \( F(x) : 1 - x \) provides the appropriate values.

There are static and dynamic ways to interpret Equation 88. In the static interpretation we replace the assignment template \( \delta_x \) with the static constraint \( x = F(x) \) using the extracted evolution function \( F(x) : 1 - x \). In this static interpretation Gödel’s axiom in Equation 87 is translated via Equation 88 into this system of equations (the constraint \( \tau_x \) could be expressed in polynomial form as \( x^2 = x \) using Boole’s representation scheme):

\[
\begin{align*}
\tau_x & : x \in \{0, 1\} \subset \mathbb{R} \\
\delta_x & : x \leftarrow 1 - (\mathcal{S}_1(x) = \{1\})
\end{align*}
\]

It is evident from inspection that Equation 89 has no solution; the solution-value set \( \mathcal{S}_{\tau_x, \delta_x}(x) \) evaluates to \( \{\} \). In fact simple substitution reveals that Equation 89 is equivalent to the constraint \( 1 = 0 \). Therefore, in the static interpretation, the truth value of Gödel’s formula \([R(q); q]\) is that it is unsatisfiable. In other words, there is no such thing as a logical formula \( x \) that is true if and only if \( x \) is simultaneously not provable, just as there is no such thing as a quadratic equation \( 2x^2 + 3x + c = 0 \) that has exactly \( c \) real solutions for \( x \). Each specification is internally inconsistent, and by algebraic analysis each inconsistency has been exposed as an infeasible system of equations.

The dynamic interpretation of the parametric system in Equation 88 explains the mechanism of the inconsistency in Gödel’s formula. In this interpretation the parameter \( x_t \in \{0, 1\} \) gives the state of the dynamical system at each time \( t \); the recurrence \( x_{t+1} \leftarrow 1 - x_t \) from the evolution function \( F(x) : 1 - x \) creates a transition from the state \( x = 0 \) to the state \( x = 1 \) and vice versa. The solution-value set for the objective \( x \) computed for each time \( t \) simply contains
the respective value of the state \( x_t \). Thus the dynamical system derived from Gödel’s formula is represented by the following graph:

\[
\begin{array}{c}
0 \\
\hline
\{0\} \\
\{1\} \\
1
\end{array}
\]

(90)

Following Definition [15] either initial value \( x_0 \) gives an alternating infinite sequence of solution-value sets for \( x \) at successive times \( t \):

\[
\mathcal{S}_\{1\}(x) \Rightarrow \begin{cases} 
  x_0 = 0 & \Rightarrow (\{0\}, \{1\}, \{\}, \{\}, \ldots) \\
  x_0 = 1 & \Rightarrow (\{\}, \{\}, \{1\}, \{0\}, \ldots)
\end{cases}
\]

(91)

This dynamical system derived from Equation [88] has a periodic orbit and no fixed points; therefore it is unsteady in the terminology of Definition [14]. In this dynamic interpretation Gödel’s self-denying formula \([R(q); q]\) specifies a discrete dynamical system that oscillates between the states of truth and falsity. The objective \( x \) describing the truth value of the formula oscillates between being necessarily false (having the solution-value set \( \{0\} \)) and being necessarily true (having the solution-value set \( \{1\} \)).

Note that the set \( A \) of constraints is empty in the dynamic interpretation of Gödel’s formula; in the parametric formulation shown in Equation [88] there are no constraint templates \( \alpha \), only an assignment template \( \delta \). In this respect Gödel’s formula is more like the Fibonacci recurrence than like the self-referential quadratic equations analyzed earlier.

**Example 28** It is illustrative to consider the complement of Gödel’s construction: a formula that asserts its own *provability*. Using the parameter \( y \) to represent the formula, we start with the declaration \( y \in \{T, F\} \) and the axiom:

\[
\vdash y = \text{Bew}(y)
\]

(92)

This type declaration and axiom are translated into parametric polynomial equations according to Definition [12]:

\[
\begin{align*}
\tau_y : y & \in \{0, 1\} \subset \mathbb{R} \\
\delta_y : y & \leftarrow (\mathcal{S}_\{1\}(y) = \{1\})
\end{align*}
\]

(93)

Following Definition [14] the identity function \( F(y) \) : \( y \) is extracted as the evolution function encoded by Equation [93]. Incorporating the constraint \( y = F(y) \) the static interpretation of Equation [93] yields:

\[
\begin{align*}
\tau_y : y & \in \{0, 1\} \subset \mathbb{R} \\
\delta_y : y & = y
\end{align*}
\]

(94)

As the constraint \( \delta \) is tautological the solution-value set \( \mathcal{S}_{\{\tau, \delta\}}(y) \) evaluates to \( \{0, 1\} \). Thus the formula that asserts its own provability is ambiguous in the static interpretation.

In the dynamic interpretation the parametric system in Equation [93] specifies the following dynamical system based on the recurrence relation \( y_{t+1} \leftarrow y_t \):

\[
\begin{array}{c}
\{0\} \\
\hline
\{\} \\
\{1\} \\
\{1\}
\end{array}
\]

(95)

Either initial value \( y_0 \) gives a monotonous sequence of solution-value sets for \( y \) at successive times \( t \):

\[
\mathcal{S}_\{1\}(y) \Rightarrow \begin{cases} 
  y_0 = 0 & \Rightarrow (\{0\}, \{0\}, \{0\}, \{0\}, \ldots) \\
  y_0 = 1 & \Rightarrow (\{1\}, \{1\}, \{1\}, \{1\}, \ldots)
\end{cases}
\]

This dynamical system has two fixed points; hence it is contingent by Definition [14]. In contrast to Gödel’s specification in Equation [87] that describes a formula that cannot be realized consistently, the complementary specification in Equation [92] describes a formula that can be realized in two different ways, either of which is consistent with the specification: in one case the formula is necessarily true (having the solution-value set \( \{1\} \)), and in the other case the formula is necessarily false (having the solution-value set \( \{0\} \)). The tautological vacuousness of self-affirmation mirrors the viciously-circular contradiction of self-denial.
Example 29 Let us consider a set of axioms that gives a slightly more complicated dynamical system than Gödel’s formula \( R(q); q \). We now specify a logical formula \( z \) with the following properties: \( z \) is not provable; if \( z \) is ambiguous, then it is not true; and if \( z \) is not provable, the negation of \( z \) is not provable, and \( z \) is not ambiguous, then \( z \) is true. Thus in logical notation we have the declaration \( z \in \{ T, F \} \) and the following axioms:

\[
\begin{align*}
& \vdash \neg \text{Bew}(z) \\
& \vdash \text{Viel}(z) \rightarrow \neg z \\
& \vdash (\neg \text{Bew}(z) \land \neg \text{Bew}(\neg z) \land \neg \text{Viel}(z)) \rightarrow z
\end{align*}
\] (96)

Here the notation \( \text{Viel}(z) \) is introduced for the proposition that \( z \) is ambiguous (‘vieldeutig’ in German), in other words that its solution-value set \( \mathcal{S}_A(z) = \{ 0, 1 \} \).

We introduce the parameter \( \theta \) to represent the solution-value set for \( z \). The domain of \( \theta \) is the power set \( \{ \{ \}, \{ 0 \}, \{ 1 \}, \{ 0, 1 \} \} \) of the elementary values \( \{ 0, 1 \} \). Following Definition 16 we translate the axioms in Equation 96 into the following parametric polynomial system (here we have delayed polynomial translation of the material implication operators and represented negations in a slightly different way for convenience):

\[
\begin{align*}
\tau_0 : & \quad z \in \{ 0, 1 \} \\
\tau_\theta : & \quad \theta \in \{ \{ \}, \{ 0 \}, \{ 1 \}, \{ 0, 1 \} \} \\
\alpha_1 : & \quad (\theta = \{ 1 \}) = 0 \\
\alpha_2 : & \quad (\theta = \{ 0, 1 \}) \rightarrow (1 - z) = 1 \\
\alpha_3 : & \quad ((\theta \neq \{ 1 \}) \land (\theta \neq \{ 0 \}) \land (\theta \neq \{ 0, 1 \}) \rightarrow z) = 1 \\
\delta_\theta : & \quad \theta \leftrightarrow \mathcal{S}_{\{ \alpha_1, \alpha_2, \alpha_3 \}}(z)
\end{align*}
\] (97)

This parametric system of polynomial equations specifies the following dynamical system (each node is labeled with a state \( \theta \), and each arc indicates the updated solution-value set calculated for \( z \) assuming the state corresponding to the originating node):

\[
\begin{array}{c}
\{ \} \\
\{ 0 \} \\
\{ 1 \} \\
\{ 0, 1 \}
\end{array}
\begin{array}{c}
\{ \} \rightarrow \{ 1 \}, \\
\{ 1 \} \rightarrow \{ \}, \\
\{ 0 \} \rightarrow \{ 0, 1 \}, \\
\{ 0, 1 \} \rightarrow \{ 0 \}
\end{array}
\] (98)

Hypothetico-deductive analysis produces the following mappings for the evolution function \( F \):

\[
\{ \} \mapsto \{ 1 \}, \quad \{ 1 \} \mapsto \{ \}, \quad \{ 0 \} \mapsto \{ 0, 1 \}, \quad \{ 0, 1 \} \mapsto \{ 0 \}
\] (99)

For example, given the hypothesis \( \theta = \{ \} \) the constraint templates in Equation 97 are instantiated as:

\[
\begin{align*}
\alpha_1 : & \quad (\{} = \{ 1 \}) = 0 \\
\alpha_2 : & \quad ((\{} = \{ 0, 1 \}) \rightarrow (1 - z)) = 1 \\
\alpha_3 : & \quad ((\{} \neq \{ 1 \}) \land (\{} \neq \{ 0 \}) \land (\{} \neq \{ 0, 1 \}) \rightarrow z) = 1
\end{align*}
\] (100)

which simplify to:

\[
(0) = 0, \quad ((0) \rightarrow (1 - z)) = 1, \quad ((1)(1) \rightarrow z) = 1
\] (101)

Translating the material implication operators according to Table 3 yields the polynomial constraints:

\[
0 = 0, \quad 1 = 1, \quad z = 1
\] (102)

These constraints lead to the solution-value set \( \mathcal{S}_{\{ 0=0,1=1,z=1 \}}(z) \rightarrow \{ 1 \} \) for \( z \), which is assigned as the new value of the parameter \( \theta \) by the updating function in template \( \delta_\theta \). Thus is established the mapping \( \{ \} \rightarrow \{ 1 \} \) for the evolution function \( F(\theta) \). The other hypotheses \( \theta = \{ 1 \}, \theta = \{ 0 \}, \) and \( \theta = \{ 0, 1 \} \) are handled in a similar way. The results are the same using coefficients in \( \mathbb{R} \) or in \( \mathbb{F}_2 \).

For this problem there is no direct representation of the evolution function \( F(\theta) \) as a polynomial with coefficients in \( \mathbb{R} \) or in \( \mathbb{F}_2 \), since the domain \( \{ \}, \{ 0 \}, \{ 1 \}, \{ 0, 1 \} \) for \( \theta \) is not a subset of either set of numbers. Nonetheless it is evident from the mappings in Equation 99 and the graph in Equation 98 that there are no fixed points in this dynamical system; there is no state of \( \theta \) that maps to itself. Hence the formula \( z \) defined by the axioms in Equation 96 is unsteady.
4.6 Quantifier Elimination

It is not necessary to use quantifiers to examine Gödel’s incompleteness argument; however for the sake of completeness let us address briefly how quantifiers should be handled in this algebraic formulation of logic. Existential and universal quantifiers add complexity to polynomial formulas and systems of constraints. In the field of computational algebraic geometry, there is a large body of work on eliminating quantifiers from polynomial formulas with real-number coefficients [29, 2]. Since Boole’s translation method allows logical formulas to be converted into polynomials with ordinary numeric coefficients, such algebraic geometry methods for managing quantifiers are applicable to problems that were originally specified in traditional logical notation. Eliminating quantifiers from an equation may change the overall system of equations from a simple conjunction (all equations are to be satisfied simultaneously) to a disjunction (various subsets of equations must be solved separately).

What is difficult about quantifiers is not the quantifiers themselves, but the indeterminate functions of infinite domains that often accompany them. Indeterminate functions of finite domains can be treated parametrically using Lemma [19]. However indeterminate predicates such as Man(x) and Mortal(x) assumed to have an infinite universe of discourse for x (as x ∈ N) are problematic. Perhaps it would be possible to choose a ‘quorum’ (sufficiently-large but finite universe of discourse) based on the number and arity of predicates declared in a logical system. Anyway there is more work to be done to connect the treatment of quantifiers in computational algebraic geometry with the treatment of quantifiers in classical first-order logic.

5 DISCUSSION

I have argued two major propositions in this essay: that logic is an application of mathematics, more specifically solving systems of polynomial equations to yield sets of feasible elementary values; and that allowing equations to refer to their own solutions creates discrete dynamical systems. In this analysis I have employed Boole’s groundbreaking algebraic formulation of logic, revised and extended with concepts from algebraic geometry and computer science. Within this framework of dynamic polynomial logic, Gödel’s special formula that asserts its own unprovability is easily expressed and is not at all paradoxical. Furthermore, Gödel’s formula is not so special after all: it is easy to modify ordinary quadratic equations to exhibit the same behavior, or to specify other logical formulas that encode more elaborate dynamical systems. Kurt Gödel provided innovative but convoluted proofs of some normal properties of a simple recurrence relation, accompanied by spectacular misinterpretations.

A familiar result from geometry may provide insight into the subtle cognitive error made by Gödel and many classical logicians: what I shall call the Pythagorean fallacy.

5.1 The Pythagorean Fallacy

The Pythagorean theorem holds an important lesson about undecidability and incompleteness. We all know the famous equation $a^2 + b^2 = c^2$ relating the lengths of the sides and hypotenuse of a right triangle. Perhaps less well known is the fact that Pythagoras and his followers initially considered the only valid numbers to be ratios of natural numbers. To them, the solution for c with some values of a and b (such as $a = 1$ and $b = 1$) was $\sqrt{2}$ (alos: $\sqrt{2}$); though we typically say ‘irrational’, a more literal translation would be ‘not reasonable’. Of course, to the modern thinker it is not a number like $\sqrt{2}$ that seems unreasonable; it is the expectation that an algebraic equation with natural-number coefficients must have a solution that is the ratio of two natural numbers.

Let us first recognize the phenomenon of the anticipated-actual type mismatch: the expectation that the value of a mathematical expression should be one type of object when it is in fact another. It is an anticipated-actual type mismatch to expect the expression $\sqrt{a^2 + b^2}$ to have a rational value for all natural-number arguments a and b. The anticipated-actual type mismatch is facilitated by expressions that use objects of one type to construct objects
of another type: for example whereas the sum or product of any integers must itself be an integer, the quotient of integers may not be. We can recognize that certain mechanisms of constructing expressions lead to certain types of mathematical values: for example if we are allowed the square root operation then we can make algebraic numbers from integers, and if we are allowed infinite series then we can make transcendental numbers from rational numbers. Notation and terminology may obscure the actual mechanisms in use. For example there is no square root sign in the equation \( a^2 + b^2 = c^2 \), yet we must invoke this operation to solve for \( c \).

Let us call it the Pythagorean fallacy to encounter an anticipated-actual type mismatch but then to misplace the blame: to decide that the actual value, rather than the incorrect expectation, is at fault. This error is facilitated by formulas that have the expected type for some but not all arguments (in other words the expected type is a special case of the actual type): for example the value of \( \sqrt{3^2 + 4^2} \) is indeed a rational number (5) but the value of \( \sqrt{1^2 + 1^2} \) is not any rational number. We may recognize certain circumscribed exceptions in actual values: for example whereas dividing one rational number by another generally yields a rational number, division by zero does not. It is not problematic to recognize such limited exceptions. But it is inappropriate to indulge the Pythagorean fallacy by declaring that some general mathematical method is fundamentally defective because of an anticipated-actual type mismatch, when the fault lies with an incorrect expectation.

Ancient Greek mathematicians (and perhaps Pythagoras himself) eventually resolved their Pythagorean fallacy in an exemplary way: they broadened their concept of number to include the irrational values necessary for the general solution of algebraic equations like \( a^2 + b^2 = c^2 \). Our modern concept of number has grown to include many more ‘unreasonable’ types of objects: not only algebraic numbers like \( \sqrt{2} \), but also transcendental numbers like \( e \) and \( \pi \) and imaginary numbers like \( \sqrt{-1} \). We have come to understand that the mechanisms of geometric and algebraic construction ought to produce irrational numbers sometimes, even when the input values are natural numbers. We would find it absurd to accuse geometry or algebra of undecidability or incompleteness because there are expressions whose values are not ratios of natural numbers.

5.2 Gödel’s Error

Returning to the main results from [15], what Gödel’s Theorem VI says is technically true but not detrimental to formal reasoning as he and many others have concluded. However it is emphatically not the case that Gödel’s special formula is semantically true; instead this formula that asserts its own unprovability is exceptional in a particular way. The types of ‘undecidability’ and ‘inconsistency’ that have been identified as exceptions—unsatisfiability, ambiguity, unsteadiness, and contingency—are semantically and syntactically appropriate features of systems of equations. Moreover, these exceptions can just as well be demonstrated in ordinary polynomial equations which had no origin in traditional logical notation. For example it is evident that the formula \( x \) has no decidable real-number value given the definition that \( x \) is a real number satisfying \( x^2 = -1 \); and that the formula \( y \) has no definite real value when \( y \) is defined as a real number satisfying \( y^2 = y \); and that the formula \( z \) has no definite integer value when \( z \) is defined as an integer satisfying \( z_{i+1} = 1 - z_i \). Each of these formulas \( x, y, \) and \( z \) is exceptional in a particular way and should not have a definite elementary value. There is no need to divorce syntax from semantics. Undecidability is not a crime!
In his famous 1931 paper Gödel described a special logical formula that asserts of itself that it cannot be proven, in the context of a formal system PM that can express logical formulas, theorems, proof, and natural numbers. Gödel claimed that this special formula must be semantically true but syntactically undecidable: impossible to prove or disprove by calculations within PM. Gödel concluded that the existence of such a true-but-undecidable formula renders any formal system like PM essentially incomplete and incapable of proving even its own consistency. In this presentation I shall introduce a new method of analysis called computational algebraic logic that proves most of these conjectures wrong. Although Gödel’s special formula is indeed ‘undecidable’ in the sense that it has a value outside of the set \{TRUE, FALSE\}, such undecidability is a feature rather than a bug in formal reasoning: the formula is not semantically true.

There are three main ideas to be discussed: exceptions, translation, and dynamical systems. First, mathematicians are unfailingly perplexed by formulas that give values outside of the sets they had originally expected; they give pejorative names to the unexpected exceptional values. Natural numbers are usually welcome. But others are called negative, fractional (broken!), irrational, or imaginary. If you knew nothing about complex numbers, you might say that \sqrt{-1} is undecidable: the value of this formula cannot be proven to be any real number, even though its argument is real. Likewise \sqrt{2}, \pi, e, and so on seem exceptional and undecidable if you are expecting rational numbers. Gödel’s undecidable formula is just another unexpected object in this litany of un-natural numbers: the best solution is to find the right data structure to represent it, not to indict formal reasoning as incomplete and inconsistent.

Second, translation from logical to algebraic notation helps to uncover the right data structure. You would not want to do your calculus homework or balance your checkbook in Roman numerals; likewise it is difficult to perform calculations within PM. Gödel concluded that the existence of such a true-but-undecidable formula renders any formal system like PM essentially incomplete and incapable of proving even its own consistency. In this presentation I shall introduce a new method of analysis called computational algebraic logic that proves most of these conjectures wrong. Although Gödel’s special formula is indeed ‘undecidable’ in the sense that it has a value outside of the set \{TRUE, FALSE\}, such undecidability is a feature rather than a bug in formal reasoning: the formula is not semantically true.

A EXECUTIVE SUMMARY

\[
\mathcal{S}(q) = \begin{cases} q : (p, q) \in \mathbb{R}^2, p = 1, pq - p + 1 = 1, p = p^2, q = q^2 \end{cases}
\]

This system of polynomial equations is easy enough to solve by inspection; the only solution is \((p, q) = (1, 1)\) which gives the solution-value set \(\mathcal{S}(q) = \{1\}\). So the only feasible value for \(q\) is 1; therefore, we call \(q\) a theorem. Voilà, modus ponens by elementary algebra. What was missing from Boole’s work was a general method to solve systems of multivariate polynomial equations. Luckily for us, Buchberger invented such a method in the 1960s; today his Gröbner-basis algorithms are widely implemented in computer algebra systems.

Next, what does Gödel’s special formula look like in algebraic notation? To define a formula \(x\) that is true if and only if \(x\) itself is not provable, we can use a real-valued variable \(x\) and the equation \(x = (\mathcal{S}(x) \neq \{1\})\) in which the idea ‘\(x\) is not provable’ is expressed by saying that the solution-value set \(\mathcal{S}(x)\) is not equal to the set \(\{1\}\) (we adopt the usual computer-programming convention that the test of inequality \(\neq\) returns the numeric value 0 if its arguments are equal and 1 if they are not equal). This self-referential equation turns out to be a recurrence relation which is equivalent to the recurrence \(x_{n+1} = 1 - x_n\) with each \(x_n \in \{0, 1\}\). The mathematical object that is defined is a discrete dynamical system with two states \((x = 0 \text{ and } x = 1)\) and a periodic orbit that oscillates between them:

\[
1 \quad 0
\]

The graph shows exactly what is exceptional about Gödel’s special formula: it defines system that has no fixed points (steady state). This dynamical system encodes for \(x\) an infinite sequence \((\ldots, \{0\}, \{1\}, \{0\}, \{1\}, \ldots)\) of alternating solution-value sets, with a phase determined by what initial condition \(x_0\) was assumed.

Dynamical systems are the ‘complex numbers’ that capture the value of Gödel’s special formula and similar self-referential systems of equations. These dynamical systems are perfectly computable from self-referential formulas.
by ordinary means and their existence does not jeopardize the consistency of the formal system within which they are derived (elementary algebra). I do not think that any mathematician or philosopher is troubled that the Fibonacci numbers are not all the same: we should understand that Gödel’s special formula is the same kind of construction and that it yields a similarly dynamic sequence of elementary values. It is no more correct to say that Gödel’s special formula is semantically true or false than it is to say that Fibonacci’s formula is semantically zero, one, or two (or three, or five, or eight; you get the idea).

Within this framework of algebraic logic, many other types of exceptions can be recognized through computation. To complement Gödel’s special formula we might introduce a new formula \( y \) that is true if and only if it is provable:

\[
y = (S(y) = \{1\}).
\]

This is equivalent to the recurrence relation \( y_{t+1} \iff y_t \) with each \( y_t \in \{0, 1\} \). What is defined is a discrete dynamical system with two fixed points:

\[
\begin{array}{c}
0 \\
\uparrow \\
1
\end{array}
\]

Such a dynamical system is contingent. It encodes for \( y \) two different infinite sequences of solution-value sets. From the initial condition \( y_0 = 0 \) that \( y \) is not a theorem the sequence \((\{0\}, \{0\}, \{0\}, \ldots)\) confirms that \( y \) is never a theorem. And from the initial condition \( y_0 = 1 \) that \( y \) is a theorem the sequence \((\{1\}, \{1\}, \{1\}, \ldots)\) confirms that \( y \) is always a theorem.

Rounding out the host of exceptional formulas are those whose solution-value sets are empty and those whose solution-value sets have multiple members. Using Boole’s translation \( 1 - y \) for \( \neg y \), we might model the liar paradox with the equations \( z = 1 - z \) and \( z^2 = z \) for \( z \in \mathbb{R} \); in this case \( S(z) = \{\} \) and the formula \( z \) is unsatisfiable (just like the equation \( 0 = 1 \)). We might add the complementary truth-teller problem using \( w = w \) and \( w^2 = w \) with \( w \in \mathbb{R} \); in this case \( S(w) = \{0, 1\} \) and the formula \( w \) is ambiguous.

What Gödel saw as undecidability does not show some foundational defect in formal reasoning any more than irrational or imaginary numbers invalidate arithmetic. The myth of incompleteness is dispelled when self-reference is unmasked as a recurrence relation that defines a dynamical system. Gödel’s incompleteness argument is commonly understood to prove that logic and mathematics are fundamentally incompatible, but in fact he demonstrated quite the opposite: for formal reasoning it is essential to recognize that problems in logic are problems in algebra, just as Boole had demonstrated.
The methods presented in this analysis of Gödel’s incompleteness theorems can be applied to another well-known problem in logic, Bertrand Russell’s paradox (introduced in a 1902 letter to Frege that is reproduced in [30]). Russell asked whether a special set—the set of all sets that do not contain themselves—contains itself or not; either answer seems paradoxical. Although Russell’s problem does not translate directly as a system of polynomial equations, it is nonetheless a system of mathematical constraints and the same insights apply. Russell’s construction is unsatisfiable in the static interpretation and unsteady in the dynamic interpretation; it behaves exactly like Gödel’s special formula. Furthermore, a bit of reflection reveals that the specific issue with Russell’s paradox is neither totality nor unrestricted comprehension but indefiniteness: his construction requires the set being defined to be used as a free variable in the predicate that defines it, which violates even the unrestricted axiom schema of comprehension. Regarding Russell’s paradox, it is sets of humans not axioms of sets that have a problem with comprehension.

B.1 Algebraic Analysis

Let us define \( S \) as a set of sets that includes at least some set \( r \), and further define \( r \) as the set of all members of \( S \) that are not members of themselves. We desire the truth value of the formula \( r \in r \) that asks whether the set \( r \) is a member of itself. The definitions of \( r \) and of \( S \) provide two constraints that constitute a generalized (beyond polynomial equations) system of constraints:

\[
\begin{align*}
 r & := \{ s \in S : s \notin s \} \\
 r & \in S
\end{align*}
\]

In the special case that \( S \) is considered to be the set of all possible sets (assuming for the moment that such a construct is meaningful), this pair of constraints amounts to the classic formulation of Russell’s paradox. But Equations 103 and 104 are not limited to that special case; in particular \( S \) may denote a countable and finite set of sets and it is not important whether \( S \) contains itself. It is, however, essential to the problem that \( S \) contains \( r \).

In order to ascertain whether the set \( r \) thus defined contains itself, we use the solution-value set for the formula \( r \in r \) subject to the constraints in Equations 103 and 104:

\[
A_{\{s \in S : s \notin s \}, \in S}(r \in r)
\]

Using the methods in Section 3 there are static and dynamic ways to interpret this solution-value set. In terms of Definition 12 we use the value of the set-membership expression \( r \in r \) as the parameter \( \theta \) for this problem. We note that the expression \( r \in r \) takes a value in the Boolean finite field \( \mathbb{F}_2 := \{0, 1\} \) according to whether the relation in it is false or true. Thus if the relation \( r \in r \) is false we have \( (r \in r) = 0 \) (equivalently \( r \notin r \)) and \( \theta = 0 \); conversely if the relation is true we have \( (r \in r) = 1 \) (equivalently \( r \in r \)) and \( \theta = 1 \).

Using a hypothetico-deductive approach we consider these two possible cases \( \theta = 0 \) and \( \theta = 1 \) for the parameter \( \theta = (r \in r) \) as illustrated in Table 4. In each instantiation \( A(\theta_i) \) the conjectured value of the formula \( r \in r \) is specified as an explicit constraint (since it cannot be substituted directly into an existing constraint as in the earlier examples). From either conjectured value we compute the other. That is, from the conjecture \( \theta = \theta_1 = 0 \) indicating \( r \notin r \) the constructed set \( r \) includes itself and hence the computed value \( \theta_1 = (r \in r) \Rightarrow 1 \). Likewise from the conjecture \( \theta = \theta_2 = 1 \) indicating \( r \in r \) the constructed set \( r \) does not include itself and so the computed value \( \theta_2 = (r \in r) \Rightarrow 0 \). Thus from Equations 103 and 104 we derive the state transitions \( 0 \mapsto 1 \) and \( 1 \mapsto 0 \) defining the following dynamical system:

\[
\begin{array}{c}
 r \notin r \\
{\{1\}} \\
\hline
 r \in r \\
{\{0\}}
\end{array}
\]

with the states labeled \( r \notin r \) and \( r \in r \) instead of \( \theta = 0 \) and \( \theta = 1 \) for convenience. Since this dynamical system has no fixed points, we conclude in this dynamic interpretation that the formula \( r \in r \) is unsteady subject to the constraints in Equation 103 and 104. Moreover, in the static interpretation the formula \( r \in r \) is unsatisfiable given the constraints in Equation 103 and 104 since its conjectured and computed values never agree.
Table 4  The state-transition worksheet for Russell’s set-inclusion query using the parameter \( \theta := (r \in r) \).

| \( i \) | \( \theta_i \) | \( A(\theta_i) \) | \( \mathcal{J}_{A(\theta_i)}(r \in r) \) | \( F(\theta_i) \) |
|---|---|---|---|---|
| 1 | 0 | \{ \( r = \{ s \in S : s \notin s \}, r \in S, r \notin r \} \} | \{ 1 \} | 1 |
| 2 | 1 | \{ \( r = \{ s \in S : s \notin s \}, r \in S, r \in r \} \} | \{ 0 \} | 0 |

Russell’s special formula specifies exactly the same dynamical system as Gödel’s special formula: an oscillator with two states. This coincidence is not surprising: when the definition of a formula includes a recursive reference to a solution-set feature that has two possible states, the only way to make an unstable/unsatisfiable system is to have the conjecture of either state lead to the computation of the other. Such instability is exactly what seems paradoxical about formulas like Gödel’s and Russell’s (for some reason the contingent/ambiguous dual of each problem is not perceived as paradoxical).

B.2 Russell’s Free Variable Revealed

In more conventional language, Russell’s construction necessarily employs the set being defined as a free variable in the predicate that defines it—which renders that predicate indefinite. It is important to recognize such indefinite predicates, which Zermelo recognized as problematic in his axiomatization of set theory \([34]\). In particular, when specifying the construction of a set by the action of a predicate upon the members of some universe of discourse, the members of that universe of discourse must be considered when evaluating the predicate for free variables.

**Lemma 30 (Revealing Free Variables in Set-Building)** Consider the definition of a set \( y \) using some universe of discourse \( z \) and some predicate \( \phi(x) \):

\[
y := \{ x \in z : \phi(x) \}
\]

Every member of the set \( z \) that is a variable is in fact a free variable in the predicate \( \phi(x) \). In particular, if the set \( y \) being defined is a member of the set \( z \) used in its own definition, then the instantiation \( \phi(y) \) of the predicate (with \( y \) as the argument) must be evaluated during the construction of \( y \). It is evident that \( y \) is a free variable in this instance \( \phi(y) \). Leaving the universe of discourse anonymous as in the unrestricted definition \( y := \{ x : \phi(x) \} \) does not change this property; if \( y \) is a possible value for \( x \) then the variable \( y \) must still be considered free in the predicate \( \phi(x) \).

Actually, it is not just the predicate \( \phi(x) \) that is at issue; it is the entire right-hand-side expression \( \{ x \in z : \phi(x) \} \) in the definition. Even with a definition such as \( y := \{ x \in z : \top \} \) whose trivial predicate \( \top \) (the elementary value true) has no variables, the definiendum \( y \) must be considered a free variable in the definiens \( \{ x \in z : \top \} \) if and only if \( y \in z \); similarly, the definiendum \( y \) is a free variable in the definiens \( \{ x : \top \} \) of the unrestricted definition \( y := \{ x : \top \} \).

It is important to recognize that Zermelo’s \([34]\) axiom schema of separation (Axiom III) contains two clauses: a separation clause and a definiteness clause. That axiom is repeated here:

Whenever the propositional function \( \mathcal{E}(x) \) is definite for all elements of a set \( M \), \( M \) possesses a subset \( M_\mathcal{E} \) containing as elements precisely those elements \( x \) of \( M \) for which \( \mathcal{E}(x) \) is true.

The idea of a definite propositional function (addressed in Lemma\([30]\)) already entails the principle of separation, in the sense that unrestricted set comprehension always implies an indefinite predicate. For example, using the unrestricted definition \( y := \{ x : \phi(x) \} \) the variable \( y \) must be free in the predicate \( \phi(x) \) by the argument above; hence the predicate is indefinite. To be sure, an expression that includes free variables could still have a definite value if other constraints in the system limit the values of those variables; for the moment we assume the absence of such additional constraints.

However, requiring separation without requiring definiteness does not close the loophole in naive set theory. Simply introducing some independently-defined set-of-sets \( z \) to restrict set comprehension as in \( y := \{ x \in z : \phi(x) \} \) is not sufficient to ensure definiteness; in the case \( y \in z \) the variable \( y \) remains free in \( \phi(x) \) and thus the predicate remains indefinite (at some point in the construction of \( y \) the predicate \( \phi(y) \) must be evaluated to determine whether to include \( y \) in itself). That is all to say that an unrestricted axiom schema of comprehension, retaining the definiteness clause from Zermelo’s Axiom III but omitting its separation clause, would be sufficient to prevent Russell’s paradox. Conversely a
restricted axiom schema of comprehension retaining the separation clause but omitting the definiteness clause would not prevent the paradox. The principle that a definition requires a definite expression in its definiendum is not at all peculiar to set theory!

Returning to Equation 103 in the definition \( r := \{ s \in S : s \notin s \} \), the variable \( r \) is free in the predicate \( s \notin s \) exactly if \( r \) is a member of \( S \). Hence using this restricted definition it would violate the definiteness clause in Zermelo’s Axiom III to consider \( r \) a set exactly if \( r \in S \). In the unrestricted definition \( r := \{ s : s \notin s \} \), the variable \( r \) is always free in the predicate \( s \notin s \) since \( r \) is considered to be a member of the universe of discourse from which the values of \( s \) are drawn. Hence both clauses of Zermelo’s Axiom III would be violated by the unrestricted definition. Using either the restricted or the unrestricted formulation, at some point the predicate to be evaluated must become the expression \( r \notin r \) in which \( r \) is plainly a free variable; thus the predicate is indefinite. The precise issue with Russell’s paradox is neither totality nor unrestricted comprehension; it is the attempt to define a set using an indefinite predicate.

One can imagine a set theory in which Zermelo’s definiteness clause is relaxed and thereby it is allowed to use the set being defined as a free variable in its own definition. In such a liberalized set theory it would be possible to use recurrence relations to define dynamical systems that give infinite sequences of sets, just as it is possible to use a recurrence relation to define the Fibonacci sequence in elementary algebra. Russell’s construction would give one of these dynamical systems, characterized by a periodic orbit that oscillates between two states as shown in the graph in Equation 106. The unrestricted ‘set of all sets’ (as from \( y := \{ x : T \} \)) would define a different dynamical system, as if by the recurrence relation \( y_{ i+1} \leftarrow y_i \cup \{ y_i \} \) (this system would have no fixed points, but orbits that are neither periodic nor convergent). As in the earlier examples, these dynamical systems could instead be interpreted as static constraints (both Russell’s special set and the ‘set of all sets’ \( y := \{ x : T \} \) would be unsatisfiable in this interpretation).

The method of algebraic analysis detailed above illustrates exactly which exceptions can occur if a set \( y \) is allowed to be a free variable in its own definition (either from the explicit circumstance \( y \in \varepsilon \) or from unrestricted comprehension): unsteadiness and contingency. (Note that a self-referential set \( y \) could also have a steady, unique value and be unexceptional.) In this sense we gain the ability to discern more details about the various paradoxes in naive set theory, for example to clarify how ‘the set of all sets that do not contain themselves’ differs mathematically from ‘the set of all sets that do not contain themselves and Godel’s ‘logical formula that asserts its own unprovability.’

B.3 Functional Programming

As a final thought on Russell’s non-paradox, it is interesting to me as a computer programmer that the passage in Frege’s Begriffsschrift that inspired Russell to create his set-inclusion problem also has a computer-programming interpretation. This is how Russell introduced his problem [23]:

There is just one point where I have encountered a difficulty. You state (p. 17) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction.

Indeed, in the referenced §9 of Begriffsschrift Frege had written, “On the other hand, it may be that the argument is determinate and the function indeterminate.” This is idea can be explored in terms of predicates and sets, as Russell did. But it is also a fairly straightforward description of functional programming—the idea that functions should be treated as first-class mathematical objects (like numbers). This is implemented in several computer programming languages (most prominently Lisp [27]). Even beyond the conceptual appeal of such an object-oriented approach, it can be practical to solve systems of equations with functions as unknowns, when the type of function is suitably restricted (e.g. to polynomials over a given set of variables with coefficients in the finite Boolean field \( F_2 \): Boole’s function development method allows parametric descriptions of arbitrary logical functions of any desired arity). Some challenging problems in logic, such as Smullyan’s puzzles about liars and truth-tellers [25], amount to solving for unknown logical functions rather than unknown elementary truth values. The calculus of infinitesimals is another place in mathematics where functions are treated as first-class objects. What is integration except solving a system of constraints (on rates of change and perhaps boundary values) in which a function is the unknown variable?
C  NOTES ON MODAL LOGIC: THINKING OUTSIDE THE BOX

We can use dynamic polynomial logic to define modal operators in terms of solution-value sets. First the alethic modes corresponding to the members of \(2^{[0,1]}\) (here using \(s\) to abbreviate the solution-value set \(S_A(p)\) of the objective \(p(x_1,x_2,\ldots,x_n)\) relative to the set \(A\) of axioms):

- \(\mathbf{□}\) (or \(\mathbf{□}_1\)) necessarily true: \(\mathbf{□}(p)\) means \(s = \{1\}\) and \(\mathbf{□}(\neg p)\) means \(s = \{0\}\)
- \(\mathbf{□}_0\) necessarily false: \(\mathbf{□}_0(p)\) means \(s = \{0\}\) and \(\mathbf{□}_0(\neg p)\) means \(s = \{1\}\)
- \(\mathbf{□}\) ambiguous (mnemonic \(\mathbf{□}\) for all truth values): \(\mathbf{□}(p)\) or \(\mathbf{□}(\neg p)\) mean \(|s| > 1\) thus \(s = \{0, 1\}\)
- \(\mathbf{□}\) unsatisfiable (mnemonic empty set \(\emptyset\)): \(\mathbf{□}(p)\) or \(\mathbf{□}(\neg p)\) mean \(s = \{\}\)

Then some hybrids, each corresponding to several members of \(2^{[0,1]}\):

- \(\mathbf{□}\) (or \(\mathbf{□}_1\)) possibly true: \(\mathbf{□}(p)\) means \(1 \in s\) thus \(s \in \{\{1\}, \{0, 1\}\}\) and \(\mathbf{□}(\neg p)\) means \(0 \in s\) thus \(s \in \{\{0\}, \{0, 1\}\}\)
- \(\mathbf{□}\) possibly false: \(\mathbf{□}_0(p)\) means \(0 \in s\) thus \(s \in \{\{0\}, \{0, 1\}\}\) and \(\mathbf{□}_0(\neg p)\) means \(1 \in s\) thus \(s \in \{\{1\}, \{0, 1\}\}\)
- \(\mathbf{□}\) definite (mnemonic ‘get to the point’): \(\mathbf{□}(p)\) or \(\mathbf{□}(\neg p)\) mean \(|s| = 1\) thus \(s \in \{\{0\}, \{1\}\}\)

Perhaps we should use \(\mathbf{□}(p|A)\) or \(\mathbf{□}(p,q_1,q_2,\ldots,q_m)\) etc. to make the axioms explicit.

These operators and their negations cover 14 of the 16 members the power set \(2^{[0,1]}\). Of the remaining 2 members \(s \in \{\}, \{0\}, \{1\}, \{0, 1\}\) is tautological and \(s \in \{\}\) is impossible (even the empty set is not a member of itself). Modal operations are recurrence relations; they must be evaluated as dynamical systems (though it is fine to report static interpretations). All these modal operators have the property that they can be evaluated during hypothetico-deductive analysis: given a computed solution-value set \(s\) it is possible to tell whether any operator is satisfied using standard set operations (equality, inequality, membership, complement, cardinality). For multiple variables \((x_1,x_2,\ldots,x_n)\) we should probably use the solution set of all variables as the state of the dynamical system.

For \(d\)-valued logic we could introduce \(d\) distinct modal necessity operators \(\mathbf{□}_0, \mathbf{□}_1, \ldots, \mathbf{□}_{d-1}\) with each \(\mathbf{□}_k\) meaning \(s = \{k\}\). The remaining \(2^d - d - 1\) non-empty solution-value sets in the power set of the set of elementary values (each necessarily with cardinality \(> 1\)) would satisfy \(\mathbf{□}\) (thus making it hybrid rather than singleton). We could likewise introduce \(d\) distinct modal possibility operators where each \(\mathbf{◊}_k\) means \(k \in s\). Whatever special value \(k^*\) is considered ‘true’ would customarily be omitted from the operator sign: thus the unadorned \(\mathbf{□}\) and \(\mathbf{◊}\) instead of \(\mathbf{□}_{k^*}\) and \(\mathbf{◊}_{k^*}\).

Note that the position of negation matters. For example \(\neg(\mathbf{□}p)\) means \(s \neq \{1\}\) thus the solution-value set \(s\) could be \(\{\}, \{0\}, \text{ or } \{0, 1\}\): whereas \(\mathbf{□}(\neg p)\) means the solution-value set \(s\) is definitely \(\{0\}\). In other words we distinguish between ‘not necessarily true’ and ‘necessarily false’. Note also that ‘possible’ here does not follow classical modal logic: \(\mathbf{◊}(p)\) is not equivalent to \(\neg\mathbf{□}(\neg p)\) because the latter includes the possibility \(s = \{\}\) that \(p\) is unsatisfiable whereas the former does not. That is, we distinguish between ‘possible’ and ‘not necessarily false’. Also \(\mathbf{□}(\neg p)\) means the same as \(\mathbf{□}(\neg p)\): the negation of an ambiguous formula is itself ambiguous.

In this framework \(\mathbf{□}(p)\) does not mean the same thing as \(p\). In fact the axiom \(\vdash \mathbf{□}(p)\) is a quite different assertion from the axiom \(\vdash p\); in isolation, the former specifies a dynamical system with a fixed point at the state that \(p\) is unsatisfiable. However the solution-value set \(S_A(\mathbf{□}(p))\) evaluates to \(\{1\}\) exactly if the solution-value set \(S_A(p)\) also yields \(\{1\}\); in this sense \(\mathbf{□}(p)\) is similar to \(p\) when these expressions are treated as objectives.

Only unsatisfiability is ‘explosive’: from a contradiction it follows that all formulas are unsatisfiable (including the constants 0 and 1: e.g. \(S_A(1) = \{1, 1 = 0\} \Rightarrow \{\}\) ). Thus if \(\mathbf{□}(p|A)\) is true for any formula \(p\) then it must be true for all formulas subject to the same axioms \(A\). Also, the sense in which algebraic proof is monotonic is that adding constraints cannot add members to solution sets; it can leave them unchanged or remove members. Thus for any sets \(A\) and \(B\) of axioms and any formula \(p\) we have \(S_{A\cup B}(p) \subseteq S_A(p)\) and \(S_{A\cup B}(p) \subseteq S_B(p)\). It is not the case that \(\mathbf{□}(p|A)\) guarantees \(\mathbf{□}(p|A,B)\) since the additional axioms \(B\) could render the whole system unsatisfiable; thus we could have \(\mathbf{□}(p|A)\) but \(\mathbf{□}(p|A,B)\). Algebraic proof is global not local; all constraints must be taken into account, and the solution set \(S\) is what is left after all infeasible solution values have been eliminated.
Sketch of commands for a proposed computer system that implements ‘Logic Query Language’ incorporating dynamical systems, Boolean translation from logic to algebra, and polynomial equations.

/* george: computational algebraic logic (LQL) */

parameter c in {0,1,2};
b := 2; // simple macro assignment, resolved at compile time
c := |$x|; // update rule: c gets size of solution-value set $x of x
x^2 + b*x + c == 0; // equation template
x^2 + b*x + |$x| == 0; // same constraint, implicit parameter definition

parameter t in FF(2);
t := $y == {1}; // update rule: t gets true if y was proved a theorem
t := ?y; // same update rule: t gets true if y was proved a theorem
y == 1 - t; // template: y true if it was not proved a theorem
|- y <-> !t; // same constraint in logical notation, |- for assert
y == {$y != {1}}; // same constraint with implicit parameter definition
y == ?y; // same with modal operator ? (necessarily true)
y = !?y; // interpret as static constraint or recurrence?

// Fibonacci-like recurrence relation: update rules but no constraints
parameter t1, t2 in NN;
t1 := t2;
t2 := t1 + t2;
t1[0]=0; t2[0]=1; // initial conditions give t1[t] : (0,1,1,2,3,5,8,...)

function F(2,2) indefinite<z>; // arity 2, each argument in FF(2)
// creates variables z[0,0], z[0,1], z[1,0], z[1,1] for coefficients
// call as F(x,y), for which polynomial function of x, y, z[i,j] substituted

parameter d;
function G(d,d,d) indefinite<w>; // arity 3, each argument in FF(d)
// creates variables w[0] .. w[d^3] for coefficients

function G(x,y) : x -> y || y -> x; // a definite function
G(x,y) : x -> y || y -> x; // a definite function

(:A x,y,z: F(x,y) -> z) // universal quantifier
( :E x : F(x,x) -> 1 ) // existential quantifier

% solve x; // runtime query solution-value set for x
% $x; // same: returns recurrence F(x), domain U={0,1,2}, objective x
% ?x; // ask if dynamic theorem: exactly one fixed pt x == F(x) with x == 1?
% $ t1 @ (0,1); // to get sequence (0,1,1,2,3,5,8,...)
% $ t1; // to get collection { {0}@0(0,0) }
% $ x @@ // collection of sequences $x@0, $x@1, $x@2

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ACHILLES had once again overtaken the tortoise, and had seated himself comfortably on its back.

“So you’ve understood the Liar?” said the Tortoise, “even though he told you he was lying? I thought some mathematician or other proved that the thing couldn’t be done?”

“It can be done,” said Achilles. “It has been done. Quod erat faciendum. You see, it was a simple matter of solving a system of simultaneous equations; and so—”

“But what if the equations did not happen all at the same time?” the Tortoise interrupted, “What then?”

“Then I shouldn’t have solved them,” Achilles modestly replied; “and you would have got several times round the world, by this time!”

“You impress me—compress, I mean,” said the Tortoise; “for you are truly dense, and no mistake! Well now, would you like to hear of some logical propositions that are supposed to be true or false, while they are really neither of the two?”

“Achilles rose to his feet and balanced atop the Tortoise, still writing furiously. “So my answer depends upon itself, rather like a turtle standing on its own shell?”

“But just what do you suppose that answer depends upon, my gallant Greek? Surely you can tell me something more particular about it.”

“Very clever, young Tortoise, very clever” said Achilles, “but it’s turtles all the way down! I dare say any particular answer would be nonsense.”

“Nonsense indeed,” replied the Tortoise, “But which kind of nonsense? There are two different methods to this madness, the unsteady kind and the contingent kind. You can tell them apart if you start from the beginning, which is after all why we call it so.”

“Let us start with the very first turtle—or urtle if you like, since it has no beginning.”

“Achilles sighed, “With such a long and wandering explanation, which will surely vex some Logicians of the Twenty-First Century—would you mind renaming yourself Tor-tu-ous?”

“As you please!” replied the rational reptile, admiring the warrior’s tidy notes, “Provided that you, for your excellent penmanship, will call yourself A Quill Ease!”
REFERENCES

[1] Juan C. Agudelo and Walter Carnielli. Polynomial ring calculus for modal logics: A new semantics and proof method for modalities. Review of Symbolic Logic, 4:150–170, 2011.

[2] Dennis S. Arnon. A bibliography of quantifier elimination for real closed fields. Journal of Symbolic Computation, 5:267–274, 1988.

[3] E. T. Bell. The Development of Mathematics. McGraw-Hill, second edition, 1945.

[4] George Boole. The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning. Macmillan, London, 1847.

[5] George Boole. An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities. Macmillan, London, 1854.

[6] Bruno Buchberger. Gröbner bases: A short introduction for systems theorists. Lecture Notes in Computer Science, 2178:1–19, 2001.

[7] Walter Carnielli. Polynomizing: Logical Inference in Polynomial Format and the Legacy of Boole, volume 64 of Studies in Computational Intelligence, pages 349–364. Springer, 2007.

[8] Lewis Carroll. A logical paradox. Mind, New Series, 3(11):436–438, 1894.

[9] Lewis Carroll. What the tortoise said to Achilles. Mind, New Series 4(14):278–280, 1895.

[10] Allan Clark. Elements of Abstract Algebra. Wadsworth, Belmont, CA, 1971. Reprinted by Dover in 1984.

[11] David Cox, John Little, and Donal O’Shea. Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra. Springer, New York, third edition, 2007.

[12] Howard Eves. Foundations and Fundamental Concepts of Mathematics. PWS-Kent, Boston, third edition, 1990. Reprinted by Dover in 1997.

[13] Gottlob Frege. Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens. Louis Nebert, Halle (Saale), 1879. English translation “Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought” appears in [30].

[14] Oded Galor. Discrete Dynamical Systems. Springer, New York, 2010.

[15] Kurt Gödel. Über formal unentscheidbare Sätze der Principia mathematica und verwandter Systeme I. Monatshefte für Mathematik und Physik, 38:173–198, 1931. English translation “On formally undecidable propositions of Principia mathematica and related systems I” appears in [30].

[16] Rebecca Goldstein. Incompleteness: The Proof and Paradox of Kurt Gödel. Norton, New York, 2005.

[17] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. Concrete Mathematics. Addison-Wesley, second edition, 1994.

[18] Richard A. Holmgren. A First Course in Discrete Dynamical Systems. Springer, New York, second edition, 1996.

[19] Deepak Kapur and Paliath Narendran. An equational approach to theorem proving in first-order predicate calculus. ACM SIGSOFT Software Engineering Notes, 10(4):63–66, 1985.

[20] Ernest Nagel and James R. Newman. Gödel's Proof. New York University Press, revised edition, 2001.

[21] Emil Leon Post. Introduction to a general theory of elementary propositions. American Journal of Mathematics, 43:163–185, 1921. Reprinted in [30].

[22] Eugenio Roanes-Lozano, Luis M. Laita, and Eugenio Roanes-Macías. A polynomial model for multi-valued logics with a touch of algebraic geometry and computer algebra. Mathematics and Computers in Simulation, 45:83–99, 1998.

[23] Bertrand Russell. Letter to Frege, 1902. First published in [30].

[24] Pieter A. M. Seuren. Eubulides as a 20th-century semanticist. Language Sciences, 27:75–95, 2005.

[25] Raymond Smullyan. What is the Name of This Book? Prentice-Hall, Englewood Cliffs, NJ, 1978.

[26] Ernst Snapper. The three crises in mathematics: Logicism, intuitionism, and formalism. Mathematics Magazine, 4:207–216, 1979.

[27] Guy L. Steele. Common Lisp: The Language. Digital Press, second edition, 1990.

[28] M. H. Stone. The theory of representation for Boolean algebras. Transactions of the American Mathematical Society, 40:37–111, 1936.
[29] Alfred Tarski. A decision method for elementary algebra and geometry. Technical Report R-109, RAND Corporation, Santa Monica, CA, 1948.

[30] Jean van Heijenoort, editor. From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931. Harvard University Press, 1967.

[31] Alfred North Whitehead and Bertrand Russell. Principia Mathematica, volume 1–3. Cambridge University Press, 1910–1913.

[32] Ludwig Wittgenstein. Tractatus Logico-Philosophicus. Routledge & Kegan Paul Ltd, London, 1922. Translated from German by C. K. Ogden; reprinted by Dover in 1999.

[33] Wolfram Research, Inc. Mathematica. Champaign, IL, 2010. Version 8.0.

[34] Ernst Zermelo. Untersuchungen über die Grundlagen der Mengenlehre I. Mathematische Annalen, 65:261–281, 1908. English translation “Investigations in the foundations of set theory I” appears in [30].