1. Introduction

Euclidean correlation functions in the static approximation are known to be very noisy, the noise to signal ratio $R_{NS}$ growing exponentially with the time separation. For the Eichten-Hill (EH) action $\Psi$, the law

$$R_{NS} = \frac{\text{noise}}{\text{signal}} \propto \exp(x_0 \Delta), \quad \Delta = E_{\text{stat}} - m_\pi,$$

is roughly fulfilled [2], with the ground state energy of a heavy meson $E_{\text{stat}}$ being linearly divergent while approaching the continuum limit.

Here we explore the possibility of reducing the exponent in eq. (1) by changing the discretisation of the static lattice action. On the other hand we want to retain some properties of the Eichten-Hill action in order to preserve the same level of $O(a)$ improvement, in particular

- heavy quark spin symmetry,
- local conservation of heavy quark flavor number.

2. Numerical Results

We mainly looked at the gain concerning noise reduction and at some scaling properties for the new actions. The setup is defined by the Schrödinger functional scheme implemented with non-perturbatively $O(a)$ improved Wilson actions together with gauge, cubic and parity invariance and locality. Writing the static lattice action

$$S_{\text{stat}}^W = a^4 \sum_x \bar{\psi}_h(x) D_0 \psi_h(x),$$

$$D_0 \psi_h(x) = \frac{1}{a} \left[ \psi_h(x) - W^\dagger(x - a\hat{0}, 0) \psi_h(x - a\hat{0}) \right],$$

we studied the cases

$$W_S(x, 0) = V \left[ \frac{g_0^2}{5} + \left( \frac{1}{3} \text{tr} V^\dagger V \right)^{1/2} \right]^{-1},$$

$$W_A(x, 0) = V,$$

$$W_{\text{HYP}}(x, 0) = V_{\text{HYP}},$$

where $V$ is the average of the 6 staples around the link $U(x, 0)$ and $V_{\text{HYP}}$ is the HYP-link $[3]$.\(^2\)

\(^2\)For the coefficients $\alpha_1, \alpha_2$ and $\alpha_3$ defining the HYP-smearing we used the values proposed in [3].
for the gauge and the light quark sectors. A more
detailed discussion of the framework and defini-
tions of the correlation functions can be found in [4]. Our aim is the computation of the \( B_s \)-
meson decay constant as well as other b-physics matrix
elements.

2.1. Noise reduction and cut-off effects

In Fig. 1 we show \( R_{NS} \) for the static-light axial
correlator \( f_A^{stat} \) as obtained from an ensemble of
2500 quenched configurations. This is one of the
ensembles used for the computation of \( F_{B_s} \sqrt{m_{B_s}} \)
in the static limit. The Figure shows that more
than an order of magnitude can be gained in
\( R_{NS}(x_0 \approx 1.5 \text{ fm}) \) with respect to the \( EH \) ac-
tion (filled circles) by using \( S_{stat}^S \) (empty squares),
which behaves similarly to \( S_{stat}^{A} \). The picture is
even better for \( S_{stat}^{HYP} \) (empty triangles).

We then studied the scaling behaviour for a set of
observables. In [4] we reported about the step
scaling function of the static-light axial current
renormalisation constant \( Z_{stat}^A \) (see [5]). Here we
want to present a study of the quantity

\[
h = \frac{f_A^{stat}(T/2)_{\theta=0}}{f_A^{stat}(T/2)_{\theta=1/2}} \quad \text{at} \quad m_q = 0,
\]

where \( \theta \) is the angle defining the periodicity of the
fermions. Our results are shown in Fig. 2; they refer to four different lattice resolutions of an \( L^4 \)
volume with \( L = 1.436r_0 \) with \( r_0 = 0.5 \text{ fm} \).

Throughout all our computations we defined the \( O(a) \) improved correlator \( f_A^{stat} \) by using the
tree level value 1/2 for \( b_A^{stat} \) [6] and the 1-loop

![Figure 1. \( R_{NS} \) for \( f_A^{stat} \) from a 243 x 36, \( \beta = 6.2 \)
lattice. See text for the symbols.](image)

![Figure 2. Scaling plot for \( h \). Symbols as in Fig. 1 (empty circles refer to \( S_{stat}^A \))](image)

results for \( c_A^{stat} \). In particular we set

\[
c_A^{stat} = -0.08237g_0^2 + O(g_0^4), \quad S_{stat} = S_{stat}^{EH},
\]

\[
c_A^{stat} = -0.1164(10)g_0^2 + O(g_0^4), \quad S_{stat} = S_{stat}^S, S_{stat}^{A},
\]

\[
c_A^{stat} = -0.090(3)g_0^2 + O(g_0^4), \quad S_{stat} = S_{stat}^{HYP}.
\]

The first two numbers have been worked out in
perturbation theory, while the last one has been
numerically estimated by solving with respect to
\( c_A^{stat}(S_{stat}^{HYP}) \) the implicit equation

\[
\frac{f_A^{stat}(T/2)_{\theta=0}}{f_A^{stat}(T/2)_{\theta=1/2}^{HYP}} = 1.
\]

This defines an improvement condition for a dis-
cretisation \( (S_{stat}^{HYP}) \), once the improvement co-
efficients for another discretisation \( (S_{stat}^{A}) \) are
known. In Fig. 3 we show our numerical results for the coefficients \( c_A^{stat,(1)}(S_{stat}^{A}) \) and
\( c_A^{stat,(1)}(S_{stat}^{HYP}) \), taking as input \( c_A^{stat}(S_{stat}^{A}) \). The
same data sets used for computing \( h \) in eq. 6
have been used here \( (\theta = 0, \theta' = 1/2) \). The lower
part of the plot can be regarded as a test of the
method.

2.2. The \( B_s \)-meson decay constant

The setup has been used for computing

\[
\Phi_{RGI} \propto Z_{RGI} f_A^{stat}(x_0) e^{(x_0 - T/2)E_{stat}(x_0)},
\]

for which we have calculated the regularisation
dependent part of the renormalisation constant
\( Z_{RGI} \) exactly as done in [4] for the \( EH \) action.
Here the \( B_s \)-meson boundary to boundary cor-
relation function \( f_1 \) enters. The proportional-
ity symbol in eq. 8 summarises volume factors
coming from the normalisation of the correlation.
functions, cmp. [4]. In addition we introduced hydrogen-like wavefunctions on the boundaries of the Schrödinger functional in order to minimise the overlap with the first excited state; we arrive at a plateau of length 0.8 fm in $\Phi_{\text{RGI}}$ and the desired ground state matrix element can be extracted with confidence.

The quantity $F_{B_s}\sqrt{m_{B_s}}$ is related to $\Phi_{\text{RGI}}$ via

$$\Phi_{\text{RGI}} = \frac{F_{B_s}\sqrt{m_{B_s}}}{C_{PS}(M_b/\Lambda_{MS})} + O(1/M_b),$$

the function $C_{PS}(M_b/\Lambda_{MS})$ [5] being known in perturbation theory up to and including $\bar{g}^4(m_b)$ corrections to the leading order [7].

We computed $\Phi_{\text{RGI}}$ for three different lattice resolutions ($\beta = 6, 6.1$ and 6.2) on a $L^3 \times T$ topology with $T/L = 3/2$ and $L = 1.5$ to 1.9 fm. The continuum limit extrapolation of $r_0^{3/2}\Phi_{\text{RGI}}$, evaluated from $S_{\text{HYP}}$ at $x_0 = T/2$, is shown in Fig. 4. The extrapolated value is

$$r_0^{3/2}\Phi_{\text{RGI}} = 1.74(13).$$

3. CONCLUSIONS

We have shown that the problem with the statistical precision of correlation functions computed with the Eichten-Hill action can be overcome by changing the lattice discretisation. In particular the largest gain is obtained by making use of the HYP-links in the static action. Cut-off effects for the proposed actions turned out to be of the same size as for the EH action, and are in general rather small. The improvements presented led to a computation of the quantity $\Phi_{\text{RGI}}$ in the continuum limit with an error of 7%. We are presently reducing this error further by adding one more lattice resolution ($\beta = 6.45$). The same data can be used for improving the results on the renormalisation group invariant $b$-quark mass along the lines described in [8]. Finally, a precise determination of $F_{B_s}$ can be obtained by combining the result in eq. (10) with data around the charm quark mass.

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