This is an electronic reprint of the original article. This reprint may differ from the original in pagination and typographic detail.

Author(s): Geiss, Christel; Steinicke, Alexander

Title: L2-variation of Lévy driven BSDEs with non-smooth terminal conditions

Year: 2016

Version: 

Please cite the original version: Geiss, C., & Steinicke, A. (2016). L2-variation of Lévy driven BSDEs with non-smooth terminal conditions. Bernoulli, 22(2), 995-1025. https://doi.org/10.3150/14-BEJ684

All material supplied via JYX is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.
L₂-variation of Lévy driven BSDEs with non-smooth terminal conditions

CHRISTEL GEISS¹ and ALEXANDER STEINICKE²

¹Department of Mathematics and Statistics, University of Jyväskylä, FI-40014, Finland.
E-mail: christel.geiss@jyu.fi
²Department of Mathematics, University of Innsbruck, 6020 Innsbruck, Austria.
E-mail: alexander.steinicke@uibk.ac.at

We consider the L₂-regularity of solutions to backward stochastic differential equations (BSDEs) with Lipschitz generators driven by a Brownian motion and a Poisson random measure associated with a Lévy process (Xₜ)ₜ∈[0,T]. The terminal condition may be a Borel function of finitely many increments of the Lévy process which is not necessarily Lipschitz but only satisfies a fractional smoothness condition. The results are obtained by investigating how the special structure appearing in the chaos expansion of the terminal condition is inherited by the solution to the BSDE.

Keywords: backward stochastic differential equations; Chaos expansion; L₂-regularity; Lévy processes; Malliavin calculus

1. Introduction

The main objective of this paper consists in studying the relation between fractional smoothness of the terminal condition of a BSDE and the L₂-variation of its according solution.

A motivation to investigate this relation arises from the fact that the convergence rate of the discretization error of BSDEs with Lipschitz generator is determined by the convergence of the discretized terminal condition to its limit and by the L₂-variation properties of the exact solution (Y, Z).

In the Brownian scenario, the discretization of BSDEs has been studied by many authors, see, for example, [4,10,11,14,27,28,36]. If the BSDE is given by

\[ Y_t = \xi + \int_t^T F(s,Y_s,Z_s) \, ds - \int_t^T Z_s \, dW_s, \quad 0 \leq t \leq T, \]

we define the Lₚ-variation

\[ \text{var}_p(\xi,F,\tau) := \sup_{1 \leq i \leq n} \sup_{t_{i-1} \leq s \leq t_i} \| Y_s - Y_{t_{i-1}} \|_p + \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \| Z_t - \hat{Z}_{t_{i-1}} \|_p^2 \, dt \right)^{1/2}, \]

where \( \tau = (t_i)_{i=0}^n \) is a deterministic time-net \( 0 = t_0 < \cdots < t_n = T \) and

\[ \hat{Z}_{t_{i-1}} := \frac{1}{t_i - t_{i-1}} E \left[ \int_{t_{i-1}}^{t_i} Z_s \, ds \mid \mathcal{F}_{t_{i-1}} \right]. \]
Gobet and Makhlouf [21] considered $L_2$-regularity of $(Y, Z)$ for a terminal condition given by $\xi = g(X_T)$ where $g$ does not need to be Lipschitz and $X$ denotes the forward process. In [14], the $L_p$-regularity of $(Y, Z)$ for $p \geq 2$ was shown if the terminal condition depends on the forward process at finitely many time points, $\xi = g(X_{r_1}, \ldots, X_{r_m})$, and satisfies a path-dependent Malliavin fractional smoothness condition which is weaker than the Lipschitz condition on $g$. Using these results and following the ideas of [11], one can show that the convergence rate of the error between the discretizations $(Y^\tau, Z^\tau)$ and the solution $(Y, Z)$ is of order $\frac{1}{2}$, that is

$$\text{Err}_{\tau, 2}(Y, Z) := \left\{ \sup_{0 < t \leq T} E |Y_t - Y^\tau_t|^2 + \int_0^T E |Z_t - Z^\tau_t|^2 \, dt \right\}^{1/2} \leq c|\tau|^{1/2}$$

provided that the time grid for the discretization is chosen in an appropriate way (like in [14]), and the discretized terminal condition converges in this order. Without any assumptions on the dependence of the terminal condition $\xi$ on a forward process $X$, Hu, Nualart and Song [22] have shown the convergence rate $\frac{1}{2}$ supposing Malliavin differentiability properties of $\xi$ (and of the generator).

For a BSDE driven by a Poisson random measure, Bouchard and Elie [9] proved that the convergence rate is of order $\frac{1}{2}$ (in the non-degenerate case) if the terminal condition is given by $\xi = g(X_T)$ where $g$ is Lipschitz.

Here we study whether the $L_2$-variation would allow to achieve the convergence rate $\frac{1}{2}$ with a terminal condition $\xi = g(X_{r_1}, \ldots, X_{r_m})$ and whether the Lipschitz condition on $g$ can be weakened to a Malliavin fractional smoothness condition. The method we use allows to answer this question in the case where $X$ is the Lévy process itself.

In the Brownian setting, in case of a zero generator it is stated in [20], relation (8), that the rate $\frac{1}{2}$ is the best possible as long as $\xi$ can not be represented as a linear function of $W_T$. Moreover, in [20], Theorem 3.5, it is shown that for equidistant grids there is a direct connection between the convergence rate and the index of fractional smoothness of the terminal condition. In [15], Theorems 5 and 6, the same results are recovered for $W$ replaced by a square integrable Lévy process $X$, even if the Lévy process does not have a Brownian part.

The paper is organised as follows. In Section 2, we describe the setting and recall some needed facts. In Section 3 we recall some basic facts about Malliavin calculus in the Lévy setting. Furthermore, we state a result about Malliavin differentiabilty of the solution $(Y, Z)$ to a BSDE. Our method to show the $L_2$-regularity of solutions to BSDEs exploits the fact that their Malliavin derivative is piece-wise constant in time. This is ensured by selecting a terminal condition which has this property. For this purpose, we introduce a space of suitable terminal conditions and investigate the chaos expansion of the according solution in Section 4. Section 5 contains our main result, equivalences and implications concerning the $L_2$-regularity of $(Y, Z)$. An important fact, which will be considered in Section 6, is a sufficient condition for the $L_2$-regularity of the solution: a certain Malliavin fractional smoothness of the terminal condition (in addition to our standing assumption that the generator is Lipschitz).
2. Setting

Let \( X = (X_t)_{t \in [0,T]} \) be an \( L_2 \)-Lévy-process on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with Lévy-measure \( \nu \). We will denote the augmented natural filtration of \( X \) by \( \mathbb{F} := (\mathcal{F}_t)_{t \in [0,T]} \) and let \( L_2 := L_2(\Omega, \mathcal{F}_T, \mathbb{P}) \).

The Lévy–Itô decomposition of an \( L_2 \)-Lévy-process \( X \) can be written as

\[
X_t = \gamma t + \sigma W_t + \int_{[0,t] \times (\mathbb{R} \setminus \{0\})} x\tilde{N}(ds, dx),
\]

where \( \sigma \geq 0, W \) is a Brownian motion and \( \tilde{N} \) is the compensated Poisson random measure corresponding to \( X \).

We define the random measure \( M \) by

\[
M(dr, dx) := \sigma dW_t \delta_0(dx) + x\tilde{N}(dt, dx).
\]

Then \( \mathbb{E}[M(B)]^2 = \int_B m(dr, dx) \) for \( B \in \mathcal{B}([0,T] \times \mathbb{R}) \) where

\[
m(dr, dx) = (\lambda \otimes \mu)(dr, dx)
\]

with

\[
\mu(dx) = \sigma^2 \delta_0(dx) + x^2 \nu(dx)
\]

and \( \lambda \) denotes the Lebesgue measure. For \( 0 \leq t \leq T \), we consider the BSDE

\[
Y_t = \xi + \int_t^T F(s, Y_s, \tilde{Z}_s) ds - \int_{[t,T] \times \mathbb{R}} Z_{s,x} M(dx, dx),
\]

where

\[
\tilde{Z}_s = \int_{\mathbb{R}} Z_{s,x} \kappa(dx)
\]

and \( \kappa(dx) := \kappa(x) \mu(dx) \) such that \( \kappa' \in L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu) \). We will use the following assumptions:

\[
(A_\xi) \quad \xi \in L_2,
\]

\[
(A_F) \quad F: \Omega \times [0, T] \times \mathbb{R}^2 \to \mathbb{R} \text{ is jointly measurable, adapted to } (\mathcal{F}_t)_{t \in [0,T]}, \text{ Lipschitz-continuous in the last two variables, uniformly in } \omega \text{ and } t, \text{ that is,}
\]

\[
|F(t, y_1, z_1) - F(t, y_2, z_2)| \leq L_f (|y_1 - y_2| + |z_1 - z_2|),
\]

and satisfies

\[
\mathbb{E} \int_0^T |F(t, 0, 0)|^2 dt < \infty.
\]

For later use, we introduce spaces of stochastic processes.
Definition 2.1. 1. Let $S$ denote the space of all $(\mathcal{F}_t)$-adapted and càdlàg processes $(y_t)_{0 \leq t \leq T}$ such that

$$\|y\|_S^2 := \mathbb{E} \sup_{0 \leq t \leq T} |y_t|^2 < \infty.$$ 

2. We define $H$ as the space of all random fields $z : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ which are measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ (where $\mathcal{P}$ denotes the predictable $\sigma$-algebra on $\Omega \times [0, T]$ generated by the left-continuous $\mathbb{F}$-adapted processes) such that

$$\|z\|_H^2 := \mathbb{E} \int_{[0,T] \times \mathbb{R}} |z_{s,x}|^2 \nu(ds, dx) < \infty.$$ 

The space $S \times H$ is equipped with the norm $\|(y, z)\|_{S \times H} := (\|y\|_S^2 + \|z\|_H^2)^{1/2}$.

A pair $(Y, Z) \in S \times H$ which satisfies (3) is called a solution to the BSDE (3).

Theorem 2.2. Assume $(\xi, F)$ satisfies $(A_\xi)$ and $(A_F)$. Then the BSDE (3) has a unique solution $(Y, Z) \in S \times H$.

This assertion can be found in Tang and Li [35], Lemma 2.4 and in Barles, Buckdahn and Pardoux [5], Theorem 2.1. We next cite the stability result of [5] comparing the distance between solutions to the BSDE (3) with different terminal conditions and generators.

Theorem 2.3 ([5], Proposition 2.2). Let $(\xi, F)$ and $(\xi', F')$ satisfy $(A_\xi)$ and $(A_F)$. For the corresponding solutions $(Y, Z)$ and $(Y', Z')$ to (3), it holds

$$\|Y - Y'\|_S^2 + \|Z - Z'\|_H^2 \leq C \left( \|\xi - \xi'\|_{L_2}^2 + \int_0^T \|F(s, Y_s, \bar{Z}_s) - F'(s, Y_s, \bar{Z}_s)\|_{L_2}^2 ds \right),$$

where $C = C(T, L_{F'}, \mu)$.

Remark 2.4. According to [34], Proposition 3 (see also [29], Proposition 3 or [2], Lemma 2.2) for any $z \in L_2(\mathbb{P} \otimes \mathcal{m})$ there exists a process

$$p_z \in L_2(\Omega \times [0, T] \times \mathbb{R}, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}), \mathbb{P} \otimes \mathcal{m})$$

such that for any fixed $x \in \mathbb{R}$ the function $(p_z)_.,x$ is a version of the predictable projection (in the classical sense) of $z_.,x$. In the following, we will always use this result to get predictable projections which are measurable w.r.t. a parameter.

3. Malliavin differentiability of $(Y, Z)$

We will use the Malliavin derivative which is defined via chaos expansions, that is series of multiple stochastic integrals. Following Itô [23], for $n \geq 1$ we define elementary functions of the
type
\[ f_n((t_1, x_1), \ldots, (t_n, x_n)) = \sum_{k=1}^{m} a_k \prod_{i=1}^{n} \mathbb{1}_{B_i^k}(t_i, x_i), \]

where \( a_k \in \mathbb{R} \), and for all \( k \) the sets \( B_i^k \in \mathcal{B}([0, T] \times \mathbb{R}) \), \( i = 1, \ldots, n \) are disjoint and satisfy \( m(B_i^k) < \infty \). The multiple stochastic integral \( I_n \) of order \( n \) of the elementary function \( f_n \) with respect to the random measure \( M \) is defined by
\[ I_n(f_n) := \sum_{k=1}^{m} a_k \prod_{i=1}^{n} M(B_i^k). \]

Since the elementary functions given above are dense in \( L_2^n := L_2^2([0, T] \times \mathbb{R}^n, m^\otimes n) \), by linearity and continuity of \( I_n \) its domain extends to \( L_2^n \). For \( n = 0 \) we set \( L_2^0 := \mathbb{R} \) and \( I_0(f_0) := f_0 \) for \( f_0 \in \mathbb{R} \). The properties of \( I_n \) are very similar to those in the Brownian case, especially it holds
\[ I_n(f_n) = I_n(\tilde{f}_n), \]
where \( \tilde{f}_n \) denotes the symmetrization of \( f \) with respect to the \( n \) pairs \( (t_1, x_1), \ldots, (t_n, x_n) \). Moreover,
\[ \mathbb{E} I_n(f_n) I_m(g_m) = \begin{cases} \langle \tilde{f}_n, \tilde{g}_n \rangle_{L_2^n}, & n = m, \\ 0, & n \neq m. \end{cases} \]

Any \( G \in L_2 \) has a chaos expansion
\[ G = \sum_{n=0}^{\infty} I_n(f_n) \]
which is unique if symmetric \( f_n \in L_2^n \) are used (which we will be our standing assumption from now on), and it holds
\[ \|G\|_2^2 := \|G\|_{L_2}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L_2^n}^2. \]

For example, for \( X_s \) from (1) we have
\[ X_s = I_0(\gamma s) + I_1(\mathbb{1}_{[0, T]} \times \mathbb{R}) = \gamma s + M([0, s] \times \mathbb{R}). \quad (5) \]

A straightforward generalisation of [30], Lemma 1.2.5, implies (\( \mathbb{E}_t \) stands for the conditional expectation \( \mathbb{E}[-|\mathcal{F}_t] \))
\[ \mathbb{E}_t G = \sum_{n=0}^{\infty} I_n(f_n \mathbb{1}_{[0,t]^n}). \]
The space $\mathbb{D}_{1,2}$ consists of all random variables $G \in L^2$ such that
\[
\|G\|_{\mathbb{D}_{1,2}}^2 := \sum_{n=0}^{\infty} (n+1)! \|f_n\|_{L^2}^2 < \infty.
\]
For $G \in \mathbb{D}_{1,2}$ the Malliavin derivative
\[
\mathcal{D}G \in L^2(\mathbb{P} \otimes m) := L^2(\Omega \times [0, T] \times \mathbb{R}, \mathcal{F}_T \otimes \mathcal{B}([0, T] \times \mathbb{R}), \mathbb{P} \otimes m)
\]
is given by
\[
\mathcal{D}_{t,x}G(\omega) := \sum_{n=1}^{\infty} n I_{n-1} \left( f_n \left( (t, x), \cdot \right) \right)(\omega),
\]
for $\mathbb{P} \otimes m$-a.e. $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}$. For example, for $X_s$ from (1) representation (5) implies
\[
\mathcal{D}_{t,x}X_s = \mathcal{D}_{t,x}I_1(1_{[0,s]} \times \mathbb{R}) = 1_{[t,T]}(s) \quad \text{for } \mathbb{P} \otimes m\text{-a.e. } (\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}.
\] (6)

For random variables in $\mathbb{D}_{1,2}$ there is an explicit expression for the integrand in the formulation of the predictable representation property (for an introduction to stochastic integration w.r.t. random measures see, e.g., [3]).

**Lemma 3.1 (Clark–Ocone–Haussmann formula [33], Theorem 10).** Assume $G \in \mathbb{D}_{1,2}$. Then
\[
G = \mathbb{E}G + \int_{[0,T] \times \mathbb{R}} \mathbb{P}(\mathcal{D}G)_{t,x} M(dr, dx).
\] (7)

The Malliavin derivative $\mathcal{D}_{t,0}$ can be interpreted as a Malliavin derivative in the Brownian setting with values in the $L^2$-space of random variables depending on the jump part of the Lévy process (see [1,32]). On the other hand, for $x \neq 0$, the Malliavin derivative $\mathcal{D}_{t,x}$ behaves like a difference quotient (see [1,32]). This is also illustrated by the next lemma which contains formulae for the Malliavin derivative of differentiable Lipschitz functions depending on random variables in $\mathbb{D}_{1,2}$.

**Lemma 3.2.** Let $f \in C^1(\mathbb{R}^n; \mathbb{R})$ with bounded partial derivatives. If $G_1, \ldots, G_n \in \mathbb{D}_{1,2}$ then $f(G_1, \ldots, G_n) \in \mathbb{D}_{1,2}$ and

(i) for $x = 0$ it holds
\[
\mathcal{D}_{t,0}f(G_1, \ldots, G_n) = \sum_{i=1}^{n} (\partial_i f)(G_1, \ldots, G_n)\mathcal{D}_{t,0}G_i,
\]
for $\mathbb{P} \otimes \lambda$-a.e. $(\omega, t)$;

(ii) for $x \neq 0$ we get the difference quotient
\[
\mathcal{D}_{t,x}f(G_1, \ldots, G_n) = \frac{f(G_1 + x\mathcal{D}_{t,x}G_1, \ldots, G_n + x\mathcal{D}_{t,x}G_n) - f(G_1, \ldots, G_n)}{x},
\]
for $\mathbb{P} \otimes m$-a.e. $(\omega, t, x)$. 


**Proof.** Assertion (i) follows immediately from [32], Proposition 3.5, combined with [32], Proposition 3.3 and [30], Proposition 1.2.3. Assertion (ii) is a straightforward extension of [16], Lemma 5.1. □

We will make use of the following properties for the Malliavin derivative [13], Lemmas 3.1–3.3.

**Lemma 3.3.** (i) Let $G \in D_{1,2}$. Then for $0 \leq s \leq T$, $E_s G \in D_{1,2}$ and

$$D_{t,x} E_s G = E_s (D_{t,x} G) 1_{[0,s]}(t), \quad \mathbb{P} \otimes m\text{-a.e.}$$

(ii) Let $F : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a product measurable and adapted process, $\rho$ a finite measure on $([0, T] \times \mathbb{R}, \mathcal{B}([0, T] \times \mathbb{R}))$ such that the conditions

(a) $\mathbb{E} \int_{[0,T] \times \mathbb{R}} |F(s,y)|^2 \rho(ds, dy) < \infty$,
(b) $F(s, y) \in D_{1,2}$ for $\rho$-a.e. $(s, y) \in [0, T] \times \mathbb{R}$,
(c) $\mathbb{E} \int_{[0,T] \times \mathbb{R}} \int_{[0,T] \times \mathbb{R}} |D_{t,x} F(s,y)|^2 \rho(ds, dy) m(dt, dx) < \infty$

are satisfied. Then

$$\int_{[0,T] \times \mathbb{R}} F(s,y) \rho(ds, dy) \in D_{1,2},$$

and the differentiation rule

$$D_{t,x} \int_{[0,T] \times \mathbb{R}} F(s,y) \rho(ds, dy) = \int_{[0,T] \times \mathbb{R}} D_{t,x} F(s,y) \rho(ds, dy)$$

holds for $\mathbb{P} \otimes m$-a.e. $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}$.

(iii) Let $F : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a predictable process satisfying

$$\mathbb{E} \int_{[0,T] \times \mathbb{R}} |F(s,y)|^2 m(ds, dy) < \infty.$$

Then conditions (a)–(c) of (ii) are satisfied for $\rho = m$ if and only if

$$\int_{[0,T] \times \mathbb{R}} F(s,y) M(ds, dy) \in D_{1,2}.$$

In this case, the formula

$$D_{t,x} \int_{[0,T] \times \mathbb{R}} F(s,y) M(ds, dy) = F(t,x) + \int_{[0,T] \times \mathbb{R}} D_{t,x} F(s,y) M(ds, dy)$$

holds $\mathbb{P} \otimes m$-a.e.
The following theorem is concerned with Malliavin differentiability of the solution to a BSDE of the form
\[
Y_t = \xi + \int_t^T f(s, X_s, Y_s, \bar{Z}_s) \, ds - \int_{[t, T] \times \mathbb{R}} Z_{s,x} M(ds, dx), \quad 0 \leq t \leq T,
\]
where we will assume
\[
(A_f) \quad f \in C([0, T] \times \mathbb{R}^3) \text{ is Lipschitz-continuous in } (x, y, z), \text{ uniformly in } t, \text{ that is,}
\[
|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq L_f \left( |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| \right).
\]
\[(A_f 1) \quad f \text{ satisfies } \left( A_f \right) \text{ and } f \in C^{0,1,1}([0, T] \times \mathbb{R}^3).
\]
Note that (8) is a special form of (3), and \(F(\omega, t, y, z) := f(t, X_t(\omega), y, z)\) satisfies \((A_f)\) if \(f\) does satisfy \((A_f)\).

**Theorem 3.4.** Let \(\xi \in D_{1,2}\) and assume \((A_f 1)\). Then the following assertions hold.

(i) For \(\mathbb{m}\)-a.e. \((r, v) \in [0, T] \times \mathbb{R}\), there exists a unique solution \((U^{r,v}, V^{r,v}) \in S \times H\) to the BSDE
\[
U_t^{r,v} = D_{r,v}\xi + \int_t^T F_{r,v}(s, U_s^{r,v}, \bar{V}_s^{r,v}) \, ds - \int_{[t, T] \times \mathbb{R}} V_{s,x}^{r,v} M(ds, dx), \quad t \in [r, T],
\]
\[
U_t^{r,v} = V_{s,x}^{r,v} = 0, \quad t \in [0, r],
\]
where \(\bar{V}_s^{r,v} := \int_{\mathbb{R}} V_{s,x}^{r,v} \kappa(dx), \quad F_{r,0}(s, U_s^{r,0}, \bar{V}_s^{r,0}) := [\nabla f(s, X_s, Y_s, \bar{Z}_s), (\mathbb{1}_{[r,T]}(s), U_s^{r,0}, \bar{V}_s^{r,0})],\)
with \(\nabla = (\partial_x, \partial_y, \partial_z)\), and
\[
F_{r,v}(s, U_s^{r,v}, \bar{V}_s^{r,v}) := \frac{1}{v} \left[ f(s, X_s + v \mathbb{1}_{[r,T]}(s), Y_s + v U_s^{r,v}, \bar{Z}_s + v \bar{V}_s^{r,v}) - f(s, X_s, Y_s, \bar{Z}_s) \right]
\]
for \(v \neq 0\).

(ii) For the solution \((Y, Z)\) of (8) it holds
\[
Y \in L_2([0, T]; D_{1,2}), \quad Z \in L_2([0, T] \times \mathbb{R}; D_{1,2}),
\]
and \(D_{r,v}Y\) admits a càdlàg version for \(\mathbb{m}\)-a.e. \((r, v) \in [0, T] \times \mathbb{R}\).

(iii) \((DY, DZ)\) is a version of \((U, V)\), that is, for \(\mathbb{m}\)-a.e. \((r, v)\) it solves
\[
D_{r,v}Y_t = D_{r,v}\xi + \int_t^T F_{r,v}(s, D_{r,v}Y_s, \int_{\mathbb{R}} D_{r,v}Z_{s,x} \kappa(dx)) \, ds
\]
\[- \int_{[t, T] \times \mathbb{R}} D_{r,v}Z_{s,x} M(ds, dx), \quad t \in [r, T].
\]
(iv) The process $\mathbb{P}((\mathcal{D}_{r,v}Y_r)_{r \in [0,T],v \in \mathbb{R}})$ is a version of $Z$ where we set
\[ \mathcal{D}_{r,v}Y_r(\omega) := \lim_{t \uparrow r} \mathcal{D}_{r,v}Y_t(\omega) \]
for all $(r, v, \omega)$ for which $\mathcal{D}_{r,v}Y$ is càdlàg and $\mathcal{D}_{r,v}Y_r(\omega) := 0$ otherwise.

In the setting of time, delayed BSDEs a similar result was proved by Delong and Imkeller [13] assuming that the time horizon $T$ or the Lipschitz constant $L_f$ of the generator are sufficiently small. For the convenience of the reader a proof of Theorem 3.4 is contained in the preprint version [17].

4. Chaotic representation of $(Y, Z)$

The goal of this section is to investigate properties of the chaos expansions of the solution $(Y, Z)$ to (8) with terminal values $\xi$ of the following form:

Let $r_0 = 0 < r_1 < \cdots < r_m = T$ be a partition of $[0, T]$. Define $\Lambda_k := ]r_{k-1}, r_k]$ for $k = 1, \ldots, m$ and $V_m^n := \{1, \ldots, m\}^n$. The set of cuboids
\[ \left\{ \Lambda_\alpha := \Lambda_{\alpha_1} \times \cdots \times \Lambda_{\alpha_n} : \alpha = (\alpha_1, \ldots, \alpha_n) \in V_m^n \right\} \]
forms a partition of $[0, T]^n$. Furthermore, we let
\[ \mathbb{H} := \left\{ \xi = \sum_{n=0}^{\infty} I_n(f_n) \in L_2 : \exists g_n^\alpha \in L_2(\mathbb{R}^n, \mu^{\otimes n}) \text{ such that} \right. \]
\[ \left. f_n((t_1, x_1), \ldots, (t_n, x_n)) = \sum_{\alpha \in V_m^n} g_n^\alpha(x_1, \ldots, x_n) 1_{\Lambda_\alpha}(t_1, \ldots, t_n) \right\}. \]

Hence, on each cuboid $\Lambda_\alpha$ the function $f_n$ is constant in $(t_1, \ldots, t_n)$. In particular, this space contains random variables of the form
\[ g(X_{r_m} - X_{r_{m-1}}, \ldots, X_{r_1} - X_0) \in L_2, \]
where $g$ is a Borel function (see [6]).

The benefit to consider terminal conditions from $\mathbb{H}$ lies in the fact that $t \mapsto \mathcal{D}_t \xi$ is a.e. constant as long as $t$ is within an interval $\Lambda_k$. This property will be used several times below, especially in the proofs of Lemmas 5.3–5.5.

Remark 4.1. By convolution with mollifiers, we construct for any function $f \in C([0, T] \times \mathbb{R}^3)$ satisfying $(A_f)$ a sequence $(f_n)_{n \geq 0}$ converging uniformly to $f$ in $C([0, T] \times \mathbb{R}^3)$, such that for all $n \in \mathbb{N}$ and $t \in [0, T]$ we have $f_n(t, \cdot) \in C^\infty(\mathbb{R}^3)$, and $f_n$ satisfies the Lipschitz-condition (9) with the same constant $L_f$ for all $n$. 

Let \((\xi_n)_{n \geq 0} \subseteq \mathbb{D}_{1,2}\) be a sequence converging to \(\xi\) in \(L_2\). By \((Y^n, Z^n)\), we will denote the solution to (8) with terminal condition \(\xi_n\) and generator \(f_n\). Then Theorem 2.3 implies that

\[
(Y^n, Z^n) \to (Y, Z) \quad \text{if } n \to \infty \text{ in } S \times H.
\] (13)

If \(\xi \in \mathbb{H}\), then the solution \((Y, Z)\) has a chaos expansion which resembles those of the elements of \(\mathbb{H}\).

**Theorem 4.2.** Let \((A_f)\) hold. For \(\xi \in \mathbb{H}\) the chaos expansion of \((Y, Z) \in S \times H\) has the representation

\[
Y_t = \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V_m^n} \varphi_n^\alpha(t) \mathbb{1}_{A_\alpha \cap [0,t]}' \right), \quad \mathbb{P} \otimes \lambda\text{-a.e.,}
\] (14)

where \(\varphi_n^\alpha : [0, T] \times \mathbb{R}^n \to \mathbb{R}\) is jointly measurable, \(\varphi_n^\alpha(t) \in L_2(\mathbb{R}^n, \mu^n)\) and

\[
Z_{t,x} = \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V_m^n} \psi_n^\alpha(t, x) \mathbb{1}_{A_\alpha \cap [0,t]}' \right), \quad \mathbb{P} \otimes \mu\text{-a.e.,}
\] (15)

where \(\psi_n^\alpha : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}\) is jointly measurable and \(\psi_n^\alpha(t) \in L_2(\mathbb{R}^{n+1}, \mu^n \otimes \nu^{n+1})\).

**Proof.** We may use Remark 4.1 and approximate \(\xi \in \mathbb{H}\) by a sequence \((\xi_n)_{n \in \mathbb{N}} \subseteq \mathbb{H} \cap \mathbb{D}_{1,2}\) and \(f\) by \((f_n)\) satisfying \((A_f)\). Since the convergence in \(S \times H\) implies convergence w.r.t. the norm

\[
\| (y, z) \| := \left( \| y \|^2_{L_2(\mathbb{P} \otimes \lambda)} + \| z \|^2_{H} \right)^{1/2},
\] (16)

and the space of processes \((y, z)\) with representations (14) and (15) is closed with respect to the norm (16) we only need to show that the assertion holds for any solution \((Y^n, Z^n)\) w.r.t. \((\xi_n, f_n)\). Hence we may assume that \(\xi \in \mathbb{H} \cap \mathbb{D}_{1,2}\) and \(f \in C^{0,1,1,1}(\mathbb{R}^3)\) such that \(\partial_x f, \partial_y f\) and \(\partial_z f\) are bounded by \(L_f\). According to Theorem 3.4, we can differentiate (8) and obtain for \(m\text{-a.e.} (t, x)\) and all \(s \in [t, T]\) that

\[
\mathcal{D}_{t,x} Y_s = \mathcal{D}_{t,x} \xi + \int_s^T \mathcal{D}_{t,x} f(r, X_r, Y_r, \bar{Z}_r) \, dr - \int_{[s,T]} \mathcal{D}_{t,x} Z_{r,y} M(dr, dy).
\]

Theorem 3.4 yields that \(Z\) is a version of \(\mathbb{P}(\mathcal{D}_{t,x} Y_t)\), hence

\[
Z_{t,x} = \mathcal{D}_{t,x} Y_t, \quad \mathbb{P} \otimes m\text{-a.e.}
\]

We define the recursion

\[
Y_0 := 0, \quad Z_{0,y} := 0,
\]

\[
Y^{k+1} := \xi + \int_s^T f(r, X_r, Y^k_r, \bar{Z}^k_r) \, dr, \quad Y^{k+1} := o(Y^{k+1}),
\] (17)
L₂-variation of Lévy driven BSDEs

where o denotes the optional projection, which is according to [12], Theorem 47 and Remark 50, càdlàg. Since \( Y^{k+1}_u = \mathbb{E}_u Y^{k+1}_u \) \( \mathbb{P} \)-a.s. one gets by Lemma 3.3

\[
D_{s,y} Y^{k+1}_u = D_{s,y} \mathbb{E}_u \xi + D_{s,y} \mathbb{E}_u \int_u^T f(r, X_r, Y^k_r, \bar{Z}_r^k) \, dr
\]

(18)

\[
= \mathbb{E}_u D_{s,y} \xi + \mathbb{E}_u \int_u^T D_{s,y} f(r, X_r, Y^k_r, \bar{Z}_r^k) \, dr, \quad u \in [s, T].
\]

Since \( D_{s,y} \xi + \int_u^T D_{s,y} f(r, X_r, Y^k_r, \bar{Z}_r^k) \, dr, u \in [s, T] \), has continuous paths for a.e. \((s, y)\) we can again apply [12], Theorem 47 and Remark 50, to get a càdlàg optional projection. Hence, we may define the set

\[
A_k := \{(s, y) \in [0, T] \times \mathbb{R} : \text{the RHS of (18) is càdlàg on } [s, T] \mathbb{P}\text{-a.s.}\}
\]

and assume a pathwise càdlàg version of \( D_{s,y} Y^{k+1}_u \) for any \((s, y) \in A_k\) while we let \( D_{s,y} Y^{k+1}_u \) be zero otherwise. In this sense, we can set

\[
\bar{Z}^{k+1}_{s,y} := \lim_{t \downarrow s} D_{s,y} Y^{k+1}_t, \quad Z^{k+1} := \mathbb{P}(\bar{Z}^{k+1})
\]

(19)

for \( k = 0, 1, 2, \ldots \).

The process \( Y^{k+1} \) has a càdlàg version, therefore, \((Y^k, Z^k) \in S \times H\) for all \( k \in \mathbb{N}\). In the proof of [35], Theorem 2.2, it is shown that \((Y^k, Z^k)\) converges to \((Y, Z)\) with respect to the norm (16).

Consequently, we only need to show that (14) and (15) hold for \((Y^k, Z^k)\).

For fixed \( t \in ]0, T[\), we describe (14) by introducing the space

\[
\mathbb{H}_t := \left\{ \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} g^{\alpha}_n \mathbb{1}_{A_{\alpha} \cap [0,t]}^n \right) \in L_2 : g^{\alpha}_n \in L_2(\mathbb{R}^n, \mu^\otimes n) \right\}.
\]

From [6], one can conclude the following fact.

**Lemma 4.3.** For any Borel function \( h : \mathbb{R}^d \to \mathbb{R} \) and \( \xi_1, \ldots, \xi_d \in \mathbb{H}_t \) such that \( h(\xi_1, \ldots, \xi_d) \in L_2 \) it holds \( h(\xi_1, \ldots, \xi_d) \in \mathbb{H}_t \).

Assume now that (14) and (15) hold for \( Y^k \) and \( Z^k \), respectively. We have

\[
\bar{Z}^k_t = \int_\mathbb{R} \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} \psi^{\alpha}_n(t,x) \mathbb{1}_{A_{\alpha} \cap [0,t]}^n \right) \kappa(dx)
\]

\[
= \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} \int_\mathbb{R} \psi^{\alpha}_n(t,x) \kappa(dx) \mathbb{1}_{A_{\alpha} \cap [0,t]}^n \right)
\]

(20)

\[
= \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} \bar{\psi}^{\alpha}_n(t) \mathbb{1}_{A_{\alpha} \cap [0,t]}^n \right) \in \mathbb{H}_t.
\]
From Lemma 4.3, it follows that \( f(t, X_t, Y^k_t, \tilde{Z}^{k}_t) \in \mathbb{H}_t \) that is,

\[
f(t, X_t, Y^k_t, \tilde{Z}^{k}_t) = \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} g^{\alpha}_n(t) \mathbb{1}_{\Lambda_\alpha \cap [0,t]^{0\otimes n}} \right),
\]

with \( g^{\alpha}_n(t) \in L_2(\mathbb{R}^n, \mu^{0\otimes n}) \). Because \( f(\cdot, X, Y^k, \tilde{Z}^k) \) is square integrable w.r.t. \( \mathbb{P} \otimes \lambda \) on \( \Omega \times [0, T] \) one can show that the \( g^{\alpha}_n \) can be chosen jointly measurable. This implies

\[
\mathbb{E}_t \int_t^T f(r, X_r, Y^k_r, \tilde{Z}^{k}_r) \, dr = \int_t^T \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} g^{\alpha}_n(r) \mathbb{1}_{\Lambda_\alpha \cap [0, t]^{0\otimes n}} \right) \, dr
\]

\[
= \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} \int_t^T g^{\alpha}_n(r) \, dr \mathbb{1}_{\Lambda_\alpha \cap [0, t]^{0\otimes n}} \right).
\]

From (17), we have that \( Y^{k+1}_t = \mathbb{E}_t Y^{k+1}_t \) \( \mathbb{P} \)-a.s. and since \( \mathbb{E}_t \xi \in \mathbb{H}_t \) we conclude representation (14) for \( Y^{k+1}_t \). To find out the representation of \( Z^{k+1}_t \), we will use (19). Let \( \alpha := (\alpha_2, \ldots, \alpha_n) \). Assuming \( \xi = \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} \hat{g}^{\alpha}_n(t) \mathbb{1}_{\Lambda_\alpha} \right) \) with symmetric \( f_n = \sum_{\alpha \in V^n_m} \hat{g}^{\alpha}_n \mathbb{1}_{\Lambda_\alpha} \) we get by Lemma 3.3 for \( \mathbb{P} \otimes m \)-a.e. \((t, y, \omega) \in [0, s] \times \mathbb{R} \times \Omega \) that

\[
\mathcal{D}_{t,y} Y^{k+1}_s = \mathcal{D}_{t,y} \mathbb{E}_s \xi + \mathcal{D}_{t,y} \mathbb{E}_s \int_s^T f(r, X_r, Y^k_r, \tilde{Z}^{k}_r) \, dr
\]

\[
= \mathcal{D}_{t,y} \mathbb{E}_s \xi + \mathcal{D}_{t,y} \int_s^T \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} g^{\alpha}_n(r) \mathbb{1}_{\Lambda_\alpha \cap [0, s]^{0\otimes n}} \right) \, dr
\]

\[
= \sum_{n=1}^{\infty} n I_{n-1} \left( \sum_{\alpha \in V^n_m} \hat{g}^{\alpha}_n(y, \cdot) \mathbb{1}_{\Lambda_\alpha_1} \mathbb{1}_{\Lambda_\alpha} \right)
\]

\[
+ \sum_{n=1}^{\infty} n I_{n-1} \left( \sum_{\alpha \in V^n_m} g^{\alpha}_n(r, y, \cdot) \mathbb{1}_{\Lambda_\alpha_1} \mathbb{1}_{\Lambda_\alpha} \right) \, dr,
\]

where we again have chosen symmetric integrands \( \sum_{\alpha \in V^n_m} g^{\alpha}_n(r) \mathbb{1}_{\Lambda_\alpha \cap [0, s]^{0\otimes n}} \). One easily checks the \( L_2 \)-convergence

\[
\lim_{s \searrow t} \mathcal{D}_{t,y} Y^{k+1}_s = \sum_{n=1}^{\infty} n I_{n-1} \left( \sum_{\alpha \in V^n_m} \hat{g}^{\alpha}_n(y, \cdot) \mathbb{1}_{\Lambda_\alpha_1} \mathbb{1}_{\Lambda_\alpha} \right)
\]

\[
+ \sum_{n=1}^{\infty} n I_{n-1} \left( \sum_{\alpha \in V^n_m} g^{\alpha}_n(r, y, \cdot) \mathbb{1}_{\Lambda_\alpha_1} \mathbb{1}_{\Lambda_\alpha} \right) \, dr.
\]
If we consider the càdlàg version of $D_{t,y} Y^{k+1}$, we obtain the same expression for the pathwise limit, that is, $\mathbb{P}$-a.s.

$$Z_{t,y}^{k+1} = \lim_{s \uparrow t} D_{t,y} Y^{k+1} = \sum_{n=1}^{\infty} n I_{n-1} \left( \sum_{\alpha \in V_n} \mathbb{E}^{(\alpha)} \left[ g^{(\alpha)}_{\alpha}(y, \cdot) + \int_t^T g^{(\alpha)}_{\alpha}(r, y, \cdot) \, dr \right] \mathbb{1}_{\Lambda_{\alpha_1}(t)} \mathbb{1}_{\Lambda_{\alpha_2} \cap [0,t]^{(\alpha_1-1)}} \right).$$

\[\square\]

5. $L_2$-variation of $(Y, Z)$

The next theorem is our main statement, which allows conclusions on the $L_2$-regularity of the solutions to BSDE (8) by observing regularity properties of $Y_{r_k}$ for fixed time points $r_0 = 0 < r_1 < \cdots < r_m = T$.

**Theorem 5.1.** Assume that $(A_{f})$ is satisfied and $\xi \in \mathbb{H}$. Let $k \in \{1, \ldots, m\}$ and $\theta_k \in [0, 1]$. For the solution $(Y, Z)$ of (8), consider the following assertions:

(i) There is some $c_1 > 0$ such that for all $s \in [r_{k-1}, r_k]$,

$$\|Y_{r_k} - E_s Y_{r_k}\|^2 \leq c_1 (r_k - s)^{\theta_k}.$$

(ii) There is some $c_2 > 0$ such that for all $r_{k-1} \leq s < t \leq r_k$,

$$\|Y_t - Y_s\|^2 \leq c_2 \int_s^t (r_k - r)^{\theta_k - 1} \, dr.$$

(iii) There is some $c_3 > 0$ such that for $\lambda$-a.e. $s \in [r_{k-1}, r_k[$,

$$\|Z_{s,\cdot}\|_{L_2(\mathbb{P} \otimes \mu)}^2 \leq c_3 (r_k - s)^{\theta_k - 1}.$$

(iv) There is some $c_4 > 0$ and a Borel set $N_k$ with $\lambda(N_k) = 0$ such that for all $s, t \in [r_{k-1}, r_k] \setminus N_k$ with $s < t$ and for all $h \in L_2(\mu)$ it holds

$$\left\| \int_{\mathbb{R}} (Z_{t,x} - Z_{s,x}) h(x) \mu(dx) \right\|^2 \leq \|h\|_{L_2(\mu)}^2 c_4 \int_s^t (r_k - r)^{\theta_k - 2} \, dr.$$

Then it holds that

(i) \iff (ii) \iff (iii) \implies (iv).

**Remark 5.2.** (i) Analogously to [14], Definition 1, we may introduce the concept of path-dependent fractional smoothness: fix $\Theta = (\theta_1, \ldots, \theta_m) \in [0, 1]^m$. If $(Y, Z)$ is the solution to BSDE (8) with generator $f$ and terminal condition $\xi \in \mathbb{H}$, we let

$$(\xi, f) \in B^{\Theta}_{2, \infty}(X)$$
provided that there is some $c > 0$ such that

$$
\|Y_{r_k} - E_s Y_{r_k}\|^2 \leq c(r_k - s)^{\theta_k}, \quad r_{k-1} \leq s < r_k, k = 1, \ldots, m.
$$

If $f = 0$ we simply write $\xi \in B_{2,\infty}^\Theta(X)$. If, moreover, $T = 1$ and $m = 1$ then the space $B_{2,\infty}^\Theta(X)$ can be understood as the real interpolation space $(L_2, D_1, 2)^{\theta_1, \infty}$ which describes fractional smoothness. For $\xi = \sum_{n=0}^{\infty} I_n(f_n) \in \mathbb{H}$ set $T_\xi(t) := \|E_t \xi\|^2 = \sum_{n=0}^{\infty} \|I_n(f_n)\|^2 t^n$, and using the ideas of [19], Proposition 3.2 and [20], formula (13), one can conclude that

$$
\|\xi\|_{(L_2, D_1, 2)^{\theta_1, \infty}} \sim_c \|\xi\| + \sup_{0 \leq t < 1} (1 - t)^{-\theta_1/2} \sqrt{T_\xi(1) - T_\xi(t)} \\
= \|\xi\| + \sup_{0 \leq t < 1} (1 - t)^{-\theta_1/2} \|\xi - E_t \xi\|.
$$

By assumption, we have $\|\xi - E_t \xi\|^2 \leq c(1 - t)^{\theta_1}$ hence the RHS is finite. From the lexicographical scale of the real interpolation spaces (see [8] or [7]), it follows

$$
(L_2, D_1, 2)^{\theta_1', 2} \supseteq (L_2, D_1, 2)^{\theta_1, \infty} \quad \text{for all } \theta_1' \in [0, \theta_1[.[20, \text{Remark A.1}]
$$

Especially, $\|\xi - E_t \xi\|^2 \leq c(1 - t)^{\theta_1}$ implies that $\sum_{n=0}^{\infty} n^{\theta_1'} \|I_n(f_n)\|^2 < \infty$ for all $\theta_1' \in ]0, \theta_1[$ (see [20], Remark A.1).

(ii) In general (iv) $\not\Rightarrow$ (iii): let $(p_n)_{n=1}^{\infty}$ be an ONB in $L_2(\mu)$. For simplicity, assume $T = 1$, $m = 1$, $f \equiv 0$ and $\xi = \sum_{n=0}^{\infty} I_n(g_n, [0, T])$ so that

$$
Z_{s,x} = \sum_{n=1}^{\infty} n I_{n-1}(g_n(x, \cdot)_{2\mathbb{Z}[0,s]})
$$

Setting $g_n := \beta_n(n!)^{-1/2} p_n^m$ we have

$$
\|Z_{s,x}\|_{L_2(\mathbb{P} \otimes \mu)}^2 = \sum_{n=1}^{\infty} n \beta_n^2 s^{n-1}.
$$

For a sequence $(\beta_n)$ such that $\beta_1^2 := 1, \beta_2^2 := 0, \beta_n^2 := \frac{1}{n^2}, n \geq 3$, we use Lemma A.1 of [31] which states that

$$
1 + \sum_{n=2}^{\infty} (\log n)^{-2} s^n \sim_c \frac{1}{(1 - s)(1 - \log(1 - s))^2}
$$

(Where for some $c \geq 1$ and $A, B > 0$ the expression $A \sim_c B$ is a short notation for $c^{-1} A \leq B \leq c A$). Hence

$$
\|Z_{s,x}\|_{L_2(\mathbb{P} \otimes \mu)}^2 \sim_c \frac{1}{(1 - \log(1 - s))^2(1 - s)^{-1}},
$$

so that there does not exist any $\theta \in [0, 1]$ for which property (iii) holds.
But for any $h = \sum_{n=1}^{\infty} \alpha_n p_n$ such that $\|h\|^2_{L_2(\mu)} = \sum_{n=1}^{\infty} \alpha_n^2 = 1$ we have

$$
\left\| \int_{\mathbb{R}} (Z_{t,x} - Z_{s,x}) h(x) \mu(dx) \right\|^2 = \sum_{n=3}^{\infty} \alpha_n^2 \frac{1}{(\log(n-1))^2} \left( t^{n-1} - s^{n-1} \right) \leq \frac{1}{(\log 2)^2},
$$

which means that (iv) is fulfilled for any $\theta \in ]0, 1]$.

We prepare some lemmas to prove Theorem 5.1.

**Lemma 5.3.** Let $\eta \in \mathbb{H} \cap \mathbb{D}_{1,2}$ and $k \in \{1, \ldots, m\}$. Then for $\lambda$-a.e. $s, t \in [r_{k-1}, r_k[$ with $s < t$ it holds

$$
\left\| \mathbb{E}_t D_t \cdot \eta - \mathbb{E}_s D_s \cdot \eta \right\|^2_{L_2(\mathbb{P} \otimes \mu)} \leq 4 \int_s^t \left\| \mathbb{E}_{r_k} \eta - \mathbb{E}_{r} \eta \right\|^2 \frac{d r}{(r_k - r)^2}.
$$

**Proof.** Let $\eta \in \mathbb{H} \cap \mathbb{D}_{1,2}$ be given by $\eta = \sum_{n=0}^{\infty} I_n \left( \sum_{\alpha \in V^n_m} g_\alpha \mathbb{1}_{\Lambda_\alpha} \right)$ where we assume that the functions $f_n((t_1, x_1), \ldots, (t_n, x_n))$ are symmetric. In the following, we use again the notation $\alpha := (\alpha_2, \ldots, \alpha_n)$. Since

$$
D_{t,x} \eta = \sum_{n=1}^{\infty} n I_{n-1} \left( \sum_{\alpha \in V^n_m} g_\alpha (x, \cdot) \mathbb{1}_{\Lambda_\alpha} (t) \mathbb{1}_{\Lambda_\alpha} \right)
$$

and since there exists a version of $D \eta$ which is constant on $]r_{k-1}, r_k[$ we get for $s, t \in [r_{k-1}, r_k[$ that

$$
\left\| \mathbb{E}_t D_t \cdot \eta - \mathbb{E}_s D_s \cdot \eta \right\|^2_{L_2(\mathbb{P} \otimes \mu)} = \sum_{n=1}^{\infty} n I_{n-1} \left( \sum_{\alpha \in V^n_m} g_\alpha \mathbb{1}_{\Lambda_\alpha} (t) \mathbb{1}_{\Lambda_\alpha} (\mathbb{1} \otimes (n-1) - \mathbb{1} \otimes (n-1)) \right)
$$

(21)

and since there exists a version of $D \eta$ which is constant on $]r_{k-1}, r_k[$ we get for $s, t \in [r_{k-1}, r_k[$ that

$$
\left\| \mathbb{E}_t D_t \cdot \eta - \mathbb{E}_s D_s \cdot \eta \right\|^2_{L_2(\mathbb{P} \otimes \mu)} = \sum_{n=1}^{\infty} n I_{n-1} \left( \sum_{\alpha \in V^n_m} g_\alpha \mathbb{1}_{\Lambda_\alpha} (t) \mathbb{1}_{\Lambda_\alpha} \right)
$$

For $\beta \in V^n_m$ and $1 \leq l \leq m$, we define

$$
\gamma_l(\beta) := \# \{ i \mid \beta_i = l, \ i = 1, \ldots, n \}.
$$

Notice that the intersection $\Lambda_\alpha \cap ([0, t]^{n-1} \setminus [0, s]^{n-1})$ is empty if $\sum_{d=k+1}^{m} \gamma_d(\alpha) > 0$. Therefore, setting

$$
\delta_\alpha := \mathbb{1}_{[0]} \left( \sum_{d=k+1}^{m} \gamma_d(\alpha) \right)
$$
we have
\[ \lambda_{\otimes(n-1)}(\Lambda_{\bar{\alpha}} \cap ([0, 1]^{n-1} \setminus [0, 1]^{n-1})) \]
\[ = \left((t - r_{k-1})^\gamma(\bar{\alpha}) - (s - r_{k-1})^\gamma(\bar{\alpha})\right) \prod_{1 \leq l < k} (r_l - r_{l-1})^{\gamma_l(\bar{\alpha})} \delta_{\bar{\alpha}}. \]

Using the symmetry of the functions in the chaos decomposition, we get that
\[ g_\alpha^n(x_1, \ldots, x_n) = g_{\pi(\alpha)}^n(x_{\pi(1)}, \ldots, x_{\pi(n)}) \]
and hence
\[ \|g_\alpha^n\|_{L_2(\mu_{\otimes n})}^2 = \|g_{\pi(\alpha)}^n\|_{L_2(\mu_{\otimes n})}^2 \]
for all \( \pi \in S(n) \) where we used the notation \( \pi(\alpha) := (\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}) \). Applying this fact, we reduce our summation over \( \alpha \in V_m^n \) to a summation over equivalence classes \([\alpha] \in V_m^n/\sim\) where
\[ \alpha \sim \beta \iff \exists \pi \in S(n): \alpha = \pi(\beta). \]

Thus, since in (21) we fixed \( \alpha_1 \), by taking equivalence classes for \( V_m^{n-1} \) we obtain
\[ \|\mathbb{E}_t^r D_t, \eta - \mathbb{E}_s^r D_s, \eta\|_{L_2(\mu_{\otimes n})}^2 \]
\[ = \sum_{n=2}^{\infty} \sum_{[\alpha] \in V_m^{n-1}/\sim} \frac{(n-1)!}{\gamma_1(\alpha)! \cdots \gamma_k(\alpha)!} \|g_n^{(k, \alpha)}\|_{L_2(\mu_{\otimes n})}^2 \]
\[ \times \left((t - r_{k-1})^{\gamma_1(\alpha)} - (s - r_{k-1})^{\gamma_k(\alpha)}\right) \prod_{1 \leq l < k} (r_l - r_{l-1})^{\gamma_l(\alpha)} \delta_{\bar{\alpha}}, \]

because the cardinality of the equivalence class \([\alpha]\) is \( (n-1)!/\gamma_1(\alpha)! \cdots \gamma_k(\alpha)! \) and \( \gamma_l(\alpha) \) is invariant of permutations of \( \alpha \). For \( \gamma \geq 1 \), we estimate
\[ (t - r_{k-1})^\gamma - (s - r_{k-1})^\gamma = \int_s^t \gamma(r - r_{k-1})^{\gamma-1} dr \]
using for the integrand on the right-hand side the inequality
\[ (r - r_{k-1})^{\gamma-1} \leq \frac{1}{(r_k - u)(u - r)} \int_u^r \int_v^\rho (v - r_{k-1})^{\gamma-1} dv d\rho, \quad r_{k-1} \leq r < u < r_k. \]

For \( u = \frac{r_k + t}{2} \) this leads to
\[ (t - r_{k-1})^\gamma - (s - r_{k-1})^\gamma \leq \frac{4}{(\gamma + 1)} \int_s^t \frac{(r_k - r_{k-1})^{\gamma+1} - (r - r_{k-1})^{\gamma+1}}{(r_k - r)^2} dr. \]
This yields
\[
\begin{align*}
\|\mathbb{E}_t \mathcal{D}_t \cdot \eta - \mathbb{E}_s \mathcal{D}_s \cdot \eta\|_{L^2_2(P \otimes \mu)}^2 & \leq 4 \int_s^t \sum_{n=1}^\infty n! \sum_{[\alpha] \in V_m^{n-1}/\sim} \gamma_1(\alpha)! \cdots \gamma_k(\alpha)! (\gamma_{k-1}(\alpha) + 1)! \left\| g_n^{(k, \alpha)} \right\|_{L^2_2(\mu \otimes \eta)}^2 \\
& \quad \times \frac{(r_k - r_{k-1})^{2\gamma(\alpha)} - (r - r_{k-1})^{2\gamma(\alpha)}}{(r_k - r)^2} \prod_{1 \leq l < k} (r_l - r_{l-1})^{\gamma_l(\alpha)} \delta_{[\alpha]} \, dr,
\end{align*}
\]
where for \( \gamma_k(\alpha) = 0 \) we have used
\[
0 = (t - r_{k-1})^{2\gamma(\alpha)} - (s - r_{k-1})^{2\gamma(\alpha)} \leq \int_s^t \frac{(r_k - r_{k-1}) - (r - r_{k-1})}{(r_k - r)^2} \, dr.
\]
Because of
\[
\gamma_l(\alpha) = \gamma_l(\alpha), \quad 0 < l < k \quad \text{and} \quad \gamma_k(\alpha) = \gamma_k(\alpha) + 1
\]
if \( \alpha = (k, \alpha) \) we finally get
\[
\begin{align*}
\|\mathbb{E}_t \mathcal{D}_t \cdot \eta - \mathbb{E}_s \mathcal{D}_s \cdot \eta\|_{L^2_2(P \otimes \mu)}^2 & \leq 4 \int_s^t \sum_{n=1}^\infty n! \sum_{[\alpha] \in V_m^{n-1}/\sim} \gamma_1(\alpha)! \cdots \gamma_k(\alpha)! \left\| g_n^{(k, \alpha)} \right\|_{L^2_2(\mu \otimes \eta)}^2 \\
& \quad \times \frac{(r_k - r_{k-1})^{2\gamma(\alpha)} - (r - r_{k-1})^{2\gamma(\alpha)}}{(r_k - r)^2} \prod_{l<k} (r_l - r_{l-1})^{\gamma_l(\alpha)} \prod_{1 \leq l < k} (r_l - r_{l-1})^{\gamma_l(\alpha)} \delta_{[\alpha]} \, dr.
\end{align*}
\]
□

**Lemma 5.4.** If \( \eta \in \mathbb{H} \cap \mathbb{D}_{1,2} \) and \( k \in \{1, \ldots, m\} \) then for \( \lambda \)-a.e. \( t \in ]r_{k-1}, r_k[ \)
\[
\|\mathbb{E}_t \mathcal{D}_t \cdot \eta\|_{L^2_2(P \otimes \mu)}^2 \leq \frac{\|\mathbb{E}_t \eta - \mathbb{E}_r \eta\|_{L^2_2(P \otimes \mu)}^2}{r_k - r}.
\]

**Proof.** Similar to the proof of the previous lemma, we get (using the same notation)
\[
\begin{align*}
\|\mathbb{E}_t \mathcal{D}_t \cdot \eta\|_{L^2_2(P \otimes \mu)}^2 & = \left\| \sum_{n=1}^\infty n I_{n-1} \left( \sum_{\alpha \in V_m^n} g_n^{(\alpha)} \mathbb{1}_{[\alpha]} (t) \mathbb{1}_{[\alpha] \in [0, t]} \mathbb{1}_{[\alpha] \in [0, t]}^{\otimes (n-1)} \right) \right\|_{L^2_2(P \otimes \mu)}^2
\end{align*}
\]
C. Geiss and A. Steinicke

\[
= \sum_{n=1}^{\infty} \frac{n!}{n!} \sum_{\alpha \in V_m} \|g_n^{\alpha}\|_{L_2(\mu \otimes m)}^2 (t - r_{k-1})^{\gamma_k(\alpha)} \prod_{l<k} (r_l - r_{l-1})^{\gamma_l(\alpha)} \delta_{\alpha}
\]

\[
\leq \sum_{n=1}^{\infty} \frac{n!}{n!} \sum_{\alpha \in V_m} \|g_n^{\alpha}\|_{L_2(\mu \otimes m)}^2 \frac{(r_k - r_{k-1})^{\gamma_k(\alpha)} - (t - r_{k-1})^{\gamma_k(\alpha)}}{r_k - t} \prod_{l<k} (r_l - r_{l-1})^{\gamma_l(\alpha)} \delta_{\alpha}
\]

\[
= \frac{\|\mathbb{E}_t \eta - \mathbb{E}_s \eta\|^2}{r_k - t}.
\]

\[= \frac{\|\mathbb{E}_t \eta - \mathbb{E}_s \eta\|^2}{r_k - t}.
\]

\[\square\]

**Lemma 5.5.** Suppose \(u \in ]r_{k-1}, T]\), \(\eta \in \mathbb{H}_u \cap \mathbb{D}_{1,2}\) and \(h \in L_2(\mu)\). Then the equation

\[
\mathbb{E}_s [\eta I_1(\mathbb{I}_{s,a})h] = \mathbb{E}_s [\int_{]s,a[ \times \mathbb{R}} h(x) M(dt, dx)] = \int_{]s,a[ \times \mathbb{R}} \mathbb{E}_s D_{s,x} \eta h(x) \mu(dx)
\]

is satisfied \(\mathbb{P}\)-a.s. for \(\lambda\)-a.e. \(r_{k-1} < s < a \leq r_k \wedge u\).

**Proof.** By the Clark–Ocone–Haussmann formula (7), we express \(\eta\) as

\[
\eta = \mathbb{E} \eta + \int_{]0,u[ \times \mathbb{R}} P(D\eta)_{t,x} M(dt, dx).
\]

Thus we can write

\[
\mathbb{E}_s \left[ \eta \int_{]s,a[ \times \mathbb{R}} h(x) M(dt, dx) \right] = \mathbb{E}_s \left[ \int_{]0,u[ \times \mathbb{R}} P(D\eta)_{t,x} M(dt, dx) \int_{]s,a[ \times \mathbb{R}} h(x) M(dt, dx) \right]
\]

(the constant \(\mathbb{E} \eta\) multiplied with \(\int_{]s,a[ \times \mathbb{R}} h(x) M(dt, dx)\) gives zero when applying \(\mathbb{E}_s\)). Using now the conditional Itô-isometry, we arrive at

\[
\mathbb{E}_s \left[ \int_{]0,u[ \times \mathbb{R}} P(D\eta)_{t,x} M(dt, dx) \int_{]s,a[ \times \mathbb{R}} h(x) M(dt, dx) \right]
\]

\[
= \mathbb{E}_s \int_{]s,a[ \times \mathbb{R}} \mathbb{E}_t D_{t,x} \eta h(x) m(dt, dx)
\]

\[
= \int_{]s,a[ \times \mathbb{R}} \mathbb{E}_t D_{t,x} \eta h(x) m(dt, dx)
\]

\[
= (a - s) \int_{\mathbb{R}} \mathbb{E}_s D_{s,x} \eta h(x) \mu(dx)
\]

since \(D\eta\) is \(\mathbb{P} \otimes m\)-a.e. constant on the interval \(]r_{k-1}, r_k \wedge u[\) with respect to the time variable because \(\eta\) is in \(\mathbb{H}_u\).  \[\square\]
Proof of Theorem 5.1. In the following, we will indicate the dependency of the constants on certain parameters but nevertheless the constants may vary from line to line.

(iii) ⇒ (ii): This step is analogous to the proof of [14], Theorem 1, (C2i) ⇒ (C3i). It holds

\[ \| Y_t - Y_s \|^2 \leq 2(t-s) \int_s^t \| f(r, X_r, Y_r, \bar{Z}_r) \|^2 \, dr + 2 \int_s^t \| Z_r \|^2_{L^2(\mathbb{P} \otimes \mu)} \, dr \]
\[ \leq c(L_f, \mu(\mathbb{R}), \kappa') \int_s^t (1 + \| Y_r \|^2 + \| Z_r \|^2_{L^2(\mathbb{P} \otimes \mu)}) \, dr \]
\[ \leq c(L_f, \mu(\mathbb{R}), \kappa', c_3) \int_s^t (r_k - r)^{\theta_k - 1} \, dr \]

since \( \int_0^T \| Y_r \|^2 \, dr < \infty \) and \( \| \bar{Z}_r \| \leq \| \kappa' \|_{L^2(\mu)} \| Z_r \|_{L^2(\mathbb{P} \otimes \mu)} \).

(ii) ⇒ (i): The argument of [14], Theorem 1, (C3i) ⇒ (C4i), works here as well so that we have

\[ \| Y_{r_k} - \mathbb{E}_s Y_{r_k} \|^2 \leq 4 \| Y_{r_k} - Y_s \|^2 \leq 4c_2 \theta_k (r_k - s)^{\theta_k}. \]

(i) ⇒ (iii): Step 1. We first assume that

\[ \xi \in \mathbb{H} \cap D_{1,2} \text{ and } f \text{ satisfies } (A_f 1). \]  

Because of the relation

\[ Y_r = \mathbb{E}_r Y_{r_k} + \mathbb{E}_r \int_r^{r_k} f(u, X_u, Y_u, \bar{Z}_u) \, du, \quad r_{k-1} < r < r_k. \]  

Lemma 3.3 and Theorem 3.4(iv) we have \( \mathbb{P} \)-a.s. for \( \mathbb{m} \)-a.e. \( (t, x) \in ]r_{k-1}, r_k[ \times \mathbb{R} \) that

\[ Z_{t,x} = \lim_{r \searrow t} \mathcal{D}_{t,x} Y_r \]
\[ = \lim_{r \searrow t} \mathcal{D}_{t,x} \left( \mathbb{E}_r Y_{r_k} + \mathbb{E}_r \int_r^{r_k} f(u, X_u, Y_u, \bar{Z}_u) \, du \right) \]
\[ = \lim_{r \searrow t} \left( \mathbb{E}_r \mathcal{D}_{t,x} Y_{r_k} + \mathbb{E}_r \mathcal{D}_{t,x} \int_r^{r_k} f(u, X_u, Y_u, \bar{Z}_u) \, du \right) \]
\[ = \lim_{r \searrow t} \left( \mathbb{E}_r \mathcal{D}_{t,x} Y_{r_k} + \mathbb{E}_r \int_r^{r_k} \mathcal{D}_{t,x} f(u, X_u, Y_u, \bar{Z}_u) \, du \right) \]
\[ = \mathbb{E}_t \mathcal{D}_{t,x} Y_{r_k} + \mathbb{E}_t \int_t^{r_k} \mathcal{D}_{t,x} f(u, X_u, Y_u, \bar{Z}_u) \, du, \]

where we assumed the right continuous versions of the according expressions: Since \( Y_{r_k} \in \mathbb{H} \cap D_{1,2} \) the expression \( \mathcal{D} Y_{r_k} \) can be realized such that it is constant in \( t \) on \( ]r_{k-1}, r_k[ \) and
\((E_uD_{t,u}Y_{rk})_{s \leq r_{k-1}, r_k}\) is a martingale (for fixed \(x\)). From Lemma 5.4, we conclude that

\[
\|Z_{t,.}\|_{L^2(\mathbb{P} \otimes \mu)} \leq \frac{\|Y_{rk} - E_r Y_{rk}\|}{\sqrt{r_k - t}} + \int_t^{r_k} \|D_{t,.} f(u, X_u, Y_u, Z_{ru})\|_{L^2(\mathbb{P} \otimes \mu)} du.
\]

(26)

Since Lemma 3.2, the Lipschitz condition and relation (6) imply

\[
|D_{t,y} f(r, X_r, Y_r, Z_r)| \leq L_f \left(\mathbb{1}_{[r, T]}(r) + |D_{t,y} Y_r| + |D_{t,y} Z_r|\right), \quad y \neq 0,
\]

and

\[
D_{t,0} f(r, X_r, Y_r, Z_r) = (\mathbb{1}_{[r, T]}(r) \partial_x + D_{t,0} Y_r \partial_y + D_{t,0} \tilde{Z}_r \partial_z) f(r, X_r, Y_r, Z_r),
\]

we have

\[
\|D_{t,.} f(r, X_r, Y_r, Z_r)\|_{L^2(\mathbb{P} \otimes \mu)} \leq L_f \left(\sqrt{\mu(\mathbb{R})} + \|D_{t,.} Y_r\|_{L^2(\mathbb{P} \otimes \mu)} + \|D_{t,.} \tilde{Z}_r\|_{L^2(\mathbb{P} \otimes \mu)}\right).
\]

(27)

We take the Malliavin derivative of (24), and by Lemmas 3.3 and 5.4, we get

\[
\|D_{t,.} Y_r\|_{L^2(\mathbb{P} \otimes \mu)} \leq \frac{\|Y_{rk} - E_r Y_{rk}\|}{\sqrt{r_k - t}} + \int_r^{r_k} \|D_{t,.} f(u, X_u, Y_u, Z_{ru})\|_{L^2(\mathbb{P} \otimes \mu)} du.
\]

(28)

In order to estimate \(\|D_{t,.} \tilde{Z}_r\|_{L^2(\mathbb{P} \otimes \mu)}\), we will use the representation

\[
\tilde{Z}_r = \mathbb{E}_r[(E_u Y_{rk}) I_1(\mathbb{1}_{[r, a]} \kappa')] \frac{1}{u - r} + \int_r^{r_k} \frac{\mathbb{E}_r[f(a, X_a, Y_a, \tilde{Z}_a) I_1(\mathbb{1}_{[r, a]} \kappa')]}{a - r} da,
\]

for \(\lambda\)-a.e. \(u\) such that \(r_{k-1} < r < u < r_k\), which is a consequence of equation (25), the fact that \(E_u Y_{rk} \in H_u\), \(f(a, X_a, Y_a, \tilde{Z}_a) \in H_a\), and Lemma 5.5.

Hence for \(r_{k-1} < t < r < u < r_k\) the ‘conditional’ Hölder inequality implies

\[
\|D_{t,.} \tilde{Z}_r\|_{L^2(\mathbb{P} \otimes \mu)} \leq \|D_{t,.} \mathbb{E}_u Y_{rk}\|_{L^2(\mathbb{P} \otimes \mu)} \|\kappa'\|_{L^2(\mu)} \frac{1}{\sqrt{u - r}}
\]

\[
+ c(L_f, \mu(\mathbb{R}), \kappa') \int_r^{r_k} \frac{1 + \|D_{t,.} Y_a\|_{L^2(\mathbb{P} \otimes \mu)} + \|D_{t,.} \tilde{Z}_a\|_{L^2(\mathbb{P} \otimes \mu)}}{\sqrt{a - r}} da,
\]

where we used that \((\mathbb{E}_r I_1(\mathbb{1}_{[r, u]} \kappa')^2)^{1/2} \leq c(\kappa') \sqrt{u - r}\) a.s., and from (27) one gets the estimate for the integral. Choosing \(u = \frac{r_k + r}{2}\), we conclude by Lemma 5.4 the inequality

\[
\frac{\|D_{t,.} \mathbb{E}_u Y_{rk}\|_{L^2(\mathbb{P} \otimes \mu)}}{\sqrt{u - r}} \leq 2 \frac{\|Y_{rk} - E_r Y_{rk}\|}{r_k - r}.
\]
From the estimate (28) for $D_t, Y_r$ and the above one for $D_t, \tilde{Z}_r$, we obtain

$$\| D_t, Y_r \|_{L^2(P \otimes \mu)} + \| D_t, \tilde{Z}_r \|_{L^2(P \otimes \mu)} \leq c(\kappa', \| Y_{rk} - E_r Y_{rk} \|_{L^2(P) \otimes \mu}) + c(L_f, \mu(\mathbb{R}), \kappa') \int_r^{rk} \frac{1 + \| D_t, Y_a \|_{L^2(P \otimes \mu)} + \| D_t, \tilde{Z}_a \|_{L^2(P \otimes \mu)}}{\sqrt{a - r}} \, da$$

which can be treated using an iteration and Gronwall’s lemma (see the proof of Lemma 4 in [14]) in order to get

$$\| D_t, Y_r \|_{L^2(P \otimes \mu)} + \| D_t, \tilde{Z}_r \|_{L^2(P \otimes \mu)} \leq c(L_f, \mu(\mathbb{R}), \kappa') \frac{\| Y_{rk} - E_r Y_{rk} \|_{L^2(P)}}{rk - r}.$$  (29)

Hence from (26) and (27), it follows

$$\| Z_t, \cdot \|_{L^2(P \otimes \mu)} \leq \frac{\| Y_{rk} - E_r Y_{rk} \|}{\sqrt{rk - t}} + c(L_f, \mu(\mathbb{R}), \kappa') \int_t^{rk} \left( 1 + \frac{\| Y_{rk} - E_r Y_{rk} \|}{rk - r} \right) \, dr.$$  (30)

Step 2. Here we use Remark 4.1 and approximate $\xi \in \mathbb{H}$ by a sequence $(\xi_n)_n \subseteq \mathbb{H} \cap D_{1,2}$ and $f$ such that $(A_f)$ is fulfilled by $(f_n)_n$ satisfying $(A_f 1)$. The convergence (13) implies that we can find a subsequence $(Z_{nk})$ which we will again denote by $(Z_n)$ such that for $\lambda$-a.e. $t \in [0, T]$

$$\| Z_{nk} - Z_t, \cdot \|_{L^2(P \otimes \mu)}^2 \to 0 \quad \text{as } n \to \infty.$$  (31)

From the first step, we conclude that (30) holds for $Z_n$ and therefore

$$\| Z_t, \cdot \|_{L^2(P \otimes \mu)} \leq \| Z_t, \cdot - Z_{nk}, \cdot \|_{L^2(P \otimes \mu)} + \| Z_{nk}, \cdot \|_{L^2(P \otimes \mu)}$$

$$\leq \| Z_t, \cdot - Z_{nk}, \cdot \|_{L^2(P \otimes \mu)} + \frac{\| Y_{nk} - E_r Y_{nk} \|}{\sqrt{rk - t}}$$

$$+ c(L_f, \mu(\mathbb{R}), \kappa') \int_t^{rk} \left( 1 + \frac{\| Y_{nk} - E_r Y_{nk} \|}{rk - r} \right) \, dr$$

$$\leq \frac{\| Y_{rk} - E_r Y_{rk} \|}{\sqrt{rk - t}} + c(L_f, \mu(\mathbb{R}), \kappa') \int_t^{rk} \left( 1 + \frac{\| Y_{rk} - E_r Y_{rk} \|}{rk - r} \right) \, dr$$

$$+ \| Z_t, \cdot - Z_{nk}, \cdot \|_{L^2(P \otimes \mu)} + \frac{2\| Y_{nk} - Y_{rk} \|}{\sqrt{rk - t}}$$

$$+ c(L_f, \mu(\mathbb{R}), \kappa') \int_t^{rk} \frac{\| Y_{rk} - E_r Y_{rk} - (Y_{rk} - E_r Y_{rk}) \|}{rk - r} \, dr.$$
For sufficiently large $n$ the terms in the second last line are arbitrarily small. For the last term, we use the relation (24) and get

$$
\int_t^{r_k} \| Y_{r_k} - \mathbb{E}_r Y_{r_k} - (Y_{r_k} - \mathbb{E}_r Y_{r_k}) \| \, dr
\leq \int_t^{r_k} \frac{f(r)}{r_k - r} \| f(u, X_u, Y_u, \bar{Z}_u) - f(u, X_u, Y^n_u, \bar{Z}^n_u) \| \, du
\leq \left[ \int_t^{r_k} \left( \int_t^u \frac{1}{r_k - r} \, dr \right)^2 \| f(u, X_u, Y_u, \bar{Z}_u) - f(u, X_u, Y^n_u, \bar{Z}^n_u) \| \, du \right]^{1/2}
\times \left[ \int_t^{r_k} \| f(u, X_u, Y_u, \bar{Z}_u) - f(u, X_u, Y^n_u, \bar{Z}^n_u) \|^2 \, du \right]^{1/2},
$$

where the last factor is arbitrarily small for sufficiently large $n$.

(i) $\Rightarrow$ (iv): Step 1. We assume first that (23) holds for $(\xi, f)$. In the following, we use the notation $f(r) := f(r, X_r, Y_r, \bar{Z}_r).$ Then equation (25) allows us to write

$\left\| \int_R (Z_{t,x} - Z_{s,x})h(x)\mu(dx) \right\|
\leq \left\| \int_R \left( \mathbb{E}_t \mathcal{D}_{t,x} Y_{r_k} - \mathbb{E}_s \mathcal{D}_{s,x} Y_{r_k} \right) h(x)\mu(dx) \right\|
+ \left\| \int_R \left[ \mathbb{E}_t \int_t^{r_k} \mathcal{D}_{t,x} f(r) \, dr - \mathbb{E}_s \int_s^{r_k} \mathcal{D}_{s,x} f(r) \, dr \right] h(x)\mu(dx) \right\|
\leq \left\| \mathbb{E}_t \mathcal{D}_t Y_{r_k} - \mathbb{E}_s \mathcal{D}_s Y_{r_k} \right\|_{L_2(\mathbb{P} \otimes \mu)} \| h \|_{L_2(\mu)}
+ \left\| \int_R \left[ \mathbb{E}_t \int_t^{r_k} \mathcal{D}_{t,x} f(r) \, dr - \mathbb{E}_s \mathbb{E}_t \int_t^{r_k} \mathcal{D}_{t,x} f(r) \, dr \right] h(x)\mu(dx) \right\|
+ \left\| \int_R \mathbb{E}_s \int_s^{t} \mathcal{D}_{s,x} f(r) \, dr h(x)\mu(dx) \right\|,
$
which is used to estimate
\[
\left\| \int_s^t \mathbb{E}_s \left( \int_s^r \mathcal{D}_{s,x} \mathbf{f}(r) \, dr \, h(x) \mu(dx) \right) \right\|
\leq \|h\|_{L^2(\mu)} \int_s^t \|\mathcal{D}_{s,r} \mathbf{f}(r)\|_{L^2(\mathbb{P} \otimes \mu)} \, dr
\leq \|h\|_{L^2(\mu)} \sqrt{c(L_f, \mu(\mathbb{R}), \kappa')} \int_s^t \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{r_k - r} \, dr.
\]

From Lemma 5.5, we conclude that
\[
\int_s^t \mathbb{E}_s \mathcal{D}_{s,x} \mathbf{f}(r) h(x) \mu(dx) = \frac{\mathbb{E}_t[I_1(\mathbb{1}_{[t,r]}h)\mathbf{f}(r)\mathbb{1}_{[r,t]}]}{r - t}.
\]

Applying the Clark–Ocone–Haussmann formula (7) and (33) yields
\[
\left\| \mathbb{E}_t \mathcal{D}_{t,x} \mathbf{f}(r) h(x) \mu(dx) - \mathbb{E}_s \mathbb{E}_t[I_1(\mathbb{1}_{[t,r]}h)\mathbf{f}(r)\mathbb{1}_{[r,t]}] \right\|_2^2
\leq \frac{1}{(r - t)^2} \int_s^t \int_{[s,t] \times \mathbb{R}} \mathbb{P}[\mathcal{D}_{u,y} \mathbb{E}_t[I_1(\mathbb{1}_{[t,r]}h)\mathbf{f}(r)] \mathbb{1}_{[r,t]}] M(du, dy)
\leq \frac{1}{(r - t)^2} \int_s^t \int_{[s,t]} \mathbb{E}[\mathbb{E}_t[I_1(\mathbb{1}_{[t,r]}h)\mathcal{D}_{u,y} \mathbf{f}(r)]^2] \mathbb{P}(du, dy)
\leq \frac{1}{r - t} \|h\|_{L^2(\mu)}^2 \int_s^t \|\mathcal{D}_{u,y} \mathbf{f}(r)\|_{L^2(\mathbb{P} \otimes \mu)}^2 \, du
\leq \frac{1}{r - t} \|h\|_{L^2(\mu)}^2 \sqrt{c(L_f, \mu(\mathbb{R}), \kappa')} \int_s^t \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|^2}{(r_k - r)^2} \, du.
\]

For the first inequality, we have used that for \( u < t < r \) it holds \( \mathbb{P} \otimes \mathbb{P}-\text{a.e.} \)
\[
\mathcal{D}_{u,y} [I_1(\mathbb{1}_{[t,r]}h)\mathbf{f}(r)] = I_1(\mathbb{1}_{[t,r]}h)\mathcal{D}_{u,y} \mathbf{f}(r)
\]
since \( \mathcal{D}_{u,y} I_1(\mathbb{1}_{[t,r]}h) = 0 \). This can be proved, for example, applying [16], Corollary 3.1, and approximation. Hence,
\[
\left\| \int_t^{r_k} \int_{[t,r]} \mathbb{E}_t \mathcal{D}_{t,x} \mathbf{f}(r) h(x) \mu(dx) - \mathbb{E}_s \int_{[t,r]} \mathbb{E}_t \mathcal{D}_{t,x} \mathbf{f}(r) h(x) \mu(dx) \right\| \, dr
\leq \|h\|_{L^2(\mu)} \sqrt{c(L_f, \mu(\mathbb{R}), \kappa')} \int_t^{r_k} \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{(r_k - r)\sqrt{r - t}} \, dr \sqrt{r - t - s}.
\]
Consequently, we infer
\[
\left\| \int_{\mathbb{R}} (Z_{t,x} - Z_{s,x}) h(x) \mu(dx) \right\|_{L^2}^2 \\
\leq \|h\|_{L^2(\mu)}^2 c(L_f, \mu(\mathbb{R}), \kappa') \left[ \int_{s}^{t} \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{(r_k - r)^2} dr \\
+ \left( \int_{t}^{r_k} \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{(r_k - r)\sqrt{r - t}} dr \right)^2 (t - s) \right].
\]

(34)

Obviously (5.1) implies
\[
\int_{s}^{t} \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{(r_k - r)^2} dr \leq c_1 \int_{s}^{t} (r_k - r)^{\theta_k - 2} dr
\]
and
\[
\left( \int_{t}^{r_k} \frac{\|Y_{r_k} - \mathbb{E}_r Y_{r_k}\|}{(r_k - r)\sqrt{r - t}} dr \right)^2 \leq c_1 \left( \int_{0}^{1} (1 - u)^{(\theta_k/2) - 1} u^{-1/2} du \right)^2 (r_k - t)^{\theta_k - 1}
\]
\[
= c_1 B^2 \left( \frac{\theta_k}{2}, \frac{1}{2} \right) (r_k - t)^{\theta_k - 1},
\]
where \(B\) denotes the beta function. For \(\theta_k < 1\) one can see that for all \(s, t \in [r_{k-1}, r_k[\) with \(s < t\) it holds
\[
(r_k - t)^{\theta_k - 1}(t - s) \leq \int_{s}^{t} \frac{r_k - r_{k-1}}{r_k - r} \left[ 1 - (1 - \varepsilon) \right]^{1-\theta_k} dr
\]
since this inequality is equivalent to
\[
t - s := \varepsilon (r_k - s) \leq \frac{r_k - r_{k-1}}{1 - \theta_k} \left[ 1 - (1 - \varepsilon)^{1-\theta_k} \right]
\]
for \(\varepsilon \in \)0, 1[ and \(s \in [r_{k-1}, r_k[, and the last inequality can be proved easily. For \(\theta_k = 1\) we have
\[
(r_k - t)^0(t - s) \leq \int_{s}^{t} \frac{r_k - r_{k-1}}{r_k - r} dr.
\]

Summarizing we get the assertion with
\[
c_4 = c_1 c(L_f, \mu(\mathbb{R}), \kappa') \left( 1 + B^2 \left( \frac{\theta_k}{2}, \frac{1}{2} \right) (r_k - r_{k-1}) \right).
\]
Step 2. Now we take the sequence \((\xi^n, f^n)_n\) from step 2 of the implication (i) \(\Rightarrow\) (iii) and proceed with (34) in the same way as we did with (30). In order to get the analogous estimate, we use the relations

\[
\int_s^t \left( f(r, u, X_u, Y_u, Z_u) - f(r, u, X_u, Y_u, Z_u^n) \right)^2 \, du \leq \int_s^t \frac{1}{r_k - r} \, dr \int_{r_{k-1}}^{r_k} \left( f(u, X_u, Y_u, Z_u) - f(u, X_u, Y_u^n, Z_u^n) \right)^2 \, du
\]

which is arbitrarily small for fixed \(s, t \in [r_{k-1}, r_k] \setminus N_k\) where \(\lambda(N_k) = 0\) and large \(n \in \mathbb{N}\), and

\[
\int_t^{r_k} \int_s^{r_k} \left( f(r, u, X_u, Y_u, Z_u) - f(r, u, X_u, Y_u^n, Z_u^n) \right)^2 \, du \, dr \leq \frac{2}{(r_k - t) \sqrt{r - t}} \int_t^{r_{k+1}/2} \frac{1}{\sqrt{r - t}} \, dr + \frac{\sqrt{2}}{\sqrt{r_k - t}} \int_t^{r_{k+1}/2} \frac{f(r, u, X_u, Y_u, Z_u) - f(r, u, X_u, Y_u^n, Z_u^n)}{r_k - r} \, du \, dr.
\]

For the last term, we can apply the estimate (32) to see that the RHS is arbitrarily small for large \(n \in \mathbb{N}\).

\[\square\]

6. A sufficient condition on \(\xi\) for fractional smoothness

Assume \((A_f)\) is satisfied for (8) and \(\xi \in \mathbb{H}\). If \(k = m\), condition (i) of Theorem 5.1 means in fact

\[
||\xi - E_s \xi||^2 \leq c_1 (T - s)^\theta_m, \quad s \in (r_{m-1}, T].
\]

Following the ideas of [14], we will formulate a condition on \(\xi \in \mathbb{H}\) which implies that (5.1) of Theorem 5.1 holds for all \(k \in \{1, \ldots, m\}\).

Assume that \(\tilde{X}\) and \(X\) are processes on \((\Omega, \mathcal{F}, \mathbb{P})\) such that \(\tilde{X}\) is an independent copy of the Lévy process \(X\). We define for \(0 \leq t < r \leq T\)

\[
X_s^{t,r} := \int_0^s 1_{[0,T] \setminus [t,r]}(u) \, dX_u + \int_0^s 1_{[t,r]}(u) \, d\tilde{X}_u, \quad s \in [0, T],
\]

that is, we obtain the process \(X^{t,r}\) from \(X\) by replacing it on the interval \([t, r]\) by its independent copy. Consequently, for the random measure \(M^{t,r}\) w.r.t. \(X^{t,r}\) we have the relation

\[
M^{t,r}(B) = M(B \setminus ([t, r] \times \mathbb{R})) + \tilde{M}(B \cap ([t, r] \times \mathbb{R})), \quad B \in \mathcal{B}([0, T] \times \mathbb{R}).
\]

By \((\tilde{\mathcal{F}}_t)_{t \in [0, T]}\) we denote the augmented natural filtration w.r.t. \((X, \tilde{X})\) and put \(L_2 := L_2(\Omega, \tilde{\mathcal{F}}_T, \mathbb{P})\) (the notation \((\mathcal{F}_t)_{t \in [0, T]}\) we keep for the augmented natural filtration w.r.t. \(X\)).
For symmetric $f_n \in L^2_2$ it holds
\[
\left\| I^{t, r}_n (f_n) - I_n (f_n) \right\|^2 = 2n! \left\| f_n (1 - 1_{([0, T] \backslash [r, t])} \times \mathbb{R}^n) \right\|^2_{L^2_2}.
\] (36)

For any $\eta \in L_2$ given by $\eta = \sum_{n=0}^{\infty} I_n (f_n)$, we define
\[
\eta^{t, r} := \sum_{n=1}^{\infty} I^{t, r}_n (f_n).
\]

**Theorem 6.1.** Assume that $\xi \in \mathbb{H}$ and $(A_f)$ is satisfied for (8). If there exist constants $c > 0$ and $\theta_k \in ]0, 1]$ such that
\[
\left\| \xi - \xi^{t, r_k} \right\|^2 \leq c(r_k - t)^{\theta_k} \quad \text{for all } t \in [r_k-1, r_k]
\]
then
\[
\left\| Y_{r_k} - \mathbb{E}_t Y_{r_k} \right\|^2 \leq C(r_k - t)^{\theta_k} \quad \text{for all } t \in [r_k-1, r_k].
\]

**Remark 6.2.** (i) For $f = 0$, it follows from Theorem 6.1 the implication
\[
\left\| \xi - \xi^{t, r_k} \right\|^2 \leq c(r_k - t)^{\theta_k} \quad \text{for all } t \in [r_k-1, r_k]
\]
\[
\implies \left\| \mathbb{E}_r \xi - \mathbb{E}_t \xi \right\|^2 \leq c(r_k - t)^{\theta_k} \quad \text{for all } t \in [r_k-1, r_k].
\] (37)

For certain $\xi$ the implication (37) is in fact an equivalence: for example, if $\xi = g(X_{r_m} - X_{r_{m-1}}, \ldots, X_{r_1} - X_{r_0}) \in L_2$ such that $g$ is a symmetric function and $r_k = \frac{kT}{m}$, for $k = 0, \ldots, m$. A more detailed discussion under which conditions equivalence holds for (37) as well as an example where $\left\| \mathbb{E}_r \xi - \mathbb{E}_t \xi \right\|^2 \leq c(r_k - t)^{\theta_k}$, for all $t \in [r_k-1, r_k]$ does not imply $\left\| \xi - \xi^{t, r_k} \right\|^2 \leq c(r_k - t)^{\theta_k}$, for all $t \in [r_k-1, r_k]$ can be found in [18].

(ii) If $\xi \in \mathbb{H}$ the case $\Theta = (1, 1, \ldots, 1)$ corresponds to Malliavin differentiability:
\[
\exists c > 0: \left\| \xi - \xi^{t, r_k} \right\|^2 \leq c(r_k - t) \quad \text{for all } t \in [r_k-1, r], k = 1, \ldots, m
\]
\[
\iff \xi \in \mathbb{D}_{1,2}.
\] (38)

Indeed, using the notation of the proof of Lemma 5.3 and setting $(\gamma (\alpha)) := \frac{n!}{\gamma_1 (\alpha)! \cdots \gamma_m (\alpha)!}$ we have
\[
\left\| \xi - \xi^{t, r_k} \right\|^2 = 2 \sum_{n=1}^{\infty} n! \sum_{[\alpha] \in V^m \backslash \sim} \left( \gamma (\alpha) \right) \left\| s_n^\alpha \right\|^2_{L^2_2 (\mu \otimes \omega)}
\]
\[
\times \left( (r_k - r_{k-1})^{\gamma_k (\alpha)} - (t - r_{k-1})^{\gamma_k (\alpha)} \right) \prod_{1 \leq l \leq m} (r_l - r_{l-1})^{\gamma_l (\alpha)}.
\]
This implies for \( r := \frac{t-r_{k-1}}{r_k-r_{k-1}} \) and \( R := \max_{1 \leq k \leq m} \frac{1}{r_k-r_{k-1}} \) that

\[
\frac{\| \xi - \xi_{t,r_k} \|^2}{r_k - t} = \frac{2}{r_k - r_{k-1}} \sum_{n=1}^{\infty} n! \sum_{[\alpha] \in V_m^*} \left( \frac{n}{\gamma(\alpha)} \right) \left\| g^\alpha_n \right\|_{L_2(\mu \otimes n)}^2 \lambda^n(\Lambda_\alpha) \\
\times \mathbb{1}_{\{\gamma_k(\alpha) \geq 1\}} \left( 1 + r + \cdots + r^{\gamma(\alpha)-1} \right)
\]

\[
\leq 2R \sum_{n=1}^{\infty} n! \sum_{[\alpha] \in V_m^*} \left( \frac{n}{\gamma(\alpha)} \right) \left\| g^\alpha_n \right\|_{L_2(\mu \otimes n)}^2 \lambda^n(\Lambda_\alpha) \gamma_k(\alpha)
\]

\[
\leq 2R \| \mathcal{D} \xi \|_{\mathbb{P}^m}^2
\]

since \( \gamma_k(\alpha) \leq n \). On the other hand, we get because of \( n = \sum_{k=1}^{m} \gamma_k(\alpha) \) for \( \alpha \in V_m^* \) from the above relation that

\[
\| \mathcal{D} \xi \|_{\mathbb{P}^m}^2 = \sum_{k=1}^{m} \sum_{n=1}^{\infty} n! \sum_{[\alpha] \in V_m^*} \left( \frac{n}{\gamma(\alpha)} \right) \left\| g^\alpha_n \right\|_{L_2(\mu \otimes n)}^2 \lambda^n(\Lambda_\alpha) \gamma_k(\alpha)
\]

\[
\leq \frac{T}{2} \sup_{1 \leq k \leq m} \sup_{r_k-1 < t < r_k} \frac{\| \xi - \xi_{t,r_k} \|^2}{r_k - t}.
\]

In [18], there is an example which shows that (38) is not necessarily true without assuming \( \xi \in \mathbb{H} \).

**Example 6.3.** If \( X \) is any square integrable Lévy process it holds for \( \xi := 1_{|K, \infty]}(X_1) \) with \( K \in \mathbb{R} \) that

\[
\xi \in \mathbb{D}_{1,2} \iff \sigma = 0 \quad \text{and} \quad \int_{|x| \leq 1} |x| \, d\nu(x) < \infty
\]

(see [26], Example 3.1). If \( X \) is a tempered \( \alpha \)-stable process given by the Lévy measure

\[
\nu_\alpha(dx) = \frac{d}{|x|^{1+\alpha}} (1 + |x|)^{-m} \mathbb{1}_{\{|x| \neq 0\}} \, dx,
\]

where \( d > 0, \alpha \in ]0, 2[ \) and \( m \in ]2 - \alpha, \infty[ \), it follows from [15], Section 4.2, that

\[
\xi \in \mathbb{B}^{1/2}_{2,\infty} := (L_2, \mathbb{D}_{1,2})_{1/2,\infty},
\]

that is, (see Remarks 5.2(i) and 6.2(i)) there exists a \( c > 0 \):

\[
\| \xi - \xi_{t,1} \|^2 \leq c(1 - t)^{1/2} \quad \text{for all} \ t \in [0, 1].
\]

Consequently, for any \( \alpha \in [1, 2[ \) the above \( \xi \) is in \( \mathbb{B}^{1/2}_{2,\infty} \) but not in \( \mathbb{D}_{1,2} \).

**Proof of Theorem 6.1.** If \((Y, Z)\) is a solution of (8), then \((Y^{t,r}, Z^{t,r})\) solves

\[
\mathcal{Y}_u = \xi^{t,r} + \int_u^T f(s, X^{t,r}_s, \mathcal{Y}_s, \tilde{Z}_s) \, ds - \int_{[u, T] \times \mathbb{R}} Z_{s,x} M^{t,r}(ds, dx).
\]
From (36), we conclude that
\[
\left\| \mathbb{E}_r I_n^r (f_n) - \mathbb{E}_r I_n (f_n) \right\|^2 = 2 \left\| \mathbb{E}_t I_n (f_n) - \mathbb{E}_r I_n (f_n) \right\|^2.
\]
Since \( Y_{r_k} \) is \( \mathcal{F}_{r_k} \)-measurable this implies for \( t \in ]r_k-1, r_k[ \) that
\[
2 \left\| Y_{r_k} - \mathbb{E}_t Y_{r_k} \right\|^2 = \left\| Y_{r_k} - Y_{t,r_k} \right\|^2.
\]
Since \( M \) and \( M_{t,r_k} \) coincide on \( ]r_k, T] \times \mathbb{R} \) we have
\[
Y_{r_k} - Y_{t,r_k} = \xi - \xi_{t,r_k}
\]
\[+ \int_{r_k}^T f(s, X_s, Y_s, \tilde{Z}_s) - f(s, X_{t,r_k}^s, Y_{r_k}^s, \tilde{Z}_s) \, ds
\]
\[- \int_{[r_k, r_k,T] \times \mathbb{R}} (Z_{s,x} - Z_{s,r_k,x}^t) M(dx, dx).
\]
By Theorem 2.3, we get
\[
\mathbb{E} \left| Y_{r_k} - Y_{t,r_k} \right|^2 + \mathbb{E} \int_{[r_k, T] \times \mathbb{R}} \left| Z_{s,x} - Z_{s,r_k,x}^t \right|^2 m(ds, dx)
\]
\[\leq C \left( \mathbb{E} \left| \xi - \xi_{t,r_k} \right|^2 + \mathbb{E} \int_{r_k}^T \left| f(s, X_s, Y_s, \tilde{Z}_s) - f(s, X_{t,r_k}^s, Y_{r_k}^s, \tilde{Z}_s) \right|^2 ds \right),
\]
which can be reduced by the Lipschitz property of \( f \) to
\[
\mathbb{E} \left| Y_{r_k} - Y_{t,r_k} \right|^2 + \mathbb{E} \int_{[r_k, T] \times \mathbb{R}} \left| Z_{s,x} - Z_{s,r_k,x}^t \right|^2 m(ds, dx)
\]
\[\leq C \left( \mathbb{E} \left| \xi - \xi_{t,r_k} \right|^2 + \mathbb{E} \int_{r_k}^T L_T^2 |X_s - X_{t,r_k}^s|^2 ds \right).
\]
By definition of \( X_{t,r_k} \) in (35), we get for \( s > r_k \)
\[
\mathbb{E} \left| X_s - X_{t,r_k}^s \right|^2 = \mathbb{E} \left| X_{r_k} - X_t + (\tilde{X}_{r_k} - \tilde{X}_t) \right|^2 = C_1 (r_k - t).
\]
Thus, there is a constant \( \tilde{C} \) such that
\[
\mathbb{E} \left| Y_{r_k} - Y_{t,r_k} \right|^2 + \mathbb{E} \int_{[r_k, T] \times \mathbb{R}} \left| Z_{s,x} - Z_{s,r_k,x}^t \right|^2 m(ds, dx)
\]
\[\leq C \mathbb{E} \left| \xi - \xi_{t,r_k} \right|^2 + \tilde{C} (r_k - t),
\]
which implies the assertion. \( \square \)
7. Concluding remarks

1. The assumption that the Lévy process $X$ is square integrable could be avoided by using a more general formulation of the Clark–Ocone–Haussmann formula and modifying the dependency of the generator $f(t, X_t, Y_t, \tilde{Z}_t)$ on $X_t$. (If $X$ is not square integrable, $X_t$ does not belong to $D_{1,2}$.)

2. A generalization to the setting of a $d$-dimensional Lévy process seems to be possible as well and similar results might be expected. For example, for a multidimensional Lévy process without Brownian part, a chaos decomposition and a Clark–Ocone–Haussmann formula can be found in [24] and [25]. This could be extended to general Lévy processes.

3. In this paper, the dependency of the driver with respect to the $Z$ process is given by the integral $\int Z_{t,x,k}(dx)$. A generalization to the dependency on finitely many integrals,

$$f(s, X_s, Y_s, \int Z_{t,x,k_1}(dx), \ldots, \int Z_{t,x,k_n}(dx)),$$

where the variables $z_1, \ldots, z_n$ in the generator underly the same assumptions as for one $z$-variable appears to be straightforward. Note that the choice $\kappa = \delta_0$ covers the case for the $Z$-variable from [5], for instance.

4. The investigation of the case where the terminal condition or the generator depends on paths of a process of a Lévy driven SDE is of major interest for further research, as well as the extension to assumptions beyond the Lipschitz generator setting like quadratic drivers.

Acknowledgement

We would like to thank S. Geiss for fruitful discussions and valuable suggestions and the unknown referee for his critical remarks.

Alexander Steinicke was partially supported by the project 133914 Stochastic and Harmonic Analysis, Interactions and Applications of the Academy of Finland.

References

[1] Alós, E., León, J.A. and Vives, J. (2008). An anticipating Itô formula for Lévy processes. ALEA Lat. Am. J. Probab. Math. Stat. 4 285–305. MR2456970

[2] Ankirchner, S. and Imkeller, P. (2008). Quadratic hedging of weather and catastrophe risk by using short term climate predictions. Preprint, HU Berlin.

[3] Applebaum, D. (2004). Lévy Processes and Stochastic Calculus. Cambridge Studies in Advanced Mathematics 93. Cambridge: Cambridge Univ. Press. MR2072890

[4] Bally, V. and Pagès, G. (2003). A quantization algorithm for solving multi-dimensional discrete-time optimal stopping problems. Bernoulli 9 1003–1049. MR2046816

[5] Barles, G., Buckdahn, R. and Pardoux, E. (1997). Backward stochastic differential equations and integral-partial differential equations. Stoch. Stoch. Rep. 60 57–83. MR1436432

[6] Baumgartner, F. and Geiss, S. (2014). Permutation invariant functionals of Lévy processes. Available at arXiv:1407.3645.
Bennett, C. and Sharpley, R. (1988). *Interpolation of Operators*. Pure and Applied Mathematics **129**. Boston, MA: Academic Press. MR0928802

Bergh, J. and Löfström, J. (1976). *Interpolation Spaces. An Introduction*. Berlin: Springer. MR0482275

Bouchard, B. and Elie, R. (2008). Discrete-time approximation of decoupled forward-backward SDE with jumps. *Stochastic Process. Appl.* **118** 53–75. MR2376252

Bouchard, B., Elie, R. and Touzi, N. (2009). Discrete-time approximation of BSDEs and probabilistic schemes for fully nonlinear PDEs. In *Advanced Financial Modelling*. Radon Ser. Comput. Appl. Math. **8** 91–124. Berlin: de Gruyter. MR2648459

Bouchard, B. and Touzi, N. (2004). Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.* **111** 175–206. MR2056536

Delong, Ł. and Imkeller, P. (2010). On Malliavin’s differentiability of BSDEs with time delayed generators driven by Brownian motions and Poisson random measures. *Stochastic Process. Appl.* **120** 1748–1775. MR2673973

Geiss, C., Geiss, S. and Laukkarinen, E. (2013). A note on Malliavin fractional smoothness for Lévy processes and approximation. *Potential Anal.* **39** 203–230. MR3102985

Geiss, C. and Laukkarinen, E. (2011). Denseness of certain smooth Lévy functionals in $D_{1,2}$. *Probab. Math. Statist.* **31** 1–15. MR2804974

Geiss, C. and Steinicke, A. $L^2$-variation of Lévy driven BSDEs with non-smooth terminal conditions. Available at arXiv:1201.3420.

Geiss, S. (2014). Interpolation on the Wiener–Lévy space and applications. *Lecture Notes*. To appear.

Geiss, S. and Hujo, M. (2003). Interpolation and approximation in $L^2(\gamma)$. Preprint 290, Dept. Mathematics and Statistics, Univ. Jyväskylä, Finland.

Geiss, S. and Hujo, M. (2007). Interpolation and approximation in $L^2(\gamma)$. Preprint 290, Dept. Mathematics and Statistics, Univ. Jyväskylä, Finland.

Hu, Y., Nualart, D. and Song, X. (2011). Malliavin calculus for backward stochastic differential equations and application to numerical solutions. *Ann. Appl. Probab.* **21** 2379–2423. MR2895419

Itô, K. (1956). Spectral type of the shift transformation of differential processes with stationary increments. *Trans. Amer. Math. Soc.* **81** 253–263. MR0077017

Lemor, J.-P., Gobet, E. and Warin, X. (2006). Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli* **12** 889–916. MR2265667

Ma, J. and Zhang, J. (2002). Path regularity for solutions of backward stochastic differential equations. *Probab. Theory Related Fields* **122** 163–190. MR1894066
[29] Meyer, P. (1979). Une remarque sur le calcul stochastique dépendant d’un paramètre. Séminaire de Probabilités (Strasbourg) 13 199–203.

[30] Nualart, D. (2006). The Malliavin Calculus and Related Topics, 2nd ed. Probability and Its Applications (New York). Berlin: Springer. MR2200233

[31] Seppälä, H. (2011). Interpolation spaces with general weight functions and approximation in $\ell_2(E)$ with application to stochastic integrals in: Interpolation spaces with parameter functions and $L_2$-approximations of stochastic integrals. Ph.D. thesis, Univ. Jyväskylä.

[32] Solé, J.L., Utzet, F. and Vives, J. (2007). Canonical Lévy process and Malliavin calculus. Stochastic Process. Appl. 117 165–187. MR2290191

[33] Solé, J.L., Utzet, F. and Vives, J. (2007). Chaos expansions and Malliavin calculus for Lévy processes. In Stochastic Analysis and Applications. Abel Symp. 2 595–612. Berlin: Springer. MR2397807

[34] Stricker, C. and Yor, M. (1978). Calcul stochastique dépendant d’un paramètre. Z. Wahrsch. Verw. Gebiete 45 109–133. MR0510530

[35] Tang, S.J. and Li, X.J. (1994). Necessary conditions for optimal control of stochastic systems with random jumps. SIAM J. Control Optim. 32 1447–1475. MR1288257

[36] Zhang, J. (2004). A numerical scheme for BSDEs. Ann. Appl. Probab. 14 459–488. MR2023027

Received December 2012 and revised October 2014