Scattering by a left-handed particle on a left-handed slab or surface

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Abstract. A small left-handed particle situated upon a slab of arbitrary electromagnetic parameter values is modelled as a superposition of electric and magnetic dipoles. The full-wave solution to the radiation problem is obtained for each of these dipoles and the far-field asymptotic forms are used to calculate the far-field scattering characteristics of the particle upon the slab. Novel scattering properties are observed in both the right- and left-handed cases, with the scattering patterns displaying enhanced transmission along with hitherto unobserved structure and directionality, offering a potential wealth of applications. The effect of the slab being left-handed is to cause a reversal in the scattering pattern below a semi-infinite slab, but to leave that of a finite slab unchanged. This reversal occurs about the plane perpendicular to both the plane of incidence and the interface.

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1. Introduction

The advent of left-handed media, which have recently been constructed at microwave frequencies [1], prompts revisiting classical problems of propagation and scattering that have been well studied in their right-handed form. The Mie scattering by left-handed spheres and cylinders has been studied in a number of papers [2]–[5] and the lensing properties of left-handed slabs has provoked much debate due to their ability to resolve below the diffraction limit [6]–[8]. In addition, there exists a large body of work that examines the effect of scattering by spherical particles in the presence of non-magnetic right-handed media [9]–[12]. Undertaken in this paper is a detailed study of the far-field scattering from a small spherical particle in the presence of bulk media that may be either right- or left-handed in nature. Since the perfect lens proposed by Pendry [6], with $\varepsilon = \mu = -1$, is impedance matched to vacuum, the slabs considered in this paper shall also possess this property, although the values of $\varepsilon$ and $\mu$ shall not take their perfect lens values in order that problems with divergent integrals are avoided [8]. A number of striking new scattering properties are observed, and these may be interpreted in terms of established left-handed phenomena. The scattering problem tackled here has additional relevance in that it can be considered a first approximation to modelling contaminants upon the surface of a left-handed lens.

In order to approach the problem of scattering by a small left-handed particle, it is first necessary to extend the Mie calculation to media for which the magnetic permeability, $\mu$, deviates from unity. The scattered field from a small spherical particle of electromagnetic parameters $\varepsilon \neq 1$ and $\mu \neq 1$ is identical in the far field to that radiated from a pair of orthogonal dipoles: an electric dipole orientated parallel to the incident electric field, and a magnetic dipole orientated parallel to the incident magnetic field [13]–[15]. One of the most interesting features of such a scatterer is the case where it is impedance matched to the surrounding medium. In this case, the scatterer is found to radiate no energy in the backscattering direction [2, 16]. This property has
only been exhibited for the isolated scatterer, however, and it is of interest to see whether it is preserved in the presence of bulk media that cause the spherical symmetry to be broken.

Although at present left-handed behaviour is only demonstrated by objects comparable in size to the wavelength, development of new materials is continuing apace: for example, substrates incorporating pairs of gold nano-particles have recently been shown to display the magnetic responses crucial to left-handed media at optical wavelengths [17], and magnifying superlenses have been realized that also operate in the optical regime [18]. Given the scale invariance of the Maxwell equations, there is no physical barrier to the construction of left-handed media on different length scales and the issue becomes one of materials science. It is shown by Monzon et al [2] that a finite sized impedance matched scatterer also displays the same zero-backscattering and enhanced forward scattering as the dipole term in the Mie expansion. The dipole model therefore extracts the essence of a finite sized scatterer and enables the problem to be solved analytically, identifying the relevant physics.

The above considerations prompt the approach towards the far field scattering problem for a left-handed particle in the presence of bulk media as being that of solving the radiation problem for the dipoles described previously. It is already commonplace [12] to treat a small non-magnetic scatterer as an electric dipole and this notion is extended in this paper to account for the modified scattering properties of a magnetic scatterer.

The relevance of considering the scattering by a left-handed particle in the presence of bulk media is clear following the result of [9] in which the standard radiation patterns of electric dipoles are significantly modified by the presence of an adjacent semi-infinite half space: a large proportion of the radiated energy is transmitted across the interface when the refractive index of the half space is large. This would suggest that significant modifications can be made also to the scattering properties of a left-handed scatterer in the presence of bulk media.

This paper brings together the problem of left-handed media and scattering by small particles in the presence of bulk media. The method of Hertz potentials used in [9], is extended in two ways. The first extension allows for the solution of the radiation problem for dipoles whose location is arbitrarily close to a bulk medium of arbitrary $\epsilon$ and $\mu$. The media considered in [9] was non-magnetic. The second extension is to account for the treatment of magnetic dipoles, in addition to the electric dipoles considered in [9]. The solutions obtained are the full-wave solutions for the radiation problem of electric and magnetic dipoles in the presence of a bulk medium of arbitrary electromagnetic parameters. In [10] the radiation problem for an electric dipole above a half space is considered, with results obtained in both the near- and far-fields. In [11] the scattering by a non-magnetic particle above a perfect surface is modelled using dipoles.

In all of the above cases the material above which the dipoles are situated is non-magnetic, and therefore necessarily right-handed. Also, only the case of a semi-infinite slab is discussed. We extend the formalism to allow for finite slabs of arbitrary electromagnetic parameters, thereby including the left-handed case. Furthermore, the formulation used here enables one to address the case where the scatterer is placed upon a planar slab, as well as upon the interface between two semi-infinite media, an extension upon the work in [9]–[11]. In addition, the expression of the solution as a superposition of cylindrical waves allows considerable physical insight into the problem, as in [8], clearly identifying the role of the surface resonances that are the hallmark of left-handed media.

In section 2 of this paper the dielectric magnetic scatterer is discussed briefly in the absence of a slab, together with its interesting scattering properties for the case of impedance matching.
The approach of [12] is then used to model the left-handed scatterer as a superposition of dipoles that are both electric and magnetic in nature, and the magnitudes of these dipoles are derived.

Having established that the scatterer may be modelled in the far-field as the above arrangement of dipoles, section 3 is concerned with the solution of the radiation problem for an arbitrary dipole on a slab with planar interfaces. The majority of this section sets out the calculation for a right-handed slab in order to introduce the formalism and notation within a familiar setting, while identifying the modifications that are required in order to account for a left-handed slab. The method of Hertz potentials [9, 19, 20] is used and the form of the solution is found to be in such a form so as to transparently illustrate the physics of the problem. It is this that enables the discussion of the scattering problem in conjunction with the plasmon resonances displayed by left-handed media. In addition, the solutions obtained are in a closed form and results may be achieved by quadrature rather than simulation.

Section 4 outlines the novel scattering properties that may be obtained from the scattering systems under discussion here: the scatterer on the surface of a semi-infinite slab and the scatterer on the surface of a finite slab of thickness $d$.

Conclusions are drawn in section 5 and the appendix details the far-field calculation, performed using the method of stationary phase, that is used to obtain the results of section 4 from the solutions obtained in section 3.

2. Modelling the scattering problem

In this section it is shown how the far-field of a magnetic scatterer may be modelled as the superposition of the fields arising from orthogonal dipoles placed upon the upper surface of a slab.

The Mie scattering of plane incident radiation by a spherical particle in vacuum is well documented [2, 14, 16]. In the small scatterer limit $k_0a \ll 1$, where $k_0$ is the wavevector of the incident radiation in vacuum and $a$ is the particle radius, the scattered far-field is identical to that produced by the superposition of the far-fields of two dipoles, one electric and one magnetic, with moments $p$ and $m$ respectively given in Gaussian units by

$$p = a^3 \epsilon_s - \frac{1}{\epsilon_s + 2} E^{(i)} ,$$

$$m = a^3 \mu_s - \frac{1}{\mu_s + 2} H^{(i)},$$

where $E^{(i)}$ and $H^{(i)}$ are the incident electric and magnetic fields respectively. The relative permittivity and permeability of the scatterer are given by $\epsilon_s$ and $\mu_s$ respectively, the subscript serving to differentiate these parameters from those of the bulk medium. For a non-magnetic particle, for which $\mu_s = 1$, only the electric dipole is retained since terms pertaining to the magnetic response are of higher order in the dimensionless particle size parameter $k_0a$.

When the scatterer is impedance matched to the surrounding vacuum, $\epsilon_s = \mu_s$ and the dipoles are of equal magnitude since in vacuum $|E^{(i)}| = |H^{(i)}|$ for Gaussian units. The scattered far-field time averaged Poynting vector is given by

$$S = \frac{c}{8\pi} \text{Re}(E \times H^*),$$

where $c$ is the speed of light in vacuum and $\text{Re}$ denotes the real part of the complex vector.

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Figure 1. Cross-section of the far-field normalized Poynting flux for the scattered wave from an isolated impedance matched scatterer. The scatterer is situated at the origin and plane radiation is incident upon it along the $z$-axis ($\theta = 0$).

and is normalized by its value in the forward scattering direction. This enables the normalized time averaged Poynting vector to be plotted as a function of the spherical polar variables $\theta$ and $\phi$ where $\theta$ is measured from the positive $z$-axis and $\phi$ is measured form the $x$-axis in the $xy$-plane [15]. The normalized far field time averaged Poynting vector for the isolated scatterer is given by

$$\bar{S} = \frac{1}{4} (1 - \cos \theta)^2,$$

and this is shown in cross-section in figure 1. The absence of radiation in the backscattering direction is clear and this arises due to the interference between the fields from the two orthogonal dipoles.

A bulk medium is now introduced, modelled as a slab of thickness $d$ in the $z$-direction, and of infinite extent in the $x$- and $y$-directions. In this paper only the case of $p$-polarization shall be considered, for which $E^{(0)}$ lies in the plane of incidence, although the case of $s$-polarization may also be treated using the same method. A scatterer illuminated with $p$-polarized light is shown in figure 2. The field incident upon the scatterer consists of two contributions: the incident plane wave (including its reflection from the surface) and the scattered wave from the particle–surface system. The latter of these is accounted for by considering an image dipole located a distance $h$ on the other side of the interface. It is assumed here that the interacting radiation strikes the interface at normal incidence, as in [21], in which case the magnitude of the image dipole is
Figure 2. Geometrical configuration of the scattering problem for \( p \)-polarization (positive \( y \)-axis into the page). The scatterer is placed upon the surface of a bulk medium of material parameters \( \epsilon \) and \( \mu \) whose upper surface is the plane \( z = 0 \). The angle of incidence relative to the normal to this plane is denoted \( \theta_0 \).

The above approach of neglecting the effect of the image dipole has also been employed with great success in modelling larger particles (\( a \) being comparable to the wavelength) within the dipole approximation [21]. In terms of the formalism here this corresponds to a value \( h > 0 \) and the approximation is valid due to the fact that the scattered field decays in amplitude between the first and second scattering events [22].

In the absence of any contributions from the image dipole, the field at a scatterer located on the slab is given by [12]

\[
E_x^{(i)} = (1 - R_p) E^{(i)} \cos \theta_0, \tag{5}
\]

\[
E_y^{(i)} = 0, \tag{6}
\]

\[
e_z^{(i)} = (1 + R_p) E^{(i)} \sin \theta_0, \tag{7}
\]

\[
H_x^{(i)} = 0, \tag{8}
\]

\[
H_y^{(i)} = -(1 + R_p) H^{(i)}, \tag{9}
\]

\[
H_z^{(i)} = 0. \tag{10}
\]
where

\[ R_p = \frac{r_p(1 - e^{2i\beta d})}{1 - r_p^2 e^{2i\beta d}} \]  \hspace{2cm} (11)

is the reflection coefficient of the slab for \( p \)-polarized light, \( \beta = nk_0 \cos \theta_0 \), and

\[ r_p = \frac{\epsilon \cos \theta_0 \mp \sqrt{\epsilon \mu - \sin^2 \theta_0}}{\epsilon \cos \theta_0 \pm \sqrt{\epsilon \mu - \sin^2 \theta_0}} \]  \hspace{2cm} (12)

is the reflection coefficient for the single interface. The upper sign is used when the bulk medium is right-handed and the lower sign when it is left-handed. It can be seen that the case of a single interface can easily treated by taking the limit \( d \to \infty \) in (11) with a vanishingly small lossy part introduced to the refractive index, \( n \), or by using (12) directly in (5)–(10).

There are therefore three dipoles (two electric and one magnetic) excited by the incident radiation and, taking the incident radiation to be such that \( E^{(i)} = H^{(i)} = 1 \), these are given by

\[ p_x = (1 - R_p)a^3 \left( \frac{\epsilon_s - 1}{\epsilon_s + 2} \right) \cos \theta_0, \]  \hspace{2cm} (13)

\[ p_z = (1 + R_p)a^3 \left( \frac{\epsilon_s - 1}{\epsilon_s + 2} \right) \sin \theta_0, \]  \hspace{2cm} (14)

\[ m_y = -(1 + R_p)a^3 \left( \frac{\mu_s - 1}{\mu_s + 2} \right), \]  \hspace{2cm} (15)

\[ m_z = p_y = m_x = 0. \]  \hspace{2cm} (16)

The magnitudes of the dipoles given above are a function of both the scatterer material parameters \( \epsilon_s \) and \( \mu_s \) and the bulk medium parameters \( \epsilon \) and \( \mu \), which appear within the reflection coefficient \( R_p \). The angular dependence of the dipole moments is affected by the sensitivity of the reflection coefficient to the incident angle and as a result the dipoles are no longer necessarily parallel to the electric and magnetic fields of the plane incident radiation.

The scattered far-fields from the scatterer on the slab are obtained by superposing the far-fields of the above dipoles and from these the time averaged Poynting vector, \( \mathbf{S} \), may be readily calculated.

3. Hertz potential solution for dipoles above a slab

Considered here is the radiation problem for a dipole situated on the \( z \)-axis a height \( h \) above a planar slab of left-handed medium as shown in figure 3. The upper and lower surfaces of the slab are the planes \( z = 0 \) and \( z = -d \) respectively. Regions 1 and 3 (\( z > 0 \) and \( z < -d \) respectively) are vacuum while region 2 (\( 0 > z > -d \)) has material parameters \( \epsilon \) and \( \mu \). Having obtained the
solution for $h > 0$, the solution for the dipole on the interface is obtained through taking the limit $h \to 0$.

Following the approaches of [8, 9, 19], the problem is solved using the Hertz vector potentials, of which there are two: one electric, denoted $\mathbf{e}$, and one magnetic, denoted $\mathbf{m}$. These can be used to obtain the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ via the relations

$$
\mathbf{E} = \nabla (\nabla \times \mathbf{e}) + \epsilon \mu k_0^2 \mathbf{e} + i k_0 \mu (\nabla \times \mathbf{m}),
$$

and

$$
\mathbf{H} = \nabla (\nabla \times \mathbf{m}) + \epsilon \mu k_0^2 \mathbf{m} - i k_0 \epsilon (\nabla \times \mathbf{e}),
$$

and satisfy the inhomogeneous wave equations

$$
\nabla^2 \mathbf{e} - \frac{1}{c^2} \frac{\partial^2 \mathbf{e}}{\partial t^2} = -\frac{4\pi}{\epsilon} \mathbf{p},
$$

and

$$
\nabla^2 \mathbf{m} - \frac{1}{c^2} \frac{\partial^2 \mathbf{m}}{\partial t^2} = -4\pi \mathbf{m}.
$$

where $\mathbf{p}$ and $\mathbf{m}$ are the source polarization and magnetization vectors respectively.

An appropriate notation is required to label the various components of the Hertz potentials arising from the different dipoles. The following convention is adopted: $\mathbf{e}_x$, for example, denotes the $x$-component of the Hertz vector in region 2 arising from an electric dipole oriented parallel to the $y$-axis. Similarly, $\mathbf{m}_y$, indicates the $y$-component of the Hertz vector in region 1 arising from a magnetic dipole oriented parallel to the $z$-axis.

The calculation may be performed as in [8] for each dipole with the additional stipulation that when considering a dipole in the plane of the interface two components of the Hertz vector are required in order to satisfy the boundary conditions [19]: one parallel to the dipole moment, the other perpendicular to the interface. The results for the dipole directly upon the slab ($h = 0$) follow.

**Figure 3.** Geometrical configuration for a dipole situated above a slab. Also shown are the cylindrical coordinates $(R, \phi, z)$. 

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3.1. Vertical dipoles

3.1.1. Electric case. In this case \( p = p_\delta \delta(R) \hat{z} \) and \( \varepsilon \Pi^z = \varepsilon \Pi^z \hat{z} \). The solution in this case is

\[
e^\varepsilon \Pi^z_{1z}(R, z) = \int_0^\infty e^{\varepsilon \chi_{1z}^z(\alpha)} J_0(\alpha R) e^{-\gamma_1^z} d\alpha,
\]

(21)

\[
e^\varepsilon \Pi^z_{2z}(R, z) = \int_0^\infty \left[ e^{\varepsilon \chi_{2z}^z(\alpha)} e^{\gamma_2^z} + e^{\varepsilon \chi_{2z}^-}(\alpha) e^{-\gamma_2^z} \right] J_0(\alpha R) d\alpha,
\]

(22)

\[
e^\varepsilon \Pi^z_{3z}(R, z) = \int_0^\infty e^{\varepsilon \chi_{3z}^z}(\alpha) J_0(\alpha R) e^{\gamma_3^z} d\alpha,
\]

(23)

where

\[
e^{\varepsilon \chi_{1z}^z}(\alpha) = \frac{p_\alpha}{\gamma_1}\left[ \left( \frac{\gamma_2 - \epsilon \gamma_1}{\gamma_2 + \epsilon \gamma_1} \right) \left( \frac{e^{-2\gamma_2d}-1}{D_\epsilon(\alpha)} \right) + 1 \right],
\]

(24)

\[
e^{\varepsilon \chi_{2z}^z}(\alpha) = \frac{p_\alpha}{\gamma_1} \left( \frac{2\gamma_1}{\gamma_2 + \epsilon \gamma_1} \right) \frac{1}{D_\epsilon(\alpha)},
\]

(25)

\[
e^{\varepsilon \chi_{2z}^-}(\alpha) = \frac{p_\alpha}{\gamma_1} \left( \frac{2\gamma_1}{\gamma_2 + \epsilon \gamma_1} \right) \left( \frac{\gamma_2 - \epsilon \gamma_1}{\gamma_2 + \epsilon \gamma_1} \right) \frac{e^{-2\gamma_2d}}{D_\epsilon(\alpha)},
\]

(26)

\[
e^{\varepsilon \chi_{3z}^z}(\alpha) = \frac{p_\alpha}{\gamma_1} \left( \frac{2\gamma_1}{\gamma_2 + \epsilon \gamma_1} \right) \left( \frac{2\epsilon \gamma_2}{\gamma_2 + \epsilon \gamma_1} \right) \frac{e^{(\gamma_1-\gamma_2)d}}{D_\epsilon(\alpha)},
\]

(27)

and

\[
D_\epsilon(\alpha) = 1 - \left( \frac{\gamma_2 - \epsilon \gamma_1}{\gamma_2 + \epsilon \gamma_1} \right)^2 e^{-2\gamma_2d}.
\]

(28)

The integration variable \( \alpha \) corresponds to the component of the wavevector parallel to the plane of the interfaces, and \( \gamma_1 \) the component normal to the interfaces (the \( z \)-component) in region \( i \). The dispersion relation dictates that \( \gamma_1 = \sqrt{n_i^2 k_0^2 - \alpha^2} \), where \( n_i = \sqrt{\epsilon_i \mu_i} \) is the refractive index of the medium occupying region \( i \). The integral solution given above can therefore be interpreted as a superposition of outgoing cylindrical waves that propagate in the \( z \)-direction for \( \alpha < n_i k_0 \) but that are evanescent for \( \alpha > n_i k_0 \).

In regions 1 and 3 the boundary conditions at infinity [23] require that the following signs of the square root must be taken:

\[
\gamma_1 = \begin{cases} 
- i \sqrt{k^2 - \alpha^2} & \text{for } \alpha < k_0, \\
\sqrt{\alpha^2 - k^2} & \text{for } \alpha > k_0.
\end{cases}
\]

(29)
In region 2 the same boundary conditions dictate that

$$\gamma_2 = \begin{cases} \mp i \sqrt{\epsilon \mu k^2 - \alpha^2} & \text{for } \alpha < \sqrt{\epsilon \mu k_0}, \\ \sqrt{\alpha^2 - \epsilon \mu k^2} & \text{for } \alpha > \sqrt{\epsilon \mu k_0}. \end{cases}$$

(30)

where the minus sign is used when the medium in region 2 is right-handed, and the plus sign when it is left-handed.

The Hertz potential solution for the dipole on an infinite half-space can be obtained from the above by taking the limit $d \to \infty$ and considering regions 1 and 2 only: 1 being the vacuum above the interface, 2 being the medium below. In order to do this a vanishingly small imaginary part needs to be introduced to the $\gamma_i$ for the propagating modes. This result shall not be presented explicitly however, as it is contained within the above.

3.1.2. Magnetic case. The solutions for the vertical magnetic dipole on a slab, for which $m = m_z \delta(z) \delta(R) \hat{z}$ and $\Pi^z = \Pi^z \hat{z}$, can be obtained from the above solution for the vertical electric dipole by replacing the polarization with the magnetization (i.e. $p_z$ with $m_z$), and by interchanging $\epsilon$ and $\mu$. In so doing, the factor $D_\epsilon$ becomes

$$D_\mu(\alpha) = 1 - \left( \frac{\gamma_2 - \mu \gamma_1}{\gamma_2 + \mu \gamma_1} \right)^2 e^{-2\gamma_2 d}. \quad (31)$$

3.2. Horizontal dipoles

3.2.1. Electric case. Here, taking the dipole to lie along the $x$-axis, $p = p_x \delta(z) \delta(R) \hat{x}$. When dealing with a dipole lying in the plane of an interface, two components of the Hertz potential are required: one parallel to the dipole moment and one perpendicular to the interface [9, 19]. In this case:

$$\epsilon \Pi^h_x = \epsilon \Pi^h_x \hat{x} + \epsilon \Pi^h_z \hat{z}, \quad (32)$$

and the solution is

$$\epsilon \Pi^x_{1x}(R, z) = \int_0^\infty \epsilon \chi^x_{1x}(\alpha) J_0(\alpha R) e^{-\gamma_1 z} \, d\alpha, \quad (33)$$

$$\epsilon \Pi^x_{2x}(R, z) = \int_0^\infty \left( \epsilon \chi^x_{2x}(\alpha) e^{\gamma_2 z} + \epsilon \chi^x_{1x}(\alpha) e^{-\gamma_1 z} \right) J_0(\alpha R) \, d\alpha, \quad (34)$$

$$\epsilon \Pi^x_{3x}(R, z) = \int_0^\infty \epsilon \chi^x_{3x}(\alpha) J_0(\alpha R) e^{\gamma_2 z} \, d\alpha, \quad (35)$$

and

$$\epsilon \Pi^x_{1z}(R, z) = \int_0^\infty \epsilon \Omega^x_{1z}(\alpha) J_1(\alpha R) e^{-\gamma_1 z} \, d\alpha, \quad (36)$$
\[ e^x \Pi_{2z}(R, z) = \int_0^\infty (e^x \Omega_{2z}^{+}(\alpha)e^{i\gamma z} + e^x \Omega_{2z}^{-}(\alpha)e^{-i\gamma z}) J_1(\alpha R) \, d\alpha, \]  
(37)

\[ e^x \Pi_{3z}(R, z) = \int_0^\infty e^x \Omega_{3z}^{+}(\alpha) J_1(\alpha R) e^{i\gamma z} \, d\alpha, \]  
(38)

where

\[ e^x \chi_{1z}^+(\alpha) = \frac{p_\alpha}{\gamma_1} \left[ \frac{2\gamma_2 + \mu \gamma_1}{\gamma_2 + \mu \gamma_1} \frac{1}{D_\mu(\alpha)} \right], \]  
(39)

\[ e^x \chi_{2z}^+(\alpha) = \frac{p_\alpha}{\epsilon \gamma_1} \left( \frac{2\gamma_1}{\gamma_2 + \mu \gamma_1} \right) \frac{1}{D_\mu(\alpha)}, \]  
(40)

\[ e^x \chi_{2z}^-(\alpha) = \frac{p_\alpha}{\epsilon \gamma_1} \left( \frac{2\gamma_2 \gamma_1}{\gamma_2 + \mu \gamma_1} \right) \frac{e^{-2\gamma_2 d}}{D_\mu(\alpha)}, \]  
(41)

\[ e^x \chi_{3z}^+(\alpha) = \frac{p_\alpha}{\gamma_1} \left( \frac{2\gamma_2}{\gamma_2 + \mu \gamma_1} \right) \frac{e^{\gamma_2 - \gamma_1 d}}{D_\mu(\alpha)}, \]  
(42)

and

\[ e^x \Omega_{1z}^{+}(\alpha) = -\frac{2p_\alpha a^2(n - 1 - \cos \phi)}{(\gamma_2 + \mu \gamma_1)(\gamma_2 + \epsilon \gamma_1)} \left( \frac{1 - e^{-2\gamma_2 d}}{D_\mu(\alpha)D_\epsilon(\alpha)} \right) \left[ 1 - \left( \frac{\gamma_2 - \mu \gamma_1}{\gamma_2 + \epsilon \gamma_1} \right) \frac{\gamma_2 - \epsilon \gamma_1}{\gamma_2 + \mu \gamma_1} e^{-2\gamma_2 d} \right], \]  
(43)

\[ e^x \Omega_{2z}^{+}(\alpha) = -\frac{2p_\alpha a^2(n - 1 - \cos \phi)}{e(\gamma_2 + \mu \gamma_1)(\gamma_2 + \epsilon \gamma_1)} \left( \frac{1}{D_\mu(\alpha)D_\epsilon(\alpha)} \right) \]  
\[ \times \left[ 1 - \left( \frac{\gamma_2 - \epsilon \gamma_1}{\gamma_2 + \mu \gamma_1} \right) \frac{2\gamma_2}{\gamma_2 + \mu \gamma_1} - \frac{\gamma_2 - \mu \gamma_1}{\gamma_2 + \mu \gamma_1} e^{-2\gamma_2 d} \right], \]  
(44)

\[ e^x \Omega_{2z}^{-}(\alpha) = -\frac{2p_\alpha a^2(n - 1 - \cos \phi)}{e(\gamma_2 + \mu \gamma_1)(\gamma_2 + \epsilon \gamma_1)} \left( \frac{e^{-2\gamma_2 d}}{D_\mu(\alpha)D_\epsilon(\alpha)} \right) \]  
\[ \times \left[ \gamma_2 - \epsilon \gamma_1 \left( \frac{1}{\gamma_2 + \epsilon \gamma_1} - \frac{2\gamma_2}{\gamma_2 + \mu \gamma_1} e^{-2\gamma_2 d} \right) \right], \]  
(45)

\[ e^x \Omega_{3z}^{-}(\alpha) = -\frac{2p_\alpha a^2(n - 1 - \cos \phi)}{(\gamma_2 + \mu \gamma_1)(\gamma_2 + \epsilon \gamma_1)} \left( \frac{1}{D_\mu(\alpha)D_\epsilon(\alpha)} \right) \]  
\[ \times \left[ \gamma_2 - \epsilon \gamma_1 \left( \frac{1 - 2\gamma_2}{\gamma_2 + \mu \gamma_1} e^{-2\gamma_2 d} \right) \frac{\gamma_2 - \mu \gamma_1}{\gamma_2 + \mu \gamma_1} \left( 1 - \frac{2\gamma_2}{\gamma_2 + \epsilon \gamma_1} e^{-2\gamma_2 d} \right) \right]. \]  
(46)

The solution for a dipole lying parallel to the y-axis is obtained from the above by making the transformation \( \phi \rightarrow \phi' = \phi - \pi/2 \), under which \( \cos \phi \rightarrow \sin \phi \), in the above.
3.2.2. Magnetic case. For a magnetic dipole on a slab orientated along the $x$-axis, $m = m_0 \delta(z) \delta(R) \hat{x}$ and two components of the Hertz potential are again required: $\Pi^{\uparrow} = \Pi^{\uparrow}_x \hat{x} + \Pi^{\uparrow}_z \hat{z}$. Each component may be obtained from its electric counterpart by means of the same transformation as before, namely the interchange of the electromagnetic parameters $\epsilon$ and $\mu$ and the replacement of the polarization with the magnetization.

The transformation to a dipole orientated along the $y$-axis is again obtained by making a transformation in the azimuthal angle.

4. Results

The radiation pattern of each individual dipole in the previous section already possesses an interesting angular distribution, as seen in [9]. Each dipole used to model the scattering problem also has a magnitude that is dictated by the reflection properties of the slab, as given in equations (13)–(16). The interplay of these two effects leads to novel scattering patterns for the left-handed scatterer upon a slab, as shall now be shown.

The expressions in the previous section represent the full-wave solution to the radiation problem for any given dipole. From the full-Hertz potential solutions given above, the method of stationary phase may be used to derive their asymptotic form in the far-field and subsequently the asymptotic form of the electromagnetic fields themselves via equations (17) and (18). This calculation is outlined in full in the appendix. Having obtained the fields, it is a simple matter to calculate the time averaged Poynting vector. The time averaged Poynting vector is again normalized by its forward scattering value for the isolated scatterer.

In the results that follow, a variety of angles of incidence are considered. Normal incidence occurs when the scatterer is illuminated from above by plane incident radiation whose wavevector is pointed along the negative $z$-axis. For non-normal incidence, the angle of incidence $\theta_0$ is measured from the positive $z$-axis and in all the results that follow, $-\pi/2 < \theta_0 < 0$, indicating that the particle is illuminated by radiation that is incident in the $xz$-plane from the region where $x < 0$ and $z > 0$, as shown in figure 2.

The values of the scatterer parameters used here are $\epsilon_s = \mu_s = -1$. However, within the dipole approximation used here, the normalization of the Poynting vector means that these values do not appear explicitly in the expressions for the normalized Poynting flux. The values taken for the slab, be it finite or semi-infinite, are $\epsilon = \mu = \pm 2$, the sign obviously dictating the handedness. It is necessary that $\mu \neq 1$ in order for magnetic effects to be present and the absolute values are equal in order to observe the effect upon the radiation patterns of the handedness alone.

In all of the following cases therefore, both the scatterer and the medium in region 2 are impedance matched to the vacuum. Since the impedance matched scatterer displays zero backscattering, and there is no reflection from an impedance matched interface at normal incidence, multiple scattering events do not occur.

In what follows, the results are given for a right-handed slab with the effect of the handedness of the slab considered last.

4.1. Normal incidence ($\theta_0 = 0$) and zero backscatter

Cross-sections of the normalized Poynting flux in the incident plane ($\phi = 0$) are shown in figures 4 and 5 for the semi-infinite and finite slabs respectively. For these cross-sections it
Figure 4. Cross-section in the incident plane of the far-field scattering pattern from a small spherical particle ($\epsilon_s = \mu_s = -1$) on a semi-infinite slab ($\epsilon = \mu = 2$) illuminated at normal incidence ($\theta_0 = 0$). Left-hand figure is plotted on a linear scale, right-hand figure on a logarithmic scale. The dashed lines indicate the directions for which $\theta = \pm (\pi - \theta_c)$.

Figure 5. Cross-section in the incident plane of the far-field scattering pattern from a small spherical particle ($\epsilon_s = \mu_s = -1$) on a slab ($\epsilon = \mu = 2$) of thickness $d = \lambda$ illuminated at normal incidence ($\theta_0 = 0$). Left-hand figure is plotted on a linear scale, right-hand figure on a logarithmic scale.

is required to take $\theta$ in the range $-\pi < \theta < \pi$ where $\theta$ is measured from the positive $z$-axis. The full patterns are azimuthally symmetric, as expected given the normal incidence geometry also shares this symmetry. The left-hand figures are plotted on a linear scale, the right-hand figures on a logarithmic scale. There are a number of interesting features to note.
The zero backscattering property of the isolated scatterer is preserved in region 1 in the case of both the semi-infinite and finite slabs and this is again expected since both the scatterer and the interface are impedance matched: the zero backscattering property is preserved in the presence of impedance matched bulk media for illumination at normal incidence. This arises as a result of the azimuthal symmetry of the normal incidence geometry, enabling the complete destructive interference of fields along the backscattering direction, as is the case for the isolated scatterer. Despite the qualitative similarity in backscatter, there are quantitative differences between the semi-infinite and finite slab cases however.

The magnitude of the scattering pattern in region 1 is far smaller for the case of the semi-infinite slab than it is for the finite slab. This can be explained simply by increased reflection due to the introduction of a second interface: scattered modes that are transmitted into the slab through the first interface may be reflected back into region 1 by the second. This is a result of taking $d = \lambda$ here, a value for which the finite slab reflection coefficient is greater than that of the semi-infinite slab for all but a narrow range of incident angles.

This situation can be reversed, by taking for example $d = \lambda/2$ for which $R_p < r_p$ for most angles, whereupon the reflection from the finite slab would diminish. This is due to interference between reflected modes from each interface. By taking larger values of $d$ the finite slab reflection coefficient can achieve small scale oscillations with incident angle and this shall be revisited later.

In the lower far-field region, the scattering pattern cross-section is greater in magnitude for the semi-infinite slab than it is for the finite slab. The comparatively small size of the pattern for the finite slab can be accounted for in two ways: the increased reflection into the upper far-field region previously discussed, and the possibility of bound modes within the slab. These bound modes propagate to infinity laterally within the slab and do not contribute to the scattered power calculated here.

It is therefore the case that a scatterer upon an impedance matched semi-infinite slab transmits the vast proportion of scattered radiation into the lower far-field region. There are two reasons for this. Firstly, the scattered field from the particle itself has a strong predominance towards the forward scattering direction, as shown in figure 1 and [2]. Secondly, the effect of the interface is to reinforce this forward scattering as discussed by Engheta and Papas [9]. The case of a scatterer upon an impedance matched finite slab displays a similar, although less marked, property.

Another clear feature of the scattering pattern in the lower far-field region is the appearance of sharp cusps located symmetrically about the z-axis for the scatterer on a semi-infinite slab. These cusps occur at $\theta = \pm(\pi - \theta_c)$ where $\theta_c$ is the critical angle given by $\sin^{-1}\left(\frac{1}{n}\right)$. For $\pi/2 < |\theta| < \pi - \theta_c$ waves from the source that are evanescent in region 1 become propagating in region 2 and so the scattering pattern in these directions retains near-field information about the dipoles. However, for $\pi - \theta_c < |\theta| < \pi$, all the radiation results from fields that are propagating in region 1. The cusp arises due to this change in the nature of the modes contributing to the radiation pattern either side of the critical angle.

The cusps do not appear in the case of the finite slab and their disappearance is directly attributable to the presence of the second interface. Propagating modes that strike the second interface at an angle to the normal greater than the critical angle undergo total internal reflection. Therefore in region 3 these modes revert to their initial evanescent nature, meaning that they no longer contribute to the far-field scattering pattern.
4.2. Non-normal incidence and extreme skewing

The general effect of non-normal incidence is to skew the scattering pattern in both the upper and lower far-field regions. This is to be expected since the strong forward scattering maximum of the scatterer is no longer directed perpendicularly into the bulk medium.

Considering first the scatterer on a semi-infinite slab, the scattering patterns for incidence at $\theta_0 = -0.1$ rads are shown in figure 6 for the upper far-field region. Despite the small deviation from normal incidence, the pattern is skewed almost entirely into the region $0 < \theta < \pi$. The maximum does not lie in the specular direction itself due to the interference effects introduced by reflections at the interface, both prior to and after the scattering event. The size of the skewed lobe, while still small, is much larger than that for normal incidence despite the slight deviation in angle of incidence from 0 to $-0.1$ rads. This suggests that the minimal backscattering seen in the case of normal incidence deteriorates rapidly with a slight increase in the angle of incidence and is attributable to increased reflection resulting from the forward scattering maximum of the isolated scatterer no longer being directed perpendicularly into the surface.

The logarithmic plot on the right-hand side of figure 6 shows that there is radiation into the region $-\pi/2 < \theta < 0$ and that there is a structure arising as the result of interference between the fields of the different dipoles used to model A scatterer. However, the patterns in this region are incredible small as compared to those in the region $0 < \theta < \pi/2$.

Further, it should be noted that this extreme skewing occurs only towards the specular direction, that is into the region $0 < \theta < \pi/2$ here. It is not possible with a semi-infinite slab to reverse the direction of this skewing to being into the region $-\pi/2 < \theta < 0$.

The pattern in region 2 for $\theta_0 = -0.1$ rads, shown on the left-hand side of figure 7, differs only slightly in both shape and size from the normal incidence result, as shown in figure 4. However, when $\theta_0 = -\pi/4$, as shown on the right-hand side of figure 7, the skewing in the lower far-field region is again pronounced, with a distinct maximum at $\theta = \pi - \theta_c$. What is more, this maximum value of the Poynting flux is approximately 10 times that of the isolated scatterer, indicating an extremely high level of transmission in this direction. It is also interesting to note that the maximum in the scattered radiation occurs at $\theta = \pi - \theta_c$ for all non-normal angles of incidence.
Figure 7. Cross-section in the incident plane of the far-field radiation pattern for the scatterer on a semi-infinite slab ($\epsilon = \mu = 2$) in lower far-field region ($\pi/2 < \pm \theta < \pi$) plotted on a logarithmic scale. The small spherical particle ($\epsilon_s = \mu_s = -1$) is illuminated with plane radiation incident at $\theta_0 = -0.1$ for the left-hand figure, and at $\theta_0 = -\pi/4$ for the right-hand figure. The dashed lines indicate the directions for which $\theta = \pm (\pi - \theta_c)$.

The asymmetric skewing is reproducible also for the case of the scatterer upon a finite slab. Figure 8 shows the scattering patterns for two angles of incidence: $\theta_0 = \pi/4$ on the left and $\theta_0 = \pi/3$ on the right. It can clearly be seen that the skewing can occur in both directions: the patterns in region 1 are highly sensitive to the angle of incidence. The converse is true for region 3 (not shown) where there is relatively little change in the scattering pattern for the same change in angle of incidence. It is again true, as it was for the semi-infinite slab, that there is an increase in radiation into the region above the slab at the expense of that into the region below due to increased reflection. However, this increased reflection is again a facet of the value of $d$ taken here, for which the reflection coefficient increases with angle of incidence.

Another means of modifying the direction of skewing is to alter the slab thickness $d$. Figure 9 shows the cross-section in the plane of incidence of the scattering pattern for a particle on a slab illuminated with light incident at $\theta_0 = \pi/4$. The left-hand figure is for a slab with thickness $d = \lambda$, whereas for the right-hand figure $d = \lambda/8$. It is found that the effect of this change in slab thickness is to reverse the direction of the skewing. In the case shown here of $d = \lambda/8$ this skewing is again pronounced with the forward lobe being several times smaller than the backward lobe. This is similar to the effect observed for the single interface for $\theta_0 \approx 0.1$ where the forward lobe dwarfed the backward lobe. It is possible to obtain a strong skewing of the scattering pattern in both directions in the presence of a slab (although not shown, strong forward skewing may be obtained with, for example, $d = 11 \lambda/20$, $\lambda/30$, keeping all other parameters constant), whereas for a single interface strong skewing was only possible towards the same side of the normal as the specular direction. With a constant angle of incidence therefore it may be possible to construct a device in which the scattered radiation is highly directional in nature by means of varying the slab thickness. As observed when considering the angle of incidence, there is only a slight skewing in the pattern below the slab for this change in $d$ along with a slight modification in the magnitude of the scattering pattern.
Figure 8. Cross-sections in the incident plane of the far-field scattering patterns for a small spherical particle ($\epsilon_s = \mu_s = -1$) on a finite slab ($\epsilon = \mu = 2, d = \lambda$) plotted on a logarithmic scale. $\theta_0 = \pi/4$ for left-hand figure, and $\theta_0 = \pi/3$ for right-hand figure. Dashed line in both cases shows the direction of incidence.

Figure 9. Cross-sections in the incident plane of the far-field scattering pattern for a small spherical particle ($\epsilon_s = \mu_s = -1$) on a finite slab ($\epsilon = \mu = 2$) illuminated at $\theta_0 = \pi/4$, plotted on a logarithmic scale. For the left-hand figure $d = \lambda$, and $d = \lambda/8$ for the right-hand figure. Dashed line in both cases shows the incident direction.

4.3. Diffraction effect in thick slab

Thus far only values of $d$ less than or equal to a wavelength have been considered, and these have resulted in scattering patterns consisting of smooth envelopes displaying only slow variation in $\theta$. As $d$ increases more rapid variations in the scattering patterns are observed as $\theta$ varies.
The cross-sections in the incident plane of the scattering patterns for $\theta_0 = -\pi/4$ and $d = 25\lambda$ are shown in figure 10. It can clearly be seen that in both far-field regions there are fine-scale variations in the scattering pattern with $\theta$. These rapid variations are in stark contrast to the slowly varying form seen for smaller values of $d$ (see figure 5, for example).

A similar effect is observed in [10] but is the result of a different phenomenon. In [10] it is claimed that the rapidly oscillating behaviour arises as a result of the interference between the homogeneous modes and the surface waves. The structures in figure 10 arise as a consequence of interference between radiation that has undergone different numbers of multiple internal reflections within the slab before emerging and contributing to the far-field. The effect is analogous to that seen in a diffraction grating, and is characterized, in part, by the small scale variations in the finite slab reflection coefficient $R_p$. This explanation also accounts for the fact that no small scale variations are noticed for smaller $d$. As for the standard diffraction grating, larger $d$ allows for more orders of interference to be observed. Here, the analogue of the grating spacing is the separation between the points at which a ray emerges from the slab surface and this varies linearly with $d$.

There are differences too between the effect observed here and that of the standard diffraction grating. A diffraction grating has a constant phase difference between orders emerging from the grating, and these orders have constant amplitude. This is clearly not the case here since each successive ray emerging from the slab differs in amplitude from that of the previous by a factor of $R_p^2$ having undergone a further two internal reflections. It is for this reason that complete cancellation is not observed as for the standard diffraction grating.

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The smooth form of the radiation patterns seen for smaller $d$ are again obtained in the limit $d \to \infty$ since in this case there are no reflections from the slab interface at $z \to -\infty$.

### 4.4. Effect of left-handed slab

All of the above results have been for the case where the bulk medium is right-handed. The case where they are left-handed is also tractable within our formalism through judicious choices of the signs of the wavevector within the bulk medium as discussed earlier, see equation (30). In the results that follow, only the signs of $\epsilon$ and $\mu$ have been changed, their absolute values remain the same in order that the effect of the change in handedness alone is clear.

There is an added complication with a left-handed interface since it can support surface resonances. These resonances manifest themselves as poles in the integrands of the Hertz vector potentials meaning that the integral becomes dominated by the region around the resonance. In this case it is the dipoles that excite the resonances and, as a consequence of considering steady state situations where all quantities vary as $e^{-i\omega t}$, the pole in the integrand represents the fact that energy is continually being fed into the resonance over an infinite period of time. While the integrand contains a pole, the integral itself can still be evaluated [8]. Moreover, the contribution from the pole decays exponentially with $z$ away from the interface and so does not contribute to the far-field.

The strong fields associated with the surface resonances do not affect the assumptions regarding the single scattering of radiation since all resonant components take the form of outgoing cylindrical waves confined to the interface, and so do not impinge again upon the scatterer. It is therefore possible to proceed as before, noting the changes in the sign of $\gamma_2$ required for a left-handed medium.

In the case of the scatterer on the semi-infinite slab, when the medium in region 2 is left-handed, the scattering pattern in region 1 is unchanged from that shown in figure 4 since the reflection coefficient (12) is unchanged: the change in sign of the material parameters is precisely countered by the differing choice in sign in equation (30).

Within the bulk medium it is found that the scattering pattern for the left-handed case may be obtained from that of the right-handed case by means of a reflection across the $z$-axis ($\theta = \pi$). This is precisely what would be expected given the fact that the pattern in the slab for a left-handed medium comprises propagating modes that have undergone negative refraction. For normal incidence, the scattering pattern is unchanged if the signs of the material parameters $\epsilon$ and $\mu$ are negated. This is due to the symmetry of the normal incidence scenario.

When considering the scatterer upon a finite slab it is found that the scattering patterns in region 3 are independent of the handedness of the medium in region 2. In the solutions for the finite slab the change of sign in $\gamma_2$ in the $e^{i2\gamma_2 d}$ term means that the integrands and reflection coefficients become sensitive to the handedness of the slab material. However, the only difference arising upon a reversal of handedness is one of phase, with the moduli of all quantities remaining invariant. Since the Poynting vector is composed solely of moduli of the fields the asymmetry will have no effect upon the far-field scattering patterns. Physically, this is as expected. The effect of a single left-handed interface is to reverse the direction of the scattering pattern in region 2 due to negative refraction. In the case of a finite slab the pattern has simply undergone two such reversals, corresponding to modes undergoing two negative refractions, and is thus unchanged.
5. Conclusions

In this paper the full-wave solution to the radiation problem for an arbitrary dipole placed upon a slab of arbitrary electromagnetic parameters has been obtained. The far-field asymptotic form of the full solution is obtained and used as a means of modelling the far-field scattering characteristics of a small dielectric magnetic spherical particle placed adjacent to the bulk medium.

It has been shown that there are a number of novel scattering properties that are demonstrated by a dielectric, magnetic scatterer placed upon a slab. The radiation patterns display hitherto unobserved structure and directionality offering a potential wealth of applications.

The first property is that of zero backscatter seen in the case of normal incidence, the same property displayed by the isolated scatterer. In addition, there is significantly reduced reflection back into the upper far-field region \((z \to \infty)\), particularly in the case of the semi-infinite slab. In the lower far-field region \((z \to -\infty)\) the radiated energy is dispersed across a wide range of angles. Such enhanced transmission may be of use in areas where it is desirable that as much incident radiation as possible is dispersed within a medium, such as solar panels for example.

Also observed is the possibility of extreme skewing of the scattering pattern in the upper far-field region, whereby the reflected radiation is highly directional, but not necessarily specular. In the case of the semi-infinite slab, the direction of this skewing is always towards the specular direction, but with a finite slab it is tunable by either varying the angle of incidence or the slab thickness. Severe skewing is also seen in the lower far-field region for the case of the semi-infinite slab, but for the finite slab no such effect is observed.

For large (but not infinite) slab thicknesses another effect is observed. Fine scale oscillations are observed in the radiation pattern, similar in nature to the pattern seen from a diffraction grating. Since there is no multiple scattering, these lobes arise as a result of interference between radiation that has undergone different multiples of internal reflection within the slab.

The effect of the slab being left-handed is to cause a reversal in the radiation pattern for each planar interface due to the negative refraction that occurs. This reversal occurs about the plane perpendicular to both the plane of incidence and the interface. This has implications when considering the problem of surface contaminants upon left-handed lenses: surface contaminants will tend to radiate a significant amount of energy towards the focus of the left-handed lens.

Although no detailed study of the polarization effects has been performed here, for the case of impedance matching both \(s\)- and \(p\)-polarizations are equivalent. The formalism outlined in this paper, however, facilitates the treatment of the case in which either one or both of the slab and the scatterer are not impedance matched, and performing the relevant calculations would provide an interesting extension to this work.

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Appendix A. Far-field asymptotics

The far-field asymptotics are performed in exactly the same way as in [9]. There are two far-field regions: above the dipole ($k_0z \to +\infty$) and below the dipole ($k_0z \to -\infty$). The nature of the solution in the latter region depends upon whether the dipole is placed upon a finite or semi-infinite slab and these two cases shall be dealt with separately. The solution for the far-field above the dipole has the same form in both cases and shall be dealt with first.

A.1. Far-field as $k_0z \to +\infty$

In region 1, the Hertz potentials are all in one of two forms: either

$$\Pi_1^a(R, z) = \int_0^\infty \chi_1(\alpha) J_0(\alpha R) e^{-\gamma_1 z} d\alpha$$ (A.1)

or

$$\Pi_1^b(R, z) = \int_0^\infty \Omega_1(\alpha) J_1(\alpha R) e^{-\gamma_1 z} d\alpha.$$ (A.2)

By transforming to spherical polar coordinates $(r, \theta, \phi)$ in the spatial domain and doing likewise in the wavevector domain, and by using the relationship

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(z \sin \beta - n\beta)} d\beta,$$ (A.3)

it can be shown using the method of stationary phase [9] that in the far-field $k_0z \to \infty$ (or, in spherical polar coordinates, $k_0r \to \infty$, $0 \leq \theta < \pi/2$) integrals of the form (A.1) vary as

$$\Pi_1^a(r, \theta) \sim -i \frac{\chi_1(k_0 \sin \theta) \cos \theta e^{ik_0r}}{r},$$ (A.4)

and integrals of the form (A.2) as

$$\Pi_1^b(r, \theta) \sim -\frac{\Omega_1(k_0 \sin \theta) \cos \theta e^{ik_0r}}{r}.$$ (A.5)

A.2. Far-field as $k_0z \to -\infty$

The nature of the far-fields as $z \to \infty$ depends upon whether the dipole is located on a finite or semi-infinite slab. Each of these shall be dealt with in turn.

A.2.1. Finite slab. When the dipole is located on the upper surface of a finite slab, it is the vacuum in region 3 that occupies the far-field domain as $z \to -\infty$. All Hertz potential integrals in this region are again in one of two forms:

$$\Pi_3^a(R, z) = \int_0^\infty \chi_3(\alpha) J_0(\alpha R) e^{\gamma_3 z} d\alpha$$ (A.6)
or

$$\Pi^{b}_{3}(R, z) = \int_{0}^{\infty} \Omega_{3}(\alpha) J_{1}(\alpha R) e^{\gamma_{3} z} d\alpha.$$  \hspace{1cm} (A.7)

Again, $\chi_{3}(\alpha)$ and $\Omega_{3}(\alpha)$ are functions of $\alpha$ that can be obtained from comparison between (A.6) and (A.7) and the relevant form of the Hertz vector given in section 3. Using the same method as before, the above can be shown to vary in the far-field ($k_{0}r \to \infty$, $\pi/2 < \theta \leq \pi$) as

$$\Pi^{a}_{3}(r, \theta) \sim \frac{i\chi_{3}(k_{0} \sin \theta) \cos \theta e^{ik_{0}r}}{\sin \theta}$$ \hspace{1cm} (A.8)

for integrals of the form (A.6), and as

$$\Pi^{b}_{3}(r, \theta) \sim \frac{\Omega_{3}(k_{0} \sin \theta) \cos \theta e^{ik_{0}r}}{\sin \theta}$$ \hspace{1cm} (A.9)

for integrals of the form (A.7).

A.2.2. Semi-infinite slab. When the dipole is located upon a semi-infinite slab it is the medium in region 2 that occupies the far field domain, and the handedness of this medium impacts directly upon the far-field calculation. In the cases considered above, the handedness of medium 2 is accounted for by inserting the relevant sign of material parameters and by taking the corresponding choice of signs within the integrand, as discussed earlier. The integrals are obtained from those in section 4 by taking the limit $d \to \infty$ whereupon only one of the terms in the integrand does not vanish. These integrals are all of the form

$$\Pi^{a}_{2}(R, z) = \int_{0}^{\infty} \chi_{2}(\alpha) J_{0}(\alpha R) e^{\gamma_{2} z} d\alpha.$$ \hspace{1cm} (A.10)

or

$$\Pi^{b}_{2}(R, z) = \int_{0}^{\infty} \Omega_{2}(\alpha) J_{1}(\alpha R) e^{\gamma_{2} z} d\alpha,$$ \hspace{1cm} (A.11)

and the handedness is accounted for by taking the appropriate sign of $\gamma_{2}$, as in equation (30). Applying the method of stationary phase again yields

$$\Pi^{a}_{2}(r, \theta) \sim \frac{i\chi_{2}(\pm nk_{0} \sin \theta) \cos \theta e^{in_{k_{0}r}}}{\sin \theta}$$ \hspace{1cm} (A.12)

for integrals of the form (A.10), and

$$\Pi^{b}_{2}(r, \theta) \sim \frac{\Omega_{2}(\pm nk_{0} \sin \theta) \cos \theta e^{in_{k_{0}r}}}{\sin \theta}$$ \hspace{1cm} (A.13)

for integrals of the form (A.11). The upper and lower signs are used when the medium in region 2 is right- and left-handed respectively.
A.3. Far-fields from Hertz potentials

Having obtained the far field forms of the Hertz potentials it is noted that, when transformed into spherical polar coordinates, they may all be written in the form

\[ \Pi_{ir} = \xi_{ir}(\theta, \phi) \frac{e^{i k_ir}}{r}, \]  
(A.14)

\[ \Pi_{i\theta} = \xi_{i\theta}(\theta, \phi) \frac{e^{i k_i r}}{r}, \]  
(A.15)

\[ \Pi_{i\phi} = \xi_{i\phi}(\theta, \phi) \frac{e^{i k_i r}}{r}, \]  
(A.16)

where the subscript \( i = 1, 2, 3 \) denotes the region and \( k_i = n_i k_0 \). In the limit \( k_i r \to \infty \), applying the field definitions (17) and (18) to the case of an electric dipole, for which only the electric potential is required, yields to leading order in \( k_0 r \)

\[ e^E_{ir} = 0, \]  
(A.17)

\[ e^E_{i\theta} = k_i^2 e^E_{i\theta} \xi_{i\theta}(\theta, \phi) \frac{e^{i k_i r}}{r}, \]  
(A.18)

\[ e^E_{i\phi} = k_i^2 e^E_{i\phi} \xi_{i\phi}(\theta, \phi) \frac{e^{i k_i r}}{r}, \]  
(A.19)

\[ e^H_{ir} = 0, \]  
(A.20)

\[ e^H_{i\theta} = -\epsilon_i k_0 k_i e^H_{i\phi} \xi_{i\phi}(\theta, \phi) \frac{e^{i k_i r}}{r}, \]  
(A.21)

\[ e^H_{i\phi} = \epsilon_i k_0 k_i e^H_{i\theta} \xi_{i\theta}(\theta, \phi) \frac{e^{i k_i r}}{r}. \]  
(A.22)

For a magnetic dipole, the fields are obtained from the magnetic potentials:

\[ m^H_{ir} = 0, \]  
(A.23)

\[ m^H_{i\theta} = k_i^2 m^H_{i\theta} \xi_{i\theta}(\theta, \phi) \frac{e^{i k_i r}}{r}, \]  
(A.24)

\[ m^H_{i\phi} = k_i^2 m^H_{i\phi} \xi_{i\phi}(\theta, \phi) \frac{e^{i k_i r}}{r}. \]  
(A.25)
\begin{equation}
mE_{ir} = 0.\tag{A.26}
\end{equation}

\begin{equation}
mE_{i\theta} = \mu_0 k_0 k_i m \xi_{i\phi}(\theta, \phi) \frac{e^{ikr}}{r}, \tag{A.27}
\end{equation}

\begin{equation}
mE_{i\phi} = -\mu_0 k_0 k_i m \xi_{i\theta}(\theta, \phi) \frac{e^{ikr}}{r}. \tag{A.28}
\end{equation}

The far fields above may readily be used to construct the time averaged Poynting vector, $\bar{S}$, which is defined as [14]

$$
\bar{S} = \frac{c}{8\pi} \text{Re} (E \times H^*). \tag{A.29}
$$

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