Applications of Generalized Special Functions in Stellar Astrophysics

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ABSTRACT

This article gives a brief outline of the applications of generalized special functions such as generalized hypergeometric functions, G-functions and H-functions into the general area of nuclear energy generation and reaction rate theory such as the energy generation in a simple stellar model and nuclear reaction rates in non-resonant and resonant as well as screened non-resonant situations.

1 Introduction

During the last few years several authors have been trying to get explicit and exact analytic representations for various problems in astrophysics. Analytical solutions are often extremely difficult to obtain and often lead to complicated intractable mathematical expressions. Due to these facts some researchers go for approximations, numerical results and computer printouts. But such numerical results usually do not give much insight into what is happening between the stage of the formulation of the problem and the final computer printout. When the experimental results do not agree with the computer printouts resulting from a proposed theory one needs explicit analytic formulation to modify the theory. Hence there has been renewed interest in getting explicit and exact theoretical results rather than numerical results.
During the past few years the authors have demonstrated that the techniques
of generalized special functions combined with statistical techniques are very
powerful in getting analytical results for many problems which were believed
to be intractable mathematically. This paper will confine itself to the applica-
tion of such techniques for studying energy generation and thermonuclear
reaction rates for stellar models.

2 Energy Generation Rate in Simple Stellar Models

Some simple stellar models for looking into the internal structure of stars have
been given by Kourganoff (1980) and Haubold and Mathai (1984, 1986b).
Consider a spherically symmetric star in quasi-static equilibrium, that is, we
exclude magnetic fields, rotation and other effects which are changing rapidly,
so that we can take the star as a gas sphere in hydrostatic equilibrium. Let
\( r \) denote the distance from the center to an interior point and \( R \) the radius
of the star under consideration.

2.1 Linear model

The simplest model one can think of for the radial density \( \rho(r) \) is the follow-
ing,

\[
\rho(r) = \rho_c \left(1 - \frac{r}{R}\right) = \rho_c (1 - x), \quad x = r/R, \quad \rho(0) = \rho_c, \quad \rho(R) = 0. \tag{1}
\]

In this case the mass

\[
M(r) = 4\pi \int_0^r t^2 \rho(t)dt = \frac{4\pi}{3} \rho_c r^3(1 - \frac{3x}{4}), \tag{2}
\]

the pressure

\[
P(r) = P(0) - G \int_0^r \frac{M(t)}{t^2} \rho(t)dt = \frac{\pi}{36} G \rho_c^2 R^2 (5 - 24x^2 + 28x^3 - 9x^4), \tag{3}
\]
where $G$ is the gravitational constant, and the temperature from the equation of state $P(r) = k\rho(r)T(r)/M_u\mu$, where $\mu$ is the mean molecular weight, $M_u$ is the atomar mass unit, $k$ is the Boltzmann constant, is then

$$
T(r) = \frac{\pi GM_u}{36k}\rho_c R^2 \frac{(5 - 24x^2 + 28x^3 - 9x^4)}{1 - x} = \frac{\pi GM_u}{36k}\rho_c R^2 \sum_{n=0}^{\infty} c_n x^n \tag{4}
$$

with $c_0 = c_1, c_2 = -19/5, c_3 = 9/5, c_3 = 9/5, c_n = 0$ for $n \geq 4$. The nuclear energy generation rate $\epsilon(r)$ will depend on $T(r)$ and $\rho(r)$. If we take a simple model,

$$
\epsilon(r) = \epsilon_0(\rho_0, T_0)\left(\frac{\rho(r)}{\rho_0}\right)^\alpha \left(\frac{T(r)}{T_0}\right)^\beta, \alpha > 0, \beta > 0
$$

then

$$
\epsilon(r) = d_1(1 - x)^\alpha(\sum_{n=0}^{\infty} c_n x^n)^\beta, \tag{5}
$$

where $d_1$ is a constant. The luminosity $L$ of the star is then,

$$
L = 4\pi \int_0^R r^2 \rho(r)\epsilon(r)dr = d_2 \int_0^1 x^2(1 - x)^{\alpha+1}(\sum_{n=0}^{\infty} c_n x^n)^\beta dx, \tag{6}
$$

where $d_2$ is a constant. We consider a general integral

$$
g(\alpha, \beta, \gamma) = \int_0^1 x^\gamma(1 - x)^{\alpha+1}(\sum_{n=0}^{\infty} c_n x^n)^\beta dx. \tag{7}
$$

The $L$ in (6) is evaluated in Haubold and Mathai (1984) for various structures of $\sum_{n=0}^{\infty} c_n x^n$. One such structure is when

$$
\sum_{n=0}^{\infty} c_n x^n = (1 + a_1 x)(1 + a_2 x) \ldots (1 + a_k x)
$$

for a fixed $k$. In this case one can show that $g(\alpha, \beta, \gamma)$ can be evaluated in terms of a Lauricella function $F_D$. Then a particular case will go in terms of Gauss’ hypergeometric function.
2.2 A general stellar model

A linear decrease of the density from the center to the surface may not be very appropriate. Faster or slower decrease can be incorporated by using one more parameter and taking the model for the density as,

\[ \rho(r) = \rho_c(1 - \left(\frac{r}{R}\right)^\delta) = \rho_c(1 - x^\delta), \quad x = r/R, \quad \rho(0) = \rho_c, \quad \rho(R) = 0. \]  

In this case the mass, pressure, and luminosity will work out to be the following.

\[ M(r) = \frac{4\pi}{3} \rho_c R^3 x^3 \left(1 - \frac{3}{\delta + 3} x^\delta\right). \]

\[ P(r) = \frac{4\pi G}{3} \rho_c^2 R^2 \left\{ \psi - \frac{x^2}{2} + \frac{(\delta + 6)}{(\delta + 2)(\delta + 3)} x^{\delta + 2} - \frac{3}{2(\delta + 1)(\delta + 3)} x^{2\delta + 2} \right\}, \]

\[ \psi = \frac{1}{2} - \frac{\delta + 6}{(\delta + 2)(\delta + 3)} + \frac{3}{2(\delta + 1)(\delta + 3)}, \]

and

\[ L(R) = 4\pi \rho_c c_0 (\rho_c, T_c) R^3 \delta \int_0^1 x^{(3/\delta) - 1}(1 - x)^{\alpha + \beta + 1} \]

\[ \left[1 - \frac{x^{2/\delta}}{2\psi} \left\{ \frac{2(\delta + 6)x}{(\delta + 2)(\delta + 3)} + \frac{3x^2}{(\delta + 1)(\delta + 3)} \right\} \right]^\beta d.x. \]

For \( \beta = 1 \) or a positive integer the evaluation of the integral in (11) is not difficult. For the general case one can look at various situations as in Section 2.1 and solve many of these in terms of generalized functions.

2.3 A further generalized stellar model

A model with more flexibility for the density is a two parameter model

\[ \rho(r) = \rho_c[1 - \left(\frac{r}{R}\right)^\delta]^\gamma, \quad \delta > 0, \quad \gamma > 0. \]

In this case the mass will lead to the form

\[ M(r) = 4\pi \rho_c \int_0^r t^2 \left[1 - \left(\frac{t}{R}\right)^\delta\right] \gamma dt, \quad 0 < \frac{t}{R} \leq \frac{r}{R} \leq 1 \]

\[ = \frac{4}{3} \pi \rho_c r^3 \, _2F_1 \left(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; \left(\frac{r}{R}\right)^\delta\right), \]

(12)
where \( \text{$_2F_1$} \) is a Gauss’ hypergeometric function. Pressure and temperature in this case can be obtained in series of hypergeometric functions. For example,

\[
P(r) = \frac{4\pi G \rho_c^2}{\delta^2} \left\{ R^2 \sum_{m=0}^{\infty} \frac{(-\gamma)_m (\frac{2}{\delta} + 1)_m}{m!(\frac{2}{\delta} + 1 + \gamma)_m (\frac{2}{\delta} + m)(\frac{2}{\delta} + m)} - \frac{r^2}{\delta^2} \sum_{m=0}^{\infty} \frac{(-\gamma)_m [(\frac{r}{R})^\delta]^m}{m! (\frac{2}{\delta} + m)(\frac{2}{\delta} + m)} \right\}.
\]

Note that when \( \gamma \) is a positive integer then \( P(r) \) is available as a finite sum.

### 3 Collision Probabilities

In Haubold and Mathai (1984a) a detailed derivation of the collision probability for nuclear reactions is given under the assumption that the nuclear velocity distribution remains Maxwell-Boltzmannian. The relation between the macroscopic nuclear reaction probability and the microscopic nuclear reaction cross section on the basis of equilibrium-thermodynamic arguments is studied in this paper as well as in Haubold and Mathai (1985). A basic integral to be evaluated to compute the reaction probability, under this approach, is the following:

\[
A = \int_0^\infty y^p e^{-y - zy^{1/2}} dy
\]

More complicated integrals will appear when dealing with thermonuclear reaction rates in various non-resonant and resonant cases, see for example, Haubold and Mathai (1986, 1986a,b,c). First we will introduce a statistical technique of tackling integrals of the type (15). Let us consider a slightly more general integral

\[
A_r(z; p, n, m) = p \int_0^\infty e^{-py} y^{-nr} e^{-zy^{-n/m}} dy.
\]

**Theorem 3.1.**
\[ p \int_0^\infty e^{-pt}t^{-nr}e^{-zt^{n/m}} \, dt = p^{nr}(2\pi)^{(2-n-m)/2}m^{1/2}n(1-2nr/2) \]

\[ C_{0,m+n,0}^{m+n,0} \left( \frac{z^{m/p}}{m^{m/n}} \right) \bigg|_{0}^{\infty} \left( \frac{m}{m/n} \right)^{1-n/m} \ldots \left( \frac{m}{m/n} \right)^{n/m} \]

for \( R(p) > 0, R(z) > 0, G(.) \) is a G-function and \( R(.) \) denotes the real part of \( . \). For a discussion of the G-function and the more generalized H-function see Mathai and Saxena (1973, 1978).

Writing the Mellin-Barnes integral representation for the G-function and then simplifying the gammas by using the multiplication formula for gamma functions, namely,

\[ \Gamma(mz) = (2\pi)^{(1-m)/2}m^{mz-1/2}\Gamma(z)\Gamma(z + \frac{1}{m})\ldots\Gamma(z + \frac{m-1}{m}), m = 1, 2 \ldots \]

one has the right side of (17) simplified to the following form,

\[ A_r(z; p, n, m) = p^{nr}m \frac{1}{n} \int_L \Gamma \left( \frac{ms}{n} \right) \Gamma(1 - nr + s) (pz^{m/n})^{-s} ds, \]

where \( L \) is a suitable contour. The integral on the right side of (19) can be written as an H-function, see Mathai and Saxena (1978), which can be reduced to a G-function due to the fact that \( m/n \) is rational. Now consider two independent positive real random variables \( x \) and \( y \) with the density functions \( f_1(x) < 0 \) for \( 0 < x < \infty \), \( f_2(y) < 0 \) for \( 0 < y < \infty \) and \( f_1(x) = 0, f_2(y) = 0 \) elsewhere. Let the \((s-1)\)st moments of \( x \) and \( y \) be denoted by \( g_1(s) \) and \( g_2(s) \) respectively. Consider \( u = xy \). From statistical independence of \( x \) and \( y \) one has

\[ E(u^{s-1}) = (Ex^{s-1})(Ey^{s-1}) = g_1(s)g_2(s), \]

where \( E \) denotes the expected value. If the density of \( u \) is denoted by \( g(u) \) then from the inverse Mellin transform

\[ g(u) = \frac{1}{2\pi i} \int_{L_1} g_1(s)g_2(s)u^{-s} \, ds, \]

where \( L_1 \) is a suitable contour. Now by using transformation of variables technique the density is given by

\[ g(u) = \int_0^\infty f_1(v)f_2(\frac{u}{v})v^{-1} dv. \]
But due to the uniqueness of the density of \(u\) the \(g(u)\) appearing in (20) and (21) must be one and the same. Let \(v = pt, u = p^z/m/n, f_1(t) = t^{1-nr}e^{-t}\) and \(f_2(t) = e^{-t^{n/m}}\). Then one has

\[
g_1(s) = \Gamma(1 - nr + s), R(1 - nr + s) > 0
\]

\[
g_2(s) = \frac{m}{n} \Gamma\left(\frac{ms}{n}\right), R(s) > 0
\]

and

\[
\frac{1}{2\pi i} \int_{L_1} g_1(s)g_2(s)u^{-s} ds = \frac{1}{2\pi i} \int_{L_1} \left(\frac{m}{n}\right)\Gamma\left(\frac{ms}{n}\right)\Gamma(1 - nr + s)(pz^{m/n})^{-s} ds
\]

\[
\int_0^\infty f_1(v)f_2\left(\frac{u}{v}\right)v^{-1}dv = \int_0^\infty e^{-pt}(pt)^{1-nr}e^{-zt^{n/m}}t^{-1}dt,
\]

which gives

\[
A(z; p, n, m) = p^{nr}\frac{m}{n} \frac{1}{2\pi i} \int_{L_1} \Gamma\left(\frac{ms}{n}\right)\Gamma(1 - nr + s)(pz^{m/n})^{-s} ds.
\]

Comparing (23) and (19) the result is established.

Thus in order to evaluate the basic collision probability integral one has to represent the G-function in Theorem 3.1 in computable forms. Explicit computable representations of the G-function for various parameter values are given in Haubold and Mathai (1984a). Computable representation of a general G-function is available in Mathai and Saxena (1973).

### 3.1 Non-resonant reaction rate

In this case the closed-form representation of the screened nuclear reaction rate can be evaluated by evaluating the following integral.

\[
I(z; t, a, v, n, m) = \int_0^\infty e^{-ay}y^ve^{-z(y+t)^{-n/m}}dy,
\]

for \(z > 0, t > 0, a > 0, m, n\) positive integers, see Haubold and Mathai (1986c) for details.

The integral in (24) can be evaluated by using the following lemmas, of these, Lemma 3.1a can be established by going through the same process as in the proof of Theorem 3.1, but Lemma 3.1b needs some modifications.
In this case one of the random variables will be having a non-zero density function in the interval \([0,d]\) and the density function will be zero outside this interval. Apart from this modification the derivation will remain more or less parallel. Hence we will list the lemmas and the theorem to follow without proofs.

**Lemma 3.1a.** For \(a > 0, z > 0, n, m\) positive integers
\[
N_1(z; a, v, n, m) = \int_0^\infty e^{-ay}y^v e^{-zy^{n/m}}dy
= a^{-(v+1)}(2\pi)^{(2m-n)/2}m^{1/2}n^{v/2} G_{0,m+n,0}(m^m m^n | 0, \ldots, m, 1, \ldots, n, n).
\]

**Lemma 3.1b.** For \(z > 0, d > 0, a > 0, m, n\) positive integers and denoting \(N_2(z; d, a, v, n, m)\) by \(N_2\),
\[
N_2 = \int_0^d y^v e^{-ay} e^{-zy^{n/m}}dy
= (2\pi)^{1/2} m^{1/2} n^{-1} d^{v+1} \sum_{r=0}^{\infty} \frac{(-ad)^r}{r!} G_{n,m+n,0}(\frac{z^m}{d^m m^n} | (a)),
\]
\[
(b) = (\frac{v}{n} + \frac{r+1}{n} + \frac{j-1}{m}, j = 1, \ldots, n, \frac{j-1}{m}, j = 1, \ldots, m),
\]
\[
(a) = (\frac{v}{n} + \frac{r+2}{n} + \frac{j-1}{m}, j = 1, \ldots, n).
\]

**Theorem 3.2.** For \(a > 0, z > 0, t > 0, m, n\) positive integers and \(v\) a non-negative integer
\[
\int_0^\infty y^v e^{-ay} e^{-z(y+t)^{n/m}}dy = t^{v+1} e^{a_1} \sum_{r=0}^{v} (-1)^r [N_1(z_1; a_1, v-r, n, m)
- N_2(z_1; 1, a_1, v-r, n, m)],
\]
where \(a_1 = at, z_1 = zt, (v) = v!/[r!(v-r)!], 0! = 1, N_1(.)\) and \(N_2(.)\) are given in (25) and (26) respectively.

### 3.2 Resonant reaction rates

Analytic representations of the reaction probability for the resonant reaction rates are discussed in detail in Haubold and Mathai (1986). The integral to
be evaluated has a more complicated structure compared to the integrals in Section 3.1. In this case the integral to be evaluated is \( N_3 \) where,

\[
N_3 = \int_0^\infty \frac{t^v e^{\left[-at - qt^{n/m}\right]}}{(b-t)^2 + g^2} dt,
\]

(27)

for \( a > 0, q > 0, m, n \) positive integers and \( v \) a non-negative integer. This will be evaluated with the help of Lemmas 3.1a, 3.1b, Theorems 3.1, 3.2 and the following result which will be stated as a lemma.

**Lemma 3.2a.** For \((b-t)^2 + g^2 > 0\),

\[
[(b-t)^2 + g^2]^{-1} = \int_0^\infty e^{\left[-((b-t)^2 + g^2) x\right]} dx.
\]

(28)

Now replacing the denominator on the right side of (27) by the integral in (28) and evaluating the double integral with the help of Lemmas 3.1a, 3.1b and Theorems 3.1 and 3.2 we have the following result.

**Theorem 3.3.**

\[
N_3 = \sum_{k=0}^\infty \frac{(-1)^k}{g^2(g^2)_k} \sum_{k_1=0}^{2k} \binom{2k}{k_1} (-1)^{k_1}
\]

\[
b^{2k-k_1} a^{-(v+k_1+1)} (2\pi)^{(2-n-m)/m} n^{v+k_1+1/2}
\]

\[
\Gamma(m+n,0) \left( \frac{a^m b^n}{m^n n^m} \right)^{1\over m,\ldots,\frac{m-1}{m},\ldots,\left(\frac{1+v+k_1}{n},\ldots,\frac{v+k_1}{n}\right)}
\]

where \( N_3(.) \) is defined in (27). One can look at structures which are mathematically more complicated than the one in (27). For example the denominator in (27) could be replaced by \([(b-t)^2 + g^2]^{\gamma} \) for \( \gamma \geq 1 \). Still the techniques described above will work. One can replace \( t^{-n/m} \) by \((1+t)^{-n/m}\) and one can also look at a finite range integral going from 0 to some number \( d < \infty \) instead of 0 to \( \infty \). Integrals with such modifications can be tackled by using the combined statistical and generalized special function techniques described in Sections 2 and 3 of this paper.
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