Spontaneous symmetry breaking induced by interaction in linearly coupled binary Bose–Einstein condensates

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Abstract The spontaneous symmetry breaking (SSB) induced by a specific component of a linearly coupled binary Bose–Einstein condensate was analyzed. The model is based on linearly coupled Schrödinger equations with cubic nonlinearity and double-well potential acting on only one of the atomic components. By numerical simulations, symmetric and asymmetric ground states were obtained, and an induced asymmetry in the partner field was observed. In this sense, it is adequately demonstrated that the linear coupling mixing the two-field component (Rabi coupling) promotes the (in)balance between the atomic species, as well as the appearance of the Josephson and SSB phases.

Keywords Spontaneous symmetry breaking · Bose–Einstein condensate · Nonlinear Schrödinger equation · Double-well potential

1 Introduction

Bose–Einstein condensates (BECs) [1–3] offer the opportunity to observe macroscopic quantum effects, providing a way to study various types of phenomena such as bright [4–9] and dark [10–12] solitons, vortices [13,14], Anderson localization of matter waves [15,16], breathers [17,18], self-trapped states supported by beyond-mean-field interactions (“quantum droplets”) [19–22], etc. In this sense, models with binary bosonic mixtures have been extensively studied and show a wide application potential. For example, in Ref. [23] by using numerical Quantum Monte Carlo simulations was demonstrated that a Bose mixture of trapped dipolar atoms of identical masses and dipole moments presents demixing for low finite temperatures. Recently, Anderson localization induced by a linear interaction between the atomic components of binary BECs (Rabi coupling) was reported in Ref. [24], even with one of the components not being directly influenced by the disordered potential.

Another important phenomenon observed in binary bosonic mixtures is the symmetry breaking. Indeed, the stability [25] and dynamic tunneling properties of binary BECs trapped by double-well potential were investigated in Ref. [26]. Specifically, in [27], the authors showed the possibility of describing the imbalance between atomic populations through effective equations, analogous to the dynamic equations of the coupled pendulum. Spontaneous symmetry breaking (SSB) was also studied in a two-component model in double-well nonlinear (pseudo)potentials [28]. In this case, cubic nonlinear Schrödinger equations with nonlinear coupling were considered, so that the nonlinear modulation function acts in the system as a double-well potential, producing asymmetric profiles under certain conditions. Moreover, SSB was found in some
other models such as Bose–Fermi mixtures trapped by double-well potential [29], nonlinear Schrödinger (NLS) equations with a linear double-well [30] and parity-time-symmetric potential [31], two-component linearly coupled system with the intrinsic cubic nonlinearity and the harmonic-oscillator confining potential (valid for BECs and optical systems) [32], linearly coupled Korteweg-de Vries systems [33], etc. Furthermore, SSB was also investigated in binary BECs confined by a triple-well potential [34].

The description of ultra-cold interacting bosonic gases based on mean-field approximations is dictated by a well-known time-dependent three-dimensional Gross–Pitaevskii equations (GPEs) [35,36]. Therefore, for BECs strongly trapped in the transverse direction, it is possible to apply a dimensional reduction method capable of producing an effective equation that describes the dynamics of the system through a quasi-one-dimensional equation for the longitudinal component of the wave function, while keeping the transversal one “frozen” [37–41]. In particular, Salasnich et al. [37] presented a time-dependent nonpolynomial Schrödinger equation that accurately described an anisotropic BEC confined strongly in the transverse direction by a harmonic trap (cigar shaped), obtained through a variational approach. This technique has been widely used for several other models [42–49].

The goal of the present work is to carry out a systematic study about the symmetry breaking induced by a specific atomic component of a linearly coupled binary BEC. To this end, differently from the previous studies, we will analyze the behavior of a specific atomic component of a linearly coupled system with the intrinsic cubic nonlinearity and the harmonic-oscillator confining potential (valid for BECs and optical systems) [32], linearly coupled Korteweg-de Vries systems [33], etc. Furthermore, SSB was also investigated in binary BECs confined by a triple-well potential [34].

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2 Theoretical model

We consider a diluted BEC of two components, close to absolute zero temperature, whose atomic species interact linearly. This type of interaction is obtained when the interspecies interaction of the binary BEC with Rabi-coupling is turned off, remaining only the linear component. Here, we are considering that with the auxiliary laser field, only one of the atomic species (component 1) is in “direct contact” with a double-well type potential.

In systems where the three-dimensional external potential is strongly confining in the transverse direction (y, z), it is convenient to use a dimensional reduction procedure, making it possible to use one-dimensional dynamic equations to describe the behavior of the binary BEC [50,51]. Therefore, the effective coupled equations that precisely describes this model can be written as

\[ i \frac{\partial f_1}{\partial t} = -\frac{1}{2} \frac{\partial^2 f_1}{\partial x^2} + V_{DW}(x) f_1 + \alpha |f_1|^2 f_1 + \kappa f_2, \]

(1a)

\[ i \frac{\partial f_2}{\partial t} = -\frac{1}{2} \frac{\partial^2 f_2}{\partial x^2} + \gamma |f_2|^2 f_2 + \kappa f_1, \]

(1b)

where \( x \) is the spatial coordinate, \( i = \sqrt{-1} \) is the unit imaginary number, \( f_j(x, t) \), \( (j = 1,2) \) is the macroscopic wave function of the binary BEC, \( \alpha \) and \( \gamma \) are the strengths of intraspecies interactions relative to atomic species 1 and 2, respectively, \( \kappa \) is the linear coupling constant that effectively mediates the interaction between the two components of bosonic gas, and \( V_{DW} \) is the external confinement (see Eq. (3)). The system is subject to joint normalization condition

\[ \int_{-\infty}^{+\infty} (|f_1|^2 + |f_2|^2) \, dx = N_1 + N_2, \]

(2)

where \( N_j (j = 1,2) \) is the individual norm of each atomic component. In the results below the fields \( f_1 \) and \( f_2 \) are initially normalized to unity. In the numerical simulations presented below, the values of \( N_{1,2} \) change according to the respective ground states \( f_1 \) and \( f_2 \), showing values between 0 and 2. Therefore, due to the conservation of the norm, the joint normalization results in \( N_1 + N_2 = 2 \).

In Ref. [32], the dynamics of linearly coupled two-component systems with intrinsic cubic nonlinearity and the confinement potential of the harmonic oscillator (HO) acting on only one of the components was studied. It was shown through approximate analytics solutions of ground state, first exited state (dipole mode) and numerical solutions that this asymmetric model (half-trapped system) has SSB, presenting imbalance between the norms of the components. Differently from Ref. [32], in the present study we consider
that component 1 is subject to the action of a double-well (DW) potential \( V_{DW}(x) \), given by

\[
V_{DW} = -V_0 \left( \frac{1}{\cosh(x-x_0)^2} + \frac{1}{\cosh(x+x_0)^2} \right),
\]

obtained from the combination of two Pöschl–Teller (PT) potentials centered on the positions \( \pm x_0 \), and separated by a barrier of height \( V_0 \). The single PT potential have been studied in systems described by the Schrödinger equation and their eigenvalues and eigenfunctions are known \[52,53\]. Effects such as the absence of reflection of the Gaussian wave packet scattered by a PT well \[54,55\], second and third harmonic generation \[56\] and intersubband absorption in optical systems \[57,58\] were reported too.

In Refs. \[27,59\], the DW potential (3) was used to study the atomic Josephson effect in binary BECs with nonlinear interaction. The system was modeled by two coupled GPE, confined by a double-well potential along the axial direction and a strong harmonic confinement in the transverse direction, where effective equations describe the dynamic behavior of atomic components. Also, in Ref. \[60\] was reported the existence of SSB in a single BEC component trapped by a double-PT potential (similar to the one we are using here, i.e., Eq. (3)). In this case, the dimensional reduction process \[37\] was employed to produce a nonpolynomial Schrödinger equation (1D) capable of accurately describing the longitudinal dynamics of the three-dimensional model, where symmetrical/asymmetrical and collapsed ground states were found. Numerical results are presented in the next section.

### 3 Numerical results

In this section, we present the numerical results obtained from the integration of Eqs. (1a) and (1b), under the joint normalization condition (2). With purpose of producing stationary states, whose configuration has the lowest energy (ground states), we use an imaginary-time simulations with split-step method based on the Runge–Kutta algorithm. To check the dynamic stability of the solutions, we use the real-time propagation method (similar algorithm) with the spatial and temporal integration steps being 0.04 and 0.0001, respectively.

The ground state presented below is obtained from the slightly asymmetrical initial condition

\[
f(x, 0) = \delta_- \exp \left[ \frac{(x-x_0)^2}{2} \right] + \delta_+ \exp \left[ \frac{(x+x_0)^2}{2} \right],
\]

with \( \delta_+ \) and \( \delta_- \) being the parameters that introduce initial asymmetry into the system. Here, we will define two initial asymmetry conditions: \( \Delta_L \equiv \delta_+ = 1.01 \) and \( \delta_- = 1.00 \) or \( \Delta_R \equiv \delta_+ = 1.00 \) and \( \delta_- = 1.01 \).

The ground states obtained from (4), by considering the DW potential (3) and on the joint normalization (2), are of two types: the symmetric (Josephson phase) and those with broken symmetry (SSB phase). Symmetric profiles have equally distributed energy density between the wells, differently from the asymmetric ones.

We started our analysis by studying the influence of the coupling parameter on localization. In Fig. 1a–c we present symmetric ground state profiles for the case of self-repulsive binary BEC. Under these conditions, i.e., \( \alpha = \gamma > 0 \), the field \( f_2 \) is delocalized in the absence of interaction between the components \( (\kappa = 0) \). On the other hand, for a weak coupling (see Fig. 1a) we verify that there is a localization of component 2, indicating that field \( f_1 \) acts on field \( f_2 \) simulating an external trap. It is important to note that the shapes of the densities are quite different, with \( |f_1|^2 \) having a depth much greater than the density \( |f_2|^2 \). This behavior disappears when increasing the coupling intensity, as can be seen in Fig. 1(b,c). Specifically, in Fig. 1c, with strong coupling \( \kappa = 15 \), the fields have practically the same configuration. Similar results are found in the weak self-attractive regime. In this case the self-attractive interaction (\( \alpha, \gamma < 0 \)) produces densities with peaks slightly higher than the repulsive cases, as observed by Fig. 1d–f.

In general, we observe that a systematic increase in coupling strength produces a unification of densities, e.g., causing the number of condensed atoms to behave practically equal with each other, as well as the shape of fields \( f_1 \) and \( f_2 \). In order to study in detail this behavior, we analyzed the influence of parameters \( \alpha, \gamma, \kappa \), and \( V_0 \) on the value of the norms of components 1 and 2. We observe that the interspecies interaction, both attractive and repulsive, do not significantly change the norm of the components, as can be seen in Fig. 2a. On the other hand, the interspecies interaction parameter \( \kappa \) has great importance in the individual norm values. Fig. 2b shows that the binary BEC is sensitive to the increment of the linear coupling strength.
Fig. 1 Density profiles $|f_1(x)|^2$ and $|f_2(x)|^2$ versus $x$ for a binary BEC subject to the normalization condition (2). The values of the nonlinearity parameters used here are $a-e$ $\alpha = \gamma = 1$ and $d-f$ $\alpha = \gamma = -0.1$. The other parameters are $V_0 = 1$, $x_0 = 2$ and $\Delta_L$.

Fig. 2 Comparison between the value of the norms $N_1$ and $N_2$, obtained via profiles $f_1(x)$ and $f_2(x)$, versus the parameters (a) $\alpha = \gamma$, (b) $\kappa$ and (c) $V_0$. The parameters used here are (a) $\kappa = 4$ and $V_0 = 1$; (b) $\alpha = \gamma = -0.1$ and $V_0 = 1$; (c) the same as in (b) but now considering $\kappa = 4$. Here, we use $\Delta_L$.

For example, with $\kappa = 0.2$, the difference between the individual norms $N_1 - N_2 = 1.68$, while for $\kappa = 15$, the difference reduces to $N_1 - N_2 = 0.04$. The difference between the individual norms presents a nonlinear pattern, which slowly goes to zero with increasing of $\kappa$. From a phenomenological point of view, this behavior makes the condensed cloud composed of component 2 equal, both in shape and in the number of condensed atoms, to component 1, in which the latter is under influence of the DW potential (see Fig. 1c, f). The height of the barrier also modifies the relationship between individual norms. However, unlike the previous case, the increment of $V_0$ promotes the linear increment of the difference $N_1 - N_2$, as shown in Fig. 2c. All results presented above were obtained by using the initial asymmetry condition $\Delta_L$. It is worth noting that these same results are obtained when we choose the initial asymmetry condition $\Delta_R$, due to the symmetrical configuration of the ground states of strongly coupled and weakly self-attracting BECs.

The results presented above were obtained in the repulsive or weak self-attractive regime. In these configurations, the profiles have a symmetrical shape $|f_{1,2}(x)|^2 = |f_{1,2}(-x)|^2$. However, for certain parameter values (detailed below) the symmetry of the profiles is broken (SSB phase) and they start to present a particle density displaced from the center of the trap. A typical example of the appearance of asymmetric profiles is shown in Fig. 3. In the strong coupling regime (Fig. 3a), for example $\kappa = 10$, the densities $|f_1|^2$ and $|f_2|^2$ have a symmetrical shape. However, by decreasing the coupling strength, the ground states of both fields start to present broken symmetry, as can be seen in Fig. 3b, c. In these cases, we observe that the left hump is larger than the right hump for both atomic components. It is noteworthy that the difference between the amplitude of the humps increases by decreasing the coupling intensity, resulting in the limit case for which the profiles have only one hump located on the left with maximum amplitude in both components. In addition, when the system presents these asymmetric states, the initial asymmetry dictates which side of the trap (left or right) will be more populated. For example, in figure Fig. 3b, c we use $\Delta_L$, which resulted in a par-
Fig. 3 Density profiles $|f_1(x)|^2$ and $|f_2(x)|^2$ versus $x$ for an attractive binary BEC subject to the normalization condition (2). The other parameter values used here were $\alpha = \gamma = -0.3$, $V_0 = 1$, $x_0 = 2$ and $\Delta_L$.

tice density with a larger left hump (region $x < 0$). In this same configuration, but choosing $\Delta_R$, the results are similar, except for the particle density that presents a higher peak in the region $x > 0$.

To study the relationship between symmetry and the degree of asymmetry of the ground states of each atomic component with the parameters $\alpha$, $\gamma$, $\kappa$ and $V_0$, we use the asymmetry rate, defined by [28, 61, 62]

$$\Theta_{1,2} = \frac{\int_0^{+\infty} |f_1|^2 dx - \int_0^{-\infty} |f_1|^2 dx}{\int_{-\infty}^{+\infty} |f_1|^2 dx}. \tag{5}$$

Note that $\Theta = 0$ indicates a symmetry (both traps equally populated with the same atomic species), while $\Theta \neq 0$ is related to the asymmetry of the ground states. Figure 4 shows some findings regarding the asymmetry rate $\Theta$ of ground states of binary BECs with a single component ($f_1$) confined by the double PT potential (3). In Fig. 4a we observe a bifurcation behavior, i.e., the profiles obtained by setting $\kappa = 0.2$ (weak coupling) present asymmetrical shapes for $|f_1|^2$ and $|f_2|^2$ in a strongly self-attractive regime, while in a weak self-attractive or self-repulsion regime the ground states present symmetrical shapes for both initial asymmetry conditions previously used. In this sense, the bifurcation point is approximately $\alpha = \gamma = -0.12$. Similar results are found when we consider the self-interaction parameter $\gamma = 0$ (see Fig. 4b).

The self-interaction of the second atomic component ($f_2$) does not promote symmetry breaking in weak coupling regime when setting $\alpha = 0$. For example, choosing $\kappa = 0.2$, we get the constant function $\Theta_{1,2}(\gamma) = 0$ for $\Delta_L$ or $\Delta_R$, indicating that only symmetric profiles are found in these configurations. In contrast, very different effects occur in the strong coupling regime. For example, in Fig. 4c we present the asymmetry rate $\Theta_{1,2}(\gamma)$, setting $\alpha = 0$ and $\kappa = 10$, were we observed that the asymmetry rate obtained with the initial condition $\Delta_L$, $\Theta_{1,2}^{\Delta L}(\gamma)$, has a positive value, while $\Theta_{1,2}^{\Delta R}(\gamma)$ has a negative value, for $\gamma < -1.37$. This indicates that initial profiles (4) with particle density presenting a small asymmetry by the conditions $\Delta_L$ or $\Delta_R$, produce ground states with a larger right (left) peak, respectively, inverting the initial asymmetry condition (asymmetry inversion). Exactly for $\gamma = -1.37$ the ground states of both components are symmetrical. By increasing $\gamma$, $\Theta(\gamma)$ continues to change, inverting again the signal and hence the “direction” of asymmetry. Finally, at $\gamma \approx -0.89$, the asymmetry rates $\Theta_{1,2}^{\Delta L, \Delta R}(\gamma > -0.89) \to 0$ abruptly, obtaining from that point only symmetric ground states $|f_1(x)|^2$ and $|f_2(x)|^2$. To conclude the bifurcation analysis, we

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Fig. 4 Asymmetry rate of ground states for binary BECs vs $\alpha$, $\gamma$, $\kappa$. The profiles $|f_1(x)|^2$ and $|f_2(x)|^2$ were obtained numerically by Eqs. (1a)-(1b), and plotted by considering different initial asymmetry conditions. The solid (gray) curve and the circles (red) represent the results provided by $|f_1(x)|^2$ and $|f_2(x)|^2$, respectively, with initial asymmetry $\Delta_L$. The dot-dashed (blue) and the squares (yellow) represent, respectively, $|f_1(x)|^2$ and $|f_2(x)|^2$ with initial asymmetry $\Delta_R$. The other parameters are $V_0 = 1$ and $x_0 = 2$.
present in Fig. 4d the results of $\Theta^{\Delta_L, \Delta_R}$ versus $\kappa$. We observe that, by setting $\alpha = \gamma = -0.2$, the ground states obtained in a weak coupling regime are asymmetric, while those from strong coupling are symmetric. Therefore, the $\Theta^{\Delta_L, \Delta_R}(\kappa)$ bifurcation takes place in $\kappa = 0.4$.

In the previous analyses considering the self-attractive regime, we observed that binary BECs with $\alpha = \gamma = -0.3$ present symmetrical profiles for $\kappa > 2.8$. For the same setting, but with $\kappa \leq 2.8$, the ground states change abruptly, starting to present an asymmetric profile. This effect indicates that the coupling has relevance in the phase presented by the ground states. Therefore, we carried out a systematic study to analyze the regions of existence of symmetric and asymmetric profiles by changing the values of the parameters $\alpha, \gamma, V_0$ and $\kappa$. The existence diagram of the SSB and Josephson phases, with respect to the values of the intra- and inter-species interaction parameters $(\alpha, \gamma)$, shows that increasing the coupling strength will generate symmetrical solutions. However, for $\alpha = \gamma < -0.3$ only asymmetric ground states are found, as can be seen in Fig. 5a. A similar behavior is observed when fixing the auto-interaction parameter of component 2 ($\gamma = 0$) and varying $\alpha$ and $\kappa$ (see Fig. 5b). In this configuration, the strong coupling region in association with a weak self-interaction favors the appearance of Josephson phase.

In Fig. 5c we verify the influence of the interspecies interaction of component 2 in the absence of self-interaction in component 1 (i.e., $\alpha = 0$), on the shape of the ground states. In general, symmetric profiles are found in strongly self-attracting systems. Contrary to the previous examples, in this case the weak coupling regime favors the appearance of symmetric profiles. For example, by choosing $\kappa < 0.69$ only symmetrical ground states are found. We observed that the coupling increment generates the SSB phase; however there is a narrow band of symmetric states between the shaded areas in Fig. 5c, which splits the SSB area into two regions. We found that this band of symmetric ground states can be approximated by the values of $\gamma = [1.18(\exp(-0.71/\kappa) - 2.09) \pm 0.02]$. Below the Josephson narrow area we get again a SSB region (orange filled area SSB(i)). Unlike the previous results, this region presents an inversion of symmetry. Therefore, this panel is an extension of Fig. 4c, where we can see not only a single point, but an inversion zone. Finally, we analyze the behavior of the ground states in relation to the coupling parameter and height of the DW barrier potential. Figure 5d shows that increasing the height of the barrier leads to symmetry breaking.

To analyze the three types of ground state behavior for $\gamma < 0$ (fixed $\alpha$) presented earlier, we display the densities $|f_1|^2$ and $|f_2|^2$ for three different values of $\gamma$, covering all the phase types. Figure 6a, b shows the behavior of the ground states with $\gamma = -1.60$ ($\alpha = 0$) obtained with $\Delta_L$ and $\Delta_R$ maintaining the same configuration. Note that in this regime, the initial asymmetry $\Delta_L$ produces asymmetric profiles with the particle density shifted to the left, while the asymmetry $\Delta_R$ produces densities shifted to the right. In Fig. 6c, d, whose profiles were obtained with $\gamma = -1.65$, exactly localized in the narrow band of Josephson phase (see Fig. refF3c), the ground states are independent of the initial asymmetry. On the other hand, contrary to Fig. 6(a,b), choosing $\gamma = -1.70$ (SSB(i) area in Fig. 5c), the direction of density displacements are now reversed to their initial asymmetry conditions.

![Figure 5](image-url)  
Fig. 5 Phase diagrams of the attractive BEC in the symmetric double-well potential $V(x)$ with two minima centered at $x_0 = 2$. The Josephson phase (J) is represented by the empty area while the spontaneous symmetry breaking (SSB) is represented by the filled area (blue and orange SSB(i)). In particular, the SSB(i) area indicates the zone where the asymmetry is inverted ($\Theta^{\Delta_L, \Delta_R}(\gamma) > 0$ and $\Theta^{\Delta_L, \Delta_R}(\gamma) < 0$). The points K–R will be used in the analyzes below (Figs. 7 and 8)
We also studied the stability of the ground states of the configurations presented above. For this, we perform the evolution of the ground state profiles randomly perturbed in a maximum of ±5% in their amplitudes, given by:

$$f_{1,2}(x, t = 0) = [1 + \sigma_{1,2}(x)]f_{1,2}(x), \quad (6)$$

where \(f_{1,2}\) are the ground states obtained numerically via imaginary-time propagation and \(\sigma_{1,2}\) is the perturbation function with random values obtained by the RAND function of the gnu-OCTAVE program. It is worth noting that the perturbation \(\sigma_{1,2}(x)\) is a “white noise” presenting \(\langle \sigma_{1,2}(x) \rangle = 0\).

In Fig. 7a, b we analyze the longtime evolution of the symmetric ground states \(|f_1|^2\) and \(|f_2|^2\) (point L in Fig. 5a). These profiles were obtained in a regime of weak self-attractive interaction with \(\alpha = \gamma = -0.2\) and strong coupling \(\kappa = 3\), and perturbed randomly via (6). Both components showed stability throughout the evolution, i.e., the initial profiles were maintained with no significant change in the values of their amplitudes or widths. These assertions can be better observed in Fig. 7c–d, where we found that the initial \(|f_1,2(x, t = 0)|^2\) and output profiles \(|f_1,2(x, t = 5000)|^2\) present very similar configurations.

To complete the analysis, we studied the evolution of asymmetric profiles. Figure 7e–h shows the stable evolution of asymmetric profiles (point K in Fig. 5a). Here, we choose a weak coupling regime with \(\kappa = 0.2\). Note that even in the presence of perturbation the system is not able to produce relevant changes in the initial configuration. Furthermore, even though the asymmetric (point K in Fig. 5a) and symmetric (point L in Fig. 5a) form of \(f_2\) was induced by \(f_1\), we see that the configuration is stable under small perturbations.

The phase diagram of \(\gamma (\alpha = 0) \) vs \(\kappa\) showed that the difference between the self-interactions of components 1 and 2 drastically interfere in the model configuration (see Fig. 5c). By choosing \(\alpha \neq \gamma\), not only the phase of the system can change, but other phenomena such as asymmetry inversion also occurs. As example, we investigated six distinct points of these diagram, observing the evolution of the \(f_2\) for all the situations presented above (symmetry, asymmetry and inversion of asymmetry). We emphasize that component 1 presents similar results when compared to those of component 2 (not shown here).

Figure 8a shows the stable evolution of the asymmetric ground state \(|f_2|^2\), obtained by setting \(\kappa = 1\), \(\gamma = -1.82\) and \(\Delta_{L}\. This profile corresponds to point M on the diagram of Fig. 5c, which presents the stability of the asymmetric profile just before the narrow band of asymmetry. Differently from the previous case, the symmetric profile obtained with \(\kappa = 1, \gamma = -1.87\) and \(\Delta_{L}\) presents an unstable evolution, just over a narrow band of symmetry represented by point N in Fig. 5c. In the first instants of propagation, the symmetrical profile changes abruptly, becoming an asymmetrical profile with a larger right peak compared to the left one. The asymmetric peak continues to oscillate, but maintaining this asymmetrical configuration throughout evolution. These results are displayed in Fig. 8b and c. By decreasing the value of the self-interaction parameter of component 2, we produce ground states
Fig. 7 Evolution of stable symmetrical and asymmetrical profiles (Points L and K of Fig. 5a). In a–b the initial profiles $|f_1|^2$ and $|f_2|^2$ were numerically obtained by considering the parameters $\alpha = \gamma = -0.2$, $\kappa = 3$, $V_0 = 1$, $\Delta_L$ and perturbed according to (6). Panels e and d show the initial perturbed ground states $|\bar{f}_1, 2(x, t = 0)|^2$ and the output profiles $|f_1, 2(x, t = 5000)|^2$, relative to panels (a, b), presented in solid lines (orange) and dot-dashed lines (black), respectively. In panels e–h the same as (a–d), but now by considering $\kappa = 0.2$. Here, we use the initial asymmetry condition $\Delta_L$.

$|f_2|^2$ with inverted asymmetry. As an example, the evolution of this profile, obtained with $\kappa = 1$, $\gamma = -1.92$, is shown in Fig. 8c. Therefore, we observe the stability of the profile with inverted asymmetry, just below the narrow range of symmetry (point O). Figure 8d shows the evolution of the asymmetry rate $\Theta_2$ for configurations (a–c). Collaborating with the previous analysis, we see that the asymmetry profiles have an approximately constant asymmetry rate, even with the initial perturbation. However, the unstable profile (Fig. 8b) shows non-periodic oscillations.

Finally, we repeat the stability analysis for other coupling intensities ($\kappa$). As an example, we present the evolution of three other points on the diagram (points P, Q and R), for a strong coupling $\kappa = 10$. The results are very similar to those presented above. Figure 8f–i shows that the asymmetric states (points P and R) are stable, while the symmetric states obtained over the narrow range of symmetry (point Q in Fig. 5c) are unstable. Therefore, we can conclude by the real-time propagation results that only unstable profiles are those found on the narrow symmetrical band.

4 Conclusion

In conclusion, by considering a binary Bose–Einstein condensate with linear interaction between species, we investigated the existence of induced symmetry breaking. In this model, only one atomic component is under the influence of a double well potential, which for certain parameters induces the appearance of asymmetric ground states both in the actuated field ($f_1$) and in the partner field ($f_2$). Thus, we observe that component 1 acts as a confining potential on component 2, simulating an asymmetric double-well potential for certain parameter values. These added effects promote symmetry breaking (SSB phase) in both atomic components, where the resulting densities have center of mass displaced in relation to the center of the trap. Symmetrical profiles (Josephson phase) were also found. They are very sensitive to variation in intraspecies interaction, as well as the intensity of the double-well potential when individual norms are considered. As observed here, the coupling strength drastically influences the phase of the ground state profiles obtained in the self-attractive regime ($\alpha, \gamma < 0$). Diagrams were used to present the region of existence of both phases for different parameters, where the difference between the self-interaction
Fig. 8 Evolution of stable and unstable profiles. The ground states $|f_2|^2$ were numerically obtained by considering the parameter values $\kappa = 1$, $\alpha = 0$, $\Delta_L$, and the attractive nonlinearities $a \gamma = -1.82$ (point M in Fig. 5c), $b \gamma = -1.87$ (point N in Fig. 5c) and $c \gamma = -1.92$ (point O in Fig. 5c). In (d), we display the asymmetry rate $(\Theta_2)$ vs $t$ for each of the situations. The dotted (black), solid (red) and dot-dashed lines, refer to panels (a), (b), and (c), respectively. Panel (e) shows the initial profile $|f_2(x, t = 0)|^2$ and output profile $|f_2(x, t = 5000)|^2$ of the unstable configuration (b), in solid line (orange) and dot-dash line (black), respectively. In panels (f–h) is displayed the same as in (a–c) but now for the points P–R of Fig. 5c, by considering $\kappa = 10$; $\gamma = -1.33, -1.38$ and $-1.43$, respectively. Panel (i) displays the evolution of asymmetry rate $(\Theta_2)$ for the results of panels (f) in the dotted (black) line, (g) in solid (red) line, and (h) dot-dashed (blue) line. Panel (j) shows the initial and output profiles of unstable configuration (g).

parameters of components 1 and 2 promotes, for certain parameters, an asymmetry inversion zone. The evolution analysis showed that only the profiles obtained under the narrow Josephson range, between the extensive SSB bands are unstable. Therefore, the asymmetric profiles $f_2$, induced by $f_1$ are stable. This work intends to advance the study of binary BECs, but also sheds light on the investigation of induction effects in partner fields for other types of coupled systems.

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Data availability All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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