Little cubes and long knots

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Abstract

This paper gives a partial description of the homotopy type of $\mathcal{K}$, the space of long knots in $\mathbb{R}^3$. The primary result is the construction of a homotopy equivalence $\mathcal{K} \simeq C_2(\mathcal{P} \sqcup \{*\})$ where $C_2(\mathcal{P} \sqcup \{*\})$ is the free little 2-cubes object on the pointed space $\mathcal{P} \sqcup \{*\}$, where $\mathcal{P} \subset \mathcal{K}$ is the subspace of prime knots, and $*$ is a disjoint base-point. In proving the freeness result, a close correspondence is discovered between the Jaco-Shalen-Johannson decomposition of knot complements and the little cubes action on $\mathcal{K}$. Beyond studying long knots in $\mathbb{R}^3$ we show that for any compact manifold $M$ the space of embeddings of $\mathbb{R}^n \times M$ in $\mathbb{R}^n \times M$ with support in $I^n \times M$ admits an action of the operad of little $(n+1)$-cubes. If $M = D^k$ this embedding space is the space of framed long $n$-knots in $\mathbb{R}^{n+k}$, and the action of the little cubes operad is an enrichment of the monoid structure given by the connected-sum operation.

Key words: spaces of knots; little cubes; operad; embedding; diffeomorphism

1 Introduction

A theorem of Morlet’s (38) states that the topological group $\text{Diff}(D^n)$ of boundary-fixing, smooth diffeomorphisms of the unit $n$-dimensional closed disc is homotopy equivalent to the $(n+1)$-fold loop space $\Omega^{n+1}(PL_n/O_n)$. Morlet’s method did not involve the techniques invented by Boardman, Vogt and May (2; 35) for recognizing iterated loop spaces, little cubes actions. This paper begins by defining little cubes operad actions on spaces of diffeomorphisms and embeddings, thus making the loop space structure explicit. In Theorem 5 it’s proved the embedding space

$$\text{EC}(k, M) = \{f \in \text{Emb}(\mathbb{R}^k \times M, \mathbb{R}^k \times M), \text{supp}(f) \subset I^k \times M\}$$

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admits an action of the operad of little \((k+1)\)-cubes. Here the support of \(f\),
\[ \text{supp}(f) = \{ x \in \mathbb{R}^k \times M : f(x) \neq x \} \text{ and } I = [-1, 1]. \]

The case \(k = 1\) and \(M = D^2\) is of primary interest in this paper as \(EC(1, D^2)\)
is the space of framed long knots in \(\mathbb{R}^3\). In section 3 the structure of \(EC(1, D^2)\)
as a little 2-cubes object is determined. It is shown in Proposition 9 that the
little 2-cubes action on \(EC(1, D^2)\) restricts to a subspace \(\hat{K}\) which is homotopy
equivalent to \(K\), the space of long knots in \(\mathbb{R}^3\). Moreover it is shown that as
little 2-cubes objects, \(EC(1, D^2) \simeq \hat{K} \times \mathbb{Z}\). In Theorem 11 it is shown that \(\hat{K}\) is a
free little 2-cubes object on the subspace of prime long knots \(\hat{K} \simeq C_2(\mathcal{P} \sqcup \{\ast\})\).

Theorems 11 and 5 are the main theorems of this paper.

The homotopy-theoretic content of Theorem 11 is that \(K \simeq C_2(\mathcal{P} \sqcup \{\ast\}) \simeq \\
\sqcup_{n=0}^{\infty}(C_2(n) \times \mathcal{P}^n)/S_n\) where \(C_2(n)\) the space of \(n\) little 2-cubes. \(C_2(n)\) as an
\(S_n\)-space has the same homotopy type as the configuration space of \(n\) labeled
points in the plane \(C_n(\mathbb{R}^2)\). \(\mathcal{P} \subset K\) is the space of prime long knots, thus it
is the union of all the components of \(K\) which consist of prime knots. \(S_n\) is
the symmetric group on \(n\) elements, acting diagonally on the product. One
interpretation of Theorem 11 is that it refines Schubert’s Theorem (44) which
states that \(\pi_0K\) is a free commutative monoid with respect to the connected-
sum operation \(\pi_0K \simeq \bigoplus_{n=0}^{\infty} \mathbb{N}\). The refinement is a space-level theorem about
\(K\) where the cubes action on \(K\) replaces the connected-sum operation on \(\pi_0K\).
The novelty of this interpretation is that the connected-sum is not a unique
decomposition in \(K\), as it is parametrized by a configuration space. Perhaps
the most interesting aspect of Theorem 11 is that it states that the homotopy
type of \(K\) is a functor in the homotopy type of the space of prime long knots
\(\mathcal{P}\). In Section 4 we mention how the results in this paper combine with res ults
of Hatcher (22) and other results of the author’s (8) to determine the full
homotopy-type of \(K\).

There are elementary consequences of the little cubes actions defined in Section 2
that are of interest. In Corollary 6 we mention how the cubes action on
\(EC(n, \{\ast\})\) endows \(\text{Diff}(D^n) \simeq EC(n, \{\ast\})\) with the structure of an \((n+1)-\)
fold loop space. This corollary is part of Morlet’s ‘Comparison’ Theorem (38). To my knowledge, it is the first explicit demonstration of the \((n + 1)\)-cubes acting on groups homotopy equivalent to \(\text{Diff}(D^n)\). In Corollary 7 the loop space recognition theorem together with the cubes action on \(\text{EC}(k, D^m)\) and some elementary differential topology tell us that \(\text{EC}(k, D^m)\) is a \((k + 1)\)-fold loop space provided \(m > 2\). This last result, to the best of my knowledge, is new. Since these results appeared, Dev Sinha (46) has constructed an action of the operad of 2-cubes on the homotopy fiber of the map \(\text{Emb}(\mathbb{R}, \mathbb{R}^n) \to \text{Imm}(\mathbb{R}, \mathbb{R}^n)\) for \(n \geq 4\). Sinha’s result has recently been extended by Paolo Salvatore (42), to construct actions of the operad of 2-cubes on both the full embedding space \(\text{Emb}(\mathbb{R}, \mathbb{R}^n)\) and the ‘framed’ long knot space \(\text{EC}(1, D^{n-1})\) for \(n \geq 4\), thus allowing for a comparison with the cubes actions constructed in this paper. Both the methods of Salvatore and Sinha use the Goodwillie Calculus of Embeddings (14; 45; 43; 52; 5) together with the techniques of McClure and Smith (37).

The existence of cubes actions on the space of long knots in \(\mathbb{R}^3\) was conjectured by Turchin (50), who discovered a bracket on the \(E^2\)-page of the Vassiliev spectral sequence for the homology of \(\mathcal{K}\) (51). Given the existence of a little 2-cubes action on \(\text{EC}(1, D^k)\) one might expect a co-bracket in the Chern-Simons approach to the de Rham theory of spaces of knots (4; 32; 31; 11) but at present only a co-multiplication is known (12). This paper could also be viewed as an extension of the work of Gramain (16) who discovered subgroups of the fundamental group of certain components of \(\mathcal{K}\) which are isomorphic to pure braid groups.

2 Actions of operads of little cubes on embedding spaces

In this section we define actions of operads of little cubes on various embedding spaces. An invention of Peter May’s, operads are designed to parametrize the multiplicity of ways in which objects can be ‘multiplied’. In the case of iterated loop spaces, the relevant operad is the operad of little \(n\)-cubes, essentially defined by Boardman and Vogt (2) as ‘categories of operators in standard form,’ and later recast into the language of operads by May (35).

**Definition 1** *The space of long knots in \(\mathbb{R}^n\) is defined to be \(\text{Emb}(\mathbb{R}, \mathbb{R}^n) = \{f : \mathbb{R} \to \mathbb{R}^n : \text{where } f \text{ is a } C^\infty\text{-smooth embedding and } f(t) = (t, 0, 0, \cdots, 0) \text{ for } |t| > 1\}\). We give \(\text{Emb}(\mathbb{R}, \mathbb{R}^n)\) the weak \(C^\infty\) function space topology (see Hirsch (23) §2.1). \(\text{Emb}(\mathbb{R}, \mathbb{R}^n)\) is considered a pointed space with base-point given by \(\mathcal{I} : \mathbb{R} \to \mathbb{R}^n\) where \(\mathcal{I}(t) = (t, 0, 0, \cdots, 0)\). Any knot isotopic to \(\mathcal{I}\) is called an unknot. We reserve the notation \(\mathcal{K}\) for the space of long knots in \(\mathbb{R}^3\), ie: \(\mathcal{K} = \text{Emb}(\mathbb{R}, \mathbb{R}^3)\).*
The connected-sum operation $\#$ gives a homotopy-associative pairing

$$\#: \text{Emb}(\mathbb{R}, \mathbb{R}^n) \times \text{Emb}(\mathbb{R}, \mathbb{R}^n) \to \text{Emb}(\mathbb{R}, \mathbb{R}^n)$$

As shown in Schubert’s work (44), this pairing turns $\pi_0\mathcal{K}$ (the path-components of $\mathcal{K}$) into a free commutative monoid with a countable number of generators (corresponding to the isotopy classes of prime long knots). Schubert’s argument that $\pi_0\mathcal{K}$ is commutative comes from the idea of ‘pulling one knot through another,’ illustrated in Figure 2.

Figure 2 suggests the existence of a map $\iota : S^1 \times \mathcal{K} \to \mathcal{K}$ such that $\iota(1, f, g) = f\#g$ and $\iota(-1, f, g) = g\#f$. Such a map would exist if the connected sum operation on $\mathcal{K}$ was induced by a 2-cubes action. Turchin’s conjecture states that such a 2-cubes action exists.

When first constructing the little 2-cubes action on the space of long knots, it was observed that it is necessary to ‘fatten’ the space $\mathcal{K}$ into a homotopy equivalent space $\hat{\mathcal{K}}$ where the little cubes act. The problem with directly defining a little cubes action on $\mathcal{K}$ is that little cubes actions are very rigid. Certain diagrams must commute (35; 34). A homotopy commutative diagram is not enough in the sense that one can not in general promote such diagrams to a genuine cubes action. All known candidates for little cubes actions on $\mathcal{K}$ that one might naively put forward have, at best, homotopy-commutative diagrams. Definition 2 provides us a ‘knot space’ $\text{EC}(k, M)$ where the connect-sum operation is given by composition of functions. The benefit of this construction is that connect-sum becomes a strictly associative function, allowing us to satisfy the rigid axioms of a cubes action.

**Definition 2**

- $D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$, where $\partial D^n = S^{n-1}$.
- A (single) little $n$-cube is a function $L : I^n \to I^n$ such that $L = l_1 \times \cdots \times l_n$.
where each $l_i : I \to I$ is affine-linear and increasing ie: $l_i(t) = a_it + b_i$ for some $a_i > 0$ and $b_i \in \mathbb{R}$.

- Let $\text{CAut}_n$ denote the monoid of affine-linear automorphisms of $\mathbb{R}^n$ of the form $L = l_1 \times \cdots \times l_n$ where $l_i$ is affine-linear and increasing for all $i \in \{1, 2, \cdots, n\}$.

- Given a little $n$-cube $L$, we sometimes abuse notation and consider $L \in \text{CAut}_n$ by taking the unique affine-linear extension of $L$ to $\mathbb{R}^n$.

- The space of $j$ little $k$-cubes $\mathcal{C}_k(j)$ is the space of maps $L : \bigsqcup_{i=1}^j I^k \to I^k$ such that the restriction of $L$ to the interior of its domain is an embedding, and the restriction of $L$ to any connected component of its domain is a little $k$-cube. Given $L \in \mathcal{C}_k(j)$, denote the restriction of $L$ to the $i$-th copy of $I^k$ by $L_i$. By convention $\mathcal{C}_k(0)$ is taken to be a point. This makes the union $\bigsqcup_{j=0}^\infty \mathcal{C}_k(j)$ into an operad, called the operad of little $k$-cubes $\mathcal{C}_k$ (35; 34).

- Given a compact manifold $M$, let $\text{Emb}(\mathbb{R}^k \times M, \mathbb{R}^k \times M)$ denote the space of $C^\infty$-smooth embeddings of $\mathbb{R}^k \times M$ in $\mathbb{R}^k \times M$. We do not demand the embeddings to be proper ie: if $f \in \text{Emb}(\mathbb{R}^k \times M, \mathbb{R}^k \times M)$ then the image of the boundary of $\mathbb{R}^k \times M$ need not lay in the boundary of $\mathbb{R}^k \times M$. We give this space the weak $C^\infty$-topology (See (23) §2.1).

- $\text{EC}(k, M)$ is defined to be the subspace of $\text{Emb}(\mathbb{R}^k \times M, \mathbb{R}^k \times M)$ consisting of embeddings $f : \mathbb{R}^k \times M \to \mathbb{R}^k \times M$ whose support is contained in $I^k \times M$ ie: they are required to restrict to the identity function outside of $I^k \times M$. We consider $\text{EC}(k, M)$ to be a based space, with base-point given by the identity function $\text{Id}_{\mathbb{R}^k \times M}$. Any knot in the path component of $\text{Id}_{\mathbb{R}^k \times M}$ is typically called an unknot.
We will show that the operad of little \((k+1)\)-cubes acts on \(EC(k, M)\), but first we define an action of the monoid \(CAut_k\) on \(Emb(\mathbb{R}^k \times M, \mathbb{R}^k \times M)\).

\[
\mu : CAut_k \times Emb(\mathbb{R}^k \times M, \mathbb{R}^k \times M) \to Emb(\mathbb{R}^k \times M, \mathbb{R}^k \times M)
\]

\[
\mu(L, f) = (L \times Id_M) \circ f \circ (L^{-1} \times Id_M)
\]

In the above formula, we consider both \(L\) and \(L^{-1}\) to be elements of \(CAut_n\). We write the above action as \(\mu(L, f) = L.f\) (see Figure 3).

**Proposition 3** The two maps

\[
\mu : CAut_k \times Emb(\mathbb{R}^k \times M, \mathbb{R}^k \times M) \to Emb(\mathbb{R}^k \times M, \mathbb{R}^k \times M)
\]

\[
\circ : Emb(\mathbb{R}^k \times M, \mathbb{R}^k \times M) \times Emb(\mathbb{R}^k \times M, \mathbb{R}^k \times M) \to Emb(\mathbb{R}^k \times M, \mathbb{R}^k \times M)
\]

are continuous, where \(\circ\) is composition.

The continuity of \(\circ\) is an elementary consequence of the weak topology. The continuity of \(\mu\) follows immediately.

**Definition 4** \(\bullet\) Given \(j\) little \((k+1)\)-cubes, \(L = (L_1, \ldots, L_j) \in C_{k+1}(j)\) define the \(j\)-tuple of (non-disjoint) little \(k\)-cubes \(L^\pi = (L^\pi_1, \ldots, L^\pi_j)\) by the rule \(L^\pi_i = l_{i,1} \times \cdots \times l_{i,k}\) where \(L_i = l_{i,1} \times \cdots \times l_{i,k+1}\). Similarly define \(L^t \in P\) by \(L^t = (L^t_1, \ldots, L^t_j)\) where \(L^t_i = l_{i,k+1}(-1)\) (see Figure 4).

\(\bullet\) The action of the operad of little \((k+1)\)-cubes on the space \(EC(k, M)\) is given by the maps \(\kappa_j : C_{k+1}(j) \times EC(k, M)^j \to EC(k, M)\) for \(j \in \{1, 2, \cdots\}\) defined by

\[
\kappa_j(L_1, \cdots, L_j, f_1, \cdots, f_j) = L^\pi_{\sigma(1)} \circ f_{\sigma(1)} \circ L^\pi_{\sigma(2)} \circ f_{\sigma(2)} \circ \cdots \circ L^\pi_{\sigma(j)} \circ f_{\sigma(j)}
\]

where \(\sigma : \{1, \cdots, j\} \to \{1, \cdots, j\}\) is any permutation such that \(L^t_{\sigma(1)} \leq L^t_{\sigma(2)} \leq \cdots \leq L^t_{\sigma(j)}\). The map \(\kappa_0 : C_{k+1}(0) \times EC(k, M)^0 \to EC(k, M)\) is the inclusion of a point \(*\) in \(EC(k, M)\), defined so that \(\kappa_0(*) = Id_{\mathbb{R}^k \times M}\) (see Figures 5 and 7).
$L_1 < L_2$ so $\sigma$ is the identity and $\kappa_2(L_1, L_2, f_1, f_2) = L_1^{\pi} \cdot f_1 \circ L_2^{\pi} \cdot f_2.$

\textbf{Theorem 5} For any compact manifold $M$ and any integer $k \geq 0$ the maps $\kappa_j$ for $j \in \{0, 1, 2, \cdots \}$ define an action of the operad of little $(k + 1)$-cubes on $\text{EC}(k, M)$.

\textbf{PROOF.} First we show the map $\kappa_j$ is well-defined. The only ambiguity in the definition is the choice of the permutation $\sigma$. If there is an ambiguity in the choice of $\sigma$ this means that a pair of coordinates $L_t^t$ and $L_l^t$ in $j$-tuple $L^t = (L_1^t, \cdots, L_j^t)$ must be equal. Since $L = (L_1, \cdots, L_j)$ are disjoint cubes, if a pair $L_p$ and $L_q$ have projections $L_p^t = L_q$, then $L_p^\pi$ and $L_q^\pi$ are disjoint. Since $\text{supp}(L_p^\pi, f_p) = (L_p^\pi \times Id_M)(\text{supp}(f_p))$ and $\text{supp}(L_q^\pi, f_q) = (L_q^\pi \times Id_M)(\text{supp}(f_q))$, $L_p^\pi \cdot f_p$ and $L_q^\pi \cdot f_q$ must have disjoint support. So the order of composition of $L_p^\pi \cdot f_p$ and $L_q^\pi \cdot f_q$ is irrelevant. This proves the maps $\kappa_j$ are well-defined.

We prove the continuity of the maps $\kappa_j$. Given a permutation $\sigma$ of the set $\{1, \cdots, j\}$ consider the function

$$\kappa_\sigma : \mathcal{C}_{k+1}(j) \times \text{EC}(k, M)^j \to \text{EC}(k, M)$$

defined by

$$(L_1, \cdots, L_j, f_1, \cdots, f_j) \mapsto L_{\sigma(1)}^\pi \cdot f_{\sigma(1)} \circ \cdots \circ L_{\sigma(j)}^\pi \cdot f_{\sigma(j)}$$

This function is continuous, since the composition operation and the action of $\text{CAut}_k$ is continuous by Proposition 3. Given a permutation $\sigma$, consider the
Figure 6

subspace $W_\sigma$ of $C_{k+1}(j) \times \text{EC}(k, M)^j$ where $L_{\sigma(1)}^j \leq \cdots \leq L_{\sigma(j)}^j$. Notice that our map $\kappa_j$ when restricted to $W_\sigma$ agrees with $\kappa_\sigma$. Thus the map $\kappa_j$ is the union of finitely many continuous functions $\kappa_\sigma$ whose definitions agree where their domains $W_\sigma$ overlap, so $\kappa_j$ is a continuous function by the pasting lemma.

We need to show the maps $\kappa_j$ satisfy the axioms of a little cubes action as described in sections 1 and 4 of (35) (or II §1.4 of (34)). There are three conditions that must be satisfied: the identity criterion, symmetry and associativity. The identity criterion is tautological, since if $\text{Id}_{k+1}$ is the identity little $(k + 1)$-cube, its projection is the identity cube, which acts trivially on $\text{EC}(k, M)$. Symmetry is similarly tautological. The associativity condition demands that the diagram in Figure 6 commutes. The commutativity of this diagram follows from the same argument given that shows that the maps are well-defined. If one chases the arrows around the diagram both ways, the two objects that you get in $\text{EC}(k, M)$ are composites of the same embeddings, perhaps in a different order. Any pair of embeddings that have their order permuted must have disjoint supports, so the change in order of composition is irrelevant.

Corollary 6 The group of boundary-fixing diffeomorphisms of the compact $n$-dimensional ball, $\text{Diff}(D^n)$, is homotopy-equivalent to an $(n + 1)$-fold loop space.

**PROOF.** Peter May’s loop space recognition theorem (36) states that a little $(n + 1)$-cubes object $X$ is (weakly) homotopy equivalent to an $(n + 1)$-fold loop space if and only if the induced monoid structure on $\pi_0X$ is a group.

Consider the monoid structure on $\pi_0\text{EC}(n, \{\ast\})$. Let $L = (L_1, L_2) \in C_{n+1}(2)$ be two little $(n + 1)$-cubes such that $L^* = (L_1^*, L_2^*) = (\text{Id}_{D^n}, \text{Id}_{D^n})$. Suppose $L^t = (L_1^t, L_2^t)$ with $L_1^t \prec L_2^t$, then $\kappa_2(L_1, L_2, f_1, f_2) = f_1 \circ f_2$. This means the induced monoid structure on $\pi_0\text{EC}(n, \{\ast\})$ is given by composition. $\text{EC}(n, \{\ast\})$ is a group under composition since it is the group of diffeomorphisms $\mathbb{R}^n$ with support contained in $\mathbf{I}^n$. Thus, $\pi_0\text{EC}(n, \{\ast\})$ is also a group, and so $\text{EC}(n, \{\ast\})$ is weakly homotopy equivalent to an $(n + 1)$-fold loop space. Since $\text{EC}(n, \{\ast\})$ has the weak (compact-open) topology (see Hirsch (23) §2.1) it satisfies the
\[ L_1^t < L_3 < L_2^t \] so \( \sigma = (23) \) and \( \kappa_3(L_1, L_2, L_3, f_1, f_2, f_3) = L_1^t.f_1 \circ L_3^t.f_3 \circ L_2^t.f_2 \), which explains why we see the figure-8 knot ‘inside’ of the trefoil.

\[ \text{Figure 7} \]

**first axiom of countability** and so the topology on \( \text{EC}(n, \{\ast\}) \) is compactly-generated in the sense of Steenrod (48). Thus by the loop space recognition theorem, the \( \text{EC}(n, \{\ast\}) \) is homotopy-equivalent to an \((n+1)\)-fold loop space.

Provided we show that \( \text{Diff}(D^n) \simeq \text{EC}(n, \{\ast\}) \) we are done. Fix a collar neighborhood of \( S^{n-1} \). There is a restriction map from \( \text{Diff}(D^n) \) to the space of collar neighborhoods of \( S^{n-1} \) in \( D^n \). This restriction map is a fibration (41) and the space of collar neighborhoods of \( S^{n-1} \) in \( D^n \) is contractible (see (23) §4.5.3). The above argument is not sufficient, because the fiber of this fibration is not \( \text{EC}(n, \{\ast\}) \). Replace the smooth collar neighborhood of \( S^{n-1} \) in \( D^n \) with a manifold-with-corners neighborhood of \( S^{n-1} \) which is the complement of an open cube in \( D^n \). In this case we get a fibration whose fiber we can identify with \( \text{EC}(n, \ast) \). The argument that the space of cubical collar neighborhoods is contractible is analogous to the proof in Hirsch’s text (see (23) §4.5.3).

May’s recognition theorem applies equally-well to spaces that have actions of the operad of (unframed) little balls (34). Thus we could have simply adapted Definition 4 to give an action of the space of unframed \((n+1)\)-balls directly on the the space \( \text{Diff}(D^n) \) and deduced the result without recourse to the intermediate homotopy-equivalence \( \text{Diff}(D^n) \simeq \text{Diff}(I^n) \).

The above corollary is also a corollary of Morlet’s ‘Comparison Theorem’ (38). Morlet’s manuscript was not widely distributed. A proof of Morlet’s Theorem can be found in Burghelea and Lashof’s paper (10), as well as in Kirby and Siebenmann’s book (28). As Siebenmann points out, the Morlet Comparison Theorem was first observed by Cerf.
Corollary 7 \( \text{EC}(k, D^n) \) is homotopy equivalent to a \((k + 1)\)-fold loop space, provided \( n > 2 \).

**PROOF.** This follows from the loop space recognition theorem (36) since we will show that \( \pi_0 \text{EC}(k, D^n) \) is a group. Consider the fibration \( \text{EC}(k, D^n) \to \text{Emb}(\mathbb{R}^k, \mathbb{R}^{k+n}) \) where \( \text{Emb}(\mathbb{R}^k, \mathbb{R}^{k+n}) \) is the space \( \{f : \mathbb{R}^k \to \mathbb{R}^{k+n} : f(t_1, t_2, \ldots, t_k) = (t_1, t_2, \ldots, t_k, 0, 0, \ldots, 0) \text{ if } |t_i| \geq 1 \text{ for any } i \in \{1, 2, \ldots, k\}\}. \) Haefliger proved (18) that \( \pi_0 \text{Emb}(\mathbb{R}^k, \mathbb{R}^{k+n}) \) is a group provided \( n > 2 \) where the group structure is induced by concatenation, thus \( \pi_0 \text{EC}(k, D^n) \) is a group as it is a monoid which is an extension of two groups (see (9) for an alternative proof of Haefliger’s theorem).

Our preferred model for \( \mathcal{K} \) will be a subspace \( \hat{\mathcal{K}} \) of \( \text{EC}(1, D^2) \), which we will relate back to the standard model \( \mathcal{K} \). Given an embedding \( f \in \text{EC}(1, D^2) \), define \( \omega(f) \in \mathbb{Z} \) to be the linking number of \( f|_{\mathbb{R} \times \{(0,0)\}} \) with \( f|_{\mathbb{R} \times \{(0,1)\}} \). One concrete way to define this integer is as the transverse intersection number of the map \( \mathbb{R}^2 \ni (t_1, t_2) \mapsto f(t_1, 0, 1) - f(t_2, 0, 0) \in \mathbb{R}^3 - \{(0,0,0)\} \) with the ray \( \{(0,t,0) : t > 0\} \subset \mathbb{R}^3 - \{(0,0,0)\} \). \( \omega(f) \) is called the framing number of \( f \).

**Definition 8** \( \hat{\mathcal{K}} \), the space of ‘fat’ long knots in \( \mathbb{R}^3 \) is defined to be the kernel of \( \omega \), \( \hat{\mathcal{K}} = \omega^{-1}\{0\} \).

**Proposition 9** The two spaces \( \hat{\mathcal{K}} \) and \( \mathcal{K} \) are homotopy equivalent.

**PROOF.** Consider the fibration \( \text{EC}(1, D^2) \to \text{Emb}(\mathbb{R}, \mathbb{R}^3) \) given by restriction \( f \mapsto f|_{\mathbb{R} \times \{(0,0)\}} \) (41). Let \( X \) denote the fiber of this fibration. By definition, \( X \) is the space of tubular neighborhoods of the unknot which are standard outside of \( I \times D^2 \). By the classification of tubular neighborhoods theorem (see for example (23) §4.5.3), \( X \) is homotopy equivalent to the space of fibrewise-linear automorphisms of \( \mathbb{R} \times D^2 \) with support in \( I \times D^2 \), ie: \( X \simeq \Omega\text{SO}_2 \simeq \mathbb{Z} \). Thus \( \omega \) defines a splitting of the fibration \( X \to \text{EC}(1, D^2) \to \text{Emb}(\mathbb{R}, \mathbb{R}^3) \), giving the two homotopy equivalences

\[
\begin{align*}
\text{EC}(1, D^2) & \simeq \text{Emb}(\mathbb{R}, \mathbb{R}^3) \times \mathbb{Z} \\
\hat{\mathcal{K}} & \overset{\simeq}{\longrightarrow} \mathcal{K} \\
\hat{f} & \overset{\psi}{\longrightarrow} f|_{\mathbb{R} \times \{(0,0)\}}
\end{align*}
\]

Combining Proposition 9 with the proof of Corollary 7 we get the following observation.
Corollary 10 There is an action of the operad of \((k+1)\)-cubes on spaces homotopy-equivalent to the ‘long embedding spaces’ \(\text{Emb}(\mathbb{R}^k, \mathbb{R}^{k+n})\) for all \(k \in \mathbb{N}\) and \(n \leq 2\).

As mentioned in the introduction, Salvatore (42) has removed the bound \(n \leq 2\) in the above corollary, provided \(k = 1\).

3 The freeness of the 2-cubes action on \(\hat{\mathcal{K}}\)

The goal of this section is to prove that \(\hat{\mathcal{K}} \simeq \mathcal{C}_2(\mathcal{P} \sqcup \{\ast\})\), where \(\mathcal{P} \subset \hat{\mathcal{K}}\) is the subspace of prime knots. \(\mathcal{P} = \{ f \in \hat{\mathcal{K}} : f \) is nontrivial and not a connected-sum of 2 or more nontrivial knots\}.

If \(X\) is a pointed space with base-point \(\ast \in X\) the free little 2-cubes object on \(X\) (35) is the space \(\mathcal{C}_2(X) = ((\sqcup_{n=0}^{\infty} \mathcal{C}_2(n) \times X^n)/S_n)/\sim\). \(S_n\) is the symmetric group, acting diagonally on the product in the standard way, and the equivalence relation \(\sim\) is generated by the relations
\[
((f_1, \cdots, f_{i-1}, f_i, f_{i+1}, \cdots, f_n), (x_1, \cdots, x_{i-1}, \ast, x_{i+1}, \cdots, x_n)) \\
\sim ((f_1, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n), (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n))
\]
If we give an arbitrary unpointed space \(X\) a disjoint base-point \(\ast\), then there is the identity \(\mathcal{C}_2(X \sqcup \{\ast\}) \equiv \sqcup_{n=0}^{\infty} (\mathcal{C}_2(n) \times X^n)/S_n\). Thus, we will prove \(\hat{\mathcal{K}} \simeq \sqcup_{n=0}^{\infty} \mathcal{C}_2(n) \times S_n\).

Theorem 11 \(\hat{\mathcal{K}} \simeq \mathcal{C}_2(\mathcal{P} \sqcup \{\ast\})\), moreover the map \(\sqcup_{n=0}^{\infty} \kappa_n : \sqcup_{n=0}^{\infty} \mathcal{C}_2(n) \times S_n \rightarrow \hat{\mathcal{K}}\) restricts to a homotopy equivalence
\[
\sqcup_{n=0}^{\infty} \mathcal{C}_2(n) \times S_n \mathcal{P}^n \rightarrow \hat{\mathcal{K}}
\]
To prove Theorem 11 we first build up a close correspondence between the little cubes action and the satellite decomposition of knots, or to be more precise, the JSJ-decomposition (26) of knot complements (also sometimes also known as the splice decomposition (13)). We then use techniques of Hatcher’s to reduce the proof of Theorem 11 to a problem about a diagram of mapping class groups of 2 and 3-dimensional manifolds.

Definition 12 • Given a long knot \(f \in \hat{\mathcal{K}}\), we denote the component of \(\hat{\mathcal{K}}\) containing \(f\) by \(\hat{\mathcal{K}}_f\).

• We say \(f\) is a connected-sum of \(f_1, \cdots, f_n\) if there exists \(\tilde{f} \in \hat{\mathcal{K}}_f\) with \(\tilde{f} = \kappa_n(L_1, L_2, \cdots, L_n, f_1, f_2, \cdots, f_n)\), for some \(n\), \((L_1, L_2, \cdots, L_n) \in \mathcal{C}_2(n)\) and \(f_i \in \hat{\mathcal{K}}\) for all \(i \in \{1, 2, \cdots, n\}\). Denote this by \(f \sim f_1 \# f_2 \# \cdots \# f_n\) and call the long knots \(\{f_i : i \in \{1, 2, \cdots, n\}\}\) summands of \(f\).
• For any long knot \( f \sim f \# Id_{\mathbb{R} \times D^2} \). If \( f \sim f_1 \# f_2 \# \cdots \# f_n \) we call the connected-sum trivial if \((n-1)\) of the long knots \( \{f_1, f_2, \cdots, f_n\} \) are in \( \mathcal{K}_{Id_{\mathbb{R} \times D^2}} \). A long knot is prime if is not in the component of the unknot, and if all connected-sum decompositions of it are trivial.

Let \( Q_i \) denote the 2-cube \([-1 + \frac{4i-2}{2n+1}, -1 + \frac{4i}{2n+1}] \times [0, \frac{2}{2n+1}]\). Choose the base-point \(*\) for \( C_2(n) \), \(* = (Q_1, Q_2, \cdots, Q_n)\) as in Figure 8.

Since \( C_2(n) \) is connected, we can choose the \( n \) little 2-cubes \((L_1, L_2, \cdots, L_n) \in C_2(n)\) in Definition 12 to be \(*\). As in Proposition 9 we can associate to \( f \in \mathcal{K}\) the long knot \( g \in \text{Emb}(\mathbb{R}, \mathbb{R}^3) \) where \( g = f|_{\mathbb{R} \times \{0,0\}} \). Define \( B_i = \{x \in \mathbb{R}^3 : |x - (\frac{4i-2}{2n+1}, 0, 0)| \leq \frac{1}{2n+1}\} \) and \( S_i = \partial B_i \) (see Figure 9). We will provide an equivalent definition for \( f \) to be a connected-sum in terms of \( g \) (see Figure 9).

We say \( g \) is a connected-sum if \( g \) is isotopic to \( g' \in \text{Emb}(\mathbb{R}, \mathbb{R}^3) \) such that:

• \( \text{supp}(g') \subset (\bigcup_{i=1}^{n} B_i) \cap (\mathbb{R} \times \{0\}^2) \)

• \( \text{img}(g') \cap S_i = (\mathbb{R} \times \{0\}^2) \cap S_i \) for all \( i \in \{1,2,\cdots, n\} \).

• There exists long knots (the summands of \( g \)) \( g_i \in \text{Emb}(\mathbb{R}, \mathbb{R}^3) \) for \( i \in \{1,2,\cdots, n\} \) such that \( \text{supp}(g_i) \subset B_i \cap (\mathbb{R} \times \{0\}^2) \) and \( g_i|_{B_i \cap (\mathbb{R} \times \{0\}^2)} = g'|_{B_i \cap (\mathbb{R} \times \{0\}^2)} \).

Non-trivial connected-sums and prime knots are defined analogously. A theorem of Schubert (44) states that up to isotopy, every non-trivial \( g \) can be written uniquely up to a re-ordering of the terms, as a connected-sum of prime knots \( g = g_1 \# \cdots \# g_n \).

We review the Jaco-Shalen-Johannson decomposition of 3-manifolds (26). This is a standard decomposition of 3-manifolds along spheres and tori, given by the connected-sum decomposition (29) followed by the torus decomposition of the prime summands (26) (see for example (21) or (39)). For us, all our 3-manifolds will be compact, and they are allowed to have a boundary. For a more exhaustive treatment of JSJ-decompositions of knot and link complements in \( S^3 \), see (7).
A 3-manifold \( M \) is a connected-sum \( M = M_1 \# M_2 \) if surgery along an embedded 2-sphere produces manifolds \( M_1 \) and \( M_2 \), called the summands of \( M \). Provided neither \( M_1 \) nor \( M_2 \) are 3-spheres we say the connect-sum is non-trivial. If a 3-manifold \( M \) is not \( S^3 \) and if all connect-sum decompositions of \( M \) are trivial, \( M \) is called prime. Kneser’s Theorem (29) states that every compact, orientable 3-manifold is a connected-sum of a unique collection of prime 3-manifolds \( M = M_1 \# M_2 \# \cdots \# M_n \), where uniqueness is up to a re-ordering of the terms.

The torus decomposition of a prime 3-manifold \( M \) consists of a minimal collection of embedded incompressible tori \( \sqcup_{i=1}^n T_i \subset M \) such that the complement \( M - \sqcup_{i=1}^n \nu T_i \) is a disjoint union of atoroidal and Seifert-fibered manifolds, where \( \nu T_i \) is an open tubular neighborhood of \( T_i \subset M \). A torus \( T_i \) is incompressible if the induced map \( \pi_1 T_i \to \pi_1 M \) is injective. A torus in a 3-manifold is peripheral if it is isotopic to a boundary torus. A 3-manifold is atoroidal if all incompressible tori are peripheral. The theorem of Jaco, Shalen and Johannson states that such a collection of tori \( \{T_1, T_2, \cdots, T_n\} \) always exists and they are unique up to isotopy (26). Given an arbitrary prime 3-manifold, there is an associated graph called the JSJ-graph of \( M \). The vertices of the JSJ-graph are the components of the manifold \( M - \sqcup_{i=1}^n \nu T_i \). The edges of the graph are the tori \( T_i \) for \( i \in \{1, 2, \cdots, n\} \).

Given a long knot \( f \in \hat{K} \), consider the compact 3-manifold \( B - N' \) where \( B \subset \mathbb{R}^3 \) is a closed 3-ball containing \( I \times D^2 \), and \( N' \) is the interior of the image of \( f \). We will call \( C = B - N' \) the knot complement. Define \( T = \partial C \). We review JSJ-splittings of knot complements. Every sphere in \( \mathbb{R}^3 \) bounds a 3-ball by the Alexander-Schoenflies Theorem (see for example (21)), thus knot complements are prime 3-manifolds, and the Jaco-Shalen-Johannson decomposition of a knot complement is simply the torus decomposition. The Generalized Jordan Curve Theorem (see for example (17)) tells us a knot complement’s associated graph is a tree. The tree is rooted, as only one component of \( C - \sqcup_{i=1}^n \nu T_i \) contains \( T \). The component of \( C - \sqcup_{i=1}^n \nu T_i \) containing \( T \) will be called the root manifold of the JSJ-splitting.

**Definition 13** Fix an embedding \( b : \sqcup_{i \in \{1, 2, \cdots, n\}} D^2 \to D^2 \) such that \( \partial D^2 \cap img(b) = \phi \). Let \( D^2_i \) denote the image of the \( i \)-th copy of \( D^2 \) under \( b \). Choose
b so that $D^2_i$ is the disc of radius $\frac{1}{2n+1}$ centered around the point $(\frac{4i-2n-2}{2n+1},0)$. Define $P_n$ to be $D^2 - \text{int}(\bigcup_{i=1}^{n} D^2_i)$. $P_n$ will be called the n-times punctured disc. $\partial D^2$ is the external boundary and $\partial(\text{img}(b))$ the internal boundary of $P_n$ (see Figure 10).

There are a few elementary facts that we will need about JSJ-splittings of knot complements and diffeomorphism groups of 2 and 3-dimensional manifolds. We assemble these facts in the following lemmas, all which are widely ‘known’ yet published proofs are elusive. A more detailed study of JSJ-decompositions of knot and link complements in $S^3$ has recently appeared (7) and could be used in place of several of these lemmas. An essential reference for the following arguments is Hatcher’s notes on 3-dimensional manifolds (21).

**Lemma 14** If $M$ is sub-manifold of $S^3$ whose boundary consists of a non-empty collection of tori, then either $M$ is a solid torus $S^1 \times D^2$ or a component of the complement of $M$ in $S^3$ is a solid torus.

**Proof.** Let $C = S^3 - \text{int}(M)$ be the complement. Since $\partial M$ consists of a disjoint union of tori, every component of $\partial M$ contains an essential curve $\alpha$ which bounds a disc $D$ in $S^3$. Isotope $D$ so that it intersects $\partial M$ transversely in essential curves. Then $\partial M \cap D \subset D$ consists of a finite collection of circles, and these circles bound a nested collection of discs in $D$. Take an innermost disc $D'$. If $D' \subset M$ then $M$ is a solid torus. If $D' \subset C$ then the component of $C$ containing $D'$ is a solid torus.

**Lemma 15** If a Seifert-fibred 3-manifold is a component of a knot complement (in $S^3$) split along its JSJ-decomposition, then it is diffeomorphic to one of the following:

- A solid torus (unknot complement).
• The complement of a non-trivial torus knot. Such a manifold is Seifert-fibred over a disc with two singular fibres.
• $S^1 \times P_n$ for $n \geq 2$ (trivially fibred over a $n$-times punctured disc).
• Fibred over an annulus with one singular fibre (The complement of a regular and singular fibre in a Seifert-fibring of $S^3$).

**PROOF.** Seifert-fibered manifolds that fiber over a non-orientable surface do not embed in $S^3$ since a non-orientable, embedded closed curve in the base lifts to a Klein bottle, which does not embed in $S^3$ by the Generalized Jordan Curve Theorem (17). Similarly, a Seifert-fibered manifold that fibers over a surface of genus $g > 0$ does not embed in $S^3$ since the base manifold contains two curves that intersect transversely at a point. If we lift one of these curves to a torus in $S^3$, it must be non-separating. This again contradicts the Generalized Jordan Curve Theorem.

Consider a Seifert-fibered manifold $M$ over an $n$-times punctured disc with $n > 0$ and with perhaps multiple singular fibers. By Lemma 14, either $M$ is a solid torus or some component $Y$ of $S^3 \setminus M$ is a solid torus. Consider the latter case. There are two possibilities.

1. The meridians of $Y$ are fibres of $M$. If there is a singular fibre in $M$, let $\beta$ be an embedded arc in the base surface associated to the Seifert-fibring of $M$ which starts at the singular point in the base and ends at the boundary component corresponding to $\partial Y$. $\beta$ lifts to a 2-dimensional CW-complex in $M$, and the endpoint of $\beta$ lifts to a meridian of $Y$, thus it bounds a disc. If we append this disc to the lift of $\beta$, we get a CW-complex $X$ which consists of a 2-disc attached to a circle. The attaching map for the 2-cell is multiplication by $\beta$ where $\beta$ is the slope associated to the singular fibre. The boundary of a regular neighbourhood of $X$ is a 2-sphere, so we have decomposed $S^3$ into a connected sum $S^3 = L^3_{\beta} \# Z$ where $L^3_{\beta}$ is a lens space with $H_1 L^3_{\beta} = \mathbb{Z}_\beta$. Since $S^3$ is irreducible, $\beta = 1$. Thus $M \simeq S^1 \times P_{n-1}$ for some $n \geq 1$.

2. The meridians of $Y$ are not fibres of $M$. In this case, we can extend the Seifert fibring of $M$ to a Seifert fibring of $M \cup Y$. Either $M \cup Y = S^3$, or $M \cup Y$ has boundary.
   • If $M \cup Y = S^3$ then we know by the classification of Seifert fibrings of $S^3$ that any fibring of $S^3$ has at most two singular fibres. If $M$ is the complement of a regular fibre of a Seifert fibring of $S^3$, then $M$ is a torus knot complement. Otherwise, $M$ is the complement of a singular fibre, meaning that $M$ is a solid torus.
   • If $M \cup Y$ has boundary, we can repeat the above argument. Either $M \cup Y$ is a solid torus, or a component of $S^3 \setminus M \cup Y$ is a solid torus, so we obtain $M$ from the above manifolds by removing a Seifert fibre. By induction, we obtain $M$ from either a Seifert fibring of a solid torus, or
Corollary 16 Given a long knot $f \in \hat{K}$ with complement $C$, if the root manifold of the JSJ-splitting of the knot complement is Seifert fibered with one singular fiber, then $f$ is a cabling of another long knot. Another way to say this is that the one-point compactification $\tilde{f} : S^1 \to S^3$ of $f_{|\mathbb{R} \times \{0\}} : \mathbb{R} \to \mathbb{R}^3$ is an essential curve in the boundary of a tubular neighborhood of some embedding $g : S^1 \to S^3$ (see Figure 11).

Thurston (49) has proved that the non-Seifert-fibered manifolds in the JSJ-splitting of a knot complement are finite-volume hyperbolic manifolds. These hyperbolic manifolds can have arbitrarily many boundary components (7). Figure 12 demonstrates a hyperbolic satellite knot (a knot such that the root manifold in the JSJ-decomposition is hyperbolic) which contains the Borromean rings complement in its JSJ-decomposition. In general, one can prove that if the root manifold is a hyperbolic manifold with $n+1$ boundary components, then it is the complement of an $(n+1)$-component hyperbolic link in $S^3$ which contains an $n$-component sublink which is the unlink.

Lemma 17 A knot is a non-trivial connected-sum if and only if the root manifold of the associated JSJ-tree is diffeomorphic to $S^1 \times P_n$ for some $n \geq 2$. In
this case, \( n \) is the number of prime summands of \( f \).

**Proof.** If \( f \in \hat{\mathcal{K}} \) is a non-trivial connected-sum, let \( n \) be the number of prime summands of \( f \), and isotope \( f \) so that \( f|_{\mathbb{R} \times \{0\}^2} \) satisfies Definition 12.

Let \( L \subset \mathbb{R}^2 \) be the closed disc of radius \( \frac{1}{2} \) centered about the origin. Let \( N' = \text{img}(f|_{\mathbb{R} \times L}) \), \( N = \text{img}(f) \) and define \( C = B - \text{int}(N') \) where \( B \) is a closed, convex ball neighborhood of \( I \times D^2 \) in \( \mathbb{R}^3 \). Let \( B_1, B_2, \cdots B_n \) be the closed 3-balls from Definition 12, with \( S_i = \partial B_i \) and \( S_i \) intersecting \( \text{img}(f) \) in two discs for all \( i \in \{1, 2, \cdots, n\} \). Define \( C_i = B_i - \text{int}(N) \) and \( T_i = \partial C_i \).

Let \( \nu T_i \) be a small open tubular neighborhood of \( T_i \), then \( C - \bigcup_{i=1}^n \nu T_i \) consists of \( n + 1 \) components. One component contains \( T = \partial C \) and the other \( n \) components are the knot complements of the prime summands of \( f \), \( C_1, C_2, \cdots, C_n \). The component containing \( T \) we will denote \( V \). \( V \) is diffeomorphic to \( S^1 \times P_n \).

By Dehn’s Lemma the tori \( \{T_i : i \in \{1, 2, \cdots, n\}\} \) are incompressible in \( C \). If \( \{T_{n+1}, \cdots, T_{n+m}\} \) are the tori of the JSJ-decomposition for \( \bigcup_{i=1}^n C_i \), the collection \( \{T_1, T_2, \cdots, T_n, T_{n+1}, \cdots, T_{n+m}\} \) is therefore the JSJ-decomposition of \( C \). Thus, \( V \simeq S^1 \times P_n \) is the root manifold in the JSJ-tree associated to \( C \).

To prove the converse, let \( V \) be the root manifold of the JSJ-splitting of \( C \). Observe that \( \partial V \simeq \partial(S^1 \times P_n) \) divides \( \mathbb{R}^3 \) into \( n + 2 \) components, only one containing the knot. Let \( T \) denote the boundary of the component which contains the knot. By Lemma 15 the fibers of \( S^1 \times P_n \) are meridians of the knot. Let \( L_1, \cdots, L_n \) be properly embedded intervals in \( P_n \) which cut \( P_n \) into the union of a disc with \( n \) once-punctured discs. Then \( \bigcup_{i=1}^n (S^1 \times L_i) \) can be extended to \( n \) disjoint, embedded 2-spheres \( S_i \subset \mathbb{R}^3 \) such that \( S_i \cap (S^1 \times P_n) = S^1 \times L_i \), and \( S_i \cap \text{img}(f|_{\mathbb{R} \times \{0\}^2}) \) consists of two points. Thus we have decomposed the long knot \( f \) into a connected-sum.

**Definition 18** In the above lemma, we call the tori \( T_1, \cdots, T_n \) the base level of the Jaco-Shalen-Johannson decomposition of the knot complement.

Lemmas 17, 16 and Thurston’s Hyperbolisation Theorem (49) gives us a canonical decomposition of knots into simpler knots via cablings, connected-sums and hyperbolic satellite operations commonly referred to as the satellite or splice decomposition of knots. This is worked out in detail in (7).

**Example 19** Figure 13 shows a knot with its JSJ-tori, and the associated JSJ-tree. In the standard terminology of knot theory, this knot would be described as a connect-sum of three prime knots: the left-handed trefoil, the figure-8 knot and the Whitehead double of the figure-8 knot. \( V = S^1 \times P_3 \) is the root manifold, \( T_1, T_2, T_3 \) are the base-level of the JSJ-decomposition of \( C \), and \( T_4 \) is the remaining torus in the JSJ-decomposition of \( C \). The leftmost summand
Figure 13

is the trefoil knot. The center summand is a figure-8 knot, whose complement is hyperbolic. The rightmost summand is the Whitehead double of the figure-8 knot, its complement is $C_3$. $C_3$ is the union of $C_3'$ (the Whitehead link complement, which is hyperbolic) and $C_3''$ (a figure-8 knot complement) where $\partial C_3' = T_3 \sqcup T_4$, and $\partial C_3'' = T_4$, $C_3'' \cap C_3' = T_4$. The interior of $C_3'$ is also a hyperbolic 3-manifold of finite volume.

**Lemma 20** The component $\hat{K}_f$ of $\hat{K}$ containing the long knot $f$ is the classifying space of $\text{Diff}(C, T)$, the group of diffeomorphisms of the knot complement which fix the boundary torus $T = \partial C$ point-wise. Moreover, $\hat{K}_f$ is a $K(\pi, 1)$.

**PROOF.** Let $B = I \times D^2$ and let $\text{Diff}(B)$ be the group of diffeomorphisms of $\mathbb{R}^3$ with support contained in $B$. The map $\text{Diff}(B) \rightarrow \hat{K}_f$ defined by restriction to $\text{img}(f)$ induces a fibration

$$\text{Diff}(C, T) \rightarrow \text{Diff}(B) \rightarrow \hat{K}_f$$

where $C = B - \text{int}(\text{img}(f))$, $T = \partial C$, and $\text{Diff}(C, T)$ is the group of diffeomorphisms of $C$ that fix $T$ point-wise. Since $\text{Diff}(B)$ is contractible (20), $\text{BDiff}(C, T) \simeq \hat{K}_f$, where $\hat{K}_f$ is the component of $\hat{K}$ containing $f$. The fact that $\text{Diff}(C, T)$ has contractible components is due to Hatcher (19).

In the above lemma, $BG = EG/G$ is the classifying space of a topological group $G = \text{Diff}(C, T)$ and $EG = \text{Diff}(B)$. Using Smale’s Theorem $\text{Diff}(D^3) \simeq \{\ast\}$ (47), an argument analogous to the above gives $C_2(n)/S_n \simeq \text{BDiff}(P_n)$
where $\text{Diff}(P_n)$ is the group of diffeomorphisms of $P_n$ that fix the external boundary of $P_n$ point-wise.

Let $\text{PDiff}(P_n)$ denote the subgroup of $\text{Diff}(P_n)$ consisting of diffeomorphisms whose restrictions to $\partial P_n$ are isotopic to the identity map $\text{Id}_{\partial P_n} : \partial P_n \to \partial P_n$. Then similarly, by Smale’s Theorem $C_2(n) \simeq B\text{PDiff}(P_n)$.

Let $\text{PFDiff}(P_n)$ be the subgroup of $\text{PDiff}(P_n)$ consisting of diffeomorphisms whose restrictions to $\partial P_n$ are equal to the identity $\text{Id}$. $\pi_0\text{Diff}(P_n)$ is called the braid group on $n$-strands. $\pi_0\text{PDiff}(P_n)$ is called the pure braid group on $n$-strands, and $\pi_0\text{PFDiff}(P_n)$ is called the pure framed braid group on $n$ strands.

Observe that $\text{PFDiff}(P_n)$ is homotopy equivalent to the subgroup $\text{PFDiff}^+(P_n)$ of $\text{PFDiff}(P_n)$ consisting of diffeomorphisms which restrict to the identity in an $\epsilon$-neighborhood $N$ of the internal boundary of $P_n$. This follows from the fact that the space of collar neighborhoods of $\partial P_n$ in $P_n$ is contractible.

**Definition 21** This definition will use the notation of Definition 13 and the previous paragraph. Every diffeomorphism in $\text{PFDiff}^+(P_n)$ can be canonically extended to a diffeomorphism of the once-punctured disc $D^2 - \text{int}(D^2_i)$ by taking the union with $\text{Id}_{D^2_j}$ for $j \neq i$. Thus, for each $i \in \{1, 2, \ldots, n\}$ there is a homomorphism $w_i : \text{PFDiff}^+(P_n) \to \pi_0\text{PFDiff}(S^1 \times I) \simeq \mathbb{Z}$ given by the above extension together with an identification $D^2 - \text{int}(D^2_i) \equiv S^1 \times I$. Here $\text{PFDiff}(S^1 \times I)$ denotes the group of boundary-fixing diffeomorphisms of $S^1 \times I$. The generator of $\pi_0\text{PFDiff}(S^1 \times I) \simeq \mathbb{Z}$ is a Dehn twist about a boundary-parallel curve (15). Let $\text{DN}_n$ denote a free abelian subgroup of $\text{PFDiff}^+(P_n)$ having rank $n$, all whose elements have support in $N$, generated by Dehn twists about $n$ curves in $N$, the $i$-th curve parallel to $\partial D^2_i$.

**Lemma 22** There is an isomorphism of groups

$$\pi_0\text{PDiff}(P_n) \times \mathbb{Z}^n \simeq \pi_0\text{PFDiff}(P_n)$$

Moreover, the subgroups $\cap_{i=1}^n \ker(w_i)$ and $\text{DN}_n$ satisfy:

- **Inclusion** $\cap_{i=1}^n \ker(w_i) \to \text{PDiff}(P_n)$ is a homotopy equivalence.
- The elements of $\text{DN}_n$ and $\cap_{i=1}^n \ker(w_i)$ commute with each other, and $\text{DN}_n \cap (\cap_{i=1}^n \ker(w_i))$ is the trivial group.
- The homomorphism $\cap_{i=1}^n \ker(w_i) \times \text{DN}_n \to \text{PFDiff}(P_n)$ is a homotopy equivalence.

**Proof.** Take $\text{Diff}(S^1)$ to be the group of orientation preserving diffeomorphisms of a circle, and consider the fibration $\text{PFDiff}(P_n) \to \text{PDiff}(P_n) \to \prod_{i=1}^n \text{Diff}(S^1)$ given by restriction to the internal boundary of $P_n$. This gives us the short exact sequence

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19
\[
0 \rightarrow \prod_{i=1}^{n} \pi_1 \Diff(S^1) \rightarrow \pi_0 \PFDiff(P_n) \rightarrow \pi_0 \PDiff(P_n) \rightarrow 0
\]

but \(\prod_{i=1}^{n} \pi_1 \Diff(S^1) \cong \mathbb{Z}^n\), which is the subgroup \(\text{DN}_n \subset \pi_0 \PFDiff(P_n)\). The map \(\prod_{i=1}^{n} w_i : \pi_0 \PFDiff(P_n) \rightarrow \mathbb{Z}^n \cong \prod_{i=1}^{n} \pi_1 \Diff(S^1)\) is a splitting of the above short exact sequence. The kernel of \(\prod_{i=1}^{n} w_i\) is \(\pi_0 \cap \prod_{i=1}^{n} \ker(w_i)\). By definition, elements in \(\cap_{i=1}^{n} \ker(w_i)\) and \(\text{DN}_n\) commute with each other, and so the result follows.

We will also need a mild variation on Lemma 22. Let \(\ast = (0, -1)\) be the basepoint of \(D^2\) and let \(\gamma_i : [0, 1] \rightarrow P_n\) for \(i \in \{1, 2, \ldots, n\}\) be the affine-linear map starting at \(\ast\) and ending at \((4i - 2n - 2n_2 + 1, -1 - 2n_2 + 1)\).

**Definition 23** Define \(K\Diff(P_n)\) to be \(\cap_{i=1}^{n} \ker(w_i)\). Define \(F\Diff(P_n)\) to be the subgroup of \(\Diff(P_n)\) such that each diffeomorphism \(f \in F\Diff(P_n)\)

- restricts to a diffeomorphism of \(N\), ie: \(f|_N : N \rightarrow N\).
- the restriction of \(f|_N\) to any connected component of \(N\) is a translation in the plane.

Observe, there is an epi-morphism \(F\Diff(P_n) \rightarrow S_n \ltimes \mathbb{Z}^n\) given by \(f \mapsto (\sigma_f, \omega_1(f), \cdots, \omega_n(f))\) where

- \(\sigma_f \in S_n\) is the permutation of \(\{1, 2, \cdots, n\}\) defined by \(\sigma_f(i) = j\) if \(f(\partial D^2_j) = \partial D^2_j\).
- \(\omega_i(f) \in \mathbb{Z}\) is the linking number of \(\overline{\gamma_j} \cdot (f \circ \gamma_i)\) with \(D^2_j\) where \(\sigma(i) = j\). Here \(\overline{\gamma_j}(t) = \gamma_j(1-t)\) and concatenation is by convention right-to-left, ie: if \(\gamma, \eta : [0, 1] \rightarrow X\) satisfy \(\eta(1) = \gamma(0)\) then \(\gamma \cdot \eta(t) = \eta(2t)\) for \(0 \leq t \leq \frac{1}{2}\) and \(\gamma \cdot \eta(t) = \gamma(2t - 1)\) for \(\frac{1}{2} \leq t \leq 1\).
- \(S_n \ltimes \mathbb{Z}^n\) is the semi-direct product of \(S_n\) and \(\mathbb{Z}^n\) where \(S_n\) acts on \(\mathbb{Z}^n\) by the regular representation ie: \(\sigma(i_1, i_2, \cdots, i_n) = (i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \cdots, i_{\sigma^{-1}(n)})\).
Call the above epi-morphism $W : \text{FDiff}(P_n) \to S_n \times \mathbb{Z}^n$, and define $\widehat{\text{KDiff}}(P_n) = W^{-1}(S_n \times \{0\}^n)$.

**Lemma 24** There is a fiber-homotopy equivalence

\[
\begin{array}{c}
\text{PDiff}(P_n) \\
\downarrow \\
\text{Diff}(P_n) \\
\downarrow \\
\text{KDiff}(P_n) \\
\downarrow \\
\text{KDiff}(P_n) \wedge \text{KDiff}(P_n) \\
\downarrow \\
S_n \\
\end{array}
\]

where all vertical arrows are inclusions.

The above lemma follows immediately from Lemma 22.

Abstractly there is a homotopy equivalence between $\text{BDiff}(P_n)$ and $C_2(n)$ given by the proof of Lemma 20. Since the properties of this homotopy equivalence will be important later, we define it precisely here.

**Definition 25** Given $f \in \text{Diff}(D^2)$, let $\zeta(f) = (L_1, L_2, \cdots, L_n) \in C_2(n)$ be $n$ little 2-cubes such that the center of $L_i$ is $f(\frac{4i-2n-2}{2n+1}, 0)$. For $\zeta(f)$ to be well-defined (and continuous) we need to choose the side lengths of $L_i$ equal to the minimum of these two numbers: $\frac{4i-2n-2}{2n+1}$ and the largest number $w$ so that the little cubes with centers $f(\frac{4i-2n-2}{2n+1}, 0)$ with width and height equal to $w$ for $i \in \{1, 2, \cdots, n\}$ have disjoint interiors. Then $\phi : \text{Diff}(D^2) \to C_2(n)$ factors to a map $\text{BDiff}(P_n) \to C_2(n)$ which is a homotopy-equivalence.

The definition below will use the conventions of Definition 13, in particular we will call $S^1 \times \partial D^2 \subset S^1 \times P_n$ the external boundary of $S^1 \times P_n$, and $\partial(S^1 \times P_n) - S^1 \times \partial D^2$ the internal boundary of $S^1 \times P_n$.

**Definition 26** Let $\eta_i : S^1 \to \partial D^2$ be a clockwise parametrization of $\partial D^2$ starting and ending at $\gamma_i(1)$. Notice that $\lambda_i = \gamma_i \eta_i$ for $i \in \{1, 2, \cdots, n\}$ are generators for $\pi_1 P_n$. Let $\{\ast\} \times \lambda_i$ and $S^1 \times \{\ast\}$ denote generators of $\pi_1(S^1 \times \partial D^2)$. Let $\text{Diff}(S^1 \times P_n)$ be the group of diffeomorphisms of $S^1 \times P_n$ whose restriction to the external boundary are equal to the identity $\text{Id}_{S^1 \times \partial D^2}$ and whose restriction to the internal boundary $S^1 \times \partial(\text{img}(b))$ sends $\{1\} \times \eta_i$ to a curve isotopic to $\{1\} \times \eta_{\sigma(i)}$ for all $i \in \{1, 2, \cdots, n\}$ where $\sigma : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ is a permutation of $\{1, 2, \cdots, n\}$. Let $\text{PDiff}(S^1 \times P_n)$ denote the group of diffeomorphisms of $S^1 \times P_n$ whose restrictions to the internal boundary are isotopic to the identity and whose restrictions to the external boundary are equal to the identity $\text{Id}_{S^1 \times \partial D^2}$. Similarly, define $\text{PDiff}(S^1 \times P_n)$ to be the group of diffeomorphisms of $S^1 \times P_n$ which restrict to the identity $\text{Id}_{S^1 \times \partial P_n}$. Let $\text{KDiff}(S^1 \times P_n)$ be the subgroup of $\text{PDiff}(S^1 \times P_n)$ consisting of diffeomorphisms having the form $\text{Id}_{S^1 \times f}$ where $f \in \text{KDiff}(P_n)$, and let $\text{KDiff}(S^1 \times P_n)$ denote the subgroup of $\text{Diff}(S^1 \times P_n)$ consisting of diffeomorphisms of the form
\( f = Id_{S^1} \times g \) for \( g \in \widehat{K\text{Diff}}(P_n) \).

**Lemma 27** There is a fiber-homotopy equivalence

\[
\begin{array}{cccc}
\text{PDiff}(S^1 \times P_n) & \longrightarrow & \text{Diff}(S^1 \times P_n) & \longrightarrow & S_n \\
\downarrow & & \downarrow & & \downarrow \\
\text{KDiff}(S^1 \times P_n) & \longrightarrow & \widehat{K\text{Diff}}(S^1 \times P_n) & \longrightarrow & S_n
\end{array}
\]

where all vertical arrows are inclusions (and homotopy equivalences).

**PROOF.** We consider \( S^1 \times P_n \) to be a Seifert fibered manifold. Hatcher (19) proves that the full group of diffeomorphisms of \( S^1 \times P_n \) is homotopy equivalent to the fiber-preserving subgroup. Let \( G \) denote the fiber-preserving subgroup of \( \text{PDiff}(S^1 \times P_n) \). Thus, the inclusion \( G \to \text{PDiff}(S^1 \times P_n) \) is a homotopy equivalence. Since the group of orientation preserving diffeomorphisms of \( S^1 \) is homotopy equivalent to \( SO_2 \), \( G \) is homotopy equivalent to the subgroup \( G' \subset G \) of fibrewise-linear diffeomorphisms of \( S^1 \times P_n \). Since every diffeomorphism in \( \text{PDiff}(S^1 \times P_n) \) restricts to a diffeomorphism of \( \partial(S^1 \times P_n) \) which is isotopic to the identity, \( G' \) is homotopy equivalent to the subgroup of diffeomorphisms of the form \( Id_{S^1} \times f \) where \( f \in \text{PDiff}(S^1 \times P_n) \). The key consideration in the above argument is whether or not \( f \) could be a Dehn twist along a vertical annulus. By Lemma 22, \( \text{PDiff}(P_n) \) is homotopy equivalent to \( \text{KDiff}(P_n) \). The remaining results follow from Lemmas 24 and 22.

As a historical note, some of Hatcher’s results on diffeomorphism groups of Haken manifolds were independently discovered by Ivanov (24; 25).

The following lemma is used to simplify the proof of Theorem 11. It is a standard variation of a construction of Borel (3) (chapter IV, §3).

**Lemma 28** If \( G \) is a topological group with \( H \) a closed normal subgroup such that \( G/H \) is a finite group, then there exists a canonical normal, finite-sheeted covering space

\[
G/H \to BH \to BG
\]

where the map \( BH \to BG \) is given by the projection \( EG/H \to EG/G \) where we make the identification \( BH = EG/H \).

First, we sketch the proof of Theorem 11. The fact that the map \( \sqcup_{n=0}^{\infty} \kappa_n \) induces a bijection

\[
\sqcup_{n \in \{0,1,2,3,\ldots\}} \pi_0 \left( (C_2(n) \times \mathcal{P}^n) / S_n \right) \to \pi_0 \hat{K}
\]

\[22\]
is due to Schubert (44). His theorem states that every long knot decomposes uniquely into a connected-sum of prime knots, up to a re-ordering of the terms. Since the map \( \sqcup_{n=0}^{\infty} \kappa_n \) is bijective on components, we need only to verify that it is a homotopy equivalence when restricted to any single connected component. By Lemma 20, the components of both the domain and range are \( K(\pi,1) \)'s. So we have reduced the theorem to checking that the induced map is an isomorphism of fundamental groups for every component. The inspiration for the proof of this is the fibration below, which we call the little cubes fibration.

\[
S_n \to \mathcal{C}_2(n) \times \mathcal{P}^n \to (\mathcal{C}_2(n) \times \mathcal{P}^n)/S_n
\]

Let \( f \in \hat{\mathcal{K}} \) with \( f = f_1 \# f_2 \# \cdots \# f_n \), where \( (f_1, \cdots, f_n) \in \mathcal{P}^n \) are the prime summands of \( f \). Let \( \hat{\mathcal{K}}_f \) denote the component of \( \hat{\mathcal{K}} \) containing \( f \), similarly define \( \hat{\mathcal{K}}_{f_i} \). Thus the above fibration, when restricted to the connected component \( \mathcal{C}_2(n) \times \prod_{i=1}^{n} \hat{\mathcal{K}}_{f_i} \) of \( \mathcal{C}_2(n) \times \mathcal{P}^n \), has the form:

\[
\Sigma_f \to \mathcal{C}_2(n) \times \prod_{i=1}^{n} \hat{\mathcal{K}}_{f_i} \to (\mathcal{C}_2(n) \times \prod_{i=1}^{n} \hat{\mathcal{K}}_{f_i})/\Sigma_f
\]

where \( \Sigma_f \subset S_n \) is the subgroup which preserves the partition \( \sim \) of \( \{1, 2, \cdots, n\} \) with \( i \sim j \iff \hat{\mathcal{K}}_{f_i} = \hat{\mathcal{K}}_{f_j} \).

By Lemma 20 the little cubes fibration gives the short exact sequence below.

\[
0 \to \pi_1 \mathcal{C}_2(n) \times \prod_{i=1}^{n} \pi_1 \hat{\mathcal{K}}_{f_i} \to \pi_1((\mathcal{C}_2(n) \times \prod_{i=1}^{n} \hat{\mathcal{K}}_{f_i})/\Sigma_f) \to \Sigma_f \to 0
\]

\( \pi_1 \hat{\mathcal{K}}_f \simeq \pi_0 \text{Diff}(C, T) \) by Lemma 20. So the idea of the proof is to find an analogous fibration for \( \hat{\mathcal{K}}_f \). So we are looking for an epimorphism \( \pi_0 \text{Diff}(C, T) \to \Sigma_f \).

Since the permutation \( \sigma_g : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\} \) by the condition that \( \sigma_g(i) = j \) if \( g(T_i) \) is isotopic to \( T_j \) where \( T_1, T_2, \cdots, T_n \) are the base-level of the JSJ-decomposition of \( C \). This is well-defined since \( g \) fixes \( T = \partial C \) and the JSJ-decomposition is unique up to isotopy. The homomorphism \( \sigma : \pi_0 \text{Diff}(C, T) \to S_n \) is onto \( \Sigma_f \) since two long knots \( f_i \) and \( f_j \) are isotopic if and only if \( C_i \) and \( C_j \) admit orientation preserving diffeomorphisms which also preserve the (oriented) meridians of \( C_i \) and \( C_j \).

The kernel of \( \sigma \) one would expect to be the mapping class group of diffeomorphisms of \( C \) which do not permute the base-level of the JSJ-splitting.
of \( C \). Such a diffeomorphism \( g \), when restricted to \( V \simeq S^1 \times P_n \) can isotoped to be in \( \text{KDiff}(S^1 \times P_n) \). Thus \( g \) restricts to diffeomorphisms \( g_{|C_i} \in \text{Diff}(C_i, T_i) \) for all \( i \in \{1, 2, \cdots, n\} \), leading us to expect the kernel of \( \sigma \) is \( \pi_0 \text{PDif}(P_n) \times \prod_{i=1}^n \pi_0 \text{Diff}(C_i, T_i) \). By Lemma 20 \( \pi_0 \text{Diff}(P_n) \simeq \pi_1 C_2(n) \) and \( \pi_0 \text{Diff}(C_i, T_i) \simeq \pi_1 K_{f_i} \) where \( f_i \) denotes the \( i \)-th summand of \( f \). So we have constructed a SES

\[ 0 \to \pi_1 C_2(n) \times \prod_{i=1}^n \pi_1 K_{f_i} \to \pi_1 \hat{K}_f \to \Sigma_f \to 0 \]

which is the analogue of the SES coming from the little cubes fibration.

In the argument below, we rigorously redo the above sketch at the space-level. We construct a fibration of diffeomorphism groups whose long exact sequence is the SES given above. We then use Lemma 28 to convert this fibration of diffeomorphism groups into a fibration which describes \( \hat{K}_f \), and this we will show is equivalent to the little cubes fibration.

**PROOF.** (of Theorem 11)

We will show that \( \sqcup_{n=0}^{\infty} \kappa_n \) is a homotopy equivalence, component by component. Let \( f \in \hat{K} \) be a knot specifying a connected component \( \hat{K}_f \) of \( \hat{K} \).

In the case of the unknot \( f = \text{Id}_{\mathbb{R} \times \mathbb{D}^2} \), we know from the proof of the Smale conjecture (20) that the component of \( \hat{K} \) containing \( f \) is contractible. \( C_2(0) \times \mathcal{P}^0 \) is a point thus the map \( \kappa_0 \) is a homotopy equivalence between these two components.

If \( f \) is a prime knot, \( n = 1 \) and the little cubes fibration \( S_1 \to C_2(1) \times \mathcal{P}^1 \to (C_2(1) \times \mathcal{P}^1)/S_1 \) is trivial, thus \( \hat{K}_f \) is a component of \( \mathcal{P} \). In this case, our map \( \kappa_1 : C_2(1) \times \mathcal{P} \to \hat{K} \) is mapping from \( C_2(1) \times \mathcal{P} \) to \( \hat{K} \). Since \( C_2(1) \) is contractible and our action satisfies the identity axiom, \( \kappa_1 \) is homotopic to the composite of the projection map \( C_2(1) \times \mathcal{P} \to \mathcal{P} \) with the inclusion map \( \mathcal{P} \to \hat{K} \), and so \( \kappa_1 \) is a homotopy equivalence between \( (C_2(1) \times \mathcal{P})/S_1 \) and \( \mathcal{P} \).

Consider the case of a composite knot \( f = f_1 \# f_2 \# \cdots \# f_n \in \hat{K} \) for \( n \geq 2 \) with \( f_i \) prime for all \( i \in \{1, 2, \cdots, n\} \). Let \( C = B - N' \) denote the knot complement, as in Lemma 17. Let \( T = \partial C \), let \( V \simeq S^1 \times P_n \) denote the root manifold of the associated tree to the JSJ-decomposition of \( C \) and let \( T_1, \cdots, T_n \) denote base-level of the JSJ-decomposition of \( C \) (see Lemma 17, Definition 18). Similarly, let \( V \simeq S^1 \times P_n, B_i \) and \( C_i \) for \( i \in \{1, 2, \cdots, n\} \) be as in Lemma 17. Let \( \text{Diff}(C, T) \) be the group of diffeomorphisms of \( C \) that fix \( T \) point-wise. Let \( \text{Diff}^V(C, T) \) denote the subgroup of \( \text{Diff}(C, T) \) consisting of diffeomorphisms which restrict to diffeomorphisms of \( V \). Let \( \text{PDiff}^V(C, T) \) denote the subgroup of \( \text{Diff}^V(C, T) \) consisting of diffeomorphisms whose restrictions to \( \partial V \) are iso-
topic to $Id_{gV}$. Let $\text{Emb}(\sqcup_{i=1}^{n} T_i, C)$ denote the space of embeddings of $\sqcup_{i=1}^{n} T_i$ in $C$. If we restrict a diffeomorphism in $\text{Diff}(C, T)$ to $\sqcup_{i=1}^{n} T_i$ and mod-out by the parametrization of the individual tori, we get a fibration (which is not necessarily onto)

$$\text{PDiff}^V(C, T) \rightarrow \text{Diff}(C, T) \rightarrow \text{Emb}(\sqcup_{i=1}^{n} T_i, C)/\prod_{i=1}^{n} \text{Diff}(T_i)$$

Since $T_i$ is incompressible in $C$, this fibration is mapping to embeddings which are also incompressible. The tori $\sqcup_{i=1}^{n} T_i$ are part of the JSJ-splitting of $C$, and the JSJ-splitting is unique up to isotopy. This means that a diffeomorphism in $\text{Diff}(C, T)$ must send $T_i$ to another torus in the JSJ-splitting (up to isotopy), but more importantly that torus must be in the base-level of the JSJ-splitting since the diffeomorphism is required to preserve $T$.

A component of $\text{Emb}(\sqcup_{i=1}^{n} T_i, C)/\prod_{i=1}^{n} \text{Diff}(T_i)$ is an isotopy class of $n$ embedded, labeled tori. Provided the tori are incompressible, such a component must be contractible (19). Consider the union $X$ of all the components of $\text{Emb}(\sqcup_{i=1}^{n} T_i, C)/\prod_{i=1}^{n} \text{Diff}(T_i)$ which correspond to embeddings whose image are the base-level of the JSJ-splitting of $C$. $X$ must have the homotopy type of the symmetric group $S_n$. Consider $S_n$ to be the subspace $S_n \equiv \text{Diff}(\sqcup_{i=1}^{n} T_i)/\prod_{i=1}^{n} \text{Diff}(T_i) \simeq X \subset \text{Emb}(\sqcup_{i=1}^{n} T_i, C)/\prod_{i=1}^{n} \text{Diff}(T_i)$.

The above argument proves that there is a fiber-homotopy equivalence, where all the vertical arrows are given by inclusion.

$$\text{PDiff}^V(C, T) \xrightarrow{\text{Diff}(C, T)} X$$

Typically it is demanded that fibrations are onto. Since the long knot $f$ is a connected-sum, and some of the summands $\{f_i : i \in \{1, 2, \cdots, n\}\}$ may be repeated, define the equivalence relation $\sim$ on $\{1, 2, \cdots, n\}$ by $i \sim j \iff f_i$ is isotopic to $f_j$. Let $\Sigma_f \subset S_n$ be the partition-preserving subgroup of $S_n$. Thus the above fibration is onto $\Sigma_f \subset S_n$.

Since every diffeomorphism $g \in \text{PDiff}^V(C, T)$ restricts to a diffeomorphism of $V$, consider the restriction to $V \simeq S^1 \times P_n$. Since the $g$ extends to a diffeomorphism of $\mathbb{R}^3$, $g_V : V \rightarrow V$ must preserve (up to isotopy) the longitudes and meridians of each $T_i$. To be precise, a meridian of each $T_i$ is an oriented closed essential curve in $T_i$ which bounds a disc in $\mathbb{R}^3 - \text{int}(C_i)$. The orientation of the meridian is chosen so that the linking number of the meridian with the knot is $+1$. A longitude in $T_i$ is an essential oriented curve in $T_i$ which bounds
a Seifert surface in \( C_i \). The orientation of the curve is chosen to agree with the orientation of \( f_i \).

Thus, if we identify \( V \) with \( S^1 \times P_n \) in a way that sends knot meridians to fibers of \( S^1 \times P_n \) and the longitude of \( f_i \) to \( \{1\} \times \eta_i \subset S^1 \times P_n \) for all \( i \in \{1, 2, \ldots, n\} \) then (by a slight abuse of notation) \( g|_{S^1 \times P_n} \in \text{PDiff}(S^1 \times P_n) \).

Define \( \text{KDiff}^V(C, T) \subset \text{PDiff}^V(C, T) \) and \( \overline{\text{KDiff}}^V(C, T) \subset \text{Diff}^V(C, T) \) to be the subgroups such that each diffeomorphism \( g \) restricts to a diffeomorphism of \( V \equiv S^1 \times P_n \), \( g|_{S^1 \times P_n} \in \text{KDiff}(S^1 \times P_n) \) and \( g|_{S^1 \times P_n} \in \overline{\text{KDiff}}(S^1 \times P_n) \) respectively. By Lemma 27, the vertical inclusion maps in the diagram below give a fiber-homotopy equivalence

\[
\begin{array}{ccc}
\text{PDiff}^V(C, T) & \longrightarrow & \text{Diff}^V(C, T) \\
\approx & & \approx \\
\text{KDiff}^V(C, T) & \longrightarrow & \overline{\text{KDiff}}^V(C, T) \\
\end{array}
\]

Analogously to Lemma 22, the inclusion \( \text{KDiff}(S^1 \times P_n) \times \prod_{i=1}^n \text{Diff}(C_i, T_i) \to \text{KDiff}^V(C, T) \) is a homotopy equivalence.

If we apply Lemma 28 to the above fibration, we get the normal covering space

\[
\begin{array}{ccc}
\Sigma_f & \longrightarrow & \text{BKDiff}(S^1 \times P_n) \times \prod_{i=1}^n \text{BDiff}(C_i, T_i) \\
\approx & & \approx \\
C_2(n) \times \prod_{i=1}^n \hat{\mathcal{K}}_{f_i} & \longrightarrow & \hat{\mathcal{K}}_{f}
\end{array}
\]

where the two vertical homotopy equivalences come from Lemma 20 and the identification \( \text{KDiff}(S^1 \times P_n) \equiv \text{KDiff}(P_n) \).

Consider \( C_2(n) \times \prod_{i=1}^n \hat{\mathcal{K}}_{f_i} \) as a \( \Sigma_f \)-space, where the \( \Sigma_f \) action is simply the restriction of the diagonal \( S_n \times (C_2(n) \times \hat{\mathcal{K}}^n) \to C_2(n) \times \hat{\mathcal{K}}^n \) to \( \Sigma_f \times (C_2(n) \times \prod_{i=1}^n \hat{\mathcal{K}}_{f_i}) \to C_2(n) \times \prod_{i=1}^n \hat{\mathcal{K}}_{f_i} \). By design, the homotopy equivalence \( \text{BKDiff}^V(S^1 \times P_n) \times \prod_{i=1}^n \text{BDiff}(C_i, T_i) \to C_2(n) \times \prod_{i=1}^n \hat{\mathcal{K}}_{f_i} \) is \( \Sigma_f \)-equivariant (see Definition 25).

Thus we know abstractly that there exists a homotopy equivalence between \( (C_2(n) \times \prod_{i=1}^n \hat{\mathcal{K}}_{f_i})/\Sigma_f \) and \( \hat{\mathcal{K}}_f \). To finish the proof, we show \( \kappa_n : (C_2(n) \times \prod_{i=1}^n \hat{\mathcal{K}}_{f_i})/\Sigma_f \to \hat{\mathcal{K}}_f \) is such a homotopy equivalence. Since both the domain and range of \( \kappa_n \) are \( K(\pi, 1) \)'s, it suffices to show that the diagram below commutes.
\[ \pi_1 \text{BKDiff}(S^1 \times P_n) \times \prod_{i=1}^n \pi_1 \text{BDiff}(C_i, T_i) \xrightarrow{\simeq} \pi_1 \text{BKDiff}^V(C, T) \]

\[ \pi_1 \mathcal{C}(n) \times \prod_{i=1}^n \pi_1 \hat{K}_{f_i} \xrightarrow{\pi_1 \kappa_n} \pi_1 \hat{K}_f \]

Fix \( i \in \{1, 2, \ldots, n\} \) and \( \phi \in \pi_0 \text{Diff}(C_i, T_i) \). Consider \( \phi \) to be an element of \( \pi_1 \text{BKDiff}(S^1 \times P_n) \times \prod_{i=1}^n \pi_1 \text{BDiff}(C_i, T_i) \) by the standard inclusion. If one chases \( \phi \) along the clockwise route around the diagram to \( \pi_1 \hat{K}_f \), one is simply converting \( \phi \) into an element \( \overline{\phi} \in \pi_1 \hat{K}_f \) using Lemma 20. This means that one is applying an isotopy to the \( i \)-th knot summand \( f_i \) of \( f \), and the isotopy has support in \( B_i \) (see Lemma 17). If one chases \( \phi \) along the counter-clockwise route around the diagram, one converts \( \phi \) into a loop in \( \pi_1 \hat{K}_{f_i} \) using Lemma 20, then the little cubes construction is applied to this loop creating a second loop \( \hat{\phi} \in \pi_1 \hat{K}_f \). The loop produced via the little cubes construction \( \hat{\phi} \) is the same loop in \( \pi_1 \hat{K}_f \) as \( \overline{\phi} \) since the little cubes and other knot summands remain fixed through the isotopy, keeping the support of the isotopy in \( B_i \).

Given \( \theta \in \pi_0 \text{Diff}(S^1 \times P_n) \) consider it as an element of \( \pi_1 \text{BKDiff}(S^1 \times P_n) \times \prod_{i=1}^n \pi_1 \text{BDiff}(C_i, T_i) \) by the standard inclusion. We will chase \( \theta \) around the diagram. This chase is a little more involved than the previous one, as it involves the little cubes action on \( \hat{K} \) in a non-trivial manner.

Our strategy for the proof is to chase \( \theta \) around the diagram in a counter-clockwise manner to get an element in \( \pi_0 \text{BKDiff}^V(C, T) \). We denote this diffeomorphism by \( C_{\theta} \). We need to show that \( C_{\theta} \) is the identity on \( \bigcup_{i=1}^n C_i \) and when restricted to \( V \), \( C_{\theta}|_{\partial V} \equiv \theta \) under our identification \( V \equiv S^1 \times P_n \). We will do this via an explicit computation. First, notice that we can simplify the problem. \( \theta \) determines a loop \( \overline{\theta} \in \pi_1 \mathcal{C}(n) \) which in turn defines an isotopy \( \kappa_n(\overline{\theta}, f_1, f_2, \ldots, f_n) \) of \( f \), which by Lemma 20 determines the diffeomorphism \( C_{\theta} \) of \( C \). Recall how \( C_{\theta} \) is constructed. Given an isotopy \( F_{\theta} : [0, 1] \times B \to B \) such that

- \( F_{\theta}(0, x) = x \) for all \( x \in B \)
- \( F_{\theta}(t, x) = x \) for all \( x \in T = \partial B \) and \( t \in [0, 1] \)
- \( F_{\theta}(t, x) = \kappa_n(\overline{\theta}(t), f_1, f_2, \ldots, f_n)(x) \) for all \( (t, x) \in [0, 1] \times B \).

Then \( C_{\theta}(x) = F_{\theta}(1, x) \) for \( x \in C \).

Define \( T_{\theta} : B \to B \) by \( T_{\theta}(x) = F_{\theta}(1, x) \) for \( x \in B \). \( \pi_0 \text{Diff}(S^1 \times P_n) \approx \pi_0 \text{Diff}(P_n) \) is the pure braid group which can be in turn thought of as a subgroup of the full braid group, \( \pi_0 \text{BKDiff}(S^1 \times P_n) \approx \pi_0 \text{BKDiff}(P_n) \approx \pi_0 \text{Diff}(P_n) \).

In \( \pi_0 \text{Diff}(P_n) \) every element can be written as a product of Artin generators \( \{\sigma_i : i \in \{1, 2, \ldots, n-1\}\} \) (see for example (1)), these are the half Dehn twists about curves bounding the \( i \)-th and \( (i+1) \)-st punctures of \( P_n \). Let
\[ \theta = \alpha_j \circ \alpha_{j-1} \circ \cdots \circ \alpha_1 \] where \( \alpha_i \in \text{Diff}(P_n) \) are either Artin generators or their inverses, thus \( T_{\theta} = T_{\alpha_j} \circ T_{\alpha_{j-1}} \circ \cdots \circ T_{\alpha_1} \). This in principle reduces our problem to studying \( T_{\sigma_i} \) for \( i \in \{1, 2, \ldots, n-1\} \).

By the definition of \( \kappa_n \), \( T_{\sigma_i} \) is the identity on the balls \( B_k \) for \( k \notin \{i, i+1\} \), and \( T_{\sigma_i} \) permutes the two balls \( B_i \) and \( B_{i+1} \), acting by translation. Thus \( T_{\theta} \) must restrict to be the identity on \( \sqcup_{i=1}^n C_i \).

Let \( * = (0, -1, 0) \in \partial B \) be the base-point of \( B \). Let \( \xi_i : [0, 1] \to B \) be the unique affine-linear function so that \( \xi_i(0) = * \) and \( \xi_i(1) = \left( \frac{4i-2n-2}{2n+1}, -\frac{1}{2n+1}, 0 \right) \in \partial B_i \). Let \( p_i : S^1 \to C_i \) be a longitude of \( C_i \) starting and ending at \( \xi_i(1) \). Since \( T_{\sigma_i} \) acts by translation on the balls \( \{B_i : i \in \{1, 2, \ldots, n\}\} \), for all \( k \in \{0, 1, 2, \ldots, j\} \) define the \( i \)-th longitude \( p^k_i \) of \( (\alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1)(C) \) to be the restriction of \( \sqcup_{s=1}^n (\alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1 \circ p_s) : \sqcup_{s=1}^n S^1 \to B \) to \( (\sqcup_{s=1}^n \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1 \circ p_s)^{-1}(B_i) \). Define \( l_i = \xi_i \cdot p_i \cdot \xi_i \) and similarly \( l^k_i = \xi_i \cdot p^k_i \cdot \xi_i \), so \( l^0_i = l_i = l^k_i \) for all \( i \in \{1, 2, \ldots, n\} \).

\[ \pi_1 \left((\alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1)(C)\right) \] therefore has a natural identification with \( \mathbb{Z} \times (\ast_{i=1}^n \mathbb{Z}) \) which has presentation \( \langle m, l^1_i, l^2_i, \ldots, l^k_i : [m, l^1_i], [m, l^2_i], \ldots, [m, l^k_i] \rangle \). Here \( m \) is a knot meridian, or equivalently a fiber of the Seifert fibering of the base-manifold of the JSJ-splitting of \( C \).

Call the above identification \( \phi_k : \pi_1 \left((\alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1)(C)\right) \to \mathbb{Z} \times (\ast_{i=1}^n \mathbb{Z}) \). \( \phi_k \) determines a diffeomorphism \( \tilde{\phi}_k : (\alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1)(C) \to S^1 \times P_n \) defined
by the condition that $\tilde{\phi}_k(l^k_i) = \{1\} \times \lambda_i$, $\tilde{\phi}_k(m) = S^1 \times \{\ast\}$.

Recall the Dehn-Nielsen theorem (40) (see (53) for a modern proof). It states that the map $\pi_0\text{Diff}(P_n) \to \text{Aut}(\pi_1 P_n)$ is injective. We compute the induced automorphism on $\mathbb{Z} \times (\ast_{i=1}^n \mathbb{Z})$ given by the composite $\tilde{\phi}_{k+1} \circ T_{\alpha_{k+1}} \circ \tilde{\phi}_{k}^{-1}$. Without loss of generality, assume $\alpha_{k+1} = \sigma_q$ for some $q \in \{1, 2, \ldots, n-1\}$, therefore $\kappa_n(\alpha_{k+1}, f_1, f_2, \ldots, f_n)$ represents an isotopy which pulls the knot summand in the ball $B_{q+1}$ through the knot summand in the ball $B_q$. Therefore, $\pi_1(\tilde{\phi}_{k+1} \circ T_{\alpha_q} \circ \tilde{\phi}_{k}^{-1})$ fixes $m$ and fixes $\lambda_i$ unless $i \in \{q, q+1\}$, in which case $(\tilde{\phi}_{k+1} \circ T_{\alpha_q} \circ \tilde{\phi}_{k}^{-1})(\lambda_{q+1}) = \lambda_{q+1}\lambda_q\lambda_{q+1}^{-1}$ and $(\tilde{\phi}_{k+1} \circ T_{\alpha_q} \circ \tilde{\phi}_{k}^{-1})(\lambda_{q+1}) = \lambda_q$.

Thus, via our identifications, $C_\theta \in \text{KDiff}^V(C, T)$ induces the same automorphism of $\pi_1 V \equiv \pi_1(S^1 \times P_n)$ as does $\theta \in \text{KDiff}(S^1 \times P_n)$, which proves the theorem.

**Corollary 29** There is a little 2-cubes equivariant homotopy equivalence

$$\text{EC}(1, D^2) \simeq C_2(\mathcal{P} \sqcup \{\ast\}) \times \Omega^2 \mathbb{C}P^\infty$$

where $\mathbb{C}P^\infty = BS^1 = B^2\mathbb{Z}$.

4 Where from here?

There are several directions one could go from here. One direction would be to ask, what is the homotopy type of the full space $\mathcal{K}$? By Theorem 11 this is equivalent to asking what is the homotopy type of $\mathcal{P}$ but Theorem 11 can be used to refine this question further.

Starting with the unknot, one can produce new knots by: using hyperbolic satellite operations, cablings, or taking the connected-sum of knots. If these
procedures are iterated, one produces all knots \((49; 26; 7)\). Theorem 11 tells us the homotopy-type of a component corresponding to a knot which is a connect-sum. If \(f \sim f_1 \# \cdots \# f_n\) is the prime decomposition of \(f\), then \(\mathcal{K}_f \simeq (\mathbb{C}_2(n) \times_{S_n} \prod_{i=1}^n \mathcal{K}_{f_i})\). To complete our understanding of \(\mathcal{K}\) all we need to understand is:

1. How the homotopy type of \(\mathcal{K}_f\) is related to the homotopy type of \(\mathcal{K}_g\) if \(f\) is a cabling of \(g\).
2. If \(f\) is obtained from knots \(\{f_i : i \in \{1, 2, \cdots, n\}\}\) via a hyperbolic satellite operation, how is the homotopy type of \(\mathcal{K}_f\) related to \(\mathcal{K}_{f_i}\) for \(i \in \{1, 2, \cdots, n\}\).

Hatcher has answered question 1.

**Theorem 30 (Hatcher) (22)** If a knot \(f\) is a cabling of a knot \(g\) then \(\mathcal{K}_f \simeq S^1 \times \mathcal{K}_g\)

More recently, a solution to question 2 has appeared in (8). Roughly, if a knot \(f\) is obtained from knots \(\{f_i : i \in \{1, 2, \cdots, n\}\}\) by a hyperbolic satellite operation then there is a fibration

\[
\prod_{i=1}^n \mathcal{K}_{f_i} \to \mathcal{K}_f \to S^1 \times S^1
\]

and the monodromy of this fibration depends on both the knots \(f_i\), their symmetry properties, and the symmetry properties of the hyperbolic manifold that is the root of the JSJ-tree of \(f\). For brevity, we skip the full statement of the result. A key theorem of Sakuma’s is used to compute the monodromy of this fibration – allowing us to show the fibration is split at the base, thus the fundamental group of any component of \(\mathcal{K}\) is an iterated semi-direct product of finite-index subgroups of braid groups.

More generally, one could ask, what is the homotopy type of other spaces of knots?

Perhaps the next simplest case is the space of embeddings of a circle in a sphere \(\operatorname{Emb}(S^1, S^n)\). As is shown in (6), there is a homotopy equivalence \(\operatorname{Emb}(S^1, S^n) \simeq \operatorname{Emb}(\mathbb{R}, \mathbb{R}^n) \times_{SO_{n-1}} SO_{n+1}\). Thus, if one knows the homotopy type of \(\operatorname{Emb}(\mathbb{R}, \mathbb{R}^n)\) as an \(SO_{n-1}\)-space, one knows the homotopy type of \(\operatorname{Emb}(S^1, S^n)\). The homotopy-type of \(\mathcal{K}\) as an \(SO_2\)-space is determined in (8).

Another interesting question is ‘what is the homotopy type of the space of closed, connected, 1-dimensional submanifolds of \(S^n\)? This space is naturally homeomorphic to \(\operatorname{Emb}(S^1, S^n)/\operatorname{Diff}(S^1)\) and has been studied recently by Hatcher (22) in the \(n = 3\) case. Studying the homotopy type of these spaces
appears to have more complications due to the delicate extension problems involved. An interesting point of Hatcher’s work is that one needs to know the answer to the Linearization Conjecture in order to understand even the homotopy type of the component of a knot as simple as a hyperbolic knot. One could go further and ask, what is the homotopy-type of the double-coset space \( SO_{n+1} \setminus \text{Emb}(S^1, S^n) / \text{Diff}(S^1) \)? This is a particularly delicate problem as the action of \( SO_{n+1} \times \text{Diff}(S^1) \) on \( \text{Emb}(S^1, S^n) \) is not free. A nice example of the kinds of problems that can arise is the paper of Kodama and Michor (30), where they prove that the figure-8 component of \( \text{Imm}(S^1, \mathbb{R}^2) / \text{Diff}(S^1) \) has the homotopy-type of \( CP^\infty \).

It would be very interesting to know more about the homotopy-type of the embedding spaces \( \text{Emb}(\mathbb{R}^j, \mathbb{R}^n) \) or \( \text{Emb}(S^j, S^n) \). Unfortunately the techniques of this paper are of limited use since it is still unknown whether or not a smooth embedded 3-sphere in \( \mathbb{R}^4 \) bounds a smooth ball (27), and very little is known about the homotopy type of \( \text{Diff}(D^4) \) other than Morlet’s ‘Comparison’ Theorem (38; 10; 28).

There are however some results known in dimension 4. Sinha and Scannell prove have computed many rational homotopy-groups of the long knot space \( \text{Emb}(\mathbb{R}, \mathbb{R}^4) \) and the corresponding framed long knot space \( \text{EC}(1, D^3) \), showing non-triviality in dimensions \( \{2, 4, 5, 6\} \). The fibration \( \text{Diff}(D^4) \to \text{EC}(1, D^3) \) has a fiber which is homotopy equivalent to \( \text{Diff}(S^2 \times D^2) \) (diffeomorphisms fixing the boundary). The homotopy LES of this fibration splits into short exact sequences \( 0 \to \pi_{i+1} \text{EC}(1, D^3) \to \pi_i \text{Diff}(S^2 \times D^2) \to \pi_i \text{Diff}(D^4) \to 0 \). We can deduce from this that \( \pi_i \text{Diff}(S^2 \times D^2) \) has non-torsion elements for \( i \in \{1, 3, 4, 5\} \). By Theorem 5 we know that \( \text{Diff}(S^2 \times D^2) \simeq \text{EC}(2, S^2) \) is a 3-fold loop space. Three-dimensional instincts might lead one to suspect that the inclusion \( \Omega^2 SO_3 \subset \text{Diff}(S^2 \times D^2) \) is a homotopy equivalence, where \( \Omega^2 SO_3 \) is thought of as the subgroup of fiber-preserving (fibrewise-linear) diffeomorphisms of \( S^2 \times D^2 \). These instincts would be wrong! We have just seen that although the inclusion \( \Omega^2 SO_3 \to \text{Diff}(S^2 \times D^2) \) admits a 3-fold de-looping, it cannot be a homotopy equivalence since the homotopy groups of the domain and range are not the same.

A possible application of the Sinha, Scannell result would be the study of ‘spun’ knots. Given \( f \in \pi_i \text{Emb}(\mathbb{R}, \mathbb{R}^n) \) one constructs a smooth embedding \( S^{i+1} \to \mathbb{R}^{n+i} \) by ‘spinning’ \( f \) about an \((n-1)\)-dimensional linear subspace of \( \mathbb{R}^{n+i} \) (this is a slight generalization of Litherland’s notion of deform twist-spun knots (33), see Figure 18). In the spirit of Markov’s Theorem (1), it would seem natural to conjecture that for some co-dimensions the ‘spinning map’

\[
\pi_i \text{Emb}(\mathbb{R}, \mathbb{R}^n) \to \pi_0 \text{Emb}(S^{i+1}, \mathbb{R}^{n+i})
\]

is an isomorphism. In the time between this article being accepted and published, some progress has been made on this problem. It turns out that, pro-
vided $2n - 3j - 3 \geq 0$ the first non-trivial homotopy group of $\text{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ is cyclic and in dimension $2n - 3j - 3$. Moreover in these cases, a spinning construction $\Omega\text{Emb}(\mathbb{R}^j, \mathbb{R}^n) \to \text{Emb}(\mathbb{R}^{j+1}, \mathbb{R}^{n+1})$ induces an epi-morphism on the first non-trivial homotopy groups of the spaces. In particular, the spinning map $\pi_2\text{Emb}(\mathbb{R}, \mathbb{R}^4) \to \pi_0\text{Emb}(S^3, \mathbb{R}^6)$ is an isomorphism – both groups are infinite-cyclic in this case (9).

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