On the theory of astronomical masers – I. Statistics of maser radiation

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ABSTRACT
In this paper, we re-analyse the amplification process of broad-band continuum radiation by astronomical masers in the one-dimensional case. The basic equations appropriate for the scalar maser and the random nature of the maser radiation field are derived from basic physical principles. Comparison with the standard radiation transfer equation allows us to examine the underlying assumptions involved in the current theory of astronomical masers. Simulations are carried out to follow the amplification of different realizations of the broad-band background radiation by the maser. The observable quantities such as intensity and spectral line profile are obtained by averaging over an ensemble of the emerging radiation corresponding to the amplified background radiation field. Our simulations show that the fluctuations of the radiation field inside the astronomical maser deviate significantly from Gaussian statistics even when the maser is only partially saturated. Coupling between different frequency modes and the population pulsing are shown to have increasing importance in the transport of maser radiation as the maser approaches saturation. Our results suggest that the standard formulation of radiation transfer provides a satisfactory description of the intensity and the line narrowing effect in the unsaturated and partially saturated masers within the framework of the one-dimensional model. However, the application of the same formulation to the strong saturation regime should be considered with caution.

Key words: line: formation – masers – polarization – radiative transfer.

1 INTRODUCTION
Since its discovery in the interstellar medium, maser emission has been used as probe to study the different aspects of astronomical environments. Strong maser emission has even provided detailed pictures of the central region of distant galaxies. There had been great expectation that masers could also provide information on the physical properties of the molecular gas at scales inaccessible by other observational means. Theoretical studies have been carried out to find the pumping mechanisms and the necessary conditions for the existence of masers. However, due to the strongly non-linear nature of masers, the relation between observed characteristics and the physical conditions of the masing medium is difficult to determine.

At the fundamental level, the physics of astronomical masers is very similar to that of laboratory lasers, which are masers operating at optical wavelengths. Most of the properties of laboratory lasers are well described by the semiclassical model of Lamb (1964). The feature that distinguishes laser radiation from classical light sources is the remarkable degree of spatial and temporal coherence largely due to the nature of stimulated emission and the use of a high-Q cavity and other means of stabilization. The statistics of laser radiation has been shown to follow a Poisson-like distribution (Arechi 1965). The similarity between the laboratory laser and the astronomical maser has prompted speculations that maser radiation might not be Gaussian. However, previous observational attempts (Evans et al. 1972; Moran 1981) to measure the non-Gaussian statistics of maser radiation have proved negative.

It was recognized very early in the study of astronomical masers (Litvak 1970) that the maser radiation field is broad-band. The coherence time set by the bandwidth of the maser radiation is therefore much shorter than the typical time-scale of molecular response. In deriving the transfer equations for maser radiation, two important assumptions are used by Litvak (1970) and later by Goldreich, Keeley & Kwan (1973) in their study of the maser polarization. The first assumption specifies that the field is stationary and different spectral components are not correlated. The second assumption concerns the Gaussian distribution of the electric field components. Goldreich et al. (1973) reason that due to the large dimension of astronomical masers, the phase shift between waves propagating in slightly different directions is so enormous

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that the waves become completely independent and random. The electric field seen by masing molecules is the superposition of independent and random electromagnetic waves and therefore will have Gaussian statistics according to the well-known central limit theorem. These two main assumptions allow considerable simplification in deriving the transfer equations. Although several theoretical studies of maser radiation are based on these assumptions, their validity has not been fully examined, especially in the saturation regime where analytical and numerical studies have been regularly carried out. We note here that the work by Field & Richardson (1984) and Field & Gray (1988) did try to take into account the partial coherence of the maser radiation in an approximate way. Their analysis suggests some difference in the predicted maser intensity in comparison with the commonly used maser theory. Furthermore, the recent debate on the theoretical descriptions of the polarization properties of astronomical masers (Watson 1994; Elitzur 1995) also underlines the urgent need to re-examine the foundations of the current maser theory.

To simulate from first principles the amplification of radiation by a masing medium is quite complicated. The radiation field has to be considered as an ensemble of all possible statistically independent realizations. Studying the amplification of maser radiation then requires a dramatic increase in computing power to follow the evolution of a large number of realizations of the incident radiation. However, from these samples, we can study the effect of amplification on the statistical distribution of the radiation field and check the validity of the above-mentioned assumptions. The physical model to describe the amplification of broad-band radiation by a one-dimensional maser has been formulated by Menegozzi & Lamb (1978). Using simple analysis they show that the Gaussian statistics cannot be maintained as the radiation is amplified and begins to affect the population inversion of the maser. Menegozzi & Lamb carried out the simulations in several cases including a medium with Doppler broadening, which is relevant to astronomical masers. Due to limited computing power, they could simulate only a handful of realizations of the radiation field, and therefore, the results are still inconclusive. It is surprising that no further study has followed the work of Menegozzi & Lamb (1978), particularly in view of the great computing power available today.

A recent work by Gray & Bewley (2003) also deals with the broad-band nature of the maser radiation field. The conclusion reached in their work indicates that different frequency components across the maser line profile are phase-locked even in the partial saturation regime. The phase-locking (or mode locking) effect also means that maser radiation is pulsating. The result is unexpected because the phase-locking of different frequency modes was not seen in the work of Menegozzi & Lamb (1978). Furthermore, in laboratory laser systems the mode locking can only be achieved through external control either with opto-electronic components (Q-switch) or with passive elements such as saturable absorbers. We could not easily find a clear explanation of the physical mechanism giving rise to this behaviour. We will discuss the model of Gray & Bewley and compare their result with that obtained from our simulations in the following sections.

In this paper, we will use the formulation worked out by Menegozzi & Lamb (1978) to study in detail the astronomical maser and simulate from first principles the amplification of the broad-band background radiation field by the maser. The characteristics of the maser radiation will be determined directly without recourse to any approximations. The generalization of our model to include the vector nature of the radiation field to study the polarization properties of masers is straightforward and will be described in a subsequent paper. We hope that our work will provide a better understanding of the underlying physical principles and the limitations of the current theory of astronomical masers.

2 BASIC THEORY

2.1 Statistical properties of radiation field

The continuum radiation which propagates along the z-axis and arrives on the boundary of the masing medium is the superposition of waves coming from many small, independent sources and therefore is a random function of time. We assume that the incident radiation field \( E(t) \) is stationary, which means that the statistical properties of the field do not change with time. We further assume that the underlying probability distribution of the random variable \( E(t) \) at any instant of time is described by a Gaussian with zero mean and variance \( \langle |E(t)|^2 \rangle = \sigma^2 \), where the brackets denote the ensemble average, or the average taken over a large number of realizations of the field. In practice, we can measure the power of the field by averaging the output of a square-law detector:

\[
I_T = \frac{1}{T} \int_{-T/2}^{T/2} |E(t)|^2 \, dt.
\]  

(1)

Because \( E(t) \) is a random process, \( I_T \) defined as above for each realization of the field during the interval \([−T/2, T/2]\) is also a random variable. As \( T \) tends to infinity, \( I_T \) will approach the average power of the radiation field:

\[
I_T = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |E(t)|^2 \, dt.
\]  

(2)

To relate the statistical properties to time average quantities, we need to restrict the random process \( E(t) \) to be ergodic, i.e. the ensemble average is equal to the time average of any member of the ensemble. Justifications for this assumption constitute the famous ergodic theorem (Doob 1953). The ergodic property is also the underlying assumption of almost all observations in radio astronomy. Therefore we can write

\[
\langle |E(t)|^2 \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |E(t)|^2 \, dt,
\]  

(3)
where the brackets denote the ensemble average. In the following sections of the paper, we shall use the concept of spectral components of the random process \( E(t) \). For that purpose, we might formally use a Fourier transform to decompose \( E(t) \) into harmonic elements \( E(\omega) \):

\[
E(t) = \int_{-\infty}^{+\infty} E(\omega) e^{-i\omega t} \, d\omega,
\]

\[
E(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E(t) e^{i\omega t} \, dt.
\]

(4)

It becomes apparent immediately that because the random function \( E(t) \) does not decay to zero at infinity and the above integrals do not exist. In other words, the function \( E(t) \) is not absolutely integrable. We can circumvent this difficulty by truncating the function \( E(t) \) on a finite interval \([-T/2, T/2]\) and define a new function \( E_T(t) \) which is equal to \( E(t) \) over \(-T/2 \leq t \leq T/2\) and periodic outside this interval. We can now expand \( E_T(t) \) into harmonic components using Fourier series:

\[
E_T(t) = \sum_{n=-\infty}^{n=+\infty} E(\omega_n) e^{-i\omega_n t},
\]

(5)

where the angular frequency \( \omega_n = 2\pi n/T \) and

\[
E(\omega_n) = \frac{1}{T} \int_{-T/2}^{T/2} E(t) e^{i\omega_n t} \, dt.
\]

(6)

Because \( E(t) \) is a random function, the coefficients \( E(\omega_n) \) are new complex random variables. In general, the coefficients \( E(\omega_n) \) are independent of each other only in the limit of large \( T \), i.e. \( \langle E(\omega_n) E(\omega_m) \rangle = 0 \) for \( n \neq m \) (Root & Pitcher 1955). For a random process with a continuum spectrum or ‘white noise’, the values of \( E(t) \) at different instants of time are completely uncorrelated, or the auto-correlation is a delta function. Therefore, the coefficients \( E(\omega_n) \) are Gaussian random variables and independent even for finite values of the interval \( T \) (Rice 1944, 1945; Goldman 1953; Root & Pitcher 1955). Using Perceval’s theorem, we obtain the following relation:

\[
\sum_{n=-\infty}^{n=+\infty} |E(\omega_n)|^2 = \frac{1}{T} \int_{-T/2}^{T/2} |E(t)|^2 \, dt.
\]

(7)

Physically, the term \( |E(\omega_n)|^2 \) can be interpreted as the power of \( E(t) \) contained within the frequency interval \( \Delta \omega = \omega_{n+1} - \omega_n = 2\pi/T \) during the interval \([-T/2, T/2]\). By taking the ensemble average, which means averaging over a large number of realizations of the radiation field, we can define

\[
\frac{c}{8\pi} \langle |E(\omega_n)|^2 \rangle = S_T(\omega) \Delta \omega,
\]

(8)

where \( c \) is the speed of light. In the limit \( T \to \infty \), the quantity \( S_T(\omega) \) approaches the power spectrum of the random process \( E(t) \):

\[
\lim_{T \to \infty} S_T(\omega) = S(\omega).
\]

(9)

The summation in the Fourier series can be formally replaced by the Fourier–Stieltjes integral in the limit that \( T \) tends to infinity by substituting \( a_n \) by \( dZ(\omega) \), the increment of an orthogonal random process \( Z(\omega) \).

\[
E(t) = \lim_{T \to \infty} \int_{-\infty}^{+\infty} e^{-i\omega t} \, dZ(\omega),
\]

(10)

where the limit should be understood in the mean square sense, namely:

\[
\lim_{T \to \infty} \left\langle \left( E(t) - \int_{-\infty}^{+\infty} e^{-i\omega t} \, dZ(\omega) \right)^2 \right\rangle = 0.
\]

(11)

If the random function \( Z(\omega) \) is differentiable, i.e. \( dZ(\omega)/d\omega \) exists, the expression above will reduce to the normal Fourier transform. As we can see from equation (8), the magnitude of \( |dZ(\omega)| \) is \( O(\sqrt{d\omega}) \), many orders of magnitude larger than \( d\omega \). Hence, the derivative of \( Z(\omega) \) does not exist. More rigorous derivation of the spectral representation theorem for random processes can be found in standard texts such as Doob (1953), Yaglom (1962) and Priestley (1981).

In the case of an emission line, the radiation is bandwidth limited around the frequency of the transition. If we take \( \omega_{ab} = 2\pi v_{ab} \), for the sake of simplicity, to be the central frequency of the maser line, the electric field of the radiation propagating along the \( z \) axis can be expressed as follows:

\[
E(z, t) = \frac{1}{2} \left\{ E(z, t) e^{-i\omega_{ab} t (z/c)} + c \cdot c \right\},
\]

(12)

where \( c \) is the speed of light and the complex amplitudes \( E(z, t) \) are slow-varying functions of both time and space. The complex function \( E(z, t) e^{-i\omega_{ab} t (z/c)} \) is called the analytic signal representation of the electric field since the function is analytic in the upper half of the complex \( \omega \) plane. In practice, we cannot simulate or measure the random process \( E(z, t) \) over an infinitely long interval of time, we shall restrict ourselves to study realizations of the radiation field during an interval \( T \) long enough to provide adequate spectral resolution \( \Delta \omega = 2\pi/T \).

Using the spectral representation theorem, we can expand the time dependent electric field in terms of Fourier series:

\[
E(z, t) e^{-i\omega_{ab} t (z/c)} = e^{-i\omega_{ab} t (z/c)} \sum_{n=-\infty}^{+\infty} E(z, \omega_n) e^{-i\omega_n t (z/c)},
\]

(13)
where the Fourier components are given by
\[
E(z, \omega_n) e^{i\omega_n t/c} = (1/T) \int_{-T/2}^{T/2} \{ E(z, t) e^{-i\omega_n t - i\omega_n z/c} \} e^{i\omega_n t} dt
\] (14)
with \( \omega_n = 2\pi n/T \). For bandwidth limited signal, the Fourier components \( E(z, \omega_n) \) will be zero for sufficiently large values of \( n \).

The complex components \( E(z, \omega_n) \) are random variables, whose real and imaginary parts \( (E', E') \) are jointly independent Gaussian with zero mean and the same variance \( \langle |E(z, \omega_n)|^2 \rangle /2 \). The joint probability distribution is (Mandel & Wolf 1965; Born & Wolf 1999)
\[
P[E(z, \omega_n)] = [1/\pi \langle |E(z, \omega_n)|^2 \rangle] \exp \left\{ -[E'(z, \omega_n)^2 + E'(z, \omega_n)^2]/\langle |E(z, \omega_n)|^2 \rangle \right\} .
\] (15)
Noting that the expression for the area element \( dE = dRe E d\text{Im} E = 1/2 d|E|^2 d\phi \) in the complex plane of \( E(z, \omega_n) \), we can see that the joint density distribution for amplitude and phase of \( E(z, \omega_n) \) as given by Menegozzi & Lamb (1978) follows directly from the above expression:
\[
\Phi[E(z, \omega_n)] = [1/2\pi \langle |E(z, \omega_n)|^2 \rangle] \exp \left\{ -|E(z, \omega_n)|^2/\langle |E(z, \omega_n)|^2 \rangle \right\}
\] (16)
with the following normalization:
\[
\int_0^{2\pi} \Phi[E(z, \omega_n)] d|E(z, \omega_n)|^2 = 1.
\] (17)
The complex components \( E(\omega_n) \) for each realization of the input continuum radiation field at the boundary of the masing medium can be generated with a random number generator. As described in Menegozzi & Lamb (1978), if \( x \) is a real random number uniformly distributed in the interval between 0 and 1 (excluding the end points), the phase \( \phi \) is simply \( 2\pi x \). The amplitude of the harmonic component is obtained by the relation
\[
|E(z, \omega_n)|^2 = -\langle |E(z, \omega_n)|^2 \rangle \ln(1 - x),
\] (18)
where \( \langle |E(z, \omega_n)|^2 \rangle \) is the variance or the intensity of the radiation field at frequency \( \omega_n \) and must be specified beforehand.

In the studies of Goldreich et al. (1973) and Deguchi & Watson (1990), the electric field is assumed to be stationary and can be decomposed into harmonic components using the usual Fourier transform in similar fashion to equation (4). The intensity or Stokes parameters, if we consider the vector nature of the radiation field, are defined as the ensemble averaged quantities involving harmonic components at the corresponding frequency, similar to our definition above. Although equation (4) has only symbolic meaning, it can be reinterpreted using a truncation technique as presented above without any change in the subsequent calculations.

The procedure used by Elitzur (1992) is markedly different because only the decomposition of a single realization of the radiation field into Fourier components is considered. By truncating the function \( E(t) \) on a finite interval \([-T, T]\) and performing the usual Fourier transform, the function \( F_T(\omega) \) is defined as
\[
F_T(\omega) = \int_{-T}^{T} E(t) e^{i\omega t} dt.
\] (19)
The power spectrum \( S(\omega) \) of the field is defined as (Elitzur 1992, equations 4.1.7, 9 and 10)
\[
\int_{0}^{\pi} S(\omega) d\omega = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t)^2 dt = \lim_{T \to \infty} \frac{\pi}{T} \int_{-\infty}^{\pi} |F_T(\omega)|^2 d\omega.
\] (20)
The last equality follows from Perceval’s theorem for the truncated function \( E(t) \) over any finite interval \( T \). The power spectrum \( S(\omega) \) of the radiation field is then identified with the limit
\[
S(\omega) = \lim_{T \to \infty} \frac{2\pi}{T} |F_T(\omega)|^2.
\] (21)
However, the existence of the average power of the stationary radiation field, i.e. the limit of \( \frac{2\pi}{T} \int_{-T}^{T} E(t)^2 dt \) as \( T \) tends to infinity, does not guarantee that the limit in the above equation exists. In fact, the value of \( 2\pi |F_T(\omega)|^2 /T \), which is called the periodogram by its originator Schuster, fluctuates wildly around the true power spectrum \( S(\omega) \). The periodogram is usually referred as the non-consistent estimator of the power spectrum in the signal processing literature. One way to achieve convergence is to perform a local averaging of the periodogram (Champeney 1989):
\[
\lim_{T \to \infty} \left\{ \lim_{T \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega \frac{2\pi}{T} |F_T(\omega)|^2 \right\} = S(\omega).
\] (22)
This theorem shows that in order to obtain a good estimate of the power spectrum, a trade-off in spectral resolution must be made, which is of course the principle of many smoothing schemes to estimate the power spectrum of random processes.

If we want to properly define the power spectrum using time average of a single fluctuating function \( E(t) \), we need to resort to the techniques of generalized harmonic analysis (Wiener 1930). Simply speaking, to obtain the power spectrum, we need to measure the auto-correlation function:
\[
R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} E(t) E^*(t + \tau) dt.
\] (23)
The Wiener–Khintchin theorem then guarantees that the function \( S(\omega) \), which can be identified with the power spectrum, exists and is related to the auto-correlation function \( R(\tau) \) by the usual Fourier transform:

\[
S(\omega) = \int_{-\infty}^{+\infty} R(\tau) e^{-i\omega \tau} \, d\tau.
\]  

(24)

This procedure serves as the basic principle of the XF or lag correlator (Thompson, Moran & Swenson 2001), which is currently more practical than the FX design and has widespread use in radio interferometry.

2.2 Matter–radiation interaction in masers

Since our aim is to simulate the spectral properties of the maser radiation, it is preferable to work from the beginning in the frequency domain. The basic equations are derived here for the idealized case of a scalar one-dimensional maser. That means the masing molecules are idealized as a system with only two (upper and lower) energy levels and the properties of the maser depend only on the \( z \) coordinate, which is also the propagation axis of the maser radiation. These assumptions are commonly used in previous works on the theory of astronomical masers. Following Menegozzi & Lamb (1978), Goldreich et al. (1973) and Deguchi & Watson (1990), we will derive the transfer equation for the radiation field within the framework of the rotating wave approximation. That means, when dealing with the frequency dependence of the electric field and polarization vector, we retain only the positive frequencies. The antiresonant (negative) frequencies are ignored. As in the previous section, we express the electric field and the induced polarization vector as

\[
E(z, \tau) = \frac{1}{2} [E(z, \tau) e^{-i\omega_{ab}(t - z/c)} + c \cdot c]
\]

(25)

and

\[
P(z, \tau) = \frac{1}{2} [P(z, \tau) e^{-i\omega_{ab}(t - z/c)} + c \cdot c],
\]

(26)

where \( \omega_{ab} \) is the centre of the maser line, corresponding to the rest frequency of the transition between the upper level \( a \) and the ground level \( b \) of the line. The amplitude of the field \( E(z, \tau) \) and polarization \( P(z, \tau) \) are assumed to vary slowly with time in comparison to the term \( \exp(-i\omega_{ab}t) \). As shown in the previous section, the amplitude of the electric field and polarization vector can be expressed in terms of Fourier series for a time interval \( T \), which corresponds to a spectral resolution of \( \Delta \omega = 2\pi / T \):

\[
E(z, \tau) = \sum_{n=-\infty}^{+\infty} E(z, \omega_n) e^{-i\omega_n(t - z/c)}
\]

(27)

and

\[
P(z, \tau) = \sum_{n=-\infty}^{+\infty} P(z, \omega_n) e^{-i\omega_n(t - z/c)}.
\]

(28)

The radiation transfer equation (Goldreich et al. 1973; Deguchi & Watson 1990) can be written as follows:

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) E(z, \omega_n) = \frac{2\pi i \omega_{ab}}{c} P(z, \omega_n).
\]

(29)

or in the frequency domain

\[
\frac{d}{dz} E(z, \omega_n) = \frac{2\pi i \omega_{ab}}{c} P(z, \omega_n).
\]

(30)

The polarization \( P \) describes how the masing medium interacts with the radiation field at the frequency of the maser line. To calculate its value, we need to use the density matrix, which describes the masing medium. The density matrix at any position \( z \) for a group of molecules moving with velocity \( v \) can be written as

\[
\rho(z, \tau) = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}.
\]

(31)

The Hermitian property of the density matrix implies that \( \rho_{ba} = \rho_{ab}^* \). When there is no possible confusion, we will drop the explicit dependence of the density matrix and the radiation field on the spatial coordinate \( z \). To work in the frequency domain, we need to expand the density matrix elements into Fourier series, in a similar fashion to the electric field:

\[
\rho_{ab}(z, \omega_n) = e^{-i\omega_{ab}(t - z/c)} \sum_{n=-\infty}^{+\infty} \rho_{ab}(z, \omega_n) e^{i\omega_n(t/c)} e^{-i\omega_n t},
\]

\[
\rho_{aa}(z, \omega_n) = \sum_{n=-\infty}^{+\infty} \rho_{aa}(z, \omega_n) e^{-i\omega_n(t - z/c)},
\]

\[
\rho_{bb}(z, \omega_n) = \sum_{n=-\infty}^{+\infty} \rho_{bb}(z, \omega_n) e^{-i\omega_n(t - z/c)}.
\]

(32)

Because the diagonal elements of the density matrix \( \rho_{aa} \) and \( \rho_{bb} \) are real, we have the following relations \( \rho_{aa}(z, \omega) = \rho_{aa}(z, -\omega)^* \). The Hamiltonian of the molecules is the sum of the Hamiltonian \( H_0 \), representing isolated molecules, and the matter–radiation field interaction
The well-known evolution equation of the density matrix \( \rho(z, \nu, t) \) (Goldreich et al. 1973; Deguchi & Watson 1990) has been shown to have the following compact form by Iqvist & Lamb (1969) and Sargent et al. (1974):

\[
\begin{align*}
\frac{\partial}{\partial t} \rho_{ab} &= -(i\omega_{ab} + \Gamma)\rho_{ab} - \frac{i}{\hbar} \mathbf{V}_{ab}(\rho_{ba} - \rho_{ab}), \\
\frac{\partial}{\partial t} \rho_{ab} &= \lambda_{a}(\nu) - \frac{i}{\hbar} \mathbf{V}_{ab}\rho_{ba} - \mathbf{V}_{ba}\rho_{ab}, \\
\frac{\partial}{\partial t} \rho_{ab} &= \lambda_{b}(\nu) - \frac{i}{\hbar} \mathbf{V}_{ba}\rho_{ab} - \mathbf{V}_{ab}\rho_{ba},
\end{align*}
\]

(33)

where \( \lambda_a \) and \( \lambda_b \) are the pumping rates into the upper and lower levels, respectively. \( \Gamma_a \) and \( \Gamma_b \) are the loss rate due to pumping and collisional decoherence (Sargent, Scully & Lamb 1974). \( \mathbf{V}_{ab} \) are the matrix elements of the interaction matrix \( \mathbf{V} \). For the sake of simplicity, we assume here that the loss rates are the same for the lower and upper levels. Substituting the Fourier expansion of density matrix elements into the above equations (except for the terms involving the interaction matrix \( \mathbf{V} \) to be written explicitly later) and collecting term by term, we obtain

\[
\rho_{ab}(\nu, \omega_b) = \frac{i}{\hbar} \{ \mathbf{V}_{ab}(\rho_{ba} - \rho_{ab})(\nu, \omega_b) \} \gamma_{\nu}^{ab}(\omega_b, \nu),
\]

\[
\rho_{ab}(\nu, \omega_a) = \left\{ \lambda_{a}(\nu) \delta_{n,0} - \frac{i}{\hbar} \{ \mathbf{V}_{ab}\rho_{ba} - \mathbf{V}_{ba}\rho_{ab}(\nu, \omega_b) \} \right\} \gamma_{\nu}^{a}(\omega_a, \nu),
\]

\[
\rho_{ab}(\nu, \omega_b) = \left\{ \lambda_{b}(\nu) \delta_{n,0} - \frac{i}{\hbar} \{ \mathbf{V}_{ba}\rho_{ab} - \mathbf{V}_{ab}\rho_{ba}(\nu, \omega_b) \} \right\} \gamma_{\nu}^{b}(\omega_b, \nu),
\]

(34)

where \( \delta_{n,0} \) is equal to 1 for \( n = 0 \) and zero otherwise. The \( \gamma \) functions are the Lorentzian response of the masing molecules moving at velocity \( \nu \) to the radiation field and given as follows:

\[
\gamma_{\nu}^{a}(\omega_a, \nu) = 1/ \left\{ \Gamma \pm i \left[ \omega_a - (\nu_b + \omega_a) \left( 1 - \frac{\nu}{c} \right) \right] \right\} \simeq 1/ \left\{ \Gamma \pm i \left( \omega_a \frac{\nu}{c} - \omega_b \right) \right\},
\]

\[
\gamma_{\nu}^{b}(\omega_b, \nu) = 1/ \left\{ \Gamma \mp i \omega_b \left( 1 - \frac{\nu}{c} \right) \right\} \simeq 1/ \left\{ \Gamma \mp i \omega_b \right\}.
\]

(35)

The functions \( \gamma_{\nu}^{ab}(\omega_b, \nu) \) peak at \( \omega_b = \omega_{ab} \cdot \nu/c \), which corresponds to the Doppler shift of the resonance frequency of the masing molecules. We note that in arriving at equations (34), we have used a similar approximation to that in section II of Menegozzi & Lamb (1978), i.e. neglecting the term involving \( e^{-i\nu t - \omega_b / c \cdot z / 2} \rho(z, \nu, \omega_b) \). Noting further that \( \nu \ll c \), we obtain the final expressions for the Lorentzian response of the masing molecules. These expressions are the same as that used in Menegozzi & Lamb (1978).

In the frequency domain, \( \omega_n \), the interaction term, i.e. the product \( \mathbf{V} \cdot \rho \), can be written as the convolution:

\[
\mathbf{V} \cdot \rho(\nu, \omega_b) = \sum_{q=-\infty}^{\infty} \mathbf{V}(\omega_q) \cdot \rho(\nu, \omega_{b-q}).
\]

(36)

The interaction matrix \( \mathbf{V} \) between radiation field and the masing molecules in the dipole approximation can be expressed as

\[
\mathbf{V} = -E \cdot d \simeq -1/2 \left\{ E(z, t) e^{-i\omega_{ab}(z - \xi/c)} \right\} \cdot d,
\]

(37)

where \( d \) is the dipole moment operator for the molecule, and in the last step we have made use of the rotating wave approximation. For simplicity, we assume that the matrix elements of the dipole moment are real, i.e. \( d_{ab} = d_{ba}^* = d \). In the frequency domain, the interaction term has the form

\[
\mathbf{V}_{ab}(z, \omega_b) = -\frac{1}{2} d E(z, \omega_b).
\]

(38)

Therefore, from equations (34), we obtain the expression for the off-diagonal element of the density matrix:

\[
\rho_{ab}(\nu, \omega_b) = \frac{i d}{2\hbar} \sum_{q} E(\omega_{b-q}) \cdot \left[ \rho_{ba}(\nu, \omega_q) - \rho_{ab}(\nu, \omega_q) \right] \cdot \gamma_{\nu}^{ab}(\omega_b, \nu).
\]

(39)

Equation (36) becomes

\[
\mathbf{V}^{\text{ab}} \cdot \rho_{ab}^{\text{ab}}(\nu, \omega_b) = \frac{i d^2}{4\hbar} \sum_{m,q} E(\omega_{m+q}) \cdot E^*(\omega_{m-q}) \cdot \left[ \rho_{ba}^{\text{ab}}(\nu, \omega_q) - \rho_{ab}^{\text{ab}}(\nu, \omega_q) \right] \cdot \gamma_{\nu}^{ab}(\omega_m, \nu).
\]

(40)

We also obtain a similar expression for the complex conjugate term

\[
\mathbf{V}^{\text{ab}} \cdot \rho_{ab}^*(\nu, \omega_b) = -\frac{i d^2}{4\hbar} \sum_{m,q} E^*(\omega_{m-q}) \cdot E(\omega_{m+q}) \cdot \left[ \rho_{ab}(\nu, \omega_q) - \rho_{ba}(\nu, \omega_q) \right] \cdot \gamma_{\nu}^{ab}(\omega_m, \nu).
\]

(41)
The population inversion of the maser can be calculated in terms of the elements of the density matrix as follows:

\[
\Delta \rho(v, \omega_n) = \rho_{aa}(v, \omega_n) - \rho_{bb}(\omega_n, v)
\]

\[
= \frac{d^2}{2\hbar^2} \left\{ \sum_{m,q} E(\omega_{m+q}) E^*(\omega_{m-q}) \left[ \rho^{\ast}_{mb}(\omega_n, v) - \rho^{\ast}_{ma}(\omega_n, v) \right] \gamma^{ab}_{+}(\omega_n, v) 
\right.

\[+ \sum_{m,q} E^*(\omega_{m+q}) E(\omega_{m-q}) \left[ \rho_{mb}(\omega_n, v) - \rho_{ma}(\omega_n, v) \right] \gamma^{ab}_{-}(\omega_n, v) 
\]

\[+ (\lambda_+(v) - \lambda_+^{(0)}(v)) \delta_{n0} \right\} \gamma_{+}(\omega_n). \tag{42}
\]

The polarization vector \( P(z, t) \) of the masing medium is the response of the molecules to the presence of the electromagnetic field. Quantum mechanically, \( P(z, t) \) is the average value of the dipole moment operator \( d \) and can be calculated as the trace of \( \rho \cdot d \):

\[ P(z, t) = \int_{-\infty}^{+\infty} dv \, tr(\rho \cdot d) \tag{43} \]

or written explicitly using the rotating wave approximation:

\[ \frac{1}{2} P(z, t) \exp \left\{ -i \omega_{ab}(t - z/c) \right\} \simeq \int_{-\infty}^{+\infty} dv \, \rho_{ab}(z, t) \cdot d. \tag{44} \]

Transforming the above equation into the frequency domain and making use of equation (39), we obtain

\[ P(\omega_n) = \frac{id^2}{\hbar} \int_{-\infty}^{+\infty} dv \sum_q E(\omega_{n-\omega}) \Delta \rho(\omega_n, v) \gamma^{ab}_{+}(\omega_n, v). \tag{45} \]

If we define the normalized profiles \( \phi_{\pm} \), which satisfy \( \int_{-\infty}^{+\infty} dv \phi_{\pm}(\omega_n, v) = 1 \), as follows,

\[ \phi_{\pm}(\omega_n, v) = \frac{\omega_{ab}}{\pi c} \gamma^{ab}_{\pm}(\omega_n, v), \tag{46} \]

the radiation transfer equation (30) becomes

\[ \frac{dE(\omega_n)}{dz} = \frac{2\pi^2 d^2}{\hbar} \int_{-\infty}^{+\infty} dv \sum_q E(\omega_{n-\omega}) \Delta \rho(\omega_n, v) \phi_{+}(\omega_n, v). \tag{47} \]

### 2.3 First order approximation

The equations derived in the previous section relating the transfer of the radiation field to the density matrix of masing molecules are rather complicated. One technique to solve them is the perturbative expansion consisting of solving the equations through a series expansion. To first order, the populations are considered constant over time within a given realization. Therefore, the population inversion can be written simply as \( \Delta \rho(v) \) with no dependence on harmonic frequency. The density matrix element connecting the upper level to the ground level therefore contains electric field to first order. The transfer equations are particularly simple:

\[ \frac{dE(\omega_n)}{dz} = \frac{2\pi^2 d^2}{\hbar} \int_{-\infty}^{+\infty} dv \, E(\omega_n) \Delta \rho(v) \phi_{+}(\omega_n, v). \tag{48} \]

The above equation can be cast into more familiar form involving the intensity of the radiation field per unit frequency \( v \) (\( v = \omega/2\pi \)) by defining \( I(\omega_n) \Delta \omega/2\pi = c/8 \pi E(\omega_n)E^*(\omega_n) \) and the real part \( \phi'(\omega_n, v) \) of \( \phi_{+}(\omega_n, v) \):

\[ \frac{dI(\omega_n)}{dz} = \frac{4 \pi^2 d^2}{\hbar} \int_{-\infty}^{+\infty} dv \, I(\omega_n) \cdot \Delta \rho(v) \cdot \phi'(\omega_n, v). \tag{49} \]

The equation for the population inversion equation (42) in this case can be written in a particularly simple form:

\[ \Gamma \Delta \rho(v) = \frac{2d^2}{c\hbar^2} \sum_n I(\omega_n) \cdot \Delta \rho(v) \cdot \left\{ \gamma^{ab}_+(\omega_n, v) + \gamma^{ab}_-(\omega_n, v) \right\} \Delta \omega + \Delta \lambda(v), \tag{50} \]

where \( \Delta \lambda(v) = \lambda_{+}^{(0)}(v) - \lambda_{-}(v) \). The coefficient \( 2d^2/c\hbar^2 \) can be converted to the Einstein coefficient \( B \) by the relation \( 8\pi^2 d^2/c\hbar^2 = 3B/4\pi^2 \).

We can simplify the equation if we define the real part of Lorentz functions \( \gamma^{ab}_+(\omega_n, v) \) as \( \gamma'(\omega_n, v) \):

\[ \Gamma \Delta \rho(v) = -\frac{3B}{2\pi^2} \sum_n I(\omega_n) \cdot \Delta \rho(v) \cdot \gamma'(\omega_n, v) \Delta \omega + \Delta \lambda(v). \tag{51} \]

We note that the function \( \gamma'(\omega_n, v) \) has the normalization \( \int_{-\infty}^{+\infty} \gamma'(\omega_n, v) \, d\omega = \pi \). From the radiation transfer equation, we can readily derive the unsaturated optical depth of the maser as

\[ \tau_0(v) = \frac{4\pi^2 d^2}{\hbar} L \frac{\Delta \lambda(v)}{\Gamma} = \left( \frac{\hbar v}{4\pi} \cdot \frac{3B}{c} \cdot \frac{\epsilon}{v} \right) L \frac{\Delta \lambda(v)}{\Gamma}, \tag{52} \]
where $L$ is the length of the maser. The appearance of the factor $v/c$ is due to the fact that $\rho(v)$ has the dimension of particle number per unit velocity ($\text{cm}^{-3} \text{cm}^{-1} \text{s}^{-1}$) and needs to be converted to particle number per unit frequency ($\text{cm}^{-3} \text{Hz}^{-1}$) with the factor $v/c$ in the above equation.

The equations (49) and (51) can be easily seen as identical to the equation derived by Casperson & Yariv (1972) to describe the amplification of optical continuum in a laser amplifier with both homogeneous and inhomogeneous broadening.

The standard radiation transfer equation for the average intensity of maser emission can be recovered by taking ensemble-averaging of the above equations and postulating that the maser medium is in steady state with fluctuations of $\Delta \rho(v)$ independent from that of $I(\omega_m)$. Of course, the last assumption is very drastic because the population inversion is driven by the radiation field. However, Goldreich et al. (1973) suggest that in the small signal regime the fluctuation of the population inversion is small and has very narrow bandwidth, or the autocorrelation time is much longer than that of the radiation field. That fact can justify the assumption of ignoring the fluctuation of the population inversion with respect to that of the radiation field. Noting further that $\gamma'$ and $\phi'$ are sharply peaked functions and satisfy the normalization $\sum_\omega \gamma'(\omega_n, \nu) \Delta \omega = \pi$ and $\int_{-\infty}^{+\infty} \phi'(\omega_n, \nu) d\nu = 1$, we obtain the standard equation describing the ensemble average intensity $\langle I(\omega_m) \rangle$ for the maser:

$$\frac{d(I(\omega_m))}{d(\nu/L)} = \frac{\tau_n(\nu)}{1 + (3B/2\pi\Gamma)(I(\omega_m))}. \quad (53)$$

This equation was also derived by Litvak (1970) using the integral equation approach, which makes use explicitly of the assumption on Gaussian statistics of the radiation field.

### 2.4 Change of radiation statistics

To characterize the deviation from Gaussian statistics of the radiation field we define the parameter $\delta$ following Menegozzi & Lamb (1978):

$$\delta(\omega_m) = \frac{\langle I^2(\omega_m) \rangle - \langle I(\omega_m) \rangle^2}{\langle I(\omega_m) \rangle^2}. \quad (54)$$

For a complex Gaussian random variable the value of $\delta$ is equal to unity. The evolution of $\delta$ in the unsaturated regime can be derived heuristically by taking the derivative of $\delta$:

$$\frac{d\delta}{dz} = \frac{1}{\langle I(\omega_m) \rangle^2} \left[ \frac{d}{dz} \langle I^2(\omega_m) \rangle - 2 \frac{d}{dz} \langle I(\omega_m) \rangle \right]. \quad (55)$$

To estimate the change of statistics of the maser radiation field, we restrict ourselves to the first order approximation as shown in the previous section, namely equation (49). In this approximation, the population inversion is considered to be constant within a given realization. We note further that because the function $\phi'(\omega_n, \nu)$ is sharply peaked, the population inversion $\Delta \rho(\nu)$ at any velocity $\nu$ is mainly affected by the Fourier component $\omega_n$ of the radiation field in resonance with the molecular transition. Thus, we may consider the function $\phi'(\omega_n, \nu)$ as a delta function in velocity and ignore the contribution of molecules having velocities not in resonance with the radiation field. In this case, the equations for the ensemble average $\langle I(\omega_m) \rangle$ and $\langle I^2(\omega_m) \rangle$ follow easily:

$$\frac{d}{dz} \langle I(\omega_m) \rangle = \frac{4\pi^2 d^2}{h} \langle I(\omega_m) \rangle \Delta \rho \quad (56)$$

and

$$\frac{d}{dz} \langle I^2(\omega_m) \rangle = \frac{8\pi^2 d^2}{h} \langle I^2(\omega_m) \rangle \Delta \rho. \quad (57)$$

It should be noted here that the fluctuations of $\Delta \rho$ are induced by the radiation field. In the unsaturated or small signal regime, we can approximate the dependence of population inversion on the intensity of the radiation field as

$$\Delta \rho \simeq \frac{\Delta \lambda}{\Gamma} \left[ 1 - \frac{3B}{2\pi\Gamma} I(\omega_m) \right]. \quad (58)$$

Substituting the above expressions into equation (55), we obtain

$$\frac{d\delta}{dz} = \frac{8\pi^2 d^2}{h} \Delta \lambda \frac{3B}{2\pi\Gamma} \left[ -\langle I(\omega_m) \rangle + \frac{\langle I^2(\omega_m) \rangle}{\langle I(\omega_m) \rangle} \right]. \quad (59)$$

In the small signal regime, the initial radiation field has Gaussian statistics. As shown in Menegozzi & Lamb (1978), for Gaussian variables we can factorize the terms involving power of intensity $I$ such as $\langle I^2(\omega_m) \rangle = 6 \langle I(\omega_m) \rangle^3$ and $\langle I^2(\omega_m) \rangle = 2 \langle I(\omega_m) \rangle^2$. Therefore, the change in statistics of the total radiation field seen by the molecules inside the maser is

$$\frac{d\delta}{dz} = -\frac{16\pi^2 d^2}{h} \Delta \lambda \frac{3B}{2\pi\Gamma} \langle I(\omega_m) \rangle. \quad (60)$$

The above equation shows that the parameter $\delta$ will decrease as the radiation is amplified by the maser. That means large fluctuations of the field are suppressed in comparison to smaller fluctuations. The result has a simple physical interpretation: large fluctuations can quickly deplete the population inversion, resulting in slower growth rate. Smaller fluctuations can grow at a different and faster rate, thus effectively reducing the range of intensity fluctuations.
We can also assess heuristically the effect of the interaction between masing molecules and the radiation field by comparing the intensity predicted by the standard radiation transfer equation (53) and our model in the first order approximation described by equations (56) and (58). The amplification of the ensemble averaged intensity of the radiation field in this approximation is

\[
\frac{d}{dt} \langle I(\omega_m) \rangle = \frac{4\pi^2 d^2}{\hbar} \frac{\Delta \lambda}{\Gamma} \left[ \langle I(\omega_m) \rangle - \frac{3B}{2\pi I} \langle I^2(\omega_m) \rangle \right] = \frac{4\pi^2 d^2}{\hbar} \frac{\Delta \lambda}{\Gamma} \left[ \langle I(\omega_m) \rangle - 2 \left( \frac{3B}{2\pi I} \right) \langle I(\omega_m) \rangle^2 \right].
\]

(61)

The appearance of factor 2 in the second term indicates that the fluctuations of the radiation field can reduce the population inversion faster than that predicted by the standard equation (53). Consequently, our simulations will give a slightly lower ensemble average intensity under the same conditions for the maser in comparison to the standard maser theory. We note that previous work by Field & Richardson (1984) and Field & Gray (1988), taking into account in an approximate manner the partial coherence of the maser radiation, also reached a qualitatively similar conclusion, namely that lower maser intensity appears in the modest saturation regime in comparison to the standard maser theory.

3 SIMULATION OF RADIATION TRANSPORT

In our simulations, we will choose the parameters appropriate to an astronomical maser. For simplicity, we choose the loss rate \( \Gamma = 1 \) s\(^{-1}\) and the normalized pump rates are assumed to have a velocity dispersion \( \sigma \) with the difference in pump rates \( \Delta \lambda(\nu) = e^{-(\nu^2/\sigma^2)} \). In our simulations, we use 200 modes with a frequency resolution of \( \Delta \nu = 0.75 \) GHz or 0.75 s\(^{-1}\) around the maser line and we assume that this frequency range covers the range \(-\sigma\) to \(+\sigma\) in the velocity domain. The actual velocity resolution \( \Delta \nu \) will depend on the frequency of the maser line, as \( \Delta \nu = c/\omega_m \Delta \omega \). For a maser line such as the 1612 MHz OH maser, \( \Delta \nu \) is approximately 2 cm s\(^{-1}\). The corresponding value for the velocity dispersion is 200 cm s\(^{-1}\). Although the velocity dispersion of the maser line in our simulations is much smaller than in real astronomical masers, the number of frequency modes and the bandwidth are large enough, i.e. the bandwidth of 150 s\(^{-1}\) is much greater than the loss rate of 1 s\(^{-1}\), to capture the main features of the broad-band radiation field produced by the astronomical masers. Since we deal with only partially saturated masers, we consider only 15 harmonic components of the population inversion \( \rho(\omega_m) \). As shown later, the number of harmonic components is enough to capture the pulsation of the molecular population inversion. We generate the background continuum radiation in a similar way to Menegozzi & Lamb (1978) using the random generator ran2 from Press et al. (1992). The phase of electric field components is random and uniformly distributed over the interval 0 to 2\(\pi\). For the sake of simplicity, we assume that the amplitude of different modes has a Gaussian distribution with zero mean and variance \( \langle I(\omega_m) \Delta \omega \rangle = 1 \) on a scale of 2\(\nu^2/c^2\). On this scale, the intensity has the unit of the photon occupation number. The intensity of the maser shown in all figures is also of the form \( \langle I(\omega_m) \Delta \nu \rangle \), where \( I(\omega_m) \) is the intensity per unit frequency \( \nu \) as defined in the previous section. We use a fourth order Runge–Kutta method with a fixed step \( h = 0.01 \) to integrate equation (47). Each realization is evolved through the maser using the rate equations (42) together with the radiation transfer equation (47) and we record the emergent radiation for later analysis.

We carry out our simulations in the partially saturated regime, which is likely relevant to most astronomical masers. The choice is also necessary because we consider only a limited number of harmonic components of the molecular population inversion. In the saturated regime, the fluctuation will be stronger and thus require the consideration of a larger number of harmonic components. The amplification of the maser is specified by the unsaturated optical depth \( \tau_0(\nu = 0) \) at the line centre. A value of \( \tau_0(\nu = 0) = 22 \) is used throughout in our simulations. The spontaneous transition rate between the upper and lower maser levels is taken to be \( 10^{-9} \) s\(^{-1}\). The Einstein coefficient \( B \) is related to the spontaneous transition rate by the well-known relation \( A = 2h\nu^3/c^2B \).

4 DISCUSSION

The new ingredient of our model is the explicit treatment of the random radiation field and the inclusion of the molecular population pulsation and the coupling between different modes of the radiation field. We can examine each of these effects on the propagation of electromagnetic waves in the maser.

If we neglect the population pulsation, as can be seen from equation (47), only the intensity of the waves enters into the radiation transfer equation. Thus, different realizations with initial constant amplitude but random phases are indistinguishable and we expect them to evolve similarly. However, when population pulsation is taken into account, different modes can interact and the random phases can induce fluctuations in the amplitude of the waves. In Fig. 1, we show the evolution of a particular realization of the field whose modes have constant amplitude but random phases. Initially, the mode coupling is negligible and the intensity across the maser line half-way into the maser medium, which corresponds to the optical depth of \( \tau/2 \), varies smoothly according to the change of the optical depth with frequency. The usual line-narrowing effect can be clearly seen because the initial FWHM of the maser line due to inhomogeneous (Doppler) broadening is 2\(\sigma \ln 2 \sim 140 \) frequency channels and the FWHM of the laser line at the mid-point of the laser (see the middle frame of Fig. 1) is only around 50 frequency channels. As the intensity grows, different modes of the radiation field inside the homogeneous Lorentzian profile can couple and drive the molecular population pulsations. The emergent maser radiation fluctuates widely around the line centre where this effect is strongest (Fig. 1). The maser in this case is only partially saturated with the saturation parameter \( R/\Gamma \sim 0.8 \), where \( R \) is the simulated emission rate. In Fig. 2, we show the amplitude of the Fourier components for the population pulsations at the line centre. The dominant component is at zero frequency or the average of population inversion. The amplitude of other Fourier components remains small and decreases at higher frequencies.
Evolution of a realization of the radiation field with different modes having the same amplitude but random phases. Intensity is on the scale of $2h\nu^3/c^2$. The left-hand panel shows the initial intensity and phases of the radiation field. The middle panel shows the intensity and phases of the field half-way into the maser medium, corresponding to $\tau_0(\nu = 0) = 11$. The right-hand panel shows the intensity and phases of the radiation field at the output of the maser.

Nevertheless, the population pulsation and mode coupling are effective in inducing the fluctuations of the amplified maser radiation. We can also see from Fig. 1 that the phases of different modes remain random as the waves propagate through the maser and experience amplification. No mode locking or phase correlation is seen in our simulations. We also checked the simulation results shown in Menegozzi & Lamb (1978) and could not find any evidence for phase correlation between different modes of the radiation field. Thus, our results do not corroborate the suggestion of Gray & Bewley (2003) that there is a strong phase correlation between frequency modes or mode locking even for partially saturated masers. The likely reason for the difference is the formulation of Gray & Bewley (2003), which follows closely that of Yariv (1989) for the amplification of monochromatic (single mode) laser emission, instead of the broad-band random radiation field. The steady state constraint imposed on equation (28) of their paper leads to the presence in subsequent equations of time-dependent terms, which are supposed to be eliminated by the same procedure. Of course, the problem can be avoided if their equation (28) is expanded into Fourier series as we have done here.

In Fig. 3, we show the input radiation field with both random amplitude and random phase of different modes. We also show in Fig. 3 the radiation field at the mid-point of the maser and the emergent radiation field. As can be seen in the figure, only modes around the line centre are amplified strongly. Because of the non-linear amplification, the relative intensity fluctuations between different modes change as the radiation is amplified in the maser medium. That is the case for the relatively strong mode seen close to the line centre in the initial input field (see Fig. 3). When the intensity of the radiation field is still small, i.e. the maser is unsaturated, all the modes are amplified by the same factor as evident in the middle frame of Fig. 3. However, as the maser approaches saturation, the adjacent modes, which are initially weaker, grow faster. As a result, the relative intensity fluctuation is reduced in the output radiation field. We also note that the phases between modes are still completely random as seen in Fig. 3.

We have followed the amplification of an ensemble of 18 000 realizations of the incident radiation field through the maser. By taking the ensemble average of the output field, we could build up the observable intensity profile of the maser line and the statistics of the radiation field. The parameter $\delta$, which measures the deviation from Gaussian statistics, is shown in Fig. 4. For the first time, we are able to determine the statistics of the maser radiation directly from numerical simulations. Even in our case of partial saturation with $R/\Gamma \sim 0.8$ at the line centre, the statistical distribution of the total radiation field deviates quite significantly from Gaussian. The decrease of $\delta$ at the line centre suggests that intensity fluctuations diminish with saturation and the statistical distribution of the radiation field becomes more centrally peaked. As discussed previously, the non-Gaussian statistics are the direct consequence of the non-linear interaction between maser molecules and the radiation field. We could expect that the deviation will be even more pronounced as the maser becomes fully saturated.
Figure 2. Amplitude of Fourier components of the population pulsation with velocity $\nu = 0$ at the output of the maser medium ($z = L$), corresponding to the simulation presented in Fig. 1.

Figure 3. Evolution of a realization of the radiation field with different modes having random amplitude and random phases. Intensity is on the scale of $2\hbar c^2 / c^2$. The left-hand panel shows the initial intensity and phases of the radiation field. The middle panel shows the intensity and phases of the field half-way into the maser medium, corresponding to $\tau_0 (\nu = 0) = 11$. The right-hand panel shows the intensity and phases of the emerging radiation field at the output of the maser.

We also compare the ensemble averaged intensity of the total electric field taken over 18,000 realizations of the field (see Fig. 4) with the prediction of the standard radiation transfer equations (equation 53). The two calculations match closely with each other in the line wings. At the centre of the maser line, our simulation produces a slightly lower ensemble average intensity than that given by the standard theory, consistent with our heuristic analysis in the previous section. Therefore, in the unsaturated and partially saturated regime, the standard theory of astronomical masers can produce results in good agreement with that from our more realistic treatment. However, in saturated masers where population pulsation and mode coupling are expected to be much stronger than considered here and the radiation field deviates even more from Gaussian statistics, the usual approaches used by Litvak (1970) and Goldreich et al. (1973) are no longer applicable. As a result, previous calculations using the standard radiation transfer equations and carried out in the full saturation regime with $R/\Gamma \gg 10$...
Figure 4. Upper frame: ensemble average power spectrum taken over 18,000 independent samples of the emerging radiation field. The maser emission calculated using the standard radiation transfer equation is shown as a dotted line. Intensity is on the scale of $2h\nu^3/c^2$. Lower frame: ensemble average of parameter $\delta$, i.e. the departure from Gaussian statistics, taken over the same number of independent samples of the emerging radiation field.

(Western & Watson 1984; Deguchi & Watson 1990; Nedoluha & Watson 1990) might not give an accurate description of the maser radiation and should be considered with caution. We also note that a fully saturated maser is of purely theoretical interest because real astronomical masers are unlikely to be found in this regime due to various constraints including the pumping of masers. Although it is difficult to determine the degree of saturation for astronomical masers, the modelling results of Watson, Sarma & Singleton (2002) suggest that recent observations of water maser features with Gaussian spectral line profiles (Sarma, Troland & Romney 2001) can only be explained if the maser is unsaturated with $R/\Gamma < 1/3$.

The deviation from Gaussian statistics seen in our simulations can be measured in practice if we can observe an isolated maser source with frequency resolution comparable to the homogeneous line width $\sim \Gamma$. When the frequency resolution is much larger than the homogenous bandwidth, the measured electric field is the sum over many independently random harmonic components and the resulting statistics are expected to become Gaussian. Large bandwidth is used in previous attempts (Evans et al. 1972; Moran 1981) to measure the statistics of the maser radiation and that might contribute to the negative results, although other complications such as the presence of several independent maser sources inside the radio telescope beam might preclude any such detection.

Finally, we should emphasize that our formulation of the maser theory, which is based on the work of Menegozzi & Lamb (1978), and the standard theory of masers of Litvak (1970) and Goldreich et al. (1973) are all based on the one-dimensional idealization of the maser medium. Thus, the maser radiation field consists of plane waves propagating along the same direction. In real astronomical masers, the masing medium is a three-dimensional object. The radiation field at any point in the medium is the superposition of waves propagating in many different directions. The formulation of the maser theory in three dimensions, although similar in principle, will be more complicated. The above-mentioned results and conclusions from our simulations will need to be reconfirmed in the future with fully three-dimensional calculations. In addition, the spontaneous emission by the masing molecules has been omitted in our model. This simplification is quite common in theoretical studies of astronomical masers. An explicit treatment of the spontaneous emission will require the reformulation of the maser theory using quantum electrodynamics, probably along the lines presented in the classic work of Sargent et al. (1974), and clearly is out of the scope of our paper. However, we note that in a number of cases the astronomical masers are known observationally to amplify the background continuum source (such as the central red giant star or the AGN). Thus, it is reasonable to expect that the omission of the spontaneous emission will not change qualitatively the results of our model.

5 CONCLUSION

The explicit incorporation of the broad-band random radiation field and the molecular population pulsation into our treatment of astronomical masers has proved very important to investigate the properties of maser emission. We have shown clearly the effect of mode coupling in driving the molecular population pulsation. The amplification of background radiation by masers is accompanied by the change of statistics of the radiation field in which the large fluctuations are suppressed relative to smaller fluctuations. Our simulation results suggest that the standard radiation transfer equation provides a good description of the maser properties such as intensity and line-narrowing in the unsaturated and partially saturated regime. However, the application of the same equation to study masers in the strong saturation regime should be considered with caution. Further studies on the effect of mode coupling and population pulsation in saturated masers are needed and we hope to address them in a future publication.
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REFERENCES

Arechi F. T., 1965, Phys. Rev. Lett., 15, 912
Born M., Wolf E., 1999, Principles of Optics, 7th edn. Cambridge Univ. Press, Cambridge
Casperson L. W., Yariv A., 1972, IEEE J. Quantum Electron., QE-8, 80
Champeney D. C., 1989, A Handbook of Fourier Theorems. Cambridge Univ. Press, Cambridge
Deguchi S., Watson W. D., 1990, ApJ, 354, 649
Doob J. L., 1953, Stochastic Processes. Wiley, New York
Elitzur M., 1992, Astronomical Maser. Kluwer Academic Publishers, Dordrecht
Elitzur M., 1995, ApJ, 440, 345
Evans N. J., Hills R. E., Rydbeck O. E. H., Kollberg E., 1972, Phys. Rev. A, 6, 1643
Field D., Gray M. D., 1988, MNRAS, 234, 353
Field D., Richardson I. M., 1984, MNRAS, 211, 799
Goldreich P., Keeley D. A., Kwan J. Y., 1973, ApJ, 179, 111
Goldman S., 1953, Information Theory. Prentice-Hall, New York
Gray M. D., Bewley S. L., 2003, MNRAS, 344, 493
Icsevgi A., Lamb W. E. Jr, 1969, Phys. Rev., 185, 517
Lamb W. E. Jr., 1964, Phys. Rev., 134, A1429
Litvak M. M., 1970, Phys. Rev. A, 2, 2107
Mandel L., Wolf E., 1965, Rev. Mod. Phys., 37, 231
Menegozzi L. N., Lamb W. E. Jr, 1978, Phys. Rev. A, 17, 701
Moran J., 1981, BAAS, 15, 508
Nedoluha G. E., Watson W. D., 1990, ApJ, 354, 660
Press W. H., Teukolsky S. A., Vetterling W. T., Flannery B. P., 1992, Numerical Recipes in Fortran 77. Cambridge Univ. Press, New York
Priestley M. B., 1981, Spectral Analysis and Time Series. Academic Press, London
Rice S. O., 1944, Bell Syst. Tech. J., 23, 282 (reproduced in Wax N., ed., 1954, Noise and Stochastic Processes. Dover, New York)
Rice S. O., 1945, Bell Syst. Tech. J., 24, 46 (reproduced in Wax N., ed., 1954, Noise and Stochastic Processes. Dover, New York)
Root W. L., Pitcher T. S., 1955, Ann. Math. Stat., 26, 313
Sargent M., Scully M., Lamb W. E. Jr, 1974, Laser Physics. Addison-Wesley, Reading, Massachusetts
Sarma A. P., Troland T. H., Romney J. D., 2001, ApJ, 554, L217
Thompson A. R., Moran J. M., Swenson G. W. Jr, 2001, Interferometry and Synthesis in Radio Astronomy. John Wiley & Sons, New York
Watson W. D., 1994, ApJ, 424, L37
Watson W. D., Sarma A. P., Singleton M. S., 2002, ApJ, 570, L37
Western L. R., Watson W. D., 1984, ApJ, 285, 158
Wiener N., 1930, Acta Math., 55, 117
Yaglom A. M., 1962, An Introduction to the Theory of Stationary Random Functions. Prentice-Hall, New York
Yariv A., 1989, Quantum Electronics, 3rd edn. John Wiley & Sons, New York

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