Density distributions for trapped one-dimensional spinor gases

Yajiang Hao,1,2 Yunbo Zhang,1 J. Q. Liang,1 and Shu Chen2

1Department of Physics and Institute of Theoretical Physics,
Shanxi University, Taiyuan 030006, P. R. China
2Institute of Physics, Chinese Academy of Sciences, Beijing 100080, P. R. China

We numerically evaluate the density distribution of a spin-1 bosonic condensate in its ground state within a modified Gross-Pitaevskii theory, which is obtained by the combination of the exact solution of the corresponding integrable model with the local density approximation. Our study reveals that atoms in the state are almost completely suppressed for the anti-ferromagnetic interactions in both weakly and strongly interacting regimes, whereas all three components remain non-vanishing for ferromagnetic interactions. Specially, when the system is in the Tonks-Girardeau (TG) regime, obvious Fermi-like distribution emerges for each component. We also discuss the possible deviation of the spatial distribution from the Fermi-like distribution when the spin-spin interaction is strong enough.

PACS numbers: 03.75.Mn, 03.75.Hh, 67.40.Db

I. INTRODUCTION

The experimental realization of Bose-Einstein condensates (BECs) of trapped alkali atomic clouds has stimulated active studies in many new regimes. From then on, BECs have become a popularly investigated platform for various effects of quantum many-body interaction in strongly correlated systems. When BECs are realized in optical traps rather than in magnetic ones, the spin degrees of freedom in alkali atoms are liberated. As a consequence, such a system with internal degrees of freedom manifests very rich physics and many fascinating phenomena have been observed in spinor condensates, e.g., quantum entanglement of spins, spinor four-wave mixing, and spin domains, etc.

On the other hand, in the research area of low-dimensional physics, the effect of dimensionality reduction in bosonic system has been investigated extensively and is being paid more and more attention. There has been tremendous experimental progress towards the realization of trapped one-dimensional (1D) cold atom systems. Very recently, several groups have reported the observation of a 1D Tonks-Girardeau (TG) gas. An array of 1D quantum gas is obtained by tightly confining the particle motion in two directions to zero point oscillations by means of two-dimensional optical lattice potentials. By loading the condensate in optical lattice and changing the trap intensities, and hence the atomic interaction strength, the atoms can be made to act either like a condensate or like a TG gas. The TG gas provides a textbook example where atom-atom interaction plays a critical role and mean-field theory fails to obtain reasonable results. The dimensionless interaction parameter governs the crossover from the weakly interacting condensate to the strongly interacting TG gas, where is an effective 1D interaction constant, is the mass of the atom, and is the 1D density of the quantum gas.

Within the mean field theory, Zhang and You recently studied the spin-1 atomic condensate in a cigar-shaped trap by solving an effective quasi-1D nonpolynomial Schrödinger equation. As is well known, the mean field theory fails to work if the system enters the strongly interacting TG regime. It is therefore desirable to investigate the 1D spinor gases within a theoretical approach valid to the strongly interacting TG regime as well as the weakly interacting Thomas-Fermi regime. In this paper, we investigate the ground state properties of the 1D spinor Bose gases under the local density approximation combining with the exact result of the solvable Bose gas model. The dependence of density profile on the magnetization and the effects of many-body interaction are studied in both the weakly and strongly interacting regimes, whereas all three components remain non-vanishing for ferromagnetic interactions. Particularly, when the system is in the TG regime, we found that the effect of spin-spin interaction could be dramatic.

The paper is organized as follows. In Sec. II, we give a brief review of the 1D spinor model trapped in an external potential and derive the modified GPEs. In Sec. III, we introduce our numerical procedure for solving the coupled GPEs. Aiming at the realistic systems which may be accessible in experiments, in Sec IV, we present results of the ground state density distributions for the trapped spinor gases in different interacting regimes. A brief summary is given in Sec. V.

II. FORMULATION OF THE MODEL AND METHOD

The s-wave scattering between two identical spin-1 bosons is characterized by the total spin of two collid-
ing bosons, 0 or 2, and we denote the corresponding scattering lengths by \( a_0 \) and \( a_2 \). When the cold atoms were trapped intensively in transverse direction with the transverse trapping frequency \( \hbar \omega_{\perp} \) greatly exceeding the chemical potential, the radial motion of atoms is essentially ‘frozen out’. The dynamics is thus governed by a 1D Hamiltonian with the effective 1D interaction strength given by \[ U_{0,2} = -\frac{2\hbar^2}{ma_{0,2}^2}, \] \[ a_{0,2}^{1D} = \frac{d^2}{2a_{0,2}} (1 - C (a_{0,2}/d_{\perp})), \]

where \( d_{\perp} = \sqrt{\hbar/m\omega_{\perp}} \) and \( C \approx 1.4603 \). For arbitrary hyperfine states in the hyperfine manifold in which there are two spin-1 atoms, the effective interaction may therefore be written as \[ U_{ij} = \delta (x_i - x_j) (U_0 P_0 + U_2 P_2) \]

where the operators \( P_0 \) and \( P_2 \) project the wave function of a pair of atoms into a state of total spin 0 and 2. The coefficients \( c_0 \) and \( c_2 \) are related to the effective 1D interaction constants \( U_0 \) and \( U_2 \) through \( c_0 = \frac{U_0 + U_2}{2} \) and \( c_2 = \frac{U_2 - U_0}{2} \). The ground state of the system is ferromagnetic if \( c_2 < 0 \), and anti-ferromagnetic if \( c_2 > 0 \). \( F_x \), \( F_y \) and \( F_z \) are three spin-1 matrices with the quantization axis taken along the z-axis direction.

For a spin-1 Bose condensate trapped in an external potential \( V_{ext} (x) \) which is independent of the internal states, the Hamiltonian can be expressed in the second quantized form as

\[ \mathcal{H} = \int dx \hat{\Psi}_i^+ \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_{ext} (x) \right) \hat{\Psi}_i \]

\[ + \frac{c_0}{2} \int dx : \left( \hat{\Psi}_i^+ \hat{\Psi}_i \right)^2 : \]

\[ + \frac{c_2}{2} \int dx \hat{\Psi}_i^+ \hat{\Psi}_j^+ \left( F_{\eta} \right)_{ij} \left( F_{\eta} \right)_{kl} \hat{\Psi}_j \hat{\Psi}_l, \]

where \( \hat{\Psi}_i \) (\( \hat{\Psi}_i^+ \)) is the field operator that annihilates (creates) an atom in the \( i \)-th internal state at location \( x \), and \( i = +, 0, - \) denotes the atomic hyperfine state \( |F = 1, m_F = \pm 1, 0, -1 \rangle \) respectively. Summation is assumed for repeated indices in the above Hamiltonian and the pair of colons denote the normal-order product. Within the mean field approach the properties of a spinor gas are determined by the following spin-dependent energy functional

\[ \mathcal{E} = \int dx \left[ \Phi_i^* \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_{ext} (x) \right) \Phi_i + \rho \epsilon (\rho) \right] \]

\[ + \int dx \left[ \frac{c_2}{2} \Phi_i^* \Phi_i \left( F_{\eta} \right)_{ij} \Phi_j \Phi_i \right] , \]

where \( \rho = \sum_i \rho_i = \sum_i |\Phi_i|^2 \) and \( \epsilon (\rho) = c_0\rho/2 \).

For a one-component Bose gas, it turns out that such a mean field approach works well in the weakly interacting regime but is not able to describe the density distribution correctly in the TG regime. However, a modified Gross-Pitaevskii theory, within which the interaction effect is properly taken into account by using the Lieb-Liniger solution, proves to be able to yield accurate ground-state density distribution even in the Tonks limit \[ 9 \ 12 \ 14 \ 27 \ 33 \]. The modified Gross-Pitaevskii theory essentially is based on the local density approximation together with the analytical solution of the homogeneous system. Within the local density approximation, the chemical potential of the inhomogeneous gas is determined by the local equilibrium condition,

\[ \mu = \mu_{hom} [\rho (x)] + V_{ext} (x), \]

with \( \mu_{hom} [\rho] \) being the chemical potential of the corresponding homogeneous system. For the one-component Bose gas, \( \mu_{hom} [\rho] \) can be obtained from the solution of the Lieb-Liniger model. In our present problem, the spin exchange strength \( c_2 \) is much smaller than the density-density interaction \( c_0 \) and the model \[ 3 \] for \( V_{ext} (x) = 0 \) and \( c_2 = 0 \) is the integrable three-component Bose gas model \[ 22 \ 30 \]. To make progress, we work in the scheme of modified Gross-Pitaevskii theory \[ 4 \ 27 \ 34 \] and take \( \mu_{hom} [\rho] \) from the corresponding integrable model of three-component Bose gas. Effectively, this is equivalent to replacing the energy density \( \epsilon (\rho) \) in Eq. (4) with the energy density of the exactly solvable three-component Bose model, which takes the values in the two limiting cases as following

\[ \epsilon (\rho) = \frac{\hbar^2}{2m} \rho \epsilon_\gamma (\gamma) = \left\{ \begin{array}{ll}
\frac{c_0\rho}{2}, & \gamma \ll 1, \\
\pi^2 \hbar^2 \rho^2/6m, & \gamma \gg 1.
\end{array} \right. \]

The ground state energy density of the three-component Bose model has a similar form as its one-component correspondence, i.e., the Lieb-Liniger model, however, here the density \( \rho = \rho_+ + \rho_0 + \rho_- \) is the total density of three components. We note that in Eq. (4) the kinetic energy term is associated with the inhomogeneity of the gas due to external confinement \( V_{ext} (x) \) \[ 14 \ 35 \] and \( \epsilon (\rho) \) represents essentially the energy density in the homogeneous system, while the last term the spin-spin interaction energy in 1D.

In the weakly interacting regime \( (\gamma \ll 1) \) the interaction energy will not change the wave function greatly because it is negligibly small compared with the characteristic kinetic energy of an individual atom. In the TG regime \( (\gamma \gg 1) \), however, the interaction between atoms is so strong that the bosonic atoms behave much like spinless fermions. In both cases, the spin dependent term can be expressed in the explicit form

\[ \Phi_i^* \Phi_j \left( F_{\eta} \right)_{ij} \Phi_j^* \Phi_i \]

\[ = \rho_+^2 + 2 \rho_+ \rho_0 + 2 \rho_0 + 2 \rho - 2 \rho_+ \rho_0 - 2 \rho_+ - 2 \rho_0 + \rho_+^2 \Phi_+ \Phi_- + 2 \Phi_0^* \Phi_- \Phi_. \]
We note that the last two terms in the above equation correspond to the processes that would change the spin states. An atom in the \( m_F = 1 \) state scatters with another atom in the \( m_F = -1 \) state, and consequently it produces two atoms in the \( m_F = 0 \) state or vice versa. Nevertheless, these processes conserve the magnetization of the system \( \mathcal{M} = \int dx \langle F \rangle = \int dx \left[ \Phi^*_+ \Phi_+ - \Phi^*_- \Phi_- \right] \) \cite{20, 36}. In order to obtain the ground state from a global minimization of \( \mathcal{E} \) with the constraints on both \( N \) and \( \mathcal{M} \), we introduce separately Lagrange multiplier \( B \) to conserve \( \mathcal{M} \) and the chemical potential \( \mu \) to conserve \( N \). The ground state is then determined by a minimization of the free-energy functional \( \mathcal{F} = \mathcal{E} - \mu N - BM \). The dynamics of \( \Phi_i \) is governed by the coupled GPEs

\[
i \hbar \partial \Phi_+/\partial t = \left[ H - B + c_2 (\rho_+ + \rho_0 - \rho_-) \right] \Phi_+ + c_2 \Phi^*_0 \Phi^*_-, \\
i \hbar \partial \Phi_0/\partial t = \left[ H + c_2 (\rho_+ + \rho_-) \right] \Phi_0 + 2c_2 \Phi^*_+ \Phi^*_0, \tag{7}
\]

\[
i \hbar \partial \Phi_-/\partial t = \left[ H + B + c_2 (\rho_- + \rho_0 - \rho_+) \right] \Phi_- + c_2 \Phi^*_0 \Phi^*_+, \\
\]

with

\[
H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_{\text{ext}} (x) + \tilde{F} (\rho) \tag{8}
\]

and

\[
\tilde{F} (\rho) = \frac{\partial}{\partial \rho} \left[ \rho \phi (\rho) \right] = \left\{ \frac{\pi^2 \hbar^2 \rho^2}{2m}, \gamma \ll 1, \frac{\rho^2}{\hbar^2} \right\} \gg 1. \tag{9}
\]

By numerically solving the above equations, we will determine the ground state density distributions for the 1D spinor Bose gases trapped in an harmonic trap \( V_{\text{ext}} (x) = \frac{1}{2} m \omega^2 x^2 \) both in the weakly interacting regime and in the TG regime.

### III. NUMERICAL METHOD

To simplify the formalism we choose a slightly different notation. Taking advantage of the fact that all distances and energies in the calculation can be scaled in units of typical length and energy of the external harmonic potential, we introduce the standard length unit

\[
a = \sqrt{\frac{\hbar}{m \omega}}, \tag{10}
\]

and rescale the spatial coordinate, the wave function, and the time variable as

\[
x = \tilde{x} a, \\
\Phi_i = \sqrt{\frac{N}{a}} \phi_i, \\
\hat{\rho}_i = |\phi_i|^2, \\
\hat{\rho} = \sum_{i=1}^{3} \hat{\rho}_i, \\
t = \frac{\tau}{\omega},
\]

The wave function is thus normalized to \( \int dx \sum_{i=1}^{3} |\phi_i|^2 = 1 \). With the above changes, our coupled nonlinear Schrödinger equations become

\[
i \hbar \partial \phi_+/\partial \tau = \left[ \tilde{H} - \tilde{B} + g_2 (\rho_+ + \rho_0 - \rho_-) \right] \phi_+ + g_2 \phi^*_0 \phi^*_-, \\
i \hbar \partial \phi_0/\partial \tau = \left[ \tilde{H} + g_2 (\rho_+ + \rho_-) \right] \phi_0 + 2g_2 \phi^*_+ \phi^-_0, \tag{11}
\]

\[
i \hbar \partial \phi_-/\partial \tau = \left[ \tilde{H} + \tilde{B} + g_2 (\rho_- + \rho_0 - \rho_+) \right] \phi_- + g_2 \phi^*_0 \phi^*_+, \\
\]

where \( \tilde{B} = NB/ah\omega \) and \( \tilde{H} = -(1/2) \frac{d^2}{dx^2} + \ddot{x}^2/2 + \tilde{F} (\rho) \) with

\[
\tilde{F} (\rho) = \frac{\tilde{g}_0 \rho}{\tilde{g}_0 \rho^2} \gamma \ll 1, \frac{\rho^2}{\hbar^2} \gg 1.
\]

The pair interaction constants are also rescaled as

\[
g_0 = N c_0/ah\omega, \\
\tilde{g}_0 = N^2 \pi^2/2, \\
g_2 = N c_2/ah\omega.
\]

By propagating the coupled GPEs Eq. \ref{11} in imaginary time, we obtain the ground state of spin-1 BECs in one dimension. In each propagating step, the wave function \( \phi_i \) is normalized to conserve the atomic number. We ensure the conservation of magnetization \( \mathcal{M} \) by adjusting the Lagrange multiplier \( B \). In our procedure the Crank-Nicholson scheme is used to discretize Eq. \ref{11} \cite{37}. We take the initial wave function to be a complex Gaussian with a constant velocity: \( \exp \left[ -x^2/2q - ikx \right] \). Here \( q \) and \( k \) are adjustable parameters that shall not affect the final converged ground state \cite{30}. 

![Figure 1](image-url)
In the weakly interacting regime, the trap frequencies are chosen as \( \omega_x = 50\text{Hz}, \omega_\perp = 10\text{kHz} \) and \( N = 1000 \) so that \( \gamma \sim 0.001 \). The parameters of the system in the TG regime are \( \omega_x = 10\text{Hz}, \omega_\perp = 2000\text{kHz} \) and \( N = 50 \) with \( \gamma \sim 15 \). The density profiles are shown in the Fig. 2 in units of \( N/a \) for \( m=0 \) and \( m=0.2 \), respectively. In the TG regime they exhibit similar fermionization behavior just as in the case of its ferromagnetic counterpart \(^{87}\text{Rb}\). The only difference is that here the population in 0 component remains completely suppressed at zero magnetic field according to the mean field theory. A straight explanation is that atoms prefer to be aligned anti-parallel due to the anti-ferromagnetic spin interaction. On the other hand, atoms in 0 component would collide into pairs with one atom in + component and the other in - component in order to lower the spin interaction energy. This result is in agreement with the calculation for a spinor condensate confined in a spherically symmetric 3D harmonic trap, in which case the condensation occurs for + and - components respectively [10].

For the condensates in the TG regime, the pair interaction coefficient \( g_0 = N^2\pi^2/2 \) is clearly a constant irrespective of the s-wave scattering length. Should the interaction between atoms be enhanced by the so-called Feshbach resonance, the pair interaction coefficient \( g_2 \) could be increased greatly to the magnitude order of the coefficient \( g_0 \). This enable us to investigate explicitly the effect of the spin-spin interaction. As an example, we consider the system of \( N = 15 \) \(^{23}\text{Na}\) atoms and \( m=0.2 \) in the harmonic trap with \( \omega_x = 10\text{Hz} \) and \( \omega_\perp = 2000\text{kHz} \). This correspond to a spin interaction parameter \( g_2 = 500 \) while \( g_0 = 1110.33 \) if the s-wave scattering length \( a_2 \) is enhanced to 104.8\( a_B \). The corresponding density profiles are given in Fig. 3. It is shown that the density pro-
files are no longer Fermi-like and most atoms are compressed to the narrower range around the center of the harmonic trap. A naïve explanation may be that pairs of atoms form singlets due to the strong spin-spin interactions and the effective interaction between the singlets is relatively very weak, therefore the picture of Tonks gas breaks down. Finally, we discuss the conditions under which the above result makes sense. For the $^{23}\text{Na}$ atoms discussed here, the spin exchange energy is always weak since $c_0$ is two order of magnitude smaller than $c_0$. Even for the case in which $c_2$ is enhanced to 104.8$e_B$, we have $c_0/c_2 = 4.14$ and the theory presented in this paper still holds. However, for a condensate which enters the regime with $g_2$ larger than $g_0$, our result is obviously not applicable.

V. SUMMARY

The density profiles of 1D spin-1 Bose gases in the ground state are evaluated in both the weakly interacting regime and the strongly interacting TG regime. The population of atoms in different components depends on the overall magnetization and the (anti-)ferromagnetism of the Bose gases. When the system is in the Tonks regime, the density profiles show obvious Fermi-like distribution. However, for strong enough spin-spin interaction, we observe apparent deviation of the density distribution from the Fermi-like distribution in the TG regime.

Acknowledgments

The authors acknowledge the NSF of China (Grant No. 90203007 and Grant No. 10574150) for financial support. SC is also supported by the "Hundred Talent" program of Chinese Academy of Sciences. We thank L. You, Y. Wang, W.-X. Zhang and W.-D. Li for useful discussions.

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E.Wieman, E. A. Cornell, Science 269, 198 (1995); K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995); C. C. Bradley, C. A. Sackett, J. J. Tollett, R. G. Hulet, ibid. 75, 1687 (1995).
[2] J. O. Andersen, U. Al Khawaja, and H. T. C. Stoof, Phys. Rev. Lett. 88, 070407 (2002).
[3] T.-L. Ho and S. K. Yip, Phys. Rev. Lett. 84, 4031 (2000).
[4] T.-L. Ho, Phys. Rev. Lett. 81, 742 (1998).
[5] C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. 81, 5257 (1998).
[6] T. Ohmi and K. Machida, J. Phys. Soc. Jpn. 67, 1822 (1998).
[7] W. Zhang, H. Pu, C. Search, and P. Meystre, Phys. Rev. Lett. 86, 06401 (2002).
[8] M. Olshanii, Phys. Rev. Lett. 81, 938 (1998).
[9] V. Dunjko, V. Lorent and M. Olshanii, Phys. Rev. Lett. 86, 5413 (2001).
[10] D. L. Luxat and A. Griffin, Phys. Rev. A. 67, 043603 (2003).
[11] P. Pedri, L. Santos, P. Öhberg, and S. Stringari, Phys. Rev. A. 68, 043601 (2003).
[12] S. Chen and R. Egger, Phys. Rev. A. 68, 063605 (2003).
[13] K. K. Das, M. D. Girardeau, and E. M. Wright, Phys. Rev. Lett. 89, 170404 (2002); K. K. Das, G. J. Lapeyre, and E. M. Wright, Phys. Rev. A. 65, 063603 (2002).
[14] P. Öhberg and L. Santos, Phys. Rev. Lett. 89, 240402 (2002).
[15] M. D. Girardeau and E. M. Wright, Phys. Rev. Lett. 84, 5230 (2000); T.-L. Ho and M. Ma, J. Low Temp Phys. 115, 61 (1999); H. Monien, M. Linn, and N. Elstner, Phys. Rev. A. 58, R3395 (1998).
[16] A. Recati, P. O. Fedichev, W. Zwerger, and P. Zoller, Phys. Rev. Lett. 90, 020401 (2003).
[17] G. E. Astrakharchik, D. Blume, S. Giorgini, and B. E. Granger, Phys. Rev. Lett. 92, 030402 (2004).
[18] A. Gürlich J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, J. R. Abo-Shaeer, A. P. Chikkatur, S. Gupta, S. Inouye, T. Rosenband, and W. Ketterle, Phys. Rev. Lett. 87, 130402 (2001).
[19] H. Moritz, T. Stöferle, M. Köhl, and T. Esslinger, Phys. Rev. Lett. 91, 250402 (2003); T. Stöferle, H. Moritz, C. Schori, M. Köhl, and T. Esslinger, ibid. 92, 130403 (2004).
[20] S. Richard, F. Gerbier, J. H. Thywissen, M. Hughart, P. Bouyer, and A. Aspect, Phys. Rev. Lett. 91, 010405 (2003).
[21] Y.-J. Wang, D. Z. Anderson, V. M. Bright, E. A. Cornell, Q. Diet, T. Kishimoto, M. Prentiss, R. A. Saravanam, S. R. Segal, and S. Wu., Phys. Rev. Lett. 94, 090405 (2005).
[22] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G. V. Shlyapnikov, T. W. Hänsch, and I. Bloch, Nature 429, 277 (2004).
[23] T. Kinoshita, T. Wenger and D. S. Weiss, Science 305, 1125 (2004).
[24] N. J. van Druten and W. Ketterle, Phys. Rev. Lett. 79, 549 (1997).
[25] M. D. Girardeau, J. Math. Phys. (N.Y.) 1, 516 (1960); L. Tonks, Phys. Rev. 50, 955 (1936); V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, Quantum inverse scattering method and correlation functions (Cambridge university press,1993).
[26] W. Zhang and L. You, Phys. Rev. A. 71, 025603 (2005).
[27] E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, Phys. Rev. Lett. 85, 1146 (2000).
[28] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963); E. H. Lieb, ibid. 130, 1616 (1963).
[29] B. Sutherland, Beautiful models (World Scientific, 2004).
[30] Y. Q. Li, S. J. Gu and Z. J. Ying, J. Phys. A: Math. Gen. 36, 2821 (2003).
[32] D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. 87, 050404 (2001); D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven, ibid. 85, 3745 (2000).

[33] Y. E. Kim and A. L. Zubarev, Phys. Rev. A. 67, 015602 (2003).

[34] A. Minguzzi, S. Succi, F. Toschi, M.P. Tosi, P. Vignolo, Phys. Rep. 395, 223 (2004).

[35] E. H. Lieb, R. Seiringer, and J. Yngvason, Phys. Rev. Lett. 91, 150401 (2003).

[36] S. Yi, O N. E. Müstecaplıoğlu, C. P. Sun, and L. You, Phys. Rev. A. 66, 011601 (2002).

[37] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Numerical Recipes in C: The Art of Scientific Computing, 2nd edition (Cambridge University Press, London, 1992).

[38] E. G. M. van Kempen, S. J. J. M. F. Kokkelmans, D. J. Heinzen, and B. J. Verhaar, Phys. Rev. Lett. 88, 093201 (2002).

[39] A. Crubellier, O. Dulieu1, F. Masnou-Seeuws, M. Elbs, H. Knöckel, and E. Tiemann, Eur. Phys. J. D 6, 211 (1999).

[40] W. Zhang, S. Yi and L. You, Phys. Rev. A 70, 043611 (2004).