PDNet: Semantic Segmentation integrated with a Primal-Dual Network for Document binarization

Kalyan Ram Ayyalasomayajula, Filip Malmberg, Anders Brun

Division of Visual Information and Interaction, Dept. of Information Technology, Uppsala University, Uppsala, 751 05, Sweden.

ABSTRACT

Binarization of digital documents is the task of classifying each pixel in an image of the document as belonging to the background (parchment/paper) or foreground (text/ink). Historical documents are often subject to degradations, that make the task challenging. In the current work a deep neural network architecture is proposed that combines a fully convolutional network with an unrolled primal-dual network that can be trained end-to-end in order to achieve state of the art binarization on four out of seven datasets. Document binarization is formulated as a energy minimization problem. A fully convolutional neural network is trained for semantic labeling of pixels to provide class labeling cost associated with each pixel. This cost estimate is refined along the edges to compensate for any over or under estimation of the under represented fore-ground class using a primal-dual approach. We provide necessary overview on proximal operator that facilitates theoretical underpinning in order to train a primal-dual network using a gradient descent algorithm. Numerical instabilities encountered due to the recurrent nature of primal-dual approach are handled. We provide experimental results on document binarization competition dataset along with network changes and hyperparameter tuning required for stability and performance of the network. The network when pre-trained on synthetic dataset performs better as per the competition metrics.

1. Introduction

The process of binarizing digital documents deals with classifying each pixel as belonging to the background (parchment/paper) or foreground (text/ink) by preserving most of the relevant visual information in the image. Binarization is a common pre-processing step in most tasks performed on the document image, such as word spotting and transcription where a high-quality and accurate binarization significantly simplifies the task at hand. In addition to the challenges due to uneven illumination and artifacts introduced by capturing devices, historical documents may have other degradations such as; bleed through; fading or paling of the ink in some areas; smudges, stains and blots covering the text; textured background and handwritten documents with heavy-feeble pen strokes for cursive or calligraphic effects to name a few. In general, this makes the task of document binarization very challenging as shown in Fig.1. The task is often subjective as there are corner cases such as considering ink blot as being part of foreground or background. The problem of document binarization garners interest in the field, which has led to the document image binarization content (DIBCO) (Pratikakis et al. (2011)) for automatic methods with minimum parameters to set.

The task of document binarization borrows techniques from de-noising, background removal, image segmentation and image in-painting, hence there exist several successful methods with individual strengths. The classical approaches have tried to separate the pixels into two classes using a single global threshold or series of finer local thresholds. The approach from Otsu (1979), which tries to maximize the gray level separation between foreground (FG) and background (BG) classes by maximizing the inter class variance to separate the classes. However, local intensity variations and other artifacts introduced when creating a digital image have led to a more locally adaptive techniques, such as the methods
Fig. 1: Examples of typical image degradations from DIBCO data-set (a) smudging of text (b) staining of the parchment (c) textured background (d) uneven pen strokes (e) scanning artifacts (f) bleed through of ink from the other side of the document (g) blotting over text (h) feeble contrast between ink and parchment (i) artifacts from document aging (j) fading of text

from [Niblack] (1986), Sauvola and Pietikäinen (2000). The techniques discussed in all these classical methods are generic and applicable to any image in general. However developing methods specific to documents has been the trend in winning entries of DIBCO in the past. These methods seek to improve binarization through modeling properties of FG/BG in documents images specifically. [Lu et al. (2010)] have for instance modeled background using polynomial smoothing followed by local thresholding on detected text strokes, [Bar et al. (2007)] iteratively grow FG and BG within a 7 × 7 window.

The basic algorithm proposed in this paper draws motivations from other ideas that employed use of a high level loss function as an energy associated with binarization. This loss is then optimized by minimizing the said energy over the image; typical examples include the use of Markov random fields (MRF) for binarization such as [Mishra et al. (2011)] and intensity variation reduction using Laplacian kernel (Howe (2011)). These methods take both the global and the local aspects of the image into consideration in order to label the pixels. The former uses Laplacian of the image to obtain invariance in BG intensity, followed by a graph-cut with suitable source-sink (seed points for FG-BG respectively) and edge estimates required to build an image graph. Although the fundamental idea of using a defined loss as employed in Howe’s method has been explored previously as separate methods, combining them into an energy function proved particularly effective. Further improvement of Howe’s approach was proposed in our previous method [Ayyalasomayajula and Brun (2014)] through detection of seeds for the source and sink estimate with effective detection of edges by exploiting the inherent topology by defining a binarization space.

The recent success of deep neural methods in vision related tasks has been broadly due to their ability to effectively encode the spatial dependencies and redundancy in an image. A fully convolutional neural network (FCNN) (Shelhamer et al. (2015)) is best suited for semantic labeling of pixels which is the primary objective in segmentation. The crucial idea is to use skip connections to combine the coarse features from deep layers with fine features from shallow layers to improve the final segmentation. Training such models on text images, however, often result in loss of finer details along edges. Hence a post processing step such as a graph-cut (Boykov and Kolmogorov (2004)) often improves the results as shown in our previous work (Ayyalasomayajula and Brun (2017)). Another strength of deep neural framework is their ability to train end-to-end networks that incorporate loss functions into a single architecture. This aspect has been exploited to design a loss function that evaluates the binarization output as per a DIBCO metrics in (Tensmeyer and Martinez (2017)). In addition the authors have successfully augmented the network with other features to improve the output form their network.

In the present work we would like to improve upon our previous approach of employing a segmentation network to provide a labeling cost associated with each pixel in the image. This cost is the employed in minimizing an associated energy function modeled as a primal-dual update network (PDUtNet). This approach facilitates end-to-end training in a single network as opposed to an additional requirement of a graph-cut as a post processing step. This framework helps in training the unary cost depending on pixel labeling, pairwise associated smoothness of neighboring pixels and cost of overlooking an edge when merging regions into final segmentation image in a single framework. The contributions can be summarized as:

- A stable framework that allows end-to-end training of an energy minimization function along with to semantic labeling network we call Primal-Dual Net (PDNet).
Fig. 2: Basic architecture of binarization network (PDNet) with unary (ENet), primal-dual update (PDuNet), loss function and finite difference scheme based edge estimation blocks is depicted. Network modules are shown in solid lines and layers are shown as dashed lines.

- Improved segmentation output from a semantic labeling network that is light in weight in terms of trainable weights.
- A numerically stable PDuNet when formulating binarization as a total-variation problem that can be extended to generic image based segmentation.
- Improved gradient propagation from PDuNet using modified class weighting in loss function.

This paper is divided into four main sections and an appendix. An overview of document binarization with relevant background is covered in the introduction section. This is followed by an overview of the network architecture and functionality of its basic blocks. The section on methodology covers all the details on architectural changes made in building the network. This is followed by an experimental section that covers the results on DIBCO dataset for the architectural choices discussed previously. The article is concluded with contributions in the current work and possible directions for future research. All the necessary details on the theoretical framework is covered in an appendix towards the end.

2. Methodology

The basic architecture of the proposed end-to-end binarization network PDNet is shown in Fig. 2. The network is built of three basic blocks:

- **Unary network**: This is a semantic segmentation network that is capable of associating each pixel in a given image with a cost of respective class. Ideally such a network is quite capable of segmenting a given image by itself. We use the efficient neural network (ENet) proposed by Paszke et al. (2016). The motivation for such an architecture is presented in the following section. However, as shown in our previous work Ayyalasomayajula and Brun (2014), Ayyalasomayajula and Brun (2017) text segmentation is sensitive to edge artifacts. Instead of using this network output directly we use the output prior to a typical soft-max like classification layer as the cost term for each pixel to further improve the segmentation result.

- **Primal-Dual network**: The design of such a network is inspired from previous methods that made use of conditional random field (CRF) Chen et al. (2014) as a post processing layer in segmentation. This allows for a way to enforce structural information between neighboring pixels. However we wanted to extend this idea further in text where the label propagation between neighbors should be encouraged but also restricted at the edges. Use of Primal-Dual network in tasks that involve total-variation formulation of energy function is already explored in depth super resolution (Riegler et al. (2016)) and multi-class labeling problem (Ochs et al. (2015)). We extend these ideas into a more stable architecture that permits end-to-end training within the intended theoretical framework of the underlying proximal operator, eliminating the exploding gradient problem due to its recurrent structure.

- **Loss function**: A typical document image has a lot of background pixels as opposed to written text, which naturally leads to class imbalance in the training data for these classes. The loss function used in the network is the weighted Spatial Cross Entropy loss Badrinarayanan et al. (2015) often used to offset any class imbalance in the training samples. However, due to the redistribution of pixel labels due to the total-variation regularization from the PDuNet these weights need to be re-evaluated. We propose an empirical approach to achieve this end.

2.0.1. ENet architecture

The ENet architecture is inspired from scene parsing CNNs based on probabilistic auto-encoders Ngiam et al. (2011), where two separate neural networks are combined together as an encoder-decoder pair. The encoder is trained to classify an input through down-sampling and the decoder is used to up-sample the encoder output. The ENet architecture also tries to make use of the bottleneck module that was introduced in the ResNets He et al. (2015a). A bottleneck structure consists of a main branch that is separated from an extension consisting of convolutions filters. These two branches are later merged using element-wise addition as shown in Fig. 4. conv is either a regular, dilated or full convolution and if the bottleneck is downsampling then max pooling layer is added at the main branch.

We conclude the section by discussing few key aspects of ENet architecture as shown in Table. ?? The bottleneck sections 1-3 constitute the encoder and 4,5 are part of decoder respectively. ENet has some important architectural details that improve the speed of training keeping the parameters quite low. The projection layers do not include bias terms, in order to reduce the number of kernel call and overall memory operations without effecting the accuracy. The network architecture heavily reduces the input size in the first two blocks allowing for small feature maps to be retained for further processing. This is based on the fact that visual information can be highly compressed as it has a lot of spatial redundancy. ENet opts for a large encoder with smaller decoder as opposed to a more symmetric design. This is motivated by the fact that encoder must be able to operate on smaller resolution data, reducing the role of the decoder is that of simple upsampling.
ENet further exploits the redundancy in convolution weights by strategic replacement of \( n \times n \) convolutions with two \( n \times 1 \) and \( 1 \times n \) convolutions as discussed in Jin et al. (2014), Szegedy et al. (2015). Strong down-sampling of feature space needs to be compensated with equally adept up-sampling. Convolutions layer were alternated with dilated convolutions to have a wide receptive field at the same time avoiding overly downsampling. Parametric rectifier linear Units (PReLU) (He et al. (2015b)) were used for to learn the negative slope of non-linearities through an additional learnable parameter. Most of these aspects on receptive fields, non-linear activation functions and concerned limitation in text segmentation were raised in our previous work (Ayyalasomayajula and Brun (2017)) making ENet worth exploring for text binarization.

### 2.0.2. Primal-Dual Update network

The primal-dual network is built on three basic concepts

- Total variation formulation of segmentation
- Proximal operator
- Smoothing based on Bregman functions

as discussed in the Appendix section. The primal-dual formulation of the segmentation problem is given by

\[
\min_{u} \max_{p} \left\{ \sum_{i=1}^{k} \langle \nabla u_i, p_i \rangle + \langle u_i, f_i \rangle \right\} + \delta_U(u) - \delta_P(p)
\]

with the primal and dual updates given by

\[
\hat{u} = \Pi_U(\bar{u} - \tau \nabla^T p - \tau f)
\]

\[
\hat{p} = \Pi_P(\bar{p} + \sigma \nabla u)
\]

where are \( u = (u_i)_{i=1}^{k}, p = (p_i)_{i=1}^{k}, f = (f_i)_{i=1}^{k} \) are the primal, dual and cost vectors for \( 1, \ldots, k \) classes. \( \delta_U(u), \delta_P(p) \) are the...
indicator functions for the primal and dual variables $u, p$ corresponding to the constraints sets $U, P$ respectively defined in Eqs [10][15]. The orthogonal projections on to $U, P$ are given by $\Pi_U, \Pi_P$ respectively. An approach to handling both the constraints implicitly as well as obtain a closed form representation for the projections in Eqs [2][3] is to use the Bregman proximity function. The updates for the $p, u$ are given in Eqs [24][21][22] respectively. In order to converge to a segmentation result, the primal and dual updates need to be iterated over. The primal-dual network used in binarization has a five unrolled iterations. A network with thrice such unrolled iteration is shown in Fig [4]. The $\hat{p}_i^k, \hat{u}_i^k$, and $u_{sum}$ are initialized to $0, \frac{1}{2}$ and $\frac{1}{2}$ respectively. The overall output of the network can be interpreted as a perturbation of the unary cost using primal-dual updates to give a more controlled segmentation. The final segmentation is obtained by using a weighted cross entropy loss on the $u_{sum}$.

2.0.3. Loss function

A cross entropy criterion (CEC) combines the logistic-softmax over the class with classwise negative log likelihood criterion to get the final classification. A common problem in classification is imbalance in the samples over classes. This can be adjusted by associating weights; to each class, typically inverse of their sample histogram to compensate for the class imbalance. However, due to the smoothing introduced by the PDuNet the weights for CEC loss estimated from the histogram alone overcompensate for the imbalance. We propose a power law over the histogram based weight calculation. The weighting used is inverse of square-root of class histograms, this is determined by the computing heuristic gradients estimates during training with normal histogram based weighing with and without PDuNet being part of the network.

The source code for the implementation of the PDNet used for binarization is made publicly available at https://github.com/krayyalasomayajula/pdNet.git

2.1. Architectural modifications

In the subsections below, we summarize the architectural changes from the basic network blocks.

2.1.1. E-Net architecture initial block changes

When carrying out experiments on the DIBCO dataset we experimented with both color and gray scale images. Although state of the art results were obtained using gray-scale images we would like to highlight some extreme cases as shown in Fig [5]. The initial block was modified to include RGB and two gray channels, with the max pooling applied to RGB channels alone. When training the network for RGB and gray-scale alone the original ENet architecture was used without any changes.

2.1.2. Clamped primal-dual updates

The typical primal-dual updates when properly initialized is usually stable. However, when training a PDuNet on images data in deep networks exploding gradient problem is commonly encountered. Implementations by [Riegler et al. (2016)] and [Ochs et al. (2015)] have dealt with this issue by limiting the gradients to specific bounds during back-propagation. We resorted to another approach of clamping the values in primal and dual updates as shown in Fig [6]. This approach has two advantages:

- The clamping as shown in Fig [6] resets the pixel where instability was encountered to their initial values $0, \frac{1}{2}$ for $\hat{p}_i^k, \hat{u}_i^k$ respectively.

- The gradients are not restricted during back-propagation thus leading to a faster training of the network making the loss converge within 10 epochs as opposed to 30 epochs in a network without clamping.
Fig. 6: The network allows to learn permissible values by setting a large maximum value of $10^3$ in positive and negative direction. At the same time primal and dual values near $|\epsilon|$ are clamped to $\epsilon = 10^{-8}$.

2.1.3. Choice of labeling

The convention often followed in semantic segmentation is to allow an unknown class, to neglect objects such as background scenery or to include classes that need to be ignored. Presence of such as class does not cause any change in the result, but does increase the training time for the PDNet. The unknown class is usually labeled 1 and then other classes are labeled incrementally. We begin by relabling the FG, BG and unknown classes to 1, 2 and 3 respectively. Once the unary network is trained, the cost tensor can be truncated to include just the FG and BG costs. Training PDNet on these costs leads to faster convergence as the size of $\bar{p}_k^i, \bar{u}_k^i$ and further tensors used in computations is reduced by 30%.

2.2. Tuning hyper-parameters

The binarization network has two hyper-parameters to tune: The index to be used in the power law for the class weight used in the CEC loss function and the weights to be associated with the edges along the class boundaries. As mentioned previously the index in the power for the weight was determined from the ratio of the gradients with and without the PDuNet, and adjusted heuristically. The weights associated with the edges is a shared parameter over all the unrolled loops and also multiplied with $\tau$ weight of each loop unroll. Training for both the edge weights and $\tau$ can cause instability. We instead initialized the edge weight to 1.0 and trained on a small training sample set to determine the convergence. This value was later used in the network. Although the edge weight determined by this approach is suboptimal, it can be compensated for by learning $\tau$ values through back-propagation in the network over the completed training sets. This method of training for the edge weight produced a better validation loss, though this mode of pre-training is not critical for the network performance it can save time required for the over all training of the network.

3. Experimental results

3.0.1. DIBCO dataset

The experiments were conducted on the DIBCO datasets by Pratikakis et al. (2011) for binarization competition from years 2009-2016 consisting of 76 images in total. The images and ground truth images were augmented by applying an identical deformation field transformation to both. These augmented images were the converted into 128 × 256 pixels of cropped images with 25% overlap horizontally and vertically to in order to create more data for training. The cropped size 128 × 256 was selected based on the encoder requirement for the width and height to be equal to $2^n$ for some $n$ and to fit the training batches into memory in Torch7 framework Collobert et al. (2011). The training set was picked by excluding the all the augmented images, except those from competition year under evaluation. The validation set was made by randomly picking 3000 original cropped images from the DIBCO datasets excluding the year under evaluation. The trained network was then used to produce binary output on the dataset of DIBCO competition year. The images were then stitched to the original size to compute the DIBCO evaluation metrics. Three main evaluation metrics for comparison are $F$-Measure, Peak signal to noise ratio (PSNR) and Distance Reciprocal Distortion Metric (DRD), definitions of which can be found in Pratikakis et al. (2011), Ntirogiannis et al. (2014).

3.0.2. Synthetic Dataset

In order to have better weight initialization and prevent over fitting the network to data, the network was pre-trained on synthetic data. Documents resembling historical handwritten and printed material were generated synthetically. Various filters were applied to resemble background textures and degradations in the parchment. The text was generated using handwriting and machine printed fonts from the Google Fonts (2011). Fig. 7 shows few cropped images from the synthetic dataset. The results from binarization on DIBCO dataset using the network pre-trained on the synthetic dataset are presented in Table 2.

3.0.3. Training

The network was trained in three stages:
- Pre-training unary network
- Training unary network
- Combined training of unary and PDuNet networks
The encoder of the unary network was pre-trained on patches of sizes $128 \times 256$ with batch-size 30 on the Synthetic dataset. The pre-trained encoder was then used to initialize the decoder weights and then the decoder was per-trained on the Synthetic dataset using the same patch size and batch size. This gives the pre-trained unary model. This model was then used to initial the weights of the unary model that is to be trained on the augmented DIBCO dataset. The gradient were trained using Adam [Kingma and Ba (2014)] with the learning rate, weight decay, and momentum set to $5 \times 10^{-4}$, $2 \times 10^{-4}$ and 0.9 respectively. The unary models were trained for 20 epochs and the best model with least loss was picked in further steps. When training the unary and PDuNet the learning rate was set to 0.0005 for the first 10 epochs and decayed to 0.0002 and 0.0001 in between 11 to 15 and 16 to 20 epochs respectively. The results on DIBCO dataset, with the proposed binarization network is summarized in Table 2.

| Year | PD$_c$ | PD$_g$ | GC | TM | DBC | PD$_c$ | PD$_g$ | GC | TM | DBC | PD$_c$ | PD$_g$ | GC | TM | DBC | PD$_c$ | PD$_g$ | GC | TM | DBC |
|------|--------|--------|-----|-----|-----|--------|--------|-----|-----|-----|--------|--------|-----|-----|-----|--------|--------|-----|-----|-----|
| 2009 | **91.50** | 90.46 | 89.24 | 89.76 | 91.24 | **19.25** | 17.82 | 17.28 | 18.43 | 18.66 | **3.06** | 3.45 | 4.05 | 4.89 | - | | |
| 2010 | 92.91 | 90.45 | 89.84 | **94.89** | 91.50 | 20.40 | 19.62 | 18.73 | **21.84** | 19.78 | 1.85 | 3.37 | 3.29 | **1.26** | - | | |
| 2011 | 91.87 | 85.68 | 88.36 | **93.60** | 88.74 | 19.07 | 17.43 | 17.22 | **20.11** | 17.97 | 2.57 | 15.45 | 4.27 | **1.85** | 5.36 | - | |
| 2012 | **93.04** | 89.64 | 91.97 | 92.53 | 92.85 | 20.50 | 19.63 | 19.80 | 20.60 | **21.80** | 2.92 | 4.78 | 2.81 | **2.48** | 2.66 | - | |
| 2013 | **93.97** | 93.20 | 90.59 | 93.17 | 92.70 | **23.10** | 20.75 | 19.05 | 20.71 | 21.29 | **1.83** | 2.23 | 3.18 | 2.21 | 3.10 | - | |
| 2014 | 89.99 | 93.79 | 92.40 | 91.96 | **96.88** | 20.52 | 20.79 | 18.68 | 20.76 | **22.66** | 7.42 | 2.30 | 2.72 | 2.72 | 0.90 | - | |
| 2016 | **90.18** | 89.89 | 88.79 | 89.52 | 88.72 | **18.99** | 18.88 | 18.05 | 18.67 | 18.45 | **3.61** | 3.68 | 4.33 | 3.76 | 3.86 | - | |

PD$_c$, PD$_g$ are the outputs from primal-dual networks trained on grayscale and color images respectively. GC is the result obtained from a graph-cut based approach as discussed in our previous work [Ayyalasomayajula and Brun (2017)]. It takes the output from the unary model as seed points along with the output from the classification layer acting as costs for pixels and Canny edges as boundaries estimates for segmentation. This approach requires three parameters a) cost associated with pixels labeling as obtained from the unary network, 2) weight associated to edges and Canny threshold to estimates the boundaries. It then employs an external graph-cut method [Boykov and Kolmogorov (2004)] to obtain the final segmentation. In contrast to this approach the current methods does all these parameter estimations and the energy minimization in end-to-end manner in a single framework. TM are the results from another CNN based approach developed by Tensmeyer and Martinez (2017) that augments the segmentation result with relative darkness feature [Wu et al. (2015)] to aid in binarization. DBC are results from winning entries in DIBCO competitions using various classical approaches in as given in Nitrogiannis et al. (2014), Pratikakis et al. (2016).

4. Conclusion

In the current work we have extended our previous approach of using graph-cut with CNN output to obtain binarized documents images. We have proposed an architecture that benefits from the use of energy minimization function and CNN based feature learning. The energy minimization framework is flexible in imposing constraints on the desired segmentation. The primal-dual network is implemented to impose total variation based energy minimization in an unrolled CNN architecture. The unary CNN learns pixel labeling cost and the combined network learns all the associated parameters within a single framework. The numerical instability caused by the recurrent nature in DIBCO competitions using graph-cut post-processing. The binarization results achieve state of the art results on four out of seven DIBCO datasets with close to best result on the rest. Further investigation in dropout layers in the unary model and robust selection of the weight in cross entropy loss are planned for future. The results from using a trained network to other historical document images and cases of transfer learning are of practical interest. Using more sophisticated unary and primal-dual schemes could yield better result on document binarization.

Acknowledgments

This project is a part of q2b, From quill to bytes, an initiative sponsored by the Swedish Research Council "Vetenskapsrådet D.Nr 2012-5743) and Riksbankens Jubileumsfond (R.Nr NHS14-2068:1) and Uppsala university. The authors would like to thank Tomas Wilkinson of Dept. of Information Tech., Uppsala University for discussions on debugging and performance optimization in Torch framework.

References

Ayyalasomayajula, K., Brun, A., 2014. Document binarization using topological clustering guided laplacian energy segmentation, in: Int. Conf. on Frontiers in Handwriting Recognition, pp. 523–528.
Ayyalasomayajula, K., Brun, A., 2017. Historical document binarization combining semantic labeling and graph cuts in: LNCS, Scandinavian Conference on Image Analysis, Vol(1), pp. 386–396.
Badrinarayanan, V., Kendall, A., Cipolla, R., 2015. Segnet: A deep convolutional encoder-decoder architecture for image segmentation. CoRR abs/1511.00561.
Bar, I., Beckman, I., Kedem, K., Dinstein, I., 2007. Binarization, character extraction, and writer identification of historical hebrew calligraphy documents. Int. Jou. on Document Analysis and Recognition 9, 89–99.
Appendix

**Total variation formulation:** Formulating image segmentation as a saddle point problem and applying proximal operator to the primal-dual variables is a well studied problem in fixed point analysis of convex functions (Chambolle and Pock (2011)). An overview of theory is provided here to make the discussion comprehensive and also provide intuition for network implementation discussed in the method. We consider segmentation of image into \( k \)-pairwise disjoint regions as a total variation on the segmentation image as:

\[
\min_{(R_i)_{i=1}^k, c} \frac{1}{2} \sum_{i=1}^k \text{Per}(R_i) + \frac{\lambda}{2} \sum_{i=1}^k \int_{R_i} |g(x) - c_i|^2 \, dx
\]

where \( \text{Per}(R_i) \) is the perimeter of the region \( R_i \) in a domain \( \Omega \), \( g : \Omega \to \mathbb{R} \) is the input image, \( c_i \in \mathbb{R} \) are the optimal mean values and the regions \( (R_i)_{i=1}^k \) form a partition of \( \Omega \) that is, \( R_i \cap R_m = \emptyset, i \neq m \) and \( \bigcup_{i=1}^k R_i = \Omega \). \( \lambda \) is a regularization weight. This is an optimization problem between the data fitting term \( |g(x) - c_i| \) and the length term \( \text{Per}(R_i, \Omega) \) where the ideal mean values of the region \( c_i = \frac{1}{|R_i|} \int_{R_i} g(x) \, dx \) are unknown apriori as they depend on the partition we seek.

By introducing a labeling function \( u = (u_i)_{i=0}^k : \Omega \to \mathbb{R} \) and viewing the data fitting term as a weighing function \( f_i = \frac{1}{2} |g(x) - c_i| \) Eq 3 can be generalized as:

\[
\min_{u \in \{0,1\}^\Omega} J(u) + \sum_{i=1}^k \int_{\Omega} u_i f_i \, dx,
\]

\[
\sum_{i=1}^k u_i(x) = 1, u_i \geq 0, \forall x \in \Omega
\]

where \( J(u) \) is the relaxation term.

**Proximal operator:** This basic structure in Eq 4 is of the form

\[
\min_{x \in X} F(x) + G(x)
\]

involving a linear map \( K : X \to Y \) with the induced norm \( \|K\| = \max \{\|Kx\| : x \in X \text{ with } \|x\| \leq 1\} \) where \( X, Y \) are the primal and dual spaces respectively. The corresponding dual formulation of this equation is a generic saddle-point problem

\[
\min_{x \in X} \max_{y \in Y} \langle K(x), y \rangle + G(x) - F^*(y)
\]

where \( \langle K(x), y \rangle \) is the inner product induced by the vector space \( Y \) and \( F^* \) is the convex conjugate of \( F \).

The advantage of such an approach is discussed further down with the segmentation example, for now we can observe that this structure readily presents a computationally tractable algorithm. The dual variable \( y \in Y \) acts like bounded slack variables introduced to ease the solution in the resulting dual space \( Y \). Introducing the dual variable \( y \) relieves the initial composition of \( F(Kx) \) making computations involving \( \langle K(x), y \rangle \) independent of \( F^*(y) \). As per the structure of segmentation problem, \( F \) is typically an indicator function for capturing the constraints on \( x \) which translates to \( F^* \) being an indicator function for capturing the constraints on its dual variable \( y \). Since the primal space has \( \|x\| \leq 1 \) if the dual variable is bounded which is most often the case then iterating repeated between the two variables should converge to a solution.
The solution takes a form involving the proximal operator or gradient of the functions \( F, G \) depending on them being convex or convex as well as differentiable, respectively. The basic idea behind a proximal operator is a generalization of projection on to a vector space. This makes it an ideal operator that can be used in a gradient descent algorithm where the iteration involves suitable step towards the solution along the gradient direction. But since the function need not be differentiable the gradient need not necessarily exist and hence the question of uniqueness along the gradient direction does not arise. This results in a set of permissible vectors that though not strictly a gradient can act as one at a given point \( x \), such a set of permissible vectors is called subgradient. The set \( \partial F \) is the subgradient it is also the set of underestimators of \( F \) at \( x \). A closely related set is the resolvent operator with the property

\[
x = (I + \tau \partial F)^{-1}(y) = \arg \min_x \left\{ \frac{||x - y||^2}{2\tau} + F(x) \right\}
\]

(7)

it can be interpreted as the closest point \( x \) in the set under consideration to the given point \( y \) under an associated error \( F(x) \). The primal-dual formulation allows for an algorithm that iterates between the primal and dual variables \( x, \lambda \) respectively in this case leading to convergence according to the forward – backward algorithm

\[
y^{n+1} = \text{prox}_x(y^n) = (I + \sigma \partial F)^{-1}(y^n + \sigma K x^n)
\]

\[
x^{n+1} = \text{prox}_x(x^n) = (I + \tau \partial G)^{-1}(x^n - \tau K^* y^{n+1})
\]

(8)

where \( \tau, \sigma \) are step lengths along dual and primal sub-gradients and \( \theta \) is the relaxation parameter in iterating the primal variable.

Considering the image discretized over the Cartesian grid of size \( M \times N \) as \( \{(i h, j h) : 1 \leq i \leq M, 1 \leq j \leq N \} \), where \( h \) is the size spacing and \( (i, j) \) the indices in discrete notation. \( X \) is a vector space in \( \mathbb{R}^{MN} \) equipped with standard inner product \( (u, v) \) for \( u, v \in X \). The gradient is defined as \( \nabla : X \to Y \), \( \nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots \right) \) with \( Y = X \times X \) equipped with the inner product defined as,

\[
\langle p, q \rangle_Y = \sum_{i,j} p_{i,j} q_{i,j}^* + p_{i,j} q_{i,j} = \langle p^1, p^2 \rangle = \langle q^1, q^2 \rangle \in Y
\]

(11)

Applying the above framework to Eq[4] with \( J(u) = \frac{1}{2} \sum_{i=1}^k \int \Omega |\nabla u_i| \), we have

\[
\min_{u = (u_i)_{i=1}^k} \frac{1}{2} \sum_{i=1}^k \int \Omega |\nabla u_i| + \langle \nabla u, f_i \rangle + \delta_{U}(u)
\]

(9)

where \( G(u) = \delta_{U}(u) \) is the indicator function for the unit simplex,

\[
U = \left\{ u \in X^k : \sum_{i=1}^k u_i(x) = 1, u_i \geq 0 \right\}
\]

(10)

\( f = (f_i)_{i=1}^k \in X^k \) is the discretized weighting function or the cost per pixel, \( u = (u_i)_{i=1}^k \) is the primal variable and \( X^k \) is the extension of the vector space for \( k \) classes. Considering

\[
\frac{1}{2} \sum_{i=1}^k \int \Omega |\nabla u_i| + \langle u_i, f_i \rangle = \max_{p = (p_i)_{i=1}^k} \left( \sum_{i=1}^k \langle \nabla u_i, p_i \rangle + \langle u_i, f_i \rangle \right) - \delta_{\{p \}}(p)
\]

(11)

where \( p \in Y^k \) is the dual variable, with \( Y^k \) is the extension of the gradient vector space for \( k \) classes, \( \delta_{\{p \}}(p) \) is the indicator function for \( p \in P \) defined as

\[
P = \left\{ p \in Y^k : ||p||_{\infty} \leq \frac{1}{2} \right\}
\]

(12)

We have the primal-dual formulation as

\[
\min_{u = (u_i)_{i=1}^k} \max_{p = (p_i)_{i=1}^k} \left( \sum_{i=1}^k \langle \nabla u_i, p_i \rangle + \langle u_i, f_i \rangle \right) + \delta_{U}(u) - \delta_{P}(p)
\]

(13)

with \( u, p \) related as \( \langle \nabla u, p \rangle_Y = -\langle u, \nabla f \rangle_X \) which is a consequence of applying Gauss Divergence theorem on the scalar function \( u \) and vector field \( p \). This gives the relation \( -\text{div} = \nabla^* \) where \( \text{div}, \nabla^* \) are divergence in \( Y \); and the conjugate of gradient \( \nabla \) respectively. Further since \( \nabla^* = -\nabla^T \) it turns out that \( \text{div} = \nabla^T \).

Bregman Proximity Functions: The sets considered so far are unit simplex and unit ball (or more strictly a ball of radius \( \frac{1}{2} \)) as defined by \( U, P \) in Eqs[10][12] respectively. The resolvent of these sets are orthogonal projections on unit simplex and point projection onto unit ball respectively. However, in the case of segmentation when using more sophisticated relaxation that yield better delineation along edges, like paired calibrations given by,

\[
J(u) = \int \Omega \Psi(Du)
\]

(14)

\[
\Psi(u) = \sup_b \left\{ \frac{1}{2} \sum_{i=1}^k \langle a_i, b_m \rangle : |a_i-b_m| \leq 1, 1 \leq l \leq m \leq k \right\}
\]

where \( a = (a_1, \ldots, a_k), b = (b_1, \ldots, b_k) \) is used in Eq[14]. The corresponding set for the dual variables is no longer a unit ball, but intersection of unit balls given by,

\[
P = \left\{ p \in Y^k : \left| p_l - p_{\text{me}} \right| \leq 1, 1 \leq l \leq m \leq k \right\}
\]

(15)

the resolvent of which is orthogonal projection on to such an intersection of unit balls. As the relaxations get more sophisticated the corresponding resolvent set becomes more complex and orthogonal projections on to them get computationally more involved. One approach towards getting a solution that can be used in a computational algorithm is to use Bregman functions. Suppose we have a convex function \( \psi(x) \) that is continuously differentiable on the interior of its domain; \( \text{int}(X) \) and continuous on its closure; \( \text{cl}(X) \), we use \( \bar{x} \) to denote a point from \( \text{int}(X) \). A Bregman proximity function \( D_{\psi} : X \times \text{int}(X) \to \mathbb{R} \) generated by \( \psi \) is defined as

\[
D_{\psi}(x, \bar{x}) = \psi(x) - \psi(\bar{x}) - \langle \nabla \psi(\bar{x}), x - \bar{x} \rangle
\]

(16)
In iterative algorithms, the Bregman proximity function can be used with the proximity operator for a convex function $g : X \to \mathbb{R}$ as

$$\text{prox}^\psi_{\arg} (\bar{x}) = \arg \min_{x \in X} \alpha g(x) + D_\psi(x, \bar{x}) \tag{17}$$

In image segmentation problems, the basic class of functions we are interested in are of the form $g(x) = \langle x, c \rangle + \delta_X(x)$ as seen in Eq. 13. The associated proximal operator is

$$\text{prox}^\psi_{\arg} (\bar{x}) = \arg \min_{x \in X} \alpha \langle x, c \rangle + D_\psi(x, \bar{x}) \tag{18}$$

The necessary and sufficient condition for optimality, which has a unique solution for Eq. 18 is

$$\nabla \psi(\bar{x}) - c = \nabla \psi(x) \tag{19}$$

This constraint is implicitly taken care by the Bregman proximity function. For further details on Bregman functions one may refer to Ochs et al. (2015). In image segmentation, the dual variables belong to the intersection of unit balls as shown in Eq. 15, so each coordinate of the dual variable $p$ should satisfy $-1 \leq p_j \leq 1$ and solve the dual problem

$$\max_{p=\{p\}_{l=1}^k} \sum_{l=1}^k (\nabla u_l, p_l) - \delta_P(p) \tag{20}$$

A suitable Bregman proximity function that encodes the dual variable constraints and the corresponding proximal solution along each coordinate $i$ are given by

$$\psi(x) = \frac{1}{2} [(1 + x) \log(1 + x) + (1 - x) \log(1 - x)]$$

$$\left(\text{prox}^\psi_{\arg} (\bar{x}) \right)_i = \frac{\exp(-2ac_i) - \frac{1 - \bar{x}_i}{1 + \bar{x}_i}}{\exp(-2ac_i) + \frac{1 - \bar{x}_i}{1 + \bar{x}_i}} \tag{21}$$

where $c_i = (\nabla u_i)_i$ is the $i$-component of the $\nabla u$.

Similarly, the primal problem deals with

$$\min_{u=\{u\}_{l=1}^k} \langle u_l, f_l \rangle + \delta_U(u) \tag{22}$$

with the primal variables $u$ restricted by $u_i \geq 0$ along each coordinate. The Bregman function encoding these constraints and the corresponding proximal solution are given by

$$\psi(x) = x \log x$$

$$\left(\text{prox}^\psi_{\arg} (\bar{x}) \right)_i = x_i \exp(-2ac_i) \tag{23}$$

to satisfy $\sum_{l=1}^k u_l(x) = 1$ in Eq. 10 it is sufficient to normalize it as

$$\left(\text{prox}^\psi_{\arg} (\bar{x}) \right)_i = \frac{x_i \exp(-2ac_i)}{\sum_{j=1}^k x_j \exp(-2ac_j)} \tag{24}$$

where $c_i = (\nabla^T p)_i - f_i$, $f_i$ denoting the cost associated with the $i^{th}$ class. Eqs. 21, 24 can now be used in Eq. 8 to converge to a solution for image segmentation.