PERMUTATIONAL BEHAVIOR OF REVERSED DICKSON POLYNOMIALS OVER FINITE FIELDS

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Abstract. In this paper, we use the method developed previously by Hong, Qin and Zhao to obtain several results on the permutational behavior of the reversed Dickson polynomial $D_{n,k}(1, x)$ of the $(k + 1)$-th kind over the finite field $\mathbb{F}_q$. Particularly, we present the explicit evaluation of the first moment $\sum_{a \in \mathbb{F}_q} D_{n,k}(1, a)$. Our results extend the known results from the case $0 \leq k \leq 3$ to the general $k \geq 0$ case.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of characteristic $p$ with $q$ elements. Associated to any integer $n \geq 0$ and a parameter $a \in \mathbb{F}_q$, the $n$-th Dickson polynomials of the first kind and of the second kind, denoted by $D_n(x, a)$ and $E_n(x, a)$, are defined for $n \geq 1$ by

$$D_n(x, a) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-2i}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}$$

and

$$E_n(x, a) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} (-a)^i x^{n-2i},$$

respectively, and $D_0(x, a) := 2, E_0(x, a) := 1$, where $\left\lfloor \frac{n}{2} \right\rfloor$ means the largest integer no more than $\frac{n}{2}$. In 2012, Wang and Yucas [7] further defined the $n$-th Dickson polynomial of the $(k + 1)$-th kind $D_{n,k}(x, a) \in \mathbb{F}_q[x]$ for $n \geq 1$ by

$$D_{n,k}(x, a) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-2ik}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}$$

and $D_{0,k}(x, a) := 2 - k$.

Hou, Mullen, Sellers and Yucas [5] introduced the definition of the reversed Dickson polynomial of the first kind, denoted by $D_n(a, x)$, as follows

$$D_n(a, x) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n-2i}{n-i} \binom{n-i}{i} (-x)^i a^{n-2i}$$

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if \( n \geq 1 \) and \( D_0(a,x) = 2 \). To extend the definition of reversed Dickson polynomials, Wang and Yucas \cite{7} defined the \( n \)-th reversed Dickson polynomial of \((k + 1)\)-th kind \( D_{n,k}(a,x) \in \mathbb{F}_q[x] \), which is defined for \( n \geq 1 \) by

\[
D_{n,k}(a,x) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k i}{n-i} (-x)^i a^{n-2i} \tag{1.1}
\]

and \( D_{0,k}(a,x) = 2 - k \).

It is well known that \( D_n(x,0) \) is a permutation polynomial of \( \mathbb{F}_q \) if and only if \( \gcd(n,q-1) = 1 \), and if \( a \neq 0 \), then \( D_n(x,a) \) induces a permutation of \( \mathbb{F}_q \) if and only if \( \gcd(n,q^2-1) = 1 \). Besides, there are lots of published results on permutational properties of Dickson polynomial \( E_n(x,a) \) of the second kind (see, for example, \cite{2}). In \cite{4}, Hou and Ly found several necessary conditions for the reversed Dickson Polynomials \( D_n(1,x) \) of the first kind to be a permutation polynomial. Recently, Hong, Qin and Zhao \cite{3} studied the reversed Dickson polynomial \( E_n(a,x) \) of the second kind that is defined for \( n \geq 1 \) by

\[
E_n(a,x) := \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} (-x)^i a^{n-2i} \tag{1.2}
\]

and \( E_0(a,x) = 1 \). In fact, they gave some necessary conditions for the reversed Dickson polynomial \( E_n(1,x) \) of the second kind to be a permutation polynomial of \( \mathbb{F}_q \). Regarding the reversed Dickson polynomial \( D_{n,2}(a,x) \in \mathbb{F}_q[x] \) of the third kind, from its definition one can derive that

\[
D_{n,2}(a,x) = E_{n-1}(a,x) \tag{1.3}
\]

for each \( x \in \mathbb{F}_q \). Using (1.2), one can deduce immediately from (3) the similar results on the permutational behavior of the reversed Dickson polynomial \( D_{n,2}(a,x) \) of the third kind. On the other hand, by using the method presented by Hong, Qin and Zhao in \cite{3}, Cheng, Hong and Qin \cite{1} obtained the results on the permutational behavior of the reversed Dickson polynomial \( D_{n,3}(a,x) \) of the fourth kind.

In this paper, our main goal is to continue to use the method developed by Hong, Qin and Zhao in \cite{3} to investigate the reversed Dickson polynomial \( D_{n,k}(a,x) \) of the \((k+1)\)-th kind which is defined by (1.1) if \( n \geq 1 \) and \( D_{0,k}(a,x) := 2 - k \). For \( a \neq 0 \), we write \( x = y(a-y) \) with an indeterminate \( y \neq \frac{a}{2} \). Then one can rewrite \( D_{n,k}(a,x) \) as

\[
D_{n,k}(a,x) = \frac{((k-1)a-(k-2)y)g^n - (a+(k-2)y)(a-y)^n}{2y-a}. \tag{1.3}
\]

We have

\[
D_{n,k}\left(a, \frac{a^2}{4}\right) = \frac{(kn-k+2)a^n}{2^n}. \tag{1.4}
\]

In fact, (1.3) and (1.4) follow from Theorem 2.2 (i) and Theorem 2.4 (i) below. It is easy to see that if \( \text{char}(\mathbb{F}_q) = 2 \), then \( D_{n,k}(a,x) = E_n(a,x) \) if \( k \) is odd and \( D_{n,k}(a,x) = D_n(a,x) \)
if $k$ is even. We also find that $D_{n,k}(a, x) = D_{n,k+p}(a, x)$, so we can restrict $p > k$. Thus we always assume $p = \text{char}(\mathbb{F}_q) > 3$ in what follows.

The paper is organized as follows. First in section 2, we study the properties of the reversed Dickson polynomial $D_{n,k}(a, x)$ of the fourth kind. Subsequently, in Section 3, we prove a necessary condition for the reversed Dickson polynomial $D_{n,k}(1, x)$ of the $k+1$-th kind to be a permutation polynomial of $\mathbb{F}_q$ and then introduce an auxiliary polynomial to present a characterization for $D_{n,k}(1, x)$ to be a permutation of $\mathbb{F}_q$. From the Hermite criterion [6] one knows that a function $f : \mathbb{F}_q \to \mathbb{F}_q$ is a permutation polynomial of $\mathbb{F}_q$ if and only if $\sum_{i=0}^{q-1} f(a)^i$ is computable. We are able to treat with this sum when $i = 1$. The final section is devoted to the computation of the first moment $\sum_{a \in \mathbb{F}_q} D_{n,k}(1, a)$.

### 2. Reversed Dickson Polynomials of the $k+1$-th kind

In this section, we study the properties of the reversed Dickson polynomials $D_{n,k+1}(a, x)$ of the fourth kind. Clearly, if $a = 0$, then

$$D_{n,k+1}(0, x) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ (-1)\frac{q}{2+1}(k-2)x^{\frac{q}{2}}, & \text{if } n \text{ is even}. \end{cases}$$

Therefore, $D_{n,k+1}(0, x)$ is a PP (permutation polynomial) of $\mathbb{F}_q$ if and only if $n$ is an even integer with $\gcd\left(\frac{q}{2}, q-1\right) = 1$. In what follows, we always let $a \in \mathbb{F}_q^*$. First, we give a basic fact as follows.

**Lemma 2.1.** [6] Let $f(x) \in \mathbb{F}_q[x]$. Then $f(x)$ is a PP of $\mathbb{F}_q$ if and only if $cf(dx)$ is a PP of $\mathbb{F}_q$ for any given $c, d \in \mathbb{F}_q^*$.

Then we can deduce the following result.

**Theorem 2.2.** Let $a, b \in \mathbb{F}_q^*$. Then the following are true.

(i). One has $D_{n,k}(a, x) = \frac{a^n}{b^n} D_{n,k}(b, \frac{b^2}{a^2} x)$.

(ii). We have that $D_{n,k}(a, x)$ is a PP of $\mathbb{F}_q$ if and only if $D_{n,k+1}(1, x)$ is a PP of $\mathbb{F}_q$.

**Proof.** (i). By the definition of $D_{n,k}(a, x)$, we have

$$\frac{a^n}{b^n} D_{n,k}\left(b, \frac{b^2}{a^2} x\right) = \frac{a^n}{b^n} \sum_{i=0}^{\frac{q}{2}} \frac{n - ki}{n - i} \binom{n - i}{i} (-1)^i b^{n-2i} \frac{b^{2i}}{a^{2i}} x^i = \sum_{i=0}^{\frac{q}{2}} \frac{n - ki}{n - i} \binom{n - i}{i} (-1)^i a^{n-2i} x^i = D_{n,k}(a, x)$$

as required. Part (i) is proved.

(ii). Taking $b = 1$ in part (i), we have

$$D_{n,k}(a, x) = a^n D_{n,k}\left(1, \frac{x}{a^2}\right).$$
Lemma 2.3. The following result is given in \cite{3} and \cite{5} without proof. For its proof, one can see \cite{1}.

and a PP of $F_n$ of the reversed Dickson polynomial $D_n(x)$ of the fourth kind. The basic properties on the reversed Dickson polynomial $D_n(x)$ of the fourth kind. The following result is given in \cite{3} and \cite{5} without proof. For its proof, one can see \cite{1}.

Theorem 2.4. Each of the following is true.

(i). For any integer $n \geq 0$, we have

$$D_n(1, x(1-x)) = x^n + (1-x)^n$$

and

$$E_n(1, x(1-x)) = \frac{x^{n+1} - (1-x)^{n+1}}{2x - 1}$$

if $x \neq \frac{1}{2}$.

(ii). If $n_1$ and $n_2$ are positive integers such that $n_1 \equiv n_2 \pmod{q^2 - 1}$, then one has $D_{n_1,k}(1, x_0) = D_{n_2,k}(1, x_0)$ for any $x_0 \in \mathbb{F}_q \setminus \{\frac{1}{2}\}$.

Proof. (i). First of all, it is easy to see that $D_{0,k}(1, \frac{1}{4}) = 2 - k = \frac{kx^0-k+2}{2^n}$ and $D_{1,k}(1, \frac{1}{4}) = 1 = \frac{kx^1-k+2}{2^n}$, the first identity is true for the cases that $n = 0$ and 1.

Now let $n \geq 2$. Then one has

$$D_{n,k}(1, \frac{1}{4}) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-k}{n-i} \binom{n-i}{i} \left(-\frac{1}{4}\right)^i$$

$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n-(k-1)i}{n-i} \binom{n-i}{i} \left(-\frac{1}{4}\right)^i + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{-i}{n-i} \binom{n-i}{i} \left(-\frac{1}{4}\right)^i$$

$$= D_{n,k-1}(1, \frac{1}{4}) + \frac{1}{4} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-i}{i} \left(-\frac{1}{4}\right)^i$$

$$= D_{n,k-1}(1, \frac{1}{4}) + \frac{1}{4} E_{n-2}(1, \frac{1}{4})$$

which follows from Theorem 2.2 (1) in \cite{3} that

$$D_{n,k}(1, \frac{1}{4}) = D_{n,1}(1, \frac{1}{4}) + (k-1) \frac{1}{4} E_{n-2}(1, \frac{1}{4})$$

$$= \frac{n+1}{2^n} + (k-1)n - (k-1) \frac{1}{2^n}$$

$$= \frac{kn-k+2}{2^n}$$

as desired. So the first identity is proved.
Now we turn our attention to the second identity. Let \( x \neq \frac{1}{4} \), then there exists \( y \in \mathbb{F}_{q^2} \setminus \{ \frac{1}{2} \} \) such that \( x = y(1 - y) \). So by the definition of the \( n \)-th reversed Dickson polynomial of the \( k \)-th kind, one has

\[
D_{n,k}(1, y(1 - y)) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n - ki}{n - i} \binom{n - i}{i} (-y(1 - y))^i
\]

\[
= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{k(n - i) - 2n}{n - i} \binom{n - i}{i} (-y(1 - y))^i
\]

\[
= k \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n - i}{i} (-y(1 - y))^i - (k - 1) \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n - i} \binom{n - i}{i} (-y(1 - y))^i
\]

\[
= kE_n(1, y(1 - y)) - (k - 1)D_n(1, y(1 - y)).
\] (2.1)

But Lemma 2.3 gives us that

\[
D_n(1, y(1 - y)) = y^n + (1 - y)^n
\] (2.2)

and

\[
E_n(1, y(1 - y)) = \sum_{i=0}^{n} y^{n-i}(1 - y)^i = \frac{x^{n+1} - (1 - x)^{n+1}}{2x - 1}.
\] (2.3)

Thus it follows from (2.1) to (2.3) that

\[
D_{n,k}(1, x) = D_{n,k}(1, y(1 - y))
\]

\[
= kE_n(1, y(1 - y)) - (k - 1)D_n(1, y(1 - y))
\]

\[
= \frac{ky^{n+1} - k(1 - y)^{n+1}}{2y - 1} - (k - 1)(y^n + (1 - y)^n)
\]

\[
= \frac{(k - 1 - (k - 2)y)y^n - (1 + (k - 2)y)(1 - y)^n}{2y - 1}
\] as required. So the second identity holds. Part (i) is proved.

(ii). For each \( x_0 \in \mathbb{F}_q \setminus \{ \frac{1}{2} \} \), one can choose an element \( y_0 \in \mathbb{F}_{q^2} \setminus \{ \frac{1}{2} \} \) such that \( x_0 = y_0(1 - y_0) \). Since \( n_1 \equiv n_2 \pmod{q^2 - 1} \), one has \( y_0^{n_1} = y_0^{n_2} \) and \( (1 - y_0)^{n_1} = (1 - y_0)^{n_2} \). It then follows from part (i) that

\[
D_{n_1,k}(1, x_0) = \frac{(k - 1 - (k - 2)y_0)y_0^{n_1} - (1 + (k - 2)y_0)(1 - y_0)^{n_1}}{2y_0 - 1}
\]

\[
= \frac{(k - 1 - (k - 2)y_0)y_0^{n_2} - (1 + (k - 2)y_0)(1 - y_0)^{n_2}}{2y_0 - 1}
\]

\[
= D_{n_2,k}(1, x_0)
\] as desired. This ends the proof of Theorem 2.4.

Evidently, by Theorem 2.2 (i) and Theorem 2.4 (i) one can derive that (1.3) and (1.4) are true.

**Proposition 2.5.** Let \( n \geq 2 \) be an integer. Then the recursion

\[
D_{n,k}(1, x) = D_{n-1,k}(1, x) - xD_{n-2,k}(1, x)
\]

holds for any \( x \in \mathbb{F}_q \).
Proof. We consider the following two cases.

CASE 1. \( x \neq \frac{1}{4} \). For this case, one may let \( x = y(1-y) \) with \( y \in \mathbb{F}_{q^2} \setminus \{ \frac{1}{2} \} \). Then by Theorem 2.4 (i), we have

\[
D_{n-1,k}(1, x) - xD_{n-2,k}(1, x) = D_{n-1,k}(1, y(1-y)) - y(1-y)D_{n-2,k}(1, y(1-y)) = \frac{(k-1 -(k-2)y)y^{n-1} - (1 + (k-2)y)(1-y)^{n-1}}{2y-1} - y(1-y)\frac{(k-1 -(k-2)y)y^{n-2} -(1 + (k-2)y)(1-y)^{n-2}}{2y-1} = \frac{(k-1 -(k-2)y)y^n - (1 + (k-2)y)(1-y)^n}{2y-1} = D_{n,k}(1, x)
\]
as required.

CASE 2. \( x = \frac{1}{4} \). Then by Theorem 2.4 (i), we have

\[
D_{n-1,k}\left(1, \frac{1}{4}\right) - \frac{1}{4}D_{n-2,k}\left(1, \frac{1}{4}\right) = \frac{k(n-1) - k + 2}{2^{n-1}} - \frac{1}{4} \frac{k(n-2) - k + 2}{2^{n-2}} = \frac{kn - k + 2}{2^n} = D_{n,k}\left(1, \frac{1}{4}\right).
\]

This concludes the proof of Proposition 2.5.

By Proposition 2.5, we can obtain the generating function of the reversed Dickson polynomial \( D_{n,k}(1, x) \) of the \( k+1 \)-th kind as follows.

**Proposition 2.6.** The generating function of \( D_{n,k}(1, x) \) is given by

\[
\sum_{n=0}^{\infty} D_{n,k}(1, x)t^n = \frac{(k-1)t - k + 2}{1 - t + xt^2}.
\]

Proof. By the recursion presented in Proposition 2.5, we have

\[
(1 - t + xt^2) \sum_{n=0}^{\infty} D_{n,k}(1, x)t^n = \sum_{n=0}^{\infty} D_{n,k}(1, x)t^n - \sum_{n=0}^{\infty} D_{n,k}(1, x)t^{n+1} + x \sum_{n=0}^{\infty} D_{n,k}(1, x)t^{n+2} = (k-1)t - k + 2 + \sum_{n=0}^{\infty} (D_{n+2,k}(1, x) - D_{n+1,k}(1, x) + xD_{n,k}(1, x))t^{n+2} = (k-1)t - k + 2.
\]

Thus the desired result follows immediately.

Now we can use Theorem 2.4 to present an explicit formula for \( D_{n,k}(1, x) \) when \( n \) is a power of the characteristic \( p \). Then we show that \( D_{n,k}(1, x) \) is not a PP of \( \mathbb{F}_q \) in this case.
Proposition 2.7. Let $p = \text{char}(\mathbb{F}_q) > 3$ and $s$ be a positive integer. Then

$$2^{p^s} D_{p^s,k}(1, x) + k - 2 = k(1 - 4x)^{\frac{p^s - 1}{p - 1}}.$$

Proof. We consider the following two cases.

CASE 1. $x \neq \frac{1}{4}$. For this case, putting $x = y(1 - y)$ in Theorem 2.4 (i) gives us that

$$D_{p^s,k}(1, x) = D_{p^s,k}(1, y(1 - y)) = \frac{(k - 1 - (k - 2)y)y^{p^s} - (1 + (k - 2)y)(1 - y)^{p^s}}{2y - 1} = \frac{k + (2 - k)u(u + 1)p^s}{u} - \frac{k + (k - 2)u(1 + 1)p^s}{u} = \frac{1}{2^{p^s+1}u}((k + (2 - k)u)(u + 1)p^s - (k + (k - 2)u)(1 - u)p^s) = \frac{1}{2^{p^s}(ku^{p^s - 1} - k + 2)},$$

where $u = 2y - 1$. So we obtain that

$$2^{p^s} D_{p^s,k}(1, x) = k(u^2)^{\frac{p^s - 1}{p - 1}} - k + 2 = k((2y - 1)^2)^{\frac{p^s - 1}{p - 1}} - k + 2,$$

which infers that

$$2^{p^s} D_{p^s,k}(1, x) + k - 2 = k(1 - 4x)^{\frac{p^s - 1}{p - 1}}$$

as desired.

CASE 2. $x = \frac{1}{4}$. By Theorem 2.4 (i), one has

$$2^{p^s} D_{p^s,k}(1, \frac{1}{4}) + k - 2 = 2^{p^s} \frac{kp^s - k + 2}{2^{p^s}} + k - 2 = 0 = k(1 - 4 \times \frac{1}{4})^{\frac{p^s - 1}{p - 1}}$$

as required. So Proposition 2.7 is proved. 

It is well known that every linear polynomial over $\mathbb{F}_q$ is a PP of $\mathbb{F}_q$ and that the monomial $x^n$ is a PP of $\mathbb{F}_q$ if and only if $\gcd(n, q - 1) = 1$. Then by Proposition 2.7, we have the following result.

Corollary 2.8. Let $p > 3$ be a prime and $q = p^e$. Let $e$ and $s$ be positive integers with $s \leq e$. Then $D_{p^s,k}(1, x)$ is not a PP of $\mathbb{F}_q$.

Proof. By Proposition 2.7, we know that $D_{p^s,k}(1, x)$ is a PP of $\mathbb{F}_q$ if and only if

$$(1 - 4x)^{\frac{p^s - 1}{p - 1}}$$

is a PP of $\mathbb{F}_q$ which is equivalent to

$$\gcd\left(\frac{p^s - 1}{2}, q - 1\right) = 1.$$

The latter one is impossible since $\frac{p^s - 1}{2} | \gcd\left(\frac{p^s - 1}{2}, q - 1\right)$ implies that

$$\gcd\left(\frac{p^s - 1}{2}, q - 1\right) \geq \frac{p - 1}{2} > 1.$$

Thus $D_{p^s,k}(1, x)$ is not a PP of $\mathbb{F}_q$. 

Proposition 2.9. Let \( p = \text{char}(\mathbb{F}_q) > 3 \) and \( s \) and \( l \) be integers such that \( 0 < s < l \). Then
\[
D_{p^r+p^s, k}(1, x) = \frac{k}{4} \left((1 - 4x)^{p^r + p^s} + (1 - 4x)^{p^r + p^s} - \frac{k - 2}{4}(1 + (1 - 4x)^{p^r + p^s})\right).
\]

Proof. We consider the following two cases.

Case 1. \( x \neq \frac{1}{2} \). For this case, putting \( x = y(1 - y) \) in Theorem 2.4 (i) gives us that
\[
D_{p^r+p^s, k}(1, x) = D_{p^r+p^s, k}(1, y(1 - y)) \]
\[
= \frac{(k - 1 - (k - 2)y)y^{p^r + p^s} - (1 + (k - 2)y)(1 - y)^{p^r + p^s}}{2y - 1} \]
\[
= \frac{k}{4} \left((u^2)^{p^r + p^s} + (1 + u^{p^r + p^s}) - \frac{k - 2}{4}(1 + (u^2)^{p^r + p^s})\right),
\]
where \( u = 2y - 1 \) and \( u^2 = 1 - 4x \). So we obtain that
\[
D_{n,k}(1, x) = \frac{k}{4} \left((1 - 4x)^{p^r + p^s} + (1 - 4x)^{p^r + p^s} - \frac{k - 2}{4}(1 + (1 - 4x)^{p^r + p^s})\right)
\]
as desired.

Case 2. \( x = \frac{1}{2} \). By Theorem 2.4 (i), one has
\[
D_{p^r+p^s, k}(1, \frac{1}{2}) = \frac{k(p^r + p^s) - k + 2}{2p^r + p^s} = \frac{-k + 2}{4}.
\]

Besides,
\[
\frac{k}{4} \left((1 - 4x)^{p^r + p^s} + (1 - 4x) \frac{k}{4} \right) - \frac{k - 2}{4}(1 + (1 - 4x)^{p^r + p^s}) = \frac{-k + 2}{4}.
\]
Thus the required result follows. So Proposition 2.9 is proved. \( \square \)

Lemma 2.10. \( \mathbb{F}_2 \) Let \( x \in \mathbb{F}_q \). Then \( x(1 - x) \in \mathbb{F}_q \) if and only if \( x^q = x \) or \( x^q = 1 - x \).

Let \( V \) be defined by
\[
V := \{ x \in \mathbb{F}_q : x^q = 1 - x \}.
\]
Clearly, \( \mathbb{F}_q \cap V = \{ \frac{1}{2} \} \). Then we obtain a characterization for \( D_{n,k}(1, x) \) to be a PP of \( \mathbb{F}_q \) as follows.

Theorem 2.11. Let \( q = p^e \) with \( p > 3 \) being a prime and \( e \) being a positive integer. Let
\[
f : y \mapsto \frac{k(1 - (k - 2)y)y^n - (1 + (k - 2)y)(1 - y)^n}{2y - 1}
\]
be a mapping on \( (\mathbb{F}_q \cup V) \setminus \{ \frac{1}{2} \} \). Then \( D_{n,k}(1, x) \) is a PP of \( \mathbb{F}_q \) if and only if \( f \) is 2-to-1 and \( f(y) \neq \frac{k_n - k + 2}{2} \) for any \( y \in (\mathbb{F}_q \cup V) \setminus \{ \frac{1}{2} \} \).

Proof. First, we show the sufficiency part. Let \( f \) be 2-to-1 and \( f(y) \neq \frac{k_n - k + 2}{2} \) for any \( y \in (\mathbb{F}_q \cup V) \setminus \{ \frac{1}{2} \} \). Let \( D_{n,k}(1, x_1) = D_{n,k}(1, x_2) \) for \( x_1, x_2 \in \mathbb{F}_q \). To show that \( D_{n,k}(1, x) \) is a PP of \( \mathbb{F}_q \), it suffices to show that \( x_1 = x_2 \) that will be done in what follows.

First of all, one can find \( y_1, y_2 \in \mathbb{F}_q \) satisfying \( x_1 = y_1(1 - y_1) \) and \( x_2 = y_2(1 - y_2) \). By Lemma 2.10, we know that \( y_1, y_2 \in \mathbb{F}_q \cup V \). We divide the proof into the following two cases.
Case 1. At least one of $x_1$ and $x_2$ is equal to $\frac{1}{2}$. Without loss of any generality, we may let $x_1 = \frac{1}{2}$. So by Theorem 2.4 (i), one derives that

$$D_{n,k}(1, x_2) = D_{n,k}(1, x_1) = D_{n,k}(1, x_1) = \frac{kn - k + 2}{2^n}. \quad (2.4)$$

We claim that $x_2 = \frac{1}{2}$. Assume that $x_2 \neq \frac{1}{2}$. Then $y_2 \neq \frac{1}{2}$. Since $f(y) \neq \frac{kn - k + 2}{2^n}$ for any $y \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$, by Theorem 2.4 (i), we get that

$$D_{n,k}(1, x_2) = \frac{(k - 1 - (k - 2)y_2)y_2^n - (1 + (k - 2)y_2)(1 - y_2)^n}{2y_2 - 1} = f(y_2) \neq \frac{kn - k + 2}{2^n},$$

which contradicts to (2.4). Hence the claim is true, and so we have $x_1 = x_2$ as required.

Case 2. Both of $x_1$ and $x_2$ are not equal to $\frac{1}{2}$. Then $y_1 \neq \frac{1}{2}$ and $y_2 \neq \frac{1}{2}$. Since $D_{n,k}(1, x_1) = D_{n,k}(1, x_2)$, by Theorem 2.4 (i), one has

$$\frac{(k - 1 - (k - 2)y_1)y_1^n - (1 + (k - 2)y_1)(1 - y_1)^n}{2y_1 - 1} = \frac{(k - 1 - (k - 2)y_2)y_2^n - (1 + (k - 2)y_2)(1 - y_2)^n}{2y_2 - 1},$$

which is equivalent to $f(y_1) = f(y_2)$. However, $f$ is a 2-to-1 mapping on $(\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$, and $f(y_2) = f(1 - y_2)$ by the definition of $f$. It then follows that $y_1 = y_2$ or $y_1 = 1 - y_2$.

Thus $x_1 = x_2$ as desired. Hence the sufficiency part is proved.

Now we prove the necessity part. Let $D_{n,k}(1, x)$ be a PP of $\mathbb{F}_q$. Choose two elements $y_1, y_2 \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$ such that $f(y_1) = f(y_2)$, that is,

$$\frac{(k - 1 - (k - 2)y_1)y_1^n - (1 + (k - 2)y_1)(1 - y_1)^n}{2y_1 - 1} = \frac{(k - 1 - (k - 2)y_2)y_2^n - (1 + (k - 2)y_2)(1 - y_2)^n}{2y_2 - 1}. \quad (2.5)$$

Since $y_1, y_2 \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$, it follows from Lemma 2.10 that $y_1(1 - y_1) \in \mathbb{F}_q$ and $y_2(1 - y_2) \in \mathbb{F}_q$. So by Theorem 2.4 (i), (2.5) implies that

$$D_{n,k}(1, y_1(1 - y_1)) = D_{n,k}(1, y_2(1 - y_2)).$$

Thus $y_1(1 - y_1) = y_2(1 - y_2)$ since $D_{n,k}(1, x)$ is a PP of $\mathbb{F}_q$, which infers that $y_1 = y_2$ or $y_1 = 1 - y_2$. Since $y_2 \neq \frac{1}{2}$, one has $y_2 \neq 1 - y_2$. Therefore $f$ is a 2-to-1 mapping on $(\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$.

Now take $y' \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\}$. Then from Lemma 2.10 it follows that $y'(1 - y') \in \mathbb{F}_q$ and

$$y'(1 - y') \neq \frac{1}{2}(1 - \frac{1}{2}).$$

Notice that $D_{n,k}(1, x)$ is a PP of $\mathbb{F}_q$. Hence one has

$$D_{n,k}(1, y'(1 - y')) \neq D_{n,k}\left(1, \frac{1}{2}(1 - \frac{1}{2})\right).$$

But Theorem 2.4 (i) tells us that

$$D_{n,k}\left(1, \frac{1}{2}(1 - \frac{1}{2})\right) = \frac{kn - k - 2}{2^n}.$$

Then by Theorem 2.4 (i) and noting that $y' \neq \frac{1}{2}$, we have

$$\frac{(k - 1 - (k - 2)y')y'^n - (1 + (k - 2)y')(1 - y')^n}{2y' - 1}.$$
which infers that \( f(y') \neq \frac{k_n-k-2}{2^n} \) for any \( y' \in (\mathbb{F}_q \cup V) \setminus \{\frac{1}{2}\} \). So the necessity part is proved.

The proof of Theorem 2.11 is complete. \( \square \)

3. A NECESSARY CONDITION FOR \( D_{n,k}(1,x) \) TO BE PERMUTATIONAL AND AN AUXILIARY POLYNOMIAL

In this section, we study a necessary condition on \( n \) for \( D_{n,k}(1,x) \) to be a PP of \( \mathbb{F}_q \).

In particular, if \( k = 3 \), then it is easy to check that

\[
D_{0,k}(1,0) = 2 - k, D_{n,k}(1,0) = 1
\]

for any \( n \geq 1 \) and

\[
D_{0,k}(1,1) = 2 - k, D_{1,k}(1,1) = 1, D_{n+2,k}(1,1) = D_{n+1,k}(1,1) - D_{n,k}(1,1)
\]

for \( n \geq 0 \), then one can easily show that the sequences \( \{D_{n,k}(1,1)|n \in \mathbb{N}\} \) are periodic with the smallest positive periods 6. In fact, one has

\[
D_{n,k}(1,1) = \begin{cases} 
2 - k, & \text{if } n \equiv 0 \pmod{6}, \\
1, & \text{if } n \equiv 1 \pmod{6}, \\
k - 1, & \text{if } n \equiv 2 \pmod{6}, \\
k - 2, & \text{if } n \equiv 3 \pmod{6}, \\
-1, & \text{if } n \equiv 4 \pmod{6}, \\
1 - k, & \text{if } n \equiv 5 \pmod{6}
\end{cases}
\]

**Theorem 3.1.** Assume that \( D_{n,k}(1,x) \) is a PP of \( \mathbb{F}_q \) with \( q = p^e \) and \( p > 3 \). Then \( n \neq 1 \pmod{6} \).

**Proof.** Let \( D_{n,k}(1,x) \) be a PP of \( \mathbb{F}_q \). Then \( D_{n,k}(1,0) \) and \( D_{n,k}(1,1) \) are distinct. Then by the above results, the desired result \( n \neq 1 \pmod{6} \) follows immediately. \( \square \)

Let \( n, k \) be nonnegative integers. We define the following auxiliary polynomial \( p_{n,k}(x) \in \mathbb{Z}[x] \) by

\[
p_{n,k}(x) := k \sum_{j \geq 0} \left( \frac{n}{2j+1} \right) x^j - (k - 2) \sum_{j \geq 0} \left( \frac{n}{2j} \right) x^j
\]

for \( n \geq 1 \) and \( p_{0,k}(x) := 2^n(2 - k) \). Then we have the following relation between \( D_{n,k}(1,x) \) and \( p_{n,k}(x) \).

**Theorem 3.2.** Let \( p > 3 \) be a prime and \( n \geq 0 \) be an even integer. Then

(i). One has

\[
D_{n,k}(1,x) = \frac{1}{2^n} f_n(1 - 4x).
\]  

(ii). We have that \( D_{n,k}(1,x) \) is a PP of \( \mathbb{F}_q \) if and only if \( p_{n,k}(x) \) is a PP of \( \mathbb{F}_q \).

**Proof.** (i). Clearly, (3.1) follows from the definitions of \( p_{0,k}(x) \) and \( D_{0,k}(1,x) \) if \( n = 0 \). Then we assume that \( n \geq 1 \) in what follows.
First, let \( x \in \mathbb{F}_q \setminus \{ \frac{1}{4} \} \). Then there exists \( y \in \mathbb{F}_{q^2} \setminus \{ \frac{1}{2} \} \) such that \( x = y(1 - y) \). Let \( u = 2y - 1 \). It then follows from Theorem 2.4 (i) that

\[
D_{n,k}(1, x) = D_{n,k}(1, y(1 - y)) = \frac{(k - 1 - (k - 2)y)y^n - (1 + (k - 2)y)(1 - y)^n}{2y - 1} = \frac{-(k-2)u+k}{u} \frac{(u+1)^n}{2} \frac{-(k-2)u+1}{2} \frac{(1-u)^n}{u} = \frac{1}{2^{n+1}u} \left( k((u+1)^n - (1-u)^n) - (k-2)u((u+1)^n + (1-u)^n) \right) = \frac{1}{2^n} \left( k \sum_{j \geq 0} \binom{n}{2j+1} x^j - (k-2) \sum_{j \geq 0} \binom{n}{2j} u^{2j} \right) = \frac{1}{2^n} p_{n,k}(u^2) = \frac{1}{2^n} p_{n,k}(1 - (4y(1-y))) = \frac{1}{2^n} p_{n,k}(1 - 4x)
\]

as desired. So (3.1) holds in this case.

Consequently, we let \( x = \frac{1}{4} \). Then by Theorem 2.4 (i), we have

\[
D_{n,k} \left( 1, \frac{1}{4} \right) = \frac{kn - k + 2}{2^n}.
\]

On the other hand, we can easily check that \( p_{n,k}(0) = kn - k + 2 \). Therefore

\[
D_{n,k} \left( 1, \frac{1}{4} \right) = \frac{1}{2^n} p_{n,k}(0) = \frac{1}{2^n} p_{n,k}(0) \left( 1 - 4 \times \frac{1}{4} \right)
\]

as one desires. So (3.1) is proved.

(ii). Notice that \( \frac{1}{4} \in \mathbb{F}_{q^2}^* \) and \( 1 - 4x \) is linear. So \( D_{n,k}(1, x) \) is a PP of \( \mathbb{F}_q \) if and only if \( p_{n,k}(x) \) is a PP of \( \mathbb{F}_q \). This ends the proof of Theorem 3.2. \( \square \)
4. The first moment $\sum_{n=1}^{\infty} D_{n,k}(1,x)$

In this section, we compute the first moment $\sum_{n=1}^{\infty} D_{n,k}(1,x)$. By Proposition 2.6, one has

$$\sum_{n=0}^{\infty} D_{n,k}(1,x)t^n = \frac{(k-1)t-k+2}{1-t+xt^2} = \frac{(k-1)t-k+2}{1-t} \frac{1}{1-\ell^2 x}$$

$$= \frac{(k-1)t-k+2}{1-t} \left( 1 + \sum_{m=1}^{\infty} \sum_{\ell=0}^{\infty} \left( \frac{t^2}{t-1} \right)^m x^m \right) (\mod x^q - x)$$

$$= \frac{(k-1)t-k+2}{1-t} \left( 1 + \sum_{m=1}^{\infty} \frac{\left( \frac{t^2}{t-1} \right)^m}{1-\ell^2 x} x^m \right)$$

$$= \frac{(k-1)t-k+2}{1-t} \left( 1 + \sum_{m=1}^{\infty} \frac{(t-1)^{q-1-k}2^m}{(t-1)^{q-1} - \ell^2(q-1)^{-1} x^m} \right). \quad (4.1)$$

Moreover, by Theorem 2.4 (ii), it follows that for any $x \in \mathbb{F}_q \setminus \{1\}$, one has

$$D_{n_1,k}(1,x) = D_{n_2,k}(1,x)$$

when $n_1 \equiv n_2 \pmod{q^2-1}$. Thus if $x \neq \frac{1}{q}$, one has

$$\sum_{n=0}^{\infty} D_{n,k}(1,x)t^n = 1 + \sum_{n=1}^{\infty} \sum_{\ell=0}^{q^2-1} D_{n+\ell(q^2-1),k}(1,x)t^{n+\ell(q^2-1)}$$

$$= 1 + \sum_{n=1}^{q^2-1} D_{n,k}(1,x) \sum_{\ell=0}^{\infty} t^{n+\ell(q^2-1)}$$

$$= 1 + \frac{1}{1-t^{q^2-1}} \sum_{n=1}^{q^2-1} D_{n,k}(1,x)t^n. \quad (4.2)$$

Then (4.1) together with (4.2) gives that for any $x \neq \frac{1}{q}$, we have

$$\sum_{n=1}^{q^2-1} D_{n,k}(1,x)t^n$$

$$= (\sum_{n=0}^{\infty} D_{n,k}(1,x)t^n - 1) \left( 1 - t^{q^2-1} \right)$$

$$= \left( \frac{(k-1)t-k+2}{1-t} - 1 \right) \left( 1 - t^{q^2-1} \right)$$

$$+ \frac{(1-t^{q^2-1})(k-1)t-k+2}{1-t} \sum_{m=1}^{q^2-1} \frac{(t-1)^{q-1-m}2^m}{(t-1)^{q-1} - \ell^2(q-1)^{-1} x^m} (\mod x^q - x)$$

$$= \frac{(kt+1-k)(1-t^{q^2-1})}{1-t} + h(t) \sum_{m=1}^{q^2-1} \frac{(t-1)^{q-1-m}2^m x^m}, \quad (4.3)$$

where
\[ h(t) := \frac{(t^{q^2-1} - 1)(k - 1)t - k + 2}{(t - 1)^q - (t - 1)t^{2(q-1)}}. \]

**Lemma 4.1.** Let \( u_0, u_1, \ldots, u_{q-1} \) be the list of the all elements of \( \mathbb{F}_q \). Then
\[
\sum_{i=0}^{q-1} u_i^k = \begin{cases} 
0, & \text{if } 0 \leq k \leq q-2, \\
-1, & \text{if } k = q-1.
\end{cases}
\]

Now by Theorem 2.4 (i), Lemma 4.1 and (4.3), we derive that
\[
\sum_{n=1}^{q^2-1} \sum_{a \in \mathbb{F}_q} D_{n,k}(1,a) t^n
= \sum_{n=1}^{q^2-1} D_{n,k}(1,\frac{1}{4}) t^n + \sum_{n=1}^{q^2-1} \sum_{a \in \mathbb{F}_q \setminus \{\frac{1}{4}\}} D_{n,k}(1,a) t^n
= \sum_{n=1}^{q^2-1} \frac{kn - k + 2}{2^n} t^n + \sum_{a \in \mathbb{F}_q \setminus \{\frac{1}{4}\}} (kt + 1 - k)(1 - t^{q^2-1}) + h(t) \sum_{m=1}^{q-1} (t - 1)^{q - 1 - m} t^{2m} a^m
= \sum_{n=1}^{q^2-1} \frac{kn - k + 2}{2^n} t^n + (q-1)(kt + 1 - k)(1 - t^{q^2-1}) + h(t) \sum_{m=1}^{q-1} (t - 1)^{q - 1 - m} t^{2m} a^m
- h(t) \sum_{m=1}^{q-1} (t - 1)^{q - 1 - m} t^{2m} \left(\frac{1}{4}\right)^m
= \sum_{n=1}^{q^2-1} \frac{kn - k + 2}{2^n} t^n - \frac{(kt + 1 - k)(1 - t^{q^2-1})}{1 - t} - h(t)t^{2(q-1)} - h(t) \sum_{m=1}^{q-1} (t - 1)^{q - 1 - m} t^{2m} \left(\frac{1}{4}\right)^m.
\]

(4.4)

Since \( (t - 1)^q = t^q - 1 \) and \( q \) is odd, one has
\[
\begin{align*}
\frac{q^2 - q}{t^q - t^{q-1}} &:= -1 - (t - t^q)^{q-1}.
\end{align*}
\]
Then by the binomial theorem applied to \((t - t^q)^{q-1}\), we can derive the following expression for the coefficient \(b_i\).

**Proposition 4.2.** For each integer \(i\) with \(0 \leq i \leq q^2 - q\), write \(i = \alpha + \beta q\) with \(\alpha\) and \(\beta\) being integers such that \(0 \leq \alpha, \beta \leq q - 1\). Then

\[
b_i = \begin{cases} 
(1-\beta+1)^{(q-1)}_{\beta}, & \text{if } \alpha + \beta = q - 1, \\
-1, & \text{if } \alpha = \beta = 0, \\
0, & \text{otherwise}.
\end{cases}
\]

For convenience, let

\[
a_n := \sum_{a \in \mathbb{F}_q} D_{n,k}(1, a).
\]

Then by (4.4) and (4.5), we arrive at

\[
\sum_{n=1}^{q^2-1} (a_n - \frac{kn-k+2}{2^n}) t^n
\]

\[
= - \frac{(kt+1-k)(1 - t^{q-1})}{1 - t} \left( t^{2(q-1)} + \sum_{m=1}^{q-1} (t - 1)^{q-1-k} t^k \left( \frac{1}{4} \right)^m \right)
\]

which implies that

\[
(t^q - t^{q-1} - 1) \sum_{n=1}^{q^2-1} (a_n - \frac{kn-k+2}{2^n}) t^n
\]

\[
= - (t^q - t^{q-1} - 1)(kt+1-k) \sum_{i=0}^{q^2-2} t^i - (2t-1) \left( t^{2(q-1)} + \sum_{k=1}^{q-1} (t - 1)^{q-1-k} t^k \left( \frac{1}{4} \right)^{q-k} \right) \sum_{i=0}^{q^2-q} b_it^i.
\]

Let

\[
\sum_{i=1}^{q^2+q-1} c_i t^i
\]

denote the right-hand side of (4.8) and let

\[
d_n := a_n - \frac{kn-k+2}{2^n}
\]

for each integer \(n\) with \(1 \leq n \leq q^2 - 1\). Then (4.8) can be reduced to

\[
(t^q - t^{q-1} - 1) \sum_{n=1}^{q^2-1} d_n t^n = \sum_{i=1}^{q^2+q-1} c_i t^i.
\]

Then by comparing the coefficient of \(t^i\) with \(1 \leq i \leq q^2 + q - 1\) of the both sides in (4.9), we derive the following relations:

\[
\begin{align*}
  c_j &= -d_j, & \text{if } 1 \leq j \leq q - 1, \\
  c_q &= -d_1 - d_q, \\
  c_{q+j} &= d_j - d_{j+1} - d_q, & \text{if } 1 \leq j \leq q^2 - q - 1, \\
  c_{q^2+j} &= d_{q^2+q+j} - d_{q^2+q+j+1}, & \text{if } 0 \leq j \leq q - 2, \\
  c_{q^2+q-1} &= d_{q^2-1}.
\end{align*}
\]
from which we can deduce that

\[
\begin{aligned}
    d_j &= -c_j, & \text{if } 1 \leq j \leq q - 1, \\
    d_q &= c_1 - c_q, \\
    d_{\ell q + j} &= d_{(\ell - 1)q + j} - d_{(\ell - 1)q + j + 1} - c_{\ell q + j}, & \text{if } 1 \leq \ell \leq q - 2 \text{ and } 1 \leq j \leq q - 1, \\
    d_{\ell q} &= d_{(\ell - 1)q} - d_{(\ell - 1)q + 1} - c_{\ell q}, & \text{if } 2 \leq \ell \leq q - 2, \\
    d_{q^2 - q + j} &= \sum_{i=j}^{q-1} c_{q^2 + i}, & \text{if } 0 \leq j \leq q - 1.
\end{aligned}
\]  

(4.10)

Finally, (4.10) together with the following identity

\[
\sum_{a \in \mathbb{F}_q} D_{n,k}(1,a) = d_n + \frac{kn - k + 2}{2^n}
\]

shows that the last main result of this paper is true:

**Theorem 4.3.** Let \( c_i \) be the coefficient of \( t^i \) in the right-hand side of (4.8) with \( i \) being an integer such that \( 1 \leq i \leq q^2 + q - 1 \). Then we have

\[
\sum_{a \in \mathbb{F}_q} D_{j,k}(1,a) = -c_j + \frac{kj - k + 2}{2j} \text{ if } 1 \leq j \leq q - 1,
\]

\[
\sum_{a \in \mathbb{F}_q} D_{q,k}(1,a) = c_1 - c_q - \frac{k - 2}{2},
\]

\[
\sum_{a \in \mathbb{F}_q} D_{\ell q + j,k}(1,a) = \sum_{a \in \mathbb{F}_q} D_{(\ell - 1)q + j,k}(1,a) - \sum_{a \in \mathbb{F}_q} D_{(\ell - 1)q + j + 1,k}(1,a) - c_{\ell q + j} + \frac{k}{2\ell + 2}
\]

if \( 1 \leq \ell \leq q - 2 \text{ and } 1 \leq j \leq q - 1,\)

\[
\sum_{a \in \mathbb{F}_q} D_{\ell q,k}(1,a) = \sum_{a \in \mathbb{F}_q} D_{(\ell - 1)q,k}(1,a) - \sum_{a \in \mathbb{F}_q} D_{(\ell - 1)q + 1,k}(1,a) - c_{\ell q} + \frac{k}{2\ell} \text{ if } 2 \leq \ell \leq q - 2
\]

and

\[
\sum_{a \in \mathbb{F}_q} D_{q^2 - q + j,k}(1,a) = \sum_{i=j}^{q-1} c_{q^2 + i} + \frac{kj - k + 2}{2j} \text{ if } 0 \leq j \leq q - 1.
\]

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