NUMERICAL DIMENSION AND LOCALLY AMPLE CURVES

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Abstract. In the paper [Lau16], it was shown that the restriction of a pseudoeffective divisor $D$ to a subvariety $Y$ with nef normal bundle is pseudoeffective. Assuming the normal bundle is ample and that $D|_{Y}$ is not big, we prove that the numerical dimension of $D$ is bounded above by that of its restriction, i.e. $\kappa_{\sigma}(D) \leq \kappa_{\sigma}(D|_{Y})$. The main motivation is to study the cycle classes of "positive" curves: we show that the cycle class of a curve with ample normal bundle lies in the interior of the cone of curves, and the cycle class of an ample curve lies in the interior of the cone of movable curves. We do not impose any condition on the singularities on the curve or the ambient variety. For locally complete intersection curves in a smooth projective variety, this is the main result of Ottem [Ott16]. The main tool in this paper is the theory of $q$-ample divisors.

1. Introduction

This paper deals with subvarieties (of projective variety) which manifest positivity property. Recall that a divisor $D$ is $q$-ample if for any $\mathcal{F}$ there is an $m_0$ such that

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mD)) = 0 \quad \forall m \geq m_0.$$ 

Let $X$ be a projective variety, let $Y$ be a subvariety of $X$ of codimension $r$ and let $\tilde{X} \to X$ be the blowup morphism of $X$ along $Y$, with exceptional divisor $E$. We call $Y$ a locally ample subvariety of $X$ if $\mathcal{O}_E(E)$ is $(r-1)$-ample. If $Y$ is l.c.i in $X$, being locally ample is equivalent to having ample normal bundle. We call $Y$ an ample subvariety of $X$ if $\mathcal{O}_{\tilde{X}}(E)$ is $(r-1)$-ample (The notion of an ample subvariety was introduced in [Ott12]). We call $Y$ a nef subvariety of $X$ if $\mathcal{O}_{\tilde{X}}(mE + A)$ is $(r-1)$-ample for $m \gg 0$, where $A$ is an ample divisor. If $Y$ is l.c.i. in $X$, being nef is the same as having nef normal bundle.

In [Lau16], we showed that the restriction of a pseudoeffective divisor to a nef subvariety is pseudoeffective. In this paper, we shall study how the numerical dimension of the classes on the boundary of $\overline{\text{Eff}}^1(X)$ behave under the restriction $\tau^* : \overline{\text{Eff}}^1(X) \to \overline{\text{Eff}}^1(Y)$, assuming $Y$ is locally ample.

Nakayama showed that if $H$ is a smooth ample divisor of a smooth projective variety $X$ and $\eta \in N^1(X)_{\mathbb{R}}$ is not big, then $\kappa_{\sigma}(\eta) \leq \kappa_{\sigma}(\eta|_{H})$ [Nak04, Proposition 2.7(5)]. On the other hand, Ottem showed that if $X$ is a smooth projective variety, $Y$ is a l.c.i. subvariety with ample normal bundle and $\eta \in N^1(X)_{\mathbb{R}}$ satisfies $\eta|_{Y} = 0$, then $\kappa_{\sigma}(\eta) = 0$ [Ott16, Theorem 1]. This was a conjecture due to Peternell [Pet12, Conjecture 4.12]. The following theorem generalizes both of the above results.

**Theorem A.** Let $\iota : Y \hookrightarrow X$ be a locally ample subvariety of codimension $r$ of a projective variety $X$. If $\eta \in N^1(X)_{\mathbb{R}}$ is a pseudoeffective class such that $\eta|_{Y}$ is not big, then $\kappa_{\sigma}(\eta) \leq \kappa_{\sigma}(\eta|_{Y})$.

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From this, we deduce the following result (see theorem 5.5). Let \( Y \) be a locally ample subvariety of \( X \) and let \( f : X \to Z \) be a morphism from \( X \) to a projective variety \( Z \). If \( \dim f(Y) < \dim Y \), then \( f|_Y : Y \to Z \) is surjective, i.e. \( f(Y) = Z \).

One can regard these results as evidence that it is natural to study the notion of locally ample subvariety.

We now turn our focus to the main application of theorem A.

It seems interesting to ask how the positivity of the normal bundle of a subvariety influences the positivity of the underlying cycle class of the subvariety. The divisor case is well-known. For example, ample divisors generate an open cone in \( \text{N}^1(X)_\mathbb{R} \), called the ample cone. The closure of the ample cone is dual to the closure of the cone generated by curves in \( X \) (Kleiman). Furthermore, an effective Cartier divisor with ample normal bundle is big [Har70, Theorem III.4.2]. In this paper, we want to see whether similar properties hold for curves. Boucksom, Demailly, Păun and Peternell [BDPP13] showed that the closure of the cone of effective divisors in \( \text{N}^1(X)_\mathbb{R} \), called the pseudoeffective cone, is dual to the closure of the cone generated by strongly movable curves, called the movable cone of curves. Using this result, one can show that the cycle class of a nef curve (in particular a curve with nef normal bundle) lies in the movable cone of curves ([DPS96, Theorem 4.1], [Lau16, Theorem 1.3]). By analogy to the divisor case, it is natural to pose the following question: given a locally ample (resp. ample) curve, does the cycle class of the curve lies in the interior of the cone of curves (resp. movable cone of curves)? In this paper, we give a positive answer to this question.

**Theorem B.** Let \( X \) be a projective variety and let \( Y \) be a locally ample curve in \( X \). Then \( [Y] \in \text{N}_1(X)_\mathbb{R} \) is big, i.e. it lies in the interior of cone of curves. Furthermore, if \( Y \) meets all prime divisors of \( X \), e.g. \( Y \) is ample, then \( [Y] \) lies in the interior of the movable cone of curves.

Following an observation of Peternell [Pet12, Conjecture 4.1], Ottem already deduced that the cycle class of a locally complete intersection curve with ample normal bundle in a smooth projective variety lies in the interior of the cone of curves ([Ott16, Theorem 2]). Indeed, if \( \eta \in \text{N}^1(X)_\mathbb{R} \) is nef and \( \eta|_Y = 0 \), then the conjecture says \( \kappa_\omega(\eta) = 0 \), which forces \( \eta = 0 \). Theorem B improves upon Ottem’s result by removing any restrictions on smoothness of \( X \) and \( Y \). Our proof is different from Ottem’s in the sense that the theory of \( q \)-ample divisors is used here.

**Notation.** We work over a field of characteristic zero. A variety is meant to be an integral scheme. A curve is meant to be an integral scheme of dimension 1.

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## 2. Preliminaries

In this section, we shall recall the necessary definitions and tools needed.

### 2.1. Dualizing sheaf.

**Definition 2.1** (Dualizing sheaf [Har77, p.241]). Let \( X \) be a projective scheme of dimension \( n \). A **dualizing sheaf** for \( X \) is a coherent sheaf \( \omega_X \), together with a trace map \( t : H^n(X, \omega) \to k \) to the ground field \( k \), such that for any coherent sheaf \( \mathcal{F} \) on \( X \) the natural pairing

\[
H^n(X, \mathcal{F}) \times \text{Hom}(\mathcal{F}, \omega_X) \to H^n(X, \omega_X),
\]
followed by \( t \), induces an isomorphism
\[
\text{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^\vee
\]
of \( k \)-vector spaces.

**Proposition 2.2.** [Har77, Proposition 7.2, 7.5] Let \( X \) be a projective scheme of dimension \( n \). Then the dualizing sheaf for \( X \) exists and is unique up to unique isomorphism.

We now show that a dualizing sheaf can be embedded into a sufficiently ample line bundle. The proof can be found in the proof of [Tot13, Theorem 9.1], but we include here for the sake of convenience.

**Lemma 2.3** (Embedding a dualizing sheaf into a line bundle). Let \( X \) be a projective variety of dimension \( n \). Given an ample divisor \( H \) on \( X \). Then \( \omega_X \) is torsion-free. Moreover, there is \( l \) such that there is an embedding \( \omega_X \hookrightarrow \mathcal{O}_X(lH) \).

**Proof.** Let us first show that \( \omega_X \) is torsion-free. Indeed, let \( T \subset \omega_X \) be the torsion subsheaf.

\[
\text{Hom}(T, \omega_X) \cong H^n(X, T) = 0,
\]

The last equality follows from the fact that \( T \) is supported at a proper closed subscheme of \( X \). As \( \omega_X \) is generically a line bundle, \( \omega_X^\vee \neq 0 \). For \( l \) large, there is a nontrivial section \( s \in H^0(X, \omega_X^\vee \otimes \mathcal{O}_X(lH)) \). This induces a nontrivial map \( \omega_X \to \mathcal{O}_X(lH) \), which has to be an injection, since \( \omega_X \) is torsion free of rank 1.

\[\square\]

### 2.2. \( q \)-ample divisors

The main tool used in this paper is the theory of \( q \)-ample divisors, developed by Totaro in [Tot13]. Let us recall its definition.

**Definition 2.4** (\( q \)-ample line bundle [DPS96],[Tot13]). Let \( X \) be a projective scheme. A line bundle bundle \( \mathcal{L} \) is \( q \)-ample if for any coherent sheaf \( \mathcal{F} \) on \( X \), there is an \( m_0 \) such that
\[
H^i(X, \mathcal{F} \otimes \mathcal{L}^m) = 0
\]
for \( i > q \) and \( m > m_0 \).

**Definition 2.5** (\( q \)-T-ampleness [Tot13, Definition 6.1]). Let \( X \) be a projective variety of dimension \( n \). We fix a \( 2n \)-Koszul-ample line bundle \( \mathcal{O}_X(1) \) on \( X \). We say that a line bundle \( \mathcal{L} \) is \( q \)-T-ample if there is a positive integer \( N \), such that
\[
H^{q+i}(X, \mathcal{L}^N \otimes \mathcal{O}_X(-n-i)) = 0,
\]
for \( 1 \leq i \leq n-q \).

The definition of a Koszul-ample line bundle can be found in [Tot13, Section 1]. Given an ample line bundle, any sufficiently large tensor power is \( 2n \)-Koszul-ample [Bac86]. The following theorem is the key technical theorem in Totaro’s paper.

**Theorem 2.6.** [Tot13, Theorem 6.3] The notion of \( q \)-ampleness and \( q \)-T-ampleness are equivalent.

**Definition 2.7** (\( q \)-ample \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisors). Let \( X \) be a projective scheme. An \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( D \) on \( X \) is \( q \)-ample if \( D \) is numerically equivalent to \( cL + A \) with \( L \) a \( q \)-ample line bundle, \( c \in \mathbb{R}_{>0} \), \( A \) an ample \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor.

Based on the work of Demailly, Peternell and Schneider, Totaro also proved that
Theorem 2.8 ([Tot13, Theorem 8.3]). An integral divisor is \( q \)-ample if and only if its associated line bundle is \( q \)-ample. The \( q \)-ample \( \mathbf{R} \)-Cartier \( \mathbf{R} \)-divisors in \( N^1(X)_\mathbf{R} \) defines an open cone (but not convex in general) and that the sum of a \( q \)-ample \( \mathbf{R} \)-Cartier \( \mathbf{R} \)-divisor and an \( r \)-ample \( \mathbf{R} \)-Cartier \( \mathbf{R} \)-divisor is \( (q + r) \)-ample.

Theorem 2.9 ([Tot13, Theorem 9.1]). Let \( X \) be a projective variety of dimension \( n \). A line bundle \( \mathcal{L} \) on \( X \) is \((n-1)\)-ample if and only if \([\mathcal{L}^\vee]\) \( \in N^1(X) \) does not lie in the pseudoeffective cone.

Definition 2.10 \((q\text{-almost ample})\). Let \( X \) be a projective scheme and let \( A \) be an ample divisor on \( X \). We say that a \( \mathbf{R} \)-Cartier \( \mathbf{R} \)-divisor \( D \) is \( q \)-almost ample if \( D + \epsilon A \) is \( q \)-ample for all \( 0 < \epsilon \ll 1 \).

2.3. \( \sigma \)-dimension. Let us start with the definition of the \( \sigma \)-dimension of an \( \mathbf{R} \)-Cartier \( \mathbf{R} \)-divisor.

Definition 2.11 \((\sigma\text{-dimension})\). Let \( X \) be a projective variety. Let \( D = \sum a_i C_i \) be an \( \mathbf{R} \)-Cartier \( \mathbf{R} \)-divisor, where \( a_i \in \mathbf{R} \) and \( C_i \)'s are integral Cartier divisors and let \( H \) be any integral Cartier divisor. We then define

\[
\kappa_\sigma(D) := \max_{H \text{ integral Cartier}} \{ \max \{ l \in \mathbb{Z} \mid \limsup_{t \to \infty} \frac{h^0(X, \mathcal{O}_X(\sum |ta_j C_i + H)|)}{t^l} > 0 \} \}.
\]

This is a measure of positivity of an \( \mathbf{R} \)-Cartier \( \mathbf{R} \)-divisor that lies on the boundary of the pseudoeffective cone. However, this definition looks slightly different from the one that appeared in the literature ([Nak04], [Leh13] and [Eck16]). We shall prove in proposition (2.13) that the definition is well-defined, i.e. independent of the decomposition \( D = \sum a_i C_i \); is a numerical invariant and agrees with the usual definition with \( X \) is smooth. Nakayama’s proof of the \( \sigma \)-dimension is a numerical invariant relies on an Angehrn-Siu type argument, which requires smoothness on \( X \). One can apply resolution of singularities on a singular \( X \) and reduce to the case when \( X \) is smooth. We shall give a proof that has no assumptions on singularities on \( X \) using \( q \)-ample divisors.

Lemma 2.12. Let \( X \) be a projective variety. Let \( \mathcal{B} \subset N^1(X)_\mathbf{R} \) be a bounded subset. Then there is an integral Cartier divisor \( H \) such that for any integral Cartier divisor \( C \) with \([C] \in \mathcal{B}, \]

\[
h^0(X, \mathcal{O}_X(H - C)) \neq 0.
\]

Proof. Let \( A \) be an ample divisor on \( X \). Fix a \((2n)\)-Koszul-ample line bundle \( \mathcal{O}_X(j) \) on \( X \). Let \( \omega_X \) be the dualizing sheaf of \( X \). There is an embedding \( \omega_X \hookrightarrow \mathcal{O}_X(j) \) for some \( j \), and that \( \dim \text{Supp}(\text{coker}(\omega_X \hookrightarrow \mathcal{O}_X(j))) \leq n - 1 \) by lemma 2.3.

One can choose a sufficiently large \( m \) such that \( \mathcal{O}_X(mA - C) \otimes \mathcal{O}_X(-j - n - 1) \) is ample for any integral Cartier divisor \( C \) with \([C] \in \mathcal{B} \). In particular, \( \mathcal{O}_X(-mA + C) \otimes \mathcal{O}_X(j + n + 1) \) is not \((n - 1)\)-ample. This implies \( h^n(X, \mathcal{O}_X(-mA + C) \otimes \mathcal{O}_X(j)) \neq 0 \) [Tot13, Theorem 6.3], and \( h^n(X, \mathcal{O}_X(mA - C)) = h^n(X, \mathcal{O}_X(-mA + C) \otimes \omega_X) \neq 0 \).

Proposition 2.13. Let \( X \) be a projective variety and let \( D \) be a pseudoeffective \( \mathbf{R} \)-Cartier \( \mathbf{R} \)-divisor on \( X \). Then

1. The definition of \( \kappa_\sigma(D) \) does not depend on the decomposition \( D = \sum a_i C_i \). In fact, if \( D \equiv D' \), then \( \kappa_\sigma(D) = \kappa_\sigma(D') \).
2. Assuming that \( X \) is smooth,

\[
\kappa_\sigma(D) = \max_{H \text{ integral Cartier}} \{ \max \{ l \in \mathbb{Z} \mid \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(|mD| + H))}{m^l} > 0 \} \}.
\]
The right hand side of this equation is the usual definition of the \( \kappa_\sigma(D) \) ([Nak04],[Leh13],[Eck16]). Here we are rounding down \( D \) as an \( \mathbb{R} \)-Weil divisor.

**Proof.** For (1), suppose \( D \equiv D' \), \( D = \sum a_i C_i \) and \( D' = \sum a'_i C'_i \). By lemma 2.12, there is an integral Cartier divisor \( H' \) such that \( \mathcal{O}_X(H' + C) \) is effective for any integral Cartier \( C \equiv \sum r_i C_i + \sum r'_j C'_j \) where \( r_i, r'_j \in [-2, 2] \). Given any integral Cartier divisor \( H \), write \( \sum [ma_i]C'_i + H + H' \) as

\[
\sum [ma_i]C_i + H + (\sum [ma_i]C'_i - mD') + (mD - \sum [ma_i]C_i) + (mD' - mD) + H'.
\]

This implies \( h^0(X, \mathcal{O}_X(\sum [ma_i]C_i + H)) \leq h^0(X, \mathcal{O}_X(\sum [ma_i]C'_i + H + H')) \). We can reverse the roles of \( D \) and \( D' \) and conclude (1).

For (2), \( D \) is expressed uniquely as \( \sum a_i \Gamma_i \), where \( \Gamma_i \)'s are prime divisors (which are Cartier by the smoothness assumption), \( a_i \in \mathbb{R} \). We have \( |mD| = \sum |ma_i| \Gamma_i \), the equality then follows from (1).

\[ \square \]

Thanks to Proposition 2.13 (1), we may refer to \( \kappa_\sigma(\eta) \), where \( \eta \in N^1(X)_{\mathbb{R}} \), without ambiguity.

Here are some of the basic properties of \( \kappa_\sigma(D) \). The proof is essentially the same as the one given in [Nak04, Proposition V.2.7].

**Proposition 2.14** (Basic properties). Let \( X \) be a projective variety of dimension \( n \) and let \( \eta \in N^1(X)_{\mathbb{R}} \) be a pseudoeffective class.

1. If \( f : X' \to X \) is a surjective morphism from a projective variety, then \( \kappa_\sigma(\eta) = \kappa_\sigma(f^*(\eta)) \).
2. \( 0 \leq \kappa_\sigma(\eta) \leq n \).
3. \( \kappa_\sigma(\eta) = n \) if and only if \( \eta \) is big.

### 2.4. Ample and Locally ample subvariety.

In this subsection, we shall first recall the definition of an ample subscheme, which was introduced by Ottem in [Ott12]. Then we introduce the notion of a locally ample subscheme, which generalizes the notion of a subvariety that is l.c.i. in the ambient variety with ample normal bundle.

**Definition 2.15** (Ample subscheme [Ott12, Definition 3.1]). Let \( X \) be a projective scheme of dimension \( n \) and let \( Y \) be a subscheme of \( X \) of codimension \( r \). Let \( E \) be the exceptional divisor of the blowup of \( X \) along \( Y \). We say that \( Y \) is an **ample subscheme** of \( X \) if \( E \) is \((r - 1)\)-ample.

This notion of ample subschemes indeed generalize the notion of an ample divisor naturally. For example, if \( Y \) is a smooth ample subvariety of a smooth projective variety, then the Lefschetz hyperplane theorem with rational coefficient holds: the natural maps

\[ H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q}) \]

are isomorphisms for \( i < n - r \) and is injective for \( i = n - r \) [Ott12, Corollary 5.3].

From the point of view of intersection theory, we also know that if \( Y \) is an l.c.i. ample subvariety of a projective variety \( X \). Then for any subvariety \( Z \) of \( X \) of complementary dimension, \( Y \cdot Z > 0 \) [FL83].

For more about ample subvarieties, c.f. [Ott12].

**Definition 2.16** (Locally ample subscheme). Let \( X \) be a projective scheme of dimension \( n \) and let \( Y \) be a subscheme of \( X \) of codimension \( r \). Let \( E \) be the exceptional divisor of
the blowup of $X$ along $Y$. We say that $Y$ is an *locally ample subscheme* of $X$ if $\mathcal{O}_E(E)$ is $(r-1)$-ample.

The following proposition shows that the concept of a locally ample subscheme generalizes the notion of an l.c.i. subvariety with ample normal bundle.

**Proposition 2.17.** [Ott12, Corollary 4.3] Let $X$ be a projective scheme of dimension $n$ and let $Y$ be a l.c.i. subscheme of $X$ of codimension $r$. Then $Y$ has ample normal bundle if and only if $Y$ is locally ample in $X$.

**Proposition 2.18** (Pullback). Let $X$ be a projective scheme and let $Y$ be a locally ample subscheme of $X$ of codimension $r$. Let $Z$ be a closed subscheme of $X$. Suppose $Y \cap Z$ has codimension $r$ in $Z$. Then $Y \cap Z$ is locally ample in $Z$.

**Proof.** Indeed, we have the following commutative diagram

$$\begin{array}{ccc}
\text{Bl}_Y X & \xrightarrow{\pi_X} & X \\
\text{Bl}_{Y \cap Z} Z & \xrightarrow{\pi_Z} & \text{Bl}_Y Z \\
\end{array}$$

Note that the exceptional divisor of $\pi_Z$, $E_Z$, is the restriction of the exceptional divisor $E$ of $\pi_X$. If $\mathcal{O}_E(E)$ is $(r-1)$-ample, so is $\mathcal{O}_{E_Z}(E)$. \hfill \square

We now show that the notion of locally ample subscheme satisfies the transitivity property. The proof is a bit involved but is very similar to the proof of transitivity of ample subschemes [Lau16, Theorem 4.10]. The following theorem demonstrates that the notion of locally ample subvarieties is a reasonable generalization of the notion of subvarieties with ample normal bundle. However, we won’t need it later. The reader may want to skip on the first reading.

**Theorem 2.19** (Transitivity of locally ample subschemes). Let $Y$ be a locally ample subscheme of $X$ of codimension $r_1$ and let $Z$ be a locally ample subscheme of $Y$ of codimension $r_2$. Then $Z$ is a locally ample subscheme of $X$ of codimension $r_1 + r_2$.

**Proof.** First, note that we have the following commutative diagram

$$\begin{array}{ccc}
\text{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X & \xrightarrow{\pi_Y} & \text{Bl}_{\mathcal{I}_Z} X \\
\text{Bl}_{\mathcal{I}_Y} X & \xrightarrow{\pi_Z} & X, \\
\end{array}$$

where $\pi_Y$ and $\pi_Z$ are induced by blowing up the ideals $\mathcal{I}_Y \cdot \mathcal{O}_{\text{Bl}_{\mathcal{I}_Z}}$ and $\mathcal{I}_Z \cdot \mathcal{O}_{\text{Bl}_{\mathcal{I}_Y}}$ respectively. Let $E'_Y$ and $E'_Z$ be the exceptional divisors of $\pi'_Y$ and $\pi'_Z$. We also let $E_Y$ be the exceptional divisor of $\pi_Z$ and let $E_Y$ be the divisor in $\text{Bl}_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X$ such that $E_Y + E_Z$ is the exceptional divisor of $\pi_Y$. Note that $\pi'_Z E'_Y = E_Y + E_Z$ and $\pi'_Y E'_Z = E_Z$. The proof of the above statements can be found in [Lau16, Lemma 4.11].

To prove that $Z$ is locally ample in $X$, it is the same as to show that $\mathcal{O}_{E'_Z}(E'_Z)$ is $(r_1 + r_2 - 1)$-ample. If we let $\hat{Y}$ be the strict transform of $Y$ in $\text{Bl}_{\mathcal{I}_Z} X$. We know that $\mathcal{O}_{E'_Z \cap \hat{Y}}(E'_Z)$ is $(r_2 - 1)$-ample. By [Lau16, Proposition 4.6], we know that $\hat{\pi'}_Y$ has fiber dimension at most $r_1 - 1$. Therefore, $\hat{\pi}_Y$ has fiber dimension at most $r_1 - 1$ as well. Let $H$ be an ample divisor on $\text{Bl}_{\mathcal{I}_Z} X$. By [Lau16, Lemma 4.9], it suffices to show that for any $l \geq 0$,

$$H^i(E_Z, \mathcal{O}_{E_Z}(lE_Z) \otimes \pi_Y^* \mathcal{O}_{\text{Bl}_{\mathcal{I}_Z} X}(lH)) = 0$$
for $i > r_1 + r_2 - 1$ and $m \gg 0$. Fix $l \in \mathbb{Z}_{\geq 0}$.

Claim 1. $(E_Z - \delta E_Y)|_{E_Z \cap E_Y}$ is $(r_2 - 1)$-ample for $0 < \delta \ll 1$.

Proof of claim. Since $-E_Y$ is $\pi_Y$-ample, $\left((\pi^*_Y E'_Z - \delta E_Y)|_{E_Z \cap E_Y} = (E_Z - \delta E_Y)|_{E_Z \cap E_Y}\right)$ is $(r_2 - 1)$-ample for $0 < \delta \ll 1$, by [Lau16, Proposition 2.8].

Claim 2. $(E_Y + E_Z - \epsilon E_Z)|_{E_Z}$ is $(r_1 - 1)$-ample for $0 < \epsilon \ll 1$.

Proof of claim. Note that $\pi_Z$ restricts to a morphism $E_Z \to E'_Y$, $(\pi^*_Z E'_Y - \epsilon E_Z)|_{E_Z} = (E_Y + E_Z - \epsilon E_Z)|_{E_Z}$ is $(r_1 - 1)$-ample for $0 < \epsilon \ll 1$ since $-E_Z$ is $\pi_Z$-ample, by [Lau16, Proposition 2.8].

By the above claims, for sufficiently large integer $k$, $\mathcal{O}_{E_Z \cap E_Y}(kE_Z - E_Y)$ is $(r_2 - 1)$-ample and $\mathcal{O}_{E_Z}(kE_Z + kE_Z)$ is $(r_1 - 1)$-ample. Fix such $k$.

Given $m_1, m_2 \in \mathbb{Z}$, write

$$m_1E_Y + m_2E_Z = \lambda_1(kE_Z - E_Y) + \lambda_2(kE_Z + (k + 1)E_Y) + j_1E_Y + j_2E_Z,$$

where $\lambda_2 = \lfloor \frac{m_1 + \lfloor \frac{m_2}{k+2} \rfloor}{k+2} \rfloor$, $\lambda_1 = \lfloor \frac{m_2}{k} \rfloor - \lambda_2$, $j_1 = \left((m_1 + \lfloor \frac{m_2}{k} \rfloor) \mod (k+2)\right)$ and $j_2 = (m_2 \mod k)$. Note that $0 \leq j_1 < k + 2$ and $0 \leq j_2 < k$. The precise formulae for $\lambda_1$ and $\lambda_2$ are not very important. The plan is to choose a big $m_2$, then let $m_1$ increase. As $m_1$ grows, $\lambda_1$ decreases and $\lambda_2$ increases. We then use the positivity of $(r_2 - 1)$-ampleness of $\mathcal{O}_{E_Z \cap E_Y}(kE_Z - E_Y)$ and $(r_1 - 1)$-ampleness of $\mathcal{O}_{E_Z}(kE_Z + (k + 1)E_Y)$ to prove the required vanishing statement.

Since $\mathcal{O}_{E_Z}(kE_Z + (k + 1)E_Y)$ is $(r_1 - 1)$-ample, we may find $\Lambda_2$ such that

$$(2.1) \quad H^i(E_Z, \mathcal{O}_{E_Z}(\lambda_2(kE_Z + (k + 1)E_Y) + j_1E_Y + j_2E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{s_X}} x(\cdot)) = 0$$

for $i > r_1 - 1$, $\lambda_2 \geq \Lambda_2$, $0 \leq j_1 < k + 2$ and $0 \leq j_2 < k$.

Applying theorem [Lau16, Theorem 3.9] to the scheme $E_Z \cap E_Y$, there is an $\Lambda'_2$ such that

$$(2.2) \quad H^i(E_Z \cap E_Y, \mathcal{O}_{E_Z}(\lambda_1(kE_Z - E_Y) + \lambda_2(kE_Z + (k + 1)E_Y) + j_1E_Y + j_2E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{s_X}} x(\cdot)) = 0$$

for $i > (r_2 - 1) + (r_1 - 1)$, $\lambda_1 \geq 0$, $\lambda_2 \geq \Lambda'_2$, $0 \leq j_1 < k + 2$ and $0 \leq j_2 < k$. This implies

$$H^i(E_Z, \mathcal{O}_{E_Z}(m_2E_Z + m_1E_Y) \otimes \pi_Y^*(\mathcal{O}_{Bl_{s_X}} x(\cdot))) \cong H^i(E_Z \cap E_Y, \mathcal{O}_{E_Z}(m_2E_Z + (m_1 + 1)E_Y) \otimes \pi_Y^*(\mathcal{O}_{Bl_{s_X}} x(\cdot)))$$

for $i > r_1 + r_2 - 1$, $0 < m_1 + 1 < (k + 1)[\frac{m_2}{k}] + k + 2$ and $[\frac{m_1 + 1 + \lfloor \frac{m_2}{k+2} \rfloor}{k+2}] \geq \Lambda'_2$.

Choose some big $M_2$ such that $[\frac{M_2}{k+2}] \geq \max\{\Lambda_2, \Lambda'_2\}$. Applying (2.2) repeatedly, we have for $m_2 > M_2$,

$$(2.3) \quad H^i(E_Z, \mathcal{O}_{E_Z}(m_2E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{s_X}} x(\cdot))) \cong H^i(E_Z, \mathcal{O}_{E_Z}(m_2E_Z + (k + 1)[\frac{m_2}{k}]E_Y) \otimes \pi_Y^*(\mathcal{O}_{Bl_{s_X}} x(\cdot)))$$

for $i > r_1 + r_2 - 1$. The above cohomology group can be rewritten as

$$H^i(E_Z, \mathcal{O}_{E_Z}([\frac{m_2}{k}](kE_Z + (k + 1)E_Y) + (m_2 - k[\frac{m_2}{k}])E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{s_X}} x(\cdot)))$$

which is 0 by (2.1). This completes the proof. \[\square\]

Corollary 2.20 (Intersection of locally ample subschemes). Let $X$ be a projective scheme. Let $Y$ and $Z$ be locally ample subschemes of $X$ of codimension $r$ and $s$ respectively and that $Y \cap Z$ is of codimension $r + s$ in $X$. Then $Y \cap Z$ is locally ample in $X$. \[\square\]
Proof. By proposition 2.18, \( Y \cap Z \) is locally ample in \( Z \). Hence, \( Y \cap Z \) is locally ample in \( X \) as well. \( \square \)

3. Numerical dominance

In this section, we prove a basic fact on Nakayama’s notion of numerical dominance, which will streamline the argument in the proof of the main theorem.

Let us first start by stating the definition of numerical dominance.

**Definition 3.1.** [Nak04, Definition 2.12] Given two classes \( \eta_1, \eta_2 \in \text{N}^1(X)_{\mathbb{R}} \). We say that \( \eta_1 \) **numerically dominates** \( \eta_2 \) if for any ample divisor \( A \) and for any \( b \in \mathbb{R} \) there are \( t_1, t_2 > b \) such that \( \eta_1 t_1 - t_2 \eta_2 + A \) is pseudoeffective.

We say that a class \( \eta \in \text{N}^1(X)_{\mathbb{R}} \) numerically dominates a closed subvariety \( Y \) of \( X \) if on the blowup \( \pi : \text{Bl}_Y X \to X \), \( \pi^* \eta \) numerically dominates the exceptional divisor \( E \).

**Lemma 3.2.** Let \( X \) be a projective variety and let \( \eta_1, \eta_2 \in \text{N}^1(X)_{\mathbb{R}} \). Then \( \eta_1 \) numerically dominates \( \eta_2 \) if there exists an ample divisor \( A \) such that for any \( b \in \mathbb{R} \) there are \( t_1, t_2 > b \) such that \( t_1 \eta_1 - t_2 \eta_2 + A \) is pseudoeffective.

**Proof.** Suppose the hypothesis in the lemma holds. Given an ample divisor \( A' \), choose a large enough integer \( a \) such that \( aA' - A \) is pseudoeffective. Given \( b > 0 \), take \( t_1, t_2 > ab \) such that \( t_1 \eta_1 - t_2 \eta_2 + A \) is pseudoeffective. Then \( \frac{t_1}{a} \eta_1 - \frac{t_2}{a} \eta_2 + A' = \frac{t_1}{a}(t_1 \eta_1 - t_2 \eta_2 + A) + (A' - \frac{1}{a}A) \) is pseudoeffective. \( \square \)

Let us relate the negation of numerical dominance and vanishing of the top cohomology group.

**Proposition 3.3.** Let \( X \) be a projective variety of dimension \( n \) and let \( Y \) be a subvariety of \( X \). Let \( E \) be the exceptional divisor on \( \tilde{X} := \text{Bl}_Y X \), the blowup of \( X \) along \( Y \). Let \( D \) be a pseudoeffective \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor on \( X \), written as \( \sum a_i C_i \), where \( a_i \in \mathbb{R} \) and \( C_i \)'s are integral Cartier divisors. Fix a \( 2n \)-Koszul-ample line bundle \( \mathcal{O}(H) \) on \( \tilde{X} \).

If there is some ample integral Cartier divisor \( A \) such that \( A - (n + 1)H \) and \( A - (n + 1)H + eE - \sum c_i C_i \) are ample for \( e, c_i \in [0, 1] \) on \( \tilde{X} \) and there is some \( b \in \mathbb{R} \) such that

\[
h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum |t_{a_i}| \pi^* C_i - A)) = 0
\]

for all \( t \in (b, +\infty) \) and for all integer \( k > b \), then \( D \) does not numerically dominate \( Y \).

On the other hand, if \( D \) does not numerically dominate \( Y \), then for any divisor \( B \), there is \( b \in \mathbb{R} \) such that

\[
h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum |t_{a_i}| \pi^* C_i - B)) = 0
\]

for all \( t \in (b, +\infty) \) and for all integer \( k > b \).

**Proof.** For the first statement, by the hypothesis,

\[
h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum |t_{a_i}| \pi^* C_i - A + (n + 1)H) \cap \mathcal{O}_{\tilde{X}}(-(n + 1)H)) = 0
\]

for \( k, t > b, k \in \mathbb{Z} \). By theorem 2.6, \( kE - \sum |t_{a_i}| \pi^* C_i - A + (n + 1)H \) is \( (n - 1) \)-ample for \( k, t > b, k \in \mathbb{Z} \). For \( t_1, t_2 > b \), we can write \( t_2 E - t_1 \pi^* D - (A - (n + 1)H) = (|t_2|E - |t_1| \pi^* D - \epsilon(A - (n + 1)H)) + ((1 - \epsilon)(A - (n + 1)H) + \epsilon(1 - \epsilon)(A - (n + 1)H) + \epsilon(1 - \epsilon)(A - (n + 1)H)) + \epsilon(1 - \epsilon)(A - (n + 1)H) \) and observe that the first term is \( (n - 1) \)-ample and the second term is ample for \( 0 < \epsilon \ll 1 \). It follows that \( t_2 E - t_1 \pi^* D - (A - (n + 1)H) \) is \( (n - 1) \)-ample for \( t_1, t_2 > b \). Thus, \( t_1 \pi^* D - t_2 E + (A - (n + 1)H) \) is not pseudoeffective for \( t_1, t_2 > b \). This proves the first assertion.
For the second statement, for sufficiently large $l$, we can embed $\omega_{\tilde{X}} \hookrightarrow \mathcal{O}(lH)$. We may also assume that $B+lh$ is ample. By lemma 3.2, there is a $b$ such that $t_1 \pi^*D - t_2 E + B + lh$ is not pseudoeffective for $t_1, t_2 > b$. Thus, for $k, t > b$ and $k \in \mathbb{Z}$,

\[
h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum [ta_i] \pi^*C_i - B)) = h^0(\tilde{X}, \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\sum [ta_i] \pi^*C_i - kE + B)) \quad \text{(Duality)}
\]

\[
\leq h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\sum [ta_i] \pi^*C_i - kE + B + lh)) \quad (\omega_{\tilde{X}} \hookrightarrow \mathcal{O}(lh))
\]

\[
= 0
\]

\[\square\]

4. Proof of Theorem A

We are now ready to demonstrate how the notion of numerical dominance come into the picture.

**Proposition 4.1.** Let $X$ be a projective variety of dimension $n$, let $Y$ be a locally ample subvariety of codimension $r$ of $X$ and let $\eta \in N^1(X)_{\mathbb{R}}$ be a pseudoeffective class such that $\eta|_Y$ is not big. Then $\eta$ does not numerically dominate $Y$.

**Proof.** Let $\tilde{X}$ be the blowup of $X$ along $Y$, with exceptional divisor $E$. We fix a Koszul-ample line bundle $\mathcal{O}_{\tilde{X}}(H)$. Take $D = \sum \alpha_i C_i$ to be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that its class equals to $\eta$. Here $\alpha_i \in \mathbb{R}$ and $C_i$’s are integral Cartier divisors. We fix an integer $l > n + 1$ such that $(l - (n + 1))H + eE - \sum c_i C_i$ is ample for any $e, c_i \in [0, 1]$. We would like to prove that for any coherent sheaf $\mathcal{F}$ on $X$, there is $k_0$ such that

\[
h^{n-1}(E, \mathcal{F} \otimes \mathcal{O}_E(kE - \sum [ta_i] \pi^*C_i - lH)) = 0
\]

for $k \geq k_0$ and $t \geq 0$. It is enough to prove that for the vanishing of cohomology groups on each of the irreducible components of $E$. In other words, letting $E'$ be an irreducible component of $E$, it suffices to prove that there is $k_0'$ such that $h^{n-1}(E', \mathcal{F} \otimes \mathcal{O}_{E'}(kE - \sum [ta_i] \pi^*C_i - lH)) = 0$ for $k \geq k_0'$ and $t \geq 0$. As there is a surjection $\mathcal{O}(B) \twoheadrightarrow \mathcal{F}$, where $\mathcal{O}(B)$ is a line bundle, it suffices to prove the vanishing assuming $\mathcal{F}$ is a line bundle $\mathcal{O}(B)$. By duality,

\[
h^{n-1}(E', \mathcal{O}_{E'}(kE - \sum [ta_i] \pi^*C_i + B - lH)) = h^0(E', \omega_{E'} \otimes \mathcal{O}_{E'}(-kE + \sum [ta_i] \pi^*C_i - B + lH)),
\]

where $\omega_{E'}$ is the dualizing sheaf of $E'$. We may embed $\omega_{E'} \hookrightarrow \mathcal{O}_{E'}(jH)$ for some $j$ by lemma 2.3. It suffices to prove that there is $k_0'$ such that

\[
h^0(E', \mathcal{O}_{E'}(-kE + \sum [ta_i] \pi^*C_i - B + (l + j)H)) = 0
\]

for $k \geq k_0'$ and $t \geq 0$.

As $D|_Y$ is not big, $-D|_Y$ is $(n-r-1)$-almost ample. By [Lau16, Proposition 2.8], $\pi^*(-D)|_E$ is also $(n-r-1)$-almost ample. Since $\mathcal{O}_E(E)$ is $(r-1)$-ample, we may take $k_0'$ such that $kE + \sum e_i \pi^*C_i + B - (l + j)H)|_{E'}$ is $(r-1)$-ample for $k \geq k_0'$ and $e_i \in [0, 1]$, thanks to the openness of the $(r-1)$-ample cone (theorem 2.8). Thus for $k \geq k_0'$ and $t \geq 0$,

\[
(kE - \sum [ta_i] \pi^*C_i + B - (l + j)H)|_{E'} = ((kE + \sum [ta_i] \pi^*C_i + B - (l + j)H) + \pi^*(-tD))|_{E'}
\]

is $(r-1) + (n-r-1) = (n-2)$-ample, by theorem 2.8. Now we have (4.2) by [Tot13, Theorem 9.1], hence also (4.1).

If we fix $t$ and take $k$ large enough, then $h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum [ta_i] \pi^*C_i - lH))) = 0$, since $E$ is $(n-1)$-ample. We tensor the short exact sequence

\[
0 \rightarrow \mathcal{O}_{\tilde{X}}(kE) \rightarrow \mathcal{O}_{\tilde{X}}((k + 1)E) \rightarrow \mathcal{O}_{\tilde{X}}((k + 1)E) \rightarrow 0
\]

\[\square\]
by \( \mathcal{O}_X(\sum_{} [ta_i] \pi^* C_i - IH) \), and consider its associated long exact sequence of cohomologies. We apply (4.1), letting \( \mathcal{F} \) to be the structure sheaf \( \mathcal{O}_E \), there is \( k_0 \) such that \( h^{n-1}(E, \mathcal{O}_E (kE - \sum [ta_i] \pi^* C_i - lH)) = 0 \) for \( k \geq k_0 \) and \( t \geq 0 \). Therefore, 
\[
h^n(X, \mathcal{O}_X(kE - \sum [ta_i] \pi^* C_i - lH)) = 0
\]
for \( k \geq k_0 \) and \( t \geq 0 \). We may now conclude the proof by applying proposition 3.3. \( \square \)

**Proposition 4.2.** Let \( X \) be a projective variety and let \( Y \) be a subvariety of \( X \). Let \( D \) be a pseudoeffective \( \mathcal{R} \)-Cartier \( \mathcal{R} \)-divisor such that \( D \) does not numerically dominate \( Y \). Let \( \pi : X \to X \) be the blowup of \( X \) along \( Y \), with exceptional divisor \( E \). Suppose \( \pi|_E : E \to Y \) is an equidimensional morphism. Then \( \kappa_\sigma(D) \leq \kappa_\sigma(D|_Y) \).

**Proof.** We use the same notations as in the proof of the preceding proposition. By proposition 2.14, \( \kappa_\sigma(D) = \kappa_\sigma(D|_Y) \). It is enough to look at the growth (in \( t \)) of \( h^0(X, \mathcal{O}_X(\sum [ta_i] \pi^* C_i + b_1 H)) \), for a large enough integer \( b_1 \). Since \( \omega_X \) is generically a line bundle, the natural map \( \mathcal{O}_X \to \omega_X \) is an injection. We have the inequality 
\[
h^0(X, \mathcal{O}_X(\sum [ta_i] \pi^* C_i + b_1 H)) \leq h^0(X, \omega_X \otimes \mathcal{O}_X(\sum [ta_i] \pi^* C_i + b_1 H)) = h^0(X, \omega_X \otimes \mathcal{O}_X(\sum [ta_i] \pi^* C_i - b_1 H)).
\]
There is some surjection \( \oplus \mathcal{O}_X(-b_2 H) \to \omega_X \). Therefore, 
\[
h^n(X, \omega_X \otimes \mathcal{O}_X(\sum [ta_i] \pi^* C_i - b_1 H)) \leq N \cdot h^n(X, \mathcal{O}_X(\sum [ta_i] \pi^* C_i - (b_1 + b_2) H))
\]
By proposition 4.1 and proposition 3.3, there is \( k_0 \) such that 
\[
h^n(X, \mathcal{O}_X(kE - \sum [ta_i] \pi^* C_i - (b_1 + b_2) H)) = 0
\]
for \( k \geq k_0 \) and \( t \geq k_0 \). Tensoring the short exact sequence 4.3 by \( \mathcal{O}_X(\sum [ta_i] \pi^* C_i - (b_1 + b_2) H) \) and considering the associated long exact sequence of cohomologies, we have 
\[
h^n(X, \mathcal{O}_X(\sum [ta_i] \pi^* C_i - (b_1 + b_2) H)) \leq \sum_{k=1}^{k_0} h^{n-1}(E, \mathcal{O}_E (kE - \sum [ta_i] \pi^* C_i - (b_1 + b_2) H))
\]
for \( t \geq k_0 \).

Note that the restriction of \( \pi : X \to X \) to the exceptional divisor \( \pi|_E : E \to Y \) is an equidimensional morphism, with fiber dimension equals to \( r - 1 \). Thus, \( R^d(\pi|_E)_* \mathcal{O}_E (kE - (b_1 + b_2) H) = 0 \) for \( d > r - 1 \). Note also that \( \dim Y = n - r \), which implies that \( h^d(Y, \mathcal{F}) = 0 \) for \( d > n - r \) and for any coherent sheaf \( \mathcal{F} \) on \( Y \). We now apply Leray spectral sequence and the above remarks to see that for \( 1 \leq k \leq k_0 \), 
\[
h^{n-1}(E, \mathcal{O}_E (kE - \sum [ta_i] \pi^* C_i - (b_1 + b_2) H)) = h^{n-r}(Y, (R^{n-r}(\pi|_E)_* \mathcal{O}_E (kE - (b_1 + b_2) H)) \otimes \mathcal{O}_Y (-[ta_i] C_i)) = h^0(Y, \omega_Y \otimes (R^{n-r}(\pi|_E)_* \mathcal{O}_E (kE - (b_1 + b_2) H)) \otimes \mathcal{O}_Y ([ta_i] C_i)) \quad \text{(Duality)}\]
Since \( (R^{n-r}(\pi|_E)_* \mathcal{O}_E (kE - (b_1 + b_2) H)) \otimes \mathcal{O}_Y ([ta_i] C_i) \) is reflexive [Har80, Corollary 1.2] and by lemma 2.3, for sufficiently large \( l \), there is an embedding \( \omega_Y \otimes (R^{n-r}(\pi|_E)_* \mathcal{O}_E (kE - (b_1 + b_2) H)) \otimes \mathcal{O}_Y ([ta_i] C_i) \to \oplus_{k=0}^{k_0} \mathcal{O}_Y (lH) \) for \( 1 \leq k \leq k_0 \). We can conclude that 
\[
h^0(X, \mathcal{O}_X(\sum [ta_i] \pi^* C_i + b_1 H)) \leq N \cdot (\sum_{k=1}^{k_0} N_k) \cdot h^0(Y, \mathcal{O}_Y ([ta_i] C_i + lH)) \quad \text{for} \quad t \gg 0.\]
This proves the proposition. \( \square \)

**Theorem A** (Numerical dimension via restriction). With the same assumptions as in proposition 4.1. Then \( \kappa_\sigma(\eta) \leq \kappa_\sigma(\eta|_Y) \).
Proof. Combine proposition 4.1 and 4.2 and note that if $Y$ is locally ample, then $E \to Y$ is equidimensional [Lau16, Proposition 4.6].

5. APPLICATIONS OF THEOREM A

We give three applications of theorem A. The first one is on positivity of cycle classes of locally ample and ample curves; the second one concerns the fact that locally ample subvarieties cannot be contracted and the third one relates numerical dimension and partial positivity.

5.1. Cycle classes of locally ample/ample curves. Peternell conjectured that if $Y$ is a smooth curve with ample normal bundle in a smooth projective variety $X$ and $\eta \in N^1(X)$ is a pseudoeffective class with $\eta|_Y = 0$, then $\kappa_\sigma(\eta) = 0$ [Pet12, Conjecture 4.12]. Ottem later showed that the conjecture is indeed true [Ott16, Theorem 1]. From there, Peternell observed that the cycle class of a smooth curve with ample normal bundle lies in the interior of the cone of curves ([Pet12, Conjecture 4.1],[Ott16, Theorem 2]). Indeed, if $\eta \in N^1(X)_\mathbb{R}$ is nef and $\eta|_Y = 0$, the conjecture says $\kappa_\sigma(\eta) = 0$. But this forces $\eta = 0$. We are able to generalize this result by removing any restrictions on smoothness on $X$ and $Y$.

Proposition 5.1. [Ott16] Let $X$ be a projective variety. Let $\eta \in N^1(X)_\mathbb{R}$ be a pseudoeffective class. If $\kappa_\sigma(\eta) = 0$ and $\eta$ is nef, then $\eta = 0$.

Proof. It follows from the argument on [Ott16, p.5]. We include the proof here for the sake of completeness.

Let $H$ be an ample divisor of $X$. Note that if we can prove that $\eta|_H = 0$, it would imply $\eta = 0$. By induction on dimension of $X$, it suffices to show that $\kappa_\sigma(\eta|_H) = 0$. Let $D = \sum a_i C_i$ be a pseudoeffective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that the numerical class of $D$ is $\eta$. Here $a_i \in \mathbb{R}$ and $C_i$’s are integral Cartier divisors. By Fujita vanishing theorem, there is a $k_1$ such that for $k \geq k_1$,

$$H^1(X, \mathcal{O}_X(kH + N)) = 0,$$

for any nef divisor $N$. Take a sufficiently large $k_1$ such that $k_1 H - \sum e_i C_i$ is ample, for any $e_i \in [0, 1]$. For $t \geq 0$, $k_1 H + \sum [t a_i] C_i = t D + (k_1 H - \sum t a_i C_i)$ is nef. Thus,

$$H^1(X, \mathcal{O}_X(kH + \sum [t a_i] D)) = 0$$

for $k \geq k_0 + k_1$. Therefore, we have the surjection

$$H^0(X, \mathcal{O}_X(\sum [t a_i] D + k H)) \to H^0(H, \mathcal{O}_H(\sum [t a_i] D + k H))$$

for $k \geq k_0 + k_1$ and $t \geq 0$. Hence $\kappa_\sigma(\eta|_H) = 0$. 

The following theorem generalizes the first half of the main theorem in Ottem’s paper [Ott16, Theorem 2].

Theorem 5.2. Let $X$ be a projective variety. Let $Y$ be a locally ample subvariety of dimension 1 of $X$. Then the cycle class of $Y$ in $N_1(X)\mathbb{R}$ is big, i.e. it lies in the interior of the cone of curves, $\overline{NE}(X)$.

Proof. Suppose there is some nef class $\eta \in N^1(X)\mathbb{R}$ such that $\eta|_Y = 0$. By theorem A, $\kappa_\sigma(\eta) = 0$. We then apply proposition 5.1 to conclude that $\eta = 0$.

We shall need the following proposition which shows that a pseudoeffective class $\eta \in N^1(X)_\mathbb{R}$ on a smooth projective variety with $\kappa_\sigma(\eta) = 0$ is in fact “effective”.

Proof. Combine proposition 4.1 and 4.2 and note that if $Y$ is locally ample, then $E \to Y$ is equidimensional [Lau16, Proposition 4.6].

□
Proposition 5.3. [Nak04, Proposition V.2.7] Let $X$ be a smooth projective variety. Let $\eta \in N^1(X)_{\mathbb{R}}$ be a pseudoeffective class. If $\kappa_\sigma(\eta) = 0$, then there is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $\sum a_iC_i$, where $a_i \in \mathbb{R}_{>0}$ and $C_i$ are prime divisors, such that its numerical class in $N^1(X)_{\mathbb{R}}$ equals to $\eta$.

We are now ready to show that the cycle class of an ample curve lies in the interior of the movable cone of curves. This strengthens the second half of [Ott16, Theorem 2].

Theorem 5.4. Let $X$ be a projective variety and let $Y$ be a locally ample curve in $X$. Suppose $Y$ meets all prime divisors of $X$. Then the cycle class $[Y]$ lies in the interior of the movable cone of curves. In particular, the cycle class of an ample subvariety of dimension 1 lies in the interior of the movable cone of curves.

Proof. Note that the second statement follows from the first. Indeed, if $Y$ is an ample curve in $X$, then $H^{n-1}(X \setminus Y, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on $X \setminus Y$ [Ott12, Proposition 5.1]. In particular, $X \setminus Y$ cannot contain any prime divisor.

Let $\pi : \tilde{X} \rightarrow X$ be the blowup of $X$ along $Y$, let $X' \xrightarrow{f} \tilde{X} = \text{Bl}_Y X$ be a resolution of singularities on $\tilde{X}$ and let $f = \pi \circ f'$ be the composition. The famous result in [BDPP13] says that the dual cone of the movable cone of curves is the pseudoeffective cone. We can apply [Lau16, Theorem 6.1] to see that $Y$ lies in the movable cone of curves. It suffices to show that for any pseudoeffective class $\eta \in N^1(X)_{\mathbb{R}}$ such that $\eta \cdot [Y] = 0$, then $\eta = 0$.

Theorem A says that $\kappa_\sigma(f^*\eta) = \kappa_\sigma(\eta) = 0$. As $f^*\eta$ is pseudoeffective, it is equal to the class of an effective $\mathbb{R}$-Cartier $\mathbb{R}$ divisor $\sum b_iB_i$ where $b_i > 0$ and $B_i$’s are prime divisors by proposition 5.3.

Suppose $\bigcup \text{Supp}(B_i) \cap f^{-1}(Y) = \emptyset$. By the projection formula, $[\eta] \equiv \sum b_if_*[B_i]$ in $N_{n-1}(X)$. But $\bigcup \text{Supp}(f(B_i)) \cap Y = \emptyset$ and the hypothesis imply all $B_i$’s are exceptional. Thus $[\eta] = 0$ in $N_{n-1}(X)$ and $\eta = 0$ by [FL17, Example 2.7].

We may assume $\bigcup \text{Supp}(B_i) \cap f^{-1}(Y) \neq \emptyset$. Applying the negativity lemma to $\sum b_iB_i$ (note that $- \sum b_iB_i$ is clearly $f$-nef), for any closed point $p \in f(\bigcup \text{Supp}(B_i))$, $f^{-1}(p) \subset \bigcup \text{Supp}(B_i)$. Take a curve $C' \subset f^{-1}(Y)$ such that $f(C') = Y$. By the previous remark, $C' \cap \bigcup \text{Supp}(B_i) \neq \emptyset$. On the other hand, $\sum b_iB_i \cdot [C'] = f^*\eta \cdot [C'] = \deg(\kappa(C) : \kappa(Y))\eta \cdot [Y] = 0$. Therefore, $C' \subset \bigcup \text{Supp}(B_i)$ and $f^{-1}(Y) \subset \bigcup \text{Supp}(B_i)$. Thus, $f^*(\pi^*\eta - \epsilon E)$ is pseudoeffective for some small $\epsilon > 0$. But proposition 4.1 says that $\eta$ does not dominate $Y$ numerically. This gives a contradiction. \qed

5.2. Locally ample subvarieties cannot be contracted. In this subsection, we show that, as a consequence of theorem A, a locally ample subvariety cannot be contracted.

Theorem 5.5. Let $X$ be a projective variety and let $Y$ be a locally ample subvariety of $X$. Suppose $f : X \rightarrow Z$ is a morphism from $X$ to a projective variety $Z$. Then if $\dim f(Y) < \dim Y$, then $f|_Y : Y \rightarrow Z$ is surjective, i.e. $f(Y) = Z$.

Proof. Let $A$ be an ample divisor on $Z$. Then $\dim f(Y) = \kappa_\sigma(A|_{f(Y)}) = \kappa_\sigma(f^*(A)|_Y) < \dim Y$. Note that $f^*(A)|_Y$ is not big. By theorem A,

$$\kappa_\sigma(f^*(A)) \leq \kappa_\sigma(f^*(A)|_Y).$$

But $\kappa_\sigma(f^*(A)) = \dim Z$. This forces the equality $\dim Z = \dim f(Y)$. \qed

Remark. The special case of theorem 5.5, where $Y$ is contracted to a point, is observed by Ottem by an elementary argument [Ott16, Proof of Lemma 12].
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