HYPERGEOMETRIC FUNCTION AND MODULAR CURVATURE I.
HYPERGEOMETRIC FUNCTIONS IN HEAT COEFFICIENTS

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ABSTRACT. We give a three-fold upgrade to the rearrangement lemma (pioneer by Connes-Tretkoff-Moscovici) base on the functional analytic ground laid by Lesch. First, we show that the building blocks of the spectral functions appeared in the lemma belong to a class of multivariable hypergeometric functions called Lauricella functions of type $D$, which includes Gauss hypergeometric functions and Appell’s $F_1$ functions as one and two variable subclass respectively. Second, we extend previous results from dimension two to arbitrary dimensions by absorbing the dimension into the parameters of those hypergeometric functions. Third, the highlight of this paper is a reduction formula for the multivariable functions to the Gauss hypergeometric functions using iterated divided differences and differentiations. As for applications, we take perturbed Laplacian operator (the degree zero part that acts on functions) in studied Connes-Moscovici’s paper as an example and extend the computation of the associated modular curvature (functional density of the second heat coefficient) two noncommutative tori of arbitrary dimension. The two spectral functions are computed first as linear combinations of Gauss and Appell’s hypergeometric functions and then as explicit functions in the dimension parameter. It allows us to perform symbolic verification (in a CAS (computer algebra system)) for the celebrated Connes-Moscovici functional relations. A surprising discovery is that functional relations holds as a continuous family with respect to the dimension parameter.

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1. Introduction

1.1. Modular Geometry. A general question behind this paper and its sequel is to explore the notion of intrinsic curvature in the framework of Connes’s noncommutative geometry. Our starting point is the recent development of the analogue of Gaussian curvature for noncommutative two tori \([CM14]\), which is shown later in \([LM16]\), to be stable inside the Morita equivalence class (another purely noncommutative feature), implemented by Heisenberg modules. Besides the underlying Riemannian geometry (the spectral geometry in particular), an essential new ingredient is the modular theory for von-Neumann algebras. It associates to a state a one-parameter group of automorphisms of the ambient algebra measuring how far away the state from being a trace. In our discussion, such a state plays the role of the volume form functional of a new type of metric whose local invariants, such as the Riemannian curvature, that we would like to study. The new phenomenon emerged in the computation is the deep interplay between the modular automorphism and the classical differentials tied to the smooth structure. Therefore, we add “modular” in front of the name of the local invariants, such as modular scalar curvature, to emphasize its noncommutative nature.

More specifically, the curved metrics on \(T^2_\theta\) are obtained by conformal change of the canonical flat one parametrized by self-adjoint elements \(h = h^*\) in the algebra \(C^\infty(T^2_\theta)\) of smooth coordinate functions, whose exponential \(k = e^h\) is called a Weyl factor, or a conformal factor. At measure theoretical level, the new volume form is a state \(\varphi\) obtained by rescaling the canonical trace (volume form of the flat metric) \(\varphi_0\) using the Weyl factor, that is \(\varphi(a) := \varphi_0(ak^{-1}), \forall a \in C^\infty(T^2_\theta)\). The new metric is quantized to a perturbed Dolbeault Laplacian \(\Delta_\varphi = \bar{\partial}^* \partial \varphi\), in which the adjoint is taken with respect to the volume state \(\varphi\), compared to the flat one \(\Delta = \bar{\partial}^* \partial = \bar{\partial}^* \partial\). The corresponding local invariants are encoded in geometric functionals as the coefficients of the small time heat kernel expansion:

\[
\text{Tr}(ae^{-t\Delta_\varphi}) \sim t^{-\gamma} \sum_{j=0}^{\infty} V_j(a, \Delta_\varphi) t^{(j-m)/2}, \forall a \in C^\infty(T^2_\theta),
\]

where \(m\) is the dimension parameter which equals 2 in the example above. Each coefficient is a local functional with its functional density \(R_j \in C^\infty(T^2_\theta)\) in the sense that

\[
V_j(a, \Delta_\varphi) = \varphi_0(a R_j), \forall a \in C^\infty(T^2_\theta).
\]

We would like to focus on the density of the second heat coefficient in great detail which resembles the scalar curvature in Riemannian geometry. Recall that, for the scalar Laplacian \(\Delta\) or the squared Dirac operator \(\bar{D}^2\) on a closed manifold, the density of second coefficient \(V_2(\cdot, \Delta)\) or \(V_2(\cdot, \bar{D}^2)\) is equal to \(S_g/6\) and \(-S_g/12\) respectively (up to a universal constant), where \(S_g\) denotes the scalar curvature function of the metric \(g\).

The full expression of \(R_2\) on \(T^2_\theta\) was first computed in \([CM14]\). Independent computation was carried out in \([FK13]\), and later in \([Fat15, FK15]\) on noncommutative four tori. For the computational aspect, the analytical technical tools, such as Connes’s pseudo-differential calculus and the rearrangement lemmas go back to \([Con80, CT11]\), see also \([FK12]\). Various
computation of this type on (low dimensional) noncommutative tori has been carried out \cite{KMS16, FGK17, DGK18, FGK16}. An non-conformal perturbation of the flat metric was studied in \cite{DS15}. There are also other approaches for Riemannian curvature \cite{Ros13} and Ricci flow \cite{BM12} on $T^2_\theta$. On the other hand, computation on $T^2_\theta$ has been pushed to the $V^4_1$-term in \cite{CF16}, in which one sees spectral functions of three and four variables.

In \cite{CM14}, for the perturbed Laplacian $\Delta \varphi = e^{h/2} \Delta e^{h/2}$, the full expression of the modular curvature is of the form:

$$R_{\Delta \varphi} = K_{CM}(\mathbf{x})(\text{Tr}(\nabla^2 h)) + H_{CM}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) (\text{Tr}(\nabla h \nabla h)).$$

(1.2)

The new input, as a consequence of the noncommutativity, is the action of the modular derivation $\mathbf{x} = -\text{ad}_h$ implemented by two modular spectral functions $K_{CM}(u)$ and $H_{CM}(u, v)$. If one would like to view the derivation $\mathbf{x}$ as a noncommutative differential, compared to the covariant differential $\nabla$, then $\mathbf{x}^{(l)}$, with $l = 1, 2$, serve as the partial version only acting on the $l$-th factor of the product. As one expects, (1.2) contains its Riemannian counterpart as the commutative limit, in which $\mathbf{x} = 0$, so that the operators $K_{CM}(\mathbf{x})$ and $H_{CM}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ degenerate to complex numbers $K(0)$ and $H(0, 0)$ respectively, served as the coefficients of a second order differential operator spanned by the contraction $\nabla^2 h$ and $(\nabla h)(\nabla h)$, which are the scalar Laplacian and squared length of one-forms with respect to the flat metric:

$$\text{Tr}(\nabla^2 h) = \sum_{\alpha=1}^{2} (\nabla^2 h)_{\alpha\alpha} = -\Delta h, \quad \text{Tr}(\nabla h \nabla h) = \sum_{\alpha=1}^{2} (\nabla h)_{\alpha}(\nabla h)_{\alpha} = \langle \nabla h, \nabla h \rangle.$$

A point of departure of our investigation is the two observations in Connes-Moscovici \cite{CM14}, which are still begging for deeper conceptual understanding. By slightly changing $K_{CM}$ and $H_{CM}$,

$$\tilde{K}_{CM}(u) = 4 \frac{\sinh(u/2)}{u} K_{CM}(u), \quad \tilde{H}_{CM}(u, v) = 4 \frac{\sinh(u + v)/2}{u + v} H_{CM}(u, v),$$

one sees that:

1) The one variation function $\tilde{K}_{CM}(u)/8$ equals the generating function of Bernoulli numbers.

2) There is a remarkable functional relation (will be referred as Connes-Moscovici relation in the rest of the paper) discovered by variational computation:

$$\frac{1}{2} \tilde{H}_{CM}(u, v) = -\tilde{K}_{CM}[-u, v] + \tilde{K}_{CM}[u, u + v] - \tilde{K}_{CM}[v, u + v],$$

(1.3)

where the squared brackets in $K[u, v]$ denote divided difference of $K$, see §2.3.

These observations has been improved to larger class which covers example of arbitrary dimension: toric noncommutative manifolds (aka Connes-Landi deformations) by the author in \cite{Liu18, Liu17}. In this paper and its sequel, we shall further pushing the two features forward by connecting the spectral functions to hypergeometric functions and exploring the variational structure behind the functional relations.

\footnote{Here the covariant derivative $\nabla$ denotes the basic derivations $\delta_\alpha$, with $\alpha = 1, 2$ on noncommutative tori: $(\nabla h)_\alpha = \nabla_\alpha h = \delta_\alpha(h)$.}

\footnote{This is a improved version due to M. Lesch, especially for bring in the divided difference feature.}
1.2. **Calculus with respect to noncommutative variables.** Besides the non-unimodular feature of the volume functional $\varphi$, the modular operator/derivation can be used to compress the ansatz caused by the noncommutative between the coordinate $h$ (or $k = e^h$) and their derivatives. Let us look at a toy example: for a derivation operator $\nabla$, one applies the Leibniz property to expand the second differential:

$$\nabla^2 k^3 = k^2(\nabla^2 k) + (\nabla k)^2 + k(\nabla k)(\nabla k) + (\nabla k)(\nabla k) + (\nabla k)(\nabla k)$$

(1.4)

$$= k^2(1 + y)(\nabla k) + k(1 + y^{(1)} + y^{(1)}y^{(2)})(\nabla k \cdot \nabla k).$$

From the rearrangement process above, one sees the spectral functions $1 + y$ and $1 + y_1 + y_1y_2$ evaluated at the modular operator $y$ and its partial version $y^{(j)}$, $j = 1, 2$, cf. §2.1. The classical formula can be recover by setting $y = y_1 = y_2 = 1$. The rearrangement lemma (first developed in [CT11]) deals with the similar process in pseudo-differential calculus that defines the building blocks of the spectral functions. The first main result of this paper is a three-fold improvement to the lemma.

First of all, we show that the building blocks in general belong to the class of hypergeometric functions knowns as Lauricella functions of type $D$. In particular, the one variable family is Gauss hypergeometric functions $\mathrm{F}_1$ and two-variable one is known as Appell’s $F_1$ functions. Those special functions functions lies in the intersection of many deep areas of mathematics, such as differential equations, algebraic geometry and number theory.

Secondly, the extended rearrangement lemma covers the most general setup so far: namely, it is designed for all $V_j$-term computations ($j = 2, 4, 6, \ldots$, as in (1.1)) on arbitrary $m$-dimensional noncommutative manifold. Furthermore, the new integral representation for the spectral functions absorbs $m$ and $j$ into the parameters of hypergeometric functions. As a consequence, the upgraded pseudo-differential algorithm works for all dimensions simultaneously. In [CT11] where the rearrangement lemma was first developed, the argument works only for $m = 2$. The argument can be extended to higher dimensions but requires repeated integration by parts (cf. [Liu17]), in which the number of times depends on the dimension.

Last but not least, the highlight of the paper (Theorem 3.13) is a reduction formula for the multivariable spectral functions to single variable ones via iterating divided differences and differentiations. In dimension two, such recursive relations was first studied in [Les17]. In the $a_4$-term computation [CF16], similar differential relations were derived to evaluated iterated integral. For the single variable case, i.e., the Gauss hypergeometric function, Mathematica provides fast evaluation based on numerous functional relations among this family. Therefore, the issue of finding explicit expressions of the integral representations is completely solved. In early works, the spectral functions were evaluated by symbolic integration in a Mathematica. The reduction formula further confirms the crucial role of divided difference in both pseudo-differential and variational calculus first observed by Lesch in [Les17].

At the end, we test our formula on the spectral functions for the functional density of $V_2(\cdot, \Delta_\varphi)$ where $\Delta_\varphi = k^{1/2}\Delta k^{1/2}$ is the degree zero Laplacian studied in [LM16] [CM14]. Spectral functions are written as linear combinations of hypergeometric functions. Moreover, we are able to obtain their expressions as an explicit function in $m$, the dimension. The

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3 It seems to be an interesting coincidence that both notions of intrinsic curvature and hypergeometric functions were pioneered by Gauss and Riemann.

4 Notice that differentiations belong to a special type called confluent divided differences, therefore at the theoretical level, one only needs divided differences to generated all the multivariable spectral functions. Nevertheless, when implementing in a CAS, one uses differentiations to compute confluent divided difference, therefore it is better to mention both.
symbolic verification gives us a surprising by product: they satisfy a family (parametrized by $m$) of Connes-Moscovici type functional relations even when $m \in [2, \infty)$ takes real-valued. We remark that the notion of dimension takes real valued in a natural way for noncommutative spaces. In such functional analytic approach, dimension is interpreted according to the Weyl’s law, that is, it is proportional to the growth rate of the eigenvalues of the Laplacian. But so far, we are not able to perform similar detailed calculation for the spectral geometry on those examples.

The paper is organized as follows. In §2, we summarize some frequently used notations for the functional calculus following [Les17]. We have seen in Eq. (1.4) how the spectral functions rise in elementary calculus with respect to a noncommutative variable. The corresponding ground rules in pseudo-differential calculus are laid by the rearrangement lemma, which is the main result in §3. We start with an outline of the pseudo-differential approach (in §3.1) to the heat asymptotic and conclude §3.1 with a qualitative statement Lemma 3.1. The computation of the spectral functions is spread from §3.2-3.4 and recapitulated in Theorem 4.2. The rest §3.5-3.6 are devoted to the differential and contiguous functional relations, which eventually leads to the reduction formula: Theorem 3.13. In §4 and §5, In the last two sections, we return to the geometric problem of studying the second heat coefficient $V_2(\cdot, \Delta \varphi)$ (see (1.1)) where the notion of scalar curvature is encoded. The new input of this section The second key result of the paper is Theorem 4.2 which extends Eq. (1.2) to arbitrary dimension, also the spectral functions are expressed in terms of hypergeometric functions. In §5, we further compute the hypergeometric functions explicitly as a function in the dimension parameter $m$, which leads to the climax (by symbolic verification) that the two spectral function satisfy a continuous family (with respect to $m$) of Connes-Moscovici relations, whose conceptual interpretation is the main theme of the sequel paper (part II).

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2. Notations

We first collect some functional analytic setups following [Les17] that will be frequently used through the paper.

2.1. Smooth functional calculus on $\mathcal{A}^\otimes(n+1)$. Let $\mathcal{A}$ be a nuclear Fréchet algebra so that the maximal and projective completion of the $n$-fold algebraic tensor product $\mathcal{A}^\otimes n$ agrees, which will be denoted by $\mathcal{A}^\otimes n$. There is no harm to take $\mathcal{A} = C^\infty (\mathbb{T}_m)$ in the paper. The key step is the contraction map: $\cdot : \mathcal{A}^\otimes(n+1) \times \mathcal{A}^\otimes n \rightarrow \mathcal{A}$, on elementary tensors:

$$(a_0, \ldots, a_n) \cdot (\rho_1, \ldots, \rho_n) = a_0 \rho_1 a_1 \ldots a_{n-1} \rho_n a_n,$$

in which we insert the $j$-th factor of the first elementary tensor into the $j$-th slot of the second one followed by the multiplication to get back to the target $\mathcal{A}$, with $0 \leq j \leq n$. For any $a \in \mathcal{A}$, it has $n+1$ partial version $a^{(j)}$ with $0 \leq j \leq n$ as operators on $\mathcal{A}^\otimes n$, namely, multiplication at the $j$-th slot:

$$(a^{(j)} = (1, \ldots, a, \ldots, 1) \in \mathcal{A}^\otimes(n+1),$$
which acts on $\mathcal{A}^\otimes n$ via Eq. (2.1). Let us assume that $a \in \mathcal{A}$ is self-adjoint, we introduce the partial version of $\text{ad}_a^{(j)}$ and $\text{Ad}_e^{(j)}$ as operators on the $n$-fold tensor $\mathcal{A}^\otimes n$ in a similar fashion:

\begin{equation}
\text{ad}_a^{(j)} = -a^{(j-1)} + a^{(j)}, \quad \text{Ad}_e^{(j)} = (1, \ldots, 1, e^{-a}, e^a, 1, \ldots, 1),
\end{equation}

where $e^{-a}$ appears in the $j - 1$-slot and $1 \leq j \leq n$. The following identities are crucial to the rearrangement lemma: for $1 \leq j \leq n$,

\begin{equation}
(e^{\alpha})^{(j)} = (e^{\alpha})^{(0)} \text{Ad}_{e^{-a}}^{(1)} \cdots \text{Ad}_{e^{-a}}^{(j)}, \quad a^{(j)} = a^{(0)} + \text{ad}_{-a}^{(1)} + \cdots + \text{ad}_{-a}^{(j)}.
\end{equation}

Let $f \in \mathcal{S}(\mathbb{R}^{n+1})$ be a Schwartz function and so is its Fourier transform $\hat{f}$, we write

$$f(x) = \int_{\mathbb{R}^{n+1}} \hat{f}(\xi)e^{i\xi \cdot x}d\xi.$$ 

For any self-adjoint $a \in \mathcal{A}$, denote by $\bar{a} = (a^{(0)}, \ldots, a^{(n)})$, we define the smooth functional calculus:

\begin{equation}
f(\bar{a}) = \int_{\mathbb{R}^{n+1}} \hat{f}(\xi)e^{i\xi \cdot \bar{a}}d\xi.
\end{equation}

The integral converges in $\mathcal{A}^{\otimes (n+1)}$ in Bochner sense and

$$e^{i\xi \cdot \bar{a}} = \exp(i\xi_0 a^{(0)}) \cdots + i\xi_n a^{(n)}) = e^{i\xi a} \otimes \cdots \otimes e^{i\xi a} \in \mathcal{A}^{\otimes (n+1)},$$

which acts on the $n$-fold tensor $\mathcal{A}^\otimes n$ by the contraction map. Notice that $a$ is a bounded operator, the functional calculus works for any $f \in C^\infty(U^{n+1})$, where $U$ is an open subset in $\mathbb{R}$ containing $\text{Spec} a$. One can use suitable cutoff function to extend $f$ to be a Schwartz function $\hat{f}$ on $\mathbb{R}^{n+1}$ and then apply the functional calculus. As long as $\hat{f} = \hat{f}$ when restricted on $(\text{Spec} a)^{n+1}$, the resulting operator are the same.

### 2.2. Modular operator and Modular derivation.

We focus on the local geometry on smooth noncommutative $m$-tori $C^\infty(T^m_\theta)$. The letter $k = e^h$ always denotes a Weyl factor with its self-adjoint logarithm $h = h^* \in C^\infty(T^m_\theta)$. The corresponding modular operator and the modular derivation are denoted by bold font letters: $\forall a \in C^\infty(T^m_\theta)$.

\begin{equation}
y(a) = \text{Ad}_k(a) = k^{-1}ak, \quad x(a) = -\text{ad}_h(h) = [a, h] = ah - ha.
\end{equation}

Two perturbed Laplacian are considered which are differ by a conjugation:

\begin{equation}
\Delta_\varphi = k^{1/2}\Delta k^{1/2}, \quad \Delta_k = k\Delta = k^{1/2}\Delta k^{-1/2}.
\end{equation}

Apply the construction in the previous section §2.1 $y$ and $x$ have different partial versions when acting on the $n$-fold tensor $C^\infty(T^m_\theta)^\otimes n$ (or a product length $n$): for $1 \leq j \leq n$:

\begin{equation}
y^{(j)} = (1, \ldots, k^{-1}, k, 1, \ldots, 1) = (k^{(j-1)})^{-1}k^{(j)},
\end{equation}

\begin{equation}
x^{(j)} = h^{(j)} - h^{(j-1)}.
\end{equation}

The essence of rearrangement is to replace the partial multiplications $k^{(j)}$ with $1 \leq j \leq n$ by the partial modular operator/derivation according to the substitution identities:

\begin{equation}
k^{(j)} = (k^{(0)})^{-1}y^{(1)} \cdots y^{(j)}, \quad h^{(j)} = h^{(0)} + x^{(1)} + \cdots + x^{(j)}.
\end{equation}

Similar to (2.5), we define the smooth functional calculus for the modular derivations using the Fourier transform. For $\rho_1, \ldots, \rho_n \in C^\infty(T^m_\theta)$, we define:

$$f(x^{(1)}, \ldots, x^{(n)})(\rho_1 \cdots \rho_n) = \int_{\mathbb{R}^n} \hat{f}(\xi)y^{i\xi_1}(\rho_1) \cdots y^{i\xi_n}(\rho_n)d\xi.$$
For partial modular operators, we set:

\[ f(y^{(1)}, \ldots, y^{(n)}) = \tilde{f}(x^{(1)}, \ldots, x^{(n)}), \]

where \( \tilde{f}(x_1, \ldots, x_n) = f(e^{x_1}, \ldots, e^{x_n}). \)

2.3. **Divided differences.** For a one-variable function \( f(z) \), the divided difference can be defined inductively as below:

\[
f[x_0] := f(x_0); \\
f[x_0, x_1, \ldots, x_n] := (f[x_0, \ldots, x_{n-1}] - f[x_1, \ldots, x_n])/(x_0 - x_n)
\]

For example,

\[ f[x_0, x_2] = (f(x_0) - f(x_1))/(x_0 - x_1). \]

Use induction, one can show in general:

\[
f[x_0, x_1, \ldots, x_n] = \sum_{l=0}^{n} f(x_l) \prod_{s=0, s \neq l}^{n} (x_l - x_s)^{-1}.
\]

(2.11)

For multivariable functions, we shall use a subscript to indicate on which variable the divided difference acts, for example:

\[ f(x, z, y)[z_1, \ldots, z_s]_z. \]

Through out the paper, we fix the variable \( z \) as the default choice for the divided difference operator: \( [\bullet, \ldots, \bullet] := [\bullet, \ldots, \bullet]_z. \)

3. **Hypergeometric functions in heat kernel expansions**

The technical tool that being used to decipher the heat kernel expansion is a pseudo-differential calculus which is suitable for studying the spectral geometry of the underlying manifolds, such as Connes’s calculus for noncommutative tori \cite{Con80} and deformation of Widom’s calculus for general toric noncommutative manifolds \cite{Lin18, Lin17}. Nevertheless, the rearrangement lemma is universal to the choice of symbolic calculus.

3.1. **Rearrangement lemma.** The goal is to study the trace of the heat operator which can be defined using holomorphic functional calculus:

\[
e^{-tP} = \frac{1}{2\pi i} \int_C e^{-t\lambda} (P - \lambda)^{-1} d\lambda,
\]

where \( C \) is a contour winding around the spectrum of \( P \). Two obstacles are presented:

i) the resolvent \((P - \lambda)^{-1}\) involving the inverse of a differential operator;

ii) a trace formula for pseudo-differential operators.

A pseudo-differential calculus moves the two hurdles above from differential operators to their symbols, which are smooth functions (or sections) on the total space of the cotangent bundle of the ambient manifold. Besides of the function natural of symbol algebra, it admits a formal product and a filtration (or grading) structure inherited from the algebra of pseudo-differential operators. The two ingredients give rise to a recursive algorithm to construction a sequence of symbols \( \{b_j\}_{j=0}^{\infty} \), whose truncated sum \( \sum_{j=0}^{N} b_j \) approximates the symbol of the resolvent \( \sigma((P - \lambda)^{-1}) \) so that \( b_j \)-term determines the \( V_j \)-coefficient in the heat asymptotic Eq. (1.1). For instance, in Connes’s pseudo-differential calculus, the
symbols are $C^\infty(\mathbb{T}_\theta^m)$-valued functions on $\mathbb{R}^m$ and $\xi \in \mathbb{R}^m$ plays the role of fiber coordinates of the total space of cotangent bundle of $\mathbb{T}_\theta^m$. We have:

$$V_j(a, P) = \varphi_0 \left( a \left[ \int_{\mathbb{R}^m} \frac{1}{2\pi i} \int_C e^{-\lambda b_j(\xi, \lambda)} d\lambda d\xi \right] \right), \quad \forall a \in C^\infty(\mathbb{T}_\theta^m).$$

To initiate the algorithm, one only needs to invert the leading symbol with respect to the “functional multiplication”. Let us denote the lead symbol of the second order elliptic operator in question by $p_2$, then $b_0 = (p_2(\xi) - \lambda)^{-1}$ is the resolvent in the algebra $C^\infty(\mathbb{T}_\theta^m)$ for fixed $\xi$. The price we have to pay is to fight against the combinatorial complexity of higher $b_j$’s. Nevertheless, $b_j$’s are conceptually quite simple in the sense that the inverse only appears in those factors consisting of powers of $b_0$. More precisely, they are finite sums (could be very lengthy) of the form:

$$b_j = \sum b_0^{l_0} \rho_1 b_0^{l_1} \rho_2 \cdots b_0^{l_{n-1}} \rho_n b_0^{l_n},$$

where the exponents $l_j$’s are non-negative integers and $\rho_j$’s are the derivatives of symbols of the elliptic operator (in particular, no inverse involved).

Notice that $b_0$ and $\rho_j$’s do not commute in general, even in the commutative world, when the operators acting on vector bundles, the symbols are matrix-valued functions on the total space of the cotangent bundle. In the commutative setting, all the powers of $b_0$ in the summands of Eq. (3.2) can be moved to the left, the contour integral simply equals the exponential of $p_2$:

$$\frac{1}{2\pi i} \int_C e^{-\lambda (p_2 - \lambda)^{-1}} d\lambda \propto e^{-p_2},$$

where $\propto$ means “equals upto a constant” or “proportional to”. The spread of $b_0$-factors creates an essential obstacle to perform integration. The first key contribution of this paper is to show that, after the contour integral, the obstacle gives rise to a hidden part besides the exponential function consisting of confluent hypergeometric functions $\, _1F_1$ and their multivariable generalization, cf. Eq. (3.13).

Notice that all the $\rho_j$’s are independent of the resolvent parameter $\lambda$. After integration over the unit sphere, we can further assume that they have no $\xi$-dependence (or $r$-dependence), where we have used spherical coordinates $\xi = \xi(r, s)$: $s$ is the coordinate for the unit sphere $S^{m-1}$ and $r = |\xi|$ is length function. As a consequence, one can assume that $\rho_j$’s appeared in Eq. (3.2) are constants with respect to the integrations in Eq. (3.1) which can be factored out by the contraction defined in Eq. (2.1):

$$b_0^{l_0} \rho_1 b_0^{l_1} \rho_2 \cdots b_0^{l_{n-1}} \rho_n b_0^{l_n} = (b_0^{l_0} \otimes \cdots \otimes b_0^{l_n}) \cdot (\rho_1 \otimes \cdots \otimes \rho_n).$$

Furthermore, the functional density of $V_j$ (in Eq. (3.1)) is a finite sum of the form:

$$R_j = \frac{1}{2\pi i} \int_{\mathbb{R}^m} e^{-\lambda b_j(\xi, \lambda)} d\lambda d\xi$$

$$= \sum \left( \frac{1}{2\pi i} \int_{\mathbb{R}^m} e^{-\lambda (b_0^{l_0} \otimes \cdots \otimes b_0^{l_n})} d\lambda d\xi \right) \cdot (\rho_1 \otimes \cdots \otimes \rho_n).$$

From now on, let us specialize our elliptic operator to $\Delta_\varphi$ and $\Delta_k$ whose the leading symbols are the same: $p_2 = k |\xi|^2 = kr^2$. In this case, the $\rho_j$’s are product of $k$ and the derivatives of $k$, that is, the components of the tensors of the form $k^l \nabla^l k$ with $l, l' \in \mathbb{N}$, where the factor $k'$ can be moved to the very left of Eq. (3.4) with the help of the partial modular operators. They constitute the classical part of $R_j$ which only involves the commutative differential $\nabla$. 
On the other hand, the new ingredient, purely coming from noncommutativity, is the operator-valued integral in the squared brackets of Eq. (3.4). The rearrangement lemma concerns the issue that how to express the integral as a functional calculus of the partial modular operators (or derivations). We first give a qualitative statement and will show in the next section that the spectral function $\hat{H}_{(l_0,\ldots, l_n)}(y_1, \ldots, y_n; m; j)$ described below is given by a Euler type integral that belong to the celebrated family of hypergeometric functions.

**Lemma 3.1** (Rearrangement Lemma, Part I). The operator-valued integral in Eq. (3.4) acting on $C^\infty((\mathbb{T}^m)^\otimes n)$ equals a functional calculus of the partial modular operators: $y^{(1)}, \ldots, y^{(n)}$, up to a power of $k^{(0)}$:

$$\left(3.5\right) \int_{\mathbb{R}^m} \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\lambda}(b_0^l \otimes \cdots \otimes b_0^n) d\lambda d\xi = \kappa^{\frac{i m}{2}} \hat{H}_{(l_0,\ldots, l_n)}(y^{(1)}, \ldots, y^{(n)}; m; j),$$

where the spectral functional $\hat{H}_{(l_0,\ldots, l_n)}(y_1, \ldots, y_n; m; j)$ is determined by the integral in Eq. (3.8).

**Proof.** Integrating over the unit sphere gives:

$$\int_{S^{m-1}} (b_0^l(\xi, \lambda) \otimes \cdots \otimes b_0^n(\xi, \lambda)) d\sigma = r^l(b_0^l(r, \lambda) \otimes \cdots \otimes b_0^n(r, \lambda))$$

where $l = 2(l_0 + \cdots + l_n - j + 1)$ due to the following homogeneity of $b_j$:

$$\left(3.6\right) b_j(c\xi, c^2\lambda) = e^{-2(j+1)}b_j(\xi, \lambda), \ \forall c > 0.$$  

We continue:

$$\left(3.7\right) \int_{\mathbb{R}^m} \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\lambda}(b_0^l \otimes \cdots \otimes b_0^n) d\lambda d\xi \propto \int_{0}^{\infty} \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\lambda} b_0^l \otimes \cdots \otimes b_0^n d\lambda r^{\hat{l}+m-1} dr$$

We remind the reader of Eq. (2.10): $k^{(j)} = (k^{(0)})^{-1}y^{(1)}\cdots y^{(j)}$, for $\leq j \leq n$, which brings in the partial modular operators. The final result follows from two substitutions: $r' = r^2$ and $r'' = k^{(0)}r'$:

$$\left(3.8\right) \int_{0}^{\infty} \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\lambda}(k^{(0)}r^2 - \lambda)^{-l_0} \cdots (k^{(n)}r^2 - \lambda)^{-l_n} d\lambda (r^{\hat{l}+m-1} dr)$$

For the second substitution $r'' = k^{(0)}r'$, one needs the smooth operator substitution lemma [Les17, Theorem 3.4], which enables us to perform the substitution as $k^{(0)}$ were a positive number. □
We end this subsection with a remark of a subtle issue of the trace formula for pseudo-differential operators which have been overlooked in the literature. In order to derive Eq. (3.1), one needs to know how to compute the trace of a pseudo-differential operator from its symbol. The trace formula, roughly speaking, states that the trace can be recovered by integrating its symbol over the total space of the cotangent bundle, up to some error which has no harm to the study of the heat asymptotic. The error is caused by the fact that the symbol (called complete symbol) is an equivalent class of smooth functions obtained by quotient out the subspace of smoothing symbols. Indeed, the construction of a symbol map is never canonical. It requires some flexible choices, such as a cut-off function, whose effect to the symbol map is restricted inside smoothing symbols. For Connes’s pseudo-differential calculus the issue was first carefully treated in Lesch-Moscovici, [LM16, Theorem 6.2] contains the precise statement. For pseudo-differential operator on manifolds, we have similar results in [Wid78, Theorem 5.7].

3.2. Contour integral for the heat operator. We start with a lemma which handles the contour integral that defines the heat operator. In our applications, the elliptic operator $P$ has its spectrum contained in the open interval $(0, \infty)$. If $P$ has nontrivial kernel, denote by $P$ the corresponding projection (onto the kernel), write:

\[ \text{Tr}(f e^{-tP}) = \text{Tr}(f e^{-tP} (1 - P_{\ker})) + \text{Tr}(f P_{\ker}), \]

where the second trace is not difficult to compute:

\[ \text{Tr}(f P_{\ker}) = \text{Tr}(P_{\ker} f P_{\ker}) = \sum_{i=1}^{n} \langle f P_{\ker} \psi_i, P_{\ker} \psi_i \rangle = \sum_{i=1}^{n} \langle f \psi_i, \psi_i \rangle = \sum_{i=1}^{n} \int f \psi_i^2, \]

where \( \{ \psi_1, \ldots, \psi_n \} \) is an orthonormal basis of the kernel. While the first term can be interpreted as restricting $P$ to the orthogonal complement of its kernel. Sum up, we are allowed to choose the contour $C$ to be the imaginary axis oriented from $-i\infty$ to $i\infty$, so that

\[ e^{-tP} = \frac{1}{2\pi i} \int_{C} e^{-t\lambda}(P - \lambda)^{-1} d\lambda = (2\pi)^{-1} \int_{0}^{\infty} e^{-itx}(P - ix)^{-1} dx. \]

**Lemma 3.2.** Let $A, B, C$ be positive real numbers and $a, b, c$ be non-negative integers, we have

\[ (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix}(A - ix)^{-a}(B - ix)^{-b} dx = \frac{1}{\Gamma(a)\Gamma(b)} \int_{0}^{1} (1 - t)^{a-1} t^{b-1} e^{-(A - (A-B)t)} dt \]

and

\[ (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix}(A - ix)^{-a}(B - ix)^{-b}(C - ix)^{-c} dx \]

\[ = \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_{0}^{1} \int_{0}^{1-t} (1 - t - u)^{a-1} u^{b-1} e^{-(A - (A-B)t - (A-C)u)} du dt. \]

**Remark.** Notice that the right hand side of (3.11) is a confluent hypergeometric function:

\[ (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix}(A - ix)^{-a}(B - ix)^{-b} dx = \frac{e^{-A}}{\Gamma(a + b)} {_1F_1}(a; a + b; B - A), \]

\footnote{Same statement holds for the deformed Widom’s pseudo-differential calculus on toric noncommutative manifolds, cf. [Lin17, Proposition 5.2].}
where the function \( _1F_1(a; b; B - A) \) has the following integral representation:

\[
_1F_1(a; b; z) = \frac{\Gamma(a - b)\Gamma(b)}{\Gamma(a)} \int_0^1 t^{a-1} (1 - t)^{b-a-1} e^{zt} \, dt.
\]

The identity \((3.13)\) appeared in the heat kernel related work \cite{Gus91} and \cite{AB01}. This was the motivation at the early stage that brought the author’s attention to hypergeometric functions.

**Proof.** We only prove \((3.12)\) and leave \((3.11)\) to the reader. Observe that powers like \((A - ix)^{-a}\) can be rewritten in terms of Mellin transform:

\[
(A - ix)^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty s^{a-1} e^{-s(A - ix)} \, ds.
\]

The rest of the computation is straightforward, denote \(\gamma(a, b, c) = [\Gamma(a)\Gamma(b)\Gamma(c)]^{-1}\), then:

\[
(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix} (A - ix)^{-a} (B - ix)^{-b} (C - ix)^{-c} \, dx
\]

\[
= (2\pi)^{-1} \gamma(a, b, c) \int_{0}^{\infty} \int_{0}^{\infty} e^{-ix} s^{a-1} t^{b-1} u^{c-1} e^{-s(A - ix) - t(B - ix) - u(C - ix)} dsdtdu
\]

\[
= \gamma(a, b, c) \int_{0}^{\infty} \int_{0}^{\infty} s^{a-1} t^{b-1} u^{c-1} e^{-sA - tB - Cu} dsdtdu \left( (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ix(s+t+u)} \, dx \right)
\]

\[
= \gamma(a, b, c) \int_{0}^{\infty} \int_{0}^{\infty} (1 - t - u)^{a-1} t^{b-1} u^{c-1} e^{-sA - tB - Cu} dsdtdu.
\]

For the last equal sign, we use the fact that \((2\pi)^{-1} \int_{-\infty}^{\infty} e^{ixy} \, dx\) equals the Dirac-delta distribution \(\delta(y)\), therefore the domain of integration is reduced from \([0, \infty)^3\) to

\[
D = \{s + t + u = 1, \ s, t, u > 0\}.
\]

This is obviously an intuitive argument and the domain of integration needs a small correction, it is the projection of the triangle \(D\) onto the \(uv\)-plane.

The rigorous proof\cite{M. Lesch} for last step goes as follows. Choose an auxiliary test function \(\psi\) with \(\psi(0) = 1\). We need to show that the limit as \(\lambda \to 0^+\)

\[
\lim_{\lambda \to 0^+} \int_{[0, \infty]^3} e^{-ix} s^{a-1} t^{b-1} u^{c-1} e^{-sA - tB - Cu} \psi(\lambda x) dsdtdu
\]

converges to the last step above. Perform the substitution for \(dsdtdu\): \(r = s + t + u, \ t' = t/r\) and \(u' = u/r\) so that

\[
\int_{[0, \infty]^3} s^{a-1} t^{b-1} u^{c-1} e^{-sA - tB - Cu} dsdtdu
\]

\[
= \int_{0}^{\infty} \int_{0}^{\infty} (1 - t - u)^{a-1} t^{b-1} u^{c-1} \left( \int_{0}^{\infty} Q(r, t, u) \, dr \right) dtdu,
\]

where

\[
Q(r, u, t) = r^{a+b+c-1} e^{-r[1 - A(B - C)u]}
\]

\textsuperscript{6}The argument was provided by M. Lesch.
To finish the proof, we let \( x' = \lambda x \) and \( r' = (r - 1)/\lambda \), then
\[
(2\pi)^{-1} \lim_{\lambda \to +0} \int_{-\infty}^{\infty} \int_{0}^{\infty} Q(r, u, t) e^{ix(r-1)} \psi(\lambda x) dx dr
= (2\pi)^{-1} \lim_{\lambda \to +0} \int_{-\infty}^{\infty} \int_{0}^{\infty} Q(\lambda r' + 1, u, t) e^{ixr'} \psi(x') dx' dr'
= \lim_{\lambda \to +0} \int_{-1/\lambda}^{\infty} Q(\lambda r' + 1, u, t) \hat{\psi}(r') dr'
= Q(1, u, t) \int_{0}^{\infty} \hat{\psi}(r') dr' = Q(1, u, t) \psi(0) = Q(1, u, t),
\]
where \( \hat{\psi} \) denotes the inverse Fourier transform of \( \psi \).

The following generalization is straightforward and the proof is left to interested readers.

**Lemma 3.3.** Let \((A_0, \ldots, A_n) \in (0, \infty)^{n+1}\) and consider
\[
\begin{equation} \tag{3.16}
G_{\alpha_0, \ldots, \alpha_n}(A_0, \ldots, A_n) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i(x-ix)^{-\alpha_0} \cdots (A_n - ix)^{-\alpha_n}} dx
\end{equation}
\]
coming from the contour integral in (3.10). The right hand side can be written as an exponential \( e^{-A_0} \) times a confluent type of hypergeometric function:
\[
\begin{equation} \tag{3.17}
G_{\alpha_0, \ldots, \alpha_n}(A_0, \ldots, A_n)
= \gamma(\alpha_0, \ldots, \alpha_n) \int_{D_{(u_1, \ldots, u_n)}} (1 - \sum_{l=1}^{n} u_l)^{\alpha_0-1} u_1^{\alpha_1-1} \cdots u_n^{\alpha_n-1} \
\cdot \exp \left( -A_0 (1 - \sum_{l=1}^{n} u_l) - \sum_{l=1}^{n} A_l u_l \right) du_n \cdots du_1,
\end{equation}
\]
where \( \gamma(\alpha_0, \ldots, \alpha_n) = \Gamma(\alpha_0) \cdots \Gamma(\alpha_n) \) and the domain of integration is the standard \( n \)-simplex parametrized as below:
\[
\begin{equation} \tag{3.18}
D_{(u_1, \ldots, u_n)} = \left\{ 1 - \sum_{l=1}^{n} u_l \geq 0, u_1, \ldots, u_n \geq 0 \right\},
\end{equation}
\]
\[
\int_{D_{(u_1, \ldots, u_n)}} = \int_{0}^{1} \int_{0}^{1-u_1} \int_{0}^{1-u_1-u_2} \cdots \int_{0}^{1-u_1-\cdots-u_{n-1}} du_n du_{n-1} \cdots du_1.
\]

3.3. **Spectral functions in terms of hypergeometric functions.** Now we are ready to compute the spectral functions in Lemma 3.3. We start with spectral functions of one and two variable which belong to the family of Gauss hypergeometric functions \( {}_2F_1 \) and Appell’s \( {}_1F_1 \) functions respectively. The two proofs are parallel, we left the easier one to the reader.

**Proposition 3.4.** Keep notations in (2) Let \( a, b, m \) be positive integers. For the modular operator \( y \), we introduce \( z = 1 - y \), then:
\[
\begin{equation} \tag{3.19}
\int_{C}^{\infty} \frac{1}{2\pi i} e^{-\lambda (k(0)r^2 - \lambda)^{-a} \cdot \rho \cdot (k(1)r^2 - \lambda)^{-b}} d\lambda (r^{2d+1} dr)
= (k(0))^{-(d+1)} H_{a,c}(z; m)(\rho), \quad \forall \rho \in C^{\infty}(\mathbb{T}_m).
\end{equation}
\]
where the spectral function equals an Euler type integral:

\[
H_{a,c}(z; m) = \Gamma(d + 1)\gamma(a, b) \int_0^1 (1 - t)^{a-1} t^{b-1} (1 - zt)^{-(d+1)} dt
= \frac{\Gamma(d + 1)}{\Gamma(a + b)} \, _2F_1(d + 1, b; b + a; z),
\]

where the exponent \(d\) depends on the dimension parameter \(m\):

\[
d_m := d_m(a, b) = a + b - 2 + (m - 2)/2, \quad \frac{\Gamma(d_m + 1)}{\Gamma(a + b)} = (a + b)_{m/2-2}
\]
due to the homogeneity of \(b_2\) described in Eq. (3.6).

**Proposition 3.5.** Keep notations in [2] Let \(a, b, c, m\) be positive integers and introduce \(z_1 = 1 - y^{(1)}\) and \(z_2 = 1 - y^{(1)} y^{(2)}\) for the partial modular operators, then \(\forall \rho_1, \rho_2 \in C^\infty(T_\theta^m),\)

\[
\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda (k^{(2)})^2} \cdot \rho_2 \cdot (k^{(2)} r^2 - \lambda)^{-c} \cdot \rho_1 \cdot (k^{(1)} r^2 - \lambda)^{-b} dr d\lambda
= (k^{(0)})^{-(d+1)} H_{a,b,c}(z_1, z_2; m)(\rho_1 \rho_2).
\]

where

\[
H_{a,b,c}(z_1, z_2; m)
= \Gamma(d + 1)\gamma(a, b, c) \int_0^1 \int_0^{1-t} (1 - t - u)^{a-1} u^{b-1} (1 - z_1 t - z_2 u)^{-(d+1)} dudt
= \frac{\Gamma(d + 1)}{\Gamma(a + b + c)} \, _2F_1(d + 1; c, a + b + c; z_2, z_1)
\]

\[
= \frac{\Gamma(d + 1)}{\Gamma(a + b + c)} \, _2F_1(d + 1; b, c, a + b + c; z_1, z_2),
\]

where \(d = d_m\) depends on the dimension

\[
d_m = a + b + c - 2 + (m - 2)/2, \quad \frac{\Gamma(d_m + 1)}{\Gamma(a + b + c)} = (a + b + c)_{m/2-2}.
\]

**Proof.** We first perform a substation \(r \mapsto r^2\) so that the volume form \(r^{2d_m+1} dr\) becomes \(r^{d_m} dr/2\). The overall factor \(1/2\) will be omitted in the rest of the proof. The contour integral has been computed in lemma 3.2, but with operator-valued input, namely, \(A = k^{(0)} r\), \(B = k^{(2)} r\) and \(C = k^{(2)}_r\), which are all bounded operators with positive spectrum.

\[
\int_C \frac{1}{2\pi i} \int e^{-\lambda (k^{(0)} r^2 - \lambda)^{-a} (k^{(1)} r^2 - \lambda)^{-b} (k^{(2)} r^2 - \lambda)^{-c} d\lambda
= \gamma(a, b, c) \int_0^1 \int_0^{1-t} (1 - t - u)^{a-1} u^{b-1} (1 - (1-t)(1-u)\lambda)^{-(d+1)} dudt.
\]

To finish the proof of Eq. (3.22), it remains to show that after integrating in \(r\), the exponential factor \(e^{-r[k^{(0)}-t(k^{(0)}-k^{(1)})-u(k^{(0)}-k^{(1))}]}\) gives rise to

\[
(1 - t(1 - y_1) - u(1 - y_1 y_2))^{-(d+1)}.
\]
Indeed,
\[
\int_0^\infty r^de^{-r[k(0)-t(k(0)-k(1))-u(k(0)-k(1))]}dr
= (k(0))^{-(d+1)} \int_0^\infty r^d e^{-r[1-t(1-y(1)) - u(1-y(1)y(2))]}dr
= (k(0))^{-(d+1)} \left(1 - t(1-y(1)) - u(1-y(1)y(2))\right)^{-(d+1)} \left(\int_0^\infty r^d e^{-r}dr\right)
= \Gamma(d+1)(k(0))^{-(d+1)} \left(1 - t(1-y_1) - u(1-y_1y_2)\right)^{-(d+1)}.
\]

To reach the first equal sign, we set \( r \mapsto rk(0) \), and apply the relations in (2.10), that is
\[ k_1 = k_0y_1, \quad k_2 = k_0y_1y_2. \]

An operator substitution lemma \cite[Thm 2.2]{Les17} has been used twice, which, Roughly speaking, allows us to treat mutually commuting positive operators \( k(j) \) with \( j = 0, 1, 2, \) and \( 1 - t(1-y(1)) - u(1-y(1)y(2)) \) as positive real numbers during the computation. For the second equal sign, we let \( r \mapsto r[1 - t(1-y(1)) - u(1-y(1)y(1))] \), which is a positive operator for \( r > 0 \) and \( 0 < t, u < 1 \).

The Euler type integral (3.22) belongs to Appell’s \( F_1 \) family according to Eq. (A.16). \( \square \)

### 3.4. Multivariable generalization.

We have seen spectral functions of three and four variable in the \( V_1 \)-term computation in \cite{CF16}. We would like to derive a general rearrangement lemma that computes the arbitrary \( n \)-variable spectral functions:

\[ H_{\alpha_0,\ldots,\alpha_n}(z_1,\ldots,z_n;m;j), \]

appeared in the computation of \( V_j \)-term on noncommutative \( m \)-tori, where \( j \) comes from the homogeneity in Eq. (3.6).

The multi-variable extension of Appell’s \( F_1 \) family is the Lauricella Functions \( F_D^{(n)} \) of type \( D \). The three variable case was first introduced by Lauricella in \cite{Lau93} and later fully by Appell and Kampe de Fériet \cite{AdF26}.

It has the power series form near the origin:

\[
F_D^{(n)}(a; \alpha_1, \ldots, \alpha_n; c; x_1, \ldots, x_n) = \sum_{\beta_1, \ldots, \beta_n \geq 0} \frac{(a)_{\beta_1+\ldots+\beta_n}(\alpha_1)_{\beta_1}(\alpha_n)_{\beta_n}x_1^{\beta_1} \ldots x_n^{\beta_n}}{(c)_{\beta_1+\ldots+\beta_n}! \beta_1! \ldots \beta_n!}.
\]

What we need in the paper is the following integral representation (cf. \cite{HK74}):

\[
\Gamma(\alpha_1) \ldots \Gamma(\alpha_n) \Gamma(c - \sum_{l=1}^n \alpha_l) F_D^{(n)}(a; \alpha_1, \ldots, \alpha_n; c; x_1, \ldots, x_n)
= \int_{D(u_1, \ldots, u_n)} (1 - \sum_{l=1}^n u_l)^{c-\sum_{l=1}^n \alpha_l+1} u_1^{\alpha_1-1} \ldots u_n^{\alpha_n-1} \cdot (1 - \sum_{j=1}^n x_j u_j)^{-a} du_n \ldots du_1.
\]
Proposition 3.6. Let $\alpha_0, \ldots, \alpha_n \in \mathbb{Z}_{>0}$ and $z_l = 1 - y^{(1)} \cdots y^{(l)}$ for $l = 1, 2, \ldots, n$. The spectral functions in Lemma 3.1 have the following integral representation which belong to the family of Lauricella functions $F_D^{(n)}$:

$$
(k^{(0)})^{d+1} \int_0^\infty G_{\alpha_0, \ldots, \alpha_n} (k^{(0)} r, \ldots, k^{(n)} r) (r^d dr)
$$

(3.26)

$$
= \Gamma(d + 1) \gamma(\alpha_0, \ldots, \alpha_n) \int_{D(u_1, \ldots, u_n)} (1 - \sum_{j=1}^n u_j)^{\alpha_0 - 1} u_1^{\alpha_1 - 1} \cdots u_n^{\alpha_n - 1}
\cdot (1 - \sum_{j=1}^n z_j u_j)^{-(d+1)} du_n \cdots du_1.
$$

(3.27)

We extend the spectral functions Eq. (3.20) and (3.22) to arbitrary $n$-variable case by defining $H_{\alpha_0, \ldots, \alpha_n}(z_1, \ldots, z_n; m; j)$ to be in integral in (3.26) with the exponent $d = d(m, j)$:

$$d(m, j) = \alpha_0 + \cdots + \alpha_n - j + (m - 2)/2.$$

Proof. First of all, $G_{\alpha_0, \ldots, \alpha_n} (k^{(0)} r, \ldots, k^{(n)} r)$ comes from the contour integral and has been computed in Eq. (3.17):

$$
G_{\alpha_0, \ldots, \alpha_n} (k^{(0)} r, \ldots, k^{(n)} r)
$$

$$
= \gamma(\alpha_0, \ldots, \alpha_n) \int_{D(u_1, \ldots, u_n)} (1 - \sum_{l=1}^n u_l)^{\alpha_0 - 1} u_1^{\alpha_1 - 1} \cdots u_n^{\alpha_n - 1}
\cdot \exp \left( -k^{(0)} r (1 - \sum_{l=1}^n u_l) - r \sum_{l=1}^n k^{(l)} u_l \right) du_n \cdots du_1.
$$

Notice that only the exponential factor involves $r$. We can finish the proof by showing the $r$-integral equals:

$$
\int_0^\infty \exp \left( -r k^{(0)} (1 - \sum_{j=1}^n u_j) - \sum_{l=1}^n r k^{(l)} u_j \right) (r^d dr)
$$

$$
= \Gamma(d + 1)(k^{(0)})^{-(d+1)} \left( 1 - \sum_{j=1}^n z_j u_j \right)^{-(d+1)},
$$

by using the operator substitution lemma (Les17 Lemma 2.1) twice: $r \to rk^{(0)}$ and then

$$r \to r(1 - \sum_{l=1}^n u_l) - \sum_{l=1}^n r y^{(l)} u_l.$$

Of course, $k^{(0)}, \ldots, k^{(n)}$ have been replaced by $y^{(1)}, \ldots, y^{(n)}$ according to the identities in (2.10).

$\square$
Now we are ready to summarize the most general spectral functions in Lemma 3.1 that compute the \( V_j \)-coefficient in the heat asymptotic \( 1.1 \) on a \( m \)-dimensional torus. We remind the reader that \( j \) is involved because of the homogeneity of \( b_j \) in Eq. (3.6).

**Theorem 3.7 (Rearrangement Lemma, Part II).** With the substitution:
\[
z_q = 1 - y_1 \cdots y_q, \quad 1 \leq q \leq n,
\]
the spectral functions \( \tilde{H}_{(l_0, \ldots, l_n)}(y_1, \ldots, y_n; m; j) \) appeared in the first part of the rearrangement lemma (Lemma 3.1) are Lauricella hypergeometric function of type \( D \):
\[
H_{l_0, \ldots, l_n}(z_1, \ldots, z_n; m; j) = (l)_{m/2-j} F_D^{(n)}(l + m/2 - j; l_1, \ldots, l_n; j; z_1, \ldots, z_n)
\]
where \( l = l_0 + \cdots + l_n \). The spectral functions that we need for the \( V_2 \)-term calculation are denoted by:
\[
H_{l_0, \ldots, l_n}(z_1, \ldots, z_n; m) := H_{l_0, \ldots, l_n}(z_1, \ldots, z_n; m; 2).
\]

**Remark.** There is another integral representation of \( F_D^{(n)} \) beside Eq. (3.25):
\[
F_D^{(n)}(a; \alpha_1, \ldots, \alpha_n; c; z_1, \ldots, z_n) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a) \Gamma(n)} \int_0^1 t^{n-1} (1-t)^{c-a-1} (1-z_1t)^{-\alpha_1} \cdots (1-z_n t)^{-\alpha_n} dt,
\]
but it requires \( \Re c > \Re a > 0 \) for the convergence of the integral.

### 3.5. Differential and contiguous functional relations.

The hypergeometric family is a bridge to many other deep areas which will be explored in future works. There are tons of functional relations for the family, among which the differential and the contiguous relations are the basic. Those relations allow us to derive some recursive algorithms to compute the explicit expressions of those integrals only using algebraic and differential operations. For instance, the dependency of the spectral functions on the dimension parameter \( m \) is clarified in Proposition 3.10.

**Proposition 3.8.** For the one variable family \( H_{a,b}(u; m) \) defined in (3.20), the following functional relations hold:
\[
(3.32) \quad H_{a,b}(u; m + 2) = (\tilde{d}_m + u d/du) H_{a,b}(u; m),
\]
\[
(3.33) \quad H_{a,b+1}(u; m) = (b^{-1} d/du) H_{a,b}(u; m),
\]
\[
(3.34) \quad H_{a,b+1}(u; m) = (1 + b^{-1} u d/du) H_{a+1,b}(u; m),
\]
where \( \tilde{d}_m := \tilde{d}_m(a, b) = a + b + m/2 - 2 \). Moreover,
\[
(3.35) \quad \frac{a + b \ H_{a+1,b}(u; m)}{\tilde{d}_m H_{a,b}(u; m)} = \frac{2F_1(\tilde{d}_m + 1; 1; a + b + 1; u)}{2F_1(\tilde{d}_m, b; a + b + u)},
\]
where the right hand side is a Gauss’s continued fraction.

**Proof.** Follows quickly from the differential relations (A.5) to (A.8). \( \Box \)

**Proposition 3.9.** Let \( \tilde{d}_m := \tilde{d}_m(a, b, c) = a + b + c + m/2 - 2 \). For the two variable family \( H_{a,b,c}(u, v; m) \) defined in (3.22),
\[
(3.36) \quad H_{a,b,c}(u, v; m + 2) = (\tilde{d}_m + u \partial_u + v \partial_v) H_{a,b,c}(u, v; m),
\]
\[
(3.37) \quad H_{a,b+1,c}(u, v; m) = b^{-1} \partial_v H_{a,b,c}(u, v; m),
\]
\[
(3.38) \quad H_{a,b,c+1}(u, v; m) = c^{-1} \partial_u H_{a,b,c}(u, v; m).
\]
When increasing the parameter \( a \) by one, we encounter a similar relation as in (3.35):

\[
\frac{H_{a+1,b,c}(u,v;m)}{H_{a,b,c}(u,v;m)} = \frac{d_m}{a+b+c} \frac{F_1(d_m+1;c,b;a+b+c+1;u,v)}{F_1(d_m;c,b;a+b+c;u,v)}.
\]

**Proof.** Follows quickly from the differential relations (A.18) to (A.22). \( \square \)

The recursive relations on the dimension \( m \) is much less straightforward to observe. They were motivated by the explicit expressions (3.49) and (3.50).}

**Proposition 3.10.** Let \( m = \dim M \geq 2 \) and \( a, b, c \) be positive integers,

\[
H_{a,b}(z;m+2) = aH_{a+1,b}(z;m) + bH_{a,b+1}(z;m)
\]

(3.39)

\[
H_{a,b,c}(u,v;m+2) = aH_{a+1,b,c}(u,v;m) + bH_{a,b+1,c}(u,v;m)
\]

(3.40)

\[
+ cH_{a,b,c+1}(u,v;m).
\]

**Proof.** Notice that (3.39) and (3.40) are equivalent to the following contiguous relations of hypergeometric functions respectively:

\[
(a+b)F_1(d_m+1,b; a+b; u) = aF_1(d_m+1,b; a+b+1; u) + bF_1(d_m+1,b+1; a+b+1; u),
\]

(3.41)

\[
(a+b+c)F_1(d_m+1,c; a+b+c; u, v) = aF_1(d_m+1,c; a+b+c+1; u, v)
\]

(3.42)

\[
+ bF_1(d_m+1,c+1; a+b+c+1; u, v)
\]

\[
+ cF_1(d_m+1,c+1, b; a+b+c+1; u, v).
\]

We prove (3.42) as an example and left (3.41) to interested readers. Indeed, let \( F = F_1(\alpha; \beta, \beta'; \gamma; u, v) \), from (A.20), (A.21) and (A.22), we can solve for \((u\partial_u + v\partial_v)F\) in two different ways:

\[
(u\partial_u + v\partial_v)F = (\gamma - 1)(F(\gamma) - F)
\]

\[
= \beta(F(\beta) - F) + \beta'(F(\beta') - F),
\]

where \( F(\beta), F(\beta') \) and \( F(\gamma) \) stand for rising or lowering the indicated parameter by one. Two sides of (3.42) appear as the two lines on the right hand side above, with \( \alpha = d_m+1, \beta = c, \beta' = \tilde{b} \) and \( \gamma = a+b+c+1 \). \( \square \)

3.6. Reduction to Gauss hypergeometric functions. The main goal of this section is to compute the explicit expressions of spectral functions in Proposition 3.4 and 3.5 with the assistance of Mathematica. First of all, asking a computer algebra system to perform symbolic integration is tremendously inefficient. The improvement is based on the fact that Mathematica provides rapid evaluation for the Gaussian hypergeometric functions \( H_{a,b}(u;m) \) (by implementing the differential and contiguous relations) even when the parameter \( m \) is symbolic.\(^7\) For the double integral \( H_{a,b,c}(u,v;m) \), we show that all the Appell \( F_1 \) functions can be reduced to linear combinations (with function coefficients) of the \( 2F_1 \)-family.

\(^7\) Found in the author’s previous work [Liu17, Eq. (3.9)].

\(^8\) To use the symbolic integration feature, one has to specify the all values of \( a, b \) and \( m \).
Proposition 3.11. For $a \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we have the following divided difference relation:

$$F_1(a; 1, 1; c; x, y) = \frac{x_2 F_1(a, 1; c; x) - y_2 F_1(a, 1; c; y)}{x - y} = (z_2 F_1(a, 1; c; z))[x, y]_z$$

(3.43) and

$$F_1(a; 1, 2; c; x, y) = c^{-1}(x - y)^{-2} \left[ c^2 2 F_1(a, 1; c; x) + c y^2 2 F_1(a, 2; c; y) + x \left( -a y^2 2 F_1(a + 1, 2; c + 1; y) - 2 c y 2 F_1(a, 1; c; y) \right) \right].$$

(3.44)

Proof. Appell $F_1$ can be written as a $F_2$ type using Eq. (A.28). For those special $F_2$ functions, there is a reduction formula (A.29) to $2 F_1$ functions. For example, set $p = 0$ and $q = 1$ in (A.29), we obtain (3.43).

In fact, (3.43) admits a more elementary proof which leads to the following generalization:

Proposition 3.12. For $a \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we have

$$F_D^{(n)}(a, 1, 1, 1, 1; c; z_1, z_2, z_3, z_4) = (z^{-1} F_1(a, 1; c; z))[z_1, \ldots, z_4]_z$$

(3.45)

Proof. For $c > a$, we use the integral representation Eq. (3.28):

$$F_D^{(n)}(a; \alpha_1, \ldots, \alpha_n; c; z_1, \ldots, z_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 t^{a-1}(1 - t)^{c-a-1}(1 - z t)^{-\alpha_1} \cdots (1 - z_n t)^{-\alpha_n} dt,$$

in particular, $2 F_1(a, 1; c; z) = F_D^{(1)}(a; 1; c; z)$. Then (3.45) follows from the fact that

$$(z^{-1}(1 - z t)^{-1})[z_1, \ldots, z_n] = (1 - z t)^{-1} \cdots (1 - z_n t)^{-1}.$$  

To remove the assumption $c > a$, we show that (3.45) is preserved with respect to the differential relation:

$$F(c) = c^{-1}(c + z \partial_z) F(c+),$$

$$F_D^{(n)}(c) = c^{-1}(c + z_1 \partial_{z_1} + \cdots + z_n \partial_{z_n}) F_D^{(n)}(c+),$$

where $F = 2 F_1$, so that (3.45) holding for $c$ implies that it holds for $c - 1$ as well. To be more specific, let us assume (3.45) holds for $c+$, we claim that

$$F_D^{(n)}(c) = c^{-1}(c + z_1 \partial_{z_1} + \cdots + z_n \partial_{z_n})(z^{-1} F(c+; z))[z_1, \ldots, z_n]_z = (z^{-1}(c - z \partial_z))F(c+; z))[z_1, \ldots, z_n]_z = (z^{-1} F(c; z))[z_1, \ldots, z_n]_z,$$

in which the non-trivial step is to show that

$$P_n(z^{-1} F(c+; z))[z_1, \ldots, z_n]_z = (z^{-1} P_n F(c+; z))[z_1, \ldots, z_n]_z,$$

where $P_n$ and $P_0$ are the following differential operators:

$$P_n = z_1 \partial_{z_1} + \cdots + z_n \partial_{z_n}, \ P_0 = z \partial_z.$$

To shorten the proof, we make use of the residue version of divided difference:

$$f[z_1, \ldots, z_n] = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{(z - z_1) \cdots (z - z_n)} dz,$$
where the contour circling around the points \( z_1, \ldots, z_n \) exactly once. One can of course use \([2.11]\) to complete the argument which does not require \( f \) to be analytic. Observe that
\[
 z_l \partial_z (1 - z_l/z)^{-1} = -z \partial_z (1 - z/z_l)^{-1},
\]
furthermore:
\[
P_n \left( \frac{1}{(1 - z_1/z) \cdots (1 - z_n/z)} \right) = -P_1 \left( \frac{1}{(1 - z_1/z) \cdots (1 - z_n/z)} \right).
\]

We can prove Eq. \((3.46)\) by passing the relation through the contour integral and apply integration by parts:
\[
P_n(z^{n-1}F(c+; z))[z_1, \ldots, z_n]_z = \frac{1}{2\pi i} \int_C P_n \left( \frac{z^{n-1}F(c+; z)}{(z-z_1) \cdots (z-z_n)} \right) dz
\]
\[
= \frac{1}{2\pi i} \int_C z^{-1}F(c+; z)P_n \left( \frac{1}{(1 - z_1/z) \cdots (1 - z_n/z)} \right) dz
\]
\[
= \frac{1}{2\pi i} \int_C z^{-1}F(c+; z)(z \partial_z) \left( \frac{1}{(1 - z_1/z) \cdots (1 - z_n/z)} \right) dz
\]
\[
= \frac{1}{2\pi i} \int_C \left( 1 - z_1/z \right) \cdots (1 - z_n/z) \partial_z F(c+; z)
\]
\[
= \frac{1}{2\pi i} \int_C z^{n-1}(z \partial_z) F(c+; z) = (z^{n-1}P_1 F(c+; z))[z_1, \ldots, z_n]_z.
\]

The inductive argument fails when \( c = 0 \), therefore we need to assume that \( c \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \). \( \square \)

We end this section with the conclusion that the \( H \)-family defined in Eq. \((3.28)\) can be derived from the one-variable case \( H_{a,1}(z; m; j) \) by applying iterated divided differences and differentials.

**Theorem 3.13.** For the spectral functions in the \( V_j \)-term computation, we have the following two-step reduction to the Gaussian hypergeometric functions \( H_{a,1}(z; m; j) \) generated by differentials and divide differentives respectively:

\[
(3.47) \quad H_{a_0,\alpha_1,\ldots,\alpha_n}(z_1, \ldots, z_n; m; j) = \frac{\partial_{z_1}^{\alpha_1-1}\cdots \partial_{z_n}^{\alpha_n-1}H_{a_0,\ldots,1}(z_1, \ldots, z_n; m; j)}{(\alpha_1-1)! \cdots (\alpha_n-1)!}
\]

\[
(3.48) \quad H_{a,1,\ldots,1}(z_1, \ldots, z_n; m; j) = (z^{n-1}H_{a+n-1,1}(z; m; j))[z_1, \ldots, z_n]_z
\]

**Remark.** We compare the result to \([Lef17] \S 4,\S 5\), in which the spectral functions are generated in the same fashion by the modified logarithm function \( L_0 \) (first observed in \([CT11]\)), which, in particular, is a hypergeometric function:

\[
H_{1,1}(1 - z; 2) = 2F_1(1, 1; 2; 1 - z) = \frac{\ln z}{z-1} = L_0(z).
\]

**Proof.** The differential reduction \((3.47)\) follows from the differential relations:

\[
F_D(\alpha_q+) = (\alpha_q)^{-1}\partial_{\alpha_q}F_D, \quad 1 \leq q \leq n,
\]

where \( F_D = F_D(a; \alpha_1, \ldots, \alpha_n; z_1, \ldots, z_n) \). While Eq. \((3.48)\) is equivalent to Eq. \((3.45)\). \( \square \)

It has been shown in the author’s previous work \([Liu17] \) Eq. \((3.9)\) that for \( m = \dim M \) even and greater than two, functions \( K_{a,b}(u; m) \) and \( H_{a,b,c}(u, v; m) \) are jets of the following functions at zero: Eq. \((3.49)\) and \((3.50)\). Using the recurrence relations in Proposition \((3.10)\), we can give a short proof based on induction as below.
Proposition 3.14. Assume that the dimension $m$ is even and greater or equal than 4, for $j_m = (m - 4)/2 \in \mathbb{Z}_{\geq 0}$ and non-negative integers $a, b, c$, we have:

\( (3.49) \quad H_{a,b}(u; m) = \frac{d^{j_m}}{dz^{j_m}} \bigg|_{z=0} (1 - z)^{-a}(1 - u - z)^{-b}, \)

\( (3.50) \quad H_{a,b,c}(u, v; m) = \frac{d^{j_m}}{dz^{j_m}} \bigg|_{z=0} (1 - z)^{-a}(1 - u - z)^{-b}(1 - v - z)^{-c}. \)

Proof. We use induction on the dimension $m$. Let us focus on the one variable case first. When $m = 4$, $d_m = a + b$, (3.49) follows from the identity:

\[ 2F_1(\alpha, \beta; \beta, z) = 2F_1(\beta, \alpha; \beta, z) = (1 - z)^{-\alpha}. \]

Now assume that (3.49) holds for some even $m$, it remains to show that

\[ H_{a,b}(u; m + 2) = \frac{d^{j_m}}{dz^{j_m}} \bigg|_{z=0} d^{j_m} \bigg[ (1 - z)^{-a}(1 - u - z)^{-b} \bigg] = aH_{a+1,b}(u; m) + bH_{a,b+1}(u; m), \]

(3.51)

which is valid due to Proposition (3.14). Therefore (3.49) has been proved by induction. Same arguments work for (3.50). □

4. Symbolic computation

The explicit computation for the $b_2$ term in the resolvent approximation (see (3.2)) has been performed several times in different settings and via different methods: [LMI16, CMI13, FK15, FK13, FK12, CT11, Liu18, Liu17]. In this section, we shall only outline some key steps following [Liu18]. The new input from this section is the explicit dependence of the functional density in the dimension parameter $m$. Therefore, we shall assume $M_\theta = \mathbb{T}_\theta^m$ to the a noncommutative $m$-torus. As far as the spectral functions are concerned, same result holds for any toric noncommutative manifold of the same dimension.

Let us quickly review the algorithm of constructing resolvent approximation using pseudo-differential calculus. We assume the pseudo-differential operators considered in the sections are scalar operators, acting on smooth functions. Symbols of pseudo-differential operators form a subalgebra inside smooth functions on the cotangent bundle which admits a filtration. The associated graded algebra is called the algebra of complete symbols. Let $P$ and $Q$ be two pseudo-differential operators with symbol $p$ and $q$ respectively. Then the symbol of their composition has a formal expansion

\[ \sigma(PQ) = p \ast q \sim \sum_{j=0}^{\infty} a_j(p, q), \]

where each $a_j(\cdot, \cdot)$ is a bi-differential operator such that $a_j(p, q)$ reduce the total degree by $j$.

Consider a second order differential operator $P$ with symbol $\sigma(P) = p_2 + p_1 + p_0$, where $p_j$ homogeneous of order $j$ with $j = 0, 1, 2$. In most of pseudo-differential calculi, $a_0$ has no differential, here we assume that $a_0(p, q) = pq$. With the initial value $b_0 = (p_2 - \lambda)^{-1}$, one can recursively construct $b_j$:

\[ b_1 = [a_0(b_0, p_1) + a_1(b_0, p_2)](-b_0) \]

\[ b_2 = [a_0(b_0, p_0) + a_0(b_1, p_1) + a_1(b_0, p_1) + a_1(b_1, p_2) + a_2(b_0, p_2)](-b_0). \]

The construction can be continued while the complexity of the right hand sides increases dramatically.
We now specialize on noncommutative tori (of arbitrary dimension \( \geq 2 \)) from deformation point of view. Let \( M = \mathbb{T}^m \), a \( m \)-dimensional torus with the flat Euclidean metric, and let \( \nabla \) be the Levi-Civita connection. For two symbols \( p = p(x, \xi) \) and \( q = q(x, \xi) \), with \((x, \xi) \in T^*\mathbb{T}^m \cong \mathbb{T}^m \times \mathbb{R}^m \), we have

\[
a_j(p, q) = \frac{(-i)^j}{j!} (D^j p) \cdot (\nabla^j q), \quad j = 0, 1, 2, \ldots,
\]

where \( D = D_\xi \) is the vertical differential so that \((D^j p)\) is a contravariant \( j \)-tensor fields. Deformed contracting (what the \( \cdot \) stands for) with a covariant \( j \)-tensor field \( \nabla^j q \) gives rise to a scalar tensor field which is also a symbol of order \( \deg p + \deg q - j \). For more explanations about the notations, see [Liu18, Liu17].

Consider the perturbed Laplacian \( \Delta_k := k\Delta \), with the heat operator as a contour integral,

\[
e^{-t\Delta_k} = \int_C e^{-t\lambda}(\Delta_k - \lambda)^{-1}d\lambda.
\]

If we ignore the zero spectrum of \( \Delta_j \), the contour \( C \) can be chosen to be the imaginary axis from \(-i\infty \) to \( i\infty \) (cf. Eq. (3.9) for more details).

Any finite sum \( \sum_{j=0}^N b_j \) will give an approximation of the resolvent \((\Delta_k - \lambda)^{-1}\) which leads to the asymptotic expansion:

\[
\text{Tr}(ae^{-t\lambda}) \asymp \sum_{j=0}^\infty t^{(j-m)/2}V_j(a, \Delta_k) = \varphi_0(aR_j), \quad \forall a \in C^\infty(\mathbb{T}_\theta^m),
\]

where \( R_j \) is the functional density which can be explicitly determined by \( b_j \):

\[
R_j = \int_{T^*_x\mathbb{T}^m} \frac{1}{2\pi i} \int_C e^{-\lambda b_j}(\xi, \lambda)d\lambda d\xi.
\]

The symbol of \( \sigma(\Delta_k) = \sigma(k\Delta) = p_2 + p_1 + p_0 \) is a degree two polynomial in \( \xi \):

\[
p_2 = k|\xi|^2, \quad p_1 = p_0 = 0.
\]

As a function on the cotangent bundle, one can compute the vertical \( D \) and horizontal \( \nabla \) differential of \( p_2 \) as below:

\[
(Dp_2)_j = 2\epsilon_j, \quad (D^2p_2)_{jl} = 21_{jl},
\]

where \( 1_{jl} \) stands for the Kronecker-delta symbol. By substituting (4.3) and the derivatives of the symbols (4.5) into general formula (4.2), we obtain the expanded \( b_2 \) term as a function on the cotangent bundle,

\[
b_2(\xi) = 4r^2\xi_j\xi_l b^{0}_{l,j} b^{0}_{k,l} k^2.(\nabla^2 k)_{l,j} b_{0}. - r^2.1_{jl}.b^{0}_{l,j} k.(\nabla^2 k)_{l,j} b_{0} + 4r^2\xi_j\xi_l b^{0}_{l,j} b^{0}_{k,l} k.(\nabla^2 k)_{l,j},
\]

where the summation is taken automatically for repeated indices from 1 to \( m = \dim \mathbb{T}^m \). Let \( r = |\xi|^2 \), where the length is associated to the flat Riemannian metric. We will compute the integration over the fiber \( T^*_x\mathbb{T}^m \) using spherical coordinates:

\[
\int_{T^*_x\mathbb{T}^m} b_2(\xi) d\xi = \int_0^\infty b_2(r)r^{m-1}dr,
\]
with
\[ b_2(r) = \int_{|\xi|^2 = 1} b_2(\xi) ds, \]
where \( ds \) is the standard volume form for the unit sphere in \( \mathbb{R}^m \).

**Lemma 4.1.** Let \( ds \) be the standard volume form for the unit sphere in \( \mathbb{R}^m \), we have
\[ \text{Vol}(S^{m-1}) = \int_{S^{m-1}} ds = \frac{2\pi^{m/2}}{\Gamma(m/2)}, \]
\[ \int_{S^{m-1}} \xi_j \xi_l ds = \frac{\pi^{m/2}}{\Gamma(1 + m/2)} \mathbf{1}_{jl} = \text{Vol}(S^{m-1}) \frac{1}{m} \mathbf{1}_{jl}. \]

**Proof.** Elementary calculus, left to the reader. \( \square \)

Upto an overall constant factor \( \text{Vol}(S^{m-1}) \), \( b_2(r) = \int_{S^{m-1}} b_2(\xi) ds \) equals
\[ b_2(r) = -r^2 \mathbf{1}_{jl} \mathbf{1}_{b_0^2} k \cdot (\nabla^2 k)_{l.j}.b_0 + \frac{4r^4 \mathbf{1}_{jl} \mathbf{1}_{b_0^2} b_2 (\nabla^2 k)_{l.j} b_0}{m}, \]
\[ + 2r^4 \mathbf{1}_{jl} \mathbf{1}_{b_0^2} k \cdot (\nabla k)_l b_0 \cdot (\nabla k)_j b_0 + \frac{4r^4 \mathbf{1}_{jl} \mathbf{1}_{b_0^2} b_0 \cdot (\nabla k)_l b_0 \cdot (\nabla k)_j b_0}{m}, \]
\[ - \frac{8r^6 \mathbf{1}_{jl} \mathbf{1}_{b_0^2} b_0^2 \cdot (\nabla k)_l b_0 \cdot (\nabla k)_j b_0}{m} - \frac{4r^6 \mathbf{1}_{jl} \mathbf{1}_{b_0^2} b_0 \cdot (\nabla k)_l b_0 \cdot (\nabla k)_j b_0}{m}. \]

The summation over \( i, j \) stands for contraction between contravariant and covariant tensors. To be precise:
\[ (\nabla^2 k) \cdot g^{-1} := \sum_{ij} \mathbf{1}_{ij} (\nabla^2 k)_{ij} = \text{Tr Hess}(k) = -\Delta k, \]
\[ (\nabla k \nabla k) \cdot g^{-1} := \sum_{ij} \mathbf{1}_{ij} (\nabla k)_i (\nabla k)_j = \langle \nabla k, \nabla k \rangle_g, \]
where \( g^{-1} \) stands for the metric on the cotangent bundle. We now give some examples on how to apply integration lemma developed at the beginning of the paper. Recall \( b_0 = (kr^2 - \lambda)^{-1} \). We first move powers of \( k \) in front, for instance,
\[ r^2 \mathbf{1}_{jl} \mathbf{1}_{b_0^2} k \cdot (\nabla^2 k)_{l.j}.b_0 = k \left( r^2 \mathbf{1}_{jl} \mathbf{1}_{b_0^2} (\nabla^2 k)_{l.j}.b_0 \right), \]
\[ r^6 \mathbf{1}_{jl} \mathbf{1}_{b_0^2} b_0^2 \cdot (\nabla k)_l b_0 \cdot (\nabla k)_j b_0 = k^2 y_1 \left( 1_{jl} \mathbf{1}_{b_0^2} (\nabla k)_l b_0 \cdot (\nabla k)_j b_0 \right), \]
where in the second line, \( y_1 \) is the conjugation operator acting on the factor \( (\nabla k)_l \), which allows us to move the \( k \) between \( (\nabla k)_l \) and \( (\nabla k)_j \) in front. Then we apply Proposition 3.4
\[ \int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda r^2} \mathbf{1}_{jl} \mathbf{1}_{b_0^2} k \cdot (\nabla^2 k)_{l.j}.b_0 d\lambda (r^{m-1} dr) = k^{-(m/2+1)} K_{2,1}(y; m)(\nabla^2 k) \cdot g^{-1}, \]
and Proposition 3.5
\[ \int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda} \mathbf{1}_{jl} \mathbf{1}_{b_0^2} (\nabla k)_l b_0 \cdot (\nabla k)_j b_0 d\lambda (r^{m-1} dr) = k^{-(m/2+3)} H_{2,2,1}(y_1, y_2; m)((\nabla k) \nabla k) \cdot g^{-1}. \]

The one variable spectral function is obtained by integrating the first two terms in (4.7):
\[ K_{\Delta_k}(y; m) = \frac{4}{m} H_{3,1}(z; m) - H_{2,1}(z; m), \quad z = 1 - y \]
and for the two variable function, it comes from the last four terms in (4.7):

\[
H_{\Delta_k}(y_1, y_2; m) = \left( \frac{4}{m} + 2 \right) H_{2,1,1}(z_1, z_2; m) - \frac{4(1 - z_1) H_{2,2,1}(z_1, z_2; m)}{m} \tag{4.11}
\]

where \( z_1 = 1 - y_1 \) and \( z_2 = 1 - y_1 y_2 \). Let us summerize the computation of the whole section as a theorem.

**Theorem 4.2.** For the perturbed Laplacian \( \Delta_k = k \Delta \), a closed form of the functional \( V_2(\cdot, \Delta_k) \) is given by

\[
V_2(a, \Delta_k) = \varphi_0(a R_{\Delta_k}), \quad \forall a \in C^\infty(\mathbb{T}_g^m),
\]

with \( R_{\Delta_k} \in C^\infty(\mathbb{T}_g^m) \). Upto an overall constant \( \text{Vol}(S^{m-1})/2 \),

\[
R_{\Delta_k} = k^{jm} K_{\Delta_k}(y; m)(\nabla^2 k) \cdot g^{-1} + k^{jm-1} H_{\Delta_k}(y_1, y_2; m)(\nabla k \nabla k) \cdot g^{-1},
\]

where \( j_m = -m/2 \). Contractions with the metric \( g^{-1} \) are explained in (4.8) and (4.9). Spectral functions \( K_{\Delta_k} \) and \( H_{\Delta_k} \) are defined in (4.10) and (4.11) respectively.

In terms of hypergeometric functions, set \( z = 1 - y \), \( z_1 = 1 - y_1 \) and \( z_2 = 1 - y_1 y_2 \):

\[
K_{\Delta_k}(y; m) = -\frac{1}{2} \Gamma(m/2 + 1) \binom{2}{m/2 + 1, 1, 1, 3, z}
\]

\[
+ \frac{2\Gamma(m/2 + 2)}{3m} \binom{2}{m/2 + 2, 1, 4, z},
\]

and

\[
H_{\Delta_k}(y_1, y_2; m) = \frac{1}{6m} \binom{2}{m + 2} \Gamma(m/2 + 2) \binom{2}{m/2 + 2, 1, 1, 4; z_1, z_2}
\]

\[
- \frac{1}{6m} \Gamma(m/2 + 3) \binom{2}{m/2 + 3, 1, 1, 5; z_1, z_2}
\]

\[
- \frac{1}{6m} \binom{2}{m/2 + 3}(1 - z_1) \binom{2}{m/2 + 3, 2, 1, 5; z_1, z_2}. \tag{4.14}
\]

5. **FUNCTIONAL RELATIONS**

The spectral functions \( K_{\Delta_k} \) and \( H_{\Delta_k} \) are subject to some functional relations which can be established via a priori arguments. We will only provide symbolic verification in this paper which strongly endorses the validation of the whole calculation. The conceptual interpretation is left to the sequel paper (pat II).

5.1. **Gauss-Bonnet theorem for \( \mathbb{T}_g^2 \).** Let us fix the dimension \( m = 2 \) in this section 5.1.

Consider the action/functional \( F(k) = \varphi_0(R_{\Delta_k}) \), which admits a local expression

\[
F(k) = \varphi_0 \left( k^{-2} T(y) (\nabla k) \cdot (\nabla k) \right) \cdot g^{-1},
\]

where

\[
T(y) = -K_{\Delta_k}(1) \frac{y^{-2} - 1}{y - 1} + H_{\Delta_k}(y, y^{-1}). \tag{5.1}
\]

The Gauss-Bonnet theorem states that \( F(k) \) is a constant functional, in particular the zero functional since \( F(1) = 0 \). Since \( \varphi_0 \) is a trace, we have

\[
F(k) = \varphi_0 \left( k^{-2} T(y) (\nabla k) \cdot (\nabla k) \right) \cdot g^{-1} = \varphi_0 \left( (\nabla k) T(y^{-1}) (\nabla k) k^{-2} \right) \cdot g^{-1}
\]

\[
= \varphi_0 \left( k^{-2} y^{-2} T(y^{-1}) (\nabla k) \cdot (\nabla k) \right) \cdot g^{-1}.
\]
Let $\bar{T}(y) = y^{-2}T(y^{-1})$, then a sufficient condition for $F(k) = 0$ is the following:

\begin{equation}
(5.2)
T(y) + \bar{T}(y) = 0.
\end{equation}

The goal of this section is to show, using the functional relations between the Gauss hypergeometric family, that $K_{\Delta_k}$ and $H_{\Delta_k}$ defined in (4.13) and (4.14) indeed give rise to the definition of $\Delta_k$. The first term in (5.1) equals $y^{-1}/6$ and the second term leads to the following three hypergeometric functions:

\begin{equation}
(5.3)
T(y) = \frac{1}{6}[1/y + 8_{2}F_{1}(3, 1; 4; 1 - y) - 6_{2}F_{1}(4, 1; 5; 1 - y) - 3y_{2}F_{1}(4, 2; 5; 1 - y)].
\end{equation}

We have used the reduction relation (A.25) for Appell $F_1$ functions.

**Proposition 5.1.** Consider the following function: The function $T(y)$ defined above (5.3) can be simplified to

\[
T(y) = K(1) \left(y^{-1} - 2_{1}F_{1}(3, 1; 5; 1 - y)\right),
\]

where $K(1) := K_{\Delta_k}(1; 2) = 1/6$. Furthermore, it satisfies the functional equation:

\begin{equation}
(5.4)
T(y) + y^{-2}T(y^{-1}) = 0.
\end{equation}

**Proof.** Apply (A.11), we see that $2_{1}F_{1}(4, 1; 5; 1 - y) = y^{-1}2_{1}F_{1}(3, 1; 5; 1 - y)$, moreover

\[
8_{2}F_{1}(3, 1; 4; 1 - y) - 6_{2}F_{1}(4, 1; 5; 1 - y) - 3y_{2}F_{1}(4, 2; 5; 1 - y) = - 2_{1}F_{1}(3, 1; 5; 1 - y).
\]

It remains to check that:

\begin{equation}
(5.5)
0 = 6 \left[T(y) + y^{-2}T(y^{-1})\right] = \frac{2}{y} - 2_{1}F_{1}(3, 1; 5; 1 - y) - y^{-2}2_{1}F_{1}(3, 1; 5; 1 - y^{-1}).
\end{equation}

Indeed,

\[
2_{1}F_{1}(3, 1; 5; 1 - y) = y_{2}F_{1}(2, 4; 5; 1 - y), \quad \text{by (A.11)},
\]

\[
y^{-2}2_{1}F_{1}(3, 1; 5; 1 - y^{-1}) = y_{2}F_{1}(3, 4; 5; 1 - y), \quad \text{by (A.10)}.
\]

Therefore (5.5) is reduced to:

\begin{equation}
(5.6)
2_{1}F_{1}(3, 4; 5; 1 - y) + 2_{1}F_{1}(2, 4; 5; 1 - y) = 2y^{-2},
\end{equation}

which is one of the contiguous relations in Prop. (A.1). In fact, take $F = 2_{1}F_{1}(3, 4; 5; 1 - s)$ then $F(a-) = 2_{1}F_{1}(2, 4; 5; 1 - y)$, and (5.6) follow from equating the third line and the fifth line in the right hand side of (A.9):

\[
2F(a-) + 2F = 4yF(b+) = 4y_{2}F_{1}(3, 5; 5; 1 - y) = 4y^{-2}.
\]

We have used (A.12) for the last equal sign. $\square$

### 5.2. The Connes-Moscovici functional relation.

**Proposition 5.2.** For any $m = \dim M > 2$, the modular curvature $R_{\Delta_k}$ in Theorem 4.2 involves the following spectral functions

\begin{equation}
(5.7)
K_{\Delta_k}(y; m) = \frac{-8y^{-\frac{m}{2}} \left((m(y - 1) - 4y)y^{m/2} + y(m(y - 1) + 4)\right) \Gamma \left(\frac{m}{2} + 2\right)}{(m - 2)m^2(m + 2)(y - 1)^4},
\end{equation}
and
\( (5.8) \)
\[ H_{\Delta_k}(y_1, y_2; m) = \frac{2}{m}(y_1 - 1)^{-2}(t - 1)^{-2}(y_1 t - 1)^{-3}\Gamma(m/2 + 1) \]
\[ \left[ 2y_1^{-m/2}(y_1 y_2 - 1)^3 + 2(y_2 - 1)^2 \left( \frac{1}{2}m(y_1 - 1)(y_1 y_2 - 1) + y_1 (1 - 2y_1)y_2 + 1 \right) \right. \]
\[ \left. - 2(y_1 - 1)^2y_2(y_1 y_2)^{-1/2} \left( \frac{1}{2}m(y_2 - 1)(y_1 y_2 - 1) + y_1 y_2^2 + y_2 - 2 \right) \right]. \]

**Remark.** One can recover \( K_{\Delta_k}(s; 2) \) and \( H_{\Delta_k}(s; t; 2) \) by setting \( m \to 2 \) in the RHS of \((5.7)\) and \((5.8)\) respectively.

**Proof.** Using **Mathematica**, one computes explicitly the following Gauss hypergeometric functions:
\[ H_{2,1}(z; m) = \frac{2(1-z)^{-\frac{m}{2}}((m-2)z+2)(1-z)^{m/2-2z+2})\Gamma(m+1)}{(m-2)m^2} \]
\[ H_{3,1}(z; m) = \frac{(1-z)^{-\frac{m}{2}}((m-2)z(m+4)+8)(1-z)^{m/2-8z+8})\Gamma(m+2)}{m(m^2-4)z^4} \]
\[ H_{4,1}(z; m) = \frac{(1-z)^{-\frac{m}{2}}((m-2)z(mz+(m+2)z+6)+24)(1-z)^{m/2-8z+8})\Gamma(m+3)}{3(m-2)m(m+2)(m+4)z^4} \]

For the two-variable family, we have the reduction using divided difference and differentials:
\[ H_{2,1,1}(z_1, z_2; m) = (z H_{3,1}(z; m))[z_1, z_2]_z, \quad H_{3,1,1}(z_1, z_2; m) = (z H_{4,1}(z; m))[z_1, z_2]_z \]
\[ H_{2,2,1}(z_1, z_2; m) = \partial_{z_1} H_{2,1,1}(z_1, z_2; m). \]

We remind the reader again \( z = 1 - y, z_1 = 1 - y_1 \) and \( z_2 = 1 - y_1 y_2 \). Symbolic simplification leads to \((5.7)\) and \((5.8)\).

With the explicit expressions \((5.7)\) and \((5.8)\), one checks via CAS that:

**Theorem 5.3.** For all \( m \in [2, \infty) \), the spectral functions \( K_{\Delta_k} \) and \( H_{\Delta_k} \) defined in \((5.7)\) and \((5.8)\) respectively satisfies the following Connes-Moscovici type functional relation:
\[ -H_{\Delta_k}(y_1, y_2; m) = y_1^{-m/2-2}K_{\Delta_k}(z; m)[y_1^{-1}, y_2^{-1}]_z - (y_1 y_2)^{-m/2-2}K_{\Delta_k}(z; m)[(y_1 y_2)^{-1}, y_2]_z \]
\[ -K_{\Delta_k}(z; m)[y_1 y_2, y_1]_z, \]
where the divided difference notation is explained in \((2.3)\).

The functional relation is a multiplicative version compared with Eq. \((1.3)\), of course, extended to all dimensions. The variational proof and geometric interpretation of the equation will be the main topic of the sequel paper.

**Appendix A. Hypergeometric Functions**

Hypergeometric functions had been studied intensively in the nineteenth century. The pioneers include Gauss (1813), Ernst Kummer (1836) and Riemann (1857), etc. The two variable extension of the hypergeometric functions has four different types known as Appell’s \( F_1 \) to \( F_4 \) functions. In this appendix, we only collect some basic knowledge of \( 2F_1 \) and \( F_1 \) functions that are related to our exploration of modular curvature. Most of the identities quoted in this section can be found in [EMOT53], [OLBC10], [App25] and [AdF26].
A.1. **Gauss Hypergeometric functions.** For \(|z| < 1\), the (Gauss) hypergeometric function \(2F_1(a, b; c; z)\) is represented by the hypergeometric series

\[
2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},
\]

(A.1)

where the coefficients are given by Pochhammer symbols:

\[
(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}.
\]

(A.2)

What we need in the paper is the following Euler type integral representation:

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-t)^{c-b-1} t^{b-1} (1-zt)^{a} \, dt.
\]

(A.3)

It is a solution of Euler’s hypergeometric differential equation

\[
z(1-z) \frac{d^2 w}{dz^2} + (c - (a + b + 1)z) \frac{dw}{dz} - abw = 0
\]

(A.4)

For \(F := 2F_1(a, b; c; z)\), there are six associated contiguous functions obtained by applying \(\pm 1\) on only one of the parameters \(a, b\) and \(c\). We abbreviate them as \(F(a+), F(b+), F(c-)\), etc. Gauss showed that \(F\) can be written as a linear combination of any two of its contiguous functions, which leads to 15 (6 choose 2) relations. They can be derived from the differential relations among the family

\[
\frac{d}{dz} (2F_1(a, b; c; z)) = \frac{ab}{c} 2F_1(a + 1, b + 1; c + 1; z),
\]

(A.5)

also

\[
F(a+) = F + \frac{1}{a} z \frac{d}{dz} F,
\]

(A.6)

\[
F(b+) = F + \frac{1}{b} z \frac{d}{dz} F,
\]

(A.7)

\[
F(c-) = F + \frac{1}{c} z \frac{d}{dz} F.
\]

(A.8)

Combine with the second order ODE [A.4], we have

**Proposition A.1.** One can read all 15 relations among the contiguous functions by equating any two lines of the right hand side of Eq. [A.9]:

\[
z \frac{dF}{dz} = \frac{ab}{c} F(a+, b+, c+)
\]

\[
= a(F(a+) - F)
\]

\[
= b(F(b+) - F)
\]

\[
= (c-1)(F(c-) - F)
\]

\[
= (c-a)F(a+) + (a-c+az)F
\]

\[
= \frac{(c-b)F(b+) + (b-c+az)F}{1-z}
\]

\[
= \frac{z}{c(1-z)} (c-a)(c-b)F(c+) + c(a+b-c)F(1-z).
\]

(A.9)
There are other types of symmetries among the hypergeometric family. For example, under fractional linear transformation
\[ 2F_1(a, b; c; z) = (1 - z)^{-b} 2F_1(c - a, b; c; \frac{z}{z - 1}) \] (A.10)
\[ 2F_1(a, b; c; z) = (1 - z)^{-a} 2F_1(a, c - b; c; \frac{z}{z - 1}). \] (A.11)

They are known as Pfaff transformations. Then the Euler transformation
\[ 2F_1(a, b; c; z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b; c; z) \] follows quickly.

At last, we recall the following special cases:
\[ 2F_1(1, 1; 2; z) = -\ln(1 - z)/z, \quad 2F_1(a, b; z) = (1 - z)^{-a}. \] (A.12)

In particular, we see that \( 2F_1(1, 1; 2; 1 - z) \) is the modified-log function, whose first appearance related to the modular geometry is in [CT11] (denoted by \( L_0(z) \)).

As a crucial example, we examine the functional equation for the Gauss-Bonnet theorem without computing the explicit expression.

The following proposition is useful to compute explicit expression of hypergeometric functions appeared in the study of modular geometry.

**Proposition A.2.** For \( p \in \mathbb{Z}_{>0} \),
\[ 2F_1(a, 1; c; z) = \frac{(1 - c)^p}{(a - c + 1)^p} 2F_1(a, 1; c - p; z) + \frac{1}{z} \sum_{k=1}^{p} \frac{(1 - c)_k}{(a - c + 1)_k} \left( \frac{z - 1}{z} \right)^{k-1}. \] (A.13)

Since \( 2F_1(a, 1; 1; z) = (1 - z)^{-a} \), when \( a \) and \( c \) belong to the natural domain of the right hand side of (A.13), it provides a symbol evaluation for \( 2F_1(a, 1; c; z) \).

**A.2. Appell Hypergeometric functions.** The hypergeometric series (A.14) has a variety of generalizations to multi-variable cases. For the two-variable case, Appell introduced four types of series \( F_1 \) to \( F_4 \). So far, \( F_1 \) is directly related to the modular curvature functions. To deal with some symbolic computation, we need \( F_2 \) as a bridge.

\[ F_1(a, b; b'; c; x; y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m} (b')_{n}}{m! n! (c)_{m+n}} x^{m} y^{n} \] (A.14)
\[ F_2(a, b; b'; c, c'; x; y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m} (b')_{n}}{m! n! (c)_{m+n} (c')_{n+m}} x^{m} y^{n}, \] (A.15)

where Pochhammer symbols (A.2) is used in the coefficients. They have a double integral representation:
\[ \frac{\Gamma(b)\Gamma(b')\Gamma(c - b - b')}{\Gamma(c)} F_1(a; b, b'; c; x, y) \]
\[ = \int_{0}^{1} \int_{0}^{1-t} u^{b-1} v^{b'-1}(1 - u - v)^{c-b-b'-1}(1 - xu - yv)^{-a} du dv. \]
\[ \frac{\Gamma(b)\Gamma(b')\Gamma(c - b)\Gamma(c' - b')}{\Gamma(c)\Gamma(c')} F_2(a; b, b'; c, c'; x, y) \]
\[ = \int_{0}^{1} \int_{0}^{1} u^{b-1} v^{b'-1}(1 - u)^{c-b-1}(1 - v)^{c'-b'-1}(1 - xu - yv)^{-a} du dv. \] (A.16) (A.17)

Parallel to the differential system for Gaussian hypergeometric functions, we have a similar relations for rising the parameters via differential operators: \( x\partial_x \) and \( y\partial_y \). Denote
by $F_1 := F_1(a; b, b', c; x, y)$ and $F_1(a+) := F_1(a + 1; b, b', c; x, y)$, same pattern applies to $F_1(b+), F_1(b'+)$ and $F_1(c+)$, then
\begin{align}
\partial_x F_1 &= F_1(a+, b+, c+), \quad \partial_y F_1 = F_1(a+, b'+, c+),
\end{align}
also
\begin{align}
F_1(a+) &= a^{-1}(a + x\partial_x + y\partial_y) F_1 \\
F_1(b+) &= b^{-1}(b + x\partial_x) F_1 \\
F_1(b'+) &= b'^{-1}(b' + y\partial_y) F_1 \\
F_1 &= c^{-1}(c + x\partial_x + y\partial_y) F_1(c+).
\end{align}

For $F_1$ itself, it is a solution of the PDE system:
\begin{align}
\left[ x(1 - x)\partial_x^2 + y(1 - x)\partial_x\partial_y + [c - (a + b + 1)]\partial_x - by\partial_y - ab \right] F_1 &= 0 \tag{A.23} \\
\left[ y(1 - y)\partial_y^2 + x(1 - y)\partial_x\partial_y + [c - (a + b' + 1)]\partial_x - b'x\partial_y - ab' \right] F_1 &= 0 \tag{A.24}
\end{align}

The $F_1$ family reduces to the hypergeometric functions in the situations:
\begin{align}
F_1(a; b, b'; c; 0, y) &= 2F_1(a, b'; c; y), \\
F_1(a; b, b'; c; x, 0) &= 2F_1(a, b; c; x).
\end{align}

In addition,
\begin{align}
F_1(a; b, b'; c; x, x) &= (1 - x)^{c-a-b-b'} 2F_1(c - a; c - b - b'; c; x) \\
&= 2F_1(a, b; b + b'; x), \\
F_1(a; b, b'; b + b'; x, y) &= (1 - y)^{-a} 2F_1(a, b + b'; \frac{x - y}{1 - y} \frac{1}{x - 1} 
\end{align}

Similar to the Pfaff transformation \[ \text{(A.10)}, \] we have:
\begin{align}
F_1(a; b, b'; c; x, y) &= (1 - x)^{-b}(1 - y)^{-b'} F_1(c - a; b, b'; c; \frac{x}{x - 1}, \frac{y}{y - 1}) \\
&= (1 - x)^{-a} F_1(a; c - b - b', b'; c; \frac{x}{x - 1}, \frac{y - x}{1 - x}) \tag{A.27}.
\end{align}

The $F_1$ family can be computed via the reduction formula, cf. [OLBC10, Sec. 16.16] or [EMOT53, Sec. 5.10, 5.11],
\begin{align}
F_1(a, b, b'; c, x, y) &= (x/y)^b F_2(b + b'; a, b'; c, b + b', x, 1 - x/y) \\
&= (y/x)^b F_2(b + b'; a, b; c, b + b', y, 1 - y/x). \tag{A.28}
\end{align}

**Proposition A.3** ([OSS05], Theorem 2). For $a \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, p, q \in \mathbb{Z}_{\geq 0}, p < q$ and $|x| + |y| < 1$,
\begin{align}
F_2(q + 1, a, p + 1; b, p + 2; x, y) &= - \frac{p!}{q!(1 - q)_p} \frac{p + 1}{y^{p+1}} 2F_1(a, q - p; b; x) \\
&= \frac{p + 1}{y^{p+1}} \sum_{k=0}^{\binom{p}{k}} \binom{-1}{k} \binom{p}{k} \frac{(-1)^{k}}{q - k} (1 - y)^{q - k} \left(\frac{p}{k}\right) \sum_{m=0}^{p-k} \binom{p - k}{m} \binom{p - k}{m} \binom{a}{m} \frac{x^{m}}{b^{m}} \cdot 2F_1 \left( a + m, q - k; b + m; \frac{x}{1 - y} \right). \tag{A.29}
\end{align}
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