Conformal scattering theory for a tensorial Fackerell-Ipser equations on Schwarzschild spacetime

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Abstract. In this paper, we show that the existing the energy and pointwise decay results for the fields satisfied the tensorial Fackerell-Ipser equations (which are obtained from the Maxwell and spin ±1 Teukolsky equations) on the Schwarzschild spacetime are sufficient to obtain a conformal scattering theory. This work is continuing the previous work on the conformal scattering theory for linearized gravity fields [43] which are arised from spin ±2 Teukolsky equations.

Keywords. Conformal scattering, black holes, scalar (tensorial) Fackerell-Ipser equations, Teukolsky equations, Schwarzschild metric, null infinity, conformal compactification.

Mathematics subject classification. 35L05, 35P25, 35Q75, 83C57.

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1 Introduction

The idea of the conformal compactification structure of spacetimes was posed initially by Penrose [40] since 1960’s. It plays an important role to study peeling and conformal scattering which are two aspects of geometric conformal analysis. Conformal scattering has a long time historical from the works of Friedlander [13, 14, 15, 16, 17], Baez et al. [5], Hörmander [23] to Mason and Nicolas [32], Joudioux [27, 28], Nicolas [38], Mokdad [35] and Taujanskas [47].

A conformal scattering theory on the asymptotic flat spacetimes such as Schwarzschild and Kerr consists three steps: first, we solve the Cauchy problem of the rescaled equations on the

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rescaled spacetime, then define and extend the trace operator; second, we prove the energy
identity up to the future timekine infinity $i^+$ and the extension of trace operator is injective;
third, to prove the extension of trace operator is subjective (this shows that the trace operator is
an isomorphic), we solve the Goursat problem with the initial data on the conformal boundary
consists the horizon and null infinity. The works of Nicolas [38] and Mokdad [35] are detailed
the method to construct a conformal scattering for the scalar wave and Maxwell equations
on the static symmetric spherical spacetimes such as Schwarzschild and Reissner-Nordström de
Sitter spacetimes respectively. In their works, the results about the bounded energy, Morawertz
estimate and pointwise decay of the fields play an important role in the second step to prove the
energy identity up to $i^+$. Besides conformal scattering, there exist some works which construct
analytic scattering theories used also the energy and decay results for wave equations on the
interior of Reissner-Nordström de Sitter and slowly Kerr spacetimes of Keller et al. [29] and
Dafermos et al. [10].

The uniformly bounded energy, Morawerts estimate and pointwise decay are obtained nat-
urally in the program to prove linear and nonlinear stability of black hole spacetimes and the
related problems (see [3, 9, 10, 11, 19, 21, 22, 26, 28, 45, 24]). There the method of $r^p$-theory of
Dafermos and Rodnianski [8] is an essential tool of the proof in a lot of work later.

Conformal scattering for linearized gravity fields has obtained in the recent work of the author
[43]. The linearized gravity fields are the solutions of Reeger-Wheeler and Zerelli equations
which arised from the spin $\pm 2$ Teukolski equations by Chandrasekka transformations. This
paper we explore the method in [38, 35, 43] to construct a conformal scattering for the tensorial
Facke~rell-Ipser equations which arised from Maxwell and spin $\pm 1$ Teukolsky equations. In detail,
the tensorial equations will established from the scalar wave equations by commuting with the
angular derivatives $r\nabla$ (see Section 2.3). In comparing with the previous work on linearized
gravity fields [43] and also for the complex Fackerell-Ipser equations in [1] the potential decays
like $r^{-3}$, whence the real potential of these equations in this work decays like $r^{-2}$. There are
some works on the decays of scalar Fackerell-Ipser equations in Schwarzschild spacetime such as
[6, 18, 34] hence we can understand the decays for tensorial equations. However, in the second
step of the construction we follow the energy and pointwise decay results which are obtained in
the recent work of Pasqualotto [44] to show that the energy across the hypersurfaces $H_{uT}$ and
$\Sigma_{vT}$ that tends vers to zero when $uT$ and $vT$ tend to infinity (see Section 3.2). This leads to
the energy indentity up to $i^+$ i.e the energy of the fields restricted on the initial hypersurface
$\Sigma_0 = \{t = 0\}$ equals to the sum of energies of the fields restricted on the future horizon $\mathcal{I}^+$
(resp. $\mathcal{I}^-$) and on the future null infinity $\mathcal{I}^+$ (resp. $\mathcal{I}^-$). As a consequence, the extended
trace operator is injective. To prove that the trace operator is subjective, we need to solve the
Goursat problem with the initial data on the conformal boundary. This work is done by the
same manner for the wave equations in [38].

Remark and notation.
• Since the Cauchy problem are well-posedness and Goursat problem can be solved similarly as
the work [38] respectively, in this work we focus to show that how the decay results are sufficient
to obtain the energy identity up to $i^+$ in the step 2 of the construction.
• We denote the space of smooth compactly supported scalar functions on $\mathcal{M}$ (a smooth manifold
without boundary) by $C^\infty_0(\mathcal{M})$ and the space of distributions on $\mathcal{M}$ by $\mathcal{D}'(\mathcal{M})$.
• We denote the bundle tangent to each 2-sphere $S^2(t, r)$ at $(t, r)$ by $\mathcal{B}$ with dual $\mathcal{B}^*$ and the
vector space of all smooth sections of $\mathcal{B}$ by $\Gamma(\mathcal{B})$.

- Let $f(x)$ and $g(x)$ be two real functions. We write $f \lesssim g$ if there exists a constant $C \in (0, +\infty)$ such that $f(x) \leq Cg(x)$ for all $x$, and write $f \simeq g$ if both $f \lesssim g$ and $g \lesssim f$ are valid.

## 2 Geometrical and analytical setting

### 2.1 Schwarzschild metric and Penrose conformal compactification

Let $(\mathcal{M} = \mathbb{R} \times [0, +\infty[\times \mathbb{S}^2_\omega, g)$ be 4-dimensional Schwarzschild manifold, with the Lorentzian metric $g$ is given by

$$g = F dt^2 - F^{-1} dr^2 - r^2 d\sigma^2, \quad F = F(r) = 1 - \mu, \quad \mu = \frac{2M}{r},$$

where $d\sigma^2$ is the Euclidean metric on the unit 2-sphere $\mathbb{S}^2$, and $M > 0$ is the mass of the black hole. In this paper, we work on the exterior of the black hole $\mathcal{B}_I = \{r > 2M\}$.

Recall the Regge-Wheeler coordinate $r_* = r + 2M \log(r - 2M)$, such that $dr = Fdr_*$. In the system $(t, r_*, \omega^2)$, the Schwarzschild metric takes the form

$$g = F(dt^2 - dr_*^2) - r^2 d\sigma^2.$$

The retarded and advanced Eddington-Finkelstein coordinates $u$ and $v$ defined by

$$u = t - r_*, \quad v = t + r_*.$$

The outgoing and incoming principal null directions are

$$L = \partial_v = \partial_t + \partial_{r_*}, \quad L = \partial_u = \partial_t - \partial_{r_*}.$$

Putting $\Omega = 1/r$ and $\dot{g} = \Omega^2 g$. We obtain a conformal compactification of the exterior domain in the coordinates $u, R = 1/r, \omega$ that is $\left(\mathbb{R}_u \times \left[0, \frac{1}{2M}\right] \times \mathbb{S}^2_\omega, \dot{g}\right)$ with the rescaled metric

$$\dot{g} = R^2(1 - 2MR)du^2 - 2dudR - d\sigma^2. \quad (1)$$

Future null infinity $\mathcal{I}^+$ and the past horizon $\mathcal{H}^-$ are null hypersurfaces of the rescaled spacetime

$$\mathcal{I}^+ = \mathbb{R}_u \times \{0\} \times \mathbb{S}^2_\omega, \quad \mathcal{H}^- = \mathbb{R}_u \times \{1/2M\} \times \mathbb{S}^2_\omega.$$

If we use the advanced coordinates $v, R, \omega$, the rescaled metric $\dot{g}$ takes the form

$$\dot{g} = R^2(1 - 2MR)dv^2 + 2dvdR - d\sigma^2. \quad (2)$$

Past null infinity $\mathcal{I}^-$ and the future horizon $\mathcal{H}^+$ described as the null hypersurfaces

$$\mathcal{I}^- = \mathbb{R}_v \times \{0\} \times \mathbb{S}^2_\omega, \quad \mathcal{H}^+ = \mathbb{R}_v \times \{1/2M\} \times \mathbb{S}^2_\omega.$$
The Penrose conformal compactification of $B_I$ is

$$\tilde{B}_I = B_I \cup \mathcal{I}^+ \cup \mathcal{I}^- \cup \mathcal{I}^+ \cup S_c^2,$$

where $S_c^2$ is the crossing sphere.

The scalar curvature of the rescaled metric $\hat{g}$ is

$$\text{Scal}_{\hat{g}} = 12MR.$$

The volume forms associated with the Schwarzschild metric $g$ and the rescaled metric $\hat{g}$ are

$$d\text{Vol}_g = r^2 F dt \wedge dr \wedge dS^2$$

and

$$d\text{Vol}_{\hat{g}} = R^2 F dt \wedge dr \wedge dS^2,$$

where $dS^2$ being the euclidean area element on unit 2-sphere $S^2$.

### 2.2 The equations arising from Maxwell system

We follow [44] to express the Maxwell, Teukolsky and tensorial Fackerell-Ipser equations. Let $\mathcal{F}$ be an antisymmetric 2-form on the exterior of Schwarzschild black hole $B_I$. The Maxwell equations take the form

$$d\mathcal{F} = 0, \quad d* \mathcal{F} = 0,$$

where $*$ denotes the Hodge dual operator. The equations can be expressed as

$$\nabla_{[\mu} \mathcal{F}_{\kappa \lambda]} = 0, \quad \nabla^\mu \mathcal{F}_{\mu \nu} = 0.$$

The Maxwell field $\mathcal{F}$ can be decomposed into extreme components $\alpha, \alpha \in \Gamma(B^*)$ and $\rho, \sigma \in C^\infty(B_I)$ which are defined as follows

$$\alpha(V) := F(V, L), \quad \alpha(V) := F(V, L) \forall V \in \Gamma(B),$$

$$\rho := \frac{1}{2} \left( 1 - \frac{2M}{r} \right)^{-1} F(L, L), \quad \sigma := \frac{1}{2} e^{cd} F_{cd},$$

where $e_{cd} \in \Lambda^2(B)$ is the volume form of 2-sphere.

The extreme components $\alpha$ and $\alpha$ of the Maxwell field $\mathcal{F}$ satisfy the spin $\pm 1$ Teukolsky equations respectively

$$\nabla_L \nabla_{L}(r\alpha_a) + \frac{2}{r} \left( 1 - \frac{3M}{r} \right) \nabla_{L}(r\alpha_a) - F\Delta (r\alpha_a) + \frac{F}{r^2} r\alpha_a = 0,$$

$$\nabla_L \nabla_{L}(r\alpha_a) + \frac{2}{r} \left( 1 - \frac{3M}{r} \right) \nabla_{L}(r\alpha_a) - F\Delta (r\alpha_a) + \frac{F}{r^2} r\alpha_a = 0,$$

where $\nabla$ and $\Delta$ are the orthogonal projection of covariant derivative $\nabla$ and covariant Laplacian operator on $B$ (i.e the bundle tangent of 2-sphere $S^2(t, r)$ at $(t, r)$) respectively.
Suppose \((\alpha,\alpha,\rho,\sigma)\) satisfy the Maxwell equation, we define

\[
\phi_a := \frac{r^2}{F} \nabla_L(r\alpha_a), \quad \bar{\phi}_a := \frac{r^2}{F} \nabla_L(r\bar{\alpha}_a).
\]

Then \(\phi\) and \(\bar{\phi}\) satisfy the vectorial Fackerell-Ipser equations

\[
\nabla_L \nabla_L (\phi_a) - F \Delta(\phi_a) + \frac{F}{r^2} \phi_a = 0,
\]

\[
\nabla_L \nabla_L (\bar{\phi}_a) - F \Delta(\bar{\phi}_a) + \frac{F}{r^2} \bar{\phi}_a = 0.
\]

We define the vectorial wave operator

\[
\nabla_g = \frac{1}{F} \nabla_L \nabla_L - \frac{1}{r^2} \Delta_{S^2},
\]

where \(\Delta_{S^2}\) is the orthogonal projection of the covariant Laplacian operator on the unit 2-sphere \(S^2\). Then we get

\[
\nabla_g (\phi_a) + \frac{1}{r^2} \phi_a = 0,
\]

\[
\nabla_g (\bar{\phi}_a) + \frac{1}{r^2} \bar{\phi}_a = 0.
\]

We have the following relations

\[
\phi_a = r^3(\nabla_a \rho + e_{ab} \nabla^b \sigma), \quad \bar{\phi}_a = r^3(-\nabla_a \rho + e_{ab} \nabla^b \sigma).
\]

Moreover \(r^2\rho\) and \(r^2\sigma\) satisfy the scalar Fackerell-Ipser equation, in precisely a scalar wave equation

\[
\Box_g \phi = \frac{1}{F} L_L \phi - \frac{1}{r^2} \Delta_{S^2}\phi = 0.
\]

Commuting this equation with the projected covariant angular derivative \(r\nabla\) with a note that \([r\nabla, \nabla_L] = [r\nabla, \nabla_L] = 0\), we derive also the vectorial Fackerell-Ipser equations (3) and (4).

**Remark 1.** • In comparing with the linearized gravity equations (see [43]), the potential of free derivative term of the equation is of order \(-3\) in \(r\), whence in Equations (3) and (4) the ones are of order \(-2\) in \(r\).
• If we projective the equations (3) and (4) on the basic frame of 2-sphere \(S^2_{(t,r)}\) we then get the scalar equations in the following form

\[
\Box_g \phi + L_1(\phi) = 0,
\]

where \(L_1\) is an operator consists the derivatives of order one and free derivative. Therefore, the conformal rescaled equation has the same form respect to rescaled metric \(\hat{g}\) and we can apply the results in [37] to solve the Goursat problem.
In the rest of paper we will construct a conformal scattering theory for the equations (3) and (4) by using the results about the uniform bounded energy and pointwise decay of the solutions obtained in [44]. Since the scalar Fackerell-Ipser equation (5) is conformal invariant under the following formula
\[ \square_g + \frac{1}{6} \text{Scal}_g = \Omega^2 (\square_g + \frac{1}{6} \text{Scal}_g) \Omega^{-1}, \]
the vectorial form is also conformal invariant
\[ \square_g + \frac{1}{6} \text{Scal}_g = \Omega^2 (\square_g + \frac{1}{6} \text{Scal}_g) \Omega^{-1}. \]
Therefore, \( \hat{\phi}_a = r \phi_a \) and \( \hat{\phi}_a = r \phi_a \) are the solutions of the rescaled equations of (3) and (4) respectively
\[ \square_g \hat{\phi}_a + (2MR + 1) \hat{\phi}_a = 0, \]
\[ \square_g \hat{\phi}_a + (2MR + 1) \hat{\phi}_a = 0, \]
where
\[ \square_g = \frac{r^2}{F} \nabla_L \nabla_L - \Delta_{S^2}. \]
The equations (6) and (7) are equivalent to
\[ \nabla_L \nabla_L \hat{\phi}_a - FR^2 \Delta_{S^2} \hat{\phi}_a + (2MR + 1) R^2 F \hat{\phi}_a = 0, \]
(8)
\[ \nabla_L \nabla_L \hat{\phi}_a - FR^2 \Delta_{S^2} \hat{\phi}_a + (2MR + 1) R^2 F \hat{\phi}_a = 0. \]
(9)

3 Energy of the scalar fields

3.1 Energy conservation

We follow the convention by Penrose and Rindler [46] about the Hodge dual of a 1-form \( \alpha \) on a spacetime \((\mathcal{M}, g)\) (i.e. a 4–dimensional Lorentzian manifold that is oriented and time-oriented)
\[ (*\alpha)_{abcd} = e_{abcd} \alpha^d, \]
where \( e_{abcd} \) is the volume form on \((\mathcal{M}, g)\), denoted simply as \( \text{dVol}_g \). If \( S \) is the boundary of a bounded open set \( \Omega \) and has outgoing orientation, using Stokes theorem, we have (see [39])
\[ \int_{\Omega} d \ast \alpha = \int_S \ast \alpha = \int_S \alpha_a N^a L \text{dVol}_g, \]
(10)
where \( L \) is a vector transverse to \( S \) and \( N \) is the normal vector field to \( S \) such that \( L^a N_a = 1 \).

The formula (8) of Fackerell-Ipser equation leads to
\[ (\nabla_L + \nabla_L) \int_{S^2} \left( |\nabla_L \hat{\phi}|^2 + |\nabla_L \hat{\phi}|^2 + 2F |R \nabla_S \hat{\phi}|^2 + 2(2MR + 1) R^2 F |\hat{\phi}|^2 \right) R^2 F \text{dS}^2 \]
\[ + (\nabla_L - \nabla_L) \int_{S^2} \left( |\nabla_L \hat{\phi}|^2 - |\nabla_L \hat{\phi}|^2 \right) R^2 F \text{dS}^2 = 0. \]
This is equivalent to
\[
\nabla_L \int_{S^2} \left( (\nabla_L \phi)^2 + F|\nabla S^2 \phi|^2 + (2MR + 1)R^2F|\phi|^2 \right) R^2F dS^2 \\
+ \nabla_L \int_{S^2} \left( (\nabla_L \phi)^2 + F|\nabla S^2 \phi|^2 + (2MR + 1)R^2F|\phi|^2 \right) R^2F dS^2 = 0. \tag{11}
\]

**Remark 2.** The formula (11) is the orthogonal projection of covariant derivatives along the incoming and outgoing directions \( L \) and \( L_0 \) on the 2-sphere \( S_{(t,r)} \). It can be regarded as \( d \ast \alpha = 0 \) with \( \alpha \) is a 1-form determined by contracting the 3-form \( \ast \alpha \) (the quantities are under the covariant derivatives \( \nabla_L \) and \( \nabla_L \)) with \( d\text{Vol}_g \).

Consider the domain \( \Omega \) bounded by the following hypersurfaces
\[
\Sigma_0 = \{ t = 0 \}, \quad H_{2uT}^+ = \{ u = 2u_T, v \leq v_T \}, \quad I_{2vT}^+ = \{ v = 2v_T, u \leq u_T \},
\]
\[
H_{uT} = \{ u = u_T, v_T \leq v \leq 2v_T \}, \quad I_v = \{ v = v_T, u_T \leq u \leq 2u_T \}.
\]
On \( \Sigma_0 \), we take
\[
\hat{L} = \frac{r^2}{F} \partial_t, \quad \hat{N} = \partial_t.
\]
On \( I_{2vT}^+ \) and \( I_v \), we take \( \hat{L} = -\partial_R \) in \( u, R, \omega \) coordinates
\[
\hat{L} = \frac{r^2 F^{-1}}{2} l|_{I_{2vT}^+}.
\]
On \( H_{2uT}^+ \) and \( H_{uT} \), we take \( \hat{L} = \partial_R \) in \( v, R, \omega \) coordinates
\[
\hat{L} = \frac{r^2 F^{-1}}{2} n|_{H_{2uT}^+}.
\]
Hence we have \( \hat{N} = \partial_t \) on both \( H_{2uT}^+ \) and \( I_{2vT}^+ \). This corresponds to \( \hat{N} = \partial_u \) on \( H_{2uT}^+ \) and \( H_{uT} \)
and \( \hat{N} = \partial_v \) on \( I_{2vT}^+ \) and \( I_v \). Since that, we can calculate the the energy fluxes across the boundary hypersurfaces of \( \Omega \) as follows
\[
\mathcal{E}_{\Sigma_0}(\phi) = \frac{1}{2} \int_{\Sigma_0} \left( |\nabla_{\partial_t} \phi|^2 + |\nabla_{\partial_v} \phi|^2 + R^2F|\nabla S^2 \phi|^2 + (2MR + 1)R^2F|\phi|^2 \right) dr_s dS^2,
\]
\[
\mathcal{E}_{H_{2uT}^+}(\phi) = \int_{u_T}^{2u_T} \int_{S^2} \left( |\nabla_L \phi|^2 + R^2F|\nabla S^2 \phi|^2 + (2MR + 1)R^2F|\phi|^2 \right) dvdS^2,
\]
\[
\mathcal{E}_{H_{uT}}(\phi) = \int_{\Sigma_0}^{2u_T} \int_{S^2} \left( |\nabla_L \phi|^2 + R^2F|\nabla S^2 \phi|^2 + (2MR + 1)R^2F|\phi|^2 \right) dvdS^2,
\]
\[
\mathcal{E}_{I_{2vT}^+}(\phi) = \int_{v_T}^{2v_T} \int_{S^2} \left( |\nabla_L \phi|^2 + R^2F|\nabla S^2 \phi|^2 + (2MR + 1)R^2F|\phi|^2 \right) du dS^2,
\]
\[
\mathcal{E}_{I_v}(\phi) = \int_{v_T}^{2v_T} \int_{S^2} \left( |\nabla_L \phi|^2 + R^2F|\nabla S^2 \phi|^2 + (2MR + 1)R^2F|\phi|^2 \right) du dS^2.
\]
We observe that the hypersurfaces $\mathcal{H}_{u_T}$ and $\mathcal{I}_{v_T}$ will tend to the future horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$ as $u_T$ and $v_T$ (as well as $T$) tend to infinity respectively. Therefore, the energy on the future horizon and null infinity are

$$
\mathcal{E}_{\mathcal{H}^+}(\hat{\phi}) = \int_{\mathcal{H}^+} |\nabla_L \hat{\phi}|_{\mathcal{H}^+}^2 \, d\nu \, dS^2,
$$

$$
\mathcal{E}_{\mathcal{I}^+}(\hat{\phi}) = \int_{\mathcal{I}^+} |\nabla_L \hat{\phi}|_{\mathcal{I}^+}^2 \, d\nu \, dS^2.
$$

Integrating the conservation law (11) (i.e. integrating $d^* \alpha = 0$) on the domain $\Omega$, then by using Stokes’s formula (10) we establish that

$$
\mathcal{E}_{\mathcal{H}^+}(\hat{\psi}) + \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}) = \mathcal{E}_{\Sigma_0}(\hat{\phi}).
$$

As a direct consequence of the energy equality (12) we have

**Proposition 1.** Consider the smooth and compactly supported initial data on $\Sigma_0$, we can define the energy fluxes of the rescaled solution $\hat{\psi}$ across the null conformal boundary $\mathcal{H}^+ \cup \mathcal{I}^+$ by

$$
\mathcal{E}_{\mathcal{H}^+}(\hat{\psi}) + \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}) := \lim_{T \to \infty} \mathcal{E}_{\mathcal{H}^+}(\hat{\psi}) + \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}).
$$

Moreover, we have

$$
\mathcal{E}_{\mathcal{H}^+}(\hat{\psi}) + \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}) \leq \mathcal{E}_{\Sigma_0}(\hat{\phi}),
$$

the equality holds if and only if

$$
\lim_{u_T, v_T \to \infty} \left( \mathcal{E}_{\mathcal{H}^+}(\hat{\psi}) + \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}) \right) = 0.
$$

By Leray’s theorem for the hyperbolic differential equations and the energy identity (12), we have the classic result on the well-posedness of Cauchy problem for the rescaled equation (6). We define the finite energy space as follows

**Definition 1.** We denote by $\mathcal{H}$ the completion of $C^\infty_0(\Sigma_0) \times C^\infty_0(\Sigma_0)$ in the norm

$$
\|(\hat{\psi}_0, \hat{\psi}_1)\|_{\mathcal{H}} = \frac{1}{\sqrt{2}} \left( \int_{\Sigma_0} \left( |\nabla_{\partial_t} \hat{\phi}|^2 + |\nabla_{\partial_r} \hat{\phi}|^2 + R^2 F |\nabla \hat{\phi}|^2 + (2M + 1)R^2 F |\hat{\phi}|^2 \right) \, dr \, dS^2 \right)^{1/2}.
$$

The Cauchy problem can be stated as follows

**Proposition 2.** The Cauchy problem for (6) on $\bar{B}_1$ (hence (3) on $B_1$) is well-posed in $\mathcal{H}$. This means that for any $(\hat{\psi}_0, \hat{\psi}_1) \in \mathcal{H}$, there exists a unique solution $\hat{\psi} \in \mathcal{D}'(\bar{M})$ of (6) such that

$$
(r\hat{\psi}, r\partial_t \hat{\psi}) \in C(\mathbb{R}_t; \mathcal{H}) : r\hat{\psi}|_{t=0} = \hat{\psi}_0; \ r\partial_t \hat{\psi}|_{t=0} = \hat{\psi}_1.
$$

Moreover, $\hat{\psi} = r\hat{\psi}$ belongs to $H^1_{loc}(\bar{M})$ (see [38] for the detailed definition of $H^1_{loc}(\bar{M})$ i.e. the Sobolev spaces defined on open sets).
3.2 Energy identity up to $i^+$ and trace operator

In this section, we will show that
\[ \lim_{u_T, v_T \to \infty} \left( \mathcal{E}_{H_T}(\hat{\psi}) + \mathcal{E}_{I_T}(\hat{\psi}) \right) = 0 \]
and obtain the energy equality between $\Sigma_0$, $\mathcal{H}^+$ and $\mathcal{I}^+$ as
\[ \mathcal{E}_{\Sigma_0}(\hat{\psi}) = \mathcal{E}_{\mathcal{H}^+}(\hat{\psi}) + \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}). \]

We shall utilise the following results about the energy decay and pointwise decay of the linear scalar fields which are satisfied Equation (3) that is obtained by Pasqualotto (see Theorems 4.2 and Lemma 5.8 in [44]).

**Theorem 1.**

i) (Energy Decay, Lemma 5.8 in [44]). There exists a positive number $(r_{FH})_s$ and a positive constant $C$ such that the following holds. Let $\psi$ be a solution to the Fackerell-Ipser equations (3), with the initial data imposed on $\mathcal{H}_0$. We have the following decay energy on $\{ u = u_T, v \geq v_T \} \cup \{ u \geq u_T, v = v_T \}$ (where $u_T \geq u_0$ and $v_T = u_T + 2(r_{FH})_s$):
\[ \mathcal{E}_{u=u_T, v \geq v_T}(\phi) + \mathcal{E}_{u \geq u_T, v = v_T}(\phi) \leq Cu^{-2}. \] (13)

ii) (Pointwise Decay, Theorem 4.2 in [44]). In the region $r_* \leq (r_{FH})_s$, the solutions of (3) decays as $v^{-1}$. In the region $r_* \geq (r_{FH})_s$, it decays as $r^{-1}(|u| + 1)^{-1/2}$ or $r^{-1/2}(|u| + 1)^{-1}$ i.e
\[ \int_{S^2} |\phi|^2 dS^2 \lesssim \frac{1}{r(|u| + 1)^{1/2}} \text{ and } \frac{1}{r^{1/2}(|u| + 1)}. \]

Using these decay results we will prove the following theorem

**Theorem 2.** The energy of the rescaled fields $\hat{\psi}$ across the hypersurface $\mathcal{H}_T$ tends to zero as $T$ tends to infinity
\[ \lim_{u_T, v_T \to \infty} \left( \mathcal{E}_{H_T}(\hat{\psi}) + \mathcal{E}_{I_T}(\hat{\psi}) \right) = 0. \]

As a consequence, the equality of the energies holds true
\[ \mathcal{E}_{\Sigma_0}(\hat{\psi}) = \mathcal{E}_{\mathcal{H}^+}(\hat{\psi}) + \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}). \] (14)
Proof. In the region \( \{r_* \geq (r_{FH})_*\} \), we have

\[
\mathcal{E}_{I_{vT}}(\hat{\psi}) = \int_{vT}^{2vT} \int_{S^2} \left( |\nabla_L \hat{\phi}|^2 + R^2 F |\nabla_{S^2} \hat{\phi}|^2 + (2MR + 1) R^2 F |\hat{\phi}|^2 \right) d\nu dS^2
\]

\[
= \int_{vT}^{2vT} \int_{S^2} \left( r^2 |\nabla_L \phi|^2 + F |\nabla_{S^2} \phi|^2 + F^2 |\phi|^2 + (2MR + 1) F |\phi|^2 \right) d\nu dS^2
\]

\[
= \int_{vT}^{2vT} \int_{S^2} \left( |\nabla_L \phi|^2 + F |\nabla_{S^2} \phi|^2 + FR^2 |\phi|^2 \right) r^2 d\nu dS^2
\]

\[
+ \int_{vT}^{2vT} \int_{S^2} \left( (F^2 - F) |\phi|^2 + (2MR + 1) F |\phi|^2 \right) d\nu dS^2
\]

\[
\lesssim \mathcal{E}_{I_{vT}}(\phi) + \int_{vT}^{\infty} \int_{S^2} |\phi|^2 d\nu dS^2
\]

\[
\lesssim cu_T^{-2} + \int_{vT}^{2vT} \frac{1}{r(u_T + 1)^2} du
\]

\[
\lesssim cu_T^{-2} + \frac{v_T}{(r_{FH})_*(u_T + 1)^2} = cu_T^{-2} + \frac{u_T + 2(r_{FH})_*}{(r_{FH})_*(u_T + 1)^2} \to 0 \text{ as } u_T \to \infty.
\]

On the other hand, in the region \( \{r_* \leq (r_{FH})_*\} \) we have

\[
\mathcal{E}_{H_{uT}}(\hat{\psi}) = \int_{uT}^{2uT} \int_{S^2} \left( |\nabla_L \hat{\phi}|^2 + R^2 F |\nabla_{S^2} \hat{\phi}|^2 + (2MR + 1) R^2 F |\hat{\phi}|^2 \right) d\nu dS^2
\]

\[
= \int_{uT}^{2uT} \int_{S^2} \left( r^2 |\nabla_L \phi|^2 + F |\nabla_{S^2} \phi|^2 + F^2 |\phi|^2 + (2MR + 1) F |\phi|^2 \right) d\nu dS^2
\]

\[
= \int_{uT}^{2uT} \int_{S^2} \left( |\nabla_L \phi|^2 + F |\nabla_{S^2} \phi|^2 + FR^2 |\phi|^2 \right) r^2 d\nu dS^2
\]

\[
+ \int_{uT}^{2uT} \int_{S^2} \left( (F^2 - F) |\phi|^2 + (2MR + 1) F |\phi|^2 \right) d\nu dS^2
\]

\[
\lesssim \mathcal{E}_{H_{uT}}(\phi) + \int_{uT}^{\infty} \int_{S^2} |\phi|^2 d\nu dS^2
\]

\[
\lesssim cu_T^{-2} + \int_{uT}^{2uT} \frac{1}{v_T^2} du
\]

\[
\lesssim cu_T^{-2} + \frac{u_T}{v_T^2} = cu_T^{-2} + \frac{v_T - 2(r_{FH})_*}{v_T^2} \to 0 \text{ as } u_T, v_T \to \infty.
\]

\[\square\]

The energy equality leads us to define the trace operator on the conformal boundary

**Definition 2.** (Trace operator). Let \((\hat{\psi}_0, \hat{\psi}_1) \in C^\infty_0(\Sigma_0) \times C^\infty_0(\Sigma_0)\). Consider the solution of the rescaled equations (6), \(\hat{\psi} \in C^\infty(M)\) such that

\[\hat{\psi}|_{\Sigma_0} = \hat{\psi}_0, \partial_t \hat{\psi}_1|_{\Sigma_0} = \hat{\psi}_1.\]
We define the trace operator $T^+$ from $C^\infty_0(\Sigma_0) \times C^\infty_0(\Sigma_0)$ to $C^\infty(\mathcal{I}^+) \times C^\infty(\mathcal{I}^+)$ as follows

$$T^+(\hat{\psi}_0, \hat{\psi}_1) = (\hat{\psi}|_{\mathcal{I}^+}, \hat{\psi}|_{\mathcal{I}^+}).$$

Then we can extend the function space for scattering data by density as the following definition.

**Definition 3.** The function space for scattering data $\mathcal{H}^+$ is the completion of $C^\infty_0(\mathcal{I}^+) \times C^\infty_0(\mathcal{I}^-)$ in the norm

$$\|(\xi, \zeta)\|_{\mathcal{H}^+} = \left(\int_{\mathcal{I}^+} |\nabla_\mathcal{L} \xi|^2 \, dv \omega + \int_{\mathcal{I}^+} |\nabla_\mathcal{L} \zeta|^2 \, d\nu \omega\right)^{1/2}.$$ 

This means that

$$\mathcal{H}^+ \simeq \dot{H}^1(\mathbb{R}^+; L^2(S^2_\omega)) \times \dot{H}^1(\mathbb{R}^-; L^2(S^2_\omega)).$$

As a direct consequence of the equality energy (14), we have the following theorem

**Theorem 3.** The trace operator extends uniquely as a bounded linear map from $\mathcal{H}$ to $\mathcal{H}^+$. The extended operator is a partial isometry i.e for any $(\hat{\psi}_0, \hat{\psi}_1) \in \mathcal{H}$,

$$\|T^+(\hat{\psi}_0, \hat{\psi}_1)\|_{\mathcal{H}^+} = \|\psi_0, \psi_1\|_{\mathcal{H}}.$$

### 4 Conformal scattering theory

To obtain the conformal scattering operator, we need to show that the trace operator is subjective. This corresponds to solve the Goursat problem for the rescaled equations (6) with the initial data on the future (resp. past) horizon and future (resp. past) null infinity i.e the conformal boundary $\mathcal{I}^+ \cup \mathcal{I}^-$ (resp. $\mathcal{I}^- \cup \mathcal{I}^-$). The Goursat problem of the rescaled equation (6) is solved by the exact way in [38] (see Section 4.1 and Appendix) for the rescaled wave equation. The method is based on the results of Hörmander [23] and extended by Nicolas in [37]. Here, we only state the result.

**Theorem 4.** The Goursat problem of the rescaled equation (6) is well-posed i.e for the initial data $(\xi, \zeta) \in C^\infty_0(\mathcal{I}^+) \times C^\infty_0(\mathcal{I}^+)$, there exists a unique solution of (6) such that

$$(\hat{\psi}, \partial_t \hat{\psi}) \in C(\mathbb{R}_t; \mathcal{H}) \text{ and } T^+(\hat{\psi}|_{\Sigma_0}, \partial_t \hat{\psi}|_{\Sigma_0}) = (\xi, \zeta).$$

This means that the trace operator $T^+: \mathcal{H} \to \mathcal{H}^+$ is subjective and combine with Theorem 3 we obtain that the trace operator $T^+$ is an isometry.

We now define the conformal scattering operator as follows.

**Definition 4.** Similarly $T^+$, we introduce the past trace operator $T^-$ and the space $\mathcal{H}^-$ of past scattering data on the past horizon and the past null infinity. We define the scattering operator $S$ as the operator that to the past scattering data associates the future scattering data, i.e.

$$S := T^+ \circ (T^-)^{-1}.$$
Remark 3. • The extension of this work to construct a conformal scattering theory for the Fackerell-Ipser equations on the other symmetric spherical spacetimes such as Reissner-Nordström-de Sitter black hole seems can be done by the same way: first, extend the work [44] to obtain the decay results (where the results of Giorgi [20] can be useful) and then using these results to the construction, where it remains useful to use the time-like Killing vector field \( \partial_t \) to obtain the energy of the fields. However, the extension in the stationary and non symmetric spherical spacetimes such as Kerr spacetimes is more complicated. In Kerr spacetimes, the existence of the orbiting null geodesics and the fact that the vector field \( \partial_t \) is no longer global time-like in the exterior of the black hole lead an issue that the conserved energies on the Cauchy hypersurfaces of the spacetime is not defined as usual. However, the recent results on the energy and pointwise decay estimates for all components of Maxwell fields obtained by S. Ma [31] can be useful. This work will be treated in the future program. Besides, the peeling property of the rescaled equations (6) is also an interesting question where the method can be done by the same manner for the linear scalar fields on Schwarzschild spacetimes in [33] and extended to Kerr spacetimes in [39, 42].

• The conformal scattering for the spin \( \pm 1 \) or \( \pm 2 \) Teukolski equations can be seem also to construct firstly in Schwarzschild spacetime.

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