Hankel matrices acting on the Hardy space $H^1$ and on Dirichlet spaces

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Abstract
If $\mu$ is a finite positive Borel measure on the interval $[0, 1)$, we let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k \geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0, 1, 2, \ldots$, $\mu_n$ denotes the moment of order $n$ of $\mu$. This matrix induces formally the operator $\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n$ on the space of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in the unit disc $\mathbb{D}$. When $\mu$ is the Lebesgue measure on $[0, 1)$ the operator $\mathcal{H}_\mu$ is the classical Hilbert operator $\mathcal{H}$ which is bounded on $H^p$ if $1 < p < \infty$, but not on $H^1$. J. Cima has recently proved that $\mathcal{H}$ is an injective bounded operator from $H^1$ into the space $\mathcal{C}$ of Cauchy transforms of measures on the unit circle. The operator $\mathcal{H}_\mu$ is known to be well defined on $H^1$ if and only if $\mu$ is a Carleson measure and in such a case we have that $\mathcal{H}_\mu(H^1) \subset \mathcal{C}$. Furthermore, it is bounded from $H^1$ into itself if and only if $\mu$ is a 1-logarithmic 1-Carleson measure. In this paper we prove that when $\mu$ is a 1-logarithmic 1-Carleson measure then $\mathcal{H}_\mu$ actually maps $H^1$ into the space of Dirichlet type $D_1^1$. We discuss also the range of $\mathcal{H}_\mu$ on $H^1$ when $\mu$ is an $\alpha$-logarithmic 1-Carleson measure ($0 < \alpha < 1$). We study also the action of the operators $\mathcal{H}_\mu$ on Bergman spaces and on Dirichlet spaces.

Keywords Hankel matrix · Generalized Hilbert operator · Hardy spaces · Cauchy transforms · Weighted Bergman spaces · Dirichlet spaces · Duality

Mathematics Subject Classification 47B35 · 30H10

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1 Introduction and main results

Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ denote the open unit disc in the complex plane $\mathbb{C}$, $\partial \mathbb{D}$ will be the unit circle. The space of all analytic functions in $\mathbb{D}$ will be denoted by $\operatorname{Hol}(\mathbb{D})$. We also let $H^p$ ($0 < p \leq \infty$) be the classical Hardy spaces. We refer to [11] for the notation and results regarding Hardy spaces.

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space $A^p_\alpha$ consists of those $f \in \operatorname{Hol}(\mathbb{D})$ such that

$$
\|f\|_{A^p_\alpha} \overset{\text{def}}{=} \left( (\alpha + 1) \int_\mathbb{D} (1 - |z|^2)^\alpha |f(z)|^p \, dA(z) \right)^{1/p} < \infty.
$$

Here, $dA$ stands for the area measure on $\mathbb{D}$, normalized so that the total area of $\mathbb{D}$ is 1. Thus $dA(z) = \frac{1}{\pi} \, dx \, dy = \frac{1}{\pi} \, r \, dr \, d\theta$. The unweighted Bergman space $A^p_0$ is simply denoted by $A^p$. We refer to [12,18,29] for the notation and results about Bergman spaces.

The space of Dirichlet type $D^p_\alpha$ ($0 < p < \infty$ and $\alpha > -1$) consists of those $f \in \operatorname{Hol}(\mathbb{D})$ such that $f' \in A^p_\alpha$. In other words, a function $f \in \operatorname{Hol}(\mathbb{D})$ belongs to $D^p_\alpha$ if and only if

$$
\|f\|_{D^p_\alpha} \overset{\text{def}}{=} |f(0)| + \left( (\alpha + 1) \int_\mathbb{D} (1 - |z|^2)^\alpha |f'(z)|^p \, dA(z) \right)^{1/p} < \infty.
$$

The Hilbert matrix is the infinite matrix $H = (\frac{1}{k+n+1})_{k,n \geq 0}$. It induces formally an operator, called the Hilbert operator, on spaces of analytic functions as follows:

If $f \in \operatorname{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$, then we set

$$
\mathcal{H}f(z) = \sum_{n=0}^\infty \left( \sum_{k=0}^\infty \frac{a_k}{n+k+1} \right) z^n, \quad z \in \mathbb{D}, \tag{1}
$$

whenever the right-hand side of (1) makes sense for all $z \in \mathbb{D}$ and the resulting function is analytic in $\mathbb{D}$. We define also

$$
\mathcal{I}f(z) = \int_0^1 \frac{f(t)}{1 - tz} \, dt, \quad z \in \mathbb{D}, \tag{2}
$$

if the integrals in the right-hand side of (2) converge for all $z \in \mathbb{D}$ and the resulting function $\mathcal{I}f$ is analytic in $\mathbb{D}$. It is clear that the correspondences $f \mapsto \mathcal{H}f$ and $f \mapsto \mathcal{I}f$ are linear.

If $f \in H^1$, $f(z) = \sum_{n=0}^\infty a_n z^n$, then by the Fejér-Riesz inequality [11, Theorem 3.13, p.46] and Hardy’s inequality [11, p.48], we have

$$
\int_0^1 |f(t)| \, dt \leq \pi \|f\|_{H^1} \quad \text{and} \quad \sum_{n=0}^\infty \frac{a_n}{n+1} \leq \pi \|f\|_{H^1}.
$$
This immediately yields that if \( f \in H^1 \) then \( \mathcal{H} f \) and \( I f \) are well defined analytic functions in \( \mathbb{D} \) and that, furthermore, \( \mathcal{H} f = I f \).

Diamantopoulos and Siskakis \cite{DS} proved that \( \mathcal{H} \) is a bounded operator from \( H^p \) into itself if \( 1 < p < \infty \), but this is not true for \( p = 1 \). In fact, they proved that \( \mathcal{H}(H^1) \not\subset H^1 \). Cima \cite{Cima} has recently proved the following result.

**Theorem A** (i) The operator \( \mathcal{H} \) maps \( H^1 \) into the space \( \mathcal{C} \) of Cauchy transforms of measures on the unit circle \( \partial \mathbb{D} \).

(ii) \( \mathcal{H} : H^1 \to \mathcal{C} \) is injective.

We recall that if \( \sigma \) is a finite complex Borel measure on \( \partial \mathbb{D} \), the Cauchy transform \( C \sigma \) is defined by

\[
C \sigma(z) = \int_{\partial \mathbb{D}} \frac{d\sigma(\xi)}{1 - \overline{\xi} z}, \quad z \in \mathbb{D}.
\]

We let \( \mathcal{M} \) be the space of all finite complex Borel measure on \( \partial \mathbb{D} \). It is a Banach space with the total variation norm. The space of Cauchy transforms is \( \mathcal{C} = \{C \sigma : \sigma \in \mathcal{M}\} \). It is a Banach space with the norm \( \|C \sigma\| = \inf \{\|\tau\| : C \tau = C \sigma\} \). We mention \cite{We} as an excellent reference for the main results about Cauchy transforms. We let \( A \) denote the disc algebra, that is, the space of analytic functions in \( \mathbb{D} \) with a continuous extension to the closed unit disc, endowed with the \( \|\cdot\|_{H^\infty} \) - norm. It turns out \cite[Chapter 4]{We} that \( A \) can be identified with the pre-dual of \( \mathcal{C} \) via the pairing

\[
\langle g, C \sigma \rangle \overset{\text{def}}{=} \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta})C \sigma(re^{i\theta}) \, d\theta.
\]

This is the basic ingredient used by Cima to prove the inclusion \( \mathcal{H}(H^1) \subset \mathcal{C} \).

Now we turn to consider a class of operators which are natural generalizations of the operators \( \mathcal{H} \) and \( I \). If \( \mu \) is a finite positive Borel measure on \( [0, 1] \) and \( n = 0, 1, 2, \ldots \), we let \( \mu_n \) denote the moment of order \( n \) of \( \mu \), that is, \( \mu_n = \int_{[0,1]} t^n \, d\mu(t) \), and we define \( \mathcal{H}_\mu \) to be the Hankel matrix \( (\mu_{n,k})_{n,k \geq 0} \) with entries \( \mu_{n,k} = \mu_{n+k} \). The measure \( \mu \) induces formally the operators \( I_\mu \) and \( \mathcal{H}_\mu \) on spaces of analytic functions as follows:

\[
I_\mu f(z) = \int_{[0,1]} f(t) \frac{1}{1 - tz} \, d\mu(t), \quad \mathcal{H}_\mu f(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} a_k \mu_{n+k} \right) z^n, \quad z \in \mathbb{D},
\]

for \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in Hol(\mathbb{D}) \) being such that the terms on the right-hand sides make sense for all \( z \in \mathbb{D} \), and the resulting functions are analytic in \( \mathbb{D} \). If \( \mu \) is the Lebesgue measure on \( [0, 1] \) the matrix \( \mathcal{H}_\mu \) reduces to the classical Hilbert matrix and the operators \( \mathcal{H}_\mu \) and \( I_\mu \) are simply the operators \( \mathcal{H} \) and \( I \).

If \( I \subset \partial \mathbb{D} \) is an interval, \( |I| \) will denote the length of \( I \). The Carleson square \( S(I) \) is defined as \( S(I) = \{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\} \).
If \( s > 0 \) and \( \mu \) is a positive Borel measure on \( \mathbb{D} \), we shall say that \( \mu \) is an \( s \)-Carleson measure if there exists a positive constant \( C \) such that

\[
\mu(S(I)) \leq C|I|^{s}, \quad \text{for any interval } I \subset \partial \mathbb{D}.
\]

A 1-Carleson measure will be simply called a Carleson measure. We recall that Carleson \([4]\) proved that \( H^p \subset L^p(d\mu) \) \((0 < p < \infty)\) if and only if \( \mu \) is a Carleson measure (see also \([11, \text{Chapter 9}]\)).

For \( 0 \leq \alpha < \infty \) and \( 0 < s < \infty \) we say that a positive Borel measure \( \mu \) on \( \mathbb{D} \) is an \( \alpha \)-logarithmic \( s \)-Carleson measure if there exists a positive constant \( C \) such that

\[
\mu(S(I)) \left( \log \frac{2\pi}{|I|} \right)^{\alpha} \leq C|I|^{s}, \quad \text{for any interval } I \subset \partial \mathbb{D}.
\]

A positive Borel measure \( \mu \) on \([0, 1)\) can be seen as a Borel measure on \( \mathbb{D} \) by identifying it with the measure \( \tilde{\mu} \) defined by

\[
\tilde{\mu}(A) = \mu(A \cap [0, 1)), \quad \text{for any Borel subset } A \text{ of } \mathbb{D}.
\]

In this way a positive Borel measure \( \mu \) on \([0, 1)\) is an \( s \)-Carleson measure if and only if there exists a positive constant \( C \) such that

\[
\mu([t, 1)) \leq C(1 - t)^{s}, \quad 0 \leq t < 1.
\]

We have a similar statement for \( \alpha \)-logarithmic \( s \)-Carleson measures.

The action of the operators \( \mathcal{I}_\mu \) and \( \mathcal{H}_\mu \) on distinct spaces of analytic functions have been studied in a number of articles (see, e.g., \([2,5,14-16,22,25,27]\)).

Combining results of \([14]\) and of \([16]\) we can state the following result.

**Theorem B** Let \( \mu \) be a finite positive Borel measure on \([0, 1)\).

(i) The operator \( \mathcal{I}_\mu \) is well defined on \( H^1 \) if and only if \( \mu \) is a Carleson measure.

(ii) If \( \mu \) is a Carleson measure, then the operator \( \mathcal{H}_\mu \) is also well defined on \( H^1 \) and \( \mathcal{I}_\mu f = \mathcal{H}_\mu f \) for all \( f \in H^1 \).

(iii) The operator \( \mathcal{H}_\mu \) is a bounded operator from \( H^1 \) into itself if and only if \( \mu \) is a 1-logarithmic 1-Carleson measure.

Galanopoulos and Peláez \([14, \text{Theorem 2.2}]\) proved the following.

**Theorem C** Let \( \mu \) be a positive Borel measure on \([0, 1)\). If \( \mu \) is a Carleson measure then \( \mathcal{H}_\mu(H^1) \subset \mathcal{C} \).

This result is stronger than Theorem A(i). In view of these results, the following question arises naturally.

**Question 1** Suppose that \( \mu \) is a 1-logarithmic 1-Carleson measure on \([0, 1)\). What can we say about the image \( \mathcal{H}_\mu(H^1) \) of \( H^1 \) under the action of the operator \( \mathcal{H}_\mu \)?
To answer Question 1, let us start noticing that it is known that, for $0 < p \leq 2$, the space of Dirichlet type $D^p_{p-1}$ is continuously included in $H^p$ (see [26, Lemma 1.4]). In particular, the space $D_{0}^1$ is continuously included in $H^1$. In fact, the estimates obtained by Vinogradov in the proof of his lemma easily yield the inequality

$$\|f\|_{H^1} \leq 2\|f\|_{D_{0}^1}, \quad f \in D_{0}^1.$$  

We shall prove that if $\mu$ is a 1-logarithmic 1-Carleson measure on $[0, 1)$ then $\mathcal{H}_\mu(H^1)$ is contained in the space $D_{0}^1$. Actually, we have the following stronger result.

**Theorem 1** Let $\mu$ be a positive Borel measure on $[0, 1)$. Then the following conditions are equivalent.

(i) $\mu$ is a 1-logarithmic 1-Carleson measure.

(ii) $\mathcal{H}_\mu$ is a bounded operator from $H^1$ into itself.

(iii) $\mathcal{H}_\mu$ is a bounded operator from $H^1$ into $D_{0}^1$.

(iv) $\mathcal{H}_\mu$ is a bounded operator from $D_{0}^1$ into $D_{0}^1$.

There is a gap between Theorem C and Theorem 1 and so it is natural to discuss the range of $H^1$ under the action of $\mathcal{H}_\mu$ when $\mu$ is an $\alpha$-logarithmic 1-Carleson measure with $0 < \alpha < 1$. We shall prove the following result.

**Theorem 2** Let $\mu$ be a positive Borel measure on $[0, 1)$. Suppose that $0 < \alpha < 1$ and that $\mu$ is an $\alpha$-logarithmic 1-Carleson measure. Then $\mathcal{H}_\mu$ maps $H^1$ into the space $D^1(\log^{\alpha-1})$ defined as follows:

$$D^1(\log^{\alpha-1}) = \left\{ f \in \mathcal{Hol}(\mathbb{D}) : \int_\mathbb{D} |f'(z)| \left( \log \frac{2}{1 - |z|} \right)^{\alpha-1} dA(z) < \infty \right\}.$$  

These results will be proved in Sect. 2. Since the space of Dirichlet type $D_{0}^1$ has showed up in a natural way in our work, it seems natural to study the action of the operators $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ on the Bergman spaces $A^p_\alpha$ and the Dirichlet spaces $D^p_{\alpha}$ for general values of the parameters $p$ and $\alpha$. This will be done in Sect. 3.

Throughout this paper the letter $C$ denotes a positive constant that may change from one step to the next. Moreover, for two real-valued functions $E_1, E_2$ we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant $C$ independent of the arguments such that $E_1 \leq CE_2$, respectively $E_1 \geq CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that $E_1$ and $E_2$ are equivalent and we write $E_1 \asymp E_2$.

2 Proofs of the theorems 1 and 2

**Proof of Theorem 1** We already know that (i) and (ii) are equivalent by Theorem B.
To prove that (i) implies (iii) we shall use some results about the Bloch space. We recall that a function \( f \in \mathcal{H}ol(D) \) is said to be a Bloch function if
\[
\| f \|_B \overset{\text{def}}{=} | f(0) | + \sup_{z \in \mathbb{D}} (1 - |z|^2) | f'(z) | < \infty.
\]
The space of all Bloch functions will be denoted by \( \mathcal{B} \). It is a non-separable Banach space with the norm \( \| \cdot \|_B \) just defined. A classical source for the theory of Bloch functions is [1]. The closure of the polynomials in the Bloch norm is the little Bloch space \( \mathcal{B}_0 \) which consists of those \( f \in \mathcal{H}ol(D) \) with the property that
\[
\lim_{|z| \to 1} (1 - |z|^2) | f'(z) | = 0.
\]
It is well known that (see [1, p. 13])
\[
| f(z) | \lesssim \| f \|_B \log \frac{2}{1 - |z|}.
\]
(4)

The basic ingredient to prove that (i) implies (iii) is the fact that the dual \( (\mathcal{B}_0)^* \) of the little Bloch space can be identified with the Bergman space \( A^1 \) via the integral pairing
\[
\langle h, f \rangle = \int_{\mathbb{D}} h(z) \overline{f(z)} dA(z), \quad h \in \mathcal{B}_0, \quad f \in A^1.
\]
(5)
(See [29, Theorem 5.15]).

Let us proceed to prove the implication (i) ⇒ (iii). Assume that \( \mu \) is a 1-logarithmic 1-Carleson measure and take \( f \in H^1 \). We have to show that \( I_{\mu} f \in \mathcal{D}_0^1 \) or, equivalently, that \( (I_{\mu} f)' \in A^1 \). Since \( \mathcal{B}_0 \) is the closure of the polynomials in the Bloch norm, it suffices to show that
\[
\left| \int_{\mathbb{D}} h(z) \overline{(I_{\mu} f)'}(z) dA(z) \right| \lesssim \| h \|_B \| f \|_{H^1}, \quad \text{for any polynomial } h.
\]
(6)

So, let \( h \) be a polynomial. We have
\[
\int_{\mathbb{D}} h(z) \overline{(I_{\mu} f)'}(z) dA(z) = \int_{\mathbb{D}} h(z) \left( \int_{[0,1]} \frac{t f(t)}{(1 - t z)^2} d\mu(t) \right) dA(z)
\]
\[
= \int_{\mathbb{D}} h(z) \int_{[0,1]} \frac{t f(t)}{(1 - t z)^2} d\mu(t) dA(z)
\]
\[
= \int_{[0,1]} \frac{t f(t)}{(1 - t z)^2} \int_{\mathbb{D}} h(z) dA(z) d\mu(t).
\]

\[\square\]
Because of the reproducing property of the Bergman kernel \[29, \text{Proposition 4.23}\],
\[
\int_D \frac{h(z)}{(1-t\bar{z})^2} \, dA(z) = h(t).
\]
Then it follows that
\[
\int_D h(z) \left(\overline{I_\mu f}\right)'(z) \, dA(z) = \int_{[0,1]} t \overline{f(t)} \, h(t) \, d\mu(t).
\]
(7)

Since \(\mu\) is a 1-logarithmic 1-Carleson measure, the measure \(\nu\) defined by
\[
d\nu(t) = \log \frac{2}{1-t} \, d\mu(t)
\]
is a Carleson measure \[15, \text{Proposition 2.5}\]. This implies that
\[
\int_{[0,1]} |f(t)| \log \frac{2}{1-t} \, d\mu(t) \lesssim \|f\|_{H^1}.
\]
This and (4) yield
\[
\int_{[0,1]} \left| t \overline{f(t)} \, h(t) \right| \, d\mu(t) \lesssim \|h\|_B \|f\|_{H^1}.
\]
Using this and (7), (6) follows.

Since \(D^1_0 \subset H^1\), the implication (iii) \(\Rightarrow\) (iv) is trivial. To prove that (iv) implies (i) we shall use the following result of Pavlović \[23, \text{Theorem 3.2}\].

**Theorem D** Let \(f \in \mathcal{Hol}(\mathbb{D})\), \(f(z) = \sum_{n=0}^{\infty} a_n z^n\), and suppose that the sequence \(\{a_n\}\) is a decreasing sequence of non-negative real numbers. Then \(f \in D^1_0\) if and only if \(\sum_{n=0}^{\infty} \frac{a_n}{n+1} < \infty\), and we have
\[
\|f\|_{D^1_0} \asymp \sum_{n=0}^{\infty} \frac{a_n}{n+1}.
\]

Now we turn to prove the implication (iv) \(\Rightarrow\) (i). Assume that \(\mathcal{H}_\mu\) is a bounded operator from \(D^1_0\) into \(D^1_0\). We argue as in the proof of Theorem 1.1 of \[16\]. For \(\frac{1}{2} < b < 1\) set
\[
f_b(z) = \frac{1-b^2}{(1-bz)^2}, \quad z \in \mathbb{D}.
\]
We have \(f'_b(z) = \frac{2b(1-b^2)}{(1-bz)^3}\) (\(z \in \mathbb{D}\)). Then, using Lemma 3.10 of \[29\] with \(t = 0\) and \(c = 1\), we see that
\[
\|f_b\|_{D^1_0} \asymp \int_D \frac{1-b^2}{|1-bz|^3} \, dA(z) \asymp 1.
\]
Since $\mathcal{H}_\mu$ is bounded on $\mathcal{D}_0^1$, this implies that
\[ 1 \gtrsim \| \mathcal{H}_\mu(f_b) \|_{\mathcal{D}_0^1}. \] (8)

We also have,
\[ f_b(z) = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad \text{with} \quad a_{k,b} = (1 - b^2)(k + 1)b^k. \]

Since the $a_{k,b}$’s are positive, it is clear that the sequence $\{\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b}\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(f_b)$ is a decreasing sequence of non-negative real numbers. Using this, Theorem D, (8), and the definition of the $a_{k,b}$’s, we obtain

\[ \begin{align*}
1 & \gtrsim \| \mathcal{H}_\mu(f_b) \|_{\mathcal{D}_0^1} \gtrsim \sum_{n=1}^{\infty} \frac{1}{n} \left( \sum_{k=0}^{\infty} \mu_{n+k} a_{k,b} \right) \\
& = \sum_{n=1}^{\infty} \frac{1}{n} \left( \sum_{k=0}^{\infty} a_{k,b} \int_{[0,1]} t^{n+k} d\mu(t) \right) \\
& \gtrsim (1 - b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left( \sum_{k=1}^{\infty} k b^k \int_{[b,1]} t^{n+k} d\mu(t) \right) \\
& \gtrsim (1 - b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left( \sum_{k=1}^{\infty} k b^{n+2k} \mu([b,1]) \right) \\
& = (1 - b^2) \mu([b,1]) \sum_{n=1}^{\infty} \frac{b^n}{n} \left( \sum_{k=1}^{\infty} k b^{2k} \right) \\
& = (1 - b^2) \mu([b,1]) \left( \log \frac{1}{1-b} \right) \frac{b^2}{(1-b^2)^2}. 
\end{align*} \]

Then it follows that
\[ \mu([b,1]) = O \left( \frac{1 - b}{\log \frac{1}{1-b}} \right), \quad \text{as} \quad b \to 1. \]

Hence, $\mu$ is a 1-logarithmic 1-Carleson measure. \qed

Before embarking on the proof of Theorem 2 we have to introduce some notation and results. Following [24], for $\alpha \in \mathbb{R}$ the weighted Bergman space $A^1(\log^\alpha)$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that
\[ \| f \|_{A^1(\log^\alpha)} \overset{\text{def}}{=} \int_{\mathbb{D}} |f(z)| \left( \log \frac{2}{1-|z|} \right)^{\alpha} dA(z) < \infty. \]
This is a Banach space with the norm $\| \cdot \|_{A^1(\log^{\alpha})}$ just defined and the polynomials are dense in $A^1(\log^{\alpha})$. Likewise, we define

$$D^1(\log^{\alpha}) = \{ f \in Hol(\mathbb{D}) : f' \in A^1(\log^{\alpha}) \}.$$  

We define also the Bloch-type space $B(\log^{\alpha})$ as the space of those $f \in Hol(D)$ such that

$$\| f \|_{B(\log^{\alpha})} \overset{\text{def}}{=} |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \left( \log \frac{2}{1 - |z|} \right)^{-\alpha} |f'(z)| < \infty,$$

and

$$B_0(\log^{\alpha}) = \left\{ f \in Hol(\mathbb{D}) : |f'(z)| = o \left( \frac{\left( \log \frac{2}{1 - |z|} \right)^{\alpha}}{1 - |z|} \right), \text{ as } |z| \to 1 \right\}.$$ 

The space $B(\log^{\alpha})$ is a Banach space and $B_0(\log^{\alpha})$ is the closure of the polynomials in $B(\log^{\alpha})$.

We remark that the spaces $D^1(\log^{\alpha})$, $B(\log^{\alpha})$, and $B_0(\log^{\alpha})$ were called $\mathfrak{B}_1^{\log^{\alpha}}$, $\mathfrak{B}_{\log^{\alpha}}$, and $b_{\log^{\alpha}}$ in [24]. Pavlović identified in [24, Theorem 2.4] the dual of the space $B_0(\log^{\alpha})$.

**Theorem E** Let $\alpha \in \mathbb{R}$. Then the dual of $B_0(\log^{\alpha})$ is $A^1(\log^{\alpha})$ via the pairing

$$\langle h, g \rangle = \int_D f(z) \overline{g(z)} \, dA(z), \quad h \in B_0(\log^{\alpha}), \quad g \in A^1(\log^{\alpha}).$$

Actually, Pavlović formulated the duality theorem in another way but it is a simple exercise to show that his formulation is equivalent to this one which is better suited to our work.

**Proof of Theorem 2** Let $\mu$ be a positive Borel measure on $[0, 1)$ and $0 < \alpha < 1$. Suppose that $\mu$ is an $\alpha$-logarithmic 1-Carleson measure. Take $f \in H^1$. We have to show that $\mathcal{I}_\mu f \in D^1(\log^{\alpha-1})$ or, equivalently, that $(\mathcal{I}_\mu f)' \in A^1(\log^{\alpha-1})$. Bearing in mind Theorem E and the fact that $B_0(\log^{\alpha-1})$ is the closure of the polynomials in $B(\log^{\alpha-1})$, it suffices to show that

$$\left| \int_D h(z) \left( \mathcal{I}_\mu f \right)'(z) \, dA(z) \right| \lesssim \| h \|_{B(\log^{\alpha-1})} \| f \|_{H^1}, \quad \text{for any polynomial } h. \quad (9)$$

So, let $h$ be a polynomial. Arguing as in the proof of the implication $(i) \Rightarrow (iii)$ in Theorem 1 we obtain

$$\int_D h(z) \left( \mathcal{I}_\mu f \right)'(z) \, dA(z) = \int_{[0,1]} i \overline{f(t)} h(t) \, d\mu(t). \quad (10)$$
Now, it is clear that
\[ |h(z)| \lesssim \|h\|_{B(\log^{\alpha-1})} \left( \log \frac{2}{1 - |z|} \right)^\alpha, \]
and then it follows that
\[
\int_{[0,1]} \left| t \overline{f(t)} h(t) \right| \, d\mu(t) \lesssim \|h\|_{B(\log^{\alpha-1})} \int_{[0,1]} |f(t)| \left( \log \frac{2}{1 - t} \right)^\alpha \, d\mu(t).
\]
Using the fact that the measure \( \left( \log \frac{2}{1 - t} \right)^\alpha \, d\mu(t) \) is a Carleson measure [15, Proposition 2.5], this implies that
\[
\int_{[0,1]} \left| t \overline{f(t)} h(t) \right| \, d\mu(t) \lesssim \|h\|_{B(\log^{\alpha-1})} \|f\|_{H^1}.
\]
This and (10) give (9). \( \square \)

3 The operators \( \mathcal{H}_\mu \) acting on Bergman spaces and on Dirichlet spaces

Jevtić and Karapetrović [20] have recently proved the following result.

**Theorem F** The Hilbert operator \( \mathcal{H} \) is a bounded operator from \( \mathcal{D}_\alpha^p \) into itself if and only if \( \max(-1, p-2) < \alpha < 2p-2 \).

Now, it is well known that \( A_\alpha^p = \mathcal{D}_\alpha^{p} \) (see [29, Theorem 4.28]). Hence, regarding Bergman spaces Theorem F says the following.

**Corollary G** The Hilbert operator \( \mathcal{H} \) is a bounded operator from \( A_\alpha^p \) into itself if and only if \(-1 < \alpha < p - 2 \).

Let us recall that Diamantopoulos [8] had proved before that the Hilbert operator is bounded on \( A_\alpha^p \) for \( p > 2 \), but not on \( A^2 \). The situation on \( A^2 \) is even worse. Dostanić, Jevtić, and Vukotić [10] proved that the Hilbert operator is not well defined on \( A^2 \). Indeed, they considered the function \( f \) defined by
\[
f(z) = \sum_{n=1}^{\infty} \frac{1}{\log(n+1)} z^n, \quad z \in \mathbb{D}, \quad (11)
\]
which belongs to \( A^2 \). However, the series defining \( \mathcal{H} f \) is \( \sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)} = \infty \) and the integral defining \( \mathcal{I} f \) is \( \int_{0}^{1} f(t) \, dt = \infty \). Hence neither \( \mathcal{H} \) nor \( \mathcal{I} \) are defined on \( A^2 \).

This result can be extended. We can assert that \( \mathcal{H} \) is not well defined on \( A_{p-2}^p \) for any \( p > 1 \). Indeed, let \( f \) be the function defined in (11). Notice that the sequence
{1/(n+1) log(n+1)} is decreasing and that \( \sum_{n=1}^{\infty} \frac{1}{n(\log(n+1))^p} < \infty \). Then (see Proposition 1 below) it follows that \( f \in A_{p-2}^p \), and we have already seen that \( \mathcal{H} f \) and \( \mathcal{I} f \) are not defined. Since \( \alpha \geq p-2 \Rightarrow A_p^p \subset A_\alpha^p \), it follows that the Hilbert operator \( \mathcal{H} \) is not defined on \( A_\alpha^p \) if \( \alpha \geq p-2 \).

In this section we shall obtain extensions of the mentioned results of Jevtić and Karapetrović considering the generalized Hilbert operators \( \mathcal{H}_\mu \).

**Theorem 3** Suppose that \( \max(-1, p-2) < \alpha < 2p-2 \) and let \( \mu \) be a finite positive Borel measure on \( [0, 1) \). If \( \mu \) is a Carleson measure then the operators \( \mathcal{H}_\mu \) and \( \mathcal{I}_\mu \) are well defined on \( D_\alpha^p \). Furthermore, \( \mathcal{I}_\mu f = \mathcal{H}_\mu f \), for all \( f \in D_\alpha^p \).

When dealing with Bergman spaces Theorem 3 reduces to the following.

**Corollary 1** Suppose that \( p > 1 \) and \( -1 < \alpha < p-2 \), and let \( \mu \) be a finite positive Borel measure on \( [0, 1) \). If \( \mu \) is a Carleson measure then the operators \( \mathcal{H}_\mu \) and \( \mathcal{I}_\mu \) are well defined on \( A_\alpha^p \). Furthermore, \( \mathcal{I}_\mu f = \mathcal{H}_\mu f \), for all \( f \in A_\alpha^p \).

**Proof of Theorem 3** Suppose that \( \mu \) is a Carleson measure and take \( f \in D_\alpha^p \). Set \( \beta = \frac{2+\alpha}{p} - 1 \). Observe that \( 0 < \beta < 1 \). Using [29, Theorem 4.14], we see that \( |f'(z)| \lesssim \frac{1}{(1-|z|)^{2+\alpha}/p} \) and, hence, \( |f(z)| \lesssim \frac{1}{(1-|z|)^\beta} \). Then it follows that

\[
\int_{[0,1]} |f(t)| \, d\mu(t) \lesssim \int_{[0,1]} \frac{d\mu(t)}{(1-t)^\beta}.
\]

Integrating by parts, using that \( \mu \) is a Carleson measure, and that \( 0 < \beta < 1 \), we obtain

\[
\int_{[0,1]} \frac{d\mu(t)}{(1-t)^\beta} = \mu([0, 1)) - \lim_{t \to 1} \frac{\mu([t, 1))}{(1-t)^\beta} + \beta \int_0^1 \frac{\mu([t, 1))}{(1-t)^{\beta+1}} \, dt
\]

\[
= \mu([0, 1)) + \beta \int_0^1 \frac{\mu([t, 1))}{(1-t)^{\beta+1}} \, dt
\]

\[
\lesssim \mu([0, 1)) + \int_0^1 \frac{1}{(1-t)^\beta} \, dt
\]

\[
< \infty.
\]

Consequently, we obtain that

\[
\int_{[0,1]} |f(t)| \, d\mu(t) < \infty. \quad (12)
\]

Clearly, this implies that the integral

\[
\int_{[0,1]} \frac{f(t) \, d\mu(t)}{1-tz}
\]

converges absolutely and uniformly on compact subsets of \( \mathbb{D} \).

\[
(13)
\]
This gives that $\mathcal{I}_\mu f$ is a well defined analytic function in $\mathbb{D}$ and that

$$\mathcal{I}_\mu f(z) = \sum_{n=0}^{\infty} \left( \int_{[0,1)} t^n f(t) \, d\mu(t) \right) z^n, \quad z \in \mathbb{D}. \quad (14)$$

Using [19, Theorem 2.1] (see also [20, Theorem 2.1]) we see that for these values of $p$ and $\alpha$ we have that if $f \in A^p_\alpha$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\sum_{k=0}^{\infty} |a_k| < \infty$. Now, since $\mu$ is a Carleson measure we have that $|\mu_n| \lesssim \frac{1}{n+1}$ ([5, Proposition 1]). Then it follows that

$$\sum_{k=0}^{\infty} |\mu_{n+k} a_k| \lesssim \sum_{k=0}^{\infty} \frac{|a_k|}{k+n+1} \lesssim \sum_{k=0}^{\infty} \frac{|a_k|}{k+1}, \quad \text{for all } n.$$}

Clearly, this implies that $H_\mu f$ is a well defined analytic function in $\mathbb{D}$ and that $\int_{[0,1)} t^n f(t) \, d\mu(t) = \sum_{k=0}^{\infty} \mu_{n+k} a_k$ for all $n$. This and (13) give that $\mathcal{I}_\mu f = H_\mu f$.

Our next result is an extension of Corollary G.

**Theorem 4** Suppose that $-1 < \alpha < p - 2$ and let $\mu$ be a finite positive Borel measure on $[0, 1]$.

The operator $H_\mu$ is well defined on $A^p_\alpha$ and it is a bounded operator from $A^p_\alpha$ to itself if and only if $\mu$ is a Carleson measure.

A number of results will be needed to prove this theorem. We start with a characterization of the functions $f \in \mathcal{H}ol(\mathbb{D})$ whose sequence of Taylor coefficients is decreasing which belong to $A^p_\alpha$.

**Proposition 1** Let $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$). Suppose that $1 < p < \infty$, $\alpha > -1$, and that the sequence $\{|a_n|\}_{n=0}^{\infty}$ is a decreasing sequence of non-negative real numbers. Then

$$f \in A^p_\alpha \iff \sum_{n=1}^{\infty} n^{p-3-\alpha} a_n^p < \infty.$$ 

Furthermore, $\|f\|_{A^p_\alpha}^p \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-3-\alpha} a_n^p < \infty$.

This result can be proved with arguments similar to those used in the proofs of [15, Theorem 3.10] and [23, Theorem 3.1] where the analogous results for the Besov spaces $B^p = D^p_{p-2}$ ($p > 1$) and for the spaces $D^p_{p-1}$ ($p > 1$) were proved. The case $\alpha = 0$ is proved in [3, Proposition 2.4]. Consequently, we omit the details.

The following lemma is a generalization of [13, Lemma 3(ii)].

**Lemma 1** Let $\mu$ be a positive Borel measure on $[0, 1)$ which is a Carleson measure. Assume that $0 < p < \infty$ and $\alpha > -1$. Then there exists a positive constant
\( C = C(p, \alpha, \mu) \) such that for any \( f \in A^p_\alpha \)
\[
\int_{[0,1]} M^p_\infty(r, f)(1-r)^{\alpha+1} d\mu(r) \leq C \| f \|_{A^p_\alpha}.
\]

Of course, \( M^p_\infty(r, f) = \sup_{|z|=r} |f(z)| \).

**Proof** Take \( f \in A^p_\alpha \) and set
\[
g(r) = M^p_\infty(r, f)(1-r)^{\alpha+1}, \\
F(r) = \mu([0, r)) - \mu([0, 1]) = -\mu([r, 1)), \quad 0 < r < 1.
\]

Integrating by parts, we have
\[
\int_{[0,1]} M^p_\infty(r, f)(1-r)^{\alpha+1} d\mu(r) = \int_{[0,1]} g(r) d\mu(r) \\
= \lim_{r \to 1} g(r) F(r) - g(0) F(0) - \int_0^1 g'(r) F(r) \, dr \\
= |f(0)|^p \mu([0, 1)) - \lim_{r \to 1} M^p_\infty(r, f)(1-r)^{\alpha+1} \mu([r, 1)) \\
+ \int_0^1 g'(r) \mu((r, 1)) \, dr. \tag{15}
\]

Since \( f \in A^p_\alpha \) we have that \( M^p_\infty(r, f) = o((1-r)^{-2-\alpha}) \), as \( r \to 1 \) (see, e.g., [18, p.54]). This and the fact that \( \mu \) is a Carleson measure imply that
\[
\lim_{r \to 1} M^p_\infty(r, f)(1-r)^{\alpha+1} \mu([r, 1)) = 0. \tag{16}
\]

Using again that \( \mu \) is a Carleson measure and integrating by parts we see that
\[
\int_0^1 g'(r) \mu((r, 1)) \, dr \lesssim \int_0^1 g'(r)(1-r) \, dr \\
= \lim_{r \to 1} g(r)(1-r) - g(0) + \int_0^1 g'(r) \, dr \\
\leq \lim_{r \to 1} M^p_\infty(r, f)(1-r)^{\alpha+2} + \int_0^1 M^p_\infty(r, f)(1-r)^{\alpha+1} \, dr \\
= \int_0^1 M^p_\infty(r, f)(1-r)^{\alpha+1} \, dr.
\]

Then, using [13, Lemma 3. (ii)], it follows that
\[
\int_0^1 g'(r) \mu((r, 1)) \, dr \lesssim \| f \|_{A^p_\alpha}.
\]
Using this and (16) in (15) readily yields
\[ \int_{[0,1]} M_\infty^{A_p}(r, f)(1 - r)^{\alpha + 1} \, d\mu(r) \lesssim \|f\|_{A_p}^p. \]
\[ \Box \]

We shall also need the following characterization of the dual of the spaces \( A^q_{\alpha} \) \( (q > 1) \). It is a special case of [21, Theorem 2.1].

**Lemma 2** If \( 1 < q < \infty \) and \( \beta > -1 \), then the dual of \( A^q_{\beta} \) can be identified with \( A^p_{\alpha} \) where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \alpha \) is any number with \( \alpha > -1 \), under the pairing
\[ \langle h, f \rangle_{A^q_{\beta}, A^p_{\alpha}} = \int_{\mathbb{D}} h(z) f(z) (1 - |z|^2)^{\frac{\beta}{q} + \frac{\alpha}{p}} \, dA(z), \quad h \in A^q_{\beta}, \ f \in A^p_{\alpha}. \]  
\[ (17) \]

Finally, we recall the following result from [13, (5.2), p. 242] which is a version of the classical Hardy’s inequality [17, pp. 244–245].

**Lemma 3** Suppose that \( k > 0, q > 1 \), and \( h \) is a non-negative function defined in \( (0,1) \), then
\[ \int_0^1 \left( \int_{1-r}^1 h(t) \, dt \right)^q (1 - r)^{k-1} \, dr \leq (\frac{q}{k})^q \int_0^1 (h(1 - r))^q (1 - r)^{q+k-1} \, dr. \]

**Proof of Theorem 4** Suppose first that \( \mathcal{H}_\mu \) is a bounded operator from \( A^p_{\alpha} \) into itself. For \( 0 < b < 1 \), set
\[ f_b(z) = \frac{(1 - b^2)^{1 - \frac{\alpha}{p}}}{(1 - b z)^{\frac{2}{p} + 1}}, \quad z \in \mathbb{D}. \]
Recall that \( p - \alpha > 2 \). Then using [29, Lemma 3.10] with \( t = \alpha \) and \( c = p - \alpha \), we obtain
\[ \|f_b\|_{A^p_{\alpha}}^p = (1 - b^2)^{p - \alpha} \int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha}{|1 - b z|^{2+p}} \, dA(z) \asymp 1. \]
Since \( \mathcal{H}_\mu \) is bounded on \( A^p_{\alpha} \), this implies
\[ 1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{A^p_{\alpha}}. \]  
\[ (18) \]
We also have
\[ f_b(z) = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad (z \in \mathbb{D}), \quad \text{with} \ a_{k,b} \asymp (1 - b^2)^{1 - \frac{\alpha}{p}} \frac{2}{k^p} b^k. \]

Since the \( a_{k,b} \)'s are positive, it is clear that the sequence \( \left\{ \sum_{k=0}^{\infty} \mu_{n+k} a_{k,b} \right\}_{n=0}^{\infty} \) of the Taylor coefficients of \( \mathcal{H}_\mu(f_b) \) is a decreasing sequence of non-negative real numbers.
Using this, Proposition 1, (18), and the definition of the \( a_{k,b} \)'s, we obtain

\[
1 \geq \| \mathcal{H}_\mu(f_b) \|_{A_0^p}^p \geq \sum_{n=1}^{\infty} n^{p-\alpha-3} \left( \sum_{k=1}^{\infty} \mu_{n+k} a_{k,b} \right)^p \\
= \sum_{n=1}^{\infty} n^{p-\alpha-3} \left( \sum_{k=1}^{\infty} a_{k,b} \int_{[0,1]} t^{n+k} d\mu(t) \right)^p \\
\geq (1 - b^2)^{p-\alpha} \sum_{n=1}^{\infty} n^{p-\alpha-3} \left( \sum_{k=1}^{\infty} \frac{2}{b} b^k \int_{[b,1]} t^{n+k} d\mu(t) \right)^p \\
\geq (1 - b^2)^{p-\alpha} \sum_{n=1}^{\infty} n^{p-\alpha-3} \left( \sum_{k=1}^{\infty} \frac{2}{b} b^{n+k} \mu([b,1]) \right)^p \\
= (1 - b^2)^{p-\alpha} \mu([b,1])^p \sum_{n=1}^{\infty} n^{p-\alpha-3} b^{np} \left( \sum_{k=1}^{\infty} \frac{2}{b} b^k \right)^p \\
\asymp (1 - b^2)^{p-\alpha} \mu([b,1])^p \frac{1}{(1 - b^3)^2 + p} \sum_{n=1}^{\infty} n^{p-\alpha-3} b^{np} \\
\asymp (1 - b^2)^{p-\alpha} \mu([b,1])^p \frac{1}{(1 - b^3)^2 + p} \cdot \frac{1}{(1 - b^2)^{p-\alpha-2}} \\
\asymp \mu([b,1])^p \frac{1}{(1 - b)^p}.
\]

Then it follows that

\[
\mu([b,1]) = O(1 - b), \quad \text{as } b \to 1,
\]

and, hence, \( \mu \) is a Carleson measure.

We turn to prove the other implication. So, suppose that \( \mu \) is a Carleson measure and take \( f \in A_0^p \). Let \( q \) be defined by the relation \( \frac{1}{p} + \frac{1}{q} = 1 \) and take \( \beta = \frac{-aq}{p} = \frac{-\alpha}{p-1} \). Observe that \( \beta > -1 \) and that with this election of \( \beta \) the weight in the pairing (17) is identically equal to 1. We have to show that \( \mathcal{H}_\mu f \in A_0^p \) which is equal to \( (A_0^p)^* \) under the pairing \( \langle \cdot, \cdot \rangle_{q, \beta, \alpha} \). So take \( h \in A_0^q \).

\[
\langle h, \mathcal{H}_\mu f \rangle_{q, \beta, \alpha} = \int_{[0,1]} h(z) \mathcal{H}_\mu f(z) dA(z) \\
= \int_{[0,1]} \frac{h(z)}{1 - tz} dA(z) d\mu(t) \\
= \int_{[0,1]} \frac{h(z)}{1 - tz} \left( \int_0^1 \frac{r}{\pi} \int_0^{2\pi} \frac{h(r e^{i\theta})}{1 - tr e^{-i\theta}} d\theta dr \right) d\mu(t) \\
= \int_{[0,1]} \frac{h(z)}{1 - tz} \left( \int_0^1 \frac{r}{\pi i} \int_{|\xi|=1} \frac{h(r \xi)}{\xi - tr} d\xi \right) dr d\mu(t) \\
= 2 \int_{[0,1]} h(r^2 t) dr d\mu(t).
\]
Thus,

\[ \left| \langle h, \mathcal{H}_\mu f \rangle_{q, \beta, \alpha} \right| \leq 2 \int_0^1 |f(t)|G(t) \, d\mu(t), \]

where \( G(t) = \int_0^1 r |h(r^2 t)| \, dr \). Using Hölder’s inequality we obtain,

\[ \int_{[0,1]} f(t)G(t) \, d\mu(t) = \int_{[0,1]} |f(t)| \left(1 - t\right)^{\frac{\alpha+1}{p}} G(t) \left(1 - t\right)^{-\frac{\alpha+1}{p}} \, d\mu(t) \]
\[ \leq \left( \int_{[0,1]} |f(t)|^p \left(1 - t\right)^{\alpha+1} \, d\mu(t) \right)^{1/p} \cdot \left( \int_{[0,1]} G(t)^q \left(1 - t\right)^{-\frac{q(\alpha+1)}{p}} \, d\mu(t) \right)^{1/q}. \]

Lemma 1 implies that

\[ \left( \int_{[0,1]} |f(t)|^p \left(1 - t\right)^{\alpha+1} \, d\mu(t) \right)^{1/p} \lesssim \| f \|_{A^p_{\alpha}}. \]

Next we will show that

\[ \int_{[0,1]} G(t)^q \left(1 - t\right)^{-\frac{q(\alpha+1)}{p}} \, d\mu(t) \lesssim \| h \|_{A^q_{\beta}}^q. \tag{19} \]

This will give that

\[ \left| \langle h, \mathcal{H}_\mu f \rangle_{q, \beta, \alpha} \right| \lesssim \| f \|_{A^p_{\alpha}} \cdot \| h \|_{A^q_{\beta}}^q. \]

By the duality theorem, this implies that \( \mathcal{H}_\mu f \in A^p_{\alpha} \).

Let us prove (19). Observe first that if \( 0 < t < 1/2 \) then \( |h(r^2 t)| \leq M_\infty \left( \frac{1}{2}, h \right) \) for each \( r \in (0,1) \), thus

\[ G(t) = \int_0^1 |h(r^2 t)| \, dr \leq M_\infty \left( \frac{1}{2}, h \right), \quad 0 < t < 1/2. \]

Clearly, this implies

\[ \int_{[0,1/2]} G(t)^q \left(1 - t\right)^{-\frac{q(\alpha+1)}{p}} \, d\mu(t) \lesssim M_\infty^q \left( \frac{1}{2}, h \right) \lesssim \| h \|_{A^q_{\beta}}^q. \tag{20} \]

Notice that \( -\frac{q(\alpha+1)}{p} = \frac{p-2-\alpha}{p-1} - 1 > -1 \). Making the change of variables \( r^2 t = s \), we obtain \( \int_0^1 r |h(r^2 t)| \, dr = \frac{1}{2t} \int_0^t |h(s)| \, ds \) and, hence,

\[ \int_{[1/2,1]} G(t)^q \left(1 - t\right)^{-\frac{q(\alpha+1)}{p}} \, d\mu(t) \]
Hankel matrices acting on the Hardy space \( H^1 \) and on Dirichlet...

\[
\begin{align*}
&= \int_{[1/2, 1)} \left( \int_0^1 |h(r^2 t)| r \, dr \right)^q (1 - t)^{-\frac{q(\alpha + 1)}{p}} \, d\mu(t) \\
&= \int_{[1/2, 1)} \frac{1}{(2t)^q} \left( \int_0^t |h(s)| \, ds \right)^q (1 - t)^{-\frac{q(\alpha + 1)}{p}} \, d\mu(t) \\
&\leq \int_{[1/2, 1)} \left( \int_0^t M_\infty(s, h) \, ds \right)^q (1 - t)^{-\frac{q(\alpha + 1)}{p}} \, d\mu(t) \\
&\leq \int_{[0, 1)} \left( \int_{1-t}^1 M_\infty(1 - s, h) \, ds \right)^q (1 - t)^{-\frac{q(\alpha + 1)}{p}} \, d\mu(t)
\end{align*}
\]

Let us call \( H(t) = \left( \int_{1-t}^1 M_\infty(1 - s, h) \, ds \right)^q (1 - t)^{-\frac{q(\alpha + 1)}{p}} \) for \( 0 \leq t < 1 \). Integrating by parts we obtain the following

\[
\int_{[0, 1)} H(t) \, d\mu(t) = H(0)\mu([0, 1)) - \lim_{t \to 1^-} H(t)\mu([t, 1)) + \int_0^1 \mu([t, 1)) H'(t) \, dt.
\]

The first term is equal to 0. Using the fact that \( \mu \) is a Carleson measure we have that

\[
H(t)\mu([t, 1)) \lesssim (1 - t)H(t)
\]

\[
= \left( \int_{1-t}^1 M_\infty(1 - s, h) \, ds \right)^q (1 - t)^{-1 - \frac{q(\alpha + 1)}{p}}
\]

\[
= \left( \int_0^t M_\infty(s, h) \, ds \right)^q (1 - t)^{-1 - \frac{q(\alpha + 1)}{p}}.
\]

Since \( h \in A_\beta^q \) we have \( M_\infty(t, h) = o \left( (1 - t)^{-\frac{\beta + 2}{q}} \right) \), as \( t \to 1 \). Then, bearing in mind that \( \frac{\beta + 2}{q} > 1 \), it follows that

\[
H(t)\mu([t, 1)) = o \left( (1 - t)^{-\beta - 2 + q} \cdot (1 - t)^{-1 - \frac{q(\alpha + 1)}{p}} \right) = o(1), \quad \text{as} \quad t \to 1.
\]

Actually, we have also proved that

\[
(1 - t)H(t) = o(1), \quad \text{as} \quad t \to 1.
\]

Using that \( \mu \) is a Carleson measure, integrating by parts, and using the definition of \( H \) and (24), we obtain

\[
\int_0^1 \mu([t, 1)) H'(t) \, dt \lesssim \int_0^1 (1 - t)H'(t) \, dt
\]

\[
= \lim_{t \to 1^-} (1 - t)H(t) - H(0) + \int_0^1 H(t) \, dt.
\]
Now, using Lemma 3 and [13, Lemma 3], we see that

\[
\int_0^1 \left( \int_{1-t}^1 M_\infty(1-s, h) \, ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}} \, dt 
\approx \int_0^1 M^q_\infty(t, h)(1-t)^{\alpha+1} \, dt \lesssim \|h\|_{A^q}^q.
\]

Using this, (25), (23), (22), and (21), it follows that

\[
\int_{[1/2, 1)} G(t)^q (1-t)^{-\frac{q(\alpha+1)}{p}} \, d\mu(t) \lesssim \|h\|_{A^q}^q.
\]

This and (20) yield (19). \qed

Our final aim in this article is to find the analogue of Theorem 4 for Dirichlet spaces. In other words, we wish give an answer to the following question.

**Question 2** If \( \max(-1, p-2) < \alpha < 2p-2 \), is it true that \( \mathcal{H}_\mu \) is a bounded operator from \( \mathcal{D}^p_\alpha \) into itself if and only if \( \mu \) is a Carleson measure?

Since \( p-1 < \alpha < 2p-2 \) implies that \( \mathcal{D}^p_\alpha = A^{p-\alpha}_p \), Theorem 4 answers the question affirmatively for these values of \( p \) and \( \alpha \). It remains to consider the case \( \max(-1, p-2) < \alpha \leq p-1 \). We shall prove the following result which gives a positive answer to Question 2 in the case \( p > 1 \).

**Theorem 5** Suppose that \( p > 1 \) and \( p-2 < \alpha \leq p-1 \), and let \( \mu \) be a finite positive Borel measure on \([0, 1)\).

The operator \( \mathcal{H}_\mu \) is well defined on \( \mathcal{D}^p_\alpha \) and it is a bounded operator from \( \mathcal{D}^p_\alpha \) into itself if and only if \( \mu \) is a Carleson measure.

The following two lemmas will be needed in the proof of Theorem 5. The first one follows trivially from Proposition 1.

**Lemma 4** Let \( f \in \text{Hol}(\mathbb{D}) \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) (\( z \in \mathbb{D} \)). Suppose that \( 1 < p < \infty \) and \( p-2 < \alpha \leq p-1 \), and that the sequence \( \{a_n\}_{n=0}^{\infty} \) is a decreasing sequence of non-negative real numbers. Then

\[
f \in \mathcal{D}^p_\alpha \iff \sum_{n=0}^{\infty} (n+1)^{2p-\alpha-3} a_n^p < \infty.
\]

The following lemma is a generalization of [13, Lemma 4].
Lemma 5 Let $\mu$ be a positive Borel measure on $[0, 1)$ which is a Carleson measure. Assume that $0 < p < \infty$ and $\alpha > -1$. Then there exists a positive constant $C = C(p, \alpha, \mu)$ such that for any $f \in D_\alpha^p$

$$\int_{[0,1]} M_p^\infty(r, f)(1 - r)^{\alpha-p+1} d\mu(r) \leq C \|f\|_{D_\alpha^p}^p.$$ 

Proof We argue as in the proof of Lemma 1. Take $f \in D_\alpha^p$ and set

$$g(r) = M_p^\infty(r, f)(1 - r)^{\alpha-p+1},$$

$$F(r) = \mu([0, r)) - \mu([0, 1)) = -\mu([r, 1)), \quad 0 < r < 1.$$ 

Integrating by parts, we have

$$\int_{[0,1]} M_p^\infty(r, f)(1 - r)^{\alpha-p+1} d\mu(r) = \int_{[0,1]} g(r) d\mu(r)$$

$$= \lim_{r \to 1} g(r) F(r) - g(0) F(0) - \int_0^1 g'(r) F(r) \, dr$$

$$= |f(0)|^p \mu([0, 1)) - \lim_{r \to 1} M_p^\infty(r, f)(1 - r)^{\alpha-p+1} \mu([r, 1))$$

$$+ \int_0^1 g'(r) \mu([r, 1)) \, dr. \quad (26)$$ 

Since $f \in D_\alpha^p$ we have that $M_p^\infty(r, f') = o((1 - r)^{-2-\alpha})$, as $r \to 1$. Hence, $M_p^\infty(r, f) = o((1 - r)^{-2-\alpha+p})$, as $r \to 1$. This and the fact that $\mu$ is a Carleson measure imply that

$$\lim_{r \to 1} M_p^\infty(r, f)(1 - r)^{\alpha-p+1} \mu([r, 1)) = 0. \quad (27)$$ 

Using again that $\mu$ is a Carleson measure and integrating by parts we see that

$$\int_0^1 g'(r) \mu([r, 1)) \, dr \lesssim \int_0^1 g'(r)(1 - r) \, dr$$

$$= \lim_{r \to 1} g(r)(1 - r) - g(0) + \int_0^1 g(r) \, dr$$

$$\leq \lim_{r \to 1} M_p^\infty(r, f)(1 - r)^{\alpha-p+2}$$

$$+ \int_0^1 M_p^\infty(r, f)(1 - r)^{\alpha-p+1} \, dr$$

$$= \int_0^1 M_p^\infty(r, f)(1 - r)^{\alpha-p+1} \, dr.$$
Then, using [13, Lemma 3], it follows that
\[ \int_0^1 g'(r) \mu((r, 1)) \, dr \lesssim \| f \|_{D^p_\alpha}. \]

Using this and (27) in (26) readily yields
\[ \int_{[0, 1]} M^p_\infty(r, f)(1 - r)^{\alpha - p + 1} \, d\mu(r) \lesssim \| f \|_{D^p_\alpha}. \]

\[ \square \]

Proof of Theorem 5 Suppose first that \( \mathcal{H}_\mu \) is a bounded operator from \( D^p_\alpha \) into itself. For \( 1/2 < b < 1 \) we set
\[ f_b(z) = \frac{(1 - b^2)^{1 - \frac{\alpha}{p}}}{(1 - bz)^{2/p}}, \quad z \in \mathbb{D}. \]

We have \( \|f_b\|_{D^p_\alpha} \approx 1. \) Then arguing as in the proof of the correspondent implication in Theorem 4 we obtain that \( \mu \) is a Carleson measure. We omit the details.

To prove the other implication, suppose that \( \mu \) is a Carleson measure and take \( f \in D^p_\alpha. \) Since \( \mathcal{H}_\mu \) and \( \mathcal{I}_\mu \) coincide on \( D^p_\alpha, \) we have to prove that \( \mathcal{I}_\mu f \in D^p_\alpha \) and that \( \| \mathcal{I}_\mu f \|_{D^p_\alpha} \lesssim \| f \|_{D^p_\alpha} \) or, equivalently, that \( (\mathcal{I}_\mu f)' \in A^p_\alpha \) and
\[ \| (\mathcal{I}_\mu f)' \|_{A^p_\alpha} \lesssim \| f \|_{A^p_\alpha}. \tag{28} \]

We shall distinguish two cases.

First case: \( \alpha < p - 1. \) Let \( q \) be defined by the relation \( \frac{1}{p} + \frac{1}{q} = 1 \) and take \( \beta = -\frac{aq}{p}. \)

In view of Lemma 2, (28) is equivalent to
\[ \left| \int_{\mathbb{D}} h(z) (\mathcal{I}_\mu f)'(z) \, dA(z) \right| \lesssim \| f \|_{D^p_\alpha} \| h \|_{A^q_\beta}, \quad h \in A^q_\beta. \tag{29} \]

So, take \( h \in A^q_\beta. \) Just as in the proof of Theorem 1, we have
\[ \int_{\mathbb{D}} h(z) (\mathcal{I}_\mu f)'(z) \, dA(z) = \int_{[0, 1]} t f(t)(1 - t)^s \, d\mu(t). \tag{30} \]

Set \( s = -1 + \frac{\alpha + 1}{p}. \) Observe that \( ps = \alpha - p + 1 \) and \(-qs = \beta + 1.\) Then, using (30), Hölder’s inequality, Lemma 1, and Lemma 5, we obtain
\[ \left| \int_{\mathbb{D}} h(z) (\mathcal{I}_\mu f)'(z) \, dA(z) \right| \leq \int_{[0, 1]} |f(t)|(1 - t)^s |h(t)|(1 - t)^{-s} \, d\mu(t) \]
\[ \leq \left( \int_{\mathbb{D}} |f(t)|^p (1 - t)^{\alpha - p + 1} \, d\mu(t) \right)^{1/p} \left( \int_{[0, 1]} |h(t)|^q (1 - t)^{\beta + 1} \, d\mu(t) \right)^{1/q} \]
\[ \leq \left( \int_{\mathbb{D}} M^p_\infty(t, f)(1 - t)^{\alpha - p + 1} \, d\mu(t) \right)^{1/p}. \]
Theorem 6

Let \( \mu \) be a finite positive Borel measure on \([0, 1]\) and \(-1 < \alpha < 0\). If \( \mu \) is a Carleson measure then the operator \( \mathcal{H}_\mu \) is a bounded operator form \( \mathcal{D}_{\alpha}^1 \) to itself.

\[
\times \left( \int_{[0,1]} M_q^\beta(t,h)(1-t)^{\beta+1} d\mu(t) \right)^{1/q} \\
\leq \| f \|_{\mathcal{D}_p^\alpha} \| h \|_{A_q^\beta}.
\]

Thus, (29) holds.

**Second case: \( \alpha = p - 1 \).** We let again \( q \) be defined by the relation \( \frac{1}{p} + \frac{1}{q} = 1 \) and take \( \beta = q - 1 \). Using Lemma 2 and arguing as in the preceding case, we have to show that

\[
\left| \int_D (1 - |z|^2) h(z) \left( \mathcal{I}_\mu f \right)'(z) dA(z) \right| \lesssim \| f \|_{\mathcal{D}_p^{p-1}} \| h \|_{A_q^{q-1}}, \ h \in A_q^{q-1}. \tag{31}
\]

We have

\[
\int_D (1 - |z|^2) h(z) \left( \mathcal{I}_\mu f \right)'(z) dA(z) = \int_{[0,1]} t f(t) \int_D (1 - |z|^2) h(z) (1 - t z)^2 dA(z) d\mu(t). \tag{32}
\]

Now, \( \int_D \frac{h(z)}{(1 - rz)} dA(z) = h(t) \) and

\[
\int_D \frac{|z|^2 h(z)}{(1 - rz)^2} dA(z) = \int_0^1 \int_0^{2\pi} \frac{h(re^{i\theta})}{(1 - tr e^{-i\theta})^2} dr d\theta = \int_0^1 \int_0^{2\pi} \frac{r^3 h(re^{i\theta})}{(e^{i\theta} - tr)^2} dr d\theta = \int_0^1 \int_0^{2\pi} \frac{r^3 h(re^{i\theta})}{(e^{i\theta} - tr)^2} dr d\theta
\]

Then it is clear that \( \left| \int_D (1 - |z|^2) h(z) (1 - rz)^2 dA(z) \right| \lesssim M_\infty(t,h) \). Using this, (32), Hölder’s inequality, Lemma 1, and Lemma 5, we obtain

\[
\left| \int_D (1 - |z|^2) h(z) \left( \mathcal{I}_\mu f \right)'(z) dA(z) \right| \lesssim \int_{[0,1]} M_\infty(t,f) M_\infty(t,h) d\mu(t)
\]

\[
\leq \left( \int_{[0,1]} M_q^\beta(t,f) d\mu(t) \right)^{1/p} \left( \int_{[0,1]} M_q^\beta(t,h) d\mu(t) \right)^{1/q}
\]

\[
\leq \| f \|_{\mathcal{D}_p^{p-1}} \| h \|_{A_q^{q-1}}.
\]

This is (31). \( \Box \)

We shall close the article with some comments about the case \( p = 1 \) in Question 2. We have the following result.

**Theorem 6** Let \( \mu \) be a finite positive Borel measure on \([0, 1]\) and \(-1 < \alpha < 0\). If \( \mu \) is a Carleson measure then the operator \( \mathcal{H}_\mu \) is a bounded operator form \( \mathcal{D}_{\alpha}^1 \) to itself.
Proof Using \[29\], we see that \(A^1_\alpha\) can be identified as the dual of the little Bloch space under the pairing
\[
\langle h, g \rangle = \int_{\mathbb{D}} (1 - |z|^2)^\alpha h(z) \overline{g(z)} \, dA(z), \quad h \in B_0, \ g \in A^1_\alpha.
\] (33)

Suppose that \(\mu\) is a Carleson measure. Using this duality relation and the fact that \(H_{\mu} = I_{\mu}\) on \(D^1_\alpha\), showing that \(H_{\mu}\) is a bounded operator from \(D^1_\alpha\) to itself is equivalent to showing that
\[
\left| \int_{\mathbb{D}} (1 - |z|^2)^\alpha h(z) \left( I_{\mu} f \right)'(z) \, dA(z) \right| \lesssim \|h\|_B \cdot \|f\|_{D^1_\alpha}, \quad h \in B_0, \ f \in D^1_\alpha.
\] (34)

Let us prove (34). Take \(h \in B_0\) and \(f \in D^1_\alpha\). We have
\[
\int_{\mathbb{D}} (1 - |z|^2)^\alpha h(z) \left( I_{\mu} f \right)'(z) \, dA(z) = \int_{[0,1]} t \overline{f(t)} \int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha h(z)}{(1 - t \overline{z})^2} \, dA(z) \, d\mu(t).
\] (35)

Using \[29\], we have that the operator \(T\) defined by
\[
T \phi(\xi) = (1 - |\xi|^2)^{-\alpha} \int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha \phi(z)}{(1 - \xi \overline{z})^2} \, dA(z)
\]
is a bounded operator from \(B\) into \(L^\infty(\mathbb{D})\). Then it follows that
\[
\left| \int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha h(z)}{(1 - t \overline{z})^2} \, dA(z) \right| \lesssim \|h\|_B (1 - t^2)^\alpha, \quad t \in [0, 1).
\]

Using this in (35), we obtain
\[
\left| \int_{\mathbb{D}} (1 - |z|^2)^\alpha h(z) \left( I_{\mu} f \right)'(z) \, dA(z) \right| \lesssim \|h\|_B \int_{\mathbb{D}} (1 - t)^\alpha |f(t)| \, d\mu(t). \quad (36)
\]

The fact that \(\mu\) is a Carleson measure readily implies that the measure \(\nu\) defined by \(d\nu(t) = (1 - t)^\alpha d\mu(t)\) is a \((1 - \alpha)\)-Carleson measure. Using Theorem 1 of \[28\] we see that then \(\nu\) is a Carleson measure for \(D^1_\alpha\), that is,
\[
\int_{[0,1]} (1 - t)^\alpha |g(t)| \, d\mu(t) \lesssim \|g\|_{D^1_\alpha}, \quad g \in D^1_\alpha.
\]

Using this in (36), (34) follows.
We do not know whether the converse of Theorem 6 is true. This is due to the fact that we do not know whether Lemma 4 remains true for $p = 1$. The inequality
\begin{equation}
\sum_{n=0}^{\infty} |a_n|(n+1)^{-\left(1+\alpha\right)} \lesssim \|f\|_{\mathcal{D}_1^\alpha}.
\end{equation}

is certainly true with no assumption on the sequence $\{a_n\}$. Indeed, by Hardy’s inequality [11, p. 48],
\begin{equation}
\sum_{n=1}^{\infty} |a_n|r^{n-1} \lesssim \int_0^{2\pi} |f'(re^{i\theta})|d\theta.
\end{equation}
Hence
\begin{equation}
\|f\|_{\mathcal{D}_1^\alpha} \asymp \int_0^1 (1-r)^\alpha \int_0^{2\pi} |f'(re^{i\theta})|d\theta dr
\geq \sum_{n=1}^{\infty} |a_n| \int_0^1 (1-r)^{\alpha} r^{n-1} dr = \sum_{n=1}^{\infty} |a_n| B(\alpha + 1, n),
\end{equation}
where $B(\cdot, \cdot)$ is the Beta function. Stirling’s formula gives $B(\alpha + 1, n) \asymp n^{-\left(\alpha+1\right)}$ and then (37) follows.

However, the proof of Theorem D in [23] does not seem to work to prove the opposite inequality when $\{a_n\}$ is decreasing.

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