SOME PROPERTIES OF 3 × 3
OCTONIONIC HERMITIAN MATRICES
WITH NON-REAL EIGENVALUES

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ABSTRACT

We discuss our preliminary attempts to extend previous work on 2 × 2 Hermitian octonionic matrices with non-real eigenvalues to the 3 × 3 case.

1. INTRODUCTION

In previous work [1,2; see also 3], we considered the real eigenvalue problem for 2 × 2 and 3 × 3 Hermitian matrices over the octonions \( \mathbb{O} \). The 2 × 2 case corresponds closely to the standard, complex eigenvalue problem, since any 2 × 2 octonionic Hermitian matrix lies in a complex subalgebra \( \mathbb{C} \subset \mathbb{O} \). The 3 × 3 case requires considerable care, resulting in some changes in the expected results. However, we also showed in [1] that there are octonionic Hermitian matrices which admit eigenvalues which are not real, and a complete treatment of the 2 × 2 case was given in [4]. Here, we discuss our preliminary results for the 3 × 3 case.

Although we are able to obtain a 3rd-order characteristic equation for the (right) eigenvalues in the 3 × 3 case, we have not been able to solve this equation, nor have we been able to extend our orthonormality results [1,2] from the real case. We therefore discuss several illustrative examples and make some conjectures regarding more general results.

The 3 × 3 case is of particular interest mathematically because it corresponds to the exceptional Jordan algebra, also known as the Albert algebra. There have been numerous attempts to use this algebra to describe quantum physics, which was in fact Jordan’s original motivation. More recently, Schray [5,6] has shown how to use the exceptional Jordan algebra to give an elegant description of the superparticle, which we have been attempting to extend to the superstring. Our dimensional reduction scheme extends naturally to this case [7], and we believe it is the natural language to describe the fundamental particles of nature.

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The paper is organized as follows. In Section 2 we briefly review the properties of octonions. We then consider 3 × 3 octonionic Hermitian matrices, deriving a characteristic equation for the eigenvalues in Section 3, and considering several examples in Section 4. Finally, in Section 5 we discuss our results. Some parts of this presentation have appeared in our previous work.

2. OCTONIONS

We summarize here only the essential properties of the octonions \( \mathbb{O} \). For a more detailed introduction, see [1] or [8,9].

The octonions \( \mathbb{O} \) are the nonassociative, noncommutative, normed division algebra over the reals. In terms of a natural basis, an octonion \( a \) can be written

\[
a = \sum_{q=1}^{8} a^q e_q
\]

where the coefficients \( a^q \) are real, and where the basis vectors satisfy \( e_1 = 1 \) and

\[
e_q^2 = -1 \quad (q = 2, \ldots, 8)
\]

The multiplication table is conveniently encoded in the 7-point projective plane, shown in Figure 1. The product of any two imaginary units is given by the third unit on the unique line connecting them, with the sign determined by the relative orientation.

**Octonionic conjugation** is given by reversing the sign of the imaginary basis units

\[
\bar{a} = a^1 e_1 - \sum_{q=2}^{8} a^q e_q
\]
Conjugation is an antiautomorphism, since it satisfies
\[ \overline{ab} = \overline{b} \overline{a} \]
The real and imaginary parts of an octonion \( a \) are given by
\[ \text{Re}(a) = \frac{1}{2}(a + \overline{a}) \quad \text{Im}(a) = \frac{1}{2}(a - \overline{a}) \] (4)

The **inner product** on \( \mathbb{O} \) is the one inherited from \( \mathbb{R}^8 \), namely
\[ a \cdot b = \sum_q a^q b^q \] (5)
which can be rewritten as
\[ a \cdot b = \frac{1}{2}(ab + b\overline{a}) = \frac{1}{2}(\overline{ba} + \overline{a}b) \] (6)
and which satisfies the identities
\[ a \cdot (xb) = b \cdot (\overline{x}a) \] (7)
\[ (ax) \cdot (bx) = |x|^2 a \cdot b \] (8)
for any \( a, b, x \in \mathbb{O} \). The **norm** of an octonion is just
\[ |a| = \sqrt{a\overline{a}} = \sqrt{a \cdot a} \] (9)
which satisfies the defining property of a normed division algebra, namely
\[ |ab| = |a||b| \] (10)

The **associator** of three octonions is
\[ [a, b, c] = (ab)c - a(bc) \] (11)
which is totally antisymmetric in its arguments, has no real part, and changes sign if any one of its arguments is replaced by its octonionic conjugate. Although the associator does not vanish in general, the octonions do satisfy a weak form of associativity known as **alternativity**, namely
\[ [b, a, a] = 0 = [b, a, \overline{a}] \] (12)
The underlying reason for alternativity is Artin’s Theorem \[10,11\], which states that that any two octonions lie in a quaternionic subalgebra of \( \mathbb{O} \), so that any product containing only two octonionic directions is associative. We will also have use for the associator identity
\[ [a, b, c]d + a[b, c, d] = [ab, c, d] - [a, bc, d] + [a, b, cd] \] (13)
for any \( a, b, c, d \in \mathbb{O} \), which is proved by writing out all the terms.

### 3. 3 × 3 OCTONIONIC HERMITIAN MATRICES

In this section, we derive a characteristic equation for the (right) eigenvalues of \( A \), which reduces to that of \[1\] for real eigenvalues. Unfortunately, we have been unable to solve this equation when the eigenvalues are not real, so that we have also been unable to investigate orthogonality and decomposition results analogous to those for real eigenvalues. We discuss this further in the next section, where we study several examples with intriguing properties.
a) Jordan matrices

The $3 \times 3$ octonionic Hermitian matrices, henceforth referred to as Jordan matrices, form the exceptional Jordan algebra (also called the Albert algebra) under the Jordan product

$$A \circ B := \frac{1}{2}(AB + BA)$$

which is commutative, but not associative. A special case of this is

$$A^2 \equiv A \circ A$$

and we define

$$A^3 := A^2 \circ A = A \circ A^2$$

Remarkably, with these definitions, Jordan matrices satisfy the usual characteristic equation

$$A^3 - (\text{tr } A) A^2 + \sigma(A) A - (\text{det } A) I = 0$$

where $\sigma(A)$ is defined by

$$\sigma(A) := \frac{1}{2} \left( (\text{tr } A)^2 - \text{tr } (A^2) \right)$$

and where the determinant of $A$ is defined abstractly in terms of the Freudenthal product.  

Concretely, if

$$A = \begin{pmatrix} p & a & b \\ a & m & c \\ b & c & n \end{pmatrix}$$

with $p, m, n \in \mathbb{R}$ and $a, b, c \in \mathbb{O}$ then

$$\text{tr } A = p + m + n$$

$$\sigma(A) = pm + pn + mn - |a|^2 - |b|^2 - |c|^2$$

$$\text{det } A = pmn + b(ac) + \frac{b(ac)}{n} - n|a|^2 - m|b|^2 - p|c|^2$$

As shown originally by Ogievetsky [15], $A$, has 6, rather than 3, real eigenvalues, which furthermore fail to satisfy the characteristic equation (17). A complete, computer-assisted [2] treatment of this case was given in [1], and a somewhat more general analytic treatment was later given by Okubo [3]. As shown there, the eigenvalues naturally belong to 2 distinct families, each containing 3 real eigenvalues. Furthermore, within each family, the corresponding eigenvectors lead to a decomposition of the form

$$A = \sum_{\alpha=1}^{3} \lambda_{\alpha} \left( v_{\alpha} v_{\alpha}^{\dagger} \right)$$

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2 The $2 \times 2$ octonionic Hermitian matrices form a special Jordan algebra since they are alternative [12].

3 The Freudenthal product of two Jordan matrices $A$ and $B$ is given by [14]

$$A \ast B = A \circ B - \frac{1}{2} \left( A \text{ tr } (B) + B \text{ tr } (A) \right) + \frac{1}{2} \left( \text{ tr } (A) \text{ tr } (B) - \text{ tr } (A \circ B) \right)$$

The determinant can then be defined as

$$\text{det}(A) = \frac{1}{3} \text{ tr } \left( (A \ast A) \circ A \right)$$
Furthermore, eigenvectors $v_\alpha$ corresponding to different eigenvalues are automatically orthogonal in the generalized sense

$$(vv^\dagger)w = 0$$

**b) Characteristic Equation**

Set

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then the (right) eigenvalue problem

$$Av = v\lambda$$

becomes

$$x(\lambda - p) = ay + \bar{b}z$$  \hspace{1cm} (25)

$$y(\lambda - m) = cz + \pi x$$  \hspace{1cm} (26)

$$z(\lambda - n) = bx + \overline{\pi}y$$  \hspace{1cm} (27)

As shown in [1], (24) admits solutions for which $\lambda$ is not real; further examples are given in Section 4.  \(^4\) Multiplying (25) on the right by $(\lambda - m)$ leads to

$$(ay)(\lambda - m) = x(\lambda - p)(\lambda - m) - (\bar{b}z)(\lambda - m)$$

whereas multiplying (26) on the left by $\pi$

$$a(y(\lambda - m)) = a(cz) + |a|^2 x$$

Subtracting these 2 equations immediately yields

$$[a, y, \lambda] = x \left( (\lambda - p)(\lambda - m) - |a|^2 \right) - (\bar{b}z)(\lambda - m) - a(cz)$$

Similarly, multiplying (26) on the right by $(\lambda - p)$ and (25) on the left by $\pi$ (or using symmetry) leads to

$$[\pi, x, \lambda] = y \left( (\lambda - p)(\lambda - m) - |a|^2 \right) - (cz)(\lambda - p) - \pi(\bar{b}z)$$

We plan to multiply (30) by $b$ on the left, (31) by $\overline{\pi}$ on the left, add, and then use (27). Before doing so, we first use (13) to write

$$[b, x, (\lambda - p)(\lambda - m)] = [b, x(\lambda - p), (\lambda - m)] - [bx, (\lambda - p), (\lambda - m)]$$

$$+ b[x, (\lambda - p), (\lambda - m)] + [b, x, (\lambda - p)] = [b, x(\lambda - p), \lambda] + [b, x, \lambda](\lambda - m)$$

$$= [b, ay + \bar{b}z, \lambda] + [b, x, \lambda](\lambda - m)$$

as well as

$$[b, \bar{b}z, \lambda] = [b\bar{b}, z, \lambda] + [b, \bar{b}, z\lambda] - b[\bar{b}, z, \lambda] - [b, \bar{b}, z]$$

$$= -b[\bar{b}, z, \lambda]$$

\(^4\) The quite different Jordan eigenvalue problem in the $3 \times 3$ case admits only real eigenvalues and was discussed in [7].
Thus, returning to (30), we obtain

\[ b [a, y, \lambda] = (bx)((\lambda - p)(\lambda - m) - |a|^2) - b(cz) \]

\[ - [b, x, (\lambda - p)(\lambda - m)] - b[b, z, \lambda] \]

\[ = (bx)((\lambda - p)(\lambda - m) - |a|^2) - b(cz) \]

\[ - [b, ay, \lambda] - [b, x, \lambda](\lambda - m) \]

(34)

Similarly, (31) becomes

\[ \bar{c} [\bar{a}, x, \lambda] = (\bar{c}y)((\lambda - p)(\lambda - m) - |a|^2) - |c|^2z(\lambda - p) - \bar{c}(\bar{a}(\bar{b}z)) \]

\[ - [\bar{c}, \bar{ax}, \lambda] - [\bar{c}, y, \lambda](\lambda - p) \]

(35)

Adding these last 2 equations, we obtain

\[ b [a, y, \lambda] + \bar{c} [\bar{a}, x, \lambda] = (bx + \bar{c}y)((\lambda - p)(\lambda - m) - |a|^2) - |b|^2z(\lambda - m) - |c|^2z(\lambda - p) \]

\[ - b(a(cz)) - \bar{c}(\bar{a}(\bar{b}z)) \]

\[ - [\bar{c}, \bar{ax}, \lambda] - [\bar{c}, y, \lambda](\lambda - p) - [b, ay, \lambda] - [b, x, \lambda](\lambda - m) \]

(36)

Finally, using (27), factoring out \( z \), and rearranging terms leads to the generalized characteristic equation in the form

\[ z \left( \lambda^3 - (\text{tr} \mathcal{A}) \lambda^2 + \sigma(\mathcal{A}) \lambda - \det \mathcal{A} \right) = b(a(cz)) + \bar{c}(\bar{a}(\bar{b}z)) - (b(ac) + (\bar{a}b))z \]

\[ + b [a, y, \lambda] + [b, ay, \lambda] + [b, x, \lambda](\lambda - m) \]

\[ + \bar{c} [\bar{a}, x, \lambda] + [\bar{c}, \bar{ax}, \lambda] + [\bar{c}, y, \lambda](\lambda - p) \]

(37)

If \( \lambda \) is real, all the associators on the right-hand-side vanish, and we recover the generalized characteristic equation given in [1]. The requirement in that case that the right-hand-side be a real multiple of \( z \) (since the left-hand-side is) then constrains \( z \), resulting in precisely 2 values for that real multiple, and reducing (37) to 2 cubic equations, one for each family of real eigenvalues.

While we find the form of (37) attractive, as there are no extraneous terms involving both \( z \) and \( \lambda \), we have so far been unable to further simplify (37) when \( \lambda \) is not real.

c) Alternate Approach

We briefly describe another possible approach to finding the eigenvalues, which relies on the associator identity

\[ [v^\dagger, v, \lambda] := (v^\dagger v)\lambda - v^{\dagger}(v\lambda) \equiv 0 \]

(38)

which follows for any octonionic vector \( v \) and \( \lambda \in \mathbb{O} \) by alternativity, and which is further discussed in the Appendix of [4]. If \( v \) is a normalized right eigenvector of \( \mathcal{A} \) with eigenvalue \( \lambda \), then

\[ v^{\dagger}(\mathcal{A}v) = v^{\dagger}(v\lambda) = (v^{\dagger}v)\lambda = \lambda \]

(39)

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5 The extra terms multiplying \( z \) on the right-hand-side come from \( \det \mathcal{A} \).
which yields an equation for $\lambda$ in terms of $\mathcal{A}$ and the components of $v$. A similar construction using the associator

$$[v^\dagger, \mathcal{A}, v] := (v^\dagger \mathcal{A})v - v^\dagger (\mathcal{A}v) = (\mathcal{A}v)^\dagger v - v^\dagger (\mathcal{A}v) = \left(v^\dagger (\mathcal{A}v)\right)^\dagger - v^\dagger (\mathcal{A}v)$$

leads for normalized eigenvectors to

$$[v^\dagger, \mathcal{A}, v] = \left(v^\dagger (v\lambda)\right)^\dagger - v^\dagger (v\lambda) = \left((v^\dagger v)\lambda\right)^\dagger - (v^\dagger v)\lambda = \overline{\lambda} - \lambda = -2 \text{Im}(\lambda)$$

Inserting (19) and (23) into (39) leads, after minor rearrangement using associators and (27), to

$$\lambda = \frac{p|x|^2 + m|y|^2 - n|z|^2 + 2x \cdot (ay)}{|x|^2 + |y|^2 - |z|^2} + \frac{[x, a, y] + [z, b, x] + [y, c, z]}{|x|^2 + |y|^2 + |z|^2}$$

which gives explicit expressions for the real and imaginary parts of $\lambda$. The first term can be rewritten using cyclic permutations of $\{x, y, z\}$ (and $\{a, c, b\}$), and the resulting expressions set equal to obtain

$$\text{Re}(\lambda) = \frac{x \cdot (ay) + z \cdot (bx) + p|x|^2}{|x|^2}$$

and similar expressions obtained by cyclic permutation. Finally, if $v$ is normalized, the imaginary part of (42) reduces to

$$\text{Im}(\lambda) = [x, a, y] + [z, b, x] + [y, c, z]$$

We had hoped to use these various expressions to impose conditions on $\mathcal{A}$ which would in turn enable us to solve for $\lambda$, but have not yet found a way to do so.

4. EXAMPLES

Without being able to solve (some version of) the characteristic equation in the $3 \times 3$ case, it is not possible in general to determine all the (non-real) eigenvalues of a given Hermitian octonionic matrix. It is therefore instructive to consider several explicit examples.

a) Example 1

Consider the matrix

$$\mathcal{B} = \begin{pmatrix} p & iq & kqs \\ -iq & p & jq \\ -kqs & -jq & p \end{pmatrix}$$

where

$$s = \cos \theta + k \ell \sin \theta$$

Note that $\mathcal{B}$ is quaternionic if $\theta = 0$.

The real eigenvalues of $\mathcal{B}$, and corresponding orthonormal bases of eigenvectors, were given in [2]. But $\mathcal{B}$ also admits eigenvectors with eigenvalues which are not real. For

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6 These expressions can also be obtained directly from (25)–(27) by taking the dot product with $x, y, z$, respectively, and using (7).
These eigenvectors and eigenvalues reduce to the ones given in [2] when θ → 0. Somewhat surprisingly, these eigenvectors (when normalized) yield a decomposition of the form (21). Remarkably, they also yield a decomposition of the form

\[ \mathcal{A} = \sum_{\alpha=1}^{3} (v_\alpha \lambda_\alpha) v_\alpha^\dagger \]  

We now describe some further properties of these eigenvectors. Each eigenspace with eigenvalues \( \lambda_\tilde{u} = p \pm 2q\bar{s} \) is 1-dimensional (over R), so that the eigenvectors \( \hat{u}_\pm \) are essentially unique. By contrast, the eigenspaces with eigenvalues \( \lambda_\tilde{v} = \lambda_\tilde{w} = p \pm q\bar{s} \) are 5-dimensional. Interestingly, though, for any given eigenvector such as \( \hat{u}_\pm \), there is again an essentially unique eigenvector, in this case \( \hat{v}_\pm \), which is orthogonal to it. Here “essentially unique” means unique up to a real multiplicative factor, and orthogonality can be defined either as \( v^\dagger w = 0 \) or as \( (vv^\dagger)w = 0 \); these turn out to be equivalent in this case.

There are also additional eigenvectors with eigenvalues of the form

\[ \lambda = (p + \rho) - \beta k\ell \]  

where \( \beta, \rho \in \mathbb{R} \). These must satisfy

\[ 32\beta^2 = \left(\sqrt{32\rho^2 - 7q^2} - 5q\right) \left(11q - \sqrt{32\rho^2 - 7q^2}\right) \]  

and therefore only exist provided that

\[ q^2 \leq \rho^2 \leq 4q^2 \]  

In the case of equality, we recover the real eigenvalues given in [2]. For each admissible \( \rho \) and each of the two corresponding choices for \( \beta \), the eigenspace is 3-dimensional. We have not explored the properties of these eigenvectors in depth.

All the eigenvalues discussed above have the form (50). While we suspect that there are no others, we have not been able to prove that this is the case.
b) Example 2

A related example is given by the matrix

$$\hat{B} = \begin{pmatrix} p & qi & \frac{q}{2}ks \\ -qi & p & \frac{q}{2}j \\ -\frac{q}{2}ks & -\frac{q}{2}j & p \end{pmatrix}$$

(53)

with $s$ again given by (46). We choose $\theta$ such that

$$s = \frac{\sqrt{3}}{3} - \frac{2}{3}k\ell$$

(54)

resulting in

$$\hat{B} = \begin{pmatrix} p & qi & \frac{q}{6}(\sqrt{5}k + 2\ell) \\ -qi & p & \frac{q}{2}j \\ -\frac{q}{6}(\sqrt{5}k + 2\ell) & -\frac{q}{2}j & p \end{pmatrix}$$

(55)

The 2 families of real eigenvalues of $\hat{B}$ turn out to be \{ $p \pm q, p \mp \frac{q}{2}(1 + \sqrt{3}), p \mp \frac{q}{2}(1 - \frac{\sqrt{3}}{2})$ \}. Some eigenvectors for $\hat{B}$ corresponding to eigenvalues which are not real are

$$\lambda_{u_1} = (p + \frac{\sqrt{5}}{2}q) - \frac{q}{2}k\ell : u_1 = \begin{pmatrix} 3k \\ \sqrt{5}j - 2i\ell \\ 1 + \sqrt{5}k\ell \end{pmatrix}$$

$$\lambda_{u_2} = (p + \frac{\sqrt{5}}{2}q) + \frac{q}{2}k\ell : u_2 = \begin{pmatrix} \sqrt{5}k + 2\ell \\ 3j \\ \sqrt{3} - k\ell \end{pmatrix}$$

$$\lambda_{v_1} = (p - \frac{\sqrt{5}}{3}q) + \frac{2q}{3}k\ell : v_1 = \begin{pmatrix} \sqrt{5}j - 2i\ell \\ 3k \\ 0 \end{pmatrix}$$

$$\lambda_{v_2} = (p - \frac{\sqrt{5}}{3}q) - \frac{2q}{3}k\ell : v_2 = \begin{pmatrix} 3j \\ \sqrt{5}k + 2\ell \\ 0 \end{pmatrix}$$

$$\lambda_{w_1} = (p - \frac{\sqrt{5}}{6}q) - \frac{q}{6}k\ell : w_1 = \begin{pmatrix} 3k \\ \sqrt{5}j - 2i\ell \\ -7 - \sqrt{5}k\ell \end{pmatrix}$$

$$\lambda_{w_2} = (p - \frac{\sqrt{5}}{6}q) + \frac{q}{6}k\ell : w_2 = \begin{pmatrix} \sqrt{5}k + 2\ell \\ 3j \\ -3\sqrt{3} - 3k\ell \end{pmatrix}$$

(56)

However, we have been unable to find any decompositions of $\hat{B}$ involving these vectors. It is intriguing that, for instance, $v_1$ is orthogonal to both $u_1$ and $w_1$ (in the sense of (22)), but that $u_1$ and $w_1$ are not orthogonal. In fact, we have shown using Mathematica that there is no eigenvector triple containing $w_1$ which is orthogonal in the sense of (22). Unless $w_1$ is special in some as yet to be determined sense, we are forced to conclude that neither (22) nor (21) are generally true for eigenvectors whose eigenvalues are not real. It is curious,
however, that the sum of the squares (outer products) of all six of these (normalized) vectors is indeed (twice) the identity! We will consider possible implications of this fact below.

c) Example 3

In all of the examples considered so far, the eigenvalues have been in the complex subalgebra of \( \mathbb{C} \) determined by the associator \([a, b, c]\) (with \( a, b, c \) as in (19)). We now give an example for which this is not the case.

Consider
\[
C = \begin{pmatrix}
p & iq & -q(j - i\ell - j\ell) \\
-iq & p & q(1 + k + l) \\
q(j - i\ell - j\ell) & -q(1 - k - l) & p \\
\end{pmatrix}
\]  

which admits an eigenvector
\[
v = \begin{pmatrix}
j \\
l \\
0 \\
\end{pmatrix}
\]  

with eigenvalue
\[
\lambda_v = p + q lk
\]

However, the associator takes the form
\[
[a, b, c]_{q^3} = [i, (j - i\ell - j\ell), (1 + k + l)] = 2(l - k)
\]

5. DISCUSSION

As pointed out in [1], the orthonormality relation (22) is equivalent to assuming that
\[
v v^\dagger + \ldots + w w^\dagger = I
\]
If we define a matrix \( U \) whose columns are just \( v, ..., w \), then this statement is equivalent to
\[
UU^\dagger = I
\]
Furthermore, the eigenvalue equation (24) can now be rewritten in the form
\[
AU = UD
\]
where \( D \) is a diagonal matrix whose entries are the eigenvalues. Decompositions of the form (49) now take the form
\[
A = (UD) U^\dagger
\]
and multiplication of (63) on the right by \( U^\dagger \) shows that
\[
(AU) U^\dagger = (UD) U^\dagger = A = A(UU^\dagger)
\]
Thus, just as in [1], decompositions of the form (21) in similar language, which leads us to suspect that (49) is more fundamental.

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\(^7\) Much of this discussion also appeared in [4].
This is somewhat less surprising when one realizes that
\[ (v\lambda) v^\dagger v = (v\lambda) \] (67)
for any (normalized) vector \(v\) and octonion \(\lambda\), due to an associator identity discussed in the Appendix of [4]. One can therefore construct matrices with arbitrary octonionic (right) eigenvalues, although such matrices will not be Hermitian. We conjecture, however, that the correct notion of orthogonality is
\[ ((v\lambda) v^\dagger) w = 0 \] (68)
not (22), to which it of course reduces if the eigenvalues are real, in which case it might be possible to decompose a matrix into non-Hermitian pieces of the form (67), and yet have the sum of these pieces be Hermitian. This is precisely what happens in the 2 \(\times\) 2 case [4], as well as in Example 1. In any case, it is intriguing that this notion of orthogonality can be written as
\[ ((A v) v^\dagger) w = 0 \] (69)
which explicitly involves \(A\).

Putting these ideas together, it would be natural to conjecture that all eigenvectors of a 3 \(\times\) 3 octonionic Hermitian matrix come in families of 3, which form a decomposition in the sense that (66) is satisfied, and which are orthogonal in the sense of (68). However, this conjecture appears to fail for Example 2, as our claim that there is no orthonormal (in the sense (22)) eigenvector triple containing \(w_1\) rules out a decomposition (involving \(w_1\)) in either the form (21) or the form (49). The reason for this is that the eigenvectors of \(\hat{B}\) do not depend on the parameters \(p\) and \(q\), which in turn means that the \(p\) and \(q\) parts of the decompositions can be treated separately. If a decomposition of either form were to exist, the terms involving \(p\) would then imply (62), contradicting our claim. (This is the reason that we did not reduce to the case \(p = 0, q = 1\).)

Nonetheless, for all known decompositions (62) does hold, and could be used to further simplify (66). One possible generalization would be for (62) to fail, but for a decomposition along the lines of (66) to hold. However, even (66) appears to fail for Example 2.

There is, however, another intriguing possibility. Example 2 suggests that the eigenvectors of 3 \(\times\) 3 octonionic Hermitian matrices may come in sets of 6 (or more), rather than in sets of 3. This would fit nicely with our recent result with Okubo [16] that, for real eigenvalues, it takes all 6 eigenvectors in order to decompose an arbitrary vector into a linear combination of eigenvectors, despite the fact that only 3 eigenvectors are needed to decompose the original matrix. Further evidence for this point of view is provided by the fact that the most general eigenvectors for the given eigenvalues of \(\hat{B}\) have at most 2 free parameters, rather than the 4 degrees of freedom shown in [1] to exist for real eigenvalues, or the 8 for complex matrices.

We therefore conjecture that, for any 3 \(\times\) 3 octonionic Hermitian matrix, (66) should hold when suitably rewritten for a set consisting of \(n\) eigenvectors, where \(n\) presumably divides 24, the number of (real) independent eigenvectors with real eigenvalues. However, we have so far been unable to write Example 2 in such a form. Whether or in what form orthogonality would hold in such a context is an interesting open question.
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