LOW MACH NUMBER LIMIT FOR A MODEL OF ACCRETION DISK

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Abstract. We study an hydrodynamical model describing the motion of thick astrophysical disks relying on compressible Navier-Stokes-Fourier-Poisson system. We also suppose that the medium is electrically charged and we include energy exchanges through radiative transfer. Supposing that the system is rotating, we study the singular limit of the system when the Mach number, the Alfvén number and Froude number go to zero and we prove convergence to a 3D incompressible MHD system with radiation with two stationary linear transport equations for transport of radiation intensity.

1. Introduction. Our motivation in this work is the study of the equations describing objects called “accretion disk” which are quasi planar structures observed in various places in the universe. From a naive point of view, if a massive object attracts matter distributed around it through Newtonian gravitation in presence of angular momentum, the matter is not accreted isotropically around the central object but forms a disk around it. As the three main ingredients claimed by astrophysicists for explaining the existence of such objects are: gravitation, angular momentum and viscosity (see [24] [28] [29] for detailed presentations), a reasonable framework for their study seems to be a viscous self-gravitating rotating fluid.

In previous works we derived thin disks models [10] [11] corresponding to limit domains $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ for $\varepsilon \to 0$. In the present one we consider a thick model where $\varepsilon$ is no more small and is replaced by 1 in the sequel.

The mathematical model we consider is basically the compressible heat conducting MHD system [6] describing the motion of a viscous charged fluid confined to the thick disk $\Omega = \omega \times (0, 1)$, where $\omega \in \mathbb{R}^2$ is a 2-D domain, moreover as we suppose a global rotation of the system, some new terms appear due to the change of frame and we also suppose that the fluid exchanges energy with radiation through radiative transfers (see [6] [9]).

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More precisely, the non-dimensional system of equations giving the evolution of the mass density \( \varrho = \varrho(t, x) \), the velocity field \( \vec{u} = \vec{u}(t, x) \), the (divergence free) magnetic field \( \vec{B} = \vec{B}(x, t) \), and the radiative intensity \( I = I(x, t, \vec{\omega}, \nu) \) as functions of the time \( t \in (0, T) \), the spatial coordinate \( x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3 \), and (for \( I \)) the angular and frequency variables \((\vec{\omega}, \nu) \in S^2 \times \mathbb{R}_+\), reads as follows

\[
\begin{align*}
\partial_t \varrho + \text{div}_x (\varrho \vec{u}) &= 0 \quad \text{in } (0, T) \times \Omega, \\
\partial_t (\varrho \vec{u}) + \text{div}_x (\varrho \vec{u} \otimes \vec{u}) + \nabla_x p + \varrho \vec{\chi} \times \vec{u} &= \text{div}_x \mathbb{S} + \varrho \nabla \Psi + \varrho \nabla_x |\vec{\chi} \times \vec{x}|^2 + \vec{j} \times \vec{B} \quad \text{in } (0, T) \times \Omega, \\
\partial_t (\varrho \vec{E}) + \text{div}_x (\varrho \vec{E} \vec{u}) + \text{div}_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - \rho \text{div}_x \vec{u} + \vec{j} \cdot \vec{E} - S_E \quad \text{in } (0, T) \times \Omega, \\
\frac{1}{c} \partial_t I + \vec{\omega} \cdot \nabla_x I &= S \quad \text{in } (0, T) \times \Omega \times (0, \infty) \times S^2, \\
\partial_t \vec{B} + \text{curl}_x (\vec{B} \times \vec{u}) + \text{curl}_x (\lambda \text{curl}_x \vec{B}) &= 0 \quad \text{in } (0, T) \times \Omega, \\
- \Delta \Psi &= 4\pi G(\eta \varrho + g) \quad \text{in } (0, T) \times \Omega.
\end{align*}
\]

In the electromagnetic source terms, electric current \( \vec{j} \) and electric field \( \vec{E} \) are interconnected by \textit{Ohm’s law}

\[
\vec{j} = \sigma (\vec{E} + \vec{u} \times \vec{B}),
\]

and \textit{Ampère’s law}

\[
\zeta \vec{j} = \text{curl}_x \vec{B},
\]

where \( \zeta > 0 \) is the (constant) magnetic permeability.

In (6) \( \Psi \) is the gravitational potential and the corresponding source term in (2) is the Newton force \( \varrho \nabla \Psi \). \( G \) is the Newton constant and \( g \) is a given function, modeling an external gravitational effect. Supposing that \( \varrho \) is extended by 0 outside \( \Omega \) we have

\[
\Psi(t, x) = G \int_{\Omega} K(x - y)(\eta \varrho(t, y) + g(y)) \, dy,
\]

where \( K(x) = \frac{1}{|x|^2} \), and the parameter \( \eta \) may take the values 0 or 1: for \( \eta = 1 \) self-gravitation is present and for \( \eta = 0 \) gravitation only acts as an external field (some astrophysicists consider self-gravitation of accretion disks as small compared to the external attraction by a given massive central object modeled by \( g \) [29]).

We also assume that the system is globally rotating at uniform velocity \( \vec{\chi} \) around the vertical direction \( \vec{e}_3 \) and we note \( \vec{\chi} = \chi \vec{e}_3 \). Then Coriolis acceleration term \( \varrho \vec{\chi} \times \vec{u} \) appears in the system, together with the centrifugal force term \( \rho \nabla_x |\vec{\chi} \times \vec{x}|^2 \) (see [3]).

In (5) \( \lambda = \lambda(\vartheta) > 0 \) is the magnetic diffusivity of the fluid.

Observe that we consider here the simplified model studied in [12] where radiation does not appear in the momentum equation. Only the source \( S_E \) appears in the energy equation

\[
S_E(t, x) = \int_{S^2} \int_0^\infty S(t, x, \vec{\omega}, \nu) \, d\vec{\omega} \, d\nu.
\]

The symbol \( p = p(\varrho, \vartheta) \) denotes the thermodynamic pressure and \( e = e(\varrho, \vartheta) \) is the specific internal energy, interrelated through Maxwell’s relation

\[
\frac{\partial e}{\partial \varrho} = \frac{1}{\vartheta^2} \left( p(\varrho, \vartheta) - \vartheta \frac{\partial p}{\partial \vartheta} \right).
\]

Furthermore, \( \mathbb{S} \) is the viscous stress tensor determined by

\[
\mathbb{S} = \mu \left( \nabla_x \vec{u} + \nabla_x^T \vec{u} - \frac{2}{3} \text{div}_x \vec{u} \right) + \eta \, \text{div}_x \vec{u} \, \mathbb{I},
\]
where the shear viscosity coefficient $\mu = \mu(\vartheta) > 0$ and the bulk viscosity coefficient $\eta = \eta(\vartheta) \geq 0$ are effective functions of the temperature. Similarly, $\bar{q}$ is the heat flux given by Fourier's law

$$\bar{q} = -\kappa \nabla_x \vartheta,$$

with the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$. Finally,

$$S = S_{a,c} + S_s,$$

where

$$S_{a,c} = \sigma_a \left( B(\nu, \vartheta) - \bar{I} \right), \quad S_s = \sigma_s \left( \bar{I} - \bar{I} \right).$$

In this formula $\bar{I} := \frac{1}{4\pi} \int_{\mathbb{S}^2} I(\cdot, \omega) \, d\omega$ and $B(\nu, \vartheta) = 2h \nu^3 c^{-2} \left( e^{\frac{h\nu}{kT}} - 1 \right)^{-1}$ is the radiative equilibrium function where $h$ and $k$ are the Planck and Boltzmann constants, $\sigma_a = \sigma_a(\nu, \vartheta) \geq 0$ is the absorption coefficient and $\sigma_s = \sigma_s(\nu, \vartheta) \geq 0$ is the scattering coefficient. More restrictions on these structural properties of constitutive quantities will be imposed in Section 2 below.

System (1 - 6) is supplemented with the boundary conditions:

$$\bar{u}|_{\partial \Omega} = 0, \quad \bar{q} \cdot \bar{n}|_{\partial \Omega} = 0, \quad \bar{B} \cdot \bar{n}|_{\partial \Omega} = 0, \quad \bar{E} \times \bar{n}|_{\partial \Omega} = 0, \quad I(t, x, \nu, \bar{\omega}) = 0 \quad \text{for} \quad x \in \partial \Omega, \quad \bar{\omega} \cdot \bar{n} \leq 0,$$

where $\bar{n}$ denotes the outer normal vector to $\partial \Omega$.

Let us mention that there are already existing works in this field but not in the case of rotating fluid with radiation. We can mention some of existing works. First one was done by Kukučka [19] when Mach and Alfven number go to zero in the case of bounded domain. In [27] Novotný and his collaborators investigated the problem in the case of strong stratification. See also work of Trivisa et al. [21], work of Wang et al.[15], or works of Jiang et al.[17, 18, 16].

**Remark 1.**

- Let us mention that we consider a simplified model without radiative momentum as explained in our reference [12] quoted in the text. In fact one can observe that the radiative momentum is of order $c^{-1}$ with respect to radiative energy $S_E$, which is very small (as $c$ is the velocity of light) and consequently can be neglected in the model.
- The realistic physical domain of our thick disk is indeed a free boundary region. As we consider a well-formed disk we can suppose that the domain is fixed and the no slip boundary condition reflects an heuristic simulation of the exterior vacuum.

The paper is organized as follows.

In Section 2, we list the principal hypotheses imposed on constitutive relations, introduce the concept of weak solution to problem (1 - 13), and state the existence result for our model. In Section 3 we compute the formal asymptotics of the problem characterized by an infra relativistic matter velocity $C \gg 1$, a low Mach number $Ma << 1$ a sound speed small with respect to the velocity of light (accretion disk regime) $Ma \cdot C \gg 1$, a small Froude number $Fr \ll 1$ and a small Alfven number $Al << 1$.

Uniform bounds imposed on weak solutions by the data are derived in Section 4. The convergence Theorem is proved in Section 5. We conclude the paper with an Appendix. In the Appendix A we perform a slightly different scaling than those one of Section 3, while existence of a solution for the target system is briefly given in the Appendix B.
2. **Hypotheses and stability result.** We consider the pressure in the form
\[
p(\varrho, \vartheta) = \varrho^{3/2} P \left( \frac{\varrho}{\varrho^{3/2}} \right) + \frac{a}{3} \varrho^4, \quad a > 0,
\]
where \( P : [0, \infty) \to [0, \infty) \) is a given function with the following properties:
\[
P \in C^1[0, \infty), \quad P(0) = 0, \quad P'(Z) > 0 \text{ for all } Z \geq 0,
\]
\[
0 < \frac{\frac{5}{3} P(Z) - P'(Z) Z}{Z} < c \text{ for all } Z \geq 0,
\]
\[
\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0.
\]
After Maxwell’s equation (7), the specific internal energy \( e \) is
\[
e(\varrho, \vartheta) = \frac{3}{2} \varrho \left( \varrho^{3/2} \right) P \left( \frac{\varrho}{\varrho^{3/2}} \right) + a \varrho^4,
\]
and the associated specific entropy reads
\[
s(\varrho, \vartheta) = M \left( \frac{\varrho}{\varrho^{3/2}} \right) + \frac{4a}{3} \varrho^3,
\]
with
\[
M'(Z) = -\frac{3}{2} \frac{\frac{5}{3} P(Z) - P'(Z) Z}{Z^2} < 0.
\]
A new feature of the present paper (see below) will be the explicit introduction of the entropy for the photon gas.

The transport coefficients \( \mu, \eta, \kappa \) and \( \lambda \) are continuously differentiable functions of the absolute temperature such that
\[
0 < c_1(1 + \vartheta) \leq \mu(\vartheta), \quad \mu'(\vartheta) < c_2, \quad 0 \leq \eta(\vartheta) \leq c(1 + \vartheta),
\]
\[
0 < c_1(1 + \vartheta^3) \leq \kappa(\vartheta), \quad \lambda(\vartheta) \leq c_2(1 + \vartheta^3)
\]
for any \( \vartheta \geq 0 \). Moreover we assume that \( \sigma_a, \sigma_s, B \) are continuous functions of \( \nu, \vartheta \) such that
\[
0 \leq \sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta), |\partial_\nu \sigma_a(\nu, \vartheta)|, |\partial_\vartheta \sigma_s(\nu, \vartheta)| \leq c_1,
\]
\[
0 \leq \sigma_a(\nu, \vartheta) B(\nu, \vartheta), |\partial_\nu |\sigma_a(\nu, \vartheta) B(\nu, \vartheta)|| \leq c_2,
\]
\[
\sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta), \sigma_a(\nu, \vartheta) B(\nu, \vartheta) \leq h(\nu), \quad h \in L^1(0, \infty).
\]
for all \( \nu \geq 0, \vartheta \geq 0 \), where \( c_{1,2,3} \) are positive constants.

**Remark 2.** We consider a hot gas and the \( \vartheta^4 \) contribution is the classical Stefan-Boltzmann term. However we wish also to keep track of non equilibrium phenomena described by the radiative transfer equation. This model has also been described in our previous work [6].

Let us recall some definitions introduced in [7].

- In the weak formulation of the Navier-Stokes-Fourier system the equation of continuity (1) is replaced by its (weak) renormalized version [5] represented by the family of integral identities
  \[
  \int_0^T \int_\Omega \left( \varrho + b(\varrho) \right) \partial_t \varphi + \left( \varrho + b(\varrho) \right) \vec{u} \cdot \nabla_x \varphi + \left( b(\varrho) - b'(\varrho) \varrho \right) \text{div}_x \vec{u} \varphi \right) \, dx \, dt
  = - \int_\Omega \left( \varrho_0 + b(\varrho_0) \right) \varphi(0, \cdot) \, dx
  \]
  \[\text{(25)}\]
satisfied for any \( \varphi \in C_c^\infty([0,T) \times \Omega) \), and any \( b \in C^\infty[0,\infty) \), \( b' \in C_c^\infty[0,\infty) \), where (25) implicitly includes the initial condition \( \varrho(0,\cdot) = \varrho_0 \).

- Similarly, the momentum equation (2) is replaced by

\[
\int_0^T \int_\Omega \left( (\varrho \vec{u}) \cdot \partial_t \varphi + (\varrho \vec{u} \otimes \vec{u}) : \nabla \vec{u} \varphi + p \, \text{div}_x \varphi + (\varrho \nabla \times \vec{u}) \cdot \varphi \right) \, dx \, dt \\
= \int_0^T \int_\Omega \left( \mathbb{S} : \nabla \varphi - \varrho \nabla \Psi \cdot \varphi - (\vec{j} \times \vec{B}) \cdot \varphi - \varrho \nabla_x |\vec{x}|^2 \cdot \varphi \right) \, dx \, dt \\
- \int_\Omega (\varrho \vec{u})_0 \cdot \varphi(0,\cdot) \, dx
\]

for any \( \varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3) \). As usual, for (26) to make sense, the field \( \vec{u} \) must belong to a certain Sobolev space with respect to the spatial variable we require

\[
\vec{u} \in L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)),
\]

where (27) already includes the no-slip boundary condition (12).

- The magnetic equation (5) is replaced by

\[
\int_0^T \int_\Omega \left( \vec{B} \cdot \partial_t \varphi - (\vec{B} \times \vec{u} + \lambda \text{curl}_x \vec{B}) \cdot \text{curl}_x \varphi \right) \, dx \, dt + \int_\Omega \vec{B}_0 \cdot \varphi(0,\cdot) \, dx = 0,
\]

to be satisfied for any vector field \( \varphi \in \mathcal{D}([0,T) \times \mathbb{R}^3) \).

Here, according the boundary conditions, one has to take

\[
\vec{B}_0 \in L^2(\Omega), \quad \text{div}_x \vec{B}_0 = 0 \text{ in } \mathcal{D}'(\Omega), \quad \vec{B}_0 \cdot \vec{n}|_{\partial \Omega} = 0.
\]

Following Theorem 1.4 in [32], \( \vec{B}_0 \) belongs to the closure of all solenoidal functions from \( \mathcal{D}(\Omega) \) with respect to the \( L^2 \)-norm.

Anticipating (see (41) below) we see that

\[
\vec{B} \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)), \quad \text{curl}_x \vec{B} \in L^2(0,T; L^2(\Omega; \mathbb{R}^3))
\]

and we deduce from (28) that

\[
\text{div}_x \vec{B}(t) = 0 \text{ in } \mathcal{D}'(\Omega), \quad \vec{B}(t) \cdot \vec{n}|_{\partial \Omega} = 0 \text{ for a.a. } t \in (0,T).
\]

In particular, using Theorem 6.1 in [13], we conclude

\[
\vec{B} \in L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)), \quad \text{div}_x \vec{B}(t) = 0, \quad \vec{B} \cdot \vec{n}|_{\partial \Omega} = 0 \text{ for a.a. } t \in (0,T).
\]

- From (2) and (3) we have the energy conservation law

\[
\partial_t \left( \frac{1}{2} \varrho |\vec{u}|^2 + \varrho e + \frac{1}{2\kappa} |\vec{B}|^2 \right) + \text{div}_x \left( \left( \frac{1}{2} \varrho |\vec{u}|^2 + \varrho e + p \right) \vec{u} + \vec{E} \times \vec{B} - \mathbb{S} \vec{u} + \vec{q} \right) \\
= \varrho \nabla_x \Psi \cdot \vec{u} + \varrho \nabla_x |\vec{x}|^2 \cdot \vec{u} - \mathbf{S}_E.
\]

Let us rearrange the right hand side.

As the gravitational potential \( \Psi \) is determined by equation (6) considered on the whole space \( \mathbb{R}^3 \), the density \( \varrho \) being extended to be zero outside \( \Omega \) we observe that

\[
\int_\Omega \varrho \nabla_x \Psi \cdot \vec{u} \, dx = \frac{\kappa}{2} \int_\Omega \varrho \Psi \, dx.
\]

In the same stroke \( \int_\Omega \varrho \nabla_x |\vec{x}|^2 \cdot \vec{u} \, dx = \frac{\kappa}{4} \int_\Omega \varrho |\vec{x}|^2 \, dx \). Denoting now by \( E^R \) the radiative energy given by

\[
E^R(t,x) = \frac{1}{c} \int_{S^2} \int_0^\infty I(t,x,\vec{\omega},\nu) \, d\vec{\omega} \, d\nu,
\]

(32)
and integrating the radiative transfer equation (4), we get
\[ \partial_t \int_\Omega E^R \, dx + \int_0^T \int_{\partial \Omega \times \mathbb{R}^2} \sum_{\omega \neq 0} \int_0^\infty \omega \cdot \hat{n} I(t, x, \omega, \nu) \, d\nu \, d\omega \, dS_x \, dt = \int_\Omega S_E \, dx. \]

Using boundary conditions, we deduce the identity
\[ \frac{d}{dt} \left( \frac{1}{2} \varrho |\vec{B}|^2 + \varrho c_\Perp \vec{B} \cdot \vec{E} \right) - \frac{1}{2} \varrho \Psi - \varrho \vec{E} \times \vec{B}^2 + E^R \right) \, dx 
+ \int_{\partial \Omega \times \mathbb{R}^2} \sum_{\omega \neq 0} \int_0^\infty \omega \cdot \hat{n} I(t, x, \omega, \nu) \, d\nu \, d\omega \, dS_x = 0. \tag{33} \]

- Finally, dividing (3) by \( \vartheta \) and using Maxwell’s relation (7), we obtain the entropy equation
\[ \partial_t (\varrho s) + \text{div}_x (\varrho s \vec{u}) + \text{div}_x \left( \frac{\vec{q}}{\vartheta} \right) = \varsigma, \tag{34} \]
where
\[ \varsigma = \frac{1}{\vartheta} \left( \mathcal{S} : \nabla_x \vec{u} - \frac{\vec{q}}{\vartheta} + \frac{1}{\kappa} \text{curl}_x \vec{B} \right) - \frac{S_E}{\vartheta}. \tag{35} \]

where the first term \( \varsigma_m := \frac{1}{\vartheta} \left( \mathcal{S} : \nabla_x \vec{u} - \frac{\vec{q}}{\vartheta} + \frac{1}{\kappa} \text{curl}_x \vec{B} \right) \) is the (positive) electromagnetic matter entropy production.

In order to identify the second term in (35), let us recall [1] the formula for the entropy of a photon gas
\[ s^R = -\frac{2k}{c^2} \int_0^\infty \int_{\mathbb{R}^2} \nu^2 [n \log n - (n + 1) \log (n + 1)] \, d\omega d\nu, \tag{36} \]
where \( n = n(I) = \frac{e^{\nu I}}{2\hbar^2} \) is the occupation number. Defining the radiative entropy flux
\[ \vec{q}^R = -\frac{2k}{c^2} \int_0^\infty \int_{\mathbb{R}^2} \nu^2 [n \log n - (n + 1) \log (n + 1)] \vec{\omega} \, d\omega d\nu, \tag{37} \]
and using the radiative transfer equation, we get the equation
\[ \partial_t s^R + \text{div}_x \vec{q}^R = -\frac{k}{\hbar} \int_0^\infty \int_{\mathbb{R}^2} \frac{1}{\nu} \log \frac{n}{n + 1} S \, d\omega d\nu =: \varsigma^R. \tag{38} \]

Checking the identity \( \log \frac{n(I)}{n(I + \delta)} = \frac{\nu}{\kappa} I \) with \( B = B(\vartheta, \nu) \) the Planck’s function, and using the definition of \( S \), the right-hand side of (38) rewrites
\[ \varsigma^R = \frac{S_E}{\vartheta} - \frac{k}{\hbar} \int_0^\infty \int_{\mathbb{R}^2} \frac{1}{\nu} \left[ \log \frac{n(I)}{n(I) + 1} - \log \frac{n(B)}{n(B) + 1} \right] \sigma_a (B - I) \, d\omega d\nu 
- \frac{k}{\hbar} \int_0^\infty \int_{\mathbb{R}^2} \frac{1}{\nu} \left[ \log \frac{n(I)}{n(I) + 1} - \log \frac{n(I)}{n(I) + 1} \right] \sigma_a (I - I) \, d\omega d\nu, \]
where we used the hypothesis that the transport coefficients \( \sigma_{a,s} \) do not depend on \( \vec{\omega} \). So we obtain finally
\[ \partial_t (\varrho s + s^R) + \text{div}_x (\varrho s \vec{u} + \vec{q}^R) + \text{div}_x \left( \frac{\vec{q}}{\vartheta} \right) = \varsigma + \varsigma^R \tag{39} \]
and equation (34) is replaced in the weak formulation by the inequality
\[ \int_0^T \int_\Omega \left( (\varrho s + s^R) \partial_t \varphi + \varrho s \vec{u} \cdot \nabla_x \varphi + \left( \frac{\vec{q}}{\vartheta} + \vec{q}^R \right) \cdot \nabla_x \varphi \right) \, dx \, dt \tag{40} \]
Concerning the transport equation (4), it can be extended to the whole physical space \( \mathbb{R}^3 \) defined on the whole physical space \( \Omega \times S^2 \). We say that \( \varphi \), \( \bar{u} \), \( \bar{B} \), \( I \) are weak solutions of problem (1 - 6) if

\[
\begin{align*}
&\int_{\Omega} \left( (1 + \frac{1}{2}) \nabla \cdot \bar{u} + \frac{1}{2} |\bar{B}|^2 - \frac{1}{2} \varphi \nabla \cdot \bar{B} + \nabla \cdot \bar{B} \right) \varphi \, dx \\
&- \frac{k}{\nu} \int_0^T \int_{\Omega} \int_0^\infty \int_{S^2} \frac{1}{\nu} \left[ \log \frac{n(I)}{n(I) + 1} - \log \frac{n(B)}{n(B) + 1} \right] \sigma_a (B - I) \, d\bar{\omega} d\nu \\
&+ \int_0^T \int_0^\infty \int_{S^2} \frac{1}{\nu} \left[ \log \frac{n(I)}{n(I) + 1} - \log \frac{n(\bar{I})}{n(\bar{I}) + 1} \right] \sigma_s (\bar{I} - I) \, d\bar{\omega} d\nu
\end{align*}
\]

for any \( \varphi \in C^\infty_0 ([0, T) \times \Omega) \), \( \varphi \geq 0 \), where the sign of all the terms in the right hand side may be controlled.

- Since replacing equation (3) by inequality (40) would result in a formally under-determined problem, system (25), (26), (40) must be supplemented with the total energy balance

\[
\int_{\Omega} \left( \frac{1}{2} \varphi |\bar{u}|^2 + \rho \varphi (\varphi, \vartheta) + \frac{1}{2} \nabla \cdot \bar{B} + \frac{1}{2} \varrho |\bar{B}|^2 - \frac{1}{2} \varrho \nabla \cdot \bar{B} \right) \varphi \, dx
\]

where \( E_0^R \) is given by

\[
E_0^R(x) = \frac{1}{c} \int_{S^2} \int_0^\infty I(0, x, \bar{\omega}, \nu) \, d\bar{\omega} \, d\nu.
\]

Concerning the transport equation (4), it can be extended to the whole physical space \( \mathbb{R}^3 \) provided we set \( \sigma_a (x, \nu, \vartheta) = 1_\Omega \sigma_a (\nu, \vartheta) \) and \( \sigma_s (x, \nu, \vartheta) = 1_\Omega \sigma_s (\nu, \vartheta) \) and take the initial distribution \( I_0 (x, \bar{\omega}, \nu) \) to be zero for \( x \in \mathbb{R}^3 \setminus \Omega \). Accordingly, for any fixed \( \bar{\omega} \in S^2 \), equation (4) can be viewed as a linear transport equation defined in \( (0, T) \times \mathbb{R}^3 \), with a right-hand side \( S \). With the above mentioned convention, extending \( \bar{u} \) to be zero outside \( \Omega \), we may therefore assume that both \( \varrho \) and \( I \) are defined on the whole physical space \( \mathbb{R}^3 \).

**Definition 2.1.** We say that \( \varrho, \bar{u}, \vartheta, \bar{B}, I \) is a weak solution of problem (1 - 6) if

\[
\begin{align*}
\varrho &\in L^\infty_\text{loc} (0, T; L^{5/3} (\Omega)), \quad \vartheta \in L^\infty (0, T; L^4 (\Omega)) \\
\bar{u} &\in L^2 (0, T; W^{1,2}_0 (\Omega; \mathbb{R}^3)) \\
\vartheta &\in L^2 (0, T; W^{1,2} (\Omega)), \quad \vartheta \in L^\infty (0, T; L^4 (\Omega)) \\
\bar{B} &\in L^2 (0, T; W^{1,2}_0 (\Omega; \mathbb{R}^3)), \\
I &\in L^\infty (0, T) \times \Omega \times S^2 \times (0, \infty), \quad I \in L^\infty (0, T; L^1 (\Omega \times S^2 \times (0, \infty)),
\end{align*}
\]

and if \( \varrho, \bar{u}, \vartheta, \bar{B}, I \) satisfy the integral identities (25), (26), (40), (28), (41), together with the transport equation (4).

The stability result of [9] reads now
Theorem 2.2. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions $\rho, \varepsilon, s$ satisfy hypotheses (14 - 19), and that the transport coefficients $\mu, \lambda, \kappa, \sigma_a, \sigma_s$ comply with (20 - 24), $g \in L^r(\Omega), r > 1$.

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon\}_{\varepsilon > 0}$ be a family of weak solutions to problem (1 - 13) in the sense of Definition 2.1 such that

$$
\varrho_\varepsilon(0, \cdot, \cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \text{ in } L^{5/3}(\Omega),
$$

$$
\int_{\Omega} \left( \frac{1}{2} \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon \varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) + \frac{1}{2} \varrho_\varepsilon \vartheta_\varepsilon \right) \, dx \leq E_0,
$$

$$
\int_{\Omega} \left( (\varrho_\varepsilon)_{0,\varepsilon} + (\varrho_\varepsilon)_{0,\varepsilon} + \varrho_\varepsilon \vartheta_\varepsilon \right) \, dx \leq E_0,
$$

$$
\int_{\Omega} \left( \varrho_\varepsilon \varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) + s^R(I_\varepsilon) \right)(0, \cdot, \cdot) \, dx \equiv \int_{\Omega} (s + s^R(0, \cdot, \cdot)) \, dx \geq S_0,
$$

and

$$
0 \leq I_\varepsilon(0, \cdot, \cdot) \equiv I_{0,\varepsilon}(\cdot) \leq I_0, \ |I_{0,\varepsilon}(\cdot, \nu)| \leq h(\nu) \text{ for a certain } h \in L^1(0, \infty).
$$

Then

$$
\varrho_\varepsilon \to \varrho \text{ in } C_{\text{weak}}([0, T]; L^{5/3}(\Omega)),
$$

$$
\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3)),
$$

$$
\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)),
$$

$$
\vec{B}_\varepsilon \to \vec{B} \text{ weakly in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3)),
$$

and

$$
I_\varepsilon \to I \text{ weakly-*(*) in } L^\infty((0, T) \times \Omega \times S^2 \times (0, \infty)),
$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, \vec{B}, I\}$ is a weak solution of problem (1 - 6).

3. Formal scaling analysis. In order to identify the appropriate limit regime we perform a general scaling, denoting by $L_{\text{ref}}, T_{\text{ref}}, U_{\text{ref}}, P_{\text{ref}}, \vartheta_{\text{ref}}, \varrho_{\text{ref}}, \varepsilon_{\text{ref}}, \mu_{\text{ref}}, \lambda_{\text{ref}}, \kappa_{\text{ref}}, \sigma_a_{\text{ref}}, \sigma_s_{\text{ref}}$ the reference hydrodynamical quantities (length, time, velocity, density, temperature, pressure, energy, viscosity, conductivity), by $I_{\text{ref}}, \varrho_{\text{ref}}, \sigma_a_{\text{ref}}, \sigma_s_{\text{ref}}$, the reference radiative quantities (radiative intensity, frequency, absorption and scattering coefficients), by $\chi_{\text{ref}}$, the reference rotation velocity, and by $\zeta_{\text{ref}}$, $B_{\text{ref}}$ the reference electrodynamic quantities (permeability and magnetic induction).

We also assume the compatibility conditions $p_{\text{ref}} = \rho_{\text{ref}} e_{\text{ref}}, \nu_{\text{ref}} = \frac{k_B \vartheta_{\text{ref}}}{\hbar}$,

$$
I_{\text{ref}} = \frac{2 \hbar e_{\text{ref}}}{c^2 \varepsilon_{\text{ref}}}, \quad \lambda = \frac{\chi_{\text{ref}}}{\sqrt{p_{\text{ref}} U_{\text{ref}}}}, \quad \text{and we denote by } S r := \frac{L_{\text{ref}}}{T_{\text{ref}} U_{\text{ref}}}, \quad M a := \frac{U_{\text{ref}}}{\sqrt{p_{\text{ref}} / \rho_{\text{ref}}}},
$$

$$
R e := \frac{U_{\text{ref}}}{\rho_{\text{ref}} P_{\text{ref}} L_{\text{ref}} / \mu_{\text{ref}}}, \quad P e := \frac{U_{\text{ref}}}{\sqrt{\rho_{\text{ref}} p_{\text{ref}} L_{\text{ref}}^2}}, \quad F r := \frac{U_{\text{ref}}}{\sqrt{\rho_{\text{ref}} p_{\text{ref}} L_{\text{ref}}}}, \quad C := \frac{c}{U_{\text{ref}}} \text{, the Strouhal, Mach, Reynolds, Péclet, Froude and "infra relativistic" dimensionless numbers corresponding to hydrodynamics, by } R o := \frac{U_{\text{ref}}}{\chi_{\text{ref}} L_{\text{ref}}} \text{ the Rossby number, by } A l := \frac{U_{\text{ref}}}{\chi_{\text{ref}} L_{\text{ref}}} \text{ the Alfvén number and by } L := L_{\text{ref}} \sigma_a_{\text{ref}}, \quad L_s := \frac{\sigma_s_{\text{ref}}}{\sigma_a_{\text{ref}}}, \quad P := \frac{2 k_B \hbar c s}{\rho_{\text{ref}} p_{\text{ref}} L_{\text{ref}}} \text{, various dimensionless numbers corresponding to radiation.}
$$

Using these scalings and using carets to symbolize renormalized variables we get

$$
S = \frac{I_{\text{ref}}}{L_{\text{ref}}} \hat{S},
$$
where
\[
\hat{S} = \mathcal{L}\hat{\sigma}_\alpha \left( B(\hat{\nu}, \hat{\vartheta}) - \hat{I} \right) + \mathcal{L}\mathcal{L}_s \hat{\sigma}_s \left( \frac{1}{4\pi} \int_{S^2} \hat{I}(\cdot, \hat{\vartheta}) \, d\hat{\vartheta} - \hat{I} \right).
\]

Omitting the carets in the following, we get first the scaled equation for \( I \), in the region \((0, T) \times \Omega \times (0, \infty) \times S^2\)
\[
\frac{Sr}{C} \partial_t I + \vec{\varpi} \cdot \nabla_x I = s = \mathcal{L}\sigma_a (B - I) + \mathcal{L}\mathcal{L}_s \sigma_s \left( \frac{1}{4\pi} \int_{S^2} I(\cdot) \, d\vartheta - I \right),
\]
where we used the same notation \( B \) for the dimensionless Planck function
\[
B(\nu, \vartheta) = \nu \frac{3}{e^\nu \vartheta - 1}.
\]

Denoting also by \( E^R = \int_{S^2} \int_0^\infty I \, d\nu \, d\vartheta \), the (renormalized) radiative energy, by \( \vec{F}^R = \int_{S^2} \int_0^\infty \vec{\varpi} I \, d\nu \, d\vartheta \), the renormalized radiative momentum, by \( s_E = \int_{S^2} \int_0^\infty S \, d\nu \, d\vartheta \), the renormalized radiative energy source, by \( s^R = -\int_{S^2} \nu^2 \left[ n \log n - (n+1) \log(n+1) \right] \, d\vartheta \, d\varphi \), the renormalized radiative entropy with \( n = n(I) = \frac{\nu^3}{e^\nu - 1} \), by \( q^E = -\int_{S^2} \nu^2 \left[ n \log n - (n+1) \log(n+1) \right] \vec{\varpi} \, d\vartheta \, d\varphi \), the renormalized radiative entropy flux, and taking the first moment of (44) with respect to \( \hat{\vartheta} \), we get first an equation for \( E^R \)
\[
\frac{1}{C} \partial_t E^R + \nabla_x \vec{F}^R = s_E.
\]

The continuity equation is now
\[
Sr \partial_t \rho + \text{div}_x (\rho \vec{u}) = 0,
\]
and the momentum equation
\[
Sr \partial_t (\rho \vec{u}) + \text{div}_x (\rho \vec{u} \otimes \vec{u}) + \frac{Ma^2}{Re} \nabla_x p + \frac{1}{Ro} \rho \vec{v} \times \vec{u} = \frac{1}{Re} \nabla_p S + \frac{1}{Fr^2} \rho \nabla \Psi + \frac{1}{Ro} \rho \nabla_x |\vec{\chi} \times \vec{u}|^2 + \frac{1}{Al^2} \vec{J} \times \vec{B},
\]

The balance of internal energy rewrites
\[
Sr \partial_t (\rho e + \mathcal{P} E^R) + \text{div}_x (\rho e \vec{u} + \mathcal{P} \vec{C} \vec{F}^R) + \frac{1}{Pe} \text{div}_x \vec{q} = \frac{Ma^2}{Re} S : \nabla_x \vec{u} - p \text{div}_x \vec{u} + \frac{Ma^2}{Al^2} \vec{J} \cdot \vec{E},
\]
and we get the balance of matter (fluid) entropy
\[
Sr \partial_t (\rho s) + \text{div}_x (\rho s \vec{u}) + \frac{1}{Pe} \text{div}_x \left( \frac{\vec{q}}{\vartheta} \right) = \varsigma,
\]
with
\[
\varsigma = \frac{1}{\vartheta} \left( \frac{Ma^2}{Re} S : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} + \frac{\lambda}{\varsigma} |\text{curl}_x \vec{B}|^2 \right) + \frac{S_E}{\vartheta},
\]
and the balance of radiative entropy
\[
\frac{Sr}{C} \partial_t s^R + \text{div}_x \vec{q}^R = \varsigma^R,
\]
The scaled equation for total energy gives finally the total energy balance
\[ \sigma_s(I - \bar{I}) \, d\omega d\nu + S_E \frac{\rho}{\sigma}. \]

The scaled equation for the electromagnetic field is
\[ Sr \partial_t \vec{B} + \text{curl}_x(\vec{B} \times \vec{u}) + \text{curl}_x(\lambda \text{curl}_x \vec{B}) = 0. \tag{50} \]

The scaled equation for total energy gives finally the total energy balance
\[ Sr \frac{d}{dt} \int_{\Omega} \left( \frac{Ma^2}{2} \vec{u} \cdot \vec{u} + \rho \vec{u} + \frac{1}{C} \rho \vec{E} + \frac{Ma^2}{A^2} |\vec{B}|^2 - 1 \frac{Ma^2}{Fr^2} \rho \Psi - \frac{Ma^2}{Ro} \rho |\vec{\chi} \times \vec{x}|^2 \right) \, dx \]
\[ + P \int_{\Gamma_o} \int_{\Gamma_o} \vec{\omega} \cdot \vec{n} I \, d\Gamma_o d\nu = 0. \tag{51} \]

In the sequel we analyze a simple asymptotic regime characterized by an infra relativistic matter velocity $C >> 1$, a low Mach number $Ma << 1$ a sound speed small with respect to the velocity of light (accretion disk regime) $Ma \cdot C >> 1$, a small Froude number $Fr << 1$ and a small Alfvén number $Al << 1$. So, given a small number $\varepsilon > 0$, we assume that the regime under study is defined by
\[ Ma = \varepsilon, \text{ Al = } \varepsilon, \text{ Fr = } \varepsilon^{1/2}, \text{ C = } \varepsilon^{-2}, \]
and we put
\[ Sr = 1, \text{ Pe = 1, Re = 1, Ro = 1, } P = 1, L = 1, L_s = 1, \]
in the previous system.

Plugging this scaling into the previous system gives
\[ \varepsilon^2 \partial_t I + \vec{\omega} \cdot \nabla_x I = \sigma_a (B - I) + \sigma_s \left( \frac{1}{4\pi} \int_{S^2} I \, d\omega - I \right), \tag{52} \]
\[ \partial_t \rho + \text{div}_x (\rho \vec{u}) = 0, \tag{53} \]
\[ \partial_t (\rho \vec{u}) + \text{div}_x (\rho \vec{u} \vec{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\rho, \vec{u}) + \rho \nabla_x \vec{u} = \text{div}_x \mathbb{S} + \frac{1}{\varepsilon} \rho \nabla \Psi + \rho \nabla_x |\vec{\chi} \times \vec{x}|^2 + \frac{1}{\varepsilon^2} \vec{j} \times \vec{B}, \tag{54} \]
\[ \partial_t (\rho \vec{e}) + \text{div}_x (\rho \vec{e} \vec{u}) + \text{div}_x \vec{q} = \varepsilon^2 \mathbb{S} : \nabla_x \vec{u} - p \text{div}_x \vec{u} + \vec{j} \cdot \vec{E} - s_E, \tag{55} \]
\[ \partial_t \left( \rho s + \varepsilon^2 s^2 \right) + \text{div}_x \left( \rho \vec{u} \vec{u} + \vec{q} \right) + \text{div}_x \left( \frac{\vec{q}}{\rho} \right) \geq \varsigma, \tag{56} \]
with
\[ \varsigma_{\varepsilon} = \frac{1}{\theta} \left( \varepsilon^2 \mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vec{q}}{\rho} + \frac{\lambda}{\varsigma} (\text{curl}_x \vec{B})^2 \right) \]
\[ + \int_{S^2} \frac{1}{\nu} \left( \frac{\rho}{\varepsilon} \frac{\rho}{n(I) + 1} - \frac{n(I)}{n(B) + 1} \right) \sigma_a (I - B) \, d\omega d\nu \]
\[ + \int_{S^2} \frac{1}{n(I) + 1} \left( \frac{\rho}{\varepsilon} \frac{\rho}{n(I) + 1} - \frac{n(I)}{n(B) + 1} \right) \sigma_s (I - \bar{I}) \, d\omega d\nu, \]
\[ \partial_t \vec{B} + \text{curl}_x (\vec{B} \times \vec{u}) + \text{curl}_x (\lambda \text{curl}_x \vec{B}) = 0, \tag{57} \]
and finally
\[ \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \varepsilon^2 \rho |\vec{u}|^2 + \rho \vec{e} + \varepsilon^2 E^R + \frac{1}{2\varsigma} |\vec{B}|^2 - \frac{1}{2} \varepsilon \rho \Psi - \rho |\vec{\chi} \times \vec{x}|^2 \right) \, dx \]
\[
+ \int_{0}^{\infty} \int_{\Gamma_{+}} \vec{\omega} \cdot \vec{n} I \, d\Gamma_{+} \, d\nu = 0
\]

(58)

where \( \Gamma_{+} = \{(x, \vec{\omega}) \in \partial \Omega \times \mathbb{S}^2 : \vec{\omega} \cdot \vec{n}_{x} > 0\} \).

In order to compute the limit system, we consider now the formal expansions

\[
(I, \varrho, \vec{u}, \vartheta, p, \vec{B}) = (I_0, \varrho_0, \vec{u}_0, \vartheta_0, p_0, \vec{B}_0) + \varepsilon(I_1, \varrho_1, \vec{u}_1, \vartheta_1, p_1, \vec{B}_1) + O(\varepsilon^2).
\]

(59)

- We first observe from (54) and using the arguments of [14] that we can choose \( \varrho_0 = Cte, \vartheta_0 = Cte, \) and \( \vec{B}_0 = 0 \) moreover

\[
\nabla_{x} p_1 = \varrho_0 \nabla_{x} \Psi(\varrho_0).
\]

(60)

From (53) we derive the incompressibility condition

\[
\text{div}_{x} \vec{u}_0 = 0,
\]

(61)

and

\[
\partial_{t} \vartheta_1 + \text{div}_{x} (\varrho_0 \vec{u}_1 + \varrho_1 \vec{u}_0) = 0.
\]

(62)

- From (52) we get now two stationary linear transport equations for the two moments \( I_0 \) and \( I_1 \)

\[
\vec{\omega} \cdot \nabla_{x} I_0 = \sigma_{a,0} (B_0 - I_0) + \sigma_{s,0} \left( \vec{I}_0 - I_0 \right),
\]

(63)

\[
\vec{\omega} \cdot \nabla_{x} I_1 = \sigma_{a,0} (\partial_{\theta} B_0 \vartheta_1 - I_1) + \partial_{\theta} \sigma_{s,0} (B_0 - I_0) \vartheta_1 + \partial_{\theta} \sigma_{s,0} \left( \vec{I}_0 - I_0 \right) \vartheta_1 + \sigma_{s,0} \left( \vec{I}_1 - I_1 \right),
\]

(64)

where \( \vec{I} := \frac{1}{4\pi} \int \nabla I \, d\vec{\omega}, \sigma_{a,0} = \sigma_{a}(\nu, \vartheta_0), \sigma_{s,0} = \sigma_{s}(\nu, \vartheta_0) \) and \( B_0 = B(\nu, \vartheta_0) \).

- The limit momentum equation is

\[
\varrho_0 (\partial_{t} \vec{u}_0 + \text{div}_{x}(\vec{u}_0 \otimes \vec{u}_0)) + \text{div}_{x} \Pi + \varrho_0 \vec{\chi} \times \vec{u}_0 = \text{div}_{x} \mathcal{S}(\vec{u}_0) + \frac{1}{\zeta} \text{curl}_{x} \vec{B}_1 \times \vec{B}_1 + \vec{F},
\]

(65)

where \( \mu_0 = \mu(\vartheta_0) \) and \( \Pi \) is an effective pressure.

In the right-hand side, the force term is

\[
\vec{F} = \varrho_1 \nabla_{x} \Psi(\varrho_0) + \varrho_0 \nabla_{x} |\vec{x} \times \vec{x}|^2 - \varrho_0 \vec{\chi} \times \vec{u}_0.
\]

We compute the first term by using (60) which gives \( \varrho_1 = \frac{\varrho_0}{\partial_{\rho} \psi(\varrho, \vec{x})} \Psi(\varrho_0) - \frac{\partial_{\rho} \psi(\varrho, \vec{x})}{\partial_{\psi} \psi(\varrho, \vec{x})} \vartheta_1 \).

Then

\[
\vec{F} = \frac{\vartheta_0 \nabla_{x} |\vec{x} \times \vec{x}|^2 - \varrho_0 \nabla_{x} \Psi(\varrho_0)}{\varrho_0} \vec{\chi} \times \vec{u}_0.
\]

- The limit magnetic field \( \vec{B}_1 \) solves

\[
\partial_{t} \vec{B}_1 + \text{curl}_{x}(\vec{B}_1 \times \vec{u}_0) + \text{curl}_{x}(\lambda_0 \text{curl}_{x} \vec{B}_1) = 0,
\]

(66)

for \( \lambda_0 = \lambda(\vartheta_0) \).

- At lowest order the energy equation gives

\[
\varrho_0 \partial_{\rho} e_0 D \vartheta_1 + (\varrho_0 \partial_{\rho} e_0 + e_0) D \vartheta_1 + (\varrho_0 e_0 + p_0) \vec{u}_1 - \text{div}_{x} (\kappa_0 \nabla_{x} \vartheta_1) = -S_{E1},
\]

(67)

where \( D \) is the transport operator \( D := \partial_{t} + \vec{u}_0 \cdot \nabla_{x} \).

Observing that from (60) we have

\[
\partial_{\rho} p_0 D \vartheta_1 + \partial_{\rho} p_0 D \vartheta_1 + \varrho_0 \vec{u}_0 \cdot \nabla_{x} \Phi_0 = 0,
\]

(68)

where \( D := \partial_{t} + \vec{u}_0 \cdot \nabla_{x} \), and from (60)

\[
\varrho_0 \text{div}_{x} \vec{u}_1 = -D \vartheta_1,
\]

for \( \lambda_0 = \lambda(\vartheta_0) \).
and after (64)

\[
S_{E1} = \int_0^{\infty} \int_{S^2} \{ \partial_0 \sigma_{a,0} (B_0 - I_0) \vartheta_1 + \sigma_s (\partial_0 B_0 \vartheta_1 - I_1) \} \, d\varsigma \, d\nu,
\]
we end with

\[
\omega \tilde{c} \overline{\Pi} (\partial_t \vartheta_1 + \text{div}_x (\vartheta_1 \tilde{u}_0)) - \text{div}_x (\kappa_0 \nabla_x \vartheta_1) = G,
\]
where \( \overline{\Pi} = \partial_0 \epsilon_0 \), and \( G = -\frac{\sigma_0}{\sigma_0 \rho_0 \delta} \tilde{u}_0 \cdot \nabla_x \Psi (\rho_0) - \int_0^{\infty} \int_{S^2} \{ \partial_0 \sigma_{a,0} (B_0 - I_0) \vartheta_1 + \sigma_s (\partial_0 B_0 \vartheta_1 - I_1) \} \, d\varsigma \, d\nu.
\]
Putting

\[
\tilde{U} = \tilde{u}_0, \quad \Theta = \vartheta_1, \quad \varrho = \varrho_1, \quad \overline{B} = \overline{B}_0, \quad \overline{\varrho} = \varrho_0, \quad \overline{\vartheta} = \vartheta_0, \quad \overline{D} (\tilde{U}) = \frac{1}{2} (\nabla \tilde{u}_0 + \nabla^T \tilde{u}_0),
\]
and the sources

\[
\tilde{F} = \frac{\overline{\varrho}}{\rho_0} + \frac{\vert \overline{\varrho} \vert}{\rho_0} \chi \times \overline{x} - \frac{\overline{\varrho}}{\rho_0} \nabla_x \Psi (\overline{\varrho}) - \frac{\overline{\varrho}}{\rho_0} \chi \times \overline{x},
\]

\[
G = -\frac{\overline{\varrho}}{\rho_0} \nabla_x \Psi (\overline{\varrho}) : \partial_t \tilde{U} + \int_0^{\infty} \int_{S^2} \sigma_s I_1 \, d\varsigma \, d\nu - \int_0^{\infty} \int_{S^2} (\partial_0 \sigma_s (B - I_0) + \sigma_s B_0) \, d\varsigma \, d\nu,
\]

\[
H = \sigma_s (B - I_0) + \sigma_s (I_0 - I_0),
\]
we obtain the limit system in \((0, T) \times \Omega\)

\[
\text{div}_x \tilde{U} = 0,
\]

\[
\overline{\varrho} (\partial_t \tilde{U} + \text{div}_x (\tilde{U} \otimes \tilde{U})) + \nabla_x \Pi = \text{div}_x (2 \Pi \overline{D} (\tilde{U})) + \frac{1}{\zeta} \text{curl}_x \overline{B} \times \overline{B} + \widetilde{F}
\]

\[
\partial_t \tilde{B} + \text{curl}_x (\tilde{B} \times \tilde{U}) + \text{curl}_x (\overline{\varrho} \text{curl}_x \tilde{B}) = 0,
\]

\[
\text{div}_x \tilde{B} = 0,
\]

\[
\overline{\varrho} \overline{\varphi} (\vartheta_1 \Theta + \text{div}_x (\Theta \tilde{U})) - \text{div}_x (\pi \nabla \Theta) = G,
\]

\[
\overline{\varrho} \text{curl}_x \partial_t \Theta = \text{curl}_x (\Theta \tilde{U}) - \text{div}_x (\pi \nabla \Theta) = G,
\]

\[
\overline{\varrho} \cdot \nabla_x I_0 = H,
\]

\[
\overline{\varrho} \cdot \nabla_x I_1 = L,
\]


together with the Boussinesq relation (60)

\[
\partial_0 \rho \nabla_x \Theta + \partial_0 \rho \nabla_x r = \overline{\varrho} \nabla_x \Psi (\overline{\varrho}),
\]

where the (linear) sources \( \tilde{F} \) and \( G \) are given by (69) and (70).

We finally consider the boundary conditions

\[
\tilde{U} \vert_{\partial \Omega} = 0, \quad \nabla \Theta \cdot \tilde{n} \vert_{\partial \Omega} = 0, \quad \overline{B} \cdot \tilde{n} \vert_{\partial \Omega} = 0, \quad \text{curl}_x \overline{B} \times \tilde{n} \vert_{\partial \Omega} = 0
\]

for (73)-(77) and

\[
I_0 (x, \nu, \tilde{\varrho}) = 0 \text{ for } x \in \partial \Omega, \quad \tilde{\varrho} \cdot \tilde{n} \leq 0
\]

\[
I_1 (x, \nu, \tilde{\varrho}) = 0 \text{ for } x \in \partial \Omega, \quad \tilde{\varrho} \cdot \tilde{n} \leq 0
\]

for (78) and (79), and the initial conditions

\[
\tilde{U} \vert_{t=0} = \tilde{U}_0, \quad \varTheta \vert_{t=0} = \varTheta_0, \quad \tilde{B} \vert_{t=0} = \tilde{B}_0, \quad I_0 \vert_{t=0} = I_{0,0}, \quad I_1 \vert_{t=0} = I_{1,0}
\]
Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.

For any $T > 0$ the initial-boundary value problem (73) - (84) has at least a weak solution $(\tilde{U}, \Theta, \tilde{B}, I_0, I_1)$ such that

1. $\tilde{U} \in L^\infty(0, T; \mathcal{H}(\Omega)) \cap L^2(0, T; \mathcal{V}(\Omega)),$
   $\tilde{B} \in L^\infty(0, T; \mathcal{V}(\Omega)) \cap L^2(0, T; \mathcal{W}(\Omega)),$
   with $\mathcal{H}(\Omega) = \{ \tilde{U} \in L^2(\Omega; \mathbb{R}^3), \text{div}_x \tilde{U} = 0 \text{ in } \Omega, \tilde{U}\big|_{\partial \Omega} = 0 \},$ $\mathcal{U}(\Omega) = \mathcal{H}(\Omega) \cap \mathcal{W}_0^{1,2}(\Omega; \mathbb{R}^3), \mathcal{V}(\Omega) = \{ \tilde{b} \in L^2(\Omega; \mathbb{R}^3) \text{div}_x \tilde{b} = 0, \tilde{b} \cdot \vec{n}\big|_{\partial \Omega} = 0 \}$ and $\mathcal{W}(\Omega) =$ $\mathcal{V}(\Omega) \cap \mathcal{W}_0^{1,2}(\Omega; \mathbb{R}^3),$

2. $\Theta \in V_2^{1,1/2}((0, T) \times \Omega),$
   where $V_2^{1,1/2}$ is the energy space defined in [22] p.6,

3. $I_0, I_1 \in L^\infty((0, T) \times \Omega) \times S^2 \times \mathbb{R}_+,$
   with $\omega \cdot \nabla_x I_0, \omega \cdot \nabla_x I_1 \in L^p((0, T) \times \Omega) \times S^2 \times \mathbb{R}_+,$
   for any $p > 1.$

In the following we introduce the convergence result from the primitive system (1)-(13) to the incompressible limit (73)-(84).

4. **Global existence for the primitive system and uniform estimates.** Let us prepare initial data such that

$$
\begin{cases}
\varrho(0, \cdot) = \varrho_{0, \varepsilon} = \overline{\varrho} + \varepsilon \varrho_{0, \varepsilon}^{(1)}, \\
\vec{u}(0, \cdot) = \vec{u}_{0, \varepsilon}, \\
\vartheta(0, \cdot) = \vartheta_{0, \varepsilon} = \overline{\vartheta} + \varepsilon \vartheta_{0, \varepsilon}^{(1)}, \\
I(0, \cdot, \cdot) = I_{0, \varepsilon} = \overline{I} + \varepsilon I_{0, \varepsilon}^{(1)}, \\
\vec{B}(0, \cdot) = B_{0, \varepsilon} = \varepsilon \vec{B}_{0, \varepsilon}^{(1)},
\end{cases}
$$

where $\overline{\varrho} > 0, \overline{\vartheta} > 0, \overline{I} > 0$ and $\int_{\Omega} \varrho_{0, \varepsilon}^{(1)} \, dx = 0$ for any $\varepsilon > 0.$

After [14], for any locally compact Hausdorff metric space $X$ we denote by $\mathcal{M}(X)$ the set of signed Borel measures on $X$ and by $\mathcal{M}^+(X)$ the cone of non-negative elements of $\mathcal{M}(X).

From Theorem 2.2 we get immediately (by combining the approximating schemes introduced in [7] and [6]) the existence of a weak solution $(\varrho_{\varepsilon, \cdot}, \vec{u}_{\varepsilon, \cdot}, \vartheta_{\varepsilon, \cdot}, I_{\varepsilon, \cdot}, \vec{B}_{\varepsilon})$ to the radiative MHD system (1 - 11).

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions $p, c, s$ satisfy hypotheses (14 - 19), and that the transport coefficients $\mu, \lambda, \kappa, \sigma_a, \sigma_s$ and the equilibrium function $\mathcal{E}$ comply with (20 - 24). $g \in L^r(\Omega), r > 1.$ Let the initial data $(\varrho_{0, \varepsilon}, \vec{u}_{0, \varepsilon}, \vartheta_{0, \varepsilon}, I_{0, \varepsilon}, \vec{B}_{0, \varepsilon})$ be given by (85), where $(\varrho_{0, \varepsilon}^{(1)}, \vec{u}_{0, \varepsilon}^{(1)}, \vartheta_{0, \varepsilon}^{(1)}, I_{0, \varepsilon}^{(1)}, \vec{B}_{0, \varepsilon}^{(1)})$ are bounded measurable functions.

Then for any $\varepsilon > 0$ small enough (in order to maintain positivity of $\varrho_{0, \varepsilon}^{(1)}$ and $\vartheta_{0, \varepsilon}^{(1)}$), there exists a weak solution $(\varrho_{\varepsilon, \cdot}, \vec{u}_{\varepsilon, \cdot}, \vartheta_{\varepsilon, \cdot}, I_{\varepsilon, \cdot}, \vec{B}_{\varepsilon})$ to the radiative Navier-Stokes
system \((1-11)\) for \((t,x,\omega,\nu) \in (0,T) \times \Omega \times S^2 \times \mathbb{R}_+,\) supplemented with the boundary conditions \((12-13)\) and the initial conditions \((85)\).

More precisely we have

\[
\int_0^T \int_\Omega \rho_\varepsilon b(\rho_\varepsilon) (\partial_t \phi + \bar{\varepsilon}_x \cdot \nabla_x \phi) \, dx \, dt = \int_0^T \int_\Omega \beta(\rho_\varepsilon) \text{div}_x u_\varepsilon \phi \, dx \, dt - \int_\Omega \rho_{0,\varepsilon} b(\rho_{0,\varepsilon}) \phi(0,\cdot) \, dx, \tag{86}
\]
for any \(\beta\) such that \(\beta \in L^\infty \cap C[0,\infty),\) \(b(\rho) = b(1) + \int_1^\rho \frac{\beta(z)}{z^2} \, dz\) and any \(\phi \in C_c^\infty((0,T) \times \Omega),\)

\[
\int_0^T \int_\Omega \left( \frac{\varepsilon^2}{2} \rho_{0,\varepsilon} |\bar{u}_{0,\varepsilon}|^2 + \rho_{0,\varepsilon} e_{\varepsilon} + \varepsilon^2 E_{R} + \frac{1}{2\zeta} |\bar{B}_{0,\varepsilon}|^2 - \frac{1}{2} \varepsilon \rho_{0,\varepsilon} \Psi - \rho_{0,\varepsilon} |x| \phi \right) \, dx \, dt
\]

for any \(\phi \in C_c^\infty((0,T) \times \Omega; \mathbb{R}^3)\) with \(p_{\varepsilon} = p(\rho_{\varepsilon}, \vartheta_{\varepsilon}), S_{\varepsilon} = S(\bar{u}_{\varepsilon}, \vartheta_{\varepsilon}),\) and \(j_{\varepsilon} = \frac{1}{\varepsilon} \text{curl}_x \bar{B}_{\varepsilon},\)

\[
\int_0^T \int_\Omega \left( \frac{\varepsilon^2}{2} \rho_{0,\varepsilon} |\bar{u}_{0,\varepsilon}|^2 + \rho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon^2 E_{R} + \frac{1}{2\zeta} |\bar{B}_{0,\varepsilon}|^2 - \frac{1}{2} \rho_{0,\varepsilon} \Psi - \rho_{0,\varepsilon} |x| \phi \right) \, dx \, dt \tag{88}
\]

for a.a. \(t \in (0,T)\) with \(\Gamma_+ = \{(x,\omega) \in \partial \Omega \times S^2 : \omega \cdot n_\varepsilon \geq 0\}\) and with \(e_{\varepsilon} = e(\rho_{\varepsilon}, \vartheta_{\varepsilon}), \Psi = \Psi(\rho_{\varepsilon}), \Psi_{0,\varepsilon} = \Psi(\rho_{0,\varepsilon})\) and \(E_{R}(t,x) = \int_0^\infty \int_{S^2} I_\varepsilon(t, x, \omega, \nu) \, d\omega \, d\nu\)

\[
\int_0^T \int_\Omega \left( \bar{B}_{\varepsilon} \partial_t \varphi - (\bar{B}_{\varepsilon} \cdot \bar{u}_{\varepsilon} + \lambda_{\varepsilon} \text{curl}_x \bar{B}_{\varepsilon}) \cdot \text{curl}_x \varphi \right) \, dx \, dt + \int_\Omega \bar{B}_{0,\varepsilon} \varphi(0,\cdot) \, dx = 0, \tag{89}
\]
for any vector field \(\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^3),\) with \(\lambda_{\varepsilon} = \lambda(\vartheta_{\varepsilon}).\)

\[
\int_0^T \int_\Omega \left( \rho_{\varepsilon} s_{\varepsilon} + \varepsilon^2 s_{\varepsilon}^R \right) \partial_t \varphi + \left( \rho_{\varepsilon} s_{\varepsilon} \bar{u}_{\varepsilon} + q_{\varepsilon}^R \right) \cdot \nabla_x \varphi \, dx \, dt + \int_0^T \int_\Omega \frac{\varepsilon}{\vartheta_{\varepsilon}} \cdot \nabla_x \varphi \, dx \, dt + \int_0^T \int_\Omega \frac{\varepsilon}{\vartheta_{\varepsilon}} \cdot \nabla_x \varphi \, dx \, dt
\]

\[
+ \left( \zeta_{m}^R + \zeta_{m} \right) \varphi \right|_{\partial T}^{C(0,T) \times \Omega} = - \int_\Omega \left( \rho_{0,\varepsilon} s_{0,\varepsilon} + \varepsilon^2 s_{0,\varepsilon}^R \right) \varphi(0,\cdot) \, dx, \tag{90}
\]
where

\[
\zeta_{m}^R \geq \frac{1}{\vartheta_{\varepsilon}} \left( \varepsilon^2 S_{\varepsilon} : \nabla_x \bar{u}_{\varepsilon} - \frac{\vartheta_{\varepsilon}}{\vartheta_{\varepsilon}} \cdot \nabla_x \vartheta_{\varepsilon} + \frac{\lambda_{\varepsilon}}{\zeta} |\text{curl}_x \bar{B}_{\varepsilon}|^2 \right),
\]
and
\[ \zeta_e^R \geq \frac{k}{h} \int_0^\infty \int_{S^2} \frac{1}{\nu} \left[ \log \frac{n(I_e)}{n(I_e) + 1} - \log \frac{n(B_e)}{n(B_e) + 1} \right] \sigma_{ae}(B_e - I_e) \, d\tilde{\omega} d\nu \]
\[ + \int_0^\infty \int_{S^2} \frac{1}{\nu} \left[ \log \frac{n(I_e)}{n(I_e) + 1} - \log \frac{n(I_e)}{n(I_e) + 1} \right] \sigma_{se}(I_e - I_e) \, d\tilde{\omega} d\nu, \]
for any \( \varphi \in C_c^\infty((0,T) \times \Omega) \) with \( \zeta_e^m \in M^+(0,T) \times \Omega) \) and \( \zeta_e^R \in M^+(0,T) \times \Omega), \) and with \( \sigma_{ae} = \sigma_a(\nu, \vartheta_e), \sigma_{se} = \sigma_s(\nu, \vartheta_e), B_e = B(\nu, \vartheta_e), \quad \tilde{q}_e = \kappa(\vartheta_e), \vartheta_e \)
\[ \nabla x \vartheta_e, s_e = s(q_e, \vartheta_e), s_e^R = s^R(I_e), \quad \sigma_e^R = \sigma^R(t) \text{ and} \]
\[ I_e := \frac{1}{4\pi} \int_{S^2} I_e(t, x, \nu, \tilde{\omega}) d\tilde{\omega}, \]
\[ \int_0^T \int_\Omega_0 \int_{S^2} (\varepsilon^2 \partial_t \psi + \tilde{\omega} \cdot \nabla x \psi) I_e \, d\tilde{\omega} \, d\nu \, dx \, dt \]
\[ + \int_0^T \int_\Omega_0 \int_{S^2} \left[ \sigma_{ae}(B_e - I_e) + \sigma_{se}(I_e - I_e) \right] \psi \, d\tilde{\omega} \, d\nu \, dx \, dt, \quad (91) \]
\[ = \int_\Omega_0 \int_{S^2} \varepsilon^2 I_{0,\varepsilon} \psi(0, x, \vartheta_e) \, d\tilde{\omega} \, d\nu \, dx + \int_0^T \int_{0}^{T} \int_{S^2} \tilde{\omega} \cdot \nabla x I_e \psi \, d\Gamma \, d\nu \, dx \, dt, \]
for any \( \psi \in C_c^\infty((0,T) \times \Omega \times S^2 \times \mathbb{R}^+). \)

4.1. Uniform estimates. We recall from [14] the necessary definitions in the formalism of essential and residual sets (see [12]).

Given three numbers \( \bar{\vartheta} \in \mathbb{R}^+, \tilde{\vartheta} \in \mathbb{R}^+ \) and \( E \in \mathbb{R}^+ \) we define \( O_{ess}^H \) the set of hydrodynamical essential values
\[ O_{ess}^H := \left\{ (\vartheta, \vartheta) \in \mathbb{R}^2 : \frac{\vartheta}{2} < \vartheta < 2\bar{\vartheta}, \frac{\tilde{\vartheta}}{2} < \vartheta < 2\tilde{\vartheta} \right\}, \quad (92) \]
and \( O_{ess}^R \) the set of radiative essential values
\[ O_{ess}^R := \left\{ E^R \in \mathbb{R} : \frac{E^R}{2} < E^R < 2E \right\}, \quad (93) \]
with \( O_{ess} := O_{ess}^H \cup O_{ess}^R \), and their residual counterparts
\[ O_{res}^H := (\mathbb{R}^+)^2 \setminus O_{ess}^H, \quad O_{res}^R := \mathbb{R}^+ \setminus O_{ess}^R, \quad O_{res} := (\mathbb{R}^+)^3 \setminus O_{ess}. \quad (94) \]
Let \( \{ \vartheta_e, \tilde{\vartheta}_e, \vartheta_e, I_e \}_{e>0} \) a family of solutions of the scaled radiative Navier-Stokes-Fourier system given in Theorem 4.1. We call \( \mathcal{M}_{ess}^e \subset (0,T) \times \Omega \) the set
\[ \mathcal{M}_{ess}^e = \left\{ (t, x) \in (0,T) \times \Omega : \left( \vartheta_e(t, x), \tilde{\vartheta}_e(t, x), E^R_e(t, x) \right) \in O_{ess} \right\}, \]
and \( \mathcal{M}_{res} = (0,T) \times \Omega \setminus \mathcal{M}_{ess}^e \) the corresponding residual set.

To any measurable function \( h \) we associate its decomposition into essential and residual parts
\[ h = [h]_{ess} + [h]_{res}, \]
where \( [h]_{ess} = h \cdot 1_{\mathcal{M}_{ess}^e} \) and \( [h]_{res} = h \cdot 1_{\mathcal{M}_{res}}. \)
Denoting by \( H_{\vartheta} \) the Helmholtz function for matter
\[ H_{\vartheta}(\vartheta, \vartheta) = \vartheta e - \tilde{\vartheta} \vartheta s, \]
and
\[ H_{\vartheta}^R(I) = E^R - \tilde{\vartheta} s^R, \]
the corresponding radiative function and using (90) we rewrite (88) as
\[
\int_{\Omega} \left( \frac{\varepsilon^2}{2} \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \mathcal{H}_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) + \varepsilon^2 \mathcal{H}_\varepsilon^R(\varrho_\varepsilon) + \frac{1}{2\varepsilon} |\vec{B}_\varepsilon|^2 - \frac{1}{2} \varepsilon \varrho_\varepsilon \varphi_\varepsilon - \varrho_\varepsilon |\vec{\chi} \times \vec{\chi}|^2 \right) \, dx
\]
\[+ \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_\varepsilon (t, x, \vec{\omega}, \nu) \, d\Gamma \, d\nu \, dt + \mathcal{F}(\varrho_\varepsilon^m + \varrho_\varepsilon^R) \left[ [0, t] \times \Omega \right]
\]
\[= \int_{\Omega} \left( \frac{\varepsilon^2}{2} \varrho_0,\varepsilon |\vec{u}_0,\varepsilon|^2 + \varrho_0,\varepsilon \varrho_0,\varepsilon + \varepsilon^2 \mathcal{E}_0,\varepsilon^R + \frac{1}{2\varepsilon} |\vec{B}_0,\varepsilon|^2 \right.
\]
\[\left. - \frac{1}{2} \varepsilon \varrho_0,\varepsilon \varphi_0,\varepsilon - \varrho_0,\varepsilon |\vec{\chi} \times \vec{\chi}|^2 - \varrho_0,\varepsilon \varrho_0,\varepsilon \right) \, dx.
\]

Observing that the total mass is a constant of motion \( M = \int_{\Omega} \varrho_\varepsilon \, dx = \varphi \Omega \) and using Hardy-Littlewood-Sobolev inequality, we get \( \int_{\Omega} \varrho_\varepsilon \varphi_\varepsilon \, dx \leq \frac{C}{\varepsilon^2} \mathcal{M}^{2/3} \| \varrho_\varepsilon \|_{L^{4/3}(\Omega)}^{4/3} \).

After (14) and (18) we have also \( \varrho_\varepsilon \varphi_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) \geq a\varrho_\varepsilon^4 + \frac{3\varrho_\varepsilon}{2} \varrho_\varepsilon^5 \), so we have the lower bound
\[
\int_{\Omega} \mathcal{H}_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{1}{2} \varepsilon \varrho_\varepsilon \varphi_\varepsilon \, dx \geq c \int_{\Omega} \mathcal{H}_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) \, dx,
\]
for \( \varepsilon \) small and a \( c(\varepsilon) < 1 \) and we deduce finally the energy-entropy inequality
\[
\int_{\Omega} \left( \frac{\varepsilon^2}{2} \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \mathcal{H}_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - \varphi \varrho_\varepsilon) \partial_\varrho \mathcal{H}_\varphi(\varphi, \varphi) - \mathcal{H}_\varphi(\varphi, \varphi) + \frac{1}{2\varepsilon} |\vec{B}_\varepsilon|^2 + \varepsilon^2 \mathcal{H}_\varepsilon^R(\varrho_\varepsilon) \right) \, dx
\]
\[
+ \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_\varepsilon (t, x, \vec{\omega}, \nu) \, d\Gamma \, d\nu \, dt + \mathcal{F}(\varrho_\varepsilon^m + \varrho_\varepsilon^R) \left[ [0, t] \times \Omega \right]
\]
\[\leq C \int_{\Omega} \left( \frac{\varepsilon^2}{2} \varrho_0,\varepsilon |\vec{u}_0,\varepsilon|^2 + \left( \mathcal{H}_\varepsilon(\varrho_0,\varepsilon, \varrho_0,\varepsilon) - (\varrho_0,\varepsilon - \varphi \varrho_0,\varepsilon) \partial_\varrho \mathcal{H}_\varphi(\varphi, \varphi) - \mathcal{H}_\varphi(\varphi, \varphi) + \frac{1}{2\varepsilon} |\vec{B}_0,\varepsilon|^2 \right)
\]
\[+ \varepsilon^2 \mathcal{H}_\varepsilon^R(\varrho_0,\varepsilon, \varrho_0,\varepsilon) \right) \, dx.
\]
(95)

Now, after Lemma 4.1 in [12] (see [14]) we have the following properties for matter and radiative Helmholtz functions

**Lemma 4.2.** Let \( \varphi > 0 \) and \( \varphi > 0 \) two given constants and let
\[
\mathcal{H}_\varphi(\varphi, \varphi) = \varphi \varphi - \varphi \varphi \varphi,
\]
and
\[
\mathcal{H}_\varphi^R(\varphi) = \mathcal{E}_\varphi - \varphi \mathcal{E}_\varphi.
\]
Let \( \mathcal{O}_{ess} \) and \( \mathcal{O}_{res} \) be the sets of essential and residual values introduced in (92-94). There exist positive constants \( C_j = C_j(\varphi, \varphi) \) for \( j = 1, \cdots, 8 \) such that
1.
\[
C_1 \left( |\varphi - \varphi|^2 + |\varphi - \varphi|^2 \right) \leq \mathcal{H}_\varphi(\varphi, \varphi) - (\varphi - \varphi) \partial_\varphi \mathcal{H}_\varphi(\varphi, \varphi) - \mathcal{H}_\varphi(\varphi, \varphi)
\]
\[\leq C_2 \left( |\varphi - \varphi|^2 + |\varphi - \varphi|^2 \right),
\]
for all \( (\varphi, \varphi) \in \mathcal{O}_{ess}^H \),
2.
\[
\mathcal{H}_\varphi(\varphi, \varphi) - (\varphi - \varphi) \partial_\varphi \mathcal{H}_\varphi(\varphi, \varphi) - \mathcal{H}_\varphi(\varphi, \varphi)
\]
\[\geq \inf_{\varrho_\varepsilon, \varphi_\varepsilon \in \mathcal{O}_{res}} \left\{ \mathcal{H}_\varphi(\varphi, \varphi) - (\varphi - \varphi) \partial_\varphi \mathcal{H}_\varphi(\varphi, \varphi) - \mathcal{H}_\varphi(\varphi, \varphi) \right\} = C_3,
\]
for all \( (\varphi, \varphi) \in \mathcal{O}_{ess}^H \),
Suppose that the initial data satisfy Lemma 4.3. Using (95) and Lemma 4.2, we get the following energy estimates

\begin{align}
\| \vartheta_0 - \vartheta \|_{L^2(\Omega)}^2 &\leq C\varepsilon^2, \quad \| \vartheta_0 - \vartheta \|_{L^2(\Omega)}^2 \leq C\varepsilon^2, \quad \| E^{R} - E \|_{L^2(\Omega)}^2 \leq C\varepsilon^2, \\
\| \tilde{B}_{0, \varepsilon} \|_{L^2(\Omega; \mathbb{R}^3)} &\leq C\varepsilon^2,
\end{align}

and

\begin{align}
\| \sqrt{\vartheta_0 - \vartheta} \|_{L^2(\Omega; \mathbb{R}^3)} &\leq C,
\end{align}

the following estimates hold

\begin{align}
\text{ess sup}_{t \in (0,T)} |M_{\varepsilon} \vartheta_0 (t)| &\leq C\varepsilon^2, \\
\text{ess sup}_{t \in (0,T)} \| \vartheta_0 - \vartheta \|_{L^2(\Omega)}^2 &\leq C\varepsilon^2, \\
\text{ess sup}_{t \in (0,T)} \| \vartheta_0 - \vartheta \|_{L^2(\Omega)}^2 &\leq C\varepsilon^2, \\
\text{ess sup}_{t \in (0,T)} \| E^{R} - E \|_{L^2(\Omega)}^2 &\leq C\varepsilon, \\
\text{ess sup}_{t \in (0,T)} \| \vartheta_0 \|_{L^1(\Omega)} &\leq C\varepsilon^2, \\
\text{ess sup}_{t \in (0,T)} \| \vartheta_0 \|_{L^1(\Omega)} &\leq C\varepsilon, \\
\text{ess sup}_{t \in (0,T)} \| s^{R} \|_{L^1(\Omega)} &\leq C\varepsilon, \\
\left( \varepsilon_{m}^{R} + \varepsilon_{c}^{R} \right) [0, t] \times \bar{\Omega} &\leq C\varepsilon^2,
\end{align}

\begin{align}
\text{ess sup}_{t \in (0,T)} \| \tilde{B}_{0, \varepsilon} \|_{L^2(\Omega; \mathbb{R}^3)} &\leq C, \\
\text{ess sup}_{t \in (0,T)} \| \sqrt{\vartheta} \|_{L^2(\Omega; \mathbb{R}^3)} &\leq C,
\end{align}

\begin{align}
\text{ess sup}_{t \in (0,T)} \int_{\Omega} \left( \left| \vartheta_0 \right|^2 + |\vartheta_0|^4 \right) (t) \, dx &\leq C\varepsilon^2.
\end{align}
\[ \int_0^T \| \vec{u}_\varepsilon(t) \|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 \, dt \leq C; \]  
\[ \int_0^T \left\| \frac{\partial_x - \overline{\vartheta}}{\varepsilon}(t) \right\|_{W^{1,2}(\Omega)}^2 \, dt \leq C, \]  
\[ \int_0^T \left\| \log(\vartheta_x) - \log(\overline{\vartheta}) \right\|_{W^{1,2}(\Omega)}^2 \, dt \leq C, \]  
\[ \int_0^T \left\| \frac{\vec{B}_\varepsilon(t)}{\varepsilon} \right\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 \, dt \leq C. \]

**Proof.** Estimate (102) follow after (97). Bounds (103), (104) and (108) follow after (96) and (99). Bounds (106) and (107) follow after (98) Bounds (109) and (110) follow after (101). Bounds (110), (111) and (112) follow after energy inequality (95). Bound (113) follows after (106) and the expression (18) of \( e \).

From (110) we see that
\[ \int_0^T \| \nabla_x \vec{u}_\varepsilon + \nabla_x^t \vec{u}_\varepsilon - \frac{2}{3} \text{div}_x \vec{u}_\varepsilon \|_{L^2(\Omega; \mathbb{R}^3 \times \mathbb{R}^3)}^2 \, dt \leq C. \]  
From (102), (112) and (118) we get (114). From (110) we get
\[ \int_0^T \left\| \nabla_x \left( \frac{\vartheta_x}{\varepsilon} \right) \right\|_{L^2(\Omega)}^2 + \left\| \nabla_x \left( \frac{\log(\vartheta_x)}{\varepsilon} \right) \right\|_{L^2(\Omega)}^2 \, dt \leq C, \]
which, using (103) and (104) gives (115) and (116).

Finally after (110) one gets
\[ \left\| \frac{\text{curl}_x \vec{B}_\varepsilon}{\varepsilon} \right\|_{L^2(\Omega; \mathbb{R}^3)}^2 \leq C, \]
and (117) follows by using Theorem 6.1 in [13].

Our goal in the next Section will be to prove that the incompressible system (73)-(84) is the limit of the primitive system (86)-(91) in the following sense

**Theorem 4.4.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain of class \( C^{2,\nu} \). Assume that the thermodynamic functions \( p, e, s \) satisfy hypotheses (14 - 19) with \( P \in C^1[0, \infty) \cap C^2(0, \infty), g \in L^r(\Omega), r > 1 \), and that the transport coefficients \( \mu, \eta, \lambda, \sigma_a, \sigma_s \) and the equilibrium function \( B \) comply with (20 - 24).

Let \( (\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon) \) be a weak solution of the scaled system (1 - 11) for \( (t, x, \omega, \nu) \in [0, T] \times \Omega \times S^2 \times \mathbb{R}_+ \), supplemented with the boundary conditions (12 - 13) and initial conditions \( (\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \vec{B}_{0,\varepsilon}, I_{0,\varepsilon}) \) given by
\[ \varrho_\varepsilon(0, \cdot) = \overline{\varrho} + \varepsilon \varrho_{0,\varepsilon}, \quad \vec{u}_\varepsilon(0, \cdot) = \vec{u}_{0,\varepsilon}, \quad \vartheta_\varepsilon(0, \cdot) = \overline{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}, \]
\[ I_\varepsilon(0, \cdot) = \overline{I} + \varepsilon I_{0,\varepsilon}, \quad \vec{B}_\varepsilon(0, \cdot) = \varepsilon \vec{B}_{0,\varepsilon}, \]
where \( \overline{\varrho} > 0, \overline{\vartheta} > 0, \overline{I} > 0 \) are constants and
\[ \int_{\Omega} \varrho_{0,\varepsilon}^{(1)} \, dx = 0, \quad \int_{\Omega} \varrho_{0,\varepsilon}^{(1)} \, dx = 0, \quad \int_{\Omega} I_{0,\varepsilon}^{(1)} \, dx = 0, \quad \int_{\Omega} \vec{B}_{0,\varepsilon}^{(1)} \, dx = 0 \quad \text{for all} \ \varepsilon > 0. \]
Assume that
\[
\begin{align*}
\theta^{(1)}_0 &\rightarrow \theta^{(1)}_0 \text{ weakly } - (*) \text{ in } L^\infty(\Omega), \\
\tilde{v}^{(1)}_0 &\rightarrow \tilde{U}_0 \text{ weakly } - (*) \text{ in } L^\infty(\Omega; \mathbb{R}^3), \\
\varphi^{(1)}_0 &\rightarrow \varphi^{(1)}_0 \text{ weakly } - (*) \text{ in } L^\infty(\Omega), \\
I^{(1)}_0 &\rightarrow I^{(1)}_0 \text{ weakly } - (*) \text{ in } L^\infty(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+), \\
\vec{B}^{(1)}_0 &\rightarrow \vec{B}^{(1)}_0 \text{ weakly } - (*) \text{ in } L^\infty(\Omega; \mathbb{R}^3),
\end{align*}
\]
Then
\[
\text{ess sup}_{t \in (0, T)} \| \varrho_\varepsilon(t) - \bar{\varrho} \|_{L^{\frac{3}{2}}(\Omega)} \leq C\varepsilon, \quad (119)
\]
and up to subsequences
\[
\begin{align*}
\tilde{u}_\varepsilon \rightarrow \tilde{U} \text{ weakly } - (*) \text{ in } L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)), \\
\frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} &\rightarrow \varrho^{(1)} \text{ weakly } - (*) \text{ in } L^2(0, T; W^{1,2}(\Omega)), \\
I_\varepsilon &\rightarrow I_0 \text{ weakly } - (*) \text{ in } L^2(0, T; L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)), \\
\frac{\vec{B}_\varepsilon}{\varepsilon} &\rightarrow \vec{B} \text{ weakly } - (*) \text{ in } L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)),
\end{align*}
\]
and
\[
I_\varepsilon - \frac{T}{\varepsilon} = I^{(1)}_1 \text{ weakly } - (*) \text{ in } L^2(0, T; L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)), \quad (124)
\]
where \((\tilde{U}, \Theta, \vec{B}, I_0, I_1)\) solves the system (73)-(79).

5. Proof of Theorem 4.4. Let us first quote the following result of [12] (see [14])

**Proposition 1.** Let \(\{\varrho_\varepsilon\}_{\varepsilon > 0}, \{\vartheta_\varepsilon\}_{\varepsilon > 0}, \{I_\varepsilon\}_{\varepsilon > 0}\) be three sequences of non-negative measurable functions such that
\[
\begin{align*}
\left[\varrho^{(1)}_\varepsilon\right]_{\text{ess}} &\rightarrow \varrho^{(1)} \text{ weakly } - (*) \text{ in } L^\infty(0, T; L^2(\Omega)), \\
\left[\vartheta^{(1)}_\varepsilon\right]_{\text{ess}} &\rightarrow \vartheta^{(1)} \text{ weakly } - (*) \text{ in } L^\infty(0, T; L^2(\Omega)), \\
\left[I^{(1)}_\varepsilon\right]_{\text{ess}} &\rightarrow I^{(1)} \text{ weakly } - (*) \text{ in } L^\infty(0, T; L^2(\Omega)), \text{ a.e. in } \mathcal{S}^2 \times \mathbb{R}_+,
\end{align*}
\]
where
\[
\begin{align*}
\varrho^{(1)}_\varepsilon &= \frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon}, \quad \vartheta^{(1)}_\varepsilon = \frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon}, \quad I^{(1)}_\varepsilon = \frac{I_\varepsilon - T}{\varepsilon}.
\end{align*}
\]
Suppose that
\[
\text{ess sup}_{t \in (0, T)} |\mathcal{M}^{\varepsilon}_{\text{res}}(t)| \leq C\varepsilon^2. \quad (125)
\]

Let \(G, G^R \in C^1(\overline{O}_{\text{ess}})\) be given functions. Then
\[
\left[G(\varrho_\varepsilon, \vartheta_\varepsilon)\right]_{\text{ess}} - G(\bar{\varrho}, \bar{\vartheta}) \rightarrow \frac{\partial G(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \varrho^{(1)} + \frac{\partial G(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \vartheta^{(1)},
\]
weakly - (*) in \(L^\infty(0, T; L^2(\Omega))\), and if we note
\[
\left[G^R(I_\varepsilon)\right]_{\text{ess}} = \left[G^R(I_\varepsilon(\cdot, \cdot, \bar{\varrho}, \bar{\vartheta}))\right]_{\text{ess}} = G^R(I_\varepsilon) \cdot \mathbb{1}_{\text{ess}}, \text{ for a.a. } (\bar{\varrho}, \bar{\vartheta}) \in \mathcal{S}^2 \times \mathbb{R}_+,
\]
we have
\[
\left[G^R(I_\varepsilon)\right]_{\text{ess}} - G^R(T) \rightarrow \frac{\partial G(T)}{\partial T} I^{(1)},
\]
weakly - (*) in \(L^\infty(0, T; L^2(\Omega))\), a.e. in \(\mathcal{S}^2 \times \mathbb{R}_+\).
Moreover if $G, G^R \in C^2(\mathcal{O}_{ess})$ then
\[
\left\| \frac{[G(\varrho_e, \vartheta_e)]_{ess} - G(\varrho, \vartheta)}{\varepsilon} - \frac{\partial G(\varrho, \vartheta)}{\partial \varrho} \right\|_{L^\infty(0,T;L^1(\Omega))} \leq C\varepsilon,
\]
and
\[
\left\| \frac{[G^R(I_e)]_{ess} - G^R(\bar{T})}{\varepsilon} - \frac{\partial G(\bar{T})}{\partial I} \right\|_{L^\infty(0,T;L^1(\Omega))} \leq C\varepsilon,
\]
for a.a. $(\bar{\omega}, \nu) \in S^2 \times \mathbb{R}^+$. 

Clearly, this result provides us with the convergence properties (119-124).

To conclude the proof of Theorem 4.4, let us prove that the limit quantities $(\bar{U}, \Theta, \bar{B}, I_0, I_1)$ solve the target system (73)-(79).

As number of terms in the equations of our model are similar to those of the radiative Navier-Stokes-Fourier analyzed in [12] we only focus on the new contributions.

### 5.1. Continuity and momentum equations.

For the continuity equation, one expects that in the low Mach number limit, it reduces to the incompressibility constraint. In fact after Lemma 4.3 we know that $\int_0^T \| \bar{u}_e(t) \|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \, dt \leq C$ so passing to the limit after possible extraction of a subsequence, we deduce that
\[
\bar{u}_e \to \bar{U}, \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)).
\]

In the same stroke $\varrho_e \to \varrho$, weakly in $L^\infty(0,T;L^{3/2}(\Omega;\mathbb{R}^3))$. So we can pass to the limit in the weak continuity equation (86) which gives $\int_0^T \int_\Omega \bar{U} \cdot \nabla \varphi \, dx \, dt = 0$ for all $\varphi \in \mathcal{D}((0,T) \times \Omega)$, which rewrites
\[
\text{div}_x \bar{U} = 0, \quad \text{a.e. in } (0,T) \times \Omega, \quad \bar{U}\big|_{\partial\Omega} = 0,
\]
provided $\partial\Omega$ is regular.

For the momentum equation one knows that due to possible strong time oscillations of the gradient component of velocity, one has only $\varrho_e \bar{u}_e \otimes \bar{u}_e \to \varrho \bar{U} \otimes \bar{U}$ weakly in $L^2(0,T;L^{3/2}(\Omega;\mathbb{R}^3))$. However one can show after the analysis in [14] that one can pass to the limit in the convective term and obtain
\[
\int_0^T \int_\Omega \varrho \bar{U} \otimes \bar{U} : \nabla_x \varphi \, dx \, dt \to \int_0^T \int_\Omega \bar{U} \otimes \bar{U} : \nabla_x \varphi \, dx \, dt.
\]

Moreover after the hypotheses on the pressure law, the temperature $\vartheta_e$ is bounded in $L^\infty((0,T);L^4(\Omega)) \cap L^2(0,T;L^6(\Omega))$, which implies that $\mathcal{S}_e \to \mu(\bar{\vartheta}(\nabla_x \bar{U} + \nabla^T_x \bar{U}))$ weakly in $L^q(0,T;L^q(\Omega;\mathbb{R}^3))$ for a $q > 1$.

So taking a divergence free test vector field $\phi$ in (87), we have
\[
\int_0^T \int_\Omega (\varrho_e \bar{u}_e \cdot \partial_t \phi + \varrho_e \bar{u}_e \otimes \bar{u}_e : \nabla_x \phi + \varrho_e \chi \times \bar{u}_e \cdot \phi) \, dx \, dt
\]
\[
= \int_0^T \int_\Omega \left( \mathcal{S}_e : \nabla_x \phi - \frac{\varrho_e - \varrho}{\varepsilon} \nabla_x \psi_e \cdot \phi - \frac{1}{\zeta} \text{curl}_x \bar{B}_e \times \varphi - \frac{\varrho_e - \varrho}{\varepsilon} \chi \times \bar{x} \cdot \varphi \right) \, dx \, dt
\]
\[
- \int_\Omega \varrho_{0,e} \bar{u}_{0,e} \cdot \phi(0,\cdot) \, dx.
\]
Moreover, using (28) together with estimates (111), (117) and Lions-Aubin lemma we get

$$\frac{\vec{B} \varepsilon}{\varepsilon} \rightarrow \vec{B}$$ weakly in $L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3))$ and strongly in $L^2((0, T) \times \Omega; \mathbb{R}^3)$, (128)

$$\frac{1}{\zeta} \varepsilon \times \frac{\vec{B} \varepsilon}{\varepsilon} \rightarrow \frac{1}{\zeta} \varepsilon \times \vec{B}$$ weakly in $L^q((0, T) \times \Omega; \mathbb{R}^3)$,

for a certain $q > 1$.

Then passing to the limit and using (120)-(124), we get

$$\int_0^T \int_\Omega \left( \bar{\varepsilon} \cdot \partial_t \phi + \bar{\varepsilon} \otimes \vec{U} : \nabla_x \phi + \bar{\varepsilon} \times \vec{U} \cdot \phi \right) \, dx \, dt = 0,$$

which is the weak formulation of

$$\partial_t \rho(\bar{\varepsilon}) \nabla_x \phi(\bar{\varepsilon}) + \partial_\varphi (\bar{\varepsilon}) \nabla_x \varphi(\bar{\varepsilon}) - \bar{\varepsilon} \nabla_x \varphi(\bar{\varepsilon}) = 0.$$

### 5.2. Radiative transfer equation. Using the $L^\infty$ bound shown in the previous sections for $I_\varepsilon$, it is clear that $I_\varepsilon \rightarrow I_0$ weakly in $L^2((0, T) \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+)$, and we have also after Lemma 4.3 $\partial_{\varphi \varepsilon} \rightarrow \varphi$ weakly in $L^2(0, T; W^{1,2}(\Omega))$.

Using the cut-off hypotheses (22)/(24), we can pass to the limit which gives

$$\int_0^T \int_\Omega \left( \bar{\varepsilon} \cdot \nabla_x \psi I_0 \right) \, dx \, dt = 0,$$

using the same notation for any time-independent test function $\psi \in C_0^\infty(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)$, which is the weak formulation of the stationary problem

$$\bar{\varepsilon} \cdot \nabla_x I_0 = S_0,$$

with the boundary condition

$$I_0 = 0 \text{ on } \Gamma_+,$$

where $S_0 = \sigma_a(\nu, \bar{\varepsilon}) (B(\nu, \bar{\varepsilon}) - I_0) + \sigma_s(\nu, \bar{\varepsilon}) (I_0 - I_0)$. 

Now from (91)
\[
\int_0^T \int_\Omega \int_0^\infty \int_{S^2} (\varepsilon^2 \partial_t \psi + \tilde{\omega} \cdot \nabla_x \psi) \frac{I_\varepsilon - I_0}{\varepsilon} d\tilde{\omega} \, d\nu \, dx \, dt \\
+ \int_0^T \int_\Omega \int_0^\infty \int_{S^2} \left[ \frac{S_\varepsilon - S_0}{\varepsilon} \right] \psi \, d\tilde{\omega} \, d\nu \, dx \, dt,
\]
\[
= \int_\Omega \int_0^\infty \int_{S^2} \varepsilon^2 \frac{I_{0,\varepsilon} - I_0}{\varepsilon} \psi(0, x, \tilde{\omega}, \nu) \, d\tilde{\omega} \, d\nu \, dx \\
+ \int_0^T \int_\Gamma_+ \int_0^\infty \tilde{\omega} \cdot \tilde{n}_x \frac{I_\varepsilon - I_0}{\varepsilon} \psi \, d\Gamma \, d\nu \, dt,
\]
for any \( \psi \in C^\infty_\varepsilon([0, T] \times \Omega \times S^2 \times \mathbb{R}_+) \), with \( S_\varepsilon - S_0 = S(I_\varepsilon) - S(I_0) \). From Proposition 1, we get
\[
\frac{S_\varepsilon - S_0}{\varepsilon} \to S_1 := \partial_\theta (\sigma_a B)(\nu, \overline{\nu}) \theta^{(1)} - \partial_\theta \sigma_a (\nu, \overline{\nu}) \theta^{(1)} I_0 - \sigma_a (\nu, \overline{\nu}) \tilde{I}_1,
\]
\[
+ \partial_\sigma \sigma_a (\nu, \overline{\nu}) \theta^{(1)} \tilde{I}_1 + \sigma_s (\nu, \overline{\nu}) \tilde{I}_1 - \partial_\phi \sigma_s (\nu, \overline{\nu}) \theta^{(1)} I_0 - \sigma_s (\nu, \overline{\nu}) \tilde{I}_1,
\]
weakly in \( L^\infty((0, T); L^2(\Omega \times S^2 \times \mathbb{R}_+)) \) with \( I_1 := I^{(1)} \).

Passing to the limit we find the limit equation
\[
\int_\Omega \int_0^\infty \int_{S^2} \tilde{\omega} \cdot \nabla_x \psi \, I_1 \, d\tilde{\omega} \, d\nu \, dx + \int_\Omega \int_0^\infty \int_{S^2} S_1 \psi \, d\tilde{\omega} \, d\nu \, dx = \int_\Gamma_+ \int_0^\infty \tilde{\omega} \cdot \tilde{n}_x \, I_1 \psi \, d\Gamma \, d\nu \, dt,
\]
using the same notation for any time-independent test function \( \psi \in C^\infty_\varepsilon(\Omega \times S^2 \times \mathbb{R}_+) \) which is the weak formulation of the stationary problem
\[
\tilde{\omega} \cdot \nabla_x I_1 = S_1,
\]
with the boundary condition
\[
I_1 = 0 \quad \text{on} \quad \Gamma_+.
\]

5.3. Entropy balance. We rewrite equation (90) as
\[
\int_0^T \int_\Omega \left\{ \theta_\varepsilon \frac{s_\varepsilon - \overline{s}}{\varepsilon} (\partial_\varepsilon \varphi + \overline{\varepsilon} \cdot \nabla_x \varphi) + \frac{s_\varepsilon R - \overline{s} R}{\varepsilon^2} \partial_\varepsilon \varphi + \frac{\overline{q} R - \overline{q} R}{\varepsilon} \cdot \nabla_x \varphi \right\} \, dx \, dt
\]
\[
+ \int_0^T \int_\Omega \kappa \frac{\partial_\varepsilon}{\partial_\varepsilon} \nabla_x \left( \frac{\theta_\varepsilon}{\varepsilon} \right) \cdot \nabla_x \varphi \, dx \, dt
\]
\[
+ \frac{1}{\varepsilon} \left\{ \left( \psi_\varepsilon^m + \psi_\varepsilon^R \right) \right\}_{|\mathcal{M} : C|([0, T] \times \overline{\Omega})} = - \int_\Omega \left\{ \theta_\varepsilon \frac{s_0, \varepsilon - \overline{s}}{\varepsilon} + \varepsilon \frac{s_\varepsilon R - \overline{s} R}{\varepsilon} \right\} \varphi(0, \cdot) \, dx,
\]
Similarly to [14], using Proposition 1 and energy estimates, we see that
\[
\theta_\varepsilon \frac{s_\varepsilon - \overline{s}}{\varepsilon} \to \overline{\theta} \left( \partial_\theta s(\overline{\nu}, \overline{\nu}) \theta^{(1)} + \partial_\theta s(\overline{\nu}, \overline{\nu}) \theta^{(1)} \right),
\]
weakly * in \( L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \),
\[
\frac{\kappa(\partial_\varepsilon)}{\partial_\varepsilon} \nabla_x \left( \frac{\theta_\varepsilon}{\varepsilon} \right) \to \frac{\kappa(\overline{\theta})}{\overline{\theta}} \nabla_x \theta^{(1)},
\]
weakly * in \( L^2(0, T; L^2(\Omega; \mathbb{R}^3)) \) and
\[
\frac{1}{\varepsilon} \left( \psi_\varepsilon^m + \psi_\varepsilon^R \right) \to 0.
\]
Moreover
\[ \varrho_\varepsilon \frac{S_\varepsilon - \varepsilon}{\varepsilon} \cdot \vec{u}_\varepsilon \to \varrho \left( \partial_\nu s(\varpi, \vartheta) \vartheta^{(1)} + \partial_\sigma s(\varpi, \vartheta) \vartheta^{(1)} \right) \cdot \vec{U}, \]
weakly * in \( L^2(0, T; L^{3/2}(\Omega; \mathbb{R}^3)) \). Now applying Proposition 1 in the same stroke, we get
\[ \varepsilon^2 \frac{s_\varepsilon - s_R}{\varepsilon} \to 0, \]
weakly * in \( L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \).

Let us compute the limit of \( \frac{q^R - q^R_\varepsilon}{\varepsilon} \). We have
\[ q^R_\varepsilon = q^R(I_\varepsilon) = - \int_0^\infty \int_{S^2} \nu^2 \{ n_\varepsilon \log n_\varepsilon - (n_\varepsilon + 1) \log(n_\varepsilon + 1) \} \ d\varpi \ dv, \]
with \( n_\varepsilon = n(I_\varepsilon) = \frac{I_\varepsilon}{1 + \varepsilon} \).

Applying once more Proposition 1 with \( G^R(I) = n(I) \log n(I) - (n(I)+1) \log(n(I)+1) \) and integrating on \( S^2 \times \mathbb{R}_+ \), we find
\[ \frac{q^R - q^R_\varepsilon}{\varepsilon} \to \int_0^\infty \int_{S^2} \frac{1}{\nu} \log \left( \frac{n(T) + 1}{n(T)} \right) \varpi^2 I^{(1)}(1) \ d\varpi \ dv, \]
and as \( \frac{n(T)+1}{n(T)} = \frac{\nu}{s} \), we have
\[ \frac{q^R - q^R_\varepsilon}{\varepsilon} \to \frac{1}{\nu} \varpi^2 F^R(I^{(1)}), \]
with the radiative momentum \( \varpi^2 F^R(I^{(1)}) = \int_0^\infty \int_{S^2} \varpi^2 I^{(1)}(1) \ d\varpi \ dv \). So
\[ \int_0^T \int_{\Omega} \left( \frac{q^R - q^R_\varepsilon}{\varepsilon} \right) \cdot \nabla \varphi \ dx \ dt \to - \int_0^T \int_{\Omega} \frac{\text{div}_{x} \varpi F^R(I^{(1)})}{\nabla} \phi \ dx \ dt. \]

As we have from (134)
\[ \text{div}_{x} \varpi F^R = \int_0^\infty \int_{S^2} \left[ \partial_\sigma \sigma_a(\nu, \vartheta) \left( B(\nu, \vartheta) - I_0 \right) \vartheta^{(1)} + \sigma_a(\nu, \vartheta) \left( \partial_\sigma B(\nu, \vartheta) \vartheta^{(1)} - I_1 \right) \right] \ d\varpi \ dv, \]
the limit contribution in the right-hand side becomes
\[ - \int_0^T \int_{\Omega} \int_0^\infty \int_{S^2} \frac{1}{\nu} \left[ \partial_\sigma \sigma_a(\nu, \vartheta) \left( B(\nu, \vartheta) - I_0 \right) \vartheta^{(1)} + \sigma_a(\nu, \vartheta) \left( \partial_\sigma B(\nu, \vartheta) \vartheta^{(1)} - I_1 \right) \right] \ d\varpi \ dv \ dx \ dt, \]
Gathering all of these terms, we find the limit equation for entropy
\[ - \int_0^T \int_{\Omega} \left( \partial_\sigma s(\varpi, \vartheta) \vartheta^{(1)} + \partial_\sigma s(\varpi, \vartheta) \vartheta^{(1)} \right) \left( \partial_\xi \phi + \vec{U} \cdot \nabla \varphi \right) \ dx \ dt \]
\[ - \int_0^T \int_{\Omega} \int_{\Omega} \kappa(\varpi, \vartheta) \nabla x \vartheta^{(1)} \cdot \nabla \phi \ dx \ dt \]
\[ + \int_0^T \int_{\Omega} \int_0^\infty \int_{S^2} \frac{1}{\nu} \left[ \partial_\sigma \sigma_a(\nu, \vartheta) \left( B(\nu, \vartheta) - I_0 \right) \vartheta^{(1)} + \sigma_a(\nu, \vartheta) \left( \partial_\sigma B(\nu, \vartheta) \vartheta^{(1)} - I_1 \right) \right] \ d\varpi \ dv \ dx \ dt \]
\[ - \int_{\Omega} \left( \partial_\sigma s(\varpi, \vartheta) \vartheta^{(1)} + \partial_\sigma s(\varpi, \vartheta) \vartheta^{(1)} \right) \phi(0, \cdot) \ dx. \]
Using (130), it is routine to check that we finally obtain the thermal equation (77).

5.4. Maxwell equation. From (126) and (128) we get
\[ \frac{\vec{B}_\varepsilon}{\varepsilon} \times \vec{u} \rightarrow \vec{B} \times \vec{U} \text{ weakly in } L^q(0,T;L^q(\Omega,\mathbb{R}^3)) \text{ for } q > 1, \]
and
\[ \lambda \text{curl}_x \frac{\vec{B}_\varepsilon}{\varepsilon} \rightarrow \lambda \text{curl}_x \vec{B} \text{ weakly in } L^2(0,T,L^2(\Omega,\mathbb{R}^3)). \]

Then it is easy to pass to the limit in (89).

Appendix.

Appendix A. A different scaling. Here, for the worth of completeness we show how the asymptotic regime characterized by a small amount of radiation in the flow and a sound speed small with respect to the velocity of light (accretion disk regime), can be defined through a different scaling than the one performed in Section 3, namely for any \( \varepsilon > 0 \) we set
\[ \mathcal{P} = \varepsilon^2, M_a = \varepsilon, F_r = \varepsilon^{1/2}, \mathcal{C} = \varepsilon^{-2}, \]
and by \( S\dot{r} = 1, Pe = 1, Re = 1, Ro = 1, Al = 1, \mathcal{L} = \mathcal{L}_s = 1 \) in the previous system.

Plugging this scaling into the system (44)-(51) we get
\[ \varepsilon^2 \partial_t I + \vec{\omega} \cdot \nabla_x I = \sigma_a (B - I) + \sigma_s \left( \frac{1}{4\pi} \int_{S^2} I \, d\vec{\omega} - I \right), \]
(136)
\[ \partial_t \rho + \text{div}_x (\rho \vec{u}) = 0, \]
(137)
\[ \partial_t (\rho \varepsilon + \varepsilon^2 E^R) + \text{div}_x (\rho \varepsilon \vec{B} + \varepsilon^2 \vec{F}^R) + \text{div}_x \vec{q} = \varepsilon^2 \varepsilon : \nabla_x \vec{u} - p \text{div}_x \vec{u} + \varepsilon^2 \vec{j} \cdot \vec{E}, \]
(138)
\[ \partial_t (\rho s + \varepsilon^2 s^R) + \text{div}_x (\rho s \vec{u} + \varepsilon^2 \vec{q}) + \text{div}_x \left( \frac{\vec{q}}{\varepsilon} \right) \geq \varsigma \varepsilon, \]
(139)
with
\[ \varsigma \varepsilon = \frac{1}{\theta} \left( \varepsilon^2 \varepsilon : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \theta}{\theta} + \frac{\lambda}{\zeta} |\text{curl}_x \vec{B}|^2 \right) \]
+ \int_0^\infty \int_{S^2} \frac{1}{\nu} \left[ \log \frac{n(I)}{n(I) + 1} - \log \frac{n(B)}{n(B) + 1} \right] \sigma_a (I - B) \, d\vec{\omega} d\nu \]
+ \int_0^\infty \int_{S^2} \frac{1}{\nu} \left[ \log \frac{n(I)}{n(I) + 1} - \log \frac{n(I)}{n(I) + 1} \right] \sigma_s (I - \tilde{I}) \, d\vec{\omega} d\nu, \]
(140)
and finally
\[ \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \varepsilon^2 \theta |\vec{u}|^2 + \varepsilon^2 E^R + \frac{\varepsilon^2}{2\varsigma} |\vec{B}|^2 - \frac{1}{2} \theta \Psi - \varepsilon^2 \theta \varepsilon^2 |\vec{\chi} \times \vec{x}|^2 \right) \, dx \]
+ \varepsilon^2 \int_0^\infty \int_{\Gamma_+} \vec{\omega} \cdot \vec{n} I \, d\Gamma_+ d\nu = 0. \]
where \( \Gamma_+ = \{(x, \bar{\omega}) \in \partial \Omega \times S^2 : \bar{\omega} \cdot \bar{n}_x > 0\} \). Note that the main differences compared with the scaled system (52)-(58) are in the momentum equation (138) and in the boundary on radiation intensity term in the total energy identity (142).

The limit analysis follows the same line of arguments of the previous section so we mention here only the points were we have the main differences.

According to this new scaling the initial data have to be taken in the following form

\[
\begin{align*}
\varrho(0, \cdot) &= \varrho_{0, \varepsilon} = \varrho + \varepsilon \varrho_{0,1}, \\
\vec{u}(0, \cdot) &= \vec{u}_{0, \varepsilon}, \\
\vartheta(0, \cdot) &= \vartheta_{0, \varepsilon} = \vartheta + \varepsilon \vartheta_{0,1}, \\
I(0, \cdot) &= I_{0, \varepsilon} = I + \varepsilon I_{0,1}, \\
\vec{B}(0, \cdot) &= \vec{B}_{0, \varepsilon} = \vec{B}_{0,1},
\end{align*}
\]

and the existence theorem is now

**Theorem A.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded Lipschitz domain. Assume that the thermodynamic functions \( p, e, s \) satisfy hypotheses (14 - 19), \( g \in L^r(\Omega), r > 1 \), and that the transport coefficients \( \mu, \lambda, \kappa, \sigma_a, \sigma_s \) and the equilibrium function \( B \) comply with (20 - 24). Let the initial data \( (\varrho_{0, \varepsilon}, \vec{u}_{0, \varepsilon}, \vartheta_{0, \varepsilon}, I_{0, \varepsilon}, \vec{B}_{0, \varepsilon}) \) be given by (143), where \( (\varrho_{0,1}, \vec{u}_{0,1}, \vartheta_{0,1}, I_{0,1}, \vec{B}_{0,1}) \) are bounded measurable functions.

Then for any \( \varepsilon > 0 \) small enough (in order to maintain positivity of \( \varrho_{0, \varepsilon} \) and \( \vartheta_{0, \varepsilon} \)), there exists a weak solution \( (\varrho_{\varepsilon}, \vec{u}_{\varepsilon}, \vartheta_{\varepsilon}, I_{\varepsilon}, \vec{B}_{\varepsilon}) \) to the radiative Navier-Stokes system (1 - 11) for \( (t, x, \bar{\omega}, \nu) \in (0, T) \times \Omega \times S^2 \times \mathbb{R}_+ \), supplemented with the boundary conditions (12 - 13) and the initial conditions (143).

More precisely we have (86), (89), (90), (91) and

\[
\begin{align*}
\int_0^T \int_\Omega \left( \varrho_{\varepsilon} \vec{u}_{\varepsilon} \cdot \partial_t \phi + \varrho_{\varepsilon} e_{\varepsilon} \varrho_{\varepsilon} \vec{u}_{\varepsilon} \otimes \vec{u}_{\varepsilon} : \nabla_x \phi + \frac{\varrho_{\varepsilon}}{\varepsilon^2} \text{div}_x \phi + \varrho_{\varepsilon} \vec{x} \times \vec{u}_{\varepsilon} \cdot \phi \right) \, dx \, dt \\
= \int_0^T \int_\Omega \left( \varepsilon_0 \varrho_{\varepsilon} \varrho_{\varepsilon} \Psi_{\varepsilon} \cdot \varphi - (\vec{j}_{\varepsilon} \times \vec{B}_{\varepsilon}) \cdot \varphi + \varrho_{\varepsilon} \varrho_{\varepsilon} |\vec{x} \times \vec{x}|^2 \cdot \varphi \right) \, dx \, dt \\
- \int_\Omega \varrho_{0, \varepsilon} \varrho_{0, \varepsilon} \varepsilon \varrho_{0, \varepsilon} \cdot \phi(0, \cdot) \, dx,
\end{align*}
\]

for any \( \phi \in C_0^\infty((0, T) \times \Omega; \mathbb{R}^3) \) with \( p_{\varepsilon} = p(\varrho_{\varepsilon}, \vartheta_{\varepsilon}), \ S_{\varepsilon} = S(\vec{u}_{\varepsilon}, \vartheta_{\varepsilon}), \) and \( \vec{j}_{\varepsilon} = \vec{\kappa} \text{curl}_1 \vec{B}_{\varepsilon} \).

\[
\begin{align*}
\int_\Omega \left( \frac{\varepsilon^2}{2} \varrho_{\varepsilon} |\vec{u}_{\varepsilon}|^2 + \varrho_{\varepsilon} e_{\varepsilon} + \varepsilon^2 E^R_{\varepsilon} + \frac{\varepsilon^2}{2 \kappa} |\vec{B}_{\varepsilon}|^2 - \frac{1}{2} \varepsilon \varrho_{\varepsilon} \Psi_{\varepsilon} - \varepsilon^2 \varrho_{\varepsilon} |\vec{x} \times \vec{x}|^2 \right) \, dx \, dt \\
+ \varepsilon^2 \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x \vec{l}_e(t, x, \bar{\omega}, \nu) \, d\vec{\omega} \, d\nu \, dt \\
= \int_\Omega \left( \frac{\varepsilon^2}{2} \varrho_{0, \varepsilon} |\vec{u}_{0, \varepsilon}|^2 + \varrho_{0, \varepsilon} e_{0, \varepsilon} + \varepsilon^2 E^R_{0, \varepsilon} + \frac{\varepsilon^2}{2 \kappa} |\vec{B}_{0, \varepsilon}|^2 - \frac{1}{2} \varepsilon \varrho_{0, \varepsilon} \Psi_{0, \varepsilon} - \varepsilon^2 \varrho_{0, \varepsilon} |\vec{x} \times \vec{x}|^2 \right) \, dx.
\end{align*}
\]
Assuming now that the initial data satisfy
\[ \|\varrho_0 - \varrho\|_{L^2(\Omega)}^2 \leq C\varepsilon^2, \quad \|\varrho_0 - \varrho\|_{L^2(\Omega)}^2 \leq C\varepsilon^2, \]
\[ \|E_{0,\varepsilon}^R - \mathcal{F}\|_{L^2(\Omega)}^2 \leq C\varepsilon^2, \quad \|\vec{B}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \]
and
\[ \|\sqrt{\varrho_{0,\varepsilon}} \bar{u}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \]
we get as in Section 4 the bounds (102)-(110) and (112)-(116), while the bounds on the magnetic field \( \vec{B}_{\varepsilon} \) are now the following
\[ \text{ess sup}_{t \in (0,T)} \|\vec{B}_{\varepsilon}(t)\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \quad (146) \]
\[ \int_0^T \|\vec{B}_{\varepsilon}(t)\|_{W^{1,2}(\Omega;\mathbb{R}^3)}^2 \, dt \leq C. \quad (147) \]
The main theorem is given by

**Theorem A.2.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded Lipschitz domain. Assume that the thermodynamic functions \( p, \epsilon, s \) satisfy hypotheses (14 - 19) with \( P \in C^1[0,\infty) \cap C^2(0,\infty), g \in L^r(\Omega), r > 1 \), and that the transport coefficients \( \mu, \eta, \kappa, \lambda, \sigma_\alpha, \sigma_s \) and the equilibrium function \( B \) comply with (20 - 24).

Let \( (\varrho_{\varepsilon}, \bar{u}_{\varepsilon}, \vartheta_{\varepsilon}, \vec{B}_{\varepsilon}, I_{\varepsilon}) \) be a weak solution of the scaled system (1 - 11) for \( (t, x, \bar{\omega}, \nu) \in [0, T] \times \Omega \times \mathbb{S}^2 \times \mathbb{R}_+ \), supplemented with the boundary conditions (12 - 13) and initial conditions \( (\varrho_{0, \varepsilon}, \bar{u}_{0, \varepsilon}, \vartheta_{0, \varepsilon}, \vec{B}_{0, \varepsilon}, I_{0, \varepsilon}) \) given by
\[ \varrho_{\varepsilon}(0, \cdot) = \varrho + \varepsilon \varrho_{0, \varepsilon}^{(1)}, \quad \bar{u}_{\varepsilon}(0, \cdot) = \bar{u}_{0, \varepsilon}, \quad \vartheta_{\varepsilon}(0, \cdot) = \vartheta + \varepsilon \vartheta_{0, \varepsilon}^{(1)}, \]
\[ I_{\varepsilon}(0, \cdot) = I + \varepsilon I_{0, \varepsilon}^{(1)}, \quad \vec{B}_{\varepsilon}(0, \cdot) = \vec{B}_{0, \varepsilon}^{(1)}, \]
where \( \varrho > 0, \vartheta > 0, I > 0 \) are constants and
\[ \int_\Omega \varrho_{0, \varepsilon}^{(1)} \, dx = 0, \quad \int_\Omega \vartheta_{0, \varepsilon}^{(1)} \, dx = 0, \quad \int_\Omega I_{0, \varepsilon}^{(1)} \, dx = 0, \quad \int_\Omega \vec{B}_{0, \varepsilon}^{(1)} \, dx = 0 \quad \text{for all } \varepsilon > 0. \]
Assume that
\[
\left\{ \begin{array}{l}
\varrho_{0, \varepsilon}^{(1)} \rightarrow \varrho_0^{(1)} \text{ weakly } - \ast \text{ in } L^\infty(\Omega), \\
\varrho_{0, \varepsilon}^{(1)} \rightarrow \varrho_0^{(1)} \text{ weakly } - \ast \text{ in } L^\infty(\Omega; \mathbb{R}^3), \\
v_{0, \varepsilon}^{(1)} \rightarrow v_0^{(1)} \text{ weakly } - \ast \text{ in } L^\infty(\Omega), \\
I_{0, \varepsilon}^{(1)} \rightarrow I_0^{(1)} \text{ weakly } - \ast \text{ in } L^\infty(\Omega \times \mathbb{S}^2 \times \mathbb{R}_+), \\
\vec{B}_{0, \varepsilon}^{(1)} \rightarrow \vec{B}_0^{(1)} \text{ weakly } - \ast \text{ in } L^\infty(\Omega; \mathbb{R}^3),
\end{array} \right.
\]
Then
\[ \text{ess sup}_{t \in (0,T)} \|\varrho_{\varepsilon}(t) - \varrho\|_{L^\frac{3}{2}(\Omega)} \leq C\varepsilon, \quad (148) \]
and up to subsequences
\[ \bar{u}_{\varepsilon} \rightarrow \bar{U} \text{ weakly } - \ast \text{ in } L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)), \quad (149) \]
\[ \vartheta_{\varepsilon} - \vartheta \frac{1}{\varepsilon} \rightarrow \vartheta^{(1)} \rightarrow \Theta \text{ weakly } - \ast \text{ in } L^2(0, T; W^{1,2}(\Omega)), \quad (150) \]
\[ I_{\varepsilon} \rightarrow I_0 \text{ weakly } - \ast \text{ in } L^2(0, T; L^2(\Omega \times \mathbb{S}^2 \times \mathbb{R}_+)), \quad (151) \]
\[ \vec{B}_{\varepsilon} \rightarrow \vec{B}^{(1)} \rightarrow \vec{B} \text{ weakly } - \ast \text{ in } L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)), \quad (152) \]
and
\[ I_ε - T \frac{I_ε}{ε} = I^{(1)} \rightarrow I_1 \text{ weakly} - (*) \text{ in } L^2(0, T; L^2(Ω × S^2 × R_+)), \]
(153)
where \((U, Θ, B, I_0, I_1)\) solves the system (73)-(79).

**Remark 3.** Concerning regularity of a domain Ω let us mention that an existence of weak solution of the barotropic case for more general domain was studied by Kukučka in the case of Lipschitz domain. It can be generalized to John domain. Important is that we need to apply the Bogovski operator, Korn inequality for more details, see [4].

**Proof.** The proof can performed as in Section 5, the only terms that deserve some attention are the ones involving the magnetic field. By using (28) together with Lions-Aubin lemma we get
\[ \frac{1}{ε} \text{curl} B_ε \rightarrow \frac{1}{ε} \text{curl} B \text{ weakly in } L^q((0, T) × Ω; R^3), \]
(154)
for a certain \( q > 1 \). And finally from (154) we obtain
\[ B_ε \rightarrow B \text{ weakly in } L^q(0, T; L^q(Ω, R^3)) \]
for \( q > 1 \), and
\[ λ \text{curl} B_ε \rightarrow λ \text{curl} B \text{ weakly in } L^2(0, T, L^2(Ω, R^3)). \]

Then it is easy to pass to the limit in (89)

**Appendix B. Proof of Theorem 3.1.**

1. The stationary radiative problem (78), (82) has a weak solution \( I_0 \in L^∞(Ω × S^2 × R_+) \) such that \( Ξ \cdot η_0 I_0 \in L^p(Ω × S^2 × R_+) \) for any \( p > 1 \), after Theorem 1 and Proposition 2 of [2].

2. One checks that the linearly coupled problem for the remaining equations rewrites
\[ \text{div}_x U = 0, \]
\[ \partial_t U + (U \cdot \nabla_x) U + \nabla_x Π - μΔU = \frac{1}{ε} \text{curl}_x B \times B + \alpha \times U + \beta Θ, \]
\[ \partial_t B + (U \cdot \nabla_x) B + (B \cdot \nabla_x) U - Θ Δ B = 0, \]
\[ \partial_t Θ + (U \cdot \nabla_x) Θ - \text{div}_x (K Θ) = \tilde{η} \cdot U + ε Θ + \int_0^∞ \int_{S^2} σ_s I_1 d\tilde{ω} dν, \]
\[ \tilde{ω} \cdot \nabla_x I_1 + σ_α I_1 + σ_s(I_1 - I_1) = ζ Θ, \]
where \( α \in (C^∞(Ω))^3, \ Ξ \in (L^∞(Ω))^3, \ Ξ \in (L^∞(Ω))^3, \ ζ \in L^∞(Ω) \) and \( \xi \in L^∞(Ω) \), together with the boundary conditions
\[ U|_{∂Ω} = 0, \ \nabla Θ \cdot \tilde{n}|_{∂Ω} = 0, \ \tilde{B} \cdot \tilde{n}|_{∂Ω} = 0, \ \text{curl}_x \tilde{B} \times \tilde{n}|_{∂Ω} = 0, \]
and
\[ I_1(x, ν, Ω) = 0 \text{ for } x \in ∂Ω, \ \tilde{ω} \cdot \tilde{n} \leq 0, \]
and the initial conditions
\[ U|_{t=0} = U_0, \ Θ|_{t=0} = Θ_0, \ \tilde{B}|_{t=0} = \tilde{B}_0, \ I_1|_{t=0} = I_{1,0}. \]
In order to apply Schauder’s fixed point method used by Nečas and Roubíček [25] (see [30] Chap. XII.2) we first consider, for $\Theta$ given, the solution $(\vec{U}, \vec{B}, I_1)$ of the “rotating-radiative-MHD problem”

$$\text{div}_x \vec{U} = 0,$$
$$\partial_t \vec{U} + (\vec{U} \cdot \nabla_x) \vec{U} + \nabla_x \Pi - \pi \Delta \vec{U} = -\frac{1}{\zeta} \text{curl}_x \vec{B} \times \vec{B} + \vec{a} \times \vec{U} + \vec{b} \Theta,$$
$$\partial_t \vec{B} + (\vec{U} \cdot \nabla_x) \vec{B} - \Delta \vec{B} = 0,$$
$$\vec{\omega} \cdot \nabla_x I_1 + \sigma_a I_1 - \sigma_s (\bar{I}_1 - I_1) = \zeta \Theta,$$

with

$$\vec{U} |_{\partial\Omega} = 0, \quad \vec{B} \cdot \vec{n} |_{\partial\Omega} = 0, \quad \text{curl}_x \vec{B} \times \vec{n} |_{\partial\Omega} = 0,$$

and

$$\vec{U} |_{t=0} = \vec{U}_0, \quad \vec{B} |_{t=0} = \vec{B}_0, \quad I_1 |_{t=0} = I_{1,0}.$$

The mhd part has a weak solution $\vec{U} \in L^2(0,T;\mathcal{U}(\Omega)), \vec{B} \in L^2(0,T;\mathcal{W}(\Omega))$ after an extension of the Leray-Hopf Theorem (see [31]). Moreover the inhomogeneous stationary radiative equation also has a weak solution $I_1 \in L^2((0,T) \times \Omega) \times S^2 \times \mathbb{R}_+^+$ after Theorem 1 and Proposition 2 of [2]. Consequently the mapping

$$A : \Theta \rightarrow (\vec{U}, \vec{B}, I_1) : L^2(0,T;W^{-1,2}(\Omega)) \rightarrow L^2(0,T;\mathcal{U}(\Omega))$$

$$\times L^2(0,T;\mathcal{W}(\Omega)) \times L^2((0,T) \times \Omega) \times S^2 \times \mathbb{R}_+^+,$$

is continuous.

Then we consider the solution $\Theta$ of the transport-diffusion equation

$$\partial_t \Theta + (\vec{V} \cdot \nabla_x) \Theta - \text{div}_x (K \nabla \Theta) - \xi \Theta = \vec{n} \cdot \vec{U} + \int_0^\infty \int_{S^2} \sigma_s I_1 \, d\omega \, d\nu,$$  \hfill (155)

with

$$\nabla \Theta \cdot \vec{n} |_{\partial\Omega} = 0 \quad \text{and} \quad \Theta |_{t=0} = \Theta_0.$$

It has a weak solution $\Theta \in V^{1,1/2}_2((0,T) \times \Omega)$ after Theorem 5.1 in [22] Chapter III, moreover $\Theta \in L^2(0,T;W^{-1,2}(\Omega))$ and the mapping

$$B : (\vec{U}, I_1) \rightarrow \Theta : L^2(0,T;\mathcal{U}(\Omega)) \times S^2 \times \mathbb{R}_+^+ \rightarrow L^2(0,T;W^{-1,2}(\Omega)),$$

is also continuous.

So we can follow verbatim the scheme of proof of Proposition 12.6 in [25] to conclude.

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