Exterior Differential Systems for Field Theories

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Abstract

Exterior Differential Systems (EDS) and Cartan forms, set in the state space of field variables taken together with four space-time variables, are formulated for classical gauge theories of Maxwell and SU(2) Yang-Mills fields minimally coupled to Dirac spinor multiplets. Cartan character tables are calculated, showing whether the EDS, and so the Euler-Lagrange partial differential equations, is well-posed. The first theory, with 22 dimensional state space (10 Maxwell field and potential components and 8 components of a Dirac field), anticipates QED. In the second, non-Abelian, case (30 Yang-Mills field components and 16 Dirac), only if three additional ”ghost” fields are included (15 more scalar variables) is a well-posed EDS found. This classical formulation anticipates the need for introduction of Fadeev-Popov ghost fields in the quantum standard model.

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I. INTRODUCTION. GAUGE FIELDS

Well posed exterior differential systems (EDS) for Maxwell’s vacuum (source free) equations, and for their gauge or Yang-Mills generalizations, have previously been given \cite{1}. In the first case the forms generating the system live in a 14 dimensional “state space”: ten variables spanning potential and field components, together with four independent space-time variables. For SU(2) Yang-Mills the state space dimension rises to thirty-four. Cartan-Kähler theory characterizes solutions of an EDS as those (in the present case) four dimensional submanifolds of state space that are the maximal null set, or solution, of a closed ideal of exterior differential forms, the EDS \cite{2}. These are general solutions of sets of first order partial differential equations.

In the Maxwell case the ideal is generated by a single 2-form $\theta$ and its exterior derivative $d\theta$, together with an additional “dynamic” 3-form $\psi$. $\theta$ and $\psi$ are functions of the state space coordinates, $F_{ij}, A_i, x, y, z, t$:

$$\theta = dA_1 \land dx + dA_2 \land dy + dA_3 \land dz + dA_4 \land dt - F_{14} dx \land dt - F_{12} dx \land dy + F_{31} dx \land dz - F_{24} dy \land dt - F_{23} dy \land dz - F_{34} dz \land dt \quad (1)$$

$$\psi = -dF_{12} \land dz \land dt + dF_{14} \land dy \land dz - dF_{23} \land dx \land dt + dF_{31} \land dy \land dt - dF_{24} \land dx \land dz - dF_{34} \land dx \land dy \quad (2)$$

In the Yang-Mills SU(2) generalization of this we have three such pairs of forms, $\theta^a$ and $\psi_a$, $a = 1, \ldots, 3$. Their definitions in terms of potentials $A^a_i$ and fields $F^a_{ij}$ are given in Ref. 1. This 2-form/3-form structure of an EDS characterizes so-called gauge theories. On the other hand, EDS’s generated only by ”contact” 1-forms and 4-forms are called multicontact systems \cite{3} \cite{4}. These include the scalar field theory we will discuss in Section II and source free Dirac theory discussed in Section III (although there no contact forms are used). In subsequent sections we discuss EDS’s for minimally coupled field theories where generating forms of all ranks occur.

The set of generators of an EDS–1-forms, 2-forms, 3-forms, and so on– must be closed under exterior differentiation; Cartan’s non-perturbative theory of the existence and structure of solutions of an EDS requires calculation of a series of so-called Cartan characters, integers determined at a single, generic, point of N-dimensional state space from the ranks of a series of nested linear sets of equations sequentially set using auxiliary vectors. The characters are denoted $s_i$, $i = 0, 1, 2 \ldots$; the theory shows how these calculations must stop at, say, n-1, giving n as the dimension of the solution submanifold. If the signature is right, the sum of these characters is the number of evolution equations in the equivalent set of partial differential equations. The characters also categorize the constraints, or integrability conditions, encountered in numerical integration. A final Cartan character is conventionally computed, denoted $s_n$, it is the number of arbitrary functions that enter the general solution; if this is non-zero, in field theory it is customarily called the degree of gauge freedom.

An EDS belongs to a variational principle if there is a Cartan n-form equivalent to a Lagrangian density \cite{5}; its exterior derivative, the so-called multisymplectic n+1-form, contracted with arbitrary vectors must give n-forms in the ideal of the generators of the EDS (this is arbitrary variation of the Cartan 4-form). If, using only these, the character $s_{n-1}$ is less than $N - n$ this signals the need for lower rank forms to generate a well posed EDS. If it can thus be completed, the EDS then codes the Euler-Lagrange partial differential equations of the variational principle, together with all their integrability conditions or constraints. \cite{6}
The multisymplectic 5-form for Maxwell is the exterior product $\theta \wedge \psi$, and that for the other free field Yang-Mills gauge theories $\theta^{a} \wedge \psi_{a}$, and these factors generate the EDSs. The multisymplectic form for multicontact EDSs can be formed from just exterior products of 1-forms and 4-forms. As we will see in Section III, the Cartan form for the Dirac equation has a special relation to the EDS, and the multisymplectic form does not factor but $s_{n-1}$ is just $N - n$; no potential fields are introduced, and the EDS is generated solely by 4-forms. There even exist systems, such as the Hilbert Lagrangian for vacuum general relativity imbedded in flat 10-space, where the multisymplectic form factors in either way, giving different well-posed EDSs that may describe the possibility of gravitational field phase change \cite{7} \cite{8}.

In calculating the various new sets of Cartan characters reported here we used a small suite of Mathematica programs written by the late H. D. Wahlquist, called AVF (Algebra Valued, or indexed, Forms). They have been carefully edited by José M Martín-Garcia and are now available on the xAct website \cite{9}. We will report the characters obtained for an EDS in an array $N(s_{0}, s_{1}, ..., s_{n-1})n + s_{n}$.

An important test that is checked sequentially during the calculation of the characters of an EDS is that those n (here four) state space variables one anticipates taking as independent when writing an equivalent set of partial differential equations remain so in the solution submanifold and can indeed be used. We have referred to this property as being ”well-posed”. Cartan calls such variables as being ”in involution” or ”involuntary”. A well-posed EDS then satisfies ”Cartan’s test” \cite{2}. In more recent language, we want solutions to be cross sections of a a bundle over an n-dimensional base. The AVF suite checks well-posedness beginning with the first n coordinates of state space that are entered in its coordinate list, so we always enter $x, y, z, t$ first in an AVF calculation.

In Ref. \cite{1} the tables of Cartan characters, $N[s_{0}, s_{1}, s_{2}, s_{3}]4 + s_{4}$ of vacuum Maxwell and SU(2) Yang-Mills theories in four dimensions were reported as respectively $14[0, 1, 3, 5]4 + 1$, and $34[0, 3, 9, 15]4 + 3$. The coordinates $x, y, z, t$ are in involution, and the degrees of gauge freedom one and three, as shown. An EDS for three coupled scalar fields is given in Section II, 19(3,3,3,6)4, and a single Dirac spinor field in Section III has 12[0,0,0,8]4.

In Sections IV and V we treat field theories with Cartan and multisymplectic forms that ”minimally” couple suitable multiplets of the free Dirac equation to the EDS’s for Maxwell and SU(2) Yang-Mills. We calculate their Cartan character tables. and check involutivity. Only the first of these, QED, immediately proves to be well-posed; it also has one degree of gauge freedom. For the SU(2) field theory we find well-posedness only if the two sets of Dirac fields are further supplemented by three scalar ”ghost” fields. The ”ghosts” enter the Cartan form (Lagrangian density) only as coupled to the Yang-Mills or gauge fields.

\section{Scalar Fields}

We treat a multicontact EDS for a multiplet of three. The 19 dimensional state space is spanned by 15 fields $\rho_{1}, \rho_{2}, \rho_{3}, \rho_{11}, \rho_{12}, \rho_{13}, \rho_{14}, \rho_{21}, \rho_{22}, \rho_{23}, \rho_{24}, \rho_{31}, \rho_{32}, \rho_{33}, \rho_{34}$ together with $x, y, z, t$. The multicontact EDS is generated by three contact 1-forms and their exterior
and three 4-forms to carry the dynamic content:

\[ \sigma_1 = d\rho_1 - \rho_{11}dx_1 - \rho_{12}dx_2 - \rho_{13}dx_3 - \rho_{14}dx_4 \quad (3) \]

\[ \sigma_2 = d\rho_2 - \rho_{21}dx_1 - \rho_{22}dx_2 - \rho_{23}dx_3 - \rho_{24}dx_4 \quad (4) \]

\[ \sigma_3 = d\rho_3 - \rho_{31}dx_1 - \rho_{32}dx_2 - \rho_{33}dx_3 - \rho_{34}dx_4 \quad (5) \]

and three 4-forms to carry the dynamic content:

\[ \Sigma_1 = \lambda^2 \rho_1 \left( -\mu^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 \right) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_1 \wedge dx_2 \wedge dx_3 \wedge d\rho_{14} \]
\[ -dx_1 \wedge dx_2 \wedge dx_4 \wedge d\rho_{13} + dx_1 \wedge dx_3 \wedge dx_4 \wedge d\rho_{12} - dx_2 \wedge dx_3 \wedge dx_4 \wedge d\rho_{11} \quad (6) \]

\[ \Sigma_2 = \lambda^2 \rho_2 \left( -\mu^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 \right) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_1 \wedge dx_2 \wedge dx_3 \wedge d\rho_{24} \]
\[ -dx_1 \wedge dx_2 \wedge dx_4 \wedge d\rho_{23} + dx_1 \wedge dx_3 \wedge dx_4 \wedge d\rho_{22} - dx_2 \wedge dx_3 \wedge dx_4 \wedge d\rho_{21} \quad (7) \]

\[ \Sigma_3 = \lambda^2 \rho_3 \left( -\mu^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 \right) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_1 \wedge dx_2 \wedge dx_3 \wedge d\rho_{34} \]
\[ -dx_1 \wedge dx_2 \wedge dx_4 \wedge d\rho_{33} + dx_1 \wedge dx_3 \wedge dx_4 \wedge d\rho_{32} - dx_2 \wedge dx_3 \wedge dx_4 \wedge d\rho_{31} \quad (8) \]

AVF calculates the Cartan integer table to be 19(3, 3, 3, 6)4 with no gauge freedom. \( x, y, z, t \) are in involution, so equivalent partial differential equations that adopt them as independent variables are well-posed. The exact multisymplectic form is

\[ d\Lambda = \sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2 + \sigma_3 \wedge \Sigma_3 \quad (9) \]

Integrating this by parts gives a Cartan form, unique up to an exact 4-form:

\[ \Lambda = -\rho_1 (d\rho_{11} \wedge \eta_1 + d\rho_{12} \wedge \eta_2 + d\rho_{13} \wedge \eta_3 + d\rho_{14} \wedge \eta_4) \]
\[ -\rho_2 (d\rho_{21} \wedge \eta_1 + d\rho_{22} \wedge \eta_2 + d\rho_{23} \wedge \eta_3 + d\rho_{24} \wedge \eta_4) \]
\[ -\rho_3 (d\rho_{31} \wedge \eta_1 + d\rho_{32} \wedge \eta_2 + d\rho_{33} \wedge \eta_3 + d\rho_{34} \wedge \eta_4) \]
\[ -(1/2) \left( \rho_{11}^2 + \rho_{12}^2 + \rho_{13}^2 + \rho_{14}^2 + \rho_{21}^2 + \rho_{22}^2 + \rho_{23}^2 + \rho_{24}^2 + \rho_{31}^2 + \rho_{32}^2 + \rho_{33}^2 + \rho_{34}^2 \right) \]
\[ + \lambda^2 \left( \rho_1^2 + \rho_2^2 + \rho_3^2 - \mu^2 \right)^2 / 2 \eta \quad (10) \]

where a useful notation for the basic 3-forms and 4-form we will use henceforth is

\[ \eta_0 = dx \wedge dy \wedge dz \quad (11) \]
\[ \eta_1 = dt \wedge dz \wedge dy \quad (12) \]
\[ \eta_2 = dz \wedge dt \wedge dx \quad (13) \]
\[ \eta_3 = dx \wedge dt \wedge dy \quad (14) \]
\[ \eta = dt \wedge dx \wedge dy \wedge dz \quad (15) \]

As we have emphasized \( \Lambda \) lives in the 19 dimensional state space. Field theorists however customarily anticipate the bundle structure, substitute back into into such Cartan forms the contact 1-forms from the EDS, \( \sigma_1, \sigma_2, \sigma_3 \), eliminating \( d\rho_1, d\rho_2, d\rho_3 \), and writing the \( \rho_{ij} \)
as pullbacks $\partial_j \rho_i = \rho_{i,j}$. So treated, the Cartan 4-form becomes an expression $L\eta$ and the Lagrangian density functional $L$, or $(-T + V)$, is used in variational calculus. In this case we have

$$L/2 = -\rho_{1,1}^2 - \rho_{1,2}^2 - \rho_{1,3}^2 - \rho_{1,4}^2 - \rho_{2,1}^2 - \rho_{2,2}^2 - \rho_{2,3}^2 - \rho_{2,4}^2 - \rho_{3,1}^2$$

$$- \rho_{3,2}^2 - \rho_{3,3}^2 - \rho_{3,4}^2 + \lambda^2 \left( \rho_1^2 + \rho_2^2 + \rho_3^2 - \mu^2 \right)^2 / 2 \quad (16)$$

All the other Cartan forms given below can be similarly treated to become Lagrangian densities.

Other important associated structures to an EDS in state space are conservation laws (or currents). These are 3-forms, not in the ideal, whose exterior derivatives are in the ideal, the EDS. Here there are three of these:

$$J_1 = (\rho_{21}\rho_3\eta_1 + \rho_{22}\rho_3\eta_2 + \rho_{23}\rho_3\eta_3 + \rho_{24}\rho_3\eta_0) - (\rho_{31}\rho_2\eta_1 + \rho_{32}\rho_2\eta_2 + \rho_{33}\rho_2\eta_3 + \rho_{34}\rho_2\eta_0) \quad (17)$$

$$J_2 = (\rho_{31}\rho_1\eta_1 + \rho_{32}\rho_1\eta_2 + \rho_{33}\rho_1\eta_3 + \rho_{34}\rho_1\eta_0) - (\rho_{11}\rho_3\eta_1 + \rho_{12}\rho_3\eta_2 + \rho_{13}\rho_3\eta_3 + \rho_{14}\rho_3\eta_0) \quad (18)$$

$$J_3 = (\rho_{11}\rho_2\eta_1 + \rho_{12}\rho_2\eta_2 + \rho_{13}\rho_2\eta_3 + \rho_{14}\rho_2\eta_0) - (\rho_{21}\rho_1\eta_1 + \rho_{22}\rho_1\eta_2 + \rho_{23}\rho_1\eta_3 + \rho_{24}\rho_1\eta_0) \quad (19)$$

### III. DIRAC FIELDS

The eight Dirac Equations (sic) are for eight fields that are functions of four independent variables. They are usually written in compressed spinor notation, but we need explicit state space coordinates, say $X_i + iY_i$ for each complex Dirac spinor component, and together with spacetime coordinates $x,y,z,t$ they span a twelve dimensional state space. As an EDS it has notably different structure from that for Maxwell and the gauge theories; it is a sub case of a multicontact system, coded as eight 4-forms, say $\Xi_i$ and $\Psi_i$, and the character table is just $12(0,0,0,8)4$ with no constraints and no gauge freedom. Unlike the Maxwell and Yang-Mills systems no potential fields were needed to be adjoined to achieve a variational
principle. Also, its generalization appears to be just the use of multiple copies, or multiplets.

\[
\Xi_1 = dX_1 \land \eta_0 + dX_0 \land \eta_1 + dY_0 \land \eta_2 + dX_3 \land \eta_3 - mY_1 \eta
\]  
(20)

\[
\Psi_1 = dY_1 \land \eta_0 + dY_0 \land \eta_1 - dX_0 \land \eta_2 + dY_3 \land \eta_3 + mX_1 \eta
\]  
(21)

\[
\Xi_2 = dX_2 \land \eta_0 + dX_3 \land \eta_1 - dY_3 \land \eta_2 - dX_0 \land \eta_3 - mY_2 \eta
\]  
(22)

\[
\Psi_2 = dY_2 \land \eta_0 + dY_3 \land \eta_1 + dX_3 \land \eta_2 - dY_0 \land \eta_3 + mX_2 \eta
\]  
(23)

\[
\Xi_3 = -dX_3 \land \eta_0 - dX_2 \land \eta_1 - dY_2 \land \eta_2 - dX_1 \land \eta_3 - mY_3 \eta
\]  
(24)

\[
\Psi_3 = -dY_3 \land \eta_0 - dY_2 \land \eta_1 + dX_2 \land \eta_2 - dY_1 \land \eta_3 + mX_3 \eta
\]  
(25)

\[
\Xi_0 = -dX_0 \land \eta_0 - dX_1 \land \eta_1 + dY_1 \land \eta_2 + dX_2 \land \eta_3 - mY_0 \eta
\]  
(26)

\[
\Psi_0 = -dY_0 \land \eta_0 - dY_1 \land \eta_1 - dX_1 \land \eta_2 + dY_2 \land \eta_3 + mX_0 \eta
\]  
(27)

The \(x, y, z, t\) are in involution so equivalent partial differential equations using them as independent variables—the Dirac equation—are well posed.

A Lagrangian density, hence a Cartan 4-form, say \(\Lambda_D\), for the Dirac partial differential set is well known; in our notation it is

\[
\Lambda_D = (X_1 \Psi_1 - Y_1 \Xi_1 + X_2 \Psi_2 - Y_2 \Xi_2 + X_3 \Psi_3 - Y_3 \Xi_3 + X_0 \Psi_0 - Y_0 \Xi_0)/2
\]  
(28)

\(\Lambda_D\) is of course not unique, but its exterior derivative \(d\Lambda_D\), the multisymplectic 5-form, is. Expanding this, for the Dirac system we have

\[
d\Lambda_D = (dX_1 \land dY_1 + dX_2 \land dY_2 + dX_3 \land dY_3 + dX_0 \land dY_0) \land \eta_0
\]

\[
+ (dX_1 \land dY_0 + dX_0 \land dY_1 + dX_3 \land dY_2 + dX_2 \land dY_3) \land \eta_1
\]

\[
+ (dX_0 \land dX_1 + dY_0 \land dY_1 + dX_2 \land dX_3 + dY_2 \land dY_3) \land \eta_2
\]

\[
+ (dX_1 \land dY_3 + dX_3 \land dY_1 - dX_2 \land dY_0 - dX_0 \land dY_2) \land \eta_3
\]

\[
+ m (X_1 dX_1 + Y_1 dY_1 + X_2 dX_2 + Y_2 dY_2 - X_3 dX_3 - Y_3 dX_3
\]

\[
- X_0 dX_0 - Y_0 dY_0) \land \eta
\]  
(29)

This can also be written as

\[
2d\Lambda_D = dX_1 \land \Psi_1 - dY_1 \land \Xi_1 + dX_2 \land \Psi_2 - dY_2 \land \Xi_2
\]

\[
+ dX_3 \land \Psi_3 - dY_3 \land \Xi_3 + dX_0 \land \Psi_0 - dY_0 \land \Xi_0
\]

\[
+ m d [X_1^2 + Y_1^2 + X_2^2 + Y_2^2 + X_3^2 + Y_3^2 + X_0^2 + Y_0^2] \land \eta/2
\]  
(30)

An arbitrary vector in state space, say \(X\), contracted on this 5-form yields a 4-form that is in the EDS generated by the \(\Xi_i\) and \(\Psi_i\). The Dirac system, spanned only by 4-forms, is also seen to be further anomalous in that its Cartan form (28) and functional Lagrangian vanish when evaluated on solutions of the EDS they generate variationally.

For the Dirac ideal the conserved current \(J\) is

\[
J = (X_1^2 + Y_1^2 + X_2^2 + Y_2^2 + X_3^2 + Y_3^2 + X_0^2 + Y_0^2) \eta_0 + 2 (X_1 X_0 + Y_1 Y_0 + X_2 X_3 + Y_2 Y_3) \eta_1
\]

\[
+ 2 (-Y_1 X_0 + X_1 Y_0 - X_2 Y_3 + Y_2 X_3) \eta_2 + 2 (X_1 X_3 + Y_1 Y_3 - X_2 X_0 - Y_2 Y_0) \eta_3
\]  
(31)

from which

\[
dJ = 2 (X_1 \Xi_1 + Y_1 \Psi_1 + X_2 \Xi_2 + Y_2 \Psi_2 + X_3 \Xi_3 + Y_3 \Psi_3 + X_0 \Xi_0 + Y_0 \Psi_0)
\]  
(32)
IV. MAXWELL-DIRAC THEORY

In the classical version of QED, the outer product of the potential 1-form \( A = A_i dx^i \) of Maxwell theory and of the conserved current 3-form \( J \) of Dirac theory is taken as a "minimal" coupling term added to the sum of the two respective Cartan 4-forms. An EDS formulating this is implied in an elegant and overlooked paper of Barut, Moore and Piron[10]. They first neatly expound how the canonical Cartan 1-form and its exterior derivative 2-form, or symplectic structure, work in classical mechanics and their generalization to the higher dimension of field theory. They discuss the straightforward use of Cartan forms in state space vs. the subtlety of variational/functional calculus. They present the Cartan \( \Lambda \) form in the 22 dimensional state space of coupled Maxwell-Dirac theory, and propose an EDS from contractions of an arbitrary vector on the multisymplectic 5-form \( d\Lambda \). It is generated by 18 4-forms and we calculate its character table to be \( 22(0, 0, 0, 15, 0, 0)7 \), and moreover not in involution. Barut et al state that it is a simple matter then to write the field theoretic partial differential equations, and indeed it is apparent from inspection that the lower rank gauge 2-form \( \theta \) and 3-forms \( d\theta \) and \( \psi + J \), like those of the pure Maxwell EDS, can be added to just 8 4-forms as generators. We calculate this specialized EDS to have characters \( 22(0, 1, 3, 13)4 + 1 \) with \( x, y, z, t \) now in involution, and one degree of gauge freedom. It of course directly yields the partial differential equations Barut et al discuss.

The motivation of Barut et al was to go on to show how the Schrödinger and Maxwell-Schrödinger equations emerge as direct limits of the relativistic field equations.

Taking note of the identity \( J \wedge (\theta - dA) = 0 \) the multisymplectic 5-form for the system can be written to show how the coupling has affected both the gauge and spinor EDSs combined in this field theory; the 3-form second factor in the Maxwell multisymplectic 5-form \( -\theta \wedge \psi \) now includes a \( J \) current form, while the 5-form of the spinor field now includes \( A \wedge dJ \). We learn from this example that finding a well-posed EDS from a given field theoretic Lagrangian or Cartan form is not quite straightforward, that the presence of lower rank generating forms is signaled by the algebraic structure of the Cartan form, and how the to-be-independent variables enter. cf. Ref.[6]. The character table of a possible EDS must then always be calculated and well-posedness confirmed. In the following simple field theory, involving textbook Yang-Mills generalizations of Maxwell-Dirac theory, and using Equation (33) as a guide, we report success of this program only if additional currents from "ghost" fields are introduced.

V. COUPLED SU(2) GAUGE AND DIRAC FIELDS

It is surprising that in addition to 3-forms like Eq. (30) multiple copies of the Dirac equations allow joint nontrivial conservation laws, though this is well known to particle
then there are three non trivial joint conserved 3-forms, \( dJ \) theorists. For example if we have two sets of variables, say \( X_{i,1}, Y_{i,1} \) and \( X_{i,2}, Y_{i,2} \) and an EDS generated by two independent sets of Dirac 4-forms, in a 20 dimensional state space, then there are three non trivial joint conserved 3-forms, \( J_a \), which we next write. It can be verified that the forms \( dJ_a \) are in the EDS generated by two independent copies of Equations (20)-(27), 20(0,0,0,16)4. We use a recipe \[ \text{structure equations} \] based on representation theory of \( U(2) \) (the Pauli matrices):

\[
J_1 = - (X_{1,1}X_{1,2} + Y_{1,1}Y_{1,2} + X_{2,1}X_{2,2} + Y_{2,1}Y_{2,2} + X_{3,1}X_{3,2} + Y_{3,1}Y_{3,2} + X_{0,1}X_{0,2} + Y_{0,1}Y_{0,2}) \eta_0 \\
- (X_{1,1}X_{0,2} + Y_{1,1}Y_{0,2} + X_{2,1}X_{3,2} + Y_{2,1}Y_{3,2} + X_{1,2}X_{0,1} + Y_{1,2}Y_{0,1} + X_{2,2}X_{3,1} + Y_{2,2}Y_{3,1}) \eta_1 \\
- (-Y_{1,1}X_{0,2} + X_{1,1}Y_{0,2} - X_{2,1}X_{3,2} + Y_{2,1}Y_{3,2} - Y_{1,2}X_{0,1} + X_{1,2}Y_{0,1} - X_{2,2}X_{3,1} + Y_{2,2}X_{3,1}) \eta_2 \\
- (X_{1,1}X_{3,2} + Y_{1,1}Y_{3,2} - X_{2,1}X_{0,2} - Y_{2,1}Y_{0,2} + X_{1,2}X_{3,1} + Y_{1,2}Y_{3,1} - X_{2,2}X_{0,1} \\
- Y_{2,2}Y_{0,1}) \eta_3 \tag{36}
\]

\[
J_2 = (-Y_{1,1}X_{1,2} + X_{1,1}Y_{1,2} - Y_{2,1}X_{2,2} + X_{2,1}Y_{2,2} - Y_{3,1}X_{3,2} + X_{3,1}Y_{3,2} - Y_{0,1}X_{0,2} + X_{0,1}Y_{0,2}) \eta_0 \\
+ (-Y_{1,1}X_{0,2} + X_{1,1}Y_{0,2} + X_{2,1}X_{3,2} - Y_{2,1}Y_{3,2} + X_{3,1}Y_{2,2} - Y_{3,1}X_{2,2} + X_{0,1}Y_{1,2} - Y_{0,1}X_{1,2}) \eta_1 \\
+ (-X_{1,1}X_{0,2} - Y_{1,1}Y_{0,2} + X_{2,1}X_{3,2} + Y_{2,1}Y_{3,2} - X_{3,1}X_{2,2} + Y_{3,1}X_{2,2} + X_{0,1}X_{1,2} + Y_{0,1}Y_{1,2}) \eta_2 \\
+ (X_{3,1}Y_{1,2} - Y_{3,1}X_{1,2} - X_{0,1}Y_{2,2} + X_{0,1}X_{2,2} + X_{1,1}Y_{3,2} - Y_{1,1}X_{3,2} - X_{2,1}Y_{0,2} \\
+ Y_{2,1}X_{0,2}) \eta_3 \tag{37}
\]

\[
J_3 = -(X_{1,1}^2 + Y_{1,1}^2 + X_{2,1}^2 + Y_{2,1}^2 + X_{3,1}^2 + Y_{3,1}^2 + X_{0,1}^2 + Y_{0,1}^2 - X_{1,2}^2 - Y_{1,2}^2 - X_{2,2}^2 - Y_{2,2}^2 \\
- X_{3,2}^2 - Y_{3,2}^2 - X_{0,2}^2 - Y_{0,2}^2) \frac{\eta_0}{2} - (X_{1,1}X_{0,1} + Y_{1,1}Y_{0,1} + X_{2,1}X_{3,1} + Y_{2,1}Y_{3,1} - X_{1,2}X_{0,2} \\
- Y_{1,2}X_{0,2} - X_{2,2}X_{3,2} - Y_{2,2}Y_{3,2}) \eta_1 - (-Y_{1,1}X_{0,1} + X_{1,1}Y_{0,1} - X_{2,1}Y_{3,1} + Y_{2,1}X_{3,1} \\
+ Y_{1,2}X_{0,2} - X_{1,2}Y_{0,2} + X_{2,2}X_{3,2} - Y_{2,2}X_{3,2}) \eta_2 - (X_{1,1}X_{3,1} + Y_{1,1}Y_{3,1} - X_{2,1}X_{0,1} \\
- Y_{2,1}X_{0,1} - X_{1,2}X_{3,2} - Y_{2,1}Y_{3,2} + X_{2,2}X_{0,2} + Y_{2,2}Y_{0,2}) \eta_3 \tag{38}
\]

More succinctly we calculate

\[
d[J_1] + X_{1,1}\Xi_{1,2} + X_{1,2}\Xi_{1,1} + Y_{1,1}\Psi_{1,2} + Y_{1,2}\Psi_{1,1} + X_{2,1}\Xi_{2,2} \\
+ X_{2,2}\Xi_{2,1} + Y_{2,1}\Psi_{2,2} + Y_{2,2}\Psi_{2,1} + X_{3,1}\Xi_{3,2} + X_{3,2}\Xi_{3,1} + \\
Y_{3,1}\Psi_{3,2} + Y_{3,2}\Psi_{3,1} + X_{0,1}\Xi_{4,2} + X_{0,2}\Xi_{4,1} + Y_{0,1}\Psi_{4,2} + Y_{0,2}\Psi_{4,1} = 0 \tag{39}
\]

\[
d[J_2] + Y_{1,1}\Xi_{1,2} - Y_{1,2}\Xi_{1,1} - X_{1,1}\Psi_{1,2} + X_{1,2}\Psi_{1,1} + Y_{2,1}\Xi_{2,2} \\
- Y_{2,2}\Xi_{2,1} - X_{2,1}\Psi_{2,2} + X_{2,2}\Psi_{2,1} + Y_{3,1}\Xi_{3,2} - Y_{3,2}\Xi_{3,1} - \\
X_{3,1}\Psi_{3,2} + X_{3,2}\Psi_{3,1} + Y_{0,1}\Xi_{4,2} - Y_{0,2}\Xi_{4,1} - X_{0,1}\Psi_{4,2} + X_{0,2}\Psi_{4,1} = 0 \tag{40}
\]

and we recognize \( d[J_3] \) to be just the difference of the two first order Dirac currents Eq.(32). Following the prescription for minimal coupling in Ref. 5, Koshelkin \[ \text{structure equations} \] has set up and discussed the equations for a threefold SU(2) Yang-Mills field, which has an EDS generated by forms \( \theta^a \) and \( \psi_a \), functions of potentials \( A_i^a \) and fields \( F_{ij}^a \), when minimally coupled to a multiplet of (two) Dirac fields using these currents. The coupling added to the sum of the
gauge and bispinor Cartan 4-forms is $A^a \wedge J_a$. Taking an exterior derivative, we find the the multisymplectic 5-form in a 50-dimensional phase space:

$$d\Lambda = -\theta^1 \wedge (\psi_1 + J_1) - \theta^2 \wedge (\psi_2 + J_2) - \theta^3 \wedge (\psi_3 + J_3) + d\Lambda_{D1} + d\Lambda_{D2} + (A^1 \wedge dJ_1 + A^2 \wedge dJ_2 + A^3 \wedge dJ_3)$$

(41)

EDS : $\theta^i, d\theta^i, (\psi_i + J_i), d(\psi_i + J_i), X \bullet d\Lambda$ (42)

From this, the EDS should be generated by 2-forms $\theta^a$, 3-forms $\psi_a + J_a$ (showing the Dirac bispinor currents as sources of the YM fields) and their closures, together with sixteen 4-forms that are two copies of $\Xi_i, \Psi_i$, ”covariantly” modified with the YM potential terms in the final parenthesis. We calculate the characters to be $50(0,3,9,32,0,0)6$, at the same time finding that $x, y, z, t$ are not in involution. Although obtained from a Cartan form, an apparently acceptable variational principle, the EDS is not well-posed. Koshelkin [12][13] found a class of solutions to this theory, approximations to which then lead on to a paradox, so he also concluded that there are in fact no general self-consistent solutions. The failure of the well-posedness, and perhaps the need for extra conditions, may not be clear in a perturbative treatment of this coupled SU(2)-bispinor field theory, but the resolution in the case of the standard model was already found by Fadeev and Slavnov.

In the present case three additional ”ghost” fields must be included, also minimally coupled to the gauge currents but without themselves directly contributing to the multi-symplectic dynamics. Using the notation of Section II, instead of Equation (41) and (42) we take

$$d\Lambda = -\theta^1 \wedge (\psi_1 + J_1 + J_{1s}) - \theta^2 \wedge (\psi_2 + J_2 + J_{2s}) - \theta^3 \wedge (\psi_3 + J_3 + J_{3s}) + d\Lambda_{D1} + d\Lambda_{D2} + A^1 \wedge (dJ_1 + dJ_{1s}) + A^2 \wedge (dJ_2 + dJ_{2s}) + A^3 \wedge (dJ_3 + dJ_{3s})$$

(43)

EDS : $\theta^i, d\theta^i, (\psi_i + J_i + J_{is}), d(\psi_i + J_i + J_{is}), X \bullet d\Lambda$ (44)

The AVF program now finds the Cartan table in 65 dimensions to be $65(0,6,12,37)4+6$, well posed, with $x, y, z$ and $t$ in involution and with 6 degrees of gauge freedom, as shown.

### A. Summary

Cartan’s formulation of sets of first order partial differential equations as Exterior Differential Systems allows determination of integrability and well-posedness while not resorting to approximation or perturbation theory. We have used it to explore and characterize some classical coupled theories that are precursors to the standard model of quantum field theory. QED, or Maxwell-Dirac theory, proved to be well-posed when written as an EDS induced from a Cartan form set on a state space of $18+4$ dimensions. Coupling a SU(2) Yang-Mills field to a Dirac multiplet required three additional scalar ”ghost” fields, for a total of $61+4$ dimensions, to achieve well-posedness.

(added note: We have subsequent to arXiv submission in 2014 found the SU(2) coupled gauge theory of Sec. V as the third auxiliary exercise posed in the online Lecture Notes on Advanced Quantum Field Theory of Claudio Scrucca, Institute of Theoretical Physics, IFPL, Lausanne)
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