ON FUNCTION THEORY IN QUANTUM DISC:
A q-ANALOGUE OF BEREZIN TRANSFORM

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Let $\alpha$ be a positive number.

Section 1 of this work contains a study of Toeplitz-Bergman operators with finite symbols in the quantum disc, and section 4 deals already with Toeplitz-Bergman operators with bounded symbols. An alternate way of producing Toeplitz-Bergman operators with polynomial symbols is described in section 7 (lemma 7.2).

Section 2 introduces a Berezin transform $B_{q,\alpha}$ for finite functions in the quantum disc; the same is done in section 4 for bounded functions. An alternate way of constructing a Berezin transform for a polynomial function is described in sections 6, 7 (proposition 6.6 and lemma 7.2).

An asymptotic expansion (3.2), (3.6) for a Berezin transform for a finite function is obtained in section 3; a similar expansion (5.2) for the case of a bounded function can be found in section 5. An application of the latter result to formal series with polynomial coefficients in section 8 affords the main result of this work (theorem 8.4).

We use the background and notation used in [8, 9, 10].

1 Toeplitz-Bergman operators with finite symbols

Consider the covariant algebra $\mathbb{C}[z]_q$ (see [6]). Algebraically it is isomorphic to the polynomial algebra $\mathbb{C}[z]$, and the $U_q\mathfrak{sl}_2$-action is determined by the relations

$$K^{\pm 1}z = q^{\pm 2}z, \quad Fz = q^{1/2}.$$ 

We also follow [12] in using a covariant (left) $\mathbb{C}[z]_q$-module with the generator $\mathbb{T}$ and the relations

$$K^{\pm 1}\mathbb{T} = q^{\pm(2\alpha+1)}\mathbb{T}, \quad F\mathbb{T} = 0.$$ 

Denote this covariant module by $\mathbb{C}[z]_{q,\alpha}$.

Let $F_{q,\alpha} \subset \text{End}(\mathbb{C}[z]_{q,\alpha})$ be the covariant algebra of linear operators $A : z^j \mapsto \sum_{m \in \mathbb{Z}_+} a_{mj}z^m$, $j \in \mathbb{Z}_+$, with finitely many nonzero matrix elements $a_{mj}$. In virtue of this definition, $F_{q,\alpha} \mapsto \mathbb{C}[z]_{q,\alpha} \otimes \mathbb{C}[z]_{q,\alpha}$

Our immediate purpose is to construct a morphism of $U_q\mathfrak{sl}_2$-modules $D(U)_q \to F_{q,\alpha}$ which is normally called a Toeplitz quantization.
Remind the notation (see [8]):

\[
\int_U q \, f \, d\nu_\alpha = \frac{1 - q^{4\alpha}}{1 - q^2} \int_U (1 - zz^*)^{2\alpha + 1} d\nu,
\]

\[
(f_1, f_2)_{q, \alpha} = \int_U f_2^* f_1 d\nu_\alpha.
\] (1.1)

Form a completion of the linear space \( D(U)_q \) with respect to the norm \( \|f\|_{q, \alpha} = (f, f)_{q, \alpha}^{1/2} \). It is easy to show that this Hilbert space admits an embedding into \( D(U)'_q \) and is canonically isomorphic to the space \( L^2_{q, \alpha} \) defined in [8].

Let \( P_{q, \alpha} \) be the orthogonal projection in \( L^2_{q, \alpha} \) onto the closure \( H^2_{q, \alpha} \) of the subspace \( \mathbb{C}[z]_{q, \alpha} \subset L^2_{q, \alpha} \). Given \( \hat{f} \in D(U)_q \), we call the linear operator

\[
\hat{f} : \mathbb{C}[z]_{q, \alpha} \to \mathbb{C}[z]_{q, \alpha}; \quad \hat{f} : \psi \mapsto P_{q, \alpha}(\hat{f} \psi), \quad \psi \in \mathbb{C}[z]_{q, \alpha}
\]
a Toeplitz-Bergman operator with the finite symbol \( \hat{f} \). This is well defined, as one can see from

**Proposition 1.1** With \( \hat{f} \in D(U)_q \), for all but finitely many \( m, j \in \mathbb{Z}_+ \) the integral

\[
I_{m,j} = \int_{U_q} z^m \hat{f} z^j d\nu_\alpha
\]
is zero.

**Proof.** It was shown in [8] that for any \( \hat{f} \in D(U)_q \) one has \( z^N \hat{f} = \hat{f} z^N = 0 \) for some \( N \in \mathbb{N} \). Hence \( I_{m,j} = 0 \) if \( \max(m, j) \geq N \).

A straightforward consequence of proposition 1.1 is that the Toeplitz-Bergman operator with a finite symbol belongs to the covariant algebra \( F_{q, \alpha} \).

**Proposition 1.2** Toeplitz quantization \( D(U)_q \to F_{q, \alpha} \), \( f \mapsto \hat{f} \), is a morphism of \( U_q \mathfrak{sl}_2 \)-modules.

**Proof.** One can deduce from the invariance of the scalar product in \( \mathbb{C}[z]_{q, \alpha} \) and the covariance of the left \( \mathbb{C}[z]_{q} \)-module \( \mathbb{C}[z]_{q, \alpha} \) that the linear map

\[
D(U)_q \otimes \mathbb{C}[z]_{q, \alpha} \to \mathbb{C}[z]_{q, \alpha}; \quad \hat{f} \otimes \psi \mapsto P_{q, \alpha}(\hat{f} \psi)
\]
is a morphism of \( U_q \mathfrak{sl}_2 \)-modules. On the other hand, we need to demonstrate that the linear map

\[
D(U)_q \to \mathbb{C}[z]_{q, \alpha} \otimes \mathbb{C}[z]_{q, \alpha}^*; \quad \hat{f} \mapsto \hat{f}
\]
(the tensor product here requires no completion due to proposition 1.1). Observe that the two statements are equivalent to \( U_q \mathfrak{sl}_2 \)-invariance of the same element of the corresponding completion of the tensor product \( F_{q, \alpha} \otimes D(U)'_q \), which is determined by the canonical isomorphisms \( \text{End}_\mathbb{C}(V_1, V_2) \simeq V_2 \hat{\otimes} V_1^* \), \( (V_1 \otimes V_2)^* \simeq V_2^* \hat{\otimes} V_1^* \).
Remind the notation $\text{Fun}(U)_q = \text{Pol}(\mathbb{C})_q + D(U)_q$. A very important construction of \[8\] was the representation $T$ of $\text{Fun}(U)_q$ in the infinitely dimensional vector space $H$. A basis in $H$ was formed by the vectors $v_j = T(z^j)v_0$, $j \in \mathbb{Z}_+$ (see \[8\]). $T$ provides a one-to-one map between the space of finite functions $D(U)_q$ and the space of linear operators in $H$ whose matrices in the basis $\{v_j\}_{j \in \mathbb{Z}_+}$ have finitely many non-zero entries. For $j \in \mathbb{Z}_+$, let $f_j$ stand for such finite function that $T(f_j)v_k = \delta_{jk}v_k$, $k \in \mathbb{Z}_+$.

The relation $(1 - zz^*)(v_j = q^{2j}v_j$, $j \in \mathbb{Z}_+$, motivates the following definition:

$$
(1 - zz^*)^\lambda \overset{\text{def}}{=} \sum_{n=0}^\infty q^{2n\lambda} f_n, \quad \lambda \in \mathbb{C}.
$$

(The series converges in the topological space $D(U)_q'$.)

The work \[9\] presents an explicit form of the invariant integral in the quantum disc. It is easy to show that for any finite function $f$

$$
\int_{U_q} f d\nu = (1 - q^2) \text{tr} T(f(1 - zz^*)^{-1}).
$$

**Remark 1.3.** Let $\hat{f}_0$ be the Toeplitz-Bergman operator with symbol $f_0$. It follows from the relations $z^* f_0 = f_0 z = 0$, $\int_{U_q} f_0 d\nu = 1 - q^2$ that

$$
\hat{f}_0 : z^j \mapsto \begin{cases} 1 - q^{4j} & , \ j = 0 \\ 0 & , \ j \neq 0 \end{cases}.
$$

Now the relation (1.1), the trace properties and the definition of $D(U)_q'$ imply

**Lemma 1.4**

1. For all $f \in D(U)_q$, $\lambda \in \mathbb{C}$

$$
\int_{U_q} f(z)(1 - zz^*)^\lambda d\nu = \int_{U_q} (1 - zz^*)^\lambda f(z) d\nu,
$$

2. $\int_{U_q} f_1(z)f_2(z)(1 - zz^*)d\nu(z) = \int_{U_q} f_2(z)f_1(z)(1 - zz^*)d\nu(z)$

for all $f_1(z) \in D(U)_q'$, $f_2(z) \in D(U)_q$. The following proposition describes an integral representation for matrix elements of Toeplitz-Bergman operator.

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Proposition 1.5 Let $\hat{\circ} f \in D(U)_q$ and $\hat{f} : \mathbb{C}[z]_{q, \alpha} \to \mathbb{C}[z]_{q, \alpha}; \hat{f} : z^j \mapsto \sum_{m \in \mathbb{Z}_+} \hat{f}_{mj} z^m$ be a Toeplitz-Bergman operator with symbol $\circ f$. Then

$$\hat{f}_{mj} = \frac{1 - q^{4\alpha}}{1 - q^2} \int_{U_q} P_{z,mj} \hat{f} (z) d\nu(z),$$

with

$$P_{z,mj} = \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} q^{2j} z^j (1 - z z^*)^{2\alpha+1} z^m.$$  \hspace{1cm} (1.2)

**Proof.** Apply the relation

$$(z^m, z^l)_{q, \alpha} = \frac{(q^2; q^2)_m}{(q^{4\alpha+2}; q^2)_m} \delta_{ml}, \quad m, l \in \mathbb{Z}_+$$

to get

$$\hat{f}_{mj} = \frac{(\hat{f} z^m, z^j)_{q, \alpha}}{(z^m, z^m)_{q, \alpha}} = \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} \int_{U_q} z^m \hat{f} z^j d\nu_{\alpha} =$$

$$= \frac{1 - q^{4\alpha}}{1 - q^2} \cdot \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} \int_{U_q} z^m \hat{f} z^j (1 - z z^*)^{2\alpha+1} d\nu.$$  \hspace{1cm} (1.3)

Hence, by lemma 1.4,

$$\hat{f}_{mj} = \frac{1 - q^{4\alpha}}{1 - q^2} \cdot \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} \int_{U_q} (1 - z z^*) z^j (1 - z z^*)^{2\alpha} z^m \hat{f} d\nu. \hspace{1cm} (1.4)$$

It remains to apply the relation

$$(1 - z z^*) z = q^2 z (1 - z z^*). \hspace{1cm} \square$$

**Remark 1.6.** The matrix $P_z = (P_{z,mj})_{m,j \in \mathbb{Z}_+}$ is a q-analogue for the matrix of a one-dimensional orthogonal projection onto the subspace generated by the vector $k_z$ from an overfull system (see [1]). A q-analogue of the overfull system itself is presented in the Appendix.

**Remark 1.7.** It follows from proposition 1.2 and the relation $U_q \mathfrak{sl}_2 \cdot \hat{f}_0 = F_{q, \alpha}$ to be proved later on (see proposition 6.4) that the map $D(U)_q \to F_{q, \alpha}, f \mapsto \hat{f}$, given by Toeplitz quantization is onto.

## 2 Berezin transform: finite functions

Consider a $U_q \mathfrak{sl}_2$-module $V$ and the covariant algebra $\text{End}_\mathbb{C}(V)_f \simeq V \otimes V^*$. There is a well known (see [2, 3]) formula for an invariant integral

$$\text{tr}_q : \text{End}_\mathbb{C}(V)_f \to \mathbb{C}, \quad \text{tr}_q : A \mapsto \text{tr}(A \cdot K^{-1}).$$
In the case $V = \mathbb{C}[z]_{q, \alpha}$ and $A : z^j \mapsto \sum_{m \in \mathbb{Z}_+} a_{mj}z^m$ being an element of the covariant algebra $F_{q, \alpha} \subset \text{End}_\mathbb{C}(\mathbb{C}[z]_{q, \alpha})$, one has $\text{tr}_q(A) = \sum_{k \in \mathbb{Z}_+} a_{kk}q^{-2k}$.

Given a linear operator $\hat{f} \in F_{q, \alpha}$, a distribution $f \in D(U)''_q$ is said to be a symbol of $\hat{f}$ if for all $\psi \in D(U)_q$
\[
\int_{U_q} f \cdot \psi \, d\nu = \frac{1 - q^{2\alpha}}{1 - q^2} \text{tr}_q(\hat{f} \psi). \tag{2.1}
\]
(Here $\hat{\psi}$ is the Toeplitz-Bergman operator with symbol $\psi$.)

This definition is a $q$-analogue of the Berezin’s definition, as one can observe from relation (3.15) in [1].

**Proposition 2.1** The covariant symbol of a linear operator $\hat{f} : z^j \mapsto \sum_{m \in \mathbb{Z}_+} \hat{f}_{jm}z^m$, $j \in \mathbb{Z}_+$, from the algebra $F_{q, \alpha}$, is given by
\[
f = \text{tr}_q(\hat{f} \cdot P_z) = \sum_{j, m \in \mathbb{Z}_+} \hat{f}_{jm}P_{z, mj}q^{-2j}. \tag{2.2}
\]

**Proof.** By a virtue of (1.2)
\[
\text{tr}_q(\hat{f} \psi) = \sum_{j, m \in \mathbb{Z}_+} \hat{f}_{jm} \psi_{mj}q^{-2j} = \sum_{j, m \in \mathbb{Z}_+} \hat{f}_{jm} \frac{1 - q^{4\alpha}}{1 - q^2} \int_{U_q} P_{z, mj} \psi \, d\nu(z) \cdot q^{-2j} = \\
= \int_{U_q} \left( \frac{1 - q^{4\alpha}}{1 - q^2} \sum_{j, m \in \mathbb{Z}_+} \hat{f}_{jm}P_{z, mj} \cdot q^{-2j} \right) \psi \, d\nu(z). \quad \square
\]

Note that the integral representation (2.2) is a $q$-analogue of the relation (3.4) in [1].

On can deduce from the covariance of algebras $D(U)''_q$, $F_{q, \alpha}$, the invariance of the integrals $\nu : D(U)_q \to \mathbb{C}$, $\text{tr}_q : F_{q, \alpha} \to \mathbb{C}$, the “integration in parts” formula [1, proposition 2.1], and proposition 1.2 the following

**Proposition 2.2** The linear map $F_{q, \alpha} \to D(U)''_q$, $\hat{f} \mapsto f$, which takes a linear operator to its covariant symbol, is a morphism of $U_q \mathfrak{sl}_2$-modules.

As in [13], we call the covariant symbol $f$ for the Toeplitz-Bergman operator $\hat{f}$ with symbol $\hat{f} \in D(U)_q$ a Berezin transform of the function $\hat{f}$. The associated transform map will be denoted by $B_{q, \alpha}$:
\[
B_{q, \alpha} : D(U)_q \to D(U)''_q; \quad B_{q, \alpha} : \hat{f} \mapsto f.
\]

Propositions 1.2 and 2.2 imply

**Proposition 2.3** The Berezin transform is a morphism of $U_q \mathfrak{sl}_2$-modules.
Example 2.4. Let $\hat{f} \in F_{q,\alpha}$ be given by $\hat{f}z^j = \begin{cases} 1, & j = 0 \\ 0, & j \neq 0 \end{cases}$. Then one has $f = (1 - zz^*)^{2\alpha + 1}$.

Hence, $B_{q,\alpha}f_0 = (1 - q^{4\alpha})(1 - zz^*)^{2\alpha + 1}$ since $\hat{f}_0 : z^j \mapsto \begin{cases} 1 - q^{4\alpha}, & j = 0 \\ 0, & j \neq 0 \end{cases}$ (see Example 1.3).

To conclude, we prove that Berezin transform is an integral operator, and find its kernel. In this way, a $q$-analogue of the relation (4.8) from [1] is to be obtained.

Proposition 2.5 For all $\hat{f} \in D(U)_q^+$, $$(\hat{B}_{q,\alpha} \hat{f})(z) = \int_{U_q} b_{q,\alpha}(z, \zeta) \hat{f}(\zeta) d\nu(\zeta),$$

with $b_{q,\alpha} \in D(U \times U)'_q$ being given by

$$b_{q,\alpha}(z, \zeta) = \frac{1 - q^{4\alpha}}{1 - q^2} (1 - zz^*)^{2\alpha + 1}(1 - \zeta \zeta^*)^{2\alpha + 1} \{(q^2 z^* \zeta; q^2)^{-2\alpha + 1} \cdot (z \zeta^*; q^2)^{-2\alpha + 1}\}.$$ (See [10] for the definition of $\{\cdot, \cdot\}$.)

Proof. Consider the linear operator

$$\hat{B}_{q,\alpha} : D(U)_q \to D(U)'_q, \quad \hat{B}_{q,\alpha} : \hat{f} \mapsto \int_{U_q} b_{q,\alpha}(z, \zeta) \hat{f}(\zeta) d\nu(\zeta).$$

Its kernel coincides up to a constant multiple to the invariant kernel $k_{22}^{-(2\alpha + 1)} \cdot k_{11}^{-(2\alpha + 1)}$ (see [10]). Hence, $\hat{B}_{q,\alpha}$ is a morphism of $U_q\mathfrak{sl}_2$-modules by [4, proposition 4.5]. Note that $B_{q,\alpha}$ possesses the same property. It was shown in [9] that $f_0 \in D(U)_q$ generates the $U_q\mathfrak{sl}_2$-module $D(U)_q$. In this context, the desired equality $\hat{B}_{q,\alpha}f_0 = B_{q,\alpha}f_0$ becomes a consequence of $\hat{B}_{q,\alpha}f_0 = (1 - q^{4\alpha})(1 - zz^*)^{2\alpha + 1} = B_{q,\alpha}f_0$. \qed

3 Berezin transform and Laplace-Beltrami operator

The following lemma is deduced from the relation

$$(1 - zz^*)^\lambda = \sum_{n=0}^{\infty} q^{2n\lambda} f_n, \quad \lambda \in \mathbb{C}$$

and (1.3):

Lemma 3.1 For all $m, j \in \mathbb{Z}_+$ the following decomposition is valid in $D(U)^+_q$:

$$P_{z,mj} = \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} \sum_{n=0}^{\infty} q^{4\alpha n} P^{(n)}_{z,mj},$$

with $P^{(n)}_{z,mj} = q^{2(j+n)} z^j \cdot f_n \cdot z^m \in D(U)_q$.  

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Let \( f, \psi \in D(U)_q \). Consider the integral \( \int_{U_q} \psi^* \cdot B_{q,\alpha} \circ f \ d\nu \) as a function of \( t = q^{4\alpha} \). Now proposition 2.1 and lemma 3.1 imply the analyticity of this function as \( t \in [0,1) \). Hence, one has

**Proposition 3.2** There exists a unique sequence of \( U_q \mathfrak{sl}_2 \)-module morphisms \( B^{(n)}_{q} : D(U)_q \rightarrow D(U)'_q, n \in \mathbb{Z}_+ \), such that for all \( \circ f \in D(U)_q \)

\[
B_{q,\alpha} \circ f = \sum_{n=0}^{\infty} q^{4\alpha n} B^{(n)}_{q} \circ f. \tag{3.2}
\]

Our purpose is to prove that the linear operators \( B^{(n)}_{q} \) are polynomials of Laplace-Beltrami operator in the quantum disc.

Let \( p_j(t) = \sum_{k=0}^{j} \frac{(q^{-2j};q^2)_k}{(q^2;q^2)_k} q^{2k} \cdot \prod_{i=0}^{k-1} \left( 1 - q^{2i} \left( (1-q^2)^2 t + 1 + q^2 \right) + q^{4i+2} \right). \tag{3.3} \)

**Lemma 3.3** \( p_j(\square)f_0 = q^{2j} \cdot f_j \) for all \( j \in \mathbb{Z}_+ \).

**Proof.** Remind [8] that for all \( l \in \mathbb{C} \) the basic hypergeometric series

\[
\varphi_l = _3\Phi_2 \left[ (1 - z z^*)^{-1}, q^{-2l}, q^{2(l+1)}; q^2, q^2 \right],
\]

converge in \( D(U)'_q \), and

\[
\square \varphi_l = - \frac{(1 - q^{-2l})(1 - q^{2l+2})}{(1 - q^2)^2} \varphi_l.
\]

By a virtue of [8, §6], it suffices to show that for all \( l \in \mathbb{C} \)

\[
q^{-2j} \cdot \int_{U_q} \varphi_l^* \cdot p_j(\square)f_0 d\nu = \int_{U_q} \varphi_l^* f_j d\nu. \tag{3.4}
\]

After substituting \( l \) by \( T \) we find out that (3.4) is equivalent to

\[
p_j \left( - \frac{(1 - q^{-2l})(1 - q^{2(l+1)})}{(1 - q^2)^2} \right) = _3\Phi_2 \left[ q^{-2j}, q^{-2l}, q^{2(l+1)}; q^2, q^2 \right].
\]

Prove this relation. By the definition of \( _3\Phi_2 \) one has

\[
= \sum_{k=0}^{j} \frac{(q^{-2j};q^2)_k}{(q^2;q^2)_k} \cdot \prod_{i=0}^{k-1} \left( 1 - q^{2i} \cdot q^{2(l+1)} \right) \cdot q^{2k} =
\]

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\[ \sum_{k=0}^{j} \left( \frac{q^{2j}}{(q^2;q^2)_k} \right) \prod_{i=0}^{k-1} \left( 1 + q^{2i} \cdot u + q^{4i+2} \right) \cdot q^{2k} \]

It remains to prove that

\[ q^j \sum_{k=0}^{j-1} \left( \frac{q^{2k}}{(q^2;q^2)_k} \right) \prod_{i=0}^{k-1} \left( 1 + q^{2i} \cdot u + q^{4i+2} \right) \cdot q^{2k} \]

For that, it suffices to exclude \( u \) by a substitution \( u = -(1-q^2) t - 1 - q^2 \).

The next statement refines essentially proposition 3.2.

**Proposition 3.4** For all \( f \in D(U)_q \) the following expansion in \( D(U)'_q \) is valid:

\[ B_{q,\alpha} f = (1 - q^{4\alpha}) \sum_{j \in \mathbb{Z}_+} q^{4\alpha \cdot j} \cdot p_j(\square) f. \] (3.5)

**Proof.** One has the relation \( B_{q,\alpha} f_0 = (1 - q^{4\alpha}) \sum_{k \in \mathbb{Z}_+} q^{(4\alpha+2)k} f_k \) (see example 2.4). Hence, in the special case \( \tilde{\circ} f = f_0 \) our statement follows from lemma 3.3. It remains to take into account that \( f_0 \) generates the \( U_q \mathfrak{sl}_2 \)-module \( D(U)'_q \), and the operators \( B_{q,\alpha}, \square \) are morphisms of \( U_q \mathfrak{sl}_2 \)-modules (see [8, proposition 2.1]). \( \square \)

**Corollary 3.5**

\[ B_q^{(n)} = \begin{cases} 
I & , n = 0 \\
p_n(\square) - p_{n-1}(\square) & , n \in \mathbb{N}
\end{cases} \] (3.6)

4 **Toeplitz-Bergman operators with bounded symbols**

It is very well known [5, 6] that the \( \ast \)-algebra \( \text{Pol}(\mathbb{C})_q \) has a unique up to unitary equivalence faithful irreducible representation. As it was described in [8], this representation \( T \) lives in a Hilbert space \( H \) constructed as a completion of the pre-Hilbert space \( H \). Let \( L(H) \) be the algebra of all bounded operators in \( H \) and \( H' \) the vector space of all bounded antilinear functionals on \( H \). One has

\[ \text{End}_{\mathbb{C}}(H) \subset L(H) \subset \text{Hom}_{\mathbb{C}}(H, H'). \]

It was demonstrated in [8] that the map \( T : \text{Pol}(\mathbb{C})_q \rightarrow L(H) \) is extendable by a continuity up to the isomorphism \( T : D(U)'_q \cong \text{Hom}_{\mathbb{C}}(H, H') \).

We call a distribution \( f \in D(U)'_q \) bounded if \( T(f) \in L(H) \). Impose the notation

\[ L^\infty_q = \{ f \in D(U)'_q \mid T(f) \in L(H) \}, \quad \| f \|_\infty = \| T(f) \|. \]

(It is easy to show that the algebra \( L^\infty \) defined in this way is isomorphic to the enveloping von Neumann algebra of the \( C^* \)-algebra of continuous functions in the quantum disk, which was considered, in particular, in [4]).
Consider the subspaces
\[ C[z]_{q,\infty} = \{ f \in D(U)_q \mid f \cdot z = 0 \}, \]
\[ H^2_{q,\infty} = \{ f \in L^2(U)_q \mid f \cdot z = 0 \}, \]
\[ \mathbb{C}[[z]]_{q,\infty} = \{ f \in D(U)'_q \mid f \cdot z = 0 \}. \]

It follows from [9, proposition 3.3] that
\[ C[z]_{q,\infty} = C[z] \cdot f_0, \quad \mathbb{C}[[z]]_{q,\infty} = \mathbb{C}[[z]] \cdot f_0, \]
and hence
\[ H \simeq C[z]_{q,\infty}, \quad \Pi \simeq H^2_{q,\infty}, \quad H' \simeq \mathbb{C}[[z]]_{q,\infty}. \]

\( T \) is unitarily equivalent to the representation \( \hat{T} \) of \( \text{Pol}(\mathbb{C})_q \) in \( H^2_{q,\infty} \) given by
\[ \hat{T} : \psi \mapsto f \cdot \psi; \quad f \in \text{Pol}(\mathbb{C})_q, \quad \psi \in H^2_{q,\infty} \subset D(U)'_q. \]

Thus, a distribution \( f \in D(U)'_q \) is bounded iff the linear operator \( \hat{T}(f) \) is in \( L(H^2_{q,\infty}) \); in this case \( \|f\|_{\infty} = \|\hat{T}(f)\|_{\infty} \).

The following proposition justifies the use of the symbol \( \infty \) in the notation for the vector spaces \( C[z]_{q,\infty}, H^2_{q,\infty}, \mathbb{C}[[z]]_{q,\infty} \).

**Proposition 4.1** For any polynomial \( \psi \in \mathbb{C}[z]_q \)
\[ \lim_{\alpha \to \infty} (\psi, \psi)_{q,\alpha} = \frac{1}{1-q^2} (\psi f_0, \psi f_0). \]

**Proof.**
\[ (\psi, \psi)_{q,\infty} \overset{\text{def}}{=} \lim_{\alpha \to \infty} \frac{1-q^2}{1-q^{4\alpha}} (\psi, \psi)_{q,\alpha} = \lim_{\alpha \to \infty} \int_{U_q} \psi^* \psi \sum_{n=0}^{\infty} q^{4n+4\alpha} f_{n\alpha} d\nu = \int_{U_q} \psi^* \psi f_0 d\nu = (\psi f_0, \psi f_0). \]

The following remark will not be used in the sequel. Proposition 4.1 allows one to prove that the covariant algebra \( D(U)_q \) is isomorphic to a "limit \( F_{q,\infty} \) of covariant algebras \( F_{q,\alpha} \) as \( \alpha \to \infty \)". This leads to an alternate scheme of producing the covariant algebra \( D(U)_q \) of finite functions in the quantum disk. Under this scheme, at the first step a unitarizable Harish-Chandra module \( V_\alpha \) with lowest weight \( \alpha > 0 \) and the covariant algebras \( V_\alpha \otimes V_\alpha^* \to \text{End}_{\mathbb{C}}(V_\alpha) \) are constructed. The second step is in "passage to the limit" \( \lim_{\alpha \to +\infty} V_\alpha \otimes V_\alpha^* \) which is to be declared the algebra of finite functions in the quantum disk.

Finally, impose the notation
\[ F_{q,\infty} \overset{\text{def}}{=} \text{End}_{\mathbb{C}}(C[z]_{q,\infty}, \mathbb{C}[[z]]_{q,\infty}). \]

It follows from the definitions that the representation \( \hat{T} \) is extendable up to a bijection \( \hat{T} : D(U)' \to F_{q,\infty} \).
It should be noted that $\text{Pol}(\mathbb{C})_q \subset L_q^\infty$. This can be deduced, for example, from the fact that the representation $\hat{T}$ of $\text{Pol}(\mathbb{C})_q$ in the pre-Hilbert space $\mathbb{C}[z]_{q,\infty}$ is a $^*$-representation of this algebra. Hence, $I - \hat{T}(z)\hat{T}(z^*) \geq 0$, $\|\hat{T}(z)\| = \|\hat{T}(z^*)\| = 1$.

Let $A$ be a compact linear operator in a Hilbert space and $|A| \overset{\text{def}}{=} (A^*A)^{1/2}$. Consider the sequence of eigenvalues of $|A|$, with their multiplicities being taken into account:

$$s_1(A) \geq s_2(A) \geq \ldots .$$

The numbers $s_p(A)$, $p \in \mathbb{N}$, are called s-values of $A$.

Remind the notation $S_\infty$ for the ideal of all compact operators in a Hilbert space, together with the notation

$$\|A\|_p = \left(\sum_{n \in \mathbb{N}} s_n(A)^p\right)^{1/p}, \quad S_p = \{A \in S_\infty | \|A\|_p < \infty\}, \quad p > 0,$$

for the normed ideals of von Neumann-Schatten (see [3]).

**Lemma 4.2** For any function $\psi \in D(U)_q$

$$\|\psi\| = (1 - q^2)^{1/2} \cdot \|\hat{T}(\psi(1 - zz^*)^{-1})\|_2,$$

with $\|\psi\| = \left(\int_{U_q} |\psi|^2 d\nu\right)^{1/2}$.

**Proof.** It follows from (1.1) and the well known tracial properties of an operator $A \in S_1$ that

$$\|\psi\|^2 = (1 - q^2)\text{tr} \hat{T}(\psi^*\psi(1 - zz^*)^{-1}) = (1 - q^2)\text{tr} \hat{T}((1 - zz^*)^{-1/2}\psi^*\psi(1 - zz^*)^{-1/2}). \quad \Box$$

**Corollary 4.3** Let $f \in L_q^\infty$, $\psi \in D(U)_q$, then $\hat{f} \psi \in L^2(d\nu)_q$ and $\|\hat{f} \psi\| \leq \|f\|_\infty \cdot \|\psi\|$. 

**Proof.**

$$\|\hat{f} \psi\| = (1 - q^2)^{1/2}\|\hat{T}(f)\hat{T}(\psi(1 - zz^*)^{-1})\|_2 \leq (1 - q^2)^{1/2}\|\hat{T}(f)\| \cdot \|\hat{T}(\psi(1 - zz^*)^{-1/2})\|_2 = \|f\|_\infty \cdot \|\psi\|. \quad \Box$$

It follows from the boundedness of the multiplication operator by a bounded function $\hat{f}$:

$$D(U)_q \rightarrow L^2(d\nu)_q, \quad \psi \mapsto \hat{f} \psi$$

that it admits an extension by a continuity onto the entire space $L^2(d\nu)_q$. This allows one to define a Toeplitz-Bergman operator $\hat{f}$ with symbol $\hat{0}f \in L_q^\infty$:

$$\hat{f} : H^2_{q,\alpha} \rightarrow H^2_{q,\alpha}; \quad \hat{f} : \psi \mapsto P_{q,\alpha}(\hat{0}f \psi).$$
By a virtue of corollary 4.3 one has

\[ \| \hat{f} \| \leq \| \hat{f} \|_\infty, \]  

with \( \| \hat{f} \| \) being the norm of the operator \( \hat{f} \) in \( H^2_{q,a} \). Thus we get a norm decreasing linear map \( L^\infty_q \to L(H^2_{q,a}), \hat{f} \mapsto \hat{f} \). This definition generalizes that of a Toeplitz-Bergman operator with finite symbol (see section 2).

5 Berezin transform: bounded functions

The \( U_q \mathfrak{sl}_2 \)-module \( \mathbb{C}[z]_{q,\alpha} \) is formed by polynomials \( \psi = \sum_{i \in \mathbb{Z}_+} a_i(\psi)z^i \). Consider a completion \( \mathbb{C}[[z]]_{q,\alpha} \) of this vector space in the topology of coefficientwise convergence, and impose the notation \( F_{q,\alpha} \overset{\text{def}}{=} \text{Hom}_\mathbb{C}(\mathbb{C}[z]_{q,\alpha}, \mathbb{C}[[z]]_{q,\alpha}) \) for the corresponding completion of \( F_{q,\alpha} \). Equip \( F_{q,\alpha} \) with the topology of pointwise (strong) convergence:

\[ \lim_{n \to \infty} A_n = A \iff \forall \psi \in \mathbb{C}[z]_{q,\alpha} \lim_{n \to \infty} A_n \psi = A \psi. \]

Evidently, \( \mathbb{C}[z]_{q,\alpha} \subset H^2_{q,a} \subset \mathbb{C}[[z]]_{q,\alpha} \), and so

\[ F_{q,\alpha} \subset L(H^2_{q,a}) \subset F_{q,\alpha}. \]

The representation operators of \( E, F, K^\pm \) in \( \mathbb{C}[z]_{q,\alpha} \) have degrees +1, −1, 0 respectively. Hence they are extendable by a continuity from \( \mathbb{C}[z]_{q,\alpha} \) onto \( \mathbb{C}[[z]]_{q,\alpha} \), and from \( F_{q,\alpha} \) onto \( F_{q,\alpha} \).

Of course, \( F_{q,\alpha} \) is a covariant bimodule over the covariant algebra \( F_{q,\alpha} \). It is easy to show that the linear functional

\[ F_{q,\alpha} \otimes F_{q,\alpha} \to \mathbb{C}, \quad \hat{f} \otimes \hat{\psi} \mapsto \text{tr}_q(\hat{f} \hat{\psi}) \]

is extendable by a continuity up to a morphism of \( U_q \mathfrak{sl}_2 \)-modules \( F_{q,\alpha} \otimes F_{q,\alpha} \to \mathbb{C} \).

Define a covariant symbol \( f \in D(U)'_q \) of a linear operator \( \hat{f} \in F_{q,\alpha} \) by (2.1). The map \( F_{q,\alpha} \to D(U)'_q \) arising this way is a \( U_q \mathfrak{sl}_2 \)-module morphism.

In the following proposition we use notation \( \hat{f} \) for a linear operator without assuming it to be a Toeplitz-Bergman operator.

**Proposition 5.1** Let \( \hat{f} \) be a linear operator

\[ \hat{f} : \mathbb{C}[z]_{q,\alpha} \to \mathbb{C}[[z]]_{q,\alpha}, \quad \hat{f} : z^j \mapsto \sum_{m \in \mathbb{Z}_+} \hat{f}_{mj} z^m, \quad j \in \mathbb{Z}_+. \]

The series \( \sum_{j,m \in \mathbb{Z}_+} \hat{f}_{jm} P_{z,mj} q^{-2j} \) converges in \( D(U)'_q \) to the covariant symbol of \( \hat{f} \).

**Proof.** It follows from the results of section 1 that for any \( \hat{\psi} \in D(U)_q \) all but finitely many of integrals \( \int_{U_q} P_{z,mj} \hat{\psi} (z) d\nu(z) \) are zero. This allows one to reproduce literally the argument used in the proof of proposition 2.1. \( \square \)
Let \( \hat{\omega} \in L_q^\infty \), and \( \hat{f} \in L(H_{q,\alpha}^2) \subset \mathcal{F}_{q,\alpha} \) be the Toeplitz-Bergman operator with symbol \( \hat{\omega} \). We follow [13] in using the term "Berezin transform of the function \( \hat{f} \)" for the covariant symbol of the linear operator \( \hat{f} \).

Our purpose is to decompose the operator-function \( B_{q,\alpha} : L_q^\infty \to D(U'_q) \) into series in powers of \( t = q^{4\alpha} \) (cf. (3.5)).

One can use again the argument of proposition 1.5 to get (1.4) for all bounded symbols \( \hat{f} \in L_q^\infty \). An application of (1.1) and the fact that \( \hat{T}((1-zz^*)^{2\alpha}) \) is a trace class operator for all \( \alpha > 0 \), yields also

**Proposition 5.2** For all \( \hat{\omega} \in L_q^\infty \), \( m, j \in \mathbb{Z}_+ \)

\[
\hat{f}_{mj} = \frac{(q^{4\alpha+2};q^2)_m}{(q^2;q^2)_m} \cdot (1 - q^{4\alpha}) \cdot \text{tr}\left( \hat{T}(z^j(1-zz^*)^{2\alpha}z^{*m})\hat{T}(\hat{f}) \right).
\]

Let \( \Theta \) be the vector space of holomorphic functions in the unit disc with values in the Banach algebra \( S_1 \) of trace class operators in \( \mathcal{M} \). (Each function \( Q(t) \) from \( \Theta \) admits an expansion into the power series \( Q(t) = \sum_{n \in \mathbb{Z}_+} t^n \cdot Q^{(n)} \) with \( \lim_{n \to \infty} ||Q^{(n)}||_1^n \leq 1 \)).

**Proposition 5.3** For all \( j, m \in \mathbb{Z}_+ \)

\[
\sum_{n \in \mathbb{Z}_+} t^n \cdot \hat{T}(z^j \cdot f_n \cdot z^{*m}) \in \Theta.
\]

**Proof.** Remind that \( \hat{T}(z) \hat{T}(z^*) = 1 - \sum_{n \in \mathbb{Z}_+} q^{2n} \hat{T}(f_n) \), and that \( \hat{T}(f_n) \) are one-dimensional projections, \( n \in \mathbb{Z}_+ \). Hence \( ||\hat{T}(z)|| = ||\hat{T}(z^*)|| = ||\hat{T}(f_n)||_1 = 1 \). Finally,

\[
||\hat{T}(z^j f_n z^{*m})||_1 \leq ||\hat{T}(z)||^j \cdot ||\hat{T}(f_n)||_1 \cdot ||\hat{T}(z^*)||^m = 1.
\]

Propositions 5.2, 5.3 and the definition of Berezin transform imply

**Corollary 5.4** Let \( \psi \in D(U)_q \). There exists a unique function \( Q_\psi(t) \in \Theta \) such that

\[
\int_{U_q} (B_{q,\alpha} \hat{\omega}) \psi d\nu = \text{tr}\left( \hat{T}(\hat{\omega}) Q_\psi(q^{4\alpha}) \right) \tag{5.1}
\]

for all \( \hat{\omega} \in L_q^\infty \).

**Proof.** The uniqueness of \( Q_\psi(t) \) is evident. In fact, given such \( A \in L(\mathcal{M}) \) that for all \( \hat{f} \in D(U)_q \) one has \( \text{tr}\left( \hat{T}(\hat{\omega}) A \right) = 0 \), then surely \( A = 0 \). The existence of \( Q_\psi \in \Theta \) follows from propositions 5.2, 5.3 and the definition of Berezin transform.

The coefficients of the Taylor series for the holomorphic function \( Q_\psi(t) \) at \( t = 0 \) are trace class operators. One can use (3.5) and (1.1) to express those coefficients via the operators \( \hat{T}(p_j(\hat{\square})\psi) \), \( j \in \mathbb{Z}_+ \). Thus we get the following
Proposition 5.5 Let $\hat{f} \in L_\infty^q$.

1. For all $\alpha > 0$ one has an expansion in $D(U)'_q$

   $$B_{q,\alpha} \hat{f} = \sum_{n \in \mathbb{Z}_+} q^{4\alpha n} B_q^{(n)} \hat{f}.$$ 

2. For all $\psi \in D(U)_q$ one has the asymptotic expansion

   $$\int_{U_q} \left( B_{q,\alpha} \hat{f} \right) \psi \, d\nu \sim_{\alpha \to +\infty} \sum_{n=0}^{\infty} q^{4\alpha n} \int_{U_q} \left( B_q^{(n)} \hat{f} \right) \psi \, d\nu. \tag{5.2}$$

Here $B_q^{(n)} : D(U)'_q \to D(U)'_q$ are polynomial functions of the Laplace-Beltrami operator, given explicitly by (3.6).

6 Covariant symbols

The notation $\hat{z}, \hat{z}^*$ in \cite{8} stand for the Toeplitz-Bergman operators with symbols $z, z^*$. Those are defined in the graded vector space $\mathbb{C}[z]_{q,\alpha}$, with $\deg(\hat{z}) = +1, \deg(\hat{z}^*) = -1$. Hence for any matrix $(a_{ij})_{i,j \in \mathbb{Z}_+}$ with numerical entries, series

$$\hat{f} = \sum_{i,j \in \mathbb{Z}_+} a_{ij} \hat{z}^i \hat{z}^{*j} \tag{6.1}$$

converge in the topological vector space $\mathcal{F}_{q,\alpha} = \text{Hom}_\mathbb{C}(\mathbb{C}[z]_{q,\alpha}, \mathbb{C}[[z]]_{q,\alpha})$.

Proposition 6.1 $\hat{f} z^n = \sum_{m \in \mathbb{Z}_+} b_{mn} z^m, \ n \in \mathbb{Z}_+$,

with $b_{mn} = \min(m,n) \sum_{j=0}^{\min(m,n)} \frac{(q^{2m};q^{-2})_n \cdot (q^{2n};q^{-2})_j}{(q^{4\alpha+2m};q^{-2})_n \cdot (q^{4\alpha+2n};q^{-2})_j} a_{m-j,n-j}$.\[\]

Proof. It suffices to apply the relations

$$\hat{z}(z^m) = z^{m+1}, \quad \hat{z}^*(z^m) = \begin{cases} \frac{1-q^{2m}}{1-q^{4\alpha+2m}} \cdot z^{m-1}, & m \neq 0 \\ 0, & m = 0 \end{cases} \tag{6.2}$$

which were established in \cite{8, section 7} (see also \cite{5}). \[\]

Corollary 6.2 For any linear operator $\hat{f} \in \mathcal{F}_{q,\alpha}$ there exists a unique decomposition (6.1).
EXAMPLE 6.3. Consider the linear operator \( \hat{f}_0 : z^j \mapsto \begin{cases} 1 - q^{4\alpha}, & j = 0 \\ 0, & j \neq 0 \end{cases} \), \( j \in \mathbb{Z}_+ \). Prove that
\[
\hat{f}_0 = (1 - q^{4\alpha}) \sum_{k=0}^{\infty} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} q^{(4\alpha+2)k} z^k \bar{z}^{*k}.
\] (6.3)

Pass from the equality of operators to the equalities of their matricial elements with respect to the base \( \{z^n\}_{n \in \mathbb{Z}_+} \). Of course, all the non-diagonal elements are zero. An identification of the diagonal elements yields
\[
\sum_{k=0}^{j} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} \cdot \frac{(q^{2j}; q^{-2})_k}{(q^{4\alpha+2}; q^{-2})_k} \cdot q^{(4\alpha+2)k} = \delta_{j0}.
\] (6.4)

It suffices to consider the case \( j > 0 \). Multiply (6.4) by \( \frac{(q^{4\alpha+2j}; q^{-2})_j}{(q^2; q^2)_j} \) to get
\[
\sum_{k=0}^{j} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} \cdot \frac{(q^{4\alpha+2}; q^2)_j-k}{(q^2; q^2)_j-k} \cdot q^{(4\alpha+2)k} = 0.
\]

That is,
\[
\sum_{k+m=j} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} \cdot q^{(4\alpha+2)k} \cdot \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} = 0.
\]

So, it remains to consider the q-binomial series (see [4]):
\[
a(t) = \sum_{k \in \mathbb{Z}_+} \frac{(q^{-4\alpha-2}; q^2)_k}{(q^2; q^2)_k} \cdot q^{(4\alpha+2)k} \cdot t^k = \frac{(t; q^2)_{\infty}}{(q^{4\alpha+2}t; q^2)_{\infty}},
\]
\[
b(t) = \sum_{m \in \mathbb{Z}_+} \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} \cdot t^m = \frac{(q^{4\alpha+2}t; q^2)_{\infty}}{(t; q^2)_{\infty}},
\]
and to observe that \( a(t) \cdot b(t) = 1 \).

It was noted in section 1 that \( \hat{f}_0 \) is a Toeplitz-Bergman operator with symbol \( f_0 \). This element generates the topological 2-module \( F_{q,\alpha} \), as one can see from

Proposition 6.4 \( U_q \mathfrak{sl}_2 \hat{f}_0 = F_{q,\alpha} \).

**Proof.** Since for all \( i, j, n \in \mathbb{Z}_+ \), \( \hat{z}^i \hat{f}_0 \hat{z}^n = z^j \mapsto (1 - q^{4\alpha}) \cdot \frac{(q^j; q^2)_n}{(q^{4\alpha}; q^2)_n} \cdot \delta_{jn} z^i \), the linear operators \( \{\hat{z}^i \hat{f}_0 \hat{z}^n\}_{i, n \in \mathbb{Z}_+} \) generate \( F_{q,\alpha} \) as a vector space. It remains to show that all those operators are in the \( U_q \mathfrak{sl}_2 \)-module generated by \( \hat{f}_0 \). For that, it suffices to reproduce the proof of [1] theorem 3.9]. One has only to alter the notation for the generators (now they are \( \hat{z}, \hat{z}^*, \hat{f}_0 \)), together with the constants in formulae which describe the action of \( X^\pm \) on \( f_0 \):
\[
X^+ \hat{f}_0 = c' \hat{z} \cdot \hat{f}_0; \quad X^- \hat{f}_0 = c'' \hat{f}_0 \cdot \hat{z}^*; \quad c', c'' \neq 0.
\]

These relations follow from proposition 1.2, [2] proposition 3.8], and
\[
C \hat{z} \hat{f}_0 = C \hat{z} \hat{f}_0, \quad C \hat{f}_0 \hat{z}^* = C \hat{f}_0 \hat{z}^*.
\]
The latter relations can be deduced from
\[ \text{Im} \ f_0 = \text{Im} \ f_0 z^* = \mathbb{C}; \quad \text{Ker} \ f_0 = \text{Ker} \ f_0 z^* = \mathbb{C}^\perp. \]

It was shown in [section 1] that for any \( f \in D(U)_q \) there exists a unique decomposition
\[
a = \sum_{j,k \in \mathbb{N}_+} a_{jk} z^j z^k \]
similar to (6.1).

**Example 6.5.** Prove that
\[
(1 - q^{4\alpha})(1 - z^*)^{2n+1} = (1 - q^{4\alpha}) \sum_{k \in \mathbb{N}_0} \left( \frac{q^{-(4\alpha+2)}; q^2}{q^2; q^2} \right)_k q^{4\alpha+2k} z^k z^k.
\]

Apply the operator \( \hat{T} \) to the both parts of (6.5) and identify the matricial elements with respect to the base \( \{ z^m \} \) (it suffices to consider the diagonal elements).

Use the relations \( \bullet \)
\[
\hat{T}(z) z^m = z^{m+1}, \quad \hat{T}(z^*) z^m = \begin{cases} (1 - q^{2m}) z^{m-1}, & m \neq 0 \\ 0, & m = 0 \end{cases}
\]
to get
\[
\sum_{k=0}^j \frac{q^{4\alpha-2}; q^2}{q^2; q^2} \cdot \frac{q^{4\alpha+2k} q^{4\alpha+2k}}{q^2; q^2} = q^{2j(2\alpha+1)},
\]
\[
\sum_{k+m=j} \frac{q^{4\alpha-2}; q^2}{q^2; q^2} \cdot \frac{1}{q^2; q^2} = q^{2j(2\alpha+1)}.
\]

It remains to pass to the q-binomial decompositions (see [4]) in the both sides of the obvious relation \( a(t) b(t) = c(t) \), with
\[
a(t) = \frac{t; q^2}{(q^{4\alpha+2t}; q^2)_\infty}, \quad b(t) = \frac{1}{(t; q^2)_\infty}; \quad c(t) = \frac{1}{(q^{4\alpha+2t}; q^2)_\infty}.
\]

**Proposition 6.6** The covariant symbol of the operator \( \hat{f} = \sum_{j,k \in \mathbb{N}_+} a_{jk} z^j z^k \) is
\[
f = \sum_{j,k \in \mathbb{N}_+} a_{jk} z^j z^k.
\]

**Proof.** Let \( S'_{\alpha,q} : F_{q,\alpha} \rightarrow D(U)'_q \) be the map which takes a linear operator \( \hat{f} \in F_{q,\alpha} \) to its covariant symbol. We have to prove that this map coincides with the map \( S''_{\alpha,q} : F_{q,\alpha} \rightarrow D(U)'_q \), given by \( S''_{\alpha,q} : \sum_{j,k \in \mathbb{N}_+} a_{jk} z^j z^k \rightarrow \sum_{j,k \in \mathbb{N}_+} a_{jk} z^j z^k \). The linear operators \( S' \), \( S'' \) are morphisms of \( U_q \mathfrak{sl}_2 \)-modules, and the element \( f_0 \) generates the topological \( U_q \mathfrak{sl}_2 \)-module \( F_{q,\alpha} \) by proposition 6.4. Thus it suffices to obtain the relation \( S'(f_0) = S''(f_0) \). It was shown in section 2 that \( S''(f_0) = (1 - q^{4\alpha})(1 - z^*)^{2n+1} \). So it remains to see that \( S''(f_0) = (1 - q^{4\alpha})(1 - z^*)^{2n+1} \).

This follows from (6.3), (6.5). \( \Box \)

\textsuperscript{1}These relations can be deduced from (6.2) via passage to the limit as \( \alpha \rightarrow \infty \).
Corollary 6.7 The map $F_{q, \alpha} \to D(U)_q'$ which takes a linear operator to its covariant symbol is one-to-one.

To conclude, we give another illustration of corollary 6.2. Our immediate purpose is to get the expansion $\hat{z}^* \hat{z} = \sum_{k \in \mathbb{Z}_+} c_k \hat{z}^k \hat{z}^*^k$ and to find a generating function $c(u) = \sum_{k \in \mathbb{Z}_+} c_k u^k$.

By (6.2), the coefficients $c_k$ can be found from the system of equations
\[
\sum_{k=0}^m c_k \frac{(q^{2m}; q^{-2})_k}{(q^{4\alpha+2m}; q^{-2})_k} = \frac{1 - q^{2(m+1)}}{1 - q^{4\alpha+2(m+1)}}, \quad m \in \mathbb{Z}_+. \tag{6.6}
\]

Apply an expansion of the right hand side of (6.6) as series:
\[
\frac{1 - q^{2(m+1)}}{1 - q^{4\alpha+2(m+1)}} = 1 + \sum_{j \in \mathbb{N}} (1 - q^{-4\alpha}) q^{(2\alpha+1+m)2j}.
\]

For a fixed $j \in \mathbb{N}$ consider the system of equations
\[
\sum_{k=0}^m \gamma_k \frac{(q^{4\alpha+2}; q^2)_i}{(q^2; q^2)_i} = q^{2mj}, \quad m \in \mathbb{Z}_+. \tag{6.7}
\]

Multiply (6.7) by $\frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m}$ and convert it to the form
\[
\sum_{i+k=m} \gamma_k \frac{(q^{4\alpha+2}; q^2)_i}{(q^2; q^2)_i} = q^{2mj} \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m}. \tag{6.8}
\]

Introduce the generating functions
\[
\alpha(u) = \sum_{m \in \mathbb{Z}_+} \frac{(q^{4\alpha+2}; q^2)_m}{(q^2; q^2)_m} q^{2mj} u^m = \frac{(q^{4\alpha+2+2j}; q^2)_\infty}{(q^2; q^2)_\infty},
\]
\[
\beta(u) = \sum_{i \in \mathbb{Z}_+} \frac{(q^{4\alpha+2}; q^2)_i}{(q^2; q^2)_i} u^i = \frac{(q^{4\alpha+2}; q^2)_\infty}{(u; q^2)_\infty}.
\]

It follows from (6.8) that
\[
\gamma(u) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}_+} \gamma_k u^k = \frac{\alpha(u)}{\beta(u)} = \frac{(u; q^2)_j}{(q^{4\alpha+2}; q^2)_j}.
\]

Turn back to the initial system (6.6) to obtain
\[
c(u) = 1 + \sum_{j \in \mathbb{N}} (1 - q^{-4\alpha}) q^{(2\alpha+1)2j} \frac{(u; q^2)_j}{(q^{4\alpha+2}; q^2)_j}. \tag{6.9}
\]
7 *- Product

Let $A$ be an algebra over $\mathbb{C}$. Impose the notation

$$C[[q^{4\alpha}]] = \left\{ \sum_{n \in \mathbb{Z}_+} q^{4\alpha} u_n | u_n \in \mathbb{C}, n \in \mathbb{Z}_+ \right\},$$

$$A[[q^{4\alpha}]] = \left\{ \sum_{n \in \mathbb{Z}_+} q^{4\alpha} a_n | a_n \in A \right\}$$

for the ring of formal series with complex coefficients and the $C[[q^{4\alpha}]]$-algebra of formal series with coefficients from $A$.

Our goal is to derive a new "distorted" multiplication in the $C[[q^{4\alpha}]]$-algebra $\text{End}(C[z; q, \infty])[[q^{4\alpha}]]$ from an ordinary multiplication in the $C[[q^{4\alpha}]]$-algebra $\text{End}(C[z; q, \infty])[[q^{4\alpha}]]$.

The presence of the base $\{z^m\}_{m=0}^{\infty}$ in each vector space $C[z; q, \alpha]$, $C[z; q, \infty]$ allows one to "identify" them via the isomorphisms $i_{i\alpha} : C[z; q, \infty] \rightarrow C[z; q, \alpha]$; $i_{i\alpha} : z^m \mapsto z, m \in \mathbb{Z}_+$.

Consider the linear operators $i_{i\alpha}^{-1} z^j z^{*k} i_{i\alpha}, j, k \in \mathbb{Z}_+$ in $C[z; q, \infty]$. It follows from (6.2) that

$$i_{i\alpha}^{-1} z^j z^{*k} i_{i\alpha} : z^m \mapsto \frac{(q^2 m; q^2)^{k}}{(q^{4\alpha+2m}; q^2)^{k}} z^{m-k+j}, \quad m \in \mathbb{Z}_+. \quad (7.1)$$

From now on we shall identify the rational function $\frac{1}{(q^{4\alpha+2m}; q^2)^{k}}$ of an indeterminate $t = q^{4\alpha}$ with its q-binomial series (see [3])

$$\left( \frac{q^{4\alpha+2m+2}; q^2}{q^{4\alpha+2m+2-2k}; q^2} \right)_{\infty} = \sum_{n \in \mathbb{Z}_+} \left( \frac{(q^{2k}; q^2)^{n}}{(q^2; q^2)^n} \cdot q^{2(m-k+1)n} \right) q^{4\alpha n}.$$

The construction of *-product will be done via the $C[[q^{4\alpha}]]$-linear map

$$Q : \text{Pol}(C)_q[[q^{4\alpha}]] \rightarrow \text{End}(C[z; q, \infty])[[q^{4\alpha}]]$$

defined as follows:

$$Q : \sum_{n \in \mathbb{Z}_+} q^{4\alpha n} \sum_{j,k=1}^{N(n)} a^{(n)}_{j,k} z^j z^{*k} \mapsto \sum_{n \in \mathbb{Z}_+} q^{4\alpha n} \sum_{j,k=1}^{N(n)} a^{(n)}_{j,k} i_{i\alpha}^{-1}(z^j z^{*k}) i_{i\alpha}$$

for all numbers $a^{(n)}_{j,k} \in \mathbb{C}$.

**Lemma 7.1** The map $Q$ is injective.

**Proof.** In the case $Q$ has a non-trivial kernel, there should be for some $j, k \in \mathbb{Z}_+$, $\sum_{n \in \mathbb{Z}_+} c_n z^j z^{*k} = 0$, with $c_n \in C[[q^{4\alpha}]]$, $n \in \mathbb{Z}_+$, and $c_0 \neq 0$. An application of the operator $\sum_{n \in \mathbb{Z}_+} c_n z^j z^{*k}$ to the vector $z^k$ yields $c_0 \cdot \frac{(q^{2k}; q^2)^{k}}{(q^{4\alpha+2k}; q^2)^{k}} \cdot z^j = 0$, which is a contradiction. \(\square\)
Lemma 7.2 Let \( j, k \in \mathbb{Z}_+ \) and \( \hat{f} = z^j z^k \). The Toeplitz-Bergman operator \( \hat{f} \) with symbol \( \hat{f} \) is \( \hat{z}^j \hat{z}^k \).

Proof. For all \( \psi_1, \psi_2 \in H^2_{q, \alpha} \) one has
\[
(\hat{f}\psi_1, \psi_2)_{q, \alpha} = (P_{q, \alpha}(z^j z^k \psi_1), \psi_2)_{q, \alpha} = (z^j z^k \psi_1, z^i \psi_2)_{q, \alpha} = (z^k \psi_1, z^i \psi_2)_{q, \alpha} = (\hat{z}^j \hat{z}^k \psi_1, \psi_2)_{q, \alpha}.
\]

The main result of this section is

Proposition 7.3 There exists a unique \( \mathbb{C}[[q^{4\alpha}]] \)-bilinear map
\[
*: \text{Pol}(\mathbb{C})_q[[q^{4\alpha}]] \times \text{Pol}(\mathbb{C})_q[[q^{4\alpha}]] \rightarrow \text{Pol}(\mathbb{C})_q[[q^{4\alpha}]]
\]
such that \( Q(f_1 \ast f_2) = (Qf_1) \cdot (Qf_2) \) for all \( f_1, f_2 \in \text{Pol}(\mathbb{C})_q[[q^{4\alpha}]] \).

Proof. The uniqueness follows from lemma 7.1. The existence of this \( \mathbb{C}[[q^{4\alpha}]] \)-bilinear map will be established via verifying an explicit formula (7.4). We start with considering the case \( f_1 = z^*, f_2 = z \).

In section 6 a generating function \( c(u) = \sum_{k \in \mathbb{Z}_+} c_k u^k \) for the coefficients of the expansion
\[
\hat{z}^* \hat{z} = \sum_{k \in \mathbb{Z}_+} c_k \hat{z}^k \hat{z}^* \tag{7.2}
\]
was derived. Prove that
\[
B_{q, \alpha}(z^* z) = \sum_{k \in \mathbb{Z}_+} c_k z^k z^* \tag{7.3}
\]
In fact, the distribution \( B_{q, \alpha}(z^* z) \) coincides with the covariant symbol of the Toeplitz-Bergman operator with symbol \( z^* \hat{z} \). This operator is \( \hat{z}^* \hat{z} \) by a virtue of corollary 7.2. Its covariant symbol is \( \sum_{k \in \mathbb{Z}_+} c_k z^k z^* \) due to proposition 6.6.

It should be noted that \( B_{q, \alpha}(z^* z) \in \text{Pol}(\mathbb{C})_q[[q^{4\alpha}]] \). In fact, (6.9) implies
\[
c(u) = c(u, q^{4\alpha}) = \sum_{n \in \mathbb{Z}_+} q^{4\alpha n} \cdot P_n(u),
\]
with \( P_n(u) \) being a polynomial of a degree at most \( n + 1 \). Now our statement in the case \( f_1 = z^*, f_2 = z \) follows from (7.2) and (7.3):
\[
z^* \ast z = B_{q, \alpha}(z^* z).
\]

In a more general setting \( f_1 = z^{m} \), \( f_2 = z^{k} \), \( m, k \in \mathbb{Z}_+ \), one can use a similar argument. One has:
\[
z^{m} \ast z^{k} = B_{q, \alpha}(z^{m} z^{k}).
\]
The relations \( Q(zf) = zQ(f), Q((zf)^*) = Q(f)z^* \), \( f \in \text{Pol}(\mathbb{C})_q[[q^{4\alpha}]] \), allow one to consider even more general case of \( f_1, f_2 \in \text{Pol}(\mathbb{C})_q \):
\[
z^i z^m \ast z^j z^k = z^i B_{q, \alpha}(z^{m} z^{k}) z^* z^j, \quad i, j, k, m \in \mathbb{Z}_+. \tag{7.4}
\]
To complete the proof of proposition 7.3, it remains to define the $*$-product of formal series:
\[
\sum_{i \in \mathbb{Z}_+} q^{4\alpha_i} f_1^{(i)} \ast \sum_{j \in \mathbb{Z}_+} q^{4\alpha_j} f_2^{(j)} \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}_+} q^{4\alpha n} \left( \sum_{i+j=n} f_1^{(i)} \ast f_2^{(j)} \right),
\]
with $f_1^{(i)}, f_2^{(j)} \in \text{Pol}(\mathbb{C})_q$, $i, j \in \mathbb{Z}_+$.

Remark 7.4. The polynomials $P_n(u), n \in \mathbb{Z}_+$, could be found without application of the explicit formula for generating function (6.9). In fact, if one sets up $c_k = \sum_{n \in \mathbb{Z}_+} q^{4\alpha n} c_k^{(n)}$,
\[
\hat{z}^* \hat{z} = \sum_{n \in \mathbb{Z}_+} q^{4\alpha n} \sum_{k=0}^{n+1} c_k^{(n)} \hat{z}^k \hat{z}^* k.
\]
The constants $c_k^{(n)}$ could be found from the relation (see [8])
\[
\hat{z}^* \hat{z} = q^2 \hat{z} z^* + 1 - q^2 + q^{4\alpha} \cdot \frac{1 - q^2}{1 - q^{4\alpha}} \cdot (1 - \hat{z}^* \hat{z}).
\]
(For example, $P_0 = c_0^{(1)} u + c_0^{(0)} = q^2 u + 1 - q^2$.) This kind of description for coefficients in (7.5) was used in [8]. We observe that the $*$-product introduced here coincides with the $*$-product considered in [8].

8 *-Product and q-differential operators

The operators $\Box, \partial^{(l)}_z, \partial^{(r)}_z, \partial^{(l)}_z, \partial^{(r)}_z$ were introduced in [8].

Lemma 8.1 Let $\varphi, \psi$ be polynomials of one indeterminate. Then
\[
\frac{\partial^{(r)}}{\partial z^*} (\varphi(z^*) \psi(z)) = \frac{\partial^{(r)} \varphi(z^*)}{\partial z^*} \cdot \psi(q^2 z).
\]

Proof. Since $dz^* \cdot z = q^2 z \cdot dz^*$, one has
\[
\frac{\partial^{(r)}}{\partial z^*} (\varphi(z^*) \psi(z)) \cdot dz^* = \partial (\varphi(z^*) \psi(z)) = (\partial \varphi(z^*)) \psi(z) = \\
\frac{\partial^{(r)}}{\partial z^*} \varphi(z^*) \cdot dz^* \cdot \psi(z) = \frac{\partial^{(r)} \varphi(z^*)}{\partial z^*} \cdot \psi(q^2 z) \cdot dz^*.
\]

Lemma 8.2 For all $\psi(z) \in \mathbb{C}[z]_q$, $\partial^{(r)} \psi(z) \frac{dz}{\partial z} = \frac{\partial^{(l)} \psi(q^2 z)}{\partial z}$.

Proof. Since $dz \cdot z = q^2 z \cdot dz$, one has
\[
\frac{\partial^{(r)} \psi(z)}{\partial z} \cdot dz = \partial \psi = dz \cdot \frac{\partial^{(l)} \psi(z)}{\partial z} = \frac{\partial^{(l)} \psi(q^2 z)}{\partial z} \cdot dz.
\]
Proposition 8.3 Let $f_1, f_2$ be polynomials of one indeterminate. Then

$$\Box((f_2(z^*)f_1(z)) = q^2 \frac{\partial^{(r)} f_2}{\partial z^*} \cdot (1 - zz^*)^2 \cdot \frac{\partial^{(l)} f_1}{\partial z}. \quad (8.1)$$

Proof. It follows from [11, corollary 2.9] that

$$\Box(f_2(z^*)f_1(z)) = q^2 \left( \frac{\partial^{(r)} f_2}{\partial z^*} \frac{\partial^{(r)} (f_2(z^*)f_1(z))}{\partial z} \right) (1 - zz^*)^2 =$$

$$= q^{-2} \left( \frac{\partial^{(r)} f_2}{\partial z^*} \frac{\partial^{(r)} f_1(z)}{\partial z} \right) (1 - z^*z)^2.$$

Apply lemmas 8.1, 8.2 to conclude that

$$\Box(f_2(z^*)f_1(z)) = q^{-2} \frac{\partial^{(r)} f_2(z^*)}{\partial z^*} \cdot \frac{\partial^{(l)} f_1(q^4z)}{\partial z} (1 - z^*z)^2.$$

It remains to apply the commutation relation $z(1 - z^*z)^2 = q^{-4}(1 - z^*z^2)z$. □

Remind the notation from [8]:

$$\tilde{\Box} = q^{-2} (1 - (1 + q^{-2})z^* \otimes z + q^{-2}z^2 \otimes z^2) \cdot \frac{\partial^{(r)}}{\partial z^*} \otimes \frac{\partial^{(l)}}{\partial z}.$$

$m : \text{Pol}(\mathbb{C})_q \otimes \text{Pol}(\mathbb{C})_q \rightarrow \text{Pol}(\mathbb{C})_q, m : \psi_1 \otimes \psi_2 \mapsto :\psi_1\psi_2.$

Now we are in a position to prove [8, theorem 7.3].

Theorem 8.4 For all $f_1, f_2 \in \text{Pol}(\mathbb{C})_q$

$$f_1 \ast f_2 = (1 - q^{4\alpha}) \cdot \sum_{j \in \mathbb{Z}_+} q^{4\alpha - j} m(p_j(\tilde{\Box}) f_1 \otimes f_2),$$

with $p_j, j \in \mathbb{Z}_+$, being the polynomials determined by (3.3).

Proof. With $f_1, f_2, f_3, f_4 \in \mathbb{C}[z]_q$, one can deduce from the results of section 7 that

$$(f_1(z)f_2(z^*) \ast (f_3(z)f_4(z)^*) = f_1(z)B_{q,\alpha}(f_2(z^*)f_3(z))f_4(z^*). \quad (8.1)$$

An application of the results of section 5 to the bounded function $f_2(z^*)f_3(z)$ yields:

$$B_{q,\alpha}(f_2(z^*)f_3(z)) \sim (1 - q^{4\alpha}) \sum_{j \in \mathbb{Z}_+} q^{4\alpha - j} p_j(\Box)(f_2(z^*)f_3(z)).$$

It remains to apply proposition 8.3 and the definition of $\tilde{\Box}$. □

Remark 8.5. One can observe from proposition 2.5 that (8.1) is a q-analogue of relation (4.7) from [8].
Appendix. Overflowing vector systems

Unlike the main text where $\alpha$ was allowed to be an arbitrary positive number, let us assume now $\alpha \in \frac{1}{2}\mathbb{N}$.

Remind the notation $\tilde{X}$ for the quantum principal homogeneous space, and $i: D(U)_q' \hookrightarrow D(\tilde{X})_q$ for the canonical embedding of distribution spaces (see [10]).

Consider the embedding of vector spaces

$$i_\alpha: \text{Pol}(\mathbb{C})_q \hookrightarrow D(\tilde{X})_q; \quad i_\alpha: f \mapsto i(f) \cdot t_{12}^{-2\alpha - 1}.$$ 

Equip $\text{Pol}(\mathbb{C})_q$ with a new $U_q\mathfrak{sl}_2$-module structure given by $i_\alpha \xi f = \xi i_\alpha f$ for all $f \in \text{Pol}(\mathbb{C})_q$, $\xi \in U_q\mathfrak{sl}_2$. Denote this $U_q\mathfrak{sl}_2$-module by $\text{Pol}(\mathbb{C})_{q, \alpha}$. There exists an embedding $\mathbb{C}[z]_{q, \alpha} \hookrightarrow \text{Pol}(\mathbb{C})_{q, \alpha}$.

The results of [11, section 6] imply

**Proposition A.1.** *The linear map $D(\tilde{X})_q \to \mathbb{C}[z]_{q, \alpha}$ given by*

$$\psi \mapsto \int_{\tilde{X}_q} \tau_{12}^{*(2\alpha - 1)} \cdot (z\zeta^*; q^2)_{2\alpha + 1}^{-1} \cdot \psi d\nu,$$

*is a morphism of $U_q\mathfrak{sl}_2$-modules.*

Proposition A.1 allows one to treat the function $\tau_{12}^{*(2\alpha - 1)} \cdot (z\zeta^*; q^2)_{2\alpha + 1}^{-1}$ as a q-analogue of a coherent state in the sense of Perelomov [12].

**Corollary A.2.** *For all $\psi \in D(U)_q$*

$$P_{q, \alpha} \psi(z) = \int_{\tilde{U}_q} (z\zeta^*; q^2)_{2\alpha + 1}^{-1} \psi(\zeta) d\nu_{\alpha}(\zeta). \quad (A.1)$$

**Proof.** Consider the integral operator

$$P: D(U)_q \to \mathbb{C}[z]_{q, \alpha}; \quad P: \psi(z) \mapsto \int_{\tilde{U}_q} (z\zeta^*; q^2)_{2\alpha + 1}^{-1} \psi(\zeta) d\nu_{\alpha}.$$ 

It is a morphism of $U_q\mathfrak{sl}_2$-modules, as one can deduce from proposition A.1. The orthoprojection $P_{q, \alpha}$ is also a morphism of $U_q\mathfrak{sl}_2$-modules, due to the invariance of the scalar product in $H^2_{q, \alpha}$. It remains to use the relations $P f_0 = 1 - q^{4\alpha}$, $P_{q, \alpha} f_0 = 1 - q^{4\alpha}$, together with the fact that $f_0$ generates the $U_q\mathfrak{sl}_2$-module $D(U)_q$ (see [11]).

**Remark A.3.** (A.1) means that the distribution $(z\zeta^*; q^2)_{2\alpha + 1}^{-1}$ is a reproducing kernel.

Let us find the kernel of the integral operator $P_{q, \alpha} \overset{\circ}{\circ} P_{q, \alpha}$. For $\overset{\circ}{\circ} f, \psi \in D(U)_q$ one has by corollary A.2

$$P_{q, \alpha} \overset{\circ}{\circ} P_{q, \alpha}: \psi(z) \mapsto \int_{\tilde{V}_q} K_q(f_{\circ}; z, z') \psi(z') d\nu_{\alpha}(z'),$$

with

$$K_q(f_{\circ}; z, z') = \int_{\tilde{V}_q} (z\zeta^*; q^2)_{2\alpha + 1}^{-1} f_{\circ}(\zeta) \cdot (\zeta z^*; q^2)_{2\alpha + 1}^{-1} d\nu_{\alpha}(\zeta).$$
Now an application of lemma 1.4 yields

\begin{equation*}
K_q(\circ f; z, z') = \frac{1 - q^{4\alpha}}{1 - q^2} \int_{U_q} (z^*; q^2)^{-1}_{2\alpha+1} (1 - \zeta^*; q^2)^{-1}_{2\alpha+1} f(\zeta^*; q^2) d\nu(\zeta) =
\end{equation*}

\begin{equation*}
= \frac{1 - q^{4\alpha}}{1 - q^2} \int_{U_q} \frac{1}{(\zeta^*; q^2)^{-1}_{2\alpha+1}} (1 - \zeta^*)^{2\alpha}(z^*; q^2)^{-1}_{2\alpha+1} f(1 - \zeta^*) d\nu(\zeta) =
\end{equation*}

\begin{equation*}
= \frac{1 - q^{4\alpha}}{1 - q^2} \int_{U_q} (1 - \zeta^*)(\zeta^*; q^2)^{-1}_{2\alpha+1} (1 - \zeta^*)^{2\alpha}(z^*; q^2)^{-1}_{2\alpha+1} \circ f d\nu(\zeta).
\end{equation*}

Finally, use the relation 

\begin{equation*}
(1 - \zeta^*)\zeta = q^2\zeta(1 - \zeta^*)
\end{equation*}

to obtain

**Proposition A.4.** \(P_{q,\alpha} \circ f P_{q,\alpha} \) is an integral operator:

\begin{equation*}
P_{q,\alpha} \circ f P_{q,\alpha} \psi(z) = \int_{U_q} K_q(\circ f; z, z')\psi(z') d\nu_{\alpha}(z'),
\end{equation*}

whose kernel is given by

\begin{equation*}
K_q(\circ f; z, z') = \frac{1 - q^{4\alpha}}{1 - q^2} \int_{U_q} k_\zeta(q^2z')^* \cdot k_\zeta(z) \circ f(\zeta) d\nu(\zeta),
\end{equation*}

with \(k_\zeta(z) = (1 - \zeta^*)^{\alpha+1/2} \cdot (\zeta^*; q^2)^{-1}_{2\alpha+1} \in D(U \times U)'_q\).

Proposition A.4 allows one to treat the distribution \(k_\zeta(z)\) as a q-analogue of an overflowing vector system.

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