Image charges revisited: a closed form solution

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Abstract

We demonstrate that the corrections to the classical Kelvin image theory due to finite electron screening length $\lambda$, recently discussed by Roulet and Saint Jean, Am. J. Phys. 68(4) 319, is amenable to an exact closed form solution in terms of an integral involving Bessel functions. An improper choice of boundary conditions is rectified as well, enabling also a complete solution for all potentials - both inside and outside the metal surface.

I. INTRODUCTION

The theory of image charges over a perfect metal surface has an early history that dates back to Lord Kelvin in 1848 [1]. It is a subject covered in most textbooks on electrostatics [2], [3], as well as an important topic in modern research [4], [5]. Hence it is of considerable importance in the undergraduate curriculum. In a recent article in this journal, Roulet and Saint Jeans (RSJ) [6] discussed the corrections beyond the classical Kelvin theory by considering the effect due to a finite electron screening length $\lambda$ in the metal. This topic has indeed hardly been addressed in any textbooks, except for more advanced texts such as Mahan [7], but only in the more general theory of electron screening and response functions. A simplified consideration based on chemical equilibrium such as employed by RSJ [6] should deepen the understanding of classical Kelvin image theory as well as stimulate advanced
students to consider the more comprehensive treatment based on many-body theory and linear response [7]. Unfortunately the otherwise excellent exposition of RSJ [8] is plagued by an improper choice of boundary conditions, and by their inability to obtain a complete solution for the potential in the metal dictated by a Helmholtz type partial differential equation (PDE), which as we shall see is in fact separable, see Appendix I and II. They resorted to a perturbative solution that is not valid for large screening lengths - which is in fact a trivial limit. In this paper we shall rectify this and consequently provide the complete solution for the potentials both inside and outside the metal surface. Naturally our results, being exact will recover the expected classical theory both in the case of small screening length ($\lambda \to 0$), as treated by RSJ [8], and the opposite case of large screening length ($\lambda \to \infty$), when the metal becomes ineffective and the classical Laplace potential ensues.

This paper is divided into four sections. In section I we shall review the classical Kelvin image theory, solved by the separation of the Laplace equation in cylindrical coordinates. This detail solution is nowadays not commonly taught in the undergraduate curriculum [8]. In section II we shall demonstrate that by using the same technique, the Helmholtz equation (Eq.(22) of ref [8]), for the screened potential inside the metal can also be solved. More importantly we shall discuss the proper boundary conditions for this problem. The standard conditions follow from the Maxwell equations [2]: ($\text{div} \ D = \rho$ and $\text{curl} \ E = 0$) on the surface edge ($z = 0$). However RSJ imposed an artificial condition based on a strict compliance of the final surface charge density (here denoted as $\sigma(r)$) with the limiting classical surface charge density (here denoted as $\sigma_0(r)$). Here we shall argue that the limiting surface charge density $\sigma_0(r)$ should only be a consequence of the complete theory and not a precondition. As a result $\sigma(r)$ should be $\lambda$ dependent, while $\sigma_0(r)$ is manifestly not.

The RSJ choice of boundary condition, as it will be shown is also amenable to a closed form solution (see Appendix II), hence their perturbative approach is unnecessary. It is interesting to note (see Appendix II) that the exact solution to their problem in fact diverges as $\lambda^2$ for large $\lambda$ (see Appendix II) and thus the perturbation method is inoperable in principle, as it can never converge to the exact solution. We believe this essentially non-perturbative
feature is an artifact of their boundary condition which is unlike the solutions presented here. Moreover, although the departures between our theory and theirs only become apparent at $O(\lambda^2)$ for small $\lambda$, their theory further leaves the outside potentials undetermined and will require additional assumptions to complete. We shall show that our solution to the problem is complete and can be expressed in terms of integrals involving Bessel functions of the type similar to and much studied in electromagnetic propagation [9], [10], beginning with Sommerfeld’s classic work of 1909 [11]. In section III shall present the exact solutions and in section IV we shall discuss the effects on and hence corrections to the classical image potentials outside the metal surface, which were omitted by RSJ [1]. Here we shall also obtain the corrections to the surface charge density as discussed above. This correction integrates to zero charge as we shall see (Appendix III). In section V we shall replace the metal in our theory by a dielectric with $\epsilon > \epsilon_0$. The reader may like to note that all the results of this paper are not new. They have in fact been derived before (unknown to the author) by Newns [12] and more recently by Krčmar et al [13] (see the end notes). As a result of our investigations, the conclusions reached by RSJ for the case of large $\lambda/h >> 1$ as in a semiconductor is subtle as naively the limit $\lambda \to \infty$ in fact approaches the classical results as required for a dielectric. A more careful comparison will require investigating the case $\lambda >> h$ under the condition that $z, r >> \lambda$ which is indeed a non-trivial limit in our theory. More detailed numerical calculations can be found in ref. [13]. We shall conclude with a brief discussion of time-dependent effects, such as the case of an oscillating charge, by which ingenious experiments based on the classical skin effect could be used to test our predictions. These experiments could be designed and possibly be inspiring to an undergraduate class.

II. CLASSICAL IMAGE THEORY

In this section we shall adopt the more common convention in which the physical charge $q$ is introduced above the metal surface at $z = h > 0$ and where the rest of the metal is defined for $z \leq 0$, see Fig 1. This is opposite to the case of RSJ but is similar to most
textbooks such as Jackson [2] and Landau et al [3]. It is well known that Kelvin image theory provides the solution for the potential outside the metal as:

$$\phi(r, z) = \frac{q}{4\pi\epsilon_0 \sqrt{r^2 + (h - z)^2}} - \frac{q}{4\pi\epsilon_0 \sqrt{r^2 + (h + z)^2}}$$  \hfill (2.1)$$

The second term is the so-called image potential due to the fictitious charge \(-q\) whose addition enables the boundary condition \(\phi(r, 0) = 0\) to be satisfied, see Fig 1. This is a Dirichlet boundary value problem whose solution Eq(2.1) is justified heuristically in most texts, see for example [3]. It is perhaps useful in a later undergraduate course to actually verify that Eq(2.1) is indeed a solution of the Laplace equation. This can be demonstrated in the following way. The Laplace equation in cylindrical coordinates with azimuthal symmetry:

$$\frac{\partial^2 \phi^0}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r}\right) \phi^0 = 0,$$  \hfill (2.2)$$

has solutions that are given in terms of Bessel functions [8], [14]:

$$\phi^0_>(r, z) = \int_0^\infty f(k) e^{-kz} J_0(kr) dk \quad \text{for} \quad z > h,$$
$$\phi^0_<(r, z) = \int_0^\infty g(k) \sinh(kz) J_0(kr) dk \quad \text{for} \quad 0 \leq z \leq h.$$  \hfill (2.3)$$

The coefficients of these expansions namely \(f(k)\) and \(g(k)\) are to be determined by appropriate boundary conditions as we shall see. Note that the second expansion has been chosen with the \(\sinh\) function in order to satisfy the required boundary condition on the metal surface \(z = 0\). We have two coefficients and thus we need two more boundary conditions to determine the solutions. As is well known from most texts [2], [3], [14], for electrostatics, these are derived from the two Maxwell equations:

$$\text{curl} \ E = 0 \quad \text{and} \quad \text{div} \ D = \rho,$$  \hfill (2.4)$$

which upon integrating around an infinitesimal surface volume imply the continuity of the potential, say at \(z = h\):

$$\phi^0_>(r, h) = \phi^0_<(r, h).$$  \hfill (2.5)$$
and the discontinuity of the slope:

$$\frac{q \delta(r)}{\epsilon_0} = \left( \frac{\partial \phi_0}{\partial z} - \frac{\partial \phi_0}{\partial z} \right)_{z=h};$$

(2.6)

where the charge density \( \rho \) consists only of the true charge and does not include any induced charges. The delta function in Eq.(2.6) is normalized for:

$$2\pi \int_0^\infty r \delta(r) dr = 1.$$  

(2.7)

Now the continuity condition Eq. (2.5) leads to:

$$f(k) = g(k)e^{kh}\sinh(kh),$$

(2.8)

while the discontinuity condition Eq. (2.6), with the use of the orthogonality property of the Bessel functions [15]:

$$\int_0^\infty r J_0(kr) J_0(k'r) dr = \frac{1}{k} \delta(k-k'),$$

(2.9)

leads to:

$$\frac{q}{2\pi\epsilon_0} = g(k)\cosh(kh) + f(k)e^{-kh}.$$  

(2.10)

The straightforward solution of the simultaneous Eq.(2.8) and Eq.(2.10) leads to:

$$f(k) = \frac{q}{2\pi\epsilon_0} \sinh(kh) \quad \text{and} \quad g(k) = \frac{q}{2\pi\epsilon_0} e^{-kh}.$$  

(2.11)

That these solutions lead to the classical Kelvin image potential Eq.(2.1) follows readily from another student exercise in the form of a mathematical identity for Bessel functions [16]:

$$1/\sqrt{r^2 + h^2} = \int_0^\infty e^{-k|h|} J_0(kr) dk.$$  

(2.12)

Notice that the final solution depends on the boundary conditions both at \( z = h \) and at \( z = 0 \), and not just on the latter alone as the heuristic argument seems to suggest.
Finally to conclude this section we shall derive the classical induced surface charge density \( \sigma_0(r) \). This is simply obtained from the required boundary condition [3], [14]:

\[
\sigma_0(r) = -\epsilon_0 \frac{\partial \phi_0}{\partial z} \bigg|_{z=0} = -\frac{q}{2\pi\epsilon_0} \int_0^\infty e^{-k|h|} k J_0(kr) dk = -\frac{q}{2\pi\epsilon_0} \frac{h}{(r^2 + h^2)^{3/2}},
\]

(2.13)
a result easily derived using Eq.(2.12). We remind the reader that this boundary condition is obtained from an infinitesimal small surface volume integral of the Maxwell equation:

\[
\text{div } E = \frac{\rho_t}{\epsilon_0},
\]

(2.14)
in the same way as Eq.(2.6) where the total charge density \( \rho_t \) contains all charges, true and induced [14]. In the section IV, as with RSJ [3], we shall derive this surface charge density by integrating the bulk interior charge density \( \rho^0(r, z) \) as we shall see.

III. BULK CHARGE DENSITY - AN EXACT SOLUTION

The (Thomas-Fermi) screening theory modifications for the potential inside the metal \( \phi_{in} \) is given by the Helmholtz equation [3]:

\[
\frac{\partial^2 \phi_{in}}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) \phi_{in} = \frac{\phi_{in}}{\lambda^2} \quad \text{for} \quad z < 0,
\]

(3.1)

where the screening length \( \lambda \) is given by:

\[
\lambda^2 = \frac{\epsilon_0}{e^2} \left( \frac{\partial \mu_0}{\partial n_0} \right)_T.
\]

(3.2)

This is expressed in terms of a thermodynamic derivative of the chemical potential with respect to the density. We note that this theory for \( \lambda \) given by eqn(3.1) is in fact quite general, it should also be valid in the case of a dielectric or a liquid surface, upon replacing \( \lambda \) by the appropriate Debye-Hückel screening length. Significant modifications to the Thomas-Fermi theory will come about only when \( \lambda \) becomes comparable to the Fermi wavelength \( \lambda_F \) which we shall briefly mention in the conclusion. In Appendix I, we shall show that the solution of the Helmholtz equation Eq.(3.1) is also expressed in terms of Bessel functions.
Hence the complete solution of boundary value problem specified by Eq.(2.2) and Eq.(3.1) is now given by:

\[ \phi_>(r, z) = \int_0^\infty f(k)e^{-kz}J_0(kr)dk \quad \text{for} \quad z > h, \]

\[ \phi_<(r, z) = \int_0^\infty (g_1(k)e^{kz} + g_2(k)e^{-kz})J_0(kr)dk \quad \text{for} \quad 0 \leq z \leq h \quad \text{and}, \]

\[ \phi_{in}(r, z) = \int_0^\infty g_3(\beta_k)e^{\beta_kz}k\beta_kJ_0(kr)dk \quad \text{for} \quad z < 0; \quad (3.3) \]

where \( \beta_k^2 = k^2 + 1/\lambda^2, \) see Appendix I. Note that the non-vanishing of the potential on the surface \((z = 0)\) for general \( \lambda \) requires that we construct solutions for \( \phi_< \) that contains two (in general) unequal coefficients \( g_1(k) \) and \( g_2(k) \). We have already discussed the boundary conditions at \( z = h \). The continuity condition Eq.(2.5) leads to:

\[ f(k) - g_2(k) = g_1(k)e^{2kh}, \quad (3.4) \]

while the discontinuity condition Eq.(2.6) leads to:

\[ g_1(k) = \frac{q}{4\pi \varepsilon_0}e^{-kh}. \quad (3.5) \]

Two more boundary conditions are required to obtain the solution. These are in fact obtained in the same way as Eq.(2.5) and Eq.(2.6), only that now we have continuity of the potentials and their slopes on the surface \( z = 0 \) in the absence of true charges on the surface. Thus from:

\[ \phi_{in}(r, 0) = \phi_<(r, 0), \quad (3.6) \]

we obtain:

\[ g_1(k) + g_2(k) = \frac{k}{\beta_k}g_3(\beta_k), \quad (3.7) \]

and from:

\[ \frac{\partial \phi_{in}}{\partial z} \bigg|_{z=0} = \frac{\partial \phi_<}{\partial z} \bigg|_{z=0}, \quad (3.8) \]

we obtain:
\[ g_1(k) - g_2(k) = g_3(\beta_k). \] (3.9)

The problem is now completely specified by equations: (3.4), (3.5), (3.7) and (3.9). A simultaneous solution of these equations leads to a complete solution for all the potentials, hence:

\[
\phi_>(r,z) = q \frac{2}{2\pi\epsilon_0} \int_0^\infty \left[ \frac{kcosh(kh) + \beta_k sinh(kh)}{k + \beta_k} \right] e^{-kz} J_0(kr) dk \quad \text{for } z > h,
\]
\[
\phi_<(r,z) = q \frac{2}{2\pi\epsilon_0} \int_0^\infty \left[ \frac{kcosh(kz) + \beta_k sinh(kz)}{k + \beta_k} \right] e^{-kh} J_0(kr) dk \quad \text{for } 0 \leq z \leq h \quad \text{and},
\]
\[
\phi_{in}(r,z) = q \frac{2}{2\pi\epsilon_0} \int_0^\infty \frac{ke^{\beta_kz}}{k + \beta_k} e^{-kh} J_0(kr) dk \quad \text{for } z < 0; \quad (3.10)
\]

In the Appendix II, we shall present in the same way the exact solution for the RSJ problem which consist of Eq.(3.1) only and the boundary condition that \( \sigma(r) = \sigma_0(r) \), which however leaves the outside potentials unspecified.

### IV. SCREENING CORRECTIONS

Naturally it would be meaningful to compare these potentials with the classical image potentials in Eq.(2.3) and Eq.(2.11). To do this each coefficient in Eq.(3.10) is rationalized by multiplying with \( (k - \beta_k)/(k - \beta_k) \) and after some simple algebra, we have:

\[
\phi_< = \phi_0^0 + \phi^\lambda \quad \text{and} \quad \phi_> = \phi_0^0 + \phi^\lambda, \quad (4.1)
\]

where \( \phi_0^0 \) and \( \phi_>^0 \) are the classical potentials given in section II. The screening corrections are given in terms of the \( \lambda \) dependent potential:

\[
\phi^\lambda(r,z) = q \frac{\lambda^2}{2\pi\epsilon_0} \int_0^\infty \left( \sqrt{k^2 + \frac{1}{\lambda^2}} - k \right) e^{-k(h+z)} J_0(kr) dk. \quad (4.2)
\]

We can easily derive that for small \( \lambda \):

\[
\phi^\lambda(r,z) \bigg|_{\lambda \to 0} \approx q \frac{\lambda}{2\pi\epsilon_0} \int_0^\infty e^{-k(h+z)} J_0(kr) dk = q \frac{\lambda}{2\pi\epsilon_0} \frac{(h+z)}{r^2 + (h+z)^2} \quad (4.3)
\]

For large \( \lambda \) we use an expansion of the square root: \( \sqrt{1+1/x^2} \approx 1 + 1/(2x^2) \) to obtain:
\[ \phi^\lambda(r, z) \bigg|_{\lambda \to \infty} \approx \frac{q}{4\pi \varepsilon_0} \int_0^\infty e^{-k(h+z)} J_0(kr) \, dk = \frac{q}{4\pi \varepsilon_0} \frac{1}{\sqrt{r^2 + (h + z)^2}}, \quad (4.4) \]

which cancels the image potential in Eq.(2.1) as required. Using the same manipulations we obtain the inside potential as:

\[ \phi_{in}(r, z) = \frac{q \lambda^2}{2\pi \varepsilon_0} \int_0^\infty \left( \sqrt{k^2 + \frac{1}{\lambda^2} - k} \right) e^{-kh} e^{z \sqrt{k^2 + \frac{1}{\lambda^2}}} k J_0(kr) \, dk, \quad (4.5) \]

from which the bulk charge density:

\[ \rho(r, z) = -\frac{\varepsilon_0}{\lambda^2} \phi_{in}(r, z) = -\frac{q}{2\pi} \int_0^\infty \left( \sqrt{k^2 + \frac{1}{\lambda^2} - k} \right) e^{-kh} e^{z \sqrt{k^2 + \frac{1}{\lambda^2}}} k J_0(kr) \, dk \quad (4.6) \]

is obtained. In particular we have the limit:

\[ \phi_{in}(r, z) \bigg|_{\lambda \to 0} \approx \frac{q \lambda}{2\pi \varepsilon_0} \int_0^\infty e^{-kh} e^{z \sqrt{k^2 + \frac{1}{\lambda^2}}} k J_0(kr) \, dk, \quad (4.7) \]

in agreement with RSJ [6]. However it is easy to show that the higher order terms differ, see Appendix II. Moreover unlike RSJ, we have no difficulty with the limit of large \( \lambda \) which again by an expansion of the square root leads to:

\[ \phi_{in}(r, z) \bigg|_{\lambda \to \infty} \approx \frac{q q}{4\pi \varepsilon_0} \int_0^\infty e^{-k(h-z)} J_0(kr) \, dk = \frac{q}{4\pi \varepsilon_0} \frac{1}{\sqrt{r^2 + (h - z)^2}} (z < 0), \quad (4.8) \]

which is of course the correct Laplace potential for the case when the metal is ineffective, since the Helmholtz equation Eq(3.1) now reduces to the Laplace equation. As promised in section I, we shall evaluate the surface charge density:

\[ \sigma(r) = \int_{-\infty}^0 \rho(r, z) \, dz = \sigma_0(r) + \sigma^\lambda(r), \quad (4.9) \]

where

\[ \sigma^\lambda(r) = \frac{q}{2\pi} \int_0^\infty \frac{e^{-kh}}{\sqrt{k^2 + 1/\lambda^2}} k^2 J_0(kr) \, dk, \quad (4.10) \]

which is the new term in our theory. This quantity integrates to zero charge, (see Appendix III), thus the total charge remains \(-q\) as before. Hence the form \( \sigma_0 \) is not the only (unique) surface charge density that integrates to a total charge of \(-q\). We note that while the limit
λ → ∞ appears straightforward, this is deceptive. Attempts to calculate the next order i.e. \(O(1/\lambda^4)\) corrections lead to divergent integrals that require careful treatment. We shall not discuss this exercise here. Suffice to say the corrections to the classical potentials due to \(\phi^\lambda\) and \(\phi_{in}\) are to smear the image charge from a point charge to a charge distribution whose weight vanishes as \(\lambda\) increases. This charge distribution can be derived from the results presented here and has been worked out by Newns [12] in terms of an order two Bessel function.

V. SEMICONDUCTOR SURFACE

The extension of our theory to a semiconductor surface, in which \(\lambda\) is finite, is now rather straightforward. The only significant modification is that the boundary condition Eq.(3.8) is now replaced by:

\[
\epsilon \frac{\partial \phi_{in}}{\partial z} \bigg|_{z=0} = \epsilon_0 \frac{\partial \phi_\infty}{\partial z} \bigg|_{z=0},
\]

where \(\epsilon > \epsilon_0\) is the dielectric constant of the material. We need not repeat the details here and present merely the solution for \(\phi_{in}\) which now takes the form:

\[
\phi_{in}(r, z) = \frac{q}{2\pi\epsilon_0} \int_0^\infty \frac{ke^{\beta_k z}}{k + \bar{\epsilon}\beta_k} e^{-kh}J_0(kr) dk \quad \text{for} \quad z < 0,
\]

where \(\bar{\epsilon} = \epsilon/\epsilon_0\). The integral can be analyzed in both limits as usual. For small \(\lambda\) we have:

\[
\phi_{in}(r, z) \bigg|_{\lambda \to 0} \approx \frac{q\lambda}{2\pi\bar{\epsilon}\epsilon_0} \frac{h}{[r^2 + (h-z)^2]^{3/2}} e^{z/\lambda} \quad (z < 0),
\]

which is analogous with the metallic case. For large \(\lambda\) however:

\[
\phi_{in}(r, z) \bigg|_{\lambda \to \infty} \approx \frac{q}{2\pi(1+\bar{\epsilon})\epsilon_0} \int_0^\infty e^{-kh}e^{kz}J_0(kr) dk = \frac{q}{2\pi(1+\bar{\epsilon})\epsilon_0} \frac{1}{\sqrt{r^2 + (h-z)^2}} (z < 0),
\]

which is the classical result. Recall that the potential inside a dielectric is equivalent to an image charge \(\tilde{q}\) given by [14]:

\[
\tilde{q} = \frac{q}{2\pi\epsilon_0} \int_0^\infty e^{-kh}J_0(kr) dk.
\]
\[
\tilde{q} = \frac{2\tilde{\epsilon}q}{(1 + \tilde{\epsilon})},
\] 

(5.5)

replacing the real charge \(q\) at the point \(h\) outside the surface. Hence for large \(\lambda >> h\), classical theory is recovered and as a matter of fact it is for \(\lambda << h\), as shown in Eq.(5.3) that screening creates departures from classical theory. Thus for dielectrics it is the case of small and finite \(\lambda\) that leads to corrections to the Kelvin theory. Finally we shall briefly mention the case of an oscillating charge. The results here being static will lead to a zero frequency contribution to the classical finite frequency skin effect. The propagation of electromagnetic waves in the substrate will now be dictated by a complex wave vector [17]:

\[
k^2 = \tilde{\epsilon}\omega^2/c^2 + 2i/\delta^2 - 1/\lambda^2
\]

(5.6)

to which we have added the screening length \(\lambda\). At low frequencies, [17], the skin depth \(\delta\) which varies as the inverse square root of the frequency dominates. Hence by extrapolating skin depth measurements to zero frequency, the results presented in this paper may be detectable [18]. This could be a novel undergraduate experiment for physics and engineering classes.

VI. CONCLUSION

In conclusion we have presented an exact solution for the corrections to the classical Kelvin image theory of electrostatics previously discussed by RSJ [1] in this journal. Some inadequacies in their analyses are rectified and a complete solution for all the potentials are obtained in closed form. We found that some care needs to be exercised with regard to statements about non-classical behaviour which are only valid in the case of finite \(\lambda\), whereas for small and large \(\lambda\) we have shown that the theory reduce to the standard textbook analysis. Nevertheless our theory shows that the non-vanishing potentials \(\phi^\lambda\) and \(\phi_{in}\), for any finite \(\lambda > 0\) in the case when the charge is right on the metal surface i.e. for \(h = 0\) as opposed to the classical theory, have an essential role for surface chemistry that should be noted in all textbooks [19].
A. Notes added:

After this work was completed we were grateful to be advised by Gabriel Barton and Bernard Roulet that the results of our paper, with the exception of the proofs in the appendices, have in fact been derived some thirty years ago by Dennis Newns [12] apart from minor differences in our definition of the induced potentials. Also unknown to the author, the results have also been rederived recently by Krčmar et al. [13]. The results of this paper are in agreement with Newns and Krčmar et al. The reader might be interested to know that the latter authors have also considered the case when the charge q becomes immersed inside the material, using extensions of the methods presented here. Neither Newns, Krčmar et al, nor the present author have considered the limit when $\lambda_F$ becomes significant. The author is further indebted to Marshall Stoneham for pointing out that the solution in the limit of large $\lambda_F >> \lambda_{TF}$ has been considered by John Willis [20], only within a static charge approximation. This identified the significance of Fermi surface effects (omitted in this work) which lead to Friedel type oscillations in the density near the surface. No doubt a more sophisticated theory such as one using a density functional approach will be needed to treat all the various issues.

B. Appendix I - solution of the Helmholtz equation

The Helmholtz equation Eq.(3.1) in azimuth symmetric cylindrical coordinates can be easily separated [16] by the ansatz $\phi(r,z) = f(r)g(z)$, such that:

$$\frac{d^2 f}{dz^2} = k^2 f$$

which has solutions: $f(z) = e^{kz}$, where $k^2$ is the separation constant. Note that we have $z < 0$ so that the other solution $f(z) = e^{-kz}$ is exponentially increasing and cannot be admitted by the boundary condition for $z \rightarrow -\infty$. The radial equation now takes the form:

$$\frac{d^2 g}{dz^2} + \frac{1}{r} \frac{dg}{dz} + \alpha_k g = 0,$$
where: \( \alpha_k^2 = (k^2 - \frac{1}{\lambda^2}) \). For \( \alpha_k^2 > 0 \) this is the differential equation for Bessel functions of order zero. For \( \alpha_k = 0 \) the solution goes as \( \log r \) which is inadmissible as is the case where \( \alpha_k^2 < 0 \) since Bessel functions of imaginary arguments namely \( I_0(r) \) and \( K_0(r) \) are also inadmissible. Thus the general solution of Eq. (3.1) is a superposition of solutions using an expansion coefficient \( g_3(k) \) given by:

\[
\phi_{in}(r, z) = \int_{1/\lambda}^{\infty} g_3(k)e^{kz}J_0(\alpha_k r)dk \quad \text{for} \quad z < 0. \tag{6.3}
\]

By a straightforward change of variable \( x = \alpha_k \) we easily obtain the solution given by the last of Eqn. (3.3).

C. Appendix II - exact solution of the RSJ problem

The solution for the RSJ boundary value problem can be obtained using the results of Appendix I. The RSJ potential and hence the bulk charge density is given by:

\[
\rho(r, z) = -\frac{\varepsilon_0}{\lambda^2} \int_0^{\infty} g_3(\beta_k) e^{\beta_k z} k \frac{J_0(kr)}{\beta_k} dk \quad \text{for} \quad z < 0. \tag{6.4}
\]

The RSJ boundary condition requires that the integral:

\[
\int_{-\infty}^{0} dz \rho(r, z) = -\frac{\varepsilon_0}{\lambda^2} \int_0^{\infty} g_3(\beta_k) \frac{k}{\beta_k^2} J_0(kr) dk
\]

\[
= \sigma_0(r) = -\frac{q}{2\pi} \int_0^{\infty} e^{-kh} k J_0(kr) dk \tag{6.5}
\]

hence we obtain:

\[
g_3(\beta_k) = \frac{q\lambda^2}{2\pi\varepsilon_0} \beta_k^2 e^{-kh}. \tag{6.6}
\]

Note that while this determines the inside potential in closed form:

\[
\phi_{in}^{RSJ}(r, z) = \frac{q\lambda^2}{2\pi\varepsilon_0} \int_0^{\infty} e^{-kh} e^{\beta_k z} \beta_k k J_0(kr) dk, \tag{6.7}
\]

it leaves the outside potentials undetermined since Eq. (3.4) and Eq. (3.5) now become two equations with \textit{three} unknowns. Moreover Eq. (3.7) is now divergent as \( \lambda^2 \) for \( \lambda \to \infty \) and
does not reduce to the classical Laplace potential as required. It is interesting to compare Eq. (6.7) with Eq. (4.2). The former is essentially non-perturbative whereas the latter may be obtainable perturbatively if some care is exercised in treating the asymptotic integrals, see the remarks at the end of section IV. It is unclear and it would be interesting to speculate if this feature of the RSJ solution has experimental significance such as when the total surface charge can be maintained constant, i.e. $\sigma_0(r) = \text{const}$. In this case, following the Eq.(6.3), we can show that the potential now goes as $\lambda e^{-z/\lambda}$. Nevertheless it is interesting to compare the small $\lambda$ expansion for the two solutions. We note that the difference between Eq.(6.7) and Eq.(1.3) is a negative term:

$$
\phi_{in}(r, z) = q\lambda^2 \frac{2\pi\epsilon_0}{\epsilon_0} \int_0^{\infty} e^{-kh} e^{\beta_k z}(\beta_k - k)k J_0(kr)dk.
$$

(6.8)

For small $\lambda$ we change variable to $k = x/h$ and carry out the expansions for the square root and exponential terms via:

$$
\sqrt{1 + \left(\frac{x\lambda}{h}\right)^2} \approx 1 + \frac{1}{2} \left(\frac{x\lambda}{h}\right)^2 + \ldots \quad \text{and}
$$

(6.9)

$$
e^{\frac{z}{\lambda}\sqrt{1 + \left(\frac{x\lambda}{h}\right)^2}} \approx e^{\frac{z}{\lambda}} \left(1 + \frac{1}{2} \frac{z\lambda x^2}{h^2} + \ldots\right).
$$

(6.10)

Inserting these into the integrands and evaluating the integrals we have the limiting expressions to $O((\lambda/h)^2)$ as:

$$
\phi_{RSJ}^{in}(r, z) \approx \frac{q\lambda e^{z/\lambda}}{2\pi\epsilon_0} \frac{h}{(h^2 + r^2)^{3/2}} \left[1 + \frac{3\lambda}{2} \frac{(2h^2 - 3r^2)}{(h^2 + r^2)^2} + \ldots\right],
$$

(6.11)

in agreement with RSJ while our solution has the expansion:

$$
\phi_{in}(r, z) \approx \frac{q\lambda e^{z/\lambda}}{2\pi\epsilon_0} \frac{h}{(h^2 + r^2)^{3/2}} \left[1 + \frac{3\lambda}{2} \frac{(2h^2 - 3r^2)}{(h^2 + r^2)^2} - \frac{\lambda}{h} \frac{(2h^2 - r^2)}{(h^2 + r^2)} + \ldots\right],
$$

(6.12)

which differs from the latter at $O((\lambda/h)^2)$ with an additional term.

D. Appendix III - proof that the integral of $\sigma^\lambda(r)$ vanishes

The proof is straightforward and it makes use of a simple trick involving a well known property of the Bessel functions. We need to obtain, denoting $1/\lambda$ by $\alpha$, the integral:
\[ Q(\alpha) = q \int_0^\infty r dr \int_0^\infty dk \frac{e^{-kh}}{\sqrt{k^2 + \alpha^2}} k^2 J_0(kr) \]

\[ = \lim_{\beta \to 0} q \int_0^\infty r dr \int_0^\infty dk \frac{e^{-kh}}{\sqrt{k^2 + \alpha^2}} k^2 J_0(\beta r) J_0(kr), \quad (6.13) \]

which follows from the property that \( \lim_{x \to 0} J_0(x) = 1 \). Making use of the orthogonality property Eq.(2.9) we now have:

\[ Q(\alpha) = \lim_{\beta \to 0} q \int_0^\infty dk \frac{e^{-kh}}{\sqrt{k^2 + \alpha^2}} k^2 \beta \delta(\beta - k) \]

\[ = \lim_{\beta \to 0} q \beta e^{-\beta h} \frac{k^2}{\sqrt{\beta^2 + \alpha^2}} \delta(\beta - k) \]

\[ = 0 \quad \text{for} \quad \alpha > 0 \]

\[ = q \quad \text{for} \quad \alpha = 0 \quad (6.14) \]

respectively. Although \( Q(\alpha) \) is a discontinuous function at \( \alpha = 0 \), the interchange of the limit with the integrals, while not rigorous, may be justified by the convergence of the integrals in Eq.(6.13) for all \( \alpha \geq 0 \) [16].
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Fig. 1: Cylindrical coordinates for the charge $q$ placed at a distance $z-h$ outside the metal or dielectric surface at $z=0$, with its image charge $q$ at $z=H$. The solution for the potentials at any point $P(r,z)$ with screening corrections is given in the text.