Stability analysis of an autonomous system in loop quantum cosmology

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We discuss the stability properties of an autonomous system in loop quantum cosmology. The system is described by a self-interacting scalar field \( \phi \) with positive potential \( V \), coupled with a barotropic fluid in the Universe. With \( \Gamma = V''/V' = \lambda \), the autonomous system is extended from three dimensions to four dimensions. We find that the dynamic behaviors of a subset, not all, of the fixed points are independent of the form of the potential. Considering the higher-order derivatives of the potential, we get an infinite-dimensional autonomous system which can describe the dynamical behavior of the scalar field with more general potential. We find that there is just one scalar-field-dominated scaling solution in the loop quantum cosmology scenario.

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I. INTRODUCTION

The scalar field plays an important role in modern cosmology. Indeed, scalar-field cosmological models are of great importance in the study of the early Universe, especially in the investigation of inflation. The dynamical properties of a scalar fields also make an interesting research topic for modern cosmological studies [1,2]. The dynamical behavior of scalar field coupled with a barotropic fluid in a spatially flat Friedmann-Robertson-Walker universe has been studied by many authors (see [1,3,4], and the first section of [2]).

The phase-plane analysis of the cosmological autonomous system is an useful method for studying the dynamical behavior of a scalar field. One always considers the dynamical behavior of a scalar field with an exponential potential in the classical cosmology [5,6] or modified cosmology [8,9]. And, if one considers the dynamical behavior of a scalar field coupled with a barotropic fluid, the exponential potential is also the first choice [10–13]. The exponential potential \( V \) leads to the fact that the variables \( \Gamma = V''/V' = \lambda \) equal 1 and that \( \lambda = V'/V \) is also a constant. Then the autonomous system is always two dimensional in classical cosmology [8], and three dimensional in loop quantum cosmology (LQC) [8]. Although one can also consider a more complex case with \( \lambda \) being a dynamically changing quantity [1,14,15], the fixed point is not a real one, and this method is not exact. Recently, Zhou et al. [16,17] introduced a new method by which one can make \( \Gamma \) a general function of \( \lambda \). Then the autonomous system is extended from two dimensions to three dimensions in classical cosmology. They found that this method can help investigate many quintessence models with different potentials. The goal of this paper is to extend this method for studying the dynamical behavior of a scalar field with a general potential coupled with a barotropic fluid in LQC.

LQC [18,19] is a canonical quantization of homogeneous spacetime based on the techniques used in loop quantum gravity (LQG) [20,21]. Owing to the homogeneity and isotropy of the spacetime, the phase space of LQC is simpler than that of LQG. For example, the connection is determined by a single parameter \( c \) and the triad is determined by \( p \). Recently, it has been shown that the loop quantum effects can be very well described by an effective modified Friedmann dynamics. Two corrections of the effective LQC are always considered: the inverse volume correction and the holonomy correction. These modifications lead to many interesting results: the big bang can be replaced by the big bounce [22], the singularity can be avoided [23], the inflation can be more likely to occur (e.g., see [24,25]), and more. But the inverse volume modification suffers from gauge dependence which cannot be cured and thus yields unphysical effects. In the effective LQC based on the holonomy modification, the Friedmann equation adds a \(-\frac{\kappa}{V}\) term, in which \( \kappa = 8\pi G \), to the right-hand side of the standard Friedmann equation [20]. Since this correction comes with a negative sign, the Hubble parameter \( H \), and then \( a \) will vanish when \( \rho = \rho_c \), and the quantum bounce occurs. Moreover, for a universe with a large scalar factor, the inverse volume modification to the Friedmann equation can be neglected and only the holonomy modification is important.

Based on the holonomy modification, the dynamical behavior of dark energy has recently been investigated by many authors [8,30,31]. The attractor behavior of the scalar field in LQC has also been studied [26,52]. It was found that the dynamical properties of dark-energy models in LQC are significantly different from those in classical cosmology. In this paper, we examine the background dynamics of LQC dominated by a scalar field with a general positive potential coupled with a barotropic fluid. By considering \( \Gamma \) as a function of \( \lambda \), we investigate scalar fields with different potentials. Since the Friedmann equation has been modified by the quantum
effect, the dynamical system will be very different from the one in classical cosmology, e.g., the number of dimensions of autonomous system will change to four in LQC. It must be pointed out that this method cannot be used to describe the dynamical behavior of scalar field with arbitrary potential. To overcome this problem, therefore, we should consider an infinite-dimensional autonomous system.

The paper is organized as follows. In Sec. II we present the basic equations and the four dimensional dynamical system, and in Sec. III we discuss the properties of this system. In Sec. IV we discuss the autonomous system in greater detail, as well as an infinite-dimensional autonomous system. We conclude the paper in the last section. The Appendix contains the analysis of the dynamical properties of one of the fixed points, $P_3$.

II. BASIC EQUATIONS

We focus on the flat Friedmann-Robertson-Walker cosmology. The modified Friedmann equation in the effective LQC with holonomy correction can be written as

$$H^2 = \frac{1}{3}\rho \left(1 - \frac{\rho}{\rho_c}\right),$$

in which $\rho$ is the total energy density and the natural unit $\kappa = 8\pi G = 1$ is adopted for simplicity. We consider a self-interacting scalar field $\phi$ with a positive potential $V(\phi)$ coupled with a barotropic fluid. Then the total energy density can be written as $\rho = \rho_\phi + \rho_\gamma$, with the energy density of scalar field $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and the energy density of barotropic fluid $\rho_\gamma$. We consider that the energy momenta of this field to be covariant conserved. Then one has

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0,$$  

(2)

$$\dot{\rho}_\gamma + 3\dot{H}\rho_\gamma = 0,$$  

(3)

where $\gamma$ is an adiabatic index and satisfies $p_\gamma = (\gamma - 1)\rho_\gamma$ with $p_\gamma$ being the pressure of the barotropic fluid, and the prime denotesthe differentiation with respect to the field $\phi$. Differentiating Eq. (1) and using Eqs. (2) and (3), one can obtain

$$\dot{H} = -\frac{1}{2} \left(\dot{\phi}^2 + \gamma \rho_\gamma \right) \left[1 - \frac{2(\rho_\phi + \rho_\gamma)}{\rho_c}\right].$$

(4)

Equations (1)-(3) and (2)-(3) characterize a closed system which can determine the cosmic behavior. To analyze the dynamical behavior of the Universe, one can further introduce the following variables [5, 6]:

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{\sqrt{V}}{\sqrt{3}H}, \quad z \equiv \frac{\rho}{\rho_c}, \quad \lambda \equiv \frac{V'}{V},$$

(5)

where the $z$ term is a special variable in LQC [see Eq. (1)]. In the LQC scenario, the total energy density $\rho$ should be less than or equal to the critical energy density $\rho_c$, and thus $0 \leq z \leq 1$. Notice that, in the classical region, $z = 0$ for $\rho \ll \rho_c$. Using these new variables, one can obtain

$$\frac{\rho_\gamma}{3H^2} = \frac{1}{1 - z} - x^2 - y^2,$$  

(6)

$$\frac{\dot{H}}{H^2} = - \left[3x^2 + \frac{3\gamma}{2} \left(\frac{1}{1 - z} - x^2 - y^2\right)\right] (1 - 2z).$$

(7)

Using the new variables (5), and considering Eqs. (6) and (7), one can rewrite Eqs. (1)-(11) in the following forms:

$$\frac{dx}{dN} = -3x - \sqrt{\frac{3}{2}}\lambda y^2 + x \left[3x^2 + \frac{3\gamma}{2} \left(\frac{1}{1 - z} - x^2 - y^2\right)\right] \times (1 - 2z),$$

(8)

$$\frac{dy}{dN} = \frac{\sqrt{6}}{2} \lambda xy + y \left[3x^2 + \frac{3\gamma}{2} \left(\frac{1}{1 - z} - x^2 - y^2\right)\right] \times (1 - 2z),$$

(9)

$$\frac{dz}{dN} = -3\gamma z - 3z (1 - z) \left(2x^2 - \gamma x^2 - \gamma y^2\right),$$

(10)

$$\frac{d\lambda}{dN} = \sqrt{6}\lambda x (\Gamma - 1),$$

(11)

where $N = \ln a$ and

$$\Gamma = \frac{VV''}{V'^2}.$$  

(12)

Note that the potential $V(\phi)$ is positive in this paper, but one can also discuss a negative potential. Just as [3] has shown, the negative scalar potential could slow down the growth of the scale factor and cause the Universe to be in a collapsing phase. The dynamical behavior of the scalar field with the positive and negative potential in brane cosmology has been discussed by [5]. In this paper we are concerned only with an expanding universe, and both the Hubble parameter and the potential are positive.

Differentiating $\lambda$ with respect to the scalar field $\phi$, we obtain the relationship between $\lambda$ and $\Gamma$,

$$\frac{d\lambda^{-1}}{d\phi} = 1 - \Gamma.$$  

(13)

If we only consider a special case of the potential, like exponential potential [6, 13], then $\lambda$ and $\Gamma$ are both constants. In this case, the four dimensional dynamical system, Eqs. (8)-(11), reduces to a 3-dimensional one, since $\lambda$ is a constant. (In the classical dynamical system, the $z$ term does not exist, and then the dynamical system is reduced from three dimensions to two dimensions.) The cost of this simplification is that the potential of the field is restricted. Recently, Zhou et al. [16, 17] considered the potential parameter $\Gamma$ as a function of another potential parameter $\lambda$, which enables one to study the fixed points for a large number of potentials. We will follow...
this method in this section and the sections that follow to discuss the dynamical behavior of the scalar field in the LQC scenario, and we have

$$\Gamma(\lambda) = f(\lambda) + 1. \quad (14)$$

In this case, Eq. (14) can cover many scalar potentials.

For completeness’ sake, we briefly review the discussion of Eq. (17) in the following. From Eq. (13), one can obtain

$$\frac{d\lambda}{\lambda f(\lambda)} = \frac{dV}{V}. \quad (15)$$

Integrating out $\lambda = \lambda(V)$, one has the following differential equation of potential

$$\frac{dV}{V\lambda(V)} = d\phi. \quad (16)$$

Then, Eqs. (15) and (16) provide a route for obtaining the potential $V = V(\phi)$. If we consider a concrete form of the potential (e.g., an exponential potential), the dynamical system is specialized (e.g., the dynamical system is reduced to three dimensions if one considers the exponential potential for $d\lambda/dN = 0$). These specialized dynamical systems are too special if one hopes to distinguish the fixed points that are the common properties of scalar field from those that are just related to the special potentials [17]. If we consider a more general $\lambda$, then we can get the more general stability properties of scalar field in the LQC scenario. We will continue the discussion of this topic in Sec. [17]. In this case, Eq. (11) becomes

$$\frac{d\lambda}{dN} = \sqrt{6}\lambda^2 f(\lambda). \quad (17)$$

Hereafter, Eqs. (13)-(10) along with Eq. (17) definitely describe a dynamical system. We will discuss the stability of this system in the following section.

### III. PROPERTIES OF THE AUTONOMOUS SYSTEM

Obviously, the terms on the right-hand side of Eqs. (8)-(10) and (17) only depend on $x, y, z, \lambda$, but not on $N$ or other variables. Such a dynamical system is usually called an autonomous system. For simplicity, we define $\frac{dx}{dN} = F_1(x, y, z, \lambda) \equiv F_1, \frac{dy}{dN} = F_2(x, y, z, \lambda) \equiv F_2, \frac{dz}{dN} = F_3(x, y, z, \lambda) \equiv F_3$. The fixed points $(x_c, y_c, z_c, \lambda_c)$ satisfy $F_1 = 0, i = 1, 2, 3, 4$. From Eq. (17), it is straightforward to see that $x = 0, \lambda = 0$ or $f(\lambda) = 0$ can make $F_3 = 0$. Also, we must consider $\lambda^2 f(\lambda) = 0$. Just as [17] argued, it is possible that $\lambda^2 f(\lambda) \neq 0$ and $\frac{d\lambda}{dN} \neq 0$ when $\lambda = 0$. Thus the necessary condition for the existence of the fixed points with $x \neq 0$ is $\lambda^2 f(\lambda) = 0$. Taking into account these factors, we can easily obtain all the fixed points of the autonomous system described by Eqs. (8)-(10) and (17), and they are shown in Tab. I.

| Fixed-points $x_c$ $y_c$ $z_c$ $\lambda_c$ | Eigenvalues $M^T = (0, -3\gamma, 3\gamma - 3 + \sqrt{\gamma})$ | Stability $U$, for all $\gamma$ |
|---------------------------------------------|------------------------------------------------|-------------------------|
| $P_1$ 0 0 0 0 | $M^T = (0, -3\gamma, 3\gamma - 3 + \sqrt{\gamma})$ | $U$, for all $\gamma$ |
| $P_2$ 0 0 0 $\lambda_c$ | $M^T = (0, -3\gamma, 3\gamma - 3 + \sqrt{\gamma})$ | $U$, for all $\gamma$ |
| $P_3$ 0 1 0 0 | $M^T = (-3, -3\gamma, 0, 0)$ | $U$, for all $\gamma$ |
| $P_4$ 1 0 0 0 | $M^T = (0, -6, 6 - 3\gamma)$ | $U$, for all $\gamma$ |
| $P_5$ -1 0 0 0 | $M^T = (0, -6, 6 - 3\gamma)$ | $U$, for all $\gamma$ |
| $P_6$ 0 0 0 $\lambda_a$ | $M^T = (0, -3\gamma, -3 + \sqrt{\gamma})$ | $U$, for all $\gamma$ |
| $P_7$ 1 0 0 $\lambda_a$ | $M^T = (-6, -3\gamma, -3 + \sqrt{\gamma})$ | $U$, for all $\gamma$ |
| $P_8$ -1 0 0 $\lambda_a$ | $M^T = (-6, -3\gamma, -3 + \sqrt{\gamma})$ | $U$, for all $\gamma$ |
| $P_9$ $\sqrt{1 - \frac{\gamma}{2}}\lambda_a$ | $M^T = (0, -3\gamma, -3 + \sqrt{\gamma})$ | $U$, for all $\gamma$ |
| $P_{10}$ $\sqrt{\frac{1}{2} \lambda_a}$ | $M^T = (-\lambda_a^2, -3 + \frac{1}{2} \lambda_a^2, \lambda_a^2 - 3\gamma, -\lambda_a^2 - f_1(\lambda_a))$ | $S$, for $f_1(\lambda_a) > \lambda_a$, and $\lambda_a < 3\gamma$ |
| $P_{11}$ See the Eq. (19) | $M^T = (0, -3\gamma, -3 + \sqrt{\gamma})$ | $U$, for all $\gamma$ |

TABLE I: The stability analysis of an autonomous system in LQC. The system is described by a self-interacting scalar field $\phi$ with positive potential $V$ coupled with a barotropic fluid $\rho$. Explanation of the symbols used in this table: $P_i$ denotes the fixed points located in the four dimensional phase space, which are earmarked by the coordinates $(x_c, y_c, z_c, \lambda_c)$. $\lambda_a$ means that $\lambda$ can be any value. $\lambda_a$ is the value that makes $f(\lambda) = 0$. $M^T$ means the inverted matrix of the eigenvalues of the fixed points. $f_1(\lambda) = \frac{df(\lambda)}{d\lambda}|_{\lambda=\Lambda}$ with $\Lambda = 0, \lambda_a$. $A = 2f(\lambda_a) + \lambda_a \left(\frac{df(\lambda)}{d\lambda}|_{\lambda=\Lambda}\right)$.
The properties of each fixed points are determined by the eigenvalues of the Jacobi matrix
\[
\mathcal{M} = \begin{pmatrix}
\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial \lambda} \\
\frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial \lambda} \\
\frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} & \frac{\partial F_3}{\partial \lambda} \\
\frac{\partial F_4}{\partial x} & \frac{\partial F_4}{\partial y} & \frac{\partial F_4}{\partial z} & \frac{\partial F_4}{\partial \lambda}
\end{pmatrix}_{(x_c, y_c, z_c, \lambda_c)}.
\] (18)

According to Lyapunov’s linearization method, the stability of a linearized system is determined by the eigenvalues of the matrix \(\mathcal{M}\) (see Chapter 3 of [34]). If all of the eigenvalues are strictly in the left-half complex plane, then the autonomous system is stable. If at least one eigenvalue is strictly in the right-half complex plane, then the system is unstable. If all of the eigenvalues are in the left-half complex plane, but at least one of them is on the \(i\omega\) axis, then one cannot conclude anything definite about the stability from the linear approximation. By examining the eigenvalues of the matrix \(\mathcal{M}\) for each fixed point shown in Table I, we find that points \(P_{1,2,4,8,9}\) are unstable and point \(P_3\) is stable only under some conditions. We cannot determine the stability properties of \(P_3\) from the eigenvalues, and we will give the full analysis of \(P_3\) in the Appendix.

Some remarks on Tab.I:

1. Apparently, points \(P_2\) and \(P_6\) have the same eigenvalues, and the difference between these two points is just on the value of \(\lambda\). As noted in the caption of Table I, \(\lambda_0\) means that \(\lambda\) can be any value, and \(\lambda_a\) is just the value that makes \(f(\lambda) = 0\). Obviously, \(\lambda_0\) is just a special value of \(\lambda_a\), and point \(P_6\) is a special case of point \(P_2\). \(P_6\) is connected with \(f(\lambda)\), but \(P_2\) is not. From now on, we do not consider separately the special case of \(P_6\) when we discuss the property of \(P_2\). Hence the value of \(\lambda_0\) is contained in our discussion of \(\lambda_a\).

2. It is straightforward to check that, if \(x_c = \lambda_c = 0\), \(y_c\) be any value \(y_s\) when it is greater than or equal 1. But, if \(y_s > 1\), then \(z_c = 1 - 1/y_s^2 < 1\), and this means that the fixed point is located in the quantum-dominated regions. Although the stability of this point in the quantum regions may depend on \(f(\lambda)\), it is not necessary to analyze its dynamical properties, since it does not have any physical meanings. The reason is the following: If the Universe is stable, it will not evolve to today’s pictures. If the Universe is unstable, it will always be unstable. We will just focus on point \(P_3\) staying in the classical regions. Then \(y_c = y_s = 1, z_c = 1 - 1/y_s^2 = 0\), i.e., for the classical cosmology region, \(\rho \ll \rho_c\).

3. Since the adiabatic index \(\gamma\) satisfies \(0 < \gamma < 2\) (in particular, for radiation \(\gamma = \frac{4}{3}\) and for dust \(\gamma = 1\)), all the terms that contain \(\gamma\) should not change sign. A more general situation of \(\gamma\) is \(0 \leq \gamma \leq 2\) [35]. If \(\gamma = 0\) or \(\gamma = 2\), the eigenvalues corresponding to points \(P_{1,2,4,5,9}\) will have some zero elements and some negative ones. To analyze the stability of these points, we need to resort to other more complex methods, just as we do in the Appendix for the dynamical properties of point \(P_3\). In this paper, we just consider the barotropic fluid which includes radiation and dust, and \(\gamma \neq 0, 2\). Notice that if one considers \(\gamma = 0\), the barotropic fluid describes the dark energy. This is an interesting topic, but will not be considered here for the sake of simplicity.

4. \(-\sqrt{6} < \lambda_a < \sqrt{6}, \lambda_a \neq 0\) should hold for point \(P_9\), hence \(-3 + \frac{1}{2} \lambda_a^2 < 0\).

5. \(\lambda_a > 0\) should hold, since \(y_c > 0\) for point \(P_{10}\). The eigenvalue of this point is

\[
\mathcal{M} = \begin{pmatrix}
-3\gamma \\
-\frac{3}{2} + \frac{3}{4} \gamma + \frac{3}{4} \sqrt{(2 - \gamma)(\lambda_a^2(2 - \gamma) + 8 \gamma + 24 \gamma^2)} \\
-\frac{3}{2} + \frac{3}{4} \gamma - \frac{3}{4} \lambda_a \sqrt{(2 - \gamma)(\lambda_a^2(2 - \gamma) + 8 \gamma + 24 \gamma^2)}
\end{pmatrix}
\] (19)

Since we just consider \(0 < \gamma < 2\) in this paper, it is easy to check that \((2 - \gamma)(\lambda_a^2(2 - \gamma) + 8 \gamma + 24 \gamma^2) > 0\) is always satisfied. And this point is unstable with \(f_1(\lambda_a) = \frac{d f(\lambda)}{d \lambda} |_{\lambda=\lambda_a}\) being either negative or positive, since \(-\frac{3}{2} + \frac{3}{4} \gamma + \frac{1}{\lambda_a} \sqrt{(2 - \gamma)(\lambda_a^2(2 - \gamma) + 8 \gamma + 24 \gamma^2)}\) is always positive.

Based on Table I and the related remarks above, we have the following conclusions:

1. Points \(P_{1,2}\): The related critical values, eigenvalues and stability properties do not depend on the specific form of the potential, since \(\lambda_c = 0\) or \(\lambda\) can be any value \(\lambda_a\).

2. Point \(P_3\): The related stability properties depend on \(f_1(0) = \frac{d f(\lambda)}{d \lambda} |_{\lambda=0}\).

3. Points \(P_{4,5}\): The related eigenvalues and stability properties do not depend on the form of the potential, but the critical values of these points should satisfy \(\lambda^2 f(\lambda) = 0\) since \(x_c \neq 0\).

4. Point \(P_6\): It is a special case of \(P_2\), but \(f(\lambda_a) = 0\) should be satisfied.

5. Points \(P_{7,8}\): Same as \(P_6\), they would not exist if \(f(\lambda_a) \neq 0\).
6. Point $P_{3,10}$: $f(\lambda_0) = 0$ should hold. The fixed values and the eigenvalues of these two points depend on $f_1(\lambda_a) = \frac{df(\lambda)}{d\lambda} |_{\lambda=\lambda_a}$.

Thus, only points $P_{1,2}$ are independent of $f(\lambda)$.

Comparing the fixed points in LQC and the ones in classical cosmology (see the Table I of [17]), we can see that, even though the values of the coordinates $(x_c, y_c, \lambda_c)$ are the same, the stability properties are very different. This is reasonable, because the quantum modification is considered, and the autonomous system in the LQC scenario is very different from the one in the classical scenario, e.g., the autonomous system is four dimensional in LQC but three dimensional in the classical scenario. Notice that all of the fixed points lie in the classical regions, and therefore the coordinates of fixed points remain the same from classical to LQC, which we also pointed out in an earlier paper [31].

Now we focus on the late time attractors: point $P_3$ under the conditions of $\gamma = 1, f_1(0) \geq 0$ and $\gamma = 4/3, f_1(0) = 0$, and point $P_0$ under the conditions of $\lambda_0^2 < 6, f_1(\lambda_a) > \lambda_a, \lambda_a < 3\gamma$. Obviously, these points are scalar-field dominated, since $\rho_a = H^2(1/(1 - z_c) - x_c^2 - y_c^2) = 0$. For point $P_3$, the effective adiabatic index $\gamma_\phi = (\rho_\phi + p_\phi)/\rho_\phi = 0$, which means that the scalar field is an effective cosmological constant. For point $P_0$, $\gamma_\phi = \lambda_0^2/2$. This describes a scaling solution that, as the universe evolves, the kinetic energy and the potential energy of the scalar field scale together. And we can see that there is not any barotropic fluid coupled with the scalar-field-dominated scaling solution. This is different from the dynamical behavior of scalar field with exponential potential $V = V_0 \exp(-\lambda \kappa \phi)$ in classical cosmology [8,13], and also is different from the properties of the scalar field in brane cosmology [8], in which $\lambda = \text{const.}$ (notice that the definition of $\lambda$ in [8] is different from the one in this paper) and $\Gamma$ is a function of $L(\rho(a))$ and $|V|$. In these models, the Universe may enter a stage dominated by a scalar field coupled with fluid when $\lambda, \gamma$ satisfy some conditions [8,13].

We discuss the dynamical behavior of the scalar field by considering $\Gamma$ as a function of $\lambda$ in this and the preceding sections. But $\Gamma$ cannot always be treated as a function of $\lambda$. We need to consider a more general autonomous system, which we will introduce in the next section.

IV. FURTHER DISCUSSION OF THE AUTONOMOUS SYSTEM

The dynamical behavior of the scalar field has been discussed by many authors (e.g., see [1,2,5,13,14]). If one wants to get the potentials that yield the cosmological scaling solutions beyond the exponential potential, one can add a $\frac{d\phi}{dt}$ term into the autonomous system [32]. All of these methods deal with special cases of the dynamical behavior of scalar fields in backgrounds of some specific forms. By considering $\Gamma$ as a function of $\lambda$, one can treat potentials of more general forms and get the common fixed points of the general potential, as shown in [16,17] and in the two preceding sections. However, as is discussed in [17], sometimes $\Gamma$ is not a function of $\lambda$, and then the dynamical behaviors of the scalar fields discussed above are still not general in the strict sense. For a more general discussion, we must consider the higher-order derivatives of the potential. We define

$$ (1) \Gamma = \frac{V V_2}{V^2}, \quad (2) \Gamma = \frac{V V_3}{V^2}, \quad (3) \Gamma = \frac{V V_5}{V^2}, \quad \cdots \quad (n) \Gamma = \frac{V V_{n+2}}{V^2}, \quad \cdots $$

in which $V_n = \frac{d^nV}{d\phi^n}, n = 3, 4, 5, \cdots$. Then we can get

$$ \frac{d\Gamma}{dN} = \sqrt{6} \Gamma \left[ (1) \Gamma - 2 \lambda^2 \right], $$

$$ \frac{d(1)\Gamma}{dN} = \sqrt{6} \left[ (1) \Gamma + (2) \Gamma - 2 \lambda^2 \left( (1) \Gamma \right) \right], $$

$$ \frac{d(2)\Gamma}{dN} = \sqrt{6} \left[ (2) \Gamma + (3) \Gamma - 2 \lambda^2 \left( (2) \Gamma \right) \right], $$

$$ \frac{d(3)\Gamma}{dN} = \sqrt{6} \left[ (3) \Gamma + (4) \Gamma - 2 \lambda^2 \left( (3) \Gamma \right) \right], \quad \cdots $$

$$ \frac{d(n)\Gamma}{dN} = \sqrt{6} \left[ (n) \Gamma + (n+1) \Gamma - 2 \lambda^2 \left( (n) \Gamma \right) \right], \quad \cdots $$

To discuss the dynamical behavior of scalar field with more general potential, e.g., when neither $\lambda$ nor $\Gamma$ is constant, we need to consider a dynamical system described by Eqs. (21)-(25) coupled with Eqs. (21)-(25). It is easy to see that this dynamical system is also an autonomous one. We can discuss the values of the fixed points of this autonomous system. Considering Eq. (25), we can see that the values of fixed points should satisfy $x_c = 0, \lambda_c = 0, or \Gamma_c = 1$. Then, we can get the fixed points of this infinite-dimensional autonomous system.

1. If $x_c = 0$, considering Eqs. (8)-(11), one can get $(y_c, z_c, \lambda_c) = (0, 0, 0)$ or $(y_c, z_c, \lambda_c) = (0, 0, \lambda_a)$, and $(n) \Gamma_c$ can be any values.

2. If $\lambda_c = 0$, considering Eqs. (8)-(10), one can see that the fixed points of $(x, y, z)$ are $(x_c, y_c, z_c) = (0, 1 - 1/y_c^2)$ and $(x_c, y_c, z_c) = (\pm 1, 0, 0)$. If $x_c = 0$, $(n) \Gamma_c$ can be any values, and if $x_c = \pm 1$, $(n) \Gamma_c = 0$.

3. If $\Gamma_c = 1$, considering Eqs. (8)-(10), one can get that the fixed points of $(x, y, z, \lambda)$ are $(x_c, y_c, z_c, \lambda_c) = (0, 0, 0, \lambda_c)$ and $(x_c, y_c, z_c, \lambda_c) = (\pm 1, 0, 0, \lambda_a)$. And $(n) \Gamma_c$ should satisfy $(n) \Gamma_c = \lambda_c^2$. There are other fixed points, which will be discussed below.

Based on the above analysis and Table I, one can find that points $P_{1-10}$ are just special cases of the fixed points.

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[References and equations are not fully transcribed due to constraints.]
of an infinite-dimensional autonomous systems. Considering the definition of $\Gamma$ (see Eq. (12)), the simplest potential is an exponential potential when $\Gamma_c = 1$. The properties of these fixed points have been discussed by many authors [5–13]. If $x_c = 0$ and $y_c = 0$, this corresponds to a fluid-dominated universe, which we do not consider here. If $x_c = \pm 1$, $\Gamma_c = 0$ and $(n)\Gamma_c = 0$, we do not need to consider the $\Gamma$ and the $(n)\Gamma$ terms. Then the stability properties of these points are the same as points $P_{4,5}$ in Table I, and there are unstable points. The last case is $(x_c, y_c, z_c, \lambda_c) = (0, y_c, 1 - 1/\gamma^2, 0)$ and $\Gamma, (n)\Gamma$ can be any value. To analyze the dynamical properties of this autonomous system, we need to consider the $(n)\Gamma_c$ terms. We will get an infinite series. In order to solve this infinite series, we must truncate it by setting a sufficiently high-order $(M)\Gamma$ to be a constant, for a positive integer $M$, so that $d ((M)\Gamma) / d\lambda = 0$. Thus we can get an $(M + 4)$-dimensional autonomous system. One example is the quadratic potential $V = \frac{1}{2}m^2\phi^2$ with some positive constant $m$ that gives a five dimensional autonomous system, and another example is the Polynomial (concave) potential $V = M^{4-n}\phi^n$ [39] that gives an $(n + 3)$-dimensional autonomous system. Following the method we used in the two preceding sections, we can get the dynamical behavior of such finite-dimensional systems.

In the remainder of this section, we discuss whether this autonomous system has a scaling solution.

If $x_c = 0$, then $\Gamma_c \neq 0, (n)\Gamma_c \neq 0$, and the stability of the fixed points may depend on the truncation. As an example, if we choose $(2)\Gamma = 0$, then we can get a six dimensional autonomous system. The eigenvalues for the fixed point $(x_c, y_c, z_c, \lambda_c, \Gamma_c, (1)\Gamma_c) = (0, 0, 0, \lambda_b, \Gamma_*, (1)\Gamma_*)$, where $\lambda_b = 0$ or $\lambda_b = \lambda_c$, is

$$M^T = (0, 0, 0, 3/2, \gamma, -3\gamma, -3 + 3/2 \gamma).$$

Obviously, this is an unstable point, and it has no scaling solution. The eigenvalues for the fixed point $(x_c, y_c, z_c, \lambda_c, \Gamma_c, (1)\Gamma_c) = (0, 1, 0, 0, \Gamma_*, (1)\Gamma_*)$ is

$$M^T = (0, 0, 0, 0, -3\gamma, -3 - 3\gamma).$$

According to the center manifold theorem (see Chapter 8 of [37], or [38]), there are two nonzero eigenvalues, and we need to reduce the dynamical system to two dimensions to get the stability properties of the autonomous system. This point may have scaling solution, but we need more complex mathematical method. We discuss this problem in another paper [40].

We discuss the last case. If $\Gamma_c = 1$, we can consider an exponential potential. Then the autonomous system is reduced to three dimensions. It is easy to check that the values $(x_{ec}, y_{ec}, z_{ec})$ of the fixed points are just the values $(x_c, y_c, z_c)$ of points $P_{6-10}$ in Table I. We focus on the two special fixed points:

$$F_1 : (x_{ec}, y_{ec}, z_{ec}) = (-\lambda/\sqrt{6}, \sqrt{1 - \lambda^2/6}, 0),$$

$$F_2 : (x_{ec}, y_{ec}, z_{ec}) = (-\sqrt{3/2} \gamma/\lambda, \sqrt{3(2 - \gamma)}/(2\lambda^2), 0).$$

Using Lyapunov’s linearization method, we can find that $F_2$ is unstable and $F_1$ is stable if $\lambda < 3\gamma$. It is easy to check that $\rho_\gamma = H^2[1/(1 - z_{ec}) - x_{ec}^2 - y_{ec}^2] = 0$ when $(x_{ec}, y_{ec}, z_{ec}) = (-\lambda/\sqrt{6}, \sqrt{1 - \lambda^2/6}, 0)$. From the above analysis, we find that there is just the scalar-field-dominated scaling solution when we consider the autonomous system to be described by a self-interacting scalar field coupled with a barotropic fluid in the LQC scenario.

V. CONCLUSIONS

The aim of this paper is two-fold. We discuss the dynamical behavior of scalar field in the LQC scenario following [16, 17]. To further analyze the dynamical properties of scalar field with more general potential, we introduce an infinite-dimensional autonomous system.

To discuss the dynamical properties of scalar field in the LQC scenario, we take $\Gamma$ as a function of $\lambda$, and extend the autonomous system from three dimensions to four dimensions. We find this extended autonomous system has more fixed points than the three dimensional one does. And we find that for some fixed points, the function $f(\lambda)$ affects either their values, e.g., for points $P_{6-10}$, or their stability properties, e.g., for points $P_{6,9}$. In other words, the dynamical properties of these points depend on the specific form of the potential. But some other fixed points, e.g., points $P_{1,2}$, are independent of the potential. The properties of these fixed points are satisfied by all scalar fields. We also find that there are two later time attractors, but the Universe is scalar-field dominated since $\rho_\gamma = 0$ at these later time attractors.

The method developed by [16, 17] can describe the dynamical behavior of the scalar field with potential of a more general form than, for example, an exponential potential [5–13]. But it is not all-encompassing. If one wants to discuss the dynamical properties of a scalar field with an arbitrary potential, one needs to consider the higher-order derivatives of the potential $V(\phi)$. Hence the dynamical system will extend from four dimensions to infinite-dimensions. This infinite-dimensional dynamical system is still autonomous, but it is impossible to get all of its dynamical behavior unless one considers $\Gamma_c = 1$ which just gives an exponential potential. If one wants to study as much as possible the dynamical properties of this infinite-dimensional autonomous system, one has to consider a truncation that sets $(M)\Gamma = \text{Const.}$, with $M$ above a certain positive integer. Then the infinite-dimensional system can be reduced to $M + 4$ dimensions.

And we find that there is just the scalar-field-dominated scaling solution for this autonomous system. We only give out the basic properties of this infinite-dimensional autonomous system in this paper, and will continue the discussion in the paper in [40].

We only get the scalar-field-dominated scaling solutions, whether we consider $\Gamma$ as a function of $\lambda$ or consider the higher-order derivatives of the potential. This
conclusion is very different from the autonomous system which is just described by a scalar field with an exponential potential \[3\]. This is an interesting problem that awaits further analysis.

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**Appendix: The stability properties of the Point \(P_3\)**

In Sec. IIIB, we point out that it is impossible to get the stability properties of the fixed point if at least one of the eigenvalues of \(\mathcal{M}\) is on the \(\omega c\) axis with the rest being in the left-half complex plane. The fixed point \(P_3\) is exactly such a point. In this appendix, we use the center manifold theorem (see Chapter 8 of \[37\], or \[38\]) to get the condition for stability of \(P_3\). The coordinates of \(P_3\) are \((0, 1, 0, 0)\) and the eigenvalues are \((-3, -3\gamma, 0, 0)\). First, we transfer \(P_3\) to \(P'_3\) \((\bar{x}_c = 0, \bar{y}_c = y - 1 = 0, z_c = 0, \lambda_c = 0)\). In this case, Eqs. \[3\]-\[10\] and \[17\] become

\[
\begin{align*}
\frac{dx}{dN} &= -3x - \frac{1}{2} \sqrt{\lambda} (\bar{y} + 1)^2 + x \left[ 3x^2 + \frac{3}{2} \gamma ((1 + z) \right. \\
&\quad - x^2 - (\bar{y} + 1)^2 \left. \right] (1 - 2z), \\
\frac{d\bar{y}}{dN} &= \frac{1}{2} \sqrt{6\lambda} x (\bar{y} + 1) + (\bar{y} + 1) \left[ 3x^2 + \frac{3}{2} \gamma ((1 + z) \right. \\
&\quad - x^2 - (\bar{y} + 1)^2 \left. \right] (1 - 2z), \\
\frac{dz}{dN} &= -3\gamma z - 3z (1 - z) \left[ 2x^2 - \gamma x^2 - (\bar{y} + 1)^2 \right], \\
\frac{d\lambda}{dN} &= \sqrt{6\lambda}^2 (f(0) + f_1(0)\lambda) x,
\end{align*}
\]

where we have considered that the related variables \((x, \bar{y}, z, \lambda)\) are small around point \((x_c, \bar{y}_c, z_c, \lambda_c) = (0, 0, 0, 0)\). Therefore the following Taylor series

\[
\frac{1}{1 - z} = 1 + z + \cdots , \\
f(\lambda) = f(0) + f_1(0)\lambda + \cdots ,
\]

can be used, where \(f_1(0) = \frac{df(\lambda)}{d\lambda} \big|_{\lambda=0}\).

We can get the Jacobi matrix \(\mathcal{M}'\) of the dynamical system Eqs. \[A1\]-\[A4\] as

\[
\mathcal{M}' = \begin{pmatrix}
-3 & 0 & 0 & -\sqrt{\gamma} \\
0 & -3\gamma & \frac{\sqrt{\gamma}}{2} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

It is easy to get the eigenvalues and eigenvectors of \(\mathcal{M}'\). Let \(A\) denote the matrix whose columns are the eigenvalues, and \(S\) denote the matrix whose columns are the eigenvectors, and then we have

\[
A = \begin{pmatrix}
-3 & 0 & 0 & 0 \\
0 & -3\gamma & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
1 & -\frac{\sqrt{\gamma}}{2} & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

With the help of \(S\), we can transform \(\mathcal{M}'\) into a block diagonal matrix

\[
S^{-1}\mathcal{M}'S = \begin{pmatrix}
-3 & 0 & 0 & 0 \\
0 & -3\gamma & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}A_1 & 0 \\
0 & A_2\end{pmatrix},
\]

where all eigenvalues of \(A_1\) have negative real parts, and all eigenvalues of \(A_2\) have zero real parts.

Now we change the variables to be

\[
\begin{pmatrix}X \\
Y \\
Z \\
\lambda\end{pmatrix} = S^{-1} \begin{pmatrix}x \\
\bar{y} \\
z \\
\lambda\end{pmatrix} = \begin{pmatrix}x + \frac{\sqrt{\gamma}}{2} \lambda \\
\bar{y} - \frac{\sqrt{\gamma}}{2} z \\
\lambda \\
z\end{pmatrix}.
\]

Then, we can rewrite the autonomous system in the form of the new variables:

\[
\begin{pmatrix}
\frac{dX}{dN} \\
\frac{dY}{dN} \\
\frac{dZ}{dN} \\
\frac{d\lambda}{dN}
\end{pmatrix} = \begin{pmatrix}
-3 & 0 & 0 & 0 \\
0 & -3\gamma & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}X \\
Y \\
Z \\
\lambda\end{pmatrix} + \begin{pmatrix}G_1 \\
G_2 \\
G_3 \\
G_4\end{pmatrix},
\]

where \(G_i = G_i(X, Y, Z, \lambda), (i = 1, 2, 3, 4)\) are functions of \(X, Y, Z, \text{and } \lambda\). It is easy to get \(G_i\) by substituting the transformations \(x = X - \frac{3}{2} \lambda, \bar{y} = Y + \frac{3}{2} \lambda, z = \lambda, \lambda = Z\) into the R.H.S. of Eqs. \[A1\]-\[A4\].

According to the center manifold theorem \[38\], there exists a \(C^\infty\)-center manifold

\[
W_{loc}^c = \{(X, Y, Z, \lambda) : X \equiv h_1(Z, \lambda), Y \equiv h_2(Z, \lambda), \quad h_1(0, 0) = 0, h_2(0, 0) = 0\}
\]

such that the dynamics of \[A9\] can be restricted to the center manifold. \(J_{h_1}\) is the Jacobi matrix of \(h_i\), and \(h_1(Z, \lambda), h_2(Z, \lambda)\) are

\[
h_1(Z, \lambda) = A_1 Z^2 + A_2 Z \lambda + A_3 \lambda^2 + \cdots , \quad (A10)
\]

\[
h_2(Z, \lambda) = B_1 Z^2 + B_2 Z \lambda + B_3 \lambda^2 + \cdots . \quad (A11)
\]

We just consider the quadratic forms of \(Z\) and \(\lambda\) in this appendix.
Considering the center manifold theorem, we have
\[
\frac{dX}{dN} = \frac{\partial h_1(Z, \bar{X})}{\partial Z} \frac{dZ}{dN} + \frac{\partial h_1(Z, \bar{X})}{\partial \bar{X}} \frac{d\bar{X}}{dN},
\]
(A12)
\[
\frac{dY}{dN} = \frac{\partial h_2(Z, \bar{Y})}{\partial Z} \frac{dZ}{dN} + \frac{\partial h_2(Z, \bar{Y})}{\partial \bar{Y}} \frac{d\bar{Y}}{dN}.
\]
(A13)
Inserting the Eqs. (A10) and (A11) into \(dX/dN, dY/dN\) in Eq. (A9) and Eqs. (A12)-(A13), and comparing the coefficients of \(dX/dN\) and \(dY/dN\), we get
\[
A_1 = 0, \quad A_2 = \sqrt[6]{\frac{6}{\gamma}}, \quad A_3 = 0, \quad B_1 = \frac{1}{12},
\]
\[
B_2 = 0, \quad B_3 = \frac{1}{8}.
\]
(A14)
Then, the dynamics near the origin is governed by the following equations,
\[
\frac{dZ}{dN} = -Z^3 f_1(0),
\]
(A15)
\[
\frac{d\bar{\lambda}}{dN} = -Z^2 \bar{\lambda} + \frac{3}{2} \gamma \bar{\lambda}^3.
\]
(A16)
We consider two different values of \(\gamma\) to get the stability properties of this system. This is because a different \(\gamma\) will give a different dynamical systems. The first one to be considered is dust, which has \(\gamma = 1\). Then, we have
\[
\frac{dZ}{dN} = -Z^3 f_1(0),
\]
(A17)
\[
\frac{d\bar{\lambda}}{dN} = -Z^2 \bar{\lambda}.
\]
(A18)
According to Lyapunov’s theorem, we can define a Lyapunov function to analyze the stability properties of a dynamical system. Different dynamical systems have different Lyapunov functions, and one dynamical system can also have different Lyapunov functions. But all the Lyapunov functions \(U\) should satisfy \(U(x) \geq 0\) at the original point (Chapter 2 of [37]). Then we can define
\[
U_1 = \frac{1}{2} \left( Z^2 + \bar{\lambda}^2 \right).
\]
(A19)
Using Eqs. (A17) and (A18), we have
\[
\frac{dU_1}{dN} = -f_1(0) Z^4 - \frac{3}{2} Z^2 \bar{\lambda}^3.
\]
(A20)
According to Lyapunov’s stability theorem, the system is stable if \(f_1(0) \geq 0\).

Now we turn to considering radiation, which has \(\gamma = \frac{4}{3}\). Eqs. (A15) and (A16) become
\[
\frac{dZ}{dN} = -Z^3 f_1(0),
\]
(A21)
\[
\frac{d\bar{\lambda}}{dN} = -2 \bar{\lambda}^3 + \frac{3}{2} Z^2 \bar{\lambda}.
\]
(A22)
We need to consider three possible cases: (a) \(f_1(0) \neq 0\), (b) \(f_1(0) = 0, Z(N = 0) = 0\), and (c) \(f_1(0) = 0, Z(N = 0) \neq 0\), since these three different cases will bring out three different dynamical systems.

If \(f_1(0) \neq 0\), the Lyapunov function can be defined as
\[
U_2 = \frac{1}{1 + Z^2/(6A) + \bar{\lambda}^2},
\]
(A23)
where \(A = f_1(0)\) if \(f(0) > 0\), and \(A = -f_1(0)\) if \(f_1(0) < 0\). Then one can get
\[
\frac{dU_2}{dN} = \frac{12 A^2 \left( Z^2 - \bar{\lambda}^2 \right)^2 + 5 \bar{\lambda}^4}{6A + 6A \bar{\lambda}^2 + Z^2} > 0.
\]
(A24)
Then this point is an unstable one.

If \(f_1(0) = 0\) and \(Z(N = 0) = 0\), Eq. (A22) becomes
\[
\frac{d\bar{\lambda}}{dN} = -2 \bar{\lambda}^3.
\]
Defining Lyapunov function,
\[
U_3 = 1 + \bar{\lambda}^2,
\]
(A25)
then
\[
\frac{dU_3}{dN} = -4 \bar{\lambda}^4 \leq 0.
\]
(A26)
If \(f_1(0) = 0\) and \(Z(N = 0) \neq 0\), one can get \(Z = C\) from Eq. (A21), with a non-zero constant \(C\). Equation (A22) becomes
\[
\frac{d\bar{\lambda}}{dN} = -2 \bar{\lambda}^3 + \frac{1}{3} C^2 \bar{\lambda},
\]
(A27)
The Lyapunov function can be defined as
\[
U_4 = \left( 1 - \frac{6}{C^2 \bar{\lambda}^2} \right)^2,
\]
(A28)
Then we have
\[
\frac{dU_4}{dN} = -\frac{8}{C^2} \bar{\lambda}^2 \left( C^2 - 6 \bar{\lambda}^2 \right)^2 \leq 0.
\]
(A29)
Obviously, according to Lyapunov’s stability theorem, this point is stable as long as \(f_1(0) = 0\), regardless of \(Z(N = 0) = 0\) or \(Z(N = 0) \neq 0\).
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