Soldering Formalism: Theory and Applications

Clovis Wotzasek

Instituto de Física
Universidade Federal do Rio de Janeiro
21945, Rio de Janeiro, Brazil

Abstract

The soldering mechanism is a new technique to work with distinct manifestations of dualities that incorporates interference effects, leading to new physical results that includes quantum contributions. To work out this formalism in different scenarios we have developed a systematic method of obtaining duality symmetric actions in any space-time dimensions. This approach was used to investigate the cases of electromagnetic dualities, and $D \geq 2$ bosonization. In the former context this technique is applied for the quantum mechanical harmonic oscillator, the scalar field theory in two dimensions and the Maxwell theory in four dimensions. In all cases there are two such distinct actions. Furthermore, by soldering these actions in any dimension a master action is obtained which is duality invariant under a much bigger set of symmetries than is usually envisaged. The concept of swapping duality is introduced and its implications are discussed. The effects of coupling to gravity are also elaborated. Finally, the extension of the analysis for arbitrary dimensions is indicated. In the later context, a technique is developed that solders the dual aspects of some symmetry following from the bosonisation of two distinct fermionic models, thereby leading to new results which cannot be otherwise obtained. Exploiting this technique, the two dimensional chiral determinants with opposite chirality are soldered to reproduce either the usual gauge invariant expression leading to the Schwinger model or, alternatively, the Thirring model. Likewise, two apparently independent three dimensional massive Thirring models with same coupling but opposite mass signatures, in the long wavelegth limit, combine by the process of bosonisation and soldering to yield an effective massive Maxwell theory. The current bosonisation formulas are given, both in the original independent formulation as well as the effective theory, and shown to yield consistent results for the correlation functions. Similar features also hold for quantum electrodynamics in three dimensions.

Keywords: Soldering Formalism, Duality symmetric actions, Bosonization

1 Based on work done in collaboration with R.Banerjee and E.M.C.Abreu
2 Talk presented at Universite de Montreal, Nov 24-27, 1997.
1 Introduction

The role of duality as a qualitative tool in the investigation of physical systems is being gradually disclosed at different context and dimensions\cite{1}. Much effort has been given in sorting out several technical aspects of duality symmetric actions. A new technique, able to work with distinct manifestations of the duality symmetry was proposed many years ago by Stone\cite{2}. The Stones’ soldering mechanism for fusing together opposite aspects of duality symmetries, provides a new formalism that includes the quantum interference effects between the independent components. This leads to a new and unique way of obtaining physical results that includes quantum contributions, and is dimension independent.

Electromagnetic duality in 2D and 4D is reexamined under the soldering point of view providing interesting new results, both explicitly dual and covariant as the result of the interference between self and anti-self dual actions displaying opposite aspects of the electromagnetic duality symmetry\cite{3}. Even in the most elementary problem in theoretical physics, the simple harmonic oscillator, the soldering technique has been able to produce new and interesting results. In fact it is precisely this simple manifestation of duality that pervades all field theoretical examples as will be explicitly shown.

Exciting new results in 2D and 3D bosonization were generated via soldering that could not be obtained by any of the known existent bosonization techniques. In 2D this technique allowed us to show how massless chiral fields combine to provide the gauge invariant massive mode present in the $QED_2$, showing that the Schwinger mechanism is the result of quantum interference between right and left moving modes, very much like in the optical double slit Young’s experiment. In fact, by equipping the soldering technique with gauge and Bose symmetry\cite{4}, it automatically selects, in a unique way, the massless sector of the chiral models displaying the Jackiw-Rajaraman parameter that reflects the bosonization ambiguity by $a = 1$. In the 3D case, the soldering mechanism was used to show the result of fusing together two topologically massive modes generated by the bosonization of two massive Thirring models with opposite mass signatures in the long wave-length limit. The bosonized modes, which are described by self and anti-self dual Chern-Simons models\cite{5, 6}, were then soldered into the two massive modes of the 3D Proca model\cite{7}.

These Lecture Notes are divided into three sections. we conclude this first section elaborating over the soldering formalism and the ideas of self and anti-self dualities. This is illustrated further in section II with a very simple example, the one-dimensional harmonic oscillator mentioned above. Indeed this example is worked out in some details to unveil the key concepts of our approach and to set the general tone of the Notes. In fact, the extension to field theory is more a matter of technique rather than introducing truly new and fundamental concepts. The remaining of this section is devoted to the discussion of electromagnetic dualities in 4D, but due to the special nature of the four-dimensional duality transformation, it becomes mandatory to discuss the even dimensional case in its full generality. Indeed, for this problem the dimensionality of space time appears to be extremely crucial. To treat all even dimensional cases in a unified way, one introduces the idea of an internal two-dimensional space. This internal space will generalize the idea of dual operation in such way that the concept of self and anti-self duality will be well defined in all even $D = 2k + 2$ dimensions. Their explicit realisation is one of the central results of our work.

The analyse of special features and applications of bosonization is the object of the third section
of the Notes. We show that two distinct models displaying dual aspects of some symmetry can be combined by the simultaneous implementation of bosonisation and soldering to yield a completely new theory. Importantly, dimensionality poses no constraint in this analyses. We start studying the two dimensional case where bosonisation is known to yield exact results. Using bosonised expressions, the soldering mechanism fuses, in a precise way, left and right chiralities. This leads to a general lagrangean in which the chiral symmetry no longer exists, but it contains two extra parameters manifesting the bosonisation ambiguities. It is shown that different parametrisations lead to different models. Our results are not restrict to the gauge invariant Schwinger model but the Thirring model is also reproduced. We take the opportunity to stress the importance of Bose symmetry in the chiral bosonization analysis. Some interesting consequences regarding the arbitrary parametrisation in the chiral Schwinger model are also charted.

In the subsequent section, the discussion of the three dimensional bosonization illuminates the full power and utility of the present approach. Recall that bosonisation in higher dimensions is not exact, nevertheless, for massive fermionic models in the large mass or, equivalently, the long wavelength limit, well defined local expressions are known to exist\[8, 9\]. Interestingly, these expressions exhibit a self or an anti self dual symmetry that is dictated by the signature of the fermion mass. It is precisely these symmetries that simulates the dual aspects of the left and right chiral symmetry in the two dimensions. Indeed, two distinct massive Thirring models with opposite mass signatures, are soldered to yield a massive Maxwell theory. A direct comparison of the current correlation functions obtained before and after the soldering process are carefully developed to confirm the utility and practical nature of this new formalism.

1.1 The Soldering Formalism

The technique of soldering essentially comprises in lifting the gauging of a global symmetry to its local version and exploits certain concepts introduced in a different context by Stone\[2\] and one of us\[10\]. The analysis is intrinsically quantal without having any classical analogue. This is easily explained by the observation that a simple addition of two independent classical lagrangeans is a trivial operation without leading to anything meaningful or significant. On the other hand, the direct sum of classical actions depending on different fields would not give anything new. It is the soldering process that leads to a new and nontrivial result.

The basic idea of the soldering procedure is to raise a global Noether symmetry of the self and anti-seld dual constituents into a local one, but for an effective composite system, consisting of the dual components and an interference term. This algorithm, consequently, defines the soldered action. Here we shall adopt an iterative Noether procedure to lift the global symmetries. Therefore, assume that the symmetries in question are being described by the local actions $S_{\pm}(\phi_{\pm})$, invariant under a global multi-parametric transformation

$$\delta \phi^\eta_{\pm} = \alpha^\eta$$  

Here $\eta$ represents the tensorial character of the basic fields in the dual actions $S_{\pm}$ and, for notational simplicity, will be dropped from now on. Now, under local transformations these actions will not remain invariant, and Noether counter-terms become necessary to reestablish the invariance, along
with appropriate compensatory soldering fields $B^{(N)},$
\[ S_{\pm}(\phi_{\pm})^{(0)} \rightarrow S_{\pm}(\phi_{\pm})^{(N)} = S_{\pm}(\phi_{\pm})^{(N-1)} - B^{(N)}J^{(N)}_{\pm} \]  
(2)

Here $J^{(N)}_{\pm}$ are the Noether currents, and $(N)$ is the iteration number. For the self and anti-self dual systems we have in mind, this iterative gauging procedure is (intentionally) construct not to produce invariant actions for any finite number of steps. However, if after $N$ repetitions the non invariant piece end up being only dependend on the gauging parameters, but not on the original fields, there will exist the possibility of mutual cancelation, if both self and anti-seld gauged systems are put together. Then, suppose that after $N$ repetitions we arrive at the following simultaneous conditions,
\[ \delta S_{\pm}(\phi_{\pm})^{(N)} \neq 0 \]
\[ \delta S_{B}(\phi_{\pm}) = 0 \]  
(3)

with
\[ S_{B}(\phi_{\pm}) = S^{(N)}_{\pm}(\phi_{\pm}) + S_{-}(\phi_{-}) + \text{Contact Terms} \]  
(4)

Then we can immediately identify the (soldering) interference term as,
\[ S_{\text{int}} = \text{Contact Terms} - \sum_{N} B^{(N)}J^{(N)}_{\pm} \]  
(5)

where the Contact Terms are generally quadratic functions of the soldering fields. Incidentally, these auxiliary fields $B^{(N)}$ may be eliminated from the resulting effective action, in favor of the physically relevant degrees of freedom. It is important to notice that after elimination of the soldering fields, the resulting effective action will not depend on either self or anti-self dual fields $\phi_{\pm}$ but only in some collective field, say $\Phi$, defined in terms of the original ones in a (Noether) invariant way.
\[ S_{B}(\phi_{\pm}) \rightarrow S_{\text{eff}}(\Phi) \]  
(6)

Once such effective action has been established, the physical consequences of the soldering are readily obtained by simple inspection. This will progressively be clarified in the especific applications to be given in the sections that follow.

2 Duality Symmetry and Soldering in Different Dimensions

In recent times the old idea [11, 12, 13, 14, 15] of electromagnetic duality has been revived with considerable attention and emphasis [16, 17, 18, 19]. Recent directions [16, 20, 21, 22] also include an abstraction of manifestly covariant forms for such actions or an explicit proof of their equivalence with the nonduality symmetric actions, which they are supposed to represent. There are also different suggestions on the possible analogies between duality symmetric actions in different dimensions. In particular it has been claimed [21] that the two dimensional self dual action given in [23] is the analogue of the four dimensional electromagnetic duality symmetric action [16]. In spite of the recent spate of papers on this subject there does not seem to be a simple clear cut way of arriving
at duality symmetric actions. Consequently the fundamental nature of duality remains clouded by
technicalities. Additionally, the dimensionality of space time appears to be extremely crucial. For
instance, while the duality symmetry in $D = 4k$ dimensions is characterised by the one-parameter
continuous group $SO(2)$, that in $D = 4k + 2$ dimensions is described by a discrete group with
just two elements \[19\]. Likewise, it has also been argued from general notions that a symmetry
generator exists only in the former case. From an algebraic point of view the distinction between the
dimensionalities is manifested by the following identities,

\[
\ast\ast F = F ; \quad D = 4k + 2 \\
= -F ; \quad D = 4k
\]

where the $\ast$ denotes a usual Hodge dual operation and $F$ is the $\frac{D}{2}$-form. Thus there is a self dual
operation in the former which is missing in the latter dimensions. This apparently leads to separate
consequences for duality in these cases.

The object of this section is to develop a method for systematically obtaining and investigating
different aspects of duality symmetric actions that embrace all dimensions. A deep unifying structure
is illuminated which also leads to new symmetries. Indeed we show that duality is not limited to field
or string theories, but is present even in the simplest of quantum mechanical examples- the harmonic
oscillator. It is precisely this duality which pervades all field theoretical examples as will be explicitly
shown. The basic idea of our approach is deceptively simple. We start from the second order action
for any theory and convert it to the first order form by introducing an auxiliary variable. Next, a
suitable relabelling of variables is done which induces an internal index in the theory. It is crucial
to note that there are two distinct classes of relabelling characterised by the opposite signatures of
the determinant of the $2 \times 2$ orthogonal matrix defined in the internal space. Correspondingly, in
this space there are two actions that are manifestly duality symmetric. Interestingly, their equations
of motion are just the self and anti-self dual solutions, where the dual field in the internal space is
defined below in (8). It is also found that in all cases there is one (conventional duality) symmetry
transformation which preserves the invariance of these actions but there is another transformation
which swaps the actions. We refer to this property as swapping duality. This indicates the possibility,
in any dimensions, of combining the two actions to a master action that would contain all the duality
symmetries. Indeed this construction is explicitly done by exploiting the ideas of soldering introduced
in \[2\] and developed by us \[24, 7\]. The soldered master action also has manifest Lorentz or general
coordinate invariance. The generators of the symmetry transformations are also obtained.

It is easy to visualise how the internal space effectively unifies the results in the different $4k + 2$
and $4k$ dimensions. The dual field is now defined to include the internal index $(\alpha, \beta)$ in the fashion,

\[
\tilde{F}^\alpha = \epsilon^{\alpha\beta\ast}F^\beta ; \quad D = 4k \\
\tilde{F}^{\alpha\ast} = \sigma^{\alpha\beta\ast}_1 F^\beta ; \quad D = 4k + 2
\]

where $\sigma_1$ is the usual Pauli matrix and $\epsilon_{\alpha\beta}$ is the fully antisymmetric $2 \times 2$ matrix with $\epsilon_{12} = 1$. Now,
irrespective of the dimensionality, the repetition of the dual operation yields,

\[
\tilde{\tilde{F}} = F
\]
which generalises the relation (7). An immediate consequence of this is the possibility to obtain self and anti-self dual solutions in all even \( D = 2k + 2 \) dimensions. Their explicit realisation is one of the central results of the paper.

This section is organised into five subsections. In section 2.1 the above ideas are exposed by considering the example of the simple harmonic oscillator. Provocatively, a close parallel with the electromagnetic notation is also developed to illuminate the connection between this exercise and those given for the field theoretical models in the next subsections. The duality of scalar field theory in two dimensions is considered in section 2.2. The occurrence of a pair of actions is shown which exhibit duality and swapping symmetries. These are the analogues of the four dimensional electromagnetic duality symmetric actions. Indeed, from these expressions, it is a trivial matter to reproduce both the self and anti-self dual actions given in [23]. Our analysis clarifies several issues regarding the intertwining roles of chirality and duality in two dimensions. The soldering of the pair of duality symmetric actions is also performed leading to fresh insights. The analysis is completed by including the effects of gravity in section 2.3. In section 2.4, the Maxwell theory is treated in great details. Following our prescription the duality symmetric action given in [16] is obtained. However, there is also a new action which is duality symmetric. Once again the soldering of these actions leads to a master action which contains a much richer structure of symmetries. Incidentally, it also manifests the original symmetry that interchanges the Maxwell equations with the Bianchi identity, but reverses the signature of the action. In section 2.5, the effects of gravity are straightforwardly included. Section 2.6 contains the concluding comments.

### 2.1 Duality in \( 0 + 1 \) dimension

The basic features of duality symmetric actions are already present in the quantum mechanical examples as the present analysis on the harmonic oscillator will clearly demonstrate. Indeed, this simple example is worked out in some details to illustrate the key concepts of our approach and set the general tone of the paper. An extension to field theory is more a matter of technique rather than introducing truly new concepts. The Lagrangean for the one-dimensional oscillator is given by,

\[
L = \frac{1}{2}(\dot{q}^2 - q^2)
\]  

leading to an equation of motion,

\[
\ddot{q} + q = 0
\]

Introducing a change of variables,

\[
E = \dot{q} \quad ; \quad B = q
\]

so that,

\[
\dot{B} - E = 0
\]

is identically satisfied, the above equations (10) and (11) are, respectively, expressed as follows;

\[
L = \frac{1}{2}(E^2 - B^2)
\]

and,

\[
\dot{E} + B = 0
\]
It is simple to observe that the transformations

\[ E \rightarrow \pm B \; ; \; B \rightarrow \mp E \]  

(16)

swaps the equation of motion (15) with the identity (13) although the Lagrangean (14) is not invariant. The similarity with the corresponding analysis in the Maxwell theory is quite striking, with \( q \) and \( \dot{q} \) simulating the roles of the magnetic and electric fields, respectively. There is a duality among the equation of motion and the ‘Bianchi’ identity (13), which is not manifested in the Lagrangean.

In order to elevate the duality to the Lagrangean, the basic step is to rewrite (10) in the first order form by introducing an additional variable,

\[ L = p\dot{q} - \frac{1}{2}(p^2 + q^2) \]

(17)

where a symmetrisation has been performed. There are now two possible classes for relabelling these variables corresponding to proper and improper rotations generated by the matrices \( R^+(\theta) \) and \( R^-(\varphi) \) with determinant \( +1 \) and \( -1 \), respectively,

\[
\begin{pmatrix}
q \\
p
\end{pmatrix}
=
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

(18)

\[
\begin{pmatrix}
q \\
p
\end{pmatrix}
=
\begin{pmatrix}
\sin \varphi & \cos \varphi \\
\cos \varphi & -\sin \varphi
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

(19)

leading to the distinct Lagrangeans,

\[ L_{\pm} = \frac{1}{2} \left( \pm x_\alpha \epsilon_{\alpha\beta} \dot{x}_\beta - x_\alpha^2 \right) \]

(20)

where we have reverted back to the ‘electromagnetic’ notation introduced in (12). By these change of variables an index \( \alpha = (1, 2) \) has been introduced that characterises a symmetry in this internal space, the complete details of which will progressively become clear. It is useful to remark that the above change of variables are succinctly expressed as,

\[ q = x_1 ; \; p = x_2 \]
\[ q = x_2 ; \; p = x_1 \]

(21)

by setting the angle \( \theta = 0 \) or \( \varphi = 0 \) in the rotation matrices (18) and (19). Correspondingly, the Lagrangean (17) goes over to (20). Now observe that the above Lagrangeans (20) are manifestly invariant under the continuous duality transformations,

\[ x_\alpha \rightarrow R^+_{\alpha\beta} x_\beta \]

(22)

\footnotetext{Note that these are just the discrete cases \((\alpha = \pm \frac{\pi}{2})\) for a general \(SO(2)\) rotation matrix parametrised by the angle \( \alpha \).}
which may be equivalently expressed as,

\[ E_\alpha \rightarrow R^+_{\alpha\beta} E_\beta \]
\[ B_\alpha \rightarrow R^+_{\alpha\beta} B_\beta \]  \hspace{1cm} (23)

where \( R^+_{\alpha\beta} \) is the usual \( SO(2) \) rotation matrix \([18]\). The generator of the infinitesimal symmetry transformation is given by,

\[ Q^\pm = \pm \frac{1}{2} x_\alpha x_\alpha \]  \hspace{1cm} (24)

so that the complete transformations \((22)\) are generated as,

\[ x_\alpha \rightarrow x'_\alpha = e^{-i\theta Q} x_\alpha e^{i\theta Q} \]
\[ = R^+_{\alpha\beta}(\theta) x_\beta \]  \hspace{1cm} (25)

This is easy to verify by using the basic symplectic brackets obtained from \((20)\),

\[ \{x_\alpha, x_\beta\} = \mp \epsilon_{\alpha\beta} \]  \hspace{1cm} (26)

Parametrising the angle by \( \theta = \frac{\pi}{2} \) the discrete transformation is obtained,

\[ E_\alpha \rightarrow \epsilon_{\alpha\beta} E_\beta \]
\[ B_\alpha \rightarrow \epsilon_{\alpha\beta} B_\beta \]  \hspace{1cm} (27)

This is the parallel of the usual constructions done in the Maxwell theory to induce a duality symmetry in the action.

Let us now comment on an interesting property, which is related to the existence of two distinct Lagrangeans \((20)\), by replacing \((23)\) with a new set of transformations,

\[ E_\alpha \rightarrow R^-_{\alpha\beta}(\varphi) E_\beta \]
\[ B_\alpha \rightarrow R^-_{\alpha\beta}(\varphi) B_\beta \]  \hspace{1cm} (28)

Notice that these transformations preserve the invariance of the Hamiltonian following from either \( L_+ \) or \( L_- \). Interestingly, the kinetic terms in the Lagrangeans change signatures so that \( L_+ \) and \( L_- \) swap into one another. This feature of duality swapping will subsequently recur in a different context and has important implications in higher dimensions.

The discretised version of \((28)\) is obtained by setting \( \varphi = 0 \),

\[ E_\alpha \rightarrow \sigma^{\alpha\beta}_1 E_\beta \]
\[ B_\alpha \rightarrow \sigma^{\alpha\beta}_1 B_\beta \]  \hspace{1cm} (29)

It is precisely the \( \sigma_1 \) matrix that reflects the proper into improper rotations,

\[ R^+(\theta) \sigma_1 = R^-(\theta) \]  \hspace{1cm} (30)

which illuminates the reason behind the swapping of the Lagrangeans in this example.
Since we have systematically developed a procedure for obtaining a duality symmetric Lagrangean, it is really not necessary to show its equivalence with the original Lagrangean, as was done in the Maxwell theory. Nevertheless, to complete the analogy, we show that (20) reduces to (14) or (10) by using the equation of motion,

\[ x_\alpha = \pm \epsilon_{\alpha\beta} \dot{x}_\beta \]  

which can be reexpressed as,

\[ B_\alpha = \pm \epsilon_{\alpha\beta} E_\beta \]  

to eliminate one component (say the variables with label 2) from (20). This immediately reproduces (14) while (27) reduces to (16).

An important point to stress is that there are actually two, and not one, duality symmetric actions \( L_{\pm} \) (20), corresponding to the signatures in the determinant of the transformation matrices. As shall be shown in subsequent sections this is also true for the scalar field theory in 1 + 1 dimensions and the electromagnetic theory in 3 + 1 dimensions. Usually, in the literature, only one of these is highlighted while the other is not mentioned. We now elaborate the implications of this property which will also be crucial in discussing field theoretical models. In the coordinate language these Lagrangeans correspond to two bi-dimensional chiral oscillators rotating in opposite directions. This is easily verified either by looking at the classical equations of motion or by examining the spectrum of the angular momentum operator,

\[ J_{\pm} = \pm \epsilon_{ij} x_i p_j = \pm H \]  

where \( H \) is the Hamiltonian of the usual harmonic oscillator. In other words the two Lagrangeans manifest the dual aspects of rotational symmetry in the two-dimensional internal space. Consequently it is possible to solder them by following the general techniques elaborated in [24, 7]. This soldering as well as its implications are the subject of the remainder of this section.

The soldering mechanism, it must be recalled, is intrinsically an operation that has no classical analogue. The crucial point is that the Lagrangeans (20) are now considered as functions of independent variables, namely \( L_+(x) \) and \( L_-(y) \), instead of the same \( x \). A naive addition of the classical Lagrangeans with the same variable is of course possible leading to a trivial result. If, on the other hand, the Lagrangeans are functions of distinct variables, a straightforward addition does not lead to any new information. The soldering process precisely achieves this purpose. Consider the gauging of the Lagrangeans under the following gauge transformations,

\[ \delta x_\alpha = \delta y_\alpha = \dot{\eta}_\alpha \]  

Then the gauge variations are given by,

\[ \delta L_{\pm}(z) = \epsilon_{\alpha\beta} \dot{\eta}_\alpha J^{(\pm)}_\beta(z) \ ; \ z = x, y \]  

where the currents are defined by,

\[ J^{(\pm)}_\alpha(z) = \pm \dot{z}_\alpha + \epsilon_{\alpha\beta} z_\beta \]  

Introducing a new field \( B_\alpha \) transforming as,

\[ \delta B_\alpha = \epsilon_{\beta\alpha} \dot{\eta}_\beta \]
that will effect the soldering, it is possible to construct a first iterated Lagrangean,

\[ L^{(1)}_{\pm} = L_{\pm} - B_{\alpha}J_{\alpha}^{\pm} \]  

(38)

The gauge variation of (38) is easily obtained,

\[ \delta L^{(1)}_{\pm} = -B_{\alpha}\delta J_{\alpha}^{\pm} \]  

(39)

Using the above results we define a second iterated Lagrangean,

\[ L^{(2)}_{\pm} = L^{(1)}_{\pm} - \frac{1}{2}B_{\alpha}^{2} \]  

(40)

which finally leads to a Lagrangean,

\[ L = L^{(2)}_{+}(x) + L^{(2)}_{-}(y) = L_{+}(x) + L_{-}(y) - B_{\alpha}(J_{\alpha}^{+}(x) + J_{\alpha}^{-}(y)) - B_{\alpha}^{2} \]  

(41)

that is invariant under the complete set of transformations (34) and (37), i.e.;

\[ \delta L = 0 \]  

(42)

It is now possible to eliminate the auxiliary \( B_{\alpha} \) field by using the equation of motion, which yields,

\[ B_{\alpha} = -\frac{1}{2}(J_{\alpha}^{+}(x) + J_{\alpha}^{-}(y)) \]  

(43)

Inserting this solution back into (41), we obtain the final soldered Lagrangean,

\[ L(w) = \frac{1}{4}(\dot{w}_{\alpha}^{2} - w_{\alpha}^{2}) \]  

(44)

which is no longer a function of \( x \) and \( y \) independently, but only on their gauge invariant combination,

\[ w_{\alpha} = (x_{\alpha} - y_{\alpha}) \]  

(45)

The soldered Lagrangean just corresponds to a simple bi-dimensional oscillator. Thus, by starting from two Lagrangeans which contained the opposite aspects of a duality symmetry, it is feasible to combine them into a single Lagrangean which has a richer symmetry. A similar phenomenon also exists in the field theoretical examples, as shall be shown subsequently.

Let us now expose all the symmetries of the above Lagrangean. It is most economically done by recasting this Lagrangean in two equivalent forms,

\[ L = \Omega_{\alpha}^{+}\Omega_{\alpha}^{-} = \bar{\Omega}_{\alpha}^{+}\bar{\Omega}_{\alpha}^{-} \]  

(46)

where,

\[ \Omega_{\alpha}^{\pm} = \frac{1}{2}(\dot{w}_{\alpha} \pm \Lambda_{\alpha\beta}w_{\beta}) \]  

\[ \bar{\Omega}_{\alpha}^{\pm} = \frac{1}{2}(\Lambda_{\alpha\beta}\dot{w}_{\beta} \pm w_{\alpha}) \]  

\[ \Lambda_{\alpha\beta} = (R_{\alpha\beta}^{+}, R_{\alpha\beta}^{-}) \]  

(47)
Now the Lagrangean (46) is manifestly symmetric under the following continuous dual transformations,

$$ w_\alpha \rightarrow R^\pm_{\alpha\beta} w_\beta $$

(48)

The transformation involving $R^+$ is just the original symmetry (23). Those involving the $R^-$ matrices are the new symmetries. Recall that the latter transformations swapped the two independent Lagrangeans $L_\pm$. The soldered Lagrangean contains both combinations and hence manifests both these symmetries. The corresponding symmetry group is therefore $O(2)$. This is a completely new phenomenon. It also occurs in field theory with certain additional subtle features.

The generator of the infinitesimal transformations that leads to the $SO(2)$ rotation in (48) is given by,

$$ Q = w_\alpha \epsilon_{\alpha\beta} \pi_\beta $$

(49)

so that,

$$ w_\alpha \rightarrow w'_\alpha = e^{-i\theta Q} w_\alpha e^{i\theta Q} = R^\alpha_{\alpha\beta}(\theta) w_\beta $$

(50)

which is verified by using the canonical brackets,

$$ \{w_\alpha, \pi_\beta\} = \delta_{\alpha\beta} $$

(51)

It is worthwhile to point out the quantum nature of the above calculation by rewriting (44), after an appropriate scaling of variables, in the form of an identity,

$$ L(x - y) = L(x) + L(y) - 2x_\alpha^+ y^-_\alpha - \alpha \zeta_\alpha^+ \pm \alpha \epsilon_{\alpha\beta} \zeta_\beta $$

(52)

This shows that the Lagrangean of the simple harmonic oscillator expressed in terms of the “gauge invariant” variables $w = x - y$ is not obtained by just adding the independent contributions. Rather, there is a contact term which manifests the quantum effect. Indeed, the above identity can be interpreted as the analogue of the well known Polyakov-Weigman [25] identity in two dimensional field theory. As our analysis shows, such identities will always occur whenever dual aspects of some symmetry are being soldered or fused to yield a composite picture, irrespective of the dimensionality of space-time [24]. In the Polyakov-Weigman case it was the chiral symmetry whereas here it was the rotational symmetry.

### 2.2 The Scalar Theory in 1+1 Dimensions

The ideas developed in the previous section are now implemented and elaborated in 1+1 dimensions. It is simple to realise that the scalar theory is a very natural example. For instance, in these dimensions, there is no photon and the Maxwell theory trivialises so that the electromagnetic field can be identified with a scalar field. Thus all the results presented here can be regarded as equally valid for the “photon” field. Indeed the computations will also be presented in a very suggestive way.
notation which illuminates the Maxwellian nature of the problem. Consequently the present analysis is an excellent footboard for diving into the actual electromagnetic duality discussed in the next section. The effects of gravity are easily included in our approach as shown in a separate subsection.

The Lagrangean for the free massless scalar field is given by,

\[ \mathcal{L} = \frac{1}{2} \left( \partial_{\mu} \phi \right)^2 \]  

(53)

and the equation of motion reads,

\[ \ddot{\phi} - \phi'' = 0 \]  

(54)

where the dot and the prime denote derivatives with respect to time and space components, respectively. Introduce, as before, a change of variables using electromagnetic symbols,

\[ E = \dot{\phi} ; \quad B = \phi' \]  

(55)

Obviously, \( E \) and \( B \) are not independent but constrained by the identity,

\[ E' - \dot{B} = 0 \]  

(56)

In these variables the equation of motion and the Lagrangean are expressed as,

\[ \dot{E} - B' = 0 \]

\[ \mathcal{L} = \frac{1}{2} \left( E^2 - B^2 \right) \]  

(57)

It is now easy to observe that the transformations,

\[ E \to \pm B ; \quad B \to \pm E \]  

(58)

display a duality between the equation of motion and the ‘Bianchi’-like identity (56) but the Lagrangean changes its signature. Note that there is a relative change in the signatures of the duality transformations (56) and (58), arising basically from dimensional considerations. This symmetry corresponds to the improper group of rotations.

To illuminate the close connection with the Maxwell formulation, we introduce covariant and contravariant vectors with a Minkowskian metric \( g_{00} = -g_{11} = 1 \),

\[ F_{\mu} = \partial_{\mu} \phi ; \quad F^\mu = \partial^\mu \phi \]  

(59)

whose components are just the ‘electric’ and ‘magnetic’ fields defined earlier,

\[ F_{\mu} = \left( E, B \right) ; \quad F^\mu = \left( E, -B \right) \]  

(60)

Likewise, with the convention \( \epsilon_{01} = 1 \), the dual field is defined,

\[ {}^*F_{\mu} = \epsilon_{\mu\nu} \partial^\nu \phi = \epsilon_{\mu\nu} F^\nu \]

\[ = \left( -B, -E \right) \]  

(61)
The equations of motion and the ‘Bianchi’ identity are now expressed by typical electrodynamical relations,

\[ \partial_\mu F^\mu = 0 \]
\[ \partial_\mu *F^\mu = 0 \] (62)

To expose a Lagrangean duality symmetry, the basic principle of our approach to convert the original second order form (57) to its first order version and then invoke a relabelling of variables to provide an internal index, is adopted. This is easily achieved by first introducing an auxiliary field,

\[ \mathcal{L} = PE - \frac{1}{2}P^2 - \frac{1}{2}B^2 \] (63)

where \( E \) and \( B \) have already been defined. The following renaming of variables corresponding to the proper and improper transformations (see for instance (18) and (19) or (21)) is used,

\[ \phi \rightarrow \phi_1 \]
\[ P \rightarrow \pm \phi'_2 \] (64)

where we are just considering the discrete sets (21) of the full symmetry (18) and (19). Then it is possible to recast (63) in the form,

\[ \mathcal{L} \rightarrow \mathcal{L}_{\pm} = \frac{1}{2} \left[ \pm \phi'_\alpha \sigma_1^{\alpha\beta} \phi_\beta - \phi'^2_\alpha \right] = \frac{1}{2} \left[ \pm B_\alpha \sigma_1^{\alpha\beta} E_\beta - B^2_\alpha \right] \] (65)

In the second line the Lagrangean is expressed in terms of the electromagnetic variables. This Lagrangean is duality symmetric under the transformations of the basic scalar fields,

\[ \phi_\alpha \rightarrow \sigma_1^{\alpha\beta} \phi_\beta \] (66)

which, in the notation of \( E \) and \( B \), is given by,

\[ B_\alpha \rightarrow \sigma_1^{\alpha\beta} B_\beta \]
\[ E_\alpha \rightarrow \sigma_1^{\alpha\beta} E_\beta \] (67)

It is quite interesting to observe that, contrary to the harmonic oscillator example or the electromagnetic theory discussed in the next section, the transformation matrix in the \( O(2) \) space is not the epsilon, but rather a Pauli matrix. This result is in agreement with that found from general algebraic arguments [16, 19] which stated that for \( d = 4k + 2 \) dimensions there is a discrete \( \sigma_1 \) symmetry. Observe that (67) is a manifestation of the original duality (58) which was also effected by the same operation. It is important to stress that the above symmetry is only implementable at the discrete level. Moreover, since it is not connected to the identity, there is no generator for this transformation.
To complete the picture, we also mention that the following rotation,

\[ \phi_{\alpha} \rightarrow \epsilon_{\alpha \beta} \phi_{\beta} \] (68)

interchanges the Lagrangeans (65),

\[ L_+ \leftrightarrow L_- \] (69)

Thus, except for a rearrangement of the matrices generating the various transformations, most features of the simple harmonic oscillator example are perfectly retained. The crucial point of departure is that now all these transformations are only discrete. Interestingly, the master action constructed below lifts these symmetries from the discrete to the continuous.

Let us therefore solder the two distinct Lagrangeans to manifestly display the complete symmetries. Before doing this it is instructive to unravel the self and anti-self dual aspects of these Lagrangeans, which are essential to physically understand the soldering process. The equations of motion following from (65), in the language of the basic fields, is given by,

\[ \partial_{\mu} \phi_{\alpha} = \mp \sigma_{1}^{\alpha \beta} \epsilon_{\mu \nu} \partial_{\nu} \phi_{\beta} \] (70)

provided reasonable boundary conditions are assumed. Note that although the duality symmetric Lagrangean is not manifestly Lorentz covariant, the equations of motion possess this property. We will return to this aspect again in the Maxwell theory. In terms of a vector field \( F_{\mu}^{\alpha} \) and its dual \( \star F_{\mu}^{\alpha} \) defined analogously to (60), (61), the equation of motion is rewritten as,

\[ F_{\mu}^{\alpha} = \pm \sigma_{1}^{\alpha \beta} \star F_{\mu}^{\beta} = \pm \tilde{F}_{\mu}^{\alpha} \] (71)

where the generalised Hodge dual \( (\tilde{F}) \) has been defined in (8). This explicitly reveals the self and anti-self dual nature of the solutions in the combined internal and coordinate spaces. The result can be extended to any \( D = 4k + 2 \) dimensions with suitable insertion of indices.

We now solder the two Lagrangeans. This is best done by using the notation of the basic fields of the scalar theory. These Lagrangeans \( L_+ \) and \( L_- \) are regarded as functions of the independent scalar fields \( \phi_{\alpha} \) and \( \rho_{\alpha} \). Consider the gauging of the following symmetry,

\[ \delta \phi_{\alpha} = \delta \rho_{\alpha} = \eta_{\alpha} \] (72)

Following exactly the steps performed for the harmonic oscillator example the final Lagrangean analogous to (11) is obtained,

\[ \mathcal{L} = \mathcal{L}_+(\phi) + \mathcal{L}_-(\rho) - B_\alpha \left( J_\alpha^+(\phi) + J_\alpha^-(\rho) \right) - B_2^2 \] (73)

where the currents are given by,

\[ J_\alpha^+(\theta) = \pm \sigma_{1}^{\alpha \beta} \dot{\theta}_{\beta} - \theta'_{\alpha} \ ; \ \theta = \phi \ , \ \rho \] (74)

The above Lagrangean is gauge invariant under the extended transformations including (72) and,

\[ \delta B_\alpha = \eta'_{\alpha} \] (75)
Eliminating the auxiliary $B_\alpha$ field using the equations of motion, the final soldered Lagrangean is obtained from (73),

\[ \mathcal{L}(\Phi) = \frac{1}{4} \partial_\mu \Phi_\alpha \partial^\mu \Phi_\alpha \]  

(76)

where, expectedly, this is now only a function of the gauge invariant variable,

\[ \Phi_\alpha = \phi_\alpha - \rho_\alpha \]  

(77)

This master Lagrangean possesses all the symmetries that are expressed by the continuous transformations,

\[ \Phi_\alpha \rightarrow R_{\alpha\beta}^\pm(\theta) \Phi_\beta \]  

(78)

The generator corresponding to the $SO(2)$ transformations is easily obtained,

\[ Q = \int dy \Phi_\alpha \epsilon_{\alpha\beta} \Pi_\beta \Phi_\alpha \rightarrow \Phi'_\alpha = e^{-i\theta Q} \Phi_\alpha e^{i\theta Q} \]  

(79)

where $\Pi_\alpha$ is the momentum conjugate to $\Phi_\alpha$. Observe that either the original symmetry in $\sigma_1$ or the swapping transformations were only at the discrete level. The process of soldering has lifted these transformations from the discrete to the continuous form. It is equally important to reemphasize that the master action now possesses the $SO(2)$ symmetry which is more commonly associated with four dimensional duality symmetric actions, and not for two dimensional theories. Note that by using the electromagnetic symbols, the Lagrangean can be displayed in a form which manifests the soldering effect of the self and anti self dual symmetries (71),

\[ \mathcal{L} = \frac{1}{8} \left( F_\mu^\alpha + \tilde{F}_\mu^\alpha \right) \left( F^\nu_\alpha - \tilde{F}^\nu_\alpha \right) \]  

(80)

where the generalised Hodge dual in $D = 4k + 2$ dimensions has been defined in (8).

An interesting observation is now made. Recall that the original duality transformation (58) switching equations of motion into Bianchi identities may be rephrased in the internal space by,

\[ E_\alpha \rightarrow \mp R^\pm_{\alpha\beta} B_\beta \]
\[ B_\alpha \rightarrow \mp R^\pm_{\alpha\beta} E_\beta \]  

(81)

which is further written directly in terms of the scalar fields,

\[ \partial_\mu \Phi_\alpha \rightarrow \pm R^\pm_{\alpha\beta} \epsilon_{\mu\nu} \partial^\nu \Phi_\beta \]  

(82)

It is simple to verify that under these transformations even the Hamiltonian for the theories described by the Lagrangeans $\mathcal{L}_\pm$ (73) are not invariant. However the Hamiltonian following from the master Lagrangean (76) preserves this symmetry. The Lagrangean itself changes its signature. This is the exact analogue of the original situation. A similar phenomenon also occurs in the electromagnetic theory. This completes the discussion on the symmetries of the master Lagrangean.
It is now straightforward to give a Polyakov-Weigman type identity, that relates the “gauge invariant” Lagrangean with the non gauge invariant structures, by reformulating (76) after a scaling of the fields $(\phi, \rho) \rightarrow \sqrt{2}(\phi, \rho)$,

\[ L(\Phi) = L(\phi) + L(\rho) - 2\partial_+ \phi_\alpha \partial_- \rho_\alpha \]  

where the light cone variables are given by,

\[ \partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1) \]  

Observe that, as in the harmonic oscillator example, the gauge invariance is with regard to the transformations introduced for the soldering of the symmetries. Thus, even if the theory does not have a gauge symmetry in the usual sense, the dual symmetries of the theory can simulate the effects of the former. This leads to a Polyakov-Wiegman type identity which has an identical structure to the conventional identity.

Before closing this sub-section, it may be useful to highlight some other aspects of duality which are peculiar to two dimensions, as for instance, the chiral symmetry. The interpretation of this symmetry with regard to duality seems, at least to us, to be a source of some confusion and controversy. As is well known a scalar field in two dimensions can be decomposed into two chiral pieces, described by Floreanini Jackiw (FJ) actions [23]. These actions are sometimes regarded [21] as the two dimensional analogues of the duality symmetric four dimensional electromagnetic actions [16]. Such an interpretation is debatable since the latter have the $SO(2)$ symmetry (characterised by an internal index $\alpha$) which is obviously lacking in the FJ actions. Our analysis, on the other hand, has shown how to incorporate this symmetry in the two dimensional case. Hence we consider the actions defined by (65) to be the true analogue of the duality symmetric electromagnetic actions to be discussed later. Moreover, by solving the equations of motion of the FJ action, it is not possible to recover the second order free scalar Lagrangean, quite in contrast to the electromagnetic theory [16]. Nevertheless, since the FJ actions are just the chiral components of the usual scalar action, these must be soldered to reproduce this result. But if soldering is possible, such actions must also display the self and anti-self dual aspects of chiral symmetry. This phenomenon is now explored along with the soldering process.

The two FJ actions defined in terms of the independent scalar fields $\phi_+$ and $\phi_-$ are given by,

\[ L_{\pm}^{FJ}(\phi_\pm) = \pm \dot{\phi}_\pm \phi'_\pm - \phi'_\pm \phi'_\pm \]  

whose equations of motion show the self and anti self dual aspects,

\[ \partial_\mu \phi_\pm = \mp \epsilon_{\mu\nu} \partial^n \phi_\pm \]  

A trivial application of the soldering mechanism leads to,

\[ L(\Phi) = L_+^{FJ}(\phi_+) + L_-^{FJ}(\phi_-) + \frac{1}{8} \left( J_+(\phi_+) + J_-(\phi_-) \right)^2 \]

\[ = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi \]  

(87)
where the currents $J_\pm$ and the composite field $\Phi$ are given by,
\[
J_\pm = 2\left( \pm \dot{\phi}_\pm - \phi'_\pm \right)
\]
\[
\Phi = \phi_+ - \phi_-
\]

\[ (88) \]

Thus the usual scalar action is obtained in terms of the composite field. The previous analysis has, however, shown that each of the Lagrangeans (65) are equivalent to the usual scalar theory. Hence these Lagrangeans contain both chiralities described by the FJ actions (85). However, in the internal space, $\mathcal{L}_\pm$ carry the self and anti self dual solutions, respectively. This clearly illuminates the ubiquitous role of chirality versus duality in the two dimensional theories which has been missed in the literature simply because, following conventional analysis in four dimensions [15, 16], only one particular duality symmetric Lagrangean $\mathcal{L}_-$ was imagined to exist.

### 2.3 Coupling to gravity

It is easy to extend the analysis to include gravity. This is most economically done by using the language of electrodynamics already introduced. The Lagrangean for the scalar field coupled to gravity is given by,
\[
\mathcal{L} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} F_\mu F_\nu
\]

\[ (89) \]

where $F_\mu$ is defined in (60) and $g = \det g_{\mu\nu}$. Converting the Lagrangean to its first order form, we obtain,
\[
\mathcal{L} = PE - \frac{1}{2\sqrt{-g} g^{00}} \left( P^2 + B^2 \right) + \frac{g^{01}}{g^{00}} PB
\]

\[ (90) \]

where the $E$ and $B$ fields are defined in (55) and $P$ is an auxiliary field. Let us next invoke a change of variables mapping $(E, B) \to (E_1, B_1)$ by means of the $O(2)$ transformation analogous to (64), and relabel the variable $P$ by $\pm B_2$. Then the Lagrangean (90) assumes the distinct forms,
\[
\mathcal{L}_\pm = \frac{1}{2} \left[ \pm B_\alpha \sigma_1^{\alpha\beta} E_\beta - \frac{1}{\sqrt{-g} g^{00}} B_\alpha \left( B_\alpha + \frac{g^{01}}{g^{00}} \sigma_1^{\alpha\beta} B_\alpha B_\beta \right) \right]
\]

\[ (91) \]

which are duality symmetric under the transformations (67). As in the flat metric, there is a swapping between $\mathcal{L}_+$ and $\mathcal{L}_-$ if the transformation matrix is $\epsilon_{\alpha\beta}$. To obtain a duality symmetric action for all transformations it is necessary to construct the master action obtained by soldering the two independent pieces. The dual aspects of the symmetry that will be soldered are revealed by looking at the equations of motion following from (91),
\[
\sqrt{-g} F_\mu^\alpha = \mp g_{\mu\nu} \sigma_1^{\alpha\beta} F_\nu^\beta
\]

\[ (92) \]

The result of the soldering process, following from our standard techniques, leads to the master Lagrangean,
\[
\mathcal{L} = \frac{1}{4} \sqrt{-g} g^{\mu\nu} F_\mu^\alpha F_\nu^\alpha
\]

\[ (93) \]
where $F^\alpha_\mu$ is defined in terms of the composite field given in (77). In the flat space this just reduces to the expression found previously in (76). It may be pointed out that, originating from this master action it is possible, by passing to a first order form, to recover the original pieces.

To conclude, we show how the FJ action now follows trivially by taking any one particular form of the two Lagrangians, say $L_+$. To make contact with the conventional expressions quoted in the literature [26], it is useful to revert to the scalar field notation, so that,

$$L_+ = \frac{1}{2}\left[\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2g^{01}g^{00}\phi_1^2 - \frac{1}{g^{00}\sqrt{-g}}\phi_1^2\phi_2^2\right]$$

This is diagonalised by the following choice of variables,

$$\phi_1 = \phi_+ + \phi_- \\
\phi_2 = \phi_+ - \phi_-$$

leading to,

$$L_+ = L_+^{(+)}(\phi_+, \mathcal{G}_+) + L_+^{(-)}(\phi_-, \mathcal{G}_-)$$

with,

$$L_+^{(\pm)}(\phi_\pm, \mathcal{G}_\pm) = \pm\dot{\phi}_\pm^2 + \mathcal{G}_\pm\phi_\pm^2$$

and

$$\mathcal{G}_\pm = \frac{1}{g^{00}\sqrt{-g}}\left(-1\pm g^{01}\right)$$

These are the usual FJ actions in curved space as given in [29]. Such a structure was suggested by gauging the conformal symmetry of the free scalar field and then confirmed by checking the classical invariance under gauge and affine transformations [26]. Here we have derived this result directly from the action of the scalar field minimally coupled to gravity.

Observe that the explicit diagonalisation carried out in (94) for two dimensions is actually a specific feature of $4k + 2$ dimensions. This is related to the basic identity (7) governing the dual operation. If, however, one works with the master (soldered) Lagrangean, then diagonalisation is possible in either $D = 4k + 2$ or $D = 4k$ dimensions since the corresponding identity (3) always has the correct signature.

2.4 The Electromagnetic Duality

Exploiting the ideas elaborated in the previous sections, it is straightforward to implement duality in the electromagnetic theory. Let us start with the usual Maxwell Lagrangean,

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

which is expressed in terms of the electric and magnetic fields as

$$L = \frac{1}{2}(E^2 - B^2)$$

[4Bold face letters denote three vectors.]
where,
\[
E_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0 \\
B_i = \epsilon_{ijk} \partial_j A_k
\] (100)

The following duality transformation,
\[
E \rightarrow \mp B ; \quad B \rightarrow \pm E
\] (101)
is known to preserve the invariance of the full set comprising Maxwell’s equations and the Bianchi identities although the Lagrangean changes its signature. To have a duality symmetric Lagrangean, we now know how to proceed in a systematic manner. The Maxwell Lagrangean is therefore recast in a symmetrised first order form,
\[
\mathcal{L} = \frac{1}{2} \left( \mathbf{P} \dot{\mathbf{A}} - \dot{\mathbf{P}} \cdot \mathbf{A} \right) - \frac{1}{2} \mathbf{P}^2 - \frac{1}{2} \mathbf{B}^2 + A_0 \nabla \cdot \mathbf{P}
\] (102)

Exactly as was done for the harmonic oscillator, a change of variable \( s \) is invoked. Once again there are two possibilities which translate from the old set \( (\mathbf{P}, \mathbf{A}) \) to the new ones \( (\mathbf{A}_1, \mathbf{A}_2) \). It is, however, important to recall that the Maxwell theory has a constraint that is implemented by the Lagrange multiplier \( A_0 \). The redefined variables are chosen which solve this constraint so that,
\[
\begin{align*}
\mathbf{P} & \rightarrow \mathbf{B}_2 ; \quad \mathbf{A} \rightarrow \mathbf{A}_1 \\
\mathbf{P} & \rightarrow \mathbf{B}_1 ; \quad \mathbf{A} \rightarrow \mathbf{A}_2
\end{align*}
\] (103)

It is now simple to show that, in terms of the redefined variables, the original Maxwell Lagrangean takes the form,
\[
\mathcal{L}_{\pm} = \frac{1}{2} \left( \pm \dot{\mathbf{A}}_\alpha \epsilon_{\alpha\beta} \mathbf{B}_\beta - \dot{\mathbf{B}}_\alpha \cdot \mathbf{B}_\alpha \right)
\] (104)

Adding a total derivative that would leave the equations of motion unchanged, this Lagrangean is expressed directly in terms of the electric and magnetic fields,
\[
\mathcal{L}_{\pm} = \frac{1}{2} \left( \pm \mathbf{B}_\alpha \epsilon_{\alpha\beta} \mathbf{E}_\beta - \mathbf{B}_\alpha \cdot \mathbf{B}_\alpha \right)
\] (105)

It is duality symmetric under the full \( SO(2) \) transformations mentioned in an earlier context. Note that one of the above structures (namely, \( \mathcal{L}_{-} \)) was given earlier in [16]. Once again, in analogy with the harmonic oscillator example, it is observed that the transformation (28) involving the \( R^- \) matrices switches the Lagrangeans \( \mathcal{L}_+ \) and \( \mathcal{L}_- \) into one another. The generators of the \( SO(2) \) rotations are given by,
\[
Q^{(\pm)} = \mp \frac{1}{2} \int d^3 x \ A^\alpha \cdot \mathbf{B}^\alpha
\] (106)

so that,
\[
A^\alpha \rightarrow A'^\alpha = e^{-iQ^\theta} A^\alpha e^{iQ^\theta}
\] (107)

This can be easily verified by using the basic brackets following from the symplectic structure of the theory,
\[
\left[ A^i_\alpha (x), \epsilon^{jkl} \partial^k A^l_\beta (y) \right] = \pm i \delta^{ij} \epsilon_{\alpha\beta} \delta (x - y)
\] (108)
It is useful to digress on the significance of the above analysis. Since the duality symmetric Lagrangeans have been obtained directly from the Maxwell Lagrangean, it is redundant to show the equivalence of the former expressions with the latter, which is an essential perquisite in other approaches. Furthermore, since classical equations of motion have not been used at any stage, the purported equivalence holds at the quantum level. The need for any explicit demonstration of this fact, which has been the motivation of several recent papers, becomes, in this analysis, superfluous.

A related observation is that the usual way of showing the classical equivalence is to use the equations of motion to eliminate one component from (104), thereby leading to the Maxwell Lagrangean in the temporal $A_0 = 0$ gauge. This is not surprising since the change of variables leading from the second to the first order form solved the Gauss law thereby eliminating the multiplier. Finally, note that there are two distinct structures for the duality symmetric Lagrangeans. These must correspond to the opposite aspects of some symmetry, which is next unravelled. By looking at the equations of motion obtained from (104),

\[ \dot{A}_\alpha = \pm \epsilon_{\alpha\beta} \nabla \times A_\beta \]  

(109)

it is possible to verify that these are just the self and anti-self dual solutions,

\[ F^\alpha_{\mu\nu} = \pm \epsilon^{\alpha\beta} F_{\mu\nu}^\beta ; \quad *F^\beta_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} F^\rho_\beta \]  

(110)

obtained by setting $A_0^\alpha = 0$. Recall that in the two dimensional motion the equation of motion naturally assumed a covariant structure. Here, on the other hand, the introduction of $A_0^\alpha$ is necessary since this term gives a vanishing contribution to the Lagrangean. This feature distinguishes a gauge theory from the non gauge theory discussed in the two dimensional example. It may be observed that the opposite aspects of the dual symmetry are contained in the internal space. Following our quantum mechanical analogy, the next task is to solder the two Lagrangeans (104). Consider then the gauging of the following symmetry,

\[ \delta H_\alpha = h_\alpha ; \quad H = P, Q \]  

(111)

where $P$ and $Q$ denote the basic fields in the Lagrangeans $L_+$ and $L_-$, respectively. The Lagrangeans transform as,

\[ \delta L_\pm = \epsilon_{\alpha\beta} (\nabla \times h_\alpha) J^\pm_\beta \]  

(112)

with the currents defined by,

\[ J^\pm_\alpha(H) = \left( \mp \dot{H}_\alpha + \epsilon_{\alpha\beta} \nabla \times H_\beta \right) \]  

(113)

Next, the soldering field $W_\alpha$ is introduced which transforms as,

\[ \delta W_\alpha = - \epsilon_{\alpha\beta} \nabla \times h_\beta \]  

(114)

Following standard steps as outlined previously, the final Lagrangean which is invariant under the complete set of transformations (111) and (114) is obtained,

\[ \mathcal{L} = L_+(P) + L_-(Q) - W^\alpha (J^+_\alpha(P) + J^-_\alpha(Q)) - W^2_\alpha \]  

(115)
Eliminating the soldering field by using the equations of motion, the effective soldered Lagrangean following from (115) is derived,

\[ \mathcal{L} = \frac{1}{4} \left( \dot{G}_\alpha \cdot \dot{G}_\alpha - \nabla \times G_\alpha \cdot \nabla \times G_\alpha \right) \] (116)

where the composite field is given by the combination,

\[ G_\alpha = P_\alpha - Q_\alpha \] (117)

which is invariant under (111). It is interesting to note that, reinstating the \( G_\alpha^0 \) variable, this is nothing but the Maxwell Lagrangean with a doublet of fields,

\[ \mathcal{L} = -\frac{1}{4} G_\alpha^{\mu \nu} G_\alpha^{\mu \nu} ; \quad G_\alpha^{\mu \nu} = \partial_\mu G_\alpha^{\nu} - \partial_\nu G_\alpha^{\mu} \] (118)

In terms of the original \( P \) and \( Q \) fields it is once again possible, like the harmonic oscillator example, to write a Polyakov-Weigman like identity,

\[ \mathcal{L}(P - Q) = \mathcal{L}(P) + \mathcal{L}(Q) - 2W^\pm_{i,\alpha}(P)W^-_{i,\alpha}(Q) \]

\[ W^\pm_{i,\alpha}(H) = \frac{1}{\sqrt{2}} \left( F_{0i}^{\alpha}(H) \pm \epsilon_{ijk} \epsilon_{\alpha \beta} F_{jk}^{\beta}(H) \right) ; \quad H = P, Q \] (119)

With respect to the gauge transformations (111), the above identity shows that a contact term is necessary to restore the gauge invariant action from two gauge variant forms. This, it may be recalled, is just the basic content of the Polyakov-Weigman identity. It is interesting to note that the “mass” term appearing in the above identity is composed of parity preserving pieces \( W^\pm_{i,\alpha} \), thanks to the presence of the compensating \( \epsilon \)-factor from the internal space.

Following the oscillator example, it is now possible to show that by reducing (118) to a first order form, we exactly obtain the two types of the duality symmetric Lagrangeans (105). This shows the equivalence of the soldering and reduction processes.

A particularly illuminating way of rewriting the Lagrangean (118) is,

\[ \mathcal{L} = -\frac{1}{8} \left( G_\alpha^{\mu \nu} + \epsilon^{\alpha \beta \gamma} G_\beta^{\mu \nu} \right) \left( G_\alpha^{\mu \nu} - \epsilon^{\alpha \beta \gamma} G_\beta^{\mu \nu} \right) \]

\[ = -\frac{1}{8} \left( G_\mu^{\alpha \nu} + \tilde{G}_\mu^{\alpha \nu} \right) \left( G_\alpha^{\mu \nu} - \tilde{G}_\alpha^{\mu \nu} \right) \] (120)

where, in the second line, the generalised Hodge dual in the space containing the internal index has been used to explicitly show the soldering of the self and anti self dual solutions. A similar situation prevailed in the two dimensional analysis. The above Lagrangean manifestly displays the following duality symmetries,

\[ A^\alpha_\mu \rightarrow R^\pm_{\alpha \beta} A^\beta_\mu \] (121)

where, without any loss of generality, we may denote the composite field, of which \( G_{\mu \nu} \) is a function, by \( A \). The generator of the \( SO(2) \) rotations is now given by,

\[ Q = \int d\mathbf{x} \epsilon^{\alpha \beta} \Pi^\alpha \cdot \mathbf{A}^\beta \] (122)
Now observe that the master Lagrangean was obtained from the soldering of two distinct Lagrangeans \((\text{104})\). The latter were duality symmetric under both \(A_\alpha \to \pm \epsilon_{\alpha\beta} A_\beta\), while the transformations involving the \(\sigma_1\) matrix interchanged \(\mathcal{L}_+\) with \(\mathcal{L}_-\). The soldered Lagrangean is therefore duality symmetric under the transformations \((\text{121})\). Furthermore, the discrete transformation related to the \(\sigma_1\) matrix has been lifted to its continuous form \(R^-\). The master Lagrangean, therefore, contains a bigger set of duality symmetries than \((\text{104})\) and, significantly, is also manifestly Lorentz invariant. Furthermore, recall that under the transformations mapping the field to its dual, the original Maxwell equations are invariant but the Lagrangean changes its signature. The corresponding transformation in the \(SO(2)\) space is given by,

\[
G^\alpha_{\mu
u} \to R^\pm_{\alpha\beta} \ast G^\beta_{\mu
u}
\]

which, written in component notation, looks like,

\[
E^\alpha \to \mp \epsilon^{\alpha\beta} B^\beta ; \quad B^\alpha \to \pm \epsilon^{\alpha\beta} E^\beta
\]

The standard duality symmetric Lagrangean fails to manifest this property. However, as may be easily checked, the equations of motion obtained from the master Lagrangean swap with the corresponding Bianchi identity while the Lagrangean flips sign. In this manner the original property of the second order Maxwell Lagrangean is retrieved. Note furthermore that the master Lagrangean possesses the \(\sigma_1\) symmetry (which is just the discretised version of \(R^-\)), a feature expected for two dimensional theories. A similar phenomenon occurred in the previous section where the master action in two dimensions revealed the \(SO(2)\) symmetry usually associated with four dimensional theories.

### 2.5 Coupling to gravity

To discuss how the effects of gravity are included, we will proceed as in the two dimensional example. The starting point is the Maxwell Lagrangean coupled to gravity,

\[
\mathcal{L} = -\frac{1}{4} \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}
\]

From our experience in the usual Maxwell theory we know that an eventual change of variables eliminates the Gauss law so that the term involving the multiplier \(A_0\) may be ignored from the outset. Expressing \((\text{125})\) in terms of its components to separate explicitly the first and second order terms, we find,

\[
\mathcal{L} = \frac{1}{2} \dot{A}_i M^{ij} \dot{A}_j + M^i \dot{A}_i + M
\]

where,

\[
M^{ij} = \sqrt{-g} \left( g^{0i} g^{0j} - g^{ij} g^{00} \right)
\]

\[
M^i = \sqrt{-g} g^{0k} g^{ji} F_{jk}
\]

\[
M = \frac{1}{4} \sqrt{-g} g^{ij} g^{km} F_{im} F_{kj}
\]
Now reducing the Lagrangean to its first order form, we obtain,

$$L = P^i E_i - \frac{1}{2} P^i M_{ij} P^j - \frac{1}{2} M^i M_{ij} M^j + P^i M_{ij} M^j + M$$  

(128)

where $\dot{A}_i$ has been replaced by $E_i$ and $M_{ij}$ is the inverse of $M^{ij}$,

$$M_{ij} = \frac{-1}{\sqrt{-gg_{00}}}g_{ij}$$  

(129)

with,

$$g_m^{\mu \nu} g_{\nu \lambda} = \delta^\mu_\lambda$$  

(130)

Next, introducing the standard change of variables which solves the Gauss constraint,

$$E_i  \rightarrow E_i^{(1)}$$

$$P^i \rightarrow \pm B^{i(2)}$$  

(131)

the Lagrangean (128) is expressed in the desired form,

$$L_{\pm} = \pm E^\alpha \epsilon^{\alpha \beta} B^i_\beta + \frac{1}{\sqrt{-gg_{00}}} g_{ij} B^i_\alpha B^j_\beta$$

$$\pm g^{0k}_{\mu0} \epsilon_{ijk} \epsilon^{\alpha \beta} B^i_\alpha B^j_\beta$$  

(132)

Once again there are two duality symmetric actions corresponding to $L_{\pm}$. The enriched nature of the duality and swapping symmetries under a bigger set of transformations, the constructing of a master Lagrangean from soldering of $L_+$ and $L_-$, the corresponding interpretations, all go through exactly as in the flat metric case. Incidentally, the structure for $L_-$ only was previously given in [16].

### 2.6 Final Discussions

The ideas developed in these sections revealed a unifying structure behind the construction of the various duality symmetric actions. The essential ingredient was the conversion of the second order action into a first order form followed by an appropriate redefinition of variables such that these may be denoted by an internal index. The duality naturally occurred in this internal space. Since the duality symmetric actions were directly derived from the original action the proof of their equivalence becomes superfluous. This is otherwise essential where such a derivation is lacking and recourse is taken to either equations of motion or some hamiltonian analysis. Obviously the most simple and fundamental manifestation of the duality property was in the context of the quantum mechanical harmonic oscillator. Since a field is interpreted as a collection of an infinite set of such oscillators, it is indeed expected and not at all surprising that all these concepts and constructions are almost carried over entirely for field theories. It may be remarked that the extension of the harmonic oscillator analysis to field theories has proved useful in other contexts and in this particular case has been really clinching. Furthermore, by invoking a highly suggestive electromagnetic notation for the harmonic oscillator analysis, its close correspondence with the field theory examples was highlighted.
A notable feature of the analysis was the revelation of a whole class of new symmetries and their interrelations. Different aspects of this feature were elaborated. To be precise, it was shown that there are actually two\footnote{Note that usual discussions of duality symmetry consider only one of these actions, namely $\mathcal{L}_-$.} duality symmetric actions ($\mathcal{L}_\pm$) for the same theory. These actions carry the opposite (self and anti self dual) aspects of some symmetry and their occurrence was essentially tied to the fact that there were two distinct classes in which the renaming of variables was possible, depending on the signature of the determinant specifying the proper or improper rotations. To discuss further the implications of this pair of duality symmetric actions it is best to compare with the existing results. This also serves to put the present work in a proper perspective. It should be mentioned that the analysis for two and four dimensions are generic for $4k+2$ and $4k$ dimensions, respectively.

It is usually observed\cite{19} that the invariance of the actions in different $D$-dimensions is preserved by the following groups,

\begin{equation}
\mathcal{G}_d = Z_2 \ ; \ D = 4k + 2 \\
\end{equation}

and,

\begin{equation}
\mathcal{G}_c = SO(2) \ ; \ D = 4k \\
\end{equation}

which are called the “duality groups”. The $Z_2$ group is a discrete group with two elements, the trivial identity and the $\sigma_1$ matrix. Observe an important difference since in one case this group is continuous while in the other it is discrete. In our exercise this was easily verified by the pair of duality symmetric actions $\mathcal{L}_\pm$. The new ingredient is that nontrivial elements of these groups are also responsible for the swapping $\mathcal{L}_+ \leftrightarrow \mathcal{L}_-$, but in the other dimensions. Thus the “duality swapping matrices” $\Sigma_s$ are given by,

$$\Sigma_s = \sigma_1 \ ; \ D = 4k$$
$$= \epsilon \ ; \ D = 4k + 2$$

(135)

It was next shown that $\mathcal{L}_\pm$ contained the self and anti-self dual aspects of some symmetry. Consequently, following the ideas developed in\cite{24, 7}, the two Lagrangeans could be soldered to yield a master Lagrangean $\mathcal{L}_m = \mathcal{L}_+ \oplus \mathcal{L}_-$. The master action, in any dimensions, was manifestly Lorentz or general coordinate invariant and was also duality symmetric under both the groups mentioned above. Moreover the process of soldering lifted the discrete group $Z_2$ to its continuous version. The duality group for the master action in either dimensionality therefore simplified to,

\begin{equation}
\mathcal{G} = O(2) \ ; \ D = 2k + 2 \\
\end{equation}

Thus, at the level of the master action, the fundamental distinction between the odd and even $N$-forms gets obliterated. It ought to be stated that the lack of usual Chern Simons terms in $D = 4k+2$ dimensions to act as the generators of duality transformations is compensated by the presence of a similar term in the internal space. Thanks to this it was possible to explicitly construct the symmetry generators for the master action in either two or four dimensions.

We also showed that the master actions in any dimensions, apart from being duality symmetric under the $O(2)$ group, were factored, modulo a normalisation, as a product of the self and anti self dual solutions,

$$\mathcal{L} = \left( F^\alpha + \tilde{F}^\alpha \right) \left( F^\alpha - \tilde{F}^\alpha \right) \ ; \ D = 2k + 2$$

(137)
where the internal index has been explicitly written and the generalised Hodge operation was defined in (8). The key ingredient in this construction was to provide a general definition of self duality that was applicable for either odd or even $N$ forms. Self duality was now defined to include the internal space and was implemented either by the $\sigma_1$ or the $\epsilon$, depending on the dimensionality. This naturally led to the universal structure (137).

Some other aspects of the analysis deserve attention. Specifically, the novel duality symmetric actions obtained in two dimensions revealed the interpolating role between duality and chirality. Furthermore, certain points concerning the interpretation of chirality symmetric action as the analogue of the duality symmetric electromagnetic action in four dimensions were clarified. We also recall that the soldering of actions to obtain a master action was an intrinsically quantum phenomenon that could be expressed in terms of an identity relating two “gauge variant” actions to a “gauge invariant” form. The gauge invariance is with regard to the set of transformations induced for effecting the soldering and has nothing to do with the conventional gauge transformations. In fact the important thing is that the distinct actions must possess the self and anti self dual aspects of some symmetry which are being soldered. The identities obtained in this way are effectively a generalisation of the usual Polyakov Weigman identity. We conclude by stressing the practical nature of our approach to duality which can be extended to other theories. This will be the object of the next section.

3 Bosonisation and Soldering of Dual Symmetries in Two and Three Dimensions

Bosonisation is a powerful technique that maps a fermionic theory into its bosonic counterpart. It was initially developed and fully explored in the context of two dimensions[27]. More recently, it has been extended to higher dimensions[8, 28, 9, 29]. The importance of bosonisation lies in the fact that it includes quantum effects already at the classical level. Consequently, different aspects and manifestations of quantum phenomena may be investigated directly, that would otherwise be highly nontrivial in the fermionic language. Examples of such applications are the computation of the current algebra[9] and the study of screening or confinement in gauge theories[30].

This section is devoted to analyse certain features and applications of bosonisation which, as far as we are aware, are unexplored even in two dimensions. The question we pose is the following: given two independent fermionic models which can be bosonised separately, under what circumstances is it possible to represent them by one single effective theory? The answer lies in the symmetries of the problem. Two distinct models displaying dual aspects of some symmetry can be combined by the simultaneous implementation of bosonisation and soldering to yield a completely new theory. This is irrespective of dimensional considerations. The technique of soldering essentially comprises in lifting the gauging of a global symmetry to its local version and exploits certain concepts introduced in a different context by Stone[2] and one of us[10]. The analysis is intrinsically quantal without having any classical analogue. This is easily explained by the observation that a simple addition of two independent classical lagrangeans is a trivial operation without leading to anything meaningful or significant.

The basic notions and ideas are first introduced in the context of two dimensions where bosoni-
sation is known to yield exact results. The starting point is to take two distinct chiral lagrangeans with opposite chirality. Using their bosonised expressions, the soldering mechanism fuses, in a precise way, the left and right chiralities. This leads to a general lagrangean in which the chiral symmetry no longer exists, but it contains two extra parameters manifesting the bosonisation ambiguities. It is shown that different parametrisations lead to different models. In particular, the gauge invariant Schwinger model and Thirring model are reproduced. As a byproduct, the importance of Bose symmetry is realised and some interesting consequences regarding the arbitrary parametrisation in the chiral Schwinger model are charted.

Whereas the two dimensional analysis lays the foundations, the subsequent three dimensional discussion illuminates the full power and utility of the present approach. While the bosonisation in these dimensions is not exact, nevertheless, for massive fermionic models in the large mass or, equivalently, the long wavelength limit, well defined local expressions are known to exist[28, 4]. Interestingly, these expressions exhibit a self or an anti self dual symmetry that is dictated by the signature of the fermion mass. Clearly, therefore, this symmetry simulates the dual aspects of the left and right chiral symmetry in the two dimensional example, thereby providing a novel testing ground for our ideas. Indeed, two distinct massive Thirring models with opposite mass signatures, are soldered to yield a massive Maxwell theory. This result is vindicated by a direct comparison of the current correlation functions obtained before and after the soldering process. As another instructive application, the fusion of two models describing quantum electrodynamics in three dimensions is considered. Results similar to the corresponding analysis for the massive Thirring models are obtained.

We conclude by discussing future prospects and possibilities of extending this analysis in different directions.

3.1 The two dimensional example

In this section we develop the ideas in the context of two dimensions. Consider, in particular, the following lagrangeans with opposite chiralities,

\[\mathcal{L}_+ = \bar{\psi}(i\partial \psi + eA \gamma_5)\psi\]

\[\mathcal{L}_- = \bar{\psi}(i\partial \psi + eA \gamma_5)\psi\]

where \(P_{\pm}\) are the projection operators,

\[P_{\pm} = \frac{1 \pm \gamma_5}{2}\]

It is well known that the computation of the fermion determinant, which effectively yields the bosonised expressions, is plagued by regularisation ambiguities since chiral gauge symmetry cannot be preserved[31]. Indeed an explicit one loop calculation following Schwinger’s point splitting method [22] yields,

\[W_+[\varphi] = -i \log \det(i\partial + eA_+) = \frac{1}{4\pi} \int d^2x \left( \partial_+ \varphi \partial_- \varphi + 2e A_+ \partial_- \varphi + a e^2 A_+ A_- \right)\]

\[W_-[\rho] = -i \log \det(i\partial + eA_-) = \frac{1}{4\pi} \int d^2x \left( \partial_+ \rho \partial_- \rho + 2e A_- \partial_+ \rho + b e^2 A_+ A_- \right)\]
where the light cone metric has been invoked for convenience,

\[ A_{\pm} = \frac{1}{\sqrt{2}} (A_0 \pm A_1) = A^\mp \ ; \ \partial_{\pm} = \frac{1}{\sqrt{2}} (\partial_0 \pm \partial_1) = \partial^\mp \]  \hspace{1cm} (141)

Note that the regularisation or bosonisation ambiguity is manifested through the arbitrary parameters \( a \) and \( b \). The latter ambiguity is particularly transparent since by using the normal bosonisation dictionary \( \bar{\psi} \gamma^\mu \psi \rightarrow \frac{1}{\sqrt{\pi}} \epsilon^\mu_{\nu} \partial^\nu \varphi \) (which holds only for a gauge invariant theory), the above expressions with \( a = b = 0 \) are easily reproduced from (138).

It is crucial to observe that different scalar fields \( \varphi \) and \( \rho \) have been used in the bosonised forms to emphasize the fact that the fermionic fields occurring in the chiral components are uncorrelated. It is the soldering process which will abstract a meaningful combination of these components\(^{24}\). This process essentially consists in lifting the gauging of a global symmetry to its local version. Consider, therefore, the gauging of the following global symmetry,

\[ \delta \varphi = \delta \rho = \alpha \]
\[ \delta A_{\pm} = 0 \]  \hspace{1cm} (142)

The variations in the effective actions (140) are found to be,

\[ \delta W_+[\varphi] = \int d^2 x \partial_- J_+ (\varphi) \]
\[ \delta W_-[\rho] = \int d^2 x \partial_+ J_- (\rho) \]  \hspace{1cm} (143)

where the currents are defined as,

\[ J_\pm (\eta) = \frac{1}{2\pi} (\partial_\pm \eta + e A_\pm) \ ; \ \eta = \varphi, \rho \]  \hspace{1cm} (144)

The important step now is to introduce the soldering field \( B_{\pm} \) coupled with the currents so that,

\[ W^{(1)}_\pm[\eta] = W_\pm[\eta] - \int d^2 x B_\mp J_\pm (\eta) \]  \hspace{1cm} (145)

Then it is possible to define a modified action,

\[ W[\varphi, \rho] = W^{(1)}_+ [\varphi] + W^{(1)}_- [\rho] + \frac{1}{2\pi} \int d^2 x B_+ B_- \]  \hspace{1cm} (146)

which is invariant under an extended set of transformations that includes (142) together with,

\[ \delta B_{\pm} = \partial_{\pm} \alpha \]  \hspace{1cm} (147)

Observe that the soldering field transforms exactly as a potential. It has served its purpose of fusing the two chiral components. Since it is an auxiliary field, it can be eliminated from the invariant action (146) by using the equations of motion. This will naturally solder the otherwise independent chiral components and justifies its name as a soldering field. The relevant solution is found to be,
\[ B_\pm = 2\pi J_\pm \]  

Inserting this solution in (146), we obtain,

\[
W[\Phi] = \frac{1}{4\pi} \int d^2x \left\{ \left( \partial_+ \Phi \partial_- \Phi + 2e A_+ \partial_- \Phi - 2e A_- \partial_+ \Phi \right) + (a + b - 2) e^2 A_+ A_- \right\} 
\]

where,

\[
\Phi = \varphi - \rho 
\]

As announced, the action is no longer expressed in terms of the different scalars \( \varphi \) and \( \rho \), but only on their specific combination. This combination is gauge invariant.

Let us digress on the significance of the findings. At the classical fermionic version, the chiral lagrangeans are completely independent. Bosonising them includes quantum effects, but still there is no correlation. The soldering mechanism exploits the symmetries of the independent actions to precisely combine them to yield a single action. Note that the soldering works with the bosonised expressions. Thus the soldered action obtained in this fashion corresponds to the quantum theory.

We now show that different choices for the parameters \( a \) and \( b \) lead to well known models. To do this consider the variation of (149) under the conventional gauge transformations, \( \delta \varphi = \delta \rho = \alpha \) and \( \delta A_\pm = \partial_\pm \alpha \). It is easy to see that the expression in parenthesis is gauge invariant. Consequently a gauge invariant structure for \( W \) is obtained provided,

\[
a + b - 2 = 0
\]

The effect of soldering, therefore, has been to induce a lift of the initial global symmetry (142) to its local form. By functionally integrating out the \( \Phi \) field from (149), we obtain,

\[
W[A_+, A_-] = -\frac{e^2}{4\pi} \int d^2x \left\{ A_+ \frac{\partial_-}{\partial_-} A_+ + A_- \frac{\partial_+}{\partial_-} A_- - 2A_+ A_- \right\}
\]

which is the familiar structure for the gauge invariant action expressed in terms of the potentials. The opposite chiralities of the two independent fermionic theories have been soldered to yield a gauge invariant action.

Some interesting observations are possible concerning the regularisation ambiguity manifested by the parameters \( a \) and \( b \). Since a single equation (151) cannot fix both the parameters, it might appear that there is a whole one parameter class of solutions for the chiral actions that combine to yield the vector gauge invariant action. Indeed, without any further input, this is the only conclusion. However, Bose symmetry imposes a crucial restriction. This symmetry plays an essential part that complements gauge invariance. Recall, for instance, the calculation of the triangle graph leading to the Adler-Bell-Jackiw anomaly. The familiar form of the anomaly cannot be obtained by simply demanding gauge invariance; Bose symmetry at the vertices of the triangle must also be imposed\[33, 34\]. Similarly, Bose symmetry\[3\] is essential in reproducing the structure of the one-cocycle that is mandatory in the analysis on smooth bosonisation\[35\]; gauge invariance alone fails.
In the present case, this symmetry corresponds to the left-right (or + -) symmetry in (140), thereby requiring \( a = b \). Together with the condition (151) this implies \( a = b = 1 \). This parametrisation has important consequences if a Maxwell term was included from the beginning to impart dynamics. Then the soldering takes place among two chiral Schwinger models having opposite chiralities to reproduce the usual Schwinger model. It is known that the chiral models satisfy unitarity provided \( a, b \geq 1 \) and the spectrum consists of a vector boson with mass,

\[
m^2 = \frac{e^2 a^2}{a - 1}
\]

and a massless chiral boson. The values of the parameters obtained here just saturate the bound. In other words, the chiral Schwinger model may have any \( a \geq 1 \), but if two such models with opposite chiralities are soldered to yield the vector Schwinger model, then the minimal bound is the unique choice. Moreover, for the minimal parametrisation, the mass of the vector boson becomes infinite so that it goes out of the spectrum. Thus the soldering mechanism shows how the massless modes in the chiral Schwinger models are fused to generate the massive mode of the Schwinger model.

Naively it may appear that the soldering of the left and right chiralities to obtain a gauge invariant result is a simple issue since adding the classical lagrangeans \( \bar{\psi} \gamma^\mu \gamma_5 \psi \) and \( \bar{\psi} \gamma^\mu \gamma_5 \psi \), with identical fermion species, just yields the usual vector lagrangean \( \bar{\psi} \gamma^\mu \gamma_5 \psi \). The quantum considerations are, however, much involved. The chiral determinants, as they occur, cannot be even defined since the kernels map from one chirality to the other so that there is no well defined eigenvalue problem. This is circumvented by working with \( \bar{\psi} (i \partial / + e A^\mu) \psi \), that satisfy an eigenvalue equation, from which their determinants may be computed. But now a simple addition of the classical lagrangeans does not reproduce the expected gauge invariant form. At this juncture, the soldering process becomes important. It systematically combined the quantised (bosonised) expressions for the opposite chiral components. Note that different fermionic species were considered so that this soldering does not have any classical analogue, and is strictly a quantum phenomenon. This will become more transparent when the three dimensional case is discussed.

It is interesting to show that a different choice for the parameters \( a \) and \( b \) in (149) leads to the Thirring model. Indeed it is precisely when the mass term exists (i.e., \( a + b - 2 \neq 0 \)), that (149) represents the Thirring model. Consequently, this parametrisation complements that used previously to obtain the vector gauge invariant structure. It is now easy to see that the term in parentheses in (149) corresponds to \( \bar{\psi} (i \partial / + e A^\mu) \psi \) so that integrating out the auxiliary \( A^\mu \) field yields,

\[
\mathcal{L} = \bar{\psi} i \partial / \psi - \frac{g}{2} (\bar{\psi} \gamma^\mu \psi)^2 \quad ; \quad g = \frac{4\pi}{a + b - 2}
\]

which is just the lagrangean for the usual Thirring model. It is known that this model is meaningful provided the coupling parameter satisfies the condition \( g > -\pi \), so that,

\[
| a + b | > 2
\]

This condition is the analogue of (151) found earlier. As usual, there is a one parameter arbitrariness. Imposing Bose symmetry implies that both \( a \) and \( b \) are equal and lie in the range

\[
1 < | a | = | b |
\]
This may be compared with the previous case where \( a = b = 1 \) was necessary for getting the gauge invariant structure. Interestingly, the positive range for the parameters in (156) just commences from this value.

Having developed and exploited the concepts of soldering in two dimensions, it is natural to investigate their consequences in three dimensions. The discerning reader may have noticed that it is essential to have dual aspects of a symmetry that can be soldered to yield new information. In the two dimensional case, this was the left and right chirality. Interestingly, in three dimensions also, we have a similar phenomenon.

### 3.2 The three dimensional example

This section is devoted to an analysis of the soldering process in the massive Thirring model in three dimensions. We shall show that two apparently independent massive Thirring models in the long wavelength limit combine, at the quantum level, into a massive Maxwell theory. This is further vindicated by a direct comparison of the current correlation functions following from the bosonization identities. These findings are also extended to include three dimensional quantum electrodynamics. The new results and interpretations illuminate a close parallel with the two dimensional discussion.

### 3.3 The massive Thirring model

In order to effect the soldering, the first step is to consider the bosonisation of the massive Thirring model in three dimensions [28, 9]. This is therefore reviewed briefly. The relevant current correlator generating functional, in the Minkowski metric, is given by,

\[
Z[\kappa] = \int D\psi D\bar{\psi} \exp \left( i \int d^3x \left[ \bar{\psi} (i\partial + m) \psi - \frac{\lambda^2}{2} j_\mu j^\mu + \lambda j_\mu \kappa^\mu \right] \right) \tag{157}
\]

where \( j_\mu = \bar{\psi} \gamma_\mu \psi \) is the fermionic current. As usual, the four fermion interaction can be eliminated by introducing an auxiliary field,

\[
Z[\kappa] = \int D\psi D\bar{\psi} Df_\mu \exp \left( i \int d^3x \left[ \bar{\psi} (i\partial + m + \lambda(f + \kappa)) \psi + \frac{1}{2} f_\mu f^\mu \right] \right) \tag{158}
\]

Contrary to the two dimensional models, the fermion integration cannot be done exactly. Under certain limiting conditions, however, this integration is possible leading to closed expressions. A particularly effective choice is the large mass limit in which case the fermion determinant yields a local form. Incidentally, any other value of the mass leads to a nonlocal structure [29]. The large mass limit is therefore very special. The leading term in this limit was calculated by various means.
and shown to yield the Chern-Simons three form. Thus the generating functional for the massive Thirring model in the large mass limit is given by,

$$Z[\kappa] = \int Df_\mu \exp \left( i \int d^3x \left( \frac{\lambda^2}{8\pi} \frac{m}{m} |\epsilon_{\mu\nu\lambda} f_\mu \partial^\nu f^\lambda + \frac{1}{2} f_\mu f^\mu + \frac{\lambda^2}{4\pi} \frac{m}{m} |\epsilon_{\mu\nu\kappa} \partial^\nu f^\kappa \right) \right)$$ \hspace{1cm} (159)$$

where the signature of the topological terms is dictated by the corresponding signature of the fermionic mass term. In obtaining the above result a local counter term has been ignored. Such terms manifest the ambiguity in defining the time ordered product to compute the correlation functions \cite{41}. The lagrangean in the above partition function defines a self dual model introduced earlier \cite{5}. The massive Thirring model, in the relevant limit, therefore bosonises to a self dual model. It is useful to clarify the meaning of this self duality. The equation of motion in the absence of sources is given by,

$$f_\mu = -\lambda^2 \frac{m}{4\pi} \frac{m}{m} |\epsilon_{\mu\nu\lambda} \partial^\nu f^\lambda$$ \hspace{1cm} (160)$$

from which the following relations may be easily verified,

$$\partial_\mu f^\mu = 0$$

$$\left( \Box + M^2 \right) f_\mu = 0 \hspace{1cm} ; \hspace{1cm} M = \frac{4\pi}{\lambda^2}$$ \hspace{1cm} (161)$$

A field dual to $f_\mu$ is defined as,

$$\tilde{f}_\mu = \frac{1}{M} \epsilon_{\mu\nu\lambda} \partial^\nu f^\lambda$$ \hspace{1cm} (162)$$

where the mass parameter $M$ is inserted for dimensional reasons. Repeating the dual operation, we find,

$$\left( \tilde{\tilde{f}}_\mu \right) = \frac{1}{M} \epsilon_{\mu\nu\lambda} \partial^\nu \tilde{f}^\lambda = f_\mu$$ \hspace{1cm} (163)$$

obtained by exploiting (161), thereby validating the definition of the dual field. Combining these results with (160), we conclude that,

$$f_\mu = -\frac{m}{m} |\tilde{f}_\mu$$ \hspace{1cm} (164)$$

Hence, depending on the sign of the fermion mass term, the bosonic theory corresponds to a self-dual or an anti self-dual model. Likewise, the Thirring current bosonises to the topological current

$$j_\mu = \frac{\lambda}{4\pi} \frac{m}{m} \epsilon_{\mu\nu\rho} \partial^\nu f^\rho$$ \hspace{1cm} (165)$$

The close connection with the two dimensional analysis is now evident. There the starting point was to consider two distinct fermionic theories with opposite chiralities. In the present instance, the analogous thing is to take two independent Thirring models with identical coupling strengths but opposite mass signatures.

$$L_+ = \bar{\psi} (i\partial + m) \psi - \frac{\lambda^2}{2} \left( \bar{\psi} \gamma_\mu \psi \right)^2$$

$$L_- = \bar{\xi} (i\partial - m') \xi - \frac{\lambda^2}{2} \left( \bar{\xi} \gamma_\mu \xi \right)^2$$ \hspace{1cm} (166)$$

31
Note that the only the relative sign between the mass parameters is important, but their magnitudes are different. From now on it is also assumed that both $m$ and $m'$ are positive. Then the bosonised lagrangeans are, respectively,

\begin{align*}
\mathcal{L}_+ &= \frac{1}{2M} \epsilon_{\mu\nu\lambda} f^\mu \partial^\nu f^\lambda + \frac{1}{2} f_\mu f^\mu \\
\mathcal{L}_- &= -\frac{1}{2M} \epsilon_{\mu\nu\lambda} g^\mu \partial^\nu g^\lambda + \frac{1}{2} g_\mu g^\mu
\end{align*}

(167)

where $f_\mu$ and $g_\mu$ are the distinct bosonic vector fields. The current bosonization formulae in the two cases are given by

\begin{align*}
j^+_\mu &= \bar{\psi} \gamma_\mu \psi = \frac{\lambda}{4\pi} \epsilon_{\mu
u\rho} \partial^\nu f^\rho \\
j^-_\mu &= \bar{\xi} \gamma_\mu \xi = -\frac{\lambda}{4\pi} \epsilon_{\mu\nu\rho} \partial^\nu g^\rho
\end{align*}

(168)

The stage is now set for soldering. Taking a cue from the two dimensional analysis, let us consider the gauging of the following symmetry,

\[ \delta f_\mu = \delta g_\mu = \epsilon_{\mu\rho\sigma} \partial^\rho \alpha^\sigma \]

(169)

Under such transformations, the bosonised lagrangeans change as,

\[ \delta \mathcal{L}_\pm = J_\pm^{\rho\sigma} (h_\mu) \partial_\rho \alpha_\sigma \quad ; \quad h_\mu = f_\mu, \ g_\mu \]

(170)

where the antisymmetric currents are defined by,

\[ J_\pm^{\rho\sigma} (h_\mu) = \epsilon^{\mu\rho\sigma} h_\mu \pm \frac{1}{M} \epsilon^{\rho\sigma\nu} \epsilon_{\mu\nu\rho} \partial^\mu h^\nu \]

(171)

It is worthwhile to mention that any other variation of the fields (like $\delta f_\mu = \alpha_\mu$) is inappropriate because changes in the two terms of the lagrangeans cannot be combined to give a single structure like (171). We now introduce the soldering field coupled with the antisymmetric currents. In the two dimensional case this was a vector. Its natural extension now is the antisymmetric second rank Kalb-Ramond tensor field $B_{\rho\sigma}$, transforming in the usual way,

\[ \delta B_{\rho\sigma} = \partial_\rho \alpha_\sigma - \partial_\sigma \alpha_\rho \]

(172)

Then it is easy to see that the modified lagrangeans,

\[ \mathcal{L}_\pm^{(1)} = \mathcal{L}_\pm - \frac{1}{2} J_\pm^{\rho\sigma} (h_\mu) B_{\rho\sigma} \]

(173)

transform as,

\[ \delta \mathcal{L}_\pm^{(1)} = -\frac{1}{2} \delta J_\pm^{\rho\sigma} B_{\rho\sigma} \]

(174)
The final modification consists in adding a term to ensure gauge invariance of the soldered lagrangean. This is achieved by,

\[ \mathcal{L}^{(2)}_\pm = \mathcal{L}^{(1)}_\pm + \frac{1}{4} B^{\rho \sigma} B_{\rho \sigma} \quad (175) \]

A straightforward algebra shows that the following combination,

\[ \mathcal{L}_S = \mathcal{L}^{(2)}_+ + \mathcal{L}^{(2)}_- = \mathcal{L}_+ + \mathcal{L}_- - \frac{1}{2} B^{\rho \sigma} \left( J^\rho_+ (f) + J^-_\rho (g) \right) + \frac{1}{2} B^{\rho \sigma} B_{\rho \sigma} \quad (176) \]

is invariant under the gauge transformations (169) and (172). The gauging of the symmetry is therefore complete. To return to a description in terms of the original variables, the auxiliary soldering field is eliminated from (176) by using the equation of motion,

\[ B_{\rho \sigma} = \frac{1}{2} \left( J^\rho_+ (f) + J^-_\rho (g) \right) \quad (177) \]

Inserting this solution in (176), the final soldered lagrangean is expressed solely in terms of the currents involving the original fields,

\[ \mathcal{L}_S = \mathcal{L}_+ + \mathcal{L}_- - \frac{1}{8} \left( J^\rho_+ (f) + J^-_\rho (g) \right) \left( J^\rho_+ (f) + J^\rho_- (g) \right) \quad (178) \]

It is now crucial to note that, by using the explicit structures for the currents, the above lagrangean is no longer a function of \( f_\mu \) and \( g_\mu \) separately, but only on the combination,

\[ A_\mu = \frac{1}{\sqrt{2} M} (g_\mu - f_\mu) \quad (179) \]

with,

\[ \mathcal{L}_S = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{M^2}{2} A_\mu A^\mu \quad (180) \]

where,

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (181) \]

is the usual field tensor expressed in terms of the basic entity \( A_\mu \). Our goal has been achieved. The soldering mechanism has precisely fused the self and anti self dual symmetries to yield a massive Maxwell theory which, naturally, lacks this symmetry.

It is now instructive to understand this result by comparing the current correlation functions. The Thirring currents in the two models bosonise to the topological currents (168) in the dual formulation. From a knowledge of the field correlators in the latter case, it is therefore possible to obtain the Thirring current correlators. The field correlators are obtained from the inverse of the kernels occurring in (167),

\[
\begin{align*}
\langle f_\mu (+k) f_\nu (-k) \rangle &= \frac{M^2}{M^2 - k^2} \left( i g_{\mu \nu} + \frac{1}{M^2} \epsilon_{\mu \rho \nu} k^\rho - \frac{i}{M^2} k_\mu k_\nu \right) \\
\langle g_\mu (+k) g_\nu (-k) \rangle &= \frac{M^2}{M^2 - k^2} \left( i g_{\mu \nu} - \frac{1}{M^2} \epsilon_{\mu \rho \nu} k^\rho - \frac{i}{M^2} k_\mu k_\nu \right)
\end{align*}
\quad (182)
\]
where the expressions are given in the momentum space. Using these in (168), the current correlators are obtained,

\[
\langle j^+(\mu) j^+(-\nu) \rangle = \frac{M}{4\pi(M^2 - k^2)} \left( ik^2 g_{\mu\nu} - ik_{\mu} k_{\nu} + \frac{1}{M} \epsilon_{\mu\nu\rho} k^\rho k^2 \right)
\]

\[
\langle j^-(-\mu) j^+(-\nu) \rangle = \frac{M}{4\pi(M^2 - k^2)} \left( ik^2 g_{\mu\nu} - ik_{\mu} k_{\nu} - \frac{1}{M} \epsilon_{\mu\nu\rho} k^\rho k^2 \right)
\]

(183)

It is now feasible to construct a total current,

\[
j_\mu = j^+_\mu + j^-_\mu = \lambda \frac{\sqrt{M^2 - k^2}}{2\pi(M^2 - k^2)} \left( g_{\mu\nu} - k_{\mu} k_{\nu} \right)
\]

(184)

Then the correlation function for this current, in the original self-dual formulation, follows from (183) and noting that \( \langle j^+_\mu j^-_\nu \rangle = 0 \), which is a consequence of the independence of \( f_\mu \) and \( g_\nu \);

\[
\langle j_\mu(\mu) j_\nu(-\nu) \rangle = \langle j^+_\mu j^+_\nu \rangle + \langle j^-_\mu j^-_\nu \rangle = \frac{iM}{2\pi(M^2 - k^2)} \left( k^2 g_{\mu\nu} - k_{\mu} k_{\nu} \right)
\]

(185)

The above equation is easily reproduced from the effective theory. Using (179), it is observed that the bosonization of the composite current (184) is defined in terms of the massive vector field \( A_\mu \),

\[
j_\mu = \bar{\psi} \gamma_\mu \psi + \bar{\xi} \gamma_\mu \xi = -\sqrt{\frac{M}{2\pi}} \epsilon_{\mu\nu\rho} \partial^\nu A^\rho
\]

(186)

The current correlator is now obtained from the field correlator \( \langle A_\mu A_\nu \rangle \) given by the inverse of the kernel appearing in (180),

\[
\langle A_\mu(\mu) A_\nu(-\nu) \rangle = \frac{i}{M^2 - k^2} \left( g_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{M^2} \right)
\]

(187)

From (186) and (187) the two point function (185) is reproduced, including the normalization.

We conclude, therefore, that two massive Thirring models with opposite mass signatures, in the long wavelength limit, combine by the process of bosonisation and soldering, to a massive Maxwell theory. The bosonization of the composite current, obtained by adding the separate contributions from the two models, is given in terms of a topological current (186) of the massive vector theory. These are completely new results which cannot be obtained by a straightforward application of conventional bosonisation techniques. The massive modes in the original Thirring models are manifested in the two modes of (180) so that there is a proper matching in the degrees of freedom. Once again it is reminded that the fermion fields for the models are different so that the analysis has no classical analogue. Indeed if one considered the same fermion species, then a simple addition of the classical lagrangeans would lead to a Thirring model with a mass given by \( m - m' \). In particular, this difference can be zero. The bosonised version of such a massless model is known [8, 29] to yield a highly nonlocal theory which has no connection with (186). Classically, therefore, there is no possibility of even understanding, much less, reproducing the effective quantum result. In this sense the application in three dimensions is more dramatic than the corresponding case of two dimensions.
3.4 Quantum electrodynamics

An interesting theory in which the preceding ideas may be implemented is quantum electrodynamics, whose current correlator generating functional in an arbitrary covariant gauge is given by,

\[ Z[\kappa] = \int D\bar{\psi} D\psi DA_\mu \exp \left\{ i \int d^3x \left( \bar{\psi} (i\partial^\mu + m + eA) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\eta}{2} (\partial_\mu A^\mu)^2 + e j_\mu \kappa^\mu \right) \right\} \tag{188} \]

where \( \eta \) is the gauge fixing parameter and \( j_\mu = \bar{\psi} \gamma_\mu \psi \) is the current. As before, a one loop computation of the fermionic determinant in the large mass limit yields,

\[ Z[\kappa] = \int DA_\mu \exp \left\{ i \int d^3x \left[ \frac{e^2}{8\pi} \left| \frac{m}{m} \right| \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2}{4\pi} \frac{m}{m} \epsilon_{\mu\nu\rho} \kappa^\mu \partial^\nu A^\rho + \frac{\eta}{2} (\partial_\mu A^\mu)^2 \right] \right\} \tag{189} \]

In the absence of sources, this just corresponds to the topologically massive Maxwell-Chern-Simons theory, with the signature of the topological term determined from that of the fermion mass term. The equation of motion,

\[ \partial^\nu F_{\nu\mu} + \frac{e^2}{4\pi} \frac{m}{m} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda = 0 \tag{190} \]

expressed in terms of the dual tensor,

\[ F_\mu = \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \tag{191} \]

reveals the self (or anti self) dual property,

\[ F_\mu = \frac{4\pi}{e^2} \frac{m}{m} \epsilon_{\mu\nu\lambda} \partial^\nu F^\lambda \tag{192} \]

which is the analogue of \((160)\). In this fashion the Maxwell-Chern-Simons theory manifests the well known \([6, 41, 42]\) mapping with the self dual models considered in the previous subsection. The difference is that the self duality in the former, in contrast to the latter, is contained in the dual field \((191)\) rather than in the basic field defining the theory. This requires some modifications in the ensuing analysis. Furthermore, the bosonization of the fermionic current is now given by the topological current in the Maxwell-Chern-Simons theory,

\[ j_\mu = \frac{e}{4\pi} \frac{m}{m} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \tag{193} \]

Consider, therefore, two independent models describing quantum electrodynamics with opposite signatures in the mass terms,

\[ L_+ = \bar{\psi} (i\partial + m + eA) \psi - \frac{1}{4} F_{\mu\nu} (A) F^{\mu\nu} (A) \]
\[ L_- = \bar{\xi} (i\partial + m' + eB) \xi - \frac{1}{4} F_{\mu\nu} (B) F^{\mu\nu} (B) \tag{194} \]
whose bosonised versions in an appropriate limit are given by,

\[
L_{\pm} = -\frac{1}{4} F_{\mu\nu}(A) + \frac{M}{2} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda; \quad M = \frac{e^2}{4\pi}
\]

\[
L_{-} = -\frac{1}{4} F_{\mu\nu}(B) - \frac{M}{2} \epsilon_{\mu\nu\lambda} B^\mu \partial^\nu B^\lambda
\]

(195)

where \(A_\mu\) and \(B_\mu\) are the corresponding potentials. Likewise, the corresponding expressions for the bosonized currents are found from the general structure (193),

\[
j_{\mu}^+ = \bar{\psi} \gamma_\mu \psi = \frac{M}{e} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda
\]

\[
j_{\mu}^- = \bar{\xi} \gamma_\mu \xi = -\frac{M}{e} \epsilon_{\mu\nu\lambda} \partial^\nu B^\lambda
\]

(196)

To proceed with the soldering of the above models, take the symmetry transformation,

\[
\delta A_\mu = \alpha_\mu
\]

(197)

Such a transformation is spelled out by recalling (169) and the observation that now (191) simulates the \(f_\mu\) field in the previous case. Under this variation, the lagrangeans (195) change as,

\[
\delta L_{\pm} = J_{\pm}^{\rho\sigma}(P) \partial_\rho \alpha_\sigma; \quad P = A, B
\]

(198)

where the antisymmetric currents are defined by,

\[
J_{\pm}^{\rho\sigma}(P) = \pm m \epsilon^{\rho\sigma\mu} P_\mu - F_{\rho\sigma}(P)
\]

(199)

Proceeding as before, the antisymmetric soldering field \(B_{\alpha\beta}\) transforming as (172) is introduced by coupling with these currents to define the first iterated lagrangeans analogous to (173),

\[
L_{\pm}^{(1)} = L_{\pm} - \frac{1}{2} J_{\pm}^{\rho\sigma}(P) B_{\rho\sigma}
\]

(200)

These lagrangeans are found to transform as,

\[
\delta L_{\pm}^{(1)} = \frac{1}{4} \delta B_{\lambda\sigma}^2 - \frac{1}{2} \left( \pm m \epsilon_{\mu\lambda\sigma} \alpha^\mu B^{\lambda\sigma} \right)
\]

(201)

It is now straightforward to deduce the final lagrangean that will be gauge invariant. This is given by,

\[
L_S = L_{\pm}^{(2)} + L_{-}^{(2)}; \quad \delta L_S = 0
\]

(202)

where the second iterated pieces are,

\[
L_{\pm}^{(2)} = L_{\pm} - \frac{1}{2} J_{\pm}^{\rho\sigma} B_{\rho\sigma} - \frac{1}{4} B_{\rho\sigma} B^{\rho\sigma}
\]

(203)
The invariance of $\mathcal{L}_S$ (202) is verified by observing that,
\begin{equation}
\delta \mathcal{L}^{(2)}_\pm = \mp \frac{1}{2} m \epsilon_{\mu\lambda\sigma} \alpha^\mu B^{\lambda\sigma}
\end{equation}

To obtain the effective lagrangean it is necessary to eliminate the auxiliary $B_{\rho\sigma}$ field by using the equation of motion following from (202),
\begin{equation}
B_{\sigma\lambda} = -\frac{1}{2} \left( J^+_{\sigma\lambda}(A) + J^-_{\sigma\lambda}(B) \right)
\end{equation}

Putting this back in (202), we obtain the final soldered lagrangean,
\begin{equation}
\mathcal{L}_S = -\frac{1}{4} F_{\mu\nu}(G) F^{\mu\nu}(G) + \frac{M^2}{2} G_{\mu} G^\mu
\end{equation}
written in terms of a single field,
\begin{equation}
G_\mu = \frac{1}{\sqrt{2}} (A_\mu - B_\mu)
\end{equation}
The lagrangean (206) governs the dynamics of a massive Maxwell theory.

As before, we now discuss the implications for the current correlation functions. These functions in the original models describing electrodynamics can be obtained from the mapping (197). The first step is to abstract the basic field correlators found by inverting the kernels occurring in (195). The results, in the momentum space, are
\begin{align}
\langle A_\mu(+k) A_\nu(-k) \rangle &= \frac{i}{M^2 - k^2} \left[ g_{\mu\nu} + \frac{M^2 - k^2(\eta + 1)}{\eta k^4} k_\mu k_\nu + \frac{i M}{k^2} \epsilon_{\mu\rho\nu\sigma} k_\sigma \right]
\langle B_\mu(+k) B_\nu(-k) \rangle &= \frac{i}{M^2 - k^2} \left[ g_{\mu\nu} + \frac{M^2 - k^2(\eta + 1)}{\eta k^4} k_\mu k_\nu - \frac{i M}{k^2} \epsilon_{\mu\rho\nu\sigma} k_\sigma \right]
\end{align}
The current correlators are easily computed by substituting (208) into (196),
\begin{align}
\langle j^+_\mu(+k) j^+_\nu(-k) \rangle &= i \left( \frac{M}{e} \right)^2 \frac{1}{M^2 - k^2} \left[ k^2 g_{\mu\nu} - k_\mu k_\nu - i M \epsilon_{\mu\rho\nu\sigma} k_\sigma \right]
\langle j^-_\mu(+k) j^-_\nu(-k) \rangle &= i \left( \frac{M}{e} \right)^2 \frac{1}{M^2 - k^2} \left[ k^2 g_{\mu\nu} - k_\mu k_\nu + i M \epsilon_{\mu\rho\nu\sigma} k_\sigma \right]
\end{align}
where, expectedly, the gauge dependent ($\eta$) contribution has dropped out. Defining a composite current,
\begin{equation}
j_\mu = j^+_\mu + j^-_\mu = \frac{M}{e} \epsilon_{\mu\nu\lambda\rho} \partial^\nu \left( A^\lambda - B^\lambda \right)
\end{equation}
it is simple to obtain the relevant correlator by exploiting the results for $j^+_\mu$ and $j^-_\mu$ from (209),
\begin{equation}
\langle j_\mu(+k) j_\nu(-k) \rangle = 2 i \left( \frac{M}{e} \right)^2 \frac{1}{M^2 - k^2} \left( k^2 g_{\mu\nu} - k_\mu k_\nu \right)
\end{equation}
In the bosonized version obtained from the soldering, \((210)\) represents the mapping,

\[
j_{\mu} = \bar{\psi} \gamma_{\mu} \psi + \bar{\xi} \gamma_{\mu} \xi = \sqrt{2} \frac{M}{e} \epsilon_{\mu \nu \lambda} \partial^{\nu} G^{\lambda}
\]

where \(G_{\mu}\) is the massive vector field \((207)\) whose dynamics is governed by the lagrangean \((206)\). In this effective description the result \((211)\) is reproduced from \((212)\) by using the correlator of \(G_{\mu}\) obtained from \((206)\), which is exactly identical to \((187)\).

Thus the combined effects of bosonisation and soldering show that two independent quantum electrodynamical models with appropriate mass signatures are equivalently described by the massive Maxwell theory. In the self dual version the massive modes are the topological excitations in the Maxwell-Chern-Simons theories. These are combined into the two usual massive modes in the effective massive vector theory. A complete correspondence among the composite current correlation functions in the original models and in their dual bosonised description was also established. The comments made in the concluding part of the last subsection naturally apply also in this instance.

### 3.5 Final Remarks

The present analysis clearly revealed the possibility of obtaining new results from quantum effects that conspire to combine two apparently independent theories into a single effective theory. The essential ingredient was that these theories must possess the dual aspects of the same symmetry. Then, by a systematic application of bosonisation and soldering, it was feasible to abstract a meaningful combination of such models, which can never be obtained by a naive addition of the classical lagrangeans.

The basic notions and ideas were particularly well illustrated in the two dimensional example where the bosonised expressions for distinct chiral lagrangeans were soldered to reproduce either the usual gauge invariant theory or the Thirring model. Indeed, the soldering mechanism that fused the opposite chiralities, clarified several aspects of the ambiguities occurring in bosonising chiral lagrangeans. It was clearly shown that unless Bose symmetry is imposed as an additional restriction, there is a whole one parameter class of bosonised solutions for the chiral lagrangeans that can be soldered to yield the vector gauge invariant result. The close connection between Bose symmetry and gauge invariance was thereby established, leading to a unique parametrisation. Similarly, using a different parametrisation, the soldering of the chiral lagrangeans led to the Thirring model. Once again there was a one parameter ambiguity unless Bose symmetry was imposed. If that was done, there was a specified range of solutions for the chiral lagrangeans that combined to yield a well defined Thirring model.

The elaboration of our methods was done by considering the massive version of the Thirring model and quantum electrodynamics in three dimensions. By the process of bosonisation such models, in the long wavelength limit, were cast in a form which manifested a self dual symmetry. This was a basic perquisite for effecting the soldering. It was explicitly shown that two distinct massive Thirring models, with opposite mass signatures, combined to a massive Maxwell theory. The Thirring current correlation functions calculated either in the original self dual formulation or in the effective massive
vector theory yielded identical results, showing the consistency of our approach. The application to quantum electrodynamics followed along similar lines.

It is evident that the present technique of combining models by the two step process of bosonisation and soldering can be carried through in higher dimensions provided the models have the relevant symmetry properties. It is also crucial to note that duality pervades the entire analysis. In the three dimensional case this was self evident since the models had a self (and anti) self dual symmetry. This was hidden in the two dimensional case where chiral symmetry was more transparent. But it may be mentioned that in two dimensions, chiral symmetry is the analogue of the duality \( \partial_\mu \phi = \pm \epsilon_{\mu\nu} \partial^\nu \phi \). Interestingly, the duality in two dimensions was manifest in the lagrangeans while that in three dimensions was contained in the equations of motion. This opens up the possibility to discuss different aspects of duality, contained either in the lagrangean or in the equations of motion, in the same framework. Consequently, the methods developed here can be relevant and useful in different contexts; particularly in the recent discussions on electromagnetic duality or the study of chiral forms which exactly possess the type of self dual symmetry considered in this paper. We will report on these and related issues in a future work.

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References

[1] L. Alvarez-Gaume and S.F. Hassan, Fortsch. Phys. 45 (1997) 159; Also see, L. Alvarez-Gaume and F. Zamora, Duality in Quantum Field Theory (String Theory), hep-th/9709180

[2] M. Stone, Illinois preprint ILL-(TH)-89-23; 1989

[3] R. Banerjee and C. Wotzasek, hep-th/9710060, Nucl.Phys.B, to appear.

[4] N. Banerjee and R. Banerjee Nucl. Phys. B445 (1995) 516.

[5] P.K. Townsend, K. Pilch and P. van Nieuwenhuizen, Phys. Lett. B136 (1984) 452.

[6] S. Deser and R. Jackiw, Phys. Lett. B139 (1984) 371.

[7] R. Banerjee and C. Wotzasek, hep-th/9709103, Nucl.Phys.B, to appear.

[8] E.C. Marino, Phys. Lett. B263 (1991) 63.

[9] R. Banerjee, Phys. Lett. B358 (1995) 297 and Nucl. Phys. B465 (1996) 157.

[10] R. Amorim, A. Das and C. Wotzasek, Phys. Rev. D53 (1996) 5810.

[11] D.I. Olive, Exact electromagnetic duality, hep-th/9508083, Nucl. Phys. B(Proc. Suppl.) 58 (1997) 43
[12] C. Gomez and R. Hernandez, Electric-magnetic duality and effective field theories, [hep-th/9510023](hep-th/9510023)

[13] J. Schwarz, Lectures on superstring and M theory dualities, [hep-th/9607201](hep-th/9607201); Nucl. Phys. Proc. Suppl. 55B (1997) 1

[14] D. Zwanziger, Phys. Rev. D3 (1971) 880

[15] S. Deser and C. Teitelboim, Phys. Rev. D13 (1976) 1592

[16] J. Schwarz and A. Sen, Nucl. Phys. B411 (1994) 35

[17] G.W. Gibbons and D.A. Rasheed, Nucl. Phys. B454 (1995) 185

[18] N. Berkovits, Phys. Lett. B388 (1996) 743; ibid B398 (1997) 79

[19] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, Phys. Lett. B400 (1997) 80

[20] A. Khoudeir and N. Pantoja, Phys. Rev. D53 (1996) 5974

[21] P. Pasti, D. Sorokin and M. Tonin, Phys. Lett. B352 (1995) 59; Phys. Rev. D52 (1995) R4277

[22] H.O. Girotti, Phys. Rev. D55 (1997) 5136

[23] R. Floreanini and R. Jackiw, Phys. Rev. Lett. 59 (1987) 1873

[24] E.M.C. Abreu, R. Banerjee and C. Wotzasek, UFRJ report-257-97; 1997 [hep-th/9707204](hep-th/9707204), to appear in Nucl.Phys. B

[25] A. Polyakov and P. Weigman, Phys. Lett. B131 (1983) 121

[26] J. Sonnenschein, Nucl. Phys. B309 (1988) 752

[27] For a review see, E. Abdalla, M.C.B. Abdalla and K.D. Rothe, Nonperturbative Methods in Two Dimensional Quantum Field Theory, World Scientific, Singapore, 1991.

[28] C. Burges, C. Lütken and F. Quevedo, Phys. Lett. B336 (1994) 18; E. Fradkin and F. Schaposnik, Phys. Lett. B338 (1994) 253; K. Ikegami, K. Kondo and A. Nakamura, Prog. Theor. Phys. 95 (1996) 203; D.Barci, C.D. Fosco and L. Oxman, Phys. Lett. B375 (1996) 267.

[29] R. Banerjee and E.C. Marino, [hep-th/9607040](hep-th/9607040) (To appear in Phys. Rev. D); [hep-th/9707033](hep-th/9707033) and 9707100.

[30] E. Abdalla and R. Banerjee, [hep-th/9704176](hep-th/9704176). Phys. Rev. Lett., to appear.

[31] See, for instance, R. Jackiw, Diverse Topics in Theoretical and Mathematical Physics, World Scientific, Singapore, 1995.

[32] R. Banerjee, Phys. Rev. Lett. 56 (1986) 1889.
[33] L. Rosenberg, Phys. Rev. 129 (1963) 2786.

[34] S.L. Adler, Lectures in Elementary Particles and Quantum Field Theory, S.Deser et al. eds., 1970, Brandeis Lectures, MIT Press, Cambridge.

[35] P.H.Damgaard, H.B.Nielsen, and R.Sollacher, Nucl. Phys. B414 (1994) 541; P.H.Damgaard and R.Sollacher, Phys. Lett. B 322 (1994) 131 and Nucl. Phys. B 433 (1995) 671.

[36] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54 (1985) 1219; 2060(E).

[37] J.Schwinger, Phys. Rev. 128 (1962) 2425.

[38] L. Alvarez-Gaume and E. Witten, Nucl.Phys.B234 (1983) 269.

[39] S. Coleman, Phys. Rev. D11 (1975) 2088.

[40] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. (NY) 140 (1982) 372; A.N. Redlich, Phys. Rev. D29 (1984) 2366.

[41] R. Banerjee, H.J. Rothe and K.D. Rothe, Phys. Rev. D52 (1995) 3750.

[42] R. Banerjee and H.J. Rothe, Nucl. Phys. B447 (1995) 183.