I. INTRODUCTION

The response of artificial composites with specially designed microstructure can fundamentally differ from the response of their constituents. Such composites are termed metamaterials, and their features span various solid facets, including electromagnetic and mechanical properties [1–6].

A prominent thrust in metamaterial design is wave control [7–10], where some of the achievements thus far are wave suppressors, cloaking, negative refraction, and super-lensing [11–24]. These phenomena are often manifestations of anomalous effective properties, such as negative refractive index and negative mass [25–29], which are analytically determined using homogenization (effective medium) theories [30–41]. Notably, Willis has developed an elastodynamic homogenization theory which predicts that the momentum and stress can be constitutively coupled to the strain and the velocity, respectively, by the now termed Willis couplings [42–48]. These effective properties constitute additional degrees of freedom to manipulate waves, as was demonstrated, e.g., to experimentally realize asymmetric reflection and scattering-free refraction [49–51].

Recently, Pernas-Salomón and Shmuel [52] have generalized the approach of Willis to account for constituents that linearly deform in response to non-mechanical fields, such as piezomagnetic- and piezoelectric materials [53, 54]. The main observation that the generalized theory delivers is the emergence of additional couplings of Willis type between the momentum and the velocity to the non-mechanical fields, as illustrated in Fig. 1. Accordingly, the momentum of piezoelectric composites is coupled with the electric field, while the velocity is coupled with the electric displacement field. These additional couplings not only enlarge the design space of metamaterials, but also reflect a novel mechanism to actively manipulate waves via non-mechanical stimuli.

The objective of this work is to determine the physical restrictions imposed by reciprocity, passivity and causality on these properties in what we refer to as generalized Willis materials. The significance of establishing these restrictions is twofold. First, they test the physical admissibility of relevant theoretical and experimental results [55, 56]. Second, they provide elementary bounds for the maximal response that potential devices may achieve.

A. Relevant developments in Willis equations

Since this work is closely related to Willis equations, as it provides physical restrictions on their generalization, a more elaborated review of their relevant developments is in order. The topics discussed next do not constitute a complete review of the works in the field, and aspects such as weighted averages [59, 61], connections with asymptotic homogenization [62, 63] etc. are not addressed here.

Willis has started to develop his formulation using a variational approach that extends the concepts of ensemble averaging and comparison media from elastostatics [42–44, 64]. His effective relations exhibit two notable features, in addition to the emergence of the cross-couplings mentioned earlier. First, they are non-local in space—as known from elastostatics—and in time, even if the response of the original composite was history-independent. (The non-local nature renders the effective relations non-unique, an issue that is discussed later.) Second, the kernel that describes the effective mass density is a second-order tensor.

More recently, Willis developed a formulation that does not rely on a comparison medium, but rather on the Green func-
momentum & 
Dynamics 
velocity

that are relevant to this paper. Milton et al. [82] identified the
ture [51, 57, 60, 71–81]. We list next some of the insights
that present experimental validation and analyze their struc-
tures. Source-driven homogenization has been adopted later
out of an equivalent class that exists when the eigenstrain van-
sion has the benefit of providing a unique effective properties
be experimentally prescribed [66], their mathematical inclu-
sion has the benefit of providing a unique effective properties
out of an equivalent class that exists when the eigenstrain van-
sishes. Source-driven homogenization has been adopted later
also in Refs. [67–69]. Having listed the main developments in
Willis theory, we can now point out the common components
with our theory for media that deform by non-mechanical stimuli: our theory also relies on ensemble averaging, incor-
porates eigenstrains as additional driving source, and delivers
unique effective properties based on the Green function of the
original composite.

The recent interest in metamaterials [70] has disseminated
to Willis effective relations, resulting with a bulk of papers
that present experimental validation and analyze their structure [51, 57, 60, 71–81]. We list next some of the insights
that are relevant to this paper. Milton et al. [82] identified the
similarity between Willis couplings and bianisotropy in elec-
 tromagnetics, see also Refs. [57, 58, 69, 77, 83, 84].Sieck
et al. [69] provided a perceptive analysis on the source of the
cross-couplings in periodic media, concluding that their non-
local part originates from multiple scattering and phase change at the mesoscale, while their local part originates from asymmetry in the unit cell. Similarly, Pernas-Salomón and Shmuel [85] pointed out the analogy with the broken inversion symmetry in piezoelectric materials at the atomic scale, which leads to microscopic electroelastic coupling.

Spatially local couplings were proposed by Milton et al. [82]. As pointed out in Ref. [84], the corresponding equations
are the limiting case of the non-local equations, referred to as the Milton-Briane-Willis equations. Such a model was
developed by Milton [59], whose stress depends on the ac-
celeration rather than the velocity. Simpler spatially local
models that report acceleration-dependent stress were given
later in Refs. [58, 60, 84, 86]. These works suggest that the
non-local nature of the operator conceals [87] a more physical
constitutive description—one which employs the strain rate and acceleration as additional input functions. Here, we adapt
and examine this suggestion to our settings, by introducing
and analyzing a description that additionally includes the time
derivative of the electric field as an input function, and find
arguments that support the use of the alternative formulation.

B. Summary of our results

The first principle we employ is passivity, which at the basic
level means that the material does not generate energy. For-
formally, we require that the power supplied by external agents
is always greater or equal to the rate of change of the energy
stored by the material. This principle delivers inequalities for
the skew-Hermitian and Hermitian parts of the Fourier trans-
forms of the effective properties, as summarized in Tab. I. If
the material exhibits major symmetries, then these inequali-
ties apply to the imaginary and real parts of the transforms.
If the material is passive and lossless, we find that the di-
rect couplings—and combinations of cross-couplings—must
be either Hermitian or skew-Hermitian.

The second principle we employ is reciprocity, which refers
to an equality between the power done by conjugate fields of
different problems. In the long-wavelength limit, it implies
major symmetries for direct couplings, and transpose rela-
tions between cross-couplings. Beyond the long-wavelength,
it requires the non-local operator to be self-adjoint with re-
spect to the spatial variables. Technically, this translates to an
interchange in the functional dependency in these variables,
in addition to the transposition relations among the couplings
(see Tab. I). From this analysis we deduce that the formulation
that does not use the time derivative of the velocity, strain and
electric field is unphysical since it corresponds to imaginary
properties in the time domain. By contrast, the modified for-
mulation that is based on these rates leads to real properties in
the time domain.

The last principle we employ is causality, which means that
an effect (e.g., momentum) cannot precede its cause (e.g.,

FIG. 1. Schematics of the cross-couplings reported by Pernas-
Salamón and Shmuel [52], in composites whose elasticity is intrin-
sically coupled with other physics, such as piezoelectric and piezo-
magnetic materials.
electric field). This principle provides a connection between the real and imaginary parts of the (time) transforms of the couplings. The process we employ is standard and straightforward, and uses the Plemelj formulas to obtain relations of the Kramers-Kröning type for the generalized effective properties [88–91]. We clarify that our study of causality is restricted to the spatially local equations, and note that the corresponding analysis supports the claim that the alternative formulation should be favored.

II. DYNAMIC HOMOGENIZATION OF PIEZOELECTRIC COMPOSITES

We consider a composite occupying the volume Ω made of piezoelectric phases, driven by time-dependent body force density f, inelastic strain η, and free charge density q. These sources generate in the composite stress σ, electric displacement D and momentum density p, which satisfy the balance equations

$$\nabla \cdot \sigma + f - p = 0, \tag{1}$$

and

$$\nabla \cdot D = q, \tag{2}$$

where the superposed dot denotes a time derivative. At each material point x, these fields are related to the displacement gradient Vμu, velocity u, and electric potential gradient [92] Vϕ through the constitutive equations of piezoelectricity [93], namely [94],

$$\begin{pmatrix} \sigma \\ D \\ p \end{pmatrix} = \begin{pmatrix} C & B^T & 0 \\ B & -A & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \nabla u - \eta \\ \nabla \phi \\ \dot{u} \end{pmatrix}, \tag{3}$$

where ρ, A and B, and C, are the spatially varying [95] local mass density, dielectric, piezoelectric and elasticity tensor fields, respectively [96]. In co-ordinates, these tensors satisfy

$$A_{ij} = A_{ji}, B_{ijk} = B_{jki}, B_{ijk}^* = B_{kij},$$

$$C_{ijkl} = C_{jikl} = C_{jkl} = C_{ikjl}, \sigma_{ij} = \sigma_{ji}. \tag{4}$$

Pernas-Salomón and Shmuel [52] have proposed an effective description with constitutive equations for the composite by extending the approach of Willis [46]. This was carried out by treating the composite as random, such that its properties are not only functions of x, but also of the particular specimen that belongs to some sample space Λ. The expectation value of any property, say ρ, is given by the ensemble average

$$\langle \rho \rangle (x) = \int_\Lambda \rho(x,y) P(y) dy, \tag{5}$$

where the parameter y is used to label the specimens, and P is the probability measure function over Λ. The governing equations of our effective description are given by the following ensemble averages of Eqs. (1) and (2)

$$\nabla \cdot \langle \sigma \rangle + \langle f \rangle - \langle p \rangle = 0, \quad \nabla \cdot \langle D \rangle = q, \tag{6}$$

in which ⟨σ⟩, ⟨D⟩ and ⟨p⟩ are the effective fields [97]. Based on the Green (tensor) function of the problem, Pernas-Salomón and Shmuel [52] obtained constitutive equations for the effective fields in the form [98]

$$\begin{pmatrix} \sigma \\ D \\ p \end{pmatrix} = \begin{pmatrix} C & B^T & \mathcal{L} \\ B & -A & \mathcal{W} \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \nabla u - \eta \\ \nabla \phi \\ \dot{u} \end{pmatrix}, \tag{7}$$

where the matrix elements are now non-local operators in time and space. (At this point, we do not endow the couplings with superscript † the meaning that this symbol usually designates, and postpone it to Sec. IV.) We denote the column vectors in the left- and right-hand sides of Eq. (7) by ⟨h⟩ and ⟨g⟩, and put the latter statement into formal footing, namely,

$$\langle h \rangle (x,t) = \mathcal{L} \langle g \rangle = \int_{-\infty}^{t} \int_{\Omega} \tilde{L}(x,\chi, t-T) \langle g \rangle (\chi, T) d\chi dT, \tag{8}$$

where L denotes the non-local effective constitutive operator and L is its kernel. In the sequel, we will denote by C the kernel of C, by S the kernel of S, by T the kernel of T, etc. The effective operator exhibits three notable features, in addition to its spatio-temporal non-local nature. First, it couples ⟨σ⟩ with ⟨u⟩, and ⟨p⟩ with ⟨∇u⟩, through the so-called Willis couplings S and T. Second, (kernel of) the effective mass density T is a second-order tensor. As mentioned, these two features—which are absent from the local constitutive equations and hence represent metamaterials—were discovered by Willis [42, 43, 44] in his studies of purely elastic composites. The third distinctive feature reported by Pernas-Salomón and Shmuel [52] is the coupling W between (D) and (u), and the coupling W† between (p) and (∇φ), which we term the electro-momentum coupling. The transition to this effective description is schematically illustrated in Fig. 2. The kernel of L is endowed with the minor symmetries

$$\tilde{C}_{ijkl} = \tilde{C}_{jikl}, \tilde{C}_{ijkl} = \tilde{C}_{jikl}, \tilde{B}_{ijkl} = \tilde{B}_{ijlk},$$

$$\tilde{S}_{ijkl} = \tilde{S}_{jikl}, \tilde{S}_{ijkl} = \tilde{S}_{jikl}, \tilde{B}_{ijkl}^* = \tilde{B}_{ijlk}^* \tag{9}$$

as they translate from the microscopic to the effective description, owing to the balance of angular momentum and independence from the anti-symmetric part of Vμu. The major symmetries of the constitutive tensors in Eq. (4) induce additional symmetries between the effective tensors (and justify the superscript † mentioned above), to be discussed in Sec. IV and the Appendix.

When the composite is statistically homogeneous, the constitutive operator becomes translation invariant, i.e., it depends only on the difference x − x'; accordingly, Eq. (8) has the form of a convolution not only in time, but also in space. Therefore, the Fourier transform with respect to both time and space yields constitutive relations in the form of simple products between the transforms of L and ⟨g⟩. It follows that such an infinite medium admits plane waves in the form of the real part of u = Uω(x − ω0t) and φ = Φω(x − ω0t), for which the non-local constitutive equations are the simple products

$$\langle h \rangle (x,t) = \tilde{L} (-\kappa_0, \omega_0) \langle g \rangle (x,t). \tag{10}$$
in the \((x,t)\) space. (Again, the real part of the equation should be taken.) We emphasize that \(\tilde{L}(\mathbf{x}, \omega)\) is the space-time Fourier transform of \(L\) according to the convention

\[
\tilde{L}(\mathbf{x}, \omega) = \int_{\Omega} d\mathbf{x} \int_{\mathbb{R}} dt \tilde{L}(\mathbf{x}, t) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)},
\]
evaluated at \((-k_0, \omega_0)\). In order not to introduce more notations to the already large set used here, we will also use \((\tilde{\mathcal{O}})\) for transforms that are applied only with respect to one of the two variables (time or space).

The objective of this work is to determine the mathematical restrictions imposed on relations (7)—and specifically on the electro-momentum coupling—by the physical principles of reciprocity, passivity, and causality. In addition to form (7), we will also analyze the form

\[
\begin{pmatrix}
\langle \mathbf{\sigma} \rangle \\
\langle \mathbf{D} \rangle \\
\langle \mathbf{p} \rangle
\end{pmatrix} =
\begin{pmatrix}
\mathbf{C} & \mathbf{B}^\top & 0 \\
\mathbf{B} & -\mathbf{A} & 0 \\
0 & 0 & \mathbf{B}
\end{pmatrix}
\begin{pmatrix}
\langle \nabla \mathbf{u} \rangle - \mathbf{\eta} \\
\langle \nabla \phi \rangle \\
\langle \mathbf{u} \rangle
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & \mathbf{I} \\
0 & 0 & \mathbf{I} \\
\mathbf{I}^\top & \mathbf{I}^\top & 0
\end{pmatrix}
\begin{pmatrix}
\langle \nabla \mathbf{u} \rangle - \mathbf{\eta} \\
\langle \nabla \phi \rangle \\
\langle \mathbf{u} \rangle
\end{pmatrix},
\]

where the kernel of the time Fourier transform of \(\mathcal{S}\) is \(-\tilde{S}/\omega\), the kernel of the transform of \(\mathcal{W}\) is \(-\tilde{W}/\omega\), etc. The motivation for this form was mentioned in Sec. IA, and elaborated next. To this end, it is useful to note that the derivations that led Willis [46] and Pernas-Salomón and Shmuel [52] to their non-local operators were carried out after applying the Fourier transform with respect to time[99], where in the frequency domain the cross-coupling terms are products that include the term \(-i\omega\). An ambiguity emerges when transforming back to the time domain: should \(-i\omega\) be identified with the kernel or with the time derivative of \((\tilde{g})\) ? The former leads to relations (7), and the latter to relations (12). The forthcoming analysis supports form (12), in agreement with Refs. [58, 69].

Before we proceed, we note that a similar ambiguity exists when the transform is applied with respect to the spatial translation [69]. In this case, spatial derivatives turn to products with \(i\mathbf{x}\), and the inverse transform has the same problem as with the inversion of products of \(i\omega\). We can now highlight the motivation for introducing \(\mathbf{\eta}\): since it is not derived from a potential, there is no way to “pull outside” the gradient operator in order to obtain the effective displacement field, and then mistake the effective velocity for the effective strain by multiplying and dividing by \(i\omega\) [60]. Evidently, such operations lead to different sets of effective properties, and particularly a set without Willis couplings[100]. Since clearly the velocity or strain cannot be derived from the electric potential, there is no need in introducing an “eigen electric field” in our theory[101]. Owing to the ambiguity associated with the non–local operator and the difficulty to measure the non–local cross-coupling, Milton [66, 84] recently advocated either the use of the local cross-coupling, or the use of a non-local operator that relates the displacements to the applied force; we do not pursue this notion here.

![Diagram](image)

**FIG. 2.** The body \(\Omega\) is composed of different piezoelectric materials whose constitutive response is given by Eq. (3), as illustrated at the top of the sketch. Effectively, the response of the body is non-local with additional cross-couplings, as given by Eq. (7).

### III. PASSIVITY

The term passivity has different uses in the literature. Here, it is interpreted as in Ref. [102], namely, a system is passive if there exists a positive-definite stored energy function for it, determined uniquely by its state variables, such that the power supplied to the system by external agents is always greater or equal to the rate of change of its stored energy. This requirement, in turn, poses restrictions on the constitutive parameters [56, 103]. The implications of passivity were employed in Refs. [56, 58] to determine the restrictions on Willis materials. In this section, we extend the analysis to piezoelectric materials that exhibit electro-momentum coupling, where by assuming passivity we derive restrictions on the constitutive tensors given in Eqs. (7) and (12).

We consider a piezoelectric solid of volume \(\Omega\) that is surrounded by air. Across its boundary \(\partial \Omega\) a surface charge density \(\mathbf{w}_e\) and traction \(\mathbf{t}\) are present, in addition to the volume densities \(q\) and \(\mathbf{f}\). For simplicity, eigenstrains are not considered here, bearing in mind that the effective properties to be used in the sequel are those identified using such eigenstrains. Assuming time-harmonic fields, we can express the complex rate of work done on the piezoelectric body by the mechanical and electrical sources, namely,

\[
P_c = \oint_{\partial \Omega} \frac{1}{2} \left( \mathbf{t} \cdot \mathbf{\dot{u}} + \mathbf{\phi} \mathbf{w}_e^* \right) da + \int_{\Omega} \frac{1}{2} \left( \mathbf{f} \cdot \mathbf{\dot{u}} + \mathbf{\phi} q^* \right) d\mathbf{x},
\]

such that the real part of \(P_c\) is the time-average power done by the sources [93]. Using the connections \(\mathbf{t} = \mathbf{\sigma} \cdot \mathbf{n}\) and \(\mathbf{D} \cdot \mathbf{n} = -\mathbf{w}_e\), where \(\mathbf{n}\) is a unit vector in the outward normal direction to \(\partial \Omega\), we obtain a restatement of the complex Poynting’s theorem for piezoelectric media in the settings of
the quasi-electrostatic approximation as [93]

\[
P_c = \int_{\partial \Omega} \left( \frac{\mathbf{\sigma} \cdot \mathbf{u}^\ast}{2} - \phi \mathbf{D}^\ast \right) \cdot \mathbf{n} \, dt + \int_{\Omega} \left( \frac{\mathbf{f} \cdot \mathbf{u}^\ast}{2} + \phi \mathbf{q}^\ast \right) \, dx
\]

\[
= \int_{\Omega} \left( \frac{\mathbf{\sigma} \cdot \nabla \mathbf{u}^\ast}{2} + \mathbf{p} \cdot \mathbf{u}^\ast - \frac{\mathbf{\nabla} \phi \cdot \mathbf{D}^\ast}{2} \right) \, dx. \tag{14}
\]

In the process, we have applied the divergence theorem and used the field equations (1) and (2) after the expansion of the divergence operator. The imaginary part of this volume integral relates to the total stored energy within \( \Omega \) (elastic, kinetic and electric energy) and its real part is the time-average power loss of the system. Since a passive material cannot generate energy, the inflow of power is always non-negative, and hence \( P'_c := \text{Re} P_c \) is non-negative too, where here and henceforth \((\cdot)'\) denotes \( \text{Re} \{\cdot\} \) and \((\cdot)''\) denotes \( \text{Im} \{\cdot\} \). This requirement imposes restrictions on the permitted values of the constitutive tensors, when \( P'_c \) is expressed using the generalized Willis relations. Invoking statistical homogeneity and considering plane wave solutions, we employ form (10) to write the condition on \( P'_c \) as

\[
P'_c = \frac{1}{2} \text{Re} \left\{ \int_{\Omega} \left( u^\ast_{i,j} \tilde{C}_{ijkl} u_{kl} + u^\ast_{i,k} \tilde{B}_{ijkl} \phi_k + u^\ast_{i,i} \tilde{S}_{ijkl} u_{kl} - \phi^\ast_{i,k} \tilde{B}_{ijkl} u_{kl} - \phi^\ast_{i,k} \tilde{A}_{ijkl} \phi_k - \phi^\ast_{i,k} \tilde{W}_{ijkl} u_{kl} + \right) \right\} \geq 0;
\]

here, we used the fact that \( \text{Re} \{ \mathbf{\nabla} \phi \cdot \mathbf{D}^\ast \} = \text{Re} \{ \nabla \phi^\ast \cdot \mathbf{D} \} \). The components of \( \mathbf{L} \) appearing in Eq. (15) are the transforms at \((-\mathbf{k}, \omega)\), and we note that by linearity there is no loss of generality when considering a single \( \mathbf{k} \) vector. Eq. (15) is simplified using the following relations. First, we introduce the skew-Hermitian parts of \( \tilde{\rho}, \tilde{A}, \) and \( \tilde{C} \), namely,

\[
\tilde{\rho}^{\text{SH}}_{ik} = \frac{1}{2} (\tilde{\rho}_{ik} - \tilde{\rho}^*_{ki}), \quad \tilde{A}^{\text{SH}}_{ik} = \frac{1}{2} (\tilde{A}_{ik} - \tilde{A}^*_{ki}),
\]

\[
\tilde{C}^{\text{SH}}_{ijkl} = \frac{1}{2} (\tilde{C}_{ijkl} - \tilde{C}^*_{klij}), \quad \tag{16}
\]

\[\text{to rewrite the terms Re} \{\tilde{u}^*_{i,k} \tilde{\rho}_{ik} u_k\}, \text{Re} \{\phi^\ast_{i,k} \tilde{A}_{ik} \phi_k\}, \text{and Re} \{\tilde{u}^\ast_{i,j,k} \tilde{C}_{ijkl} u_{kl}\}\]

as

\[
\text{Re} \{\tilde{u}^*_{i,k} \tilde{\rho}_{ik} u_k\} = \frac{1}{2} (\tilde{u}^*_{i,k} \tilde{\rho}_{ik} u_k + \tilde{u}^\ast_{i,k} \tilde{\rho}_{ik} u_k)
\]

\[= -i \omega \tilde{\rho}^{\text{SH}}_{ik} \tilde{u}^\ast_{i,k} u_k, \quad \tag{17a}\]

\[
\text{Re} \{\phi^\ast_{i,k} \tilde{A}_{ik} \phi_k\} = \frac{1}{2} (\phi^\ast_{i,k} \tilde{A}_{ik} \phi_k + \phi_{i,k} \tilde{A}^\ast_{ik} \phi^\ast_{i,k}
\]

\[= -i \omega \tilde{A}^{\text{SH}}_{ik} \phi^\ast_{i,k}, \quad \tag{17b}\]

\[
\text{Re} \{\tilde{u}^\ast_{i,j,k} \tilde{C}_{ijkl} u_{kl}\} = \frac{1}{2} (\tilde{u}^\ast_{i,j,k} \tilde{C}_{ijkl} u_{kl} + \tilde{u}^\ast_{i,j,k} \tilde{C}^\ast_{ijkl} u_{kl})
\]

\[= i \omega \tilde{C}^{\text{SH}}_{ijkl} \tilde{u}^\ast_{i,j,k} u_{kl}. \quad \tag{17c}\]

We also note that if

\[
\tilde{\rho}_{ik} = \tilde{\rho}_{ki}, \quad \tilde{A}_{ik} = \tilde{A}_{ki}, \quad \tilde{C}_{ijkl} = \tilde{C}_{klij},
\]

\[\text{for all} \mathbf{k}, \quad \text{then their skew-Hermitian part is equal to their imaginary part (and the Hermitian part is equal to the real part). The remaining terms can be written as}
\]

\[
\text{Re} \{\tilde{u}^\ast_{i,j,k} \tilde{B}_{ijkl} \phi_k - \phi^\ast_{i,k} \tilde{B}_{ijkl} u_k\} = \frac{i \omega}{2} \left( \tilde{B}^\ast_{ijkl} - \tilde{B}^\ast_{klij}\right) \phi^\ast_{i,j} u^\ast_{i,j} - \frac{i \omega}{2} \left( \tilde{B}^\ast_{ijkl} - \tilde{B}^\ast_{klij}\right) \phi^\ast_{i,j} u^\ast_{i,j},
\]

\[
- \omega \text{Re} \left\{ \left( i \left( \tilde{B}^\ast_{ijkl} - \tilde{B}^\ast_{klij}\right) \phi^\ast_{i,j} u^\ast_{i,j} \right) \right\} = 2 \omega \text{Re} \left\{ \phi^\ast_{i,j,k} \tilde{Q}^{\text{SH}}_{ijkl} u^\ast_{i,j}\right\}, \quad \tag{19a}\]

\[
\text{Re} \{\tilde{u}^\ast_{i,j,k} \tilde{S}_{ijkl} u_k + \tilde{S}^\ast_{ijkl} u^\ast_{i,j}\} = \frac{i \omega}{2} \left( \tilde{S}^\ast_{ijkl} + \tilde{S}^\ast_{klij}\right) \phi^\ast_{i,j} u^\ast_{i,j} - \frac{i \omega}{2} \left( \tilde{S}^\ast_{ijkl} + \tilde{S}^\ast_{klij}\right) \phi^\ast_{i,j} u^\ast_{i,j},
\]

\[
= \omega \text{Re} \left\{ - i \left( \tilde{S}^\ast_{ijkl} + \tilde{S}^\ast_{klij}\right) \phi^\ast_{i,j} u^\ast_{i,j} \right\} = 2 \omega \text{Re} \left\{ - i u^\ast_{i,j,k} \tilde{Q}^{\text{SH}}_{ijkl} u^\ast_{i,j}\right\}, \quad \tag{19b}\]

\[
\text{Re} \{\tilde{u}^\ast_{i,j,k} \tilde{W}_{ijkl} \phi_k - \phi^\ast_{i,k} \tilde{W}_{ijkl} u_k\} = \frac{i \omega}{2} \left( \tilde{W}^\ast_{ijkl} + \tilde{W}^\ast_{klij}\right) \phi^\ast_{i,j} u^\ast_{i,j} - \frac{i \omega}{2} \left( \tilde{W}^\ast_{ijkl} + \tilde{W}^\ast_{klij}\right) \phi^\ast_{i,j} u^\ast_{i,j},
\]

\[
= \omega \text{Re} \left\{ - i \left( \tilde{W}^\ast_{ijkl} + \tilde{W}^\ast_{klij}\right) \phi^\ast_{i,j} u^\ast_{i,j} \right\} = 2 \omega \text{Re} \left\{ - i \phi^\ast_{i,j,k} \tilde{W}^{\text{SH}}_{ijkl} u^\ast_{i,j}\right\}, \quad \tag{19c}\]

\[\text{for all} \mathbf{k} \text{ then} (\cdot)^{\text{QSH}} \text{ is equivalent to the imaginary part of} (\cdot), \text{while} (\cdot)^{\text{OH}} \text{ is equivalent to the real part. Using these}\]

\[
\text{If the following symmetries hold}
\]

\[
\tilde{B}^\ast_{ijkl} = \tilde{B}^\ast_{klij}, \tilde{S}^\ast_{ijkl} = \tilde{S}^\ast_{klij}, \tilde{W}^\ast_{ijkl} = \tilde{W}^\ast_{klij} \tag{20}
\]
Expression (21) depends on the strain, velocity and electric fields, which are arbitrary and independent variables, owing to the arbitrariness of the sources. Accordingly, we can recover first the conclusions of Srivastava [56] and Muhlestein et al. [58] in the limiting elastic case, by considering a configuration where the electric field vanishes, for which

\[
P'_c = \frac{\omega}{2} \int_{\Omega} \left( u_i^* \left( i \mathbf{C}^{SH}_{ijkl} u_{k,l} + 2 \text{Re} \left\{ \phi_k i \mathbf{B}^{QSH}_{ki} u_{i,j} \right\} \right) - 2 \text{Re} \left\{ u_i^* i \mathbf{B}^{QSH}_{ki} u_k \right\} - \phi_j^* i \mathbf{A}^{SH}_{ik} \phi_k \right) \mathbf{d}x \geq 0.
\]

(22)

and in terms of \( \hat{S} \)

\[
P'_c = \frac{\omega}{2} \int_{\Omega} \left( u_i^* \left( i \mathbf{C}^{SH}_{ijkl} u_{k,l} + 2 \text{Re} \left\{ u_i^* i \mathbf{S}^{QSH}_{ijkl} \right\} \right) - 2 \text{Re} \left\{ u_i^* i \mathbf{S}^{QSH}_{ijkl} u_k \right\} \right) \mathbf{d}x \geq 0.
\]

(23)

Following similar arguments, by setting the velocity to zero we obtain

\[
\int_{\Omega} u_i^* \left( i \mathbf{C}^{SH}_{ijkl} u_{k,l} \right) \mathbf{d}x \geq 0,
\]

(24)

where the case of a vanishing strain provides

\[
\int_{\Omega} u_i^* i \mathbf{P}^{SH}_{ik} u_k \mathbf{d}x \leq 0.
\]

(25)

Eqs. (24)-(25) hold for arbitrary strain and velocity fields if and only if the Hermitian [104] forms \( i \mathbf{C}^{SH} \) and \( i \mathbf{P}^{SH} \) are positive- and negative-definite, respectively. If the medium is not only passive but also lossless, then the inequalities become equalities which imply that \( \mathbf{C} \) and \( \mathbf{P} \) are Hermitian, and the real part of the actions of \( \hat{S}^{QH} \) and \( -i \mathbf{S}^{QSH} \) is negative definite. This agrees with the notion that Hermiticity implies energy conservation [105–107].

As mentioned, this analysis recovers the results of Srivastava [56] and Muhlestein et al. [58] for Willis materials (albeit there only spatially local Willis materials were analyzed). To develop the restrictions on the couplings that arise in the electroelastic setting, we first assume a combination of sources for which the only non-vanishing field is the electric field. In this setting, Eq. (21) provides

\[
\int_{\Omega} \phi_j^* i \mathbf{A}^{SH}_{ik} \phi_k \mathbf{d}x \leq 0.
\]

(26)

Since \( \nabla \phi \) is arbitrary, this condition holds if and only if the Hermitian form \( i \mathbf{A}^{SH} \) is negative-definite, and in the lossless case implies that \( \mathbf{A} \) is Hermitian, again, in agreement with the association of Hermiticity with energy conservation. If only the velocity vanishes, we have that

\[
\int_{\Omega} \left( u_i^* \left( i \mathbf{C}^{SH}_{ijkl} u_{k,l} + 2 \text{Re} \left\{ \phi_k i \mathbf{B}^{QSH}_{ki} u_{i,j} \right\} - \phi_j^* i \mathbf{A}^{SH}_{ik} \phi_k \right) \right) \mathbf{d}x \geq 0.
\]

(27)

If only the strain is zero

\[
\int_{\Omega} \left( -u_i^* i \mathbf{P}^{SH}_{ik} u_k - \phi_j^* i \mathbf{A}^{SH}_{ik} \phi_k - 2 \text{Re} \left\{ \phi_k i \mathbf{W}^{QH}_{ki} u_i \right\} \right) \mathbf{d}x \geq 0,
\]

(28)

from which we obtain

\[
-2 \text{Re} \left\{ \phi_k i \mathbf{B}^{QSH}_{ki} u_{i,j} \right\} \leq u_i^* \left( i \mathbf{C}^{SH}_{ijkl} u_{k,l} - \phi_j^* i \mathbf{A}^{SH}_{ik} \phi_k \right),
\]

(29)

\[
2 \text{Re} \left\{ \phi_k i \mathbf{W}^{QH}_{ki} u_i \right\} \leq -u_i^* i \mathbf{P}^{SH}_{ik} u_k - \phi_j^* i \mathbf{A}^{SH}_{ik} \phi_k,
\]

(30)

and the latter is replaced by

\[
-2 \text{Re} \left\{ \phi_k i \mathbf{W}^{QH}_{ki} u_i \right\} \leq -u_i^* i \mathbf{P}^{SH}_{ik} u_k - \phi_j^* i \mathbf{A}^{SH}_{ik} \phi_k,
\]

(31)

when expressed in terms of \( \hat{W} \). Eqs. (29)-(31) thus provide bounds for \( -i \mathbf{B}^{QSH} \) and \( i \mathbf{W}^{QH} \), and in the lossless case imply they are negative definite.

**IV. RECIPROCITY**

Consider a time-invariant piezoelectric body, two arbitrary time-harmonic source distributions, and denote these sources and the fields they excite by superscripts 1 and 2, respectively. The body is reciprocal if the power that distribution 1 does along the fields excited by distribution 2 is equal to the power that distribution 2 does along the fields excited by distribution 1. A schematic illustration of this property is given in Fig. 3.

The principle of reciprocity is independent of the level of isotropy and homogeneity of the body [108], however it requires that at each point the symmetry conditions

\[
A_{ij} (x) = A_{ji} (x), \quad C_{ijkl} (x) = C_{klij} (x)
\]

(32)

are satisfied [109]. In the homogenization process of a heterogeneous reciprocal body, it is thus required that the resultant effective properties will also satisfy the reciprocity relation. Muhlestein et al. [58] have shown that this requirement imposes the following conditions on the effective properties of (spatially) local Willis materials

\[
\bar{\mathbf{p}}_k = \hat{\mathbf{p}}_k, \quad \bar{\mathbf{S}}_{ijk}^* = \hat{\mathbf{S}}_{kji}, \quad \bar{\mathbf{C}}_{ijkl} = \hat{\mathbf{C}}_{klij},
\]

(33)

where the symmetry between \( \mathbf{S}^\dagger \) and \( \mathbf{S} \) is transmitted to the modified couplings, namely,

\[
\bar{\mathbf{S}}_{ijk}^\dagger = \hat{\mathbf{S}}_{kji}.
\]

(34)

Here, we first derive the generalization of these conditions to local materials exhibiting the piezoelectric behavior, and then analyze the general (non-local) case. Our departure point towards this end is the equations that govern the
response of the body when subjected to two independent and arbitrary distributions of force \( \{ f^{(1)}_i, f^{(2)}_i \} \) and charge densities \( \{ q^{(1)}, q^{(2)} \} \), namely,

\[
\begin{pmatrix}
\sigma_{ij}^{(1)} - p^{(1)}_i \\
\frac{1}{2} \nabla^2 q^{(1)}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial f^{(1)}_j}{\partial x_i} \\
0
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{35}
\]

and

\[
\begin{pmatrix}
\sigma_{ij}^{(2)} - p^{(2)}_i \\
\frac{1}{2} \nabla^2 q^{(2)}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial f^{(2)}_j}{\partial x_i} \\
0
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{36}
\]

We denote the resultant displacement and electric potential fields by \( w^{(1)} := \{ u^{(1)}_i, \phi^{(1)} \} \) and \( w^{(2)} := \{ u^{(2)}_i, \phi^{(2)} \} \), respectively. Next, we left-multiply Eqs. (35) and (36) by \( w^{(2)} \) and \( w^{(1)} \), respectively. The difference between the two products is

\[
\begin{pmatrix}
\sigma_{ij}^{(1)} u^{(2)}_i - \sigma_{ij}^{(2)} u^{(1)}_i \\
\phi^{(1)} D^{(2)} - \phi^{(2)} D^{(1)}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial f^{(2)}_j}{\partial x_i} \\
0
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{37}
\]

which can be rearranged as

\[
\begin{array}{c}
p^{(1)}_i u^{(2)}_i - p^{(2)}_i u^{(1)}_i + \sigma_{ij}^{(1)} u^{(2)}_{ij} - \sigma_{ij}^{(2)} u^{(1)}_{ij} \\
\phi^{(2)} D^{(1)} - \phi^{(1)} D^{(2)}
\end{array}
= \Delta P, \tag{38}
\]

using the identities

\[
\begin{align*}
\sigma_{ij}^{(1)} u^{(2)}_i - \sigma_{ij}^{(2)} u^{(1)}_i &= \left\{ \sigma_{ij}^{(1)} (u^{(2)}_i - u^{(1)}_i) \right\}, \\
\sigma_{ij}^{(1)} u^{(2)}_i - \sigma_{ij}^{(2)} u^{(1)}_i &= \left\{ \sigma_{ij}^{(1)} (u^{(2)}_i - u^{(1)}_i) \right\}, \tag{39a}
\end{align*}
\]

\[
\begin{align*}
\phi^{(2)} D^{(1)} - \phi^{(1)} D^{(2)} &= \left\{ \phi^{(2)} (D^{(1)} - D^{(2)}) \right\}, \\
\phi^{(2)} D^{(1)} - \phi^{(1)} D^{(2)} &= \left\{ \phi^{(2)} (D^{(1)} - D^{(2)}) \right\}. \tag{39b}
\end{align*}
\]

where

\[
\Delta P = \left\{ \sigma_{ij}^{(1)} u^{(2)}_i + D^{(1)} - \phi^{(1)} \right\}, \\
\Delta P = \left\{ \sigma_{ij}^{(1)} u^{(2)}_i + D^{(1)} - \phi^{(1)} \right\}. \tag{40}
\]

The term \( \Delta P \) is the differential form of the difference between the power that distribution 1 does along the fields excited by distribution 2 and the power that distribution 2 does along the fields excited by distribution 1, hence vanishes if the body is reciprocal. The global form is obtained by volume integration, conversion of the first and third terms in the integral into surface integrals using the divergence theorem, and identification of the boundary sources \( t^{(l)}_i = a^{(l)}_i n_j \) and \( -w^{(l)}_e = D^{(l)} n_j \) of distribution \( l \).

We now expand the terms on the left side of Eq. (38) using the effective constitutive equations (10) in their spatially local
form \( (\mathbf{x} = 0) \) to obtain

\[
\begin{align*}
\hat{S}_{i j k} - \hat{S}_{k i j} &= \left( u_{k}^{(1)} u_{i j}^{(2)} - u_{k}^{(2)} u_{i j}^{(1)} \right) + \left( \hat{\rho}_{k} \hat{\rho}_{i j} \right) u_{k}^{(1)} u_{k}^{(1)} \\
\hat{W}_{i k} - \hat{W}_{k i} &= \left( \phi_{k}^{(1)} \phi_{i j}^{(2)} - \phi_{k}^{(2)} \phi_{i j}^{(1)} \right) + \left( \hat{\Lambda}_{i k} - \hat{\Lambda}_{k i} \right) \phi_{k}^{(1)} \phi_{k}^{(1)} \\
\hat{B}_{i j k} - \hat{B}_{k i j} &= \left( \phi_{k}^{(1)} u_{i j}^{(2)} - \phi_{k}^{(2)} u_{i j}^{(1)} \right) + \left( \hat{\Lambda}_{i k} - \hat{\Lambda}_{k i} \right) \phi_{k}^{(1)} \phi_{k}^{(1)} \\
\hat{C}_{i j k l} - \hat{C}_{k l i j} &= \left( u_{k l}^{(1)} u_{i j}^{(2)} - u_{k l}^{(2)} u_{i j}^{(1)} \right) + \left( \hat{\Lambda}_{i k} - \hat{\Lambda}_{k i} \right) u_{k l}^{(1)} u_{k l}^{(1)} = 0.
\end{align*}
\]

The arbitrariness of the sources implies that the strain, electric and velocity fields are arbitrary too. Accordingly, for Eq. (41) to hold for any \( \nabla \mathbf{u}, \mathbf{u}, \) and \( \nabla \phi \), the effective constitutive tensors must satisfy

\[
\hat{\Lambda}_{i k} = \hat{\Lambda}_{k i}, \quad \hat{B}_{i j k} = \hat{B}_{k i j}, \quad \hat{W}_{k i} = \hat{W}_{i k},
\]

in addition to restrictions (33). It is clear that the modified couplings \( \hat{W}^{\dagger} \) and \( \hat{W} \) exhibit the same symmetry between \( \mathbf{W}^{\dagger} \) and \( \mathbf{W} \), such that

\[
\hat{W}^{\dagger}_{k i} = \hat{W}_{i k}.
\]

In view of Eqs. (33) and (42), we can now revisit the conclusions in Sec. (III) and replace the conditions on the Hermitian and skew-Hermitian parts of the tensors in the long-wavelength limit by the conditions on their real and imaginary parts, respectively.

The foregoing analysis was obtained in the long-wavelength limit. We derive next the general result for arbitrary wavelengths, and show that restrictions (33) and (42) are its specialization. This will be carried out by showing that if the body is reciprocal, then the governing equations are self-adjoint, and in turn so is the Green function, which renders the constitutive operator \( \mathcal{L} \) self-adjoint too. The latter property was remarked only in passing by Willis [46, 48] and Pernas-Salomón and Shmuel [52] in their respective problems, perhaps because the notion that reciprocity and self-adjointness are closely related is somewhat known [110, 111]. However, since the self-adjoint structure of the constitutive operator clearly depends on the definition of the effective description, it is discussed in more detail here.

To proceed, it is useful to employ the formulation of Barnett and Lothe [112], who formulated the piezoelectric problem in a generalized space using the following definitions

\[
K_{\alpha \beta j} = \begin{cases} 
C_{\alpha \beta j}, & \alpha, \beta \in \{1, 2, 3\}, \\
B_{\alpha \beta j}^{\dagger}, & \beta = 4, \alpha \in \{1, 2, 3\}, \\
B_{\beta j}^{\dagger}, & \alpha = 4, \beta \in \{1, 2, 3\}, \\
-A_{\alpha j}, & \alpha = \beta = 4,
\end{cases}
\]

\[
\Lambda_{\alpha \beta} = \begin{cases} 
\delta_{\alpha \beta}, & \alpha, \beta \in \{1, 2, 3\}, \\
0, & \alpha = \beta = 4,
\end{cases}
\]

\[
b_{\alpha} = \begin{cases} 
f_{\alpha}, & \alpha \in \{1, 2, 3\}, \\
-q, & \alpha = 4,
\end{cases}
\]

where the range of latin subscripts is limited to \( \{1, 2, 3\} \). The unified governing equations in terms of \( \mathbf{K}, \Lambda \) and \( \mathbf{b} \) read in index notation

\[
\left\{ K_{\alpha \beta j} \mathbf{w}_{\beta, j} \right\}_{, \alpha} + \rho \mathbf{w}^{2} \Lambda_{\alpha \beta} \mathbf{w}_{\beta} = -b_{\alpha},
\]

which define the components \( G_{\beta \gamma}(x, X) \) of the Green matrix via

\[
\left\{ K_{\alpha \beta j} G_{\beta \gamma}(x, X) \right\}_{, \alpha} + \rho \mathbf{w}^{2} \Lambda_{\alpha \beta} G_{\beta \gamma} = -\delta_{\gamma \delta} (x - X),
\]

where \( \delta(x - X) \) is the Dirac delta. Eq. (46) spells out explicitly the components of the symbolic Eq. (9) by Pernas-Salomón and Shmuel [52]. In the Appendix we describe the standard procedure to obtain the adjoint equations and corresponding adjoint Green tensor, and verify its components satisfy

\[
G_{\gamma \beta}^{\dagger}(x, X) = G_{\beta \gamma}(X, x);
\]

we thus recover the known result that if the body satisfies \( \rho^{*} = \rho \) and \( K_{\alpha \beta j}^{\dagger} = K_{\alpha \beta j} \), where

\[
K_{\alpha \beta j}^{\dagger} = \begin{cases} 
C_{\alpha \beta j}^{*}, & \alpha, \beta \in \{1, 2, 3\}, \\
\beta = 4, \alpha \in \{1, 2, 3\}, \\
\alpha = 4, \beta \in \{1, 2, 3\}, \\
-A_{\alpha j}^{*}, & \alpha = \beta = 4,
\end{cases}
\]

which is the case by virtue of Eq. (4), then the piezoelectric problem is self-adjoint [113, 114]. As explained in the Appendix, in this case \( G_{\gamma \beta}^{\dagger}(x, X) = G_{\beta \gamma}(x, X) \), which together with the previous result implies that

\[
G_{\gamma \beta}(x, X) = G_{\beta \gamma}^{\dagger}(X, x).
\]

We recall next the expression for the kernel \( \hat{L} \) obtained by Pernas-Salomón and Shmuel [52], namely,

\[
\hat{L} = \langle L \rangle - \left\langle \mathbf{L} \left( \mathbf{BG} \right)^{T} \mathbf{L} \right\rangle + \left\langle \mathbf{L} \left( \mathbf{BG}^{T} \right)^{T} \left( \mathbf{BG} \right)^{T} \mathbf{L} \right\rangle;
\]

the symbolic matrix formulation for \( \hat{L} \) translates to the following components

\[
\hat{L}_{\alpha \beta j} = \begin{cases} 
C_{\alpha \beta j}, & \alpha, \beta \in \{1, 2, 3\}, \\
B_{\alpha \beta j}^{*}, & \beta = 4, \alpha \in \{1, 2, 3\}, \\
B_{\beta j}^{*}, & \alpha = 4, \beta \in \{1, 2, 3\}, \\
-A_{\alpha j}, & \alpha = \beta = 4,
\end{cases}
\]

Inspecting the components of Eq. (50), and employing the symmetries of \( \mathbf{G}, \mathbf{K} \) and the fact that \( L = L^{*} \), verify the symmetry

\[
\hat{L}_{\alpha \beta j}(x, X) = \hat{L}_{\beta, j \alpha}(X, x),
\]
where terms associated with the conventional couplings \(\hat{A}, \hat{\beta}, \hat{C}\), and \(\hat{\rho}\) and the modified couplings \(\hat{W}, \hat{W}'\), \(\hat{S}\) and \(\hat{S}'\) also satisfy

\[
\hat{L}_{\alpha i j} (\mathbf{x}, \mathbf{X}) = \hat{L}_{\beta j i}^* (\mathbf{x}, \mathbf{X}),
\]

while the couplings of Willis type in their original form satisfy

\[
\hat{L}_{\alpha i j} (\mathbf{x}, \mathbf{X}) = -\hat{L}_{\beta j i}^* (\mathbf{x}, \mathbf{X}),
\]

\[
\alpha \in \{1, 2, 3, 4\}, \quad \beta = 5 \& \beta \in \{1, 2, 3, 4\}, \quad \alpha = 5. \tag{54}
\]

It is important to note that symmetry (52)—which delivers the self-adjoint property of \(\hat{L}\) as explained later—originates from the symmetry \(\hat{L} = \hat{L}^T\), and does not require the composite to be lossless; symmetries (53) and (54) originate from the assumption that \(\hat{L}\) is also real.

Interestingly, the modified cross-couplings \(\hat{S}, \hat{S}'\), \(\hat{W}, \hat{W}'\) and \(\hat{S}', \hat{W}^\dagger\) are related via the same symmetry as the conventional couplings, i.e., Eq. (53). Combining Eqs. (52)-(54) implies that the conventional couplings and the modified cross-couplings are real, while those of Willis type are pure imaginary. Together with the fact that according to Eq. (46) the Green tensor is an even function of \(\Omega\) [115], this result implies that in the space-time domain \(\hat{A}, \hat{\beta}, \hat{C}, \hat{\rho}\) and \(\hat{S}, \hat{S}', \hat{W}\) and \(\hat{W}'\) are real—as they should since they relate real physical quantities. By contrast, in the space-time domain the cross-couplings \(\hat{S}, \hat{S}', \hat{W}\) and \(\hat{W}'\) are pure imaginary—an unphysical result. This observation agrees with the analysis of Norris et al. [68] in the purely elastic case, who showed that \(\hat{C}\) and \(\hat{\rho}\) are real in the space-time domain, while \(\hat{S}\) and \(\hat{S}'\) are pure imaginary.

For statically homogeneous media, we can employ the Fourier transform with respect to the translation \(\mathbf{x} - \mathbf{X}\), and write these symmetries using indices in the transformed domain as

\[
\hat{C}_{kij}^{*} (\mathbf{k}, \omega) = \hat{C}_{ijk} (\mathbf{k}, \omega) = \hat{C}_{kij} (\mathbf{k}, \omega), \tag{55a}
\]

\[
\hat{B}_{kij}^{*} (\mathbf{k}, \omega) = -\hat{B}_{ijk} (\mathbf{k}, \omega) = \hat{B}_{kij} (\mathbf{k}, \omega), \tag{55b}
\]

\[
\hat{\Lambda}_{kij}^{*} (\mathbf{k}, \omega) = -\hat{\Lambda}_{ijk} (\mathbf{k}, \omega) = \hat{\Lambda}_{ijk} (\mathbf{k}, \omega), \tag{55c}
\]

\[
\hat{\rho}_{kij}^{*} (\mathbf{k}, \omega) = \hat{\rho}_{ijk} (\mathbf{k}, \omega) = \hat{\rho}_{ijk} (\mathbf{k}, \omega), \tag{55d}
\]

for terms associated with conventional couplings; the modified cross-couplings terms satisfy the same form of symmetries, such that

\[
\hat{S}_{kij}^{*} (\mathbf{k}, \omega) = \hat{S}_{ikj}^{*} (\mathbf{k}, \omega) = \hat{S}_{kij} (\mathbf{k}, \omega), \tag{56a}
\]

\[
\hat{W}_{kij}^{*} (\mathbf{k}, \omega) = \hat{W}_{ijk}^{*} (\mathbf{k}, \omega) = \hat{W}_{kij} (\mathbf{k}, \omega), \tag{56b}
\]

while when they are in their original form they satisfy

\[
-\hat{S}_{kij} (\mathbf{k}, \omega) = \hat{S}_{ijk}^{*} (\mathbf{k}, \omega) = \hat{S}_{kij} (\mathbf{k}, \omega), \tag{57a}
\]

\[
-\hat{W}_{kij}^{*} (\mathbf{k}, \omega) = \hat{W}_{ijk}^{*} (\mathbf{k}, \omega) = \hat{W}_{kij} (\mathbf{k}, \omega). \tag{57b}
\]

The symmetry between \(\hat{W}\) and \(\hat{W}'\) is shown in detail in the Appendix. It is clear that in the limit \(\mathbf{k} = \mathbf{0}\), the symmetries (55) and (57) recover symmetries (33) and (42). Eqs. (55)-(57) also endow the adjoint notion to the symbol \(^\dagger\) for the non-local operators \(\mathcal{S}^\dagger\) and \(\mathcal{W}^\dagger\), since these symmetries imply that

\[
\int_{\Omega} \mathcal{S}^\dagger (\mathbf{u} (\mathbf{X})) \cdot \nabla \mathbf{u} (\mathbf{x}) \, d\mathbf{x} = \int_{\Omega} \mathbf{u} (\mathbf{x}) \cdot \mathcal{S}^\dagger (\nabla \mathbf{u} (\mathbf{X})) \, d\mathbf{x}, \tag{58a}
\]

\[
\int_{\Omega} \mathcal{W}^\dagger (\mathbf{u} (\mathbf{X})) \cdot \nabla \phi (\mathbf{x}) \, d\mathbf{x} = \int_{\Omega} \mathbf{u} (\mathbf{x}) \cdot \mathcal{W}^\dagger (\nabla \phi (\mathbf{X})) \, d\mathbf{x}. \tag{58b}
\]

as well as for \(\mathcal{B}^\dagger\), which satisfies

\[
\int_{\Omega} \mathcal{B}^\dagger (\mathbf{u} (\mathbf{X})) \cdot \nabla \phi (\mathbf{x}) \, d\mathbf{x} = \int_{\Omega} \mathbf{u} (\mathbf{x}) \cdot \mathcal{B}^\dagger (\nabla \phi (\mathbf{X})) \, d\mathbf{x}; \tag{59}
\]

the non-local operators \(\mathcal{R}, \mathcal{A}, \mathcal{C}\) are self-adjoint in the sense above. At the cost of repetition, we clarify that the equality between the middle- and most-right terms in Eqs. (55)-(57) was obtained by relying only on the fact that \(\hat{L} = \hat{L}^T\). This property leads to a self-adjoint effective operator \(\mathcal{L}\) in the sense above, and specifically render \(\mathcal{S}^\dagger\) and \(\mathcal{W}^\dagger\) (resp. \(\mathcal{R}^\dagger\) and \(\mathcal{A}^\dagger\)) the adjoints of \(\mathcal{S}\) and \(\mathcal{W}\) (resp. \(\mathcal{R}\) and \(\mathcal{A}\)); the same goes for \(\mathcal{B}^\dagger\) and \(\mathcal{C}\) via Eq. (59). Otherwise, it would by replaced to justify the \(^\dagger\) notation to distinguish them from the adjoint operators.

We further clarify that the equality between the left- and middle terms in Eqs. (55)-(57) relies only on the fact that in the frequency domain the properties of the composite satisfy \(\hat{L} = \hat{L}^+\), or in other words they are Hermitian. This does not necessarily imply that \(\hat{L}\) is symmetric (although it can be), and immediately satisfies Eqs. (24)-(26) as equalities. A case where \(\hat{L} = \hat{L}^+\) and \(\hat{L} \neq \hat{L}^T\) corresponds to a non-reciprocal medium whose losses are compensated by the energy it generates, such that on average the material is lossless and passive (no energy loss or gain).

V. CAUSALITY

The principle of causality states that an effect must follow its cause. This principle implies the analyticity of the response functions of linear systems and vice versa, namely, analyticity implies causality [116, 117]. With the interpretation of the constitutive properties of linear materials as response functions, causality through analyticity allows relations between their real and imaginary parts of their (time) Fourier transforms. These relations were first obtained in electromagnetics for the permeability and permittivity tensors, where they are known as the Kramers-Krönig relations [88–91]. This concept was later on applied in other branches of physics—and specifically in mechanics—to obtain conditions on the pertinent constitutive properties [56, 118]. Alù [55, 67] has shown that in certain cases the bianisotropic tensor is essential for respecting causality in passive media. Indeed, some of the electromagnetic homogenization schemes from which this cross-coupling tensor is absent violate causality in such media [1, 119, 120]. Analogously, Sieck et al. [69] recognized the need in Willis coupling to satisfy causality by the effective constitutive properties in elastodynamics. We emphasize
that the information from the Kramers-Krönig relations is limited for active media, since it is possible to realize anomalous responses for real frequencies (such as antiresonance) using suitable causal polynomials, see for example Ref. [121]. To get useful results, it is thus necessary to couple causality with passivity [122].

In this Sec., we develop the restrictions placed by causality on the effective properties (3), i.e., when microscopically the medium exhibits the intrinsic piezoelectric effect, and macroscopically exhibits also the effective electro-momentum coupling. The framework developed in Ref. [52] constitutes a platform to carry out this task with respect to the effective operator \( L \), which we recall is non-local both in space and time. To facilitate the analysis, we here focus on the long-wavelength limit \( \kappa = 0 \), and neglect spatially non-local effects on causality [123]. Accordingly, we omit the spatial dependency of the fields in Eq. (8), and rewrite it as

\[
h(t) = \int_{-\infty}^{t} \hat{L}(t-T)g(T)dT, \tag{60}\]

bearing in mind it holds at each material point. Let \( \tau = t-T \); causality implies that \( \hat{L} \) must satisfy

\[
\hat{L}(\tau) = 0 \text{ for } \tau < 0. \tag{61}\]

The approach taken here to relate the real and imaginary parts of its Fourier transform is standard, and employs the Plemelj formulas see., e.g., Ref. [124]. This approach is summarized next to provide a self-contained analysis. As discussed by Nussenzveig [124], certain assumptions regarding \( \hat{L}(\tau) \) are required in order for \( \hat{L}'(\omega) \) and \( \hat{L}''(\omega) \) to be related. In view of Eq. (61), Eq. (11) obtains the from

\[
\hat{L}(\omega) = \int_{0}^{\infty} \hat{L}(\tau)e^{i\omega\tau}d\tau, \tag{62}\]

where the integral is only over \( \mathbb{R}_+ \), implying that \( \hat{L}(\omega) \) has an analytic continuation in the upper half of the complex plane. If we further assume at first that \( \hat{L}(\tau) \) is square integrable, then through the Parseval-Plancherel theorem, we have that \( \hat{L}(\omega) \) is square integrable along any line in the upper half of the complex plane that is parallel to the real axis, such that [124]

\[
\lim_{\alpha \to \pm \infty} \hat{L}(\omega' + i\omega'') = 0, \quad \omega'' > 0. \tag{63}\]

This property is employed in the application of Cauchy’s integral formula to a closed curve \( \Gamma \) about an arbitrary point \( \omega_0 \) in the upper half of the complex plane (\( \omega'' > 0 \))

\[
\hat{L}(\omega_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\hat{L}(z)}{z-\omega_0}dz, \tag{64}\]

in order to show it reduces to an integration along the real axis

\[
\hat{L}(\omega_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{L}(z)}{z-\omega_0}dz, \quad \omega'' = 0. \tag{65}\]

The case of real \( \omega_0 \) is obtained using a closed contour that avoids \( \omega_0 \) by a semi-circle of radius \( \epsilon \), and taking the limit \( \epsilon \to 0 \) to show that

\[
\hat{L}(\omega) = \frac{1}{i\omega} \int_{-\infty}^{\infty} \frac{\hat{L}(z)}{z-\omega_0}dz, \quad \omega'' = 0, \tag{66}\]

where

\[
\int_{-\infty}^{\infty} = \lim_{\epsilon \to 0} \left( \int_{-\infty}^{0} - \int_{\infty}^{0} \right) \tag{67}\]

denotes the Cauchy’s principal value. The real and imaginary parts of Eq. (66) provide the following relations

\[
\hat{L}'(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{L}''(z)}{z-\omega}dz, \tag{68a}\]

\[
\hat{L}''(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{L}'(z)}{z-\omega}dz \tag{68b}\]

between the \( \hat{L}'(\omega) \) and \( \hat{L}''(\omega) \) at any real frequency \( \omega \). An alternative form of relations (68) is obtained using the fact that \( \hat{L}(t) \) is real, and hence

\[
\hat{L}'(\omega) = \left( \int_{0}^{\infty} \hat{L}(t)e^{i\omega t}dt \right)' = \int_{0}^{\infty} \hat{L}(t)e^{-i\omega t}dt = \hat{L}(-\omega), \tag{69}\]

leading to

\[
\hat{L}'(\omega) = \hat{L}'(-\omega), \tag{70a}\]

\[
\hat{L}''(\omega) = -\hat{L}''(-\omega). \tag{70b}\]

Employing these symmetries leads to

\[
\hat{L}'(\omega) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\omega \hat{L}''(z)}{z^2-\omega^2}dz, \tag{71a}\]

\[
\hat{L}''(\omega) = -\frac{2}{\pi} \int_{0}^{\infty} \frac{\omega \hat{L}'(z)}{z^2-\omega^2}dz. \tag{71b}\]

We examine next the case where \( \hat{L}(\omega) \) is not square integrable but a bounded function. In this case, property (63) that is needed to obtain Eq. (65) no longer holds, and consequently the relation between \( \hat{L}'(\omega) \) and \( \hat{L}''(\omega) \) can be determined only up to an arbitrary real constant. To determine this constant, a knowledge of the value of \( \hat{L}(\omega) \) at some real frequency is needed. Say \( \hat{L}(\omega) \) is differenciable and known at \( \omega_0 \); then, we can repeat the procedure that led to Eqs. (68) and (71), only now we replace \( \hat{L}(\omega) \) by the function

\[
\frac{\Delta \hat{L}}{\Delta \omega}(\omega) := \frac{\hat{L}(\omega) - \hat{L}(\omega_0)}{\omega - \omega_0}, \tag{72}\]
since it is bounded for \( \omega \rightarrow \omega_0 \), analytic in the upper half of the complex plane and square integrable. If we further assume that \( \omega_0 \rightarrow \infty \), the end result can be put in the form

\[
\hat{L}'(\omega) = \frac{2}{\pi \sqrt{\omega}} \int_0^\infty \frac{\omega \hat{L}''(z)}{z^2 - \omega^2} \, dz + \hat{L}'(\infty), \tag{73a}
\]

\[
\hat{L}''(\omega) = -\frac{2}{\pi \sqrt{\omega}} \int_0^\infty \frac{\omega \hat{L}'(z)}{z^2 - \omega^2} \, dz, \tag{73b}
\]

cf. Eq. (4.7) in Ref. [58]. Eq. (73) should hold for all the tensor-valued (history) functions that comprise \( \hat{L}(\omega) \), i.e., \( \hat{A}, \hat{B}, \hat{B}^\dagger, \hat{C} \), as well as \( \hat{S}, \hat{W} \), their adjoints and their modified versions \( \tilde{S} \) and \( \tilde{W} \). This conclusion thus generalizes previous works by requiring that the electro-momentum tensor is also subjected to relations of the Kramer-Krönig type, as expected. We recall that Sec. IV showed that the Fourier transforms of the constitutive tensors of reciprocal, passive and lossless media are all real, including the modified cross-couplings \( \hat{S}, \hat{S}^\dagger, \hat{W} \) and \( \hat{W}^\dagger \), and excluding \( \tilde{S}, \tilde{S}^\dagger, \tilde{W} \) and \( \tilde{W}^\dagger \) which are pure imaginary. When this conclusion is combined with conditions (73), we obtain in the long-wavelength limit of reciprocal, passive and lossless media that

\[
\hat{A}(\omega) = \hat{A}'(\infty), \quad \hat{B}(\omega) = \hat{B}'(\infty), \quad \hat{B}^\dagger(\omega) = \hat{B}^{\dagger'}(\infty),
\]

\[
\hat{C}(\omega) = \hat{C}'(\infty), \quad \hat{p}(\omega) = \hat{p}'(\infty), \tag{74}
\]

and similarly for the modified cross-couplings \( \tilde{S}, \tilde{S}^\dagger, \tilde{W} \) and \( \tilde{W}^\dagger \)

\[
\tilde{S}(\omega) = \tilde{S}'(\infty), \quad \tilde{S}^\dagger(\omega) = \tilde{S}^{\dagger'}(\infty),
\]

\[
\tilde{W}(\omega) = \tilde{W}'(\infty), \quad \tilde{W}^\dagger(\omega) = \tilde{W}^{\dagger'}(\infty), \tag{75}
\]

where notably, the couplings of Willis type in their original representation must be null. This is clear from Eq. (73b), as the integrand in the right side is identically zero since these couplings are pure imaginary, which then implies that the left side—which is their imaginary part—also vanishes.

VI. CLOSURE

Piezoelectric- and piezomagnetic materials exhibit intrinsic coupling with non-mechanical fields. Recently, it was shown that the effective response of composites made of such constituents exhibit additional cross-couplings that are absent from the response of the constituents, and are of Willis type [52]. These meta-properties not only enlarge the design space for wave manipulation, but also reflect a novel mechanism to actively manipulate waves by application of electric stimuli. The recent development of such generalized Willis materials comes with the question: What are the mathematical restrictions that their constitutive properties should satisfy in order to respect passivity, reciprocity and causality? Through this study, we expect to promote the use of these metamaterials for wave manipulation in the same way that corresponding elastic studies promoted the use of Willis materials in purely mechanical devices [57, 78].

By adapting standard methodologies used in electromagnetics, elastodynamics and mathematics [56, 58, 105, 124], we arrived at the following findings. From passivity we obtained several inequality conditions on the skew-Hermitian and Hermitian parts of the Fourier transforms of the effective properties. From reciprocity we found certain symmetry and adjoint relations that the effective operator satisfies. Causality delivers relations of the Kramer-Kröning type between the real and imaginary parts of the operator in the frequency domain. A summary of the mathematical restrictions are given in Tab. I. Our results generalize those reported by Srivastava [56] and Muhlestein et al. [58] for local Willis material.

ACKNOWLEDGMENTS

We thank Graeme Milton for enriching discussions on non-locality and uniqueness, his insights on causality, and for useful references. We thank Doron Shilo for his observation regarding the similarity with the broken inversion symmetry at the atomic scale in piezoelectric materials. This research was supported by the Israel Science Foundation, funded by the Israel Academy of Sciences and Humanities (Grant no. 2061/20), the United States-Israel Binaional Science Foundation (Grant no. 2014358), and Ministry of Science and Technology (grant no. 880011).

APPENDIX A. DERIVATIONS RELATED TO THE ADJOINT OPERATOR AND GREEN TENSOR

The left side of Eq. (45) defines the action of an operator \( \mathcal{M} \) on the vector field \( \mathbf{w} \). Accordingly, the left side of Eq. (45) is the \( \alpha \)-component of the action of \( \mathcal{M} \) on the vector field \( \mathbf{G}(\gamma) \), whose components are \( \mathbf{G}_{\beta}(\gamma), \beta = 1, 2, 3, 4 \). The adjoint operator \( \mathcal{M}^\dagger \) is defined via the Green’s identity

\[
\langle \mathcal{M}(\mathbf{w}), \mathbf{v} \rangle = \left. \int_\Omega \mathcal{M}(\mathbf{w}) \cdot \mathbf{v} \, d\mathbf{x} \right|_{\partial \Omega} = \mathbf{B.T.} \tag{A.1}
\]

where

\[
\langle \mathcal{M}(\mathbf{w}), \mathbf{v} \rangle := \int_\Omega \mathcal{M}(\mathbf{w}) \cdot \mathbf{v} \, d\mathbf{x} = \int_\Omega \{ \mathcal{M}(\mathbf{w}) \alpha \} v_\alpha \, d\mathbf{x}, \tag{A.2}
\]

and B.T. denotes a surface integral of some bilinear function of \( \mathbf{w}, \mathbf{v} \) and their derivatives, where the superscript * denotes complex conjugate operation. Setting \( \mathbf{w} = w^{(1)} \) and \( \mathbf{v} = w^{(2)} \), we obtain...
The mathematical restrictions that the effective description of piezoelectric composites with generalized Willis couplings satisfy, owing to reciprocity and passivity. The restrictions that result from causality are of the Kramer-Krönig type, namely, 

\[ \tilde{\mathcal{L}}''(\omega) = -\frac{1}{\pi} \int \frac{\delta(\omega)}{\omega - \omega'} \, dz. \]  

Table I. The mathematical restrictions that the effective description of piezoelectric composites with generalized Willis couplings satisfy, owing to reciprocity and passivity. The restrictions that result from causality are of the Kramer-Krönig type, namely, 

\[ \tilde{\mathcal{L}}''(\omega) = -\frac{1}{\pi} \int \frac{\delta(\omega)}{\omega - \omega'} \, dz. \]  

using integration by parts, where the boundary terms that result in the process are indeed bilinear functions of \( w, v \), and were omitted from Eq. (A.3) for brevity. This identifies \( \mathcal{M}^\dagger \) with the adjoint equations

\[ K^\dagger_{\alpha \beta} = \begin{cases} C_{T^{\alpha \beta}} & \alpha, \beta \in \{1, 2, 3\}, \\ B_{T^{\alpha \beta}} & \beta = 4, \alpha \in \{1, 2, 3\}, \\ B_{T^{\beta \alpha}} & \alpha = 4, \beta \in \{1, 2, 3\}, \\ -A_{T^{\alpha \beta}} & \alpha = \beta = 4. \end{cases} \]  

Accordingly, the components of the adjoint green matrix \( G^\dagger_{\alpha \gamma}(x, X) \) are defined by

\[ \left\{ K^\dagger_{\alpha \beta} G^\dagger_{\gamma \mu} \right\}_{\mu} + \rho^\ast \omega^2 \lambda_{\alpha \beta} \rho_{\gamma} = -\delta_{\alpha \gamma} \delta(x-X). \]  

Following the standard procedure, we set \( w_{\alpha}(x, X) = G_{\alpha(\gamma)}(x, X) \) and \( v_{\alpha}(x, X) = G^\dagger_{\alpha(\beta)}(x, X) \), and employ Eqs. (45)-(A.1), (A.4) and (A.6) to show that

\[ \sum_{\mu} \left\{ K^\dagger_{\alpha \beta} w_{\mu} \right\}_{\mu} + \rho^\ast \omega^2 \lambda_{\alpha \beta} w_{\beta} = m_{\alpha}. \]  

| Property                      | Real-space Reciprocity | Fourier-space Reciprocity | Passivity                  |
|-------------------------------|------------------------|----------------------------|----------------------------|
| Elasticity                    | \( \tilde{C}(x, X) = \tilde{C}^T(X, x) \) | \( \tilde{C}(\kappa) = \tilde{C}^T(-\kappa) \) | \( i\tilde{C}^SH \) positive definite |
| Mass density                  | \( \tilde{\rho}(x, X) = \tilde{\rho}^T(X, x) \) | \( \tilde{\rho}(\kappa) = \tilde{\rho}^T(-\kappa) \) | \( i\tilde{\rho}^SH \) negative definite |
| Willis Coupling               | \( \tilde{S}^T(x, X) = \tilde{S}^T(X, x) \) | \( \tilde{S}^T(\kappa) = \tilde{S}^T(-\kappa) \) | \( 2\text{Re}\{i\tilde{S}^OSH\} \) bounded |
| Modified Willis coupling      | \( \tilde{S}^T(x, X) = \tilde{S}^T(X, x) \) | \( \tilde{S}^T(\kappa) = \tilde{S}^T(-\kappa) \) | \( 2\text{Re}\{i\tilde{S}^OSH\} \) bounded |
| Permittivity                  | \( \tilde{A}(x, X) = \tilde{A}^T(X, x) \) | \( \tilde{A}(\kappa) = \tilde{A}^T(-\kappa) \) | \( i\tilde{A}^SH \) negative definite |
| Piezoelectric                 | \( \tilde{B}^T(x, X) = \tilde{B}^T(X, x) \) | \( \tilde{B}^T(\kappa) = \tilde{B}^T(-\kappa) \) | \( 2\text{Re}\{i\tilde{B}^OSH\} \) bounded |
| Electro-momentum coupling     | \( \tilde{W}^T(x, X) = \tilde{W}^T(X, x) \) | \( \tilde{W}^T(\kappa) = \tilde{W}^T(-\kappa) \) | \( 2\text{Re}\{i\tilde{W}^OSH\} \) bounded |
| Modified EM coupling          | \( \tilde{W}^T(x, X) = \tilde{W}^T(X, x) \) | \( \tilde{W}^T(\kappa) = \tilde{W}^T(-\kappa) \) | \( 2\text{Re}\{i\tilde{W}^OSH\} \) bounded |
\[
\langle \mathcal{M}(w), v \rangle_{\Omega} - \langle w, \mathcal{M}^T(v) \rangle_{\Omega} = \\
\int_{\Omega} \{ \mathcal{M} G_{\gamma}(\chi, x) \}_\alpha G_{\alpha(\beta)}^* (\chi, X) d\chi - \int_{\Omega} G_{\alpha(\beta)}(\chi, x) \{ \mathcal{M} G_{\alpha(\beta)}^T(\chi, X) \}^* d\chi = \\
\int_{\Omega} \delta_{\gamma(\beta)}(\chi - x) G_{\alpha(\beta)}^* (\chi, X) d\chi - \int_{\Omega} G_{\alpha(\beta)}(\chi, x) \delta_{\alpha(\beta)}(\chi - x) d\chi = \\
G_{\gamma(\beta)}^* (x, X) - G_{\beta(\alpha)}(x, X) = 0, \tag{A.7}
\]

and hence \( G_{\gamma(\beta)}^* (x, X) = G_{\beta(\gamma)}(x, X) \). If the body satisfies \( K_{\alpha(\beta)} = K_{\beta(\alpha)} \) and \( \rho^* = \rho \)—which is the case by virtue of Eq. (4)—then the problem is self-adjoint, as Eq. (A.4) is identical to Eq. (A.7). In this case \( \mathcal{M} = \mathcal{M}^T \), hence \( G_{\gamma(\beta)}^* (x, X) = G_{\beta(\gamma)}(x, X) \), which together with the previous result implies that

\[
G_{\gamma(\beta)}(x, X) = G_{\beta(\gamma)}^*(x, X). \tag{A.8}
\]

As discussed in Sec. IV, the symmetries of \( G \) are required in showing that \( \mathcal{L} \)—which is a function of \( G \)—satisfies the symmetries that are given by Eqs. (52)-(54), i.e., it is self-adjoint. The components of \( \mathcal{L} \) involve lengthy expressions, which we omitted here. We choose however to provide expression for \( \mathbf{W} \) and \( \mathbf{W}^T \), and show the symmetry we reported. We begin by writing the result for \( \mathbf{W}^T \) from Eq. (50) as

\[
\mathbf{W}^T(x, X) = -\alpha_{32} + \gamma_{32}, \tag{A.9}
\]

where \( \alpha_{32} \) and \( \gamma_{32} \) are the (3,2) entries of the symbolic \( 3 \times 3 \) block matrices \( \alpha = \langle L B G \rangle^T L \) and \( \gamma = \langle L B G \rangle^{-T} \langle (B G)^T L \rangle \), which read

\[
\begin{align*}
\alpha_{32}(x, X) &= s \langle \rho(x) \langle V_x G_{\gamma 1}^T \rangle B^T(X) \rangle - s \langle \rho(x) \langle V_x G_{\gamma 2}^T \rangle A(X) \rangle, \\
\gamma_{32}(x, X) &= s \langle \rho(x) \langle V_x G_{\gamma 2}^T \rangle B^T(X) \rangle - s \langle \rho(x) \langle V_x G_{\gamma 1}^T \rangle A(X) \rangle + s \langle \rho(x) \langle V_x G_{12}^T \rangle B^T(X) \rangle - s \langle \rho(x) \langle V_x G_{11}^T \rangle A(X) \rangle \\
&\quad + s \langle \rho(x) \langle V_x G_{21}^T \rangle B^T(X) \rangle - s \langle \rho(x) \langle V_x G_{22}^T \rangle A(X) \rangle + s \langle \rho(x) \langle V_x G_{12}^T \rangle B^T(X) \rangle - s \langle \rho(x) \langle V_x G_{11}^T \rangle A(X) \rangle \\
&\quad + s \langle \rho(x) \langle V_x G_{21}^T \rangle B^T(X) \rangle - s \langle \rho(x) \langle V_x G_{22}^T \rangle A(X) \rangle,
\end{align*}
\]

where

\[
V \equiv G^{-1} = \begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix}. \tag{A.10}
\]

Note that if \( G \) satisfy the symmetry \( G^{T*}(x, X) = G(X, x) \), then \( G^{T*}(x, X)^{-1} = G(X, x)^{-1} \) and consequently, \( V^{T*}(x, X) = V(X, x) \), so

\[
V_{\beta(\gamma)}^{T*}(x, X) = V_{\gamma(\beta)}(X, x). \tag{A.11}
\]

On the other hand, from Eq. (50) we have

\[
\mathbf{W}^{T*}(x, X) = -\alpha_{23}^{T*}(X, x) + \gamma_{23}^{T*}(X, x), \tag{A.12}
\]

where

\[
\alpha_{23}^{T*}(x, X) = s^* \langle \rho^{T*}(x) \langle V_x G_{\gamma 1}^T \rangle B^T(X) \rangle - s^* \langle \rho^{T*}(x) \langle V_x G_{\gamma 2}^T \rangle A^{T*}(X) \rangle, \\
\gamma_{23}^{T*}(x, X) = s^* \langle \rho^{T*}(x) \langle V_x G_{\gamma 2}^T \rangle B^T(X) \rangle - s^* \langle \rho^{T*}(x) \langle V_x G_{\gamma 1}^T \rangle A^{T*}(X) \rangle + s^* \langle \rho^{T*}(x) \langle V_x G_{12}^T \rangle B^T(X) \rangle - s^* \langle \rho^{T*}(x) \langle V_x G_{11}^T \rangle A^{T*}(X) \rangle \\
&\quad + s^* \langle \rho^{T*}(x) \langle V_x G_{21}^T \rangle B^T(X) \rangle - s^* \langle \rho^{T*}(x) \langle V_x G_{22}^T \rangle A^{T*}(X) \rangle + s^* \langle \rho^{T*}(x) \langle V_x G_{12}^T \rangle B^T(X) \rangle - s^* \langle \rho^{T*}(x) \langle V_x G_{11}^T \rangle A^{T*}(X) \rangle \\
&\quad + s^* \langle \rho^{T*}(x) \langle V_x G_{21}^T \rangle B^T(X) \rangle - s^* \langle \rho^{T*}(x) \langle V_x G_{22}^T \rangle A^{T*}(X) \rangle.
\]

Note that if \( L^{T*} = L \), since \( G^{T*}(X, x) = G(X, x) \), Eq. (A.13) holds and \( \alpha_{32}(x, X) = -\alpha_{23}^{T*}(X, x), \gamma_{32}(x, X) = -\gamma_{23}^{T*}(X, x) \).
for $s = -i\omega$, indicating that $\mathbf{W}^\dagger(\mathbf{x}, \mathbf{X}) = -\mathbf{W}^{T\ast}(\mathbf{x}, \mathbf{X})$. We assume statistically homogeneous media, so $\mathbf{W}^\dagger(\mathbf{x} - \mathbf{X}) = -\mathbf{W}^{T\ast}(\mathbf{x} - \mathbf{X})$ and its Fourier transform leads to the relation $\mathbf{W}^\dagger(\mathbf{\kappa}) = -\mathbf{W}^{T\ast}(\mathbf{\kappa})$ or

$$\mathbf{W}^\dagger(\mathbf{\kappa}, \omega) = -\mathbf{W}^{T\ast}(\mathbf{\kappa}, \omega), \tag{A.17}$$

which is the first relation in Eq. (57b). If $L$ is real, from the Fourier transform of Eq. (52) we have

$$\mathbf{W}^\dagger(\mathbf{\kappa}, \omega) = \mathbf{W}^\dagger(\mathbf{-\kappa}, \omega). \tag{A.18}$$

Finally, from Eqs. (A.17) and (A.18) we obtain the relation

$$\mathbf{W}(\mathbf{\kappa}, \omega) = -\mathbf{W}^\ast(\mathbf{-\kappa}, \omega). \tag{A.19}$$

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[94] Note5. Dependency in other areas of science. 2016.

[95] Note6. If the response of the composite depends on the history, this approximation means that the difference between time scales of the mechanical and electromagnetic effects allows to neglect time derivatives and magnetic fields in Maxwell equations.

[96] Note7. Note that in Eq. 3 we amend our notation for the single contraction, etc.

[97] Note8. If the response of the composite depends on the history, this approximation means that the difference between time scales of the mechanical and electromagnetic effects allows to neglect time derivatives and magnetic fields in Maxwell equations.
Shmuel [52], by changing the superscript T to †.

[99] Note9. More precisely, the Laplace transform with the variable $s$ was used, which is connected to the Fourier transform via $s = -i\omega$.

[100] Note10. The problem goes beyond ambiguity between velocity and strain, since it also follows that even in non-local elastostatics the compatibility of the effective strain with the effective displacement field implies that there are infinitely many kernels that equivalently relate the stress and the strain [48].

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