Characterisation of Gram matrices of multi-mode coherent states

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Abstract

Quantum communication protocols are typically formulated in terms of abstract qudit states, leaving the question of an experimental realization open. Direct translation of these states, say into single photons with some d-dimensional degree of freedom, are typically challenging to realize. Multi-mode coherent states, on the other hand, can be easily generated experimentally. In this work, we characterise the Gram matrices of multi-mode coherent states in an attempt to understand the class of communication protocols, which can be implemented using such states. We use this characterisation to show that the Hadamard exponential of a Euclidean distance matrix is positive semidefinite. We also derive the closure of the Gram matrices, which can be implemented in this way, so that we also characterise those matrices, which can be approximated arbitrarily well using multi-mode coherent states.

1 Introduction

It has been shown that the use of quantum resources offers significant qualitative and quantitative advantages over classical communication for several problems. Quantum cryptographic protocols are a prominent example of the qualitative advantage, while numerous protocols demonstrating the quantitative advantage in terms of communication or information complexity have been developed [1-4]. These quantum communication protocols are usually formulated in terms of d-dimensional quantum systems, or qudits. However manipulation and control of these systems still remains a challenge. Due to this reason the question of experimental implementation of these protocols is also left open. The most successfully implemented quantum protocols are ones which have been reformulated in terms of coherent states, which can be produced using lasers, and linear optics. Widely celebrated quantum key distribution [5,6] and quantum fingerprinting protocols [8,10] fall in this category, and as always there is a great impetus towards realising more protocols
using these tools [11,13].

Consider the quantum fingerprinting protocol as an example. Fingerprinting is a problem in communication complexity. Two parties have to check whether the strings they hold are equal or not in the absence of shared randomness and with as little communication as possible. In 2001, Buhrman et al. gave a quantum protocol for this problem [2], which required exponentially less communication than the optimal classical protocol for this task. It was not until 2015 [9], though, that the protocol was implemented experimentally. This implementation became possible after the protocol had been reformulated such that it required only coherent states and linear optics, in way which also preserved the communication cost [8]. The original protocol mapped the set of n-bit strings, \( \{s_i\}_{i=1}^{2^n} \) to qudit states, \( \{|\psi_i\rangle\}_{i=1}^{2^n} \subset \mathcal{H}_q \), such that for \( i \neq j \), \(|\langle \psi_i | \psi_j \rangle| \leq \delta \) for some \( \delta \), and \( \dim(\mathcal{H}_q) = \mathcal{O}(n) \). Using this mapping, they showed that one could decide if two strings were equal or unequal by communicating only \( \mathcal{O}(\log n) \) qubits. Arrazola et al. [8] showed that one could instead choose multi-mode coherent states which satisfy the requirements on the overlap for the protocol. This naturally leads us to ask under what conditions can the vectors forming a Gram matrix be chosen as multi-mode coherent states, so that a protocol implemented by them may be run using coherent states.

This is the question we answer in this paper. We characterise the set of Gram matrices of coherent states and their closure. Not only will such a characterisation help us in reformulating quantum protocols in terms of coherent states, but it will also shed light on the fundamental properties of sets of coherent states, which are the most classical states of light [14–16]. A Gram matrix of a set of quantum states encodes the information about their orientation relative to each other. For this reason we can very often exchange one set of states with another in a protocol if their Gram matrices are the same though the measurements also need to be changed correspondingly. Additionally, the set of states attainable by applying physical transformations on an initial set of states also depends on the Gram matrix of the initial set of states [17,18]. Therefore, our work also characterises all the set of states attainable from a set of multi-mode coherent states.

The paper is organised as follows. In section 2, we establish the notation for the paper. In Section 3, we completely characterise the set of Gram matrices, which can be constructed using multi-mode coherent states. We provide a simple test to check if a matrix belongs in this set. We show that the Hadamard exponential of Euclidean distance matrices can be written as a Gram matrix of multi-mode coherent states, which proves that they are positive semidefinite. Moreover in Section 4, we derive the closure of this set to characterise the Gram matrices, which can be approximated arbitrarily well using multi-mode coherent states.
2 Notation

In this paper, vectors will be denoted by alphabets, and should be identified by their spaces. For example, \( v \in \mathbb{C}^n \) denotes a vector. Coherent states, on the other hand, will be represented using kets, for example \(|\alpha\rangle\) denotes a coherent state with complex amplitude \(\alpha\). Recall that a coherent state is

\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle
\]

for \(\alpha \in \mathbb{C}\). A multi-mode coherent state \(\alpha_k\) is simply

\[
\bigotimes_{k=1}^{n} |\alpha_k\rangle
\]

for \(\{\alpha_k\}_{k=1}^{n} \in \mathbb{C}\). We will use \(\mathcal{C}_n\) to denote the set of n-mode coherent states.

This notation can be simplified, by using the map

\[
\| \cdot \| : \mathbb{C}^n \rightarrow \mathcal{C}_n
\]

for an amplitude vector, \(\alpha = (\alpha_1 \alpha_2 \cdots \alpha_n)^T \in \mathbb{C}^n :\)

\[
\|\alpha\| := \bigotimes_{k=1}^{n} |\alpha_k\rangle
\]

and hence,

\[
\mathcal{C}_n = \{e^{i\phi} \|\alpha\| : \alpha \in \mathbb{C}^n, \phi \in \mathbb{R}\}
\]

It will be seen that this notation arises naturally during the proof of Theorem 1. The following notation will be used to denote the standard inner product on a Hilbert Space, \(\mathcal{H}\).

\[
\langle \cdot , \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}
\]

As an example, the inner product between two multi-mode coherent states, \(\|\alpha\|\) and \(\|\beta\|\) will be denoted as \(\langle \|\alpha\| , \|\beta\| \rangle\).

For a matrix, \(P\), the notation \(P \geq 0\) indicates that the matrix is positive semidefinite. We will also define a function which takes an n-tuple of vectors and maps them to their Gram matrix.
Table 1: Notation for frequently used sets

| Notation   | Description                                                                 |
|------------|-----------------------------------------------------------------------------|
| $L(\mathcal{X}, \mathcal{Y})$ | The set of linear operators from complex Euclidean space, $\mathcal{X}$ to complex Euclidean space, $\mathcal{Y}$. |
| $L(\mathcal{X})$ | The set of linear operators from complex Euclidean space, $\mathcal{X}$ to itself. |
| $U(\mathcal{X}, \mathcal{Y})$ | The set of isometries from complex Euclidean space, $\mathcal{X}$ to complex Euclidean space, $\mathcal{Y}$. |
| $\text{Herm}(\mathcal{X})$ | The set of Hermitian operators in $L(\mathcal{X})$. |
| $\text{Pos}(\mathcal{X})$ | The set of positive semidefinite operators in $L(\mathcal{X})$. |
| $[n]$ | The set $\{1, 2, \ldots, n\}$. |

**Definition 1.** Let $G$ be the function which takes vectors $(v_1, v_2, \ldots, v_n) \in \mathcal{H}$ (a Hilbert Space) and maps them to their Gram matrix.

$$G : \mathcal{H}^n \to \text{Pos}(\mathbb{C}^n)$$

$$(G(v_1, v_2, \ldots, v_n))_{ij} := \langle v_i, v_j \rangle \text{ for } 1 \leq i, j \leq n \tag{1}$$

Given this definition of $G$, we can introduce the notation,

$$G(S^n) = \{G(v_1, v_2, \ldots, v_n) : v_1, v_2, \ldots, v_n \in S\}$$

Lastly, in Table 1, we list the notation for frequently used sets from linear algebra.

### 3 Characterisation of Gram matrices of multi-mode coherent states

In this section, we will prove a theorem which characterises Gram matrices of multi-mode coherent states. Namely, we will answer the question: can we write a Gram matrix, $P$, as $P = G(e^{i \phi_1} || a_1 \rangle, e^{i \phi_2} || a_2 \rangle, \ldots, e^{i \phi_n} || a_n \rangle)$?

For the statement of the theorem, we need the Hadamard logarithm, which is defined as

$$(\log \odot (P))_{ij} := \log P_{ij}$$

We also define the vector, $u \in \mathbb{C}^n$ to be the vector of all ones, i.e.,

$$u := (1 \ 1 \ \ldots \ \ 1)^T$$

In the following theorem, we restrict ourselves to Gram matrices of unit vectors, i.e., $P \in \text{Pos}(\mathbb{C}^n)$ such that $i \in [n]$, $P_{ii} = 1$, since quantum states are represented by unit vectors. However, our theorem can be easily extended to characterise the Gram matrices of non-normalised coherent states.
Theorem 1. For a matrix, $P \in L(\mathbb{C}^n)$ such that for $i \in [n]$, $P_{ii} = 1$, the following are equivalent:

1. $\exists m \in \mathbb{N}, \{\alpha_{ik} : i \in [n], k \in [m]\} \subseteq \mathbb{C}$, and $\{\phi_i : i \in [n]\} \subseteq \mathbb{R}$ such that for $i,j \in [n],\frac{e^{i\phi_i} \bigotimes_{k=1}^{m} |\alpha_{ik}\rangle}{\bigotimes_{k=1}^{m} |\alpha_{jk}\rangle} = P_{ij}$ \hspace{1cm} (2)

Which simply states that you can choose multi-mode coherent states such that their Gram matrix is the same as $P$, or more succinctly

$\exists m \in \mathbb{N}$ such that $P \in G(\mathbb{C}^m)$ \hspace{1cm} (3)

2. $\exists x \in \mathbb{C}^n$ such that

$log \circ (P) + xu^\dagger + ux^\dagger \geq 0$ \hspace{1cm} (4)

3. $\forall s \in \mathbb{C}^n$ such that $\langle u,s \rangle = 1$, it holds that

$(1 - us^\dagger) (log \circ (P)) (1 - su^\dagger) \geq 0$ \hspace{1cm} (5)

4. $\exists s \in \mathbb{C}^n$ such that $\langle u,s \rangle = 1$ and

$(1 - us^\dagger) (log \circ (P)) (1 - su^\dagger) \geq 0$ \hspace{1cm} (6)

5. $\forall y \in \mathbb{C}^n$ such that $\langle u,y \rangle = 0$, it holds that

$y^\dagger (log \circ (P)) y \geq 0$ \hspace{1cm} (7)

Proof. In the characterisation presented here, $log \circ (P)$ behaves similar to a Euclidean distance matrices \cite{20, 21}. The equation of the inner product of multi-mode coherent states in terms of their amplitude vectors is similar to the equation for an element of a distance matrix. The proofs of these statements are almost the same as the ones given by Gower in Ref. \cite{20} for the characterisation of Euclidean distance matrices. It should be noted that $log \circ (P)$ is not a distance matrix (since its entries may be complex). We will prove the statements of the theorem in the order

$(1) \Rightarrow (2) \Rightarrow (1)$

$(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (2)$

$(3) \Rightarrow (5) \Rightarrow (3)$

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First we will reduce statement (1) to a simpler form,
\[
P_{ij} = \left( e^{i\phi_i} \otimes_{k=1}^m |\alpha_{ik}\rangle, e^{i\phi_j} \otimes_{k=1}^m |\alpha_{jk}\rangle \right) \\
= e^{i(\phi_j - \phi_i)} \prod_{k=1}^m \langle |\alpha_{ik}\rangle, |\alpha_{jk}\rangle \rangle \\
= e^{i(\phi_j - \phi_i)} \prod_{k=1}^m \exp\left( -\frac{1}{2} (|\alpha_{ik}|^2 + |\alpha_{jk}|^2 - 2\alpha_{ik}^*\alpha_{jk}) \right) \\
= \exp\left( i(\phi_j - \phi_i) - \frac{1}{2} \sum_{k=1}^m (|\alpha_{ik}|^2 + |\alpha_{jk}|^2 - 2\alpha_{ik}^*\alpha_{jk}) \right).
\]

Define, \( \alpha_i := (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{im})^T \in \mathbb{C}^m \), and \( \phi := (\phi_1, \phi_2, \ldots, \phi_n)^T \in \mathbb{R}^n \). This naturally gives rise to the notation mentioned earlier for multi-mode coherent states. The inner product can be written as
\[
\log(P_{ij}) = -\frac{1}{2} \left( \|\alpha_i\|^2 + \|\alpha_j\|^2 - 2\langle \alpha_i, \alpha_j \rangle \right) + i(\phi_j - \phi_i) \tag{8}
\]
or equivalently,
\[
\log(P_{ij}) = -\frac{1}{2} \|\alpha_i - \alpha_j\|^2 + i(\text{Im}\{\langle \alpha_i, \alpha_j \rangle\} + (\phi_j - \phi_i)). \tag{9}
\]

It can be seen that statement (1) is equivalent to the existence of vectors \( \{\alpha_i\}_{i=1}^n \in \mathbb{C}^m \), and \( \phi \in \mathbb{R}^n \) such that Eq. (8) or Eq. (9) is satisfied for all \( i, j \in [n] \). It is clear from Eq. (8) that the collection of vectors, \( \{\alpha_i\}_{i=1}^n \), is isometrically invariant, i.e., if \( \{\alpha_i\}_{i=1}^n \) satisfies Eq. (8) then \( \forall U \in U(\mathbb{C}^m, \mathbb{C}^n) \), \( \{U\alpha_i\}_{i=1}^n \) also satisfy Eq. (8). Therefore, one can simply constrain the vectors, \( \{\alpha_i\}_{i=1}^n \), to be in an \( n \)-dimensional space (since there are only \( n \) vectors), i.e., we can set \( m = n \) in statement (1). We will later state this result as Corollary \( \square \).

Now we will prove that statement (1) \( \Rightarrow \) statement (2). Let \( G \) be the Gram matrix of the amplitude vectors, \( \{\alpha_i\}_{i=1}^n \), i.e., \( G_{ij} := \langle \alpha_i, \alpha_j \rangle \). Then, we have that \( G \in \text{Pos}(\mathbb{C}^n) \), since the set of Gram matrices and positive semidefinite matrices is equivalent. We can write Eq. (8) as
\[
\log(P_{ij}) = i(\phi_j - \phi_i) - \frac{1}{2} (G_{ii} + G_{jj} - 2G_{ij}) \tag{10}
\]

Without loss of generality, \( G = \log \circ (P) + X \) for some \( X \). Then, \( G_{ii} = \log(P_{ii}) + X_{ii} = \log(1) + X_{ii} = X_{ii} \). We substitute this into Eq. (10) to derive a consistency equation for \( X \).
\[
\log(P_{ij}) = i(\phi_j - \phi_i) - \frac{1}{2} (X_{ii} + X_{jj} - 2\log(P_{ij}) - 2X_{ij}) \\
\Rightarrow X_{ij} = \left( \frac{1}{2} X_{ii} + i\phi_i \right) + \left( \frac{1}{2} X_{jj} - i\phi_j \right)
\]
Define $x \in \mathbb{C}^n$, as $x_i = \frac{X_i}{2} + i\phi_i$ ($X_{ii} = \|\alpha_i\|_2^2 \in \mathbb{R}$, $\phi_i \in \mathbb{R}$). Then, we may write

$$X_{ij} = x_i + x_j^*$$

$$X = xu^\dagger + ux^\dagger.$$  \hfill (11)

Therefore, if statement (1) is true, then $G = \log \odot (P) + xu^\dagger + ux^\dagger \geq 0$, which is what statement (2) states.

For the converse, statement (2) \implies statement (1), assume that $\exists x \in \mathbb{C}^n : \log \odot (P) + xu^\dagger + ux^\dagger \geq 0$. Let $G = \log \odot (P) + xu^\dagger + ux^\dagger$. Then, $\exists \{\alpha_i\}_{i=1}^n : G_{ij} = \langle \alpha_i, \alpha_j \rangle$, since $G$ is positive semi-definite. Consequently, we can define a set of amplitude vectors that realize this Gram matrix, $G$. We can now show that these amplitude vectors satisfy Eq. 8 for an appropriate definition of $\phi$. To see this, observe that

$$G_{ij} = \log P_{ij} + x_i + x_j^*$$

$$G_{ii} = x_i + x_i^*.$$  

Now let’s evaluate the following expression.

$$\|\alpha_i\|_2^2 + \|\alpha_j\|_2^2 - 2(\alpha_i, \alpha_j) = G_{ii} + G_{jj} - 2G_{ij}$$

$$= x_i + \bar{x}_i + x_j + \bar{x}_j - 2\log P_{ij} - 2x_i - 2\bar{x}_j$$

$$= -2\log P_{ij} - (x_i - \bar{x}_i) + (x_j - \bar{x}_j)$$

$$= -2\log P_{ij} - 2i\text{Im}\{x_i\} + 2i\text{Im}\{x_j\}$$

Define, $\phi \in \mathbb{R}^n : \phi_i = \text{Im}\{x_i\} \in \mathbb{R}$. Then, the RHS of Eq. 8 is

$$-\frac{1}{2} (\|\alpha_i\|_2^2 + \|\alpha_j\|_2^2 - 2(\alpha_i, \alpha_j)) + i(\phi_j - \phi_i)$$

$$= (\log P_{ij} + i\text{Im}\{x_i\} - i\text{Im}\{x_j\}) + i(\text{Im}\{x_j\} - \text{Im}\{x_i\})$$

$$= \log P_{ij}.$$  

Hence, given statement (2) one can define $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{C}^m$, and $\phi \in \mathbb{R}^m$ such that Eq. 8 is satisfied for all $i, j \in [n]$. Therefore, (1) \iff (2).

To see that statement (2) \implies statement (3), choose $s \in \mathbb{C}^n$ such that $\langle u, s \rangle = 1$. We note that for any such choice

$$\begin{align*}
(\mathbbm{1} - us^\dagger) xu^\dagger (\mathbbm{1} - su^\dagger) &= (xu^\dagger - us^\dagger xu^\dagger)(\mathbbm{1} - su^\dagger) \\
&= xu^\dagger - u(s^\dagger x) u^\dagger - x(u^\dagger s) u^\dagger + (s^\dagger x) u(u^\dagger s) u^\dagger \\
&= xu^\dagger - u(s^\dagger x) u^\dagger - xu^\dagger + (s^\dagger x) uu^\dagger \\
&= 0.  \hfill (12)
\end{align*}$$
Using the fact that the expression in Eq. 12 and its conjugate are zero, we can now show, starting with statement (2),

\[
G := \log \odot (P) + xu^\dagger + ux^\dagger \geq 0
\]

\[
\Rightarrow (\mathbb{1} - us^\dagger)^\dagger G (\mathbb{1} - su^\dagger) \geq 0
\]

\[
\Rightarrow (\mathbb{1} - us^\dagger)^\dagger (\log \odot (P)) (\mathbb{1} - su^\dagger) \geq 0.
\]

This proves that statement (2) ⇒ statement (3). Statement (3) ⇒ statement (4) trivially.

For statement (4) ⇒ statement (2), assume that \( \exists s \in \mathbb{C}^n \), such that, \( \langle u, s \rangle = 1 \), and

\[
(\mathbb{1} - us^\dagger)^\dagger (\log \odot (P)) (\mathbb{1} - su^\dagger) \geq 0.
\]

Expanding this out, we get

\[
\log \odot (P) - \log \odot (P) su^\dagger - us^\dagger \log \odot (P) + (s^\dagger \log \odot (P) s) uu^\dagger \geq 0
\]

\[
\Rightarrow \log \odot (P) + xu^\dagger + ux^\dagger \geq 0,
\]

for the choice \( x = \frac{1}{2} (s^\dagger \log (P) s) u - \log \odot (P)s \). Hence, we have shown the equivalence of the first four statements.

For statement (3) ⇒ statement (5), we have that \( (\mathbb{1} - us^\dagger)^\dagger (\log \odot (P)) (\mathbb{1} - su^\dagger) \geq 0 \) using statement (3), then \( \forall y \in \mathbb{C}^n \), such that, \( \langle u, y \rangle = 0 \),

\[
y^\dagger (\mathbb{1} - us^\dagger)^\dagger (\log \odot (P)) (\mathbb{1} - su^\dagger) y \geq 0
\]

\[
\Rightarrow y^\dagger (\log \odot (P)) y \geq 0.
\]

For statement (5) ⇒ statement (3), we have that \( \forall y \in \mathbb{C}^n \), such that, \( \langle u, y \rangle = 0 \),

\[
y^\dagger (\log \odot (P)) y \geq 0.
\]

If we choose any vector \( s \in \mathbb{C}^n \) such that \( \langle u, s \rangle = 1 \), and any vector \( v \in \mathbb{C}^n \), then we can construct a vector \( y \) with \( \langle u, y \rangle = 0 \) by setting \( y = (\mathbb{1} - su^\dagger)v \). Using statement (5) with this choice of vector \( y \) then gives

\[
v^\dagger (\mathbb{1} - us^\dagger)^\dagger (\log \odot (P)) (\mathbb{1} - su^\dagger) v \geq 0
\]

\[
\Rightarrow (\mathbb{1} - us^\dagger)^\dagger (\log \odot (P)) (\mathbb{1} - su^\dagger) \geq 0,
\]

which proves statement (5) ⇒ statement (3).
Further, we established the following corollary while proving statement (1) ⇒ statement (2).

**Corollary 1.** \( \forall m \in \mathbb{N} : G(C_n^m) = G(C_n) \).

That is, no more than \( n \)-modes are required to represent a Gram matrix of \( n \)-vectors.

This corollary allows us to restrict the number of modes of coherent states to the number of vectors forming the Gram matrix. It tells us that one cannot do better by simply increasing the number of modes. With this result in hand, we can see that the set of matrices in \( \text{Pos}(\mathbb{C}^n) \), which can be constructed using coherent states, is just \( G(C_n) \). Therefore, in Section 4 where we study the closure of the set of Gram matrices of multi-mode coherent states, we will just consider the set \( G(C_n) \).

We can also use our characterisation to prove that the Hadamard exponential of a Euclidean distance matrices is positive semidefinite, which we will formulate as Corollary 2. A matrix, \( D \in L(\mathbb{R}^n) \) is a Euclidean distance matrix if there exist \( \{ \alpha_i \}_{i=1}^n \subset \mathbb{R}^n \), such that for \( i,j \in [n] \)

\[
D_{ij} = -\frac{1}{2}\|\alpha_i - \alpha_j\|_2^2.
\]

In Ref. [20] it has been proven that this condition is equivalent to the existence of a vector \( s \in \mathbb{R}^n \), such that \( \langle u, s \rangle = 1 \), and

\[
(\mathbb{1} - us\dagger)D(\mathbb{1} - su\dagger) \geq 0.
\]  
(13)

Lastly, we define the Hadamard exponential of a matrix, \( X \) as

\[
(\exp \otimes (X))_{ij} := \exp(X_{ij}).
\]

**Corollary 2.** The Hadamard exponential of a Euclidean distance matrix, \( D \) is positive semidefinite. That is

\[
\exp \otimes (D) \geq 0.
\]

**Proof.** Let \( P := \exp \otimes (D) \). Since, \( D \) is a Euclidean distance matrix, \( \exists s \in \mathbb{R}^n \), such that \( \langle u, s \rangle = 1 \) and

\[
(\mathbb{1} - us\dagger)D(\mathbb{1} - su\dagger) \geq 0
\]
\[
\Rightarrow (\mathbb{1} - us\dagger)(\log \otimes (P))(\mathbb{1} - su\dagger) \geq 0
\]
\[
\Rightarrow P \in G(C_n)
\]
\[
\Rightarrow P \geq 0
\]
\[
\Rightarrow \exp \otimes (D) \geq 0.
\]
Where we have used the equivalence of statements (4) and (1) of Theorem 1 in the second step and the fact that Gram matrices are positive semidefinite in the third step.

We would like to point out that one can use Theorem 1 to check in polynomial time (in the size of the matrix) if a given square matrix, \( P \in L(\mathbb{C}^n) \) (such that \( \forall \ i \in [n]: P_{ii} = 1 \)) can be represented as a Gram matrix of coherent states or not. One can use the following algorithm for example.

1. Form the matrix, \( X = (\mathbb{I} - u e_1^\dagger) (\log \odot (P)) (\mathbb{I} - e_1 u^\dagger) \).

2. Check if \( X \) is positive semidefinite or not.

   (a) If it is positive semidefinite, then statement (4) of Theorem 1 guarantees that you can write your matrix as a Gram matrix of multi-mode coherent states. Moreover, these coherent states can be found by using the columns of \( X^{1/2} \),

   \[
   X^{1/2} = (\alpha_1 \alpha_2 \ldots \alpha_n)
   \]

   where \( \forall \ i \in [n] : \alpha_i \in \mathbb{C}^n \). Then, for \( \phi_i = -\text{Im}\{\log P_{ii}\} \), we have \( P = G(e^{i\phi_1\|\alpha_1\|}, e^{i\phi_2\|\alpha_2\|} \ldots e^{i\phi_n\|\alpha_n\|}) \).

   (b) If it is not positive semidefinite, then according to Statement (3) of Theorem 1, \( P \notin G(\mathbb{C}_n^n) \), and it cannot be written as a Gram matrix of multi-mode coherent states.

**Observation:** If we replace \( e_1 \) with \( u/n \) in the above algorithm, then \( X \) and \( X^{1/2} \) will have rank at most \( (n-1) \), which means that one could in fact choose vectors in \( \mathbb{C}^{n-1} \) to satisfy Eq. 8. Thus, we can choose \( (n-1) \)-mode coherent states to construct any matrix in \( G(\mathbb{C}_n^n) \), i.e., \( G(\mathbb{C}_n^n) = G(\mathbb{C}_{n-1}^{n-1}) \). One can also see this by observing that for \( s \in \mathbb{C}^n \),

\[
G(e^{i\phi_1\|\alpha_1\|} \ldots e^{i\phi_n\|\alpha_n\|}) = G(e^{i\phi_1\|\alpha_1+s\|} \ldots e^{i\phi_n\|\alpha_n+s\|})
\]

for \( \phi'_i = \phi_i - \text{Im}\{\langle \alpha_i, s \rangle\} \), which allows us to reduce the dimension of the span of the amplitude vectors by at least 1, say by choosing \( s = -\alpha_1 \). We state this result as Corollary 1, a slightly strengthened form of Corollary 1.

**Corollary 1.** For \( n \geq 2 \) and \( \forall \ m \in \mathbb{N} : G(\mathbb{C}_{n-1+m}^n) = G(\mathbb{C}_{n-1}^n) \).

However, we will continue to use \( G(\mathbb{C}_n^n) \) to represent the Gram matrices of \( n \) multi-mode coherent states for notational convenience.

We can also connect our work to the coherent state mapping presented in Ref. [22]. We will assume that the entries of the Gram matrix, \( P \) are close to 1 (and \( \forall \ i \in [n] : P_{ii} = 1 \)). That is, the angles between the vectors forming the Gram matrix are small. Then we
need to find \( \{\alpha_i\}_{i=1}^n \subset \mathbb{C}^n \) and \( \phi \in \mathbb{R}^n \) such that Eq. 8 is satisfied. We will choose, \( \forall i \in [n] : \|\alpha_i\|_2 = 1, \) and \( \phi_i = 0. \) For this choice, we have

\[
\log(P_{ij}) = -\frac{1}{2} \left( \|\alpha_i\|^2 + \|\alpha_j\|^2 - 2(\alpha_i, \alpha_j) + i(\phi_j - \phi_i) \right)
\]

\[
= -\frac{1}{2} (2 - 2(\alpha_i, \alpha_j)).
\]

Using the assumption that the entries of \( P \) are close to 1,

\[
P_{ij} - 1 + O((P_{ij} - 1)^2) = -1 + (\alpha_i, \alpha_j)
\]

\[
\Rightarrow P_{ij} \approx (\alpha_i, \alpha_j).
\]

Thus, the problem of finding coherent states in this case reduces to finding unit vectors forming the Gram matrix, \( P. \) This can be done easily by choosing \( \{\alpha_i\}_{i=1}^n \) to be the columns of \( B \) for any \( B \) such that \( P = B^\dagger B. \) Ref. [22] studies exactly this mapping of qudit states \( \{\{\alpha_i\}\}_i \) to multi-mode coherent states \( \{\{\alpha_i\}\}_i \). Our results show that this is indeed well motivated.

### 4 Closure of the set of Gram matrices of coherent states

The set of Gram matrices of \( n \) multi-mode coherent states, \( G(\mathcal{E}_n^m) \) is not closed. For example, one may construct Gram matrices arbitrarily close to the identity matrix using coherent states, but the identity matrix itself cannot be constructed, since the inner products between any two coherent states is never zero. In this section, we will characterise the closure of \( G(\mathcal{E}_n^m) \), which we will represent as \( G(\mathcal{E}_n^m) \). This set consists of the Gram matrices which can be approximated arbitrarily well using Gram matrices of coherent states. Experimentally \( G(\mathcal{E}_n^m) \) is more relevant than \( G(\mathcal{E}_n^m) \). One expects that block diagonal Gram matrices, where each of the blocks is a Gram matrix of coherent states, would lie in \( G(\mathcal{E}_n^m) \). Each block could be realized by the set of corresponding coherent states, and one could displace the amplitudes between the sets relative to each other with a sufficiently large amplitude vector to achieve this. In fact, we will show that all the matrices in \( G(\mathcal{E}_n^m) \) can be put into such a block diagonal form. To prove this, we will require two intermediate results, lemmas [1] and [2]. Lemma [1] relates the distance between the amplitude vectors of two coherent states with their inner-product with each other and a third coherent state. Lemma [2] shows that if a Gram matrix with non-zero entries belongs in \( G(\mathcal{E}_n^m) \), then it also belongs in \( G(\mathcal{E}_n^m) \). Together these two will allow us to characterise \( G(\mathcal{E}_n^m) \).

**Lemma 1.** If \( \|\alpha\|, \|\beta\|, \|\gamma\| \in \mathcal{E}_n \), are such that \( \|\langle\alpha\rangle, \|\beta\|\rangle\| = p_{\alpha\beta} \) and \( \|\langle\alpha\rangle, \|\gamma\|\rangle\| = p_{\alpha\gamma} \), then

\[
(2 \log p_{\alpha\gamma})^{1/2} + (2 \log p_{\alpha\beta})^{1/2} \geq ||\beta - \gamma||_2 \geq \left(2 \log p_{\alpha\gamma}\right)^{1/2} - \left(2 \log p_{\alpha\beta}\right)^{1/2}.
\]

(15)
Proof. From Eq. 9 the following can be deduced,

\[ \| \alpha - \beta \|_2 = (-2 \log (p_{\alpha \beta}))^{1/2} \]
\[ \| \alpha - \gamma \|_2 = (-2 \log (p_{\alpha \gamma}))^{1/2}. \]

To establish the lower bound, we use the triangle inequality in the following manner.

\[ \| \gamma - \beta \|_2 = \| (\gamma - \alpha) - (\beta - \alpha) \|_2 \]
\[ \geq 2 \| \gamma - \alpha \|_2 - \| \beta - \alpha \|_2 \]
\[ \geq 2 \log (p_{\alpha \gamma})^{1/2} - 2 \log (p_{\alpha \beta})^{1/2} \]

We use the triangle inequality again to establish the upper bound.

\[ \| \gamma - \beta \|_2 \leq \| \gamma - \alpha \|_2 + \| \beta - \alpha \|_2 \]
\[ = 2 \log (p_{\alpha \gamma})^{1/2} + 2 \log (p_{\alpha \beta})^{1/2} \]

Before we go on further, we would like to point out that if \( P \in G(\mathcal{C}_n) \), then \( \forall i, j \in [n] : P_{ij} \neq 0 \). This can be seen from Eq. 8. In the lemma that follows, we prove that if a matrix with non-zero entries belongs in \( G(\mathcal{C}_n) \), then it also belongs in \( G(\mathcal{E}_n) \).

**Lemma 2.** For \( P \in L(\mathbb{C}^n) \) such that \( \forall i \in [n] : P_{ii} = 1 \), if \( \forall i, j \in [n] : P_{ij} \neq 0 \), then \( P \in G(\mathcal{E}_n) \) iff \( P \in G(\mathcal{C}_n) \).

**Proof.** \( P \in G(\mathcal{C}_n) \Rightarrow P \in G(\mathcal{E}_n) \) is trivial. For the other direction, we have that, \( P \in G(\mathcal{E}_n) \Rightarrow \exists \{ P_k \}_{k=1}^\infty \subseteq G(\mathcal{C}_n) \) such that \( P_k \stackrel{\| \cdot \|_2}{\rightarrow} P \) (where \( \| \cdot \|_2 \) is defined as \( \| X \|_2 = (\text{Tr}(X^\dagger X))^{1/2} \); all norms are equivalent in a finite dimensional space, so we can choose \( \| \cdot \|_2 \) WLOG). Firstly, observe that \( P \in \text{Pos}(\mathbb{C}^n) \) such that \( \forall i \in [n] : P_{ii} = 1 \), since this set is closed. Next, we will use continuity and statement (5) of Theorem 1 to prove the rest of the claim. For each \( P_k \) and vector \( y \in \mathbb{C}^n \) such that \( \langle u, y \rangle = 0 \), we have that

\[ y^\dagger (\log \circ P_k) y \geq 0 \]
\[ \Rightarrow \lim_{k \to \infty} y^\dagger (\log \circ P_k) y \geq 0 \]
\[ \Rightarrow y^\dagger \left( \lim_{k \to \infty} (\log \circ P_k) \right) y \geq 0 \]
\[ \Rightarrow y^\dagger (\log \circ \left( \lim_{k \to \infty} P_k \right)) y \geq 0 \]
\[ \Rightarrow y^\dagger (\log \circ P) y \geq 0. \]
Where we have used the continuity of the functions $f_y(X) = y^TXy$, and $f(X) = \log \circ X$. \log \circ X is continuous only when none of the elements $X_{ij}$ lie on the branch cut of the log function. For a $P$, such that $\forall i, j \in [n]: P_{ij} \neq 0$, one can always choose the branch cut of the log such that for sufficiently large $k$, $(P_k)_{ij}$ doesn’t lie on the branch cut. Therefore, using the equivalence of Theorem 1, this establishes that,

$$P \in \overline{G(C_2^n)} \text{ and } \forall i, j : P_{ij} \neq 0 \Rightarrow P \in G(C_2^n).$$

\[ \square \]

We need one final notion to characterise $\overline{G(C_2^n)}$. Observe that if

$$G(v_1, v_2, \ldots, v_n) \in G(C_2^n)$$

$$\Rightarrow G(v_{\pi^{-1}(1)}, v_{\pi^{-1}(2)}, \ldots, v_{\pi^{-1}(n)}) = P_{\pi}G(v_1, v_2, \ldots, v_n)P_{\pi}^\dagger \in G(C_2^n),$$

for any permutation $\pi$ (where $P_{\pi}$ represents the permutation matrix associated with $\pi$).

If one can construct a Gram matrix, $G$ using coherent states, then all one needs to do to construct the Gram matrix, $P_{\pi}G_PP_{\pi}^\dagger$ is to permute the order of the coherent states forming $G$ by $\pi$. Therefore, $G \in G(C_2^n) \iff P_{\pi}G_PP_{\pi}^\dagger \in G(C_2^n)$. In fact, because of the isometric invariance [23] of the 2-norm, $G \in G(C_2^n) \iff P_{\pi}G_PP_{\pi}^\dagger \in G(C_2^n)$. We can use this fact to simplify our analysis of matrices in $G(C_2^n)$. In the rest of the paper, we will refer to a Gram matrix of the form, $G' = G(v_{\pi^{-1}(1)}, v_{\pi^{-1}(2)}, \ldots, v_{\pi^{-1}(n)})$ as a rearrangement of the vectors forming the Gram matrix, $G = G(v_1, v_2, \ldots, v_n)$.

**Theorem 2.** For a $P \in L(C^n)$, $P \in \overline{G(C_2^n)}$, iff $P$ can be written as

$$P_{\pi}P_{\pi}^\dagger = \bigoplus_{i=1}^m P_i = \begin{pmatrix} P_1 & & \\ & P_2 & \\ & & \ddots \\ & & & P_m \end{pmatrix}$$

(17)

where, $\bigoplus_{i=1}^m P_i$ represents a direct sum of matrices, $\{P_i\}_{i=1}^m$ and $\forall i \in [m]: P_i \in G(C_2^{n_i})$ for some $n_i \in \mathbb{N}$, and $P_{\pi}$ is a permutation matrix.

In other words, $P \in \overline{G(C_2^n)}$, iff up to a rearrangement of the vectors forming it, $P$ can be written as a block-diagonal matrix where each block is a Gram matrix that can be realized by multi-mode coherent states.

**Proof.** We will first prove that if $P \in \overline{G(C_2^n)}$, then it has the aforementioned block diagonal form. This will be done in two steps. In the first step, we will establish two properties of elements of such a matrix, $P$. In the second step, which primarily relies on linear algebra, we will show that these two properties suffice to prove that the matrix, $P$, has the required block diagonal structure.

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Step 1: For a $P \in \mathcal{G}(C^n)$, and indices $i, k \in [n]$ such that $P_{ik} \neq 0$ we will prove that, if $j \in [n]$ such that $P_{ij} = 0$, then $P_{kj} = 0$ and if $j \in [n]$ such that $P_{ij} \neq 0$, then $P_{kj} \neq 0$. The idea is that if two multi-mode coherent states have a non-zero inner product then their amplitude vectors have to be a finite distance away from each other, but if their inner product approaches zero then the distance between these vectors has to grow infinitely large.

We will first show that if $P_{ik} \neq 0$ and $j \in [n]$ such that $P_{ij} = 0$, then $P_{kj} = P_{jk} = 0$. To prove this pick any $\{P_u\}_{u=1}^{\infty} \subseteq G(C^n)$ such that,

$$P_u \rightarrow P \iff \forall a, b \in [n] : (P_u)_{ab} \rightarrow P_{ab}.$$ 

Since, $P_u \in G(C^n), \exists \{e^{i\phi_u} \alpha_i^u\}_{i=1}^{n} \subseteq C_n$ such that $\forall a, b \in [n] : (P_u)_{ab} = \langle e^{i\phi_u} \alpha_i^u, e^{i\phi_b} \alpha_b^u \rangle$. Using Lemma 2 we have,

$$\left( -2 \log |(P_u)_{jk}| \right)^{1/2} = \|\alpha_k^u - \alpha_j^u\|_2 \geq \left( -2 \log |(P_u)_{ij}| \right)^{1/2} - \left( -2 \log |(P_u)_{ik}| \right)^{1/2}.$$ 

Taking the limit, $u \rightarrow \infty$, we have

$$\lim_{u \rightarrow \infty} \left( -2 \log |(P_u)_{jk}| \right)^{1/2} \geq \left( -2 \log \lim_{u \rightarrow \infty} |(P_u)_{ij}| \right)^{1/2} - \left( \lim_{u \rightarrow \infty} -2 \log |(P_u)_{ik}| \right)^{1/2},$$

where we have used the continuity of $|\cdot|$, $\log(\cdot)$, and $(\cdot)^{1/2}$ functions. The limit on the RHS tends to $\infty$ since $\lim_{u \rightarrow \infty} |(P_u)_{ij}| = |P_{ij}| = 0$ and $P_{ik} \neq 0$, therefore

$$\left( -2 \log \lim_{u \rightarrow \infty} |(P_u)_{jk}| \right)^{1/2} = \infty \Rightarrow \lim_{u \rightarrow \infty} |(P_u)_{jk}| = |P_{jk}| = 0.$$ 

Hence, for $P_{ik} \neq 0$, we have

$$\forall j \in [n] : P_{ij} = 0 \Rightarrow P_{kj} = P_{jk} = 0.$$ 

(since, $P \in \text{Herm}(C^n)$)

Now, we will prove that given $P_{ik} \neq 0, \forall j \in [n] : P_{ij} \neq 0 \Rightarrow P_{kj} \neq 0$. For $j$ such that $P_{ij} \neq 0$, using the upper bound given in Lemma 2, we have

$$\left( -2 \log |(P_u)_{jk}| \right)^{1/2} = \|\alpha_k^u - \alpha_j^u\|_2 \leq \left( -2 \log |(P_u)_{kl}| \right)^{1/2} + \left( -2 \log |(P_u)_{ij}| \right)^{1/2}.$$ 

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Taking the limit, $u \to \infty$, we have

$$\lim_{u \to \infty} \left( -2 \log |(P_u)_{jk}| \right)^{1/2} \leq \lim_{u \to \infty} \left( -2 \log |(P_u)_{ki}| \right)^{1/2} + \lim_{u \to \infty} \left( -2 \log |(P_u)_{ij}| \right)^{1/2} \leq (-2 \log |P_{ki}|)^{1/2} + (-2 \log |P_{ij}|)^{1/2} := M < \infty$$

where we have defined $M$ to be the upper bound. This gives us

$$\Rightarrow |P_{jk}| \geq \exp \left( -\frac{1}{2} M^2 \right) > 0.$$ 

For $P_{ik} \neq 0$, we have

$$\forall j \in [n] : P_{ij} \neq 0 \Rightarrow P_{jk} = P_{kj}^* \neq 0.$$ 

(19)

Therefore, we have proven that for any $n \in \mathbb{N}$ and $P \in \mathbb{G}(\mathbb{C}_n)$ and $i, j, k \in [n]$ Eq. 18 and Eq. 19 hold. These two properties are sufficient to establish the block diagonal structure of $P$.

**Step 2:** We will use induction on the size of the Gram matrices to prove the statement that if $P \in \mathbb{G}(\mathbb{C}_n)$, then $P$ has the block diagonal form given in Eq. 17 up to a rearrangement of the vectors forming it. First observe that since the sets, $\{ X \in \text{L}(\mathbb{C}^n) : X_{ii} = 1 \ \forall \ i \in [n] \}$ and $\text{Pos}(\mathbb{C}^n)$ are closed, $P$ will belong in these sets. The induction hypothesis is clearly true for $n = 1$ as $P = (1)$ is the only Gram matrix in this case and $P = G(\{0\})$. We assume that our hypothesis is true for all $p \leq n$. For $P \in \mathbb{G}(\mathbb{C}_{n+1}^n)$, $P$ can always be put into the form

$$P = \begin{pmatrix} P' & x \\ x^\dagger & 1 \end{pmatrix}$$

where $x \in \mathbb{C}^n$. It can be shown that $P' \in \mathbb{G}(\mathbb{C}_n)$. Since, $P \in \text{Pos}(\mathbb{C}^{n+1})$, we can write $P = G(v_1, v_2, \ldots, v_{n+1})$ for vectors $\{v_i\}_{i=1}^{n+1}$. Then, $P' = G(v_1, v_2, \ldots, v_n)$. By the induction hypothesis, there exists a permutation, $\pi$ such that

$$P_{\pi} P' P_{\pi}^\dagger = G(v_{\pi^{-1}(1)}, v_{\pi^{-1}(2)}, \ldots, v_{\pi^{-1}(n)}) = \begin{pmatrix} P'_1 & \cdots & P'_m \\ \vdots & \ddots & \vdots \\ P'_m & \cdots & P'_1 \end{pmatrix}$$

(20)

where $\forall \ i \in [m] : P'_{i} \in \mathbb{G}(\mathbb{C}_{n_i})$. We can transform $P$ as

$$P \rightarrow \begin{pmatrix} P_{\pi} & 0 \\ 0 & 1 \end{pmatrix} P \begin{pmatrix} P_{\pi}^\dagger & 0 \\ 0 & 1 \end{pmatrix}$$

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and prove our claim for this matrix without loss of generality. So, from now on we will assume that $P$ is block diagonal in the first $n \times n$ entries.

If $\forall \ i \in [n] : P_{(n+1)i} = 0$, then our matrix is already in the required block diagonal form. So, we will assume that $\exists \ i \in [n] : P_{(n+1)i} \neq 0$. Observe that the block structure of the first $n \times n$ entries divides the vectors forming the Gram matrix into orthogonal subspaces. So, we may associate a subspace with each Gram matrix, $P'_l$ (Eq. 20). Further assume WLOG that the vector, $v_i$ (where $P = G(v_1, v_2, \ldots, v_{n+1})$) is in the subspace associated with the Gram matrix, $P'_m$ (if not one can always permute the vectors forming $P$ such that this is true). Then using the fact that for all $j$ such that $P_{ij} = 0$, $P_{(n+1)j} = 0$ (Eq. 18), one can see that $P$ also has a block diagonal form if one includes the $(n + 1)^{th}$ row and column in $P'_m$ (See Fig. 1 for a schematic representation of this fact). We will call this new last block $P_m$. All the other blocks remain the same.

![Figure 1](image-url)

Figure 1: In this figure, we schematically represent the matrix, $P \in G(\mathcal{E}_{n+1}^n)$, which is diagonal in its first $n \times n$ entries. We consider the case where $P_{(n+1)i} \neq 0$. The facts proved in Step 1 of the proof show that the zero (white) and non-zero (gray) terms in the $i^{th}$-row (column) coincide with zero and non-zero terms respectively in the $(n + 1)^{th}$-row (column).

Further, $\forall \ l \in [m - 1] : P'_l \in G(\mathcal{E}_{n_l}^{n_l})$, and if we prove that the new block, $P_m \in G(\mathcal{E}_{n_{m+1}}^{n_{m+1}})$ then we would be done. Observe that $P'_m \in G(\mathcal{E}_{n_{m}}^{n_{m}})$, which
means that $\forall u, v \in [n_m] : (P_m')_{uv} \neq 0$. In addition, using Eq. 19, we have that $\forall u, v \in [n_m + 1] : (P_m)_{uv} \neq 0$. Moreover, $P_m \in G(\mathcal{E}_{n_m+1})$. Using Lemma 2 these two imply that $P_m \in G(\mathcal{E}_{n_m+1})$. Therefore, if $P \in G(\mathcal{E}_n)$, then $P$ has the block diagonal form given in Eq. 17 up to a rearrangement of the vectors forming it.

We will prove the converse of the statement by construction. We will present a construction for Gram matrices with 2 blocks, which can be generalised to $m$ blocks easily. Suppose, we have $P_1 \in G(\mathcal{E}_{n_1})$ and $P_2 \in G(\mathcal{E}_{n_2})$, then we wish to prove that

$$
P = \begin{pmatrix} P_1 & \end{pmatrix} \in G(\mathcal{E}_n)
$$

for $n = n_1 + n_2$.

For $i = 1, 2$ let the coherent states, $\{ e^{i\phi_{ij}} \parallel \alpha_{ij} \}^{|n_i|}_{j=1}$ be such that

$$(P_i)_{uv} = \{ e^{i\phi_{iu}} \parallel \alpha_{iu} \}, e^{i\phi_{iv}} \parallel \alpha_{iv} \}.$$ 

There are at least two ways in which this can be accomplished. One way would be to put both the sets of amplitude vectors into the same space, say $X = \mathbb{C}^n$ for $n' = \max\{n_1, n_2\}$, and to displace one of the sets by a large amplitude vector, $A \in X$. This way the inner products of the coherent states belonging to the same set of states remains invariant for appropriately defined phases, but the inner product of the coherent states belonging to different sets would tend to zero as the norm of the vector, $A$, tends to $\infty$. The second way, which we present here, is similar but it puts the amplitude vectors in different subspaces, and makes the distance between these subspaces go to $\infty$.

We will define coherent states $\{ e^{i\Phi_j} \parallel \beta_{j} \}^{|n|}_{j=1}$ dependent on a parameter, $A \in \mathbb{R}$, such that their Gram matrix will approach $P$ as $A \to \infty$. Define,

$$
n = n_1 + n_2$$

$$
\{ \beta_j \}^{|n|}_{j=1} \subset (\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}) \oplus (\mathbb{C} \oplus \mathbb{C})$$

$$
\{ \Phi_j \}^{|n|}_{j=1} \subset \mathbb{R}
$$

$$
\beta_j = \begin{cases} 
\alpha_j^1 \oplus 0 \oplus A \oplus 0 & j \leq n_1 \\
0 \oplus \alpha_j^2 \oplus 0 \oplus A & j' = j - n_1 > 0
\end{cases}
$$

$$
\Phi_j = \begin{cases} 
\phi_{1j} & j \leq n_1 \\
\phi_{2j'} & j' = j - n_1 > 0
\end{cases}
$$
Given these, one can check that the following hold for $u, v \in [n_1]$

$$
\begin{align*}
\|\beta_u - \beta_v\|_2^2 &= \|\alpha_u^1 - \alpha_v^1\|_2^2 \\
\text{Im}\{\langle\beta_u, \beta_v\rangle\} &= \text{Im}\{\langle\alpha_u^1, \alpha_v^1\rangle\}
\end{align*}
$$

$$
\Phi_u - \Phi_v = \phi_{1u} - \phi_{1v}.
$$

$$
\Rightarrow \langle e^{i\Phi_u} \|\beta_u\rangle, e^{i\Phi_v} \|\beta_v\rangle\rangle = \langle e^{i\phi_{1u}} \|\alpha_u^1\rangle, e^{i\phi_{1v}} \|\alpha_v^1\rangle\rangle.
$$

Similar relations hold for the case when $u, v > n_1$, although one needs to replace $u$ with $u' = u - n_1$ and $v$ with $v' = v - n_1$ on the RHS of these equations.

For $u \leq n_1$ and $v > n_1$, we have

$$
\begin{align*}
\|\beta_u - \beta_v\|_2^2 &= \|\alpha_u^1\|_2^2 + \|\alpha_v^2\|_2^2 + 2A^2 \geq 2A^2 \\
\Rightarrow |\langle e^{i\Phi_u} \|\beta_u\rangle, e^{i\Phi_v} \|\beta_v\rangle\rangle| \leq \exp(-A^2).
\end{align*}
$$

Therefore, for $P(A) = G(e^{i\Phi_1} \|\beta_1\rangle, \ldots, e^{i\Phi_n} \|\beta_n\rangle) \in G(C^n_{n+2}) = G(C^n_n)$ (by Corollary 1), and we have that

$$
\lim_{A \to \infty} P(A) = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \in G(C^n_n).
$$

This together with the observation that $P \in G(C^n_n) \iff P \pi P^\dagger \in G(C^n_n)$ for a permutation matrix, $P_\pi$, completes our proof.

5 Conclusion

In this paper, we have successfully characterised the set of Gram matrices of multi-mode coherent states and its closure. We provide simple tests to check if a Gram matrix belongs to either of these sets. We proved that no more than $(n - 1)$-modes are required to represent a Gram matrix of $n$-vectors. These results will hopefully serve as a toolbox for formulating quantum protocols in terms of coherent states, and facilitate their experimental implementation. They also add to our theoretical knowledge of coherent states, and completely describe sets of states attainable from them. We also expect our results to be beneficial towards understanding the kind of quantum resources a communication protocol requires.

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