A Calabi-Yau threefold with non-Abelian fundamental group

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The aim of this note is to answer a question of I. Dolgachev by constructing a Calabi-Yau threefold whose fundamental group is the quaternionic group $H$ with 8 elements. The construction is very reminiscent of Reid’s unpublished construction of a surface with $p_g = 0$, $K^2 = 2$ and $\pi_1 = H$; I explain below the link between the two problems.

1. The example

Let $H = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternionic group, and $V$ its regular representation. We denote by $\hat{H}$ the group of characters $\chi : H \to \mathbb{C}^*$; it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The group $H$ acts on $\mathbb{P}(V)$\footnote{I use Grothendieck’s notation, i.e. $\mathbb{P}(V)$ is the space of hyperplanes in $V$.} and on $S^2 V$; for each $\chi \in \hat{H}$, we denote by $(S^2 V)_\chi$ the eigensubspace of $S^2 V$ with respect to $\chi$, i.e. the space of quadratic forms $Q$ on $\mathbb{P}(V)$ such that $h \cdot Q = \chi(h) Q$ for all $h \in H$.

**Theorem.** For each $\chi \in \hat{H}$, let $Q_\chi$ be a general element of $(S^2 V)_\chi$. The subvariety $\tilde{X}$ of $\mathbb{P}(V)$ defined by the 4 equations $Q_\chi = 0 (\chi \in \hat{H})$ is a smooth threefold, on which the group $H$ acts freely. The quotient $X := \tilde{X}/H$ is a Calabi-Yau threefold with $\pi_1(X) = H$.

Let me observe first that the last assertion is an immediate consequence of the others. Indeed, since $\tilde{X}$ is a Calabi-Yau threefold, one has $h^{1,0}(\tilde{X}) = h^{2,0}(\tilde{X}) = \chi(\mathcal{O}_{\tilde{X}}) = 0$, hence $h^{1,0}(X) = h^{2,0}(X) = \chi(\mathcal{O}_X) = 0$. This implies $h^{1,0}(X) = 1$, so there exists a nonzero holomorphic 3-form $\omega$ on $X$; since its pull-back to $\tilde{X}$ is everywhere nonzero, $\omega$ has the same property, hence $X$ is a Calabi-Yau threefold. Finally $\tilde{X}$ is a complete intersection in $\mathbb{P}(V)$, hence simply connected by Lefschetz’ theorem, so the fundamental group of $X$ is isomorphic to $H$.

So the problem is to prove that $H$ acts freely and $\tilde{X}$ is smooth. We will need to write down explicitly the elements of $(S^2 V)_\chi$. As a $H$-module, $V$ is the direct sum of the 4 one-dimensional representations of $H$ and twice the irreducible two-dimensional representation $\rho$. Thus there exists a system of homogeneous coordinates $(X_1, X_\alpha, X_\beta, X_\gamma; Y, Z; Y', Z')$ such that

$$g \cdot (X_1, X_\alpha, X_\beta, X_\gamma; Y, Z; Y', Z') = (X_1, \alpha(g)X_\alpha, \beta(g)X_\beta, \gamma(g)X_\gamma; \rho(g)(Y, Z); \rho(g)(Y', Z'))$$.

To be more precise, I denote by $\alpha$ (resp. $\beta$, resp. $\gamma$) the nontrivial character which is $+1$ on $i$ (resp. $j$, resp. $k$), and I take for $\rho$ the usual representation via
Pauli matrices:

\[ \rho(i)(Y,Z) = (\sqrt{-1}Y, -\sqrt{-1}Z) \quad \rho(j)(Y,Z) = (-Z,Y) \quad \rho(k)(Y,Z) = (-\sqrt{-1}Z, -\sqrt{-1}Y). \]

Then the general element \( Q_\chi \) of \((S^2V)_\chi\) can be written

\[
Q_1 = t_1^1 X_1^1 + t_2^1 X_\alpha^2 + t_3^1 X_\beta^2 + t_4^1 X_\gamma^2 + t_5^1 (YZ' - Y'Z),
\]

\[
Q_\alpha = t_1^\alpha X_1 X_\alpha + t_2^\alpha X_\beta X_\gamma + t_3^\alpha YZ + t_4^\alpha Y'Z' + t_5^\alpha (YZ' + ZY'),
\]

\[
Q_\beta = t_1^\beta X_1 X_\beta + t_2^\beta X_\alpha X_\gamma + t_3^\beta (Y^2 + Z^2) + t_4^\beta (Y'^2 + Z'^2) + t_5^\beta (YY' + ZZ'),
\]

\[
Q_\gamma = t_1^\gamma X_1 X_\gamma + t_2^\gamma X_\alpha X_\beta + t_3^\gamma (Y^2 - Z^2) + t_4^\gamma (Y'^2 - Z'^2) + t_5^\gamma (YY' - ZZ').
\]

For \( t := (t_1^X) \) fixed, let \( \mathcal{X}_t \) be the subvariety of \( P(V) \) defined by the equations \( Q_\chi = 0 \). Let us check first that the action of \( H \) on \( \mathcal{X}_t \) is fixed point free for \( t \) general enough. Since a point fixed by an element \( h \) of \( H \) is also fixed by \( h^2 \), it is sufficient to check that the element \(-1 \in H\) acts without fixed point, i.e. that \( \mathcal{X}_t \) does not meet the linear spaces \( L_+ \) and \( L_- \) defined by \( Y = Z = Y' = Z' = 0 \) and \( X_1 = X_\alpha = X_\beta = X_\gamma = 0 \) respectively.

Let \( x = (0,0,0,0; Y,Z; Y',Z') \in \mathcal{X}_t \cap L_+ \). One of the coordinates, say \( Z \), is nonzero; since \( Q_1(x) = 0 \), there exists \( k \in \mathbb{C} \) such that \( Y' = kY \), \( Z' = kZ \). Substituting in the equations \( Q_\alpha(x) = Q_\beta(x) = Q_\gamma(x) = 0 \) gives

\[
(t_3^\alpha + t_5^\alpha k + t_4^\alpha k^2) YZ = (t_3^\beta + t_5^\beta k + t_4^\beta k^2) (Y^2 + Z^2) = (t_3^\gamma + t_5^\gamma k + t_4^\gamma k^2) (Y^2 - Z^2) = 0
\]

which has no nonzero solutions for a generic choice of \( t \).

Now let \( x = (X_1; X_\alpha, X_\beta, X_\gamma; 0,0,0,0) \in \mathcal{X}_t \cap L_+ \). As soon as the \( t_i^X \)'s are nonzero, two of the \( X \)-coordinates cannot vanish, otherwise all the coordinates would be zero. Expressing that \( Q_\beta = Q_\gamma = 0 \) has a nontrivial solution in \( (X_\beta, X_\gamma) \) gives \( X_\alpha^2 \) as a multiple of \( X_1^2 \), and similarly for \( X_\beta^2 \) and \( X_\gamma^2 \). But then \( Q_1(x) = 0 \) is impossible for a general choice of \( t \).

Now we want to prove that \( \mathcal{X}_t \) is smooth for \( t \) general enough. Let \( Q = \bigoplus_{\chi \in \hat{H}} (S^2V)_\chi \); then \( t := (t_1^X) \) is a system of coordinates on \( Q \). The equations \( Q_\chi = 0 \) define a subvariety \( \mathcal{X} \) in \( Q \times P(V) \), whose fibre above a point \( t \in Q \) is \( \mathcal{X}_t \). Consider the second projection \( p : \mathcal{X} \to P(V) \). For \( x \in P(V) \), the fibre \( p^{-1}(x) \) is the linear subspace of \( Q \) defined by the vanishing of the \( Q_\chi \)'s, viewed as linear forms in \( t \). These forms are clearly linearly independent as soon as they do not vanish. In other words, if we denote by \( B_\chi \) the base locus of the quadrics in \((S^2V)_\chi\) and put \( B = \cup B_\chi \), the map \( p : \mathcal{X} \to P(V) \) is a vector bundle fibration above \( P(V) \to B \); in particular \( \mathcal{X} \) is non-singular outside \( p^{-1}(B) \). Therefore it is enough to prove that \( \mathcal{X}_t \) is smooth at the points of \( B \cap \mathcal{X}_t \).
Observe that an element $x$ in $B$ has two of its $X$-coordinates zero. Since the equations are symmetric in the $X$-coordinates we may assume $X_\beta = X_\gamma = 0$. Then the Jacobian matrix \( \frac{\partial Q_x}{\partial X_\psi}(x) \) takes the form
\[
\begin{pmatrix}
2t_1^1 X_1 & 2t_2^1 X_\alpha & 0 & 0 \\
t_1^\alpha X_\alpha & t_1^\beta X_1 & 0 & 0 \\
0 & 0 & t_1^\beta X_1 & t_2^\beta X_\alpha \\
0 & 0 & t_2^\gamma X_\alpha & t_1^\gamma X_1
\end{pmatrix}.
\]
For generic $t$ this matrix is of rank 4 except when all the $X$-coordinates of $x$ vanish; but we have seen that this is impossible when $t$ is general enough.

2. Some comments

As mentioned in the introduction, the construction is inspired by Reid’s example of a surface of general type with $p_g = 0$, $K^2 = 2$, $\pi_1 = H$ [R]. This is more than a coincidence. In fact, let $\tilde{S}$ be the hyperplane section $X_1 = 0$ of $\tilde{X}$. It is stable under the action of $H$ (so that $H$ acts freely on $\tilde{S}$), and one can prove as above that it is smooth for a generic choice of the parameters. The surface $S := \tilde{S}/H$ is a Reid surface, embedded in $X$ as an ample divisor, with $h^0(X, O_X(S)) = 1$.

In general, let us consider a Calabi-Yau threefold $X$ which contains a rigid ample surface — i.e. a smooth ample divisor $S$ such that $h^0(O_X(S)) = 1$. Put $L := O_X(S)$. Then $S$ is a minimal surface of general type (because $K_S = L|_S$ is ample); by the Lefschetz theorem, the natural map $\pi_1(S) \to \pi_1(X)$ is an isomorphism. Because of the exact sequence
\[
0 \to O_X \to L \to K_S \to 0
\]
the geometric genus $p_g(S) := h^0(K_S)$ is zero.

One has $K_S^2 = L^3$; the Riemann-Roch theorem on $X$ yields
\[
1 = h^0(L) = \frac{L^3}{6} + \frac{L \cdot c_2}{12}
\]
by Miyaoka theorem [Mi] one has $L \cdot c_2 > 0$ (the strict inequality requires playing around a little bit with the index theorem), hence $K_S^2 \leq 5$.

With a few exceptions, the possible fundamental groups of surfaces with $p_g = 0$ and $K_S^2 = 1$ or 2 are known (see [B-P-V] for an overview). In the case $K_S^2 = 1$, the fundamental group is cyclic of order $\leq 5$; if $K_S^2 = 2$, it is of order $\leq 9$; moreover the dihedral group $D_8$ cannot occur. I believe that the symmetric group $S_3$ cannot occur either, though I do not think the proof has been written down. If this is true, the quaternionic group $H$ is the only non-Abelian group which occurs in this range.
On the other hand, little is known about surfaces with \( p_g = 0 \) and \( K_\mathcal{S}^2 = 3, 4 \) or 5. Inoue has constructed examples with \( \pi_1 = H \times (\mathbb{Z}_2)^n \), with \( n = K^2 - 2 \) (loc. cit.); I do not know if they can appear as rigid ample surfaces in a Calabi-Yau threefold.

Let us denote by \( \tilde{X} \) the universal cover of \( X \), by \( \tilde{L} \) the pull back of \( L \) to \( \tilde{X} \), and by \( \rho \) the representation of \( G \) on \( H^0(\tilde{X}, \tilde{L}) \). One has \( \text{Tr} \rho(g) = 0 \) for \( g \neq 1 \) by the holomorphic Lefschetz formula, and \( \text{Tr} \rho(1) = \chi(\tilde{L}) = |G| \chi(L) = |G| \). Therefore \( \rho \) is isomorphic to the regular representation. Looking at the list in loc. cit. one gets a few examples of this situation, for instance:
- \( G = \mathbb{Z}_5 \), \( \tilde{X} \) is a quintic hypersurface in \( \mathbb{P}^4 \);
- \( G = (\mathbb{Z}_2)^3 \) or \( \mathbb{Z}_4 \times \mathbb{Z}_2 \), \( \tilde{X} \) is an intersection of 4 quadrics in \( \mathbb{P}^7 \) as above;
- \( G = \mathbb{Z}_3 \times \mathbb{Z}_3 \), \( \tilde{X} \) is a hypersurface of bidegree \((3, 3)\) in \( \mathbb{P}^2 \times \mathbb{P}^2 \).

Of course when looking for Calabi-Yau threefolds with interesting \( \pi_1 \) there is no reason to assume that it contains an ample rigid surface. Observe however that if we want to use the preceding method, i.e. find a projective space \( \mathbb{P}(V) \) with an action of \( G \) and a smooth invariant linearly normal Calabi-Yau threefold \( \tilde{X} \subset \mathbb{P}(V) \), then the line bundle \( \mathcal{O}_{\tilde{X}}(1) \) will be the pull-back of an ample line bundle \( L \) on \( X \), and by the above argument the representation of \( G \) on \( V \) will be \( h^0(L) \) times the regular representation. This leaves little hope to find an invariant Calabi-Yau threefold when the product \( h^0(L) |G| \) becomes large.

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