Future stability of the $1+3$ Milne model for the Einstein–Klein–Gordon system

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Abstract
We study the small perturbations of the $1+3$-dimensional Milne model for the Einstein–Klein–Gordon (EKG) system. We prove the nonlinear future stability, and show that the perturbed spacetimes are future causally geodesically complete. For the proof, we work within the constant mean curvature (CMC) gauge and focus on the $1+3$ splitting of the Bianchi–Klein–Gordon equations. Moreover, we treat the Bianchi–Klein–Gordon equations as evolution equations and establish the energy scheme in the sense that we only commute the Bianchi–Klein–Gordon equations with spatially covariant derivatives while normal derivative is not allowed. We propose some refined estimates for lapse and the hierarchies of energy estimates to close the energy argument.

Keywords: Milne model, Einstein–Klein–Gordon, Cauchy problem, stability

Contents

1. Introduction 2
   1.1. The EKG system 2
   1.2. Main result 5
   1.3. Strategy of the proof 5
      1.3.1. Energy scheme 5
      1.3.2. The coupling structures 6
      1.3.3. Hierarchies of energy estimates 6
2. Preliminaries 8
   2.1. Notations and conventions 8
   2.2. The background materials 8
      2.2.1. Lorentzian geometric equations 9
      2.2.2. The background spacetime 10
   2.3. Local existence 10
3. $1+3$ splitting and the energy scheme 11
1. Introduction

1.1. The EKG system

The Einstein–Klein–Gordon (EKG) system takes the form of

\[ \hat{\mathcal{R}}_{\alpha\beta} - \frac{1}{2} \hat{\mathcal{R}} \hat{g}_{\alpha\beta} = \mathcal{T}_{\alpha\beta}(\phi), \]  
\[ \mathcal{T}_{\alpha\beta}(\phi) = \hat{D}_\alpha \phi \hat{D}_\beta \phi - \frac{1}{2} \hat{g}_{\alpha\beta} \left( \hat{D}^\mu \phi \hat{D}_\mu \phi + m^2 \phi^2 \right), \]

(1.1a)

(1.1b)
where $\tilde{R}_{\alpha\beta}$ and $\tilde{R}$ denote the Ricci and scalar curvature of an unknown Lorentzian metric $\tilde{g}_{\alpha\beta}$, respectively, and $T_{\alpha\beta}(\phi)$ is the energy momentum tensor for a massive scalar field $\phi$. The Bianchi identities imply that the scalar field $\phi$ satisfies the Klein–Gordon (KG) equation

$$\Box g \phi - m^2 \phi = 0.$$  \hspace{1cm} (1.2)

$D$ denotes the covariant derivative associated to $\tilde{g}_{\alpha\beta}$ and the geometric wave operator is $\Box_g = \tilde{D}_\alpha \tilde{D}^\alpha$. We use $m^2$ to denote the mass of the KG field.

Our first motivation for studying the EKG system comes from the problem of nonlinear stability of the Kaluza–Klein spacetime, which takes the form of $M = \mathbb{R}^{1+3} \times S^1$ [31]. Then the five-dimensional metric depending on $S^1$ periodically can be Fourier-expanded. The projections of the five-dimensional Einstein equations onto the zero (massless) mode and the non-zero (massive) modes give rise to a wave-Klein–Gordon system [32]. Subject to the zero mode perturbation, both of the works [43] by Wyatt and [10] by Branding–Fajman–Kroencke, have recently contributed on this aspect. [43] is related to the nonlinear stability of the Kaluza–Klein spacetime with the base on the Minkowski space, while [10] is concerning with the one based on the Milne model. There is also relevant result [30] by Choquet-Bruhat and Moncrief, who establish the stability of $U(1)$-symmetric spacetimes to $1+3$-dimensions.

The backgrounds that we consider are the following family of cosmological vacuum spacetimes. Let $\bar{M}$ be a 4-manifold of the form $I \times \Sigma$, where $I \subset \mathbb{R}$ is an interval, $\Sigma$ is a compact 3-manifold admitting an Einstein metric $\gamma$ with negative Einstein constant $\lambda$. We choose $\lambda = -2$ so that the Einstein metric $\gamma$ is a hyperbolic metric with sectional curvature $-1$. Then $(\bar{M}, \bar{\gamma})$ with $\bar{\gamma}$ given by

$$\bar{\gamma} = -dt^2 + t^2 \gamma$$

is a solution to the vacuum Einstein equations and known as an $1+3$-dimensional Milne model. This model undergoing accelerated expansion in the future direction is locally isometric to the $k = -1$ vacuum Friedmann–Lemaitre–Robertson–Walker (FLRW) model. Andersson and Moncrief [7] first consider the nonlinear stability of this model (called hyperbolic cone spacetime there), and show that for the constant mean curvature (CMC) Cauchy data for the vacuum Einstein equations close to the standard data for the hyperbolic cone spacetime, the maximal future Cauchy development is globally foliated by the CMC Cauchy surfaces and causally geodesically complete to the future. This also motivates the current project. We refer to [1, 3, 4, 20] and references therein for more backgrounds on the existence of CMC foliations.

There are several results concerning the decay of the KG field [22, 23, 37, 38]. In contrast to the wave equation, the KG equation is not conformal invariant and hence does not commute well with the symmetry of scaling. Due to this fact, Klainerman [23] employs the hyperboloid foliations which respect the Lorentz invariance of the KG operator, and derives the asymptotic behavior of the KG field by the energy method alone. Furthermore, Katayama [22] somehow overcomes the incompatibility between the wave and the KG field. Other method such as using Fourier analysis for the wave-Klein–Gordon system is established by Ionescu–Pausader [21]. Besides, the global solutions to the semilinear KG equations in the FLRW spacetimes are studied by Galstian–Yagdjian [19].

Let us review some results on the nonlinear stability of the Minkowski spacetime. Christodoulou and Klainerman [14] initiate a covariant proof based on the Bel–Robinson energy. Their proof relies upon the geometric foliations of spacetime, including the maximal $t = \text{const}$ slices and a family of outgoing null cones, and the null condition hidden inside the Bianchi equations. After that, Lindblad and Rodnianski [24, 25] devise another proof based on
the wave coordinate gauge, under which the vacuum Einstein equations satisfy the weak null condition. This method is extended to the massive Einstein–Vlasov system by Lindblad and Taylor [26] lately. All of these works require the full symmetries of the Minkowski spacetime, including the conformal symmetry. It suggests that these methods do not apply to the EKG system straightforwardly. Recently, inspired by [22] and [24, 25], Ma and LeFloch [27, 28] make use of the hyperboloid method to address an energy argument for both of the wave and KG equations uniformly, and then prove the nonlinear stability of the Minkowski spacetime for the EKG system. In addition to the \( L^\infty \to L^\infty \) estimate for the wave equation [22, 24, 25], Ma and LeFloch [27] also demonstrate an \( L^\infty \to L^\infty \) estimate for the KG equation, which is crucial in dealing with the nonlinear couplings [28]. Through this hyperboloid method, the massive Einstein–Vlasov system is alternatively understood by Fajman, Joudioux and Smulevici [15]. Similar to the KG field, the massless Vlasov field is lack of the conformal symmetry as well. Taylor [39] proves the stability for the massless Einstein–Vlasov system with some compact support assumptions, which in fact reduces the problem to a semi-global one since the matter is shown to be supported in a strip going to null infinity.

The works [7, 8] also constitute the results on the nonlinear stability for the vacuum Einstein equations. The proof of [7] is based on the Bel–Robinson energy and its higher-order generalization. Our proof possesses some similarities to that of [7] as far as we both treat the Bianchi equations as evolution equations to derive the energy estimates for the Weyl fields. Compared to [7], the backgrounds considered in [8] are extended to a family of \( 1 + n \)-dimensional Milne model. And for the proof in [8], the authors turn to the Einstein evolution equations and use a wave equation type of energy introduced in [6] for the energy estimates. Related to this, Andersson and Fajman [5] show the future stability of the \( 1 + 3 \)-dimensional Milne model for the massive Einstein–Vlasov system. We remark that the works [5, 7, 8] are all based on the constant mean curvature, spatially harmonic (CMCSH) gauge, with which the local existence theorem for the vacuum Einstein equations is proved in [6].

We would also like to mention some contributions related to the future-global stability of spacetimes featuring accelerated expansion with a positive cosmological constant. We only list some of them to see the rough picture, [2, 18, 34, 36].

In this paper, we are concerning the nonlinear stability of the \( 1 + 3 \)-dim Milne model for the EKG system. We work within the CMC gauge with zero shift. The local existence theorem then follows by adjusting the argument in [14] where the maximal gauge with zero shift is considered. Our main difficulty lies in the fact that the KG field is not scale invariant and hence scaling does not qualify as a commutator vector field. For this reason, we recast the KG equation in a first order form, focusing on the \( 1 + 3 \) form of the Bianchi–Klein–Gordon equations, and propose an energy argument in which the high order energies contain only high order of spatially covariant derivatives. Consequently, the (high order) normal derivative of the KG field possesses no good estimate, and hence is referred as a bad derivative. Meanwhile, our covariant proof avoids some unnecessarily nonlinear couplings which will arise when working in coordinates (see [28] for the Minkowski case). Besides, the ‘genuinely’ coupling structures admit some cancellations, such that in the Bianchi equations, the nonlinear couplings involve no high order of bad (normal) derivatives of the KG field. However, our difficulty does not disappear now, because if we proceed to the top order energy estimate, some nonlinear couplings become borderline terms due to the restriction of regularity. Another difficulty arises from the resulted borderline terms when commuting the spatially covariant derivatives with the KG equation. Fortunately, these borderline terms exhibit linearizable feature. Therefore, motivated by [13] (see also [29, 40–42]), we resolve these difficulties by the method of linearization and hierarchies of energy estimates. In particular, we use the \( L^\infty \to L^\infty \) estimate for the KG equation [27] in the procedure of linearization. Additionally, we reveal the relation
between the lapse and the KG field, with which we manage to improve the estimates step by step and finally reduce the nonlinear borderline terms into linear ones.

Based on our work, Fajman and Wyatt [17] alternatively achieve such stability result with the CMCSH gauge as well.

1.2. Main result

Before stating the main result, we introduce some notations. Let \((\tilde{M}, \tilde{g})\) be the maximal development of the EKG data. \((\tilde{M}, \tilde{g})\) is diffeomorphic to \(\mathbb{R} \times \Sigma\), where \(\Sigma\) is a compact Cauchy surface and it is globally foliated by a CMC foliation \(\{\Sigma_\tau, \tau \in I\}\). We rescale the CMC time function \(\tau\) and define \(t := -\frac{1}{\tau}\). Then, as \(\tau \to 0^+\), there is \(t \to +\infty\). Let \(\bar{\partial}_t\) denote the coordinate vector field corresponding to \(t\), then \(\bar{g}(\bar{\partial}_t, \bar{\partial}_t) = -N^2\), where \(N\) is called lapse, and we set \(N = N - 1\). Let \(\bar{g}_i\) be the induced Riemannian metric on \(\Sigma_t\) (we always reparametrize \(\{\Sigma_\tau, \tau \in I\}\) by \(t\) in what follows), \(\bar{\nabla}\) the corresponding covariant derivative. \(T\) is the unit time-like vector field normal to \(\Sigma_t\), and we have \(\bar{\partial}_t = NT\). The second fundamental form is denoted by \(\bar{k}_i\), while \(\bar{k}_i\) is the traceless part of \(\bar{k}_i\). In the CMC gauge, \(\bar{k}_i\) is future causally geodesic.

Let \(g_0 = t^{-2}\bar{g}_i\) be the spatially normalized metric, \(\nabla\) the corresponding connection. As shown in (4.20a)–(4.20c) in [7], the following variables are scale-free (the indices refer to a coordinate frame):

\[
\begin{align*}
g_{ij} &= t^{-2}\bar{g}_{ij}, & g^{ij} &= \bar{g}^{ij}, \\
k_{ij} &= t^{-1}\bar{k}_{ij}, & \text{tr}k &= t\text{tr}\bar{k}, \\
T &= tT, & \text{d}k_g &= t^{-3}\text{d}\bar{k}_g.
\end{align*}
\]

\(\text{tr}\), \(\text{tr}\) mean taking trace with respect to \(\bar{g}, g\), respectively. It is convenient to introduce the logarithmic time \(\bar{T} := \ln t\), which has the property that \(\partial_{\bar{T}} = i\partial_t\) is scale-free. We also define the rescaled mass for the KG field,

\[
m(t) = mt. \tag{1.4}
\]

We remark that the rescaling (1.4) is transitional, aiming to make the KG energy well-behaved with respect to the above scale-free variables. Finally, we let \(\phi\) be the KG field, and \(E_0, H_0\) be the electric and magnetic parts of the Weyl field \(W_{\mu\nu\lambda\beta}\), refer to (3.62). Note that, \(E_0, H_0\) are scale-free as well.

**Theorem 1.1.** Let \(\Sigma\) be a compactly hyperbolic 3-manifold without boundary. Assume that \((\Sigma, g_0, k_0, \phi_0, \phi_1)\) is a rescaled data set for the EKG system (1.1a), (1.1b) and (1.2), where \(g_{0ij}, k_{0ij}\) are rescaled as (1.3) and \(\phi_0 = \phi|_{t=0}, \phi_1 = T\phi|_{t=0}\). Let \(t_0 > \text{max}\{1, 3m^{-1}\}\). Then there is an \(\varepsilon > 0\), so that if

\[
\begin{align*}
\|R_{0ij} + 2g_{0ij}\|^2_{H^1(\Sigma_{t_0})} + \|k_{0ij} + g_{0ij}\|^2_{H^1(\Sigma_{t_0})} &\leq \varepsilon^2, \\
\|\phi_0\|^2_{H^1(\Sigma_{t_0})} + \|\phi_1\|^2_{H^1(\Sigma_{t_0})} &\leq \varepsilon^2,
\end{align*}
\]

where \(R_{0ij}\) denotes the Ricci curvature of \(g_0\), the maximal development \((\tilde{M}, \tilde{g})\) of the EKG data has a global CMC foliation in the expanding direction and \((\tilde{M}, \tilde{g})\) is future causally geodesically complete.

1.3. Strategy of the proof

1.3.1. Energy scheme. As explained before, the scaling \(\partial_{\bar{T}}\) (and hence \(T\)) does not qualify as a commutator vector field for the KG equation. Therefore, we only commute the KG
equation with the spatially covariant derivative $\nabla$. Technically, we address an $1 + 3$ splitting for the KG equation, which takes the form of

$$\mathcal{L}_\phi \hat{T} \phi - (\text{tr}N + 1) \hat{T} \phi - N\nabla\nabla \phi + \bar{m}^2(t)N\phi = \nabla_i N\nabla_i \phi.$$ 

$\mathcal{L}$ means taking Lie derivative. Thus in such an $1 + 3$ form, the high order KG equation (3.32) can be derived by calculating the commutator $[\mathcal{L}_\phi, \nabla]$ (see section 3.1), which involves only $N, \bar{k}$ and their spatial derivatives. The Bianchi equations are treated in an analogous fashion. We then establish the energy inequalities for these $1 + 3$ evolution equations (see sections 3.2 and 3.3). The energies are defined as below: fixing an integer $I$, for $t > t_0 > 3m^{-1}$, the energy norms for $\phi$ and $W(E, H)$ are

$$E_i(\phi, t) = \|\nabla^i \hat{T} \phi\|_{L^2(\Sigma_t)}^2 + \|\nabla^2 \nabla \phi\|_{L^2(\Sigma_t)}^2 + \|\nabla^l (\bar{m}(t)\phi)\|_{L^2(\Sigma_t)}^2, \quad I \leq I,$n

$$E_i(W, t) = \|\nabla^i E\|_{L^2(\Sigma_t)}^2 + \|\nabla^1 H\|_{L^2(\Sigma_t)}^2, \quad I \leq I - 1.$$ 

The $L^2(\Sigma_t)$ norm is defined with respect to the volume form $d\mu_t$ (1.3) and the notation $\nabla^l$ means taking the derivative $\nabla$ $l$ times. The energy (1.6) in fact comes from the modified energy (3.27).

In other words, we give up the standard method of energy-momentum tensor. Alternatively, we recast the KG equation in a first order form and treat it as a transport equation rather than a wave equation. This turns out to be more compatible with the energy scheme, in which we commute the KG equation merely with spatial derivatives.

### 1.3.2. The coupling structures.

In the strategy of the $1 + 3$ splitting, the normal derivative $\hat{T}$ is referred as a bad derivative, while the spatial derivative $\nabla$ is a good derivative. This is implied by the following facts: the energy (1.6) does not control $\hat{T}^2\phi$. However, making use of the KG equation which expresses $T^2\phi$ in terms of $\hat{T}\phi$ and $\nabla^l \phi$, $0 \leq i \leq 2$, in turn yields almost $t_i^2$ growth for $T^2\phi$. On the other hand, implied by the Sobolev inequalities, terms involving the spatial derivatives $\nabla^i T \phi, \nabla^i \nabla \phi, i \leq I - 2$ decay almost like $t_i^{-2}$, and $\nabla^i \phi \sim t^{-1} \nabla^i (\bar{m}(t)\phi)$, $i \leq I - 2$ decay almost like $t_i^{-4}$.

One of our key observations is that the nonlinear couplings avoid $\hat{T}^2\phi$, due to some cancellations (refer to (6.16) and (6.18)). Instead, we get couplings such as $\nabla \hat{T} \phi \nabla E$ and $\nabla^2 \phi \hat{T} E$, which generally decay fast enough. Now, let us pay more attention to the estimates for $\nabla^2 \hat{T} E$ and its higher order versions $\nabla^l (\nabla^2 \hat{T} E)$, $I \leq I - 1$. Notice that, for the lower order cases: $I \leq I - 2$, $\|\nabla^2 \hat{T} E\|_{L^2(\Sigma_t)}$ is estimated as $t^{-1} \|\nabla^i (\bar{m}(t)\phi)\|_{L^2(\Sigma_t)}$. Unfortunately, this estimate is forbidden for the top order case: $I = I - 1$, since it requires one more derivative than our purposes. That is to say, $\|\nabla^I \nabla^2 \phi\|_{L^2(\Sigma_t)}$ has no additional $t_i^{-1}$ factor. As a result, the top order coupling $\nabla^I - 1 (\nabla^2 \phi \hat{T} E)$ will provide a borderline term $\nabla^I - 1 \nabla^2 \hat{T} E$ which requires more work.

### 1.3.3. Hierarchies of energy estimates.

We start with some weak bootstrap assumptions (with $\delta, 0 < \delta < \frac{1}{9}$ loss in the decay rate of $t$), and finally close the bootstrap argument by sharpening these estimates (section 7.1). In particular, the estimates for $\bar{k}$ are retrieved by the transport equation and the elliptic system for $\bar{k}$ (section 7.1.2).

When commuting $\nabla^l (1 \leq l \leq I)$ with the KG equation, we encounter the borderline term $BL_\lambda$, taking the form of $BL_\lambda = \sum_{a=0}^{l-1} BL_a^\lambda$, with

$$BL_a^\lambda = \bar{m}(t) \nabla^{l-a-1} \hat{N} \ast \nabla^l (\bar{m}(t)\phi) \ast \nabla^a \hat{T} \phi + m(t) \nabla^{l-a} \hat{N} \ast \nabla^l \phi \ast \nabla^{a-1} (\bar{m}(t)\phi).$$

(1.8)
We confirm the inevitability of these borderline terms in remark 3.10. Meanwhile, we should notice that $BL_i$ will vanish if the scalar field is massless.

Because of the weak bootstrap assumptions, $BL_i, 1 \leq I \leq I$ bear $\delta$ loss in the decay rate of $t$. We make use of the $L_\infty - L_\infty$ estimate for the KG field [27] to improve $\|T\phi\|_{L_\infty}, \|\tilde{m}(t)\phi\|_{L_\infty}$ (refer to corollary 5.1), and then retrieve the $\delta$ loss in $BL_i, 1 \leq I \leq I$. To do this, it costs the regularity: the $L_\infty - L_\infty$ estimate requires the strongest decay estimate for $\nabla^2 \phi$, which further forces us to perform two more orders of energy estimates. That is, in (1.6) and (1.7), we take $I = 4$ rather than $I = 2$.

We observe the linear structure in (1.8): $\nabla^{a-1}(\tilde{m}(t)\phi), \nabla^{a-1}T\phi$ are of lower orders (less than $l$ orders) and also $\nabla^{2-a+1}\hat{N}$ depends mainly on the lower order energies of the KG field.

In more detail, the refined estimates for the lapse (lemma 4.2) tell that $\|\nabla^i\hat{N}\|_{L^2(\Sigma_t)}, i \leq 2$ rely crucially on the zeroth order energy $E_0(\phi, t)$, and $\|\nabla^i\hat{N}\|_{L^2(\Sigma_t), 2 < I \leq I}$ are primarily related to $E_{I-2}(\phi, t)$, which is again of lower order. Enlightened by these features, we proceed by an argument of hierarchies of energy estimates for the KG field, which is carried out in section 5.3. The idea is that, the information already obtained for lower order energies will help to sharpen the estimates for $\nabla^{I-1-a+1}\hat{N}, \nabla^{a-1}(\tilde{m}(t)\phi), \nabla^{a-1}T\phi$ and reduces the nonlinear structure of (1.8) to a linear one. In addition, this idea of linearization also applies to the borderline terms (like $\nabla^{I-1}\nabla^2 \phi T\phi \hat{E}$) that arise in the top order energy estimate for the Weyl field, since we can now take advantage of the improved estimates for $\phi$, see (6.27).

We here sketch the linearization in the energy argument. Generically, the energy inequalities for the KG field before linearization take the form of ($l \leq 4$)

$$\partial_t E_l(\phi, t) + E_l(\phi, t) \lesssim t^{-1}E_l(\phi, t) + tE_l(\phi, t)E^\frac{1}{2}_l(\phi, t),$$

where the second term on the right hand side is a rough bound for the borderline terms (1.8), and $E_l(\phi, t)$ means the energies of the KG field up to four orders. Here the first quadratic term $E_l(\phi, t)$ is due to the fact that $\nabla^{I-1-a+1}\hat{N}, 1 \leq a \leq I \leq 4$ in (1.8) depend quadratically on $\phi$ (refer to lemma 4.2 and remark 4.3). We aim to linearize the borderline terms via the hierarchies of energy estimates and the $L_\infty - L_\infty$ estimate for the KG field, so that the energy inequality is improved as (see section 5.3): $l \leq 4$,

$$\partial_t E_l(\phi, t) + E_l(\phi, t) \lesssim t^{-1}E_l(\phi, t) + t^{-1}E_l(\phi, t)E^\frac{1}{2}_l(\phi, t)$$

$$\lesssim \left( t^{-1} + \epsilon M \right) E_l(\phi, t) + \epsilon^2 I^2 \epsilon M t^{-1}.$$  

Here $I$ is a constant depending on the initial data. Specifically, the previously quadratic term is now replaced by a linear one. Changing to $t$, we have

$$\partial_t (tE_l(\phi, t)) \lesssim \left( t^{-2} + \epsilon M t^{-1} \right) (tE_l(\phi, t)) + \epsilon^2 I^2 \epsilon M t^{-1}.$$ 

The Grönwall inequality then yields that $tE_l(\phi, t) \lesssim \epsilon^2 I^2 \epsilon M \omega$, where $C_M$ depends linearly on $M$. Thus the energy argument is closed.

The paper is organized as follows. In section 2, we describe some conventions and notations, and introduce the background materials. In section 3, we conduct the $1 + 3$ splitting for the Bianchi–Klein–Gordon system and establish the energy scheme. In section 4, we introduce the bootstrap assumptions, and present some preliminary estimates that follow from the Sobolev inequalities. The energy estimates for the KG field and the Weyl field are carried out in sections 5 and 6. Section 7 is devoted to close the bootstrap argument and prove the global existence theorem, geodesic completeness. Finally, a number of useful definitions, identities and ODE estimates are collected in the appendix.
2. Preliminaries

2.1. Notations and conventions

Indices:
- $\alpha, \beta, \ldots, \mu, \nu, \ldots$: Greek indices (spacetime indices) run over $0, \cdots, 3$.
- $i, j, \ldots, p, q, \ldots$: Latin indices (spatial indices) run over $1, \cdots, 3$.

Geometry related:
- $(M, \bar{g}_{\mu\nu}, D)$: the rescaled spacetime manifold with $D$ the connection.
- $(\Sigma_t, g_{ij}, \nabla)$: the (rescaled) induced spatial manifold with $\nabla$ the connection.
- $\Box = D^\mu D_\mu$: the rescaled spacetime wave operator.
- $\Delta = \nabla^i \nabla_i$: the (rescaled) spatial Laplacian operator.
- $d\mu_\xi$, $d\mu_\psi$: the volume form with respect to $\bar{g}$ or $g$.
- $R_{\mu\alpha\nu\beta}, \bar{R}_{\mu\nu}, R$: the components of the Riemann, Ricci, scalar curvature tensor of $(M, \bar{g}_{\mu\nu})$.
- $R_{\mu\nu\alpha\beta}, R_{\mu\nu}$: the components of the Riemann, Ricci, scalar curvature tensor of $(\Sigma_t, g_{ij})$.
- $R_{\mu\nu\alpha\beta}$: projections of $R_{\mu\nu\alpha\beta}, \bar{R}_{\mu\nu}$ to $(\Sigma_t, g_{ij})$.

Foliation related:
- $\tau, t, \tilde{\tau}$: $\tau$ is the CMC time; $t = -\frac{3}{2}\bar{\tau} = \ln t$.
- $\partial_\tau, \partial_t$: coordinate vector fields corresponding to $t, \tau$.
- $T, \bar{T}$: $T$ denotes the future directed, unit vector normal to $\Sigma_t$; $\bar{T}$ is the rescaled normal vector field, $\bar{T} = iT$.

Simplified conventions:
- $\nabla_i \psi$: the $i$th order covariant derivative $\nabla i \cdots \nabla \psi$ and the multi index $I_j = \{i_1 \cdots i_l\}$ is used.
- $\nabla_i \cdots \tilde{\nabla}_j \psi$: $\nabla_i \cdots \nabla_j \psi$ with the $j$th $\nabla_j$ being replaced by $\tilde{\nabla}_j$.
- $D\phi$: any element in $\{T\phi, \nabla \phi, m(t)\phi\}$.
- $f_1 \lesssim f_2$: $f_1 \leq C f_2$ with some universal constant $C$.
- $f_1 \sim f_2$: $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$.
- $M$: a constant depending linearly on some controlling quantity $\Lambda$.
- $A \ast B$: any finite sum of products of $A$ and $B$, with each product being a contraction (with respect to $g$) between the two $\Sigma_t$-tensors $A$ and $B$.

2.2. The background materials

Let $\Sigma$ be a spacelike hypersurface and let $M = I \times \Sigma$ be an $1 + n$-dim manifold with Lorentz metric $\bar{g}$ of signature $- + \cdots +$. We introduce local coordinates $(\tau, x^i, i = 1, \cdots, n)$ on $M$ so that $\tau$ is a time function and $x^i$ are coordinates on the level sets $\Sigma_\tau$ of $\tau$.

Let $\partial_\tau = \partial / \partial \tau$ be the coordinate vector field corresponding to $\tau$. The lapse function $\gamma N$ and the shift vector field $X_\tau$ of the foliation $\{\Sigma_\tau\}$ are defined by $\partial_\tau = \gamma N T + X_\tau$, where $T$ is the unit timelike vector field normal to $\Sigma_\tau$. Assume $T$ is future directed so that $\gamma N > 0$. We

\footnote{In the estimates, we will only employ the formula $\|A \ast B\| \lesssim \|A\| \|B\|$ which allows ourselves to ignore the detailed product structure at this point. $\| \cdot \|$ denotes the norm associated to $g$.}
make the gauge choice \( X_t = 0 \), so that the spacetime metric \( \bar{g} \) with vanishing shift takes the form of

\[
g_{\mu\nu} = -\gamma N^2 d\tau^2 + \bar{g}_{ij} dx^i dx^j.
\]  

(2.1)

The second fundamental form \( k_{ij} \) of \( \Sigma_\tau \) is given by \( \tilde{k}_{ij} = -\frac{1}{2} \mathcal{L}_T \bar{g}_{ij} \). The CMC gauge is referred to as

\[
\text{tr}_g \tilde{k} = \tau.
\]  

(2.2)

We call \( \tau \) the CMC time function. For notational convenience, we define a rescaled time function \( t := -\frac{3}{\tau} \) so that the corresponding lapse \( \bar{N} \) takes

\[
iN = \frac{3}{\tau^2} N, \quad \partial_t = iN \tau.
\]  

(2.3)

We will drop the subscript \( t \), and denote the lapse for the \( t \)-foliation \( \{ \Sigma_t \} \) by \( N \). We also let

\[
\bar{N} = N - 1.
\]  

(2.4)

In terms of the \( t \)-foliation \( \{ \Sigma_t \} \),

\[
\text{tr}_g \tilde{k} = -\frac{3}{t}, \quad \tilde{k}_{ij} = -\frac{1}{t} \bar{g}_{ij} + \tilde{k}_{ij},
\]  

(2.5)

where \( \tilde{k}_{ij} \) is the trace free part of \( \tilde{k}_{ij} \). We also introduce the logarithmic time \( \bar{\tau} := \ln t \), so that \( \partial_{\bar{\tau}} = t \partial_t \) is scale-free.

Performing the rescaling (1.3), the rescaled curvature is related to the original one by

\[
\bar{R}_{imjn} = t^{-2} \bar{R}_{imjn}, \quad \bar{R}_{ij} = \bar{R}_{ij}, \quad \bar{R} = t^2 \bar{R}.
\]

Here \( \bar{R}_{imjn}, \bar{R}_{ij}, \bar{R} \) denote the components of the Riemannian, Ricci, scalar curvature tensor of \( (\Sigma_t, \bar{g}_0) \). And we introduce the rescaled spacetime metric as well:

\[
\bar{g}_{\mu\nu} = t^{-2} \bar{g}_{\mu\nu}, \quad \bar{g}^{\mu\nu} = t^2 \bar{g}^{\mu\nu}, \quad \bar{T} = t\bar{T},
\]

\[
\bar{R}_{\mu\alpha\nu\beta} = t^{-2} \bar{R}_{\mu\alpha\nu\beta}, \quad \bar{R}_{\mu\nu} = \bar{R}_{\mu\nu}, \quad \bar{R} = t^2 \bar{R}.
\]

We remark that, if \( \tau \in [\tau_0, 0) \) with \( \tau_0 < 0 \), then \( t \in [t_0, +\infty) \), where \( t_0 = -\frac{3}{\tau_0} \). Letting \( m^2 \) be the mass of the KG field (1.2), we require that \( |\tau_0| < m \), i.e. \( t_0 > 3m^{-1} \).

2.2.1. Lorenzian geometric equations. Recall the first and the second variation equations for the rescaled metric (in CMC gauge with zero shift)

\[
\mathcal{L}_{\partial_{\bar{\tau}}} g_{ij} = 2\bar{N} g_{ij} - 2N \tilde{k}_{ij},
\]  

(2.6a)

\[
\mathcal{L}_{\partial_{\bar{\tau}}} \tilde{k}_{ij} + \tilde{k}_{ij} = \bar{N} g_{ij} - \nabla_i \nabla_j N + N \left( \bar{R}_{ij\beta} - \bar{k}_{i\beta} \bar{k}^\beta_j \right).
\]  

(2.6b)

Here the indices \( i, j \) refer to the frame \( \{ \partial_i, \ i = 1, 2, 3 \} \). And the Gauss–Codazzi equations are

\[
R_{imjn} = -\frac{1}{2} (g \circ g)_{imjn} + (g \circ \tilde{k})_{imjn} - \frac{1}{2} (\tilde{k} \circ \tilde{k})_{imjn} + \bar{R}_{imjn},
\]  

(2.7)

where the Kulkarni–Nomizu product \( \circ \) is defined by: let \( \xi, \eta \) be any symmetric \((0, 2)\)-tensors,

\[
(\xi \circ \eta)_{imjn} = \xi_{ij} \eta_{mn} = \xi_{im} \eta_{jn} + \eta_{im} \xi_{jn} = \eta_{in} \xi_{jm} - \eta_{jm} \xi_{in}.
\]  

(2.8)
\[ \nabla_i \hat{k}_jm - \nabla_j \hat{k}_im = R_{Tmj}. \]  
(2.9)

The traces of the Gauss–Codazzi equations (2.7) and (2.9) are
\[ R_{ij} + 2g_{ij} = \hat{k}_ip \hat{k}_pj + \hat{k}_{ij} + \bar{R}_{i} \bar{T}_j + \bar{R}_{ij}, \]  
(2.10a)
\[ \nabla_j \hat{k}_{ij} = -\bar{R}_{ij}. \]  
(2.10b)

The double trace of the Gauss equation (2.7) is
\[ R - \hat{k}_{ij} \hat{k}^{ij} + 6 = 2\bar{R}_{T} \bar{T} + \bar{R}. \]  
(2.11)

2.2.2. The background spacetime. Let \((\Sigma, \gamma)\) be a compact manifold without boundary which is of hyperbolic type, with \(\gamma\) being the standard hyperbolic metric of sectional curvature \(-1\). The hyperbolic cone spacetime (or 1+3-dim Milne model) \((\bar{M}, \bar{\gamma})\) is the Lorentzian cone over \((\Sigma, \gamma)\), i.e.
\[ \bar{M} = (0, \infty) \times \Sigma, \quad \bar{\gamma} = -d\rho^2 + \rho^2 \gamma. \]  
(2.12)

The family of hyperboloids \(\Sigma_\rho\) given by \(\rho = \text{constant}\) has the normal \(T = \partial_\rho\). The vector field \(\rho \partial_\rho\) is a timelike homothetic killing field. A calculation shows that \(\bar{k}_{ij} = -\frac{1}{\rho} \tilde{g}_{ij}\), where \(\tilde{g}_{ij} = \rho^2 \gamma_{ij}\), and the mean curvature \(\text{tr} \bar{k} = -\frac{3}{\rho}\). The \(\rho\)-foliation has the lapse \(N = 1\).

2.3. Local existence

It is known that in the CMC gauge with zero shift, the vacuum Einstein equations are non-strictly hyperbolic. However, following the work of Christodoulou–Klainerman [14], where they use the maximal gauge with zero shift, we can prove the local existence theorem for the EKG system. The method is also analogous to the one based on wavelike coordinates [11] or CMCSH gauge [6], in the sense that one can show that to solve the local existence theorem for the original EKG system, it suffices to solve a ‘reduced’ system. Moreover, this approach is irrelevant to matter field. We sketch the proof in appendix A.

**Theorem 2.1.** Assume that \(\Sigma\) is a compact Riemannian manifold of hyperbolic type without boundary. Let \((g_0, k_0, \phi_0, \phi_1)\) be the rescaled CMC data (making the normalization \(\text{tr} g_0 k_0 = -3\)) for the EKG system and verify the following conditions:

1. \(R_{0ij}\) the Ricci curvature of \(g_0\) satisfies \(R_{0ij} \in H^3(\Sigma, g_0)\),
2. \(k_{0ij}\) is a symmetric \((0, 2)\)-tensor verifying \(k_{0ij} \in H^4(\Sigma, g_0)\),
3. \((\phi_0, \phi_1) \in H^2(\Sigma, g_0) \times H^4(\Sigma, g_0)\).

Then there are \(0 < t_- < t_0 < t_+\) such that there is a unique, local-in-time smooth development \((\bar{M}, \bar{g})\) of \((g_0, k_0, \phi_0, \phi_1)\), foliated by CMC hypersurfaces, and \(\bar{M} = (t_-, t_+) \times \Sigma, t = t_0\) corresponding to the initial slice \(\Sigma\).
3. 1 + 3 splitting and the energy scheme

In the quantitative computations throughout this paper, we employ the conventions: $R_{ijn}$ whose four indices $i, m, j, n$ range over $1, \ldots, 3$ is viewed as a $(0, 4)$-tensor (not the special component) on $\Sigma_t$.

**Definition 3.1.** A $(0, l)$-tensor $\Psi_{\alpha_1 \cdots \alpha_l}$ is called $T$-tangent if

$$T^a \Psi_{\alpha_1 \cdots \alpha_{i-1} \beta \alpha_{i+1} \cdots \alpha_l} = 0, \quad \forall i \in \{1, \ldots, l\}. \quad (3.1)$$

Here if $i = 1$ or $l$, then $\alpha_0$ and $\alpha_{l+1}$ are interpreted as being absent.

If $\Psi_{\alpha_1 \cdots \alpha_l}$ is $T$-tangent, then $\mathcal{L}_{\partial_t} \Psi_{\alpha_1 \cdots \alpha_l}$ and $\mathcal{L}_{\bar{T}} \Psi_{\alpha_1 \cdots \alpha_l}$ are both $T$-tangent. However, $D_{\beta} \Psi_{\alpha_1 \cdots \alpha_l}$ and $D_{\bar{\beta}} \Psi_{\alpha_1 \cdots \alpha_l}$ are no longer $T$-tangent. Any $T$-tangent tensors can be viewed as tensors on $\Sigma_t$. And we denote $\nabla_t \Psi_{\alpha_1 \cdots \alpha_l}$ the projection of $\nabla_\bar{T} \Psi$ to $\Sigma_t$ (evaluated on the corresponding component). Suppose $\Psi_{\bar{t}}$ is a $(0, l)$-tensor on $\Sigma_t$, then the zero-extension of $\Psi_{\bar{t}}$ becomes a $T$-tangent tensor on $M$, which we will still denote by $\Psi_{\bar{t}}$. The following commuting lemma 3.2 holds for any $T$-tangent tensors as well.

Let $\Gamma^a_{\bar{t}j}$ be the connection coefficient of $\nabla$. Then the Lie derivative $\mathcal{L}_{\partial_t} \Gamma^a_{\bar{t}j}$ is a tensor field (C.1),

$$\mathcal{L}_{\partial_t} \Gamma^a_{\bar{t}j} = \frac{1}{2} g^{ab} \left( \nabla_t \mathcal{L}_{\partial_j} g_{\bar{t}b} + \nabla_j \mathcal{L}_{\partial_t} g_{\bar{t}b} - \nabla_b \mathcal{L}_{\partial_t} g_{\bar{t}j} \right). \quad (3.2)$$

We remind ourselves that $\mathcal{L}_{\partial_t} g_{\bar{t}j} = 2 \bar{N} \bar{g}_{\bar{t}j} - 2 \bar{N} k_{\bar{t}j}$.

A commuting identity between $\nabla$ and $\mathcal{L}_{\partial_t}$ is given below.

**Lemma 3.2.** Let $V$ be an arbitrary $(0, l)$-tensor field on $(\Sigma, g_{\bar{t}})$. The following commuting formula is true:

$$\mathcal{L}_{\partial_t} \nabla_j V_{a_1 \cdots a_l} = \nabla_j \mathcal{L}_{\partial_t} V_{a_1 \cdots a_l} - \sum_{i=1}^{l} \mathcal{L}_{\partial_t} \Gamma^p_{ja_i} V_{a_1 \cdots a_{p-i}}. \quad (3.3)$$

This lemma can be proved by straightforward calculations. Meanwhile, we note that for any $T$-tangent tensor $V_{a_1 \cdots a_l}$,

$$\mathcal{L}_{\bar{T}} \nabla_j V_{a_1 \cdots a_l} = \nabla_j \mathcal{L}_{\bar{T}} V_{a_1 \cdots a_l} + N^{-1} \nabla_j \bar{N} \mathcal{L}_{\bar{T}} V_{a_1 \cdots a_l}$$

$$+ N^{-1} \left( \mathcal{L}_{\partial_t} \nabla_j V_{a_1 \cdots a_l} - \nabla_j \mathcal{L}_{\partial_t} V_{a_1 \cdots a_l} \right). \quad (3.4)$$

3.1. Commuting identities

We shall introduce a commuting lemma, based on which we can derive the high order energy identities.

**Lemma 3.3 (Commuting lemma for scalar field).** Let $l \geq 1$, then

$$\mathcal{L}_{\partial_t} \nabla_j \psi = \nabla_j (\partial_t \psi) + \mathcal{K} \mathcal{N}_b (\psi), \quad (3.5a)$$

$$\mathcal{L}_{\bar{T}} \nabla_j \psi = \nabla_j \bar{T} \psi + \mathcal{K} \mathcal{N} \mathcal{T}_b (\psi), \quad (3.5b)$$
where when \( l = 1 \), \( \mathcal{K} \mathcal{N}_1(\psi) = 0 \), and

\[
\mathcal{K} \mathcal{N}_l(\psi) = \pm \sum_{a+b+1=l} \nabla_{\xi_a} \nabla \left( \hat{N} k + \hat{N} \right) * \nabla_{\xi_b} \nabla \psi, \quad l \geq 2,
\]

(3.6a)

\[
\mathcal{K} \mathcal{N} T_l(\psi) = N^{-1} \mathcal{N}_l(\bar{T} \psi) + N^{-1} \mathcal{K} \mathcal{N}_l(\psi), \quad l \geq 1,
\]

(3.6b)

with \( \mathcal{N}_l(T \psi) \) defined as below

\[
\mathcal{N}_l(T \psi) = \sum_{a+b+1=l} \nabla_{\xi_a} \nabla \hat{N} * \nabla_{\xi_b} \bar{T} \psi.
\]

(3.7)

**Proof.** We will prove the following commuting identities:

\[
\mathcal{L}_{\partial_r} \nabla_{i_1} \cdots \nabla_{i_l} \psi = \nabla_{i_1} \cdots \nabla_{i_l} \partial_r \psi
\]

\[
= - \sum_{a=0}^{l-2} \sum_{m=a+2}^{l} \nabla_{i_1} \cdots \nabla_{i_a} \left( \mathcal{L}_{\partial_r} \Gamma_{i_1+1}^p \cdots \nabla_{p+1} \cdots \nabla_{i_a} \psi \right),
\]

(3.8)

where we use the convention for the right hand side: when \( l = 1 \), it is identically zero; when \( l \geq 2 \), the term corresponding to \( a = 0 \) is interpreted as \(- \sum_{m=2}^{l} \mathcal{L}_{\partial_r} \Gamma_{i_1}^p \cdots \nabla_{p+1} \cdots \nabla_{i_m} \psi \).

Namely, the index of \( \nabla_{i_1} \cdots \nabla_{i_l} \{i_1 \cdots i_a \} \) is always in a proper order.

For \( l = 1 \), we have by lemma 3.2,

\[
\mathcal{L}_{\partial_r} \nabla_{i_1} \psi = \nabla_{i_1} \partial_r \psi.
\]

(3.9)

Hence (3.8) is true for \( l = 1 \). Next, we will prove (3.8) by induction. Suppose (3.8) holds for \( l \leq n - 1 \), we wish to prove that it also holds for \( l = n \). By the commuting lemma 3.2,

\[
\mathcal{L}_{\partial_r} \nabla_{i_1} \cdots \nabla_{i_l} \psi
\]

\[
= \nabla_{i_1} \left( \mathcal{L}_{\partial_r} \nabla_{i_1} \cdots \nabla_{i_l} \psi \right) = \sum_{m=2}^{n} \mathcal{L}_{\partial_r} \Gamma_{i_1+1}^p \cdots \nabla_{p+1} \cdots \nabla_{i_m} \psi.
\]

(3.10)

Apply (3.8) to \( l = n - 1 \),

\[
\mathcal{L}_{\partial_r} \nabla_{i_1} \cdots \nabla_{i_l} \psi = \nabla_{i_1} \cdots \nabla_{i_l} \partial_r \psi
\]

\[
= - \sum_{a=1}^{n-2} \sum_{m=a+2}^{n} \nabla_{i_1} \cdots \nabla_{i_a} \left( \mathcal{L}_{\partial_r} \Gamma_{i_1+1}^p \cdots \nabla_{p+1} \cdots \nabla_{i_m} \psi \right).
\]

(3.11)

Note that we set the inverted order one \( \nabla_{i_1} \nabla_{i_2} = 1 \). Substituting (3.11) into (3.10), and making some rearrangements, we prove that (3.8) holds for \( l = n \). Noting the formula (3.2) for \( \mathcal{L}_{\partial_r} \Gamma_{i_1}^p \), we derive (3.5a) and (3.6a).

Based on (3.4), we have

\[
\mathcal{L}_T \nabla_i \nabla_{L_k} \psi = \nabla_i \mathcal{L}_T \nabla_{L_k} \psi + N^{-1} \nabla_j N \mathcal{L}_T \nabla_{L_k} \psi
\]

\[
- \sum_{k=1}^{n} N^{-1} \mathcal{L}_{\partial_r} \Gamma_{j_1}^p \cdots \nabla_{p+1} \cdots \nabla_{i_k} \psi.
\]

(3.12)

We will use (3.12) to prove (3.5b) and (3.6b) by induction. For \( l = 1 \), viewing (3.9) and (3.4), we have
\[ \mathcal{L}_T \nabla \psi = \nabla \mathcal{T} \psi + N^{-1} \nabla_i \hat{N} \cdot \hat{\mathbf{T}}. \]

That is, (3.5b) and (3.6b) hold for \( l = 1 \). Suppose (3.5b) and (3.6b) hold for \( l \leq n - 1 \), in view of (3.12), we now consider by induction

\[
\mathcal{L}_T \nabla \nabla \cdots \nabla \psi = \nabla \mathcal{T} \nabla \cdots \nabla \psi
+ N^{-1} \nabla_i N \mathcal{L}_T \nabla \cdots \nabla \psi \pm N^{-1} \nabla (\hat{N} + \hat{\mathbf{T}}) \nabla_{l-1} \psi
= \nabla \left( \nabla \cdots \nabla \mathcal{T} \psi + \sum_{a+b=n-2} N^{-1} \nabla_i N \nabla \cdots \nabla \hat{\mathbf{T}} \psi \right)
+ N^{-1} \nabla_i N \left( \nabla \cdots \nabla \mathcal{T} \psi + \sum_{a+b=n-2} N^{-1} \nabla_i \nabla \cdots \nabla \hat{\mathbf{T}} \psi \right)
+ N^{-1} \nabla \left( \mathcal{K}_i \nabla_{l-1} (\psi) \right) \pm N^{-1} \nabla (\hat{N} + \hat{\mathbf{T}}) \nabla_{l-1} \psi.
\]

That is,

\[
\mathcal{L}_T \nabla_i \psi = \nabla_i \mathcal{T} \psi + N^{-1} \nabla \nabla \cdots \nabla \psi \pm \nabla \nabla \cdots \nabla \psi 
+ N^{-1} \nabla_i N \nabla \cdots \nabla \hat{\mathbf{T}} \psi + N^{-1} \nabla \left( \mathcal{K}_i \nabla_{l-1} (\psi) \right) \pm N^{-1} \nabla (\hat{N} + \hat{\mathbf{T}}) \nabla_{l-1} \psi,
\]

which further implies (3.5b) and (3.6b).

Lemma 3.4 (Commuting lemma for \( T \)-tangent tensor). Let \( \Psi_{l} \) be a \( T \)-tangent \((0,n)\)-tensor field and let \( l \geq 1 \), then,

\begin{align}
\mathcal{L}_{\partial_r} \nabla_l \Psi_{l} &= \nabla_{l} \mathcal{L}_{\partial_r} \Psi_{l} + \mathcal{K}_{l} N_{l} (\Psi_{l}), \\
\mathcal{L}_T \nabla_l \Psi_{l} &= \nabla_{l} \mathcal{L}_T \Psi_{l} + \mathcal{K}_T T_{l} (\Psi_{l}),
\end{align}

where \( \mathcal{K}_l N_{l} (\Psi_{l}) \) and \( \mathcal{K}_l T_{l} (\Psi_{l}) \) are defined as follows,

\[
\mathcal{K}_l N_{l} (\Psi_{l}) = \pm \sum_{a+b=l} \nabla_l \nabla (\hat{N} + \hat{\mathbf{T}}) \nabla \Psi_{l},
\]

\[
\mathcal{K}_l T_{l} (\Psi_{l}) = N^{-1} \mathcal{N}_l (\mathcal{L}_T \Psi_{l}) + N^{-1} \mathcal{K}_l N_{l} (\Psi_{l}),
\]

with

\[
\mathcal{N}_l (\Psi_{l}) = \sum_{a+b=l} \nabla_l \nabla \nabla \cdots \nabla \Psi_{l}.
\]

Proof. In fact, we can prove

\[
\mathcal{L}_{\partial_r} \nabla \cdots \nabla \Psi_{l} = \nabla \cdots \nabla \mathcal{L}_{\partial_r} \Psi_{l} \Psi_{l-1} - \nabla \cdots \nabla \mathcal{L}_{\partial_r} \Psi_{l} \Psi_{l-1}
= - \sum_{a=0}^{l-1} \sum_{b=a+2}^{l} \nabla_l \cdots \nabla_a \left( \mathcal{L}_{\partial_r} \Gamma_{l-a+1}^{p} \nabla_{l-a+2} \cdots \nabla \Psi_{l-1} \right)
= - \sum_{a=0}^{l} \sum_{b=1}^{n} \nabla_l \cdots \nabla_a \left( \mathcal{L}_{\partial_r} \Gamma_{l-a+1}^{p} \nabla_{l-a+2} \cdots \nabla \Psi_{l-1} \right)
\]

(3.17)
Similar commuting formula between $\mathcal{L}_\tau$ and $\nabla_L$ holds as well. Noting the formula (3.2) for $\mathcal{L}_\partial$, (3.4) and (3.4), we can prove (3.14)–(3.16).

We also present a commuting identity between $\nabla$ and $\Delta$, which can be easily proved by induction.

**Lemma 3.5.** Let $l \geq 1$. For any scalar field $\psi$, there is

\[
\Delta \nabla_L \psi = \nabla_L \Delta \psi + \mathcal{R}_L(\psi),
\]

(3.18)

\[
\mathcal{R}_L(\psi) = \pm \sum_{a+b=l, a \leq l-1} \nabla_L R_{mja} \ast \nabla_L \psi.
\]

(3.19)

Generally, for any $T$-tangent $(0, n)$-tensor $\Psi_L$, we have

\[
\Delta \nabla_L \Psi_L = \nabla_L \Delta \Psi_L + \mathcal{R}_L(\Psi_L),
\]

(3.20)

\[
\mathcal{R}_L(\Psi_L) = \pm \sum_{a+b+l = l} \nabla_L R_{mja} \ast \nabla_L \Psi_L.
\]

(3.21)

We collect some definitions which will be used later. For any scalar function or $T$-tangent tensor $A$, we define

\[
\hat{\mathcal{K}}_N(\mathcal{A}) = \pm \sum_{a+b+c=l} \nabla_L N \ast \nabla_L \hat{k} \ast \nabla_L A.
\]

(3.22)

At last, we show an identity that is useful in the calculations followed.

**Proposition 3.6.** For any scalar function $f$, there is an identity

\[
\partial_\tau \int_{\Sigma_t} f \, d\mu_g = \int_{\Sigma_t} \left( \partial_\tau f + 3 \hat{\mathcal{N}}_f \right) \, d\mu_g.
\]

(3.23)

### 3.2. Energy identities for the $1 + 3$ KG equation

The KG equation can be decomposed in the following $1 + 3$ form:

\[
\mathcal{L}_\partial \nabla_L \phi - (\text{tr} N + 1) \nabla_L \phi - N \Delta \phi + m^2(t) N \phi - \nabla_L \nabla L \phi = 0.
\]

(3.24)

Define the $i$th order energy norm for the KG field $\phi$ as follows,

\[
E_i(\phi, t) = \int_{\Sigma_t} \left( |\nabla_L \nabla_L \phi|^2 + |\nabla_L \nabla L \phi|^2 + m^2(t) |\nabla_L \phi|^2 \right) \, d\mu_g.
\]

(3.25)

where $|\nabla_L \phi|^2 := g^{ij} \ldots g^{ij} \nabla_L \phi \nabla_L \phi$. And $E_i(\phi, t)$ is defined via replacing the rescaled $g$ and $T$, $m(t)$ in (3.25) by $\tilde{g}$ and $\tilde{T}, \tilde{m}$. Then,

\[
E_i(\phi, t) = t^{-1+2j} E_i^\tilde{g}(\phi, t).
\]

(3.26)

We additionally define the following energy density for the KG field,

\[
\rho_i(\phi) = |\nabla_L \nabla_L \phi|^2 + |\nabla_L \nabla L \phi|^2 + m^2(t) |\nabla_L \phi|^2.
\]

In what follows, for any $\Sigma_t$-tensor $\Psi_L$, we also use $|\Psi_L|$ to denote $|\Psi_L|_e$ for simplicity. As we can see in the following sections, the ($i$th order) $T$ energy of the KG field is given by
\[ \bar{E}_i(\phi, t) = \int_{\Sigma_t} \left( \rho_i(\phi) - \text{tr} k T \nabla_i \phi \nabla^i \phi \right) d\mu_g. \] (3.27)

For \(|\text{tr} k| < \bar{m}(t)\) (or \(t > 3\bar{m}^{-1}\)), these two energy norms are indeed equivalent: \(\frac{3}{2}E_i(\phi, t) < \bar{E}_i(\phi, t) < \frac{3}{2}E_i(\phi, t)\).

3.2.1 The zero order case.

**Theorem 3.7.** Let \(\phi\) be a solution to the KG equation (1.2), then the 0th-order energy admits

\[ \partial_t \bar{E}_0(\phi, t) + \bar{E}_0(\phi, t) + \int_{\Sigma_t} 2N|\nabla \phi|^2 d\mu_g + \int_{\Sigma_t} \omega LK d\mu_g = 0, \] (3.28)

where the error term \(\omega LK\) is given by

\[ \omega LK = 3\phi T \phi - 3\bar{N} \phi T \phi - 2N\bar{k}_g \nabla^i \phi \nabla_i \phi - 2\nabla_i N \nabla^i \phi T \phi. \] (3.29)

**Remark 3.8.** A similar energy inequality for the KG equation is also introduced by Galstian–Yagdjian [19] in the FLRW spacetimes. However, they pursue global results with low regularities, and hence do not need higher order energy identities.

**Proof.** We multiply \(2T \phi\) on (3.24), noticing that

\[ 2Nm^2(t)\phi T \phi = \partial_t(\bar{m}^2(t)\phi^2) - 2\bar{m}^2(t)\phi^2, \]

and

\[ -2N\nabla^i \nabla_i \phi T \phi = -\nabla_i \left( 2N\nabla^i \phi T \phi \right) + 2\nabla^i \phi \nabla_i (N \phi) \]

\[ = -\nabla_i \left( 2N\nabla^i \phi T \phi \right) + 2\nabla^i \phi L_{\partial_i \phi}. \]

where in the second identity, we have used the commuting identity \((3.5b)\) with \(l = 1\). Furthermore, making use of the formula \(L_{\partial_i \phi} g^{ij} = -2\bar{N} g^{ij} + 2N\bar{k}^{ij}\), and hence

\[ 2\nabla^i \phi L_{\partial_i \phi} \nabla_i \phi = \partial_t |\nabla \phi|^2 - 2Nk_i \nabla^i \phi \nabla_i \phi + 2N|\nabla \phi|^2. \]

we derive

\[ \partial_t \rho_0(\phi) - \bar{m}^2(t)\phi^2 + 4N(\bar{T} \phi)^2 + 2\bar{N}(\bar{T} \phi)^2 + 2N|\nabla \phi|^2 \]

\[ - 2Nk_i \nabla^i \phi \nabla_i \phi - 2\nabla_i N \nabla^i \phi T \phi - 2\nabla_i (N \nabla^i \phi T \phi) = 0. \] (3.30)

Now let us focus on the quadratic terms \(4N(\bar{T} \phi)^2 - 2\bar{m}^2(t)\phi^2\) in (3.30). We will additionally address an integration by part (or rather extract a divergence form) and use the KG equation to make the sign (of this quadratic term) right. Performing an integration by part,

\[ N(\bar{T} \phi)^2 = \partial_t (\phi \bar{\nabla} \phi) - \phi L_{\partial \phi} \bar{T} \phi. \]

We then substitute the KG equation (3.24) into the following formula,

\[ 4N(\bar{T} \phi)^2 - 2\bar{m}^2(t)\phi^2 = N(\bar{T} \phi)^2 + 3\partial_t (\phi \bar{T} \phi) - 3\phi L_{\partial \phi} \bar{T} \phi - 2\bar{m}^2(t)\phi^2 \]

\[ = N(\bar{T} \phi)^2 + \bar{m}^2(t)\phi^2 + \partial_t (3\phi \bar{T} \phi) + 2\bar{m}^2(t)\bar{N} \phi \]

\[ + 3\phi \left( 2N \bar{T} \phi + \bar{N} \bar{T} \phi - N \Delta \phi - \nabla_i N \nabla^i \phi \right). \]
where the first two terms (in the second identity) now have good signs, and in regard of the extra quadratic term \(-3N\phi \Delta \phi\),

\[
-3N\phi \Delta \phi = -3\nabla_i \left(N\phi \nabla^i \phi\right) + 3N|\nabla \phi|^2 + 3\phi \nabla^i N \nabla_i \phi.
\]

Putting the two identities above together, we have

\[
\partial_t \left(\rho_0(\phi) + 3\phi \tilde{T}(\phi)\right) + N\rho_0(\phi) + 2N|\nabla \phi|^2 + 2\tilde{N} \rho_0(\phi)
- 3\nabla_i \left(N\phi \nabla^i \phi\right) - 2\nabla_i \left(N\nabla^i \phi \tilde{T}(\phi)\right) + 6N\phi \tilde{T}(\phi)
+ 3\tilde{N} \phi \tilde{T}(\phi) - 2Nk_1 \nabla^i \phi \nabla_i \phi - 2\nabla_i N \nabla^i \phi \tilde{T}(\phi) = 0.
\]

We remark that the error terms arising in this procedure are all of lower orders: they indeed have an additional \(r^{-1}\) factor, for instance, \(N\phi \tilde{T}(\phi)\) can be rewritten as \(m^{-1} r^{-1} N(m(t) \phi) \tilde{T}(\phi)\).

Integrating on \(\Sigma_t\) and making use of the identity \((3.23)\), there is

\[
\partial_t \int_{\Sigma_t} \left(\rho_0(\phi) + 3\phi \tilde{T}(\phi)\right) d\mu_g + \int_{\Sigma_t} 2N|\nabla \phi|^2 d\mu_g
+ \int_{\Sigma_t} N \left(\rho_0(\phi) + 3\phi \tilde{T}(\phi)\right) d\mu_g + \int_{\Sigma_t} 3N \phi \tilde{T}(\phi) - 6N \phi \tilde{T}(\phi)
+ \int_{\Sigma_t} -\tilde{N} \rho_0(\phi) - 2Nk_1 \nabla^i \phi \nabla_i \phi - 2\nabla_i N \nabla^i \phi \tilde{T}(\phi) = 0.
\]

We then achieve \((3.28)\) and \((3.29)\). \(\square\)

As a remark, \((3.28)\) and \((3.29)\) can also be deduced by the method of energy-momentum tensor. However, if we proceed to the higher order cases, it will be natural and simpler to commute the spatially covariant derivative \(\nabla\) with the KG equation in 1 + 3 form \((3.24)\). And this turns out to be more compatible with the energies involving only high order of spatial derivatives.

### 3.2.2. The higher order cases.

**Lemma 3.9.** Let \(l \geq 1\), the \(l\)th-order KG equation \(\nabla_i \left(N \left(\Box_g \phi - \bar{m}^2(t) \phi\right)\right) = 0\) can be decomposed in the following \(1 + 3\) form

\[
\mathcal{L}_{\partial_t} \nabla_i \tilde{T}(\phi) - (Ntr + 1) \nabla_i \tilde{T}(\phi) - N \nabla_i \Delta \phi + \bar{m}^2(t) N \nabla_i \phi
+ \bar{m}^2(t) N_{i_0}(\phi) - \mathcal{K}N^j_{i_0}(\tilde{T}(\phi)) = 0,
\]

where \(N_{i_0}(\phi), \bar{N}_{i_0}(\tilde{T}(\phi))\) are defined by \((3.16)\), and \(\mathcal{K}N^j_{i_0}(\tilde{T}(\phi))\) is defined by

\[
\mathcal{K}N^j_{i_0}(\tilde{T}(\phi)) = \mathcal{K}N^j_{i_0}(\tilde{T}(\phi)) + \text{tr}kN^j_{i_0}(\tilde{T}(\phi)),
\]

with \(\mathcal{K}N^j_{i_0}(\tilde{T}(\phi)), N^j_{i_0}(\tilde{T}(\phi))\) defined in \((3.5a)\) and \((3.7)\).

**Remark 3.10** (Inevitability of the borderline terms in the high order KG equation). We first remark that

\[
\bar{m}^2(t) N_{i_0}(\phi) = \bar{m}^2(t) \left(\nabla_i (N\phi) - N \nabla_i \phi\right),
\]

is one of the borderline terms in the energy estimates for the KG equation.
One may argue that this borderline term is artificial, since if we commute $\nabla_h$ straightforwardly with the original KG equation (without multiplying $N$),
\[
\nabla_h \left( \mathcal{L}_T \nabla \phi - \left( N^{-1} + \text{tr} k \right) \nabla \phi - \Delta \phi + m^2(t) \phi - N^{-1} \nabla_h N \nabla'h \phi \right) = 0, \quad (3.34)
\]
then the aforementioned borderline term $m^2(t) N_b(\phi)$ is seemingly absent. However, this borderline term will alternatively arise from the commuting term $[\mathcal{L}_T, \nabla_h] T\phi = K N T\phi$, see (3.6b). After commuting $\mathcal{L}_T$ with $\nabla_h$ and multiplying $N$ on (3.34), one has
\[
L_{\phi}, \nabla_h \nabla^{i} \nabla^{j} \nabla^{k} \nabla^{l} \nabla \nabla \phi - (N \text{tr} + 1) \nabla_h \nabla \phi - N \nabla_h \nabla \phi + m^2(t) N \nabla_h \nabla \phi = N [\mathcal{L}_T, \nabla_h] T\phi + N \nabla_h \left( N^{-1} \nabla_h N \nabla'h \phi \right) - N N \left( N^{-1} \nabla_h T\phi \right).
\]
This almost takes the same form as (3.32) (ignoring the lower order terms), except that the previous term $m^2(t) N_b(\phi)$ is now replaced by $-N [\mathcal{L}_T, \nabla_h] T\phi$. Now, taking a closer look at $-N [\mathcal{L}_T, \nabla_h] T\phi$, we will find out that this is exactly the missing borderline term. Refer to (3.5b) and (3.6b),
\[
-N [\mathcal{L}_T, \nabla_h] T\phi = -N_b(T^2 \phi) - K_N (T\phi), \quad l \geq 1,
\]
where
\[
-N_b(T^2 \phi) = \sum_{a+b=1=l} -\nabla_h \nabla N \ast \nabla_h T^2 \phi.
\]
We further make use of the KG equation to substitute $\nabla_h T^2 \phi$, then
\[
-N_b(T^2 \phi) = \sum_{a+b=1=l} m^2(t) \nabla_h \nabla N \ast \nabla_h \phi + \text{1.o.t.},
\]
where the first one on the right hand side of the equality is exactly the borderline term $m^2(t) N_b(\phi)$. Note that, in the energy argument, this term will be multiplied with $2 \nabla^h T\phi$, and hence the whole term takes the form of $2 m^2(t) \nabla^h T\phi \ast N_b(\phi)$.

Secondly, let us remark that the other borderline term originate from the commutator $[\mathcal{L}_T, \nabla_h] \phi$. As we will see in the proof of corollary 3.11, to establish the high order energy identity for the KG field, we multiply $2 \nabla^h T\phi$ on (3.32). Focus on the massive term (3.45) for the moment,
\[
2 m^2(t) N \nabla_h \phi \nabla^h T\phi = \partial_{\phi} \left( m^2(t) | \nabla_h \phi |^2 \right) - 2 m^2(t) N \nabla^h \phi [\mathcal{L}_T, \nabla_h] \phi - 2 m^2(t) | \nabla_h \phi |^2 + 2 \nabla^2(t) N \nabla_h \phi \ast \nabla^h \phi \ast \nabla_h \phi,
\]
where the $\partial_{\phi} \left( m^2(t) | \nabla_h \phi |^2 \right)$ will contribute to the energy norm, $-2 m^2(t) | \nabla_h \phi |^2$ will be absorbed (refer to (3.51)), and the other terms in the 3rd line are cubic and of lower order. While the rest $-2 m^2(t) \nabla^h \phi [\mathcal{L}_T, \nabla_h] \phi$, as shown before, leads to the borderline term $-2 m^2(t) \nabla^h \phi \ast N_b(\phi)$, which can be viewed as somehow ‘dual’ to the previous $2 m^2(t) \nabla^h T\phi \ast N_b(\phi)$.

The terms discussed above are the only two borderline cases. More explicitly, they take the forms of
\[
\sum_{a+b=1=l} 2 m^2(t) \nabla_h \nabla N \ast (\nabla_h \phi \ast \nabla^h T\phi - \nabla_h T\phi \ast \nabla^h \phi),
\]
which can not be cancelled in any case.

**Proof of lemma 3.9.** Take the derivative $\nabla_i$ on the KG equation (3.24), by the commuting lemma 3.3 (with $l = 1$ in (3.5a)),

\[
0 = L_{\partial r} \nabla_i T \phi - (N t k + 1) \nabla_i T \phi - N \nabla_i \Delta \phi + \bar{m}^2(t) N \nabla_i \phi \\
+ \bar{m}^2(t) \nabla_i N \phi - \text{tr} \nabla_i N T \phi - \mathcal{K} \mathcal{N}_{l-1}(T \phi) - (\nabla_i N \Delta \phi + \nabla_i (\nabla^p N \nabla_p \phi)).
\]

Then (3.32) and (3.33) with $l = 1$ follows.

Suppose (3.32) and (3.33) holds true for $l = n - 1$. That is, the $(n - 1)$th order KG equation takes the form of

\[
\mathcal{L}_{\partial r} \nabla_i \nabla \phi - (N T k + 1) \nabla_i \nabla \phi - N \nabla_i \Delta \phi + \bar{m}^2(t) N \nabla_i \phi \\
+ \bar{m}^2(t) N \nabla_i \phi + \bar{m}^2(t) \mathcal{N}_{l-1}(\phi) - \mathcal{K} \mathcal{N}_{l-1}(T \phi) + \mathcal{N}_{l}(\nabla \phi) = 0.
\]

(3.35)

Now taking $\nabla_i$ on (3.35), and using the commuting lemma 3.2 with $l = n - 1$, we have

\[
\nabla_i (\mathcal{L}_{\partial r} \nabla_i \nabla \phi) = \mathcal{L}_{\partial r} \nabla_i \nabla \phi + \sum_{m=2}^{n} \mathcal{L}_{\partial r} \Gamma_{l+m}^{p} \nabla_i \nabla_i \nabla \phi.
\]

Hence, we obtain

\[
\mathcal{L}_{\partial r} \nabla_i \nabla \phi - (N T k + 1) \nabla_i \nabla \phi - N \nabla_i \Delta \phi + \bar{m}^2(t) N \nabla_i \phi \\
+ \sum_{m=2}^{n} \mathcal{L}_{\partial r} \Gamma_{l+m}^{p} \nabla_i \nabla_i \nabla \phi - \text{tr} \nabla_i N \nabla_i \phi - \mathcal{K} \mathcal{N}_{l-1}(T \phi) + \mathcal{N}_{l}(\nabla \phi) = 0.
\]

(3.36)

In view of the definition for $\mathcal{K} \mathcal{N}_{l}^{p}(\nabla \phi)$, $\mathcal{N}_{l+1}(\nabla \phi)$, $\mathcal{N}_{l}(\phi)$ and $\mathcal{K} \mathcal{N}_{l}^{p}(T \phi)$, we rearrange the 2nd–4th lines in (3.36), and it shows that (3.32) and (3.33) holds true for $l = n$.

We next show that the high order KG equations (3.32) and (3.33) leads to the high order energy identities.

**Corollary 3.11.** Let $\phi$ be a solution to the KG equation (1.2), and $l \geq 1$, then the $l$th-order modified energy $\tilde{E}_{l}(\phi, t)$ (see definition (3.27)) admits

\[
\partial_r \tilde{E}_{l}(\phi, t) + \tilde{E}_{l}(\phi, t) + \int_{\Sigma} 2N|\nabla_{k+1} \phi|^2 d\mu_g \\
= \int_{\Sigma} (B L_{l} + i L_{K_{1}} + i L_{K_{2}} + i L_{K_{3}}) d\mu_g,
\]

(3.37)

where the lower order terms $i L_{K_{1}}$, $i L_{K_{2}}$, $i L_{K_{3}}$ are given as follows,

\[
i L_{K_{1}} = \left(\hat{\nabla} + N \hat{k} \right) |\nabla \phi| \pm \left( N - \hat{\nabla} \right) \nabla \phi \nabla \phi + \nabla \phi \nabla \phi \\
- N \nabla \phi \nabla \phi R_{k}(\phi) - N \nabla \phi \nabla \phi R_{k}(\phi).
\]

(3.38)
with \( D\phi \in \{ T\phi, \nabla\phi, \bar{m}(t)\phi \} \), and
\[
\begin{align*}
iLK_2 &= \nabla^h D\phi \left( N_{l+1} D\phi + N_{l} D\phi + K N_{l} D\phi \right), \\
iLK_3 &= \nabla^h \bar{T}\phi K N_{l} \phi + \nabla^h \phi (K N_{l} \bar{T}\phi) + N_{l} \bar{T}\phi + N_{l+1} (\nabla \phi),
\end{align*}
\] (3.39)
(3.40)
and the borderline term \( BL_t \) is given by
\[
BL_t = 2m^2 (t) \left( \nabla^h \bar{T}\phi N_{l} \phi - \nabla^h \phi N_{l} \bar{T}\phi \right). 
\] (3.41)
We refer to (3.7) and (3.15)–(3.22) for definitions of \( K N_{l} (D\phi) \) · · · Note that, we always ignore irrelevant constants.

**Remark 3.12.** If working within the CMCSH gauge, \( \partial_t = NT + X_t, \ \partial_r = NT + X, \) where \( X = iX_t \) denotes the rescaled shift. And extra terms as \( -\nabla h \left( LX T\phi \right) + \nabla h \left( X^2 \cdot \bar{T}\phi \right) \) must be added in the KG equation (3.24). In the energy argument, these terms yield the leading terms \( \bar{T} \left( X \left| \nabla h T\phi \right|^2 \right) \pm X \left| \nabla h T\phi \right|^2 \pm \nabla L_{\tau} \left( X^2 \nabla \phi \right) \right) \bar{T} \phi, \) where \( X \tau = \nabla X_t + \nabla \phi \phi \) denotes the deformation tensor of \( X. \) The first term vanishes after integrating on \( \Sigma_t, \) while the other two terms can never be borderline terms, for we can expect that \( X \) shares the same (or better) estimate with \( \hat{N}, \) see remark 4.4.

**Proof.** The proof is analogous to that of theorem 3.7. We multiply \( 2\nabla h \bar{T}\phi \) on the \( h \)th-order KG equations (3.32) and (3.33). At first, for \( 2\nabla^h \bar{T}\phi L_{\delta_t} \nabla h \bar{T}\phi, \)
\[
2\nabla^h \bar{T}\phi L_{\delta_t} \nabla h \bar{T}\phi = \partial_r \left( \left| \nabla h \bar{T}\phi \right|^2 \right) + 2\bar{N} \left| \nabla h \bar{T}\phi \right|^2 - 2N \phi \nabla^h \bar{T}\phi * \nabla^h \bar{T}\phi. 
\] (3.42)
Secondly, for \( -2\nabla^h \bar{T}\phi N \nabla L_{\delta_t} \Delta \phi, \) by the commuting identity (3.18),
\[
-2\nabla^h \bar{T}\phi N \nabla L_{\delta_t} \Delta \phi = -2\nabla^h \bar{T}\phi N \left( \Delta \nabla \phi + R_k(\phi) \right) \\
= -\nabla_i \left( 2N \nabla^h \nabla_{h+1} \phi \nabla^h \bar{T}\phi \right) + 2N \nabla^h \nabla_{h+1} \phi \nabla_{h+1} \bar{T}\phi \\
+ 2N \nabla^h \bar{T}\phi * N \nabla \phi \nabla_{h+1} \phi - 2N \nabla^h \nabla \phi R_k(\phi). 
\] (3.43)
Applying the commuting identity (3.5b) to \( \nabla h_{h+1} \bar{T}\phi, \) in (3.43), the term \( 2N \nabla^h \nabla_{h+1} \phi \nabla_{h+1} \bar{T}\phi \) can be further calculated in an analogous way as (3.42). Therefore,
\[
-2\nabla^h \bar{T}\phi N \nabla L_{\delta_t} \Delta \phi = -\nabla_i \left( 2N \nabla^h \nabla_{h+1} \phi \nabla^h \bar{T}\phi \right) \\
+ \partial_r \left( \left| \nabla_{h+1} \phi \right|^2 \right) + 2N \left| \nabla^h \nabla_{h+1} \phi \right|^2 - 2N \phi \nabla^h \nabla_{h+1} \phi * \nabla_{h+1} \phi \\
- 2N \nabla^h \nabla_{h+1} \phi * K N_t \bar{T}_{h+1} \phi) + \nabla^h \bar{T}\phi \left( N \nabla \phi \nabla_{h+1} \phi - N R_k(\phi) \right). 
\] (3.44)
Thirdly, \( 2\bar{m}^2(t) \nabla^h \bar{T}\phi N \nabla L_{\delta_t} \phi \) can be treated in the same way, thus
\[
2\bar{m}^2(t) \nabla^h \bar{T}\phi N \nabla L_{\delta_t} \phi = \partial_r \left( \bar{m}^2(t) \left| \nabla h \bar{T}\phi \right|^2 \right) - 2\bar{m}^2(t) N \nabla^h \phi * K N_t \bar{T}_{h+1} \phi) \\
- 2\bar{m}^2(t) \left| \nabla h \phi \right|^2 - \bar{m}^2(t) \left| N \nabla h \phi \right|^2 - 2\bar{m}^2(t) \nabla \phi * \nabla^h \phi. 
\] (3.45)
Therefore, putting (3.42)–(3.45) together,
\[
\partial_r \rho_l(\phi) + 4N \left| \nabla h \bar{T}\phi \right|^2 - 2\bar{m}^2(t) \left| \nabla h \phi \right|^2 \\
- \nabla_i \left( 2N \nabla^h \nabla L \phi \bar{T}\phi \right) + \text{error} = 0,
\]
where the error terms are given by
\[
\text{error} = 2N|\nabla_k D\phi|^2 - 2N\hat{k} * |\nabla^k D\phi|^2 + \nabla^k T\phi \nabla N * \nabla_{k+i} \phi - N \nabla^k T\phi R_k(\phi)
- 2\hat{N}\left(\nabla^{k+1} \phi * K\nabla T\phi(\phi) + \hat{m}^2(t) \nabla^k \phi * K\nabla T\phi(\phi)\right)
+ 2\nabla^k T\phi \left(\hat{m}^2(t) N_{k}(\phi) - K\nabla T\phi(\phi) - N_{k+i}(\nabla \phi)\right).
\]
(3.46)

Notice that, \(K\nabla T\phi(\phi)(3.6b)\) can be split into two parts:
\[
K\nabla T\phi(\phi) = N^{-1}N_{k}(\nabla \phi) + N^{-1}K\nabla N(\phi), \quad I \geq 1,
\]
where the definitions of \(N_{k}(\nabla \phi), K\nabla N(\phi)\) are shown in (3.7) and (3.6a). Hence, the 3rd line of (3.46) contains
\[
-2\nabla^{k+1} \phi * N_{k+i}(\nabla \phi) - 2\hat{m}^2(t) \nabla^k \phi * N_{k}(\nabla \phi).
\]
We note that \(\hat{m}^2(t) = m^2 \hat{t}^2\). The extra \(\hat{t}^2\) makes \(\hat{m}^2(t) \nabla^k \phi * N_{k+i}(\nabla \phi)\) a borderline term, while \(\nabla^{k+1} \phi * N_{k+i}(\nabla \phi)\) is of lower order. The other borderline term is \(2\hat{m}^2(t) \nabla^k T\phi * N_{k}(\nabla \phi)\), which is originated from \(\hat{m}^2(t)|\nabla k, N(\phi)\). Now, we rearrange the identity as
\[
\partial_{\tau} \rho_k(\phi) + 4N|\nabla_k T\phi|^2 - 2\hat{m}^2(t)|\nabla_k \phi|^2 - \nabla_k \left(2N\nabla^k \nabla_k \nabla^{k+1} T\phi\right) - BL_d + \text{l.o.t.} = 0,
\]
(3.47)
where we separate the error terms into two parts: the borderline term
\[
BL_d = -2\hat{m}^2(t) \nabla^k \phi * N_{k+i}(\nabla \phi) + 2\hat{m}^2(t) \nabla^k T\phi * N_{k}(\nabla \phi)
\]
(3.48)
and the lower order terms
\[
\text{l.o.t.} = 2N|\nabla_k D\phi|^2 - 2N\hat{k} * |\nabla_k D\phi|^2 - 2\nabla^{k+1} \phi * N_{k+i}(\nabla \phi)
- 2\left(\nabla^{k+1} \phi * K\nabla N_{k+i}(\phi) + \hat{m}^2(t) \nabla^k \phi * K\nabla N_{k+i}(\phi)\right)
- \nabla^k T\phi \left(\hat{K}N_{k+i}(\nabla \phi) + N_{k+i}(\nabla \phi) + N R_k(\phi)\right).
\]
(3.49)
Here we note that \(\nabla^k T\phi \nabla N * \nabla_{k+i} \phi\) in (3.46) has been covered by \(\nabla^{k+1} \phi * N_{k+i}(\nabla \phi)\) in (3.49).

In the second step, we will concentrate on the quadratic terms in (3.47),
\[
4N|\nabla_k T\phi|^2 - 2\hat{m}^2(t)|\nabla_k \phi|^2.
\]
(3.50)
We again apply the commuting identity (3.5b) to \(\nabla_k T\phi\) and use the high order KG equation (3.32),
\[
N|\nabla_k T\phi|^2 = \partial_{\tau} \left(\nabla^k T\phi \nabla_k \phi\right) + \hat{m}^2(t) N|\nabla_k \phi|^2 + 2\hat{N}\nabla^k \phi \nabla_k T\phi
- N\hat{k} \nabla^k T\phi * \nabla_k \phi - N \nabla^k T\phi * K\nabla N_{k+i}(\nabla \phi) + 2N\nabla^k \phi \nabla_k T\phi
+ \nabla^k \phi \left(-N\nabla_{k+i} \Delta \phi + \hat{m}^2(t) N_{k+i}(\nabla \phi) - K\nabla T\phi(\phi) - N_{k+i}(\nabla \phi)\right).
\]
Therefore, (3.50) becomes
\[
4N|\nabla_k T\phi|^2 - 2\hat{m}^2(t)|\nabla_k \phi|^2 = N|\nabla_k T\phi|^2 + \partial_{\tau} \left(3\nabla^k T\phi \nabla_k \phi\right)
+ \hat{m}^2(t) N|\nabla_k \phi|^2 + 3N\nabla^k \phi \nabla_k T\phi - 3N\nabla^k \phi \nabla_k \Delta \phi + \text{l.o.t.},
\]
(3.51)
where the lower order terms are given by

\[
\text{l.o.t.}_2 = (3N + 6\dot{N})\nabla^j \phi \nabla_j \phi - \nabla^j \phi \nabla_j \phi - N\dot{N} \nabla^j \phi \nabla_j \phi - N\nabla^j \phi \nabla_j \phi + 2m^2(t)N|\nabla_L \phi|^2
\]

\[
+ \nabla^j \phi \left( m^2(t)\dot{N}_L(\phi) - K\dot{N}_L(\phi) - \dot{N}_{L_3}(\nabla_L \phi) \right). \tag{3.52}
\]

Apply the commuting identity between $\Delta$ and $\nabla_L$ (3.18) to $\nabla_L \Delta \phi$ above,

\[
- 3N\nabla^j \phi \nabla_L \nabla_L \phi = -3N\nabla^j \phi (\Delta \nabla_L \phi - R_L(\phi))
\]

\[
= -3\nabla^j (N\nabla^k \phi \nabla_j \nabla_L \phi + 3N\nabla^k \phi \phi \nabla_L \phi) + 1\text{l.o.t.}_3, \tag{3.53}
\]

where the lower order terms are given by

\[
1\text{l.o.t.}_3 = 3\nabla^j N\nabla^k \phi \nabla_j \phi + 3N\nabla^k \phi \nabla_L \phi. \tag{3.54}
\]

Putting (3.51)–(3.54) together, we obtain

\[
4N|\nabla_L \phi|^2 - 2m^2(t)|\nabla_L \phi|^2
\]

\[
= N|\nabla_L \phi|^2 + \dot{m}(t)N|\nabla_L \phi|^2 + 3N|\nabla_L \phi|^2 + 3N \nabla^j \phi \nabla_L \phi
\]

\[
+ \dot{\tau} (3\nabla^j \phi \nabla_L \phi) - 3\nabla^j (N\nabla^k \phi \nabla_L \phi) + 1\text{l.o.t.}_2 + 1\text{l.o.t.}_3. \tag{3.55}
\]

Combining (3.55) with (3.47)–(3.49), we arrive at

\[
\dot{\tau} (\rho(\phi) + 3\nabla^j \phi \nabla_L \phi) + N (\rho(t(\phi) + 3\nabla^j \phi \nabla_L \phi)
\]

\[
+ 2N|\nabla_L \phi|^2 - 3\nabla^j (3N\nabla^k \phi \nabla_L \phi + 2N\nabla^k \phi \nabla_L \phi)
\]

\[
- BLd + 1\text{l.o.t.}_1 + 1\text{l.o.t.}_2 + 1\text{l.o.t.}_3, \tag{3.56}
\]

where $BLd, 1\text{l.o.t.}_1, \ldots, 1\text{l.o.t.}_3$ are defined in (3.48), (3.49), (3.52) and (3.54). Integrating (3.56) on $\Sigma_t$, we note the splitting for $N\dot{N}_L(\nabla_L \phi)$ (3.33), then

\[
\partial_t \int_{\Sigma_t} (\rho(\phi) + 3\nabla^j \phi \nabla_L \phi) \, d\mu_g + \int_{\Sigma_t} (\rho(t(\phi) + 3\nabla^j \phi \nabla_L \phi) \, d\mu_g
\]

\[
+ \int_{\Sigma_t} 2N|\nabla_L \phi|^2 \, d\mu_g - \int_{\Sigma_t} (BLd + \mu Lk_1 + \mu Lk_2 + \mu Lk_3) \, d\mu_g = 0,
\]

where $\mu Lk_i$ are defined in (3.38)–(3.40). As we can see from the above formula, the term $\int_{\Sigma_t} 2N|\nabla_L \phi|^2 \, d\mu_g$ has a favorable sign and contributes a positive integral in the energy estimates. We then achieve (3.37)–(3.40).

### 3.3. Energy identities for the $1 + 3$ Bianchi equations

#### 3.3.1. The Weyl fields and the Bianchi equations.

The Weyl tensor $W_{\alpha\beta\gamma\delta}$ being the traceless part of the curvature tensor is

\[
W_{\alpha\beta\gamma\delta} = \tilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{2} (g \circ Ric(g))_{\alpha\beta\gamma\delta} + \frac{1}{12} \tilde{R}(g \circ g)_{\alpha\beta\gamma\delta}, \tag{3.57}
\]

where $Ric(g)$ is the Ricci tensor of $g$, and $\circ$ is defined in (2.8). We define the left and right Hodge duals of $W$ by
\[ *W_{\alpha\beta\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} W^{\mu\nu}_{\gamma\delta}, \quad W_{\alpha\beta\gamma\delta}^* = \frac{1}{2} W^\mu_{\alpha\beta} \epsilon^{\mu\nu\gamma\delta}. \] (3.58)

However, by virtue of the algebraic properties of \( W \), the two duals coincide: \( *W = W^* \). The Bianchi identities entail the divergence equations

\[ D^\alpha W_{\alpha\beta\gamma\delta} = J_{\beta\gamma\delta}, \] (3.59a)

\[ D^\alpha * W_{\alpha\beta\gamma\delta} = J_{\beta\gamma\delta}^*, \] (3.59b)

where

\[ J_{\beta\gamma\delta} = \frac{1}{2} (D_{\tau} \bar{R}_{\beta\delta} - D_{\delta} \bar{R}_{\beta\tau}) - \frac{1}{12} (\bar{g}_{\beta\delta} D_{\tau} \bar{R} - \bar{g}_{\beta\tau} D_{\delta} \bar{R}), \] (3.60)

\[ J_{\beta\gamma\delta}^* = \frac{1}{2} J_{\beta\gamma\delta}^{\mu\nu} \epsilon_{\mu\nu\gamma\delta}. \] (3.61)

The source terms \( J_{\beta\gamma\delta}, J_{\beta\gamma\delta}^* \) are related to the matter field, since the Ricci tensor \( \bar{R}_{\alpha\beta} \) is related to the KG field, see (A.5).

The electric and magnetic parts \( E(W), H(W) \) of the Weyl field \( W \), with respect to the foliation \( \{ \Sigma_t \} \) are defined by

\[ E(W)_{\alpha\beta} = W_{\alpha\mu\beta\nu} \bar{T}^\mu_{\nu}, \quad H(W)_{\alpha\beta} = *W_{\alpha\mu\beta\nu} \bar{T}^\mu_{\nu}. \] (3.62)

\( E \) and \( H \) are symmetric and \( T \)-tangent, i.e. \( E_{\alpha\beta} T^\alpha = H_{\alpha\beta} T^\alpha = 0 \) and trace-free \( \bar{g}^{\alpha\beta} E_{\alpha\beta} = \bar{g}^{\alpha\beta} H_{\alpha\beta} = 0 \). It follows that \( g^{ij} E_{ij} = g^{ij} H_{ij} = 0 \). The following identities concern \( W, *W \) and \( E = E(W), H = H(W) \) (see page 144 in [14]),

\[ W_{i\nu p} = - \epsilon_{ij} \eta^p H_{\nu p}, \quad *W_{i\nu p} = \epsilon_{ij} \eta^p E_{\nu p}, \]

\[ W_{i\nu pq} = - \epsilon_{ij} \eta_{pq} E_{\nu}, \quad *W_{i\nu pq} = \epsilon_{ij} \eta_{pq} H_{\nu}. \]

We remark that, the spacetime hodge dual *, and \( J_{\alpha\beta\gamma}, E_{ij}, H_{ij} \) are all scale-free.

The Gauss and Codazzi equations can be written in terms of \( E \) and \( H \)

\[ R_{ij} + 2 \bar{g}_{ij} - \hat{k}_{im} \hat{k}^m_j - \hat{k}_{ij} = E_{ij} + \frac{1}{2} R_{ij} + \frac{1}{6} (3 \bar{R}_{ij} + \bar{R}) g_{ij}, \] (3.63a)

\[ \text{curl}_{k_{ij}} = - H_{ij}. \] (3.63b)

For the energy scheme of the Weyl field, we turn to the following (rescaled) \( 1 + \) splitting of the Bianchi equations (3.59a) and (3.59b) (see proposition 7.2.1 in [14] or corollary 3.2 in [7]),

\[ \mathcal{L}_{\partial_t} E_{ij} = N \text{curl} H_{ij} - (\nabla N \wedge H)_{ij} - \frac{5N}{2} (E \times k)_{ij} - \frac{2N}{3} (E \cdot k) g_{ij} - \frac{N}{2} \text{trk} E_{ij} - NJ_{ij}, \] (3.64a)

\[ \mathcal{L}_{\partial_t} H_{ij} = - N \text{curl} E_{ij} + (\nabla N \wedge E)_{ij} - \frac{5N}{2} (H \times k)_{ij} - \frac{2N}{3} (H \cdot k) g_{ij} - \frac{N}{2} \text{trk} H_{ij} - NJ_{ij}. \] (3.64b)

Refer to the appendix for the definitions of \( \wedge, \times, \text{div} \) and \( \text{curl} \).

We shall allow ourselves to use \( W \) to represent the electric part \( E \) or the magnetic part \( H \). Define the \( l \)-th order energy norm for \( W \)
\[ E_i(W, t) = \int_{\Sigma_t} \left( |\nabla I_i E|^2 + |\nabla I_i H|^2 \right) d\mu_g. \quad (3.65) \]

Let \( E^\delta(W, t) \) be the energy norm associated to \( \delta g \), i.e. replacing \( g \) in (3.65) by \( \delta g \). Then, \( t^2 E_i(W, t) = \delta^\delta t^2E_i(W, t) \).

And define the energy density
\[ \rho_0(W) = |\nabla I_i E|^2 + |\nabla I_i H|^2. \quad (3.66) \]

### 3.3.2. The zeroth order energy identity.

**Theorem 3.13.** For a solution to the Bianchi equations (3.64a) and (3.64b), there is the energy identity
\[ \partial_t \bar{\tau} E_0(W, t) + 2E_0(W, t) = \int_{\Sigma_t} (aLW + S_0) d\mu_g, \quad (3.67) \]
where the error term \( aLW \) and the source term \( S_0 \) are given by
\[ aLW = \tilde{N} \left( |E|^2 + |H|^2 \right) + \nabla \tilde{N} \cdot (E \times H), \]
\[ S_0 = -N \left( \tilde{J}_i E_i^0 + \tilde{J}_i^* H_i^0 \right). \quad (3.68) \]

**Proof.** We multiply \( 2E_i^0 \) on (3.64a), and \( 2H_i^0 \) on (3.64b) to derive
\[ \partial_t \left( |E|^2 + |H|^2 \right) = \frac{2N}{3} \text{trk} \left( |E|^2 + |H|^2 \right) + \text{div} \left( 2N \cdot (E \times H) \right) + 6\nabla N \cdot (E \times H), \]
\[ - 4\tilde{N} \left( E \times (E + H \times H) \right) \]
\[ - 4\tilde{N} (E^2 + H^2) \]
\[ - 3N \left( E \times \tilde{k} \right) \cdot E - 3N \left( H \times \tilde{k} \right) \cdot H \]
\[ - 2NJ_i E_i^0 - 2NJ_i^0 H_i^0. \quad (3.69) \]

Here we have used the identity (C.9) which relates curl, \( \cdot \), div and \( \wedge \), and the following splitting,
\[ (E \times k)_{ij} = \left( E \times \tilde{k} \right)_{ij} - \frac{\text{trk}}{3} E_{ij}. \quad (3.70) \]

Integrating on \( \Sigma_t \), we have
\[ \partial_t \int_{\Sigma_t} \rho_0(W) d\mu_g + \int_{\Sigma_t} \left( 2N\tilde{\rho}_0(W) - 3\tilde{N}\tilde{\rho}_0(W) \right) d\mu_g \]
\[ = \int_{\Sigma_t} \left( 4\tilde{N} \left( E \times (E + H \times H) \right) - 4\tilde{N} (E^2 + H^2) \right) \]
\[ + \left( 6\nabla N \cdot (E \times H) - 2NJ_i E_i^0 - 2NJ_i^0 H_i^0 \right) d\mu_g. \quad (3.71) \]

This gives (3.67) and (3.68). \( \square \)
\subsection{Higher order energy identities.} In this section, we always require \( l \leq 3 \). To continue with the higher order cases, we need a high order version of the identity \((C.9)\).

**Lemma 3.14.** Let \( l \geq 1 \), and \( H_{ij}, E_{ij} \) be any two symmetric and trace-free \((0, 2)\)-tensors on \( \Sigma_t \). Then, there is the identity

\[
\nabla^l (\text{curl } H)_{ij} \nabla^l E^{ij} - \nabla^l (\text{curl } E)_{ij} \nabla^l H^{ij} = \sum_{a \leq l} \nabla^a (\nabla^l H \ast \nabla^l E)^a + \left( \hat{\mathcal{R}}_{l-1} (H) \ast \nabla^l E + \hat{\mathcal{R}}_{l-1} (E) \ast \nabla^l H \right),
\]

(3.72)

where \( \hat{\mathcal{R}}_{l-1} (A) \) is defined as

\[
\hat{\mathcal{R}}_{l-1} (A) = \pm \sum_{a + b = l - 1} \nabla^a \hat{R}_{imjn} \ast \nabla^b A.
\]

(3.73)

with \( \hat{R}_{imjn} = R_{imjn} + \frac{1}{2} (g \circ \hat{k})_{imjn} \) being the error term of the Riemannian curvature. By the Gauss equation \((2.7)\),

\[
\hat{R}_{imjn} = (g \circ \hat{k})_{imjn} - \frac{1}{2} ( \hat{k} \circ \hat{k})_{imjn} + \bar{R}_{imjn}.
\]

(3.74)

The proof for this lemma is presented in appendix C.

**Theorem 3.13** can be generalized to the higher order cases as follows.

**Corollary 3.15.** For a solution to the Bianchi equations \((3.64a)\) and \((3.64b)\) with \( l \geq 1 \), there is the energy identity

\[
\partial_t E_l (W, t) + 2 E_l (W, t) = \int_{\Sigma_t} (\mathcal{L}_{\partial_t} \nabla^l E_{ij} + \mathcal{L}_{\partial_t} (W, t)) \, d\mu_g,
\]

(3.75)

where

\[
\mathcal{L}_{\partial_t} \nabla^l E_{ij} = \nabla^l \left( \mathcal{K} \mathcal{N}_l (W) + \hat{\mathcal{K}} \mathcal{N}_l (W) \right) \nabla^l W,
\]

\[
\mathcal{L}_{\partial_t} (W, t) = \nabla^l W \nabla^l W + \sum_{a \leq l} \mathcal{N}_{a+l} (W) \nabla^l W,
\]

\[
S_l = -\nabla^l \left( \hat{N} \mathcal{N}_l (W) \right) \nabla^l E_{ij} - \nabla^l \left( \hat{N} \mathcal{N}_l (E) \right) \nabla^l H_{ij}.
\]

(3.76)

In (3.76), \( W \) is taken to be \( E \) or \( H \). We refer to \((3.15)-(3.22)\) and \((3.73)\) for the definitions of \( \mathcal{K} \mathcal{N}_l (E), \hat{\mathcal{K}} \mathcal{N}_l (E), \mathcal{R}_{l-1} (E) \cdots \)

**Proof.** Perform \( \nabla^l \) on \((3.64a)\) and apply the commuting identity \((3.14a)\) to \( \mathcal{L}_{\partial_t} \nabla^l E_{ij} \). Then,

\[
\mathcal{L}_{\partial_t} \nabla^l E_{ij} = \nabla^l \left( N \text{curl } H_{ij} + \frac{N}{3} \text{trk } E_{ij} \right) + \mathcal{K} \mathcal{N}_l (E_{ij})
\]

\[
- \nabla^l \left( (\nabla N \wedge H)_{ij} + \frac{5N}{2} (E \times \hat{k})_{ij} \right) + \frac{2N}{3} \left( E \cdot \hat{k} \right) g_{ij} + N \mathcal{N}_l (E_{ij}).
\]

(3.76)
Here the splitting (3.70) is used. We rearrange and collect similar terms,
\[
\mathcal{L}_{\partial_t} \nabla_t E_j = N \nabla_t \text{curl} H_j + \frac{N}{3} \text{tr} \nabla_t E_j - \nabla_t \left( NJ\tilde{T}_j \right)
\]
\[
\begin{aligned}
&+ KN_{\Gamma}(E_j) \pm N\tilde{T}_j(H_j) + \tilde{K}N_{\Gamma}(E_j).
\end{aligned}
\]  

(3.77)

Multiplying \(2\nabla_b E_j\) on (3.77), we derive the transport equation for \(|\nabla_t E|^2\),
\[
\begin{aligned}
\partial_t (|\nabla_t E|^2) &= 2N \nabla_t \text{curl} H_j \nabla_t E_j - \nabla_t \left( NJ\tilde{T}_j \right) \nabla_b E^b - \hat{N}\nabla_t E_\ast \nabla_b E \\
&+ \left( KN_{\Gamma}(E_j) \pm N\tilde{T}_j(H_j) + \tilde{K}N_{\Gamma}(E_j) \right) \nabla^b E^b
\end{aligned}
\]

(3.77)

Similar argument applies to (3.64b) as well.

Combining the transport equations for \(|\nabla_t E|^2\) and \(|\nabla_t H|^2\), we have
\[
\begin{aligned}
\partial_t (|\nabla_t E|^2 + |\nabla_t H|^2) &= 2N \left( \nabla_t \text{curl} H_j \nabla_t E_j - \nabla_t \text{curl} E_j \nabla_t E^b \right) \\
&+ \left( KN_{\Gamma}(E_j) \pm N\tilde{T}_j(W) + \tilde{K}N_{\Gamma}(E_j) \right) \nabla_b W - \hat{N}\nabla_t W_\ast \nabla_b W \\
&- \nabla_t \left( NJ\tilde{T}_j \right) \nabla_b E^b - \nabla_t \left( NJ\tilde{T}_j \right) \nabla_b H^b.
\end{aligned}
\]

Due to the identity (3.72), the second line above turns into
\[
N \left( \nabla_t \text{curl} H_j \nabla_t E_j - \nabla_t \text{curl} E_j \nabla_t E^b \right) \\
= \text{div} + \hat{R}_{\tilde{T}_{\Gamma}}(W) \nabla_b W + \sum_{a \leq l} \nabla N \ast \nabla_l E \ast \nabla_l H.
\]

We always use div to denote some divergence forms. Integrate on \(\Sigma_t\),
\[
\begin{aligned}
&\partial_t \int_{\Sigma_t} \rho_t(W) d\mu_\Sigma + \int_{\Sigma_t} 2N \rho_t(W) d\mu_\Sigma \\
&= \int_{\Sigma_t} \left( LW_1 + LW_2 + LW_3 + S_l \right) d\mu_\Sigma
\end{aligned}
\]

(3.78)

where the lower order terms \(LW_1, \cdots, LW_3\) and source term \(S_l\) are defined in (3.76).

\(\square\)

4. Preliminary estimates

Let us recall the energy norms \(E_i(\phi, t)\) (3.25) and \(E_i(W, t)\) (3.65). We also define the energy norm up to \(l\) orders as follows,
\[
\mathcal{E}_l(\phi, t) = \sum_{i \leq l} E_i(\phi, t), \quad \mathcal{E}_l(W, t) = \sum_{i \leq l} E_i(W, t).
\]

(4.1)

Besides, the energies for \(\hat{k}\) are defined by
\[
E_l(\hat{k}, t) = \int_{\Sigma_t} |\nabla_t \hat{k}|^2 d\mu_\Sigma, \quad \mathcal{E}_l(\hat{k}, t) = \sum_{i \leq l} E_i(\hat{k}, t).
\]

(4.2)
4.1. The continuity argument

Fix a constant $0 < \delta < \frac{1}{9}$ and let $M$ be a large constant to be determined. We start with the following weak assumptions: The bootstrap assumptions for the KG field $\phi$,

$$t E_i(\phi, t) \leq \varepsilon^2 M^2 \bar{t}^\delta, \quad i = 0, \ldots, 4.$$  \hspace{1cm} (4.3)

The bootstrap assumptions for the Weyl tensor $E(W), \, H(W)$,

$$t^2 E_i(W, t) \leq \varepsilon^2 M^2 \bar{t}^\delta, \quad i = 0, \ldots, 3.$$  \hspace{1cm} (4.4)

The bootstrap assumptions for $\hat{k}$ and $\hat{N} = N - 1$,

$$t^2 E_i(\hat{k}, t) \leq \varepsilon^2 M^2 \bar{t}^\delta (0 \leq i \leq 4), \quad t^2 \|\hat{N}\|_{L^\infty}^2 \leq \varepsilon^2 M^2 \bar{t}^\delta.$$ \hspace{1cm} (4.5)

The smallness condition for the KG data (2nd line in (1.5)) given in theorem 1.1 entails that (4.3) holds for $t = t_0$. Conducting the elliptic estimates for $\hat{N}$ (refer to (4.10)) on the initial slice, we further show that the data (1.5) imply that (4.5) holds on $\{t = t_0\}$. Moreover, via the Gauss equations (2.7)–(2.10b) and the Weyl tensor (3.57) and (1.5) implies that (4.4) holds for $t = t_0$. Let $[t_0, \, t^*]$ be the largest time interval on which (4.3)–(4.5) still holds. We shall show that if $\varepsilon$ is sufficiently small (depending on initial data and $\delta$), then on $[t_0, \, t^*]$, (4.3)–(4.5) implies the same inequality with the constant $\varepsilon^2 M^2$ being replaced by $\varepsilon^2 M^2$, see section 7.1.

It will then follow that the solution and the estimate (4.3)–(4.5) can be extended to a larger time interval, thus contradicting the maximality of $t^*$. This will imply that $t^* = \infty$ and the solution is global (see theorem 7.3). We will in fact prove that for a sufficiently small $\varepsilon$, the stronger estimate (7.22) (theorem 7.3) holds true on the interval $[t_0, \, t^*]$. From now on, we always assume $t \in [t_0, \, t^*].$

We use the following convention for initial data:

$$\varepsilon^2 I_i^2(\psi) := I_0^i E_i(\psi, t_0),$$  \hspace{1cm} (6.6)

where, in application, $i = 1$ if $\psi$ is taken as $\phi$, and $i = 2$ if $\psi$ is $\hat{k}$ or $W$. At the first stage, we choose $M$ large enough so that $I_k(\phi), \, I_0(W), \, I_0(\hat{k}) < 2M$, and $\varepsilon$ is chosen small enough such that $\varepsilon M < 1$.

4.2. Sobolev inequality

Recall that $g_{ij} = \bar{t}^{-2} \tilde{g}_{ij}$ is scale-free, with $\nabla$ the corresponding connection. Throughout the paper, we simplify the notation $L^p(\Sigma, \tilde{g})$ by $L^p(\Sigma_\varepsilon)$, $p > 0$, and $\int_{\Sigma_\varepsilon} d\mu_\varepsilon$ by $\int_{\Sigma_\varepsilon}$. By the Sobolev embedding theorem on $(\Sigma, \tilde{g})$, we derive the following preliminary estimates: for $\phi$ (recall that $\mathcal{D} \phi \in \{ \tilde{T}, \nabla \phi, \tilde{m}(t) \phi \})$,

$$\sum_{j \in \mathbb{N}} \|\nabla^j \mathcal{D} \phi\|_{L^\infty} \leq \varepsilon \varepsilon^2 \sum_{j \in \mathbb{N}} \|\nabla^j \mathcal{D} \phi\|_{L^1(\Sigma_\varepsilon)} + \|\nabla^j \mathcal{D} \phi\|_{L^1(\Sigma_\varepsilon)} \lesssim E_{1,1}(\phi, t).$$ \hspace{1cm} (4.7)

For $W \in \{E, H\}$,

$$\sum_{j \in \mathbb{N}} \|\nabla^j W\|_{L^\infty} \leq \varepsilon \varepsilon^2 \sum_{j \in \mathbb{N}} \|\nabla^j W\|_{L^1(\Sigma_\varepsilon)} + \|\nabla^j W\|_{L^1(\Sigma_\varepsilon)} \lesssim E_{1,1}(W, t).$$ \hspace{1cm} (4.8)

And for $\hat{k}$,

$$\sum_{j \in \mathbb{N}} \|\nabla^j \hat{k}\|_{L^\infty} \leq \varepsilon \varepsilon^2 \sum_{j \in \mathbb{N}} \|\nabla^j \hat{k}\|_{L^1(\Sigma_\varepsilon)} + \|\nabla^j \hat{k}\|_{L^1(\Sigma_\varepsilon)} \lesssim E_{1,1}(\hat{k}, t).$$ \hspace{1cm} (4.9)
Remark 4.1. Let $g_0 = t_0^{-2} \tilde{g}_0$ be the initially normalized metric. Using the estimates for $\|\tilde{h}\|_{L^\infty}, \|\tilde{N}\|_{L^\infty}$ (4.5) and (4.9) and the evolution equation for $g_{ij}$ (2.6a), we are able to show that $g$ is close to $g_0$: $|g_{ij} - g_{0ij}| \lesssim \varepsilon M$. And the volumes of $\Sigma$ associated to $g_{ij}$ and $g_{0ij}$ are equivalent: $\exp(-\varepsilon M) \lesssim \frac{d\mu}{d\mu_{0}} \lesssim \exp(\varepsilon M)$. Thus, there exists a uniform constant bounding all the Sobolev constants for $g_{ij}$.

4.3. The lapse

In this section, we present the elliptic estimates for the lapse. In particular, we highlight the subtle relation between the lapse and the KG field, which shall serve as a key point in establishing the hierarchies of energy estimates for the KG field, see section 5.3.

The equation for the lapse $\tilde{N}$ can be derived by taking the trace of the transport equation for $\tilde{k}_{ij}$ (2.6b), $\tilde{N}$ defined in (2.4) should be regarded as small quantity in the small data situation we are considering.

$$\Delta \tilde{N} - 3\tilde{N} = N \left( R_{TT} + |\tilde{k}_{ij}|^2 \right).$$

(4.10)

Let us turn to the elliptic estimates for $\tilde{N}$. Define the energy for $\tilde{N}$:

$$E_i(\tilde{N}, t) = \int_{\Sigma} |\nabla \tilde{N}|^2 d\mu_\Sigma, \quad \tilde{E}_i(\tilde{N}, t) = \sum_{\ell \leq i} E_i(\tilde{N}, t).$$

(4.11)

Again we have $E_i(\tilde{N}, t) = r^{-3+2\ell} E^{\tilde{N}}_i(\tilde{N}, t)$, where $E^{\tilde{N}}_i(\tilde{N}, t)$ is defined by replacing the $g$ in $E_i(\tilde{N}, t)$ (4.11) by $\tilde{g}$.

Lemma 4.2. With the bootstrap assumption (4.3)–(4.5), we have

$$r^2 E_i(\tilde{N}, t) \lesssim \varepsilon^2 M^2 r^\delta, \quad 0 < \delta < \frac{1}{6}, \quad i \leq 5.$$  

(4.12)

Moreover, $\tilde{N}$ depends on the KG field in an exact manner as follows: letting

$$r^2(\phi, t) := \|\tilde{T}\phi\|_{L^\infty}^2 + \|\tilde{m}(t)\phi\|_{L^\infty}^2,$$

(4.13)

we have

$$E_2(\tilde{N}, t) \lesssim \tilde{E}_2^2(\tilde{k}, t) + r^2(\phi, t)E_0(\phi, t),$$

(4.14)

$$E_3(\tilde{N}, t) \lesssim \tilde{E}_3^2(\tilde{k}, t) + r^2(\phi, t)E_1(\phi, t),$$

(4.15)

$$E_4(\tilde{N}, t) \lesssim \tilde{E}_4^2(\tilde{k}, t) + E_2^2(\phi, t) + r^2(\phi, t)E_2(\phi, t),$$

(4.16)

$$E_5(\tilde{N}, t) \lesssim \tilde{E}_5^2(\tilde{k}, t) + E_2(\phi, t)E_3(\phi, t) + r^2(\phi, t)E_3(\phi, t).$$

(4.17)

Remark 4.3. Provided that we do not take advantage of the special structure of the source term, the ellipticity of the equation for lapse only yields the rough bound (4.12). Or in other words, $E_5(\tilde{N}, t) \lesssim \tilde{E}_5^2(\phi, t)$, ignoring lower order terms.

Remark 4.4. If one works within the CMCSH gauge, the elliptic equation for the shift $X$ interacts with the matter current $\tilde{T}_\phi(\phi) = \nabla^i \phi \tilde{T}_\phi:$

27
\[
\Delta X^i + R^i_j X^j = -2\nabla_j N \dddot{k}^j - \frac{4\kappa}{3} \nabla^i N + 2N \nabla^i \phi T \phi \\
- \left(2\dot{k}^{pq} - \nabla^p \phi \right) \left(\Gamma^i_{pq}(g) - \Gamma^i_{pq}(\gamma)\right). 
\]  

(4.18)

Notice that, \(\nabla^i \phi T \phi \sim \tau^{-1} \nabla^i (\dot{m}(t) \phi) T \phi\). Therefore, the leading term on the right hand side of (4.18) is \(\nabla^i N\), and we can expect the following estimate

\[
\mathcal{E}_5 (X, t) \lesssim \mathcal{E}_4 (\dot{N}, t) + \tau^{-2} \mathcal{E}_4^2 (\phi, t) + \text{l.o.t.},
\]

(4.19)

where \(\mathcal{E}_5 (X, t)\) is defined in the same way as (4.11).

**Proof.** We multiply \(\dot{N}\) on (4.10) to derive

\[
\int_{\Sigma_t} |\nabla \dot{N}|^2 + 3|\dot{N}|^2 + |\dddot{\dot{N}}| \lesssim \int_{\Sigma_t} |\dot{N}|^2 \left(\|T \phi\|^2 + |\dot{\phi} | \right) \lesssim \int_{\Sigma_t} \epsilon^2 \mathcal{M}^2 \tau^{-1+2t} |\dot{N}|^2.
\]

Making use of the \(L^\infty\) estimate (4.7) and noting that \(R_{TT} = (T \phi)^2 - \frac{\dot{m}(t)}{2} \phi^2\), the last term admits

\[
\int_{\Sigma_t} |\dot{N}|^2 |R_{TT}| \lesssim \int_{\Sigma_t} |\dot{N}|^2 \left(\|T \phi\|^2 + |\dot{m}(t) \phi | \right) \lesssim \int_{\Sigma_t} \epsilon^2 \mathcal{M}^2 \tau^{-1+2t} |\dot{N}|^2.
\]

And the Cauchy–Schwarz inequality shows that for some constant \(c\),

\[
\int_{\Sigma_t} \dot{N} \left( |\dddot{\dot{N}}| + |R_{TT}| \right) \lesssim \int_{\Sigma_t} \epsilon |\dot{N}|^2 + \int_{\Sigma_t} c^{-1} \left( |\dddot{\dot{N}}| + |R_{TT}| \right)^2.
\]

We choose the constant \(c < 2\) so that \(\int_{\Sigma_t} \left( \epsilon + \epsilon^2 \mathcal{M}^2 \tau^{-1+2t} \right) |\dot{N}|^2\) is absorbed by the left hand side of (4.20). Thus,

\[
\int_{\Sigma_t} |\nabla \dot{N}|^2 + |\dot{N}|^2 \lesssim \int_{\Sigma_t} \left( |\dddot{\dot{N}}| + |R_{TT}| \right)^2
\]

\[
\lesssim \mathcal{E}_4^2 (\dddot{\dot{N}}, t) + \int_{\Sigma_t} \left( (\|T \phi|_{L^\infty}^2 + \|\ddot{m}(t) \phi |_{L^\infty}^2) (\|T \phi\|^2 + |\ddot{m}(t) \phi |^2 \right) \]

(4.21)

Substituting the bootstrap assumptions for \(\dddot{\dot{N}}, \ddot{m}(t) \phi\) and \(\dddot{\dot{k}}\), we prove (4.12) up to one order. Concerning the higher order cases, we wish to prove by induction that

\[
\int_{\Sigma_t} \epsilon^2 |\nabla_i \dot{N}|^2 \lesssim \epsilon^2 \mathcal{M}^2 \tau^{4i}, \quad 0 \leq i \leq 5.
\]

(4.22)

It has been proved that (4.22) holds for \(i = 0, 1\). Suppose (4.22) holds for \(i \leq l + 1, l \leq 3\), we shall now prove that (4.22) holds for \(i = l + 2\) as well. Apply \(\nabla_i (0 \leq l \leq 3)\) to (4.10),

\[
\nabla_i \Delta N = 3 \nabla_i \dot{N} + \sum_{a+b=i} \nabla_i \nabla_a N \ast \nabla_b \left( R_{TT} + |\dot{\dot{N}}|^2 \right).
\]

(4.23)
Noting the commuting identity between $\nabla_k$ and $\Delta$ (3.18) and (3.19), we integrate by parts to derive
\[
\int_{\Sigma_t} |\nabla_{k\perp} \hat{N}|^2 = \int_{\Sigma_t} |\nabla_k \Delta \hat{N}|^2 + \mathcal{R}_{k}(\hat{N}) \nabla_k \Delta \hat{N} - \mathcal{R}_{k+1}(\hat{N}) \nabla_{k+1} \hat{N}.
\]
Thus, by the Cauchy–Schwarz inequality,
\[
\int_{\Sigma_t} |\nabla_{k\perp} \hat{N}|^2 \lesssim \int_{\Sigma_t} |\nabla_k \Delta \hat{N}|^2 + |\nabla_{k+1} \hat{N}|^2 + |\mathcal{R}_{k}(\hat{N})|^2 + |\mathcal{R}_{k+1}(\hat{N})|^2.
\]
See (3.19) for the definition of $\mathcal{R}_{k}(\hat{N})$. Using (4.23),
\[
\int_{\Sigma_t} |\nabla_{k\perp} \hat{N}|^2 \lesssim \int_{\Sigma_t} |\nabla_k \hat{N}|^2 + |\nabla_{k+1} \hat{N}|^2 + |\mathcal{R}_{k}(\hat{N})|^2 + |\mathcal{R}_{k+1}(\hat{N})|^2
\]
\[
+ \int_{\Sigma_t} \sum_{a+b=l} |\nabla_k \hat{N} \ast \nabla_l \hat{R}_{TT}|^2 + |\nabla_k \hat{N} \ast \nabla_{l\perp} \hat{k}|^2.
\]
More explicitly,
\[
\int_{\Sigma_t} |\nabla_{k\perp} \hat{N}|^2 \lesssim \varepsilon_{i+1}(\hat{N}, t) + \int_{\Sigma_t} \sum_{i=1}^3 SN_i + \sum_{i=1}^2 RN_i,
\]
where
\[
RN_1 = \sum_{a+b+l\leq l-1} |\nabla_k \mathcal{R}_{a+b} \ast \nabla_k \hat{N}|^2,
\]
\[
RN_2 = \sum_{a+b=l+1, a\leq l} |\nabla_k \mathcal{R}_{a+b} \ast \nabla_k \hat{N}|^2,
\]
\[
SN_1 = \sum_{a+b+c=l} |\nabla_k \hat{N} \ast \nabla_{l\perp} T \phi \ast \nabla_k T \phi|^2,
\]
\[
SN_2 = \sum_{a+b+c=l} |\nabla_k \hat{N} \ast \nabla_k \hat{m}(t) \phi \ast \nabla_k \hat{m}(t) \phi|^2,
\]
\[
SN_3 = \sum_{a+b+c=l} |\nabla_k \hat{N} \ast \nabla_{l\perp} \hat{k} \ast \nabla_k \hat{k}|^2.
\]
As a result of (2.7)–(2.10a) and (3.57), the expansion for $R_{a+b}$ is simplified as
\[
R_{a+b} = g \ast g \pm \hat{k} \ast g \pm \hat{k} \ast \hat{k} + W_{a+b} \pm R_{a+b} \ast g \pm R \ast g \ast g.
\]
In the following estimates, we only need the formula \(|A \ast B| \lesssim ||A|| ||B||\), while the detailed product structure and constants are irrelevant. As a $(0,4)$-tensor on $\Sigma_t$, $R_{a+b}$ can be further simplified as
\[
R_{a+b} \sim 1 \pm \hat{k} \pm |\hat{k}|^2 + E \pm |D \phi|^2.
\]
By the inductive assumption, (4.22) holds for $i \leq l+1$, $l \leq 3$, namely,
\[
\int_{\Sigma_t} |\nabla_{k\perp} \hat{N}|^2 \lesssim \varepsilon^2 M^2 \varepsilon^2 \delta^k, \quad \text{for each } l \leq 3.
\]
Now we will make use of (4.27) to bound the error terms (4.25). Since $SN_1$ and $SN_2$ can be treated in the same way, we only take $SN_2$ for instance.
Case $SN_{21}$: $a = 0$. We always apply $L^4$ to all of the four factors in $SN_2$. That is, $SN_{21}$ $(b + c = l)$ is bounded by
\[
\int_{\Sigma_t} |\nabla_h (m(t)\phi) \ast \nabla_L (\tilde{m}(t)\phi)|^2 \lesssim \mathcal{E}^2_{l+1}(\phi, t).
\]
This estimate will be treated in a more subtle way later.

Case $SN_{22}$: $a \geq 1$. Then $0 \leq b, c \leq l - 1 \leq 2$. We can apply $L^4$ to all of the six factors in $SN_2$, then in this case $(a + b + c = l, a \geq 1)$,
\[
\int_{\Sigma_t} |\nabla_L \tilde{N} \ast \nabla_h (m(t)\phi) \ast \nabla_L (\tilde{m}(t)\phi)|^2 \lesssim \mathcal{E}^2_{l+1}(\tilde{N}, t)\mathcal{E}^2_{l}(\phi, t).
\]
Similar procedures apply to $SN_3$. Roughly speaking, $\hat{k}_i$ enjoys better estimate than $\tilde{m}(t)\phi$, and we have the bound
\[
\int_{\Sigma_t} |SN_i| \lesssim \mathcal{E}^2_{l+1}(\hat{k}, t) + \mathcal{E}^2_{l+1}(\tilde{N}, t)\mathcal{E}^2_{l}(\hat{k}, t). \tag{4.28}
\]
We summarize these estimates as $(l \leq 3)$
\[
\int_{\Sigma_t} \sum_{i=1}^3 |SN_i| \lesssim \mathcal{E}^2_{l+1}(\phi, t) + \mathcal{E}^2_{l+1}(\hat{k}, t) + \mathcal{E}^2_{l+1}(\tilde{N}, t) \tag{4.29}
\]
\lesssim \varepsilon^2 M^2 l^{-2+4\delta}.

For terms involving curvature, we first look at the higher order term $RN_2$.

Case $RN_{21}$: $a = 0, b = l + 1$. Due to the expansion for $R_{ijn}$ (4.26), and the $L^\infty$ estimates for $\hat{k}, \phi$, $RN_{21}$ is estimated as
\[
\int_{\Sigma_t} \left( 1 + ||\hat{k}||^2_{L^\infty} + ||\tilde{k}||^2_{L^\infty} + ||D\phi||^2_{L^\infty} + ||E||^2_{L^\infty} \right) |\nabla_L \tilde{N}|^2 \lesssim \mathcal{E}^2_{l+1}(\tilde{N}, t) \left( 1 + \mathcal{E}^2_{l+1}(\hat{k}, t) + \mathcal{E}^2_{l+1}(\phi, t) \right).
\]

Case $RN_{22}$: $1 \leq a \leq l$. Then $1 \leq b \leq l$, and $RN_{22}$ $(a + b = l + 1, 1 \leq a, b \leq l, l \leq 3)$ turns into
\[
|\nabla_L (\hat{k} + |\hat{k}|^2 + |D\phi|^2 + E) |^2 \ast |\nabla_L \tilde{N}|^2.
\]
If $1 \leq a \leq 2$, we can always apply $L^4$ to all of the four factors (such as $|\nabla_L E|^2 |\nabla_L \tilde{N}|^2$) or $L^6$ to all of the six factors (such as $|\nabla_L |\hat{k}|^2|^2 |\nabla_L \tilde{N}|^2$) in each term of $RN_{22}$, so that for $1 \leq a \leq 2, l \leq 3$,
\[
\int_{\Sigma_t} |RN_{22}| \lesssim \mathcal{E}^2_{l+1}(\tilde{N}, t) \left( \mathcal{E}^2_{a+1}(\phi, t) + \mathcal{E}^2_{a+1}(\hat{k}, t) \right) + \mathcal{E}^2_{l+1}(\tilde{N}, t) \left( \mathcal{E}^2_{a+1}(W, t) + \mathcal{E}^2_{a+1}(\hat{k}, t) \right).
\]
If $a = 3$, then $l = 3$ and $b = 1$. We apply $L^2, L^2, L^\infty, L^\infty$ to the four factors (such as $|\nabla_k E|^2 |\nabla_k N|^2$) or $L^b$ to all of the six factors (such as $|\nabla_k |k|^2|\nabla_k N|^2$) in each term of $R N_{22}$ to derive (noting that here $l = 3$)

\[
\int_{\Sigma_i} |R N_{22}| \lesssim E_i(\hat{N}, t) \left( E_i(W, t) + E_i(\hat{k}, t) \right)
+ E_{i-1}(\hat{N}, t) \left( E_{i-1}^2(\phi, t) + E_{i-1}^2(\hat{k}, t) \right).
\]

Compared to $R N_2$, $R N_1$ involves lower order derivatives and it can be estimated in the same (in fact easier) way. In summary, $R N_1$ and $R N_2$ share the following estimate ($l \lesssim 3$)

\[
\int_{\Sigma_i} |R N_1| + |R N_2| \lesssim E_{i+1}(\hat{N}, t) \lesssim \varepsilon^2 M^2 t^{-2+\delta}.
\] (4.30)

Putting the estimates (4.29) and (4.30) together, we close the inductive argument and complete the proof for (4.12).

In the above proof, we have derived that for $0 \leq l \leq 3$,

\[
E_{i+2}(\hat{N}, t) \lesssim \int_{\Sigma_i} \sum_{b+c=l} |N \nabla_k T \phi \ast \nabla_k T \phi|^2 \\
+ \int_{\Sigma_i} \sum_{b+c=l} |N \nabla_k (\hat{m}(t) \phi) \ast \nabla_k (\hat{m}(t) \phi)|^2 \\
+ E_{i+1}(\hat{N}, t) + E_{i+1}^2(\hat{k}, t).
\] (4.31)

Following the proof, we also find out that when $l = 0$, both of $R N_1$ and $R N_{22}$ are absent. More precisely, we have, in view of (4.21),

\[
E_2(\hat{N}, t) \lesssim E_1(\hat{N}, t) + E_1^2(\hat{k}, t) + r^2(\phi, t) E_0(\phi, t) \\
\lesssim E_1^2(\hat{k}, t) + r^2(\phi, t) E_0(\phi, t).
\] (4.32)

When $l = 1$, (4.31) and the estimates for $E_2(\hat{N}, t)$ ((4.21) and (4.32)) suggest

\[
E_3(\hat{N}, t) \lesssim E_2^2(\hat{k}, t) + r^2(\phi, t) E_0(\phi, t) \\
+ \int_{\Sigma_i} r^2(\phi, t) \left( |\nabla T \phi|^2 + |\nabla (\hat{m}(t) \phi)|^2 \right),
\] (4.33)

which yields (4.15). When $l = 2$, we combine (4.31) with the estimates for $\hat{N}$ up to three order derivatives (4.21), (4.32) and (4.33) to deduce

\[
E_4(\hat{N}, t) \lesssim E_3^2(\hat{k}, t) + r^2(\phi, t) E_1(\phi, t) + \int_{\Sigma_i} \left|\nabla T \phi\right|^4 + |\nabla (\hat{m}(t) \phi)|^4 \\
+ \int_{\Sigma_i} r^2(\phi, t) \left( |\nabla^2 T \phi|^2 + |\nabla^2 (\hat{m}(t) \phi)|^2 \right).
\] (4.34)
This gives rise to (4.16), due to the Sobolev inequality. When \( l = 3 \), by (4.21) and (4.31)–(4.34), there is
\[
E_0(\hat{N}, t) \lesssim E^2_0(\hat{k}, t) + r^2(\phi, t)E_2(\phi, t) + E_3^2(\phi, t)
\]
\[
+ \int_{\Sigma_t} gre [\nabla T \phi * \nabla^2 \bar{T} \phi]^2 + |\nabla (\hat{m}(t) \phi) * \nabla^2 (\bar{m}(t) \phi)|^2
\]
\[
+ \int_{\Sigma_t} r^2(\phi, t) \left( |\nabla^3 \bar{T} \phi|^2 + |\nabla^3 (\bar{m}(t) \phi)|^2 \right).
\]

Thus, (4.17) follows by the Sobolev inequality. We remark that the top order refined estimate for lapse (4.17) is not needed in the hierarchies of energy estimates for the KG field.

**Remark 4.5.** In view of the definition for the KG energies (3.25), there is no energy norm involving two order of normal derivatives \( \| \nabla^2 \bar{T} \phi \|_{L^2(\Sigma_t)} \). Hence we lose the control for \( \| T^2 \phi \|_{L^\infty} \) as well. However, the 1 + 3 form of the KG equation (3.24) leads to
\[
\| \bar{T} \phi \|_{L^\infty} \lesssim \| \bar{T} \phi \|_{L^\infty} + \| \hat{m}(t) \phi \|_{L^\infty} + \| \nabla^2 \phi \|_{L^\infty}
\]
\[
+ \| N^{-1} \nabla N \|_{L^\infty} \| \nabla \phi \|_{L^\infty}.
\]

Thus, by the bootstrap assumption (4.3) and the \( L^\infty \) estimate (4.7) followed, we achieve
\[
\| T^2 \phi \|_{L^2(\Sigma_t)} \lesssim \| \hat{m}(t) \phi \|_{E_1^2(\phi, t)} + E_1^2(\phi, t)E_1^2(\hat{N}, t) \lesssim \varepsilon M \hat{t}^{\frac{1}{2} + \delta},
\]
\[
\| T^2 \phi \|_{L^\infty(\Sigma_t)} \lesssim \| \hat{m}(t) \phi \|_{E_1^2(\phi, t)} + E_1^2(\phi, t)E_1^2(\hat{N}, t) \lesssim \varepsilon M \hat{t}^{\frac{1}{2} + \delta}.
\]

This will be used to derive the estimates for \( N' \), see lemma 4.6 below.

From now on, we set
\[
E_i(g, t) = E_i(W, t) + E_{1+i}(\hat{k}, t) + E_{2+i}(\hat{N}, t).
\]

The hierarchies of energy estimates for the KG field (namely, the idea of linearization), also rely deeply on an \( L^\infty - L^\infty \) estimate for the KG field, which helps to retrieve the \( \delta \) loss for \( \| \hat{m}(t) \phi \|_{L^\infty}, \| \bar{T} \phi \|_{L^\infty} \). This will be explained in section 5.1. To proceed, we first need \( \| N' \|_{L^\infty} \).

Recall the definition
\[
N' = \partial_t N,
\]
and the elliptic equation for \( N' \)
\[
\Delta N' - 3N' = N' \left( R_{TT} + |\hat{k}|^2 \right) + N \partial_{\hat{k}} \left( R_{TT} + |\hat{k}|^2 \right)
\]
\[
- |\nabla N|^2 - 2\hat{k} \cdot \left( N \nabla \nabla_{\hat{k}} N + \nabla_{\hat{k}} \bar{N} \nabla_{\hat{k}} \bar{N} \right) + 2N \Delta \bar{N} + 2N \nabla^2 \bar{N} R_{TT}.
\]

**Lemma 4.6.** Fixing \( 0 < \delta < \frac{1}{6} \), and assuming (4.3)–(4.5), we have for \( N' \),
\[
\int_{\Sigma_t} \| N' \|^2 + |\nabla N'|^2 + |\nabla^2 N'|^2 \lesssim \varepsilon^2 M \hat{t}^{4\delta}.
\]

And hence by the Sobolev inequality,
\[
\| N' \|_{L^\infty} \lesssim \varepsilon M \hat{t}^{2\delta}.
\]
Proof. Multiply $N'$ on the elliptic equation for $N'$, and integrate by parts,
\[
\int_{\Sigma_{t}} |\nabla N'|^2 + 3|N'|^2 = \int_{\Sigma_{t}} \left( |\nabla \hat{N}|^2 - 2\hat{N}\Delta \hat{N} \right) N'
\]
\[
- \int_{\Sigma_{t}} |N'|^2 \left( |\hat{k}|^2 + R_{TT} \right) + N'\hat{N}\partial_{\tau} \left( |\hat{k}|^2 + R_{TT} \right)
\]
\[
+ \int_{\Sigma_{t}} 2N'\hat{k} \left( N\nabla_{\Sigma} \nabla \hat{N} + \nabla_{\tau} \hat{N} \nabla_{\tau} \hat{N} \right) - 2N'\nabla_{\Sigma} \nabla \hat{N} R_{T\tau}.
\]
(4.42)

Then by the Cauchy–Schwarz inequality, and the $L^{\infty}$ estimate for $\hat{k}$, $\mathcal{D}\phi$,
\[
\int_{\Sigma_{t}} |\nabla N'|^2 + |N'|^2 \lesssim \int_{\Sigma_{t}} \left( \partial_{\tau} |\hat{k}|^2 + \partial_{\tau} R_{TT} \right)^2 + |\nabla \hat{N}|^2 \left( 1 + |\hat{k}|^2 \right)
\]
\[
+ |\nabla N|^2 (1 + |\hat{k}|^2) + |\nabla^2 \hat{N}|^2 (|\hat{N}|^2 + |\hat{k}|^2).
\]

Using the evolution equation for $\hat{k}$ (2.6b), and substituting the estimates for $\hat{N}$ (see lemma 4.2) into the above formula, we arrive at
\[
\int_{\Sigma_{t}} |\nabla N'|^2 + |N'|^2 \lesssim \mathcal{E}^2_{1}(g, t) (1 + \mathcal{E}_{1}(g, t)) + \mathcal{E}_{1}(g, t) \mathcal{E}^2_{1}(\phi, t)
\]
\[
+ \int_{\Sigma_{t}} N^2 |T\phi|^2 |\nabla_{\Sigma} \phi|^2 + N^2 \hat{m}^2(t)|\hat{m}(t)\phi|^2 + |\hat{m}(t)\phi|^2.
\]
(4.43)

Thanks to remark 4.5 for $\|T^2\phi\|_{\tilde{L}^{2}(\Sigma_{t})}$, there is
\[
\int_{\Sigma_{t}} |\nabla N'|^2 + |N'|^2 \lesssim \mathcal{E}^2_{2}(\phi, t) + \text{1.o.t.} \lesssim \varepsilon^2 M^2 \mathcal{I}^{4t}.
\]
(4.44)

where 1.o.t. means lower order terms.

Proceeding to the second order derivative, we make use of the Böchner identity,
\[
\int_{\Sigma_{t}} |\nabla^2 N'|^2 = \int_{\Sigma_{t}} |\nabla N|^2 - R_{ij} \nabla_{\Sigma} N' \nabla_{\Sigma} N'.
\]

Noting the expansion for $R_{ij}$ (4.26), $R_{ij}$ can be expanded in the same way. Therefore, with the help of the Cauchy–Schwarz inequality, we obtain
\[
\int_{\Sigma_{t}} |\nabla^2 N'|^2 \lesssim \int_{\Sigma_{t}} \left( \partial_{\tau} R_{TT} + \partial_{\tau} |\hat{k}|^2 \right)^2 + |N'|^2 \left( |T\phi|^4 + |\hat{m}(t)\phi|^4 + |\hat{k}|^4 \right)
\]
\[
+ |\nabla N|^2 \left( |\nabla_{\Sigma} \phi|^2 \left( 1 + |\hat{k}|^2 \right) + |T\phi \nabla \phi|^2 \right) + |\nabla^2 N|^2 \left( |\hat{N}|^2 + |\hat{k}|^2 \right)
\]
\[
+ |\nabla N|^2 \left( 1 + |\hat{k}| + |\hat{k}|^2 + |\mathcal{D}\phi|^2 + |E| \right).
\]

Substituting the estimates for $\hat{N}$ (lemma 4.2) and the result (4.44), we derive
\[
\int_{\Sigma_{t}} |\nabla^2 N'|^2 \lesssim \mathcal{E}^2_{2}(\phi, t) + \text{1.o.t.} \lesssim \varepsilon^2 M^2 \mathcal{I}^{4t}.
\]

We complete the proof for (4.40) and hence (4.41). □
5. Energy estimate for the KG field

5.1. The $L^\infty - L^\infty$ estimate for the KG field

We begin with the $L^\infty$ estimate below which follows from the weak bootstrap assumption (4.3):

$$\sum_{i \leq 2} \|\nabla^i D\phi\|_{L^\infty} \lesssim \epsilon M t^{-1/2 + \delta}.$$ 

To retrieve the $\delta$ loss above, we turn to a technical ODE estimate (see lemma 3.5 in [27]), of which a specific case for application is exhibited in the appendix, lemma B.2.

**Corollary 5.1 (Improved $L^\infty$ estimate for the KG field).** With the bootstrap assumption (4.3)–(4.5), we have the improved estimate for $\phi$,

$$|t^{1/2} \dot{m}(t)\phi| + |t^{1/2} T \phi| \lesssim \epsilon I_4(\phi) + \epsilon M.$$  \hspace{1cm} (5.1)

**Proof.** Set

$$\Phi = t^{3/2} \phi.$$ \hspace{1cm} (5.2)

Then a straightforward calculation with respect to the $1 + 3$ form of KG equation (3.24) leads to the following second order ODE

$$T^2 \Phi + m \Phi = t^{-1/2} \left( \Delta \phi + N^{-1} \nabla_p \nabla^p \phi \right) - \frac{3}{2m} t^{-1/2} N^{-1} \dot{N} r^{-1} \dot{m}(t) \phi - \frac{3}{4m} t^{-1/2} N^{-2} t^{-1} m(t) \phi.$$ \hspace{1cm} (5.3)

We should perform $L^\infty$ on $\Delta \phi$, which is

$$\|\Delta \phi\|_{L^\infty} \lesssim t^{-1} \|\Delta (m(t) \phi)\|_{L^\infty} \lesssim t^{-1} I_4^2(\phi, t).$$

Thus, the right hand side of (5.3) can be bounded by

$$\|T^2 \Phi + m \Phi\|_{L^\infty} \lesssim t^{-1/2} \left( \epsilon I_4^2(\phi, t) + \epsilon I_2^2(N', t) \epsilon I_2^2(\phi, t) \right) + t^{-1} \epsilon I_4^2(\dot{N}, t) \epsilon I_2^2(\phi, t) \lesssim \epsilon M t^{-2 + \delta}.$$ 

We integrate along the integral curves of $T$, which are the same as that of $\partial_t$ but parametrized by arc length $s$, i.e. $t = \frac{\partial_s}{m}$. Along this curve, $ds = N \sqrt{t}$.

It follows from the technical ODE estimate [27] (or lemma B.2) that

$$|\Phi(t)| + |T \Phi(t)| \lesssim \epsilon I_4(\phi) + \int_{t_0}^{t} \epsilon M t^{-2 + \delta} N \sqrt{t} dt.$$ \hspace{1cm} (5.4)

This further leads to the improved $L^\infty$ estimate (5.1). \hfill \Box

5.2. The lower order terms

In this section, we will prove the following estimates for the lower order terms. Recall the definition of $\mathcal{E}_i(g, t)$ (4.38).
Corollary 5.2. With the bootstrap assumption (4.3)–(4.5), the lower order terms \( \partial \mathcal{L}K \) (3.29) and \( \mathcal{L}K_1, \ldots, \mathcal{L}K_3 \) (1 \( \leq l \) \( \leq 4 \)) (3.38)–(3.40) in the energy identities for the KG field share the following estimates

\[
\int_{\Sigma_t} |\partial \mathcal{L}K| \lesssim t^{-1} E_0(\phi, t) + \mathcal{E}_{1}^{\frac{1}{2}}(g, t) E_0(\phi, t),
\]

(5.5)

\[
\int_{\Sigma_t} \sum_{i=1}^{3} |\mathcal{L}K_i| \lesssim t^{-1} E_i(\phi, t) + \mathcal{E}_{1}^{\frac{1}{2}}(g, t) E_4(\phi, t).
\]

(5.6)

5.2.1 The estimate for \( \partial \mathcal{L}K \). The formula for \( \partial \mathcal{L}K \) is given in (3.29). We should note that \( \phi \sim t^{-1} \bar{m}(t) \), hence \( \partial \mathcal{L}K \) can be rewritten as

\[
\partial \mathcal{L}K = \left( t^{-1} N + t^{-1} \tilde{N} + N \tilde{k} + \nabla \tilde{N} \right) |\mathcal{D} \phi|^2.
\]

Therefore, due to \( \| \tilde{N} \|_{L^\infty} \) (4.5) and \( \| \tilde{k} \|_{L^\infty} \) (4.9), we obtain (5.5).

5.2.2 The estimate for \( \mathcal{L}K_1 \) (1 \( \leq l \) \( \leq 4 \)). Since \( \nabla^h \phi \sim t^{-1} \nabla^h (\bar{m}(t) \phi) \), the first line in \( \mathcal{L}K_1 \) (3.38) can be rewritten as

\[
\mathcal{L}K_1^1 = \left( \tilde{N} + \bar{N} \tilde{k} + t^{-1} N + t^{-1} \tilde{N} + t^{-1} \tilde{N} \right) |\nabla_k \mathcal{D} \phi|^2,
\]

while the second line in \( \mathcal{L}K_1 \) (3.38) is

\[
\mathcal{L}K_1^2 = - N \nabla^h \tilde{T} \phi \mathcal{R}_k(\phi) - m^{-1} t^{-1} N \nabla^h (\bar{m}(t) \phi) \mathcal{R}_k(\phi).
\]

Making use of \( \| \tilde{N} \|_{L^\infty} \) (4.5) and \( \| \tilde{k} \|_{L^\infty} \) (4.9), we have

\[
\int_{\Sigma_t} |\mathcal{L}K_1^1| \lesssim t^{-1} E_i(\phi, t) + \mathcal{E}_{1}^{\frac{1}{2}}(g, t) E_4(\phi, t), \quad 1 \leq l \leq 4.
\]

(5.7)

For \( \mathcal{L}K_1^2 \) (1 \( \leq l \) \( \leq 4 \)), compared to the first term, the second term in \( \mathcal{L}K_1^2 \) taking the form of \( t^{-1} N \nabla^h \mathcal{D} \phi \mathcal{R}_k(\phi) \) is of lower order. We now consider the general form \( N \nabla^h \mathcal{D} \phi \mathcal{R}_k(\phi) \), which can be further rewritten as

\[
t^{-1} \nabla^h \mathcal{D} \phi \sum_{a+b=l, \ a \leq l-1} N \nabla_k \mathcal{R}_{mn} \ast \nabla_k (\bar{m}(t) \phi).
\]

(5.8)

Case \( \mathcal{L}K_1^2 \)-I: \( a = 0 \). Then \( b = l \). In view of the expansion for \( \mathcal{R}_{mn} \) (4.26), we have

\[
\int_{\Sigma_t} t^{-1} |\nabla^h \mathcal{D} \phi \ast \mathcal{R}_{mn} \ast \nabla_k \mathcal{D} \phi| \lesssim t^{-1} E_i(\phi, t) \left( 1 + \mathcal{E}_{1}^{\frac{1}{2}}(g, t) + \mathcal{E}_2(\phi, t) \right), \quad 1 \leq l \leq 4.
\]

(5.9)

Case \( \mathcal{L}K_1^2 \)-II: \( a \geq 1 \). Then \( 1 \leq a, b \leq l - 1 \). If \( a \leq 2 \), we can apply \( L^4, L^4 \) to \( \nabla_k \mathcal{R}_{mn} \ast \nabla_k \mathcal{D} \phi \). To estimate \( \| \nabla_k \mathcal{R}_{mn} \|_{L^4(\Sigma_t)} \), we note that \( \nabla_k \mathcal{R}_{mn} = \nabla_k (k \ast k \ast k \ast E \ast \mathcal{D} \phi \ast \mathcal{D} \phi) \). Furthermore, to estimate \( \| \nabla_k (\mathcal{D} \phi \ast \mathcal{D} \phi) \|_{L^4(\Sigma_t)} \) or \( \| \nabla_k (k \ast k) \|_{L^4(\Sigma_t)} \), noticing that \( \frac{3}{2} + 2 \leq \left[ \frac{1}{2} \right] + 2 \leq 3 < 4 \) for \( l \leq 4 \), we can apply \( L^4, L^4 \) to the two factors. Finally, we derive for \( 1 \leq a \leq 2 \).
\[
\| \nabla_l R_{\text{symn}} \|_{L^2(\Sigma_t)} \| \nabla_k D \phi \|_{L^2(\Sigma_t)} \lesssim \left( E_3^i (g, t) + E_4 (\phi, t) \right) E_3^i (\phi, t).
\]

If \( a = 3 \), which occurs only when \( l = 4 \), and therefore \( b = 1 \). In this case, we apply \( L^2, L^\infty \) to \( \nabla^3 R_{\text{symn}} * \nabla D \phi \), then
\[
\| \nabla^3 R_{\text{symn}} \|_{L^2(\Sigma_t)} \| \nabla D \phi \|_{L^\infty} \lesssim \left( E_3^i (g, t) + E_3 (\phi, t) \right) E_3^i (\phi, t).
\]

Thus, it follows that for Case \( iLK^2_1 \) (\( a \geq 1, 1 \leq l \leq 4 \)),
\[
\int_{\Sigma_t} t^{-1} |\nabla^k D \phi * \nabla_l R_{\text{symn}} * \nabla_k D \phi| \lesssim t^{-1} E_3 (\phi, t) \left( E_3^i (g, t) + E_4 (\phi, t) \right).
\]

Equations (5.9) and (5.10) yields that for \( iLK^2_1 \) (\( 1 \leq i \leq 4 \)),
\[
\int_{\Sigma_t} |iLK^2_i| \lesssim t^{-1} E_3 (\phi, t) + t^{-1} E_4 (\phi, t) \left( E_3^i (g, t) + E_4 (\phi, t) \right).
\]

Combine this with (5.7), then for \( 1 \leq l \leq 4 \),
\[
\int_{\Sigma_t} |iLK_i| \lesssim t^{-1} E_3 (\phi, t) + E_4 (\phi, t) \left( E_3^i (g, t) + t^{-1} E_4 (\phi, t) \right).
\]

5.2.3. The estimate for \( iLK_2(1 \leq l \leq 4) \). \( iLK_2 \) (3.39) can be written as \( iLK_2 = iLK_{21} + iLK_{22} \), where (we have always ignored irrelevant constants) \( iLK_{2i} \), \( i = 1, 2 \) are given as follows: \( iLK_{21} = iLK^1_{21} + iLK^2_{21} \), with
\[
iLK^1_{21} := \nabla^k D \phi \sum_{a+b+1=l+1} \nabla_k \nabla^l N \ast \nabla_{k} D \phi,
\]
\[
iLK^2_{21} := \nabla^k D \phi \sum_{a+b=l} \nabla_k \nabla^l N \ast \nabla_{k} D \phi.
\]

And \( iLK_{22} = iLK^1_{22} + iLK^2_{22} \), with
\[
iLK^1_{22} := \nabla^k D \phi \sum_{p+q+l} \nabla_{p+q} \nabla^l \hat{k} \ast \nabla_{k} D \phi,
\]
\[
iLK^2_{22} := \nabla^k D \phi \sum_{p+b=l, p \geq 1} \nabla_{p} \nabla^l \hat{k} \ast \nabla_{k} D \phi.
\]

For \( iLK^1_{21} \), \( iLK^2_{21} \), \( a + b \leq l \) and \( \left[ \frac{l}{2} + 2 \right] < 4 (l \leq 4) \), we can always apply \( L^2, L^\infty \) or \( L^\infty, L^2 \) to the two factors in \( \nabla_k \nabla^l N \ast \nabla_{k} D \phi \) to derive
\[
\int_{\Sigma_t} |iLK_{21}| \lesssim E_3^i (\phi, t) E_3^i (N, t) E_4^i (\phi, t) \lesssim \varepsilon^4 M^4 t^{-2+4\delta}.
\]

In the case of \( iLK^2_{21} \), since \( b, q < l - 1 \), and \( p + 1 \leq l \) \((l \leq 4)\), we can apply \( L^2, L^6, L^6, L^6 \) to the four factors, so that
\[
\int_{\Sigma_t} |iLK^2_{21}| \lesssim E_3 (\phi, t) E_{l-1} (g, t) \lesssim \varepsilon^4 M^4 t^{-3+5\delta}.
\]
As for $iLK_{22}^3$, $\varepsilon^4 + 2(\varepsilon^4) + 4$, we can always apply $L^2$, $L^2$, $L^\infty$ or $L^2$, $L^\infty$, $L^2$ to the three factors in $iLK_{22}^3$, then

$$\int \left| iLK_{22}^3 \right| \lesssim E_t^4(\phi, t)E_t^1(g, t)E_t^1(\phi, t) \lesssim \varepsilon^3 M^3 r^{-2-4\delta}. \quad (5.16)$$

In summary, we have

$$\int \left| iLK_{22} \right| \lesssim E_t^4(\phi, t)E_t^1(g, t), \quad 1 \leq l \leq 4. \quad (5.17)$$

Combining (5.14) with (5.17), we conclude that for $iLK_2$

$$\int \left| iLK_2 \right| \lesssim E_t^4(\phi, t)E_t^1(g, t), \quad 1 \leq l \leq 4. \quad (5.18)$$

5.2.4. The estimate for $iLK_3^1 (1 \leq l \leq 4)$. Generally speaking, compared to $iLK_2$, $iLK_3 (3.40)$ has an extra factor $r^{-1}$, for $\nabla_i \phi \sim r^{-1} i_i (m(t) \phi)$. To be more specific, there is

$$\left| iLK_3^1 \right| \lesssim r^{-1} |\nabla_i D\phi| \left( |KN_i(D\phi)| + |N_i(D\phi)| + |N_{i+1}(D\phi)| \right) \sim r^{-1} (iLK_{22} + iLK_{21}). \quad (5.19)$$

Referring to the estimate for $iLK_2 (5.18)$, we bound $iLK_3$ as

$$\int \left| iLK_3 \right| \lesssim r^{-1} E_t^4(\phi, t)E_t^1(g, t), \quad 1 \leq l \leq 4. \quad (5.20)$$

At last, comparing the bound in (5.11), (5.18) and (5.20), we achieve corollary 5.2.

5.3. The borderline term and the hierarchies of energies

In this section, we investigate the borderline term $BL_t (3.41)$. We will take advantage of the refined estimates for the lapse (lemma 4.2), and the enhanced $L^\infty$ estimate (5.1) to establish the hierarchies of energy estimates. Then each order energy for the KG field and the lapse will be improved step by step, based on which the nonlinear term $BL_t$ is reduced to a linear one. The ideas of linearization and hierarchies of energies work together to close the energy argument for the KG field. For the calculations followed, we here remind ourselves the definition of $E_t(g, t)$ (4.38).

5.3.1. The zeroth order energy estimate.

**Lemma 5.3.** We have the improved estimate for the zeroth order energy

$$t\tilde{E}_0(\phi, t) \lesssim \varepsilon^3 t_0^2(\phi). \quad (5.21)$$

**Proof.** According to the zeroth order energy identity (3.28) (theorem 3.7) and the estimate for $\partial_t LK$ (5.5), there is

$$\partial_t \tilde{E}_0(\phi, t) + \tilde{E}_0(\phi, t) \lesssim r^{-1} E_0(\phi, t) + \varepsilon^4 E_t^2(\phi, t).$$

We then change to the time function $t$ to derive

$$\partial_t (t\tilde{E}_0(\phi, t)) \lesssim r^{-2} (tE_0(\phi, t)) + t^{-1} E_t^2(\phi, t)(tE_0(\phi, t)). \quad (5.22)$$
Substitute the bound for $E_l(g, t)$ (4.4), (4.5) and (4.12) and integrate along $\partial_t$, noting that $tE_0(\phi, t) \sim tE_0(\phi, t)$ and $0 < \delta < \frac{1}{2}$,

$$tE_0(\phi, t) \lesssim \varepsilon^2 I_0^2(\phi) + \int_0^t \left( \frac{1}{r^2} + \frac{\varepsilon M}{r^2 - 2\varepsilon} \right) (t'E_0(\phi, t')) dt'.$$  \hspace{1cm} (5.23)

An application of the Grönwall inequality yields (5.21).

With the result of lemma 5.3, we can further sharpen the estimates for the lapse up to two order derivatives.

**Corollary 5.4.** The estimate for $E_2(\tilde{N}, t)$ is improved as

$$\hat{r}^2 E_2(\tilde{N}, t) \lesssim \varepsilon^2 I_2^2(\phi) \varepsilon^2 M^2. \hspace{1cm} (5.24)$$

**Proof.** By virtue of (4.14) (lemma 4.2), we substitute the enhanced estimates (5.1) and (5.21) for $\|\bar{m}(t)\phi\|_{L^\infty}$, $\|\bar{T}\phi\|_{L^\infty}$, $E_0(\phi, t)$ and the bootstrap assumption for $E_l(k, t)$ into (4.14), thus

$$\hat{r}^2 E_2(\tilde{N}, t) \lesssim \varepsilon^2 I_2^2(\phi) \left( \varepsilon^2 M^2 + \varepsilon^2 I_2^2(\phi) \right) + \varepsilon^4 M^4 r^{-2 + \delta} \lesssim \varepsilon^2 I_2^2(\phi) \varepsilon^2 M^2. \hspace{1cm} \square$$

**5.3.2. The higher order energy estimates.** Now we make use of the previously improved $\phi$ and $\tilde{N}$ to refine the estimates for $E_l(\phi, t)$, $l \leq 4$ step by step.

**Lemma 5.5.** There is a constant $C_M$ depending linearly on $M$, such that

$$tE_l(\phi, t) \lesssim \left( \varepsilon^2 I_l^2(\phi) + \varepsilon^3 M^3 \right) \ell^C \varepsilon, \hspace{1cm} 1 \leq l \leq 4. \hspace{1cm} (5.25)$$

Meanwhile, the lapse admits the enhanced estimates

$$\hat{r}^2 E_{l+2}(\tilde{N}, t) \lesssim \varepsilon^2 M^2 \left( \varepsilon^2 I_l^2(\phi) + \varepsilon^3 M^3 \right) \ell^C \varepsilon, \hspace{1cm} 1 \leq l \leq 2. \hspace{1cm} (5.26)$$

**Proof.** After substituting the estimates for the lower order terms $\ell L_k^1, \cdots, L_k^3$ (corollary 5.2) into the energy identity for $\phi$ (3.37) (corollary 3.11), we arrive at, for $1 \leq l \leq 4$ (noting that $E_l(\phi, t) \sim E_l(\phi, t)$),

$$\partial_\nu E_l(\phi, t) + \bar{E}_l(\phi, t) \lesssim t^{-1} E_l(\phi, t) + \varepsilon^2 \dot{I}_l(g, t) \varepsilon_4(\phi, t) + \int_{\Sigma_0} |B_{L_l}|, \hspace{1cm} (5.27)$$

where the borderline term $B_{L_l}$ (3.41) takes the form: $B_{L_l} = B_{L_l}^1 + \cdots + B_{L_l}^i$, with $B_{L_i}$, $i = 1, \cdots, l$ given by

$$B_{L_i} = \bar{m}(t) \nabla^{i-1} N \ast \left( \nabla^i (\bar{m}(t) \phi) \ast \nabla^i \bar{T} \phi \ast \nabla^i \bar{T} \phi \ast \nabla^i \bar{m}(t) \phi \right). \hspace{1cm} (5.28)$$

The structure of (5.28) is essential to the refinement for $E_l(\phi, t)$.

**Step 1: the improvement for $E_1(\phi, t)$.** With the help of corollary 5.4 and the enhanced $\|\bar{m}(t)\phi\|_{L^\infty}$, $\|\bar{T}\phi\|_{L^\infty}$ (5.1), the borderline term $B_{L_1} = B_{L_1}^1$, which reads

$$\bar{m}(t) \nabla N \ast \nabla (\bar{m}(t) \phi) \ast \bar{T} \phi - \bar{m}(t) \nabla N \ast \nabla \bar{T} \phi \ast (\bar{m}(t) \phi),$$
is in fact a linear term. Applying $L^2, L^3, L^\infty$ to the three factors in each term of $BL_1$, we have, by the Cauchy–Schwarz inequality,

$$\int_{\Sigma} |BL_1| \lesssim t \|\hat{\phi}\|_{L^\infty} \|\nabla N\|_{L^2(\Sigma)} \|\nabla (\hat{m}(t)\phi)\|_{L^2(\Sigma)}$$

$$+ t \|\hat{m}(t)\phi\|_{L^\infty} \|\nabla T\phi\|_{L^2(\Sigma)}$$

$$\lesssim t^2 \varepsilon M^2 E_1^2 (\hat{N}, t) E_1^2 (\phi, t) \lesssim t^{-1} \varepsilon^2 M I_4 (\phi) E_1^2 (\phi, t). \quad (5.29)$$

Hence, adding (5.29) to (5.27) with $l = 1$,

$$\partial_t \tilde{E}_1 (\phi, t) + \tilde{E}_1 (\phi, t) \lesssim (t^{-1} + \varepsilon M) E_1 (\phi, t) + \varepsilon^2 I_4^2 (\phi) \varepsilon M^{-1}$$

$$+ \varepsilon^3 M^3 t^{-1} \lesssim (t^{-2} + \varepsilon M^{-1}) (t E_1 (\phi, t)) + \varepsilon^2 I_4^2 (\phi) \varepsilon M^{-1}$$

$$+ \varepsilon^3 M^3 t^{-1} \lesssim (t E_1 (\phi, t)) + \varepsilon^2 M^3 \lesssim t^M. \quad (5.30)$$

Noting that $\tilde{E}_1 (\phi, t) \sim E_1 (\phi, t)$, by the Grönwall inequality, there is a constant $C_M$ depending linearly on $M$, such that

$$t E_1 (\phi, t) \lesssim (\varepsilon^2 I_4^2 (\phi) + \varepsilon^3 M^3) t^M. \quad (5.31)$$

Therefore, (5.25) with $l = 1$ holds.

**Step II: the improvement for $E_2 (\phi, t)$**. The borderline term $BL_2 = BL_2^1 + BL_2^2$, where $BL_2^1$ and $BL_2^2$ read as below:

$$BL_2^1 = \hat{m}(t) \nabla^2 (\hat{m}(t)\phi) * \nabla^2 N * T\phi + \hat{m}(t) \nabla^2 T\phi * \nabla^2 N * \hat{m}(t)\phi,$$

$$BL_2^2 = \hat{m}(t) \nabla^2 (\hat{m}(t)\phi) * \nabla N * \nabla T\phi + \hat{m}(t) \nabla T\phi * \nabla N * \nabla (\hat{m}(t)\phi).$$

Now, in addition to the enhanced $E_0 (\phi, t)$ (5.23), $E_2 (\hat{N}, t)$ (5.24), and $\|\hat{m}(t)\phi\|_{L^\infty}, \|\hat{T}\phi\|_{L^\infty}$ (5.1), we should also utilize the newly improved $E_1 (\phi, t)$.

For $BL_2^1$, we follow the estimate for $BL_1$ (5.29) to deduce

$$\int_{\Sigma} |BL_2^1| \lesssim t^{-1} \varepsilon^2 M I_4 (\phi) E_1^2 (\phi, t) \lesssim \varepsilon M E_2 (\phi, t) + \varepsilon^2 I_4 (\phi) M^{-1}. \quad (5.32)$$

For $BL_2^2$, we apply $L^2, L^3, L^1$ to the three factors. By the Sobolev inequalities, $\|\nabla D\phi\|_{L^1(\Sigma)} \lesssim E_1^2 (\phi, t)$. Note that $E_2^1 (\phi, t) \lesssim E_1^2 (\phi, t) + E_2^1 (\phi, t)$, and $E_1^2 (\phi, t)$ has been improved via Step I. Thus, $BL_2^2$ is indeed a linear term:

$$\int_{\Sigma} |BL_2^2| \lesssim ||\nabla N||_{L^1(\Sigma)} ||\nabla T\phi||_{L^1(\Sigma)} ||\nabla^2 (\hat{m}(t)\phi)\|_{L^2(\Sigma)}$$

$$+ t \||\nabla N||_{L^1(\Sigma)} \|\nabla (\hat{m}(t)\phi)\|_{L^2(\Sigma)} \|\nabla^2 T\phi\|_{L^2(\Sigma)}$$

$$\lesssim \varepsilon E_1^2 (\hat{N}, t) (E_1^2 (\phi, t) + E_2^1 (\phi, t)) E_2^1 (\phi, t)$$

$$\lesssim \varepsilon I_4 (\phi) \left( (\varepsilon I_4 (\phi) + \varepsilon^2 M^3) t^{-1} + \varepsilon^2 I_4 (\phi) \right) E_2 (\phi, t)$$

$$\lesssim \varepsilon^2 I_4^2 (\phi) t^{-1} + \varepsilon^2 (3 M^3 + \varepsilon I_4 (\phi)) E_2 (\phi, t). \quad (5.33)$$
With (5.32) and (5.33), the energy inequality (5.27) with \( l = 2 \) becomes
\[
\partial_t \tilde{\mathcal{E}}_2(\phi, t) + \tilde{\mathcal{E}}_2(\phi, t) \lesssim (t^{-1} + \varepsilon M) E_2(\phi, t) + \varepsilon^2 \tilde{I}_2^3(\phi) t^{-1} + C \varepsilon + \varepsilon^3 M^3 t^{-2 + 4\delta}. \tag{5.34}
\]
Change to \( t \),
\[
\partial_t (t E_2(\phi, t)) \lesssim (t^{-2} + \varepsilon M t^{-1}) (t E_2(\phi, t)) + \varepsilon^2 \tilde{I}_2^3(\phi) t^{-1} + C \varepsilon + \varepsilon^3 M^3 t^{-2 + 4\delta}.
\]
As before, the Grönwall inequality gives rise to (5.25) with \( l = 2 \).

**Step III: the improvement for \( \dot{N} \).** In general, the refinement for \( \mathcal{E}_2(\phi, t) \) leads to improvement for \( \mathcal{E}_{n+2}(\dot{N}, t) \). We substitute the previously improved \( \mathcal{E}_2(\phi, t) \) and \( \| \tilde{m}(t) \|_{L^\infty} \), \( \| T \phi \|_{L^\infty} \) into the corresponding inequality for \( \dot{N} (4.15) \) and (4.16) \( (l = 1 \text{ corresponds to } (4.15), \ l = 2 \text{ corresponds to } (4.16)) \). Then
\[
\int |BL_{l+2}| \lesssim \varepsilon M \left( \varepsilon I_4(\phi) + \varepsilon^2 M^2 \right) t^{-\frac{1}{2} + 4\delta} + \varepsilon^2 \tilde{I}_2^3(\phi) + \varepsilon^3 M^3 \int dt \phi, \ l = 1, 2, 3.
\]
where \( 0 < \delta < \frac{1}{2} \). Hence (5.26) follows.

**Step IV: the improvement for \( E_3(\phi, t) \).** We make use of the refined \( \mathcal{E}_2(\phi, t) \), \( \mathcal{E}_3(\dot{N}, t) \) and \( \| \tilde{m}(t) \|_{L^\infty} \), \( \| T \phi \|_{L^\infty} \) (5.1) to linearize the borderline term \( BL_3 \). Note that \( BL_3 = BL_3^1 + BL_3^2 + BL_3^3 \), where \( BL_3^i, i = 1, 2, 3 \) are shown in (5.28).

For \( BL_3^1 \), we follow the argument for \( BL_4^1, BL_4^2 \) to derive
\[
\int |BL_{l+1}| \lesssim \varepsilon M \left( \varepsilon I_4(\phi) + \varepsilon^2 M^2 \right) t^{-\frac{1}{2} + 4\delta} E_3^1(\phi, t) + \varepsilon^2 \tilde{I}_2^3(\phi) + \varepsilon^3 M^3 \int dt \phi, \ l = 1, 2, 3.
\]

For \( BL_3^2 \), we apply \( L^2, L^4, L^4 \) to the three factors, then
\[
\int |BL_{l+2}| \lesssim \varepsilon M \left( \varepsilon I_4(\phi) + \varepsilon^2 M^2 \right) t^{-\frac{1}{2} + 4\delta} + \varepsilon \tilde{I}_2^3(\phi) + \varepsilon^2 \tilde{I}_2^3(\phi) + \varepsilon^3 M^3 \int dt \phi, \ l = 1, 2, 3.
\]

With regard to \( BL_3^3 \), the argument for \( BL_3^2 \) can be adapted here to conclude, noting that \( \mathcal{E}_3^1(\phi, t) \lesssim \varepsilon \tilde{I}_2^3(\phi, t) + E_3^1(\phi, t) \),
\[
\int |BL_{l+3}| \lesssim \varepsilon \tilde{I}_2^3(\phi) t^{-1 + 4\delta} + \left( \varepsilon^3 M^3 + \varepsilon^3 I_2^3(\phi) \right) E_3(\phi, t).
\]

Combining (5.35)–(5.37) and the energy inequality (5.27) with \( l = 3 \) yields
\[
\partial_t \tilde{E}_3(\phi, t) + \tilde{E}_3(\phi, t) \lesssim (t^{-1} + \varepsilon M) E_3(\phi, t) + \left( \varepsilon \tilde{I}_2^3(\phi) + \varepsilon^3 M^3 \right) t^{-1 + 4\delta}.
\]

Then (5.25) with \( l = 3 \) follows by the Grönwall inequality.
Step V: the improvement for $E_4(\phi, t)$. Compared to Step III, improvement for $E_4(\phi, t)$ also relies on the previously improved $E_3(\phi, t)$ and $E_4(\tilde{N}, t)$.

It follows from (5.28) that $BL_4 = BL_4^1 + BL_4^2 + BL_4^3 + BL_4^4$. $BL_4^1$ can be argued in the same way as that for $BL_4$ (5.29), $BL_4^2$ (5.32), or $BL_4^3$ (5.35). Thus, there is

$$\int_{\Sigma_t} |BL_4^1| \lesssim \varepsilon^2 M^2 E_4(\phi, t) + (\varepsilon^2 I_3^2(\phi) + \varepsilon^3 M^3) t^{-1+\varepsilon}. \tag{5.39}$$

Analogous to $BL_3$ (5.36), $BL_4^2$ and $BL_4^3$ share the following estimates:

$$\int_{\Sigma_t} |BL_4^2| + |BL_4^3| \lesssim \varepsilon^2 M^2 E_4(\phi, t) + (\varepsilon^2 I_3^2(\phi) + \varepsilon^3 M^3) t^{-1+\varepsilon}. \tag{5.40}$$

Finally, following the argument for $BL_3^2$ (5.33) or $BL_3^3$ (5.37), and noting that $E_4^1(\phi, t) \lesssim E_4^1(\phi, t) + E_4^1(\phi, t)$, we obtain for $BL_4^4$,

$$\int_{\Sigma_t} |BL_4^4| \lesssim \varepsilon^2 I_3^2(\phi) t^{-1+\varepsilon} + (\varepsilon^3 M^3 + \varepsilon^2 I_3^2(\phi)) E_4(\phi, t). \tag{5.41}$$

Summarizing the above estimates, we achieve

$$\partial_t \tilde{E}_4(\phi, t) + \tilde{E}_4(\phi, t) \lesssim (t^{-1} + \varepsilon M) E_4(\phi, t) + (\varepsilon^2 I_3^2(\phi) + \varepsilon^3 M^3) t^{-1+\varepsilon}. \tag{5.42}$$

Making use of the Grönwall inequality and $\tilde{E}_4(\phi, t) \sim E_4(\phi, t)$, we accomplish the proof for (5.25) with $l = 4$.

6. Energy estimate for the Weyl field

With the improved energies for the KG field (5.21) and (5.25), we are ready to improve the energy estimates for the Weyl tensor.

**Lemma 6.1.** For some constant $C_M$ depending linearly on $M$, there is

$$r^2 E_l(W, t) \lesssim \varepsilon^2 I_3^2(W) + \varepsilon^3 M^3, \quad 0 \leq l \leq 2, \tag{6.1}$$

$$r^2 E_3(W, t) \lesssim (\varepsilon^2 I_3^2(\phi) + \varepsilon^3 I_3^2(W) + \varepsilon^3 M^3) t^{\varepsilon}. \tag{6.2}$$

We start with the following corollary which will be proved later in sections 6.1 and 6.2.

**Corollary 6.2.** For the nonlinear terms $lLW_1$, $lLW_2$ and source term $S_l$ in the Bianchi identities (3.67) and (3.75) with $l \leq 3$, we have

$$\int_{\Sigma_t} |\rho LW| + \sum_{l=1}^3 \sum_{i=1}^3 |lLW_i| + \sum_{l=0}^3 |S_l| \lesssim E^3_3(g, t) + \left( r^{-1} E^3_3(g, t) + E_3(g, t) \right) E_4(\phi, t). \tag{6.3}$$
And the top order source term $S_3$ obeys

$$\int_{\Sigma_t} |S_3| \lesssim \varepsilon ME_3(W, t) + \varepsilon M^{-1} E_4(\phi, t).$$  \tag{6.4}$$

With the help of corollary 6.2, we now prove lemma 6.1.

**Proof of lemma 6.1.** Substituting the result of (6.3), the bootstrap assumption (4.4) and (4.5), and the estimates for $\phi, \hat{N}$ (see section 5.3) into the $l$th order energy identity (3.75), we derive, for $0 \leq l \leq 2$ and $0 < \delta < \frac{1}{6}$,

$$\partial_t E_l(W, t) + 2E_l(W, t) \lesssim \varepsilon^3 M^3 t^{-3+4\delta}.$$  \tag{6.5}$$

We then change to the time function $t$ and multiply $t$ on (6.5) to obtain: for $0 \leq l \leq 2$ and $0 < \delta < \frac{1}{6}$,

$$\partial_t (t^2 E_l(W, t)) \lesssim \varepsilon^3 M^3 t^{-2+4\delta}.$$  \tag{6.5}$$

Then (6.1) follows. Substituting the results (6.3) and (6.4), and the estimate for $E_4(\phi, t)$ (lemma 5.5) into the energy identity (3.75) with $l = 3$, we have

$$\partial_t E_3(W, t) + 2E_3(W, t) \lesssim \varepsilon ME_3(W, t) + \varepsilon^3 M^3 t^{-3+4\delta} + \varepsilon M \left( \varepsilon^2 I_2^2(\phi) + \varepsilon^3 M^3 \right) t^{-2+C\varepsilon}.$$  \tag{6.6}$$

Changing to $t$ and multiplying $t$ on (6.6), then

$$\partial_t (t^2 E_3(W, t)) \lesssim \varepsilon M t^{-1} (t^2 E_3(W, t)) + \left( \varepsilon^2 I_2^2(\phi) + \varepsilon^3 M^3 \right) t^{-1+C\varepsilon}.$$  \tag{6.6}$$

An application of the Grönwall inequality then yields (6.2). $\Box$

### 6.1. The nonlinear error terms

In this section, we present the estimates for the nonlinear error terms $\partial LW, \partial LW_1, \cdots, \partial LW_3 (1 \leq l \leq 3)$.

**Corollary 6.3.** For the nonlinear error terms $\partial LW$ and $\partial LW_1, \cdots, \partial LW_3 (1 \leq l \leq 3)$, there is

$$\int_{\Sigma_t} |\partial LW| + \sum_{l=1}^{3} \sum_{i=1}^{3} |\partial LW_i| \lesssim E_3^2(g, t) + E_3(g, t) E_3(\phi, t).$$  \tag{6.7}$$

**Proof.** The error terms $\partial LW_1, \partial LW_2$ can be rearranged as

$$\partial LW_1 = \sum_{a+b+c=l} \nabla_{l} N * \nabla_{l} \hat{k} * \nabla_{l} W * \nabla^b W,$$

$$\partial LW_2 = \sum_{c \leq l \atop a+b+c+1} \sum_{a+b=c+1} \nabla_{l+c} \hat{N} * \nabla_{l} W * \nabla^b W,$$

$$\partial LW_3 = \sum_{a+b=l} \nabla_{l} \hat{N} * \nabla_{l} W * \nabla^b W.$$  \tag{6.8}$$
where $|LW_1| \lesssim |LW_{11}| + |LW_{22}| + |LW_2| \leq \sum_{i=1}^3 |LW_i|$. And $LW_3$ is

$$LW_3 = \sum_{a+b=l-1} N\nabla_k \tilde{R}_{min} * \nabla_b W * \nabla_k W.$$  \hspace{1cm} (6.9)

Note that, when $l = 0$, the three terms in (6.8) have covered $LW_3 (6.8)$. Hence, we take $0 \leq l \leq 3$ in (6.8), while $1 \leq l \leq 3$ in (6.9).

For $LW_{11}$, if $a = 0$, then $b + c = l$. Noting that $[\frac{l}{2}] + 2 \leq 3(l \leq 3)$, then we can perform $L^\infty, L^2, L^2$ or $L^2, L^\infty, L^2$ on the three factors $\nabla_k \tilde{k} * \nabla_k W * \nabla_k W$. If $a \geq 1$, then $b, c \leq l - 1 \leq 2$ and $a \leq l \leq 3$, we can perform $L^6, L^6, L^6, L^2$ on the four factors. Putting these two cases together, we have the estimate

$$\int_{\Sigma_t} |LW_{11}| \lesssim \mathcal{E}_3^2 (g, t) + \mathcal{E}_3^2 (g, t), \hspace{0.5cm} 0 \leq l \leq 3.$$  \hspace{1cm} (6.10)

For $LW_{21}$ and $LW_{22}$, noting that $[\frac{l}{2}] + 2 \leq 3(l \leq 3)$ and $[\frac{l}{2}] + 1 + 2 \leq 4$ (to perform $L^\infty$ on $\nabla_k \tilde{n} \tilde{N}$ in $LW_{21}$), we can always perform $L^2, L^\infty, L^2$ or $L^2, L^\infty, L^2$ on the three factors in each of $LW_{21}, LW_{22}$, so that

$$\int_{\Sigma_t} |LW_{21}| + |LW_{22}| \lesssim \mathcal{E}_3^2 (g, t), \hspace{0.5cm} 0 \leq l \leq 3.$$  \hspace{1cm} (6.11)

Finally, for $LW_3$, we recall that $\tilde{R}_{min} = \tilde{k} \pm \tilde{k} \pm E \pm |D\phi|^2$. Since $a, b \leq l - 1 \leq 2$, we can always perform $L^4, L^4, L^2$ on the three factors (such as $\nabla_k E * \nabla_k W * \nabla_k W$) or $L^4, L^6, L^4, L^2$ on the four factors (such as $\nabla_k \tilde{k} \tilde{k} * \nabla_k W * \nabla_k W$) in $LW_3$. As a consequence,

$$\int_{\Sigma_t} |LW_3| \lesssim \mathcal{E}_3^2 (g, t) + \mathcal{E}_3 (g, t) \mathcal{E}_3 (\phi, t), \hspace{0.5cm} 1 \leq l \leq 3.$$  \hspace{1cm} (6.12)

Summarizing the estimate (6.10)–(6.12), we achieve corollary 6.3.

\hfill \square

### 6.2. The nonlinear coupling terms

In this section, we will prove the following estimates for the nonlinear couplings in the Bianchi equations.

**Corollary 6.4.** For the coupling terms $S_l$, $0 \leq l \leq 3$ (see (3.67) and (3.75)), we have

$$\int_{\Sigma_t} |S_l| \lesssim \left( r^{-1} \mathcal{E}_3^1 (g, t) + \mathcal{E}_3 (g, t) \right) \mathcal{E}_4 (\phi, t), \hspace{0.5cm} 0 \leq l \leq 2.$$  \hspace{1cm} (6.13)

$$\int_{\Sigma_t} |S_3| \lesssim \varepsilon ME_3 (W, t) + \varepsilon M r^{-1} \mathcal{E}_4 (\phi, t).$$  \hspace{1cm} (6.14)

We postpone the proof to section 6.2.2 and explore the structures of the couplings first.

#### 6.2.1. The structures of the couplings.

The source terms $S_l$, $0 \leq l \leq 3$ can be split into the electric and magnetic parts: $S_l = S_{l1} + S_{l2}$, where

$$S_{l1} = \nabla_k \left( NJ_{(i)} \right) \nabla^k E^i,$$

$$S_{l2} = \nabla_k \left( NJ_{(i)}^* \right) \nabla^k H^i.$$  \hspace{1cm} (6.15)
Due to the fact that $E_{ij}$ is trace-free and $T$-tangent, we calculate

$$J_{[ij]} E^{ij} = \frac{1}{2} \left( D_I R_{ij} - D_j R_{iI} \right) E^{ij} = \frac{1}{2} \left( D_I D_I \phi - D_I D_J \phi \right) E^{ij} = \left( \nabla T \phi \pm k^I_i \phi \right) \ast E \pm \left( \nabla^2 \phi \ast T \phi \pm \left( \nabla T \phi \right)^2 \hat{k} \right) \ast E$$

(6.16)

and

$$J_{*ij} H^{ij} = \frac{1}{4} \left( D_I \epsilon_{[ij]} \epsilon_{\mu \nu} H^\mu \right) \cdot \epsilon_{\nu \mu}^i H^j = \frac{1}{4} \left( D_I D_I \phi \ast D_J \phi \right) \epsilon_{ij}^k H^j \ast \nabla T \phi \ast \nabla \phi \ast E$$

(6.17)

where

$$T_1 = m^2 \epsilon_{ij} \phi \epsilon^i_j H^j - \frac{1}{4} m^2 \epsilon_{ij} \phi \epsilon^i_j H^j,$$

$$T_2 = - \frac{1}{24} D_I \epsilon_{ij} \epsilon^i_j H^j \ast \nabla \phi \ast E + \frac{1}{24} D_I \epsilon_{ij} \epsilon^i_j H^j.$$

Notice that, $\epsilon_{ij}^i$, $\epsilon_{ij}^j$ are both antisymmetric in $i$ and $j$, while $H_{ij}$ is symmetric in $i$, $j$. Hence, $T_1 = T_2 = 0$. This is one of the main cancellations. With the same reason, $J_{*ij} H^{ij}$ can be further reduced as

$$J_{*ij} H^{ij} = \frac{1}{4} \left( D_I D_I \phi \ast D_J \phi \right) \epsilon_{ij}^k H^j \ast \nabla T \phi \ast \nabla \phi \ast E$$

(6.18)

The fact that $E_{ij}$ is trace-free and $T$-tangent can be generalized to the cases of higher order derivatives. Obviously, $\nabla_{[ij]} E_{ij}$, which is projected to $\Sigma_t$, is $T$-tangent. And there is $g^{ij} \nabla_i E_{ij} = 0$. Hence, the results of (6.16) and (6.18) can be generalized as

$$\nabla_I \left( NJ_{[ij]} \right) \nabla^I E^{ij} \ast \nabla T \phi \ast \nabla \phi \pm N \nabla^2 \phi \ast T \phi \ast \nabla^I E \ast \nabla \phi \ast E \ast \nabla \phi$$

(6.19)

and

$$\nabla_I \left( NJ_{*ij} \right) \nabla^I H^{ij} = \nabla_I \left( N \left( \nabla^2 \phi \ast T \phi \right) \ast \nabla \phi \ast \nabla \phi \ast E \ast \nabla \phi \ast E \right)$$

(6.20)
6.2.2. The estimates for the coupling terms. We rewrite (6.19) and (6.20) as
\[
S_{l1} = \sum_{a+b=l} \nabla I_{a} N * \nabla I_{b} (\nabla T \phi * \nabla \phi) \nabla^{h} E^{j}
\]
\[
+ \nabla I_{a} N * \nabla I_{b} (\nabla^{2} \phi * \bar{T} \phi) \nabla^{h} E^{j}
\]
\[
+ \nabla I_{a} N * \nabla I_{b} (\nabla \phi * \nabla k_{j} \phi) \nabla^{h} E^{j}
\]
\[
+ \nabla I_{a} N * \nabla I_{b} ((\bar{T} \phi)^{2} * \bar{k}) \nabla^{h} E^{j},
\]
and
\[
S_{l2} = \sum_{a+b=l} \nabla I_{a} N * \nabla I_{b} (\nabla^{2} \phi * \nabla \phi) \nabla^{h} H^{j}
\]
\[
+ \nabla I_{a} N * \nabla I_{b} (k * T \phi * \nabla \phi) \nabla^{h} H^{j}.
\]

We begin with the estimates for the electric part \(S_{l1}(0 \leq l \leq 3)\).

**Case \(S_{l1}-I\):** \(a = 0\). Then \(b = l\), \(S_{l1}\) consists
\[
S_{l1}^{1} = N \nabla I_{l} (\nabla T \phi * \nabla \phi) * \nabla^{h} E,
\]
\[
S_{l1}^{2} = N \nabla I_{l} (\nabla^{2} \phi * T \phi) * \nabla^{h} E,
\]
\[
S_{l1}^{3} = N \nabla I_{l} (k * \nabla \phi * \nabla \phi) * \nabla^{h} E,
\]
\[
S_{l1}^{4} = N \nabla I_{l} (\bar{T} \phi^{2} \bar{k}) * \nabla^{h} E.
\]

For \(S_{l1}^{1}\), we further rewrite it as
\[
S_{l1}^{1} = \sum_{p+q=l} (mt)^{-1} N \nabla I_{p+1} \bar{T} \phi * \nabla I_{q+1} (\bar{m}(t) \phi) * \nabla^{h} E.
\]

Knowing that \([\frac{l}{2}] + 1 + 2 \leq 4(l \leq 3)\), we can apply \(L^{2}, L^{\infty}, L^{2}\) or \(L^{\infty}, L^{2}, L^{2}\) to the three factors in (6.24) and derive
\[
\int_{\Sigma} |S_{l1}^{1}| \lesssim t^{-1} E_{3}^{l} (W, t) E_{4} (\phi, t) \lesssim \varepsilon^{3} M^{l} t^{-3+38}, \ l \leq 3.
\]

For \(S_{l1}^{2}\), it can be rewritten as
\[
S_{l1}^{2} = \sum_{p+q=l} N \nabla I_{p+1} \nabla \phi * \nabla I_{q+1} \bar{T} \phi * \nabla^{h} E.
\]

In the case of \(0 \leq l \leq 2\), we rearrange it as
\[
\sum_{p+q=l \leq 2} (mt)^{-1} N \nabla I_{p+1} (\bar{m}(t) \phi) * \nabla I_{q+1} \bar{T} \phi * \nabla^{h} E.
\]

We apply \(L^{4}, L^{4}, L^{2}\) (if \(p \leq 1\)) or \(L^{2}, L^{\infty}, L^{2}\) (if \(p = 2\)) to the three factors. As \(S_{l1}^{1}\) (6.25), there is the estimate
\[
\int_{\Sigma} |S_{l1}^{2}| \lesssim r^{-1} E_{3}^{l} (W, t) E_{4} (\phi, t), \ l \leq 2.
\]
In the case of \( l = 3 \),

- If \( p \leq l - 1 \leq 2 \), then \( \nabla_{h+2}(\dot{m}(t)\phi) \) will never exceed the top (fourth) order derivative. We can conduct the same argument as that in the case of \( 0 \leq l \leq 2 \) (6.26). Hence, the same bound \( t^{-1}\mathcal{E}_3^4(W, t)\dot{E}_4(\phi, t) \) can be obtained as well.
- If \( p = l = 3 \), the argument for (6.26) is no longer valid now. Because of the regularity, \( \nabla^2(\dot{m}(t)\phi) \) is not allowed, hence \( \nabla^4\phi \) can not absorb an extra \( t \) (or \( \dot{m}(t) \)) as in the previous case, which makes itself the borderline case. Nevertheless, as in the proof leading to lemma 5.5, the improved \( \|\tilde{T}\phi\|_{L^\infty} \) and \( E_4(\phi, t) \)(5.25) will help to linearize this borderline term. That is, when \( p = l = 3 \),

\[
\int_{\Sigma_t} |S_{111}^3| = \int_{\Sigma_t} |N\nabla^4\phi \ast \tilde{T}\phi \ast \nabla^3 E| \\
\lesssim \|\nabla^4\phi\|_{L^2(\Sigma_t)}\|\tilde{T}\phi\|_{L^\infty}\|\nabla^3 E\|_{L^1(\Sigma_t)} \\
\lesssim E_3^2(\phi, t) \ast \varepsilon Mt^{-\frac{1}{2}} \cdot E_3^1(W, t) \\
\lesssim E_3(W, t) + \varepsilon Mt^{-1}E_4(\phi, t).
\]  

(6.27)

Relative to \( S_{111}^1 \) and \( S_{111}^2 \), \( S_{111}^3 \) and \( S_{111}^4 \) are indeed lower order terms. \( S_{111}^3 \) can be rewritten as

\[
S_{111}^3 = \sum_{p+q+h=1} N\nabla_{l_1} k_j^l \ast \nabla_{l_{p+1}} \phi \ast \nabla_{l_{q+1}} \phi \ast \nabla^h E.
\]

If \( h = 0 \), it becomes \( \sum_{p+q=1} Nk_j^l \ast \nabla_{l_{p+1}} \phi \ast \nabla_{l_{q+1}} \phi \ast \nabla^h E \), which is equivalent to noting that \( p + 1, q + 1 \leq l + 1 \leq 4 \)

\[
\sum_{p+q=l} t^{-2}N\nabla_{l_{p+1}}(\dot{m}(t)\phi) \ast \nabla_{l_{q+1}}(\dot{m}(t)\phi) \ast \nabla^h E.
\]  

(6.28)

Quantitatively, (6.28) is again equivalent to \( t^{-1}S_{111}^3 \) (6.24). Thus, similar to (6.25) and (6.28) can be bounded by \( t^{-2}E_3^4(W, t)\dot{E}_4(\phi, t) \). If \( h \geq 1 \), we get

\[
\sum_{p+q+h=l+1} Nt^{-2}\nabla_{l} \hat{k} \ast \nabla_{l_{p+1}}(\dot{m}(t)\phi) \ast \nabla_{l_{q+1}}(\dot{m}(t)\phi) \ast \nabla^h E.
\]  

(6.29)

Knowing that \( 1 \leq h \leq l \leq 3 \), and \( 0 \leq p, q \leq l - 1 \leq 2 \), we can apply \( L^6, L^6, L^6, L^2 \) to (6.29) and deduce the bound \( t^{-2}E_3^2(g, t)\dot{E}_4(\phi, t) \), which is of lower order, compared to (6.28). As a summary, we conclude

\[
\int_{\Sigma_t} |S_{111}^3| \lesssim t^{-2}E_3^2(g, t)\dot{E}_4(\phi, t), \quad l \leq 3.
\]  

(6.30)

\( S_{111}^4 \) can be further rewritten as

\[
S_{111}^4 = \sum_{p+q+h=1} N\nabla_{l} T\phi \ast \nabla_{l} T\phi \ast \nabla_{l} \hat{k} \ast \nabla^h E.
\]  

(6.31)

Noting that \( p, q, h \leq l \leq 3 \), we can apply \( L^6, L^6, L^6, L^2 \) to the four factors in (6.31) to derive

\[
\int_{\Sigma_t} |S_{111}^4| \lesssim E_3(g, t)\dot{E}_4(\phi, t), \quad l \leq 3.
\]  

(6.32)
Case $S_{11}$-II: $a \geq 1$. Letting $a = c + 1$, $c \geq 0$, $S_{11}$-II composes of

$$S_{112}^1 = \sum_{c+1+b=l} \nabla_{l+c} \tilde{N} * \nabla_{b} (\nabla T \phi * \nabla \phi) * \nabla^6 E,$$

$$S_{112}^2 = \sum_{c+1+b=l} \nabla_{l+c} \tilde{N} * \nabla_{b} (\nabla^2 \phi * \tilde{T} \phi) * \nabla^6 E,$$

$$S_{112}^3 = \sum_{c+1+b=l} \nabla_{l+c} \tilde{N} * \nabla_{b} (k^l_1 * \nabla \phi * \nabla \phi) * \nabla^6 E,$$

$$S_{112}^4 = \sum_{c+1+b=l} \nabla_{l+c} \tilde{N} * \nabla_{b} \left( (\tilde{T} \phi)^2 \hat{k} \right) * \nabla^6 E. \quad (6.33)$$

Actually, $S_{11}$-II (6.33) enjoys better estimate than $S_{11}$-I (6.23). This can be roughly verified by comparing $N$ with $\tilde{N}$.

$S_{11}^1$ can be rearranged as $(for \; b = b_1 + b_2 \leq l - 1 \leq 2, \; then \; b_1 + b_2 + b_3 + b_4 = l \leq 3)$,

$$\sum_{c+1+b_1+b_2=b_3} m^{-1} r^{-1} \nabla_{l+c} \tilde{N} * \nabla_{b_1} (\tilde{m}(k^l_1) * \nabla^6 E. \quad (6.34)$$

Here $c + 1 \leq 3$ and $b_1 + 1, b_2 + 1 \leq 3$, we can apply $L^5, L^6, L^6, L^2$ to the four factors in (6.34) to derive

$$\int_{\Sigma} |S_{112}^1| \lesssim t^{-1} E_3 (g, t) E_4 (\phi, t), \quad l \leq 3. \quad (6.35)$$

$S_{11}^2$ can be rearranged as $(for \; b = b_1 + b_2 \leq l - 1 \leq 4, \; then \; b_1 + b_2 + b_3 + b_4 = l \leq 3)$,

$$\sum_{c+1+b_1+b_2=b_3} m^{-1} r^{-1} \nabla_{l+c} \tilde{N} * \nabla_{b_1} (\tilde{m}(k^l_1) * \nabla^6 E. \quad (6.36)$$

Here $c + 1 \leq 3$ and $b_1, b_2 \leq 2$. If $b_1 \leq 1$, then $b_1 + 2 \leq 3$, and we can apply $L^6, L^6, L^6, L^2$ to the four factors in (6.36); If $b_1 = 2$, then $c = b_2 = 0$. We use $L^\infty, L^2, L^\infty, L^2$ alternatively. In any case, $S_{11}^2$ admits the estimate

$$\int_{\Sigma} |S_{112}^2| \lesssim t^{-1} E_3 (g, t) E_4 (\phi, t), \quad l \leq 3. \quad (6.37)$$

$S_{11}^3$ can be rearranged as $(c + 1 + b_1 + b_2 + b_3 = l$ and $b_2 + 1, b_1 + 1 \leq 3)$,

$$m^{-2} l^{-2} \nabla_{l+c} \tilde{N} * \nabla_{b_1} (k^l_1) * \nabla_{b_2+1} (\tilde{m}(k^l_1) * \nabla^6 E. \quad (6.38)$$

Here $c + 1, b_1 + 1, b_2 + 1 \leq l \leq 3$. If $b_1 = 0$, $k^l_1 \sim 1$, we perform $L^6, L^\infty, L^6, L^6, L^2$ on the five factors in (6.38) and derive the bound $t^{-2} E_3 (g, t) E_4 (\phi, t)$; If $l \leq b_1$, noting that $b_1 \leq l - 1 \leq 2$, we apply $L^6, L^\infty, L^6, L^6, L^2$ to the five factors as well and the bound is replaced by $t^{-2} E_3 (g, t) E_4 (\phi, t)$, which is of lower order. In a word, we arrive at

$$\int_{\Sigma} |S_{112}^3| \lesssim t^{-2} E_3 (g, t) E_4 (\phi, t), \quad l \leq 3. \quad (6.39)$$

$S_{11}^4$ can be rearranged as $(c + 1 + b_1 + b_2 + b_3 = l)$,

$$\nabla_{l+c} \tilde{N} * \nabla_{b_1} \tilde{T} \phi * \nabla_{b_2} \tilde{T} \phi * \nabla_{b_3} \hat{k} * \nabla^6 E. \quad (6.40)$$

Noting that, $c + b_1 + b_2 + b_3 = l - 1 \leq 2$, we can always apply $L^2, L^\infty, L^\infty, L^\infty, L^2$ to the five factors in (6.40) and obtain

47
\[
\int_{\Sigma_l} |\mathcal{S}_{1l2}| \lesssim \mathcal{E}_3^2(g, t)\mathcal{E}_4(\phi, t), \quad l \leq 3.
\]  
(6.41)

Putting **Case S_{1l}** and **Case S_{1l}** together, we have for \(S_{1l}\),
\[
\int_{\Sigma_l} |\mathcal{S}_{1l}| \lesssim \left( r^{-1} \mathcal{E}_3^2(g, t) + \mathcal{E}_3(g, t) \right) \mathcal{E}_4(\phi, t), \quad 0 \leq l \leq 2,
\]  
(6.42)
\[
\int_{\Sigma_l} |\mathcal{S}_{2l1}| \lesssim \varepsilon ME_3(W, t) + \varepsilon M r^{-1} \mathcal{E}_4(\phi, t).
\]  
(6.43)

We next turn to the magnetic part \(S_{2l}\) \((0 \leq l \leq 3)\).

**Case S_{2l}-I:** \(a = 0\). Then \(b = l\), and \(S_{2l}\) consists of
\[
S_{2l1} = N \nabla_h \left( \nabla^2 \phi \ast \nabla \phi \right) \ast \nabla^l H,
\]
\[
S_{2l2} = N \nabla_h \left( \hat{k} \ast \nabla \phi \ast \bar{T} \phi \right) \ast \nabla^l H.
\]  
(6.44)

Notice that, \(S_{2l1}^1, S_{2l2}^2\) possess the same structures as \(S_{1l1}^1, S_{1l1}^2\) \((6.23)\), respectively if we replace \(\nabla^2 \phi \ast \nabla \phi\) in \(S_{2l1}\) by \(\nabla T \phi, \nabla \phi\) in \(S_{2l2}\) by \(T \phi\). Specifically, the structures in \(S_{2l1}, S_{2l2}\) are better, since spatial derivative is in general better than normal derivative \(T\). Anyway, an argument analogous to that for \(S_{1l1}, S_{1l2}\) can be applied to \(S_{2l1}, S_{2l2}\). Therefore,
\[
\int_{\Sigma_l} |\mathcal{S}_{2l1}| + |\mathcal{S}_{2l2}| \lesssim \left( r^{-1} \mathcal{E}_3^2(g, t) + \mathcal{E}_3(g, t) \right) \mathcal{E}_4(\phi, t), \quad l \leq 3.
\]  
(6.45)

**Case S_{2l}-II:** \(a \geq 1\). Letting \(a = c + 1, c \geq 0\), \(S_{2l}\) consists
\[
S_{2l1}^1 = \sum_{c+1+b=l} \nabla_{k+c+b} \hat{N} \ast \nabla_h \left( \nabla^2 \phi \ast \nabla \phi \right) \ast \nabla^l H,
\]
\[
S_{2l2}^2 = \sum_{c+1+b=l} \nabla_{k+c+b} \hat{N} \ast \nabla_h \left( \bar{T} \phi \ast \nabla \phi \ast \hat{k} \right) \ast \nabla^l H.
\]  
(6.46)

Again, if replacing \(\nabla^2 \phi\) in \(S_{2l1}\), \(\nabla \phi\) in \(S_{2l2}\), by \(\nabla T \phi, \bar{T} \phi\), respectively, one gets the same structures as \(S_{1l2}, S_{1l2}\). Thus, the estimates for \(S_{1l2}, S_{1l2}\) can be adapted to \(S_{2l1}, S_{2l2}\), and we obtain
\[
\int_{\Sigma_l} |\mathcal{S}_{2l1}| + |\mathcal{S}_{2l2}| \lesssim \left( r^{-1} \mathcal{E}_3(g, t) + \mathcal{E}_3^2(g, t) \right) \mathcal{E}_4(\phi, t), \quad l \leq 3.
\]  
(6.47)

This is of course of lower order, compared to \((6.45)\).

Viewing the results for **Case S_{2l}-I** and **Case S_{2l}-II**, we have for \(S_{2l}\),
\[
\int_{\Sigma_l} |\mathcal{S}_{2l}| \lesssim \left( r^{-1} \mathcal{E}_3^2(g, t) + \mathcal{E}_3(g, t) \right) \mathcal{E}_4(\phi, t), \quad l \leq 3.
\]  
(6.48)
7. Global existence

7.1. Close the bootstrap argument

7.1.1. For the KG field and the Weyl field. In lemmas 5.3, 5.5 and 6.1, we have showed that, for some universal constant $C$, and some constant $C_M$ depending linearly on $M$,

\[ tE_0(\phi, t) \leq C\varepsilon^2 I_0^2(\phi), \]

\[ tE_i(\phi, t) \leq C (\varepsilon^2 I_0^2(\phi) + \varepsilon^3 M^3)^{\varepsilon^6}, \quad 1 \leq i \leq 4. \]

and

\[ \tau^2 E_i(W, t) \leq C (\varepsilon^2 I_0^2(W) + \varepsilon^3 M^3)^{\varepsilon^6}, \quad 0 \leq i \leq 2, \]

\[ \tau^3 E_3(W, t) \leq C (\varepsilon^2 I_0^2(W) + \varepsilon^3 I_1^2(W) + \varepsilon^3 M^3)^{\varepsilon^6}. \]

Then we choose $M$ satisfying $C(I_0^2(W) + I_1^2(\phi)) \leq \frac{M^3}{4}$ and $\varepsilon$ small enough such that $\varepsilon C_M \leq \frac{1}{4}$. $C_M \varepsilon \leq \delta$. Hence, for $0 \leq i \leq 4$, $0 \leq j \leq 3$,

\[ tE_i(\phi, t) \leq \frac{\varepsilon^2 M^2}{2} \varepsilon^6, \quad \tau^2 E_j(W, t) \leq \frac{\varepsilon^3 M^2}{2} \varepsilon^6, \quad (7.1) \]

which improves the bootstrap assumption (4.3) and (4.4).

7.1.2. For the second fundamental form. We will use the transport equation to derive a bound for $\|\hat{k}\|_{L^2(\Sigma_t)}$. And its higher order bound can be deduced through the elliptic estimates. These help to improve the weak bootstrap assumption for $\hat{k}$ (4.5).

**Lemma 7.1.** Fixing $0 < \delta < \frac{1}{6}$, we have, for some universal constant $C$,

\[ \tau^2 E_0(\hat{k}, t) \leq C \varepsilon^2 I_0^2(\hat{k}) + C (\varepsilon I_0(\phi) + \varepsilon I_0(W) + \varepsilon^3 M^2) \varepsilon^6. \quad (7.2) \]

**Proof.** Recalling the evolution equation for $\hat{k}_g$ (2.6b), and noticing that

\[ \bar{R}_{ij}\hat{k}^{ij} = \left( E_{ij} - \frac{1}{2} R_{ij} \right) \hat{k}^{ij} = \left( E_{ij} - \frac{1}{2} \nabla_i \phi \nabla_j \phi \right) \hat{k}^{ij}, \]

we multiply $2\hat{k}^{ij}$ and integrate on $\Sigma_t$ to obtain

\[ \partial_t E_0(\hat{k}, t) + 2E_0(\hat{k}, t) = \int_{\Sigma_t} (\hat{k}^{ij} (-2\nabla_i \nabla_j N + 2NE_{ij}) - \bar{N}|\hat{k}|^2 + 2N|\hat{k}|^3 - N\nabla_i \phi \nabla_j \phi \hat{k}^{ij}. \quad (7.3) \]

The terms in the second line of (7.3) are cubic, and hence easier. We make use of $\|\hat{k}\|_{L^\infty(\Sigma_t)}$ (4.9) and $\|\bar{N}\|_{L^\infty(\Sigma_t)}$ (4.5), then

\[ \int_{\Sigma_t} |\bar{N}| |\hat{k}|^2 + N|\hat{k}|^3 \leq E_1^2 (g, t) E_0(\hat{k}, t) \leq \varepsilon M^{r-1+2\delta} E_0(\hat{k}, t). \]

Rewrite $\nabla_i \phi \nabla_j \phi \hat{k}^{ij}$ as $(mt)^{-3} \nabla_i (\bar{m}(t) \phi) \nabla_j (\bar{m}(t) \phi) \hat{k}^{ij}$ and perform $L^4, L^4, L^2$ on the three factors,
\[ \int_{\Sigma_t} |\nabla_i \phi \nabla_j \phi \hat{k}^i| \lesssim t^{-2} \mathcal{E}_2(\phi, t)E_0^4(\hat{k}, t) \lesssim \varepsilon M t^{-1+2\delta} E_0(\hat{k}, t) + \varepsilon^3 M^3 t^{-5+2\delta}. \]

The first line on the right hand of (7.3) involves quadratic terms. We utilize the refined \( \mathcal{E}_2(\hat{N}, t) \) (5.24) and \( E_0(W, t) \) (6.1) to linearize them:

\[ \int_{\Sigma_t} |\nabla_i \nabla_j N \hat{k}^i| \lesssim E_0^2(\hat{N}, t) E_0^4(\hat{k}, t) \lesssim \varepsilon I_0(\phi) t^{-1} E_0^4(\hat{k}, t), \]
\[ \int_{\Sigma_t} |\hat{E}_i \hat{k}^i| \lesssim E_0^4(W, t) E_0^4(\hat{k}, t) \lesssim \left( \varepsilon I_0(W) + \varepsilon^3 M^3 \right) t^{-1} E_0^4(\hat{k}, t). \]

Now, we substitute the bootstrap assumption for \( E_0(\hat{k}, t) \) (4.5) into the above formulae, then (7.3) turns into

\[ \partial_\tau E_0(\hat{k}, t) + 2E_0(\hat{k}, t) \lesssim \varepsilon M t^{-1+2\delta} E_0(\hat{k}, t) + \left( \varepsilon I_0(\phi) + \varepsilon I_0(W) + \varepsilon^3 M^3 \right) \varepsilon M t^{-2+\delta}. \]

Changing to \( t \) and multiplying \( t \) on the above inequality, we deduce

\[ \partial_\tau (t^2 E_0(\hat{k}, t)) \lesssim \varepsilon M t^{-2+2\delta} (t^2 E_0(\hat{k}, t)) + \left( \varepsilon I_0(\phi) + \varepsilon I_0(W) + \varepsilon^3 M^3 \right) \varepsilon M t^{-1+\delta}, \tag{7.4} \]

where \( 0 < \delta < \frac{1}{5} \). By the Grönwall inequality,

\[ t^2 E_0(\hat{k}, t) \lesssim \exp(\varepsilon M) \left( \varepsilon^2 I_0^2(\hat{k}) + \left( \varepsilon I_0(\phi) + \varepsilon I_0(W) + \varepsilon^3 M^3 \right) \varepsilon M t^\delta \right), \]

where \( \varepsilon \) is chosen such that \( \varepsilon M < 1 \). Thus (7.2) is implied. \( \square \)

\( \hat{k}_{ij} \) being the trace free part of \( k_{ij} \) (tr\( \hat{k} \) = 0) satisfies the following elliptic system,

\[ (\text{div} \hat{k}) = -R_{fi}, \]
\[ \text{curl} \hat{k}_{ij} = -H_{ij}. \tag{7.5} \]

See appendix C for the definitions for div and curl.

**Lemma 7.2.** Fixing \( 0 < \delta < \frac{1}{5} \), we have, for \( 0 \leq i \leq 3 \) and some universal constant \( C \),

\[ t^2 E_{i+1}(\hat{k}, t) \leq C \varepsilon^2 I_0^2(\hat{k}) + C \left( \varepsilon I_0(W) + \varepsilon I_0(\phi) + \varepsilon^3 M^3 \right) \varepsilon M t^\delta. \tag{7.6} \]

**Proof.** The elliptic system (7.5) gives the identity (refer to proposition 4.4.1 in [14]),

\[ \int_{\Sigma_t} |\nabla \hat{k}|^2 + 3R_{mn} \hat{k}^m \hat{k}^n \hat{k}_i + \frac{R}{2} |\hat{k}|^2 = \int_{\Sigma_t} |H_{ij}|^2 + \frac{1}{2} |R_{fi}|^2. \tag{7.7} \]

We should note that here \( 3R_{mn} \hat{k}^m \hat{k}^n \hat{k}_i - \frac{R}{2} |\hat{k}|^2 \sim -3|\hat{k}|^2 \) does not offer a good sign. Viewing the formulae (2.11) and (4.26) for \( R, R_{mn} \), we also have the expansions,
\[ R_{ij} \sim 1 \pm \hat{k}^2 + E \pm |\mathcal{D}\phi|^2, \quad R \sim 1 \pm |\hat{k}|^2 + E \pm |\mathcal{D}\phi|^2. \] (7.8)

Using the Cauchy–Schwarz inequality, there is
\[
\int_{\Sigma_t} |\nabla \hat{k}|^2 \lesssim \int_{\Sigma_t} \left( 1 + \|\mathcal{D}\phi\|^2_{L^\infty} + \|\hat{k}\|_{L^\infty} + \|E\|_{L^\infty} \right) |\hat{k}|^2
\]
\[+ \int_{\Sigma_t} |H|^2 + \|\bar{T}\phi\|^2_{L^\infty} |\nabla \phi|^2. \] (7.9)

That is,
\[
\int_{\Sigma_t} |\nabla \hat{k}|^2 \lesssim \left( 1 + \mathcal{E}_2(\phi, t) + \mathcal{E}_2^2(g, t) + \mathcal{E}_2^2(g, t) \right) E_0(\hat{k}, t)
\]
\[+ E_0(W, t) + \mathcal{E}_2(\phi, t)\mathcal{E}_0(\phi, t). \]

By the bootstrap assumption (4.3) and the improved \( W, \phi \) (see section 7.1.1), we infer (7.6) with \( i = 0 \), and
\[
\int_{\Sigma_t} \tau^2 |\nabla \hat{k}|^2 \lesssim \tau^2 E_0(\hat{k}, t) + (\varepsilon^2 \mathcal{E}_0^2(W) + \varepsilon^2 \mathcal{E}_0^2(\phi) + \varepsilon^3 M^3) \tau^2 u^2. \] (7.10)

Proceeding to the higher order cases, we wish to prove that for \( 0 \leq i \leq 3 \),
\[
\int_{\Sigma_t} \tau^2 |\nabla \Delta \hat{k}|^2 \lesssim \tau^2 E_0(\hat{k}, t) + (\varepsilon^2 \mathcal{E}_0^2(W) + \varepsilon^2 \mathcal{E}_0^2(\phi) + \varepsilon^3 M^3) \tau^2 u^2. \] (7.11)

Now (7.10) suggests that (7.11) holds for \( i = 0 \). Suppose (7.11) holds for \( i \leq l(l \leq 2) \), we have to show that (7.11) also holds for \( i = l + 1(l \leq 2) \).

Using the commuting identity between \( \nabla_k \) and \( \Delta (3.20) \) and (3.21), after an integration by parts, we calculate, for \( 0 \leq l \),
\[
\int_{\Sigma_t} |\nabla_{l+1} \hat{k}|^2 = \int_{\Sigma_t} |\nabla_k \Delta \hat{k}|^2 + |\nabla_l \Delta \hat{k}|^2 \pm |\mathcal{R}_l(\hat{k}_i)|^2 \pm |\mathcal{R}_{l+1}(\hat{k}_i)|^2.
\]

Apply the Cauchy–Schwarz inequality, then
\[
\int_{\Sigma_t} |\nabla_{l+1} \hat{k}|^2 \lesssim \int_{\Sigma_t} |\nabla_k \Delta \hat{k}|^2 + |\nabla_{l+1} \hat{k}|^2 + |\mathcal{R}_l(\hat{k}_i)|^2 + |\mathcal{R}_{l+1}(\hat{k}_i)|^2.
\]

In the mean time, there are
\[
\nabla_l \Delta \hat{k}_i = \nabla_k \left( \nabla^m \mathcal{R}_{mi}^r - \nabla_j \mathcal{R}_{pj} + R_{qj}^p \hat{k}_p - R_{jq}^p \hat{k}_p \right)
\]
and \( \nabla^m \mathcal{R}_{mi}^r = \epsilon_{mli} \nabla^m H_{pj} - \frac{1}{2} \nabla (R_{pj}^i) + \frac{1}{2} \nabla (R_{pj}^i) \). As a result,
\[
\int_{\Sigma_t} |\nabla_{l+1} \hat{k}|^2 \lesssim \int_{\Sigma_t} |\nabla_k \Delta \hat{k}|^2 + |\nabla_{l+1} H|^2 + |\mathcal{R}_l(\bar{R})|^2
\]
\[+ |\mathcal{R}_l(\hat{k})|^2 + |\mathcal{R}_{l+1}(\hat{k})|^2. \]
Note that, \( R_{ij} = \nabla_i / \phi T + \frac{m_i(t)}{\phi^2} \tilde{g}_{ij}, \nabla_k (\tilde{T}^\mu_{\nu \mu}) = 0 \), then \( \nabla_k R_{ij} = \nabla_k (\nabla_j / \phi T) \). And by virtue of the expansion for \( R_{\text{min}} \),

\[
\int \nabla_{b+l+1} \hat{k}^2 \lesssim \int \nabla_{b+l} \hat{k}^2 + |\nabla_{b+l} H|^2 + \sum_{a+b=l+1} |\nabla_k \nabla / \phi \nabla_k T|^2
\]

\[+ \quad R K_1 + R K_2 + R K_3 + R K_4, \]

where \( R K_1, \ldots, R K_4 \) are defined as below,

\[
R K_1 = \left(1 + |\hat{k}|^2 + |\hat{k}|^4 + |E|^2 + |D \phi|^4\right) |\nabla_k \hat{k}|^2,
\]

\[
R K_2 = \sum_{a+b=l} |\nabla_{l+1} \left(\hat{k} + |\hat{k}|^2 + E + |D \phi|^2\right)|^2 |\nabla_k \hat{k}|^2,
\]

\[
R K_3 = \left(1 + |\hat{k}|^2 + |E|^2 + |D \phi|^2\right) |\nabla_{l+1} \hat{k}|^2,
\]

\[
R K_4 = \sum_{a+b=l+1} |\nabla_{l+1} \left(\hat{k} + |\hat{k}|^2 + E + |D \phi|^2\right)|^2 |\nabla_k \hat{k}|^2.
\]

We should always remind ourselves that \( l \leq 2 \) here. First of all,

\[
\int \nabla_{b+l} \hat{k}^2 + |\nabla_{b+l} H|^2 \lesssim E_{l+1}(\hat{k}, t) + E_{l+1}(W, t), \quad l \leq 2.
\] (7.12)

And viewing \( [\frac{|\hat{k}|^2}{2} + \frac{|E|^2}{2}] \leq 3(l \leq 2) \), we have the bound for the source term,

\[
\int \nabla_k \nabla \phi \nabla_k \hat{T} \lesssim |\nabla_k \hat{k}|^2 \quad l \leq 2.
\] (7.13)

For \( R K_1 \) and \( R K_3 \), it is apparent that

\[
\int \nabla_{l+1} \hat{k}^2 + |\nabla_{l+1} H|^2 \lesssim E_{l+1}(\hat{k}, t) \quad l \leq 2.
\] (7.14)

As for \( R K_2 \), since \( 1 \leq a + 1 \leq l + 2, b \leq l + 1 \leq 1 \), we can always apply \( L^2 \) to each of the four factors (such as \( |\nabla_{l+1} E|^2 |\nabla_k \hat{k}|^2 \)) or \( L^6 \) to each of the six factors (such as \( |\nabla_{l+1} |D \phi|^2|^2 |\nabla_k \hat{k}|^2 \)). Then

\[
\int \nabla_{l+1}^2 \lesssim (E_{l+1}(g, t) + E_{l+1}^2(\phi, t)) E_{l+1}(\hat{k}, t), \quad l \leq 2.
\] (7.15)

In the case of \( R K_4 \), there are two cases:

- If \( 1 \leq a + 1 \leq l \leq 2, \) then \( 1 \leq b \leq l \leq 2 \). The above argument for \( R K_2 \) (7.15) can be adapted here to derive the bound

\[
(E_{l+1}(g, t) + E_{l+1}^2(\phi, t)) E_{l+1}(\hat{k}, t), \quad l \leq 2.
\]

- If \( a + 1 = l + 1 \leq 3, \) then \( b = 0 \), we get

\[
|\nabla_{l+1} \left(\hat{k} + |\hat{k}|^2 + E + |D \phi|^2\right)|^2 |\hat{k}|^2.
\]


For four factors (such as $\|\nabla_{\vec{h}_{\vec{1}}} E|^2 |\vec{k}|^2$), we apply $L^2, L^2, L^\infty, L^\infty$; for six factors (such as $\|\nabla_{\vec{h}_{\vec{1}}} |\vec{k}|^2^2 |k|^2$), we apply $L^\infty, L^\infty$ to $|\vec{k}|^2$, and meanwhile, to estimate the rest, like $\|\nabla_{\vec{h}_{\vec{1}}} |\vec{k}|^2^2 |\nabla_{\vec{h}_{\vec{1}}} (\vec{k})^2|^2|_{E_{I1}}$), we can perform $L^2, L^\infty$ or $L^\infty, L^2$ on the two factors in $\nabla_{\vec{h}_{\vec{1}}} |\vec{k}|^2$, since $[\hat{\Sigma}]^{2} + 2 \leq 5 (l \leq 2)$. Finally, we obtain the following bound in this case,

\[
\left( \mathcal{E}_{l+1}(W,t) + \mathcal{E}_{l+1}(\hat{k},t) \right) \mathcal{E}_2(\hat{k},t)
+ \left( \mathcal{E}_{l+1}(\hat{k},t) \mathcal{E}_3(\hat{k},t) + \mathcal{E}_{l+1}(\phi,t) \mathcal{E}_3(\phi,t) \right) \mathcal{E}_2(\hat{k},t).
\]

As a summary, we have by the Cauchy–Schwarz inequality, for $l \leq 2$,

\[
\int_{\Sigma} RK_4 \lesssim (\mathcal{E}_{l+1}(g,t) + \mathcal{E}_3(\phi,t) \mathcal{E}_{l+1}(\phi,t)) \mathcal{E}_2(\hat{k},t).
\] (7.16)

Through (7.12)–(7.16), we draw the conclusion that for $l \leq 2$,

\[
\int_{\Sigma} |\nabla_{\hat{h}_{\vec{1}}} (\hat{k})^2| \lesssim \mathcal{E}_{l+1}(\hat{k},t) + \mathcal{E}_{l+1}(W,t) + \mathcal{E}_3(\phi,t) \mathcal{E}_{l+1}(\phi,t).
\] (7.17)

Combining the estimates for $W, \phi$ (see section 7.1.1) with the result of lemma 7.1, we prove (7.11) by induction and hence (7.6) follows.

We choose $M$ additionally satisfying $C \left( I_0(\hat{k}) + I_3(W) + I_4(\phi) \right) \leq M/8$, $C \hat{I}_0(\hat{k}) \leq M/8$ and $\varepsilon$ sufficiently small such that $C \varepsilon^2 M^2 \leq 1/4$, then (7.2) and (7.6) give

\[
\hat{k}^i E_i(\hat{k},t) \lesssim \frac{1}{2} \varepsilon^2 M^2 \delta, \quad 0 \leq i \leq 4,
\] (7.18)

which improves the bootstrap assumption for $\hat{k}$ (4.5).

7.13. For the lapse. Due to the Sobolev inequalities, the improved energy for $\hat{N}$ (5.24) (corollary 5.4) implies that, for some universal constant $C$,

\[
||\hat{N}||_{L^\infty} \leq C \varepsilon I_4(\phi) \mathcal{T}^{-1}.
\] (7.19)

We have chosen $C \hat{J}_2(\phi) \leq M^2/2$, then

\[
||\hat{N}||_{L^\infty} \leq \frac{1}{2} \varepsilon^2 M^2 \mathcal{T}^{-1+4\delta}, \quad 0 < \delta < \frac{1}{6},
\] (7.20)

which improves the bootstrap assumption for $||\hat{N}||_{L^\infty}$ (4.5).

We are now in a position to state the global existence theorem.

**Theorem 7.3.** Let $(\Sigma, g_0, k_0, \phi_0, \phi_1)$ be the rescaled CMC data for the EKG equations (1.1a), (1.1b) and (2.1) (see also the $l + 3$ form: (3.64a), (3.64b) and (3.24)), where $\phi_0 = \phi_1 |_{\hat{g}_0 = 0}, \phi_1 = \hat{T} \hat{g}_0 |_{\hat{g}_0 = 0}$. Assume that $\Sigma$ is a compact hyperbolic 3-manifold without boundary and $\text{tr}_{g_0 \hat{k}_0} = -3$, with the initial time $t_0 > \max \{3 \mathcal{M}^{-1}, 1 \}$. Then there is a constant $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and

\[
t_0 \mathcal{E}_4(\phi, t_0) + \hat{I}_0^2 \mathcal{E}_3(W, t_0) + \hat{I}_0^2 \mathcal{E}_2(\hat{k}, t_0) \leq \varepsilon^2,
\] (7.21)

a uniquely global solution $(M, \hat{g}, \phi)$ exists for $t \geq t_0$ with the property that for some universal constant $C$ and some fixed number $0 < \delta < \frac{1}{6}$.
\[ t\mathcal{E}_4(\phi, t) + \hat{r}^2 \mathcal{E}_3(W, t) \lesssim \varepsilon^2 \hat{r}^2, \quad \hat{r}^2 \mathcal{E}_4(\hat{k}, t) \lesssim \varepsilon^2 \hat{r}^6. \]  

(7.22)

In the end, we can follow the argument presented in [7] (theorem 6.2) to prove the geodesic completeness.

**Remark 7.4.** Let us explore the asymptotic state of the rescaled metric \( g \). As we can see from the Gauss equation (2.7), the decay estimates for \( \hat{k}, D\phi, H \) lead to

\[ \|R_{imjn} + \frac{1}{2} (g \odot g)_{imjn}\|_{L^\infty} \lesssim \varepsilon r^{-1+\delta/2}, \quad 0 < \delta < \frac{1}{6}. \]

It shows that the sectional curvature \( K \) of \( g \) tends to \(-1\).

**Remark 7.5.** Restrict the Gauss–Codazzi equations (2.9) and (2.10a) to the initial slice, the following estimates for data

\[ t_0^2 \|R_{ij} + 2g_{ij}\|_{H^1(\Sigma, g_0)} + t_0^2 \|k_0 + g_{ij}\|_{H^1(\Sigma, g_0)} \lesssim \varepsilon^2, \]

\[ t_0 \|\phi_0\|_{H^1(\Sigma, g_0)} + t_0 \|\phi_1\|_{H^1(\Sigma, g_0)} \lesssim \varepsilon^2, \]  

(7.23)

will enforce (7.21), up to some universal constants.

**Remark 7.6 (Existence of CMC data).** We also refer to [12] for results regarding the existence of CMC data in the current setting. We assume \( \|\phi_0\|_{L^\infty} \) to be small enough, so that \( 6 - 2\phi_0^2 > 0 \) (that is, \( B_{\text{min}} > 0 \) by the conventions in [12]). In our case, \( \Sigma \) has negative Yamabe invariant and \( B_{\text{min}} > 0 \), then the existence of CMC solutions to the constraint equations for the EKG system on these compact manifolds is confirmed in [12].

### 7.2. Generalize to non-CMC data

We can further remove the CMC restriction on initial data. Considering generalized data, which are not necessarily CMC, but close to the initially CMC data induced by the background solution, the existence of a maximal globally hyperbolic development of this initial data is assured by Ringström [35]. Under the smallness condition, the existence of a CMC surface in such a development is shown in the vacuum case by Fajman–Kroencke [16] (theorem 2.9). Based on these results, we can now generalize our main theorem to the case of non-CMC data.

To match the condition in [16], one needs the smallness for \( \|g_0 - \gamma\|_{H^s(\Sigma, \gamma)} + \|k_0 + \gamma\|_{H^{s-1}(\Sigma, \gamma)} \), \( s > 3/2 + 1 \), where \( \gamma \) is the hyperbolic metric. Defining \( E_i(g) := \|R_i + 2g\|_{H^1(\Sigma, g)} \) \( F_i(k) := \|k_i + g\|_{H^1(\Sigma, g)} \), we will confirm that

\[ \|g_0 - \gamma\|_{H^s(\Sigma, \gamma)}^2 + \|k_0 + \gamma\|_{H^{s-1}(\Sigma, \gamma)}^2 \lesssim E_{s-2}(g_0) + F_{s-2}(k_0), \]  

(7.24)

if \( g_0 \) is sufficiently close to \( \gamma \). We take \( s = 5 \) and choose the harmonic coordinates with respect to \( g_0 \) and \( \gamma \) on the initial surface. By the analysis in section 4.2 of [8], \( g_0 - \gamma = u^\text{TT} + z \), \( k_0 + g = \hat{u}^\text{TT} + w \), where \( u^\text{TT}, \hat{u}^\text{TT} \) are TT (divergence-free and trace-free) tensors with respect to \( \gamma \), and \( z, w \) are \( L^2 \)-orthogonal to the space of TT tensors defined with respect to \( \gamma \). And an estimate of the form \( \|z\|_{H^s(\Sigma, \gamma)} + \|w\|_{H^{s-1}(\Sigma, \gamma)} \lesssim \|u^\text{TT}\|_{H^s(\Sigma, \gamma)} + \|\hat{u}^\text{TT}\|_{H^{s-1}(\Sigma, \gamma)} \) holds. Denote \( A = D(R_{ij} + 2g_{ij}) \), where \( D \) is the Frechet derivative at \( \gamma \). We recall from [7, 8] that,
2\mathcal{A}_\eta = -\Delta_\gamma \eta - 2\eta - 2\delta^* \text{div}_\gamma \eta - \nabla d(\text{tr}_\gamma \eta),
\]
where \text{div}_\gamma = \nabla^i \gamma_{ij} \delta^* \gamma_{ij} = -1/2(\nabla_i \xi_j + \nabla_j \xi_i). Hence, for a TT tensor \( h \),
\[
2 \mathcal{A}_h = 2\mathcal{A}_0 h := -\Delta_\gamma h - 2h.
\]
Let \( \lambda \) be any eigenvalue of \( \mathcal{A}_0 \). It is known that, the hyperbolic metric \( \gamma ( n \geq 3 ) \) is strictly stable, i.e. \( \lambda > 0 \) [9 § 12 H]. Then it follows that \( \| h \|_{H^2(\Sigma, \gamma)} \lesssim \| \mathcal{A}_0 h \|_{H^2(\Sigma, \gamma)} \). Meanwhile, the Hessian of \( E_\gamma(g) \) at \( \gamma \) takes the form of \( \mathbf{D}^2 E_\gamma(h) = 2\| \mathcal{A}_h \|^2_{H^2(\Sigma, \gamma)} \). As a consequence, for any TT tensor \( h \),
\[
\| h \|^2_{H^2(\Sigma, \gamma)} \lesssim \mathbf{D}^2 E_\gamma(h).
\]
Note that, \( \gamma \) is a critical point of \( E_\gamma \). It follows from the above argument and Taylor’s theorem that there is an \( \varepsilon > 0 \) such that \( \| u^{TT} \|_{H^2(\Sigma, \gamma)} \lesssim E_\gamma^2(u^{TT} + \gamma) \) \( \lesssim E_\gamma^2(g_0) + \| u^{TT} \|_{H^2(\Sigma, \gamma)} + \| \nabla^{TT} \|^2_{H^2(\Sigma, \gamma)} \) holds if \( \| g_0 - \gamma \|_{H^2(\Sigma, \gamma)} \lesssim \varepsilon \). And hence (7.24) holds as well if \( g_0 \) is sufficiently close to \( \gamma \).

Now we can prove along the lines of the corresponding argument presented in [16, 35] to show the existence of a CMC surface in the non-vacuum setting. Therefore, the data can be generalized to be the non-CMC data satisfying (7.23). We then achieve theorem 1.1.

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Appendix A. Local in time development of CMC data

In this section, we return to the original spacetime metric \( \bar{g}_{\mu\nu} \) and spatial metric \( \bar{g}_{ij} \). The \( \bar{g}_{ij} \) will be denoted by \( g_{ij} \) (dropping the tilde) here for the sake of simplicity. These conventions (dropping the tilde) apply to \( \nabla, k_{ij}, R_{ij}, \cdots \) as well.

The EKG system (1.1a) and (1.1b) has the structure equations (2.6a)–(2.10b) taking the following forms: the evolution equations
\[
\mathcal{L}_{\partial_i} \bar{g}_{ij} = -2Nk_{ij}, \quad (A.1a)
\]
\[
\mathcal{L}_{\partial_i} k_{ij} = -\nabla_i \nabla_j N + N \left( R_{ij} - 2k_{ij}k_{ij} + \text{tr}k^2ij - \bar{R}_{ij}(\phi) \right). \quad (A.1b)
\]
The KG equation
\[
\Box \phi - m\phi = 0. \quad (A.2)
\]
The equation for the lapse \( N \)
\[
\Delta N + \partial_t \text{tr}k - N \left( \bar{R}_{TT}(\phi) + |k|^2 \right) = 0. \quad (A.3)
\]
The constraint equations
\[ \text{tr}k + 3r^{-1} = 0, \quad (A.4a) \]
\[ R - |k|^2 + (\text{tr}k)^2 = 2\hat{R}_{TT}(\phi) + \hat{R}(\phi), \quad (A.4b) \]
\[ \nabla^i k_j - \nabla^j k_i = -\hat{R}_{i}^j(\phi). \quad (A.4c) \]

In (A.1a)–(A.4c), the coupling \( \hat{R}_{\alpha\beta}(\phi) \) is indeed the Ricci tensor \( \hat{R}_{\alpha\beta} \) (associated to \( \hat{g} \)), which, via the EKG equations (1.1a) and (1.1b), is related to the KG field by
\[ \hat{R}_{\alpha\beta}(\phi) = \hat{D}_\alpha \phi \hat{D}_\beta \phi + \frac{m}{2} \hat{g}_{\alpha\beta}. \quad (A.5) \]

As shown in [14] (page 307), we introduce
\[ A = \text{tr}k + 3r^{-1}, \quad (A.6a) \]
\[ B = R - |k|^2 + (\text{tr}k)^2 - 2\hat{R}_{TT}(\phi) - \hat{R}(\phi), \quad (A.6b) \]
\[ C_i = \nabla^i k_j - \nabla^j k_i + \hat{R}_{i}^j(\phi) + 1/2\nabla_i A, \quad (A.6c) \]
\[ D_{ij} = N^{-1} L_{ij} k_{ij} + N^{-1} \nabla_i \nabla_j N - R_{ij} + 2k_i k_j - \text{tr}k k_{ij} + \hat{R}_{ij}(\phi). \quad (A.6d) \]

The definitions of \( A, B, C, D \) are almost identical (except the couplings) to the ones introduced in [14] or [6].

The EKG system (A.1a)–(A.3) and the constraint equations (A.4a)–(A.4c) imply \( A = B = 0, C_i = 0, D_{ij} = 0 \) and the following second-order hyperbolic equations for \( k_{ij} \) and \( \phi \):
\[ -\left( N^{-1} L_{ij} \right) k_{ij} + \Delta k_{ij} = F_{ij}, \]
\[ \square_{\hat{g}} \phi - m \phi = 0, \quad (A.7) \]

and
\[ -N^{-1} L_{ij} \hat{g}_{ij} = 2k_{ij}, \quad (A.8) \]

and \( N' = \partial_t N \),
\[ \Delta N' - 3r^{-2} N' = -6r^{-3} \dot{N} + N' \left( \hat{R}_{TT}(\phi) + |\dot{\phi}|^2 \right) + N \partial_i \left( \hat{R}_{TT}(\phi) + |\dot{\phi}|^2 \right) \]
\[ -t^{-1} |\nabla N|^2 - 2\dot{\phi} \left( N \nabla_i \nabla_j N + \nabla_i N \nabla_j N \right) + 2t^{-1} N \Delta N + 2N \nabla^i \nabla_i \hat{R}_{TT}(\phi). \quad (A.9) \]

The inhomogeneous term \( F_{ij} \) is given by
\[ F_{ij} = N_{ij} - \nabla_i \hat{R}_{ij}(\phi) - \nabla_j \hat{R}_{ii}(\phi) + N^{-1} L_{ij} \left( -\text{tr}k k_{ij} + \hat{R}_{ij}(\phi) \right), \quad (A.10) \]

where \( N_{ij} \) and \( H_{ij} \) are the same as that in [14] (page 308):
\[ N_{ij} = L_{ij} - H_{ij}, \]
\[ N^2 L_{ij} = \nabla_i \nabla_j N' - N^{-1} N' \nabla_i \nabla_j N - L_{ij} \Gamma_{ipq}^j \nabla_p N \]
\[ + 2N (k_i^p L_{ij} k_{jp} + k_j^p L_{ij} k_{ip}) + 4N^2 k_p k_{ip} k_{jp}, \quad (A.11) \]
with

\[
NH_{ij} = NI_{ij} + \nabla^p N \left( 2\nabla_j k_{pj} - \nabla_i k_{pj} - \nabla_j k_{pi} \right)
- \nabla_i N \nabla_j k_{pj} - \nabla_j N \nabla_p k_{ji} + \nabla_i N \nabla_j \text{tr} k
- \frac{k_i}{\nabla_j} \nabla_i N - \frac{k_j}{\nabla_i} \nabla_j N + \text{tr} k \nabla_i \nabla_j N + k_{ij} \Delta N,
\]

\[
I_{ij} = - 3 \left( k^p R_{jp} + k^p R_{ip} \right) + 2 g_{ij} k^p R_{pq} + \text{tr} k \text{R}_{ij} + \left( k_{ij} - g_{ij} \text{tr} k \right) \text{R}.
\]

By a computation analogous to [14], using the 1 + n-decomposition of the divergence of the energy-momentum tensor, \( \tilde{D}^\mu T_{\mu\nu} = 0 \), as presented in [33] (page 17), one derives the following system of evolution equations for \( A, B, C_i, D_{ij} \) which holds for any given solution of the system (A.7) and (A.8):

\[
N^{-1} \partial_t A = F := \text{tr} D + B,
\]

\[
N^{-1} \partial_t F = \Delta A + 2 N^{-1} \nabla^j N \nabla_j A - 4 N^{-1} \nabla^j N C_i,
\]

\[
N^{-1} \partial_t C_i = \left( \nabla^j D_{ji} - \frac{1}{2} \nabla^j \text{tr} D \right) + N^{-1} \nabla^j N \text{tr} D_{ij}
+ \text{tr} C_i - \frac{1}{2} \text{tr} k \nabla_i A - \frac{1}{2} N^{-1} \nabla_j N F,
\]

\[
N^{-1} \partial_t D_{ij} = \nabla_i C_j + \nabla_j C_i.
\]

Generically, (A.13a)–(A.13d) always holds true and the proof is irrelevant to any matter field.

Therefore, as [14], one has

**Lemma A.1.** Any solution of the coupled elliptic-hyperbolic system (A.7) and (A.8) whose initial data \( g_{ij}, k_{ij}, \mathcal{L}_C k_{ij}, \phi, T \phi \) to verify the original constraint equations \( A = 0, B = 0, C_i = 0 \) as well as \( D_{ij} = 0 \) is a solution to the original system (A.1a)–(A.3).

An existence result for the reduced system can be done roughly in the same way as that for the reduced wavelike system [11] (see [14]). The only differences lie in that \( g_{ij} \) is tied to \( k_{ij} \) and \( N \) through the integral equation in (A.8). One can proceed precisely as [14], estimating \( k_{ij}, \phi \) by energy estimates, \( g_{ij} \) by the integral equation in (A.8) and \( N, N' \) by elliptic estimates. We also remark that in the elliptic equations for \( N - 1 \) and \( N' \), both of the \( \partial_t \text{tr} k \) and \( -3t^{-2} N' \) admit good signs. They will contribute to the estimates for \( \| t^{-2} (N - 1) \|_{L^2(\Sigma_t)} \) and \( \| t^{-2} N' \|_{L^2(\Sigma_t)} \). This can be seen in section 4.3. Besides, we are working in the setting of compact manifold without boundary, hence we do not have to care about the asymptotic decay at spatially infinity as [14], and the regularity we are considering is one order higher than that in [14]. In general, the local existence theorem in our setting is easier than that in [14]. At the end, we make the rescaling and theorem 2.1 follows.

**Appendix B. ODE estimates**

The following Grönnwall inequality is used throughout this paper.

**Lemma B.1 (Differential form).** Let \( a(s), b(s), \psi(s) \) be non-negative functions, and

\[
\psi'(s) \leq b(s) + a(s) \psi(s),
\]

where
then
\[ |\psi(s)| \leq \exp\left(\int_{s_0}^{s} a(\tau) d\tau \right) \left( \psi(s_0) + \int_{s_0}^{s} b(\tau) d\tau \right). \] (B.1)

The technical ODE estimate (refer to lemma 3.5 in [27]) specializes to the situation considered as follows.

Lemma B.2. Let \( c^2 > 0 \) and
\[ \frac{\partial^2}{\partial s^2} \psi + c^2 \psi = F(s), \]
with the initial data
\[ \psi|_{s=s_0} = \psi_0, \quad \frac{\partial}{\partial s} \psi|_{s=s_0} = \psi_1. \]

There is the inequality
\[ |\psi(s)| + |\partial_s \psi(s)| \lesssim |\psi_0| + |\psi_1| + \int_{s_0}^{s} |F(\tau)| d\tau. \] (B.2)

Appendix C. Basic geometric conventions and identities

Curvature. We clarify the conventions for curvature. The Riemann tensor of \((\bar{M}, \bar{g})\) is defined by
\[ \bar{R}(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \]
for any vector fields \( X, Y, Z \). We define in a coordinate system
\[ \bar{R}_{\alpha\beta\mu\nu} = \bar{g} \left( \partial_{\mu} R(\partial_{\alpha}, \partial_{\beta}) \partial_{\nu} \right). \]
The Riemannian tensor satisfies the Bianchi identities
\[ D_{[\sigma} \bar{R}_{\alpha\beta]\mu\nu} = \frac{1}{3} \left( D_{\sigma} \bar{R}_{\alpha\beta\mu\nu} + D_{\alpha} \bar{R}_{\beta\sigma\mu\nu} + D_{\beta} \bar{R}_{\sigma\alpha\mu\nu} \right) = 0. \]

Volume element. Let \( \epsilon_{\mu\alpha\beta\gamma} \) be the coefficient of the volume element of an \( 1+3 \) dimensional Lorentz manifold \((\bar{M}, \bar{g})\). Then \( \epsilon_{\mu\alpha\beta\gamma} \) is a totally anti-symmetric tensor. In an O.N. frame adapted to a spacelike hypersurface \( \Sigma \) in \((\bar{M}, \bar{g})\), the coefficient of the induced volume element is (page 144 in [14])
\[ \epsilon_{ijk} = \epsilon_{\bar{T}ijk}. \]

Identities. Let \((\Sigma, g_{ij})\) be the spatially 3-dimensional Riemann manifold, and \( \nabla \) is the covariant derivative with respect to \( g_{ij} \). Let \( \Gamma^i_{ij} \) be the connection coefficient associated to \( \nabla \), the time derivative is a tensorfield:
\[ \mathcal{L}_\partial \Gamma^a_{ij} = \frac{1}{2} g^{ab} \left( \nabla_i \mathcal{L}_\partial g_{bj} + \nabla_j \mathcal{L}_\partial g_{bi} - \nabla_b \mathcal{L}_\partial g_{ij} \right). \]  
(C.1)

Then for any \((0,2)\)-tensor \(A_{ij}\) on \(\Sigma\),
\[ \mathcal{L}_\partial \nabla_i A_{ab} = \nabla_i \mathcal{L}_\partial A_{ab} - \mathcal{L}_\partial \Gamma^p_{ia} A_{pb} - \mathcal{L}_\partial \Gamma^p_{ib} A_{ap}. \]
(C.2)

Define the following operations on symmetric 2-tensors on \(\Sigma\),
\[ A \cdot B = A_{ij}B_{ij}, \]
(C.3)
\[ (\text{div} A)_i = \nabla^j A_{ij}, \]
(C.4)
\[ \text{curl} A_{ij} = \frac{1}{2} \left( \epsilon_{ijpq} \nabla^q A_{pj} + \epsilon_{jpq} \nabla^q A_{ij} \right), \]
(C.5)
\[ (A \land B)_{ij} = \epsilon_{ijab} \epsilon_{pq} A_{ap} B_{bq} + \frac{1}{3} A \cdot B g_{ij} - \frac{1}{3} \text{tr} A \text{tr} B g_{ij}, \]
(C.6)
\[ (A \times B)_{ij} = \epsilon_{ijab} \epsilon_{pq} A_{ap} B_{bq} + \frac{1}{3} A \cdot B g_{ij} - \frac{1}{3} \text{tr} A \text{tr} B g_{ij}. \]
(C.7)

The operation \(\land\) is skew symmetric, while \(\times\) is symmetric.

We derive some identities which relates \(\text{div}\) and \(\text{curl}\). A computation shows that
\[ \text{div}(A \land B) = -\text{curl} A \cdot B + A \cdot \text{curl} B. \]  
(C.9)

The \(l\)th \((l \geq 1)\) order analogous of (C.9) is given below (or lemma 3.14).
\[ \nabla^I_l \left( \text{curl} A_{ij} \right) = \nabla^I_l \left( \epsilon_{ijpq} \nabla^q A_{pj} + \epsilon_{jpq} \nabla^q A_{ij} \right) \]
\[ = \sum_{a \leq l} \nabla^I_a \left( \nabla^I_l A_{ap} \land \nabla^I_l B_{bq} \right) + \hat{R}_l A_{ij} + \hat{R}_l B_{ij} + \hat{R}_l A_{ij} + \hat{R}_l B_{ij}. \]
(C.10)

where \(\hat{R}_l A \) is defined in (3.73).

**Proof of lemma 3.14.** As far as this paper is concerned, we can restrict our attention to the \(l \leq 3\) case for simplicity. Denote
\[ C_l = \nabla^I_l \left( \text{curl} H_{ij} \right) - \nabla^I_l \left( \text{curl} E_{ij} \right) \]
\[ = \epsilon_{ijpq} \nabla^q H_{pj} - \epsilon_{ijpq} \nabla^q E_{pj} + \epsilon_{pq} \nabla^q H_{ij} - \epsilon_{pq} \nabla^q E_{ij}. \]
\[ C_l = \epsilon_{ijpq} \nabla^q H_{pj} - \epsilon_{ijpq} \nabla^q E_{pj} + \frac{1}{3} \text{tr} A \text{tr} B g_{ij} - \frac{1}{3} \text{tr} A \text{tr} B g_{ij}. \]
\[ \text{div}(A \land B) = -\text{curl} A \cdot B + A \cdot \text{curl} B. \]  
(C.9)

We first note that, for any symmetric tensors \(H_{ij}, E_{ij}\), their derivatives \(\nabla^I_l H_{ij}\) and \(\nabla^I_l E_{ij}\) are still symmetric in \(i,j\). Therefore,
\[ C_l = \epsilon_{ijpq} \nabla^q H_{pj} - \epsilon_{ijpq} \nabla^q E_{pj} + \frac{1}{3} \text{tr} A \text{tr} B g_{ij} - \frac{1}{3} \text{tr} A \text{tr} B g_{ij}. \]

To prove this lemma, it suffices to keep track of the principle part of \(R_{imjn}\) (without derivatives). By the Gauss equation (2.7), we know that the principle part of \(R_{imjn}\) is \(- (g_{ij} g_{mn} - g_{in} g_{jm})\). In what follows, we shall introduce the notation \(\simeq\), which means equalling in the principle part. For example, \(R_{imjn} \simeq - (g_{ij} g_{mn} - g_{in} g_{jm})\).
Case I: $\epsilon^{pq} \nabla_q \nabla_i H_{pj} \cdot \nabla^i E^q - \epsilon^{pq} \nabla_q \nabla_i E_{pj} \cdot \nabla^i H^q$. This will cover the $l = 0$ case, i.e. (C.9).

$$
\epsilon^{pq} \left( \nabla_q \nabla_i \cdots \nabla_i H_{pj} \cdot \nabla^i E^q \right) = \nabla_q \left( \epsilon^{pq} \nabla_q \nabla_i \cdots \nabla_i E_{pj} \cdot \nabla^i H^q \right) = \nabla_q \left( \epsilon^{pq} \nabla_q \nabla_i \cdots \nabla_i H_{pj} \cdot \nabla^i E^q \right) + \epsilon^{pq} \nabla_q \nabla_i \cdots \nabla_i E_{pj} \cdot \nabla^i H^q = \nabla_q \left( \epsilon^{pq} \nabla_i \cdots \nabla_i H_{pj} \cdot \nabla^i E^q \right),
$$

where we have exchanged the indices $p$ and $i$ in the fourth line. Note that, the sum of the fourth and the fifth lines above which equals to the product $\epsilon^{pq} \nabla_i H_{pj} \cdot \nabla^i E^q$, for $\epsilon^{pq}$ is antisymmetric in $i$ and $p$, while the other one is symmetric in $i$ and $p$.

Case II: $\epsilon^{pq} \nabla_q \nabla_i H_{pj} \cdot \nabla^i E^q - \epsilon^{pq} \nabla_q \nabla_i E_{pj} \cdot \nabla^i H^q$. In particular, this will cover $C_{1i}, l = 1$, i.e. (C.10) with $l = 1$. Combined with Case I, Case II reduces to $\epsilon^{pq} \nabla_i \nabla_q [\nabla_i H_{pj} \cdot \nabla^i E^q] - \epsilon^{pq} \nabla_i \nabla_q [\nabla_i E_{pj} \cdot \nabla^i H^q]$, as shown in the following calculations:

$$
\epsilon^{pq} \nabla_q \nabla_i \nabla_j \cdots \nabla_i H_{pj} \cdot \nabla^i E^q = \epsilon^{pq} \nabla_q \nabla_i \nabla_j \cdots \nabla_j E_{pj} \cdot \nabla^i H^q = \epsilon^{pq} \nabla_q \nabla_i \nabla_j \cdots \nabla_i E_{pj} \cdot \nabla^i H^q - \epsilon^{pq} \nabla_i \nabla_j \cdots \nabla_i E_{pj} \cdot \nabla^i H^q,
$$

where the third line belongs exactly to Case I, and the last line equals to

$$
\epsilon^{pq} \nabla_i \nabla_q [\nabla_i H_{pj} \cdot \nabla^i E^q] - \epsilon^{pq} \nabla_i \nabla_q [\nabla_i E_{pj} \cdot \nabla^i H^q] = \epsilon^{pq} \left( R_{ij} \lambda^\alpha \nabla_i \lambda^\beta \cdots \nabla_i H_{pj} + R_{ij} \lambda^\alpha \nabla_j \lambda^\beta \cdots \nabla_i H_{pj} \lambda^\gamma \right) \nabla^i E^q - \epsilon^{pq} \left( R_{ij} \lambda^\alpha \nabla_i \lambda^\beta \cdots \nabla_j E_{pj} + R_{ij} \lambda^\alpha \nabla_j \lambda^\beta \cdots \nabla_i E_{pj} \lambda^\gamma \right) \nabla^i H^q.
$$

Substituted the principle part of $R_{ij\lambda\alpha}$, the above formula is identical to

$$
- \epsilon^{pq} \left( g_{i\lambda} \lambda^\beta \cdots \nabla_i H_{pj} \cdot \nabla^i E^q \right) - \epsilon^{pq} \left( g_{i\lambda} \lambda^\beta \cdots \nabla_j H_{pj} \cdot \nabla^i E^q \right) - \epsilon^{pq} \left( g_{i\lambda} \lambda^\beta \cdots \nabla_i E_{pj} \cdot \nabla^i H^q \right) - \epsilon^{pq} \left( g_{i\lambda} \lambda^\beta \cdots \nabla_j E_{pj} \cdot \nabla^i H^q \right).
$$

Because of the symmetries and cancellations, it further turns into

$$
\sum_{i_1 = i_2}^{n} \left( - \epsilon^{pq} g_{i_1 i_2} \nabla_q \cdots \nabla_i \lambda \cdots \nabla_i H_{pj} \cdot \nabla^i E^q + \epsilon^{pq} g_{i_1 i_2} \nabla_q \cdots \nabla_i \lambda \cdots \nabla_i E_{pj} \cdot \nabla^i H^q - \epsilon^{pq} g_{i_1 i_2} \nabla_q \cdots \nabla_i \lambda \cdots \nabla_j H_{pj} \cdot \nabla^i E^q + \epsilon^{pq} g_{i_1 i_2} \nabla_q \cdots \nabla_i \lambda \cdots \nabla_j E_{pj} \cdot \nabla^i H^q - \epsilon^{pq} g_{i_1 i_2} \nabla_q \cdots \nabla_i \lambda \cdots \nabla_i E_{pj} \cdot \nabla^i H^q + \epsilon^{pq} g_{i_1 i_2} \nabla_q \cdots \nabla_i \lambda \cdots \nabla_j H_{pj} \cdot \nabla^i E^q \right),
$$

(C.12)
which are denoted by $S_i$, $i = 1, \ldots, 6$. First of all,
\[
S_1 + S_4 = -\nabla b \left( \epsilon^{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q \right) 
+ \epsilon_{qb} \nabla_{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q 
\allowdisplaybreaks[4] 
- \epsilon_{qb} \nabla_{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} E^q \nabla_{q} \cdots \nabla_{b\lambda} H^q 
= \text{div} + \epsilon_{qb} \nabla_{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q 
- \epsilon_{qb} \nabla_{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} E^q \nabla_{q} \cdots \nabla_{b\lambda} H^q ,
\]
where the last two terms cancel. That is,
\[
S_1 + S_4 = -\nabla b \left( \epsilon^{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q \right) .
\]

Next, looking at $S_2 + S_3$, we change the index $i$ to $\lambda$, then
\[
S_2 + S_3 = \epsilon^{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q 
+ \epsilon^{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} E^q \nabla_{q} \cdots \nabla_{b\lambda} H^q 
\allowdisplaybreaks[4] 
= \nabla q \left( \epsilon^{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q \right) 
- \epsilon^{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q 
+ \epsilon^{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} E^q \nabla_{q} \cdots \nabla_{b\lambda} H^q ,
\]
where the last two terms cancel as well. As a result,
\[
S_2 + S_3 = \nabla q \left( \epsilon^{b\lambda} \nabla_{b\lambda} \cdots \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q \right) .
\]

Finally,
\[
S_5 + S_6 = -\epsilon_{bq} \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q \nabla_{b\lambda} H_{bq} \nabla_{bq} \cdots \nabla_{b\lambda} E^q ,
\]
belongs exactly to Case I. As a summary, we have proved in Case II:
\[
\epsilon^{pq} \nabla_{p\lambda} \nabla_{q} \cdots \nabla_{b\lambda} E^q \cdots \nabla_{p\lambda} H^q = \nabla (\nabla_{b\lambda} H_{bq} \cdot \nabla_{q} \cdots \nabla_{b\lambda} E^q \cdots \nabla_{p\lambda} H^q ) .
\]

Case III: $\epsilon^{pq} \nabla_{p\lambda} \nabla_{q} \cdots \nabla_{b\lambda} H_{bq} \cdot \nabla_{q} \cdots \nabla_{b\lambda} E^q \cdots \nabla_{p\lambda} H^q$.
We begin with the subcase: $l = 2$, i.e. $\nabla_{b\lambda} (\text{curl} H)_{b\lambda} \cdot \nabla_{b\lambda} E^q = \nabla_{b\lambda} (\text{curl} E)_{b\lambda} \cdot \nabla_{b\lambda} H^q$. It can be further calculated as follows,
\[= \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} \nabla^q H^{ij} - \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla^q E^{ij} \]
\[= \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} \nabla^q H^{ij} - \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla^q E^{ij} \]
\[+ \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} \nabla^q E^{ij} - \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla^q H^{ij}, \]
where the second line reduces to Case II, and for the third line, as computed in Case II (C.12), the principle part is
\[\simeq \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} H^{ij} - \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} E^{ij}, \]
which in fact reduces to Case II with \(l = 1\). This completes the proof for Case III with \(l = 2\), and it covers \(C_1, l = 2\), i.e. (C.10) with \(l = 2\).

In general, combined with Case II, Case III reduces to \(\epsilon_{ij} \nabla_i [\nabla_{ij}, \nabla_q] \nabla_{ij} H_{pq} \cdot \nabla^q E^{ij} - \epsilon_{ij} \nabla_i [\nabla_{ij}, \nabla_q] \nabla_{ij} H_{pq} \cdot \nabla^q H^{ij}, \) which we shall call the reduced Case III.

As we only need \(l \leq 3\) in this paper, we will focus on the reduced Case III with \(l = 3\) (the desired conclusion for the reduced Case III with \(l = 2\) also holds true, as we can see from the above argument for the subcase: \(l = 2\)): \(\epsilon_{ij} \nabla_i [\nabla_{ij}, \nabla_q] \nabla_{ij} H_{pq} \cdot \nabla^q E^{ij} - \epsilon_{ij} \nabla_i [\nabla_{ij}, \nabla_q] \nabla_{ij} E_{pq} \cdot \nabla^q H^{ij}, \) whose principle part is
\[\simeq - \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} E^{ij} + \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} H^{ij} \]
\[+ \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} H^{ij} - \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} H^{ij} \]
\[- \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} E^{ij} + \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} H^{ij}, \]
and it can be rewritten as
\[\simeq - \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla_q E^{ij} + \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla_q H^{ij} \]
\[+ \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} \nabla_q H^{ij} - \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} \nabla_q H^{ij} \]
\[- \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla_q E^{ij} + \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla_q H^{ij} + \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla_q H^{ij} \]
We index the six terms above by \(A_1, \ldots, A_6\). Note that, \(A_5 + A_6\) belongs precisely to Case II, while \(A_1, A_3\) can be further computed as
\[A_1 = - \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla_q E^{ij} - \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla_q E^{ij} \]
\[:= A_{11} + A_{12}; \]
\[A_3 = \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} \nabla_q H^{ij} + \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} \nabla_q H^{ij} \]
\[:= A_{31} + A_{32}; \]
\(A_{11}\) and \(A_{31}\) can be further calculated as
\[A_{11} = - \nabla_q \left( \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla_q E^{ij} \right) + \epsilon_{ij} \nabla_i \nabla_j H_{pq} \nabla^{ij} \nabla_q E^{ij} \]
\[= \text{div} + \epsilon_{ij} \nabla_i \nabla_j \nabla_q \nabla_p H_{pq} \nabla^{ij} \nabla_q E^{ij} + \epsilon_{ij} \nabla_i \nabla_j \nabla_q \nabla_p H_{pq} \nabla^{ij} \nabla_q E^{ij}, \]
\[A_{31} = \nabla_q \left( \epsilon_{ij} \nabla_i \nabla_j E_{pq} \nabla^{ij} \nabla_q H^{ij} \right) - \epsilon_{ij} \nabla_i \nabla_j \nabla_q \nabla_p E_{pq} \nabla^{ij} \nabla_q H^{ij} \]
\[= \text{div} - \epsilon_{ij} \nabla_i \nabla_j \nabla_q \nabla_p E_{pq} \nabla^{ij} \nabla_q H^{ij} - \epsilon_{ij} \nabla_i \nabla_j \nabla_q \nabla_p H_{pq} \nabla^{ij} \nabla_q H^{ij}. \]

We note that the second term in the second line of \(A_{11}\) is cancelled with \(A_4\) and the second term in the second line of \(A_{31}\) is cancelled with \(A_2\). Hence \(A_1 + \cdots + A_4\), up to some divergence forms, equals to
\[ -\epsilon_{\lambda}^{\rho} \nabla_{\lambda} H_{p} \nabla_{q} E_{ij} \nabla_{p} E_{ij} + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} E_{ij} \nabla_{p} E_{ij}^\rho \]
\[ + \epsilon_{\lambda}^{\rho} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] - \epsilon_{\rho}^{\lambda} \nabla_{\lambda} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho, \]

whose principal part is, after a straightforward calculation,
\[ -\epsilon_{\mu}^{\rho} \nabla_{\mu} H_{p} \nabla_{q} E_{ij} \nabla_{p} E_{ij} \nabla_{q} H_{ij}^\rho + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} E_{ij} \nabla_{p} E_{ij} \nabla_{q} H_{ij}^\rho \]
\[ - \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho \]
\[ = \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho. \]

Again, we calculate the principle part of the above formula:
\[ -\epsilon_{\mu}^{\rho} \nabla_{\mu} H_{p} \nabla_{q} E_{ij} \nabla_{p} E_{ij} \nabla_{q} H_{ij}^\rho + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} E_{ij} \nabla_{p} E_{ij} \nabla_{q} H_{ij}^\rho \]
\[ - \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho \]
\[ = \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho. \]

Up to some divergence forms, it is identical to
\[ + \epsilon_{\mu}^{\rho} \nabla_{\mu} E_{p} \nabla_{q} H_{ij}^\rho + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} H_{ij}^\rho (= 0) \]
\[ + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} E_{ij} \nabla_{p} E_{ij} \nabla_{q} H_{ij}^\rho (= 0) \]
\[ + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho (= 0) \]
\[ + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho (= 0) \]
\[ + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho (= 0) \]
\[ + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho (= 0) \]
\[ + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] + \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} E_{p} \nabla_{q} E_{ij}^\mu] \nabla_{p} E_{ij}^\rho (= 0), \]

where each line vanishes. Hence, we have proved in Case III (\( l \leq 3 \)):
\[ \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} [\nabla_{\mu} H_{p} \nabla_{q} E_{ij}^\mu] \nabla_{\nu} H_{ij}^\rho \]
\[ \simeq \sum_{\nu = 1}^{l} \nabla_{\nu} (\nabla_{\nu} H_{ij} \nabla_{\nu} E_{ij}). \]

Let us come to the final Case IV: \( \epsilon_{\rho}^{\lambda} \nabla_{\lambda} E_{p} \nabla_{q} H_{ij}^\rho \nabla_{\nu} E_{ij}^\rho \nabla_{\nu} E_{ij} \nabla_{\nu} H_{ij}^\rho. \)
Combined with the previous Case III, it reduces to consider
\[ \epsilon^p q \nabla_i [\nabla_{i, q} H_{ij}, \nabla^l E^{lj}] - \epsilon^p q \nabla_i [\nabla_{i, q} E_{pj}, \nabla^l H^{lj}] \]

\[ \simeq - \epsilon^p \lambda \nabla_i H_{ij} \cdot \nabla^l p E^{lj} + \epsilon^p \lambda \nabla_i E_{ij} \cdot \nabla^l p H^{lj} \]

\[ = - \epsilon^p \lambda \nabla_i H_{ij} \cdot [\nabla_{ij}, \nabla^l p E^{lj}] + \epsilon^p \lambda \nabla_i E_{ij} \cdot [\nabla_{ij}, \nabla^l p H^{lj}] \]

where the last line is already covered by Case III, and the remaining parts equal to

\[ - \epsilon^p \lambda \nabla_i H_{ij} \cdot [\nabla_{ij}, \nabla^l p E^{lj}] + \epsilon^p \lambda \nabla_i E_{ij} \cdot [\nabla_{ij}, \nabla^l p H^{lj}], \]

Again, the second line above is covered by Case II. Hence we are left with

\[ - \epsilon^p \lambda \nabla_i H_{ij} \cdot \nabla_{ij} \nabla^l p E^{lj} + \epsilon^p \lambda \nabla_i E_{ij} \cdot \nabla_{ij} \nabla^l p H^{lj}, \]

which belongs exactly to Case I \( (l = 2) \). Therefore, we complete the proof for Case IV and hence (C.10) with \( l \leq 3 \) follows.

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