Resolution Limits of Non-Adaptive Querying for Noisy 20 Questions Estimation

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Abstract—We study fundamental limits of estimation accuracy for the noisy 20 questions problem with measurement-dependent noise and introduce optimal non-adaptive procedures that achieve these limits. The minimal achievable resolution is defined as the absolute difference between the estimated and the true values of the target random variable, given a finite number of queries constrained by the excess-resolution probability. Inspired by the relationship between the 20 questions problem and the channel coding problem, we derive non-asymptotic bounds on the minimal achievable resolution. Furthermore, applying the Berry–Esseen theorem to our non-asymptotic bounds, we obtain a second-order asymptotic approximation to finite blocklength performance, specifically the achievable resolution of optimal non-adaptive query procedures with a finite number of queries subject to the excess-resolution probability constraint.

I. INTRODUCTION

The noisy 20 questions problem (cf. [1]–[7]) arises when one aims to accurately estimate an arbitrarily distributed random variable $S$ by successively querying an oracle and using its noisy responses to form an estimate $\hat{S}$. A central goal in this problem is to find optimal query strategies that yield a good estimate $\hat{S}$ for the unknown target random variable $S$.

Depending on the query design strategy adopted, the 20 questions problem can either be adaptive or non-adaptive. In adaptive query procedures, the design of a subsequent query depends on all previous queries and noisy responses to these queries from the oracle. In non-adaptive query procedures, all the queries are designed independently in advance. For example, the bisection policy [5] Section 4.1) is an adaptive query procedure and the dyadic policy [5] Section 4.2] is a non-adaptive query procedure. Compared with adaptive query procedures, non-adaptive query procedures have the advantage of lower computation cost, parallelizability and no need for feedback. Depending on whether or not the noisy responses depend on the queries, the noisy 20 questions problem is classified into two categories: querying with measurement-independent noise (e.g., [5], [6]); and querying with measurement-dependent noise (e.g., [7], [8]). As argued in [8], measurement-dependent noise better models practical applications. For example, for target localization in a sensor network, the noisy response to each query can depend on the size of the query region due to possible presence of clutter. Another example is in human query systems where personal biases about the state may affect the response.

In earlier works on the noisy 20 questions problem, e.g., [5], [9], [10], the queries were designed to minimize the entropy of the posterior distribution of the target variable $S$. As pointed out in later works, e.g., [6]–[8], [11], other accuracy measures, such as the estimation resolution and the quadratic loss are often better criteria for localization, where the resolution is defined as the absolute difference between $S$ and its estimate $\hat{S}$, $|\hat{S} - S|$, and the quadratic loss is $(\hat{S} - S)^2$.

Motivated by the scenario of limited resources, computation and response time, we obtain new results on the non-asymptotic tradeoff among the number of queries $n$, the achievable resolution $\delta$ and the excess-resolution probability $\varepsilon$ of optimal adaptive and non-adaptive query procedures for noisy 20 questions estimation of an arbitrarily distributed random variable $S$ taking values in the alphabet $S = [0, 1]$.

Our contributions for non-adaptive querying are as follows. First, we derive non-asymptotic bounds of optimal non-adaptive query procedures for any number of queries $n$ and any excess-resolution probability $\varepsilon$. Secondly, applying the Berry–Esseen theorem, under mild conditions on the measurement-dependent noise, we obtain a second-order asymptotic approximation to the achievable resolution of optimal non-adaptive query procedures with a finite number of queries. As a corollary of our result, we establish a phase transition in the excess-resolution probability as a function of the resolution decay rate for optimal non-adaptive query procedures. Finally, we specialize our second-order analyses to measurement-dependent versions of the binary symmetric channel.

Finally, we clarify the differences between our work and [8]. First of all, our results hold for arbitrary discrete channels under mild conditions, while the results in [8] focused only on a measurement-dependent binary symmetric channel. Furthermore, our proof techniques are significantly different from [8]. The authors in [8] used large deviations analysis to prove the achievability part and the Fano’s inequality for the converse part. In contrast, our proofs use recent advances in finite blocklength information theory [12], [13]. It is important to note that our non-asymptotic bounds for non-adaptive query schemes are novel and there are no such comparable results in the previous literature including [8]. Finally, our second-order asymptotic result in Theorem 3 refined [8]. Theorem 3 refined [8].
1]. In particular, Theorem 3 provides an approximation to the performance of optimal query procedures employing a finite number of queries while [8, Theorem 1] only characterizes the asymptotic performance when the number of queries tends to infinity.

II. PROBLEM FORMULATION

Notation

Random variables and their realizations are denoted by upper case variables (e.g., \(X\)) and lower case variables (e.g., \(x\)), respectively. All sets are denoted in calligraphic font (e.g., \(\mathcal{X}\)). Let \(X^n := (X_1, \ldots, X_n)\) be a random vector of length \(n\). We use \(\Phi^{-1}(\cdot)\) to denote the inverse of the cumulative distribution function (cdf) of the standard Gaussian. We use \(\mathbb{R}, \mathbb{R}_+, \text{ and } \mathbb{N}\) to denote the sets of real numbers, positive real numbers and integers respectively. Given any integers \((m, n) \in \mathbb{N}^2\), we use \(m : n\) to denote the set of integers \([m, m + 1, \ldots, n]\) and use \([m]\) to denote \([1 : m]\). Given any \((m, n) \in \mathbb{N}^2\), for any matrix \(A\) of size \(m \times n\), we use \(A_{ij}\) to denote the element at the \(i\)-th row and \(j\)-th column of \(A\). The set of all probability distributions on a finite set \(\mathcal{X}\) is denoted as \(\mathcal{P}(\mathcal{X})\) and the set of all conditional probability distributions from \(\mathcal{X}\) to \(\mathcal{Y}\) is denoted as \(\mathcal{P}(\mathcal{Y}|\mathcal{X})\). Furthermore, we use \(\mathcal{F}(\mathcal{S})\) to denote the set of all probability density functions on a set \(\mathcal{S}\). All logarithms are base \(e\) unless otherwise noted. Finally, we use \(1(\cdot)\) to denote the indicator function.

A. Noisy 20 Questions Problem

Let \(S\) be a continuous random variable defined on the unit interval \(S = [0, 1]\) with arbitrary probability density function (pdf) \(f_S \in \mathcal{F}(\mathcal{S})\). In the noisy 20 questions problem, a player aims to accurately estimate the value of the target random variable \(S\) by posing a sequence of queries \(A^n = (A_1, \ldots, A_n) \subseteq [0, 1]^n\) to an oracle knowing \(S\). After receiving the queries, the oracle finds binary answers \(Y_i = 1(S \in A_i)\) and passes them to the player. The player uses a decoding function \(g : \mathcal{Y}^n \to \hat{S}\) to obtain an estimate \(\hat{S}\) of the target variable \(S\). Throughout the paper, we assume that the alphabet \(\mathcal{Y}\) for the noisy response is finite.

A query procedure for the noisy 20 questions problem consists of the queries \(A^n \subseteq [0, 1]^n\) and the decoder \(g : \mathcal{Y}^n \to \hat{S}\). In general, these procedures can be classified into two categories: non-adaptive and adaptive querying. In a non-adaptive query procedure, the player needs to first determine the number of queries \(n\) and then design all the queries \(A^n\) simultaneously. In contrast, in an adaptive query procedure, the design of queries is done sequentially and the number of queries is a variable. In particular, when designing the \(i\)-th query, the player can use the previous queries and the noisy responses from the oracle to these queries, i.e., \(\{X_j, Y_j\}_{j \in [i-1]}\), to formulate the next query \(A_i\). Furthermore, the player needs to choose a stopping criterion, which may be random, determining the number of queries to make.

In subsequent sections, we clarify the notion of the measurement-dependent channel including concrete examples and we present specific definitions of non-adaptive and adaptive query procedures.

B. The Measurement-Dependent Channel

In this subsection, we describe succinctly the measurement-dependent channel scenario [8], also known as a channel with state [14, Chapter 7]. Given a sequence of queries \(A^n \subseteq [0, 1]^n\), the channel from the oracle to the player is a memoryless channel whose transition probabilities are functions of the queries. Specifically, for any \((x^n, y^n) \in \{0, 1\}^n \times \mathcal{Y}^n\),

\[
P_{Y^n|X^n}(y^n|x^n) = \prod_{i \in [n]} P_{A_i|Y^n}(y_i|x_i),
\]

(1)

where \(P_{A_i|Y^n}\) denotes the transition probability of the channel which depends on the \(i\)-th query \(A_i\). Given any Lebesgue measurable query \(A \subseteq [0, 1]\), define the size \(|A|\) of \(A\) as its Lebesgue measure, i.e., \(|A| = \int_{A} dt\). Throughout the paper, we consider only Lebesgue measurable queries and assume that the measurement-dependent channel \(P_{Y^n|X^n}\) depends on the query \(A\) only through its size, i.e., \(P_{Y^n|X^n}\) is equivalent to a channel with state \(P_{Y^n|X^n}|A\) where the state \(q = |A| \in [0, 1]\).

For any \(q \in [0, 1]\), any \(\xi \in (0, \min(q, 1 - q))\) and any subsets \(A, A^+\) and \(A^-\) of \([0, 1]\) with sizes \(|A| = q, |A^+| = q + \xi\) and \(|A^-| = q - \xi\), we assume the measurement-dependent channel is continuous in the sense that there exists a constant \(c(q)\) depending on \(q\) only such that

\[
\max \left\{ \left\| \log \frac{P_{Y^n|X^n}|A}{P_{Y^n|X^n}|A^+} \right\|_\infty, \left\| \log \frac{P_{Y^n|X^n}|A}{P_{Y^n|X^n}|A^-} \right\|_\infty \right\} \leq c(q)\xi.
\]

(2)

A particular example of a measurement-dependent channel satisfying the continuous constraint in (2) is given in Definition 1. See [15] for other examples.

Definition 1. Given any \(A \subseteq [0, 1]\), a channel \(P_{Y^n|X^n}\) is said to be a measurement-dependent Binary Symmetric Channel (BSC) with parameter \(\nu \in [0, 1]\) if \(X = Y = \{0, 1\}\) and for any \((x, y) \in \mathcal{X} \times \mathcal{Y}\),

\[
P_{Y^n|X^n}(y|x) = (\nu|A|)^{(y \neq x)}(1 - \nu|A|)^{(y = x)}
\]

(3)

This definition generalizes [8, Theorem 1], where the authors considered a measurement-dependent BSC with parameter \(\nu = 1\). Note that the crossover probability (cf. (3)) increases as the size \(|A|\) of the query region increases.

C. Non-Adaptive Query Procedures

A non-adaptive query procedure with resolution \(\delta\) and excess-resolution constraint \(\varepsilon\) is defined as follows.

Definition 2. Given any \(n \in \mathbb{N}\), \(\delta \in \mathbb{R}_+\) and \(\varepsilon \in [0, 1]\), an \((n, \delta, \varepsilon)\)-non-adaptive query procedure for the noisy 20 questions problem consists of

- \(n\) queries \(A_1, \ldots, A_n \subseteq [0, 1]^n\), and a decoder \(g : \mathcal{Y}^n \to \hat{S}\).
that the excess-resolution probability satisfies
\[
P_e(n, \delta) := \sup_{f_S \in \mathcal{F}(S)} \Pr[|g(Y^n) - S| > \delta] \leq \varepsilon. \tag{4}
\]

We remark that the definition of the excess-resolution probability with respect to \( \delta \) is inspired by rate-distortion theory \cite{16, 17}. Our formulation differs from that of \cite{8} where the authors constrained the \( s \)-dependent maximum excess-resolution probability, where \( s \in S \) is the target variable.

Motivated by practical applications where the number of queries are limited (e.g., due to the high cost of queries and low-delay requirement), we are interested in the following non-asymptotic fundamental limit on achievable resolution \( \delta \):
\[
\delta^*(n, \varepsilon) := \inf \{ \delta : \exists \text{ an } (n, \delta, \varepsilon)-\text{non-adaptive--procedure} \}. \tag{5}
\]

Note that \( \delta^*(n, \varepsilon) \) denotes the minimal resolution one can achieve with probability at least \( 1 - \varepsilon \) using a non-adaptive query procedure with \( n \) queries. In other words, \( \delta^*(n, \varepsilon) \) is the achievable resolution of optimal non-adaptive query procedures tolerating an excess-resolution probability of \( \varepsilon \) in \([0,1]\). Dual to \( \delta^* \) is the sample complexity, determined by the minimal number of queries required to achieve a resolution \( \delta \) with probability at least \( 1 - \varepsilon \), i.e.,
\[
n^*(\delta, \varepsilon) := \inf \{ n : \exists \text{ an } (n, \delta, \varepsilon)-\text{non-adaptive--procedure} \}. \tag{6}
\]

One can easily verify that for any \((\delta, \varepsilon) \in \mathbb{R}_+ \times [0,1],\)
\[
n^*(\delta, \varepsilon) = \inf \{ n : \delta^*(n, \varepsilon) \leq \delta \}. \tag{7}
\]

Thus, it suffices to focus on the fundamental limit \( \delta^*(n, \varepsilon) \).

### III. MAIN RESULTS

The proof of all results are omitted due to space limitation. Details are available in our extended version \cite{15}.

#### A. Non-Asymptotic Bounds

We first present an upper bound on the error probability of optimal non-adaptive query procedures. Given any \((p, q) \in [0,1]^2\), let \( P_{Y^n_q}^p \) be the marginal distribution on \( \mathcal{Y} \) induced by the Bernoulli distribution \( P_X = \text{Bern}(p) \) and the measurement-dependent channel \( P_{Y|X}^q \). Furthermore, define the following information density
\[
\iota_{p,q}(x; y) := \log \frac{P_{Y|X}^q(y|x)}{P_Y^n(y)}, \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}. \tag{8}
\]

Correspondingly, for any \((x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n\), we define
\[
\iota_p(x^n; y^n) := \sum_{i=1}^n \iota_{p,p}(x_i; y_i) \tag{9}
\]
as the mutual information density between \(x^n\) and \(y^n\).

**Theorem 1.** Given any \((n, M) \in \mathbb{N}^2\), for any \( p \in [0,1] \) and any \( \eta \in \mathbb{R}_+ \), there exists an \((n, \frac{1}{M\eta}, \varepsilon)\)-non-adaptive query procedure such that
\[
\varepsilon \leq 4n \exp(-2M\eta^2) + \exp(n\eta(p)) \mathbb{E} \left[ \min \left\{ 1, M \Pr\{\iota_p(\hat{X}^n; Y^n) \geq \iota_p(X^n; Y^n) | X^n, Y^n \} \right\} \right], \tag{10}
\]
where the tuple of random variables \((X^n, \hat{X}^n, Y^n)\) is distributed as \( P_X^n(X^n)P_{\hat{X}|X}^p(X^n)P_{Y|X}^q(Y^n|X^n) \) with \( P_X \) defined as the Bernoulli distribution with parameter \( p \) (i.e., \( P_X(1) = p \)).

Consider the measurement-independent channel where \( P_{Y|X} = P_{Y|X}^q = P_{Y|X} \) for all \( q \in [0,1] \). It is straightforward to verify that for any \( p \in [0,1] \), there exists an \((n, \frac{1}{M\eta}, \varepsilon)\)-non-adaptive query procedure such that
\[
\varepsilon \leq \mathbb{E} \left[ \min \left\{ 1, M \Pr\{\iota_p(\hat{X}^n; Y^n) \geq \iota_p(X^n; Y^n) | X^n, Y^n \} \right\} \right], \tag{11}
\]
where the tuple of random variables \((X^n, \hat{X}^n, Y^n)\) is distributed as \( P_X^n(X^n)P_{\hat{X}|X}^p(X^n)P_{Y|X}^q(Y^n|X^n) \), the information density \( \iota_p(x^n; y^n) \) is defined as
\[
\iota_p(x^n; y^n) := \log \frac{P_{Y|X}^q(y^n|x^n)}{P_Y^n(y^n)}, \tag{12}
\]
and \( P_Y \) is induced by \( P_X \) and \( P_{Y|X} \). Comparing the measurement-independent case \( \text{(11)} \) with the measurement-dependent case \( \text{(10)} \), the non-asymptotic upper bound \( \text{(10)} \) in Theorem \( \text{1} \) differs from \( \text{(11)} \) in two aspects: an additional additive term and an additional multiplicative term in \( \text{(10)} \).

As is made clear in the proof of Theorem \( \text{1} \) the additive term \( 4n \exp(-2M\eta^2) \) results from the atypicality of the measurement-dependent channel and the multiplicative term \( \exp(n\eta(p)) \) appears due to the change-of-measure we use to replace the measurement-dependent channel \( P_{Y|X}^{\delta^n} \) with the measurement-independent channel \( P_{Y|X}^{\delta^n} \).

We next provide a non-asymptotic converse bound to complement Theorem \( \text{1} \). For simplicity, for any query \( A \subseteq [0,1] \) and any \((x, y) \in \mathcal{X} \times \mathcal{Y} \), we use \( \iota_A(x, y) \) to denote \( \iota_{|A|}(x, y) \).

**Theorem 2.** Set \((n, \delta, \varepsilon) \in \mathbb{N} \times \mathbb{R}_+ \times [0,1]. \) Any \((n, \delta, \varepsilon)-\text{non-adaptive query procedure satisfies the following. For any } \beta \in (0, \frac{1}{2\varepsilon}) \) and any \( \kappa \in (0, 1 - \varepsilon - 2\beta), \)
\[
-\log \delta \leq -\log \beta - \log \kappa + \sup_{A \subseteq [0,1]} \sup_{\|n\| \leq [0,1]^n} \left\{ t \left| \Pr\left\{ \sum_{i=1}^n \iota_{A_i}(X_i; Y_i) \leq t \right\} \right| \leq \varepsilon + 2\beta + \kappa \right\}. \tag{13}
\]

The proof of Theorem \( \text{2} \) is decomposed into two steps:

i) we use the result in \( \text{8} \) which states that the excess-resolution probability of any non-adaptive query procedure can be lower bounded by the error probability associated with channel coding over the measurement-dependent channel with uniform message distribution, minus a certain term depending
We remark that phase transition only appears in the second-order asymptotic analysis and is not revealed by the first-order asymptotic resolution decay rate with vanishing worst-case excess-resolution probability, i.e., \( \lim_{n \to 0} \lim_{n \to \infty} \left( - \log \delta^*(n, \varepsilon) \right) \). Second, our results hold for any measurement-dependent channel satisfying (2) while [8] Theorem 1] only considers the measurement-dependent BSC.

B. Second-Order Asymptotic Approximation

In this subsection, we present the second-order asymptotic approximation to the achievable resolution \( \delta^*(n, \varepsilon) \) of optimal non-adaptive query procedures after \( n \) queries subject to a worst case excess-resolution probability of \( \varepsilon \in [0, 1) \).

Given measurement-dependent channels \( \{ P_{Y|X}^{q} \}_{q \in [0,1]} \), the channel “capacity” is defined as

\[
C := \max_{q \in [0,1]} \mathbb{E}[\nu_{q,\nu}(X;Y)],
\]

where \( (X,Y) \sim \text{Bern}(q) \times P_{Y|X}^{q} \).

Let the capacity-achieving set \( \mathcal{P}_c \) be the set of optimizers achieving (14). Then, for any \( \varepsilon \in [0,1) \), define the following “dispersion” of the measurement-dependent channel

\[
V_{\varepsilon} := \left\{ \inf_{q \in \mathcal{P}_c} \text{Var}[\nu_{q,\nu}(X;Y)] \right\} \text{ if } \varepsilon < 0.5,
\]

\[
\left\{ \sup_{q \in \mathcal{P}_c} \text{Var}[\nu_{q,\nu}(X;Y)] \right\} \text{ if } \varepsilon \geq 0.5.
\]

The case of \( \varepsilon < 0.5 \) will be the focus of the sequel of this paper. We assume that for any \( q \in \mathcal{P}_c \), the third absolute moment of \( \nu_{q,\nu}(X;Y) \) is finite. Under this assumption, we obtain the second-order asymptotic result.

**Theorem 3.** For any \( \varepsilon \in (0,1) \), the achievable resolution \( \delta^*(n, \varepsilon) \) of optimal non-adaptive query procedures satisfies

\[
- \log \delta^*(n, \varepsilon) = nC + \sqrt{nV_{\varepsilon}} \Phi^{-1}(\varepsilon) + \Theta(\log n),
\]

where the remainder satisfies that \(-\frac{1}{2} \log n \leq \Theta(\log n) \leq \log n + O(1)\).

We make the following remarks.

Firstly, Theorem 3 implies a phase transition in a machine learning sense [18], [19], which we interpret in Figure 1. We remark that phase transition only appears in the second-order asymptotic analysis and is not revealed by the first-order asymptotic analysis, e.g., that developed in [8] Theorem 1.

Secondly, Theorem 3 refines [8] Theorem 1] in several directions. First, Theorem 3 is a second-order asymptotic result that provides good approximation for the finite blocklength performance while [8] Theorem 1] only characterizes the asymptotic resolution decay rate with vanishing worst-case excess-resolution probability, i.e., \( \lim_{n \to 0} \lim_{n \to \infty} (- \log \delta^*(n, \varepsilon)) \).

Thirdly, the dominant event which leads to an excess-resolution in noisy 20 questions problem with measurement-dependent noise, using the example of a measurement-dependent BSC with parameter \( \nu = 0.2 \). On the one hand, when the resolution decay rate is strictly greater than the capacity \( C \), then as the number of the queries \( n \to \infty \), the excess-resolution probability tends to one. On the other hand, when the resolution decay rate is strictly less than the capacity \( C \), then the excess-resolution probability vanishes as the number of the queries increases.

Finally, we remark that any real number \( s \in \mathcal{S} = [0,1] \) has the binary expansion \((b_0,b_1,b_2,\ldots)\). We can thus interpret the result in Theorem 3 as follows: using optimal non-adaptive query procedures, after \( n \) queries, with probability of at least \( 1 - \varepsilon \), one can extract the first \([-log_2 \delta^*(n,\varepsilon)]\) bits of the binary expansion of the target variable \( S \).

C. Case of Measurement-Dependent BSC

In the following, we specialize Theorem 3 to a measurement-dependent BSC. Given any \( \nu \in (0,1) \) and any \( q \in [0,1] \), let \( \beta(\nu, q) := q(1 - \nu q) + (1 - q)\nu q \). For any \( (x,y) \in \{0,1\}^2 \), the information density of a measurement-dependent BSC with parameter \( \nu \) is

\[
\nu_{q,\nu}(x,y) = \mathbb{I}(x \neq y) \log(\nu q) + \mathbb{I}(x = y) \log(1 - \nu q) - \mathbb{I}(y = 1) \log(\beta(\nu, q)) - \mathbb{I}(y = 0) \log(1 - \beta(\nu, q)).
\]

It can be verified that the capacity of the measurement-dependent BSC with parameter \( \nu \) is given by

\[
C(\nu) = \mathbb{E}[\nu_{q,\nu}(X;Y)] = \max_{q \in [0,1]} \left( h_b(\beta(\nu, q)) - h_b(\nu q) \right),
\]

where \( h_b(p) = -p \log(p) - (1 - p) \log(1 - p) \) is the binary entropy function.

Depending on the value of \( \nu \in (0,1) \), the set of capacity-achieving parameters \( \mathcal{P}_c \) may or may not be singleton. In particular, for any \( \nu \in (0,1) \), the capacity-achieving parameter \( q^* \) is unique. When \( \nu = 1 \), there are two capacity-achieving parameters \( q_1^* \) and \( q_2^* \) where \( q_1^* + q_2^* = 1 \). It can be verified easily...
that $V(1,q^*_i) = V(1,1 - q^*_i)$. As a result, for any capacity-achieving parameter $q^*$ of the measurement-dependent BSC with parameter $\nu \in (0, 1]$, the dispersion of the channel is

$$V(\nu) = \text{Var}[q^*,\nu q^*(X;Y)].$$

(19)

**Corollary 4.** Set any $\nu \in (0, 1)$. If the channel from the oracle to the player is a measurement-dependent BSC with parameter $\nu$, then Theorem 3 holds with $C = C(\nu)$ and $V_\varepsilon = V(\nu)$ for any $\varepsilon \in (0, 1)$.

We make the following observations.

First, if we let $\nu = 1$, then for any $\varepsilon \in (0, 1)$,

$$\lim_{n \to \infty} -\log \frac{\delta^*_{\text{mi}}(n, \varepsilon)}{n} = \max_{q \in [0, 1]} \left( h_b(\beta(1, q)) - h_b(q) \right).$$

(20)

This strengthens [8, Theorem 1] with strong converse.

Second, when one considers the measurement-independent BSC with parameter $\nu \in (0, 1)$, then it can be shown that the achievable resolution $\delta^*_{\text{mi}}(n, \varepsilon)$ of optimal non-adaptive query procedures satisfies

$$-\log \delta^*_{\text{mi}}(n, \varepsilon) = n(1 - h_b(\nu)) + \sqrt{n\nu(1 - \nu)} \log_2 \frac{1 - \nu}{\nu} \Phi^{-1}(\varepsilon) + O(\log n).$$

(21)

To compare the performances of optimal non-adaptive query procedures under measurement-dependent and measurement-independent channels, we plot in Figure 2 the second-order approximation to the average number of bits in the binary expansion of the target random variable $S$ extracted per query after $n$ queries, i.e., $-\log_2 \delta^*(n, \varepsilon)$ and $-\log_2 \delta^*_{\text{mi}}(n, \varepsilon)$ for $\varepsilon = 0.001$ and different values of $\nu$ (the remainder $O(\log n)$ is ignored). We observe an interesting phenomenon. When $\nu < 0.5$, optimal query procedures under a measurement-dependent channel achieve a higher resolution than their counterpart in the measurement-independent case. Intuitively, this is because the probability of receiving wrong answers in the measurement-dependent channel is smaller compared with the measurement-independent channel with the same parameter. However, when $\nu > 0.5$, we find that the relative performances can be reversed. The reasons for this phenomenon are two fold: i) BSC is a symmetric channel, thus under the measurement-independent setting, having a BSC with crossover probability $\nu > 0.5$ is equivalent to having a BSC with parameter $1 - \nu < 0.5$ since one can easily flip all bits; ii) under the measurement-dependent setting, the probability of receiving wrong answers depends on the size of the query, this symmetric nature of BSC is lost.

**IV. Numerical Illustration**

In this section, we numerically illustrate the minimal achievable resolution of non-adaptive procedures over a measurement-dependent BSC with parameter $\nu = 0.4$. We consider the case where the target random variable $S$ is uniformly distributed over the alphabet $S = [0, 1]$ and set the target excess-resolution probability $\varepsilon = 0.1$. The simulation results for this case is provided in Figure 3 which demonstrate strong agreement with the theoretical result in Corollary 4.

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![Fig. 2. Plot of the number of bits in the binary expansion of the target variable $S$ extracted by optimal non-adaptive query procedures for both measurement-dependent ($-\log_2 \delta^*(n, \varepsilon)$) and measurement-independent ($-\log_2 \delta^*_{\text{mi}}(n, \varepsilon)$) versions of BSC with different parameters.](image)

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![Fig. 3. Minimal achievable resolution of non-adaptive query procedures for estimating a uniformly distributed target $S \in [0, 1]$ under a measurement-dependent BSC with parameter $\nu = 0.4$. The red line corresponds to the theoretical value asserted in Corollary 2 and the blue dots denote the performance of the non-adaptive query procedure using Monte Carlo simulation. The error bar for the simulated result denotes thirty standard deviations above and below the mean.](image)

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**V. Conclusion**

We derived the minimal achievable resolution of non-adaptive query procedures for the noisy 20 questions problem where the channel from the oracle to the player is a measurement-dependent discrete channel. In our extended version [15], we generalize our results to estimate a multi-dimensional target over the unit cube and to simultaneously estimate multiple targets. Furthermore, we establish a lower bound on the resolution gain associated with adaptive querying for estimating a target over the unit interval.

In this paper, we were interested in fundamental limits of optimal query procedures. It would be interesting to explore low-complexity practical query procedures and compare the performances of proposed query procedures to our derived benchmarks.
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