\section*{Abstract}

An evergreen theme in topological graph theory is the study of graph complexes, \cite{1,15,16,17}. The majority of these complexes are $\mathbb{Z}_2$-spaces and the associated $\mathbb{Z}_2$-index $\text{Ind}_{\mathbb{Z}_2}(X)$ is an invariant of great importance for estimating the chromatic numbers of graphs. We introduce $W_I$-posets (Definition 2) as intermediate objects and emphasize the importance of Bredon’s theorem (Theorem 9) which allows us to use standard tools of topological combinatorics for comparison of $\mathbb{Z}_2$-homotopy types of $\mathbb{Z}_2$-posets. Among the consequences of general results are known and new results about $\mathbb{Z}_2$-homotopy types of graph complexes. It turns out that, in spite of great variety of approaches and definitions, all graph complexes associated to $G$ can be viewed as avatars of the same object, as long as their $\mathbb{Z}_2$-homotopy types are concerned. Among the applications are a proof that each finite, free $\mathbb{Z}_2$-complex is a graph complex and an evaluation of $\mathbb{Z}_2$-homotopy types of complexes $\text{Ind}(C_n)$ of independence sets in a cycle $C_n$.

\section*{Introduction}

By a deep observation of L. Lovász \cite{15}, the chromatic number $\chi(G)$ of a graph can be approximated from below by integers reflecting the topological complexity of associated graph complexes. The impact of this observation cannot be overestimated and this direction of topological graph theory has been for decades a vital part of topological combinatorics, see \cite{16,17} and the references therein.

The first in the series of graph complexes is the so called neighborhood complex $N(G) := \{S \subset V_G \mid \text{CN}(S) \neq \emptyset\}$, where $\text{CN}(S)$ is the set of all common neighbors of $S$ in $G$. Currently there exist a dozen of graph complexes, see the references \cite{1,2,4,8,10,14,17,19,21,24}. Many of them originated from the neighborhood complex $N(G)$ and all of them are used to produce lower bounds for the chromatic number of $G$ in terms of other numerical invariants. A central among these invariants is the equivariant index $\text{Ind}_{\mathbb{Z}_2}(K)$, which applies to graph complexes $K$ with fixed point free involutions $\omega : K \to K$. $\text{Ind}_{\mathbb{Z}_2}(K)$ is defined as the minimum integer $n$ such that there exists a $\mathbb{Z}_2$-equivariant map $f : K \to S^n$. This integer is an invariant of the $\mathbb{Z}_2$-homotopy type of the $\mathbb{Z}_2$-complex $K$ and it is not a surprise that much of the current research is focused on clarifying the mutual relationship of different graph complexes \cite{1,10,16,17}. 

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In this paper we develop a unified approach to the problem of comparing $\mathbb{Z}_2$-homotopy types of graph complexes. The so called W1-posets (Definition 2) are designed to capture the essential features of the neighborhood complex (lattice) $N(G)$ and to serve as a basis for construction of graph posets. The idea to use posets (lattices) as intermediate objects in the construction of graph complexes is not new. J.W. Walker introduced ortholattices in [24] precisely for this purpose and was the first to emphasize the functoriality of such a construction. The novelty of our approach is in the systematic use of Bredon’s theorem (Theorem 9) which allows us to shift from $\mathbb{Z}_2$-homotopy types to the ordinary homotopy types of posets. This change of perspective brings in powerful and elegant tools of topological combinatorics, notably Quillen fiber theorem and its relatives. It also accounts for the greater generality and conceptual simplicity achieved by the introduction of W1-posets. Using this approach we obtain new $\mathbb{Z}_2$-homotopy equivalences between graph complexes and posets (Sections 3 and 4) and as a consequence derive new conceptual proofs of related results of Matoušek and Ziegler [17], Csorba et al. [8] [10], and Lovász (Sect. 5). Among the highlights are a proof (Section 6) of the fact that each finite, free $\mathbb{Z}_2$-complex is a graph complex (earlier proved by Csorba [9]) and an analysis of the $\mathbb{Z}_2$-homotopy types of complexes $\text{Ind}(C_n)$, yielding results originally used by Bobson and Kozlov [4] in their solution of Lovász conjecture (Section 8).

The notation used in the paper is standard [6]. $G = (V_G, E_G)$ is a finite graph with $V_G$ and $E_G$ as the sets of vertices and edges. All graphs are simple and undirected. The collection of all chains in a (finite) posets $P$ forms a simplicial complex called the order complex $\Delta(P)$ of $P$. A $\mathbb{Z}_2$-space is a topological space $X$ equipped with a continuous involution $\omega : X \to X$, $\omega^2 = 1_X$. A $\mathbb{Z}_2$-equivariant map $f : X \to Y$ between two $\mathbb{Z}_2$-spaces $X$ and $Y$ is a continuous map satisfying the condition $f(\omega x) = \omega f(x)$. A $\mathbb{Z}_2$-equivariant map, or a $\mathbb{Z}_2$-map for short, is a $\mathbb{Z}_2$-equivalence if there exists a $\mathbb{Z}_2$-map $g : Y \to X$ such that $g \circ f$ is $\mathbb{Z}_2$-homotopic to $1_X$ and $f \circ g$ is $\mathbb{Z}_2$-homotopic to $1_Y$. A general reference for $G$-spaces, $G$-equivariant maps and related concepts and facts is [11]. Expositions oriented towards applications in combinatorics can be found in [16], [26] and [27].

1 Involutive and weakly involutive posets

Definition 1 A poset $(Q, \leq)$ is involutive (I-poset) if it is equipped with an involution $C : Q \to Q$ which is either monotone or antitone, i.e. which satisfies either the condition $x \leq y \Rightarrow C(x) \leq C(y)$ or the dual condition $x \leq y \Rightarrow C(x) \geq C(y)$. We also say that $(Q, \leq)$ admits a $\mathbb{Z}_2$-action or that $(Q, \leq)$ is a $\mathbb{Z}_2$-poset.

Definition 2 A weakly involutive poset $(P, C)$, or a W1-poset for short, is a finite poset $P$ equipped with a function $C : P \to P$ such that

\[ x \leq y \Rightarrow C(y) \geq C(x) \]  

\[ x \leq C(C(x)) = C^2 x \]
Remark 3 The theories of antitone and monotone $I$-posets are similar but there are also some important differences. For example only antitone $I$-posets are WI-posets in the sense of Definition 4. Some results in the paper are sensitive to this difference so whenever necessary, it will emphasized what kind of $I$-posets we are dealing with.

Definition 4 Suppose that $(P, C)$ is a WI-poset. Then the associated (antitone) involutive poset $L(P)$ is a subposet of $P$ defined by $L(P) = \{x \in P \mid C^2 x = x\}$.

An easy consequence of equations 11 and 12 is the equality $C^3 x = C x$ which implies that $L(P)$ is non-empty and that $C$, restricted to $L(P)$, is a genuine antitone involution turning $(L(P), C)$ into an involutive poset in the sense of Definition 11. The involutive poset $(L(P), C)$ is often called the Lovász poset associated to $(P, C)$ for the reasons explained in Section 5.

Definition 5 The box poset $B(P)$ associated to a WI-poset $(P, C)$ is a subposet of $P \times P$ defined by

$$B(P) = B(P, C) = \{(x, y) \in P \times P \mid x \leq C y \land y \leq C x\}.$$ 

It is desirable to isolate the “correct” notion of a morphism of WI-posets which would turn $P \rightarrow L(P)$ and $P \rightarrow B(P)$ into genuine functors. If $f : (P, C) \rightarrow (Q, C)$ is a monotone map of WI-posets such that $f(C(x)) = C(f(x))$ then obviously there exists a monotone map $\bar{f} : L(P) \rightarrow L(Q)$ of associated Lovász posets. This condition is unfortunately too restrictive. Here is a natural condition on a monotone map $f : (P, C) \rightarrow (Q, C)$ of WI-posets guaranteeing that the associated map $F : B(P) \rightarrow B(Q)$, $(x, y) \mapsto (f(x), f(y))$, is well defined and monotone.

Definition 6 A monotone map $f : (P, C) \rightarrow (Q, C)$ of WI-posets is a WI-morphism if $f(C(x)) \leq C(f(x))$ for each $x \in P$.

Definition 7 The poset of intervals $(\text{Int}(Q), \preceq)$, associated to a poset $(Q, \leq)$, is by definition

$$\text{Int}(Q) = \{(x, y) \in P \times P \mid x \leq y\}$$

where $(x, y) \preceq (x', y') \iff x \leq x' \leq y' \leq y$. The elements of $\text{Int}(Q)$ may be interpreted as the intervals $(y)_{x_Q} = [x, y]_Q$ in the poset $Q$ and $\preceq$ as the reversed containment relation.

As usual, $\Delta(Q)$ is the order complex of a poset $(Q, \leq)$. Given a simplicial complex $K$, more generally a polyhedral or a regular CW-complex, the associated face poset is $\Phi(K) = (\Phi(K), \supseteq)$. Note that $\Phi(K)$ is ordered by the reversed inclusion, i.e. $F_1 \leq F_2$ is equivalent to $F_1 \supseteq F_2$.

Definition 8 The poset $\text{Chain}(Q) = \Phi(\Delta(Q))$ is called the chain poset associated to $(Q, \leq)$. Its elements are chains $A = \{x_1 \leq \ldots \leq x_k\}$ in $Q$ and $A \preceq B$ if $B$ is a subchain of $A$. 

3
2 Bredon’s theorem

A fundamental tool in the theory of transformation groups is a theorem of Bredon which gives a necessary and sufficient conditions on a $G$-map $f : X \to Y$ to be a $G$-homotopy equivalence, cf. [7] Ch. II or [11] Section II.2. In this paper we need a $\mathbb{Z}_2$-version of this result. Here and elsewhere throughout the paper we consistently assume that all spaces are simplicial $\mathbb{Z}_2$-complexes (polyhedral, CW) and that the $\mathbb{Z}_2$-maps are simplicial (cellular). Bredon’s theorem holds in higher generality [12] than stated/used in this paper but in combinatorial applications we can usually restrict our attention to narrower and more manageable classes of spaces.

Theorem 9 Suppose that $f : X \to Y$ is a (simplicial) $\mathbb{Z}_2$-map of simplicial $\mathbb{Z}_2$-complexes $X$ and $Y$. Let $X^{\mathbb{Z}_2}$ and $Y^{\mathbb{Z}_2}$ be the associated subspaces of fixed points and $f^{\mathbb{Z}_2} : X^{\mathbb{Z}_2} \to Y^{\mathbb{Z}_2}$ the map induced by $f$. Then $f$ is a $\mathbb{Z}_2$-homotopy equivalence if and only if both $f : X \to Y$ and $f^{\mathbb{Z}_2} : X^{\mathbb{Z}_2} \to Y^{\mathbb{Z}_2}$ are homotopy equivalences.

Corollary 10 If in Theorem 9 the actions of $\mathbb{Z}_2$ on both $X$ and $Y$ are free, i.e. if $X^{\mathbb{Z}_2} = Y^{\mathbb{Z}_2} = \emptyset$, then a $\mathbb{Z}_2$-map $f : X \to Y$ is a $\mathbb{Z}_2$-equivalence if and only if it is an ordinary homotopy equivalence.

Corollary 11 Suppose that $P$ and $Q$ are two involutive posets (Definition 1) and let $f : P \to Q$ be a $\mathbb{Z}_2$-equivariant map of posets. Let $P^{\mathbb{Z}_2}$ and $Q^{\mathbb{Z}_2}$ be the associated subposets of fixed elements. Then $f : P \to Q$ is a $\mathbb{Z}_2$-equivalence if and only if both $f : P \to Q$ and $f^{\mathbb{Z}_2} : P^{\mathbb{Z}_2} \to Q^{\mathbb{Z}_2}$ are homotopy equivalences of posets.

Once we reduced the question of $\mathbb{Z}_2$-equivalence to the problem of verifying ordinary homotopy equivalences, we have on our disposal all the usual combinatorial tools, cf. [6] and [28]. Our main tool in this paper is the well known Quillen fiber theorem [6] [20] [28] which says that a monotone map $f : P \to Q$ of posets is a homotopy equivalence if $f^{-1}(Q \leq q)$ is contractible for each $q \in Q$. Equally important and useful is the following result widely known as Order Homotopy Theorem, [22] [20] [6], see also [25] for subsequent developments and related references.

Proposition 12 Suppose that $f$ and $g$ are two monotone maps of posets $P$ and $Q$ such that $f(x) \leq g(x)$ for each $x \in P$. Then there is a homotopy equivalence $\Delta(f) \simeq \Delta(g) : \Delta(P) \to \Delta(Q)$ between the induced maps of associated order complexes. Moreover, if $P = Q$ and $g = 1_P$ is the identity map, then the subcomplex $\text{Im}(f) \subseteq \Delta(P)$ is a deformation retract of $\Delta(P)$.

Example 13 The well known fact that the inclusion map $\mathcal{L}(P) \to P$ is a homotopy equivalence, actually an inverse to a deformation retraction, is easily deduced from the second half of Proposition 12. Indeed, it is sufficient to define $f$ as the map $C^2 : P \to P$.
3 \( \mathbb{Z}_2 \)-homotopy equivalences of \( \mathbb{Z}_2 \)-posets

Suppose that \((P, C)\) is a WI-poset (Definition 2). Then the associated Lovász poset \( L(P) \) is involutive with the action (antitone involution) \( \omega : L(P) \rightarrow L(P) \) defined by \( \omega(x) := Cx \). The box poset \( \mathfrak{B}(P) \) (Definition 5) also admits a \( \mathbb{Z}_2 \)-action defined by \( \omega(x, y) := (y, x) \). If \((Q, \leq)\) is a \( \mathbb{Z}_2 \)-poset with an antitone involution \( \omega : Q \rightarrow Q \), then both the poset of intervals \( Int(Q) \) (Definition 7) and the chain poset \( Chain(Q) \) (Definition 8) admit natural \( \mathbb{Z}_2 \)-actions. More precisely, if \((y, x) \in Int(Q)\) then \( \omega(y, x) := (\omega(x), \omega(y)) \) and for \( A = \{x_1, \ldots, x_k\} \in Chain(Q), \omega(A) = B \) where \( B = \{\omega(x_k) : x_k \in A\} \).

Consequently for each WI-poset \((P, C)\) there arise four different \( \mathbb{Z}_2 \)-posets \( L(P), \mathfrak{B}(P), Int(L(P)), \) and \( Chain(L(P)) \). Our objective is to demonstrate that all these posets are \( \mathbb{Z}_2 \)-homotopy equivalent.

**Proposition 14** Assume that \((Q, \leq)\) is an I-poset with an antitone involution \( C : Q \rightarrow Q \). Then the \( \mathbb{Z}_2 \)-map \( \Omega : Int(Q) \rightarrow \mathfrak{B}(Q) \) defined by \( \Omega(y, x) := (x, Cy) \) is a \( \mathbb{Z}_2 \)-isomorphism of \( \mathbb{Z}_2 \)-posets.

**Proof.** Define the inverse map \( \Omega' : \mathfrak{B}(Q) \rightarrow Int(Q) \) by the formula \( \Omega'(a, b) := (Ca, b) \). Note that both \( \Omega \) and \( \Omega' \) are well defined. It remains to be shown that one of them, say \( \Omega \), is both monotone and \( \mathbb{Z}_2 \)-equivariant. Indeed, \( \Omega \) is monotone since \((y, x) \leq (y', x') \) if and only if \( x \leq x' \leq y \leq y' \) implies \( x \leq x' \) and \( Cy \leq Cy' \), i.e. \((x, Cy) \leq (x', Cy')\) in \( \mathfrak{B}(Q) \). It is \( \mathbb{Z}_2 \)-equivariant since
\[
\Omega(\omega(y, x)) = \Omega((Cx, Cy)) = (Cy, C^2x) = (Cy, x) = \omega(x, Cy) = \omega(\Omega(y, x)) \quad \blacksquare.
\]

**Proposition 15** Let \((P, C)\) be a WI-poset. Then the \( \mathbb{Z}_2 \)-map \( \Theta : \mathfrak{B}(L(P)) \rightarrow \mathfrak{B}(P) \) of \( \mathbb{Z}_2 \)-posets, induced by the inclusion map \( L(P) \rightarrow P \), is a \( \mathbb{Z}_2 \)-equivalence.

**Proof.** By Bredon’s theorem (Theorem 9), we are supposed to show that the following two conditions are satisfied,

- \( \beta : \mathfrak{B}(L(P)) \rightarrow \mathfrak{B}(P) \) is a homotopy equivalence,
- \( \beta^{\mathbb{Z}_2} : \mathfrak{B}(L(P))^{\mathbb{Z}_2} \rightarrow \mathfrak{B}(P)^{\mathbb{Z}_2} \) is a homotopy equivalence.

Suppose that \((u, v) \in \mathfrak{B}(P)\) and \((u, v) \in \mathfrak{B}(L(P))\) such that \((u, v) \geq (x, y) \). Then \( \beta^{-1}(\mathfrak{B}(L(P))^{\mathbb{Z}_2}) \) has a minimum element \((C^2x, C^2y) \geq (C^2u, C^2v) \). Hence it is contractible. By Quillen fiber theorem \( \beta : \mathfrak{B}(L(P)) \rightarrow \mathfrak{B}(P) \) is a homotopy equivalence.

Let us start with an observation that \( \mathfrak{B}(L(P))^{\mathbb{Z}_2} = \{(x, x) \mid x \in L(P)\} \cong L(P) \) and \( \mathfrak{B}(P)^{\mathbb{Z}_2} = \{(u, u) \mid u \in P\} \cong P \). It follows from Example 13 that \( \beta^{\mathbb{Z}_2} \) is also an equivalence of posets. \( \blacksquare \)
**Proposition 16** Suppose that $(Q, C)$ is an (antitone) involutive poset (I-poset) in the sense of Definition \[.\] Let $\Sigma : \operatorname{Chain}(Q) \to \operatorname{Int}(Q)$ be the map of the associated chain and interval posets defined by $\Sigma(A) = \left(\frac{a}{x_i} \right)$ where $A = \{x_1 \leq \ldots \leq x_m\} \in \operatorname{Chain}(Q)$. Then $\Sigma$ is a $\mathbb{Z}_2$-equivalence.

**Proof.** As before, owing to Bredon’s theorem, it is sufficient to show that both $\Sigma : \operatorname{Chain}(Q) \to \operatorname{Int}(Q)$ and $\Sigma^{\mathbb{Z}_2} : \operatorname{Chain}(Q)^{\mathbb{Z}_2} \to \operatorname{Int}(Q)^{\mathbb{Z}_2}$ are homotopy equivalences. Given $(\frac{b}{a}) \in \operatorname{Int}(Q)$ and $A = \{x_1 \leq \ldots \leq x_m\} \in \operatorname{Chain}(Q)$, we observe that

$$\left(\frac{b}{a}\right) \leq \Sigma(A) \iff a \leq x_1 \leq \ldots \leq x_m \leq b.$$ 

Let $D := \Sigma^{-1}(\operatorname{Int}(Q)_{\geq (\frac{b}{a})})$. Define two monotone maps $\lambda, \mu : D \to D$, by the formulas

$$\lambda(A) = A' := \{a \leq x_1 \leq \ldots \leq x_m \leq b\} \text{ and } \mu(A) := \{a \leq b\}.$$ 

Let $1_D : D \to D$ be the identity map. Then $1_D(A) \succcurlyeq \lambda(A) \preccurlyeq \mu(A)$ for each $A \in D$. By Proposition \[12\] the poset $D$ is contractible, so by Quillen fiber theorem $\Sigma$ is a homotopy equivalence. Let us establish now a similar fact for the map $\Sigma^{\mathbb{Z}_2} : \operatorname{Chain}(Q)^{\mathbb{Z}_2} \to \operatorname{Int}(Q)^{\mathbb{Z}_2}$. We start with an observation that $(\frac{b}{a}) \in \operatorname{Int}(Q)^{\mathbb{Z}_2}$ if and only if $(\frac{b}{a}) = (\frac{C}{x})$ for some $x \in Q$ such that $x \leq Cx$. Similarly, $A \in \operatorname{Chain}(Q)^{\mathbb{Z}_2}$ if and only if there exist elements $x_1 \leq \ldots \leq x_k$ in $Q$ such that $x_k \leq Cx_k$, in which case $A := \{x_1 \leq \ldots \leq x_k \leq Cx_k \leq \ldots \leq Cx_1\}$. Note that the inequality $x_k \leq Cx_k$ is not necessarily strict. Since $\Sigma(A) = (\frac{C}{x})$, we observe that $(\frac{Ca}{a}) \leq \Sigma(A)$ if and only if $a \leq x_1$ and $Cx_1 \leq Ca$. Define $\lambda_1(A) = A' := \{a \leq x_1 \leq \ldots \leq x_k \leq Cx_k \leq \ldots \leq Cx_1 \leq Ca\}$ and $\mu_1(A) := \{a \leq Ca\}$ as monotone maps on the poset $D_1 := (\Sigma^{\mathbb{Z}_2})^{-1}(\operatorname{Int}(Q)_{\geq (\frac{Ca}{a})})$. Since $1_{D_1}(A) \succcurlyeq \lambda_1(A) \preccurlyeq \mu_1(A)$ for each $A \in D_1$ we deduce from Proposition \[12\] that $D_1$ is contractible. Hence, by Quillen fiber theorem, $\Sigma^{\mathbb{Z}_2}$ is a homotopy equivalence which completes the proof of the proposition. \[\square\]

**Corollary 17** Let $(P, C)$ be a WI-poset and $\mathcal{L}(P)$ the associated Lovász subposet. Then $\Sigma : \operatorname{Chain}(\mathcal{L}(P)) \to \operatorname{Int}(\mathcal{L}(P))$ is a $\mathbb{Z}_2$-homotopy equivalence.

**Proposition 18** Suppose that $(Q, C)$ is an involutive poset (I-poset). Then there is a $\mathbb{Z}_2$-homotopy equivalence of $\mathbb{Z}_2$-complexes $\Delta(Q)$ and $\Delta(\operatorname{Chain}(Q))$.

**Proof.** Note that $\Delta(\operatorname{Chain}(Q))$ is just the first baricentric subdivision of the simplicial complex $\Delta(Q)$. Hence, there is a well known canonical homeomorphism $\Gamma : |\Delta(Q)| \to |\Delta(\operatorname{Chain}(Q))|$ of the associated geometric realizations of these complexes. What remains to be done is to show that $\Gamma$ is $\mathbb{Z}_2$-equivariant. Recall that the $\mathbb{Z}_2$-actions on $Q$ and $\operatorname{Chain}(Q)$ are given by $\omega(q) = Cq$ and $\omega\{x_1 \leq \ldots \leq x_k\} = \{Cx_k \leq \ldots \leq Cx_1\}$ respectively. The homeomorphism $\Gamma$ is explicitly defined as follows. Let $t = t_1x_1 + \ldots + t_kx_k \in |\Delta(Q)|$, where $x_1 < \ldots < x_k$, $t_1 + \ldots + t_k = 1$, and $t_j \geq 0$. Put the sequence $(t_j)_{j=1}^k$ in the descending order which means that for some permutation $\pi : [n] \to [n]$ we have inequalities $t_{\pi_1} \geq t_{\pi_2} \geq \ldots \geq t_{\pi_k}$. Then $X_{\pi_1} \succcurlyeq X_{\pi_2} \succcurlyeq \ldots \succcurlyeq X_{\pi_k}$, where $X_{\pi} := \{x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_j}\}$, is a chain in the poset $\operatorname{Chain}(Q)$ and
$$\Gamma(t) = s_1 x_{\pi_1} + s_2 x_{\pi_2} + \ldots + s_k x_{\pi_k} \in |\Delta(Chain(Q))|$$, where the relation between sequences \((s_j)_{j=1}^k\) and \((t_j)_{j=1}^k\) is determined by the following equality

$$s_1 x_{\pi_1} + s_2 x_{\pi_2} + \ldots + s_k x_{\pi_k} = t_1 x_1 + \ldots + t_k x_k.$$  

Note that \(\omega(t) = t_1 C(x_1) + \ldots + t_k C(x_k)\), and \(\Gamma \omega(t) = s_1(\omega X_{\pi_1}) + s_2(\omega X_{\pi_2}) + \ldots + s_k(\omega X_{\pi_k}) = \omega(\Gamma(t))\) which implies that \(\Gamma\) is indeed \(\mathbb{Z}_2\)-equivariant. ■

All results in this section together imply that there exists essentially a unique \(\mathbb{Z}_2\)-homotopy type associated to a given WI-poset \((P, C)\).

**Corollary 19** For a WI-poset \((P, C)\), the order complexes of \(\mathbb{Z}_2\)-posets

\[ L(P) \quad \mathfrak{B}(P) \quad \mathfrak{B}(L(P)) \quad \text{Int}(L(P)) \quad \text{Chain}(L(P)) \]

are all \(\mathbb{Z}_2\)-homotopy equivalent.

### 4 Relatives of the box poset

As a variation on a theme, motivated by applications in Section 3, we introduce two more relatives of the box poset \(\mathfrak{B}(P)\).

**Definition 20** Assume that \((P, C)\) is a WI-poset. Define \(\tilde{P} := P \cup \{\tilde{0}\}\) as a new poset obtained by adding to \(P\) a possibly new minimum element \(\tilde{0}\). The extended box poset \(\mathfrak{B}_{ex}(P)\), associated to the WI-poset \(P\), is a subposet of \(\tilde{P} \times \tilde{P}\) defined by

\[ \mathfrak{B}_{ex}(P) := \mathfrak{B}(P) \cup \{(p, \tilde{0}) \mid p \in P\} \cup \{\tilde{0}, q\} \mid q \in P\} \]

**Theorem 21** Suppose that \((P, C)\) is a WI-poset and \(\mathfrak{B}(P), \mathfrak{B}_{ex}(P)\) the box poset, respectively the extended box poset associated to \(P\). Then the inclusion map \(e : \mathfrak{B}(P) \rightarrow \mathfrak{B}_{ex}(P)\) is a \(\mathbb{Z}_2\)-equivalence of posets.

**Proof.** As before, we ought to show that both \(e : \mathfrak{B}(P) \rightarrow \mathfrak{B}_{ex}(P)\) and \(e^{\mathbb{Z}_2} : \mathfrak{B}(P)^{\mathbb{Z}_2} \rightarrow \mathfrak{B}_{ex}(P)^{\mathbb{Z}_2}\) are homotopy equivalences of posets. Let us show that for each \((p, q) \in \mathfrak{B}_{ex}(P)\) the poset \(D_{p,q} := e^{-1}(\mathfrak{B}_{ex}(P)_{\geq (p,q)})\) is contractible. This is obvious if \(p \neq \tilde{0} \neq q\) since in that case \(D_{p,q} = \mathfrak{B}(P)_{\geq (p,q)}\). Let us establish the contractibility of \(D_{p,\tilde{0}}\), the case of \(D_{\tilde{0},q}\) is treated similarly. By definition \((x, y) \in D_{p,\tilde{0}}\) if and only if \(x \geq p, x \leq C y\), and \(y \leq C x\). As a consequence we have the inequalities \(y \leq C x \leq C p\). This means that \((p, y) \in \mathfrak{B}(P)\) and, since \((p, y) \geq (p, \tilde{0})\), we conclude that \((p, y) \in D_{p,\tilde{0}}\). The identity map \(1_{D_{p,\tilde{0}}}\) and the map \(\mu : D_{p,\tilde{0}} \rightarrow D_{p,\tilde{0}}\) defined by \(\mu(x, y) = (p, y)\) satisfy the condition \(\mu(x, y) = (p, y) \leq (x, y) = 1_{D_{p,\tilde{0}}}(x, y)\) hence, by Proposition 12, \(E_{p,\tilde{0}} = \text{Im}(\mu) = \{(p, y) \mid y \leq C p \ & \ p \leq C y\}\) is a deformation retract of \(D_{p,\tilde{0}}\). On the other hand, since \((p, C p)\) is the maximum element of \(E_{p,\tilde{0}}\), we conclude that \(E_{p,\tilde{0}}\) is contractible, so the same holds for \(D_{p,\tilde{0}}\).

The case of the map \(e^{\mathbb{Z}_2} : \mathfrak{B}(P)^{\mathbb{Z}_2} \rightarrow \mathfrak{B}_{ex}(P)^{\mathbb{Z}_2}\) is simpler since \(\mathfrak{B}(P)^{\mathbb{Z}_2} = \mathfrak{B}_{ex}(P)^{\mathbb{Z}_2} = \{(p, p) \mid p \in P\}\) and \(e^{\mathbb{Z}_2}\) is an identity map. ■
Definition 22 Suppose that \((P, C)\) is a WI-poset and that \(P\) is a subposet of an auxiliary poset \(S\). Define the box poset of \(P\) enriched over \(S\) as the \(\mathbb{Z}_2\)-subposet of \(\hat{S} \times \hat{S}\) described by the equality

\[
\mathcal{B}_S(P) := \mathcal{B}(P) \cup \{(p, \hat{0}) \mid p \in S\} \cup \{\hat{0}, q \mid q \in S\}.
\]

Theorem 23 Suppose that \((P, C)\) is a WI-poset, \(S\) a superposet of \(P\), and \(\mathcal{B}_S(P)\) the box poset of \(P\) enriched over \(S\). If \(S\) is contractible then the geometric realization \(|\Delta(\mathcal{B}_S(P))|\) of this poset is a \(\mathbb{Z}_2\)-space which is \(\mathbb{Z}_2\)-homotopy equivalent to the suspension \(\text{Susp}(|\Delta(\mathcal{B}(P))|)\) of the geometric realization of the box poset \(\mathcal{B}(P)\).

Proof. Let \(\mathcal{B}(P) = \mathcal{B}(P) \cup \{a_1, a_2\}\) be the poset obtained from the box poset \(\mathcal{B}(P)\) by adding two new incomparable minimal elements \(a_1\) and \(a_2\). Extend the involution \(\omega\) from \(\mathcal{B}(P)\) to \(\mathcal{B}(P)\) by the requirement that \(\omega(a_1) = a_2\) and \(\omega(a_2) = a_1\). Note that each chain \(A\) in \(\mathcal{B}(P)\) can be extended to chains \(A_1 = A \cup \{a_1\}\) and \(A_2 = A \cup \{a_2\}\).

Since \(a_1\) and \(a_2\) are incomparable, we observe that the geometric realization of the order complex \(\Delta(\mathcal{B}(P))\) is, as a \(\mathbb{Z}_2\)-space, homeomorphic to the suspension \(\text{Susp}(|\Delta(\mathcal{B}(P))|)\).

Define a monotone, \(\mathcal{Z}_2\)-map \(\Psi : \mathcal{B}_S(P) \rightarrow \mathcal{B}(P)\) of posets as follows. If \(p, q \in P\) then \(\Psi(p, q) = (p, q)\). Otherwise \(\Psi(p, \hat{0}) = a_1\) and \(\Psi(\hat{0}, p) = a_2\) for each \(p \in S\). The map \(\Psi\) is obviously \(\mathbb{Z}_2\)-equivariant. Let us show that it is a \(\mathbb{Z}_2\)-equivalence. In light of Theorem 9 we ought to show that both \(\Psi\) and \(\Psi^{\hat{2}}\) are ordinary homotopy equivalences. The map \(\Psi^{\hat{2}}\) turns out to be essentially an identity map so we focus our attention on \(\Psi\). Let \(D_{x,y} := \Psi^{-1}(\mathcal{B}(P)_{\geq (x,y)})\). If \((x, y) \in \mathcal{B}(P)\) then \(D_{x,y} = \mathcal{B}_S(P)_{\geq (x,y)}\), hence it is contractible. If \((x, y) = (p, \hat{0})\) for some \(p \in S\), then \(D_{p,\hat{0}}\) can be deformed to its subposet \(S_1 := \{(p, \hat{0}) \mid p \in S\}\). Indeed, such a deformation is provided by the map \(\mu : D_{p,\hat{0}} \rightarrow S_1\), where \(\mu(x, y) := (x, \hat{0})\). Since \(S_1 \cong S\), and by assumption \(S\) is contractible, we conclude that \(D_{p,\hat{0}}\) is contractible. By a similar argument \(D_{\hat{0},q}\) is also contractible and finally, by Quillen fiber theorem, \(\Psi\) is a homotopy equivalence of posets. \(\blacksquare\)

5 Applications to graph complexes

Suppose that \(G = (V_G, E_G)\) is a finite graph. The poset \((P_G, \subseteq)\), where by definition \(P_G := \{A \subset V_G \mid CN(A) \neq \emptyset\}\), is weakly involutive (a WI-poset) where the weak involution \(C : P_G \rightarrow P_G\) is defined by \(C(A) := CN(A)\). This is precisely the example which served as a motivation for introducing WI-posets and the development of the associated \(\mathbb{Z}_2\)-posets (\(\mathbb{Z}_2\)-complexes). By specialization, each of the \(\mathbb{Z}_2\)-posets from Sections 3 and 4 yields the corresponding graph \(\mathbb{Z}_2\)-complex. Here is a partial list of these complexes

\[
\mathcal{L}(G) := \mathcal{L}(P_G) \quad \mathcal{B}(G) := \mathcal{B}(P_G) \quad \mathcal{B}_{ex}(G) := \mathcal{B}(P_G) \quad \mathcal{B}_S(G) := \mathcal{B}_S(P_G).
\]

One of our objectives in this section is to compare these complexes with the existing graph complexes listed in [17]. More importantly, we demonstrate that in virtually all
cases analyzed in [17] (Theorems 1 and 3), the equality \( \text{Ind}_{\mathbb{Z}_2}(K_1) = \text{Ind}_{\mathbb{Z}_2}(K_2) \) of \( \mathbb{Z}_2 \)-indices of graph complexes is a consequence of the stronger statement that \( K_1 \) and \( K_2 \) are \( \mathbb{Z}_2 \)-homotopy equivalent. Similarly, all the inequalities \( \text{Ind}_{\mathbb{Z}_2}(K_1) \leq \text{Ind}_{\mathbb{Z}_2}(K_1) + 1 \) are found to be consequences of the \( \mathbb{Z}_2 \)-equivalence \( K_1 \cong \text{Susp}(K_2) \).

**Caveat** We interchangeably use the words posets and complexes for the same objects. This should not cause any ambiguity since one already talks about the homology and the homotopy of a poset \( Q \), having in mind the homology and homotopy of the associated order complex \( \Delta(Q) \).

The complex \( \mathcal{L}(G) \) is of course the Lovász original \( \mathbb{Z}_2 \)-poset (\( \mathbb{Z}_2 \)-complex), denoted by \( L(G) \) in [17]. The poset \( \mathfrak{B}(G) \) is easily identified as the box complex \( B_{\text{chain}}(G) \), while \( \mathfrak{B}_{ex}(G) \) is clearly the box complex \( B(G) \) from [17]. The complex \( B_0(G) \) is recognized as our complex \( \mathfrak{B}_S(G) \) where \( S = \mathcal{P}'(V_G) := \mathcal{P}(V_G) \setminus \{ \emptyset \} \) is the poset of all non-empty subsets of \( V_G \).

All \( \mathbb{Z}_2 \)-equivalences between these complexes (and their suspensions) are immediate consequences of results from Sections 3 and 4. The complexes from [17] that do not automatically fit into this scheme are complexes \( B_{\text{edge}}(G), B_{\text{KG}}^K(G), B_{\text{KG}}^G(F) \), listed as complexes no. 4, 5, and 6 in Section 5 of [17]. Note that the complexes

\[
B_{\text{KG}}^K(F) := \Delta\{ B' \cup B'' \mid B', B'' \subseteq [n], B' \cap B'' = \emptyset, (\exists X \in F) X \subseteq A \text{ or } X \subseteq B \}
\]

\[
B_{\text{KG}}^G(F) := \Delta\{ B' \cup B'' \mid B', B'' \subseteq [n], B' \cap B'' = \emptyset, (\exists X, Y \in F) X \subseteq A \text{ and } Y \subseteq B \}
\]

are defined in terms of the chosen Kneser representative \( F \) of the graph \( G = KG(F) \). This explains why they cannot be immediately expressed in terms of the associated WI-posets \( (P_G, C) \). Nevertheless, the approach based on Bredon’s theorem (Theorem 9) is equally efficient and elegant.

**Proposition 24** Let \( G = KG(F) \) be the Kneser graph associated to a finite family of sets \( F \). Let \( \mathfrak{B}(G) = B_{\text{chain}}(G) \) and \( B_{\text{KG}}^G(F) \) be the associated box complexes (no. 3 and no. 6 from the list in Section 5 of [17]). Then the map

\[
\Phi : B_{\text{chain}}(G) \longrightarrow B_{\text{chain}}^G(F)
\]

defined by \( \Phi(a \uplus b) := (A \uplus b) \) where \( A := \cup a \) and \( B := \cup b \), is a \( \mathbb{Z}_2 \)-homotopy equivalence of \( \mathbb{Z}_2 \)-posets (\( \mathbb{Z}_2 \)-spaces).

**Proof.** The action of \( \mathbb{Z}_2 \) on both \( B_{\text{chain}}^G(F) \) and \( B_{\text{chain}}(G) \) is free hence, in light of Theorem 9 it is sufficient to show that \( \Phi \) is a homotopy equivalence. Again, the Quillen fiber theorem proves to be a very convenient tool. Given \( A \uplus B \in B_{\text{chain}}^G(F) \), let

\[
D_{A,B} := \Phi^{-1}(B_{\text{chain}}^G(F)_{A \uplus b}) := \{ a \uplus b \in B_{\text{chain}}^G(F) \mid \cup a \subseteq A \text{ and } \cup b \subseteq B \}.
\]

Note that both \( a' := \{ X \in F \mid X \subseteq A \} \) and \( b' := \{ Y \in F \mid Y \subseteq B \} \) are non-empty. Moreover, \( a' \uplus b' \) is the maximum element in \( D_{A,B} \), hence \( D_{A,B} \) is contractible. It immediately follows that \( \Phi \) is a homotopy equivalence and, a posteriori by Bredon’s theorem, \( \Phi \) is a \( \mathbb{Z}_2 \)-homotopy equivalence.
Proposition 25 Assume that $G = KG(F)$ is the Kneser graph associated to $F$ and let $B_0(G) \cong \mathcal{B}_{P(G)}(G)$ and $B_{Sark}^G(F)$ be the box complexes (posets) no. 2 and no. 6 from the list in Section 5 of [17]. Then the map

$$\Psi : B_0(G) \to B_{Sark}^G(F)$$

defined by $\Psi(a \uplus b) := (A \uplus B)$ is a $\mathbb{Z}_2$-homotopy equivalence.

Proof. The proof is similar to the proof of Proposition 24. If $A \uplus B \in B_{Sark}^G(F)$, then by definition at least one of the sets $A$ and $B$ contains an element $X \in F$ as a subset. If both $A$ and $B$ satisfy this condition then, as in the proof of Proposition 24, the set $D_{A,B} := \Psi^{-1}(B_{Sark}^G(F) \geq A \uplus B)$ has a maximum element and must be contractible.

Suppose that $A \supseteq X \in F$ but $B$ does not contain elements from $F$ as subsets. Let $c := \{Y \in F | Y \subseteq A\}$. Define $\mu : D_{A,B} \to D_{A,B}$ as the monotone map such that $\mu(x \uplus y) := (c \uplus y)$. Since always $x \uplus y \geq c \uplus y$, we conclude that $\text{Im}(\mu)$ is a deformation retract of $D_{A,B}$. On the other hand $\text{Im}(\mu)$ has the maximum element $c \uplus \emptyset$, hence it is contractible. This again allows us to use Quillen fiber theorem to conclude that $\Psi$ is a homotopy equivalence. Bredon’s theorem as before implies that $\Psi$ is actually a $\mathbb{Z}_2$-homotopy equivalence. ■

For completeness we formulate one more result involving the complex $B_{edge}(G)$, listed as no. 4 in the list in Section 5 of [17]. Recall that

$$B_{edge}(G) := \{F \subset A' \times A'' | \emptyset \neq A', A'' \subset V, A' \cap A'' = \emptyset, G[A', A''] \text{ is complete}\}.$$

Proposition 26 Let $G = KG(F)$ be the Kneser graph associated to a finite family of sets $F$. Let $\mathcal{B}(G) = B_{chain}(G)$ and $B_{edge}(G)$ be the associated box complexes (no. 3 and no. 4 from the list in Section 5 of [17]). Then the map

$$\Lambda : B_{edge}(G) \to B_{chain}(G)$$

defined by $\Lambda(F) := (A' \uplus A'')$ for $F \subset A' \times A''$, is a $\mathbb{Z}_2$-homotopy equivalence of $\mathbb{Z}_2$-posets.

Proof. The proof is similar in spirit to the proofs in this and earlier sections so the details are omitted. ■

We have convinced ourselves that all $\mathbb{Z}_2$-complexes

$$\mathcal{L}(P_G) \quad \mathcal{B}(P_G) \quad \mathcal{B}_{ex}(P_G) \quad B_{chain}(G) \quad B_{edge}(G)$$

have the same $\mathbb{Z}_2$-homotopy type.

Definition 27 Given a graph $G = (V_G, E_G)$, let $\Lambda(G)$ the common $\mathbb{Z}_2$-homotopy type of each of the complexes listed in (3). We occasionally, by a slight abuse of language, refer to $\Lambda(G)$ as to the graph complex associated to $G$. 

10
6 Which $\mathbb{Z}_2$-complexes are graph complexes?

Suppose that $\hat{L} = L \cup \{\hat{0}, \hat{1}\}$ is a finite lattice with $L$ as its proper part. Section 3. The proper part $L$ of $\hat{L}$ is a semilattice in the sense that each subset $A \subset L$, bounded from above, has a least upper bound, similarly each $B \subset L$ bounded from below has a greatest lower bound. Conversely, each semilattice $L$ is the proper part of the lattice $\hat{L} := L \cup \{0, 1\}$ where $\hat{0}$ and $\hat{1}$ are added minimum (maximum) elements. Assume that $(L, \leq)$ is a semilattice which is also an $I$-poset (Definition 11) with a monotone involution $C : L \to L$. Note that $\hat{L} = L \cup \{0, 1\}$ is also an $I$-poset where $C : \hat{L} \to \hat{L}$ is an extension the old involution, $C(\hat{0}) = 0, C(\hat{1}) = 1$.

If $L$ is free in the sense that $C$ satisfies an additional condition, $x \leq C(x) \Rightarrow x = \hat{0}$ or $x = \hat{1}$, or equivalently if the involution $C : L \to L$ is fixed-point-free, then we call $L$ a free $I$-semilattice. The ortholattices used by Walker, Section 4, are very similar to our free $I$-semilattices, the main difference being that the involution $C : L \to L$ in an ortholattice is antitone, rather than monotone. The condition $x \leq y \Rightarrow C(y) \leq C(x)$ implies that the “orthogonality relation”, $x \perp y \Leftrightarrow x \leq C(y)$, is symmetric which leads to an “orthogonality graph” $G^\perp = (V^\perp, E^\perp)$ associated to $L$ defined by $V^\perp = L$ and $(x, y) \in E^\perp \Leftrightarrow x \perp y$. The associated (neighborhood) graph complex (lattice) turns out to be closely related to the original ortholattice $L$ and among the consequences is the result that each ortholattice arises as the graph complex (lattice) of some graph.

Each free $I$-semilattice also can be associated a natural graph $G_L = (V_L, E_L)$ and our main objective in this section is to analyze its graph complex $\Lambda(G_L)$.

Definition 28 Suppose that $(L, \leq)$ is a free $I$-semilattice i.e. a semilattice which is a monotone $I$-poset with a fixed-point-free involution $C : L \to L$. Define the associated “compatibility graph” $G_L = (V_L, E_L)$ as the graph on the ground set $V_L := L$ such that $(x, y) \in E_L \Leftrightarrow y \leq C(x)$ or $x \leq C(y)$.

The “fat” semilattices or $F$-semilattices for short, are particularly well behaved and admit a short and transparent description of its “compatibility graph” $G_L$.

Definition 29 A semilattice $(L, \leq)$ is a $F$-semilattice if its intervals $[x, y]_L$ are “fat” in the sense that for each strict chain $x < z < y$ in $L$ there is an element $z' \in [x, y]_L$, incomparable to $z$.

Suppose from here on that $(L, \leq)$ is a free $I$-semilattice with “fat” intervals. Let $N(G_L)$ be the neighborhood complex of $G_L$ and $\mathcal{L}(G_L)$ the associated Lovász complex. By definition

$$N(\{x\}) = L_{\geq C(x)} \cup L_{\leq C(x)} = C(L_{\geq x} \cup L_{\leq x}) = C(\text{Comp}(\{x\}))$$

where $\text{Comp}(B)$ is the set of all elements in $L$ which are $\leq$-comparable with all elements $y \in B$. Let us observe that for each $A \subset L$, if $N(A) \neq \emptyset$ then there exists a chain $a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_k \leq b_k$ in $\hat{L}$ such that

$$N(A) = [a_1, b_1]_L \cup [a_2, b_2]_L \cup \ldots \cup [a_k, b_k]_L.$$  

(4)

Note that we allow elements in this chain to be $\hat{0}$ or $\hat{1}$, however the intervals are always taken in $L$ so for example $[\hat{0}, x]_L = L_{\leq x}$ and $[y, \hat{1}]_L = L_{\geq y}$. The observation follows
by an easy induction on the size of $A \subset L$. Indeed, if $A' = A \cup \{x'\}$, then $N(A') = N(A) \cap N(\{x'\}) = N(A) \cap (L_{\geq C(x')} \cup L_{\leq C(x')})$ and, if $N(A)$ admits a decomposition (4), it is easily checked that $N(A')$ also admits such a decomposition. Note that here we did not use the fact that $L$ has “fat” intervals. This hypothesis is essentially used in the proof of the following lemma.

**Lemma 30** If $N(A)$ admits the decomposition (4) then

$$N(N(A)) = [\mathbf{0}, C(a_1)]_L \cup [C(b_1), C(a_2)]_L \cup \ldots \cup [C(b_{k-1}), C(a_k)]_L \cup [C(b_k), \mathbf{1}].$$  \hspace{1cm} (5)

**Proof.** Let $\text{Comp}(N(A))$ be the set of all elements in $L$ comparable to all elements in $N(A)$. Since the intervals in $L$ are “fat”, we observe that

$$\text{Comp}(N(A)) = [\mathbf{0}, a_1]_L \cup [b_1, a_2]_L \cup \ldots \cup [b_{k-1}, a_k]_L \cup [b_k, \mathbf{1}]$$

and the Lemma is deduced from the fact that $N(N(A)) = C(\text{Comp}(N(A))$.

**Theorem 31** Suppose that $(L, \leq)$ is a free $F$-semilattice with “fat” intervals, $F$-semilattices in the sense of Definition 24 and let $G_L = (V_L, E_L)$ be the associated “compatibility graph”. Then the graph complex $\Lambda(G_L)$ (Definition 27) of $G_L$ is $\mathbb{Z}_2$-homotopy equivalent to the order complex $\Delta(L)$.

**Proof.** Let $\hat{L}(G_L) = \{ \diamond \subset L \mid N(N(\diamond)) = \diamond \}$ be the Lovász lattice and $\mathcal{L}(G_L) = \hat{L}(G_L) \setminus \{ \mathbf{0}, \mathbf{1} \}$ its proper part. We already know that $\diamond \in \mathcal{L}(G_L)$ if and only if $L \neq \diamond \neq \emptyset$ and $\diamond = N(A)$ admits a decomposition into a union of $L$-intervals, described in (4).

The poset $\mathcal{L}(G_L)$ is ordered by the reversed inclusion, i.e. $\diamond_1 \leq \diamond_2 \iff \diamond_1 \supseteq \diamond_2$. Our objective is to compare the semilattice $\mathcal{L}(G_L)$ and the original semilattice $L$. Let $\text{Chain}(L) = \Phi(\Delta(L))$ be the chain poset associated to $(L, \leq)$, Definition 8. By Proposition 18 posets $L$ and $\text{Chain}(L)$ are $\mathbb{Z}_2$-homotopy equivalent. Define the map $\Omega : \mathcal{L}(G_L) \to \text{Chain}(L)$ by the formula

$$\Omega([a_1, b_1]_L \cup [a_2, b_2]_L \cup \ldots \cup [a_k, b_k]_L) = (a_1 \leq b_1 \leq \ldots \leq a_k \leq b_k) \in \text{Chain}(L).$$

Here as before, some of the elements are allowed to be $\mathbf{0}$ or $\mathbf{1}$ but in the chain itself they are neglected. Alternatively, one can agree that each chain in $L$ is enriched by elements $\mathbf{0}$ and $\mathbf{1}$. The map $\Omega$ is $\mathbb{Z}_2$-equivariant in the sense that for each $\diamond \in \mathcal{L}(G_L)$, $C\Omega(\diamond) = \Omega C(\diamond)$. Unfortunately the map $\Omega$ is not monotone (antitone). This is not a surprise since $\mathcal{L}(G_L)$ is an antitone while $\text{Chain}(L)$ is a monotone $I$-poset, hence there does not exists a $\mathbb{Z}_2$-equivariant monotone (antitone) map of these posets. In order get around this difficulty we pass to the $I$-poset $\text{Chain}(\mathcal{L}(G_L))$ which is a monotone $I$-poset and which, according to Proposition 18, retains the $\mathbb{Z}_2$-homotopy type of the poset $\mathcal{L}(G_L)$. The map $\Omega$ can be extended to a $\mathbb{Z}_2$-equivariant, monotone map $\Omega^I : \text{Chain}(\mathcal{L}(G_L)) \to \text{Chain}(L)$ of posets as follows. Given a chain $\diamond_1 \leq \diamond_2 \leq \ldots \leq \diamond_k$ in $\mathcal{L}(G_L)$, the associated elements $\Omega(\diamond_1), \Omega(\diamond_2), \ldots, \Omega(\diamond_k)$ are not necessarily elements of a chain in $\text{Chain}(L)$. The obstacle is that they may not be comparable. However, their union is a well defined chain in $L$ so by definition

$$\Omega^I((\diamond_j)_{j=1}^k) := \bigcup_{j=1}^k \Omega(\diamond_j).$$
We claim that $\Omega^2$ is a $\mathbb{Z}_2$-homotopy equivalence. By Theorem 9 it is sufficient to show that $\Omega^2$ is an ordinary homotopy equivalence. As before, the Quillen fiber theorem is a convenient tool. Given a chain $\Gamma = (c_1 \leq c_2 \leq \ldots \leq c_k) \in \text{Chain}(L)$, our objective is to show that $D_\Gamma := (\Omega^2)^{-1}(\text{Chain}(L)_{\geq \Gamma})$ is a contractible subposet of $\text{Chain}(\mathcal{L}(G_L))$.

Note that $D_\Gamma$ is itself a chain poset, $D_\Gamma = \text{Chain}(E_\Gamma)$. Indeed, $E_\Gamma$ is characterized by the condition $\Diamond \in E_\Gamma$ if and only if $\Omega(\Diamond)$ is a subchain of $\Gamma$. Since $D_\Gamma$ and $E_\Gamma$ have the same homotopy type, it is sufficient to show that $E_\Gamma$ is contractible. One way to establish this fact is to observe that $E_\Gamma$ is a semilattice and that $\Diamond_0 := [c_1, c_k]$ is an element in $E_\Gamma$ with an empty set of complements, cf. 8 Theorem 10.15. One can also note that $E_\Gamma$ is isomorphic to the poset $F_\Gamma$ where $I \in F_\Gamma$ if $I = \Diamond \cap \Gamma$ for some $\Diamond \in E_\Gamma$. In other words elements of $F_\Gamma$ are unions of intervals in $\Gamma$. So there are alternative proofs that $D_\Gamma \cong F_\Gamma$ is contractible, for example one can rely on the Order homotopy theorem, Proposition 12. ■

As a consequence of Theorem 81 we obtain the following result answering the question from the title to this section. Almost at the same time, actually a few days earlier, this result was announced by Péter Csorba, 9.

**Theorem 32** (9) For each finite, free $\mathbb{Z}_2$-complex $K$ there exists a graph $G$ such that the associated graph complex $\Lambda(G)$ is $\mathbb{Z}_2$-homotopy equivalent to $K$.

**Proof.** The result is a consequence of Theorem 81 since obviously the face semilattice $\Phi(K)$ of $K$ is a free $I$-semilattice with “fat” intervals. ■

## 7 Complexes $\text{Hom}(G, H)$ and the Lovász conjecture

The notion of a box poset associated to a $WI$-poset $(P, C)$, Definition 5, admits several generalizations in different directions. Here is one of the possibilities which relates this construction to complexes $\text{Hom}(G, H)$. Recall that these objects were introduced by L. Lovász whose well known conjecture about chromatic numbers of graphs $G$ with $k$-connected complexes $\text{Hom}(C_{2r-1}, G)$ was recently confirmed by Bobson and Kozlov in [4].

**Definition 33** Suppose that $G = (V_G, E_G)$ is a graph on the ground set $[n]$, $V_G \subseteq [n]$. The $G$-box poset $G\mathfrak{B}(P)$ associated to a $WI$-poset $(P, C)$ is a subposet of $P^n$ defined by

$$G\mathfrak{B}(P) := \{(x_1, \ldots, x_n) \in P^n \mid (\forall i \neq j) \{i, j\} \in E_G \Rightarrow x_i \leq C(x_j) \& x_j \leq C(x_i)\}.$$ 

If $G = K_2$ is the complete graph on two vertices, the $G$-box poset $G\mathfrak{B}(P)$ reduces to the box poset $\mathfrak{B}(P)$ from Section 1. More importantly, if $P = P_H = \{B \subset V_H \mid C N(B) \neq \emptyset\}$ is the $WI$-poset associated to a graph $H = (V_H, E_H)$, then $G\mathfrak{B}(P)$ is the face poset associated to the polyhedral complex $\text{Hom}(G, H)$, 3 4. The fact that $\text{Hom}(K_2, G)$ is one of avatars of the graph complex $\Lambda(G)$, Definition 27, is already an indication of the importance of the complex $\text{Hom}(G, H)$. Lovász conjectured that if $\text{Hom}(C_{2r+1}, G)$ is $k$-connected for some $r \geq 1$, where $C_d$ is the $d$-cycle, then $\chi(G) \geq k + 4$. This conjecture
was recently proved by Bobson and Kozlov, [4]. The proof is reasonably long and quite intricate involving a variety of different techniques. In particular it required a detailed combinatorial and homological analysis of polyhedral complexes $\text{Hom}(G, K_n)$ with a special emphasis on the complex $\text{Hom}(C_{2r+1}, K_n)$, [3, 4]. Having in mind that the existence of different models for the graph complex $\Lambda(G)$ makes them more accessible, it is interesting to ask if $\text{Hom}(G, H)$, and in particular the complex $\text{Hom}(C_{2r+1}, G)$, also have different incarnations. Even if the answer is negative, it may be of some interest to establish a “hierarchy theorem” in the spirit of Theorem 1 in [17].

Let us start with the observation that most of the complexes and posets from Sections 3–5 do have their analogs in the broader context of $\text{Hom}(G, H)$ complexes. We will not attempt to give a complete analysis here. Instead, we select some model cases and give examples which illuminate potential use of these more general objects. For example the poset $G \mathcal{B}_{\text{ex}}(P)$ is a relative of $G \mathcal{B}(P)$ obtained if in the Definition 33 we allow some, but not all entries in the vector $(x_1, \ldots, x_n)$ to be equal to an added new minimum element $\emptyset$. The following definition is just a repetition of the definition of $G \mathcal{B}_{\text{ex}}(P)$ in the case of the WI-poset $P = P_H$. The notation emphasizes the fact that the new complexes are relatives of the poset (complex) $\text{Hom}(G, H)$.

**Definition 34** The extended $\text{Hom}$-poset $\text{Hom}_{\text{ex}}(G, H)$ is a poset whose elements are all functions $\phi : V_G \to 2^{V_H}$, such that $\phi(i) \neq \emptyset$ for some $i \in V_G$, for each edge $\{i, j\} \in E_G$, $\phi(i) \cap \phi(j) = \emptyset$ and $\forall x \in \phi(i) \forall y \in \phi(j) \{x, y\} \in E_H$.

The reader familiar with [3] will notice right away that our $\text{Hom}_{\text{ex}}(G, H)$ is nothing but the complex $\text{Hom}_{\text{ex}}(G, H)$ which plays a very important role in the analysis leading eventually to the proof of Lovász conjecture! Note that $\text{Hom}(G, H)$ is a subposet of $\text{Hom}_{\text{ex}}(G, H)$ and, as a consequence of the analysis from [4], one cannot expect that these two complexes are homotopy equivalent in general. Note also that in the case of a complete graph $H = K_n$, $\phi \in \text{Hom}_{\text{ex}}(G, K_n)$ iff $\phi(i) \neq \emptyset$ for some $i$ and $\phi(i) \cap \phi(j) = \emptyset$ for each edge $\{i, j\} \in E_G$. In this case $\text{Hom}_{\text{ex}}(G, H)$ can be seen as a subposet (subcomplex) of a join $(\Delta^{n-1})^{\times V_G} = \Delta^{n-1} \ast \ldots \ast \Delta^{n-1}$ of $|V_G|$-copies of the $(n-1)$-simplex $\Delta^{n-1}$ spanned by vertices of the graph $K_n$. In order to simplify notation, from here on we assume that $V_G = [m], V_K = [n]$ and to each function $\phi \in \text{Hom}_{\text{ex}}(G, K_n)$ we associate its “graph” $\Gamma(\phi) \subset [m] \times [n]$, where $\Gamma(\phi) \cap \{(i) \times [n]\} = \{i\} \times \phi(i)$. In this notation, $\text{Hom}_{\text{ex}}(G, K_n) \subset (\Delta^{n-1})^{\times [m]}$. More importantly, the condition $\phi(i) \cap \phi(j) = \emptyset$ for each edge $\{i, j\} \in E_G$ indicates that $\text{Hom}_{\text{ex}}(G, K_n)$ is, as a simplicial complex, a $G$-deleted join of simplices $\Delta^{n-1}$ in the sense of the following definition.

**Definition 35** Suppose that $G$ is a graph on $[n]$ as a ground set, $V_G \subset [n]$. Let $\{K_i\}_{i=1}^n$ be a collection of $n$-copies of a simplicial complex $K$. Then the $G$-deleted join of $K$ is the simplicial subcomplex $K_G^{\star n}$ of $K \ast \ldots \ast K = K^{\star n}$, where $\theta_1 \ast \ldots \ast \theta_n \in K_G^{\star n}$ iff $\theta_i \cap \theta_j = \emptyset$ for each edge $\{i, j\} \in E_G$.

If $G = K_m$ is a complete graph then $G$-deleted join $K_G^{\star m} = K_G \Delta^{m}$ reduces to the usual deleted join operation of simplicial complexes, [16] Section 5.5. [21], [27]. The well known relation $(K \ast L)^* \Delta^k \approx K^{*k} \ast L^{*k} \Delta^k$ easily generalizes to the following result
Lemma 36 Suppose that $K$ and $L$ are simplicial complexes and let $G = (V_G, E_G)$ be a graph on the ground set $[m]$, $V_G \subset [m]$. Then,

$$(K \ast L)^{\ast m}_G \cong K^{\ast m}_G + L^{\ast m}_G.$$ 

An immediate consequence of Lemma 36 is the relation

$$Hom_{ex}(G, K_n) = (\Delta^{n-1})^{\ast m}_G \cong ((pt)^{\ast m}_G)^{\ast m}_G \cong ((pt)^{\ast m}_G)^{\ast m}. \tag{6}$$

The complex $(pt)^{\ast m}_G$ is well known as the complex $Ind(G)$ of all independent sets in a graph $G$. Hence the equation (6) is nothing but (a half of) the Proposition 3.2. from Bibliography 1 in disguise. This shows that the study of complexes $Hom_{ex}(C_m, K_n)$ is reduced to the study of complexes $Ind(C_n)$, which is the subject of our next section.

8  $Ind(L_n)$ and $Ind(C_n)$ as $\mathbb{Z}_2$-complexes

Let us denote by $\mathbb{1}$ and $\varepsilon$ respectively the trivial and nontrivial real representations of $\mathbb{Z}_2$. Given an Euclidean vector space $V$, let $S(V)$ be the associated unit sphere. If $V$ is an orthogonal representation of $\mathbb{Z}_2$, the sphere $S(V)$ is a $\mathbb{Z}_2$-space. For example $S(\mathbb{1})$ and $S(\varepsilon)$ are both 2-element sets, the first with trivial and the second with non-trivial action of $\mathbb{Z}_2$. Recall the well known fact that $S(U \oplus V) \cong S(U) \ast S(V)$. For example if $V = p \mathbb{1} \oplus q \varepsilon$ then $S(V)$ is the sphere in $\mathbb{R}^{p+q}$ equipped with the action of $\mathbb{Z}_2 = \{1, \omega\}$ such that $\omega(x_1, \ldots, x_p, y_1, \ldots, y_q) = \omega(x_1, \ldots, x_p, -y_1, \ldots, -y_q)$.

Definition 37 Define graphs $L_n$ and $C_n$ on $[n] = \{1, \ldots, n\}$ as the ground set by the conditions

$$(i, j) \in E_{L_n} \iff |i - j| = 1 \quad \text{and} \quad (i, j) \in E_{C_n} \iff |i - j| = 1 \pmod{n}.$$ 

Given an interval $[p, q]$ in $[n]$, let $L_{[p, q]} \cong L_{q-p+1}$ be the complete subgraph of $L_n$ on $[p, q]$ as the set of vertices. Define $\mathbb{Z}_2$-actions on both $L_n$ and $C_n$ by the involution $\omega : [n] \to [n]$ which sends $i$ to $n - i + 1$. Let $Ind(L_n)$ and $Ind(C_n)$ be the associated complexes of independent sets with inherited $\mathbb{Z}_2$-actions.

Homotopy types of spaces $Ind(L_n)$ and $Ind(C_n)$ were determined in Bibliography 13. The question of finding the associated $\mathbb{Z}_2$-homotopy types appeared as a natural step in the approach of Bobson and Kozlov to the solution of Lovász conjecture, notably in the evaluation of the height of the first Stiefel-Whitney class of the $\mathbb{Z}_2$-complex $Hom(C_{2r+1}, K_n)$, Bibliography 4 Sections 2.2 and 4.1. Their methods permitted them to evaluate only the homotopy types of the associated orbit spaces $Ind(C_n)/\mathbb{Z}_2$ but this turned out to be sufficient for the intended application.

In this section we strengthen this result of Bobson and Kozlov by demonstrating how the $\mathbb{Z}_2$-homotopy types of these complexes can be determined, again relying on the Bredon’s theorem. We restrict ourselves to the analysis of two important special cases. In the other cases, corresponding to other values of $n$, the proofs are similar in spirit and rely on similar ideas.
Proposition 38 \([3]\) Assume that \(n = 6p - 1\) and let \(S(\|)\) and \(S(\varepsilon)\) be \(2\)-element sets (0-dimensional spheres) respectively with trivial and non-trivial action of \(\mathbb{Z}_2\). Then \(\text{Ind}(L_n)\) is a \(\mathbb{Z}_2\)-complex which is \(\mathbb{Z}_2\)-homotopy equivalent to the join of \(p\) copies of \(S(\|)\) and \(p\) copies of \(S(\varepsilon)\),

\[
\text{Ind}(L_{6p-1}) \simeq_{\mathbb{Z}_2} S(\|)^p \ast S(\varepsilon)^p.
\]  

Before we commence the proof of the proposition let us introduce some auxiliary definitions and useful lemmas. Given a graph \(G = (V_G, E_G)\) and a subset \(K \subset V_G\), define \(G \setminus K\) as the graph obtained from \(G\) by removing \(K\) and all edges incident to vertices in \(K\). For example \(G \setminus v\) is obtained from \(G\) by removing a vertex \(v\), while \(G \setminus \text{St}(v)\) is the graph obtained from \(G\) if \(K = \text{St}(v)\) is the star of \(v\), \(\text{St}(v) = \{v\} \cup \{w \in V_G \mid (v, w) \in E_G\}\).

Lemma 39 There is a decomposition \(\text{Ind}(G) = X \cup Y\) where \(X = \text{Ind}(G \setminus v)\) and \(Y = \{v\} \ast \text{Ind}(G \setminus \text{St}(v))\) where \(X \cap Y = \text{Ind}(G \setminus \text{St}(v))\).

Lemma 40 Suppose that \(a, b, v \in V_G\) are three distinct vertices in a graph \(G = (V_G, E_G)\) such that both \((a, b) \in E_G\) and \((b, v) \in E_G\). Moreover we assume that \(a\) is not connected with any other vertex in \(G\), i.e. \(\deg(a) = 1\). Then \(\text{Ind}(G) \simeq \text{Ind}(G \setminus v)\).

Proof. Note that \(Y\) in the decomposition \(\text{Ind}(G) = X \cup Y\) in Lemma 39 is contractible, being a cone with vertex \(v\). The space \(X \cap Y\) is also a cone since by assumption \(G \setminus \text{St}(v)\) has an isolated vertex \(a\). The proof is completed by invoking an easily established fact that if both \(Y\) and \(X \cap Y\) are contractible complexes then \(X \cup Y \simeq X\).

Proof of Proposition 38. By successive applications of Lemma 39 we are able to remove all vertices from the set \(K = \{3, 6, \ldots, 3p, \ldots, 6p - 3\}\) without changing the homotopy type of \(\text{Ind}(L_{6p-1})\). In other words, \(\text{Ind}(L_{6p-1}) \simeq \text{Ind}(L_{6p-1} \setminus K)\). Let us show that the inclusion map \(e : \text{Ind}(L_{6p-1} \setminus K) \to \text{Ind}(L_{6p-1})\) is actually a \(\mathbb{Z}_2\)-homotopy equivalence. By Theorem 3 it is sufficient to show that the inclusion map \(e^{\mathbb{Z}_2} : \text{Ind}(L_{6p-1} \setminus K)^{\mathbb{Z}_2} \to \text{Ind}(L_{6p-1})^{\mathbb{Z}_2}\) of the associated spaces of fixed points is also a homotopy equivalence. Both \(\text{Ind}(L_{6p-1})\) and \(\text{Ind}(L_{6p-1} \setminus K)\) are subcomplexes of the simplex \(\Sigma\) spanned by vertices \(1, \ldots, n\). Identifying \(\Sigma\) with its geometric realization \([\Sigma]\), assume that vertices of \(\Sigma\) are points \(v_1, \ldots, v_n\) in some vector space \(V\). Note that \(\Sigma\) is also a \(\mathbb{Z}_2\)-space with the linear action which is on vertices defined by \(\omega(v_i) = v_{n+1-i}\). It is not difficult to check that

\[
x \in \Sigma^{\mathbb{Z}_2} \iff x = t_1 \frac{v_1 + v_{6p-1}}{2} + \ldots + t_{3p-1} \frac{v_{3p-1} + v_{3p+1}}{2} + t_{3p} x_{3p}
\]

where \(t_j \geq 0\) and \(\Sigma t_j = 1\). We conclude that \(\Sigma^{\mathbb{Z}_2}\) is a simplex isomorphic to the face \(\Sigma_1\) of \(\Sigma\) spanned by the vertices \(\{v_1, v_2, \ldots, v_{3p}\}\), where the isomorphism \(I : \Sigma_1 \to \Sigma\) is the linear extension of the map \(v_i \mapsto (v_i + v_{6p-1})/2\). The fixed point spaces \(\text{Ind}(L_{6p-1} \setminus K)^{\mathbb{Z}_2}\) and \(\text{Ind}(L_{6p-1})^{\mathbb{Z}_2}\) are subspaces of \(\Sigma^{\mathbb{Z}_2}\) which can be viewed, via isomorphism \(I\), as subspaces of \(\Sigma_1\). It immediately follows that \(\text{Ind}(L_{6p-1})^{\mathbb{Z}_2} \simeq \text{Ind}(L_{[1,3p]}) \simeq \text{Ind}(L_{3p})\) and \(\text{Ind}(L_{6p-1} \setminus K)^{\mathbb{Z}_2} \simeq \text{Ind}(L_{[1,3p]} \setminus K')\) where \(K' = \{3, 6, \ldots, 3p\}\). Again, by applications of Lemma 40 and successive removal of vertices in \(K'\), we conclude that the inclusion map \(\text{Ind}(L_{[1,3p]} \setminus K') \hookrightarrow \text{Ind}(L_{[1,3p]})\) is a homotopy equivalence, hence

\[\text{Ind}(L_{6p-1}) \simeq_{\mathbb{Z}_2} S(\|)^p \ast S(\varepsilon)^p.\]
and the result follows. Proof. By assumption 
\(K_\sigma\) is maximal in \(K\). In other words, \(\sigma\) is a generating simplex for \(K\). Lemma 41 provides a criterion for \(K\) to be contractible: if \(\sigma\) is maximal in \(K\), then \(K\) is contractible. This, together with the fact that \(Ind(L_{6p-1})\) is isomorphic to the following join of circles, 
\[
Ind(L_{6p-1} \setminus \sigma) \cong Ind(L[\{1,2\} \cup L[6p-2,6p-1]) * \ldots * Ind(L[3p-2,3p-1] \cup L[3p+1,3p+2]).
\]
This, together with the fact that 
\[
Ind(L_{[j,j+1]} \cup L[6p-j,6p-j]) \cong S(\mathbb{I}) * S(\varepsilon)
\]
finally completes the proof of Proposition 38. 

Let \(K\) be a finite simplicial complex and assume that \(\sigma \in K\) is a simplex which is maximal in the sense that it is not a proper face of any other simplex \(\tau \in K\). Let \(\dim(\sigma) = k\). If \(K' := K \setminus \{\sigma\}\) then the geometric realization of \(K'\) is obtained from the geometric realization of \(K\) by removing the interior \(\sigma\) of \(\sigma\), \(|K'| = |K| \setminus \sigma\). If \(K'\) is contractible then \(K \cong K'/K' \cong \sigma/\partial \sigma \cong S^k\). In this case we call \(\sigma\) a generating simplex of \(K\). Of course, it is not true that a complex homotopy equivalent to a sphere must have a generating simplex. For example \(S^{k-1} \times I\) is a pure \(k\)-dimensional complex homotopic to \(S^{k-1}\) which consequently cannot have a \((k-1)\)-dimensional generating simplex. The following lemma gives a sufficient condition for the existence of generating simplices.

**Lemma 41** Let \(K\) be a finite simplicial complex and assume that \(L \subset K\) is a subcomplex of \(K\) simplicially isomorphic to a triangulation of a \(k\)-sphere \(S^k\). Assume that the inclusion map \(e : L \to K\) is a homotopy equivalence and let \(\sigma \in L\) be a \(k\)-simplex which is maximal in \(K\), i.e. such that \(\sigma\) is not a proper face of a simplex \(\tau \in K\). Then \(\sigma\) is a generating simplex for \(K\) in the sense that the complex \(K' = K \setminus \{\sigma\}\) is contractible and \(K/K' \cong \sigma/\partial \sigma \cong S^k\).

**Proof.** By assumption \(L\) is a weak deformation retract of \(K\) hence a strong deformation retract, Section 1.4. Since \(\sigma\) is maximal in \(K\) we observe that \(L \setminus \{\sigma\}\) is a strong deformation retract of \(K \setminus \{\sigma\}\). Since \(L \setminus \{\sigma\}\) is contractible, \(K \setminus \{\sigma\}\) is also contractible and the result follows.

**Example 42** The proof of Proposition 38 reveals that the complexes \(Ind(L_{6p-1})\) and \(L = Ind(L_{6p-1} \setminus K)\) satisfy the conditions of Lemma 41. A simplex \(\sigma\) in \(Ind(L_{6p-1} \setminus K)\) is maximal in \(Ind(L_{6p-1})\) if and only if there are at most two vertices from the ground set \([6p-1]\) separating two consecutive vertices in \(\sigma\). Hence an example of a generating simplex is 
\[
\tau = \{2, 5, \ldots, 3p - 4, 3p - 1, 3p + 1, 3p + 4, \ldots, 6p - 5, 6p - 2\}.
\]

**Proposition 43** The complex \(Ind(C_{6p-1})\) is \(\mathbb{Z}_2\)-homotopy equivalent to the sphere \(S^{2p-1} \subset \mathbb{R}^{2p}\) with the action of \(\mathbb{Z}_2 = \{1, \omega\}\) given by the formula 
\[
\omega(x_1, \ldots, x_p, y_1, \ldots, y_p) = (x_1, \ldots, x_p, -y_1, \ldots, -y_p).
\]
In other words, 
\[
Ind(C_{6p-1}) \cong_{\mathbb{Z}_2} S(\mathbb{I})^p * S(\varepsilon)^p.
\]

(9)
**Proof.** It was shown in [13] that $\text{Ind}(C_{6p-1}) \simeq S^{2p-1}$. Moreover it was shown that the simplex

$$\sigma = \{2, 5, \ldots, 3p - 4, 3p - 1, 3p + 1, 3p + 4, \ldots, 6p - 5, 6p - 2\}$$

is a generating simplex for the complex $\text{Ind}(C_{6p-1})$. The reader is invited to prove this fact along the lines of proofs of Proposition 38 and Example 42. As a consequence we know that $\text{Ind}(C_{6p-1}) \setminus \sigma$ is contractible. Let us note that $\sigma$ is $\mathbb{Z}_2$-invariant with respect to the $\mathbb{Z}_2$-action on $\text{Ind}(C_{6p-1})$ which, as we recall, arises from the involution $\omega : [n] \to [n], \omega(j) := n + 1 - j$. It follows that $\sigma / \partial \sigma \cong S^k$ is a $\mathbb{Z}_2$-space and there is an obvious $\mathbb{Z}_2$-equivariant collapsing map $f : \text{Ind}(C_{6p-1}) \to \sigma / \partial \sigma$. Let us show that this map is a $\mathbb{Z}_2$-homotopy equivalence. Since $f$ is a homotopy equivalence, by Theorem 9 it is sufficient to show that $f^{\mathbb{Z}_2} : \text{Ind}(C_{6p-1})^{\mathbb{Z}_2} \to (\sigma / \partial \sigma)^{\mathbb{Z}_2}$ is a homotopy equivalence. We follow the same strategy as in the proof of Proposition 38 in particular we use the map $I$ to relate the fixed point sets to (subspaces) of independence complexes. For example, as in the proof of Proposition 38 $\text{Ind}(C_{6p-1})^{\mathbb{Z}_2}$ is isomorphic to the complex $\text{Ind}(L_{[1,3p]})$. Similarly, $(\sigma / \partial \sigma)^{\mathbb{Z}_2}$ is isomorphic to the space $\sigma_0 / \partial \sigma_0$, where $\sigma_0$ is the simplex in $\text{Ind}(L_{[1,3p]})$ spanned by vertices $\{2, 5, \ldots, 3p - 4, 3p - 1\}$. Note that $\sigma_0$ is a maximal simplex in $\text{Ind}(L_{[1,3p]})$. Moreover $\sigma_0$ is a $(p-1)$-dimensional simplex in the $(p-1)$-sphere

$$S^{p-1} \cong \text{Ind}(L_{[1,2]}) \ast \ldots \ast \text{Ind}(L_{[3p-2,3p-1]})$$

which is a deformation retract of $\text{Ind}(L_{[1,3p]})$. Hence $\sigma_0$ is a generating simplex in the complex $\text{Ind}(L_{[1,3p]})$ which shows that the collapsing map $f_0 : \text{Ind}(L_{[1,3p]}) \to \sigma_0 / \partial \sigma_0$ is a homotopy equivalence. This in turn implies that $f^{\mathbb{Z}_2} : \text{Ind}(C_{6p-1})^{\mathbb{Z}_2} \to (\sigma / \partial \sigma)^{\mathbb{Z}_2}$ is a homotopy equivalence and by Bredon’s theorem $\text{Ind}(C_{6p-1})$ is $\mathbb{Z}_2$-homotopy equivalent to the $\mathbb{Z}_2$-space $\sigma / \partial \sigma$.

In order to determine the $\mathbb{Z}_2$-structure of the $\mathbb{Z}_2$-space $\sigma / \partial \sigma$, note that it was already done in the proof of Proposition 38. Indeed, the simplex $\sigma$ was shown there to be a generating simplex of the complex $\text{Ind}(L_{6p-1} \setminus K)$ and the collapsing map $\text{Ind}(L_{6p-1} \setminus K) \to \sigma / \partial \sigma$ is a $\mathbb{Z}_2$-homotopy equivalence, again by an applications of Bredon’s theorem. This finally establishes the decomposition \(\square\).

**Remark 44** The fact that both $\text{Ind}(L_{6p-1})$ and $\text{Ind}(C_{6p-1})$ have identical $\mathbb{Z}_2$ decompositions, Propositions 38 and 43, is not an accident. Given a graph $G = (V_G, E_G)$ and an edge $e = (u, v) \in E_G$, define $G \setminus e$ and $G \setminus St(e)$ as the graphs $G \setminus e = (V_G, E_G \setminus \{e\})$ and $G \setminus St(e) = G \setminus \{u, v\}$. Then there is a decomposition, cf. [13], $\text{Ind}(G \setminus e) = \text{Ind}(G) \cup \{u, v\} \ast \text{Ind}(G \setminus St(e))$ where $\text{Ind}(G) \cap \{u, v\} \ast \text{Ind}(G \setminus St(e)) \cong \text{Ind}(G \setminus St(e))$. If $G = C_{6p-1}$ and $e = (1, 6p - 1)$, then

$$\text{Ind}(G \setminus St(e)) = \text{Ind}([6p-1] \setminus \{1, 2, 6p-2, 6p-1\}) \cong \text{Ind}(L_{6p-5})$$

is contractible. It follows, along the lines of the proof of Lemma 44, that the natural inclusion map $e : \text{Ind}(C_{6p-1}) \to \text{Ind}(L_{6p-1})$ is a homotopy equivalence. The map $e$ is $\mathbb{Z}_2$-equivariant and a repetition of the argument already used in the proofs of Propositions 38 and 43 allows us to conclude that $e$ is a $\mathbb{Z}_2$-homotopy equivalence.
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