Universal angular probability distribution of three particles near zero-energy threshold

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Abstract

We study bound states of a three-particle system in $\mathbb{R}^3$ described by the Hamiltonian $H(\lambda_n) = H_0 + v_{12} + \lambda_n(v_{13} + v_{23})$, where the particle pair $\{1, 2\}$ has a zero-energy resonance and no bound states, while other particle pairs have neither bound states nor zero-energy resonances. It is assumed that for a converging sequence of coupling constants $\lambda_n \to \lambda_{cr}$ the Hamiltonian $H(\lambda_n)$ has a sequence of levels with negative energies $E_n$ and wavefunctions $\psi_n$, where the sequence $\psi_n$ totally spreads in the sense that $\lim_{n \to \infty} \int_{|\xi| \leq R} |\psi_n(\xi)|^2 d\xi = 0$ for all $R > 0$. We prove that for large $n$ the angular probability distribution of three particles determined by $\psi_n$ approaches the universal analytical expression, which does not depend on pair-interactions. The result has applications in Efimov physics and in the physics of halo nuclei.

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1. Introduction

Consider the Hamiltonian of the three-particle system in $\mathbb{R}^3$

$$ H(\lambda) = H_0 + v_{12} + \lambda(v_{13} + v_{23}), \quad (1) $$

where $H_0$ is the kinetic energy operator with the center of mass removed, $\lambda > 0$ is the coupling constant and none of the particle pairs has negative energy bound states. The detailed requirements on pair-potentials will be listed in section 3. Suppose that for a converging sequence of coupling constants $\lambda_n \to \lambda_{cr}$ there exists a sequence of bound states $\psi_n \in D(H_0)$ such that $H(\lambda_n)\psi_n = E_n\psi_n$, where $E_n < 0$, $\|\psi_n\| = 1$ and $E_n \to 0$. The question whether the sequence $\psi_n$ totally spreads has been recently considered in [1, 2]. In [1] it was shown that $\psi_n$ does not spread if the two-particle subsystems of $H(\lambda_n)$, $H(\lambda_{cr})$ have no zero-energy resonances. The results of [1] were generalized to many-particle systems [2], where, in particular, the restriction on the sign of pair-potentials was removed. In [1] under certain conditions on pair-potentials it was proved that if the pair of particles $\{1, 2\}$ has a zero-energy resonance and $\psi_n$ for each $n$ is the ground state then the sequence $\psi_n$ totally spreads.
Here we focus again on the situation where the pair of particles \{1, 2\} has a zero-energy resonance and the sequence \(\psi_n(x, y)\) \((not\ necessarily\ ground\ states!)\) totally spreads. \(\text{(For the definition of Jacobi coordinates } x, y \in \mathbb{R}^3\) see \[1\] or section 3 of this paper.) Recall that by definition in \[1\] the total spreading means that

\[
\lim_{n \to \infty} \int_{|x| + |y| \leq R} |\psi_n(x, y)|^2 \, d^3x \, d^3y \to 0 \quad \text{for all } R > 0.
\]  

(2)

Thereby, the angular probability distribution of three particles for large \(n\) is especially interesting, which we define below. Let us rewrite the wavefunction in the form \(\psi_n(\rho, \theta, \hat{x}, \hat{y})\), where the arguments are the so-called hyperspherical coordinates \[3\]

\[
\rho := \sqrt{|x^2| + |y|^2}, \quad \theta := \arctan(|y|/|x|), \quad \theta \in [0, \pi/2]\text{ and } \hat{x}, \hat{y}\text{ are unit vectors in the directions of } x, y, \text{ respectively.}
\]

Then by definition the angular probability distribution is

\[
D_n(\theta, \hat{x}, \hat{y}) := \cos^2 \theta \sin^2 \theta \int \rho^5 \, |\psi_n(\rho, \theta, \hat{x}, \hat{y})|^2 \, d\rho.
\]  

(3)

The normalization \(\|\psi_n\| = 1\) implies that

\[
\int_0^{\pi/2} d\theta \int d\Omega_x \, D_n(\theta, \hat{x}, \hat{y}) = 1,
\]  

(4)

where \(\Omega_{x,y}\) are the body angles associated with the unit vectors \(\hat{x}, \hat{y}\). The main result of this paper (proved in theorem \[3\]) states that

\[
D_\infty(\theta, \hat{x}, \hat{y}) := \lim_{n \to \infty} D_n(\theta, \hat{x}, \hat{y}) = \frac{1}{4\pi^3} \sin^2 \theta,
\]  

(5)

where the convergence is in measure. Equation (5) means that all acceptable pair-potentials produce the same limiting angular probability distribution, which depends solely on \(\theta\). This is another example of the so-called universality in three-particle systems, which, in particular, manifested in the universal asymptotic form of the infinite discrete spectrum appearing the Efimov effect \[4, 5\]. Apart from the results in \[1, 2\] the proof resides on the ideas expressed in \[4–7\]. In the next section we shall discuss the two-particle case; this material will also be needed in the analysis of the three-particle case in section 3. At the end of section 3 we show how the distribution in (5) can be derived fairly easy on a physical level of rigor (this derivation was proposed by one of the referees). In section 4 we discuss physical applications.

2. The two-particle case revisited

Let us consider the two-particle Hamiltonian in \(L^2(\mathbb{R}^3)\)

\[
h(\lambda) = -\Delta_x + \lambda v(x),
\]  

(6)

where \(\lambda > 0\) is a coupling constant. For the pair potential we assume that

\[
\gamma := \max \left[ \int d^3x \left( |x|^2 \right)^2 \int d^3x \left( 1 + |x|^2 \right)^2 v(x) \right] < \infty,
\]  

(7)

where \(0 < \delta < 1\) is some constant.

The next theorem (which must be known in some form) states that a totally spreading sequence of bound state wavefunctions approaches the expression, which is independent of the details of the pair-interaction.

**Theorem 1.** Suppose there is a sequence of coupling constants \(\lambda_n > 0\) such that \(\lim_{n \to \infty} \lambda_n = \lambda_{cr} > 0\), and \(h(\lambda_n)\psi_n = E_n\psi_n\), where \(\psi_n \in D(H_0), \|\psi_n\| = 1, E_n < 0, \lim_{n \to \infty} E_n = 0\). If \(\psi_n\) totally spreads then

\[
\left| \psi_n - e^{i\varphi_n} \frac{\sqrt{k_n} e^{-|x|}}{\sqrt{2\pi |x|}} \right| \to 0,
\]  

(8)

where \(\varphi_n \in [0, 2\pi)\) are phases and \(k_n := \sqrt{|E_n|}\).
A few remarks are in order. If one takes for $\psi_n$ the ground states, then the sequence $\psi_n$ always totally spreads, see the discussion in [6, 8]. In the spherically symmetric potential $s$-states always spread, and states which have a non-zero angular momentum do not spread [6]. (This can also be seen from (8), which shows that the wavefunction must approach the spherically symmetric form.) Let us also note that $\psi_n$ does not spread if $v(x) \geq |x|^{-2+\epsilon}$ for $|x| \geq R_0$ and $\epsilon \in (0, 1)$, see [8–10].

**Proof of theorem 1.** Obviously, $R_n := (\psi_n, (1 + |x|^4)^{-1}\psi_n) \to 0$ because $\psi_n$ totally spreads. The Schrödinger equation in the integral form reads

$$\psi_n = \frac{\lambda_n}{4\pi} \int \! d^3x' \frac{e^{-k_n|x-x'|}}{|x-x'|} v(x') \tilde{\psi}_n(x'),$$

(9)

where $\tilde{\psi}_n := \psi_n/R_n^{1/2}$ is the renormalized wavefunction. Let us set

$$f_n := \frac{\lambda_n}{4\pi} e^{-k_n|x|} \int \! d^3x' v(x') \tilde{\psi}_n(x').$$

(10)

Our aim is to prove that $\|\tilde{\psi}_n - f_n\| = O(1)$. The direct calculation gives

$$\|\tilde{\psi}_n - f_n\|^2 = \frac{\lambda_n^2}{16\pi^2} \int \! d^3x d^3x' d^3x'' \left| e^{-k_n|x-x'|} - e^{-k_n|x|} \right| \left| e^{-k_n|x-x''|} - e^{-k_n|x|} \right| v(x') v(x'') \tilde{\psi}_n^*(x') \tilde{\psi}_n(x'').$$

(11)

This can be transformed into

$$\|\tilde{\psi}_n - f_n\|^2 = \frac{\lambda_n^2}{16\pi^2} \int \! d^3x d^3x' d^3x'' \left| k_n \{W(k_n(x'' - x')) + W(0) - W(k_n x') - W(k_n x'')v(x') v(x'') \tilde{\psi}_n^*(x') \tilde{\psi}_n(x'') \right|,$$

(12)

where we defined

$$W(y) := \int \! d^3z \frac{e^{-|z|} e^{-|z-y|}}{|z| |z-y|} = 2\pi e^{-|y|}.$$

(13)

The integral in (13) can be evaluated using the confocal elliptical coordinates, see e.g. appendix 9 in [14]. Next, by the obvious inequality $|W(y) - W(0)| \leq 2\pi |y|$ 

$$\|\tilde{\psi}_n - f_n\|^2 \leq \frac{\lambda_n^2}{8\pi} \int \! d^3x d^3x'' d^3x' [\|v(x')\| \|v(x'')\| \|v(x'')\|] \leq \frac{\lambda_n^2}{2\pi} \int \! d^3x d^3x'' [\|v(x')\| \|v(x'')\|] \|\tilde{\psi}_n(x')\| \|\tilde{\psi}_n(x'')\|.$$

(14)

Inserting into the rhs of (14) the identities $1 = (1 + |x'|^4)^{1/2}(1 + |x'|^4)^{-1/2}$ and the same for $x''$ and applying the Cauchy–Schwarz inequality gives

$$\|\tilde{\psi}_n - f_n\|^2 \leq \frac{\lambda_n^2}{2\pi} \gamma,$$

(15)

where $\gamma$ is defined in (7). Thus $\|\tilde{\psi}_n - f_n\| = O(1)$ and by (10) we have

$$\psi_n = \frac{\lambda_n}{4\pi} R_n^{1/2} d_n e^{-k_n|x|} + o(1),$$

(16)

where $d_n := \int \! d^3x' v(x') \tilde{\psi}_n(x')$ and $o(1)$ denotes the terms that go to zero in norm. Using that $\|\psi_n\| = 1$, we recover the statement of the theorem. ☐
3. The three-particle case

We shall consider the Hamiltonian (1). Let \( m_i \) and \( r_i \in \mathbb{R}^3 \) denote particles’ masses and position vectors. The reduced masses we shall denote as \( \mu_{ij} := m_im_j / (m_i + m_j) \). The pair-interactions \( V_{ik} \) are operators of multiplication by real \( V_{ik}(r_i − r_k) \). We shall make the following assumptions.

**R1.** The pair potentials satisfy the following requirement:

\[
y_0 := \max_{i=1,2} \max \left[ \int d^3r |V_{i3}(r)|^2, \int d^3r |V_{i3}(r)| (1 + |r|)^{23} \right] < \infty, \tag{17}\]

where \( 0 < \delta < 1/8 \) is a fixed constant. And

\[
- b_{1,2} e^{-b_{1,2}|r|} \leq V_{12}(r) \leq 0, \tag{18}\]

where \( b_{1,2} > 0 \) are some constants.

**R2.** There is a converging sequence of coupling constants \( \lambda_n > 0 \), \( \lim_{n \to \infty} \lambda_n = \lambda_{cr} > 0 \) such that \( H(\lambda_n)\psi_n = E_n\psi_n \), where \( \psi_n \in D(H_0) \), \( \|\psi_n\| = 1 \), \( E_n < 0 \), \( \lim_{n \to \infty} E_n = 0 \).

**R3.** The Hamiltonian \( H_0 = \sum_{i=1,2} V_{i2} \) is at critical coupling (for the definition of critical coupling see [2]). The Hamiltonians \( H_0 + \lambda_{v_{13}} \) and \( H_0 + \lambda_{v_{23}} \) are positive and are not at critical coupling for \( \lambda = \lambda_{n}, \lambda_{cr} \).

Again, let us stress that given that R1 is satisfied one can always tune the coupling constants so that R2, R3 would be satisfied with \( \psi_n \) being ground states. Besides, the sequence \( \psi_\infty \) in this case would totally spread; this is discussed in detail in section 6 in [1].

In the Jacobi coordinates \( x := [\sqrt{2\mu_{12}}/\hbar](r_2 − r_1) \) and \( y := [\sqrt{2M_{12}}/\hbar](r_3 − m_1/(m_1 + m_2)r_1 − m_2/(m_1 + m_2)r_2) \), where \( M_{ij} = (m_i + m_j)m_j/(m_1 + m_2 + m_3) \) \( (i, j, k) \) is a permutation of \( \{1, 2, 3\} \) the kinetic energy operator takes the form \([1, 2]\)

\[
H_0 = -\Delta_x - \Delta_y. \tag{19}\]

In the following \( \chi_\Omega : \mathbb{R} \to \mathbb{R} \) denotes the characteristic function of the interval \( \Omega \subset \mathbb{R} \) (for instance, \( \chi_{[1,\infty)}(x) \) is equal to 1 if \( x \in [1, \infty) \) and is zero otherwise). The next theorem is the analogue of theorem 1 for the three-particle case.

**Theorem 2.** Suppose \( H(\lambda) \) defined in (1) satisfies R1–3. If \( \psi_n \) totally spreads then

\[
\left| \psi_n - \frac{e^{i\varphi_n} \chi_{[1,\infty)}(\rho)}{2\pi} \frac{|x| \sin(k_n|y|) + |y| \cos(k_n|y|)}{|k_n||y| |y|} e^{-k_n|x|} \right| \to 0, \tag{20}\]

where \( \varphi_n \in [0, 2\pi] \) are phases, \( \rho := |x|^2 + |y|^2 \) and \( k_n := \sqrt{|E_n|} \).

**Remark.** Theorem 2 shows that similar to the two-particle case, total spreading in the considered three-body case is possible only for states with zero angular momentum (irrespective of the values of particles’ masses). This can be seen from (20), where in the limit the wavefunction depends only on \( |x|, |y| \) and is thus invariant under rotations of \( x, y \). This fact is not as trivial as it may seem. In theorem 2, we consider only the situation when a single particle pair has a zero-energy resonance. If two particle pairs have such resonances, one arrives at the Efimov effect, see [11, 4, 5], where there exists an infinite sequence of energy levels \( E_n \to 0 \) with orthonormal wavefunctions \( \phi_n \). This sequence of wavefunctions also totally spreads, see [2]. However, in the case of Efimov effect it is possible to choose mass ratios in the system in such a way that the sequence \( \phi_n \) would have a non-zero angular momentum, see [12, 13, 5].

Theorem 2 has a useful practical corollary.
Theorem 3. Suppose $H(\lambda)$ satisfies R1–3. If $\psi_n$ totally spreads then the angular probability distribution $D_n(\theta, \hat{x}, \hat{y})$ defined in (3) converges in measure to $D_\infty(\theta, \hat{x}, \hat{y}) = (4\pi^3)^{-1} \sin^2 \theta$.

Proof. Let us rewrite (20) in hyperspherical coordinates
\[
\|\psi_n - \Theta_n\| \to 0,
\]
where
\[
\Theta_n := \frac{e^{t_0} \chi_{[1, \infty)}(\rho)}{2\pi^{3/2}} \frac{e^{-k_n \rho \cos \theta} \sin(\theta + k_n \rho \sin \theta)}{\rho^3 \cos \theta \sin \theta}.
\]
One can easily check that $\|\Theta_n\| \to 1$. If we denote by $D_n^\Theta(\theta, \hat{x}, \hat{y})$ the angular probability distribution given by $\Theta_n$, then the limiting angular probability distribution is
\[
D_\infty(\theta, \hat{x}, \hat{y}) = \lim_{n \to \infty} D_n^\Theta = \frac{1}{4\pi^3} \lim_{n \to \infty} \frac{1}{\ln k_n} \int_1^\infty e^{-2k_n \rho \cos \theta} \rho^3 \cos \theta \sin \theta \, d\rho,
\]
where the limit is pointwise. Changing the integration variable in the last integral for $t = k_n \rho \sin \theta$ and expanding around $t = 0$ we obtain
\[
D_\infty(\theta, \hat{x}, \hat{y}) = \frac{1}{4\pi^3} \lim_{n \to \infty} \frac{1}{\ln k_n} \int_0^\infty e^{-2\rho \cos \theta} \frac{\sin^2(\theta + t) \, dt}{t} = \frac{1}{4\pi^3} \sin^2 \theta.
\]
Note that $D_n^\Theta \to D_\infty$ pointwise and uniformly. Now, we show that $\|D_n - D_n^\Theta\|_1 \to 0$. To make the notation shorter we set $d\Omega := \cos^2 \theta \sin^2 \theta \, d\theta \, d\Omega_x \, d\Omega_y$,
\[
\|D_n - D_n^\Theta\|_1 \leq \int_0^{2/\pi} d\theta \int d\Omega_x \, d\Omega_y |D_n - D_n^\Theta|
\]
\[
= \int d\Omega \left| \int |\psi_n|^2 \rho^3 \, d\rho - \int |\Theta_n|^2 \rho^3 \, d\rho \right|
\]
\[
\leq \int d\Omega \left| \rho^3 |\psi_n|^2 - |\Theta_n|^2 |\rho^5 \right| \, d\rho
\]
\[
\leq \left( \|\psi_n - \Theta_n\| \left( \int d\Omega \int \rho^5 |\psi_n|^2 + |\Theta_n|^2 \right) \right)^{1/2} \leq 2 \|\psi_n - \Theta_n\|.
\]

Remark. Theorem 3 states that the angular probability distribution converges in measure, which is equivalent to $\|D_n - D_\infty\|_1 \to 0$, whereby $D_n$ corresponds to $\psi_n$ in R2 and the meaning of $\|\cdot\|$ is explained in (25). It should be stressed here that $D_n$ does not converge to $D_\infty$ pointwise everywhere. Indeed, for smooth interactions one expects that $\psi_n$ at $|x| = 0$ should be finite; by definition (3) this immediately implies that $D_n = 0$ if $\theta = \pi/2$. Nevertheless, the limiting angular probability distribution $D_\infty$ is not zero at $\theta = \pi/2$, on the contrary, it has its maximum at this point! At $\theta \neq \pi/2$ one should expect a pointwise convergence.
Lemma 1. Suppose $H(\lambda)$ defined in (1) satisfies R1–3. If $\psi_n$ totally spreads then

$$\psi_n = \left[ H_0 + k_n^2 \right]^{-1} |v_{12}| \psi_n + o(1),$$

where $o(1)$ denotes the terms that go to zero in norm.

Proof. Rearranging in different ways the terms in the Schrödinger equation for $\psi_n$ we derive three integral equations, see [2]

$$\psi_n = \left[ H_0 + \lambda_n(v_{13})_+ + \lambda_n(v_{23})_+ + k_n^2 \right]^{-1} |v_{12}| + \lambda_n(v_{13})_- + \lambda_n(v_{23})_- \psi_n,$$

(28)

$$\psi_n = \left[ H_0 + \lambda_n(v_{13})_+ + \lambda_n(v_{23})_+ + k_n^2 \right]^{-1} |v_{12}| \psi_n,$$

(29)

where $(v_{ik})_\pm = \max[0, \pm v_{ik}]$. By (27) the lemma would be proved if we can show that

$$F_n := \lambda_n \left[ H_0 + k_n^2 \right]^{-1} v_{13} \psi_n = o(1),$$

(30)

$$\lambda_n \left[ H_0 + k_n^2 \right]^{-1} v_{23} \psi_n = o(1).$$

(31)

Below we prove (30), equation (31) is proved analogously. Substituting (28) into (30) we split $F_n$ into three parts

$$F_n = \sum_{i=1}^{3} F_n^{(i)}$$

(32)

where

$$F_n^{(1)} = \left[ H_0 + k_n^2 \right]^{-1} v_{13} \left[ H_0 + \lambda_n(v_{13})_+ + \lambda_n(v_{23})_+ + k_n^2 \right]^{-1} |v_{12}| \psi_n,$$

(33)

$$F_n^{(2)} = \lambda_n \left[ H_0 + k_n^2 \right]^{-1} v_{13} \left[ H_0 + \lambda_n(v_{13})_+ + \lambda_n(v_{23})_+ + k_n^2 \right]^{-1} |v_{12}| \psi_n,$$

(34)

$$F_n^{(3)} = \lambda_n \left[ H_0 + k_n^2 \right]^{-1} v_{13} \left[ H_0 + \lambda_n(v_{13})_+ + \lambda_n(v_{23})_+ + k_n^2 \right]^{-1} (v_{13})_+ \psi_n.$$

(35)

We introduce another pair of Jacobi coordinates $\eta = [\sqrt{\mu_{13}/\hbar}] (r_2 - r_3)$ and $\xi = [\sqrt{2M_{13}/\hbar}] (r_2 - m_1/(m_1 + m_3) r_2 - m_3/(m_1 + m_3) r_3)$. The coordinates $(\eta, \xi)$ and $(x, y)$ are connected through the orthogonal linear transformation

$$x = m_{1\eta} \eta + m_{1\xi} \xi,$$

(36)

$$y = m_{2\eta} \eta + m_{2\xi} \xi,$$

(37)

where $m_{1\eta}, m_{1\xi}, m_{2\eta}, m_{2\xi}$ are real and can be expressed through mass ratios in the system. $\mathcal{F}_{13}$ denotes the partial Fourier transform, which acts on $f(\eta, \xi)$ as

$$\mathcal{F}_{13} f := \hat{f}(\eta, p_\xi) = \frac{1}{(2\pi)^{3/2}} \int d^3 \xi \ e^{-ip_\xi \cdot \xi} f(\eta, \xi).$$

(38)

Let us introduce the operator function

$$\tilde{\mathcal{B}}_{13}(k_\eta) := \mathcal{F}_{13}^{-1} \mathcal{F}_{13}(p_\xi) \mathcal{F}_{13},$$

(39)

where

$$\tilde{\mathcal{B}}_{13}(k_\eta) = \begin{cases} |p_\xi|^{1-\delta} + (k_\eta)^{1-\delta} & \text{if } |p_\xi| \leq 1 \\ 1 + (k_\eta)^{1-\delta} & \text{if } |p_\xi| \geq 1. \end{cases}$$

(40)
We set a tilde over the operator in order to distinguish it from the one defined in equation (18) in [1]. Note that \( \tilde{B}_{13}(k_n) \) and \( \tilde{B}_{13}^{-1}(k_n) \) for each \( n \) are bounded operators.

Using the inequalities from [2] (see equations (17)–(24) in [2]), we obtain
\[
|F_n^{(1)}| \leq |v_{13}|^{1/2}[H_0 + k_n^2]^{-1}(v_{12}) |\psi_n| = \frac{|H_0 + k_n^2|^{-1}(v_{12}) |\psi_n|}{v_{13}} \leq \left[ \frac{|H_0 + k_n^2|^{-1}(v_{12})}{|\psi_n|} \right] v_{13}^{1/2},
\]
(41)
where
\[
\psi_n^{(1)} := |v_{13}|^{1/2}[H_0 + k_n^2]^{-1}(v_{12}) |\psi_n|, \quad \psi_n^{(2)} := \lambda_n|v_{13}|^{1/2}[H_0 + k_n^2]^{-1}(v_{12}) |\psi_n|.
\]
(42)

To write the upper bound on \( |F_n^{(3)}| \) we use the following expression, which follows from (29), cf equation (15) in [2]:
\[
|v_{13}|^{1/2} |\psi_n| = Q_n(v_{13})^{1/2}[H_0 + \lambda_n(v_{13}) + k_n^2]^{-1}(v_{12}) |\psi_n|, \quad \text{where we defined}
\]
\[
Q_n := \left( 1 - \lambda_n(v_{13})^{1/2}[H_0 + \lambda_n(v_{13}) + k_n^2]^{-1}(v_{12}) \right)^{-1}.
\]
(46)

\( Q_n \) is a positivity preserving operator and \( \sup_n \|Q_n\| < \infty \), see lemma 1 in [2] and lemma 12 in [1]. Substituting (45) into (35) and using the positivity preserving property of the operators (see the discussion after equation (16) in [2]) we obtain
\[
|F_n^{(3)}| \leq \lambda_n|v_{13}|^{1/2}[H_0 + k_n^2]^{-1}(v_{13})^{1/2} Q_n(v_{13})^{1/2} \times [H_0 + k_n^2]^{-1}(v_{12}) |\psi_n| = \left[ H_0 + k_n^2 \right]^{-1}(v_{13})^{1/2} \tilde{B}_{13}(k_n) |\psi_n|, \quad (47)
\]
where
\[
\psi_n^{(3)} := \lambda_n|v_{13}|^{1/2}[H_0 + k_n^2]^{-1}(v_{13})^{1/2} Q_n(\tilde{B}_{13}(k_n))^{1/2} \times [H_0 + k_n^2]^{-1}(v_{12}) |\psi_n|.
\]
(48)

Summarizing, (41), (42) and (47) can be expressed through the inequality
\[
|F_n^{(i)}| \leq L_n \psi_n^{(i)} \quad (i = 1, 2, 3),
\]
(49)
where
\[
L_n := [H_0 + k_n^2]^{-1}(v_{13})^{1/2} \tilde{B}_{13}(k_n).
\]
(50)
From lemma 2 it follows that \( \|F_n^{(i)}\| \to 0. \)

**Lemma 2.** The operators \( L_n \) are uniformly norm-bounded and \( \|\psi_n^{(i)}\| \to 0 \) for \( i = 1, 2, 3 \).

**Proof.** The proof that \( L_n \) are uniformly norm-bounded is similar to lemma 6 in [1]. Indeed, \( K_n := F_{13}L_nF_{13}^{-1} \) is an integral operator with the kernel
\[
k_n(\eta, \eta'; p_\xi) = e^{-\sqrt{\mu_{13}} |\eta - \eta'|} V_{13}(\alpha') |\eta - \eta'|^{-1/2} \tilde{I}_{13}n(p_\xi),
\]
(51)
where \( \alpha' := h/\sqrt{\mu_{13}} \), which acts on \( f(\eta, p_\xi) \in L^2(\mathbb{R}^6) \) as follows:
\[
K_n f = \int d^3\eta' k_n(\eta, \eta'; p_\xi)f(\eta', p_\xi).
\]
(52)
Therefore, we can estimate the norm as

$$\|L_n\|^2 = \|K_n\|^2 \leq \sup_{p_t} \int |k_n(\eta, \eta', p_t)|^2 \, d^3 \eta' \, d^3 \eta = C_0 \sup_{p_t} \frac{|\hat{h}_n(p_t)|^2}{p_t^2 + k_n^2}, \quad (53)$$

where

$$C_0 := \frac{1}{16\pi^2} \int \frac{e^{-2|\eta|}}{|s|^2} \, d^3 s \int |V_{13}(\alpha' \eta)| \, d^3 \eta \leq \frac{\gamma_0}{8\pi}, \quad (54)$$

and $\gamma_0$ was defined in (17). Substituting (40) into (53) it is easy to see that $\|L_n\|$ is uniformly bounded. Let us rewrite (43) as

$$\Psi^{(1)}_n := [M^{(1)}_n + M^{(2)}_n] |v_{12}|^{1/2} |\psi_n|, \quad (55)$$

where

$$M^{(1)}_n := |v_{13}|^{1/2} \left[ \hat{B}_{13}^{-1} (k_n) - (1 + (k_n)^{1-\delta})^{-1} \right] [H_0 + k_n^2]^{-1} |v_{12}|^{1/2}, \quad (56)$$

$$M^{(2)}_n := (1 + (k_n)^{1-\delta})^{-1} |v_{13}|^{1/2} [H_0 + k_n^2]^{-1} |v_{12}|^{1/2}. \quad (57)$$

By the no-clustering theorem $\|v_{12}|^{1/2} |\psi_n| \rightarrow 0$, see the appendix in [2]. Thus to prove that $\|\Psi^{(1)}_n\| \rightarrow 0$ it is enough to show that $\sup_{p_t} \|M^{(1,2)}_n\| < \infty$. It is easy to see that $\|M^{(2)}_n\|$ is uniformly norm-bounded, see e.g. the proof of lemma 7 in [1]. Next, $\|M^{(1)}_n\| = \|K''_n\|$, where $K''_n := \mathcal{F}_{13} M'_n \mathcal{F}^{-1}_{13}$ is the integral operator with the kernel

$$k''_n(\eta, p_t, \eta', p_t') = \frac{1}{2\sqrt{\pi} \sqrt{\omega}} [p_t^{-1} (p_t')^{-1}]^{-1} [V_{13}(\alpha' \eta)|^{1/2}$$

$$\times \frac{e^{-\sqrt{p_t + k_n^2} |\eta - \eta'|}}{|\eta - \eta'|} \exp \left\{ \frac{\beta}{\omega} (p_t - p_t') \right\} |V_{12}|^{1/2} (\eta, p_t); \quad (58)$$

$\beta := -m_3/((m_1 + m_3) \sqrt{2 \mu_{13}})$ and $\omega := h/\sqrt{2M_{13}}$ (see the proof of lemma 9 in [1]). In (58) $|V_{12}|^{1/2}$ denotes merely the Fourier transform of $|V_{12}|^{1/2} \in L^2(\mathbb{R}^3)$. Calculation of the Hilbert–Schmidt norm gives

$$\|M^{(1)}_n\|^2 \leq C'_0 \int_{|p_t| \leq 1} \frac{|p_t|^{1-\delta} + (k_n)^{1-\delta}}{\sqrt{p_t^2 + k_n^2}} \, d^3 p_t \leq \frac{C'_0}{8\omega^3 \pi^2} \int_{|p_t| \leq 1} \frac{d^3 p_t}{|p_t|^{3-\pi}}, \quad (59)$$

where

$$C'_0 := C_0 \int d^3 s |\hat{V}_{12}|^{1/2} (s)|^2. \quad (60)$$

From (59) it follows that $\|M^{(1)}_n\|$ is uniformly bounded and, hence, $\|\Psi^{(1)}_n\| \rightarrow 0$. The fact that $\|\Psi^{(2)}_n\| \rightarrow 0$ is proved analogously. To prove that $\|\Psi^{(3)}_n\| \rightarrow 0$, let us look at (48). We can write

$$\Psi^{(3)}_n = \lambda_n T^{(1)}_n Q_n [T^{(2)}_n |v_{12}|^{1/2} |\psi_n| + T^{(3)}_n |v_{23}|^{1/2} |\psi_n|], \quad (61)$$

where we defined the operators

$$T^{(1)}_n := |v_{13}|^{1/2} \left[ H_0 + k_n^2 \right]^{-1} (v_{13})^{1/2}, \quad (62)$$

$$T^{(2)}_n := \hat{B}_{13}^{-1} (k_n) (v_{13})^{1/2} \left[ H_0 + k_n^2 \right]^{-1} |v_{12}|^{1/2}, \quad (63)$$

$$T^{(3)}_n := \lambda_n \hat{B}_{13}^{-1} (k_n) (v_{13})^{1/2} \left[ H_0 + k_n^2 \right]^{-1} |v_{23}|^{1/2}. \quad (64)$$

The operators $Q_n$ are uniformly norm-bounded. The operators $T^{(1)}_n$ are also uniformly norm-bounded. Note that $T^{(2)}_n = M^{(1)}_n + M^{(2)}_n$, where $M^{(1,2)}_n$ is defined exactly as $M^{(1,2)}_n$. 

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except that \(|v_{13}|\) gets replaced with \((v_{13})_0\). Thus from the above analysis it follows that \(\|T_n^{(3)}\|\) is uniformly bounded. By similar arguments \(\|T_n^{(3)}\|\) is uniformly bounded. Thus due to \(\|v_{ik}\|^1/2\|\hat{\psi}_n\| \to 0\) (see the no-clustering theorem in [2]) the expression on the rhs of (61) goes to zero in norm. \[\square\]

**Proof of theorem 2.** Instead of (20) it suffices to prove that

\[
\left\| \hat{\psi}_n - \frac{\sqrt{2} e^{i p_0}}{4\pi |\ln k_n|^{1/2}} \hat{x}(k_n,1)(|p_0|) e^{-|p_0||x|} \right\| \to 0,
\]

(65)

where the hat denotes the action of the partial Fourier transform \(\mathcal{F}_{12}\), see equation (17) in [1].

Indeed, computing explicitly the inverse Fourier transform

\[
\mathcal{F}_{12}^{-1}\left( \frac{\hat{x}(k_n,1)(|p_0|) e^{-|p_0||x|}}{|x||p_0|} \right) = \frac{1}{2\pi |\ln k_n|^{1/2}} \frac{1}{
\end{equation}

\[\hat{x}(k_n,1)(|p_0|) e^{-|p_0||x|}}{|x||p_0|} \right)
\end{equation}

\[\times e^{-|p_0|\sqrt{k_n}} [-|x| \sin |y| - |y| \cos |y|] - e^{-k_n|y|} [-|x| \sin(k_n|y|) - |y| \cos(k_n|y|)].\]

(66)

Now (20) follows directly from (66), (65) after dropping those terms, whose norm goes to zero.

By lemma 1 \(\|\hat{\psi}_n - \hat{\psi}_n^{(1)}\| \to 0\), where we have set \(\hat{\psi}_n^{(1)} := \hat{[H_0 + k_n^2]^{-1}} |v_{12}| \hat{\psi}_n\). From the Schrödinger equation for the term \(\sqrt{|v_{12}|}\hat{\psi}_n\) we obtain

\[
\sqrt{|v_{12}|} \hat{\psi}_n = -\left(1 - \sqrt{|v_{12}|(H_0 + k_n^2)^{-1} \sqrt{|v_{12}|}} \right)^{-1} \sqrt{|v_{12}|} \hat{[H_0 + k_n^2]^{-1}} (\lambda_n v_{13} + \lambda_n v_{23}) \hat{\psi}_n.
\]

(67)

Substituting (67) into the expression for \(\hat{\psi}_n^{(1)}\) results in

\[
f^{(1)}_n = \hat{[H_0 + k_n^2]^{-1}} \sqrt{|v_{12}|} \left[1 - \sqrt{|v_{12}|(H_0 + k_n^2)^{-1} \sqrt{|v_{12}|}} \right]^{-1} \Phi_n,
\]

(68)

where

\[
\Phi_n := -\lambda_n \sqrt{|v_{12}|(H_0 + k_n^2)^{-1} (v_{13} + v_{23})} \hat{\psi}_n.
\]

(69)

From the proofs of lemmas 6 and 9 in [1] it follows that the operators \(\sqrt{|v_{12}|(H_0 + k_n^2)^{-1} \sqrt{|v_{12}|}}\) and \(B_{12}^{-1}(k_n)\sqrt{|v_{12}|(H_0 + k_n^2)^{-1} \sqrt{|v_{12}|}}\), where \(B_{12}(k_n)\) is defined in equations (18)–(19) in [1], are uniformly norm-bounded for \(s = 1, 2\). Thus, by (69) and theorem 3 in [2] \(\|\Phi_n\| \to 0\) and \(\|B_{12}^{-1}(k_n)\Phi_n\| \to 0\). Acting with \(\mathcal{F}_{12}\) on (68) gives

\[
\hat{f}^{(1)}_n = \left[-\Delta_x + p_3^2 + k_n^2\right]^{-1} \sqrt{|v_{12}|} \left[1 - \sqrt{|v_{12}|(-\Delta_x + p_3^2 + k_n^2)^{-1} \sqrt{|v_{12}|}} \right]^{-1} \Phi_n.
\]

(70)

Because \(\|\hat{\Phi}_n\| \to 0\) we can write

\[
\hat{f}^{(1)}_n = \hat{f}^{(2)}_n + o(1),
\]

(71)

where

\[
\hat{f}^{(2)}_n := \hat{x}(0,0,1)(\sqrt{p_3^2 + k_n^2}) \hat{f}^{(1)}_n.
\]

(72)

and \(\rho_0\) is a constant defined in lemma 11 in [1]. Now using lemma 11 in [1] (see also discussion around equation (111) in [1]) we obtain

\[
\hat{f}^{(2)}_n = \hat{f}^{(3)}_n + \hat{x}(0,0,1)(\sqrt{p_3^2 + k_n^2}) \mathcal{F}_{12} \mathcal{A}_{12}(k_n) \mathcal{F}_{12}^{-1} Z(\sqrt{p_3^2 + k_n^2}) \mathcal{B}_{12}^{-1}(k_n) \hat{\Phi}_n.
\]

(73)

where \(\mathcal{A}_{12}(k_n) := \left[H_0 + k_n^2\right]^{-1} \sqrt{|v_{12}|} \mathcal{B}_{12}(k_n)\) and \(Z\) defined in [1] remain uniformly norm-bounded for all \(n\), see lemmas 6 and 11 in [1]. The function \(\hat{f}^{(3)}_n\) is defined as follows:

\[
\hat{f}^{(3)}_n := \hat{x}(0,0,1)(\sqrt{p_3^2 + k_n^2}) \left[-\Delta_x + |p_3|^2 + k_n^2\right]^{-1} \frac{\sqrt{|v_{12}|}}{a\sqrt{|p_3|^2 + k_n^2}} \rho_0 \hat{\Phi}_n.
\]
where $a$ and $\rho_0$ are defined in equation (80) and lemma 11 in [1]. Therefore, since $\|B^{12}(k_n)\Phi_n\| \to 0$

$$\hat{f}_n^{(2)} = \hat{f}_n^{(3)} + o(1).$$  

(75)

It makes sense to introduce

$$g_n(y) := \int d^3x \phi_0(x)\Phi_n(x, y),$$  

(76)

where $\phi_0$ was defined in equation (77) in [1]. The following inequality trivially follows from the exponential bound on $V_{12}$ and the definition of $\phi_0$:

$$\phi_0(x) \leq b'_1 e^{-|x|}$$  

(77)

where $b'_1 > 0$ are constants. From the pointwise exponential fall-off of $\psi_n$ it follows that $g_n \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for each $n$. We rewrite (74) with the help of (76),

$$\hat{g}_n^{(3)} = \chi_{[0, \rho_1]}\left(\sqrt{p_\gamma^2 + k_n^2}\right) \left[-\Delta_x + p_\gamma^2 + k_n^2\right]^{-1} \sqrt{V_{12}} \phi_0(x)\hat{g}_n(p_n)$$

$$\times \int d^3x e^{-\sqrt{p_\gamma^2 + k_n^2}|x-x'|} \phi_0(x')|V_{12}(\alpha x')|^{1/2},$$  

(78)

where $\alpha := h/\sqrt{2\mu_{12}}$. Next, let us define

$$\hat{f}_n^{(4)} := \chi_{[0, \rho_1]}\left(\sqrt{p_\gamma^2 + k_n^2}\right) \hat{g}_n(0)$$

$$\times \int d^3x^{'} W\left(\sqrt{p_\gamma^2 + k_n^2}(x^{'} - x')\right) \phi_0(x')|V_{12}(\alpha x')|^{1/2} \phi_0(x'')|V_{12}(\alpha x'')|^{1/2}$$

$$\leq \int d^3p_\gamma \chi_{[0, \rho_1]}\left(\sqrt{p_\gamma^2 + k_n^2}\right)$$

$$\times \int d^3x^{'} W\left(\sqrt{p_\gamma^2 + k_n^2}(x^{'} - x')\right) \phi_0(x')|V_{12}(\alpha x')|^{1/2} \phi_0(x'')|V_{12}(\alpha x'')|^{1/2}$$

$$\leq \frac{e^{2\gamma^2}}{16\pi^2 a^2} \int d^3p_\gamma \chi_{[0, \rho_1]}\left(\sqrt{p_\gamma^2 + k_n^2}\right)$$

$$\times \int d^3x^{'} W\left(\sqrt{p_\gamma^2 + k_n^2}(x^{'} - x')\right) \phi_0(x')|V_{12}(\alpha x')|^{1/2} \phi_0(x'')|V_{12}(\alpha x'')|^{1/2}$$

(80)

where we used $W(s) \leq 2\pi$ and set

$$\delta := \int d^3x^{'} \phi_0(x')|V_{12}(\alpha x')|^{1/2}.$$  

(81)

The constant in (81) is bounded, hence, by lemma 4 $\|\hat{f}_n^{(4)} - \hat{f}_n^{(3)}\| \to 0$. As the next step we introduce

$$\hat{f}_n^{(5)} := \chi_{[0, \rho_1]}\left(\sqrt{p_\gamma^2 + k_n^2}\right)$$

$$\times \int d^3x^{'} W\left(\sqrt{p_\gamma^2 + k_n^2}(x^{'} - x')\right) \phi_0(x')|V_{12}(\alpha x')|^{1/2} \phi_0(x'')|V_{12}(\alpha x'')|^{1/2}$$

$$\leq \frac{e^{2\gamma^2}}{16\pi^2 a^2} \int d^3p_\gamma \chi_{[0, \rho_1]}\left(\sqrt{p_\gamma^2 + k_n^2}\right)$$

$$\times \int d^3x^{'} W\left(\sqrt{p_\gamma^2 + k_n^2}(x^{'} - x')\right) \phi_0(x')|V_{12}(\alpha x')|^{1/2} \phi_0(x'')|V_{12}(\alpha x'')|^{1/2}$$

(82)

where

$$R(s) := \int d^3x^{'} e^{-|x^{'}|} \phi_0(x')|V_{12}(\alpha x')|^{1/2}.$$  

(83)
Like in the proof of theorem 1 we evaluate the square of the norm of the difference
\[
\| \hat{f}_n^{(5)} - \hat{f}_n^{(4)} \|^2 \leq \int d^3p_x \chi(0,0,1)(\sqrt{p_x^2 + k_n^2}) \frac{|\hat{g}_n(0)|^2}{16\pi^2a^2(p_x^2 + k_n^2)^{3/2}} \\
\times \int d^3x' d^3x'' \left[ W(\sqrt{p_x^2 + k_n^2}(x'' - x')) + W(0) - W(\sqrt{p_x^2 + k_n^2}x') \right] \\
\times \int d^3x' \left| \hat{\varphi}_n(x') \right| |V_{12}(\alpha x')|^{1/2} |\varphi_0(x')| |V_{12}(\alpha x'')|^{1/2}.
\]
(84)

On account of R1 and (77) we conclude that \(\| \hat{f}_n^{(5)} - \hat{f}_n^{(4)} \| \to 0\) since \(\| \hat{g}_n(0) \| \to 0\) by lemma 4. Observe that
\[
|R(s) - R(0)| \leq s \vartheta,
\]
(85)
where \(\vartheta\) is defined in (81). Hence, \(\| \hat{f}_n^{(6)} - \hat{f}_n^{(5)} \| \to 0\), where by definition
\[
\hat{f}_n^{(6)} := \chi(0,0,1)(\sqrt{p_x^2 + k_n^2}) \frac{R(0)|\hat{g}_n(0)| e^{-\sqrt{p_x^2 + k_n^2}|x|}}{4\pi a \sqrt{p_x^2 + k_n^2}}.
\]
(86)
Simplifying the argument of the exponential function we define
\[
\hat{f}_n^{(7)} := \chi(0,0,1)(\sqrt{p_x^2 + k_n^2}) \frac{R(0)|\hat{g}_n(0)| e^{-|p_x||x|}}{4\pi a \sqrt{p_x^2 + k_n^2}}.
\]
(87)
After straightforward calculation we obtain
\[
\| \hat{f}_n^{(7)} - \hat{f}_n^{(6)} \|^2 = \int d^3p_x \chi(0,0,1)(\sqrt{p_x^2 + k_n^2}) \frac{R^2(0)|\hat{g}_n(0)|^2}{4\pi^2a^2(p_x^2 + k_n^2)} \\
\times \left[ \frac{1}{2\sqrt{p_x^2 + k_n^2}} + \frac{1}{|p_x|} - \frac{2}{\sqrt{p_x^2 + k_n^2} + |p_x|} \right].
\]
(88)
Replacing in the last fraction \(|p_x|\) with \(\sqrt{p_x^2 + k_n^2}\) results in the following inequality:
\[
\| \hat{f}_n^{(7)} - \hat{f}_n^{(6)} \|^2 \leq \frac{R^2(0)|\hat{g}_n(0)|^2}{8\pi^2a^2} \int d^3p_x \chi(0,0,1)(\sqrt{p_x^2 + k_n^2}) \left[ \frac{1}{|p_x|} - \frac{1}{\sqrt{p_x^2 + k_n^2}} \right].
\]
(89)
The integrals can be calculated explicitly, see [16], which results in \(\| \hat{f}_n^{(7)} - \hat{f}_n^{(6)} \| \to 0\). At last, we simplify the expression setting
\[
\hat{f}_n^{(8)} := \frac{R(0)|\hat{g}_n(0)| \chi(0,0,1)|p_x| e^{-|p_x||x|}}{4\pi a \sqrt{|x||p_x|}}.
\]
(90)
Again, one easily finds that \(\| \hat{f}_n^{(8)} - \hat{f}_n^{(7)} \| \to 0\). Summarizing, we have \(\| \hat{f}_n^{(i+1)} - \hat{f}_n^{(i)} \| \to 0\) for \(i = 1, \ldots, 7\). Thus from \(\|\hat{\psi}_n - \hat{f}_n^{(8)}\| \to 0\) it follows that \(\|\hat{\psi}_n - \hat{f}_n^{(8)}\| \to 0\). Using that \(\|\hat{\psi}_n\| = 1\) we obtain (65).
Lemma 3. There exists a sequence \( c_n > 0, c_n \to 0 \) such that
\[
|\hat{g}_n(p_y) - \hat{g}_n(0)| \leq c_n|p_y|^4,  \tag{91}
\]
where \( \delta \) is defined in (17).

Proof. The trivial inequality \( |e^{iy\cdot}\delta - 1| \leq |p_y|^4|y|^4 \) implies that
\[
|\hat{g}_n(p_y) - \hat{g}_n(0)| \leq \int d^3y|e^{iy\cdot}\delta - 1||g_n(y)| \leq |p_y|^4 c_n,  \tag{92}
\]
where \( c_n = \int d^3y|y|^4|g_n(y)| \) goes to zero by lemma 4.

The following lemma makes use of the absence of zero-energy resonances in particle pairs \( \{1, 3\} \) and \( \{2, 3\} \).

Lemma 4. The sequence \( c_n = \int d^3y(1 + |y|^4)|g_n(y)| \) is well defined and goes to zero.

Proof. By definitions (76) and (69) we have \( |g_n(y)| \leq |g_n^{(1)}(y)| + |g_n^{(2)}(y)| \), where
\[
g_n^{(1)}(y) := \lambda_n \int d^3x \phi_0(x)|v_{12}(x)|^{1/2}[H_0 + k_n^2]^{-1}v_{13}\psi_n,  \tag{93}
g_n^{(2)}(y) := \lambda_n \int d^3x \phi_0(x)|v_{12}(x)|^{1/2}[H_0 + k_n^2]^{-1}v_{23}\psi_n.  \tag{94}
\]
Consequently \( c_n \leq c_n^{(1)} + c_n^{(2)} \), where
\[
c_n^{(i)} := \int d^3y(1 + |y|^4)|g_n^{(i)}(y)|.  \tag{95}
\]

Below we shall prove that \( c_n^{(1)} \to 0 \), the fact that \( c_n^{(2)} \to 0 \) is proved analogously. Let us mention that integrals appearing below and the interchanged order of integration can be easily justified using the pointwise exponential fall-off of \( \psi_n \) [17].

We have
\[
|\hat{g}_n^{(1)}(y)| \leq \int d^3x |V_{12}\phi_0(x)|^{1/2}F_n(x, y),  \tag{96}
\]
where \( F_n \) was defined in (32). On account of R1 and (77) it follows that
\[
|\hat{g}_n^{(1)}(y)| \leq \tilde{b}_1 \int d^3x e^{-\tilde{k}_1|y|}F_n(x, y),  \tag{97}
\]
where \( \tilde{b}_{1,2} > 0 \) are constants. Using (32) and (49)–(50) gives
\[
|F_n| \leq \sum_{i=1}^{3} |F_n^{(i)}| \leq \sum_{i=1}^{3} \tilde{F}_n^{(i)},  \tag{98}
\]
\[
\tilde{F}_n^{(i)} := [H_0 + k_n^2]^{-1}|v_{13}|^{1/2}\tilde{B}_n(k_n)|\psi_n^{(i)}.  \tag{99}
\]
Substituting (97), (98) into (95) we obtain
\[
c_n^{(1)} \leq \tilde{b}_1 \sum_{i=1}^{3} \int d^3\eta d^3\xi (1 + |m_3\eta + m_1\xi|^4) e^{-\tilde{k}_1|m_3\eta + m_1\xi|^4} \tilde{F}_n^{(i)}(\eta, \xi).  \tag{100}
\]

Let us consider the term \( \tilde{F}_n^{(i)}(\eta, \xi) \). Acting on it with direct and inverse partial Fourier transforms (38) we obtain
\[
\tilde{F}_n^{(i)} = \mathcal{F}_n^{-1}\left[-\Delta_\eta + p_\xi^2 + k_n^2\right]^{-1}|v_{13}|^{1/2}\tilde{t}_n(p_\xi)\tilde{g}_n^{(i)},  \tag{101}
\]
\[ \hat{F}_n^{(i)}(\eta, \zeta) = \frac{1}{2^{7/2}\sqrt{\pi}} \int d^3\eta' d^3p_\xi e^{ip_\xi \cdot \xi'} |V_{13}(\alpha' \eta')|^{1/2} e^{\frac{-\sqrt{p_\xi^2 + k_n^2} |\eta - \eta'|}{|\eta - \eta'|}} \hat{f}_n(p_\xi) \hat{\Psi}_n^{(i)}(\eta', p_\xi). \] (102)

Hence,
\[ |\hat{F}_n^{(i)}(\eta, \zeta)| \leq \frac{1}{2^{7/2}\sqrt{\pi}} \int d^3\eta' d^3p_\xi |V_{13}(\alpha' \eta')|^{1/2} e^{\frac{-\sqrt{p_\xi^2 + k_n^2} |\eta - \eta'|}{|\eta - \eta'|}} \hat{f}_n(p_\xi) |\hat{\Psi}_n^{(i)}(\eta', p_\xi)|. \] (103)

Substituting (103) into (100) and interchanging the order of integration we obtain the inequality
\[ c_n^{(i)} \leq \frac{\hat{b}_1}{2^{7/2}\sqrt{\pi}} \sum_{i=1}^{3} \int d^3\eta' \int d^3p_\xi |V_{13}(\alpha' \eta')|^{1/2} \hat{f}_n(p_\xi) |\hat{\Psi}_n^{(i)}(\eta', p_\xi)| J(\eta', p_\xi). \] (104)

where we define
\[ J(\eta', p_\xi) := \int d^3\eta \int d^3\xi e^{\frac{-\sqrt{p_\xi^2 + k_n^2} |\eta - \eta'|}{|\eta - \eta'|}} (1 + |m_\eta \eta + m_\xi \xi|^{1/2}) e^{-k_n |m_\eta \eta + m_\xi \xi|}. \] (105)

Applying the Cauchy–Schwarz inequality to (104) gives
\[ c_n^{(i)} \leq \frac{\hat{b}_1}{2^{7/2}\sqrt{\pi}} \sum_{i=1}^{3} \|\Psi_n^{(i)}\| \left( \int d^3\eta' \int d^3p_\xi |V_{13}(\alpha' \eta')|^{1/2} \hat{f}_n(p_\xi) J^2(\eta', p_\xi) \right)^{1/2}. \] (106)

Inserting the estimate from lemma 5 we finally obtain
\[ c_n^{(i)} \leq \frac{\hat{b}_1 C}{2^{7/2}\sqrt{\pi}} \sum_{i=1}^{3} \|\Psi_n^{(i)}\| \left( \int_0^1 \frac{s^2(s^{1/2} + k_n^{1/2})^2 ds}{(s^2 + k_n^2)^{2/3}} + \int_{k_n}^\infty \frac{s^2(1 + k_n^{1-\delta})^2 ds}{(s^2 + k_n^2)^{1+\delta/2}} \right)^{1/2}. \] (107)

where \( C := \int d^3\eta' |V_{13}(\alpha' \eta')|(1 + |\eta'|)^{28} \) is finite by (17). The last integral in (107) is clearly uniformly bounded for all \( n \). To see that the first integral in (107) is uniformly bounded we use the following inequality:
\[ (s^{1-\delta} + k_n^{1-\delta})^2 \leq 2(s^{1-\delta})^2 + 2(k_n^{1-\delta})^2 \leq 4(s^2 + k_n^2)^{1-\delta}. \] (108)

where we used \( a^\alpha + b^\beta \leq (a+b)^\alpha \) for any \( a, b \geq 0 \) and \( 0 \leq \alpha \leq 1 \). Hence,
\[ \int_0^1 \frac{s^2(s^{1-\delta} + k_n^{1-\delta})^2 ds}{(s^2 + k_n^2)^{2/3}} \leq 4 \int_0^1 \frac{s^2 ds}{(s^2 + k_n^2)^{1/3}} \leq 4 \int_0^1 \frac{s^2 ds}{s^2 + 2k_n^2} \leq 8. \] (109)

Thus, the rhs of (107) goes to zero by lemma 2.

\[ \square \]

**Lemma 5.** The following estimates hold:
\[ J(\eta', p_\xi) \leq \frac{c(1 + |\eta'|)^\delta}{p_\xi^2 + k_n^2} \quad \text{for} \quad |p_\xi| \geq 1, \] (110)
\[ J(\eta', p_\xi) \leq \frac{c(1 + |\eta'|)^\delta}{(p_\xi^2 + k_n^2)^{1+\delta/2}} \quad \text{for} \quad |p_\xi| \leq 1, \] (111)

where \( c > 0 \) is a constant.
Proof. Using the trivial inequality $|z + z'|^4 \leq |z|^4 + |z'|^4$ for any $z, z' \in \mathbb{R}^3$ it is easy to see that

$$
\int d^3 \xi \left( 1 + |m_{y\rho} + m_{x\xi}|^3 \right) e^{-|m_{y\rho} + m_{x\xi}|^3} \leq c' (1 + |\eta|)^4, \tag{112}
$$

where $c' > 0$ is some constant. Using (105) and (112) we obtain

$$
J(\eta', p_\xi) \leq c' \int d^3 \eta e^{-\sqrt{|p_\xi + k_\eta|^2}} (1 + |\eta|)^4 \leq c' \int d^3 \eta e^{-\sqrt{|p_\xi + k_\eta|^2}} |(1 + |\eta| + |\eta'|)^4|
\leq c' \int d^3 \eta e^{-\sqrt{|p_\xi + k_\eta|^2}} |1 + |\eta'|^4 + |\eta|^4|.
\tag{113}
$$

Now the statement easily follows. 

Remark. The proof of lemma 3 is not just a mathematical formality, as can be illustrated by the following example. Suppose that two particles 2, 3 are identical and $V_\delta \leq 0$ for all $1 \leq i < k \leq 3$. Suppose also that $H \equiv H(1) \geq 0$, where $H(1)$ is defined through (1), and particle pairs $[1, 2]$ and $[1, 3]$ have zero-energy resonances. In this case there exists $[4, 5, 11]$ an orthonormal sequence $\phi_n$ such that $H\phi_n = E_n \phi_n$, where $E_n < 0$, $E_n \to 0$. Similar to lemma 1, one can prove that $\phi_n = f_n^{(12)} + f_n^{(13)}$, where $f_n^{(ik)} := [H_0 + k_\eta + k_\eta']^{-1} v_{ik} |\phi_0$ and the sequences $f_n^{(12)}$, $f_n^{(13)}$ must totally spread. However, a relation like (20) for $f_n^{(12)}$ (and a similar relation for $f_n^{(13)}$ with rotated Jacobi coordinates) would be wrong. Indeed, as we have already mentioned in the remark after theorem 2 one can choose the mass ratios in such a way that the sequence $\phi_n$ would have an angular momentum different from zero. At the same time, the limiting expression in (20) always has zero angular momentum. Additionally, one can prove that $(\phi_n, \phi_{n+1}) = 0$ would not hold in this case in the limit of large $n$. This example demonstrates that the condition that only one particle pair has a zero-energy resonance is crucial to the proof of lemma 3.

Finally, let us show how the angular probability distribution in (5) can be derived using a less rigorous but more physical approach. The derivation below was proposed by one of the referees, whose contribution is gratefully acknowledged. Suppose that the interaction between particles 1, 2 depends on $|x|$ and is resonant, while other pair-interactions are non-resonant. Let us consider the ground state wavefunction $\psi_\infty(x, y) > 0$ of the Hamiltonian (1) for $\lambda = \lambda_{\infty}$, which as we know from [1] is not normalizable. The wavefunction $\psi_\infty$ obeys the equation $[H_0 + v_{12} + v_{13} + v_{23}] \psi_\infty = 0$, where the interactions $v_{12}, v_{23}$ can be dropped because they are non-resonant (cf lemma 1). Since the remaining term in the Hamiltonian is invariant with respect to independent rotations of vectors $x$ and $y$, the ground state should possess the same symmetry, that is, we can write the wavefunction as $\psi_\infty(|x|, |y|)$. Following the recipe in [11, 18] we can replace the resonant interaction $v_{12}$ through the boundary condition $\partial(|x| \psi_\infty)/\partial|x| = 0$ and solve instead the equation $H_0 \psi_\infty = 0$ using this boundary condition. Setting $\psi_0(|x|, |y|) := |x||y|\psi_\infty(|x|, |y|)$ we obtain the following equation:

$$
\left( \frac{\partial^2}{\partial|x|^2} + \frac{\partial^2}{\partial|y|^2} \right) \psi_0(|x|, |y|) = 0, \tag{114}
$$

where $\psi_0(|x|, |y|)$ should satisfy boundary conditions $\partial \psi_0/\partial|x| = 0$ and $\psi_0(|x|, 0) = 0$. In polar coordinates (114) reads

$$
\frac{1}{\rho} \frac{\partial \psi_0(\rho, \theta)}{\partial \rho} + \frac{\partial^2 \psi_0(\rho, \theta)}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 \psi_0(\rho, \theta)}{\partial \theta^2} = 0, \tag{115}
$$

where $\rho, \theta$ were defined in section 1. Separating radial and angular variables one easily finds that the solution of (115), which satisfies the aforementioned boundary conditions, is given by
\(\rho^{-n}\sin(n\theta)\) for \(n = 1, 3, 5, \ldots\). The non-normalizable wavefunction corresponds to \(n = 1\), which gives \(\psi_0(\rho, \theta) = \rho^{-1}\sin(\theta)\). Returning back to the original wavefunction \(\psi_\infty\) results in \(\psi_\infty(\rho, \theta) = [\rho^{-3}\cos(\theta)]^{-1}\). This is the expression in (22) that we obtain after removing the normalization factor and setting \(k_n = 0\). This angular dependence in \(\psi_\infty(\rho, \theta)\) leads to the universal angular probability distribution (5).

4. Physical applications

In nuclear physics one encounters nuclei [19], which effectively possess the three-particle Borromean structure consisting of two neutrons and a tightly bound core. In most applications the core can be well treated as a structureless particle. Borromean in this context means that the three constituents are pairwise unbound rather like heraldic Borromean rings. The ground states in some of these nuclei are weakly bound and two neutrons form a dilute halo around the core. Thereby, a substantial part of the wavefunction is located in the classically forbidden region so that resulting inter-particle distances exceed by far the range of the interaction. Typical examples of such halo nuclei are weakly bound \(^{6}\text{He}\) and \(^{11}\text{Li}\). The calculated density correlation plots in [19, 20] reveal the formation of the so-called ‘dineutron peak’ in the ground state. There is another peak called a cigar-like peak but the substantial part of the wavefunction that is responsible for the halo formation concentrates in the dineutron peak. The dineutron peak is remarkably well fitted by the angular probability distribution in (5).

Additional applications could be found in Efimov physics. The so-called three-particle Efimov states predicted in [11] appear when two binary subsystems either have very large scattering lengths or bound states close to zero-energy threshold. Efimov states were found experimentally in the ultracold Bose gas of cesium atoms [21]. In [1] we predicted the existence of very spatially extended halo-like states for three atoms near zero-energy threshold, if one pair of atoms has a large scattering length (that is, it is close to the zero-energy resonance). These states can be looked for in ultracold gas mixtures prepared through the appropriate Feshbach tuning. The reported result shows that the density distribution in such system of three atoms would have a universal form described by (5), which at a sufficiently large distance should match the nucleon density in nuclear halos.

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References

[1] Gridnev D K 2012 J. Phys. A: Math. Theor. 45 175203 (arXiv:1111.6788v2)
[2] Gridnev D K 2012 J. Phys. A: Math. Theor. 45 395302 (arXiv:1112.0112v2)
[3] Petrov S V, Jarovoy S S and Babaev Yu A 1987 J. Phys. B: At. Mol. Phys. 20 4679
[4] Yafaev D R 1974 Math. USSR Sb. 23 535
Yafaev D R 1975 Notes LOMI Semin. 51 203 (Russian)
[5] Sobolev A V 1993 Commun. Math. Phys. 156 101
[6] Klaus M and Simon B 1980 Ann. Phys., NY 130 251
[7] Klaus M and Simon B 1980 Commun. Math. Phys. 78 153
[8] Gridnev D K and Garcia M E 2007 J. Phys. A: Math. Theor. 40 9005
[9] Bolle D, Gesztesy F and Schweiger W 1985 J. Math. Phys. 26 1661
[10] Gridnev D K 2012 J. Math. Phys. 53 102108
[11] Efimov V 1970 Phys. Lett. B 33 563
Efimov V 1971 Sov. J. Nucl. Phys. 12 589
[12] Helfrich K and Hammer H-W 2011 J. Phys. B: At. Mol. Opt. Phys. 44 215301
[13] Efimov V 1973 Nucl. Phys. A 210 157
[14] Bransden B H and Joachain C J 1990 Physics of Atoms and Molecules (Essex: Longman Scientific and Technical)
[15] Bartle R G 1995 The Elements of Integration and Lebesgue Measure (New York: Wiley)
[16] Gradshteyn I S and Ryzhik I M 1994 Table of Integrals, Series, and Products (London: Academic)
[17] Reed M and Simon B 1975 Methods of Modern Mathematical Physics vol 2 (New York: Academic)
[18] Reed M and Simon B 1978 Methods of Modern Mathematical Physics vol 4 (New York: Academic)
[18] Bethe H A and Peierls R 1935 Proc. R. Soc. Lond. A 148 146
[19] Zhukov M V, Danilin B V, Fedorov D V, Bang J M, Thompson I J and Vaagen J S 1993 Phys. Rep. 231 151
[20] Oganessian Y T, Zagrebaev V I and Vaagen J S 1999 Phys. Rev. Lett. 82 4996
[21] Kraemer T et al 2006 Nature 440 315