A CHARACTERIZATION OF QUASIPOSITIVE SEIFERT SURFACES (CONSTRUCTIONS OF QUASIPOSITIVE KNOTS AND LINKS, III)

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§0. INTRODUCTION; STATEMENT OF RESULTS

Let \( T_{n, n} \subset S^3 \) be a fiber surface for the torus link \( O\{n, n\} \); say, to be concrete, \( T_{n, n} = \{(z, w) \in S^3 : z^n + w^n \geq 0\} \subset S^3 = \{(z, w) : |z|^2 + |w|^2 = 1\} \).

Characterization Theorem. A Seifert surface \( S \) is quasipositive if and only if, for some \( n \), \( S \) is ambient isotopic to a full subsurface of \( T_{n, n} \). □

This should be contrasted with the following.

Lyon’s Theorem [4]. Any Seifert surface is ambient isotopic to a (full) subsurface of the fiber surface of \( O\{n, n\} \neq O\{n, n\} \) for some \( n \). □

Here, a surface is smooth, compact, oriented, and has no component with empty boundary. A Seifert surface is a surface embedded in \( S^3 \). A subsurface \( S \) of a surface \( T \) is full if each simple closed curve on \( S \) that bounds a disk on \( T \) already bounds a disk on \( S \). The definition of quasipositivity is recalled in §1 after a review of braided surfaces.

The “only if” statement of the Characterization Theorem is proved in [2]. Some results about graphs on braided surfaces (stated, with one eye on other applications [6], in somewhat more generality than needed here) are obtained in [3] and used in [4] to prove the “if” statement of the Characterization Theorem. A conjectural extension to ribbon surfaces in the 4-disk is discussed in [5].

Remark. The relation “\( S \) is a full subsurface of \( T \)” is transitive, so the Characterization Theorem has the interesting corollary (applied in [3] and [7]) that any full subsurface of any quasipositive Seifert surface is quasipositive.

§1. BRAIDS, BRAIDED SURFACES, AND QUASIPOSITIVITY

Let \( \sigma_1, \ldots, \sigma_{n-1} \) be the usual generators of the \( n \)-string braid group \( B_n \). For \( 1 \leq i < j \leq n \), let \( \sigma_{i,j} := (\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_i \cdots \sigma_{j-2})^{-1} \). A positive (resp., negative) embedded band in \( B_n \) is any \( \sigma_{i,j} \) (resp., \( \sigma_{i,j}^{-1} \)). An embedded band representation in \( B_n \), of length \( k \), is a \( k \)-tuple \( b = (b(1), \ldots, b(k)) \) of embedded bands. If each \( b(t) \) is positive, then \( b \) is

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quasipositive. If each $b(t)$ is some usual generator $\sigma_i = \sigma_{i-1}$ or inverse $\sigma_i^{-1}$, then $b$ is a braidword. (A quasipositive braidword is customarily simply called positive.)

Given $b$, a braided surface $S(b)$ is constructed as in Fig. 1 (cf. [2–4]). More precisely, if $b(t) = \sigma_{i(t)}, j(t) \in B_n, 1 \leq i(t) < j(t) \leq n, \varepsilon(t) = \pm 1, 1 \leq t \leq k$, then $S(b) \subset \mathbb{R}^3 \subset S^3$ is a Seifert surface with handle decomposition $\bigcup_{s=1}^{n} h_s^{(0)} \cup \bigcup_{t=1}^{k} h_t^{(1)}$ such that, for appropriate rectangular coordinates $(x,y,z)$ on $\mathbb{R}^3$,

1. $h_0^{(0)}$ is contained in the vertical half plane $\{(s,y,z): y \geq 0, z \in \mathbb{R}\}$,
2. $h_1^{(1)}$ is contained in the box $\{(x,y,z): i(t) \leq x \leq j(t), y \leq 0, t-1 \leq z \leq t\}$,
3. $h_t^{(1)}$ joins $h_0^{(0)}$ and $h_j^{(0)}$ with a single half-twist of sign $\varepsilon(t)$.

According to [2], this construction has a converse: if $S$ is any Seifert surface, then there exists $b$ (highly nonunique!) such that $S$ is ambient isotopic to $S(b)$. $S$ is quasipositive if some such $b$ is quasipositive.

Note that (after enlarging its 0-handles in their half-planes as necessary) the link $\partial S(b)$ is presented, with respect to a horizontal braid axis (omitted from the figure) as the closed braid $\hat{\beta}(b)$, where $\hat{\beta}(b) := b(1) \cdots b(k)$.

§2.

The “only if” statement in the Characterization Theorem follows immediately from [2.1] and [2.2] plus the observation that the relation “$S$ is a full subsurface of $T$” is transitive.

**Lemma 2.1.** If $p$ is a positive braidword in $B_m$, then, for some $n$, $S(p)$ is ambient isotopic to a full subsurface of $T_{n,n}$.

**Proof.** Let $\nabla_n$ be the positive braidword of length $n(n-1)$ in $B_n$ with $\nabla(s) = \sigma_d$ if $s = (n-1)c + d$, $0 \leq c \leq n-1$, $1 \leq d \leq n-1$. Then $\hat{\beta}(\nabla_n)$ is a torus link $O\{n,n\}$ of type $(n,n)$, and $S(\nabla_n)$ is a fiber surface $T_{n,n}$ (cf. [3]). Let $n$ be the greater of $m$ and the length of the braidword $p$; make the usual identification of $B_m$ with a subgroup of $B_n$. By lavishly inserting letters into $p$, pad it out to $\nabla_n$. Now $S(p)$ may be constructed as a (manifestly full) subsurface of $S(\nabla_n)$.

**Lemma 2.2.** If $S$ is a quasipositive Seifert surface, then $S$ is ambient isotopic to a full subsurface of $S(p)$ for some positive braidword $p$ in some $B_n$.

‡ The orientation conventions in [2–4] are opposed to those of [5], [6], and the present paper.
Proof. We may assume that $S = S(b)$, where $b = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ is a quasipositive embedded braid word in some $B_n$. By inspection (cf. the 1-handle pictured in Fig. 2) $S(b)$ is ambient isotopic to a (full) subsurface of $S(p)$ where $p$ is the positive braid word $(\sigma_1, \sigma_2, \ldots, \sigma_k)$. □

§3. GRAPHS ON BRAIDED SURFACES

A graph is a polyhedron $G$ of dimension $\leq 1$. If $g \in G$, then $g$ is an isolated point (resp., endpoint; ordinary point; intrinsic vertex) if the link of $g$ in $G$ consists of 0 (resp., 1; 2; $v(g) \geq 3$) points. $G$ is trivalent if $v(g) = 3$ for each intrinsic vertex $g \in G$.

A graph $E$ contained in a vertical half-plane $\{(s,y,z) : y \geq 0, z \in R\}$ is a comb if $E$ is the union of a vertical interval $\{s,y,z) : y = y_0, z_1 \leq z \leq z_m\}$ and $m \geq 2$ horizontal intervals $\{(s,y,z) : x = s, y_0 \geq y \geq 0, z_0 \leq z \leq z_m\}$. $E$ is trivalent, with $m$ endpoints (on the boundary of the half-plane) and $m - 2$ intrinsic vertices.

Let $S$ be a surface in which a graph $G$ is embedded. Let $N_S(G)$ be a regular neighborhood of $G$. $G$ is full if no simple closed curve $C \subset G$ bounds a disk on $S$ (so $G$ is full if and only if $N_S(G)$ is), and $G$ is proper if $G \cap \partial S$ is precisely the set of endpoints of $G$. If $N \subset S$ is a subsurface such that each component of $N \cap \partial S$ is an arc, then $N$ is a regular neighborhood $N_S(G)$ of some proper trivalent graph $G \subset S$.

Let $G'$ be another graph embedded in $S$. $G$ and $G'$ are isotopic if there exists a piecewise-smooth isotopy of $S$ carrying $G$ onto $G'$. (It may not be possible to find such an isotopy which is smooth near intrinsic vertices.) $G$ and $G'$ are Whitehead-equivalent if $N_S(G)$ and $N_S(G')$ are ambient isotopic on $S$.

Given a handle decomposition $S = \bigcup_{s=1}^{n} h_s^{(0)} \cup \bigcup_{t=1}^{k} h_t^{(1)}$, set $G^{(0)} := G \cap \bigcup_{s=1}^{n} h_s^{(0)}$, $G^{(1)} := G \cap \bigcup_{t=1}^{k} h_t^{(1)}$. $G$ is well-placed if each point of $G^{(1)}$ is an ordinary point of $G$, and each component of $G^{(1)}$ is isotopic to a core arc of some $h_t^{(1)}$. If $S = S(b)$ is braided and $\bigcup_{s=1}^{n} h_s^{(0)} \cup \bigcup_{t=1}^{k} h_t^{(1)}$ is its given handle decomposition, $G$ is combed if it is well-placed and $G^{(0)}$ is a disjoint union of combs and isolated points.

Lemma 3.1. Any graph on $S$ is isotopic to a well-placed graph.

Proof. Obvious. □

Lemma 3.2. Any trivalent, full, proper graph on $S(b)$ is isotopic, by an isotopy supported on the 0-handles, to a combed graph.
Proof. Let \( G \) satisfy the hypotheses. By \ref{lem:combed} we can assume \( G \) is well-placed. Let 
\[ S(b) \cap \{(x,y,z): y \geq 0, z \in \mathbb{R}\} = h^0 \] be a 0-handle. By fullness of \( G \) and the Jordan Curve Theorem, each component of \( G \cap h^0 \) is an isolated point or a tree. By the properness of \( G \), if \( e \) is an endpoint of \( G \cap h^0 \), then \( e \in \partial h^0 \), and \( e \) is interior to an attaching arc of some 1-handle (resp., exterior to all the attaching arcs) iff \( e \) is an ordinary point (resp., an endpoint) of \( G \). Thus after a preliminary isotopy supported in a collar of \( \partial h^0 \), we can assume that every endpoint of \( G \cap h^0 \) is an interior point of the interval \( J := S(b) \cap \{(s,0,z): z \in \mathbb{R}\} \).

For each tree component \( E \) of \( G \cap h^0 \), let \( I \subset J \) be the smallest subinterval containing all the endpoints of \( E \), let \( A \) be the subarc of \( E \) joining the endpoints of \( I \), and let \( D(E) \) be the subdisk of \( h^0 \) bounded by \( I \cup A \). The set of disks \( D(E) \) is partially ordered by inclusion. Using trivalence of \( G \), we can construct the desired isotopy by induction over this poset.

\[ \square \]

**Lemma 3.3.** Let \( G \subset S(b) \) be combed. If for some \( s, t \), there is a comb \( E \subset G \cap h^0 \) which has two or more endpoints in the attaching arc \( h^0 \cap h^1 \), then there is a combed graph \( G' \) Whitehead-equivalent to \( G \) such that \( G'^{(1)} \) has fewer components than \( G^{(1)} \).

**Proof.** Let \( E = \{(s,y_0,z): z_1 \leq z \leq z_m\} \cup \bigcup_{i=1}^m \{(s,y,z): y_0 \geq y \geq 0, z = z_i\} \) be such a comb. Then, for some \( i \), adjacent endpoints \( (s,0,z_i), (s,0,z_{i+1}) \) of \( E \) are in \( h^1 \). Altering \( G \) in a neighborhood of \( h^0 \cap \bigcup \{(s,y,z): 0 \leq y \leq y_0, z_i \leq z \leq z_{i+1}\} \), we can replace \( G \) with a Whitehead-equivalent graph \( G'' \) which is still trivalent, full, proper, and well-placed, such that \( G''^{(1)} \) has fewer components than \( G^{(1)} \). (Cf. Fig. 3) the “Whitehead move” we use is the equivalent, on the level of graphs, of a handle-slide on the level of regular neighborhoods.) Then we apply \ref{lem:combed} to \( G'' \) to obtain \( G' \).

\[ \square \]

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**§4.**

At this point, \( S(\nabla_n) \) is no longer the best choice of a braided surface ambient isotopic to \( T_{n,n} \). Instead, for \( \nu := (n-1)^2 + 1 \), let \( q \) be the quasipositive band representation of length \( 2(\nu-1) \) in \( B_\nu \) defined as follows: for \( 1 \leq s \leq \nu-1 \), set \( q(s) = \sigma_{1,\nu-s+1} \); for \( \nu \leq s \leq 2(\nu-1) \), if \( s-\nu = (n-1)c+d \), \( 0 \leq c,d \leq n-2 \), then set \( q(s) = \sigma_{1,\nu-c-(n-1)d} \). (The case \( n = 3 \) is illustrated in Fig. 4.)
LEMMA 4.1. $S(q)$ is ambient isotopic to $T_{n,n}$.

Proof. First show that $\hat{\beta}(q)$ is ambient isotopic to $O\{n,n\}$ (this is a straightforward exercise in braid relations and Markov moves, given that $O\{n,n\} = \hat{\beta}(\nabla_n)$). Then calculate

$$\chi(S(q)) = v - 2(v - 1) = 2 - v = 1 - (n - 1)^2 = n - n(n - 1) = \chi(T_{n,n}).$$

Finally, use the familiar fact that up to ambient isotopy a fiber surface is the unique Seifert surface of maximal Euler characteristic for its boundary.

In addition to its given handle decomposition $S(q) = \bigcup_{s=1}^{v} h_s^{(0)} \cup \bigcup_{t=1}^{2(v-1)} h_t^{(1)}$, which will now be called fine, we need a coarse handle decomposition with a single 0-handle $H_1^{(0)} := h_1^{(0)}$ and $v - 1$ 1-handles $H_i^{(1)} := h_i^{(1)} \cup h_{v-s+1}^{(0)} \cup h_{v+s}^{(1)}$ where $1 \leq s = 1 + c + (n - 1)d \leq v - 1$, $0 \leq c, d \leq n - 1$, $s' = d + (n - 1)c$. Note that, if $G$ is well-placed with respect to the coarse handle decomposition, then $G$ is certainly well-placed with respect to the fine one.

THEOREM 4.2. Any full subsurface of $T_{n,n}$ is quasipositive.

Proof. Let $S \subset S(q)$ be a full subsurface. We may assume $S \cap \partial S(q) = \emptyset$. Among all graphs $G \subset S(q)$ with $N_{S(q)}(G)$ isotopic to $S$ on $S(q)$, such that $G$ is proper, full, trivalent, and well-placed with respect to the coarse handle decomposition, let $G_0$ be such that $G_0^{(1)}$ (also with respect to the coarse handle decomposition) has the minimal number of components. Now apply 3.2 with respect to the fine handle decomposition, to find a combed graph $G_1$ isotopic to $G_0$. Let $E \subset G_1$ be a comb; then $E$ has no more than one endpoint in any attaching arc of a 1-handle of the fine handle decomposition, for otherwise 3.3 would contradict the assumed minimality of $G_0$.

We are nearly done. Of course $N_{S(q)}(G_1)$ is isotopic to $S$ on $S(q)$. On the other hand, Fig. 5 shows how to perform local moves (in the vicinity of each coarse 1-handle) which effect an ambient isotopy of $N_{S(q)}(G_1)$ is in $\mathbb{R}^3$ that pushes $N_{S(q)}(G_1)$ off $S(q)$ and onto a quasipositive braided surface. (The fact that the coarse decomposition has a single
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Fig. 5. The black surface on the left is $N_{x_0}(G_i)$; it is isotopic, in $\mathbb{R}^3$, to the black surface on the right, which can be braided by expanding the thin vertical pieces into fat 0-handles.

0-handle ensures that the local moves do not interfere with each other. It was for this reason that the coarse decomposition was introduced. □

Taken with §2.4.2 completes the proof of the Characterization Theorem.

§5. A CONJECTURAL GENERALIZATION

Let $d : \mathbb{C}^2 \to [0, \infty] : (z, w) \mapsto (|z|^2 + |w|^2)^{1/2}$. A smoothly embedded surface $S \subset D^4(r) := d^{-1}([0, r])$, $r > 0$, is ribbon-embedded if $\partial S = S \cap \partial D^4(r)$ and the restriction $d^2|S$ is a Morse function which has no local maxima on Int $S$. If $S \subset D^4(r)$ is ribbon-embedded and $r' \in [0, r]$ is a regular value of $d^2|S$, then $S' := S \cap D^4(r')$ is ribbon-embedded in $D^4(r')$, and $S'$ is full on $S$; call $(S, S')$ a ribbon-embedded pair. A surface ambient isotopic to a ribbon embedded surface is a ribbon; a subsurface $S' \subset S$ of a ribbon is a subribbon if the pair $(S, S')$ is ambient isotopic to a ribbon-embedded pair.

Let $\Gamma_n(\varepsilon)$ be the complex plane curve \{$(z, w) \in \mathbb{C}^2 : z^n + w^n = \varepsilon$\}. Then, for all sufficiently small $\varepsilon \neq 0$, the sets $\Gamma_n(\varepsilon) \cap D^4(1)$ are mutually isotopic ribbon-embedded surfaces. Let $\tilde{T}_{n, n}$ denote any one of them.

**Conjecture.** A ribbon is quasipositive if and only if, for some $n$, it is ambient isotopic to a subribbon of $\tilde{T}_{n, n}$.

Here, by appealing to results in [2], we can define *quasipositive ribbon* as follows (the original definition involves braids, and “band representations” more general than the embedded band representations used in this paper). Let $D$ denote the bidisk \{$(z, w) : |z| \leq 1, |w| \leq 1$\}, $\partial_1 D := \{(z, w) : |z| = 1, |w| \leq 1$\}. Let $\Gamma$ be a non-singular complex plane curve which intersects $\partial_1 D$ transversely and the rest of $\partial D$ not at all. Let $\eta : D \to D^4(1)$ be a smoothing (a homeomorphism which is a diffeomorphism except along the corner torus $\partial(\partial_1 D)$). Then $\eta(\Gamma \cap D)$ is a quasipositive ribbon, and every quasipositive ribbon arises this way.
The following facts are known, cf. [1], [2], [3]. (1) One may “push in” the interior of a Seifert surface in $S^3 = \partial D^4(1)$ to obtain a ribbon with the same boundary; if the Seifert surface is quasipositive, so is the ribbon. (2) In particular, $T_{n,n}$ itself is a quasipositive ribbon, for it is the push-in of the 3-sphere—of the Milnor fiber of $z^n + w^n$. (3) If $S$ is a quasipositive ribbon, then for some $n$, $S$ is a subribbon of $\tilde{T}_{n,n}$. (This can be seen as follows. Realize $S$ by $\eta(\Gamma \cap D)$ as above. Without loss of generality, the completion of $\Gamma$ in $\mathbb{CP}^2 \supset \mathbb{C}^2$ is non-singular and transverse to the line at infinity. Let $f(z,w)$ be the defining polynomial of $\Gamma$. For all sufficiently large $R > 0$, $\eta(\Gamma \cap D)$ is isotopic to a union of components of the ribbon-embedded surface $\{(z,w) \in \mathbb{C}^2 : f(z,Rw) = 0\} \cap D^4(1)$. This is in turn a subribbon of $\{(z,w) \in \mathbb{C}^2 : f(z,Rw) = 0\} \cap D^4(r)$ for generic $r \geq 1$. Because $\Gamma$ is transverse to the line at infinity, $\{(z,w) \in \mathbb{C}^2 : f(z,Rw) = 0\} \cap D^4(r)$ is isotopic to $\{(z,w) \in \mathbb{C}^2 : z^n + w^n = 1\} \cap D^4(r)$ for all sufficiently large $r$, where $n$ is the degree of $f$. Finally, by rescaling, this last ribbon is isotopic to $\tilde{T}_{n,n}$.)

Thus what remains conjectural is that every subribbon of $\tilde{T}_{n,n}$ is quasipositive. If this is true, then (up to ambient isotopy) an oriented link $L$ in $S^3 = \partial D^4(r)$ bounds a piece of complex plane curve in $D^4(r)$ if and only if $L$ has some representation as the closure of a quasipositive braid. The Characterization Theorem gives some hope that this is so, but it seems likely that entirely different techniques will be needed for a proof.

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ADDENDA

Typographical errors in the original publication have been corrected without notice; it is to be hoped that no new ones have been introduced. The following notes provide updates on various points.

1. Boileau and Orevkov [10] have proved that, indeed, up to ambient isotopy an oriented link $L$ in $S^3$ bounds a piece of complex plane curve in $D^4$ if and only if $L$ has some representation as the closure of a quasipositive braid. In fact their proof shows that many subribbons of $\tilde{T}_{n,n}$ are quasipositive; as far as I am aware, the Conjecture in full generality remains open (but is perhaps of somewhat less interest).

2. [5] was published as [11].

3. A considerably expanded version of [6] was published as [12].

Additional References

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