LIMIT PERIODIC JACOBI MATRICES WITH A SINGULAR CONTINUOUS SPECTRUM AND THE RENORMALIZATION OF PERIODIC MATRICES

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Abstract. For all hyperbolic polynomials we proved in [11] a Lipschitz estimate of Jacobi matrices built by orthogonalizing polynomials with respect to measures in the orbit of classical Perron-Frobenius-Ruelle operators associated to hyperbolic polynomial dynamics (with real Julia set). Here we prove that for all sufficiently hyperbolic polynomials this estimate becomes exponentially better when the dimension of the Jacobi matrix grows. In fact, our main result asserts that a certain natural non-linear operator on Jacobi matrices built by a hyperbolic polynomial with real Julia set is a contraction in operator norm if the polynomial is sufficiently hyperbolic. This allows us to get for such polynomials the solution of a problem of Bellissard, in other words, to prove the limit periodicity of the limit Jacobi matrix. This fact does not require the iteration of the same fixed polynomial, and therefore it gives a wide class of limit periodic Jacobi matrices with singular continuous spectrum.

1. Introduction

Let $T$ be an expanding polynomial with the real Julia set $\text{Julia}(T)$, $\deg T = d$. We recall that $\text{Julia}(T)$ is a nonempty compact set of points which do not go to infinity under forward iterations of $T$. Under the normalization

$$T^{-1} : [-1, 1] \to [-1, 1]; \pm 1 \in \text{Julia}(T)$$

such a polynomial is well-defined by the position of its critical values

$$\text{CV}(T) := \{t_i = T(c_i) : T'(c_i) = 0, c_i > c_j \text{ for } i > j\}.$$ 

Expanding, or hyperbolic polynomials are those, for which

$$c_i \notin \text{Julia}(T), \forall i,$$

which is the same as to say that $\text{CV}(T) \cap \text{Julia}(T) = \emptyset$ (just use the fact that $\text{Julia}(T)$ is invariant under taking full preimage $T^{-1}$). The term “expanding” is deserved because for expanding polynomials one has the following inequality

$$\exists Q > 1, |(T^n)'(x)| \geq cQ^n, \forall x \in \text{Julia}(T).$$

Here and in everything what follows $T^n$ means $n$-th iteration of $T$, $T^n = T \circ T \circ \ldots \circ T$. 

Date: December 15, 2021.

Partially supported by NSF grant DMS-0200713, the Austrian Science Fund FWF, project number: P16390-N04 and Marie Curie International Fellowship within the 6-th European Community Framework Programme, Contract Number: MIF1-CT-2005-006966.

AMS subject classification codes: 42B20, 42C15, 42A50, 47B35, 47B38.

Key words: almost periodic Jacobi matrices, singular continuous spectrum, hyperbolic polynomials, harmonic measure.
Let us mention that for $T$ with a real Julia set one has $|T(c_i)| > 1$ since all solutions of $T(x) = \pm 1$ should be real.

We will need to consider the notion of “sufficiently expanding” (“sufficiently hyperbolic”) polynomials. As we saw, the expanding property is the same (in our normalization) as $\text{dist}(\text{CV}(T), [-1, 1]) > 0$. The polynomial $T$ with normalization (1.1) will be called sufficiently hyperbolic (or sufficiently expanding) if

$$\text{dist}(\text{CV}(T), [-1, 1]) \geq A,$$

where $A$ is a large absolute constant to be specified later (but $A = 10$ will work). Notice that the definition of sufficient hyperbolicity does not involve the degree of $T$. In particular, $T$ and any of its iterative powers $T^2, T^3, \ldots$ are sufficiently hyperbolic simultaneously.

A Jacobi matrix $J : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is called almost periodic if the family

$$\{S^{-k} J S^k\}_{k \in \mathbb{Z}},$$

where $S$ is the shift operator in $l^2(\mathbb{Z})$, $S|m| = |m + 1|$, is a precompactum in the operator topology.

**Example.** Let $G$ be a compact abelian group, $p(\alpha), q(\alpha)$ be continuous functions on $G$, $p(\alpha) \geq 0$. Then $J(\alpha)$ with the coefficient sequences $\{p(\alpha + k\mu)\}_k, \{q(\alpha + k\mu)\}_k$, $\mu \in G$, is almost periodic.

Let us show that in fact this is a general form of almost periodic Jacobi matrices. For a given almost periodic $J$ define the metric on $\mathbb{Z}$ by

$$\rho_J(k) := ||S^{-k} J S^k - J||.$$

Evidently $\rho_J(k + m) \leq \rho_J(k) + \rho_J(m)$. Then $J = J(0)$, where $G = I_J$, $I_J$ is the closure of $\mathbb{Z}$ with respect to $\rho_J$, and $\mu = 1 \in I_J$.

Recall that for a given system of integers $\{d_k\}_{k=1}^\infty$ one can define the set

$$I = \lim_{l \to \infty} \{\mathbb{Z}/d_1 \ldots d_l \mathbb{Z}\},$$

(1.4)

that is $\alpha \in I$ means that $\alpha$ is a sequence $\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ such that

$$\alpha_k \in \mathbb{Z}/d_1 \ldots d_{k+1} \mathbb{Z} \quad \text{and} \quad \alpha_{k+1} \mod d_1 \ldots d_k = \alpha_k - 1.$$

The addition in $I$ is defined as addition in the $l$-th entry. The metric $\text{dist}(\alpha, \beta) = \kappa^l$, where $\kappa \in (0, 1)$, $l$ is the smallest integer such that $\alpha_l \neq \beta_l$, makes $I$ a compact abelian group. In particular, if $p$ is a prime number and $d_k = p$ we get the ring of $p$-adic integers, $I = \mathbb{Z}_p$.

In this work we build a certain machinery that allows to construct almost periodic Jacobi matrices with singularly continuous spectrum such that $I_J = I$.

The key element of the construction is the following

**Theorem 1.1.** Let $\bar{J}$ be a Jacobi matrix with the spectrum on $[-1, 1]$. Then the following Renormalization Equation has a solution $J = \bar{J}$ (J; $T$) with the spectrum on $T^{-1}([-1, 1])$:

$$V^*(z - J)^{-1} V = (T(z) - \bar{J})^{-1} T'(z)/d,$$

where $V|k| = |dk|$. Moreover, if $\min_{t_i} |t_i| \geq 10$ then

$$||J(\bar{J}_1) - J(\bar{J}_2)|| \leq \kappa ||\bar{J}_1 - \bar{J}_2||,$$

with an absolute constant $\kappa < 1$ (does not depend on $T$ also).

This theorem, for example, will result in the following consequence:
**Theorem 1.2.** Let $T$ be sufficiently hyperbolic in the sense of (1.3). Let $J_\omega$ be the Jacobi matrix obtained by orthogonalizing polynomials with respect to the balanced (equilibrium) measure $\omega$ on the Julia set of $T$. Then $J_\omega$ is a limit periodic matrix. In other words, the sequences that give the diagonal and the below (above) diagonal entries are uniform limits of periodic sequences.

**Remarks.**

1) A known problem (due to Bellissard) asks to prove this statement for all hyperbolic polynomials normalized as in (1.1). Here we do it for all sufficiently hyperbolic polynomials (recall that we always tacitly assume that Julia($T$) is on the real line). Our result about sufficiently hyperbolic polynomials explains, in a sense, the earlier results in [2], [3], where it has been shown that polynomials $T(z) := \varepsilon^{-n}T_n(\varepsilon z)$ generate limit periodic Jacobi matrices if $\varepsilon$ is sufficiently small, here $T_n$ is the $n$-th Tchebyshef polynomial. Smallness of $\varepsilon$ obviously makes $T = \varepsilon^{-n}T_n(\varepsilon z)$ sufficiently hyperbolic in our sense (1.3).

2) In the thesis of Herndon [7] Theorem 1.2 is proved by another method. We regret that it has not been published, that might have clarified the proof, which seems to be quite involved.

3) One can wonder after analyzing the results of [2], [3] and the present paper, that may be there is a threshold of hyperbolicity: before it $J_\omega$ is not limit periodic, and after it it is limit periodic. However, we do not believe in this sort of behavior, but at this stage we cannot prove the conjecture of Bellissard for all hyperbolic polynomial with real Julia set.

4) Let us mention that, in fact, (1.5) has $2^{d-1}$ solutions such that the spectrum of $J$ is on $T^{-1}([-1, 1])$. Here we use only one of them.

We note that the real output of Theorem 1.1 is much wider than Theorem 1.2. It shows that

- roughly speaking, constructing in a regular iterative way a Cantor set $E$, $E \subset \cdots \subset E_{n+1} \subset E_n \ldots$, that may support the spectrum of a limit–periodic Jacobi matrix it is enough to follow the strategy: on each step the approximating set $E_n$ should have a form of an inverse polynomial image, i.e.:
  $$E_n = U_n^{-1}[-1, 1], \quad U_n \text{ is a polynomial},$$
  or, what is the same, $E_n$ should be the spectrum of a periodic Jacobi matrix;

- the above statement becomes a theorem if on each step we remove from the previous set a sufficiently large part (using sufficiently expanding polynomials), i.e. if $T_1, T_2, \ldots$, is a sequence of polynomials with sufficiently large critical values, then $E_n := U_n^{-1}[-1, 1]$, with $U_n = T_n \circ \ldots \circ T_2 \circ T_1$;

- the set $E$, that was constructed in the above described way, is the spectral set of infinitely many (uncountable set) of different limit periodic Jacobi matrices, that means that each of the matrices does not belong to the hull of another one (can’t be obtained as a limit of shifts). The problem: to describe the set of all limit periodic Jacobi matrices with spectrum $E$ or certain subclasses (or at least to try to understand how these sets look like), is a very interesting and challenging problem.
Let us outline a proof of claim b). First, we point out the following two properties of the function $J(\tilde{J}; T)$ in Theorem 1.1. Due to the commutant relation $VS = S^d V$ one gets $J(S^{-m} \tilde{J} S^m) = S^{-dm} J(\tilde{J}) S^{dm}$. The second property is

\[ J(J(\tilde{J}; T_2); T_1) = J(\tilde{J}; T_2 \circ T_1), \]

that is, the chain rule holds.

Now, we produce a limit periodic Jacobi matrix with singularly continuous spectrum and such that $I_J = I$. For the chosen system of polynomials $T_1, T_2, \ldots$, $\deg T_k = d_k$, with sufficiently large critical values, define $J_m = J(J; T_m \circ \ldots \circ T_2 \circ T_1)$. By Theorem 1.1, the limit $I_J = \lim_{m \to \infty} J_m$ exists and does not depend on $J$. Moreover,

\[ \forall j, \ |J - S^{-d_1 \ldots d_j} JS^{d_1 \ldots d_j}| \leq ||J - S^{-j} JS^{j}|| \leq 2 \kappa. \]

That is, $\rho_J$ defines on $\mathbb{Z}$ the standard $p$–adic topology in this case. This proves that $J$ is a limit periodic matrix, in particular, it is almost periodic.

Notice that for the case $T_1 = T_2 = \ldots = T_m =: T$, we get the limit periodic matrix with the spectrum on Julia($T$).

2. Renormalization Equation

In this section it is convenient to assume that

\[ T(z) = z^d - qdz^{d-1} + \ldots \]

is a *monic* expanding polynomial. Under this normalization $T^{-1} : [-\xi, \xi] \to [-\xi, \xi]$ for a certain $\xi > 0$.

Let $\tilde{J} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be a Jacobi matrix with the spectrum on $[-\xi, \xi]$. We describe the set of solutions of the Renormalization Equation

\[ V^*(z - J)^{-1} V = (T(z) - \tilde{J})^{-1} T'(z)/d, \quad V|k) = |kd), \]

(2.1)

here $J$ is a Jacobi matrix with the spectrum on $T^{-1}([-\xi, \xi])$ that should satisfy (2.1).

In what follows by $l^2_{\pm}(s)$ we denote the spaces which are formed by $\{|s + k|\}$ with $k \leq 0$ and $k \geq 0$ respectively, that is $l^2(\mathbb{Z}) = l^2_+(s) \oplus l^2_+(s + 1)$. Correspondingly to these decompositions we set $\bar{J}(s) = P_{l^2_+(s)} J_{l^2_+(s)}$.

Recall that a (finite or infinite) one–sided Jacobi matrix is uniquely determined by its so called resolvent function

\[ \tilde{r}_\pm(z, s) = \langle s | (\bar{J}_\pm(s) - z)^{-1} | s \rangle, \]

(2.2)

for which the following decomposition in the continued fraction holds true

\[ \tilde{r}_+(z, s) = \frac{-1}{z - \tilde{q}_s} = \frac{-1}{z - \tilde{q}_s + \tilde{p}_s \tilde{r}_+(z, s + 1)}. \]

(2.3)

**Lemma 2.1.** Assume that a matrix $J$ satisfies (2.1). Then

\[ p_{sd+1} \ldots p_{sd+d} = \tilde{p}_{s+1} \quad \text{and} \quad q_{sd} = q. \]

(2.4)
Further, let \( J^{(s)} \) be the \( s \)-th \( d \times d \) block of \( J \), that is,

\[
J^{(s)} = \begin{bmatrix}
q_{sd} & p_{sd+1} & & \\
p_{sd+1} & q_{sd+1} & p_{sd+2} & \\
 & \ddots & \ddots & \ddots \\
p_{sd+d-2} & q_{sd+d-2} & p_{sd+d-1} & p_{sd+d} \\
& & & q_{sd+d} &
\end{bmatrix}.
\]  

Then its resolvent function is of the form

\[
\langle 0 \left| (z - J^{(s)})^{-1} \right| 0 \rangle = \frac{T'(z)/d}{T^{(s)}(z)},
\]

where \( T^{(s)}(z) \) is a monic polynomial of degree \( d \). Moreover, at the critical points \( \{ c : T'(c) = 0 \} \) the following recurrence relation holds

\[
T^{(s)}(c) + \frac{\tilde{p}_s^2}{T^{(s-1)}(c)} = T(c) - \tilde{q}_s.
\]

**Proof.** We write the Jacobi matrix \( J \) as a \( d \times d \) block matrix (each block is of infinite size), that is, we are just reordering the standard basis:

\[
J = \begin{bmatrix}
Q_0 & P_1 & \cdots & SP_d \\
P_1 & Q_1 & P_2 & \\
 & \ddots & \ddots & \ddots \\
P_{d-2} & Q_{d-2} & P_{d-1} & P_d \\
& & & Q_d &
\end{bmatrix}.
\]  

Here \( P_k \) (respectively \( Q_k \)) is a diagonal matrix \( P_k = \text{diag}\{ p_{k+sd} \}_{s \geq 0} \) and \( S \) is the shift operator. With respect to this reordering \( V^* \) is the projection on the first–place block–component.

Using this representation and the well known identity for block matrices

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1} = \begin{bmatrix}(A - BD^{-1}C)^{-1} & * \\
* & *
\end{bmatrix},
\]

we get

\[
\frac{T(z) - \tilde{J}}{T'(z)/d} = z - Q_0 - \{ P_1, \ldots, SP_d \} \{ z - J_1 \}^{-1} \begin{bmatrix} P_1 \\
P_d S^* \end{bmatrix},
\]

where \( J_1 \) is the matrix that we obtain from \( J \) by deleting the first block–row and the first block–column in (2.8). Thus the second relation in (4.1) is already proved, we just compare the leading terms in the decomposition over powers of \( 1/z \) in the right and left hand sides and note that the third term on the right is of order \( 1/z \).

But the most important remark is that in \( (z - J_1) \) each block is a diagonal matrix (means all diagonals are the main diagonals in each block, on the contrary to \( J \) that contains \( SP_d \) and \( P_d S^* \)). That’s why we can easily get an inverse matrix in terms of the scalar orthogonal polynomials.

Let us introduce the following notations: everything related to \( J^{(s)} \) has superscript \( s \). For instance: \( p_k^{(s)} = p_{sd+k}, 1 \leq k \leq d \), respectively \( P_d^{(s)} \) and \( Q_d^{(s)} \) mean...
orthonormal polynomials of the first and second kind, in particular,

\[(0)(z - J^{(s)})^{-1}|0) = \frac{Q_d^{(s)}(z)}{P_d^{(s)}(z)}\]  \hspace{1cm} (2.10)

Let \(J_1^{(s)}\) denote the matrix that we obtain from \(J^{(s)}\) (see (2.5)) by deleting the first row and the first column. Then, for \(J_1^{(s)}, Q_d^{(s)}\) are orthogonal polynomials of the first kind and we denote by \(R_d^{(s)}\) corresponding to them orthogonal polynomials of the second kind. Note that \(P_d^{(s)}\) and \(R_d^{(s)}\) are related by

\[
\frac{P_d^{(s)}(z)}{Q_d^{(s)}(z)} = z - q_0^{(s)} - (p_1^{(s)})^2 \frac{R_d^{(s)}(z)}{Q_d^{(s)}(z)}.
\]  \hspace{1cm} (2.11)

In these terms the four interesting for us elements of the resolvent of \(J_1^{(s)}\) are:

\[
(z - J_1^{(s)})^{-1} = \begin{bmatrix}
\frac{R_d^{(s)}}{Q_d^{(s)}} & \cdots & \frac{1}{p_1^{(s)} p_d^{(s)} Q_d^{(s)}} \\
\vdots & \ddots & \vdots \\
\frac{1}{p_1^{(s)} p_d^{(s)} Q_d^{(s)}} & \cdots & \frac{Q_d^{(s)}/p_d^{(s)}}{Q_d^{(s)}}
\end{bmatrix}
\]  \hspace{1cm} (2.12)

Recalling again that \(J_1\) is just a block-diagonal matrix we substitute (2.12) in (2.9). As result in the RHS (as well as in the LHS) we get a three-diagonal matrix. On the main diagonal we have

\[
z - Q_0 - P_1 \{(z - J_1)^{-1}\}_{1,1} P_1 - S P_d \{(z - J_1)^{-1}\}_{d-1,d-1} P_d S^*,
\]

and each entry on the diagonal, due to (2.12) and then (2.11), is

\[
z - q_0^{(s+1)} - (p_1^{(s+1)})^2 \frac{R_d^{(s+1)}(z)}{Q_d^{(s+1)}(z)} - (p_d^{(s)})^2 \frac{Q_d^{(s)}(z)}{Q_d^{(s)}(z)}.
\]

Comparing this with the LHS (2.9) we get

\[
\frac{T(z) - \tilde{q}_{s+1}}{T'(z)/d} = \frac{P_d^{(s+1)}(z)}{Q_d^{(s+1)}(z)} - (p_d^{(s)})^2 \frac{Q_d^{(s)}(z)}{Q_d^{(s)}(z)}.
\]  \hspace{1cm} (2.13)

Similarly, below the main diagonal on the right we have

\[-S P_d \{(z - J_1)^{-1}\}_{d-1,1} P_1.
\]

So, using (2.12), we get from (2.9)

\[
\frac{\tilde{p}_{s+1}}{T'(z)/d} = \frac{1}{Q_d^{(s)}(z)} = \frac{p_1^{(s)} \ldots p_d^{(s)}}{z^{d-1} + \ldots}.
\]  \hspace{1cm} (2.14)

Thus the first relation in (4.1) is also proved, moreover all \(Q_d^{(s)}(z)\) (independently on \(s\), being normalized to the leading coefficient one, coincides with \(T'(z)/d\).

We define \(T^{(s)}(z) = z^d + \ldots\) by the same normalization

\[
T^{(s)}(z) := \tilde{p}_{s+1} P_d^{(s)}(z).
\]  \hspace{1cm} (2.15)

Then (2.10) implies (2.6).
Now we use the (last) well known fact on orthogonal polynomials — the Wronskian identity:
\[ p_d^{(s)} Q_d^{(s)}(z) P_{d-1}^{(s)}(z) - p_d^{(s)} P_d^{(s)}(z) Q_{d-1}^{(s)}(z) = 1, \]
due to which, if \( T'(c) = 0 \) then
\[ -p_d^{(s)} Q_{d-1}^{(s)}(c) = \frac{1}{P_d^{(s)}(c)}, \quad (2.16) \]
So, combining (2.16) with (2.14), we get from (2.13) the recurrence relation
\[ T(c) - \tilde{q}_{k+1} = T^{(s+1)}(c) + \frac{\tilde{\beta}_{s+1}^2}{T'(c)}, \quad (2.17) \]
Thus the lemma is completely proved. \( \square \)

Now we are in a position to show that the Renormalization Equation has \( 2^{d-1} \) solutions. Then we show that they are the only possible solutions. This set of solutions we parametrize by a collection of vectors
\[ \delta := \{ \delta_c \}_c, \quad (2.18) \]
where each component \( \delta_c \) can be chosen as plus or minus one.

**Theorem 2.2.** Fix a vector \( \delta \) of the form (2.18). For a given \( \tilde{J} \) with the spectrum on \( [-\xi, \xi] \) define the Jacobi matrix \( J \) according to the following algorithm:
For \( s \in \mathbb{Z} \) we put
\[ \frac{1}{T'(c)} = \tilde{r}_-(T(c), s), \quad \text{if } \delta_c = -1, \quad (2.19) \]
and
\[ T^{(s)}(c) = \tilde{\beta}_{s+1}^2 \tilde{r}_+(T(c), s + 1), \quad \text{if } \delta_c = 1, \quad (2.20) \]
where the functions \( \tilde{r}_\pm(z, s) \) are defined by (2.2). Then define the monic polynomial \( T^{(s)}(z) \) of degree \( d \) by the interpolation formula
\[ T^{(s)}(z) = (z - q)T'(z)/d + \sum_{c \in T'(c) = 0} \frac{T'(z)}{(z - c)T''(c)} T^{(s)}(c). \quad (2.21) \]
Define the block \( J^{(s)} \) (see (2.5)) by its resolvent function according to (2.6). Finally define the entry \( p_{sd+1} \) by (4.1).

We claim that the matrix \( J = J(\delta, \tilde{J}) \), combined with such blocks and entries over all \( s \), satisfies (2.1).

**Example.** The solution related to the vector
\[ \delta_- = \{-1, \ldots, -1\}, \]
that is all \( T^{(s)}(c) \) are defined by (2.19), plays the most important role in what follows. Precisely for this solution \( J(\tilde{J}) := J(\delta_-, \tilde{J}) \) we prove the contractibility property (our main Theorem 1.1).

**Proof.** First of all let us mention that for all \( c, |T(c)| > \xi \), that is \( T^{(s)}(c) \) is well defined by either (2.19) or (2.20), moreover this value is of the same sign as \( T(c) \). It means that the rational function \( \frac{T'(z)}{T''(z)} \) is the Stieltjes transform of a positive measure (supported at the zeros of \( T^{(s)}(z) \)), and hence there exists a unique \( d \times d \) Jacobi matrix defined by (2.5). Note that (4.2) implies immediately that \( q_{sd} = q \),
again we look at the leading term in the decomposition of the resolvent function into the continued fraction.

Now, we see that (2.14) holds, it’s just a matter of the definition of \( p_d^{(s)} \). We have to check (2.13). Using the form of \( Q_d^{(s)}(z) \) we have equivalently

\[
T(z) - \tilde{q}_{s+1} = T^{(s)}(z) + \tilde{p}_{s+1} p_d^{(s)} Q_d^{(s)}(z). \tag{2.22}
\]

Subtracting \((z-q)T'(z)/d\) from both parts we arrive at the question of the identity of two polynomials of degree \( d-2 \). Thus, we need to check (2.22) only at the critical points. Using the Wronskian identity (2.16) we get (2.17). Of course, the main point is that \( T^{(s)}(c) \), being defined by either (2.19) or (2.20), satisfy the recursion (2.7) (this is the continued fraction decomposition for \( \tilde{r}_\pm(z,s) \), see (2.3)).

Thus having (2.9), we proved (2.1).

\[\square\]

**Theorem 2.3.** Theorem 2.2 describes the whole set of solutions of the Renormalization Equation.

**Proof.** We need to show that (2.19), (2.20) give the complete possible choice of the values \( T^{(s)}(c) \), say for \( s = 0 \). Then all other values are of the same form due to Lemma 2.1.

We claim that any other choice of \( T^{(0)}(c) \) contradicts to the regularity of the resolvent of \( J \) at \( c \).

Using standard formulas for orthogonal polynomials for two–sided Jacobi matrices (see Appendix, Corollary 7.2) we have from the Renormalization Equation

\[
\tilde{r}_-^{-1}(T(z),0) = r_-^{-1}(z,0)T'(z)/d + p_1^2 \tilde{p}_1 R_d^{(0)}(z),
\]

\[
\tilde{p}_1^2 \tilde{r}_+ (T(z),1) = p_1^2 r_+(z,1)T'(z)/d + p_1^2 \tilde{p}_1 R_d^{(0)}(z), \tag{2.23}
\]

where \( r_\pm(z,s) \) are the resolvent functions of \( J \).

In the same time both functions \( r_+(z,1) \) and \( r_-^{-1}(z,0) \) can not have a pole at \( c \) simultaneously. This contradicts to (see (7.5))

\[
\langle 1 | (J - z)^{-1} | 1 \rangle = r_+(z,1) r_-^{-1}(z,0) \langle 0 | (J - z)^{-1} | 0 \rangle; \tag{2.24}
\]

\( \langle 0 | (J - z)^{-1} | 0 \rangle \) can not have zero multiplicity more than one and \( \langle 1 | (J - z)^{-1} | 1 \rangle \) should be regular at \( c \).

By (2.11) \( T^{(0)}(c) = -p_1^2 \tilde{p}_1 R_d^{(0)}(c) \), so we get (2.19) from (2.23) if \( r_-^{-1}(z,0) \) is regular and (2.20) in the case when \( r_+(z,1) \) is regular at \( c \).

\[\square\]

**Remark.** Formulas (2.19), (2.20) play the main role in solving the Renormalization Equation. Actually, we proved or found them in Theorem 2.3 for \( s = 0 \), or, in the same way, for any other fixed \( s \). However, there is another important ingredient: it should be also shown that if we choose this or that definition of \( T^{(s)}(c) \) for any particular \( s \) we have to be consistent with this definition for all other values of \( s \), that is we can not define, for instance, \( T^{(0)}(c) \) by (2.19) and \( T^{(1)}(c) \) by (2.20) for the same \( c \). Precisely this claim is the main output of Lemma 2.1.

Let us mention that the Renormalization Equation can be rewritten equivalently in the form of polynomials equations.

**Lemma 2.4.** Equation (2.1) is equivalent to

\[
V^* T(J) = \tilde{J} V^*. \tag{2.25}
\]
\[
V^* \frac{T(z) - T(J)}{z - J} V = T'(z)/d. \quad (2.26)
\]

**Proof.** Starting with (2.25), (2.26) we get
\[
(T(z) - \tilde{J})V^*(z - J)^{-1}V = V^* \{T(z) - T(J)\} \{z - J\}^{-1}V = T'(z)/d.
\]

Having (2.1) we get
\[
V^* \frac{T(z) - T(J)}{z - J} V = T(z) V^* (z - J)^{-1} V - V^* \frac{T(J)}{z - J} V = T'(z)/d.
\]

Since the left hand side in (2.27) is a polynomial of \( z \) we obtain two relations
\[
V^* \frac{T(z) - T(J)}{z - J} V = T'(z)/d
\]
and
\[
\{ \tilde{J}V^* - V^* T(J) \} \{(z - J)^{-1} V\} = 0.
\]

Since vectors of the form \((z - J)^{-1} V f, f \in l^2\), are complete in \( l^2 \) the last relation implies (2.26). \( \square \)

### 3. Proof of the main theorem

We start with

**Lemma 3.1.** Let \( J^{(s)} \) be the \( s \)-the block of \( J = J(\delta, \tilde{J}) \). Define the measure \( \sigma^{(s)} \) by
\[
\frac{T^{(s)}(z)}{T'(z)/d} = z - q - \int \frac{d\sigma^{(s)}(x)}{x - z}, \quad (3.1)
\]
that is, \( \sigma^{(s)} \) is the spectral measure of the obliterated matrix \( J_{1}^{(s)} \) normalized by
\[
\int d\sigma^{(s)}(x) = (p_{1}^{(s)})^2.
\]

Then
\[
\int \frac{d\sigma^{(s)}(x)}{T^{(s)}(c)^2} = \frac{p_{1}^{2}(s+1)d}{\hat{p}_{s+1}^2}. \quad (3.2)
\]

**Proof.** Note that \( Q_{d-1}^{(s)} \) is the orthonormal polynomial with respect to \( \sigma^{(s)} \),
\[
\int (Q_{d-1}^{(s)}(c))^2 d\sigma^{(s)}(c) = 1.
\]
Using (2.16) and the normalization (2.15) we get (3.2). \( \square \)

The proof of the theorem is based on the following well-known and simple lemma.
Lemma 3.2. Assume that two non-normalized measures $\sigma$ and $\hat{\sigma}$ are mutually absolutely continuous. Moreover, that $d\hat{\sigma} = f \, d\sigma$ and $(1 + \epsilon)^{-1} \leq f \leq (1 + \epsilon)$. Let us associate with these measures Jacobi matrices $J = J(\sigma)$, $\tilde{J} = J(\hat{\sigma})$. Then for their coefficients we have

$$|\hat{p}_s - p_s| \leq \epsilon ||J||, \quad s \geq 0.$$  

Proof. Assume that $p_s \geq \hat{p}_s$. Let us use an extremal property of orthogonal polynomials,

$$(1 + \epsilon)\hat{p}_0^2 \cdots \hat{p}_s^2 = (1 + \epsilon) \int p_0^2 \cdots p_s^2 \hat{p}_s^2 \, d\hat{\sigma} \geq \int \{z^s + \ldots\}^2 \, d\sigma$$

$$\geq \inf_{\{P = z^s + \ldots\}} \int P^2 \, d\sigma = p_0^2 \cdots p_s^2.$$  

Similarly

$$(1 + \epsilon)p_0^2 \cdots p_{s-1}^2 \geq \hat{p}_0^2 \cdots \hat{p}_{s-1}^2.$$  

Therefore

$$p_s^2 \leq \hat{p}_s^2 \leq (1 + \epsilon)^2 p_s^2$$

and hence

$$0 \leq \hat{p}_s - p_s \leq \epsilon p_s.$$  

Proof of Theorem 1.1. Given $\tilde{J}_I$ and $\tilde{J}_{II}$ let us compare the blocks $J_I^{(s)}$ and $J_{II}^{(s)}$ of the matrices $J_I := J(\delta_-, \tilde{J}_I)$ and $J_{II} := J(\delta_-, \tilde{J}_{II})$. Actually we will apply Lemma 3.2 to non-normalized spectral measures $\sigma_I^{(s)}$ and $\sigma_{II}^{(s)}$, see (3.1), corresponding to the obliterated matrices $(J_I^{(s)})_1$ and $(J_{II}^{(s)})_1$.

Note that both measures are supported on the critical points $\{c : T'(c) = 0\}$ and, therefore, they are mutually absolutely continuous. Moreover, the density of the second measure with respect to the first one is of the form

$$f(c) = f^{(s)}(c) := \frac{T_{II}^{(s)}(c)}{T_I^{(s)}(c)}.$$  

Assuming $f(c) \geq 1$ let us estimate $f(c) - 1$ from above.

$$f(c) - 1 = \frac{1/T_I^{(s)}(c) - 1/T_{II}^{(s)}(c)}{1/T_{II}^{(s)}(c)} = \frac{(s)(T(c) - \tilde{J}_{II,-}(s))^{-1}(\tilde{J}_{II,-}(s) - \tilde{J}_{I,-}(s))(T(c) - \tilde{J}_{I,-}(s))^{-1}|s|}{1/T_{II}^{(s)}(c)}.$$  

(3.3)

Since the spectrum of $\tilde{J}_{II,-}(s)$ is on $[-\xi, \xi]$ we get, by definition (2.19),

$$\left|\frac{1}{T_{II}^{(s)}(c)}\right| \geq \frac{1}{|T(c)| + \xi},$$

and, for the same reason,

$$||(T(c) - \tilde{J}_{i,-})^{-1}|s|| \leq \frac{1}{|T(c)| - \xi}, \quad i = I \text{ or } i = II.$$
Therefore
\[ 0 \leq f(c) - 1 \leq \frac{||\tilde{J}_1 - \tilde{J}_{II}||}{(|T(c)| + \xi)(|T(c)| - \xi)^2}. \tag{3.4} \]
Thus, by Lemma 3.2, we obtain
\[ |(p_I)_{sd+k} - (p_{II})_{sd+k}| \leq \kappa ||\tilde{J}_1 - \tilde{J}_{II}||, \quad 1 \leq k \leq d - 1, \tag{3.5} \]
where
\[ \kappa := \max_c \frac{|T(c)|/\xi + 1}{(|T(c)|/\xi - 1)^2}. \]
We have to estimate \(|(p_I)_{sd+d} - (p_{II})_{sd+d}|. \) Note that due to (4.1) and Lemma 3.1
\[ \frac{1}{(p_I)_{sd+1} \cdots (p_I)_{sd+d-1} - (p_{II})_{sd+1} \cdots (p_{II})_{sd+d-1}} \leq \frac{1}{|T(c)|/\xi - 1}, \quad i = I \text{ or } II. \tag{3.6} \]
Now,
\[ (p_I)_{sd+d} - (p_{II})_{sd+d} = \frac{(p_I)_{s+1}}{(p_I)_{sd+1} \cdots (p_I)_{sd+d-1}} - \frac{(p_{II})_{s+1}}{(p_{II})_{sd+1} \cdots (p_{II})_{sd+d-1}} \]
\[ = \frac{(\tilde{p}_I)_{s+1} - (\tilde{p}_{II})_{s+1}}{(p_I)_{sd+1} \cdots (p_I)_{sd+d-1}} \]
\[ + \frac{(\tilde{p}_{II})_{s+1}}{(p_I)_{sd+1} \cdots (p_I)_{sd+d-1}} \left( 1 - \frac{(p_I)_{sd+1} \cdots (p_I)_{sd+d-1}}{(p_{II})_{sd+1} \cdots (p_{II})_{sd+d-1}} \right) \]
Using (3.6), (3.4) and Lemma 3.2 we obtain
\[ |(p_I)_{sd+d} - (p_{II})_{sd+d}| \leq \max_c \frac{1}{|T(c)|/\xi - 1} ||\tilde{J}_1 - \tilde{J}_2|| \max_c \frac{1}{|T(c)|/\xi - 1} \left( 1 + \kappa/2 \right) \]
\[ = ||\tilde{J}_1 - \tilde{J}_2|| \max_c \frac{1}{|T(c)|/\xi - 1} \left( 1 + \kappa/2 \right). \tag{3.7} \]
Thus (3.5) and (3.7) show that say for \( \min_c |T(c)|/\xi \geq 10 \) the renormalization is a contraction. \( \square \)

4. Proof of Theorem 1.2

Proof. For a given sufficiently hyperbolic polynomial \( T \) we define
\[ J_{n+1} = J(\delta - J_n) \]
starting from an arbitrary initial \( J_0 = \tilde{J} \) with the spectrum on \([-\xi, \xi]\). Due to the contractibility of the renormalization \( J_n \) converges in the operator norm to \( J \), moreover \( J \) has the spectrum on Julia\( (T) \) and it is limit periodic
\[ ||J - S^{-d^{n+1}} JS^{d^{n+1}}|| \leq 2\xi \kappa^n, \quad \kappa < 1. \]

We claim that \( J \) is an orthogonal sum of two one-sided Jacobi matrices
\[ J = \begin{bmatrix} J_-(0) & 0 \\ 0 & J_+(0) \end{bmatrix}. \tag{4.1} \]
That is, we claim that \( p_0 = 0 \). Indeed, by Lemma 3.1 we have
\[ p(n+1)_{sd} \leq \max_c \frac{1}{|T(c)|/\xi - 1} p(n)_{s} \leq \kappa p(n)_{s}, \tag{4.2} \]
where \( p(n) \) is the \( s \)-th coefficient in the matrix \( J_n \). Therefore, all \( sd^l \)-th coefficients of \( J \) are uniformly small

\[
p_{sd^l} \leq 2\xi \kappa^l,
\]

and, in particular, \( p_0 = 0 \).

Thus \( J_+ := J_+(0) \) is a one–sided Jacobi limit–periodic matrix with the spectrum on \( \text{Julia}(T) \), moreover, its spectral measure \( \sigma_+ \) (supported on \( \text{Julia}(T) \)) possesses the renormalization property

\[
\int \frac{T'(z)/d}{T(z) - x} d\sigma_+(x) = \int \frac{1}{z - x} d\sigma_+(x).
\]

This means that \( \sigma_+ \) is an eigen–measure for the Ruelle operator \( L^* \sigma_+ = \sigma_+ \), where the operator \( L \) acts on a continuous function \( f \) on \( \text{Julia}(T) \) by

\[
(Lf)(x) = \frac{1}{d} \sum_{T(y)=x} f(y).
\]

In other words \( \sigma_+ \) is the balanced measure on \( \text{Julia}(T) \).

\[\Box\]

Note that due to the Renormalization Equation the spectral measure of \( J_- \) is the eigen–measure for the Ruelle operator

\[
(L_2 f)(x) = \sum_{T(y)=x} \frac{f(y)}{T'(y)^2},
\]

i.e. \( L_2^* \sigma_- = \rho \sigma_- \), \( \rho > 0 \). In the case of quadratic polynomials this fact was proved in [10].

5. The renormalization of periodic matrices

The renormalization (2.1) acts in the most natural way on periodic Jacobi matrices. We recall some basic facts from the spectral theory of such matrices.

The spectrum \( E \) of any periodic matrix \( J \) is an inverse polynomial image

\[
E = U^{-1}[-1, 1]
\]

the polynomial \( U \) of degree \( g + 1 \) should have all critical points \( \{c_U\} \) real and for all critical values \( |U(c_U)| \geq 1 \). For simplicity we will assume \( |U(c_U)| > 1 \). Then the spectrum of \( J \) consists of \( g \) intervals

\[
E = [b_0, a_0] \setminus \cup_{j=1}^g (a_j, b_j).
\]

Also it would be convenient for us to normalize \( U \) by a linear change of the variable such that \( b_0 = -1 \) and \( a_0 = 1 \).

Having the set \( E \) of the above form fixed, let us describe the whole set of periodic Jacobi matrices \( J(E) \) with the given spectrum. To this end we associate with \( U \) the hyper–elliptic Riemann surface

\[
X = \{ Z = (z, \lambda) : \lambda^2 - 2U(z)\lambda + 1 = 0 \}.
\]

The involution on it we denote by \( \tau \),

\[
\tau Z := \left( z, \frac{1}{\lambda} \right) \in X.
\]

The set

\[
X_+ = \{ Z \in X : |\lambda(Z)| < 1 \}
\]
we call the upper sheet of $X$. Note $X_+ \simeq \mathbb{C} \setminus E$, in fact, $z(Z) \in \mathbb{C} \setminus E$ if $Z \in X_+$.

The following well known theorem describes $J(E)$ in terms of real divisors on $X$. The Jacobian variety of $X$, $\text{Jac}(X)$, is a $g$ dimensional complex torus, $\text{Jac}(X) \simeq \mathbb{C}^g/L(X)$, where $L$ is a lattice (that can be chosen in the form $L = \mathbb{Z}^g + \Omega \mathbb{Z}^g$ with $\text{Im} \Omega > 0$). Consider the $g$ dimensional real subtorus consisting of divisors of the form

$$D(E) = \{D = D_+ - D_C, \quad D_+ := \sum_{i=1}^{g} Z_i : Z_i \in X, \quad z(Z_i) \in [a_i, b_i]\},$$

here $D_C$ is a point of normalization that we choose of the form

$$D_C := \sum_{i=1}^{g} C_i : C_i \in X, \quad z(C_i) = (c_U)_i, |\lambda(C_i)| > 1,$$

—the collections of the points on the lower sheet with the $z$–coordinates at the critical points. (At least topologically, it is evident $D(E) \simeq \mathbb{R}^g/\mathbb{Z}^g$).

**Theorem 5.1.** For given $E$ of the form (5.1) there exists an one–to–one correspondence between $J(E)$ and $D(E)$.

Let now $\tilde{U}$ be a polynomial of the above described form, we restore the normalization $T^{-1} : [-1, 1] \to [-1, 1]$ for the expanding polynomial, and we define $U = \tilde{U} \circ T$. Then we have a covering $\pi$ of the Riemann surface $\tilde{X}$ associated to $\tilde{U}$ by the surface $X$ associated to $U$:

$$\pi(z, \lambda) = (T(z), \lambda), \quad (5.3)$$

note $\pi : X_+ \to \tilde{X}_+$.

According to the general theory, this covering generates different natural mappings [9], in particular,

$$\pi_* : \text{Jac}(X) \to \text{Jac}(\tilde{X}), \quad (5.4)$$

and

$$\pi^* : \text{Jac}(\tilde{X}) \to \text{Jac}(X). \quad (5.5)$$

In this section we solve equations (2.25), (2.26) using this language, see Theorem 5.8. Note that (2.25) already guarantied that $J \in J(E)$ implies $\tilde{J} \in J(\tilde{E})$.

To continue we need to recall some special functions on hyper–elliptic Riemann surfaces. The first object is the Complex Green’s function. Note that the function $\lambda$ in $X_+$ has no zeros except for infinity, where it has a zero of multiplicity $g + 1$, moreover $|\lambda| = 1$ on $\partial X_+$. We define the Complex Green’s function (with respect to infinity) by $b^{g+1} = \lambda$. It is not single valued in $X_+$ but it has the only simple zero at infinity. Note that

$$G(z) = \log \frac{1}{|b(z)|},$$

where $G(z) = G(z, \infty)$ is the standard Green’s function for the domain $\mathbb{C} \setminus E$. Generally,

$$G(z, z_0) = \log \frac{1}{|b_{z_0}(z)|},$$

defines the Complex Green’s function $b_{z_0}$ with the only zero at $Z_0 \in X_+, z(Z_0) = z_0 \in \mathbb{C} \setminus E$. 

\[ \text{LIMIT PERIODIC JACOBI MATRICES} \]
Since $\tilde{\lambda} \circ \pi = \lambda$ we have the relation
\[
\tilde{b} \circ \pi = b^d
\] (5.6)

The differential $\frac{1}{2\pi i} d \log b$, being restricted on $\partial X_+$, is the harmonic measure $d\omega$ of the domain $\mathbb{C} \setminus E$ with pole at infinity.

The space $L^p(\partial X_+)$, in a sense, is the $L^p$ space with respect to the harmonic measure, but it should be mentioned that $\partial X_+ = (E - i0) \cup (E + i0)$, i.e., an element $f$ of $L^p(\partial X_+)$ may have different values $f(x + i0)$ and $f(x - i0)$, $x \in E$.

Having in mind (5.6) we get
\[
\int_{\partial X_+} f d\omega = \int_{\partial \tilde{X}_+} \frac{1}{d} \left( \sum_{\pi(Z) = \tilde{Z}} f(Z) \right) (\tilde{Z}) d\tilde{\omega}
\] (5.7)

for every $f \in L^1(\partial X_+)$.

**Definition 5.2.** The Hardy space $H^2(X_+)$ consists of functions $f$ holomorphic on $X_+$ (or what is the same in the domain $\mathbb{C} \setminus E$) having harmonic majorant
\[
|f(z)|^2 \leq u(z), \ z \in \mathbb{C} \setminus E,
\] (5.8)
where $u(z)$ is harmonic in $\mathbb{C} \setminus E$. The norm of $f$ is defined by
\[
\|f\|^2 := \inf_u u(\infty),
\]
where $u$ runs over all harmonic functions satisfying (5.8)

An equivalent way to define $H^2(X_+)$ is to close the set of holomorphic functions uniformly bounded in $X_+$ with respect to the norm
\[
\|f\|^2 := \int_{\partial X_+} |f|^2 \, d\omega.
\] (5.9)

As it follows directly from (5.7), the covering (5.3) generates an isometrical enclosure
\[
v_+ : H^2(\tilde{X}) \to H^2(X_+)
\] (5.10)
acting in a natural way
\[
(v_+ f)(Z) = f(\pi(Z)).
\] (5.11)

Now we have to describe the most complicated but the most important element of the construction: we have to introduce a very natural orthonormal basis in $H^2(X_+)$. The multiplication operator by $z$, with respect to this basis, will lead us to Jacobi matrices, the substitution (5.11) to the isometry $V$ and so on... This basis is a counterpart of the standard basis of $\{\zeta^n\}_{n \geq 0}$ in the standard Hardy space $H^2(\mathbb{D})$, $\mathbb{D} = \{ |\zeta| < 1 \}$.

Note that $1 \in H^2(X_+)$, moreover,
\[
\langle f, 1 \rangle = f(\infty)
\]
for every $f \in H^2(X_+)$. Therefore the orthogonal complement to 1 consists of functions with $f(\infty) = 0$. Let us give an alternative description of
\[
H^2_0(X_+) = \{ f \in H^2(X_+) : f(\infty) = 0 \}.
\]

Any function from $H^2_0(X_+)$, having zero at infinity, is the form $f = b\tilde{f}$. However $b$ is not single-valued, thus so is $\tilde{f}$. We need to generalize slightly Definition 5.2.
Definition 5.3. Let $\Gamma = \Gamma(E)$ be the fundamental group of the domain $\hat{\mathbb{C}} \setminus E$. Let $\alpha$ be an element of the dual group of characters $\Gamma^*$, that is, for any contour $\gamma \in \Gamma$ in the domain, $\gamma \mapsto \alpha(\gamma)$, where $\alpha(\gamma)$ is a number of absolute value one, and for any two contours $\gamma_1, \gamma_2$

$$
\alpha(\gamma_1 \gamma_2) = \alpha(\gamma_1) \alpha(\gamma_2).
$$

The Hardy space $H^\infty(X_+), \alpha)$ consists of holomorphic multivalued functions $f$ uniformly bounded in the domain $\hat{\mathbb{C}} \setminus E$ such that

$$
f(\gamma z) = \alpha(\gamma)f(z),
$$

and $H^2(X_+, \alpha)$ is the closure of $H^\infty(X_+, \alpha)$ with respect to the norm (5.9).

Note that the absolute value of a function from $H^2(X_+, \alpha)$ is single valued and $\alpha$ fixes, actually, the ramification of the argument of the function.

Example. As it was mentioned, the function $b$ is not single valued but $|b(z)|$ is a single valued function. We define the character $\mu \in \Gamma^*$ by

$$
b(\gamma z) = \mu(\gamma)b(z).
$$

Let $\gamma_j$ be the contour, that starts at infinity (or any other real point bigger than 1), go in the upper half–plane to the gap $(a_j, b_j)$ and then go back in the lower half–plane to the initial point. Assuming that $b_0 < \ldots < a_j < b_j < a_{j+1} < \ldots < a_0$, we have $\mu(\gamma_j) = e^{-2\pi i \frac{a_{j+1} - a_j}{a_{j+1} - a_j}}$, equivalently $\omega([b_j, a_0]) = \frac{q+1-j}{q+1}$.

Remark. Note that the system of the above contours $\gamma_j$ is a generator of the free group $\Gamma^*(E)$. In other words a character $\alpha$ is uniquely defined by the vector

$$
[\alpha(\gamma_1), \alpha(\gamma_2), \ldots, \alpha(\gamma_9)] \in \mathbb{T}^9.
$$

This sets an one–to–one correspondence between $\Gamma^*(E)$ and $\mathbb{T}^9$.

Proposition 5.4. Using the above definitions we get the orthogonal decomposition

$$
H^2(X_+) = \{1\} \oplus H^2_0(X_+) = \{1\} \oplus bH^2(X_+, \mu^{-1}). \tag{5.12}
$$

Now we can iterate (5.12). Let $k^\alpha$ be the reproducing kernel of $H^2(X_+, \alpha)$ with respect to infinity, that is, the vector from $H^2(X_+, \alpha)$, which is uniquely defined by the condition

$$
(f, k^\alpha) = f(\infty), \forall f \in H^2(X_+, \alpha).
$$

Then

$$
H^2(X_+) = \{1\} \oplus \{bk^\mu\} \oplus b^2H^2(X_+, \mu^{-2}),
$$

and so on...

Theorem 5.5. Let $\alpha \in \Gamma^*$. The system

$$
\{e_n^\alpha\}_{n \in \mathbb{Z}_+}, \quad e_n^\alpha := b^nK^{\alpha\mu^{-n}}, \quad K^{\alpha\mu^{-n}} := \frac{k^{\alpha\mu^{-n}}}{\|k^{\alpha\mu^{-n}}\|} \tag{5.13}
$$

forms an orthonormal basis of $H^2(X_+, \alpha)$. The same system with $n \in \mathbb{Z}$ is an orthonormal basis in $L^2(\partial X_+)$. Moreover, the multiplication operator by $b$ is a periodic Jacobi matrix with spectrum $E$.

Theorem 5.5 indicates a special importance of the functions $k^\alpha$. They are very well studied [6]. First of all, they have analytic continuation (as multivalued functions) on the whole $X$, so we can write $k^\alpha(Z)$.
Theorem 5.6. For every $\alpha \in \Gamma^*$ the reproducing kernel $k^\alpha(Z)$ has on $X$ exactly $g$ simple poles that do not depend on $\alpha$ and $g$ simple zeros. The divisor $D_+ = \sum_j Z_j$ of zeros
\[
k^\alpha(Z_j) = 0
\]
with the divisor of poles form the divisor
\[
div(k^\alpha) = D_+ - D_C
\]
that belongs to $D(E)$, moreover (5.15) sets an one–to–one correspondence between $D(E)$ and $\Gamma^*(E)$.

The functions $k^\alpha$ possess different representations, in particular, in terms of theta–functions [9], and the map $D \mapsto \alpha$ can be written explicitly in terms of abelian integrals (the Abel map).

Summary. The three objects $J(E), D(E)$ and $\Gamma^*(E)$ are equivalent. Both maps $\Gamma^*(E) \to D(E)$ and $\Gamma^*(E) \to J(E)$ can be defined in terms of the reproducing kernels of the spaces $H^2(X, \alpha), \alpha \in \Gamma^*(E)$. The first one is given by (5.15). It associates to the given $k^\alpha(Z)$ the sets of its zeros and poles (the poles are fixed and the zeros vary with $\alpha$). The matrix $J(\alpha) \in J(E)$ is defined as the matrix of the multiplication operator by $Z$ with respect to the basis (5.13):
\[
z(Z)e_\alpha(Z) = p^\alpha_x e_{x-1}(Z) + q^\alpha_x e_\alpha(Z) + p^\alpha_{x+1} e_{x+1}(Z), Z \in X, s \in \mathbb{Z}.
\]

It’s really easy to see that $J(\alpha)$ is periodic: just recall that $b^{g+1}$ is single valued, that is, $\mu^{g+1} = 1$, and therefore the spaces $H^2(X, \alpha)$ and $H^2(X, \alpha \mu^{-g+1}$) (and their reproducing kernels) coincide.

Now we can go back to the Renormalization Equation. Note that $\pi$ acts naturally on $\Gamma(E)$:
\[
\pi \gamma = \{\pi(Z), Z \in \gamma\} \in \Gamma(\tilde{E}), \quad \text{for } \gamma \in \Gamma(E).
\]
The map $\pi^* : \Gamma^*(\tilde{E}) \to \Gamma^*(E)$ is defined by duality:
\[
(\pi^* \hat{\alpha})(\gamma) = \hat{\alpha}(\pi \gamma).
\]

Theorem 5.7. Let $T, T^{-1} : [-1, 1] \to [-1, 1]$, be an expanding polynomial. Let $\tilde{J}$ be a periodic Jacobi matrix with spectrum $\tilde{E} \subset [-1, 1]$, and therefore there exists a polynomial $\tilde{U}$ such that $\tilde{E} = U^{-1}[-1, 1]$ and a character $\hat{\alpha} \in \Gamma^*(\tilde{E})$ such that $\tilde{J} = J(\hat{\alpha})$. Then
\[
J := J(\pi^* \alpha)
\]
is the periodic Jacobi matrix with spectrum $E = U^{-1}[-1, 1], U := \tilde{U} \circ T$, that satisfies the Renormalization Equation (2.1).

Proof. First we note, that for the operator multiplication by $z(Z)$ in $L^2(\partial X_+)$, the operator multiplication by $\tilde{z}(\tilde{Z})$ in $L^2(\partial \tilde{X}_+)$, the spectral parameter $z_0$ and the isometry
\[
(vf)(Z) = f(\pi(Z)), \quad v : L^2(\partial X_+) \to L^2(\partial X_+),
\]
we have
\[
\int_{\partial X_+} \frac{1}{z_0 - z(Z)} |(vf)(Z)|^2 \omega = \int_{\partial \tilde{X}_+} \left( \frac{1}{d} \sum_{\pi(Z) = \tilde{Z}} \frac{1}{z_0 - z(Z)} \right) |f(\tilde{Z})|^2 d\tilde{\omega}.
\]
It is evident, that
\[ \frac{1}{d} \sum_{T(y)=x} \frac{1}{z_0 - y} = \frac{T'(z_0)/d}{T(z_0) - x}. \]

Thus
\[ v^*(z_0 - z(Z))^{-1} v = (T'(z_0)/d)(T(z_0) - \tilde{z}(\tilde{Z}))^{-1}. \]

It remains to show that \( v \) transforms the basis vector
\[ e_{\tilde{\alpha}}^\alpha = \tilde{b}_n K^\tilde{\alpha} \tilde{\mu} - n \]
into
\[ e_{\tilde{\alpha}}^\pi = b^n K^{\tilde{\alpha} \tilde{\mu}} - n \]

Or, what is the same, that \( K^{\tilde{\alpha} \circ \pi} = K^{\tilde{\alpha} \circ \pi} \) for all \( \tilde{\alpha} \in \Gamma^*(\tilde{E}) \). Note that both functions are of norm one in the same space \( H^2(X_+, \pi^* \tilde{\alpha}) \), in particular, they have the same character of automorphy \( \pi^* \tilde{\alpha} \in \Gamma^*(E) \). Note, finally, that the divisor
\[ \text{div}(k^{\tilde{\alpha} \circ \pi}) = \frac{\pi^{-1}(\tilde{D}_+)}{\pi^{-1}(\tilde{D}_C)}, \]
where \( \text{div}(k^{\tilde{\alpha}}) = \tilde{D}_+ - \tilde{D}_C \), belongs to \( D(E) \), therefore \( k^{\tilde{\alpha} \circ \pi} \) is the reproducing kernel and the theorem is proved. \( \Box \)

To find all other solutions of (2.1) let us look a bit more carefully at the above proof.

Note that the same identity (5.19) holds for any isometry \( v \) of the form
\[ vf = v_0 f = \theta(f \circ \pi), \]
where \( \theta \) is a unimodular (|\( \theta \)| = 1) function on \( \partial X_+ \).

Concerning the second part of the proof, let us mention that the set of critical points of \( U \) splits in two sets:
\[ \{ cv \} = T^{-1}\{ c_\delta \} \cup \{ c_T \}. \]

Correspondingly,
\[ \sum (C_U)_j = \sum_k \sum_{\pi(C_U)_{k,j} = (C_\delta)_k} (C_U)_{k,j} + \sum (C_T)_j, \]
and the divisor of \( k^{\tilde{\alpha} \circ \pi} \) consists of two parts, that one that depends on \( \tilde{\alpha} \)
\[ \pi^{-1}(\tilde{D}), \]
and that part that corresponds to the critical points of the polynomial \( T \)
\[ \{(C_T)_j\}^{d-1}_{j=1}, \]
since
\[ D = \text{div}(k^{\tilde{\alpha} \circ \pi}) = \frac{\pi^{-1}(\tilde{D}) + \sum_{j=1}^{d-1} (C_T)_j - \pi^{-1}(\tilde{D}_C) - \sum_{j=1}^{d-1} (C_T)_j}{d}. \]

Thus we can fix an arbitrary system of points \( \{ Z_{c,j} \}^{d-1}_{j=1} \) such that \( z(Z_{c,j}) \) belongs to the same gap in the spectrum \( E \) as the critical point \( (C_T)_{j} \). If \( \theta \) is the canonical product on \( X \) with the divisor
\[ \text{div}(\theta) = \sum_{j=1}^{d-1} Z_{c,j} - \sum_{j=1}^{d-1} (C_T)_j, \]
then \( \theta k^\alpha \circ \pi \) is the reproducing kernel simultaneously for all \( \hat{\alpha} \in \Gamma^*(\bar{E}) \). But to make \( \theta \) unimodular (zeros and poles are symmetric) our choice is restricted just to \( Z_{e,j} = (C_T)_j \) or \( Z_{c,j} = \tau(C_T)_j \). Note that \( \tau(C_T)_j - (C_T)_j \) is the divisor of the Complex Green function \( b_{(C_T)_j} \). In this way we arrive at

**Theorem 5.8.** For an expanding polynomial \( T \), and a periodic Jacobi matrix \( \hat{J} = J(\hat{\alpha}) \), \( \hat{\alpha} \in \Gamma^*(E) \) as in Theorem 5.7 there exist \( 2d-1 \) solutions of the Renormalization Equation (2.1). Denote by \( \mu_{(C_T)_j} \) the character generated by the Green’s function \( b_{(C_T)_j} \), \( b_{(C_T)_j} \circ \gamma = \mu_{(C_T)_j}(\gamma)b_{(C_T)_j} \). Then these solutions are of the form

\[
J := J(\eta_\delta \pi^* \hat{\alpha}), \quad \eta_\delta := \prod_{j=1}^{d-1} \mu_{(C_T)_j}^{(1+\delta_{(C_T)_j})}, \quad (5.21)
\]

as before

\[
\delta = \{\delta_{(C_T)_j}\}, \quad \delta_{(C_T)_j} = \pm 1.
\]

*Proof.* We define the isometry

\[
(vf)(Z) = \left( \prod_{j=1}^{d-1} b_{(C_T)_j}^{(1+\delta_{(C_T)_j})}(Z) \right) f(\pi(Z))
\]

and then repeat the arguments of the proof of Theorem 5.7.

Concluding this section note that the central part in the proof of Theorem 1.2 (the claim that the limit matrix has a form of the orthogonal sum) also can be reduced to an another well known fact from the theory of Hardy spaces on Riemann surfaces. Namely to the statement that \( H^2 \) is trivial, i.e.,

\[
H^2(\mathbb{C} \setminus E) = \{\text{const}\} \quad (5.22)
\]

in a domain of the form \( \mathbb{C} \setminus E \), where the Lebesgue measure of \(|E| = 0\).

An alternative proof of (4.1). Since we can start with an arbitrary \( \hat{J} \), we start with a periodic matrix related to a certain \( H^2(\hat{X}_+) \), e.g., with the matrix with constant coefficients \( J = \frac{\delta z + z^*}{2}, \quad E = [-1,1] \). Then, under inverse iterations of the polynomial \( T \) according to Theorem 5.7, we will get spaces of the same nature (i.e., the character is trivial, equals one on every contour). Let \( J_n \) be the matrix with spectrum \( E_n = (T^n)^{-1}[-1,1] \). We have

\[
z(Z)e_1(Z,n) = p(n)_1 e_0(Z,n) + q(n)_1 e_1(Z) + p_2 e_2(Z,n), \quad Z \in X_n,
\]

here \( n \) is related to the number of iterations and the position of the element of the matrix is fixed. Recall that \( e_0(Z,n) = 1 \) (the initial basic vector, see (5.12)) and we have, putting \( Z = \infty \),

\[
p(n)_1 = (zb_n)(\infty)K^{n^{-1}}(\infty, n).
\]

\((zb_n)(\infty)\) is the so called capacity of \( E_n \); if it goes to zero even better, in fact it does not, but in any case it is uniformly bounded. Then, assuming that \( K^{n^{-1}}(\infty, n) \) does not go to zero, by compactness arguments, we can find a subsequence

\[
(b_{n_j})(z)K^{n_j^{-1}}(z, n_j)
\]

that converges pointwise in the domain to a non–trivial holomorphic function from \( H^2(\mathbb{C} \setminus E) \), \( E = \lim E_n = \text{Julia}(T) \), that equals zero at \( \infty \). But this contradicts to (5.22). Thus \( p(n)_1 \to 0 \). \( \square \)
Note that this proof is valid for expanding polynomials (we do not require that $T$ is sufficiently hyperbolic). Note also the flip in notations of the matrices’ elements: in this section a basis of holomorphic functions substitutes the standard polynomial basis (instead of the multiplicity of the pole at infinity we enlarge the multiplicity of zero). That is, $p_1$ in this section is the same as $p_0$ in Section 4 (we are just unable to enumerate the elements, related to holomorphic functions, by negative integers).

6. Concluding remarks

Our concluding remarks concern basically other solutions of the Renormalization Equation.

6.1. The duality $\delta \mapsto -\delta$. In Theorem 1.1 we proved contractibility of only one of the solutions of the renormalization equation corresponding to $\delta = \delta_-$, but it means that at least one more solution has the same property.

**Theorem 6.1.** The dual solution of the Renormalization Equation $J(\tilde{J}, -\delta)$, possesses the contractibility property simultaneously with $J(\tilde{J}, \delta)$.

It deals with the following universal involution acting on Jacobi matrices

$$ J \rightarrow J_{\tau} := U_{\tau}JU_{\tau}, \quad \text{where } U_{\tau}|l| = |1 - l|. \quad (6.1) $$

Obviously $VU_{\tau} = U_{\tau}S^{1-d}V$. Thus, having $J$ as a solution of the renormalization equation corresponding to $\tilde{J}$ we have simultaneously that $S^{d-1}J_{\tau}S^{1-d}$ solves the equation with the initial $\tilde{J}_{\tau}$. The following lemma describes which branch corresponds to which in this case.

**Lemma 6.2.** Let $J = J(\tilde{J}, \delta)$ then

$$ S^{d-1}J_{\tau}S^{1-d} = J(\tilde{J}_{\tau}, -\delta). \quad (6.2) $$

**Proof.** We give a proof using the language of Sect. 5, so formally we prove the claim only for periodic matrices.

Note that the involution (6.1) is strongly related to the standard involution $\tau$ (5.2) on $X$. Indeed, the function $K(\tau Z, \alpha)$ has the divisor $\tau D_+ + \tau D_- C = (\tau D_+ - D_C)$, that is,

$$ K(\tau Z, \alpha) = \frac{K(Z, \beta)}{b_{c_1}(Z) \cdots b_{c_g}(Z)}, $$

and $\beta = \nu\alpha^{-1}$, where $\nu = \mu_{c_1} \cdots \mu_{c_g}$. Due to this remark and the property $z(\tau Z) = z(Z)$ we have

$$ (J(\alpha))_{\tau} = J(\nu\mu\alpha^{-1}). \quad (6.3) $$

Now we apply (6.3) to prove (6.2). Let $\tilde{J}_{\tau} = J(\tilde{\alpha})$ with $\tilde{\alpha} \in \Gamma^*(\tilde{X}_+)$. Or, in other words, $\tilde{J} = J(\tilde{\mu}\tilde{\nu}\tilde{\alpha}^{-1})$. Then by (5.21)

$$ J(\tilde{J}, \delta) = J(\eta\pi^*(\tilde{\mu}\tilde{\nu}\tilde{\alpha}^{-1})), \quad \eta := \prod_{j=1}^{d-1} \mu_{(c_1 \tau)}^{\delta_j + (1, \delta_{c_1 \tau})}. $$

But $\pi^*\tilde{\mu} = \mu^d$ and $\pi^*\tilde{\nu} = \nu\eta_{\tilde{\alpha}}^{-1}$ (just to look at the characters of the corresponding Blaschke products). Thus, having in mind that $\eta\eta_{-\delta} = \eta_{\delta_+}$, we obtain

$$ J(\tilde{J}, \delta) = J(\mu^d\nu\eta_{-\delta}^{-1}\pi^*(\tilde{\alpha}^{-1})). $$
Using again (6.3) we get
\[
(J(\tilde{J}, \delta))_T = J(\mu^{1-d} \eta^{-d} \pi^*(\tilde{\alpha})) = S^{1-d} J(\eta^{-d} \pi^*(\tilde{\alpha})) S^{d-1},
\]
and the lemma and Theorem 6.1 are proved. □

Having two different contractive branches of solutions of the renormalization equation, following [8], to an arbitrary sequence
\[\epsilon = \{\epsilon_0, \epsilon_1, \epsilon_2 \ldots\}, \quad \epsilon_j = \delta_{\pm} .\]
we can associate a limit periodic matrix \(J\) with the spectrum on Julia\((T)\). For a fixed sufficiently hyperbolic polynomial \(T\), we define \(J\) as the limit of
\[J_n := J(\eta_{\epsilon_0} \pi^* \eta_{\epsilon_1} \ldots \pi^* \eta_{\epsilon_{n-1}}). \quad (6.4)\]

6.2. Other solutions of the Renormalization Equation and the Ruelle operators. We conjecture that actually all branches of solutions of the renormalization equation are contractions for sufficiently hyperbolic \(T\). At least the previous remark looks as a quite strong indication in this direction: considering, instead of initial \(T\), \(T^2 = T \circ T\) or its bigger powers, we get, as in (6.4), several \(\delta\)'s, \(\eta_{\delta} = \eta_{\epsilon_0} \pi^* \eta_{\epsilon_1} \ldots \pi^* \eta_{\epsilon_{n-1}}\), possessing the contractibility property with respect to the polynomial \(T^n\) and different from \(\delta_{\pm}\) (related to \((\pi^*)^n\)).

Similarly to (4.4), (4.5) we formulate

**Conjecture 6.3.** Let \(T(z)\) be an expanding polynomial and let \(T'(z) = A_1(z) A_2(z)\) be an arbitrary (polynomial) factorization of the derivative. Denote by \(\sigma_{1,2}\), the (nonnegative) eigen–measures, corresponding to the Ruelle operators
\[
(L_{A_i} f)(x) = \sum_{T(y) = x} \frac{f(y)}{A_i(y)^2}, \quad (6.5)
\]
i.e., \(L_{A_i} \sigma_i = \rho_i \sigma_i\). Finally let \(J_{1,2}\) be the one-sided Jacobi matrices associated with \(\sigma_{1,2}\). Then the block matrix \(J = J_- \oplus J_+\) with \(J_- = J_1\) and \(J_+ = J_2\) is limit periodic.

Note that by the same reason as above the conjecture holds true, say, for \(T^2(z)\) and \(A_1(z) = T'(z), A_2 = T' (T(z))\).

6.3. Shift transformations with the Lipschitz property. We say that the direction \(\eta \in \Gamma^*\) has the Lipschitz property with a constant \(C(\eta)\) if for all \(\alpha, \beta \in \Gamma^*\)
\[
||J(\eta \alpha) - J(\eta \beta)|| \leq C(\eta)||J(\alpha) - J(\beta)||. \quad (6.6)
\]
Then, one can get the contractibility of the map \(\eta \pi^*\) in two steps:
\[
||J(\eta \pi^* \tilde{\alpha}) - J(\eta \pi^* \tilde{\beta})|| \leq C(\eta)||J(\pi^* \tilde{\alpha}) - J(\pi^* \tilde{\beta})||
\leq C(\eta)\kappa ||J(\tilde{\alpha}) - J(\tilde{\beta})||. \quad (6.7)
\]

Note, that in fact the situation is a bit more involved because we should be able to compare Jacobi matrices with different spectral sets, for example, when \(E_i = T^{-1} E_i, E_i \neq E_2\). But we just wanted to indicate the general idea, in particular, for directions \(\eta_{\delta}\) of the form (5.21) such a comparison is possible. Of course, for our goal the constant \(C(\eta)\) should be uniformly bounded when we increase the level of sufficient hyperbolicity of \(T\) making \(\kappa\) smaller.
However the key point of this remark (this way of proof) is that, actually, we do not need to constrain ourselves by the form of the vector \( \eta \). Combining a “Lipschitz” shift by \( \eta \) (the direction is restricted just by this property) with a sufficiently contractive pull–back \( \pi^* \) we arrive at an iterative process that produces a limit periodic Jacobi matrix with the spectrum on the same Julia\((T)\). In the next subsection we give examples of directions with the required property, see Corollary 6.6.

We do not have a proof of the Lipschitz property of \( \eta \)'s, but there is a good chance to generalize the result of the next subsection in a way that at least some of the directions \( \eta \) will be also available.

Finally, we would be very interested to know, whether there is in general a relation between the form of the “weight” vector \( \eta \) and the corresponding weights of the Ruelle operators (if any exists).

6.4. Quadratic polynomials and the Lipschitz property of the Darboux transform. Consider the simplest special case \( T(z) = \rho(z^2 - 1) + 1, \rho > 2 \). Note that the spectral set \( E = T^{-1} \tilde{E} \) is symmetric, moreover the matrix related to \( H^2(\pi^*\tilde{\alpha}) \) has zero main diagonal (as well as a one–sided matrix related to a symmetric measure). Now we introduce a decomposition of \( H^2(\pi^*\tilde{\alpha}) \) which is very similar to the standard decomposition into even and odd functions.

We define the two–dimensional vector–function representation of \( f \in H^2(\pi^*\tilde{\alpha}) \)

\[
\begin{pmatrix}
g_1(\tilde{Z}) \\
g_2(\tilde{Z})
\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} f(Z_1(\tilde{Z})) + f(Z_2(\tilde{Z})) \\ f(Z_1(\tilde{Z})) - f(Z_2(\tilde{Z})) \end{pmatrix},
\]

(6.8)

where

\[
\begin{pmatrix}
g_1(\tilde{Z}) \\
g_2(\tilde{Z})
\end{pmatrix} = \frac{1}{2}\begin{pmatrix} f(Z_1(\tilde{Z})) + f(Z_2(\tilde{Z})) \\ f(Z_1(\tilde{Z})) - f(Z_2(\tilde{Z})) \end{pmatrix},
\]

the first component, in a sense, is even and the second is odd. To be more precise, let us describe analytical properties of this object in \( \partial \tilde{X}_+ \).

Note that due to

\[
\int_{\partial X_+} |f|^2 d\omega = \int_{\partial \tilde{X}_+} \frac{1}{2} \sum_{\pi(Z) = \tilde{Z}} |f|^2 d\tilde{\omega}
\]

metrically it is of \( L^2 \) with respect to \( \tilde{\omega} \), moreover the transformation is norm–preserved.

It is evident that the function \( g_1 \) belongs to \( H^2(\tilde{X}_+, \tilde{\alpha}) \). Consider the second function. Note that the critical points of \( T \) are zero and infinity. For a small circle \( \gamma \) around the point \( T(0) = -\rho + 1 \) we have \( g_2 \circ \gamma = -g_2 \) and the same property for a contour \( \gamma \) that surrounds infinity. Let us introduce

\[
\Delta^2 := \tilde{b}_{T(0)} \tilde{b}
\]

Note that for the above contours we have \( \Delta \circ \gamma = -\Delta \). We are going to represent \( g_2 \) in the form \( g_2 = \Delta \hat{g}_2 \) and to claim that \( \hat{g}_2 \) has nice automorphic properties in \( \tilde{X}_+ \). Let us note that

\[
\frac{\hat{g}_2 T(0)}{\tilde{b}_{T(0)}} = \frac{\hat{g}_2 \tilde{Z} - T(0)}{\Delta^2}
\]
is an outer function in the domain $\mathbb{C} \setminus \hat{E} \simeq \tilde{X}_+$. So, the square root of this function is well defined. We put
\[
\hat{b}_\phi := \sqrt{\hat{b}^2 \hat{z} - T(0) \rho \Delta^2}
\]
and denote by $\hat{\eta}$ the character generated by $\phi$, $\phi \circ \gamma = \eta(\gamma)\phi$. Thus (6.9) reduces the ramification of the function $\Delta$ to the function $\phi$, which is well defined in the domain, and to the elementary function $\sqrt{\hat{z} - T(0)}$.

**Theorem 6.4.** The transformation $f \mapsto g_1 \oplus \hat{g}_2$ given by (6.8) is a unitary map from $H^2(\pi^*\hat{\alpha})$ to $H^2(\hat{\alpha}) \oplus H^2(\hat{\alpha}\hat{\eta})$. Moreover with respect to this representation
\[
zf f \mapsto \begin{bmatrix} 0 & \phi \\ \hat{\phi} & 0 \end{bmatrix} \begin{bmatrix} g_1 \\ \hat{g}_2 \end{bmatrix}
\]
and
\[
v_+ f \mapsto f \oplus 0, \quad f \in H^2(\hat{\alpha}),
\]
where the isometry $v_+: H^2(\hat{\alpha}) \to H^2(\pi^*\hat{\alpha})$ is defined by (5.11).

**Proof.** By the definition of $\Delta$ we have
\[
g_2 = \Delta \hat{g}_2, \quad \text{where } \hat{g}_2 \in H^2(\hat{\alpha}\hat{\eta}).
\]
Further, since
\[
z_{1,2} = \pm \sqrt{\frac{\hat{z} - T(0)}{\rho}},
\]
we have, say for the second component,
\[
\frac{1}{\Delta} \left( \frac{(zf)(Z(\hat{Z}_1)) - (zf)(Z(\hat{Z}_2))}{2} \right) = \sqrt{\frac{\hat{z} - T(0)}{\rho \Delta^2}} \frac{f(Z(\hat{Z}_1)) + f(Z(\hat{Z}_2))}{2} = \phi g_1.
\]
Since on the boundary of the domain
\[
\phi^2 \Delta^2 = \frac{\hat{z} - T(0)}{\rho} = |\phi|^2
\]
(the second expression is positive on $\partial \tilde{X}_+$) we have
\[
\phi \Delta^2 = \phi \text{ on } \hat{E}.
\]
Using this relation, similarly to (6.12), we prove the identity of the first components in (6.10).

\[\square\]

**Theorem 6.5.** The multiplication operator $\phi: L^2(\partial \tilde{X}_+) \to L^2(\partial \tilde{X}_+)$ with respect to the basis systems (5.13) related to $\hat{\alpha}$ and $\hat{\eta}\hat{\alpha}$, respectively, is a two diagonal matrix $\Phi$. Moreover,
\[
\Phi^* \Phi = \frac{J(\hat{\alpha}) - T(0)}{\rho}, \quad \Phi \Phi^* = \frac{J(\hat{\eta}\hat{\alpha}) - T(0)}{\rho}.
\]
In other words, the transformation $J(\hat{\alpha}) \mapsto J(\hat{\eta}\hat{\alpha})$ is the Darboux transform.
Proof. First of all φ is a character–automorphic function with the character ˜η with a unique pole at infinity (bφ is an outer function). Therefore the multiplication operator acts from \(bH^2(\tilde{\alpha}\tilde{\omega}^{-1})\) to \(H^2(\tilde{\gamma}\tilde{\omega})\). Therefore, the operator Φ has only one non–trivial diagonal above the main diagonal. The adjoint operator has the symbol \(\bar{\Phi}\). According to (6.13) it has holomorphic continuation from the boundary inside the domain. Thus Φ∗ is a lower triangular matrix. Combining these two facts we get that Φ has only two non–trivial diagonals. Then, just comparing symbols of operators on the left and right parts of (6.14), we prove these identities. □

Corollary 6.6. Let \(\tilde{J}_{1,2}\) be periodic Jacobi matrices with the spectrum on \([-1, 1]\). Let \(\text{Darb}(\tilde{J}_{1,2}, \rho)\) be their Darboux transforms. Then

\[ \|\text{Darb}(\tilde{J}_1, \rho) - \text{Darb}(\tilde{J}_2, \rho)\| \leq C(\rho)\|\tilde{J}_1 - \tilde{J}_2\|. \]  

(6.15)

Proof. For the given \(\tilde{J}_{1,2}\) we define \(J_{1,2}\) via the quadratic polynomial \(T(z) = \rho(z^2 - 1) + 1\). Being decomposed into even and odd indexed subspaces they are of the form

\[ J_{1,2} = \begin{bmatrix} 0 & \Phi_{1,2}^* \\ \Phi_{1,2} & 0 \end{bmatrix}. \]  

(6.16)

Due to the main theorem, that gives the uniform estimate for \(\|J_1 - J_2\|\), we have

\[ \|\Phi_1 - \Phi_2\| \leq \kappa(\rho)\|\tilde{J}_1 - \tilde{J}_2\|, \]  

(6.17)

with \(\kappa(\rho) = \frac{C}{\rho^{1/2}}, \) \(C\) is an absolute constant. Using (6.14) we get (6.17) with \(C(\rho) = \frac{2\kappa C}{\rho^{1/2}}\). □

7. Appendix

Here we recall some basic facts on two–sided Jacobi matrices. Let \(J\) define a bounded selfadjoint operator on \(L^2(\mathbb{Z})\). The resolvent matrix–function is defined by the relation

\[ W(z) = W(z, J) = \begin{bmatrix} \langle 0| (J - z)^{-1} |0\rangle & \langle 0| (J - z)^{-1} |1\rangle \\ \langle 1| (J - z)^{-1} |0\rangle & \langle 1| (J - z)^{-1} |1\rangle \end{bmatrix}. \]  

(7.1)

This matrix–function has an integral representation

\[ W(z) = \int \frac{d\sigma(x)}{x - z} \]  

(7.2)

with \(2 \times 2\) matrix–measure having a compact support on \(\mathbb{R}\). \(J\) is unitary equivalent to the multiplication operator by an independent variable on

\[ L^2_\sigma = \left\{ f = \begin{bmatrix} f_0(x) \\ f_1(x) \end{bmatrix} : \int f^* d\sigma f < \infty \right\}, \]  

(7.3)

moreover, under this unitary mapping from \(L^2 \to L^2_\sigma\) we have

\[ |0\rangle \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]  

(7.4)

Let \(r_-(z) = r_-(z, 0), r_+(z) = r_+(z, 1)\) be resolvent functions of \(J_+ = J_+(0)\) and \(J_+ = J_+(1)\) respectively (see (2.2)), and \(r_{\pm}\) be the corresponding (scalar!) spectral measures. Then

\[ W(z) = \begin{bmatrix} r^{-1}_-(z) & p_1 \\ p_1 & r^{-1}_+(z) \end{bmatrix}^{-1}. \]  

(7.5)
Recall that according to our notation \( p_1 Q_d \) is the orthonormal polynomial of the degree \( d - 1 \) for the measure \( \sigma_+ \) and \( p_1 R_d \) is the related polynomial of the second kind:

\[
p_1 R_d(z) = \int d\sigma_+(x) \frac{p_1 Q_d(x) - p_1 Q_d(z)}{x - z}. \tag{7.6}
\]

In this notation

\[
|d\rangle \mapsto \mathcal{E}_d(x) := \begin{bmatrix} -p_1^2 R_d(x) \\ p_1 Q_d(x) \end{bmatrix}, \tag{7.7}
\]

moreover

\[
\begin{bmatrix} 0 \\ p_1 R_d(x) \end{bmatrix} = \int d\sigma(x) \frac{\mathcal{E}_d(x) - \mathcal{E}_d(z)}{x - z}. \tag{7.8}
\]

**Lemma 7.1.** Let

\[
\mathcal{F}(x) := \begin{bmatrix} 1 & -p_1^2 R_d(x) \\ 0 & p_1 Q_d(x) \end{bmatrix}, \quad \mathcal{G}(x) := \begin{bmatrix} 0 & 0 \\ 0 & p_1 R_d(x) \end{bmatrix}, \tag{7.9}
\]

Then

\[
\begin{bmatrix} \langle 0 | (J - z)^{-1} | 0 \rangle & \langle 0 | (J - z)^{-1} | d \rangle \\ \langle d | (J - z)^{-1} | 0 \rangle & \langle d | (J - z)^{-1} | d \rangle \end{bmatrix} = \mathcal{F}^*(\bar{z}) W(z) \mathcal{F}(z) + \mathcal{F}^*(\bar{z}) \mathcal{G}(z). \tag{7.10}
\]

**Proof.** This is a standard trick from the theory of orthogonal polynomials. Due to the unitary mapping onto \( L^2_\sigma \), equivalently we have to calculate

\[
\int \frac{\mathcal{F}(x)^* d\sigma(x) \mathcal{F}(x)}{x - z}. \tag{7.11}
\]

Therefore, using orthogonality and (7.8), we continue

\[
= \int \frac{\mathcal{F}(x)^* - \mathcal{F}(\bar{z})^*}{x - z} d\sigma(x) \mathcal{F}(x) + \mathcal{F}(\bar{z})^* \int \frac{d\sigma(x) \mathcal{F}(x)}{x - z}
\]

\[
= \mathcal{F}(\bar{z})^* \int d\sigma(x) \frac{\mathcal{F}(x) - \mathcal{F}(z)}{x - z} + \mathcal{F}(\bar{z})^* \int \frac{d\sigma(x) \mathcal{F}(z)}{x - z} \tag{7.12}
\]

\[
= \mathcal{F}^*(z) \mathcal{G}(z) + \mathcal{F}^*(\bar{z}) W(z) \mathcal{F}(z).
\]

\[\square\]

**Corollary 7.2.** Combining (7.5) with (7.10) we get (2.23) from

\[
\begin{bmatrix} \langle 0 | (J - z)^{-1} | 0 \rangle & \langle 0 | (J - z)^{-1} | d \rangle \\ \langle d | (J - z)^{-1} | 0 \rangle & \langle d | (J - z)^{-1} | d \rangle \end{bmatrix} = \begin{bmatrix} \langle 0 | (\bar{J} - T(z))^{-1} | 0 \rangle & \langle 0 | (\bar{J} - T(z))^{-1} | 1 \rangle \\ \langle 1 | (\bar{J} - T(z))^{-1} | 0 \rangle & \langle 1 | (\bar{J} - T(z))^{-1} | 1 \rangle \end{bmatrix} T'(z)/d,
\]

which is a part of the Renormalization Equation.

**Proof.** A straightforward computation.

\[\square\]

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