Diagram groups and directed 2-complexes: homotopy and homology

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Abstract

We show that diagram groups can be viewed as fundamental groups of spaces of positive paths on directed 2-complexes (these spaces of paths turn out to be classifying spaces). Thus diagram groups are analogs of second homotopy groups, although diagram groups are as a rule non-Abelian. Part of the paper is a review of the previous results from this point of view. In particular, we show that the so called rigidity of the R. Thompson’s group $F$ and some other groups is similar to the flat torus theorem. We find several finitely presented diagram groups (even of type $F_\infty$) each of which contains all countable diagram groups. We show how to compute minimal presentations and homology groups of a large class of diagram groups. We show that the Poincaré series of these groups are rational functions. We prove that all integer homology groups of all diagram groups are free Abelian.

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1 Introduction

The first definition of diagram groups was given by Meakin and Sapir in terms of string rewriting systems (semigroup presentations). Some results about diagram groups were obtained by Meakin’s student Vesna Kilibarda [22, 23]. Further results about diagram groups have been obtained by the authors of this paper [16, 17, 18], D. Farley [14], and B. Wiest [34].

The definition of diagram groups in terms of string rewriting systems does not reflect the geometry of diagram groups and geometrical nature of the constructions that can be applied to diagram groups. In this paper we introduce a more geometric definition of diagram groups in terms of directed 2-complexes.

A directed 2-complex (see [33, 27]) is a directed graph equipped with 2-cells each of which is bounded by two directed paths (the top path and the bottom path). With any directed 2-complex one can associate the set of (directed) homotopies or 2-paths which is defined similar to the set of combinatorial homotopies between 1-paths in ordinary combinatorial 2-complexes. Equivalence classes of 2-paths form a groupoid with respect to the natural concatenation of homotopies. The local groups of that groupoid are the diagram groups of the directed 2-complex. Thus, from this point of view, the diagram groups are “directed” analogs of the second homotopy groups.

The new point of view gave us an opportunity to revisit some earlier results about diagram groups. We show that (a multi-dimensional version of) the Squier complex of a semigroup presentation has a natural realization as the space of positive paths in a directed 2-complex. We also show that several facts about diagram groups proved earlier have a natural topological interpretation in terms of directed 2-complexes.

The only major part of our previous work that is not revisited here is the combinatorics on diagrams ([16, Section 15]): the description of centralizers of elements in diagram groups, solving conjugacy problem, etc. All these results can be easily generalized to the case of diagrams over directed 2-complexes.

The largest part of the paper is devoted to completely new results. In particular, we find several diagram groups of type $F_{\infty}$ each of which contains all countable diagram groups. One of them has only 3 generators and 6 defining relations. Recall that a group $G$ is said to be of type $F_{\infty}$ if there is some $K(G,1)$ CW complex having a finite $n$-skeleton in each dimension $n$.

We show that the universal cover of the space of positive paths of a directed 2-complex $K$ is homeomorphic to the the space of positive paths of what we call the universal 2-cover of $K$ which is again a directed 2-complex. The universal 2-covers possess some remarkable properties which make them 2-dimensional analogs of rooted trees. We call such directed 2-complexes rooted 2-trees.

We study complete directed 2-complexes (they are analogs of complete string rewriting systems). We show how to construct a minimal $K(G,1)$ CW complex (with respect to the number of cells in each dimension) for a diagram group $G$ of a complete directed 2-complex. We compute integer homology of such groups, and show that in the case when the groups are of type $F_{\infty}$ (that happens very often), the Poincaré series are rational. We answer Pride’s question by showing that the integer homology groups of arbitrary diagram groups are free Abelian. We also study the cohomological dimension of diagram groups of complete directed 2-complexes. In particular, we show that the cohomological dimension of a group in that class is $\geq n$ if and only if the group contains a copy of $\mathbb{Z}^n$ (for any natural number $n$).

It was shown by Farley [14] that diagram groups of finite semigroup presentations act freely cellularly by isometries on CAT(0) cubical complexes. One of the important results in the
theory of CAT(0) groups is the flat torus theorem that shows a rigid connection between a
group acting “nicely” on a CAT(0) space, and a geometric property of the space. The algebraic
property is “to contain a copy of \( \mathbb{Z}^n \)”, and the geometric property is “to contain a \( \mathbb{Z}^n \)-invariant
copy of \( \mathbb{R}^n \)”. We show that similar rigid connection exists in our situation between, say, the
R. Thompson group \( F \) and the universal cover of the space of positive paths of the Dunce hat.

The results of this paper are used in our next paper [19] to show that all diagram groups
are totally orderable.

2 Combinatorial definition

We start by giving a precise definition of directed 2-complexes. Our definition differs insignifi-
cantly from the original definition in [33] and is close to the definition of [27].

**Definition 2.1.** For every directed graph \( \Gamma \) let \( P \) be the set of all (directed) paths in \( \Gamma \), including
the empty paths. A directed 2-complex is a directed graph \( \Gamma \) equipped with a set \( F \) (called the
set of 2-cells), and three maps \( [\cdot] : F \to P \), \( [\cdot] : F \to P \), and \( -1 : F \to F \) called top, bottom, and
inverse such that

- for every \( f \in F \), the paths \([f]\) and \( [f] \) are non-empty and have common initial vertices
  and common terminal vertices,
- \( -1 \) is an involution without fixed points, and \([f^{-1}] = [f] \), \([f^{-1}] = [f] \) for every \( f \in F \).

We shall often need an orientation on \( F \), that is, a subset \( F^+ \subseteq F \) of positive 2-cells, such
that \( F \) is the disjoint union of \( F^+ \) and the set \( F^- = (F^+)^{-1} \) (the latter is called the set of
negative 2-cells).

If \( \mathcal{K} \) is a directed 2-complex, then paths on \( \mathcal{K} \) are called 1-paths (we are going to have 2-paths
later). The initial and terminal vertex of a 1-path \( p \) are denoted by \( \iota(p) \) and \( \tau(p) \), respectively.
For every 2-cell \( f \in F \), the vertices \( \iota([f]) = \iota([f]) \) and \( \tau([f]) = \tau([f]) \) are denoted \( \iota(f) \) and
\( \tau(f) \), respectively.

A 2-cell \( f \in F \) with top 1-path \( p \) and bottom 1-path \( q \) will be called a 2-cell of the form
\( p = q \). We shall use a notation \( \mathcal{K} = \langle E \mid [f] = [f], f \in F^+ \rangle \) for a directed 2-complex with
one vertex, the set of edges \( E \) and the set of 2-cells \( F \).

For example, the complex \( \langle x \mid x^2 = x \rangle \) is the Dunce hat obtained by identifying all edges
in the triangle (Figure 1) according to their directions. It has one vertex, one edge, and one
positive 2-cell. The remarkable feature of the Dunce hat is that the famous R. Thompson’s
group \( F \) is its diagram group (see Section 6 below). (The survey [13] collects some known
results about \( F \). See also [9, 5, 6, 16, 17, 4, 15] for other results about this group.)

![Figure 1](image)

There exists a natural way to assign a directed 2-complex to every semigroup presentation
(to a string rewriting system). It is similar to assigning a 2-complex to any group presentation.
If $\mathcal{P} = \langle X \mid u_i = v_i \ (i \in I) \rangle$ is a string rewriting system, then the corresponding directed complex $K_{\mathcal{P}}$ has one vertex, one edge $e$ for each generator from $X$, and one positive 2-cell for each relation $u_i = v_i$. The top path of this cell is labelled by $u_i$ and the bottom path labelled by $v_i$.

Every directed 2-complex $K = \langle E \mid [f] = [f], \ f \in F^+ \rangle$ with one vertex can be considered as a rewriting system with the alphabet $E$ and the set of rewriting rules $F^+$. The difference between these directed 2-complexes and string rewriting systems is that there may be several 2-cells in $K$ with the same top and bottom 1-paths, hence a rewriting rule $p = q$ can repeat many times. We shall observe later that directed 2-complexes with one vertex provide the same class of diagram groups as all directed 2-complexes. But sometimes it is convenient to consider complexes with more than one vertex.

With the directed 2-complex $K$, one can associate a category $\Pi(K)$ whose objects are 1-paths, and morphisms are 2-paths (or homotopies), i.e. sequences of replacements of $[f]$ by $[f]$ in 1-paths, $f \in F$. More precisely, an atomic 2-path (an elementary homotopy) is a triple $(p, f, q)$, where $p$, $q$ are 1-paths in $K$, and $f \in F$ such that $\tau(p) = \iota(f)$, $\tau(f) = \iota(q)$. If $\delta$ is the atomic 2-path $(p, f, q)$, then $p \ [f] \ q$ is denoted by $[\delta]$, and $p \ [f] \ q$ is denoted by $[\delta]$; these are called the top and the bottom 1-paths of the atomic 2-path. Every nontrivial 2-path $\delta$ on $K$ is a sequence of atomic paths $\delta_1, \ldots, \delta_n$, where $[\delta_i] = [\delta_{i+1}]$ for every $1 \leq i < n$. In this case $n$ is called the length of the 2-path $\delta$. The top and the bottom 1-paths of $\delta$, denoted by $[\delta]$ and $[\delta]$, are $[\delta_1]$ and $[\delta_n]$, respectively. We say that $\delta$ is positive if each $\delta_i$ corresponds to a positive 2-cell in $K$. Every 1-path $p$ is considered as a trivial 2-path with $[p] = [p] = p$. These are the identity morphisms in the category $\Pi(K)$. The composition of 2-paths $\delta$ and $\delta'$ is called concatenation and is denoted $\delta \circ \delta'$.

We say that 1-paths $p$, $q$ in $K$ are (directly) homotopic whenever there exists a 2-path $\delta$ such that $[\delta] = p$ and $[\delta] = q$. We also say that $\delta$ connects $p$ to $q$ in $K$.

As in the standard homotopy theory (see, for example, [31]), we need to introduce a homotopy relation on the set of 2-paths and identify homotopic 2-paths in $\Pi(K)$. To do this, we choose a computation-friendly way, similar to the one developed by Peiffer, Reidemeister and Whitehead for the second homotopy group of a combinatorial 2-complex, and later simplified by Huebschmann, Sieradski and Fenn (see Bogley and Pride [2]). The idea is to represent the elements of the second homotopy groups in terms of the so-called pictures. We are going to use the dual objects called diagrams (for a picture version, see [16]).

With every atomic 2-path $\delta = (p, f, q)$, where $\ [f] = u$, $\ [f] = v$ we associate the labelled plane graph $\Delta$ on Figure 2. An arc labelled by a word $w$ is subdivided into $|w|$ edges\(^1\). All edges are oriented from the left to the right. The label of each oriented edge of the graph is a symbol from the alphabet $E$, the set of edges in $K$. As a plane graph, it has only one bounded face; we label it by the corresponding cell $f$ of $K$. This plane graph $\Delta$ is called the diagram of $\delta$. Such diagrams are called atomic. The leftmost and rightmost vertices of $\Delta$ are denoted by $\iota(\Delta)$ and $\tau(\Delta)$, respectively. The diagram $\Delta$ has two distinguished paths from $\iota(\Delta)$ to $\tau(\Delta)$ that correspond to the top and bottom paths of $\delta$. Their labels are $puq$ and $pvq$, respectively. These are called the top and the bottom paths of $\Delta$ denoted by $[\Delta]$ and $[\Delta]$.

\(^1\)In this paper, we denote the length of a word or a path $w$ by $|w|$.
The diagram corresponding to the trivial 2-path $p$ is just an arc labelled by $p$; it is called a trivial diagram and it is denoted by $\varepsilon(p)$.

Let $\delta = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_n$ be a 2-path in $K$, where $\delta_1, \ldots, \delta_n$ are atomic 2-paths. Let $\Delta_i$ be the atomic diagram corresponding to $\delta_i$. Then the bottom path of $\Delta_i$ has the same label as the top path of $\Delta_{i+1}$ ($1 \leq i < n$). Hence we can identify the bottom path of $\Delta_i$ with the top path of $\Delta_{i+1}$ for all $1 \leq i < n$, and obtain a plane graph $\Delta$, which is called the diagram of the 2-path $\delta$.

It is clear that the above diagram $\Delta$, as a plane graph, has exactly $n$ bounded faces or cells.

Two diagrams are considered equal if they are isotopic as plane graphs. The isotopy must take vertices to vertices, edges to edges, it must also preserve labels of edges and inner labels of cells. Two 2-paths are called isotopic if the corresponding diagrams are equal.

For example, consider the diagram on Figure 3 below. It is clear that it corresponds to the 2-path

$$(p, f_1, q u_2 r) \circ (p v_1 q, f_2, r)$$

as well as the 2-path

$$(p u_1 q, f_2, r) \circ (p, f_1, q v_2 r).$$

In that case we call the atomic 2-paths $(p, f_1, qu_2 r)$ and $(pu_1 q, f_2, r)$ independent.

The proof of the following lemma in the case of semigroup presentations can be extracted from the proof of [16, Lemma 6.2]. We leave it to the reader to generalize the proof to the case of directed 2-complexes.

**Lemma 2.2.** The isotopy relation on 2-paths of a directed 2-complex $K$ is the smallest equivalence relation containing all pairs of 2-paths of the form (1) and (2) and invariant under concatenation.

Concatenation of 2-paths corresponds to the concatenation of diagrams: if the bottom path of $\Delta_1$ and the top path of $\Delta_2$ have the same labels, we can identify them and obtain a new diagram $\Delta_1 \circ \Delta_2$.

Note that for any atomic 2-path $\delta = (p, f, q)$ in $K$ one can naturally define its inverse 2-path $\delta^{-1} = (p, f^{-1}, q)$. The inverses of all 2-paths and diagrams are defined naturally. The inverse diagram $\Delta^{-1}$ of $\Delta$ is obtained by taking the mirror image of $\Delta$ with respect to a horizontal line, and replacing labels of cells by their inverses.
Let us identify in the category $\Pi(K)$ all isotopic 2-paths and also identify each 2-path of the form $\delta \delta^{-1} \delta''$ with $\delta' \delta''$. The quotient category is obviously a groupoid (i.e. a category with invertible morphisms). It is denoted by $D(K)$ and is called the diagram groupoid of $K$.

Two 2-paths are called homotopic if they correspond to the same morphism in $D(K)$. For each 1-path $p$ of $K$, the local group of $D(K)$ at $p$ (i.e. the group of equivalence classes of 2-paths connecting $p$ with itself) is called the diagram group of the directed 2-complex $K$ with base $p$ and is denoted by $D(K, p)$. Notice that if $p$ is empty, then $D(K, p)$ is trivial by definition. In this paper, we shall usually ignore these diagram groups.

We shall give a much easier (equivalent) definition of the diagram groupoid after Theorem 2.6 below.

Remark 2.3. Notice first that the diagram groups of a directed 2-complex do not depend on the orientation on the set of 2-cells of that complex. Notice also that if a directed 2-complex $K'$ is obtained from $K$ by identifying vertices, then the diagram groupoid of $K'$ may differ from the diagram groupoid of $K$ because the set of 1-paths may increase, but the diagram groups of $K$ will be diagram groups of $K'$ as well.

One can easily check that if $K = K_P$ for some semigroup presentation $P$ and $w$ is a word over $X$ (that is, the corresponding path in $K_P$), then the diagram group $D(K, w)$ we just defined coincides with the diagram group $D(P, w)$ over $P$ defined in [16]. Clearly, if $K = K_P$ then 2-paths are just the derivations over the semigroup presentation $P$.

It is convenient to define diagrams over a directed 2-complex $K$ in an “abstract” way, without referring to 2-paths of $K$. Such a definition was given by Kashintsev [21] and Remmers [28] in the case of semigroup presentations. Here we basically repeat their definition and result.

Definition 2.4. A diagram over $K = \langle E \mid [f] = [f], f \in F^+ \rangle$ is a finite plane directed and connected graph $\Delta$, where every edge is labelled by an element from $E$, and every bounded face is labelled by an element of $F$ such that:

- $\Delta$ has exactly one vertex-source $\iota$ (which has no incoming edges) and exactly one vertex-sink $\tau$ (which has no outgoing edges);
- every 1-path in $\Delta$ is simple;
- each face of $\Delta$ labelled by $f \in F$ is bounded by two 1-paths $u$ and $v$ such that the label of $u$ is $[f]$, the label of $v$ is $[f]$, and the loop $uv^{-1}$ on the plane goes around the face in the clockwise direction.

It is easy to see [16] that every plane graph satisfying the conditions of Definition 2.4 is situated between two positive paths connecting $\iota$ and $\tau$. These paths are $[\Delta]$ and $[\Delta]$.

We say that a diagram $\Delta$ over a directed 2-complex $K$ is a $(u, v)$-diagram whenever $u$ is the top label and $v$ is the bottom label of $\Delta$. If $u$ and $v$ are the same, then the diagram is called spherical (with base $u = v$).

The following lemma (see [16, Lemma 3.5]) shows that diagrams over $K$ in the sense of Definition 2.4 are exactly diagrams that correspond to 2-paths in $K$. We will often use this fact without reference.

Lemma 2.5. Let $K$ be a directed 2-complex. Then 1-paths $u, v$ are homotopic in $K$ if and only if there exists a $(u, v)$-diagram over $K$ (in the sense of Definition 2.4).
Diagrams over $K$ corresponding to homotopic 2-paths are called equivalent. The equivalence relation on the set of diagrams (and the homotopy relation on the set of 2-paths of $K$) can be defined very easily as follows. We say that two cells $\pi_1$ and $\pi_2$ in a diagram $\Delta$ over $K$ form a dipole if $[\pi_1]$ coincides with $[\pi_2]$ and the labels of the cells $\pi_1$ and $\pi_2$ are mutually inverse. Clearly, if $\pi_1$ and $\pi_2$ form a dipole, then one can remove the two cells from the diagram and identify $\lceil \pi_1 \rceil$ with $\lfloor \pi_2 \rfloor$. The result will be some diagram $\Delta'$. As in [16], it is easy to prove that if $\delta$ is a 2-path corresponding to $\Delta$, then the diagram $\Delta'$ corresponds to a 2-path $\delta'$, which is homotopic to $\delta$. We call a diagram reduced if it does not contain dipoles. A 2-path $\delta$ in $K$ is called reduced if the corresponding diagram is reduced. The following is an analog of Kilibarda’s lemma. The proof coincides with the proof of [16, Theorem 3.17] and we omit it here.

**Theorem 2.6.** Every equivalence class of diagrams over a directed 2-complex $K$ contains exactly one reduced diagram. Every 2-path in $K$ is homotopic to a reduced 2-path, every two homotopic reduced 2-paths have equal diagrams and so they contain the same number of atomic factors.

Thus one can define morphisms in the diagram groupoid $D(K)$ as reduced diagrams over $K$ with operation “concatenation + reduction” (that is, the product of two reduced diagrams $\Delta$ and $\Delta'$ is the result of removing all dipoles from $\Delta \circ \Delta'$ step by step).

The diagram groupoid $D(K)$ has another natural operation, addition: if $\Delta'$ and $\Delta''$ are diagrams over $K$ and $\tau(\lceil \Delta' \rceil) = i(\lceil \Delta'' \rceil)$ in $K$ then one can identify $\tau(\Delta')$ with $i(\Delta'')$ to obtain the new diagram denoted by $\Delta' + \Delta''$ and called the sum of $\Delta'$ and $\Delta''$. If $K$ has only one vertex, then this operation is everywhere defined. In that case the operation of addition makes $D(K)$ a tensor groupoid in the sense of [20].

Figure 4 below illustrates the concepts of the concatenation of diagrams and the sum of them.

![Figure 4](image)

**3 Topological definition**

We have seen that diagram groups are directed analogs of the second homotopy groups. Recall that one can define the second homotopy groups of a topological space as the fundamental group of a space of paths. On the other hand, in the case of semigroup presentations, the diagram groups can be defined as fundamental groups of the so called 2-dimensional Squier complexes associated with the presentations (see [16]).

In this section, we show that the Squier complex of a directed 2-complex $K$ can be considered as the space of certain positive paths in $K$.

First we define a multi-dimensional version of the Squier complex $\text{Sq}(K)$ from [16] as a semi-cubical complex. Recall [30, 10] that a semi-cubical complex is a family of disjoint sets $\{M_n; n \geq 0\}$ (the elements of $M_n$ are called $n$-cubes) with face maps $\lambda^k_i: M_n \to M_{n-1}$ ($1 \leq i \leq n$, $k = 0, 1$) satisfying the semi-cubical relations (the rightmost map act first):

$$\lambda^k_i \lambda^k_j = \lambda^k_{j-1} \lambda^k_i \quad (i < j).$$  \hspace{2cm} (3)
A realization of a semi-cubical complex \( \{M_n; n \geq 0\} \) can be obtained as a factor-space of the disjoint union of Euclidean cubes, one \( n \)-cube \( c(x) \) for each element \( x \in M_n, n \geq 0 \). The equivalence relation identifies (point-wise) the cube \( \lambda^k_x(c(x)) \) with the cube \( c(\lambda^k_x(x)) \) for all \( i, k, x \). Here \( \lambda^k_x(I^n) \) is the corresponding \((n-1)\)-face of the Euclidean \( n \)-cube \( I^n \) (that is, \( \lambda^k_x(I^n) = I^{n-1} \times \{k\} \times I^{n-i} \)).

**Definition 3.1.** The semi-cubical complex \( \text{Sq}(K) \) is defined as follows. For every \( n \geq 0 \), let \( M_n \) be the set of thin diagrams \([14]\) of the form

\[
\varepsilon(u_0) + f^{(1)} + \varepsilon(u_1) + \cdots + f^{(n)} + \varepsilon(u_n)
\]

where \( f^{(i)} \) are negative\(^2\) 2-cells of \( K \) and \( u_i \) are 1-paths in \( K \) (Figure 3 thus shows a thin diagram with two cells). The face map \( \lambda^k_i \) takes the thin diagram \( c \) of the form (4) to

\[
[c]_i = \varepsilon(u_0) + f^{(1)} + \cdots + \left[ f^{(i)} \right] + \cdots + f^{(n)} + \varepsilon(u_n)
\]

if \( k = 0 \) and

\[
[c]_i = \varepsilon(u_0) + f^{(1)} + \cdots + \left[ f^{(i)} \right] + \cdots + f^{(n)} + \varepsilon(u_n)
\]

if \( k = 1 \). These faces are called the top and the bottom \( i \)th faces of \( c \), respectively. It is easy to check that the conditions (3) are satisfied, so \( \text{Sq}(K) \) is a semi-cubical complex.

**Remark 3.2.** Note that the realization of \( \text{Sq}(K) \) does not depend on the orientation \( F^+ \) on \( K \). If we change the orientation, then some cells \( f_i \) in the thin diagram (4) are replaced by their inverses. This means that we simply change the reference point of a cube. So we will think about every thin diagram as of a cube with fixed reference point.

Thus the vertices of \( \text{Sq}(K) \) are 1-paths of \( K \), the edges correspond to negative atomic 2-paths \((u, f, v)\), 2-cells correspond to pairs of independent atomic 2-paths, etc. For example, the thin diagram \( \varepsilon(u) + f + \varepsilon(v) + g + \varepsilon(w) \) determines a square with contour

\[(u, f, v [g] w) \circ (u [f] v, g, w) \circ (u, f^{-1}, v [g] w) \circ (u [f] v, g^{-1}, w).\]

It is convenient to enrich the structure of the Squier complex \( \text{Sq}(K) \) by introducing inverse edges: \((u, f, v)^{-1} = (u, f^{-1}, v)\). Then the edges \((u, f, v)\) will be called positive if \( f \) is a positive 2-cell of \( K \), and negative if \( f \) is negative. As a result, the 1-skeleton of \( \text{Sq}(K) \) turns into a graph in the sense of Serre [29], and the 2-skeleton of \( \text{Sq}(K) \) coincides with the Squier complex defined in [16] provided \( K = K_P \) for some \( P \). Hence, in particular, the fundamental groups of \( \text{Sq}(K_P) \) coincide with fundamental groups of the Squier complex in [16].

Clearly the complex \( \text{Sq}(K) \) is in general disconnected. If \( p \) is a 1-path in \( K \), then by \( \text{Sq}(K, p) \) we will denote the connected component of the Squier complex that contains \( p \).

**Example 3.3.** Figure 5 shows a part of the Squier complex of the Dunce hat on Figure 1. The thick line shows the boundary of one of the 2-cells in this complex.

![Figure 5](image-url)

\(^2\)Taking negative edges instead of positive simplifies some computations later.
The following theorem is similar to Kilibarda’s statement (see [16, Theorem 6.1]).

**Theorem 3.4.** Let $\mathcal{K}$ be a directed 2-complex, $p$ be a 1-path in $\mathcal{K}$. Then the diagram group $\mathcal{D}(\mathcal{K}, p)$ is isomorphic to the fundamental group $\pi_1(\text{Sq}(\mathcal{K}), p)$ of the semi-cubical complex $\text{Sq}(\mathcal{K})$ with the basepoint $p$.

The proof of Kilibarda’s theorem carries without any essential changes. As an immediate corollary of Theorem 3.4, we obtain the following

**Corollary 3.5.** Let $\mathcal{K}$ be a directed 2-complex, $p$ and $q$ be homotopic 1-paths in $\mathcal{K}$. Then $\mathcal{D}(\mathcal{K}, p)$ is isomorphic to $\mathcal{D}(\mathcal{K}, q)$.

**Proof.** Indeed, $p$ and $q$ belong to the same connected component of $\text{Sq}(\mathcal{K})$.

The diagram groups with different bases can be very different but there exists the following useful relationship between them.

**Corollary 3.6.** If $p = p_1 p_2$ is a 1-path in $\mathcal{K}$, then $\mathcal{D}(\mathcal{K}, p_1) \times \mathcal{D}(\mathcal{K}, p_2)$ is embedded into $\mathcal{D}(\mathcal{K}, p_1 p_2)$.

**Proof.** Indeed, the map $(\Delta_1, \Delta_2) \to \Delta_1 + \Delta_2$ from $\mathcal{D}(\mathcal{K}, p_1) \times \mathcal{D}(\mathcal{K}, p_2)$ to $\mathcal{D}(\mathcal{K}, p_1 p_2)$ is an injective homomorphism (see [16], Remark 2 after Lemma 8.1).

Now let us introduce a natural topological realization of the semi-cubical complex $\text{Sq}(\mathcal{K})$.

We expand the set of paths in $\mathcal{K}$ allowing paths that go “inside” 2-cells.

Let $\mathcal{K}$ be a directed 2-complex. Attaching a 2-cell $f \in F^+$ with $p = [f], q = [f]$ can be done as follows. Let $D = [0,1] \times [0,1]$ be a unit square. For any $t \in [0,1]$ we have the path $d_t$ in $D$ defined by $d_t(s) = (s, t) \in D$ ($s \in [0,1]$). We attach this square to $\mathcal{K}$ in such a way that $d_0$ is identified with $p$, $d_1$ is identified with $q$, all points of the form $(0, t) \in D$ are collapsed to $\iota(p) = \iota(q)$, all points of the form $(1, t)$ are collapsed to $\tau(p) = \tau(q)$ ($t \in [0,1]$).

Now for any $t \in [0,1]$, the image of $d_t$ in $\mathcal{K}$ becomes a path inside the 2-cell $f$. This path will be denoted by $f_t$. Clearly, $f_0 = p$, $f_1 = q$. So we have a continuous family of paths $\{f_t\}$ ($t \in [0,1]$) that transforms $p$ into $q$ (see Figure 6).

For any $f \in F^+$, $t \in [0,1]$ one can also define $(f^{-1})_t = f_{1-t}$. So $f_t$ makes sense for any $f \in F$.

![Figure 6.](image_url)
By definition, a positive path in a directed 2-complex $K$ is a finite product of the form $p_1 \cdots p_n (n \geq 1)$, where each factor $p_i$ ($1 \leq i \leq n$) is either a 1-path on $K$, or a path of the form $f_t$ for some $f \in F$, $t \in [0, 1]$. Of course, we assume that the terminal point of $p_i$ coincides with the initial point of $p_{i+1}$ for any $1 \leq i < n$. The set of all positive paths in $K$ defined in this way will be denoted by $\Omega_+(K)$. Note that every 1-path in $K$ is a positive path in this sense.

Note that every positive path $p$ in $\Omega_+(K)$ can be uniquely written in the normal form $p = u_0 f_{t_1}^{(1)} u_1 \cdots f_{t_n}^{(n)} u_n$, where $u_i$ are 1-paths in $K$ ($0 \leq i \leq n$), $f_i \in F^-$, $t_i \in (0, 1)$ ($1 \leq i \leq n$). To the path $p$ we assign a point $\xi(p)$ with coordinates $(t_1, \ldots, t_n)$ in the $n$-cube $\varepsilon(u_0) + f^{(1)} + \cdots + f^{(n)} + \varepsilon(u_n)$ (see Figure 7).

**Figure 7.**

**Lemma 3.7.** The map $\xi$ is a bijection between $\Omega_+(K)$ and a realization of $\text{Sq}(K)$. For any positive path

$$p = u_0 f_{t_1}^{(1)} u_1 \cdots f_{t_n}^{(n)} u_n$$

in $\Omega_+(K)$, where $u_i$ are 1-paths in $K$ ($0 \leq i \leq n$) and $t_i \in [0, 1]$ ($1 \leq i \leq n$), the point $\xi(p)$ has coordinates $(t_1, \ldots, t_n)$ in the $n$-cube $\varepsilon(u_0) + f^{(1)} + \cdots + f^{(n)} + \varepsilon(u_n)$.

**Proof.** Any point $x$ in a realization of $\text{Sq}(K)$ is an inner point of a unique cube (the point of a cube of dimension 0 is inner in itself by definition). Let $x$ be a point in the $n$-cube $\varepsilon(u_0) + f^{(1)} + \cdots + f^{(n)} + \varepsilon(u_n)$ with coordinates $(t_1, \ldots, t_n)$, where $f_i \in F^-$, $t_i \in (0, 1)$ ($1 \leq i \leq n$). One can assign to it the positive path $p = u_0 f_{t_1}^{(1)} u_1 \cdots f_{t_n}^{(n)} u_n$ (written in the normal form). This defines a map $\zeta$ from a realization of $\text{Sq}(K)$ to $\Omega_+(K)$. It is obvious that $\xi$ and $\zeta$ are mutually inverse. Thus $\xi$ is a bijection.

Now we have to check that if $p$ is written in the form (7), where some subscripts $t_i$ are equal to 0 or 1, then the image of $p$ under $\xi$ is defined by the same rule. Changing the reference point of a cube, we may assume that $f_i \in F^-$ for all $i$. Now one can pass from (7) to the normal form of $p$ step by step replacing subpaths of the form $f_0^{(i)}$ by $\lfloor f^{(i)} \rfloor$ and $f_1^{(i)}$ by $\lceil f^{(i)} \rceil$. Let us analyze what happens at one elementary step. Without loss of generality assume that a subpath of the form $f_0$ is replaced by $\lfloor f \rfloor$. So our path is written in two forms: $\cdots u_i \cdots$ and $\cdots u_i f_0 u'' \cdots$, where $u_i = u' \lfloor f \rfloor u''$. The rule defining $\xi$ assigns two points to these forms. One of them belongs to the cube $c' = \cdots + \varepsilon(u_i) + \cdots$ and has coordinates $(t_1, \ldots, t_n)$ in it. The other point belongs to the cube $c = \cdots + \varepsilon(u') + f + \varepsilon(u'') + \cdots$ and has coordinates $(t_1, \ldots, 0, \ldots, t_n)$, where 0 is inserted after $t_i$. The cube $c'$ is the $(i+1)$th face of $c$. By definition of $\text{Sq}(K)$, we glue $c'$ and $\lceil c \rceil$ isometrically. This means that these two points in a realization of $\text{Sq}(K)$ coincide.

Lemma 3.7 shows that $\Omega_+(K)$ is a natural realization of the semi-cubical complex $\text{Sq}(K)$. Note that the empty paths correspond to isolated points in the Squier complex.

Thus we have an equivalent definition of diagram groups of $K$ as the fundamental groups of the space of positive paths in $K$. One possible way of generalizing the definition of diagram groups and of defining “continuous versions” of the diagram groups would be to consider a more general spaces than directed 2-complexes, define “positive paths” in a suitable way, and then
to consider the fundamental groups of the spaces of positive paths. They may have certain properties in common with diagram groups.

Usually we shall not distinguish between the Squier complex $\text{Sq}(K)$ and its geometric realization.

We shall return to the Squier complexes and consider their universal covers in Section 8.

4 Theorems about isomorphism. The class of diagram groups

In this section, we show that diagram groups do not change much if we do certain surgeries on directed 2-complexes.

The following useful statement contains a directed 2-complex analog of Tietze transformations for group and semigroup presentations.

**Theorem 4.1.** Let $K$ be a directed 2-complex.

1) Let $u$ be a non-empty 1-path in $K$. Let $K'$ be the directed 2-complex obtained from $K$ by adding a new edge $e$ with $\iota(e) = \iota(u)$, $\nu(e) = \nu(u)$ and a new 2-cell $f$ of the form $u = e$ (and also the inverse 2-cell $f^{-1}$). Then for every 1-path $w$ in $K$, the diagram groups $D(K, w)$ and $D(K', w)$ are isomorphic.

2) Suppose that $K$ is a union of two directed 2-complexes $K_1$ and $K_2$ such that all vertices and edges of $K$ are both in $K_1$ and in $K_2$. Suppose that the top path $[f]$ (bottom path $[f]$) of each positive 2-cell $f$ of $K_1$ is homotopic in $K_2$ to some path $u_f$ (resp., $v_f$). Let us consider the directed complex $K'$ with the same vertices and edges as $K$ and 2-cells from $K_2$ together with all positive 2-cells $u_f = v_f$ for all positive 2-cells $f$ from $K_1$ (plus the corresponding negative cells $v_f = u_f$). Then the diagram groups $D(K, w)$ and $D(K', w)$ are isomorphic for every 1-path $w$ in $K$.

**Proof.** 1. Indeed, there exists a natural embedding of $D(K, w)$ into $D(K', w)$ which maps every reduced $(w, w)$-diagram over $K$ to itself. In order to show that this map is surjective, notice that if a $(w, w)$-diagram $\Delta$ over $K'$ does not contain edges labelled by $e$ then it is a diagram over $K$. If $\Delta$ contains an edge labelled by $e$ then this edge cannot be on $[\Delta]$ or $[\Delta]$. Hence this edge is a common edge of the contours of two cells $\pi$ and $\pi'$ in $K$. Since $K'$ has only two 2-cells with $e$ on the boundary (namely, $f$ and $f^{-1}$), one of the two cells $\pi$ or $\pi'$ is labelled by $f$ and another by $f^{-1}$ (the edge labelled by $e$ is the top path of one of these cells and the bottom path of another one). Hence $\pi$ and $\pi'$ form a dipole. This implies that every reduced $(w, w)$-diagram over $K'$ contains no edges labelled by $e$, and so it is a diagram over $K$. Hence by Theorem 2.6 the natural embedding of $D(K, w)$ into $D(K', w)$ is surjective.

2. By Theorem 2.5, for any 2-cell $f$ of $K_1$ there exist diagrams $\Gamma_f$ and $\Delta_f$ over $K_2$ such that the label of $[\Gamma_f]$ is $[f]$, the label of $[\Gamma_f]$ is $u_f$, the label of $[\Delta_f]$ is $[f]$, the label of $[\Delta_f]$ is $v_f$.

By $D_w$ (respectively, $D_w'$) we denote the set of all $(w, w)$-diagrams over $K$ (respectively $K'$). We are going to define two maps $\phi: D_w \to D_w'$, $\psi: D_w' \to D_w$.

Let $\Xi \subseteq D_w$. Let $\textbf{F}_1^+$ be the set of positive 2-cells in $K_1$. For every cell $\pi$ in $\Xi$ with inner label $f \in \textbf{F}_1^+$, we do the following operation. First we cut $\pi$ into three parts by connecting the initial vertex of $\pi$ with the terminal vertex of $\pi$ by two simple curves, $p_1$ and $p_2$, that have no intersections other than at the endpoints. We enumerate the three parts from top to bottom and assume that $p_1$ is above $p_2$. Then we subdivide $p_1$ into edges and give them labels such that $p_1$ will have label $u_f$. Similarly, we turn $p_2$ into a path labelled by $v_f$. Now we insert the diagram $\Gamma_f$ between the top path of $\pi$ and $p_1$. Analogously, we insert $\Delta_f$ between $p_2$ and the
The space between $p_1$ and $p_2$ becomes a cell $u_f = v_f$, which is a cell $\mathcal{K}'$. We can assign the inner label $f$ to it. If the inner label of a cell $\pi$ of $\Xi$ is $f^{-1} \in F_1$, then we subdivide it in the same way to get the mirror image of the diagram we had for cells with inner label $f$. (The inner label for the cell in the middle will be $f^{-1}$.)

Every diagram $\Xi$ over $\mathcal{P}$ now becomes a diagram over $\mathcal{P}'$. We denote it by $\phi(\Xi)$.

The map $\psi$ is defined similarly. Now if we have a diagram $\Xi$ over $\mathcal{K}'$, then we replace each of its cells $\pi$ of the form $u_f = v_f$ ($f \in F_1$) by the concatenation of three diagrams. The first of them is $\Gamma_f^{-1}$, the third is $\Delta_f$, and the second one is a cell with inner label $f$. We do similar transformation with cells of the form $v_f = u_f$ whose inner labels are negative.

The result of these replacements will be a diagram over $\mathcal{K}$ denoted by $\psi(\Xi)$.

It follows from our construction that for any diagram $\Xi$ over $\mathcal{K}$, the diagram $\psi(\phi(\Xi))$ over $\mathcal{K}$ is equivalent to $\Xi$. This is so because after applying $\phi$ and then $\psi$ to $\Xi$, we get a diagram with a number of subdiagrams of the form $\Gamma^{\pm 1}\Gamma^{\mp 1}$ or $\Delta^{\pm 1}\Delta^{\mp 1}$. Cancelling all the dipoles, we get the diagram $\Xi$ we had in the beginning. Analogously, for any diagram $\Xi$ over $\mathcal{K}'$, the diagram $\phi(\psi(\Xi))$ over $\mathcal{K}'$ will be equivalent to $\Xi$. It is also clear that $\phi$ and $\psi$ preserve the operation of concatenation of diagrams.

This means that maps $\phi$, $\psi$ induce homomorphisms of diagram groups $\tilde{\phi}: \mathcal{D}(\mathcal{K}, w) \to \mathcal{D}(\mathcal{K}', w)$ and $\tilde{\psi}: \mathcal{D}(\mathcal{K}', w) \to \mathcal{D}(\mathcal{K}, w)$. The fact about equivalence of diagrams means that $\tilde{\phi}$ and $\tilde{\psi}$ are mutually inverse. Thus they are isomorphisms and $\mathcal{D}(\mathcal{K}, w) \cong \mathcal{D}(\mathcal{K}', w)$. □

As an immediate application of Theorem 4.1, we obtain the following statement about subdivisions of directed 2-complexes. Let $\mathcal{K}$ be a directed 2-complex and let $f$ be its 2-cell. Let us add a new edge $e$ to the complex with $\iota(e) = \iota([f]) = \iota(\lfloor f \rfloor)$ and $\tau(e) = \tau([f]) = \tau(\lfloor f \rfloor)$, remove the 2-cells $f^{\pm 1}$, and add new 2-cells $f_1^{\pm 1}$, $f_2^{\pm 1}$, where $f_1$, $f_2$ have the form $\lfloor f \rfloor = e$ and $\lfloor f \rfloor$, respectively. This operation can be done for several positive 2-cells of $\mathcal{K}$ at once. This simply means that we cut some 2-cells of $\mathcal{K}$ into two parts. The resulting directed 2-complex $\mathcal{K}'$ is called a subdivision of $\mathcal{K}$.

**Lemma 4.2.** If $\mathcal{K}$ is a directed 2-complex, $w$ is a non-empty 1-path in $\mathcal{K}$ and $\mathcal{K}'$ is a subdivision of $\mathcal{K}$, then the diagram groups $\mathcal{D}(\mathcal{K}, w)$ and $\mathcal{D}(\mathcal{K}', w)$ are isomorphic.

**Proof.** We use the notation from the paragraph preceding the formulation of the lemma. Let $\mathcal{K}''$ be the directed 2-complex obtained from $\mathcal{K}$ by adding the edge $e$ and the 2-cells $f_1^{\pm 1}$. By part 1) of Theorem 4.1, $\mathcal{D}(\mathcal{K}, w) = \mathcal{D}(\mathcal{K}'', w)$. Now represent $\mathcal{K}''$ as the union of $\mathcal{K}_1$ and $\mathcal{K}_2$, where $\mathcal{K}_1$, $\mathcal{K}_2$ have the same vertices and edges, $\mathcal{K}_1$ contains exactly two 2-cells $f$, $f^{-1}$, and $\mathcal{K}_2$ contains all other 2-cells of $\mathcal{K}''$. Notice that $[f]$ is homotopic to $e$ in $\mathcal{K}_2$ (because $\mathcal{K}_2$ contains the 2-cell $f_1$). Hence by part 2) of Theorem 4.1, we can replace $f^{\pm 1}$ in $\mathcal{K}''$ by $f_2^{\pm 1}$, where $f_2$ has the form $e = \lfloor f \rfloor$ without changing the diagram group with the base $w$. But the resulting directed 2-complex is precisely $\mathcal{K}'$. Hence $\mathcal{D}(\mathcal{K}, w) \cong \mathcal{D}(\mathcal{K}', w)$. □

As an immediate corollary of Lemma 4.2 we get the following statement.

**Theorem 4.3.** The classes of diagram groups over semigroup presentations and diagram groups of directed 2-complexes coincide.

**Proof.** Notice that complexes of the form $\mathcal{K}_P$ corresponding to semigroup presentations considered in [16] are precisely the directed 2-complexes with one vertex in which different 2-cells cannot have the same top and bottom paths. We have already mentioned that we can only consider directed 2-complexes with one vertex (if we identify vertices we preserve existing diagram groups but the set of diagram groups can increase since the set of 1-paths can increase).
It is obvious that if we subdivide each 2-cell of $K$ twice (into three parts instead of two) then we turn $K$ into a $K_P$ for some $P$. It remains to apply Lemma 4.2.

\[\square\]

5 Morphisms of complexes and universal diagram groups

Let $K, K'$ be directed 2-complexes. A morphism $\phi$ from $K$ to $K'$ is a map that takes vertices to vertices, edges to non-empty 1-paths and 2-cells to 2-paths and preserves the functions $\iota, \tau, [\cdot], [\cdot], -1$:

M1 For every edge $e$, $\phi(\iota(e)) = \iota(\phi(e))$, $\phi(\tau(e)) = \tau(\phi(e))$.

M2 For every 2-cell $f$ of $K$, $\phi([f]) = [\phi(f)]$, $\phi([f]) = [\phi(f)]$; here for every 1-path $p = e_1e_2 \cdots e_k$ we set $\phi(p) = \phi(e_1)\phi(e_2) \cdots \phi(e_n)$, where $e_i$ are edges (the latter product exists because of M1).

M3 For every 2-cell $f$ of $K$, $\phi(f^{-1}) = \phi(f)^{-1}$.

Every morphism $\phi: K \to K'$ induces a functor from the category $\Pi(K)$ to $\Pi(K')$: the image of an atomic 2-path $(p, f, q)$ is $(\phi(p), \phi(f), \phi(q))$, where $(u, \delta_1 \circ \cdots \circ \delta_n, v)$ for atomic 2-paths $\delta_i = (p_i, f_i, q_i)$ is short for $(up_1, f_1, q_1v) \circ \cdots \circ (up_n, f_n, q_nv)$.

A morphism $\phi: K \to K'$ also induces a functor from the diagram groupoid $D(K)$ to $D(K')$. The image of a $(p, q)$-diagram $\Delta$ over $K$ is the $(\phi(p), \phi(q))$-diagram obtained from $\Delta$ by a) replacing each edge that has label $e$ by a path labelled by $\phi(e)$, and b) replacing each cell that has label $f$ by the diagram over $K'$ corresponding to the 2-path $\phi(f)$. Both of these functors will be denoted by $\phi$ as well. The restriction of $\phi$ onto a diagram group $D(K, p)$ is a homomorphism that will be denoted by $\phi_p$.

For a non-empty 1-path $p$ in $K$, we say that a morphism $\phi: K \to K'$ is $p$-nonsingular if the induced homomorphism $\phi_p$ is injective on $D(K, p)$. In that case the subgroup $\phi_p(D(K, p))$ of $D(K')$ is called naturally embedded. If the morphism $\phi$ is $p$-nonsingular for every $p$, we call it nonsingular.

Lemma 5.1. Let $\phi: K \to K'$ be a morphism of two directed 2-complexes.

1) If $\phi(\delta)$ is reduced for every reduced 2-path $\delta$, and $\phi(f)$ is not empty for every 2-cell $f$ of $K$, then $\phi$ is nonsingular.

2) Suppose that $\phi$ is injective on the set of 2-cells and $\phi(f)$ is a 2-cell for every 2-cell $f$ of $K$. Then $\phi$ is nonsingular.

3) If a directed 2-complex $K'$ is obtained by adding 2-cells to a directed 2-complex $K$, then diagram groups of the form $D(K, p)$ are naturally embedded into the diagram groups $D(K', p)$, for every 1-path $p$.

4) If a directed 2-complex $K'$ is obtained by adding 2-cells to a directed 2-complex $K$, and $[f]$ is homotopic to $[f]$ in $K$ for every 2-cell $f \in K' \setminus K$, then for every 1-path $p$ in $K$, the diagram group $D(K, p)$ is a retract of the diagram group $D(K', p)$.

Proof. 1) Indeed, suppose that the kernel of $\phi_p$ is not trivial. Then it contains a reduced 2-path $\delta \neq p$ by Theorem 2.6. By the assumption, $\phi(\delta)$ is reduced, and non-empty, so $\phi_p(\delta) \neq 1$ by Theorem 2.6, a contradiction.

2) It is easy to see that for every reduced $(p, p)$-diagram $\Delta$, the diagram $\phi_p(\Delta)$ does not contain dipoles. It remains to use part 1) of this lemma.

3) Immediately follows from 2).
4) The retraction $\psi$ is given as follows. Fix a $(\lceil f \rceil, \lfloor f \rfloor)$-diagram $\Gamma_f$ over $K$ for every 2-cell $f \in K' \setminus K$ in such a way that inverse diagrams correspond to inverse 2-cells. Then for every $(p, p)$-diagram $\Delta$ over $K'$, the diagram $\psi(\Delta)$ is obtained from $\Delta$ by inserting $\Gamma_f$ instead of every cell in $\Delta$ labelled by $f \in K \setminus K'$. Clearly, $\psi^2 = \psi$.

Let us call a directed 2-complex $K$ universal if every finite or countable directed 2-complex maps nonsingularly into $K$. A diagram group is called universal if it contains copies of all countable diagram groups. A directed 2-complex is said to be 2-path connected if all its non-empty 1-paths are homotopic to each other. By Theorem 3.4 it has at most one non-trivial diagram group up to isomorphism. Notice that if a universal directed 2-complex is 2-path connected then its nontrivial diagram group is universal because every at most countable diagram group is (obviously) a diagram group of at most countable directed 2-complex.

**Lemma 5.2.** Let $U$ be the directed 2-complex

$$\langle x \mid x^m = x^n, x^m = x^n, \ldots \text{ for all } 1 \leq m < n \rangle$$

(every equality appears countably many times). Then $U$ is universal.

**Proof.** Let $K$ be at most countable directed 2-complex and let $K_1$ be obtained from $K$ by identifying all its edges and vertices. By $\phi$ we denote the natural morphism from $K$ to $K_1$. Then $\phi$ is nonsingular by Lemma 5.1, part 2. It is easy to see that $U$ can be obtained from $K_1$ by adding 2-cells. Hence $K_1$ is a subcomplex of $U$ and $\phi: K \to U$ is nonsingular by Lemma 5.1, part 3. □

We can simplify the universal directed 2-complex $U$ by using part 2) of Theorem 4.1. For every $1 \leq n \leq \infty$ let

$$H_n = \langle x \mid x^2 = x, x = x, x = x, \ldots \text{ (n times)} \rangle.$$

This complex contains one vertex, one edge, and $n + 1$ positive 2-cells, one of which is the Dunce hat and all others are spheres. It is obtained from the plane diagram on Figure 8 by identifying all edges according to their directions. For example, the complex $H_0$ is the Dunce hat (see Figure 1).

**Lemma 5.3.** The directed 2-complex $H_\infty$ is universal. In particular, every countable diagram group embeds into the diagram group $D(H_\infty, x)$.

**Proof.** In fact we shall show that $D(U, x)$ is isomorphic to $D(H_\infty, x)$. Let us denote by $Q$ the complex obtained from $U$ by removing the cell $x^2 = x$ and its inverse. Then $H_0 \cup Q = U$. It is easy to see that every non-empty 1-path in $H_0$ is homotopic to $x$. Therefore, let us replace each
2-cell \(x^m = x^n\) in \(U\) by a cell \(x = x\), and obtain a complex \(H_\infty\). By part 2) of Theorem 4.1, the diagram groups of \(U\) and \(H_\infty\) are isomorphic. The proof of part 2) of Theorem 4.1 actually gives us a nonsingular morphism from \(U\) into \(H_\infty\). It remains to use Lemma 5.2.

The diagram group \(D(H_\infty, x)\) is not even finitely generated (see Example 6.8 below).

Our next goal is to show that already the group \(D(H_1, x)\) is universal. This will follow from Lemma 5.3 and the fact that there exists a nonsingular morphism from \(H_\infty\) to \(H_1\). The group \(D(H_1, x)\) is finitely presented (see Example 6.8) and has a nice structure (see [19]).

**Lemma 5.4.** There exists a nonsingular morphism from \(H_\infty\) to \(H_2\).

**Proof.** Let us label the positive 2-cells of \(H_\infty\) by \(f_0, f_1, \ldots\), where \(f_0\) is the cell \(x^2 = x\). The complex \(H_2\) has positive 2-cells \(f_0, f_1, f_2\). Consider the morphism \(\phi\) from \(H_\infty\) to \(H_2\) which takes the edge \(x\) to \(x\), \(f_0\) to \(f_0\), and each \(f_i, i \geq 1\), to the 2-path

\[(1, f_1, 1)^i \circ (1, f_2, 1) \circ (1, f_1, 1)^i.\] (8)

We need to show that \(\phi_x\) is injective (then every \(\phi_x^k\) will be injective too because all local groups in the diagram groupoid \(D(H_\infty)\) are conjugate by Corollary 3.5).

Indeed, let \(\Delta\) be a nontrivial reduced diagram over \(H_\infty\). Then \(\bar{\Delta} = \phi(\Delta)\) is obtained by replacing every cell with label \(f_i\) by the diagram \(\Gamma_i\) corresponding to the 2-path (8) (see Figure 9) and each cell with label \(f_i^{-1}\) by the diagram \(\Gamma_i^{-1}\).

![Figure 9](image_url)

It is sufficient to show that \(\bar{\Delta}\) is also nontrivial. Any diagram over \(H_n\) (for any \(n\)) can be uniquely decomposed into subdiagrams of the following two types. Each subdiagram of the first type is an \((x^2, x)\)-cell or its mirror image. Each subdiagram of the second type is a maximal \((x, x)\)-subdiagram, which is a product of \((x, x)\)-cells only.

If we decompose \(\Delta\) in such a way, then we see that each subdiagram of the second type is the image of a subdiagram of the second type in \(\Delta\) under the mapping \(\phi\). Since \(\Delta\) has no dipoles, each of its subdiagram of the second type is a product of \((x, x)\)-cells, where the word formed by their labels is a freely irreducible group word \(v\) over the alphabet \(\{f_i^{\pm 1} \mid i \geq 1\}\). Notice that the \(\phi\)-image of this subdiagram is a product of \((x, x)\)-cells such that the word formed by their inner labels is \(\bar{v}\), where \(\bar{f}_i = f_1^i f_2 f_1^{-i}\).

Since the map \(f_i \mapsto \bar{f}_i\) is an embedding of the free group with generators \(f_i (i \geq 1)\) into the free group with two generators \(f_1, f_2\), the word \(\bar{v}\) will be non-empty after all free cancellations are made in it. Thus if we reduce dipoles in each subdiagram of the second type in \(\Delta\), the resulting diagram \(\Delta'\) will be reduced. Indeed, we have cancelled all the dipoles formed by \((x, x)\)-cells. No dipoles between cells that correspond to \(x^2 = x\) may appear because each
subdiagram of the second type in \( \tilde{\Delta} \) remains nontrivial after all cancellations. The diagram \( \Delta' \)
contains the same number of cells labelled by \( f_0^{\pm 1} \) as \( \Delta \) and at least as many \((x,x)\)-cells as \( \Delta \). Hence \( \Delta' \) is nontrivial. \( \square \)

**Lemma 5.5.** There exists a nonsingular morphism from \( H_2 \) into \( H_1 \).

**Proof.** The idea is similar to the one of the previous lemma. We keep notation for 2-cells of \( H_2 \)
from that lemma. Letters \( f_i \) \((i = 1, 2)\) will be also used to denote the atomic 2-paths \((1, f_i, 1)\)
and the corresponding \((x,x)\)-diagrams that consist of one cell labelled by \( f_i \). By \( a \) we denote
any nontrivial reduced \((x,x)\)-diagram over \( H_0 = \langle x \mid x^2 = 1 \rangle \) and one of the corresponding
2-paths in \( H_0 \). Any two \((x,x)\)-diagrams can be concatenated. So each word in \( f_1^{\pm 1}, f_2^{\pm 1}, a^{\pm 1} \)
denotes some \((x,x)\)-diagram.

Now we use the morphism \( \psi: H_2 \to H_1 \) that takes the edge \( x \) to \( x \), \( f_1 \) to \( f_1 a f_1 \), \( f_2 \) to \( f_2^2 a f_2^2 \).

Let \( \Delta \) be a nontrivial reduced diagram over \( H_2 \). Let \( \tilde{\Delta} \) be the diagram obtained from \( \psi(\Delta) \)
by cancelling all dipoles of \((x,x)\)-cells. As in Lemma 5.4, we subdivide \( \Delta \) into subdiagrams of
the two types. Let \( \Gamma \) be a subdiagram of the second type. It is a product of cells with inner
labels \( f_i^{\pm 1}, f_2^{\pm 1} \). Let \( v \) be the word that is the product of these labels. Clearly, this is a freely
reduced word in \( f_1^{\pm 1}, f_2^{\pm 1} \). After we replace \( v \) by \( \bar{v} \), where \( \bar{f}_1 = f_1 a f_1 \), \( \bar{f}_2 = f_2^2 a f_2^2 \)
and then freely reduce the result, we get a word of the form

\[
f_1^{s_0} a^{k_1} f_1^{s_1} \cdots a^{k_r} f_1^{s_r},
\]

where \( r \) is the length of \( v \). Note that \( s_0 \neq 0, s_r \neq 0, k_1, \ldots, k_r = \pm 1 \). Note also that none
of the occurrences of letters \( a^{\pm 1} \) in \( \bar{v} \) disappears after the reduction, hence only occurrences of
the letter \( f_1^{\pm 1} \) can disappear. Thus the \( \psi \)-image of any subdiagram of the second type after
cancelling all dipoles of \((x,x)\)-cells becomes reduced and of the form (9) as well. After we cancel
all \((x,x)\)-dipoles in the subdiagrams \( \psi(\Gamma) \) for all maximal subdiagrams \( \Gamma \) of \( \Delta \) of the second
type, we would not have any more \((x,x)\)-dipoles. Hence we shall get the diagram \( \tilde{\Delta} \).

Since \( s_0 \) and \( s_r \) are always non-zero, the 2-cells labelled by \( f_0^{\pm 1} \) cannot form a dipole in
\( \tilde{\Delta} \). Therefore, \( \tilde{\Delta} \) is reduced. Since the number of cells in \( \tilde{\Delta} \) is at least the same as in \( \Delta \), the
diagram \( \tilde{\Delta} \) is nontrivial. \( \square \)

**Theorem 5.6.** The directed 2-complex \( H_1 \) is universal. Hence the group \( G_1 = D(H_1, x) \) contains
copies of all countable diagram groups. This group is finitely presented and even of type \( F_\infty \).

**Proof.** The first statement follows from Lemmas 5.3, 5.4, 5.5. The fact that the group \( G_1 \) is
of type \( F_\infty \) follows from Theorem 9.3 below. It can also be deduced from results of [14]. In
Example 6.8 below, we shall compute a presentation of \( G_1 \). It has 6 generators and 18 defining
relations. \( \square \)

Theorem 5.6 gives an example of a universal directed 2-complex with one edge and two
positive 2-cells. The following theorem shows that a complex with one edge and one positive
2-cell can be universal as well.

**Theorem 5.7.** The directed 2-complex \( V = \langle y \mid y^3 = y^2 \rangle \) is universal (this complex is obtained
from Figure 10 by identifying all edges according to their directions). Its diagram group \( D(V, y^2) \)
is also a universal group of type \( F_\infty \). It has the following Thompson-like presentation:

\[
\langle x_0, x_1, \ldots, y_0, y_1, \ldots \mid x_j^{x_i} = x_{j+1}, y_j^{x_i} = y_{i+1}, 0 \leq i < j - 1 \rangle.
\]
Proof. Let us consider the two diagrams $A$ and $B$ over $V$ on Figure 11. Here $A$ is a $(y^8, y^4)$-diagram and $B$ is a $(y^4, y^4)$-diagram. These diagrams correspond to the following 2-paths on $V$. Let $p_{i,j}$ $(i, j \geq 0)$ denote the atomic 2-path $(y^i, y^3 = y^2, y^j)$. Then $A$ corresponds to the 2-path $\alpha = p_{-1}^{-33} p_{15}^{-1} p_{41}^{-1} p_{04}^{-1} p_{30}^{-1} p_{11}^{-1}$, whereas $B$ corresponds to $\beta = p_{01}^{-1} p_{10}^{-1}$. Let us consider the morphism $\gamma$ from $H_1$ to $V$ which takes the edge $x$ to the 1-path $y^4$, the 2-cell $f_0$ to $\alpha$, and the 2-cell $f_1$ to $\beta$. We are going to show that $\gamma$ is nonsingular. As before, it is enough to show that it is $x$-nonsingular.

Let $\Delta$ be any reduced $(x, x)$-diagram over $H_1$. The diagram $\gamma(\Delta)$ is obtained as follows. First we subdivide each edge labelled by $x$ into 4 parts and label each of them by $y$. Then each $(x^2, x)$-cell becomes a $(y^8, y^4)$-cell. Every $(x^2, x)$-cell with inner label $f_0$ is replaced by $A$. A mirror image of such a cell is replaced by $A^{-1}$. Similarly, any $(x, x)$-cell of $\Delta$ becomes a $(y^4, y^4)$-cell, so we replace by $B^{\pm 1}$ all $(x, x)$-cells labelled by $f^{\pm 1}_1$. After all these replacements, we get a $(y^4, y^4)$-diagram $\tilde{\Delta}$ over $V$.

We shall show that $\tilde{\Delta}$ is reduced, and then apply Lemma 5.1, part 1). Note that each of the diagrams $A, B$ has no dipoles so a dipole in $\tilde{\Delta}$, if it occurs, must belong to different subdiagrams of the form $A^{\pm 1}, B^{\pm 1}$. Suppose that the upper cell of the dipole is contained in $A^{\pm 1}$. From the structure of $A$ it is obvious that this cell must be a $(y^3, y^2)$-cell. So it cannot form a dipole with a cell from $B^{\pm 1}$. The lower cell of the dipole is a $(y^2, y^3)$-cell.

There are two types of vertices in $\tilde{\Delta}$. Vertices of the first type (we call them red) are the images of vertices of $\Delta$. The other vertices are called green. It is easy to see that $B$ has only two red vertices, $\iota(B)$ and $\tau(B)$. The diagram $A$ has exactly three red vertices: $\iota(A), \tau(A)$, and the middle point of the top path of $A$. The middle point of the bottom path of $A$ is green. So we have to consider two cases for the two cells that form the dipole. In the first case the middle point of the common part of the boundary of the cells forming a dipole is red, in the second case it is green.

In the first case the upper cell of the dipole is contained in a copy of $A^{-1}$ and the lower cell is contained in a copy of $A$. Denote these copies by $\Gamma_1, \Gamma_2$, respectively. We claim that the bottom path of $\Gamma_1$ coincides with the top path of $\Gamma_2$. Indeed, $\Gamma_1$ is the $\gamma$-image of an $(x, x^2)$-cell $\pi_1$ in $\Delta$ and $\Gamma_2$ is the $\gamma$ image of an $(x^2, x)$-cell $\pi_2$ in $\Delta$. The bottom path of $\pi_1$ was subdivided into 8 parts. The same is true for the top path of $\pi_2$. The product of the 4th and the 5th of these parts is the same for both $\pi_1$ and $\pi_2$ since it is the common boundary of the cells forming the dipole. This can happen only if the bottom path of $\pi_1$ coincides with the top path of $\pi_2$. But in this case we have a dipole in $\Delta$. This contradicts the assumption that $\Delta$ is reduced.
In the second case the upper cell of the dipole is contained in a copy of \( A \) and the lower cell is contained in a copy of \( A^{-1} \). We also denote these copies by \( \Gamma_1, \Gamma_2 \), respectively. The bottom path of \( \Gamma_1 \) is the image of an edge in \( \Delta \). This edge is subdivided into 4 parts. The product of its second and third part is the common boundary of the cells of the dipole. The same is true for the top path of \( \Gamma_2 \). So the bottom of \( \Gamma_1 \) coincides with the top of \( \Gamma_2 \) since they must be images of the same edge in \( \Delta \). In this case an \((x^2,x)\)-cell forms a dipole in \( \Delta \) with an \((x,x^2)\)-cell, a contradiction.

If the lower cell of the dipole is contained in \( A^{\pm 1} \), then the same arguments are applied. So to finish the proof, let us assume that the dipole in \( \Delta \) is formed by two cells that are contained in copies of \( B^{\pm 1} \). Suppose that the upper cell of the dipole belongs to a copy of \( B \). Thus it is a \((y^2,y^3)\)-cell. Note that the leftmost point of it is green and the rightmost point is red. The lower cell must be a \((y^3,y^2)\)-cell with the corresponding points of the same colour. Thus the lower cell is contained in a copy of \( B^{-1} \). As in the previous paragraph, we see from this fact that the last 3 of 4 sections of some edges in \( \Delta \) coincide. Then these edges also coincide and so \( \Delta \) has a dipole that consists of two \((x,x)\)-cells. The case when the upper cell of the dipole belongs to a copy of \( B^{-1} \) is quite analogous. Thus \( \gamma \) is nonsingular.

That diagram groups of \( \mathcal{V} \) are of type \( F_\infty \) follows directly from [14]. The presentation of \( D(\mathcal{V},y^3) \) is found in [16, page 114]. It can be found using Theorem 6.5 below.

In the next section we shall construct a universal directed 2-complex whose diagram groups have very simple presentations.

### 6 Presentations of diagram groups

In [16, Section 9], we showed how to find nice presentations of diagram groups of the so called complete string rewriting systems. Here we shall generalize these results for diagram groups of directed 2-complexes.

We start with a definition of a complete directed 2-complex. Throughout this section, \( K \) is a directed 2-complex with the set of edges \( E \), set of 2-cells \( F \) and a fixed set of positive 2-cells \( F^+ \).

Let \( p, q \) be 1-paths in \( K \). We write \( p \xrightarrow{\delta} q \) if \( p \neq q \) and there exists a positive 2-path \( \delta \) with \([\delta]=p \) and \([\delta]=q \).

We say that \( K \) is Noetherian if every sequence of 1-paths \( p_1 \xrightarrow{\delta_1} p_2 \xrightarrow{\delta_2} \cdots \) terminates.

We say that \( K \) is confluent, if for every two positive 2-paths \( \delta_1, \delta_2 \) with \([\delta_1]=[\delta_2] \) there exist two positive 2-paths \( \delta_1 \circ \delta'_1 \) and \( \delta_2 \circ \delta'_2 \) such that \([\delta'_1]=[\delta'_2] \). In that case we say that \( \delta_1 \) and \( \delta_2 \) can be extended to a diamond.

If a directed 2-complex is Noetherian and confluent, then we say that \( K \) is complete.

It is easy to see that if \( K = K_P \) for some complete string rewriting system \( P \), then \( K \) is complete.

Let \( \delta_1, \delta_2 \) be two positive atomic 2-paths on \( K \). Assume that \([\delta_1]\neq[\delta_2]\) for each \( i = 1, 2 \). Suppose that one of the two cases hold:

1. \( \delta_1 = (1,f_1,q), \delta_2 = (p,f_2,1) \), where \([f_1]=ps, [f_2]=sq\) for some non-empty 1-path \( s \);
2. \([f_2]\) is a subpath of \([f_1]\) and \( f_1 \neq f_2 \).

Then we say that \( \delta_1 \) and \( \delta_2 \) form a critical pair. The diagrams representing these cases are shown on Figure 12.
We say that the critical pair can be resolved if it can be extended to a diamond.

For string rewriting systems, it is known (Newman’s lemma [16]) that a Noetherian string rewriting system is complete if and only if every critical pair can be resolved. One can similarly prove that a Noetherian directed 2-complex is complete if and only if every critical pair of its positive atomic 2-paths can be resolved.

A 1-path \( p \) in a complete directed 2-complex \( K \) is called irreducible if \( p \to q \) is impossible (that is, \( p \) cannot be changed by any positive 2-path). It is easy to see that every 1-path \( p \) in a complete directed 2-complex \( K \) is homotopic to a unique irreducible 1-path \( \bar{p} \), which is called the irreducible form of \( p \).

From Lemma 5.1, part 4), one can almost immediately deduce the following important statement.

**Lemma 6.1.** Every diagram group of a directed 2-complex \( K \) is a retract of a diagram group of a complete directed 2-complex \( K' \supseteq K \). The number of classes of homotopic 1-paths in \( K' \) is the same as in \( K \). If \( K \) is finite and it has finitely many classes of homotopic 1-paths, then \( K' \) is also finite.

**Proof.** Let \( G = D(K, p) \). Let us fix some total well ordering on the set of edges of \( K \). Then we can introduce the ShortLex order on 1-paths of \( K \).

Let us change the orientation on the set of 2-cells of \( K \) as follows. For every positive cell \( f \in K \), if \( \lceil f \rceil \) is smaller than \( \lfloor f \rfloor \) in ShortLex, we call \( f \) negative and \( f' \) positive. This operation does not change the diagram groups of the 2-complex and the classes of homotopic 1-paths (see Remark 2.3).

In every class \( W \) of homotopic 1-paths of \( K \) choose the ShortLex smallest 1-path \( p(W) \). Now add cells to \( K \) as follows. First for every edge \( e \) in \( K \), we add a positive 2-cell \( f_e \) of the form \( e = p(W) \), where \( W \) is the class of homotopic 1-paths containing \( e \). We also add the inverse of that 2-cell. Now let \( U \) and \( V \) be two classes of homotopic 1-paths in \( K \) such that the product \( p(U)p(V) \) exists (that is, \( \tau(p(U)) = \iota(p(V)) \)) and let \( W \) be the class of homotopic 1-paths that contains \( p(U)p(V) \). Add a positive cell \( f_{U,V} \) with top path \( p(U)p(V) \) and bottom path \( p(W) \) (also add the corresponding negative 2-cell). As a result of these operations, the number of classes of homotopic 1-paths does not change.

The resulting complex \( K' \) is clearly Noetherian. Indeed, for every positive 2-cell \( f \) of \( K' \), \( \lceil f \rceil \geq \lfloor f \rfloor \) in the ShortLex order.

Every 1-path of a class \( W \) is connected to \( p(W) \) by a positive 2-path, which consists of atomic 2-paths corresponding to the cells of the form \( f_e \) and \( f_{U,V} \). (This can be easily proved by induction on the length of the 1-path.) Hence the complex \( K' \) is confluent. Indeed, for every two positive atomic 2-paths \( \delta_1, \delta_2 \) in \( K \) with \( \lceil \delta_1 \rceil = \lceil \delta_2 \rceil \), their bottom 1-paths are homotopic. So one can reduce each of them to the same 1-path and complete the diamond. Thus \( K' \) is a complete directed 2-complex.
The complex $K'$ is also confluent. Indeed, for every two positive atomic 2-paths $\delta_1$, $\delta_2$ in $K$ with $[\delta_1] = [\delta_2]$, the 1-paths $[\delta_1]$ and $[\delta_2]$ are in the same class $W$ of homotopic 1-paths. Now using the new cells $f_e$ and $f_{L,V}$, one can reduce each of these 1-paths to $p(W)$ and complete the diamond. Thus $K'$ is a complete directed 2-complex.

By Lemma 5.1, part 4), $G$ is a retract of $D(K', p)$. The last statement of the theorem obviously holds because we add only finitely many cells. 

Since we are looking for nice presentations of diagram groups, which are fundamental groups of Squier complexes, it is natural to start with finding nice spanning forests in $Sq(K)$. It can be done in the case when $K$ is complete (and in some other cases which we do not discuss here). Recall that a spanning forest of a 1-complex $S$ is a forest whose intersection with every connected component of $S$ is a spanning tree in that component.

**Definition 6.2.** Let $K$ be a complete directed 2-complex. A spanning forest $T$ in $Sq(K)$ is called a left forest whenever the following two conditions hold:

**F1** for any edge $e = (p, f, q)$ in $T$, the 1-path $p$ is irreducible;

**F2** if an edge $e = (p, f, q)$ belongs to $T$, then any edge of the form $(p, f, q')$ also belongs to $T$.

Analogously one can define a right forest.

Because of the property F2, we will often use the notation $(u, f, *)$ when we mention an edge of a left forest. Analogously, $(*, f, v)$ will be used for edges from a right forest.

**Lemma 6.3.** If $K$ is a complete directed 2-complex, then $Sq(K)$ has a left forest and a right forest.

**Proof.** In each connected component of $Sq(K)$ we choose the vertex which is an irreducible 1-path. If $p$ is not irreducible, then we find its shortest initial segment $p'$, which is not irreducible. Let $p = p'v$ for some $v$. By definition, $p'$ can be reduced so it has a subpath of the form $[f]$ for some negative cell $f$ of $K$, where $[f] \neq [f]$ (there are possibly many ways to choose $f$ with the above properties but we choose one of them arbitrarily). Obviously, this subpath is a suffix of $p'$. Hence $p' = u[f]$, where $u$ must be irreducible. Thus to every vertex $p$ of $Sq(K)$ that is not reducible, we can assign an edge $e = (u, f, v)$. Let us consider the subgraph $T$ of the 1-skeleton of $Sq(K)$ that contains all vertices and all the edges of the form $e^{\pm 1}$, where $e$ was assigned to some $p$. We leave it as an exercise for the reader to check that $T$ is a spanning forest and that it satisfies conditions F1, F2 (see the proof of [16, Lemma 9.4]). A right forest in $Sq(K)$ is constructed in a similar way. 

**Remark 6.4.** Let $K$ be a complete directed 2-complex. In general, the way to construct a left (right) forest from the proof of Lemma 6.3 is not unique. However, if the second case of the critical pair from its definition never occurs (that will be the case in all the examples considered below), then it is not difficult to prove that the left forest in $Sq(K)$ is unique and consists of all edges $(u, f, v)^{\pm 1}$, where $f \in F^-$, $[f] \neq [f]$, and every proper initial subpath of $u [f]$ is irreducible.

Let us fix a left forest $T_l$ and a right forest $T_r$ in $Sq(K)$. Then for every vertex $p$ in $Sq(K)$, where $p \neq \bar{p}$, there exists a unique negative edge $e \in T_l$ (resp., $e \in T_r$) going into $p$. Indeed, otherwise there would be two different paths in $T_l$ (resp., $T_r$) that consist of positive edges and connect $p$ with $\bar{p}$. We shall say that $e$ is assigned to $p$. 

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The following theorem is a translation of [16, Theorem 9.5]. It gives a Wirtinger-like presentation of any diagram group of a complete directed 2-complex. The translation of the proof from [16] is straightforward.

**Theorem 6.5.** Let $\mathcal{K}$ be a complete directed 2-complex with the set of negative 2-cells $\mathbf{F}^-$, and a distinguished non-empty 1-path $w$. Then the diagram group $\mathcal{D}(\mathcal{K}, w)$ admits the following presentation. The generating set $S$ consists of all the negative edges in $\text{Sq}(\mathcal{K}, w)$ excluding edges from the left forest $T_l$. The defining relations are all relations of the form

$$(p, f_1, q [f_2] r) = (p, f_1, q [f_2] r)^{[p,f_1|q,f_2,r]} \quad (10)$$

if the edge $e = (p [f_1] q, f_2, r)$ is not in $T_l$, or of the form

$$(p, f_1, q [f_2] r) = (p, f_1, q [f_2] r) \quad (11)$$

if $e \in T_l$. Here $f_1, f_2 \in \mathbf{F}^-$, and all edges involved in these relations are from the generating set $S$.

In most cases, this presentation can be simplified.

As in [16], with every negative edge $(u, f, v)$ in $\text{Sq}(\mathcal{K})$ we associate a group word $[u, f, v]$ in the alphabet of negative edges of $\text{Sq}(\mathcal{K})$ and their inverses, defined by the Noetherian induction on the strict order generated by $\ddownarrow$ and the relation $\text{suff}$, where $(u, u') \in \text{suff}$ if and only if $u'$ is a proper suffix of $u$:

- If $u \neq \bar{u}$, then $[u, f, v] = [\bar{u}, f, v]$.
- If $u = \bar{u}$ and $(u, f, v)$ is in $T_l$, then $[u, f, v] = 1$.
- If $u = \bar{u}$, $v = \bar{v}$ and $(u, f, v)$ is not in $T_l$, then $[u, f, v] = (u, f, v)$.
- If $u = \bar{u}$, $v \neq \bar{v}$ and $(u, f, v)$ is not in $T_l$, then take the negative edge $(p, g, q)$ from the right forest $T_r$ that is assigned to $v$ (thus, $g \in \mathbf{F}^-$, $v = p [g] q$, and $q = \bar{g}$). By the induction hypothesis, we can assume that the word $[u, f, p [g] q]$ is already defined. Then let

$$[u, f, v] = [u, f, p [g] q]^{[p,f_1|g,p,q]}.$$ 

Notice that every letter (or its inverse) in any word $[u, f, v]$ has the form $(p, g, q)$, where $p$, $q$ are irreducible, $g \in \mathbf{F}^-$, and $(p, g, q)$ is not in $T_l$.

Finally, let us present the translation of [16, Theorem 9.8] into the language of directed 2-complexes (we are correcting some misprints in the formulation of that theorem as well). The translation of the proof of that theorem is straightforward.

**Theorem 6.6.** Let $\mathcal{K}$ be a complete directed 2-complex and let $w$ be a non-empty 1-path in $\mathcal{K}$. The group $\mathcal{D}(\mathcal{K}, w)$ is generated by the set $X$ of all edges $(u, f, v)$ in $\text{Sq}(\mathcal{K}, w)$, where $u$, $v$ are irreducible, $f \in \mathbf{F}^-$, and $(u, f, v)$ is not in the left forest $T_l$, subject to the following defining relations:

$$[p, f_1, q [f_2] r] = [p, f_1, q [f_2] r]^{[p,f_1|q,f_2,r]} \quad (12)$$

where $x^y$ means $y^{-1}xy$.  

\[\text{Here and below } x^y \text{ means } y^{-1}xy.\]
• $f_1, f_2 \in F^-$,
• $p, q, r$ are irreducible,
• $p[f_1] q [f_2] r$ is homotopic to $w$ in $K$,
• $(p, f_1, *)$ is not in $T_1$ and $(*, f_2, r)$ is not in $T_r$.

Remark 6.7. a) Notice that every relation in Theorem 6.6 is a conjugacy relation of the form $x^y = x^z$ for some generator $x$ and words $y, z$. Therefore, the set $X$ in Theorem 6.6 is a minimal generating set of the diagram group (because it freely generates the abelianization of $D(K, w)$).

In Section 9 we will show that the number of defining relations given by Theorem 6.6 is also minimal possible.

b) One can check that in the formulation of Theorem 6.6, we can replace $T_r$ by $T_l$. The numbers of generators and relations will be the same (see Remark 9.8 below), but the relations in general will be more complicated.

Example 6.8. As we mentioned before, the directed 2-complex $H_n$ is complete for every $n$. This complex has only two irreducible 1-paths, 1 and $x$. Let $g_0$ be the negative 2-cell of $H_n$ of the form $x = x^2$ and let $g_1, g_2, \ldots, g_n$ be the negative 2-cells of $H_n$ of the form $x = x$. Then the left forest consists of edges of the form $(1, g_0, *)^{\pm 1}$. The right forest consists of edges of the form $(*, g_0, 1)^{\pm 1}$.

By Theorem 6.6, the diagram group $D(H_n, x)$ is generated by the edges of the forms $(x, g_0, 1)$, $(x, g_0, x)$, and $(p, g, q), 1 \leq i \leq n$, where $p, q \in \{1, x\}$ (the number of generators is $4n + 2$) subject to the conjugacy relations (12), where there are $2n + 1$ choices for the pair $(p, f_1), 2n + 1$ choices for the pair $(f_2, r)$, and $q$ can be equal to 1 or $x$ (the number of relations is $2(2n + 1)^2$).

In particular, if $n = \infty$, the diagram group is not finitely generated. For any integer $n$, the group $D(H_n, x)$ is finitely presented. In case $n = 0$, it has two generators $x_0 = (x, g_0, 1), x_1 = (x, g_0, x)$ and two defining relations $x_1^2 = x_1^{x_0 x_1}, x_1^3 = x_1^{x_2 x_1}$. This is one of the classical presentations of R. Thompson’s group $F$ [13]. In case $n = 1$ we get a presentation of the group $G_1$ with 6 generators and 18 defining relations.

Now we are going to use Theorems 6.5 and 6.6 to give an example of a universal diagram group with a very simple presentation.

Theorem 6.9. Let $K = (a, y \mid ay = a, y^3 = y^2)$. Then $K$ is a universal directed 2-complex. The group $H = D(K, a)$ is universal. It can be given by the following Thompson-like group presentation

$$\langle x_0, x_1, x_2, \ldots \mid x_j^{x_i} = x_{j+1}, 0 \leq i < j - 1 \rangle. \quad (13)$$

The group $H$ also has the following finite presentation with three generators and six defining relations:

$$\langle x_0, x_1, x_2 \mid x_2^x = x_2^{x_3^{x_4^x}}, x_2^{x_3^x} = x_2^{x_3^{x_4^x}}, x_2^{x_3^{x_4^x}}, (j = 3, 4, 5) \rangle. \quad (14)$$

Proof. It is easy to see that $K$ contains $V$, whence $K$ is universal, and $D(V, y^2)$ is embedded into $D(K, y^2)$. Hence $D(K, y^2)$ contains copies of all countable diagram groups. By Corollary 3.6, $D(K, a) \times D(K, y^2)$ is embedded into $D(K, ay^2)$. Since $ay^2$ and $a$ are homotopic in $K$, by Corollary 3.5, we can conclude that $D(K, y^2)$ is embedded into $D(K, a)$. Hence $D(K, a)$ also contains copies of all countable diagram groups.

It is easy to check that the directed 2-complex $K$ is complete.
Theorem 6.5 implies that $H$ can be generated by the edges of the form $(u, a = ay, v)$ or $(u, y^2 = y^3, v)$, where $u, v$ are 1-paths in $K$, $u$ is irreducible, and $uvv$ (resp., $uyv^2v$) is homotopic to $a$ in $K$.

If $uvv$ is homotopic to $a$, then clearly $u$ is empty, which implies that $(u, a = ay, v)$ is in the left forest. Hence $H$ is generated by the edges of the form $(u, y^2 = y^3, v)$ only. If $(u, y^2 = y^3, v)$ is one of our generators, then $uy^2v$ must be homotopic to $a$. Then $u = ay^k, v = y^l$ for some $k, l \geq 0$. Since $u$ must be an irreducible 1-path, we have $k = 0$. So this generator has the form $(a, y^2 = y^3, y'')$. We denote it by $x_i$. By Theorem 6.5 the defining relations of $H$ are the following

$$(a, y^2 = y^3, vy^3w) = (a, y^2 = y^3, vy^2w)^{ay^v, y^2v = y^3, w}.$$ 

Note that $ay^2v = a$. Let $i = |w|, j = |vy^2w| = i + 2 + |v|$. Then our defining relation has the form $x_{j+1} = x_j^{x_i}$, where $j \geq i + 2$. This leads to (13).

To describe a finite presentation of $H$, we use Theorem 6.6. Our set of generators now consists of the elements $x_j = (a, y^2 = y^3, y')$, where $j = 0, 1, 2$ since the word $y'$ is irreducible.

The defining relations have the form

$$[a, y^2 = y^3, py^3q] = [a, y^2 = y^3, py^2q]^{[a, y^2 = y^3, q]}$$

since $ay^2p = a$, where the words $p, q$ are irreducible. Also we have a restriction that $(a, y^2 = y^3, q)$ is not in the right forest. Hence $q$ is non-empty. Therefore $p = 1, y, y^2$ and $q = y, y^2$. Let $z_j = [a, y^2 = y^3, y']$. According to Definition 6.2, $z_j$ can be expressed as follows in terms of the generators: $z_0 = (a, y^2 = y^3, 1) = x_0, z_1 = x_1, z_2 = x_2, z_3 = [a, y^2 = y^3, y^3] = [a, y^2 = y^3, y^2)^{[a, y^2 = y^3, 1]} = x_2^{x_0}$ and analogously $z_4 = z_5 = z_6 = z_7 = z_8 = z_9 = z_0 = x_2^{x_0} = x_2^{x_0} = z_5 = x_2^{x_0}$, and so on. Thus we have 6 defining relations in terms of the $z_j$’s: $z_4 = z_5 = z_6 = z_7 = z_8 = z_9 = z_0 = x_2^{x_0} = z_5 = x_2^{x_0}$. If we now rewrite them in terms of the generators $x_0, x_1, x_2$, we obtain (14).

\section{7 Rooted 2-trees}

We shall need a special class of directed 2-complexes called \textit{rooted 2-trees}, which are 2-dimensional analogs of rooted trees. Let $w$ be a simple arc subdivided into subarcs (edges). One can regard $w$ as a directed 2-complex with no 2-cells. Let us denote it by $K_0$. Consider any ascending family of directed 2-complexes

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$$

obtained by the following inductive procedure. For every $n \geq 0$, in order to construct $K_{n+1}$, we add new 2-cells to $K_n$. For each positive 2-cell $f$ from $K_{n+1} \setminus K_n$, the top 1-path $[f]$ must belong to $K_n$, and the bottom 1-path $[f]$ must be a simple arc that meets $K_n$ at the endpoints only. We also assume that the arcs of the form $[f]$, for all positive 2-cells from $K_{n+1} \setminus K_n$, are disjoint except for their endpoints.

Then the union $K = \bigcup_{n \geq 0} K_n$ is a directed 2-complex called a \textit{rooted 2-tree} (with the root 1-path $w$).

We say that 1-paths $p, q$ of a directed 2-complex have the same endpoints if $\iota(p) = \iota(q)$, $\tau(p) = \tau(q)$.

Let us consider the following three conditions for a directed 2-complex $K$ and a distinguished 1-path $w$ in it:

\textbf{T1} For any vertex $o$ of $K$, there exist a 1-path from $\iota(w)$ to $\tau(w)$ containing $o$. 

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T2 Every two 1-paths in \( K \) with the same endpoints are homotopic in \( K \).

T3 The diagram group \( D(K, w) \) is trivial.

Lemma 7.1. Any rooted 2-tree \( K \) with the root \( w \) satisfies conditions T1 – T3.

Proof. Let \( K \) be constructed using the ascending chain (15). It is easy to see that \( K \) satisfies the conditions T1 – T3 provided each \( K_n \) \((n \geq 0)\) does.

We proceed by induction on \( n \). All three conditions are obvious for \( K_0 \). So for \( n \geq 0 \), assume that \( K_n \) satisfies conditions T1 – T3.

Let us check that \( K_{n+1} \) satisfies T1 – T3. If \( o \) is a vertex that belongs to \( K_{n+1} \setminus K_n \) then it belongs to the bottom path \([f]\) of some positive 2-cell \( f \). The endpoints of \([f]\) belong to \( K_n \) so there are 1-paths \( q', q'' \) in \( K_n \) from \( \iota(w) \) to \( \iota(f) \) and from \( \tau(f) \) to \( \tau(w) \), respectively. Hence the 1-path \( q'[f]q'' \) contains \( o \). Therefore, T1 holds for \( K_{n+1} \).

To check T2, let us consider 1-paths \( p, q \) in \( K_{n+1} \), where \( \iota(p) = \iota(q), \tau(p) = \tau(q) \). Let \( \rho_n(s) \) denote the number of edges in \( s \) that belong to \( K_{n+1} \setminus K_n \). We proceed by induction on \( \rho_n(p) + \rho_n(q) \). If this number is zero, then both 1-paths belong to \( K_n \) so they are homotopic. So we can assume that \( \rho_n(q) \neq 0 \). This means that \( q \) contains an edge \( e \) from \( K_{n+1} \setminus K_n \). The edge \( e \) is contained in the bottom 1-path \( r = [f] \) of a positive 2-cell \( f \notin K_n \) such that \( u = [f] \) belongs to \( K_n \). By definition, no edges of \( v \) belong to \( K_n \). Notice that by definition, the graph \( K_{n+1} \) is obtained from \( K_n \) by adding a union of simple arcs of the form \([g], g \in K_{n+1} \setminus K_n \) which are pairwise disjoint except for the endpoints. Therefore, every 1-path in \( K_{n+1} \) that connects two vertices in \( K_n \) and contains \( e \), must contain the whole 1-path \( v = [f] \). Hence \( q \) contains \( v \).

Thus we can represent \( q \) in the form \( q'vq'' \). Therefore, \( q \) is homotopic to the 1-path \( r = q'vq'' \). Since (by definition) \( u \) is contained in \( K_n \), we have \( \rho_n(r) < \rho_n(q) \). By the inductive assumption, the 1-paths \( p \) and \( r \) are homotopic in \( K_{n+1} \). So \( p \) and \( q \) are also homotopic, that is, T2 holds for \( K_{n+1} \).

It remains to check T3. Suppose that the group \( D(K_{n+1}, w) \) is not trivial. Then there exists a reduced nontrivial \((w, w)\)-diagram \( \Delta \) over \( K_{n+1} \). If all labels of the cells in \( \Delta \) belong to \( K_n \), then \( \Delta \) is a diagram over \( K_n \) and so it represents a nontrivial element of \( D(K_n, w) \), a contradiction. So let \( \pi \) be a cell in \( \Delta \) that belongs to \( K_{n+1} \) but not to \( K_n \). Without loss of generality, the label \( f \) of \( \pi \) is a positive 2-cell \( f \in K_{n+1} \). Since \([f]\) is a simple arc in \( K_{n+1} \), the path in \( \Delta \) labelled by \( v \) must be the common boundary of \( \pi \) and a cell of the form \([f] = [f]\). The only possible label of this cell is \( f^{-1} \). Thus \( \Delta \) has a dipole, a contradiction.

Now we are going to prove the converse to Lemma 7.1. This gives a characterization of rooted 2-trees.

Theorem 7.2. Let \( K \) be a directed 2-complex and let \( w \) be a 1-path in it. Then \( K \) satisfies the conditions T1 – T3 if and only if \( K \) is a rooted 2-tree with the root \( w \).

Proof. The “if” part is given by Lemma 7.1. To prove the “only if” part, let us define a sequence (15) of subcomplexes in \( K \). Notice that by T2, any 1-path in \( K \) is a simple arc because an empty 1-path cannot be homotopic to a nonempty 1-path. Hence we can set \( K_0 \) to be equal to the simple arc \( w \). If \( K_n \) is already defined for some \( n \geq 0 \), let \( K_{n+1} \) be the subcomplex formed by \( K_n \) and all the 2-cells \( f \notin K_n \) such that \([f]\) is contained in \( K_n \). Conditions T1 and T2 guarantee that \( K \) is equal to the union of the \( K_n \), \( n \geq 0 \). By definition, it is enough to show that each of the directed 2-complexes \( K_n \) \((n \geq 0)\) is a rooted 2-tree with \( w \) as the root.
So we prove that $K_n$ is a rooted 2-tree (with $w$ as the root) by induction on $n \geq 0$. This is obvious if $n = 0$. So we assume that $K_n$ is a rooted 2-tree and prove the same for $K_{n+1}$.

In fact a stronger claim is true: for every subcomplex $M$ of $K$, which is a rooted 2-tree with root $w$, and any positive 2-cell $f \in K \setminus M$ with $[f]$ in $M$, no internal points of $[f]$ can belong to $M$. (Clearly, if this claim is true, then the directed 2-complex $M'$ obtained from $M$ by adding $f$, is again a rooted 2-tree with root $w$. Since $K_{n+1}$ can be obtained from $K_n$ by adding 2-cells one by one, the claim implies that $K_{n+1}$ is a rooted 2-tree.)

By contradiction, let $f$ be a 2-cell in $K \setminus M$, where $[f]$ is contained in $M$. Suppose that there exists an internal vertex $o$ of $[f]$ that belongs to $M$. Since (by Lemma 7.1) $M$ satisfies T1, we can find a 1-path $p$ in $M$ from $o(w)$ to $o(w)$ that is subdivided by $o$ into a product of two factors, $p = p'p''$. Since the endpoints $o(f)$, $o(f)$ are also in $M$, there exist 1-paths $q'$ (respectively, $q''$) in $M$ from $o(f)$ to $o(f)$ (from $o(f)$ to $o(f)$). The 1-path $v = [f]$ is subdivided by $o$ into a product of the form $v = v'v''$. Now we have two 1-paths $p'$ and $q'v'$ that go from $o(w)$ to $o$. By T2, they are homotopic in $K$. Let $\Delta'$ be a $(q'v', p)$-diagram over $K$. Analogously, there exists a $(v''q'', p'')$-diagram $\Delta''$ over $K$. Now we can form a $(w, w)$-diagram $\Delta$ over $K$ as follows. The 1-paths $p'p''$ and $q'uv''$ belong to $M$ and have the same endpoints as $w$. Since $M$ satisfies T2, there exists a $(w, q'uv'')$-diagram $\Delta_1$ and a $(p'p'', w)$-diagram $\Delta_2$. Both of them are diagrams over $M$. Now let

$$\Delta = \Delta_1 \circ (\varepsilon(q') + \pi + \varepsilon(q'')) \circ (\Delta' + \Delta'') \circ \Delta_2,$$

where $\pi$ is a cell of $\Delta$ labelled by $f$.

Suppose that $\Delta$ has a cell labelled by $f^{-1}$. This cell cannot belong to $\Delta_1$ or $\Delta_2$ because they are diagrams over $M$. Then this cell is in $\Delta'$ or in $\Delta''$. These cases are symmetric so let $\pi'$ be a cell of $\Delta'$ labelled by $f^{-1}$. There is a (directed) path in $\Delta$ from $\tau(\pi')$ to $\tau(\Delta')$. Let us extend it by the path labelled by $v''$ from $\tau(\Delta')$ to $\tau(\pi)$. If we project the resulting path into $K$, then we get a loop at $\tau(f)$. This contradicts T2. Thus $\Delta$ contains no cells labelled by $f^{-1}$. Therefore, after reducing all the dipoles in $\Delta$, we get a nontrivial reduced $(w, w)$-diagram over $K$ which contradicts T3. 

Given a rooted 2-tree $K$, the ascending sequence $K_n$ ($n \geq 0$) of rooted 2-trees defined in the proof of Theorem 7.2, will be called the natural filtration of $K$. Notice that for each 2-cell $f$ of $K$, there exists a unique $n \geq 1$ such that $f$ or its inverse has the form $u = v$, where $u$ belongs to $K_n$ and $v$ is a simple arc in $K_{n+1} \setminus K_n$. This allows us to choose the natural orientation on the set of 2-cells of $K$: we call $f$ positive if $[f]$ is contained in $K_n$ but $[f]$ is not contained in $K_n$ (for some $n \geq 0$).

8 Universal 2-covers of directed 2-complexes

Let $K$ be a directed 2-complex. Consider any directed 2-complex $M$ and a morphism $\phi: M \to K$ that sends edges to edges and 2-cells to 2-cells. For every edge or 2-cell, its image under $\phi$ will be called its label. In this situation, we shall call $M$ a directed 2-complex over $K$. The morphism $\phi$ will be sometimes called a labelling map. For every 1-path $p$ in $M$, the set of all atomic 2-paths in $M$ with top $p$ will be called the 2-star of $p$. Clearly, for any 1-path $p$ in $M$, we have an induced map from the 2-star of $p$ into the 2-star of $\phi(p)$. This induced map will be called the local map of $M$ at $p$.

Let $p$ be any 1-path in $K$. A universal 2-cover of $K$ with base $p$ is a directed 2-complex $M$ over $K$ with a labelling map $\phi$ which satisfy the following properties:
U1 \( \mathcal{M} \) is a rooted 2-tree with root \( \tilde{p} \), where \( \phi(\tilde{p}) = p \);

U2 for any 1-path \( \tilde{q} \) in \( \mathcal{M} \) from \( \iota(\tilde{p}) \) to \( \tau(\tilde{p}) \), the local map of \( \mathcal{M} \) at \( \tilde{q} \) is bijective.

The following theorem shows the existence and uniqueness of universal 2-covers. We say that two directed 2-complexes over \( \mathcal{K} \) are isomorphic whenever there exists an isomorphism between these complexes that preserves all labels.

**Theorem 8.1.** For every directed 2-complex \( \mathcal{K} \) and every 1-path \( p \) in \( \mathcal{K} \), there exists a universal 2-cover of \( \mathcal{K} \) with base \( p \). Every two universal 2-covers of \( \mathcal{K} \) with base \( p \) are isomorphic.

**Proof.** Let us construct a directed 2-complex \( \mathcal{M} \) over \( \mathcal{K} \) defined as a rooted 2-tree with natural filtration \( \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots \), and the labelling map \( \phi \). Both \( \mathcal{M}_n \) and \( \phi \) are defined by induction on \( n \).

By definition, \( \mathcal{M}_0 \) is a simple arc \( \tilde{p} \) subdivided into subarcs (edges) and labelled by \( p \). The labelling of \( \tilde{p} \) gives the restriction of \( \phi \) onto \( \mathcal{M}_0 \).

Suppose that a rooted 2-tree \( \mathcal{M}_n \) with root \( \tilde{p} \) and the restriction \( \phi \) onto \( \mathcal{M}_n \) are already defined for some \( n \geq 0 \).

Let us consider all pairs of the form \((\tilde{r}, f)\), where \( \tilde{r} \) is a 1-path in \( \mathcal{M}_n \) and \( f \) is a 2-cell of \( \mathcal{K} \) such that \( [f] = \phi(\tilde{r}) \). Suppose that \( \mathcal{M}_n \) does not have a 2-cell labelled by \( f \) whose top path is \( \tilde{r} \). Then we add a new 2-cell \( \tilde{f} \) to \( \mathcal{M}_n \) as follows. First we add a simple arc \( \tilde{s} \) with the same endpoints as \( \tilde{r} \), subdivide it into subarcs and label it by \([f]\). Then we add a 2-cell \( \tilde{f} \) of the form \( \tilde{r} = \tilde{s} \), labelled by \( f \). (Note that \( \tilde{f} \) depends on the pair \((\tilde{r}, f)\).) Applying this operation to all possible pairs \((\tilde{r}, f)\) as above, we obtain \( \mathcal{M}_{n+1} \) and the restriction of the labelling function \( \phi \) on \( \mathcal{M}_{n+1} \). We assume that all the new arcs we add are disjoint from each other and from \( \mathcal{M}_n \), except for their endpoints. Clearly then, \( \mathcal{M}_{n+1} \) is a rooted 2-tree with root \( \tilde{p} \).

By definition, \( \mathcal{M} = \bigcup_{n \geq 0} \mathcal{M}_n \) is also a rooted 2-tree with root \( \tilde{p} \) so it satisfies U1. We claim that \( \mathcal{M} \) satisfies the following property

U2' for any 1-path \( \tilde{r} \) in \( \mathcal{M} \) and for any 2-cell \( f \) of \( \mathcal{K} \) such that \([f] = \phi(\tilde{r})\), there is exactly one 2-cell \( \tilde{f} \) of \( \mathcal{M} \) labelled by \( f \) with top path \( \tilde{r} \).

Indeed, let \( n \geq 0 \) be the smallest integer such that \( \tilde{r} \) is contained in \( \mathcal{M}_n \). If there is no 2-cell \( \tilde{f} \) labelled by \( f \) with top path \( \tilde{r} \) in \( \mathcal{M}_n \), then such a 2-cell will appear in \( \mathcal{M}_{n+1} \) by definition.

Let us prove by induction that for any \( n \geq 0 \), there is at most one cell \( \tilde{f} \) in \( \mathcal{M}_n \) labelled by \( f \) with the given top 1-path \( \tilde{r} \). If \( n = 0 \), then \( \mathcal{M}_0 \) has no 2-cells at all. Suppose that our assumption is true for \( \mathcal{M}_n \). By the definition of \( \mathcal{M}_{n+1} \), if \( \mathcal{M}_n \) has a 2-cell labelled by \( f \) with top \( \tilde{r} \), then no 2-cell with the same properties appears in \( \mathcal{M}_{n+1} \). Otherwise, only one such a 2-cell may appear. Hence our claim is true for \( \mathcal{M}_{n+1} \).

Now let us take any 1-path \( \tilde{q} \) in \( \mathcal{M} \) with the same endpoints as \( \tilde{p} \). We prove that the local map of \( \mathcal{M} \) at \( \tilde{q} \) is surjective. Suppose that \( \delta \) belongs to the 2-star of \( q = \phi(\tilde{q}) \) in \( \mathcal{K} \). Then \( \delta = (u, f, v) \) for some 1-paths \( u, v \) in \( \mathcal{K} \) and some 2-cell \( f \). We also have \( q = [\delta] = u [f] v \). The 1-path \( \tilde{q} \) can be decomposed as \( \tilde{q} = \tilde{u} \tilde{r} \tilde{v} \), where \( \phi(\tilde{u}) = u, \phi(\tilde{r}) = [f], \phi(\tilde{v}) = v \). Since property U2' holds for \( \mathcal{M} \), the complex \( \mathcal{M} \) has a 2-cell \( \tilde{f} \) labelled by \( f \) with top \( \tilde{r} \). So the atomic 2-path \( \delta = (\tilde{u}, \tilde{f}, \tilde{v}) \) belongs to the 2-star of \( \tilde{q} \) and maps onto \( \delta \) under \( \phi \).

Now we need to prove that the local map of \( \mathcal{M} \) at \( \tilde{q} \) is injective. Assume the contrary. Then there are two different atomic 2-paths in \( \mathcal{M} \) with the same top \( \tilde{q} \) and the same image \( \delta = (u, f, v) \) in \( \mathcal{K} \). In this case the 2-cells of these atomic 2-paths are also different. But they have the same top \( \tilde{r} \) and the same label \( f \), which contradicts U2'.
We proved that the local map of $\mathcal{M}$ at any 1-path from $\iota(\bar{p})$ to $\tau(\bar{p})$ is bijective, that is, property U2 holds. So $\mathcal{M}$ is a universal 2-cover of $\mathcal{K}$ with base $p$.

Now let us prove the uniqueness. Suppose that $\mathcal{M}'$ is a universal 2-cover of $\mathcal{K}$ with base $p$. We will show that $\mathcal{M}'$ and $\mathcal{M}$ are isomorphic as directed 2-complexes over $\mathcal{K}$.

First of all, let us show that $\mathcal{M}'$ also satisfies U2'. If $\tilde{r}$ is a 1-path in $\mathcal{M}'$ such that $[\tilde{f}] = \phi(\tilde{r})$ for some 2-cell $\tilde{f}$ in $\mathcal{K}$, then we can include $\tilde{r}$ into a 1-path $\tilde{q} = \tilde{u}r\tilde{v}$ in $\mathcal{M}$ from its $\iota(\bar{p})$ to $\tau(\bar{p})$. The 1-path $q = \phi(\tilde{q})$ has the form $urv$, where $r = \phi(\tilde{r}) = [f]$. So we have an atomic 2-path $\delta = (u, f, v)$ in $\mathcal{K}$ with $[\delta] = q$. Since $\mathcal{M}'$ satisfies U2, the local map of $\mathcal{M}'$ at $\tilde{q}$ is surjective and so $\delta$ has a preimage $\tilde{\delta}$ with top $\tilde{q}$. The 2-cell of $\tilde{\delta}$ is labelled by $f$ and its top is $\tilde{r}$. If there were two 2-cells in $\mathcal{M}'$ with the same top $\tilde{r}$ and the same label $f$, then we would have a contradiction with the injectivity of that local map.

Let us take the natural filtration of $\mathcal{M}'$, that is, the ascending union $\mathcal{M}'_0 \subseteq \mathcal{M}'_1 \subseteq \cdots$ defined in Section 7. Recall that $\mathcal{M}'_0$ is the root arc of $\mathcal{M}'$ and for any $n \geq 0$, $\mathcal{M}'_{n+1}$ is the subcomplex in $\mathcal{M}'$ formed by $\mathcal{M}'_n$ and all the 2-cells $f \notin \mathcal{M}'_n$ such that $[f]$ is contained in $\mathcal{M}'_n$. All the subcomplexes $\mathcal{M}'_n$ are rooted 2-trees with the same root. It suffices to show that $\mathcal{M}_n$ and $\mathcal{M}'_n$ are isomorphic as directed 2-complexes over $\mathcal{K}$ for any $n \geq 0$. We prove this fact by induction.

If $n = 0$, then both $\mathcal{M}_0$ and $\mathcal{M}'_0$ are simple arcs labelled by $p$. Assume that $\mathcal{M}_n$ and $\mathcal{M}'_n$ are isomorphic for some $n \geq 0$ (as complexes over $\mathcal{K}$). For simplicity, we can identify $\mathcal{M}_n$ and $\mathcal{M}'_n$. We are going to establish a natural bijection between the 2-cells of $\mathcal{M}_{n+1} \setminus \mathcal{M}_n$ and the 2-cells of $\mathcal{M}'_{n+1} \setminus \mathcal{M}'_n$. Each pair of the 2-cells in the bijection, will have the same label and the same top contained in $\mathcal{M}_n = \mathcal{M}'_n$. This will clearly induce the desired isomorphism between $\mathcal{M}_{n+1}$ and $\mathcal{M}'_{n+1}$.

Let $\tilde{f}$ be a 2-cell of $\mathcal{M}_{n+1} \setminus \mathcal{M}_n$, where $\tilde{r} = [\tilde{f}]$ is contained in $\mathcal{M}_n$ and $\phi(\tilde{f}) = f$. By U2', there is a 2-cell $\tilde{f}'$ in $\mathcal{M}'$ labelled by $f$ with the top $\tilde{r}$. This cell cannot belong to $\mathcal{M}'_n$ since otherwise it also belongs to $\mathcal{M}_n$ and so $\mathcal{M}$ will have two different 2-cells with the same label and the same top. Thus it belongs to $\mathcal{M}_{n+1}$ by definition of the natural filtration. In fact the map $\tilde{f} \mapsto \tilde{f}'$ is the desired bijection. Indeed if $\tilde{f}'$ is a 2-cell of $\mathcal{M}'_{n+1} \setminus \mathcal{M}'_n$, then $\mathcal{M}$ must contain a 2-cell $\tilde{f}$ labelled by $f$ with top $\tilde{r}$. By the same arguments, $\tilde{f}$ does not belong to $\mathcal{M}_n = \mathcal{M}'_n$. So it belongs to $\mathcal{M}_{n+1}$ by definition.

**Remark 8.2.** The proof of Theorem 8.1 also shows that a directed 2-complex $\mathcal{M}$ over $\mathcal{K}$ with a labelling map $\phi$ is a universal 2-cover of $\mathcal{K}$ if and only if it satisfies U1 and U2' (thus U2 can be replaced by U2'). The condition U2' is sometimes easier to verify than U2.

The universal 2-cover of $\mathcal{K}$ with base $p$ will be denoted by $\tilde{\mathcal{K}}_p$. There are two useful orientations on $\tilde{\mathcal{K}}_p$. First, we have the natural orientation on $\tilde{\mathcal{K}}_p$ because it is a rooted 2-tree. The set of its positive 2-cells will be denoted by $\tilde{\mathcal{F}}^+$. Second, given an orientation on the set of 2-cells of $\mathcal{K}$, we have an *induced orientation* on the set of 2-cells of $\tilde{\mathcal{K}}_p$. Namely, a 2-cell of $\tilde{\mathcal{K}}_p$ is positive in that orientation whenever its image under $\phi$ is a positive 2-cell of $\mathcal{K}$.

In order to illustrate the process of constructing $\tilde{\mathcal{K}}_p$, let us take $\mathcal{K}$ to be the Dunce hat $\mathcal{H}_0 = \langle x \mid x^2 = x \rangle$ and let $p = x$. Then the natural filtration $\mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \cdots$ of $\tilde{\mathcal{K}}_p$ is constructed as follows. The complex $\mathcal{K}_0$ is just an edge $e$ labelled by $x$. The complex $\mathcal{K}_1$ consists of three vertices, three edges, and one positive 2-cell (in the natural orientation) of the form $e = e_1e_2$ labelled by the cell $x = x^2$ (all edges of $\tilde{\mathcal{K}}_p$ are labelled by $x$). Now we get a new path $\tilde{q} = e_1e_2$ connecting $\iota(e)$ and $\tau(e)$. Its image $q$ in $\mathcal{H}_0$ is the 1-path $x^2$. There are three atomic 2-paths in $\mathcal{H}_0$ with top 1-path $q$: $(1, x^2 = x, 1)$, $(1, x = x^2, x)$ and $(x, x = x^2, 1)$. The first of them has a preimage in $\tilde{\mathcal{K}}_1$, the other two do not have preimages. Thus we need
Lemma 8.3. For every 2-path $\delta$ in $K$ with $|\delta| = p$, there exists a unique 2-path $\tilde{\delta}$ in the universal 2-cover $\tilde{K}_p$ such that $|\tilde{\delta}| = \tilde{p}$ and $\phi(\tilde{\delta}) = \delta$.

For any $(p, q)$-diagram $\Delta$ over $K$, where $q$ is a 1-path in $K$, there exists a unique 1-path $\tilde{q}$ in $\tilde{K}_p$ and a unique $(\tilde{p}, \tilde{q})$-diagram $\tilde{\Delta}$ over $\tilde{K}_p$ such that the morphism $\phi$ maps $\tilde{\Delta}$ onto $\Delta$.

Proof. The first part of the lemma immediately follows from condition U2.

Let us consider any $(p, q)$-diagram $\Delta$ over $K$. It corresponds to some 2-path $\delta$ in $K$ with the top 1-path $p$. We can lift $\delta$ to a 2-path $\tilde{\delta}$ in $\tilde{K}_p$. The diagram $\tilde{\Delta}$ of this 2-path is a preimage of $\Delta$ under $\phi$. Clearly, any $(\tilde{p}, \tilde{q})$-diagram $\tilde{\Delta}'$ over $\tilde{K}_p$ such that $\phi(\tilde{\Delta}') = \Delta$ corresponds to a lift $\tilde{\delta}'$ of $\delta$.

All 2-paths assigned to the same diagram $\Delta$ are isotopic (see Section 3). Notice that the local maps preserve the property of a pair of atomic 2-paths to be independent. Therefore, by Lemma 2.2 isotopic 2-paths in $K$ lift to isotopic 2-paths of $\tilde{K}_p$. Hence the diagram $\tilde{\Delta}$ with the desired properties is unique. \hfill $\square$

We shall say that the diagram $\tilde{\Delta}$ from the statement of Lemma 8.3 is a lift of $\Delta$.

The following theorem shows a connection between the universal cover of the Squier complex $\text{Sq}(K)$ and the universal 2-cover of $K$.

We are going to use Farley’s [14] description of the universal cover $\tilde{\text{Sq}}(K)$ of $\text{Sq}(K)$. Farley proved, in the case of semigroup presentations, that $\tilde{\text{Sq}}(K)$ can be described in just the same way as $\text{Sq}(K)$: the vertices of $\tilde{\text{Sq}}(K)$ are arbitrary diagrams over $K$, and for arbitrary $n \geq 1$ the $n$-cubes are pairs where the first component is an arbitrary diagram $\Delta$ over $K$, and the second component is an arbitrary thin diagram $\Theta$ over $K$ with $n$ cells such that $\Delta \circ \Theta$ exists and has no dipoles. The face maps are defined in the natural way. The restriction of the covering map $\tilde{\text{Sq}}(K) \to \text{Sq}(K)$ to the vertices is $\lfloor \cdot \rfloor$. Any realization of the cubical complex $\tilde{\text{Sq}}(K)$ will be also denoted by $\text{Sq}(K)$. Farley’s proof carries without any change to the case of arbitrary directed 2-complexes.

Theorem 8.4. The connected component $\text{Sq}(\tilde{K}_p, \tilde{p})$ of the Squier complex of the universal 2-cover $\tilde{K}_p$ is homeomorphic to the universal cover of the connected component $\text{Sq}(K, p)$ of the Squier complex of $K$.

Proof. We are using the fact that the space of positive paths in $\tilde{K}_p$ is a realization of the Squier complex $\text{Sq}(\tilde{K}_p)$ (Lemma 3.7). Consider the connected component of $\Omega_+(\tilde{K}_p)$ that contains
\[ q = u_0 f_1^{(1)} u_1 \cdots f_m^{(m)} u_m, \]  
(16)

where \( m \geq 0 \), \( u_0, \ldots, u_m \) are 1-paths, \( f^{(1)}, \ldots, f^{(m)} \) are 2-cells in \( \bar{F} \) and the parameters \( t_1, \ldots, t_m \) belong to the open interval \((0,1)\). It is clear that these and only these paths form the connected component of \( \tilde{p} \) in \( \Omega_+ (\tilde{K}_p) \). Let \( \tilde{r} = u_0 \left[ f^{(1)} \right] u_1 \cdots \left[ f^{(m)} \right] u_m \). We assign to \( \tilde{q} \) an ordered pair \((\Delta, \Psi)\), where \( \Delta \) is the reduced \((\tilde{p}, \tilde{r})\)-diagram over \( \tilde{K}_p \) (this diagram exists by the property T2 of the rooted 2-tree \( \tilde{K}_p \) and is unique by property T3). By \( \Psi \) we denote the thin diagram \( \varepsilon(u_0) + f^{(1)} + \varepsilon(u_1) + \cdots + f^{(m)} + \varepsilon(u_m) \). If we apply the morphism \( \phi \) to \( \Delta \) and \( \tilde{\Psi} \), we get a pair \((\Delta, \Psi)\), where \( \Delta \) is a \((p,r)\)-diagram over \( K \) and \( \Psi \) is a thin diagram over \( K \). It is obvious that the concatenation \( \Delta \circ \Psi \) exists, moreover, it follows from the construction of \( \tilde{K}_p \) that it has no dipoles. So the pair \((\Delta, \Psi)\) defines an \( m \)-dimensional cube in \( \tilde{S} q(K,p) \) according to [14]. Take the point with coordinates \((t_1, \ldots, t_m)\) in this \( m \)-cube and assign it to the element \( \tilde{q} \). It is not hard to show that this defines a homeomorphism between the connected component of the path \( \tilde{p} \) in \( \Omega_+ (\tilde{K}_p) \) and the space \( \tilde{S} q(K,p) \) from Farley [14].

The following important theorem by Farley [14] also can be translated into the language of directed 2-complexes without difficulty.

**Theorem 8.5.** ([14]) The universal cover \( \tilde{S} q(K,p) \) of \( S q(K,p) \) is contractible. Hence \( S q(K,p) \) is a \( K(G,1) \) CW complex for the diagram group \( G = D(K,p) \), for every directed 2-complex \( K \) and every 1-path \( p \) in \( K \).

**Remark 8.6.** The proof of contractibility of \( \tilde{S} q(K,p) \) given in [14] is based on some classical topological facts such as Whitehead’s theorem. We give a direct geometric proof using Theorem 8.4 here.

We can represent the space \( \Omega_+ (\tilde{K}_p) \) as the ascending union \( \Omega_0 \subseteq \Omega_1 \subseteq \cdots \), where \( \Omega_n \) \((n \geq 0)\) is formed by those positive paths from the connected component of \( \tilde{p} \) which belong to \( \mathcal{K}_n \). Since \( \Omega_0 \) consists of the single point \( \tilde{p} \), it suffices to show that \( \Omega_n \) is a deformation retract of \( \Omega_{n+1} \) for all \( n \geq 0 \). To show this, we take any element in \( \Omega_{n+1} \). This is a positive path in \( \tilde{K}_p \) from \( i \) to \( \tau \). It can be uniquely represented in the form \( h = v_0 g_{s_1}^{(1)} v_1 \cdots g_{s_k}^{(k)} v_k \), where \( k \geq 0 \), \( v_0, \ldots, v_k \) are positive paths in \( \mathcal{K}_n \), the 2-cells \( g^{(1)}, \ldots, g^{(k)} \) belong to \( \mathcal{K}_{n+1} \ \backslash \ \mathcal{K}_n \) and \( s_1, \ldots, s_k \in (0,1] \). Here we also assume that all the 2-cells \( g^{(i)} \) \((1 \leq i \leq k)\) are in \( \bar{F}^+ \), that is, they have their top paths in \( \mathcal{K}_n \) and the bottom paths in \( \mathcal{K}_{n+1} \ \backslash \ \mathcal{K}_n \). For any \( s \in [0,1] \), we define the positive path

\[ h_s = v_0 g_{s_1}^{(1)} v_1 \cdots g_{s_k}^{(k)} v_k. \]  
(17)

It is obvious that \( h_0 \) belongs to \( \mathcal{K}_n \) and \( h_1 = h \). It is also clear that \( h_s = h \) for all \( s \in [0,1] \) provided \( h \) belongs to \( \mathcal{K}_n \). The map \( (h, s) \mapsto h_s \) from \( \Omega_{n+1} \times [0,1] \) to \( \Omega_{n+1} \) is clearly continuous. Thus \( \Omega_n \) is a deformation retract of \( \Omega_{n+1} \).

The fact that \( \tilde{K}_p \) is a covering space for \( S q(K,p) \) can be proved easily as well. The covering map can be constructed as follows. We extend \( \phi \) to the set of all positive paths in \( \tilde{K}_p \) by sending the paths of the form \( f_t \), where \( f \) is a 2-cell of \( \tilde{K}_p \) and \( t \in [0,1] \), to the positive path \( f_t \) in \( K \) \((f = \phi(f)) \) is the label of \( f \). Now \( \phi \) maps the connected component of \( \tilde{p} \) in \( \Omega_+ (\tilde{K}_p) \) onto \( S q(K,p) \). We leave it to the reader as an exercise to check that \( \phi \) defines the desired covering map. (In fact, this is essentially contained in Lemma 8.3 and its proof.)
Remark 8.7. Recall that Farley proved that every connected component $\tilde{\text{Sq}}(K)$ is a CAT(0) cubical complex provided, for example, $K$ is finite. Thus is this case the diagram groups of $K$ act discretely, cellurally, by isometries on CAT(0) cubical complexes.

Remark 8.8. There exists an alternative method of constructing the universal 2-cover $\tilde{K}_p$. Similarly to the Stallings foldings in graphs [32], let us define an elementary 2-folding in a complex $M$ over $K$ as follows. If two 2-cells $\pi_1$ and $\pi_2$ in $M$ have a common top path and the same label, then we identify their bottom paths and remove one of the cells. The result of this operation is again a directed 2-complex over $K$.

Consider all possible diagrams over $K$ with top 1-path labelled by $p$. Identify the top paths of all these diagrams. Clearly, we get a directed 2-complex $N$ over $K$. Now do all (possibly infinitely many) elementary 2-foldings in $N$. It can be shown that the resulting directed 2-complex over $K$ is $\tilde{K}_p$. We shall prove this in our next paper where foldings are used in a more general situation.

9 Homology

Theorem 8.5 shows that the components $\text{Sq}(K,w)$ are $K(G,1)$ spaces for diagram groups $G = D(K,w)$. In fact in most cases $\text{Sq}(K)$ is too large. Here we will use the technique of collapsing schemes [9, 8, 12] to find a “smaller” CW complex, which is homotopy equivalent to $\text{Sq}(K)$ (at least in the case when $K$ is complete).

We recall the concept of collapsing scheme from [8, 12, 9]. Let $X$ be a semi-cubical complex. We say that we have a collapsing scheme for $X$ if the following is true:

- there exists a subdivision of the set of all cubes of $X$ into three disjoint subsets: essential, collapsible, and redundant cubes;
- there exists a strict partial order $\succ$ on the set of all redundant $n$-cubes ($n \geq 0$) that satisfies the descending chain condition (that is, any sequence $c_1 \succ c_2 \succ \cdots$ terminates);
- there exists a bijection $c \mapsto \hat{c}$ between the set of all redundant $n$-cubes and the set of all collapsible $(n+1)$-cubes (for every $n \geq 0$);
- any redundant $n$-cube $c$ occurs exactly once among the $n$-faces of $\hat{c}$ and all the other redundant $n$-faces $c'$ of $\hat{c}$ precede $c$ in the order $\succ$ (that is, $c \succ c'$); the redundant $n$-cube $c$ is called the free face of the collapsible $(n+1)$-cube $\hat{c}$.

The next lemma is proved in almost the same way as [8, Proposition 1].

Lemma 9.1. Given a collapsing scheme for a semi-cubical complex $X$, one can construct a CW complex $Y$ which is homotopy equivalent to $X$ in such a way that the $n$-cells of $Y$ are in one-to-one correspondence with the essential $n$-cubes of $X$.

As in [8], for each $n \geq 0$ one has to do an infinite number of elementary steps, one for each collapsible $n$-cube. The free face of a collapsible cube is identified (homeomorphically) with the union of the other faces and the collapsible cube disappears. The space $Y$ is a end result of the process.

Now let $K$ be a complete directed 2-complex. Let $T_i$ be a left forest in $K$. Recall that for any 1-path $p$, the irreducible form of $p$ is denoted by $\tilde{p}$.
Let \( c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n) \) be an \( n \)-cube of \( X \) (a thin diagram with \( n \) cells). For any \( 0 \leq i \leq n \), we say that the term \( u_i \) in \( c \) is special provided it is not an irreducible 1-path, that is, \( \bar{u}_i \neq u_i \). For \( 1 \leq i \leq n \), we say that the term \( f_i \) in \( c \) is special provided \((u_{i-1}, f_i, \ast)\) is in \( T_1 \).

The \( n \)-cube \( c \) is called essential if it has no special terms. If \( c \) is not essential, then we find its leftmost special term. If this is one of the \( u_i \)'s, then we call \( c \) redundant. Otherwise we call \( c \) collapsible (in this case the special term is one of the \( f_i \)'s).

To describe the strict partial order \( \succ \), we need to introduce one technical concept. Let \( \Psi = \varepsilon(u_0) + g_1 + \cdots + g_k + \varepsilon(u_k) \) \((k \geq 0)\) be a thin diagram over \( \mathcal{K} \). Suppose that for some \( i, j \), where \( 0 \leq i \leq j \leq k \), we have \( u_j = u' \cdot [g] \cdot u'' \), for some 1-paths \( u', u'' \) and \( g \in \mathbb{F}^- \).

Then it is possible to move the cell \( g_i \) to the right replacing \( \Psi \) by a new thin diagram \( \Psi' = \varepsilon(u_0) + \cdots + \varepsilon(u_{i-1}) + \varepsilon([g_i]) + \cdots + \varepsilon(u') + g + \varepsilon(u'') + \cdots \) (we replace the term \( g_i \) by \( \varepsilon([g_i]) \) and the term \( \varepsilon(u_j) \) by \( \varepsilon(u') + g + \varepsilon(u'') \), thus we remove the cell \( g_i \) and insert a cell \( g \)). It is easy to see that this process of moving cells to the right always terminates. Indeed, \( \Psi' \) also has \( k \) cells and it can be represented in the form \( \Psi' = \varepsilon(u_0) + g_1' + \cdots + g_k + \varepsilon(u_k') \). If we compare the \((k + 1)\)-vectors \( b = (|u_k|, \ldots, |u_0|) \) and \( b' = (|u_k'|, \ldots, |u_0'|) \), then it follows from our description that \( b' \) strictly precedes \( b \) in the lexicographical order. Since these vectors have non-negative coordinates, the process of moving cells to the right must terminate.

Now we can define \( \succ \). Let \( c, c' \) be redundant \( n \)-cubes of \( X \). If \([c] \not\succ [c']\) then we set \( c \succ c' \). Otherwise, if \([c] = [c']\), then we set \( c \succ c' \) whenever \( c' \) can be obtained from \( c \) by a (non-zero) number of moving cells to the right. The fact that \( \mathcal{K} \) is Noetherian and the remark from the previous paragraph imply that \( \succ \) is a strict partial order satisfying the descending chain condition.

Let \( c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n) \) be a redundant \( n \)-cube. This means that \( u_i \) is not irreducible for some \( 0 \leq i \leq n \) whereas all the terms to the left of \( u_i \) are not special. Let us find the edge \((p, f, q)\) in \( T_1 \) assigned to \( u_i \). Thus \( u_i = p \cdot [f] \cdot q \). Note that \( p \) is irreducible by definition. Thus the \((n + 1)\)-cube \( \varepsilon(u_0) + f_1 + \cdots + f_i + \varepsilon(p) + f + \varepsilon(q) + \cdots \) is collapsible. We denote it by \( \hat{c} \) and check that \( c \) is the free face of \( \hat{c} \). We consider all \( n \)-faces \([\hat{c}]_j \) and \([\hat{c}]_i \) of \( c \), \( 1 \leq j \leq n + 1 \). Let \( j = i + 1 \). Then \([\hat{c}]_{j-1} = c \) and \( c' = [\hat{c}]_j \) is obtained from \( c \) by replacing \([f] \) by \([f] \). Since \( f \) participates in an edge from \( T_1 \), one has \([f] \not\succ [f] \). Hence \( c \succ c' \) whenever \( c' \) is redundant.

Now suppose that \( j > i + 1 \). Then \([\hat{c}]_{j-1} \) and \([\hat{c}]_{j-1} \) are collapsible \( n \)-cubes. Thus we may skip this case because we compare \( c \) with redundant cubes only.

Finally, take \( j \leq i \). We can only consider the case when \( c' = [\hat{c}]_j \) since \([f] \not\succ [f] \) or \([f] = [f] \). Suppose that \( c' \) is a redundant \( n \)-cube. To compare \( c \) and \( c' \), notice that their bottom paths are the same. The diagram \( c' \) is obtained by moving one cell of \( c \) to the right (the cell \( f_j \) has been deleted and the cell \( f \) has been added). Thus \( c \succ c' \).

It remains to check that we have a bijection between redundant \( n \)-cubes and collapsible \((n + 1)\)-cubes. We already assigned a collapsible \((n + 1)\)-cube \( \hat{c} \) to each redundant \( n \)-cube \( c \). If we start with a collapsible \((n + 1)\)-cube, then we can find its leftmost special term. This is some \( f \in \mathbb{F}^- \). The cube then has the form \( \cdots + \varepsilon(p) + f + \varepsilon(q) + \cdots \), where \((p, f, q)\) is in \( T_1 \). Replacing \( f \) by \([f] \) gives us a redundant cube \( c \). It follows directly from definitions that the cube \( \hat{c} \) assigned to \( c \) is exactly the collapsible \((n + 1)\)-cube we started with. This completes our proof that we have defined a collapsible scheme for \( X \).

Summarizing and taking into account Lemma 9.1, we get the following.

**Theorem 9.2.** Suppose that \( \mathcal{K} \) is a complete directed 2-complex and let \( G = \mathcal{D}(\mathcal{K}, w) \), where \( w \) is a non-empty 1-path in \( \mathcal{K} \). Then there exists a \( K(G, 1) \) CW complex \( Y_w \) whose \( n \)-dimensional cells
are in one-to-one correspondence with thin diagrams of the form \( c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n) \), where \( n \geq 0 \), the 1-paths \( u_i \) are irreducible for all \( 0 \leq i \leq n \), the edges \((u_{i-1}, f_i, *)\) are not in \( T_i \) for all \( 1 \leq i \leq n \), and \([c]\) is homotopic to \( w \) in \( \mathcal{K} \).

Note that each thin diagram \( c \) described in the statement of Theorem 9.2 is an essential cube in the Squier complex \( \text{Sq}(\mathcal{K}) \).

Recall that a directed 2-complex is called 2-path connected if all non-empty 1-paths in it are homotopic. Let us call a directed 2-complex almost 2-path connected if the number of classes of homotopic 1-paths is finite (that is, \( \text{Sq}(\mathcal{K}) \) has finitely many connected components). A complex of the form \( \mathcal{K}_p \), where \( p \) is a semigroup presentation, is almost 2-path connected if and only if the semigroup given by \( p \) is finite. Thus the following statement generalizes a result from [14] and strengthens [16, Theorem 10.7].

**Theorem 9.3.** Let \( \mathcal{K} \) be a finite almost 2-path connected 2-complex. Then all diagram groups of \( \mathcal{K} \) are of type \( F_\infty \).

**Proof.** First suppose that \( \mathcal{K} \) is complete. Let \( C \) denote the number of homotopy classes of 1-paths in \( \mathcal{K} \) (the empty 1-paths are included) and let \( N \) be the number of positive 2-cells of \( \mathcal{K} \). It is clear that the number of \( n \)-cells in the \( K(G, 1) \) space \( Y_w \) does not exceed \( C(CN)^n \). In particular, it is finite so \( G \) has type \( F_\infty \).

Now suppose that \( \mathcal{K} \) is not necessarily complete. By Lemma 6.1, \( \mathcal{K} \) is contained in a finite complete almost 2-path connected directed 2-complex \( \mathcal{K}' \), and the diagram groups of \( \mathcal{K} \) are retracts of the diagram groups of \( \mathcal{K}' \). It remains to recall that a retract of an \( F_\infty \) group is of type \( F_\infty \). \( \square \)

By [16, Theorem 10.3], if \( \mathcal{P} \) is a finite complete rewrite system such that all diagram groups over it are finitely generated, then all of them are finitely presented. Now we can deduce a much stronger result. In fact we can even eliminate the assumption that the presentation is finite.

**Theorem 9.4.** Let \( \mathcal{K} \) be a complete directed 2-complex. Suppose that all diagram groups of \( \mathcal{K} \) are finitely generated. Then all of them are of type \( F_\infty \).

**Proof.** By Theorem 9.2 it is enough to prove that for any \( n \geq 1 \), each connected component of \( \text{Sq}(\mathcal{K}) \) has only finitely many essential \( n \)-cubes. We proceed by induction on \( n \). Let \( n = 1 \). By definition, the set of essential cubes of dimension 1 is in one-to-one correspondence with the generating set of the corresponding diagram group described in Theorem 6.6. Since this set is minimal by Remark 6.7, it is finite.

Now let \( n > 1 \). For any essential 1-cube \( \varepsilon(p) + f + \varepsilon(q) \) of a connected component \( \text{Sq}(\mathcal{K}, w) \) of \( \text{Sq}(\mathcal{K}) \), let us consider the set of all essential \((n-1)\)-cubes in \( \text{Sq}(\mathcal{K}, q) \). By the inductive assumption, it is finite. It remains to note that any essential \( n \)-cube \( c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n) \) of \( \text{Sq}(\mathcal{K}, w) \) is determined uniquely by an essential 1-cube \( \varepsilon(u_0) + f_1 + \varepsilon(q) \) of \( \text{Sq}(\mathcal{K}, w) \) and an essential \((n-1)\)-cube \( \varepsilon(u_1) + f_2 + \cdots + f_n + \varepsilon(u_n) \) from \( \text{Sq}(\mathcal{K}, q) \). \( \square \)

**Remark 9.5.** Note that one can extract a stronger fact from the proof of Theorem 9.4. Suppose that \( \mathcal{K} \) is a complete directed 2-complex. If some diagram group \( \mathcal{D}(\mathcal{K}, w) \) is not of type \( F_\infty \), then there are two 1-paths \( w'w_1 \) and \( w_2w'' \) homotopic to \( w \) such that the diagram groups \( \mathcal{D}(\mathcal{K}, w') \) and \( \mathcal{D}(\mathcal{K}, w'') \) are not finitely generated.

Now we are going to prove that for any complete directed 2-complex \( \mathcal{K} \), the complex \( Y_w \) described in the statement of Theorem 9.2 is in fact “minimal”. Namely, for every \( n \geq 0 \), we shall compute the integer \( n \)th homology group of every diagram group \( \mathcal{D}(\mathcal{K}, w) \) of \( \mathcal{K} \) and show
that it is a free Abelian group whose rank coincides with the number of \( n \)-cells in \( Y_w \) (that is, the number of essential \( n \)-cubes in \( \text{Sq}(K, w) \)).

Let \( G = D(K, w) \). Since the homology groups of a group \( G \) coincide with the homology groups of any \( K(G, 1) \) CW complex, let us consider the complex \( X = \text{Sq}(K, w) \) (it is a \( K(G, 1) \) by Theorem 8.5). As usual, let \( T_l \) be a left forest in \( X \).

Denote by \( P_n \) the free Abelian group with the set of \( n \)-cubes of \( X \) as a free basis. The boundary maps \( \partial_n : P_n \to P_{n-1} \) \((n \geq 1)\) are given by the formulas of Serre [30, p. 440] (see also [9]):

\[
\partial_n(c) = \sum_{i=1}^{n}(-1)^i([c]_i - [c]_{i+1}),
\]

where \( c \) is an \( n \)-cube. Since the maps (18) form a chain complex [30], the \( n \)th integer homology group \( H_n(G; \mathbb{Z}) \) coincides with the \( n \)th homology group of that chain complex (that is, \( \text{Ker} \partial_n / \text{Im} \partial_{n+1} \)).

As in [8, 12], we define an endomorphism \( \phi \) of the chain complex \( P = (P_n, \partial_n) \). This endomorphism maps every \( P_n \) into the subgroup \( Q_n \) of \( P_n \) freely generated by the essential \( n \)-cubes. Let \( c \) be an \( n \)-cube (a generator of \( P_n \)). If \( c \) is collapsible then we set \( \phi(c) = 0 \). If \( c \) is essential then we set \( \phi(c) = c \). Finally suppose that \( c \) is redundant. In that case we proceed by the Noetherian induction on \( \succ \).

Since \( c \) is redundant, there exists a collapsible \((n+1)\)-cube \( \hat{c} \) such that \( c \) is the free face of \( \hat{c} \). Then \( \partial_{n+1}(\hat{c}) = \pm c + \Sigma \) for some linear combination \( \Sigma \) of cubes that are either essential or collapsible but smaller than \( c \) with respect to \( \succ \). Thus we can assume that \( \phi(c') \) has been defined already for all \( c' \) occurring in \( \Sigma \). So we can set \( \phi(c) = \mp \phi(\Sigma) \).

It is shown in [12] (see also [8, p. 150]), that \( \phi \) indeed is an endomorphism of the chain complex (that is, it commutes with the boundary maps), and that the chain complex \( Q \) formed by the groups \( Q_n \) and boundary maps \( \partial_n = \phi \partial_n \) is chain-equivalent to the initial chain complex. Thus the homology groups of \( Q \) coincide with the homology groups of \( P \).

We are going to prove that \( \partial_n \) is a zero map. For that we need the following statement.

**Lemma 9.6.** For any \( n \)-cube \( c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n) \) of \( \text{Sq}(K, w) \), we let \( \bar{c} = \varepsilon(\bar{u}_0) + f_1 + \cdots + f_n + \varepsilon(\bar{u}_n) \). Then \( \phi(c) = \bar{c} \) if \( \bar{c} \) is essential and \( \phi(c) = 0 \) if \( \bar{c} \) is collapsible (note that \( \bar{c} \) cannot be redundant by definition).

**Proof.** Note that if \( c \) is essential then \( c = \bar{c} \) and \( \phi(c) = c \) as required.

Suppose that \( c \) is collapsible. Then \( \phi(c) = 0 \) by definition. Thus we only need to check that \( \bar{c} \) is also collapsible. Since \( c \) is collapsible, the edge \((u_{i-1}, f_i, *)\) is in the left forest \( T_l \) for some \( i \leq n \) and all the \( u_j \)'s are irreducible for \( 0 \leq j < i \). Then \( \bar{c} \) possesses similar properties whence \( \bar{c} \) is collapsible.

Now suppose that \( c \) is redundant. Take the smallest number \( i \leq n \) such that \( u_i \) is not irreducible, and consider the edge \((p, f, q)\) from \( T_l \) assigned to \( u_i \). This \( i \) will be called the index of \( c \). Consider also the collapsible \((n+1)\)-cube \( \hat{c} = \varepsilon(u_0) + \cdots + f_i + \varepsilon(p) + f + \varepsilon(q) + \cdots \) whose free cell is \( c \). Since \( K \) is Noetherian, we can assume without loss of generality that the statement of the lemma does not hold for \( c \) but holds for all \( n \)-cubes \( c' \) such that \([c] \xrightarrow{\partial} [c']\). We can also assume that among all such counterexamples, \( c \) has the smallest index \( i \).

Notice that for every \( j \), \( 1 \leq j \leq n+1 \),

\[
|\hat{c}|_j = [\hat{c}]_{j'},
\]

(19)
Also notice that \( c = [\hat{c}]_{i+1} \) and \([c] \xrightarrow{\hat{c}} [c']\), where \( c' = [\hat{c}]_{i+1} \). By (19), \( \bar{c} = \bar{c}' \). By the
definition of \( \phi \), we have \( \phi(c) = \phi(c') + \phi(\Sigma) \), where \( \Sigma \) is the sum of \([\hat{c}]_j - [\hat{c}]_j\), \( j \neq i + 1 \) by (18).

It remains to check that

\[
\phi([\hat{c}]_j - [\hat{c}]_j) = 0
\]

(20)

for every \( j \neq i + 1 \). If \( j > i + 1 \), then both cells \([\hat{c}]_j\) and \([\hat{c}]_j\) are collapsible, so \( \phi([\hat{c}]_j) = \phi([\hat{c}]_j) = 0 \) and (20) holds.

Let \( 1 \leq j \leq i \). The thin diagrams \( c' = [\hat{c}]_j \) and \( e = [\hat{c}]_j \) are obtained from \( \hat{c} \) by replacing \( f_j \) by its top and bottom path, respectively. Suppose that \( e' \neq e \) (otherwise there is nothing to prove). Thus \( [f_j] \xrightarrow{\hat{c}} [f_j] \) and the statement of the lemma holds for \( e' \) because \([c] = [e] \xrightarrow{\hat{c}} [e']\).

It is easy to see from definition that the cube \( e \) is redundant. The index of \( e \) is \( j - 1 < i \). Hence the statement of the lemma holds for \( e \) as well. By (19), \( \bar{e} = \bar{c} \). Therefore, \( \phi(e' - e) = 0 \) and (20) holds.

Now it is easy to show that all boundary maps \( \partial_n, n \geq 1 \), in the chain complex \( Q \) are zero. Indeed, by Lemma 9.6, the value \( \phi(c) \) depends only on \( \bar{c} \). By (19) and (18), for every \( n \)-cube \( c \),

\[
\delta_n(c) = \phi \partial_n(c) = \phi \left( \sum_{i=1}^{n} (-1)^i ([c]_i - [c]_i) \right) = \sum_{i=1}^{n} (-1)^i (\phi([c]_i) - \phi([c]_i)) = 0.
\]

Thus \( \ker \partial_n = Q_n \) and \( \operatorname{im} \partial_{n+1} = 0 \). Hence for \( H_n(G; \mathbb{Z}) \cong Q_n \) is free Abelian, \( n \geq 1 \).

If \( n = 0 \), then \( H_0(G; \mathbb{Z}) \cong \mathbb{Z} \) and we have only one essential cell of dimension 0 — this is the vertex corresponding to the irreducible 1-path of \( \operatorname{Sq}(\mathcal{K}, w) \). Thus we proved

**Theorem 9.7.** Let \( \mathcal{K} \) be a complete directed 2-complex and let \( w \) be a non-empty 1-path in \( \mathcal{K} \). The \( n \)th integer homology \( H_n(G; \mathbb{Z}) \) \((n \geq 0)\) of the diagram group \( G = \mathcal{D}(\mathcal{K}, w) \) is free Abelian. Its free basis consists of all essential \( n \)-cubes from \( \operatorname{Sq}(\mathcal{K}, w) \).

Theorem 9.7 implies that the CW complex \( Y_w \) from Theorem 9.2 gives a minimal presentation of \( G = \mathcal{D}(\mathcal{K}, w) \) in terms both the number of generators and the number of relations. In fact, it gives a minimal set of generators of homology groups in all dimensions.

**Remark 9.8.** It is not difficult to prove that the presentation given by \( Y_w \) is precisely the presentation from Theorem 6.6, where \( T_r \) is replaced by \( T_i \) (see Remark 6.7, part b). We already know (Remark 6.7, part a) that the presentation from Theorem 6.6 involves the minimal possible number of generators. Let us show that it contains the minimal number of relations as well.

For any 1-path \( p \), let us denote by \( \mu(p) \) the minimal number of generators for the diagram group \( G = \mathcal{D}(\mathcal{K}, w) \). This is exactly the number of edges of the form \((u, f, v) \notin T_i \) that belong to \( \operatorname{Sq}(\mathcal{K}, w) \), where \( u, v \) are irreducible, \( f \in F^- \) (these are the essential 1-cubes of \( \operatorname{Sq}(\mathcal{K}, w) \)). If we replace here \( T_i \) by \( T_r \), then we again have a minimal generating set of \( G \) because of a symmetry. It is easy to give a formula to compute the number of the defining relations of \( G \) given by Theorem 9.2. Let \( s_1, \ldots, s_m \) be the third components of the essential 1-cubes of \( G \) \((m = \mu(w)) \). Each of the defining relations corresponds to an essential 1-cube in \( \operatorname{Sq}(\mathcal{K}, s_i) \) for some \( i \). Thus the sum \( \mu(s_1) + \cdots + \mu(s_m) \) is exactly the number of the defining relations of \( G \) given by Theorem 9.2. Clearly, it will be the same if we use \( T_r \) instead of \( T_i \) in the definition of essential cubes. Thus the number of defining relations given by Theorem 6.6 and Theorem 9.2 are the same.
Now consider the homology groups of arbitrary diagram groups. Let \( H \) be a diagram group of an arbitrary directed 2-complex. We know from Lemma 5.1, part 4 that \( H \) is a retract of a diagram group \( G \) of a complete directed 2-complex. Notice that the retraction can be described in the language of group homomorphisms: \( H \) is a retract of \( G \) if and only if there are two homomorphisms \( \phi: G \to H \) and \( \psi: H \to G \) such that \( \phi \psi = \text{id}_H \) (\( \psi \) acts first). Since \( H_n(-, \mathbb{Z}) \) is a covariant functor \([7]\), this implies that \( H_n(H; \mathbb{Z}) \) is also free Abelian and its rank does not exceed the rank of \( H_n(G; \mathbb{Z}) \). So we proved

**Theorem 9.9.** For any \( n \geq 0 \) and for any diagram group \( G \), the \( n \)th integer homology group \( H_n(G; \mathbb{Z}) \) is free Abelian.

This theorem answers a question by S. Pride.

If \( G \) is a group of type \( F_\infty \), then one can consider its Poincaré series

\[
P_G(t) = \sum_{n=0}^{\infty} r_n t^n,
\]

where \( r_n \) denotes the rank of the \( n \)th integer homology group of \( G \). Note that \( r_0 = 1 \).

**Example 9.10.** a) Let \( \mathcal{K} = \langle x \mid x^r = x \rangle \) (\( r \geq 2 \)). It is proved in \([16]\) that \( D(\mathcal{K}, x) \cong F_r \), where \( F_r \) is a generalization of the R. Thompson group \( F = F_2 \) defined in \([8]\). That complex is complete so we can use Theorems 9.2 and 9.7. The essential \( n \)-cells of this complex \( (n \geq 1) \) have the form \( c = \varepsilon(x^k_0) + (x = x^i) + \varepsilon(x^k_1) + \cdots + (x = x^r) + \varepsilon(x^k_n) \), where \( 1 \leq i < n \), \( 0 \leq k_i < r \) for \( 0 \leq i < n \), \( 0 \leq k_n < r \). These cells may belong to different components of \( \text{Sq}(\mathcal{K}) \). Clearly, \( c \) belongs to the component of \( x \) if and only if the sum \( n + k_0 + k_1 + \cdots + k_n \) equals 1 modulo \( r - 1 \). This condition determines uniquely the number \( k_0 \) by the other numbers. So there are exactly \( r \cdot (r - 1)^{n-1} \) ways to choose \( c \) in \( \text{Sq}(\mathcal{K}, x) \). (For instance, in the case \( r = 2 \) we have the R. Thompson’s group \( F \), which has \( \mathbb{Z}^2 \) as its \( n \)th integer homology group for \( n \geq 1 \), which was proved in \([9]\).)

The Poincaré series for \( F_r \) has the form

\[
P(t) = 1 + rt + r(r - 1)t^2 + \cdots + r(r - 1)^{n-1} t^n + \cdots = \frac{1 + t}{1 - (r - 1)t}.
\]

b) Now let \( \mathcal{V} = \langle y \mid y^3 = y^2 \rangle \) be the complex on Figure 9 (by Theorem 5.7, the diagram group \( D(\mathcal{V}, y^2) \) is universal). The complex \( \mathcal{V} \) is complete as well. The essential \( n \)-cubes in \( \text{Sq}(\mathcal{V}, x^2) \) have the form \( c = \varepsilon(y^k_0) + (y^2 = y^3) + y^k_1 + \cdots + (y^2 = y^3) + \varepsilon(y^k_n) \), where \( k_i = 1, 2 \) for \( 0 \leq i < n \), \( k_n = 1, 2, 3 \). All of them for \( n \geq 1 \) belong to \( \text{Sq}(\mathcal{K}, y^2) \). Hence the rank of the \( n \)th homology group of \( D(\mathcal{V}, x^2) \) is \( 3 \cdot 2^n \) (\( n \geq 1 \)) and so the Poincaré series is

\[
P(t) = 1 + 6t + 12t^2 + \cdots + 3 \cdot 2^n t^n + \cdots = \frac{1 + 4t}{1 - 2t}.
\]

Notice that the Poincaré series of \( D(\mathcal{V}, x^2) \) coincides with the Poincaré series of the free product \( F_3 * F_3 \) but these diagram groups are not isomorphic (which can be proved by using Kurosh’s theorem).

c) One more universal diagram group is given by the complete directed 2-complex \( \mathcal{H}_1 = \langle x \mid x^2 = x, x = x \rangle \) (Theorem 5.6). Let us find the number of the essential \( n \)-cubes in \( \text{Sq}(\mathcal{H}_1, x) \). Let \( c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n) \) be one of these cubes. Then there are three possibilities for each of the pairs \((u_{i-1}, f_i)\), namely, \((x, x = x^2)\), \((1, x = x)\), or \((x, x = x)\) (\( 1 \leq i \leq n \)).
are two possibilities for the 1-path $u_n$, namely, 1 and $x$. So the rank of the $n$th homology group of $G_1 = D(H_1, x)$ equals $2 \cdot 3^n$ for $n \geq 1$. It is interesting to mention that both universal groups considered in b), c) have the same minimal number of generators (equal to 6) and we know that they are embeddable into each other because of their universal property. However, they are not isomorphic because their second homology groups have ranks 12 and 18, respectively. Thus the Poincaré series of $G_1$ is

$$P(t) = 1 + 6t + 18t^2 + \cdots + 2 \cdot 3^n t^n + \cdots = \frac{1 + 3t}{1 - 3t}.$$  

We see that all these Poincaré series are rational. This can be explained by the following

**Theorem 9.11.** Let $K$ be a complete finite almost 2-path connected directed 2-complex. Then the Poincaré series of any of its diagram groups is rational.

**Proof.** We refer to [25] for the well-known properties of rational languages. Let $A = P \cup (P \times F^-)$, where $P$ consists of all irreducible 1-paths in $K$, including the empty 1-paths, $F^-$ consists of all negative 2-cells of $K$.

Let 0 be a symbol not in $P$, and let us define a binary operation $\cdot$ on $S = P \cup \{0\}$: $p \cdot q = pq$ if $\tau(p) = \iota(q)$, $p, q \in P$; all other products are equal to 0. It is easy to see that $S$ is a semigroup.

Let $\phi$ be a homomorphism from the free semigroup $A^+$ to $S$ induced by the map that takes each $p \in P$ to itself and each pair $(p, f) \in P \times F^-$ to $p [f] = p \bar{f}$. Notice that for every $s \in P$, every word $w$ of the form $(u_0, f_1) \cdots (u_{n-1}, f_n)u_n$ from the rational language $\phi^{-1}(s) \subseteq A^+$ corresponds to an $n$-cube in $Sq(K, s)$. This cube is essential if and only if $w$ does not contain letters of the form $(p, f)$, where $p \in P$, $f \in F^-$, and $(p, f, *)$ belongs to the fixed left forest $T_i$ of $Sq(K)$. Thus let $L$ be the sublanguage of $\phi^{-1}(s)$ consisting of all words from $\phi^{-1}(s) \cap A^+P$ which do not contain the letters described in the previous paragraph. Clearly, $L$ is a rational language. There exists a one-to-one correspondence between words of length $n + 1$ in $L$ and essential $n$-cubes in $Sq(K, s)$. Since the generating function of a rational language $L$ is rational (see for example [11]), the Poincaré series of $Sq(K, s)$ is a rational function as well.

Recall [7] that for any group $G$, its geometric dimension, $gd(G)$, is the smallest dimension of a $K(G, 1)$. Its cohomological dimension, $cd(G)$, is the length of the shortest projective resolution of the trivial $ZG$-module $Z$. It is easy to see [7] that

$$cd(G) \leq gd(G). \quad (21)$$

By the Eilenberg – Ganea theorem [24], $cd(G) = gd(G)$ provided $cd(G) \neq 2$ or $gd(G) \neq 3$.

Theorems 9.2 and 9.7 immediately imply that for diagram groups over complete directed 2-complexes these two dimensions coincide.

**Theorem 9.12.** For every diagram group $G$ over a complete directed 2-complex,

$$cd(G) = gd(G).$$

**Proof.** Let $G = D(K, w)$. By Theorem 9.7 the length of any projective resolution of the trivial $ZG$-module $Z$ cannot be smaller than the highest dimension $n$ of an essential cube of $Sq(K, w)$. By Theorem 9.2, $n \geq gd(G)$. Therefore, $cd(G) \geq n \geq gd(G)$. Hence by (21), $gd(G) = cd(G)$.

$\square$
The following result gives an algebraic characterization of groups of finite cohomological dimension among diagram group of complete directed 2-complexes.

**Theorem 9.13.** Let $G$ be a diagram group over a complete directed 2-complex $\mathcal{K}$, and $n$ be a natural number. Then $\text{cd}(G) \geq n$ if and only if $G$ contains a copy of $\mathbb{Z}^n$.

**Proof.** Let $G = \mathcal{D}(\mathcal{K}, w)$. Let $T_i$ be a left forest in $\text{Sq}(\mathcal{K})$. The “if” statement is well known [7]. So suppose that $\text{cd}(G) \geq n$. Then by Theorem 9.7, $\text{Sq}(\mathcal{K}, w)$ contains an essential cube $c = \varepsilon(u_0) + f_1 + \cdots + f_n + \varepsilon(u_n)$.

By definition, for every $1 \leq i \leq n$, the edge $(u_{i-1}, f_i,*)$ does not belong to $T_i$. Hence the connected component $\text{Sq}(\mathcal{K}, u_{i-1}[f_i])$ contains edges not in $T_i$. Therefore, by Theorem 6.6 (or Theorem 9.7), the group $G_i = \mathcal{D}(\mathcal{K}, u_{i-1}[f_i])$ is non-trivial. Since all diagram groups are torsion-free [16], $G_i$ contains a copy of $\mathbb{Z}$.

By Corollary 3.6, the diagram group $G_1 \times \cdots \times G_n$ embeds into $H = \mathcal{D}(\mathcal{K}, p)$, where $p = u_0[f_1] \cdots [f_n] u_n$. Therefore, a copy of $\mathbb{Z}^n$ is contained in $H$. But by the choice of $c$, the 1-paths $p$ and $w$ are in the same connected component of $\text{Sq}(\mathcal{K})$. Hence by Corollary 3.5, $H$ is isomorphic to $G$, so $G$ contains a copy of $\mathbb{Z}^n$, as required. 

Notice that Theorem 10.4 below gives a characterization of directed 2-complexes $\mathcal{K}$ such that $\mathcal{D}(\mathcal{K}, w)$ contains a copy of $\mathbb{Z}^n$.

Theorem 9.13 immediately implies the following result.

**Theorem 9.14.** A diagram group $G$ over a complete directed 2-complex is free if and only if $G$ does not contain a copy of $\mathbb{Z}^2$. In particular, a hyperbolic group can be a diagram group of a complete directed 2-complex if and only if it is free.

**Proof.** Indeed, by Theorem 9.12, if $G$ does not contain $\mathbb{Z}^2$ then $\text{cd}(G) \leq 1$, and one can use the well known result of Stallings and Swan [24] (or, easier, one can use Remark 9.8, and conclude that $G$ has a presentation with no relations).

**Problem 9.15.** Is it possible to drop the completeness restriction from the formulations of Theorems 9.11, 9.12, 9.13, 9.14?

**Remark 9.16.** Recall that by Lemma 3.7 the space of positive paths $\Omega^+(\mathcal{K})$ is a realization of $\text{Sq}(\mathcal{K})$. By Theorem 8.5, the homology of a connected component of that space coincides with the homology of the corresponding diagram group. Hence by Theorem 9.11, the Poincaré series of the space of positive paths of a complete almost 2-path connected directed 2-complex is rational. This resembles the well known result of Serre (see, for example, [1]) that the Poincaré series of the loop space of a simply connected CW 2-complex is always rational.

**Remark 9.17.** Notice that the completeness restriction in the statements of this paper can be replaced by the condition “there exists a left forest”. Say, let $\mathcal{K} = \langle a, b \mid ab = a, ba = b \rangle$. It is not hard to check that $\mathcal{K}$ is not complete. However, one can construct a spanning forest satisfying conditions F1 and F2 of the left forest (it is formed by all edges of the form $(1, a = ab, *)$, $(a, b = ba, *)$, $(1, b = ba, *)$, $(b, a = ab, *)$ and the inverse edges). Using that forest, as above, one can compute the presentation of the corresponding diagram groups, and their homology groups. The Poincaré series of the diagram groups of this complex are rational.
10 Rigidity

Recall that the flat torus theorem [3, Theorem 7.1] says, in particular, that if $X$ is a metric space with CAT(0) universal cover $\tilde{X}$ and the fundamental group of $X$ contains a copy of $\mathbb{Z}^n$, then $X$ contains a $\pi_1$-embedded torus $\mathbb{R}^n/\mathbb{Z}^n$.

Results of this section are of similar spirit: we prove that a diagram groupoid of a directed 2-complex $K$ contains certain diagram group $G = D(S, p)$ if and only if there exists a $p$-nonsingular morphism from $S$ into $K$. Since every $p$-nonsingular morphism $\phi : S \to K$ induces a $\pi_1$-injective continuous map $\text{Sq}(S, p) \to \text{Sq}(K, \phi(p))$, these results (and Theorem 3.4) imply that if $\pi_1(\text{Sq}(K), u)$ contains a copy of $G = \pi_1(\text{Sq}(S), p)$ then there exists a $\pi_1$-injective continuous map from $\text{Sq}(S, w)$ into $\text{Sq}(K)$.

In general we say that a triple (diagram group $G$, directed 2-complex $K$, 1-path $p$ in $K$) is rigid if $G = D(K, p)$ and for every directed complex $K'$ such that $D(K')$ contains a copy of $G$ there exists a $p$-nonsingular morphism of $K$ into $K'$.

![Diagram](https://example.com/diagram.png)

Figure 13.

For example, consider the directed 2-complex $K$ on Figure 13 with two vertices $\iota$ and $\tau$, one edge $x$ connecting $\iota$ with $\tau$ and positive 2-cells of the form $x = x$ labelled by elements of some set $A$. By Theorem 6.6 (or a straightforward computation), the diagram group $G = D(K, x)$ is the free group of rank $|A|$. It is easy to see (exercise) that the triple $(G, K, x)$ is rigid.

A much more nontrivial example of a rigid triple involves the R. Thompson group $F$. Example 6.8 shows that $F$ is the diagram group of the Dunce hat $H_0$.

The following proposition shows a remarkable property of the Dunce hat.

**Theorem 10.1.** Every morphism from the Dunce hat to any directed 2-complex $K$ is nonsingular.

**Proof.** Let us consider any morphism $\phi$ from $H_0$ into a directed 2-complex $K$. We need to show that it is $p$-nonsingular for every non-empty 1-path $p$ in $H_0$. Since for every such $p$, the diagram groups $D(H_0, p)$ and $D(H_0, x^5)$ are conjugate, it is enough to show that $\phi$ is $x^5$-nonsingular. Since $x^5$ and $x$ are homotopic, we have $D(H_0, x^5) \cong D(H_0, x) \cong F$ (see Example 6.8).

Since all proper homomorphic images of the group $F$ are Abelian [13], it suffices to find an element in the derived subgroup of $D(H_0, x^5)$ that is not in the kernel of $\phi_{x^5}$. Let $\Pi$ be the diagram over $H_0$ corresponding to the atomic 2-path $(1, f, 1)$, where $f$ is the positive 2-cell $x^2 = x$ in $H_0$. Consider the diagram $\Psi = \varepsilon(x) + \Pi + \Pi^{-1} + \varepsilon(x)$ from $D(H_0, x^5)$. By Theorem 11.3 from [16], $\Psi$ is in the derived subgroup of $D(H_0, x^5)$ and is nontrivial. By Theorem 3.4, we can assume that the diagram $\Delta = \phi_{x^5}(\Pi)$ is reduced. Since it is a $(\phi(x)^2, \phi(x))$-diagram, it is nontrivial. Then $\phi_{x^5}(\Psi) = \varepsilon(\phi(x)) + \Delta + \Delta^{-1} + \varepsilon(\phi(x))$ is also a reduced and nontrivial diagram, hence by Theorem 3.4, $\phi_{x^5}(\Psi) \neq 1$.

Note that the same property is true for directed 2-complexes $\langle x \mid x^r = x \rangle$ that correspond to groups $F_r$ ($r \geq 2$), the generalizations of $F$ (see [6, 16]). The proof is based on the same idea (all proper homomorphic images of these groups are also Abelian).
Here is a reformulation of the main result of [18] which shows that the triple \((F, \mathcal{H}_0, x)\) is rigid.

**Theorem 10.2.** ([18]) Let \(K\) be a directed complex. Then the following conditions are equivalent.

1. A diagram groupoid \(K\) contains an isomorphic copy of the R. Thompson group \(F\).
2. The complex \(K\) contains a non-empty 1-path which is homotopic to its square.
3. There exists a (nonsingular) morphism from the Dunce hat to \(K\).

Thus if the diagram groupoid of \(K\) contains a copy of \(F\) then it contains a naturally embedded copy of \(F\).

Another example of a rigid triple is given by [17, Theorem 24]. Let \(Q\) be the directed 2-complex with three vertices, three edges \(x, y, z\) and three positive cells of the forms \(xy = x, y = y, yz = z\) on Figure 14 (to obtain the complex from the diagram, we identify all edges having the same labels).

It is proved in [17] that the diagram group \(\mathcal{D}(Q, xyz)\) is isomorphic to the restricted wreath product \(\mathbb{Z} \wr \mathbb{Z}\). Theorem 24 of [17] shows that the triple \((\mathbb{Z} \wr \mathbb{Z}, Q, xyz)\) is rigid.

![Figure 14.](image)

The free Abelian group \(\mathbb{Z}^n\) can participate in a rigid triple too (for every \(n \geq 1\)). In fact, using a description of commuting diagrams ([17, Theorem 17]) one can obtain a much more precise result (Theorem 10.4 below).

Let \(S_n\) be the following directed 2-complex: take a simple path labelled by the word \(w_n = x_1 \cdots x_n\), where \(x_i\) are letters and let us attach \(n\) positive 2-cells \(x_1 = x_1, \ldots, x_n = x_n\) to it. Thus \(S_n\) is a chain of \(n\) spheres (Figure 15).

![Figure 15.](image)

Then \(\text{Sq}(S_n, w_n)\) is the \(n\)-dimensional torus (this is easy to check). Thus by Theorem 3.4, the group \(\mathcal{D}(S_n, w_n)\) is isomorphic to \(\mathbb{Z}^n\).

The next result is one of the most useful technical facts about diagram groups. In [17], it is formulated and proved for diagrams over semigroup presentations (see [17, Theorem 24]). The proof for directed complexes is completely similar (in fact it can be deduced from the result of
by using subdivisions of complexes). It is similar to the well known theorem that commuting matrices over an algebraically closed field are simultaneously conjugate to their Jordan forms.

**Lemma 10.3.** Let $\mathcal{K}$ be a directed 2-complex and $w$ be a non-empty 1-path in $\mathcal{K}$. Suppose that $A_1, \ldots, A_n$ are spherical $(w, w)$-diagrams that pairwise commute in $G$. Then there exist a 1-path $v = v_1 \cdots v_m$, spherical $(v_j, v_j)$-diagrams $\Delta_j$ ($1 \leq j \leq m$) over $\mathcal{K}$, integers $d_{ij}$ ($1 \leq i \leq n$, $1 \leq j \leq m$) and some $(w, v)$-diagram $\Gamma$ over $\mathcal{K}$ such that

$$\Gamma^{-1}A_i\Gamma = \Delta_1^{d_{i1}} + \cdots + \Delta_m^{d_{i_m}}$$

for all $1 \leq i \leq n$.

**Theorem 10.4.** The triple $(\mathbb{Z}^n, S_n, w_n)$ is rigid for every $n \geq 1$. In addition, let $\mathcal{K}$ be a directed 2-complex. Then every copy of $\mathbb{Z}^n$ in $D(\mathcal{K})$ is conjugate in $D(\mathcal{K})$ to a subgroup of a naturally embedded copy of $\mathbb{Z}^m$ for some $m \geq n$.

**Proof.** Suppose that a diagram group $D(\mathcal{K}, p)$ of some directed 2-complex contains a copy $G = \langle A_1, \ldots, A_n \rangle$ of $\mathbb{Z}^n$. We use the notation from Lemma 10.3. Let us also assume that $m$ is chosen to be minimal. Then all diagrams $\Delta_1, \ldots, \Delta_m$ are nontrivial. Indeed, if we assume the contrary, then $m > 1$ because $G \neq 1$. If $\Delta_i$ is trivial for some $i$, then one has $i < m$ or $i > 1$. Without loss of generality we assume that $i < m$. But now it would be possible to replace $\Delta_i$ by $\Delta_{i+1}$ taking into account that the power of a sum is the sum of powers and all powers of a trivial diagram coincide.

Clearly, $m \geq n$ (otherwise the rank of the subgroup generated by $A_1, \ldots, A_n$ would be less than $n$).

Now the map $\phi$ from $S_m$ to $\mathcal{K}$ that sends the positive 2-cell $x_i = x_i$ to the 2-path corresponding to $\Delta_i$ ($1 \leq i \leq m$), defines a morphism. The image of $D(S_m, w_m)$ under $\phi_{w_m}$ is generated by the diagrams $\varepsilon(v_1 \cdots v_{i-1}) + \Delta_i + \varepsilon(v_{i+1} \cdots v_m)$. Thus the image of $\phi_{w_m}$ is isomorphic to $\mathbb{Z}^m$. Therefore, $\phi$ is $w_m$-nonsingular. Clearly, the naturally embedded copy $\phi(D(S_m, w_m))$ of $\mathbb{Z}^m$ contains $\Gamma^{-1}GT$. This proves the second statement of the theorem.

In order to prove the rigidity statement, we just note that $S_n$ is a subcomplex of $S_m$ so it maps into it nonsingularly. But we already have a nonsingular morphism of $S_m$ into $\mathcal{K}$. It suffices to compose the morphisms.

**Problem 10.5.** It is interesting to characterize other diagram groups that can participate in rigid triples. In particular, in view of rigidity of the triple $(F, \mathcal{H}_0, x)$ it is natural to ask if the analog is true for $\mathcal{H}_n$. By Theorem 5.6, it is enough to prove that for $n = 1$. This would give a characterization of universal directed 2-complexes as those admitting a nonsingular morphism from $\mathcal{H}_1$.

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