Type IIB Solutions with Interpolating Supersymmetries

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Abstract

We study type IIB supergravity solutions with four supersymmetries that interpolate between two types widely considered in the literature: the dual of Becker and Becker’s compactifications of M-theory to 3 dimensions and the dual of Strominger’s torsion compactifications of heterotic theory to 4 dimensions. We find that for all intermediate solutions the internal manifold is not Calabi-Yau, but has $SU(3)$ holonomy in a connection with a torsion given by the 3-form flux. All 3-form and 5-form fluxes, as well as the dilaton, depend on one function appearing in the supersymmetry spinor, which satisfies a nonlinear differential equation. We check that the fields corresponding to a flat bound state of D3/D5-branes lie in our class of solutions. The relations among supergravity fields that we derive should be useful in studying new gravity duals of gauge theories, as well as possibly compactifications.

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I. INTRODUCTION

Supersymmetric $\mathcal{N} = 1$ warped solutions of IIB supergravity have played an extensive role in gauge/gravity duality and string compactification. Unfortunately, the general supersymmetric solution is not known. Recent work has involved two special cases with four-dimensional Poincaré invariance, which can be characterized by the form of the ten-dimensional supersymmetry spinor $\epsilon = (\epsilon^1, \epsilon^2)$. These Majorana-Weyl spinors can be decomposed as

$$\epsilon^1 = \zeta \otimes \chi_1 + \zeta^* \otimes \chi_1^*$$
$$\epsilon^2 = \zeta \otimes \chi_2 + \zeta^* \otimes \chi_2^*.$$  

Here $\zeta$ is a four-dimensional chiral spinor, $\Gamma^{(4)} \zeta = \zeta$, and $\chi_{1,2}$ are six-dimensional chiral spinors, $\Gamma^{(6)} \chi_i = -\chi_i$. Each independent pair $(\chi_1, \chi_2)$ gives rise to one $D = 4$ supersymmetry. The two special cases are then

Type A(ndy): $\chi_2 = 0$
Type B(eker): $\chi_2 = i\chi_1$.  

(2)

The behavior of the spinor correlates with that of the complex three-form flux $G_{(3)}$. In type A solutions there is only NS-NS 3-form flux, which means that $G_{(3)}$ must be imaginary. In fact, only the NS-NS background is nontrivial. In type B solutions, the 5-form is non-vanishing. Also, $G_{(3)}$ must be imaginary self-dual; more specifically, it must be of $(2,1)$ and primitive with respect to the complex structure of the transverse space. Vanishing $G_{(3)}$ also gives type B solutions. Pure brane systems are of one or the other of these types: the NS5-brane is of type A (and the D5-brane is of the S-dual type), and the D3-brane and D7-brane are of type B.

The type A solutions are closely related to the warped heterotic solutions found by Strominger [1]. (Among other papers, the IIB supersymmetry conditions for type A solutions were discussed in detail in [2, 3].) The Maldacena-Nuñez solution [4] is a notable AdS/CFT example of this type. Conditions to have $\mathcal{N} = 2$ supersymmetry in this class of solutions are discussed in [5], and an $\mathcal{N} = 2$ AdS/CFT example is [6]. Type A compactifications have been reconsidered recently in [7, 8, 9], because they are related to type B compactifications by a series of U-dualities.

The type B solutions are dual to M theory solutions found by Becker and Becker [10, 11]. In the M theory form, the corresponding restriction on the supersymmetry spinor is that it
have definite eight-dimensional chirality. The explicit IIB form was obtained in [12, 13] for the special case of a constant dilaton-axion, and in [14, 15] for nonconstant dilaton-axion. Such solutions have played an important role in gauge/gravity duality. Along with the standard $AdS_5 \times S^5$ solution, the $\mathcal{N} = 1$ conifold fractional brane solution [16] is of this form, as well as its $\mathcal{N} = 2$ generalization [17, 18]. Type B compactifications have been the focus of intense interest; simple compact manifolds were studied in [19, 20, 21, 22, 23, 24] and general cases were examined in [25, 26], including supersymmetry breaking cases. Giddings, Kachru and Polchinski [26] showed that compact solutions involving D3-branes, O3 planes, D5-branes wrapped on collapsed 2-cycles and D7 branes wrapped on $K3$ are all of type B form. Notably, [27] has constructed a de Sitter solution of string theory by adding nonperturbative effects to type B compactifications.

In this paper, we will describe a class of solutions that interpolates between type A and B solutions (effectively tracing out a curve in spinor space), studying the fermion supersymmetry transformations directly. Our solutions would then correspond to D3/NS5 bound states and eventually D7-branes when we include a nontrivial axion, where in one extrema (type A) the D3 (and D7)-charge vanishes, while in the other (type B) the 5-brane wraps a vanishing 2-cycle. We found it easiest to describe solutions that interpolate between the S-dual of type A solutions, which we will call type C (for convenience) and correspond to D5-branes with spinors $\chi_2 = -\chi_1$, and the type B solutions, since for both of them the spinors $\chi_1$ and $\chi_2$ have the same norm. We can then S-dualize to find solutions that interpolate between the NS5-brane type A and type B solutions. We start in the next section by reviewing characteristics of type A, B, and C solutions; then in section III we describe the actual interpolating solution, keeping a trivial R-R scalar for simplicity. In section IV we explain how to include a nontrivial R-R scalar (as sourced by D7-branes) and also how to generalize the type of 5-brane charge (by duality transformations). Because bound states of branes should correspond to some intermediate type of supersymmetry solution, we examine the supergravity solution of a D3/D5-brane bound state [28, 29] in section V and show that it is in the class described in III. Finally, we discuss our results in section VI.
II. KNOWN SOLUTIONS OF IIB SUPERGRAVITY

All the solutions we study preserve $SO(3, 1)$ invariance, so we start with the most general ansatz for the background metric and 5-form flux that preserve such invariance:

$$
\begin{align*}
    ds^2 &= e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu + ds_6^2 \\
    F_{\mu\nu\lambda\rho m} &= e^{-4A} \epsilon_{\mu\nu\lambda\rho} \partial_m h .
\end{align*}
$$

The 3-forms are completely on the transverse space; we allow branes that are extended in the four $x^\mu$ directions. Greek subindices take values from 0 to 3, and indicate directions of the $SO(3, 1)$ symmetry, while Roman subindices take values from 4 to 9. $\epsilon_{\mu\nu\lambda\rho}$ has coordinate indices, i.e. $\epsilon_{0123} = e^{4A}$. $A$ and $h$ in (3) are functions of $x^m$.

In the following two subsections, we will write the known type A, C and B solutions. We will not show how to get these solutions from supersymmetry constraints, since that will become clear when working out the interpolating solutions.

A. Types A and C

The type A solution is the dual of Strominger’s heterotic “superstrings with torsion” [1]. The IIB version of it is the supergravity solution describing NS5 branes wrapped on 2-cycles. We work in the string frame, in which the metric is unwarped, $A = 0$ with the internal metric Hermitean on a complex manifold. Such a manifold has torsion, i.e. $SU(3)$ invariant tensors are parallel in a connection given by the Levi Civita connection plus a torsion. This torsion is nothing else than the NS-NS 3-form flux, and it is related to the complex structure of the manifold by

$$
    H = i(\bar{\partial} - \partial) J
$$

The manifold is endowed with a holomorphic (3,0) form $\Omega$, whose norm fixes the dilaton

$$
    \phi = \phi_0 - \frac{1}{2} \ln |\Omega|
$$

For Calabi-Yau manifolds $|\Omega|$ is constant, and $J$ is closed. In this case, we have instead

$$
    d^\dagger J + i(\bar{\partial} - \partial) \ln |\Omega| = 0
$$

We will see where these equations come from when working out the interpolating solution.
In the type C, or D5-brane solution, the form is much the same, with $H_{(3)} \rightarrow F_{(3)}$ and now $A = \phi/2$ and $ds_{6}^{2} = e^{\phi}ds_{6}^{2}$ with $ds_{6}^{2}$ the complex manifold with torsion. The solutions corresponding to $(p, q)5$-branes are somewhat more complicated, as they include a nontrivial R-R scalar. We do not describe them here.

B. Type B

Type B solutions, corresponding to regular D3 branes, are of the “warped Calabi-Yau” form:

$$ds_{6}^{2} = e^{-2A} \hat{ds}_{6}^{2}$$

(7)

where $\hat{ds}_{6}^{2}$ is a metric on a Calabi-Yau manifold. The square of the warp factor, usually called $Z$, obeys a Poisson equation:

$$-\tilde{\nabla}^{2}(e^{-4A}) = (2\pi)^{4}g\alpha'^{2}\rho_{3}$$

(8)

where $\rho_{3}$ is the density of D3-branes. There is 5-form flux of the form stated in the ansatz (3), where $h$ is

$$h = \frac{1}{g}e^{4A}$$

(9)

and the dilaton is constant $e^{\phi} = e^{\phi_{0}} = g$. (In fact, the entire complex dilaton-axion $\tau = C + ie^{-\phi}$ is constant.)

We can add 3-form flux to this solution, which can be sourced by fractional branes (D5-branes wrapped on collapsed 2-cycles). The condition that we get from supersymmetry is that the combination $G_{(3)} = F_{(3)} - \tau H_{(3)}$ should be $(2, 1)$ and primitive with respect to the complex structure of the Calabi Yau space, which means

$$G_{ijk} = G_{ij}^{j} = G_{ijk} = G_{ijk} = 0$$

(10)

This 3-form flux acts like a source in the Poisson equation for the warp factor, which gets modifies to

$$-\tilde{\nabla}^{2}(e^{-4A}) = (2\pi)^{4}g\alpha'^{2}\rho_{3} + g^{2}G_{pqrr}G^{pqrr}$$

(11)

where $\rho_{3}$ is the 6-form number density of D3-brane charge.

It is also known how to include a nontrivial axion-dilaton, as in F-theory, in the presence of D7-branes [14, 15]. The axion-dilaton is holomorphic $\tilde{\partial}\tau = 0$, and the five-form now satisfies $h = e^{4A-\phi}$. We will see how this comes about below.
III. INTERPOLATING SOLUTION

In this section we will show how to get interpolating solutions between type B and type C solutions.

The type IIB supersymmetry variations in the string frame for bosonic backgrounds are

\[
\delta \lambda = \frac{1}{2} \Gamma^M \partial_M \phi \varepsilon - \frac{e^\phi}{2} \Gamma^M F_M (i \sigma^2) \varepsilon - \frac{e^\phi}{24} \Gamma^{MNP} F'_{MNP} \sigma^1 \varepsilon - \frac{1}{24} \Gamma^{MNP} H_{MNP} \sigma^3 \varepsilon
\]

\[
\delta \psi_M = \nabla_M \varepsilon + \frac{1}{8} e^\phi \Gamma^N \Gamma_M F_N (i \sigma^2) \varepsilon - \frac{1}{8} \Gamma^{PQ} H_{MPQ} \sigma^3 \varepsilon + \frac{1}{48} e^\phi \Gamma^{PQRST} F_{PQRST} \Gamma_M (i \sigma^2) \varepsilon.
\]

As before, \( \varepsilon = (\varepsilon^1, \varepsilon^2) \) are two Majorana-Weyl spinors of negative 10d chirality, and \( \sigma^i \) are the Pauli matrices that act on the column vector \( (\varepsilon^1, \varepsilon^2) \). We have defined \( F'_M = F_M - CH_M \).

These variations vanish in supersymmetric solutions. We use the following decomposition for the gamma matrices:

\[
\Gamma^\mu = \gamma^\mu \otimes 1, \quad \Gamma^m = \Gamma^{(4)} \otimes \gamma^m
\]

with \( \Gamma^{(4)} = (i/4!) \epsilon_{\mu \nu \lambda \rho} \gamma^{\mu \nu \lambda \rho} \).

A. Solving Supersymmetry Conditions

For convenience, we will set the axion \( C \) to zero and reintroduce it later. A constant nonzero \( C \) trivially replaces \( F \rightarrow F' \) in the below results. From \( \delta \psi_M = 0 \), we get

\[
\gamma^\mu \Gamma^{(4)} \left( \frac{1}{2} \gamma^n \partial_n A + \frac{i}{8} e^{-4A} e^\phi \gamma^n h \Gamma^{(4)} i \sigma^2 - \frac{1}{48} e^\phi F \sigma^1 \right) \varepsilon = 0
\]

where \( F \equiv F_{mnp} \gamma^{mnp} \). This means

\[
\left( \frac{1}{2} \gamma^n \partial_n A + \frac{i}{8} e^{-4A} e^\phi \gamma^n h \Gamma^{(4)} i \sigma^2 - \frac{1}{48} e^\phi F \sigma^1 \right) \varepsilon = 0
\]

From \( \delta \psi_m \), we get

\[
\nabla_m \varepsilon = -\frac{1}{8} H_m \sigma^3 \varepsilon + \frac{1}{8} e^\phi F_m \sigma^1 \varepsilon - \frac{1}{48} e^\phi \gamma_m F \sigma^1 \varepsilon + \frac{i}{8} e^{-4A} e^\phi \gamma^m \gamma_m \partial_n h \Gamma^{(4)} i \sigma^2 \varepsilon = 0
\]

using (10), the term involving \( F \) can be written in terms of derivatives of the warp factor and 4-form flux. Using \( \gamma^n \gamma_m = 2 \gamma^n_m + \gamma_m \gamma^n \) in the last term, everything combines to give

\[
\nabla_m \varepsilon = -\frac{1}{2} \partial_m A - \frac{1}{2} \gamma^m \partial_n A \varepsilon - \frac{i}{4} e^{-4A} e^\phi \gamma^m \gamma_m \partial_n h \Gamma^{(4)} i \sigma^2 \varepsilon - \frac{1}{8} H_m \sigma^3 \varepsilon + \frac{1}{8} F_m \sigma^1 \varepsilon = 0
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\]
where $F_m \equiv F_{mnp} \gamma^{np}$ and similarly for $H_m$.

The term that does not contain gamma matrices can be canceled by the term $\partial_m \varepsilon$ if we define $\varepsilon = e^{A/2} \varepsilon'$. We get for $\varepsilon'$

$$
\nabla_m \varepsilon' - \frac{1}{2} \gamma_m \partial_n A \varepsilon' - \frac{i}{4} e^{-4A} e^{\phi} \gamma_m \partial_n h \Gamma^{(4)} i \sigma^2 \varepsilon' - \frac{1}{8} H_m \sigma^3 \varepsilon' + \frac{1}{8} F_m \sigma^1 \varepsilon' = 0 \quad (19)
$$

To go any further, we should make an ansatz for the spinors, and this is the point where general (Poincaré invariant) solutions become particular ones. The ansatz that we make is the simplest one interpolating between type B and type C spinors:

$$
\varepsilon_{1'} = \zeta \otimes \chi + \zeta^* \otimes \chi^*,
$$

$$
\varepsilon_{2'} = i e^{i \alpha} \zeta \otimes \chi - i e^{-i \alpha} \zeta^* \otimes \chi^*. \quad (20)
$$

Following the notation of the introduction, $\chi_1 = e^{A/2} \chi, \chi_2 = i e^{i \alpha} \chi_1$. Type B corresponds to $\alpha = 0$ and type C to $\alpha = \frac{\pi}{2}$. We will let $\alpha$ vary over the compact manifold, which, as we will see, is indeed necessary to get solutions other than B and C.

Inserting this ansatz in (19) and using (29) below, we get

$$
\left( \nabla_m + \frac{1}{2} e^{2i \alpha} \gamma_m \partial_n A - \frac{1}{8} H_m + \frac{i}{8} e^{i \alpha} e^{\phi} F_m \right) \chi = 0 \quad (21)
$$

$$
\left( \nabla_m + \frac{1}{2} e^{-2i \alpha} \gamma_m \partial_n A + \frac{1}{8} H_m - \frac{i}{8} e^{-i \alpha} e^{\phi} F_m + i \partial_m \alpha \right) \chi = 0 \quad (22)
$$

Now add up (21,22) to get

$$
\left( \nabla_m + \frac{1}{2} \cos(2\alpha) \partial_n A \gamma_m n - \frac{1}{8} \sin \alpha e^{\phi} F_m + \frac{i}{2} \partial_m \alpha \right) \chi = 0 \quad . \quad (23)
$$

Again, the term that does not involve gamma matrices can be canceled by taking $\chi = e^{-i \alpha/2} \chi'$. If we assume for now that

$$
\cos(2\alpha) \partial_m A = \partial_m A' \quad (24)
$$

is a total derivative, then we can rescale the internal metric

$$
ds_6^2 = e^{-2A'} ds_6^2 \quad (25)
$$

to eliminate the warp factor terms, which clearly agrees with type B at $\alpha = 0$ ($A = A'$ in type B, as can be seen from (17)). Such a function $A'$ does not logically have to exist. In

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1 We don’t need to use (29) here, but it makes things look a little simpler.
fact, we can distinguish three (nonexclusive) cases: first, that $A'$ does exist; second, that the manifold is complex; and, third, that $A'$ does not exist and the internal manifold is not complex. The calculations presented in this section demonstrate that the first two cases are equivalent, but we do not address the possibility of the third case.

Then we have

$$\left(\tilde{\nabla}_m - \frac{1}{8} \sin \alpha e^{\phi + 2A'} F_{mnp} \gamma^{np}\right) \chi' = 0,$$

(26)

This means that there is a normalized spinor that is constant with respect to a connection equal to the Levi-Civita connection $\tilde{\nabla}_m$ plus a torsion $-(1/2) \sin \alpha e^{\phi + 2A'} F_m$. This suffices to show that the manifold is complex (but in general not Kähler). The complex structure, being built out of the spinor, will also be covariantly constant with respect to this connection with torsion. We will comment on this later. For now, we just need that the manifold is complex and that we can take $\gamma^i \chi = 0$ for our negative chirality spinor.

The relationship between $h$ and the warp factor that we used to get Eqs (21, 22), can be obtained from (16) using our particular ansatz for the spinors. By doing so, we get

$$\left(\frac{1}{2} \gamma^n \partial_n A - \frac{1}{8} e^{-4A} e^{i\alpha} e^{\phi} \gamma^n \partial_n h - \frac{i}{48} e^{i\alpha} e^{\phi} F\right) \chi = 0$$

along with the complex conjugate equations. Multiplying the second by $-i e^{i\alpha}$ and adding both up, we get

$$e^{i\alpha} e^{-4A} e^{\phi} \partial_m h = 2(e^{2i\alpha} + 1) \partial_m A$$

(28)

or

$$\partial_m h = 4 e^{4A} e^{-\phi} \cos \alpha \partial_m A$$

(29)

(which means the rhs is a total derivative, at least in patches). We also see from (27) that

$$F_{ijk} = F_{ijk} = 0.$$  

(30)

The other equation that we get from the set of Eqs (27) is

$$e^\phi F_{ij}^j = 4 \sin \alpha \partial_i A.$$  

(31)

The complex conjugates of these equations follow from the $\chi^*$ eqns.
Now, let us turn to the two equations coming from the variation of the dilatino (12), 
\[ \delta \lambda = 0: \]
\[ \frac{1}{2} \gamma^m \partial_m \phi \chi - \frac{1}{24} H \chi - \frac{i}{24} e^{i \alpha} e^\phi F \chi = 0 \]  
(32)
\[ \frac{1}{2} \gamma^m \partial_m \phi \chi + \frac{1}{24} H \chi + \frac{i}{24} e^{-i \alpha} e^\phi F \chi = 0 \]  
(33)
First of all, knowing that there is no (3,0) R-R 3-form flux, we can see that there should not be (3,0) (or (0,3)) NS-NS flux. So the 3-form flux \( G_{(3)} \) is only (2,1) and (1,2), as happens in both type B and C solutions.

Now, subtracting (33) from (32), we get
\[ H \chi + i \cos \alpha e^\phi F \chi = 0 \]  
(34)
which means
\[ \cos \alpha F_{ij}^j - i e^{-\phi} H_{ij}^j = 0. \]  
(35)
For type B, this is the primitivity condition for \( G_{(3)} \), \( G_{ij}^j = 0. \)

Finally, adding (33) and (32), we get the equation
\[ \frac{1}{2} \gamma^m \partial_m \phi \chi + \frac{1}{24} \sin \alpha e^\phi F \chi = 0 \]  
(36)
from what we derive
\[ \sin \alpha e^\phi F_{ij}^j = 2 \partial_i \phi = 4 \sin^2 \alpha \partial_i A. \]  
(37)
where the last equality comes from (31). This is the correct relation between the warp factor and dilatino both for type B and type C compactifications (see section II). In all solutions except type B, the dilatino varies over the 6-dimensional space.

The only equation that we have not used so far is the one we get by subtracting (22) from (21):
\[ \left( i \frac{\sin(2 \alpha)}{2} \gamma_m n \partial_n A - \frac{1}{8} H_m + i \frac{\cos \alpha e^\phi F_m}{8} - \frac{i}{2} \partial_m \alpha \right) \chi = 0. \]  
(38)
For \( m = i \), we end up with
\[ \left( H - i \cos \alpha e^\phi F \right)_{ij}^j = 2i \partial_i \alpha - 2i \sin(2 \alpha) \partial_i A, \]  
(39)
and, for \( m = \bar{i} \), we get
\[ \left( H - i \cos \alpha e^\phi F \right)_{i\bar{j}}^{\bar{j}} = 2i \partial_i \alpha + 2i \sin(2 \alpha) \partial_i A \]  
(40)
and
\[
\left[ -(H - i \cos \alpha e^\phi F)_{ijk} \gamma^{jk} + 4i \sin(2\alpha) \partial_j A \gamma^j \right] \chi = 0 .
\]
(41)

Taking the complex conjugate of (40) and comparing with (35), we get a relationship between the function \( \alpha \) in the spinor ansatz and the warp factor:
\[
\partial_i \alpha = -\sin(2\alpha) \partial_i A
\]
(42)
while adding (35) and (39) we get an equation for the trace of \( H(3) \):
\[
H_{ij}^j = 2i \partial_i \alpha
\]
(43)
Finally, we get from (41)
\[
\cos \alpha F_{ijk} + ie^{-\phi} H_{ijk} = -4e^{-\phi} \sin(2\alpha) g_{[ij]} \partial_k A
\]
(44)

The complex conjugate of this equation for the case \( \alpha = 0 \) is the type B condition of having no (1,2) piece of \( G^{(3)} \).

From these equations we can see that \( \alpha \) being constant gives nontrivial solutions only when this constant is zero or \( \pi/2 \), which corresponds to type B and C. So in the class of solutions with supersymmetry parameters of the form (20), type B and C are very special ones.

**B. Scalar Relations and Complex Structure**

We can collect the information we have so far to write the 4D and 6D warp factors, the 4-form potential, and the dilaton in terms of \( \alpha \). After we have done this, we will be able to relate the 3-form fluxes to the complex structure, and finally get \( \alpha \) in terms of the complex structure.

Let us first obtain \( A, A', \phi \) and \( h \) in terms of \( \alpha \). From (42) we get \( A \) in terms of \( \alpha \). Then, (29) gives \( h \) as a function of \( \alpha \). We use the last equality in (37) to get the dilaton in terms of \( \alpha \), and finally we get \( A' \) from (24). The result is
\[
A = -\frac{1}{2} \ln \tan \alpha + A_0
\]
(45)
\[
h = e^{4A_0 - \phi_0} \cot^2 \alpha + h_0 = e^{4A - \phi_0} + h_0
\]
(46)
\[
\phi = \ln \cos \alpha + \phi_0
\]
(47)
\[
A' = -\frac{1}{2} \ln \sin(2\alpha) + A'_0
\]
(48)
So indeed the only possible solutions with $\alpha$ constant for our spinor ansatz are type B and C. Note that these scalar solutions appear to be singular when $\alpha \to 0, \pi/2$, which are the type B and C limits. Still, we recover the known solutions by looking at the relations between $A, h, \phi$ and $A'$ implied by (45-48), i.e.: when $\alpha \to 0$ (type B), we get $A' \sim A$, $h \sim e^{4A-\phi}$, $\phi = \phi_0$, which is what we expect for type B; $\alpha \to \pi/2$ (type C) gives $A' \sim -A$, $h = 0$, $\phi = 2A$, which is what we expect for type C. (And these relations are implied by the differential equations, also, even if you disbelieve the solutions above.)

Let us make the type B and C limits more precise. We can include the type B and C cases if we allow the integration constants (denoted by subscript 0) to be infinite, so that they cancel the divergences due to the dependence on $\alpha$. Let us consider first the type B limit, where $\alpha(x) = \delta \beta(x)$ with $\delta \to 0$ a constant. Then the scalar equations (45,46, 47,48)

\[
\begin{align*}
A &= -\frac{1}{2} \ln \beta + \hat{A}_0, \quad A_0 = \hat{A}_0 + \frac{1}{2} \ln \delta \\
h &= e^{4A_0} e^{-\phi_0} \beta^{-2} + h_0 = -e^{-\phi} e^{4A} + h_0 \\
\phi &= \phi_0 = \text{constant} \\
A' &= -\frac{1}{2} \ln \beta + \hat{A}_0', \quad A'_0 = \hat{A}_0' + \frac{1}{2} \ln(2\delta), \quad A' = A + \text{constant} .
\end{align*}
\]

(49)

These are, in fact, the expected relations among the scalars for type B. In particular, note that $2A = -2A' = \phi$, as in section. Additionally, the NS-NS 3-form vanishes because
of (44) and its conjugate (along with the fact that no (0, 3), (3, 0) fluxes are allowed). This is just what we would expect for a D5-brane type background.

Solutions that go to the type B or C limit at some position pose an interesting problem. In some cases, the divergent scalars give what we expect due to the presence of a source, such as the dilaton in the presence of a D5-brane. Then we would not want to renormalize out the divergence. We will examine this case in more detail in the example of the D5/D3 bound state. In cases where the solution approaches a limit at infinity, the situation is much more complicated, depending on the expected solution. For example, if the solution should go to type B AdS at infinity, the warp factor is expected to diverge in a certain way, so, again, we might not have to renormalize out that divergence.

Let us now get the 3-form fluxes and $\alpha$ in terms of the complex structure and holomorphic (3,0) form of the manifold. This derivation follows that of Strominger’s [1], since our equations (26) and (36) have the same form as those that appear in the heterotic case, with the NS-NS flux replaced by the R-R flux combined with a function of $\alpha$.

The Killing spinor equation (26) implies that the supersymmetry parameters feel a torsion in the metric $d\tilde{s}_6^2$: the torsion is equal to

$$T = -\frac{1}{2} \sin \alpha \ e^{\phi + 2A'} F = -\frac{1}{4} e^{\phi_0 + 2A_0} F .$$

(51)

Clearly, if we go to the type B limit, the torsion vanishes, which means that $d\tilde{s}_6^2$ is a Calabi-Yau manifold, which we also expect. The following considerations do not apply in that limit because of division by zero problems, but they do apply in all other solutions. Note that the only solution without torsion is type B.

First of all, equation (26) implies that there is an almost complex structure

$$\tilde{J}_m^n = i \chi^m \bar{\chi}_n \chi'.
(52)$$

The complex structure is covariantly constant with respect to the connection with the torsion given in (51). Therefore it is possible to show that $\tilde{J}$ is an integrable complex structure and satisfies $\tilde{J} = i\tilde{\gamma}_{ij} dz^i d\bar{z}^j$. Additionally, there is an expression for the RR 3-form flux in terms of the fundamental two form, as in Strominger (see subsection IIA for appropriate normalizations in IIB):

$$e^{\phi_0 + 2A_0} F = -2i(\bar{\partial} - \partial)\tilde{J}
(53)$$

Again, in the type B solutions, the left hand side of (53) vanishes due to the divergence of $A'_0$, so we cannot divide by zero and use the following results.
With this $F_{(3)}$, let's plug into equation (44). Using (45) we get

$$H = -2e^{-2A_0} \left( \cos^2 \alpha d\tilde{J} - \sin(2\alpha)\tilde{J} \wedge d\alpha \right).$$

(54)

To get $\alpha$ in terms of the complex structure and the $(3,0)$ form, we use equation (36). It is a standard procedure to multiply this equation to the left and to the right with $\tilde{\gamma}_n$ and subtract the resulting equations, to get

$$\chi'^{\dagger} \left( [\tilde{\gamma}_n, \tilde{\gamma}^m] \partial_m \phi + \frac{1}{12} e^{\phi_0 + 2A_0} F_{mpq} \{\tilde{\gamma}_n, \tilde{\gamma}^{mpq}\} \right) \chi'^* = 0$$

(55)

This, as in Strominger's case, leads to the equation

$$-2\nabla_m \phi + J_m \nabla_q J_n q = 0$$

(56)

where we have used the fact that $\tilde{J}$ is "$F$-covariantly constant", i.e. $\nabla_m \tilde{J}_n^p = -\frac{1}{4} e^{\phi_0 + 2A_0} F_{sm} \tilde{J}_n^s - \frac{1}{4} e^{\phi_0 + 2A_0} F_{mn} \tilde{J}_n^p = 0$.

Now construct the holomorphic $(3,0)$ form $\Omega$ as follows

$$\Omega = e^{2\phi} \chi'^{\dagger} \tilde{\gamma}_{ijk} \chi'^* dz^i dz^j dz^k$$

(57)

(to be a $(3,0)$ form we need to construct it with the positive chirality spinor $\chi'^*$, which obeys the same two equations (26) and (36) as $\chi$).

The covariant antiholomorphic derivative gives

$$\nabla_i \Omega_{ijk} = (2\nabla_i \phi - \nabla_i \phi) \Omega_{ijk}$$

(58)

where the second term comes from derivatives acting on $\chi'^*$. The right hand side of this, using (56) to write the derivative of the dilaton in terms of the complex structure, is exactly the difference between the covariant and the regular antiholomorphic derivatives of $\Omega$. This means that $\Omega$ is holomorphic (for more details, see [1]).

The norm of $\Omega$ can be obtained by

$$\nabla_i |\Omega| = 2\nabla_i |\Omega|$$

(59)

where we have used $\nabla_i \Omega_{ijk} = 3\nabla_i \phi |\Omega|$. Then

$$|\Omega| = e^{2\phi + \phi_0}$$

(60)
and then
\[ \phi = \frac{1}{2} (\ln |\Omega| - \Omega_0). \]  \hspace{1cm} (61)

Comparing with (17) we get
\[ \cos \alpha = |\Omega|^{1/2}, \quad \phi_0 = -\frac{\Omega_0}{2}. \]  \hspace{1cm} (62)

The relationship between \( \alpha \) and the norm of the (3,0) form \( \Omega \) is not valid in type C, where \( \alpha = \pi/2 \) is constant, but the norm of \( \Omega \) is not. In that case, Eq. (61) is still valid, but we should not use (17) to relate \( \Omega \) and \( \alpha \).

Finally, the dilatino implies then:
\[ d^\dagger \tilde{J} + i(\bar{\partial} - \partial) |\Omega| = 0. \]  \hspace{1cm} (63)

C. Bianchi Identities

Now we turn to Bianchi identities
\[ dF = (2\pi)^2 \alpha' \rho_5, \quad dH = 0, \quad dF_{(5)} = H \wedge F + (2\pi)^4 \alpha'^2 \rho_3. \]  \hspace{1cm} (64)

where \( F \) is as before the 3-form RR flux. Note that we do not include any NS5-brane sources because they should lie outside this ansatz; in general, one could add those sources. There are some subtleties, however. As has been known, anomaly relations require a modified 3-form Bianchi identity, a fact which has been studied recently and extensively in [9]. In the type IIB string theory, these modifications arise from \( \alpha' \) corrections to the D9-brane (and O9-plane) world-volume action. Additionally, complications can arise without D9-branes; the pure 5-brane systems presented in [8] appear to have too few D5-branes to cancel the charge from the O5-planes. The solution appears to be that the space transverse to the 5-brane world-volumes is a 4-chain rather than a 4-cycle. The integral over the boundary of the 4-chain allows precisely the right amount of charge non-conservation. It seems that \( \alpha' \) corrections are not necessary to understand D5-brane charge conservation in the absence of D9-branes.

The Bianchi identity for \( F \) gives
\[ dF = e^{-\phi_0 - 2A_0} \partial \bar{\partial} \tilde{J} = (2\pi)^7 \alpha'^4 \rho_5, \]  \hspace{1cm} (65)

We thank M. Schulz for bringing this problem, and its solution, to our attention.
which is just as we would get following Strominger [1]. The corresponding equation for $H$ gives

$$dH = 2 \sin(2\alpha) e^{-2A_0} \left( d\alpha \wedge d\tilde{J} + d\tilde{J} \wedge d\alpha \right) = 0$$

which is satisfied automatically because of the wedge product.

The Bianchi identity for $F(5)$ leads to the equation

$$e^{-4A + 2A'} \left( \hat{\nabla}^2 h - 4 \partial^m(A + A') \partial_m h \right) = 8i e^{-\phi_0 - 4A_0} \cos \alpha \tilde{\nabla} \tilde{J} \wedge \tilde{\partial} \tilde{J} + \sin \alpha \tilde{J} \wedge (\tilde{\partial} - \partial) \tilde{J} \wedge d\alpha \right) + (2\pi)^4 \alpha^2 \tilde{\nabla}_6 \rho_3 \right).$$

It is straightforward but unilluminating to plug in for the scalars in terms of $\alpha$. We get a rather nonlinear equation for $\alpha$ that controls the entire geometry. In the type B limit, it reduces to equation (66).

We should note that the Bianchi identities are troubling for the prospects of supersymmetric compactifications due to the no-go theorem of [26]. All compactifications of type IIB string theory with localized sources that satisfy a certain BPS-like inequality are subject to the no-go theorem. D5-branes on 2-cycles satisfy the inequality, but O5-planes violate it and avoid the no-go theorem. If the inequality is satisfied by all the local sources in the compactification, then the no-go theorem implies that the inequality must be saturated, and the solution (if supersymmetric) must be of pure type B. Therefore, an interpolating type compactification – that is, one that is not pure type B – should have O5-planes (or 9-branes perhaps). However, it is hard to see how D3-brane charge, which is present in all but pure type C solutions, could be conserved in a supersymmetric way, as the supersymmetries of O3-planes are incompatible with those of O5-planes.

### D. Summary of results for BC interpolating solution

We summarize here the results obtained so far for solutions that interpolate between type B and type C, i.e. where 5-brane sources are pure D5.

The spinor anstaz used was

$$\varepsilon^{1'} = \zeta \otimes \chi + \zeta^* \otimes \chi^*$$

$$\varepsilon^{2'} = ie^{i\alpha} \zeta \otimes \chi - ie^{-i\alpha} \zeta^* \otimes \chi^*.$$
The metric and 5-form flux are of the form

\[
d s^2 = e^{2A} \eta_{\mu \nu} dx^\mu dx^\nu + e^{-2A'} \tilde{d}s_6^2
\]

\[
F_{\mu \nu \lambda \rho m} = e^{-4A} \epsilon_{\mu \nu \lambda \rho \delta} \partial_m h.
\]

where \( \tilde{d}s_6^2 \) is a metric for a complex manifold for which the connection \( \tilde{\nabla}_m = \frac{1}{8} \sin \alpha e^{\phi+2A'} F_{mnkp} \tilde{\gamma}^{np} \) has \( SU(3) \) holonomy. The functions \( A, A', h \) and the dilaton \( \phi \) are related to the function \( \alpha \) in the spinor ansatz by

\[
A = -\frac{1}{2} \ln \tan \alpha + A_0
\]

\[
h = e^{4A_0-\phi_0} \cos^2 \alpha + h_0 = e^{4A-\phi_0} + h_0
\]

\[
\phi = \ln \cos \alpha + \phi_0
\]

\[
A' = -\frac{1}{2} \ln \sin(2\alpha) + A'_0.
\]

(70)

The fluxes obey the equations

\[
H_{ij} = 2i \partial_i \alpha
\]

\[
e^\phi F_{ij} = 4 \sin \alpha \partial_i A
\]

\[
(\cos \alpha F - ie^{-\phi} H)_{ij} = 0
\]

\[
(\cos \alpha F + ie^{-\phi} H)_{ijk} = -4e^{-\phi} \sin(2\alpha) g_{ij} \partial_k A.
\]

(71)

The complex structure \( \tilde{J} \) and the holomorphic (3,0) form \( \Omega \) of the manifold with metric \( \tilde{d}s_6^2 \) obey

\[
d \bar{d} \tilde{J} + i(\bar{\partial} - \partial) |\Omega| = 0.
\]

(72)

where the norm of \( \Omega \) is related to the function \( \alpha \) by

\[
\cos \alpha = |\Omega|^{1/2}
\]

(73)

Finally, Bianchi identities turn into equations for \( \tilde{J} \) and \( |\Omega| \) of the form

\[
e^{-\phi_0-2A_0} \bar{\partial} \partial \tilde{J} = 2 \kappa_{10}^2 \rho_5
\]

\[
e^{-4A+2A'} (\bar{\nabla}^2 h - 4 \bar{\partial} \bar{\partial} (A + A') \partial_m h) = 8ie^{-\phi_0-4A_0} \cos \alpha \bar{\star}_6 (\cos \alpha \partial \tilde{J} \wedge \bar{\partial} \tilde{J}
\]

\[
+ \sin \alpha \tilde{J} \wedge (\bar{\partial} - \partial) \tilde{J} \wedge d\alpha) + (2\pi)^4 \alpha^{r3} \bar{\star}_6 \rho_3
\]

(74)

where we wrote the second equation as an equation for \( h \) instead of \( |\Omega| \) for simplicity.
IV. GENERALIZING THE SOLUTION

A. Inclusion of R-R Scalar

We know that type B solutions can include a nonzero R-R scalar $C$, as long as the complex dilaton-axion $\tau$ is holomorphic, and it is also evident from the equations of motion that arbitrary constant $C$ is compatible with type C solutions. Therefore, we expect that nontrivial behavior of $C$ is compatible with our spinor ansatz (20). Here we discuss the changes to the relations given in section III A without going into the details of the derivations.

First, the internal gravitino variation (23) now gives

$$\left[ \nabla_m + \frac{i}{4} \cos \alpha e^\phi \partial_m C - \left( \frac{1}{2} \partial_n A \gamma^m_n - \frac{1}{4} \cos \alpha e^{-4A} \partial_n h \gamma^m_n \right) - \frac{1}{8} \sin \alpha e^\phi F_m \right] \chi = 0 , \quad (75)$$

so we rescale by $e^{-2A'}$, as before, with

$$-\partial_m A + \frac{1}{2} \cos \alpha e^{\phi-4A} \partial_m h = \partial_m A' . \quad (76)$$

This is the same as we used before, but now equation (29) will be modified. The differential equation for the spinor is now

$$\left( \tilde{\nabla}_m + \frac{i}{4} \cos \alpha e^\phi \partial_m C - \frac{1}{8} \sin \alpha e^{\phi+2A'} F^p_{mnp} \tilde{\gamma}^{np} \right) \chi' = 0 . \quad (77)$$

The new feature is the derivative term from $C$; it is a $U(1)$ connection on the internal manifold. It is very important now to see that there is a complex structure, which actually follows exactly as before. That is, the almost complex structure $\tilde{J}$ defined in (52) is neutral under the $U(1)$, so the proof of its integrability is unchanged.

Now we turn to the external components of the gravitino equation. As before, the $(3, 0)$ and $(0, 3)$ parts of $F' = F - CH$ vanish, and equation (31) is unchanged except for taking $F \rightarrow F'$. The other equation, (29) becomes

$$\partial_i h = 4 e^{4A} e^{-\phi} \cos \alpha \partial_i A + i e^{4A} \partial_i C \quad (78)$$

We should note (anticipating that we recover $\partial \bar{\tau} = 0$ for type B) that this is $\partial h = \partial e^{4A-\phi}$ in the type B limit, which is exactly what is needed for force cancellation on a D3-brane. Since D3-branes are BPS in type B backgrounds, that result is as expected.

The dilatino equations are derived as before. Equations (35) and (37) become respectively

$$\cos \alpha F^i_{ij} - i e^{-\phi} H_{ij}^i = 2 i \sin \alpha \partial_i C . \quad (79)$$
and
\[ \sin \alpha e^\phi F_{ij}^j = 2 \partial_i \phi - 2i \cos \alpha e^\phi \partial_i C \quad . \] (80)

In particular, we see from (80) that \( \tau \) is holomorphic as \( \alpha \to 0 \).

Finally, we consider the other information that we get from the internal gravitino variation. Equations (39, 40) are unchanged except to take \( F \to F' \) because the \( C \) term cancels precisely with a contribution from \( h \). Also, we find
\[ \cos \alpha F_{ijk} + ie^{-\phi}H_{ijk} = -4e^{-\phi} \sin(2\alpha)g_{ij}d_jA + 2i \sin \alpha e^\phi g_{ij}\partial_kC \quad . \] (81)

To go with equations (76, 78), (39, 40, 81) give two more independent equations, which we can write using relations for the fluxes as
\[ \partial_i \alpha = -\sin(2\alpha)\partial_i A + i \sin \alpha e^\phi \partial_i C \]
\[ \partial_i \phi = i \cos \alpha e^\phi \partial_i C + 2 \sin^2 \alpha \partial_i A \quad . \] (82)

There are not enough equations to determine all the scalars in terms of a single function now. This fact is not entirely surprising; in type B solutions, the holomorphic \( \tau \) is after all independent of the warp factor. We can, however, determine
\[ \phi = \ln \sin \alpha + 2A + \phi_0 \quad . \] (83)

There is a troubling aspect to the fact that all the scalars are not given by a single function. Namely, we can no longer prove that the combination (76) is a total derivative. After manipulation of the various scalar relations, it is possible to show that a necessary and sufficient condition is that
\[ \cos \alpha e^\phi (\partial - \bar{\partial}) C = dB \] (84)
is a total derivative. This is true, at least locally, as long as the left hand side is closed, or
\[ 2\partial \bar{\partial}C = -d(\phi + \ln \cos \alpha) \wedge (\partial - \bar{\partial}) C \quad . \] (85)

This condition seems to be rather complicated to solve, but it is easily demonstrated to be true in the type C (\( C \to constant \)) and type B (\( \alpha \to 0, \tau \) holomorphic) limits.

Finally, we touch on the equations governing the complex structure. The equations (53, 54) can be modified simply. They become
\[ F' = -i \csc \alpha e^{-2A' - \phi} (\partial - \bar{\partial}) J \quad . \] (86)

\[ H = e^{-2A'} \left( -\frac{1}{2} \cot \alpha d\bar{J} - 4 \sin(2\alpha) dA \wedge \bar{J} + 2i \sin \alpha e^\phi \partial - \partial \wedge C \wedge J \right) \quad . \] (87)
These look more complicated because we do not have simple relations among the scalars, but they reduce to (53, 54) for constant $C$. The $(3, 0)$ form defined as in (57) is no longer holomorphic, however. To get the holomorphic $(3, 0)$ form, we must assume (84); then $\Omega' = e^{-iB/2}\Omega$ is holomorphic. In that case, it is possible to derive the relations

$$
\nabla_i \left( |\Omega'|^{-1/2}\Omega'_{j_1j_2j_3} \right) = -i\partial_i B|\Omega'|^{-1/2}\Omega'_{j_1j_2j_3}, \quad \nabla_i \left( |\Omega'|^{-3/2}\Omega'_{j_1j_2j_3} \right) = -i\partial_i B|\Omega'|^{-3/2}\Omega'_{j_1j_2j_3}
$$

(88)

for the R-R scalar. It would be interesting to see if these results provide a topological constraint on the behavior of the R-R axion.

B. S-duality and $(p, q)5$-branes

Type A solutions, corresponding to NS5-branes, are much more well-studied than the type C solutions, so it is useful to examine how our interpolating solutions dualize to solutions that interpolate between type B and type A solutions. These solutions correspond to NS5/D3-brane bound states. Let us start by noting the action of the $SL(2, \mathbb{Z})$ duality (hats denote the dual variables) (see [33]):

$$
\hat{\tau} = \frac{a\tau + b}{c\tau + d},
$$

$$
\hat{g}_{EMN} = g_{EMN}, \quad \hat{g}_{MN} = e^{-\phi/2}g_{MN}
$$

$$
\hat{F}_(5) = F_5, \quad \hat{H}_(3) = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \begin{bmatrix} H_3 \\ F_3 \end{bmatrix}.
$$

(89)

To get the pure type A-B interpolating solution, then, start with the $C = 0$ solution of section III A and dualize taking $b = -c = 1$. Then the scalars become

$$
\hat{A} = A - \frac{\phi}{4} + \frac{\hat{\phi}}{4} = -\frac{1}{2}\sin \alpha + \hat{A}_0, \quad \hat{A}_0 \equiv A_0 - \frac{\phi_0}{2}
$$

$$
\hat{h} = h = e^{4\hat{A}_0}e^{-\hat{\phi}_0} \cot^2 \alpha + h_0
$$

$$
\hat{\phi} = -\phi = -\ln \cos \alpha + \hat{\phi}_0, \quad \hat{\phi}_0 \equiv -\phi_0
$$

$$
\hat{A}' = -\frac{1}{2}\ln \sin \alpha + \hat{A}'_0, \quad \hat{A}'_0 \equiv A'_0 - \frac{1}{2}\ln 2 + \frac{\phi_0}{2}.
$$

(90)
The fluxes obey the equations

\[
\begin{align*}
\sin \alpha e^{-\hat{\phi}} \hat{H}_{ij}^j &= 2 \partial_i \hat{\phi} \\
\hat{F}_{ij}^j &= 2i \partial_i \alpha \\
\left( \cos \alpha \hat{H} + ie^{\hat{\phi}} \hat{F} \right)_{ij} &= 0 \\
\left( \cos \alpha \hat{H} - ie^{\hat{\phi}} \hat{F} \right)_{ijk} &= 4 \sin(2\alpha) \hat{g}_{[ij} \partial_{k]} \left( \hat{A} - \hat{\phi}/2 \right).
\end{align*}
\] (91)

It is very easy to check that these correspond to a spinor ansatz \( \varepsilon = e^{\hat{A}/2} \varepsilon' \),

\[
\begin{align*}
\hat{\varepsilon}^1 &= (e^{-i\alpha/2} - ie^{i\alpha/2}) \zeta \otimes \chi + (e^{i\alpha/2} + ie^{-i\alpha/2}) \zeta^* \otimes \chi^* \\
\hat{\varepsilon}^2 &= (e^{-i\alpha/2} + ie^{i\alpha/2}) \zeta \otimes \chi + (e^{i\alpha/2} - ie^{-i\alpha/2}) \zeta^* \otimes \chi^*.
\end{align*}
\] (92)

Here \( \chi \) is covariantly constant with respect to the torsional connection.

This is what we expect from the \( SL(2, \mathbb{R}) \) transformation of the superparameter that can be derived from \( [30, 32] \),

\[
\hat{\varepsilon}^1 - i\hat{\varepsilon}^2 = e^{\hat{\phi}/8 - \hat{\phi}/8} \left( \frac{c\tau + d}{c\tau + d} \right)^{1/2} (\varepsilon^1 - i\varepsilon^2)
\] (93)

for \( \tau \) purely imaginary. The dilaton prefactor comes from the transformation of the metric.

Naively, we could continue the \( SL(2, \mathbb{Z}) \) dualities by shifting \( \tau \) to get \( (p,q)5/D3 \)-brane bound states, but those all have nonzero asymptotic R-R scalar, and they have vanishing \( F' = F - CH \) as \( \alpha \to \pi/2 \). Indeed, the spinors are the same as those for the NS5-brane, so they are not the \( (p,q)5 \)-branes we want. Fortunately, the low-energy supergravity has the full \( SL(2, \mathbb{R}) \) as a symmetry, and there are \( SL(2, \mathbb{R}) \) transformations that take NS5-branes to \( (p,q)5 \)-branes with vanishing asymptotic R-R scalar \( [34, 35, 36] \). These transformations can similarly be used to generate the \( (p,q)5/D3 \)-brane bound state from the NS5/D3-brane bound state given by (92). We should note that the \( (p,q)5 \)-brane states have a nonconstant R-R scalar, but it is determined as all other scalars are.

We might think it is possible to generate the solutions of section \( IV \) in this manner. However, let us note that the \( SL(2, \mathbb{R}) \) transformations acting on the solutions of section \( III \) those with D5-brane charge and vanishing R-R scalar, can only yield a constant R-R scalar if they return D5-brane charge. Therefore, the solutions of section \( IV \) cannot be reached by \( SL(2, \mathbb{R}) \) transformations. This is because the varying axion in \( IV \) can be thought of as back-reaction to 7-branes. It would be interesting to study states dual to the solutions with varying axion, but the duality transformations would be somewhat messy.
V. BOUND STATE OF D3/D5 BRANES AS A PARTICULAR INTERPOLATING SOLUTION

Breckenridge, Michaud, and Myers [28] and Costa and Papadopoulos [29] found the supergravity solution corresponding to a bound state of D3/D5-branes by applying T-duality to the solution for a $D_4$ brane located at an angle with respect to the direction of T-duality.

The supergravity solution they found corresponds to a D5 brane expanded in the $(x^0, x^1, x^2, x^3, x, y)$ directions, with D3 brane flux spread uniformly in the $x - y$ plane directions. The ratio of D3 to D5 charge densities, which is the tangent of the angle of the original D4-brane in the $x - y$ plane, is

$$\frac{\tilde{q}_3}{q_5} = -\tan \varphi$$

where both $\tilde{q}_3$ and $q_5$ are charge densities in the $(0, 1, 2, 3, x, y)$ volume.

The metric, dilaton, 2-form and 4-form potentials for the bound state configuration are as follows

$$ds^2 = \frac{1}{\sqrt{H}} \left( -dx^0 dx^0 + \sum_{i=1}^{3} dx^i dx^i \right) + \frac{\sqrt{H}}{1 + (H - 1) \cos^2 \varphi} \left\{ dx^2 + dy^2 + (1 + (H - 1) \cos^2 \varphi) (dr^2 + r^2 (d\theta^2 + \sin^2 \theta (d\phi_1^2 + \sin^2 \phi_2 d\phi_2^2))) \right\}$$

$$C_{(4)} = \pm \mu l^2 \sin \varphi \left( 1 + \frac{1}{2} (H - 1) \cos^2 \varphi \right) \sin^2 \theta \cos \phi_1 dy \wedge dx \wedge d\theta \wedge d\phi_2$$

$$\pm \frac{\sin \varphi}{H} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$C_{(2)} = \pm \mu l^2 \cos \varphi \sin^2 \theta \cos \phi_1 d\theta \wedge d\phi_2$$

$$B_{(2)} = \frac{(H - 1) \cos \varphi \sin \varphi}{1 + (H - 1) \cos^2 \varphi} dx \wedge dy$$

$$e^{2\phi} = \left( 1 + (H - 1) \cos^2 \varphi \right)^{-1}$$

where $H = 1 + \frac{\mu}{2l}$, $\mu$ is some dimensionless constant proportional to the number of D5 branes, $l$ is a length scale determined by the string length and number of branes, $l = (4\pi g N)^{1/4} l_s$, and $r$ is the transverse coordinate distance to the D5-branes. The $\pm$ signs correspond to the choice of D3/D5 or D3/D5 bound state.
Comparing both metrics, (95) for the bound state one and (3) for the interpolating solution, and using (45), we get the bound state warp factor $H$ in terms of the parameter $\alpha$ in the interpolating solutions:

$$\sqrt{H} = \tan \alpha e^{-2A_0}$$  \hfill (100)

From the bound state dilaton (99), comparing it with (47), we get another relationship that involves the parameter $\varphi$:

$$\frac{1}{1 + (H - 1) \cos^2 \varphi} = \cos^2 \alpha e^{2\phi_0}$$  \hfill (101)

These two are consistent if we choose

$$e^{-\phi_0} = \sin \varphi, \quad e^{-2A_0} = \tan \varphi$$  \hfill (102)

Then, going back to (100), we can get the radial behavior of $\alpha$ as

$$\sqrt{H} = \sqrt{1 + \frac{\mu}{2r^2}} = \tan \alpha \tan \varphi$$  \hfill (103)

$\alpha$ has the expected behavior in the limits of small and big $r$, i.e.

$$\alpha \rightarrow_{r \rightarrow 0} \pi/2, \quad \alpha \rightarrow_{r \rightarrow \infty} \pi/2 - \varphi$$  \hfill (104)

This means that when we are close to the D5 branes, we see the D5-brane solution (type C, $\alpha = \pi/2$), and when we are far away we see the D3-brane solution (type B, $\alpha = 0$) mixing with the D5, with a “strength” related to $\varphi$. $\varphi \rightarrow \pi/2$, which corresponds in the bound state to no D5-charge, gives a type B solution without 3-form flux.

We want to check now that the whole solution for a D3/D5 bound state lies in the class of solutions we found. Let us start with the 4-form potential. From the $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ term in (96), it is easy to extract the function $h$ appearing in our ansatz (3)

$$h = \pm \frac{\sin \varphi}{H} = \pm e^{-\phi_0 + 4A_0} \cot^2 \alpha$$  \hfill (105)

where in the second equality we have used (100) and (102). The result is in perfect agreement with (46) if we choose $h_0 = 0$ (and the upper sign).

To check the agreement in the 3-form fluxes, we need to split the 6-dimensional metric in (95) in a warp factor and $\tilde{g}_{mn}$. Everything is consistent if we consider the splitting

$$e^{-2A'} = \frac{\sqrt{H}}{1 + (H - 1) \cos^2 \varphi}$$  \hfill (106)

$$ds_6^2 = dx^2 + dy^2 + (1 + (H - 1) \cos^2 \varphi) \left\{dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta (d\phi_1^2 + \sin^2 \phi_2 d\phi_2^2)\right)\right\}.$$
Then, comparing with our form for the 6D warp factor (18), and using (100-102) we get the constant in $A'$ in terms of the parameter $\varphi$ as

$$e^{2A'_0} = \sin(2\varphi).$$

(107)

The RR 3-form flux is, according to (97)

$$F^{(3)} = dC^{(2)} = -\mu l^2 \cos \varphi \sin^2 \theta \sin \phi_1 d\phi_1 \wedge d\theta \wedge d\phi_2$$

(108)

where we have used the upper sign. This would agree with our $F^{(3)}$ (53) if

$$\mu l^2 \cos^2 \varphi \sin^2 \theta \sin \phi_1 d\phi_1 \wedge d\theta \wedge d\phi_2 = i(\bar{\partial} - \partial) \tilde{J}$$

(109)

The metric $d\tilde{s}_6^2$ in (106) splits into a flat 2-dimensional metric times a conformally flat 4-dimensional one. Then, the only nonzero derivatives of the complex structure $\tilde{J}$ are in these 4 dimensions, and are proportional to $H' \cos^2 \varphi$ (prime denotes a derivative with respect to $r$), which is proportional to $\mu l^2 \cos^2 \varphi$. After a long but straightforward calculation that we will not reproduce here, we find precisely (109).

The agreement between NS-NS fluxes is easier to check. From (98), the NS-NS 3-form flux for the D3/D5 system is

$$H^{(3)} = dB^{(2)} = \frac{H'}{(1 + (H - 1) \cos^2 \varphi)^2} \cos \varphi \sin \varphi dr \wedge dx \wedge dy$$

(110)

For us, $H^{(3)}$ is (Eq. (54))

$$H^{(3)} = -2e^{-2A'_0} \left( \cos^2 \alpha d\tilde{J} - \sin(2\alpha) \tilde{J} \wedge d\alpha \right)$$

$$= -\frac{\tan \varphi}{1 + (H - 1) \cos^2 \varphi} d\tilde{J} + 2 \frac{\sqrt{H}}{1 + (H - 1) \cos^2 \varphi} \tilde{J} \wedge d\alpha$$

(111)

The first term in this equation involves only the 4-dimensional $(r, \theta, \phi_1, \phi_2)$ conformally flat part of the metric, where

$$d\tilde{J} = H' \cos^2 \varphi dr \wedge \tilde{J}_4$$

(112)

The second term involves the whole complex structure in 6 dimensions and a derivative of $\alpha$ which, using (100) and (102), is

$$d\alpha = \frac{1}{2\sqrt{H}(1 + (H - 1) \cos^2 \varphi)} \cos \varphi \sin \varphi H' dr$$

(113)
Inserting (112) and (113) in (111), we get

\[
H(3) = -\frac{H'}{(1 + (H - 1)\cos^2 \varphi)^2} \cos \varphi \sin \varphi (dr \wedge \tilde{J}_4 - dr \wedge \tilde{J}_4),
\]

exactly (110).

We can conclude that the whole D3/D5 bound state solution of [28, 29] is in the class of interpolating solutions found. This constitutes a nontrivial check to our solutions.

VI. DISCUSSION

Using our spinor ansatz, we found that there are supersymmetric solutions of type IIB supergravity that interpolate between type B and type A (or C) solutions. When the axion is zero, we were able to write all the functions parametrizing the solutions (4 and 6D warp factors \(A\) and \(A'\), the “potential” \(h\) for the 5-form, and dilaton \(\phi\)), in terms of a single function \(\alpha(x^m)\). To find the actual form of the solutions, we need to solve a nonlinear equation for the function \(\alpha\). These functions of \(\alpha\) are smooth for the interval \(\alpha \in (0, \pi/2)\), but they diverge at the extrema (where it is possible to renormalize away the divergence). We might have anticipated this behavior, from the fact that type B and C correspond to special limits of the sources: no D3 charge (type C) or D5-branes wrapping a vanishing 2-cycle (type B). Also, is only at these extremal points in spinor space that a solution with constant \(\alpha\) is possible.

We have also demonstrated how the axion affects the solution; the scalars are no longer determined in terms of a single function. We expect this behavior from type B systems with D7-branes, in which the complex dilaton-axion \(\tau\) is a holomorphic function independent of the warp factor. Therefore, we have generalized D7-brane solutions.

Our work is closely related to the recent warped M-theory backgrounds with G-flux of Martelli and Sparks [37]. In this paper, the authors generalize the spinor of Becker and Becker’s compactifications to 3 dimensions [10] allowing for 8-dimensional spinors with both chiralities. These backgrounds correspond to space filling M2-branes as well as M5-branes. Their background fields depend on a function \(\zeta\), which is related to the norm of the spinors of each chirality. Doing a reduction to 10 dimensions and T-dualizing, we get the same features of our AB interpolating solutions, if we identify their \(\sin \zeta\) with our \(\cos \alpha\). The M2/M5 bound state solution of [38] is in the class of compactifications studied by Martelli.
and Sparks, while the D3/D5 bound states of \[28, 29\] are in our class.

There are two points to make, however. The first is that we have used a very particular spinor ansatz, and it is unclear if our solutions are as general as those of \[37\], even including \(SL(2, \mathbb{R})\) transformations. The other is related to the R-R axion as discussed in section IV A. In the type B case, the nontrivial axion is sourced by D7-branes, which are dual to Kaluza-Klein monopoles in the M theory. To be consistent with the results of \[37\], these monopoles are related to a nontrivial \(U(1)\) fibration in the manifold of \(G_2\) structure. It would be interesting to see how the relations \[88\] arise in that context.\(^3\)

From the AdS/CFT viewpoint, it would be interesting to study the gauge theory duals of these interpolating solutions, in which \(\alpha\) regulates the renormalization flow. For example, a solution could interpolate between the Klebanov-Strassler solution \[16\] in the UV and the Maldacena-Nuñez solution \[4\] in the IR (or vice-versa). Such a solution could provide interesting insights into the relation of the theories on the branes (especially since the UV theory in \[4\] is a little string theory), as well as providing interesting gauge theory dynamics.

There are some other known AdS renormalization flows that are related to our ansatz. For example, the \(\mathcal{N} = 2\) flow of \[39, 40\] seems to be a related to our interpolating solutions with \((p, q)5\)-brane charge. It has been argued \[37\] that the \(\mathcal{N} = 2\) M-theory flow of \[41\] is such an interpolating solution. Also of interest is the \(\mathcal{N} = 1^*\) flow of Polchinski and Strassler \[42\], which is only known perturbatively. This solution is a 5-brane/D3-brane bound state, but the perturbative solution includes \((0, 3)\) fluxes in the usual complex coordinates \[12\]. It is possible that the complex coordinates should be chosen differently, or the supersymmetry spinors for \(\mathcal{N} = 1^*\) may be more complicated than our ansatz. Nevertheless, we hope that our work will be a step toward finding the exact solution.

Solutions that are not pure type B are also interesting for phenomenology. It was shown in \[43\] that the fields on a D3-brane do not couple to a type B background. For standard-like models made out of D3-branes, the only way to get fermion masses from the fluxes is by embedding the branes on backgrounds that are not pure type B.

In conclusion, we have made progress toward finding a unified description of all supersymmetric solutions of type IIB supergravity. More progress in that direction would be very helpful in finding exact gravity duals for gauge theories, as we have indicated.

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