GKZ HYPERGEOMETRIC SYSTEMS AND APPLICATIONS TO MIRROR SYMMETRY

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We analyze GKZ(Gel'fand, Kapranov and Zelevinski) hypergeometric systems and apply them to study the quantum cohomology rings of Calabi-Yau manifolds. We will relate properties of the local solutions near the large radius limit to the intersection rings of a toric variety and of a Calabi-Yau hypersurface.

1 Introduction

Mirror symmetry of Calabi-Yau manifolds is one of the most beautiful aspects of string theory. It has been applied with great success to do non-perturbative calculation of quantum cohomology rings. More recently, new ideas have been developed to apply mirror symmetry to study the moduli space of the type II string vacua compactified on a Calabi-Yau manifold. Some of the recent work on verifying the so-called heterotic-type II string duality relies heavily on these new ideas.

One of the key ingredients for studying families of Calabi-Yau manifolds is the so-called Picard-Fuchs equations. They are differential equations which govern the period integrals of a Calabi-Yau manifold. In this report, we will review several aspects of the Picard-Fuchs equations which arise in mirror symmetry. We define the flat coordinates and use them to give a natural description of the quantum cohomology ring. We relate the flat coordinates to the general solutions of the Picard-Fuchs equations at the so-called point of maximally unipotent monodromy. The general solutions turn out to be in a subspace of the solutions to a Gel'fand-Kapranov-Zelevinski (GKZ) hypergeometric system. A GKZ system is therefore reducible in our case.

2 Mirror symmetry and quantum cohomology ring

To see the essence of the mirror symmetry, we should go back to a toy model (c = 3) of a string theory with target space given by a complex 1-torus $T$. We can write it as $T = \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ is a rank 2 lattice in $\mathbb{C}$. We
introduce the metric and the antisymmetric tensor on the torus by \( G_{ij} := e_i \cdot e_j \) and \( B_{ij} = B e_{ij} \). Then it is natural to introduce the complexified Kähler modulus \( \lambda = 2(B + i\sqrt{G}) \) and the complex structure modulus \( \rho = \frac{G_{12}}{G_{22}} + i\sqrt{G} \). The partition function \( Z_T(\lambda, \rho) \) is determined exactly as a function of moduli and is known to have the following symmetries

1) \( Z_T(\lambda, \rho) = Z_T(\lambda, \rho + 1) = Z_T(\lambda, -1/\rho) \),
2) \( Z_T(\lambda, \rho) = Z_T(-\lambda, -\rho) \),
3) \( Z_T(\lambda, \rho) = Z_T(\rho, \lambda) \).

The first two symmetry is the result of modular invariance and orientation invariance. The last symmetry, however, comes from the invariance under the exchange of momenta and winding numbers. It is the simplest example of mirror symmetry: an invariance under the exchange of the complexified Kähler moduli and the complex structure moduli.

If we combine the symmetries 1) and 3), we may derive the relation \( Z_T(\lambda, \rho) = Z_T(-1/\lambda, \rho) \), which is the complex analogue of the famous symmetry, \( R \leftrightarrow 1/R \) duality found by Kikkawa and Yamazaki.

When the complex 1-torus is replaced by a Calabi-Yau 3-fold \( M \), the symmetry 3) becomes a little more involved. First we need a new manifold \( W \) which is "mirror" to \( M \). Then we have

\[
Z_M(f(\rho), g(\lambda)) = Z_W(\lambda, \rho)
\]

where the parameters \( \lambda, \rho \) represent local coordinates of the appropriate moduli spaces. The functions \( f, g \) are mirror maps which we will later describe in terms of the flat coordinates of the Gauss-Manin system.

There are two local operator algebras, which are called type A, B respectively, associated to a string model compactified along \( M \). The type A algebra is associated to the Kähler deformation \( \lambda \) and it receives non-perturbative quantum corrections from the \( \sigma \)-model instantons. The type B algebra depends on the complex structure but receives no quantum correction. Then the symmetry expressed by (1) implies that there is an isomorphism between the type A algebra of \( M \) and the type B algebra of \( W \) and vice versa. This turns out to be a very powerful tool for computing the quantum correction appearing in the type A algebra, say, of \( M \). This is the quantum cohomology ring of \( M \).

The type A and B algebras may be described in more geometrical terms as follows. In classical geometry, the (complexified) Kähler moduli of a Calabi-Yau manifold \( M \) can be described in terms of the cohomology group \( H^{3,1}(M, \mathbb{C}) \), regarded as a subspace of the commutative algebra \( \bigoplus_{i=0}^{3} H^{3-i,i}(M, \mathbb{C}) \). On the other hand, the complex structure deformation of the mirror \( W \) may be described by the variation of the Hodge structures, for which the space \( H^3(W, \mathbb{C}) = \bigoplus_{i=0}^{3} H^{3-i,i}(W, \mathbb{C}) \) and its Hodge filtrations play an essential role. When \( M \) is a Calabi-Yau hypersurface in a weighted projective space
the mirror $W$ can be represented by polynomial deformations of the defining equation $P(z)$ of $W$. In fact we have an isomorphism to the Jacobian ring: $\oplus_{i=0}^{3} H^{3-i,i}(W,\mathbb{C}) \cong \mathbb{C}[z_1, \ldots, z_5]/(\partial P(z)/\partial z_i)$. Now mirror symmetry predicts that the quantum cohomology ring of $M$ is given by

$$\oplus_{i=0}^{3} H^{3-i,i}_q(M,\mathbb{C}) \cong \oplus_{i=0}^{3} H^{3-i,i}(W_\psi,\mathbb{C}),$$

where $\psi = (\psi_1, \ldots, \psi_{n^{2,1}(W)})$ are parameters in the polynomial deformation of the mirror $W$, and the right hand side is given the structure of the Jacobian ring $J_\psi$ above.

In their original work, Candelas et al determined the quantum cohomology ring starting from the Jacobian ring $J_\psi$ for the mirror of a quintic hypersurface in $\mathbb{P}^4$. In general we may define the quantum cohomology ring through a specific basis (flat coordinate) of the Jacobian ring, $\{1, O_a, O_b^O, O^{(3)} \}$ ($a, b = 1, \ldots, h^{2,1}(W)$), where $O_a, O_b^O$ and $O^{(3)}$ represent the elements with charge one, two and three, respectively, in the Jacobian ring. The flat coordinate is characterized by the properties that $O_a O_b^O \equiv \delta_a^b O^{(3)}$, $O_a O^{(3)} \equiv 0$ in the Jacobian ring. Then the relations $O_a O_b^O \equiv \sum_{c} K_{a b c}(t(\psi)) O^{c}$ determines the coupling as a function in flat coordinates. We may compute the quantum cohomology ring at the so-called large radius limit where we have non-trivial $q$-expansion for the coupling:

$$K_{a b c}(t) \sim \int_M h_a \wedge h_b \wedge h_c + \sum N_{i_1 i_2 \cdots i_n} q_1^{i_1} q_2^{i_2} \cdots q_n^{i_n}.$$

The first term in the expansion are the classical intersection numbers for the elements $h_a \in H^{1,1}(M,\mathbb{Z})$ ($a = 1, \ldots, h^{1,1}(M)$) and $q_i := e^{2\pi i t_i}$ ($i = 1, \ldots, n = h^{2,1}(W)$). It has been verified in numerous examples that the numbers $N_{i_1 i_2 \cdots i_n}$ are integers, possibly negative, which "count" the instantons appearing in the quantum correction.

**3 Gauss-Manin system and flat coordinates**

In this section we will characterize the flat coordinates through the analysis of the Gauss-Manin system.

Let us consider the quintic hypersurface $M$ in $\mathbb{P}^4$ and its mirror $W$ obtained by orbifoldizing $M$ by $G = (\mathbb{Z}_5)^3$. We take the defining equation for $W_\psi$ as $P_\psi = \frac{1}{5} z_1^5 + \cdots + \frac{1}{5} z_5^5 - \psi \phi$ with $\phi = z_1 z_2 z_3 z_4 z_5$. The deformed Jacobian ring $J_\psi$ is given by

$$J_\psi := C[z_1, \ldots, z_5]^G/(\partial P_\psi/\partial z_i) \cap C[z_1, \ldots, z_5]^G.$$


We fix a basis of $J_\psi$ as \{\(\varphi(0), \varphi(1), \varphi(2), \varphi(3)\)\} indicating the charges by the superscripts (\(\varphi^{(i)}\) refers to the element with charge \(i\) or homogeneous degree \(i\)). Then the Gauss-Manin system is a set of first order differential equations satisfied by the period integrals. They are given by

$$w_j = \frac{i}{1} \int_{\gamma_j} \text{Res}_{P_\psi} \left( \frac{\varphi^{(i)}}{P_\psi + 1} \right) ,$$

where \(\gamma_j\)'s are cycles in \(H_3(W_\psi, \mathbb{Z})\) and \(d\mu := \sum_k (-1)^{k+1} z_k dz_1 \wedge \cdots \wedge dz_k \wedge \cdots \wedge dz_5\). We can derive, using the reduction pole order argument, the differential equation satisfied by (5) as

$$\frac{\partial}{\partial \psi} w = \begin{pmatrix}
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 \\
 \frac{\varphi}{1-\varphi} & \frac{15\varphi^2}{1-\varphi} & \frac{25\varphi^3}{1-\varphi} & \frac{10\varphi^4}{1-\varphi}
\end{pmatrix} w ,$$

(6)

We notice that we can do a degree-preserving change of basis on the Jacobian ring by \(\varphi^{(i)} \rightarrow Q_i(\psi) \varphi^{(i)} + \sum_j R_{ij}(z, \psi) \partial P_\psi / \partial z_j\), where \(Q_i(\psi)\) and \(R_{ij}(z, \psi)\) are arbitrary. We may also change the normalization of the defining equation by an arbitrary function \(r(\psi): P_\psi \rightarrow r(\psi) P_\psi\). It is easy to see that these changes result in a new period \(v\) related to the original one by \(w = M(\psi)v\) with lower triangular matrix \(M(\psi)\). By a change of local parameter \(\psi = \psi(t)\) to the flat coordinate \(t\), the Gauss-Manin system (6) becomes

$$\frac{\partial}{\partial t} v = \left( \frac{\partial r}{\partial \psi} \right) \left( M^{-1} G_\psi M - M^{-1} \frac{\partial}{\partial \psi} M \right) v + \left( K_{\psi}(t) \right) v ,$$

(7)

where \(G_\psi\) is the connection matrix in the right hand side of (6) and \(K_{\psi}(t)\) is a function which will be determined by this form of the Gauss-Manin system. We can determine the matrix \(M(\psi)\) explicitly as

$$M(\psi) = \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 \\
 r' s + \frac{r'}{r} s & \frac{1}{s} & \frac{C}{1-\varphi} & 0
\end{pmatrix} ,$$

(8)
where the functions \( r = r(\psi) \) and \( s = s(\psi) \) satisfy differential equations (, with \( r' = \partial r/\partial \psi \) e.t.c.,)

\[
\begin{align*}
\rho_{\mu}^{r'''} &= -\frac{10 \psi^4}{1-\psi^4} \rho_{\mu}^{r''} - \frac{25 \psi^3}{1-\psi^4} \rho_{\mu}^{r''} - \frac{15 \psi^2}{1-\psi^4} \rho_{\mu}^{r'} - \frac{\psi}{r} \rho_{\mu}^r = 0 \\
\phi_{\nu}'' - \frac{5 \psi^4}{1-\psi^4} \phi_{\nu}' - \left( \frac{5 \psi^3}{1-\psi^4} + \frac{5 \psi^4}{1-\psi^4} \rho_{\mu}^{r'} + \frac{2 \phi_{\nu}}{r} + \frac{3 \phi_{\nu}'}{r} \right) \phi_{\nu} = 0.
\end{align*}
\]

The differential equation for \( r(\psi) \) coincides with the Picard-Fuchs equation. The relation (8) determines the coupling and the flat coordinate \( t = t(\psi) \) as follows;

\[
K_{\mu}(t) = \frac{1}{r^2} \frac{C}{1-\psi^5} \left( \frac{\partial \psi}{\partial t} \right)^3, \quad \frac{\partial \psi}{\partial t} = \frac{s(\psi)}{r(\psi)}.
\]

The Picard-Fuchs equation satisfied by \( r(\psi) \) in (8) can be arranged to

\[
\{ \theta_1^2 - 5z(5\theta + 4)(5\theta + 3)(5\theta + 2)(5\theta + 1) \} w(z) = 0,
\]

where \( z = \frac{1}{(5\psi)^2} \) and \( w(z) = -5\psi r(\psi) \). It is evident that the indices of the Picard-Fuchs equation are zero at the large radius limit \( z = 0 \), and the monodromy becomes maximally unipotent. All the solutions can be determined by the standard Frobenius method starting from the series \( w_0(z, \rho) := \sum \frac{\Gamma(5(n+\rho)+1)}{\Gamma(n+\rho+1)^2} z^{n+\rho} \). One can verify that the ratio \( t(z) = \frac{1}{2\pi i} \frac{w_1(z)}{w_0(z)} \), where \( w_0(z) := w_0(z, \rho) \big|_{\rho=0} \) and \( w_1(z) := \frac{\partial}{\partial \rho} w_0(z, \rho) \big|_{\rho=0} \), coincides with the flat coordinate.

In fact the functions \( r(\psi) = -\frac{1}{\psi} w_0(z) \) and \( s(\psi) = -\frac{1}{\psi} w_0(z) \frac{\partial t}{\partial \psi} \) solve the equations (8). Because of the behavior \( t \sim \frac{1}{2\pi i} \log(z) \) near the large radius limit, we obtain the desired \( q \)-expansion (3) with \( q = e^{2\pi it} \) and \( C = \frac{5^2}{(2\pi i)^5} \).

We can read off the prepotential \( F(t) \) for the quantum coupling \( K_{\mu}(t) \) from the form of the Gauss-Manin system in the flat coordinate (8). To see this note that the first order system (8) is equivalent to \( \frac{1}{K_{\mu}(t)} \frac{\partial^2}{\partial \psi^2} \theta_1^2 \partial^2_\psi v^{(0)} = 0 \), where \( v^{(0)} \) is the first row of the periods \( v = (v^{(1)}) \). Then it is easy to see that the following \( v^{(0)} \) s constitute the solutions; \( v_0^{(0)} = 1, v_1^{(0)} = t, \partial^2_\psi v_2^{(0)} = K_{\mu}, \partial^2_\psi v_3^{(0)} = -t K_{\mu} \). These relations are sufficient to determine the prepotential

\[
F(t) := \frac{1}{2} \left( v_0^{(0)} v_3^{(0)} + v_1^{(0)} v_2^{(0)} \right),
\]

such that \( K_{\mu}(t) = \partial^2_\psi F(t) \). Since we have a relation \( v_j^{(0)} = \frac{1}{r(\psi)} w^{(0)}_j \) from the matrix \( M(\psi) \) in (8), we can write the prepotential in terms of the solutions of the Picard-Fuchs equation.
4 GKZ hypergeometric system and the flat coordinate

In this section, we will consider a Calabi-Yau hypersurface $X_d(w)$ in a weighted projective space $\mathbf{P}^4(w)$ (where $d$ represents the homogeneous degree of the surface). We will consider two a priori different objects: the intersection ring for $X_d(w)$ and the GKZ hypergeometric system for the periods of $X_d^*(w)$. We will find a close relationship between the two.

4.1 Intersection ring

According to Batyrev, we can construct the mirror pairing $(M,W) = (X_d(w), X_d^*(w))$ of Calabi-Yau manifolds starting from a pairing of the reflexive polyhedra ($\Delta(w), \Delta^*(w)$) in $\mathbf{R}^4$. Here the polyhedron $\Delta(w)$ is defined through the Newton polyhedron of the defining equation (the potential) of $X_d(w)$ and may be written as

$$\Delta(w) = \text{Conv.} \left\{ \{ x \in \mathbf{Z}^5 | \sum w_i x_i = 0, (x_i \geq -1) \} \right\}. \quad (13)$$

We see that all vertices of this polyhedron are integral by definition and the origin is the only interior integral point in $\Delta(w)$ (in fact this is the defining property of the reflexive polyhedron). The (polar) dual of $\Delta(w)$ is defined by

$$\Delta^*(w) := \{ y \in \Lambda(w)_\mathbf{R}^* | \langle y, x \rangle \geq -1 \}$$

and turns out to be reflexive if $\Delta(w)$ is reflexive. Here $\Lambda(w)_\mathbf{R}$ is the scalar extension of the lattice $\Lambda(w) = \{ x \in \mathbf{Z}^5 | \sum w_i x_i = 0 \}$. The cones over the faces of the respective polyhedra define the complete fans $\Sigma(\Delta(w))$ and $\Sigma(\Delta^*(w))$. These fans define compact toric varieties $\mathbf{P}_{\Sigma(\Delta(w))}$ and $\mathbf{P}_{\Sigma(\Delta^*(w))}$. Then the Calabi-Yau hypersurfaces $X_d(w)$ and $X_d^*(w)$ are crepant resolutions of zero loci of certain Laurent polynomials in the ambient spaces $\mathbf{P}_{\Sigma(\Delta(w))}$ and $\mathbf{P}_{\Sigma(\Delta^*(w))}$, respectively.

If all singularities of the ambient space $\mathbf{P}_{\Sigma(\Delta^*(w))}$ are Gorenstein, the subdivision of $\Sigma(\Delta^*(w))$ using all integral points on the faces makes $\Sigma(\Delta^*(w))$ regular and results in the smooth ambient space $\mathbf{P}_{\Sigma(\Delta^*(w))}$. This is the case for the models of type I and II in the classification given in refs. For simplicity, we will restrict our attention to this case.

The classical cohomology $\oplus H^{i,i}(X_d(w), \mathbf{Z})$ may be understood as the restrictions of the cohomology of the ambient space $\mathbf{P}_{\Sigma(\Delta^*(w))}$. The cohomology ring of the smooth toric variety is generated by the toric divisors associated to each one dimensional cones. We denote the integral points in $\Delta^*$ as $\nu_1^*, \ldots, \nu_p^*$ and the corresponding divisors as $D_1, \ldots, D_p$. Then the cohomology ring called an intersection ring, of the toric variety can be described by

$$A^*(\mathbf{P}_{\Sigma(\Delta^*(w))}, \mathbf{Z}) = \mathbf{Z}[D_1, \ldots, D_p]/(SR_{\Sigma(\Delta^*(w))} + I) \quad , \quad (14)$$
where $SR_{\Sigma(\Delta^*(w))}$ is the Stanley-Reisner ideal for the fan $\Sigma(\Delta^*(w))$ and $I$ represents the divisors of the rational functions. These two ideals are generated, respectively, by

\[ D_1 \cdots D_k \text{ for } \nu_i^* \text{ not in a cone of } \Sigma(\Delta^*(w)), \]
\[ \sum_{i=1}^p \langle u_i, \nu_i^* \rangle D_i \text{ for } u \in \mathbb{Z}^4. \]

The degree four element in $A^*(P_{\Sigma(\Delta^*(w))}, \mathbb{Z})$ should give the 'volume' form of $P_{\Sigma(\Delta^*(w))}$. Since we know that the total Chern class is $c(T_{P_{\Sigma(\Delta^*(w))}}) = \prod_{i=1}^p (1 + D_i)$, and $\chi(P_{\Sigma(\Delta^*(w))}) = \# \text{ of } 4\text{-dimensional cones in } \Sigma(\Delta^*(w))$, we can normalize the volume form using the relation

\[ \int_{P_{\Sigma(\Delta^*(w))}} c(T_{P_{\Sigma(\Delta^*(w))}}) = \chi. \]

With this normalization, we may define the intersection couplings

\[ \langle D_{i_1} D_{i_2} D_{i_3} D_{i_4} \rangle := \int_{P_{\Sigma(\Delta^*(w))}} J_{D_{i_1}} J_{D_{i_2}} J_{D_{i_3}} J_{D_{i_4}}. \]

Since the divisor of the hypersurface has an expression $[X_d(w)] = D_1 + \cdots + D_p$, the intersection ring of the Calabi-Yau hypersurface $X_d(w)$ may be described by $A^*(X_d(w), \mathbb{Z}) = A^*(P_{\Sigma(\Delta^*(w))}, \mathbb{Z})/Ann(D_1 + \cdots + D_p)$, where $Ann(x)$ consists of those elements which vanish after multiplication by $x$ in the ring $A^*(P_{\Sigma(\Delta^*(w))}, \mathbb{Z})$. Then the intersection coupling of $X_d(w)$ may be written as $\langle D_{i_1} D_{i_2} D_{i_3} [X_d(w)] \rangle$.

### 4.2 GKZ hypergeometric system

As remarked by Batyrev in $\mathbb{P}^4$, a period integral of $X_d(w)$ satisfies the GKZ hypergeometric system. This system is defined by the integral points $A = \{ \tilde{\nu}_i^* = (1, \nu_i^*) \}_{i=0, \ldots, p}$ in $\Delta^*$ placed on a hyperplane in $\mathbb{R}^5$, and an exponent $\beta \in \mathbb{R}^5$. The set $A$ is not linearly independent but has affine relations expressed by a lattice $L = \{ (l_0, \cdots, l_p) \in \mathbb{Z}^{p+1} \mid \sum_i l_i \tilde{\nu}_i^* = 0 \}$. Then the GKZ hypergeometric system for the period integral $\Pi(a)$ is given by

\[ D_l \Pi(a) = 0 \quad (l \in L), \quad Z_u \Pi(a) = 0 \quad (u \in \mathbb{Z}^5), \]

where

\[ D_l = \prod_{i, \geq 0} \left( \frac{\partial}{\partial a_i} \right)^{l_i} - \prod_{i, \leq 0} \left( \frac{\partial}{\partial a_i} \right)^{l_i}, \quad Z_u = \sum_i \langle u_i, \nu_i^* \rangle \theta_{a_i} - \beta_u, \]

where $\beta_u = \langle u, \beta \rangle$ with $\beta = (-1, 0, 0, 0, 0)$. This system has a formal solution

\[ \Pi(a, \gamma) = \sum_{l \in L} \frac{1}{\prod_{0 \leq i \leq p} \Gamma(\bar{l}_i + \gamma_i + 1)} a^{l+\gamma}, \]
for $\gamma \in \mathbb{R}^{p+1}$ satisfying $\sum_i \gamma_i \bar{\nu}^*_i = \beta$. It is shown in [4] that convergent power
series solutions can be constructed from this formal solution for each regular triangulation of the polyhedron $P = \text{Conv.}\{(0, \bar{\nu}^*_0, \ldots, \bar{\nu}^*_p)\}$. The regular triangulation determines local variables $x_k = a^{(k)}$ through a compatible basis $\{l^{(1)}, \ldots, l^{(p-d)}\}$ of the lattice $L$.

The $q$-series expansion of the quantum coupling [3] is valid at the large radius limit where the monodromy becomes maximally unipotent. There is in fact a regular triangulation which realizes this property. It is a triangulation for which we have only one power series solution and all other solutions have logarithmic singularities. This regular triangulation has the property that all 4-simplices contain the point $\bar{\nu}^*_0 (\text{the origin in } \Delta^*)$ and have volume one (where the volume should be normalized so that the unit 'cubic' in $n$-dimensions has volume $n!$). In [4], we called the regular triangulation with this property the maximal triangulation. Note that the maximal triangulation is a triangulation we can associate with the complete fan $\Sigma(\Delta^*(w))$ and thus with the desingularization of the ambient space $P_{\Sigma(\Delta^*(w))}$.

The relation between the maximal triangulation $T_0$ of the polyhedron $P$ and the ambient space $P_{\Sigma(\Delta^*(w))}$ has an important implication on the quantum cohomology. To see this, let us write the power series solution $w_0(x, \rho) = a_0 \Pi(a, \gamma)$ in terms of the integral basis $\{l^{(1)}, \ldots, l^{(p-n)}\}$ of $L$ compatible with $T_0$, where we define the indices $\rho$ by $\gamma = \sum_k \rho_k l^{(k)} + (-1, 0, \ldots, 0)$ and $x_k = (-1)^{\rho_k} a^{(k)}$. Explicitly, this series has the form

$$w_0(x, \rho) = \sum_{m_1, \ldots, m_k \geq 0} \frac{\Gamma(-\sum_{i \leq k} (m_i + \rho_i) l^{(k)}_i + 1)}{\prod_{1 \leq i \leq p} \Gamma(\sum_{i \leq k} (m_i + \rho_i) l^{(k)}_i + 1)} x^{m+\rho}.$$  \hspace{1cm} (20)

Because of the difference of the 'gauge' factor $a_0$ between $w_0(x, \rho)$ and $\Pi(a, \gamma)$, the first order differential operator $\mathcal{Z}_u$ takes the form $\mathcal{Z}_u = \sum_{k} \langle u, \bar{\nu}^*_k \rangle \theta_{a_k}$ for $w_0(x, \rho)$. In this form, this differential operator coincides with the linear relation ii) in (13) under the identification $\theta_{a_i} \leftrightarrow D_i (i = 1, \ldots, p)$. Under this identification, the generators in i) of (13) also correspond to the leading terms of certain differential operators $\mathcal{D}_l$ as follows. Consider the toric ideal $\langle \mathcal{D}_l \mid l \in L \rangle \subset \mathbb{C}[\frac{\partial}{\partial x_{l_0}}, \ldots, \frac{\partial}{\partial x_{l_p}}]$, construct a Gröbner basis of this ideal (with respect to a term order defined by the maximal triangulation), and consider the leading terms of the basis elements. After a suitable multiplication by a monomial of the form $a^l_+ \text{ or } a^l_-$ with $l = l_+ - l_-$, the leading term of a basis element $\mathcal{D}_l$ becomes the principal part of the operator near $x_k = 0$. The basis elements together determines the local solutions completely. It can be shown that the principal parts of the Gröbner basis elements coincides with
the generators i) in (15) for the Stanley-Reisner ideal $SR_{\Sigma(\Delta^*)}$ under the identification $\theta_n$ with $D_{a_i}^n$.

Now we can determine the solutions with logarithmic singularities from the principal parts of the GKZ system using the Frobenius method. We observe that the solutions are given by

$$w_0(x,0), \partial_{\rho_i} w_0(x,0), \sum_{k,l} C_{ijkl} \partial_{\rho_k} \partial_{\rho_l} w_0(x,0), \sum_{i,j,k,l} C_{ijkl} \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k} \partial_{\rho_l} w_0(x,0),$$

where $C_{ijkl} := \langle \theta_x \theta_y \theta_z \theta_w \rangle$ is the coupling (16) under the identification above and $\partial_{\rho_i} := \frac{1}{2\pi i} \partial_{\rho_i}$.

It has been reported that our GKZ system is reducible and we can extract canonically the solutions to a subsystem by considering the restriction of the intersection ring to the hypersurfaces $X_d^i(w)$. That is, we use in (21) the cubic coupling $K^{cl}_{ijk} = \langle \theta_x \theta_y \theta_z \theta_w [X_d^i(w)] \rangle$ instead of $C_{ijkl}$ for the ambient space. It has been verified experimentally that the system with these solutions coincides with the Picard-Fuchs equations derived from the reduction argument of Dwork-Griffiths-Katz. In terms of the local solutions near the large radius limit, we can write the prepotential in a concise form

$$F(t) = \frac{1}{6} \sum_{ijk} K^{cl}_{ijk} t_i t_j t_k - \frac{1}{6} \sum_{i} K^{cl}_{i} t_i - \frac{16}{3} \pi^2 \chi(X_d^i(w)) + O(q),$$

where $K^{cl}_{i} := \partial_{\rho_i}|_{w_0(x,0)}$, $D^{(1)}_i := \partial_{\rho_i}$, $D^{(2)}_i := \sum_{j,k} K_{ijk}^{cl} \partial_{\rho_j} \partial_{\rho_k}$, $D^{(3)} := -\frac{1}{6} \sum_{ijk} K^{cl}_{ijk} \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k}$.

The mirror map is then defined by $t_k := \frac{D^{(1)}_k w_0(x,0)}{w_0(x,0)}$. It turns out that the asymptotic form of the prepotential is $F(t) = \frac{1}{6} \sum_{ijk} K^{cl}_{ijk} t_i t_j t_k - \frac{1}{6} \sum_{i} K^{cl}_{i} t_i - i \frac{c_2}{10} \chi(X_d^i(w)) + O(q)$.

5 Summary

We have discussed the flat coordinates of the Gauss-Manin system in the context of the mirror symmetry. In these coordinates, we can compute the quantum cohomology ring $\oplus_i H_q^{i,i}(M, \mathbb{C})$ using the Jacobian ring $J_q$ of the mirror.

In case of the toric realization of the mirror symmetry, we can trace the structure of the quantum cohomology ring to the logarithmic solutions to the GKZ hypergeometric system near the large radius limit. The closed formula for the prepotential is written completely in terms of the data of the reflexive polyhedron $\Delta^*$. It is an interesting and important problem to relate the $q$-series expansion of the prepotential to the axiomatic definition of the quantum cohomology ring.
We thank S.T. Yau for his collaboration. S.H. is supported in part by Grant-in-Aid for Science Research on Priority Area 231 "Infinite Analysis".

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