WEIGHTED A PRIORI ESTIMATES FOR THE SOLUTION OF THE HOMOGENEOUS DIRICHLET PROBLEM FOR THE POWERS OF THE LAPLACIAN OPERATOR.

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Abstract. Let $u$ be a weak solution of $(-\Delta)^m u = f$ with Dirichlet boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. Then, the main goal of this paper is to prove the following a priori estimate:

$$\|u\|_{W^{2m,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)},$$

where $\omega$ is a weight in the Muckenhoupt class $A_p$.

1. Introduction

We will use the standard notation for Sobolev spaces and for derivatives, namely, if $\alpha$ is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ we denote $|\alpha| = \sum_{j=1}^n \alpha_j$, $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and

$$W^{k,p}(\Omega) = \{ v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \quad \forall |\alpha| \leq k \}.$$ 

For $u \in W^{k,p}(\Omega)$, its norm is given by

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}.$$ 

We consider the homogeneous problem

(1.1) \[ \begin{align*} (-\Delta)^m u &= f \quad \text{in } \Omega, \\ \left( \frac{\partial}{\partial \nu} \right)^j u &= 0 \quad \text{in } \partial \Omega \quad 0 \leq j \leq m - 1, \end{align*} \]

where $\frac{\partial}{\partial \nu}$ is the normal derivative.

In the classic paper [1], the authors obtained a priori estimates for solutions of (1.1) for smooth domain $\Omega$ given by

$$\|u\|_{W^{2m,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$
Where a key tool to prove those estimates was the Calderón-Zygmund theory for singular integral operators.

On the other hand, after the pioneering work of Muckenhoupt [7], a lot of work on continuity in weighted norms has been developed. In particular, weighted estimates for a wide class of singular integral operators has been obtained for weights in the class of Muckenhoupt $A_p$. Therefore, it is a natural question whether analogous weighted a priori estimates can be proved for the derivatives of solutions of elliptic equations.

For the Laplace equation ($m = 1$), it was proved in [5] that for a weight $\omega$ belonging to the Muckenhoupt class $A_p$

$$\|u\|_{W^{2,p}_p(\Omega)} \leq C \|f\|_{L^p_\omega(\Omega)}$$

on a bounded domain $\Omega$ with $\partial \Omega \in C^2$.

The goal of this paper is to extend the results of [5] for powers of the Laplacian operator with homogeneous Dirichlet boundary conditions, i.e. it is to prove that

(1.2)  \[ \|u\|_{W^{2m,p}_p(\Omega)} \leq C \|f\|_{L^p_\omega(\Omega)}, \]

for $\omega \in A_p$, where the constant $C$ depends on $\Omega$, $m$, $n$ and the weight $\omega$.

The main ideas for the proof of these estimates are similar to those given in [5]. However, non trivial technical modifications are needed because, for $m \geq 2$, the Green function is not positive in general and therefore, we cannot apply the maximum principle.

2. Preliminaries

We consider the problem in a bounded domain $\Omega$ with $\partial \Omega \in C^{6m+4}$ for $n = 2$ and $\partial \Omega \in C^{5m+2}$ for $n > 2$ (the regularity on the boundary is necessary in order to use the results of the Green function given in [6]).
The solution of (1.1) is given by

\[ (2.1) \quad u(x) = \int_{\Omega} G_m(x, y) f(y) \, dy \]

where \( G_m(x, y) \) is the Green function of the operator \((-\Delta)^m\) in \(\Omega\) which can be written as

\[ (2.2) \quad G_m(x, y) = \Gamma(x - y) + h(x, y) \]

where \(\Gamma(x - y)\) is a fundamental solution and \(h(x, y)\) satisfies

\[
\begin{cases}
(-\Delta)^m h(x, y) = 0 & x \in \Omega \\
\left(\frac{\partial}{\partial n}\right)^j h(x, y) = -\left(\frac{\partial}{\partial n}\right)^j \Gamma(x - y) & x \in \partial\Omega \quad 0 \leq j \leq m - 1
\end{cases}
\]

for each fixed \(y \in \Omega\).

Then

\[ (2.3) \quad h(x, y) = -\sum_{j=0}^{m-1} \int_{\partial\Omega} K_j(y, P) \left(\frac{\partial}{\partial n}\right)^j \Gamma(P - x) \, dS \]

where \(K_j(y, P)\) are the Poisson kernels and \(dS\) denotes the surface measure on \(\partial\Omega\).

We recall that any fundamental solution associated to (1.1) is smooth away from the origin and it is homogeneous of degree \(2m - n\) if \(n\) is odd or if \(2m < n\) and the logarithmic function appears if \(n\) is even and \(2m \geq n\). However, in both cases we have the known estimates of the Green function \(G_m(x, y)\) and the Poisson kernels \(K_j(x, y)\). In what follows the letter \(C\) will denote a generic constant not necessarily the same at each occurrence.

\[ (2.4) \quad |D_\alpha x G_m(x, y)| \leq C \quad \text{for } |\alpha| < 2m - n, \]

\[ (2.5) \quad |D_\alpha x G_m(x, y)| \leq C \log \left(\frac{2 \text{diam}(\Omega)}{|x - y|}\right) \quad \text{for } |\alpha| = 2m - n, \]

\[ (2.6) \quad |D_\alpha x G_m(x, y)| \leq C |x - y|^{2m-n-|\alpha|} \quad \text{for } |\alpha| > 2m - n, \]

\[ (2.7) \quad |D_\alpha x G_m(x, y)| \leq C \frac{1}{|x - y|^n} \min \left\{1, \frac{d(y)}{|x - y|}\right\}^m \quad \text{for } |\alpha| = 2m, \]
(2.8) \[ |K_j(x, y)| \leq C \frac{d(x)^m}{|x - y|^{n-j+m-1}} \quad \text{for } 0 \leq j \leq m - 1, \]
where \( d(x) := \text{dist}(x, \partial \Omega) \) (see [6] for (2.4), (2.5) and (2.6) and [4] for (2.7) and (2.8)).

3. The estimates for the derivatives of \( u \)

In this section we state pointwise estimates for the first \( 2m - 1 \) derivatives of the function \( u \) and a weak estimate for the \( 2m \) derivative. These estimates will be allow to proof the main result of this work.

Lemma 3.1. Let \( u(x) \) be solution of the problem (1.1). Then, for \( |\alpha| \leq 2m - 1 \) we have

\[ |D^{\alpha}_x u(x)| \leq C Mf(x), \]
where \( Mf(x) \) is the usual Hardy- Littlewood maximal function of \( f \).

Proof:

\[ |D^{\alpha}_x u(x)| \leq \int_\Omega |D^{\alpha} G_m(x, y)||f(y)| dy \]

\[ \leq C \int_\Omega \frac{|f(y)|}{|x - y|^{n-1}} dy \leq C Mf(x), \]
by (2.4), if \( 2m - n + 1 \leq |\alpha| \leq 2m - 1 \) and by (2.5) and (2.6), if \( |\alpha| \leq 2m - n \).

\[ \square \]

Proposition 3.2. Given two measurable functions \( f \) and \( g \) in \( \Omega \), for \( |\alpha| = 2m \) we have that

\[ \int_D |D^{\alpha}_x G_m(x, y) f(y) g(x)| dy dx \leq C \left( \int_\Omega Mf(x) |g(x)| dx + \int_\Omega Mg(y) |f(y)| dy \right), \]
where \( D := \{(x, y) \in \Omega \times \Omega : |x - y| > d(x)\} \).
**Proof:** We write $D = D_1 \cup D_2$, where

$$D_1 = \{(x, y) \in D : d(y) \leq 2d(x)\} \quad \text{and} \quad D_2 = \{(x, y) \in D : d(y) > 2d(x)\}.$$ 

Then, using (2.7) we have

$$\int_D |D_x G_m(x, y) f(y) g(x)| \, dy \, dx \leq \int_D \frac{d(y)^m}{|x - y|^{n+m}} |f(y)| |g(x)| \, dy \, dx$$

$$\leq 2^n \int_{D_1} \frac{d(x)^m}{|x - y|^{n+m}} |f(y)| |g(x)| \, dy \, dx$$

(3.1)

$$+ \int_{D_2} \frac{d(x)^m}{|x - y|^{n+m}} |f(y)| |g(x)| \, dy \, dx = I + II.$$

Calling $\Omega_k(x) = \{z \in \Omega : 2^k d(x) \leq |x - z| < 2^{k+1} d(x)\},$

$$\int_{D_1} \frac{d(x)^m}{|x - y|^{n+m}} |f(y)| |g(x)| \, dy \, dx \leq \int \sum_{k=1}^\infty \int_{\Omega_k(x)} \frac{d(x)}{|x - y|^{n+1}} |f(y)| \, dy \, g(x) \, dx$$

$$= \int A(x) |g(x)| \, dx$$

with

$$A(x) \leq \sum_{k=1}^\infty \int_{|x - y| < 2^{k+1} d(x)} \frac{d(x)}{|x - y|^{n+1}} |f(y)| \, dy \leq 2^n \sum_{k=1}^\infty \frac{1}{2^k} Mf(x) = 2^n Mf(x).$$

In order to estimate the term II in (3.1), we first observe that for $(x, y) \in D_2$, we have that $|x - y| \geq \frac{1}{2} d(y)$. Then

$$\int_{D_2} \frac{d(y)^m}{|x - y|^{n+m}} |f(y)| |g(x)| \, dy \, dx \leq C \int \sum_{k=1}^\infty \int_{\Omega_{k-1}(y)} \frac{d(y)}{|x - y|^{n+1}} |g(x)| \, dx \, |f(y)| \, dy$$

$$= \int B(y) |f(y)| \, dy$$

and therefore, by the same arguments used before we have that

$$B(y) \leq 2^{n+1} Mg(y)$$
and the Proposition is proved.

In order to see how to estimate in $\Omega \setminus D$, we consider separately the function $h$ and $\Gamma$ involved in $G_m$.

**Proposition 3.3.** If $|\alpha| \geq 2m - n + 1$, there exists a constant $C$ such that

\[
|D^\alpha h(x, y)| \leq C d(x)^{2m-n-|\alpha|}
\]

for $|x - y| \leq d(x)$.

**Proof:** In view of (2.3) we must find estimates for $D_x^\alpha \left( \frac{\partial}{\partial y} \right)^j \Gamma(P - x)$ and $K_j(y, P)$.

From the general properties of the fundamental solution $\Gamma(x - y)$ we have that

\[
|D_x^\alpha \left( \frac{\partial}{\partial y} \right)^j \Gamma(P - x)| \leq C |P - x|^{2m-n-|\alpha|-j}
\]

for $|\alpha| + j \geq 2m - n + 1$, and for $0 \leq j \leq m - 1$, by (2.8) we have that

\[
|K_j(y, P)| \leq C \frac{d(y)^m}{|y - P|^{n-j+m-1}}
\]

for $y \in \Omega$ and $P \in \partial \Omega$.

Then by (3.3), (3.4) and the fact that if $|x - y| \leq d(x)$ then $d(y) < 2d(x)$, we have for $|\alpha| + j \geq 2m - n + 1$

\[
|D_x^\alpha h(x, y)| \leq C \sum_{j=0}^{m-1} \int_{\partial \Omega} \frac{d(y)^m}{|y - P|^{n-1+m-j}} |P - x|^{2m-n-|\alpha|-j} dS
\]

\[
\leq C d(x)^{2m-n-|\alpha|} \sum_{j=0}^{m-1} \int_{\partial \Omega} \frac{d(y)^{m-j}}{|y - P|^{n-1+m-j}} dS.
\]

In order to see that each integral is finite we write $\partial \Omega = F_1 \cup F_2$, with

$F_1 = \{ P \in \partial \Omega : |P_0 - P| > 2d(y) \}$ and $F_2 = \{ P \in \partial \Omega : |P_0 - P| \leq 2d(y) \}$,

where $P_0 \in \partial \Omega$ is that $|y - P_0| = d(y)$. And now, the convergence of these integrals follow in a standard way.
It follows from the previous Proposition that for each $x \in \Omega$ and $|\alpha| \geq 2m - n + 1$ we have that $D^\alpha_\Omega h(x, y)$ is bounded uniformly in a neighborhood of $x$ and so

$$(3.5) \quad D^\alpha_\Omega \int_\Omega h(x, y) f(y) \, dy = \int_\Omega D^\alpha_\Omega h(x, y) f(y) \, dy.$$  

On the other hand, although $D^\alpha_\Omega \Gamma$ is a singular kernel for $|\alpha| = 2m$, taking $\beta$ such that $|\beta| = 2m - 1$, we have that

$$(3.6) \quad D_x, \int_\Omega D^\beta_\Omega \Gamma(x - y) f(y) \, dy = Kf(x) + c(x)f(x)$$

where $c$ is a bounded function and $K$ is a Calderón - Zygmund operator given by

$$(3.7) \quad Kf(x) = \lim_{\epsilon \to 0} K_\epsilon f(x), \quad \text{with} \quad K_\epsilon f(x) = \int_{|x - y| > \epsilon} D^\alpha_\Omega \Gamma(x - y) f(y) \, dy.$$  

Here and in what follows we consider $f$ defined in $\mathbb{R}^n$ extending the original $f$ by zero.

Now we are in conditions to give the following estimate:

**Theorem 3.4.** Given $g$ a measurable function and $|\alpha| = 2m$. Then there exists a constant $C$ depending only on $n$, $m$ and $\Omega$ such that, for any $x \in \Omega$,

$$\int_\Omega |D^\alpha_\Omega u(x) g(x)| \, dx \leq C \left( \int_\Omega \widetilde{K} f(x) |g(x)| \, dx + \int_\Omega M f(x) |g(x)| \, dx + \int_\Omega M g(y) |f(y)| \, dy + \int_\Omega |g(x)| \, dx \right)$$

where $\widetilde{K} f(x) = \sup_{\epsilon > 0} |K_\epsilon f(x)|$. 

Proof: Using the representation formula for $u$, by (3.5), (3.6) and (3.7) we have that

$$D^a_x u(x) = \lim_{\epsilon \to 0} \int_{\epsilon < |x-y| \leq d(x)} D^a_x \Gamma(x-y) f(y) \, dy + c(x)f(x)$$

$$+ \int_{|x-y| \leq d(x)} D^a_x h(x, y) f(y) \, dy + \int_{|x-y| > d(x)} D^a_x G(x, y) f(y) \, dy$$

(3.8)

$$=: I + II + III + IV.$$

By the results given above, for $I$, $II$ and $III$ we have pointwise estimates, and obtain (in the same way that in [5]) that

$$|I + II + III| \leq C \left( \tilde{K} f(x) + |f(x)| + M f(x) \right).$$

However, for $IV$ we have just a weak estimate. Indeed, for the Proposition 3.2 we have

$$\int_{\Omega} |IV| |g(x)| \, dx \leq C \left( \int_{\Omega} Mf(x) |g(x)| \, dx + \int_{\Omega} Mg(y) |f(y)| \, dy \right)$$

and the Theorem is proved.

□

4. Main result

We can now state and prove our main result. First we recall the definition of the $A_p$ class for $1 < p < \infty$. A non-negative locally integrable function $\omega$ belongs to $A_p$ if there exists a constant $C$ such that

$$\left( \frac{1}{|Q|} \int_Q \omega(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C$$

for all cube $Q \subset \mathbb{R}^n$. 
For any weight \( \omega \), \( L^p_\omega(\Omega) \) is the space of measurable functions \( f \) defined in \( \Omega \) such that
\[
\|f\|_{L^p_\omega(\Omega)} = \left( \int_{\Omega} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty
\]
and \( W^{k,p}_\omega(\Omega) \) is the space of functions such that
\[
\|f\|_{W^{k,p}_\omega(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p_\omega(\Omega)}^p \right)^{1/p} < \infty.
\]

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain such that \( \partial \Omega \) is of class \( C^{6m+4} \) for \( n = 2 \) and \( \partial \Omega \) is of class \( C^{5m+2} \) for \( n \geq 2 \). If \( \omega \in A_p \), \( f \in L^p_\omega(\Omega) \) and \( u \) a weak solution of (1.1), then there exists a constant \( C \) depending only on \( n, m, \omega \) and \( \Omega \) such that
\[
\|u\|_{W^{2m,p}_\omega(\Omega)} \leq C \|f\|_{L^p_\omega(\Omega)}.
\]

**Proof:** Since \( M \) is a bounded operator in \( L^p_\omega(\Omega) \), by Lemma 3.1 it follows that
\[
\sum_{|\alpha| \leq 2m-1} \|D^\alpha u\|_{L^p_\omega(\Omega)} \leq C \|f\|_{L^p_\omega(\Omega)}.
\]

Therefore, it only remains to estimate \( \|D^\alpha u\|_{L^p_\omega(\Omega)} \) for \( |\alpha| = 2m \).

Let \( \omega \in A_p \) and \( g(x) := (D^\alpha u(x))^{p-1} \omega(x) \). By Theorem 3.1 we see that
\[
\int_{\Omega} |D^\alpha u(x)|^p \omega(x) \, dx = \int_{\Omega} |D^\alpha u(x)| g(x) \, dx
\]
\[
\leq C \left( \int_{\Omega} K f(x) |g(x)| \, dx + \int_{\Omega} M f(x) |g(x)| \, dx \right)
\]
\[
+ \int_{\Omega} M g(y) |f(y)| \, dy + \int_{\Omega} |f(x)| |g(x)| \, dx \right).
\]

(4.1)
Since $\tilde{K}$ and $M$ are bounded operators in $L^p_\omega(\Omega)$, applying the Hölder inequality, it follows that
\[
\int_\Omega \tilde{K} f(x) |g(x)| \, dx = \int_\Omega \tilde{K} f(x) |g(x)| \frac{1}{\omega(x)^{1/p}} \omega(x)^{1/p} \, dx
\leq \left( \int_\Omega \tilde{K} f(x)^p \omega(x) \, dx \right)^{1/p} \left( \int_\Omega |g(x)|^q \frac{1}{\omega(x)^{q/p}} \, dx \right)^{1/q}
\leq \|f\|_{L^p_\omega(\Omega)} \left( \int_\Omega |g(x)|^q \frac{1}{\omega(x)^{q/p}} \, dx \right)^{1/q},
\]
(4.2)
where $\frac{1}{p} + \frac{1}{q} = 1$.

In the same way, we obtain that
\[
\int_\Omega M f(x) |g(x)| \, dx \leq \|f\|_{L^p_\omega(\Omega)} \left( \int_\Omega |g(x)|^q \frac{1}{\omega(x)^{q/p}} \, dx \right)^{1/q},
\]
(4.3)
and
\[
\int_\Omega |f(x)| |g(x)| \, dx \leq \|f\|_{L^p_\omega(\Omega)} \left( \int_\Omega |g(x)|^q \frac{1}{\omega(x)^{q/p}} \, dx \right)^{1/q}.
\]
(4.4)
For the last term in (4.1), taking into account that $\omega^{-q/p} \in A_q$, we have that
\[
\int_\Omega M g(y) |f(y)| \, dy \leq \|f\|_{L^p_\omega(\Omega)} \left( \int_\Omega M g(y)^q \frac{1}{\omega(y)^{q/p}} \, dy \right)^{1/q}
\leq \|f\|_{L^p_\omega(\Omega)} \left( \int_\Omega |g(x)|^q \frac{1}{\omega(x)^{q/p}} \, dx \right)^{1/q}.
\]
(4.5)
Then, by (4.2), (4.3), (4.4) and (4.5) we have
\[
\|D^p_x u\|_{L^p_\omega(\Omega)} \leq C \|f\|_{L^p_\omega(\Omega)} \left( \int_\Omega |g(x)|^q \frac{1}{\omega(x)^{q/p}} \, dx \right)^{1/q}.
\]
By the definition of $g(x)$,
\[
\left( \int_\Omega |g(x)|^q \frac{1}{\omega(x)^{q/p}} \, dx \right)^{1/q} = \left( \int_\Omega |D^p_x u|^{(p-1)q} \omega(x)^q \frac{1}{\omega(x)^{q/p}} \, dx \right)^{1/q}
= \left( \int_\Omega |D^p_x u|^p \omega(x) \, dx \right)^{1/q} = \|D^p_x u\|_{L^p_\omega(\Omega)}.
\]
Then we obtain
\begin{equation}
\|D^\alpha u\|_{L^p_\omega(\Omega)}^p \leq C \|f\|_{L^p_\omega(\Omega)} \|D^\alpha u\|_{L^p_\omega(\Omega)}^{p/q}
\end{equation}
and the Theorem is proved for \( u \in W^{2m,p}_\omega(\Omega) \).

Finally, we will show that the weak solution \( u \) of (1.1) belong to \( W^{2m,p}_\omega(\Omega) \):

We have that \((-\Delta)^m u = f\), with \( f \in L^p_\omega(\Omega) \), then there exists a sequence \( f_k \in C^\infty(\mathbb{R}^n) \) such that \( \lim_{k \to \infty} f_k = f \) in \( L^p_\omega(\Omega) \).

For each \( k \), there exists \( u_k \in C^\infty(\Omega) \) satisfying
\[
\begin{cases}
(-\Delta)^m u_k = f_k & \text{in } \Omega \\
(\partial_\nu)^j u_k = 0 & \text{in } \partial \Omega \quad 0 \leq j \leq m - 1.
\end{cases}
\]

It is easily to see, from Lemma 3.1 that \( u_k \in W^{2m-1,p}_\omega(\Omega) \), and obviously \( u_k \in W^{2m,p}_{\omega,loc}(\Omega) \). Moreover for all compact set \( K \subset \Omega \), we have
\[
\|u_k\|_{W^{2m,p}_\omega(K)} \leq C(K),
\]
where \( C(K) \) is a constant depending on the measure of \( K \). Indeed, taking \( v_k = u_k \varphi \) with \( \varphi \in C^\infty_0(K) \), it follows that \( v_k \in W^{2m,p}_\omega(\Omega) \), satisfies (1.1) with \( f = g_k \in L^p_\omega(\Omega) \), and we can use (4.6).

Then, it follows by the dominated convergence theorem that \( u_k \in W^{2m,p}_\omega(\Omega) \) and applying (4.6), we have that
\[
\|u_k\|_{W^{2m,p}_\omega(\Omega)} \leq C \|f_k\|_{L^p_\omega(\Omega)}.
\]

Therefore, \( \{u_k\} \) is a Cauchy sequence in \( W^{2m,p}_\omega(\Omega) \) and there exists \( v \in W^{2m,p}_\omega(\Omega) \) such that \( \lim_{k \to \infty} u_k = v \) in \( W^{2m,p}_\omega(\Omega) \). Let see now that \( v \) solves (1.1).

Obviously, \( f = \lim_{k \to \infty} f_k = \lim_{k \to \infty} (-\Delta)^m u_k = (-\Delta)^m v \) in \( L^p_\omega(\Omega) \) and by the classical trace theorems in Sobolev spaces and the definition of \( \omega \in A_p \), it follows that \( v \) satisfies the homogeneous boundary conditions and by uniqueness of the solution, the Theorem is proved.
Remark 4.2. The result of Theorem 4.1 is valid also for \( u \) a weak solution of

\[
\begin{cases}
  Lu = f & \text{in } \Omega \\
  B_j u = 0 & \text{in } \partial \Omega \quad 0 \leq j \leq m - 1
\end{cases}
\]

when \( L := \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha \) is uniformly elliptic and \( B_j := \sum_{|\alpha| \leq j} b_\alpha D^\alpha, 0 \leq j \leq m - 1 \) are the boundary operators defined in [1].

Indeed, we define \( l_1 > \max_j (2m - j) \) and \( l_0 = \max_j (2m - j) \). If the coefficients \( a_\alpha \in C^{l_1+1}(\Omega), b_j \in C^{l_1+1}(\partial \Omega) \) and \( \partial \Omega \in C^{l_1+2m+1} \) we have that the Green function \( G_m \) and the Poisson kernels \( K_j \) for \( 0 \leq j \leq m - 1 \) exist whenever \( l_1 > 2(l_0 + 1) \) for \( n = 2 \) and \( l_1 > \frac{3}{2} l_0 \) for \( n \geq 3 \).

Moreover, wherever they are defined, the Green function and the Poisson kernels of the operator \( L \) with these boundary conditions satisfy the estimates (2.4), (2.5), (2.6), (2.7) and (2.8) (see [4] and [6]).

Remark 4.3. Using the fact that \( d(x)^\beta \in A_p \) for \(-1 < \beta < p-1\) and some imbedding Theorems for weighted Sobolev spaces (see [5]) we have as a consequence of the main result

\[
\text{Theorem 4.4. Let } \Omega \subset \mathbb{R}^n \text{ be a bounded domain as above, } f \in L_p^\gamma(\Omega), \text{ with } \gamma = k\beta,
\]

where \( k \in \mathbb{N} \) and \( 0 \leq \beta \leq 1 \). If \( u \) be the solution of problem (1.1), \( 0 \leq \gamma < p - 1 \) and \( \frac{1}{p} - \frac{1}{q} \leq \frac{2m}{n + k} \) (with \( q < \infty \) when \( 2mp = n + k \), then there exists a constant \( C \) depending only on \( \gamma, p, q, n \) and \( \Omega \) such that

\[
\| u \|_{L_q^\gamma(\Omega)} \leq C \| f \|_{L_q^\gamma(\Omega)}.
\]

Finally, as a particular case of (4.7) taking \( \gamma = m \) we have that

\[
\| u \|_{L_m^p(\Omega)} \leq C \| f \|_{L_m^p(\Omega)}
\]
for \( p > m + 1 \) and \( \frac{1}{p} - \frac{1}{q} \leq \frac{2m}{n+1} \) (with \( q < \infty \) when \( 2mp = n + m \)).

This result is proved in [4] using different arguments for the case \( \frac{1}{p} - \frac{1}{q} < \frac{2m}{n+m} \).

Our results shows that, at least in the case \( p > m + 1 \), the estimate remains valid when \( \frac{1}{p} - \frac{1}{q} = \frac{2m}{n+m} \).

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