A Proof of Schiffer’s Conjecture in Starlike Domain
by Far-Field Patterns

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Abstract

We formulate the Schiffer’s conjecture in spectral geometry in the context of scattering theory. The problem is equivalent to finding a non-trivial solution in an interior transmission problem. We compare the back-scattering data of the perturbation along all incident angles. The uniqueness of the inverse scattering problem along each incident direction proves the Schiffer’s conjecture.

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1 Introduction

In this paper, we study the following inverse spectral problem:

\[
\begin{aligned}
\Delta u + k^2 u &= 0, & & \text{in } D, \, k^2 \in \mathbb{R}^+; \\
\frac{\partial u}{\partial \nu} &= 0, & & \text{on } \partial D, \\
u &= 1, & & \text{on } \partial D.
\end{aligned}
\] (1.1)

where \(\nu\) is the unit outer normal; \(D\) is a starlike domain in \(\mathbb{R}^3\) containing the origin with Lipschitz boundary \(\partial D\). We interpret the model as the plane waves perturbed by the boundary condition specified by \(D\) and satisfies the Helmholtz equation outside \(D\). Let \(u\) be a non-trivial eigenfunction with some \(k^2 \in \mathbb{R}^+\). We want to show that \(D\) are actually balls.

Here we prove the result as a special case of interior transmission problems \([2, 4, 10, 11, 12, 13, 14, 15, 16, 18, 21, 22, 25, 26, 28]\). In interior transmission problems, we look for a frequency so that a stationary wave behaves like a spherical Bessel function outside the perturbation. In Schiffer’s conjecture, we ask if there is a frequency so that a perturbed wave can stay in its initial shape travelling to infinity in constant speed. We recommend \([1, 25]\) and the reference there for the connections of interior transmission problem to other questions in mathematical science.

Let us start with the Rellich’s representation in scattering theory. We expand the possible solution \(u\) of (1.1) in a series of spherical harmonics near infinity by Rellich’s lemma \([15\text{, p. 32, p. 227}]\):

\[
u(x; k) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m}(r) Y_{l,m}(\hat{x}),
\] (1.2)

where \(r := |x|, \, r \geq R_0\) with a sufficiently large \(R_0\); \(\hat{x} = (\theta, \varphi) \in S^2\). The summations converge uniformly and absolutely on suitable compact subsets away from \(D\). The spherical harmonics

\[
Y_{l,m}(\theta, \varphi) := \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_{|m|}^{l}(\cos \theta) e^{im\varphi}, \, m = -l, \ldots, l; \, l = 0, 1, 2, \ldots,
\] (1.3)

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form a complete orthonormal system in $L^2(S^2)$, in which

$$P^n_m(t) := (1 - t^2)^{m/2} d^n P_n(t) \frac{dt}{dm}, \quad m = 0, 1, \ldots, n,$$

(1.4)

where the Legendre polynomials $P_n$, $n = 0, 1, \ldots$, form a complete orthogonal system in $L^2[-1, 1]$. We refer this to [13, p. 25]. By the orthogonality of the spherical harmonics, the family of functions

$$\{u_{l,m}(x;k)\}_{l,m} := \{a_{l,m}(r)Y^m_l(\hat{x})\}_{l,m}$$

(1.5)

satisfy the first equation in (1.1) independently for each $(l, m)$ in $r \geq R_0$ for sufficiently large $R_0$.

Now we consider the boundary condition as given by the second and third equations in (1.1), and extend the solutions $u_{l,m}(x;k)$ into $r \leq R_0$ as follows. Let $\hat{x} \in S^2$ be a given incident direction intersects $\partial D$ at $\hat{R}$. For any given $\hat{x}$, we impose the differential operator

$$\Delta_{S^2} = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \varphi} \frac{\partial}{\partial \varphi} \sin \varphi \frac{\partial}{\partial \varphi} + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2}{\partial \theta^2}$$

on $u_{l,m}(x;k)$ and, accordingly, we have the following ODE:

$$\frac{d^2 a_{l,m}(r)}{dr^2} + \frac{2}{r} \frac{da_{l,m}(r)}{dr} + (k^2 - \frac{l(l+1)}{r^2})a_{l,m}(r) = 0,$$

(1.6)

which is solved by spherical Bessel functions and spherical Neumann functions. Let

$$y_l(r) := ra_{l,m}(r),$$

(1.7)

so we obtain

$$\begin{cases}
    \frac{y''_{l,m}(r)}{r} + (k^2 - \frac{l(l+1)}{r^2})y_{l,m}(r) = 0; \\
    y_{l,m}(0) = 0.
\end{cases}$$

(1.8)

To give an initial condition, we apply the boundary conditions in (1.1) at the intersection points $\hat{R}$ along $\hat{x}$. Hence,

$$\left[ \frac{y_{l,m}(r;k)}{r} \right]_{r=\hat{R}} = 0;$$

(1.9)

$$\left[ \frac{y_{l,m}(r;k)}{r} \right]_{r=\hat{R}} = 1.$$  

(1.10)

The solutions $y_{l,m}(r;k)$ are independent of $m$, so we write $y_{l,m}(r;k)$ as $y_l(r;k)$. Hence, now we have following boundary conditions.

$$\frac{y'_l(\hat{R};k)}{\hat{R}} - \frac{y_l(\hat{R};k)}{\hat{R}^2} = 0;$$

(1.11)

$$y_l(\hat{R};k) - \hat{R} = 0.$$  

(1.12)

If $k$ satisfies (1.11) and (1.12), then $y'_l(\hat{R};k) = 1$. Thus, (1.9) and (1.10) equivalently satisfy

$$F_l(k;\hat{R}) := y_l(\hat{R};k) - \hat{R} = 0;$$

(1.13)

$$G_l(k;\hat{R}) := y'_l(\hat{R};k) - 1 = 0.$$  

(1.14)

In the initial state, $y_l(\hat{R};k)$ is the boundary defining function of $D$. Combining (1.8), (1.13), and (1.14), we consider the following eigenvalue problem at $\hat{R}$ for each fixed $\hat{x} \in S^2$ and all $l \geq 0$:

$$\begin{cases}
    y''_l(r;k) + (k^2 - \frac{l(l+1)}{r^2})y_l(r;k) = 0, \quad 0 < r < \infty; \\
    y_{l,m}(0;k) = 0; \\
    F_l(k;\hat{R}) = 0; \\
    G_l(k;\hat{R}) = 0.
\end{cases}$$

(1.15)
This is a two-way initial value problem starting at \( r = \hat{R} \) inward and outward. The eigenvalue \( k \) passes through to the infinity by the uniqueness of the ODE and define the far-field patterns near infinity. There is an one-to-one correspondence between the far-field pattern and the radiating solution of the Helmholtz equation. The \( y_l(r;k) \) depends on the incident angle \( \hat{x} \). Most important of all, we will examine the zero set of \( y_{l,m}(0;k) = 0 \) which constitutes the eigenvalues of \((1.15)\). The solutions \( \{y_l(r;k)\}_{l \geq 0} \) is a family of entire functions of exponential type \([5, 6, 7, 27]\). For each \( l \geq 0 \), it behaves like a sine function in complex plane with zero set asymptotically approaching the zero set of sine functions for each incident direction. The Weyl’s law of the eigenvalues of \((1.15)\) in many settings are found in \([10, 11, 12]\) as a direct consequence of the Cartwright-Levinson theory in value distribution theory \([3, 8, 9, 20, 23, 24]\). In particular, one can show that the density of the zero set for each incident direction is connected to the direct consequence of the Cartwright-Levinson theory in value distribution theory \([3, 8, 9, 20, 23, 24]\). In particular, one can show that the density of the zero set for each incident direction is connected to the radius \( \hat{R} \) as a spectral invariant. Rellich’s lemma indicates that all perturbations behave like spherical waves near the infinity by which we prove a special case of Schiffer’s conjecture.

**Theorem 1.1.** Let \( D \) be a starlike domain assumed as in \((1.1)\). If there is an eigenvalue \( k_0^2 \in \mathbb{R}^+ \), \( k_0^2 \geq 1 \), then Schiffer’s conjecture holds in starlike domain.

### 2 Singular Sturm-Liouville Theory

Here we collect some following asymptotic behaviors for \( y_l(r;k) \) and \( y_l'(r;k) \). For \( l \geq 0 \), we apply the results from \([5, 6, 7, 27]\): Let \( z_l(\xi;k) \) be the solution of

\[
\begin{cases}
-\xi^2 z''(\xi) + \frac{(l+1)z'(\xi)}{\xi} + p(\xi)z(\xi) = k^2 z(\xi); \\
z_l(1;k) = -b; z_l'(1;k) = a, a, b \in \mathbb{R},
\end{cases}
\]  

(2.1)

where \( p(\xi) \) is square integrable; the real number \( t \geq -1/2 \). In general,

\[|z_l(\xi;k) + b \cos k(1-\xi) + a \frac{\sin k(1-\xi)}{k}| \leq \frac{K(\xi)}{|k|} \exp\{|3k|[1-\xi]|, |k| \geq 1,\]  

(2.2)

where

\[K(\xi) \leq \exp\left\{ \int_{|\xi|}^{1} \frac{|l(l+1)|}{t^2} + |p(t)|dt \right\}, 0 \leq |\xi| \leq 1.\]  

(2.3)

This explains the behaviors of solutions \( z_l(\xi;k) \) and \( z_l'(\xi;k) \) for all \( l \) in unit interval. For our application in to \((1.13)\) and \((1.14)\), we take

\[b = -\hat{R}; a = 1\]

for each incident direction, and the problem \((2.1)\) in interval \([0, \hat{R}]\).

Outside the domain \( D \), we consider \((1.15)\) as an initial problem starting at \( \hat{R} \) to the infinity. If \( p(\xi) \equiv 0 \), then we consider the following special case:

\[v_l''(\xi) + \frac{|k^2 - \frac{l(l+1)}{\xi^2}|v_l(\xi) = 0.\]  

(2.4)

The solutions of \((2.4)\) are essentially Bessel’s functions with a basis of two elements. The variation of parameters formula leads to the following asymptotic expansions: For \( \xi > 0 \) and \( \Re k \geq 0 \), there is a constant \( C \) so that

\[|v_l(\xi, k) - \frac{\sin\{k\xi - \frac{l\pi}{2}\}}{k^{l+1}}| \leq C|k|^{-l} \exp\{|3k|\xi|};\]  

(2.5)

\[|v_l'(\xi, k) - \frac{\cos\{k\xi - \frac{l\pi}{2}\}}{k^l}| \leq C|k|^{-l} \exp\{|3k|\xi|}.\]  

(2.6)

We refer these estimates to \([6, Lemma 3.2, Lemma 3.3]\). A solution of the initial value problem of \((2.4)\) is a linear combination of \((2.5)\) and \((2.6)\).
3 Cartwright-Levinson Theory

We review the following vocabularies from entire function \( \{3, 8, 9, 20, 23, 24\} \) to describe the asymptotic behavior of the eigenvalues of (1.15).

**Definition 3.1.** Let \( f(z) \) be an integral function of order \( \rho \) and \( N(f, \alpha, \beta, r) \) be the number of the zeros of \( f(z) \) inside the angle \( [\alpha, \beta] \), and \( |z| \leq r \). We define the density function

\[
\Delta_f(\alpha, \beta) := \lim_{r \to \infty} \frac{N(f, \alpha, \beta, r)}{r^\rho},
\]

(3.1)

and

\[
\Delta_f(\beta) := \Delta_f(\alpha_0, \beta),
\]

(3.2)

with some fixed \( \alpha_0 \notin E \), in which \( E \) is at most a countable set \( [23, 24] \).

Let us define

\[
\hat{\Delta}(\xi) := \Delta_{y_l(\xi; k)}(-\epsilon, \epsilon), b = -\hat{R},
\]

(3.3)

as the density of the zero set along \( \hat{x} \).

**Lemma 3.2.** The entire functions \( y_l(\xi; k) \) and \( y'_l(\xi; k) \) are of order one and of type \( \xi \).

**Proof.** From (2.2), we have

\[
y_l(\xi; k) = -\hat{R} \cos k(\hat{R} - \xi) - \frac{\sin k(\hat{R} - \xi)}{k} + O\left(\frac{K(\xi)}{|k|} \exp\{|\Im k||\hat{R} - \xi|\}\right), \quad |k| \geq 1.
\]

(3.4)

To find the type of an entire function, we compute the following definition of Lindelöf’s indicator function \( [23, 24] \).

**Definition 3.3.** Let \( f(z) \) be an integral function of finite order \( \rho \) in the angle \( [\theta_1, \theta_2] \). We call the following quantity as the indicator of the function \( f(z) \).

\[
h_f(\theta) := \lim_{r \to \infty} \frac{\ln |f(re^{i\theta})|}{r^\rho}, \quad \theta_1 \leq \theta \leq \theta_2.
\]

(3.5)

We find that if \( k = |k|e^{i\theta} \), then

\[
h_{y_l(\xi; k)}(\theta) = |(\hat{R} - \xi)\sin \theta|, \quad \theta \in [0, 2\pi], \quad 0 < \xi < \hat{R}.
\]

(3.6)

We refer more details to \( [10, 11, 12, 9, 23, 24] \). Some examples are found in \( [27, p. \, 70] \). The maximal value of \( h_{y_l(\xi; k)}(\theta) \) gives the type of an entire function \( [23, p. \, 72] \), which is \( (\hat{R} - \xi) \). A similar proof holds for \( y'_l(\xi; k) \).

Most important of all, the indicator function (3.6) leads to the following Cartwright’s theory \( [23, p. \, 251] \).

**Lemma 3.4.** We have the following asymptotic behavior of the zero set of \( y_l(\xi; k) \).

\[
\hat{\Delta}(\xi) = \frac{\hat{R} - \xi}{\pi}.
\]

**Proof.** We observe (3.4) that \( |y_l(\xi; k)| \) is bounded on the real axis. Hence, it is in Cartwright’s class. All of the properties in \( [23, p. \, 251] \) hold.

Letting \( \xi = 0 \), we obtain the eigenvalue density of \( [1, 5] \) in \( \mathbb{C} \). Moreover, they are all real.

**Lemma 3.5.** The eigenvalues \( k \) of (1.15) are all real.

**Proof.** For \( l = 0 \), the result is classic \( [6, 27] \). For our case, \( y_l(\xi; k) \) is real for \( k \in 0i + \mathbb{R} \). Furthermore, the asymptotic behavior of (3.4) proves the lemma, which is a special case of Bernstein’s theorem in entire function theory \( [17, \text{Theorem 1}] \). A step-by-step proof is provided in \( [12, \text{Lemma 2.6}] \).
4 Proof of Theorem 1.1

Proof. Let \( k_0^2 \) be an eigenvalue of (1.1) as the assumption of Theorem 1.1. Particularly, from (1.2) we have

\[
\begin{align*}
  u(x; k_0) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m}(r; k_0) Y_l^m(\hat{x}); \\
  u_{l,m}(x; k_0) &= a_{l,m}(r; k_0) Y_l^m(\hat{x}), \quad \hat{x} \in \mathbb{S}^2,
\end{align*}
\]

in which the coefficient \( a_{l,m}(r; k_0) \) does not depend on the incident direction \( \hat{x} \in \mathbb{S}^2 \) for sufficiently large \(|x| := r\). The functions as in (1.1) solve the Helmholtz equation in \( r \geq R_0 \). As a result of uniqueness of the ODE (1.15), the solutions \( y_l(r; k_0) \) extend both outward to the infinity and inward to the origin for all \( l \geq 0 \). For the given eigenvalue \( k_0^2 \), the equation (1.15) holds for all incident directions \( \hat{x} \in \mathbb{S}^2 \) and for all \( l \geq 0 \).

The representation (1.1) is unique: Suppose that there is another eigenvalue \( k' \) of (1.15) from incident angle \( x' \neq \hat{x} \in \mathbb{S}^2 \) with the solution

\[
\begin{align*}
  u'_{l,m}(x; k') := a'_{l,m}(r; k) Y_l^m(x').
\end{align*}
\]

The analytic continuation of Helmholtz equation [15, p. 18] implies that

\[
\Delta(0) = \frac{\hat{R}}{\pi}, \quad \hat{x} \in \mathbb{S}^2.
\]

The ODE (1.16) holds for all \( \xi \geq \hat{R} \) and \( l \geq 0 \). In particular, we apply the estimates (2.5) and (2.6):

\[
\begin{align*}
  |v_l(\xi, k) - \sin\{k(\xi - \hat{R}) - \frac{l\pi}{2}\} / k^{l+1}| &\leq C|k|^{-l+1}\exp\{|3k|\xi\} / |k\xi|, \\
  |v'_l(\xi, k) - \cos\{k(\xi - \hat{R}) - \frac{l\pi}{2}\} / k^l| &\leq C|k|^{-l}\exp\{|3k|\xi\} / |k\xi|.
\end{align*}
\]

Therefore, the initial value problem (2.1) with \( p \equiv 0, b = \hat{R} \), and \( a = 1 \) provides the asymptotic behavior for the solution:

\[
\begin{align*}
  y_l(\xi, k) &= \hat{R} \cos\{k(\xi - \hat{R}) - \frac{l\pi}{2}\} + O\{1 / k^{l/2}\xi\}, \quad 0 < \xi < \infty,
\end{align*}
\]

That is,

\[
\begin{align*}
  k^l y_l(\xi, k) &= \hat{R} \cos\{k(\xi - \hat{R}) - \frac{l\pi}{2}\}[1 + O\{1 / k_0^l\xi\}], \quad 0 < \xi < \infty,
\end{align*}
\]

outside the zeros of \( \cos\{k(\xi - \hat{R}) - \frac{l\pi}{2}\} \). This is classic in Sturm-Liouville theory [3, 6, 27].

The given eigenvalue \( k_0 \) satisfies (1.15), for all \( l \geq 0 \) and all \( \hat{x} \in \mathbb{S}^2 \), and (1.2) consequently. Therefore,

\[
\begin{align*}
  k^l_0 y_l(\xi, k_0) &= \hat{R} \cos\{k_0(\xi - \hat{R}) - \frac{l\pi}{2}\}[1 + O\{1 / k_0^l\xi\}], \quad 0 < \xi < \infty.
\end{align*}
\]

We choose \( l \uparrow \infty \) and so \( \xi \uparrow \infty \) such that

\[
\begin{align*}
  k^l_0 y_l(\xi, k_0) &= \hat{R} + O\{1 / k_0^l\xi\}, \quad \xi = \hat{R} + \frac{l\pi}{2k_0}, \quad |k_0| \geq 1.
\end{align*}
\]

Using (1.7), (1.2) and the uniqueness of the Helmholtz equation, as shown in (4.3), the far-field patterns are asymptotically the same periodic functions for each \( \hat{x} \in \mathbb{S}^2 \). In particular, the boundary defining function \( \hat{R} \) is constant to \( \hat{x} \in \mathbb{S}^2 \) and Theorem 1.1 is thus proven.

\[\square\]
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