A New Approach to Construct the Operator on Lattice for the Calculation of Glueball Masses

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Abstract

We develop a new approach to construct the operator on lattice for the calculation of glueball mass, which is based on the connection between the continuum limit of the chosen operator and the quantum number $J^{PC}$ of the state studied. The spin of the state studied in this approach is then determined uniquely and directly in numerical simulation. Furthermore, the approach can be applied to calculate the mass of glueball states (ground or excited states) with any spin $J$ including $J \geq 4$. Under the quenched approximation, we present a pre-calculation result for the mass of $0^{++}$ state and $2^{++}$ state, which are $1754(85)(86)\text{MeV}$ and $2417(56)(117)\text{MeV}$, respectively.
1 Introduction

During the past two decades, extensive Monte Carlo simulations were carried out to calculate the glueball mass spectra on lattice. Most of these papers carried out two key steps: one is the choice of glueball operators with certain quantum number $J^{PC}$ basing on the method introduced in Ref. [1] and the other is the application of variational principle. Meanwhile, with a great amount of the improvement, such as fuzzying and smearing, etc., these approaches surely work well and many results of well-controlled errors are obtained. However, the simulations are still puzzled by an ambiguity, i.e., how to identify the definite spin $J$ from the irreducible representation $R$ of the cubic point group, according to which the glueball operators transform, because the corresponding between this irreducible representation $R$ of cubic point group and spin $J$ is not one-to-one.

Meanwhile, basing on the representation theory of $O(4)$ group and the hypercubic group, Mandula et. al. [12] developed an elegant scheme for the choice of glueball operators. With the definition of lattice gauge field, they utilized the lattice color electric and magnetic field to construct operators with definite $J^{PC}$ through the decomposition of the composition of lattice color electric and magnetic field into certain representation of hypercubic group. But, the correspondence between irreducible representation of hypercubic group and spin $J$ is also not one-to-one. The 'leading spin' is then assumed when $a \rightarrow 0$. But, this assumption cannot determine the spin uniquely. For example, one don’t know how to separate 'leading spin' $J = 1^-$ from 'leading spin' $J = 2^+$ in $6^{(+)}$ representation and one can not get the content of the non-leading spin $[12]$.

We would like to show a possible solution to these troubles in this paper. Unlike above references, we start our discussion with the asymptotic expansion of the operator. By expanding the chosen operator according to power of lattice spacing $a$, we require that the leading term of the expansion of the chosen operator belongs to the irreducible representation $J^{PC}$ of $SO(3)^{PC}$ group. We assume that the leading term of the expansion will give the main effect to the state studied when $a \rightarrow 0$, and the contribution should be only given by the leading term in the continuum. Therefore, the spin of the corresponding state is uniquely determined by the leading term of the expansion when the lattice tends to continuum.

We claim that there are two advantages in this approach: (1) Appending to the application of variational method one can also study the contribution of the different current ( with the same $J^{PC}$) to the definite state; (2) One can determine the spin of the corresponding state unambiguously and directly. In this way, we can distinguish state with definite $J^{PC}$ from other $J^{PC}$ states.

Some observations are shown in the forthcoming section. We introduce our method in section 3 and give an example of pre-calculation to verify our method in section 4. Section 5 is a short summary.
2 Some Observations

In the continuum, the glueball states with definite quantum number $J^{PC}$ make up of the bases of certain irreducible representation $J^{PC}$ of $SO(3)^{PC}$ group. But, on lattice, there is only its finite point subgroup, $O^{PC}$, and its corresponding irreducible representations $R^{PC}(R = A_1, A_2, E, T_1 \text{ and } T_2)$. Then to measure glueball mass in lattice QCD, there arises such problem: how we get correct results by only utilizing $O^{PC}$ group. To solve the problem, authors make the continuum limit assumption($\beta = \infty$) since Berg and Billoire\cite{4}, i.e., denoting masses of the states extracted from operators in the irreducible representation $\hat{R}$ by $m(\hat{R})$ and masses of the states with certain spin $J$ by $m(J)$, they assume:\cite{4, 2, 4, 5}

$$m(0^{PC}) = m(A_1^{PC}),$$
$$m(1^{PC}) = m(T_1^{PC}),$$
$$m(2^{PC}) = m(E^{PC}) = m(T_2^{PC}),$$
$$m(3^{PC}) = m(A_2^{PC}).$$ \hspace{1cm} (1a, 1b, 1c, 1d)

But as Ref. \cite{5} shows, this assumption is not always right. From simulation results, in Ref. \cite{5}, for example, $T_{1^{++}}$ channel was not interpreted as $J^{PC} = 1^{++}$ states but most likely as $J^{PC} = 3^{++}$ state ( less likely $J = 6, 7, 9, \ldots$ interpretation cannot rule out ), since it seems this channel degenerated with $A_{2}^{++}$ channel in the continuum.

As to the $T_{1^{+-}}$ channel, it was interpreted as $J^{PC} = 1^{+-}$ state instead of $3^{+-}$. But, $A_{2}^{+-}, T_{1}^{+-*}$ and $T_{2}^{+-}$ channels become degenerated in the continuum and they were interpreted as $3^{+-}$ state. Therefore, we hope to develop an approach to determine spin $J^{PC}$ of states studied directly and uniquely in lattice simulation.

Meanwhile, on a $D = 2 + 1$ lattice, Johnson and Teper\cite{13} found that in $A_{1}^{++}$ channel there exist two states with different masses. They interpret the higher one as $4^{++}$ state and lower one as $0^{++}$ state. Therefore, they also consider that one needs a systematic and general procedure to construct operators of arbitrary spin as $a \rightarrow 0$\cite{13}.

Now, we present a possible procedure to solve these problems here. Let us begin the discussion with some observations.

A.

An arbitrary state $|\psi>$ can be regarded as generated by current $o$ acting on vacuum $|0>$:

$$|\psi> = o|0>.$$ \hspace{1cm} (2)

Since $|0>$ is invariance under Poincare group and $SU_C(3)$ group, the character of $|\psi>$ can be described by $o$. For simplicity, we only consider currents with mass dimension 4 here, saying $B^a_i(x)B^b_j(x)$, where $B^a_i$ is color magnetic field and $i, j = 1, 2, 3$. 


Since $|\psi>$ is color singlet, we also require $o$ is color singlet. One can get 6 color singlet currents from above combinations:

$$2Tr(B_i B_j) = \sum_{a=1}^{8} B_i^a B_j^a,$$

where $B_i = \sum_{a=1}^{8} B_i^a \lambda_a^2$.

We also require $|\psi>$ and $o$ transform as certain representation $J^{PC}$ under $SO(3)^{PC}$ group. Since $\bar{B}$ transforms as $1^{+-}$ under this group, The $P, C$ of $Tr(B_i B_j)$ are $++$. Using C-G coefficients, we can decompose these bases consisting of $Tr(B_i B_j)$ into $J = 0$ and $J = 2$. Then, in the subduced representation $J \downarrow O$ of the rotation group $SO(3)$ restricted to subgroup $O$, we find that the base in $J = 0$ is the basis of representation $A_1$, and we can further reduce another five bases in $J = 2$ according to irreducible representations $E$ and $T_2$ of the cubic point group. So, we can categorize these bases as:

$$J = 0: \quad a_{11} = Tr(B_1 B_1 + B_2 B_2 + B_3 B_3); \quad (3a)$$

$$J = 2: \quad e_1 = Tr(B_1 B_1 - B_2 B_2), \quad e_2 = Tr(B_1 B_1 + B_2 B_2 - 2B_3 B_3); \quad (3b)$$

$$t_{21} = Tr(B_2 B_3), \quad t_{22} = Tr(B_1 B_3), \quad t_{23} = Tr(B_1 B_2). \quad (3c)$$

Here $a_{11}$ is the basis of representation $A_1$, $e_1$ and $e_2$ construct bases of representation $E$, while $t_{21}$, $t_{22}$ and $t_{23}$ just make up of bases of representation $T_2$.

This is just what table 3 (see Appendix) tells us: the subduced representation $J = 2$ of the rotation group can be decomposing into representation $E$ and $T_2$ in the cubic group; the subduced representation $J = 0$ is just representation $A_1$.

We can make similar analysis for higher mass-dimensional gauge invariant operator consisting of color magnetic field and its covariant derivative.

B.

Now, we consider how to construct a glueball operator on lattice. By expanding the chosen operator according to power of spacing $a$, we require that the leading term of the expansion of the chosen operator belongs to and only belongs to the irreducible representation $J^{PC}$ of $SO(3)^{PC}$ group. This is the key point of this paper. Therefore, we first consider the perturbative expansion of Wilson loops on lattice according to power of lattice spacing $a$. A simple example is plaquette operator (We denote unit vector in the positive $i$-direction by $\hat{i}$):

$$O_{ij} = \sum_n O_{ij}(n) = \sum_n Tr[1 - U(n, i) U(n + \hat{i}, j) U^{-1}(n + \hat{j}, i) U^{-1}(n, j)], \quad (4)$$

where the link variable $U$ is a connector and defined by

$$U(n, i) = P \exp(i \int_0^a dt A_i(an + it)), \quad (5)$$

\footnote{The systemic decomposition was well discussed by Jaffe et al. in the study of the qualitative features of the glueball spectrum. They suggest to construct glueball operators for certain $J^{PC}$ states with color magnetic and electric fields in the continuum case. But, we should go further to study the construction of operators on lattice as the rest of the paper points out.}
Where $P$ is path-order operator.

There are two methods to expand the operator. One is the application of non-Abelian Stokes theorem\cite{7,8} and the other one is introduced by Luscher and Weisz in Ref. \cite{6}. We use both methods and get the same results:

$$O_{ij} = \sum_n \left\{ \frac{a^4}{2} \text{Tr}(F_{ij} F_{ij})(n) + \frac{a^6}{6} \text{Tr}(F_{ij} F_{ij} F_{ij})(n) ight\} + O(a^8),$$

where $F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]$ is gauge field and $D_{i\cdot} = \partial_i - i[A_i, \cdot]$ is covariant derivative.

We now consider the $PC = ++$ sector of operators, or real part of operators in Eq. (6) with ignoring the second term of r.h.s. in Eq. (6). Due to $O_{ji} = O_{ij}^*$, there are three non-zero independent operators $\text{Re}O_{12}$, $\text{Re}O_{23}$, $\text{Re}O_{31}$. Restricting oneself into the cubic group, one can combine these operators into representation $A_{1^+}$ and $E_{2^+}$:

$$A_{1^+} : \quad \text{Re}(O_{23} + O_{13} + O_{23}) = \frac{a^4}{2} \sum_n \text{Tr}(B_1 B_1 + B_2 B_2 + B_3 B_3)(n) + O(a^6);$$
$$E_{2^+} : \quad \text{Re}(O_{23} - O_{13}) = \frac{a^4}{2} \sum_n \text{Tr}(B_1 B_1 - B_2 B_2)(n) + O(a^6),$$

where color magnetic field is $B_i = -\frac{1}{2} \sum_{jk} \epsilon_{ijk} F_{jk}$.

We suppose that the leading term gives the most contribution of the operator when $a$ is small enough, or, only the leading term gives the contribution in the continuum. While comparing Eq. (6) to Eq. (3), it is assured that, in the continuum limit, the state extracted from such operator $A_{1^+}$ is $J = 0$, and the states extracted from the operator in $E_{2^+}$ corresponds to $J = 2$ state.

We should emphasis again, in the general case, the continuum limits of the operator in representation $E$ or in $T_2$ is not always corresponding to $J = 2$, i.e., the parallelism in Eq. (1) does not always hold. Only after expanding the chosen operator as we do above, we are then able to affirm or disaffirm the parallelism.

By the way, we should point out here that the non-leading terms in the expansion of the operator do not always belong to the same $J^{PC}$ as that of leading term, which will bring up the mixing with different spin $J$. But, as argued above, we expect that this artificial mixing will decrease with the decreasing of lattice spacing $a$ so that the mixing should vanish when $a \rightarrow 0$ although it will affect our error estimate. On the other hand, we can utilize non-leading terms to explore high-spin states.

These two examples tell us that to calculate the mass of the definite $J^{PC}$ state, we should require the continuum limits of our operators belong to and only belong to $J^{PC}$ representation of $SO(3)^{PC}$ group. One can get this aim by using the combination of the different operators which belong to the same $R^{PC}$. We will present an example to construct the operator in the following section and then show the simulation results and discuss the errors in section 4.
3 The Construction of the Operator

We exemplify here how to construct operator $0^{++}$ and $2^{++}$ up to $a^4$. On lattice, gauge-invariant current $o$ with $0^{++}$ corresponding to the scalar glueball can be written as

\[ o = \sum_n \left( a^4 \sum_{i=1}^3 Tr(B_i B_i)(n) + a^6 \times \text{(current with mass dimension 6)} + \cdots \right), \]

where the current with mass dimension 6 is the combination of $\sum_{i,j}^3 Tr(D_i F_{ij} D_i F_{ij})$, $\sum_{i,j,k}^3 Tr(D_i F_{jk} D_i F_{jk})$ and $\sum_{i,j,k}^3 Tr(D_i F_{ik} D_j F_{jk})$. For simplicity, we only consider the current $o$ up to mass dimension 4 in this paper. Then, let us observe the sum of the planar special $2 \times 1$ rectangular over all lattice sites:

\[
O'_{ij} = \sum_n O'_{ij}(n) = \frac{1}{2} \sum_n \text{Tr}\{[1 - U(n, i)U(n + \hat{i}, i)U(n + 2\hat{i}, j)U^{-1}(n + \hat{i} + \hat{j}, i) U^{-1}(n + \hat{j}, i)U^{-1}(n, j)] + [1 - U(n, i)U(n + \hat{i}, j)U(n + \hat{i} + \hat{j}, j) U^{-1}(n + 2\hat{j}, i)U^{-1}(n + \hat{j}, j)U^{-1}(n, j)]\}. 
\]

The real part of the expansion for the operator $O'_{ij}$ is

\[
\text{Re}O'_{ij} = \sum_n \left( \frac{a^4}{2} 4\text{Tr}(F_{ij}F_{ij})(n) - \frac{a^6}{24} 10\text{Tr}(F_{ij}(D_i^2 + D_j^2)F_{ij})(n) \right) + O(a^8). \tag{10}
\]

Then, we define

\[
\Theta_{ij}(n) = \text{Re}(O_{ij}(n) - \frac{1}{10} O'_{ij}(n)). \tag{11}
\]

Continuum limit of operator $\Theta_{ij}$ is

\[
\Theta_{ij} = \sum_n \frac{3a^4}{10} \text{Tr}(F_{ij}F_{ij}) + O(a^8). \tag{12}
\]

Decomposing $\Theta_{ij}$ into $A_1^{++}$ according to traditional method, we get the basis of representation $A_1^{++}$:

\[
F \equiv \Theta_{12} + \Theta_{13} + \Theta_{23} = \sum_n \frac{3a^4}{10} \text{Tr}(B_1 B_1 + B_2 B_2 + B_3 B_3) + O(a^8). \tag{13}
\]
Apparently, the quantum number of continuum limit of $F$ is $0^{++}$. In other words, $F$ transforms as $0^{++}$ under $SO(3)^{PC}$ group up to $a^4$. We expect that the symmetry of $SO(3)$ has been restored when $a \to 0$, the extracted state should be mainly given by the leading term of $F$, so the extracted state is $0^{++}$ one in the continuum limit. Operator $F$ is our aimed operator for $0^{++}$ state.

We may also choose the bases $G_1$ and $G_2$ of representation $E^{++}$ to measure tensor glueball mass as follows. The operators and their expansion are:

$$G_1 = Re(\Theta_{23} - \Theta_{13}) = \sum_n \frac{3a^4}{10} Tr(B_1B_1 - B_2B_2) + O(a^8), \quad (14)$$

and

$$G_2 = Re(\Theta_{23} + \Theta_{13} - 2\Theta_{12}) = \sum_n \frac{3a^4}{10} Tr(B_1B_1 + B_2B_2 - 2B_3B_3) + O(a^8). \quad (15)$$

According to (3b), they belong to bases of representation $2^{++}$ up to $O(a^4)$.

### 4 Simulation Results

Under the quenched approximation, we perform our calculation on anisotropic lattice with improved gluonic action as chosen in Ref. $[9]$:

$$S_{II} = \beta \left\{ \frac{5\Omega_{sp}}{3\xi u_s^4} + \frac{4\xi\Omega_{tp}}{3u_s^2u_t^2} - \frac{\Omega_{sr}}{12u_s^6} - \frac{\xi\Omega_{str}}{12u_s^4u_t^2} \right\}, \quad (16)$$

where $\beta = 6/g^2$, $g$ is the QCD couple constant, $u_s$ and $u_t$ are mean link renormalization parameters (we set $u_t = 1$), $\xi = a_s/a_t$ is the aspect ratio, and $\Omega_{sp}$ includes the sum over all spatial plaquettes on the lattice, $\Omega_{tp}$ indicates the temporal plaquettes, $\Omega_{sr}$ denotes the planar $2 \times 1$ spatial rectangular loops and $\Omega_{str}$ refers to the short temporal rectangles (one temporal and two spatial links). More detail is given in Ref. $[9]$. In each $\beta$ calculation, we set 2800 sweeps to make configurations reach to equilibrium and make measurement once after every four sweeps. Our calculation spends 80 bins in which there are 70 measurements after reaching equilibrium. The method to set the scale used here is introduced by ref $[11]$. For each $\beta$, we have measured $u_s^4$ and found they coincide with those in Ref. $[3]$ in the error bound. So we adopt the data in Ref. $[3]$ as our energy scale. Table 1 shows the simulation parameters.

| $\beta$ | $\xi$ | $u_s^4$ | Lattice | $r_s/r_0$ | $a_s(fm)$ |
|---------|------|-------|---------|---------|----------|
| 1.7     | 5    | 0.295 | $6^4 \times 30$ | 0.8169  | 0.39     |
| 1.9     | 5    | 0.328 | $8^4 \times 40$ | 0.727   | 0.35     |
| 2.2     | 5    | 0.378 | $8^4 \times 40$ | 0.5680  | 0.27     |
| 2.4     | 5    | 0.409 | $8^4 \times 40$ | 0.459   | 0.22     |
| 2.5     | 5    | 0.424 | $10^4 \times 50$ | 0.407   | 0.20     |

*Table 1* The glueball simulation parameters. Here we assume $r_0 = 410(20)MeV$. 


As argued above, we choose operator $F$ to calculate scalar glueball mass and operator $G_1$ and $G_2$ to calculate tensor glueball mass. As usual, we calculate the average over sample vacua of a correlation $C(t) = <0|o^R(t)o^R(0)|0>$, where $o^R(t) = o(t) - <0|o(t)|0>$ is the vacuum-subtracted form of the chosen operator, to determine masses of the corresponding glueball states with the improvements such as fuzzying and smearing. At the same time, following the mean field theory\[10\], we also replace link variant $U$ by $U/u_s$ in the chosen operators due to the tadpole correction. After such programmes, we get our results in each $\beta$ which are shown in table 2:

| $\beta$ | 1.7 | 1.9 | 2.2 | 2.4 | 2.5 |
|---------|-----|-----|-----|-----|-----|
| scalar  | 0.609(4) | 0.515(8) | 0.412(7) | 0.315(6) | 0.322(2)) |
| tensor  | 1.019(3) | 0.95(1) | 0.71(2) | 0.548(6) | 0.519(4) |

Table. 2 Glueball energy $m_{G_\ell}$ for each $\beta$.

The numerals in the brackets are the error estimates.

Now we comment a little on the error estimate.

First, our action breaks the rotation symmetry to $O(a_s^4, a_s^2)$, i.e., the upper limit of the precision in the calculation is $O(a_s^4, a_s^2)$. Since as argued by many authors, the contribution of $O(a_s^2)$ can be ignored, the upper limit of the precision here is $o(a_s^4)$.

Second, we ignore terms (currents) with mass dimension 6 in Eq. (8). Due to dimensional analysis, the contribution of the terms to error should have a square mass suppression\[14\], which will make two effects on our mass measurement. One is that we should include it in systematic error in the continuum limit, which needs further calculation to get its accurate value. Here we simply expect that it is about $(\Lambda_{QCD}/M)^2$, where we set $\Lambda_{QCD} \approx 250MeV$ and $M$ is measured mass. The second is that it will takes $O(a_s^2)$ error when $a_s \neq 0$. Since it is not statistical error, its contribution to error can be fitted by $c_2a_s^2 + c_4a_s^4 + \cdots$.

From the argument and calculated data, we use the formula $m(0^{++}, a_s) = 1.754 - 1.514(a_s/r_0) + 1.773(a_s/r_0)^2$ and $m(2^{++}, a_s) = 2.417 + 0.783(a_s/r_0)^2 - 0.787(a_s/r_0)^4$ (unit: GeV) to fit our data. We present our data and fitting curves in Fig. 1.

The statistical error is 0.076GeV for scalar glueball and 0.044GeV for tensor glueball. According to Ref. \[3\], systematic error is 1 percent (from aspect ratio). But since our method also gives about 2 and 1 percent systematic error for $0^{++}$ and $2^{++}$ states respectively, the total systematic error is about 2.2 percent (39MeV) and 1.4 percent (34MeV) respectively. Therefore, the mass of scalar glueball is $1.754(85)GeV$ and the mass of tensor glueball is $2.417(56)GeV$. Including the uncertainty in $r_0^{-1} = 410(20)MeV$, Our final results are: $M_G(0^{++}) = 1754(85)(86)MeV$ and $m_G(2^{++}) = 2417(56)(117)MeV$.

5 Conclusion

Basing on the connection of the asymptotic expansion of the operator and the quantum number $J^{PC}$ of the extracted state, we present a new approach to construct...
operator on lattice for the calculation of the glueball mass, which may solve the ambiguity in the simulation. This approach points out that, in general, to calculate the mass of definite $J^{PC}$ glueball states, first one should write out these currents which transform as the representation $J^{PC}$ under the $SO(3)^{PC}$ group in continuum and decompose them into irreducible representations $R^{PC}$ of the group $O^{PC}$ in the subduced representation which is obtained by trivially embedding the $O^{PC}$ group into the $SO(3)^{PC}$ group; then one should construct corresponding operators which belongs to the representation $R^{PC}$ of the $O^{PC}$ group on lattice, its continuum limits should be those currents mentioned above.

To verify our approach, we calculate the scalar and tensor glueball mass under the quenched approximation in this approach. Since the continuum limit of operator $F$ is actual $0^{++}$, we affirm the mass extracted from the operator $F$ is scalar glueball mass and its value is $1754(85)(86)\, MeV$. With the same reason, the mass extracted from operator $G_i (i = 1, 2)$ is the tensor glueball one and its value is $2417(56)(117)\, MeV$. These results are consistent with those obtained in references [3, 5, 9, 15].

Apparently, there is no radical obstacle to prevent us to calculate the mass of states with any spin $J$ including $J \geq 4$ in this approach. We will make such study systematically. We have first calculated the mass of the ground $4^{++}$ states in the $E^{++}$ channel with $2^{++}$ stete under the quenched approximation in this approach. It will be shown elsewhere.

Of course, the operator, its continuum limit transform as $J^{PC}$ of $SO(3)^{PC}$ group, is not unique. For example, one can also construct the operator including color electric field. With these operators, one can determine their relative weights of contribution by variational principle. But, we did not make such treatment here.

To compare our results with the experiments, we need to calculate it on an unquenched lattice. We should also study the mixing between glueball states and normal mesons with the same $J^{PC}$. Such works should be finished in future.

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Appendix

In the cubic group, the subduced representation $J$ of irreducible representation $J$ can be decomposed into irreducible representation $R$. Table 3 shows the multiplicity of decomposing $J$ up to $J = 6$. It is well known that the subduced representation with $J \geq 2$ are reducible and only up to $J = 3$ do new irreducible representation of the cubic group show up.

| $R \setminus J$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------------|---|---|---|---|---|---|---|
| $A_1$           | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| $A_2$           | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| $E$             | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| $T_1$           | 0 | 1 | 0 | 1 | 1 | 1 | 2 |
| $T_2$           | 0 | 0 | 1 | 1 | 1 | 1 | 2 |

Table 3 The Subduced representations of rotation group up to $J = 6$ for the cubic group.

Figure Caption

**Figure 1** Masses of scalar and tensor glueball against the lattice spacing $(a_s/r_0)^2$. The fitting curve are $m(0^{++}, a_s) = 1.754 - 1.514(a_s/r_0)^2 + 1.773(a_s/r_0)^4$ for scalar glueball mass and $m(2^{++}, a_s) = 2.417 + 0.783(a_s/r_0)^2 - 0.787(a_s/r_0)^4 (unit : GeV)$ for tensor glueball mass. The masses in continuum limit are $1.754(76)GeV$ and $2.417(44)GeV$ if we only consider the statistical error. The top data and curve: $m(4^{++}, a_s) = 3.65 - 1.22(a_s/r_0)^2 + 2.74(a_s/r_0)^4 (unit : GeV)$ is for $4^{++}$ glueball mass.
