Hyers–Ulam Stability of Two-Dimensional Flett’s Mean Value Points

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Received: 14 May 2019; Accepted: 8 August 2019; Published: 11 August 2019

Abstract: If a differentiable function \( f : [a, b] \rightarrow \mathbb{R} \) and a point \( \eta \in [a, b] \) satisfy \( f(\eta) - f(a) = f'(\eta)(\eta - a) \), then the point \( \eta \) is called a Flett’s mean value point of \( f \) in \([a, b] \). The concept of Flett’s mean value points can be generalized to the 2-dimensional Flett’s mean value points as follows: For the different points \( \hat{r} \) and \( \hat{s} \) of \( \mathbb{R} \times \mathbb{R} \), let \( L \) be the line segment joining \( \hat{r} \) and \( \hat{s} \). If a partially differentiable function \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) and an intermediate point \( \hat{\omega} \in L \) satisfy \( f(\hat{\omega}) - f(\hat{r}) = \langle \hat{\omega} - \hat{r}, f'(\hat{\omega}) \rangle \), then the point \( \hat{\omega} \) is called a 2-dimensional Flett’s mean value point of \( f \) in \( L \). In this paper, we will prove the Hyers–Ulam stability of 2-dimensional Flett’s mean value points.

Keywords: Hyers-Ulam stability; mean value theorem; Flett’s mean value point; two-dimensional Flett’s mean value point

MSC: 39B82; 39B22; 26D10

1. Introduction

In 1940, Ulam [1] raised an interesting question regarding the stability of group homomorphisms:

*Under what conditions is the approximate solution of an equation necessarily close to the exact solution of the equation?*

In the following year, Hyers [2] positively solved the Ulam’s question only in the case of additive functional equations under the additional assumption that \( G_1 \) and \( G_2 \) are all Banach spaces. Indeed, Hyers proved that every solution of inequality \( \|f(x + y) - f(x) - f(y)\| \leq \varepsilon \) for all \( x \) and \( y \), can be approximated by an exact solution (an additive function). In this case, the Cauchy additive functional equation, \( f(x + y) = f(x) + f(y) \), is said to satisfy the Hyers–Ulam stability, or it is called stable in the sense of Hyers and Ulam.

Mean value theorem is one of the most important theorems in real analysis. According to Lagrange’s mean value theorem, for any planar ‘regular’ curve between two endpoints, there exists at least one point between these endpoints at which the tangent to the curve is parallel to the secant through those endpoints. More precisely, if \( f : [a, b] \rightarrow \mathbb{R} \) is a continuous function on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), then there exists a point \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

In 1958, Flett proved another mean value theorem which is a variant of the Lagrange’s mean value theorem (see [3]):

*Let \( f : [a, b] \rightarrow \mathbb{R} \) be a differentiable function satisfying \( f'(a) = f'(b) \). Then there exists a point \( \eta \in (a, b) \) that satisfies \( f(\eta) - f(a) = f'(\eta)(\eta - a) \).*
In particular, the point $\eta$ in the Flett’s mean value theorem is called a Flett’s (mean value) point of $f$ in $[a, b]$.

In Section 3 of the present paper, the Hyers–Ulam stability of 2-dimensional Flett’s mean value points will be proved (We may refer to Theorem 2 for the exact definition of 2-dimensional Flett’s points). Theorem 3 of this paper is a generalization as well as an extension of ([4] Theorem 2.2) and at the same time it is a counterpart of Theorem 2 for the 2-dimensional Flett’s points.

2. Preliminaries and Historical Backgrounds

We can apply the concept of Hyers–Ulam stability to other mathematical objects. Ulam and Hyers [5] appear to be the first ones who apply the Hyers–Ulam stability concept to differential expressions. The Ulam and Hyers’ theorem is essential to establish the main result of the present paper.

**Theorem 1.** (Ulam and Hyers [5]) Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is $n$ times differentiable in a neighborhood $N$ of a point $t_0$, $f^{(n)}(t_0) = 0$, and that $f^{(n)}(t)$ changes sign at $t_0$. Then, for any given $\varepsilon > 0$, there is a $\delta > 0$ with the property that if a function $g : \mathbb{R} \rightarrow \mathbb{R}$ is $n$ times differentiable in $N$ and $g$ satisfies $|f(t) - g(t)| < \delta$ for $t \in N$, then there exists a point $t_1 \in N$ satisfying $g^{(n)}(t_1) = 0$ and $|t_1 - t_0| < \varepsilon$.

One similar question, such as the question of Ulam, may be raised for various kinds of mean value points:

If a function $f$ has a mean value point $\eta$ and $g$ is a function very close to $f$, will $g$ have a mean value point near $\eta$?

Indeed, Das, Riedel and Sahoo studied Hyers–Ulam stability of Flett’s points (see [6]). However, there was unfortunately some incompleteness in their proof. Thereafter, Lee, Xu and Ye [7] created a counterexample to show that the proof of [6] was incorrect, and succeeded in proving the Hyers–Ulam stability of the Sahoo-Riedel’s points.

3. Main Theorem

According to the generalization of Lagrange’s mean value theorem (see ([8] Theorem 4.1)), for every function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ with continuous partial derivatives $f_x$ and $f_y$ and for all distinct pairs $(p, q)$ and $(u, v)$ in $\mathbb{R} \times \mathbb{R}$, there exists an intermediate point $(\eta, \xi)$ on the line segment connecting the points $(p, q)$ and $(u, v)$ that satisfies

$$f(u, v) - f(p, q) = (u - p)f_x(\eta, \xi) + (v - q)f_y(\eta, \xi).$$

For any points $(p, q)$ and $(u, v)$ of $\mathbb{R} \times \mathbb{R}$, the Euclidean inner product is denoted by $\langle (p, q), (u, v) \rangle$ and is defined as

$$\langle (p, q), (u, v) \rangle = pu + qv$$

and the norm $|(p, q)|$ of $(p, q)$ is defined as

$$|(p, q)| = \sqrt{\langle (p, q), (p, q) \rangle} = \sqrt{p^2 + q^2}.$$

For notational simplicity, we denote $(f_x, f_y)$ by $f'$ if $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function of two variables with partial derivatives $f_x$ and $f_y$. Using the above definition and notation, we can rewrite (1) as

$$f(u, v) - f(p, q) = \langle (u - p, v - q), f'(\eta, \xi) \rangle.$$

This can be further simplified to

$$f(s) - f(r) = \langle s - r, f'(\omega) \rangle,$$
where \( \hat{s} = (u, v) \), \( \hat{r} = (p, q) \), and \( \hat{\omega} = (\eta, \xi) \).

Now we present a generalization of Flett’s mean value theorem for real valued functions of two variables (see ([8] Theorem 5.4)).

**Theorem 2.** (Two-dimensional Flett’s mean value theorem) Given the different points \( \hat{r} \) and \( \hat{s} \) of \( \mathbb{R} \times \mathbb{R} \), let \( L \) be the line segment connecting \( \hat{r} \) and \( \hat{s} \). For each function \( f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) that has continuous partial derivatives \( f_x, f_y \) and satisfies \( f'(\hat{r}) = f'(\hat{s}) \), there exists an intermediate point \( \hat{\omega} \in L \) that satisfies

\[
f(\hat{\omega}) - f(\hat{r}) = \langle \hat{\omega} - \hat{r}, f'(\hat{\omega}) \rangle.
\]

Such a point \( \hat{\omega} \) will be called a 2-dimensional Flett’s (mean value) point of \( f \) in \( L \).

We define

\[
\Gamma := \{ f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \mid f \text{ is continuously partial differentiable and } f'(\hat{r}) = f'(\hat{s}) \}
\]

and we remind that \( f'(x, y) = (f_x(x, y), f_y(x, y)) \). Then by Theorem 2, for any \( f \in \Gamma \), there exists a 2-dimensional Flett’s point of \( f \) in \( L \).

In ([9] Theorem 3.1), we proved the Hyers–Ulam stability of the 2-dimensional Lagrange’s points by making use of Theorem 1. In a context similar to the preceding task, using a theorem of [6] and the proof of ([9] Theorem 3.1), we will prove a Hyers–Ulam type stability for the 2-dimensional Flett’s points.

**Theorem 3.** Given the different points \( \hat{r} \) and \( \hat{s} \) of \( \mathbb{R} \times \mathbb{R} \), let \( L \) be the line segment connecting \( \hat{r} \) and \( \hat{s} \). Assume that \( f \) belongs to \( \Gamma \) and \( \hat{\omega}_0 \) is the unique 2-dimensional Flett’s point of \( f \) in \( L \). For any given \( \varepsilon > 0 \), there is a \( \delta > 0 \) with the property that if \( g \) belongs to \( \Gamma \) and \( g \) satisfies the inequality \( \|f(\hat{r}) - g(\hat{r}) - f(\hat{s}) + g(\hat{s})\| < \delta \) for all \( \hat{s} = (x, y) \in L \), then \( g \) has a 2-dimensional Flett’s point \( \hat{\omega}_3 \) in \( L \) satisfying \( |\hat{\omega}_0 - \hat{\omega}_3| < \varepsilon \).

**Proof.** Let \( \hat{r} = (p, q) \) and \( \hat{s} = (u, v) \) be two different points of \( \mathbb{R} \times \mathbb{R} \). By putting \( h = u - p \) and \( k = v - q \), the coordinates of every point on \( L \) are expressed by \( \hat{\omega} = (p + ht, q + kt) \) for some \( t \in [0, 1] \).

An auxiliary function \( F: \mathbb{R} \to \mathbb{R} \) will be defined by \( F(t) = f(\hat{\omega}) = f(p + ht, q + kt) \), and we calculate the derivative of this function as

\[
F'(t) = hf_x(p + ht, q + kt) + kf_y(p + ht, q + kt) = \langle (h, k), f'(\hat{\omega}) \rangle.
\]

Since \( f \) belongs to \( \Gamma \), we obtain

\[
F'(0) = \langle (h, k), f'(\hat{r}) \rangle = \langle (h, k), f'(\hat{s}) \rangle = F'(1).
\]

We will now define another auxiliary function \( \mathcal{F}: \mathbb{R} \to \mathbb{R} \) by

\[
\mathcal{F}(t) = \begin{cases} 
\frac{F(t) - F(0)}{t} & \text{(for } t \neq 0 \text{),} \\
F'(0) & \text{(for } t = 0 \text{).}
\end{cases}
\]

It then follows from (5) that

\[
\mathcal{F}'(t) = \frac{F(t) - F(0)}{t^2} + \frac{F'(t)}{t} = \frac{\mathcal{F}(t) - F(t)}{t} + \frac{F'(t)}{t}
\]

for all \( t \in (0, 1] \). It is clear that \( \mathcal{F} \) is continuous on \([0, 1]\) and continuously differentiable on \((0, 1] \).
We now assert that there is a \( t_0 \in (0,1) \) satisfying
\[
F'(t_0) = 0. \tag{7}
\]

From (5), we know that \( F(0) = F'(0) \). If \( F(1) = F'(0) \), then by Rolle’s theorem, there exists a \( t_0 \in (0,1) \) such that \( F'(t_0) = 0 \) and our assertion is established. If \( F(1) \neq F'(0) \), then either \( F(1) > F'(0) \) or \( F(1) < F'(0) \). Suppose \( F(1) > F'(0) \). It then follows from (4) and (6) that \( F'(1) = -F(1) + F'(1) = -F(1) + F'(0) < 0 \). Since \( F' \) is continuous on \([0,1]\) and \( F'(1) < 0 \), there exists a \( t_1 \in (0,1) \) such that \( F(t_1) > F(1) \). Hence, we have \( F(0) = F'(0) < F(1) < F(t_1) = F'(1) \) and by the intermediate value theorem, there exists a \( t_2 \in (0,t_1) \) such that \( F(t_2) = F(1) \). Now applying Rolle’s theorem to the function \( F \) on the interval \([t_2,1]\), we have \( F'(t_0) = 0 \) for some \( t_0 \in (t_2,1) \subset (0,1) \). A similar argument applies if \( F(1) < F'(0) \), and then, we have \( F'(t_0) = 0 \) for some \( t_0 \in (0,1) \).

We will prove the uniqueness of \( t_0 \) satisfying \( F'(t_0) = 0 \). Assume that there is a \( t^* \in (0,1) \) satisfying \( F'(t^*) = 0 \) and that \( t^* \) is different from \( t_0 \). Then, according to (5) and (6) and previous assumption, the function \( F \) satisfies
\[
F'(t^*) = \frac{F(t^*) - F(0)}{t^*},
\]
i.e., \( t^* \) is the Flett’s point of \( F \). Furthermore, by (3), we obtain
\[
\langle (h,k), f'(t^*) \rangle = \frac{f(\hat{t}^*) - f(\hat{t})}{t^*},
\]
where we set \( \hat{t}^* = (p + ht^*, q + kt^*) \). And the last equality can be rewritten as
\[
f(\hat{t}^*) - f(\hat{t}) = \langle \hat{t}^* - \hat{t}, f'(\hat{t}^*) \rangle.
\]

In view of (2), \( \hat{t}^* \) is the 2-dimensional Flett’s point of \( f \), which would be contrary to our hypothesis that \( \hat{t} \) is the unique 2-dimensional Flett’s point of \( f \). Therefore, we conclude that \( t^* = t_0 \).

In addition, if \( F(1) = F'(0) \), it then follows from (5) that \( F(0) = F(1) \). In view of (5), when \( F(1) > F'(0) (= F(0)) \) or \( F(1) < F'(0) (= F(0)) \), it follows from (4) and (6) that \( F'(1) < 0 \) or \( F'(1) > 0 \), respectively. These facts imply that \( F \) is not monotone on \((0,1)\) for each case, and hence, there exists \( t_1, t_2 \in (0,1) \) such that \( F'(t_1) > 0 \) and \( F'(t_2) < 0 \). Let us here consider the case \( F(1) > F'(0) \) only, and other cases can be treated analogously. Further, we assume that \( t_1 < t_2 \) without loss of generality. Since \( F' \) is continuously differentiable on \((0,1)\), there has to exist a \( \hat{t} \in (t_1, t_2) \) such that \( F'(\hat{t}) = 0 \). By the uniqueness of the point \( t_0 \) satisfying condition (7), we conclude that \( \hat{t} = t_0 \) and we can impose the following condition on \( t_0 \):
\[
F' \text{ changes sign at } t_0.
\]

Let \( \varepsilon > 0 \) be an arbitrary constant and let \( N = (l_0 - r, t_0 + r) \) be an open subinterval of \([0,1]\) for some \( 0 < r \leq \min\{t_0, 1 - t_0\} \). Further, let \( \varepsilon = \min\{t_0 - r, 1 - (t_0 + r)\} \).

With reference to Table 1, Theorem 1 can be translated into Statement (8).

\begin{table}[h]
\centering
\caption{The substitution table.}
\begin{tabular}{cccccccc}
\hline
Theorem 1 & \( n \) & \( N \) & \( t_0 \) & \( f \) & \( g \) & \( \varepsilon \) & \( \delta \) & \( t_1 \) \\
\hline
(8) below & 1 & \( N \) & \( t_0 \) & \( F \) & \( G \) & \( \tilde{\varepsilon} \) & \( \tilde{\delta} \) & \( t_3 \) \\
\hline
\end{tabular}
\end{table}
Indeed, referring to Table 1, Theorem 1 can be translated into:

For any given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) with the property that if a function \( G : \mathbb{R} \rightarrow \mathbb{R} \) is differentiable in \( N \) and satisfies \( |F(t) - G(t)| < \delta \) for each \( t \in N \),

then there is a \( t_3 \in N \) satisfying \( G'(t_3) = 0 \) and \( |t_3 - t_0| < \varepsilon \). 

Since \( F \) is continuously differentiable in \( (0, 1) \), it is also continuously differentiable in \( N \). Now, using (6) and (7), we obtain

\[
F'(t_0) = \frac{-F(t_0) + F(0)}{t_0} = 0
\]

and using the last equality and (5), we get

\[
F'(t_0) = F(t_0) = \frac{F(t_0) - F(0)}{t_0},
\]

i.e., \( t_0 \) is a Flett’s point of \( F \) in \([0, 1]\). Obviously,

\[
F(t_0) = f(p + ht_0, q + kt_0) = f(\eta_0, \xi_0)
\]

and

\[
F'(t_0) = hf\eta_0(p + ht_0, q + k) + k\eta_0 q + k t_0
\]

\[
= hf\eta_0(\eta_0, \xi_0) + k\eta_0(\eta_0, \xi_0),
\]

where \( \eta_0 = p + ht_0 \) and \( \xi_0 = q + kt_0 \). We define \( \hat{\omega}_0 = (\eta_0, \xi_0) = (p + ht_0, q + kt_0) \). Indeed, \( \hat{\omega}_0 \in L \), \( \min \{p, u\} \leq \eta_0 \leq \max \{p, u\} \) and \( \min \{q, v\} \leq \xi_0 \leq \max \{q, v\} \).

By (9), (10), and (11), we have

\[
f(\eta_0, \xi_0) - f(p, q) = F(t_0) - F(0) = t_0 F'(t_0)
\]

\[
= h t_0 f\eta_0(\eta_0, \xi_0) + k t_0 f\eta_0(\eta_0, \xi_0)
\]

\[
= (\eta_0 - p)f\eta_0(\eta_0, \xi_0) + (\xi_0 - q)f\eta_0(\eta_0, \xi_0).
\]

From the last equality, we obtain

\[
f(\hat{\omega}_0) = f(p) = \langle \hat{\omega}_0 - p, f'(\hat{\omega}_0) \rangle,
\]

where \( f'(\hat{\omega}_0) = f'(\eta_0, \xi_0) = (f\eta_0(\eta_0, \xi_0), f\eta_0(\eta_0, \xi_0)) \), which implies that \( \hat{\omega}_0 \) is the unique 2-dimensional Flett’s mean value point of \( f \), which was given in the statement of this theorem.

Assume that \( g \) belongs to \( \Gamma \) and \( g \) satisfies the inequality

\[
|f(p + ht, q + kt) - f(p, q) - g(p + ht, q + kt) + g(p, q)| < \delta
\]

for any \( t \in [0, 1] \), where the value of \( \delta \) will be determined in the end of the proof. An auxiliary function \( G : \mathbb{R} \rightarrow \mathbb{R} \) will now be defined by \( G(t) = g(\hat{\omega}) = g(p + ht, q + kt) \) and the derivative of \( G \) is calculated as follows:

\[
G'(t) = hg\eta_0(\eta_0, \xi_0) + kg\eta_0(\eta_0, \xi_0)
\]

Since \( g \in \Gamma \), it holds that \( G'(0) = \langle (h, k), g'(\hat{s}) \rangle = \langle (h, k), g'(s) \rangle = G'(1) \). Assume that \( \mathcal{G} : \mathbb{R} \rightarrow \mathbb{R} \) is another auxiliary function corresponding to \( G \) defined as

\[
\mathcal{G}(t) = \begin{cases} \frac{G(t) - G(0)}{t} & \text{for } t \neq 0, \\ G'(0) & \text{for } t = 0. \end{cases}
\]
It then follows from (13) that
\[ G'(t) = -\frac{G(t) - G(0)}{t^2} + \frac{G'(t)}{t} = -\frac{G(t)}{t} + \frac{G'(t)}{t} \] (14)
for all \( t \in (0,1] \). Analogously to the case of \( F \), \( G \) is also continuous on \([0,1]\) and continuously differentiable on \((0,1]\).

By using (5), (12), and (13), we get
\[
|F(t) - G(t)| = \left| \frac{F(t) - F(0)}{t} - \frac{G(t) - G(0)}{t} \right| = \frac{1}{t} \left| f(p + ht, q + kt) - f(p, q) - g(p + ht, q + kt) + g(p, q) \right| \leq \frac{\delta}{c} < \frac{1}{c},
\] (15)
for all \( t \in N \). In view of definitions of \( N = (t_0 - r, t_0 + r) \) for some \( 0 < r \leq \min\{t_0, 1 - t_0\} \) and \( c = \min\{t_0 - r, 1 - (t_0 + r)\} \), we get \( \frac{1}{c} < \frac{1}{r} < \frac{1}{c} \). Thus, it follows from (15) and the last inequality that
\[ |F(t) - G(t)| < \frac{\delta}{c} = \delta \] (16)
for all \( t \in N \), where we choose the value of \( \delta \) so that the paragraph (8) is true for a previously given \( \xi = \frac{\epsilon}{\sqrt{1 + c^2}} > 0 \). Then, according to (8), there exists a point \( t_3 \in N \) such that
\[ G'(t_3) = 0 \] (17)
and
\[ |t_3 - t_0| < \xi. \] (18)

Since \( G \) is continuously differentiable on \((0,1]\), it is also continuously differentiable on \( N \). Due to (14) and (17), we have
\[ G'(t_3) = -\frac{G(t_3)}{t_3} + \frac{G'(t_3)}{t_3} = 0 \]
and by the last equality and (13), we get
\[ G'(t_3) = G(t_3) = \frac{G(t_3) - G(0)}{t_3}, \] (19)
which implies that \( t_3 \) is a Flett’s point of \( G \) in \([0,1]\).

Clearly, since \( g \in \Gamma \), we obtain
\[ G(t_3) = g(p + ht_3, q + kt_3) = g(\eta_3, \xi_3) \] (20)
and
\[ G'(t_3) = h g_x(p + ht_3, q + kt_3) + k g_y(p + ht_3, q + kt_3) = h g_x(\eta_3, \xi_3) + k g_y(\eta_3, \xi_3), \] (21)
where \( \eta_3 = p + ht_3 \) and \( \xi_3 = q + kt_3 \). Let us set \( \hat{w}_3 = (\eta_3, \xi_3) = (p + ht_3, q + kt_3) \). Then, \( \hat{w}_3 \in L \), \( \min\{p, u\} \leq \eta_3 \leq \max\{p, u\} \) and \( \min\{q, v\} \leq \xi_3 \leq \max\{q, v\} \).
On account of (19), (20) and (21), we get
\[ g(\eta_3, \xi_3) - g(p, q) = G(t_3) - G(0) = t_3 G'(t_3) = h t_3 g_x(\eta_3, \xi_3) + k t_3 g_y(\eta_3, \xi_3) = (\eta_3 - p) g_x(\eta_3, \xi_3) + (\xi_3 - q) g_y(\eta_3, \xi_3). \]

By using the last equality, we have
\[ g(\hat{\omega}_3) - g(\hat{\tau}) = \langle \hat{\omega}_3 - \hat{\tau}, g'(\hat{\omega}_3) \rangle, \]
where \( g'(\hat{\omega}_3) = g'(\eta_3, \xi_3) = (g_x(\eta_3, \xi_3), g_y(\eta_3, \xi_3)) \). Hence, \( \hat{\omega}_3 \) is a 2-dimensional Flett’s point of \( g \).

Furthermore, by (18) we obtain
\[
|\hat{\omega}_0 - \hat{\omega}_3| = |(p + ht_0, q + kt_0) - (p + ht_3, q + kt_3)|
\quad = |(h(t_0 - t_3), k(t_0 - t_3))|
\quad = \sqrt{h^2 + k^2} |t_0 - t_3|
\quad < \sqrt{h^2 + k^2} \varepsilon = \varepsilon. \tag{22}
\]

Finally, for the value of \( \varepsilon = \frac{\delta}{\sqrt{h^2 + k^2}} \), we select the value of \( \delta \) according to the paragraph (8). Note that the value of \( \varepsilon \) has been given in the paragraph just before (8). Next, we set \( \delta = c \delta \) by using the selected constant \( \delta \). Then, by (12), (16), and (22), we complete our proof. \( \square \)

**Example 1.** Let \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be the function defined as follows
\[ f(x, y) = 4 \sin \left( \frac{x}{2} \cos \left( \frac{y}{4} \right) \right) + x + 2y. \]

Then \( f_x(x, y) = 2 \cos \left( \frac{x}{2} \cos \left( \frac{y}{4} \right) \right) + 1 \) and \( f_y(x, y) = -\sin \left( \frac{x}{2} \sin \left( \frac{y}{4} \right) \right) + 2 \) are continuous, \( f_x(\hat{\tau}) = f_x(\hat{s}) = 1 \) and \( f_y(\hat{\tau}) = f_y(\hat{s}) = 2 \), where we set \( \hat{\tau} = (2\pi, 2\pi) \) and \( \hat{s} = (\pi, 4\pi) \) (see Figure 1).

![Figure 1. Two-dimensional Flett’s point in the line segment.](image)

Moreover, we put \( \hat{\omega} = (2\pi - \pi t, 2\pi + 2\pi t) \) for some \( t \in (0, 1) \). Then we obtain
\[
\begin{align*}
     f(\hat{\omega}) - f(\hat{\tau}) &= -4 \sin^2 \left( \frac{\pi}{2} t \right) + 3\pi t, \\
     f'(\hat{\omega}) &= \left( 2 \sin \left( \frac{\pi}{2} t \cos \left( \frac{\pi}{2} t \right) + 1, -\sin \left( \frac{\pi}{2} t \cos \left( \frac{\pi}{2} t \right) + 2 \right) .
\end{align*}
\]
According to Theorem 2, the point \( \hat{\omega} \) has to satisfy the equation
\[
f(\hat{\omega}) - f(\hat{\tau}) = \langle \hat{\omega} - \hat{\tau}, f'(\hat{\omega}) \rangle
\]
in order for \( \hat{\omega} \) to become a 2-dimensional Flett’s point of \( f \) in the line segment \( L \) connecting two points \( \hat{\tau} \) and \( \hat{\delta} \).

And the last equation is equivalent to
\[
4 \sin^2 \frac{\pi}{2} t = 4 \pi t \sin \frac{\pi}{2} t \cos \frac{\pi}{2} t \quad \text{or} \quad \tan \frac{\pi}{2} t = \pi t
\]
for some \( t \in (0, 1) \). We know that there exists only one solution of the previous equation and we denote this solution by \( t_0 \). Then \( \hat{\omega}_0 = (2 \pi - \pi t_0, 2 \pi + 2 \pi t_0) \) is the unique 2-dimensional Flett’s point of \( f \). We can estimate \( \hat{\omega}_0 = (3.95206 \ldots, 10.9454 \ldots) \) by the numerical calculation using Wolfram Alpha.

Thus by Theorem 3, if a function \( g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is very close to \( f \), then \( g \) has a 2-dimensional Flett’s point near \( \hat{\omega}_0 \). More precisely, for any given \( \epsilon > 0 \), there is a \( \delta > 0 \) with the property that if \( g \) belongs to \( \Gamma \) and satisfies
\[
-4 \sin^2 \frac{\pi}{2} t + 3 \pi t - g(2 \pi - \pi t, 2 \pi + 2 \pi t) + g(2 \pi, 2 \pi) < \delta
\]
for all \( t \in [0, 1] \), then there is a 2-dimensional Flett’s point \( \hat{\omega}_3 = (x_3, y_3) \) of \( g \) in \( L \) with \( (x_3 - 3.95206 \ldots)^2 + (y_3 - 10.9454 \ldots)^2 < \epsilon^2 \).

4. Discussion

The Flett’s mean value theorem is a variant of the Lagrange’s mean value theorem. If a differentiable function \( f : [a, b] \rightarrow \mathbb{R} \) and a point \( \eta \in [a, b] \) satisfy \( f(\eta) - f(a) = f'(\eta)(\eta - a) \), then the point \( \eta \) is called a Flett’s (mean value) point of \( f \) in \( [a, b] \). This concept of Flett’s mean value points can be generalized to 2-dimensional Flett’s mean value points as follows: For the different points \( \hat{\tau} \) and \( \hat{\delta} \) of \( \mathbb{R} \times \mathbb{R} \), let \( L \) be the line segment joining \( \hat{\tau} \) and \( \hat{\delta} \). If a partially differentiable function \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) and an intermediate point \( \hat{\omega} \in L \) satisfy \( f(\hat{\omega}) - f(\hat{\tau}) = \langle \hat{\omega} - \hat{\tau}, f'(\hat{\omega}) \rangle \), then the point \( \hat{\omega} \) is called a 2-dimensional Flett’s (mean value) point of \( f \) in \( L \).

According to the 2-dimensional Flett’s mean value theorem (Theorem 2), for every function \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) with continuous partial derivatives \( f_x \) and \( f_y \) and satisfying \( f'(\hat{\tau}) = f'(\hat{\delta}) \), there is a 2-dimensional Flett’s point \( \hat{\omega} \) of \( f \) in \( L \). Assume that \( g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) has continuous partial derivatives \( g_x \) and \( g_y \) and satisfies \( g'(\hat{\tau}) = g'(\hat{\delta}) \). Then, due to Theorem 2, there is a 2-dimensional Flett’s point \( \hat{\omega} \) of \( g \) in \( L \). If the surface of \( z = g(x, y) \) almost overlaps the surface of \( z = f(x, y) \), the question arises whether each 2-dimensional Flett’s mean value point \( \hat{\omega} \) of \( g \) is always in the vicinity of a 2-dimensional Flett’s mean value point \( \hat{\omega} \) of \( f \).

The answer to this question was discussed in Section 3. Indeed, we proved the Hyers–Ulam stability of 2-dimensional Flett’s points under the assumption that \( f \) has exactly one 2-dimensional Flett’s point in \( L \). We may regard Theorem 3 of the present paper as a generalization as well as an extension of ([4] Theorem 2.2) and at the same time it is a counterpart of Theorem 2 for the 2-dimensional Flett’s points.

The 2-dimensional Flett’s mean value theorem is mainly used in the field of functional equations, which will be omitted in this paper because of time and space constraints. We recommend that readers who are interested in usability of 2-dimensional Flett’s mean value theorem read ([8] Chapter 5).

Now we are going to raise two open problems.

(a) Can we prove Theorem 3 without assuming the uniqueness of 2-dimensional Flett’s point?
(b) Can we prove Theorem 3 for the \( n \)-dimensional Flett’s points?

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript. Writing original draft, S.-M.J. and J.-H.K.; Writing review & editing, S.-M.J. and Y.W.N.

Funding: This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B03931061).
Acknowledgments: This work was supported by 2019 Hongik University Research Fund.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this article.

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