An example of $C^1$-generically wild homoclinic classes with index deficiency

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Abstract
Given a four-dimensional smooth closed manifold, we construct a diffeomorphism that has a homoclinic class whose continuation locally generically satisfies the following condition: it does not admit any kind of dominated splittings whereas any periodic point belonging to it never has index (the dimension of the unstable manifold) one.

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1. Introduction
Let $M$ be a closed (compact and boundaryless) smooth manifold equipped with a Riemannian metric. We denote the group of $C^1$-diffeomorphisms of $M$ by $\text{Diff}^1(M)$ and furnish it with the $C^1$-topology. Let $P$ be a hyperbolic periodic point of $f \in \text{Diff}^1(M)$. We denote by $W^s(P, f)$ (respectively $W^u(P, f)$) the stable (respectively unstable) manifold of $P$. We also use the simplified notation $W^s(P)$ and $W^u(P)$.

The homoclinic class of $P$, denoted by $H(P, f)$ (or simply $H(P)$), is defined to be the closure of the set of points of transversal intersections of $W^u(P)$ and $W^s(P)$, i.e. we define $H(P) := \overline{W^u(P) \cap W^s(P)}$. One can prove that homoclinic classes are topologically transitive (have a dense orbit) and hyperbolic periodic points are dense in homoclinic classes. We already have enough understanding about the structure of homoclinic classes in uniformly hyperbolic dynamical systems.

In the sphere of non-hyperbolic dynamics, one can find homoclinic classes that are robustly non-hyperbolic. Some of them exhibit a weak form of hyperbolicity such as partial hyperbolicities or dominated splittings. But, in the ‘worst’ case, homoclinic classes can fail to have any kind of dominated splittings robustly. Let us review the notion of dominated splittings. Given an $f$-invariant subset $\Sigma$ of $M$, we say that $\Sigma$ admits an $l$-dominated splitting if there
exist two non-trivial $df$-invariant subbundles $F$ and $G$ of $TM|\Sigma$ such that $TM|\Sigma = F \oplus G$ and for all $x \in \Sigma$ the following holds:

$$\|d(f^t)(x)|_{F(x)}\| \cdot \|d(f)|_{G(f(x))}\| < 1/2,$$

where $\| \cdot \|$ denotes the operator norm derived from the Riemannian metric. We say that $\Sigma$ does not admit dominated splitting if $\Sigma$ does not admit $n$-dominated splitting for all $n > 0$.

It is an intriguing subject to study the effects of the robust absence of dominated splittings on homoclinic classes. (Here the meaning of the word ‘robust’ is not clear. We will discuss the notion of the robust absence of dominated splittings later.) For example, in [BDP], Bonatti, Díaz and Pujals proved, $C^1$-generically, robust absence of dominated splittings implies the $C^1$-Newhouse phenomenon, i.e. locally generic coexistence of infinitely many sinks or sources. Bonatti and Díaz in [BD] showed that, under some conditions on Jacobian, the robust absence of dominated splittings on a homoclinic class implies the creation of complicated dynamics called universal dynamics. These results tell us that the study of robustly non-dominated homoclinic classes is able to produce new ideas.

In [S], inspired by [ABCDW], aiming at understanding the mechanisms of the absence of dominated splitting, the index sets of non-dominated homoclinic classes are studied. The index of a hyperbolic periodic point $P$ (denoted by $\text{ind}(P)$) is defined to be the dimension of the unstable manifold of $P$. For a homoclinic class $H(P)$, its index set, denoted by $\text{ind}(H(P))$, is defined to be the collection of integers that appear as the index of some hyperbolic periodic point in $H(P)$, namely, we put

$$\text{ind}(H(P)) := \{\text{ind}(Q) \in \mathbb{N} | Q \in \text{Per}_h(f) \cap H(P)\},$$

where $\text{Per}_h(f)$ is the set of hyperbolic periodic points of $f$. In [S], it is proved that, under some assumptions on Jacobian, $C^1$-generically, the index set of three-dimensional robustly non-dominated homoclinic classes contains all possible indices, namely 1 and 2. The result sounds reasonable if one reviews the idea of the argument in [BDP]. The robust absence of a homoclinic class scatters its hyperbolicity to any directions. Hence, it is plausible that one can construct periodic points with prescribed indices by mixing the hyperbolicity with small perturbations.

Intuitively speaking, the lack of domination on a homoclinic class seems to imply that its index set would be large. In this paper, we give an example which tells us that this naive idea has some limitations in higher dimensional cases.

Let us give the precise statement of our result. For $g \in \text{Diff}^1(M)$ sufficiently close to $f$, one can define the continuation of $P$. We denote the continuation of $P$ for $g$ by $P(g)$. We use the notation $H(P, g)$ in the sense of $H(P(g), g)$. A homoclinic class $H(P, f)$ is said to be wild if there exists a $C^1$-neighbourhood $\mathcal{U}$ of $f$ such that for all $g \in \mathcal{U}$, the continuation $H(P, g)$ of $H(P, f)$ does not admit any kind of dominated splittings. It is said to be generically wild if there exists a neighbourhood $\mathcal{U}$ of $f$ and residual subset $\mathcal{R}$ of $\mathcal{U}$ such that for every $g \in \mathcal{R}$ the continuation $H(P, g)$ of $H(P, f)$ does not admit dominated splitting.

Then, our result is the following.

**Theorem.** For every four-dimensional smooth closed manifold $M$, there exists $f \in \text{Diff}^1(M)$ that satisfies the following: there exist a hyperbolic fixed point $P$ of $f$, a $C^1$-neighbourhood $\mathcal{U}$ of $f$ and a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that for every $g \in \mathcal{R}$, $H(P, g)$ does not admit any kind of dominated splittings and $\text{ind}(H(P, g)) = \{2, 3\}$.

This theorem says that a generically wild homoclinic class may have index deficiency, more precisely, it is not always true that one can construct a saddle with any prescribed index, by giving a small perturbation to a generically wild homoclinic class.
Let us introduce the idea of the proof. The idea of ensuring that the homoclinic class is generically wild is not novel. One can find a similar idea in [BD]. The novelty of our argument is the way we ensure the non-existence of periodic points whose indices are equal to 1. The idea is as follows: if a diffeomorphism strongly expands some two-dimensional subspace of the tangent space at each point in the homoclinic class, then each three-dimensional subspace must be volume-expansive. In particular, no periodic point inside the homoclinic class has index 1.

In this paper, we work on four-dimensional manifolds. This is only because it is enough for our purpose to present an example of a generically wild homoclinic class with index deficiency. It is natural to wonder what would happen on higher dimensional manifolds. However, the complete argument that covers general cases requires complicated descriptions and most of them do not seem to be essential. So, we concentrate on this special environment.

Let us discuss a technical matter. Theorem 1 says that we can construct an example of generically wild homoclinic class. In papers such as [BD] and [S], they treated wild homoclinic classes instead of generically wild homoclinic classes. The difference is not so serious for the following reason: we can prove many results that hold for wild homoclinic classes starting from generically wild homoclinic classes. For example, we can get generically wild homoclinic class version of the result in [S] with slight modifications. Yet, the following question might be interesting in itself:

**Question.** Are generically wild homoclinic classes wild?

This paper is organized as follows. In section 2, we reduce our problem to a local problem and we describe an abstract condition that is sufficient to guarantee the properties we claimed in the theorem. Section 3, we construct a diffeomorphism that satisfies the conditions given in section 2.

2. A sufficient condition

In this section, we describe an abstract condition that is sufficient for the conclusion of our theorem, and to prove the theorem assuming the existence of the diffeomorphism that satisfies such a condition.

We introduce some notation. We denote the tangent bundle of $M$ by $T M$ and by $\Lambda^k(T M)$ the exterior product of $T M$ with degree $k$. We furnish this bundle with the metric canonically induced from the Riemannian metric on $M$. For $f \in \text{Diff}^1(M)$, we denote by $\Lambda^k(df)$ the bundle map of $\Lambda^k(T M)$ canonically induced from $df$. For $f : V \to W$, where $V$ and $W$ are two finite dimensional Euclidean spaces and $f$ is a linear map, we define the value $m(f)$ to be the minimum of $\| f(v) \|$, where $v$ ranges over all the unit vectors in $V$.

The following proposition provides the sufficient conditions for the theorem.

**Proposition 1.** Let $M$ be a closed four-dimensional smooth Riemannian manifold and $f \in \text{Diff}^1(M)$. Suppose that $f$ satisfies all the conditions below:

(W1) There are two compact sets $A$ and $B$ in $M$ such that $B \subset A$, $f(A) \subset \text{int}(A)$ and $f(B) \subset \text{int}(B)$ (for $U \subset M$, we denote its topological interior by $\text{int}(U)$).

(W2) There exist two hyperbolic fixed points $P$ and $Q$ of $f$ in $C \coloneqq A \setminus B$.

(W3) $f$ has a heterodimensional cycle associated with $P$ and $Q$.

(W4) $\text{ind}(P) = 3$, and let $\sigma(P)$, $\mu_1(P)$, $\mu_2(P)$ and $\mu_3(P)$ be the eigenvalues of $df(P)$ in non-decreasing order of their absolute values. Then $\mu_1(P)$ and $\mu_2(P)$ are in $\mathbb{C} \setminus \mathbb{R}$.

(W5) $\text{ind}(Q) = 2$, and every eigenvalue of $df(Q)$ is in $\mathbb{C} \setminus \mathbb{R}$.
There exists a constant $K > 1$ such that $m(\Lambda^3(df)) > K$ on $C$. In other words, $df$ expands every three-dimensional subspace of $T_x M$ for all $x \in C$ in the volume with an expansion rate greater than $K$.

Then, there exists an open neighbourhood $U \subset \text{Diff}^1(M)$ of $f$ and a residual subset $\mathcal{R}$ of $U$ satisfying the following: for every $g \in \mathcal{R}$, the homoclinic class $H(P, g)$ does not admit dominated splittings and $\text{ind}(H(P, g)) = [2, 3]$.

For the proof, we prepare three lemmas. To state the first lemma, we review the definition of robust cycles. Let $f \in \text{Diff}^1(M)$, and $\Gamma$ and $\Sigma$ be two transitive hyperbolic invariant sets of $f$. We say that $f$ has a heterodimensional cycle associated with $\Gamma$ and $\Sigma$ if the following holds:

1. The indices (the dimension of the unstable manifolds) of the sets $\Gamma$ and $\Sigma$ are different.
2. The stable manifold of $\Gamma$ meets the unstable manifold of $\Sigma$ and the same holds for the unstable manifold of $\Sigma$ and the unstable manifold of $\Gamma$.

We say that the heterodimensional cycle associated with $\Gamma$ and $\Sigma$ is $\text{C}^1$-robust if there exists a $\text{C}^1$-neighbourhood $U$ of $f$ such that for each $g \in U$, $g$ has the heterodimensional cycle associated with the continuations $\Gamma(g)$ of $\Gamma$ and $\Sigma(g)$ of $\Sigma$. Then, we have the following (for details, see [BDK]).

**Lemma 1 (Theorem 2 in [BDK]).** Consider a diffeomorphism $f$ of a four-dimensional closed smooth manifold having a heterodimensional cycle associated with two hyperbolic fixed points $P$ and $Q$ with $\text{ind}(P) = 1$ and $\text{ind}(Q) = 2$. Let $\sigma(P)$, $\mu_1(P)$, $\mu_2(P)$ and $\mu_3(P)$ be the eigenvalues of $df(P)$ in non-decreasing order of their absolute values. Suppose $df(Q)$ has a non-real eigenvalue with absolute value less than one and $\mu_1(P)$, $\mu_2(P)$ are non-real.

Then, there are diffeomorphisms $g$ arbitrarily $\text{C}^1$-close to $f$ with robust heterodimensional cycles associated with transitive hyperbolic sets $\Gamma(g)$ and $\Sigma(g)$ containing the continuations of $P(g)$ and $Q(g)$ of $P$ and $Q$.

To state the second lemma, we prepare some definitions. Let $\varepsilon$ be a positive real number, $x, y \in M$ and $d(\cdot, \cdot)$ be a metric of $M$. An $\varepsilon$-chain from $x$ to $y$ for $f \in \text{Diff}^1(M)$ is a sequence $(x_i)_{i=1}^n$ ($n \geq 2$) in $M$ satisfying $d(f(x_i), x_{i+1}) < \varepsilon$ for $1 \leq i < n$, $x_1 = x$ and $x_n = y$. Two points $x, y$ are said to be chain equivalent if for given $\varepsilon > 0$, there exist an $\varepsilon$-chain from $x$ to $y$ and an $\varepsilon$-chain from $y$ to $x$. A point $x \in M$ is said to be a chain recurrent point if $x$ is chain equivalent to itself. For a chain recurrent point $x$, its chain recurrence class $\mathcal{C}(x)$ is defined to be the set of the points that is chain equivalent to $x$.

**Lemma 2 ([BC]).** There exists a residual subset $\mathcal{R}_1$ in $\text{Diff}^1(M)$ such that the following holds: for every $f \in \mathcal{R}_1$ and every chain recurrence class $\mathcal{C}(x)$ of $f$, if $\mathcal{C}(x)$ contains a hyperbolic periodic point $P$, then $\mathcal{C}(x)$ coincides with $H(P)$.

The following lemma is easy to prove, so the proof is left to the reader.

**Lemma 3.** Let $P$ and $Q$ be hyperbolic periodic points and there exists a heterodimensional cycle associated with two transitive hyperbolic invariant sets $\Gamma$ and $\Sigma$ containing $P$ and $Q$, respectively. Then $P$ and $Q$ belong to the same chain recurrence class.

Let us give the proof of proposition 1.

**Proof of proposition 1.** Let $f$ be a diffeomorphism of a four-dimensional manifold such that $f$ satisfies (W1)–(W6). By applying lemma 1 to $P$ and $Q$, we can take a diffeomorphism $g$, an open neighbourhood $U$ of $g$ and two hyperbolic transitive invariant sets $\Gamma$, $\Sigma$ such that
for every $h \in \mathcal{U}$, the continuation $\Gamma(h)$ contains $P(h)$, $\Sigma(h)$ contains $Q(h)$ and $f$ has a heterodimensional cycle associated with $\Gamma(h)$ and $\Sigma(h)$. Note that by taking $\mathcal{U}$ sufficiently close to $f$, we can assume, for every $h \in \mathcal{U}$, all the properties (W1)-(W6) hold except (W3).

Let us put $\mathcal{R} = \mathcal{U} \cap \mathcal{R}_1$, where $\mathcal{R}_1$ is the residual set in lemma 2. We prove that the conclusions of proposition 1 hold for every $h \in \mathcal{R}$. First, we show that $H(P, h)$ does not admit dominated splittings. Since $h \in \mathcal{U}$, there exists a heterodimensional cycle associated with $\Gamma(h)$ and $\Sigma(h)$ with $P \in \Gamma(h)$ and $Q \in \Sigma(h)$. Then, lemma 3 implies that $P(h)$ and $Q(h)$ belong to the same chain recurrence class. Since $h \in \mathcal{R}_1$, this chain recurrence class coincides with $H(P, h)$ and simultaneously $H(Q, h)$, in particular $H(P, h) = H(Q, h)$. By (W4), we can see that $H(P, h)$ does not admit dominated splitting $E \oplus F$ with $\dim E = 2$. We can also see that $H(P, h)$ does not admit dominated splitting $E \oplus F$ where $\dim E = 1$ or 3, for $H(P, h)$ contains $Q(h)$ and $h$ satisfies (W5). Thus, $H(P, h)$ does not admit any kind of dominated splittings.

Let us show that $\text{ind}(H(P, h)) = [2, 3]$. Since $H(P, h)$ contains $P(h)$ and $Q(h)$, $\text{ind}(H(P, h))$ contains 2 and 3. We need to prove that $\text{ind}(H(P, h))$ does not contain 1. To check this, it is enough to prove that $H(P, h) \subset C$. Indeed, (W6) says that every iteration of $dh$ expands every three-dimensional subspace of the tangent space on $H(P, h)$ in volume. So the indices of any periodic point in $H(P, h)$ cannot be 1.

To see $H(P, h) \subset C$, we first show $W^s(P, h) \cap W^u(P, h) \subset C$. Let us take $x \in W^s(P, h) \cap W^u(P, h)$. Since $x \in W^u(P, h)$, there exists $n > 0$ such that $h^{-n}(x) \in A$. So we have $x \in h^n(A)$. Since $A$ satisfies $h(A) \subset \text{int}(A)$, we have $h^n(A) \subset A$ for $n > 0$ and this implies $x \in A$. We show $x \notin B$. Since $h(B) \subset \text{int}(B)$, we get $h^n(B) \subset B$. If $x \in B$, then for all $n > 0$ we have $h^n(x) \in B$. By definition, $x \in W^u(P, h)$ and therefore $f^n(x)$ converges to $P(h)$ as $n \to \infty$. This contradicts the fact $P(h) \notin B$. Thus we have $W^s(P, h) \cap W^u(P, h) \subset C$.

In the following we show $H(P, h) \subset C$. Take $y \in H(P, h)$. By definition there exists a sequence $(x_n) \subset W^s(P, h) \cap W^u(P, h)$ converging to $y$ as $n \to \infty$. Since $W^s(P, h) \cap W^u(P, h) \subset A$ and $A$ is compact, we have $H(P, h) \subset A$, in particular $y \in A$. To prove $y \notin B$, let us assume that $y \in B$. Then $h(y)$ belongs to $\text{int}(B)$ and hence there exists a neighbourhood $U$ of $h(y)$ contained in $\text{int}(B)$. Since $h$ is continuous, the sequence $(h(x_n))$ converges to $h(y)$. This implies that there exists $N$ such that $h(x_N)$ belongs to $U$, in particular to $B$. Since $x_N \in W^s(P, h) \cap W^u(P, h)$, we have $h(x_N) \in W^s(P, h) \cap W^u(P, h)$. This is a contradiction, because we have already proved that $W^s(P, h) \cap W^u(P, h)$ is disjoint from $B$. Therefore, we have proved $H(P, h) \subset C$ and completed the proof of proposition 1.

The following proposition will be proved in section 3.

**Proposition 2.** There exists $f \in \text{Diff}(\mathbb{R}^4)$ satisfying the following properties:

(w1) The support of $f$ is compact, where the support of $f$ is defined to be the closure of the set $\{x \in M | f(x) \neq x\}$.

(w2) There are two compact sets $A, B \subset \mathbb{R}^4$ with $B \subset A$, $h(A) \subset \text{int}(A)$ and $h(B) \subset \text{int}(B)$.

(w3) There exist two fixed points $P$ and $Q$ of $f$ in $C := A \setminus B$.

(w4) $f$ has a heterodimensional cycle associated with $P$ and $Q$.

(w5) $\text{ind}(P) = 3$, and let $\sigma(P), \mu_1(P), \mu_2(P)$ and $\mu_3(P)$ be the eigenvalues of $df(P)$ in non-decreasing order of their absolute values. Then $\mu_1(P)$ and $\mu_2(P)$ are in $\mathbb{C} \setminus \mathbb{R}$.

(w6) $\text{ind}(Q) = 2$, and every eigenvalue of $df(Q)$ is in $\mathbb{C} \setminus \mathbb{R}$.

(w7) There exists a constant $K > 1$ such that $m(\Lambda^3(df)) > K$ on $C$ (we furnish $\mathbb{R}^4$ with the standard Riemannian structure). In other words, $df$ expands every three-dimensional subspace of $T_x\mathbb{R}^4$ in the volume for all $x \in C$ with an expansion rate greater than $K$. 

Let us prove our theorem using propositions 1 and 2.

**Proof of the theorem.** Let us take a diffeomorphism $f$ of $\mathbb{R}^4$ which satisfies $(w1)$–$(w7)$ of proposition 2. Given a four-dimensional smooth closed manifold $M$, we take a point $x \in M$ and a coordinate, chart $\phi : U \rightarrow \mathbb{R}^4$ around $x$. By changing the coordinate, if necessary, we can assume $\phi(U)$ contains the support of $f$. Then, we define a diffeomorphism $F$ of $M$ as follows: $F(x) = (\phi^{-1} \circ f \circ \phi)(x)$ for $x \in U$, otherwise $F(x) = x$. Let us denote the standard Riemannian metric on $\mathbb{R}^4$ by $g$. By the partition of unity, we construct a Riemannian metric $\tilde{g}$ on $M$ that coincides with the pullback of $g$ by $\phi$ at every point in $\phi^{-1}(A)$. Now, by applying proposition 1 to the triplet $(M, F, \tilde{g})$, we can find a diffeomorphism and its open neighbourhood in $\text{Diff}^1(M)$ that satisfy the statement of the theorem. \hfill $\square$

Now, for the proof of our theorem, we only need to prove proposition 2. We will give the proof in the next section.

3. **Construction of the diffeomorphism**

In this section, we give the proof of proposition 2.

### 3.1. Notation and sketch of the proof

In this subsection, we introduce some notation and provide the idea of our proof of proposition 2.

We identify every point $X$ of $\mathbb{R}^4$ with the vector that starts from the origin and ends at $X$. Under this identification, we define the addition between any two points of $\mathbb{R}^4$ and multiplication by real numbers. We denote the standard Euclidean distance of $\mathbb{R}^4$ as $d(\cdot, \cdot)$. We also use the $\ell^\infty$-distance of $\mathbb{R}^4$ and denote it as $d_1(\cdot, \cdot)$. More precisely, given $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, we define

$$d(X, Y) = \left(\sum_{i=1}^{4}(x_i - y_i)^2\right)^{1/2}, \quad d_1(X, Y) = \max_{i=1,\ldots,4} |x_i - y_i|.$$  

Let us define some points in $\mathbb{R}^4$ as follows:

- $P = (0, 0, 0, 0)$, $Q = (0, 10, 0, 0)$, $C_1 = (0, 10, 5, 0)$, $C_2 = (10, 10, 5, 0)$, $C_3 = (10, 0, 5, 0)$, $C_4 = (10, 0, 0, 0)$.

We also define three subsets in $\mathbb{R}^4$ as follows:

- $\ell_1 := \{(10 + x, 0, 0, 0) \mid |x| < 0.2\}$, $\ell_2 := \{(0, 10 + y, 0, 0) \mid |y| < 1\}$,
- $\ell_3 := \{(10, 10, 5 + z, w) \mid |z|, |w| < 0.2\}$.

For $X \in \mathbb{R}^4$ and $l > 0$, we denote by $B(X, l)$ the four-dimensional cube with edges of length $2l$ centred at $X$. More precisely, we put

$$B(X, l) := \{Z \in \mathbb{R}^4 \mid d_1(X, Z) \leq l\}.$$  

With this notation, we define the following:

- $B_h(P) := B(P, 1/100)$, $B_h(Q) := B(Q, 1/100)$,
- $B_s(P) := B(P, 1/300)$, $B_s(Q) := B(Q, 1/300)$.

Given $X, Y \in \mathbb{R}^4$ and $l < 2d(X, Y)$, we denote by $C(X, Y, l)$ the four-dimensional box defined as the collection of points whose $\ell^\infty$-distance from the segment joining $X$ and $Y$ is less than $l$. More precisely, we put

$$C(X, Y, l) := \bigcup_{0 \leq t \leq 1} B(tX + (1 - t)Y, l).$$
With this notation, we define
\[ D := \mathcal{C}(C_1, C_2, 1) \cup \mathcal{C}(C_2, C_3, 0.7) \cup \mathcal{C}(C_3, C_4, 0.4). \]

Throughout this section, \( \lambda \) denotes a real number greater than 20. We define
\[ A := [-\lambda^2, \lambda^2]^4, \quad C := [-\lambda^2, \lambda^2] \times (-1/2, 1/2)^2, \quad C' := [-\lambda^2, \lambda^2]^2 \times (-7, 7)^2, \]
\[ B := A \setminus C, \quad B' := A \setminus C'. \]

Note that these sets depend on the value of \( \lambda \), while the points and sets defined above are independent of \( \lambda \).

Let us give the idea of the proof of proposition 2. For the proof, we construct two diffeomorphisms \( \Phi \) and \( \Upsilon \) of \( \mathbb{R}^4 \). The composition \( \Upsilon \circ \Phi \) gives a diffeomorphism that satisfies \((w1)-(w7)\) of proposition 2 except \((w4)\). Let us explain the role of each diffeomorphism. \( \Phi \) has two hyperbolic fixed points \( P \) and \( Q \) whose eigenvalues satisfy \((w5)\) and \((w6)\), and there is a non-empty intersection between \( W^u(P) \) and \( W^s(Q) \) (see figure 1). Furthermore, \( \Phi \) is constructed so that once a point escapes, it never returns close to \( P \) and \( Q \) (this corresponds to \((w2)\)).

To obtain property \((w4)\), we need to connect \( W^u(Q) \) and \( W^s(P) \). The diffeomorphism \( \Upsilon \) is used for this purpose. \( \Upsilon \) pushes \( W^u(Q, \Phi) \) so that it has an intersection with \( W^s(P) \). We need to guarantee that this perturbation has little effect on the connection between \( W^u(P) \) and \( W^s(Q) \) and the whole structure of the dynamics. So the push makes a detour (see figure 2).

The diffeomorphism \( \Upsilon \) is constructed independent of the value of \( \lambda \). After the construction of \( \Upsilon \), we choose \( \lambda \) sufficiently large so that the effect of \( d\Upsilon \) is ignorable when we check \((w7)\) for \( \Upsilon \circ \Phi \).

Finally, we prepare a lemma that we frequently use throughout this section. We omit its proof.

**Lemma 4.** Given two closed interval \([a, b] \subset [c, d]\) \(a, b, c, d \text{ may be } \pm \infty \text{ and we assume that } [a, b] \text{ is a proper subset of } [c, d]\), there exists a \( C^\infty \)-function \( \rho[a, b, c, d](t) : \mathbb{R} \to \mathbb{R} \) that satisfies the following properties:

- For all \( t \in \mathbb{R} \), \( 0 \leq \rho[a, b, c, d](t) \leq 1 \).
- If \( t \in [a, b] \), then \( \rho[a, b, c, d](t) = 1 \).
- If \( t \notin [c, d] \), then \( \rho[a, b, c, d](t) = 0 \).
In this section, we construct the diffeomorphism $\Phi$.

**Proposition 3.** There exists a diffeomorphism $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ that has the following properties:

1. The support of $\Phi$ is contained in $[-\lambda^3 - 1, \lambda^3 + 1]^4$.
2. $\Phi(A) \subset \text{int}(A)$ and $\Phi(B) \subset \text{int}(B)$.
3. $P$ and $Q$ are hyperbolic fixed points of $\Phi$.
4. Ind$(P) = 3$, and let $\sigma(P), \mu_1(P), \mu_2(P)$ and $\mu_3(P)$ be the eigenvalues of $df(P)$ in non-decreasing order of their absolute values. Then $\mu_1(P)$ and $\mu_2(P)$ are in $\mathbb{C} \setminus \mathbb{R}$.
5. $\ell_1 \subset W^s(P, \Phi)$ and $\bigcup_{n \geq 1} \Phi^{-n}(\ell_1) \cap D = \emptyset$.
6. $\ell_2 \subset W^u(P, \Phi)$ and $\bigcup_{n \geq 0} \Phi^n(\ell_2) \cap D = \emptyset$.
7. Ind$(Q) = 2$, and every eigenvalue of $df(Q)$ is in $\mathbb{C} \setminus \mathbb{R}$.
8. $\ell_2 \subset W^s(Q, \Phi)$ and $\bigcup_{n \geq 0} \Phi^{-n}(\ell_2) \cap D = \emptyset$.
9. $\ell_3 \subset W^u(Q, \Phi)$ and $\bigcup_{n \geq 1} \Phi^{-n}(\ell_3) \cap D = \emptyset$.
10. There exists a constant $c_\Phi > 0$ (independent of $\lambda$) such that the inequality $m(\Lambda^3(d\Phi)(X)) > c_\Phi \lambda$ holds for every $X \in C$.

To prove proposition 3, we create auxiliary diffeomorphisms $F$, $G$ and $H$ of $\mathbb{R}$ and $\Theta$ of $\mathbb{R}^4$. After that, we give the proof of proposition 3.

**Lemma 5.** There exists a $C^\infty$-diffeomorphism $F$ of $\mathbb{R}$ that satisfies the following properties:

1. The support of $F$ is contained in $[-\lambda^3, \lambda^3]$.
2. For any $x \in [-\lambda^2, \lambda^2] \setminus \{0\}$, the inequality $0 < F(x)/x < 1/9$ holds.
3. There is a constant $c_1 > 0$ (independent of $\lambda$) such that the inequality $\min_{x \in [-\lambda^2, \lambda^2]} F'(x) > c_1$ holds.

**Proof.** Let us consider the vector field on $\mathbb{R}$ given by $\dot{x}(t) = f(x)$, where $f(x)$ is a $C^\infty$-function on $\mathbb{R}$ that has the following properties:

- If $|x| < \lambda^3 - 2$, $f(x) = -\log(10)x$.
- If $|x| > \lambda^3 - 1$, $f(x) = 0$.

We can construct such an $f$ as follows:

$$f(x) = -\log(10)x[(-\lambda^3 - 2), \lambda^3 - 2, -(\lambda^3 - 1), \lambda^3 - 1](x).$$
Let us take the time-1 map of this vector field and denote it by $F(x)$ (we can consider the time-1 map for all $x \in \mathbb{R}$ since $f$ has compact support and is Lipschitz continuous). Then, it is not difficult to check that $F(x)$ satisfies all the properties (F1)–(F3) and details are left to the reader. \hfill \square

**Remark 1.** Conditions (F1)–(F3) imply that $F$ satisfies the following conditions:

(F4) 0 is an attracting fixed point of $F$.

(F5) $F([-\lambda^2, \lambda^2]) \subset (-\lambda^2, \lambda^2)$.

**Lemma 6.** There exists a $C^\infty$-diffeomorphism $G$ of $\mathbb{R}$ satisfying the following properties:

(G1) The support of $G$ is contained in $[-\lambda^3, \lambda^3]$.

(G2) $G([-\lambda^2, \lambda^2]) \subset (-\lambda^2, \lambda^2)$.

(G3) For any $y \in [-1/100, 1/100] \setminus \{0\}$, the inequality $9 < G(y)/y < 10$ holds.

(G4) For any $y \in [-1/100, 1/100] \setminus \{0\}$, the inequality $0 < (G(y + 10))/y < 1/10$ holds.

(G5) For any $y \in (0, 10)$, $\lim_{n \to \infty} G^{-n}(y) = 0$ and $\lim_{n \to \infty} G^{-n}(y) = 10$.

(G6) There is a constant $c_2 > 0$ (independent of $\lambda$) such that the inequality $\min_{y \in [-\lambda^3, \lambda^3]} G'(y) > c_2$ holds.

**Proof.** Let us consider the vector field $\dot{y}(t) = g(y)$, where $g(y)$ is a $C^\infty$-function on $\mathbb{R}$ satisfying the following properties:

- If $|y| < \lambda^3 - 2$, $g(y) = -\log(10)/100(y^3 - 100y)$.
- If $|y| > \lambda^3 - 1$, $g(y) = 0$.

We can construct such a $g$ as follows:

$$g(y) = -\log(10)/100(y^3 - 100y)\rho(-\lambda^3 - 2, \lambda^3 - 2, -\lambda^3 - 1, \lambda^3 - 1)(y).$$

Let us take the time-1 map of this vector field and denote it by $\tilde{G}(y)$ (we can consider the time-1 map for all $x \in \mathbb{R}$ by the same reason for $F(x)$). Then, it is not difficult to check that $\tilde{G}(y)$ satisfies (G1)–(G6) except (G3) and (G4). (G3) can be proved by comparing the dynamics of $\dot{y}(t) = g(y)$ with the linear dynamics $\ddot{y}(t) = (\log 10)y$ and $\dot{y}(t) = (\log 9)y$. (G4) can be proved by similar techniques. Details are left to the reader. \hfill \square

**Remark 2.** Conditions (G1)–(G6) imply that $G$ satisfies the following conditions:

(G7) 0 is a repelling fixed point of $G$.

(G8) 10 is an attracting fixed point of $G$.

**Lemma 7.** There exists a $C^\infty$-diffeomorphism $H$ of $\mathbb{R}$ satisfying the following properties:

(H1) For every $z$, $H(-z) = -H(z)$.

(H2) The support of $H(z)$ is contained in $[-\lambda^3, \lambda^3]$.

(H3) $H([-\lambda^2, \lambda^2]) \subset (-\lambda^2, \lambda^2)$.

(H4) $H([1/2, \lambda^2]) \subset (7, \lambda^2)$.

(H5) For $0 \leq z \leq 1$, $H(z) = \lambda z$. In particular, 0 is a repelling fixed point of $H$.

**Proof.** First, we construct a $C^\infty$-function $h(t)$ on $\mathbb{R}_{\geq 0}$ that satisfies the following properties:

(h1) For all $t \geq 0$, $h(t) > 0$.

(h2) For $0 \leq t \leq 1$, $h(t) = \lambda$.

(h3) $\int_1^\infty h(t) \, dt = 1$. 

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(h4) For $2 \leq t \leq \lambda^2$, $h(t) = \lambda^2$.
(h5) $\int_0^{\lambda^2} h(t) \, dt = \lambda^3 - 1$.
(h6) For $t \geq \lambda^3 - 1$, $h(t) = 1$.

Let us discuss how to construct such an $h(t)$. We put

- $\rho_1(t) := (\lambda - \lambda^2) \rho[-\infty, 1, -\infty, 1 + \lambda^{-2}](t)$,
- $\rho_2(t; \alpha) := \alpha \rho[\frac{7}{5}, \frac{8}{5}, \frac{6}{5}, \frac{9}{5}](t)$,
- $\rho_3(t) := (1 - \lambda^{-2}) \rho[\lambda^3 - 1, +\infty, \lambda^3 - 2, +\infty](t)$,
- $\rho_4(t; \beta) := \beta \rho[\lambda^3 - 8/5, \lambda^3 - 7/5, \lambda^3 - 9/5, \lambda^3 - 6/5](t)$.

and take a function $\eta(t; \alpha, \beta) = \rho_1(t) + \rho_2(t; \alpha) + \rho_3(t) + \rho_4(t; \beta) + \lambda^{-2}$. Note that, for all $\alpha, \beta \geq 0$ the function $\eta(t; \alpha, \beta)$ satisfies conditions (h1)–(h6) except (h3) and (h5). We show there exist positive real numbers $\alpha_0$ and $\beta_0$ for which $\eta(t; \alpha_0, \beta_0)$ satisfies (h3) and (h5).

First, let us see how to take $\alpha_0$ for which $\eta(t; \alpha_0, \beta)$ satisfies (h3). Let us put $\Xi(\alpha) = \int_0^\alpha \eta(t; \alpha, \beta) \, dt$. Then, $\eta(\alpha)$ is independent of $\beta$, continuous, monotone increasing and $\Xi(\alpha) \to \infty$ as $\alpha \to +\infty$. Moreover, $\Xi(0) < \lambda \cdot \lambda^{-2} + 1 \cdot \lambda^{-2} < 1$ (remember that we assume $\lambda > 20$). Hence, by the intermediate value theorem, we get $\alpha_0$ for which $\eta(t; \alpha_0, \beta)$ satisfies (h3). In a similar way, we can find $\beta_0$ so that (h5) holds for $\eta(t; \alpha_0, \beta_0)$.

Then, by putting $H(z) := \int_0^z h(t) \, dt$ for $z \geq 0$ and $H(z) := -H(-z)$ for $z < 0$, we construct a map $H(z)$. It is easy to see that $H(z)$ is a $C^\infty$-diffeomorphism and enjoys (H1), (H2) and (H5). Let us check (H3) and (H4). We have $H(1/2) = \lambda/2 > 7$ and $H(\lambda^2)$ can be estimated as follows:

\[ H(\lambda^2) = H(2) + (\lambda^2 - 2)\lambda^{-2} = \lambda + 1 + 1 - 2/\lambda^2 < \lambda - 2 < \lambda^2. \]

Hence we have proved (H3) and (H4) for $H(z)$.

\[ \square \]

Lemma 8. There exists a $C^\infty$-diffeomorphism $\Theta$ of $\mathbb{R}^4$ that has the following properties:

- (Θ1) The support of $\Theta$ is contained in $B_6(P) \cup B_6(Q)$.
- (Θ2) For $X = (x, y, z, w) \in B_6(P)$, $\Theta(x, y, z, w) = (x, -z, y, w)$, in particular, $P$ is a fixed point of $\Theta$.
- (Θ3) $\Theta$ fixes every point in the $x$-axis, more precisely, for $X = (x, 0, 0, 0)$, $\Theta(X) = X$.
- (Θ4) For any $X \in B_6(P)$, $\Theta$ preserves the $d$-distance between $P$ and $X$, more precisely, $d(\Theta(X), \Theta(X)) = d(P, X)$.
- (Θ5) In $B_{6}(P)$, $\Theta$ preserves the $yz$-plane, more precisely, for $X = (0, y, z, 0) \in B_{6}(P)$, the $x$-coordinate and $w$-coordinate of $\Theta(X)$ are 0.
- (Θ6) For $(x, y + 10, z, w) \in B_{6}(Q)$, $\Theta(x, y + 10, z, w) = (y, x + 10, -w, z)$, in particular, $Q$ is a fixed point of $\Theta$.
- (Θ7) For any $X \in B_{6}(Q)$, $\Theta$ preserves the $d$-distance between $Q$ and $X$.
- (Θ8) In $B_{6}(Q)$, $\Theta$ preserves the $xy$-plane. More precisely, for $X = (x, y, 0, 0) \in B_{6}(Q)$, the $z$-coordinate and $w$-coordinate of $\Theta(X)$ are 0.
- (Θ9) In $B_{6}(Q)$, $\Theta$ preserves the plane that passes through $Q$ and parallel to $zw$-plane, more precisely, for $X = (0, 10, z, w) \in B_{6}(Q)$, the $x$-coordinate of $\Theta(X)$ is 0 and the $y$-coordinate of $\Theta(X)$ is 10.

Proof. First we define three functions $\rho_1(t)$, $\omega_1(X)$, $\omega_2(X)$ as follows:

- $\rho_1(t) := \rho[-1/300, 1/300, -1/200, 1/200](t)$,
- $\omega_1(x, y, z, w) := (\pi/2)\rho_1(x)\rho_1(y)\rho_1(z)\rho_1(w)$,
- $\omega_2(x, y, z, w) := (\pi/2)\rho_1(x)\rho_1(y - 10)\rho_1(z)\rho_1(w)$. 

We also define a map $R[\alpha] : \mathbb{R}^2 \to \mathbb{R}^2$ to be the rotation of angle $\alpha$, more precisely, for $(x, y) \in \mathbb{R}^2$ we put

$$R[\alpha](x, y) := (\cos(\alpha)x - \sin(\alpha)y, \sin(\alpha)x + \cos(\alpha)y).$$

Then, we define $\Theta$ as follows:

- For $X = (x, y, z, w) \in B_R(P)$, $\Theta(X) := (x, R[\omega_1(X)](y, z), w)$.
- For $X = (x, y + 10, z, w) \in B_R(Q)$, $\Theta(X) := (R[\omega_2(X)](x, y), R[\omega_2(X)](z, w)) + Q$.
- Otherwise, $\Theta(X) := X$.

Now, it is not difficult to see that $\Theta$ is a $C^\infty$-diffeomorphism satisfying all the properties $(\Theta1)\sim(\Theta9)$.

**Lemma 9.** There is a $C^\infty$-diffeomorphism $\Psi$ of $\mathbb{R}^4$ that has the following properties:

(Ψ1) The support of $\Psi$ is contained in $[-\lambda^3 - 1, \lambda^3 + 1]^4$.

(Ψ2) For $(x, y, z, w) \in [-\lambda^3, \lambda^3]^4$, $\Psi(x, y, z, w) = (F(x), G(y), H(z), H(w))$.

**Proof.** We define the functions $\rho_i$ on $\mathbb{R}$ and $R_i (i = 1, \ldots, 4)$ on $\mathbb{R}^4$ as follows:

$$\rho_i(t) := \sin(-\lambda^3, \lambda^3, -\lambda^3, \lambda^3 + 1) + t,$$

and for $i = 1, 2, 3, 4$:

$$R_i(X) := \rho_i(x_1)\rho_i(x_2)\rho_i(x_3)\rho_i(x_4),$$

where $X = (x_1, x_2, x_3, x_4)$. Then, we define a map $\Psi : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\Psi_i(X) := R_i(X)F_i(x_i) + (1 - R_i(X))x_i, \quad \text{for } i = 1, \ldots, 4,$$

where $\Psi_i(X)$ denotes the $i$th coordinate of $\Psi(X)$ and we put $F_1 := F$, $F_2 := G$ and $F_3 = F_4 := H$ as a matter of convenience.

It is easy to see that $\Psi$ is a $C^\infty$-map and satisfies (Ψ1) and (Ψ2). The perplexing part is to confirm that $\Psi$ is a diffeomorphism. To see this, we put $I_0 := [-\lambda^3, \lambda^3], I_1 := \mathbb{R} \setminus I_0$, $S := \{\sigma_i \in S \mid \sigma_i = 0, 1\}$ and for every $\sigma_i \in S$ we put $I[\sigma_i] := \prod_{i=1}^{4} I_{\sigma_i}$.

Then, divide $\mathbb{R}^4$ into 16 subsets as follows:

$$\mathbb{R}^4 = \bigsqcup_{(\sigma_i) \in S} I[\sigma_i].$$

We claim that, for every $(\sigma_i) \in S$, the restriction of $\Psi$ to $I[\sigma_i]$ is a diffeomorphism. Let us fix $(\sigma_i) \in S$ and put $j := \sum_{i=1}^{4} \sigma_i$. We define

$$P[\sigma_i] := \{x_i \in \mathbb{R}^4 \mid x_i \in I_1\},$$

and for $Y = (\sigma_i, y_i) \in P[\sigma_i]$,

$$S((\sigma_i), Y) := \{y + ((1 - \sigma_i)z_i) \mid z_i \in I_0\}.$$ 

Then, we have the following decomposition:

$$I[\sigma_i] = \bigsqcup_{Y \in P[\sigma_i]} S((\sigma_i), Y).$$

Intuitively speaking, we divided $I[\sigma_i]$ into $(4 - j)$-dimensional cubes that are parametrized by $j$-dimensional parameters.

First, we prove that $\Psi_{I[\sigma_i]}(X) = x_k$ for $k$ with $\sigma_k = 1$. One important observation is that each $F_k$ is the identity map on $I_1$ by (F1), (G1) and (H2). So, for $k$ with $\sigma_k = 1$, we have $\Psi_{I[\sigma_i]}(X) = R_k(X)x_k + (1 - R_k(X))x_k = x_k$ independent of the value of $R_k(X)$. 
Second, we investigate the behaviour of the restriction of $\Psi$ to $S(\{\sigma_i\}, Y)$. We show that $\Psi_k|_{S(\{\sigma_i\}, Y)}(X) = \tilde{F}_k(x_k)$ for $k$ with $x_k \neq 0$, where $\tilde{F}_k(x_k)$ is some diffeomorphism of $I_0$. The key point is that the change of $X$ in $S(\{\sigma_i\}, Y)$ never varies $R_k(X)$. Indeed, for $l$ that satisfies $\sigma_l = 1$, the $l$th coordinate of $X$ is fixed, and for $l$ that satisfies $\sigma_l = 0$, $l$th coordinate of $X$ is in $I_0$. Hence the change of $X$ in $S(\{\sigma_i\}, Y)$ has no effect on $R_k(X)$. Now the formula $\Psi_k|_{S(\{\sigma_i\}, Y)}(X) = R_k(X)F_k(x_k) + (1 - R_k(X))x_k$ tells us that this map is a diffeomorphism of $I_0$, and it is a convex combination of diffeomorphisms of $I_0$. Therefore, $\Psi|_{S(\{\sigma_i\}, Y)}$ is a diffeomorphism of $S(\{\sigma_i\}, Y)$.

We have proved that $\Psi|_{I([\sigma_i])}$ is a diffeomorphism when it is restricted to $S(\{\sigma_i\}, Y)$, and behaves as the identity map in the $P(\{\sigma_i\})$ direction. These two facts confirm that $\Psi|_{I([\sigma_i])}$ is a diffeomorphism of $I([\sigma_i])$. □

Now, let us give the proof of proposition 3.

**Proof of proposition 3.** We put $\Phi := \Theta \circ \Psi$. From the properties of $F$, $G$, $H$, $\Theta$ and $\Psi$, we can see $\Phi$ satisfies (F1)–(F10). Indeed,

- (F1) follows from (Ψ1) and (Θ1).
- (F2) follows from (F5), (G2), (H3), (H4) and (Θ1).
- (F3) follows from (F4), (G7), (G8), (H5), (Θ2) and (Θ6).
- (F4) follows from (F2), (G3), (H5) and (Θ2).
- (F5) follows from (F2), (Θ1) and (Θ3).
- (F6) follows from (F2), (G3), (G5), (Θ1), (Θ4) and (Θ5).
- (F7) follows from (F4), (G8), (H5) and (Θ6).
- (F8) follows from (F2), (G4), (G5), (Θ1), (Θ7) and (Θ8).
- (F9) follows from (H5), (Θ1), (Θ7), and (Θ9).

Let us check (F10). We investigate the action of $\Lambda^3(d\Psi)(X)$ (we put $X = (x, y, z, w)$) to the orthonormal basis $\langle e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 \rangle$, where $\langle e_i \rangle$ denotes the standard orthonormal basis of $T\mathbb{R}^4|_X$. They are given as follows:

$$\Lambda^3(d\Psi)(X)(e_1 \wedge e_2 \wedge e_3) = F'(x)G'(y)H'(z)e_1 \wedge e_2 \wedge e_3,$$

$$\Lambda^3(d\Psi)(X)(e_1 \wedge e_2 \wedge e_4) = F'(x)G'(y)H'(w)e_1 \wedge e_2 \wedge e_4,$$

$$\Lambda^3(d\Psi)(X)(e_1 \wedge e_3 \wedge e_4) = F'(x)H'(z)H'(w)e_1 \wedge e_3 \wedge e_4,$$

$$\Lambda^3(d\Psi)(X)(e_2 \wedge e_3 \wedge e_4) = G'(y)H'(z)H'(w)e_2 \wedge e_3 \wedge e_4.$$

This calculation tells us that $m(\Lambda^3(d\Psi)(X))$ is the minimum of the lengths of these four vectors. Let us fix a real number $c > 0$ which satisfies the following inequality:

$$\min_{(x,y) \in [-\lambda,\lambda]^2} \left[ |F'(x)G'(y)|, |F'(x)|, |G'(y)| \right] > c.$$

We can take such a $c$ by (F3) and (G6). Note that $c$ can be chosen independent of $\lambda$. Since $H'(z) = H'(w) = \lambda$ for any $(x, y, z, w) \in C$, we have proved the lengths of these four vectors are bounded below by $c\lambda$, and $m\left(\Lambda^3(d\Psi)(X)\right) > c\lambda$ for all $X \in C$.

Let us put $c_\Theta := \min_{x \in \mathbb{R}^2} m(\Lambda^3(d\Theta)(x))$. Since $\Theta$ is a diffeomorphism and has compact support, $c_\Theta$ is a positive number. Now, for any $X \in C$, the inequality

$$m(\Lambda^3(d\Phi)(X)) \geq \min_{x \in \mathbb{R}^2} m(\Lambda^3(d\Theta)(x)) \cdot m(\Lambda^3(d\Psi)(X)) = c_\Theta c_\lambda,$$

holds and this implies (F10) for $c_F = c_\Theta c_\lambda$.

Therefore, the proof is completed. □
3.3. Construction of \( \Upsilon \)

**Proposition 4.** There exists a \( C^\infty \)-diffeomorphism \( \Upsilon \) of \( \mathbb{R}^4 \) satisfying the following properties:

(\( \Upsilon 1 \)) The support of \( \Upsilon \) is contained in \( D \).

(\( \Upsilon 2 \)) For \( X \in B(C_1, 0.2) \), \( \Upsilon(X) = X + (10, -10, -5, 0) \).

To construct \( \Upsilon \), we need the following auxiliary diffeomorphism \( \chi(X) \).

**Lemma 10.** Given two points \( Y, Z \in \mathbb{R}^4 \) and three real numbers \( a > b > 0 \) with \( d(Y, Z) > 2a \), there exists a diffeomorphism \( \chi(Y, Z, a, b)(X) = \chi(X) \) of \( \mathbb{R}^4 \) satisfying the following properties:

(\( \chi 1 \)) The support of \( \chi(X) \) is contained in \( C(Y, Z, a) \).

(\( \chi 2 \)) For \( X \in B(Y, b) \), \( \Upsilon(X) = X + Z - Y \).

Before the proof of lemma 10, let us see how one can construct \( \Upsilon \) from \( \chi \). We put \( l_i := 1.15 - 0.15i \), \( \chi(X) := \chi(C_1, C_i, l_{2i-1}, l_{2i}) \) for \( i = 1, 2, 3 \) and \( \Upsilon := \chi_3 \circ \chi_2 \circ \chi_1 \). Then, it is clear that \( \Upsilon \) satisfies (\( \Upsilon 1 \)) and (\( \Upsilon 2 \)).

Now, let us give the proof of lemma 10.

**Proof of Lemma 10.** By changing the coordinate, we can assume \( Y \) is the origin of \( \mathbb{R}^4 \) and \( Z = (\xi, 0, 0, 0) \) where \( \xi := d(Y, Z) \). We put \( c := (a + b)/2 \).

First, we construct a diffeomorphism \( \kappa(x) \) of \( \mathbb{R}^4 \) satisfying the following conditions:

(\( \kappa 1 \)) The support of \( \kappa \) is contained in \([-a, c + a] \).

(\( \kappa 2 \)) For \( x \in [-b, b] \), \( \kappa(x) = x + \xi \).

We will explain how to construct such a diffeomorphism later. Then, put \( \rho_1(t) = \rho[-b, b, -c, c](t), R(x, y, z, w) = \rho_1(y)\rho_1(z)\rho_1(w) \). Finally, for \( X = (x, y, z, w) \in \mathbb{R}^4 \), we define

\[
\chi(Y, Z, a, b)(X) := (R(X)\kappa(x) + (1 - R(X))x, y, z, w).
\]

It is not difficult to see \( \chi \) satisfies the required conditions.

Let us see how to construct \( \kappa(x) \). We prepare a \( C^\infty \)-function \( \eta(t) \) on \( \mathbb{R} \) that satisfies the following properties:

(\( \eta 1 \)) \( \eta(t) > 0 \) for all \( t \in \mathbb{R} \).

(\( \eta 2 \)) \( \eta(t) = 1 \) for \( t < -c, -b < t < b, \) or \( t > \xi + c \).

(\( \eta 3 \)) \( \int_{-c}^{b} \eta(x) \, dx = \xi + c - b \).

(\( \eta 4 \)) \( \int_{-c}^{b} \eta(x) \, dx = c - b \).

Then, \( \kappa(x) := \xi + \int_{-c}^{b} \eta(t) \, dt \) is a \( C^\infty \)-diffeomorphism satisfying (\( \kappa 1 \)) and (\( \kappa 2 \)). So, let us construct the function \( \eta(t) \). We fix a positive real number \( e < (b - c)/2 \) and define

\[
\rho_2(t) = \rho[-c + e, -b - e, -c, -b](t), \quad \rho_3(t) = \rho[b + e, \xi + c - e, b, \xi + c](t),
\]

and

\[
\eta(t; \alpha, \beta) = \exp(\alpha \rho_2(t) + \beta \rho_3(t)),
\]

where \( \alpha, \beta \) are some real numbers.

We show there exist \( \alpha_1, \beta_1 \) such that \( \eta(t; \alpha_1, \beta_1) \) satisfies (\( \eta 1 \))–(\( \eta 4 \)). The function \( \eta(t; \alpha, \beta) \) satisfies properties (\( \eta 1 \)) and (\( \eta 2 \)) for all \( \alpha \) and \( \beta \). Let us consider (\( \eta 3 \)) and (\( \eta 4 \)). For \( t \in [-c, -b] \), \( \eta(t; \alpha, \beta) \) is equal to \( \exp(\alpha \rho_2(t)) \). We put \( J(\alpha) := \int_{-c}^{b} \eta(t; \alpha, \beta) \, dt \). Then, one can check that \( J(\alpha) \to 0 \) as \( \alpha \to -\infty \), \( J(\alpha) \to +\infty \) as \( \alpha \to +\infty \), and \( J(\alpha) \) is continuous and monotone increasing. So, the intermediate value theorem says there exists \( \alpha_1 \) satisfying \( J(\alpha_1) = \xi + c - b \). In a similar way, one can find \( \beta_1 \) such that \( \int_{-c}^{b} \eta(t; \alpha_1, \beta_1) \, dt = c - b \). Hence \( \eta(t; \alpha_1, \beta_1) \) satisfies (\( \eta 1 \))–(\( \eta 4 \)). \( \square \)
3.4. Proof of proposition 2

Finally, let us give the proof of proposition 2.

Proof of proposition 2. Let us put \( c_{\Upsilon} := \min_{x \in R^4} m(\Lambda^3(d\Upsilon)(x)) \). Since \( \Upsilon \) is a diffeomorphism and has compact support, \( c_{\Upsilon} > 0 \). We fix \( \lambda_0 > 0 \) so that \( K := c_{\Phi_1} c_{\Upsilon} \lambda_0 > 1 \) holds. We put \( \Omega := \Upsilon \circ \Phi_1 \) and show \( \Omega, A, B, P, Q \) and \( K \) satisfy properties \((w1)–(w7)\) when \( \lambda = \lambda_0 \). Indeed,

- \((w1)\) follows from \((\Phi1)\) and \((\Upsilon1)\).
- \((w2)\) follows from \((\Phi2)\) and \((\Upsilon1)\).
- \((w3)\) follows from \((\Phi3), (\Phi4), (\Phi7)\) and \((\Upsilon1)\).
- \((w4)\) follows from \((\Phi5), (\Phi6), (\Phi8), (\Phi9), (\Upsilon1)\) and \((\Upsilon2)\).
- \((w5)\) follows from \((\Phi4)\) and \((\Upsilon1)\).
- \((w6)\) follows from \((\Phi7)\) and \((\Upsilon1)\).
- \((w7)\) follows from \((\Phi10)\) and the definition of \( \lambda_0 \).

So, the proof is completed. \( \square \)

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