Avalanche dynamics and nonexponential relaxation

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The theory of SOC, and related avalanche dynamics, is proposed as the origin of the ubiquitous nonexponential relaxation observed in complex systems. Introducing some scaling laws and relations we have obtained that the normalized relaxation function follows an stretched exponential decay and that the frequency spectrum follows an "1/f" noise. Moreover, in the MF approach the relaxation is found to be exponential.

I. INTRODUCTION

Self-Organized criticality (SOC) was proposed as a general theory to understand the ubiquitous "1/f" noise and self-similar fractal structures in dynamical systems with many degrees of freedom. The SOC idea has been illustrated by a large variety of computer models, including sandpiles, evolution, interface depinning, and more, in which a slow driving field leads to a stationary state with avalanches with widely distributed sizes. However, most of the research has been devoted to the study of the stationary critical state.

Another puzzle seeking a physical explanation is the empirical observation that relaxation phenomena in nature is not exponential. Nonexponential relaxation has been observed in different phenomena like mechanical, dielectric and magnetic relaxation, and in a large variety of materials, including amorphous polymers, supercooled liquids, glasses, spin glasses, and amorphous semiconductors. These are complex systems where relaxation phenomena offer a large variety of challenging problems.

Many theoretical models has been proposed to explain the ubiquitous nonexponential relaxation. The physical origin of the nonexponential decay is attributed to the existence of fractal clusters, cooperative dynamics, hierarchical constraints, or anomalous diffusion. All these models have their advantages and limitations and may be applied to some, but not all, experimental situations.

In the present work we propose the SOC theory, and the related avalanche dynamics, as a physical mechanism leading to the nonexponential relaxation, often observed in nature. The present approach may be enclosed among those models which attribute the nonexponential behavior to the existence of fractal clusters, however the clusters (avalanches) in the SOC have an intrinsic dynamical nature, and correlations are present not only in space but also in time.

II. CRITICAL RELAXATION

In this section we analyze the relaxation of a self-organizing system near the critical state. We first start with a mean-field approach to the problem and later we analyze some scaling laws for the different magnitudes characterizing the dynamics of the system.

A. Mean field theory

The first step towards a comprehensive understanding of SOC and related phenomena is provided by mean-field (MF) theories, which give insight into the fundamental physical mechanism of the problem. The first MF analysis to sandpile models is due to Tang and Bak. They mapped a critical height (Abelian) sandpile model in a two state cellular automaton and took the driving rate and the average height as control parameters. Later Caldarelli, Maritan and Vondruscolo generalized the MF approach of Tang and Bak in order to include stochastic rules. On the other hand, Vespignani and Zapperi mapped the sandpile automata in a three state cellular automata on a d-dimensional lattice, and took the driving field and the dissipation rates as control parameters. In this way, they obtained that the system is critical in the double limit and h/ε → 0. The approach developed here is in the same line as that of Vespignani and Zapperi.

We divide the states that each site can assume in stable, unstable and active, and denote their average densities by ρs, ρu and ρa, respectively. Stable sites are those that cannot become active by addition of grains. Unstable sites are those that may become active by addition of grains. Active sites are relaxing and transfer energy to its nearest neighbors. Sites may receive energy either from the driving field or from its nearest active neighbors. The driving field is characterized by the driving rate h, the probability per unit time that a site will receive a grain of energy from the driving field. On the
other hand, the probability per unit time that a site receives a grain from nearest active neighbors is given by $1 - (1 - \rho_a)^2 \simeq g \rho_a$, where $g$ is the effective number of nearest neighbors and $\rho_a$ is small. Then, taking into account both contributions, the local driving field is given by

$$h_{\text{loc}} = h + g \rho_a.$$  

(1)

This local field gives the probability per unit time that a site receives a grain, either from the driving field or from nearest active neighbors.

Let $u$ be the fraction of stable sites that become unstable by addition of a grain, $p$ the fraction of unstable sites that become active by addition of a grain, and $q$ the fraction of active sites that become stable. With these definitions we write the following rate equations for the average densities

$$\frac{d}{dt} \rho_a = -[1 - q - (1 - q)ph_{\text{loc}}] \rho_a + ph_{\text{loc}} \rho_u ,$$  

(2)

$$\frac{d}{dt} \rho_s = q \rho_a - u h_{\text{loc}} \rho_s ,$$  

(3)

plus the normalization condition

$$\rho_s + \rho_u + \rho_a = 1 .$$  

(4)

The first term in the right hand side of eq. (2) characterizes the transition of active sites to stable or unstable sites. The correction $(1 - q)ph_{\text{loc}}$ takes into account that the fraction $1 - q$ of active sites that become unstable may remain active by addition of grains from the local field. The second term of the same equation characterizes the transition of unstable sites to active induced by the local driving field. On the other hand, the first term in the right hand side of eq. (3) gives the fraction of active sites that become stable, and the second one, the transition of stable sites to unstable ones induced by the local driving field.

After imposing stationarity ($\frac{d}{dt} \rho_a = 0, \frac{d}{dt} \rho_s = 0$), from eqs. (3,4) one obtains

$$upqg \rho_a^2 + [pq + u(1 - pg + pqh)] \rho_a - uph = 0 ,$$  

(5)

$$\rho_s = \frac{q}{u + h + g \rho_a} \rho_a ,$$  

(6)

$$\rho_u = 1 - \rho_a - \rho_s .$$  

(7)

The first expression is a quadratic equation for $\rho_a$. Its solution gives the average density of active sites as a function of the driving field. We can expand $\rho_a(h)$ for small values of $h$, i.e.

$$\rho_a = \rho_a^{(0)} + \chi h + O(h^2) ,$$  

(8)

where $\rho_a^{(0)}$ is the zero field average density of active sites and $\chi$ the susceptibility, characterizing the linear response of the system to the external field. Substituting eq. (5) in (8), comparing terms with similar order in $h$, and taking the physically admissible solutions, we obtain

$$\rho_a^{(0)} = \left\{ \begin{array}{ll}
\frac{1}{ug} (\theta - \theta_c) , & \theta < \theta_c \\
\frac{1}{ug} (\theta - \theta_c)^{-1} , & \theta > \theta_c
\end{array} \right. ,$$  

(9)

$$\chi = \frac{u}{q} (\theta - \theta_c)^{-1} ;$$  

(10)

where

$$\theta = \frac{u(pg - 1)}{pq} , \quad \theta_c = 1 ;$$  

(11)

Moreover, just at $\theta = \theta_c$ and $h \to 0$, from eq. (5) it follows that

$$\rho_a = \frac{1}{\sqrt{qg}} h^{1/2} .$$  

(12)

Taking into account these results, from eqs. (3,4) we obtain

$$\rho_s = \frac{q}{ug} + O(h, |\theta - \theta_c|) ,$$  

(13)

$$\rho_u = \frac{ug - q}{ug} + O(h, |\theta - \theta_c|) .$$  

(14)

Thus, for $\theta < \theta_c$, the zero field average density of active sites is zero while, for $\theta > \theta_c$, it is proportional to $(\theta - \theta_c)^{\beta}$, with $\beta = 1$. The susceptibility diverges when $\theta \to \theta_c, \theta \neq \theta_c$, according to $|\theta - \theta_c|^{-\gamma}$, with $\gamma = 1$. Just at $\theta_c$, the average density of active sites scales with the driving field as $h^{1/\delta}$, with $\delta = 2$. These features are reminiscent of ordinary critical phenomena. $h$ is the external field, $\theta$ plays the role of temperature and $\rho_a$ is the order parameter.

It is believed that conservation is a necessary condition to obtain SOC in sandpile models. The global conservation law states that the average input flux $hL^d$ must balance, in average, the dissipated flux $\rho_a \epsilon L^d$. $\epsilon$ is the dissipation rate per toppling event. It is an effective parameter that takes into account the dissipation, either in the bulk or at the boundary. Then, in the stationary state

$$h = \rho_a \epsilon .$$  

(15)

From this equation one determines the average density of active sites as a function of the driving field, obtaining $\rho_a = h/\epsilon$. But, to be consistent with eq. (5), we must have $\rho_a^{(0)} = 0$ and $\chi = 1/\epsilon$. Moreover, using the expression for the susceptibility obtained above, eq. (10), one obtains

$$\theta = \theta_c - \frac{u}{q} \epsilon .$$  

(16)
In systems with dissipation in the bulk $\epsilon$ is a fixed parameter and, therefore, the system is in a subcritical regime $\theta < \theta_c$. To obtain criticality ($\theta = \theta_c$) we have to fine-tune $\epsilon$ to zero. Hence, in this case, there is no difference with ordinary non-equilibrium critical phenomena, we have to fine-tune $\epsilon$ to reach the critical state. On the other hand, in systems with dissipation at the boundary, $\epsilon$ decreases with increasing lattice site $L$. The number of boundary lattice sites, where dissipation may take place, grows slower than the total number of lattice sites. One can therefore assume that $\epsilon \sim L^{-\mu}$, where $\mu$ is a scaling exponent. In the thermodynamic limit ($L \to \infty$) $\epsilon = 0$ and, from eq. (10), it follows that $\theta = \theta_c$. In this case, the system self-organize itself into the critical state. Hence, the conservation law is not a sufficient condition to obtain SOC. The way in which energy is dissipated plays a determinant role.

We have investigated the properties of the stationary state and the conditions to obtain criticality. Now we proceed to analyze the relaxation of the system after a switch off of the external field. If the system is close to equilibrium, $h = 0$ and $\epsilon \approx 0$ then form eqs. (3), (4) and (10) it follows that

$$\rho(t) \sim \exp(-pct). \quad (17)$$

Thus the relaxation towards equilibrium is exponential, with a relaxation time $\tau_e \sim \epsilon^{-1}$. If the system is at the critical state ($\epsilon = 0$) then the system takes an infinite time to reach equilibrium. This phenomenon is characteristic of critical systems and it is known as critical relaxation.

**B. Scaling laws**

The MF theory leads to an exponential relaxation, however it is well known that relaxation in real systems is nonexponential. The spreading of an avalanche in MF theory can be described by a front consisting of non-interacting particles that can either trigger subsequent activity or die out. The MF theory thus neglects correlations among the particles. Next we investigate the propagation of a deltaic perturbation, in space and time, though a self-organizing system close to the critical state, but using some scaling laws.

Let us calculate the average number of perturbed particles $\psi(t)$ due to a deltaic perturbation, in space and time. $\psi(t)$ is then the response function of the system. These particles are causally connected in space and time, thus forming an avalanche with average size

$$\langle s \rangle = \int_0^\infty dt \psi(t) . \quad (18)$$

We can express the average response $\psi(t)$ as a superposition of the distribution of avalanche sizes, i.e.

$$\psi(t) = \int_0^\infty ds P(s) \psi(s,t) , \quad (19)$$

where $P(s)$ is the distribution of avalanche sizes and $\psi(s,t)$ is the response function of $s$-avalanches. In order to be consistent with eq. (18) $\psi(s,t)$ must satisfy

$$s = \int_0^\infty dt \psi(s,t) . \quad (20)$$

Now we are going to assume scaling. If $l$ is the linear dimension of an $s$-avalanche then its characteristic size and duration should scale as $l^D$ and $l^z$, respectively, where $D$ is the fractal dimension of the avalanches and $z$ is a dynamic scaling exponent. Therefore, the characteristic time scales as $s^{z/D}$ and $\psi(s,t)$ may be written as

$$\psi(s,t) = t^q f_1(t s^{-z/D}) . \quad (21)$$

where $f(x)$ is a cutoff function. The exponent $q$ is determined from the normalization condition in eq. (21), obtaining $q = D/z - 1$. Moreover, $\psi(s,t)$ is independent of $s$ for $ts^{-z/D} \ll 1$ then $f(x) \sim $ cte for $x \ll 1$. On the other hand, $\psi(s,t) \ll 1$ for $ts^{-z/D} \gg 1$ and, therefore, $f(x) \ll 1$ for $x \gg 1$.

The distribution of avalanche sizes near the SOC state satisfy the scaling relation

$$P(s) = s^{-\tau} f_2(s e^{1/\sigma}) , \quad (22)$$

where $\tau$ and $\sigma$ are scaling exponents and $f_2(x)$ is a cutoff function with similar asymptotic behavior as $f_1(x)$. $\epsilon$ is the control parameter which determine how close is the system to the SOC state, i.e. the dissipation rate in the previous MF approach.

Substituting eqs. (21) and (22) in eq. (19) one obtains

$$\psi(t) = t^{\kappa-1} f_3(t^{1/\Delta}) , \quad (23)$$

where

$$\Delta = D/z \sigma , \quad \kappa = D(2 - \tau)/z , \quad (24)$$

and $f_3(x)$ is a cutoff function.

Finally, the normalized relaxation function $\phi(t)$ defined through the expression

$$\phi(t) = \int_0^\infty dt' \psi(t'), \quad \int_0^\infty dt' \psi(t') , \quad (25)$$

may be approximated, for $te^{1/\Delta} \ll 1$, by

$$\phi(t) \sim \exp \left[-\left(\frac{t}{\tau_e}\right)^\kappa\right] . \quad (26)$$

where $\tau_e \sim \epsilon^{-1/\Delta}$. Thus, under the scaling laws assumed above the relaxation function follows the Kohlrausch stretched exponential.
The stretched exponent \( \kappa \) is defined in eq. (24) through the scaling exponents \( \tau, D \) and \( z \). However, using some scaling relations we can obtain a simpler expression. In systems at a SOC state the correlation length scales as the system size \( \xi \sim L \) and \( \xi \sim \epsilon^{-\nu} \), where \( \nu \) is a scaling exponent. It is also possible to show that in homogeneous SOC systems \( \langle s \rangle \sim L^2 \). Moreover, from eq. (23) it follows that \( \langle s \rangle \sim \epsilon^{-2-\tau}/\sigma \). Hence,

\[
\langle s \rangle \sim L^2 \sim \xi^2 \sim \epsilon^{-2\nu} \sim \epsilon^{-(2-\tau)/\sigma},
\]
and therefore

\[
2 - \tau = 2\sigma \nu.
\]

On the other hand, the cutoff avalanche size is given by

\[
s_c \sim \xi^D \sim \epsilon^{-D\nu} \sim \epsilon^{-1/\sigma},
\]
and therefore

\[
D\nu \sigma = 1.
\]

Then, using (24), (28) and (30), one obtains the scaling relations

\[
\Delta = \frac{1}{2\nu}, \quad \kappa = \frac{2}{z}.
\]

The stretched exponent \( \kappa \) only depends on the dynamic scaling exponent \( z \) (the duration of an avalanche scales with its linear dimension as \( l^z \)). In the MF approach \( z = 2 \) in order to recover the exponential relaxation obtained in the preceding subsection. However, if one considers correlations among the different branches forming an avalanche then one expects that the duration of the avalanche grows faster with its linear dimension and, therefore, \( z > 2 \).

### III. DISCUSSION

We have obtained that the normalized relaxation function follows a stretched exponential decay, with a stretched exponent given by the second scaling relation in eq. (31). However, this functional dependence is limited to the time scale \( 1 \ll t \ll \tau_c \sim \epsilon^{1/\Delta} \). In the SOC state \( \epsilon = 0 \) and, therefore, the stretched exponential decay will be observed for infinitely long times.

This temporal behavior leads to the frequency spectrum

\[
S(f) = \int_0^\infty dt \cos(2\pi ft)\phi(t) \sim f^{-1-\kappa},
\]
in the frequency range \( \epsilon^{-1/\Delta} \ll f \ll 1 \). We thus found that the avalanche dynamics leads to a "1/f" noise in the intermediate frequency window. This result seems to be similar to that reported by Bak, Tang and Wiesenfeld in their pioneering work [1], however this is not the case. Bak, Tang and Wiesenfeld obtained a "1/f" noise assuming a power law distribution \( P(T) \sim T^{-\alpha} \) of avalanche durations, which is equivalent to assume a power law distribution of avalanche sizes, but they also assume that each avalanche relaxes exponentially, i.e.

\[
\phi(t) = \int_0^\infty P(T)\exp(t/T) \sim t^{-(\alpha-1)},
\]
thus obtaining \( S(f) \sim f^{-2+\alpha} \). The exponent \( \alpha \) may be expressed in terms of of the scaling exponents introduced here. Since \( T \sim s^{z/D} \) and \( P(s)ds = P(t)dt \) then

\[
\alpha = 1 + \frac{D}{z}(\tau - 1) = 1 + \frac{D - 2}{z},
\]
where the second equality is obtained using the scaling relations in equations (28) and (31).

In both cases the power spectrum is of the form \( 1/f^\beta \), but \( \beta = 1 + \kappa \) is larger than one in our model, while \( \beta = 1 - (D - 2)/z \) is smaller than one in the Bak, Tang and Wiesenfeld approach. Moreover, the functional dependence of the normalized relaxation function is different, we obtain a stretched exponential while they obtain a power law decay. This discrepancy is a consequence of the assumption made by Bak, Tang and Wiesenfeld that the avalanches relax exponentially, with a relaxation time equal to its duration. Such an assumption does not take into account the intrinsic dynamic nature of an avalanche, which may lead by itself to a nonexponential relaxation.

The normalized relaxation function associated with an avalanche of duration \( T \), or equivalently of size \( s \) \( (T \sim s^{z/D}) \), can be obtained from eqs. (22) and (23), resulting

\[
\phi(T,t) = \int_0^\infty dx x^{1-z/D} f_1(x) \int_0^{\infty} dx x^{1-z/D} f_1(x).
\]

Hence, the characteristic time is the avalanche duration, but the normalized relaxation function is not necessarily exponential. It is well known that cooperative dynamics leads, without assuming any disorder, to a nonexponential relaxation dynamics.

### IV. CONCLUSIONS

We have proposed the avalanche dynamics as the origin of the ubiquitous nonexponential relaxation observed in complex systems. Introducing some scaling laws and relations we have obtained that the normalized relaxation function follows an stretched exponential decay with a stretched exponent \( \kappa = 2/z \), where \( z \) is the dynamic scaling exponent. Moreover, in the MF approach \( \kappa = 1 \) and the relaxation is exponential.

The stretched exponential decay leads to the "1/f" noise spectrum \( S(f) \sim f^{-1-\kappa} \). This result is more general than the one reported by Bak, Tang, and Wiesenfeld [1], since it takes into account the intrinsic dynamic nature of the avalanches.
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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. 59 (1987) 381.
[2] M. W. Weissman, Rev. Mod. Phys. 60 (1988) 537.
[3] B. Mandelbrot, The fractal geometry of nature (Fremann, New York, 1983).
[4] P. Bak and K. Sneppen, Phys. Rev. Lett. 71 (1993) 4083.
[5] K. Sneppen, Phys. Rev. Lett. 69 (1992) 3539.
[6] P. Bak, How nature works: The science of self-organized criticality (Copernicus, New York, 1996).
[7] Cooperative dynamics in complex systems, J. Non-Cryst. Solids 131-133 (1991); Dynamics of glass transition and related topics, Prog. Theor. Phys. Suppl. 126 (1997).
[8] M. H. Cohen and G. Grest, Phys. Rev. B 24 (1981) 4091; R. V. Chamberlin, J. Appl. Phys. 76 (1994) 6401; A. Vázquez, O. Sotolongo-Costa, and F. Brouers, J. Phys. Soc. Jpn. 66, 2324 (1997).
[9] K. L. Ngai, Comment. Sol. Stat. Phys. 9 (1979) 127; 9 141 (1980); W. Götze and L. Sjögren, Rep. Prog. Phys. 55, 241 (1992).
[10] R. G. Palmer, D. L. Stein, E. Abrahams, and P. W. Anderson, Phys. Rev. Lett. 53 (1984) 958; C. H. Chou, M. H. Tu, and C. L. Wu, Chinese J. Phys. 34 (1996) 143.
[11] J.-P. Bouchaud and A. Georges, Phys. Rep. 195:127 (1990); and references therein.
[12] C. Tang and P. Bak, Phys. Rev. Lett. 60, 2347 (1988); C. Tang and P. Bak, J. Stat. Phys. 51 (1988) 797.
[13] G. Caldarelli, A. Maritan, and M. Vendruscolo, Europhys. Lett. 35 (1996) 481.
[14] A. Vespignani and S. Zapperi, Phys. Rev. Lett. 78 (1997) 4793.
[15] T. Hwa and M. Kardar, Phys. Rev. Lett. 62 (1989) 1813; Phys. Rev. A 45 (1992) 7001.
[16] G. Grinstein, D.-H. Lee, and S. Sachdev, Phys. Rev. Lett. 64 (1990) 1927.