Two-dimensional Yang-Mills Theories Are String Theories

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ABSTRACT

We show that two-dimensional SO($N$) and Sp($N$) Yang-Mills theories without fermions can be interpreted as closed string theories. The terms in the $1/N$ expansion of the partition function on an orientable or nonorientable manifold $\mathcal{M}$ can be associated with maps from a string worldsheet onto $\mathcal{M}$. These maps are unbranched and branched covers of $\mathcal{M}$ with an arbitrary number of infinitesimal worldsheet cross-caps mapped to points in $\mathcal{M}$. These string theories differ from SU($N$) Yang-Mills string theory in that they involve odd powers of $1/N$ and require both orientable and nonorientable worldsheets.

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1. Introduction

Calculating hadron physics directly from QCD remains tantalizingly elusive after nearly two decades of effort. The strong interactions exhibit stringy characteristics at low momentum transfers and this provided the original impetus for the development of string theory as a theory of hadrons. Formulating QCD as a string theory would be an important step in connecting it to hadron physics and, perhaps, in explaining confinement.

In a series of ground-breaking papers, Gross, Taylor, and Minahan have made progress in this direction by arguing that two-dimensional QCD is a string theory. This identification was possible because the partition function of Yang-Mills gauge theory on an arbitrary two-dimensional manifold is known in closed form. On a compact orientable surface $\mathcal{M}_G$ with area $A$ and genus $G$ (so that $\chi = 2 - 2G$ is the Euler characteristic), the partition function is given by

$$Z_{\mathcal{M}_G} = \int [DA^\mu] \exp \left[ -\frac{1}{4e^2} \int_{\mathcal{M}_G} d^2x \sqrt{g} \text{Tr} F_{\mu\nu}F^{\mu\nu} \right] = \sum_R (\dim R)^{2-2G} e^{-\lambda C_2(R)/2N}. \quad (1.1)$$

The sum runs over all irreducible representations $R$ of the gauge group, $\dim R$ and $C_2(R)$ denote the dimension and quadratic Casimir of $R$, and $\lambda = e^2N$, where $e$ is the gauge coupling constant. The claim that two-dimensional SU($N$) gauge theory without fermions is equivalent to a closed string theory (with string coupling $1/N$ and string tension $\lambda/2$) is verified by relating the expansion of (1.1) in powers of $1/N$ to the genus expansion of a string theory. The terms in the expansion of the partition function can be interpreted as a weighted counting of maps from (possibly disconnected) orientable worldsheets $\mathcal{W}$ onto the target space $\mathcal{M}_G$. The power of $N$ gives the Euler characteristic of the worldsheet, and the coefficients are related to the number of inequivalent maps from $\mathcal{W}$ to $\mathcal{M}_G$. The string theory is therefore determined by the set of worldsheets to be included in its genus expansion, the types of maps counted, and the weights of those maps in the expansion of the SU($N$) Yang-Mills partition function.

In this paper we show that two-dimensional Yang-Mills theories with gauge groups SO($N$) and Sp($N$) can also be interpreted as closed string theories. A significant difference between these theories and SU($N$) Yang-Mills theory emerges immediately. The $1/N$ expansion of the partition function for SU($N$) involves only even powers of $1/N$, whereas we shall see that the partition functions for SO($N$) and Sp($N$) necessarily involve odd powers of $1/N$. In a string interpretation, these terms correspond to worldsheets of odd Euler characteristic. Since we are considering gauge theories without fermions, these worldsheets do not have boundaries and must necessarily be nonorientable. Thus the closed string theories corresponding to SO($N$) and Sp($N$) Yang-Mills theory include both orientable and nonorientable worldsheets.
At first sight, the necessity for nonorientable worldsheets would appear to pose difficulties for a string interpretation of SO($N$) and Sp($N$) Yang-Mills theories on an orientable surface, since there are no coverings of orientable target spaces by nonorientable surfaces. As we will see, however, the odd $1/N$ terms in the partition function for orientable surfaces are associated not with true coverings, but with pinch maps, for which the theorem just cited does not hold. Thus, we are able to give a consistent string interpretation of (1.1) for SO($N$) and Sp($N$) on $M_G$.

The SO($N$) and Sp($N$) Yang-Mills theories on a nonorientable surface can also be given a string interpretation. An arbitrary compact nonorientable two-manifold can be constructed as the connected sum of $q$ projective planes, or as a sphere with the insertion of $q$ cross-caps. (A cross-cap is a projective plane with a disc removed.) The resulting surface $M_q$, where $q$ is referred to as the genus, has Euler characteristic

\[ \chi = 2 - q . \]  

The partition function of Yang-Mills theory on $M_q$ is given by

\[ Z_{M_q} = \sum_{R=\overline{R}} (\varepsilon_R \dim R)^{2-q} e^{-\lambda AC_2(R)/2N} , \]

where the sum only runs over self-conjugate representations ($R = \overline{R}$) of the gauge group, and where $\varepsilon_R = 1(-1)$ if there exists a symmetric (anti-symmetric) invariant in $R \otimes \overline{R} \to \mathbb{C}$.

We will demonstrate that all the leading terms in the $1/N$ expansion of the SO($N$) and Sp($N$) Yang-Mills partition functions on an arbitrary surface $\mathcal{M}$ (either orientable or nonorientable) can be associated with surface maps from orientable and nonorientable worldsheets $\mathcal{W}$ onto the target space $\mathcal{M}$. In general, these maps are compositions of branched coverings of the target space with pinch maps. The pinch maps send infinitesimal cross-caps on the worldsheet to points on the target space. When the Euler characteristic of the target space satisfies $\chi \neq 0$, the $1/N$ expansion contains subleading terms whose geometric interpretation remains obscure. For the torus and Klein bottle ($\chi = 0$), however, the subleading terms vanish, and all the terms in the partition function can be given a geometric interpretation.

In section 2, we present the $1/N$ expansion of the Yang-Mills partition function for SO($N$) and Sp($N$). In section 3, we clarify the nature of the Young tableau transposition symmetry of the SU($N$) Yang-Mills theory that guarantees that only even powers of $1/N$ appear in the partition function. The lack of a similar symmetry for SO($N$) and Sp($N$) leads to odd powers of $1/N$ in the partition functions of these theories. However, certain odd terms are absent in the partition functions of all three theories due to the presence of a partial transposition symmetry, and this fact has a common string interpretation as the evenness of the number of branch points. In sections 4 and 5, we present the string interpretation of SO($N$) and Sp($N$) Yang-Mills theories.

† The subtleties of quantizing and solving a gauge theory on a nonorientable manifold are explained in ref. [3].
Intuitively it is easy to see why odd powers of $N$ occur for SO($N$) and Sp($N$). In the double line picture, a closed gluon propagator contains both a ribbon and a Möbius band, due to the self-conjugacy of the fundamental representations of SO($N$) and Sp($N$), and thus gives contributions proportional to both $N^2$ and $N$. In contrast, the closed gluon propagator in SU($N$) contains only a ribbon in the double line formalism, because the fundamental representation of SU($N$) is not self-conjugate, so that only even powers of $N$ appear.

2. The $1/N$ Expansion of the Yang-Mills Partition Function

In this section we formulate the $1/N$ expansion of the partition function for SO($N$) and Sp($N$) Yang-Mills theory on an orientable or nonorientable surface. We begin by expressing the quadratic Casimirs and dimensions of the representations of SO($N$) and Sp($N$) as polynomials in $N$.

All irreducible representations of Sp($N$) and all irreducible tensor representations of SO($N$) can be represented by Young tableaux with at most $n$ rows, where $n$ is the rank of the group. (For Sp($N$), $N$ is even, and the rank is $n = \frac{1}{2}N$. For SO($N$), the rank $n$ is the integer part of $\frac{1}{2}N$.) We will denote the $i$th row length of the tableau by $\ell_i$ so that $\ell_i \geq \ell_{i+1} \geq 0$, and the $j$th column length by $k_j$ so that $k_j \geq k_{j+1} \geq 0$. (For the relation between row lengths and Dynkin indices, see, for example, ref. [14].)

We denote by

$$r = \sum_{i=1}^{n} \ell_i = \sum_{j=1}^{k} k_j$$

(2.1)

the number of cells (or boxes) in the tableau associated with a given representation.

The quadratic Casimirs for SO($N$) and Sp($N$) are given by

$$C_2(R) = fN \left[ r + \frac{T(R)}{N} - \frac{\sigma r}{N} \right]$$

(2.2)

where the long roots of each group satisfy $(\alpha, \alpha) = 2$, and where\[ T(R) = \sum_{i=1}^{n} \ell_i (\ell_i + 1 - 2i) = \sum_{i=1}^{k_1} \ell_i^2 - \sum_{j=1}^{k_2} k_j^2

(2.3)

with

$$f = 1, \quad \sigma = 1, \quad \text{for SO}(N),$$

$$f = \frac{1}{2}, \quad \sigma = -1, \quad \text{for Sp}(N).$$

The permutation sign $\varepsilon_R$ is given by

$$\varepsilon_R = \sigma^r$$

(2.4)

\[T(R)\] is the quantity denoted $\hat{C}(R)$ in refs. [3] and [4], $\bar{n}$ in ref. [4], and $X + r$ in ref. [14].
for tensor representations of $\text{SO}(N)$ and all representations of $\text{Sp}(N)$.\textsuperscript{10}

The dimension of a representation of $\text{SU}(N)$ corresponding to a Young tableau $R$ is conveniently given by Robinson’s celebrated hook length formula:\textsuperscript{3}

$$
\left(\text{dim } R\right)_{\text{SU}(N)} = \prod_{x \in R} \frac{N + a(x)}{h(x)}.
$$

(2.5)

The product in (2.5) runs over all cells $x$ of $R$, each of which is identified by its row $i$ and column $j$. The hook length of a cell is given by

$$
h(x) = h(i, j) = \ell_i + k_j - i - j + 1
$$

(2.6)

and $a(x) = a(i, j) = j - i$. The hook length product is related to the dimension $d_R$ of the representation of the symmetric group $S_r$ specified by the tableau $R$ by\textsuperscript{7}

$$
\frac{1}{r!} \prod_{x \in R} h(x) = \frac{d_R}{r!}.
$$

(2.7)

The analogs of Robinson’s formula for $\text{SO}(N)$ and $\text{Sp}(N)$ are\textsuperscript{8}

$$
\left(\text{dim } R\right)_{\text{Sp}(N)} = \prod_{x \in R} \frac{N - c(x)}{h(x)}
$$

(2.8)

and

$$
\left(\text{dim } R\right)_{\text{SO}(N)} = \prod_{x \in R} \frac{N + s(x)}{h(x)}
$$

(2.9)

where

$$
c(x) = c(i, j) = \begin{cases} k_i + k_j - i - j, & i \leq j, \\ -\ell_i - \ell_j + i + j - 2, & i > j, \end{cases}
$$

(2.10)

$$
s(x) = s(i, j) = \begin{cases} \ell_i + \ell_j - i - j, & i \geq j, \\ -k_i - k_j + i + j - 2, & i < j. \end{cases}
$$

(2.11)

A straightforward calculation gives

$$
\sum_{x \in R} a(x) = -\sum_{x \in R} c(x) = \sum_{x \in R} s(x) = \frac{1}{2} T(R).
$$

(2.12)

The hook length formulae are very convenient for computing the $1/N$ expansion of the dimension of $R$. The leading terms of this expansion for representations of $\text{SU}(N)$, $\text{SO}(N)$, and $\text{Sp}(N)$ are given by

$$
\text{dim } R = \frac{d_R N^r}{r!} \left[ 1 + \frac{T(R)}{2N} + \mathcal{O}\left(\frac{1}{N^2}\right) \right],
$$

(2.13)

\textsuperscript{\dagger} \varepsilon_R$ equals the sign in eq. 2.20 of ref. \textsuperscript{15} where an extensive discussion of these signs may be found.
where the $\mathcal{O}(1/N^2)$ terms are different for each group.

We now have the ingredients to write the $1/N$ expansion of the partition function (1.1) or (1.3) on a surface $\mathcal{M}$. From (2.2), we see that tableaux with $r \ll N$ have quadratic Casimirs of $\mathcal{O}(N)$, and so will contribute perturbatively to the partition function. It is straightforward to prove that if $r$ is not $\ll N$, then the Casimir of the representation is of $\mathcal{O}(N^2)$ or higher. Spinor representations of $\text{SO}(N)$ also have Casimirs $\geq \mathcal{O}(N^2)$. Consequently, the contribution of these representations to the partition function is exponentially suppressed, and corresponds to effects non-perturbative in $1/N$. (This is in contrast to $\text{SU}(N)$, where certain representations with $\mathcal{O}(N)$ boxes, namely, composite representations, have Casimirs of $\mathcal{O}(N)$, and so contribute perturbatively.) We will neglect non-perturbative effects in this paper, and consider only Young tableaux with $r \ll N$, denoting the set of such tableaux by $\mathcal{Y}$. Let $\mathcal{A} = f\lambda\lambda$ denote the dimensionless area of the surface $\mathcal{M}$. Using (2.2), (2.4), and (2.13), we write the perturbative Yang-Mills partition function on an orientable (1.1) or a nonorientable (1.3) two-manifold $\mathcal{M}$ with Euler character $\chi$ as

\[
Z_{\mathcal{M}} = \sum_{R \in \mathcal{Y}} (\sigma^r \text{dim } R) \chi \frac{e^{-Ar/2}}{e^{-\sigma Ar/2N}} e^{-AT(R)/2N} 
\]

\[
= \sum_{R \in \mathcal{Y}} \sum_{i=0}^{\infty} \left( \frac{\sigma^r N r d_R}{r!} \right) \chi \frac{1}{i!} \left( \frac{-AT(R)}{2N} \right)^i e^{-Ar/2} e^{\sigma Ar/2N} \left[ 1 + \mathcal{O}(1/N^2) \right]. \tag{2.15}
\]

(All the tableaux in $\mathcal{Y}$ are self-conjugate, and so contribute to the partition function on nonorientable surfaces.) The subleading terms in the square brackets result solely from the subleading contributions to the dimension of $R$ in (2.13), and have no dependence on $\mathcal{A}$.

For each of the tableaux $R \in \mathcal{Y}$, there exists a transposed tableau $\tilde{R}$, also in $\mathcal{Y}$, obtained by interchanging rows and columns. From (2.3), we see that

\[
T(\tilde{R}) = -T(R). \tag{2.16}
\]

Rewriting $\sum_R = \frac{1}{2} \left[ \sum_R + \sum_{\tilde{R}} \right]$ and using (2.16), we find

\[
Z_{\mathcal{M}} = \sum_{R \in \mathcal{Y}} \sum_{i=0}^{\infty} \left( \frac{\sigma^r N d_R}{r!} \right) \chi \frac{1}{(2i)!} \left( \frac{AT(R)}{2N} \right)^{2i} e^{-Ar/2} e^{\sigma Ar/2N} \left[ 1 + \mathcal{O}(1/N^2) \right]. \tag{2.17}
\]

Now the subleading terms in the square brackets are all of the form $\mathcal{A}^m/N^n$ with $m < n$. They are absent in the case of Yang-Mills theory on a torus or Klein bottle ($\chi = 0$).

In sections 4 and 5, we will interpret the leading order terms in (2.17) (all the terms in the case of the torus or Klein bottle) as weighted multiplicities of maps from (possibly disconnected) orientable and nonorientable worldsheets onto the target space $\mathcal{M}$. 

5
3. Young Tableau Transposition Symmetry

Before presenting the string interpretation of the partition function, we briefly comment on a transposition symmetry present in Yang-Mills theory. In SU($N$) Yang-Mills theory, there is a symmetry under which

\[ C_2(\bar{T}, N) = -C_2(T, -N), \quad \left| \dim(\bar{T}, N) \right| = \left| \dim(T, -N) \right|, \quad \text{for SU}(N) \]  

(3.1)

where $T$ is the composite representation $\bar{S}R$ (see ref. 5), and $\bar{T} = \bar{S}R$. This symmetry was noted in ref. 3 in the case of chiral representations ($S = 1$), but it is easy to prove that the more general symmetry (3.1) holds for representations of SU($N$). Since both $T$ and $\bar{T}$ contribute perturbatively to the partition function sum, this symmetry guarantees the absence of odd powers of $1/N$ in the partition function. As noted in the introduction, this means that SU($N$) requires only orientable worldsheets. The symmetry (3.1) also ensures that only even powers of $AT(R)/N$ appear, which in the string interpretation is related to the evenness of the number of branch points on a surface without boundary.

In the case of SO($N$) and Sp($N$), the analogous symmetry for the Casimirs does not hold because of the last term in (2.2). While (2.13) shows that

\[ \dim(\bar{R}, N) = (-1)^r \dim(R, -N) \quad \text{to} \quad \mathcal{O}(1/N), \]  

(3.2)

for SO($N$) and Sp($N$) representations, this transposition symmetry breaks down at $\mathcal{O}(1/N^2)$, as seen in the example

\[ (\dim \boxdot)_{\text{Sp}(N)} = \frac{N^4}{4!} \left( 1 + \frac{6}{N} + \frac{11}{N^2} + \frac{6}{N^3} \right), \]  

(3.3)

\[ (\dim \boxdot)_{\text{Sp}(N)} = \frac{N^4}{4!} \left( 1 - \frac{6}{N} - \frac{1}{N^2} + \frac{6}{N^3} \right). \]  

(3.4)

As a result of the breakdown of this symmetry, the partition function contains odd powers of $1/N$, and therefore implies the existence of nonorientable worldsheets.

However, the transposition symmetry (2.16) that does remain valid in the case of SO($N$) and Sp($N$) has the effect that only even powers of $AT(R)/N$ appear in the leading order terms of (2.17). In the following section, we interpret these terms as arising from branched coverings with simple (order two) branch points, and it is a basic topological fact that the number of such branch points is even on any surface without boundary. From the vantage point of Yang-Mills theory, the transposition symmetry (2.16) enforces this evenness, and thus makes possible for all three groups the string theory interpretation in terms of branched coverings.
4. Nonorientable Target Spaces

In this section we show that $\text{SO}(N)$ or $\text{Sp}(N)$ Yang-Mills theory on a nonorientable surface $\mathcal{M}_q$ is equivalent to a closed string theory with target space $\mathcal{M}_q$. Each term in the Yang-Mills free energy corresponds to some surface map from a connected worldsheet $\mathcal{W}$ onto the nonorientable target space $\mathcal{M}_q$. Since the free energy corresponds to the sum over maps from connected worldsheets, the partition function corresponds to the sum over maps from all worldsheets, both connected and disconnected. Inequivalent maps from $\mathcal{W}$ to $\mathcal{M}_q$ give distinct contributions to the free energy, but the precise nature of the equivalence is somewhat unclear. Topologically inequivalent maps clearly count as distinct. The presence of geometric moduli, however, seems to indicate that some topologically equivalent maps are also distinguished, and that the equivalence relation is a refinement of the purely topological classification. While we intend to return to this question in the near future, in this paper we will adopt the combinatorial procedure of ref. 5 to count distinct maps.

We begin by considering (possibly disconnected) $r$-fold unbranched covers of $\mathcal{M}_q$. They are characterized by the fact that exactly $r$ points of the cover are mapped to each point of the target space. The generators $a_j$, $j = 1, \ldots, q$, of the fundamental group $\pi_1(\mathcal{M}_q)$ satisfy the single relation

$$a_1 a_1 \cdots a_q a_q = 1.$$  \hspace{1cm} (4.1)

Let $\nu$ be a (possibly disconnected) $r$-fold unbranched covering of $\mathcal{M}_q$. Choose a point $p$ of $\mathcal{M}_q$, and label the $r$ sheets of $\nu$ over $p$ by the integers in $I = \{1, 2, \ldots, r\}$. With each element $t \in \pi_1(\mathcal{M}_q)$, we associate the permutation of $I$ that results from the transport of the labels on sheets around the path obtained by lifting $t$ to $\nu$. This procedure defines a homomorphism $H_\nu$ from $\pi_1(\mathcal{M}_q)$ to the permutation group $S_r$. Homeomorphisms of the covering surface can permute the labeling of the sheets, but they leave the homomorphism $H_\nu$ invariant. For a given covering $\nu$, there are $r!$ different labelings of the sheets. Relabeling the $r$ sheets with the permutation $\rho$ gives the conjugate homomorphism $\rho H_\nu \rho^{-1}$. If $\rho$ belongs to $S_\nu \subset S_r$, the group of permutations produced by homeomorphisms of the covering surface, then $\rho H_\nu \rho^{-1} = H_\nu$. Thus, the number of distinct homomorphisms corresponding to $\nu$ is $r!/|S_\nu|$, where $|S_\nu|$ is the number of elements of $S_\nu$.

Next consider an $r$-fold branched covering of $\mathcal{M}_q$ with $2i$ branch points $b_1, \ldots, b_{2i}$. We will only need the generic case of simple branched coverings, in which exactly $r - 1$ points of the worldsheet are mapped to each branch point on $\mathcal{M}_q$ and $r$ points are sent to every other point of $\mathcal{M}_q$. It is a topological fact that the number of such branch points is necessarily even. Choose a set of $2i$ curves $\{c_j\}$, each of which encircles one of the branch points $b_j$. The $c_j$ together with the $a_j$ form a set of generators for the fundamental group $\pi_1(\mathcal{M} \setminus \{b_j\})$, defined by the single relation

$$c_1 c_2 \cdots a_1 a_1 \cdots a_q a_q = 1.$$  \hspace{1cm} (4.2)
As before, an $r$-fold covering $\nu$ with branch points $b_1, \ldots, b_{2i}$ defines a homomorphism from $\pi_1(\mathcal{M}\setminus\{b_j\})$ to $S_r$, but each of the generators $c_j$ is associated with a permutation $p_j$ belonging to $P_r$, the conjugacy class of permutations that interchange only two elements, since the branched cover is simple.

Let $\Sigma(q, r, 2i)$ denote the set of (connected and disconnected) $r$-fold covers of $\mathcal{M}_q$ with $2i$ branch points. We wish to count each covering with a weight of $1/|S_r|$. In light of the previous discussion, this is equivalent to counting distinct homomorphisms $H_\nu$ with a weight of $(1/r!)$. The weighted sum over coverings is given by

$$\sum_{\nu \in \Sigma(q, r, 2i)} \frac{1}{|S_\nu|} = \sum_{p_1 \cdots p_{2i} \in P_r} \sum_{t_1 \cdots t_q \in S_r} \frac{1}{r!} \delta(p_1 \cdots p_{2i} t_1^2 \cdots t_q^2)$$

(4.3)

where the delta function $\delta(\rho)$, defined by

$$\delta(\rho) = \begin{cases} 1 & \text{if } \rho = \text{identity} \\ 0 & \text{if } \rho \neq \text{identity} \end{cases}$$

(4.4)

enforces the relation (4.2). Let $D_R(\rho)$ be the matrix associated with $\rho \in S_r$ in the representation of $S_r$ specified by the Young tableau $R$. The character of $\rho$ is given by $\chi_R(\rho) = \text{Tr} D_R(\rho)$. By the orthogonality of characters, we have

$$\delta(\rho) = \frac{1}{r!} \sum_R d_R \chi_R(\rho)$$

(4.5)

where $d_R = \text{Tr} I_R$ is the dimension of $R$. ($I_R$ is the identity matrix in the representation $R$.) Thus (4.3) can be expanded as

$$\sum_{\nu \in \Sigma(q, r, 2i)} \frac{1}{|S_\nu|} = \sum_{p_1 \cdots p_{2i} \in P_r} \sum_{t_1 \cdots t_q \in S_r} \left(\frac{1}{r!}\right)^2 d_R \chi_R(p_1 \cdots p_{2i} t_1^2 \cdots t_q^2).$$

(4.6)

To compute this, consider

$$\sum_{p_1 \cdots p_{2i} \in P_r} \sum_{t_1 \cdots t_q \in S_r} D_R(p_1 \cdots p_{2i} t_1^2 \cdots t_q^2)$$

$$= \sum_{p_1 \in P_r} D_R(p_1) \cdots \sum_{p_{2i} \in P_r} D_R(p_{2i}) \sum_{t_1 \in S_r} D_R(t_1^2) \cdots \sum_{t_q \in S_r} D_R(t_q^2).$$

(4.7)

Using Schur’s lemma and the fact that all the representations of $S_r$ are real, it follows that

$$\sum_{t \in S_r} D_R(t^2) = \frac{r!}{d_R} I_R.$$

(4.8)

In ref. [5] it is shown that

$$\sum_{p \in P_r} D_R(p) = \frac{T(R)}{2} I_R.$$

(4.9)
Therefore,

\[
\sum_{p_1 \cdots p_{2i} \in P_r} \sum_{t_1 \cdots t_q \in S_r} D_R(p_1 \cdots p_{2i} t_1^2 \cdots t_q^2) = \left( \frac{T(R)}{2} \right)^{2i} \left( \frac{r!}{d_R} \right)^q I_R. \tag{4.10}
\]

Thus, the weighted sum \([4.10]\) over the coverings of \(M_q\) is finally given by

\[
\sum_{\nu \in \Sigma(q,r,2i)} \left( \frac{1}{|S_{\nu}|} \right) = \sum_R \left( \frac{T(R)}{2} \right)^{2i} \left( \frac{r!}{d_R} \right)^q \tag{4.11}
\]

The representations \(R\) summed over in \([4.11]\) correspond to the set of Young tableaux with \(r\) boxes.

Using \([4.11]\), we may rewrite the \(SO(N)\) and \(Sp(N)\) Yang-Mills partition function on \(M_q\) \([2.17]\) as a sum over branched coverings of \(M_q\)

\[
Z_{M_q} = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\nu \in \Sigma(q,r,2i)} \frac{1}{|S_{\nu}|} \sigma^{(2-q)r} e^{-Ar/2} e^{\sigma Ar/2N} \frac{A^{2i}}{(2i)!} \left( \frac{1}{N} \right)^{(q-2)r+2i} \left[ 1 + O\left( \frac{1}{N^2} \right) \right]. \tag{4.12}
\]

All the leading terms in the partition function \([4.12]\) can be interpreted in terms of surface maps from worldsheets onto \(M_q\), as follows.

The leading \(1/N\) term

\[
\sum_{r=0}^{\infty} \sigma^{(2-q)r} e^{-Ar/2} \sum_{\nu \in \Sigma(q,r,0)} \frac{1}{|S_{\nu}|} N^{r(2-q)}, \tag{4.13}
\]

corresponds to a sum over unbranched \(r\)-fold coverings (local homeomorphisms). For such coverings, we necessarily have

\[
\chi_W = r \chi_{M_q} = r(2-q),
\]

and the area of the worldsheet is \(rA\), so these maps contribute with a factor \(N^{(2-q)r}\) and with the action given by the string coupling times the area of the worldsheet. The coefficient of this term is the weighted sum over coverings \(\sum_{\nu \in \Sigma(q,r,0)} 1/|S_{\nu}|\). The covering spaces can be either connected or disconnected, and either orientable or nonorientable (but must be nonorientable when \((2-q)r\) is odd). For example, consider coverings of the Klein bottle \((q = 2)\). The weighted sum over \(r\)-fold coverings is given by

\[
\sum_{\nu \in \Sigma(q=2,r,0)} \frac{1}{|S_{\nu}|} = \sum_R (1) = p(r), \tag{4.14}
\]

where \(p(r)\) is the number of partitions of \(r\). Using the relation between the partition function and the free energy discussed above, the number of connected \(r\)-fold coverings of the Klein bottle is therefore \(q(r) = \sum_{d|r} d\), the sum of divisors of \(r\). If \(r\) is odd, the covering surface must be nonorientable, \textit{i.e.}, a Klein bottle. If \(r\)
is even, however, the Klein bottle can also be covered by the torus. In fact, for even \( r \), we count \( q(r/2) \) connected \( r \)-fold coverings of the Klein bottle by the torus, and \( q(r) - q(r/2) \) connected \( r \)-fold coverings by the Klein bottle, using a procedure analogous to that described in ref. [4].

Terms in (4.12) with \( i \neq 0 \) correspond to branched coverings. The addition of each pair of branch points decreases the Euler characteristic of the worldsheet by two, so that the Euler characteristic of the world sheet is \((2 - q)r - 2i\) for an \( r \)-fold branched covering with \( 2i \) branch points. Integrating over the positions of \( 2i \) branch points on the target space and accounting for their indistinguishability gives the factor \( A^{2i}/(2i)! \).

Finally, the sum (4.12) contains a factor

\[
\exp \left( \frac{\sigma A r}{2N} \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{\sigma A r}{2N} \right)^k.
\]

The \( k \)th term in this sum corresponds to an \( r \)-fold covering of \( M_q \) with \( 2i \) branch points and with the insertion of \( k \) infinitesimal cross-caps. The insertion of one or more cross-caps renders the worldsheet nonorientable. The surface maps from these worldsheets are compositions of unbranched or branched coverings with pinch maps, in which the cross-caps on the worldsheets are mapped to points on the target space. Each cross-cap decreases the Euler characteristic of the worldsheets by one, contributing an overall factor \( 1/N^k \). Integrating the positions of the \( k \) cross-caps over the area of the worldsheet, and taking account of their indistinguishability, gives the factor \( (r A)^k/k! \). From (4.15) we see that each cross-cap contributes a factor \( 1/2 \) for \( \text{SO}(N) \) and \(-1/2\) for \( \text{Sp}(N) \).

We have exhibited all the leading terms as a genus expansion of weighted multiplicities of surface maps. The remaining \( O(1/N^2) \) terms in (4.12), for which we cannot yet account, are due to the subleading corrections to the dimension of \( R \). For Yang-Mills theory on a Klein bottle these subleading terms vanish, and we have succeeded in giving a string interpretation for all the terms in the partition function.

5. Orientable Target Spaces

In this section, we show that there appears to be no obstacle to a string theory interpretation of \( \text{SO}(N) \) or \( \text{Sp}(N) \) Yang-Mills theory on an orientable surface, analogous to that for nonorientable surfaces given in the last section.

It was shown in ref. [4] that the weighted sum over \( r \)-fold coverings of an orientable genus \( G \) target space \( M_G \) with \( 2i \) branch points is given by

\[
\sum_{\nu \in \Sigma (G, n, 2i)} \frac{1}{|S_\nu|} = \sum_R \frac{1}{T(R)} \left( \frac{r!}{d_R} \right)^{2G-2},
\]

where \( T(R) \) is the dimension of the target space.

\[ (5.1) \]
Using this, the partition function on $\mathcal{M}_G$ \[(2.17)\] may be written as a weighted sum over coverings of $\mathcal{M}_G$

$$Z_{\mathcal{M}_G} = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\nu \in \Sigma(G,r,2i)} \frac{1}{|S_{\nu}|} e^{-A r/2} e^{\sigma A r/2N} \frac{A^{2i}}{(2i)!} \left( \frac{1}{N} \right)^{(2G-2)r+2i} \left[ 1 + \mathcal{O}\left( \frac{1}{N^2} \right) \right]. \quad (5.2)$$

As before, the leading terms in this partition function can be interpreted in terms of surface maps from worldsheets onto $\mathcal{M}_G$.

The leading $1/N$ term corresponds to a weighted sum over unbranched $r$-fold coverings of $\mathcal{M}_G$. In contrast to the last section, the covering spaces here must be orientable. For example, the weighted sum over $r$-fold coverings of the torus is

$$\sum_{\nu \in \Sigma(G=1,r,0)} \frac{1}{|S_{\nu}|} = \sum_{R}(1) = p(r). \quad (5.3)$$

Reasoning as before, the number of connected $r$-fold coverings of the torus is $q(r)$, and the covering spaces here are all tori.

Terms with $i \neq 0$ correspond to branched coverings, and the counting is the same as before. We again interpret the term $\exp(\sigma A r/2N)$ as due to surface maps constructed by inserting infinitesimal cross-caps into (possibly branched) covering spaces. This necessarily renders the worldsheet nonorientable. Note that such maps from nonorientable worldsheets onto orientable target spaces do exist, because the maps are not true covers but compositions of covering maps with pinch maps. Therefore, there appears to be no reason to exclude them from the genus expansion of an unoriented string, and indeed they make possible a string interpretation of $SO(N)$ and $Sp(N)$ gauge theories on orientable target spaces, including the physical case of the torus.

6. Concluding Remarks

We have shown that two-dimensional $SO(N)$ and $Sp(N)$ Yang-Mills theories without fermions can be understood as closed string theories. All the leading terms in the $1/N$ expansion of the partition function on a manifold $\mathcal{M}$ can be interpreted in terms of maps from a string worldsheet onto a target space $\mathcal{M}$. These maps include unbranched and branched coverings of $\mathcal{M}$ with an arbitrary number of infinitesimal worldsheet cross-caps mapped to points in $\mathcal{M}$.

These string theories differ from $SU(N)$ Yang-Mills string theory in that the worldsheets need not be orientable. In particular, terms in the expansion of the partition function with odd powers of $1/N$ necessarily correspond to nonorientable worldsheets.

It is intriguing that although the (perturbative) Yang-Mills partition function has exactly the same form on an orientable and a nonorientable manifold with the
same Euler characteristic, the terms have different string theory interpretations. For example, the leading $1/N$ term of the partition function on the Klein bottle corresponds to maps from the Klein bottle and torus to the Klein bottle, whereas the identical term in the partition function on the torus corresponds to maps from the torus to the torus.

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