Small energy stabilization for 1D Nonlinear Klein Gordon Equations

Scipio Cuccagna, Masaya Maeda, Stefano Scrobogna

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Abstract

We give a partial extension to dimension 1 of the result proved by Bambusi and Cuccagna [1] on the absence of small energy real valued periodic solutions for the NLKG in dimension 3. We combine the framework in Kowalczyk and Martel [14] with the notion of "refined profile".

1 Introduction

Let $m > 0$ and $V \in S(\mathbb{R}, \mathbb{R})$ (Schwartz function) with set of eigenvalues

$$\sigma_{\delta}(L_1) = \{\lambda_j^2 \mid j = 1, \ldots, N\}$$

with $0 < \lambda_1 < \cdots < \lambda_N < m$, where $L_1 = -\partial_x^2 + V + m^2$. (1.1)

We assume there exist $C > 0$ and $a_1 > 0$ such that

$$|V^{(l)}(x)| \leq Ce^{-a_1|x|} \text{ for all } 0 \leq l \leq N + 1. \quad (1.2)$$

Let $f \in C^\infty(\mathbb{R}, \mathbb{R})$ s.t. $f(0) = f'(0) = 0$. We consider the nonlinear Klein-Gordon (NLKG) equation

$$\dot{u} = J(L_1 u + f[u]), \quad u = (u_1 u_2) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2,$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} L_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f[u] = \begin{pmatrix} f(u_1) \\ 0 \end{pmatrix}. \quad (1.3)$$

Denoting by $\phi_j$ a real valued eigenfunction with $L^2(\mathbb{R})$ norm equal to 1 of $L_1$ associated to $\lambda_j^2$, setting

$$\Phi_j := \begin{pmatrix} \phi_j \\ i\lambda_j \phi_j \end{pmatrix} \text{ for } j = 1, \cdots, N, \quad (1.4)$$

we have

$$JL_1 \Phi_j = i\lambda_j \Phi_j \text{ and } JL_1 \overline{\Phi_j} = -i\lambda_j \overline{\Phi_j}. \quad (1.5)$$

In fact the $\Phi_j$ and their complex conjugates $\overline{\Phi_j}$ generate all the eigenfunctions of the linearization $JL_1$ of our NLKG (1.3).
Our NLKG (1.3) is a Hamiltonian system for the symplectic form
\[
\Omega(u,v) := \langle J^{-1}u, v \rangle, \quad \text{where} \quad \langle u, v \rangle := \text{Re}(u, \nabla)
\]
and the Hamiltonian or energy function is given by
\[
E(u) = \frac{1}{2} \langle L_1 u, u \rangle + \int_{\mathbb{R}} F(u_1) \, dx, \quad \text{where} \quad F(u) = \int_0^s f(\tau) \, d\tau.
\]
The local well-posedness of (1.3) is well known. From the conservation of the energy, we have that for sufficiently small \(\delta > 0\), if \(\|u_0\|_{\mathcal{H}^1} \leq \delta\), then \(\|u\|_{L^\infty(\mathbb{R}, \mathcal{H}^1)} \lesssim \delta\) and in particular we obtain the global well-posedness for small data, where
\[
\|u\|_{\mathcal{H}^1}^2 = \|u_1\|_{\mathcal{H}^1}^2 + \|u_2\|_{L^2}^2.
\]
Given a constant \(a > 0\), we consider the space defined by the norm
\[
\|u\|_{\mathcal{H}_a} := \|\text{sech}(ax) u\|_{\mathcal{H}^1}.
\]
We denote by \(\phi[z]\) the refined profile, introduced below in Sect. 1.1, where
\[
z = (z_1, ..., z_N),
\]
encodes the discrete modes and where \(\phi[z] = \sum z_j \Phi_j + \text{O}(\|z\|)\), where by \(g + c.c.\), we mean \(g + \bar{g}\) and \(\|z\|^2 := \sum_{j=1}^N |z_j|^2\).

The main result in this paper is the following theorem.

**Theorem 1.1.** Under Assumptions 1.3, 1.6 and 1.9 given below, for any \(a > 0\) and \(\epsilon > 0\) there exists \(\delta_0 > 0\) such that if \(\|u_0\|_{\mathcal{H}^1} =: \delta < \delta_0\), then we have a global representation
\[
u(t) = \phi[z(t)] + \eta(t) \quad \text{for appropriate} \quad z \in C^1(\mathbb{R}, \mathbb{C}^N) \quad \text{and} \quad \eta \in C^0(\mathbb{R}, \mathcal{H}^1),
\]
and, for \(I = \mathbb{R}\),
\[
\int_I \|\eta(t)\|_{\mathcal{H}_a}^2 \, dt \leq \epsilon,
\]
and
\[
\lim_{t \to \infty} z(t) = 0.
\]

The result of this paper is a partial extension to dimension 1 of the result, on local decay to zero for small real valued solutions of an NLKG with a trapping potential and, in particular, on the absence of small energy real valued periodic solutions, proved for dimension 3 by Bambusi and Cuccagna [1]. The latter was an extension, to cases with quite general spectral configurations, of a result proved by Soffer and Weinstein [30] under rather restrictive spectral hypotheses. There is a substantial literature on the asymptotic stability of patterns for wave like equations, partially reviewed for the case of the Nonlinear Schrödinger Equation (NLS) in [6]. In particular, in a series of papers referenced in [6], we have expanded the result of [1] to various contexts where dispersion can be proved using Strichartz estimates. The crux of these papers consisted in proving a form
of radiation induced damping on the discrete modes of the system (the so called Nonlinear Fermi
Golden Rule, or FGR), due to the spilling of the energy in the discrete modes in the radiation
component of the solutions, where dispersion occurs because of linear dispersion. Recently, thanks
to the notion introduced in [7], of Refined Profile, we have been able to simplify significantly the
proofs, see also [8, 9], eliminating the normal forms arguments required to find a coordinates system
where the FGR can be seen. In fact, an ansatz involving the Refined Profile yields automathically
a framework adequate to prove the FGR, as we will see later.

Lately, in the literature there has been considerable attention on low dimensional problems,
especially in 1D, where, due to the relative strength of the nonlinearities, the Strichartz estimates
are not sufficient to prove dispersion. Various papers like for example [11]–[29], [31] and [33] have
recently dealt with asymptotic stability problems in the context of long range nonlinearities. In
[4, 5] use is made of the theory of Virial Inequalities developed by Kowalczyk et al. [14]–[18]. In
this paper we will follow closely Kowalczyk and Martel [14]. So, as in [14]–[18], we will need two
distinct sets of Virial Inequalities. We follow the Kowalczyk and Martel [14] idea of proving the
FGR utilizing the initial sets of coordinates, contrary to what is done in [4, 5]. In particular, in
the proof of the FGR we use a functional derived from Kowalczyk and Martel [14], instead of the
localized energy $E(\phi[z])$. The proof simplifies, avoiding the use of the smoothing estimates, which
played a significant role in [4, 5]. We highlight that our result works under a somewhat restrictive
hypothesis on the potential $V$, specifically that the potential $V_D$ obtained after eliminating all the
eigenvalues of $L_1$ with a sequence of Darboux transformations, must be a repulsive potential, in the
sense of Assumption 1.9.

1.1 Assumptions and refined profile

Notation 1.2. We write $a \lesssim b$ to mean that there exists a constant $C > 0$ s.t. $a \leq Cb$. The positive
number $C$ omitted is called the implicit constant.

We set $\lambda = (\lambda_1, \cdots, \lambda_N, -\lambda_1, \cdots, -\lambda_N) \in \mathbb{R}^{2N}$ and

$\mathbf{R} := \{ \mathbf{m} = (\mathbf{m}_+, \mathbf{m}_-) \in (\mathbb{N} \cup \{0\})^{2N} | |\mathbf{m} \cdot \lambda| > m\}$,

$\mathbf{R}_{\min} := \{ \mathbf{m} \in \mathbf{R} | \nexists \mathbf{n} \in \mathbf{R} \text{ s.t. } \mathbf{n} < \mathbf{m} \}$,

$I := \{ \mathbf{m} \in (\mathbb{N} \cup \{0\})^{2N} | \exists \mathbf{n} \in \mathbf{R}_{\min} \text{ s.t. } \mathbf{n} < \mathbf{m} \}$,

where

$n = (n_+, n_-) < m = (m_+, m_-)$

$\iff \forall j = 1, \cdots, N, \ n_{+, j} + n_{-, j} \leq m_{+, j} + m_{-, j}$ and $\|\mathbf{n}\| < \|\mathbf{m}\|$, where

$\|\mathbf{m}\| := \sum_{j=1}^{N} \sum_{\pm} m_{\pm,j}$.

We also set $\mathbf{e}^j = (\delta_{1j}, \cdots, \delta_{Nj}, 0, \cdots, 0)$ where $\delta_{jk}$ is the Kronecker’s delta, $\overline{\mathbf{m}} = (\overline{\mathbf{m}_+}, \overline{\mathbf{m}_-}) := (\overline{\mathbf{m}_-}, \overline{\mathbf{m}_+})$ and

$\mathbf{NR} := (\mathbb{N} \cup \{0\})^{2N} \setminus (\mathbf{R}_{\min} \cup \mathbf{I})$,

$\Lambda_j := \{ \mathbf{m} \in \mathbf{NR} | \mathbf{m} \cdot \lambda = \lambda_j \}$,

$\overline{\Lambda}_j := \{ \overline{\mathbf{m}} | \mathbf{m} \in \Lambda_j \}$

$\Lambda_0 := \{ \mathbf{m} \in \mathbf{NR}\setminus\{0\} | \lambda \cdot \mathbf{m} = 0 \}$.

We assume the following, which is true for generic $L_1$. 

Assumption 1.3. For $M$ the largest number in $\mathbb{N}$ such that $(M-1)\lambda_1 < m$, then for a multi-index $m \in \mathbb{N}^2$ we assume
\[
||m|| \leq M \implies (m \cdot \lambda)^2 \neq m^2. \tag{1.15}
\]
We also assume that for $m = (m_+, m_-) \in \mathbb{N}^2$ then
\[
||m|| \leq 2M \text{ and } m \cdot \lambda = 0 \implies m_+ = m_- \tag{1.16}
\]

Lemma 1.4. The following facts hold.
1. If $||m|| > M$, with $M$ the constant in Assumption 1.3, then $m \in I$.
2. $R_{\text{min}}$ and $NR$ are finite sets.
3. If $m \in NR$, then $|\lambda \cdot m| < m$ and if $m \in R_{\text{min}}$, then $m_+ = 0$ or $m_- = 0$.
4. If $m \in \Lambda_j$ then there is a $n \in \Lambda_0$ with $m = e^j + n$.

Proof. The proof is taken from [5]. If $||m|| > M$, we can write $m = \alpha + \beta$ with $||\alpha|| = M$. If $\alpha = (\alpha_+, \alpha_-)$ and if we set $n = (n_+, n_-)$ with $n_+ = \alpha_+ + \alpha_-$ and $n_- = 0$, then $n \cdot \lambda \equiv M\lambda_1 > m$. This implies that $n \in R$ and that there exists $a \in R_{\text{min}}$ with $a \leq n$. From $||\beta|| > 1$ it follows that $a \prec m$ and so $m \in I$.

Obviously, from the first claim it follows that if $m \in R_{\text{min}} \cup NR$ then $||m|| \leq M$. Next we observe that $m \in NR$ implies $||m|| \leq M$ and $|\lambda \cdot m| < m$ and, by Assumption 1.3, $|\lambda \cdot m| < m$. If $m \in R_{\text{min}}$ with, say, $m \cdot \lambda > m$, then obviously we have $m_+ \cdot \lambda > m$ and it is elementary that $m = (m_+, 0)$. Finally, from the first claim we know that if $m \in \Lambda_j$ then $||m|| \leq M$. From $m \cdot \lambda - \lambda_j = 0$ it follows from (1.16) that we have the last claim. \hfill \square

For $z = (z_1, \cdots , z_N) \in \mathbb{C}^N$ and $m \in (\mathbb{N} \cup \{0\})^{2N}$, we set
\[
z^m = \prod_{j=1}^N z_j^{m_+} \bar{z}_j^{m_-}. \tag{1.17}
\]
Notice that we have $\overline{z^m} = z^m$.

Notice that $\sum_{j=1}^N (z_j \Phi_j + c.c.)$, satisfies (1.3) up to $O(||z||^2)$ error if $\tilde{z}_j = i \lambda_j z_j$. The refined profile is a generalization of this kind of approximate solution of (1.3).

We set $|| \cdot ||_{\Sigma^*} := || \cdot ||_{H_{\Sigma^2}} := ||e^{a_2(z)} \cdot ||_{H^*}$ where $a_2 = \frac{1}{2} \sqrt{m^2 - \lambda_N}$ and denote by $\Sigma^*$ the corresponding spaces. We set
\[
||u||_{\Sigma^*}^2 := ||u_1||_{\Sigma^1}^2 + ||u_2||_{\Sigma^2}^2. \tag{1.16}
\]

Let $\Sigma^\infty = \cap_{t \in \mathbb{R}} \Sigma^t$.

Proposition 1.5. There exist $\{\Phi_m\}_{m \in NR}$ in $\Sigma^\infty$, $\{G_m\}_{m \in R_{\text{min}} \subset \Sigma^\infty}$, $\{\lambda_{nj}\}_{n \in \Lambda_0 \cup \{0\} \subset \mathbb{R}}$ for $j = 1, \cdots , N$ with $\Phi_m^* = \Phi_j$ and $\lambda_{0j} = \lambda_j$, a $\delta > 0$ s.t. there exists $\tilde{z}_2 \in C^\infty(B_{\mathbb{C}^N}(0, \delta_1), \mathbb{C}^N)$ satisfying
\[
||\tilde{z}_2||_{\Sigma^N} \lesssim \sum_{m \in R_{\text{min}}} |z^m|, \tag{1.17}
\]
s.t. for any $l$

$$
\|R[z]\|_{\Sigma^l} \lesssim l \|z\|_{CN} \sum_{m \in R_{\min}} |z^m|,
$$

(1.18)

where $R[z]$ is defined by the equality

$$
D\phi[z]\tilde{z} = JL_1\phi[z] + f[\phi[z]] - \sum_{m \in R_{\min}} z^mG_m - R[z],
$$

(1.19)

(where (1.18) and (1.19) define the $G_m$) and

$$
\phi[z] := \left(\begin{array}{c}
\phi_1[z] \\
\phi_2[z]
\end{array}\right) = \sum_{m \in NR} z^m\phi_m,
$$

(1.20)

$$
\phi_m = \overline{\phi_m}
$$

(1.21)

$$
\tilde{z} = z_0 + z_1 + \tilde{z}_2 \text{ with}
$$

(1.22)

$$
\tilde{z}_0 = (i\lambda_1 z_1, ..., i\lambda_N z_N) =: i\lambda z,
$$

(1.23)

$$
\tilde{z}_1 = i \sum_{n \in \Lambda_0} \lambda_m z_{n}z_1, ..., i \sum_{m \in \Lambda_0} \lambda_N z_{n}z_N,
$$

(1.24)

$$
\lambda_m = \lambda_m \in \mathbb{R}^{2N}
$$

(1.25)

where $\lambda_m := (\lambda_m^1, ..., \lambda_m^N, -\lambda_m^1, ..., -\lambda_m^N)$, such that, setting

$$
\mathcal{H}_c[z] := \{u \in \mathcal{H}^1 \mid \Omega(u, Dz\phi[z]w) = 0 \text{ for all } w \in \mathbb{C}^N\}
$$

(1.26)

and

$$
\tilde{R}[z] = \sum_{m \in R_{\min}} z^mG_m + R[z],
$$

(1.27)

we have

$$
J\tilde{R}[z] \in \mathcal{H}_c[z].
$$

(1.28)

Proof. We begin observing that $JL_1$ leaves the following decomposition invariant,

$$
L^2(\mathbb{R}, \mathbb{C}^2) = L^2_{\text{discr}} \oplus L^2_{\text{disp}} \text{ where } L^2_{\text{discr}} := \oplus_{\lambda \in \sigma_p(JL_1)} \ker (L_1 - \lambda),
$$

(1.29)

where $L^2_{\text{disp}}$ is the $\langle J, \cdot \rangle$–orthogonal of $L^2_{\text{discr}}$.

We insert (1.20) in (1.19), using (1.22)–(1.24). We expand

$$
f(\phi_1[z]) = \sum_{\ell=2}^M \frac{f^{(\ell)}(0)}{\ell!} \phi_1^{(\ell)}[z] + O(\|z\|^{M+1}),
$$

Then, for $i = (1, 0)$,

$$
\sum_{\ell=2}^M \frac{f^{(\ell)}(0)}{\ell!} \phi_1^{(\ell)}[z] i = \sum_{m \in NR} z^mh_m + \sum_{m \in R \cup I \mid |m| \leq M} z^mh_m + O(\|z\|^{M+1})
$$
where, for $\phi_m = \psi(\phi_1, \phi_2)$,

$$h_m = \sum_{\ell=2}^{M} \frac{f^{(\ell)}(0)}{\ell!} \sum_{m_1, \ldots, m_{\ell} \in \text{NR}} \phi_1 m_1 \cdots \phi_1 m_{\ell} \cdot i.$$

(1.30)

Using

$$(D_0 z^m) \bar{z}_0 = i(m \cdot \lambda) z^m,$$

where $\lambda z := (\lambda_1 z_1, \ldots, \lambda_N z_N)$,

and recalling (1.22), we obtain

$$D_0 \phi[z] := i \sum_{m \in \text{NR}} (m \cdot \lambda) z^m \phi_m + i \sum_{m \in \text{NR}, n \in \Lambda_0} (m \cdot \lambda_n) z^m \phi_m + D_0 \phi[z] \bar{z}_2.$$

Let us set

$$\mathcal{R}[z] := J (L_1 \phi[z] + f(\phi[z])) - D_0 \phi[z] \bar{z} - \bar{z}_2)
\quad = J \left( L_1 \phi[z] + f(\phi[z]) \right) - D_0 \phi[z] \bar{z} - \bar{z}_2).$$

We expand now to get

$$\mathcal{R}[z] = \sum_{m \in \text{NR}} z^m \mathcal{R}_m + \sum_{m \in \mathcal{R} \mathcal{J} \mid m \leq M} z^m \mathcal{R}_m + O(\|z\|^{M+1}),$$

(1.32)

where

$$\mathcal{R}_m = (J L_1 - i \lambda \cdot \mathbf{m}) \phi_m + \mathcal{E}_m$$

with

$$\mathcal{E}_m = J h_m - \sum_{m' + n' = m \in \mathbf{m}' \mathbf{m'} \mid n' \in \Lambda_0} i(\lambda_n \cdot \mathbf{m}') \phi_{m'}.$$

We seek $\mathcal{R}_m \equiv 0$ for $\mathbf{m} \in \text{NR}$. For $\|\mathbf{m}\| = 1$ the equation reduces to $(J L_1 - i \lambda \cdot \mathbf{m}) \phi_m = 0$, so that we can set $\phi_{\mathbf{m}} = \Phi$ and $\phi_{\mathbf{m}'} = \overline{\Phi}_{\mathbf{m}'}$. Let us consider now $\|\mathbf{m}\| \geq 2$ with $\mathbf{m} \not\in \bigcup_{j=1}^{N} (\Lambda_j \cup \overline{\Lambda}_j)$. In this case, let us assume by induction that $\phi_{m'}$ and $\lambda_{m'}$ have been defined for $\|\mathbf{m}'\| < \|\mathbf{m}\|$ and that they satisfy (1.21)–(1.25). Then, from (1.30) we obtain $h_{\mathbf{m}} = \overline{\mathcal{R}}_m$ and $\mathcal{E}_{\mathbf{m}} = \overline{\mathcal{E}}_m$. We can solve $\mathcal{R}_m = 0$ writing $\phi_m = (J L_1 - i \lambda \cdot \mathbf{m})^{-1} \mathcal{E}_m$. By $\lambda \cdot \mathbf{m} = -\lambda \cdot \mathbf{m}$, we conclude $\phi_{\mathbf{m}} = \overline{\phi}_{\mathbf{m}}$.

We now consider $\mathbf{m} \in \Lambda_j$. We assume by induction that $\phi_{m'}$ have been defined for $\|\mathbf{m}'\| < \|\mathbf{m}\|$ and so too $\lambda_{m'}$ for $\|n'\| < \|\mathbf{m}\| - 1$. Then, for $\mathbf{m} = \mathbf{n} + \mathbf{e}^j$ where $\mathbf{n} \in \Lambda_0$, $\mathcal{R}_m = 0$ becomes

$$(J L_1 - i \lambda_j) \phi_m = \mathcal{E}_m = i\lambda_n \cdot \mathbf{e}^j \Phi_{\mathbf{j}} - K_m$$

with

$$K_m := J h_m - \sum_{m' + n' = m \in \mathcal{R} \mathcal{J} \mid n' \in \Lambda_0} i\lambda_{n'} \cdot \mathbf{m}' \phi_{m'}.$$

(1.33)

This equation can be solved if we impose $(J \mathcal{E}_m, \overline{\Phi}_{\mathbf{j}}) = 0$, that is, for $\lambda_{nj} := \lambda_{n} \cdot \mathbf{e}^j$, if

$$-i\lambda_{nj} (J \Phi_{\mathbf{j}}, \overline{\Phi}_{\mathbf{j}}) = -2\lambda_{nj} \lambda_j = (J K_m, \overline{\Phi}_{\mathbf{j}}).$$
which is true for \( \lambda_{nj} = -2^{-1}\lambda_j^{-1} (J\kappa_m, \Phi_j) \). Then we can solve for \( \phi_m = -(JL_1 - i\lambda_j)^{-1}\kappa_m \) in the complement, in (1.29), of \( \ker(JL_1 - i\lambda_j) \).

We want to show that \( \lambda_{nj} \in \mathbb{R} \). For the corresponding \( \overline{m} \in \overline{\Lambda}_j \), we have

\[
(JL_1 + i\lambda_j) \phi_{\overline{m}} = i\lambda_n \cdot \overline{\Phi_j} - \kappa_{\overline{m}} \text{ with }
\]

\[
\kappa_{\overline{m}} := JL_m - \sum_{\overline{m} \in N\Lambda_2, n \in \Lambda_0} i\lambda_n \cdot \overline{m} \phi_{\overline{m}}.
\]

(1.34)

Notice that by induction \( \kappa_{\overline{m}} = \overline{\kappa}_m \). Since \( \lambda_n \cdot \overline{\Phi_j} = -\lambda_{nj} \), taking the complex conjugate of (1.33) we obtain

\[
(JL_1 + i\lambda_j) \phi_{\overline{m}} = i\lambda_{nj} \overline{\Phi_j} - \overline{\kappa}_m \text{ and }
\]

\[
(JL_1 + i\lambda_j) \overline{\phi}_m = i\lambda_{nj} \overline{\Phi_j} - \overline{\kappa}_m.
\]

(1.35)

Applying \( (J, \Phi_j) \) on both the last two equations, we obtain

\[
i\lambda_{nj} (J\overline{\Phi_j}, \Phi_j) = (J\overline{\kappa}_m, \Phi_j) \text{ and } i\lambda_{nj} (J\overline{\Phi_j}, \Phi_j) = (J\overline{\kappa}_m, \Phi_j).
\]

Hence \( \lambda_{nj} = \overline{\lambda}_{nj} \) and we have proved that \( \lambda_{nj} \in \mathbb{R} \).

Since the equations in (1.35) are the same, we conclude \( \phi_{\overline{m}} = \overline{\phi}_m \).

We consider now

\[
JR[z] = \mathcal{R}[z] - D_\omega \phi[z]z_2,
\]

(1.36)

where we seek \( z_2 \) so that (1.28) is true. This will follow from (here \( J^{-1} = -J \))

\[
\langle J\mathcal{R}[z], D_\omega \phi[z]w \rangle - \langle JD_\omega \phi[z]z_2, D_\omega \phi[z]w \rangle = 0 \text{ for the standard basis } w = e_1, ie_1, ..., e_N, ie_N.
\]

Since the restriction of \( (J, \cdot) \) in \( L^2_{discr} \) is a non–degenerate symplectic form and from \( \phi_{\overline{m}} = \Phi_j \) and \( \overline{\phi}_{\overline{m}} = \overline{\Phi_j} \), the Implicit Function Theorem guarantees the existence of \( z_2 \in C^\infty(B_{CN}(0, \delta_1), C^N) \) with \( z_2(0) = 0 \) for a sufficiently small \( \delta_1 > 0 \). Furthermore, from the last formula and from the fact that in the expansion (1.32) we have \( \mathcal{R}_m = 0 \) for all \( m \in N\mathbb{R} \), we obtain the bound (1.18).

Solving in (1.36) for \( \mathcal{R}[z] = J^{-1}\mathcal{R}[z] - J^{-1}D_\omega \phi[z]z_2 \), exploiting the fact that we have \( \mathcal{R}_m \) for all \( m \in N\mathbb{R} \) and by (1.17), by Taylor expansion in the variable \( z \), we obtain expansion (1.27), with the estimate (1.18).

We assume the following.

**Assumption 1.6** (Fermi Golden Rule). For any \( m \in \mathbb{R}_{min} \), there exists a bounded solution \( \mathbf{g}_m \) of

\[
JL_1 \mathbf{g}_m = i(\mathbf{m} \cdot \lambda) \mathbf{g}_m \text{ s.t. }
\]

\[
\langle \mathbf{G}_m, \mathbf{g}_m \rangle = \gamma_m > 0.
\]

(1.37)

**Remark 1.7.** Notice that all it matters in (1.37) is to have \( \gamma_m \neq 0 \), since by replacing \( \mathbf{g}_m \) with \( -\mathbf{g}_m \), we can then obtain \( \gamma_m > 0 \).

Recall now from Sect. 3 Deift-Trubowitz [10], the following result on Darboux transformations, here stated with stricter hypotheses than in [10].

**Proposition 1.8.** Let \( W \in \mathcal{S}(\mathbb{R}, \mathbb{R}) \) s.t. \( \sigma_d(-\partial_x^2 + W) \neq \emptyset \) and let \( \omega = \inf \sigma_d(-\partial_x^2 + W) \). Let \( \psi \) be a ground state of \( -\partial_x^2 + W \), that is a generator of \( \ker(-\partial_x^2 + W - \omega) \), and set \( A_W = \frac{1}{\psi} \partial_x(\psi \cdot) \) (recall that \( \psi(x) \neq 0 \) for all \( x \in \mathbb{R} \)). Then, there exists \( W_1 \in \mathcal{S}(\mathbb{R}, \mathbb{R}) \) s.t.

\[
A_W A_{W_1} = -\partial_x^2 + W - \omega, \quad A_{W_1} A_W = -\partial_x^2 + W_1 - \omega
\]

and \( \sigma_d(-\partial_x^2 + W_1) = \sigma_d(-\partial_x^2 + W) \setminus \{\omega\} \).

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Using Proposition 1.8, we inductively define $V_j \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ ($j = 1, \cdots, N + 1$) by

1. $V_1 := V, \ L_1 := -\partial_x^2 + V_1 + m^2, \ \psi_1 = \phi_1$ and $A_1 = A_{V_1}$.

2. Given $V_k$, we define

$$A_k := A_{V_k} \text{ and } L_{k+1} := -\partial_x^2 + V_{k+1} + m^2 := A_k^* A_k + \lambda_k^2,$$

and, by Proposition 1.8, we have $L_k = -\partial_x^2 + V_k + m^2 = A_k A_k^* + \lambda_k^2$.

From Proposition 1.8, we have

$$\sigma_d(L_k) = \{\lambda_j^2 \mid j = k, \cdots, N\}, \ k = 1, \cdots, N, \ \text{and} \ \sigma_d(L_{N+1}) = \emptyset.$$  

If $\psi_k$ is the ground state of $L_k$ and $A_k = \frac{1}{\psi_k} \partial_x (\psi_k \cdot)$ then, from

$$A_j^* L_j = A_j^* (A_j A_j^* + \lambda_j^2) = (A_j^* A_j + \lambda_j^2) A_j^* = L_{j+1} A_j^*,$$

we have the conjugation relation

$$A^* L_1 = L_{N+1} A^*,$$  

where

$$A = A_1 \cdots A_N \text{ and } A^* = A_N^* \cdots A_1^*.$$  

We write $L_D := L_{N+1}$ and $V_D := V_{N+1}$. We assume that $V_D$ is repulsive with respect to the origin, specifically the following.

**Assumption 1.9.** We assume $x V_D' \leq 0$ and $x V_D''(x) \neq 0$.

The main point for us is that $L_1$ has eigenvalues, we have the orthogonal decomposition

$$L^2(\mathbb{R}, \mathbb{C}) = (\oplus_{j=1}^N \ker (L_1 - \lambda_j^2)) \oplus L_c^2(L_1),$$

where $L_c^2(L_1)$ is the continuous spectrum component associated to $L_1$. We denote by $P_c$ the orthogonal projection onto $L_c^2(L_1)$.

## 2 Main estimates and proof of Theorem 1.1.

Using the refined profile given in Proposition 1.5, we first decompose the solution by appropriate orthogonality condition.

**Lemma 2.1 (Modulation).** There exists $\delta_1 > 0$ s.t. there exists $z \in C^\infty(B_{\mathcal{H}^1}(0, \delta_1), \mathbb{C}^N)$ s.t. $z(0) = 0$ and

$$\eta[u] := u - \phi[z(u)] \in \mathcal{H}_{c}[z(u)].$$

Furthermore, we have

$$\|u\|_{\mathcal{H}^1} \sim \|\eta[u]\|_{\mathcal{H}^1} + \|z(u)\|.$$

**Proof.** Standard. \qed
In the following, we fix a solution $u$ of (1.3) with $u(0) = u_0$ satisfying the assumption of Theorem 1.1 (with $\delta_0 > 0$ to be determined). We write $z(t) = z(u(t))$ and $\eta(t) = \eta[u(t)]$. By the conservation of energy and by (2.2) we have

$$\|z\|_{L^\infty(R_+ C^N)} + \|\eta\|_{L^\infty(R_+ H^1)} \lesssim \delta.$$  \hspace{1cm} (2.3)

Substituting $u = \phi[z] + \eta$ into (1.3), we have

$$\dot{\eta} + D_x \phi[z](\dot{z} - \zbar) = J \left( L[z] \eta + F[z, \eta] + \sum_{m \in \mathbb{R}_{min}} z^m G_m + R[z] \right).$$  \hspace{1cm} (2.4)

where, for $df$ the Fréchet derivative of $f$,

$$L[z] = L_1 + df[\phi[z]],$$  \hspace{1cm} (2.5)

$$F[z, \eta] = f' \phi[z] + \eta - f[\phi[z]] - df[\phi[z]] \eta.$$  \hspace{1cm} (2.6)

Notice that $F[z, \eta] = \iota(F_1[z, \eta]) 0$ where

$$F_1[z, \eta] = f(\phi[z] + \eta) - f(\phi[z]) - f' \phi[z] \eta.$$  \hspace{1cm} (2.7)

We will consider constants $A, B, \varepsilon > 0$ satisfying

$$\log(\delta^{-1}) \gg \log(\varepsilon^{-1}) \gg A \gg B^2 \gg B \gg \exp(\varepsilon^{-1}) \gg 1.$$  \hspace{1cm} (2.8)

We will denote by $o_\varepsilon(1)$ constants depending on $\varepsilon$ such that

$$o_\varepsilon(1) \xrightarrow{\varepsilon \to 0^+} 0.$$  \hspace{1cm} (2.9)

Let

$$\kappa \in (0, \min(m - \lambda_N, a_1)/10).$$  \hspace{1cm} (2.10)

We will consider the norms

$$\|\eta\|_{\Sigma_A} := \left\| \sech \left( \frac{2}{A} x \right) \eta_1 \right\|_{L^2} + A^{-1} \left\| \sech \left( \frac{2}{A} x \right) \eta \right\|_{L^2}$$

and

$$\|\eta\|_{L^2_\kappa} := \left\| \sech(\kappa x) \eta \right\|_{L^2}.$$  \hspace{1cm} (2.11, 2.12)

We will prove the following continuation argument.

**Proposition 2.2.** Under the assumptions 1.3, 1.6 and 1.9, for any small $\varepsilon > 0$ there exists a $\delta_0 = \delta_0(\varepsilon)$ s.t. if in $I = [0, T]$ we have

$$\|\dot{z} - \zbar\|_{L^2(I)} + \sum_{m \in \mathbb{R}_{min}} \|z^m\|_{L^2(I)} + \|\eta\|_{L^2(I, \Sigma_A \cap L^2_\kappa)} \leq \varepsilon$$  \hspace{1cm} (2.13)

then for $\delta \in (0, \delta_0)$ and $\delta = \|u_0\|_{H^1}$ inequality (2.13) holds for $\varepsilon$ replaced by $o_\varepsilon(1)\varepsilon$ where $o_\varepsilon(1) \xrightarrow{\varepsilon \to 0^+} 0$.

Notice that Proposition 2.2 implies by standard continuation arguments Theorem 1.1.

We will prove Proposition 2.2 from the following statements.
Proposition 2.3. We have
\[ \| \dot{z} - \tilde{z} \|_{L^2(I)} = o_\varepsilon(1) \| \eta \|_{L^2(I, L^2_{-\infty})}. \]  
\( (2.14) \)

Proposition 2.4 (FGR estimate). We have
\[ \sum_{m \in \mathbb{R}_{\min}} \| z^m \|_{L^2(I)} \lesssim \delta + A^{-1/4} \| \eta \|_{L^2(I, \Sigma_A)}. \]  
\( (2.15) \)

Proposition 2.5 (1st virial estimate). We have
\[ \| \eta \|_{L^2(I, \Sigma_A)} \lesssim \delta + \| \eta \|_{L^2(I, L^2_{-\infty})} + \sum_{m \in \mathbb{R}_{\min}} \| z^m \|_{L^2}. \]  
\( (2.16) \)

Proposition 2.6 (2nd virial estimate). We have
\[ \| \eta \|_{L^2(I, L^2_{-\infty})} \lesssim B \varepsilon^{-N} \delta + A^{-1/4} \| \eta \|_{L^2(I, \Sigma_A)} + \sum_{m \in \mathbb{R}_{\min}} \| z^m \|_{L^2}. \]  
\( (2.17) \)

Proof of Proposition 2.2 assuming Propositions 2.3–2.6. From Propositions 2.3 and 2.4 and from (2.8) we have
\[ \| \dot{z} - \tilde{z} \|_{L^2(I)} + \sum_{m \in \mathbb{R}_{\min}} \| z^m \|_{L^2(I)} = o_\varepsilon(1). \]  
\( (2.18) \)

Entering this in (2.17) we get
\[ \| \eta \|_{L^2(I, L^2_{-\infty})} = o_\varepsilon(1). \]  
\( (2.19) \)

Entering (2.18) and (2.19) in (2.16) we get \( \| \eta \|_{L^2(I, \Sigma_A)} = o_\varepsilon(1) \). This completes the proof of Proposition 2.2.

Proof of Theorem 1.1. By continuity, Proposition 2.2 implies that inequality (2.13) is valid with \( I = \mathbb{R}_+ \). This implies (1.13) (adjusting \( \epsilon \)). From the equation for \( z \), see (3.5) below, we have \( \dot{z} \in L^\infty(\mathbb{R}, \mathbb{C}^N) \). By \( z^m \in L^2(\mathbb{R}) \) for any \( m \in \mathbb{R}_{\min} \), so in particular \( z^m_j \in L^2(\mathbb{R}) \) for \( m_j \) the largest \( m_j \in \mathbb{N} \) such that \((m_j - 1)\lambda_j < m \), we have \( \lim_{t \to +\infty} z(t) = 0 \).

3 Proof of Proposition 2.3

Proof of Proposition 2.3. We fix an even function \( \chi \in C_0^\infty(\mathbb{R}, [0, 1]) \) satisfying
\[ 1_{[-1, 1]} \leq \chi \leq 1_{[-2, 2]} \) and \( x \chi'(x) \leq 0 \) and set \( \chi_A := \chi(\cdot/A) \). \( (3.1) \)

Lemma 3.1. For the \( F_1 \) in (2.7), we have
\[ \| \text{sech}(\kappa x) F_1[z, \eta] \|_{L^2} \lesssim \delta \| \text{sech}(\kappa x) \eta_1 \|_{L^2}, \]  
\( (3.2) \)
\[ \| \chi_A F_1[z, \eta] \|_{L^2} \lesssim A^{1/2} \delta \| \text{sech} \left( \frac{2 x}{A} \right) \eta_1 \|_{L^2}. \]  
\( (3.3) \)
Proof. By Taylor expansion, \( F_1[z, \eta] = \int_0^1 (1 - t) f''(\phi_1(z) + t\eta_1)\eta_1^2\ dt \). Thus,
\[
\|\text{sech}(\kappa x) F_1[z, \eta]\|_{L^2} \lesssim \sup_{|u| \leq 1} |f''(u)| \|\eta_1\|_{L^\infty} \|\text{sech}(\kappa x)\|_{L^2} \lesssim \delta \|\text{sech}(\kappa x)\|_{L^2},
\]
\[
\|\chi_A F_1[z, \eta]\|_{L^1} \lesssim \sup_{|u| \leq 1} |f''(u)| \|\eta_1\|_{L^\infty} \|\eta_1 \chi_A\|_{L^1} \lesssim A^{1/2} \|\text{sech}\left(\frac{2}{A}x\right)\|_{L^2},
\]
where we have used \( \text{sech}\left(\frac{2}{A}x\right) \sim 1 \) in \( \text{supp}\chi_A \), (2.3) and the embedding \( H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \). \( \square \)

Lemma 3.2. We have
\[
\|\tilde{z} - \bar{z}\| \lesssim \delta \|\text{sech}(\kappa x)\eta\|_{L^2}. \tag{3.4}
\]

Proof. Recalling (1.27) and (2.5), differentiating (1.19) we have for \( w \in \mathbb{C}^N \)
\[
D^2_{\tilde{z}}\phi[z](\tilde{z}, w) + D_{\tilde{z}}\phi[z]D_{\tilde{z}}\tilde{z}(z)w + \mathcal{J} D_{\tilde{z}}\bar{R}[z]w = \mathcal{J} L[z] D_{\tilde{z}}\phi[z]w.
\]
We apply \( \Omega(\cdot, D_{\tilde{z}}\phi[z]w) \) to (2.4), obtaining
\[
\Omega(\eta, D_{\tilde{z}}\phi[z]w) + \Omega(D_{\tilde{z}}\phi[z](\tilde{z} - \bar{z}), D_{\tilde{z}}\phi[z]w) \]
\[
= \langle L[z]\eta, D_{\tilde{z}}\phi[z]w \rangle + \langle F[z, \eta], D_{\tilde{z}}\phi[z]w \rangle,
\]
where we used \( \Omega(\mathcal{J} \bar{R}[z], D_{\tilde{z}}\phi[z]w) = 0 \), that is (1.28). Using \( \eta \in \mathcal{H}_c[z] \), we have
\[
\langle L[z]\eta, D_{\tilde{z}}\phi[z]w \rangle = \langle \eta, L[z] D_{\tilde{z}}\phi[z]w \rangle = \langle \eta, J^{-1} D^2_{\tilde{z}}\phi[z](\tilde{z}, w) + D_{\tilde{z}}\bar{R}[z]w \rangle
\]
\[
= -\Omega(\eta, D^2_{\tilde{z}}\phi[z](\tilde{z}, w)) + \langle \eta, D_{\tilde{z}}\bar{R}[z]w \rangle
\]
and
\[
\Omega(\eta, D_{\tilde{z}}\phi[z]w) = -\Omega(\eta, D^2_{\tilde{z}}\phi[z](\tilde{z}, w)) = -\Omega(\eta, D^2_{\tilde{z}}\phi[z](\tilde{z} - \bar{z}, w)) - \Omega(\eta, D^2_{\tilde{z}}\phi[z](\tilde{z}, w)).
\]
Thus
\[
\Omega(D_{\tilde{z}}\phi[z](\tilde{z} - \bar{z}), D_{\tilde{z}}\phi[z]w) = \Omega(\eta, D^2_{\tilde{z}}\phi[z](\tilde{z} - \bar{z}, w)) + \langle \eta, D_{\tilde{z}}\bar{R}[z]w \rangle + \langle F[z, \eta], D_{\tilde{z}}\phi[z]w \rangle. \tag{3.5}
\]
Since \( \Omega(D_{\tilde{z}}\phi[z], D_{\tilde{z}}\phi[z]) \) is a symplectic form for \( \mathbb{C}^N \), taking \( \|w\| = 1 \) in an appropriate direction we obtain
\[
\|\tilde{z} - \bar{z}\| \lesssim \delta \|\text{sech}(\kappa x)\eta\|_{L^2} + \|\text{sech}(\kappa x)F[z, \eta]\|_{L^2}.
\]
By (3.2), we have the conclusion. \( \square \)

Lemma 3.2 completes the proof of Proposition 2.3, recalling (2.12). \( \square \)

4 Technical lemmas I

The following is a slight refinement of a result in [4].
Lemma 4.1. Let $U \geq 0$ be a non-zero potential $U \in L^1(\mathbb{R}, \mathbb{R})$. Then there exists a constant $C_U > 0$ such that for any function $0 \leq W$ such that $\langle x \rangle W \in L^1(\mathbb{R})$ then
\[
\langle Wf, f \rangle \leq C_U \left( \| \langle x \rangle W \|_{L^1(\mathbb{R})} \| f' \|_{L^2(\mathbb{R})}^2 + \| W \|_{L^1(\mathbb{R})} \langle Uf, f \rangle \right). \tag{4.1}
\]
In particular, we have
\[
\| \text{sech} \left( \frac{2}{A} x \right) f \|_{L^2(\mathbb{R})}^2 \lesssim A^2 \| f' \|_{L^2(\mathbb{R})}^2 + A \| \text{sech} (\kappa x) f \|_{L^2(\mathbb{R})}^2. \tag{4.2}
\]

Proof. Let $J$ be a compact interval where $I_U := \int_J U(x) dx > 0$. Let then $x_0 \in J$ s.t.
\[
|f(x_0)|^2 \leq I_U^{-1} \int_J |f(x)|^2 U(x) dx.
\]
Then,
\[
|f(x)| \leq |x - x_0|^\frac{1}{2} \| f' \|_{L^2(\mathbb{R})} + |f(x_0)| \leq |x - x_0|^\frac{1}{2} \| f' \|_{L^2(\mathbb{R})} + I_U^{-1/2} \langle Uf, f \rangle^\frac{1}{2}.
\]
Taking second power and multiplying by $W$ it is easy to conclude the following, which after integration yields (4.1),
\[
W(x)|f(x)|^2 \leq 2 \left( 1 + |x_0| \right) \langle W(x) \rangle \langle f' \|_{L^2(\mathbb{R})}^2 + 2W(x)I_U^{-1} \langle Uf, f \rangle. \tag{5.1}
\]

We will need the following related technical result.

Lemma 4.2. There exists $A_0 > 0$ such that for any $A \geq A_0$,
\[
\| \text{sech} (\kappa x) f \|_{L^2} \leq A \left( \| \text{sech} \left( \frac{2}{A} x \right) \right) f' \|_{L^2} + A^{-1} \| \text{sech} \left( \frac{2}{A} x \right) \| f \|_{L^2} \right) \text{ for any } f. \tag{4.3}
\]

Proof. Taking $A_0 = 2/\kappa$, we have $\text{sech}(\kappa x) \leq \text{sech}(2/\kappa x)$. Thus, we have the conclusion by
\[
\| \text{sech} (\kappa x) f \|_{L^2} \leq A \cdot A^{-1} \| \text{sech} \left( \frac{2}{A} x \right) \| f ' \|_{L^2} \leq A \det \left( \frac{2}{A} x \right) \| f ' \|_{L^2} + A^{-1} \| \text{sech} \left( \frac{2}{A} x \right) \| f \|_{L^2} \right).
\]
Therefore, we have the conclusion.

\[
\]

5 Proof of Proposition 2.4: the Fermi Golden Rule

To prove Proposition 2.4, for the $g_m$ in Assumption 1.6, we consider
\[
\mathcal{J}_{\text{FGR}} := \Omega(\eta, \chi_A) \sum_{m \in \mathbb{R}_{\text{min}}} z^m g_m. \tag{5.1}
\]
Computing the time derivative of $\mathcal{J}_{\text{FGR}}$, we have the following estimate.

Lemma 5.1. We have
\[
\left| \dot{\mathcal{J}}_{\text{FGR}} - \left( \sum_{m \in \mathbb{R}_{\text{min}}} z^m G_m, \sum_{m \in \mathbb{R}_{\text{min}}} z^m g_m \right) \right| \lesssim A^{-1/2} \left( \sum_{m \in \mathbb{R}_{\text{min}}} |z|^2 + \| \eta \|_{\Sigma_A}^2 \right). \tag{5.2}
\]
Proof. Differentiating $\mathcal{J}_{\text{FGR}}$ and using (2.4), we have

$$\dot{\mathcal{J}}_{\text{FGR}} = \Omega(\dot{\eta}, \chi_A \sum_{m \in \mathbb{R}_{\min}} z^m g_m) + \Omega(\eta, \chi_A \sum_{m \in \mathbb{R}_{\min}} D_z z^m \tilde{z} g_m)$$

$$+ \Omega(\eta, \chi_A \sum_{m \in \mathbb{R}_{\min}} D_z z^m (\dot{z} - \tilde{z}) g_m) =: A_1 + A_2 + A_3.$$

By Lemma 3.2 and Lemma 4.2 and by (2.8), $A_3$ can be bounded by

$$|A_3| \lesssim \|(\chi_A)\|_{L^1} \delta \|z - \tilde{z}\|_{C^N} \lesssim \delta^2 \|\text{sech} \left(\frac{2}{A} x\right)\eta\|_{L^2} \|\text{sech} (\kappa x) \eta_1\|_{L^2}$$

$$\lesssim \delta^2 A^2 \|\eta\|_{\Sigma A}^2 \lesssim A^{-1/2} \|\eta\|_{\Sigma A}^2.$$

By Equation (2.4), we have

$$A_1 = \Omega(-D_z \phi[z](\dot{z} - \tilde{z}), \chi_A \sum_{m \in \mathbb{R}_{\min}} z^m g_m) + \left\langle L_1 \eta, \chi_A \sum_{m \in \mathbb{R}_{\min}} z^m g_m \right\rangle$$

$$+ \left\langle d[f(\phi[z])\eta + F[z, \eta] + R[z], \chi_A \sum_{m \in \mathbb{R}_{\min}} z^m g_m \right\rangle + \left\langle \sum_{m \in \mathbb{R}_{\min}} z^m G_m, \chi_A \sum_{m \in \mathbb{R}_{\min}} z^m g_m \right\rangle$$

$$=: A_{11} + A_{12} + A_{13} + A_{14}.$$

By Lemma 3.2 and Lemma 4.2 and by (2.8) we have

$$|A_{11}| \lesssim \|\chi_A\|_{L^1} \delta \|z - \tilde{z}\|_{C^N} \lesssim \delta \left(\|\text{sech}(\kappa x)\eta_1\|_{L^2}^2 + \sum_{m \in \mathbb{R}_{\min}} |z^m|^2\right)$$

$$\lesssim A^{-1/2} \left(\sum_{m \in \mathbb{R}_{\min}} |z^m|^2 + \|\eta\|_{\Sigma A}^2\right).$$

By (1.18) and Lemma 3.1 we have

$$|A_{13}| \lesssim \sum_{m \in \mathbb{R}_{\min}} |z^m| \left(||d[f(\phi[z])\eta]||_{L^1} + ||F[z, \eta]|_{\chi_A}||_{L^1} + ||R[z]||_{L^1}\right)$$

$$\lesssim A^{1/2} \delta \left(\|\text{sech} \left(\frac{2}{A} x\right)\eta_1\|_{L^2}^2 + \sum_{m \in \mathbb{R}_{\min}} |z^m|^2\right) \leq A^{-1/2} \left(\sum_{m \in \mathbb{R}_{\min}} |z^m|^2 + \|\eta\|_{\Sigma A}^2\right).$$

The term $A_{12}$ can be further decomposed as

$$A_{12} = \left\langle \eta, \chi_A \sum_{m \in \mathbb{R}_{\min}} z^m L_1 g_m \right\rangle + \left\langle \eta, [L_1, \chi_A] \sum_{m \in \mathbb{R}_{\min}} z^m g_m \right\rangle =: A_{121} + A_{122}.$$
Finally, by the antisymmetry of $[L_1, \chi_A] = \left( -\chi_A' - 2\chi_A'\partial_x, 0 \right)$, we have the bound

\[
|A_{122}| \lesssim \sum_{m \in \mathbb{R}_{\min}} |z^m| \left( \|\chi_A''\eta_1\|_{L^1} + \|\chi_A'\eta_1'\|_{L^1} \right)
\lesssim \sum_{m \in \mathbb{R}_{\min}} |z^m| (A^{-3/2}\|\sech \left( \frac{2}{A} x \right)\eta_1\|_{L^2} + A^{-1/2}\|\sech \left( \frac{2}{A} x \right)\eta_1'\|_{L^2})
\lesssim A^{-1/2} \left( \sum_{m \in \mathbb{R}_{\min}} |z^m|^2 + \|\sech \left( \frac{2}{A} x \right)\eta_1^2\|_{L^2} + A^{-2}\|\sech \left( \frac{2}{A} x \right)\eta_1\|_{L^2}^2 \right),
\]

while we have, see Assumption 1.6,

\[
A_{121} = \left\langle \eta, \chi_A \sum_{m \in \mathbb{R}_{\min}} z^m i(m \cdot \lambda) J^{-1} g_m \right\rangle.
\]

The term $A_{14}$ can be decomposed as

\[
A_{14} = \left\langle \sum_{m \in \mathbb{R}_{\min}} z^m G_m, \sum_{m \in \mathbb{R}_{\min}} z^m g_m \right\rangle - \left\langle \sum_{m \in \mathbb{R}_{\min}} z^m G_m, (1 - \chi_A) \sum_{m \in \mathbb{R}_{\min}} z^m g_m \right\rangle
= \left\langle \sum_{m \in \mathbb{R}_{\min}} z^m G_m, \sum_{m \in \mathbb{R}_{\min}} z^m g_m \right\rangle + A_{141},
\]

where the 1st term of line (5.4) is the main term appearing in (5.2). Recalling $a_2 = \frac{1}{2}\sqrt{m^2 - \lambda_N}$,

\[
|A_{141}| \lesssim e^{-a_2 A/2} \left| \sum_{m \in \mathbb{R}_{\min}} z^m \right|^2 \lesssim A^{-1/2} \sum_{m \in \mathbb{R}_{\min}} |z^m|^2.
\]

By the elementary identity $D_z z^m \bar{z}_0 = im \cdot \lambda z^m$, the term $A_2$ can be decomposed as

\[
A_2 = \left\langle J^{-1} \eta, \chi_A \sum_{m \in \mathbb{R}_{\min}} im \cdot \lambda z^m g_m \right\rangle + \Omega \left( \eta, \chi_A \sum_{m \in \mathbb{R}_{\min}} D_z z^m (\bar{z} - \bar{z}_0) g_m \right) =: A_{21} + A_{22},
\]

where

\[
|A_{22}| \lesssim \delta \|\chi_A \eta\|_{L^1} \sum_{m \in \mathbb{R}_{\min}} |z^m| \lesssim A^{-1/2} \left( \sum_{m \in \mathbb{R}_{\min}} |z^m|^2 + A^{-2}\|\sech \left( \frac{2}{A} x \right)\eta_1\|_{L^2}^2 \right).
\]

By the antisymmetry of $J^{-1}(= -J)$ we have the cancellation $A_{121} + A_{21} = 0$. Collecting all the estimates, we obtain (5.2). \hfill \Box

We next take out the nonresonant terms from the main part of $\hat{J}_{FGR}$.

**Lemma 5.2.** Let $m, n \in \mathbb{R}_{\min}$ and $m \neq n$. Then,

\[
z^m z^n = \frac{1}{i(m \cdot \lambda - n \cdot \lambda)} \frac{d}{dt} (z^m z^n) + r_{m,n}\text{ where}
\]

\[
|r_{m,n}| \lesssim \delta \sum_{m \in \mathbb{R}_{\min}} |z^m|^2 + \delta \|\bar{z} - \bar{z}\|^2.
\]

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Proof. We have
\[
\frac{d}{dt} (z^m \bar{z}^n) = i (m \cdot \lambda - n \cdot \lambda) z^m \bar{z}^n + D_z (z^m \bar{z}^n) (\bar{z} - \bar{z}_0) + D_{\bar{z}} (z^m \bar{z}^n) (\bar{z} - \bar{z}).
\]
The estimate of \( r_{m,n} \) follows from Proposition 1.5. \( \square \)

Lemma 5.3. We have
\[
\left| \left\langle \sum_{m \in R_{\min}} z^m G_m, \sum_{m \in R_{\min}} z^m g_m \right\rangle - \sum_{m \in R_{\min}} \gamma_m |z^m|^2 - \frac{d}{dt} \Gamma \right| \lesssim \delta \sum_{m \in R_{\min}} |z_m|^2
\]
where
\[
\Gamma := \sum_{m, n \in R_{\min}, m \neq n} \left\langle \frac{z^m \bar{z}^n}{i (m \cdot \lambda - n \cdot \lambda)} G_m, g_n \right\rangle.
\]

Proof. It is immediate from Lemma 5.2. \( \square \)

Proof of Proposition 2.4. The proof follows from Lemmas 5.1 and 5.3 and the following estimates, due to (2.3),
\[
|J_{FGR}| \lesssim \|\eta\|_{L^2} \|\chi_A\|_{L^2} \sum_{m \in R_{\min}} |z^m| \lesssim \sqrt{A} \delta^3 \lesssim \delta^2
\]
and
\[
|\Gamma| \lesssim \sum_{m \in R_{\min}} |z^m|^2 \lesssim \delta^2.
\]

6 Proof of Proposition 2.5.

We set, for the \( \chi \) in (3.1),
\[
\zeta_A(x) := \exp \left( -\frac{|x|}{A} (1 - \chi(x)) \right), \quad \varphi_A(x) := \int_0^x \zeta_A^2(y) dy \quad \text{and} \quad S_A := \frac{1}{2} \varphi_A + \varphi_A \partial_x.
\]
We will consider the functionals
\[
I_{1st,1} := \frac{1}{2} \Omega(\eta, S_A \eta), \quad I_{1st,2} := \frac{1}{2} \Omega (\eta, \sigma_A \zeta_A^4 \eta),
\]
where both \( S_A \) and \( \sigma_A \zeta_A^4 \) are anti-symmetric w.r.t. \( \Omega \).

Lemma 6.1. We have
\[
\|\text{sech} \left( \frac{2}{A} x \right) \eta_1 \|_{L^2}^2 + A^{-2} \|\text{sech} \left( \frac{2}{A} x \right) \eta_1 \|_{L^2}^2 \lesssim -\dot{I}_{1st,1} + A^2 \|\eta\|_{\Sigma_A}^2 + \|\text{sech}(\kappa x) \eta\|_{L^2}^2 + \sum_{m \in R_{\min}} |z^m|^2.
\]

(6.2)
Proof. We have
\[ \hat{T}_{1st,1} = -\Omega(D\phi[z]|z, S_A\eta) + (L_1\eta, S_A\eta) + (f[\phi[z] + \eta] - f[\phi[z]], S_A\eta) + \left( \mathbf{R}[z], S_A\eta \right) \]
\[ =: B_1 + B_2 + B_3 + B_4, \]
where \( \mathbf{R} \) is defined in (1.27) and
\[ \mathbf{F}[z, \eta] := f[\phi[z] + \eta] - f[\phi[z]]. \]

The main term, \( B_2 \), can be decomposed as
\[ B_2 = (L_1\eta_1, S_A\eta_1) \]
\[ = -\| (\zeta_A\eta) \|^2_{L^2} - \frac{1}{2} \int \varphi_A V' \eta^2 dx - \frac{1}{2} \int A^{-1} \left( \chi'' |x| + 2\chi' \frac{x}{|x|} \right) \zeta_A \eta^2 dx \]
\[ = -\| (\zeta_A\eta) \|^2_{L^2} + B_{21} + B_{22}, \]
where, \( |\varphi_A V'| \lesssim |x V'| \lesssim |xe^{-a|x|}| \) and (2.10) imply
\[ |B_{21}| \lesssim \| \text{sech}(\kappa x) \eta \|^2_{L^2}, \]
and by (3.1)
\[ |B_{22}| \lesssim A^{-1} \| \text{sech}(\kappa x) \eta \|^2_{L^2}. \]

By Lemma 3.2, we have
\[ |B_1| \leq \| z - \bar{z} \|_{L^\infty} \| \eta \|_{L^2} \lesssim \delta \| \eta \|^2_{L^2}. \]

By (1.18) and (1.27) we have
\[ |B_4| \lesssim \| \eta \|^2_{L^2} + \sum_{m \in \mathbb{R}_{\min}} |z^m|^2. \]

By \( f(\phi_1[z] + \eta_1) - f(\phi_1[z]) = \left[ \int_0^1 \int_0^1 f''(s_1 \phi[z]_1 + s_2 \eta_1) \phi[z] \eta_1 ds_1 ds_2 + f(\eta_1) \right] \), we have
\[ B_3 = \left( \int_0^1 \int_0^1 f''(s_1 \phi[z]_1 + s_2 \eta_1) \phi[z] \eta_1 ds_1 ds_2, S_A \eta_1 \right) + (f(\eta_1), S_A \eta_1) = B_{31} + B_{32}. \]

By integration by parts,
\[ B_{31} = -\frac{1}{2} \langle \int_0^1 \int_0^1 \partial_x (f''(s_1 \phi[z] + s_2 \eta_1) \phi[z] \eta_1 ds_1 ds_2, \varphi_A \eta_1) \rangle. \]

Therefore, we have
\[ |B_{31}| \lesssim \| \cosh(\kappa x) \int_0^1 \int_0^1 \partial_x (f''(s_1 \phi[z] + s_2 \eta_1) \phi[z] \eta_1 ds_1 ds_2 \|_{L^\infty} \| \text{sech}(\kappa x) \eta \|^2_{L^1} \]
\[ \lesssim \| \phi[z] \|_{L^2} \| \text{sech}(\kappa x) \eta \|^2_{L^2} \lesssim A^2 \| \eta \|^2_{S_A}, \]
where the last inequality follows from Lemma 4.2.
For the pure in $\eta_1$ nonlinear term $B_{32}$, by Lemma 2.7 of [3], which follows [17], taking $A$ sufficiently large and $\delta_0$ sufficiently small, we have

$$|B_{32}| \leq \omega(1)(\zeta A \eta_1)'^2.$$

Collecting the estimates, we have

$$\|\zeta A \eta_1\|^2_{L^2} \lesssim -\mathcal{I}_{1st, A} + \|\text{sech}(\kappa x)\eta_1\|^2_{L^2} + A^2 \delta \|\eta\|^2_A \|\text{sech}(\kappa x)\eta\|^2_{L^2} + \sum_{m \in R_{\min}} |z^m|^2.$$

Finally, we claim the following, which is analogous to (19) of [14],

$$\|\text{sech}\left(\frac{2}{A}x\right)\eta_1\|^2_{L^2} \lesssim \|\zeta A \eta_1\|^2_{L^2} + A^{-1}\|\text{sech}(\kappa x)\eta_1\|^2_{L^2}. \quad (6.5)$$

This yields (6.2). To prove (6.5), we set $w_1 := \zeta A \eta_1$. We have

$$\int \zeta_A^2 |w_1'|^2 dx = \int \zeta_A^2 |\zeta A \eta_1 + \zeta A \eta_1|^2 dx = \int \left(\zeta_A^4 \eta_1^2 + \zeta_A^4 \zeta_A (\eta_1')^2 + \zeta_A^2 \zeta_A^2 \eta_1^2\right) dx$$

$$= \int \left(\zeta_A^4 \eta_1^2 - \zeta_A^4 \zeta_A \eta_1^2 - 2\zeta_A^2 \zeta_A \eta_1^2\right) dx.$$

This implies

$$\int \zeta_A^4 \eta_1^2 \lesssim \int \zeta_A^2 w_1^2 dx + A^{-2} \int \zeta_A^2 w_1^2 dx.$$

Since by (4.2) we have

$$A^{-2} \int \zeta_A^2 w_1^2 dx \lesssim \|w_1'\|^2_{L^2(\mathbb{R})} + A^{-1}\|\text{sech}(2\kappa x)\zeta A \eta_1\|^2_{L^2(\mathbb{R})} \lesssim \|w_1'\|^2_{L^2(\mathbb{R})} + A^{-1}\|\text{sech}(\kappa x)\eta_1\|^2_{L^2(\mathbb{R})},$$

we obtained the desired bound on the first term in the left hand side of (6.5). We have

$$A^{-2}\|\text{sech}\left(\frac{2}{A}x\right)\eta_1\|^2_{L^2} \lesssim A^{-2} \int \zeta_A^2 w_1^2 dx \lesssim \|w_1'\|^2_{L^2(\mathbb{R})} + A^{-1}\|\text{sech}(\kappa x)\eta_1\|^2_{L^2(\mathbb{R})}$$

and hence we conclude the proof of (6.5). \qed

**Lemma 6.2.** There exist $\delta_0 > 0$ and $A_0 > 0$ s.t. if $\delta < \delta_0$, for any $A > A_0$, we have

$$\|\text{sech}\left(\frac{2}{A}x\right)\eta_2\|^2_{L^2}$$

$$\lesssim -\mathcal{I}_{1st, 2} + \|\text{sech}\left(\frac{2}{A}x\right)\eta_1\|^2_{L^2} + \|\text{sech}\left(\frac{2}{A}x\right)\eta_2\|^2_{L^2} + \|\text{sech}(\kappa x)\eta\|^2_{L^2} + \sum_{m \in R_{\min}} |z^m|^2. \quad (6.6)$$

**Proof.** We have

$$\mathcal{I}_{1st, 2}$$

$$= -\Omega(D\phi[z](\hat{z} - \bar{z}), \sigma_3 \zeta_A^2 \eta) + \langle L_1 \eta, \sigma_3 \zeta_A^4 \eta \rangle + \langle f[\phi[z] + \eta] - f[\phi[z]], \sigma_3 \zeta_A^4 \eta \rangle + \langle \bar{R}[z], \sigma_3 \zeta_A^4 \eta \rangle$$

$$=: C_1 + C_2 + C_3 + C_4.$$
For the main term $C_2$, we have

$$C_2 = -\|\zeta^2 \eta_2\|_{L^2}^2 + \langle L_1 \eta_1, \zeta^4 \eta_1 \rangle$$

and

$$| \langle L_1 \eta_1, \zeta^4 \eta_1 \rangle | \lesssim \| \text{sech} \left( \frac{2}{A} x \right) \eta_1' \|_{L^2}^2 + \| \text{sech} \left( \frac{2}{A} x \right) \eta_1 \|_{L^2}^2.$$ 

For the remainder terms, we have

$$|C_1| \lesssim \| \dot{z} - \tilde{z} \| \| \text{sech} (\kappa x) \eta \|_{L^2} \lesssim \delta \| \text{sech} (\kappa x) \eta \|_{L^2}^2,$$

$$|C_3| \lesssim \delta \| \text{sech} \left( \frac{2}{A} x \right) \eta_1 \|_{L^2}^2,$$

$$|C_4| \lesssim \| \text{sech} (\kappa x) \eta \|_{L^2}^2 + \sum_{m \in \mathbb{R}_{\min}} |z^m|^2.$$

Collecting the estimates, we have the conclusion.

**Proof of Proposition 2.5.** From $|I_{1st,1}| \lesssim A \delta^2$, $|I_{1st,2}| \lesssim \delta^2$, we have the conclusion from Lemmas 6.1 and 6.2.

### 7 Technical lemmas II

We consider

$$\mathcal{T} := (i \xi \partial_x)^{-N} A^*.$$  \hspace{1cm} (7.1)

The following lemma, where $P_c$ is the orthogonal projection on the continuous spectrum component of $L_1$, see (1.42), is proved in [4, Sect. 9].

**Lemma 7.1.** We have

$$u = \prod_{j=1}^{N} R_{L_1}(\lambda_j^2) P_c A (i \xi \partial_x)^N \mathcal{T} u \text{ for all } u \in L^2_c(L_1).$$  \hspace{1cm} (7.2)

**Proof.** We provide the simple proof for completeness. We claim that we have

$$\mathcal{A} \mathcal{A}^* = A_1 \circ \cdots \circ A_N \circ A_N^* \circ \cdots \circ A_1^* = \prod_{j=1}^{N} (L_1 - \lambda_j^2).$$  \hspace{1cm} (7.3)

To prove (7.3), we begin with the following, see the line below (1.38),

$$A_N \circ A_N^* = L_N - \lambda_N^2.$$ 

For $2 \leq j \leq N$, we assume (notice that the Schrödinger operator $L_j$ is fixed)

$$A_j \circ \cdots \circ A_N \circ A_N^* \circ \cdots \circ A_j^* = \prod_{k=j}^{N} (L_j - \lambda_k^2).$$  

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Then, by
\[
A_{j-1}(L_j - \lambda_k^2) = A_{j-1}(A_{j-1}^* A_{j-1} + \lambda_j^2 - \lambda_k^2) = (A_{j-1} A_{j-1}^* + \lambda_j^2 - \lambda_k^2) A_{j-1} = (L_j - \lambda_k^2) A_{j-1},
\]
we have
\[
A_{j-1} \circ \cdots \circ A_N \circ A_N^* \circ \cdots A_{j-1} = A_{j-1} \prod_{k=j}^{N} (L_j - \lambda_k^2) A_{j-1} = \prod_{k=j}^{N} (L_j - \lambda_k^2) A_{j-1} \circ A_{j-1}^*
\]
\[
= \prod_{k=j}^{N} (L_j - \lambda_k^2) (L_j - \lambda_{j-1}^2) = \prod_{k=j-1}^{N} (L_j - \lambda_k^2).
\]
Therefore, we have (7.3) by induction. Using it, from (7.1) and \( u \in L^2_c(L_1) \) we have
\[
\prod_{j=1}^{N} R_{L_j}(\lambda_j^2) P_c A (i \epsilon \partial_x)^N T u = \prod_{j=1}^{N} R_{L_j}(\lambda_j^2) P_c A_1 \cdots A_N \circ A_N^* \circ \cdots A_{j-1}^* u
\]
\[
= \prod_{j=1}^{N} R_{L_j}(\lambda_j^2) P_c \prod_{j=1}^{N} (L_j - \lambda_j^2) u = P_c u = u.
\]
\[
\]
In [4, Sect. 5] the following lemma was proved.

**Lemma 7.2.** Suppose that a Schwartz function \( V \in S(\mathbb{R}, \mathbb{C}) \) has the property that for \( M \geq N + 1 \) its Fourier transform satisfies
\[
|\hat{V}(k_1 + ik_2)| \leq C_M |\langle k_1 \rangle|^{-M-1} \text{ for all } (k_1, k_2) \in \mathbb{R} \times |\mathbf{b}, \mathbf{b}| \text{ and (7.4)}
\]
\[
\hat{V} \in C^0(\mathbb{R} \times [-\mathbf{b}, \mathbf{b}]) \cap H(\mathbb{R} \times (-\mathbf{b}, \mathbf{b})) \text{ with } H(\Omega) \text{ the set of holomorphic functions in an open subset } \Omega \subseteq \mathbb{C} \text{ and with a number } \mathbf{b} > 0.
\]

Then, for multiplicative operators \( \mathbb{b}(\mathbf{b}x) \) and \( \mathbb{b}(\mathbf{b}x) \), we have
\[
||i \epsilon \partial_x)^N [V, (i \epsilon \partial_x)^N] \cosh(b \mathbf{x})||_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C_b \varepsilon, \tag{7.5}
\]
\[
|| \cosh(\mathbf{b} \mathbf{x}) (i \epsilon \partial_x)^N [V, (i \epsilon \partial_x)^N] \cosh(\mathbf{b} \mathbf{x})||_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C_b \varepsilon. \tag{7.6}
\]

**Proof.** For completeness we give the proof. We start with (7.5), repeating the proof from [4]. We have for \( \sigma = 0 \)
\[
\langle i \epsilon \partial_x \rangle^{-N} [V, (i \epsilon \partial_x)^N] f = \int_{\mathbb{R}} dy K^\sigma(x, y) f(y),
\]
where we set
\[
K^\sigma(x, y) = \int_{\mathbb{R}^2} e^{ixk - iy\ell} \langle \varepsilon k \rangle^{-\sigma} H(k, \ell) dk d\ell \text{ with (7.7)}
\]
\[
H(k, \ell) = \langle \varepsilon k \rangle^{-N} \hat{V}(k - \ell) \langle \varepsilon \ell \rangle^N.
\]
Notice that

$$H(k, \ell) = \varepsilon H_1(k, \ell)$$

where $H_1(k, \ell) = \langle \varepsilon k \rangle^{-N} \sqrt{N}(k - \ell) - \ell \frac{P(\varepsilon k, \varepsilon \ell)}{\langle \varepsilon k \rangle^N + \langle \varepsilon \ell \rangle^N},$ \hfill (7.8)

where $P$ is a $2N - 1$ degree polynomial. Hence the generalized integral in (7.7) is absolutely convergent for $\sigma > 0$. But also for $\sigma = 0$ the operator

$$T_\sigma f(x) = \int_R dy f(y) \int_{\mathbb{R}^2} e^{ixk - iy\ell} \langle \varepsilon k \rangle^{-\sigma} H_1(k, \ell) dkd\ell$$

defines an operator $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ of norm uniformly bounded in $\sigma \geq 0$. Let us focus now on $k = k_1 + i0$ and $\ell = \ell_1 - i0$

$$T_\sigma(\chi_{\mathbb{R}^+} f)(x) = \int_{\mathbb{R}^+} dy f(y) e^{-yb} \int_{\mathbb{R}^2} e^{ixk_1 - iy\ell_1} \langle \varepsilon k_1 \rangle^{-\sigma} H_1(k_1, \ell_1 - i0) dkd\ell_1.$$ 

Now we claim that there exists $C > 0$ such that

$$\|T_\sigma \chi_{\mathbb{R}^+} f\|_{L^2(\mathbb{R})} \leq C \|e^{-|x|b} f\|_{L^2(\mathbb{R}^+)}$$

for all $\sigma > 0$ and for all $f$. \hfill (7.9)

Set $g(y) = \chi_{\mathbb{R}^+}(y) f(y) e^{-yb}$. Then

$$T_\sigma(\chi_{\mathbb{R}^+} f)(k_1) = \int_{\mathbb{R}} \langle \varepsilon k_1 \rangle^{-\sigma} H_1(k_1, \ell_1 - i0) \hat{g}(\ell_1) d\ell_1.$$ 

We claim that we have

$$\sup_{k_1 \in \mathbb{R}} \int_{\mathbb{R}} \langle \varepsilon k_1 \rangle^{-\sigma} |H_1(k_1, \ell_1 - i0)| d\ell_1 < C,$$ \hfill (7.10)

$$\sup_{\ell_1 \in \mathbb{R}} \int_{\mathbb{R}} \langle \varepsilon k_1 \rangle^{-\sigma} |H_1(k_1, \ell_1 - i0)| dk_1 < C,$$ \hfill (7.11)

for a fixed constant $C > 0$. We have

$$\int_{\mathbb{R}} |H_1(k_1, \ell_1 - i0)| d\ell_1 \lesssim \int_{|\ell_1| \in \left[ \frac{|k_1|}{2}, |k_1| \right]} \langle \varepsilon k_1 \rangle^{-N} \langle k_1 - \ell_1 \rangle^{-M} \left( \langle \varepsilon k_1 \rangle^{N-1} + |\langle \varepsilon \ell_1 - i\varepsilon b \rangle|^{N-1} \right) d\ell_1$$

$$+ \int_{|\ell_1| \notin \left[ \frac{|k_1|}{2}, |k_1| \right]} \langle \varepsilon k_1 \rangle^{-N} \langle k_1 - \ell_1 \rangle^{-M} \left( \langle \varepsilon k_1 \rangle^{N-1} + |\langle \varepsilon \ell_1 - i\varepsilon b \rangle|^{N-1} \right) d\ell_1.$$ 

The first integral can be bounded above by

$$\int_{|\ell_1| \in \left[ \frac{|k_1|}{2}, |k_1| \right]} \langle \varepsilon k_1 \rangle^{-1} \langle k_1 - \ell_1 \rangle^{-M} d\ell_1 \leq \| \langle x \rangle^{-M} \|_{L^1(\mathbb{R})},$$

while the second can be bounded above by

$$\int_{\mathbb{R}} \langle \varepsilon k_1 \rangle^{-N} \frac{\langle \varepsilon k_1 \rangle^{N-1} + |\langle \varepsilon \ell_1 - i\varepsilon b \rangle|^{N-1}}{\langle k_1 \rangle^M + \langle \ell_1 \rangle^M} d\ell_1 \leq \| \langle x \rangle^{-M-1+N} \|_{L^1(\mathbb{R})}.$$
and correspondingly we have there exists \( \exists \). This is equivalent to (7.5).

Now we show that this remains true for \( \sigma \). For a sequence \( \sigma_n \rightarrow 0^+ \) then \( T_{\sigma_n} f \xrightarrow{n \rightarrow +\infty} T_0 f \) point–wise for \( f \in C_0^0(\mathbb{R}) \). Then by the Fatou lemma and by the density of \( C_0^0(\mathbb{R}) \) in \( L^2(\mathbb{R}) \)

\[
\|T_{\sigma} f\|_{L^2(\mathbb{R})} \leq C\|e^{-|x|b}f\|_{L^2(\mathbb{R})} \quad \text{for all } \sigma > 0 \quad \text{and for all } f.
\]

So (7.11) is true for \( C = \| (x)^{-2} \|_{L^1(\mathbb{R})} \). By Young’s inequality, see Theorem 0.3.1 [32], we conclude that (7.9) is true \( C = \| (x)^{-2} \|_{L^1(\mathbb{R})} \). Proceeding similarly we can show

\[
\|T_{\sigma} \chi_{\mathbb{R}^+} f\|_{L^2(\mathbb{R})} \leq C\|e^{-|x|b}f\|_{L^2(\mathbb{R})} \quad \text{for all } \sigma > 0 \quad \text{and for all } f,
\]

concluding, for \( C = \| (x)^{-2} \|_{L^1(\mathbb{R})} \),

\[
\|T_{\sigma} f\|_{L^2(\mathbb{R})} \leq C\|e^{-|x|b}f\|_{L^2(\mathbb{R})} \quad \text{for all } \sigma > 0 \quad \text{and for all } f.
\]

Now we show that this remains true for \( \sigma = 0 \). For a sequence \( \sigma_n \rightarrow 0^+ \) then \( T_{\sigma_n} f \xrightarrow{n \rightarrow +\infty} T_0 f \) point–wise for \( f \in C_0^0(\mathbb{R}) \). Then by the Fatou lemma and by the density of \( C_0^0(\mathbb{R}) \) in \( L^2(\mathbb{R}) \)

\[
\|T_0 f\|_{L^2(\mathbb{R})} \leq C\|e^{-|x|b}f\|_{L^2(\mathbb{R})} \quad \text{for all } f.
\]

This is equivalent to (7.5).

The proof of (7.6) is similar, with the difference that for example

\[
\chi_{\mathbb{R}^+} T_{\sigma} (\chi_{\mathbb{R}^+} f)(x) = e^{-\frac{y b}{2}} \int_{\mathbb{R}^+} dy f(y) e^{-y \frac{b}{2}} \int_{\mathbb{R}^2} e^{i \xi k_1 - iy \ell_1} \left( e^{i \xi \ell_1 + i \frac{b}{2}} \right)^{-\sigma} H_1 (k_1 + \frac{b}{2}, \ell_1 - ib) dk_1 d\ell_1,
\]

and correspondingly we have there exists \( C > 0 \) such that

\[
\|e^{\frac{-y}{2} b} \chi_{\mathbb{R}^+} T_{\sigma} \chi_{\mathbb{R}^+} f\|_{L^2(\mathbb{R})} \leq C\|e^{-|x|b}f\|_{L^2(\mathbb{R})} \quad \text{for all } \sigma \geq 0 \quad \text{and for all } f,
\]

which can be proved like (7.9), and so similarly the rest of the proof of (7.6). \( \square \)

We will need the following analogue of Lemma 7.2.

**Lemma 7.3.** Suppose that a Schwartz function \( V \in \mathcal{S}(\mathbb{R}, \mathbb{C}) \) has the property that its Fourier transform satisfies

\[
|\hat{V}(k_1 + ik_2)| \leq C_M (k_1)^{-2} \quad \text{for all } (k_1, k_2) \in \mathbb{R} \times [b, b] \quad \text{and}
\]

\[
\hat{V} \in C^0(\mathbb{R} \times [-b, b]) \cap H(\mathbb{R} \times (-b, b)),
\]

with a number \( b > 0 \). Then

\[
\|\{\chi_{\mathbb{R}^+} (-i \nabla)^{-N} \cosh(b y)\} \|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C_b.
\]
\textit{Proof.} The proof is similar to that of Lemma 7.2. We have for \( \sigma = 0 \)
\[
|\mathcal{V}, \{i\bar{z}\partial_z\}^{-N}|f = \int_{\mathbb{R}} dy L^\sigma(x, y)f(y),
\]
where we set
\[
L^\sigma(x, y) = \int_{\mathbb{R}^2} e^{i(xk - iy\ell)} M_\sigma(k, \ell)dkd\ell \quad \text{with} \quad (7.15)
\]
\[
M_\sigma(k, \ell) = \hat{\mathcal{V}}(k - \ell) \left( \langle \varepsilon k \rangle^{-N-\sigma} - \langle \varepsilon \ell \rangle^{-N-\sigma} \right). \]
Hence the generalized integral in (7.15) is absolutely convergent for \( \sigma > 0 \). But also for \( \sigma = 0 \) the operator
\[
S_\sigma f(x) = \int_{\mathbb{R}} dy f(y) \int_{\mathbb{R}^2} e^{i(xk - iy\ell)} M_0(k, \ell),
\]
defines an operator \( L^2(\mathbb{R}) \to L^2(\mathbb{R}) \), and the norm is uniformly bounded in \( \sigma \geq 0 \). Let us focus now on \( k = k_1 + i0 \) and \( \ell = \ell_1 - i\mathbf{b} \)
\[
S_\sigma (\chi_{\mathbb{R}_+} f)(x) = \int_{\mathbb{R}_+} dy f(y)e^{-y\mathbf{b}} \int_{\mathbb{R}^2} e^{i(xk_1 - iy\ell_1)} M_\sigma(k_1, \ell_1 - i\mathbf{b})dk_1d\ell_1.
\]
Now we claim that there exists \( C > 0 \) such that
\[
\|S_\sigma \chi_{\mathbb{R}_+} f\|_{L^2(\mathbb{R})} \leq C\|e^{-|x|\mathbf{b}} f\|_{L^2(\mathbb{R}_+)} \quad \text{for all} \quad \sigma > 0 \quad \text{and for all} \quad f. \quad (7.16)
\]
Set like before \( g(y) = \chi_{\mathbb{R}_+}(y)f(y)e^{-y\mathbf{b}} \). Then
\[
\widehat{S_\sigma (\chi_{\mathbb{R}_+} f)}(k_1) = \int_{\mathbb{R}} M_\sigma(k_1, \ell_1 - i\mathbf{b})\hat{g}(\ell_1)d\ell_1.
\]
We claim that for a fixed constant \( C > 0 \) we have
\[
\sup_{k_1 \in \mathbb{R}} \int_{\mathbb{R}} |M_\sigma(k_1, \ell_1 - i\mathbf{b})|d\ell_1 < C, \quad (7.17)
\]
\[
\sup_{\ell_1 \in \mathbb{R}} \int_{\mathbb{R}} |M_\sigma(k_1, \ell_1 - i\mathbf{b})|dk_1 < C. \quad (7.18)
\]
We have
\[
\int_{\mathbb{R}} |M_\sigma(k_1, \ell_1 - i\mathbf{b})|d\ell_1 \lesssim \int_{\mathbb{R}} \langle k_1 - \ell_1 \rangle^{-2} \left( \langle \varepsilon k_1 \rangle^{-N-\sigma} + \langle \varepsilon \ell_1 - i\mathbf{b} \rangle^{-N-\sigma} \right) d\ell_1 
\]
\[
\lesssim \int_{\mathbb{R}} \langle k_1 - \ell_1 \rangle^{-2} d\ell_1 = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})}. 
\]
So (7.17) is true for \( C = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})} \). Next we prove (7.18). Proceeding as above
\[
\int_{\mathbb{R}} |H_1(k_1, \ell_1 - i\mathbf{b})|dk_1 \lesssim \int_{\mathbb{R}} \langle k_1 - \ell_1 \rangle^{-2} \left( \langle \varepsilon k_1 \rangle^{-N-\sigma} + \langle \varepsilon \ell_1 - i\mathbf{b} \rangle^{-N-\sigma} \right) dk_1 
\]
\[
\lesssim \int_{\mathbb{R}} \langle k_1 - \ell_1 \rangle^{-2} dk_1 = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})}. 
\]
So (7.17)–(7.18) are true for $C = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})}$ and by Young’s inequality we conclude that (7.16) is true $C = \| \langle x \rangle^{-2} \|_{L^1(\mathbb{R})}$. Proceeding like above we conclude

$$\| S_\sigma f \|_{L^2(\mathbb{R})} \leq C \| e^{-|x|^b} f \|_{L^2(\mathbb{R})}$$

for all $\sigma > 0$ and for all $f$, which in turn, proceeding as above yields

$$\| S_0 f \|_{L^2(\mathbb{R})} \leq C \| e^{-|x|^b} f \|_{L^2(\mathbb{R})}$$

and yields (7.14).

We now apply Lemma 7.2 to obtain the following result.

**Lemma 7.4.** We have

$$\| \prod_{j=1}^N R_{L_j}(\lambda_j^2)P_c A \langle i\varepsilon \partial_x \rangle^N w \|_{L^2_{\infty}} \lesssim \| w \|_{L^2_{\varepsilon}}.$$  

(7.20)

**Proof.** We sketch the proof. By a standard discussion in [4, Appendix A] which we skip here, we have

$$\prod_{j=1}^N R_{L_j}(\lambda_j^2)P_c = K_1 \ldots K_N,$$

with integral operators with kernels satisfying $|K_j(x,y)| \leq C \langle x-y \rangle^{m^2-\lambda_j^2}$ for a fixed $C > 0$. Then, by

$$\kappa \leq \frac{m - \lambda_N}{10} < \frac{\sqrt{m^2 - \lambda_j^2}}{10},$$

we have

$$\| \text{sech}(\kappa x) \prod_{j=1}^N R_{L_j}(\lambda_j^2)P_c A \langle i\varepsilon \partial_x \rangle^N v \|_{L^2} \lesssim \| \prod_{j=1}^N R_{L_j}(\lambda_j^2)P_c \text{sech}(\kappa x) A \langle i\varepsilon \partial_x \rangle^N v \|_{L^2}.$$  

We have

$$\text{sech}(\kappa x) A = P_N(x, i\partial_x) \text{sech}(\kappa x),$$

for an $N$–th order differential operator with smooth and bounded coefficients. Next, we write

$$\text{sech}(\kappa x) \langle i\varepsilon \partial_x \rangle^N = \langle i\varepsilon \partial_x \rangle^N \text{sech}(\kappa x) + \langle i\varepsilon \partial_x \rangle^N \langle i\varepsilon \partial_x \rangle^{-N} \left[ \text{sech}(\kappa x), \langle i\varepsilon \partial_x \rangle^N \right],$$

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so that

\[
\left\| \text{sech}(\kappa x) \prod_{j=1}^{N} R_{L_1}(\lambda_j^2) P_c A \langle i\varepsilon \partial_x \rangle^{N} v \right\|_{L^2(\mathbb{R})} \\
\lesssim \left\| \prod_{j=1}^{N} R_{L_1}(\lambda_j^2) P_c P_N(x, i\varepsilon \partial_x) \langle i\varepsilon \partial_x \rangle^{N} \text{sech}(\kappa x) v \right\|_{L^2(\mathbb{R})} \\
+ \left\| \prod_{j=1}^{N} R_{L_1}(\lambda_j^2) P_c P_N(x, i\varepsilon \partial_x) \langle i\varepsilon \partial_x \rangle^{N} \langle i\varepsilon \partial_x \rangle^{-N} \left[ \text{sech}(\kappa x), \langle i\varepsilon \partial_x \rangle^{N} \right] v \right\|_{L^2(\mathbb{R})} \\
=: I + II.
\]

We have

\[
I \leq \left\| \prod_{j=1}^{N} R_{L_1}(\lambda_j^2) P_c P_N(x, i\varepsilon \partial_x) \langle i\varepsilon \partial_x \rangle^{N} \right\|_{L^2(\mathbb{R})} \left\| \text{sech}(\kappa x) v \right\|_{L^2(\mathbb{R})} \leq C \left\| \text{sech}(\kappa x) v \right\|_{L^2(\mathbb{R})},
\]

with a fixed constant \(C\) independent from \(\varepsilon \in (0, 1)\). Next, we have

\[
II \leq \left\| \langle i\varepsilon \partial_x \rangle^{-N} \left[ \text{sech}(\kappa x), \langle i\varepsilon \partial_x \rangle^{N} \right] v \right\|_{L^2(\mathbb{R})} \leq C\varepsilon \left\| \text{sech}(2^{-1}\kappa x) v \right\|_{L^2(\mathbb{R})},
\]

by Lemma 7.2, because \(\int e^{-ikx} \text{sech}(x) dx = \pi \text{ sech}(\frac{k}{2})\), so that in the strip \(k = k_1 + ik_2\) with \(|k_2| \leq b := \kappa/2\), then \(\text{sech}(\frac{\kappa}{2} x)\) satisfies the estimates required on \(\hat{V}\) in (7.4). This completes the proof of (7.20).

As an application of (7.14), we prove the following.

**Lemma 7.5.** For any \(u \in H^1\) we have

\[
\left\| \text{sech} \left( \frac{4}{A} x \right) Tu \right\|_{L^2} \lesssim \varepsilon^{-N} \left\| \text{sech} \left( \frac{2}{A} x \right) u \right\|_{L^2}, \tag{7.21}
\]

\[
\left\| \text{sech} \left( \frac{4}{A} x \right) \partial_x Tu \right\|_{L^2} \lesssim \varepsilon^{-N} \left\| \text{sech} \left( \frac{2}{A} x \right) u' \right\|_{L^2} + \left\| \text{sech}(\kappa x) u \right\|_{L^2}. \tag{7.22}
\]

**Proof.** We have

\[
\left\| \text{sech} \left( \frac{4}{A} x \right) Tu \right\|_{L^2} \leq \left\| \langle i\varepsilon \partial_x \rangle^{-N} \text{sech} \left( \frac{4}{A} x \right) A^* u \right\|_{L^2} + \left\| \text{sech} \left( \frac{4}{A} x \right) \langle i\varepsilon \partial_x \rangle^{-N} A^* u \right\|_{L^2}
\]

\[=: I + II.\]

We have

\[
\text{sech} \left( \frac{4}{A} x \right) A^* = P_N(\partial_x) \text{sech} \left( \frac{4}{A} x \right),
\]

for an \(N\)-th order differential operator with smooth and bounded coefficients, uniformed bounded in \(A \gg 1\), so that

\[
I \leq \left\| \langle i\varepsilon \partial_x \rangle^{-N} P_N(\partial_x) \text{sech} \left( \frac{4}{A} x \right) u \right\|_{L^2} \lesssim \varepsilon^{-N} \left\| \text{sech} \left( \frac{4}{A} x \right) u \right\|_{L^2}.
\]
We have
\[ II = \left\| \left[ \text{sech} \left( \frac{4}{A} x \right), \langle i \varepsilon \partial_x \rangle^{-N} \right] A^* u \right\|_{L^2} \]
\[ \leq \left\| \left[ \text{sech} \left( \frac{4}{A} x \right), \langle i \varepsilon \partial_x \rangle^{-N} \right] \cos \left( \frac{2}{A} x \right) \right\|_{L^2 \to L^2} \left\| \text{sech} \left( \frac{2}{A} x \right) A^* u \right\|_{L^2} \lesssim \left\| \text{sech} \left( \frac{2}{A} x \right) A^* u \right\|_{L^2}, \]
by Lemma 7.2, because \( \int e^{-ikx} \text{sech}(x) dx = \pi \text{sech} \left( \frac{\pi}{2} k \right) \), so that in the strip \( k = k_1 + ik_2 \) with \( |k_2| \leq b = 2/A \), then \( \text{sech} \left( 2A \frac{\pi}{2} k \right) \) satisfies the estimates required on \( \hat{V} \) in (7.14). This completes the proof of (7.21). Now we turn to the proof of (7.22). We have
\[ T u = T \partial_x u + \langle i \varepsilon \partial_x \rangle^{-N} [\partial_x, A^*] u. \]

By (7.21) we have
\[ \left\| \text{sech} \left( \frac{4}{A} x \right) T \partial_x u \right\|_{L^2} \lesssim \varepsilon^{-N} \left\| \text{sech} \left( \frac{2}{A} x \right) \partial_x u \right\|_{L^2}. \]

We have
\[ [\partial_x, A^*] = \sum_{j=1}^{N} \prod_{i=0}^{N-1-j} \left[ \frac{1}{j} - i \right] \prod_{i=1}^{j-1} A^* \partial_x \text{sech}(kx), \]
with the convention \( \prod_{i=0}^{l} B_i = B_0 \circ \ldots \circ B_l \), with \( \psi_k \) the ground state of \( L_k \) and with \( P_N(\partial_x) \) and \( N \)-th order differential operator with bounded coefficients. We then have
\[ \left\| \text{sech} \left( \frac{2}{A} x \right) \langle i \varepsilon \partial_x \rangle^{-N} [\partial_x, A^*] u \right\|_{L^2} \lesssim \left\| \langle i \varepsilon \partial_x \rangle^{-N} P_N(\partial_x) \text{sech}(kx) u \right\|_{L^2} \lesssim \varepsilon^{-N} \left\| \text{sech}(kx) u \right\|_{L^2}. \]

As an application of Lemma 7.3 we have the following.

**Lemma 7.6.** For any \( u \in H^1 \),
\[ \left\| \langle i \varepsilon \partial_x \rangle^{-N}, V_D \right\|_{L^2} \lesssim \varepsilon \left\| \text{sech}(kx) T u \right\|_{L^2}, \quad (7.23) \]
\[ \left\| \cosh \left( \frac{k}{2} x \right) \langle i \varepsilon \partial_x \rangle^{-N}, V_D \right\|_{L^2} \lesssim \varepsilon \left\| \text{sech} \left( \frac{k}{2} x \right) T u \right\|_{L^2}. \quad (7.24) \]

**Proof.** We have
\[ \left\| \langle i \varepsilon \partial_x \rangle^{-N}, V_D \right\|_{L^2} = \left\| \langle i \varepsilon \partial_x \rangle^{-N}, V_D, \langle i \varepsilon \partial_x \rangle^{-N} \right\|_{L^2} \]
Notice that
\[ V_D = V - 2 \sum_{j=1}^{N} (\log \psi_j)^{''}. \]

By (1.2) and by the proof of Lemma 6 p.156 and Theorem 2 p. 167 [10] it then follows
\[ |V_D^{(l)}(x)| \leq C e^{-10\alpha|x|} \text{ for all } 0 \leq l \leq N + 1. \quad (7.25) \]
This implies by an elementary integration by parts
\[ |\tilde{V}_D(k_1 + ik_2)| \leq C \langle k_1 \rangle^{-N-1} \] in the strip \( |k_2| \leq 9\kappa. \) \hspace{1cm} (7.26)

Then in particular, from (7.5) we obtain
\[
\| (i\varepsilon\partial_x)^{-N} [V_D, (i\varepsilon\partial_x)^N] \cosh(\kappa x) \sech(\kappa x) T u \|_{L^2} \lesssim \varepsilon \| \sech(\kappa x) T u \|_{L^2} \text{ and similarly}
\]
\[
\| \cosh \left( \frac{K}{2} x \right) (i\varepsilon\partial_x)^{-N} [V_D, (i\varepsilon\partial_x)^N] \cosh \left( \frac{K}{2} x \right) \sech \left( \frac{K}{2} x \right) T u \|_{L^2} \lesssim \varepsilon \| \sech \left( \frac{K}{2} x \right) T u \|_{L^2}.
\]

\[
8 \text{ Proof of Proposition 2.6}
\]

Using the operator \( T \) in (7.1), we consider the transformed variable
\[
v := T \eta. \hspace{1cm} (8.1)
\]

Then, for \( L_D := \begin{pmatrix} L_D & 0 \\ 0 & 1 \end{pmatrix} \) the variable \( v \) satisfies
\[
\dot{v} = -T \phi[z](\dot{z} - \tilde{\beta}) + J \left( L_D v + \left( \begin{pmatrix} [i \varepsilon \partial_x]^{-N}, V_D \end{pmatrix} 0 \end{pmatrix} A^* \eta \right) + J^2 \left( f[\phi[z] + \eta] - f[\phi[z]] + \sum_{m \in \mathbb{R}_{\min}} z^m G_m + R[z] \right).
\]

From Lemma 7.4, we have
\[
\| \sech(\kappa x) \eta \|_{L^2} \lesssim \| \sech(2^{-1}\kappa x) v \|_{L^2}.
\]

Set
\[
\psi_{A,B} = \chi_{A} \varphi_{B}, \quad \tilde{S}_{A,B} = \frac{1}{2} \psi_{A,B} + \psi_{A,B} \partial_x,
\]
and consider the functionals
\[
\mathcal{I}_{2nd,1} := \frac{1}{2} \Omega(v, \tilde{S}_{A,B} v), \quad \mathcal{I}_{2nd,2} := \frac{1}{2} \Omega(v, \sigma_3 e^{-\kappa(x)} v).
\]

**Lemma 8.1.** We have
\[
\| \sech(2^{-1}\kappa x) v \|_{L^2}^2 + \mathcal{I}_{2nd,2} \lesssim \left( \varepsilon^{-N} A^2 \delta + A^{-1/2} \right) \| \eta \|_{L^2}^2 + \sum_{m \in \mathbb{R}_{\min}} |z|^2.
\]

**Proof.** We have
\[
\mathcal{I}_{2nd,1} = -\Omega(T \phi[z](\dot{z} - \tilde{\beta}), \tilde{S}_{A,B} v) + \left< L_D v, \tilde{S}_{A,B} v \right> + \left< \begin{pmatrix} [i \varepsilon \partial_x]^{-N}, V_D \end{pmatrix} 0 \end{pmatrix} A^* \eta, \tilde{S}_{A,B} v \right> + \left< T \left( f[\phi[z] + \eta] - f[\phi[z]] \right), \tilde{S}_{A,B} v \right> + \left< T R[z], \tilde{S}_{A,B} v \right>
\]
\[
= D_1 + D_2 + D_3 + D_4 + D_5.
\]

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Following [14], for the main term $D_2$ we have

$$D_2 = \left( L_D v_1, S_{A,B} v_1 \right) = - \int \left( \xi_1'^2 + V_B \xi_1 \right) \, dx + D_{21} \text{ where } \xi_1 = \chi_A \zeta_B v_1,$$

and where

$$V_B = \frac{1}{2} \left( \zeta_B'' - \left( \zeta_B' \right)^2 \right) - \frac{1}{2} \frac{\varphi_B}{\zeta_B^2} \quad \text{and} \quad D_{21} = \frac{1}{4} \int \left( \chi_A'' \right)' \left( \zeta_B' \right)^2 v_1^2 + \frac{1}{2} \int \left( 3 \left( \chi_A' \right)^2 + \chi_A'' \chi_A \right) \zeta_B^2 v_1^2 - \int \left( \chi_A'' \varphi_B \left( v_1' \right)^2 + \frac{1}{4} \int \left( \chi_A'' \right)'' \varphi_B v_1^2. \right.$$

We claim

$$\int \left( \xi_1'^2 + V_B \xi_1 \right) \, dx \gtrsim \left( \| \text{sech} \left( \frac{K}{2} x \right) \|_{L^2} ^2 + \| \text{sech} \left( \frac{K}{2} x \right) \|_{L^2} ^2 \right) - A^{-1} \| \eta \|_{L^2} ^2. \quad (8.5)$$

The proof is like in [14, Lemma 3]. We have

$$\int_{|x| \leq A} \text{sech} \left( kx \right) v_1^2 \leq \int_{|x| \leq A} \text{sech} \left( \frac{K}{2} x \right) \zeta_B^2 v_1^2 \leq \int_{|x| \leq A} \text{sech} \left( \frac{K}{2} x \right) \xi_1^2.$$

We have

$$\int_{|x| \leq A} \text{sech} \left( kx \right) v_1'^2 \leq \int_{|x| \leq A} \text{sech} \left( \frac{K}{2} x \right) \left( \xi_1' - \zeta_B v_1 \right)^2 \leq \int_{|x| \leq A} \text{sech} \left( \frac{K}{2} x \right) \left( \xi_1'^2 + \xi_1^2 \right).$$

We have

$$\int_{|x| \geq A} \text{sech} \left( kx \right) \left( v_1'^2 + v_1^2 \right) \leq \text{sech} \left( \frac{K}{2} A \right) \int_{\mathbb{R}} \text{sech} \left( \frac{8}{A} x \right) \left( v_1'^2 + v_1^2 \right) \, dx \lesssim \text{sech} \left( \frac{K}{2} A \right) \varepsilon^{-N} \int_{\mathbb{R}} \text{sech} \left( \frac{4}{A} x \right) \left( \eta_1'^2 + \eta_1^2 \right) \, dx \leq A^{-1} \| \eta \|_{L^2} ^2.$$

Finally, Lemma 4.1 and Assumption 1.9 imply

$$\int_{\mathbb{R}} \text{sech} \left( \frac{K}{2} x \right) \left( \xi_1'^2 + \xi_1^2 \right) \lesssim \int_{\mathbb{R}} \left( \xi_1'^2 + V_B \xi_1 \right) \, dx,$$

completing the proof of (8.5).

We next claim the following, which is [14, Lemma 4],

$$|D_{21}| \lesssim A^{-1/2} \left( \| \eta \|_{L^2} ^2 + \| \text{sech} \left( kx \right) \eta_1 \|_{L^2} ^2 \right) \lesssim A^{-1/2} \left( \| \eta \|_{L^2} ^2 + \varepsilon^{-N} \| \text{sech} \left( \frac{K}{2} x \right) \eta_1 \|_{L^2} ^2 \right), \quad (8.6)$$

where the 2nd inequality follows from (7.21). Now we prove the first inequality.

Notice that $\chi_A(x)$ is constant for $|x| \notin [A,2A]$, so that

$$|\left( \chi_A'' \right)' \left( \zeta_B' \right)| \lesssim A^{-1} e^{-\frac{1}{B}}, \quad |\left( \chi_A' \right)^2 + \chi_A'' \chi_A \zeta_B^2| \lesssim A^{-2} e^{-\frac{1}{B}}$$

and since $|\varphi_B| \lesssim B$ we have $|\chi_A''|, \varphi_B| \lesssim A^{-3} B$ and $|\chi_A''|, \varphi_B| \lesssim A^{-2} B$, we have

$$\left| \frac{1}{4} \left( \chi_A'' \right)' \left( \zeta_B' \right)^2 v_1^2 + \frac{1}{2} \left( 3 \left( \chi_A' \right)^2 + \chi_A'' \chi_A \right) \zeta_B^2 v_1^2 - \left( \chi_A'' \right)' \varphi_B \left( v_1' \right)^2 + \frac{1}{4} \left( \chi_A'' \right)'' \varphi_B v_1^2 \right| \lesssim \frac{B}{A} \text{sech} \left( \frac{8}{A} x \right) \left( v_1'^2 + \frac{1}{A^2} v_1^2 \right),$$

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by Lemma 7.5 we have
\[
|D_{21}| \lesssim A^{-1/2} \left( \| \text{sech} \left( \frac{4}{A} x \right) v_1' \|_{L^2} + A^{-2} \| \text{sech} \left( \frac{4}{A} x \right) v_1 \|_{L^2} \right)
\]
\[
\lesssim A^{-1/2} \varepsilon^{-N} \left( \| \text{sech} \left( \frac{2}{A} x \right) \eta_1' \|_{L^2}^2 + A^{-2} \| \text{sech} \left( \frac{2}{A} x \right) \eta_1 \|_{L^2}^2 + \| \text{sech} (kx) \eta_1 \|_{L^2}^2 \right),
\]
which yields the desired inequality (8.6).

By Lemma 3.2 and by an analogue to (7.21), we have
\[
|D_1| \lesssim |\hat{z} - \bar{z}| \| \text{sech} (2kx) v_1 \|_{L^2} \lesssim \delta \| \text{sech} (kx) \eta_1 \|_{L^2} \| \text{sech} (2kx) v_1 \|_{L^2} \lesssim \delta \varepsilon^{-N} \| \text{sech} (kx) \eta_1 \|_{L^2}^2.
\]

By Lemma 7.6, we have
\[
|D_3| = \left| \left\langle \left[ (ix \partial_x)^{-N}, V_D \right] A^* \eta_1, \tilde{S}_{A,B} v_1 \right\rangle \right|
\]
\[
\leq \| \text{cosh} \left( \frac{K}{2} x \right) \left[ (ix \partial_x)^{-N}, V_D \right] A^* \eta_1 \|_{L^2} \| \text{sech} \left( \frac{K}{2} x \right) \tilde{S}_{A,B} v_1 \|_{L^2}
\]
\[
\leq \varepsilon \| \text{sech} \left( \frac{K}{2} x \right) v_1 \|_{L^2} \left( \| \text{sech} \left( \frac{K}{2} x \right) v_1' \|_{L^2} + \| \text{sech} \left( \frac{K}{2} x \right) v_1 \|_{L^2} \right)
\]
\[
\lesssim \varepsilon \left( \| \text{sech} \left( \frac{K}{2} x \right) v_1' \|_{L^2}^2 + \| \text{sech} \left( \frac{K}{2} x \right) v_1 \|_{L^2}^2 \right),
\]
where the upper bound can be absorbed inside the left hand side of (8.4).

Like in Lemma 6.1, we have
\[
D_4 = \left( \int_0^1 \int_0^1 f''(s_1 \phi_1[z] + s_2 \eta_1) \phi_1[z] \eta_1 \, ds_1 \, ds_2, \tilde{S}_{A,B} v_1 \right) + \left\langle f(\eta_1), \tilde{S}_{A,B} v_1 \right\rangle =: D_{41} + D_{42}.
\]

Ignoring the irrelevant $ds_1 ds_2$ integral, we have
\[
|D_{41}| \lesssim \| \text{cosh} (2kx) \| f''(s_1 \phi_1[z] + s_2 \eta_1) \phi_1[z] \| \text{sech} (kx) \eta_1 \| \| \text{sech} (kx) \| \tilde{S}_{A,B} v_1 \|_{L^2}
\]
\[
\lesssim \| z \| \| \text{sech} (kx) \eta_1 \|_{L^2} \left( \| \text{sech} (kx) \| \eta_1' \|_{L^2} + \| \text{sech} (kx) \| v_1 \|_{L^2} \right)
\]
\[
\lesssim \delta \varepsilon^{-N} \left( \| \text{sech} (kx) \| \eta_1' \|_{L^2}^2 + \| \text{sech} (kx) \| v_1 \|_{L^2}^2 \right),
\]
which can be absorbed inside the left hand side of (8.4). Next, we have
\[
|D_{42}| = \left| \left\langle \text{sech} \left( \frac{2}{A} x \right) f(\eta_1), \cosh \left( \frac{2}{A} x \right) \left( \frac{1}{2} (\chi_A^2 \varphi_B)' + \chi_A^2 \varphi_B \partial_x \right) v_1 \right\rangle \right|
\]
\[
\lesssim \| \eta_1 \|_{L^\infty} \| \text{sech} \left( \frac{2}{A} x \right) \eta_1 \|_{L^\infty} \times
\]
\[
\left( \| \text{cosh} \left( \frac{6}{A} x \right) \psi_{A,B} \|_{L^\infty} \| \text{sech} \left( \frac{4}{A} x \right) v_1 \|_{L^2} + \| \cosh \left( \frac{6}{A} x \right) \psi_{A,B} \|_{L^\infty} \| \text{sech} \left( \frac{4}{A} x \right) v_1' \|_{L^2} \right)
\]
\[
\lesssim A\delta \| \eta_1 \|_{\Sigma_A} \left( \| \text{sech} \left( \frac{4}{A} x \right) v_1 \|_{L^2} + \| \text{sech} \left( \frac{4}{A} x \right) v_1' \|_{L^2} \right)
\]
\[
\lesssim \varepsilon^{-N} A\delta \| \eta_1 \|_{\Sigma_A} \left( \| \text{sech} \left( \frac{4}{A} x \right) \eta_1 \|_{L^2} + \| \text{sech} \left( \frac{4}{A} x \right) \eta_1' \|_{L^2} \right) \lesssim \varepsilon^{-N} A^2 \delta \| \eta_1 \|_{\Sigma_A}^2.
\]

Finally, we consider
\[
D_5 = \left( \sum_{m \in \mathbb{R}_{\min}} z^m T \mathbf{G}_m, \tilde{S}_{A,B} v \right) + \langle TR[z], \tilde{S}_{A,B} v \rangle =: D_{51} + D_{52}.
\]

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We focus on $D_{51}$ which is the main term. We have

$$\left| \langle \bar{z}^m \mathcal{T} G_m, \bar{S}_{A,B} v \rangle \right| \leq |z|^2 \|\cosh (\kappa x) \bar{S}_{A,B} \mathcal{T} G_m\|_{L^2} \|\text{sech} (\kappa x) v\|_{L^2} \lesssim \frac{1}{\mu} |z|^2 + \mu \|\text{sech} (\kappa x) v\|_{L^2}^2,$$

where for $\mu$ small enough the last term can be absorbed in the left hand side of (8.4).

Collecting the estimates, we have the conclusion. $\square$

**Lemma 8.2.** We have

$$\| e^{-\kappa(x)/2} v_2 \|_{L^2} + \hat{I}_{2,2} \lesssim \| e^{-\kappa(x)/2} v_1' \|_{L^2}^2 + \| e^{-\kappa(x)/2} v_1 \|_{L^2}^2 + \sum_{m \in \mathbb{R}_{\text{min}}} |z|^2 + \delta A \| \eta \|^2_{\mathcal{S}_A}. \quad (8.7)$$

**Proof.** Differentiating $\hat{I}_{2,2}$, we have

$$\hat{I}_{2,2} = -\Omega (TD\phi[z] (\dot{z} - \ddot{z}), \sigma_3 e^{-\kappa(x)} v) + \left\langle L_D v, \sigma_3 e^{-\kappa(x)} v \right\rangle$$

$$+ \left\langle [\{i \tilde{\epsilon} \partial_x\}^{-N}, \mathcal{A}_v] \eta_1, \sigma_3 e^{-\kappa(x)} v_1 \right\rangle + \left\langle \mathcal{T} (f(\phi[z] + \eta) - f(\phi[z]), e^{-\kappa(x)} v_1 \right\rangle$$

$$+ \left\langle \mathcal{T} \bar{R}[z], \sigma_3 e^{-\kappa(x)} v \right\rangle =: E_1 + E_2 + E_3 + E_4 + E_5.$$

The main term is

$$E_2 = -\|e^{-\kappa(x)/2} v_2\|_{L^2}^2 + \left\langle L_D v_1, e^{-\kappa(x)} v_1 \right\rangle = -\|e^{-\kappa(x)/2} v_2\|_{L^2}^2 + E_{21},$$

with

$$|E_{21}| \lesssim \|e^{-\kappa(x)/2} v_1'\|_{L^2}^2 + \|e^{-\kappa(x)/2} v_1\|_{L^2}^2.$$

By Lemma 3.2, we have

$$|E_1| \lesssim \delta \|e^{-\kappa(x)/2} v\|_{L^2} \|e^{-\kappa(x)} \eta\|_{L^2} \lesssim \delta \varepsilon^{-N} \|e^{-\kappa(x)/2} v\|_{L^2}^2.$$ 

By (7.24), we have

$$|E_3| = \left| \left\langle [\{i \tilde{\epsilon} \partial_x\}^{-N}, \mathcal{A}_v] \eta_1, \sigma_3 e^{-\kappa(x)} v_1 \right\rangle \right| \lesssim \varepsilon \|e^{-\mathcal{F}(x)} v_1\|_{L^2} \|e^{-\kappa(x)} v_1\|_{L^2} \lesssim \varepsilon \|e^{-\mathcal{F}(x)} v_1\|_{L^2}^2.$$ 

We write

$$E_4 = \left\langle \int_0^1 \int_0^1 f''(s_1 \phi[z] + s_2 \eta_1) \phi[z] \eta_1 ds_1 ds_2, e^{-\kappa(x)} v_1 \right\rangle + \left\langle f(\eta_1), \sigma_3 e^{-\kappa(x)} v_1 \right\rangle =: E_{41} + E_{42}.$$ 

Ignoring the irrelevant $ds_1 ds_2$ integral, we have

$$|E_{41}| \lesssim \| f''(s_1 \phi[z] + s_2 \eta_1) \cos (\kappa x) \phi[z] \eta_1 \|_{L^2} \|e^{-\kappa(x)} v_1\|_{L^2} \lesssim \|z\| \|\text{sech} (\kappa x) \eta_1\|_{L^2} \|e^{-\kappa(x)} v_1\|_{L^2} \lesssim \delta \|\text{sech} (\frac{\kappa}{2}) v_1\|_{L^2}^2.$$ 

We have

$$|E_{42}| = \left| \left\langle f(\eta_1), e^{-\kappa(x)} v_1 \right\rangle \right|$$

$$\lesssim \| \eta_1 \|_{L^\infty} \|\text{sech} (\frac{2}{A}) \| \|\text{sech} (\frac{\kappa}{2}) v_1\|_{L^2} \lesssim \delta A \left( \|\text{sech} (\frac{\kappa}{2}) v_1\|_{L^2}^2 + \|\eta\|^2_{\mathcal{S}_A} \right).$$
We have
\[ E_5 = \left\langle \sum_{m \in \mathbb{R}_{\min}} z^m T G_m, \sigma_3 e^{-\kappa(x)} v \right \rangle + \left\langle TR[z], \sigma_3 e^{-\kappa(x)} v \right \rangle =: E_{51} + E_{52}. \]

We focus on $D_{51}$ which is the main term, the other being simpler. We have
\[
| \left\langle z^m T G_m, \sigma_3 e^{-\kappa(x)} v \right \rangle | \leq |z^m| \|T G_m\|_{L^2} \|\text{sech} (\kappa x) v\|_{L^2} \lesssim \frac{1}{\mu} |z^m|^2 + \mu \|\text{sech} (\kappa x) v\|_{L^2}^2
\]
\[
= \frac{1}{\mu} |z^m|^2 + \mu \|\text{sech} (\kappa x) v_1\|_{L^2}^2 + \mu \|\text{sech} (\kappa x) v_2\|_{L^2}^2,
\]
where for $\mu$ small enough the very last term in $v_2$ can be absorbed in the left hand side of (8.7)

Collecting the estimates, we have the conclusion.

**Lemma 8.3.** For any $\mu > 0$, we have
\[
\int_0^T \left( \|\text{sech} \left( \frac{\kappa}{2} x \right) v_1\|_{L^2}^2 + \|\text{sech} \left( \frac{\kappa}{2} x \right) v_2\|_{L^2}^2 \right) \lesssim B \varepsilon^{-N} \delta^2
\]
\[
+ \left( \varepsilon^{-1} A^2 \delta + A^{-1/2} \right) \int_0^T \|\eta\|_{L^2}^2 + \sum_{m \in \mathbb{R}_{\min}} \|z^m\|_{L^2(0,T)}^2.
\]

**Proof.** The claim follows from Lemmas 8.1 and 8.2 and
\[
|I_{2\text{nd},1}| \lesssim B \varepsilon^{-N} \delta^2, \quad |I_{2\text{nd},2}| \lesssim \varepsilon^{-N} \delta^2. \tag{8.8}
\]

**Proof of Proposition 2.6.** It is a consequence of Lemma 8.3 and inequality (8.3).

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Department of Mathematics and Geosciences, University of Trieste, via Valerio 12/1 Trieste, 34127 Italy. E-mail Address: scuccagna@units.it

Department of Mathematics and Informatics, Graduate School of Science, Chiba University, Chiba 263-8522, Japan. E-mail Address: maeda@math.s.chiba-u.ac.jp

Department of Mathematics and Geosciences, University of Trieste, via Valerio 12/1 Trieste, 34127 Italy. E-mail Address: STEFANO.SCRIBOGNA@units.it