THE LAPLACIAN ON THE UNIT SQUARE IN A SELF-SIMILAR MANNER

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Abstract. In this paper, we show how to construct the standard Laplacian on the unit square in a self-similar manner. We rewrite the familiar mean value property of planar harmonic functions in terms of average values on small squares, from which we could know how the planar self-similar resistance form and the Laplacian look like. This approach combines the constructive limit-of-difference-quotients method of Kigami for p.c.f. self-similar sets and the method of averages introduced by Kusuoka and Zhou for the Sierpinski carpet.

1. Introduction

Let $K$ denote a self-similar set which is non-empty compact and generated by a finite family of contraction similarity mappings $\{F_i\}_{i=1}^N$ on $\mathbb{R}^n$ such that

$$K = \bigcup_{i=1}^N F_i K.$$ 

If $w = (w_1, \ldots, w_m)$ is a finite word with each $w_j \in \{1, \ldots, N\}$, we define the mapping

$$F_w = F_{w_1} \circ \cdots \circ F_{w_m},$$ 

and call $F_w K$ a $m$-cell of $K$.

$K$ is called post-critically finite (p.c.f.) if $K$ is connected, and there exists a finite set $V_0 \subseteq K$ called the boundary, such that

$$F_w K \cap F_{w'} K \subseteq F_w V_0 \cap F_{w'} V_0, \quad (F_w V_0 \cap F_{w'} V_0) \cap V_0 = \emptyset,$$

for $w \neq w'$ with $|w| = |w'|$,

where $|w|$ is the length of $w$. Moreover, we require that each boundary point is the fixed point of one of the mappings in $\{F_i\}_{i=1}^N$. Without loss of generality we assume that $V_0 = \{q_1, \ldots, q_{N_0}\}$ for $N_0 \leq N$ with

$$F_i q_i = q_i, \quad \text{for } i = 1, \ldots, N_0.$$

A more general definition of p.c.f. self-similar sets can be found in [2,3] introduced by Kigami. The unit interval($I$) and the Sierpinski gasket($SG$) are two typical examples. However, the unit square ($S$, which we mainly discuss in this paper) and the Sierpinski carpet($SC$) are non-p.c.f. self-similar sets.

An analytic construction of a Laplacian on $SG$ was given by Kigami [1] as a renormalized limit of difference quotients, which he later extended to p.c.f. self-similar sets [2]. Please also refer to [6] to find a detailed introduction to this topic. At about the same time, Kusuoka and Zhou [4] developed a method of average for defining a Laplacian on $SC$, which uses average values of functions over cells

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rather than pointwise values in the definition. This method was later proved to be equivalent to Kigami’s definition for symmetric Laplacian (with respect to the standard self-similar measure) by Strichartz [5] on \( SG \), which brings the possibility to define Laplacians on more general self-similar sets.

Recall that \( SG \) is the invariant set of a family of 3 contraction mappings \( F_i x = \frac{1}{3} (x + q_i), i = 1, 2, 3 \), where \( q_1, q_2, q_3 \) are the vertices of an equilateral triangle in \( \mathbb{R}^2 \). Let \( \mu \) be the standard regular probability measure with weights \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) on \( SG \). Define the average value for a function \( f \) on the cell \( F_w SG \) as

\[
B_w(f) = \frac{1}{\mu(F_w SG)} \int_{F_w SG} f d\mu.
\]

It is easy to check that

\[
B_w(f) = \frac{1}{3} \sum_{i=1}^{3} B_{wi}(f),
\]

where \( wi \) denotes \((w_1, \cdots, w_m, i)\).

According to [5], any harmonic function \( h \) on \( SG \) satisfies the mean value property

\[
B_w(h) = \frac{1}{3} \sum_{w' \sim_m w} B_{w'}(h)
\]

for any finite word \( w \) with \( F_w SG \cap V_0 = \emptyset \), where \( |w'| = |w| \) and \( w' \sim_m w \) means that \( F_{w'} SG \cap F_w SG \) is non-empty.

The resistance form \( \mathcal{E}(\cdot, \cdot) \) on \( SG \) is defined as

\[
\mathcal{E}(f, g) = \lim_{m \to \infty} \mathcal{E}_m(f, g),
\]

with

\[
\mathcal{E}_m(f, g) = \frac{3}{2} r^{-m} \sum_{w' \sim_m w} (B_{w'}(f) - B_w(f))(B_{w'}(g) - B_w(g)),
\]

where \( r = \frac{3}{2} \) is the renormalization factor.

The symmetric Laplacian was proved to be the limit of a sequence of discrete Laplacians \( \Delta_m \), in the sense that

\[
\Delta f(x) = \lim_{m \to \infty} \Delta_m f(x)
\]

uniformly for every \( f \) in the domain of \( \Delta \), where \( \Delta_m \) is defined as

\[
\Delta_m f(x) = \frac{9}{2} r^{-m} \left( \frac{1}{3} \sum_{w' \sim_m w} B_{w'}(f) - B_w(f) \right), \text{ for } x \in F_w K, |w| = m,
\]

for each integer \( m \geq 0 \).

In essential, the method of average studies first the analysis on cell graphs \( \Gamma_m \) whose vertices are the words \( w \) of length \( m \), and edge relation, denoted by \( w \sim_m w' \) is defined by the condition that \( F_w SG \cap F_{w'} SG \neq \emptyset \), then passes the approximation to the limit as \( m \to \infty \) to establish the analysis on \( SG \).

In this paper, we will investigate that to what extent can we extend Kusuoka and Zhou’s method to the planar unit square \( S \), which could be viewed as a non-p.c.f. self-similar set, generated by a family of contraction mappings \( \{F_i\}_{i=1}^4 \) on \( \mathbb{R}^2 \) with

\[
F_i(x) = \frac{1}{2} (x + q_i), \ i = 1, 2, 3, 4,
\]
and

\[ q_1 = (0, 0), \ q_2 = (1, 0), \ q_3 = (1, 1), \ q_4 = (0, 1). \]

In particular, the mean value property for planar harmonic functions on cell graphs (several choices) will not hold exactly. However, it will be close to holding exactly for large \( m \). We will provide a sharp estimate for it. The analysis established in this paper provides a self-similar viewpoint of the familiar classical analysis on \( \mathbb{R}^2 \). It will be nice to be able to transform the classical analytical results via this self-similar construction, which maybe will bring insights to the analysis on \( SC \).

2. Mean Value Property of Harmonic Functions

From now on, we use \( \mu \) to denote the Lebesgue measure on \( \mathbb{R}^2 \). For \( x = (x_1, x_2) \in \mathbb{R}^2, \ l \in \mathbb{R}^+ \), denote

\[ \mathcal{D}(x, l) = \{ (\xi_1, \xi_2) \in \mathbb{R}^2 | x_1 - \frac{l}{2} \leq \xi_1 \leq x_1 + \frac{l}{2}, x_2 - \frac{l}{2} \leq \xi_2 \leq x_2 + \frac{l}{2} \} \quad (2.1) \]
a \( l \)-square centered at \( x \). For a function \( f \) integrable on \( \mathcal{D}(x, l) \), write

\[ I(f, x, l) = \frac{1}{l^2} \int_{\mathcal{D}(x, l)} f \, d\mu. \quad (2.2) \]

**Definition 2.1.** Let \( p = (p_1, p_2) \) be an integer pair with \( 0 \leq p_1 \leq p_2 \). For two squares \( \mathcal{D}(x, l) \) and \( \mathcal{D}(x', l) \), write \( \mathcal{D}(x, l) \sim_p \mathcal{D}(x', l) \) if

\[ |x_1 - x'_1| = p_1 l, \ |x_2 - x'_2| = p_2 l, \text{ or} \]
\[ |x_1 - x'_1| = p_2 l, \ |x_2 - x'_2| = p_1 l, \quad (2.3) \]
call them \( p \)-neighbors.

Suppose \( f \) is integrable on every \( l \)-squares \( p \)-neighbor to \( \mathcal{D}(x, l) \), denote

\[ I_p(f, x, l) = \sum_{\sim_p \mathcal{D}(x, l)} I(f, x', l). \quad (2.4) \]

For simplicity, denote a constant \( c_p \) as

\[ c_p = \begin{cases} 
\frac{1}{8}, & \text{if } p_1 = p_2 = 0, \\
\frac{1}{2}, & \text{if } p_1 = 0, p_2 \neq 0, \text{ or } p_1 = p_2 \neq 0, \\
1, & \text{if } 0 < p_1 < p_2. 
\end{cases} \quad (2.5) \]

It is easy to check that \( 8c_p \) is the number of \( p \)-neighbors of any fixed \( \mathcal{D}(x, l) \).

**Lemma 2.2.** Let \( x \in \mathbb{R}^2, \ n \) be a non-negative integer. For functions \( f_n^x, g_n^x \) defined by

\[ f_n^x(\xi) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{(\xi_1 - x_1)^{n-2j}(\xi_2 - x_2)^{2j}}{(n-2j)!(2j)!}, \]
\[ g_n^x(\xi) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \frac{(\xi_1 - x_1)^{n-2j-1}(\xi_2 - x_2)^{2j+1}}{(n-2j-1)!(2j+1)!}, \quad (2.6) \]
and integer pair \( p = (p_1, p_2) \) with \( 0 \leq p_1 \leq p_2 \), we have

\[
I_p(f^*_{n, x, l}) = \frac{1}{(4k + 2)!} T_p^{(k)} \left( \frac{l}{2} \right)^{4k}, \text{ if } 4 \mid n \text{ with } n = 4k,
\]

\[
I_p(f^*_{n, x, l}) = 0, \text{ if } 4 \nmid n,
\]

\[
I_p(g^*_{n, x, l}) = 0,
\]

where constants \( T_p^{(k)} \) are

\[
T_p^{(k)} = 2c_p \left( \left( (2p_1 + 1)^2 + (2p_2 + 1)^2 \right)^{2k+1} \sin \left( (4k + 2) \arctan \frac{2p_2 + 1}{2p_1 + 1} \right) 
- \left( (2p_1 - 1)^2 + (2p_2 + 1)^2 \right)^{2k+1} \sin \left( (4k + 2) \arctan \frac{2p_2 + 1}{2p_1 - 1} \right) 
+ \left( (2p_1 - 1)^2 + (2p_2 - 1)^2 \right)^{2k+1} \sin \left( (4k + 2) \arctan \frac{2p_2 - 1}{2p_1 + 1} \right) 
- \left( (2p_1 + 1)^2 + (2p_2 - 1)^2 \right)^{2k+1} \sin \left( (4k + 2) \arctan \frac{2p_2 - 1}{2p_1 - 1} \right) \right).
\]

with the estimates that

\[
|T_p^{(k)}| \leq 8c_p \| 2p + (1, 1) \|^{4k+2}.
\]

**Proof.** By direct calculation, we obtain

\[
I_p(f^*_{n, x, l}) = \frac{c_p}{l^2} \sum_{j=0}^{[\frac{l}{2}]} \left[ (-1)^j \left( \left( (2p_1 + 1) \frac{l}{2} \right)^{n-2j+1} + \left( - (2p_1 - 1) \frac{l}{2} \right)^{n-2j+1} 
- \left( (2p_2 + 1) \frac{l}{2} \right)^{2j+1} - \left( - (2p_2 - 1) \frac{l}{2} \right)^{2j+1} \right) \left[ \left( (2p_1 + 1) \frac{l}{2} \right)^{n-2j+1} + \left( - (2p_1 - 1) \frac{l}{2} \right)^{n-2j+1} 
+ \left( (2p_2 - 1) \frac{l}{2} \right)^{2j+1} + \left( - (2p_2 - 1) \frac{l}{2} \right)^{2j+1} \right] 
+ \left( (2p_2 + 1) \frac{l}{2} \right)^{2j+1} + \left( - (2p_2 + 1) \frac{l}{2} \right)^{n-2j+1} 
- \left( (2p_2 - 1) \frac{l}{2} \right)^{n-2j+1} - \left( - (2p_2 + 1) \frac{l}{2} \right)^{2j+1} 
- \left( (2p_1 + 1) \frac{l}{2} \right)^{2j+1} - \left( - (2p_1 + 1) \frac{l}{2} \right)^{n-2j+1} \right].
\]
Then an easy calculation yields (2.8) and (2.9).

Obviously the summation is zero unless \( n \) is even. If \( n \) is even, let \( n = 2m \), then

\[
I_p(f^x_n, x, l) = \frac{1}{(2m + 2)!} \left( \frac{l}{2} \right)^{2m} c_p \sum_{j=0}^{m} (-1)^j C_{2m+2}^{j} \gamma^{m-j,j}_p,
\]

with

\[
\gamma^{m-j,j}_p = ((2p_1 + 1)^{2m-2j+1} - (2p_1 - 1)^{2m-2j+1})((2p_2 + 1)^{2j+1} - (2p_2 - 1)^{2j+1})
\]

\[+ ((2p_2 + 1)^{2m-2j+1} - (2p_2 - 1)^{2m-2j+1})((2p_1 + 1)^{2j+1} - (2p_1 - 1)^{2j+1}).\]

Using the symmetry of \( \sum_{j=0}^{m} (-1)^j C_{2m+2}^{j} \gamma^{m-j,j}_p \), it is easy to check that the summation is zero unless \( m \) is even. If \( m \) is even, let \( m = 2k \) \( (n = 2m = 4k) \), then we obtain that

\[
I_p(f^x_n, x, l) = \frac{1}{(4k + 2)!} T_p^{(k)} \left( \frac{l}{2} \right)^{4k},
\]

and

\[
T_p^{(k)} = c_p \sum_{j=0}^{2k} (-1)^j C_{4k+2}^{j} \gamma^{2k-j,j}_p
\]

\[= c_p \sum_{j=0}^{2k} (-1)^j C_{4k+2}^{j} \left( ((2p_1 + 1)^{4k-2j+1} - (2p_1 - 1)^{4k-2j+1})((2p_2 + 1)^{2j+1} - (2p_2 - 1)^{2j+1})
\]

\[+ ((2p_2 + 1)^{4k-2j+1} - (2p_2 - 1)^{4k-2j+1})((2p_1 + 1)^{2j+1} - (2p_1 - 1)^{2j+1}) \right)\]

\[= c_p \left( Im(2p_1 + 1 + i(2p_2 + 1))^{4k+2} - Im(2p_1 - 1 + i(2p_2 + 1))^{4k+2}
\]

\[+ Im(2p_1 + 1 + i(2p_2 - 1))^{4k+2} - Im(2p_1 + 1 + i(2p_2 - 1))^{4k+2}
\]

\[+ Im(2p_2 + 1 + i(2p_1 + 1))^{4k+2} - Im(2p_2 + 1 + i(2p_1 + 1))^{4k+2}
\]

\[+ Im(2p_2 - 1 + i(2p_1 - 1))^{4k+2} - Im(2p_2 + 1 + i(2p_1 - 1))^{4k+2} \right)\].

Then an easy calculation yields (2.8) and (2.9).

Analogously, for \( g^x_n \), a similar argument will give \( I_p(g_n^x, x, l) = 0 \). \( \square \)

In fact, \( \{f_n^x\} \) and \( \{g_n^x\} \) are polynomial harmonic functions on \( \mathbb{R}^2 \), which form a "basis" of harmonic functions near \( x \).
Lemma 2.3. Let \( p \) be an integer pair as before, \( \Omega \) be an open set in \( \mathbb{R}^2 \) and \( x \in \Omega \). Then there exists a positive constant \( l_{x,p} \), such that for any harmonic function \( h \) on \( \Omega \) and \( l < l_{x,p} \),

\[
I_p(h, x, l) = 8c_p h(x) + \sum_{k=1}^{\infty} \frac{1}{(4k + 2)!} \frac{\partial^{4k} h}{\partial x_1^{4k}}(x) T_p^{(k)}(\frac{l}{2})^{4k},
\]

with the same constants \( T_p^{(k)} \) as in (2.8).

Proof. Since \( h \) is harmonic on \( \Omega \), it is real analytic in \( \Omega \). So we can expand \( h \) near \( x \) as

\[
h(x) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{\partial^n h}{\partial x_1^{n-i} \partial x_2^i}(x) (\xi_1 - x_1)^{n-i}(\xi_2 - x_2)^i.
\]

(2.11)

Noticing that

\[
\Delta h(x) = \frac{\partial^2 h}{\partial x_1^2}(x) + \frac{\partial^2 h}{\partial x_2^2}(x) = 0,
\]

we have

\[
\frac{\partial^n h}{\partial x_1^{n-i} \partial x_2^i}(x) = -\frac{\partial^n h}{\partial x_1^{n-1} \partial x_2^{i+1}}(x), \text{ for } i = 0, 1, \ldots, n-2.
\]

By iterating we get

\[
\frac{\partial^n h}{\partial x_1^{n-2j} \partial x_2^{2j}}(x) = (-1)^j \frac{\partial^n h}{\partial x_1^n}(x), \text{ for } j = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,
\]

\[
\frac{\partial^n h}{\partial x_1^{n-2j-1} \partial x_2^{2j+1}}(x) = (-1)^j \frac{\partial^n h}{\partial x_1^{n-1} \partial x_2^{i+1}}(x), \text{ for } j = 0, 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor.
\]

Then (2.11) gives

\[
h(\xi) = h(x) + \sum_{n=1}^{\infty} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\partial^n h}{\partial x_1^{n-2j} \partial x_2^{2j}}(x) (\xi_1 - x_1)^{n-2j}(\xi_2 - x_2)^{2j}
\]

\[
+ \sum_{n=1}^{\infty} \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{\partial^n h}{\partial x_1^{n-2j-1} \partial x_2^{2j+1}}(x) (\xi_1 - x_1)^{n-2j-1}(\xi_2 - x_2)^{2j+1}
\]

\[
= h(x) + \sum_{n=1}^{\infty} \left( \frac{\partial^n h}{\partial x_1^n}(x) \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j (\xi_1 - x_1)^{n-2j}(\xi_2 - x_2)^{2j}
\]

\[
+ \sum_{n=1}^{\infty} \left( \frac{\partial^n h}{\partial x_1^{n-1} \partial x_2}(x) f_n^x(\xi) + \frac{\partial^n h}{\partial x_1^{n-1} \partial x_2}(x) g_n^x(\xi) \right),
\]

where \( f_n^x \) and \( g_n^x \) are same as those in Lemma 2.2.

Choose

\( l_{x,p} = \sup \{ l > 0 \mid \text{All } p\text{-neighbors of } D(x, l) \text{ are contained in the convergence domain of (2.11)} \} \).
Then for any \( l < l_{x,p} \), we have
\[
I_p(h, x, l) = 8c_ph(x) + \sum_{n=1}^{\infty} \left( \frac{\partial^n h}{\partial \xi_1^n} (x) I_p(f^n, x, l) + \frac{\partial^n h}{\partial \xi_1^{n-1} \partial \xi_2} (x) I_p(g^n, x, l) \right).
\] (2.13)

By using Lemma 2.2, this gives (2.10). □

The following is the mean value property of planar harmonic functions in terms of average values on \( l \)-squares.

**Theorem 2.4.** Suppose \( \mathcal{P} \) is a finite set consisting of integer pairs \( p = (p_1, p_2) \) with \( 0 \leq p_1 \leq p_2 \) and \( p_2 \neq 0 \), and \( \{A_p\}_{p \in \mathcal{P}} \) is a collection of real numbers satisfies
\[
8 \sum_{p \in \mathcal{P}} c_p A_p = 1.
\] (2.14)

Let \( \Omega \) be an open set in \( \mathbb{R}^2 \) and \( x \in \Omega \). Then for any harmonic function \( h \) on \( \Omega \) and \( l < l_{x,p} \), we have
\[
|I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l)| \leq \sum_{k=1}^{\infty} \frac{1}{(4k + 2)!} \left| \frac{\partial^{4k} h}{\partial \xi_1^{4k}} (x) \right| \left( 2^{2k+1} + \|\mathcal{P}\|^4k+2 \right) \left( \frac{l}{2} \right)^{4k},
\] (2.15)

where
\[
\|\mathcal{P}\| = \max_{p \in \mathcal{P}} \|2p + (1, 1)\|,
\] (2.16)

and \( l_{x,p} = \min_{p \in \mathcal{P}} l_{x,p} \) with \( l_{x,p} \) being the same as in Lemma 2.3.

**Proof.** Noticing that
\[
I(h, x, l) = I_\theta(h, x, l), \text{ with } \theta = (0, 0),
\]
by using (2.10), we obtain that
\[
|I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l)|
= \left| (1 - 8 \sum_{p \in \mathcal{P}} c_p A_p) h(x) + \sum_{k=1}^{\infty} \frac{1}{(4k + 2)!} \left| \frac{\partial^{4k} h}{\partial \xi_1^{4k}} (x) \right| \left( T^{(k)}_{\theta} - \sum_{p \in \mathcal{P}} A_p T^{(k)}_p \right) \left( \frac{l}{2} \right)^{4k} \right|
= \sum_{k=1}^{\infty} \frac{1}{(4k + 2)!} \left| \frac{\partial^{4k} h}{\partial \xi_1^{4k}} (x) \right| \left| T^{(k)}_{\theta} - \sum_{p \in \mathcal{P}} A_p T^{(k)}_p \right| \left( \frac{l}{2} \right)^{4k}.
\]

By calculation \( T^{(k)}_{\theta} = (-1)^k 2^{2k+1} \) and using the estimate (2.9) we then have
\[
\left| T^{(k)}_{\theta} - \sum_{p \in \mathcal{P}} A_p T^{(k)}_p \right|
\leq 2^{2k+1} + \sum_{p \in \mathcal{P}} 8c_p A_p ||2p + (1, 1)||^{4k+2}
\leq 2^{2k+1} + \|\mathcal{P}\|^{4k+2},
\]
which gives (2.15). □

**Remark.** In fact, given \( \mathcal{P} \), by choosing \( A_p \) properly, we can get “higher” rate of convergence for (2.15). Define
\[
N = \inf \{k \in \mathbb{N} | \sum_{p \in \mathcal{P}} A_p T^{(k)}_p \neq T^{(k)}_{\theta} \},
\] (2.17)
called the mean value level of $\mathcal{P}$ with coefficients $\{A_p\}_{p \in \mathcal{P}}$. Then (2.15) becomes
\[
|I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l)| \leq \sum_{k=N}^{\infty} \frac{1}{(4k + 2)!} \left| \frac{\partial^{4k} h}{\partial \xi^k} (x) \right| \left( 2^{2k+1} + \|\mathcal{P}\|^{4k+2} \right) \left( \frac{l}{2} \right)^{4k}. \tag{2.18}
\]
We will discuss more on the mean value level in Section 4.

3. The Resistance form and the Laplacian on the unit square $S$

In this section, we will show the expressions of the resistance form and the (symmetric) Laplacian on the unit square $S$ in terms of average values on cells.

**Lemma 3.1.** Let $\mathcal{P}$ and $\{A_p\}_{p \in \mathcal{P}}$ be defined as in Theorem 2.4, $\Omega$ be an open set in $\mathbb{R}^2$. Then for any $f \in C^1(\Omega)$ and $x \in \Omega$, we have
\[
|\nabla f(x)|^2 = \frac{1}{2} \mathcal{M}_\mathcal{P} \lim_{l \to 0} \frac{1}{l^2} \sum_{p \in \mathcal{P}} \left( A_p \sum_{(x', l) \sim_p (x, l)} (I(f, x', l) - I(f, x, l))^2 \right), \tag{3.1}
\]
where
\[
\mathcal{M}_\mathcal{P} = \frac{1}{2} \left( \sum_{p \in \mathcal{P}} \|p\|^2 c_p A_p \right)^{-1}. \tag{3.2}
\]

**Proof.** Let $l < l_x, p$ where $l_x, p$ is the same as defined in Theorem 2.4. Then for each $p \in \mathcal{P}$, there are $8c_p$ $p$-neighbors of the $l$-square $\mathcal{D}(x, l)$. It is easy to check that $\mathcal{D}(x', l)$ with $x' = (x_1 + p_1 l, x_2 + p_2 l)$ is one of them. Using the mean value theorem for integral we have
\[
I(f, x', l) - I(f, x, l) = \frac{1}{l^2} \int_{\mathcal{D}(x', l)} f(\xi_1, \xi_2) d\mu(\xi) - \frac{1}{l^2} \int_{\mathcal{D}(x, l)} f(\xi_1, \xi_2) d\mu(\xi)
\]
\[
= \frac{1}{l^2} \int_{\mathcal{D}(x', l)} \left( f(\xi_1 + p_1 l, \xi_2 + p_2 l) - f(\xi_1, \xi_2) \right) d\mu(\xi)
\]
\[
= f(\eta_1 + p_1 l, \eta_2 + p_2 l) - f(\eta_1, \eta_2)
\]
for some $(\eta_1, \eta_2) \in \mathcal{D}(x, l)$. Hence
\[
\lim_{l \to 0} \frac{1}{l^2} \int_{\mathcal{D}(x', l)} (I(f, x', l) - I(f, x, l)) = p_1 \frac{\partial f}{\partial \xi_1} (x) + p_2 \frac{\partial f}{\partial \xi_2} (x).
\]
Dealing with other $p$-neighbors similarly, and summing over all the $8c_p$ terms, we get
\[
\lim_{l \to 0} \frac{1}{l^2} \int_{\mathcal{D}(x', l)} (I(f, x', l) - I(f, x, l))^2
\]
\[
= c_p \left( \left( p_1 \frac{\partial f}{\partial \xi_1} (x) + p_2 \frac{\partial f}{\partial \xi_2} (x) \right)^2 + \left( p_2 \frac{\partial f}{\partial \xi_1} (x) + p_1 \frac{\partial f}{\partial \xi_2} (x) \right)^2 \right)
\]
\[
+ \left( -p_2 \frac{\partial f}{\partial \xi_1} (x) + p_1 \frac{\partial f}{\partial \xi_2} (x) \right)^2 + \left( -p_1 \frac{\partial f}{\partial \xi_1} (x) - p_2 \frac{\partial f}{\partial \xi_2} (x) \right)^2
\]
\[
+ \left( p_1 \frac{\partial f}{\partial \xi_1} (x) - p_2 \frac{\partial f}{\partial \xi_2} (x) \right)^2 + \left( p_2 \frac{\partial f}{\partial \xi_1} (x) - p_1 \frac{\partial f}{\partial \xi_2} (x) \right)^2 \right)
\]
\[
= 4\|p\|^2 c_p |\nabla f(x)|^2.
\]
Summing over all $p \in \mathcal{P}$ with the coefficient $A_p$, we obtain (3.1). \qed
Now we turn to the resistance form on the unit square $S$. Notice that the Lebesgue measure $\mu$ restricted to $S$ becomes a regular probability measure with equal weights. Analogous to the $SG$ case, for a finite word $w = (w_1, \cdots, w_m)$ with each $w_j \in \{1, 2, 3, 4\}$, define the average value for a function $f$ on $F_w S$ as

$$B_w (f) = \frac{1}{\mu(F_w S)} \int_{F_w S} f \, d\mu.$$  

**Theorem 3.2.** Let $\mathcal{P}$ and $\{A_p\}_{p \in \mathcal{P}}$ be defined as before. For any $f, g \in C^1(S)$, $m \geq 0$, define

$$\mathcal{E}_m (f, g) = \mathcal{M}_\mathcal{P} \sum_{p \in \mathcal{P}} \left( A_p \sum_{|w| = |w'| \leq m, \ F_w S \cap F_{w'} S \neq \emptyset} (B_{w'} (f) - B_w (f))(B_{w'} (g) - B_w (g)) \right),$$  

(3.3)

where $\mathcal{M}_\mathcal{P}$ is the same as (3.2). Then we have

$$\mathcal{E}(f, g) := \lim_{m \to \infty} \mathcal{E}_m (f, g) = \int_S \nabla f \cdot \nabla g \, d\mu.$$  

(3.4)

**Proof.** For a $m$-cell $F_w S$ in the unit square $S$, let $l_m$ denote the side length and $x_w$ the center of $F_w S$. Then

$$l_m = \frac{1}{2m}, \ \mu(F_w S) = \frac{1}{4m}, \ \text{and} \ x_w = F_w \left( \frac{1}{4} \sum_i q_i \right),$$

hence

$$B_w (f) = I(f, x_w, l_m), \ F_w S = \mathcal{D}(x_w, l_m).$$

Noticing that for $p \in \mathcal{P}$, the $p$-neighbors of $F_w S$ may not be within $S$, we define

$$S_m = \bigcup \{F_w S : |w| = m \text{ and all } p \text{-neighbors of } F_w S \text{ are contained in } S \},$$

and $S_m^c = S \setminus S_m$. Then we have

$$\mathcal{E}_m (f, f) = \mathcal{M}_\mathcal{P} \sum_{p \in \mathcal{P}} \left( A_p \sum_{|w| = |w'| \leq m, \ F_w S \cap F_{w'} S \neq \emptyset} (B_{w'} (f) - B_w (f))^2 \right)$$

$$= \frac{1}{2} \mathcal{M}_\mathcal{P} \sum_{|w| = m, p \in \mathcal{P}} \sum_{F_w S \subseteq S_m} \left( A_p \sum_{|w'| \leq m} (B_{w'} (f) - B_w (f))^2 \right)$$

$$= \frac{1}{2} \mathcal{M}_\mathcal{P} \sum_{|w| = m, F_w S \subseteq S_m} \left( \frac{1}{l_m} \sum_{p \in \mathcal{P}} \left( A_p \sum_{|w'| \leq m} (I(f, x_w, l_m) - I(f, x_{w'}, l_m))^2 \mu(F_w S) \right) \right)$$

$$+ \frac{1}{2} \mathcal{M}_\mathcal{P} \sum_{|w| = m, F_w S \subseteq S_m^c} \left( \frac{1}{l_m} \sum_{p \in \mathcal{P}} \left( A_p \sum_{|w'| \leq m} (I(f, x_w, l_m) - I(f, x_{w'}, l_m))^2 \mu(F_w S) \right) \right).$$

It is obvious that $\lim_{m \to \infty} \mu(S_m^c) = 0$. So by Lemma 3.1 we know that when $m$ goes to $\infty$, the first term converges to $\int_S |\nabla f|^2 \, d\mu$ and the second term converges to $0$. Hence

$$\mathcal{E}(f, f) = \lim_{m \to \infty} \mathcal{E}_m (f, f) = \int_S |\nabla f|^2 \, d\mu.$$
Then by using the polarization identity

\[ E_m(f, g) = \frac{1}{4} \left( E_m(f + g, f + g) - E_m(f - g, f - g) \right), \]  

(3.5)

we obtain

\[ E(f, g) = \lim_{m \to \infty} E_m(f, g) \]

\[ = \frac{1}{4} \left( \int_K |\nabla (f + g)|^2 d\mu - \int_K |\nabla (f - g)|^2 d\mu \right) \]

\[ = \int_K \nabla f \cdot \nabla g d\mu. \]

\[ \square \]

We should remark that (3.3) and (3.4) imply that the renormalization factor of the resistance form equals to 1 in this case.

Next we come to the Laplacian on \( S \).

**Lemma 3.3.** Let \( \mathcal{P} \) and \( \{ A_p \}_{p \in \mathcal{P}} \) be defined as before. \( \Omega \) be an open set in \( \mathbb{R}^2 \). Then for any \( f \in C^2(\Omega) \) and \( x \in \Omega \), we have

\[ \Delta f(x) = M_{\mathcal{P}} \lim_{l \to 0} \frac{1}{l^2} \left( \sum_{p \in \mathcal{P}} A_p I_p(f, x, l) - I(f, x, l) \right), \]  

(3.6)

where \( M_{\mathcal{P}} \) is the same as (3.2).

**Proof.** Since \( 8 \sum_{p \in \mathcal{P}} c_p A_p = 1 \), we have

\[ \sum_{p \in \mathcal{P}} A_p I_p(f, x, l) - I(f, x, l) = \sum_{p \in \mathcal{P}} A_p \left( I_p(f, x, l) - 8c_p I(f, x, l) \right) \]

\[ = \sum_{p \in \mathcal{P}} A_p \left( \sum_{D(x', l) \sim_p D(x, l)} (I(f, x', l) - I(f, x, l)) \right). \]

Let \( p \in \mathcal{P} \). As in the proof of Lemma 3.1, there are \( 8c_p \) \( p \)-neighbors of the \( l \)-square \( D(x, l) \), for example, \( D(x', l) \) and \( D(x'', l) \) with \( x' = (x_1 + p_1 l, x_2 + p_2 l) \) and \( x'' = (x_1 - p_1 l, x_2 - p_2 l) \) are two of them. By the mean value theorem for integral, we have

\[ I(f, x', l) - I(f, x, l) + I(f, x'', l) - I(f, x, l) \]

\[ = f(\eta_1 + p_1 l, \eta_2 + p_2 l) - f(\eta_1, \eta_2) + f(\eta_1 - p_1 l, \eta_2 - p_2 l) - f(\eta_1, \eta_2), \]  

for some \( (\eta_1, \eta_2) \in D(x, l) \).

Thus

\[ \lim_{l \to 0} \frac{1}{l^2} \left( I(f, x', l) - I(f, x, l) + I(f, x'', l) - I(f, x, l) \right) = p_1^2 \frac{\partial^2 f}{\partial \xi_1^2}(x) + 2p_1 p_2 \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2}(x) + p_2^2 \frac{\partial^2 f}{\partial \xi_2^2}(x). \]
Dealing with other \( p \)-neighbors similarly, and summing over all of them, we obtain

\[
\lim_{t \to \infty} \frac{1}{t^2} \sum_{(x',t') \in \mathcal{P}_t(x)} (I(f, x', t) - I(f, x, t)) = \Delta f(x).
\]

\[
\Delta f(x) = \lim_{m \to \infty} \Delta_m f(x)
\]

uniformly.

**Proof.** Similarly, we define \( S_m \) and \( S_m^c \) as we did in the proof of Theorem 3.2. Since \( x \in S \setminus \partial S \) we know that there exists an integer \( m_0 \) such that when \( m \geq m_0 \) (that is, \( l_m \leq 1/2^m \)), \( x \in F_w S \subseteq S_m \) for some \( w \) of length \( m \). Obviously, \( x_w \) (the center of \( F_w S \)) will go to \( x \) as \( m \) goes to \( \infty \). Then from (3.6) we get

\[
\lim_{m \to \infty} \Delta_m f(x) = \mathcal{M}_\mathcal{P} \lim_{m \to \infty} 4^m \left( \sum_{p \in \mathcal{P}} \left( \sum_{F_w \in S} A_p \sum_{I_p(f, x_w, l_m)} - I(f, x_w, l_m) \right) \right)
\]

\[
= \mathcal{M}_\mathcal{P} \lim_{t \to \infty} \frac{1}{l_m} \left( \sum_{p \in \mathcal{P}} A_p I_p(f, x_w, l_m) - I(f, x_w, l_m) \right)
\]

\[
= \Delta f(x).
\]

The uniform convergence comes from the fact that \( S \) is compact.

### 4. The Level of Mean Value Property

In Section 2 we have seen that the rate of convergence of (2.15) is decided by the mean value level of \( (\mathcal{P}, \{A_p\}_{p \in \mathcal{P}}) \). For given \( \mathcal{P} \) and positive integer \( N \), from (2.17), we may find \( \{A_p\}_{p \in \mathcal{P}} \) by solving equations
\[
8 \sum_{p \in \mathcal{P}} c_p A_p = 1,
\]
\[
\sum_{p \in \mathcal{P}} A_p T_p^{(k)} = T_\theta^{(k)}, \text{ for } k = 0, 1, \ldots, N-1,
\]

such that \(N\) is the mean value level of \((\mathcal{P}, \{A_p\}_{p \in \mathcal{P}})\).

Here are some solutions:

For \(\mathcal{P} = \{(0,1)\}, N = 1\), we have a unique solution
\[
A_{(0,1)} = \frac{1}{4}, \quad \mathcal{M}_\mathcal{P} = 4;
\]

For \(\mathcal{P} = \{(0,1), (1,1)\}, N = 1\), we have infinite solutions satisfying
\[
A_{(0,1)} + A_{(1,1)} = \frac{1}{4}.
\]

For \(\mathcal{P} = \{(0,1), (1,1)\}, N = 2\), we have a unique solution
\[
A_{(0,1)} = \frac{1}{5}, \quad A_{(1,1)} = \frac{1}{20}, \quad \mathcal{M}_\mathcal{P} = \frac{10}{3};
\]

For \(\mathcal{P} = \{(0,1), (1,1), (0,2)\}, N = 3\), we have a unique solution
\[
A_{(0,1)} = \frac{16}{75}, \quad A_{(1,1)} = \frac{1}{25}, \quad A_{(0,2)} = -\frac{1}{300}, \quad \mathcal{M}_\mathcal{P} = \frac{25}{7};
\]

For \(\mathcal{P} = \{(0,1), (1,1), (0,2), (1,2)\}, N = 4\), we have a unique solution
\[
A_{(0,1)} = \frac{38}{183}, \quad A_{(1,1)} = \frac{103}{2379}, \quad A_{(0,2)} = -\frac{17}{9516}, \quad A_{(1,2)} = \frac{1}{2379}, \quad \mathcal{M}_\mathcal{P} = \frac{793}{231}.
\]

On the other hand, for a given \(\mathcal{P}\) with coefficients \(\{A_p\}_{p \in \mathcal{P}}\), it is natural to ask which kind of harmonic functions does satisfy the mean value property exactly.

**Theorem 4.1** Let \((\mathcal{P}, \{A_p\}_{p \in \mathcal{P}})\) be defined as before with the mean value level \(N\), \(\Omega\) be a connected open set in \(\mathbb{R}^2\), and \(h\) be a harmonic function on \(\Omega\). Suppose there is a connected open subset \(U\) of \(\Omega\) such that for any \(x \in U\), the identity
\[
I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l) = 0
\]
holds for sufficiently small \(l\). Then \(h\) is a polynomial harmonic function on \(\Omega\) with degree no more than \(4N\).

**Proof.** By applying (2.10) and (2.17), we know that for any \(x \in U\) and sufficiently small \(l\),
\[
I(h, x, l) - \sum_{p \in \mathcal{P}} A_p I_p(h, x, l) = \sum_{k=0}^{\infty} \frac{1}{(4k + 2)!} \left( \frac{\partial^{4k} h}{\partial \xi_1^{4k}}(x) \left( T_\theta^{(k)} - \sum_{p \in \mathcal{P}} A_p T_p^{(k)} \right) \left( \frac{l}{2} \right)^{4k} \right) = 0.
\]

From the arbitrariness of \(l\), we know that for any \(k \geq N\),
\[
\frac{\partial^{4k} h}{\partial \xi_1^{4k}}(x) \left( T_\theta^{(k)} - \sum_{p \in \mathcal{P}} A_p T_p^{(k)} \right) = 0.
\]

Then by the definition of \(N\),
\[
T_\theta^{(N)} - \sum_{p \in \mathcal{P}} A_p T_p^{(N)} \neq 0,
\]
thus
\[
\frac{\partial^{4N} h}{\partial \xi_1^{4N}}(x) = 0, \forall x \in U.
\]

Hence from (2.12), we can expand \( h \) near \( x \) as a polynomial harmonic function of degree no more than \( 4N \) as follows
\[
h(\xi) = h(x) + \sum_{n=1}^{4N-1} \left( \frac{\partial^n h}{\partial \xi_1^n} \right) \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \frac{(\xi_1 - x_1)^{n-2j}}{(n-2j)!} \frac{(\xi_2 - x_2)^{2j}}{(2j)!}
\]
\[
+ \sum_{n=1}^{4N} \left( \frac{\partial^n h}{\partial \xi_1^{n-1} \partial \xi_2} \right) \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j \frac{(\xi_1 - x_1)^{n-2j-1}}{(n-2j-1)!} \frac{(\xi_2 - x_2)^{2j+1}}{(2j+1)!}.
\]

Since \( h \) is harmonic on \( \Omega \), \( h \) is real analytic in \( \Omega \). Then from the principle of analytic continuation, we know that the above identity is valid for any \( x \in \Omega \). \( \square \)

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