ON FINITE GROUPS WITH EXACTLY TWO NON-ABELIAN CENTRALIZERS

SEKHAR JYOTI BAISHYA

Abstract. In this paper, we characterize finite group $G$ with unique proper non-abelian element centralizer. This improves [5, Theorem 1.1]. Among other results, we have proved that if $C(a)$ is the proper non-abelian element centralizer of $G$ for some $a \in G$, then $\frac{G}{Z(G)}$ is the Fitting subgroup of $\frac{C(a)}{Z(G)}$, $C(a)$ is the Fitting subgroup of $G$ and $G' \subseteq C(a)$, where $G'$ is the commutator subgroup of $G$.

1. Introduction

Throughout this paper $G$ is a finite group with center $Z(G)$ and commutator subgroup $G'$. Given a group $G$, let Cent($G$) denote the set of centralizers of $G$, i.e., $\text{Cent}(G) = \{ C(x) \mid x \in G \}$, where $C(x)$ is the centralizer of the element $x$ in $G$. The study of finite groups in terms of $|\text{Cent}(G)|$, becomes an interesting research topic in last few years. Starting with Belcastro and Sherman [15] in 1994 many authors have been studied and characterised finite groups $G$ in terms of $|\text{Cent}(G)|$. More information on this and related concepts may be found in [1, 3, 4, 6–14, 17, 21–23].

Amiri and Rostami [5] in 2015 introduced the notion of nacent($G$) which is the set of all non-abelian centralizers of $G$. Schmidt [19] characterized all groups $G$ with $|\text{nacent}(G)| = 1$ which are called CA-groups. The authors in [5] initiated the study of finite groups with $|\text{nacent}(G)| = 2$ and proved the following result ([5, Theorem 1.1]):

Theorem 1.1. Let $G$ be a finite group such that $|\text{nacent}(G)| = 2$. If $C(a)$ is a proper non-abelian centralizer for some $a \in G$, then one of the following assertions hold:

(a) $\frac{G}{Z(G)}$ is a $p$-group for some prime $p$.
(b) $C(a)$ is the Fitting subgroup of $G$ of prime index $p$, $p$ divides $|C(a)|$ and $|\text{Cent}(G)| = |\text{Cent}(C(a))| + j + 1$, where $j$ is the number of distinct centralizers $C(g)$ for $g \in G \setminus C(a)$.

(c) $\frac{G}{Z(G)}$ is a Frobenius group with cyclic Frobenius complement $\frac{C(x)}{Z(G)}$ for some $x \in G$.

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In this paper, we revisit finite groups $G$ with $|\text{nacent}(G)| = 2$ and improve this result. Among other results, we have also proved that if $C(a)$ is the proper non-abelian element centralizer of $G$ then $\frac{C(a)}{Z(G)}$ is the Fitting subgroup of $\frac{G}{Z(G)}$. $C(a)$ is the Fitting subgroup of $G$ and $G' \in C(a)$.

2. The main results

In this section, we prove the main results of the paper. We make the following Remark from [24, Pp. 571–575] which will be used in the sequel.

**Remark 2.1.** A collection $\Pi$ of non-trivial subgroups of a group $G$ is called a partition if every non-trivial element of $G$ belongs to a unique subgroup in $\Pi$. If $|\Pi| = 1$, the partition is said to be trivial. The subgroups in $\Pi$ are called components of $\Pi$. Following Miller, if $\Pi$ is a non-trivial partition of a non-abelian $p$ group $G$ ($p$ a prime), then all the elements of $G$ having order $> p$ belongs to the same component of $\Pi$.

A partition $\Pi$ of a group $G$ is said to be normal if $g^{-1}Xg \in \Pi$ for every $X \in \Pi$ and $g \in G$. A non-trivial partition $\Pi$ of a group $G$ is said to be elementary if $G$ has a normal subgroup $N$ such that all cyclic subgroups which are not contained in $N$ have order $p$ ($p$ a prime) and are components of $\Pi$. All normal non-trivial partitions of a $p$ group of exponent $> p$ are elementary.

A non-trivial partition $\Pi$ of a group $G$ is said to be non-simple if there exists a proper normal subgroup $N$ of $G$ such that for every component $X \in \Pi$, either $X \leq N$ or $X \cap N = 1$. Let $G$ be a group and $\Pi$ a normal non-trivial partition. Suppose $\Pi$ is not a Frobenious partition and is non-simple. Then $G$ has a normal subgroup $K$ of index $p$ ($p$ a prime) in $G$ which is generated by all elements of $G$ having order $\neq p$. So $\Pi$ is elementary.

Let $G$ be a group and $p$ be a prime. We recall that the subgroup generated by all the elements of $G$ whose order is not $p$ is called the Hughes subgroup and denoted by $H_p(G)$. The group $G$ is said to be a group of Hughes-Thompson type if $G$ is not a $p$ group and $H_p(G) \neq G$ for some prime $p$. In such a group we have $|G : H_p(G)| = p$ and $H_p(G)$ is nilpotent.

We now determine the structure of finite groups $G$ with $|\text{nacent}(G)| = 2$ which improves ([11, Theorem 1.1]).

**Theorem 2.2.** Let $G$ be a finite group and $a \in G \setminus Z(G)$. Then $\text{nacent}(G) = \{G, C(a)\}$ if and only if one of the following assertions hold:

(a) $\frac{G}{Z(G)}$ is a non-abelian $p$-group of exponent $> p$ ($p$ a prime), $|\frac{G}{Z(G)} : H_p(\frac{G}{Z(G)})| = p$, $H_p(\frac{G}{Z(G)}) = \frac{C(a)}{Z(G)}$, $|\frac{C(a)}{Z(G)}| = p$ for any $x \in G \setminus C(a)$ and $C(a)$ is a CA-group.

(b) $\frac{G}{Z(G)}$ is a group of Hughes-Thompson type, $H_p\left(\frac{G}{Z(G)}\right) = \frac{C(a)}{Z(G)}$ ($p$ a prime), $|\frac{C(a)}{Z(G)}| = p$ for any $x \in G \setminus C(a)$ and $C(a)$ is a CA-group.
(c) $\frac{G}{Z(G)} = \frac{C(a)}{Z(G)} \times \frac{C(x)}{Z(G)}$ is a Frobenius group with Frobenius Kernel $\frac{C(a)}{Z(G)}$, cyclic Frobenius Complement $\frac{C(x)}{Z(G)}$ for some $x \in G \setminus C(a)$ and $C(a)$ is a CA-group.

Proof. Let $G$ be a finite group such that $nacent(G) = \{G, C(a)\}$, $a \in G \setminus Z(G)$. Then clearly $C(a)$ is a CA-group.

Note that we have $C(s) \subseteq C(a)$ for any $s \in C(a) \setminus Z(G)$, $C(a) \cap C(x) = Z(G)$ for any $x \in G \setminus C(a)$ and $C(x) \cap C(y) = Z(G)$ for any $x, y \in G \setminus C(a)$ with $C(x) \neq C(y)$. Hence $\Pi = \{\frac{C(a)}{Z(G)}, \frac{C(x)}{Z(G)} | x \in G \setminus C(a)\}$ is a non-trivial partition of $\frac{G}{Z(G)}$. In the present scenario we have $(gZ(G))^{-1}\frac{C(a)}{Z(G)}gZ(G) = \frac{g^{-1}C(a)g}{Z(G)} = \frac{C(g^{-1}ag)}{Z(G)}$ for any $gZ(G) \in \frac{G}{Z(G)}$ and $\frac{C(a)}{Z(G)} \cap \frac{G}{Z(G)}$. Therefore $(gZ(G))^{-1}XgZ(G) \in \Pi$ for every $X \in \Pi$ and $gZ(G) \in \frac{G}{Z(G)}$. Hence $\Pi$ is a normal non-simple partition of $\frac{G}{Z(G)}$.

In the present scenario, if $\Pi$ is a Frobenius partition of $\frac{G}{Z(G)}$, then $\frac{G}{Z(G)} = \frac{C(a)}{Z(G)} \times \frac{C(x)}{Z(G)}$ is a Frobenius group with Frobenious Kernel $\frac{C(a)}{Z(G)}$ and cyclic Frobenius Complement $\frac{C(x)}{Z(G)}$ for some $x \in G \setminus C(a)$.

Next, suppose $\Pi$ is not a Frobenius partition. Then in view of Remark 2.1, $\frac{G}{Z(G)}$ has a normal subgroup of index $p$ ($p$ a prime) in $\frac{G}{Z(G)}$ which is generated by all elements of $\frac{G}{Z(G)}$ having order $\neq p$.

In the present situation if $\frac{G}{Z(G)}$ is not a $p$ group ($p$ a prime), then in view of Remark 2.1 $\frac{G}{Z(G)}$ is a group of Hughes-Thompson type and $\Pi$ is elementary. That is $\frac{G}{Z(G)}$ has a normal subgroup $\frac{K}{Z(G)}$ such that all cyclic subgroups which are not contained in $\frac{K}{Z(G)}$ have order $p$ ($p$ a prime) and are components of $\Pi$. In the present scenario we have $\frac{K}{Z(G)} = H_p(\frac{G}{Z(G)})$. Therefore $\Pi$ has $\frac{|G|}{p}$ components of order $p$ and these are precisely $\frac{C(x)}{Z(G)}, x \in G \setminus C(a)$. Consequently, we have $H_p(\frac{G}{Z(G)}) = \frac{C(a)}{Z(G)}$.

On the other hand, if $\frac{G}{Z(G)}$ is a $p$ group ($p$ a prime), then in view of Remark 2.1 $\frac{G}{Z(G)}$ is non-abelian of exponent $> p$ and $\frac{G}{Z(G)} : H_p(\frac{G}{Z(G)}) = p$. In the present situation by Remark 2.1 $\Pi$ is elementary. Therefore using Remark 2.1 again, $H_p(\frac{G}{Z(G)}) = \frac{C(a)}{Z(G)}$ and all cyclic subgroups which are not contained in $\frac{C(a)}{Z(G)}$ have order $p$ and are components of $\Pi$. Therefore $\Pi$ has $\frac{|G|}{p}$ components of order $p$ and these are precisely $\frac{C(x)}{Z(G)}, x \in G \setminus C(a)$. Consequently, we have $H_p(\frac{G}{Z(G)}) = \frac{C(a)}{Z(G)}$.

Conversely, suppose $G$ is a finite group such that one of (a), (b) or (c) holds. Then it is easy to see that $nacent(G) = \{G, C(a)\}$ for some $a \in G \setminus Z(G)$. □

As an immediate consequence we have the following result. Recall that for a finite group $G$, the Fitting subgroup denoted by $F(G)$ is the largest normal nilpotent subgroup of $G$. 
Theorem 2.3. Let $G$ be a finite group with a unique proper non-abelian centralizer $C(a)$ for some $a \in G$. Then we have

\begin{itemize}
  \item[(a)] $|\text{Cent}(G)| = |\text{Cent}(C(a))| + \frac{|G|}{p} + 1$, (p a prime) or $|\text{Cent}(C(a))| + \frac{|C(a)|}{Z(G)} + 1$.
  \item[(b)] $G' \subseteq C(a)$.
  \item[(c)] $\frac{C(a)}{Z(G)}$ is the Fitting subgroup of $\frac{G}{Z(G)}$.
  \item[(d)] $C(a)$ is the Fitting subgroup of $G$.
  \item[(e)] $C(a) = P \times A$, where $A$ is an abelian subgroup and $P$ is a CA-group of prime power order.
  \item[(f)] $\frac{G}{C(a)}$ is cyclic.
\end{itemize}

Proof. (a) In view of Theorem 2.2 if $\frac{G}{Z(G)}$ is a Frobenius group then by [3, Proposition 3.1], we have $|\text{Cent}(G)| = |\text{Cent}(C(a))| + \frac{|G|}{p} + 1$.

On the otherhand, if $\frac{G}{Z(G)}$ is not a Frobenius group, then it follows from the proof of Theorem 2.2 that the non-trivial partition $\Pi = \{\frac{C(x)}{Z(G)} \mid x \in G \setminus C(a)\}$ of $\frac{G}{Z(G)}$ has $\frac{|G|}{p}$ components of order $p$ and these are precisely $\frac{C(x)}{Z(G)}$, $x \in G \setminus C(a)$. Hence $|\text{Cent}(G)| = |\text{Cent}(C(a))| + \frac{|G|}{p} + 1$.

(b) If $\frac{G}{Z(G)}$ is a Frobenius group, then by Theorem 2.2, $\frac{G}{Z(G)} = \frac{C(a)}{Z(G)} \rtimes \frac{C(x)}{Z(G)}$ with cyclic Frobenius complement $\frac{C(x)}{Z(G)}$ for some $x \in G \setminus C(a)$. Therefore $\frac{G}{C(a)}$ is cyclic and hence $G' \subseteq C(a)$.

On the otherhand, if $\frac{G}{Z(G)}$ is not a Frobenius group, then it follows from Theorem 2.2 that $C(a) \triangleleft G$ and $\frac{G}{C(a)} = p$ (p a prime). Hence $G' \subseteq C(a)$.

(c) If $\frac{G}{Z(G)}$ is a Frobenius group, then by Theorem 2.2, $\frac{G}{Z(G)} = \frac{C(a)}{Z(G)} \rtimes \frac{C(x)}{Z(G)}$ with cyclic Frobenius complement $\frac{C(x)}{Z(G)}$ for some $x \in G \setminus C(a)$. In the present scenario by [18, Pp. 3], we have $\frac{C(a)}{Z(G)}$ is the Fitting subgroup of $\frac{G}{Z(G)}$.

On the otherhand, if $\frac{G}{Z(G)}$ is not a Frobenius group, then it follows from Theorem 2.2 that $H_p(\frac{G}{Z(G)}) = \frac{C(a)}{Z(G)}$ and $\frac{G}{Z(G)} : H_p(\frac{G}{Z(G)}) = p$. Hence $\frac{C(a)}{Z(G)}$ is the Fitting subgroup of $\frac{G}{Z(G)}$, noting that we have $C(a) \triangleleft G$.

(d) It follows from (c) noting that $F(\frac{G}{Z(G)}) = \frac{F(G)}{Z(G)} = \frac{C(a)}{Z(G)}$.

(e) Using (d) we have $C(a)$ is a nilpotent CA-group. Therefore using [2, Theorem 3.10 (5)], $C(a) = P \times A$, where $A$ is an abelian subgroup and $P$ is a CA-group of prime power order.

(f) It is clear from Theorem 2.2 that if $\frac{G}{Z(G)}$ is a Frobenius group, then $\frac{G}{C(a)}$ is cyclic and $|\frac{G}{C(a)}| = p$, (p a prime) otherwise. \qed
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S. J. Baishya, Department of Mathematics, Pandit Deendayal Upadhyaya Adarsha Mahavidyalaya, Behali, Biswanath-784184, Assam, India.

Email address: sekharnehu@yahoo.com