THE FORWARD SELF-SIMILAR SOLUTIONS OF THE FRACTIONAL NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we are devoted to the study of forward self-similar solutions to the 3-D Navier-Stokes equations with the fractional diffusion \((-\Delta)^\alpha\). First, we construct a global-time forward self-similar solutions to the fractional Navier-Stokes equations with \(5/6 < \alpha \leq 1\) for arbitrarily large self-similar initial data by making use of the blow-up argument. Moreover, we prove this solution is smooth in \(\mathbb{R}^3 \times (0, +\infty)\). In particular, when \(\alpha = 1\), we prove that the solution constructed by Korobkov-Tsai [23, Anal. PDE 9 (2016), 1811-1827] satisfies the decay estimate by establishing the precisely regularity of solution for the corresponding elliptic system, which implies this solution enjoys the same property with that solution was constructed in [17, Jia and Šverák, Invent. Math. 196 (2014), 233-265].

1. Introduction

In this paper, we consider the generalized incompressible Navier-Stokes equations with fractional Laplace operator:

\[
\begin{align*}
\frac{du}{dt} + (-\Delta)^\alpha u + u \cdot \nabla u + \nabla p &= 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^3 \\
\text{div} u &= 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^3
\end{align*}
\]

with initial condition

\[
u(x, 0) = u_0(x) \quad x \in \mathbb{R}^3.
\]

Here the column vector \(u = (u_1, u_2, u_3)^t\) denotes the velocity field, the scalar function \(p\) stands for the pressure which can be recovered at least formally from \(u\) via Calderón-Zygmund operators, and the fractional Laplacian \((-\Delta)^\alpha\) is a non-local operator defined in terms of the Fourier transform

\[
(-\Delta)^\alpha u(\xi) = (|\xi|^{2\alpha}) \hat{u}(\xi).
\]

Recently, there are increasing interests for studying the fractional Navier-Stokes equations (1.1), since they naturally appear in hydrodynamics, statistical mechanics, physiology, certain combustion models, and so on [32, 40]. As a simplified model of Eqs. (1.1), the following fractional Burgers equation has been studied by many mathematicians and physicists such as Biler-Funaki-Woyczynski [4] and Kiselev-Nazarov-Scherenber [20],

\[
u_t + (-\Delta)^\alpha u + uu_x = 0, \quad t > 0, \quad u(x, 0) = \varphi(x),
\]

where \(u(x, t) : \mathbb{R} \times (0, +\infty) \to \mathbb{R}\). For the fractional Navier-Stokes equations (1.1), the existence and uniqueness of solutions have been established by Wu [41] in the framework of Besov spaces, and by Zhang [42] via probabilistic approach. More recently, Tang-Yu [37] established the partial regularity of the suitable weak solution of problem (1.1) in the \(L^2\)-framework, which can be viewed as a generalization of the CKN regularity criterion [9] for the following classical Navier-Stokes system:

\[
\begin{align*}
\frac{du}{dt} - \Delta u + u \cdot \nabla u + \nabla p &= 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^3 \\
\text{div} u &= 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^3
\end{align*}
\]
Similar to the above classical Navier-Stokes equations (1.3)-(1.4), the fractional Navier-Stokes system (1.1)-(1.2) also enjoys the scaling property. Specifically, if \((u, p)\) is the solution of equations (1.1)-(1.2), then, for all \(\lambda > 0\), \((u_\lambda, p_\lambda)\) is also the solution of equations (1.1) corresponding to the initial data \(u_{0\lambda}\), where

\[
u_\lambda(x, t) = \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t), \quad p_\lambda(x, t) = \lambda^{4\alpha-2} p(\lambda x, \lambda^{2\alpha} t)
\]

and

\[
u_{0\lambda}(x) = \lambda^{2\alpha-1} u_0(x).
\]

According to this scaling property, we want to investigate the solution which is invariant under the scaling. And we call this solution the self-similar solution which has two types: one is the forward self-similar solution, another is the backward self-similar solution. A forward self-similar solution is a solution on \(\mathbb{R}^3 \times (0, +\infty)\) such that for every \(\lambda > 0\),

\[
u(x, t) = \nu_\lambda(x, t) \quad \text{and} \quad p(x, t) = p_\lambda(x, t).
\]

A backward self-similar solution is a solution on \(\mathbb{R}^3 \times (-\infty, 0)\) such that for every \(\lambda > 0\),

\[
u(x, t) = \nu_\lambda(x, t) \quad \text{and} \quad p(x, t) = p_\lambda(x, t).
\]

The problem concerning existence of the self-similar blow-up solution of (1.3) was initially proposed by Leray [28]. According to the above definition, it is easy to verify that the null solution is a trivial backward self-similar solution. However, the nontrivial self-similar blow-up solution with finite energy does not exist, which was firstly proved by Nečas-Rǎuţicǎ-Šverák in [30]. Later, Tsai [39] further proved the nonexistence of the backward self-similar solutions with local finite energy. Also, an alternative proof of Leary’s open problem was shown in Escuriaza-Seregin-Šverák [11] which solved a famous problem concerning on regularity of \(L_{3,\infty}\)-solutions.

In contrast with the case of backward self-similar solutions, several results of nontrivial forward self-similar solutions were established in the past years. In [7, 8], Cannone-Meyer-Planchon firstly proved the existence and uniqueness of the small forward self-similar solutions in the framework of homogeneous Besov spaces, see also for examples Barraza [3] in Lorentz space \(L^{3,\infty}(\mathbb{R}^3)\), and Koch and Tataru [23] in \(\text{BMO}^{-1}(\mathbb{R}^5)\).

For the large initial values with scale-invariant, it seems to us that the perturbation argument such as the contraction mapping no longer works, and one attempts to seek other methods to establish existence of solution. Recently, Jia and Šverák [17] constructed a scale-invariant solution by developing so called local-in-space regularity estimates near the initial time and Leray-Schauder fixed-point theorem proposed by Schauder in 1927 and developed by Leray in 1933. Later, Korobkov-Tsai [23] gave another new construction method on this scale-invariant solution without pointwise bound via blow-up argument. For the sake of convenience, we first recall the framework which was developed in [17] and [23].

Formally, one can reduce the study of the problem (1.3) into that of the corresponding integral equation. More precisely, seeking a self-similar solution \(\nu(x, t)\) of (1.3) is equivalent to find a self-similar solution of

\[
u = e^{4\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}(\nabla \cdot (u \otimes u)) \, ds,
\]

where \(\mathbb{P} = \text{Id} - \nabla (\Delta)^{-1} \text{div}\) is called the Leray-Hopf projection onto the divergence-free vector fields. Using the scale-invariant of \(u_0\) and the self-similarity of \(u = t^{-1/2} U(t^{-1/2} x)\), the above problem is equivalent to find a solution \(U\) of

\[
U = V_0 - T(U)
\]

with

\[
V_0 = \int_{\mathbb{R}^3} G_1(x - z) u_0(z) \, dz
\]
and
\[ T(U) = \sum_{j=1}^{3} \int_{0}^{1} \int_{\mathbb{R}^3} \frac{1}{(1-\tau)^2} \partial_j \mathcal{O} \left( \frac{x-z}{(1-\tau)^{\frac{2}{3}}} \right) \cdot \tau^{-1} U_j \left( \frac{z}{\tau^{\frac{2}{3}}} \right) U \left( \frac{z}{\tau^{\frac{2}{3}}} \right) \, dz \, d\tau. \]

Here \( G_1(x) \) is the profile of heat kernel at \( t = 1 \) and \( \mathcal{O} = (\mathcal{O}_{j,k}) \) is Oseen’s kernel with
\[ \mathcal{O}_{j,k}(x) = \delta_{jk} G_1(x) + \Gamma \partial_j \partial_k G_1, \quad \Gamma(x) \text{ is Newton potential.} \]

To solve (1.5), it suffices to verify \( T \) satisfies all the requirements of the Leray-Schauder principle in some selected Banach space \( X \):

(i) \( T : X \rightarrow X \) is a continuous and compact operator.

(ii) There exists a constant \( C \) such that, for every \( \lambda \in [0, 1] \),
\[ U = \lambda(U_0 - T(U)) \implies \|U\|_{X} \leq C. \]

For (i), the key point is to find the suitable functional-analytic setup to obtain compactness of operator. For (ii), the most difficulty of this step to establish the a-priori estimate of solutions, which help us to apply the continuation method to solve (1.5). In order to overcome these difficulties, Jia and Šverák [17] developed the so-called “local-in-space regularity estimates near the initial time \( t = 0 \)” to establish the Hölder estimate for local-Leray solutions constructed by Lemarié-Rieusset [26]. This estimate enables them to obtain regularity of self-similar solutions outside the ball, and then they got a good asymptotic behavior of such solution for large \( x \), which ensures the operator \( T \) is compact in the suitable setting.

Our goal in this paper is to apply Leray-Schauder fixed-point theorem to construct a forward self-similar solution \( u(x, t) \) of (1.1) which takes the form
\[ u(x, t) = t^{\frac{2\alpha}{2\alpha - 3}} u \left( \frac{x}{t^{\frac{2}{3}}}, 1 \right) \triangleq t^{\frac{2\alpha}{2\alpha - 3}} U \left( \frac{x}{t^{\frac{2}{3}}} \right) \text{ with } U(x) = u(x, 1), \]
when the corresponding initial value \( u_0(x) \) satisfying following scale-invariant:
\[ u_0(x) = \lambda^{2\alpha - 1} u_0(\lambda x) \quad \text{for all } \lambda > 0. \]

Setting \( U_0 = e^{-(-\Delta)^{\alpha}} u_0 \), we easily find that the different \( V \triangleq U - U_0 \) solves the following hyperviscosity perturbation of the fractional elliptic equation
\[ (-\Delta)^{\alpha} V + \nabla P = \frac{2\alpha - 1}{2\alpha} V(x) + \frac{1}{2\alpha} x \cdot \nabla V - U_0 \cdot \nabla U_0 - (U_0 + V) \nabla V - V \cdot \nabla U_0. \]

According to the Leray-Schauder fixed point theorem, the main task is now to establish the regularity estimates for the solution of problem (1.6). To do this, we will meet two difficulties. One is that the argument of Tang-Yu [37] seems infeasible for Lemarié-Rieusset’s solution [26] in the framework of uniformly locally square space \( L^2_{\text{loc}}(\mathbb{R}^3) \). This leads to that the methods used in [17] doesn’t work for our problem. Another is that the fractional diffusion operator is a nonlocal operator. To overcome the both difficulties, we will adopt the following approximate regularity system of the fractional Navier-Stokes equations (1.1) by adding an artificial diffusion \( \epsilon \Delta V \)
\[ -\epsilon \Delta V + (-\Delta)^{\alpha} V + \nabla P = \lambda \left( \frac{2\alpha - 1}{2\alpha} V(x) + \frac{1}{2\alpha} x \cdot \nabla V - U_0 \cdot \nabla U_0 - (U_0 + V) \nabla V - V \cdot \nabla U_0 \right) \]

By the blow-up argument used in Korobkov-Tsai [23], we firstly show that \( V(x) \) of (1.7) satisfies the following a priori estimate
\[ \int_{\mathbb{R}^3} \left( \epsilon |\nabla V|^2 + |(-\Delta)^{\frac{\alpha}{2}} V|^2 + \frac{5 - 4\alpha}{4\alpha} |V|^2 \right) \, dx \leq C(U_0). \]

This a priori estimate helps us to prove that the equation (1.7) possess at least one self-similar distributional weak solution of problem (1.1)-(1.2). However, this uniform estimate is not enough to obtain the natural pointwise estimate of self-similar solution for \( \alpha = 1 \) which established in [17]. In fact, since \(|x|\) is not bounded, it is impossible to get the high regularity by the classical elliptic theory directly. This requires us to develop a new technique to obtain the high regularity.
of solution $V$. Firstly, choose a appropriate test function $\varphi$ in the weighted-$H^1(\mathbb{R}^3)$, and then we derive the following key estimate

$$|x|V(x) \in H^1(\mathbb{R}^3).$$

Based on this regularity, we can show the behavior of $V$ for large $|x|$, that is,

$$|V(x)| \leq \frac{C}{1 + |x|} \quad \text{for all } x \in \mathbb{R}^3.$$ 

With this decay estimate in hand, we eventually get by the property of the fundamental solution that

$$|V(x)| \leq \frac{C}{(1 + |x|)^3 \log(1 + |x|)} \quad \text{for all } x \in \mathbb{R}^3.$$ 

Now we state our main result as follows:

**Theorem 1.1.** Assume $\frac{5}{6} < \alpha \leq 1$. Let $u_0 = \frac{\sigma(x)}{|x|^2} = \sigma(x) = \sigma(|x|) \in L^\infty(S^2)$, which satisfies $\text{div} u_0 = 0$ in $\mathbb{R}^3 \setminus \{0\}$. Then problem (1.1)-(1.2) admits at least one forward self-similar solution $u \in BC_w([0, +\infty), L^{\frac{3}{2}}(\mathbb{R}^3))$ such that

- for each $p \in [2, \frac{6}{\alpha - 2\alpha}]$, $\|u(t) - e^{-t(-\Delta)^\alpha}u_0\|_{L^p(\mathbb{R}^3)} \leq Ct \frac{1}{(1 + \frac{3}{p})} \leq 1$ for $t \geq 0$,
- $u(x, t)$ is smooth in $\mathbb{R}^3 \times (0, +\infty)$,
- for $\alpha = 1$, then we have the following pointwise estimates

$$|u(x, t)| \leq \frac{C}{|x| + \sqrt{t}} \quad \text{and} \quad |u(x, t) - e^{t\Delta}u_0| \leq \frac{Ct}{|x|^3 + t^2} \log \left(1 + \frac{|x|}{\sqrt{t}}\right)$$

for all $(x, t) \in \mathbb{R}^3 \times (0, +\infty)$.

**Remark 1.1.** When $\alpha = 1$, the existence of forward self-similar solution was shown in [23] via the blowup argument. But they did not show that the solution enjoys the decay estimate (1.8) as proved in [17]. In Theorem 1.1 we obtain this estimate by developing the new weighted-$H^1(\mathbb{R}^3)$ of weak solution $V(x)$ to system (1.6). This answers the problem purposed in Korobkov-Tsai [23]. In other words, we give an alternative construction method of the existence of forward self-similar solution.

**Remark 1.2.** According to the scaling analysis, it is well-known that system (1.1) has the same scaling with the following nonlinear equation

$$u_t + (-\Delta)^\alpha u = |u|^{\frac{4\alpha - 1}{\alpha - 1}} - u \quad \text{in } \mathbb{R}^3 \times (0, +\infty).$$

Now we consider the stationary solution $U$ of problem (1.9), which solves

$$(-\Delta)^\alpha U = |U|^{\frac{4\alpha - 1}{\alpha - 1}} - U \quad \text{in } \mathbb{R}^3.$$ 

Denote kinetic energy $K \triangleq \|U\|_{H^\alpha(\mathbb{R}^3)}$ and potential energy $P \triangleq \|U\|_{L^{\frac{2(3\alpha - 1)}{2\alpha - 1}}(\mathbb{R}^3)}$. Thanks to the embedding theorem, we find that the kinetic energy can not control the potential energy if $\alpha < \frac{5}{4}$. Inspired by this analysis, we call system (1.9) is supercritical if $\alpha < \frac{5}{4}$. Because supercritical usually implies that the kinetic energy can not control the nonlinearity, we give a roughly explanation on condition $\alpha \in (\frac{5}{6}, 1]$ in Theorem 1.1.

**Remark 1.3.** The second decay in (1.8) has a logarithmic loss which is caused by the nonlocal term $p$. Whether it is optimal or not, which can leave a reader to think only.

**Remark 1.4.** In fact, we also proved that system (1.1)-(1.2) admits at least one forward self-similar solution $u = u_L + v$ for $\frac{5}{6} < \alpha \leq \frac{5}{4}$ such that $u_L \in BC_w([0, +\infty), L^{\frac{3}{2\alpha - 1}}(\mathbb{R}^3))$ and

$$\|v(t)\|_{L^p(\mathbb{R}^3)} \leq Ct \frac{1}{(1 + \frac{3}{p})} \leq 1$$

for each $p \in [2, \frac{6}{3 - 2\alpha}]$. 
Since $\frac{5}{8} < \alpha \leq \frac{5}{6}$, we see that $\frac{6}{3-2\alpha} \leq \frac{3}{2\alpha-1}$ and $\frac{1}{2\alpha}(1 + \frac{3}{p}) - 1 > 0$. This implies $v \in L^p_{\text{loc}}([0, +\infty); L^p(\mathbb{R}^3))$ for $p \in [2, \frac{6}{3-2\alpha}]$. And the solution $u$ satisfies problem (1.1)-(1.2) in the sense of distribution.

**Notation:** We first agree that $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. We denoted by $M^3$ the space of all real $3 \times 3$ matrices. Adopting summation over repeated Latin indices, running from 1 to 3, we denote

$$A : B = A_{ij}B_{ij}, \quad |A| = \sqrt{A : A}, \quad A = (A_{ij}), \quad B = (B_{ij}) \in M^3;$$

$$u \otimes v = (u_iv_j) \in M^3, \quad Au = (A_{ij}u_j) \in \mathbb{R}^3, \quad u, v \in \mathbb{R}^3 \text{ and } A \in M^3.$$

We define

$$\|f\|_{p, \lambda} \triangleq \sup_{x \in \mathbb{R}^3} \left( \int_{|x-y|<\lambda} |f(y)|^p \, dy \right)^{\frac{1}{p}},$$

$$L^p_{ul}(\mathbb{R}^3) \triangleq \{ f \in L^p_{\text{loc}}(\mathbb{R}^3); \|f\|_{p, 1} < \infty \}, \quad \|f\|_{L^p_{ul}(\mathbb{R}^3)} \triangleq \|f\|_{p, 1}.$$

The rest of the paper is structured as follows: Section 2 contains preliminaries which consist of some basic functional spaces, notations and some standard facts on non-local heat operator. In Section 3, we study the existence and regularity of solution for of the corresponding elliptic problem by establishing some a priori estimates, which is the core of our paper. In Section 4, we give the proof of the Theorem 1.1.

2. Preliminaries

2.1. Functional spaces, Littlewood-Paley theory and several useful lemmas. In this subsection, we firstly review the statement of functional spaces, see for example [12]. Let us begin by defining the space weak-$L^p(\mathbb{R}^3)$, denoted by $L^{p, \infty}(\mathbb{R}^3)$ as follows:

$$L^{p, \infty}(\mathbb{R}^3) \triangleq \{ u : m \{ x \in \mathbb{R}^3 : |u(x)| > s \} \leq \frac{A}{sp} \text{ with some } A > 0 \text{ and all } s > 0 \}$$

with norm

$$\| \cdot \|_{L^{p, \infty}(\mathbb{R}^3)} = \left( \sup_{s>0} sp \, m \{ x \in \mathbb{R}^3 : |u(x)| > s \} \right)^{\frac{1}{p}}.$$

Here $m$ denotes the Lebesgue measure on $\mathbb{R}^3$.

Next, we recall some basic function spaces in bounded domain. Let $\Omega$ be an open set in $\mathbb{R}^3$ and $D'(\Omega) = C^\infty_c(\Omega)$. Let $\alpha \in (0, 1)$ and define the fractional Sobolev space $H^\alpha(\Omega)$ as following

$$H^\alpha(\Omega) \triangleq \left\{ u \in D'(\Omega) : u(x) \in L^2(\Omega) \text{ and } \frac{|u(x) - u(y)|}{|x-y|^{\frac{3}{2}+\alpha}} \in L^2(\Omega \times \Omega) \right\}$$

with norm

$$\|u\|_{H^\alpha(\Omega)} \triangleq \left( \int_\Omega |u|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{3+2\alpha}} \, dx \, dy \right)^{\frac{1}{2}}.$$

In particular, when $\alpha = 1$, the Sobolev space $H^1(\Omega)$ can be defined as

$$H^1(\Omega) \triangleq \left\{ u(x) \in D'(\Omega) : u \in L^2(\Omega) \text{ and } \nabla u \in L^2(\Omega) \right\}$$

with norm

$$\|u\|_{H^1(\Omega)} \triangleq \left( \int_\Omega |\nabla u|^2 + |u|^2 \, dx \right)^{\frac{1}{2}}.$$

One easily see that $C^\infty_c(\Omega)$ is dense in $H^\alpha(\Omega)$. If $\Omega$ is a domain with Lipschitz boundary, then there exists a bounded linear extension operator from $H^\alpha(\Omega)$ to $H^\alpha(\mathbb{R}^3)$. Note that $H^\alpha(\mathbb{R}^3)$ with the norm $\| \cdot \|_{H^\alpha(\mathbb{R}^3)}$ is equivalent to the space

$$\left\{ u \in L^2(\mathbb{R}^3) : |\xi|^\alpha \mathcal{F}(u)(\xi) \in L^2(\mathbb{R}^3) \right\}$$

with the norm

$$\| \cdot \|_{L^2(\mathbb{R}^3)} + \| |\xi|^\alpha \mathcal{F}(\cdot)(\xi) \|_{L^2(\mathbb{R}^3)}.$$
where $F$ denotes the Fourier transform. It is known that (see [25]) there exists $C > 0$ depending only on $\alpha$ such that for $U \in H^1(\mathbb{R}^4_+, t^{-2\alpha} \, dx \, dt) \cap C(\mathbb{R}^4_+)$
\[
\|U(\cdot, 0)\|_{H^\alpha(\mathbb{R}^3)} \leq C\|U\|_{H^1(\mathbb{R}^4_+, t^{-2\alpha} \, dx \, dt)};
\]
where $H^1(\mathbb{R}^4_+, t^{-2\alpha} \, dx \, dt)$ is defined as
\[
H^1(\mathbb{R}^4_+, t^{-2\alpha} \, dx \, dt) \triangleq \{ U : U \in L^2(\mathbb{R}^4_+, t^{-2\alpha} \, dx \, dt) \text{ and } \nabla U \in L^2(\mathbb{R}^4_+, t^{-2\alpha} \, dx \, dt) \}\]
with
\[
\|U\|_{L^2(\mathbb{R}^4_+, t^{-2\alpha} \, dx \, dt)} \triangleq \left( \int_0^{+\infty} \int_{\mathbb{R}^3} t^{2\alpha} U^2(x, t) \, dx \, dt \right)^{\frac{1}{2}}.
\]
We know that every $U \in H^1(\mathbb{R}^4_+, t^{-2\alpha} \, dx \, dt)$ has well-defined trace $u \triangleq U(\cdot, 0) \in H^\alpha(\mathbb{R}^3)$ by a standard density argument.

We define $H^\alpha_0(\Omega)$ as the completion of $C^\infty_0(\Omega)$ under the norm \( \| \cdot \|_{H^\alpha(\mathbb{R}^3)} \). When $\alpha = 1$, $H^1_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ under the following equivalent norm
\[
\|u\|_{H^1_0(\Omega)} \triangleq \left( \int_{\Omega} |\nabla u|^2(x) \, dx \right)^{\frac{1}{2}}.
\]
We also denote
\[
H^1_{0,\sigma}(\Omega) \cap H^\alpha_{0,\sigma}(\Omega) = \{ f \in H^1_0(\Omega) \cap H^\alpha_0(\Omega) : \text{ div } f = 0 \}.
\]
Let $X$ be a Banach space, we denote by $X'$ the dual space of $X$ with respect to the norm
\[
\|f\|_{X'} \triangleq \sup_{\varphi \in X, \|\varphi\|_{X} \leq 1} \left| \int_{\Omega} f \varphi \, dx \right|, \text{ for any } f \in X'.
\]
For $q > 1$ denote by $D^{1,q}(\Omega)$ to be
\[
D^{1,q}(\Omega) \triangleq \left\{ f : f \in W^{1,q}_{\text{loc}}(\Omega) \text{ and } \|f\|_{D^{1,q}(\Omega)} = \|\nabla f\|_{L^q(\Omega)} < +\infty \right\}.
\]
Further, denote by $D^{1,2}(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $D^{1,2}(\Omega)$ and
\[
H(\Omega) \triangleq \{ u : u \in D^{1,2}(\Omega) \text{ and } \text{ div } u = 0 \}.
\]
Besides, when $\Omega$ is bounded and locally Lipschitz, $u \in D^{1,q}(\Omega)$ implies $u \in W^{1,q}(\Omega)$, for details one refers to [12]. Finally, we denote the Bessel potential space by
\[
H^\alpha_p(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^3) : (I - \Delta)^{\frac{\alpha}{2}} u \in L^p(\mathbb{R}^3) \right\},
\]
which is equipped with the norm
\[
\|u\|_{H^\alpha_p(\mathbb{R}^3)} = \| (I - \Delta)^{\frac{\alpha}{2}} u \|_{L^p(\mathbb{R}^3)},
\]
and the homogeneous space by
\[
\dot{H}^\alpha_p(\mathbb{R}^3) = \left\{ u \in D'(\mathbb{R}^3) : (-\Delta)^{\frac{\alpha}{2}} u \in L^p(\mathbb{R}^3) \right\}
\]
with the semi-norm
\[
\|u\|_{\dot{H}^\alpha_p(\mathbb{R}^3)} = \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^p(\mathbb{R}^3)}.
\]
Next, we review the so-called Littlewood-Paley decomposition described, e.g., in [6]. Suppose that $(\chi, \varphi)$ be a couple of smooth functions with values in $[0,1]$ such that supp $\chi \subset \{ \xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3} \}$, supp $\varphi \subset \{ \xi \in \mathbb{R}^3 : \frac{4}{3} \leq |\xi| \leq \frac{8}{3} \}$ and
\[
\chi(\xi) + \sum_{j \in \mathbb{N}} \varphi(2^{-j} \xi) = 1 \quad \forall \xi \in \mathbb{R}^3.
\]
For any $u \in S'(\mathbb{R}^3)$, let us define
\[
\Delta_{-1} u \triangleq \chi(D) u \quad \text{and} \quad \Delta_j u \triangleq \varphi(2^{-j} D) u \quad \forall j \in \mathbb{N}.
\]
Moreover, we can define the low-frequency cut-off:
\[ S_j u \triangleq \chi(2^{-j} D) u. \]

So, we easily find that
\[ u = \sum_{j \geq -1} \Delta_j u, \quad \text{in} \ S'(\mathbb{R}^3) \]
which corresponds to the \textit{inhomogeneous Littlewood-Paley decomposition}. In usual, we always use the following properties of quasi-orthogonality:
\[ \Delta_j \Delta_{j'} u \equiv 0 \quad \text{if} \quad |j - j'| \geq 2. \]
\[ \Delta_j (S_{j'-1} u \Delta_{j'} v) \equiv 0 \quad \text{if} \quad |j - j'| \geq 5. \]

We shall also use the \textit{homogeneous Littlewood-Paley operators} governed by
\[ \dot{S}_j u \triangleq \chi(2^{-j} D) u \quad \text{and} \quad \Delta_j u \triangleq \varphi(2^{-j} D) u \quad \forall j \in \mathbb{Z}. \]

We denoted by \( S_k' (\mathbb{R}^3) \) the space of tempered distributions \( u \) such that
\[ \lim_{j \to -\infty} \dot{S}_j u = 0 \quad \text{in} \ S'(\mathbb{R}^3). \]

The \textit{homogeneous Littlewood-Paley decomposition} can be written as
\[ u = \sum_{j \in \mathbb{Z}} \Delta_j u, \quad \text{in} \ S_k' (\mathbb{R}^3) \]

**Definition 2.1.** Assume that \( s \in \mathbb{R}, \ (p, q) \in [1, \infty]^2 \) and \( u \in S'(\mathbb{R}^3) \). Then we can define the \textit{homogeneous Besov spaces} as
\[ B^s_{p, q}(\mathbb{R}^3) \triangleq \{ u \in S'(\mathbb{R}^3) \mid \| u \|_{B^s_{p, q}(\mathbb{R}^3)} < +\infty \}, \]
where
\[ \| u \|_{B^s_{p, q}(\mathbb{R}^3)} \triangleq \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jq} \| \Delta_j u \|_{L^p(\mathbb{R}^3)}^q \right)^{1/q} & \text{if} \ q < +\infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \| \Delta_j u \|_{L^p(\mathbb{R}^3)} & \text{if} \ q = +\infty. \end{cases} \]

Before we conclude this section, we recall a useful Sobolev embedding theorem:

**Lemma 2.1** ([12]). Let \( 2 \leq q \leq 2^*_a = \frac{6}{3 - 2a} \) and \( u \in H^a(\Omega) \), then
\[ \| u \|_{L^q(\Omega)} \leq C \| u \|_{H^a(\Omega)}, \quad \Omega \subset \mathbb{R}^3 \quad \text{or} \quad \Omega = \mathbb{R}^3. \]

If \( 2 \leq q < 2^*_a \), and \( \Omega \subset \mathbb{R}^n \) is a bounded domain, then we have the following compact embedding
\[ H^a(\Omega) \hookrightarrow \hookrightarrow L^q(\Omega), \]
in other words, for every bounded sequence \( \{ u_k \} \subset H^a(\Omega) \), there exists a converging subsequence, still denote by \( \{ u_k \} \), such that
\[ \lim_{k \to +\infty} u_k(x) = u(x) \in L^q(\Omega). \]

**Lemma 2.2** ([15] [31]).

(i) Let \( 1 < p, q, r < \infty, \ 0 < s_1, s_2 \leq \infty, \ \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \) and \( \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}. \) Then there holds
\[ \| f * g \|_{L^{r, s}(\mathbb{R}^3)} \leq C(p, q, s_1, s_2) \| f \|_{L^{p, s_1}(\mathbb{R}^3)} \| g \|_{L^{p, s_2}(\mathbb{R}^3)}. \]

(ii) Let \( 0 < p, q, r \leq \infty, \ 0 < s_1, s_2 \leq \infty, \ \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \) and \( \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}. \) Then we have the \textit{Hölder inequality} for Lorentz spaces
\[ \| fg \|_{L^{r, s}(\mathbb{R}^3)} \leq C(p, q, s_1, s_2) \| f \|_{L^{p, s_1}(\mathbb{R}^3)} \| g \|_{L^{p, s_2}(\mathbb{R}^3)}. \]
Lemma 2.3 (M). (i) Let \( \varphi \in \mathcal{S}(\mathbb{R}^3) \), then there holds
\[
\| \varphi \ast f \|_{L^\infty(\mathbb{R}^3)} \leq C \| f \|_{1,1}, \quad \forall f \in L^1_{ul}(\mathbb{R}^3),
\]
where \( C \) is a positive constant independent of \( f \).

(ii) If \( m \geq 1 \), then
\[
\| f \|_{q,m}\lambda \leq (Cm^3)^{\frac{1}{q}} \| f \|_{q,\lambda}, \quad \forall f \in L^q_{ul}(\mathbb{R}^3) \quad \text{and} \quad \forall \lambda > 0.
\]

Lastly, we show some properties of solutions to the stationary Euler system, which are the key point of the blowup argument.

Lemma 2.4. Let \( \Omega \) be a connected domain in \( \mathbb{R}^3 \) with Lipschitz boundary, and the functions \( v \in H(\Omega) \) and \( p \in D^{1,2}(\Omega) \) satisfy the stationary Euler system
\[
\begin{align*}
&v \cdot \nabla v + \nabla p = 0 \quad \text{in} \quad \Omega, \\
&\text{div} \ v = 0 \quad \text{in} \quad \Omega, \\
&v = 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
Then
\[
\exists \ c \in \mathbb{R} \ \text{such that} \ p(x) \equiv c \ \text{for \ } H^2 \text{-almost all} \ x \in \partial \Omega,
\]
where \( H^m \) is denoted by the \( m \)-dimensional Hausdorff measure.

Proof. The proof just follows the ideas developed in [1, 2]. We present a proof in some detail of this lemma for the reader’s convenience.

Let \( z_0 \in \partial \Omega \) and choose a new orthogonal coordinate system \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \) centered at \( z_0 \) with the \( x_3 \)-axis pointing along the inner normal to \( \partial \Omega \) at \( z_0 \). Then \((v, p)\) satisfies
\[
v \cdot \tilde{\nabla} v + \tilde{\nabla} p = 0 \quad \text{in} \quad \Omega',
\]
where \( \tilde{\nabla} \triangleq \left( \frac{\partial}{\partial \tilde{x}_1}, \frac{\partial}{\partial \tilde{x}_2}, \frac{\partial}{\partial \tilde{x}_3} \right) \) and \( \Omega' \) is the domain of \( \Omega \) in the new coordinate system \( \tilde{x} \). For sufficient small \( \epsilon > 0 \), the boundary component \( \partial \Omega' \) is given locally by
\[
\tilde{x}_3 = g(\tilde{x}_1, \tilde{x}_2), \quad (\tilde{x}_1, \tilde{x}_2) \in \mathbb{B}_\epsilon(0) \subset \mathbb{R}^2, \quad \text{with} \quad g \in C^{0,1}(\mathbb{B}_\epsilon(0)).
\]
Let \( \delta > 0 \) be sufficiently small such that
\[
\mathcal{A} = A(\epsilon, \delta) = \left\{ (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) : (\tilde{x}_1, \tilde{x}_2) \in \mathbb{B}_\epsilon(0), \ \tilde{x}_3 \in (g(\tilde{x}_1, \tilde{x}_2), g(\tilde{x}_1, \tilde{x}_2) + \delta) \right\} \subset \Omega'.
\]
One easily estimates
\[
\int_{\mathcal{A}} \frac{|v \cdot \tilde{\nabla} v|}{|\tilde{x}_3 - g(\tilde{x}_1, \tilde{x}_2)|} \, d\tilde{x} \leq \left\| \tilde{\nabla} u \right\|_{L^2(\mathcal{A})} \left\| \frac{v}{|\tilde{x}_3 - g(\tilde{x}_1, \tilde{x}_2)|} \right\|_{L^2(\mathcal{A})}
\]
and
\[
\int_{\mathcal{A}} \frac{v^2}{|\tilde{x}_3 - g(\tilde{x}_1, \tilde{x}_2)|^2} \, d\tilde{x} = \int_{\mathbb{B}_\epsilon(0)} \frac{v^2}{|g(\tilde{x}_1, \tilde{x}_2) + \delta|} \, d\tilde{x} + \int_{g(\tilde{x}_1, \tilde{x}_2)}^{|g(\tilde{x}_1, \tilde{x}_2) + \delta|} \frac{v^2}{|\tilde{x}_3 - g(\tilde{x}_1, \tilde{x}_2)|^2} \, d\tilde{x}_3
\]
\[
\leq \int_{\mathbb{B}_\epsilon(0)} \frac{v^2}{|g(\tilde{x}_1, \tilde{x}_2) + \delta|} \, d\tilde{x} + \frac{1}{\left| g(\tilde{x}_1, \tilde{x}_2) - g(\tilde{x}_1, \tilde{x}_2) \right|^2} \left( \int_{g(\tilde{x}_1, \tilde{x}_2)}^{\delta} |\partial_\eta v(\tilde{x}', \eta)| \, d\eta \right)^2 \, d\tilde{x}_3 \leq 4 \int_{\mathcal{A}} |\partial_\eta v(\tilde{x}', \eta)|^2 \, d\tilde{x},
\]
where we have used the fact \( u(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = 0 \) at \( \tilde{x}_3 = g(\tilde{x}_1, \tilde{x}_2) \) and the following Hardy inequality that
\[
\int_a^b \left( \frac{w(x)}{x-a} \right)^2 \, dx \leq 4 \int_a^b (w'(x))^2 \, dx
\]
for all functions \( w(x) \in C^1((a, b)) \) which vanish at \( x = a \).

For any scalar function \( \phi(\tilde{x}_1, \tilde{x}_2) \in C^0_{\infty}(\mathbb{B}_\epsilon(0)) \) and \( g(\tilde{x}_1, \tilde{x}_2) \in W^{1,\infty}(\mathbb{B}_\epsilon(0)) \), we have that for \( i = 1, 2 \)
\[
\int_\mathcal{A} \partial_{\tilde{x}_i} \phi p(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \, d\tilde{x}_1 \, d\tilde{x}_2 \, d\tilde{x}_3' = \int_{\mathbb{B}_\epsilon(0)} \int_0^\delta \partial_{\tilde{x}_i} \phi p(\tilde{x}_1, \tilde{x}_2, g(\tilde{x}_1, \tilde{x}_2) + x_3') \, d\tilde{x}_1 \, d\tilde{x}_2 \, dx_3'
\]

In this subsection, we first focus on the following linear equation:

\[ \Delta U - \frac{1}{2} (x \cdot \nabla U + U) = f(x) \quad \text{in} \quad \mathbb{R}^3. \]

In order to find the solution of (2.2), we define inspired by the homogeneous principle that

\[ U(x) = \int_0^1 \frac{1}{(4\pi s)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4(1-s)^2}} f \left( \frac{x-y}{\sqrt{1-s}} \right) dyds. \]

Here and what in follows, we denote

\[ \Phi(x,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4t}} \quad \text{for all} \quad (x,t) \in \mathbb{R}^3 \times (0, +\infty). \]

Setting \( \tilde{f}(x,t) = \frac{1}{t^2} f \left( \frac{x}{\sqrt{t}} \right) \), one has

\[ U(x) = \int_0^1 \int_{\mathbb{R}^3} \Phi(y,s) \tilde{f}(x-y,1-s) dyds. \]

Consequently this convolution should be solution of equation (2.2).

**Proposition 2.5.** Let \( U \) defined in (2.3), and \( f \in C^2(\mathbb{R}^3) \) satisfying

\[ \sup_{x \in \mathbb{R}^3} |x|^3 |f(x)| < +\infty. \]

Then \( U \in C^2(\mathbb{R}^3) \) solves the linear elliptic equation

\[ -\Delta U - \frac{1}{2} (x \cdot \nabla U + U) = f(x) \quad \text{in} \quad \mathbb{R}^3. \]

**Proof.** According to the representation (2.3), it is easy to show that \( U \in C^2(\mathbb{R}^3) \). So we need to show that \( u(x,t) \) is the solution of equation (2.2). We compute

\[ -\Delta U = \int_0^1 \int_{\mathbb{R}^3} ( -\Delta_y ) \Phi(y,s) \tilde{f}(x-y,1-s) dyds. \]
and
\[
\frac{1}{2} (x \cdot \nabla U + U) = \int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) \left( -\frac{1}{2} x \cdot \nabla_x - \frac{1}{2} \right) \tilde{f}(x - y, 1 - s) \, dy \, ds \\
= -\frac{1}{2} \int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) ((x - y) \cdot \nabla_x) \tilde{f}(x - y, 1 - s) \, dy \, ds \\
- \frac{1}{2} \int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) (y \cdot \nabla_x) \tilde{f}(x - y, 1 - s) \, dy \, ds \\
- \frac{1}{2} \int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) \tilde{f}(x - y, 1 - s) \, dy \, ds.
\]
\[(2.6)\]

Since
\[
\frac{1}{2} (x - y) \cdot \nabla_x \left( \tilde{f}(x - y, 1 - s) \right) = \frac{(x - y)}{2(1 - s)^2} \cdot \nabla f \left( \frac{x - y}{\sqrt{1 - s}} \right) \\
= (1 - s) \partial_s \left( (1 - s) \tilde{f}(x - y, 1 - s) \right) - \frac{3}{2} \tilde{f}(x - y, 1 - s) \\
= \partial_s \left( (1 - s) \tilde{f}(x - y, 1 - s) \right) - \frac{1}{2} \tilde{f}(x - y, 1 - s),
\]
we readily have
\[(2.7)\]
\[
-\frac{1}{2} (x - y) \cdot \nabla_x \left( \tilde{f}(x - y, 1 - s) \right) - \frac{1}{2} \tilde{f}(x - y, 1 - s) = -\partial_s \left( (1 - s) \tilde{f}(x - y, 1 - s) \right).
\]

On the other hand, we see that
\[(2.8)\]
\[
-\frac{1}{2} \int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) (y \cdot \nabla_y) \tilde{f}(x - y, 1 - s) \, dy \, ds \\
= \frac{1}{2} \int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) \tilde{f}(x - y, 1 - s) \, dy \, ds \\
- \frac{3}{2} \int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) \tilde{f}(x - y, 1 - s) \, dy \, ds - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^3} (y \cdot \nabla_y) \Phi(y, s) \tilde{f}(x - y, 1 - s) \, dy \, ds.
\]

A simple calculation yields
\[(2.9)\]
\[
\frac{1}{2} (y \cdot \nabla_y) \Phi(y, s) = -\frac{1}{(4\pi s)^{\frac{3}{2}}} \frac{|y|^2}{4s} e^{-\frac{|y|^2}{4s}} = -s \Delta_y \Phi(y, s) - \frac{3}{2} \Phi(y, s).
\]

Plugging (2.9) into (2.8) leads to
\[(2.10)\]
\[
-\frac{1}{2} \int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) (y \cdot \nabla_x) \tilde{f}(x - y, 1 - s) \, dy \, ds = \int_0^1 \int_{\mathbb{R}^3} (s \Delta_y) \Phi(y, s) \tilde{f}(x - y, 1 - s) \, dy \, ds.
\]

Inserting (2.7) and (2.10) into (2.6), we readily have
\[(2.11)\]
\[
-\frac{1}{2} (x \cdot \nabla U + U) = -\int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) \partial_s \left( (1 - s) \tilde{f}(x - y, 1 - s) \right) \, dy \, ds \\
+ \int_0^1 \int_{\mathbb{R}^3} (s \Delta_y) \Phi(y, s) \tilde{f}(x - y, 1 - s) \, dy \, ds.
\]

With (2.5) and (2.11) in hand, we find that
\[(2.12)\]
\[
-\Delta U - \frac{1}{2} (x \cdot \nabla U + U) = -\int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) \partial_s \left( (1 - s) \tilde{f}(x - y, 1 - s) \right) \, dy \, ds \\
- \int_0^1 \int_{\mathbb{R}^3} (\Delta_y) \Phi(y, s) \left( (1 - s) \tilde{f}(x - y, 1 - s) \right) \, dy \, ds.
\]
Integrating by parts with respect to time \( t \), we have
\[
- \int_0^1 \int_{\mathbb{R}^3} \Phi(y, s) \partial_s \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy \, ds
\]
\[
= \int_0^1 \int_{\mathbb{R}^3} \left( \partial_s \right) \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy \, ds
\]
\[
- \lim_{s \to 1^-} \int_{\mathbb{R}^3} \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy
\]
\[
+ \lim_{s \to 0^+} \int_{\mathbb{R}^3} \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy.
\]
Plugging this estimate in (2.12) gives
\[
-\Delta U - \frac{1}{2} \left( x \cdot \nabla U + U \right) = \int_0^1 \int_{\mathbb{R}^3} \left( \partial_s - \Delta_y \right) \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy \, ds
\]
\[
- \lim_{s \to 1^-} \int_{\mathbb{R}^3} \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy
\]
\[
+ \lim_{s \to 0^+} \int_{\mathbb{R}^3} \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy
\]
\[
= - \lim_{s \to 1^-} \int_{\mathbb{R}^3} \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy
\]
\[
+ \lim_{s \to 0^+} \int_{\mathbb{R}^3} \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy.
\]
Thanks to the condition (2.4), we know that
\[
\lim_{s \to 1^-} \left( (1 - s) \bar{f}(x - y, 1 - s) \right) = 0.
\]
This enables us to conclude
\[
\lim_{s \to 1^-} \int_{\mathbb{R}^3} \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy = 0.
\]
On the other hand, we observe that
\[
\lim_{s \to 0^+} \int_{\mathbb{R}^3} \Phi(y, s) \left( (1 - s) \bar{f}(x - y, 1 - s) \right) \, dy = \int_{\mathbb{R}^3} \delta(y) \bar{f}(x - y, 1) \, dy = f(x).
\]
Therefore we finally obtain
\[
-\Delta U - \frac{1}{2} \left( x \cdot \nabla U + U \right) = f(x) \quad \text{for all } x \in \mathbb{R}^3.
\]
So we finish the proof of the proposition. \( \square \)

Next, we will investigate some properties of the linear fractional diffusion equation. Let \( 0 < \alpha \leq 1 \), and \( u \) be the solution to the fractional diffusion equation
\[
\partial_t u + (-\Delta)^\alpha u = f(x, t) \quad \text{in } \mathbb{R}^3 \times (0, +\infty)
\]
\[
u(x, 0) = \varphi(x) \quad \text{in } \mathbb{R}^3
\]
where \( f \in C_0^\infty(\mathbb{R}^3 \times [0, +\infty)) \) and \( \varphi(x) \in C_0^\infty(\mathbb{R}^3) \).

By Duhamel formula, one writes
\[
(2.13) \quad u(x, t) = G_t^\alpha \ast \varphi + \int_0^t G_t^\alpha(x - y) f(s, y) \, ds, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3,
\]
where
\[
G_t^\alpha(x) = \mathcal{F}^{-1} \left( e^{-t|\xi|^{2\alpha}} \right) \quad \text{for all } t > 0
\]
where $\mathcal{F}^{-1}$ denote the inverse Fourier transform. The function $G_t^\alpha$ is the probability density function of a spherically symmetric $2\alpha$-stable process whose generator corresponds to the fractional Laplacian $(-\Delta)^\alpha$:

$$\int_{\mathbb{R}^3} G_t^\alpha(x) \, dx = 1 \quad \text{for all } t > 0.$$ 

**Lemma 2.6** ([5] [20]).

(i) For $(x, t) \in \mathbb{R}^n \times (0, +\infty)$, we have

$$G_t^\alpha(x) = t^{-\frac{\alpha}{2\gamma}} G_t^\alpha\left(\frac{x}{t^{\frac{\gamma}{2\alpha}}}\right),$$

where $G_t^\alpha(x)$ is a smooth strictly positive radial function on $\mathbb{R}^n$, and

$$G_t^\alpha(x) = \left((2\pi)^{n} |x|^{\frac{n-1}{2}}\right)^{-1} \int_0^\infty e^{-t^{2\alpha} \frac{\alpha}{2\gamma} J_{\mu}(|x|t)} dt,$$

where $J_{\mu}$ denotes the Bessel function of first kind of order $\mu$.

(ii) $\lim_{|x| \to +\infty} |x|^{n+2\alpha} G_t^\alpha(x) = C_{\alpha,n} \sin \alpha \pi.$

(iii) $|\nabla^k G_t^\alpha(x)| \leq t(t^{\frac{\gamma}{2\alpha}} + |x|)^{-n-2\alpha-k}.$

(iv) $|(-\Delta)^{\alpha} G_t^\alpha(x)| \leq (t^{\frac{\gamma}{2\alpha}} + |x|)^{-n-2\alpha}.$

**Lemma 2.7.** Let $\varphi \in L^{r,\infty}(\mathbb{R}^n)$ with $1 < r < +\infty$. Then we have

a) for each $p \geq r$, $\|G_t^\alpha \ast \varphi\|_{L^p(\mathbb{R}^n)} \leq C(n, p, r) t^{-\frac{n}{p}} \|\varphi\|_{L^{r,\infty}(\mathbb{R}^n)}$;

b) $u(x, t) = G_t^\alpha \ast \varphi(x) \in BC((0, +\infty), L^{r,\infty}(\mathbb{R}^n))$.

**Proof.** a) Since

$$\|G_t^\alpha\|_{L^p(\mathbb{R}^n)} \leq C \|G_t^\alpha\|_{L^1(\mathbb{R}^n)} = Ct^{-(1-\frac{1}{p})\frac{n}{2\gamma}} \|\nabla^\alpha G_t^\alpha\|_{L^p(\mathbb{R}^n)}$$

we obtain by the generalized Young inequality in Lemma 2.2 that for all $p \in [r, +\infty)$,

$$\|G_t^\alpha \ast \varphi\|_{L^{p,\infty}(\mathbb{R}^n)} \leq \|G_t\|_{L^{p,\infty}(\mathbb{R}^n)} \|\varphi\|_{L^{r,\infty}(\mathbb{R}^n)} \leq C \|\nabla^\alpha G_t^\alpha\|_{L^p(\mathbb{R}^n)} \|\varphi\|_{L^{r,\infty}(\mathbb{R}^n)},$$

where $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r}$. This inequality together with the interpolation theorem yields the first desired result.

b) Now let $v \in L^{(r',1)}(\mathbb{R}^n)$ which is the dual space of $L^{(r,\infty)}(\mathbb{R}^n)$. We observe that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\langle G_t^\alpha \ast \varphi - \varphi, v \rangle = \|\varphi \|_{L^{r,\infty}(\mathbb{R}^n)} \|G_t^\alpha \ast v - v\|_{L^{r',1}(\mathbb{R}^n)},$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{(r',1)}(\mathbb{R}^n)$, we have that $\varepsilon > 0$, there exists a function $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|v - \tilde{\varphi}\|_{L^{r',1}(\mathbb{R}^n)} < \varepsilon.$$ 

On the other hand, we have by the fact that $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$ that $G_t^\alpha \ast \tilde{\varphi} \in L^\infty((0, +\infty), H_t^s(\mathbb{R}^n))$ for all $s \geq 0$. Since $G_t^\alpha \ast \tilde{\varphi}$ solves

$$\partial_t \tilde{u} = -(-\Delta)^\alpha \tilde{u},$$

we immediately get that $\|\partial_t G_t^\alpha \ast \tilde{\varphi}\|_{H_t^s(\mathbb{R}^n)} \leq L^\infty((0, +\infty))$ for all $s \geq 0$. This implies that $u \in C((0, +\infty), H_t^s(\mathbb{R}^n))$ for all $s > 0$, and then we have $u \in C((0, +\infty), L^{r',1}(\mathbb{R}^n))$. Combining this fact with (2.15) yields

$$\|G_t^\alpha \ast v - v\|_{L^{r',1}(\mathbb{R}^n)} \to 0 \quad \text{as } t \to 0+$$

It follows from (2.14) that $\langle G_t \ast \varphi - \varphi, v \rangle \to 0$ as $t \to 0+$, from which we obtain $u(x, t)$ is weak * continuous at 0 in the sense of $L^{r,\infty}(\mathbb{R}^n)$. Similarly, we can show that $u(x, t)$ is weak * continuous for all $t > 0$ in the sense of $L^{r,\infty}(\mathbb{R}^n)$.

**Proposition 2.8.** Let $\varphi(x) = \frac{1}{|x|^{2\alpha}}$ with $\alpha \in (1/2, 1]$, and $u_\alpha(x, t) = G_t^\alpha \ast \varphi(x)$. Then we have
(i) $u_0 \in BC_w \left( [0, +\infty), L^{\frac{2}{2\alpha-1}}(\mathbb{R}^3) \right)$.

(ii) for all $s \in (\frac{3}{2\alpha-1}, +\infty)$, $\|u_0(\cdot, t)\|_{L^p(\mathbb{R}^3)} < +\infty$, and for all $s > 0$ and $p > \frac{3}{2\alpha}$,

$$\|\nabla u_0(\cdot, 1)\|_{L^p(\mathbb{R}^3)} < +\infty.$$ 

(iii) $\sup_{x \in \mathbb{R}^3} \langle x \rangle^{2\alpha - 1 + |\beta|} |D^\beta u_0(x, 1)| < +\infty$ for every $\beta$.

**Proof.** Since $\varphi(x) = \frac{1}{|x|^{\alpha-1}}$ with $\alpha \in (1/2, 1]$, it is easy to check that $\varphi \in L^{\frac{3}{2\alpha-1}}(\mathbb{R}^3)$. Thus, we can get the first two results by Lemma 2.6 and Lemma 2.7.

For $\alpha = 1$, we see that

$$u_1(x, 1) = \frac{1}{(4\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4}} \varphi(y) \, dy.$$ 

We calculate

$$\|x| u_1(x, 1) \| \leq \frac{1}{(4\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |x-y| e^{-\frac{|x-y|^2}{4}} |\varphi(y)| \, dy + \frac{1}{(4\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4}} \|y| \varphi(y) \| \, dy.$$ 

On one hand,

$$\frac{1}{(4\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4}} \|y| \varphi(y) \| \, dy \leq \sup_{x \in \mathbb{R}^3} \|x| \varphi(x) \| \frac{1}{(4\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4}} \, dy \leq \sup_{x \in \mathbb{R}^3} \|x| \varphi(x) \|.$$ 

On the other hand, we obtain by the generalized Young inequality that

$$\frac{1}{(4\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |x-y| e^{-\frac{|x-y|^2}{4}} |\varphi(y)| \, dy \leq \| \Phi(\cdot) \|_{L^{2,1}(\mathbb{R}^3)} \| \varphi \|_{L^{3,1}(\mathbb{R}^3)}.$$ 

Combining both estimates yields the third result for the case $|\beta| = 0$ and $\alpha = 1$. Repeating the same process, we can show the third result for each $\beta$ and $\alpha \in (1/2, 1]$. \qed

3. **Existence and regularity of solutions to the corresponding elliptic system**

3.1. **Existence of solutions in $H^\alpha(\mathbb{R}^3)$**. In this subsection, we are devoted to establish the existence of the solution $U(x)$ of (3.2) by the Leray-Schauder principle. In this subsection, we always assume that $\alpha \in (\frac{5}{8}, 1]$. From (1.1), we know that the profile $U(x)$ of $u(x, t)$ satisfies

$$\begin{cases} 
(-\Delta)^\alpha U + U \cdot \nabla U + \nabla P = -2\alpha - 1 \frac{1}{2\alpha} U(x) - \frac{1}{2\alpha} x \cdot \nabla U = 0, \\
\text{div } U = 0,
\end{cases}$$

in $\mathbb{R}^3$.

Letting $U_0 = G_1 \ast u_0$, there exists a pressure $P_0(x)$ such that

$$(-\Delta)^\alpha U_0 + \nabla P_0 - 2\alpha - 1 \frac{1}{2\alpha} U_0(x) - \frac{1}{2\alpha} x \cdot \nabla U_0 = 0.$$ 

We decompose $U = U_0 + V$, then the difference part $V(x)$ satisfies for $x \in \mathbb{R}^3$

*\begin{align*}
(-\Delta)^\alpha V & - 2\alpha - 1 \frac{1}{2\alpha} V(x) - \frac{1}{2\alpha} x \cdot \nabla V + \nabla P = -U_0 \cdot \nabla U_0, -(U_0 + V) \cdot \nabla V - V \cdot \nabla U_0, \\
\text{div } V & = 0
\end{align*}*

with a suitable scalar $P$. Thus the problem to solve $U$ is equivalent to solving (3.2). For this purpose, we introduce the following hyperviscosity perturbation of (3.2):

$$\begin{cases} 
-\epsilon \Delta V + (-\Delta)^\alpha V + \nabla P = \lambda \left( \frac{2\alpha - 1}{2\alpha} V(x) + \frac{1}{2\alpha} x \cdot \nabla V + F(V) \right), \\
\text{div } V = 0,
\end{cases}$$

where

$$F(V) = -U_0 \cdot \nabla U_0 - (U_0 + V) \nabla V - V \cdot \nabla U_0, \quad \lambda \in [0, 1].$$

To overcome the loss of compactness of $H^{1,\sigma}(\mathbb{R}^3) \cap H^{\alpha,\sigma}(\mathbb{R}^3)$, we will approximate $\mathbb{R}^3$ by an increasing sequence of concentric balls, construct solutions of (3.3) in these balls with zero
boundary condition, and take a limit of the approximate solution sequence to obtain a desired solution in $\mathbb{R}^3$ to (3.3) at $\lambda = 1$. Letting $\epsilon \to 0$, we finally obtain the existence of solution to problem (3.2), and then this solution is converted into a self-similar solution of (1.1).

Now we construct a weak solution $V_{R,\epsilon}$ of (3.3) in the following space

$$\mathcal{X}_R \triangleq \left\{ u : u \in H^1_{0,\sigma}(\mathbb{B}_R) \cap H^0_{0,\sigma}(\mathbb{B}_R) \quad \text{and} \quad u \equiv 0 \quad \text{for} \ x \in \mathbb{R}^3 \setminus \mathbb{B}_R \right\}.$$ 

This means that for all $\varphi \in \mathcal{X}_R$, to look for $V_{R,\epsilon}$ satisfying

$$\int_{\mathbb{R}^3} \left( \epsilon \nabla V_{R,\epsilon} \cdot \nabla \varphi + (\Delta)_{2}^{\alpha} V_{R,\epsilon} \cdot (\Delta)_{2}^{\alpha} \varphi \right) \, dx = \lambda \int_{\mathbb{B}_R} \frac{2\alpha - 1}{2\alpha} V_{R,\epsilon} + \frac{1}{2\alpha} x \cdot \nabla V_{R,\epsilon} + F(V_{R,\epsilon}) \cdot \varphi \, dx,$$

for $\lambda \in [0, 1]$. Here the second term of the left hand side is defined via Fourier transform

$$\int_{\mathbb{R}^3} (-\Delta)_{2}^{\alpha} V_{R,\epsilon} \cdot (\Delta)_{2}^{\alpha} \varphi \, dx = \int_{\mathbb{R}^3} |\xi|^{2\alpha} \hat{\varphi}(\xi) \hat{V}_{R,\epsilon}(\xi) \, d\xi.$$ 

Let $u, v \in \mathcal{X}_R$, we introduce inner product as follows

$$\langle u, v \rangle_{\mathcal{X}_R} = \int_{\mathbb{R}^3} \epsilon \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^3} (-\Delta)_{2}^{\alpha} u \cdot (-\Delta)_{2}^{\alpha} v \, dx.$$ 

Then equation (3.4) can be rewritten as

$$\langle V, \varphi \rangle_{\mathcal{X}_R} = \lambda \int_{\mathbb{B}_R} \left( \frac{2\alpha - 1}{2\alpha} V_{R,\epsilon} + \frac{1}{2\alpha} x \cdot \nabla V + F(V) \right) \cdot \varphi \, dx, \quad \forall \ \varphi \in \mathcal{X}_R.$$ 

By the Riesz representation theorem, for any $f \in \mathcal{X}^*_R$ there exists a unique linear mapping $T(f) \in \mathcal{X}_R$ such that

$$\langle T(f), \varphi \rangle_{\mathcal{X}_R} = \int_{\mathbb{B}_R} f \cdot \varphi \, dx, \quad \forall \ \varphi \in \mathcal{X}_R,$$

with

$$\|T(f)\|_{\mathcal{X}_R} \leq \|f\|_{\mathcal{X}^*_R}.$$ 

According to (3.4), we define the following operator

$$V_{R,\epsilon} = \lambda T \left( \frac{2\alpha - 1}{2\alpha} V_{R,\epsilon} + \frac{1}{2\alpha} x \cdot \nabla V_{R,\epsilon} + F(V_{R,\epsilon}) \right) \triangleq \lambda S(V_{R,\epsilon}).$$

To prove the existence of a solution $V_R$ of integral equation (3.5) at $\lambda = 1$, we first have to prove that the set

$$\left\{ x \in \mathcal{X}_R : x = \lambda Sx \quad \text{for some} \ \lambda \in [0, 1] \right\}$$

is bounded in $X$, and then prove the operator $S$ is continuous and compact.

**Step 1: a priori bound**

**Lemma 3.1** (a priori estimate). Let $V_{R,\epsilon}$ be the solution of (3.2), we have

$$\int_{\mathbb{R}^3} \left( \epsilon \|
abla V_{R,\epsilon}\|_2^2 + |(-\Delta)_{2}^{\alpha} V_{R,\epsilon}|^2 + \frac{5 - 4\alpha}{4\alpha} |V_{R,\epsilon}|^2 \right) \, dx \leq C(U_0, R, \epsilon).$$

**Proof.** We will give a proof of Lemma 3.1 by contradiction. Now let us suppose that there exists a sequence $\lambda_k \in [0, 1]$ and functions $V_k \triangleq V_{R,\epsilon}^{(k)} \in \mathcal{X}_R$ such that

$$\begin{cases}
-\epsilon \Delta V_k + (\Delta)_{2}^{\alpha} V_k + \nabla P_k = \lambda_k \left( \frac{2\alpha - 1}{2\alpha} V_k + \frac{1}{2\alpha} x \cdot \nabla V_k - U_0 \cdot \nabla U_0 \right) \\
\text{div } V_k = 0,
\end{cases}$$

and

$$L_k^2 \triangleq \int_{\mathbb{R}^3} \left( \epsilon \|
abla V_k\|_2^2 + |(-\Delta)_{2}^{\alpha} V_k|^2 \right) \, dx \to +\infty, \quad \lambda_k \to \lambda_0 \in [0, 1].$$
Multiplying (3.6) by $V_k$ and integrating by parts in $\mathbb{B}_R$, we obtain
\begin{equation}
L_k^2 + \frac{\lambda_k(5 - 4\alpha)}{4\alpha} \int_{\mathbb{B}_R} |V_k|^2 \, dx = \lambda_k \int_{\mathbb{B}_R} \left( - U_0 \cdot \nabla U_0 - V_k \cdot \nabla U_0 \right) \cdot V_k \, dx,
\end{equation}
where we have used the fact that
\[ \int_{\mathbb{B}_R} (U_0 + V_k) \cdot \nabla V_k \cdot V_k \, dx = 0. \]

Now we consider the normalized sequence of functions
\[
\tilde{V}_k \triangleq \frac{1}{L_k} V_k \quad \text{and} \quad \tilde{P}_k \triangleq \frac{1}{\lambda_k L_k^2} P_k,
\]
such that
\[
\int_{\mathbb{R}^3} \left( \epsilon |\nabla \tilde{V}_k|^2 + |(-\Delta)^{\frac{\alpha}{2}} \tilde{V}_k|^2 \right) \, dx = 1.
\]
Therefore, we can extract a subsequence still denoted by $\tilde{V}_k$ such that
\[
\tilde{V}_k \rightharpoonup V \quad \text{in} \quad H^1_0(\mathbb{B}_R) \cap H^2_0(\mathbb{B}_R).
\]
This means
\[
\tilde{V}_k \rightarrow V \quad \text{in} \quad L^3(\mathbb{B}_R).
\]
Multiplying identity (3.7) by $\frac{1}{L_k^2}$ and taking a limit as $k \to +\infty$, we have
\begin{equation}
1 + \frac{\lambda_0(5 - 4\alpha)}{4\alpha} \int_{\mathbb{B}_R} |V|^2 \, dx = -\lambda_0 \int_{\mathbb{B}_R} (V \cdot \nabla U_0) V \, dx = \lambda_0 \int_{\mathbb{B}_R} (V \cdot \nabla V) U_0 \, dx,
\end{equation}
this relation yields $\lambda_0 > 0$.

Multiplying equation (3.6) by $\frac{1}{\lambda_k L_k}$, we have
\[
\tilde{V}_k \cdot \nabla \tilde{V}_k + \nabla \tilde{P}_k = \frac{1}{L_k} \left( \frac{\epsilon}{\lambda_k} \Delta \tilde{V}_k + \frac{1}{\lambda_k} (-\Delta)^{\alpha} \tilde{V}_k + \frac{2\alpha - 1}{2\alpha} \tilde{V}_k + \frac{1}{2\alpha} x \cdot \nabla \tilde{V}_k \right.
\]
\[
\left. - \frac{1}{L_k} U_0 \cdot \nabla U_0 - U_0 \cdot \nabla \tilde{V}_k - \tilde{V}_k \cdot \nabla U_0 \right).\]

Multiplying the above equation by $\varphi$ and integrating the resulting equality over $\mathbb{B}_R$, we can show that
\[
\int_{\mathbb{B}_R} (V \cdot \nabla V) \cdot \varphi \, dx = 0 \quad \text{for all} \quad \varphi \in C_0^{\infty}(\mathbb{B}_R).
\]
Hence, we have by the Rham Theorem (for example, see [34]) that there exists a pressure
\[ P \in D^{1,\frac{3}{2}}(\mathbb{B}_R) \cap L^3(\mathbb{B}_R) \]
such that $(V, P)$ solves
\[
\begin{cases}
V \cdot \nabla V + \nabla P = 0 & \text{in} \quad \mathbb{B}_R, \\
\text{div} \, V = 0 & \text{in} \quad \mathbb{B}_R, \\
V = 0 & \text{in} \quad \mathbb{R}^3 \setminus \mathbb{B}_R.
\end{cases}
\]
By Lemma 2.1 there exists a constant $c \in \mathbb{R}$ such that $P(x) = c$ on $\partial\mathbb{B}_R$. This fact helps us to get
\[
1 + \frac{\lambda_0(5 - 4\alpha)}{4\alpha} \int_{\mathbb{B}_R} |V|^2 \, dx = -\lambda_0 \int_{\mathbb{B}_R} U_0 \cdot \nabla P \, dx
\]
\[
= -\lambda_0 \int_{\mathbb{B}_R} \text{div}(PU_0) \, dx
\]
\[
= c\lambda_0 \int_{\partial\mathbb{B}_R} U_0 \cdot \vec{n} \, ds
\]
\[
= -c\lambda_0 \int_{\mathbb{B}_R} \nabla \cdot U_0 \, dx = 0.
\]
This is a contradiction, and so we complete proof of Lemma 3.1. \hfill \Box
Step 2: Continuity and compactness

**Lemma 3.2.** The operator

\[ S : X_R \ni v \mapsto T \left( \frac{2\alpha - 1}{2\alpha} v + \frac{1}{2\alpha} x \cdot \nabla v + F(v) \right) \in X_R \]

is continuous and compact.

**Proof.** First we prove \( S \) is continuous. Let \( v_1, v_2 \in X_R \), then

\[
\| S(v_1) - S(v_2) \|_{X_R} = \left\| T \left( \frac{2\alpha - 1}{2\alpha} v_1 + \frac{1}{2\alpha} x \cdot \nabla v_1 + F(v_1) \right) - T \left( \frac{2\alpha - 1}{2\alpha} v_2 + \frac{1}{2\alpha} x \cdot \nabla v_2 + F(v_2) \right) \right\|_{X_R} \\
\leq C \left( \| v_1 - v_2 \|_{X_R} + \frac{1}{2\alpha} \| x \cdot \nabla v_1 - x \cdot \nabla v_2 \|_{X_R} + \| F(v_1) - F(v_2) \|_{X_R}' \right).
\]

Since \( L^2(\mathbb{B}_R) \subset L^\frac{6}{5}(\mathbb{B}_R) \subset C'_R \), we have

\[
\| v_1 - v_2 \|_{X_R} \leq \| v_1 - v_2 \|_{L^2(\mathbb{B}_R)}, \quad \| x \cdot \nabla v_1 - x \cdot \nabla v_2 \|_{X_R} \leq C \| v_1 - v_2 \|_{L^2(\mathbb{B}_R)},
\]

and

\[
\| F(v_1) - F(v_2) \|_{X_R} \leq \| U_0 \|_{L^3(\mathbb{B}_R)} \| \nabla (v_1 - v_2) \|_{L^2(\mathbb{B}_R)} + \| \nabla U_0 \|_{L^2(\mathbb{B}_R)} \| v_1 - v_2 \|_{L^3(\mathbb{B}_R)} + \| \nabla v_1 \|_{L^2(\mathbb{B}_R)} \| v_1 - v_2 \|_{L^2(\mathbb{B}_R)} + \| \nabla v_2 \|_{L^2(\mathbb{B}_R)} \| \nabla (v_1 - v_2) \|_{L^2(\mathbb{B}_R)}.
\]

So

\[
\| S(v_1) - S(v_2) \|_{X_R} \leq C \left( 1 + \| v_1 \|_{X_R} + \| v_2 \|_{X_R} \right) \| v_1 - v_2 \|_{X_R},
\]

which implies that \( S \) is continuous.

Now we prove \( S \) is compact, it suffices to show that: for any bounded sequence \( v_k \), there exists a subsequence \( v_{k_l} \) such that

\[ (3.9) \quad \| S(v_{k_l}) - S(v) \|_{X_R} \to 0 \quad \text{as} \quad l \to +\infty, \quad \text{for some} \quad v \in X_R. \]

Indeed, if \( \| v_k \|_{X_R} < C \), then by the Sobolev embedding theorem, there exists a subsequence, still denoted by \( v_k \), such that

\[ v_k \to v \quad \text{in} \quad L^q(\mathbb{B}_R) \quad \text{for} \quad 1 \leq q < 6, \]

from which we immediately have for any vector \( \psi \in C_0^\infty(\mathbb{R}^3) \),

\[ \langle x \cdot \nabla (v_k - v), \psi \rangle = -\langle x \cdot \nabla \psi, v_k - v \rangle - \langle v_k - v, \psi \rangle \lesssim \| v_k - v \|_{L^2} \| \psi \|_{X_R}, \]

and

\[ \langle F(v_k) - F(v), \psi \rangle = -\langle U_0 \cdot \nabla (v_k - v), \psi \rangle - \langle v_k \cdot \nabla (v_k - v), \psi \rangle - \langle (v_k - v) \cdot \nabla v, \psi \rangle - \langle (v_k - v) \cdot \nabla U_0, \psi \rangle \]

\[ = +\langle U_0 \cdot \nabla \psi, v_k - v \rangle + \langle v_k \cdot \nabla \psi, v_k - v \rangle - \langle (v_k - v) \cdot \nabla v, \psi \rangle - \langle (v_k - v) \cdot \nabla U_0, \psi \rangle \]

\[ \leq C \| v_k - v \|_{L^1} \| \psi \|_{X_R}. \]

The above inequalities imply \( (3.9) \). \( \square \)

From Lemma 3.1 and Lemma 3.2, we conclude that the operator \( S \) satisfies all the requirements of the Leray-Schauder principle, and so we have:

**Proposition 3.3** (Existence in \( \mathbb{B}_R \)). The system \( (3.4) \) has a solution \( V_{R,\epsilon} \in X_R \).

Now, we wish to extend statements of Proposition 3.3 from \( B_R \) to \( \mathbb{R}^3 \). We will establish the following uniform bound independent of \( \epsilon, R \):

**Lemma 3.4.** Let \( V_{R,\epsilon} \) be the solution of \( (3.4) \), we have the a priori bound

\[ \int_{\mathbb{R}^3} \left( \epsilon |\nabla V_{R,\epsilon}|^2 + |(-\Delta)^{\frac{\alpha}{2}} V_{R,\epsilon}|^2 + \frac{5 - 4\alpha}{4\alpha} |V_{R,\epsilon}|^2 \right) dx \leq C(U_0). \]
Proof. Since the a priori bound is independent of \( \lambda \) by Lemma 3.1, we suppose \( \lambda = 1 \) at the moment. Now we proceed, as previously, by a contradiction argument. In fact, suppose that its assertion is not true. Then there exist sequences \( B_k \triangleq B_{R_k}, \epsilon_k \) and \( V_k \triangleq V_{R_k, \epsilon_k} \in \mathcal{X}_{R_k} \) such that

\[
L_k^2 \triangleq \int_{\mathbb{R}^3} \left( \epsilon_k |\nabla V_k|^2 + |(-\Delta)^{\frac{\alpha}{2}} V_k|^2 + \frac{5 - 4\alpha}{4\alpha} |V_k|^2 \right) \, dx \to +\infty,
\]

where \( \epsilon_k \to \epsilon_0 \in [0, 1] \).

Multiplying the equation (3.4) by \( V_k \) and integrating by parts in \( B_k \), we have

\[
(3.10) \quad L_k^2 = \int_{B_k} \left( -U_0 \cdot \nabla U_0 - V_k \cdot \nabla U_0 \right) V_k \, dx,
\]

where we have used the fact that

\[
\int_{B_k} \left( (U_0 + V_k) \nabla V_k \right) V_k \, dx = 0.
\]

Now let

\[
\tilde{V}_k \triangleq \frac{1}{L_k} V_k \quad \text{and} \quad \tilde{P}_k \triangleq \frac{1}{L_k^2} P_k,
\]

then they satisfy

\[
\tilde{V}_k \cdot \nabla \tilde{V}_k + \nabla \tilde{P}_k = \frac{1}{L_k} \left( \epsilon_k \Delta \tilde{V}_k + (-\Delta)^{\alpha} \tilde{V}_k + \frac{2\alpha - 1}{2\alpha} \tilde{V}_k + \frac{1}{2\alpha} x \cdot \nabla \tilde{V}_k \right.
\]

\[
- \frac{1}{L_k} U_0 \cdot \nabla U_0 - U_0 \cdot \nabla \tilde{V}_k - \tilde{V}_k \cdot \nabla U_0 \right)
\]

and

\[
\int_{B_k} \left( \epsilon_k |\nabla \tilde{V}_k|^2 + |(-\Delta)^{\alpha} \tilde{V}_k|^2 + \frac{5 - 4\alpha}{4\alpha} |\tilde{V}_k|^2 \right) \, dx = 1.
\]

Thus we could extract a subsequence still denoted by \( \tilde{V}_k \) such that

\[
\tilde{V}_k \to V \quad \text{in} \quad H_0^\alpha (B_k);
\]

\[
\tilde{V}_k \to V \quad \text{in} \quad L^q(\Omega') \quad \text{for any bounded} \ \Omega' \subset \mathbb{R}^3 \quad \text{and} \quad 1 \leq q < \frac{6}{3 - 2\alpha}.
\]

Thanks to Proposition 2.8, one has that for \( \alpha > \frac{5}{8} \),

\[
(3.12) \quad \|U_0 \cdot \nabla U_0\|_{L^2(\mathbb{R}^3)} \leq C \|\langle \rangle^{-(4\alpha - 1)}\|_{L^2(\mathbb{R}^3)} < +\infty.
\]

Therefore we have by the Cauchy-Schwarz inequality that for \( \alpha > \frac{5}{8} \),

\[
(3.13) \quad \int_{\mathbb{R}^3} U_0 \cdot \nabla U_0 \cdot \tilde{V}_k \, dx \leq \left( \int_{\mathbb{R}^3} |U_0 \cdot \nabla U_0|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\tilde{V}_k|^2 \, dx \right)^{\frac{1}{2}} < +\infty.
\]

Multiplying (3.10) by \( \frac{1}{L_k} \) and taking a limit as \( k \to +\infty \), we have by (3.13) that

\[
(3.14) \quad 1 = \int_{\mathbb{R}^3} (-V \cdot \nabla U_0) V \, dx = - \int_{\mathbb{R}^3} (V_i V_j) \partial_i U_{0,j} \, dx.
\]

From (3.12), we have \( U_0 \in W^{1,q}(\mathbb{R}^3) \) with some large enough \( q \) and \( \text{div} \ U_0 = 0 \). By the density argument, there exists a sequence \( \varphi_n \in C_{0,\sigma}^\infty (\mathbb{R}^3) \) such that

\[
\|\nabla (\varphi_n - U_0)\|_{L^q(\mathbb{R}^3)} \to 0.
\]

From this limitation and the fact \( V \in L^m(\mathbb{R}^3) \) for \( 1 \leq m \leq \frac{6}{3 - 2\alpha} \), we immediately have

\[
0 = \int_{\mathbb{R}^3} (V \otimes V) \cdot \nabla \varphi_n \, dx \to \int_{\mathbb{R}^3} (V \otimes V) \cdot \nabla U_0 \, dx = -1,
\]

which deduces a contradiction. \( \square \)
Proposition 3.5 (Existence in $\mathbb{R}^3$). The system \((3.3)\) has a solution $V_\varepsilon \in H^1_0(\mathbb{R}^3) \cap H^\alpha_0(\mathbb{R}^3)$ for $\lambda = 1$.

**Proof.** Let $\mathbb{B}_R$ be the ball in $\mathbb{R}^3$ with radius, then by Lemma 3.4 there exists a solution $V_{R_\varepsilon} \in \mathcal{X}_R$ of the system \((3.3)\) in $\mathbb{B}_R$, which satisfies

$$
\int_{\mathbb{R}^3} \left( \epsilon |\nabla V_{R_\varepsilon}|^2 + |(-\Delta)^{\frac{\alpha}{2}} V_{R_\varepsilon}|^2 + \frac{5 - 4\alpha}{4\alpha} |V_{R_\varepsilon}|^2 \right) dx \leq C(U_0).
$$

Due to the above uniform regularity estimate, there exists a converging subsequence $\{V_{R_j,\varepsilon}\}_{j=1}^\infty$ (where $R_j \uparrow +\infty$) such that

\begin{equation}
V_{R_j,\varepsilon} \to V_\varepsilon \quad \text{in } H^1_0(\mathbb{R}^3) \cap H^\alpha_0(\mathbb{R}^3);
\end{equation}

and for any $0 < R < +\infty$

\begin{equation}
V_{R_j,\varepsilon} \to V_\varepsilon \quad \text{in } L^q(\mathbb{R}^3) \quad \text{with } 2 \leq q \leq 6,
\end{equation}

where we have used the fact that $V_{R_j,\varepsilon} \equiv 0$ in $\mathbb{R}^3 \setminus \mathbb{B}_{R_j}$.

We first show that $V_\varepsilon$ satisfies \((3.3)\) in the sense of distribution. For $\forall \varphi \in C_0^\infty(\mathbb{R}^3)$, seeding $R_j \to +\infty$ we easily derive

\begin{equation}
\int_{\mathbb{R}^3} \epsilon \nabla V_{R_j,\varepsilon} \nabla \varphi \, dx + \int_{\mathbb{R}^3} (-\Delta)\varphi V_{R_j,\varepsilon} (-\Delta)\varphi \, dx \to \int_{\mathbb{R}^3} \epsilon \nabla V_\varepsilon \nabla \varphi \, dx + \int_{\mathbb{R}^3} x \cdot \nabla V_{R_j,\varepsilon} \varphi \, dx
\end{equation}

and for any $0 < R < +\infty$

\begin{equation}
V_{R_j,\varepsilon} \to V_\varepsilon \quad \text{in } L^q(\mathbb{R}^3) \quad \text{for } 1 \leq q < 6,
\end{equation}

where $\varphi \in C_0^\infty(\mathbb{R}^3)$, we may assume that $\supp \varphi \subset \mathbb{B}_R$. The simple calculation yields

\begin{equation}
\int_{\mathbb{R}^3} V_{R_j,\varepsilon} \cdot \nabla V_{R_j,\varepsilon} \varphi(x) \, dx - \int_{\mathbb{R}^3} V_\varepsilon \cdot \nabla V_\varepsilon \varphi(x) \, dx
\end{equation}

\begin{equation}
= \int_{\mathbb{R}^3} (V_{R_j,\varepsilon} \otimes (V_{R_j,\varepsilon} - V_\varepsilon)) : \nabla \varphi(x) \, dx + \int_{\mathbb{R}^3} ((V_{R_j,\varepsilon} - V_\varepsilon) \otimes V_{R_j}) : \nabla \varphi(x) \, dx
\end{equation}

\begin{equation}
\triangleq I_1 + I_2.
\end{equation}

We choose $R_j$ larger than $R$, by the Hölder inequality, we immediately obtain

\begin{equation}
I_1 = \int_{\mathbb{R}^3} (V_{R_j,\varepsilon} \otimes (V_{R_j,\varepsilon} - V_\varepsilon)) : \nabla \varphi(x) \, dx \leq \|V_{R_j,\varepsilon}\|_{L^2(\mathbb{R}^3)} \|V_{R_j,\varepsilon}\|_{L^2(\mathbb{B}_R)} \|\nabla \varphi\|_{L^2(\mathbb{R}^3)}.
\end{equation}

This estimate together with the strong convergence \((3.16)\) yields

\begin{equation}
I_1 \to 0 \quad \text{as } R_j \to +\infty.
\end{equation}

Similarly, we have

\begin{equation}
I_2 \to 0 \quad \text{as } R_j \to +\infty.
\end{equation}

Inserting \((3.19)\) and \((3.20)\) into \((3.18)\) leads to

\begin{equation}
\int_{\mathbb{R}^3} V_{R_j,\varepsilon} \cdot \nabla V_{R_j,\varepsilon} \cdot \varphi(x) \, dx \to \int_{\mathbb{R}^3} V_\varepsilon \cdot \nabla V_\varepsilon \cdot \varphi(x) \, dx
\end{equation}

as $j$ goes to infinity.

We observe that

\begin{equation}
\int_{\mathbb{R}^3} U_0 \cdot \nabla V_{R_j,\varepsilon} \cdot \varphi(x) \, dx = - \int_{\mathbb{R}^3} V_{R_j,\varepsilon} \otimes U_0 \cdot \nabla \varphi(x) \, dx.
\end{equation}

Since $|U_0| \leq (1 + |x|)^{1-2\alpha}$ and $\varphi \in C_0^\infty(\mathbb{R}^3)$, it is easy to check that $U_0 \cdot \nabla \varphi(x) \in L^2(\mathbb{R}^2)$. Thus the weak convergence \((3.15)\) enables us to infer that

\begin{equation}
\int_{\mathbb{R}^3} V_{R_j,\varepsilon} \otimes U_0 \cdot \nabla \varphi(x) \, dx \to \int_{\mathbb{R}^3} V_\varepsilon \otimes U_0 \cdot \nabla \varphi(x) \, dx \quad \text{as } j \to +\infty.
\end{equation}
Plugging \((3.23)\) into \((3.22)\) gives

\[
(3.24) \quad \int_{\mathbb{R}^3} U \cdot \nabla V_{R_j, \varepsilon} \cdot \varphi(x) \, dx = - \int_{\mathbb{R}^3} V_{\varepsilon} \otimes U_0 \cdot \nabla \varphi(x) \, dx \quad \text{as} \quad R_j \to +\infty.
\]

In the same fashion as used in \((3.24)\), one can conclude that

\[
(3.25) \quad \int_{\mathbb{R}^3} V_{R_j, \varepsilon} \cdot \nabla U_0 \cdot \varphi(x) \, dx = - \int_{\mathbb{R}^3} U_0 \otimes V_{\varepsilon} \cdot \nabla \varphi(x) \, dx \quad \text{as} \quad R_j \to +\infty.
\]

Collecting \((3.21)\), \((3.24)\) and \((3.25)\), we readily get

\[
\int_{\mathbb{R}^3} F(V_{R_j, \varepsilon}) \varphi(x) \, dx \to \int_{\mathbb{R}^3} F(V_\varepsilon) \varphi(x) \, dx \quad \text{as} \quad R_j \to +\infty.
\]

Thus, we obtain that \(V_\varepsilon\) satisfies \((3.3)\) in the sense of distribution. By density argument, for \(\forall \varphi \in H^1_s(\mathbb{R}^3) \cap H^\alpha(\mathbb{R}^3)\) satisfying \(\|\cdot \varphi\|_{L^2(\mathbb{R}^3)} < +\infty\), we have

\[
\int_{\mathbb{R}^3} \left( \epsilon \nabla V_\varepsilon : \nabla \varphi + (-\Delta)^{\frac{\alpha}{2}} V_\varepsilon \cdot (-\Delta)^{\frac{\alpha}{2}} \varphi \right) \, dx = \int_{\mathbb{R}^3} \left( \frac{2\alpha - 1}{2\alpha} V_\varepsilon + \frac{1}{2\alpha} x \cdot \nabla V_\varepsilon + F(V_\varepsilon) \right) \cdot \varphi \, dx.
\]

Finally, according to the weak low semi-continuity of the norm, the limit function \(V_\varepsilon\) satisfies

\[
(3.26) \quad \int_{\mathbb{R}^3} \left( \epsilon \nabla V_\varepsilon \right)^2 + \left( (-\Delta)^{\frac{\alpha}{2}} V_\varepsilon \right)^2 + \frac{5 - 4\alpha}{4\alpha} |V_\varepsilon|^2 \right) \, dx \leq C(U_0).
\]

This estimate implies the desired result. \(\square\)

Letting \(\epsilon \to 0\), we obtain the main result of this subsection by the classical diagonalization argument.

**Theorem 3.6.** There is a function \(V \in H^\alpha(\mathbb{R}^3)\) satisfies system \((3.2)\) in the sense of distribution.

**Remark 3.1.** i) When \(\alpha \leq \frac{5}{2}\), we do not know whether \(V \in H^\alpha(\mathbb{R}^3)\) or not due to \(U_0 \cdot \nabla U_0 \notin L^2(\mathbb{R}^n)\), and so we can not construct solutions of \((1.6)\) by the blow-up argument for this case.

(ii) For \(\alpha = 1\), the distributional solution of system \((3.2)\) established in Theorem 3.6 that for any \(\varphi \in H^1(\mathbb{R}^3)\) with

\[\|\cdot \varphi\|_{L^2(\mathbb{R}^3)} < +\infty,\]

we have

\[
(3.27) \quad \int_{\mathbb{R}^3} \nabla V \cdot \nabla \varphi \, dx - \int_{\mathbb{R}^3} P\text{div} \varphi \, dx = \int_{\mathbb{R}^3} \left( \frac{1}{2} V + \frac{1}{2} x \cdot \nabla V + F(V) \right) \cdot \varphi \, dx.
\]

This is a start point for studying the decay estimate of the solution \(V(x)\).

### 3.2. Improved regularity of \(V(x)\).

First of all, we review the following fractional Leibniz estimate which was shown in \([4]\):

Let

\[\alpha > 0, \ 1 \leq p \leq +\infty, \ 1 < p_1, p_1, q_1, q_2 \leq +\infty, \ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2},\]

then for \(f, g \in C^\infty_0(\mathbb{R}^n)\), we have

\[
(3.28) \quad \|(-\Delta)^{\frac{\alpha}{2}} (fg)\|_{L^p(\mathbb{R}^n)} \leq C \|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{q_1}(\mathbb{R}^n)} + C \|(-\Delta)^{\frac{\alpha}{2}} g\|_{L^{p_2}(\mathbb{R}^n)} \|f\|_{L^{q_2}(\mathbb{R}^n)}.
\]

Next we consider some basic properties of the following non-local stokes operator:

\[
(3.29) \quad \begin{cases}
(-\Delta)^{\alpha} u + \lambda u + \nabla q = f(x), & x \in \mathbb{R}^n, \\
\text{div} u = 0, & x \in \mathbb{R}^n,
\end{cases}
\]

where \(\lambda > 0\).
Lemma 3.7. Let \( f(x) \in C_0^\infty(\mathbb{R}^n) \), then system (3.29) admits a solution \((u, q)\) such that
\[
\|(-\Delta)^{\alpha} u\|_{L^p(\mathbb{R}^n)} + \lambda^{\frac{\alpha}{2}} \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^p(\mathbb{R}^n)} + \lambda \|u\|_{L^p(\mathbb{R}^n)} + \|\nabla q\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}
\]
with \(1 < p < +\infty\).

Proof. We introduce
\[
q \equiv -\partial_i \Gamma \ast f_i; \quad u = (u_i) \equiv \{(\delta_{ij} \mathcal{B}_\alpha + \partial_i \partial_j \Gamma \ast \mathcal{B}_\alpha) \ast f_j\}
\]
where \(\Gamma, \mathcal{B}_\alpha\) are the fundamental solutions of the operator \(-\Delta\) and \((-\Delta)^{\alpha} + \lambda I\) respectively, i.e.,
\[
\Gamma(x) = \mathcal{F}^{-1}[|\xi|^2], \quad \mathcal{B}_\alpha(x) = \mathcal{F}^{-1}\left[\frac{1}{\lambda + |\xi|^\alpha}\right].
\]
Obviously, \((u, q)\) fulfills equation (3.29). In addition, we have
\[
(-\Delta)^{\alpha} u_j = \mathcal{F}^{-1} \left[ \frac{|\xi|^\alpha}{\lambda + |\xi|^\alpha} \left( \delta_{ij} + \frac{\xi_i \xi_j}{|\xi|^2} \right) \mathcal{F} f_j \right] \equiv \mathcal{F}^{-1}[m(\xi) \mathcal{F} f_j]
\]
with \(m(\xi) \in L^\infty(\mathbb{R}^n)\), then by the Calderon-Zygmund inequality, we derive
\[
\|(-\Delta)^{\alpha} u_j\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.
\]
On the other hand, we note
\[
\lambda u_j = \mathcal{F}^{-1} \left[ \frac{\lambda}{\lambda + |\xi|^\alpha} \left( \delta_{ij} + \frac{\xi_i \xi_j}{|\xi|^2} \right) \mathcal{F} f_j \right] \equiv \mathcal{F}^{-1}[m_1(\xi) \mathcal{F} f_j]
\]
with \(m_1(\xi) \in L^\infty(\mathbb{R}^n)\), and derive, by using the Calderon-Zygmund inequality again
\[
\|\lambda u_j\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.
\]
Due to the following interpolation equality
\[
\|u\|_{H^p_p(\mathbb{R}^n)} \leq C \|u\|_{H^{2\alpha}_p(\mathbb{R}^n)}^{\frac{1}{2}} \|u\|_{L^p(\mathbb{R}^n)}^{\frac{1}{2}},
\]
we obtain
\[
\lambda^{\frac{\alpha}{2}} \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.
\]
Finally, by the elliptic estimates, we derive
\[
\|\nabla q\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.
\]
Combining the above discussion, we complete the proof. \(\square\)

From the above Lemma, we immediately have by density

Corollary 3.8. Let \( f \in L^p(\mathbb{R}^n) \) with \(1 < p < +\infty\), then equation (3.29) admits a unique strong solution \((u, q)\) in \(H^{2\alpha}_p(\mathbb{R}^n) \times \dot{H}^1(\mathbb{R}^n)\) satisfying (3.30).

Following the argument of [21], we present the following regularity lemma of the nonlocal elliptic operator, which plays the key role in improving regularity of solution \(V(x)\).

Lemma 3.9. Let \(p \in (1, +\infty)\), \(g, f_1, ..., f_n \in L^p(\mathbb{R}^n)\) and \(\lambda > 0\), the equation
\[
(-\Delta)^{\alpha} u(x) + \lambda u + \nabla q = \partial_i f_i(x) + g(x) \quad x \in \mathbb{R}^n
\]
has a solution \(u \in H^{2\alpha-1}_p(\mathbb{R}^n)\) such that
\[
\|(-\Delta)^{\frac{2\alpha-1}{2}} u\|_{L^p(\mathbb{R}^n)} + \lambda^{\frac{2\alpha-1}{2\alpha}} \|u\|_{L^p(\mathbb{R}^n)} \leq C \sum_{i=1}^n \|f_i\|_{L^p(\mathbb{R}^n)} + \lambda^{-\frac{1}{2\alpha}} \|g\|_{L^p(\mathbb{R}^n)}.
\]

Furthermore, this equation can only have one solution in \(L^p(\mathbb{R}^n)\).
Proof. We consider
\[ (-\Delta)^{\alpha} v_i(x) + \lambda v_i(x) + \nabla q_i = f_i(x) \quad \text{in} \quad \mathbb{R}^n, \]
and
\[ (-\Delta)^{\alpha} v(x) + \lambda v(x) + \nabla q = g(x) \quad \text{in} \quad \mathbb{R}^n. \]

By the Corollary 3.8, the above two equations have solution \( v_i \) and \( v \) respectively, such that
\[
\begin{align*}
((\Delta)^{\alpha} v_i)(x) + \lambda^\frac{2}{\alpha} ((\Delta)^{\alpha} f_i)(x) + \lambda || v_i ||_{L^p(\mathbb{R}^n)} &\leq C || f_i ||_{L^p(\mathbb{R}^n)}, \\
((\Delta)^{\alpha} v)(x) + \lambda^\frac{2}{\alpha} ((\Delta)^{\alpha} g)(x) + \lambda || v ||_{L^p(\mathbb{R}^n)} &\leq C || g ||_{L^p(\mathbb{R}^n)}.
\end{align*}
\]

Letting \( u \triangleq \partial_i v_i + v \), we easily obtain (3.32) by the interpolation theory.

To prove \( u \) is a unique solution to (3.31), we only prove that for any \( w \in L^p(\mathbb{R}^n) \) solves (3.31) with \( f_i = 0 \), \( g = 0 \) in sense of distribution, then \( u \equiv 0 \). To do this, we introduce
\[
\psi(y) = \int_{\mathbb{R}^n} \varphi(y + x) w(x) \, dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).
\]

Since \( \varphi \in C_0^\infty(\mathbb{R}^n) \) and \( w \in L^p(\mathbb{R}^n) \), the function \( \psi(y) \) is infinitely smooth and tends to zero as \( y \to +\infty \). By a simple calculation, we derive
\[
(-\Delta)^{\alpha} \psi(y) + \lambda \psi(y) = \int_{\mathbb{R}^n} w(x) ((-\Delta)^{\alpha} \varphi(y + x) + \lambda \varphi(y + x)) \, dx = 0.
\]

By the Helmholtz decomposition, we see that for \( \forall \varphi \in C_0^\infty(\mathbb{R}^n) \)
\[
\varphi = \varphi_1 + \nabla \varphi_2
\]
with \( \text{div} \varphi_1 = 0 \). And then
\[
(-\Delta)^{\alpha} \psi(y) + \lambda \psi(y) = \int_{\mathbb{R}^n} w(t) ((-\Delta)^{\alpha} \varphi(y + t) + \lambda \varphi(y + t)) \, dt
\]
\[
= \int_{\mathbb{R}^n} \left((-\Delta)^{\alpha} w(t) + \lambda w(t)\right) \left(\varphi_1(y + t) + \nabla \varphi_2(y + t)\right) \, dt
\]
\[
= \int_{\mathbb{R}^n} \left((-\Delta)^{\alpha} w(t) + \lambda w(t)\right) \varphi_1(y + t) \, dt = 0.
\]

Now we claim \( \psi(y) \equiv 0 \). Indeed, suppose by contradiction that \( \text{sup}_{y \in \mathbb{R}^n} \psi(y) > 0 \). Since \( \psi(y) \to 0 \) as \( |y| \to +\infty \), we can find \( y_0 \in \mathbb{R}^n \) such that \( u(y_0) = \text{sup}_{y \in \mathbb{R}^n} \psi(y) > 0 \).

We see that,
\[
(-\Delta)^{\alpha} \psi(y_0) + \lambda \psi(y_0) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{\psi(y_0) - \psi(y)}{|y_0 - y|^{n+\alpha}} \, dy + \lambda \psi(y_0) > 0
\]
which contradicts with (3.32). Therefore we must have \( \text{sup}_{y \in \mathbb{R}^n} \psi(y) \leq 0 \). Similarly, we also infer \( \text{inf}_{y \in \mathbb{R}^n} \psi(y) \geq 0 \). Thus we have \( \psi(y) \equiv 0 \). The arbitrariness of \( \varphi \) with \( \psi(0) = 0 \) leads to the conclusion that \( w = 0 \). The lemma is thus proved. \( \square \)

**Corollary 3.10.** Let \( f_i (i = 1, 2, ..., n) \), \( g \in H^m_p(\mathbb{R}^n) \), then (3.31) has a solution \( u \in H^{2\alpha-1+m}_p(\mathbb{R}^n) \), which is a unique solution in \( L^p(\mathbb{R}^n) \).

**Definition 3.1.** Let \( 0 < \alpha \leq 1 \), we say that \( u \in D'(\mathbb{R}^n) \) satisfies \( (-\Delta)^{\alpha} u + u = 0 \) in an open set \( \Omega \) if for every \( \phi \in C_0^\infty(\Omega) \) such that
\[
\langle u, (-\Delta)^{\alpha} \phi + \phi \rangle = 0.
\]

**Lemma 3.11 (24, 33).** Let \( u \in L^1_{\text{loc}}(\mathbb{R}^n) \), and it satisfies (3.36) and
\[
\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\alpha}} \, dx < +\infty.
\]

Then \( u \) is smooth in \( \mathbb{R}^n \).

**Theorem 3.12.** Let \( \frac{\alpha}{2} < \alpha \leq 1 \), then the distributional solution \( V(x) \) of system (3.2) established in Theorem 3.6 is smooth.
Proof. We now rewrite system \((3.22)\) as
\[
(-\Delta)^{\alpha} V + \frac{2 - \alpha}{\alpha} V + \nabla P = \text{div} \left( G(U_0, V) \right), \quad \text{div} V = 0,
\]
where
\[
G(U_0, V) = \frac{1}{2\alpha} x \otimes V - U_0 \otimes U_0 - U_0 \otimes V - V \otimes U_0 - V \otimes V.
\]

**Step 1.** Now we consider
\[
(-\Delta)^{\alpha} V_1 + \frac{2 - \alpha}{\alpha} V_1 + \nabla P_1 = \text{div}(G_k(U_0, V)), \quad \text{div} V_1 = 0,
\]
where
\[
G_k(U_0, V) = \frac{1}{2\alpha} \varphi_k \left( x \otimes V - U_0 \otimes U_0 - U_0 \otimes V - V \otimes U_0 - V \otimes V \right),
\]
and \(\varphi_k \in C_0^\infty(\mathbb{B}_k R), \varphi_k \equiv 1\) in \(\mathbb{B}_{(k-1)R}\) with large integer \(k\).

Due to \(V \in H^{\alpha}(\mathbb{R}^3)\), we derive by \((3.28)\) that \(V \otimes V \in H_\alpha^\frac{3\alpha}{3-\alpha}(\mathbb{R}^3)\), and then we have
\[
G_k(U_0, V) \in H_\alpha^\frac{3\alpha}{3-\alpha}(\mathbb{R}^3).
\]
This inclusion together with Corollary \((3.10)\) implies \(V_1 \in H_\alpha^\frac{3\alpha-1}{3-\alpha}(\mathbb{R}^3)\).

Letting \(w_1 = V - V_1\), we obtain
\[
(-\Delta)^{\alpha} w_1 + \frac{2 - \alpha}{\alpha} w_1 + \nabla \tilde{P}_1 = \tilde{f}(x) \quad \text{in} \quad \mathbb{R}^3,
\]
where
\[
\tilde{f}(x) = 0 \quad \text{for} \quad x \in \mathbb{B}_{(k-1)R} \quad \text{and} \quad \tilde{P}_1 = P - P_1.
\]
Since \(\tilde{P}_1\) is harmonic in \(\mathbb{B}_{(k-1)R}\), we have \(\tilde{P}_1 \in C^\infty(\mathbb{B}_{(k-1)R})\). Now let \(\tilde{w}_1\) be the solution of
\[
(-\Delta)^{\alpha} \tilde{w}_1 + \tilde{w}_1 = \nabla \tilde{P}_1 \eta_1(x) \in C_0^\infty(\mathbb{R}^3),
\]
where \(\eta_1(x) \in C_0^\infty(\mathbb{B}_{(k-1)R})\) and \(\eta_1(x) \equiv 1\) for \(x \in \mathbb{B}_{(k-1)R}\). By the classical theory, we know that \(\tilde{w}_1 \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\). Again letting \(\tilde{v}_1 = w_1 - \tilde{w}_1\), we see that
\[
(-\Delta)^{\alpha} \tilde{v}_1 + \tilde{v}_1 = 0 \quad \text{in} \quad \mathbb{B}_{(k-2)R}.
\]
In terms of Lemma \((3.11)\) we derive \(\tilde{v}_1 \in C^\infty(\mathbb{B}_{(k-2)R})\). Therefore we have \(w_1 \in C^\infty(\mathbb{B}_{(k-2)R})\).

Since \(V = V_1 + w_1\), we infer from the fact \(\alpha > \frac{5}{6}\) and the Sobolev embedding theorem that
\(V \in H_\alpha^\frac{3\alpha-1}{3-\alpha}(\mathbb{B}_{(k-2)R}) \subset H^\alpha(\mathbb{B}_{(k-2)R})\).

**Step 2. Bootstrapping Arguments.** Again, consider
\[
(-\Delta)^{\alpha} V_2 + \frac{2 - \alpha}{\alpha} V_2 + \nabla P_2 = \text{div} \left( G_k(U_0, V) \right), \quad \text{div} V_2 = 0
\]
where
\[
G_{k-1}(U_0, V) = \frac{1}{2\alpha} \varphi_{k-1} \left( x \otimes V - U_0 \otimes U_0 - U_0 \otimes V - V \otimes U_0 - V \otimes V \right),
\]
and \(\varphi_{k-1} \in C_0^\infty(\mathbb{B}_{(k-1)R}), \varphi_{k-1} \equiv 1\) in \(\mathbb{B}_{(k-2)R}\).

The fact
\[
V \in H_\alpha^\frac{3\alpha-1}{3-\alpha}(\mathbb{B}_{(k-2)R}) \hookrightarrow \left\{ \begin{array}{ll} L_3^{\frac{3\alpha}{3-\alpha}}(\mathbb{B}_{(k-2)R}) & \text{if} \ \alpha < 1, \\ L^q(\mathbb{B}_{(k-2)R}) & \text{for each} \ q > 1 \ \text{if} \ \alpha = 1, \end{array} \right.
\]
and
\[
\nabla V \in H_\alpha^\frac{3\alpha-2}{3-\alpha}(\mathbb{B}_{(k-2)R}) \hookrightarrow L_3^{\frac{3\alpha}{3-\alpha}}(\mathbb{B}_{(k-2)R})
\]
allow us to derive
\[ \text{div} \left( G_{k-1}(U_0, V) \right) \in L^p(\mathbb{R}^3) \quad \text{with} \quad p = \frac{3}{9 - 8\alpha} > 1, \]
where we have used the fact that \( \alpha > \frac{5}{14} \).

By Theorem 2.1 of [10], we have that \( V_2 \in H^2_p(\mathbb{R}^3) \) with \( p = \frac{3}{9 - 8\alpha} \). Performing the same argument as Step 1, we have
\[ V \in H^2_p(\mathbb{R}^3(\mathbb{R}^3) \subset H^{3\alpha - 1}_p(\mathbb{R}^3(\mathbb{R}^3). \]

By the Sobolev embedding theorem, we see
\[ V \in \begin{cases} \frac{3}{9 - 8\alpha}(\mathbb{R}^3) & \text{if } \alpha < \frac{9}{10}, \\ L^q(\mathbb{R}^3) & \text{if } \alpha = \frac{9}{10}, \\ L^\infty(\mathbb{R}^3) & \text{if } \alpha > \frac{9}{10}. \end{cases} \]

Now repeating our previous argument finite times, we can get \( V \in L^\infty(\mathbb{R}^3(\mathbb{R}^3) \) with some integer \( k_0 > 0 \). Once \( V \in L^\infty(\mathbb{R}^3(\mathbb{R}^3) \) derive that \( V \cdot \nabla V \in H^{2\alpha - 1}_p(\mathbb{R}^3(\mathbb{R}^3) \) by (3.28) and immediately obtain by using the argument of Step 1 that
\[ V \in H^{2(2\alpha - 1)}_p(\mathbb{R}^3(\mathbb{R}^3), \]

Furthermore, repeating the process in Step 1, we can show by induction that
\[ V \in H^{m(2\alpha - 1)}_p(\mathbb{R}^3(\mathbb{R}^3) \quad \text{for all } m > 0. \]

This implies \( V \) is smooth. \( \square \)

3.3. Decay estimate for \( V(x) \) when \( \alpha = 1 \). In this subsection, we will prove a few decay estimates of the weak solution to equations (3.2), which is the key point in the proof of the case \( \alpha = 1 \). Specifically,

**Theorem 3.13.** Assume that \( V \in H^1(\mathbb{R}^3) \) is the weak solution of problem (3.2) established in Theorem 3.6. Then \( V \) is smooth and there exists a constant \( C > 0 \) such that
\[ \left\| \left\langle \cdot \right\rangle \nabla P(\cdot) \right\|_{L^2(\mathbb{R}^3)} < +\infty, \]
and
\[ |V|(x) \leq C(1 + |x|)^{-3} \log(1 + |x|) \quad \text{for all } x \in \mathbb{R}^3, \]

By using a profound result De Rham (See for example [Proposition 1.1, [38]]), there exists a pressure \( P \in L^2(\mathbb{R}^3) \) governed by
\[ P = \sum_{i,j=1}^{3} \frac{1}{4\pi} \delta_{x,x} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \left( V^i V^j + U_0^i V^j + V^i U_0^j + U_0^i U_0^j \right) dy \]
such that for all vector fields \( \varphi \in H^1(\mathbb{R}^3) \) satisfying \( \left\| \cdot \right\|_{L^2(\mathbb{R}^3)} < +\infty \), the couple \( (V, P) \) fulfills
\[ \int_{\mathbb{R}^3} \nabla V : \nabla \varphi \, dx - \frac{1}{2} \int_{\mathbb{R}^3} x \cdot \nabla V \cdot \varphi \, dx - \frac{1}{2} \int_{\mathbb{R}^3} V \cdot \varphi \, dx = -\int_{\mathbb{R}^3} P \text{div} \varphi \, dx \]
\[ = \int_{\mathbb{R}^3} V \cdot \nabla \varphi \cdot V \, dx - \int_{\mathbb{R}^3} U_0 \cdot \nabla V \cdot \varphi \, dx - \int_{\mathbb{R}^3} V \cdot \nabla U_0 \cdot \varphi \, dx - \int_{\mathbb{R}^3} U_0 \cdot \nabla U_0 \cdot \varphi \, dx. \]

From Theorem 3.12 it follows that \( V \in C^\infty(\mathbb{R}^3) \).

Since \( P \) satisfies
\[ -\Delta P = \text{div} \left( V \cdot \nabla V + U_0 \cdot \nabla V + V \cdot \nabla U_0 + U_0 \cdot \nabla U_0 \right), \]
we have by the elliptic theory that \( P \in C^\infty(\mathbb{R}^3) \).
From Proposition 2.8 we can conclude that \( U_0 \in L^p(\mathbb{R}^3) \) for all \( p \in (3, +\infty) \), \( \nabla U_0 \in L^2(\mathbb{R}^3) \) and
\[
\sup_{x \in \mathbb{R}^3} \left| \langle x \rangle \nabla U_0 \right|(x) + \sup_{x \in \mathbb{R}^3} \left| \langle x \rangle U_0 \right|(x) < +\infty.
\]
These estimates together with the properties of singular operator and the imbedding theorem imply that for each \( \frac{3}{2} < p \leq 3 \),
\[
\| P \|_{L^p(\mathbb{R}^3)} \leq C \| V \|_{L^{2p}(\mathbb{R}^3)}^2 + C \| V \|_{L^p(\mathbb{R}^3)} \| U_0 \|_{L^{2p}(\mathbb{R}^3)} + C \| U_0 \|_{L^2(\mathbb{R}^3)}^2 < +\infty.
\]
To accomplish the decay estimate, we first prove the \( H^1(\mathbb{R}^3) \)-estimate of \( |x|V \), which is the key estimate in our proof.

**Proposition 3.14.** Let the couple \((V, P) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) satisfying (3.37). Then we have \( W(x) \triangleq |x|V(x) \in H^1(\mathbb{R}^3) \).

**Proof.** Denoting \( h_\varepsilon(x) \triangleq \frac{|x|}{(1+\varepsilon|x|^2)^{\frac{3}{2}}} \) with \( \varepsilon > 0 \), it is easy to check that \( h_\varepsilon^2(x)V(x) \in H^1(\mathbb{R}^2) \) and satisfies
\[
\| |h_\varepsilon^2V(\cdot)| \|_{L^2(\mathbb{R}^3)} \leq C(\varepsilon).
\]
Now, choosing \( \varphi(x) \triangleq \frac{|x|^2}{(1+\varepsilon|x|^2)^2}V(x) = h_\varepsilon^2(x)V(x) \) in equality (3.37), we easily find that the vector field \( W_\varepsilon(x) \triangleq h_\varepsilon(x)V(x) \) fulfills
\[
\begin{align*}
\int_{\mathbb{R}^3} \nabla V : \nabla \left( h_\varepsilon W_\varepsilon \right) dx &- \frac{1}{2} \int_{\mathbb{R}^3} x \cdot \nabla \cdot \left( h_\varepsilon W_\varepsilon \right) dx - \frac{1}{2} \int_{\mathbb{R}^3} W_\varepsilon^2 dx \\
= &\int_{\mathbb{R}^3} \nabla \cdot \left( \nabla h_\varepsilon \otimes W_\varepsilon \right) dx + \int_{\mathbb{R}^3} h_\varepsilon \nabla V : \nabla W_\varepsilon dx \\
= &\| \nabla W_\varepsilon \|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \nabla V : \left( \nabla h_\varepsilon \otimes W_\varepsilon \right) dx - \int_{\mathbb{R}^3} \left( \nabla h_\varepsilon \otimes V \right) : \nabla W_\varepsilon dx.
\end{align*}
\]
A simple calculation yields that
\[
\begin{align*}
\int_{\mathbb{R}^3} \nabla V : \nabla \left( h_\varepsilon W_\varepsilon \right) dx &\triangleq \int_{\mathbb{R}^3} \nabla V : \left( \nabla h_\varepsilon \otimes W_\varepsilon \right) dx + \int_{\mathbb{R}^3} h_\varepsilon \nabla V : \nabla W_\varepsilon dx \\
&= \| \nabla W_\varepsilon \|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \nabla V : \left( \nabla h_\varepsilon \otimes W_\varepsilon \right) dx - \int_{\mathbb{R}^3} \left( \nabla h_\varepsilon \otimes V \right) : \nabla W_\varepsilon dx.
\end{align*}
\]
Since
\[
x \cdot \nabla \left( \frac{|x|}{(1+\varepsilon|x|^2)^{\frac{3}{2}}}V \right) = \frac{|x|}{(1+\varepsilon|x|^2)^{\frac{3}{2}}} x \cdot \nabla V + \frac{|x|}{(1+\varepsilon|x|^2)^{\frac{3}{2}}} V - \frac{3}{2} \frac{\varepsilon |x|^3}{(1+\varepsilon|x|^2)^{\frac{3}{2}}} V
\]
\[
= h_\varepsilon \left( x \cdot \nabla V + \frac{\varepsilon |x|^2}{2 \left( 1+\varepsilon|x|^2 \right)} W_\varepsilon, \right.
\]
\[
\text{we have}
\begin{align*}
-\frac{1}{2} \int_{\mathbb{R}^3} x \cdot \nabla \cdot \left( h_\varepsilon W_\varepsilon \right) dx - \frac{1}{2} \int_{\mathbb{R}^3} W_\varepsilon^2 dx
&= - \frac{1}{2} \int_{\mathbb{R}^3} x \cdot \nabla W_\varepsilon \cdot W_\varepsilon dx - \frac{3}{4} \int_{\mathbb{R}^3} \frac{\varepsilon |x|^2}{1+\varepsilon|x|^2} W_\varepsilon \cdot W_\varepsilon dx \\
&\triangleq \frac{3}{4} \int_{\mathbb{R}^3} W_\varepsilon^2 dx - \frac{3}{4} \int_{\mathbb{R}^3} \frac{\varepsilon |x|^2}{1+\varepsilon|x|^2} W_\varepsilon^2 dx.
\end{align*}
\]
Here we used the fact that \( x(W_\varepsilon)^2 \in H^1_\varepsilon(\mathbb{R}^3) \).
Setting \( g_\varepsilon(x) \triangleq \frac{1}{\sqrt{1+\varepsilon|x|^2}} \) and plugging estimates (3.40) and (3.41) in (3.39), we immediately have

\[
\int_{\mathbb{R}^3} V \cdot \nabla (h_\varepsilon W_\varepsilon) \cdot V \ dx = \int_{\mathbb{R}^3} V \cdot \nabla W_\varepsilon \cdot V \ dx + \int_{\mathbb{R}^3} V \cdot (\nabla h_\varepsilon \otimes W_\varepsilon) \cdot V \ dx
\]

(3.42)

Thanks to \( \text{div} \ V = 0 \), we have that

\[
\int_{\mathbb{R}^3} V \cdot \nabla (h_\varepsilon W_\varepsilon) \cdot V \ dx = \int_{\mathbb{R}^3} V \cdot \nabla W_\varepsilon \cdot V \ dx + \int_{\mathbb{R}^3} V \cdot (\nabla h_\varepsilon \otimes W_\varepsilon) \cdot V \ dx
\]

By the Hölder inequality and the Young inequality, one has

\[
\int_{\mathbb{R}^3} V \cdot \left( \frac{x}{|x|(1+\varepsilon|x|^2)^{\frac{3}{4}}} \otimes W_\varepsilon \right) \cdot V \ dx \leq \|V\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)} \|W_\varepsilon\|_{L^{6}(\mathbb{R}^3)} \leq C\|V\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} + \frac{1}{128} \|\nabla W_\varepsilon\|^2_{L^{2}(\mathbb{R}^3)}
\]

and

\[
- \frac{3}{2} \int_{\mathbb{R}^3} V \cdot \left( \frac{\varepsilon|x|x}{(1+\varepsilon|x|^2)^2} \otimes W_\varepsilon \right) \cdot V \ dx \leq 2\|V\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)} \|W_\varepsilon\|_{L^{6}(\mathbb{R}^3)} \leq C\|V\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} + \frac{1}{128} \|\nabla W_\varepsilon\|^2_{L^{2}(\mathbb{R}^3)}.
\]

Thus, we have

\[
\int_{\mathbb{R}^3} V \cdot \nabla (h_\varepsilon W_\varepsilon) \cdot V \ dx \leq C\|V\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} + \frac{1}{64} \|\nabla W_\varepsilon\|^2_{L^{2}(\mathbb{R}^3)}.
\]

Next, we obtain by some computations that

\[
- \int_{\mathbb{R}^3} \nabla V : (\nabla h_\varepsilon \otimes W_\varepsilon) \ dx
\]

\[
= - \int_{\mathbb{R}^3} \nabla V : \left( \frac{x}{|x|(1+\varepsilon|x|^2)^{\frac{3}{4}}} \otimes W_\varepsilon \right) \ dx + \frac{3}{2} \int_{\mathbb{R}^3} \nabla V : \left( \frac{\varepsilon|x|x}{(1+\varepsilon|x|^2)^2} \otimes W_\varepsilon \right) \ dx.
\]

By the Hölder inequality and the Young inequality, one has

\[
- \int_{\mathbb{R}^3} \nabla V : \left( \frac{x}{|x|(1+\varepsilon|x|^2)^{\frac{3}{4}}} \otimes W_\varepsilon \right) \ dx \leq \|\nabla V\|_{L^2(\mathbb{R}^3)} \|g_\varepsilon W_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C\|\nabla V\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{128} \|g_\varepsilon W_\varepsilon\|^2_{L^2(\mathbb{R}^3)}
\]

and

\[
\frac{3}{2} \int_{\mathbb{R}^3} \nabla V : \left( \frac{\varepsilon|x|x}{(1+\varepsilon|x|^2)^2} \otimes W_\varepsilon \right) \ dx \leq 2\|\nabla V\|_{L^2(\mathbb{R}^3)} \|g_\varepsilon W_\varepsilon\|_{L^2(\mathbb{R}^3)}.
\]
\[
\leq C \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{128} \|g_e W_\varepsilon\|_{L^2(\mathbb{R}^3)}^2.
\]

Therefore we have
\[
- \int_{\mathbb{R}^3} \nabla V : (\nabla h_\varepsilon \otimes W_\varepsilon) \, dx \leq C \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{64} \|g_e W_\varepsilon\|_{L^2(\mathbb{R}^3)}^2.
\]

Note that
\[
\int_{\mathbb{R}^3} (\nabla h_\varepsilon \otimes V) : \nabla W_\varepsilon \, dx = \int_{\mathbb{R}^3} \left( \frac{|x|}{|x| (1 + \varepsilon |x|^2)^{\frac{3}{4}}} \otimes V \right) : \nabla W_\varepsilon \, dx - \frac{3}{2} \int_{\mathbb{R}^3} \left( \frac{\varepsilon |x| x}{(1 + \varepsilon |x|^2)^{\frac{3}{4}}} \otimes V \right) : \nabla W_\varepsilon \, dx,
\]
we have by the Hölder inequality and the Young inequality that
\[
\int_{\mathbb{R}^3} \left( \frac{x}{|x| (1 + \varepsilon |x|^2)^{\frac{3}{4}}} \otimes V \right) : \nabla W_\varepsilon \, dx \leq \|V\|_{L^2(\mathbb{R}^3)} \|\nabla W_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C \|V\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{128} \|\nabla W_\varepsilon\|^2_{L^2(\mathbb{R}^3)}
\]
and
\[
- \frac{3}{2} \int_{\mathbb{R}^3} \left( \frac{\varepsilon |x| x}{(1 + \varepsilon |x|^2)^{\frac{3}{4}}} \otimes V \right) : \nabla W_\varepsilon \, dx \leq 2 \|V\|_{L^2(\mathbb{R}^3)} \|\nabla W_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C \|V\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{128} \|\nabla W_\varepsilon\|^2_{L^2(\mathbb{R}^3)}.
\]

So we have
\[
\int_{\mathbb{R}^3} (\nabla h_\varepsilon \otimes V) : \nabla W_\varepsilon \, dx \leq C \|V\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{64} \|\nabla W_\varepsilon\|^2_{L^2(\mathbb{R}^3)}.
\]

By the Hölder inequality and the Young inequality again, one has that
\[
- \int_{\mathbb{R}^3} U_0 \cdot V_0 \cdot (h_\varepsilon W_\varepsilon) \, dx = - \int_{\mathbb{R}^3} \frac{|x|}{(1 + \varepsilon |x|^2)^{\frac{3}{4}}} V_0 \cdot \nabla U_0 \cdot W_\varepsilon \, dx \leq \sup_{x \in \mathbb{R}^3} \|V_0\|_{L^2(\mathbb{R}^3)} \|\nabla U_0\|_{L^2(\mathbb{R}^3)} \|g_e W_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C \left( \sup_{x \in \mathbb{R}^3} \|V_0\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{128} \|g_e W_\varepsilon\|^2_{L^2(\mathbb{R}^3)} \right)
\]
and
\[
- \int_{\mathbb{R}^3} U_0 \cdot \nabla U_0 \cdot (h_\varepsilon W_\varepsilon) \, dx = - \int_{\mathbb{R}^3} \frac{|x|}{(1 + \varepsilon |x|^2)^{\frac{3}{4}}} U_0 \cdot \nabla U_0 \cdot W_\varepsilon \, dx \leq \sup_{x \in \mathbb{R}^3} \|V_0\|_{L^2(\mathbb{R}^3)} \|\nabla U_0\|_{L^2(\mathbb{R}^3)} \|g_e W_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C \left( \sup_{x \in \mathbb{R}^3} \|V_0\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{128} \|g_e W_\varepsilon\|^2_{L^2(\mathbb{R}^3)} \right).
\]

Similarly, one has
\[
- \int_{\mathbb{R}^3} U_0 \cdot \nabla V \cdot (h_\varepsilon W_\varepsilon) \, dx = - \int_{\mathbb{R}^3} \frac{|x|}{(1 + \varepsilon |x|^2)^{\frac{3}{4}}} U_0 \cdot \nabla V \cdot W_\varepsilon \, dx \leq \sup_{x \in \mathbb{R}^3} \|V_0\|_{L^2(\mathbb{R}^3)} \|\nabla V\|_{L^2(\mathbb{R}^3)} \|g_e W_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq 16 \left( \sup_{x \in \mathbb{R}^3} \|V_0\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{128} \|g_e W_\varepsilon\|^2_{L^2(\mathbb{R}^3)} \right).
\]
Integrating by parts, we have
\[
\int_{\mathbb{R}^3} P \div (h_\varepsilon W_\varepsilon) \, dx = \int_{\mathbb{R}^3} \frac{x}{|x|(1 + \varepsilon|x|^2)^{\frac{3}{2}}} \cdot W_\varepsilon P \, dx - \frac{3}{2} \int_{\mathbb{R}^3} \frac{\varepsilon |x|}{(1 + \varepsilon|x|^2)^{\frac{3}{2}}} \cdot W_\varepsilon P \, dx.
\]
Since \( P \in L^2(\mathbb{R}^3) \), in terms of the Hölder inequality, we have
\[
\int_{\mathbb{R}^3} \frac{x}{|x|(1 + \varepsilon|x|^2)^{\frac{3}{2}}} \cdot W_\varepsilon P \, dx \leq \| P \|_{L^2(\mathbb{R}^3)} \| g_\varepsilon W_\varepsilon \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \| P \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{32} \| g_\varepsilon W_\varepsilon \|^2_{L^2(\mathbb{R}^3)}
\]
and
\[
- \frac{3}{2} \int_{\mathbb{R}^3} \frac{\varepsilon |x|}{(1 + \varepsilon|x|^2)^{\frac{3}{2}}} \cdot W_\varepsilon P \, dx \leq 2 \| P \|_{L^2(\mathbb{R}^3)} \| g_\varepsilon W_\varepsilon \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \| P \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{32} \| g_\varepsilon W_\varepsilon \|^2_{L^2(\mathbb{R}^3)}.
\]
Plugging both estimates above in (3.42), we obtain
\[
\int_{\mathbb{R}^3} P \div (h_\varepsilon W_\varepsilon) \, dx \leq \frac{C}{\varepsilon} \| P \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{16} \| g_\varepsilon W_\varepsilon \|^2_{L^2(\mathbb{R}^3)}.
\]
Collecting all these estimates and using the fact that \((V, P) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\), we eventually have
\[
\| g_\varepsilon W_\varepsilon \|^2_{L^2(\mathbb{R}^3)} + \| \nabla W_\varepsilon \|^2_{L^2(\mathbb{R}^3)} \leq C(U_0, V).
\]
By the Lebesgue dominated convergence theorem, we get by taking \( \varepsilon \to 0+ \) in the above inequality that
\[
\| W \|^2_{L^2(\mathbb{R}^3)} + \| \nabla W \|^2_{L^2(\mathbb{R}^3)} \leq C(U_0, V).
\]
This implies the desired result in Proposition 3.14. 

With this weighted \( H^1 \)-estimate, we are going to improve the regularity of solution. Before doing this, we need to establish the following regularity estimate.

**Lemma 3.15.** Let \( f \in L^2(\mathbb{R}^3) \) and the divergence-free vector field \( \nabla \in H^1(\mathbb{R}^3) \). Assume that \((V, P)\) is a weak solution of the following problem

\[
\begin{cases}
- \Delta V + \nabla \cdot \nabla V - \frac{1}{2} (x \cdot \nabla V + V) + \nabla P = f & \text{in } \mathbb{R}^3 \\
div V = 0
\end{cases}
\]

that is, \( V \in L^2(\mathbb{R}^3) \), \( |x|V(x) \in L^2(\mathbb{R}^3) \), \( P \in L^2(\mathbb{R}^3) \), and for all vector fields \( \varphi \in H^1(\mathbb{R}^3) \),

\[
\int_{\mathbb{R}^3} \nabla V : \nabla \varphi \, dx - \frac{1}{2} \int_{\mathbb{R}^3} (x \cdot \nabla V + V) \cdot \varphi \, dx = \int_{\mathbb{R}^3} P \div \varphi \, dx - \int_{\mathbb{R}^3} \nabla \cdot \nabla \varphi \cdot V \, dx - \int_{\mathbb{R}^3} f \cdot \varphi \, dx.
\]

Then there exists a positive constant \( C \) such that

\[
\| V \|_{H^2(\mathbb{R}^3)} \leq C \left( \| \nabla \|^2_{H^1(\mathbb{R}^3)} \| V \|_{H^1(\mathbb{R}^3)} + \| f \|_{L^2(\mathbb{R}^3)} \right).
\]

**Proof.** We denoted by \( D^h_k u \) the difference quotient

\[
D^h_k u = \frac{u(x + he_k) - u(x)}{h} \quad \text{with} \quad h \in \mathbb{R}\ \backslash \{0\}.
\]

We take \( \varphi(x) \triangleq D^h_k (D^h_k V) \), we compute

\[
- \int_{\mathbb{R}^3} \nabla V : \nabla D^h_k (D^h_k V) \, dx = - \int_{\mathbb{R}^3} \nabla V : D^{-h}_k (D^h_k \nabla V) \, dx
\]

\[
= \int_{\mathbb{R}^3} (D^h_k \nabla V) : (D^h_k \nabla V) \, dx = \| D^h_k \nabla V \|^2_{L^2(\mathbb{R}^3)}.
\]
Integrating by parts, we get

\[
\frac{1}{2} \int_{\mathbb{R}^3} \mathbf{x} \cdot \nabla \mathbf{V} \cdot D^{-h}_k \left( D_h^k \mathbf{V} \right) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{V} \cdot D^{-h}_k \left( D_h^k \mathbf{V} \right) \, dx
\]

\[
= - \frac{1}{2} \int_{\mathbb{R}^3} D_h^k \mathbf{R} \cdot D_h^k \mathbf{V} \, dx - \frac{1}{2} \int_{\mathbb{R}^3} D_h^k \mathbf{V} \cdot D_h^k \mathbf{V} \, dx
\]

\[
= - \frac{1}{2} \int_{\mathbb{R}^3} \left( \mathbf{x} + h \mathbf{e}_k \right) \cdot \nabla D_h^k \mathbf{V} \cdot D_h^k \mathbf{V} \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \left( D_h^k \mathbf{V} \right) \cdot \nabla \mathbf{V} \cdot D_h^k \mathbf{V} \, dx - \frac{1}{2} \int_{\mathbb{R}^3} D_h^k \mathbf{V} \cdot D_h^k \mathbf{V} \, dx
\]

\[
= - \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_k} V \cdot D_h^k \mathbf{V} \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_k} \mathbf{V} \cdot D_h^k \mathbf{V} \, dx - \frac{1}{2} \int_{\mathbb{R}^3} D_h^k \mathbf{V} \cdot D_h^k \mathbf{V} \, dx
\]

\[
= - \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_k} \mathbf{V} \cdot D_h^k \mathbf{V} \, dx + \| D_h^k \mathbf{V} \|^2_{L^2(\mathbb{R}^3)}.
\]

Next, we turn to deal with the term involving pressure. Integrating by parts gives

\[
- \int_{\mathbb{R}^3} \mathbf{P} \text{div} \ D^{-h}_k \left( D_h^k \mathbf{V} \right) \, dx = \int_{\mathbb{R}^3} D_h^k \mathbf{P} \text{div} \ D_h^k \mathbf{V} \, dx = 0.
\]

According to the fact that div \( \mathbf{V} = 0 \), we obtain

\[
- \int_{\mathbb{R}^3} \nabla \cdot \nabla D^{-h}_k \left( D_h^k \mathbf{V} \right) \cdot \mathbf{V} \, dx = \int_{\mathbb{R}^3} \mathbf{V} \cdot \nabla \left( \mathbf{x} + h \mathbf{e}_k \right) \cdot \nabla D_h^k \mathbf{V} (\mathbf{x}) \cdot D_h^k \mathbf{V} (\mathbf{x}) \, dx
\]

\[
+ \int_{\mathbb{R}^3} D_h^k \mathbf{V} (\mathbf{x}) \cdot \nabla D_h^k \mathbf{V} (\mathbf{x}) \cdot \mathbf{V} (\mathbf{x}) \, dx
\]

\[
= \int_{\mathbb{R}^3} D_h^k \mathbf{V} (\mathbf{x}) \cdot \nabla D_h^k \mathbf{V} (\mathbf{x}) \cdot \mathbf{V} (\mathbf{x}) \, dx.
\]

So, we have from (3.43) that

\[
\| D_h^k \mathbf{V} \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2} \| D_h^k \mathbf{V} \|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} f(\mathbf{V}) D^{-h}_k D_h^k \mathbf{V} (\mathbf{x}) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_k} \mathbf{V} \cdot D_h^k \mathbf{V} \, dx
\]

\[
+ \int_{\mathbb{R}^3} D_h^k \mathbf{V} (\mathbf{x}) \cdot \nabla D_h^k \mathbf{V} (\mathbf{x}) \cdot \mathbf{V} (\mathbf{x}) \, dx.
\]

Also, we see that

\[
\int_{\mathbb{R}^3} D^{-h}_k \left( D_h^k \mathbf{V} \right) \cdot D^{-h}_k \left( D_h^k \mathbf{V} \right) \, dx = \int_{\mathbb{R}^3} D_h^k \left( D_h^k \mathbf{V} \right) \cdot D_h^k \left( D_h^k \mathbf{V} \right) \, dx \leq C \| \nabla D_h^k \mathbf{V} \|^2_{L^2(\mathbb{R}^3)}.
\]

This estimate enables us to conclude that

\[
\int_{\mathbb{R}^3} f(\mathbf{V}) D^{-h}_k D_h^k \mathbf{V} (\mathbf{x}) \, dx \leq \| f \|_{L^2(\mathbb{R}^3)} \| D^{-h}_k D_h^k \mathbf{V} \|_{L^2(\mathbb{R}^3)}
\]

\[
\leq C \| f \|_{L^2(\mathbb{R}^3)} \| \nabla D_h^k \mathbf{V} \|_{L^2(\mathbb{R}^3)}
\]

\[
\leq C \| f \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{16} \| \nabla D_h^k \mathbf{V} \|^2_{L^2(\mathbb{R}^3)}.
\]

By the Hölder inequality, one has

\[
\frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_k} \mathbf{V} \cdot D_h^k \mathbf{V} \, dx \leq \| \partial_{x_k} \mathbf{V} \|_{L^2(\mathbb{R}^3)} \| D_h^k \mathbf{V} \|_{L^2(\mathbb{R}^3)} \leq C \| \nabla \mathbf{V} \|^2_{L^2(\mathbb{R}^3)}.
\]

By the interpolation inequality, we see that

\[
\int_{\mathbb{R}^3} D_h^k \mathbf{V} (\mathbf{x}) \cdot \nabla D_h^k \mathbf{V} (\mathbf{x}) \cdot \mathbf{V} (\mathbf{x}) \, dx \leq \| \mathbf{V} \|_{L^6(\mathbb{R}^3)} \| D_h^k \mathbf{V} \|_{L^3(\mathbb{R}^3)} \| D_h^k \nabla \mathbf{V} \|_{L^2(\mathbb{R}^3)}
\]

\[
\leq C \| \mathbf{V} \|_{L^6(\mathbb{R}^3)} \| D_h^k \mathbf{V} \|^\frac{3}{2}_{L^2(\mathbb{R}^3)} \| D_h^k \nabla \mathbf{V} \|^\frac{3}{2}_{L^2(\mathbb{R}^3)}
\]

\[
\leq C \| \nabla \mathbf{V} \|^\frac{3}{2}_{L^2(\mathbb{R}^3)} \| \nabla \mathbf{V} \|^\frac{3}{2}_{L^2(\mathbb{R}^3)} + \frac{1}{16} \| D_h^k \nabla \mathbf{V} \|^2_{L^2(\mathbb{R}^3)}.
\]
Collecting all estimates yields
\[
\|D_h^k \nabla V\|_{L^2(\mathbb{R}^3)}^2 + \|D_h^k V\|_{L^2(\mathbb{R}^3)}^2 \leq C \|\nabla V\|_{L^2(\mathbb{R}^3)}^4 \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 + C \|f\|_{L^2(\mathbb{R}^3)}^2.
\]
Taking \( h \to 0 \) in the above inequality, we readily have
\[
\|\nabla^2 V\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 \leq C \|\nabla V\|_{L^2(\mathbb{R}^3)}^4 \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 + C \|f\|_{L^2(\mathbb{R}^3)}^2.
\]
This completes the proof of the lemma. \( \square \)

According to Lemma 3.15, we will show the \( H^2(\mathbb{R}^3) \)-estimate for \( V \) and the \( \dot{H}^1(\mathbb{R}^3) \)-estimate for \( |x|P \).

**Proposition 3.16.** Let the couple \((V, P) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) satisfying (3.37). Then we have \( V(x) \in H^2(\mathbb{R}^3) \) and
\[
Q(x) \triangleq |x|P \in \dot{H}^1(\mathbb{R}^3).
\]

**Proof.** By Proposition 3.14, we have that \( W \in H^1(\mathbb{R}^3) \), moreover, we obtain by Lemma 3.15 that \( W \in H^2(\mathbb{R}^3) \).

Thanks to (3.38), we see that \( Q = |x|P \) satisfies
\[
-\Delta Q = |x| \text{div} (V \cdot \nabla V + U_0 \cdot \nabla V + V \cdot \nabla U_0 + U_0 \cdot \nabla U_0) - 2 \frac{x}{|x|} \cdot \nabla P - \frac{2}{|x|} P.
\]
By the Cauchy-Schwarz inequality and the Hardy inequality, we get that for any \( \varphi \in C_0^\infty(\mathbb{R}^3) \)
\[
\left\langle \frac{2}{|x|} P, \varphi \right\rangle \leq C \|P\|_{L^2(\mathbb{R}^3)} \|\nabla \varphi\|_{L^2(\mathbb{R}^3)}
\]
and
\[
\left\langle \frac{x}{|x|} \cdot \nabla P, \varphi \right\rangle = -\int_{\mathbb{R}^3} P \partial_i \left( \frac{x_i}{|x|} \varphi \right) \, dx \leq C \|P\|_{L^2(\mathbb{R}^3)} \|\nabla \varphi\|_{L^2(\mathbb{R}^3)} + \|P\|_{L^2(\mathbb{R}^3)} \left\| \frac{\varphi}{|x|} \right\|_{L^2(\mathbb{R}^3)} \leq C \|P\|_{L^2(\mathbb{R}^3)} \|\nabla \varphi\|_{L^2(\mathbb{R}^3)}.
\]
Similarly, one has that for any \( \varphi \in C_0^\infty(\mathbb{R}^3) \),
\[
\left\langle |x| \text{div} (V \cdot \nabla V + U_0 \cdot \nabla V + V \cdot \nabla U_0 + U_0 \cdot \nabla U_0), \varphi \right\rangle = -\int_{\mathbb{R}^3} (V \cdot \nabla V + U_0 \cdot \nabla V + V \cdot \nabla U_0 + U_0 \cdot \nabla U_0) \nabla(|x| \varphi) \, dx \leq C \left( \|W\|_{L^4(\mathbb{R}^3)} (\|\nabla V\|_{L^4(\mathbb{R}^3)} + \|\nabla U_0\|_{L^4(\mathbb{R}^3)}) + \|\cdot \cdot \cdot\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla V\|_{L^2(\mathbb{R}^3)} \right\| \right)
\]
Combining these results and using the density argument yield the required estimate in Proposition 3.15. \( \square \)

With these regularity estimates in hand, we are going to show \( H^2 \)-estimate for \( |x|V(x) \) which implies that \( |x| |V(x) \) is bounded.

**Proposition 3.17.** Let the couple \((V, P) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) satisfying (3.37). Then we have \( W(x) = |x|V(x) \in H^2(\mathbb{R}^3) \).

Moreover,
\[
\| \langle \cdot \rangle V(\cdot) \|_{\mathcal{B}_{p,1}^2(\mathbb{R}^3)} < +\infty.
\]
Proof. Taking $\varphi(x) = -D_k^{-h} h^2 \cdot D_k^h V$ in equality \((3.37)\), we immediately have

$$
\int_{\mathbb{R}^3} \nabla V : \nabla \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx - \frac{1}{2} \int_{\mathbb{R}^3} x \cdot \nabla \cdot \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx \\
- \frac{1}{2} \int_{\mathbb{R}^3} V \cdot \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx - \int_{\mathbb{R}^3} P \, \text{div} \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx \\
= \int_{\mathbb{R}^3} V \cdot \nabla \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx - \int_{\mathbb{R}^3} U_0 \, \nabla V \cdot \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx \\
- \int_{\mathbb{R}^3} V \cdot \nabla U_0 \cdot \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx - \int_{\mathbb{R}^3} U_0 \, \nabla U_0 \cdot \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx.
$$

Some calculations yield

$$
\int_{\mathbb{R}^3} \nabla V : \nabla \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx \\
= \int_{\mathbb{R}^3} D_k^h \nabla V : \nabla \left( h^2 \cdot D_k^h V \right) \, dx \\
= \left\| h \cdot D_k^h \nabla V \right\|^2_{L^2(\mathbb{R}^3)} + 2 \int_{\mathbb{R}^3} h \cdot D_k^h \nabla V : \left( \frac{x}{|x|(1+|x|^2)^{\frac{3}{4}}} \otimes D_k^h V \right) \, dx \\
- 3 \int_{\mathbb{R}^3} h \cdot D_k^h \nabla V : \left( \frac{\varepsilon |x|x}{(1+\varepsilon |x|^2)^{\frac{3}{4}}} \otimes D_k^h V \right) \, dx.
$$

By the Hölder inequality and the Young inequality, one has

$$
-2 \int_{\mathbb{R}^3} h \cdot D_k^h \nabla V : \left( \frac{x}{|x|(1+|x|^2)^{\frac{3}{4}}} \otimes D_k^h V \right) \, dx \\
+ 3 \int_{\mathbb{R}^3} h \cdot D_k^h \nabla V : \left( \frac{\varepsilon |x|x}{(1+\varepsilon |x|^2)^{\frac{3}{4}}} \otimes D_k^h V \right) \, dx \\
\leq C \left\| \nabla V \right\|_{L^2(\mathbb{R}^3)} \left\| h \cdot D_k^h \nabla V \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| \nabla V \right\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{64} \left\| h \cdot D_k^h \nabla V \right\|^2_{L^2(\mathbb{R}^3)}.
$$

By the fact that $\text{div} V = 0$, we obtain

$$
- \frac{1}{2} \int_{\mathbb{R}^3} x \cdot \nabla \cdot \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx - \frac{1}{2} \int_{\mathbb{R}^3} V \cdot \left( -D_k^{-h} h^2 \cdot D_k^h V \right) \, dx \\
= - \frac{1}{2} \int_{\mathbb{R}^3} (x + h \cdot e_k) \nabla D_k^h V \cdot h^2 \cdot D_k^h V \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_k} V \cdot (h^2 \cdot D_k^h V) \, dx - \frac{1}{2} \int_{\mathbb{R}^3} h^2 \cdot D_k^h V \cdot D_k^h V \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^3} (x \cdot \nabla h) \cdot D_k^h V \cdot (h \cdot D_k^h V) \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_k} D_k^h V \cdot h^2 \cdot D_k^h V \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x_k} D_k^h V \cdot (h^2 \cdot D_k^h V) \, dx \\
+ \frac{1}{4} \int_{\mathbb{R}^3} h^2 \cdot D_k^h V \cdot D_k^h V \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^3} (x \cdot \nabla h) \cdot D_k^h V \cdot (h \cdot D_k^h V) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \partial_{x_k} h \cdot D_k^h V \cdot D_k^h V \, dx + \frac{1}{4} \int_{\mathbb{R}^3} h^2 \cdot D_k^h V \cdot D_k^h V \, dx.
$$

We see that

$$
\frac{1}{2} \int_{\mathbb{R}^3} (x \cdot \nabla h) \cdot D_k^h V \cdot (h \cdot D_k^h V) \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^3} \frac{|x|^3}{(1+|x|^2)^{\frac{3}{4}}} (h \cdot D_k^h V) \cdot D_k^h V \, dx - \frac{3}{4} \int_{\mathbb{R}^3} \frac{\varepsilon |x|^3}{(1+\varepsilon |x|^2)^{\frac{3}{4}}} (h \cdot D_k^h V) \cdot D_k^h V \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^3} h^2 \cdot D_k^h V \cdot D_k^h V \, dx - \frac{3}{4} \int_{\mathbb{R}^3} \frac{\varepsilon |x|^2}{1+\varepsilon |x|^2} h^2 \cdot D_k^h V \cdot D_k^h V \, dx.
$$
Thus, we have
\[
- \frac{1}{2} \int_{\mathbb{R}^3} x \cdot \nabla \cdot ( - D_{k}^{-h} h_{k}^{2} D_{k}^{h} V ) \, dx = \frac{1}{2} \int_{\mathbb{R}^3} V \cdot ( - D_{k}^{-h} h_{k}^{2} D_{k}^{h} V ) \, dx
\]
\[
= \frac{3}{4} \int_{\mathbb{R}^3} \frac{\varepsilon |x|^2}{1 + \varepsilon |x|^2} h_{k}^{2} D_{k}^{h} V \cdot D_{k}^{h} V \, dx + \frac{(1 + h)}{4} \int_{\mathbb{R}^3} (\partial_{x_{k}} h_{k}^{2}) D_{k}^{h} V \cdot D_{k}^{h} V \, dx.
\]
We calculate
\[
\frac{1}{4} \int_{\mathbb{R}^3} (\partial_{x_{k}} h_{k}^{2}) D_{k}^{h} V \cdot D_{k}^{h} V \, dx = \frac{1}{2} \int_{\mathbb{R}^3} \frac{x_{k}}{|x|(1 + \varepsilon |x|^2)\frac{3}{2}} (h_{k} D_{k}^{h} V ) \cdot D_{k}^{h} V \, dx
\]
\[
- \frac{3(1 + h)}{4} \int_{\mathbb{R}^3} \frac{\varepsilon |x|x_{k}}{(1 + \varepsilon |x|^2)\frac{3}{2}} (h_{k} D_{k}^{h} V ) \cdot D_{k}^{h} V \, dx.
\]
Since $|h| \leq 1$, we have by the Hölder inequality and the Young inequality that
\[
\leq C \|\nabla V\|_{L^2(\mathbb{R}^3)} \|g_{\varepsilon} (h_{k} D_{k}^{h} V )\|_{L^2(\mathbb{R}^3)} \leq C \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{64} \|g_{\varepsilon} (h_{k} D_{k}^{h} V )\|_{L^2(\mathbb{R}^3)}^2
\]
and
\[
\leq C \|\nabla V\|_{L^2(\mathbb{R}^3)} \|g_{\varepsilon} (h_{k} D_{k}^{h} V )\|_{L^2(\mathbb{R}^3)} \leq C \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{64} \|g_{\varepsilon} (h_{k} D_{k}^{h} V )\|_{L^2(\mathbb{R}^3)}^2.
\]
Therefore, we have
\[
- \frac{(1 + h)}{4} \int_{\mathbb{R}^3} (\partial_{x_{k}} h_{k}^{2}) D_{k}^{h} V \cdot D_{k}^{h} V \, dx \leq C \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{32} \|g_{\varepsilon} (h_{k} D_{k}^{h} V )\|_{L^2(\mathbb{R}^3)}^2.
\]
A simple calculation yields
\[
\int_{\mathbb{R}^3} P \, \text{div} ( - D_{k}^{-h} h_{k}^{2} D_{k}^{h} V ) \, dx = \int_{\mathbb{R}^3} D_{k}^{h} P \, \text{div} (h_{k}^{2} D_{k}^{h} V ) \, dx
\]
\[
= \int_{\mathbb{R}^3} D_{k}^{h} P \, \nabla h_{k}^{2} \cdot D_{k}^{h} V \, dx
\]
\[
= 2 \int_{\mathbb{R}^3} D_{k}^{h} P \, \frac{x}{|x|(1 + \varepsilon |x|^2)\frac{3}{2}} \cdot (h_{k} D_{k}^{h} V ) \, dx
\]
\[
= 2 \int_{\mathbb{R}^3} D_{k}^{h} P \, \frac{x}{|x|(1 + \varepsilon |x|^2)\frac{3}{2}} \cdot (h_{k} D_{k}^{h} V ) \, dx
\]
\[
- 3 \int_{\mathbb{R}^3} D_{k}^{h} P \, \frac{\varepsilon |x|x}{(1 + \varepsilon |x|^2)\frac{3}{2}} \cdot (h_{k} D_{k}^{h} V ) \, dx.
\]
By the Hölder inequality and the Young inequality, one has
\[
2 \int_{\mathbb{R}^3} D_{k}^{h} P \, \frac{x}{|x|(1 + \varepsilon |x|^2)\frac{3}{2}} \cdot (h_{k} D_{k}^{h} V ) \, dx - 3 \int_{\mathbb{R}^3} D_{k}^{h} P \, \frac{\varepsilon |x|x}{(1 + \varepsilon |x|^2)\frac{3}{2}} \cdot (h_{k} D_{k}^{h} V ) \, dx
\]
\[
\leq 2 \|P\|_{L^2(\mathbb{R}^3)} \|D_{k}^{h} V\|_{L^2(\mathbb{R}^3)} + 3 \|P\|_{L^2(\mathbb{R}^3)} \|D_{k}^{h} V\|_{L^2(\mathbb{R}^3)} \leq C \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla V\|_{L^2(\mathbb{R}^3)}.
\]
For the convection term, we get by integration by parts that
\[
\int_{\mathbb{R}^3} V \cdot \nabla (- D_{k}^{-h} h_{k}^{2} D_{k}^{h} V ) \cdot V \, dx
\]
\[
= \int_{\mathbb{R}^3} (V + h_{k}^2) \cdot \nabla (h_{k}^{2} D_{k}^{h} V ) \cdot D_{k}^{h} V \, dx + \int_{\mathbb{R}^3} D_{k}^{h} V \cdot \nabla (h_{k}^{2} D_{k}^{h} V ) \cdot V \, dx.
Similarly, we have
\[
\int_{\mathbb{R}^3} (\varepsilon D_k h V) \cdot \nabla (h \varepsilon D_k h V) \cdot V \, dx 
\leq C \|V\|_{L^2(\mathbb{R}^3)} \|\nabla V\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{36} \|g_\varepsilon (h \varepsilon D_k h V)\|^2_{L^2(\mathbb{R}^3)}.
\]
A simple calculation yields
\[
\int_{\mathbb{R}^3} h \varepsilon D_k h V \cdot \nabla (h \varepsilon D_k h V) \cdot V \, dx
\leq \frac{1}{36} \|g_\varepsilon (h \varepsilon D_k h V)\|^2_{L^2(\mathbb{R}^3)}.
\]
By the Hölder inequality, we find that
\[
\leq C \|V\|_{L^\infty(\mathbb{R}^3)} \|\nabla V\|_{L^2(\mathbb{R}^3)} \|g_\varepsilon (h \varepsilon D_k h V)\|_{L^2(\mathbb{R}^3)}.
\]
Similarly, we have
\[
\int_{\mathbb{R}^3} (h \varepsilon D_k h V) \cdot \nabla (h \varepsilon D_k h V) \cdot V \, dx
\leq C \|V\|_{L^\infty(\mathbb{R}^3)} \|\nabla V\|_{L^2(\mathbb{R}^3)} \|g_\varepsilon (h \varepsilon D_k h V)\|^2_{L^2(\mathbb{R}^3)}.
\]
We observe that
\[- \int_{\mathbb{R}^3} V \cdot \nabla U_0 \cdot (- D_k^{-h} \partial^2 D_k h^2 V) \, dx \]
\[= - \int_{\mathbb{R}^3} V(x + h \epsilon) \cdot \nabla D_k U_0 \cdot (h^2 D_k h^2 V) \, dx - \int_{\mathbb{R}^3} D_k h^2 V \cdot \nabla U_0 \cdot (h^2 D_k h^2 V) \, dx \]
\[\leq C \| \cdot \|^2 \nabla^2 U_0(\cdot) \|_{L^\infty(\mathbb{R}^3)} \| \nabla V \|_{L^2(\mathbb{R}^3)} \| V \|_{L^2(\mathbb{R}^3)} + C \| \cdot \|^2 \nabla U_0(\cdot) \|_{L^\infty(\mathbb{R}^3)} \| \nabla V \|_{L^2(\mathbb{R}^3)}^2.\]

Similarly, we can show that
\[- \int_{\mathbb{R}^3} U_0 \cdot \nabla U_0 \cdot (- D_k^{-h} \partial^2 D_k h^2 V) \, dx \]
\[= - \int_{\mathbb{R}^3} U_0(x + h \epsilon) \cdot \nabla D_k h^2 U_0 \cdot (h^2 D_k h^2 V) \, dx - \int_{\mathbb{R}^3} D_k h^2 U_0 \cdot \nabla U_0 \cdot (h^2 D_k h^2 V) \, dx \]
\[\leq C \| \cdot \|^2 \nabla^2 U_0(\cdot) \|_{L^\infty(\mathbb{R}^3)} \| \nabla V \|_{L^2(\mathbb{R}^3)} \| U_0 \|_{L^2(\mathbb{R}^3)} + \| \nabla U_0 \|_{L^2(\mathbb{R}^3)} \| \nabla V \|_{L^2(\mathbb{R}^3)}.\]

Collecting all estimates implies
\[\| h \epsilon \partial^2 D_k h^2 V \|^2_{L^2(\mathbb{R}^3)} + \| g_\epsilon (h \epsilon D_k h^2 V) \|^2_{L^2(\mathbb{R}^3)} \leq C \langle V, U_0 \rangle.\]

Taking \( h \to 0 \) entails
\[\| \Delta W \|^2_{L^2(\mathbb{R}^3)} + \| W \|^2_{L^2(\mathbb{R}^3)} < +\infty.\]

This estimate together with the embedding theorem that \( H^2(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{\infty,1}(\mathbb{R}^3) \) entails
\[\| \cdot \|_{\dot{B}^0_{\infty,1}(\mathbb{R}^3)} < +\infty.\]

This combined with the fact that \( \| V \|_{\dot{B}^0_{\infty,1}(\mathbb{R}^3)} < +\infty \) enables us to conclude the desired result in the proposition.

Next, we will further improve the regularity for the couple \((V, P)\) by using the bootstrapping argument.

**Proposition 3.18.** Let the couple \((V, P) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) satisfying (3.37). Then we have \( V \in H^3(\mathbb{R}^3) \) and \( E \triangleq |x| \nabla V \in H^2(\mathbb{R}^3) \).

**Proof.** By Theorem 3.12, Proposition 3.16 and Proposition 3.17, we know that \( V \) solves
\[- \Delta V = \frac{1}{2} x \cdot \nabla V + \frac{1}{2} V - P \langle V \cdot \nabla V - U_0 \cdot \nabla V - V \cdot \nabla U_0 - U_0 \cdot \nabla U_0 \rangle.\]

By the Hörmander inequality, one has
\[\| \cdot \|_{H^1(\mathbb{R}^3)} \leq \| \nabla W \|_{H^1(\mathbb{R}^3)} + \| V \|_{H^1(\mathbb{R}^3)} \leq \| W \|_{H^2(\mathbb{R}^3)} + \| V \|_{H^1(\mathbb{R}^3)}.\]

Similarly, we have
\[\| V \cdot \nabla V \|_{H^1(\mathbb{R}^3)} \leq \| \nabla V \|_{L^2(\mathbb{R}^3)} + C \| V \|_{L^\infty(\mathbb{R}^3)} \| V \|_{H^2(\mathbb{R}^3)}\]
and
\[\| V \cdot \nabla U_0 \|_{H^1(\mathbb{R}^3)} + \| U_0 \cdot \nabla V \|_{H^1(\mathbb{R}^3)} \leq 2 \| \nabla V \|_{L^2(\mathbb{R}^3)} \| \nabla U_0 \|_{L^2(\mathbb{R}^3)} + C \| V \|_{L^\infty(\mathbb{R}^3)} \| U_0 \|_{H^2(\mathbb{R}^3)} + C \| U_0 \|_{L^\infty(\mathbb{R}^3)} \| V \|_{H^2(\mathbb{R}^3)}.\]

By the Hörmander inequality and the Young inequality again, we obtain
\[\| U_0 \cdot \nabla U_0 \|_{H^1(\mathbb{R}^3)} \leq \| \nabla U_0 \|_{L^2(\mathbb{R}^3)} + C \| U_0 \|_{L^\infty(\mathbb{R}^3)} \| U_0 \|_{H^2(\mathbb{R}^3)}.\]

By the elliptic regularity theory, we immediately obtain
\[\| U \|_{H^3(\mathbb{R}^3)} < +\infty.\]
Setting $E_k \triangleq |x|\partial_{x_k} V$ and $P_k \triangleq |x|\partial_{x_k} P$, we immediately find that
\begin{equation}
- \Delta E_k - V \cdot \nabla E_k - \frac{1}{2} x \cdot \nabla E_k - \frac{1}{2} E_k + \nabla P_k
\end{equation}
\begin{equation}
= - |x|\partial_{x_k} \left( V \cdot \nabla U_0 + U_0 \cdot \nabla V + U_0 \cdot \nabla U_0 \right) - E_k \cdot \nabla V - V \cdot x\partial_{x_k} V - E_k
\end{equation}
\begin{equation}
- 2 \frac{x}{|x|} \cdot \nabla \partial_{x_k} V - 2 \frac{x}{|x|} \partial_{x_k} V + \frac{x}{|x|} \partial_{x_k} P.
\end{equation}
We see that
\begin{equation}
\| E_k \|_{L^2(\mathbb{R}^3)} \leq \| W \|_{L^2(\mathbb{R}^3)} + \| V \|_{L^2(\mathbb{R}^3)}
\end{equation}
and
\begin{equation}
\left\| \frac{x}{|x|} \cdot \nabla \partial_{x_k} V \right\|_{L^2(\mathbb{R}^3)} \leq C \| V \|_{H^2(\mathbb{R}^3)}.
\end{equation}
By the Hardy inequality, one has
\begin{equation}
\left\| \left(\frac{1}{| \cdot |} \right) \partial_{x_k} V \right\|_{L^2(\mathbb{R}^3)} \leq C \| V \|_{H^2(\mathbb{R}^3)}.
\end{equation}
With the help of the Hölder inequality, we obtain
\begin{equation}
\| V \cdot x\partial_{x_k} V \|_{L^2(\mathbb{R}^3)} \leq C \| W \|_{L^\infty(\mathbb{R}^3)} \| V \|_{H^1(\mathbb{R}^3)}
\end{equation}
and
\begin{equation}
\left\| \frac{x}{|x|} \cdot \partial_{x_k} \left( V \cdot \nabla U_0 + U_0 \cdot \nabla V + U_0 \cdot \nabla U_0 \right) \right\|_{L^2(\mathbb{R}^3)}
\leq C \left\| \frac{x}{|x|} \cdot V \right\|_{L^\infty(\mathbb{R}^3)} \| U_0 \|_{H^2(\mathbb{R}^3)} + C \left\| \frac{x}{|x|} \cdot V \right\|_{L^\infty(\mathbb{R}^3)} \| V \|_{H^1(\mathbb{R}^3)}
\end{equation}
\begin{equation}
+ C \left\| \frac{x}{|x|} \cdot U_0 \right\|_{L^\infty(\mathbb{R}^3)} \| V \|_{H^2(\mathbb{R}^3)} + C \left\| \frac{x}{|x|} \cdot \nabla U_0 \right\|_{L^\infty(\mathbb{R}^3)} \| U_0 \|_{H^1(\mathbb{R}^3)}.
\end{equation}
By resorting to Lemma \[ \text{[3.15]} \] we know that
\begin{equation}
\left\| \frac{x}{|x|} \cdot \partial_{x_k} V \right\|_{H^2(\mathbb{R}^3)} < +\infty.
\end{equation}
This estimate together with the embedding theorem leads to
\begin{equation}
\sup_{x \in \mathbb{R}^3} |x|\|\nabla V\|(x) < +\infty.
\end{equation}
We finish the proof of the proposition.

\textbf{Proposition 3.19.} Let the couple $(V, P) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfying \[ \text{[3.37]} \]. Then we have
\begin{equation}
\sup_{x \in \mathbb{R}^3} |x|^2 \|\nabla V\|(x) + \sup_{x \in \mathbb{R}^3} |x|^2 \|\nabla P\|(x) \leq C(U_0, V).
\end{equation}
\textbf{Proof.} With the help of Proposition \[ \text{[2.5]} \] we write
\begin{equation}
V^\varepsilon(x) = \int_0^1 \int_{\mathbb{R}^3} \Phi(x - y, 1 - s)s^{-\frac{1}{2}} F^\varepsilon(y/\sqrt{s}) \, dy \, ds,
\end{equation}
where
\begin{equation}
F^\varepsilon(x) \triangleq \frac{1}{1 + \varepsilon|x|^2} \left( \nabla P + V \cdot \nabla V + U_0 \cdot \nabla V + V \cdot \nabla U_0 + U_0 \cdot \nabla U_0 \right).
\end{equation}
From Proposition \[ \text{[3.15]} \] there exists a constant $C > 0$ such that
\begin{equation}
|\nabla V + V \cdot \nabla P + U_0 \cdot \nabla V + V \cdot \nabla U_0 + U_0 \cdot \nabla U_0| \leq C(1 + |x|)^{-2}.
\end{equation}
Therefore,
\begin{equation}
|x|^2 |V^\varepsilon|(x) \leq 2 \int_0^1 \int_{\mathbb{R}^3} \Phi(x - y, 1 - s)s^{-\frac{1}{2}} \left| \frac{y}{y^0} \right|^2 F^\varepsilon(y/\sqrt{s}) \, dy \, ds
\end{equation}
\begin{equation}
+ 2 \int_0^1 \int_{\mathbb{R}^3} |x - y|^{2} \Phi(x - y, 1 - s)s^{-\frac{3}{2}} \left| F^\varepsilon(y/\sqrt{s}) \right| \, dy \, ds.
\end{equation}
On one hand,
\[
\int_0^1 \int_{\mathbb{R}^3} \Phi(x - y, 1 - s) s^{-\frac{2}{3}} \left| |y|^2 F'(y/\sqrt{s}) \right| \, dy \, ds \leq \sup_{x \in \mathbb{R}^3} |x|^2 F'(x) \int_0^1 s^{-\frac{1}{3}} \, ds < +\infty.
\]

On the other hand,
\[
\int_0^1 \int_{\mathbb{R}^3} |x - y|^2 \Phi(x - y, 1 - s) s^{-\frac{2}{3}} \left| F'(y/\sqrt{s}) \right| \, dy \, ds \\
\leq 4 \int_0^1 (1 - s) \int_{\mathbb{R}^3} \frac{|x - y|^2}{4(1 - s)} \Phi(x - y, 1 - s) s^{-\frac{2}{3}} \left| F'(y/\sqrt{s}) \right| \, dy \, ds \\
\leq \int_0^1 (1 - s) s^{-\frac{3}{2}} \left\| \frac{1}{4(1 - s)} \Phi(\cdot, 1 - s) \right\|_{L^2(\mathbb{R}^3)} \left\| F'(y/\sqrt{s}) \right\|_{L^2(\mathbb{R}^3)} \, ds \\
\leq C \left\| |\cdot|^2 \Phi(\cdot, 1) \right\|_{L^2(\mathbb{R}^3)} \left\| F \right\|_{L^2(\mathbb{R}^3)} \int_0^1 (1 - s)^{\frac{3}{2}} s^{-\frac{3}{4}} \, ds,
\]
where
\[
F \triangleq \nabla P + V \cdot \nabla V + U_0 \cdot \nabla V + V \cdot \nabla U_0 + U_0 \cdot \nabla U_0.
\]

By the Hölder inequality, we find that
\[
\left\| F \right\|_{L^2(\mathbb{R}^3)} \leq \left\| \nabla P \right\|_{L^2(\mathbb{R}^3)} + \left\| V \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla V \right\|_{L^2(\mathbb{R}^3)} + \left\| U_0 \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla U_0 \right\|_{L^2(\mathbb{R}^3)} \\
+ \left\| V \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla U_0 \right\|_{L^2(\mathbb{R}^3)} + \left\| U_0 \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla U_0 \right\|_{L^2(\mathbb{R}^3)} \\
\leq C \left\| V \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla V \right\|_{L^2(\mathbb{R}^3)} + C \left\| U_0 \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla U_0 \right\|_{L^2(\mathbb{R}^3)} \\
+ C \left\| V \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla U_0 \right\|_{L^2(\mathbb{R}^3)} + C \left\| U_0 \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla U_0 \right\|_{L^2(\mathbb{R}^3)}.
\]

Combining all these estimates, we finally obtain
\[
\sup_{x \in \mathbb{R}^3} |x|^2 |V'(x)| \leq C(U_0, V),
\]
and then we get by taking \( \varepsilon \to 0^+ \) that the limit \( \overline{V} = \int_0^1 \int_{\mathbb{R}^3} \Phi(x - y, 1 - s) s^{-\frac{2}{3}} F'(y/\sqrt{s}) \, dy \, ds \) satisfies
\[
\sup_{x \in \mathbb{R}^3} |x|^2 |\overline{V}(x)| \leq C(U_0, V).
\]

First, we consider that \( W \in H^1(\mathbb{R}^3) \) with \( |x| W \in H^1(\mathbb{R}^3) \) solves the following linear equations with \( f \in L^2(\mathbb{R}^3) \):
\[
-\Delta W - \frac{1}{2} x \cdot \nabla W - \frac{1}{2} W = f.
\]
It is obvious that \( W \) is unique in the sense of \( H^1(\mathbb{R}^3) \). Indeed, suppose that \( \overline{W} \in H^1(\mathbb{R}^3) \) with \( |x| \overline{W} \in H^1(\mathbb{R}^3) \) is another solution of the above linear equations. Then the difference \( \delta W \triangleq W - \overline{W} \) satisfies
\[
-\Delta \delta W - \frac{1}{2} x \cdot \nabla \delta W - \frac{1}{2} \delta W = 0
\]
This implies \( \| \delta W \|_{H^1(\mathbb{R}^3)} = 0 \), and then we get the uniqueness. This yields that \( V = \overline{V} \), and we have
\[
\sup_{x \in \mathbb{R}^3} |x|^2 |V(x)| \leq C(U_0, V).
\]
In the same fashion as in proving the above estimates, we can show that
\[
\sup_{x \in \mathbb{R}^3} |x|^2 |\nabla V(x)| \leq C(U_0, V).
\]
Next we show the decay estimate for the pressure \( P \). Recall that
\[
-\Delta P = \text{div} \, \text{div} \left( V \otimes V + V \otimes U_0 + U_0 \otimes V + U_0 \otimes U_0 \right),
\]
one writes
\[
P = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \text{div} \left( V \otimes V + V \otimes U_0 + U_0 \otimes V + U_0 \otimes U_0 \right) (y) \, dy
\]
\[
= \sum_{i,j=1}^{3} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \left( \partial_{x_j} V^i \partial_{x_i} V^j + \partial_{x_j} V^i \partial_{x_i} U^j_0 + \partial_{x_j} U^i_0 \partial_{x_i} V^j + \partial_{x_j} U^i_0 \partial_{x_i} U^j_0 \right) (y) \, dy.
\]
Furthermore, we have
\[
\partial_{x_k} P = \sum_{i,j=1}^{3} K_{i,k}(x,y) \left( \partial_{x_j} V^i \partial_{x_i} V^j + \partial_{x_j} V^i \partial_{x_i} U^j_0 + \partial_{x_j} U^i_0 \partial_{x_i} V^j + \partial_{x_j} U^i_0 \partial_{x_i} U^j_0 \right) (y) \, dy.
\]
Multiplying the above equality by $|x|^2$, we readily have
\[
|x|^2 \partial_{x_k} P(x) \leq C \sum_{i,j=1}^{3} \left( \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} |y|^2 \left| \partial_{x_j} V^i \partial_{x_i} V^j + \partial_{x_j} V^i \partial_{x_i} U^j_0 + \partial_{x_j} U^i_0 \partial_{x_i} V^j + \partial_{x_j} U^i_0 \partial_{x_i} U^j_0 \right| (y) \, dy \right.
\]
\[
+ \left. \int_{\mathbb{R}^3} \left| \partial_{x_j} V^i \partial_{x_i} V^j + \partial_{x_j} V^i \partial_{x_i} U^j_0 + \partial_{x_j} U^i_0 \partial_{x_i} V^j + \partial_{x_j} U^i_0 \partial_{x_i} U^j_0 \right| (y) \, dy \right).
\]
By the Hölder inequality, we see that
\[
\int_{\mathbb{R}^3} \left| \partial_{x_j} V^i \partial_{x_i} V^j + \partial_{x_j} V^i \partial_{x_i} U^j_0 + \partial_{x_j} U^i_0 \partial_{x_i} V^j + \partial_{x_j} U^i_0 \partial_{x_i} U^j_0 \right| \, dy \leq C \|V\|_{H^1(\mathbb{R}^3)}^2 + C \|U_0\|_{H^1(\mathbb{R}^3)}^2.
\]
By the generalized Young inequality in Lemma 2.2, we get
\[
\int_{\mathbb{R}^3} \frac{1}{|x-y|^2} |y|^2 \left| \partial_{x_j} V^i \partial_{x_i} V^j + \partial_{x_j} V^i \partial_{x_i} U^j_0 + \partial_{x_j} U^i_0 \partial_{x_i} V^j + \partial_{x_j} U^i_0 \partial_{x_i} U^j_0 \right| (y) \, dy \leq C \|\nabla V\|_{L^6(\mathbb{R}^3)} \|\nabla U_0\|_{L^3(\mathbb{R}^3)} + \|\nabla V\|_{L^3(\mathbb{R}^3)}
\]
\[
\leq C \|E\|_{H^2(\mathbb{R}^3)}^2 + C \|\cdot \|_{L^2(\mathbb{R}^3)} \|\cdot \|_{L^\infty(\mathbb{R}^3)} \left( \|U_0\|_{H^2(\mathbb{R}^3)} + \|V\|_{H^2(\mathbb{R}^3)} \right).
\]
Combining both estimates, we get
\[
\sup_{x \in \mathbb{R}^3} |x|^2 |\nabla P|(x) \leq C(U_0, V).
\]
So we complete the proof the proposition. \hfill \Box

**Proof of Theorem 2.14** We calculate
\[
-\Delta \left( |x|^3 \partial_{x_k} P \right) = -12 |x| \partial_{x_k} P - 6 |x| x \cdot \nabla \partial_{x_k} P - |x|^3 \Delta \partial_{x_k} P
\]
\[
= 12 |x| \partial_{x_k} P - 6 \text{div} \left( |x| x \partial_{x_k} P \right) - \partial_{x_k} \left( |x|^3 \Delta P \right) + 3 |x| x_k \Delta P.
\]
Thus, we have
\[
\left\| \left( | \cdot |^3 \partial_{x_k} P \right) \right\|_{\dot{B}^0_{\infty, \infty}(\mathbb{R}^3)} 
\leq C \left\| 1 \partial_{x_k} P \right\|_{\dot{B}^2_{\infty, \infty}(\mathbb{R}^3)} + C \left\| | \cdot |^2 \partial_{x_k} P \right\|_{\dot{B}^1_{\infty, \infty}(\mathbb{R}^3)} + \left\| | \cdot |^3 \Delta P \right\|_{\dot{B}^1_{\infty, \infty}(\mathbb{R}^3)} + \left\| | \cdot |^2 \Delta P \right\|_{\dot{B}^2_{\infty, \infty}(\mathbb{R}^3)}
\]
\[
\leq C \left\| 1 \partial_{x_k} P \right\|_{L^2(\mathbb{R}^3)} + C \left\| | \cdot |^2 \partial_{x_k} P \right\|_{L^3(\mathbb{R}^3)} + \left\| | \cdot |^3 \Delta P \right\|_{L^1(\mathbb{R}^3)} + \left\| | \cdot |^2 \Delta P \right\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \left\| | \cdot |^2 \partial_{x_k} P \right\|_{L^\infty(\mathbb{R}^3)} + \left\| | \cdot |^3 \Delta P \right\|_{\dot{B}^1_{\infty, \infty}(\mathbb{R}^3)} + \left\| | \cdot |^2 \Delta P \right\|_{\dot{B}^2_{\infty, \infty}(\mathbb{R}^3)}.
\]
Since
\[
-\Delta P = \text{div} \left( V \cdot \nabla V + U_0 \cdot \nabla V + V \cdot \nabla U_0 + U_0 \cdot \nabla U_0 \right),
\]
we readily have by Proposition 3.19 that
\[
|\Delta P|(x) \leq C |x|^{-4}.
\]
Furthermore, we get from Lemma 2.3 that
\[
\| | \cdot |^3 \Delta P \|_{B^{-1}_{\infty, \infty}(\mathbb{R}^3)} \leq C \| | \cdot |^3 \Delta P \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| | \cdot |^4 \Delta P \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| | \cdot |^4 \Delta P \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| | \cdot |^4 \Delta P \|_{L^{3, \infty}(\mathbb{R}^3)} \cdot
\]
Similarly, we have
\[
\| | \cdot |^2 \Delta P \|_{B^{-2}_{\infty, \infty}(\mathbb{R}^3)} \leq C \| | \cdot |^3 \Delta P \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| | \cdot |^4 \Delta P \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| | \cdot |^4 \Delta P \|_{L^{3, \infty}(\mathbb{R}^3)} \cdot
\]
We see that
\[
|x|^2 \partial_{x_k} P(x) = \sum_{i,j=1}^{3} \left( \int_{\mathbb{R}^3} K_{i,k}(x, y) |y|^2 \left( \partial_{x_j} V^i V^j + \partial_{x_j} V^i U^j_0 + \partial_{x_j} U^i_0 V^j + \partial_{x_j} U^i_0 U^j_0 \right) dy \right.
\]
\[+ \int_{\mathbb{R}^3} K_{i,k}(x, y) (|x| - |y|)^2 \left( \partial_{x_j} V^i V^j + \partial_{x_j} V^i U^j_0 + \partial_{x_j} U^i_0 V^j + \partial_{x_j} U^i_0 U^j_0 \right) dy \]
\[+ 2 \int_{\mathbb{R}^3} K_{i,k}(x, y) (|x| - |y|)|y| \left( \partial_{x_j} V^i V^j + \partial_{x_j} V^i U^j_0 + \partial_{x_j} U^i_0 V^j + \partial_{x_j} U^i_0 U^j_0 \right) dy \]
\[\triangleq I_1 + I_2 + I_3.\]
The property of Calderón-Zygmund singular operator enables us to conclude
\[
\| I_1 \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| \| \cdot |^2 \left( \partial_{x_j} V^i V^j + \partial_{x_j} V^i U^j_0 + \partial_{x_j} U^i_0 V^j + \partial_{x_j} U^i_0 U^j_0 \right) \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| \| \cdot |^2 \partial_{x_j} V^i \|_{L^{\infty}(\mathbb{R}^3)} \left( \| V^j \|_{L^{3, \infty}(\mathbb{R}^3)} + \| U^j_0 \|_{L^{3, \infty}(\mathbb{R}^3)} \right) + C \| \| \cdot |^2 \partial_{x_j} U^i_0 \|_{L^{\infty}(\mathbb{R}^3)} \left( \| V^j \|_{L^{3, \infty}(\mathbb{R}^3)} + \| U^j_0 \|_{L^{3, \infty}(\mathbb{R}^3)} \right).\]
Integrating by parts leads to
\[
I_2 = - \int_{\mathbb{R}^3} \partial_{x_j} \left( K_{i,k}(x, y) (|x| - |y|)^2 \right) \left( V^i V^j + V^i U^j_0 + U^i_0 V^j + U^i_0 U^j_0 \right) dy \leq C \int_{\mathbb{R}^3} \frac{1}{|x - y|^2} \left| V^i V^j + V^i U^j_0 + U^i_0 V^j + U^i_0 U^j_0 \right| dy.
\]
By the generalized Young inequality in Lemma 2.2, we obtain
\[
\| I_2 \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| V^i V^j + V^i U^j_0 + U^i_0 V^j + U^i_0 U^j_0 \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| V^i \|_{L^{3, \infty}(\mathbb{R}^3)} + \| U^i_0 \|_{L^{3, \infty}(\mathbb{R}^3)}.
\]
Since
\[
|I_3| \leq \int_{\mathbb{R}^3} \frac{1}{|x - y|^2} |y| \left| \partial_{x_j} V^i V^j + \partial_{x_j} V^i U^j_0 + \partial_{x_j} U^i_0 V^j + \partial_{x_j} U^i_0 U^j_0 \right| dy,
\]
we have
\[
\| I_3 \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| \| \cdot |^2 \left( \partial_{x_j} V^i V^j + \partial_{x_j} V^i U^j_0 + \partial_{x_j} U^i_0 V^j + \partial_{x_j} U^i_0 U^j_0 \right) \|_{L^{3, \infty}(\mathbb{R}^3)} \leq C \| \| \cdot |^2 \partial_{x_j} V^i \|_{L^{3, \infty}(\mathbb{R}^3)} \left( \| V^j \|_{L^{3, \infty}(\mathbb{R}^3)} + \| U^j_0 \|_{L^{3, \infty}(\mathbb{R}^3)} \right) + C \| \| \cdot |^2 \partial_{x_j} U^i_0 \|_{L^{3, \infty}(\mathbb{R}^3)} \left( \| V^j \|_{L^{3, \infty}(\mathbb{R}^3)} + \| U^j_0 \|_{L^{3, \infty}(\mathbb{R}^3)} \right).\]
Collecting all these estimates and using the property of Beta function, we eventually obtain that
\[
\| | \cdot | \nabla P \|_{L^{3, \infty}(\mathbb{R}^3)} < + \infty.
\]
From Proposition 3.19 it follows that
\[ \sup_{x \in \mathbb{R}^3} \langle x \rangle^2 (|V| + |\nabla V|)(x) < +\infty. \]
Moreover, by [17 Equality (4.11)], we get
\[ |V|(x) \leq C|x|^{-3}\log(1 + |x|) \quad \text{for all } x \in \mathbb{R}^3 \]
and
\[ |DV|(x) \leq C|x|^{-3} \quad \text{for all } x \in \mathbb{R}^3. \]
We finish the proof of Theorem 3.13.

4. PROOF OF THEOREM 1.1

In this section, we are devoted to proof of Theorem 1.1. First of all, we focus on the existence of the forward self-similar solution. Letting
\[ u \ni \square \]
we finish the proof of Theorem 3.13.

Next, we want to show that
\[ u \ni \square \]
and
\[ \|u\|_{L^{\infty}} \leq 2 \quad \text{as long as} \quad \|p\|_{L^{\infty}} \leq 2 \]
we know that for each \( 2 \leq p \leq 6 \leq 2\alpha \),
\[ \|v(t)\|_{L^{\frac{3}{2\alpha}}(\mathbb{R}^3)} = \|V\|_{L^{\frac{3}{2\alpha}}(\mathbb{R}^3)} \leq C\|V\|_{H^\alpha(\mathbb{R}^3)} < +\infty \]
as long as \( \alpha \in (5/6, 1] \). On the other hand, by Proposition 2.8 we have
\[ \|u_L(t)\|_{L^{\frac{3}{2\alpha}}(\mathbb{R}^3)} \leq C\|u_0\|_{L^{\frac{3}{2\alpha}}(\mathbb{R}^3)}. \]
Combining both estimates yields that \( \|u\|_{L^{\infty}}(0, +\infty); L^{\frac{3}{2\alpha}}(\mathbb{R}^3)} \) is bounded. Now, we begin to show the weak continuous with respect to time \( t \). For the linear part \( u_L(x, t) \), it is obvious that \( u_L \in C_w([0, +\infty); L^{\frac{3}{2\alpha}}(\mathbb{R}^3)} \) and \( u_L \to u_0 \) in the weak sense of \( L^{\frac{3}{2\alpha}}(\mathbb{R}^3)} \) as \( t \) goes to \( 0^+ \).
So we need to show that \( v \in C_w([0, +\infty); L^{\frac{3}{2\alpha}}(\mathbb{R}^3)} \). Since \( \alpha \in (5/6, 1] \) and \( V \in H^\alpha(\mathbb{R}^3)} \), we know that for each \( 2 \leq p \leq \frac{6}{3-2\alpha} \),
\[ \|V\|_{L^p(\mathbb{R}^3)} \leq C\|V\|_{H^\alpha(\mathbb{R}^3)}. \]
This estimate together with the fact \( \|v(t)\|_{L^p} = t^{\frac{1}{2\alpha}(1+\frac{1}{p})-1}\|V\|_{L^p(\mathbb{R}^3)} \) means that for every \( 2 \leq p \leq \frac{6}{3-2\alpha} \),
\[ \|v(t)\|_{L^p(\mathbb{R}^3)} \leq Ct^{\frac{1}{2\alpha}(1+\frac{1}{p})-1}. \]
It follows that for $2 \leq p < \frac{3}{2n-4}$,
\[
\|v(t)\|_{L^p(\mathbb{R}^3)} \to 0 \quad \text{as} \quad t \to 0 + .
\]

With this property in hand, we can infer that $v(x, t) \to 0$ in $L^{\frac{3}{2n-4}}(\mathbb{R}^3)$ as $t \to 0+$, which implies that $u(x, t) \to u_0(x)$ as $t \to 0+$ in the weak sense of $L^{\frac{3}{2n-4}}(\mathbb{R}^3)$.

Finally, we are going to show the higher decay estimate of solution $v$ for the case $\alpha = 1$. From Theorem 3.13 we know that
\[
(1 + |x|^3)|V|(x) \leq C \log (1 + |x|) \quad \text{for all} \quad x \in \mathbb{R}^3.
\]

Combining this estimate with the relation that $v(x, t) = \sqrt{t}V(x/\sqrt{t})$ enables us to conclude that
\[
\sqrt{t} \left(1 + \frac{|x|^3}{(\sqrt{t})^3}\right) v(x, t) = \left(1 + \frac{|x|^3}{(\sqrt{t})^3}\right) V \left(\frac{x}{\sqrt{t}}\right) \leq C \log \left(1 + \frac{|x|}{\sqrt{t}}\right).
\]

Thus we have
\[
|v(x, t)| \leq C \frac{t}{t^{\frac{3}{4}} + |x|^3} \log \left(1 + \frac{|x|}{\sqrt{t}}\right) \quad \text{for all} \quad (x, t) \in \mathbb{R}^3 \times (0, +\infty).
\]

Thanks to Proposition 2.8 we know that $\sup_{x \in \mathbb{R}^3} (1 + |x|)|U_0(x)| < +\infty$. Since
\[
u_1(x, t) = \frac{1}{\sqrt{t}}U_0 \left(\frac{x}{\sqrt{t}}\right),
\]
we readily have that
\[
|\nu_1(x, t)| \leq C \frac{\sqrt{t}}{\sqrt{t} + |x|} \quad \text{for all} \quad (x, t) \in \mathbb{R}^3 \times (0, +\infty).
\]

So we finish the proof of Theorem 1.1.

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