A primal dual formulation through a proximal approach for non-convex variational optimization

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Abstract

This article develops a primal dual formulation for a primal proximal approach suitable for a large class of non-convex models in the calculus of variations. The results are established through standard tools of functional analysis, convex analysis and duality theory and are applied to a Ginzburg-Landau type model. Finally, in the last two sections, we present concerning optimality conditions and another related duality principle for the model in question.

1 Introduction

We start this article by justifying the suitability of the proximal approach for the concerning model.

Consider a domain \( \Omega \subset \mathbb{R}^3 \) and the functional \( J : U \to \mathbb{R} \) where

\[
J(u) = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}, \quad \forall u \in U = W^{1,2}_0(\Omega).
\]

We could write such a functional as

\[
J(u) = G_1(u, 0) + F_1(u), \quad \forall u \in U,
\]

where

\[
G_1(u, v) = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla v \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 \, dx - \frac{\varepsilon}{2} \int_{\Omega} u^2 \, dx,
\]

and

\[
F_1(u) = \frac{\varepsilon}{2} \int_{\Omega} u^2 \, dx - \langle u, f \rangle_{L^2}.
\]

Among other possibilities, we could define the dual functional as

\[
J^*(v^*, v_0^*) = -G^*_1(v^*, v_0^*) - F^*(v^*),
\]
where
\[ G_1^*(v^*, v_0^*) = \frac{1}{2} \int_{\Omega} \frac{(v^*)^2}{(-\gamma \nabla^2 + 2v_0^* - \varepsilon)} \, dx + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx + \beta \int_{\Omega} v_0^* \, dx, \]

and
\[ F_1^*(v^*) = \frac{1}{2\varepsilon} \int_{\Omega} (v^* - f)^2 \, dx \]

Through the variation in \( v_0^* \) we obtain
\[ \frac{(v^*)^2}{(-\gamma \nabla^2 + 2v_0^* - \varepsilon)^2} - \frac{v_0^*}{\alpha} - \beta = 0, \]
intending to obtain conditions for a solution \( v_0^*(v^*) \) and thus to obtain a final functional as a function of \( v^* \) with a possible large region of convexity (in fact concavity) due the term
\[ F_1^*(v^*) = \frac{1}{2\varepsilon} \int_{\Omega} (v^* - f)^2 \, dx \]
with a small value for \( \varepsilon > 0 \).

The issue is that if the term
\[ -\gamma \nabla^2 + 2v_0^* - \varepsilon \]
corresponds to an undefined matrix (this is a common situation for the case of local minima for the primal formulation) we may not have the hypothesis of the implicit function theorem satisfied so that critical points of the dual formulation may not correspond to critical points of the primal one and reciprocally.

Indeed, we may obtain for the second variation of \( J^* \) in \( v_0^* \)
\[ \frac{\partial^2 J^*(v_0^*)}{\partial (v_0^*)^2} = -4 \frac{(v^*)^2}{(-\gamma \nabla^2 + 2v_0^* - \varepsilon)^2} - \frac{1}{\alpha}, \]

Observe that for a critical point denoting
\[ u = \frac{(v^*)}{(-\gamma \nabla^2 + 2v_0^* - \varepsilon)} \]
we have
\[ v_0^* = \alpha \left( \frac{(v^*)}{(-\gamma \nabla^2 + 2v_0^* - \varepsilon)} \right)^2 - \beta = \alpha (u^2 - \beta), \]
so that
\[ \frac{\partial^2 J^*(v_0^*)}{\partial (v_0^*)^2} = -4 \frac{u^2}{(-\gamma \nabla^2 + 2v_0^* - \varepsilon)} - \frac{1}{\alpha} \]
and thus
\[ \frac{\partial^2 J^*(v_0^*)}{\partial (v_0^*)^2} = -4\alpha u^2 + \gamma \nabla^2 - 2v_0^* + \varepsilon \]
\[ = \frac{-\delta^2 J(u) + \varepsilon}{(-\gamma \nabla^2 + 2v_0^* - \varepsilon)\alpha}. \]

Therefore if for a critical point where
\[ \delta^2 J(u) - \varepsilon > 0 \]
the term
\[-\gamma \nabla^2 + 2v_0^* - \epsilon\]
corresponds to an undefined matrix, we have that
\[
\frac{\partial^2 J^*(v_0^*)}{\partial (v_0^*)^2}
\]
is also undefined and the hypothesis of the implicit function theorem may not be satisfied, in order to obtain \(v_0^*(v^*)\). The other issue is that
\[
\delta^2 J(v^*, v_0^*)
\]
may also be undefined at a critical point, so that we do not have a qualitative correspondence between the primal and dual critical points.

So this may lead us, for a large class of similar models, through such a formulation, to wrong results concerning the equivalence of critical points for the primal and dual formulations.

In order to solve this problem, in this article we propose a kind of proximal variational formulation with exact penalization. Thus, with such facts in mind, we propose as the primal dual equivalent formulation for the original primal problem in question, the following functional
\[
\hat{J}(u, p) = \gamma \int_\Omega \nabla u \cdot \nabla u \, dx + \alpha \int_\Omega (u^2 - \beta)^2 \, dx + K \int_\Omega (u - p)^2 \, dx - \langle u, f \rangle_{L^2}
\]
We highlight the proximal term
\[
\frac{K}{2} \int_\Omega (u - p)^2 \, dx
\]
makes the primal formulation convex in \(u\) for appropriate values of \(K > 0\).

In the next section we present the theoretical results for a duality principle concerning such a proximal formulation. We believe through an analysis of the proof of the next theorem the suitability of such a proximal formulation will be clarified.

**Remark 1.1.** About the references, in our work we have been greatly influenced by the works of J.J. Telega and W.R. Bielski, in particular by [3, 4]. The duality principle here developed for the proximal approach is also inspired by the works J.F. Toland [12] and Ekeland and Temam [10]. Related problems are addressed in [7, 6, 9]. About the physics of the problem in question we would cite [2] and [11]. Details on the Sobolev spaces involved may be found in [1, 7].

**Remark 1.2.** Even though we have not relabeled the functionals and operators, we shall consider a finite dimensional approximation for the model in question, in a finite elements or finite differences context.

In such a finite elements or finite differences context, we emphasize that the notation
\[
\int_\Omega \frac{(v_0^*)^2}{-\gamma \nabla^2 + K + \epsilon} \, dx
\]
stands for
\[ \langle (-\gamma \nabla^2 + K I_d + \varepsilon I_d)^{-1} v^*_1, v_1^* \rangle \]
where \( I_d \) denotes the identity matrix in an appropriate finite dimensional approximate space.

**Remark 1.3.** Finally we highlight that for invertible \( n \times n \) matrices or invertible linear operators \( A \) and \( B \) we have
\[ -A^{-1} + B^{-1} = A^{-1}(B - A)B^{-1} \]
and sometimes, as the meaning is clear, we may simply denote
\[ -A^{-1} + B^{-1} = \frac{B - A}{AB}. \]

## 2 The main duality principle

In this section we present the main result in this article, which is summarized by the next theorem.

At this point we highlight that the optimality criterion presented in the item 1c in the next theorem, namely
\[ -\gamma \nabla^2 + 2v^*_0 > 0, \]
may be found in analogous form in the article [8] which was published in 2010, but in fact submitted in August of the year 2007, as indicated in the concerning Journal web-site. Related results on duality theory may be originally found in [9].

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^3 \) be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by \( \partial \Omega \). Even though we have not relabeled the functionals and operators, consider a finite dimensional approximation for the model in question, in a finite elements or finite differences context, where we define the functionals \( \hat{J} : U \times Y \rightarrow \mathbb{R} \) and \( J : U \rightarrow \mathbb{R} \), by

\[
\hat{J}(u, p) = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx \\
+ \frac{K}{2} \int_{\Omega} (u - p)^2 \, dx - \langle u, f \rangle_{L^2} \tag{3}
\]

and

\[ J(u) = \hat{J}(u, u), \]

where

\[ U = W_0^{1,2}(\Omega), \]
\[ Y = Y^* = L^2(\Omega), \]
\[ \alpha > 0, \beta > 0, \gamma > 0, K > 0 \text{ and } f \in C^1(\overline{\Omega}). \]

Furthermore, for a sufficiently small parameter \( \varepsilon > 0 \), define \( G : U \times Y \times Y \rightarrow \mathbb{R} \) by

\[
G(u, v, p) = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 \, dx \\
- \langle u, Kp \rangle_{L^2} + \frac{K}{2} \int_{\Omega} u^2 \, dx + \frac{\varepsilon}{2} \int_{\Omega} u^2 \, dx, \tag{4}
\]

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\( F : U \to \mathbb{R} \) by
\[
F(u) = \frac{\varepsilon}{2} \int_{\Omega} u^2 \, dx + \langle u, f \rangle_{L^2}
\]
and \( H : Y \to \mathbb{R} \) by
\[
H(p) = \frac{K}{2} \int_{\Omega} p^2 \, dx,
\]
so that
\[
\hat{J}(u, p) = G(u, 0, p) - F(u) + H(p), \quad \forall (u, p) \in U \times Y.
\]
Define also, \( G^* : Y^* \times Y^* \times Y \to \mathbb{R} \) by
\[
G^*(v^*, v^*_0, p) = \sup_{u \in U} \sup_{v \in Y} \{ \langle u, v^* \rangle_{L^2} + \langle v, v^*_0 \rangle_{L^2} - G(u, v, p) \}
\]
\[
= \frac{1}{2} \int_{\Omega} \frac{(v^* + Kp)^2}{\gamma \nabla^2 + 2v^*_0 + K + \varepsilon} \, dx
\]
\[
+ \frac{1}{2\alpha} \int_{\Omega} (v^*_0)^2 \, dx + \beta \int_{\Omega} v^* \, dx,
\]
if \( v^*_0 \in B^* \) where
\[
B^* = \left\{ v^*_0 \in Y^*: \gamma \nabla^2 + 2v^*_0 + K + \varepsilon > \frac{K}{2} \right\},
\]
\( F^*: Y^* \to \mathbb{R} \) where
\[
F^*(v^*) = \sup_{u \in Y} \{ \langle u, v^* \rangle_{L^2} - F(u) \}
\]
\[
= \frac{1}{2\varepsilon} \int_{\Omega} (v^* - f)^2 \, dx.
\]
and \( J^*: Y^* \times B^* \times Y \to \mathbb{R} \) by
\[
J^*(v^*, v^*_0, p) = -G^*(v^*, v^*_0, p) + F^*(v^*) + H(p), \quad \forall (v^*, v^*_0, p) \in Y^* \times B^* \times Y.
\]

Under such hypotheses,

1. Assume \( u_0 \in U \) is such that \( \delta J(u_0) = 0 \) and define
\[
\hat{v}^*_0 = \alpha(u_0^2 - \beta),
\]
\[
\hat{v}^* = \varepsilon u_0 + f,
\]
\[
\hat{p} = u_0
\]
under such assumptions,
\[
\delta J^*(\hat{v}^*, \hat{v}^*_0, \hat{p}) = 0.
\]
(a) Assume also \( \delta^2 J(u_0) > 0 \) and \( \hat{v}^*_0 \in B^* \). Under such additional hypotheses, there exist \( r_1, r_2, r_3 > 0 \) such that
\[
J(u_0) = \inf_{u \in B_{r_1}(u_0)} J(u)
\]
\[
= \inf_{v^* \in B_{r_2}(\hat{v}^*), \hat{p} \in B_{r_3}(\hat{p})} \left\{ \inf_{v^*_0 \in B^*} J^*(v^*, v^*_0, p) \right\}
\]
\[
= J^*(\hat{v}^*, \hat{v}^*_0, \hat{p}).
\]
Moreover, defining \( J_3^* : B_{r_3}(\hat{v}^*) \rightarrow \mathbb{R} \) by

\[
J_3^*(v^*) = \inf_{p \in B_{r_2}(\hat{p})} \left\{ \sup_{v_0^* \in B^*} J^*(v^*, v_0^*, p) \right\}
\]

we have that

\[
\delta J_3^*(\hat{v}^*) = 0 \\
\delta^2 J_3^*(\hat{v}^*) > 0
\]

so that

\[
J(u_0) = \inf_{u \in B_{r_1}(u_0)} J(u) \\
= \inf_{v^* \in B_{r_3}(\hat{v}^*)} J_3^*(v^*) \\
= J_3^*(\hat{v}^*). \tag{8}
\]

(b) Suppose \( \delta^2 J(u_0) < 0 \) and \( \hat{v}_0^* \in B^* \). Under such additional hypotheses, there exist \( r_1, r_2, r_3 > 0 \) such that

\[
J(u_0) = \sup_{u \in B_{r_1}(u_0)} J(u) \\
= \inf_{v^* \in B_{r_3}(\hat{v}^*)} \left\{ \sup_{p \in B_{r_2}(\hat{p})} \left\{ \sup_{v_0^* \in B^*} J^*(v^*, v_0^*, p) \right\} \right\} \\
= J^*(\hat{v}^*, \hat{v}_0^*, \hat{p}). \tag{9}
\]

Moreover, defining \( J_5^* : B_{r_3}(\hat{v}^*) \rightarrow \mathbb{R} \) by

\[
J_5^*(v^*) = \sup_{p \in B_{r_2}(\hat{p})} \left\{ \sup_{v_0^* \in B^*} J^*(v^*, v_0^*, p) \right\}
\]

we have

\[
\delta J_5^*(\hat{v}^*) = 0 \\
\delta^2 J_5^*(\hat{v}^*) > 0
\]

so that

\[
J(u_0) = \sup_{u \in B_{r_1}(u_0)} J(u) \\
= \inf_{v^* \in B_{r_3}(\hat{v}^*)} J_5^*(v^*) \\
= J_5^*(\hat{v}^*). \tag{10}
\]

(c) For this item define \( A^+ \) by

\[
A^+ = \{ v_0^* \in Y^* : -\gamma \nabla^2 + 2v_0^* > 0 \}.
\]

Assume \( \hat{v}_0^* \in A^+ \cap B^* \).
Under such additional assumptions and definitions, we have

\[ J(u_0) = \inf_{u \in U} J(u) \]

\[ = \inf_{(v^*, p) \in Y^* \times Y} \left\{ \sup_{v_0^* \in A^+ \cap B} J^*(v^*, v_0^*, p) \right\} \]

\[ = J^*(\hat{v}^*, \hat{v}_0^*, \hat{p}). \quad (11) \]

Moreover, defining \( J^*_7 : Y^* \times Y \to \mathbb{R} \) by

\[ J^*_7(v^*, p) = \left\{ \sup_{v_0^* \in A^+ \cap B} J^*(v^*, v_0^*, p) \right\} \]

we have

\[ \delta J^*_7(\hat{v}^*, \hat{p}) = 0 \]

\[ \delta^2 J^*_7(\hat{v}^*, \hat{p}) > 0 \]

so that

\[ J(u_0) = \inf_{u \in U} J(u) \]

\[ = \inf_{(v^*, p) \in Y^* \times Y} J^*_7(v^*, p) \]

\[ = J^*_7(\hat{v}^*, \hat{p}). \quad (12) \]

**Proof.** Suppose \( u_0 \in U \) is such that \( \delta J(u_0) = 0 \).

We shall start by proving that

\[ \delta J^*(\hat{v}^*, \hat{v}_0^*, \hat{p}) = 0. \]

Observe that from

\[ \delta J(u_0) = 0 \]

we have that

\[ -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = 0, \text{ in } \Omega, \]

so that

\[ -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - \varepsilon u_0 + Ku_0 + \varepsilon u_0 - Ku_0 - f = 0, \]

that is

\[ \hat{v}^* + K\hat{p} = \varepsilon u_0 + f + Ku_0 = -\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 + \varepsilon u_0 + Ku_0. \quad (13) \]

Thus,

\[ u_0 = \frac{\hat{v}^* + K\hat{p}}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon}, \]

so that

\[ u_0 = \frac{\hat{v}^* - f}{\varepsilon} = \frac{\hat{v}^* + K\hat{p}}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon}. \]
Therefore
\[
\frac{\dot{v}^* - f}{\varepsilon} - \frac{\dot{v}^* + K\dot{p}}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon} = 0,
\]
and consequently we may infer that
\[
\frac{\partial J^*(\dot{v}^*, \hat{v}_0^*, \hat{p})}{\partial v^*} = 0.
\]
On the other hand
\[
\frac{\hat{v}_0^*}{\alpha} = (u_0^2 - \beta) = \left(\frac{\dot{v}^* + K\dot{p}}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon}\right)^2 - \beta,
\]
so that
\[
-\frac{\hat{v}_0^*}{\alpha} + \left(\frac{\dot{v}^* + K\dot{p}}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon}\right)^2 - \beta = 0,
\]
that is,
\[
\frac{\partial J^*(\dot{v}^*, \hat{v}_0^*, \hat{p})}{\partial \hat{v}_0^*} = 0.
\]
Moreover
\[
K\dot{p} = Ku_0 = K \left(\frac{\dot{v}^* + K\dot{p}}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon}\right),
\]
so that
\[
K\dot{p} - K \left(\frac{\dot{v}^* + K\dot{p}}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon}\right) = 0,
\]
that is,
\[
\frac{\partial J^*(\dot{v}^*, \hat{v}_0^*, \hat{p})}{\partial p} = 0.
\]
From these last results, we have that
\[
\delta J^*(\dot{v}^*, \hat{v}_0^*, \hat{p}) = 0.
\]
Also
\[
\frac{\partial J^*_5(\dot{v}^*)}{\partial v^*} = \frac{\partial J^*(\dot{v}^*, \hat{v}_0^*, \hat{p})}{\partial v^*} + \frac{\partial J^*(\dot{v}^*, \hat{v}_0^*, \hat{p})}{\partial \hat{v}_0^*} \frac{\partial \hat{v}_0^*}{\partial v^*} + \frac{\partial J^*(\dot{v}^*, \hat{v}_0^*, \hat{p})}{\partial \hat{p}} \frac{\partial \hat{p}}{\partial v^*} = 0.
\]
(14)
Similarly we may obtain
\[
\frac{\partial J^*_5(\dot{v}^*)}{\partial v^*} = 0,
\]
and
\[
\delta J^*_7(\dot{v}^*, \hat{p}) = 0.
\]
From the relations between the primal and dual variables, as a by-product of the Legendre transform proprieties we may obtain

\[
J^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p}) = -G^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p}) + F^*(\tilde{v}^*) + H(\tilde{p}) = J(u_0, \tilde{p}) = J(u_0) = J(u_0).
\]

Suppose now \(\delta^2 J(u_0) > 0\).

Define \(J_8^* : Y^* \times Y \to \mathbb{R}\) by

\[
J_8^*(v^*, p) = \sup_{\tilde{v}_0^* \in B^*} J^*(v^*, v_0^*, p).
\]

In particular we have got

\[
J_8^*(\tilde{v}^*, \tilde{p}) = \sup_{\tilde{v}_0^* \in B^*} J^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p}) = J^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p}).
\]

Observe that

\[
\frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{p})}{\partial p^2} = \frac{\partial^2 J^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p})}{\partial p^2} + \frac{\partial^2 J^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p})}{\partial p \partial \tilde{v}_0^*} \frac{\partial \tilde{v}_0^*}{\partial p}.
\]

At this point we recall that

\[
\frac{\partial J^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p})}{\partial \tilde{v}_0^*} = 0,
\]

so that

\[
\left( \frac{\tilde{v}^* + K\tilde{p}}{-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \varepsilon} \right)^2 - \frac{\tilde{v}_0^*}{\alpha} - \beta = 0
\]

Hence, taking the variation in \(p\) of such a last equation, we obtain

\[
\frac{2K(\tilde{v}^* + K\tilde{p})}{(-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \varepsilon)^2} - 4 \frac{(\tilde{v}^* + K\tilde{p})^2}{(-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \varepsilon)^3} \frac{\partial \tilde{v}_0^*}{\partial p} - \frac{1}{\alpha} \frac{\partial \tilde{v}_0^*}{\partial p} = 0.
\]

so that

\[
\frac{2Ku_0}{(-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \varepsilon)} - 4 \frac{(u_0)^2}{(-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \varepsilon)} \frac{\partial \tilde{v}_0^*}{\partial p} - \frac{1}{\alpha} \frac{\partial \tilde{v}_0^*}{\partial p} = 0.
\]
and thus
\[
\frac{\partial \tilde{v}_0^*}{\partial p} = \frac{2\alpha K u_0}{(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\tilde{v}_0^* + K + \epsilon)}.
\]

From this we have
\[
\frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{p})}{\partial p^2} = \frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p})}{\partial p^2}
+ \frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p})}{\partial \tilde{v}_0^* \partial \tilde{v}_0^*} \frac{\partial \tilde{v}_0^*}{\partial p}
= K - \frac{K^2}{(-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \epsilon)}
+ \frac{2(\tilde{v}^* + K \tilde{p})K}{(-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \epsilon)^2 (-\gamma \nabla^2 + 4\alpha u_0^2 + 2\tilde{v}_0^* + K + \epsilon)}.
\] (19)

Hence,
\[
\frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{p})}{\partial p^2} = K - \frac{K^2}{(-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \epsilon)}
+ \frac{1}{4\alpha K^2 u_0^2 (-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \epsilon)}
\] (20)

so that
\[
\frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{p})}{\partial p^2} = K - \frac{K^2}{(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\tilde{v}_0^* + K + \epsilon)}
= \frac{K(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\tilde{v}_0^* + \epsilon)}{(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\tilde{v}_0^* + K + \epsilon)}
\] (21)

Summarizing,
\[
\frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{p})}{\partial p^2} > 0.
\]

Similarly,
\[
\frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{p})}{\partial (v^*)^2} = \frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p})}{\partial (v^*)^2}
+ \frac{\partial^2 J_8^*(\tilde{v}^*, \tilde{v}_0^*, \tilde{p})}{\partial v^* \partial \tilde{v}_0^*} \frac{\partial \tilde{v}_0^*}{\partial v^*}.
\] (22)

As above indicated,
\[
\left(\frac{\tilde{v}^* + K \tilde{p}}{-\gamma \nabla^2 + 2\tilde{v}_0^* + K + \epsilon}\right)^2 - \frac{\tilde{v}_0^*}{\alpha} - \beta = 0
\]
Hence, taking the variation in \( v^* \) of such a last equation, we obtain

\[
\frac{2(\hat{v}^* + K\hat{p})}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon)^2} \frac{\partial v^*}{\partial v^*} - 4 \frac{\hat{v}^*}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon)^2} \frac{\partial^2 v^*}{\partial v^*} - \frac{1}{\alpha} \frac{\partial \hat{v}_0^*}{\partial v^*} = 0. \tag{23}
\]

so that

\[
\frac{2u_0}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon)} \frac{\partial v^*}{\partial v^*} - 4 \frac{(u_0)^2}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon)} \frac{\partial v^*}{\partial v^*} - \frac{1}{\alpha} \frac{\partial \hat{v}_0^*}{\partial v^*} = 0. \tag{24}
\]

so that

\[
\frac{\partial \hat{v}_0^*}{\partial v^*} = \frac{2\alpha u_0}{(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\hat{v}_0^* + K + \varepsilon)}. \tag{25}
\]

From this we have

\[
\frac{\partial^2 J^*_{\hat{v}}(\hat{v}^*, \hat{p})}{\partial (v^*)^2} = \frac{\partial^2 J^*_{\hat{v}}(\hat{v}^*, \hat{v}_0^*, \hat{p})}{\partial (v^*)^2} + \frac{\partial^2 J^*_{\hat{v}}(\hat{v}^*, \hat{v}_0^*, \hat{p})}{\partial v^* \partial v_0^*} \frac{\partial \hat{v}_0^*}{\partial v^*} = \frac{1}{\varepsilon} - \frac{1}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon)} \frac{2(\hat{v}^* + K\hat{p})}{2u_0} \frac{2\alpha u_0}{((-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon)^2) (-\gamma \nabla^2 + 4\alpha u_0^2 + 2\hat{v}_0^* + K + \varepsilon)}. \tag{25}
\]

Hence,

\[
\frac{\partial^2 J^*_{\hat{v}}(\hat{v}^*, \hat{p})}{\partial (v^*)^2} = \frac{1}{\varepsilon} - \frac{1}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon)} \frac{4\alpha u_0^2}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon)} \frac{1}{(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\hat{v}_0^* + K + \varepsilon)}. \tag{26}
\]

so that

\[
\frac{\partial^2 J^*_{\hat{v}}(\hat{v}^*, \hat{p})}{\partial (v^*)^2} = \frac{1}{\varepsilon} - \frac{1}{(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\hat{v}_0^* + K + \varepsilon)} \frac{1}{(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\hat{v}_0^* + K + \varepsilon)} \frac{1}{(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\hat{v}_0^* + K + \varepsilon)} > 0. \tag{27}
\]

Summarizing,

\[
\frac{\partial^2 J^*_{\hat{v}}(\hat{v}^*, \hat{p})}{\partial (v^*)^2} > 0. \tag{27}
\]
Finally,

\[
\frac{\partial^2 J^*_3(v^*)}{\partial(v^*)^2} = \frac{\partial^2 J^*(\hat{u}^*, \hat{v}_0^*, \hat{p})}{\partial(v^*)^2} \\
+ \frac{\partial^2 J^*(\hat{u}^*, \hat{v}_0^*, \hat{p})}{\partial v^* \partial \hat{v}_0^*} \frac{\partial \hat{v}_0^*}{\partial v^*} \\
+ \frac{\partial^2 J^*(\hat{u}^*, \hat{v}_0^*, \hat{p})}{\partial v^* \partial \hat{v}_0^*} \frac{\partial \hat{p}}{\partial v^*} \\
+ \frac{\partial^2 J^*(\hat{u}^*, \hat{v}_0^*, \hat{p})}{\partial v^* \partial \hat{p}} \frac{\partial \hat{v}_0^*}{\partial v^*} \\
= \frac{\partial^2 J^*_8(\hat{v}^*, \hat{v}_0^*, \hat{p})}{\partial(v^*)^2} \\
+ \frac{\partial^2 J^*(\hat{v}^*, \hat{v}_0^*, \hat{p})}{\partial v^* \partial \hat{v}_0^*} \frac{\partial \hat{v}_0^*}{\partial v^*} \\
+ \frac{\partial^2 J^*(\hat{v}^*, \hat{v}_0^*, \hat{p})}{\partial v^* \partial \hat{p}} \frac{\partial \hat{p}}{\partial v^*} \\
= \frac{\partial^2 J^*_8(\hat{v}^*, \hat{v}_0^*, \hat{p})}{\partial(v^*)^2} \\
+ \frac{\partial^2 J^*(\hat{v}^*, \hat{v}_0^*, \hat{p})}{\partial v^* \partial \hat{v}_0^*} \frac{\partial \hat{v}_0^*}{\partial v^*} \\
+ \frac{\partial^2 J^*(\hat{v}^*, \hat{v}_0^*, \hat{p})}{\partial v^* \partial \hat{p}} \frac{\partial \hat{p}}{\partial v^*}
\]

(28)

where from (27),

\[
\frac{\partial^2 J^*_8(\hat{v}^*, \hat{v}_0^*, \hat{p})}{\partial(v^*)^2} = \frac{1}{\varepsilon} \left( -\gamma \nabla^2 + 4\alpha u_0^2 + 2\hat{v}_0^* + K + \varepsilon \right) > 0.
\]

(29)

From this and (28) we obtain

\[
\frac{\partial^2 J^*_8(\hat{v}^*, \hat{p})}{\partial(v^*)^2} = \frac{1}{\varepsilon} \left( -\gamma \nabla^2 + 4\alpha u_0^2 + 2\hat{v}_0^* + K + \varepsilon \right) \left( \delta^2 J(u_0) + K + \varepsilon \right) \frac{1}{\partial v^*} \frac{\partial \hat{p}}{\partial \hat{p}} \\
- \frac{4K \alpha u_0^2}{(-\gamma \nabla^2 + 4\alpha u_0^2 + 2\hat{v}_0^* + K + \varepsilon)} \frac{1}{\partial \hat{v}_0^*} \frac{\partial \hat{v}_0^*}{\partial v^*} \\
- \frac{K}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon} \frac{\partial \hat{p}}{\partial v^*}.
\]

(30)

However, from the variation of \( J^* \) in \( p \) we have

\[
K \frac{\partial \hat{p}}{\partial v^*} - \frac{K(\hat{v}^* + K\hat{p})}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon} = 0,
\]

so that taking the variation in \( v^* \) of this last equation, we get

\[
K \frac{\partial \hat{p}}{\partial v^*} - \frac{K}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon} \frac{\partial \hat{v}_0^*}{\partial v^*} \\
- \frac{K^2}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon} \frac{\partial \hat{p}}{\partial v^*} \\
+ \frac{2(\hat{v}^* + K\hat{p})K}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon)^2} \frac{\partial \hat{v}_0^*}{\partial v^*} + \frac{\partial \hat{v}_0^*}{\partial p} \frac{\partial \hat{p}}{\partial v^*} = 0.
\]

(31)
so that

\[
K \frac{\partial \hat{p}}{\partial v^*} - \frac{K}{-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon} \frac{\partial \hat{p}}{\partial v^*} - \frac{K^2}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon) \partial v^*} + \frac{4\alpha K^2 u_0^2}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon) (\delta^2 J(u_0) + K + \varepsilon) \partial v^*} \\
+ \frac{4\alpha K u_0^2}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon) (\delta^2 J(u_0) + K + \varepsilon)} \\
= 0,
\]

(32)

Summarizing,

\[
\frac{\partial \hat{p}}{\partial v^*} = \frac{1}{(\delta^2 J(u_0) + K + \varepsilon)}
\]

so that, considering that \( K \gg \varepsilon \), we may obtain

\[
\frac{\partial^2 J^*(\hat{v}^*)}{\partial (v^*)^2} = \frac{1}{\varepsilon} - \frac{1}{(\delta^2 J(u_0) + K + \varepsilon)} \frac{4K\alpha u_0^2}{(-\gamma \nabla^2 + 2\hat{v}_0^* + K + \varepsilon) (\delta^2 J(u_0) + K + \varepsilon) (\delta^2 J(u_0) + \varepsilon)} \\
+ \frac{1}{(\delta^2 J(u_0) + K + \varepsilon)} \frac{K}{\varepsilon} - \frac{1}{(\delta^2 J(u_0) + \varepsilon)} \frac{1}{(\delta^2 J(u_0) + K + \varepsilon)} \\
= \frac{1}{\varepsilon} - \frac{1}{(\delta^2 J(u_0) + \varepsilon)} \frac{O\left(\frac{1}{\varepsilon}\right)}{> 0},
\]

(33)

in \( B_{r_3}(\hat{v}) \) for an appropriate not relabeled \( r_3 > 0 \), for a sufficiently small \( \varepsilon > 0 \).

From such results, we may infer that there exist not relabeled \( r_1, r_2, r_3 > 0 \) such that

\[
J(u_0) = \inf_{u \in B_{r_1}(u_0)} J(u) \\
= \inf_{v^* \in B_{r_3}(\hat{v}^*)} \left\{ \inf_{\hat{p} \in B_{r_2}(\hat{p})} \left\{ \sup_{\hat{v}_0^* \in \hat{v}^*} J^*(v^*, \hat{v}_0^*, \hat{p}) \right\} \right\} \\
= J^*(\hat{v}^*, \hat{v}_0^*, \hat{p}).
\]

(34)

Moreover,

\[
\delta J^*_3(\hat{v}^*) = 0 \\
\delta^2 J^*_3(\hat{v}^*) > 0
\]
so that
\[
J(u_0) = \inf_{u \in B_{r_1}(u_0)} J(u) = \inf_{v^* \in B_{r_3}(\hat{v}^*)} J^*(v^*) = J^*_5(\hat{v}^*).
\] (35)

The proof of the item (1a) is complete.
For the item (1b), suppose \( u_0 \in U \) is such that \( \delta J(u_0) = 0 \) and \( \delta^2 J(u_0) < 0 \).

Similarly as obtained above we may get
\[
\frac{\partial J^*_a(\hat{v}^*, \hat{p})}{\partial p^2} < 0,
\]
and
\[
\frac{\partial^2 J^*_a(\hat{v}^*)}{\partial (v^*)^2} > 0.
\]

Hence, there exist not relabeled real constants \( r_1, r_2, r_3 > 0 \) such that
\[
J(u_0) = \sup_{u \in B_{r_1}(u_0)} J(u) = \inf_{v^* \in B_{r_3}(\hat{v}^*)} J^*(v^*) = J^*_5(\hat{v}^*, \hat{v}_0^*, \hat{p}).
\] (36)

Moreover,
\[
\delta J^*_5(\hat{v}^*) = 0
\]
\[
\delta^2 J^*_5(\hat{v}^*) > 0
\]
so that
\[
J(u_0) = \sup_{u \in B_{r_1}(u_0)} J(u) = \inf_{v^* \in B_{r_3}(\hat{v}^*)} J^*_5(v^*) = J^*_5(\hat{v}^*).
\] (37)

The proof of the item (1b) is complete. For the item (1c) we recall that
\[
J^*_7 : Y^* \times Y \to \mathbb{R}
\]
is defined by
\[
J^*_7(v^*, p) = \sup_{v_0^* \in A^+ \cap B^*} J^*(v^*, v_0^*, p).
\]
Observe that through a direct computation we may obtain that the Hessian

$$\left\{ \frac{\partial^2 J^* (v^*, v_0^*, p)}{\partial v^* \partial p} \right\}$$

is positive definite in $Y^* \times (A^+ \cap B^*) \times Y$ so that $J^*_7$ is convex as the supremum of a family of convex functionals. Summarizing, we have got

$$\delta^2 J^*_7 (v^*, p) > 0$$

in $Y^* \times Y$.

From these results we may obtain

$$J^*_7 (\hat{v}^*, \hat{p}) = \inf_{(v^*, p) \in Y^* \times Y} J^*_7 (v^*, p)$$

$$= \inf_{(v^*, p) \in Y^* \times Y} \left\{ \sup_{v_0^* \in A^+ \cap B^*} J^* (v^*, v_0^*, p) \right\}$$

$$= J^* (\hat{v}^*, \hat{v}_0^*, \hat{p})$$

$$= J (u_0). \quad (38)$$

On the other hand

$$J (u_0) = J^* (\hat{v}^*, \hat{v}_0^*, \hat{p})$$

$$= -G^* (\hat{v}^*, \hat{v}_0^*, \hat{p}) - F^* (\hat{v}^*) + H (\hat{p})$$

$$= \inf_{(v^*, p) \in Y^* \times Y} \left\{ \sup_{v_0^* \in A^+ \cap B^*} J^* (v^*, v_0^*, p) \right\}$$

$$\leq \left\{ \sup_{v_0^* \in A^+ \cap B^*} \left\{ \frac{\gamma}{2} \int \nabla u \cdot \nabla u \, dx + \int v_0^* u^2 \, dx \right. \right.$$

$$- \frac{1}{2 \alpha} \int \nabla u \cdot \nabla u \, dx - \beta \int v_0^* \, dx$$

$$\left. \int K u^2 \, dx - \int K p u \, dx + \frac{K}{2} \int p^2 \, dx - \langle u, v^* \rangle_{L^2} + F^* (v^*) \right\} \right\}$$

$$\leq \left\{ \sup_{v_0^* \in Y^*} \left\{ \frac{\gamma}{2} \int \nabla u \cdot \nabla u \, dx + \int v_0^* u^2 \, dx \right. \right.$$

$$- \frac{1}{2 \alpha} \int \nabla u \cdot \nabla u \, dx - \beta \int v_0^* \, dx$$

$$\left. \frac{K + \varepsilon}{2} \int u^2 \, dx - \int K p u \, dx + \frac{K}{2} \int p^2 \, dx - \langle u, v^* \rangle_{L^2} + F^* (v^*) \right\} \right\}, \quad (39)$$

\forall u \in U, p \in Y, v^* \in Y^*.

From this, in particular for $v^* = \varepsilon u + f$ we may infer that

$$J (u_0) \leq \frac{\gamma}{2} \int \nabla u \cdot \nabla u \, dx$$

$$+ \frac{\alpha}{2} \int (u^2 - \beta)^2 \, dx + \frac{K}{2} \int (u - p)^2 \, dx$$

$$- \langle u, f \rangle_{L^2}$$

$$= J (u, p), \ \forall u \in U, p \in Y. \quad (40)$$
Consequently, from such a result and \((38)\) we may infer that
\[
J(u_0) = \inf_{u \in U} J(u)
\]
\[
= \inf_{(v^*, p) \in Y^* \times Y} \left\{ \sup_{v_0^* \in A^+ \cap B^*} J^*(v^*, v_0^*, p) \right\}
\]
\[
= J^*(\hat{v}^*, \hat{v}_0^*, \hat{p}).
\]
(41)
Moreover, considering as previously indicated, that \(J^*_7 : C^* \rightarrow \mathbb{R}\) is defined by
\[
J^*_7(v^*, p) = \left\{ \sup_{v_0^* \in A^+ \cap B^*} J^*(v^*, v_0^*, p) \right\}
\]
we get also
\[
\delta J^*_7(\hat{v}^*, \hat{p}) = 0
\]
\[
\delta^2 J^*_7(\hat{v}^*, \hat{p}) > 0
\]
so that
\[
J(u_0) = \inf_{u \in U} J(u)
\]
\[
= \inf_{(v^*, p) \in Y^* \times Y} J^*_7(v^*, p)
\]
\[
= J^*_7(\hat{v}^*, \hat{p}).
\]
(42)

The proof is complete.

\[\square\]

3 A criterion for global optimality

In this section we present a new concerning optimality criterion.

**Theorem 3.1.** Let \(\Omega \subset \mathbb{R}^3\) be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by \(\partial \Omega\).

Consider the functionals \(\hat{J} : U \times Y \rightarrow \mathbb{R}\) and \(J : U \rightarrow \mathbb{R}\) where
\[
J(u, p) = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx \\
+ \frac{K}{2} \int_{\Omega} (u - p)^2 \, dx - \langle u, f \rangle_{L^2},
\]
(43)

and
\[
J(u) = \hat{J}(u, u), \, \forall u \in U.
\]

where \(\alpha > 0, \beta > 0, \gamma > 0\) and \(f \in C^1(\overline{\Omega})\).

Assume either
\[
f(x) \geq 0, \, \forall x \in \overline{\Omega}
\]
or
\[
f(x) \leq 0, \, \forall x \in \overline{\Omega}.
\]
Suppose also, in a matrix sense

\[-\gamma \nabla^2 - 2\alpha\beta \leq 0,\]

assuming from now and on a finite dimensional approximation for the model in question, in a finite elements or finite differences context, even though the spaces, functionals and operators have not been relabeled.

Moreover define,

\[A^+ = \{ u \in U : uf \geq 0, \text{ in } \Omega \}\]

and

\[B^+ = \{ u \in U : \delta^2 J(u) \geq 0 \}.\]

Under such hypotheses,

\[\inf_{u \in U} J(u) = \inf_{u \in A^+} J(u).\]

Furthermore,

\[A^+ \cap B^+\]

is convex.

Proof. Define

\[\alpha_1 = \inf_{u \in U} J(u).\]

Let \(\varepsilon > 0\).

Thus we may obtain \(u_\varepsilon \in U\) such that

\[\alpha_1 \leq J(u_\varepsilon) < \alpha_1 + \varepsilon.\]

Define \(v_\varepsilon \in A^+\) by

\[v_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & \text{if } u_\varepsilon(x)f(x) \geq 0, \\ -u_\varepsilon(x), & \text{if } u_\varepsilon(x)f(x) < 0, \end{cases}\]

\(\forall x \in \Omega.\)

Observe that

\[J(v_\varepsilon) = \frac{\gamma}{2} \int_{\Omega} \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx + \frac{\alpha}{2} \int_{\Omega} (v_\varepsilon^2 - \beta)^2 \, dx - \langle v_\varepsilon, f \rangle_{L^2} \]

\[\leq \frac{\gamma}{2} \int_{\Omega} \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx + \frac{\alpha}{2} \int_{\Omega} (u_\varepsilon^2 - \beta)^2 \, dx - \langle u_\varepsilon, f \rangle_{L^2} = J(u_\varepsilon).\]

Hence

\[\alpha_1 \leq J(v_\varepsilon) \leq J(u_\varepsilon) < \alpha_1 + \varepsilon.\]

From this, since \(v_\varepsilon \in A^+\), we obtain

\[\alpha_1 \leq \inf_{u \in A^+} J(u) < \alpha_1 + \varepsilon.\]
Since \( \varepsilon > 0 \) is arbitrary, we may infer that
\[
\inf_{u \in U} J(u) = \alpha_1 = \inf_{u \in A^+} J(u).
\]

Finally, observe also that
\[
\delta^2 J(u) = -\gamma \nabla^2 + 6 \alpha u^2 - 2 \alpha \beta \geq 0,
\]
if, and only if
\[
H(u) \geq 0,
\]
where
\[
H(u) = \sqrt{6 \alpha |u|} - \sqrt{\gamma \nabla^2 + 2 \alpha \beta} \geq 0.
\]
Hence, if \( u_1, u_2 \in A^+ \cap B^+ \) and \( \lambda \in [0, 1] \), then
\[
H(|u_1|) \geq 0,
\]
\[
H(|u_2|) \geq 0
\]
and also since
\[
\text{sign } u_1 = \text{sign } u_2, \text{ in } \Omega,
\]
we get
\[
|\lambda u_1 + (1 - \lambda)u_2| = \lambda |u_1| + (1 - \lambda) |u_2|,
\]
so that,
\[
H(|\lambda u_1 + (1 - \lambda)u_2|) = H(\lambda |u_1| + (1 - \lambda) |u_2|) = \lambda H(|u_1|) + (1 - \lambda) H(|u_2|) \geq 0
\]
and thus,
\[
\delta^2 J(\lambda u_1 + (1 - \lambda)u_2) \geq 0.
\]
From this, we may infer that \( A^+ \cap B^+ \) is convex.

The proof is complete.

\[
\square
\]

4 Another related duality principle

In this subsection we develop a duality principle concerning the last optimality criterion established.

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^3 \) be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by \( \partial \Omega \).

Consider the functionals \( \hat{J} : U \times Y \to \mathbb{R} \) and \( J : U \to \mathbb{R} \) where
\[
J(u, p) = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx
+ \frac{K}{2} \int_{\Omega} (u - p)^2 \, dx - \langle u, f \rangle_{L^2},
\]
and
\[
J(u) = \hat{J}(u, u), \forall u \in U,
\]

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where $\alpha, \beta, \gamma$ are positive real constants, $U = W^{1,2}_0(\Omega)$, $f \in C^1(\Omega)$ and we also denote $Y = Y^* = L^2(\Omega)$.

Here we assume

$$-\gamma \nabla^2 - 2\alpha \beta \leq 0$$

in an appropriate matrix sense considering, as above indicated, a finite dimensional not relabeled model approximation, in a finite differences or finite elements context.

Assume also either

$$f(x) \geq 0, \forall x \in \bar{\Omega}$$

or

$$f(x) \leq 0, \forall x \in \bar{\Omega}.$$  

Define $G : U \times Y \to \mathbb{R}$ by

$$G(u, p) = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx + \frac{K + \varepsilon}{2} \int_{\Omega} u^2 \, dx - \langle u, Kp \rangle_{L^2}$$  

(47)

$F : U \to \mathbb{R}$ by

$$F(u) = \frac{\varepsilon}{2} \int_{\Omega} u^2 \, dx - \langle u, f \rangle_{L^2}$$

and $H : Y \to \mathbb{R}$ by

$$H(p) = \frac{K}{2} \int_{\Omega} p^2 \, dx.$$  

so that

$$\hat{J}(u, p) = G(u, p) - F(u) + H(p).$$

Furthermore, define $G^* : Y^* \times Y \to \mathbb{R}$ by

$$G^*(v^* + Kp) = \sup_{u \in U} \{ \langle u, v^* \rangle_{L^2} - G(u, p) \},$$

$F^* : Y^* \to \mathbb{R}$ by

$$F^*(v^*) = \sup_{u \in U} \{ \langle u, v^* \rangle_{L^2} - F(u) \} = \frac{1}{2\varepsilon} \int_{\Omega} (v^* - f)^2 \, dx.$$  

(48)

and $J^* : Y^* \times Y \to \mathbb{R}$ as

$$J^*(v^*, p) = -G^*(v^* + Kp) + F^*(v^*) + H(p).$$

Define also,

$$A^+ = \{ u \in U : uf \geq 0, \text{ in } \bar{\Omega} \},$$

$$B^+ = \{ u \in U : \delta^2 J(u) \geq 0 \},$$

$$E = A^+ \cap B^+,$$
Moreover, define
\[ \hat{v}_0^* = \alpha (u_0^2 - \beta), \]
\[ \hat{v}^* = \varepsilon u_0 + f, \]
\[ \hat{p} = u_0, \]
and assume \( u_0 \in U \) is such that \( \delta J(u_0) = 0 \), and
\[ u_0 \in E, \]
Under such hypothesis, assuming also \( \hat{v}_0^* \in B^* \) we have
\[ J(u_0) = \inf_{u \in E} J(u) \]
\[ = \inf_{u \in U} J(u) \]
\[ = \inf_{(v^*, p) \in Y^* \times Y} J^*(v^*, p) \]
\[ = J^*(\hat{v}^*, \hat{p}). \] (49)

Proof. Define
\[ \alpha_1 = \inf_{u \in U} J(u). \]
Hence
\[ \alpha_1 \leq J(u, p) \]
\[ = G(u, p) - F(u) + H(p) \]
\[ \leq -\langle u, v^* \rangle_{L^2} + G(u, p) + H(p) \]
\[ + \sup_{u \in U} \{\langle u, v^* \rangle_{L^2} - F(u)\} \]
\[ = -\langle u, v^* \rangle_{L^2} + G(u, p) + H(p) + F^*(v^*) \] (50)
\[ \forall u \in U, v^* \in Y^*, p \in Y. \]
Thus,
\[ \alpha_1 \leq \inf_{u \in U} \{ -\langle u, v^* \rangle_{L^2} + G(u, p) \} + H(p) + F^*(v^*) \]
\[ = G^*(v^* + Kp) + F^*(v^*) + H(p) \] (51)
\[ \forall v^* \in Y^*, p \in Y. \]
Summarizing
\[ \alpha_1 = \inf_{u \in U} J(u) \leq \inf_{(v^*, p) \in Y^* \times Y} J^*(v^*, p). \] (52)
From Theorem 3.1 we have that
\[ \alpha_1 = J(u_0) = \inf_{u \in U} J(u) = \inf_{u \in E} J(u). \]
Similarly as in the proof of Theorem 2.1 we may obtain
\[ \delta J^*(\hat{v}^*, p) = 0 \]
and
\[ J^*(\hat{v}^*, \hat{p}) = \hat{J}(u_0, \hat{p}) = \hat{J}(u_0, u_0) = J(u_0). \]

From this and (52) we may infer that
\[
J(u_0) = \inf_{u \in E} J(u) = \inf_{u \in U} J(u) = \inf_{(v^*, p) \in Y^* \times Y} J^*(v^*, p) = J^*(\hat{v}^*, \hat{p}).
\]

(53)

The proof is complete.

5 A convex dual variational formulation

Let \( \Omega \) be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by \( \partial \Omega \).

In this section we define \( G : U \to \mathbb{R} \) by
\[
G(u) = \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{K}{2} \int_\Omega u^2 \, dx - \langle u, f \rangle_{L^2},
\]
and \( F : U \to \mathbb{R} \) by
\[
F(u) = -\frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx + \frac{K}{2} \int_\Omega u^2 \, dx,
\]
where \( \alpha, \beta, \gamma > 0, f \in L^2(\Omega) \) and \( U = W^{1,2}_0(\Omega) \).

Moreover we define
\[ U_1 = \{ u \in U : ||u||_{\infty} \leq \sqrt[4]{K} \}, \]
where \( K > 0 \) is such that \( G \) and \( F \) are convex in \( U_1 \).

Define also
\[ B^+ = \{ u \in U : \delta^2 J(u) \geq 0 \} \]
where \( J : U \to \mathbb{R} \) is given by
\[
J(u) = G(u) - F(u) = \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}.
\]

(56)

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Finally, define
\[ D^* = \{v^* \in Y^* = L^2(\Omega) : \|v^*\|_\infty \leq 3K\}, \]

\[ G^* : D^* \to \mathbb{R} \text{ by} \]
\[ G^*(v^*) = \sup_{u \in U_1} \{\langle u, v^* \rangle_{L^2} - G(u)\} = \frac{1}{2} \int_\Omega \frac{(v^* + f)^2}{(-\gamma \nabla^2 + K)} \, dx, \]  
\[ (57) \]

\[ F^* : D^* \to \mathbb{R} \text{ by} \]
\[ F^*(v^*) = \sup_{u \in U_1 \cap B^+} \{\langle u, v^* \rangle_{L^2} - F(u)\} \]

and \( J^* : D^* \to \mathbb{R} \) by
\[ J^*(v^*) = -G^*(v^*) + F^*(v^*) \]

Assume now either
\[ f(x) > 0, \forall x \in \Omega \]
or
\[ f(x) < 0, \forall x \in \Omega. \]

Define
\[ A^+ = \{u \in U_1 : u f > 0 \text{ in } \Omega\} \]

and define also
\[ D_1^* = \left\{v^* \in D^* : \hat{u} = \frac{\partial F^*(v^*)}{\partial v^*} \in A^+\right\}, \]

where
\[ F_1^*(v^*) = \sup_{u \in U} \{\langle u, v^* \rangle_{L^2} - F(u)\}. \]

**Theorem 5.1.** Under the hypotheses state above \( J^* \) is convex on \( D_1^* \).

**Proof.** Let \( v^* \in D_1^* \).

Thus
\[ F^*(v^*) = \sup_{u \in U_1 \cap B^+} \{\langle u, v^* \rangle_{L^2} - F(u)\} = \sup_{u \in U} \{\langle u, v^* \rangle_{L^2} - F(u)\} \]
\[ + \gamma \int_\Omega \nabla \varphi \cdot \nabla \varphi \, dx + 6\alpha \int_\Omega u^2 \varphi^2 \, dx \]
\[ - 2\alpha \beta \int_\Omega \varphi^2 \, dx - \int_\Omega \varphi_1^2(u^2 - \sqrt{2K}) \, dx, \]
\[ (58) \]

for some appropriate Lagrange multipliers \((\varphi, \varphi_1) \in W^{1,2}(\Omega) \times L^2(\Omega)\).

The last supremum is attained for some \( \hat{u} \in U \) such that
\[ v^* - \frac{\partial F(\hat{u})}{\partial u} + \varphi(12)\alpha \hat{u} - \varphi_1^2(2\hat{u}) = 0, \quad \text{in } \Omega. \]
Taking the variation in $v^*$ in this last equation, we get
\[
\frac{\partial v^*}{\partial v^*} - \frac{\partial^2 F(\hat{u})}{\partial u^2} \frac{\partial \hat{u}}{\partial v^*} + \varphi^2(12\alpha) \frac{\partial \hat{u}}{\partial v^*} - \varphi^2_{12} \frac{\partial \hat{u}}{\partial v^*} + 24\alpha \varphi \partial_{v^*} \varphi \hat{u} - 12\varphi_1 \partial_{v^*} \varphi_1 \hat{u} = 0, \quad \text{in } \Omega.
\]
(59)

On the other hand we have the following necessary condition to be satisfied
\[
\gamma \int_{\Omega} \nabla \varphi \cdot \nabla \varphi \, dx + 6\alpha \int_{\Omega} u^2 \varphi^2 \, dx - 2\alpha \beta \int_{\Omega} \varphi^2 \, dx = 0,
\]
(60)
so that
\[
2\varphi \partial_{u^*}(\varphi^2 - 6\alpha \hat{u}^2 - 2\alpha \beta) + \varphi^2(12\alpha \hat{u}) = 0
\]
in $\Omega$ so that
\[
\varphi^2(12\alpha \hat{u}) = 0 \text{ in } \Omega.
\]

And also, we must have
\[
\int_{\Omega} \varphi^2_{12}(u^2 - \sqrt{K}) \, dx = 0,
\]
so that
\[
2\varphi_1 \partial_{u^*}(\varphi^2 - \sqrt{K}) + 2\varphi^2_1 \hat{u} = 0, \quad \text{in } \Omega,
\]
and thus
\[
2\varphi^2_1 \hat{u} = 0, \quad \text{in } \Omega.
\]
Since $v^* \in D^*$ we may assume $\varphi_1 = 0$ and thus
\[
\frac{\partial v^*}{\partial v^*} - \frac{\partial^2 F(\hat{u})}{\partial u^2} \frac{\partial \hat{u}}{\partial v^*} + \varphi^2(12\alpha) \frac{\partial \hat{u}}{\partial v^*} - \varphi^2_{12} \frac{\partial \hat{u}}{\partial v^*} + 24\alpha \varphi \partial_{v^*} \varphi \hat{u} - 12\varphi_1 \partial_{v^*} \varphi_1 \hat{u} = 1 - \frac{\partial^2 F(\hat{u})}{\partial u^2} \frac{\partial \hat{u}}{\partial v^*} + \varphi^2(12\alpha) \frac{\partial \hat{u}}{\partial v^*} = 0, \quad \text{in } \Omega.
\]
(61)

Therefore,
\[
\frac{\partial \hat{u}}{\partial v^*} = \frac{1}{\frac{\partial^2 F(\hat{u})}{\partial u^2} - 12\alpha \varphi^2 - 6\alpha \hat{u}^2 + 2\alpha \beta - 12\alpha \varphi^2 + K} > 0.
\]
(62)

On the other hand
\[ F^*(v^*) = \langle \hat{u}, v^* \rangle_{L^2} - F(\hat{u}) \]
\[ + \int_{\Omega} \nabla \varphi \cdot \nabla \varphi \, dx + \int_{\Omega} 6\alpha \hat{u}^2 \varphi^2 \, dx \]
\[ -2\alpha \beta \int_{\Omega} \varphi^2 \, dx - \int_{\Omega} \varphi_1^2 (\hat{u}^2 - \sqrt{K}) \, dx, \]  
(63)

Hence

\[ \frac{\partial F^*(v^*)}{\partial v^*} = \hat{u} + \left(v^* - \frac{\partial F(\hat{u})}{\partial u} \right) \]
\[ + \varphi^2 12\alpha \hat{u} - \varphi_1^2 (2\hat{u}) \]
\[ 2\varphi \partial_1 \varphi (-\gamma \nabla^2 + 6\alpha \hat{u}^2 - 2\alpha \beta) \]
\[ -2\varphi_1 \partial_1 \varphi_1 (\hat{u}^2 - \sqrt{K}) \]
\[ = \hat{u}. \]  
(64)

Therefore, we may infer that

\[ \frac{\partial^2 F^*(v^*)}{\partial (v^*)^2} = \frac{\partial \hat{u}}{\partial v^*} \]
\[ = \frac{1}{-6\alpha \hat{u}^2 + 2\alpha \beta - 12\alpha \varphi^2 + K} \]
\[ > 0. \]  
(65)

Thus,

\[ \frac{\partial^2 J^*(v^*)}{\partial (v^*)^2} = - \frac{\partial^2 G^*(v^*)}{\partial (v^*)^2} + \frac{\partial^2 F^*(v^*)}{\partial (v^*)^2} \]
\[ = - \frac{1}{-\gamma \nabla^2 + K} + \frac{1}{-6\alpha \hat{u}^2 + 2\alpha \beta - 12\alpha \varphi^2 + K} \]
\[ = \frac{(-\gamma \nabla^2 + K)(-6\alpha \hat{u}^2 + 2\alpha \beta - 12\alpha \varphi^2 + K)}{(-\gamma \nabla^2 + K)(-6\alpha \hat{u}^2 + 2\alpha \beta - 12\alpha \varphi^2 + K)} \]
\[ = \frac{\delta^2 J(\hat{u})}{\delta \varphi^2} + 12\alpha \varphi^2 \]
\[ > 0, \forall v^* \in D_1^* \]  
(66)

From this we may infer that \( J^* \) is convex on \( D_1^* \).

\[ \square \]

In the next lines we present our main result.

**Theorem 5.2.** Let \( \hat{v}^* \in D_1^* \) be such that

\[ \delta J^*(\hat{v}^*) = 0. \]
Assume either
\[ f(x) > 0, \forall x \in \Omega \]
or
\[ f(x) < 0, \forall x \in \Omega. \]

Define
\[ A^+ = \{ u \in U_1 : u f > 0 \text{ in } \Omega \} \]
and
\[ u_0 = (-\gamma \nabla^2 + K)^{-1} \hat{v}^* \]

Assume also
\[ u_0 \in A^+ \cap B^+ \]
and recall that
\[ D_1^* = \left\{ v^* \in D^* : \hat{u} = \frac{\partial F_1^*(v^*)}{\partial v^*} \in A^+ \right\}. \]

Under such hypotheses

\[
J(u_0) = \inf_{u \in U_1} J(u) \\
= \inf_{v^* \in D_1^*} J^*(v^*) \\
= J^*(\hat{v}^*). \tag{67}
\]

**Proof.** From the last theorem \( J^* \) is convex in \( D_1^* \) so that
\[ J^*(\hat{v}^*) = \inf_{v^* \in D_1^*} J^*(v^*). \]

Therefore,
\[
J^*(\hat{v}^*) \leq J^*(v^*) \\
= -G^*(v^*) + F^*(v^*) \\
\leq -\langle u, v^* \rangle_{L^2} + G(u) + F_1^*(v^*) \forall u \in U_1, v^* \in D_1^*. \tag{68}
\]

Hence
\[
J^*(\hat{v}^*) \leq \inf_{v^* \in D_1^*} \{ -\langle u, v^* \rangle_{L^2} + G(u) + F_1^*(v^*) \} \\
= G(u) - F(u) \\
= J(u), \forall u \in A^+. \tag{69}
\]

Similarly as in the previous theorems proofs we may obtain
\[
\inf_{u \in U_1} J(u) = \inf_{u \in A^+} J(u) \geq \inf_{v^* \in D_1^*} J^*(v^*). \tag{70}
\]

On the other hand, also similarly as the proofs of the previous theorems we may obtain
\[ \delta J(u_0) = 0 \]

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and
\[ J(u_0) = J^*(\hat{v}^*). \]

From this and (70) we may infer that
\[ J(u_0) = \inf_{u \in U_1} J(u) \]
\[ = \inf_{v^* \in D^*_1} J^*(v^*) \]
\[ = J^*(\hat{v}^*). \]
(71)

The proof is complete. \(\square\)

6 A final dual variational formulation

This final duality principle is summarized by the following theorem.

**Theorem 6.1.** Let \( U, Y \) be a Banach spaces such that \( Y = Y^* \) and let \( \Lambda : U \to Y \) be a bounded linear operator.

Consider the functional \( J : U \to \mathbb{R} \) expressed by
\[ J(u) = G_K(\Lambda u) - F(\Lambda u) - \langle u, f \rangle_U, \]
where \( G_K : Y \to \mathbb{R} \) is defined by \( G_K(\Lambda u) = G(\Lambda u) + \frac{K}{2} \langle \Lambda u, \Lambda u \rangle_Y \) where \( G : Y \to \mathbb{R} \) is a coercive, Fréchet differentiable and possibly non-convex functional. Moreover \( f \in U^* \) and \( F : Y \to \mathbb{R} \) is such that
\[ F(\Lambda u) = \frac{K}{2} \langle \Lambda u, \Lambda u \rangle_Y, \]
so that
\[ J(u) = G(\Lambda u) - \langle u, f \rangle_U. \]

Assume
\[ \inf_{u \in U} J(u) = \alpha \in \mathbb{R} \]
and \( K > 0 \) is such that \( G_K \) is convex.

Define the polar functionals \( G^*_K : Y^* \to \mathbb{R} \) and \( F^* : Y^* \to \mathbb{R} \) by
\[ G^*_K(v^* + z^*) = \sup_{v \in Y} \{ \langle v, v^* + z^* \rangle_Y - G_K(v) \}, \]
and
\[ F^*(z^*) = \sup_{v \in Y} \{ \langle v, z^* \rangle_Y - F(v) \}, \]
respectively

Define also \( A^* = \{ v^* \in Y^* : \Lambda^* v^* - f = 0 \} \)
\[ J^*(v^*, z^*) = -G^*_K(v^* + z^*) + F^*(z^*) \]
Suppose \( (u_0, v_0^*, z_0^*) \in U \times Y^* \times Y^* \) is such that
\[ \delta(J^*(v_0^*, z_0^*) + \langle u_0, \Lambda^* v_0^* - f \rangle_U) = 0. \]
Under such hypotheses, we have

\[ \delta J(u_0) = 0 \]

and

\[
J(u_0) = \min_{u \in U} \left\{ J(u) + \frac{K}{2} \langle \Lambda u - \Lambda u_0, \Lambda u - \Lambda u_0 \rangle \right\}
\]

\[
= \sup_{v^* \in A^*} \{ J^*(v^*, z_0^*) \}
\]

\[
= J^*(v_0^*, z_0^*). \tag{72}
\]

**Proof.** Observe that from the variation of \( J^* \) in \( u \) we obtain

\[ \Lambda^* v_0^* - f = 0 \]

so that \( v_0^* \in A^* \).

Moreover from the variation of \( J^* \) in \( v^* \) we have

\[ \frac{\partial G^*_K(v_0^* + z_0^*)}{\partial v^*} = \Lambda u_0. \]

Also, from the variation of \( J^* \) in \( z^* \) we have

\[ -\frac{\partial G^*_K(v_0^* + z_0^*)}{\partial z^*} + \frac{\partial F^*(z_0^*)}{\partial z^*} = 0. \]

Therefore

\[ \Lambda u_0 = \frac{\partial F^*(z_0^*)}{\partial z^*}, \]

so that from the Legendre transform properties

\[ z_0^* = \frac{\partial F(\Lambda u_0)}{\partial v} = K \lambda u_0 \]

where \( v = \Lambda u \). Hence,

\[ F^*(z_0^*) = \langle \Lambda u_0, z_0^* \rangle - F(\Lambda u_0) = \frac{K}{2} \langle \Lambda u_0, \Lambda u_0 \rangle. \]

Also from the Legendre transform properties we may obtain

\[ v_0^* + z_0^* = \frac{\partial G_K(\Lambda u_0)}{\partial v}, \]

so that

\[
\begin{align*}
v_0^* &= \frac{\partial G_K(\Lambda u_0)}{\partial v} - z_0^* \\
&= \frac{\partial G_K(\Lambda u_0)}{\partial v} - K \lambda u_0 \\
&= \frac{\partial G(\Lambda u_0)}{\partial v}. \tag{73}
\end{align*}
\]
From this and 
\[ \Lambda^* v_0^* - f = 0 \]
we have
\[ \Lambda^* \left( \frac{\partial G(\Lambda u_0)}{\partial v} \right) - f = 0, \]
that is
\[ \delta J(u_0) = 0. \]

Once more through the Legendre transform properties, we get
\[ G^*_K(v_0^* + z_0^*) = \langle \Lambda u_0, v_0^* + z_0^* \rangle_Y - G_K(\Lambda u_0), \]
and
\[ F^*(z_0^*) = \langle \Lambda u_0, z_0^* \rangle_Y - F(\Lambda u_0), \]
so that
\[ J^*(v_0^*, z_0^*) = -G^*_K(v_0^* + z_0^*) + F^*(z_0^*) \]
\[ = -(u_0, \Lambda^* v_0^*)_U + G_K(\Lambda u_0) - F(\Lambda u_0) \]
\[ = -(u_0, f)_U + G(\Lambda u_0) \]
\[ = J(u_0). \quad (74) \]

Moreover, we have
\[ J^*(v_0^*, z_0^*) \]
\[ \leq \inf_{u \in U} \left\{ J(u) + K\langle \Lambda u - \Lambda u_0, \Lambda u - \Lambda u_0 \rangle_Y \right\} \quad (75) \]

Summarizing, we have got
\[ J^*(v_0^*, z_0^*) \]
\[ \leq \inf_{u \in U} \left\{ J(u) + \frac{K}{2} \langle \Lambda u - \Lambda u_0, \Lambda u - \Lambda u_0 \rangle_Y \right\} \quad (76) \]

Therefore, from
\[ \delta J(u_0) = 0, \]
\[ J(u_0) = J^*(v_0^*, z_0^*), \]
\[ 28 \]
from (76) and the concavity of $J^*$ in $v^*$, we have

$$
J(u_0) = \min_{u \in U} \left\{ J(u) + \frac{K}{2} \langle \Lambda u - \Lambda u_0, \Lambda u - \Lambda u_0 \rangle_Y \right\}
$$

$$
= \sup_{v^* \in A^*} J^*(v^*, z_0^*)
$$

$$
= J^*(v_0^*, z_0^*). \tag{77}
$$

The proof is complete. \qed

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